Local cubic vertex functions for three massless higher even spin fields on spaces $AdS_D$: An analytic approach

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Abstract

Local cubic vertex functions of three higher even spin fields on $AdS_D$ are constructed from the Green function of three conserved currents that are dual to the higher spin fields. Conservation of the currents implies lowest order gauge invariance. These vertex functions appear by the UV divergence as the residue of the highest order pole in the dimensional regularization parameter $\epsilon$. In fact $N$-point Green functions of such conserved currents produce a series of poles up to the order $N - 1$. The method works for even $D$ and maintains covariance at any step. The resulting formula is quite concise.

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# Introduction

It is the aim of this article to construct vertex functions for three massless higher spin fields of even spin on AdS spaces. This construction is done in such fashion that these vertex functions are local and observe gauge invariance of lowest order. The history of cubic vertex functions on flat or (anti)deSitter spaces is long and has put forward different aspects of interest. First we must mention the seminal work of Fradkin and Vasiliev [3]. The frame-like approach to vertex functions of any order and non-abelian gauge symmetry proposed by these authors has developed in recent decades into several directions (for a recent source see [4]).

In flat spaces the general 3-vertex functions were constructed in [8] and restrictions on the form of these vertex function that were achieved earlier [7] could be verified. Connection with string theory was emphasized in [9]. Contrary to the very satisfactory results for flat spaces, the situation for AdS spaces is still not completely satisfactory at least in the sense that the resulting formulae do not exhibit such a simple and beautiful form as in the flat case. There are different methods applied to solve the AdS cases. Using flat ambient spaces and then reducing the dimension by one looks as a convincing ansatz but technical complications show up when this is done explicitly [5], [6]. Instead of this ansatz we use now a method based on quantum field theory and the regularization techniques of UV divergences leading to a compact final formula.

We contract a higher spin field of a symmetric tensor representation of rank ("spin") $s$ with a conserved current of the same representation. This current is built bilinearly from a real scalar conformal free field admitting only even rank tensors. The vacuum expectation value of three such currents is UV divergent, and this divergence can be characterized by a polynomial in $\epsilon^{-1}$, where this parameter $\epsilon$ can be introduced by a deformation of the dimension $D$ of the AdS space. For $N$ such currents the $N$-point Green function yields a UV divergence described by a polynomial of degree $N-1$. We are interested here only in the residue of the highest order pole term. It is known to preserve a symmetry such as gauge invariance. This method has been developed for flat spaces [1], in which case we were also able to successfully compare the results with the general known formulas [8]. It seems that the postulate of AdS covariance introduces new problems. The reader will find, however, that the choice of an appropriate mathematical formalism simplifies the whole task dramatically. This formalism was developed in [2] a few years ago when the quantum one loop trace anomaly of the same fields and currents was analysed for $AdS_4$. The restriction to the dimension $D = 4$ was motivated by the desire to compare the results with known anomalies in gravitation theory. In the present context such motivation does not seem to exist and we can as well and will deal with cases of any even space dimension. The relevant formulæ of [2] are generalized for this purpose to any even dimension $D$.

All traces of the higher spin fields are neglected, so that our result holds for traceless higher spin fields. This can be justified by two arguments: The physically
relevant terms are still present in the result, and the trace terms are uniquely determined by the traceless parts. The $A_dS$ space radius $L$ is, as is often done, fixed to the value 1. An arbitrary value $L$ can be recovered by replacing the square bracket of $D_k$ defined in (4.14) and in the final formula (5.1) by $\Box + L^{-2}[\Delta(\Delta + 1) - n(n + 1)]$.

An essential part of the formalism developed in this article uses a general basis of bitensors [10] to express the two-point functions of the currents. The same technique was developed to present two-point functions of higher spin fields in [11] and applied to two-point functions of currents in [2].

2 Scalar fields and conserved currents

It is remarkable though of course natural that we go back to the analysis of anomalies of 2-loop functions on $A_dS_D$ in [2] when we study the singular part of N-loop functions. The scalar field and the currents are in fact the same. The free scalar conformal field $\sigma(z)$ has the two point function [13]

$$<\sigma(z_1)\sigma(z_2)> = w(\zeta), \quad \zeta - 1 = \frac{(z_1 - z_2)^2}{2z_{1,0}z_{2,0}} = u \quad (2.1)$$

where Poincare coordinates are used. The function $w(\zeta)$ is a Legendre function of the second kind with the desired asymptotic behaviour which in terms of a Gaussian hypergeometric function is

$$w(\zeta) = \frac{\Gamma(\Delta)}{(2\pi)^{D/2}} \zeta^{-\Delta} F\left(\frac{\Delta}{2}, \frac{\Delta + 1}{2}; \frac{1}{2}; \zeta^{-2}\right) \quad (2.2)$$

and the parameter $\Delta$ denotes

$$\Delta = \frac{D}{2} - 1 \quad (2.3)$$

where $\Delta$ is the conformal dimension of the scalar field $\sigma$. For arbitrary $D$ the two-point function (2.2) is identical with the binomial expansion of

$$w(\zeta) = \frac{\Gamma(\Delta)}{2(2\pi)^{D/2}} \left[ \frac{1}{(\zeta - 1)^{\Delta}} + \frac{1}{(\zeta + 1)^{\Delta}} \right] \quad (2.4)$$

that is rational for even $D$. For odd $\Delta(D = 4n)$ replacing $\zeta$ by $-\zeta$ changes the sign of the function $w(\zeta)$. This symmetry is denoted antipodal symmetry, the sign change antipodal parity. We recognize that the 2-point function has two singular points at $\zeta = +1$ and $\zeta = -1$, such pair is called an antipodal pair. Correspondingly a 3-point function of three currents has eight maximal singular points when

$$\zeta_{1,2} = \pm 1, \quad \zeta_{2,3} = \pm 1, \quad \zeta_{3,1} = \pm 1 \quad (2.5)$$

Analogously the 3-vertex using (2.4) consists of eight parts. We shall choose that one which belongs to

$$\zeta_{1,2} = \zeta_{2,3} = \zeta_{3,1} = 1 \quad (2.6)$$
and concentrate our work on this alone.

For the conserved current we can take the expression \( J^{(s)}(z; a) \) from \([2]\), equ. (1) or from \([12]\), \( s \) is even. The complete expression of \( J^{(s)}(z; a) \) is a polynomial in \( a^2 \) of degree \( \frac{s}{2} \) and each factor of \((a^2)^n\) is a polynomial in \( L^{-2} \) (\( L \) is the radius of the AdS space) of degree \( n \). Since we want to consider only the traceless terms of the cubic vertex, we concentrate only on the \((a^2)^0\) part

\[
J^{(s)}(z; a) = \frac{1}{2} \sum_{p=0}^{s} A_p^{(s)}(a \nabla)^{s-p}\sigma(z)(a \nabla)^p\sigma(z) + a^2 \text{ terms} \tag{2.7}
\]

The coefficients are

\[
A_p^{(s)} = (-1)^p \binom{s}{p} \frac{(D/2 - 2)! (s + D/2 - 2)!}{(p + D/2 - 2)! (s - p + D/2 - 2)!} \tag{2.8}
\]

We are interested in the loop Green function

\[
\langle J^{(s_1)}(z_1; a_1) J^{(s_2)}(z_2; a_2) J^{(s_3)}(z_3; a_3) \rangle \tag{2.9}
\]

This Green function is evaluated by Wick’s theorem using (2.1).

3 Evaluating the Green function

Let \( F(\zeta) \) be an analytic function of \( \zeta \). Then we can use the formulae (24) and (25) of \([2]\),

\[
(a, \nabla_1)^p(b, \nabla_2)^q F(\zeta(z_1, z_2)) = \sum_{n=0}^{q-p} \frac{p!q!}{n!(q-n)!(p-q+n)!} I_{1/2}^{q-n} I_1(1, 2)^p I_2(1, 2)^q F^{(p+q)}(\zeta(z_1, z_2)) + a^2, b^2 \text{ terms} \tag{3.1}
\]

using the bitensor basis

\[
I_1(1, 2) = (a, \partial_1)\zeta(z_1, z_2) \tag{3.2}
\]

\[
I_2(1, 2) = (b, \partial_2)\zeta(z_1, z_2) \tag{3.3}
\]

\[
I_{1,2} = (a, \partial_1)(b, \partial_2)\zeta(z_1, z_2) \tag{3.4}
\]

where

\[
F^{(k)}(\zeta) = \frac{d^k}{d\zeta^k} F(\zeta) \tag{3.5}
\]

For the Green function we get in this fashion (up to trace terms)

\[
\sum_{p_1=0}^{s_1} \sum_{p_2=0}^{s_2} \sum_{p_3=0}^{s_3} A_{p_1}^{(s_1)} A_{p_2}^{(s_2)} A_{p_3}^{(s_3)} (a, \nabla_1)^{p_1}(b, \nabla_2)^{p_2} w(\zeta_{1,2})
\]

\[
\times (b, \nabla_2)^{p_2}(c, \nabla_3)^{p_3} w(\zeta_{2,3}) (c, \nabla_3)^{p_3}(a, \nabla_1)^{p_1} w(\zeta_{3,1}) \tag{3.6}
\]
which by equs. (3.1) to (3.4) yields
\[ \sum_{p_1=0}^{s_1} \sum_{p_2=0}^{s_2} \sum_{p_3=0}^{s_3} A(p_1) A(p_2) A(p_3) Q(p_1,p_2,p_3) \]
\[ \times w^{(p_1+s_2-p_2)}(\zeta_{1,2}) w^{(p_2+s_3-p_3)}(\zeta_{2,3}) w^{(p_3+s_1-p_1)}(\zeta_{3,1}) \]  
(3.7)

where the newly introduced function Q depends besides the parameters \( s_i, p_i \) on the three points \( z_1, z_2, z_3 \). It encodes the tensorial structure of the 3-point vertex function. The number of derivatives of the functions \( w \) in (3.7) are denoted by
\[ m_{i,i+1} = p_i + s_{i+1} - p_{i+1} \]  
(3.8)

A simple formula for \( Q \) is
\[ Q(p_1,p_2,p_3) = \sum_{n_{1,2}=\max\{0,s_2-p_2-p_1\}}^{s_2-p_2} \sum_{n_{2,3}=\max\{0,s_3-p_3-p_2\}}^{s_3-p_3} \sum_{n_{3,1}=\max\{0,s_1-p_1-p_2\}}^{s_1-p_1} \]
\[ (\Delta)_{m_{12}} (\Delta)_{m_{23}} (\Delta)_{m_{31}} \frac{(s_1-p_1)!(s_2-p_2)!(s_3-p_3)!}{n_{1,2}! n_{2,3}! n_{3,1}!} \]
\[ \times \left( \begin{array}{c} p_1 \\ s_2 - p_2 - n_{1,2} \end{array} \right) \left( \begin{array}{c} p_2 \\ s_3 - p_3 - n_{2,3} \end{array} \right) \left( \begin{array}{c} p_3 \\ s_1 - p_1 - n_{3,1} \end{array} \right) \]
\[ \times [I_1(1,2)I_2(1,2)/I_{1,2}]^{n_{1,2}} [I_1(1,2)^{p_1+p_2-s_2} I_2(2,3)/I_{2,3}]^{p_2} [I_2(2,3)/I_{2,3}]^{p_3} \]
\[ \times [I_1(3,1)I_3(3,1)/I_{3,1}]^{n_{3,1}} I_3(3,1)^{p_3} I_3^{p_3} \]  
(3.9)

Obviously the sums over the parameters \( n_{1,2}, n_{2,3}, n_{3,1} \) can be performed in terms of hypergeometric \( _1F_1 \) polynomials. The result has a cyclic order
\[ Q(p_1,p_2,p_3) = (\Delta)_{m_{12}} (\Delta)_{m_{23}} (\Delta)_{m_{31}} \]
\[ \times \left( \begin{array}{c} s_2 - p_2 \\ s_2 - p_1 - p_2 \end{array} \right) \frac{I_2(1,2)^{s_2-p_1-p_2} I_{1,2}^{p_1}} {I_{1,2}} \]
\[ \times _1F_1(-p_1; s_2 - p_1 - p_2 + 1; \frac{I_1(1,2) I_2(1,2)}{I_{1,2}}) \]
\[ \times \left( \begin{array}{c} s_3 - p_3 \\ s_3 - p_2 - p_3 \end{array} \right) \frac{I_3(2,3)^{s_3-p_2-p_3} I_{2,3}^{p_2}} {I_{2,3}} \]
\[ \times _1F_1(-p_2; s_3 - p_2 - p_3 + 1; \frac{I_2(2,3) I_3(2,3)}{I_{2,3}}) \]
\[ \times \left( \begin{array}{c} s_1 - p_1 \\ s_1 - p_1 - p_3 \end{array} \right) \frac{I_1(3,1)^{s_1-p_3-p_1} I_{3,1}^{p_3}} {I_{3,1}} \]
\[ \times _1F_1(-p_3; s_1 - p_3 - p_1 + 1; \frac{I_3(3,1) I_1(3,1)}{I_{3,1}}) \]  
(3.10)

If, say, \( s_2 - p_2 - p_1 \) is negative, in the first factor the function \( _1F_1 \) starts at the term
\[ [I_1(1,2)I_2(1,2)]^{p_1+p_2-s_2}, \]
thus replacing essentially the factor \( I_2(1,2)^{s_2-p_1-p_2} \) in front of \( _1F_1 \) by \( I_1(1,2)^{p_1+p_2-s_2} \). In either expression a zero at \( z_1 - z_2 \) of order \( | s_2 - p_1 - p_2 | \) is contained. A closer look at the zeros of \( Q \) will be presented in Section 5.
4 The regularization of the \( w \)-functions

Remember the definition of the UV divergent part of (2.1) with default normalization

\[
w(\zeta) = (\zeta - 1)^{-\Delta}, \quad \Delta = \frac{D}{2} - 1
\]

\[
u = \zeta - 1
\]

\[
w^{(n)}(\zeta) = (-1)^n (\Delta)_n (\zeta - 1)^{-\Delta - n}
\]

where \( D \) is even and \((\Delta)_n\) is a Pochhammer symbol. Moreover we restrict \( D \) to \( \geq 4 \). We intend to use the method of “dimensional regularization” by introducing a parameter \( \epsilon \) interpreted as a deformation of the dimension \( D \) of the \( AdS_D \) space. Then the regularized 3-vertex function appears as a rational function of \( \epsilon \) with a pole of maximal order 2 (for an \( N \)-vertex loop function it would be \( N - 1 \)). We select this pole from our vertex function (3.7) - (3.10) since it delivers us the local differential operator defining the interaction Lagrangian density.

According to [2] equ. (47) we have

\[
\frac{1}{u_{\Delta + n - \epsilon}} = \frac{(-1)^{\Delta + n - 1}}{\epsilon(\Delta + n - 1)!} \delta^{(\Delta + n - 1)}(u) + O(1)
\]

where \( \epsilon \) is thought of being hidden in \( \Delta = D/2 - 1 \) as a deformation of \( D \). This justifies the term “dimensional” regularization. Moreover we use [2] equ. (75) (\( \Omega_{D-1} \) is the area of the unit sphere in \( D \) dimensions)

\[
d\mu(z) = (2\zeta_0)^{-D} d^D z = [u(u + 2)]^\Delta du d\Omega_{D-1}, \quad u = \frac{(z_1 - z_2)^2}{2z_1^0 z_2^0}
\]

where the polar coordinates in \( AdS_D \) are defined by \( z_2 = z \) and \( z_1 \) as the pole (reference point)

\[
(2\zeta_1^0)^D \delta(z_2 - z_1) = \frac{(-1)^{\Delta} \delta^{(\Delta)}(u)}{\Delta! (u + 2)^{\Delta} \Omega_{D-1}}
\]

We must treat the distribution \( w \) with the normalization from (4.1))

\[
\lim_{\epsilon \to 0^+} \epsilon^2 u_{12}^{m_{12}} (\zeta_{12}) w^{(m_{23})}(\zeta_{23}) w^{(m_{31})}(\zeta_{31})
\]

\[
= (-1)^{m_{12} + m_{23} + m_{31}} (\Delta)_{m_{12}} (\Delta)_{m_{23}} (\Delta)_{m_{31}}
\]

\[
\times \lim_{\epsilon \to 0^+} \epsilon^2 u_{12}^{-m_{12} - \Delta + \epsilon} u_{23}^{-m_{23} - \Delta + \epsilon} u_{31}^{-m_{31} - \Delta + \epsilon}
\]

This distribution resulting in the limit is determined only by the geometry of the \( AdS \) space. We treat it in coordinate space. In flat space Fourier transforms are used and momenta are integrated over. Instead of applying corresponding harmonic analysis on \( AdS \), we will be content, however, with presenting the structure of the distribution and not all the explicit algebraic expressions. Then define

\[
r_{12} = \Delta + m_{12} - 1
\]
so that (4.4) goes into

\[ u_{12}^{-r_{12} - 1 + \epsilon} = \frac{1}{\epsilon} \frac{\delta^{(r_{12})}(u_{12})}{r_{12}!} \]  \hspace{1cm} (4.9)

We conclude from (4.9) that a delta function of \( u \) has the minimal derivative \( \Delta \) in order to define a distribution on AdS\( _D \) space. Moreover (4.9) shows that negative \( r_{12} \) do not contribute.

In order to express delta functions with argument \( u_{12} \) by delta functions with argument \( z_2 - z_1 \) we use a method developed in [2]. We start from

\[ \Phi_n(u) = \frac{\delta^{(n)}(u)}{(u + 2)\Delta} \]  \hspace{1cm} (4.10)

and apply the differential operator \((u + 2)^{\Delta} \Box\) (with the scalar Laplacian)

\[ (u + 2)^{\Delta} \Box \Phi_n(u) = A_n \delta^{(n+1)}(u) + B_n \delta^{(n)}(u) \]  \hspace{1cm} (4.11)

\[ A_n = 2(\Delta - n - 1), \quad B_n = n(n + 1) - \Delta(\Delta + 1) \]  \hspace{1cm} (4.12)

This allows us to formulate the recursion

\[ \Phi_{n+1}(u) = -D_n \Phi_n(u) \]  \hspace{1cm} (4.13)

\[ D_n = \frac{1}{2(n + 2) - (\Box + \Delta(\Delta + 1) - n(n + 1))} \]  \hspace{1cm} (4.14)

In the case of the variables \( u_{12} \) or \( u_{31} \) we place the pole of the coordinate system at \( z_1 \), define corresponding differential operators \( D_n \) and \( \Box \) acting on these coordinates (\( n = 2 \) respectively \( n = 3 \)) and denote them correspondingly by \( D_n(2), \Box(2) \) (respectively \( D_n(3), \Box(3) \)). Then we get e.g. by solving the recursion (4.13), (4.14) and starting from \( \Phi_\Delta \) using (4.6)

\[ \Phi_{n+\Delta}(u_{12}) = (-1)^{n+\Delta} \Delta! \{ \prod_{k=\Delta}^{n+\Delta-1} D_k(2) \} (2z_1^0)^D \delta(z_2 - z_1) \Omega_{D-1} \]  \hspace{1cm} (4.15)

Now we return from \( \Phi_n \) to the delta functions

\[ \delta^{(n+\Delta)}(u_{12}) = (u_{12} + 2)^{\Delta} \Phi_{n+\Delta}(u_{12}) \]

\[ = \sum_{\ell=0}^{\Delta} \frac{(-1)^\ell \Delta! 2^{\Delta-\ell}(n + \Delta)!}{\ell!(\Delta - \ell)!} \Phi_{n+\Delta-\ell}(u_{12}) \]  \hspace{1cm} (4.16)

\[ = (\Delta!)^2 \sum_{\ell=0}^{\Delta} \frac{(-1)^{n+\Delta-\ell}(n + \Delta)!}{\ell!(\Delta - \ell)!} \Phi_{n+\Delta-\ell}(u_{12}) \]

\[ \times \{ \prod_{k=\Delta}^{n+\Delta-\ell-1} D_k(2) \} (2z_1^0)^D \delta(z_2 - z_1) \Omega_{D-1} \]  \hspace{1cm} (4.17)

\[ \text{In order to recover the } L \text{ dependence we replace in (4.14) } \Box + \Delta(\Delta + 1) - n(n + 1) \text{ by } \Box + L^{-2}[\Delta(\Delta + 1) - n(n + 1)] \]
and we consider the case of $\zeta = \zeta_{1,2}$. In this case we have to replace $n + \Delta$ by $r_{12} = m_{12} + \Delta - 1$ and to multiply with (see (4.8), (4.9))

$$\frac{(-1)^{r_{12}}}{r_{12}!}$$

(4.18)

to obtain the distribution part of $w^{(m_{12})}(\zeta_{12})$. We introduce then a shorthand for the differential operator (4.17)

$$\Lambda_{r_{12}}(2) = \sum_{\ell \geq 0} \frac{2^{\Delta-\ell}}{\ell!(\Delta-\ell)!(r_{12}-\ell)!} \left\{ \prod_{k_1=\Delta} D_{k_1}(2) \right\}$$

(4.19)

that acts on the deltafunction

$$(2z_1^0)D\delta(z_2 - z_1)$$

(4.20)

Now we integrate partially with the result (where $*_a$ denotes contraction over $a$)

$$(2z_1^0)D\delta(z_2 - z_1)\Lambda_{r_{12}}(2)\{u_{23}^{-m_{23}-\Delta}u_{31}^{-m_{31}-\Delta} \times Q_{*a_1} *_{a_2} *_{a_3} h^{(s_1)} h^{(s_2)} h^{(s_3)}(z_1, z_2, z_3)\}$$

(4.21)

and denote from now on the factors behind the multi-cross by the shorthand

$$R_{p_1, p_2, p_3}(z_1, z_2, z_3) = Q_{*a_1} *_{a_2} *_{a_3} h^{(s_1)} h^{(s_2)} h^{(s_3)}(z_1, z_2, z_3)$$

(4.22)

First we study the case that $\Lambda_{r_{12}}$ acts on $R$. Then there results, replacing $u_{23}$ by $u_{31}$ using the deltafunction $\delta(z_2 - z_1)$

$$(2z_1^0)D\delta(z_2 - z_1)u_{31}^{-2\Delta-m_{23}-m_{31}} \times \Lambda_{r_{12}} R$$

(4.23)

This power of $u_{31}$ is denoted $-r_{31} - 1$ and deformed by adding $2\epsilon$. Then the leading UV divergent term is

$$\frac{1}{2\epsilon} \frac{(-1)^{r_{13}}}{r_{31}!} \delta^{(r_{31})}(u_{31})$$

(4.24)

and we can proceed as before and define a new differential operator

$$\Lambda_{r_{31}}(3) = \sum_{\ell \geq 0} \frac{2^{\Delta-\ell}}{\ell!(\Delta-\ell)!(r_{31}-\ell)!} \left\{ \prod_{k_2=\Delta} D_{k_2}(3) \right\}$$

(4.25)

so that we end up in this case after partial integration with

$$(2z_1^0)D\delta(z_2 - z_1)(2z_1^0)D\delta(z_3 - z_1)\Lambda_{r_{12}}(2)\Lambda_{r_{31}}(3)R(z_1, z_2, z_3)$$

(4.26)

However there is the more general case that

$$\Lambda_{r_{12}}(2)\{u_{23}^{-m_{23}-\Delta}R\} = \sum_{\kappa} \Theta^{(\kappa)}_{\epsilon} u^{-m_{23}-\Delta} \Theta^{(\kappa)}_{\epsilon} R$$

(4.27)
by Leibniz’s rule, where the case that for a special $\kappa_0$ we have
\[
\Theta^{(I)}_{\kappa_0} = \text{const}
\] (4.28)
has already been dealt with.

Consider the expression
\[
\Theta^{(I)}_{\kappa} u_{23}^{-m_{23}-\Delta}
\] (4.29)
In the resulting sum we consider the terms with the same factor
\[
u_{23}^{-m_{23}-\Delta-q}, q \geq 0
\] (4.30)
Then we get
\[
\Theta^{(I)}_{\kappa} u_{23}^{-m_{23}-\Delta} = \sum_{q \geq 0} P_{\kappa,q}(z_2-z_3, z_0^0) u_{23}^{-m_{23}-\Delta-q}
\] (4.31)
where $P$ is a polynomial in $z_2-z_3$, but squares $(z_2-z_3)^2$ are excluded. This negative power of $u_{23}$ has to be regularized as before in (4.19) and leads to the differential operator $\Lambda_{m_{23}+\Delta+q-1}(2)$.

Thus after the usual partial integration we end up with the expression
\[
(2z_0^0)^D \delta(z_2-z_1)(2z_0^0)^D \delta(z_3-z_1) \sum_q \Lambda_{m_{23}+\Delta+q-1}(2)
\]
\[
\sum_{\kappa} P_{\kappa,q}(z_2-z_1, z_0^0) \Theta^{(I)}_{\kappa}(2) \Lambda_{m_{31}+\Delta-1}(3) R_{p_1,p_2,p_3}
\] (4.32)

5 Discussion

By partial integration the polynomials of the Laplacians $\square(2), \square(3)$ (see [4.20]) acting on delta functions can be brought to act on the product of $Q$ and the higher spin fields. The result is
\[
\int \frac{d^Dz}{(2z_0^0)^D} \sum_{p_1=0}^{s_1} \sum_{p_2=0}^{s_2} \sum_{p_3=0}^{s_3} A_{p_1}^{(s_1)} A_{p_2}^{(s_2)} A_{p_3}^{(s_3)} (\Delta)_{m_{12}}(\Delta)_{m_{23}}(\Delta)_{m_{31}}
\]
\[
\times \sum_{q \geq 0} \Lambda_{m_{23}+\Delta+q-1}(2) \sum_{\kappa} P_{\kappa,q}(z_2-z_1, z_0^0) \Theta^{(I)}_{\kappa}(2) \Lambda_{m_{31}+\Delta-1}(3)
\]
\[
\times Q^{(s_1,s_2,s_3)}_{p_1,p_2,p_3} \ast a_1 \ast a_2 \ast a_3 \ h^{(s_1)}(z, a_1) h^{(s_2)}(z_2, a_2) h^{(s_3)}(z_3, a_3) \big|_{z_2=z_3=z}
\] (5.1)
where the shorthands (see (3.8))
\[
m_{i,i+1} = p_i + s_{i+1} - p_{i+1}
\] (5.2)
and (4.25), (4.27), (4.31) have been used. The asterisk symbols denote contractions that produce a scalar function of $Q$ and $h^{(s_1)} h^{(s_2)} h^{(s_3)}$. 

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The Laplacians and the gradients (from $\Theta^{\kappa}_{\epsilon}(2)$) act only on the variables $z_2$ and $z_3$. This is the effect of a partial integration eliminating all differentiations with respect to $z_1$. Other approaches to these vertex functions may give results symmetric in the three variables $z_1, z_2, z_3$, which makes any comparison with our result troublesome. In any case differentiations on the factors contained in $Q$ in (3.9) or (3.10) seem to necessitate an algorithmic computer program.

The maximal number of differentiations is

$$2(r_{12} + r_{31} - 2) = 3D + 2S - 4, \quad S = s_1 + s_2 + s_3 \quad (5.3)$$

However, in $Q$ there are zeros which have to be cancelled first before the differentiations act on the fields. These zeros are hidden in $I_i(i, i + 1), I_{i+1}(i, i + 1)$ and are each of order one. Thus the total number of zeros in $Q$ is

$$\Psi = |s_1 - p_1 - p_3| + |s_2 - p_2 - p_1| + |s_3 - p_3 - p_2| \quad (5.4)$$

Now assume that the three numbers $s_1, s_2, s_3$ satisfy triangular inequalities, then we can solve $\Psi = 0$ by

$$p_1 = \frac{1}{2}(s_1 + s_2 - s_3), \quad p_2 = \frac{1}{2}(s_2 + s_3 - s_1), \quad p_3 = \frac{1}{2}(s_3 + s_1 - s_2) \quad (5.5)$$

In this case the maximal number of derivatives acting on the fields is the one given by (5.3).

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