Role of transverse gluon in SSA

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It is known the single transverse spin asymmetry in semi-inclusive deep inelastic scattering can be factorized by a twist-3 distribution function $T_F$, which contains a gluon field strength tensor. With transverse gluon included in the power expansion, we find the gluon field strength tensor can be recovered definitely in soft-gluon-pole contribution at leading order of $\alpha_s$ expansion. This conclusion holds in Feynman and light-cone gauges.

I. INTRODUCTION

The single transverse spin asymmetry (SSA) in high-$P_T$ pion production is discovered in 70’s[1], and the asymmetry is of order 10%. A possible explanation in perturbative QCD for this large asymmetry is Efremov-Teryaev-Qiu-Sterman (ETQS) mechanism, which has been proposed for many years[2],[3]. In this mechanism, SSA is proportional to ETQS matrix element, which is a correlation function of quark and gluon fields defined on light-cone. How to obtain the twist-3 hard coefficients before ETQS matrix element has been discussed thoroughly in literatures. One of the remaining problems is how to recover gluon field strength tensor appearing in ETQS matrix element consistently. A clear algorithm to get the twist-3 hard coefficients is first given by Qiu and Sterman[3], in which the subcross section is expanded to $O(k_\perp)$. After the proof that only the matrix element $\langle \bar{\psi}(\partial^\perp_\perp) G^+ \psi \rangle$ appears after transverse momentum expansion, the ETQS matrix element $T_F$ then is obtained by the replacement $\partial^\perp_\perp G^+ \rightarrow -G^\perp_\perp^\perp$, where $G^\perp_\perp^\perp$ is gluon field strength tensor. This replacement is applied in almost all following works, see for example [4–9]. In [10–15] even loop correction is calculated in this formalism, and the twist-3 factorization formula is justified at one-loop level. Thus, this algorithms is reliable to give correct answer. But, since the contribution of $G^\perp$ is not calculated, there is still a problem whether the contribution of $G^\perp$ can be incorporated into ETQS matrix element. For a complete calculation, one has to also calculate the contribution of $G^\perp$ and to see whether the combination $\partial^\perp_\perp G^+ - \partial^+ G^\perp$ appears. This problem is studied by Eguchi, Koike and Tanaka in[16], where a group of consistence relations are derived in order to make sure gluon field strength tensor $G^\perp_\perp^\perp$ is correctly (completely) reproduced. For hard-gluon-pole and soft-fermion-pole contributions, these conditions are satisfied easily due to Ward Identities, and it is confirmed that $G^\perp$ expansion gives the same hard coefficients as that obtained from $G^+ \expansion$. However, for soft-gluon-pole(SGP) contribution, although the conditions are satisfied by analyzing the detailed cancellation between mirror diagrams, a direct calculation based on $G^\perp$ expansion is still missing. For this reason how to obtain SGP in light-cone gauge is not described in [16] either. It is argued that $G^\perp$ contribution contains some ambiguities and some hard coefficients may be lost due to $x\delta(x) = 0$. The analysis of [16] is very clear and thorough. But, as we will show in this paper, $G^\perp$ expansion is definite and gives the same hard coefficients as that obtained from $G^+$ expansion. The price for using $G^\perp$ expansion is one has to incorporate the contribution from another twist-3 distribution function $q_0(x)$, besides ETQS matrix element $T_F(x,x)$. Also, the coefficient of $q_0(x)$ can be determined definitely. As an example, we will consider the SSA for high $P_T$ pion production in semi-inclusive deep inelastic scattering(SIDIS), which is considered in [8],[9],[16]. The generalization of this proof to other processes is not difficult.

The paper is organized as follows: In Sec.II, our notations and the kinematics of SIDIS are introduced; In Sec.III, the calculation including $G^\perp$ expansion is performed and how to get the gluon fields strength tensor $G^\perp_\perp^\perp$ is shown explicitly, and a formula is given for the corresponding hard coefficients; In Sec.IV, the explicit expressions of hard coefficients for quark and gluon fragmentations are given; In Sec.V, we shortly discuss the generalization of our proof to higher orders of $\alpha_s$ expansion and make a summary.

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II. NOTATIONS AND KINEMATICS

We work with light-cone coordinates throughout this paper, for which the components of an arbitrary vector $a^\mu$ are $(a^+ , a^- , a^\perp_\perp)$. The transverse direction is defined by two light-like vectors $l^\mu , n^\mu$, and the transverse metric is

$$g^\perp_{\perp\perp} = g^{\mu \nu} - \frac{l^\mu n^\nu + n^\mu l^\nu}{n \cdot l}. \tag{1}$$

Then, $a^\pm$ are defined as $a^+ = n \cdot a$, $a^- = l \cdot a$ and the transverse component is $a^\perp_{\perp} = g^{\mu \nu} a_\perp \nu$.

For a hadron(proton) moving along $Z$-axis with a large $p^+_{\lambda A}$, twist-3 distribution functions are defined as

$$\int \frac{d^2 \xi}{(2\pi)^2} e^{i \xi \cdot x^+ + i \xi \cdot x^+ s^+} \langle PS_{\perp} | \bar{\psi}_3 (0) [g_s G^p_{a \perp\perp}(\xi) \psi(\xi_1)] | PS_{\perp} \rangle = \frac{[\gamma^- T^a \bar{s}^0_\perp T_F (x_1, x_2) + i \gamma^5 \gamma^- T^a \bar{s}^0_\perp T_{\Delta F} (x_1, x_2)]_{ij}}{4 \pi N_c C_F} \tag{2},$$

with $x_2 = x_1 + x$ and $\bar{s}^0_\perp = \epsilon^{+\perp\perp}$ $s_{\perp\perp}^0$.

For a hadron(proton) moving along $-Z$-axis with a large $p^-_{\lambda H}$, the fragmentation functions for quark and gluon are[17]

$$\frac{1}{z} D_q(z) = \int \frac{d^2 \xi}{2 \pi} e^{-ik^- \cdot \xi^+} (0|\bar{\psi}_1 (0)|p_H X)(Xp_H |\bar{\psi}_3 (\xi^+)|0),$$

$$\frac{1}{z} D_g(z) = \frac{1}{(N_c^2 - 1)2k^-} \int \frac{d^2 \xi}{2 \pi} e^{-ik^- \cdot \xi^+} (0|G_0 (0) -\alpha |p_H X)(p_H X|G_0 (\xi^+, 0, 0) -\beta g_{\perp\perp} 0, 0) - 1/z p^-_{\lambda H}. \tag{3}$$

The gauge link is defined as

$$\mathcal{L}_a (\xi^-) = P e^{-i g_s \int_0^\infty d\lambda^- G^a (\xi^- + \lambda^-)} \tag{4},$$

For simplicity, the gauge link is suppressed in the above definitions, if there is no derivative acting on the gauge link. In Feynman gauge, if there is no gauge link, from time-reversal and parity symmetries one can show very easily that $q_0 = 0$. In this calculation, the covariant derivative is $D^\mu = \partial^\mu + i g_s T^a c_\mu^a$, and the anti-symmetric tensor satisfies $c_{123} = 1$.

As an example, in this work we consider the SSA for pion production in SIDIS. The process is

$$e(l_e) + h_A (p_A, s_{\perp\perp}) \rightarrow e(l'_e) + (p_H) + X, \tag{5}$$

where $X$ represents undetected hadrons. The momenta and spin of particles are written in the brackets. Further, in our case the exchanged vector boson is virtual photon only. For kinematics, we mainly use the notations in [16]. We work in hadron frame, where initial hadron and virtual photon are moving along $+Z$ and $-Z$-axis, respectively. In this frame we demand the final hadron has a large transverse momentum with respect to $Z$-axis, i.e., $p_{\lambda H} \gg \Lambda_{QCD}$. With respect to the hadron plane expanded by $p_A$ and $p_H$, the azimuthal angle of final lepton is $\phi$ and the azimuthal angle of spin vector $s_{\perp\perp}$ is $\Phi_s$. Other invariants are standard:

$$x_B = \frac{Q^2}{2p_A \cdot q}, \quad z_f = \frac{p_A \cdot p_H}{p_A \cdot q}, \quad y = \frac{p_A \cdot q}{p_A \cdot l_e}, \quad s_{\epsilon p} = (l_e + p_A)^2, \quad Q^2 = -q^2, \quad q = l_e - l'_e. \tag{6}$$

To define $\pm$ components of momenta, we choose $p_A$ and $p_H$ as the two light-like vectors, with $p^+_{\lambda A} = p^+_{\lambda H} = 0$. Then, virtual photon has a transverse momentum $q^\perp_{\perp\perp}$, i.e.,

$$q^\perp = \frac{q \cdot p_H}{p_A \cdot p_H} p^\perp_{\perp\perp} + \frac{q \cdot p_A}{p_A \cdot p_H} p^\perp_{\perp\perp} + q^\perp_{\perp\perp} \tag{7}.$$
All $\phi$ distributions are contained in $A_k$. It is found in [16] that only four distributions are relevant in SSA. They are

$$A_1 = Q^2[1 + \cos^2 \psi], \quad A_2 = -2Q^2, \quad A_3 = -Q^2 \cos \phi \sinh 2\psi, \quad A_4 = Q^2 \cos 2\phi \sinh^2 \psi,$$

where $\cos \psi \equiv 2x_BS_{ep}/Q^2 - 1$. $\tilde{\gamma}^{\mu\nu}_k$ are list in Appendix B. For more details, please consult Ref [16] and reference therein.

The typical hard scale of this process is $Q$, with $Q \gg \Lambda_{QCD}$. $q_T \equiv \sqrt{-q_T^2} \sim Q$ is also taken as a hard scale. Since there is no soft scale in $d\sigma$, the differential cross section or hadronic tensor is expected to be factorized in collinear formalism. In this paper we only consider the contribution of $q_\beta$ and $T_F$. Extending our calculation to include $q_\beta$ and $T_{\Delta_F}$ is straightforward.

### III. SGP CONTRIBUTION

With the help of fragmentation function, the hadronic tensor can be written as [16]

$$W^{\mu\nu}(p_A, q, p_H) = \sum_{j=q,g} \int \frac{dz}{z^2} D_j(z) w^{\mu\nu}_{j}(p_A, q, p_h), \quad p_h = \frac{1}{z} p_H,$$

where $D_j(z)$ is the fragmentation function for final hadron $h_B$ in parton $j$. Here parton $j$ can be quark or gluon. Generally, $w^{\mu\nu}$ is

$$w^{\mu\nu} = \int \frac{d^n k_1}{(2\pi)^4} \frac{d^n k}{(2\pi)^4} H^{\mu\nu}_{a,\rho}(k_1, k) \int d^n \xi_1 d^n \xi d^ik \xi d^ik_1 \xi_1 (p_A, s_\perp) \bar{\psi}(0) g_s G^a(\xi) \psi(\xi_1)|p_A, s_\perp),$$

where $H^{\mu\nu}_{a,\rho}$ is the hard part. The main contribution is given by collinear region, where

$$k^\mu = (k^+, k^-, k_\perp) \sim Q(1, \lambda^2, \lambda), \quad k_\perp^\mu \sim Q(1, \lambda^2, \lambda).$$

Then we do power expansion in hard part $H^{\mu\nu}_{a,\rho}$. At twist-3 level, $O(\lambda)$, both $G^+ + G_\perp$ contribute. Consider the case with quark as fragmentation parton first. In this case the momentum of final gluon should be integrated. It is clear from previous studies that only pole contributions give a real cross section[3]. In this paper we just consider the soft-gluon-pole(SGP) contribution. The diagrams containing explicit SGP are shown in Fig.1. The hard part of Fig.1(a) is

$$H^{\mu\nu}_{a,\rho} G^a_\perp(\xi_1) = \int \frac{d^n k_g}{(2\pi)^n} \frac{d^n k}{(2\pi)^4} \delta(k_g^2) \delta^n(k + k_1 + q - p_h - k_\perp) P^{\beta\gamma}(k_g)$$

$$\cdot \left[ \frac{i g T^\alpha_{\gamma\delta}}{p_h - q - i\epsilon} \gamma^\nu \gamma^\rho \left( -ig T^\alpha_{\gamma\delta} G_{\alpha\rho}(\xi_1) \right) \frac{i}{p_h - q + i\epsilon} \gamma^\mu \frac{i}{p_h - q + i\epsilon} \left( -ig T^\delta_{\gamma\beta} \right) \right],$$

where $P^{\beta\gamma}(k_g)$ is the tensor for the summation of polarization of final gluon. Firstly, we work in Feynman gauge $\partial_\mu G^\mu = 0$. For convenience we constrain final gluon to be physically polarized, then $P^{\beta\gamma}$ is chosen to be

$$P^{\beta\gamma}(k_g) = -g^{\beta\gamma} + \frac{p^{\beta}_A k^\gamma_g + p^{\gamma}_A k^\beta_g}{p_A \cdot k_g},$$

where leptonic tensor is $L^{\mu\nu} = 2(\epsilon^t_R^\mu \epsilon^t_L^\nu + \epsilon^t_L^\mu \epsilon^t_R^\nu - g^{\mu\nu} Q^2)$, and hadronic tensor is

$$W^{\mu\nu} = \int \frac{d^4 x}{(2\pi)^4} e^{iq\cdot x} \sum_X (p_A, s_\perp) \left| j^{\mu}(x) |p_H X \rangle \langle X p_H | j^{\nu}(0) |p_A, s_\perp) \right|,$$

with electro-magnetic current given by $j^\mu = \bar{\psi} \gamma^\mu \psi$.

As done in [16], one can introduce tensors $\gamma^{\mu\nu}_k$ to project out lepton azimuthal angle distributions. This gives

$$L^{\mu\nu} W^{\mu\nu} = \sum_k A_k B_k, \quad A_k = L^{\mu\nu} \gamma^{\mu\nu}_k, \quad B_k = W^{\mu\nu} \tilde{\gamma}^{\mu\nu}_k.$$
with $p_A$ the reference vector. In this way, $k_g P^\beta \epsilon^\gamma = p_A \delta^\beta \epsilon^\gamma = 0$. The SGP is given by the propagator with momentum $p_h - k$. The expansion to twist-3 for following quantity is

$$
\psi_h (-i g T^a G^a_\rho) \frac{i}{\not p_h - \not k + i \epsilon} = g T^a G^a_\rho \psi_h \left[ \frac{1}{-k^+ + i \epsilon} \right] + \frac{g T^a}{-2p_h k^+ + i \epsilon} \psi_h \not \gamma^\rho \gamma^\rho (-k_\perp G^a_\perp + k^+ G_{a \perp \rho}) + O(\lambda^2).
$$

(17)

The first term is $O(1)$, and the second term is $O(\lambda)$, in which gluon field strength tensor appears naturally. The SGP contribution then is obtained by using following formula

$$
\frac{1}{-k^+ + i \epsilon} = P_1 - \frac{1}{k^+} - i \pi \delta(k^+).
$$

(18)

Now the first term in eq.(17), which is $O(1)$, gives $O(\lambda)$ contribution by expanding the other parts of $H^\nu_{a \rho}$ in $k_\perp$ to $O(\lambda)$ or $O(k_\perp)$. Here we ignore the transverse momentum $k_1^\perp$ first. We will come to $k_1^\perp$ expansion later. The expansion of $k_\perp$ contains two parts: one is from intermediate fermion propagator $i/\not p_h - \not k - \not q$, the other is from $\delta(k_\rho^2) P_{\beta\beta}(k)\epsilon$ for the final gluon, with $k_g = k_1 + k + q - p_h$. Expressed by diagrams, the expansion reads(for the rules of these diagrams, please see Appendix.A)

$$
\text{Term-1} \equiv \frac{k^\rho}{-k^+ + i \epsilon} \frac{\partial}{\partial k_\perp^\rho},
$$

(19)

and

$$
k_\perp \frac{\partial}{\partial k_\perp^\rho} = \frac{-i g T^a \gamma^\rho}{-k^+ + i \epsilon} \frac{\partial}{\partial k_\perp^\rho} + k^\rho \frac{\partial}{\partial k_\perp^\rho} G^\rho_\perp.
$$

(20)

For the expansion of fermion propagator, we have used the formula

$$
\frac{i}{\not p_h - \not k - \not q + i \epsilon} \equiv \frac{i}{\not p_h - \not q + i \epsilon} (-i) \frac{\not k_1}{\not p_h - \not q + i \epsilon} \frac{i}{\not p_h - \not q + i \epsilon}.
$$

(21)
The expansion is equivalent to an insertion of transverse momentum. Now, with the factor \( gT^a \) included, the contribution is proportional to \(-igT^a_{\gamma\lambda\mu}k_\mu^+G_a^+\). This vertex is expressed in the diagram as a cross on fermion propagator.

Next, we consider the SGP from final gluon, Fig.1(b). The SGP appears in the propagator \(-i/((k_g - k)^2 + i\epsilon)\). Explicitly, Fig.1(b) is proportional to

\[
-gf^{abc}\Gamma_{\alpha\beta\gamma}(k, -k_g, k_g - k)G_a^\alpha \frac{-i}{(k_g - k)^2 + i\epsilon}.
\]

The three-gluon vertex can be written into two parts:

\[
\Gamma_{\alpha\beta\gamma}(k, -k_g, k_g - k) = \left( g_{\alpha\beta}(k_g - k)\gamma + g_{\beta\gamma}k_\alpha + g_{\gamma\alpha}k_{\beta} \right) + 2\left( g_{\alpha\beta}k_\gamma - g_{\beta\gamma}k_\alpha - g_{\gamma\alpha}k_{\beta} \right).
\]

The first part is identified as scalar gluon contribution, which cancels between different diagrams due to Ward identity \( \langle M|\partial\cdot G|N\rangle = 0 \) for physical states \( |M\rangle, |N\rangle \). For example, \((k_g - k)_\gamma\) term of Fig.1(b) will be cancelled by the same term of Fig.1(d). In addition, when \(k_\alpha\) is contracted with \(G_a^\alpha\), it produces a contribution of order \(\lambda^2\), i.e., \(k_\alpha G_a^\alpha = O(\lambda^2)\). When \(k_\beta\) contracts with \(P^{\beta\gamma}(k_g)\), it just vanishes because \(k_\beta P^{\beta\gamma} = 0\). Now, it is clear that the scalar part in three-gluon vertex can be expanded as well. Then,

\[
-gf^{abc}P^{\beta\gamma}(k_g)\Gamma_{\alpha\beta\gamma}(k, -k_g, k_g - k)G_a^\alpha \frac{-i}{(k_g - k)^2 + i\epsilon} = igf^{abc}P^{\beta\gamma}(k_g) \left[ \frac{G_{\alpha\beta}k_\gamma - G_{\alpha\gamma}k_\beta}{-k_g^+ + i\epsilon} - g_{\beta\gamma}\frac{k_\alpha}{-k_g \cdot k + i\epsilon} \right].
\]

Now \(k_\gamma \cdot G_a\) contains longitudinal and transverse gluon contributions, because \(k_{g\perp} = q_\perp + k_\perp\) is an \(O(1)\) quantity and \(k_{g\perp} \cdot G_{a\perp}\) is still \(O(\lambda)\). For the same reason, the denominator should be expanded as well. The result is very interesting,

\[
\frac{k_\gamma \cdot G_a}{-k_g \cdot k + i\epsilon} = \frac{k_\gamma G_a^+}{-k_g^+ + i\epsilon} - k_\gamma k_{g\perp} k^+ G_{a\parallel} - k_{g\perp} G_a^+ \left( -k_g^+ + i\epsilon \right)^2 + O(\lambda^2).
\]

The transverse component of gluon, \(G_{a\perp}\), becomes a part of gluon field strength tensor automatically. The associated pole is a double pole. Apparently, the first term should be a part of gauge link. Due to the commutation relation of color matrix \(if^{abc}T^c = [T^a, T^b]\), we have

\[
= G_a^+ k^+ \frac{\partial}{-k^+ + i\epsilon \partial k^+} \begin{pmatrix}
\text{field strength term.}
\end{pmatrix}
\]

As expected, the second term in the bracket of above equation cancels the second term on RHS of eq.(20). Now, for
SGP contribution with quark fragmentation we have

\[
\begin{align*}
G^{\mu\nu}_a & = \frac{G_1^a(\xi^-) k_1^\mu}{-k^+ + i\epsilon} \\
 & + \frac{G^+_a k_1^\rho}{-k^+ + i\epsilon} \partial_{\rho} \left( -ig_{\gamma_{\perp}} \cdot k_{\perp} \right) + \text{field strength tensor term.}
\end{align*}
\]

\[
\begin{align*}
\frac{-G^+_a k_1^\rho}{-k^+ + i\epsilon} \partial_{\rho} \left( -ig_{\gamma_{\perp}} \cdot k_{\perp} \right) + \text{field strength tensor term.}
\end{align*}
\]

Now, only the second term is not in a gauge invariant form, which indicates that our calculation for SGP is not complete. Actually, the transverse gluon also contributes by coupling to the intermediate fermion, as shown in Fig. 2(a). The SGP appears by writing \( G_\perp^a \) into gluon field strength tensor, i.e.,

\[
G_\perp^a = \frac{1}{-k^+ + i\epsilon} ( -k^+ G_\perp^a ).
\]

The quantity in the bracket should be viewed as a part of gluon field strength tensor, and the SGP is given by \( 1/( -k^+ + i\epsilon ) \). Taking Fig. 2(a) into account, we have

\[
\begin{align*}
\frac{G^+_a k_1^\rho}{-k^+ + i\epsilon} \partial_{\rho} \left( -ig_{\gamma_{\perp}} \cdot k_{\perp} \right) & + \frac{-G^+_a k_1^\rho}{-k^+ + i\epsilon} \partial_{\rho} \left( -ig_{\gamma_{\perp}} \cdot k_{\perp} \right) \\
& = \frac{k^+ G_\perp^a + k_1^\rho G^+_a}{-k^+ + i\epsilon} \left( -ig_{\gamma_{\perp}} \cdot k_{\perp} \right).
\end{align*}
\]
In this way, gluon field strength tensor is retained by adding transverse gluon contribution. Finally, we have

\[ \frac{\partial}{\partial k_1^\perp} \]

\[ = k_1^\perp \frac{\partial}{\partial k_1^\perp} + \text{field strength tensor term.} \tag{30} \]

In the above, the first term with transverse derivative acting on gauge link can be written as

\[ w^{\mu \nu} \supset \int dk_1^+ \int dk^+ \frac{1}{-k^+ + i\epsilon} \frac{\partial H^{\mu \nu}(k_1 + k)}{\partial k_1^+} \int \frac{d\xi^- d\xi^+}{(2\pi)^2} e^{ik^+ \xi^+ + i k_1^+ \xi_1^+} g_s \langle Ps_\perp | \bar{\psi}(0) \gamma^+ [i \partial_\perp^\rho G^+(\xi^-)] \psi(\xi_1^-) | Ps_\perp \rangle, \]

\[ H^{\mu \nu} = \frac{1}{4N_c} \int \frac{d^3k_1^\perp}{(2\pi)^3} (2\pi) \delta(k_1^\perp) \delta(k_1^0 + q - p_h - k_g) P^{\beta \gamma}(k_g) \times \text{Tr} \left[ \gamma^-(igT^b \gamma^\beta) \frac{-i}{\not{p}_h - \not{g} - i\epsilon} \gamma^\mu \frac{i}{\not{p}_h - \not{g} + i\epsilon} (-igT^b \gamma^\beta) \right]. \tag{31} \]

The factor \( 1/(4N_c) \) comes from the projection of quark-gluon-quark correlation function, and the trace in \( H^{\mu \nu} \) includes color trace. Since \( H^{\mu \nu} \) depends on \( k_1^+ + k^+ \) only, one can make a variable transformation \( k_1^+ \rightarrow k_1^+ - k^+ \) to eliminate \( k^+ \) in \( H^{\mu \nu} \). Then integrating over \( k^+ \) gives

\[ w^{\mu \nu} \supset (-2\pi i) \int dk_1^+ \frac{\partial H^{\mu \nu}(k_1)}{\partial k_1^+} \int \frac{d\xi_1^-}{2\pi} e^{ik_1^+ \xi_1^-} \langle Ps_\perp | \bar{\psi}(0) \gamma^+ \left[ \partial_\perp^\rho G^+(\xi^-) \right] \psi(\xi_1^-) | Ps_\perp \rangle = i \int dk_1^+ \frac{\partial H^{\mu \nu}(k_1)}{\partial k_1^+} \int \frac{d\xi_1^-}{2\pi} e^{ik_1^+ \xi_1^-} \langle Ps_\perp | \bar{\psi}(0) \gamma^+ \left[ \partial_\perp^\rho \mathcal{L}_n(\xi_1^-) \right] \psi(\xi_1^-) | Ps_\perp \rangle, \tag{32} \]

where the gauge link \( \mathcal{L}_n \) is defined in eq.(4).

Next we consider the expansion in \( k_1^\perp \). Since \( k_1^\perp \sim O(\lambda) \), only \( G^+ \) contributes. Because \( k_\perp = 0 \), it is clear that such \( G^+ \) in Fig.1(a,b) are combined into a gauge link. Then, \( k_1^\perp \) expansion gives

\[ w^{\mu \nu} \supset \int dk_1^+ \int dk^+ \frac{1}{-k^+ + i\epsilon} \frac{\partial H^{\mu \nu}(k_1 + k)}{\partial k_1^\perp} \int \frac{d\xi^- d\xi^+}{(2\pi)^2} e^{ik^+ \xi^+ + i k_1^\perp \xi_1^\perp} g_s \langle Ps_\perp | \bar{\psi}(0) \gamma^+ G^+(\xi^-) i \partial_\perp^\rho \psi(\xi_1^-) | Ps_\perp \rangle \]

\[ = i \int dk_1^+ \frac{\partial H^{\mu \nu}(k_1)}{\partial k_1^\perp} \int \frac{d\xi_1^-}{2\pi} e^{ik_1^\perp \xi_1^\perp} \langle Ps_\perp | \bar{\psi}(0) \gamma^+ \left[ \partial_\perp^\rho \mathcal{L}_n(\xi_1^-) \right] \psi(\xi_1^-) | Ps_\perp \rangle, \tag{33} \]

with the same \( H^{\mu \nu} \) in eq.(31). Then, eq.(31) and eq.(33) can be combined together to give a gauge invariant matrix element, i.e.,

\[ w^{\mu \nu} \supset - \int dk_1^+ \frac{\partial H^{\mu \nu}(k_1)}{\partial k_1^\perp} \int \frac{d\xi_1^-}{2\pi} e^{ik_1^\perp \xi_1^\perp} \langle Ps_\perp | \bar{\psi}(0) \gamma^+ \left[ \partial_\perp^\rho \mathcal{L}_n(\xi_1^-) \psi(\xi_1^-) \right] | Ps_\perp \rangle. \tag{34} \]

The correlation function appearing on RHS above is rightly \( g_0(x_1) \), which is gauge invariant.

On the other hand, the field strength tensor term in eq.(30) is proportional to \( T_F(x_1, x_1 + x) \). But since Fig.1 contains a double pole, a derivative in \( x \) acting on \( T_F(x_1, x_1 + x) \) will appear. The double pole contribution is

\[ \int dx T_F(x_1, x_1 + x) H(x) \frac{1}{(x - i\epsilon)^2}, \tag{35} \]
where $H(x)$ is the hard coefficient. Other variables are suppressed for simplicity. By using integration by part, it becomes

\[
\int dx \left\{ \frac{dT_F(x_1, x_1 + x)}{dx} \frac{1}{x - i\epsilon} \right\} = i\pi d \left[ T_F(x_1, x_1 + x)H(x) \right]_{x=0} = i\pi \left[ \frac{1}{2} H(x = 0) \frac{d}{dx_1} T_F(x_1, x_1) + T_F(x_1, x_1) \frac{dH(x = 0)}{dx} \right].
\]

Thus $dT_F(x_1, x_1)/dx_1$ appears. In the above we have used the symmetry $T_F(x_1, x_2) = T_F(x_2, x_1)$, and the principal integration is ignored because it does not contribute to the real part of $w^{\mu\nu}$.

The same derivation can be applied to Fig.1(e,f) directly. The sum of Fig.2(c) and Fig.1(e,f) can be decomposed into gauge link contribution and gluon field strength tensor contribution, similar to eq.(30). For Fig.1(c,d), the $k_\perp$ expansion gives

\[
\begin{align*}
\begin{array}{c}
\includegraphics{fig1}\end{array}
\end{align*}
\]

Notice that there is a minus sign before the second diagram on RHS of the second line. This minus sign and the vertex $igT^a \bar{q}_\perp$ together give the standard quark-gluon vertex. In this way, this diagram is combined with Fig.2(c) to give a field strength tensor. The final result is

\[
\begin{align*}
\begin{array}{c}
\includegraphics{fig2}\end{array}
\end{align*}
\]

The expansion in $k_{1\perp}$ is the same as Fig.1(a,b). As a result, the contribution from Fig.1(c,d) and Fig.2(b) can be expressed by $q_\perp$ and $T_F$. Fig.1(g,h) and Fig.2(d) can be treated in the same way.

Now, we have shown that the twist expansion can generate $q_\perp$ and $T_F$ in a very natural way. Especially, $T_F$ part can be recovered by transverse gluon $G_{1\perp}$ definitely. Our formula for SGP contribution with quark fragmenting thus is

\[
u^{\mu\nu} = \int dx_1 \left\{ \frac{\partial H_1^{\mu\nu}(k_{1\perp})}{\partial k_{1\perp}^\rho} s_\perp^\rho q_\perp(x_1) - \frac{i}{\pi} \int dx \frac{H_2^{\mu\nu}(x_1, x)}{(x - i\epsilon)^2} s_\perp^\rho T_F(x_1, x_1 + x) \right\},
\]

where $H_1,2$ are particular hard coefficients defined as follows:

\[
\begin{align*}
H_1^{\mu\nu} &= H_1^{\mu\nu} \left( \frac{1}{2} \gamma^\rho p_1^\perp \right)_i j, \\
H_2^{\mu\nu} &= -\pi x \left[ H_2^{\mu\nu} \right]_i j \frac{\gamma^\rho}{4\pi N_c C_F} T_a^\perp.
\end{align*}
\]

where $H_1^{\mu\nu}$ is the hard part of Fig.3, which contains no initial gluon, and $[H_2^{\mu\nu}]_i j$ is the hard part of Fig.2, in which the initial gluon is a transverse one. This is our main formula. From this formula, it is clear that if we use $G_{1\perp}$ only to derive factorization formula, $q_\perp$ contribution is missing. And if we use $q_\perp$ only, $T_F$ contribution is missing. But if we use $G_{1\perp}$ to derive, both contributions can be included. This is the reason why $G_{1\perp}$ can give correct hard coefficients.

In addition, the same derivation can be applied to the case where gluon is the fragmenting parton, where the double poles are given by Fig.1(a,c,e,g). The hard coefficients can be obtained in a transparent way, by changing the projection operators to those for gluon fragmentation functions. Eq.(40) is still right for gluon as fragmentation parton.
Then, projected hadronic tensors

\[ w^{\mu\nu} = \int dx_1 \left\{ \frac{\partial [\tilde{H}^{\mu\nu}(k_g^2)]}{\partial k_{1\perp}^\rho} q_\rho(x_1) - \frac{i}{\pi} \int dx \frac{\tilde{H}^{\mu\nu}_2(x_1, x) \delta(k_g^2)}{(x - i\epsilon)^2} \delta_1 T_F(x_1, x_1 + x) \right\} \]

\[ = \int dx_1 \delta(x) \left\{ \frac{\partial [\tilde{H}^{\mu\nu}(k_g^2)]}{\partial k_{1\perp}^\rho} q_\rho(x_1) - \frac{i}{\pi} \int dx \frac{1}{x - i\epsilon} \frac{\partial}{\partial x} \left[ \tilde{H}^{\mu\nu}_2(x_1, x) \delta(k_g^2) T_F(x_1, x_1 + x) \right] \right\} . \]  

In the first term, \( k_g = x_1 p_A + q - p_h + k_{1\perp} \); in the second term, \( k_g = (x_1 + x) p_A + q - p_h \). Since there is a derivative in \( x \) in the second term, it is better to make a variable transformation \( x_1 \to x_1 - x \) so that there is no \( x \) contained in \( \delta(k_g^2) \) any more. This trick simplifies the calculation greatly. Then,

\[ w^{\mu\nu} = \int dx_1 \delta(x) \left\{ \frac{\partial [\tilde{H}^{\mu\nu}(k_g^2)]}{\partial k_{1\perp}^\rho} q_\rho(x_1) - \frac{i}{\pi} \int dx \frac{1}{x - i\epsilon} \frac{\partial}{\partial x} \left[ \tilde{H}^{\mu\nu}_2(x_1, x) \delta(\Delta) T_F(x_1 - x, x_1) \right] \right\} , \]

with

\[ \Delta = (x_1 p_A + q - p_h)^2 = \hat{z} \left[ \frac{(1 - \hat{z})(1 - \hat{\epsilon} x_1)}{\hat{\epsilon} x_1} - q_\rho^2 \right] . \]

Taking the pole contribution in \( x \) integration, we have

\[ w^{\mu\nu} = \int dx_1 \delta(x) \left\{ \frac{\partial [\tilde{H}^{\mu\nu}(k_g^2)]}{\partial k_{1\perp}^\rho} q_\rho(x_1) + \frac{\partial}{\partial x} \left[ \tilde{H}^{\mu\nu}_2(x_1, x) \delta(\Delta) T_F(x_1 - x, x_1) \right] \right\}_{x=0} . \]

In the first term the derivative also acts on the delta function. Following equation is helpful:

\[ \frac{\partial \delta(\Delta + 2k_{1\perp} \cdot q_\perp)}{\partial k_{1\perp}^\rho} = 2q_\perp^\rho \delta'(\Delta) = \frac{q_\perp^\rho}{p_A \cdot (q - p_h)} \frac{\partial \delta(\Delta)}{\partial x_1} . \]

After integration by part, the final formula is

\[ w^{\mu\nu} = \int dx_1 \delta(\Delta) \delta_1 \left\{ \frac{\partial [\tilde{H}^{\mu\nu}_1]}{\partial k_{1\perp}^\rho} - \frac{\partial}{\partial x_1} \left( \frac{q_\perp^\rho [\tilde{H}^{\mu\nu}_1]}{p_A \cdot (q - p_h)} \right) \right\} q_\rho(x_1) - \frac{q_\perp^\rho \tilde{H}^{\mu\nu}_1}{p_A \cdot (q - p_h)} \frac{\partial q_\rho(x_1)}{\partial x_1} + \left[ T_F(x_1, x_1) \left( \frac{\partial [\tilde{H}^{\mu\nu}_2]}{\partial x_1} \right) \right]_{x=0} - \frac{1}{2} \tilde{H}^{\mu\nu}_2(x_1, 0) \frac{\partial}{\partial x_1} T_F(x_1, x_1) . \]

Then, projected hadronic tensors \( B_k \) defined in eq.(10) are

\[ B_k = \int \frac{dz}{z^2} D(z) \int dx_1 \delta(\Delta) q_\perp \cdot \delta_1 \left\{ C_1^k q_\rho(x_1) + C_2^k \frac{\partial}{\partial x_1} q_\rho(x_1) + C_3^k T_F(x_1, x_1) + C_4^k \frac{\partial}{\partial x_1} T_F(x_1, x_1) \right\} . \]
with $k = 1, \cdots, 4$ and
\[
C_1^k = -\frac{q_\perp^\rho}{q_T^2} \left[ \frac{\partial H_1^k}{\partial x_1} - \frac{\partial}{\partial x_1} \left( \frac{q_\perp \cdot H_1^k}{p_A \cdot (q - p_\perp)} \right) \right],
\]
\[
C_2^k = -\frac{\beta}{p_A \cdot (q - p_\perp)},
\]
\[
C_3^k = -\frac{1}{q_T^2} q_\perp^\rho \left( \frac{\partial H_2^{1, \rho}(x_1 - x, x)}{\partial x} \right)_{x=0},
\]
\[
C_4^k = \frac{1}{q_T^2} q_\perp^\rho H_{2, \rho}(x_1, 0).
\]

There is a relation between $q_\perp(x)$ and $T_F(x, x)$, i.e., $-2q_\perp(x) = T_F(x, x)$, which was found by many authors in different ways[18–21]. With this relation, $B_k$ can be expressed by $T_F$ solely. That is,
\[
B_k = \int \frac{dz}{z^2} D(z) \int dx_1 \delta(\Delta) q_\perp \cdot \tilde{s}_\perp \left\{ E_1^k T_F(x_1, x_1) + E_2^k \frac{\partial}{\partial x_1} T_F(x_1, x_1) \right\},
\]
with
\[
E_1^k = -\frac{1}{2} C_1^k + C_3^k, \quad E_2^k = -\frac{1}{2} C_2^k + C_4^k.
\]

Our results are
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{z}{zf_{x_1}} \right]^{-1} E_1^1 = -\frac{\hat{x}_1 \hat{z} \left( 2\hat{x}_1^2 (6\hat{z}^2 - 6\hat{z} + 1) + \hat{x}_1^2 \left( -24\hat{z}^2 + 26\hat{z} - 5 \right) + 4\hat{x}_1 \left( 3\hat{z}^2 - 4\hat{z} + 1 \right) - \hat{z}^2 + 2\hat{z} - 2 \right)}{(\hat{x}_1 - 1)^2 (\hat{z} - 1)^2},
\]
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{z}{zf_{x_1}} \right]^{-1} E_1^2 = -\frac{8\hat{x}_1 \hat{z}^2}{\hat{z} - 1},
\]
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{z}{zf_{x_1}} \right]^{-1} E_1^3 = -\frac{Q \hat{x}_1 \hat{z} \left( \hat{x}_1^2 (8\hat{z} - 4) + \hat{x}_1 (7 - 12\hat{z}) + 3(\hat{z} - 1) \right)}{q_T (\hat{x}_1 - 1) (\hat{z} - 1)},
\]
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{z}{zf_{x_1}} \right]^{-1} E_1^4 = \frac{4\hat{x}_1 \hat{z}^2}{\hat{z} - 1};
\]

And
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{zx_B}{zf_{x_1}} \right]^{-1} E_2^1 = -\frac{\hat{x}_1 \hat{z} \left( \hat{x}_1^2 (6\hat{z}^2 - 6\hat{z} + 1) + \hat{x}_1 \left( -6\hat{z}^2 + 8\hat{z} - 2 \right) + \hat{z}^2 - 2\hat{z} + 2 \right)}{(\hat{x}_1 - 1) (\hat{z} - 1)^2},
\]
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{zx_B}{zf_{x_1}} \right]^{-1} E_2^2 = \frac{4\hat{x}_1 \hat{z}^2}{\hat{z} - 1},
\]
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{zx_B}{zf_{x_1}} \right]^{-1} E_2^3 = \frac{Q \hat{x}_1 (2\hat{z} - 1) - \hat{z} + 1}{q_T (\hat{z} - 1)},
\]
\[
\left[ \frac{\alpha_s}{2\pi^2 N_c Q^2} \frac{zx_B}{zf_{x_1}} \right]^{-1} E_2^4 = \frac{2\hat{x}_1 \hat{z}^2}{\hat{z} - 1},
\]

with
\[
q_T = Q \sqrt{\frac{(1 - \hat{z})(1 - \hat{x}_1)}{\hat{z} \hat{x}_1}}, \quad \hat{x} = \frac{\hat{x}_1}{\hat{z}}, \quad \hat{x}_1 = \frac{x_B}{x_1}.
\]

These results are the same as those given in [16].

For gluon fragmentation, the formulas are the same, but now the momentum of final gluon is related to observed
FIG. 4. Contact diagrams which are not included in hard coefficient \( H_{2\perp}^{\mu\nu} \), in our calculation. The initial gluon is transverse in these diagrams. Conjugated diagrams give the same result and are not shown.

hadron by \( k_g = p_H/z \), and the corresponding hard coefficients are

\[
\begin{align*}
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_1^1 &= \frac{\hat{x}_1 (2\hat{x}_1 - 6\hat{z} + 1) + 2\hat{x}_1 (2\hat{z} - 3\hat{z})\hat{z} + \hat{z}^2 + 1}{(\hat{x}_1 - 1)^2(\hat{z} - 1)}, \\
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_1^2 &= -8\hat{x}_1^2\hat{z}, \\
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_1^3 &= -\frac{Q}{q_T}\frac{\hat{x}_1 (2\hat{x}_1 - 6\hat{z} + 1) + 2\hat{x}_1 (2\hat{z} - 3\hat{z})\hat{z} + \hat{z}^2 + 1}{(\hat{x}_1 - 1)^2(\hat{z} - 1)}, \\
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_1^4 &= -4\hat{x}_1^2\hat{z},
\end{align*}
\]

and

\[
\begin{align*}
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_2^1 &= \frac{\hat{x}_1 (2\hat{x}_1 - 6\hat{z} + 1) + 2\hat{x}_1 (2\hat{z} - 3\hat{z})\hat{z} + \hat{z}^2 + 1}{(\hat{x}_1 - 1)^2(\hat{z} - 1)}, \\
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_2^2 &= 4\hat{x}_1\hat{z}, \\
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_2^3 &= \frac{Q}{q_T}\left[ \hat{x}_1 (4\hat{x}_1 - 2) - 2\hat{z} \right], \\
\left[ \frac{\alpha_s N_c}{2\pi^2 Q^2} \frac{z}{x_f x_1} \right]^{-1} E_2^4 &= 2\hat{x}_1\hat{z}.
\end{align*}
\]

These results are also the same as those given in \[16\].

We also do calculations in light-cone gauge with \( \tilde{G}^+ = 0 \) or \( p_H \cdot G = 0 \). For the case of gluon fragmentation, the final gluon is transversely polarized in both Feynman and light-cone gauges. Thus, in this case the hard coefficients in these two gauges are the same. For the case of quark fragmentation, \( C_2^k \) and \( C_3^k \) in light-cone gauge are the same as those in Feynman gauge, but \( C_2^k \) and \( C_3^k \) are different. The difference \( \Delta C^k = C_3^k_{\text{LC}} - C_3^k_{\text{Feynman}} \) are shown in Appendix.C. Since \( C_2 \) is determined by the on-shell \( \gamma^*q \) amplitude, \( C_2 \) is gauge independent naturally. The reason that \( C_3 \) depends on gauge is \( \tilde{H}_{2\perp}^{\mu\nu} \) is not given by a complete amplitude for the scattering of \( \gamma^* \) and \( qg \), because the contact diagrams, Fig.4, are not included. In collinear expansion, the contribution of these contact diagrams has been included in the contribution of \( qg \), because the intermediate propagator with momentum \( k_1 + k \) is collinear. Since \( qg \) contribution has been taken into account, including these contact diagrams in \( \tilde{H}_{2\perp}^{\mu\nu} \) is a double counting. These contact diagrams are regular at \( x = 0 \), and thus vanish when \( x \to 0 \) in \( C_3^k \) because of eq.(40), where \( x \) is multiplied to the amplitude in \( \tilde{H}_{2\perp}^{\mu\nu} \). So, \( C_3^k \) is gauge invariant, even though the on-shell amplitude is incomplete. For \( C_3^k \), due to the derivative in \( x \), contact diagrams give nonzero contribution. Moreover, this contribution depends on gauge. The gauge dependence can be seen from following simple analysis.

\[
\begin{align*}
\tilde{H}_{2\perp}^{\mu\nu}_{\text{contact}} \propto \text{Tr} \left[ p^+_{A}(\gamma^* - igT^a) \xi_{a\perp}(\xi^-) \right] \\
&\times \left( \frac{i}{\hat{k}_1 + \hat{k} + i\epsilon} \right) \\
&\times \left( \frac{i\gamma^+}{2(k_1 + k)^+} \right),
\end{align*}
\]

where \( (\cdots) \) represent \( \gamma^*q \) scattering cross section. If \( (\cdots) \) is projected by \( p^+_{A} \gamma^* \), the resulting \( \gamma^*q \) cross section is on-shell and thus gauge independent. But here, as we can see, it is projected by \( p^+_{A} \gamma^* \gamma_{\perp} \gamma^+ \), which gives a kind of
off-shell contribution, thus the trace depends on gauge. Since the complete amplitude is gauge invariant, the absence of compact diagrams gives gauge dependent $C_k^3$. However, the gauge dependence in $C_k^1$ and $C_k^3$ cancel each other, and the final coefficient $E^3_3$ does not depend on gauge. This also supports the conclusion that $q_0(x)$ and $T_F(x, x)$ are not independent.

V. DISCUSSION AND SUMMARY

Before the summary we want to compare our formalism with that given in [16]. The main difference is in our calculation the non-pole diagrams in Fig.2 for $G_{\perp}$ contribution are included. In [16], these diagrams are ignored. Really, if one changes $G_{\perp}$ in these diagrams to $G^{\perp}$, these diagrams do not give pole contribution and should be dropped. In addition, in our calculation, the cancellation between mirror diagrams are not used. The diagram with coherent gluon on RHS of the cut is taken as conjugated diagrams, and from PT symmetry one can show these conjugated diagrams give the same results, thus are not needed to be calculated again. But in the formalism of [16], these conjugated diagrams give different results and have to be calculated separately. At last, in our calculation it is shown explicitly how the gauge link is reproduced at twist-3 level. Different from twist-2 factorization, the gauge link cannot be obtained by using Ward identity solely: Part of $G^{\perp}$ has to be combined with $G_{\perp}$ to form gluon field strength tensor.

The equivalence between transverse momentum expansion of fermion propagator and the insertion of a transverse gluon is very general. Thus, the derivation is possible to be applied to higher orders of $\alpha_s$ expansion. But to higher order in Feynman gauge, one has to deal with the $k_{\perp}$ expansion not only for fermion propagator, but also for gluon propagator and three-gluon vertex, gluon-ghost vertex. The complicated color algebra makes the analysis difficult. It is very likely that the use of Ward identities can simplify the analysis and give a very general conclusion. Moreover, to higher order, collinear and soft divergences will appear. For a complete analysis, one has to provide a systematic subtraction scheme to get finite hard coefficients. These issues are beyond the scope of this paper. Works on these aspects are ongoing. When this work is finished, it is found that in[22], a similar formula eq.(35) is obtained with a different method. Ward identity is used in [22] before pole condition is taken. The derivation and conclusion there seems more general than ours. But as we known, the Ward identity $\langle M|\partial_{\mu}G^{\mu}\rangle = 0$ holds for complete physical amplitude, which contains compact diagrams in Fig.4. The treatment of these contact diagrams at twist-3 is not transparent[23], which may deserve further study in the language of Ward identities. In our calculation, gluon field strength tensor simply appears as a consequence of eq.(29). The calculation seems more transparent.

As a summary, we have shown how to write SGP contribution of SSA into a gauge invariant form by including transverse gluon in the power expansion. The crucial step is to include Fig.2 in the calculation. By including $G_{\perp}$ in the calculation, $G^{\perp}$ can be absorbed into gauge link and gluon field strength tensor in a definite way. The resulting formula contains two parts: one is related to $q_0(x)$, the other is to $T_F(x, x)$. The coefficient of $q_0(x)$ can be obtained from calculating quark-photon scattering amplitude with quark transverse momentum preserved. The coefficient of $T_F$ can be obtained by calculating the subcross section for $\gamma^* + qg$ with initial gluon on-shell and transversely polarized. Since initial gluon is on-shell and physically polarized, many tricks for on-shell amplitudes can be applied, please see [24] for example. As a check, we calculate the hard coefficients in SIDIS for pion production with Feynman and light-cone gauges, respectively. The results are the same as those given in [16]. Generalizing this derivation to higher order of $\alpha_s$ expansion is interesting and can help us to understand the structure of twist-3 factorization formula.

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Appendix A: Rules for special vertices

Some special vertices appear in our derivation due to expansion of $k_\perp$. The rules for these vertices are

\[ gT^a k = gT^a \frac{i}{\not{p} + i\epsilon}, \]

\[ gT^a p + k = gT^a \frac{i}{\not{p} + \not{k} + i\epsilon}, \]

\[ (−igT^a \gamma_\perp \cdot k_\perp)p + k = i \frac{1}{\not{p} + \not{k} + i\epsilon} (−igT^a \gamma_\perp \cdot k_\perp)p + k. \]  \hspace{1cm} (A1)

Appendix B: Angular distributions

The projection tensors in [16] for hadronic tensor are

\[ 1 \frac{1}{2} (2T^\mu T^\nu + X^\mu X^\nu + Y^\mu Y^\nu), \]

\[ 1 \frac{1}{2} T^\mu T^\nu, \]

\[ 1 \frac{1}{2} (T^\mu X^\nu + X^\mu T^\nu), \]

\[ 1 \frac{1}{2} (X^\mu X^\nu − Y^\mu Y^\nu). \]  \hspace{1cm} (B1)

The momenta are

\[ T^\mu = \frac{1}{Q} (q^\mu + 2x_{BJ} P_A^\mu), \]

\[ X^\mu = \frac{1}{q_T} \left( \frac{1}{z_f} P_H^\mu - q^\mu - (1 + \frac{q_T^2}{Q^2}) x_{BJ} P_A^\mu \right), \]

\[ Y^\mu = \epsilon_{\mu\nu\rho\sigma} Z^\nu X^\rho T_\sigma, \]

\[ Z^\mu = − q^\mu / Q. \]  \hspace{1cm} (B2)

which satisfy $T^2 = X^2 = Y^2 = Z^2 = 1$. Metric can be expressed as

\[ g^{\mu\nu} = T^\mu T^\nu − X^\mu X^\nu − Y^\mu Y^\nu − Z^\mu Z^\nu. \]  \hspace{1cm} (B3)

So, $Y^\mu Y^\nu$ can be eliminated.

With $p_A$ and $p_H$ chosen as two light-like reference vectors to define light-cone coordinates, the momentum of virtual photon is decomposed as

\[ q^\mu = \frac{q \cdot p_H}{p_A \cdot p_H} p_H^\mu + \frac{q \cdot p_A}{p_A \cdot p_H} p_H^\mu + q_\perp^\mu. \]  \hspace{1cm} (B4)

Because

\[ q \cdot p_H = q^+ p_H^− = \frac{−Q^2 + q_T^2}{2q_T} p_H^− = z_f \frac{−Q^2 + q_T^2}{2}, \]

\[ q_T = \sqrt{−q_\perp^2}, \]  \hspace{1cm} (B5)

$X^\mu$ can be simplified as

\[ X^\mu = \frac{1}{q_T} \left[ − q_\perp^\mu − 2 \frac{q_T^2}{Q^2} x_{BJ} P_A^\mu \right]. \]  \hspace{1cm} (B6)
Interestingly, it has a longitudinal component. The projection tensors are convenient to be rewritten as
\[
\hat{\gamma}_1^{\mu
u} = \frac{1}{2} (3T^\mu T^\nu - Z^\mu Z^\nu - g^{\mu\nu}), \quad \hat{\gamma}_2^{\mu
u} = T^\mu T^\nu,
\]
\[
\hat{\gamma}_3^{\mu
u} = -\frac{1}{2} (T^\mu X^\nu + X^\mu T^\nu), \quad \hat{\gamma}_4^{\mu
u} = \frac{1}{2} (2X^\mu X^\nu + g^{\mu\nu} - T^\mu T^\nu + Z^\mu Z^\nu).
\] (B7)

Appendix C: Gauge dependence of \( C_i^k \)

At this order, \( O(\alpha) \), the gauge dependence only appears in the case with quark fragmentation. In the following, \( C_i^k \) are calculated in Feynman gauge and light-cone gauge with \( G^+ = 0 \), respectively. The difference is defined as
\[
\Delta C_i^k = C_i^k|_{LC} - C_i^k|_{Feynman}.
\] (C1)

The results are
\[
\Delta C_2^k = \Delta C_4^k = 0, \quad \Delta C_3^k = \frac{1}{2} \Delta C_1^k,
\] (C2)

and
\[
\Delta C_1^1 = 2 \frac{\alpha_s C_F}{\pi^2 Q^2} x_B z f \left( x_B \left( -5 z z f + 3 z f^2 + z^2 \right) - x_B \left( -6 z z f + 6 z f^2 + z^2 \right) \right),
\]
\[
\Delta C_1^2 = 2 \frac{\alpha_s C_F}{\pi^2 Q^2} - \frac{2}{x_B^2} \left( x_B z f \right),
\]
\[
\Delta C_1^3 = 2 \frac{\alpha_s C_F}{\pi^2 Q^2} \left( 2 x_B z f + x_B \left( z - z f \right) \right),
\]
\[
\Delta C_1^4 = 2 \frac{\alpha_s C_F}{\pi^2 Q^2} \left( x_B \left( z - z f \right) \right). \] (C3)

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1. R. Klem, J. Bowers, H. Courant, H. Kagan, M. Marshak, E. Peterson, K. Ruddick, W. Dragoset, and J. Roberts, Phys. Rev. Lett. 36, 929 (1976).
2. A. Efremov and O. Teryaev, Sov. J. Nucl. Phys. 36, 140 (1982); Phys. Lett. B 150, 383 (1985).
3. J.-w. Qiu and G. F. Sterman, Phys. Rev. Lett. 67, 2264 (1991); Nucl. Phys. B 378, 52 (1992).
4. J.-w. Qiu and G. F. Sterman, Phys. Rev. D 59, 014004 (1999), arXiv:hep-ph/9806356.
5. Y. Kanazawa and Y. Koike, Phys. Rev. D 64, 034019 (2001), arXiv:hep-ph/0012225.
6. C. Kourvaris, J.-W. Qiu, W. Vogelsang, and F. Yuan, Phys. Rev. D 74, 114013 (2006), arXiv:hep-ph/0609238.
7. X. Ji, J.-W. Qiu, W. Vogelsang, and F. Yuan, Phys. Rev. D 73, 094017 (2006), arXiv:hep-ph/0604023.
8. X. Ji, J.-W. Qiu, W. Vogelsang, and F. Yuan, Phys. Lett. B 638, 178 (2006), arXiv:hep-ph/0604128.
9. H. Eguchi, Y. Koike, and K. Tanaka, Nucl. Phys. B 752, 1 (2006), arXiv:hep-ph/0604003.
10. W. Vogelsang and F. Yuan, Phys. Rev. D 79, 094010 (2009), arXiv:0904.0410 [hep-ph].
11. A. Chen, J. Ma, and G. Zhang, Phys. Rev. D 95, 074005 (2017), arXiv:1607.08676 [hep-ph]; Phys. Rev. D 97, 054003 (2018), arXiv:1708.09091 [hep-ph].
12. Z.-B. Kang, I. Vitev, and H. Xing, Phys. Rev. D 87, 034024 (2013), arXiv:1212.1221 [hep-ph].
13. L.-Y. Dai, Z.-B. Kang, A. Prokudin, and I. Vitev, Phys. Rev. D 92, 114024 (2015), arXiv:1409.5851 [hep-ph].
14. S. Yoshida, Phys. Rev. D 93, 054048 (2016), arXiv:1601.07737 [hep-ph].
15. S. Benic, Y. Hatta, H.-m. Li, and D.-J. Yang, Phys. Rev. D 100, 094027 (2019), arXiv:1909.10684 [hep-ph].
16. H. Eguchi, Y. Koike, and K. Tanaka, Nucl. Phys. B 763, 198 (2007), arXiv:hep-ph/0610314.
17. J. C. Collins and D. E. Soper, Nucl. Phys. B 194, 445 (1982).
18. D. Boer, P. Mulders, and F. Pijlman, Nucl. Phys. B 667, 201 (2003), arXiv:hep-ph/0303034.
19. J. Ma and Q. Wang, Eur. Phys. J. C 37, 293 (2004), arXiv:hep-ph/0310245.
20. A. Bacchetta, in International Workshop on Transverse Polarization Phenomena in Hard Processes (2005) pp. 181–187, arXiv:hep-ph/0511085.
21. J. Ma and G. Zhang, JHEP 02, 163 (2015), arXiv:1409.2938 [hep-ph].
22. H. Xing and S. Yoshida, Phys. Rev. D 100, 054024 (2019), arXiv:1904.02287 [hep-ph].
23. D. Boer and J.-w. Qiu, Phys. Rev. D 65, 034008 (2002), arXiv:hep-ph/0108179.
24. H. Elvang and Y.-t. Huang, (2019), arXiv:1308.1697 [hep-th].