Efficient Representation and Counting of Antipower Factors in Words

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Abstract

A \(k\)-antipower (for \(k \geq 2\)) is a concatenation of \(k\) pairwise distinct words of the same length. The study of antipower factors of a word was initiated by Fici et al. (ICALP 2016) and first algorithms for computing antipower factors were presented by Badkobeh et al. (Inf. Process. Lett., 2018). We address two open problems posed by Badkobeh et al. Our main results are algorithms for counting and reporting factors of a word which are \(k\)-antipowers. They work in \(O(nk \log k)\) time and \(O(nk \log k + C)\) time, respectively, where \(C\) is the number of reported factors. For \(k = o(\sqrt{n} / \log n)\), this improves the time complexity of \(O(n^2/k)\) of the solution by Badkobeh et al. Our main algorithmic tools are runs and gapped repeats. We also present an improved data structure that checks, for a given factor of a word and an integer \(k\), if the factor is a \(k\)-antipower.

1 Introduction

Antipowers are a new type of regularity of words, based on diversity rather than on equality, that has been recently introduced by Fici et al. in \cite{7,8}. Typical types of regular words are powers. If equality is replaced by inequality, other versions of powers are obtained.

Let us assume that \(x = y_1 \cdots y_k\), where \(k \geq 2\) and \(y_i\) are words of the same length \(d\). We then say that:

\(\bullet\) \(x\) is a \(k\)-power if all \(y_i\)'s are the same;

\(\bullet\) \(x\) is a \(k\)-antipower (or a \((k,d)\)-antipower) if all \(y_i\)'s are pairwise distinct;

\(\bullet\) \(x\) is a weak \(k\)-power (or a weak \((k,d)\)-power) if it is not a \(k\)-antipower, that is, if \(y_i = y_j\) for some \(i \neq j\);

\(\bullet\) \(x\) is a gapped \((q,d)\)-square if \(y_1 = y_k\) and \(q = k - 2\).

In the first three cases, the length \(d\) is called the base of the power or antipower \(x\).

If \(w\) is a word, then by \(w[i \cdots j]\) we denote a word composed of letters \(w[i] \ldots w[j]\) called a factor of \(w\). A factor can be represented in \(O(1)\) space by the indices \(i\) and \(j\). Badkobeh et al. \cite{1} considered factors of a word that are antipowers and obtained the following result.

Fact 1.1 (\cite{1}). The maximum number of \(k\)-antipower factors in a word of length \(n\) is \(\Theta(n^2/k)\), and they can all be reported in \(O(n^2/k)\) time. In particular, all \(k\)-antipower factors of a specified base \(d\) can be reported in \(O(n)\) time.

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Badkobeh et al. [I] asked for an output-sensitive algorithm that reports all \(k\)-antipower factors in a given word. We present such an algorithm. En route to enumerating \(k\)-antipowers, we (complementarily) find weak \(k\)-powers. Also gapped \((q,d)\)-squares play an important role in our algorithm.

For a given word \(w\), an antipower query \((i,j,k)\) asks to check if a factor \(w[i..j]\) is a \(k\)-antipower. Badkobeh et al. [I] proposed the following solutions:

**Fact 1.2** ([I]). Antipower queries can be answered (a) in \(O(k)\) time with a data structure of size \(O(n)\); (b) in \(O(1)\) time with a data structure of size \(O(n^2)\).

In either case, answering \(n\) antipower queries using Fact 1.2 requires \(\Omega(n^2)\) time in the worst case. We show a trade-off between the data structure space and query time that allows answering any \(n\) antipower queries more efficiently.

**Our results.** Our first main result is an algorithm that computes the number \(C\) of factors of a word of length \(n\) that are \(k\)-antipowers in \(O(nk \log k)\) time and reports all of them in \(O(nk \log k + C)\) time. We assume an integer alphabet \(\{1,\ldots,n\}\).

Our second main result is a construction in \(O(n^2/r)\) time of a data structure of size \(O(n^2/r)\), for any \(r \in \{1,\ldots,n\}\), which answers antipower queries in \(O(r)\) time. Thus, any \(n\) antipower queries can be answered in \(O(n\sqrt{n})\) time and space.

**Structure of the paper.** Our algorithms are based on a relation between weak powers and two notions of periodicity of words: gapped repeats and runs. In Section 2 we recall important properties of these notions. Section 3 shows a simple algorithm that counts \(k\)-antipowers in a word of length \(n\) in \(O(nk^2)\) time. In Section 4 it is improved in three steps to an \(O(nk \log k)\)-time algorithm. One of the steps applies static range trees that are recalled in Appendix A. Finally, algorithms for reporting \(k\)-antipowers and answering antipower queries are presented in Section 5. The reporting algorithm makes a more sophisticated application of the static range tree that is also described in Appendix A.

## 2 Preliminaries

The length of a word \(w\) is denoted by \(|w|\) and the letters of \(w\) are numbered 0 through \(|w| - 1\), with \(w[i]\) representing the \(i\)th letter. Let \([i..j]\) denote the integer interval \(\{i, i + 1, \ldots, j\}\) and \([i..j]\) denote \([i..j - 1]\). By \(w[i..j]\) we denote the factor \(w[i\ldots j]\); if \(i > j\), it denotes the empty word. Let us further denote \(w[i..j] = w[i\ldots j - 1]\). We say that \(p\) is a period of the word \(w\) if \(w[i] = w[i + p]\) holds for all \(i \in [0\ldots|w| - p]\).

An \(\alpha\)-gapped repeat \(\gamma\) (for \(\alpha \geq 1\)) in a word \(w\) is a factor \(uvw\) of \(w\) such that \(|uv| \leq \alpha |u|\). The two occurrences of \(u\) are called arms of the \(\alpha\)-gapped repeat and \(|uv|\), denoted \(\text{per}(\gamma)\), is called the period of the \(\alpha\)-gapped repeat. Note that an \(\alpha\)-gapped repeat is also an \(\alpha'\)-gapped repeat for every \(\alpha' > \alpha\). An \(\alpha\)-gapped repeat is called maximal if its arms can be extended simultaneously with the same character neither to the right nor to the left. In short, we call maximal \(\alpha\)-gapped repeats \(\alpha\)-MGRs and the set of \(\alpha\)-MGRs in a word \(w\) is further denoted by \(\text{MGReps}_\alpha(w)\). The first algorithm for computing \(\alpha\)-MGRs was proposed by Kolpakov et al. [14]. It was improved by Crochemore et al. [8], Tanimura et al. [13], and finally Gawrychowski et al. [9], who showed the following result.

**Fact 2.1** ([9]). Given a word \(w\) of length \(n\) and a parameter \(\alpha\), the set \(\text{MGReps}_\alpha(w)\) can be computed in \(O(n\alpha)\) time and satisfies \(|\text{MGReps}_\alpha(w)| \leq 18\alpha n\).

A run (a maximal repetition) in a word \(w\) is a triple \((i,j,p)\) such that \(w[i..j]\) is a factor with the smallest period \(p\), \(2p \leq j - i + 1\), that can be extended neither to the left nor to the right preserving the period \(p\). Its exponent \(e\) is defined as \(e = (j - i + 1)/p\). Kolpakov and Kucherov [11] showed that a word of length \(n\) has \(O(n)\) runs, with sum of exponents \(O(n)\), and that they can be computed in \(O(n)\) time. Bannai et al. [2] recently refined these combinatorial results.

**Fact 2.2** ([2]). A word of length \(n\) has at most \(n\) runs, and the sum of their exponents does not exceed \(3n\). All these runs can be computed in \(O(n)\) time.
A generalized run in a word $w$ is a triple $\gamma = (i, j, p)$ such that $w[i \ldots j]$ is a factor with a period $p$, not necessarily the shortest one, $2p \leq j - i + 1$, that can be extended neither to the left nor to the right preserving the period $p$. By $\text{per}(\gamma)$ we denote $p$, called the period of the generalized run $\gamma$. The set of generalized runs in a word $w$ is denoted by $G\text{Runs}(w)$.

A run $(i, j, p)$ with exponent $e$ corresponds to $\lfloor \frac{e}{p} \rfloor$ generalized runs $(i, j, p)$, $(i, j, 2p)$, $(i, j, 3p)$, \ldots, $(i, j, \lfloor \frac{e}{p} \rfloor p)$. By Fact 2.2 we obtain the following

**Corollary 2.3.** For a word $w$ of length $n$, the set $G\text{Runs}(w)$ satisfies $|G\text{Runs}(w)| \leq 1.5n$, and it can be computed in $O(n)$ time.

Our algorithm uses a relation between weak powers, $\alpha$-MGRs, and generalized runs; see Fig. 1 for an example presenting the interplay of these notions.

![Figure 1: To the left: all weak (4, 2)-powers and one (4, 2)-antipower in a word of length 16. An asterisk denotes any character. The first five weak (4, 2)-powers are generated by the run $ababa$ with period 2, and the last three are generated by the 1.5-MGR $bacbacb$, whose period (6) is divisible by 2. To the right: all weak (4, 3)-powers in the same word are generated by the same MGR because its period is a multiple of 3.](image)

An interval representation of a set $X$ of integers is

$$X = [i_1 \ldots j_1] \cup [i_2 \ldots j_2] \cup \cdots \cup [i_t \ldots j_t],$$

where $i_1 \leq j_1$, $j_1 + 1 < i_2$, $i_2 \leq j_2$, \ldots, $j_{t-1} + 1 < i_t$, $i_t \leq j_t$; the value $t$ is called the size of the representation. The following simple lemma allows implementing unions on interval representations.

**Lemma 2.4.** Assume that $X_1, \ldots, X_r$ are non-empty families of subintervals of $[0 \ldots n)$. The interval representations of $\bigcup X_1, \bigcup X_2, \ldots, \bigcup X_r$ can be computed in $O(n + m)$ time, where $m$ is the total size of the families $X_i$.

**Proof.** We start by sorting the endpoints of the intervals and grouping them by the index $i$ of the family $X_i$. This can be done in $O(n + m)$ time using bucket sort. Next, to compute the interval representation of $\bigcup X_i$, we scan the endpoints left to right maintaining the number of intervals containing the current point. We start an interval when this number becomes positive and end one when it drops to 0. This processing takes $O(m)$ time. $\square$

Let $J$ be a family of subintervals of $[0 \ldots m)$, initially empty. Let us consider the following operations on $J$, where $I$ is an interval: $\text{insert}(I) := J \cup \{I\}$; $\text{delete}(I) := J \setminus \{I\}$; and $\text{count}$, which returns $|\bigcup J|$. It is folklore knowledge that all these operations can be performed efficiently using a static range tree (sometimes called a segment tree; see [12]). In Appendix A we prove the following lemma for completeness.

**Lemma 2.5.** There exists a data structure of size $O(m)$ that, after $O(m)$-time initialization, supports $\text{insert}$ and $\text{delete}$ in $O(\log m)$ time and $\text{count}$ in $O(1)$ time.

Let us introduce another operation $\text{report}$ that returns all elements of the set $A = [0 \ldots m) \setminus \bigcup J$. We also show in Appendix A that a static range tree can support this operation efficiently.

**Lemma 2.6.** There exists a data structure of size $O(m)$ that, after $O(m)$-time initialization, supports $\text{insert}$ and $\text{delete}$ in $O(\log m)$ time and $\text{report}$ in $O(|A|)$ time.
Figure 2: An occurrence of a gapped \((q, d)\)-square generated by a gapped repeat with period \((q + 1)d\). Gray rectangles represent equal words.

3 Computing a compact representation of weak \(k\)-powers

Let us denote by \(\text{Squares}(q, d)\) the set of starting positions of occurrences of gapped \((q, d)\)-squares in the input word \(w\).

We say that an occurrence at position \(i\) of a gapped \((q, d)\)-square is \emph{generated} by a gapped repeat \(uvu\) if the gapped repeat has period \(p = (q + 1)d\) and \(w[i..i + d], w[i + p..i + p + d]\) are contained in the first arm and in the second arm of the gapped repeat, respectively; cf. Fig. 2. In other words, \(u = u_1u_2u_3, |u_2| = d, |u_3vu_4| = qd\), and \(uvu\) starts in the input word at position \(i - |u_1|\).

Similarly, an occurrence in \(w\) of a \((q, d)\)-square is \emph{generated} by a generalized run with period \(p = (q + 1)d\) if it is fully contained in this generalized run. See Fig. 3 for a concrete example.

![Figure 3: An occurrence of a gapped (2, 4)-square acaabcaabbcc generated by a generalized run with period 12. Note that the generalized run has its origin in a (generalized) run with period 6 (depicted below) that does not generate this gapped square.](image)

**Lemma 3.1.**

(a) Every gapped \((q, d)\)-square is generated by a \((q + 1)\)-MGR with period \((q + 1)d\) or by a generalized run with period \((q + 1)d\).

(b) Each gapped repeat and each run \(\gamma\) with period \((q + 1)d\) generates a single interval of positions where gapped \((q, d)\)-squares occur, denoted \(\text{Squares}(q, d, \gamma)\) (see Fig. 2). Moreover, this interval can be computed in constant time.

**Proof.** (a) Let \(i\) be the starting position of an occurrence of a gapped \((q, d)\)-square \(x\) of length \(\ell := |x| = (q + 2)d\). Observe that \(x\) has period \(p := (q + 1)d\). We denote by \(\gamma = w[i..i + \ell']\) the longest factor with period \(p\) that contains \(x\) (i.e., such that \(i' \leq i\) and \(i + \ell' - 1 \leq j'\)).

If \(|y| < 2p\), then \(\gamma\) is a gapped repeat with period \(p\), and it is maximal by definition. Moreover, it is a \((q + 1)\)-MGR since its arms have length at least \(d\). Otherwise (if \(|\gamma| \geq 2p\)), the factor \(\gamma\) corresponds to a generalized run \((i', j', p)\) that generates the gapped square \(x\). In particular, this happens for \(q = 0\).

(b) Let \(\gamma\) be a gapped repeat or a generalized run with length \(\ell\) and period \(p = (q + 1)d\) that starts at position \(i\) in \(w\). Then \(\gamma\) generates gapped \((q, d)\)-squares that start at positions in \([i..i + \ell - (p + d)]\).

Let us denote

\[
\text{Chain}_k(q, d, i) = \{i, i - d, i - 2d, \ldots, i - (k - q - 2)d\}
\]

This definition can be extended to intervals \(I\). To this end, let us introduce the operation

\[
I \ominus r = \{i - r : i \in I\}
\]

and define \(\text{Chain}_k(q, d, I) = I \cup (I \ominus d) \cup (I \ominus 2d) \cup \cdots \cup (I \ominus (k - q - 2)d)\). This set is further referred to as an \emph{interval chain}; it can be stored in \(O(1)\) space.
We denote by $\text{WeakPow}_k(d)$ the set of starting positions in $w$ of weak $(k, d)$-powers. A chain representation of a set of integers is its representation as a union of interval chains. The size of the chain representation is the number of chains. The following lemma shows how to compute small chain representations of the sets $\text{WeakPow}_k(d)$.

**Lemma 3.2.**

(a) $\text{WeakPow}_k(d) = \bigcup_{q=0}^{k-2} \bigcup_{i \in \text{Squares}(q, d)} \text{Chain}_k(q, d, i) \cap [0 \ldots n - kd]$.

(b) $\text{WeakPow}_k(d) = \bigcup_{q=0}^{k-2} \bigcup\{\text{Chain}_k(q, d, I) : \gamma \in \text{MGReps}_{q+1}(w) \cup \text{GRuns}(w), \text{ where per}(\gamma) = (q+1)d \text{ and } I = \text{Squares}(q, d, \gamma)\} \cap [0 \ldots n - kd]$.

(c) For $d = 1, \ldots, \lfloor n/k \rfloor$, the sets $\text{WeakPow}_k(d)$ have chain representations of total size $O(nk^2)$ which can be computed in $O(nk^2)$ time.

**Proof.** As for point (a), $x = y_1 \ldots y_k$ for $|y_1| = \cdots = |y_k| = d$ is a weak $(k, d)$-power if and only if $y_i \cdots y_j$ is a gapped $(j - i - 1, d)$-square for some $1 \leq i < j \leq k$. Conversely, a gapped $(q, d)$-square occurring at position $i$ implies occurrences of weak $(k, d)$-powers at positions in the set $\text{Chain}_k(q, d, i)$, limited to the interval $[0 \ldots n - kd]$ due to the length constraint; see Fig. 5.

The formula in (b) follows from point (a) by Lemma 3.1. Indeed, Lemma 3.1(a) shows that every gapped $(q, d)$-square is generated by a $(q+1)$-MGR with period $(q+1)d$ or a generalized run with period $(q+1)d$. By Lemma 3.1(b), the starting positions of all such gapped squares that are generated by an MGR or a generalized run γ form an interval $I = \text{Squares}(q, d, \gamma)$. Hence, it yields an interval chain $\text{Chain}_k(q, d, I)$ of starting positions of weak $(k, d)$-powers by point (a).

Finally, we obtain point (c) by applying the formula from point (b) to compute the chain representations of sets $\text{WeakPow}_k(d)$ for $d = 1, \ldots, \lfloor n/k \rfloor$. This is also shown in the first part of the following SimpleCount algorithm, where the resulting chain representations are denoted as $\mathcal{C}_d$. The total number of interval chains in these representations is $O(nk^2)$ because, for each $q \in [0 \ldots k - 2]$, the number of $(q+1)$-MGRs and generalized runs $\gamma$ is bounded by $O(nk)$ due to Facts 2.1 and 2.2, respectively. □

Lemma 3.2 lets us count $k$-antipowers by computing the size of the complementary sets $\text{WeakPow}_k(d)$. Thus, we obtain the following preliminary result.

**Proposition 3.3.** The number of $k$-antipower factors in a word of length $n$ can be computed in $O(nk^3)$ time.

**Proof.** See Algorithm 1. We use Lemma 3.2 points (b) and (c) to express the sets $\text{WeakPow}_k(d)$ for $d = 1, \ldots, \lfloor n/k \rfloor$ as a union of $O(nk^2)$ interval chains. That is, the total size of the sets $\mathcal{C}_d$ is $O(nk^2)$. Each of the interval chains consists of at most $k$ intervals. Hence, Lemma 2.4 can be applied to compute interval representations of the sets $\text{WeakPow}_k(d)$ in $O(nk^2)$ total time. Finally, the size of the complement of the set $\text{WeakPow}_k(d)$ (in $[0 \ldots n - kd]$) is the number of $(k, d)$-antipowers. □

Next, we improve the time complexity of this algorithm to $O(nk \log k)$.

![Figure 4: An interval, represented as a sequence of four consecutive positions (black dots), of starting positions of occurrences of gapped $(q, d)$-squares generated by a gapped repeat with period $(q + 1)d$.](image)
Figure 5: The fact that \( i \in Squares(q, d) \) is a witness of inclusion \((\text{Chain}_k(q, d, i) \cap [0..n - kd]) \subseteq \text{WeakPow}_k(d)\).

Algorithm 1: SimpleCount\((w, n, k)\)

\[
\begin{align*}
(C_d)_{d=1}^{\lceil n/k \rceil} := (\emptyset, \ldots, \emptyset) \\
\text{for } q := 0 \text{ to } k - 2 \text{ do} \\
\quad \text{foreach } (q + 1)\text{-MGR or generalized run } \gamma \text{ in } w \text{ do} \\
\quad\quad p := \text{per}(\gamma) \\
\quad\quad \text{if } (q + 1) \mid p \text{ then} \\
\quad\quad\quad d := \left\lfloor \frac{p}{q + 1} \right\rfloor \\
\quad\quad\quad I := \text{Squares}(q, d, \gamma) \\
\quad\quad\quad C_d := C_d \cup \{ \text{Chain}_k(q, d, I) \} \\
\text{antipowers} := 0 \\
\text{for } d := 1 \text{ to } \lfloor n/k \rfloor \text{ do} \\
\quad \text{WeakPow}_k(d) := (\bigcup C_d) \cap [0..n - kd] \\
\quad \text{antipowers} := \text{antipowers} + (n - kd + 1) - |\text{WeakPow}_k(d)| \\
\text{return } \text{antipowers}
\end{align*}
\]

4 Counting \( k \)-antipowers in \( O(nk \log k) \) time

We improve the algorithm SimpleCount threefold. First, we show that the chain representation of weak \( k \)-powers actually consists of only \( O(nk) \) chains. Then, instead of processing the chains by their interval representations, we introduce a geometric interpretation that reduces the problem to computing the area of the union of representations, we introduce a geometric interpretation that reduces the problem to computing the area of the union of rectangles. This area could be computed directly in \( O(nk \log n) \) time, but we improve this complexity to \( O(nk \log k) \) by exploiting properties of the dimensions of the rectangles.

4.1 First improvement of SimpleCount

First, we improve the \( O(nk^2) \) bounds of Lemma 3.2\((c)\). By inspecting the structure of MGRs, we actually show that the formula from Lemma 3.2\((b)\) generates only \( O(nk) \) interval chains. A careful implementation lets us compute such a chain representation in \( O(nk) \) time.

We say that an \( \alpha \)-MGR for integer \( \alpha \) with period \( p \) is nice if \( \alpha \mid p \) and \( p \geq 2\alpha^2 \). Let \( \text{NMGRs}_\alpha(w) \) denote the set of nice \( \alpha \)-MGRs in the word \( w \). The following lemma provides a combinatorial foundation of the improvement.

Lemma 4.1. For a word \( w \) of length \( n \) and an integer \( \alpha > 1 \), \(|\text{NMGRs}_\alpha(w)| \leq 54n\).

Proof. Let us consider a partition of the word \( w \) into blocks of \( \alpha \) letters (the final \( n \mod \alpha \) letters are not assigned to any block). Let \( uvu \) be a nice \( \alpha \)-MGR in \( w \). We know that \( 2\alpha^2 \leq |uv| \leq \alpha|w| \), so \( |u| \geq 2\alpha \).

Now, let us fit the considered \( \alpha \)-MGR into the structure of blocks. Since \( \alpha \mid |uv| \), the indices in \( w \) of the occurrences of the left and the right arm are equal modulo \( \alpha \). We shrink both arms to \( u' \) such that \( u' \) is the maximal inclusion-wise interval of blocks which is encompassed by each arm \( u \). Then, let us expand \( v \) to \( v' \) so that it fills the space between the two occurrences of \( u' \).

Let us notice that \( |uv| = |u'v'| \). Moreover, \( |u'| \geq \frac{3}{2}|u| \) since \( u \) encompasses at least one full block of \( w \). Consequently, \( |u'| \leq 3\alpha|w'| \).

Let \( t \) be a word whose letters correspond to whole blocks in \( w \) and \( u'' \), \( v'' \) be factors of \( t \) that correspond to \( u' \) and \( v' \), respectively. We have \( |u''| = |u'|/\alpha \) and \( |v''| = |v'|/\alpha \), so \( u''v''u'' \) is a 3\( \alpha \)-gapped
repeat in $t$. It is also a 3α-MGR because it can be expanded by one block neither to the left nor to the right, as it would contradict the maximality of the original nice α-MGR. This concludes that every nice α-MGR in $w$ has a corresponding 3α-MGR in $t$. Also, every 3α-MGR in $t$ corresponds to at most one nice α-MGR in $w$, as it can be translated into blocks of $w$ and expanded in a single way to a 3α-MGR (that can happen to be a nice α-MGR).

We conclude that the number of nice α-MGRs in $w$ is at most the number of 3α-MGRs in $t$. As $|t| \leq n/\alpha$, due to Fact 2.1 the latter is at most $54n$. \qed

Lemma 4.2. For $d = 1, \ldots, \lceil n/k \rceil$, the sets $\text{WeakPow}_k(d)$ have chain representations of total size $O(nk)$ which can be computed in $O(nk)$ time.

Proof. The chain representations of sets $\text{WeakPow}_k(d)$ are computed for $d < 2k - 2$ and for $d \geq 2k - 2$ separately.

From Fact 1.1 we know that all $(k, d)$-antipowers can be found in $O(n)$ time. This lets us compute the set $\text{WeakPow}_k(d)$ (and its trivial chain representation) in $O(n)$ time. Across all $d < 2k - 2$, this gives $O(nk)$ chains and $O(nk)$ time.

Henceforth we consider the case that $d \geq 2k - 2$. Let us note that if a gapped $(q, d)$-square with $d \geq 2(q + 1)$ is generated by a $(q + 1)$-MGR, then this $(q + 1)$-MGR is nice. Indeed, by Lemma 3.1 this $(q + 1)$-MGR has period $p = (q + 1)d \geq 2(q + 1)^2$. This observation lets us express the formula of Lemma 3.1 for $d \geq 2k - 2$ equivalently using $\text{MGR}_q(w)$ instead of $\text{MGReps}_{q + 1}(w)$.

By Fact 2.2 and Lemma 4.1 for every $q$ we have only $|\text{MGR}_q(w) \cup \text{GR}_q(w)| = O(n)$ MGRs and generalized runs to consider. Hence, the total size of chain representations of sets $\text{WeakPow}_k(d)$ for $d \geq 2k - 2$ is in $O(nk)$ as well. The last piece of the puzzle is the following claim.

Claim 4.3. The sets $\text{MGR}_q(w)$ for $q = 1 \ldots k - 1$ can be built in $O(nk)$ time.

Proof. The union of those sets is a subset of $\text{MGR}_k(w)$, so we can consider each $(k - 1)$-MGR $uvu$ with period $p = |uv|$ and report all $\alpha \in [\alpha_L \ldots \alpha_R]$ such that $\alpha | p$, where

$$\alpha_L = \left\lfloor \frac{p}{|w|} \right\rfloor, \quad \alpha_R = \min(k - 1, \left\lfloor \sqrt{\frac{k}{2}} \right\rfloor).$$

We will use an auxiliary table $\text{next}_p$ such that

$$\text{next}_p[\alpha] = \min\{\alpha' \in [\alpha + 1 \ldots k] : \alpha' | p\}. \tag{1}$$

This table has size $O(nk)$. For every $p \in [1 \ldots n]$, all values $\text{next}_p[\alpha]$ for $\alpha \in [1 \ldots k]$ can be computed, right to left, in $O(k)$ time. Then, all values $\alpha$ for which $uvu$ is a nice α-MGR can be computed by iterating $\alpha := \text{next}_p[\alpha]$ until a value greater than $\alpha_R$ is reached, starting from $\alpha = \alpha_L - 1$. Thus, the total time of constructing the sets $\text{MGR}_q(w)$ is $O(|\text{MGR}_k(w)| + \sum_{q = 1}^{k - 1} |\text{MGR}_q(w)|) = O(nk)$. \qed

This concludes the proof.

\section{Second improvement of SimpleCount}

We reduce the problem to computing unions of sets of orthogonal rectangles with bounded integer coordinates.

For a given value of $d$, let us fit the integers from $[0 \ldots n - kd]$ into the cells of a grid of width $d$ so that the first row consists of numbers 0 through $d - 1$, the second of numbers $d$ to $2d - 1$, etc. Let us call this grid $G_d$. The main idea behind the lemma presented below is shown in Fig. 6.

Lemma 4.4. The set $\text{Chain}_k(q, d, I)$ is a union of $O(1)$ orthogonal rectangles in $G_d$, each of height at most $k$ or width exactly $d$. The coordinates of the rectangles can be computed in $O(1)$ time.

Proof. Translating the set $\text{Chain}_k(q, d, I)$ onto our grid representation, it becomes a union of horizontal strips, each corresponding to an interval $I \in \mathbb{Z}$, for $a \in [0 \ldots k - q - 2]$, that possibly wrap around into the subsequent rows. Those strips have their beginnings in the same column, occupying consecutive positions. Depending on the column index of the beginning of a strip and its length, we have three cases:

\footnote{We assume that $\min \emptyset = \infty$.}
Figure 6: Examples of decompositions of various interval chains $\text{Chain}_k(q, d, I)$ into orthogonal rectangles in the grid $G_d$ for $d = 5$, $k = 5$, $n = 52$.

- The strip does not wrap around at all (Fig. 6(a)). Then, the union of all strips is simply a single rectangle. Its height is exactly $k - q - 1$.

- The strip’s length is smaller than the length of the row, but it wraps around at some point (Fig. 6(b)). Then, there exists a column which does not intersect with any strip. The strips’ parts that have wrapped around (that is, to the left of the column) form a rectangle and similarly the strips’ parts that have not wrapped around form a rectangle as well. Both of these rectangles have height equal to $k - q - 1$.

- The strip’s length is greater than or equal to the length of the row. In this case, excluding the first and the last row, the union of the strips is actually a rectangle fully encompassing all columns (Fig. 6(c)). Therefore the union of all strips can be represented as a union of three rectangles: the first row, the last row and what is in between. Both the first and the last row have height equal to 1 and the rectangle in between has width equal to $d$.

In some cases, such decomposition into orthogonal rectangles may include some cells that are not on the grid (negative numbers or numbers greater than $n - kd$); see Fig. 6(d). In that case, we consider the first and the last included rows as individual rectangles; the remaining part of the decomposition corresponds to one of the cases mentioned before.

Thus, by Lemma 4.2, our problem reduces to computing the area of unions of rectangles in subsequent grids $G_d$. In total, the number of rectangles is $O(nk)$.

### 4.3 Third improvement of SimpleCount

Assume that $r$ axis-aligned rectangles in the plane are given. The area of their union can be computed in $O(r \log r)$ time using a classic sweep line algorithm (see Bentley [4]). This approach would yield an $O(nk \log n)$-time algorithm for counting $k$-antipowers. We refine this approach in the case that the rectangles have bounded height or maximum width and their coordinates are bounded.

**Lemma 4.5.** Assume that $r$ axis-aligned rectangles in $[0..d]^2$ with integer coordinates are given and that each rectangle has height at most $k$ or width exactly $d$. The area of their union can be computed in $O(r \log k + d)$ time and $O(r + d)$ space.
Proof. We assume first that all rectangles have height at most \( k \).

Let us partition the plane into horizontal strips of height \( k \). Thus, each of the rectangles is divided into at most two. The algorithm performs a sweep line in each of the strips.

Let the sweep line move from left to right. The events in the sweep correspond to the left and right sides of rectangles. The events can be sorted left-to-right, across all strips simultaneously, in \( O(r + d) \) time using bucket sort \([5]\).

For each strip, the sweep line stores a data structure that allows insertion and deletion of intervals with integer coordinates in \([0 \ldots k]\) and querying for the total length of the union of the intervals that are currently stored. This corresponds to the operations of the data structure from Lemma 4.4 for \( m = k \) (with elements corresponding to unit intervals), which supports insertions and deletions in \( O(\log k) \) time and queries in \( O(1) \) time after \( O(k) \)-time preprocessing per strip. The total preprocessing time is \( O(d) \) and, since the total number of events in all strips is at most \( 2r \), the sweep works in \( O(r \log k) \) time.

Finally, let us consider the width-\( d \) rectangles. Each of them induces a vertical interval on the second component. First, in \( O(r + d) \) time the union \( S \) of these intervals represented as a union of pairwise disjoint maximal intervals can be computed by bucket sorting the endpoints of the intervals. Then, each maximal interval in \( S \) is partitioned by the strips and the resulting subintervals are inserted into the data structures of the respective strips before the sweep. In total, at most \( 2r + d/k \) additional intervals are inserted so the time complexity is still \( O((r + d/k) \log k + d) = O(r \log k + d) \).

We arrive at the main result of this section.

**Theorem 4.6.** The number of \( k \)-antipower factors in a word of length \( n \) can be computed in \( O(nk \log k) \) time and \( O(nk) \) space.

**Proof.** We use Lemma 4.2 to express the sets \( \text{WeakPow}_k(d) \) for \( d = 1, \ldots, \lfloor n/k \rfloor \) as sums of \( O(nk) \) interval chains. This takes \( O(nk) \) time. Each chain is represented on the corresponding grid \( G_d \) as the union of a constant number of rectangles using Lemma 1.4. This gives \( O(nk) \) rectangles in total on all the grids \( G_d \), each of height at most \( k \) or width exactly \( d \), for the given \( d \).

As the next step, we renumber the components in the grids by assigning consecutive numbers to the components that correspond to rectangle vertices. This can be done in \( O(nk) \) time, for all the grids simultaneously, using bucket sort \([5]\). The new components store the original values. After this transformation, rectangles with height at most \( k \) retain this property and rectangles with width \( d \) have maximal width. Let the maximum component in the grid \( G_d \) after renumbering be equal to \( M_d \) and the number of rectangles in \( G_d \) be \( R_d \); then \( \sum_d R_d = O(nk) \) and \( \sum_d M_d = O(nk) \).

As the final step, we apply the algorithm of Lemma 4.5 to each grid to compute \( |\text{WeakPow}_k(d)| \) as the area of the union of the rectangles in the grid. One can readily verify that it can be adapted to compute the areas of the rectangles in the original components. The algorithm works in \( O(\sum_d R_d \log k + \sum_d M_d) = O(nk \log k) \) time. In the end, the number of \((k, d)\)-antipower factors equals \( n - kd + 1 - |\text{WeakPow}_k(d)| \).

## 5 Reporting antipowers and answering antipower queries

The same technique can be used to report all \( k \)-antipower factors. In the grid representation, they correspond to grid cells of \( G_d \) that are not covered by any rectangle, as shown in the figure to the right. Hence, in Lemma 4.5 instead of computing the area of the rectangles with the aid of Lemma 2.5, we need to report all grid cells excluded from rectangles using Lemma 4.2. The computation takes \( O(r \log k + d + C_d) \) time where \( C_d \) is the number of reported cells. By plugging this routine into the algorithm of Theorem 4.6 we obtain

**Theorem 5.1.** All factors of a word of length \( n \) being \( k \)-antipowers can be computed in \( O(nk \log k + C) \) time and \( O(nk) \) space, where \( C \) is the size of the output.

Finally, we present our data structure for answering antipower queries that introduces a smooth trade-off between the two data structures of Badkobeh et al. \([11]\) (see Fact 1.2). Let us recall that an antipower query \((i, j, k)\) asks to check if a factor \( w[i \ldots j] \) of the word \( w \) is a \( k \)-antipower.
Theorem 5.2. Assume that a word of length $n$ is given. For every $r \in [1..n]$, there is a data structure of size $O(n^2/r)$ that can be constructed in $O(n^2/r)$ time and answers antipower queries in $O(r)$ time.

Proof. Let $w$ be a word of length $n$ and let $r \in [1..n]$. If an antipower query $(i, j, k)$ satisfies $k \leq r$, we answer it in $O(k)$ time using Fact 1.2(a). This is always $O(r)$ time, and the data structure requires $O(n)$ space.

Otherwise, if $w[i..j]$ is a $k$-antipower, then its base is at most $n/r$. Our data structure will let us answer antipower queries for every such base in $O(1)$ time.

Let us consider a positive integer $b \leq n/r$. We group the factors of $w$ of length $b$ by the remainder modulo $b$ of their starting position. For a remainder $g \in [0..b-1]$ and an index $i \in [0..\lceil \frac{n-g}{b} \rceil]$, we store, as $A^b_g[i]$, the smallest index $j > i$ such that $w[jb+g..(j(b+1)+g) = w[ib+g..(i(b+1)+g)$ ($j = \infty$ if it does not exist). We also store a data structure for range minimum queries over $A^b_g$ for each group; it uses linear space, takes linear time to construct, and answers queries in constant time (see 3). The tables take $O(n)$ space for a given $b$, which gives $O(n^2/r)$ in total. They can also be constructed in $O(n^2/r)$ total time, as shown in the following claim.

Claim 5.3. The tables $A^b_g$ for all $b \in [1..m]$ and $g \in [0..b-1]$ can be constructed in $O(nm)$ time.

Proof. Let us assign to each factor of $w$ of length at most $m$ an identifier in $[0..n)$ such that two factors of the same length are equal if and only if their identifiers are equal. For length-$1$ factors, this requires sorting the alphabet symbols, which can be done in $O(n)$ time for an integer alphabet. For factors of length $\ell > 1$, we construct pairs that consist of the identifiers of the length-$1$ prefix and length-$1$ suffix and bucket sort the pairs. This gives $O(nm)$ time in total.

To construct the tables $A^b_g$ for a given $b$, we use an auxiliary array $D$ that is indexed by identifiers in $[0..n)$. Initially, all its elements are set to $\infty$. For a given $g$, the indices $i$ are considered in descending order. For each $i$, we take as $x$ the identifier of the factor $w[ib+g..i(b+1)+g)$, set $A^b_g[i]$ to $D[x]$ and then $D[x]$ to $i$. Afterwards, in the same loop, all such values $D[x]$ are reset to $\infty$. For any $b$ and $g$, both loops take $O(n/b)$ time.

Given an antipower query $(i, j, k)$ such that $(j - i + 1)/k = b$, we set

$$g = i \mod b, \quad i' = \lfloor \frac{i}{b} \rfloor, \quad j' = \lfloor \frac{j+1}{b} \rfloor - 2,$$

and ask a range minimum query on $A^b_{g'[i']},..,A^b_{g'[j']}$.

A Applications of Static Range Tree

Let $m$ be a power of two. A basic interval is an interval of the form $[a..a+2^i]$ that is a subinterval of $[0..m-1]$ and such that $i \geq 0$ is an integer and $2^i \mid a$. For example, the basic intervals for $m = 8$ are $[0..1], [7..8], [0..2], [2..4], [4..6], [6..8], [0..4], [4..8], [0..8]$. In a static range tree (sometimes called a segment tree; see 12) each node is identified with a basic interval. The children of a node $J = [a..a+2^i)$, for $i > 0$, are $lchild(J) = [a..a+2^{i-1})$ and $rchild(J) = [a+2^{i-1}..a+2^i)$. Thus, a static range tree is a full binary tree of size $O(m)$. The root of the tree, root, corresponds to $[0..m)$.

Every interval $I \subseteq [0..m)$ can be decomposed into a disjoint union of at most $2 \log m$ basic intervals. The decomposition can be computed in $O(\log m)$ time recursively starting from the root. Let $J$ be a node considered in the algorithm. If $J \subseteq I$, the algorithm adds $J$ to the decomposition. Otherwise, for each child $J'$ of the node $J$, if $J' \cap I \neq \emptyset$, the algorithm makes a recursive call to the child. At each level of the tree, the algorithm makes at most two recursive calls. The resulting set of basic intervals is denoted by $Decomp(I)$; see Fig. 7.

Proofs of the lemmas from Section 2 follow.

Instead of Lemma 2.5, we show an equivalent lemma with an operation $count'$ which returns $|\{0..m\} \setminus \bigcup J|$.

Lemma A.1. There exists a data structure of size $O(m)$ that, after $O(m)$-time initialization, supports insert and delete in $O(\log m)$ time and $count'$ in $O(1)$ time.
Figure 7: A static range tree for $m = 8$ with the set of nodes that comprises $\text{Decomp}(\{1..7\})$. The paths visited in the recursive decomposition algorithm are shown in bold.

Figure 8: A static range tree for $m = 8$ that stores the family $\mathcal{J} = \{\{2..3\}, \{3..5\}, \{4..7\}, \{6..7\}\}$. The values $\text{val}(J)$ are shown in bold. The arrows present selected jump pointers (cf. Lemma A.2).

**Proof.** Let $m'$ be the smallest power of two satisfying $m' \geq m$. Observe that the data structure for $m$ can be simulated by an instance constructed for $m'$: it suffices to insert an interval $[m..m')$ in the initialization phase to make sure that integers $i \geq m$ will not be counted when $\text{count}'$ is invoked. Henceforth, we may assume without loss of generality that $m$ is a power of two.

We apply a static range tree. Every node $J$ of the tree stores two values (see Fig. 8):

- $b_i(J) = | \{ I \in J : J \in \text{Decomp}(I) \} |$
- $\text{val}(J) = | J \setminus \bigcup \{ J' : J' \subseteq J, J' \in \text{Decomp}(I), I \in \mathcal{J} \} |$

The value $\text{val}(J)$ can also be defined recursively:

- If $b_i(J) > 0$, then $\text{val}(J) = 0$.
- Otherwise, $\text{val}(J) = 1$ if $J$ is a leaf and $\text{val}(J) = \text{val}(l\text{child}(J)) + \text{val}(r\text{child}(J))$ if it is not.

This allows computing $\text{val}(J)$ from $b_i(J)$ and the values stored in the children of $J$.

The data structure can be initialized bottom-up in $O(m)$ time. The respective operations on the data structure are now implemented as follows:

- $\text{insert}(I)$: Compute $\text{Decomp}(I)$ recursively. For each node $J \in \text{Decomp}(I)$, increment $b_i(J)$. For each node $J$ encountered in the recursive computation, recompute $\text{val}(J)$.
- $\text{delete}(I)$: Similar to $\text{insert}$, but we decrement $b_i(J)$ for each node $J \in \text{Decomp}(I)$.
- $\text{count}'$: Return $\text{val}(\text{root})$.

The complexities of the respective operations follow.

**Lemma A.2.** There exists a data structure of size $O(m)$ that, after $O(m)$-time initialization, supports $\text{insert}$ and $\text{delete}$ in $O(\log m)$ time and $\text{report}$ in $O(|A|)$ time.
Proof. As in the proof of Lemma A.1, we assume without loss of generality that $m$ is a power of two. Again, the data structure applies a static range tree. We also reuse the values $bi(J)$ for nodes; we generalize the $val(J)$ values, though.

If $J$ and $J'$ are basic intervals and $J' \subseteq J$, then we define $val_J(J')$ as 0 if there exists a basic interval $J''$ on the path from $J$ to $J'$ (i.e., such that $J' \subseteq J'' \subseteq J$) for which $bi(J'') > 0$, and as $val(J')$ otherwise. These values satisfy the following properties.

Observation A.3. For every node $J$, (a) $val_J(J) = val(J)$; and (b) $val_{root}(J) = |J \setminus \bigcup J|$.

By point (b) of the observation, our goal in a report query is to report all leaves $J$ such that $val_{root}(J) = 1$. The first idea how to do it would be to recursively visit all the nodes $J'$ of the tree such that $val_{root}(J') > 0$. However, this approach would work in $\Omega(|A| \log m)$ time since for every leaf all the nodes on the path to the root would need to be visited.

In order to efficiently answer report queries, we introduce jump pointers, stored in each node $J$, such that $jump(J)$ is the lowest such node $J'$ in the subtree of $J$ such that $val_J(J') = val_J(J)$; see Fig. 8.

The pointer $jump(J)$ can be computed in $O(1)$ time from the values in the children of $J$:

$$jump(J) = \begin{cases} J & \text{if } J \text{ is a leaf or } 0 < val(lchild(J)) < val(J), \\ jump(lchild(J)) & \text{if } val(lchild(J)) = 0, \\ jump(rchild(J)) & \text{otherwise.} \end{cases}$$

This formula allows recomputing the jump pointers on the paths visited during a call to insert or delete without altering the complexity.

Let us consider a subtree that is composed of all the nodes $J$ with positive $val_{root}(J)$. Using jump pointers, we make a recursive traversal of the subtree that avoids visiting long paths of non-branching nodes of the subtree. It visits all the leaves and branching nodes of the subtree and, in addition, both children of each branching node. With this traversal, a report query is therefore answered in $O(|A|)$ time.

\[\square\]

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