MINIMAL MODEL PROGRAM FOR SEMI-STABLE THREEFOLDS IN MIXED CHARACTERISTIC

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ABSTRACT. In this paper, we study the minimal model theory for threefolds in mixed characteristic. As a generalization of a result of Kawamata, we show that the MMP holds for strictly semi-stable schemes over an excellent Dedekind scheme $V$ of relative dimension two without any assumption on the residue characteristics of $V$. We also prove that we can run a $\left(K_{X/V} + \Delta\right)$-MMP over $Z$, where $\pi: X \to Z$ is a projective birational morphism of $\mathbb{Q}$-factorial quasi-projective $V$-schemes and $(X, \Delta)$ is a three-dimensional dlt pair with $\text{Exc}(\pi) \subset \lfloor \Delta \rfloor$.

1. Introduction

The Minimal model program (MMP, for short), which is a higher-dimensional analog of the classification method of surfaces, is a tool to find a “simplest” variety in each birational equivalence class. In characteristic zero, this program holds for threefolds and varieties of general type (see [BCHM10]). The minimal model theory recently has been studied also in positive characteristic, and the program is now known to hold for threefolds over a perfect field of characteristic $p > 3$ (see [HX15], [CTX15], [Bir16], [GNT19], [BW17], [HW19b]).

The MMP is also studied for schemes not necessarily defined over a field. Such a generalization of the MMP plays an important role to construct a nice model $W$ over an integer ring whose generic fiber is isomorphic to a given variety $W$ (see [Mat15], [LM18], [CLL19], [CL16]). The MMP is known to hold for excellent surfaces ([Tan14]) and strictly semi-stable schemes over an excellent Dedekind scheme of relative dimension two whose each residue characteristic is neither 2 nor 3 ([Kaw94], [Kaw99]). In this paper, we study the MMP for threefolds over an excellent Dedekind scheme.

Our first main result is a generalization of the result of Kawamata. He used the classification of singularities, which depends on residue characteristic, to prove the existence of flips. We use a completely different approach to prove the existence of flips without assumption on the residue characteristics.

Theorem 1.1. (Theorem 5.10) Let $V$ be an excellent Dedekind scheme. Let $X$ be a scheme which is strictly semi-stable over $V$ of relative dimension two. Let $X \to Z$ be a projective morphism to a quasi-projective scheme $Z$ over $V$. Then we can run a $\left(K_{X/V}\right)$-MMP over $Z$ which terminates with a minimal model or a Mori fiber space. Furthermore, this program preserves good conditions (see Assumption 5.1), for example, the output $Y$ of this MMP is Cohen-Macaulay and every irreducible component of each closed fiber of $Y \to V$ is geometrically normal.
Kawamata’s result is used in several studies of reductions of varieties over an integer ring. Therefore, we also generalize such studies to the case where the residue characteristic is 2 or 3, as the following.

**Good reduction criterion for K3 surfaces** ([Mat15], [LM18], [CLL19]):

Let $K$ be a complete discrete valuation field with perfect residue field of characteristic $p$ and $X$ a K3 surface over $K$. Suppose that $X$ admits potentially semi-stable reduction. If the $G_K$-representation $H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ is unramified for some $\ell \neq p$, then $X$ admits good reduction after an unramified extension of $K$.

**Abelian surfaces have potentially combinatorial reduction** ([CL16]):

Let $K$ be a complete discrete valuation field with perfect residue field of characteristic $p$ and $X$ an abelian surface over $K$. Then $X$ admits potentially combinatorial reduction in the sense of [CL16, Definition 10.1]. Such a model give a compactification of a Néron model, and the dual graph of the special fiber can be classified (see Theorem 5.13 and Proposition 5.14).

Our second main result is a generalization of [HW19a, Theorem 1.1] to the mixed characteristic case.

**Theorem 1.2.** (Theorem 4.6) Let $V$ be an excellent Dedekind scheme. Let $(X, \Delta)$ be a dlt pair, where $X$ is a $\mathbb{Q}$-factorial integral scheme which is flat and quasi-projective over $V$ of relative dimension two. Assume that there exists a projective birational morphism $\pi: X \to Z$ to a normal $\mathbb{Q}$-factorial variety $Z$ with $\text{Exc}(\pi) \subset [\Delta]$. Then we can run a $(K_{X/V} + \Delta)$-MMP over $Z$ which terminates with a minimal model.

The existence of dlt modifications and the inversion of adjunction for klt pairs follow from Theorem 1.2 as a corollary (see Corollary 4.9 4.10).

One of the key ingredients of the proofs of Theorems 1.1 and 1.2 is to prove the existence of pl-flips with ample divisor in the boundary. Indeed, all flips appearing in the proof of Theorem 1.2 are of this type and the existence of necessary flips for Theorem 1.1 is reduced by an argument in [HW19a] to the existence of flips of this type. In positive characteristic, [HW20, Theorem 1.3] proved the existence of pl-flips with ample divisor in the boundary using global $F$-regularity and the vanishing theorem up to Frobenius twist. We employ the same strategy in mixed characteristic, replacing global $F$-regularity with global $T$-regularity and the Frobenius morphism with alterations. The global $T$-regularity of a log pair $(X, \Delta)$ over an excellent Dedekind scheme $V$ is defined by the surjectivity of the map

$$H^0_{\omega_{Y/V}}([-\pi^*(K_{X/V} + \Delta)]) \to H^0(O_X)$$

induced by the Grothendieck trace map for every alteration $\pi: Y \to X$. The vanishing theorem up to alterations is obtained as a corollary of [Bha20, Theorem 6.28] (see Corollary 3.6). As a consequence, we obtain the following theorem which is an analog of [HW20, Theorem 1.3].

**Theorem 1.3.** (Theorem 3.30, cf. [HW20, Theorem 1.3]) Let $V$ be the spectrum of a complete discrete valuation ring. Let $(X, S + A + B)$ be a dlt pair such that $S$ is an anti-ample $\mathbb{Q}$-Cartier Weil divisor and $A$ is an ample $\mathbb{Q}$-Cartier Weil divisor. Let $f: X \to Z$ be a $(K_{X/V} + S + A + B)$-flipping contraction with $\rho(X/Z) = 1$ to an affine
$V$-scheme $Z$. Assume that $(S^N, (1 - \varepsilon)A_S + B_S)$ is globally $T$-regular for all $0 < \varepsilon < 1$ after localizing at all points of $f(\text{Exc}(f))$. Further assume that the ring

$$R(K_{S^N/V} + A_S + B_S) := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(O_X(m(K_{S^N/V} + A_S + B_S)))$$

is a finitely generated $O_Z$-algebra, where $B_S := \text{Diff}_{S^N}(B)$ and $A_S := A|_{S^N}$. Then the flip of $f$ exists.

Theorem 1.3 can be applied for three-dimensional pl-flips with ample divisor in the boundary. Indeed, the finite generation of $R(K_{S^N/V} + A_S + B_S)$ is a consequence of the MMP for excellent surfaces, and the global $T$-regularity of $(S^N, (1 - \varepsilon)A_S + B_S)$ follows from the inversion of adjunction for global $T$-regularity (see Proposition 3.22) and an argument similar to the proof of [HW19a, Lemma 3.3].

Remark 1.4. After completing this work, Jakub Witaszek told us that he and Bhargav Bhatt, Linquan Ma, Zsolt Patakfalvi, Karl Schwede, Kevin Tucker, and Joe Waldron are also writing a paper on MMP in mixed characteristics (see [BMP+20]). In their article, they independently show that the MMP holds for threefolds whose each residue characteristic is greater than 5. They also define and study the notion of global $+\$-regularity which is very closely related to our global $T$-regularity (see [BMP+20, Lemma 4.7]). Furthermore, we can obtain an analog of Theorem 1.2 and Theorem 1.3 in the case where $V$ is a spectrum of a regular excellent finite-dimensional domain by the argument in [BMP+20]. Indeed, we can show the results by using [BMP+20, Proposition 3.6] instead of Theorem 3.3 and the existence of regular alterations instead of the existence of Cohen-Macaulayfications.

Acknowledgements. The authors wish to express their gratitude to their supervisors Naoki Imai and Shunsuke Takagi for their encouragement, valuable advice, and suggestions. They thank Jakub Witaszek for valuable comments on the resolution in mixed characteristic and informing us of their work on the MMP in mixed characteristic. They are also grateful to Hiromu Tanaka, Kenta Sato, Kenta Hashizume, Yohsuke Matsuzawa, Ken Sato, Tatsuro Kawakami, Yoshinori Gongyo, Yuya Matsumoto, Tetsushi Ito, Teruhisa Koshikawa, and Sándor Kovács for their helpful comments and suggestions. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan. The first author was supported by JSPS KAKENHI Grant number JP19J22795. The second author was also supported by JSPS KAKENHI Grant number JP20J11886. Moreover, the authors thank the referee for careful proofreading and many comments.

2. Preliminaries

2.1. Notations. In this subsection, we summarize notations used in this paper.

- We will freely use the notation and terminology in [KM98] and [Kol13].
- A morphism of schemes is alteration if it is projective, surjective, and generically finite.
- A Noetherian scheme $X$ is locally irreducible if every connected component of $X$ is irreducible.
• A scheme $V$ is a Dedekind scheme if $V$ is a Noetherian excellent 1-dimensional regular scheme.

• For a Dedekind scheme $V$, we say $X$ is a variety over $V$ or a $V$-variety if $X$ is an integral scheme that is separated and of finite type over $V$. We say $X$ is a curve over $V$ or a $V$-curve (respectively a surface over $V$ or a $V$-surface) if $X$ is a $V$-variety of (absolute) dimension one (respectively two).

• Let $V$ be a Dedekind scheme. Let $\alpha : X \to V$ be a quasi-projective $V$-variety. The dualizing complex $\omega_{X/V}^*$ is defined by $\alpha^!\mathcal{O}_V$, where $\alpha^!$ is defined as in [Har66, Ch. III, Theorem 8.7]. The canonical sheaf $\omega_{X/V}$ is defined by $(-d)$-th cohomology $h^{-d}(\omega_{X/V}^*)$ of the dualizing complex, where $d$ is the integer such that $(-d)$-th cohomology is the lowest non-zero cohomology of $\omega_{X/V}^*$, thus there exists a natural map $\omega_{X/V}[d] \to \omega_{X/V}^*$. We note that if $\alpha$ is flat, then $d$ coincides with the relative dimension of $X$ over $V$. If $X$ is normal, there is a Weil divisor $K_{X/V}$, called a canonical divisor, such that $\omega_{X/V} \cong \mathcal{O}_X(K_{X/V})$. Note that $K_{X/V}$ is uniquely determined up to linear equivalence. We note that $\omega_{X/V}$ satisfies the condition $(S_2)$ by [Har07, Lemma 1.3]. If the image of $\alpha$ is a closed point, then we denote the induced morphisms by

$$X \xrightarrow{\beta} \text{Spec} \ k \xrightarrow{\theta} V.$$  

Since $\theta^!\mathcal{O}_V[1] \cong k$, we have $\omega_{X/V}[1] \cong \omega_{X/k}^*$. In particular, we have $\omega_{X/V} \cong \omega_{X/k}$. In this case, we denote $\omega_{X/V}[1], \omega_{X/V}, K_{X/V}$ by $\omega_X^*, \omega_X, K_X$, respectively, for simplicity.

• Let $V$ be a Dedekind scheme. We say that $(X, \Delta)$ is a log pair over $V$ if $X$ is a quasi-projective normal $V$-variety and $\Delta$ is an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_{X/V} + \Delta$ is $\mathbb{Q}$-Cartier. We will use the singularities of the MMP defined in [Kol13, Definition 2.8]. Let $S$ be a reduced divisor on $X$ such that $\Delta' := \Delta - S$ and $S$ have no common component. Then we denote the different of the pair $(X, \Delta)$ by $\text{Diff}_\mathbb{Q}(\Delta')$ defined by [Kol13, Definition 4.2], where $S^N$ is the normalization of $S$. We will freely use the adjunction results in [Kol13, Section 4].

• Let $V$ be a Dedekind scheme. Let $\pi : Y \to X$ be a proper morphism of $V$-schemes. The trace map $R\pi_*\omega_{Y/V}^* \to \omega_{X/V}^*$ is defined as the following. Applying $R\mathcal{H}\text{om}(\underline{\_}, \omega_{X/V}^*)$ to the natural map $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$, we have

$$R\mathcal{H}\text{om}(R\pi_*\mathcal{O}_Y, \omega_{X/V}^*) \to R\mathcal{H}\text{om}(\mathcal{O}_X, \omega_{Y/V}^*) \cong \omega_{X/V}^*.$$  

The left hand side is isomorphic to

$$R\pi_*R\mathcal{H}\text{om}(\mathcal{O}_Y, \omega_{Y/V}^*) \cong R\pi_*\omega_{Y/V}^*$$  

by the Grothendieck duality, thus we obtain the trace map $R\pi_*\omega_{Y/V}^* \to \omega_{X/V}^*$. If $X$ is of relative dimension $d$ and $\pi$ is an alteration, taking the $(-d)$-th cohomology and the composition with the natural map, we obtain the map $\pi_*\omega_{Y/V} \to \omega_{X/V}$ is also called the trace map by abuse of notation. Let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $X$ and we further assume that $X$ and $Y$ are
normal. Then we can extend the map \( \pi_\ast \omega_{Y/X}(\lceil \pi^*D \rceil)|_U \to \omega_{X/Y}(\lceil D \rceil)|_U \) on the regular locus \( U \) of \( X \) to the map \( \pi_\ast \omega_{Y/X}(\lceil \pi^*D \rceil) \to \omega_{X/Y}(\lceil D \rceil) \) on \( X \) because \( \omega_{X/Y}(\lceil D \rceil) \) satisfies the condition \((S_2)\).

- Let \( R \) be a discrete valuation ring and \( V = \text{Spec} R \). \( R \) is of characteristic \((0, p)\) if the fractional field \( K \) is of characteristic zero and the residue field \( k \) is of characteristic \( p > 0 \). Let \( X \) be a \( V \)-variety. The \emph{closed fiber} of \( X \) is \( X \times_V \text{Spec} k \) denoted by \( X_\eta \) and the \emph{generic fiber} of \( X \) is \( X \times_V \text{Spec} K \) denoted by \( X_\eta \).

2.2. Negativity lemma and finite generation of the Picard rank. In this subsection, we remark that the negativity lemma holds for the general setting. Originally, the negativity lemma follows from the Bertini’s theorem and the negativity lemma for surfaces. However, in general setting, the Bertini type theorem is much harder. Thus, we use the alternative proof by [BdFF12].

**Proposition 2.1.** (Negativity lemma) Let \( \pi : Y \to X \) be a projective morphism of Noetherian normal schemes. Let \( D \) be an \( R \)-Cartier \( R \)-Weil \( \pi \)-nef divisor on \( Y \). If \( \pi_\ast D \leq 0 \), then \( D \leq 0 \).

**Proof.** The proof of [BdFF12, Proposition 2.12] also works in our setting. Thus, we obtain an analogous statement of [BdFF12, Proposition 2.12]. The negativity lemma follows from this statement. \(\square\)

**Proposition 2.2.** Let \( X \to Y \) be a proper morphism of Noetherian schemes. Then, the relative Picard number \( \text{rank}_{\mathbb{R}} N^1(X/Y) \) is finite, where \( N^1(X/Y) \) is defined by

\[
(\text{Pic}(X)/\text{numerical equivalence over } Y) \otimes_{\mathbb{Z}} \mathbb{R}.
\]

**Proof.** The proof of [Kle66, IV, §4] also works in our setting. \(\square\)

2.3. Base change. In the proof of the existence of flips, we reduce to the case where \( V \) is the spectrum of a complete discrete valuation ring with an infinite residue field. To do this, we observe properties preserved under the base change via strictly henselization and completion. We note that \( \mathbb{Q} \)-factoriality is not preserved under the above base change.

**Lemma 2.3.** Let \( V \) be an excellent Dedekind scheme. Let \((X, \Delta)\) be a dlt pair. Then every component \( D \) of \( |\Delta| \) is normal up to universal homeomorphism if \( D \) is \( \mathbb{Q} \)-Cartier.

**Proof.** We may assume that \( D = |\Delta| \). The assertion follows from the same argument as in the proof of [HW20, Lemma 2.1]. \(\square\)

**Lemma 2.4.** Let \( V \) be the spectrum of an excellent discrete valuation ring. Let \( i : V' \to V \) be the completion of the strict henselization of \( V \). Let \((X, \Delta)\) be a dlt pair over \( V \) such that the dimension of \( X \) is less than or equal to three. Let \( S \) be a component of \( |\Delta| \) such that \( S \) is \( \mathbb{Q} \)-Cartier. Let \((X', \Delta')\) and \( S' \) be the base change of \((X, \Delta)\) and \( S \) via \( i \), respectively. Then \((X', \Delta')\) is dlt and \( S' \) is locally irreducible. In particular, every irreducible component of \( S' \) is \( \mathbb{Q} \)-Cartier.
Proof. We note that $\iota$ is the composition of formally étale morphism and completion. Via both base changes, being dlt and normality are preserved, we note that a strict henselization of an excellent local ring is also excellent [Gre79, Corollary 5.6]. Indeed, being dlt is characterized by using log resolution, which exists by [CJS09] and [CP19]. Furthermore, by [Tan18b, Lemma 2.5], relative canonical sheaf is compatible with such base changes. By Lemma 2.3 the normalization $S^N \to S$ is a universal homeomorphism. By the base change via $\iota$, we have $(S^N)' \to S'$, then $(S^N)'$ is also normal and this map is a universal homeomorphism. Since normal schemes are locally irreducible and locally irreducible is preserved by homeomorphisms, $S'$ is also locally irreducible. Since $S$ is $\mathbb{Q}$-Cartier, $S'$ is also $\mathbb{Q}$-Cartier. Being $\mathbb{Q}$-Cartier is a local property and $S'$ is locally irreducible, so every irreducible component is also $\mathbb{Q}$-Cartier. □

2.4. Alterations.

Proposition 2.5. Let $V$ be an excellent Dedekind scheme. Let $X$ be a $V$-variety and $x \in X$ be a point. Let $f_x: U' \to U := \text{Spec} \mathcal{O}_{X,x}$ be an alteration from an integral scheme. Then there exists an alteration $f: X' \to X$ of $V$-varieties such that we have the following Cartesian diagram:

$$
\begin{tikzcd}
U' \ar[r] \ar[d] & U \ar[d] \\
X' \ar[r] & X.
\end{tikzcd}
$$

Proof. Since $f_x$ is projective, $U'$ is embedded in some projective space $\mathbb{P}^N_\mathbb{P}$. It extends to some open subset of $X$, thus we may assume that $U$ is an open subset of $X$. We denote the closure of $U'$ in $\mathbb{P}^N_\mathbb{P}$ by $X'$. Then the natural morphism $X' \to \mathbb{P}^N_\mathbb{P} \to X$ is projective. Since $U'$ is integral and $\dim U' = \dim X$, the morphism $X' \to X$ is also an alteration. By the construction, this satisfies the desired conditions. □

Proposition 2.6. Let $V$ be an excellent Dedekind scheme. Let $X$ be a $V$-variety. Let $f_i: Y_i \to X$ be alterations of $V$-varieties for $i = 1, \ldots, r$. Then there exists an alteration $f: Y \to X$ of $V$-varieties which factors through $f_i$ for all $i$.

Proof. By induction on $i$, we may assume that $i = 2$. We denote the fiber product of $Y_1$ and $Y_2$ over $X$ by $Y' := Y_1 \times_X Y_2$. Take an irreducible component $Y$ of $Y'$ which dominates $X$, then $f: Y \to X$ is an alteration of $V$-varieties and $f$ factors through $f_i$. □

Proposition 2.7. Let $V$ be an excellent Dedekind scheme. Let $X$ be a $V$-variety and $S_1, \ldots, S_r$ be closed sub-$V$-varieties of $X$. Let $f_i: T_i \to S_i$ be an alteration of $V$-varieties. Then there exists an alteration $g: Y \to X$ of $V$-varieties and closed sub-$V$-varieties $S_{Y,1}, \ldots, S_{Y,r}$ of $Y$ with $g(S_{Y,i}) = S_i$ for all $i$ such that the induced morphism $g_{S,i}: S_{Y,i} \to S_i$ factors through $f_i$.

Proof. First, we consider the case of $r = 1$. We set $f := f_1$, $S := S_1$, and $T := T_1$. We denote the generic point of $S$ by $x$, then $\mathcal{O}_{X,x}$ is a local domain with residue field $K(S)$. Since $K(S) \subset K(T)$ is a finite extension of fields, it is a finite sequence of simple extensions. Thus we have a finite extension of domains $\mathcal{O}_{X,x} \to A$ such that the
residue field of some maximal ideal of $A$ coincides with $K(T)$. It extends to a finite surjective morphism $\pi: X' \to X$ such that it factors through $O_{X,x} \to A$ after localizing at $x$ by taking the normalization of $X$ in $K(A)$. In particular, there exists a closed sub-$V$-variety $S'$ in $X'$ such that $\pi(S') = S$ and there exists a rational dominant map $S' \to T$ over $S$. Then there exists a birational projective morphism $S'' \to S'$ with the blow-up ideal $I_S$ which eliminates the indeterminacy of $S' \to T$. We take an ideal $I$ of $O_X$ with $I \cdot O_S = I_S$, and we denote the blow-up of $X'$ in the ideal $I$ by $Y$. Then $g: Y \to X$ satisfies the desired conditions.

Next, we consider the general case. By the above case, we can find alterations $g_i: Y_i \to X$ and closed sub-$V$-schemes $S_{Y_i}$ of $Y_i$ such that $g(S_{Y_i}) = S_i$ and $g_{Y_i}: S_{Y_i} \to S_i$ factors through $f_i$. By Proposition 2.6, there exists $g: Y \to X$ which factors through $g_i: Y_i \to X$. Since $Y \to Y_i$ is surjective, there exists closed sub-$V$-variety $S_{Y,i}$ of $Y$ with $g(S_{Y,i}) = S_{Y_i}$. Therefore, $g$ and $S_{Y,i}$ satisfy the desired conditions.

**Proposition 2.8.** Let $V$ be an excellent Dedekind scheme. Let $X$ be a $V$-variety and $D$ be a Cartier divisor on $X$. Then for any integer $n$, there exists a finite cover $f: Y \to X$ such that $f^*D = nD'$ for some Cartier divisor $D'$ on $Y$.

**Proof.** By Proposition 2.5, we may assume that $D = \text{div}(\phi)$ for some non-zero section $0 \neq \phi \in K(X)$ and $X$ is affine by shrinking $X$. Let $f: Y \to X$ be the normalization of $X$ in $K(X)[\phi^{1/n}]$, then $g^*D = n\text{div}(\phi^{1/n})$. □

**Proposition 2.9.** Let $V$ be an excellent Dedekind scheme. Let $X$ be a $V$-variety and $S \subset X$ be a closed $V$-variety. Let $D$ be a Cartier divisor on $S$. Then for every positive integer $n$, there exists an alteration $f: Y \to X$ from a $V$-variety and closed sub-$V$-variety $S_Y$ of $Y$ with $f(S_Y) = S$ such that $f^*_SD = nD'$ for some Cartier divisor $D'$ on $S_Y$, where $f_S: S_Y \to S$ is the induced morphism.

**Proof.** By Proposition 2.8, there exists a finite cover $g: T \to S$ such that $g^*D = nD'$ for some Cartier divisor $D'$ on $T$. The assertion follows from Proposition 2.7. □

**Definition 2.10.** (strictly semi-stable)

1. Let $V$ be the spectrum of a discrete valuation ring $R$. Let $\varpi$ be a uniformizer of $R$. A flat $V$-variety $X$ of relative dimension $n$ is called strictly semi-stable if the following hold.
   - The generic fiber $X_\eta$ is smooth, where $\eta \in V$ is the generic point.
   - For any closed point $x$ in the special fiber $X_s$, there exists an open neighborhood $U$ of $x$ such that $U$ is étale over the scheme $\text{Spec } R[X_0, \ldots, X_n]/(X_0 \cdots X_m - \varpi)$ for some $m \leq n$.

   In this case, we also say that $(X, X_s)$ is a strictly semi-stable pair. As in [dJon96, p. 2.16] if $R$ has a perfect residue field, the above definition is equivalent to that $(X, X_s)$ is a simple normal crossing pair.

2. Let $V$ be a Dedekind scheme. An integral flat quasi-projective $V$-variety $X$ of relative dimension $n$ is called strictly semi-stable if $X_{O_{V,s}} \to \text{Spec } O_{V,s}$ is strictly semi-stable for any closed point $s \in V$. 
Then there exists a finite surjective morphism $V' \to V$ and an alteration $\phi: X' \to X$ from a $V'$-variety such that the pair $(X', \phi^{-1}(Z)_{\text{red}})$ is a strictly semi-stable pair, and in particular, $(X', \phi^{-1}(Z)_{\text{red}})$ is a simple normal crossing pair.

Remark 2.12. If $X$ is not flat over $V$, then $X$ is defined over a field. Thus, in this case, the last assertion of Theorem 2.11 follows from [dJon96, Theorem 4.1].

2.5. Adjunction and Bertini type theorem. For the proof of the existence of flips, we discuss the adjunction of singularities related to local irreducibility. The reader is referred to [Kol13, Section 4] for more details.

Proposition 2.13. Let $V$ be an excellent Dedekind scheme. Let $(X, S + A + B)$ be a dlt pair over $V$ such that $S$ and $A$ are locally irreducible Weil divisors and $|B| = 0$. Then $(S^N, \text{Diff}_{S^N}(A + B))$ is plt.

Proof. Let $\pi: S^N \to X$ be the composition of the normalization of $S$ and the closed immersion $S \to X$. Let $E$ be an exceptional prime divisor over $S^N$ centered at $x \in S^N$. If the log discrepancy $a_E(S^N, \text{Diff}_{S^N}(A + B))$ is equal to 0, then $\pi(x)$ is the generic point of a stratum of $|S + A|$. Since $E$ is exceptional, $\pi(x)$ is contained in at least three components of $S + A$. Since $S$ and $A$ are locally irreducible, this is the contradiction. □

Lemma 2.14. Let $V$ be the spectrum of a discrete valuation ring with an infinite residue field. Let $(X, \Delta)$ be a dlt surface over $V$. Let $H$ be an ample $\mathbb{Q}$-Cartier divisor. Then there exists an effective divisor $D \sim_\mathbb{Q} H$ such that $(X, \Delta + D)$ is also dlt.

Proof. First, we note that if $X$ is not flat over $V$, then $X$ is a surface over the residue field $k$ of $V$. Then the assertion is well-known. Thus, we may assume that $X$ is flat over $V$. We take a positive integer $m \geq 2$ such that $mH$ is very ample. If there exists an effective divisor $D' \sim mH$ such that $(X, \Delta + D')$ is dlt, then $\frac{1}{m}D'$ is what we want. Let $B_1$ be the sum of the components of $\Delta$ contained in the closed fiber $X_s$ and we write $B_2 := \Delta - B_1$. Then $B_2$ and $X_s$ have no common components. Let $\Sigma_s$ be the union of $B_2 \cap X_{s, \text{red}}$ and the non-regular locus of $X_{s, \text{red}}$. We note that $\Sigma_s$ is a finite set as $X_s$ is one-dimensional. Thus, for a general member $D$ of $|mH|_{X_s, \text{red}}$, $D$ has no intersection with $\Sigma_s$, the pair $(X_{s, \text{red}}, D)$ is a simple normal crossing pair and $D$ has no common components with $B_1$ by [FOV95, Corollary 3.4.14]. By the same argument, for a general member $D$ of $|mH|_{X_s}$, $D$ has no intersection with the non-regular locus of $B_2|_{X_s}$. By the proof of [JS12, Theorem 0], we find an effective divisor $D'$ such that $D' \sim mH$ and $D'|_{X_s}$ and $D'|_{X_{s, \text{red}}}$ satisfy the conditions as above. Thus $(X, \Delta + D')$ is dlt. Indeed, around the support of $D$, the pair $(X, \Delta + D)$ is a simple normal crossing pair. □

2.6. Rational singularities. In this subsection, we study properties of rational singularities. We will use Proposition 2.16 and 2.17 in Section 5 to prove the Cohen-Macaulayness of flips. The reader is referred to [Kov17] for more details.

Definition 2.15. ([Kov17]) Let $X$ be a Noetherian excellent scheme. $X$ has rational singularities if $X$ is normal and Cohen-Macaulay, and for every birational projective
We take a projective birational morphism

Proof. We take a projective birational morphism \( g : Y \to Z \) from a normal Cohen-Macaulay scheme \( Y \). We prove that the map \( \mathcal{O}_Z \to \mathcal{O}_Y \) is an isomorphism. By [Kov17, Lemma 7.4] and [Kov17, Theorem 8.6], we may assume that \( g \) factors through \( \pi \). We denote the induced morphism by \( f : Y \to X \). We note that \( f_* \mathcal{O}_Y = \mathcal{O}_X \) and \( \pi_* \mathcal{O}_X = \mathcal{O}_Z \). First we assume that \( X \) has rational singularities and \( \mathcal{O}_Z \simeq R\pi_* \mathcal{O}_X \), then we have \( \mathcal{O}_X \simeq Rf_* \mathcal{O}_Y \). By the spectral sequence, we have \( \mathcal{O}_Z \simeq R\pi_* \mathcal{O}_X \simeq Rg_* \mathcal{O}_Y \), so \( Z \) has rational singularities.

**Proposition 2.17.** Let \( \pi : X \to Z \) be a projective birational morphism from a normal klt scheme \( X \) to an excellent normal Cohen-Macaulay scheme \( Z \) admitting a normalized dualizing complex \( \omega^*_Z \). If \( X \) has rational singularities except for finitely many closed points and \( Z \) has rational singularities, then \( X \) has rational singularities.

Proof. We take a projective birational morphism \( f : Y \to X \) from a normal Cohen-Macaulay scheme \( Y \). We put \( g : \pi \circ f : Y \to Z \). First, we prove that

\[
\pi_* R^i f_* \mathcal{O}_Y = 0
\]

for all \( j > 0 \). We consider the spectral sequence

\[
E^{i,j}_2 = R^i \pi_* R^j f_* \mathcal{O}_Y \Rightarrow E^{i+j}_2 = R^i g_* \mathcal{O}_Y.
\]

Since \( \dim \text{Exc}(\pi) \leq 1 \), we have \( E^{i,j}_2 = 0 \) for \( i > 1 \), Therefore, we have \( \pi_* R^i f_* \mathcal{O}_Y = E^{i,j}_2 \simeq E^{0,j}_\infty \) for all \( j \), and \( E^{i+1,j}_\infty = 0 \) if \( i \neq 0,1 \). Thus, we have

\[
R^i g_* \mathcal{O}_Y = E^{i+1}_\infty \simeq E^{i,0}_\infty = \pi_* R^i f_* \mathcal{O}_Y
\]

for \( j > 1 \). Since \( Z \) has rational singularities, it is zero. Thus, it is enough to show that

\[
E^{i,j}_\infty = 0. \quad \text{Since} \quad E^{i,j}_\infty = 0 \quad \text{if} \quad i + j = 1 \quad \text{and} \quad i \neq 0,1, \quad \text{we obtain the exact sequence}
\]

\[
0 \to E^{0,1}_\infty \to E^{1,0}_\infty \to 0.
\]

Since \( Z \) has rational singularities, \( E^{1,0}_\infty = R^1 g_* \mathcal{O}_Y = 0 \). Therefore, we have \( E^{0,1}_\infty = \pi_* R^i f_* \mathcal{O}_Y = 0 \). Since \( X \) has rational singularities except for finite closed points, the support of \( R^i f_* \mathcal{O}_Y \) is isolated, so we have \( R^i f_* \mathcal{O}_Y = 0 \) for every \( i > 0 \). Thus, it is enough to show that \( X \) is Cohen-Macaulay. To do this, we prove the natural map \( Rf_* \omega_Y \to \omega_X \) is an isomorphism. Since \( X \) has rational singularities except for finite closed points, \( R^i f_* \omega_Y \) has the isolated support for \( i > 0 \) by [Kov17, Theorem 1.4]. Since \( X \) is klt, we have \( f_* \omega_Y \simeq \omega_X \). Thus, by replacing the structure sheaves into canonical sheaves in the above argument, we have the natural isomorphism \( Rf_* \omega_Y \simeq \omega_X \). By the Grothendieck duality

\[
Rf_* R\mathcal{H}om(\mathcal{O}_Y, \omega_Y^*) \simeq R\mathcal{H}om(Rf_* \mathcal{O}_Y, \omega_X^*),
\]
we have

\[ Rf_*\omega_Y \cong \omega_X. \]

Since \( Y \) is Cohen-Macaulay and \( f \) is birational, \( \omega_Y \) is locally isomorphic to the shift of \( \omega_Y \) over \( X \), so \( \omega_X \) is locally isomorphic to the shift of \( \omega_X \). Thus \( X \) is Cohen-Macaulay.

3. Existence of pl-flip with ample divisor in the boundary

In this section, we prove the existence of pl-flips with ample divisor in the boundary (cf. Theorem 3.30 and Corollary 3.34). In the first subsection, we study the vanishing theorem up to alterations. Next, we introduce the notion of global \( T \)-regularity and study properties of it, for example, the adjunction and the inversion of adjunction. Combining such arguments, we obtain the existence of flips in the special setting (cf. Theorem 3.30).

In this section, we basically work over a scheme \( V \), satisfying the following properties.

**Assumption 3.1.** \( V \) is the spectrum of a complete discrete valuation ring of characteristic \((0, p)\).

**Remark 3.2.** Let \( V \) be a scheme satisfying Assumption 3.1. Let \( X \) be a \( V \)-variety. Then it is possible that \( X \) is a variety over a field, and in such a case, \( X = X_s \) and \( X_\eta = \emptyset \), or \( X = X_\eta \) and \( X_s = \emptyset \).

3.1. Kodaira type vanishing up to alterations. Bhatt \[Bha20\] proved the vanishing of local cohomology up to finite covers in mixed characteristic. Using this theorem, we obtain the Kodaira type vanishing up to alterations for semiample and big divisors (Corollary 3.6). This theorem plays an essential role to prove the existence of flips.

**Theorem 3.3.** (\[Bha20\], Theorem 6.28, cf. \[Bha10\], Proposition 5.5.3) Let \( V \) be a scheme satisfying Assumption 3.1. Let \( f : X \rightarrow Z \) be a projective surjective morphism of \( V \)-varieties to an affine scheme \( Z \). Let \( L \) be a semiample and \( f \)-big line bundle on \( X \). Let \( z \in Z \) be a closed point with residue characteristic \( p > 0 \). Then there exists a finite surjective morphism \( \pi : Y \rightarrow X \) from a \( V \)-variety such that

\[ R\Gamma (\pi, L^{-1}) \rightarrow R\Gamma (\pi, *L^{-1}), \]

is zero on \( h^i \) for all \( i < \dim (X_p) \), where \( X_p \) and \( Y_p \) are closed subscheme of \( X \) and \( Y \) defined by \( p = 0 \).

**Proof.** If \( X \) is flat over \( V \), it is \[Bha20\], Theorem 6.28. Otherwise, it follows from a similar argument to the argument in the proof of \[Bha10\], Proposition 5.5.3. \( \square \)

**Remark 3.4.** If \( X \) is flat over \( V \), the relative dimension of \( X \) over \( V \) coincides with \( \dim (X_p) \) if the closed fiber is non-empty.

**Proposition 3.5.** Let \( V \) be a scheme satisfying Assumption 3.1. Let \( f : X \rightarrow Z \) be a projective surjective morphism from a flat \( V \)-variety \( X \) to an affine flat \( V \)-variety \( Z \). We assume that the closed fiber of \( Z \rightarrow V \) is non-empty. Let \( L \) be a semiample and \( f \)-big line bundle on \( X \). Then, for every positive integer \( m \), there exists a finite
surjective morphism $\pi : Y \to X$ from a $V$-variety such that the image of the following map

$$\text{Tr}_z^i : R^{i-d}\Gamma(\omega_{Y/V}(\pi^*L)) \to R^{i-d}\Gamma(\omega_{X/V}(L)),$$

is contained in $\varpi^m R^{i-d}\Gamma(\omega_{X/V}(L))$ for all $i > 0$, where $d$ is the relative dimension of $X$ over $Z$ and $\varpi$ is a uniformizer of $V$.

**Proof.** We take a closed point $z \in Z_p$. We set $A := \mathcal{O}_{Z,z}$. Let $E$ be the injective hull of the residue field of $A$. By Theorem 3.3 there exists a finite surjective morphism $\pi : Y \to X$ from a $V$-variety such that

$$h^i R\Gamma_z R\Gamma(X_p, L^{-1}) \to h^i R\Gamma_z R\Gamma(Y_p, \pi^* L^{-1})$$

is zero for all $i < d$. It is enough to show that $\pi$ satisfied the desired condition around $z$, because any finitely many finite covers from $V$-varieties are factored by some finite cover from a $V$-variety. We take base changes via $\text{Spec} \mathcal{O}_{Z,z} \to Z$ and we use the same notations by abuse of notations. We may replace $Z$ with $\text{Spec} \mathcal{O}_{Z,z}$. By the local duality and the Grothendieck duality, we have

$$R \text{Hom}_A(R\Gamma_z R\Gamma(X_p, L^{-1}), E) \simeq R\Gamma \omega_{X_p/V}(\hat{L})[1],$$

where the right hand side is the completion of the 1-shift of $R\Gamma \omega_{X_p/V}(L)$. By the same argument for $Y_p$, we have the following diagram;

$$\begin{array}{ccc}
R \text{Hom}_A(R\Gamma_z R\Gamma(Y_p, \pi^* L^{-1}), E) & \longrightarrow & R \text{Hom}_A(R\Gamma_z R\Gamma(X_p, L^{-1}), E) \\
\simeq & & \simeq \\
R\Gamma \omega_{X_p/V}(\pi^* \hat{L})[1] & \longrightarrow & R\Gamma \omega_{X_p/V}(\hat{L})[1].
\end{array}$$

Taking the $(i - d)$-th cohomology, we obtain that the trace map

$$\text{Tr}_p^i : R^{i+1-d} \Gamma(\omega_{Y_p/V}(\pi^* L)) \to R^{i+1-d} \Gamma(\omega_{X_p/V}(L)),$$

is zero for all $i > 0$. Next, we consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X_p} \to 0,$$

and applying $R\Gamma R\mathcal{H}om(-, \omega_{X/V}(L))$ and taking $(i - d)$-th cohomology, we have the exact sequence

$$R^{i-d} \Gamma(\omega_{X/V}(L)) \to R^{i-d} \Gamma(\omega_{X/V}(L)) \to R^{i+1-d} \Gamma(\omega_{X_p/V}(L)).$$

Thus, we have the commutative diagram of exact sequences

$$\begin{array}{ccc}
R^{i-d} \Gamma(\omega_{X/V}(L)) & \longrightarrow & R^{i-d} \Gamma(\omega_{X/V}(L)) \\
\rightarrow & & \rightarrow \\
R^{i-d} \Gamma(\omega_{Y/V}(\pi^* L)) & \longrightarrow & R^{i-d} \Gamma(\omega_{Y/V}(\pi^* L)) \to R^{i+1-d} \Gamma(\omega_{Y_p/V}(\pi^* L)).
\end{array}$$
Therefore, Im(Tr^i) is contained in pR^{i-d}Γ(ω^*_X/Y(L)). By the same argument for Y, there exists a finite surjective morphism g: W → Y such that Im(Tr^i) is contained in pR^{i-d}Γ(ω^*_Y/V(π* L)), thus we have

\[ \text{Im}(Tr^i_{\pi g}) \subseteq p^2R^{i-d}Γ(ω^*_X/V(L)). \]

Repeating such a process, we obtain a finite surjective morphism of V-varieties such that the image of the trace map is contained in \( p^mR^{i-d}Γ(ω^*_X/V(L)). \) Since \( p \) is a multiple of \( \varpi \), we have the assertion.

\[ \square \]

**Corollary 3.6.** (cf. [BST15, Theorem 5.5]) Let \( V \) be a scheme satisfying Assumption 3.1. Let \( f: X \to Z \) be a projective surjective morphism from a normal flat V-variety \( X \) to an affine flat V-variety \( Z \). Let \( D \) be a semiample \( \pi \)-big Cartier divisor on \( X \). Then there exists an alteration \( \pi: Y \to X \) such that the trace map

\[ \text{Tr}^i_{\pi}: R^{i-d}Γ(ω^*_Y/V(π* D)) \to R^{i-d}Γ(ω^*_X/V(D)) \]

is zero for all \( i > 0 \), where \( d \) is the relative dimension of \( X \) over \( V \). Furthermore, if \( X \) is Cohen-Macaulay, there exists an alteration \( \pi: Y \to X \) such that the trace map

\[ \text{Tr}^i_{\pi}: H^i(ω_Y/V(π* D)) \to H^i(ω_X/V(D)) \]

is zero for all \( i > 0 \).

**Proof.** We may assume that \( X \) is regular by taking an alteration by Theorem 2.11. Then the generic fiber \( X_\eta \) is a variety over a field of characteristic zero and \( D_\eta \) is semiample and big over \( Z \), thus we have

\[ R^{i-d}Γ(ω^*_X/V(D))_\eta = 0 \]

by Kawamata-Viehweg vanishing. Therefore, we obtain the case where the closed fiber of \( X \) is empty. Thus, we assume that the closed fiber of \( X \) is non-empty. Furthermore, \( R^{i-d}Γ(ω^*_X/V(D)) \) is a \( \varpi \)-torsion finite \( O_Z \)-module. Thus, for some positive integer \( m \), we have

\[ \varpi^mR^{i-d}Γ(ω^*_X/V(D)) = 0 \]

By Proposition 3.5, there exists a finite surjective morphism \( \pi: Y \to X \) from a V-variety such that the image of the following map

\[ \text{Tr}^i_{\pi}: R^{i-d}Γ(ω^*_Y/V(π* D)) \to R^{i-d}Γ(ω^*_X/V(D)), \]

is contained in \( \varpi^mR^{i-d}Γ(ω^*_X/V(D)) = 0 \) for all \( i > 0 \). Thus, we obtain the first assertion.

Next, we assume that \( X \) is Cohen-Macaulay. By the first assertion, there exists an alteration \( \pi: Y \to X \) such that the trace map

\[ \text{Tr}^i_{\pi}: R^{i-d}Γ(ω^*_Y/V(π* D)) \to R^{i-d}Γ(ω^*_X/V(D)) \]

is zero for all \( i > 0 \). Since \( X \) is Cohen-Macaulay, we have \( R^{i-d}Γ(ω^*_X/V(D)) \cong H^i(ω_X/V(D)) \). Thus, the trace map

\[ H^i(ω_Y/V(π* D)) \to R^{i-d}(ω^*_Y/V(π* D)) \to H^i(ω_X/V(D)) \]

is zero for all \( i > 0 \). \[ \square \]
Remark 3.7. In positive characteristic, an analogous statement of Corollary 3.6 holds and we can take $Y$ in Corollary 3.6 as a finite cover by $\text{BST15}$, Theorem 5.5.

3.2. Global $T$-regularity. In positive characteristic, global $F$-regularity is important in the proof of the existence of flips (see $\text{HX15}$, $\text{HW19a}$, $\text{HW20}$). In mixed characteristic, we use global $T$-regularity instead of it.

Definition 3.8. Let $\pi: Y \to X$ be an alteration of normal schemes.

- For a prime divisor $E$ on $X$, a prime divisor $E_Y$ is called a strict transform of $E$ if $\pi(E_Y) = E$.
- For an $\mathbb{R}$-Weil divisor $D = \sum_i a_i E_i$, where $E_i$ is a prime divisor, an $\mathbb{R}$-Weil divisor $D_Y$ is called a strict transform of $D$ if $D_Y$ is denoted by $D_Y = \sum_i a_i E_{Y,i}$ such that each $E_{Y,i}$ is a strict transform of $E_i$.

Remark 3.9.

- Since $\pi$ is an alteration, $\pi|_{E_Y}$ is also an alteration.
- If $D$ is $\mathbb{Q}$-Weil or $\mathbb{Z}$-Weil, then so is $D_Y$.
- If $D$ is a locally irreducible reduced divisor, then so is $D_Y$.
- If $\pi$ is birational, then $D_Y$ is the strict transform of $D$ in the usual sense.

Definition 3.10. (Globally $T$-regular, Purely globally $T$-regular) Let $V$ be an excellent Dedekind scheme. Let $(X, \Delta)$ be a log pair, or a localization of a log pair over $V$.

- Let $L$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $X$. Then the submodule $T^0(X, \Delta; L)$ of $H^0(\mathcal{O}_X([L]))$ is defined by

$$\bigcap_{\pi: Y \to X} \text{Im}(H^0(\omega_{Y/V}([\pi^*(L - K_{X/V} - \Delta})))) \to H^0(\mathcal{O}_X([L])), $$

where the map is a composition of a trace map and the natural injection and $\pi$ runs over all alterations from a normal $V$-variety.
- The pair $(X, \Delta)$ is globally $T$-regular (globally Trace-regular) if for every alteration $\pi: Y \to X$ from a normal $V$-variety, the trace map

$$H^0(\omega_{Y/V}([\pi^*(K_{X/V} + \Delta}])) \to H^0(\mathcal{O}_X)$$

is surjective, that is, $T^0(X, \Delta) := T^0(X, \Delta; 0) = H^0(\mathcal{O}_X)$.

- We set $|\Delta| = S$. Then $(X, \Delta)$ is purely globally $T$-regular if for every alteration $\pi: Y \to X$ from a normal $V$-variety and every strict transform $S_Y$ of $S$ via $\pi$, the trace map

$$H^0(\omega_{Y/V}(S_Y + [\pi^*(K_{X/V} + \Delta}))) \to H^0(\mathcal{O}_X)$$

is surjective. We note that the map in (3.2) is well-defined. Indeed, it is enough to show that the map

$$\pi_*\omega_{Y/V}(S_Y + [\pi^*(K_{X/V} + \Delta}]) \to \mathcal{O}_X$$

is well-defined on the regular locus of $X$. On the regular locus of $X$, $S_Y \leq \pi^*S$, so the left hand side is contained in

$$\omega_Y([\pi^*(K_{X/V} + \Delta}]),$$

where $\Delta' := \Delta - S$. 

Proof. We fix a pair with Proposition 3.15. Let a projective morphism $F$ denote the strict transform of $S$. Then $\pi$ is defined as in Proposition 3.11, and they proved the equivalence of the definitions in positive characteristic. However, in characteristic zero, it does not hold in general. Indeed, let $X$ be a normal non-klt variety over a field of characteristic zero. By Proposition 3.14, $X$ is not globally $T$-regular, so $T^0$ is not $H^0(O_X)$. On the other hand, since every finite cover of normal variety splits, the right hand side in Proposition 3.11 is $H^0(O_X)$.\phantomsection
\begin{proposition}
\label{prop:3.11}
Let $X$ be a quasi-projective variety over an $F$-finite field. Let $(X, \Delta)$ be a log pair and $L$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $X$. Then
\[ T^0(X, \Delta; L) = \bigcap_{\pi} \text{Im}(H^0(\omega_Y([\pi^*(L - K_X - \Delta)])) \to H^0(O_X([L])), \]
where $\pi$ runs over all finite covers from a normal $V$-variety.
\end{proposition}

Remark 3.12. In [BST15], $T^0$ is defined as in Proposition 3.11 and they proved the equivalence of the definitions in positive characteristic. However, in characteristic zero, it does not hold in general. Indeed, let $X$ be a normal non-klt variety over a field of characteristic zero. By Proposition 3.14, $X$ is not globally $T$-regular, so $T^0$ is not $H^0(O_X)$. On the other hand, since every finite cover of normal variety splits, the right hand side in Proposition 3.11 is $H^0(O_X)$.\phantomsection

\begin{proposition}
\label{prop:3.13}
Let $V$ be a scheme satisfying Assumption 3.7. Let $f : X \to Z$ be a projective surjective morphism of $V$-varieties. Let $(X, \Delta)$ be log pair over $V$. If $(X, \Delta)$ is globally $T$-regular, then so is $(X', \Delta')$, where $(X', \Delta')$ is one of the following.
\begin{itemize}
\item A restriction of $(X, \Delta)$ over on some open subset $X'$ of $X$, or
\item A localization of $(X, \Delta)$ at some point of $Z$.
\end{itemize}
Furthermore, the same assertion holds for purely globally $T$-regular log pairs.
\end{proposition}

Proof. We can extend every alteration of $X'$ to one of $X$ by Proposition 3.15, and the surjectivity is equivalent to the property that image of the trace map contains $1 \in H^0(O_X)$. It does not change after localization or restriction. \hfill $\square$\phantomsection

\begin{proposition}
\label{prop:3.14}
Let $V$ be an excellent Dedekind scheme. Let $(X, \Delta)$ be a log pair over $V$. If $(X, \Delta)$ is globally $T$-regular, then it is klt, and in particular, $|\Delta| = 0$. If $(X, \Delta)$ is purely globally $T$-regular, then it is plt, and in particular, $|\Delta|$ is reduced.
\end{proposition}

Proof. By Proposition 3.13 we may assume that $X$ is affine. We take a birational projective morphism $\pi : Y \to X$ from a normal $V$-variety. If $(X, \Delta)$ is globally $T$-regular, then we have $\pi_*\omega_{Y/V}([-\pi^*(K_{X/V} + \Delta)]) = O_X$, as trace map is injective in the birational case. Thus, $(X, \Delta)$ is klt. If $(X, \Delta)$ is purely globally $T$-regular, we denote $S := |\Delta|$, then we have $\omega_{Y/V}(S_Y + [-\pi^*(K_{X/V} + \Delta)]) = O_X$, where $S_Y$ is the strict transform of $S$. Thus, $(X, \Delta)$ is plt. \hfill $\square$\phantomsection

\begin{proposition}
\label{prop:3.15}
Let $V$ be an excellent Dedekind scheme. Let $(X, S + \Delta)$ be a log pair with $|S + \Delta| = S$. Assume that $(X, S + \Delta)$ is purely globally $T$-regular and $S$ is $\mathbb{Q}$-Cartier. Then $(X, (1 - \varepsilon)S + \Delta)$ is globally $T$-regular for all $0 < \varepsilon < 1$ with $\varepsilon \in \mathbb{Q}$.
\end{proposition}

Proof. We fix a $0 < \varepsilon < 1$ and a positive integer $m$ with $m\varepsilon > 1$. We note that for every alteration $\mu : W \to X$, we can find an alteration $\pi : Y \to X$ and a strict transform $S_Y$.
such that $\pi^*S \geq mS_Y$ and $\pi$ factors through $\mu: W \to X$ by applying Proposition 2.8 to $W$. Indeed, we take a positive integer $r$ such that $r\mu^*S$ is Cartier. By Proposition 2.8, we take a finite cover $\pi': Y \to W$ such that $r\pi^*S = mrD$ for some Cartier divisor $D$ on $Y$, where $\pi = \pi' \circ \mu$. We take a strict transform $S_Y$ of $S$ with $S_Y \leq D$. Then we have $mS_Y \leq mD = \pi^*S$. We take an alteration $\pi: Y \to X$ from a normal $V$-variety and a strict transform $S_Y$ of $S$ such that $\pi^*S \geq mS_Y$ and $\pi^*(K_X + \Delta)$ is Cartier, the existence follows from Proposition 2.8. It is enough to show that the trace map

$$H^0(\omega_{Y/V}([-\pi^*(K_{X/V} + (1-\varepsilon)S + D)])) \to H^0(\mathcal{O}_X)$$

is surjective. Since $\pi^*(1-\varepsilon)S \leq \pi^*S - \varepsilon mS_Y \leq \pi^*S - S_Y$, the left hand side is larger than the left hand side of the map (3.2). Thus, if $(X, S + \Delta)$ is purely globally $T$-regular, then $(X, (1-\varepsilon)S_Y + \Delta)$ is globally $T$-regular. □

Proposition 3.16. Let $V$ be a scheme satisfying Assumption 3.1. Let $f: X \to Z$ be a projective surjective morphism from a normal $V$-variety $X$ to an affine $V$-variety $Z$. Let $(X, \Delta)$ be a globally $T$-regular log pair over $V$. Then the following conditions hold.

1. For every alteration $\pi: Y \to X$ from a normal $V$-variety, the trace map

$$\pi_*\omega_{Y/V}([-\pi^*(K_{X/V} + \Delta)]) \to \mathcal{O}_X$$

is a splitting surjection.

2. For every finite surjective morphism $\pi: Y \to X$ from a normal $V$-variety, the canonical map

$$\mathcal{O}_X \to \pi_*\mathcal{O}_Y([-\pi^*\Delta])$$

splits.

Proof. By the equation (3.1), there exists a global section

$$\alpha \in H^0(\omega_{Y/V}([-\pi^*(K_{X/V} + \Delta)]))$$

mapped to $1 \in H^0(\mathcal{O}_X)$. The section $\alpha$ defines a morphism

$$\mathcal{O}_X \to \pi_*\omega_{Y/V}([-\pi^*(K_{X/V} + \Delta)])$$

mapping $1$ to $\alpha$, thus it gives a splitting and we obtain (1). Applying $\mathcal{H}om(\underline{\_}, \mathcal{O}_X)$ for the trace map in (1), we obtain the canonical map in (2) when $\pi$ is finite surjective, thus we have (2). □

Lemma 3.17. Let $V$ be a scheme satisfying Assumption 3.1. Let $(X, \Delta)$ be a globally $T$-regular log pair over $V$. Let $D$ be a Weil $\mathbb{Q}$-Cartier divisor on $X$. Let $f: Y \to X$ be a finite surjective morphism from a normal $V$-variety such that $f^*D$ is Cartier. Then $\mathcal{O}_X(D) \to f_*\mathcal{O}_Y(f^*D)$ splits.

Proof. By Proposition 3.16 (2), $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits. Thus, $\mathcal{O}_X(D) \to f_*\mathcal{O}_Y(f^*D)$ splits on the regular locus of $X$. Since $\mathcal{O}_X(D)$ and $f_*\mathcal{O}_X(f^*(D))$ are reflexive, the splitting extends to the splitting on $X$. □

Proposition 3.18. Let $V$ be a scheme satisfying Assumption 3.1. Let $f: X \to Z$ be a projective surjective morphism from a normal $V$-variety $X$ to an affine $V$-variety $Z$. Let $(X, \Delta)$ be a globally $T$-regular log pair over $V$. Let $D$ be a Weil $\mathbb{Q}$-Cartier divisor. Then the following hold.
(1) If $D$ is semiample, then $(R^i f_* (\mathcal{O}_X(D)))_x = 0$ for all $x \in Z$ and $i > 0$, where $Z$ is a closed fiber.

(2) $\mathcal{O}_X(D)$ is maximal Cohen-Macaulay.

(3) If $D$ is semiample and $f$-big, then $H^i(\omega_{X/V}(D)) = 0$ for all $i > 0$.

Proof. By [Bha20, Theorem 6.28], there exists a finite surjective morphism $\pi: Y \to X$ such that $\pi^* D$ is Cartier and
\[
H^i(\mathcal{O}_{X, p}(D)) \to H^i(\mathcal{O}_{Y, p}(\pi^* D))
\]
is zero for all $i > 0$. Indeed, the assertion is reduced to the case where $D$ is Cartier. By Lemma 3.17, we have $H^i(\mathcal{O}_{X, p}(D)) = 0$. Thus, we have the map
\[
H^i(\mathcal{O}_X(D)) \xrightarrow{\pi} H^i(\mathcal{O}_X(D))
\]
is surjective for all $i > 0$. Thus, By Nakayama’s Lemma, $H^i(\mathcal{O}_X(D)) = 0$ at all points of $Z$.

Next, we consider (2). Since the assertion (2) is a local question, we may assume that $f$ is the identity map. By [Bha20, Theorem 6.28], for a closed point $x \in X$, there exists a finite surjective morphism $\pi: Y \to X$ such that $\pi^* D$ is Cartier and the map
\[
R^i \Gamma_x R\Gamma(\mathcal{O}_{X, p}(-D)) \to R^i \Gamma_x R\Gamma(\mathcal{O}_{Y, p}(-\pi^* D))
\]
is zero for all $i < \dim X$. By Lemma 3.17, we have $R^i \Gamma_x R\Gamma(\mathcal{O}_{X, p}(-D)) = 0$. Therefore, $\mathcal{O}_X(D)$ is maximal Cohen-Macaulay around $X$. Since $(X, \Delta)$ is klt, $\mathcal{O}_X(D)$ is maximal Cohen-Macaulay on the generic fiber by [KM98, Corollary 5.25], thus we obtain the assertion (2).

Finally, we consider (3). By [Bha20, Theorem 6.28], for a closed point $z \in Z$, there exists a finite surjective morphism $\pi: Y \to X$ such that $\pi^* D$ is Cartier and the map
\[
R^i \Gamma_z R\Gamma(\mathcal{O}_{X, p}(-D)) \to R^i \Gamma_z R\Gamma(\mathcal{O}_{Y, p}(-\pi^* D))
\]
is zero for all $i < \dim X$. By Lemma 3.17, we have $R^i \Gamma_z R\Gamma(\mathcal{O}_{X, p}(-D)) = 0$. By the argument of the proof of Proposition 3.5, we have
\[
H^i(\mathcal{H}om(\mathcal{O}_{X, p}(-D), \omega_{X, p/V})) = 0
\]
for all $i > 0$. Thus, we have that
\[
H^i(\omega_{X/V}(D)) \xrightarrow{\pi} H^i(\omega_{X/V}(D))
\]
is surjective for all $i > 0$. Thus, $H^i(\omega_{X/V}(D)) = 0$ around $Z$. By Kawamata-Viehweg vanishing for the generic fiber $X$, the vanishing holds on $Z$. Thus we obtain the assertion (3). \square

**Proposition 3.19.** Let $V$ be an excellent Dedekind scheme. Let $g: Y \to X$ be a projective birational morphism of normal $V$-varieties. Let $(X, \Delta)$ and $(Y, \Gamma)$ be log pairs such that $g^*(\Delta) = K_Y + \Gamma$. Then $(X, \Delta)$ is globally $T$-regular if and only if $(Y, \Gamma)$ is globally $T$-regular. The same assertion holds for purely globally $T$-regular case if $[\Gamma]$ is the strict transform of $[\Delta]$. 

Proof. First, we consider the globally $T$-regular case. We note that $(X, \Delta)$ is klt if and only if $(Y, \Gamma)$ is klt. By Proposition 3.11, we may assume that $(X, \Delta)$ and $(Y, \Gamma)$ are klt, and in particular, $[\Gamma] = 0$. Thus, the trace map coincides with the following isomorphism

$$H^0(\omega_{Y/V}([-g^*(K_{X/V} + \Delta)]) = H^0(\mathcal{O}_Y) \simeq H^0(\mathcal{O}_X).$$

Take an alteration $\pi: W \to Y$. Then the composition of trace maps

$$H^0(\omega_{W/V}([-\pi^*(K_{Y/V} + \Gamma)])) \to H^0(\mathcal{O}_Y) \simeq H^0(\mathcal{O}_X)$$

is the trace map with respect to $g \circ \pi$. Then the surjectivities of two trace maps are equivalent to each other.

Next, we consider the purely globally $T$-regular case. We denote $[\Delta]$ and $[\Gamma]$ by $S$ and $T$, respectively. Then the corresponding trace map coincides with natural isomorphic $H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_X)$. Indeed, $[K_{Y/V} + T - g^*(K_{X/V} + \Delta)] = [T - \Gamma] = 0$. Thus, by the same argument as above, we obtain the equivalence. \hfill \Box

Lemma 3.20. Let $V$ be a scheme satisfying Assumption [3.7]. Let $(X, S + B)$ be a log pair over $V$ such that $S$ is a reduced divisor and $S$ and $B$ have no common components. Let $\pi: X' \to X$ be an alteration of normal $V$-varieties and $S'$ be a strict transform of $S$ on $X'$. Then there exists an alteration $f: Y \to X$ from a normal $V$-variety and a strict transform $S_Y$ of $S$ such that the following hold.

- $f$ factors through $\pi$ and $S_Y$ is a strict transform of $S'$,
- $S_Y$ is locally irreducible,
- $(Y, S_Y)$ is a simple normal crossing pair,
- $f^*(K_{X/V} + S + B)$ and $f^!_S(K_{S_N/V} + B_S)$ are Cartier, and
- The following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{O}_X & \longrightarrow & \mathcal{O}_X \\
\downarrow & & \downarrow \\
j_* f_* f^!_S \omega_{S_Y/V}(-f^!_S(K_{S_N/V} + B_S)) & \longrightarrow & j_* \mathcal{O}_{S_N},
\end{array}
$$

where $B_S$ is the different of $(X, S + B)$, $f_S: S_Y \to S_N$ is the induced morphism, $j: S_N \to X$ is the composition of the normalization and the inclusion, the left vertical map follows from the adjunction formula and horizontal maps are induced by trace maps.

Proof. By the definition of different, we have $(K_{X/V} + S + B)|_{S_N} \sim_Q K_{S_N/V} + B_S$. By Theorem 2.11 there exists an alteration $f: Y \to X$ and a strict transform $S_Y$ of $S$ such that $(Y, S_Y)$ is a simple normal crossing pair and $f$ factors through $\pi$. By taking blowing up along the stratum of $S_Y$, we may assume that $S_Y$ is locally irreducible, thus $S_Y$ is regular. In particular, $S_Y \to S$ factors through the normalization $S_N \to S$ denoted by $f_S: S_Y \to S_N$. By Proposition 2.9 we may assume that $f^*(K_{X/V} + S + B)$ and $f^!(K_{S_N/V} + B_S)$ is Cartier and

$$f^*(K_{X/V} + S + B)|_{S_Y} \sim f^!_S((K_{X/V} + S + B)|_{S_N}) \sim f^!_S(K_{S_N/V} + B_S).$$
Thus, we define the morphism \( \omega_{Y/V}(S_Y - f^*(K_{X/V} + S + B)) \) induced by the adjunction formula \( \omega_{Y/V}(S_Y)|_{S_Y} \simeq \omega_{S_Y/V} \). Then it is enough to show that the diagram

\[
\begin{array}{ccc}
f_*\omega_{Y/V}(S_Y - f^*(K_{X/V} + S + B)) & \to & \mathcal{O}_X \\
\downarrow & & \downarrow \\
j_*j_*\omega_{S_Y/V}(-f_*^*(K_{S_Y/V} + B_S)) & \to & j_*\mathcal{O}_{S_N},
\end{array}
\]

commutes. Since \( \mathcal{H}om(f_*\omega_{Y/V}(S_Y - f^*(K_{X/V} + S + B)), j_*\mathcal{O}_{S_N}) \) is torsion-free as \( \mathcal{O}_{S_N} \)-module, it is enough to show that the above diagram commutes at every generic point of \( S \). Thus, we may assume that \( (X, S) \) is a simple normal crossing pair and \( B = 0 \). By the construction of the residue map \( \omega_{X/V}(S) \to \omega_{S/V} \) and the trace map, we have the commutative diagram

\[
\begin{array}{ccc}
f_*\omega_{Y/V}(S_Y) & \to & \omega_{X/V}(S) \\
\downarrow & & \downarrow \\
f_*\omega_{S_Y/V} & \to & \omega_{S/V}.
\end{array}
\]

Applying \( \otimes \mathcal{O}_X(-(K_{X/V} + S)) \), we have the assertion. \( \square \)

**Proposition 3.21.** (Adjunction for global \( T \)-regularity) Let \( V \) be a scheme satisfying Assumption \( 3.1 \). Let \( f: X \to Z \) be a projective surjective morphism from a normal \( V \)-variety \( X \) to an affine \( V \)-variety \( Z \) and \( (X, S + B) \) a purely globally \( T \)-regular pair with \( |S + B| = S \). Then \( (S^N, B_S) \) is globally \( T \)-regular, where \( B_S = \text{Diff}_{S^N}(B) \) is the different of \( (X, S + B) \).

**Proof.** We take an alteration \( g: T \to S^N \). Combining Proposition \( 2.7 \) and Lemma \( 3.20 \), there exists an alteration \( f: Y \to X \) and a strict transform \( S_Y \) of \( S \) as in Lemma \( 3.20 \) and \( f_S: S_Y \to S^N \) factors through \( g \). Then we have the commutative diagram

\[
\begin{array}{ccc}
H^0(\omega_{Y/V}(S_Y - f^*(K_{X/V} + S + B))) & \to & H^0(\mathcal{O}_X) \\
\downarrow & & \downarrow \\
H^0(\omega_{S_Y/V}(-f_*^*(K_{S_Y/V} + B_S))) & \to & H^0(\mathcal{O}_{S^N}).
\end{array}
\]

Thus, the image of the bottom map contains \( 1 \in H^0(\mathcal{O}_{S^N}) \), and in particular, it is surjective. \( \square \)

**Proposition 3.22.** (Inversion of Adjunction for global \( T \)-regularity) Let \( V \) be a scheme satisfying Assumption \( 3.1 \). Let \( f: X \to Z \) be a projective surjective morphism from a normal \( V \)-variety \( X \) to an affine \( V \)-variety \( Z \) and \( (X, S + \Delta) \) a log pair such that \( |S + \Delta| = S \) is reduced and \( S \) has no common component with \( \Delta \). Let \( X \to Z' \to Z \) be the Stein factorization and we denote the induced morphisms by \( f': X \to Z' \) and \( \phi: Z' \to Z \). Let \( z \in Z \) be a point with \( \phi^{-1}(z) \subset f'(S) \). We assume that \( -(K_{X/V} + S + \Delta) \) is semiample and \( f \)-big. If \( (S^N, D) \) is globally \( T \)-regular over \( z \), then \( (X, S + \Delta) \) is
purely globally $T$-regular over $z$ and $S$ is normal, where $S^N$ is the normalization of $S$ and $D = \text{Diff}_{S^N}(\Delta)$ is the different of the pair $(X, S + \Delta)$.

**Proof.** We denote $H := -(K_X + S + \Delta)$ and $H_S := -(K_{S^N} + D)$. We take base changes via $\text{Spec} \mathcal{O}_{Z, z} \to \hat{Z}$ and we use the same notations by abuse of notations. By the assumption $\phi^{-1}(z) \subset f'(S)$, the ideal $H^0(\mathcal{O}_X(-S))$ of $H^0(\mathcal{O}_X) = H^0(\mathcal{O}_{Z'})$ is contained in all maximal ideals of $H^0(\mathcal{O}_X)$. We take an alteration $g : Y \to X$ and a strict transform $S_Y$ of $S$ as in Lemma 3.20. In order to show that $(X, S + \Delta)$ is purely globally $T$-regular, it is enough to show that the image $I_g \subset H^0(\mathcal{O}_X)$ of the map

$$H^0(\omega_{Y/V}(S_Y + g^*H)) \to H^0(\mathcal{O}_X)$$

is $H^0(\mathcal{O}_X)$. Since $H^0(\mathcal{O}_X(-S))$ is contained in all maximal ideals, it is enough to show $\psi(I_g) = H^0(\mathcal{O}_{S^N})$, where $\psi : H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_{S^N})$. We take an element $\alpha \in H^0(\mathcal{O}_{S^N})$. Since $g^*H$ is semiample and big over $Z$, there exists an alteration $h : W \to Y$ and a strict transform $S_W$ of $S$ as in Lemma 3.20 and the trace map

$$H^1(\omega_{W/V}(h^*g^*H)) \to H^1(\omega_{Y/V}(g^*H))$$

is zero by Corollary 3.6 and Remark 3.7. Since $(S^N, D)$ is globally $T$-regular, there exists a section

$$\alpha_W \in H^0(\omega_{S_W/V}(h^*g^*H_S))$$

mapped to $\alpha \in H^0(\mathcal{O}_{S^N})$ by the trace map. Let $\alpha_Y \in H^0(\omega_{S_Y/V}(g^*_SH_S))$ be the image of $\alpha_W$ via the trace map. By the exact sequence

$$0 \to \omega_{Y/V}(g^*H) \to \omega_{Y/V}(S_Y + g^*H) \to \omega_{S_Y/V}(g^*_SH_S) \to 0,$$

we have the exact sequence

$$H^0(\omega_{Y/V}(S_Y + g^*H)) \to H^0(\omega_{S_Y/V}(g^*_SH_S)) \to H^1(\omega_{Y/V}(g^*H)).$$

Thus we have the following commutative diagram

$$\begin{array}{ccc}
H^0(\omega_{S_W/V}(h^*g^*H_S)) & \longrightarrow & H^1(\omega_{W/V}(h^*g^*H)) \\
\downarrow & & \downarrow 0\text{-map} \\
H^0(\omega_{S_Y/V}(g^*_SH_S)) & \longrightarrow & H^1(\omega_{Y/V}(g^*H))
\end{array}$$

by Lemma 3.20. Thus the image of $\alpha_Y$ via the connection map is zero, so $\alpha_Y$ extends to a section $\gamma \in H^0(\omega_{Y/V}(S_Y + g^*H))$ and its image is $\alpha$ in $H^0(\mathcal{O}_{S^N})$. Thus we have $\alpha \in \psi(I_g)$, and the equation $\psi(I_g) = H^0(\mathcal{O}_{S^N})$ holds.

Next, we prove the normality of $S$. We take an open affine covering $\{U_i\}$ of $X$. By Proposition 3.13 each $(U_i|_{S^N}, D|_{U_i|_{S^N}})$ is globally $T$-regular. This is a local problem, we may assume that $X$ is affine by Proposition 3.13. By the above argument, we have $\psi(\mathcal{O}_X) = \mathcal{O}_{S^N}$, in particular, $S$ is normal. □

**Corollary 3.23.** Let $V$ be a scheme satisfying Assumption 3.4. Let $X$ be a normal affine $V$-variety and $(X, S + \Delta)$ be a log pair such that $|S + \Delta| = S$ is reduced. Then $(X, S + \Delta)$ is purely $T$-regular if and only if $(S^N, \text{Diff}_{S^N}(\Delta))$ is $T$-regular. Furthermore, in both cases, $S$ is normal, and in particular, $S$ is locally irreducible.
Corollary 3.24. Let $V$ be a scheme satisfying Assumption [3.1]. Let $(X, \Delta)$ be a simple normal crossing pair with $|\Delta| = 0$, where $X$ is an affine $V$-variety. Then $(X, \Delta)$ is globally $T$-regular.

**Proof.** We prove Corollary 3.24 by the induction on $d := \dim X$. We take an alteration $\pi: Y \to X$ from a normal $V$-variety $Y$. It is enough to show that the trace map is surjective at each closed point $x \in X$. First, we consider the case where $x$ is not contained in $\text{Supp}(\Delta)$. By [Bha18, Theorem 1.2],

$$\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$$

splits. By the Grothendieck duality, the map $\pi_*\omega_{Y/V} \to \omega_{X/V}$ is surjective. Since $X$ is Gorenstein, the trace map

$$\pi_*\omega_{Y/V}(-\pi^*K_{X/V}) \to \mathcal{O}_X$$

is also surjective. Next, we assume that $x$ is contained in a component $S$ of $\text{Supp}(\Delta)$. We take a positive rational number $a$ with $\text{ord}_S(\Delta + aS) = 1$. By Proposition 3.15, it is enough to show that $(X, \Delta + aS)$ is purely $T$-regular at $x$. By Corollary 3.23, it is enough to show that $(S, (\Delta - (1 - a)S)|_S)$ is $T$-regular at $x$. Since this pair is a simple normal crossing pair, this pair is $T$-regular by the induction hypothesis on $d$. In conclusion, $(X, \Delta)$ is $T$-regular. □

Remark 3.25. By the proof of [MS18, Theorem 6.21], if a log pair $(X, \Delta)$ is BCM-regular at $x$ after completion, then $(X, \Delta)$ is $T$-regular at $x$. Thus, Corollary 3.24 also follows from [MST+19, Theorem 4.1].

Proposition 3.26. Let $V$ be a scheme satisfying Assumption [3.1]. Let $f: X \to Z$ be a projective surjective morphism from a normal $V$-variety $X$ to an affine $V$-variety $Z$ and $(X, S + \Delta)$ a log pair such that $|S + \Delta| = S$ is a reduced divisor and $S$ has no common component with $\Delta$. Let $L$ be a $Q$-Cartier Weil divisor such that $L - (K_{X/V} + S + \Delta)$ is semiample and $f$-big and $L$ is Cartier at all codimension two points of $X$ contained in $S$. Let $g: Y \to X$ be an alteration from a normal $V$-variety $Y$ and $S_Y$ a strict transform of $S$. Then we have

$$T^0(S^N, D; L|_S) \subset I_g|_{S^N},$$

where $D := \text{Diff}_{S^N}(\Delta)$ is the different, $I_g$ is the image of the trace map

$$I_g := \text{Im}(H^0(\omega_{Y/V}(S_Y + [g^*(L - (K_{X/V} + S + \Delta)]))) \to H^0(\mathcal{O}_X(L)))$$

and the right hand side is the image of $I_g$ via the natural map

$$H^0(\mathcal{O}_X(L)) \to H^0(\mathcal{O}_{S^N}(L|_{S^N})).$$

**Proof.** We denote $H := L - (K_{X/V} + S + \Delta)$ and $H_S := L|_{S^N} - (K_{S^N/V} + D)$. We note that $\text{Diff}_{S^N}(\Delta - L) = D - L|_{S^N}$ since $L$ is Cartier at all codimension two points of $X$ contained in $S$. We may assume that $(Y, S_Y)$ is as in Lemma 3.20. By the proof of Proposition 3.22, we can construct an alteration $h: W \to Y$ and a strict transform $S_W$. □
of $S$ as in Lemma 3.20 such that the trace map on the first cohomology is zero. We obtain the commutative diagram

$$
\begin{array}{c}
H^0(\omega_{W/V}(S_W + h^*g^*H)) \rightarrow H^0(\omega_{S_W/V}(h_S^*g_S^*H_S)) \rightarrow H^1(\omega_{W/V}(h^*g^*H)) \\
| \quad \quad | \\
H^0(\omega_{Y/V}(S_Y + g^*H)) \rightarrow H^0(\omega_{S_Y/V}(g_S^*H_S)) \rightarrow H^1(\omega_{Y/V}(g^*H)) \\
| \quad \quad | \\
H^0(\mathcal{O}_X(L)) \rightarrow H^0(\mathcal{O}_{SN}(L|_{SN})).
\end{array}
$$

Thus the proof is same as the proof of Proposition 3.22.

\begin{proof}
Let $H$ be an ample Cartier divisor on $X$ such that $\mathcal{O}_X(K_X + B + H)$ is globally generated. We take an effective divisor $B'$ which is linearly equivalent to $K_X + B + H$. Then for divisible enough $e$, $(X, B + \frac{1}{p^e-1}B')$ is globally $F$-regular, the Cartier index of

$$(p^e - 1)(K_X + B + \frac{1}{p^e-1}B') \sim p^e(K_X + B) + H$$

is prime to $p$, and $[-\pi^*(K_X + B + \frac{1}{p^e-1}B')] = [-\pi^*(K_X + B)]$. Thus, we may assume that the Cartier index of $K_X + B$ is prime to $p$. By BMP+20, Lemma 6.14 and Proposition 3.11, the generic fiber of $f$ is globally $T$-regular. In particular, for a given alteration $\pi: Y \rightarrow X$, the trace map

$$H^0(\omega_Y([-\pi^*(K_X + B)]) \rightarrow H^0(\mathcal{O}_X)$$

is non-zero. We take an element $\alpha \in H^0(\omega_Y([-\pi^*(K_X + B)]))$ such that $\beta := \text{Tr}(\alpha) \in H^0(\mathcal{O}_X)$ is non-zero. Since $(X, B)$ is globally $F$-regular, there exists a positive integer $e$ such that $(p^e - 1)(K_X + B)$ is Cartier and the natural map

$$\mathcal{O}_X \rightarrow F^e_*\mathcal{O}_X([p^e - 1]B] + \text{div}(\beta))$$

splits. Let $\Gamma := (1 - p^e)(K_X + B)$. We take a section $\gamma \in H^0(\mathcal{O}_X(\Gamma))$ corresponding to the splitting. Then the trace map

$$H^0(\mathcal{O}_X(\Gamma)) \rightarrow H^0(\mathcal{O}_X)$$

maps $\gamma \beta$ to $1$. We regard $\alpha \gamma$ as an element of

$$H^0(\omega_Y([-p^e\pi^*(K_X + B)])) \simeq H^0(\pi_*\omega_Y([-\pi^*(K_X + B)]) \otimes \mathcal{O}_X(\Gamma)).$$

Then the composition of morphisms

$$H^0(\omega_Y([-p^e\pi^*(K_X + B)])) \rightarrow H^0(\mathcal{O}_X(\Gamma)) \rightarrow H^0(\mathcal{O}_X)$$

maps $\alpha \gamma$ to $1$, where the first map is induced by

$$\text{Tr} \otimes \mathcal{O}_X(\Gamma): \pi_*\omega_Y([-\pi^*(K_X + B)]) \otimes \mathcal{O}_X(\Gamma) \rightarrow \mathcal{O}_X(\Gamma).$$

\end{proof}
Thus, the map
\[ H^0(\omega_Y([-\pi^*(K_X + B)]) \to H^0(\mathcal{O}_X) \]

is surjective. \hfill \Box

3.3. Restriction theorem. The goal of this subsection is to prove the existence of three-dimensional pl-flips with ample divisor in the boundary (Corollary 3.34). First, we prove the existence by assuming the global $T$-regularity of the boundary even in the higher-dimensional case (Theorem 3.30). It follows from the restriction theorem (Proposition 3.29) and Shokurov’s reduction to pl-flips. Next, we show this condition in the three-dimensional case.

Lemma 3.28. (cf. [HW20, Lemma 3.2]) Let $V$ be a scheme satisfying Assumption 3.1. Let $f: X \to Z$ be a projective birational morphism from a normal $V$-variety to an affine $V$-variety. Let $(X, B)$ be a log pair which is globally $T$-regular over a point $z$ of $Z$. Let $L$ be a Weil $\mathbb{Q}$-Cartier divisor on $X$ and $\Gamma$ an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $X$. If $g \in H^0(X, \mathcal{O}_X(L))$ corresponds to a divisor $G \in |L|$ such that $G \geq \Gamma$, then $g$ is contained in $T^0(X, \tilde{B} + \Gamma; L)$ after localizing at $z$.

Proof. We take base changes via Spec $\mathcal{O}_{Z,z} \to Z$ and we use the same notations by abuse of notations. It is enough to consider alterations $h: Y \to X$ from a normal $V$-variety such that $h^*(K_{X/V} + B)$, $h^*L$, $h^*G$ and $h^*\Gamma$ are Cartier. We consider the commutative diagram

\[
\begin{array}{ccc}
H^0(\omega_Y(h^*(L - K_{X/V} - B - \Gamma))) & \longrightarrow & H^0(\mathcal{O}_X(L)) \\
\downarrow & & \downarrow \\
H^0(\omega_Y(h^*(L - K_{X/V} - B - G))) & \longrightarrow & H^0(\mathcal{O}_X(L - G)) \\
\uparrow g & & \uparrow g \\
H^0(\omega_Y(-h^*(K_{X/V} + B))) & \longrightarrow & H^0(\mathcal{O}_X),
\end{array}
\]

where the horizontal maps are trace maps and the surjectivity of the bottom map follows from the global $T$-regularity of $(X, B)$. Thus, we have a section of $H^0(\omega_Y(h^*(L - K_{X/V} - B - \Gamma)))$ mapped to $g$. \hfill \Box

Proposition 3.29. (cf. [HW20, Proposition 3.1]) Let $V$ be a scheme satisfying Assumption 3.1. Let $f: X \to Z$ be a projective morphism from a normal $V$-variety $X$ to an affine $V$-variety $Z$. Let $(X, S + A + B)$ be a dlt pair such that $S$ and $A$ are $\mathbb{Q}$-Cartier Weil divisors. Assume that $A$ is ample and $|B| = 0$. Let $z \in Z$ be a point. If $S$ is normal and $(S, (1 - \varepsilon)A_S + B_S)$ is globally $T$-regular over $z$ for all $0 < \varepsilon < 1$, then for every $k \geq 1$ such that $k(K_{X/V} + S + A + B)$ is Cartier, we have

\[ |k(K_{X/V} + S + A + B)|_S = |k(K_{S/V} + A_S + B_S)|_S \]

after localization at $z$, where $B_S := \text{Diff}_S(B)$ and $A_S := A|_S$.

Proof. We take base changes via Spec $\mathcal{O}_{Z,z} \to Z$ and we use the same notations by abuse of notations. We note that since $(X, S + A + B)$ is dlt and $S$ and $A$ are $\mathbb{Q}$-Cartier, $(X, S + A + B)$ is a simple normal crossing pair around $S \cap A$, and in particular, $A$ is
Cartier at all codimension two points on $X$ contained in $S$. We take $F \in |k(K_{S/V} + A_S + B_S)|$. By the descending induction, we prove that there exists divisors $G_m \in |k(K_{X/V} + S + A + B) + mA|$ such that $G_m|_S = F + mA_S$. First, since $A$ is ample, such $G_m$ exists for large enough $m$ by Serre vanishing. We assume that such $G_{m+1}$ exists. We set

$$L := k(K_{X/V} + S + A + B) + mA$$

$$= K_{X/V} + S + B + (k-1)(K_{X/V} + S + A + B) + (m+1)A$$

$$\sim_Q K_{X/V} + S + B + \frac{k-1}{k}G_{m+1} + \frac{m+1}{k}A.$$

We write $H := L - (K_{X/V} + S + B + \frac{k-1}{k}G_{m+1})$, then it is ample. Since $(S, \frac{k-1}{k}A_S + B_S)$ is globally $T$-regular and

$$F + mA_S \geq \frac{k-1}{k}(F + mA_S) = \frac{k-1}{k}G_{m+1}|_S - \frac{k-1}{k}A_S,$$

a section $g \in H^0(L|_S)$ corresponding to $F + mA_S$ is contained in

$$T^0(S, B_S + \frac{k-1}{k}G_{m+1}|_S; L|_S)$$

by Lemma 3.28 By Proposition 3.26 $g$ is contained in the image of $H^0(O_X(L))$. □

The following theorem is the existence of pl-flips with ample divisor in the boundary in the special setting. The proof is an analog of the proof of [HW20, Theorem 1.3].

**Theorem 3.30.** (cf. [HW20, Theorem 1.3]) Let $V$ be a scheme satisfying Assumption 3.7. Let $(X, S + A + B)$ be a dlt pair such that $S$ is an anti-ample $\mathbb{Q}$-Cartier Weil divisor and $A$ is an ample $\mathbb{Q}$-Cartier Weil divisor. Let $f : X \to Z$ be a $(K_{X/V} + S + A + B)$-flipping contraction with $\rho(X/Z) = 1$ to an affine $V$-variety. Furthermore, we assume that $(S^N, (1 - \varepsilon)A_S + B_S)$ is globally $T$-regular for all $0 < \varepsilon < 1$ over all points of $f(\text{Exc}(f))$ and $R(K_{S/V} + A_S + B_S)$ is finitely generated, where $B_S := \text{Diff}_SN(B)$ and $A_S := A|_S$. Then the flip of $f$ exists.

**Proof.** Take a point $z \in Z$ contained in $f(\text{Exc}(f))$. Since $A$ is ample, we have $z \in f(A)$. We take base changes via Spec $O_{Z,z} \to Z$ and we use the same notations by abuse of notations. We may assume that $[B] = 0$. By Proposition 3.22 the scheme $S$ is normal. By Proposition 3.29 the restriction algebra

$$R_{S}(k(K_{X/V} + S + A + B)) := \text{Im}(R(k(K_{X/V} + S + A + B)) \to R(k(K_{S/V} + A_S + B_S)))$$

coincides with $R(k(K_{S/V} + A_S + B_S))$ for some positive integer $k$, and in particular, $R_{S}(k(K_{X/V} + S + A + B))$ is finitely generated. By Shokurov’s reduction to finite generation (see [Cor07, Lemma 2.3.6]), the flip of $f$ exists. □

**Remark 3.31.** If $(S^N, (1 - \varepsilon)A_S + B_S)$ is globally $F$-regular over $Z$, then it is globally $T$-regular over $Z$ by Proposition 3.27. Thus, applying Theorem 3.30 for the case $X = X_s$, we obtain [HW20, Theorem 1.3].

In order to use Theorem 3.30 for threefolds, we will show the pure global $T$-regularity of $(S^N, A_S + B_S)$. 
Lemma 3.32. (cf. [HW19a, Lemma 3.3]) Let $V$ be a scheme satisfying Assumption $\mathcal{A}$. We assume that the residue field of $V$ is infinite. Let $f : S \to T$ be a projective birational morphism from a normal $V$-surface $S$ to an affine $V$-surface $T$. Let $(S, C + B)$ be a plt pair with $|C + B| = C$. Assume that $-(K_{S/V} + C + B)$ and $C$ are ample. Further assume that $f$ has connected fibers. Then $(S, C + B)$ is purely globally $T$-regular over all points of $f(\text{Exc}(f))$.

Proof. Since $f$ has connected fibers, the Stein factorization of $f$ induces a homeomorphism $\phi : T' \to T$. Thus we have $\phi^{-1}(f(\text{Exc}(f))) = f'(\text{Exc}(f'))$, so we may assume that $T$ is normal and $f_*\mathcal{O}_X = \mathcal{O}_T$ by replacing $T$ into $T'$, where $f' : X \to T'$ is the induced morphism. We take a point $t \in f(\text{Exc}(f))$. First, we prove that $C$ is irreducible after shrinking $T$ around $t$. By [Tan18b, Theorem 5.2], the intersection $C \cap f^{-1}(t)$ is connected, and in particular, the scheme $C$ is irreducible. Furthermore, as $C$ is ample, $C$ is not an exceptional divisor of $f$. By adjunction, $(C, \text{Diff}_C(B + C))$ is normal klt one-dimensional pair, in particular, it is simple normal crossing. Since $C$ is not exceptional and $T$ is affine, $C$ is also affine. Therefore, $(C, \text{Diff}_C(B + C))$ is $T$-regular by Corollary 3.24. Since $-(K_{S/V} + C + B)$ is ample, $(S, B + C)$ is purely globally $T$-regular by Proposition 3.22.

Lemma 3.33. Let $V$ be a scheme satisfying Assumption $\mathcal{A}$. Assume that the residue field of $V$ is infinite. Let $f : X \to Z$ be a small projective birational morphism from a normal $V$-variety $X$ of dimension three to an affine $V$-variety $Z$. Let $(X, S + A + B)$ be a plt pair such that $-(K_{X/V} + S + A + B)$ is ample, and $S$ and $A$ are locally irreducible $\mathbb{Q}$-Cartier Weil divisors and $|B| = 0$. Assume that $-S$ and $A$ are ample. Then $(S^N, \text{Diff}_{S^N}(A + B))$ is purely globally $T$-regular over all points of $f(\text{Exc}(f))$. In particular, $S$ is normal over a neighborhood of $f(\text{Exc}(f))$.

Proof. Since $K_{X/V} + S + B$ is $\mathbb{Q}$-Cartier, $\text{Diff}_{S^N}(A + B) = D + A|_{S^N}$, where $D := \text{Diff}_{S^N}(B)$. We note that $A$ is Cartier on the codimension two points of $X$ contained in $S$. Since $f$ is small, $f|_{S^N} : S^N \to T$ is birational, where $T := f(S)$. Since $-S$ is ample, all exceptional curves of $f$ are contained in $S$, thus $f|_S : S \to T$ has connected fibers. Since $S^N \to S$ is a universal homeomorphism by Lemma 2.3, $f|_{S^N}$ also has connected fibers. By Proposition 2.13, the pair $(S^N, D + A|_{S^N})$ is plt. By Lemma 3.32, $(S^N, D + A|_{S^N})$ is purely globally $T$-regular over all points of $f(\text{Exc}(f))$. In particular, $S$ is normal over a neighborhood of $f(\text{Exc}(f))$ by Proposition 3.22.

Corollary 3.34. (cf. [HW19a, Proposition 3.4]) Let $V$ be an excellent Dedekind scheme. Let $(X, S + A + B)$ be a three-dimensional dlt pair over $V$. Let $f : X \to Z$ be a $(K_{X/V} + S + A + B)$-flipping contraction with $p(X/Z) = 1$. Assume that $S$ and $A$ are locally irreducible Weil divisors such that $-S$ and $A$ are ample $\mathbb{Q}$-Cartier. Then the flip of $f$ exists.

Proof. We may assume that $V$ is the spectrum of an excellent discrete valuation ring. We may assume $|B| = 0$. Since the existence of flip is a local problem on $Z$, we may assume that $Z$ is affine. By Shokurov’s reduction to pl-flip (see [Cor07], Lemma...
2.3.6], it is enough to show that $R_S(k(K_X + S + A + B))$ is finitely generated. This statement can be reduced to the case where $V$ is complete and the residue field is infinite taking a strict henselization and completion. By Lemma 2.4, the assumption is preserved except for the condition that the relative Picard rank is one. By Lemma 3.33, $S$ is normal and $(S, A_S + B_S)$ is purely globally $T$-regular over all points of $f(Exc(f))$, where $B_S = \text{Diff}_S(B)$ and $A_S = A|_S$. By Theorem 3.30, it is enough to show that $R(K_{S/V} + A_S + B_S)$ is finitely generated. We take an effective divisor $A_1$ on $S$ with $A_S \sim \text{Q} A_1$ such that $A_S$ and $A_1$ have no common component. Since $p_{S, A_S + B_S}$ is plt, $p_{S, A_S + B_S}$ is klt for small enough positive rational number $\varepsilon$. By [Tan18b, Theorem 1.1, Theorem 4.2, Corollary 4.11], the canonical ring $R(K_{S/V} + (1 - \varepsilon)A_S + B_S + \varepsilon A')$ is finitely generated. Since $K_{S/V} + (1 - \varepsilon)A_S + B_S + \varepsilon A'$ is also finitely generated.

Remark 3.35. The existence of necessary flips for [Kol20, Theorem 6] follows from Corollary 3.34 over an excellent Dedekind scheme.

4. Proof of Theorem 1.2 and its applications

The goal of this section is to prove Theorem 1.2 and its applications. Proposition 4.12 is one of the applications and it will be used to prove Theorem 1.1. To prove these theorems, we establish the cone theorem for pseudo-effective pairs (Proposition 4.2), by following the method given by [Kee99] [Tan18a]. We also prove the cone theorem for more general settings (Proposition 4.4) by using the method given by [Kaw94]. If every relative curve is contained in the special fiber, then the cone theorem is easily reduced to the case of surfaces, but in the relative setting, relative curves contained in the generic fiber may exist. Therefore, we should treat such cases carefully.

Proposition 4.1. Let $V$ be an excellent Dedekind scheme. Let $(X, S + B)$ be a three-dimensional dlt pair over $V$ such that $S$ is a Q-Cartier Weil divisor. Let $\rho: X \to U$ be a projective morphism over $V$. Let $\Sigma$ be a $(K_{X/V} + S + B)$-negative extremal ray contracted by $\rho$. Let $L$ be a $\rho$-nef Cartier divisor on $X$ with $\text{supp}(L) = \text{R}[\Sigma]$. Assume that $S$ is a prime divisor and $S \cdot \Sigma < 0$. Then $L$ is semiample over $U$.

Proof. It follows from a similar argument to the argument in the proof of [HW, Proposition 4.4] by replacing [HW, Lemma 2.1] with Lemma 2.3 and using [Wit20, Theorem 1.2].

Proposition 4.2. Let $V$ be an excellent Dedekind scheme. Let $\pi: X \to U$ be a projective $V$-morphism from a normal Q-factorial quasi-projective $V$-threefold $X$ to a quasi-projective $V$-variety $U$. Let $B$ be an effective $\text{R}$-Weil divisor on $X$ satisfying the following.

- all coefficients $c$ of $B$ satisfy $0 \leq c \leq 1$, and
- $K_{X/V} + B$ is pseudo-effective.
Let $A$ be a $\pi$-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. Then there exist finitely many $\pi$-relative curves $C_1, \ldots, C_r$ on $X$ such that

$$\overline{\text{NE}}(X/U) = \overline{\text{NE}}(X/U)_{K_{X/U} + B + A} + \sum_{i=1}^{r} \mathbb{R}_{\geq 0}[C_i].$$

**Proof.** The assertion is proved by the same method as in [Tan18a, Theorem 7.6]. Here, we use [Tan18b, Theorem 2.14] after the reduction to the case of surfaces.

In the proof of Proposition 4.12, we run an MMP with scaling. In order to do this, we prepare the following corollary.

**Corollary 4.3.** Let $V$ be an excellent Dedekind scheme. Let $(X, \Delta)$ be a dlt $\mathbb{Q}$-factorial pair over $V$ satisfying that $|\Delta| = X_s$ as sets. Let $\pi : X \to U$ be a projective birational morphism over $V$ from $X$ to a quasi-projective $V$-variety $U$. Let $H$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor such that $K_{X/U} + \Delta + H$ is $\pi$-nef. We put

$$\lambda_H := \inf \{ \lambda \in \mathbb{R}_{\geq 0} \mid K_{X/U} + \Delta + \lambda H \text{ is } \pi \text{-nef} \}.$$

Then there exists a $(K_{X/U} + \Delta)$-negative extremal ray $R \subset \overline{\text{NE}}(X/U)$ satisfying that

$$(K_{X/U} + \Delta + \lambda_H H) \cdot R = 0.$$ 

**Proof.** Take a rational number $a \in \mathbb{Q}_{>0}$ with $\Delta - aX_s \geq 0$. Since $K_{X/U} + \Delta$ is $\pi$-big, we have

$$K_{X/U} + \Delta \sim_{\mathbb{Q}, \pi} A + E$$

for a $\pi$-ample $\mathbb{Q}$-Cartier divisor $A$ and an effective $\mathbb{Q}$-Cartier divisor $E$. Take a rational number $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon << 1$ such that the coefficients of $\Delta - aX_s + \varepsilon E$ are less than one. Note that

$$K_{X/U} + \Delta - aX_s + \varepsilon E + \varepsilon A \sim_{\mathbb{R}, \pi} (1 + \varepsilon)(K_{X/U} + \Delta).$$

Therefore, by using Proposition 4.12 for $B = \Delta - aX_s + \varepsilon E$, it finishes the proof. □

On the other hand, if the base scheme is local, then we can prove the cone theorem in a more general situation by reducing the problem to the special fiber. In relative setting, since relative curve is not necessarily contained in the special fiber (e.g. the fiber of $X := \mathbb{A}^1_{Z_p} \times \mathbb{P}^1_{Z_p} \to \mathbb{A}^1_{Z_p}$ over the closed point corresponding to $(pX + 1) \subset Z_p[X]$), we have to give an additional argument.

**Proposition 4.4.** Let $V$ be an excellent Dedekind scheme. Let $\pi : X \to U$ be a projective $V$-morphism from a normal $\mathbb{Q}$-factorial quasi-projective flat $V$-variety $X$ of relative dimension two to a quasi-projective $V$-variety $U$. Let $B$ be an effective $\mathbb{R}$-divisor on $X$ such that every coefficient $c$ of $B$ satisfies $0 \leq c \leq 1$. Let $A$ be a $\pi$-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. We suppose one of the following.

1. The scheme $V$ is the spectrum of a discrete valuation ring.
2. The scheme $X$ is smooth over every generic point $\eta$ of $V$, and $B$ has no horizontal components.
Then there exist finitely many \( \pi \)-relative curves \( C_1, \ldots, C_r \) on \( X \) such that

\[
\overline{\text{NE}}(X/U) = \overline{\text{NE}}(X/U)_{K_{X/V} + B + A \geq 0} + \sum_{i=1}^{r} \mathbb{R}_{\geq 0}[C_i].
\]

**Proof.** We may assume that \( V \) is connected. Moreover, replacing \( \pi \) by its Stein factorization, we may assume that \( \pi_* (\mathcal{O}_X) = \mathcal{O}_U \). Therefore we may assume \( U \) is normal and flat over \( V \). First, we prove the assertion in the case (1). If \( U \to V \) is not surjective, then \( X \to U \) is normal surface over a field, so the assertion follows from the cone theorem for surfaces (cf. [Tan18b, Theorem 2.14]). Therefore we may assume \( U \to V \) is surjective. We denote the closed point of \( V \) and flat over \( V \).

Here, we put \( D \). First, we prove the assertion in the case (1). If \( s \) is the restriction of \( \eta \) to the generic fiber. Then the generic fiber of \( X \) is contained in \( U \), which maps to the closed point in \( V \). Therefore, any \( \pi \)-relative curves are contained in \( X_s \). Therefore we have

\[
\overline{\text{NE}}(X/U) = \sum_{i=1}^{n} \overline{\text{NE}}(S_i^N/U) \quad \overline{\text{NE}}(S_i^N/U)_{K_{S_i^N} + D_i + A_{S_i^N} \geq 0} + \sum_{i,j} \mathbb{R}_{\geq 0}[\Gamma_{i,j}].
\]

Now we will divide the case by the dimension of \( U \). First, consider the case where \( U \) is of relative dimension 0 over \( V \). In this case, a closed curve in \( X \) maps to a closed point in \( U \), which maps to the closed point in \( V \). Therefore, any \( \pi \)-relative curves are contained in \( X_s \).

Next, we consider the case where \( U \) is of relative dimension 1 over \( V \). Let \( \pi_\eta : X_\eta \to U_\eta \) be the restriction of \( \eta \) to the generic fiber. Then the generic fiber of \( \pi_\eta \) is geometrically irreducible (cf. [Tan18a, Lemma 2.2]). Take an open subset \( U_0 \subset U_\eta \) where \( \pi_\eta \) have geometrically irreducible fibers over \( U_0 \). Take a closed point \( P_1 \in U_0 \) which is also closed in \( U \) if exists. Let \( P_2, \ldots, P_l \in U_\eta \backslash U_0 \) be all the closed points which are also closed in \( U \). Let \( C_{s,t} \) be the irreducible components of \( \pi_\eta^{-1}(P_s) \). Then any \( \pi \)-relative curve which is contained in \( X_\eta \) is generated by \( [C_{s,t}] \subset \overline{\text{NE}}(X/U) \). Therefore, we have

\[
\overline{\text{NE}}(X/U) = \overline{\text{NE}}(X/U)_{K_{X/V} + B + A \geq 0} + \sum_{i,j} \mathbb{R}_{\geq 0}[\Gamma_{i,j}] + \sum_{s,t} \mathbb{R}_{\geq 0}[C_{s,t}].
\]

Finally, we consider the case where \( U \) is of relative dimension 2 over \( V \). In this case, \( \pi \) is birational morphism. Let \( C_1, \ldots, C_l \) be all the exceptional divisors of \( \pi \) which are
contained in the generic fiber $X_{\eta}$. Then $C_i$ are $\pi$-relative curves on $X$. Then we have
\[ \overline{NE}(X/U) = \overline{NE}(X_s/U) + \sum_s \mathbb{R}_{\geq 0}[C_s] \]
\[ = \overline{NE}(X/U)_{K_{X/V} + B + A_{\geq 0}} + \sum_{i,j} \mathbb{R}_{\geq 0}[\Gamma_{i,j}] + \sum_s \mathbb{R}_{\geq 0}[C_s]. \]

It finishes the proof of (1). The assertion in the case (2) follows from the argument in (1) and the lifting method in the proof of [Kaw94, Theorem 1.3]. Indeed, since
\[ \overline{NE}(X/U) \]
(where $X^{(s)}$ and $U^{(s)}$ are localization of $X$ and $U$ at a closed point $s$), by the argument in (1), it suffices to show that any extremal ray in $\overline{NE}(X_s/U_s)$ can be lifted to the generic fiber. This follows from the deformation theory as in the proof of [Kaw94, Theorem 1.3].

**Proposition 4.5.** (cf. [Fuj07, Theorem 4.2.1]) Let $V$ be an excellent Dedekind scheme. Let $(X, B)$ be a $\mathbb{Q}$-factorial three-dimensional dlt pair over $V$. Consider a sequence of log flips starting from $(X, B) = (X_0, B_0)$:

\[ (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots, \]

where $\phi_i: X_i \to Z_i$ is a flipping contraction associated to an extremal ray and $\phi^+: X_i^+ = X_{i+1} \to Z_i$ is the log flip. Then, after finitely many flips, the flipping locus is disjoint from $[B]$. 

**Proof.** It follows from a similar argument to the argument in the proof of [Fuj07, Theorem 4.2.1].

**Theorem 4.6.** (Theorem 1.2, cf. [HW19a, Theorem 1.1]) Let $V$ be an excellent Dedekind scheme. Let $(X, \Delta)$ be a three-dimensional $\mathbb{Q}$-factorial dlt pair over $V$. Assume that there exists a projective birational morphism $\pi: X \to Z$ to a normal $\mathbb{Q}$-factorial variety $Z$ with $\text{Exc}(\pi) \subset [\Delta]$. Then we can run a $(K_{X/V} + \Delta)$-MMP over $Z$ which terminates with a minimal model.

**Proof.** It follows from the same argument as in the proof of [HW19a, Theorem 1.1] using the cone theorem (Proposition 4.2), the contraction theorem (Proposition 4.1), the existence of flips (Corollary 3.34), and the termination of flips (Proposition 4.5).

**Lemma 4.7.** Let $f: Y \to X$ be a projective birational morphism of three dimensional separated excellent integral schemes. Let $W$ be a closed subscheme of $Y$. If the singular locus of $X$ is contained in some affine open subset of $X$, then there exists a projective birational morphism $\nu: Y' \to Y$ such that $Y'$ is regular and $\text{Exc}(\nu) \cup \nu^{-1}(W)$ has simple normal crossing support.

**Proof.** By [CP19, Theorem 1.1], the scheme $X$ admits a projective resolution $X_0$. By [CP19, Theorem 4.4] and [CJS09], an analog of [HW20, Conjecture 5.4] holds for regular three dimensional separated excellent integral schemes and its closed subschemes. By an analogous argument of the proof of [HW20, Proposition 5.5] for $X_0 \dashrightarrow Y$, we obtain
the assertion. We note that we can take an elimination of $X_0 \rightarrow Y$ which is projective over $X_0$ and $Y$. □

Remark 4.8. Let $U$ be a regular open subset of $Y$ such that $f|_U$ is an isomorphism and $\text{Exc}(\nu) \cup \nu^{-1}(W)$ has simple normal crossing support on $U$. We can take a morphism $\nu$ in Lemma 4.7 as a morphism whose isomorphic locus contains $U$. Indeed, there exists an elimination of $X_0 \rightarrow Y$ whose isomorphic locus contains $U$.

Corollary 4.9. ([HW19a, Corollary 1.4]) Let $V$ be an excellent Dedekind separated scheme. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial three-dimensional log pair over $V$ such that any coefficient of $\Delta$ is at most one. Assume that $X$ is projective birational over an affine $V$-variety. Then there exists a projective birational morphism $\pi: Y \rightarrow X$ such that the pair $(Y, \Delta_Y := \pi^{-1}_* \Delta + \text{Exc}(\pi))$ satisfies the following conditions.

1. $(Y, \Delta_Y)$ is a $\mathbb{Q}$-factorial dlt pair over $V$, and
2. $K_{Y/V} + \Delta_Y$ is nef over $X$.

Proof. By Lemma 4.7, we have a log resolution $f: W \rightarrow X$ of $(X, \Delta)$. We write $\Delta_W := \pi^{-1}_* \Delta + \text{Exc}(\pi)$, then $(W, \Delta_W)$ is dlt. By Theorem 1.2, we can run a $(K_{W/V} + \Delta_W)$-MMP over $X$, and we get a minimal model $\pi: Y \rightarrow X$. Then $(Y, \Delta_Y := \pi^{-1}_* \Delta + \text{Exc}(\pi))$ is dlt and $K_{Y/V} + \Delta_Y$ is nef over $X$. □

Corollary 4.10. (cf. [HW19a, Corollary 1.5]) Let $V$ be an excellent Dedekind separated scheme. Let $(X, S + B)$ be a $\mathbb{Q}$-factorial three-dimensional log pair over $V$ such that $S$ is a locally irreducible Weil divisor. Assume that $X$ is projective birational over an affine $V$-variety. Then $(X, S + B)$ is plt on a neighborhood of $S$ if and only if $(S^N, B_S)$ is klt, where $S^N$ is the normalization of $S$ and $B_S := \text{Diff}_{S^N}(B)$ is the different.

Proof. It follows from the same argument as in the proof of [HW19a, Corollary 1.5] replacing [HW19a, Corollary 1.4] into Corollary 1.9. □

Lemma 4.11. Let $V$ be an excellent Dedekind scheme. Let $(X, \Delta)$ be a three-dimensional dlt pair over $V$ with $|\Delta| = S + S'$, where $S$ and $S'$ are locally irreducible $\mathbb{Q}$-Cartier divisors. Let $f: X \rightarrow Z$ be a projective morphism over $V$ such that $f(S)$ is two-dimensional. Let $C$ be an irreducible component of $S \cap S'$. If $f(C)$ is a point, then $(S' \cdot C)$ is negative.

Proof. We set $D := \text{Diff}_{S^N}(\Delta - S)$, then $(S^N, D)$ is plt by Proposition 2.13. As $S^N$ is two-dimensional, $|D|$ is normal, and in particular, $|D|$ is locally irreducible. Since $(X, \Delta)$ is a simple normal crossing pair on the generic point of $S \cap S'$, we have $|D| = S'|_{S^N}$. By Lemma 2.2, $S^N \rightarrow S$ is a universal homeomorphism, thus $S \cap S'$ is also locally irreducible. In particular, $C|_S$ is $\mathbb{Q}$-Cartier and $(C|_{S^N} \cdot |D|) = (C|_{S^N})^2$. Thus, we have

$$(C \cdot S') = (C|_S \cdot S'|_S) = (C|_{S^N} \cdot |D|) = (C|_{S^N})^2 < 0,$$

because $C|_{S^N}$ is a contracted curve of $S^N$ via generically finite morphism $f|_{S^N}: S^N \rightarrow f(S)$. □
Proposition 4.12. (cf. [HW19a, Proposition 4.1]) Let $V$ be the spectrum of an excellent discrete valuation ring. Let $X$ be a flat $V$-variety of relative dimension two. Let $(X, \Delta)$ be a dlt $\mathbb{Q}$-factorial log pair over $V$. Let $f : X \to Z$ be a $(K_X/V + \Delta)$-flipping contraction with $\rho(X/Z) = 1$. Suppose that $X_s$ is contained in $|\Delta|$ as sets and every irreducible component of $X_s$ is numerically trivial over $Z$. Then the flip of $f$ exists.

Proof. We may assume that $|\Delta| = X_{s,\text{red}}$. We take a point $z \in f(\text{Exc}(f))$. By shrinking $Z$, we may assume that $Z$ is affine. First, we prove that $f^{-1}(z)$ intersects with only one irreducible component of $X_s$. Otherwise, there exist two irreducible components $S$ and $S'$ intersecting $f^{-1}(z)$. By the connectedness of $f^{-1}(z)$, there exists a flipping curve $C$ intersecting with $S$ and $S'$. By the assumption, $S$ and $S'$ are numerically trivial over $Z$. Since $(S \cdot C) = 0$ and $(S' \cdot C) = 0$, $C$ is contained in $S$ and $S'$. It contradicts Lemma 4.11.

Thus, we may assume that $X_s$ is irreducible by shrinking $Z$ around $z$, and in particular, $(X, \Delta)$ is plt. We take a reduced $\mathbb{Q}$-Cartier divisor $H$ on $X$ as in [HW20, Lemma 5.3], then $H \equiv_Z 0$ and it satisfies the conditions in the proof of [HW20, Theorem 1.2]. We note that as $X_\eta$ is a surface and the quotient field of $V$ is infinite, a general hyperplane preserves dlt singularities on the generic fiber. We take a dlt modification $Y \to X$ of $(X, \Delta + H)$ by Corollary 4.9. We note that $f : X \to Z$ is an isomorphism over the generic point of $V$, we may assume that $Y \to X$ is an isomorphism over the generic point by Remark 4.8. We run a $(K_{Y/V} + \Delta_Y + H_Y)$-MMP by the same argument as in [HW20, Theorem 1.2]. Replacing $(Y, \Delta_Y + H_Y)$ into a minimal model, we may assume that $K_{Y/V} + \Delta_Y + H_Y$ is nef over $Z$. By Corollary 4.3 and the same argument as in the proof of [HW20, Theorem 1.2], we can run a $(K_{Y/V} + \Delta_Y)$-MMP over $Z$ with scaling of $H_Y$. Replacing $(Y, \Delta_Y)$ into a minimal model, we may assume that $K_{Y/V} + \Delta_Y$ is nef over $Z$. We denote the map $Y \to Z$ by $h$. Since $(X, \Delta)$ is plt, $h$ is small by the negativity lemma (Proposition 2.1). Since the relative Picard rank of $X$ over $Z$ is one, $Y$ is the flip of $f$. \hfill \Box

5. Proof of Theorem 1.1 and its applications

Our goal of this section is to prove Theorem 1.1, which is a generalization of the result of Kawamata ([Kaw94], [Kaw99]). In this section, we deal with schemes satisfying the following conditions (Assumption 5.1), which are preserved under MMP-steps (cf. Proposition 5.7, 5.9). Kawamata proved this fact by the construction of flips, but it does not follow from our construction. Therefore, to prove the preservation, we precisely observe extremal ray contractions (cf. Proposition 5.6).

Assumption 5.1. Let $V$ be an excellent Dedekind scheme. $X$ is a $V$-variety satisfying the following conditions.

1. $X$ is flat over $V$ of relative dimension two.
2. Every generic fiber $X_s$ is smooth.
3. The fibers $X_s$ for the closed points $s \in V$ are geometrically reduced and satisfy the condition $(S_2)$.
4. Each irreducible component $S$ of every fiber $X_s$ is geometrically irreducible, geometrically normal and a $\mathbb{Q}$-Cartier divisor on $X$. 


(5) \((X, X_s)\) is dlt for all closed points \(s \in V\).

(6) For each closed point \(s \in V\) and dominant morphism \(\iota: V' = \text{Spec} A \to V\) with reduced fiber such that \(A\) is a complete discrete valuation ring with algebraically closed residue field \(k\) and \(s\) is contained in the image of \(\iota\), the base change \(X' := X \times_V V'\) satisfies the condition (5).

**Remark 5.2.**
- A strictly semi-stable scheme over an excellent Dedekind scheme of relative dimension 2 satisfies Assumption 5.1.
- The existence of an extension as in Assumption 5.1 (6) follows from [Mat89, Theorem 29.1].
- Assumption 5.1 is preserved by taking a base change as in Assumption 5.1 (6).

**Remark 5.3.** In [Kaw94], it is additionally assumed that \(\mathcal{O}_X(mK_{X/V})\) is maximal Cohen-Macaulay in order to prove the existence of flips. Kawamata proved that the condition is preserved under MMP-steps if each residue characteristic is larger than 3. However, in this paper, we do not need this assumption. We note that if each residue characteristic is larger than 5, then such a condition is induced by Assumption 5.1. Indeed, we take a closed point \(x \in X\) contained in an irreducible component \(S\) of some closed fiber \(X_s\). Then by Assumption 5.1 (4) and (5), \((X, S)\) is plt. Thus, by the adjunction, \(S\) is a klt surface. Since the characteristic of \(S\) is larger than 5, \(S\) is strongly F-regular. By Proposition 4.10, \(X\) is T-regular at \(x\). In conclusion, \(X\) is T-regular. By Proposition 3.18, \(\mathcal{O}_X(mK_{X/V})\) is maximal Cohen-Macaulay.

**Lemma 5.4.** Let \(V\) be an excellent Dedekind scheme and \(X\) be a \(V\)-variety satisfying Assumption 5.1. Let \(s \in V\) be a closed point. Let \(S_1, \ldots, S_r\) be the irreducible components of \(X_s\). We set \(X_i := S_1 \cup \cdots \cup S_i\) with reduced structure and the scheme-theoretic intersection \(C_i := X_{i-1} \cap S_i\) for \(1 \leq i \leq r\). Then \(X_i\) is reduced and satisfies the condition \((S_2)\) and \(C_i\) is reduced and pure one-dimensional for every \(i\).

**Proof.** Since \(X_s\) is reduced and \(X_s = X_r\) as sets, we have \(X_s = X_r\). In particular, \(X_r\) satisfies the condition \((S_2)\). Since \(X_{i-1}\) and \(S_i\) are \(\mathbb{Q}\)-Cartier divisors on \(X\), the scheme-theoretic intersection \(C_i := X_{i-1} \cap S_i\) is pure one-dimensional. In particular, each generic point of \(C_i\) is codimension two point in \(X\). Since \((X, X_s)\) is a simple normal crossing pair at each generic point of \(C_i\), the scheme \(C_i\) satisfies the condition \((R_0)\). Thus, in order to prove \(C_i\) is reduced, it is enough to show that \(C_i\) satisfies the condition \((S_1)\).

We take a closed point \(P\) of \(C_i\). Then \(P\) is contained in at least two components \(S_i\) and \(S_j\) for some \(j < i\). If \(P\) is contained in three components, then \((X, X_s)\) is a simple normal crossing pair at \(P\), and in particular, \(C_i\) is reduced at \(P\). Thus, we may assume that \(P\) is contained in only two components, so \(X_s = S_i \cap S_j\) around \(P\) and we obtain the exact sequence

\[0 \to \mathcal{O}_{X_s} \to \mathcal{O}_{S_i} \oplus \mathcal{O}_{S_j} \to \mathcal{O}_{C_i} \to 0\]

around \(P\). Since \(X_s\), \(S_i\) and \(S_j\) satisfy the condition \((S_2)\), \(C_i\) satisfies the condition \((S_1)\), so \(C_i\) is reduced for all \(i\). The exact sequence

\[0 \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_{i-1}} \oplus \mathcal{O}_{S_i} \to \mathcal{O}_{C_i} \to 0\]
implies that if $X_{i-1}$ satisfies the condition $(S_2)$, then so is $X_i$ by the reducedness of $C_i$.

\textbf{Proposition 5.5.} Let $\pi: S \to Z$ be a projective morphism from a surface to a variety over an algebraically closed field $k$. Let $(S, D)$ be a dlt pair and $L$ be a $\pi$-nef Cartier divisor such that $L - (K_S + D)$ is $\pi$-ample and $C$ be a reduced Weil divisor with $C \leq D$. Then the following hold.

1. $\pi_*\mathcal{O}_S(mL) \to \pi_*\mathcal{O}_C(mL)$ is surjective for all $i$ and divisible enough $m$.
2. $R^i\pi_*\mathcal{O}_S(mL) = R^i\pi_*\mathcal{O}_C(mL) = 0$ for every $i > 0$ and divisible enough $m$.
3. $L$ is semiample over $Z$.

\textit{Proof.} We note that the semiampleness follows from the abundance, since $L - (K_S + D)$ is ample over $\pi$ and $k$ is infinite. By [Tan20, Theorem 1.1], $L$ is semample over $Z$, thus we may assume that $L$ is a pullback of an ample Cartier divisor on $Z'$, where $\pi': S \to Z'$ is the morphism defined by $L$ over $Z$. In particular, we may assume that $L$ is trivial by replacing $\pi$ into $\pi'$. First, we consider the case where the dimension of $\pi(S)$ is at least one. By a perturbation of coefficients of $D$ and [Kaw94, Lemma 2.1], we have

$$R^i\pi_*\mathcal{O}_S = R^i\pi_*\mathcal{O}_S(-C) = 0$$

for all $i > 0$, thus we obtain the assertion. Next, we assume that $\pi(S)$ is a point. By [Kaw94, Lemma 2.2], we have $H^i(\mathcal{O}_S) = 0$ for all $i > 0$. By [Tan18b, Theorem 5.2], the scheme $C$ is connected, so we have $H^0(\mathcal{O}_C) \simeq k$. Since we have $H^0(\mathcal{O}_S(-C)) = 0$ and $H^0(\mathcal{O}_S) \simeq k$, the map $H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_C)$ is surjective. Thus we have $H^1(\mathcal{O}_S(-C)) = 0$. Since $-(K_S + C)$ is big, we have $H^2(\mathcal{O}_S(-C)) = 0$, so we have $H^1(\mathcal{O}_C) = 0$. \hfill \square

The following theorem is discussed in [Kaw94, Theorem 2.3]. However, we need more detailed observation of contractions, thus we use a bit different method form the method of [Kaw94, Theorem 2.3].

\textbf{Proposition 5.6.} (cf. [Kaw94, Theorem 2.3]) Let $V$ be an excellent Dedekind scheme and $X$ is a $V$-variety satisfying Assumption 5.4. Let $\pi: X \to U$ be a projective morphism to a $V$-variety. Let $s \in V$ be a closed point. Let $S_1, \ldots, S_r$ be the irreducible components of $X_s$. Let $L$ be a $\rho$-nef Cartier divisor with $L - K_{X/V}$ is $\pi$-ample. Then $L$ is semimalple. Furthermore, the map $f: X \to Z$ defined by $L$ satisfies the following conditions.

1. $R^j f_*\mathcal{O}_X = 0$ for all $j > 0$.
2. $\pi_*\mathcal{O}_{S_i} = \mathcal{O}_{f(S_i)}$, where the images are equipped with the reduced structure.
3. $Z$ is Cohen-Macaulay.

\textit{Proof.} Taking a base change via $\iota: V' \to V$ as in Assumption 5.4 (6), we may assume that $V$ is the spectrum of a complete discrete valuation ring with algebraically closed residue field. We set $X_i := S_1 \cup \cdots \cup S_i$ and $C_i := X_{i-1} \cap S_i$ for $1 \leq i \leq r$. We may assume that the conditions (i), (ii) in Proposition 5.5 for $S_i$ are satisfied for $m = 1$ and $R^i\pi_*\mathcal{O}_{X_0}(L) = 0$ by replacing $L$ into some power of $L$. We note that $X_i$ is a pushout of $X_{i-1}$ and $S_i$ with $C_i$, and $C_i$ is reduced and $X_i$ satisfies $(S_2)$ for all $i$ by Lemma 5.4. Thus, we have the surjection $\pi_*\mathcal{O}_{X_i}(L) \to \pi_*\mathcal{O}_{X_{i-1}}(L)$ and the isomorphism
$R^j\pi_*\mathcal{O}_{X_i}(L) \simeq R^j\pi_*\mathcal{O}_{X_{i-1}}(L)$ by Proposition 5.6. By the induction on $i$ and changing the order of $S_1, \ldots, S_r$, we have $R^j\pi_*\mathcal{O}_{X_i}(L) = 0$ for all $j > 0$ and the surjection $\pi_*\mathcal{O}_{X_i}(L) \to \pi_*\mathcal{O}_{S_i}(L)$ for all $i$. By the surjectivity of $\pi_*\mathcal{O}_{X_i}(L) \to \pi_*\mathcal{O}_{S_i}(L)$ and $\pi_*\mathcal{O}_{X_i}(L) \to \pi_*\mathcal{O}_{X_{i-1}}(L)$, if $L|_{S_i}$ and $L|_{X_{i-1}}$ is globally generated over $U$, then so is $L|_{X_i}$. By the induction on $i$, we may assume that $L|_{X_i}$ is globally generated.

By the exact sequence

$$0 \to \mathcal{O}_X(L) \to \mathcal{O}_X(L) \to \mathcal{O}_{X_i}(L) \to 0,$$

where the first map is the multiplication by a uniformizer $\varpi$, we have the surjection

$$R^j\pi_*\mathcal{O}_X(L) \xrightarrow{\varpi} R^j\pi_*\mathcal{O}_X(L)$$

for all $j > 0$. Thus, this vanishes around the closed fiber and the generic fiber, we have $R^j\pi_*\mathcal{O}_X(L) = 0$ for all $j > 0$. Since $L|_{X_s}$ and $L|_{X_n}$ is semiample and we have the surjection $\pi_*\mathcal{O}_X(L) \to \pi_*\mathcal{O}_{X_n}(L)$, $L$ is semiample over $U$.

Next, we prove the assertions (1) and (2), so we may assume that $L$ is trivial. By the above argument, we have $R^j f_*\mathcal{O}_X = 0$ for all $j > 0$ and $f_*\mathcal{O}_X \to f_*\mathcal{O}_{S_i}$ is surjective for all $i$. By the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{\simeq} & \mathcal{O}_Z \\
\downarrow & & \downarrow \\
f_*\mathcal{O}_{S_i} & \xrightarrow{\simeq} & f_*(\mathcal{O}_{S_i})
\end{array}$$

we have $f_*\mathcal{O}_{S_i} = \mathcal{O}_{f(S_i)}$. By the same argument as above, we have $f_*\mathcal{O}_{X_i} = \mathcal{O}_{f(X_i)}$ and $f_*\mathcal{O}_{C_i} = \mathcal{O}_{f(C_i)}$ for all $i$. By the vanishing $R^1 f_*\mathcal{O}_{X_i} = 0$ for all $i$, we have the exact sequence

$$0 \to \mathcal{O}_{f(X_i)} \to \mathcal{O}_{f(S_i)} \oplus \mathcal{O}_{f(X_{i-1})} \to \mathcal{O}_{f(C_i)} \to 0.$$ 

By the induction and the condition $(S_1)$ on $f(C_i)$, the scheme $f(X_i)$ satisfies the condition $(S_2)$. In particular, $Z_s$ satisfies the condition $(S_2)$, thus $Z$ is Cohen-Macaulay.

\begin{proposition}
Let $V$ be an excellent Dedekind scheme and $X$ is a $\mathbb{Q}$-factorial $V$-variety satisfying Assumption 5.1. Let $f: X \to Z$ be a $K_X$-negative extremal ray contraction which is a divisorial contraction. Then $Z$ also satisfies Assumption 5.1.
\end{proposition}

\begin{proof}
By Proposition 5.6, $Z$ is Cohen-Macaulay, thus $Z$ satisfies Assumption 5.1 (3). The other conditions follow from the standard argument.
\end{proof}

\begin{proposition}
Let $V$ be the spectrum of an excellent discrete valuation ring and $X$ is a $\mathbb{Q}$-factorial $V$-variety satisfying Assumption 5.1. Let

$$X \xrightarrow{\phi} Z \xleftarrow{\phi^+} Y$$

be a $K_X$-flip with $\rho(X/Z) = 1$. Assume that there exists irreducible components $S$ and $A$ of the closed fiber $X_s$ such that $-S$ and $A$ are ample. Then the strict transforms $S'$ and $A'$ of $S$ and $A$ on $Y$, respectively, are geometrically normal.
\end{proposition}
Proof. Taking a base change via $V' \to V$ in Assumption 5.1, we may assume that $V$ is the spectrum of a complete local valuation ring with algebraically closed residue field. Since $A$ and $S$ are geometrically irreducible, the irreducibility is preserved. We take a point $z \in \phi(\text{Exc}(\phi))$, by shrinking $Z$ around $z$, we may assume that $\phi(\text{Exc}(\phi)) = \{z\}$ and $Z$ is affine. We denote the image of $S$ and $A$ via $\phi$ by $T$ and $B$, respectively. By Proposition 5.6, $\phi_*O_S = O_T$ and $\phi_*O_A = O_B$, so $T$ and $B$ are normal. By the proof of Lemma 3.32, we may assume that the intersection $C := A \cap S$ is irreducible and not contracted by shrinking $Z$ around $z$. By the perturbation of coefficients, there exists a dlt pair $(X, \Delta)$ such that $|\Delta| = A + S$ and $-(K_{X/V} + \Delta)$ is ample. The strict transform of $\Delta$ on $Y$ is denoted by $\Delta_Y$. Since $X$ and $Y$ have isolated singularities, the different coincides with $(\Delta - S)|_S$ and $(\Delta_Y - S')|_{S'}$. In particular, $(\phi|_S)_*(\Delta - S)|_S = (\phi^+|_{S'})_*(\Delta_Y - S')|_{S'}$, it is denoted by $D_T$. Furthermore, the divisor

$$K_{S'} + (\Delta_Y - S')|_{S'} = (K_{Y/V} + \Delta_Y)|_{S'}$$

is ample. We denote $A|_S$ by $C'$ and the image of $C$ by $C'$ via $\phi|_S$. Since $C \to C'$ is an isomorphism by the proof of Lemma 3.32, $(C', \text{Diff}(D_T))$ is globally T-regular. Therefore, by Corollary 4.10, $(T, D_T)$ is purely globally T-regular. Since $K_{S'} + (\Delta_Y - S')|_{S'}$ is ample, there exists an effective divisor $\Gamma$ of $S'$ such that

$$(\phi^+|_{S'})^*(K_T + D_T) = K_{S'} + (\Delta_Y - S')|_{S'} + \Gamma.$$

Since Proposition 3.19, the pair $(S', K_{S'} + (\Delta_Y - S')|_{S'})$ is globally T-regular, thus $(S', K_{S'} + (\Delta_Y - S')|_{S'})$ is also globally T-regular. Therefore, $S'$ is normal.

Next, we consider the normality of $A'$. By the above argument, $S'|_A$ is not exceptional, irreducible and anti-ample over $B$. Thus, $\phi|_A$ is finite, and in particular, $\phi|_A$ is an isomorphism because of $\phi_*O_A = O_B$. By the same argument as above, $(A'^N, (\Delta_Y - S')|_{A'})$ is globally T-regular, thus $A'$ is normal. $\square$

**Proposition 5.9.** Let $V$ be an excellent Dedekind scheme and $X$ is a $\mathbb{Q}$-factorial $V$-variety satisfying Assumption 5.1. Let

$$X \xrightarrow{\phi} Z \xrightarrow{\phi^+} Y$$

be a $K_X$-flip with $\rho(X/Z) = 1$. Then $Y$ is a scheme satisfying Assumption 5.1.

Proof. It is obvious that $Y$ satisfies Assumption 5.1 except for the condition $(S_2)$ for closed fibers in (3) and the geometric normality of irreducible components of closed fibers in (4). First, we prove that the $(S_2)$ condition of closed fibers, it is enough to show that $Y$ is Cohen-Macaulay. We take a point $z \in \phi(\text{Exc}(\phi))$, by shrinking $Z$ around $z$, we may assume that $\phi(\text{Exc}(\phi)) = \{z\}$ and $Z$ is affine. In particular, we may assume that $V$ is a discrete valuation ring by localizing at the image $s$ of $z$. By Proposition 5.6, we have $R^1\phi_*O_X = 0$ and $Z$ is Cohen-Macaulay. Thus $Z$ has rational singularities by Proposition 2.16. Since $X$ is terminal, so is $Y$, and in particular, $Y$ has isolated singularities. Thus $Z$ is Cohen-Macaulay.

Next, we prove the geometric normality of irreducible components of $X_s$. First, we consider the case where $f^{-1}(z)$ is contained in only one irreducible component of $X_s$. Thus, $Y$ is Cohen-Macaulay.
Then, we may assume that $X_s$ is irreducible. Thus, $X_s$ is geometrically irreducible, and so is $Y_s$. After taking base change $V' \to V$ as in Assumption 5.1 (6), the pair $(Y, Y_s)$ is dlt and $Y_s$ is irreducible. Thus, $Y_s$ is normal in codimension one. Since $Y$ is Cohen-Macaulay, so is $Y_s$, thus $Y_s$ is normal. In conclusion, $Y_s$ is geometrically normal.

Next, we consider the other case. The extremal ray contracted by $\phi$ is denoted by $\Sigma$. We take an irreducible component $R$ of $Y_s$ such that $\phi^+(R)$ contains $z$. We write $T := \phi^+_s R$ and $S := \phi^{-1}_s T$. Suppose $(S \cdot \Sigma) = 0$, then $f^{-1}(z)$ is contained in $S$. By Lemma 4.11 every flipping curve is not contained in any other components. By the assumption, $f^{-1}(z)$ intersects another component $S'$ of $X_s$. If $(S' \cdot \Sigma) \leq 0$, then $S'$ contains flipping curves, so we have a contradiction. If $(S' \cdot \Sigma)$ is positive, then replacing $S'$, we have $(S' \cdot \Sigma)$ is negative because of $X_s \sim 0$. In conclusion, we obtain $(S \cdot \Sigma) \neq 0$.

First, we consider the case where $(S \cdot C)$ is positive. Then there exists an irreducible component $S'$ of $X_s$ such that $(S' \cdot C)$ is negative. In particular, the divisors $S$ and $-S'$ are ample. By Proposition 5.8, the scheme $R$ is geometrically normal. On the other hand, if $(S \cdot C)$ is negative, then there exists an irreducible component which is ample. Thus, by Proposition 5.8, the scheme $R$ is geometrically normal.

**Theorem 5.10.** (Theorem 1.1, cf. [Kaw94]) Let $V$ be an excellent Dedekind scheme. Let $X$ be a $V$-variety satisfying Assumption 5.1. Then we can run a $K_{X/V}$-MMP over $Z$ preserving Assumption 5.1 which terminates with a minimal model or a Mori fiber space.

*Proof.* We note that $(K_{X/V} + X_s)$-MMP is also $K_{X/V}$-MMP because $X_s$ is linearly trivial. By the cone theorem (Proposition 4.4) and the contraction theorem (Proposition 5.6), we can contract any $K_{X/V}$-negative extremal ray. Let $f : X \to Z$ be a $K_X$-negative extremal ray contraction. If $f$ is divisorial contraction, $Z$ also satisfies Assumption 5.1 by Proposition 5.8. If $f$ is flipping contraction, the extremal ray contracted by $f$ is denoted by $\Sigma$. In order to prove the existence of the flip, we may assume that $V$ has the unique closed point $s$. If $(S \cdot \Sigma) = 0$ for every irreducible component $S$ of $X_s$, then the flip of $f$ exists by Proposition 4.12. Otherwise, as $X_s$ is linearly trivial, there exists irreducible components $S$ and $A$ of $X_s$ such that $(S \cdot \Sigma) < 0$ and $(A \cdot \Sigma) > 0$. By Corollary 3.34, the flip of $f$ exists.

Then the flip $X \dashrightarrow Y$ of $f$ exists and $Y$ satisfies Assumption 5.1 by Proposition 5.9. Since $X$ has terminal singularities, a sequence of flip terminates by the argument in [KM98, Theorem 6.17]. Thus, $K_{X/V}$-MMP terminates with a minimal model or a Mori fiber space. □

In the following, we review applications of Theorem 1.1 which are discussed in [CL16].

**Definition 5.11** (cf. [CL16, Definition 5.1]). Let $O_K$ be an excellent Henselian discrete valuation ring with perfect residue field $k$. Let $K$ be the fraction field of $O_K$. Let $X$ be a K3 surface over $K$ or an abelian surface over $K$. Here, we note that an abelian surface does not necessarily admit a section. Then a minimal strictly semi-stable model of $X$ is a proper algebraic space $\mathcal{X}$ over $O_K$ satisfying the following.

(1) The generic fiber $\mathcal{X}_K$ is isomorphic to $X$.

(2) The special fiber $\mathcal{X}_s$ is a scheme whose irreducible components are smooth over $k$. 

□
(3) There exists an étale surjection $U \to X$ such that $U$ is a strictly semi-stable scheme in the sense of Definition 2.10.

(4) The relative dualizing sheaf $\omega_{X/O_K}$ is trivial. Here, see [CL16, Section 5] for the definition of the relative dualizing sheaf.

**Theorem 5.12.** Let $O_K$, $K$, $k$ and $X$ be as in Definition 5.11. Suppose one of the following.

1. The scheme $X$ is an abelian surface over $K$.
2. The scheme $X$ is a K3 surface over $K$ satisfying that $X$ admits a projective strictly semi-stable scheme model over $O_K$.

Then there exists a finite separable extension $K'/K$ such that there exists a minimal strictly semi-stable model over $O_{K'}$ of $X_{K'}$. Moreover, in the case (1), we can take a finite separable extension $K'/K$ such that there exists a minimal strictly semi-stable scheme model over $O_{K'}$.

**Proof.** By Theorem 1.1 the first half of this theorem follows from the proof of [CL16, Theorem 10.3] and the proof of [LM18, Proposition 2.1]. Now we will consider the case (1). By [Kim98], there exists a finite separable extension $K'/K$ such that there exists a strictly semi-stable scheme $Y$ over $O_{K'}$ whose smooth locus is isomorphic to the smooth locus of the Néron model of $X_{K'}$. Then we can run a $K'/O_{K'}$-MMP by Theorem 1.1. We will see that flips do not occur in this MMP. Let $Y = Y_0 \to \cdots \to Y_r \to Y_{r+1}$ be a part of the MMP, where $Y_0 \to \cdots \to Y_r$ is the composition of divisorial contractions.

Let $C \subset Y_{r+1}$ be a flipped curve of $Y_r \to Y_{r+1}$. Since the morphism $(Y_{r+1})_{K'} \to (Y_r)_{K'}$ extends over the smooth locus $Y_{r+1,sm}$, the generic point of $y \in C$ does not contained in $Y_{r+1,sm}$. Since $(Y_{r+1}, Y_{r+1,sm})$ is geometrically dlt, $y$ is contained in the intersection of two irreducible components of $Y_{r,sm}$. Therefore, there exists an exceptional prime divisor $E$ over $Y_{r+1}$ such that $a_E(Y_r, Y_{r,sm}) = 0$ and the center of $E$ on $Y$ is $C$. Since $C$ is a flipped curve, we have $a_E(Y_r, Y_{r,sm}) < a_E(Y_{r+1}, Y_{r+1,sm})$. This is the contradiction since $(Y_r, Y_{r,sm})$ is geometrically dlt. Let $Y'$ be the output variety of this MMP. By the Néron mapping property, we can show that

$$\#\text{irreducible components of } Y_s = \#\text{irreducible components of } Y'_s.$$ 

Therefore, we have $Y = Y'$ and $K_{Y/O_{K'}}$ is nef. By the argument in [Mau14, Lemma 4.7], $K_{Y/O_{K'}}$ is trivial. □

The dual graph of the special fiber of a minimal strictly semi-stable model is classified in [CL16] in the case where char $k \neq 2$. We will verify that their result holds even in char $k = 2$.

**Theorem 5.13.** Let $O_K$, $K$, $k$, and $X$ be as in Definition 5.11. Let $X$ be a minimal strictly semi-stable model over $O_K$. Then the special fiber $X_k$ is combinatorial in the sense of [CL16, Definition 5.4, Definition 5.6].

**Proof.** If $X$ is a K3 surface, it follows from the same argument as in the proof of [CL16, Proposition 5.3, Theorem 6.1]. Therefore, we will treat the case where $X$ is an abelian surface. We note that the case where char $k \neq 2$ follows from the proof of [CL16, Proposition 5.3, Theorem 8.1]. In Chiarellotto and Lazda’s argument, the assumption
char $k \neq 2$ is used only in the case where (2) (b) in [CL16, p.2253] holds for some irreducible component of the special fiber $\mathcal{X}_k$. We will review their arguments in this case. They show that the dual graph $\Gamma$ of the special fiber $\mathcal{X}_k$ is a triangulation of a compact real surface $M$ without border. The spectral sequence of coherent cohomologies shows that

$$\dim_k H^i_{\text{sing}}(\Gamma, k) = \begin{cases} 1 & \text{if } i = 0, 2, \\ 2 & \text{otherwise.} \end{cases}$$

It implies that $M$ is a torus if char $k \neq 2$ since the left hand side is equal to $C$-Betti number by the classification of real surfaces. On the other hand, in the weight spectral sequence as in the proof of [CL16, Theorem 8.3], we have $\dim_{\mathbb{Q}_\ell} E_1^{1,0} = 2$. Moreover, we have $E_1^{0,1} = 0$ by [CL16, Lemma 4.2]. Since this spectral sequence degenerates at $E_2$, we have $\dim_{\mathbb{Q}_\ell} E_2^{1,0} = 2$. Thus, $E_2^{1,0} = H^1_{\text{sing}}(\Gamma, \mathbb{Q}_\ell)$ as in the proof of [CL16, Theorem 8.3], we have $\dim_{\mathbb{Q}_\ell} H^1_{\text{sing}}(\Gamma, \mathbb{Q}_\ell) = 2$. Therefore, the surface $M$ is a torus even in char $k = 2$. □

In the case where $X$ is an abelian surface, the strictly semi-stable scheme model which we obtained in the proof of Theorem 5.12 gives a compactification of a Néron model of $X$. Conversely, the following proposition, which is proved in [JM94, Theorem 1.4] in the case where $X$ is a scheme, shows that a minimal strictly semi-stable model gives a compactification of a Néron model.

**Proposition 5.14.** Let $\mathcal{O}_K$, $K$, and $k$ be as in Definition 5.11. Let $X$ be an abelian surface over $K$. Let $\mathcal{X}$ be a minimal strictly semi-stable model of $X$ over $\mathcal{O}_K$, and $\mathcal{X}^{\text{sm}}$ the smooth locus of $\mathcal{X}$. Then $\mathcal{X}^{\text{sm}}$ is a scheme and a Néron model of $X$ over $\mathcal{O}_K$.

**Proof.** Let $\mathcal{Y}$ be a Néron model of $X$ over $\mathcal{O}_K$. By the descent argument, one can show that the Néron mapping property holds for algebraic spaces. Therefore, we have a morphism $f : \mathcal{X}^{\text{sm}} \to \mathcal{Y}$ which extends an identity on $X$. It is enough to show that $f$ is an isomorphism. We note that we have $\mathcal{X}(\mathcal{O}_K^{\text{sh}}) \neq \emptyset$ by Hensel’s Lemma, where $\mathcal{O}_K^{\text{sh}}$ is the strict henselization of $\mathcal{O}_K$. By [BLR90, Section 7.2, Theorem 1], we may assume that $X$ admits a section. First, we will show that $f$ is an étale morphism. It suffices to show that $f \circ u$ is étale for any étale morphism $u : U \to \mathcal{X}^{\text{sm}}$. By [BLR90, Section 2.2, Corollary 10], we want to show that $\Phi : (f \circ u)^* \Omega^2_{\mathcal{Y}/\text{Spec } \mathcal{O}_K} \to \Omega^2_{U/\text{Spec } \mathcal{O}_K}$ is an isomorphism. Since $\mathcal{X}^{\text{sm}}$ is a scheme in codimension 1, by using [BLR90, Section 4.3, Lemma 1], we have $\Phi$ is an isomorphism in codimension 1, so $\Phi$ is an isomorphism. Now we have $f$ is étale. By Zariski’s main theorem, the morphism $f$ is an open immersion. Let $Z$ be a complement $\mathcal{Y} \setminus \mathcal{X}^{\text{sm}}$. Suppose that $Z \neq \emptyset$. Take a valued point $z \in Z(\overline{\mathbb{K}})$. By Hensel’s lemma, the valued point $z$ lifts to $\overline{z} \in \mathcal{Y}(\mathcal{O}_K^{\text{sh}}) = X(K^{\text{sh}}) = \mathcal{X}(K^{\text{sh}}) = \mathcal{X}(\mathcal{O}_K^{\text{sh}})$, but it contradicts to the choice of $z$ (cf. the proof of [JM94, Theorem 1.4]). Therefore, $f$ is an isomorphism. □

6. More general relative MMP

The goal of this section is to prove the relative MMP in more general setting (Theorem 6.2) and the finite generation of relative canonical ring (Theorem 6.3). The first one is an analog of [HW19a, Theorem 1.6] in mixed characteristic. In this section,
\[ \text{[Wit20, Theorem 1.8]} \] plays an essential role to prove such theorems. We note that \[ \text{[Wit20, Theorem 1.8]} \] will be proved in the upcoming paper \[ \text{[Wit21]} \].

**Proposition 6.1.** Let \( V \) be the spectrum of an excellent discrete valuation ring. Let \( X \) be a flat \( V \)-variety of relative dimension two. Let \( (X, \Delta) \) be a \( \mathbb{Q} \)-factorial dlt log pair with \( X_s \subset [\Delta] \) as sets, where \( X \) is flat \( V \)-variety of relative dimension two. Let \( \rho: X \to U \) be a projective morphism to a quasi-projective \( V \)-variety \( U \). Let \( L \) be a \( \rho \)-nef Cartier divisor on \( X \) with \( L^+ = \mathbb{R}[\Sigma] \), where \( \Sigma \) is a \((K_X + \Delta)\)-negative extremal ray over \( \rho \). Then \( L \) is semiample.

**Proof.** Replacing \( L \) with \( mL \) for large enough \( m \), we may assume that \( L - (K_{X/V} + \Delta) \) is ample over \( U \). Let \( S_1, \ldots, S_r \) be the irreducible components of \( X_s \). We denote the different \( \text{Diff}_{S_i}(\Delta - S_i) \) by \( D_i \), then \((S_i^N, D_i)\) is dlt and \( L - (K_{S_i^N} + D_i) \) is ample over \( U \). We note that the normalization \( S_i^N \to S_i \) is a universal homeomorphism by Lemma 2.3. By \[ \text{[Tan20, Theorem 1.1]} \] and the proof of \[ \text{[Kee99, Lemma 1.4]} \], \( L_{|S_i} \) is semiample for all \( i \). We denote the map induced by \( L_{|S_i} \) by \( \phi_i: S_i \to T_i \). Since \(- (K_{S_i^N} + D_i) \) is ample over \( T_i \), the morphism \( \phi_i|_{S_iU_{\emptyset \cup \cdots \cup U_{\emptyset \cup \cdots \cup S_i^N}}} \) has connected fibers by \[ \text{[Tan18b, Theorem 1.4]} \]. By \[ \text{[Kee99, Corollary 2.9]} \] and the induction on \( i \), \( L_{|X_s} \) is semiample. Furthermore, \( L_{|X_s} \) is semiample by the base point free theorem. Thus, by \[ \text{[Wit20, Theorem 1.8]} \], \( L \) is also semiample.

**Theorem 6.2.** (cf. \[ \text{[HW19a, Theorem 1.6]} \]) Let \( V \) be the spectrum of an excellent discrete valuation ring. Let \( (X, \Delta) \) be a \( \mathbb{Q} \)-factorial dlt pair over \( V \), where \( X \) is a flat \( V \)-variety of relative dimension two. Let \( X \to Z \) be a projective morphism over \( V \) to a quasi-projective \( V \)-variety \( Z \). Assume that \( [\Delta] \) contains the closed fiber \( X_s \) as sets. Then we can run a \((K_{X/V} + \Delta)\)-MMP over \( Z \) which terminates with a minimal model or a Mori fiber space.

**Proof.** By the cone theorem (Proposition 4.4) and the contraction theorem (Proposition 6.1), we can contract any \((K_{X} + \Delta)\)-negative extremal ray. If the contraction is a flipping contraction, then the flip exists by the argument in the proof of Theorem 5.10. The termination of flips follows from the special termination (Proposition 4.5). □

**Theorem 6.3.** Let \( V \) be an excellent Dedekind scheme whose each residue field is perfect. Let \( X \) be a flat projective \( V \)-variety of relative dimension two. Let \( (X, \Delta) \) be a dlt pair over \( V \) such that for each closed point \( x \in V \), the pair \( (X, \Delta + X_s) \) is dlt or \( X_s \) is contained in \( [\Delta] \) as sets. Then
\[
R(K_{X/V} + \Delta) := \bigoplus_m H^0(X, \mathcal{O}_X(m(K_{X/V} + \Delta)))
\]
is finitely generated.

**Proof.** We may assume that \( V \) is the spectrum of an excellent discrete valuation ring and \([\Delta]\) contains the closed fiber \( X_s \) as sets. If \( K_{X/V} + \Delta \) is not pseudo-effective, the theorem is trivial, thus, we may assume that \( K_{X/V} + \Delta \) is pseudo-effective. By Theorem 6.2, we can run a \((K_{X/V} + \Delta)\)-MMP over \( V \), thus, we may assume that \( L := K_{X/V} + \Delta \) is nef over \( V \). By \[ \text{[Wit20, Theorem 1.8]} \], it is enough to show that \( L_{|X_s} \) is semiample over \( V \). By the argument of the proof of \[ \text{[HW19a, Theorem 1.6]} \], the pair \((W, \Delta_W)\) is a
sdlt surface and \( \pi \) is a universal homeomorphism, where \( \pi : W \to X_s \) is the \( S_2 \)-ification and \( L_W := K_W + \Delta_W = \pi^*((K_X + \Delta)|_{X_s}) \), thus it is enough to show that \( L_W \) is semiample over \( U \). By [Tan16, Theorem 0.1], it is semiample. \( \square \)

**Remark 6.4.** Theorem 6.3 holds in the relative setting if \( p \neq 2, 3, 5 \). More precisely, if \( (X, \Delta), V \) are the same as in Theorem 6.3 and \( \rho : X \to U \) is a projective morphism to a projective \( V \)-variety \( U \), then

\[
R(K_{X/V} + \Delta/U) := \bigoplus_m \rho_*O_X(m(K_{X/V} + \Delta))
\]

is finitely generated \( O_U \)-algebra. Indeed, by the same argument, we may assume that \( L := K_{X/V} + \Delta \) is nef over \( U \). By [BMP+20, Theorem H], there exists an ample Cartier divisor \( H \) on \( U \) such that \( L + \rho^*H \) is nef over \( V \). By the proof of Theorem 6.3 and [Tan17, Theorem 1], we obtain the semiampleness of \( L + \rho^*H \).

**Corollary 6.5.** Let \( V \) be an excellent Dedekind scheme. Let \( X \) be a projective \( V \)-variety satisfying Assumption 5.1. Then

\[
R(K_{X/V}) := \bigoplus_m H^0(X, O_X(mK_{X/V}))
\]

is finitely generated.

**Proof.** By Assumption 5.1 (6), we may assume that \( V \) is the spectrum of a discrete valuation ring with perfect residue field. Thus, Theorem 6.5 follows from Theorem 6.3. \( \square \)

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