Renormalization-group improved Higgs to two gluons decay rate

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Abstract

We investigate the renormalization-group scale and scheme dependence of the $H \rightarrow gg$ decay rate at the order $N^4$LO in the renormalization-group summed perturbative theory, which employs the summation of all renormalization-group accessible logarithms including the leading and subsequent four sub-leading logarithmic contributions to the full perturbative series expansion. Moreover, we study the higher-order behaviour of the $H \rightarrow gg$ decay width using the asymptotic Padé approximant method in four different renormalization schemes. Furthermore, the higher-order behaviour is independently investigated in the framework of the asymptotic Padé-Borel approximant method where generalized Borel-transform is used as an analytic continuation of the original perturbative expansion. The predictions of the asymptotic Padé-Borel approximant method are found to be in agreement with that of the asymptotic Padé approximant method. Finally, we provide the $H \rightarrow gg$ decay rate at the order $N^5$LO in the fixed-order $\Gamma_{N^5LO} = \Gamma_0 (1.8375 \pm 0.047^{\alpha_s(M_Z)}_{\square}, 1\% \pm 0.0004 M_t \pm 0.0066 M_H \pm 0.0002 \mu \pm 0.0036 p \pm 0.007_s \pm 0.0005 s_c)$ and $\Gamma_{RGSN^5LO} = \Gamma_0 (1.841 \pm 0.047^{\alpha_s(M_Z)}_{\square}, 1\% \pm 0.0005 M_t \pm 0.0066 M_H \pm 0.0002 \mu \pm 0.0027 p \pm 0.001 s_c)$ in the renormalization-group summed perturbative theories.
1 Introduction

The discovery of the Higgs boson at the Large Hadron Collider (LHC) is the first step towards a better understanding of the electroweak symmetry breaking of the standard model (SM) \cite{1, 2}. Moreover, studying the phenomenological behaviour of this particle is immensely important for providing crucial information for physics beyond the SM. In the SM, the Higgs boson dominantly decays to a pair of bottom quarks, that is, $H \rightarrow bb(\text{hadrons})$ followed by the second leading hadronic decay channel $H \rightarrow gg$, which is predominantly mediated by the top quark.

There has been an extensive theoretical investigation of the decay $H \rightarrow gg$ in literature\cite{3, 4, 5}. For instance, it has been computed in an effective theory approach by considering $M_H \ll 2m_t$ in reference \cite{6}. The $1/m_t$ corrections are evaluated up to N$^3$LO in references \cite{7, 8}. The effective Higgs coupling to gluons is computed up to N$^3$LO in references \cite{9, 10, 11, 12, 13, 14, 15}, up to N$^3$LO in the limit of heavy top quarks in references \cite{16, 17, 18}, and up to NLO with the full quark mass dependence in references \cite{19, 20}. The subleading finite top-quark mass effects in the large top mass expansion are known at N$^2$LO \cite{21}, and the top-bottom interference at NLO accuracy is computed in reference \cite{22, 23}.

We observe that a low energy theorem can also yield the QCD corrections to the $H \rightarrow gg$ decay width at N$^4$LO in the heavy quark limit \cite{3, 24}. The effective Higgs-gluons Lagrangian in the heavy top-quark limit in this approach is derived by keeping the terms that depend only on the Higgs field, and using the transformation of the gluonic field strength operator and the strong coupling constant from $\mu \rightarrow H$ may cause theoretical uncertainty in a prediction. The ambiguity arising due to the RG scale dependence in the presently unknown $1/m_{top}$ effect at N$^2$LO, and larger than the subleading finite top-quark mass effects in the large top mass expansion are known at N$^3$LO \cite{22, 23}. Moreover, this computation has reduced the uncertainty due to the truncation of the perturbation series, thus providing an improvement in the $H \rightarrow gg$ decay rate.

The hadronic processes such as $H \rightarrow gg$ are usually calculated up to finite order within perturbative QCD. Therefore, these calculations are plagued by a significant dependence on the renormalization (RG) scale parameter $\mu$. There are several prescriptions for dealing with this dependence in literature so that a physical prediction can be obtained. For instance, this scale in practice, can be identified with the mass of the decaying particle. One possibility is to use a value of $\mu$ such that the known calculated terms in perturbation expansion show a local insensitivity to this value. In addition, a value of $\mu$ may also be chosen, which minimizes the highest-order known term of the perturbative expansion \cite{34}. It is obvious that using different prescriptions may cause theoretical uncertainty in a prediction. The ambiguity arising due to the RG scale dependence in $H \rightarrow gg$ decay at the order N$^4$LO is of particular interest for the precision physics within the SM as well as beyond the SM physics \cite{34, 35, 36, 37}.

We note that the renormalization group equation (RGE) is extensively used as a tool to improve the behaviour of the perturbative expansions, and extend their domain of applicability in perturbative quantum field theory. For instance, if the perturbative expansion is known to $k$-subleading orders, the renormalization group equation can be used to sum the leading and subsequent ($k-1$) subleading log contributions to the full perturbative expansion. Such logarithms are known as “RG-accessible” in literature\cite{38, 39}.

In this work, we discuss the closed-form summation of these RG-accessible logarithms for the $H \rightarrow gg$ decay rate in the so-called “renormalization-group summed perturbation theory” (RGSPT) in the four different RG schemes, namely, $\overline{MS}$, SI, OS, and miniMOM. Perturbative series in the RGSPT is expanded in terms of coupling constant and logarithms such that coefficient of every such term can be determined through RG-invariance in terms of its leading coefficient. This is a generalization of the method of the leading logarithm summation and provides the RG-summed perturbative expansion of the series to any given order of the perturbation theory. One of the salient features of the new RGSPT expansions is the reduced sensitivity to RG scale $\mu$ in spite of the presence of large logarithms \cite{38}.

In literature, RGE is found to be helpful in the extraction of the divergent parts of the bare parameters using the determination of higher-order terms \cite{40, 41}. However, the incorporation of all available higher-order RG-accessible terms to a given order in perturbation theory was first discussed by Maxwell to deal with unphysical
RG scale dependence \cite{42}. This was further applied to moments of QCD lepto-production structure functions, and to N^2\text{LO} correlation functions for the summation of the leading logarithms in references \cite{43,44}. This formalism is also used by McKeon to extract one-loop RG functions of φ^4- and φ^6-field theories by performing the summation of leading logarithms to all orders \cite{45}. The summation procedure of McKeon was extended to study the semileptonic B-decay rate in reference \cite{38}.

The RGSPT expansions have been employed in various decays and observables in literature, and are shown to exhibit a remarkable improvement in the sensitivity to the renormalization scale. For instance, these are used to study the e^+e^- hadronic cross-section \cite{39}, extraction of the strong coupling constant from the hadronic width of the \tau-decays \cite{40,47}, extraction of the strange quark mass from moments of the \tau-decay spectral function \cite{48} and investigate the QCD static energy at the four-loop in reference \cite{49}. Moreover, RGSPT expansions can further be improved by the method of conformal mapping \cite{46,47,50,51,52,53,54,55}. In addition to the RG improvement of the H \rightarrow gg decay rate, we also investigate the higher-order behaviour of the perturbative expansion of the H \rightarrow gg decay rate using the asymptotic Padé approximant (APAP) method \cite{56,57,58,59,60,61,62,63,64}. Moreover, the higher-order estimate of the beta and gamma function coefficients are also discussed using the same method. These predictions are further independently obtained through the asymptotic Padé-Borel approximant (PBA) method, which provides an additional test of the asymptotic Padé approximant predictions.

Our paper will be presented along the following track: We first briefly review the theoretical formalism, and state-of-the-art computation of the one-loop RG functions of φ^4- and φ^6-field theories by performing the summation of leading logarithms to all orders \cite{45}. This is followed by section \ref{sec:2} where we discuss the H \rightarrow gg decay width in the standard “fixed-order perturbation theory” (FOPT), and extract the expansion coefficients in various RG schemes. In section \ref{sec:3}, we discuss the H \rightarrow gg decay rate in the RGSPT at N^4\text{LO}. The expansion functions of the RGSPT are derived in an analytic closed-form by solving the differential equation using the method of iteration in this section. We present a thorough discussion of the scale and scheme dependence of the H \rightarrow gg decay rate at N^4\text{LO} in the FOPT in section \ref{sec:5}. This is followed by the discussion of scale and scheme dependence of the H \rightarrow gg decay rate at N^4\text{LO} in the RGSPT in section \ref{sec:6}. The higher-order behaviour of the H \rightarrow gg decay rate using the APAP formalism is discussed in section \ref{sec:7}. The study of the higher-order corrections to the H \rightarrow gg decay rate in the framework of the PBA is presented in section \ref{sec:8}. We determine the H \rightarrow gg decay width using the FOPT and the new RGSPT expansions discussed in this work in section \ref{sec:9} in the \overline{\text{MS}}, scale invariant (SI), on shell (OS), and the minimal momentum subtraction (mini\text{MOM}) schemes. We summarize our results in section \ref{sec:10}.

\section{Inclusive decay of the Higgs-boson to gluons}

In this section, we briefly review the calculation of the inclusive decay of the Higgs-boson to two gluons. The inclusive decay of the Higgs-boson to gluons is calculated in the limit of a large top-quark massweffectively

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD}(n_f)} - \frac{1}{2!} \mathcal{G}_F^{\mu\nu} C_1 H G_a^{\mu\nu} G_a^{\mu\nu}, \]

where H represents the Higgs field, the renormalized gluon field-strength tensor is denoted by G_a^{\mu\nu} with n_f flavours, the Lagrangian \mathcal{L}_{\text{QCD}(n_f)} is the QCD Lagrangian, the renormalized coefficient C_1 parametrizes the top-mass dependence, and \mathcal{G}_F is the Fermi coupling constant.

The partial decay width of the Higgs-boson to gluons \Gamma_{H \rightarrow gg} decay can be computed using the imaginary part of the Higgs-boson self-energy and written as,

\[ \Gamma_{H \rightarrow gg} = \frac{\sqrt{2} G_F}{M_H} |C_1|^2 \text{Im} \Pi^{GG} (-M_H^2 - i\delta), \]

where M_H is the mass of the Higgs boson, \Pi^{GG} represents the contribution to the self-energy of the Higgs boson due to its effective coupling to gluons, and \delta is a positive real parameter, which is infinitesimally small.

The coefficient C_1 is known up to N^4\text{LO} and its perturbative expansion is given by,

\[ C_1 = -\frac{1}{3} a_s \left( 1 + \sum_{n=1} \epsilon_n a_s^n (\mu^2) \right), \]
where \( a_c \equiv \frac{a_s}{\pi} \), where \( n_f \) are number of light flavours.

The known expansion coefficients \( c_n \) in the SI scheme at the renormalization scale \( \mu = \mu_t \), where \( \mu_t = m_t(\mu_t) \) is the \( \overline{\text{MS}} \) top quark mass evaluated at scale \( \mu_t \), in the \( \overline{\text{MS}} \) and OS schemes are written as \cite{4}

\[
c_1 = 11, \quad c_2 = 154.278 + 19L_t + \left(-11.1667 + 5.33333L_t\right)n_f, \\
c_{3,\text{SI}} = 3031.7 + 537.111L_t + 209L_t^2 + \left(-492.139 + 107.852L_t + 46L_t^2\right)n_f \nonumber + \left(-14.1255 + 2.85185L_t - 3.55556L_t^2\right)n_f^2, \\
c_{4,\text{SI}} = 79815.7 + 12340.2L_t + 9831.33L_t^2 + 2299L_t^3 \nonumber + \left(-15966.1 - 1482.59L_t + 1394.11L_t^2 + 366.667L_t^3\right)n_f \\
+ \left(413.572 - 621.891L_t - 94.5741L_t^2 - 69.7778L_t^3\right)n_f^2 \nonumber + \left(-8.79068 + 23.7531L_t - 2.85185L_t^2 + 2.37037L_t^3\right)n_f^3, \\
c_{3,\overline{\text{MS}}} = c_{3,\text{SI}} - 152L_t - 42.6667L_t n_f, \\
c_{4,\overline{\text{MS}}} = c_{4,\text{SI}} - 5576.22L_t - 4230.67L_t^2 + \left(-1158.59L_t - 934.222L_t^2\right)n_f + \left(-5.03704L_t + 71.1111L_t^2\right)n_f^2, \\
c_{3,\text{OS}} = c_{3,\text{SI}} + 202.667 + 56.8889n_f, \\
c_{4,\text{OS}} = c_{4,\text{SI}} + 13362.3 + 6688L_t + \left(2659.9 + 1472L_t\right)n_f + \left(-147.307 - 113.778L_t\right)n_f^2, \\
\]

where \( L_t = \ln(\mu^2/m_t^2) \), \( \mu \) is the RG scale and \( m_t \) is the definition of the top quark mass in the corresponding RG scheme.

The absorptive part of the vacuum polarization is computed at N^4LO in reference \cite{4}, thus, making the Higgs-boson to gluons decay width complete at the N^4LO order. The absorptive part of the vacuum polarization is written in the following form,

\[
\frac{4\pi}{N_A q^4} \text{Im} \Pi^{GG}(q^2) \equiv G(q^2) = 1 + \sum_{n=1}^\infty g_n a_n^q, 
\]

where \( N_A = 8 \) in QCD.

The coefficients \( g_n \) of the renormalization of the absorptive part in the \( \overline{\text{MS}} \) scheme corresponding to the self-energy of the Higgs boson up to N^4LO are \cite{4},

\[
g_1 = 73 - \frac{14}{3} n_f - \frac{46}{3} L_q \\
g_2 = 3887.57 - 629.982n_f + 14.4283n_f^2 - \left[23 \left(73 - \frac{14}{3} n_f\right) + \frac{464}{3}\right] L_q + \frac{529}{3} L_q^2 \\
g_3 = 163394 - 49409.6n_f + 2974.39n_f^2 - 34.4213n_f^3 - \left[9769 \frac{9}{9} + \frac{580}{3} \left(73 - \frac{14}{3} n_f\right)\right] L_q + \left[\frac{34684}{9} + \frac{1058}{3} \left(73 - \frac{14}{3} n_f\right)\right] L_q^2 - \frac{48668}{27} L_q^3 \\
g_4 = 5.45154 \times 10^6 - 2.81728 \times 10^6 n_f + 318324. n_f^2 - 9921.43n_f^3 + 64.359n_f^4 + \left[-38609.3 \\
- 232 \left[3887.57 - 629.982n_f + 14.4283n_f^2\right] - \frac{115}{3} \left(163394. - 49409.6n_f + 2974.39n_f^2 - 34.4213n_f^3\right)\right] \right] L_q + \left[\frac{3924805}{81} + \frac{57362}{9} \left(73 - \frac{14}{3} n_f\right) + \frac{5290}{9} \left(3887.57 - 629.982n_f\right) \\
+ 14.4283n_f^2\right] L_q^2 + \left[\frac{5093212}{81} - \frac{121670}{27} \left(73 - \frac{14}{3} n_f\right)\right] L_q^3 + \frac{1399205}{81} L_q^4,
\]
where \( L_q = \ln \left( \frac{q^2}{m_t^2} \right) \).

The logarithm in equation \(2.6\) can also be written in terms of the pole mass of the top quark by defining,

\[
L_q = T - \ln \left( \frac{\mu^2}{M_t^2} \right),
\]

where \( T \equiv \ln \left( M_H^2 / M_t^2 \right) \) and \( M_t \) is the pole mass of the top quark, where we have set \( q^2 = M_H^2 \).

## 3 Fixed-order-perturbation theory

In the RGSPT, predicting RG-accessible next-order coefficients becomes more effective if we express the perturbative expansion in terms of the running fermion mass \(37\). Therefore, we rewrite the \( H \rightarrow gg \) decay rate expansion in terms of the running top-quark mass. For this purpose, we use the relation between the running and pole mass \(65\) \(66\).

\[
m(\mu) = 1 + \left[ -1 - \frac{4}{3} l_{\mu M} \right] x(\mu) \tag{3.8}
\]

\[
+ \left[ -14.3444 - \frac{445}{72} l_{\mu M} - \frac{19}{24} l_{\mu M}^2 + \left( 1.04137 + \frac{13}{36} l_{\mu M} + \frac{1}{12} l_{\mu M}^2 \right) n_f \right] x(\mu)^2
\]

\[
+ \left[ -198.707 - 78.9409 l_{\mu M} - \frac{11779}{864} l_{\mu M}^2 - \frac{475}{432} l_{\mu M}^3 + \left( 26.9239 + 12.3257 l_{\mu M} \right) \right] n_f + \left( -0.652692 - 0.380301 l_{\mu M}^2 - \frac{1}{108} l_{\mu M}^3 \right) x(\mu)^3,
\]

where \( l_{\mu M} = \ln \mu^2 / M_t^2 \), \( x(\mu) = \frac{\alpha_s(\mu)}{\pi} \) and \( n_f \) are number of active flavours.

Using above relation, the logarithm \( L_q \) in the equation \(2.7\) can be expressed in terms of \( \ln \frac{\mu^2}{m_t^2(\mu)} \). This can be done by writing the logarithm \( \ln \left( \frac{\mu^2}{M_t^2} \right) \) as,

\[
\ln \left( \frac{\mu^2}{m_t^2(\mu)} \right) = \ln \frac{\mu^2}{m_t^2(\mu)} + 2 x^{(\nu_f)}(\mu) \left[ -1.33333 - \ln \frac{\mu^2}{m_t^2(\mu)} \right] \tag{3.9}
\]

\[
+ 2 x^{(\nu_f)}(\mu)^2 \left[ -12.5666 + 1.04137 n_f + \left( -5.51389 + 0.36111 n_f \right) \ln \frac{\mu^2}{m_t^2(\mu)} \right]
\]

\[
+ \left( -1.29167 + 0.083333 n_f \right) \left( \ln \frac{\mu^2}{m_t^2(\mu)} \right)^2 + 2 x^{(\nu_f)}(\mu)^3 \left[ -173.452 + 25.2666 n_f - 0.652691 n_f^2 \right]
\]

\[
+ \left( -70.3593 + 11.9597 n_f \right) \ln \frac{\mu^2}{m_t^2(\mu)} + \left( -14.4252 + 2.08102 n_f - 0.380301 n_f^2 \right) \left( \ln \frac{\mu^2}{m_t^2(\mu)} \right)^2
\]

\[
+ \left( -2.22454 + 0.287037 n_f - 0.00925926 n_f^2 \right) \left( \ln \frac{\mu^2}{m_t^2(\mu)} \right)^3.
\]

Thus, the decay width of \( H \rightarrow gg \) acquires the following form in terms of the running top quark mass,

\[
\Gamma = \sqrt{2} \alpha_s M_H^3 / 72 \pi \ x^2(\mu) S \left[ x(\mu), L(\mu) \right],
\]

where the perturbative expansion \( S[x(\mu), L(\mu)] \) in the FOPT is written as,

\[
S_{FOPT}[x(\mu), L(\mu)] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_{n,k} x^n L^k.
\]

5
We work with $n_f = 5$ flavours in this work. The coefficients of expansion $T_{n,k}$ in the $\overline{\text{MS}}$ scheme are,

\begin{equation}
\begin{aligned}
T_{0,0}^{\overline{\text{MS}}} &= 1, \quad T_{1,0}^{\overline{\text{MS}}} = 17.9167 - 3.83333 T, \quad T_{1,1}^{\overline{\text{MS}}} = 3.83333, \quad T_{2,0}^{\overline{\text{MS}}} = 146.586 - 102.146 T + 11.0208 T^2, \\
T_{2,1}^{\overline{\text{MS}}} &= 100.188 - 22.0417 T, \quad T_{2,2}^{\overline{\text{MS}}} = 11.0208, \quad T_{3,0}^{\overline{\text{MS}}} = 123.647 - 1156.52 T + 394.514 T^2 - 28.1644 T^3, \\
T_{3,1}^{\overline{\text{MS}}} &= 1034.83 - 766.826 T + 84.4931 T^2, \quad T_{3,2}^{\overline{\text{MS}}} = 376.545 - 84.4931 T, \quad T_{3,3}^{\overline{\text{MS}}} = 28.1644, \\
T_{4,0}^{\overline{\text{MS}}} &= -11815.8 - 2506.94 T + 5777.77 T^2 - 1274.79 T^3 + 67.477 T^4, \\
T_{4,1}^{\overline{\text{MS}}} &= 45.6137 - 10430.6 T + 3718.31 T^2 - 269.908 T^3, \quad T_{4,2}^{\overline{\text{MS}}} = 4609.02 - 3615.6 T + 404.863 T^2, \\
T_{4,3}^{\overline{\text{MS}}} &= 1184.31 - 269.908 T, \quad T_{4,4}^{\overline{\text{MS}}} = 67.4771,
\end{aligned}
\end{equation}

and the logarithm is $L(\mu) = \ln(\mu^2/m_t^2(\mu))$.

We can also write the FOPT expansion in terms of SI, OS, and miniMOM schemes. For this purpose, we write equation 2.6 in terms of the SI, OS, and miniMOM mass of the top quark,

\begin{equation}
L_q = T - \ln \left( \frac{\mu^2}{M_{p}^2} \right),
\end{equation}

where $T \equiv \ln \left( \frac{M_H^2}{M_{p}^2} \right)$ and $p$ stands for SI, OS or miniMOM scheme. In the SI scheme, the coefficients of the FOPT expansion are,

\begin{equation}
\begin{aligned}
T_{0,0}^{\text{SI}} &= 1, \quad T_{1,0}^{\text{SI}} = 17.9167 - 3.83333 T, \quad T_{1,1}^{\text{SI}} = 3.83333, \quad T_{2,0}^{\text{SI}} = 156.808 - 102.146 T + 11.0208 T^2, \\
T_{2,1}^{\text{SI}} &= 107.854 - 22.0417 T, \quad T_{2,2}^{\text{SI}} = 11.0208, \quad T_{3,0}^{\text{SI}} = 452.461 - 1215.3 T + 394.514 T^2 - 28.1644 T^3, \\
T_{3,1}^{\text{SI}} &= 1337.74 - 810.91 T + 84.4931 T^2, \quad T_{3,2}^{\text{SI}} = 427.337 - 84.4931 T, \quad T_{3,3}^{\text{SI}} = 28.1644, \\
T_{4,0}^{\text{SI}} &= -6502.1 - 4935.467 T + 6003.09 T^2 - 1274.79 T^3 + 67.477 T^4, \\
T_{4,1}^{\text{SI}} &= 6041.54 - 12666.6 T + 3887.29 T^2 - 269.908 T^3, \quad T_{4,2}^{\text{SI}} = 6947.57 - 3992.14 T + 404.863 T^2, \\
T_{4,3}^{\text{SI}} &= 1400.62 - 269.908 T, \quad T_{4,4}^{\text{SI}} = 67.4771.
\end{aligned}
\end{equation}

In a similar manner, we write the coefficients of expansion in the OS scheme,

\begin{equation}
\begin{aligned}
T_{0,0}^{\text{OS}} &= 1, \quad T_{1,0}^{\text{OS}} = 17.9167 - 3.83333 T, \quad T_{1,1}^{\text{OS}} = 3.83333, \quad T_{2,0}^{\text{OS}} = 156.808 - 102.146 T + 11.0208 T^2, \\
T_{2,1}^{\text{OS}} &= 107.854 - 22.0417 T, \quad T_{2,2}^{\text{OS}} = 11.0208, \quad T_{3,0}^{\text{OS}} = 467.684 - 1215.3 T + 394.514 T^2 - 28.1644 T^3, \\
T_{3,1}^{\text{OS}} &= 1337.74 - 810.91 T + 84.4931 T^2, \quad T_{3,2}^{\text{OS}} = 427.337 - 84.4931 T, \quad T_{3,3}^{\text{OS}} = 28.1644, \\
T_{4,0}^{\text{OS}} &= -6091.71 - 4993.81 T + 6003.09 T^2 - 1274.79 T^3 + 67.477 T^4, \\
T_{4,1}^{\text{OS}} &= 6187.42 - 12666.6 T + 3887.29 T^2 - 269.908 T^3, \quad T_{4,2}^{\text{OS}} = 6947.57 - 3992.14 T + 404.863 T^2, \\
T_{4,3}^{\text{OS}} &= 1400.62 - 269.908 T, \quad T_{4,4}^{\text{OS}} = 67.4771.
\end{aligned}
\end{equation}

We also investigate the Higgs to gluon decay width in the miniMOM version of the OS scheme [67, 68]. In this scheme, the strong coupling is fixed by the knowledge of the gluon and ghost propagators [67, 68]. The expansion coefficients for this scheme can be written using the expansion given in reference [33],

\begin{equation}
\begin{aligned}
T_{0,0}^{\text{MM}} &= 1, \quad T_{1,0}^{\text{MM}} = 13.6528 - 3.83333 T, \quad T_{1,1}^{\text{MM}} = 3.83333, \quad T_{2,0}^{\text{MM}} = 46.9335 - 83.3368 T + 11.0208 T^2, \\
T_{2,1}^{\text{MM}} &= 89.0451 - 22.0417 T, \quad T_{2,2}^{\text{MM}} = 11.0208, \quad T_{3,0}^{\text{MM}} = -624.885 - 495.626 T + 333.353 T^2 - 28.1644 T^3, \\
T_{3,1}^{\text{MM}} &= 569.388 - 710.471 T + 84.4931 T^2, \quad T_{3,2}^{\text{MM}} = 388.058 - 84.4931 T, \quad T_{3,3}^{\text{MM}} = 28.1644, \\
T_{4,0}^{\text{MM}} &= -7041.01 + 5157.947 T + 2696.63 T^2 - 1100.39 T^3 + 67.477 T^4, \\
T_{4,1}^{\text{MM}} &= -5111.88 - 6155.32 T + 3510.88 T^2 - 269.909 T^3, \quad T_{4,2}^{\text{MM}} = 3626.15 - 3615.73 T + 404.863 T^2, \\
T_{4,3}^{\text{MM}} &= 1226.21 - 269.909 T, \quad T_{4,4}^{\text{MM}} = 67.4771.
\end{aligned}
\end{equation}
where MM represents the miniMOM scheme. The running of the strong coupling in the miniMOM is given by
\[ \alpha_{s,MM} = \alpha_s + 0.67862\alpha_s^2 + 0.91231\alpha_s^3 + 1.5961\alpha_s^4 + 3.1629\alpha_s^5 + \mathcal{O}(\alpha_s^6). \]

\[ (3.17) \]

### 4 Renormalization-group-summed perturbation theory

Now, we discuss the perturbative expansion of the Higgs to gluons decay rate in the RGSPT. In the RGSPT, the FOPT expansion of the function \( S(x(\mu), L(\mu)) \) is equivalent to writing the following new expansion,\[ S(x, L) = \sum_{n=0}^{\infty} x^n S_n(u), \]
where \( u = xL \), and the function \( S_n(u) \) is defined by,\[ S_n(u) \equiv \sum_{k=n}^{\infty} T_{k,k-n} u^{k-n}. \]

The main feature of the RGSPT is the explicit all-orders summations of all RG-accessible logarithms in the functions \( S_n(u) \) \[38, 39\]. Moreover, the functions \( S_n(u) \) can be derived in a closed analytical form \[38, 39\].

We present a derivation of the functions \( S_n(u) \) in a closed analytical form using the RG invariance. The decay width of the \( H \rightarrow gg \) decay defined in equation 3.10 is scale independent order-by-order in perturbation theory, and satisfies the RG equation,\[ \mu^2 \frac{d}{d\mu^2} \{\Gamma_{H \rightarrow gg}\} = 0. \]

The above equation can be written in the following form:\[ \left[ (1 - 2\gamma_m(x)) \frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} + \frac{2\beta(x)}{x} \right] S(x, L) = 0, \]
where the \( \overline{\text{MS}} \) \( \beta \)- and \( \gamma_m \)-functions are,\[ \beta(x) = \mu^2 \frac{dx}{d\mu^2} x(\mu) = -\left( \beta_0 x^2 + \beta_1 x^3 + \beta_2 x^4 + \ldots \right), \]
\[ \mu^2 \frac{d\gamma_m}{d\mu^2} = m\gamma_m(x(\mu)) = -m(\gamma_0 x + \gamma_1 x^2 + \gamma_2 x^3 + \ldots). \]

The coefficients of 5-loop QCD \( \beta \)-function in the \( \overline{\text{MS}} \) scheme read \[14\],\[ \beta_0 = 2.75 - 0.166667n_f, \]
\[ \beta_1 = 6.375 - 0.791667n_f, \]
\[ \beta_2 = 22.3203 - 4.36892 n_f + 0.0940394 n_f^2, \]
\[ \beta_3 = 114.23 - 27.1339 n_f + 1.58238 n_f^2 + 0.0058567 n_f^3, \]
\[ \beta_4 = 524.558 - 181.799 n_f + 17.156 n_f^2 - 0.225857 n_f^3 - 0.00179929 n_f^4. \]

The coefficients of the \( \gamma_m \)-function are given as \[70\],\[ \gamma_0 = 1, \]
\[ \gamma_1 = 4.20833 - 0.138889n_f, \]
\[ \gamma_2 = 19.5156 - 2.28412 n_f - 0.0270062 n_f^2, \]
\[ \gamma_3 = 98.9434 - 19.1075 n_f + 0.276163 n_f^2 + 0.00579322 n_f^3, \]
\[ \gamma_4 = 559.707 - 143.686 n_f + 7.48238 n_f^2 + 0.108318 n_f^3 - 0.0000853589 n_f^4. \] (4.24)

By substituting the expansion in equation 4.18 into the equation 4.21 we write the following equation,

\[ 0 = (1 - 2\gamma_m(x)) \sum_{n=0}^{\infty} k T_{n,k} x^n L^{k-1} + \beta(x) \sum_{n=0}^{\infty} n T_{n,k} x^{n-1} L^k + \frac{2\beta(x)}{x} \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_{n,k} x^n L^k. \]

The following recursion formula is derived by extracting the aggregate coefficient of \(x^n L^{n-p}\) for \(n \geq p:\)

\[ 0 = (n - p + 1) T_{n,n-p+1} - \sum_{\ell=1}^{p-1} (n - \ell + 1) \beta_r T_{n-\ell-1,n-p} + 2 \sum_{\ell=1}^{p-1} (n - p + 1) \gamma_{\ell-1} T_{n-\ell,n-p+1}. \] (4.25)

After this, we multiply both sides of equation (4.25) by \(u^{n-p}\) and sum from \(n = p\) to \(\infty.\) This results in a set of first-order linear differential equations for the functions defined in equation (4.19),

\[ 0 = (1 - \beta_0 u) \frac{dS_{p-1}}{du} - u \sum_{\ell=0}^{p-2} \beta_{\ell+1} \frac{dS_{p-\ell-2}}{du} + 2 \sum_{\ell=0}^{p-2} \gamma_{\ell} \frac{dS_{p-\ell-2}}{du} - \sum_{\ell=0}^{p-1} (p - \ell + 1) \beta_{\ell} S_{p-\ell-1}. \] (4.26)

By substituting \(n = p - 1\), the above equation can be written as,

\[ (1 - \beta_0 u) \frac{dS_n}{du} - u \sum_{\ell=1}^{n-1} \beta_{\ell+1} \frac{dS_{n-\ell-1}}{du} + 2 \sum_{\ell=0}^{n-1} \gamma_{\ell} \frac{dS_{n-\ell-1}}{du} - \sum_{\ell=0}^{n} (n - \ell + 2) \beta_{\ell} S_{n-\ell} = 0, \] (4.27)

where \(n, \ell \geq 0\) with the boundary condition \(S_n(0) = T_{n,0}0\).

We can now solve the system of equations (4.27) iteratively in an analytical closed-form. The solutions for \(n = 0, 1, 2, 3\) are,

\[ S_0(u) = \frac{T_{00}}{w_1}, \]

\[ S_1(u) = \frac{1}{\beta_0 w_1} \left[ -2T_{00}(\beta_1 - 2\beta_0 \gamma_0) \log w_1 + \beta_0 T_{00} \right], \]

\[ S_2(u) = \frac{1}{\beta_0^2 w_1} \left[ \beta_0 \left( -4\beta_0^2 \gamma_1 T_{00} u + \beta_0 (2T_{00} u (2\beta_1 \gamma_0 + \beta_2) + T_{20}) - 2\beta_0^2 T_{00} u \right) - (\beta_1 - 2\beta_0 \gamma_0) \log w_1 \right. \]

\[ \left. \left( -4\beta_0 \gamma_0 T_{00} + 3\beta_0 T_{00} + 2\beta_1 T_{00} + 3T_{00} (\beta_1 - 2\beta_0 \gamma_0)^2 \log^2 w_1 \right) \right], \]

\[ S_3(u) = \frac{1}{\beta_0^3 w_1} \left[ -\beta_0^2 \left( -2\beta_0^3 \gamma_2 T_{00} u^2 + \beta_0^2 u (2\beta_1 \gamma_0 T_{00} u + 2\beta_2 \gamma_0 T_{00} u + \beta_3 T_{00} u - 8\gamma_0 \gamma_1 T_{00} + 6\gamma_1 T_{20} + 4\gamma_2 T_{00} \right) \right. \]

\[ \left. - \beta_0 \left( u \left( 2\beta_1^2 \gamma_0 T_{00} u + 2\beta_1 \beta_2 T_{00} u - 8\beta_1 \gamma_0^2 T_{00} + 6\beta_1 \gamma_1 T_{20} + 3\beta_2 T_{20} + 2\beta_3 T_{00} \right) \right) \right. \]

\[ \left. + \beta_1 u (\beta_1 T_{00} u - 4\gamma_0 T_{00} + 3T_{20}) + 2\beta_2 T_{00} \right) - \beta_0 (\beta_1 - 2\beta_0 \gamma_0) \log w_1 \left( 4\beta_1 \gamma_0 T_{00} (1 - 3\beta_0 u) \right) \]

\[ - 2\beta_2 (3\beta_0 T_{00} u + T_{00}) + 2\beta_0 \left( 6\beta_0 \gamma_1 T_{00} u - 4\beta_0^2 T_{00} + 3\gamma_0 T_{20} + 2\gamma_1 T_{00} - 2T_{20} + 6\beta_2^2 T_{00} u - 3\beta_1 T_{20} \right) \]

\[ - (\beta_1 - 2\beta_0 \gamma_0)^2 \log^2 w_1 \left( -14\beta_0 \gamma_0 T_{00} u + 6\beta_0 T_{00} + 7\beta_1 T_{00} + 4T_{00} (\beta_1 - 2\beta_0 \gamma_0)^3 \log^3 w_1 \right), \] (4.28)

where \(w_1 = 1 - \beta_0 u.\)

The new RGS expansions now can be written as,

\[ S^{N^{\text{LO}}}_{\text{RGSPT}} = S_0(xL) + xS_1(xL), \] (4.29)

\[ S^{N^{\text{LO}}}_{\text{RGSPT}} = S_0(xL) + xS_1(xL) + x^2 S_2(xL), \]

\[ S^{N^{\text{LO}}}_{\text{RGSPT}} = S_0(xL) + xS_1(xL) + x^2 S_2(xL) + x^3 S_3(xL), \]
\[ S_{SPT}^{N+} = S_0(xL) + xS_1(xL) + x^2S_2(xL) + x^3S_3(xL) + x^4S_4(xL). \]

In the SI, OS, and miniMOM schemes, mass appearing in the logarithms does not depend on the RG scale \( \mu \). Therefore, the closed-form analytic expressions of the functions \( S_n(u) \) in these schemes are obtained by solving equation 4.27 after substituting the coefficient of the anomalous \( \gamma \) functions \( \gamma_i \) to be zero.

5 Scale and scheme dependence in the FOPT

We now investigate the perturbative behaviour of the Higgs to gluons decay width by studying the scale dependence in the FOPT in the \( \overline{\text{MS}} \), SI, OS, and miniMOM schemes. Our numerical inputs are given in table 1. Moreover, for running the strong coupling and quark masses, we use the Mathematica package “RunDec” [71].

| Parameter | Value          |
|----------------|----------------|
| \( M_H \)      | 125.25 ± 0.17  |
| \( G_F \)      | 1.1663787 × 10^{-5} GeV |
| \( \alpha_s(M_Z) \) | 0.1179 ± 0.0009 |
| \( M_{IOP} \)   | 173 GeV        |
| \( M_{IS1} \)   | 164 GeV        |

Table 1: The numerical values of masses and couplings used in this work.

The Higgs to gluons decay width is normalized by the first term in the expansion, i.e. \( \Gamma_0 \) given by,

\[ \Gamma_0 = \frac{G_F M_H^2}{36\pi^3 \sqrt{2}} (\alpha_s(M_H^2))^2 = 0.00018378, \]

where \( \alpha_s(M_H^2) = 0.112602 \) is calculated using the Mathematica package “RunDec” [71]. We use the above value of \( \Gamma_0 \) in all our predictions.
Figure 1: The variation of $\Gamma_{\text{MS}}/\Gamma_{\text{scheme}}$ at RG scales $\mu = \frac{1}{3}M_H, M_H,$ and $3M_H$ in the FOPT in the (a)SI (b)OS and (c)miniMOM schemes up to order $n = 4$.

We show the RG scale dependence of the ratio of the Higgs to gluons decay width in two different RG schemes in the FOPT in figure 1. To compare the predictions of different RG schemes, we have chosen the $\overline{\text{MS}}$ scheme as the reference scheme. The behaviour of the OS scheme is closer to that of the $\overline{\text{MS}}$ scheme due the same mass of the top quark used in both schemes.

In the $\overline{\text{MS}}$ scheme, the contribution of the $N^4$LO correction (defined by $\Gamma_{N^4\text{LO}} \times 100/\Gamma$) to the $\Gamma(H \rightarrow gg)$ decay width at the renormalization scale $\mu = M_H$ for the on shell top quark mass is $-0.6\%$. We now vary the scale for a range of $\mu = \frac{1}{3}M_H$ to $\mu = 3M_H$, and find that the contribution of the $N^4$LO corrections to the $\Gamma(H \rightarrow gg)$ in the $\overline{\text{MS}}$ scheme is 0.2% and 2%, respectively. The decay width in this range is $\Gamma(H \rightarrow gg) = 1.836\Gamma_0$ at the RG scale $\mu = \frac{1}{3}M_H$, $\Gamma(H \rightarrow gg) = 1.842\Gamma_0$ at the RG scale $\mu = M_H$, and $\Gamma(H \rightarrow gg) = 1.838\Gamma_0$ at the RG scale $\mu = 3M_H$. The $N^4$LO contribution in the SI and OS schemes is approximately identical to that of the $\overline{\text{MS}}$ scheme. The decay width in this range is $\Gamma(H \rightarrow gg) = 1.834\Gamma_0$ at the RG scale $\mu = \frac{1}{3}M_H$, $\Gamma(H \rightarrow gg) = 1.843\Gamma_0$ at the RG scale $\mu = M_H$, and $\Gamma(H \rightarrow gg) = 1.841\Gamma_0$ at the RG scale $\mu = 3M_H$ in the SI and OS schemes.

On the other hand, the $N^4$LO contribution in the miniMOM scheme is $-1.06\%$ at RG scale $\mu = M_H$. At
RG scale \( \mu = M_H/3 \), the \( N^4 \text{LO} \) correction in the miniMOM scheme is 5.55% which is quite large. The \( N^4 \text{LO} \) contribution at RG scale \( \mu = 3M_H \) is 0.68%. The decay width in this range is \( \Gamma(H \to gg) = 1.847 \Gamma_0 \) at the RG scale \( \mu = \frac{1}{3} M_H \), \( \Gamma(H \to gg) = 1.836 \Gamma_0 \) at the RG scale \( \mu = M_H \), and \( \Gamma(H \to gg) = 1.879 \Gamma_0 \) at the RG scale \( \mu = 3M_H \) in this scheme. We observe that the miniMOM scheme is quite sensitive to the scale variation, and relatively away from the predictions of the \( \overline{\text{MS}} \) scheme.

6 Scale and scheme dependence in the RGSPT

In this section, we discuss the behaviour of the RGSPT expansions of the Higgs to gluons decay width in the \( \overline{\text{MS}}, \text{SI}, \text{OS}, \) and miniMOM schemes. The variation of the ratio of the Higgs to gluons decay width in two different RG schemes at different orders is shown in figure 2. There is a relatively small difference between the FOPT and RGSPT predictions of the \( \Gamma_{\overline{\text{MS}}} / \Gamma_{\text{SI}} \) upto order \( n = 3 \) due to the different numerical values of the top-quark mass used in these schemes.

We begin our discussion by studying the behaviour of the RGSPT expansions in the \( \overline{\text{MS}} \) scheme at the RG scale \( \mu = M_H \) using the on shell top quark mass. The contribution of the \( N^4 \text{LO} \) correction to the \( \Gamma(H \to gg) \) decay width at the RG scale \( \mu = M_H \) is found to be \(-0.06\% \) in the \( \overline{\text{MS}} \) scheme. At the RG scale \( \mu = \frac{1}{3} M_H \), this contribution becomes \(-0.18\% \) and changes to \(0.04\% \) at the RG scale \( \mu = 3M_H \). The decay width at the RG scales \( \mu = \frac{1}{3} M_H, \mu = M_H, \) and \( \mu = 3M_H \) is \( \Gamma(H \to gg) = 1.845 \Gamma_0 \). This actually shows that the RGSPT expansion is considerably less sensitive to the RG scale \( \mu \). We observe that at the RG scale \( \mu = M_H \), the \( N^4 \text{LO} \) correction in the SI and OS schemes is \(-0.16\% \) and \(-0.03\% \), respectively. This correction becomes \(-0.28\% \) in the SI scheme and \(-0.15\% \) in the OS scheme at the RG scale \( \mu = \frac{1}{3} M_H \), and is \(-0.06\% \) and \(0.07\% \), respectively, at the RG scale \( \mu = 3M_H \).

Now we turn our attention towards the miniMOM scheme, and discuss our results in this scheme for the RGSPT expansions. We have already observed that in the FOPT, this scheme is relatively more sensitive to the RG scale \( \mu \). The \( N^4 \text{LO} \) contribution in this scheme at \( \mu = M_H \) is \(-1.01\% \), which is large as compared to other used schemes. The \( N^4 \text{LO} \) contribution at RG scales \( \mu = \frac{1}{3} M_H \) and \( \mu = 3M_H \) is \(-1.05\% \) and \(-0.96\% \) respectively. The important observation is that this \( N^4 \text{LO} \) contribution remains stable relative to the RG scale in the RGSPT expansion even at \( \mu = \frac{1}{3} M_H \) and \( \mu = 3M_H \). The decay width in this range is \( \Gamma(H \to gg) = 1.855 \Gamma_0 \) at the RG scale \( \mu = \frac{1}{3} M_H, \Gamma(H \to gg) = 1.853 \Gamma_0 \) at the RG scale \( \mu = M_H, \) and \( \Gamma(H \to gg) = 1.851 \Gamma_0 \) at the RG scale \( \mu = 3M_H \) in this scheme.

7 Asymptotic Padé approximant improved \( H \to gg \) decay rate

The Padé approximant is a nonlinear method of summing series that can be considered as an approximate analytic continuation \([73]\). It is widely applied to determine higher-order terms in a number of field-theoretical perturbative expansions, including \( \beta \)- and \( \gamma \)-functions of QCD at four- and five-loops \([56, 57, 58, 59, 60, 61, 62, 63, 64]\).

We begin by considering a generic perturbative expansion of the form,

\[
S \equiv 1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + \cdots ,
\]

(7.31)

where the coefficients \( \{R_1, R_2, R_3, R_4\} \) are known and the coefficients \( \{R_5, \cdots\} \) are unknown.
The Padé approximant to a generic perturbative expansion is denoted by,

\[
S_{\{N|M\}} \equiv \frac{1 + a_1 x + a_2 x^2 + \cdots + a_N x^N}{1 + b_1 x + b_2 x^2 + \cdots + b_M x^M}
= 1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + \cdots + R_{N+M+1} x^{N+M+1} + \cdots.
\]

The asymptotic error in the Padé approximant prediction is given by [59],

\[
\delta_{N+M+1} = \frac{R_{N+M+1}^{Padé} - R_{N+M+1}}{R_{N+M+1}} = -\frac{M! A^M}{[N + M + aM + b]^M} = \delta_i,
\]

where \( R_{N+M+1}^{Padé} \) is the \([N|M]\) Padé approximant prediction and \( R_{N+M+1} \) is the exact value of this coefficient. The free parameters \( \{A, a, b\} \) are chosen to produce the best results [59]. In this section, we choose \( a = -b = 10^6 \) which provide the best predictions. Moreover, for such large values of \( a \) and \( b \), the error \( \delta_i \) becomes very small.
If the coefficient $R_1$ is known, we can choose $N = 0$ and $M = 1$ and write the [0|1] Padé approximant as,

$$S_{[0|1]} = \frac{1}{1 + b_1 x} = 1 - b_1 x + b_1^2 x^2 + \cdots = 1 + R_1 x + R_2^{\text{Padé}} x^2 + \cdots,$$

(7.34)

where $R_2^{\text{Padé}} = R_1^2$ is the Padé approximant prediction for the next unknown higher-order term in the generic perturbative series in equation 7.31.

The asymptotic error formula in equation 7.33 provides the following equation for the [0|1] Padé approximant,

$$\frac{R_2^{\text{Padé}} - R_2}{R_2} = \frac{R_2^2 - R_2}{R_2} = -\frac{A}{1 + (a + b)},$$

(7.35)

which gives the value of $A$ as,

$$A = (1 + a + b)\left(1 - \frac{R_2^2}{R_2}\right).$$

(7.36)

We can determine the [2|2] Padé approximant given the knowledge of only $R_1$, $R_2$, $R_3$ and $R_4$. It is given by,

$$S_{[2|2]} = \frac{1 + a_1 x + a_2 x}{1 + b_1 x + b_2 x} = 1 + (a_1 - b_1) x + \left(a_2 - a_1 b_1 + b_1^2 - b_2\right) x^2 + \cdots$$

$$= 1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + R_5^{\text{Padé}} x^5 + \cdots.$$

(7.37)

The [2|2] Padé approximant predicts,

$$R_5^{\text{Padé}} = \frac{R_3^2 - 2 R_2 R_3 R_4 + R_1 R_4^2}{-R_2^2 + R_1 R_3}.$$

(7.38)

The asymptotic error can be written as,

$$\frac{R_5^{\text{Padé}} - R_5}{R_5} = -\frac{2A^2}{(4 + 2a + b)^2} \equiv \delta_5.$$

(7.39)

This error allows us to estimate an improved value of the true value $R_5$, which is referred to as asymptotic Padé approximant prediction (APAP) in literature [59]. Thus, our APAP estimate of the true value $R_5$ is,

$$R_5 = \frac{(-R_5^2 + 2 R_2 R_3 R_4 - R_1 R_4^2)}{(1 + \delta_5)(R_2^2 - R_1 R_3)} = \frac{8 R_2^2 (R_3^2 - 2 R_2 R_3 R_4 + R_1 R_4^2)}{(R_4^2 - R_1^2 R_4 - 2 R_1 R_3) (R_2^2 - R_1 R_3)}.$$

(7.40)

Now, we apply the APAP formalism to predict $N^{5}\text{LO}$ term of the $\Gamma(H \to gg)$ decay width in the FOPT. For the sake of clarity and reliability, we present our analysis in the MS̄, SI, OS and miniMOM schemes. The $\Gamma(H \to gg)$ decay width is given in equation 3.10, where the perturbative expansion $S[x(\mu), L(\mu)]$ now can be written as,

$$S_{\text{FOPT}}[x(\mu), L(\mu)] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_{n,k} x^n L^k = \sum_{n=0}^{\infty} R_n x^n,$$

(7.41)

where

$$R_n[\mu] = \sum_{k=0}^{n} T_{n,k} L^k.$$

(7.42)
such that

\[ R_s[L] = T_{5,0} + T_{5,1}L + T_{5,2}L^2 + T_{5,3}L^3 + T_{5,4}L^4 + T_{5,5}L^5, \]  

(7.43)

where \( T_{5,0}, T_{5,1}, T_{5,2}, T_{5,3}, T_{5,4}, \) and \( T_{5,5} \) are the unknown higher-order coefficients. The coefficients \( T_{5,1} - T_{5,5} \) can be predicted with the help of the RG invariance of \( \Gamma(H \to gg) \) decay width. The coefficient \( T_{5,0} \) is not available through the RG invariance.

Thus, we use equation 4.21 obtained from the RG invariance of \( \Gamma(H \to gg) \) decay width to predict the coefficients \( T_{5,1} - T_{5,5} \). This means that the perturbative validity of equation 4.21 to order \( x^3L^3 \) can determine the six-loop coefficients provided the \( x^3 \) contribution of equation 4.21 vanishes. Thus, in the \( \overline{\text{MS}} \) scheme we obtain,

\[
\begin{align*}
T_{5,1}^{\overline{\text{MS}}} & = -0.251403n_f^4 + 52.4477n_f^4 - 2974.99n_f^3 + 6496.7n_f^2 - 569877n_f + 1.58072 \times 10^6, \\
T_{5,2}^{\overline{\text{MS}}} & = 2.23573n_f^4 - 306.398n_f^3 + 11315.1n_f^2 - 148924n_f + 600692, \\
T_{5,3}^{\overline{\text{MS}}} & = -0.035782n_f^4 - 7.47604n_f^3 + 752.153n_f^2 - 16703.8n_f + 104439, \\
T_{5,4}^{\overline{\text{MS}}} & = -0.00462963n_f^4 + 0.319155n_f^3 + 12.6003n_f^2 - 784.131n_f + 8181.75, \\
T_{5,5}^{\overline{\text{MS}}} & = 222.674 - 13.4954n_f. 
\end{align*}
\]

(7.44)

In the SI scheme, the coefficients are as follows,

\[
\begin{align*}
T_{5,1}^{\text{SI}} & = -0.251403n_f^4 + 51.897n_f^4 - 2904.65n_f^3 + 62487.4n_f^2 - 538466n_f + 1.45824 \times 10^6, \\
T_{5,2}^{\text{SI}} & = 2.23573n_f^4 - 302.321n_f^3 + 11022.4n_f^2 - 143316n_f + 569814, \\
T_{5,3}^{\text{SI}} & = -0.035782n_f^4 - 7.30602n_f^3 + 737.153n_f^2 - 16307.8n_f + 101326, \\
T_{5,4}^{\text{SI}} & = -0.00462963n_f^4 + 0.319155n_f^3 + 12.6003n_f^2 - 776.921n_f + 8062.79, \\
T_{5,5}^{\text{SI}} & = 222.674 - 13.4954n_f. 
\end{align*}
\]

(7.45)

The coefficients obtained in the OS scheme are,

\[
\begin{align*}
T_{5,1}^{\text{OS}} & = -0.251403n_f^4 + 52.4682n_f^4 - 2966.69n_f^3 + 64629.9n_f^2 - 566720n_f + 1.57759 \times 10^6, \\
T_{5,2}^{\text{OS}} & = 2.23573n_f^4 - 305.229n_f^3 + 11256.9n_f^2 - 148434n_f + 602130, \\
T_{5,3}^{\text{OS}} & = -0.035782n_f^4 - 7.29085n_f^3 + 744.961n_f^2 - 16652.6n_f + 104721, \\
T_{5,4}^{\text{OS}} & = -0.00462963n_f^4 + 0.319155n_f^3 + 12.6003n_f^2 - 784.131n_f + 8181.75, \\
T_{5,5}^{\text{OS}} & = 222.674 - 13.4954n_f. 
\end{align*}
\]

(7.46)

whereas the coefficients obtained in the miniMOM scheme are,

\[
\begin{align*}
T_{5,1}^{\text{MM}} & = -0.00359858n_f^4 - 0.168327n_f^3 + 150.512n_f^2 + 6047.03n_f - 136552, \\
T_{5,2}^{\text{MM}} & = 0.033676n_f^4 + 28.524n_f^2 - 3364.45n_f + 25534.1, \\
T_{5,3}^{\text{MM}} & = 1.38186n_f^2 - 2691.8n_f + 38749.9, \\
T_{5,4}^{\text{MM}} & = 6001.76 - 378.012n_f, \\
T_{5,5}^{\text{MM}} & = 222.675 - 13.4954n_f. 
\end{align*}
\]

(7.47)

Now, we use the APAP method to predict the six-loop coefficients \( \{T_{5,0} - T_{5,5}\} \) in the \( \overline{\text{MS}}, \text{SI}, \text{OS}, \) and miniMOM schemes. The reliability of these APAP predicted coefficients will be estimated by calculating the uncertainty in their predictions against the RGE predicted \( \{T_{5,1} - T_{5,5}\} \) coefficients. For this purpose, we define
the moments of $R_5(w)$, where $w = \frac{m_t^2(\mu)}{\mu^2} ; (L = -ln(w))$, over the perturbative region $0 < w \leq 1$ in the following manner,

$$N_k \equiv (k + 2) \int_0^1 dw\, w^{k+1} R_5(w). \quad (7.48)$$

We substitute equation [7.43] into equation [7.48] which results in a set of equations for moments given by,

$$N_{-1} = T_{5,0} + T_{5,1} + 2 \left( T_{5,2} + 3T_{5,3} + 12T_{5,4} + 60T_{5,5} \right),$$

$$N_0 = \frac{1}{4} \left( 4T_{5,0} + 2T_{5,1} + 2T_{5,2} + 3T_{5,3} + 6T_{5,4} + 15T_{5,5} \right),$$

$$N_1 = \frac{1}{81} \left( 81T_{5,0} + 27T_{5,1} + 18T_{5,2} + 18T_{5,3} + 24T_{5,4} + 40T_{5,5} \right),$$

$$N_2 = \frac{1}{128} \left( 128T_{5,0} + 32T_{5,1} + 16T_{5,2} + 12T_{5,3} + 12T_{5,4} + 15T_{5,5} \right),$$

$$N_3 = \frac{1}{625} \left( 625T_{5,0} + 125T_{5,1} + 50T_{5,2} + 30T_{5,3} + 24T_{5,4} + 24T_{5,5} \right),$$

$$N_4 = \frac{1}{324} \left( 324T_{5,0} + 54T_{5,1} + 18T_{5,2} + 9T_{5,3} + 6T_{5,4} + 5T_{5,5} \right). \quad (7.49)$$

The above moments can be numerically computed by substituting equation [7.40] into the integrand of equation [7.48] with $L = -ln(w)$. The six-loop coefficients $T_{5,0}^{Padé} - T_{5,0}^{RGE}$ are determined by substituting these numerical values in equation [7.49]. The APAP predictions are compared to the RGE predictions for $n_f = 5$ flavours obtained from equations [7.44]-[7.47]. We estimate the relative error of the APAP predictions of the coefficients $T_{5,1} - T_{5,5}$ by computing $\Delta T_{5,i}^{Padé} \equiv (T_{5,i}^{Padé} - T_{5,i})/T_{5,i}$. Our predictions for the $\overline{MS}$, SI, OS, and the miniMOM schemes are given in tables 2 and 3.

| $\overline{MS}$ | $T_{5,0}$ | $T_{5,1}$ | $T_{5,2}$ | $T_{5,3}$ | $T_{5,4}$ | $T_{5,5}$ |
|-----------------|----------|----------|----------|----------|----------|----------|
| $Padé$          | -110686  | 12383.7  | 100812   | 39138.2  | 4417.21  | 199.466  |
| $RGE$           | -        | 15570.1  | 102046   | 38766.7  | 4613.1   | 155.197  |
| $\Delta T_{5,i}^{Padé}$ | -        | 20.5%    | 1.2%     | 1.0%     | 4.2%     | 28.5%    |

Table 2: The APAP predictions of the coefficients $R_5$ using the $S[1|3]$ Padé approximant in the $\overline{MS}$ scheme with errors.

| SI   | $T_{5,0}$ | $T_{5,1}$ | $T_{5,2}$ | $T_{5,3}$ | $T_{5,4}$ | $T_{5,5}$ |
|------|----------|----------|----------|----------|----------|----------|
| $Padé$ | -110378  | -4477.15 | 92857.5  | 37021    | 4684.3   | 139.191  |
| $RGE$ | -        | -3333.48 | 92400.3  | 37279.7  | 4530.19  | 155.197  |
| $\Delta T_{5,i}^{Padé}$ | -        | 34.3%    | 0.5%     | 0.7%     | 3.4%     | 10.3%    |

Table 3: The APAP predictions of the coefficients $R_5$ using the $S[1|3]$ Padé approximant in the SI scheme with errors.
| Scheme   | $R_{5,0}^{P_{\text{Padé}}}$ | $R_{5,1}^{R\text{GE}}$ | $\delta R_{5,0}^{P_{\text{Padé}}}$ | $R_{5}^{P_{\text{Padé}}}$ | $R_{5}^{R\text{GE}}$ | $R_{5}^{P_{\text{Padé}}}$ |
|----------|--------------------------|---------------------|-------------------------------|-------------------|-------------------|-------------------|
| $\overline{\text{MS}}$ | -86422.1 | -87826.6 | 1.6% | -285341 | 3.3315 x 10^6 | 4.83632 x 10^7 |
| SI       | -86387.5 | -87191.2 | 0.9% | -266735 | 3.53838 x 10^6 | 4.82802 x 10^7 |
| OS       | -84249.7 | -84689.2 | 0.5% | -280858 | 3.14976 x 10^6 | 4.5776 x 10^7 |
| miniMOM  | 2761.63 | -8253.07 | 133% | 496045 | 2.82424 x 10^6 | -1.66589 x 10^7 |
|          |           |          |      |         | -2.93284 x 10^8 |                   |

Table 6: The APAP predicted values of $R_{5}$ at $\mu = M_H$ with their relative error $\delta R_{5}^{\text{Padé}}$, where $\delta R_{5}^{\text{Padé}} = (R_{5}^{\text{Padé}} - R_{5}^{R\text{GE}})/R_{5}^{R\text{GE}} \times 100$ in the $\overline{\text{MS}}$, SI, OS, and miniMOM schemes.

We notice from table 6 that the overall coefficient $R_{5}$ is negative in the $\overline{\text{MS}}$, SI, and OS schemes, and is in agreement with that predicted from the use of RGE. This can be seen in the fourth column where the uncertainty $\delta R_{5}^{\text{Padé}}$ between the Padé and RGE predictions is computed. The largest uncertainty is 1.6% in the $\overline{\text{MS}}$ scheme. However, the coefficient $R_{5}$ is positive in the miniMOM scheme, and the uncertainty is quite large. Therefore, the miniMOM scheme is not working well in the APAP formalism, and its predictions are not reliable. This particular feature of the miniMOM scheme continues to hold in the PBA formalism as well.

The asymptotic errors $\lambda_i$, defined in equation (7.33) on our predictions presented in 6 are computed in the four different schemes, and given in table 7.
Table 7: Asymptotic errors for $R_{5-9}$ in the MS, SI, OS, and miniMOM schemes.

The prediction of the coefficients $T_{4,i}$ and their relative errors $\delta T^{Padé}_{4,i} \equiv \left( \frac{T^{Padé}_{4,i} - T_{4,i}}{T_{4,i}} \right) \times 100$ are obtained using the [0|3] Padé approximant. These predictions in the MS, SI, OS, and miniMOM schemes are given in tables 8-11.

| Scheme | $\delta_5$ | $\delta_6$ | $\delta_7$ | $\delta_8$ | $\delta_9$ |
|--------|------------|------------|------------|------------|------------|
| MS     | $9.89 \times 10^{-19}$ | $-2.40 \times 10^{-12}$ | $-2.40 \times 10^{-12}$ | $-2.40 \times 10^{-12}$ | $-2.40 \times 10^{-12}$ |
| SI     | $9.66 \times 10^{-19}$ | $-2.66 \times 10^{-12}$ | $-2.37 \times 10^{-12}$ | $-2.37 \times 10^{-12}$ | $-2.37 \times 10^{-12}$ |
| OS     | $-9.89 \times 10^{-19}$ | $-2.40 \times 10^{-12}$ | $-2.40 \times 10^{-12}$ | $-2.40 \times 10^{-12}$ | $-2.40 \times 10^{-12}$ |
| miniMOM| $-2.19 \times 10^{-11}$ | $-2.19 \times 10^{-11}$ | $-2.19 \times 10^{-11}$ | $-2.19 \times 10^{-11}$ | $-2.19 \times 10^{-11}$ |

Table 7: Asymptotic errors for $R_{5-9}$ in the MS, SI, OS, and miniMOM schemes.

| Scheme | $T_{4,0}$ | $T_{4,1}$ | $T_{4,2}$ | $T_{4,3}$ | $T_{4,4}$ |
|--------|-----------|-----------|-----------|-----------|-----------|
| Padé   | -1543.32  | 14291.6   | 9654.97   | 1577.93   | 67.477    |
| Known  | -453.772  | 15627.2   | 9595.85   | 1574.97   | 67.4771   |
| $\delta T^{Padé}_{4,i}$ | 240.1% | 8.5% | 0.6% | 0.2% | 0.0002% |

Table 8: The APAP predictions of the coefficients $R_{4}$ in the MS scheme with errors.

| Scheme | $T_{4,0}$ | $T_{4,1}$ | $T_{4,2}$ | $T_{4,3}$ | $T_{4,4}$ |
|--------|-----------|-----------|-----------|-----------|-----------|
| Padé   | -2911.26  | 12890.6   | 9263.72   | 1549.09   | 67.4769   |
| Known  | -1891.18  | 14042.3   | 9217.44   | 1546.13   | 67.4771   |
| $\delta T^{Padé}_{4,i}$ | 53.9% | 8.2% | 0.5% | 0.2% | 0.0003% |

Table 9: The APAP predictions of the coefficients $R_{4}$ in the SI scheme with errors.

| Scheme | $T_{4,0}$ | $T_{4,1}$ | $T_{4,2}$ | $T_{4,3}$ | $T_{4,4}$ |
|--------|-----------|-----------|-----------|-----------|-----------|
| Padé   | -922.469  | 14873.9   | 9742.49   | 1577.93   | 67.477    |
| Known  | -5.68386  | 16064.3   | 9695.27   | 1574.97   | 67.4771   |
| $\delta T^{Padé}_{4,i}$ | 16129% | 7.4% | 0.5% | 0.2% | 0.0001% |

Table 10: The APAP predictions of the coefficients $R_{4}$ in the OS scheme with errors.
Table 11: The APAP predictions of the coefficients $R_4$ in the miniMOM scheme with errors.

We note that the APAP formalism in this case is not able to reproduce the leading coefficient $T_{4,0}$ in the $\overline{MS}$, SI, and OS schemes. In the case of the miniMOM scheme, this happens with the coefficient $T_{4,1}$. However, we also notice that the overall coefficient $R_4$ of the perturbative expansion is in a good agreement with the APAP predictions as shown in Table 11. Moreover, we shall show in the next section that the asymptotic Padé-Borel approximant is capable of improving the APAP predictions of the coefficients $T_{4,i}$, therefore reproducing the known coefficient $R_4$ with a better accuracy.

Table 12: The APAP predicted values of $R_4$ at $\mu = M_H$ with relative error $\delta R_4$ and asymptotic error $\delta_4$, where $\delta R_4 = (R_4 - \hat{R}_4^{Pad\acute{e}}) / R_4 \times 100$.

We observe that the accuracy obtained in the predictions of the six-loop coefficients $T_{5,1}^{Pad\acute{e}} - T_{5,5}^{Pad\acute{e}}$ using the APAP method, in particular for the $\overline{MS}$ and OS schemes imply that we can employ the APAP formalism with confidence elsewhere. Therefore, we now use the APAP algorithm to predict the coefficients $\beta_5 - \beta_9$ and $\gamma_5 - \gamma_9$, the higher-order loop corrections to the $\beta$ and $\gamma$ functions. These coefficients will be used to compute the RGSPT function $S_5(u) - S_9(u)$.

The $\beta$ function defined in equation (4.22) can be written as,

$$\beta(x) = -\beta_0 x^2 \sum_{i=0}^{\infty} R_i x^i,$$

(7.50)

where $R_i \equiv \beta_i / \beta_0$. Using the already known values of $\beta_0$ to $\beta_4$ given in equation (4.23) we estimate the values of $\beta_5, \beta_6, \beta_7, \beta_8$ and $\beta_9$ for $n_f = 5$ using $[1|3], [3|2], [4|2], [5|2]$ and $[6|2]$ Padé approximants respectively. The values of $\beta_5 - \beta_9$ obtained are as follows,

$$\beta_5 = 54.8149, \beta_6 = 61.6663, \beta_7 = 166.748, \beta_8 = 228.033, \beta_9 = 524.048.$$

(7.51)

The $\gamma_m$ function as defined in equation (4.22) can be written as

$$\gamma_m(x) = x \sum_{i=0}^{\infty} R_i x^i,$$

(7.52)

where $R_i \equiv \gamma_i$.

We use $[1|3], [3|2], [4|2], [5|2]$ and $[6|2]$ Padé approximants to predict the values of $\gamma_5, \gamma_6, \gamma_7, \gamma_8$ and $\gamma_9$ respectively using the already known values of $\gamma_0 - \gamma_4$ for $n_f = 5$. The predictions for $\gamma_5 - \gamma_9$ are as follows,
Now we discuss the implications of the APAP determination of the coefficients \( T_{n,k} \), \( \beta_n \) and \( \gamma_n \) (\( n = 5 - 9 \)) to the \( \Gamma(H \rightarrow gg) \) decay width in the FOPT and RGSPT. For a better understanding, we present the higher-order behaviour of the ratio of the Higgs to gluons decay width in two different RG schemes at different orders in the FOPT and RGSPT in figure 3. The \( \overline{\text{MS}} \) and the OS schemes are showing very similar behaviour. The RGSPT predictions reside within the FOPT predictions in this case. In the case of the SI scheme, this occurs beyond \( n = 3 \). The miniMOM scheme is again not performing well. However, it becomes stable within the framework of the RGSPT beyond \( n = 3 \).

We provide order-by-order perturbative evaluation of the \( \Gamma(H \rightarrow gg) \) decay width up to \( N^5\text{LO} \) at \( \mu = M_H \) in the FOPT using the APAP formalism as,
by this work, we apply the generalized Borel transform to the perturbative expansion given in equation 7.31. This formalism is applied to the gluons decay width. This method is found to have a better convergence, and is referred to as Padé-Borel approximant (PBA) in literature [58].

In this section, we apply the APAP formalism to the Borel transform of the FOPT expansion of the Higgs to gluons decay width. This class of functions called the asymptotic Padé-Borel approximant improved [75]. The formalism are,

\[ \frac{\Gamma_{Padé}(M_H)}{\Gamma_{LO}(M_H)} = 1 + 0.641657 + 0.196393 + 0.0176989 - 0.011448 - 0.00742209 \]  
\[ \text{(7.54)} \]

Similar predictions of the \( \Gamma(H \rightarrow gg) \) decay width up to N^{5}LO at \( \mu = M_H \) in RGSPT using the APAP formalism are,

\[ \frac{\Gamma_{Padé}(M_H)}{\Gamma_{LO}(M_H)} = 1 + 0.695577 + 0.263334 + 0.0546671 - 0.00126852 - 0.00527992 \]  
\[ \text{(7.55)} \]

8 Asymptotic Padé-Borel approximant improved \( H \rightarrow gg \) decay rate

In this section, we apply the APAP formalism to the Borel transform of the FOPT expansion of the Higgs to gluons decay width. This method is found to have a better convergence, and is referred to as Padé - Borel approximant (PBA) in literature [58]. This formalism is applied to the \( H \rightarrow b \bar{b} \) decay rate in reference [74]. In this work, we apply the generalized Borel transform to the perturbative expansion given in equation 7.31.

We first briefly review the formalism of the generalized Borel transform [75]. Let \( \Psi(t) \) be a function given by,

\[ \Psi(t) = \sum_{n=0}^{\infty} \Psi_n t^n. \]  
\[ \text{(8.56)} \]

For our purpose, \( \Psi(t) \) is a comparison function restricted by auxiliary conditions \( \Psi_n > 0 \) and \( \lim_{n \to \infty} \Psi_{n+1}/\Psi_n = 0 \). We note that a comparison function \( \Psi(t) \) is necessarily an entire function ensured by the ratio test of convergence. A function \( f \) is referred to as \( \Psi \)-type if there exists the following relation,

\[ |f(r \exp(i\theta))| \leq M \Psi(\tau r), \]  
\[ \text{(8.57)} \]

where \( M \) and \( \tau \) are some numbers. The infimum of numbers \( \tau \) for which equation 8.57 holds defines the class of functions called \( \Psi \)-type \( \tau \) [75]. The \( \Psi \)-type of a function can be obtained from the coefficients in its power series expansion using the Nachbin’s theorem [75].

Nachbin’s theorem: A function \( f(z) = \sum_{n=0}^{\infty} f_n z^n \) is of \( \Psi \)-type \( \tau \) if and only if \( \lim_{n \to \infty} |f_n / \Psi_n|^{1/n} = \tau \).  
\[ \text{(8.58)} \]

Now we can define the generalized Borel transform given by,

\[ B[f(z)](u) = \sum_{n=0}^{\infty} \frac{f_n}{\Psi_n} u^n. \]  
\[ \text{(8.59)} \]
Since the function $f$ is $\Psi$-type $\tau$, the domain of convergence of the function $B[f(z)]$ is $|u| \leq \tau$. Moreover, we can define $f(z) = \frac{1}{2\pi i} \int_{\Gamma} \Psi(zu) B[f(z)](u) \, du$. \hfill (8.60)

The standard Borel transform is recovered by $\Psi(t) = e^t$.

We define the generalized Borel transform of the perturbative expansion (7.31) as,

$$B[S](u) = \sum_{n=0}^{\infty} \left( \frac{d_1}{n!} + \frac{d_2}{n!2} + \frac{d_3}{n!3} + \frac{d_5}{n!5} \right) R_n u^n,$$ \hfill (8.61)

where $d_{1,2,3,5}$ are the scheme-dependent real constants given in table 13. This is a phenomenological model in the sense that it reproduces the coefficient $R_4$ of the series (7.31) in particular, when the APAP formalism is applied to this Borel transform. The coefficient $R_3$ cannot be reproduced through the APAP formalism since there are three unknowns $A, a$ and $b$ to be fixed through the two known coefficients of the $R_1,2$ of the series (7.31).

| Schemes | $d_1$ | $d_2$ | $d_3$ | $d_5$ |
|---------|-------|-------|-------|-------|
| MS      | 0.5   | 1.5   | 0     | 1.2   |
| OS      | 1     | 0     | 1.623 | 0     |
| SI      | 0.87  | 0     | 1.6   | 0     |
| miniMOM | 0.68  | 1.5   | 0     | 1.2   |

Table 13: The numerical values of the constants $d_{1,2,3,5}$.

The APAP formalism is now applied to the generalized Borel transformation defined in equation (8.61). We use the $S[1,2]$ Padé approximant to predict the coefficient $R_4$, and $S[1,3]$ $S[2,2]$ for the coefficient $R_5$. The unknown constants of the error given in equation (7.33) are chosen to be $b = -a$. Moreover, our predictions are stable for the large values of the constants $a$ and $b$, such as $10^6$. This means that as the constants $a$ and $b$ approach a large value, our predictions become independent of these large values. This also means from equation (7.33) that the error on our predictions practically vanishes since the constant $A$ is independent of the values of $a$ and $b$ for the choice $b = -a$. Thus, from equation (7.33) for very large values of $a$ and $b$ we have,

$$R_{N+M+1}^{Padé} = R_{N+M+1}.$$ \hfill (8.62)

The PBA predictions of the coefficient $R_4$ in the MS, SI, OS, and miniMOM schemes for $n_f = 5$ flavours are given in table 14. The very first observation is the improvement of the prediction of the coefficient $T_{4,0}$ over that of the APAP predictions. This improvement continues even in the SI and OS schemes as shown in tables 15 and 16.

| Schemes | $T_{4,0}$ | $T_{4,1}$ | $T_{4,2}$ | $T_{4,3}$ | $T_{4,4}$ |
|---------|-----------|-----------|-----------|-----------|-----------|
| PBA     | -559.149  | 15801.3   | 9837.78   | 1579.97   | 68.6471   |
| Known   | -453.772  | 15627.2   | 9595.85   | 1574.97   | 67.4771   |
| $\delta T_{4,i}$ | 23.2%     | 1.1%      | 2.5%      | 0.3%      | 1.7%      |

Table 14: The PBA predictions of the coefficients $R_4$ in the MS scheme with errors where, $\delta T_{4,i}^{PBA} \equiv (T_{4,i}^{PBA} - T_{4,i})/T_{4,i} \times 100$ and $i = 0 - 4$. 

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In the case of the miniMOM scheme as well, we notice a good improvement in the prediction of the coefficient $T_{4,1}$ over the APAP prediction of the same coefficient as shown in table 17. Thus, we can conclude that the PBA formalism is performing relatively well for our perturbative expansion.

In table 18 we show our overall predictions of the coefficient $R_4$ in the four different schemes. We observe a good agreement with that of already computed values of the coefficient $R_4$ in the four different schemes, and conclude that the coefficient $R_4$ is better predicted by the PBA formalism in the MS, SI and OS schemes.

In a similar manner, we predict the coefficient $T_{5,i}$ for $n_f = 5$ flavours and compare it to that predicted by the RGE. Our predictions for the MS, SI, OS, and miniMOM schemes in tables 19 and 22.
| \( \text{MS} \) | \( T_{5,0} \)  | \( T_{5,1} \)  | \( T_{5,2} \)  | \( T_{5,3} \)  | \( T_{5,4} \)  | \( T_{5,5} \)  |
|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| PBA             | -126999        | 11998.4        | 105921         | 40231.5        | 4813.81        | 159.693        |
| RGE             | -              | 15570.1        | 102046         | 38766.7        | 4613.1         | 155.197        |
| \( \delta T_{5,i}^{PBA} \) | -              | 22.9%          | 3.8%           | 3.8%           | 4.4%           | 2.9%           |

Table 19: The PBA predictions of the coefficients \( R_5 \) using the \( S[1|3] \) Padé approximant in the \( \text{MS} \) scheme with errors.

| \( \text{SI} \) | \( T_{5,0} \)  | \( T_{5,1} \)  | \( T_{5,2} \)  | \( T_{5,3} \)  | \( T_{5,4} \)  | \( T_{5,5} \)  |
|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| PBA             | -150441        | -5962.74       | 113068         | 45018.4        | 5508.34        | 185.327        |
| RGE             | -              | -3333.48       | 92400.3        | 37279.7        | 4530.19        | 155.197        |
| \( \delta T_{5,i}^{PBA} \) | -              | 78.9%          | 22.4%          | 20.8%          | 21.6%          | 19.4%          |

Table 20: The PBA predictions of the coefficients \( R_5 \) using the \( S[2|2] \) Padé approximant in the \( \text{SI} \) scheme with errors.

| \( \text{OS} \) | \( T_{5,0} \)  | \( T_{5,1} \)  | \( T_{5,2} \)  | \( T_{5,3} \)  | \( T_{5,4} \)  | \( T_{5,5} \)  |
|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| PBA             | -149714        | 12139.2        | 122438         | 45832.4        | 5490.66        | 180.862        |
| RGE             | -              | 20907.1        | 104628         | 39147.8        | 4613.1         | 155.197        |
| \( \delta T_{5,i}^{PBA} \) | -              | 41.9%          | 17.0%          | 17.1%          | 19.0%          | 16.5%          |

Table 21: The PBA predictions of the coefficients \( R_5 \) using the \( S[2|2] \) Padé approximant in the \( \text{OS} \) scheme with errors.

| \( \text{miniMOM} \) | \( T_{5,0} \)  | \( T_{5,1} \)  | \( T_{5,2} \)  | \( T_{5,3} \)  | \( T_{5,4} \)  | \( T_{5,5} \)  |
|----------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| PBA                  | -69998         | -129042        | 12610.1        | 24334          | 2963.44        | 208.129        |
| RGE                  | -              | -102578        | 9429.12        | 25325.4        | 4111.7         | 155.197        |
| \( \delta T_{5,i}^{PBA} \) | -              | 25.8%          | 33.7%          | 3.9%           | 27.9%          | 34.1%          |

Table 22: The PBA predictions of the coefficients \( R_5 \) using the \( S[2|2] \) Padé approximant in the \( \text{miniMOM} \) scheme with errors.
Table 23: The PBA predicted values of $R_{5-9}$ at $\mu = M_H$ and RGE predicted value of $R_5$ at $\mu = M_H$ with their relative error $\delta R^{PBA}_5$, where $\delta R^{PBA}_5 = (R^{RGE}_5 - R^{PBA}_5) / R^{RGE}_5 \times 100$ in the $\overline{MS}$, SI, OS, and miniMOM schemes.

| Scheme | $R^{PBA}_5$ | $R^{RGE}_5$ | $\delta R^{PBA}_5$ | $R^{PBA}_6$ | $R^{PBA}_7$ | $R^{PBA}_8$ | $R^{PBA}_9$ |
|--------|-------------|-------------|---------------------|-------------|-------------|-------------|-------------|
| $\overline{MS}$ | -100576 | -104140 | 3.4% | -579764 | 3.14089 x 10^6 | 1.28987 x 10^8 | 1.55196 x 10^9 |
| SI | -120962 | -127255 | 4.9% | -927464 | 4.86118 x 10^6 | 3.10397 x 10^8 | 5.48341 x 10^9 |
| OS | -117885 | -129328 | 8.8% | -893860 | 5.32098 x 10^6 | 3.12672 x 10^8 | 5.64975 x 10^9 |
| miniMOM | 12553.1 | -5930.1 | 311.7% | 785972 | 4.58537 x 10^6 | -9.36415 x 10^7 | -1.96327 x 10^9 |

Table 24: The asymptotic errors for $R_{5-9}$ in the $\overline{MS}$, SI, OS, and miniMOM schemes.

| Scheme | $\delta^{PBA}_{5,0}$ | $\delta^{PBA}_{5,1}$ | $\delta^{PBA}_{5,2}$ | $\delta^{PBA}_{5,3}$ | $\delta^{PBA}_{5,4}$ | $\delta^{PBA}_{5,5}$ |
|--------|------------------|------------------|------------------|------------------|------------------|------------------|
| $\overline{MS}$ | 5.70 x 10^{-13} | -1.67 x 10^{-8} | -1.67 x 10^{-8} | -1.67 x 10^{-8} | -1.67 x 10^{-8} | -1.67 x 10^{-8} |
| SI | -1.01 x 10^{-10} | -1.01 x 10^{-18} | -1.01 x 10^{-18} | -1.01 x 10^{-10} | -1.01 x 10^{-10} | -1.01 x 10^{-10} |
| OS | -9.31 x 10^{-19} | -9.31 x 10^{-19} | -9.31 x 10^{-19} | -9.31 x 10^{-19} | -9.31 x 10^{-19} | -9.31 x 10^{-19} |
| miniMOM | -6.74 x 10^{-18} | -6.74 x 10^{-18} | -6.74 x 10^{-18} | -6.74 x 10^{-18} | -6.74 x 10^{-18} | -6.74 x 10^{-18} |

Table 25: Difference between the APAP predictions and PBA predictions for the coefficients $R_5$ at $\mu = M_H$ in the $\overline{MS}$, SI, OS, and miniMOM schemes where $\Delta T_{5,i} = T^{APAP}_{5,i} - T^{PBA}_{5,i}$.

The predictions of the overall coefficients $R^{PBA}_5$ with $i = 5-9$ in four different schemes are given in Table 23. The predictions for the coefficients $R_{6-9}$ are obtained using the $S[3,2], S[4,2], S[5,2], S[6,2]$ Padé approximants, respectively in the $\overline{MS}$, SI, OS, and miniMOM schemes. The asymptotic errors on the predictions of the overall coefficients $R_{5-9}$ in the $\overline{MS}$, SI, OS, and miniMOM schemes are extremely small, and are given in Table 24. The RGE prediction of the coefficient $R_5$ is obtained assuming that the coefficient $T_{5,0}$ in the RGE is identical to that of the PBA.

For a conservative estimate of the coefficient $R_5$, we associate the difference of the APAP and the PBA predictions of the coefficients $T_{5,i}$ as an uncertainty to the coefficient $R_5$. This uncertainty is calculated by adding the errors on the coefficients $T_{5,i}$ given in Table 25 in quadrature. This results in the following values of the coefficient $R_5$ at $\mu = M_H$:

$$R^{\overline{MS}}_5 = -86422.1 \pm 16456.8,$$
$$R^{SI}_5 = -86387.5 \pm 40519.1,$$
$$R^{OS}_5 = -84249.7 \pm 45568.8$$
\[ R_5^{MM} = 2761.63 \pm 7238.89. \]

Figure 4: The variation of \( \Gamma_{\text{MS}} / \Gamma_{\text{scheme}} \) at RG scales \( \mu = \frac{1}{3} M_H, M_H, \) and \( 3M_H \) in the FOPT in the (a)SI (b)OS and (c)miniMOM schemes up to order \( n = 9 \).

The higher-order behaviour of the ratio of the Higgs to gluons decay width in two different RG schemes at different orders predicted by the PBA formalism at three different RG scales in the FOPT is shown in figure 4 along with that of the APAP formalism. There is a good agreement between the predictions of the APAP and the PBA formalisms for the MS, SI, and the OS schemes. We show the higher-order behaviour of the ratio of the Higgs to gluons decay width in two different RG schemes at different orders predicted by the PBA and the APAP formalisms in the RGSPT in the MS, SI, OS, and miniMOM schemes in figure 5 at three different scales. As expected, the RGSPT expansions are less sensitive to the RG scale dependence.
Figure 5: The variation of $\Gamma_{\text{MS}}/\Gamma_{\text{scheme}}$ at RG scales $\mu = \frac{1}{3}M_H, M_H, \text{and } 3M_H$ in the RGSPT in the (a)SI (b)OS and (c)miniMOM schemes up to order $n = 9$.

We summarize our predictions using the APAP formalism using the four different RG schemes in the FOPT and in the RGSPT in figure [C]. We show the the central curves of the normalised higgs to gluons decay width at RG scale $\mu = M_H$ using four different RG schemes in the FOPT on the top-left panel. On the other hand, the central curves of the ratio $\Gamma_{\text{MS}}/\Gamma_{\text{scheme}}$ using four different RG schemes in the FOPT are shown in the top-right panel. The same curves in the RGSPT are shown in the bottom-left and the bottom-right panels. We do not show similar results in the PBA formalism due to their very similarity to the APAP predictions in the $\overline{\text{MS}}$, OS and SI schemes.
Figure 6: The variation of the normalised higgs to gluons decay width at RG scale $\mu = M_H$ with scale bands at $\mu = \frac{1}{3}M_H$ and $3M_H$ in the (a)FOPT and the (b)RGSPT in the $\overline{\text{MS}}$, SI, OS and miniMOM schemes up to order $n = 9$. The variation of $\Gamma_{\overline{\text{MS}}}/\Gamma_{\text{scheme}}$ at RG scale $\mu = M_H$ in the (b)FOPT and the (d)RGSPT in the SI, OS and miniMOM schemes up to order $n = 9$.

We now provide order-by-order perturbative evaluation of $\Gamma(H \rightarrow gg)$ decay width up to the $N^5\text{LO}$ at $\mu = M_H$ in the FOPT using the PBA formalism. The $N^5\text{LO}$ contribution at the scale $\mu = M_H$ is highlighted inside the box.

\[
\frac{\Gamma_{PBA}^{\overline{\text{MS}}}(M_H)}{\Gamma_{LO}(M_H)} = 1 + 0.641657 + 0.196393 + 0.0176989 - 0.0114448 - 0.00825599 \quad (8.64)
\]

\[
\frac{\Gamma_{PBA}^{\overline{\text{SI}}}(M_H)}{\Gamma_{LO}(M_H)} = 1 + 0.641657 + 0.197175 + 0.0178975 - 0.0115469 - 0.00953425
\]

\[
\frac{\Gamma_{PBA}^{\overline{\text{OS}}}(M_H)}{\Gamma_{LO}(M_H)} = 1 + 0.641657 + 0.196393 + 0.0180594 - 0.0111039 - 0.00914739
\]
Similarly, the order-by-order perturbative evaluation of $\Gamma(H \to gg)$ decay width up to $N^5$LO at $\mu = M_H$ in the RGSPT using the PBA formalism is,

$$\frac{\Gamma^{\text{PBA}}_{N^5\text{LO}}(M_H)}{\Gamma_{\text{LO}}(M_H)} = 1 + 0.695577 + 0.263334 + 0.0546671 - 0.00128652 - 0.0060536, \quad \text{(8.65)}$$

$$\frac{\Gamma^{\text{PBA}}_{N^5\text{LO}}(M_H)}{\Gamma_{\text{LO}}(M_H)} = 1 + 0.686965 + 0.253127 + 0.0487537 - 0.00318015 - 0.00739534, \quad \text{(8.65)}$$

$$\frac{\Gamma^{\text{PBA}}_{N^5\text{LO}}(M_H)}{\Gamma_{\text{LO}}(M_H)} = 1 + 0.695577 + 0.263334 + 0.055281 - 0.00655924 - 0.00713806, \quad \text{(8.65)}$$

$$\frac{\Gamma^{\text{PBA}}_{N^5\text{LO}}(M_H)}{\Gamma_{\text{LO}}(M_H)} = 1 + 0.884649 + 0.167738 - 0.0114242 - 0.0203406 - 0.00567677, \quad \text{(8.65)}$$

9 Determination of the Higgs to gluons decay rate

We present our predictions of the $\Gamma(H \to gg)$ decay width in the $\overline{\text{MS}}$, SI, OS, and miniMOM schemes in this section. The central value of the $\Gamma(H \to gg)$ decay width is evaluated at RG scale $\mu = M_H$. Assuming the $\Gamma_{N^5\text{LO}}$ approximately to be the exact result, the uncertainty due to the series expansion is estimated by the difference $(\Gamma_{N^5\text{LO}} - \Gamma_{N^5\text{LO}})/\Gamma_0$ at RG scale $\mu = M_H$. Our predictions for the $\Gamma(H \to gg)$ decay width at the order $N^5$LO in the APAP formalism in the FOPT are,

$$\Gamma^{\text{PBA}}_{N^5\text{LO}} = \Gamma_0 \left( 1.837 \pm 0.047 \alpha_s(M_Z), 1\% \pm 0.0004 M_t \pm 0.0066 M_H \pm 0.0009 p \pm 0.007 \right) \quad \text{(9.66)}$$

$$\Gamma^{\text{PBA}}_{N^5\text{LO}} = \Gamma_0 \left( 1.837 \pm 0.046 \alpha_s(M_Z), 1\% \pm 0.0004 M_t \pm 0.0066 M_H \pm 0.0026 p \pm 0.007 \right) \quad \text{(9.66)}$$

$$\Gamma^{\text{PBA}}_{N^5\text{LO}} = \Gamma_0 \left( 1.838 \pm 0.047 \alpha_s(M_Z), 1\% \pm 0.0004 M_t \pm 0.0066 M_H \pm 0.0023 p \pm 0.007 \right) \quad \text{(9.66)}$$

$$\Gamma^{\text{PBA}}_{N^5\text{LO}} = \Gamma_0 \left( 1.836 \pm 0.042 \alpha_s(M_Z), 1\% \pm 0.0001 M_t \pm 0.0066 M_H \pm 0.0007 p \pm 0.002 \right) \quad \text{(9.66)}$$

where P stands for the uncertainty due to the PBA predictions, and s denotes the uncertainty due to the series expansion.

Similarly, the $\Gamma(H \to gg)$ decay width using the APAP predictions at the order $N^5$LO in the RGSPT are found to be,

$$\Gamma^{\text{RGSPT}}_{N^5\text{LO}} = \Gamma_0 \left( 1.840 \pm 0.047 \alpha_s(M_Z), 1\% \pm 0.0005 M_t \pm 0.0066 M_H \pm 0.0002 p \pm 0.0007 \right) \quad \text{(9.67)}$$

$$\Gamma^{\text{RGSPT}}_{N^5\text{LO}} = \Gamma_0 \left( 1.841 \pm 0.047 \alpha_s(M_Z), 1\% \pm 0.0005 M_t \pm 0.0060 M_H \pm 0.0002 p \pm 0.0018 \right) \quad \text{(9.67)}$$

$$\Gamma^{\text{RGSPT}}_{N^5\text{LO}} = \Gamma_0 \left( 1.842 \pm 0.047 \alpha_s(M_Z), 1\% \pm 0.0005 M_t \pm 0.0066 M_H \pm 0.0002 p \pm 0.0019 \right) \quad \text{(9.67)}$$

$$\Gamma^{\text{RGSPT}}_{N^5\text{LO}} = \Gamma_0 \left( 1.847 \pm 0.043 \alpha_s(M_Z), 1\% \pm 0.0005 M_t \pm 0.0066 M_H \pm 0.0023 p \pm 0.0002 \right) \quad \text{(9.67)}$$

We notice a few important observations in our results. The uncertainty due to the mass of the top quark in the $\overline{\text{MS}}$, SI and OS schemes turns out to be smaller, for instance, a change of 4 GeV in the top quark mass causes 0.02% change in the $\Gamma(H \to gg)$ decay width. In the miniMOM scheme uncertainty due to the 4 GeV change in the top quark mass is 0.01% in the FOPT whereas it is 0.03% in the RGSPT. The dependence of the $\Gamma(H \to gg)$ decay width on the mass of the Higgs boson is of the order 0.36% in the $\overline{\text{MS}}$, SI, OS, and miniMOM scheme. The largest uncertainty in the $\Gamma(H \to gg)$ decay width at the $N^5$LO originates from the change in the
strong coupling \(\alpha_s(M_Z^2)\). For instance, a change of 1\% in the \(\alpha_s(M_Z^2)\) causes an uncertainty \((2.5 - 2.6)\%\) in the \(\overline{\text{MS}},\) SI and OS schemes. For the miniMOM scheme, it is slightly less in the range \((2.3 - 2.4)\%\).

As observed in our previous discussion, the miniMOM scheme is not showing a stable behaviour at higher orders in the APAP and PBA formalism. Therefore, excluding the prediction of this scheme, we provide our final prediction of the \(\Gamma(H \rightarrow gg)\) decay width at the order \(N^3\text{LO}\) in the FOPT as,

\[
\Gamma_{N^3\text{LO}} = \Gamma_0 \left( 1.8375 \pm 0.047\alpha_s(M_Z)\% \pm 0.0004M_t \pm 0.0066M_H \pm 0.0036\mu \pm 0.007s \pm 0.0005sc \right),
\]

(9.68)

where \(sc\) shows the uncertainty introduced due to scheme dependence, and the error due to PBA, is obtained by adding the uncertainties due to PBA in the \(\overline{\text{MS}},\) SI and OS schemes in quadrature.

In the RGSPT, the \(\Gamma(H \rightarrow gg)\) decay width at \(N^3\text{LO}\) is,

\[
\Gamma_{\text{RGSPT}} = \Gamma_0 \left( 1.841 \pm 0.047\alpha_s(M_Z)\% \pm 0.0005M_t \pm 0.0066M_H \pm 0.0002\mu \pm 0.0027s \pm 0.001sc \right).
\]

(9.69)

10 Summary

In this work, we have investigated an important issue of the renormalization scale and scheme dependence of the \(\Gamma(H \rightarrow gg)\) decay width in the FOPT and in the RGSPT at the order \(N^4\text{LO},\) and beyond it. The RGSPT exploits the method of summation of all RG-accessible logarithms, which was first proposed in reference [42]. In the RGSPT, the RG equation is utilized to derive the summation of the leading and subsequent finite subleading logarithms to all orders in the perturbation theory. This results in a closed-form summation of the RG accessible leading and subsequent finite subleading logarithms. It is found that the dependence of the perturbative expansions on the RG scale \(\mu\) is considerably reduced in the RGSPT expansions.

This work is motivated by a recent advancement in the computation of the \(H \rightarrow gg\) decay rate in the limit of a heavy top quark and any number of massless light flavours at \(N^4\text{LO}\) in reference [33]. We investigate the \(\Gamma(H \rightarrow gg)\) decay width in four different renormalization schemes, namely, \(\overline{\text{MS}},\) SI, OS, and miniMOM. We first discuss our predictions in the FOPT, and then compare these predictions to those obtained in the RGSPT. In the case of the FOPT expansions, the \(\Gamma(H \rightarrow gg)\) decay width is highly sensitive to the RG scale up to the order \(N^2\text{LO},\) and stabilizes at the order \(N^4\text{LO}.\) We observe that the summation of leading logarithms in the RGSPT expansions exhibits good stability and reduced sensitivity to RG scale \(\mu\). This is starkly obvious up to the order \(N^3\text{LO}\). The FOPT expansions begin to catch up with the behaviour of the RGSPT expansions from \(N^3\text{LO}\) onward. Moreover, the RGSPT expansions show a stable behaviour in different RG schemes as well. The largest uncertainty in our predictions for the \(\Gamma(H \rightarrow gg)\) decay width arises due to the 1\% change in the strong coupling \(\alpha_s(M_Z^2)\), and is in the range \((2.5 - 2.6)\%\) in the \(\overline{\text{MS}},\) SI and OS schemes. The corresponding range in the miniMOM scheme is \((2.3 - 2.4)\%\), which is slightly less than those obtained in the \(\overline{\text{MS}},\) SI and OS schemes.

We have also estimated the higher-order effects on the \(\Gamma(H \rightarrow gg)\) decay width using the APAP formalism. The higher-order behaviour of the perturbative expansions is alternatively determined by the PBA formalism, and found to be reasonably in agreement with that of the APAP formalism for the \(\overline{\text{MS}},\) SI, and OS schemes. The \(\Gamma(H \rightarrow gg)\) decay width is showing stability at higher-orders in the APAP as well as in the PBA frameworks, and it becomes less dependent on the higher-order corrections in all schemes. The RGSPT expansions continue to show greater stability against the RG scale at higher-orders in the APAP as well as in the PBA frameworks.

Finally, we provide our estimate of the \(H \rightarrow gg\) decay rate at \(N^4\text{LO}\) in the framework of the FOPT as well as the RGSPT. We have added the difference between the APAP and the PBA predictions of the \(H \rightarrow gg\) decay rate at \(N^3\text{LO}\) as an error to the final predictions of the \(H \rightarrow gg\) decay rate. This uncertainty is approximately 0.19\% in the FOPT, and 0.15\% in the RGSPT. The uncertainty due to the truncation of series is approximately 0.6\% at \(N^4\text{LO},\) and reduces to 0.4\% at \(N^3\text{LO}\) in the FOPT. Thus, by adding \(N^3\text{LO}\) correction to the \(H \rightarrow gg\) decay rate, the truncation error is reduced by 33\% at \(N^3\text{LO}.\) The uncertainty due to the truncation is much smaller than the error introduced by the 1\% uncertainty in the strong coupling constant. We notice that the uncertainty of the order 1\% in the \(\Gamma(H \rightarrow gg)\) decay width may not be in the reach of the LHC. However, this precision may be accessible to a future \(e^+e^-\) linear collider [71].

We emphasize that the uncertainty due to the missing higher-order effects of QCD corrections beyond \(N^3\text{LO}\) is an important issue in the Higgs physics for the upcoming high-luminosity phase of the LHC [78]. This
uncertainty reveals itself in the form of scale-dependence. The uncertainty due to scale is remarkably reduced in the framework of the RGSPT, and is approximately 0.01%. Our final predictions for the $H \to gg$ decay rate at N$^5$LO in the FOPT and in the RGSPT includes the uncertainty entering due to the scheme dependence, which is approximately 0.03% in the FOPT, and 0.06% in the RGSPT. This prediction is obtained by excluding the miniMOM scheme, which is not working well within the APAP and PBA formalisms, and requires further future investigation.

There are other sources of uncertainties to the $\Gamma(H \to gg)$ decay width. For instance, the electroweak corrections cause the enhancement of the $\Gamma(H \to gg)$ decay width by about 5% [79, 90, 91, 92]. The missing electroweak corrections beyond NLO introduce the residual theoretical uncertainties of the order 1% [93]. Additionally, corrections due to a finite bottom-quark mass induce a 12% effect at leading order, and 6% effect at NLO to the effective Higgs coupling to gluons [22, 23]. A comprehensive analysis of these effects in the framework of the RGSPT is beyond the scope of this paper, and will be presented in future work.

**Acknowledgments**

We are extremely grateful to the referee for the highly constructive feedback on this work. We are also grateful to Prof. Irinel Caprini, Prof. B. Ananthanarayan and M. S. A. Alam Khan for very important comments and suggestions on the manuscript. We are also very thankful to Prof. M. Spira for very useful comments and suggestions on the first arXiv version of the manuscript. This work is supported by the Council of Science and Technology, Govt. of Uttar Pradesh, India through the project “A new paradigm for flavour problem” no. CST/D-1301, and Science and Engineering Research Board, Department of Science and Technology, Government of India through the project “Higgs Physics within and beyond the Standard Model” no. CRG/2022/003237.

**Data availability statements**

The used data is explicitly quoted in the manuscript itself, and there is no need to deposit it separately.

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