L₁ LIMIT SOLUTIONS FOR CONTROL SYSTEMS

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ABSTRACT. For a control Cauchy problem
\[ \dot{x} = f(t, x, u, v) + \sum_{\alpha=1}^{m} g_{\alpha}(x) \dot{u}_{\alpha}, \quad x(a) = \bar{x}, \]
on an interval \([a, b]\), we propose the notion of limit solution \(x\) that verifies the following properties: i) \(x\) is defined for \(L^1\) (impulsive) inputs \(u\) and for standard, bounded measurable, controls \(v\); ii) in the commutative case (i.e. when \([g_{\alpha}, g_{\beta}] \equiv 0\), for all \(\alpha, \beta = 1, \ldots, m\), \(x\) coincides with the solution constructed via multiple fields’ rectification; iii) \(x\) subsumes former concepts of solution valid for the generic, noncommutative case. In particular, when \(u\) has bounded variation, we investigate the relation between limit solutions and (single-valued) graph completion solutions. Furthermore, we prove consistency with the classical Carathéodory solution when \(u\) and \(x\) are absolutely continuous.

Even though some specific problems are better addressed by means of special representations of the solutions, we believe that various theoretical and practical issues call for a unified notion of trajectory. For instance, this is the case of optimal control problems, possibly with state and end-point constraints, for which no extra assumptions (like e.g. coercivity, boundedness, commutativity) are made in advance.

1. INTRODUCTION

1.1. Limit solutions. Because of the presence of the derivative \(\dot{u}\), control systems of the form
\[ \dot{x} = f(t, x, u, v) + \sum_{\alpha=1}^{m} g_{\alpha}(x) \dot{u}_{\alpha}, \quad t \in [a, b], \]
\[ x(a) = \bar{x} \]
are sometimes called impulsive. This is connected with the need of implementing non regular, possibly discontinuous, controls \(u\) – a need raised e.g.

This version does not contain proofs. All the detailed proofs are included in the original version of the article, that is under review.

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by lack of coercivity (in $\dot{u}$) in a minimum problem. Actually, the issue of giving a notion of solution for a control system like (E) becomes non standard as soon as $u$ is not absolutely continuous (while $v$ can be an ordinary bounded measurable control). As is well-known, the core of the question resides in the interaction between the $x$-dependence of the vector fields $g_1, \ldots, g_m$ and the unboundedness of the derivative $\dot{u}$. A first difficulty, occurring already when $u$ is scalar-valued, is due to the fact the trajectory $x$ is expected to jump at atoms $\tau$ of $\dot{u}$, so depriving the expression $\sum g_\alpha(x(\tau))\dot{u}_\alpha(\tau)$ of a standard meaning of measure, even if the (possibly discontinuous) control $u$ has bounded variation. A second, crucial, complication shows up as soon as $u$ is vector-valued and the Lie algebra generated by $\{g_1, \ldots, g_m\}$ is non trivial.

Let us point out that several physical control settings ([11, 28, 19, 6, 12, 15]) are naturally modelled by an equation like (E) –possibly, as in some mechanical applications, (E) being a first order reduction of a higher order equation.

Notions of output are well-established at least in the following two situations:

(i) The case of commutative systems, characterized by the triviality of the associated Lie algebra. This means that $[g_\alpha, g_\beta] \equiv 0$ for all $\alpha, \beta = 1, \ldots, m$ (see e.g. [8, 32, 14]). Actually, under such hypothesis, a notion of solution to (E)-(IC) has been established for controls $u \in L^1$, possibly with unbounded variation. These solutions, whose definition essentially relies on a state space diffeomorphism induced by the family of vector fields $\{g_1, \ldots, g_m\}$, are pointwise defined and verify nice properties of well-posedness.

(ii) The case when the controls $u$ have bounded variation (and the Lie algebra is allowed to be non trivial). In order to verify standard robustness properties, this notion of solution requires a specification on how the discontinuities of $u$ are bridged. This type of solutions (see e.g. [31, 7, 13, 33, 27]), which are not single-valued at the jump instants, are described by different authors in fairly equivalent ways, and will be called here graph completion solutions. A characterization of graph completion solutions was also provided in [33], where it is shown equivalence of the latter with robust solutions for a class of differential inclusions with scalar impulsive control.

To our knowledge, no available definition of solution is applicable to the union of the above-described subclasses of equations. On the other hand, let us point out that a general notion of solution (including the above cases but not limited to them) is of interest for those questions where specific properties (of either the controls or the vector fields) are not known a priori.

\[\text{But a single-valued version is here proposed, via selections of multi-valued maps.}\]
The main objective of the present paper consists in the discussion of a unified concept of solution to (E)-(IC). In fact, we will adopt a density characterization of the output for the commutative case as a notion of solution for the general case. This concept will be denominated “limit solution”.

Let us give immediately the notion of limit solution, postponing the precise specification of the regularity and growth assumptions on the vector fields to the end of the current section.

**Definition 1.1.** Consider an initial value \( \bar{x} \in \mathbb{R}^n \), controls \( u \in L^1([a,b]; U) \) and \( v \in L^1([a,b]; V) \).

1. **(Limit Solution)** We say that a map \( x : [a,b] \to \mathbb{R}^n \) is a limit solution of the Cauchy problem (E)-(IC) corresponding to \( \bar{x} \) and \( (u,v) \) if, for every \( \tau \in [a,b] \), there exists a sequence of absolutely continuous controls \( (u^\tau_k) \) from \( [a,b] \) into \( U \) such that:
   - (i) for each \( k \in \mathbb{N} \), there exists a (Carathéodory) solution \( x^\tau_k : [a,b] \to \mathbb{R}^n \) of (E)-(IC) corresponding to the initial value \( \bar{x} \) and the control \( (u^\tau_k, v) \), and the sequence \( (x^\tau_k) \) is bounded;
   - (ii) one has
     \[
     |(x^\tau_k; u^\tau_k)(\tau) - (x,u)(\tau)| + \|(x^\tau_k, u^\tau_k) - (x,u)\|_1 \to 0.
     \]

2. **(Simple Limit Solution)** We say that a limit solution \( x : [a,b] \to \mathbb{R}^n \) is simple if \( (u^\tau_k) \) can be chosen independently of \( \tau \). In this case we write \( (u_k) \) to refer to the approximation sequence.

3. **(BV Simple Limit Solution)** We say that a simple limit solution \( x : [a,b] \to \mathbb{R}^n \) is a BV (bounded variation) simple limit solution if the approximating inputs \( u_k \) have equibounded variation.

The idea of defining control-trajectories pairs as limits of regular trajectories is obviously not new (we refer to e.g. [8, 35, 32] and the bibliographies therein). If attention is confined to the case of continuous inputs \( u \), maybe the subtler instrument of investigation is represented by T. Lyons’ rough paths ([17, 18]), which -via iterated integrals- are introduced as metric extensions of continuous input-output maps (see 6.2).

Let us point out that the topologies underlying our definition of limit solutions are not metrizable. Furthermore, the fact that in the definition of limit solution the approximating sequences \( (u^\tau_k) \) do depend on the considered time \( \tau \) is essential if we aim to a notion of everywhere defined solution corresponding to \( L^1 \)-controls \( u \) (see Remark 2.1 below).

Actually, our primary aim is not uniqueness, since, in the presence of \( u \)-jumps and unless the Lie algebra generated by the \( g_\alpha \)'s is trivial, an irremediable multiplicity of solutions has to be considered. Nevertheless, our definition meet the following properties:

a) **(Generality)** The notion of limit solution subsumes (at least) the two concepts of solutions described above.
b) **(Representability of BV simple limit solutions as graph completion solutions)** When $u$ has bounded variation, the notions of graph completion solution (in its single-valued version, see Definition 4.5) and that of BV simple limit solution turn out to be equivalent.

c) **(Consistency with the classical case)** As we have already remarked, if $u$ is absolutely continuous, any absolutely continuous BV simple limit solution $x$ corresponding to $u$ and to a given $v$ coincides with the classical Carathéodory solution (Theorem 2.2).

As for the generality of the considered system, it is worth reminding a trivial but important fact: a system of the form

$$
\dot{x} = f(t, x, u, v) + \sum_{\alpha=1}^{m} g_{\alpha}(x, u) \dot{u}_{\alpha},
$$

where the maps $g_{\alpha}$ depend on $u$ as well, can be reduced to an equation like (E) by adding extra state variables $x_{n+1}, \ldots, x_{n+m}$ with dynamics $\dot{x}_{n+\alpha} = \dot{u}_{\alpha}$, for $\alpha = 1, \ldots, m$. (This is also suggested by some applications with second order equations, like in mechanics –see e.g. [11, 19]– where one regards $u$ in the double role of input and output).

Let us conclude this introduction with a structural outline of the paper. In Section 2 we first discuss properties of uniqueness and consistency (with Carathéodory solutions). Existence and uniqueness are then proved in the commutative case. Finally, the existence of BV simple limit solutions associated to controls with bounded variation is stated (the proof being provided in Section 5).

In Section 3 we address the notion of limit solution in the case where the vector fields $g_{1}, \ldots, g_{m}$ commute (see [8, 2, 1], and also [14, 32]). We observe that, unlike the noncommutative case, uniqueness and continuous dependence is obtained also for discontinuous, $L^{1}$ controls $u$. Let us remark, however, that if the state space were restricted to an open subset $\Omega \subset \mathbb{R}^{n}$, (or to a submanifold) the mere local commutativity –namely the vanishing of the Lie brackets– would not be enough for uniqueness (see [1]).

Section 4 deals with the noncommutative case. After introducing a single-valued version of the notion of graph completion solution, we prove a result which shows how this concept can be embedded in the general notion of limit solution. Precisely: a trajectory $x$ is a (single-valued) graph completion solution if and only if it coincides pointwise with a BV simple limit solution (Theorem 4.3). By proving that for each $u$ with bounded variation a single-valued graph completion solution actually exists, one gets the existence result of BV simple limit solutions stated in Section 2. This is done in Section 5. Incidentally, as a byproduct, one obtains that every function with bounded variation (and values on a set with the Whitney property) can be pointwise approximated by means of (absolutely) continuous functions with equibounded variation (see Theorem 2.7).
In Section 6 we briefly illustrate some open issues concerning, in particular, the case where the vector fields $g_\alpha$ depend on $v$ as well, and the case where these fields are not differentiable.

1.2. **Some notation.** Let $I, E$ be a closed interval and a subset of the Euclidean space $\mathbb{R}^d$, respectively. For any $L \in [0, +\infty[$, we let $BV_L(I; E)$ be the set of functions $h : I \to E$ with total variation bounded by $L$, and we set $BV(I; E) = \bigcup_{L \geq 0} BV_L(I; E)$. We shall write $\text{Var}_I(h)$ to refer to the variation of $h$ in the interval $I$. Moreover, we use $\mathcal{L}^1([a, b]; E)$ to denote the set of pointwise defined Lebesgue integrable functions from $I$ to $\mathbb{R}^d$ with values in $E$, while $L^1([a, b]; E)$ will denote the corresponding family of equivalence classes (with respect to the Lebesgue measure). With $AC(I; E)$ we refer to the set of absolutely continuous maps from $I$ to $E$, and we let $AC_L(I; E) \subset AC(I; E)$ represent the subset made of the absolutely continuous functions having variation bounded by $L$, namely $AC_L(I; E) := AC(I; E) \cap BV_L(I; E)$. The notation $\text{Lip}(I; E)$ will indicate the set of Lipschitz continuous functions from $I$ to $E$. $\mathcal{C}^k(\mathbb{R}^n; \mathbb{R}^d)$ will denote the space of $k$-times continuously differentiable functions defined on $\mathbb{R}^n$, with values in $\mathbb{R}^d$. The elements in $\mathcal{C}^k(\mathbb{R}^n; \mathbb{R}^d)$ will be sometimes called *functions of class $\mathcal{C}^k$*.

We will say that a real function $\varphi_0 : [a, b] \to \mathbb{R}$ is *increasing* (respectively, *decreasing*) if for every pair $t_1, t_2 \in [a, b]$ verifying $t_1 < t_2$, one has $\varphi_0(t_1) \leq \varphi_0(t_2)$ (respectively, $\varphi_0(t_1) \geq \varphi_0(t_2)$). We use strict *increasing* (respectively, *strictly decreasing*) when the corresponding inequality is strict.

1.3. **Structural hypotheses.** Throughout the paper we shall assume the following hypotheses on the control sets $U, V$ and the functions $f, g_1, \ldots, g_m$:

(i) $U$ is a compact subset of $\mathbb{R}^m$ such that, for every bounded interval $I \subset \mathbb{R}$, for each $\tau \in I$, and for every function $u \in \mathcal{L}^1(I; U)$, there exists a sequence $(u^k_\tau) \subset AC(I; U)$ such that

$$|u^k_\tau(\tau) - u(\tau)| + \|u^k_\tau - u\|_1 \to 0, \quad \text{when } k \to \infty.$$ 

(Convex sets verify this hypothesis. Furthermore, it is not difficult to show that also the closure of any open, bounded subset with a Lipschitz boundary meets (i)).

(ii) The set $V \subset \mathbb{R}^d$ is compact.

(iii) For each $(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times V$, the map $t \mapsto f(t, x, u, v)$ is measurable on $[a, b]$; for each $t \in [a, b]$, the map $(x, u, v) \mapsto f(t, x, u, v)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m \times V$ and, moreover, the map

$$(x, u) \mapsto f(t, x, u, v)$$

is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$, uniformly in $(t, v) \in [a, b] \times V$.

(iv) The vector fields $g_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ are of class $\mathcal{C}^1$.

(v) There exists $M > 0$ such that

$$\left|\left(f(t, x, u, v), g_1(x), \ldots, g_m(x)\right)\right| \leq M(1 + |(x, u)|),$$

for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ uniformly in $(t, v) \in [a, b] \times V$. 
Notice that, under conditions (ii)-(v) above, for any initial value \( \bar{x} \in \mathbb{R}^n \) and each control pair \((u, v) \in AC([a, b]; U) \times L^1([a, b]; V)\), there exists a unique Carathéodory solution to the Cauchy problem \((E)-(IC)\). We let \( x[\bar{x}, u, v] \) denote this solution.

1.3.1. **Commutativity and completeness.** For some results we shall assume the hypothesis described below.

**Definition 1.2.**
(i) We say that \( g_1, \ldots, g_m \) verifies the commutativity hypothesis if
\[
[g_\alpha, g_\beta](x) = 0, \quad \text{for all } x \in \mathbb{R}^n,
\]
where \([g_\alpha, g_\beta] := Dg_\beta g_\alpha - Dg_\alpha g_\beta\) is the Lie bracket of \( g_\alpha \) and \( g_\beta \).

(ii) The vector field \( g_\alpha \) is complete if the solution of the Cauchy problem
\[
\dot{x}(t) = g_\alpha(x(t)), \quad x(0) = \bar{x},
\]
is (uniquely) defined on \( \mathbb{R} \).

The following assumption will be adopted for some results of this article.

**Hypothesis (CC):** We say that the functions \( g_1, \ldots, g_m \) satisfy the Hypothesis (CC) if they verify the commutativity hypothesis and the vector fields \( g_1, \ldots, g_m \) are complete.

2. **LIMIT SOLUTIONS**

Let us consider the Cauchy problem \((E)-(IC)\). In the Definition 1.1 of limit solution, the fact that the choice of \((u^\tau_k)\) depends on the time \( \tau \) might appear awkward. However, it is essential to incorporate the case \( u \in L^1([a, b]; U) \), as shown in Example 2.1 below. On the other hand, some important results are valid for the case where the choice of \((u^\tau_k)\) is actually independent of the point \( \tau \), which explains why we have introduced the notion of (possibly BV) simple limit solution.

**Example 2.1.** Let us consider the Cauchy problem for scalar state and control given by
\[
\dot{x} = x\dot{u}, \quad x(0) = 1, \quad t \in [0, 1].
\]
It is easy to verify that, for every bounded control \( u \in L^1([0, 1]; \mathbb{R}) \), the map \( x(t) = e^{u(a) - u(t)} \) is a limit solution corresponding to \( u \). I.e. formally, we have the same formula as in the regular case. Incidentally, due to commutativity (the control is scalar), \( u \) is also the unique limit solution (see Section 3). Yet in general \( x \) is not a simple limit solution, i.e. \((u, x)\) may fail to be a pointwise limit of absolutely continuous functions: for instance, if \( u := 1_{Q \cap [0, 1]} \) is the Dirichlet function, there is no way of pointwise approximating \( u \) with a sequence of absolutely continuous \( u_k \).

\[^2\text{This is a consequence of a theorem due to Baire (see e.g. [25], Theorem 7.3), according to which, for such an approximation to be possible, the discontinuity set of } u \text{ should be a residual set.}\]
2.1. Consistency and uniqueness. If $u$ is absolutely continuous, then for every control $v \in L^1([a,b]; V)$, the corresponding Carathéodory solution is trivially a simple limit solution. A related question, namely whether the limit extension does or does not insert new absolutely continuous solutions, is of obvious importance. From the Example 2.3 below it follows that even simple limit solutions may actually differ from the Carathéodory solution. However, consistency of limit solutions with Carathéodory solutions is achieved in two important cases: i) when the system is commutative and, ii) when the limit solution is BV simple. Of course, this consistency implies uniqueness. More precisely:

**Theorem 2.2.** Let $\bar{x} \in \mathbb{R}^n$, $(u,v) \in AC([a,b]; U) \times L^1([a,b]; V)$ and let us assume one of the following two conditions.

(A) The system verifies Hypothesis (CC) and $x$ is a limit solution corresponding to the initial condition $\bar{x}$ and the control $(u,v)$.

(B) $x \in AC([a,b]; \mathbb{R}^n)$ is a BV simple limit solution corresponding to $\bar{x}$ and $(u,v)$.

Then $x$ coincides with $x[\bar{x}, u, v]$, the (unique) Carathéodory solution corresponding to $\bar{x}$ and $(u,v)$.

The statement associated with condition (B) will be proved later, at the end of Section 4. Two crucial assumptions of this result, namely the equiboundedness of the variation of the approximating controls and the continuity of the output, are discussed in the remarks and examples below. Let us insist on the fact that these assumptions are needed because of the noncommutativity of the vector fields $g_1, \ldots, g_m$. In fact, the statement of Theorem 2.2 corresponding to item (A) is a straightforward consequence of a much more general uniqueness result, valid for all limit solutions in the commutative case (see Theorem 3.1).

**Remark 2.1.** Notice that the notion of BV simple limit solution requires that the approximating regular controls $u_k$ have equibounded variation. In fact, even for a quite regular (even constant) control $u$, if we let the variation of the approximating controls go to infinity, it may happen that there exist more than one absolutely continuous (simple) limit solution, as illustrated in the following example, which is an ad hoc reformulation of an example in [35].

**Example 2.3.** Consider the vector fields on $\mathbb{R}^3$

$$g_1(x) := (1, 0, x_2)^\top, \quad g_2(x) := (0, 1, -x_1)^\top,$$

and the control Cauchy problem on $[0, 1]$,

$$\dot{x} = g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \quad x(0) = 0. \quad (2)$$

For any $k \in \mathbb{N}$, the Carathéodory solution $x_k$ of (2) corresponding to the control

$$u_k(t) = (k^{-1/2}\cos kt - 1, k^{-1/2}\sin kt)^\top,$$
is given by
\[ x_k(t) = \left( k^{-1/2} \cos kt - 1, k^{-1/2} \sin kt, -t + k^{-1} \sin kt \right)^\top. \]

Therefore, since
\[ \lim u_k(t) = (0, 0)^\top, \quad \lim x_k(t) = (0, 0, -t)^\top, \]
uniformly on \([0, 1]\), the map \(x(t) := (0, 0, -t)^\top\) turns out to be a \((C^\infty)\) limit solution corresponding to the constant control \(u(t) \equiv (0, 0)^\top\). In particular \(x\) does not coincide with the Carathéodory solution \(x[0, u] \equiv (0, 0, 0)^\top\). Actua lly, \(x\) cannot be a \(BV\) simple limit solution, otherwise, in view of Theorem 2.2, it would coincide with \(x[0, u]\).

The previous example has made evident the importance, among the hypotheses that guarantee uniqueness, of a uniform bound on the variation of the approximating controls \(u_k\). Let us point out that the fact that \(x \in AC([a, b]; \mathbb{R}^n)\) is crucial as well, as shown in the following example.

**Example 2.4.** Consider the same system (2) as in Example 2.3 and, for each \(k \in \mathbb{N}\), set
\[
\hat{u}_k(t) := \begin{cases} 
(\cos kt - 1, \sin kt)^\top, & \text{for all } t \in [0, 2\pi/k], \\
(0, 0)^\top, & \text{for all } t \in ]2\pi/k, 1].
\end{cases}
\]

It is trivial to check that the corresponding Carathéodory solutions \(x_k\) of (2) are given by
\[
\hat{x}_k(t) := \begin{cases} 
(\cos kt - 1, \sin kt, kt)^\top, & \text{for all } t \in [0, 2\pi/k], \\
(0, 0, 2\pi)^\top, & \text{for all } t \in ]2\pi/k, 1].
\end{cases}
\]

Hence
\[
\lim \hat{u}_k(t) = (0, 0)^\top, \quad \hat{x}(t) := \lim \hat{x}_k(t) = (0, 0, 2\pi \cdot 1_{[0,1]})^\top, \text{ pointwise in } [0, 1].
\]

Notice that, unlike the controls \(u_k\) in Example 2.3, the controls \(\hat{u}_k\) have variation uniformly bounded (by 2\(\pi\)). In particular, the limit \(\hat{x}\) is a \(BV\) simple limit solution corresponding to the constant control \((0, 0)^\top\), yet it does not coincide with the Carathéodory solution \(x := (0, 0, 0)^\top\). With reference to Theorem 2.2, here the missing hypothesis is the absolute continuity of \(\hat{x}\).

2.2. **Existence of limit solutions.** Existence of limit solutions will be achieved as consequence of the representation results stated in the next sections. Let us begin with the commutative case:

**Theorem 2.5.** Let us assume that the commutativity hypothesis (CC) holds true. Then, for every \(\bar{x} \in \mathbb{R}^n\) and \((u, v) \in L^1([a, b]; U) \times L^1([a, b]; V)\), there exists a limit solution of (E).
The result is contained in Theorem 3.1 below.

The noncommutative case is more involved. Yet, for a control with bounded variation, not only Theorem 2.6 below establishes the existence of a limit solution $x$, but this $x$ turns out to be also a BV simple limit solution. To state this result we need the definition of Whitney property for a subset of an Euclidian space, which relates the geodesic distance with the Euclidian distance.

**Definition 2.1.** We say that a compact subset $U$ has the Whitney property (see [36]) if there is $M \geq 1$ such that, for every pair $(u_1, u_2) \in U \times U$, there exists an absolutely continuous path $\gamma : [0, 1] \to U$ verifying

$$
\gamma(0) = u_1, \quad \gamma(1) = u_2, \quad \text{Var}_{[0,1]}(\gamma) \leq M|u_1 - u_2|.
$$

A trivial example of a subset with the Whitney property is a compact star-shaped subset. Instead, the arcwise connected subset

$$
\{(x, y) \in \mathbb{R}^2 : x \in [0, 1], y = x \sin(1/x)\} \cup \{(0, 0)\}
$$

does not have the Whitney property.

**Theorem 2.6.** Let us assume that $U$ has the Whitney property. Then, for every initial value $\bar{x} \in \mathbb{R}^n$ and control pair $(u, v) \in BV([a, b]; U) \times L^1([a, b]; V)$ there exists an associated BV simple limit solution of $(E) - (IC)$.

Theorem 2.6 will be proved in Section 5 as a consequence of a representation property stated in Theorem 4.3 below.

Let us observe that, as a byproduct of Theorem 2.6, we get the following density result for BV functions in the Tychonoff topology of pointwise convergence.

**Theorem 2.7.** Let us assume that $U$ has the Whitney property. Then, every map $u \in BV([a, b]; U)$ is the pointwise limit of a sequence $(u_k)$ of absolutely continuous maps $u_k : [a, b] \to U$ with equibounded variation.

Indeed, the latter result follows by Theorem 2.6 since the existence of a BV simple limit solution corresponding to $u$ implies the existence of a sequence with the mentioned properties.

**3. The commutative case**

In this section we mainly recall some results from [1], where the commutativity Hypothesis (CC) were assumed. In fact, as already stated in the Introduction, the notion of limit solution coincides with a characterization of the concept of pointwise defined solution formerly proved for the particular case of commutative systems [8, 1]. Let us also refer to [32, 11, 23] for other references dealing with systems where the Lie algebra generated by $\{g_1, \ldots, g_m\}$ is trivial.
3.1. Existence, uniqueness, continuous dependence.

Theorem 3.1 (Existence and Uniqueness). Assume that the Hypothesis (CC) holds. Then, for every initial value \( \bar{x} \in \mathbb{R}^n \) and each control pair \((u, v) \in \mathcal{L}^1([a, b]; U) \times \mathcal{L}^1([a, b]; V)\), there exists a unique limit solution of the Cauchy problem (E)-(IC).

Let us use \( x[\bar{x}, u, v] \) to denote the unique limit solution of (E)-(IC) corresponding to the initial value \( \bar{x} \in \mathbb{R}^n \), and a control pair \((u, v) \in \mathcal{L}^1([a, b]; U) \times \mathcal{L}^1([a, b]; V)\).

Remark 3.1. In a more general situation when the state domain is a subset of \( \mathbb{R}^n \), one can see that the only null bracket hypothesis \([g_\alpha, g_\beta] \equiv 0\) does not guarantee the uniqueness of a limit solution (see an example in [1]). This is due to the fact that the vanishing of the brackets is not enough for the global commutativity of the vector fields \( g_1, \ldots, g_m \).

Theorem 3.2 (Dependence on the data). Let us assume Hypothesis (CC). Then the following assertions hold true.

(i) For each \( \bar{x} \in \mathbb{R}^n \) and \( u \in \mathcal{L}^1([a, b]; U) \), the function \( v \mapsto x[\bar{x}, u, v] \) is continuous from \( \mathcal{L}^1([a, b]; V) \) to \( \mathcal{L}^1([a, b]; \mathbb{R}^n) \).

(ii) For any \( r > 0 \) there exists a compact subset \( K' \subset \mathbb{R}^n \), such that the trajectories \( x[\bar{x}, u, v] \) have values in \( K' \), whenever we consider \( |\bar{x}| \leq r \), \( u \in \mathcal{L}^1([a, b]; U) \) and \( v \in \mathcal{L}^1([a, b]; V) \).

(iii) For each \( r > 0 \), there exists a constant \( M > 0 \) such that, for every \( t \in [a, b] \), for all \( |\bar{x}_1|, |\bar{x}_2| \leq r \), for all \( u_1, u_2 \in \mathcal{L}^1([a, b]; U) \) and for every \( v \in \mathcal{L}^1([a, b]; V) \), one has

\[
|x_1(t) - x_2(t)| + \|x_1 - x_2\|_1 \leq M \left[ |\bar{x}_1 - \bar{x}_2| + |u_1(a) - u_2(a)| + |u_1(t) - u_2(t)| + \|u_1 - u_2\|_1 \right],
\]

where \( x_1 := x[\bar{x}_1, u_1, v] \), \( x_2 := x[\bar{x}_2, u_2, v] \).

Since the limit solution depends on the pointwise definition of \( u \), it is interesting to investigate the effects of a change of \( u \) on a measure-zero subset of \([a, b]\). The following notation is used: let \( h \) be a locally Lipschitz vector field on \( \mathbb{R}^n \), \( \bar{x} \in \mathbb{R}^n \), and let \( x \) denote the solution of the Cauchy problem

\[
\dot{x}(t) = h(x(t)), \quad h(0) = \bar{x}.
\]

We use

\[
\bar{x} e^{\tau h}
\]

to denote the value of this solution at time \( t = \tau \).

Theorem 3.3. Assume that Hypothesis (CC) is satisfied. Let us consider an initial state \( \bar{x} \in \mathbb{R}^n \) and an ordinary control \( v \in \mathcal{L}^1([a, b]; V) \). Let \( u, \hat{u} \in \mathcal{L}^1([a, b]; U) \) be impulsive controls that coincide a.e. in \([a, b]\) and that verify \( u(a) = \hat{u}(a) \). Then, setting \( x := x[\bar{x}, u, v] \), \( \hat{x} := x[\bar{x}, \hat{u}, v] \), one has

\[
x(t) = \hat{x}(t)e^{\sum_{\alpha=1}^{m}(u_\alpha(t) - \hat{u}_\alpha(t))g_\alpha}, \quad \text{for all } t \in [a, b].
\]
In particular,
\[ x(t) = \dot{x}(t) \]
for every \( t \in [a, b] \) such that \( u(t) = \dot{u}(t) \), that is, almost everywhere.

4. The Noncommutative Case with BV Controls

As soon as \([g_\alpha, g_\beta] \neq 0\), the knowledge of a jump of the input \( u \) is not enough to determine the jump of the corresponding solution. Let us illustrate this well-known fact with the following simple example.

Example 4.1. Consider the Cauchy problem
\[ \dot{x} = g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \quad x_1(0) = x_2(0) = 1, \]
with \( g_1(x) := (x_1, x_2)^\top \), and \( g_2(x) := (0, x_1^2)^\top \), that we know do not commute. Let the controls \( u_1 \) and \( u_2 \) be given by
\[
 u_1(t) := \begin{cases} 
 0, & \text{on } [0, \frac{1}{2}], \\
 1, & \text{on } \left[\frac{1}{2}, 1\right], 
\end{cases} \\
 u_2(t) := \begin{cases} 
 0, & \text{on } [0, \frac{1}{2}], \\
 1, & \text{on } \left[\frac{1}{2}, 1\right], 
\end{cases}
\]
and consider the sequence \((u_k)\) defined by
\[
 u_{k,1}(t) := \begin{cases} 
 0, & \text{on } \left[0, \frac{1}{2} - \frac{1}{k+1}\right], \\
 (k + 1) \left(t - \frac{1}{2} + \frac{1}{k+1}\right), & \text{on } \left[\frac{1}{2} - \frac{1}{k+1}, \frac{1}{2}\right], \\
 1, & \text{on } \left[\frac{1}{2}, 1\right], 
\end{cases} \\
 u_{k,2}(t) := \begin{cases} 
 0, & \text{on } \left[0, \frac{1}{2}\right], \\
 (k + 1) \left(t - \frac{1}{2}\right), & \text{on } \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{k+1}\right], \\
 1, & \text{on } \left[\frac{1}{2} + \frac{1}{k+1}, 1\right]. 
\end{cases}
\]
Hence, \((u_{k,1}, u_{k,2})\) converges to \((u_1, u_2)\) pointwise everywhere and in the \(L^1\)-topology. One can see that the limit solution corresponding to \( u \) and the sequence \((u_k)\) is given by
\[
 x_1(t) := \begin{cases} 
 1, & \text{on } [0, \frac{1}{2}], \\
 e, & \text{on } \left[\frac{1}{2}, 1\right], 
\end{cases} \\
 x_2(t) := \begin{cases} 
 1, & \text{on } [0, \frac{1}{2}], \\
 e, & \text{on } t = \frac{1}{2}, \\
 e + e^2, & \text{on } \left[\frac{1}{2}, 1\right]. 
\end{cases}
\]
Let us now invert the roles of the controls, namely let us set \( \tilde{u}_1 := u_2, \)
\( \tilde{u}_2 := u_1, \tilde{u}_{k,1} := u_{k,2}, \tilde{u}_{k,2} := u_{k,1} \), and let \( \tilde{x}_{k,1}, \tilde{x}_{k,2} \) be the corresponding Carathéodory solutions. The limit solution associated with \( \tilde{u} \) and the sequence \((\tilde{u}_k)\) is given by
\[
 \tilde{x}_1(t) := \begin{cases} 
 1, & \text{on } [0, \frac{1}{2}], \\
 e, & \text{on } \left[\frac{1}{2}, 1\right], 
\end{cases} \\
 \tilde{x}_2(t) := \begin{cases} 
 1, & \text{on } [0, \frac{1}{2}], \\
 2, & \text{on } t = \frac{1}{2}, \\
 2e, & \text{on } \left[\frac{1}{2}, 1\right]. 
\end{cases}
\]
Hence \((x_1, x_2)\) and \((\tilde{x}_1, \tilde{x}_2)\) differ in \(L^1\), even if \((u_1, u_2)(t) = (\tilde{u}_1, \tilde{u}_2)(t)\) for all \( t \in [0, 1]\backslash\{1/2\} \).
The previous example illustrates an important feature of noncommutative impulsive systems, at least in the case when $u$ has bounded variation: at the discontinuity points of $u$, the non-drift dynamics and the $u$-path during the jump determine the corresponding discontinuity of the output. This fact is the common outcome of several investigations on the subject, which share a notion of solution here referred as graph completion solution (see e.g. \cite{31,7,13,24,33,23,27}). Graph completion solutions are set-valued at a countable subsets of instants: their selection will be called here single-valued graph completion solutions. The main result of this section is Theorem 1.3 which establishes that the concepts of single-valued graph completion solution and that of BV simple limit solution are in fact equivalent.

4.1. Graph completion solutions. For any $L > 0$, we use $\mathcal{U}_L$ to denote the subset of $L$-Lipschitz maps $(\varphi_0, \varphi)$ belonging to $\text{Lip}([0, 1]; [a, b] \times U)$ such that $\varphi_0 : [0, 1] \to [a, b]$ is increasing and surjective. Furthermore, with $\mathcal{U}_L^+ \subset \mathcal{U}_L$ we will refer to the subset made of elements $(\varphi_0, \varphi)$ for which $\varphi_0$ is strictly increasing. Finally, we set

$$
\mathcal{U} := \bigcup_{L \geq 0} \mathcal{U}_L, \quad \mathcal{U}^+ := \bigcup_{L \geq 0} \mathcal{U}_L^+.
$$

**Definition 4.1** (Space-time control). The elements $(\varphi_0, \varphi, \psi)$ of the set $\mathcal{U} \times L^1([0, 1]; V)$ will be called space-time controls.

Let us consider the space-time control system in the interval $[0, 1]$ defined as

$$
\begin{align*}
  y'_0(s) &= \varphi'_0(s), \\
  y'(s) &= f(y_0(s), y(s), \varphi(s), \psi(s)) + \sum_{\alpha=1}^{m} g_{\alpha}(y(s)) \varphi'_\alpha(s), \\
  (y_0, y)(0) &= (a, \bar{x}),
\end{align*}
$$

where the apex ' denotes differentiation with respect to the pseudo-time $s \in [0, 1]$, and the (graph) controls $(\varphi_0, \varphi, \psi)$ belong to $\mathcal{U} \times L^1([0, 1]; V)$. For the sake of simplicity, we shall fix the initial value $\bar{x} \in \mathbb{R}^n$. For a given initial value $\bar{x} \in \mathbb{R}^n$, let $\mathcal{I}$ denote the input-output map associated with the system (6), which assigns to each $(\varphi_0, \varphi, \psi) \in \mathcal{U} \times L^1([0, 1]; V)$ verifying $(\varphi_0, \varphi)(0) = (a, \bar{u})$, the corresponding solution $(y_0, y)[\varphi_0, \varphi, \psi]$ of the Cauchy problem (6). The following property holds:

**Proposition 4.2.** For every $L > 0$, the map $\mathcal{I}$ is continuous from $\mathcal{U}_L \times L^1([0, 1]; V)$ to $\text{Lip}([0, 1]; [a, b] \times \mathbb{R}^n)$, where the domain and the range are endowed with the product topology of $C^0([0, 1]; [a, b] \times U) \times L^1([0, 1]; V)$ and the $C^0([0, 1]; [a, b] \times \mathbb{R}^n)$-topology, respectively.

The previous result is an extension of \cite[Theorem 1]{7} (where $f$ is independent of $v$). For a proof we refer to the first part of the proof of \cite[Theorem 4.1]{24}.
Remark 4.1. Let us point out that the space-time system (6) has parameter free character. We mean that if $\hat{s} : [0, 1] \to [0, 1]$ is a bijective, increasing, bi-Lipschitz map then one has

$$(y_0, y)[\varphi_0, \varphi, \psi] \circ \hat{s} = (y_0, y)[\varphi_0 \circ \hat{s}, \varphi \circ \hat{s}, \psi \circ \hat{s}].$$

Consider now $(u, v) \in AC([a, b]; U) \times L^1([a, b]; V)$, and $\varphi_0 : [0, 1] \to [a, b]$ a Lipschitz, bijective map such that $\varphi_0(0) = a$ and $\varphi := u \circ \varphi_0$ is Lipschitz continuous. It is not restrictive to assume that $v$ is Borel measurable, since in every $L^1$ equivalence class there are always Borel measurable representatives. Then, with $v$ being $L^1$ and Borel measurable, the composition $\psi := v \circ \varphi_0$ belongs to $L^1([0, 1]; V)$; and the unique corresponding solution $(y_0, y)[\varphi_0, \varphi, \psi]$ of (6) verifies

$$y(s) = x \circ \varphi(s), \quad \text{for all } s \in [0, 1],$$

where $x$ is the Carathéodory solution of the graph completion solution (24)-[14] corresponding to $(u, v)$. By means of the notion of graph completion, this argument can be used to extend the notion of output of the system (E) to controls $u$ with bounded variation (and possibly discontinuous).

Definition 4.2 (Graph completion [7]). Consider a control $u \in BV([a, b]; U)$. A graph completion (shortly g.c.) of $u$ is a space-time control $(\varphi_0, \varphi) \in U$ such that, for all $t \in [a, b]$, there exists $s \in [0, 1]$ verifying $(t, u(t)) = (\varphi_0(t), \varphi(s))$.

Consider $(u, v) \in BV([a, b]; U) \times L^1([a, b]; V)$, let $(\varphi_0, \varphi) \in U$ be a g.c. of $u$ and set $\psi := v \circ \varphi_0$. Then either $(y_0, y)[\varphi_0, \varphi, \psi]$ or $(y_0, y)[\varphi_0, \varphi; v]$ shall denote the solution of (6) corresponding to $(\varphi_0, \varphi, \psi)$.

Definition 4.3 (Graph completion solution [21]). Consider $(\varphi_0, \varphi)$ a g.c. of $u \in BV([a, b]; U)$, $v$ a control in $L^1([a, b]; V)$, and $(y_0, y)[\varphi_0, \varphi; v]$ the corresponding solution of (6). We say that the (possibly set-valued) map defined by

$$x(t) := y \circ \varphi^{-1}_0(t), \quad \text{for all } t \in [a, b],$$

is the graph completion solution to (24)-[14], shortly g.c. solution, corresponding to the control triple $(\varphi_0, \varphi; v)$. Here $\varphi^{-1}_0(t)$ denotes the pre-image of the singleton $\{t\}$. When the initial condition is meant by the context this solution will be denoted by $x[\varphi_0, \varphi; v]$.

Remark 4.2. A g.c. solution $x = x[\varphi_0, \varphi; v]$ is single-valued at almost every $t$, for there are at most countably many intervals $I \subset [0, 1]$ on which $\varphi_0$ is constant and $\varphi$ is not constant (this being a consequence of $(\varphi_0, \varphi) \in BV^3$). The graph completion solution $x[\varphi_0, \varphi; v]$ is a set-valued map with compact values and compact graph. This was pointed out in [7] where

Uniqueness is a consequence of the uniqueness of solutions to Cauchy problem (5).

Let us point out that $x$ may well be set-valued even at times $t$ where $u$ is continuous: unless the system is globally integrable for this to happen, it is necessary that $\varphi_0^{-1}(t) = [s_1, s_2], s_1 < s_2$, and $\varphi |_{[s_1, s_2]}$ is a non trivial closed curve.
$f$ is independent of $v$, but the argument can be trivially extended to the present case. So, the set-valued map $t \mapsto x[\varphi_0, \varphi; v]$ is Hausdorff upper semicontinuous (see e.g. [3] for the definition of u.s.c. map).

4.2. Equivalence between single-valued g.c. solutions and BV simple limit solutions. Let us give the definition of single-valued g.c. solution by means of a selection of the inverse (set-valued) map $t \mapsto (\varphi_0, \varphi)$. We call such a selection a $(\varphi_0, \varphi)$-clock.

**Definition 4.4.** Let us consider $u \in BV([a,b]; U)$ and let $(\varphi_0, \varphi)$ be a graph completion of $u$. Any map $\sigma : [a,b] \to [0,1]$ such that, for all $t \in [a,b]$,

$$(\varphi_0, \varphi) \circ \sigma(t) = (t, u(t))$$

will be called a clock corresponding to $(\varphi_0, \varphi)$.

**Remark 4.3.** Notice that if $L > 0$ is a Lipschitz constant for $\varphi_0$, then any clock $\sigma$ verifies

$$\sigma(t_2) - \sigma(t_1) \geq \frac{1}{L}(t_2 - t_1),$$

for all $t_1, t_2 \in [a,b]$ such that $t_1 < t_2$.

**Definition 4.5.** Let $(\varphi_0, \varphi)$ be a g.c. of a control $u$ in $BV([a,b]; U)$, and let $v \in L^1([a,b]; V)$. Let $\sigma$ be a clock corresponding to $(\varphi_0, \varphi)$. The map $x(t) := y[\varphi_0, \varphi; v] \circ \sigma(t)$, for all $t \in [a,b]$, will be called the single valued g.c. solution corresponding to $(\varphi_0, \varphi; v)$ and to the clock $\sigma$. We shall use $x[\varphi_0, \varphi; v]_{\sigma}$ to denote this solution.

The notion of single-valued g.c. solution in fact coincides with that of limit solution, as stated in the following theorem.

**Theorem 4.3.** Let $\bar{x} \in \mathbb{R}^n$ and $(u, v) \in BV([a,b]; U) \times L^1([a,b]; V)$. A map $x : [a,b] \to \mathbb{R}^n$ is a single-valued g.c. solution of $[E] - [IC]$ corresponding to $(\bar{x}, u, v)$ if and only if $x$ is a BV simple limit solution of $[E] - [IC]$.

We postpone the proof of this theorem to the next section.

4.3. Proof of Theorem 2.2

5. Proofs of Theorem 4.3 and Proposition 2.6

6. Likely developments

We have mainly analyzed the commutative case for $L^1$ inputs and the noncommutative case for controls $u$ with bounded variation. In particular we have seen how some existing notions of solution (for discontinuous inputs $u$) can be embedded within the class of limit solution or within some specific subclass. Certainly, further issues can be explored in this direction. Below we briefly mention just a few situations on which we think it would be worth investigating.

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5 This version does not contain proofs. All the detailed proofs are included in the original version of the article, that is under review.
6.1. **Unbounded variation and noncommutative fields.** There exists a definition of solution for the case where neither the commutativity hypothesis is assumed nor the controls $u$ have bounded variation (see e.g. [9]). This solution is based on a fibration of the state space by means of the leaves of the ideal generated by the Lie brackets of order $\geq 2$ of the vector fields $g_1, \ldots, g_m$. Approximation results of the flows generated by the brackets are used to prove that the so-called *looping controls* can be regarded as limit of ordinary controls. In particular, the quotient system resulting from a local factorization turns out to be commutative. We conjecture that the solutions resulting by this approach can be proved to be (perhaps simple) limit solutions in the sense of the present paper.

6.2. **Continuous controls** $u$. The theory of *rough paths* begun in 1994 by T.J. Lyons and nowadays variously developed (see e.g. [17, 18, 16]), is widely recognized as an effective construction to deal with *continuous* inputs $u$. It is impossible to give in the restricted space of this subsection even an approximative idea of the notion of solution to (E) when the inputs $u$ are identified with rough paths. Let us only remind that the notion of rough paths was introduced as a *nonlinear* development of the concept of Young’s integral and successively has become a powerful tool in the field of stochastic differential equations. Let us also point out that two main notions are crucial within this theory:

1) the *$k$-iterated integrals*, namely

$$I_{[s,t]}^{i_1, \ldots, i_k} := \int_{s<t_1<\cdots<t_k<t} du_{i_1}(t_1) \cdots du_{i_k}(t_k)$$

which, in a sense, provide extra information besides the mere 1-iterated integral $u(t) - u(s)$ to single out the proper solution.$^7$

2) the pseudometric (on the of inputs’ and outputs’ spaces) induced by the $p$-variation. Solutions are defined by means of continuous extension of input-output functionals with respect to these pseudo-metrics (the latter are rendered metrics by fixing initial points). In particular, these solutions turn out to be unique and also approachable by suitable Picard iterations.

Moreover, the use of rough paths involves Lie algebraic notions, incidentally demanding a sufficient amount of regularity of vector fields to *compensate the roughness* of the paths.

A somehow different but connected approach, concerning generalized solutions on the Heisenberg group corresponding to Hölder-continuous inputs $u$, can be found in [20].

---

$^6$ We are not aware of extension including an ordinary, measurable, control $v$ besides $\dot{u}$, maybe an issue not interesting the major fields of application of rough paths.

$^7$ Notice, in particular, that $I_{[s,t]}^{i_1, i_2} - I_{[s,t]}^{i_2, i_1}$ is twice the *area spanned* by the curve $(u_{i_2}, u_{i_1})$ during the interval $[t_1, t_2]$. 
Since the notions of solution given in the above-quoted literature were obtained as extensions of input-output maps in Banach spaces, they are likely simple limit solutions in the sense proposed here. It would be perhaps instructive to study sharper connections between these issues.

6.3. **Bounded versus unbounded inputs.** A case to which the notion of limit solution could likely be extended is that of systems where the $g_\alpha$ are $v$-dependent, namely systems of the form

$$\dot{x} = f(t, x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x, u, v)\dot{u}_\alpha, \quad t \in [a, b].$$

These kind of equations are important in mechanical applications—e.g. when $u$ is a shape parameter and $v$ is a control representing an external force or torque—and in min-max control context where, for instance, the adjoint equations may contain a $v-$dependent term multiplied by an unbounded control, like in (7) (see e.g. [4]). Let us observe that the dependence of $g_\alpha$ on $v$ is much more critical than the $v$-dependence of $f$, already in the case of $u$ with bounded variation, in that a simultaneous jump of $u$ and $v$ would make the determination of the corresponding jump of $x$ quite delicate (see e.g. [21] [24] [5]). The issue could be actually extended to more general equations of the form

$$\dot{x} = \Phi(t, x, u, v, \dot{u}).$$

See e.g. [22] for the latter general case, and [10] [29] [26] for the case when $\Phi$ is polynomial in $\dot{u}$.

6.4. **Nonsmooth vector fields.** Another direction of study could be represented by weakening the regularity assumptions on the vector fields $g_\alpha$. For instance, if the latter were just locally Lipschitz continuous, conditions like vanishing Lie brackets would still make sense in view of the results in [30] or [34], so the commutative case could be likely addressed. It should be noted, however, that the homeomorphism utilized to represent limit solutions would be not differentiable, so further technical questions should be addressed to mimic the theory of the regular case.

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