Asymptotic analysis of the two-layer plates bending with nonlinear interlayer friction

Yu I Dimitrienko¹, E A Gubareva², and M O Makovskaya³

¹,²Bauman Moscow State Technical University, 2nd Baumanskaya street, Moscow, Russia
E-mail: ¹dimit.bmstu@gmail.com, ²elena.a.gubareva@yandex.ru, ³mmakovscaymargarita@gmail.com

Abstracts. The article is devoted to the study of the features of the occurrence of slip between layers of a thin two-layer plate with an initial ideal contact under the influence of critical pressure. A three-dimensional problem of the linear theory of elasticity for a thin two-layer plate with sliding conditions with friction at the interface is considered. To solve this problem, the method of asymptotic averaging is applied. An equation is established for calculating the critical pressure for a two-layer plate at which the layers begin to slip. A numerical example is considered for a two-layer fiberglass for different conditions of fixing at the ends. It is shown that the fixing conditions affect the critical pressure of the beginning of sliding.

1. Introduction
In engineering, multilayer composite materials are often used due to their high strength, stiffness, low density, and high fatigue strength. The conditions for ideal contact are considered in the vast majority of problems. [1-4]. Separately, the conditions of ideal, and nonideal contact were considered in [5, 6], but the case of the transition of an ideal contact to a nonideal one has been little studied, therefore, the study of this transition is relevant. The aim of the study is to study the conditions for the occurrence of friction between the layers of a two-layer plate under the action of uniform pressure at ideal contact, as well as to determine the area where friction occurs, and the critical pressure at which friction occurs between the layers.

2. Statement of the problem of the three-dimensional theory of elasticity for a two-layer anisotropic plate
Let’s consider a thin two-layer plate under the action of a uniformly distributed pressure. One end of the plate is rigidly clamped, and the second is hinged. The materials of the plate layers are assumed to be anisotropic [5-7]. At the initial moment of time, the layers of the plate are connected by the conditions of ideal contact, however, with an increase in pressure, the ideal contact can be broken, and the layers begin to slip. The challenge is to determine the ultimate pressure at which slippage begins.

The problem is considered in a 3-dimensional setting. To solve it, the method of asymptotic averaging (homogenization method) is used, which is widely used now [8–15], and has a high calculation accuracy for thin plates [3-5,16,17].

The three-dimensional problem of the linear theory of elasticity has the form [7]:

$$\begin{align*}
\nabla_j \sigma_{ij}^{[m]} &= 0 \\
\varepsilon_{ij}^{[m]} &= \frac{1}{2} \left( \nabla_j u_{ij}^{[m]} + \nabla_i u_{ij}^{[m]} \right) \\
\sigma_{ij}^{[m]} &= C_{ijkl}^{[m]} \varepsilon_{kl}^{[m]}, \\
\Sigma_5 : \sigma_{ij}^{[5]} &= \Sigma_3^{[2]} = S_i^{[2]} , \\
\Sigma_3 : \sigma_{ij}^{[3]} &= S_i^{[3]} , \\
\Sigma_T : u_{ij}^{[m]} &= u_{ij} ,
\end{align*}$$

Here $\sigma_{ij}^{[m]}$ - stress tensor components, $\varepsilon_{ij}^{[m]}$ - small strain tensor components; $u_{ij}^{[m]}$ - displacement vector components, $\nabla_j = \frac{\partial}{\partial \xi_j}$ - operator of differentiation in Cartesian coordinates $\xi$ (axis $O\xi_3$ directed normal to the plate surfaces), $C_{ijkl}^{[m]}$ - components of the tensor of elastic moduli of layers, $\{m\}$ - layer index $m=1,2$, $\Sigma_5 = \Sigma_3 = \{\pm h_2\}$ - outer, and inner surface of the plate, $h$ – total thickness of two-layer plate; $\Sigma_T$ – end
surface of the plate on which the displacement vectors are specified $u_{x_1}, \Sigma_3^S$ - contact surface of layers, $\theta$ - coefficient of sliding of layers relative to each other, $\theta \geq 0$. At $\theta = 0$ we obtain the conditions for the ideal contact of the layers.

The system of equations (1) includes the equilibrium equations, Cauchy relations, the generalized Hooke's law, conditions on the contact surface of layers $\Sigma_3^S$ (nonideal contact with the presence of slip), and conditions on the outer surfaces of the plate $\Sigma_{3\pm}$ - vectors of efforts are set on them $S_i^{[1]}$, and $S_i^{[2]}$.

$$ S_i^{[m]} = -\kappa^2 p^{[m]} \delta_{i3}, $$

where $p^{[m]}$ - pressure, $\kappa = h/L$ - small parameter, $L$ - characteristic plate length.

3. Asymptotic solution of the problem

The solution to problem (1), (2) is sought in the form of an asymptotic expansion in the parameter $\kappa \ll 1$, in the form of functions depending on the global $x_3 = \tilde{x}_3 / L$, and local coordinates $\xi = \tilde{x}_3 / \kappa [2-5]$: $u^{[m]}_k = u^{(0)}_k(x_3) + k u^{[1]}_k(x_3, \xi) + k^2 u^{[2]}_k(x_3, \xi) + \cdots$,

$$ \sigma_{ij}^{[m]} = \sigma_{ij}^{[0]} + k \sigma_{ij}^{[1]} + k^2 \sigma_{ij}^{[2]} + \cdots, $$

where $I$ - the index of the global coordinates $I = 1, 2$; the coordinate $\xi$ along the thickness of the plate changes in the range $-0.5 < \xi < 0.5$; the interface between the layers has the coordinate $\xi = \alpha_1$.

When deriving expressions (4), we used the boundary conditions on the surfaces of the plate $\xi = \pm 0.5$. (6)

Here the operations of averaging over thickness are introduced:

$$ \left\langle f^{[m]}(\xi) \right\rangle_{\pm 0.5} = \frac{\int^{0.5} - \int^{-0.5} f^{[m]}(\xi) d\xi}{0.5} \left( f^{[1]}(\xi) \right)_{\pm 0.5} = \int^{0.5} f^{[2]}(\xi) - f^{[m]}(\xi) d\xi, $$

$$ \left\langle u^{[m]}_i(\xi) \right\rangle_{\pm 0.5} = \int^{0.5} - \int^{-0.5} u^{[m]}_i(\xi) d\xi = \left\langle u^{[1]}_i(\xi) \right\rangle_{1} + \left\langle u^{[2]}_i(\xi) \right\rangle_{2}, $$

and $\psi_m = (-1)^{m+1} \Delta p = p^{[1]} - p^{[2]}$ - pressure drop.

When deriving expressions (4), we used the boundary conditions on the surfaces of the plate $\xi = \pm 0.5$. Expressions (4) for stresses are written out up to the principal terms of the asymptotic expansions.

To calculate the displacements of the zero approximation $u_1^{(0)}$, and $u_3^{(0)}$ the averaged problem of the theory of plate bending is solved. For the case of hinged fastening of the ends of the plate, $x = 0$, and $x = 1 (x_1 = x)$ we have

$$ u_1^{(0)} = -\Delta \bar{p} A (4x^3 - 6x^2), u_3^{(0)} = -\Delta \bar{p} B (x^4 - 2x^3 + x), $$

where

$$ B = \frac{\bar{C}_{1111}}{24\bar{q}}, \quad A = \frac{\bar{B}_{1111}}{24\bar{q}}, \quad \Delta \bar{p} = \kappa^2 \Delta p. $$

Here the $C_{ijkl}$- membrane, $B_{ijkl}$ - mixed, $D_{ijkl}$ - bending stiffness of the plate is indicated

$$ Q = D_{ijkl} C_{ijkl} = B_{ijkl} - B_{ijkl} C_{ijkl}, $$

For the case of rigid fixing of the ends $x = 0$, and $x = 1$ we obtain the following solution

$$ u_1^{(0)} = -\Delta \bar{p} A (4x^3 - 6x^2 + 2x), u_3^{(0)} = -\Delta \bar{p} B (x^4 - 2x^3 + x). $$

(6)
Substituting (5) or (6) in (4), we find an expression for the stresses.

4. Checking the conditions for the beginning of sliding layers

With ideal contact between the layers of a two-layer plate, the condition is satisfied:

\[ \sigma_T < \tau_S \]  \hspace{1cm} (7)

Where \( \sigma_T \) - maximum value of shear stresses in the slip plane:

\[ \sigma_T(x) = \left( \left( \sigma^{(1)}_{13}(a_1, x) \right)^2 + \left( \sigma^{(1)}_{23}(a_1, x) \right)^2 \right)^{0.5} \]  \hspace{1cm} (8)

\( \tau_S \) - limit value of sliding friction onset stress. There are various models for \( \tau_S \). It is generally believed that \( \tau_S \) depends on transverse normal stresses \( \sigma_N \), and \( \sigma_T \):

\[ \tau_S = \tau_S(\sigma_N, \sigma_T) \]  \hspace{1cm} (9)

Where

\[ \sigma_N(x) = \left[ \sigma^{(1)}_{33}(a_1, x) \right] \]  \hspace{1cm} (10)

In the simplest model \( \tau_S \) is a constant: \( \tau_S = \tau_S^* \).

When the condition is met:

\[ \sigma_T = \tau_S \]  \hspace{1cm} (11)

sliding begins with friction of the layers relative to each other.

Substituting expressions (8), and (9) into (11), we obtain an additional equation for finding the slip function \( \theta \).

If the simplest model is used, then condition (11) takes the form:

\[ \sigma_T(x) = \tau_S^* \]  \hspace{1cm} (12)

For the problem of bending a plate \( \sigma^{(1)}_{23} = 0 \), then, substituting (8) into (12), we obtain the following equation:

\[ \left[ \sigma^{(1)}_{13}(a_1, x^*) \right] = \tau_S^*. \]  \hspace{1cm} (13)

This equation allows you to find the limit value of the pressure \( \Delta \bar{p} \), at which sliding begins in the plate, as well as the value of the coordinate \( x^* \), that determines the zone of the beginning of sliding.

Substituting expressions (4), (5) or (6), we find the corresponding dependences \( \Delta \bar{p}(x^*) \) for various types of boundary conditions for fixing the plate.

5. Numerical results

In numerical calculations, a two-layer plate was considered, each plate of which corresponds to two types of fiberglass, the values of the elastic characteristics of which are presented in Table 1.

| Characteristic | Layer 1 | Layer 2 |
|---------------|---------|---------|
| \( E_1 \), GPa | 14      | 21      |
| \( E_2 \), GPa | 14      | 21      |
| \( E_{31} \), GPa | 5,3     | 7,95    |
| \( G_{12} \), GPa | 1,8     | 2,7     |
| \( G_{13} \), GPa | 0,75    | 1,25    |
| \( G_{23} \), GPa | 0,75    | 1,25    |
| \( v_{12} \) | 0,08    | 0,12    |
| \( v_{31} \) | 0,14    | 0,21    |
| \( v_{23} \) | 0,15    | 0,225   |
Where \( E_i \) - Young's modulus (modulus of elasticity), \( G_{ij} \) - shear modulus, \( \nu_{ij} \) - Poisson's ratio.

The coordinate of the boundary of the contact of the layers was taken equal to \( a_1 = 0.1 \). The parameter \( \kappa \) was chosen as follows \( \kappa = 0.04 \). The limit value of the stress of the beginning of sliding friction was chosen as follows: \( \tau_{SS} = 17 \text{MPa} \).

For a plate with hinged fastening of both ends of the plate, the critical pressure function graph \( \Delta \tilde{p}_s(x^*) \) is shown in Figure 1.

![Figure 1. Critical pressure for hinged fastening at the edges of the plate](image)

When \( x^* = 0 \), and \( x^* = 1 \) the critical function has minimum values equal to \( \tilde{p}_s(x^*) = 1.653 \). This means that at such a value of dimensionless pressure, sliding friction occurs between the layers of the plate, and sliding begins at the ends of the plate at \( x^* = 0 \), and \( x^* = 1 \).

For a plate with rigid attachment at the left end \( x^* = 0 \), and with hinge attachment at the right end \( x^* = 1 \) the graph of this function (8) is shown in Figure 2. At \( x^* = 0 \) the critical function has a minimum value equal to \( \tilde{p}_s(x^*) = 1.282 \). This means that at such a value of the dimensionless pressure, sliding friction arises between the layers of the plate, and sliding begins at the left rigidly fixed end of the plate at \( x^* = 0 \).

![Figure 2. Critical pressure for plate with rigid clamping at the left end, and with a free edge at the right end](image)
The calculations showed that a two-layer fiberglass plate with the boundary conditions of rigid attachment at the ends will maintain ideal contact at a higher pressure than the same plate with the conditions of hinged attachment at the ends.

6. Conclusion
In this work, a model of deformation of a two-layer anisotropic plate with the condition of sliding friction between the layers is constructed. The model is constructed within the framework of a three-dimensional problem of the theory of elasticity with nonlinear slip conditions for layers, using the method of asymptotic averaging. A solution is found for the problem of bending a two-layer orthotropic plate for two different cases of fixing the plate at the ends. The values of the critical pressure at which the sliding of the plate layers relative to each other begins are calculated.

References
[1] Dimitrienko Yu I, Yurin Yu V 2018 Timoshenko-type asymptotic theory for thin multi-layered plates shells Mathematical Modeling and Computational Methods 1 pp 16–40
[2] Dimitrienko Yu I, Gubareva E A, Yurin Yu 2015 Variational equations of asymptotic theory for multilayer thin plates Vestnik MGTU im N.E.Baumana. Ser. Estestvennye nauki [Herald of the Bauman Moscow State Technical University. Natural Sciences] 4 pp 67-87
[3] Dimitrienko Yu I 2012 Asymptotic theory of thin multilayer plates Vestnik MGTU im N.E.Baumana. Ser. Estestvennye nauki [Herald of the Bauman Moscow State Technical University. Natural Sciences] 3 pp 86-100
[4] Dimitrienko Yu I, Gubareva E A, Pichugina A E 2019 Theory of composite cylindrical shells under quasistatic vibrations, based on an asymptotic analysis of the general viscoelasticity theory equations IOP Conference Series: Material Science and Engineering 683 012013
[5] Dimitrienko Yu I, Gubareva E A 2019 Asymptotic theory of thin two-layer elastic plates with layer slippage Mathematical Modeling and Computational Methods 1 pp 3–26
[6] Lekhnitsky S G 1957 Anizotropnye plastinki Moscow Gostekhizdat Publ. 463 p
[7] Dimitrienko Yu I 2013 Osnovi Mekhaniki Tverdikh Sred/ Mekhanika Sploshnoy Sredy [Bases of Solid Mechanics/ Continuum Mechanics] 4 624 p.
[8] Bakhtvalov N S, Panasenko G P 1989 Homogenisation: Averaging Processes in Periodic Media. Mathematical Problems in the Mechanics of Composite Materials Springer Publ. 352 p
[9] Sanchez-Palencia E 1980 Non-homogeneous media and vibration theory. Lecture Notes in Physics Berlin, Springer Publ. 127 398 p
[10] Pobedrya B E 1984 Mekhanika kompozitisonnykh materialov [Mechanics of Composite Materials] Moscow, MGU Publ. 352 p
[11] Mingyang W, Jianjun L, Xiangfeng L, Di S, Zhengwen Z 2018 A study on homogenization equations of fractal porous media J. Geophys. Eng. 15 pp 2388-2398
[12] Moslerab J, Shchygloc O, Montazer Hojjata H 2014 A novel homogenization method for phase field approaches based on partial rank-one relaxation Journal of the Mechanics and Physics of Solids 68 pp 251-266
[13] Shixin X, Xingye Yu, Changrong Zh 2016 Homogenization: In mathematics or physics? Discrete & Continuous Dynamical Systems – S 9(5) pp 1575-1590
[14] Rohan E, Lukeš V 2019 Homogenization of the vibro-acoustic transmission on perforated plates Applied Mathematics and Computation 361 pp 821-845
[15] Agalovyan L A 1997 Asimptoticheskaya teoriya anizotropnykh plastin i obolochek Moscow, Nauka Publ. 414 p
[16] Dimitrienko Yu I, Yakovlev D O 2015 The asymptotic theory of thermostaticity of multilayer composite plates Composites: Mechanics, Computations, Applications: An International Journal 6(1) pp 13-51
[17] Dimitrienko Yu I, Dimitrienko I D 2016 Asymptotic Theory for Vibrations of Composite Plates Applied Mathematical Sciences 10(60) pp 2993 – 3002