LONG-TIME BEHAVIOR OF SOLUTIONS OF THE GENERALIZED KORTEWEG–DE VRIES EQUATION

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Abstract. In this paper, we study the large-time behavior of solutions to the initial-value problem for the generalized Korteweg–de Vries equation. We show that for initial data in some weighted space, the asymptotic behavior of the solution can be improved. In addition, we give the asymptotic profile of the fundamental solution of the linearized model. We extend and improve the results in [3] and [2].

1. Introduction. The Korteweg–de Vries equation and its generalized form describes various physical systems such as classical water waves, lattice waves and plasma waves. See the review paper [11] and references therein. The simple form of the the Korteweg–de Vries equation

\[ u_t + u_{xxx} + uu_x = 0 \]  

(1)

is well known to possess single and multiple soliton solutions. Two-soliton solution behaves asymptotically for large (positive and negative) time like two independent solitons of different speeds. See Zabusky–Kruskal [22], Lax [14], Bowtell–Stuart [4] and Hodnett–Moloney [9].

When we take into account the dissipation effect, the KdV equations (1) takes the form

\[ u_t + u_{xxx} + uu_x - u_{xx} = 0. \]  

(2)

Equation (2) represents the interaction of nonlinearity, dispersion and dissipation phenomena and arises in modeling unidirectional propagation of planar waves. Here \( u = u(x,t) \) is a real-valued function of two real variables \( x \) and \( t \) which may represents a displacement of the underlying medium or a velocity.

Global existence and smoothing effect for the Cauchy problem associated to (2) was proved in [18]. More precisely, it has been shown that there exists a unique solution \( u(x,t) \in C_{\text{loc}}((0,\infty); H^s(\mathbb{R})) \) to the Cauchy problem (2) with initial data \( u_0 \in H^s(\mathbb{R}) \) for \( s > -1/2 \). Concerning the asymptotic behavior of the solution of (2), it has been proved in [17] that for small initial data \( u_0 = u(x,0) \in L^1(\mathbb{R}) \cap H^7(\mathbb{R}) \) with \( xu_0(x) \in L^1(\mathbb{R}) \), then the solution of (2) has the following asymptotic behavior

\[ u(t) = t^{-1/2}f_A((.)t^{-1/2}) + O(t^{-1/2-\beta}), \]  

(3)

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where $\beta \in (0, 1/2)$, $A = \int_R u_0(x)dx$ and

$$f_A(x) = -2\partial_x \log \left( \frac{\cosh(A/4) - \sinh(A/4)}{\sqrt{\pi}} \int_0^{x/2} e^{-y^2} dy \right).$$

Using the Hopf–Cole transformation, Amick et al. [1] showed that for $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$, then the solution of (2) satisfies the decay estimate

$$\|u(t)\|_{L^2(\mathbb{R})} \leq C(1 + t)^{-1/4}, \quad \|u(t)\|_{L^\infty(\mathbb{R})} \leq C(1 + t)^{-1/2}. \quad (4)$$

For $A \neq 0$, the estimates in (4) seem optimal, since an explicit limit (depending on $u_0$) to the term $t^{1/2}\|u(t)\|_{L^2(\mathbb{R})}$ has been established in [1, Theorem 5.5].

Applying the standard scaling argument, Karch [12] showed that for $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and for $q \in [1, \infty]$, then

$$t^{(1-1/p)/2}\|u(\cdot, t) - U_A(\cdot, t)\|_{L^q(\mathbb{R})} \to 0, \quad as \quad t \to \infty, \quad (5)$$

where

$$U_A(x, t) = t^{-1/2}U_A(xt^{-1/2}, 1) \quad (6)$$

with

$$U_A(\eta, 1) = \frac{e^{-\eta^2/4}}{K + \frac{1}{2} \int_0^\infty e^{-z\zeta^2/4}d\zeta}. \quad \text{Here the constant} \quad K \quad \text{is uniquely determined as a function of} \quad A \quad \text{using the condition} \quad \int_\mathbb{R} U_A(\eta, 1)d\eta = A. \quad \text{The reader is also referred to the paper} \quad [8] \quad \text{where the asymptotic behavior} \quad (3) \quad \text{has been shown for less regularity on the initial data.}$$

If we consider the damping term $u$ instead of $-u_{xx}$ in (2), see [13] and [6], then we can easily show the following exponential decay rate

$$\|u(t)\|_{L^2(\mathbb{R})} \leq Ce^{-t}, \quad (7)$$

where $C$ is a constant depending on $u_0$. A localized dissipative term of the form $a(x)u$ instead of $-u_{xx}$ has been also considered in [15] and [5].

The main goal of this paper is the study of asymptotic behavior as $t$ goes to infinity of solutions to the Cauchy problem

$$\begin{cases}
  u_t + uu_{xx} + u_{xx} + Mu = 0, \\
  u(x, 0) = u_0(x)
\end{cases} \quad x \in \mathbb{R}, \quad t \geq 0 \quad (8)$$

where $p$ is a positive integer and $M$ is the dissipative operator defined by

$$Mu(\xi) = |\xi|^{2\nu} \hat{a}(\xi), \quad 0 < \nu \leq 1. \quad (9)$$

For the linearized equation, we show that under the assumption $A = 0$, the following decay result holds: (see Theorem 2.1 below)

$$\|D^k u(t)\|_{L^2(\mathbb{R})}^2 \leq C(1 + t)^{-\frac{1}{4}} \left( \|u_0\|^2_{L^1, \gamma(\mathbb{R})} + \|\partial_x^k u_0\|^2_{L^2} \right), \quad (10)$$

for any $k \geq 0$, $\gamma \in [0, 1]$ and $D^k$ is the operator of the form (9) with $k = 2\nu$. The decay estimate (10) improves the result in [2] and [21]. In addition, we give the asymptotic profile of the fundamental solution of the linearized model. See Theorem 2.2 below. For the nonlinear model (8), we show in Theorem 3.1 that for $0 < \nu < 3/4$, the same estimate (10) is satisfied.

This paper is organized as follows: In Section 2, we treat the linearized model, while Section 3 is devoted to the nonlinear model.

Before closing this section, we introduce some notations and some technical lemmas to be used in the sequel. Throughout this paper, $\| \cdot \|_q$ and $\| \cdot \|_{H^s}$ stand for the
\[ L^q(\mathbb{R}) \]-norm \((1 \leq q \leq \infty)\) and the \(H^s(\mathbb{R})\)-norm. Also, for \(\gamma \in [0, +\infty)\), we define the weighted function space \(L^{p, \gamma}(\mathbb{R})\), \(1 \leq p < \infty\), as follows: \(u \in L^{p, \gamma}(\mathbb{R})\) iff
\[
\|u\|_{p, \gamma} = \int_{\mathbb{R}} (1 + |x|^\gamma)|u(x)|^pdx < +\infty.
\]

The following lemma is useful in the proof of our main result and can be found, for example in [16, 20], cp. also Lemma 7.4 in [19].

**Lemma 1.1.** Let \(a > 0\) and \(b > 0\) be constants. If \(\max(a, b) > 1\), then
\[
\int_0^t (1 + t - s)^{-a} (1 + s)^{-b}ds \leq C (1 + t)^{-\min(a, b)}.
\]

The local existence result of (8) has been proved in [2]. We recall their result here.

**Theorem 1.2.** Let \(0 < \nu \leq 1\) and \(p = 1, 2\) or 3 be given. If the initial data satisfies \(u_0 \in H^s(\mathbb{R})\) for \(s \geq 2\), then there is a unique solution \(u\) of (8) satisfying
\[
u \in L^\infty([0, \infty); \mathcal{C}^s(\mathbb{R})) \quad \text{and} \quad D^\nu u \in L^2([0, \infty); \mathcal{C}^s(\mathbb{R})).
\]

2. **The linearized equation.** In this section, we prove the decay rate and the asymptotic expansions for the linearized initial-value problem
\[
\begin{aligned}
&u_t + u_{xxx} + Mu = 0, \\
&u(x, 0) = u_0(x),
\end{aligned}
\]
\((11)\)

The large time asymptotic behavior of (11) is given below.

**Theorem 2.1.** Let \(\gamma \in [0, 1]\) and \(s \geq 2\). Assume that \(u_0 \in H^s(\mathbb{R}) \cap L^{1, \gamma}(\mathbb{R})\) such that \(\int_\mathbb{R} u_0(x)dx = 0\), then the solution of (11) satisfies
\[
\left\|D^k u\right\|_{L^2(\mathbb{R})} \leq C (1 + t)^{-\frac{k}{4s}} \left(\|u_0\|_{L^{1, \gamma}(\mathbb{R})} + \|\partial_x^k u_0\|_{L^2}\right),
\]
\((12)\)

for any \(0 \leq k \leq s\).

**Proof.** Taking the Fourier transform of (11), we get
\[
\begin{aligned}
&\hat{u}_t (\xi, t) + (i\xi^3 + |\xi|^{2\nu}) \hat{u} (\xi, t) = 0, \\
&\hat{u} (\xi, 0) = \hat{u}_0 (\xi).
\end{aligned}
\]
\((13)\)

Solving (13), we obtain
\[
\hat{u} (\xi, t) = \exp \left[-\left(i\xi^3 + |\xi|^{2\nu}\right)t\right] \hat{u}_0 (\xi).
\]
\((14)\)

Applying Plancherel’s theorem, we find
\[
\left\|D^k u\right\|_{L^2(\mathbb{R})}^2 = \int_\mathbb{R} |\xi|^{2k} \exp \left[-2|\xi|^{2\nu}t\right] |\hat{u}_0 (\xi)|^2 d\xi.
\]
\[
= \int_{|\xi| \leq 1} |\xi|^{2k} \exp \left[-2|\xi|^{2\nu}t\right] |\hat{u}_0 (\xi)|^2 d\xi
\]
\[
+ \int_{|\xi| \geq 1} |\xi|^{2k} \exp \left[-2|\xi|^{2\nu}t\right] |\hat{u}_0 (\xi)|^2 d\xi
\]
\[
= L_1 + L_2.
\]
\((15)\)

Now, using [10], and for \(\int_{\mathbb{R}^N} u_0(x)dx = 0\), \(0 \leq \gamma \leq 1\), we have
\[
|\hat{u}_0 (\xi)| \leq C_\gamma |\xi|^{\gamma} \|u_0\|_{L^{1, \gamma}(\mathbb{R})}.
\]
\((16)\)
Consequently, we get
\[ L_1 \leq C_{\gamma} \| u_0 \|_{L^1,\gamma}(\mathbb{R})^2 \int_{|\xi| \leq 1} |\xi|^{2k+2\gamma+2\nu} \exp \left[ -2 |\xi|^{2\nu} t \right] d\xi, \] where we have used the inequality
\[ \int_{|\xi| \leq 1} |\xi|^\sigma e^{-ct|\xi|^{2\nu}} d\xi \leq C(1+t)^{-(\sigma+1)/(2\nu)}. \] On the other hand, we have
\[ L_2 = \int_{|\xi| \geq 1} |\xi|^{2k} \exp \left[ -2 |\xi|^{2\nu} t \right] |\hat{u}_0(\xi)|^2 d\xi \leq e^{-2\nu t} \int_{|\xi| \geq 1} |\xi|^{2k} |\hat{u}_0(\xi)|^2 d\xi \leq e^{2\nu t} \| \partial_x^k u_0 \|^2_{L^2}, \] where we have used Plancherel’s theorem. Collecting (17) and (19), we deduce that (12) holds true.

**Remark 1.** The decay rate in (12) extends the one given in [2], in the sense that the assumption \( \int_\mathbb{R} u_0(x) dx = 0 \) leads to a faster decay rate.

In the next theorem, we show the self-similar asymptotic form of equation (11). For this and following [2], we define \( G_{\nu}(x,t) \) the fundamental solution of the equation
\[
\left\{ \begin{array}{l}
u u + Mu = 0, \\
u (x,0) = u_0(x),
\end{array} \right. \quad x \in \mathbb{R}, \ t \geq 0,
\] with \( u_0(x) = \delta_0(x) \), and is given by
\[ \hat{G}_{\nu}(\xi,t) = e^{-|\xi|^{2\nu} t}. \] Thus, we have the following result which generalizes [2, Proposition 4.1].

**Theorem 2.2.** Let \( \gamma \in [0,1], \ 0 < \nu \leq 1 \) and \( s \geq 2 \) be an integer. Assume that \( u_0 \in H^s(\mathbb{R}) \cap L^1,\gamma(\mathbb{R}) \). Then, the solution \( u \) of (11) satisfies,
\[ \left\| D^k \left\{ u - AG_{\nu}(x,t) \right\} \right\|_{L^2} \leq C(1+t)^{-\frac{1+2s+2k}{4}}, \quad \text{for} \quad k = 0, \ldots, s, \] where \( A = \int_\mathbb{R} u_0(x) dx. \)

**Proof.** From the proof of lemma 3.1 in [10], we deduce that for \( \gamma \in [0,1] \),
\[ |\hat{u}_0(\xi) - \hat{u}_0(0)| = \left| \hat{u}_0(x) - \left( \int_\mathbb{R} u_0(x) dx \right) \right| \leq C_{\gamma} |\xi|^{\gamma} \| u_0 \|_{L^1,\gamma}. \] On the other hand the Fourier transform for the solution \( u \) of (11) is given by (14) and satisfies (see [2])
\[ |\hat{u}(\xi,t) - \left( \int_\mathbb{R} u_0(x) dx \right) e^{-|\xi|^{2\nu} t}| \leq \left( A \left| 1 - e^{-|\xi|^{2\nu} t} \right| + C_{\gamma} |\xi|^{\gamma} \| u_0 \|_{L^1,\gamma} \right) e^{-|\xi|^{2\nu} t} \leq C_{\gamma} \| u_0 \|_{L^1,\gamma} \left( |\xi|^{3} t + |\xi|^\gamma \right) e^{-|\xi|^{2\nu} t}. \]
Thus, the proof of Theorem 2.2 is completed.

\[ \int_{\mathbb{R}} |\xi|^{2k} \left| \hat{u} (\xi, t) - \left( \int_{\mathbb{R}} u_0 (x) \, dx \right) \hat{G}_u (x, t) \right|^2 \, d\xi \]
\[ = \int_{\mathbb{R}} |\xi|^{2k} \left( \left( \int_{\mathbb{R}} u_0 (x) \, dx \right) e^{-2|\xi|^{2t}} \right)^2 \, d\xi \]
\[ \leq C_\gamma \| u_0 \|^2_{L^{1, \gamma} (\mathbb{R})} \int_{\mathbb{R}} |\xi|^{2k} \left( |\xi|^{6t + |\xi|^{2\gamma}} \right) e^{-2|\xi|^{2t}} \, d\xi \]
\[ \leq C_\gamma \| u_0 \|^2_{L^{1, \gamma} (\mathbb{R})} (1 + t)^{-\frac{4+2\gamma+2k}{2\nu}}. \]

Thus, the proof of Theorem 2.2 is completed.

**Remark 2.** The result in Theorem 2.2 has been proved in [2] for the case \( \gamma = 1 \). Here we have extended it to the case \( \gamma \in [0, 1] \).

3. **The nonlinear model.** The main goal of this section is to consider the nonlinear problem (8) and extend the decay estimate given for the linearized problem in Theorem 2.1 to the nonlinear problem (8). Our main result reads as follows:

**Theorem 3.1.** Let \( \gamma \in [0, 1] \), \( p = 1, 2 \) or 3 and \( 0 < \nu < 3/4 \). Assume that \( u_0 \in H^s (\mathbb{R}) \cap L^{1, \gamma} (\mathbb{R}) \) for \( s \geq 2 \) and \( \int_{\mathbb{R}} u_0 (x) \, dx = 0 \). Assume further that there exists a positive constant \( \delta \) such that if \( \| u_0 \|_{H^s} + \| u_0 \|_{L^{1, \gamma}} \leq \delta \), then the solution of the Cauchy problem (8) satisfies the estimate

\[ \| D^k u (t) \|^2_{L^2 (\mathbb{R})} \leq C (1 + t)^{-\frac{2\nu + 2k + p + \gamma \nu}{2}}. \]

(21)

with \( k = 0, \ldots, \min (s, [p + \gamma (p + 1) - 3/2]) \).

The proof of Theorem 3.1 is a direct consequence of the result in Proposition 1. See [7, Theorem 1] for more details.

Now, applying the Fourier transform to (8), we get

\[
\left\{ \begin{array}{l}
\hat{u}_t (\xi, t) + (i \xi^3 + |\xi|^{2\nu}) \hat{u} (\xi, t) = F (\xi, s), \\
\hat{u} (\xi, 0) = \hat{u}_0 (\xi),
\end{array} \right.
\]

(22)

where \( F (\xi) = \hat{w} \hat{u}_x \). The solution of (22) is given by

\[
\hat{u} (\xi, t) = e^{-A (\xi) t} \hat{u}_0 (\xi) + \int_0^t e^{-A (\xi) (t-s)} F (\xi, s) \, ds,
\]

(23)

where \( A (\xi) = \left( |\xi|^{2\nu} + i \xi^3 \right) \). Using the inverse Fourier transform, we get,

\[
u (x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi \cdot x} e^{-A (\xi) t} \hat{u}_0 (\xi) \, d\xi + \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}} e^{i\xi \cdot x} e^{-A (\xi) (t-s)} F (\xi, s) \, ds.
\]

(24)

For any \( k \in \mathbb{N} \), we have

\[
\partial_x^k u (x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} (i \xi)^k e^{i\xi \cdot x} e^{-A (\xi) t} \hat{u}_0 (\xi) \, d\xi
\]
\[+ \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}} (i \xi)^k e^{i\xi \cdot x} e^{-A (\xi) (t-s)} F (\xi, s) \, ds.
\]

(25)
Now, we define for $\gamma \in [0, 1]$ and a fixed $T > 0$,

$$ M(T) := \sup_{0 \leq t \leq T} \sum_{k=0}^{\min(s, [p+\gamma(p+1)-\frac{4}{\nu}] + 1)} (1 + t)^{\frac{k}{p} + \frac{k+1}{2v}} \|D^k u(t)\|_{L^2(\mathbb{R})}. $$

(26)

Thus, we have the following lemma which will be used later.

**Lemma 3.2.** Let $\gamma \in [0, 1]$. Then the following estimate holds:

$$ \sup_{\xi \in \mathbb{R}} F(\xi) \leq C|\xi| M(T)^{p+1} (1 + t)^{-\frac{p}{2v} - \frac{2(p+1)}{2v}}. $$

(27)

**Proof.** First, it is clear that

$$ |F(\xi)| = \left|\frac{w^p u_x}{(p+1)}\right| = \left|\frac{\partial_x (w^{p+1}/(p+1))}{x}\right| \leq \frac{1}{p+1} |\xi| u^{p+1}. $$

On the other hand, we have by using the inequality

$$ \|u\|_{L^\infty} \leq \sqrt{2} \|u\|_{L^2} \|u_x\|_{L^2}, $$

the following estimate:

$$ \int_{\mathbb{R}^N} |u|^{p+1} \, dx \leq \|u\|_{L^\infty}^{p-1} \|u\|_{L^2}^2 \\
\leq C \|u\|_{L^2}^{(p-1)/2} \|u_x\|_{L^2}^{(p-1)/2} \|u\|_{L^2}^2 \\
\leq CM(T)^{p+1} (1 + t)^{-\frac{1}{2} - \frac{4}{2v} - \frac{1}{4}} \times (1 + t)^{-\frac{1}{2} - \frac{4}{2v} - \frac{1}{4}}. \\
= CM(T)^{p+1} (1 + t)^{-\frac{p+1}{2v} - \frac{2(p+1)}{2v}}. $$

This completes the proof of Lemma 3.2.

**Lemma 3.3.** Let $\gamma \in [0, 1]$ and $0 < \nu \leq 1$. Assume that $u_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ for $s \geq 2$ and $\int_{\mathbb{R}} u_0(x) \, dx = 0$, then we have

$$ \left\| \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi)^k e^{i\xi.x} e^{-A(\xi)(t-s)} F(\xi) \, d\xi \right\|_{L^2} \leq CM(T)^{p+1} (1 + t)^{-\frac{p}{2v} - \frac{2(p+1)}{2v}} (1 + t - s)^{-\frac{k+1}{2v} - \frac{1}{2v}}. $$

(28)

for $k = 0, 1, \ldots, s$.

**Proof.** Using Parseval’s equality and (27), we have

$$ \left\| \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi)^k e^{i\xi.x} e^{-A(\xi)(t-s)} F(\xi) \, d\xi \right\|_{L^2} \\
= \left( \int_{\mathbb{R}} |\xi|^{2k} e^{-2(|\xi|^2 + i\xi)(t-s)} \|F(\xi)\|^2 \, d\xi \right)^{1/2} \\
\leq \left( \int_{\mathbb{R}} |\xi|^{2k} e^{-2(|\xi|^2(t-s))} \|F(\xi)\|^2 \, d\xi \right)^{1/2} \\
\leq CM(T)^{p+1} (1 + t)^{-\frac{p}{2v} - \frac{2(p+1)}{2v}} \left( \int_{\mathbb{R}} |\xi|^{2k+2} e^{-2|\xi|^2(t-s)} \, d\xi \right)^{1/2} \\
\leq CM(T)^{p+1} (1 + t)^{-\frac{p}{2v} - \frac{2(p+1)}{2v}} (1 + t - s)^{-\frac{k+1}{2v} - \frac{1}{2v}}.$$

This completes the proof of Lemma 3.3.
Lemma 1.1, and since for $\gamma \in (0, \frac{3}{4})$, Proposition 1. Thus, multiplying (32) by $(1 + t)^{\frac{1}{2\nu} + \frac{k+\gamma}{2\nu}}$ and summing up for $0 \leq j \leq \min\left(\left\lceil \frac{p+\gamma(p+1)-\frac{3}{2}}{2} \right\rceil\right)$, we arrive at

$$M(T) \leq C \left(\|u_0\|_{L^{1,\gamma}(\mathbb{R})} + M(T)^{p+1}\right).$$

The remaining part of the proof follows exactly as in [7, Proposition 2]. We omit the details.

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