On structure of topological metagroups.

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Abstract

In this article topologies on metagroups are studied. They are related with generalized $C^*$-algebras over $\mathbb{R}$ or $\mathbb{C}$. Homomorphisms and quotient maps on them are investigated. Structure of topological metagroups is scrutinized. In particular, topologies on smashed products and smashed twisted wreath products of metagroups are scrutinized, which are making them topological metagroups. Moreover, their inverse homomorphism systems are studied.

1 Introduction.

A topological group structure plays very important role in mathematics and its applications [2, 12, 30, 32].

On the other side, noncommutative analysis is a very important part of mathematical analysis and it interacts with operator theory, operator algebras and algebraic analysis. In particular, analysis over quaternions, octonions and generalized Cayley-Dickson algebras develops fast in recent years.
(see [3, 8, 10]-[11, 13]-[23, 33] and references therein). It also plays a huge role in mathematics, analysis of noncommutative geometry, mathematical physics, quantum field theory and quantum gravity, partial differential equations (PDEs), particle physics, operator theory, etc.

It appears that a multiplicative law of their canonical bases is nonassociative and leads to a more general notion of a metagroup instead of a group [8, 24, 26]. They were used in [24, 21, 22, 25, 26] for investigations of partial differential operators and other unbounded operators over quaternions and octonions, also for automorphisms, derivations and cohomologies of generalized $C^*$-algebras over $\mathbb{R}$ or $\mathbb{C}$. They certainly have a lot of specific features in their derivations and (co)homology theory [24, 26]. It was shown in [25] that an analog of the Stone theorem for one parameter groups of unitary operators for the generalized $C^*$-algebras over quaternions and octonions becomes more complicated and multiparameter. The generalized $C^*$-algebras arise naturally while decompositions of PDEs or systems of PDEs of higher order into PDEs or their systems of order not higher than two [10, 11, 17, 23, 27, 28], that permits to integrate them subsequently or simplify their analysis.

In [16] different types of products of metagroups were studied such as smashed products and smashed twisted wreath products. That permitted also to construct their abundant families different from groups. But topologies on such metagroups were not investigated. We recall their definition.  

**Definition 1.1.** Let $G$ be a set with a single-valued binary operation (multiplication) $G^2 \ni (a, b) \mapsto ab \in G$ defined on $G$ satisfying the conditions:

(1.1.1) for each $a$ and $b$ in $G$ there is a unique $x \in G$ with $ax = b$

(1.1.2) and a unique $y \in G$ exists satisfying $ya = b$, which are denoted by $x = a \setminus b = \text{Div}_l(a, b)$ and $y = b/a = \text{Div}_r(a, b)$ correspondingly,

(1.1.3) there exists a neutral (i.e. unit) element $e_G = e \in G$: $eg = ge = g$ for each $g \in G$.

If the set $G$ with the single-valued multiplication satisfies the conditions (1.1.1) and (1.1.2), then it is called a quasigroup. If the quasigroup $G$ satisfies also the condition (1.1.3), then it is called an algebraic loop (or unital quasigroup, or shortly a loop).

The set of all elements $h \in G$ commuting and associating with $G$:  


(1.1.4) $Com(G) := \{a \in G : \forall b \in G, \ ab = ba\}$,
(1.1.5) $N_l(G) := \{a \in G : \forall b \in G, \forall c \in G, \ (ab)c = a(bc)\}$,
(1.1.6) $N_m(G) := \{a \in G : \forall b \in G, \forall c \in G, \ (ba)c = b(ac)\}$,
(1.1.7) $N_r(G) := \{a \in G : \forall b \in G, \forall c \in G, \ (bc)a = b(ca)\}$,
(1.1.8) $N(G) := N_l(G) \cap N_m(G) \cap N_r(G)$;

$C(G) := Com(G) \cap N(G)$ is called the center $C(G)$ of $G$.

We call $G$ a metagroup if a set $G$ possesses a single-valued binary operation and satisfies the conditions (1.1.1)-(1.1.3) and

(1.1.9) $(ab)c = t(a, b, c)a(bc)$

for each $a$, $b$ and $c$ in $G$, where $t(a, b, c) = t_G(a, b, c) \in C(G)$.

Then the metagroup $G$ will be called a central metagroup, if it satisfies also the condition:

(1.1.10) $ab = t_2(a, b)ba$

for each $a$ and $b$ in $G$, where $t_2(a, b) \in C(G)$.

If $H$ is a submetagroup (or a subloop) of the metagroup $G$ (or the loop $G$) and

(1.1.11) $gH = Hg$ for each $g \in G$, then $H$ will be called almost invariant (or algebraically almost normal). If in addition

(1.1.12) $(gH)k = g(Hk)$ and $k(gH) = (kg)H$ for each $g$ and $k$ in $G,$

then $H$ will be called an invariant (or algebraically normal) submetagroup (or subloop respectively).

Henceforward it will be used a notation $Inv_l(a) = Div_l(a, e)$ and $Inv_r(a) = Div_r(a, e)$.

Elements of a metagroup $G$ will be denoted by small letters, subsets of $G$ will be denoted by capital letters. If $A$ and $B$ are subsets in $G$, then $A - B$ means the difference of them $A - B = \{a \in A : a \notin B\}$. Henceforward, maps and functions on metagroups are supposed to be single-valued if something other will not be specified.

If $T_G$ is a topology on the metagroup (or quasigroup, or loop) $G$ such that multiplication, $Div_l$ and $Div_r$ are (jointly) continuous from $G \times G$ into $G$, then $(G, T_G)$ is called a topological metagroup (or quasigroup, or loop respectively).

Notice also that a loop is a quite different object than a loop group consid-
ered in geometry or mathematical physics. Certainly loops are more general objects, than metagroups. Note that metagroups are commonly nonassociative and having many specific features in comparison with loops and groups. On the other hand, if a loop $G$ is simple, then a subloop generated by all elements of the form $((ab)c)/(a(bc))$ for all $a, b, c$ in $G$ coincides with $G$ \[11\,6\].

Metagroups are intermediate between groups and quasigroups. Functions on topological metagroups were studied in \[29\].

We recall, that according to Chapter 2 and Sections 4.6, 4.10, 4.13 in \[30\] and Section 6 in \[12\] the compact connected $T_0$ topological group $G$ can be presented as the limit of an inverse homomorphism system (that is, projective limit) $G = \lim\{G_j, \pi^j_k, \Omega\}$ of compact finite-dimensional Lie groups of manifolds over $\mathbb{R}$, where $\Omega$ is a directed set, $\pi^j_k : G_j \rightarrow G_k$ is a continuous homomorphism for each $j > k$ in $\Omega$, $\pi^j_j$ is the identity map, $\pi^j_j(g_j) = g_j$ for each $g_j \in G_j$, $\pi^l_k \circ \pi^k_l = \pi^l_j$ for each $l < k < j$ in $\Omega$. This arise questions for a subsequent research. Whether a nonassociative analog of a topological group has this property or not? How weak may a nonassociative structure be that do not satisfy this property? This article answers these questions. In it analogs of topological groups are scrutinized with a rather mild nonassociative metagroup structure.

In this article topologies on metagroups are studied. They have specific features in comparison with groups because of nonassociativity in general of metagroups. Homomorphisms and quotient maps on them are investigated. In particular, topologies on smashed products and smashed twisted wreath products of metagroups are scrutinized in this article, which are making them topological metagroups. Structure of topological metagroups is scrutinized. Moreover, their inverse homomorphism systems are studied.

All main results of this paper are obtained for the first time. They can be used for further studies of topological metagroups, noncommutative mathematical analysis and function theory, structure of the generalized operator algebras and $C^*$ algebras over octonions, Cayley-Dickson algebras, operator theory and spectral theory over octonions and Cayley-Dickson algebras, PDE, noncommutative geometry, mathematical physics, their applications in the sciences.
2 Topologies on metagroups and transversal sets.

Definition 2.1. Let $G$ be a quasigroup and let $H$ be its subquasigroup. Let $V = V_{G,H}$ be a subset in $G$ such that

(2.1.1) $G = \bigcup_{v \in V} Hv$ and

(2.1.2) $(Hv_1) \cap (Hv_2) = \emptyset$ for each $v_1 \neq v_2$ in $V$.

Then $V$ is called a transversal set of $H$ in $G$. It also will be denoted by $V_{D,A}$ (using such notation in order to distinguish it with $Div_r(a/b) = a/b$).

Remark 2.2. For a metagroup $D$ and a submetagroup $A$ there exists the transversal set $V = V_{D,A}$ by Corollary 1 in [16]. According to Formulas (53) in [16] there exist single-valued surjective maps

(2.2.1) $\psi_A^D : D \to A$ and

(2.2.2) $\tau_A^D : D \to V_{D,A}$

such that $\psi_A^D(d) = a$, $\tau_A^D(d) = v$, where $d \in D$, $d = av$, $a \in A$, $v \in V$. It also is denoted by $d^\psi = \psi(d)$ and $d^\tau = \tau(d)$ respectively, if $D$ and $A$ are specified.

Let $C_A(D) = C(D) \cap A$ and let $C_1$ be a subgroup in $C(D)$ such that $C_m(D) \subset C_1$, where $C_m(D)$ denotes a minimal subgroup in $D$ such that $t_D(a,b,c) \in C_m(D)$ for each $a, b$ and $c$ in $D$. For a topological metagroup $D$ using the (joint) continuity of multiplication, $Div_l$ and $Div_r$ on $D$ one can consider without loss of generality that $C_1$ is closed in $D$. This and Condition (1.1.9) imply that $AC_1$ is a submetagroup in $D$, since $C_1$ is the commutative (Abelian) group and $C_m(D) \subset C_1$, where $PB = \{c \in D : \exists p \in P, \exists b \in B, c = pb\}$ for subsets $B$ and $P$ in $D$. Hence $D/\cdot C_1$ and $(AC_1)/\cdot C_1$ are the quotient groups by Theorem 1 in [16] (see also Definition 2.1 above). Therefore

(2.2.3) $\psi_A^D(d) = \psi_{AC_1}^D \circ \psi_{AC_1}^D(d)$ and

(2.2.4) $\tau_A^D(d) = \tau_{AC_1}^A(\psi_{AC_1}^D(d)) \tau_{AC_1}^D(d)$

for each $d \in D$. Moreover, the transversal set $V_{AC_1,A}$ can be chosen such that $V_{AC_1,A} \subset V_{D,A}$ (see Remark 3 in [16]).

Theorem 2.3. Let $G$ be a topological quasigroup with a topology $\mathcal{T}_G$,
let $G_1, G_2$ be subquasigroups such that $G_2 \subset G_1 \subset G$ and let $V_1, V_2, V_{1,2}$ be transversal sets of $G_1, G_2$ in $G$ and of $G_2$ in $G_1$ respectively. Then there exists a coarsest topology $\mathcal{T}_{G,G_1,G_2}$ on $G$ such that $\mathcal{T}_{G,G_1,G_2} \supset \mathcal{T}_G; \langle G, \mathcal{T}_{G,G_1,G_2} \rangle$ is the topological quasigroup and the maps $\psi_{G_1}^G, \psi_{G_2}^G, \psi_{G_2}^{G_1}, \tau_{G_1}^G, \tau_{G_2}^G, \tau_{G_2}^{G_1}$ are continuous, where $G$ is supplied with the topology $\mathcal{T}_{G,G_1,G_2}; G_1, G_2, V_1, V_2, V_{1,2}$ are considered in topologies inherited from $\langle G, \mathcal{T}_{G,G_1,G_2} \rangle$.

**Proof.** Note that the maps $\psi_{G_1}^G : G \rightarrow G_1, \tau_{G_1}^G : G \rightarrow V_1$ and similarly $\psi_{G_2}^{G_1}, \tau_{G_2}^{G_1}, \tau_{G_2}^G$ are surjective.

If $X$ is a topological space with a topology $\mathcal{T}_X, f : X \rightarrow Y$ is a surjective (single-valued) map, then it induces an equivalence relation $\Xi_f$ on $X$ such that $x\Xi_f z$ if and only if $f(x) = f(z)$, where $x$ and $z$ belong to $X$. Therefore, there exists a bijection $g_Y : X/\Xi_f \rightarrow Y$, where $X/\Xi_f$ denotes the quotient topological space obtained from $X$ with the equivalence relation $\Xi_f$. This means that $g_Y$ and the quotient topology $\mathcal{T}_{X,f}$ on $X/\Xi_f$ induce the corresponding (quotient) topology $\mathcal{T}_{Y}^{f}(\mathcal{T}_X) = \mathcal{T}_Y^{f}$ on $Y$ by Proposition 2.4.3 [9]. Therefore, $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y^{f})$ is continuous. More generally, if $f_j : X \rightarrow Y$ is a surjective map for each $j \in \Lambda$, where $\Lambda$ is a set, then there exists a coarsest topology $\mathcal{T}_Y^{(f_j : j \in \Lambda)}(\mathcal{T}_X) = \mathcal{T}_Y^{(f_j : j \in \Lambda)}$ such that $f_k : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y^{(f_j : j \in \Lambda)})$ is continuous and $\mathcal{T}_Y^{f_k} \subset \mathcal{T}_Y^{(f_j : j \in \Lambda)}$ for each $k \in \Lambda$ by Proposition 2.4.14 [9].

Using this we construct a new topology on $G$ by induction. At first we take the topologies $Q_{G_1,1}^G = \mathcal{T}_{G_1,1}^{\psi_{G_1}^G}(\mathcal{T}_{G_1,1}); Q_{V_1,1}^G = \mathcal{T}_{V_1,1}^{\tau_{G_1}^G}(\mathcal{T}_{G_1,1}); Q_{G_2,1}^G = \mathcal{T}_{G_2,1}^{\psi_{G_2}^G}(\mathcal{T}_{G_1,1}); Q_{V_2,1}^G = \mathcal{T}_{V_2,1}^{\tau_{G_2}^G}(\mathcal{T}_{G_1,1}); Q_{G_2,1}^{G_1} = \mathcal{T}_{G_2,1}^{\psi_{G_2}^{G_1}}(\mathcal{T}_{G_1,0}); Q_{V_2,1}^{G_1} = \mathcal{T}_{V_2,1}^{\tau_{G_2}^{G_1}}(\mathcal{T}_{G_1,0});$ where $\mathcal{T}_{G_1,0} = \mathcal{T}_{G_1,1} \cap G_1; \mathcal{T}_{G_2,0} = \mathcal{T}_{G_1,1} \cap G_2; \mathcal{T}_{G,1} = \mathcal{T}_G$. We put $\mathcal{T}'_{G_1,1}, \mathcal{T}_{V_1,1}, \mathcal{T}_{G_2,1}, \mathcal{T}_{V_2,1}, \mathcal{T}_{V_{1,2},1}$ to be the coarsest topologies generated by $Q_{G_1,1}^G \cup \mathcal{T}_{G_1,0}; Q_{V_1,1}^G \cup (\mathcal{T}_{G,1} \cap V_1); Q_{G_2,1}^G \cup Q_{G_2,1}^{G_1} \cup \mathcal{T}_{G_2,0}; Q_{V_2,1}^G \cup (\mathcal{T}_{G_1,1} \cap V_2); Q_{V_{1,2},1}^G \cup (\mathcal{T}_{G_1,0} \cap V_{1,2})$ respectively. Then we consider $f_1 = m_G, f_2 = Div_1, f_3 = Div_r$, where $m_G$ denotes multiplication on $G$, $m_G(a, b) = ab$ for each $a$ and $b$ in $G$, also $f_1 : G_1 \times V_1 \rightarrow G, f_1 : G_2 \times V_2 \rightarrow G, f_1 : G_2 \times V_{1,2} \rightarrow G_1$, etc, where $G_1 \times V_1, G_2 \times V_2, G_2 \times V_{1,2}$ are in the Tychonoff product topologies $\mathcal{T}_{G_1 \times V_1,1}; \mathcal{T}_{G_2 \times V_2,1}; \mathcal{T}_{G_2 \times V_{1,2},1}$ of the topological spaces $(G_1, \mathcal{T}'_{G_1,1}) \times (V_1, \mathcal{T}_{V_1,1}); (G_2, \mathcal{T}_{G_2,1}) \times (V_2, \mathcal{T}_{V_2,1}); (G_2, \mathcal{T}_{G_2,1}) \times (V_{1,2}, \mathcal{T}_{V_{1,2},1})$ respectively. This induces
the topologies $\mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_1 \times V_1,1})$; $\mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_2 \times V_2,1})$; $\mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_1 \times V_1,2,1})$ on $G$ and $G_1$ respectively and hence the coarsest topology $Q_{G,1}$ on $G$ such that $Q_{G,1} \supset \mathcal{T}_{G,1} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_1 \times V_1,1}) \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_2 \times V_2,1})$, consequently, the coarsest topology $\mathcal{T}_{G,2}$ on $G$ such that $\mathcal{T}_{G,2} \supset \mathcal{T}_{G,1} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(Q_{G \times G,1})$, where $Q_{G \times G,1}$ is the Tychonoff product topology on $G \times G$ induced by $Q_{G,1}$.

Then there exists the coarsest topology $Q'_{G,1}$ on $G_1$ such that $Q'_{G,1} \supset \mathcal{T}'_{G,1} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_2 \times V_2,1})$ and hence the coarsest topology $\mathcal{T}_{G,1}$ such that $\mathcal{T}_{G,1} \supset \mathcal{T}'_{G,1} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(Q'_{G_1 \times G_1})$, where $Q'_{G_1 \times G_1}$ is the Tychonoff product topology on $G_1 \times G_1$ induces by $Q'_{G,1}$.

Assume that on the $n$-th step the topologies $\mathcal{T}_{G,n}$, $\mathcal{T}_{G,n-1}$, $\mathcal{T}_{G_2,n-1}$ on $G$, $G_1$, $G_2$ respectively are constructed, where $n \geq 2$ is the natural number. Then we take the topologies $Q_{G,n}^G = T_{G_1}^{G_1}(\mathcal{T}_{G,n})$; $Q_{V,n}^G = T_{V_1}^{G_1}(\mathcal{T}_{G,n})$; $Q_{G_2,n}^G = T_{G_2}^{G_2}(\mathcal{T}_{G,n})$; $Q_{V_2,n}^G = T_{V_2}^{G_2}(\mathcal{T}_{G,n})$; $Q_{G_1}^{G_1} = T_{V_1}^{G_1}(\mathcal{T}_{G,n-1})$; $Q_{V_1,n}^{G_1} = T_{V_1}^{G_1}(\mathcal{T}_{G,n-1})$; where $\mathcal{T}_{G,n-1}$ is the coarsest topology on $G_1$ generated by $\mathcal{T}_{G,n-1} \cup (\mathcal{T}_{G,n} \cap G_1)$. Then we put $\mathcal{T}_{G,n}$, $\mathcal{T}_{V,n}$, $\mathcal{T}_{G_2,n}$, $\mathcal{T}_{V_2,n}$, $\mathcal{T}_{V_1,n}$ to be the coarsest topologies generated by $Q_{G,n}^G \supset \mathcal{T}_{G,n} \cup (\mathcal{T}_{G,n} \cap V_1)$; $Q_{V,n}^G \supset (\mathcal{T}_{G,n} \cap G_1)$; $Q_{G_2,n}^G \supset Q_{G_2,n} \cup (\mathcal{T}_{G,n-1} \cap G_2)$; $Q_{V_2,n}^G \supset (\mathcal{T}_{G,n} \cap V_2)$; $Q_{V_1,n}^{G_1} \supset (\mathcal{T}_{G,n-1} \cap V_1)$ respectively. Using these topologies we get the topologies $\mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_1 \times V_1,n})$; $\mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_2 \times V_2,n})$; $\mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_1 \times V_1,2,n})$ on $G$ and $G_1$ respectively, consequently, the coarsest topology $Q_{G,n}$ on $G$ such that $Q_{G,n} \supset \mathcal{T}_{G,n} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_1 \times V_1,n}) \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_2 \times V_2,n})$, where $\mathcal{T}_{G_1 \times V_1,n}$ is the Tychonoff product topology on the topological space $(G_1, \mathcal{T}_{G_1 \times V_1,n})$. $\mathcal{T}_{G_2 \times V_2,n}$ corresponds to $(G_2, \mathcal{T}_{G_2,n}) \times (V_2, \mathcal{T}_{V_2,n})$, etc. This induces the coarsest topology $\mathcal{T}_{G,n+1}$ on $G$ such that $\mathcal{T}_{G,n+1} \supset \mathcal{T}_{G,n} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(Q_{G \times G,n})$, where $Q_{G \times G,n}$ is the Tychonoff product topology on $G \times G$ induced by $Q_{G,n}$.

Similarly let $Q'_{G,1,n}$ be the coarsest topology on $G_1$ such that $Q'_{G,1,n} \supset \mathcal{T}'_{G,1,n} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(\mathcal{T}_{G_2 \times V_2,1,n})$ and let $\mathcal{T}_{G,1}$ be the coarsest topology such that $\mathcal{T}_{G,1} \supset \mathcal{T}'_{G,1,n} \cup \mathcal{T}_G^{(f_1,f_2,f_3)}(Q'_{G_1 \times G_1,n})$. This process is repeated by induction in $n \in \mathbb{N}$. Therefore $\mathcal{T}_{G,n+1} \supset \mathcal{T}_{G,n}$; $\mathcal{T}_{G,n} \supset \mathcal{T}_{G_n} \supset \mathcal{T}_{G_j,n+1} \supset \mathcal{T}_{V_j,n}$ and $\mathcal{T}_{V_1,n} \supset \mathcal{T}_{V_1,n+1}$ for each $n \in \mathbb{N}$ and $j \in \{1,2\}$. Hence there exists the coarsest topology $\mathcal{T}_{G,\infty}$ on $G$ such that $\mathcal{T}_{G,\infty} \supset \bigcup_{n=1}^{\infty} \mathcal{T}_{G,n}$. This implies that the coarsest topology $\mathcal{T}_{G,G_1,G_2}$ exists such that $\mathcal{T}_{G,\infty} \supset \mathcal{T}_{G,G_1,G_2} \supset \mathcal{T}_G$ relative
to which the maps \( f_i : G \times G \to G \) are continuous for each \( i \in \{1, 2, 3\} \) and \( \psi_{G_i}^1 : G \to G_1, \psi_{G_2}^1 : G \to G_2, \tau_{G_1}^2 : G \to V_1; \tau_{G_2}^2 : G \to V_2; \tau_{G_1}^2 : G_1 \to V_{1,2} \) are continuous, where \( G_1, G_2, V_1, V_2, V_{1,2} \) are considered in the topologies inherited from \((G, \mathcal{T}_{G,G_1,G_2})\).

**Corollary 2.4.** Let \( D \) be a \( T_1 \) topological metagroup with a topology \( \mathcal{T}_D \) and let the conditions of Remark 2.2 be satisfied. Then there exists the coarsest topology \( \mathcal{T}_{D,AC_1,A} \) on \( D \) such that \( \mathcal{T}_{D,AC_1,A} \supset \mathcal{T}_D \), \((D, \mathcal{T}_{D,AC_1,A})\) is the topological metagroup and the maps \( m_D, \text{Div}_1, \text{Div}_r, \psi_A^D, \tau_A^D, \psi_{AC_1}^D, \tau_{AC_1}^D \) are (jointly) continuous, where \( A, AC_1, V_D, AC_1, V_D, V_{AC_1} \) are considered in the topologies inherited from \((D, \mathcal{T}_{D,AC_1,A})\).

**Proof.** If in Theorem 2.3 \((G, \mathcal{T}_G)\) is a topological metagroup, then from Formula (1.1.9) it follows that \( t_G : G^3 \to G \) also is (jointly) continuous, consequently, \((G, \mathcal{T}_{G,G_1,G_2})\) is the topological metagroup. Therefore, the assertion of this corollary follows from Theorem 2.3 by taking \( G = D, G_1 = AC_1, G_2 = A, V_1 = V_{D,AC_1}, V_2 = V_{D,A}, V_{1,2} = V_{AC_1,A} \).

**Corollary 2.5.** Let \( G \) be a topological quasigroup and let \( A \) be a sub-quasigroup in \( G \) and \( V_{G,A} \) be a transversal set of \( A \) in \( G \). Then there exists the coarsest topology \( \mathcal{T}_{G,A} \) on \( G \) such that the maps \( m_G, \text{Div}_1, \text{Div}_r, \psi_A^G, \tau_A^G \) are continuous, where \( A, V_{G,A} \) are considered in the topologies inherited from \((G, \mathcal{T}_{G,A})\).

The assertion of this corollary is the particular case of Theorem 2.3 with \( G_2 = G_1 = A \) and \( \mathcal{T}_{G,A} = \mathcal{T}_{G,G_1,G_2} \).

**Remark 2.6.** In view of Corollary 1 in [16] and Remark 2.2

\[
(2.6.1) \psi_A^D(\psi_A^D(d)) = \psi_A^D(d) \quad \text{and} \quad \tau_A^D(\tau_A^D(d)) = \tau_A^D(d) \quad \text{and} \\
(2.6.2) d = d^{\psi_A^D}d^{\tau_A^D} \quad \text{for each} \quad d \in D.
\]

In particular, \((d^{\psi_A^D})^{\psi_A^D} = e \quad \text{and} \quad (d^{\tau_A^D})^{\tau_A^D} = e \quad \text{for each} \quad d \in D, \) where \( D \) is the metagroup. Denoting \( a = d^{\psi} \quad \text{and} \quad v = d^{\tau}, \quad a = a_d, \quad v = v_d, \) where \( \psi = \psi_A^D, \tau = \tau_A^D, \) one gets \( a = d/v. \)

From Theorem 1 in [16] it follows that \( C_1^r \) is isomorphic with \( C_1/cC_{1,A} \), where \( B^r = \{ b^r : b \in B \} \) for \( B \subset D. \) Moreover, \((AC_1)/cC_{1,A} \) and \( A/cC_{1,A} \) are the quotient groups such that \( A/cC_{1,A} \leftrightarrow (AC_1)/cC_{1,A}, \) where \( C_{1,A} = C_1 \cap A. \)

If \( a_1n_1 = b(a_2n_2), \) where \( a_1, a_2, b \) belong to \( A, \) \( n_1, n_2 \) are in \( C_1, \) then \((a_2 \backslash (b \backslash a_1))n_1 = n_2, \) hence \((a_2 \backslash (b \backslash a_1)) \in A \cap C_1 = C_{1,A}. \) Vive versa if
\( n_1 = \beta n_2 \) with \( \beta \in \mathcal{C}_{1,A} \), \( n_1, n_2 \) in \( \mathcal{C}_1 \), then for each \( a_1, a_2 \) in \( A \) there exists \( b = (a_1\beta)/a_2 \in A \) such that \( a_1\beta = ba_2 \) and consequently, \( a_1n_1 = a_1\beta n_2 = b(a_2n_2) \). Thus the quotient groups \((AC_1)/_cA \) and \( \mathcal{C}_1/_cC_{1,A} \) are isomorphic. From the latter isomorphism, Remark 3 in [16] it follows that \( V_{D,A}, V_{C_1,C_{1,A}} \) and \( V_{D,AC_1} \) can be chosen such that

\[(2.6.3) \quad V_{C_1,C_{1,A}}V_{D,AC_1} = V_{D,A}; \quad V_{C_1,C_{1,A}} = V_{AC_1,A}, \]

since \( \mathcal{C}_1 \) is the invariant subgroup in \( D \) and \( p_D(a, b, c) = e \) if \( e \in \{a, b, c\} \subset D \). Hence

\[(2.6.4) \quad (d^\psi \gamma)^\psi = d^\psi \gamma \psi \text{ and } (d^\psi \gamma)^\tau = \gamma^\tau \text{ for each } d \in D \text{ and } \gamma \in \mathcal{C}_1, \]

\[d^\psi \gamma = d^\psi (\gamma^\psi \gamma^\tau) = (d^\psi \gamma^\psi)^\gamma^\tau, \text{ since } \gamma^\psi \in \mathcal{C}_{1,A} \text{ and } \gamma^\tau = \gamma^\psi \backslash \gamma \in \mathcal{C}_1. \]

We remind that

\[(2.6.5) \quad (a^\tau)c := (a^\tau)c\tau =: \nu_{D,A}(a, c) \]

for each \( a \) and \( c \) in \( D \) (see Formula (68) in [16]). Suppose that the conditions of Remark 4 in [16] are satisfied. Let on the Cartesian product \( C = D \times F \) (or \( C^* = D \times F^* \)) for each \( d, d_1 \) in \( D, f, f_1 \) in \( F \) (or \( F^* \) respectively) a binary operation be:

\[(2.6.6) \quad (d_1, f_1)(d, f) = (d_1d, \xi((d_1^\psi, f_1), (d^\psi, f)) f_1 f^{(d_1)}), \]

where \( \xi((d_1^\psi, f_1), (d^\psi, f))(v) = \xi((d_1^\psi, f_1(v)), (d^\psi, f(v))) \in \mathcal{C}_1 \) for each \( v \in V_{D,A} \) (see Formula (86) in [16]).

**Theorem 2.7.** Let \( G \) be a \( T_1 \) topological quasigroup and let \( \mathcal{B} = \bigcup_{g \in G} \mathcal{B}_g \), where \( \mathcal{B}_g \) is an open base at \( g \) in \( G \). Then \( \mathcal{B} \) satisfies the following properties

\[(2.7.1)-(2.7.8): \]

\[(2.7.1) \quad \forall g \in G, \forall h \in G, \mathcal{B}_{hg} = h\mathcal{B}_g \text{ and } \mathcal{B}_{gh} = \mathcal{B}_g h; \]

\[(2.7.2) \quad \forall g \in G, \forall h \in G, \mathcal{B}_{h \backslash g} = h \backslash \mathcal{B}_g \text{ and } \mathcal{B}_{g/h} = \mathcal{B}_g / h; \]

\[(2.7.3) \quad \forall g \in G, \forall B \in \mathcal{B}_g, \forall B \in \mathcal{B}_g; \]

\[(2.7.4) \quad \forall g \in G, \forall U \in \mathcal{B}_g, \forall a \in G, \exists U_a \in \mathcal{B}_a, \exists b = a \backslash g \in G, \exists U_b \in \mathcal{B}_b, \]

\[(2.7.5) \quad \forall g \in G, \forall U \in \mathcal{B}_g, \forall a \in G, \exists U_a \in \mathcal{B}_a, \exists b = g \backslash a \in G, \exists U_b \in \mathcal{B}_b, \]

\[(2.7.6) \quad \forall g \in G, \forall U \in \mathcal{B}_g, \forall a \in G, \exists U_a \in \mathcal{B}_a, \exists b = a / g \in G, \exists U_b \in \mathcal{B}_b, \]

\[(2.7.7) \quad \forall g \in G, \forall U \in \mathcal{B}_g, \forall V \in \mathcal{B}_g, \exists W \in \mathcal{B}_g, W \subset U \cap V; \]

\[(2.7.8) \quad \forall g \in G, \bigcap_{U \in \mathcal{B}_g} = \{g\} \].
Conversely, let \( G \) be a quasigroup and let \( \mathcal{B} \) be a family of subsets in \( G \) satisfying (2.7.1)-(2.7.8). Then the family \( \mathcal{B} \) is a base for a \( T_1 \) topology \( \mathcal{T}_\mathcal{B} = \mathcal{T}_\mathcal{B}(G) \) on \( G \) and \((G, \mathcal{T}_\mathcal{B})\) is a topological quasigroup.

**Proof.** Assume that \( G \) is a topological quasigroup with a topology \( \mathcal{T} \). For each \( g \in G \) one can take

\[ (2.7.9) \quad B_g = \{ U : U \in \mathcal{T}, g \in U \} \text{ and put } \mathcal{B} = \bigcup_{g \in G} B_g. \]

Let \( L_q : G \to G \) be a left shift map, \( L_qg = qg \), \( R_q : G \to G \) be a right shift map, \( R_qg = qg \) for each \( g \in G \), where \( q \in G \). Then we put \( \tilde{L}_qg = q \setminus g \), \( \tilde{R}_qg = g/q \) for each \( g \in G \), where \( q \in G \). Hence

\[ (2.7.10) \quad L_q(\tilde{L}_qg) = g, \quad \tilde{L}_q(L_qg) = g, \quad R_q(\tilde{R}_qg) = g, \quad \tilde{R}_q(R_qg) = g \text{ for each } g \in G. \]

The maps \( m_G, \text{Div}_l \) and \( \text{Div}_r \) from \( G \times G \) into \( G \) are (jointly) continuous, where \( m_G \) denotes multiplication on \( G \). From (2.7.9) it follows that \( L_q, \tilde{L}_q, R_q, \tilde{R}_q \) are homeomorphisms from \( G \) onto \( G \) as the topological spaces for each \( q \in G \). This implies (2.7.1), (2.7.2), while (2.7.3) follows from (2.7.9). Then (2.7.4)-(2.7.6) follow from the continuity of \( m_G, \text{Div}_l, \text{Div}_r \).

Certainly, (2.7.9) implies (2.7.7). Property (2.7.8) follows from \( G \) being \( T_1 \) as the topological space and from \( B_g \) being the open base at \( g \) in \( G \).

Vice versa, let \( \mathcal{B} \) be a family of subsets of \( G \) satisfying conditions (2.7.1)-(2.7.8). Let \( \mathcal{T} \) be a family such that

\[ (2.7.11) \quad \forall A \in \mathcal{T}, \exists \mathcal{E} \subset \mathcal{B}, \bigcup_{U \in \mathcal{E}} U = A. \]

Assume that \( \mathcal{F} \subset \mathcal{T} \), hence \( \bigcup_{U \in \mathcal{F}} U \in \mathcal{T} \), since \( \forall C \in \mathcal{F}, \exists \mathcal{F}_C \subset \mathcal{F}, \bigcup_{U \in \mathcal{F}_C} U = C \). Then we take any fixed \( W_1 \in \mathcal{T} \) and \( W_2 \in \mathcal{T} \) and put \( W = W_1 \cap W_2 \). For each \( g \in W \) there exists \( B \in \mathcal{B}_g \) such that \( g \in B \subset W \), consequently, there exists \( \mathcal{E} \subset \mathcal{T} \) with \( \bigcup_{U \in \mathcal{E}} U = W \). Thus \( \mathcal{T} \) is a topology on \( G \). Then for each \( A \) and \( \mathcal{E} : \forall U \in \mathcal{E}, \forall g \in U, \exists U_g \in \mathcal{B}_g, g \in U_g \subset U, \forall x \in G, U_gx \in \mathcal{B}_{gx}, xU_g \in \mathcal{B}_{xg}, x \setminus U_g \in \mathcal{B}_{x \setminus g}, U_g/x \in \mathcal{B}_{g/x} \) by (2.7.1), (2.7.2) and (2.7.8), similarly to (2.7.11).

Thus the family \( \mathcal{B} \) is a base for the topology \( \mathcal{T} \). From (2.7.11) and (2.7.4)-(2.7.6) we infer that \( m_G, \text{Div}_l, \text{Div}_r \) are (jointly) continuous maps with respect to \( \mathcal{T} \). Then (2.7.11) and (2.7.1), (2.7.2) imply that \( \forall g \in G, \forall h \in G, T_{gh} = hT_g, T_{gh} = T_gh, T_{gh} = h \setminus T_g, T_{gh} = T_g/h \), where \( T_g = \{ A \in \mathcal{T} : g \in A \} \). Thus \((G, \mathcal{T})\) is the topological quasigroup. From (2.7.11) and (2.7.8) it follows that \((G, \mathcal{T})\) is \( T_1 \) as the topological space.
Lemma 2.8. Let $B$ be a quasigroup, let $V$ be a set, $F = B^V = \{ f : V \to B \}$ be a family of all (single-valued) maps from $V$ into $B$, $W(S, Q) := \{ f \in F : f(S) \subset Q \}$, where $V \supset S \neq \emptyset$, $B \supset Q \neq \emptyset$. Then for nonvoid subsets $S$, $S_1, S_2, S_i$ in $V$, $Q, Q_1, Q_i$ in $B$:

(2.8.1) $\forall S \subset V$, $\forall Q_2 \subset B$, $\forall Q_1 \subset Q_2$, $W(S, Q_1) \subset W(S, Q_2)$;

(2.8.2) $\forall b \in B$, $\forall S \subset V$, $\forall Q \subset B$, $W(S, b/Q) = b/W(S, Q) \& W(S, Q \setminus b) = W(S, Q) \setminus b$;

(2.8.3) $\forall S \subset V$, $\forall Q \subset B$, $\forall Q_1 \subset B$, $W(S, Q) \setminus W(S, Q_1) \subset W(S, Q \setminus Q_1)$
& $W(S, Q)/W(S, Q_1) \subset W(S, Q/Q_1)$ & $W(S, Q)W(S, Q_1) \subset W(S, QQ_1)$;

(2.8.4) $\forall S_2 \subset V$, $\forall S_1 \subset S_2$, $\forall Q \subset B$, $W(S_2, Q) \subset W(S_1, Q)$;

(2.8.5) for each set $\Lambda$, $\forall i \in \Lambda$, $\forall S_i \subset V$, $\forall Q \subset B$, $W(\bigcup_{i \in \Lambda} S_i, Q) = \bigcap_{i \in \Lambda} W(S_i, Q)$;

(2.8.6) for each set $\Lambda$, $\forall i \in \Lambda$, $\forall Q_i \subset B$, $(\bigcap_{i \in \Lambda} Q_i \neq \emptyset)$, $\forall S \subset V$,
$W(S, \bigcap_{i \in \Lambda} Q_i) = \bigcap_{i \in \Lambda} W(S, Q_i)$.

Proof. (2.8.1). From $f(S) \subset Q_1$ it follows that $f(S) \subset Q_2$.

(2.8.2). $(\forall f \in W(S, b/Q), \forall s \in S, \exists q \in Q, (f(s) = b/q \leftrightarrow q = f(s) \setminus b))$
$\to (\exists f_1 \in W(S, Q), (f = b/f_1 \leftrightarrow f_1 = f \setminus b))$;

$(\forall f_1 \in W(S, Q), \forall s \in S, \exists q_1 \in Q, f_1(s) = q_1) \to (\exists f \in W(S, b/Q),$
$(f_1 = f \setminus b \leftrightarrow f = b/f_1))$.

Thus $W(S, b/Q) = b/W(S, Q)$. Symmetrically it is proved $W(S, Q \setminus b) = W(S, Q) \setminus b$.

(2.8.3). $(\forall f \in W(S, Q), \forall f_1 \in W(S, Q_1), \forall s \in S, f(s) \setminus f_1(s) \in Q \setminus Q_1)$
$\to (\forall f \in W(S, Q), \forall f_1 \in W(S, Q_1), f \setminus f_1 \in W(S, Q \setminus Q_1))$;

$(\forall f \in W(S, Q), \forall f_1 \in W(S, Q_1), \forall s \in S, f(s)/f_1(s) \in Q/Q_1) \to (\forall f \in W(S, Q), \forall f_1 \in W(S, Q_1), f/f_1 \in W(S, Q/Q_1))$;

$(\forall f \in W(S, Q), \forall f_1 \in W(S, Q_1), \forall s \in S, f(s)f_1(s) \in QQ_1) \to (\forall f \in W(S, Q), \forall f_1 \in W(S, Q_1), f f_1 \in W(S, QQ_1))$,

since $Q \setminus Q_1 = \bigcup (b \in B : \exists q \in Q, \exists q_1 \in Q_1, b = q \setminus q_1)$,

$Q/Q_1 = \bigcup (b \in B : \exists q \in Q, \exists q_1 \in Q_1, b = q/q_1)$,

$QQ_1 = \bigcup (b \in B : \exists q \in Q, \exists q_1 \in Q_1, b = qq_1)$.

(2.8.4). $(\forall f \in W(S_2, Q), f(S_2) \subset Q) \to (f(S_1) \subset Q)$,

hence $W(S_2, Q) \subset W(S_1, Q)$, since $S_1 \subset S_2 \subset V$.

(2.8.5). $\forall f \in W(\bigcup_{i \in \Lambda} S_i, Q), ((f(\bigcup_{i \in \Lambda} S_i) \subset Q) \leftrightarrow (\forall i \in \Lambda, f(S_i) \subset Q)$;
\[ \forall f \in \bigcap_{i \in A} W(S_i, Q), ((\forall i \in A, f(S_i) \subset Q) \leftrightarrow (f(\bigcup_{i \in A} S_i) \subset Q)). \]

(2.8.6) \[ \forall s \in S, ((\forall i \in A, f(s) \in Q_i) \leftrightarrow (f(s) \in \bigcap_{i \in A} Q_i)), \]

hence \[ W(S, \bigcap_{i \in A} Q_i) = \bigcap_{i \in A} W(S, Q_i). \]

**Remark 2.9.** Let the conditions of Theorem 5 in [16] be satisfied. We consider the topology \( T \) on the topological metagroup \( D \), also \( A, C \) and the transversal set \( V = V_{D, A} \) in the topology inherited from \((D, T_{D, AC_1, A})\).

Let \( C(V, B) \) denote a family of all continuous maps \( f : V \to B \), where \( B \) is a topological metagroup with a topology \( T_B \). Assume that \((D, T_{D, AC_1, A})\) and \((B, T_B)\) are \( T_1 \) as topological spaces.

As usually \( U \) is a canonical closed subset (i.e. a closed domain) in \( V \) if and only if \( U = cl_V(\text{Int}_V U) \), where \( \text{Int}_V U \) denotes the interior of \( U \) in \( V \), while \( cl_V S \) denotes the closure of \( S \) in \( V \), where \( S \subset V \). Let \( \mathcal{F} \) be a family of nonvoid canonical closed subsets \( U \) in \((V, T_{D, AC_1, A} \cap V)\) such that

\[ \forall U_1 \in \mathcal{F}, \forall U_2 \in \mathcal{F}, U_1 \cup U_2 \in \mathcal{F}; \]

\[ \forall v \in V, \forall v \in P \in (T_{D, AC_1, A} \cap V), \exists U_1 \in \mathcal{F}, v \in \text{Int}(U_1) \& U_1 \subset P. \]

We put \( C_t = \bigcup \{(d, f) \in C : d \in D, f \in C(V, B)\} \), where \( C = D \Delta^\phi_\eta \kappa \xi F = D \Delta^\phi_\eta \kappa \xi F \) (see Definition 5 in [16]). Let \( \mathcal{W} \) be a family of all subsets \( W(S, Q) \) in \( C(V, B) \) such that \( S \in \mathcal{F} \) and \( Q \neq \emptyset \) is open in \((B, T_B)\).

**Theorem 2.10.** Let the conditions of Remark 2.9 be satisfied and let the maps \( \phi, \eta, \kappa, \xi \) be jointly continuous (see Remark 1 in [16]). Then \( T_{D, AC_1, A} \times \mathcal{W} \) is a base of a topology \( T_{C_t} \) on \( C_t \) relative to which \( C_t \) is a topological \( T_1 \cap T_3 \) loop.

**Proof.** Evidently each constant map \( f_b : V \to B \) belongs to \( C(V, B) \), where \( b \) in \( B \) is arbitrary fixed, \( f_b(v) = b \) for each \( v \in V \). This induces the natural embedding of \( B \) into \( C(V, B) \), consequently, \( C(V, B) \) is the nonvoid metagroup with pointwise multiplication and \( \text{Div}_l \) and \( \text{Div}_r \). If \( U_1 \) and \( U_2 \) belong to \( \mathcal{F} \), then \( U_1 \cup U_2 \in \mathcal{F} \) (see §1.8.8 in [15], 1.1.C [9]).

By the conditions of Remark 2.9 \( \{\text{Int}_V(U) : U \in \mathcal{F}\} \) is a base of the topology on \((V, T_{D, AC_1, A} \cap V)\). From \( D \) and \( B \) being the \( T_1 \) topological metagroups and hence topological quasigroups, it follows that they are \( T_3 \) as the topological spaces. Therefore for each open subset \( S_0 \) in \( D \) (or \( V \)) and each \( d_0 \in S_0 \) there exists a canonical closed subset (i.e. closed domain) \( S \) in \( D \) (or \( V \) respectively) such that \( d_0 \in \text{Int}_D S \) (or \( d_0 \in \text{Int}_V S \) respectively) and
By virtue of Theorem 5 in [16] and Corollary 2.4 the maps $\tau_A^0$ and $\psi_A^0$ are continuous on $D$ and $(D, \mathcal{T}_{D,AC_1,A})$ is the topological metagroup.

In view of Lemma 6 in [16] the map $w_j : D \times D \times V_{D,A} \to C_1$ is (jointly) continuous as the composition of jointly continuous maps for each $j \in \{1, 2, 3\}$. According to Remark 4 in [16] $f^{(d)}(v) = f^{s(d,v)}(v^{[d,v]}) = \phi(s(d,v))f(v^{[d,v]})$ for each $d \in D$, $v \in V$, $f \in F$. The map $A \times B \mapsto \phi(a)b \in B$ is jointly continuous by the conditions of this theorem. Lemma 2.8 imply that $W$ is the base of a topology on $C(V, B)$. We take the topology $\mathcal{T}_{C_1}$ generated by the base $\mathcal{B}_{D,AC_1,A} \times W$, where $\mathcal{B}_{D,AC_1,A}$ denotes the base of the topology $\mathcal{T}_{D,AC_1,A}$.

Hence Lemma 2.8 and Formula (2.6.6) imply that multiplication $m_{C_1} : C_1 \times C_1 \to C_1$ is (jointly) continuous. From Formulas in the proof of Theorem 5 in [16] it follows that $Div_l$ and $Div_r$ are jointly continuous from $C_1 \times C_1$ into $C_1$.

The base $W$ generates the $T_1$ topology $\mathcal{T}_W$ on $C(V, B)$ by Lemma 2.8 and Remark 2.9. Hence $\mathcal{T}_{C_1}$ is the $T_1$ topology on $C_t$, since $(D, \mathcal{T}_{D,AC_1,A})$ is $T_1$ by the conditions of this theorem. This implies that $(C_t, \mathcal{T}_{C_1})$ is the $T_3$ topological loop.

**Theorem 2.11.** Let the conditions of Theorem 2.10 be satisfied, let $(D, \mathcal{T}_{D,AC_1,A})$ be locally compact (or compact) and $\mathcal{F}$ be a family of all canonical closed compact subsets in $(V, \mathcal{T}_{D,AC_1,A} \cap V)$, let $F_0$ be closed in $(C(V, B), \mathcal{T}_W)$, let also $cl_{C(V, B)}F_0(v)$ be compact for each $v \in V$, where $F_0(v) = \{f(v) : f \in F_0\}$, let for each compact subset $Z$ in $V$ the restriction $F_0|_Z$ be evenly continuous, let $D\Delta^{\phi,\eta,\kappa,\xi}_{A}F_0 = C_0$ be a subloop in $C_1$. Then $C_0$ is a locally compact (or compact respectively) loop.

**Proof.** Since $V = \psi^{-1}(e)$ and $D$ is $T_1 \cap T_3$ and locally compact, then $V$ is closed in $D$ and hence locally compact by Theorem 3.3.8 [9], consequently, $V$ is a $k$-space. From Lemma 2.8, Remark 2.9 and the conditions on $\mathcal{F}$ it follows that $W$ induces a compact-open topology $\mathcal{T}_W$ on $C(V, B)$. For each compact subset $Z$ in $V$ and open $U_D$ in $D$ the conditions of this theorem imply that $\bigcup\{F_0^{(d)} : d \in U_D\}|_Z$ is evenly continuous, since $F_0^{(d)} \subset F_0$ for each $d \in D$. By virtue of Theorem 3.4.21 [9] $F_0$ is compact. Since $(D, \mathcal{T}_{D,AC_1,A})$ is locally compact, it is sufficient to take any open $U_D$ in $D$ with the compact closure

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$cl_D(U_D)$ in the $T_{D,A}$ topology. Moreover, $cl_{C(V,B)}(\cup \{F_0^{(d)}(v) : d \in U_D\})$ is closed in $F_0$ and hence compact for each $v \in V$. From $B$ being $T_1$ it follows that $B$ is $T_3$.

From the compactness of $F_0$ and Theorem 2.10 it follows that $C_0$ is either the locally compact loop, if $D$ is locally compact, or the compact loop if $D$ is compact.

**Proposition 2.12.** Let the conditions of Theorem 5 in [10] be satisfied and let $i : D \to D$ and $j : B \to B$ be automorphisms of the metagroups $D$ and $B$ such that $i|c_1 = j|c_1$. Then there exists a loop $C_{i,j}$ and an isomorphism

\[
(2.12.1) \theta_{i,j} : C \to C_{i,j} \text{ of } C \text{ onto } C_{i,j} \text{ such that } \theta_{i,j}|D = i \text{ and } \theta_{i,j}|B = j.
\]

**Proof.** By the conditions of this proposition $i(ab) = i(a)i(b)$, $i(a/b) = i(a)/i(b)$ and $i(a \setminus b) = i(a) \setminus i(b)$ for each $a$ and $b$ in $D$, similarly for $i^{-1}$. Therefore, $i : A \to i(A)$ is an isomorphism of metagroups and $i : C_1 \to i(C_1)$ is an isomorphism of groups such that $i(C_1) = j(C_1)$. In view of Corollary 1 in [16] $V_{D,i(A)} = V_{i(D),i(A)} = i(V_{D,A})$, $V_{i(C_1),i(C_1)} = i(V_{C_1,C_1,A})$, $V_{D,i(A)} = i(V_{D,A})$, $i(C_1,A) = i(C_1) \cap i(A)$.

We put

\[
(2.12.2) C_{i,j} = D_\Delta \phi_{i,j} \eta_{i,j} \kappa_{i,j} \xi_{i,j} F_{i,j}
\]

with

\[
F_{i,j} = \{j(f(i(v))) : v \in V_{D,A}, f \in F\}, \phi_{i,j} = j \circ \phi \circ i^{-1}, \phi_{i,j} : i(A) \to A(B),
\]

\[
\eta_{i,j} : i(A) \times i(A) \times B \to i(C), \kappa_{i,j} : i(A) \times B \times B \to i(C),
\]

\[
\xi_{i,j} : (i(A) \times B) \times (i(A) \times B) \to i(C) \text{ such that }
\]

\[
\eta_{i,j}(u', u', b') = i(\eta(i^{-1}(u'), i^{-1}(u'), j^{-1}(b'))),
\]

\[
\kappa_{i,j}(u', c', b') = i(\kappa(i^{-1}(u'), j^{-1}(c'), j^{-1}(b'))),
\]

\[
\xi_{i,j}(u', c'), (v', b') = i(\xi((i^{-1}(u'), j^{-1}(c'))), (i^{-1}(v'), j^{-1}(b')))) \text{ for each } u',
\]

\[
v' \text{ in } i(A), b', c' \text{ in } B; \psi_i : D \to i(A), \tau_i : D \to V_{D,i(A)} \text{ such that }
\]

\[
\psi_i(d') = i(\psi(i^{-1}(d'))), \tau_i(d') = i(\tau(i^{-1}(d'))) \text{ for each } d' \in D. \text{ Let }
\]

\[
(2.12.3) \theta_{i,j}((d, f)(v)) = (i(d), j(f))(i(v)) \text{ for each } (d, f) \in C \text{ and } v \in V_{D,A}, \text{ where } d \in D, f \in F. \text{ Therefore, (2.6.6) and (2.12.3) imply that }
\]

\[
(\theta_{i,j}(d_1, f_1(v)))(\theta_{i,j}(d, f(v))) = (i(d_1d), \xi_{i,j}((i(d_1)) \psi_i, j(f_1)), ((i(d)) \psi_i, j(f)))j(f)(j(f))j(f)(j(f))\{i(d_1)\})(i(v)) \text{ and }
\]

\[
\theta_{i,j}((d_1, f_1(v)))(d, f(v)) = (i(d_1d), i(\xi((d_1) \psi_i, f_1), (d_1, f))j(f_1)(j(f_1)j(f_1)))j(f)(f_1)(j(f_1)))(i(v))
\]

for each $v \in V_{D,A}$, consequently,
\[(2.12.4) \theta_{i,j}((d_1, f_1)(d, f)) = (\theta_{i,j}(d_1, f_1))\theta_{i,j}(d, f)\]

for each \((d, f) = g\) and \((d_1, f_1) = g_1\) in \(C\). Since \(C\) and \(C_{i,j}\) are loops, then (2.12.2) and (2.12.4) imply that \(\theta_{i,j}(g/g_1) = \theta_{i,j}(g)/\theta_{i,j}(g_1)\) and \(\theta_{i,j}(g \backslash g_1) = \theta_{i,j}(g) \backslash \theta_{i,j}(g_1)\) for each \(g\) and \(g_1\) in \(C\). Thus \(\theta_{i,j} : C \to C_{i,j}\) is the isomorphism of these loops.

**Corollary 2.13.** Assume that the conditions of Proposition 2.12 are satisfied and \(i, i^{-1}, j, j^{-1}\) are continuous relative to \(T_{D,A}, T_{B}^{C_1, A}\) and \(T_{B}^{C_1, A}\) topologies on \(D\) and \(B\) respectively. Then \(\theta_{i,j}\) and \(\theta_{i,j}^{-1}\) are continuous relative to \(T_{C_1, C_1}\) and \(T_{C_{i,j}, C_{i,j}}\) topologies on \(C_t\) and \(C_{i,j,t}\) respectively.

**Proof.** This follows from Proposition 2.12, Formula (2.12.3) and Theorem 2.10.

**Remark 2.14.** If the conditions of Theorem 6 instead of that of Theorem 5 in [16] are satisfied, then in Theorems 2.10 and 2.11, Corollary 2.13 \(C_t\), \(C_0\), \(C_{i,j,t}\) are topological metagroups; in Proposition 2.12 \(C\) and \(C_{i,j}\) are metagroups.

### 3 Homomorphisms of topological metagroups.

**Definition 3.1.** A topological space \(X\) is called homogeneous, if for each \(x \in X\) and each \(y \in X\), there exists a homeomorphism \(f\) of \(X\) onto itself such that \(f(x) = y\).

If \(G\) is a metagroup and \(H\) is a submetagroup in \(G\), \(b \in G\), then \(bH\) (or \(Hb\)) is called a left coset (or a right coset respectively).

**Lemma 3.2.** Let \(G\) be a quasigroup and \(H\) be a subquasigroup in \(G\) such that

\[(3.2.1) (Hb)a = H(ba)\] for each \(a\) and \(b\) in \(G\).

Then there exists a set \(G/eH\) of all right cosets and a single-valued quotient map \(\pi : G \to G/eH\).

**Proof.** For each \(a, b\) in \(G\):

\[(Ha) \cap (Hb) \neq \emptyset \iff (\exists h_1 \in H, \exists h_2 \in H, h_1a = h_2b) \iff (\exists h_1 \in H, \exists h_2 \in H, (h_1a)/b = h_2).\]

Notice that Condition (3.2.1) implies \(Hb = (H(ba))/a\) for each \(a\) and \(b\) in \(G\), hence
since for each $a$ and $c$ in $G$ there exists $b \in G$ with $ba = c$, that is, $b = c/a$. Therefore, for these pairs.

Therefore, there exists a single-valued map $\pi$ such that $\pi(b) = Hb$ for each $b \in G$ by (3.2.1) and (1.1.1). This implies that a set $G/_{c}H = \{Hb : b \in G\}$ of all right cosets $Hb$ exists and there exists a single-valued map $\pi : G \rightarrow G/_{c}H$ such that $\pi(b) = Hb$ for each $b \in G$.

**Example 3.3.** If the conditions of Corollary 1 in [16] are satisfied, either $(G = D, H = A\mathcal{C}_1)$ or $(G = A\mathcal{C}_1, H = A)$, then Condition (3.2.1) is satisfied for these pairs.

**Theorem 3.4.** Assume that $G$ is a topological $T_1$ quasigroup with a topology $\mathcal{T}_G$. Assume also that $H$ is a closed subquasigroup satisfying Condition (3.2.1) in $G$ and $G/_{c}H$ is supplied with the quotient topology with respect to the quotient map $\pi : G \rightarrow G/_{c}H$ (see Lemma 3.2). Then for each arbitrary fixed $b \in G$ and $g \in G$ the family $\{\pi(V g) : V \in \mathcal{T}_G, b \in V\}$ is a local base of $G/_{c}H$ at $H(bg) \in G/_{c}H$. Moreover, the map $\pi$ is open and $G/_{c}H$ is a homogeneous $T_1$-space.

**Proof.** In view of Lemma 3.2 $\pi^{-1}(\pi(H(V g))) = H(V g) = (HV)g$ for each $V \in \mathcal{T}_G$, consequently, $\pi(H(V g)) = \pi((HV)g)$ is open in $G/_{c}H$, since $\pi$ is the quotient map and $(G, \mathcal{T}_G)$ is the topological quasigroup. Consider any fixed $g, b \in G$ and an open neighborhood $P$ of $H(bg)$ in $G/_{c}H$. Then $\pi^{-1}(P) =: Q$ is open and $H(bg) \subset Q$, since $\pi$ is continuous and $H(Hg) = Hg$.

For each $V \in \mathcal{T}_G$ such that $b \in V$ and $Vg \subset Q$ one gets $\pi(V g) \subset P$, consequently, $\pi^{-1}(\pi(V g)) \subset Q$ and hence $\pi(H(V g)) \subset P$, since $H(V g) = \pi^{-1}(\pi(V g))$. Thus the map $\pi$ is open and $\{\pi(V g) : V \in \mathcal{T}_G, b \in V\}$ is the local base of $G/_{c}H$ at $H(bg)$ in $G/_{c}H$. Then we put $S_b(Hg) = (Hg)b$ for each $b \in G$, where $g \in G$. Hence $S_b : G/_{c}H \rightarrow G/_{c}H$, since $(Hg)b = H(gb)$ by (3.2.1). Therefore, $S_bS_b = I$ and $S_bS_b = I$ for each $b \in G$, where $\hat{S}_b(Hg) = (Hg)/b, I g = g$ for each $g \in G$.

This implies that $S_b$ is a bijection for each $b \in G$. For each $a, b$ in $G$ and an open neighborhood $V$ of $g$, where $g \in G$, $\pi(H((V a)b))$ is a basic neighborhood of $H((ga)b) = (H(ga))b$ and $\pi(H(V a))$ is a basic neighborhood.
of \(H(ga)\) in \(G/eH\). Therefore, \(S_b : G/eH \to G/eH\) is a homeomorphism, since 
\[S_b \pi(H(Va)) = \pi((H(Va)b) = \pi(H(Va)b)).\]

For each \(a, b \in G\) we deduce from (3.2.1) that \(S_b(H(a/b)) = Ha\), consequently, the quotient space \(G/eH\) is homogeneous. The space \(G/eH\) is \(T_1\), since \(Hb\) is closed in \(G\) and \(\pi\) is the quotient map.

**Definition 3.5.** The space \(G/eH\) in Theorem 3.4 is called the right coset space, \(S_b\) is called the right translation of \(G/eH\) by \(b\).

**Corollary 3.6.** Let \(G\) be a topological quasigroup and let \(H\) be a closed subquasigroup in \(G\) satisfying Condition (3.2.1). Then \(R_b\) and \(S_b\) are homeomorphisms of \(G\) and \(G/eH\) respectively and 
\[(3.6.1) \pi \circ R_b = S_b \circ \pi \text{ for each } b \in G.\]

**Corollary 3.7.** Assume that \(G\) is a topological quasigroup and \(H\) is a closed invariant subquasigroup in \(G\). Then \(G/eH\) with the quotient topology, multiplication, \(\text{Div}_1\) and \(\text{Div}_r\) induced from \(G\), is a topological quasigroup and the quotient map \(\pi : G \to G/eH\) is an open continuous homomorphism.

**Corollary 3.8.** Suppose that the conditions of Corollary 3.7 are satisfied. Then \(G/eH\) with the quotient topology is discrete if and only if \(H\) is open in \(G\).

The assertions of Corollaries 3.6-3.8 follow from Theorem 3.4.

**Lemma 3.9.** Suppose that \(G\) is a topological loop, \(H\) is a closed subquasigroup in \(G\) satisfying Condition (3.2.1). Then \(cI_G/eH \pi(V) \subset \pi(U/g2)\) for each \(g2 \in G\) and open subsets \(U\) and \(V\) in \(G\) such that \(g2 \in U, e \in V\) and \((V/V)g2 \subset U\).

**Proof.** We take any fixed \(g\) in \(G\) such that \(\pi(g) \in cI_G/eH \pi(V)\), where the quotient map \(\pi\) is as in Lemma 3.2. Notice that \(gV\) is an open neighborhood of \(g\) in \(G\). Note also that the quotient map \(\pi : G \to G/eH\) is open by Theorem 3.4. Therefore \(\pi(gV)\) is an open neighborhood of \(\pi(g)\) in \(G/eH\), since \(H(gV) = (Hg)V = \bigcup \{hgV : h \in H\}\) is open in \(G\). Hence \(\pi(gV) \cap \pi(V) \neq \emptyset\). This implies that there exist \(a\) and \(b\) in \(V\) with \(\pi(ga) = \pi(b)\), consequently, there exist \(h_1\) and \(h_2\) in \(H\) with \(h_1(ga) = h_2b\).

Then we infer that \(ga = h_1 \setminus (h_2b)\). From (3.2.1) it follows that there exists \(h_3 \in H\) such that \(h_1 \setminus (h_2b) = h_3b\), hence \(g = (h_3b)/a\), consequently, there exists \(h_4 \in H\) such that \(g = h_4(b/a) \in H(b/a)\) by (3.2.1). On the other
hand, \(H(b/a) \subset H(V/V) \subset \pi(U/g_2)\), consequently, \(\pi(g) \in \pi(U/g_2)\). Thus \(cl_{G/eH}\pi(V) \subset \pi(U/g_2)\).

**Theorem 3.10.** Let \(G\) be a topological \(T_1\) loop, \(H\) be a closed subquasi-group of \(G\) satisfying Condition (3.2.1). Then the quotient space \(G/eH\) is regular.

**Proof.** In view of Lemma 3.2 the quotient space \(G/eH\) and the quotient map \(\pi : G \rightarrow G/eH\) exist. Take any fixed \(g_2 \in G\) and an open neighborhood \(P\) of \(\pi(g_2)\) in \(G/eH\). From the continuity of the map \(\pi\) it follows that there exists an open neighborhood \(U\) of \(g_2\) in \(G\) such that \(\pi(U) \subset P\). Since \(G\) is the topological loop, then for each fixed \(g_2 \in G\) there exists an open neighborhood \(V\) of \(e\) in \(G\) such that \(V/V \subset U/g_2\). By virtue of Lemma 3.9 \(cl_{G/eH}\pi(V) \subset \pi(U/g_2)\), that is \(cl_{G/eH}\pi(Vg_2) \subset \pi(U)\). On the other hand, \(\pi(Vg_2)\) is an open neighborhood of \(\pi(g_2)\) and \(G/eH\) is homogeneous and \(T_1\) by Theorem 3.4. This implies that \(G/eH\) is regular.

**Definition 3.11.** If \(X\) and \(Y\) are topological spaces and \(f : X \rightarrow Y\) is a continuous map such that \(f\) is closed and a preimage \(f^{-1}(y)\) is compact for each \(y \in Y\), then \(f\) is called a perfect map.

**Lemma 3.12.** Assume that \(G\) is a quasigroup, \(A, B, C\) are subsets in \(G\). Then \((AB) \cap C = \emptyset\) if and only if \(A \cap (C/B) = \emptyset\) and if and only if \((A \setminus C) \cap B = \emptyset\).

**Proof.** For each \(a \in A, b \in B, c \in C, (ab \neq c) \leftrightarrow (a \neq c/b) \leftrightarrow (a \setminus c \neq b)\).

**Theorem 3.13.** Let \(G\) be a topological \(T_1\) loop, \(F\) be a compact subset in \(G\), \(P\) be a closed subset in \(G\) and \(F \cap P = \emptyset\). Then an open neighborhood \(V\) of \(e\) in \(G\) exists such that \(FV \cap P = \emptyset\) and \(VF \cap P = \emptyset\).

**Proof.** The left translation \(L_g : G \rightarrow G\) is the homeomorphism of \(G\) onto \(G\). Therefore, for each \(f \in F\) an open neighborhood \(V_f\) of \(e\) in \(G\) exists such that \(fV_f \cap P = \emptyset\). From the joint continuity of multiplication on \(G\) it follows that an open neighborhood \(W_f\) of \(e\) in \(G\) exists with \((fW_f)W_f \subset fV_f\). Hence \(F \subset \bigcup_{f \in F} fW_f\) and from the compactness of \(F\) it follows that these open covering of \(F\) has a finite subcovering \(F \subset \bigcup_{f \in \Lambda} fW_f\), where \(\Lambda \subset F\), \(\text{card}(\Lambda) < \aleph_0\).

We put \(S = \cap_{f \in \Lambda} W_f\), consequently, \(S\) is an open neighborhood of \(e\) in \(G\). For each \(g \in F\) an element \(f\) in \(\Lambda\) exists with \(g \in fW_f\), consequently,
\( gS \subset (fW_f)W_f \subset fV_f \subset G - P \) and hence \( FS \cap P = \emptyset \).

Symmetrically an open neighborhood \( Q \) of \( e \) in \( G \) exists such that \( QP \cap F = \emptyset \). Taking \( V = S \cap Q \), we get the assertion of this theorem.

**Theorem 3.14.** Let \( G \) be a topological \( T_1 \) loop, \( F \) be a compact subset in \( G \), \( P \) be a closed subset in \( G \). Then the sets \( FP \) and \( PF \) are closed in \( G \).

**Proof.** In view of Lemma 3.12 for each \( V \subset G \) the condition \((Va) \cap (FP) = \emptyset \) is equivalent to \((V \setminus (FP)) \cap \{a\} = \emptyset \). From the joint continuity of \( Div \) and multiplication on \( G \) it follows that for each \( f \in F \) an open neighborhood \( V_f \) of \( e \) in \( G \) exists such that \((V_f \setminus ((V_f)P)) \cap \{a\} = \emptyset \).

The subset \( F \) in \( G \) is compact, consequently, the open covering \( \{V_{f} : f \in F\} \) of \( F \) contains a finite subcovering \( \{V_{f_{\lambda}} : \lambda \in \Lambda\} \), \( F \subset \bigcup_{f \in \Lambda} V_{f} \), where \( \Lambda \) is a finite subset in \( F \). Therefore \( S = \bigcap_{f \in \Lambda} V_{f} \) is an open neighborhood of \( e \) in \( G \). Hence \( S \setminus (FP) \subset S \setminus ((\bigcup_{f \in \Lambda} (V_{f}f))P) \subset \bigcup_{f \in F} V_{f} \setminus ((V_{f}f)P) \) and \((S \setminus (FP)) \cap \{a\} = \emptyset \), since \( a \notin \bigcup_{f \in F} V_{f} \setminus ((V_{f}f)P) \). Therefore \((Sa) \cap (FP) = \emptyset \) by Lemma 3.12. Thus \( FP \) is closed. Symmetrically is proven that \( PF \) is closed in \( G \).

**Theorem 3.15.** Assume that \( G \) is a topological \( T_1 \) loop and \( H \) is a compact subquasigroup in \( G \) satisfying Condition (3.2.1). Then the quotient map \( \pi : G \to G/cH \) is perfect.

**Proof.** From Theorem 3.14 it follows that \( HP \) is closed in \( G \) for each closed subset \( P \) in \( G \). On the other hand, we have \( HP = \pi^{-1}(\pi(P)) \) and \( \pi(P) \) is closed in \( G/cH \), consequently, the quotient map \( \pi \) is closed. If \( y \in G/cH \) and \( g \in G \) with \( \pi(g) = y \), then \( \pi^{-1}(y) = Hg \) is compact in \( G \), since \( H \) is compact and \( R_g : G \to G \) is the homeomorphism of \( G \). Hence the fibers of \( \pi \) are compact. Thus the quotient map \( \pi \) is perfect.

**Corollary 3.16.** Suppose that the conditions of Theorem 3.15 are satisfied and \( G/cH \) is compact. Then \( G \) is compact.

**Proof.** According to Theorem 3.15 the quotient map \( \pi : G \to G/cH \) is perfect. Since \( G/cH \) is compact, then \( G \) is compact by Theorem 3.7.2 in [10].

**Corollary 3.17.** Let the conditions of Corollary 1 in [10] be satisfied and let \( C_1 \) and \( A \) and \( AC_1 \) be closed in \( G \), then \( D/c(AC_1) \) and \( (AC_1)/cA \) are homogeneous \( T_1 \cap T_3 \) spaces and the quotient maps \( \pi^D_{AC_1} : D \to D/c(AC_1) \) and \( \pi^A_{AC_1} : AC_1 \to (AC_1)/cA \) are open.
Proof. This follows from Theorems 3.4, 3.10 and Example 3.3.

Corollary 3.18. Let the conditions of Theorem 2.3 be satisfied and let \((G, T_{G,G_1,G_2})\) be compact and \((G, T_G)\) be \(T_1\) as the topological quasigroup. Then \(T_{G,G_1,G_2} = T_G\).

Proof. Since \((G, T_G)\) is the \(T_1\) topological quasigroup, then it is regular. In view of Corollary 3.1.14 in [9] \(T_{G,G_1,G_2} = T_G\).

Corollary 3.19. Assume that the conditions of Corollary 3.4 are satisfied and \((D, T_{D,AC_1,A})\) is compact. Then \(T_D = T_{D,AC_1,A}\); moreover, \(A\) and \(AC_1\) are compact relative to the topologies \(T_D \cap A\) and \(T_D \cap (AC_1)\) respectively inherited from \(G\).

Proof. Corollary 3.18 implies that \(T_D = T_{D,AC_1,A}\). The maps \(\tau_A^D\) and \(\tau_{AC_1}^D\) are continuous by Corollary 2.4. On the other hand, \(A = (\tau_A^D)^{-1}(e)\) and \(AC_1 = (\tau_{AC_1}^D)^{-1}(e)\), consequently, \(A\) and \(AC_1\) are closed in \(D\), hence \(A\) and \(AC_1\) are compact by Theorem 3.1.2 in [9].

Proposition 3.20. Assume that the conditions of Remark 1 in [10] are satisfied, \(G = A \star^{\phi, \eta, \kappa, \xi} B\) is a smashed twisted product of metagroups \(A\) and \(B\) with smashing factors \(\phi, \eta, \kappa, \xi\). Then embeddings \(\theta_A^G : A \hookrightarrow G\) and \(\theta_B^G : B \hookrightarrow G\) exist, and \(\theta_B^G(B)\) in \(G\) is invariant. Moreover, a transversal set \(V_{G,B}\) exists such that \(V_{G,B} = \theta_A^G(A)\).

Proof. We shortly denote \(\theta_A^G\) as \(\theta_A\), because \(G\) is specified, and we put \(\theta_A(a) = (a, e)\) with \(e = e_B\) for each \(a \in A\); \(\theta_B(b) = (e, b)\) with \(e = e_A\) for each \(b \in B\). From Formula (37) in [10] it follows that \((e, b)(a, e) = (a, b\xi((e, b), (a, e)))\) for each \(a \in A\) and \(b \in B\). Therefore, for each \(g = (a_1, b_1)\) in \(G\) there exist unique \(a \in A\) and \(b \in B\) such that

\[
(3.20.1) \quad (e, b)(a, e) = g \quad \text{with} \quad a = a_1 \quad \text{and} \quad b = b_1/\xi((e, b_1), (a_1, e)),
\]

since \(\xi((e, b), (a, e)) = \xi((e, b_1), (a, e))\) by (35) in [10]. Certainly the maps \(a_1 \mapsto a\) and \((a_1, b_1) \mapsto b\) provided by (3.20.1) are single-valued.

For each \(g_1 = (e, b_1) \in G, g_2 = (a_2, b_2) \in G, g_3 = (a_3, b_3) \in G\) we deduce that

\[
I_1 = (g_1g_2)g_3 = (a_2a_3, b^{\xi^2}_2b_2b_1\xi((e, b_1), (a_2, b_2))\xi((a_2, b_2b_1), (a_3, b_3)))\]
\[
I_2 = g_1(g_2g_3) = (a_2a_3, (b^{\xi^2}_2b_2)\xi((a_2, b_2), (a_3, b_3))b_1\xi((e, b_1), (a_2a_3, b^{\xi^2}_2b_2)))
\]

by Conditions (31), (32), (34) in [10]. Hence \(I_1 = tI_2\) with \(t = t(g_1, g_2, g_3) \in \theta_B(C)\), consequently, \(\theta_B(B)\) satisfies Condition (3.2.1), since \(C \hookrightarrow C(B)\) by
Remark 1 in [16].

In view of Lemma 3.2 and Formula (3.20.1) the transversal set \( V_{G,B} = \theta_A(A) \) and the maps \( \psi = \psi_B^G : G \to \theta_B(B) \) and \( \tau = \tau_B^G : G \to \theta_A(A) \) exist such that

\[
(3.20.2) \quad g = g^\psi g^\tau \text{ with }
\]

\[
(3.20.3) \quad g^\psi = (e, b) \text{ and } g^\tau = (a, e) \text{ for each } g = (a_1, b_1) \in G, \text{ where }
\]

\[
(3.20.4) \quad a = a_1 \text{ and } b = b_1/\xi((e, b_1), (a_1, e)).
\]

It remains to prove that \( \theta_B(B) \) is invariant in \( G \). For this it is sufficient to prove that

\[
(3.20.5) \quad g_1 \theta_B(B) = \theta_B(B) g_1 \text{ and }
\]

\[
(3.20.6) \quad (g_1 \theta_B(B))g_2 = g_1(\theta_B(B)g_2) \text{ for each } g_1 \text{ and } g_2 \text{ in } G,
\]

since Properties (3.20.5) and (3.20.6) imply that \( (g_1g_2)\theta_B(B) = g_1(g_2\theta_B(B)) \) for each \( g_1 \) and \( g_2 \) in \( G \).

For each \( g_1 = (a_1, b_1) \) in \( G \) and \( b_2 \in B \), \( b_3 \in B \) we get that

\[
(a_1, b_1)(e, b_2) = (a_1, b_2^a_1 b_1 \xi((a_1, b_1), (e, b_2))) \text{ and }
\]

\[
(e, b_3)(a_1, b_1) = (a_1, b_1 b_3 \xi((e, b_3), (a_1, b_1)))
\]

according to (37) in [16]. The following equation

\[
b_2^a_1b_1 \xi((a_1, b_1), (e, b_2)) = b_1 b_3 \xi((e, b_3), (a_1, b_1))
\]

has a unique solution

\[
b_3 = [b_1 \setminus (b_2^a_1 b_1)] \xi((a_1, b_1), (e, b_2))/\xi((e, b_1 \setminus (b_2^a_1 b_1)), (a_1, b_1))
\]

for given \( g_1 = (a_1, b_1) \) and \( g_2 = (e, b_2) \), since \( \xi \) satisfies Condition (35) in [16]. From \( \xi(g_1, g_3) \in \mathcal{C} \) for each \( g_1 \) and \( g_3 \) in \( G \) and \( \mathcal{C} \to \mathcal{C}(B) \subset B \) it follows that \( b_3 \in B \). Thus \( G \) satisfies Condition (3.20.5).

Then we consider \( I_1 = (g_1(e, b_2))g_3 \) and \( \bar{I}_2 = g_1((e, \bar{b}_2)g_3) \) for any \( g_1 \) and \( g_3 \) in \( G, b_2 \) and \( \bar{b}_2 \) in \( B \). Then we infer that

\[
I_1 = (a_1 a_3, b_2^a_1(b_2^a_1 b_1) \xi((a_1, b_1), (e, b_2)) \xi((a_1, b_2^a_1 b_1), (a_3, b_3)))
\]

and

\[
\bar{I}_2 = (a_1 a_3, (b_3^a_1 \bar{b}_2^a_1) \kappa(a_1, b_3, \bar{b}_2) \xi((e, \bar{b}_2), (a_3, b_3)) b_1 \xi((a_1, b_1), (a_3, b_3 \bar{b}_2)))
\]

by (33) and (37) [16]. The following equation \( I_1 = \bar{I}_2 \) is satisfied if and only if

\[
b_2^a_1b_1 \gamma = \bar{b}_2^a_1 b_1 \alpha \rho(b_3^a_1, \bar{b}_2^a_1, b_1 \alpha) \text{ with }
\]

\[
\gamma = \xi((a_1, b_1), (e, b_2)) \xi((a_1, b_2^a_1 b_1), (a_3, b_3));
\]

\[
\alpha = \kappa(a_1, b_3, \bar{b}_2) \xi((e, \bar{b}_2), (a_3, b_3)) \xi((a_1, b_1), (a_3, b_3 \bar{b}_2))
\]

by (1.1.1) and (1.1.9). Using (34), (35) and Lemma 2 in [16] we deduce that there exists a unique solution
(3.20.7) \( \tilde{b}_2^{a_1} = (b_2^{a_1} b_1 \gamma)/(b_1 \delta) \) for the given \( g_1 = (a_1, b_1) \) and \( g_3 = (a_3, b_3) \) in \( G, b_2 \in B \) with
\[
\delta = \alpha p(b_2^{a_1}, b_1 \alpha) \quad \text{and} \quad \alpha = \kappa(a_1, b_3, b_2)\xi((e, b_2), (a_3, b_3))\xi((a_1, b_1), (a_3, b_3 b_2)).
\]

Since \( \gamma \in C \) and \( \delta \in C, C \hookrightarrow C(B) \), then \( \tilde{b}_2^{a_1} \in B \). From (31) and (33) [16] it follows that
\[
(3.20.8) \tilde{b}_2 = (\tilde{b}_2^{a_1})^{e/a_1}/\eta(e/a_1, a_1, (\tilde{b}_2^{a_1})^{e/a_1}).
\]
Hence (3.20.7) and (3.20.8) imply that \( \tilde{b}_2 \in B \), consequently, \( G \) satisfies Condition (3.20.6). Thus \( \theta_B(B) \) is the invariant submetagroup in \( G \).

**Corollary 3.21.** If the conditions of Remark 1 in [16] are satisfied, \( A \) and \( B \) are topological \( T_1 \) metagroups, the topology on \( G \) is induced by the Tychonoff product topology on \( A \times B \) and the smashing factors \( \phi, \eta, \kappa, \xi \) are (jointly) continuous, then the maps \( \psi : G \rightarrow \theta_B(B) \) and \( \tau : G \rightarrow V_{G,B} = \theta_A(A) \) are continuous relative to the topology \( \mathcal{T}_G \) on the topological metagroup \( G = A \star^{\phi,\eta,\kappa,\xi} B \).

**Proof.** This follows from Formulas (3.20.2)-(3.20.4) and the (joint) continuity of the smashing factors \( \phi, \eta, \kappa, \xi \) and hence of \( \text{Div}_r \) and \( t_G \) on \( (G, \mathcal{T}_G) \), where the topology \( \mathcal{T}_G \) on \( G \) is induced by the Tychonoff product topology on \( A \times B \).

**Corollary 3.22.** Let for pairs of metagroups \( A_j, B_j \) the conditions of Remark 1 in [16] be satisfied for each \( j \in \{1, 2\} \), where \( B_1 = B_2 \) such that \( \mathcal{C}_m(A_j) \subset C \hookrightarrow B_j \hookrightarrow C(A_j) \) for each \( j \in \{1, 2\} \). Let \( \phi_2(a)b = id(b) = b \) for each \( a \in C \) and \( b \in B_2 \), and \( \xi_2((a,e),(e,b)) = \xi_2((e,b),(a,e)) \) for each \( a \in A_2 \) and \( b \in B_2 \). Let \( A' = A_1 \star^{\phi_1,\eta_1,\kappa_1,\xi_1} B_1 \) and \( B = A_2 \star^{\phi_2,\eta_2,\kappa_2,\xi_2} B_2 \) and let \( \phi_3, \eta_3, \kappa_3, \xi_3 \) for the pair \( (A', B) \) with \( \mathcal{C}'_1 = \theta_B^{B_2}(B_2) \) satisfy the conditions of Remark 1 in [16] (with \( \mathcal{C}' \) instead of \( C \)) and let \( D = A' \star^{\phi_3,\eta_3,\kappa_3,\xi_3} B \), where \( \theta_B^{B_2} : B_2 \rightarrow B \) is the embedding provided by Proposition 3.20. Then there are embeddings \( \theta_{A_j} : A_j \hookrightarrow D, \theta_{B_j} : B_j \hookrightarrow D \) for each \( j \in \{1, 2\} \), \( \theta_B : B \hookrightarrow D \) such that \( D \) with \( A = \theta_{A_2}(A_2) \) and \( \mathcal{C}_1 = \theta_{B_2}(B_2) \) satisfy Condition (28) in [16] and \( xA = Ax \) for each \( x \in C_1 \).

**Proof.** By virtue of Theorem 4 in [16] \( B, A' \) and \( D \) are metagroups and there are embeddings \( \theta_{A_j} : A_j \hookrightarrow D, \theta_{B_j} : B_j \hookrightarrow D \) for each \( j \in \{1, 2\} \), \( \theta_B : B \hookrightarrow D \) such that \( \theta_B(B) = \theta_{A_2}(A_2)\theta_{B_2}(B_2) \), since \( B_2 \hookrightarrow C(A_2) \).
For each \((e, b) \in B, (a, e) \in B\) and \((a_2, e) \in B\) with \(b \in B_2, a \in A_2, a_2 \in A_2\) we deduce that
\[
(a, e)(e, b) = (a, b^\alpha \xi((a, e), (e, b))) \text{ and }
(e, b)(a_2, e) = (a_2, b\xi((e, b), (a_2, e))).
\]
Therefore \((a, e)(e, b) = (e, b)(a_2, e)\) if and only if \(a = a_2\) and \(b^\alpha \xi((a, e), (e, b)) = b\xi((e, b), (a_2, e))\). From (31) and (35) in [16] and the conditions of this corollary it follows that \(xA = Ax\) for each \(x \in C_1\), since \(\phi_2(a)b = b^\alpha = b\) for each \(a \in A_2\) and \(b \in B_2\). In view of Proposition 3.20 the subgroup \(C_1\) is invariant in \(\theta_B(B)\) and \(\theta_A(A')\).

Certainly, \(\theta_B_1(B_1)\) and \(\theta_B_2(B_2)\) are isomorphic subgroups in \(D\), since \(B_1 = B_2\). Hence each \(d \in D\) can be presented in the following form: \(d = a_1a_2b\) with \(a_1 \in \theta_A_1(A_1)\), \(a_2 \in \theta_A_2(A_2)\) and \(b \in \theta_B_1(B_1)\). From \(a_1 \theta_B_2(B_2) = \theta_B_2(B_2) a_1\) and \(a_2 \theta_B_2(B_2) = \theta_B_2(B_2) a_2\) it follows that \(d \theta_B_2(B_2) = \theta_B_2(B_2) d\) for each \(d \in D\). On the other hand, \(C_m(D) \subset C_1\), since \(C_m(A_j) \subset C \leftrightarrow B_j \leftrightarrow C(A_j)\) for each \(j \in \{1, 2\}\). Consequently, the subgroup \(\theta_B_2(B_2)\) is invariant in \(D\).

**Corollary 3.23.** Assume that the conditions of Corollary 3.22 are satisfied, \(A_j, B_j\) are \(T_1\) topological metagroups for each \(j \in \{1, 2\}\) and \(\phi_i, \eta_i, \kappa_i, \xi_i\) are jointly continuous for each \(i \in \{1, 2, 3\}\). Then \(D, A, C_1\) provided by Corollaries 3.21 and 3.22 are \(T_1 \cap T_3\) topological metagroups and satisfy the conditions of Theorem 6 in [16] and \(C_1\) is closed in \(D\).

**Proof.** This follows from Theorem 4 in [16] and Corollaries 3.21 and 3.22.

**Definition 3.24.** Let \(\Lambda\) be a directed set, \(G_j\) be a topological metagroup (or loop), \(\pi_i^j : G_j \rightarrow G_i\) be a continuous homomorphism for each \(i \leq j\) in \(\Lambda\) such that \(\pi_i^j \circ \pi_j^k = \pi_i^k\) for each \(i \leq j \leq k\) in \(\Lambda\), and \(\pi_i^i = id_{G_i}\) for each \(i \in \Lambda\), where \(id_{G_i}(g_i) = g_i\) for each \(g_i \in G_i\). Then \(S = \{G_j, \pi_i^j, \Lambda\}\) is called an inverse continuous homomorphism system of topological metagroups (or loops respectively). If a topological metagroup \(G\) is a limit of \(S, G = \lim S\), then it is said that \(G\) is decomposed into \(S\).

**Theorem 3.25.** There exists an infinite family \(\mathcal{F}\), each \(G \in \mathcal{F}\) is a topological \(T_1\) metagroup such that \(G\) is compact, locally connected and cannot be decomposed into the inverse continuous homomorphism system \(S_G = \{G_j, \pi_i^j, \Lambda\}\) of topological metagroups \(G_j\) with \(\dim(G_j) < \infty\) for each \(j \in \Lambda\).
Proof. We take any locally connected $T_1$ compact metagroups $A$, $B$ and their invariant closed subgroup $C$ with positive covering dimensions $\dim(A) > 0$, $\dim(B) > 0$, $\dim(A/C) > 0$, $\dim(B/C) > 0$ such that the conditions of Theorem 6 in [16] are satisfied. Evidently, such triples $(A, B, C)$ exist and their family is infinite. Indeed, in particular, they may be direct products $A = K_1 \times C$, $B = P_1 \times C$ or semidirect products $A = K_1 \ltimes C$, $B = P_1 \ltimes C$ with topological $T_1$ metagroups $K_1$, $P_1$ and a topological $T_1$ group $C$; or particularly $A$, $B$ may be topological $T_1$ groups.

Therefore, $\theta_B(B)$ is invariant in the smashed twisted product $G = A \ast^{\phi,\eta,\kappa,\xi} B$ such that $G$ is a topological $T_1$ metagroup and a transversal set exists $V_{G,B} = \theta_A(A)$ by Corollary 1 in [16] and Proposition 3.20. By virtue of Corollary 3.21 the maps $\psi : G \to \theta_B(B)$ and $\tau : G \to V_{G,B}$ are continuous. For compact $A$ and $B$ the metagroup $G$ is compact by the Tychonoff theorem 3.2.4 in [9].

This implies that there are triples $(A_1, B_1, C)$ and $(A_2, B_2, C)$ satisfying the conditions of Corollary 3.23 with locally connected $T_1$ compact metagroups $A_1$, $A_2$, $B_1 = B_2$ and their invariant closed subgroup $C$ with positive covering dimensions $\dim(A_1) > 0$, $\dim(A_2) > 0$, $\dim(B_1) > 0$, $\dim(A_1/C) > 0$, $\dim(A_2/C) > 0$, $\dim(B_1/C) > 0$. Then $D$, $A$, $C_1$ provided by Corollaries 3.21 and 3.22 satisfy the conditions of Corollary 1 in [16] or Proposition 3.20 such that $A = \theta_{A_2}(A_2)$. By virtue of Theorem 6 in [16], Theorem 1 in [29], Theorem 3.15 and Corollary 3.23 $D$, $A$, $B$ are locally connected $T_1$ compact metagroups with a closed invariant subgroup $C_1 = \theta_{B_2}(B_2)$ and $\dim(D) > 0$, $\dim(A) > 0$, $\dim(B) > 0$. Moreover, $V_{D,AC_1} = \theta_{A'}(A')$, with $A' = A_1 \ast^{\phi_1,\eta_1,\kappa_1,\xi_1} B_1$, and there is a bijection from $V_{AC_1,A} = V_{C_1,\cdot,\cdot,\cdot,\cdot}$ onto $(AC_1)/_cA$ by Remark 2.6 and $V_{C_1,\cdot,\cdot,\cdot,\cdot}V_{D,AC_1} = V_{D,A}$ by Formula (2.6.3). Therefore, $V_{C_1,\cdot,\cdot,\cdot,\cdot}$ and $V_{D,AC_1}$ can be chosen compact, consequently, $V_{D,A}$ is compact by the Tychonoff theorem 3.2.4 in [9]. Corollaries 3.17 and 3.21 imply that the maps $\psi^D_A : D \to A$ and $\tau^D_A : D \to V_{D,A}$ are continuous relative to the topology $T_D$.

In view of Theorems 2.10 and 2.11 there exists a $T_1 \cap T_3$ compact metagroup $C_0 = D\Delta^\phi_{\cdot,\cdot,\cdot,\cdot,\cdot} F_0$, where $F_0$ is closed in $(C(V,B), T_W)$, where $F_0 \subset C(V,B) \subset F = B^V$, $V = V_{D,A}$. Hence $\dim(C_0) > 0$ and $C_0$ is locally con-
nected. In view of Theorem 3.1.9 [9] $D, A, B, C_1, C_0$ are $T_1 \cap T_4$ topological spaces. By the construction above $\text{card}(V) \geq \aleph_0$.

On the other hand, a family $\text{Hom}_{c,C} = \text{Hom}_{c,C}((A \times F_0) \times (A \times F_0), C_1)$ of all continuous homomorphisms from $(A \times F_0) \times (A \times F_0)$ into $C_1$ satisfying (35) in [16] is a proper closed subset in a family $C_C = C_C((A \times F_0) \times (A \times F_0), C_1)$ of all continuous maps from $(A \times F_0) \times (A \times F_0)$ into $C_1$ satisfying (35) in [16]. Since $\dim(B) > 0$ and $\dim(V) > 0$, then $\dim(C(V,B)) = \infty$, where $C(V,B)$ is in the $T_W$ topology.

We choose $F_0$ with $\dim(F_0) = \infty$ and the map $\xi \in C - \text{Hom}_{c,C}$ with values in $C_1$ such that $\xi((d_1^\psi, f_1), (d^\psi, f))(v)$ depends nontrivially on infinite number of coordinates $v \in V$ for an infinite family of $(f_1, f) \in F_0 \times F_0$ for each $d_1^\psi \neq e$ and $d^\psi \neq e$, where $d$ and $d_1$ belong to $D, f \in F_0, f_1 \in F_0$, since $\dim(C_1) > 0$ and $\dim(A_2) > 0$. This implies that there exists the topological metagroup $G = C_0$ with $\dim(G) = \infty$ which can not be decomposed into the inverse continuous homomorphism system $S_G = \{G_j, \pi_i^j, \Lambda\}$ of topological metagroups $G_j$ with $\dim(G_j) < \infty$ for each $j \in \Lambda$. From the proof above it follows that the family of such topological metagroups is infinite.

**Remark 3.26.** If instead of Theorem 6 use Theorem 5 in [16], then Theorem 3.25 will be for topological loops $G$.

**3.27. Conclusion.** The obtained results in this paper can be used for subsequent investigations of topological metagroups, quasigroups, loops, topological algebras, generalized $C^*$-algebras, operator algebras over octonions and the Cayley-Dickson algebras and noncommutative geometry associated with them [6]. Certainly, metagroups can be realized using deterministic functions, which are important in algorithm theory and informatics. Besides these applications of metagroups and those outlined in the introduction, it is interesting to indicate possible applications in mathematical coding theory and related reliability of software and hardware systems [4, 31, 34], because frequently codes are based on topological binary systems.

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