A Theoretical Study of (Full) Tabled Constraint Logic Programming *

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Abstract

Logic programming with tabling and constraints (TCLP, \textit{tabled constraint logic programming}) has been shown to be more expressive and, in some cases, more efficient than LP, CLP, or LP with tabling. In this paper we provide insights regarding the semantics, correctness, completeness, and termination of top-down execution strategies for full TCLP, i.e., TCLP featuring entailment checking in the calls and in the answers. We present a top-down semantics for TCLP and show that it is equivalent to a fixpoint semantics. We study how the constraints that a program generates can effectively impact termination, even for constraint classes that are not constraint compact, generalizing previous results. We also present how different variants of constraint projection impact the correctness and completeness of TCLP implementations. All of the presented characteristics are implemented (or can be experimented with) in Mod TCLP, a modular framework for Tabled Constraint Logic Programming, part of the Ciao Prolog logic programming system.

\textit{KEYWORDS:} Constraints, Tabling, Logic programming, Foundations, Implementation.

1 Introduction and Motivation

Constraint Logic Programming (CLP) (Jaffar and Maher 1994) extends Logic Programming (LP) with variables that can belong to arbitrary constraint domains and the ability to incrementally solve equations involving these variables. CLP brings additional expressive power to LP, since constraints can very concisely capture complex relationships. Also, shifting from “generate-and-test” to “constraint-and-generate” patterns reduces the search tree and therefore brings additional performance, even if constraint solving is in general more expensive than first-order unification.

Tabling (Tamaki and Sato 1986; Warren 1992) is an execution strategy for logic programs that suspends repeated calls which could cause infinite loops. Answers from non-looping branches are used to resume suspended calls which can, in turn, generate more answers. Only new answers are saved, and evaluation finishes when no new answers can be generated. Tabled evaluation always terminates for calls/programs with the bounded term depth property (those that can only generate terms with a finite bound on their depth) and can improve efficiency for terminating programs that

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repeat computations, as it automatically implements a variant of dynamic programming. Tabling has been successfully applied in a variety of contexts, including deductive databases, program analysis, semantic Web reasoning, and model checking (Warren et al. 1988; Dawson et al. 1996; Zou et al. 2005; Ramakrishna et al. 1997; Charatonik et al. 2002).

The integration of tabling and constraint solving, Tabled Constraint Logic Programming (TCLP), makes it possible to exploit their synergy in several application fields of which we highlight a few:

**Abstract interpretation:** Tabling can be used naturally to compute fixpoints (Kanamori and Kawamura 1993; Janssens and Sagonas 1998), but, additionally, by implementing abstract domain operations as constraints (Arias and Carro 2019b), entailment will automatically detect more particular calls and suspend their execution to reuse analysis results from most general calls, thereby speeding up the fixpoint computation. Constraints can also be used to state preconditions to the analysis results before the analysis starts in a powerful yet flexible fashion. These preconditions can propagate during the evaluation and help solve some verification problems faster.

**Reasoning on ontologies:** An ontology formalizes types, properties, and interrelationships among entities. They can be expressed as a lattice constraint system and, with TCLP, evaluation in ontologies can benefit from entailment of instances which are more particular than other entities, in a fashion similar to OWL (www.w3.org/owl), but in potentially richer domains and/or more complex scenarios (e.g., stream data analysis (Arias 2016)).

**Constraint-based verification:** Verification conditions can be encoded as constraint systems, and the tabling engine can use entailment to guarantee termination and save execution time (Charatonik et al. 2002; Jaffar et al. 2004; Gange et al. 2013).

**Incremental evaluation of aggregates:** For aggregates that can be embedded into a lattice (e.g., minimum), the aggregation operation can be expressed based on the partial order of the lattice. In these cases, the aggregate operations in the lattice can be seen as a counterpart of the operations among constraints defined in TCLP (Arias and Carro 2019c).

In order to highlight some of the advantages of TCLP vs. LP, tabling, and CLP with respect to declarativeness and logical reading, in (Arias and Carro 2019a) we compared how different versions of a program to compute distances between nodes in a graph behave under these three approaches. Each version was adapted to a different paradigm, but trying to stay as close as possible to the original code, so that the additional expressiveness can be solely attributed to the
evaluation strategy rather than to differences in the code itself. Their behaviors are summarized in Table 1 and explained below:

- **LP**: The code in Fig. 1a is the Prolog version of a program used to find the distance between two nodes in a graph. The distance between two nodes is calculated by adding variables \( D_1 \) and \( D_2 \), corresponding to distances to and from an intermediate node, once they are instantiated. The figure also shows a query used to determine which node(s) \( Y \) is/are within a distance \( K \) from node \( a \). This query does not terminate as left recursion makes the recursive clause enter an infinite loop. If we convert the program to a right-recursive version by swapping the calls to \( \text{edge/3} \) and \( \text{dist/3} \), the program will still not terminate in a cyclic graph.

- **CLP(R)**: Fig. 1b is the CLP(R) version of the same code where addition is modeled as a constraint and placed at the beginning of the clause. Since the total distance \( D \) is bound by the constraint \( D \leq K \) in the query, the search would be expected to be pruned if \( D \) exceeds the maximum distance, \( K \). However, the constraints placed before the recursive call do not cause this bound to be violated, and therefore it would enter a loop even for graphs without loops. The right-recursive version of the CLP(R) program in Fig. 1c will however finish because the initial bound to the distance eventually causes the constraint store to become inconsistent, which provokes a failure in the search. Note that this transformation is easy in this case, but it would not have the same effect should the clause be written with a (logically equivalent) double recursion. This is optional in this example, but it may be necessary or more natural in other cases, such as in parsing applications, language interpreters, algorithms on trees, or divide-and-conquer algorithms.

- **Tabling**: Tabling records the first occurrence of each call to a tabled predicate (the generator) and its answers. In variant tabling, the most usual form of tabling, when a call equal up to variable renaming to a previous generator is found (a variant), its execution is suspended, and it is marked as a consumer of the generator. For example, \( \text{dist}(a, Y, D) \) is a variant of \( \text{dist}(a, Z, D) \) if \( Y \) and \( Z \) are free variables. When a generator finitely finishes exploring all of its clauses and its answers are collected, its consumers are resumed and are fed the answers of the generator. This may make consumers produce new answers that will in turn cause more resumptions. Tabling is a complete strategy for all programs with the bounded term-depth property, which in turn implies that the Herbrand model is finite. Therefore, left- or right-recursive reachability terminates in finite graphs with or without cycles. However, the program in Fig. 1a has an infinite minimum Herbrand model for cyclic graphs: every cycle can be traversed an unbound number of times, giving rise to an unlimited number of answers with a different distance each. The query \( \text{?- dist}(a, Y, D), D < K \) will therefore not terminate under variant tabling.

- **TCLP**: The program in Fig. 1b can be executed with tabling and using constraint entailment to suspend calls which are more particular than previous calls and, symmetrically, to keep only the most general answers returned. Entailment can be seen as a generalization of subsumption for the case of general constraints; in turn, subsumption was shown to enhance termination and performance in tabling (Swift and Warren 2010). When a goal \( G_1 \) entails another goal \( G_0 \), the solutions for \( G_1 \) are a subset of the solutions for \( G_0 \). To make the entailment relationship explicit, we define a TCLP goal as \((g, c_g)\)

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1 This is a typical query for the analysis of social networks (Swift and Warren 2010).
where \( g \) is the call (a literal) and \( c_g \) is the projection of the current constraint store onto the variables of the call. Then, a goal \( G_0 = (\text{dist}(X, Y, D), D < 150) \) is entailed by another goal \( G_1 = (\text{dist}(X, Y, D), D > 0 \land D < 75) \) because the solutions for \( D > 0 \land D < 75 \) are contained in the solutions for \( D < 150 \land D < 75 \). We write \( G_1 \sqsubseteq G_0 \). We say that \( G_1 \), the more particular goal, is the consumer, and \( G_0 \), the most general goal, is the generator. The key observation behind the use of entailment in TCLP is that calls to more particular goals can suspend their execution and later recover the answers collected by the most general call and continue execution. The solutions for the consumer are a subset of that for the generator. However, some answers for a generator may not be valid for a consumer. For example, \( D > 125 \land D < 135 \) is a solution for \( G_0 \) but not for \( G_1 \), since \( G_1 \) has a constraint store more restrictive than the \( G_0 \). Therefore, the tabling engine should check and filter, via the constraint solver, that answers from generators are consistent with the constraint store of consumers.

The use of entailment in calls and answers enhances termination properties. Column “TCLP” in Table 1 summarizes the termination characteristics of \( \text{dist}/3 \) under TCLP, and shows that a full integration of tabling and CLP makes it possible to find all the solutions and finitely terminate in all the cases. Additionally, in (Arias and Carro 2019a) we experimentally show that Mod TCLP, a framework that fully implements entailment in the call and answer entailment phase, can improve performance.

The theoretical basis of Tabled Constraint Logic Programming (TCLP) were established in (Toman 1997) using a framework of bottom-up evaluation of Datalog systems and presenting the basic operations (projection and entailment checking) that are necessary to ensure completeness w.r.t. the declarative semantics. In this work, we present the theoretical basis of TCLP for a top-down execution on which Mod TCLP (Arias and Carro 2019a) is based. In Section 2 we present the operational semantics of a top-down execution of TCLP programs with generic constraint solvers. In Section 3 we extend the soundness, completeness, and termination proofs. In Section 4 we explain the benefits of using entailment checking with more relaxed notions projections.

## 2 Fixpoint and Top-Down Semantics of TCLP

In this section we present a bottom-up fixpoint semantics of TCLP that used constraint entailment for the answers and a top-down semantics that extends (Toman 1997) by explicitly modeling entailment both in the answers and in the calls. This semantics uses objects that mimic the construction of forests of trees in implementations of tabling.
2.1 Syntax of TCLP Programs

A (tabled) constraint logic program consists of clauses of the form:

\[ h : - c, l_1, \ldots, l_k. \]

where \( h \) is an atom, \( c \) is an atomic constraint or conjunction of constraints, \( l_i \) are literals, ‘\(-\)’ represents the logical implication ‘\( \leftarrow \)’, and ‘,’ represents the logical conjunction ‘\( \land \)’. The head of the clause is \( h \) and the rest is called the body, denoted by \( \text{body}(h) \). We will assume throughout this paper that the program has been rewritten so that clause heads are linearized (all the variables are different) and all head unifications take place in \( c \). The constraint \( c \) or the literals \( l_i \) or both may be absent. In the last case the rule is called a fact and it is customarily written omitting the body. We will assume that we are dealing with definite programs, i.e., programs where the literals in the body are always positive (non-negated) atoms.

A query to a TCLP program is a clause with the head \( false \), usually written \( ?- c_q, q \), where \( c_q \) is an atomic constraint or a conjunction of constraints and \( q \) is a literal. \(^2\)

2.2 Constraint Solvers

We follow (Jaffar and Maher 1994) in this section. Constraint logic programming introduces constraint solving methods in logic-based programming languages. During the evaluation of a CLP program, the inference engine generates constraints whose consistency with respect to the current constraint store are checked by the constraint solver. If the check fails, the engine backtracks to a previous choice and takes a pending, unexplored branch of the search tree. In the next sections we will review the fixpoint and operational semantics of CLP and will extend them to TCLP.

**Definition 1.** A constraint solver, \( \text{CLP}(\mathcal{X}) \), is a (partial) executable implementation of a constraint domain \( (\mathcal{D}, \mathcal{L}) \). The parameter \( \mathcal{X} \) stands for the 4-tuple \( (\Sigma, \mathcal{D}, \mathcal{L}, \mathcal{T}) \) where:

- \( \Sigma \) is a signature which determines the predefined predicates and function symbols and their arities.
- \( \mathcal{D} \) is a \( \Sigma \)-structure: the constraint domain over which the computation is performed.
- \( \mathcal{L} \) is the class of \( \Sigma \)-formulas: the class of constraints that can be expressed with \( \Sigma \). It should be closed under variable renaming, conjunction, and existential quantification.
- \( \mathcal{T} \) is a first-order \( \Sigma \)-theory: an axiomatization of the properties of \( \mathcal{D} \), which determines what constraints hold and what constraints do not hold. \( \mathcal{D} \) and \( \mathcal{T} \) should agree on satisfiability of constraints, and every unsatisfiability in \( \mathcal{D} \) has to be detected by \( \mathcal{T} \), i.e., for every constraint \( c \in \mathcal{L} \), \( \mathcal{D} \vdash c \) iff \( \mathcal{T} \vdash c \).

A constraint can be an atomic constraint or a conjunction of (simpler) constraints. We denote constraints with lower case letters, e.g. \( c \), and sets of constraints with uppercase letters, e.g. \( S \).

**Example 1.**

The Herbrand domain \( \text{CLP}(\mathcal{H}) \) used in logic programming is the constraint domain over finite trees, where \( \Sigma \) contains constants, function symbols, and the predicate \( =/2 \); \( \mathcal{D} \) is the set of finite trees, where each node is labeled by a constant (if it does not have children) or a function symbol of arity \( n \) (if it has \( n \) children). \( \mathcal{L} \) is the set of constraints generated by the

\(^2\) This covers as well the case of a conjunction of literals since we can always add a rule to that effect to the program.
primitive constraints (i.e., equality) between trees (terms). Typical constraints are \( X = g(a) \) and \( X = f(Z, Y) \land Z = a \).

**Definition 2** (Valuation). Let \( S = \{X_1, \ldots, X_n\} \) be a set of variables. A valuation \( v \) is a mapping from variables in \( S \) to values in \( D \). We write \( v = \{X_1 \mapsto d_1, \ldots, X_n \mapsto d_n\} \) to indicate that the value \( d_i \) is assigned to variable \( X_i \).

For convenience, and where it is not ambiguous, we will denote the value \( d_i \) assigned to a variable \( X_i \) by the valuation \( v \) as \( v(X_i) \) (e.g., \( X_i \mapsto d_i \in v \)). Likewise, for a literal \( l \) we will denote by \( v(l) \) the literal obtained by substituting the variables in \( l \) for their associated values in the valuation \( v \) (for those variables that appear in \( v \)) and, for a constraint \( c \), we define similarly \( v(c) \).

**Definition 3** (Solution of a constraint). Let \( c \) be a constraint, \( \text{vars}(c) \) the set of variables occurring in \( c \), and \( v \) a valuation over \( \text{vars}(c) \) on the constraint domain \( D \). Then \( v \) is a solution for the constraint \( c \) if \( v(c) \) holds in the constraint domain.

**Definition 4** (Projection). Let \( c \) be a constraint, \( S \subseteq \text{vars}(c) \) a set of variables occurring in \( c \), and \( T = \text{vars}(c) \setminus S \) the rest of the variables of \( c \). The projection of \( c \) over \( S \), denoted \( \text{Proj}(c, S) \), is another constraint \( c_s \) such that \( c_s \equiv \exists T \cdot c \), i.e.:
- Any solution \( v_s \) for \( c_s \) can be extended to be a solution for \( c \).
- Any solution \( v \) for \( c \) can be restricted to the variables in \( S \) and the restricted valuation is a solution for \( c_s \).

The minimal set of operations that we expect a constraint solver to support, in order to interface it successfully with a tabling system (Arias and Carro 2019a), are:

- Test for consistence or satisfiability. A constraint \( c \) is consistent in the constraint domain \( D \), denoted \( D \models c \), if it has a solution in \( D \).
- Test for entailment (\( \sqsubseteq_D \)). We say that a constraint \( c_0 \) is entailed by another constraint \( c_1 \) (\( c_0 \sqsubseteq_D c_1 \)) if any solution of \( c_0 \) is also a solution of \( c_1 \). We extend the notion of constraint entailment to a set of constraints: a set of constraints \( C_0 \) is entailed (or covered) by another set of constraints \( C_1 \) (and we write it as \( C_0 \sqsubseteq_D C_1 \)) if \( \forall c_i \in C_0 \exists c_j \in C_1, c_i \sqsubseteq_D c_j \).
- An operation to compute the projection of a constraint \( c \) onto a finite set of variables \( S \). \( \text{Proj}(S, c) \).

### 2.3 Fixpoint Semantics

The canonical model of a Prolog program is the minimal Herbrand model. Similarly, the fixpoint semantics of a CLP program \( P \) over a constraint domain \( D \) is the least \( D \)-S-model, which we define next. The presence of variables in \( D \)-S-models makes it possible to use entailment to discard subsumed constraints in the bottom-up construction of the fixpoint.

We can define the least \( D \)-S-model of a program using the S-semantics (Falaschi et al. 1989; Jaffar and Maher 1994) for languages with constraints (Gabrielli and Levi 1991). It differs from the standard model (van Emden and Kowalski 1976) essentially due to the presence of variables in interpretations and models.

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3 We may omit the subscript \( D \) if there is no ambiguity.
Definition 5 (D-S-interpretation). Let the pair \((l, c)\) be a constraint literal, where \(l\) is a literal and \(c \in D\) an atomic constraint or a conjunction of constraints such that \(\text{vars}(c) \subseteq \text{vars}(l)\). A D-S-interpretation is a set of constraint literals.

Definition 6 (D-S-model). Let \(P\) be a program. A D-S-model of \(P\) is a D-S-interpretation that is logically consistent with the clauses in \(P\).

The CLP fixpoint S-semantics is defined as the smallest fixpoint of the immediate consequence operator, \(S^D_P\), where all the operations behave as defined in the constraint domain \(D\).

Definition 7 (Operator \(S^D_P\) (Falaschi et al. 1989; Toman 1997)). Let \(P\) be a CLP program and \(I\) a D-S-interpretation. The immediate consequence operator \(S^D_P\) is defined as:

\[
S^D_P(I) = I \cup \{ (h, c) \mid h :- c_h, l_1, \ldots, l_k \text{ is a clause of } P, \ (a_i, c_i) \in I, \ 0 < i \leq k, \ c' = \text{Proj}(\text{vars}(h), c_h \land \bigwedge_{i=1}^{k} (a_i = l_i \land c_i)), \ D \models c', \ \
\text{if } c' \sqsubseteq c'' \text{ for some } (h, c'') \in I \text{ then } c = c'' \text{ else } c = c' \}
\]

Note that \(S^D_P\) may not add a pair (literal, constraint) when a constraint more general is already present in the interpretation being enlarged. However, to guarantee monotonicity, it does not remove existing more particular constraints. The operational semantics of TCLP (Definition 10) will do that.

2.4 Operational Semantics of TCLP

In this section we first present a top-down semantics for CLP without tabling/suspension (Jaffar and Maher 1994) and then we extend it to capture the operational semantics of TCLP. The operational semantics is given in terms of a transition system that computes the least model defined by the CLP fixpoint semantics (Section 2.3). The evaluation of a query is a sequence of steps from the initial state to a final state.

Definition 8. A state is a tuple \(\langle R, c \rangle\) where:

- \(R\), the resolvent, is a multiset of literals and constraints that contains the collection of as-yet-unseen literals and constraints of the program.
- \(c\), the constraint store, is an atomic constraint or a conjunction of constraints. It is acted upon by the solver.

In (Jaffar and Maher 1994) the constraint store is divided into a collection of awake constraints and a collection of asleep constraints. This separation is ultimately motivated by implementation issues and we will not make this distinction here.

Given a query \((q, c_q)\), the initial state of the evaluation is \(\langle \{q\}, c_q \rangle\). Every transition step between states resolves literals of the resolvent against the clauses of the program and adds constraints to the constraint store. A derivation is successful if it is finite and the final state has the form \(\langle \emptyset, c \rangle\) (i.e., the resolvent becomes empty). The answer for the query is \(\text{Proj}(\text{vars}(q), c)\).

As it is customary, we assume that the transitions due to constraint handling are deterministic (there is only one possible children per node), while the transitions due to literal matching may be non-deterministic (there are as many children as clauses whose head matches some literal in the resolvent). As a result, query evaluation takes the shape of a search tree, constructed following Def. 9. The order in which literals are selected is not relevant. In practice, implementations would
use a computation rule that is in charge of deciding the new constraint/literal to be resolved among the set of pending literals. A common rule is to follow the left-to-right order in which literals are written in the body of clauses.

In what follows we will assume that variables in clauses are renamed apart before they are used in order to avoid clashes with existing variable names.

**Definition 9** (CLP tree). Let $P$ be a CLP definite program and $(q, c_q)$ a query. A CLP tree of $(q, c_q)$ for $P$, denoted by $\mathcal{T}_P(q, c_q)$, is a tree such that:

1. The root of $\mathcal{T}_P(q, c_q)$ is $\langle \{q\}, c_q \rangle$, the initial state.
2. The nodes of $\mathcal{T}_P(q, c_q)$ are labeled with its corresponding state $\langle L, c \rangle$, where $L$ is a set containing the constraints and literals pending to be solved.
3. The child/children of a node $\langle L \cup \{l\}, c \rangle$, where $l$ is a literal, is/are:
   - A node/nodes $\langle \text{body}(h_i) \cup L, c \land (l = h_i) \rangle$ obtained by resolution of $l$ against the matching clause(s) $h_i \leftarrow \text{body}(h_i)$ in $P$ where $l = h_i$ is an abbreviation for the conjunction of equations between the arguments of $l$ and $h_i$. There is one node for each matching clause. Matching clauses are assumed to be renamed apart.
   - Or a leaf node $\text{fail}$ if there are no clauses in $P$ which matching heads for the literal $l$.
4. The child of a node $\langle c' \cup L, c \rangle$, where $c'$ is a constraint, is:
   - The node $\langle L, c \land c' \rangle$ if $D \models c \land c'$.
   - Or a leaf node $\text{fail}$ if $D \not\models c \land c'$.
5. A leaf node $\langle \emptyset, c \rangle$ is the final state of a successful derivation. $c$ is the final constraint store.
6. The set of answers of $\mathcal{T}_P(q, c_q)$ (i.e., the answers to the query $(q, c_q)$), denoted by $\text{Ans}(q, c_q)$, is the set of constraints $c'_i$ obtained as the projection of the final constraint stores $c_i$ onto $\text{vars}(q)$:

$$\text{Ans}(q, c_q) = \{ c'_i \mid c'_i = \text{Proj}(\text{vars}(q), c_i), \langle \emptyset, c_i \rangle \in \mathcal{T}_P(q, c_q) \}$$

We denote the set of tabled predicates in a TCLP program by $\text{Tab}_P$. The most general calls to predicates in $\text{Tab}_P$ are called generators and are resolved against program clauses. The set of generators created during the evaluation of a query $(q, c_q)$ is denoted by $\text{Gen}(q, c_q)$. The answers for a generator are collected and associated to that generator; see below how entailment is used to keep only the relevant answers. Calls to tabled predicates that are more particular than a previously created generator become consumers and are not resolved against program clauses. Instead, they are resolved by consuming the answers collected from a generator; this is termed answer resolution.

The execution of a query w.r.t. a TCLP program is represented as a forest of derivation trees, and contains the tree corresponding to the initial query and the trees corresponding to each of the generators. The evaluation of each generator corresponds to one of the trees of the forest. During execution, call entailment (Def. 10.2.b) detects when a goal is entailed/subsumed by a previous goal (its generator) and if so, it suspends their execution and eventually reuses the answers from the generator. During answer entailment, answers that are entailed by another (more general) answer are discarded/removed (Def. 10.2.f).

**Definition 10** (TCLP forest). Let $P$ be a TCLP definite program, $\text{Tab}_P$ the set of tabled predicates, and $(q, c_q)$ a query. A TCLP forest of $(q, c_q)$ for $P$, denoted as $\mathcal{F}_P(q, c_q)$ is the set of TCLP trees such that:
1. The initial tree, $\tau_p(q,c_q)$, is the TCLP tree of the query, and the rest of the trees, $\tau_p(g,c_g)$, are the TCLP trees of the generators $(g_i, c_{g_i}) \in \text{Gen}(q,c_q)$:

$$\mathcal{F}_p(q,c_q) = \{ \tau_p(q,c_q), \tau_p(g_i,c_{g_i}), \ldots \} \text{ with } i \geq 0$$

2. A TCLP tree, denoted by $\tau_p(q,c_q)$ (resp. $\tau_p(g,c_g)$), is similar to a CLP tree where:
   (a) The root of the TCLP tree $\tau_p(g,c)$ is $\langle \{g\}, c \rangle$, its initial state.
   (b) The descendants of a node $\langle t \cup L, c \rangle$ where $t$ is a tabled literal are obtained by obtaining answers for $t$ through answer resolution (i.e., consuming existing answers) in one of the two following ways:
      - If $(t, c)$ is a consumer of a previous generator $(g, c_g) \in \text{Gen}(q,c_q)$, we use the answers $c_i \in \text{Ans}(g,c_g)$ to construct its children. In this case, $g$ and $t$ match and $(g, c_g)$ is entailed by $(t, c)$, i.e., $c \land (t = g) \sqsubseteq c_g$. As a reminder, $t = g$ denotes the conjunction of equality constraints between the corresponding arguments of $t$ and $g$ and $\text{Ans}(g,c_g)$ is the set of recorded answers for $(g, c_g)$.
      - Otherwise, $(t, c)$ will produce a new generator $(t, c')$ and we use the answers $c_i \in \text{Ans}(t,c')$. In this case, a new TCLP tree $\tau_p(t,c')$, where $c' = \text{Proj}(\text{vars}(t), c)$, is created and added to the current forest. The goal $(t, c')$ is then marked as a generator and added to $\text{Gen}(q,c_q)$.

   From the possible answers $c_i$ to $(t, c)$, children nodes are constructed as follows:
   - A node $\langle c_i \cup L, c \rangle$, one for each answer $c_i$.
   - Or a leaf fail if there is no answer $c_i$.
   (c) The transitions for non-tabled literals and for new generators are as in the CLP tree (Def. 9.3).
   (d) The transitions for constraints are as in the CLP tree (Def. 9.4).
   (e) A leaf node $(\emptyset, c)$ is the final state of a successful derivation and $c$ is its final constraint store.
   (f) The set of answers of $\tau_p(g,c_g)$, the TCLP tree of the generator $(g, c_g)$, denoted by $\text{Ans}(g,c_g)$, is the set constraints $c_i'$ obtained as the projection of the final constraint stores $c_i$ onto $\text{vars}(g)$ that do not entail any other constraint $c_j$, i.e., they are the most general answers.

   $$\text{Ans}(c,c_g) = \{ c_i' \mid c_i' = \text{Proj}(\text{vars}(g), c_i), (\emptyset, c_i) \in \tau_p(g,c_g), \forall c_j : (\emptyset, c_j) \in \tau_p(g,c_g), c_i \neq c_j, c_i' \sqsubseteq \text{Proj}(\text{vars}(g), c_j) \}$$

3. The set of the answers of the forest $\mathcal{F}_p(q,c_q)$, denoted by $\text{Ans}(q,c_q)$, is the set of answers of $\tau_p(q,c_q)$ that are obtained as in the CLP tree (Def. 9.6).

The answer management strategy used in Def. 10.2.f aims at keeping only the most general answers. Since implementations incrementally save answers as they are found, some previous proposals used simpler answer management strategies. For example, (Cui and Warren 2000; Chico de Guzmán et al. 2012) checked entailment when adding answers to the previously generated ones and only discarded answers which were more particular than a previous one. This reduces the number of saved answers, but older answers that are more particular than newer answers were still kept. It could also be possible to remove previous answers that are more particular than new answers but still add answers that are more particular than previous ones. The choice among them does not impact soundness or completeness properties. However, discarding
and removing redundant answers, despite extra cost, has been shown to greatly increase the efficiency of the implementation (Arias and Carro 2019a).

Example 2. TCLP forest of \texttt{dist/3}

This example illustrates how the algorithm works with mutually dependent generators, i.e., generators that consume answers from each other, and to see why not all the answers from a generator may be directly used by its consumers.\footnote{This example also appears in the Supplementary Material of (Arias and Carro 2019a).} Fig. 2 shows the TCLP forest corresponding to querying the right-recursive \texttt{dist/3} program (Fig. 1c). Unlike the left-recursive version, which generates only one TCLP tree, the right-recursive version generates two TCLP trees, one for each generator. The reason is that the left-recursive version only seeks paths from the node \texttt{a}, but the right-recursive version creates a new TCLP tree at the state \texttt{s4} to collect the paths from the node \texttt{b}, since \texttt{edge(a, b)} had been previously evaluated at state \texttt{s3}. We explain now how we obtain some of the states; the rest are obtained similarly.

\begin{itemize}
  \item \texttt{s1} the TCLP tree \( \tau_p(\texttt{dist(a, V0, V1), V1 < 150}) \) is created.
  \item \texttt{s4} is obtained by resolving the literal \( \texttt{edge(a, Z1, D1)} \).
  \item \texttt{Ans(s5)} the tabled literal \( \texttt{dist(b, V0, D2)} \) is a new generator and a new TCLP tree \( \tau_p(\texttt{dist(b, V2, V3), V3 > 0 \land V3 < 100}) \) is created (Def. 10.2.b).
  \item \texttt{s5} is the root node of the new TCLP tree.
  \item \texttt{s6i/ii} are obtained by resolving the literal \( \texttt{dist(b, V2, V3)} \) against the clauses of the program.
  \item \texttt{s8} is resolved by resolving the literal \( \texttt{edge(b, Z1, D1)} \).
    In the state \texttt{s8}, the call \( \texttt{dist(a, V2, D2)} \), \( D2 > 0 \land D2 < 75 \) is suspended because it entails the former generator \( \texttt{dist(a, V0, V1), V1 < 150} \).
  \item \texttt{Ans(s1)} the tabled literal \( \texttt{dist(a, V2, D2)} \) is resolved with answer resolution (Def. 10.2.f) using the answers from the previous TCLP tree \( \tau_p(\texttt{dist(a, V0, V1), V1 < 150}) \) because the renamed projection\footnote{The projection of \( V3 > 0 \land V3 < 100 \land D1 > 0 \land D2 > 0 \land V3=D1+D2 \land Z1=a \land D1 > 25 \land D1 < 35 \) onto \( D2 \) is \( D2 > 0 \land D2 < 75 \). After renaming \( D2=V1 \), the resulting projection is \( V1 > 0 \land V1 < 75 \).} of the current constraint store onto the variable of the literal entails the projected constraint store of the generator: \( V1 > 0 \land V1 < 75 \). Since the initial TCLP forest is under construction and depends on itself, the current branch derivation is suspended.
  \item This suspension also causes the former generator to suspend at the state \texttt{s4}.
  \item \texttt{s9} is a final state obtained upon backtracking to the state \texttt{s6ii}.
  \item \texttt{b1} is the first answer of the second generator.
    At this point the suspended calls can be resumed by consuming the answer \texttt{b1} or by evaluating \texttt{s2ii}. The algorithm first tries to evaluate \texttt{s2ii} and then it will resume \texttt{s4} consuming \texttt{b1}.
  \item \texttt{s10} is a final state obtained upon backtracking to the state \texttt{s2ii}.
  \item \texttt{a1} is the first answer of the first generator: \( V0=b \land V1=50 \).
  \item \texttt{s11} is a final state obtained from the state \texttt{s4} by consuming \texttt{b1}.
  \item \texttt{a2} is the second answer of the first generator: \( V0=a \land V1 > 75 \land V1 < 85 \).
  \item \texttt{s12} is a final state obtained from the state \texttt{s8} by consuming \texttt{a1}.
  \item \texttt{b2} is the second answer of the second generator.
\end{itemize}
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Fig. 2: TCLP forest of \( \neg D \# < 150 \), dist(a, Y, D) with right recursion.
**s13** is a failed derivation obtained from **s8** by consuming **a2**. It fails because the constraints \( V_0 = a \land V_1 > 75 \land V_1 < 85 \) are inconsistent with the current constraint store. Note that the projection of the constraint store of **s8** onto \( V_1 \) is \( V_1 > 0 \land V_1 < 75 \). Its child is a fail node.

**s14** is a final state obtained from the state **s4** by consuming **b2**.

**a3** is the third answer of the first generator: \( V_0 = b \land V_1 > 125 \land V_1 < 135 \).

**s15** is a failed derivation obtained from **s8** by consuming **a3**. Its child is a fail node.

The comparison of this forest (with two trees) with the forest obtained for the left-recursive version (with one tree) illustrates why left recursion reduces the execution time and memory requirements when using tabling / TCLP: left recursion will usually create fewer generators. We have also seen that using answers from a most general call, as in the answer resolution of state **s8** (i.e., the constraint store of the consumer \( V_1 > 0 \land V_1 < 75 \) is more particular than the constraint store of the generator \( V_1 < 150 \)), makes it necessary to filter the correct ones (i.e., answer resolution for **a2** and **a3** failed). This is not required in variant tabling because the answers from a generator are always valid for its consumers.

### 3 Soundness, Completeness, and Termination

In this section we prove the soundness and completeness of the operational semantics for the top-down execution of tabled constraint logic programs previously presented. Then, we present some additional results on termination properties for arbitrary constraint solvers that are not necessarily constraint-compact, extending the results in (Toman 1997).

#### 3.1 Soundness and Completeness

(Toman 1997) proves soundness and completeness of \( SLG^C \) for TCLP Datalog programs by reduction to soundness and completeness of bottom-up evaluation. It is possible to extend these results to prove the soundness and completeness of our proposal: they only differ in the answer management strategy and the construction of the TCLP forest. The strategy used in \( SLG^C \) only discards answers which are more particular than a previous answer, while in our proposal we in addition remove previously existing more particular answers (Def. 10.2.f). The result of this is that only the most general answers are kept. In \( SLG^C \), the generation of the forest is modeled as the application of rewriting rules. In our proposal, the TCLP forest is defined as a transition system (Def. 10), where the different cases in the definition can be seen as rules which make the TCLP forest evolve.

The lemma, theorems, and their proofs are reformulated taking in consideration these differences. First we prove that answer resolution using entailment is correct w.r.t. SLD resolution; and although only the most general answers are kept, answer resolution using entailment is complete w.r.t. SLD resolution. Then we use these results to prove soundness and completeness of TCLP with entailment w.r.t. the least fixed point semantics.

**Lemma 1** (Application of derivations with most general constraint stores). Let \( \langle \{l_i, l_{i+1}, \ldots, l_k\}, cs_i \rangle \mapsto \langle \{l_{i+1}, \ldots, l_k\}, cs_{i+1} \rangle \) be a derivation and \( (l_i, c) \) a goal with \( cs_i \subseteq c \). Then:

\[ \exists \langle \{l_i\}, c' \rangle \mapsto \langle \emptyset, c' \rangle \quad \text{with} \quad cs_{i+1} = cs_i \land c' \]

Intuitively, if there is an SLD derivation that gives a solution for a goal \( (l_i, cs_i) \), this solution
can be obtained using the solution for a more general goal \((l_i, c)\) without the need to resolve the more particular one.

**Proof.** We will see that there exists a derivation \(\langle \{l_i\}, c \rangle \rightsquigarrow \langle \emptyset, c' \rangle\) that follows the same steps as 
\(\langle \{l_1, \ldots, l_k\}, cs_i \rangle \rightsquigarrow \langle \{l_1, \ldots, l_k\}, cs_{i+1} \rangle\):

1. if \(\langle \{l_1, \ldots, l_k\}, cs_i \rangle\) is resolved against a clause \(l_i \leftarrow c_h\), then its resulting constraint store is 
\(cs_{i+1} = cs_i \land c_h\) (plus head unification). Since \(cs_i \subseteq c\), we can apply the same rule to \(\langle \{l_i\}, c \rangle\) and its resulting constraint store is 
\(c' = c \land c_h\). Also, since \(cs_i \subseteq c\), we have \(cs_i \leftarrow cs_i \land c\). Therefore, 
\(cs_{i+1} = cs_i \land c \land c'\) (contracting \(c \land c_h\)).

2. if \(\langle \{l_1, \ldots, l_k\}, cs_i \rangle\) is resolved against a clause \(l_i \leftarrow c_h, a_1, \ldots, a_m\), the next state is 
\(\langle \{a_1, \ldots, a_m, l_1, \ldots, l_k\}, cs_i \land c_h\rangle\) (resp. \(\{a_1, \ldots, a_m, c \land c_h\}\)). By induction, since \(cs_i \subseteq c\) (resp. \(c \subseteq true\)), there exist \(m\) derivations 
\(\langle \{a_j\}, true \rangle \rightsquigarrow \langle \emptyset, c_{hj} \rangle\) such that the resulting constraint store of the path is 
\(cs_{i+1} = cs_i \land c_h \land \bigwedge_{j=1}^m c_{hj}\) (resp. \(c' = c \land c_h \land \bigwedge_{j=1}^m c_{hj}\)). Since \(cs_i \subseteq c\), we have \(cs_i \leftarrow cs_i \land c\). Therefore, 
\(cs_{i+1} = cs_i \land c \land c'\) (contracting \(c \land c_h\))

\(\square\)

We will use this lemma to prove correctness of answer resolution. We model the answers obtained for a generator with the derivation \(\langle \{l_i\}, c \rangle \rightsquigarrow \langle \emptyset, c' \rangle\), while \((l_i, cs_i)\) would be a consumer for the generator \((l_i, c)\). Note that the condition \(cs_i \subseteq c\) precisely captures the generator / consumer relationship.

**Corollary 1** (Correctness of answer resolution using entailment). As an immediate consequence of Lemma 1, using answer resolution with entailment (Def. 10.2.b) gives correct results. Answer resolution of \(\langle \{l_1, \ldots, l_k\}, cs_i \rangle\) consumes an answer \(c'\) from a previous derivation \(\langle \{l_i\}, c \rangle \rightsquigarrow \langle \emptyset, c' \rangle\) where \((l_i, c)\) is the generator of the derivation and, by the definition of generator, \(cs_i \subseteq c\). When \(D \models cs_i \land c'\) (Def. 10.2.d), it generates the state \(\langle \{l_{i+1}, \ldots, l_k\}, cs_i \land c' \rangle\).

**Corollary 2** (Completeness of answer resolution using entailment). Recall that \(Ans(l, c)\) is the set containing the most general answers for a generator goal \((l, c)\) (Def. 10.2.f), and if there are two goals \((l, c_0)\) and \((l, c_0')\) with \(c_0 \subseteq c_0'\), only the answers for the most general goal \(c_0'\) need to be kept. Therefore, for any derivation of a generator \(\langle \{l_i\}, c \rangle \rightsquigarrow \langle \emptyset, c_i \rangle\) we have that 
\(\exists c'_i \in Ans(l_i, c_i)\) for some \(c_i\) s.t. \(c_i \subseteq c'\). Let us take a (partial) clause derivation \(\langle \{l_i, \ldots, l_k\}, c \rangle \rightsquigarrow \langle \{l_{i+1}, \ldots, l_k\}, c \land c_i \rangle\). If \(c' \in Ans(l_i, c_i)\) for some \(c'_i\) (which is the entailment condition necessary to use the saved answer constraints), then \(c_i \subseteq c'_i\). If we use \(c_i\) to perform answer resolution with \((l_i, c)\), we have \(\langle \{l_i, \ldots, l_k\}, c \rangle \rightsquigarrow \langle \{l_{i+1}, \ldots, l_k\}, c \land c_i \rangle\). Given that \(c_i \subseteq c'_i\), we have that \(c \land c_i \subseteq c \land c'_i\), and any answer returned by clause resolution is contained in some answer returned by answer resolution with entailment. The same reasoning can be applied to the derivation of \(l_{i+1}\) and so on. Therefore, answer resolution with entailment does not lose answers w.r.t. clause resolution even if not all the goals and answers are memorized.

**Theorem 1** (Soundness w.r.t. the fixpoint semantics). Let \(P\) be a TCLP definite program and \((q, c_q)\) a query. Then for any answer \(c'\) of the TCLP forest \(F_p(q, c_q)\)
\[c' \in Ans(q, c_q) \Rightarrow \exists (q, c) \in \text{lfp}(S^n_p) (q, c) = c_q \land c\]
I.e., any answer derived from the forest construction can also be derived from the bottom-up computation.

**Proof.** For any answer \(c' \in Ans(q, c_q)\) there exists a successful derivation \(\langle \{q\}, c_q \rangle \rightsquigarrow \langle \emptyset, c' \rangle\). Since \(c_q \subseteq true\), by Lemma 1 there exists \(\langle \{q\}, true \rangle \rightsquigarrow \langle \emptyset, c \rangle\). \(c' = c_q \land c\). We know that for
any successful derivation \( \langle \{ q \}, \text{true} \rangle \rightarrow \langle \emptyset, c \rangle \) against the clauses of the program there is an answer derived from the bottom-up computation \( (q, c) \in \text{lfp}(P_{D}(\emptyset)) \). Therefore, by Corollary 1 if answer resolution is used instead of clause resolution, the result is also correct and for any answer \( c' \in \text{Ans}(q, c_q) \) there exists \( (q, c) \in \text{lfp}(P_{F}(\emptyset)) \). \( c' = c_q \land c \). 

**Theorem 2** (Completeness w.r.t. the fixpoint semantics). Let \( P \) be a TCLP definite program and \( (h, \text{true}) \) a query. Then for every \((h, c) \in \text{lfp}(P_{D}(\emptyset))\): 

\[ (h, c) \in \text{lfp}(P_{D}(\emptyset)) \Rightarrow \exists c' \in \text{Ans}(h, \text{true}). \ c \subseteq c' \] 

I.e., all the answers derived from the bottom-up computation are also derived by the forest construction or entailed by answers inferred in the forest.

**Proof.** We know that for any answer derived from the bottom-up computation \((h, c) \in \text{lfp}(P_{D}(\emptyset))\) there exists a successful derivation \( \langle \{ h \}, \text{true} \rangle \rightarrow \langle \emptyset, c \rangle \) against the clauses of the program. By Corollary 2 if answer resolution is used instead of clause resolution, the results is also complete. Therefore, since the answer management strategy only keeps the most general answers (Def. 10.2.f), we have that \( \exists c' \in \text{Ans}(h, \text{true}). \ c \subseteq c' \). 

### 3.2 Termination

The next definition is a fundamental property of some constraint domains that plays a key role in the termination of the evaluation of queries to TCLP programs (Toman 1997).

**Definition 11** (Constraint-compact). Let \( D \) be a constraint domain, and \( D \) the set of all constraints expressible in \( D \). Then \( D \) is constraint-compact iff:

- for every finite set of variables \( S \), and
- for every subset \( C \subseteq D \) such that \( \forall c \in C. \text{vars}(c) \subseteq S \), there is a finite subset \( C_{fin} \subseteq C \) such that \( \forall c \in C. \exists c' \in C_{fin}. c \sqsubseteq D c' \)

Intuitively speaking, a constraint domain \( D \) is constraint-compact if for any (potentially infinite) set of constraints \( C \) expressable in \( D \) using a finite number of variables, there is a **finite** set of constraints \( C_{fin} \subseteq C \) that covers \( C \) in the sense of \( \sqsubseteq D \). In other words, \( C_{fin} \) is as general as \( C \). Additionally, in a constraint-compact constraint domain, if an infinite set of constraints is unsatisfiable, then there is a finite subset which is unsatisfiable, therefore guaranteeing the existence of finite unsatisfiability proofs.

**Example 3.**

The **gap-order constraints** (Revesz 1993) is a constraint-compact domain generated from the set \( C_{\leq} = \{ x < u : u \in A \} \cup \{ u < x : u \in A \} \cup \{ x + k < y : k \in \mathbb{Z}^{+} \} \) where \( A \subseteq \mathbb{Z}^{+} \) is finite.

First, we see that the set \( C_{=\leq} \) (resp. \( C_{<\leq} \)) of possible constraints of the form \( x < u \) (resp. \( u < x \)), where \( x \in S \), is finite, because \( A \) and \( S \) are finite. Therefore, it is trivial to define a finite set that covers \( C_{=\leq} \cup C_{<\leq} \). Second, for every pair of variables \( x, y \in S \), the set \( C_{x+k<y} \) of possible constraints of the form \( x + k < y, k \in \mathbb{Z}^{+} \) can be covered by a finite subset of itself. Although for a given pair of variables \( x, y \) one can generate an infinite number of constraints \( x + k_i < y \) choosing different \( k_i \in \mathbb{Z}^{+} \), the constraint \( x + k_0 < y \) having the smallest \( k_0 \) among all the \( k_i (\forall k_i, k_0 \leq k_i) \) subsumes all the rest of the constraints \( (x + k_i < y \subseteq x + k_0 < y) \). Note that \( k_0 \) always exists, since \( k_i \in \mathbb{Z}^{+} \), which has a minimum. Since \( S \) is finite, we only have to check it for two given \( x, y \); we can repeat the same process for every pair of variables, since
there is only a finite number of them. Therefore, the infinite set $C_{x+k<y}$ has a finite subset $C_{\text{fin}} = \{x+k_0 < y\}$ which covers it ($C_{x+k<y} \subseteq C_{\text{fin}}$).

**Example 4.**
The Herbrand domain is not constraint-compact. Take the infinite set of constraints $C = \{X = a, X = f(a), X = f(f(a)), \ldots\}$. No finite subset of $C$ using only constraints in $C$ can cover $C$.

The termination of TCLP Datalog programs under a top-down strategy when the constraint system is constraint-compact is proven in (Toman 1997). In that case, the evaluation will suspend the exploration of a call whose constraint store is less general than or comparable to a previous call. Eventually, the program will generate a set of call constraint stores that can cover any infinite set of constraints in the constraint domain, therefore finishing evaluation.

Many TCLP applications require constraint domains that are not constraint-compact because constraint-compact domains in general have a limited expressiveness. We refine here the termination theorem (Toman 1997, Theorem 23) for Datalog programs with constraint-compact domains to cover cases where the constraint domain is not constraint-compact, but in which the program evaluation generates only a constraint-compact subset of all the constraints expressable in the constraint domain.

**Theorem 3** (Termination in non-constraint-compact domains). Let $P$ be a TCLP($D$) definite program and $(q, c_q)$ a query. Then the TCLP execution for that query terminates if:

- For every goal $(g, c_i)$ in the forest $F(q, c_q)$, the set $C_g$ is constraint-compact, where $C_g$ is the set of all the constraint stores $c_i$, projected and renamed w.r.t. the arguments of $g$.
- For every goal $(g, c_g)$ in the forest $F(q, c_q)$, the set $A_{\{g\}, c_g}$ is constraint-compact, where $A_{\{g\}, c_g}$ is the set of all the answer constraints $c'$, projected and renamed w.r.t. the arguments of $g$, s.t. $c'$ is a successful derivation of $(g, c_i)$ in the forest $F(q, c_q)$.

**Proof.** (Toman 1997) proves termination by observing that the SLG$^C$ rewriting rules can be applied only finitely many times. We extend this proof to ensure that the TCLP forest generated is finite and therefore the program execution terminates.

1. The execution can only generate a finite number of literals, up to variable renaming, because they are linearized (unifications take place in the constraints in the body) and the number of predicates in the program is finite.
2. The execution can only generate a finite number of TCLP forests $\tau_p(g, c_g)$ because the number of possible literals is finite (point 1) and for each literal $g$, the set $C_g$ of its possible active constraint stores is constraint-compact. That means that, for every subset of active constraint stores $C \subseteq C_g$, there exists a finite subset, $C_{\text{fin}} \subseteq C$ of possible most general calls, such that $\forall c \in C. \exists c' \in C_{\text{fin}}. c \subseteq_D c'$. Therefore, at some point every new call will be entailed by some previous generator (this is checked in Def. 10.2.b).
3. The set of answers $\text{Ans}(g, c_g)$ (Def. 10.2.f) is finite because the set of possible most general answer constraints is finite. The justification similar to that in point 2.
4. The number of children from a node resolved against clauses in $P$ (Def. 10.2.c) is finite because the number of clauses in $P$ is finite.
5. The number of children from a node resolved by answer resolution (Def. 10.2.b) is finite because, by point 3, the set of answers $\text{Ans}(g, c_g)$ is finite.

The intuition here is that for every subset $C$ from the set of all possible constraint stores $C_g$...
that can be generated when evaluating a call to \( P \), if there is a finite subset \( C_{\text{fin}} \subseteq C \) that covers (i.e., is as general as) \( C \), then, at some point, any call will be entailed by previous calls, thereby allowing its suspension to avoid loops. Similarly, for every subset \( A \) from the set of all possible answer constraints \( A_{\{g,c_g\}} \) that can be generated by a call, if there is a finite subset \( A_{\text{fin}} \subseteq A \) that covers \( A \), then, at some point, any answer will be entailed by a previous one, ensuring that the class of answers \( \text{Ans}(g,c_g) \) which entail any other possible answer returned by the program is finite.\(^6\) Note that this result implies the classical result that programs with the bounded depth term property always finish under tabling with variant tabling, since the bounded depth term property means that the number of possible constraints is finite and therefore any constraint set covers itself.

**Example 5.** The Herbrand domain (with constants and function symbols) and syntactic equality is not constraint-compact, and therefore termination of TCLP(H) programs is not guaranteed. However, in the case of programs which have only constants, the number of constraints that can be generated is finite, and therefore termination is ensured. Termination is also ensured (even with variant tabling) when a program can only generate terms with a bounded depth. In this case, the number of distinct terms (and therefore of equality constraints) that can be generated is finite as well.

**Example 6.** Fig. 3a shows a program which loops in tabled Prolog and under variant tabling. The unification appears explicitly in the body for clarity. Although CLP(H) is not constraint-compact, the constraints generated by that program under the query `?- p(X)` can make it finish. Let examine its behavior from two points of view:

**Compactness of the call constraint stores** The set of all the constraint stores generated for the predicate \( p/1 \) under the query \( (p(X), \text{true}) \) is \( C_{p(X)} = \{ \text{true}, V = f(X), V = f(f(X)), \ldots \} \).\(^7\) It is constraint-compact because for every subset \( C \) there is a finite set, e.g. \( C_{\text{fin}} = \{ \text{true} \} \), that covers \( C \).

**Compactness of the answer constraints** Additionally, the set of all answer constraints for the query, \( A_{\{p(V), \text{true}\}} = \{ V = a \} \), is also constraint-compact because it is finite. Since both are constraint-compact, the execution terminates.

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\(^6\) Note that a finite answer set does not imply a finite domain for the answers: the set of answers \( \text{Ans}(q,c_q) = \{ V > 5 \} \) is finite, but the answer domain of \( V \) is infinite.

\(^7\) The syntax \( C_{p(X)} \) means that (i) we are projecting all the calls to predicate \( p/1 \) on the variables that call, and (ii) we are renaming these variables to be \( V \) in all the calls. We could associate with every constraint store the names of the variables in the call in order to be able to compare different constraints stores (which is unnecessary after projection if there is only one variable in the call, but it would be needed if more than one variable is involved). In order to avoid such an overload, and without loss of generality, we preferred to project and rename to a unique set of variables.
Suspension due to call entailment The first recursive call is \((p(Y_1), Y_1 = f(X))\) and its projected and renamed constraint store is entailed by the initial store: \(V = f(X) \subseteq true\). Therefore, TCLP evaluation suspends the recursive call, shifts execution to the second clause, and generates the answer \(X = a\). This answer is given to the suspended recursive call, results in the inconsistent constraint store \(Y_1 = f(X) \land Y_1 = a\) and the execution terminates.

Example 7. Using the previous example (Fig. 3a) under the query \(?- p(a)\), the set of all the generated constraint stores is \(C_{p(V)} = \{V = a, V = f(a), V = f(f(a)), \ldots\}\). It is not constraint-compact and the execution does not terminate. Let us examine its behavior:

The call constraint stores are not compact The first recursive call is \((p(Y_1), X = a \land Y_1 = f(X))\) and the projection of its constraint store, \(Y_1 = f(a)\), is not entailed by the initial one after renaming: \(V = f(a) \not\subseteq V = a\). Then this call is evaluated and produces the second recursive call, \((p(Y_2), X = a \land Y_1 = f(X) \land Y_2 = f(f(X)))\). Its projected constraint store, \(Y_2 = f(f(a))\), is not entailed by any of the previous constraint stores, and so on with the rest of the recursive calls. Therefore, the evaluation loops without terminating.

Let us show the termination properties of the examples used in (Arias and Carro 2019a). These examples show under what conditions programs would terminate even if the constraint domain is not constraint-compact.

Example 8. Fig. 3b shows a program which generates all the natural numbers using TCLP(Q). Although CLP(Q) is not constraint-compact, the constraint stores generated by that program for the query \(?- X \#< 10, \text{nat}(X)\) are constraint-compact and the program finitely finishes. Let us look at its behavior from two points of view:

Compactness of the call constraint stores and answer constraints The set of all constraint stores generated for the predicate \(\text{nat}/1\) under the query \((\text{nat}(X), X < 10)\) is \(C_{\text{nat}(V)} = \{V < 10, V < 9, \ldots, V < -1, V < -2, \ldots\}\). It is constraint-compact because every subset \(C \in C_{\text{nat}(V)}\) is covered by \(C_{\text{fin}} = \{V < 10\}\). The set of all possible answer constraints for the query, \(A_{\text{nat}(V), V < 10} = \{V = 0, \ldots, V = 9\}\), is also constraint-compact because it is finite. Therefore, the program terminates.

Suspension due to call entailment The first recursive call is \((\text{nat}(Y_1), X < 10 \land X = Y_1 + 1)\) and the projection of its constraint store after renaming is entailed by the initial one since \(V < 9 \subseteq V < 10\). Therefore, TCLP evaluation suspends in the recursive call, shifts execution to the second clause and generates the answer \(X = 0\). This answer is given to the recursive call, which was suspended, produces the constraint store \(X < 10 \land X = Y_1 + 1 \land Y_1 = 0\), and generates the answer \(X = 1\). Each new answer \(X_n = n\) is used to feed the recursive call. When the answer \(X = 9\) is given, it results in the (inconsistent) constraint store \(X < 10 \land X = Y_1 + 1 \land Y_1 = 9\) and the execution terminates.

Example 9. The program in Fig. 3b does not terminate for the query \(?- X \#> 0, X \#< 10, \text{nat}(X)\). Let us examine its behaviour:

The call constraint stores are not compact The set of all constraint stores generated by the query \((\text{nat}(X), X > 0 \land X < 10)\) is \(C_{\text{nat}(V)} = \{V > 0 \land V < 10, V > - 1 \land V < 9, \ldots, V > - n \land V < (10 - n), \ldots\}\), which it is not constraint-compact. Note that \(V\) is, in successive calls, restricted to a sliding interval \([k, k + 10]\) which starts at \(k = 0\) and decreases \(k\) in each recursive call. No finite set of intervals can cover any subset of the possible intervals.
The evaluation loops. The first recursive call is \((\text{nat}(Y_1), X > 0 \land X < 10 \land X = Y_1 + 1)\) and the projection of its constraint store is not entailed by the initial one after renaming since \((V > -1 \land V < 9) \not\subseteq (X > 0 \land X < 10)\). Then this call is evaluated and produces the second recursive call, \((\text{nat}(Y_2), X > 0 \land X < 10 \land X = Y_1 + 1 \land Y_1 = Y_2 + 1)\). Again, the projection of its constraint store, \(Y_2 > -2 \land Y_2 < 8\), is not entailed by any of the previous constraint stores, and so on. The evaluation therefore loops.

Example 10. The program in Fig. 3b does not terminate with the query \(?- \text{nat}(X)\).

Compactness of the call constraints stores. The set of all constraint stores generated by the query \((\text{nat}(X), \text{true})\) is \(C_{\text{nat}(V)} = \{\text{true}\}\). The set \(C_{\text{nat}(V)}\) is constraint-compact because it is finite.

The answer constraints are not compact. However, the answer constraint set \(A_{\text{(nat}(V), \text{true})} = \{V = 0, V = 1, \ldots, V = n, \ldots\}\) is not constraint-compact, and therefore the program does not terminate.

The evaluation does not terminate. The first recursive call is \((\text{nat}(Y_1), X = Y_1 + 1)\) and the projection of its constraint store is entailed by the initial store. Therefore, the TCLP evaluation suspends the recursive call, shifts execution to the second clause, and generates the answer \(X = 0\). This answer is used to feed the suspended recursive call, resulting in the constraint store \(X = Y_1 + 1 \land Y_1 = 0\) which generates the answer \(X = 1\). Each new answer \(X = n\) is used to feed the suspended recursive call. Since the projection of the constraint stores on the call variables is \text{true}, the execution tries to generate infinitely many natural numbers.

Example 11. Unlike what happens in pure Prolog/variant tabling, adding new clauses to a program under TCLP can make it terminate.\(^8\) As an example, Fig. 3c is the same as Fig. 3b with the addition of the clause \(\text{nat}_k(X) : -X > 1000\). Let us examine its behavior under the query \(?- \text{nat}_k(X)\):

Compactness of call/answer constraint stores. The set of all constraint stores generated remains \(C_{\text{nat}_k(V)} = \{\text{true}\}\). But the new clause makes the answer constraint set become \(A_{\text{(nat}_k(V), \text{true})} = \{V = 0, V = 1, \ldots, V = n, \ldots, V > 1000, V > 1001, \ldots, V > n, \ldots\}\), which is constraint-compact because a constraint of the form \(V > n\) entails infinitely many constraints, i.e., it covers the infinite set \(\{V = n + 1, \ldots, V > n + 1, \ldots\}\). Therefore, since both sets are constraint-compact, the program terminates.

First search, then consume. The first recursive call \((\text{nat}_k(Y_1), X = Y_1 + 1)\) is suspended and the TCLP evaluation shifts to the second clause which generates the answer \(X = 0\). Then, instead of feeding the suspended call, the evaluation continues the search and shifts to the added clause, \(\text{nat}_k(X) : -X > 1000\), and generates the answer \(X > 1000\). Since no more clauses remain to be explored, the answer \(X = 0\) is used, generating \(X = 1\). Then \(X > 1000\) is used, resulting in the constraint store \(X = Y_1 + 1 \land Y_1 > 1000\), which generates the answer \(X > 1001\). However, \(X > 1001\) is discarded because \(X > 1001 \not\subseteq X > 1000\). Then, one by one each answer \(X = n\) is used, generating \(X = n + 1\). But when the answer \(X = 1000\) is used, the resulting answer \(X = 1001\) is discarded because \(X = 1001 \not\subseteq X > 1000\).

\(^8\) The equation in the body of the clause \(X = Y_1 + 1\) defines a relation between the variables but, since the domain of \(X\) is not restricted, its projection onto \(Y_1\) returns no constraints (i.e., \(\text{Proj}(Y_1, X = Y_1 + 1) = \text{true}\)).

\(^9\) This depends on the strategy used by the TCLP engine to resume suspended goals. An implementation that gathers all the answers for goals that can produce results first, and then these answers are used to feed suspended goals, makes the exploration of the forests proceed in a breadth-first fashion.
evaluation terminates because there are no more answers to be consumed. The resulting set of answers is 
\[ \text{Ans}(\text{nat}_k(X), \text{true}) = \{X=0, X > 1000, X=1, \ldots, X=1000\}. \]

4 The Role of Projection in TCLP

The detection of more particular calls and answers is performed by checking entailment of the current constraint store of calls (resp., answers) against the projected constraint store of a previous call. Some previous frameworks (Schrijvers et al. 2008; Cui and Warren 2000) did not implement a precise projection due to performance and implementation issues. Given that in some cases approximate projections can be more efficient and/or easier to implement, it is worth exploring how relaxing projection impacts soundness and completeness. Let \( c \) be a constraint store and let \( c_s \) be a projection of \( c \) on some set of variables \( S \).\(^{10}\) Let us also recall (Def. 2) that a valuation is a mapping from variables to domain constants and that a solution for a constraint is a valuation that is consistent with the interpretation of the constraint in its domain. We distinguish three possible projection variants:

**Precise projection** (denoted \( c \equiv c_s \)) \( c_s \) is a projection of \( c \) over some set of variables \( S \), as defined in Def. 4.

**Over-approximating projection** (denoted \( c \sqsubseteq c_s \)) The projected constraint \( c_s \) is more general than the precise projection, e.g., some solutions for \( c_s \) are not partial solutions for \( c \). Any solution for \( c \) is still a solution for \( c_s \).

**Under-approximating projection** (denoted \( c \sqsupseteq c_s \)) \( c_s \) is less general than the precise projection, e.g., there may be solutions for \( c \) that are not solutions for \( c_s \). Any solution of \( c_s \) is still a (partial) solution for \( c \).

Let us explain how these projection variants interact with the three phases of the operational semantics described in Section 2.4:

- During the call entailment check (see Def. 10.2.b), if a new goal \( (t, c) \), where \( t \) is a tabled literal, does not entail a previous generator then, a new TCLP forest \( F_P(t, c_s) \) is created and \( (t, c_s) \) is a new generator, where \( c_s = \text{Proj}(\text{vars}(t), c) \). Therefore, depending on the projection variant used, we have that:
  
  — Using a precise projection, as already shown, the evaluation of the generator \( (t, c_s) \) would generate the same answers as the evaluation of the goal \( (t, c) \).
  
  — Using an over-approximating projection, the generator \( (t, c_s) \) is more general than \( (t, c) \), and therefore the evaluation of \( (t, c_s) \) may generate answers that are not consistent with the constraint store \( c \). Note, however, that these answers will be filtered: when they are recovered and applied to a consumer (or to their generator) they will be checked for consistency against the constraint store of the call for which they are used.
  
  — Using an under-approximating projection, the generator \( (t, c_s) \) is more particular than the goal \( (t, c) \), and, therefore, its evaluation may not generate answers that \( (t, c) \) would. Note that all of them would be consistent with \( c \).

On the other hand, if a new goal \( (t, c') \) entails a previous generator \( (t, c_s) \), the goal \( (t, c') \) is as usual marked as a consumer and would consume the answers generated by \( (t, c_s) \).

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\(^{10}\) In all cases the projected constraint store \( c_s \) only has the variables in \( S \) in common with the original store \( c \).
• During the answer entailment check (Def. 10.2.f), the final constraint store $a$ of each successful derivation of the evaluation of a generator $(t, c_s)$ is projected to obtain the answer constraint $a_s$, i.e., $a_s = \text{Proj}(\text{vars}(t), a)$. Depending on the projection variant used we have that:

- Using a precise projection (denoted $a \equiv a_s$), as already proved, the resulting set of answer constraints for a generator does not add or exclude any valuation w.r.t. the set of its final constraint stores.
- Using an over-approximating projection (denoted $a \sqsubseteq a_s$), the projected answer constraint $a_s$ may add valuations that are not consistent with the final constraint store $a$.
- Using an under-approximating projection (denoted $a_s \sqsubseteq a$), $a_s$ may exclude valuations that are contained in the constraint store $a$.

• During the application of the answers (Def. 10.2.d), each answer constraint $a_s$ obtained during the evaluation of a generator is added to the constraint store $c$ of the goal that created the generator and the goals that were marked as consumers of that generator. If $a_s$ is consistent with $c$, i.e., $D \vDash c \land a_s$, the evaluation continues under the constraint store $c \land a_s$. Otherwise, it fails and the next answer constraint is retrieved.

We will now summarize how using non-precise projections impacts the soundness and completeness of TCLP. Tables 2a and 2b summarize whether soundness and completeness (resp.) are preserved when using over- and under-approximations for the projections in the call (column) and answer (row) entailment check: ‘✓’ in a location of each table means that the corresponding combination of projection variants preserves soundness (resp., completeness), while ‘×’ means the opposite. As expected, some combinations do not preserve soundness / completeness. Let us give an intuition behind these tables.

• In the top row of Table 2a, the only combination that may be unsound is the one that uses an over-approximation for the call projection: the answers may be more general than what a precise approximation would produce. However, as mentioned before, when an answer is applied to a goal, a conjunction with the call constraint of that goal is made. That balances the use of an over-approximation in the call. This is in fact similar to the case of a consumer that uses answers from a more general generator.
• The combinations in the middle row of Table 2a are not sound because over-approximations can produce answer constraints that allows for more valuations than a correct solution.
• The cases in the bottom row of Table 2a are clearly sound as the projection of the answer constraints is more restrictive than a precise projection, and therefore it cannot introduce unwanted solutions.
• The combinations in the rightmost column and the bottom-most row of Table 2b may not be complete because they either restrict the projected store for a call or they restrict the answers. In both cases, solutions may be missed.
• The rest of the cases in Table 2b may use projections more relaxed than a precise one, so additional solutions can be generated, but no solution should be removed.

Some approximate projections can be more efficient and/or easier to implement than precise projections, and that justifies their use in specific scenarios. For brevity, let us comment on the
Table 2: Combinations of precise, over- and under-approximation (‘≡’, ‘⊑’ and ‘⊒’) for the call and answer entailment check.

(a) Soundness preservation.

|       | $c ≡ c_s$ | $c ⊑ c_s$ | $c ⊒ c_s$ |
|-------|-----------|-----------|-----------|
| $a ≡ a_s$ | ✓         | ✓         | ✓         |
| $a ⊑ a_s$ | ×         | ×         | ×         |
| $a ⊒ a_s$ | ✓         | ✓         | ✓         |

(b) Completeness preservation.

|       | $c ≡ c_s$ | $c ⊑ c_s$ | $c ⊒ c_s$ |
|-------|-----------|-----------|-----------|
| $a ≡ a_s$ | ✓         | ✓         | ×         |
| $a ⊑ a_s$ | ✓         | ✓         | ×         |
| $a ⊒ a_s$ | ✓         | ✓         | ×         |

combinations that preserve soundness and completeness, $≡ / ≡$ and $⊑ / ≡$, and a combination that over-approximates the answers while using a precise projection in the calls, $≡ / ⊑$:

- $≡ / ≡$: Precise projection ‘≡’ in the call and answer entailment check. This is optimal in the sense that it guarantees soundness and completeness, removes redundant answers, and reduces the search space. It has been used in (Arias and Carro 2019a).
- $⊑ / ≡$: Over-approximate projection ‘⊑’ for the calls and precise projection ‘≡’ for the answers. In this case, generators may generate answers that a precise projection would not, since they start with a more relaxed constraint store (which can turn terminating queries into non-terminating ones). This of course preserves completeness. Soundness is preserved because answer constraints that are not consistent with the initial goal constraint store $c$ will be discarded.

Example 12.

Call abstraction (Schrijvers et al. 2008) is an extreme example, where the constraint store associated with the tabled call is not taken into account for the execution of the call (i.e., the projection of a constraint store is always the constraint $true$). Therefore, a generator with $true$ as constraint store will be entailed by any subsequent call because $c ⊑ true$ for any constraint $c$. As mentioned above (see Example 10), this loses several benefits of tabling with constraints because we have to compute all the possible results for an unrestricted call and then filter them through the constraint store active at call-time. However, soundness is preserved.

- $≡ / ⊑$: Precise projection ‘≡’ for the calls and over-approximate projection ‘⊑’ for the answers. This combination is relevant because applications such as program analyzers based on abstract interpretation can be seen as performing an execution in an abstract domain that over-approximates the values of the concrete domain to guarantee termination. This over-approximation can be implemented with a constraint system that reflects the operations of abstract domain and whose answer projections are as well over-approximated. Such an over-approximation can increase performance because a more general answer would be more frequently entailed by other answers, reducing the number of answers stored and the number of resumptions.

However, using an over-approximation in the answer projections may make answer resolution to lose precision arbitrarily. When an answer constraint $a$ for a generator $(t, c_s)$ is projected to obtain the over-approximated answer constraint $a_s$, this answer is saved in case it can be reused later on.
When a (more concrete) consumer \( (t, c') \) performs answer resolution consuming \( a_s \), the resulting answer would be \( c' \land a_s \). Depending on how the over-approximation is performed, \( c' \land a_s \) can be arbitrarily less precise (or even incomparable) than what would have been the result of executing \( (t, c') \) against program clauses and then abstracting it. However, there are some cases where by putting some conditions on when an answer is reused, this problem can be worked around.

**Example 13.**
The implementation of PLAI with TCLP presented in (Arias and Carro 2019b) is an example of this option. In that paper, an abstract interpreter is built using TCLP where the abstract domain and its operations are modeled using a constraint system. One of these computes the lowest upper bound of different abstract substitutions resulting from the analysis of each clause of a predicate, to return the abstract substitution corresponding to the predicate. If \( a_1 \) and \( a_2 \) are the abstract substitutions at the end of the bodies of two (normalized) clauses \( p_1 \) and \( p_2 \), one would like to calculate \( \text{Proj}(\text{var}(p), a_1 \lor a_2) \), where \( \text{Proj} \) may be an overapproximation. When answer substitutions for each clause are projected and stored separately, composing them is done by computing \( \text{Proj}(\text{var}(p), a_1) \sqcup \text{Proj}(\text{var}(p), a_2) \), which can be less precise than \( \text{Proj}(\text{var}(p), a_1 \lor a_2) \). That makes the predicate-level abstract substitution for \( p \) to possibly be an overapproximation of the more precise abstract version.

The tabled abstract substitution for goal \( p \) can be retrieved and used to compute the exit substitution for another goal \( p' \) when \( p' \sqsubseteq p \), using answer resolution. In that case, the exit substitution for \( p' \) can be arbitrarily less precise than what would have been obtained by analyzing directly \( p' \) using clause resolution and then abstracting. We worked around this issue by reusing substitutions only in the case that \( p \) and \( p' \) correspond to the same point in the lattice, i.e., when their entry substitutions are (semantically) equal modulo variable renaming. This ensures that the abstract substitution for \( p \) can be used for \( p' \) without incurring in additional loss of precision, because the analysis results for \( p' \) and \( p \) should be the same.

To the best of our knowledge, there are no examples where under-approximate projections \( \sqsubseteq \) are used. However, since they preserve soundness (except when an over-approximation is used for answer projection, which is neither sound nor complete), they can be useful in scenarios where the existence of a solution is enough to answer a question. This would be the case, for example, for program verification: a solution for a query to a TCLP program that uses underapproximations and looks for counterexamples to the correctness of a program would demonstrate the existence of an error in the program, even if the answer only shows a subset of the domain of the variables for which the program exhibits a wrong behavior.

### 5 Conclusions

We have extended the theoretical basis of tabled constraint logic programming for a top-down execution. We have characterized the properties that the constraint solver should hold in order to guarantee soundness and completeness. For non constraint-compact constraint systems, we define sufficient conditions for queries to terminate. For constraint domains without a precise implementation of the projection of constraint stores, we evaluate how relaxing the projection impacts soundness, completeness, and termination.
From our point of view, the new formalization in terms of soundness, completeness and termination would facilitate the implementation of new tabled constraint logic programming systems and their integration with a larger number of constraint domain (e.g., constraint solvers over finite domains).

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