DESIGNING PRORATED LIFETIME WARRANTY STRATEGY FOR HIGH-VALUE AND DURABLE PRODUCTS UNDER TWO-DIMENSIONAL WARRANTY

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ABSTRACT. This paper considers a prorated lifetime warranty strategy for high-value and durable products under two-dimensional warranty. Different from previous studies, the factor of capital time value is introduced into this study for a long-period lifetime warranty coverage. Besides, the mathematical model considers the warranty coverage into three situations and adopts minimal repair and complete/minimal maintenance strategy in the proposed model. To illustrate the proposed model, this paper analyzes the manufacturer cost, consumer cost, additional warranty service price and profit. The main goal of this paper is to provide a comprehensive warranty service to consumers. Through a numerical example, it is found that the proposed lifetime warranty strategy can provide consumers warranty service throughout product lifetime and reduce the expected cost expenditure to consumers. In addition, it can also provide additional profit sources to manufacturers and meet the realization of warranty objectives, while combined maintenance strategy is adopted in lifetime warranty coverage.

1. Introduction. For most products, manufacturers need to provide base warranty service to consumers for free, and consumers can perceive the quality of products through warranty service duration and service quality (Mai et al. 2017)[18]. Offering attractive warranty service can highlight the products’ reliability, enhance product competitiveness and maintain the relationship between manufacturers and consumers (Huang et al. 2017)[12]. However, the increasing strict warranty service content has brought a huge warranty cost pressure to manufacturers (Huang et al. 2015[11]; Su and Wang 2016[28]; Wang and Su 2016[32]; Wang et al. 2015[33]), and the warranty cost can range from 2% to 15% of the net sales (Murthy and Djamaludin 2002). To reduce the warranty cost borne by manufacturers, many scholars have formulated maintenance models under one- and two-dimensional warranty (see details in Banerjee and Bhattacharjee 2012[2]; Chien 2012[6]; Huang et al. 2013[10]; Khojandi et al. 2014[15]; Lin and Yeh 2010[17]; Shang et al. 2016[26]; Varnosafaderani and Chukova 2012[31]; Ye and Murthy 2016[36]). Generally, previous studies have considered age as the variable of one-dimensional warranty and adopted age and usage as the variables of two-dimensional warranty, except for Park et al. (2017)[22] who considered failure time and repair time as variables.

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At present, the reliability and lifetime of high-value and durable products have been increasing dramatically with the development of design and manufacturing technologies, and the relatively short base warranty period can hardly meet the warranty service demand of consumers. To meet the enhanced warranty service demand of consumers, some manufacturers have extended the service period of the base warranty. However, expanding the base warranty coverage to meet consumers’ enhanced warranty service demand will increase the additional cost pressure to manufacturers (Huang et al. 2017[12]; Tong et al. 2017[30]). For example, the base warranty period of vehicle should not be less than two years or 50,000 km according to the law of China which promulgated in 2013 (https://baike.baidu.com/item/%E6%B1%BD%E8%BD%A6%E4%B8%89%E5%8C%85%E6%94%BF%E7%AD%96/3684911?fr=aladdin). Obviously, the legal-formulated warranty period is extremely short compared with the lifetime of vehicles and the consumer after-sale service demand. To meet consumers warranty service demand, several automobile manufacturers have increased base warranty coverage to two years/60,000 km or three years/100,000 km for free. In addition, providing extended warranty service with additional cost to consumers have attracted much attention in recent years(Huang et al. 2017[12]; Su and Wang 2016[28]), and consumers can decide whether to purchase extended warranty service when they purchase products. However, consumers still cannot fully enjoy the warranty service throughout the product lifetime and the warranty period is relatively short for most high-value and durable products. Besides, for most of high-value and durable products, the warranty period is two-dimensional; consumers can hardly recognize or predict their usage rate of a product and decide which type of additional warranty service is most suitable for their use behavior, when they purchase high-value and durable products under a two-dimensional warranty.

Under lifetime warranty, consumers can enjoy a complete warranty service throughout the product lifetime without considering whether their usage rate matches with the additional prepaid warranty coverage. Meanwhile, lifetime warranty service develops a new service project, obtains a new source of profit and maintains the brand loyalty for warranty service providers. In the field of lifetime warranty, Wu et al. (2006) [34] presented a decision model to determine the price and warranty coverage based on the pre-determined lifetime. At the 2007 IEEE International Conference on Industrial Engineering and Engineering Management, Rahman and Chattopadhyay proposed the concept of lifetime warranty and analyzed the risk faced by consumers and warranty service providers under lifetime warranty. Subsequently, for the high-value and durable products, Chattopadhyay and Rahman (2008)[5] proposed a free lifetime warranty strategy and estimated the number of failures and expected warranty cost in lifetime warranty period. However, the model lacks the analysis of the impact of different maintenance strategies on warranty cost. Thereafter, Rahman and Chattopadhyay (2012)[24] established a cost analysis model of lifetime warranty after excluding the special components and analyzed the impact of uncertain information on the decision of consumers and service providers during the lifetime warranty period. At the 2016 IEEE International Conference on Industrial Engineering and Engineering Management, Rahman further analyzed the expected failure rate and maintenance cost of consumers without purchasing lifetime warranty service. In addition, the product lifetime is prolonged, and considerable attention has been further paid recently to product lifetime estimation(Chang et al. 2016[4]; Park and Kim 2016[23]), reliability and cost analysis(Hu and Du 2013[9]). To the best of our knowledge, the existing related studies have mainly focused on
base and extended warranties, where only few studies considered lifetime warranty strategies, such as the one-dimensional lifetime warranty strategies (Chang and Lin 2012[14]; Wu and Longhurst 2011[35]).

In this study, we attempt to propose a prorated lifetime warranty strategy for high-value and durable products sold under two-dimensional warranty. As we know, price is an important part of manufacturers’ marketing strategy, and long-period warranty service results in huge warranty cost. Although some scholars suggest that additional warranty service should not generate profit directly (Lei et al. 2017[16]), the warranty cost of additional warranty service must still be calculated and the price should be determined on this basis. Generally, product reliability is related to maintenance strategy, and maintenance strategy affects product failure function and consumer claims further. Thus, we consider the maintenance strategy and reliability into the lifetime warranty cost, consumer warranty cost, and additional warranty service price functions. Furthermore, price and payment method affect consumers’ willingness to purchase lifetime warranty service. On the basis of the warranty policy proposed by Mo et al. (2017[19]), who encourage buyers to invest preventive maintenance to improve maintenance efficiency, this paper try to offer a lifetime warranty strategy with discount payment to increase consumers’ desire to buy. Consumers can decide whether to purchase additional warranty. If consumers purchase the lifetime warranty service after the base warranty expires, then the additional warranty service is paid prorated to encourage consumers to purchase the lifetime warranty service. Otherwise, consumers have no discount on additional warranty service.

As previously stated, we conduct the research on the prorated lifetime warranty strategy for high-value and durable products under two-dimensional warranty. The remainder of this paper is outlined as follows. In section 2, the notations and assumptions of the mathematical model are derived. In section 3, we formulate the failure and warranty cost models that consider maintenance strategies and additional warranty-buying patterns. Section 4 presents an illustrative example to validate the formulated model. In Section 5, we provide the concluding remarks and suggestions for future use.

2. Notation and assumption.

3. Mathematical model.

3.1. Maintenance model. On the basis of previous studies on maintenance strategies under warranty, minimal repair and complete repair are commonly used in mathematical models and actual warranty service. Minimal repair is a type of repair that only repair failure without affect product performance. After adopting complete repair, the performance is equivalent to that of a new one (Chukova and Johnston 2006)[7]. For example, Baik et al. (2004)[1] extended minimal repair for one-dimensional failure model to two-dimensional failure model and discussed the two cases where a failed system is either always minimally repaired or replaced by a new system. Similarly, some scholars have adopted this maintenance method to formulate failure models in their studies (Chukova and Johnston 2006[7]; Jack et al. 2009[14]; Park et al. 2013[21]; Sheu and Yu 2005[27]). This paper considers the failure recovery into two parts; that is, minimal repair is adopted, and combined maintenance strategy, which combines minimal and complete repairs, is adopted in the warranty period.
can be expressed as follows:

\[ F(x; \alpha_r) = F_0((r/r_0)^x; \alpha_0) \]  

### Notations and definitions

| Notation | Description |
|----------|-------------|
| \( r \)  | usage rate |
| \( W_1 \) | age limit of the two-dimensional base warranty |
| \( U_1 \) | usage limit of the two-dimensional base warranty |
| \( W_e \) | age limit of the two-dimensional extended warranty |
| \( U_e \) | usage limit of the two-dimensional extended warranty |
| \( P_e \) | price of extended warranty service |
| \( P_a \) | price of after-warranty service |
| \( \rho \) | rebate rate |
| \( \pi \) | manufacturer profit of extra warranty service |
| \( p' \) | rate of extra warranty service profit |
| \( C_d \) | consumer cost that considers the capital time value |
| \( F_0(x; \alpha_0) \) | failure distribution function with design usage rate \( r_0 \), where \( \alpha_0 \) is the scale parameter of \( F_0(x; \alpha_0) \) |
| \( F_0(x; \alpha_r) \) | failure distribution function with usage rate \( r \) |
| \( f(x; \alpha_r) \) | failure density function with usage rate \( r \) |
| \( \tilde{F}(x; \alpha_r) \) | survivor function with usage rate \( r \) |
| \( b(x; \alpha_r) \) | hazard function with usage rate \( r \) |
| \( \hat{H}(x; \alpha_r) \) | cumulative hazard function with usage rate \( r \) |
| \( \alpha \) | accelerated parameter of the accelerated failure model |
| \( P_e, P_a \) | price of after-warranty service, price of extended warranty service |
| \( C_3 \) | warranty cost when minimal repair is adopted |
| \( C_2 \) | warranty cost of combined maintenance strategy |
| \( K_r, L_r \) | mapping coordinate values of cutoff points on the age axis, when combined maintenance strategy is adopted |
| \( K'_r, L'_r \) | optimal mapping coordinate values, where warranty coverage is \( (0, W_1) \times (0, U_1) \) |
| \( K''_r, L''_r \) | optimal mapping coordinate values, where warranty coverage is \( (0, W_1) \times (0, U_1) \) |
| \( W_e \) | abscissa value of combined maintenance strategy |
| \( C_a \) | manufacturer cost in warranty |
| \( P \) | price of extra warranty service |
| \( W_s \) | upper limit value calculated interval of \( C_2 \) |
| \( W_z \) | upper limit value calculated interval considering the capital time value |

### Assumptions

Through literature review, it can be seen that almost all studies of warranty strategy believe that consumers’ usage habit remains unchanged during warranty period, that is, consumer usage rate remains the same during warranty period. Similar to previous studies this paper also assumes that consumer usage rate is approximately constant in the product lifetime.

If consumer does not purchase extra warranty service from manufacturers, then manufacturers incur no warranty cost after the base warranty expires for that consumers purchase after-sale service from the third party after-sale departments. The rate of extra warranty service profit of manufacturers and third party after-sale departments is the same.

#### 3.1.1. Minimal repair model

In this part, minimal repair is adopted when failure occurs in the warranty coverage under normal operation. Given that usage rate has a significant effect on product reliability, and the trend of product reliability change decreases or increases with the increase or decrease in usage rate, respectively (Jack et al. 2009[14]; Tong et al. 2014[29]).

Under such circumstances, this paper adopts the accelerated failure model, which can be expressed as follows:

\[ F(x; \alpha_r) = F_0((r/r_0)^x; \alpha_0) \]  

(1)
where $\tau$ is the accelerated parameter of the accelerated failure model and $\tau \geq 1$. In practice, the accelerated parameters of different types of products are different, and manufacturers can obtain the parameters through product test.

Similarly, the hazard and cumulative hazard functions can be obtained as follows:

$$h(x; \alpha_r) = \frac{f(x; \alpha_r)}{F(x; \alpha_r)}$$

$$H(x; \alpha_r) = \int_0^x h(x; \alpha_r)dx.$$  

The warranty cost function of minimal repair model can be obtained as follows:

$$C_1 = C_m \int_0^x h(x; \alpha_r)dx.$$  

3.1.2. Combined maintenance model. In this part, complete and minimal repairs are adopted to construct the combined maintenance model. Generally, combined maintenance model separates the warranty period into three disjointed sub-regions (Jack et al. 2009[14]). After undergoing complete repair, the performance of the product can be reset. Similarly, this paper adopts the accelerated failure model, and the failure model, hazard function and cumulative hazard function are the same as that in Section 3.1.1.

The warranty cost function of the combined maintenance strategy can be expressed as follows:

$$C_2 = C(K_r, L_r, W_x, W_y, (W_Z; \gamma))$$

where $K_r$ and $L_r$ are the mapping coordinate values of cutoff points on the age axis. When the first failure occurs in the interval $[K_r, L_r]$, complete repair is adopted. Otherwise, minimal repair is adopted to rectify the failure. $W_x$ is the abscissa value of the combined maintenance strategy. Hence, we can calculate the optimal value of $K_r$ and $L_r$ in the interval $[0, W_x]$. $W_y$ is the upper limit value of the calculated interval of warranty coverage; hence, the calculated interval of the cost function is $[0, W_y]$. $W_z$ is the upper limit value of the calculated interval which considers the discount rate, and $W_z = W_y$ or $W_z = W_x$. In other words, the warranty cost function considers the discount rate $Y$ in the interval $[0, W_z]$. The schematic of the combined maintenance model is shown in Figure 1.

3.2. Mathematical model of two-dimensional lifetime warranty strategy. As previously mentioned, manufacturers should provide consumers base warranty service for free, and the two-dimensional base warranty coverage can be described as $(0, W_t) \times (0, U_t)$. Some studies have proposed two-dimensional extended warranty strategies, the coverage of which can be expressed as $(0, W_c) \times (0, U_c) - (0, W_t) \times (0, U_t)$.

In this paper, the product lifetime under two-dimensional warranty is set as $(0, W_t) \times (0, U_t)$, where $W_t > W_c > W_l$ and $U_t > U_c > U_l$. To provide consumers with a flexible warranty service, we consider the model into three situations, namely, Situation 1, where consumers only obtain the two-dimensional base warranty service; Situation 2, where consumers purchase the two-dimensional extended warranty service at the time of purchasing the product; and Situation 3, where consumers purchase the lifetime warranty services, at the time of purchasing the product. To encourage consumers to purchase the after-warranty service on the basis of purchasing the extended warranty service, this paper adopts the prorated warranty
strategy in the third situation, and defines the rebate rate as $\mu$. Moreover, if consumers purchase the after-warranty service on the basis of purchasing the extended warranty service, then this paper adopts the prorated warranty strategy in the extended warranty region to reduce the impact on manufacturers’ additional warranty profit.

In the remainder of this section, this paper analyzes the manufacturer warranty service cost, profit, additional warranty service price and consumer cost of the aforementioned three situations. Due to the additional warranty service is prepaid, calculating the expected warranty cost in the additional warranty period and pricing for the additional warranty service without considering the capital time value is evidently unreasonable. Accordingly, different from previous studies, this paper considers the capital time value in the proposed mathematical model and defines the discount rate as $\gamma$. The three disjointed sub-regions of the two-dimensional lifetime warranty can be shown in Figure 2.

Given the differences of usage rate $r$, the coordinate values of $(W_1, U_1)$, $(W_2, U_2)$ and $(W_3, U_3)$ are different, where $(W_1, U_1)$, $(W_2, U_2)$ and $(W_3, U_3)$ are the coordinate values of the expiry points of base warranty, extended warranty and after warranty, respectively. The coordinate values of $(W_1, U_1)$, $(W_2, U_2)$ and $(W_3, U_3)$ are listed in Table 1.

| $r$      | $W_1$ | $U_1$ | $r$      | $W_2$ | $U_2$ | $r$      | $W_3$ | $U_3$ |
|----------|-------|-------|----------|-------|-------|----------|-------|-------|
| $r \geq r_1$ | $W_r$ | $W_r/r$ | $r \leq r_2$ | $W_e$ | $W_e/r$ | $r \leq r_3$ | $W_l$ | $W_l/r$ |
| $r \leq r_1$ | $U_r/r$ | $U_r$ | $r > r_2$ | $U_e/r$ | $U_e$ | $r > r_3$ | $U_l/r$ | $U_l$ |
3.2.1. Situation 1: Consumers only obtain the base warranty service in \((0, W_1) \times (0, U_t)\). In this situation, consumers buy no extra warranty service. Instead, consumers purchase maintenance service from the third party after-sale department when product failure occurs. In practice, the third party after-sale department usually employs minimal repair to rectify the failures.

Case A. Minimal repair adopted in \((0, W_1) \times (0, U_t)\)

In this case, manufacturers provide free base warranty service to consumers with minimal repair. The manufacturer warranty cost in Case A with usage rate \(r\) can be obtained as follows:

\[
C_o = \int_0^{W_1} \frac{C_m}{(1 + \gamma)} h(x; \alpha_r) dx. \tag{6}
\]

As consumers do not purchase any additional warranty service in this case, we can obtain \(\pi = 0\)

As the third party after-sale department adopts minimal repair to rectify failures and the rate of additional warranty service profit is \(p'\), we can obtain

\[
C_d = \frac{C_m}{(1 - p')} \int_{W_1}^{W_3} \frac{1}{(1 + \gamma)} h(x; \alpha_r) dx, \tag{7}
\]

\[
C_a = \frac{C_m}{1 - p'} \int_{W_1}^{W_3} h(x; \alpha_r) dx. \tag{8}
\]

Case B. Complete/minimal maintenance strategy adopted in \((0, W_1) \times (0, U_t)\)

In Case B, manufacturers provide free maintenance service which combines with complete repair and minimal repair in base warranty. To optimize the warranty cost function, we obtain the mapping coordinate values of cutoff points on the age axis are \(K'_r\) and \(L'_r\) in the region of \((0, W_1) \times (0, U_t)\). Using equation (5), the manufacturer warranty cost in Case B can be obtained as follows:

\[
C_o = C(K'_r, L'_r, W_1, W_1, (W_1, \gamma)) \tag{9}
\]

where \(K'_r\) and \(L'_r\) are the optimal age coordinate values when the optimal warranty region is \((0, W_1) \times (0, U_t)\) and \(W_x = W_y = W_z = W_1\).
Similar to Case A, we obtain \( \pi = 0 \).

Considering the discount rate and the consumer feeling cost or actual payment, we obtain consumer total warranty cost of the product lifetime as follows:

\[
C_d = (C(K_e', L_e', W_1, W_3, (W_3, \gamma)) - C(K_e', L_e', W_1, W_1, (W_1, \gamma)))/(1 - p'), \tag{10}
\]

\[
C_a = (C(K_e', L_e', W_1, W_3, (W_1, \gamma)) - C(K_e', L_e', W_1, W_1, (W_1, \gamma)))/(1 - p'). \tag{11}
\]

### 3.2.2. Situation 2: Consumers purchase the extended warranty service in \((0, W_e) \times (0, U_e)\) - \((0, W_1) \times (0, U_1)\), and the total warranty period is \((0, W_e) \times (0, U_e)\)

In this situation, consumers purchase the two-dimensional extended warranty service. In contrast to Situation 1, the warranty cost in the base and extended warranties shall be borne by manufacturers, and manufacturers can profit in the extended warranty period. After the extended warranty expires, consumers can obtain after-warranty service from the third party after-sale department. The consumer total cost includes the extended warranty service price and after-warranty cost. Similar to Situation 1, we also consider minimal repair and complete/minimal maintenance strategy in this situation.

**Case C. Minimal repair adopted in \((0, W_e) \times (0, U_e)\)**

In this case, consumers can obtain the base warranty service for free and the extended warranty service at an additional cost. When minimal repair is adopted in this case, the manufacturer warranty cost in \((0, W_e) \times (0, U_e)\) can be obtained as follows:

\[
C_a = \int_0^{W_2} \frac{C_m}{(1 + \gamma)^x} h(x; \alpha_r) dx. \tag{12}
\]

The additional warranty service price in Case C is obtained as follows:

\[
P = P_e = \frac{C_m}{1 - p'} \int_{W_1}^{W_2} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) dx. \tag{13}
\]

The manufacturer profit in Case C is obtained as follows:

\[
\pi = \frac{C_m}{1 - p'} p' \int_{W_1}^{W_2} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) dx. \tag{14}
\]

Similar to Equations (7) and (8), we can obtain

\[
C_d = \frac{C_m}{1 - p'} \int_{W_1}^{W_2} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) dx + \frac{C_m}{1 - p'} \int_{W_2}^{W_3} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) dx \tag{15}
\]

\[
= \frac{C_m}{1 - p'} \int_{W_1}^{W_3} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) dx,
\]

\[
C_a = \frac{C_m}{1 - p'} \int_{W_1}^{W_2} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) dx + \frac{C_m}{1 - p'} \int_{W_2}^{W_3} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) dx. \tag{16}
\]

**Case D. Complete/minimal maintenance strategy adopted in \((0, W_e) \times (0, U_e)\)**

In this case, manufacturers provide consumers with complete/minimal maintenance service in \((0, W_e) \times (0, U_e)\). Similar to Case C, the service in base warranty is free, and the service in extended warranty is prepaid.
The manufacturer warranty cost in Case D with usage rate $r$ can be obtained as follows:

$$C_d = C(K_r^*, L_r^*, W_2, W_2, (W_2, \gamma)), \quad (17)$$

where $K_r^*$ and $L_r^*$ are the optimal age coordinate values when the warranty coverage is $(0, W_e) \times (0, U_e)$ and $W_e = W_y = W_z = W_2$.

As the rate of additional warranty service profit is $p$, the additional warranty service price and the manufacturer profit in Case D can be obtained as follows:

$$P = P_e = \frac{C(K_r^{**, L_r^*, W_2, W_2, (W_2, \gamma)}) - C(K_r^{**}, L_r^{**}, W_2, W_1, (W_1, \gamma))}{(1 - p')} \quad (18)$$

$$\pi = P_p' = \frac{C(K_r^{**, L_r^*, W_2, W_2, (W_2, \gamma)}) - C(K_r^{**}, L_r^{**}, W_2, W_1, (W_1, \gamma))}{(1 - p')} \quad (19)$$

$$C_o = \frac{C(K_r^{**}, L_r^{**}, W_2, W_2, (W_2, \gamma)) - C(K_r^{**}, L_r^{**}, W_2, W_1, (W_1, \gamma))}{(1 - p')} + \frac{C(K_r^{**}, L_r^{**}, W_2, W_3, (W_2, \gamma)) - C(K_r^{**}, L_r^{**}, W_2, W_1, (W_1, \gamma))}{(1 - p')} \quad (20)$$

$$= \frac{C(K_r^{**}, L_r^{**}, W_2, W_3, (W_2, \gamma)) - C(K_r^{**}, L_r^{**}, W_2, W_1, (W_1, \gamma))}{(1 - p')} \quad (21)$$

### 3.2.3. Situation 3: Consumers purchase the lifetime warranty service, and the total warranty period is $(0, W_i) \times (0, U_i)$.

In this situation, manufacturers can profit in the extended warranty and after-warranty period, and the manufacturer cost occurs in the lifetime warranty period, including the base, extended, and after warranties. As previously mentioned, if consumers purchase the lifetime warranty service, then the prorated extended warranty is adopted.

**Case E. Minimal repair adopted in $(0, W_i) \times (0, U_i)$**

Manufacturers provide consumers with extended warranty service and after-warranty service with extra cost, and minimal repair is adopted in this case. The manufacturer warranty cost in Case E can be obtained as follows:

$$C_o = \int_0^{W_3} \frac{C_m}{(1+\gamma)x} h(x; \alpha_r)dx \quad (22)$$

Due to consumers purchase the prorated extended warranty service and after-warranty service, we can obtain

$$P = (1 - \mu)P_e + P_a = \frac{C_m(1-\mu)}{1-p'} \int_{W_1}^{W_2} \frac{1}{(1+\gamma)x} h(x; \alpha_r)dx + \frac{C_m}{1-p'} \int_{W_2}^{W_3} \frac{1}{1+\gamma} x h(x; \alpha_r)dx. \quad (23)$$
usage rate $r$ as follows:

$$
\pi = (1 - \mu)P_e + P_a - \int_{W_i}^{W_2} \frac{C_m}{(1 + \gamma)^x} h(x; \alpha_t) \, dx
$$

$$
= \frac{C_m(1 - \mu)}{1 - P'} \int_{W_1}^{W_2} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) \, dx + \frac{C_m}{1 - P'} \int_{W_2}^{W_3} \frac{1}{(1 + \gamma)^x} h(x; \alpha_r) \, dx
$$

$$
- \int_{W_1}^{W_3} \frac{C_m}{(1 + \gamma)^x} h(x; \alpha_r) \, dx \tag{24}
$$

As the consumer cost of the product lifetime considering the discount rate is equal to the additional warranty service price, the consumer cost of the product lifetime in Case E can be obtained as follows:

$$
C_a = C_d = P \tag{25}
$$

Case F. Complete/minimal maintenance strategy adopted in $(0, W_i) \times (0, U_l)$

Similar to Case E, we can obtain the manufacturer warranty cost in Case F with usage rate $r$ as follows:

$$
C_a = C (K''_r, L''_r, W_3, W_3, (W_3, Y)) \tag{26}
$$

The additional warranty service price can be obtained as follows:

$$
P = (1 - \mu)P_e + P_a = \frac{1 - \mu}{1 - P'} \left( C (K''_r, L''_r, W_3, W_2, (W_3, Y)) - C (K''_r, L''_r, W_3, W_1, (W_3, Y)) \right)
$$

$$
+ \frac{1}{1 - P'} \left( C (K''_r, L''_r, W_3, (W_3, r)) - C (K''_r, L''_r, W_3, W_2, (W_3, r)) \right)
$$

$$
- \frac{1}{1 - P'} C (K''_r, L''_r, W_3, W_1, (W_3, Y)) \tag{27}
$$

The manufacturer profit in Case F can be obtained as follows:

$$
\pi = (1 - \mu)P_e + P_a - \frac{P'}{1 - P'} \cdot C (K''_r, L''_r, W_3, W_3, (W_3, Y))
$$

$$
- \frac{\mu P'}{1 - P'} C (K''_r, L''_r, W_3, W_2, (W_3, Y)) - \frac{1 - \mu}{1 - P'} P' C (K''_r, L''_r, W_3, W_1, (W_3, Y)) \tag{28}
$$

Similar to Case E, we can obtain the consumer cost of the product lifetime in Case F as follows:

$$
C_a = C_d = P. \tag{29}
$$

4. Mathematical model. Previous studies show that the Weibull distribution is widely used in warranty strategy studies (Banerjee and Bhattacharjee 2012[2]; Jack et al. 2009[14]; Shahanaği et al. 2013[25]; Tong et al. 2014[29]; Tong et al. 2017[30]). In this study, we also adopt the Weibull distribution function as the failure distribution, which is expressed as follows:

$$
F_0 (x; \alpha_0) = 1 - \exp (-x/\alpha_0)^\varepsilon \tag{30}
$$

where $\varepsilon$ is the shape parameter of the Weibull distribution function.

Using Equations (1) and (30), we can obtain the following expression:

$$
F (x; \alpha_t) = 1 - \exp \left( -x \left( \frac{r}{\alpha_0} \right)^T / \alpha_0 \right)^\varepsilon \tag{31}
$$
If minimal repair is adopted in warranty coverage, we can calculate the manufacturer cost, consumer cost, additional warranty service price and profit using Equations (2), (31) and equations in Cases A, C and E. However, for combined maintenance strategy, it is essential to calculate the optimal coordinate values of the cutoff points $K_r$ and $L_r$. Through literature review, we adopt the approach of calculating the optimal values, which was developed by Jack et al. (2009) (see (Jack et al. 2009) for detailed proof)[14].

If $(W; \alpha(r)) - 2H(W/2; \alpha(r)) \leq \rho - 1$, then $L^*_r = K^*_r$. Minimal repair is adopted in the warranty period, where $W \in W_1, W_2, W_3$ $L^*_r \in \{ L^r_1, L^r_2, L^r_3 \}$ and $K^*_r \in \{ K^r_1, K^r_2, K^r_3 \}$. If $H(W; \alpha(r)) - 2H > \rho - 1 L^*_r$ is the large root of the equation $H(W - L_r; \alpha_r) - H(W; \alpha_r) + H(L_r; \alpha_r) + \rho - 1 = 0$ and $K^*_r$ is the small root of the equation $F(K_r; \alpha_r) - F(L^*_r; \alpha_r) - \int_{K^*_r}^{L^*_r} h(W - x; \alpha_r) F(x; \alpha_r) dx = 0$ where $W \in \{ W_1, W_2, W_3 \}$, $L^*_r \in \{ L^1_1, L^1_2, L^1_3 \}$ $K^*_r \in \{ K^1_1, K^1_2, K^1_3 \}$, $r^*_r \in [0, W/2]$ and $r^*_r \in (W/2, W]$.

$$C_2 = C_m \left( \int_0^{k^*_1} \frac{2r^4x}{(1 + \gamma)^{x}} dx + \int_{k^*_1}^{L^*_1} \left( \rho + \int_0^{W-x} \frac{2r^4x}{(1 + \gamma)^{x}} dx \right) \frac{-\exp(-r^4x^2)}{\exp(-r^4K^2_1)} dx' \right) + \left( \int_0^{W} \frac{2r^4x}{(1 + \gamma)^{x}} dx - \int_{L^*_1}^{K^*_1} \frac{2r^4x}{(1 + \gamma)^{x}} dx \right) \frac{-\exp(-r^4L^*_1)}{\exp(-r^4K^2_1)}.$$

To illustrate the procedure of the proposed model, an automobile component produced in China is analyzed. For the warranty date is confidential, this paper use A to represent the component. As the base warranty period varies with different component of automobile, this paper obtain the base warranty coverage of A as two years or 60,000 kilometers from a warranty service provider. As mentioned previously, warranty service providers can provide extended warranty service with different coverage, and the lifetime of different types of products is also different. If manufacturers provide lifetime warranty service to consumers, they need to set the lifetime warranty coverage according to the characteristics of the product. To simplify the calculation, this paper set $W_e = 4t(\text{Year}), U_e = 12(10^4 \text{Km})$ $W_l = 6(\text{Year})$ and $U_l = 18(10^4 \text{Km})$ and $U_1 = 18(10^4 \text{Km})$. Hence, we can obtain $r_1 = r_2 = r_3 = 3(10^4 \text{Km/Year})$. Through an investigation of an automobile dealer which provide warranty service shows that the expected minimal and complete repair costs of A can be approximated to 600 yuan and 1200 yuan, respectively. Hence, we obtain $\rho = 2$. Moreover, the rate of additional warranty service profit, the rebate rate and the discount rate can be set as 0.5, 0.2 and 0.1 in this numerical example. Besides, this study set $\alpha_0 = 1, \varepsilon = 2$ and $\tau = 2$, which is the same as the parameter setting in (Jack et al. 2009[14]). As previously stated, the parameters of this numerical example can be listed in Table 2.

| $W_t$ | $U_t$ | $r_1$ | $W_e$ | $U_e$ | $r_2$ | $W_1$ | $U_1$ | $r_3$ | $C_m$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2     | 6     | 3     | 4     | 12    | 3     | 6     | 18    | 3     | 600   |

Table 2. Parameter setting

| $\rho$ | $\gamma$ | $\mu$ | $\rho'$ | $\alpha_0$ | $\varepsilon$ | $\tau$ |
|-------|-----------|-------|---------|-------------|--------------|-------|
| 0.1   | 0.2       | 0.5   | 1       | 2           | 2            | 2     |
In the following subsection, we select usage rate \( r = 0.9(10^4 \text{KmYear}) \) for the numerical example.

### 4.1. Model validation (\( r = 0.9(10^4 \text{KmYear}) \)).

Using the proposed model and parameters provided in Tables 1 and 2, we obtain \((W_1, U_1) = (2, 1.8)\), \((W_2, U_2) = (4, 3.6)\) and \((W_3, U_3) = (6, 5.4)\) when \( r = 0.9 \). On this basis, we can calculate the numerical values listed in Tables 3–7 using the previously formulated mathematical model and theorem.

### Table 4. Consumer cost of the product lifetime (\( r = 0.9 \))

| Warranty period | \((0, W_t) \times (0, U_t)\) | \((0, W_e) \times (0, U_e)\) | \((0, W_l) \times (0, U_l)\) |
|-----------------|-----------------------------|-----------------------------|-----------------------------|
| Minimal Complete /minimal | Minimal Complete /minimal | Minimal Complete /minimal |
| \( C_0 \)       | 1388.16                     | 1416.48                     | 4905.06                     |
| \( C_a \)       | 25194.24                    | 15735.00                    | 16763.64                    |
| \( K_r' / K_r' / K_r'' \) | 0.80                        | 1.67                        | 2.76                        |
| \( L_r' / L_r' / L_r'' \) | 1.48                        | 3.80                        | 5.87                        |

### Table 5. Revised additional warranty service price (\( r = 0.9 \))

| Warranty period | \((0, W_t) \times (0, U_t)\) | \((0, W_e) \times (0, U_e)\) | \((0, W_l) \times (0, U_l)\) |
|-----------------|-----------------------------|-----------------------------|-----------------------------|
| Minimal Complete /minimal | Minimal Complete /minimal | Minimal Complete /minimal |
| \( P \)         | 7033.8                      | 4422.48                     | 15356.88                    |
| \( K_r' / K_r' / K_r'' \) | 0.80                        | 1.67                        | 2.76                        |
| \( L_r' / L_r' / L_r'' \) | 1.48                        | 3.80                        | 5.87                        |

### Table 6. Manufacturer additional warranty service profit (\( r = 0.9 \))

| Warranty period | \((0, W_t) \times (0, U_t)\) | \((0, W_e) \times (0, U_e)\) | \((0, W_l) \times (0, U_l)\) |
|-----------------|-----------------------------|-----------------------------|-----------------------------|
| Minimal Complete /minimal | Minimal Complete /minimal | Minimal Complete /minimal |
| \( \pi \)       | 0                            | 2128.74                     | 880.5                       |
| \( K_r' / K_r' / K_r'' \) | 0.80                        | 1.67                        | 2.76                        |
| \( L_r' / L_r' / L_r'' \) | 1.48                        | 3.80                        | 5.87                        |
4.2. Validation analysis. Through the analysis of above tables, some rules can be easily determined. For example, the manufacturer cost and extra warranty service price increase with the extension of the warranty period for a longer warranty period, which incurs further warranty cost (i.e., 1388.16 < 4905.06 < 9769.98, 1416.48 < 3541.5 < 6643.12, 7033.8 < 15356.88, and 4422 < 8411.88; see Tables 3 and 5 for details). Meanwhile, it can be found that the manufacturers’ additional warranty service profit increase with the increase of the additional warranty period (i.e., 0 < 2128.74 < 5586.9 and 0 < 880.5 < 1768.74) despite the prorated warranty strategy being adopted (see Table 6 for details). Moreover, if manufacturers simply make the maintenance decision from the profit index, then the minimal repair should be adopted for 2128.74 > 880.5 and 5586.9 > 1768.74. However, warranty is an important part of marketing strategy, and it focuses on maintaining customer loyalty and highlighting the quality of products. The purpose of the extra warranty is to reduce the huge warranty cost pressure caused by extending the base warranty period, and making profit as its main goal is unreasonable (Lei et al. 2017)[16]. Moreover, adopting minimal repair strategy will considerably increase the consumer cost of product lifetime, which is not conducive to the quality signal transmission of manufacturers and the quality approval of consumers. In this regard, this paper recommends a combined maintenance strategy in the two-dimensional lifetime warranty coverage. Further analysis of the tables shows that the manufacturer cost, consumer cost and additional warranty service price of combined maintenance strategy are smaller than those of the minimal repair, except for an abnormal phenomenon (see Tables 3–5 for details), in which if consumers purchase no extra warranty service, then the manufacturer cost under combined maintenance strategy is larger than the case of minimal repair (i.e., 1416.48 > 1388.16). The result is different from that of Jack et al. (2009)[14] for the difference in cost cash flow that leads to this phenomenon, considering the capital time value in this proposed model. From the angle of base warranty cost reduction, manufacturers should adopt minimal repair for 1388.16 < 1416.48. However, consumers will bear further warranty cost or feeling cost in the remaining product warranty period. In addition to the above analysis, this paper conducts a detailed analysis of the consumer cost when the combined maintenance strategy is adopted (see Table 7 for details). Using the same calculation steps proposed in Section 3, Table 7 is obtained. The optimal value of $K_{r}$ and $L_{r}$ in three situations can be calculated as (0.80, 1.48), (1.67, 3.80) and (2.76, 5.87). Consequently, the product lifetime is divided into five subintervals (see Table 7 for details).
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**Table 7.** Base warranty cost and consumer cost of the product lifetime \((r = 0.9)\)

| Warrant period | \((0, W_t) \times (0, U_t)\) | \((0, W_e) \times (0, U_e)\) | \((0, W_l) \times (0, U_l)\) |
|----------------|-------------------------------|-------------------------------|-------------------------------|
|                | Base warranty                 | After base warranty            | Base warranty                 |
|                | Complete/Minimal              | Complete/Minimal              | Complete/Minimal              |
| Subinterval    |                               |                               |                               |
|               | \([0, 0.8)\)                 | \([1.48, 2\)                 | \([2, 4)\)                 |
| Cost of subinterval | 239 [.52 | 958 [.98 | 217 [.98 | 565 [.26 | 886 [.98 | 988 [.02 | 1554 [.48 | 1995 [.52 | 0.48 | 885 | 138 | 181 | 46 | 6579 | 19 | 328 | e-5 |
| \( C_d \)      | 165522.16                     | 10854.12                     | 8411.88                      |
| \( K_r' / K_r'' \) | 0.80 | 0.80 | 0.80 | 0.80 | 1.67 | 1.67 | 1.67 | 1.67 | 2.76 | 2.76 | 2.76 | 2.76 |
| \( L_r' / L_r'' \) | 1.48 | 1.48 | 1.48 | 1.48 | 3.80 | 3.80 | 3.80 | 3.80 | 5.87 | 5.87 | 5.87 | 5.87 |

As the warranty cost that occurs in the region of base warranty shall be borne by manufacturers, this paper separates product lifetime into two parts, namely, base warranty period and after base warranty period. Using the proposed equations, we can calculate the cost in each subinterval (see Table 7 for details). Manufacturers can hardly predict the usage rate of consumers when products are sold, which has been observed in our previous studies and other works (Ye and Murthy 2016)[36]. Meanwhile, minimal repair is the most commonly used in the actual operation of base warranty period. To avoid the additional warranty cost to manufacturers
caused by the combined maintenance strategy, it is necessary to modify the determination of the boundary parameter rules as follows.

If \( K^*_r < W_1 \), then \( K^*_r = W_1 \), where \( K^*_r \leq L^*_r \cdot K^*_r < W_1 \).

Using the revised rules, the mapping coordinate values of the optimal cutoff points on the age axis can be obtained as follows:

- For the region of \((0, W_1) \times (0, U_1)\), \( K^*_r = 0.8 < W_1 = 2 \) and \( L^*_r = 1.48 < W_1 = 2 \), then \( K^*_r = 2 \) and \( L^*_r = 2 \).
- For the region of \((0, W_1) \times (0, U_1)\), \( K^*_r = 1.67 < W_1 = 2 \) and \( L^*_r = 3.80 > W_1 = 2 \), then \( K^*_r = 2 \) and \( L^*_r = 3.80 \).
- For the region of \((0, W_1) \times (0, U_1)\), \( K^*_r = 2.76 > W_1 = 2 \) and \( L^*_r = 5.87 > W_1 = 2 \), then, \( K^*_r = 2.76 \) and \( L^*_r = 5.87 \).

Using the same calculation steps proposed above, the revised values can be obtained (See details in Tables 8–11).

### 5. Conclusions and suggestions.

With the development of design and manufacturing technology, consumers have been demanding for enhanced warranty service, which includes long warranty period and low failure rate. On the basis of previous related warranty studies, a prorated lifetime warranty strategy for high-value and durable products has been proposed.

In view of the interaction between maintenance strategy, failure function and warranty cost, combined maintenance strategy and minimal repair are introduced.

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**Table 8. Revised manufacturer cost in the base warranty and additional warranty period \( (r = 0.9) \)**

| Warranty period | \((0, W_t) \times (0, U_t)\) | \((0, W_e) \times (0, U_e)\) | \((0, W_l) \times (0, U_l)\) |
|-----------------|-------------------------------|-------------------------------|-------------------------------|
| \( C_0 \)       | 1388.16                       | 1388.16                       | 4905.06                       |
| \( K'_r/K''_r \) | 2                             | 2                             | 2.76                          |
| \( L'_r/L''_r \) | 2                             | 3.80                          | 5.87                          |

**Table 9. Revised consumer cost of the product lifetime \( (r = 0.9) \)**

| Warranty period | \((0, W_t) \times (0, U_t)\) | \((0, W_e) \times (0, U_e)\) | \((0, W_l) \times (0, U_l)\) |
|-----------------|-------------------------------|-------------------------------|-------------------------------|
| \( C_d \)       | 16763.64                      | 16763.64                      | 10860.24                      |
| \( C_a \)       | 25194.24                      | 22780.2                       | 13077.12                      |

**Table 10. Revised additional warranty service price \( (r = 0.9) \)**

| Warranty period | \((0, W_t) \times (0, U_t)\) | \((0, W_e) \times (0, U_e)\) | \((0, W_l) \times (0, U_l)\) |
|-----------------|-------------------------------|-------------------------------|-------------------------------|
| \( P \)         | 7033.8                        | 4422.48                       | 15356.88                      |
| \( K'_r/K''_r \) | 2                             | 2                             | 2.76                          |
| \( L'_r/L''_r \) | 2                             | 3.80                          | 5.87                          |
Table 11. Revised manufacturers’ additional warranty service profit \((r = 0.9)\)

| Warranty period | \((0, W_t) \times (0, U_t)\) | \((0, W_c) \times (0, U_c)\) | \((0, W_l) \times (0, U_l)\) |
|-----------------|----------------------------|----------------------------|----------------------------|
|                 | Minimal / minimal          | Minimal / minimal          | Minimal / minimal          |
| \(\pi\)         | 0                          | 0                          | 2128.74                    |
| \(K' / K'' / K''''\) | 2                          | 2                          | 823.08                     |
| \(L' / L'' / L''''\) | 2                          | 2                          | 3.80                       |

into the failure model. Meanwhile, consumer can decide whether to purchase additional warranty service or not, the warranty coverage is divided into three situations, that is \((0, W_t) \times (0, U_t)\), \((0, W_c) \times (0, U_c)\) and \((0, W_l) \times (0, U_l)\). Different from previous studies, the factor of capital time value and prorated warranty are considered into the mathematical model for the long-period lifetime warranty coverage. To illustrate the procedure and compare different situations and cases, this paper have analyzed and compared the manufacturer cost (See detail in table 3), consumer cost (See detail in table 4), additional warranty service price (See detail in table 5), additional warranty service profit (See detail in table 6). Some findings have been discovered through the validation analysis. Firstly, the proposed lifetime warranty strategy can effectively increase the profit of manufacturer (i.e., \(0 < 2128.74 < 5586.9\) and \(0 < 880.5 < 1768.74\)), See Table 11 for details. Secondly, although adopting minimal repair throughout lifetime warranty period would increase manufacturers’ profit, it would also greatly increase consumer spending (i.e., \(10854.12 < 16763.64\) and \(8411.88 < 15356.88\)). As the purpose of additional warranty is to reduce the huge cost borne by consumers and manufacturers, the combined maintenance strategy should be adopted. Thirdly, the proposed lifetime warranty strategy can reduce the expected cost expenditure of consumers throughout product lifetime (i.e., \(15356.88 < 16763.64\) and \(8411.88 < 10854.12 < 16763.64\)), See Table 9 for details. Finally, to analyze the abnormal phenomenon in Table 3 (1416.48 > 1388.16), this paper further analyze the consumer cost of the product lifetime when the combined maintenance strategy is adopted (See detail in Table 7) and modify the determination of the boundary parameter rules to facilitate the rational implementation of the proposed model.

The main objective of this study is to provide consumers with a comprehensive warranty service throughout product lifetime. The proposed prorated lifetime warranty strategy can provide lifetime warranty service and reduce the expected cost expenditure to consumers. Meanwhile, the research results show that the lifetime warranty service can provide additional profit sources to manufacturers, and combined maintenance strategy can better meet the realization of warranty objectives in the long-period lifetime warranty coverage.

In this study, usage rate \(r = 0.9\) is selected to illustrate and analyze the model, and the parameter variation trend is similar after the calculation of several other sets of parameters. In order simplify the article and increase the readability, this paper does not calculate other usage rate data. Besides, some scholars have assumed that the usage rate is subject to a uniform distribution (Iskandar and Murthy 2003[13]; Wang et al. 2015[33]). Uniform distribution is easy to use, but it is difficult to describe the true distribution of consumer usage rate. Besides, there are also some scholars obtain the usage rate distribution function by parameter fitting. Using the
proposed model and usage rate distribution function, the lifetime warranty strategy of different consumer groups can be further analyzed. In practical application, the parameters, such as usage rate, minimal cost, complete cost coordinate values of warranty region and the rate of extra warranty service profit should be determined in conjunction with specific products. For example, the rate of additional warranty service profit of manufactures and third-party after-sale may not be the same in practice. To simplify the calculation, this paper assumes that the two are equal. Moreover, to satisfy the needs of different consumer groups, differentiated warranty strategy need to be further studied.

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