Improved hydrodynamic pulsation models for the pulsating extreme helium star V652 Herculis

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ABSTRACT

New non-linear hydrodynamic models have been constructed to simulate the radial pulsations observed in the extreme helium star V652 Her. These use a finer zoning to allow higher radial resolution than in previous simulations. Models incorporate updated OPAL and OP opacity tables and adopt a composition based on the best atmospheric analyses to date. Key pulsation properties including period, velocity amplitude and shock acceleration are examined as a function of the mean stellar parameters (mass, luminosity, and effective temperature). The new models confirm that, for large amplitude pulsations, a strong shock develops at minimum radius, and is associated with a large phase delay between maximum brightness and minimum radius. Using the observed pulsation period to constrain parameter space in one dimension, other pulsation properties are used to constrain the model space further, and to critically discuss observational measurements. Similar models may be useful for the interpretation of other blue large amplitude pulsators, which may also exhibit pulsation-driven shocks.

Key words: stars: chemically peculiar, stars: individual (V652 Her), stars: oscillations, shock waves

1 INTRODUCTION

V652 Her is a unique object for several reasons: a) it is an extreme helium star, belonging to a group of approximately 15 low-mass giants with effective temperatures in the range 10 000 to 30 000 K and very low surface hydrogen abundances (Berger & Greenstein 1963), b) it shows large-amplitude radial pulsations which have allowed direct measurement of its radius and the possibility to explore the physical properties of its interior (Landolt 1975; Hill et al. 1981; Lynas-Gray et al. 1984; Jeffery et al. 2001), c) lying outside the classical instability strip, the pulsations could only be explained by the introduction of additional opacity from iron-group elements (Saio 1993), d) the pulsation period is decreasing, which implies a secular radius contraction (Kilkenny & Lynas-Gray 1984), and e) it has a nitrogen-rich, carbon-poor surface which cannot be easily explained by a single-star evolution model (Jeffery et al. 1999; Przybylla et al. 2005). A comprehensive review of these properties was most recently given by Jeffery et al. (2015); the most precise measurements to date give a mean radius \( \langle R \rangle = 2.31 \pm 0.02 \) R\(_\odot\), effective temperature \( \langle T_\text{eff} \rangle = 20\,950 \pm 70 \) K, luminosity \( \langle L \rangle = 919 \pm 14 \) L\(_\odot\) and mass \( M = 0.59 \pm 0.18 \) M\(_\odot\) (Jeffery et al. 2001). A summary is shown in Table 1.

Such a combination of features makes this peculiar object extremely interesting and presents challenges for theories of stellar evolution and pulsation. Models of the post-merger evolution of two helium white dwarfs match almost all of its observational properties (Saio & Jeffery 2000; Zhang & Jeffery 2012). Other models, such as the evolution of a post-giant branch star following a very-late core helium flash (Brown et al. 2001; Byrne et al. 2018) cannot be ruled out.

Hydrodynamic models of the pulsations in V652 Her have been calculated by Fadeyev & Lynas-Gray (1996) and Montañés Rodríguez & Jeffery (2002). These provided the first such calculations using opacities enhanced by contributions from iron-group elements, using OP and OPAL opacities respectively (Seaton et al. 1994; Iglesias & Rogers 1996); Montañés Rodríguez & Jeffery (2002) also provided models for the similar pulsating helium star BX Cir.

Meanwhile an extensive new spectroscopic study of the pulsation has defined the radial velocity curve very precisely, mapping the motion of the mean surface, and providing a depth-dependent description of motion as a function of optical depth within the photosphere (Jeffery et al. 2015). These observations indicate that the photosphere is compressed by a factor \( \approx 2 \) at minimum radius, and show that the phase of minimum radius is a function of optical depth, travelling through the photosphere at a speed between 140 and 240 km s\(^{-1}\). The acceleration at minimum radius is largest very deep in the photosphere, where individual line profiles imply jump discontinuities of over 70 km s\(^{-1}\) in 150 s and provide evidence of a pulsation-driven shock. The fundamental observable remains the amplitude of the surface velocity. This depends marginally on which point of the photosphere is being measured, and also on the transformation of this quantity from the observers frame to the stellar rest frame. With an observed amplitude of 68 km s\(^{-1}\) and a projection factor \( p \approx 1.31 \) (Montañés Rodríguez & Jeffery 2001), the
Table 1. Published surface, pulsation and other properties of V652 Her. Notes:
1: Kilkenny et al. (2005), 2: Jeffery et al. (2001), 3: Przybilla et al. (2005), 4: Jeffery et al. (1999), 5: Jeffery et al. (2015).

| Property       | Value                                | Units | Notes |
|----------------|--------------------------------------|-------|-------|
| $P_0$ (1974)   | $0.107,992,82 \pm 0.000,000,002$     | d     | 1     |
| $\Pi/\Pi$     | $-7.426 \pm 0.005 \times 10^{-8}$   | d d^{-1}| 1     |
| $(T_{\text{eff}})$ (ion equil) | $22.930 \pm 10$ | K | 2     |
| $(\log g)$ (He I lines) | $3.46 \pm 0.05$ | cm s^{-2} | 2     |
| $(T_{\text{eff}})$ (total flux) | $20.950 \pm 70$ | K | 2     |
| $(R)$          | $2.31 \pm 0.02$ | R_0 | 2     |
| $(L)$          | $919 \pm 0.14$ | L_0 | 2     |
| $M$            | $0.59 \pm 0.18$ | M_0 | 2     |
| $d$            | $1.70 \pm 0.02$ | kpc | 2     |
| $T_{\text{eff}}$ (NLTE) | $22.000 \pm 500$ | K | 3     |
| $\log g$ (NLTE) | $3.20 \pm 0.10$ | cm s^{-2} | 3     |
| $n_0$ (NLTE)  | $0.005 \pm 0.0005$ | 3     |
| $T_{\text{eff}}$ (mean) | $24.550 \pm 500$ | K | 4     |
| $\log g$ (mean) | $3.68 \pm 0.10$ | cm s^{-2} | 4     |
| $\log n_{\text{He}}$ | $-2.16 \pm 0.07$ | 4     |
| $\log n_{\text{C}}$ | $-4.40 \pm 0.27$ | 4     |
| $\log n_{\text{N}}$ | $-2.61 \pm 0.06$ | 4     |
| $\log n_{\text{O}}$ | $-4.00 \pm 0.08$ | 4     |
| $\log n_{\text{Fe}}$ | $-4.14 \pm 0.15$ | 4     |
| $\Delta \nu$   | $68$ km s^{-1} | 5     |
| $\Delta u = p \Delta \nu$ | $89$ km s^{-1} | 5     |
| $\Delta \phi$  | $+0.13 \pm 0.01$ | cycles | 5     |
| $T_{\text{eff}}$ (R_{max}) | $22.500$ | K | 5     |
| $\log g$ (R_{max}) | $3.3$ | cm s^{-2} | 5     |

†: projection factor $p = 1.31$; Montañés Rodriguez & Jeffery (2001)
*: adopted model
‡: light maximum defined by optical light curve: Kilkenny et al. (2005)

un-projected amplitude obtained from the strong Nii4621Å line is $\Delta \nu = 89$ km s^{-1}.

In order to obtain higher resolution in the hydrodynamic calculations and to address the questions raised by the latest observations, we set out to compute a new generation of hydrodynamic radial pulsation models for V652 Her. The aim of these is to reproduce the observed period and amplitude, and also the detailed features of the velocity curve. In doing so, we investigate how much the fundamental properties of the star are constrained by the pulsation models, and by how much these are at variance with measurements of these properties obtained by other means. We investigate if and how the pulsation properties vary with period, since the latter has decreased by $\approx 10$% since discovery. The models are also required in order to construct dynamical model atmospheres necessary for the accurate interpretation of the observed spectrum and hence to deduce more reliably the mean atmospheric properties.

The computation assumptions, code and parameter space are described in § 2. Results for pulsation periods, velocity amplitudes and curve shapes are presented in § 3. Internal properties of the pulsation are explored and compared with the observed properties of V652 Her in § 4. § 5 concludes.

2 HYDRODYNAMIC CALCULATIONS

The models were computed with the Montañés Rodríguez & Jeffery (2002) non-linear radial pulsation code, which is based on a prescription by Christy (1967) and an implementation by Bridger (1983).

![Figure 1.](image-url) The position $r$ of mass zones in the model interior throughout a pulsation cycle. This figure shows model 963666N1, having $M = 0.66 M_\odot$, log $L/L_\odot = 2.96$, log $T_{\text{eff}}/K = 4.36$ and mixture N1 (see text). The mass zones are colour-coded for temperature. The base of the photosphere, defined by Rosseland mean optical depth $\tau_\text{Ross} = 2/3$, is shown by a bold dark line. The position of every 20th zone is marked by a thin black line. Fig. 2 shows parts of this model in more detail.

The Lagrangian coordinate is used, in which the mass within each zone $\delta m_{i-1/2} = m_i - m_{i-1}$. We define $i$ to increase outward and $\delta m_{i-1/2} = a \delta m_{i+1/2}$ with a constant and $\leq 1.1$ to yield a logarithmic spacing in mass (cf. Christy 1967).

The stellar core is not computed. Instead, the mean energy flux from the core is assumed constant, with no nuclear energy released in the envelope. With this approximation, the radial pulsation is described through the momentum equation and the heat flow equation. Radiation diffusion is regarded as the only energy transport mechanism. Convective energy transport is assumed to be negligible, which is reasonable for the envelopes of stars with effective temperatures greater than 20 000 K. This provides four first-order differential equations in radius $r$, energy flux $l$, density $\rho$ and temperature $T$ in terms of mass $m$ and time $t$.

Pressure $P$, electron density $n_e$, internal energy $E$ and their first derivatives are computed as functions of temperature and density using an equation of state (EOS) (Bridger 1983) based on the formalism given by (Mihalas 1978, chapter 5). Opacity $\kappa$ and its derivatives are interpolated as a function of temperature and density from the OPAL95 opacity tables (Iglesias & Rogers 1996).

The Bridger EOS solves the Saha ionization equation for 4 species, hydrogen, helium, nitrogen and magnesium, with the latter representing 80% and 20% by mass of the metals (Z), respectively. It treats ions up to the 6th state (5+) and is therefore valid up to log $T \approx 5.3.$
Above this temperature it starts to underestimate the ionisation fraction, leading to a maximum error of 0.3% in the mean mass per particle when full ionization is achieved (at \( \approx 10^7 \) K), and hence in quantities which depend on this value. More sophisticated EOS required for opacity calculations (e.g. Hummer & Mihalas 1988, et seq.) have minor influence on the values of major state variables in hydrogen or helium-rich mixtures except at electron-degenerate densities (Rogers et al. 1996). Molecules do not feature in V652 Her.

Four boundary conditions are required. At the surface, we require i) the gas pressure to be zero and ii) the usual relation between radius, luminosity and effective temperature. At the inner boundary, defined where the radius \( R/10 < r < R/5 \) or the temperature \( T = 10^8 \) K, the dynamical model should be stable and therefore iii) \( \rho_1 = 0 \), \( r_1 = 0 \), \( t_1 = 0 \) and \( T_1 = 0 \) at the inner boundary.

In the non-linear treatment, the complete system of differential equations can be written in a finite-difference form and can be solved as an initial value problem. As initial value, we compute a finely-zoned envelope in hydrostatic equilibrium; this is subsequently rezoned onto a Lagrangian grid with \( n \) mass zones. A small velocity perturbation

\[
u_1^4 = 0.1 \left( \frac{r_1}{r_n} \right)^4 \text{ km s}^{-1}
\]

is applied to the initial envelope model, where \( r_1 \) and \( r_n \) are the initial radii of the \( i \)th and \( n \)th zones, respectively. Then we follow the growth of the perturbation until a stable pulsation period is obtained.

A limit cycle is deemed to have been reached when strict periodicity is achieved. We follow the subsequent temporal evolution step by step until pulsations with nearly constant amplitude and period are achieved (successive cycles differ by less than 0.5%). This is considered to be the solution for the initial value problem, and constitutes a hydrodynamic description of the pulsation through the stellar envelope (Fig. 1). In the current study, between 4000 and 10,000 pulsation cycles for stellar envelopes with 200–250 mass zones were computed to reach this final level of convergence, in contrast to the 200–500 cycles for envelopes with 40–50 mass zones computed in the previous models (Montañés Rodríguez & Jeffery 2002).

An artificial viscosity dissipation following Stellingwerf (1975) was used to deal with the shocks produced in outer layers during a supersonic contraction and bounce. This is especially important around minimum radius, as shown in Fig. 2. The artificial viscosity, which is proportional to the square of compression rate, is included where matter is compressed to prevent sharp discontinuities appearing in the physical variables between adjacent zones. The latter can lead to numerical singularities causing the calculation to fail. Stellingwerf (1975) introduced a cut-off compression rate, below which the artificial viscosity is set to be zero. This cut-off prevents nonphysical dissipation in slowly contracting layers. We adopted default values \( C_Q = 2.0 \) and \( \alpha_v = 0.1 \) for the constant of proportionality and the cut-off compression rate, respectively. The adopted \( C_Q \) is relatively low (Grott et al. 2005), but convergence was achieved for nearly all models of interest.

### 2.1 Model grid

We have set out to identify a model which most closely approximates the observed properties of V652 Her as outlined in Table 1. The best constrained observable is the pulsation period \( P \) which is correlated with the square root of the mean density \( \langle \rho \rangle \propto \sqrt{(M/R^3)} \). Although the radius has been measured directly from observation, a larger error in the mass arises because the surface gravity \( g \propto M/R^2 \) is measured spectrosopically, with a fitting error of \( \pm 0.10 \) dex, and a less well determined systematic error.

With these constraints, we have chosen combinations of \( T_{\text{eff}} \) and \( L \) which yield approximately the correct values of \( R \) and \( \Pi \), but which also lie on a rectangular grid, i.e. \( \log T_{\text{eff}}/K \in \{4.30, 4.32, 4.34, 4.36, 4.38 \} \) and \( \log L/L_\odot \in \{2.71, 0.05, 3.06, 2.96, 2.86, 2.76, 2.71 \} \). Those combinations close to the leading diagonal and having \( \Pi = 0.11 \pm 0.01 \) d were investigated in greatest detail. Some outlying models were computed in order to explore the behaviour of amplitude towards the boundaries of the instability zone (Fig. 3). The grid boundaries together with the instability boundary obtained from a linear non-adiabatic calculation (Jeffery & Saio 2016). The pulsation topology of the \( L/M - T_{\text{eff}} \) plane is relatively insensitive to \( M \) (Jeffery & Saio 2016).

In addition to these, the chemical composition of the stellar envelope (§ 2.2) represents a free parameter which can be partially constrained by observation. This guides the choice of opacity (§ 2.3). Other choices include the mass of the stellar envelope \( M_{\text{env}} \) included in the calculation. The latter is a computational and not a physical parameter.

For convenience and without ambiguity, each model is given a label in the form \( \text{ilmnmNn} \) where \( ill = \text{frac}(\log L/L_\odot) \), \( mm = \text{frac}(\log T_{\text{eff}}/K) \), \( nn = \text{letter and integer combination identifying the chemical composition, the choice of opacity and other properties of the calculation. For example, the label 963666N1 represents a model with log \( L/L_\odot = 2.96 \), log \( T_{\text{eff}}/K = 4.36 \), \( M = 0.66 \) \( M_\odot \), and mixture N1=J01 (Table 3). For clarity: log \( L/L_\odot \propto 3.6 \) if \( ill \leq 16 \) and log \( L/L_\odot = 2.6 \) if \( ill \geq 66 \).

### 2.2 Chemical composition

The choice of initial hydrostatic model makes assumptions about the internal structure and previous evolution of the star. For the current calculations, the envelope is assumed to be chemically homogeneous and free of strong energy sources. The first leads us to adopt the chemical composition from the surface abundance analyses by Jeffery et al. (1999, 2001); Przybilla et al. (2005) (Table 3). Potential conditions which might compromise these assumptions are: (i) the adopted surface composition was based on static model atmospheres and a mean spectrum around maximum radius; one consequence of continuing work will enable an abundance analysis based on dynamical model atmospheres, (ii) apart from iron, the abundances of elements with atomic number \( Z > 20 \) have not yet been measured for V652 Her, (iii) diffusion-induced chemical stratification in the stellar interior might already have commenced, particularly in regions of high spe-

| Table 2. Illustration of model grid for \( M = 0.66 \) \( M_\odot \) and mix N1 showing pulsation periods derived from the models. |
|-----------------------|-----------------|-----------------|-----------------|-----------------|
| \( \log L/L_\odot \)   | \( 4.30 \)       | \( 4.32 \)       | \( 4.34 \)       | \( 4.36 \)       |
| \( \log T_{\text{eff}}/K \) | \| | | |
| 3.16                  | 0.165           | 0.121           |                |                |
| 3.06                  | 0.216           | 0.183           | 0.156           | 0.133           | 0.114           | 0.098           |
| 3.01                  | 0.175           | 0.150           | 0.127           | 0.108           | 0.094           | 0.083           |
| 2.96                  | 0.117           | 0.105           | 0.086           | 0.079           | 0.069           |                |
| 2.91                  | 0.144           | 0.122           | 0.105           | 0.090           | 0.079           | 0.069           |
| 2.86                  | 0.130           | 0.113           | 0.097           | 0.083           | 0.073           |                |
| 2.76                  | 0.119           | 0.102           | 0.089           | 0.077           |                |                |
| 2.71                  | 0.108           | 0.093           | 0.081           | 0.070           |                |                |
| 2.66                  | 0.085           |                |                |                |                |                |

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Figure 2. As Fig. 1 for (left) outer layers including the opacity driving zone and (right) layers close to the surface during passage through minimum radius.

Table 3. Normalized relative metal abundances for V652 Her given by mass and based on Jeffery et al. (1999) (J99) and Jeffery et al. (2001) (J01). Elements marked ★ are measured; others are scaled relative to iron. J01 is the metal abundance distribution adopted for the OPAL opacity tables in grid N1. A few test models were calculated with relative abundances as shown in columns FeNi2 (OPAL) and FeNi3. Abundances relative to N1 are shown in columns 6 and 8. Test models calculated with mixture J01 and OP opacities were labelled N5.

| Mix Label | J99 | J01 | J99/J01 | FeNi2 | FeNi2/J01 | FeNi3 | FeNi3/J01 |
|-----------|-----|-----|---------|-------|-----------|-------|-----------|
| C★        | 0.01066 | 0.02559 | 0.4166  | 0.024884 | 0.9724    | 0.025163 | 0.983314  |
| N★        | 0.65740 | 0.64285 | 1.0226  | 0.633460 | 0.9854    | 0.618150 | 0.961586  |
| O★        | 0.02678 | 0.04990 | 0.5367  | 0.049171 | 0.9854    | 0.050205 | 1.00613   |
| Ne★       | 0.18528 | 0.11698 | 1.5839  | 0.115271 | 0.9854    | 0.112218 | 0.959300  |
| Na         | 0.00107 | 0.00079 | 1.3544  | 0.000778 | 0.9848    | 0.001466 | 1.85570   |
| Mg★       | 0.04445 | 0.08282 | 0.5367  | 0.081610 | 0.9854    | 0.081815 | 1.06788   |
| Al★       | 0.00239 | 0.00173 | 1.3815  | 0.001705 | 0.9855    | 0.001588 | 0.917919  |
| Si★       | 0.02387 | 0.03874 | 0.6162  | 0.038174 | 0.9854    | 0.039879 | 1.02940   |
| P★        | 0.00017 | 0.00345 | 0.0493  | 0.003400 | 0.9855    | 0.003168 | 0.918261  |
| S★        | 0.02127 | 0.01473 | 1.4440  | 0.014515 | 0.9854    | 0.015876 | 1.07780   |
| Cl         | 0.00019 | 0.00014 | 1.3571  | 0.000138 | 0.9857    | 0.000217 | 1.55000   |
| Ar★       | 0.00415 | 0.00573 | 0.7243  | 0.005646 | 0.9853    | 0.005021 | 0.876265  |
| K          | 0.00007 | 0.00005 | 1.4000  | 0.000049 | 0.9800    | 0.000091 | 1.82000   |
| Ca         | 0.00134 | 0.00100 | 1.3400  | 0.000985 | 0.9850    | 0.001572 | 1.57200   |
| Ti         | 0.00005 | 0.00004 | 1.2500  | 0.000039 | 0.9875    | 0.000075 | 1.87500   |
| Cr         | 0.00027 | 0.00020 | 1.3500  | 0.000197 | 0.9850    | 0.000236 | 1.63000   |
| Mn         | 0.00017 | 0.00012 | 1.4167  | 0.000118 | 0.9833    | 0.000168 | 1.40000   |
| Fe★        | 0.01940 | 0.01439 | 1.3482  | 0.028360 | 1.9708    | 0.042991 | 2.98756   |
| Ni         | 0.00102 | 0.00076 | 1.3421  | 0.001498 | 1.9710    | 0.002257 | 2.96974   |

cific opacity (Byrne et al. 2018); however this will be disrupted by pulsations and occasionally by helium shell flashes, (iv) the secular decrease in pulsation period implies that the stellar envelope is contracting and releasing gravitational potential energy as heat; the ratio between the pulsation period and secular period change, $\alpha_{\text{dotacc}} = \Pi / \Pi_0$ is so small as likely to have little effect on the current hydrodynamic models.

In the case of V652 Her, various internal structures have been proposed (Jeffery 1984; Saio & Jeffery 2000), but have the common feature of a hydrogen-contaminated but predominantly helium-rich envelope surrounding a helium-burning core. In the case of the post-merger model, nuclear helium-burning occurs in a thin shell surrounding a degenerate helium core; the radius of the latter can be no more than a few $\times 10^3$ km, compared with the 1.6 $\times 10^6$ km radius of the star. In this regard it is therefore safe to integrate down to $r \approx 0.05 R$ before the energy equation becomes an issue (at $< 0.01 R$).

One question concerns the chemical homogeneity of the envelope. In post-merger models, the hydrogen originates in the residual hydrogen envelopes of the pre-merger helium white dwarfs. The merger episode completely disrupts the entire donor white dwarf, which is then mixed with a fraction of the envelope of its more massive companion, combining some unprocessed hydrogen with CNO-processed helium. The composite merger model involves prompt accretion of part of the donor, followed by slow accretion of the remainder from a high entropy "shell". Zhang & Jeffery (2012) showed that flash-driven convection following the first helium shell ignition.
will mix the post-merger envelope, adding some carbon, but this is then overlaid by nitrogen-rich material accreted from the "shell". Whether carbon is subsequently mixed through the star depends on the total mass and whether the flash-mixed and surface convection zones interact (Zhang & Jeffery 2012). In low-mass cases, mixing does not occur and the surface remains nitrogen-rich, as in V652 Her. For the 0.25+0.25 $M_\odot$ composite merger model computed by Zhang & Jeffery (2012), the outer 40% of the mass remains nearly homogeneous; only the first four helium-shell flashes inject a small amount of carbon into the bottom 0.07 $M_\odot$ of the helium-rich envelope (op cit., Fig. 14). In all current calculations, the envelope mass $M_{\text{eff}} < 0.2 M_\odot$ and satisfies the above constraint. It remains to be determined whether chemical diffusion driven by radiative levitation could significantly influence the chemical structure in V652 Her; we touch on this briefly in §5.

2.3 Opacity

Pulsation models are critically dependent on the opacity and hence on the assumed chemical composition in the stellar envelope (§ 2.2). Opacity tables have been prepared using data from the OPAL calculations (Rogers & Iglesias 1992; Iglesias & Rogers 1996). Since each opacity table covers a wide range of hydrogen and helium abundances, only the detailed distribution of elements with $Z > 2$ is necessary to build the tables.

The primary composition adopted is based on that measured by Jeffery et al. (2001) $J01 \equiv N1$. Here, the abundances of several elements including sodium, chlorine, argon, potassium, calcium, titanium, chromium, manganese and nickel are based on a scaled solar abundance relative to iron (Asplund et al. 2009). The normalized abundances by mass of elements with $Z > 2$ are shown in Table 3.

To investigate possible errors in the envelope metallicity (factors of two are within the experimental uncertainties), we also prepared tables for mixtures with iron and nickel abundances enhanced by factors of two and three. These mixtures are labelled FeNi2 $\equiv N2$ and FeNi3 $\equiv N6$. For mixture N6, abundances of elements sodium, chlorine and calcium to manganese were also increased by $\approx 50 – 80\%$.

The same mixtures were also assumed for the composition of the envelope, so that the equation of state and opacity are as far as possible consistent.

For all of our calculations, we have defined the hydrogen abundance (mass fraction) to be $X = 0.00125$ (Przybilla et al. 2005). For the standard mixture $J01$ the metal mass fraction $Z = 0.0159$ (Jeffery et al. 2001), and the helium mass fraction $Y = 1 - X - Z = 0.98285$. For the iron and nickel enhanced mixtures FeNi2 and FeNi3, $Z$ is increased to 0.0161 and 0.01638 respectively and $Y$ is adjusted accordingly.

A small number of calculations were carried out using OPAL opacities (Seaton et al. 1994) instead of OPAL opacities. These assumed the $J01$ mixture and were labelled N5.

Table 4. The primary grid of completed models with mixture N1 defined in Table 3. Labels HittinMNa are defined in the text.

| Mass          | Models              |
|---------------|---------------------|
| 0.56 $M_\odot$| 063856N1 064056N1   |
|               | 013856N1 014056N1   |
|               | 963656N1 963856N1   |
|               | 913456N1 913656N1   |
|               | 863256N1 863456N1   |
|               | 813256N1 813456N1   |
| 0.66 $M_\odot$| 763056N1 763256N1   |
|               | 713056N1 713256N1   |
| 0.76 $M_\odot$| 063066N1 063266N1   |
|               | 063466N1 063666N1   |
|               | 063866N1 064066N1   |
|               | 013666N1 013866N1   |
|               | 014066N1 014266N1   |
|               | 963066N1 963266N1   |
|               | 963466N1 963666N1   |
|               | 963866N1 963966N1   |
|               | 913066N1 913266N1   |
|               | 913466N1 913666N1   |
|               | 913866N1 914066N1   |
|               | 863066N1 863266N1   |
|               | 863466N1 863666N1   |
|               | 863866N1 864066N1   |
|               | 813066N1 813266N1   |
|               | 813466N1 813666N1   |
|               | 813866N1 814066N1   |
|               | 763066N1 763266N1   |
|               | 763466N1 763666N1   |
|               | 713066N1 713266N1   |
|               | 713466N1 713666N1   |

Figure 3. Unstable pulsation modes in stars with homogeneous envelopes for composition $X = 0.002$, $Z = 0.02$, $M = 0.50 M_\odot$. The number of unstable radial modes in the linear non-adiabatic approximation is represented by grey scale contours, with the lightest shade marking the instability boundary (one unstable mode), and the darkest shade representing ten or more unstable modes. Broken maroon diagonal lines represent contours of constant surface gravity at $\log g = 8, 7, 6, \ldots, 1$. The spectroscopic positions of pulsating extreme helium stars V652 Her and BX Cir are marked with red triangles. The fine grid of non-linear pulsation models is indicated by a solid blue border; the boundary of the complete grid is indicated by a dashed blue border. The diagram is only weakly mass dependent. Adapted from Jeffery & Saio (2016).
each figure, individual panels show results on the grid of $T_{\text{eff}}$ and $L$ defined above. To limit computational time, work focused on models having pulsation period close to 0.11 d, with some extension to higher and lower $T_{\text{eff}}$ to explore systematics. Phase has been defined to be zero at luminosity maximum on the assumption that this is co-eval with maximum light in the Johnson V-band (Landolt 1975). This assumption should be verified.

Figure 5 and Figs. A.1–A.4 demonstrate how shape, amplitude, and phase-shift vary as functions of model parameters. Data used to prepare these figures, sampled at 600 phase points, together with procedures for reading and displaying them, are available as supplementary online material (Table B.1); their format is illustrated in Table B.2. Most models show the characteristic saw-tooth radial velocity curve observed in V652 Her (Fig. 5: lower right), with a steep acceleration phase lasting $\sim$ 0.1 cycles and a slow and nearly constant deceleration phase lasting $\sim$ 0.8 cycles. There are three principal features. 1) For models having the same mass and luminosity, the amplitude of the oscillation increases with effective temperature, with some levelling off at $\log T_{\text{eff}}/K \geq 4.35$. There is some evidence that the amplitude starts to drop for $\log T_{\text{eff}}/K \geq 4.40$, but model convergence becomes increasingly difficult beyond this value. 2) For models having approximately the same period, the velocity amplitude increases with both effective temperature and luminosity, since the two are tied. The same limiting amplitude applies. 3) Structure appears in the deceleration phase of some radial-velocity curves. This appears to be related to amplitude and hence to opacity, since these secondary structures are more common in a grid computed with iron and nickel enhanced chemistry (mixture N3, Fig. A.4).

Figures 1 and 2 represent the internal displacement and temperature of layers close to the surface through the pulsation cycle for models 963666N1. The temperature is represented by a colour scale.

### 3 PERIODS, SURFACE AND INTERNAL PROPERTIES

Figure 4 shows the behaviour of radius, luminosity and radial velocity as a function of phase for the representative model 963666N1. It explains the scales and demonstrates how the individual panels in Fig. 5 and Figs. A.1–A.4 are constructed. The latter show the radii, luminosities and radial velocities as a function of pulsation phase for many models of a given mass and chemical mixture. Within...
Figure 5. Luminosity ($100\delta L/(L)$, blue), radius ($1000\delta R/(R)$: red) and radial velocity ($U \equiv dR/dt$ km s$^{-1}$: black) as a function of phase for models with $M/M_\odot = 0.66$ and mix N1. The input parameters $M$, $T_{\text{eff}}$, $L$ (and $L/M$) associated with each model are shown along the top and right hand axes respectively. Phase zero is defined to be at maximum light. The period for each model is shown at the bottom of each panel. The number shown top left of each panel represents the number of runs completed for each model, each run representing 24,000 time steps. Similar panels for models with $M/M_\odot = 0.56$ and 0.76 and over an extended range of $T_{\text{eff}}$ and $L$ are given in the Appendix. The axes and scales for all panels are identical; for clarity only axes in the lower-left panel are labelled. The bottom right panel shows the equivalent quantities as observed in V652 Her and reported in Figs. 3 and 7 of Jeffery et al. (2015), assuming a mean radius $\langle R \rangle = 2.31 R_\odot$ (Table 1).
Figure 6. As Fig. 2 for mixtures with 2× (N3: top) and 3× (N6: middle) the iron and nickel abundances and with OP opacities (N5: bottom).
eration observed in V652 Her might not provide an upper constraint on the iron+nickel abundance.

The slightly larger amplitude obtained from the model using OP opacities implies the latter to be slightly larger than the OPAL opacities as implemented.

3.2 Luminosity effects

Figure 7 compares models with the same composition and roughly similar pulsation period but with lower and higher luminosities than the reference model, i.e. with \( \log L/L_\odot = 2.81 \) and 3.06 (models 813466N1 and 064066N1, \( \log L/M/(L_\odot/M_\odot) = 2.99 \) and 3.24). Including the reference model, the radial amplitude ranges from \( 0.11 R_\odot \) (813466N1), through \( 0.16 R_\odot \) (963666N1) to \( 0.21 R_\odot \) (064066N1). Since the \( L/M \) range covered by all three models is less than a factor 2 (\( \delta \log L/M = 0.25 \)), pulsation amplitude is clearly strongly correlated with luminosity.

3.3 Shocks

Jeffery et al. (2001) argued that the surface of V652 Her must be heated by the passage of the strong shock at minimum radius. Figures 6 and 7 show a very steep change in temperature running outward through the star, commencing shortly before minimum radius, in models 963666N3, 963666N6 and 064066N1. A radial pulsation is a pressure wave propagating outward from the driving zone into a region of lower density (and sound speed). Due to adiabatic compression and the consequent increase in sound speed, the compression front will catch up with the preceding pressure trough; if the former overruns the latter, a shock forms. So factors affecting shock formation include the amplitude of the pulsation and the sound-speed gradient in the stellar envelope. From Figs. 1 and 7, no shock is generated in the low \( L/M \) model (813466N1), the surface layers of the reference model (963666N1) are highly compressed at minimum radius (Fig. 1), and the high \( L/M \) model (064066N1) shows a strong shock.

Figures 8 and 9 present the models of Fig. 2, 6 and 7 in a different format. For each model, three panels show the radial velocity relative
Figure 8. The runs of radial velocity $u$, luminosity $l$ and temperature $T$ with phase for selected zones in the outer part of the stellar envelope. Velocity has been scaled to the local sound speed. Each shell is colour-coded identically in each panel, red being coolest and violet (black) being warmest. The panels include the reference model (963666N1: top left), and with mixtures having $2 \times$ (N3: bottom left) and $3 \times$ (N6: bottom right) the iron and nickel abundances and with OP opacities (N5: top right).
to the local sound speed\(^1\), luminosity and temperature of selected zones.

Note the behaviour of model 813466N1 \((M = 0.66 M_{\odot}, \log T_{\text{eff}}/K = 4.34, \log L/L_{\odot} = 2.81)\) in Figs. 7 and 9. In this relatively low \(L/M\) model, surface motion is almost sinusoidal and there are no discontinuities in the radial velocity or temperature at any point in the models. Temperature maximum occurs almost simultaneously at all layers, except the outermost, where it coincides with light maximum and precedes radius minimum.

For models with higher \(L/M\), the development of a shock at minimum radius is signalled by a spike in temperature which corresponds to shock heating. For milder cases (e.g. 963666N1), the temperature throughout the envelope behaves as in the non-shocked case, except around minimum radius where a temperature spike occurs in outer layers. The photosphere itself is not significantly affected (cf. Fig. 8).

In the most extreme cases (e.g. the high opacity models 963666N3 and 963666N6 and the high \(L/M\) case 064066N1), temperature and hence luminosity maxima again occur first on the surface. In these cases, however, the shock develops in deeper layers and hence propagates outwards (Figs. 6 and 8). Radius minimum at the surface is substantially delayed relative to radius minimum in deeper layers. It appears to be the action of the shock which delays contraction of the outer layers and hence has the strongest influence on the phase delay between light maximum and radius minimum.

3.4 Phase delay

Observations of V652 Her clearly show a phase difference of \(\approx +0.13\) cycles between maximum light and minimum radius (Hill et al. 1981; Jeffery et al. 2015). A similar phase difference in the models was found by Montañés Rodríguez & Jeffery (2002). Phase differences exist in classical Cepheids, but have the opposite sense \((-0.25\) cycles). They have been interpreted in terms of linear non-adiabatic effects as associated with the thin hydrogen ionization zone moving through mass layers almost as a discontinuity (Castor 1968; Szabó et al. 2007). Since the hydrogen and first helium ionization zones play no rôle in V652 Her (too little hydrogen and too hot), it is suggested that the second helium ionization zone will be important. Fig. 6 suggests that the phase delay depends on the intensity of the shock, ranging from \(\sim 0.055\) in the reference model (Fig. 6) to \(\sim 0.12\) in the most extreme example found (963666N6). Since shock strength appears to depend on both opacity (model 963666N6) and luminosity (064066N1), phase delay would appear to be a nonlinear function of both. Some insight on their separate effects is obtained from the N3 model grid (Fig. A.4 and § 4).

4 COMPARISON WITH V652 HER

The primary parameters chosen for the model computations are mass \(M\), effective temperature \(T_{\text{eff}}\), luminosity \(L\), and chemical composition. The strongest observational constraint is provided by the pulsation period of 0.108 d; this immediately constrains the radius, luminosity and surface gravity to be a slowly varying function of
mass and either constant or linear in effective temperature. These constraints are model-independent and a direct consequence of the period mean-density relation for stars pulsating in the fundamental radial mode (Eddington 1918). Figure 10 shows these quantities for the three model grids, as well as the phase delay, velocity amplitude and maximum acceleration at minimum radius, also interpolated to Π = 0.108 d.

Several results are noteworthy. Some concern measurements relating to the stellar spectrum. Others concern properties of the models. The radius of V652 Her has been measured using the Baade (1926) method. In simple terms this gives the radius as

\[ R = \frac{\delta R}{\delta \theta} \frac{\theta}{\delta \theta}, \]

where

\[ \delta R = \int \delta R \, dt. \]

\( \delta R \equiv dR/dt \) is the velocity of the stellar photosphere relative to the stellar centre and is obtained from measurements of observed and mean observed radial velocities (v and v0) via a projection factor p (Parsons 1972; Montañés Rodriguez & Jeffery 2001) such that \( R = -p(v - v_0) \). The angular radius \( \theta \) and its variation \( \delta \theta \) are obtained from measurements of the total flux around the pulsation cycle. The value for the mean radius \( \langle R \rangle = 2.31 \pm 0.02 \, R_\odot \) cited in Table 1 (Jeffery et al. 2001) appears too large by about 50%. An earlier value of 1.98 ± 0.21 \( R_\odot \) (Lynas-Gray et al. 1984) is more consistent with theory. Whether this is a consequence of assumptions in the model atmosphere analyses will be reviewed in a subsequent study.

The luminosity is derived from the distance and apparent brightness, but may also be determined from the radius and effective temperature. Technically the two are equivalent, since the distance is related to the radius by the angular diameter, and the latter is determined by fitting the flux distribution for a given effective temperature an observed flux. Assuming the Baade radius above, Jeffery et al. (2001) obtained contradictory distances (and hence masses and luminosities) of 0.963 ± 0.006 and 1.693 ± 0.115 kpc.
when normalising to visual fluxes (Baade-vis in Fig. 10) or ultraviolet fluxes (Baade-uv), respectively. From Gaia Data Release 2 (Gaia Collaboration et al. 2018), a parallax of 0.88 ± 0.09 mas led Martin (2019) to obtain a distance of 1.170 +0.110 −0.130 kpc. In the Gaia Early Data Release 3 (Gaia Collaboration 2021), the parallax is reduced to 0.636 ± 0.048 mas. Bailler-Jones et al. (2021) consequently find a distance 1.542 +0.114 −0.107 kpc, which is remarkably similar to the distance 1.5 ± 0.1 kpc inferred by Lynas-Gray et al. (1984). The pulsation period indicates that only luminosities log L/L⊙ ≤ 3.1 (L < 1260 L⊙) are permitted for the observed effective temperatures, which is entirely consistent with the luminosities inferred by Lynas-Gray et al. (1984) and Jeffery et al. (2001) (1072 +341 −258, and 919 ± 14 L⊙, respectively). With sufficient precision, temperature and luminosity could be a mass discriminant.

The surface gravity has been measured spectroscopically on a number of occasions. Fitting the profiles of neutral helium lines using LTE model atmospheres consistently gave log g/cm s−2 ≈ 3.7 ± 0.2 (Hill et al. 1981; Lynas-Gray et al. 1984; Jeffery et al. 1999). Using higher quality data, Jeffery et al. (2001) obtained the lower value log g/cm s−2 = 3.46 ± 0.05, averaged over the pulsation cycle. In an effort to resolve discrepancies in the cores of the helium lines and between different hydrogen lines, Przybilla et al. (2005) used a non-LTE model to find log g/cm s−2 ≈ 3.2 ± 0.1. This was based on a sub-sample of spectra obtained around maximum radius. For line identification, Jeffery et al. (2015, Appendix B) adopted a χ2-by-eye model with log g/cm s−2 ≈ 3.3 to approximate a median spectrum obtained around maximum radius. Pulsation arguments posit that a mean value of log g/cm s−2 ≈ 3.7±0.05 would be consistent with the pulsation period (assuming a star in hydrostatic equilibrium). If g is measured at Rmax, the effective gravity g eff felt by the local plasma is reduced by the downward acceleration of Δg/0.8Π ≈ 1200 cm s−2. This suggests a reduced value for log ρ eff (Rmax)/cm s−2 = 3.6. A subsequent paper will address the question of whether adiabatic expansion following shock heating might further reduce the density of the photosphere, the temperature which is actually measured by the plasma diagnostics.

Turning to the models, several observables may be discussed in terms of model properties.

The origin and behaviour of the phase delay between light maximum and radius minimum in the hydrodynamic models is discussed above. The observed value (+0.13 cycles) is larger than found in any of the models. A difference between visual light maximum, which is used to define the ephemeris (Kilkenny et al. 2005), and bolometric light maximum, which is poorly defined by current ultraviolet observations, might contribute some of the difference, and could be as much as 0.05 cycles. The remainder might be embedded in the treatment of the shock by the models or in the microphysics, including the equation of state and opacity. Notably, while luminosity maximum at the surface precedes radius minimum, luminosity maximum at deep layers follows radius minimum by a roughly similar amount. The extreme non-linearity of the problem makes it difficult to quantify specific dependencies. For example, a minor change to the equation of state whereby the mean atomic weight in the deep interior was reduced by ≈ 0.02% resulted in a reduction of the radial amplitude and an increase in the phase delay of a few per cent.

Pulsation amplitude, as measured by the peak-to-peak velocity amplitude, increases steadily with effective temperature up to a maximum around where V652 Her lies. As noted earlier, this increase correlates directly with the increase in luminosity. The observed and theoretical amplitudes match well at the effective temperature obtained by Przybilla et al. (2005). Two questions are what limits the model amplitude, and why does that limit exceed the observed amplitude over most of the permissible range of effective temperature? One conjectures that the first is dissipation in an increasingly violent shock, and the second is that our one-dimensional model inadequately describes that dissipation. As a check, we reduced the viscosity parameter in the reference model by 25% to Π = 1.5. The phase delay Δϕ increased from 0.062 to 0.064 cycles; the luminosity and radial-velocity amplitudes increased by a similar amount (= 2.5%). Thus a significantly larger viscosity is required to reduce the theoretical amplitude, suggesting a greater contribution from turbulence might arise in a multi-dimensional treatment. However, this would give the wrong sign for the phase delay. The alternative possibility that the observations do not sufficiently capture the amplitude of the same zones as described by the models was largely eliminated by the high-speed spectroscopy reported by Jeffery et al. (2015).

Another way of framing the question is to ask how the pulsation properties might change with period. This is moot since the period of V652 Her has decreased by ≈ 1% since first measured (Kilkenny et al. 2005). Fig. 10 was generated by interpolating in the model grids for pulsation period Π = 0.108 d. By varying Π, dependency on period can be computed easily. Taking into account numerical noise amongst the models and the coarseness of the grid, the phase delay is not substantially changed for a 10% change in period, but the velocity amplitude and maximum surface acceleration are both reduced by ≈ 10 − 20%.

Recognising that the mass of V652 Her was poorly constrained by observation, the original goal of these calculations was to obtain the best possible hydrodynamic model of pulsations in V652 Her. Neither the effective temperature nor the radius obtained spectroscopically are fully consistent with the models. From the pulsation period alone, the surface gravity log g/cm s−2 ≈ 3.7 ± 0.05, but this assumes hydrostatic equilibrium. Additional heating could reduce this more in line with the spectroscopic value ≈ 3.4. With some latitude, our recommended initial estimate for the next phase of calculations should be M ≈ 0.66 M⊙, log T eff/K ≈ 4.34, and log L/L⊙ ≈ 2.9.

Models computed around these parameters will be used as input to a time-dependent formal solution of the emergent spectrum, and will hence be applied to the exquisite observations reported by Jeffery et al. (2015). Together with refined distance measurement from Gaia, these should lead, in turn, to a more precise direct measurement of the radius and mass.

A similar comparison should be carried out for the companion star BX Cir for which excellent data are also available (Kilkenny et al. 1999; Woolf & Jeffery 2000, 2002; Martin 2019). The corresponding radial velocity curve indicates no shock is present and therefore that the L/M ratio may be smaller than in V652 Her. However, with a different chemical composition containing less hydrogen and more carbon (Drilling et al. 1998), the current models might not be applicable; more calculations will be required.

5 CONCLUSION

Precise observations of the unique pulsating helium star V652 Her have provided stellar theory with a spectrum of challenges. This investigation set out to explore non-linear pulsation models as a function of the primary parameters luminosity, effective temperature, and mass, as well as secondary parameters including opacity and viscosity. The object was to learn how these parameters might be adjusted to best-fit the observed pulsation properties, including its light and radial-velocity curves.

The models demonstrate how pulsation amplitude increases with
luminosity, or more generally, with luminosity-to-mass ratio. Amplitude is less sensitive to effective temperature except towards the cool and warm boundaries of the instability region, where the amplitude is diminished.

The development of a shock during the acceleration correlates closely with amplitude and is the primary factor determining the shape of the velocity curve. For a given period and effective temperature, there is also an inverse correlation with mass, possibly associated with the increased inertia of the envelope.

The origin and magnitude of the phase delay between maximum luminosity and minimum radius has proved harder to explain. Theoretical phase delays are generally smaller than observed by a factor of at least 2. There is a moderate reduction in phase delay with increasing effective temperature, but a value at the low limit of that observed cannot resolve the problem. Reducing the artificial viscosity produces a small but insufficient improvement.

It is conceivable that the chemical composition in the driving zone in V652 Her is richer in iron and nickel than observed on the surface. Byrne & Jeffery (2020) showed that radiative levitation enriches the iron-bump driving zone in models of contracting low-mass pre-white dwarfs at $T_{\text{eff}} \gtrsim 20,000$K, and earlier for more massive stars, sufficiently to trigger pulsations in stars observed as blue large amplitude pulsators (BLAPs: Pietrukowicz et al. 2017; Kupfer et al. 2019). Whether such enrichment also occurs in post-white dwarf merger stars has yet to be tested. Meanwhile it will be interesting to extend the non-linear calculations presented here to the case of the BLAPs in the hope that high-quality observations will in future demonstrate the ubiquity and nature of shocks amongst these stars.

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DATA AVAILABILITY

The summary files and internal structures for the last 240,000 timesteps of each model are stored as binary files. They will be provided on reasonable request to the authors. The surface variation of velocity, luminosity and radius as a function of phase over 1.4 pulsation cycles are available as ASCII files as supplementary online material.

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APPENDIX A: ADDITIONAL FIGURES

This appendix includes Figs. A.1-A.3 which are equivalent to Fig. 5 for additional masses and mixtures.

APPENDIX B: SUPPLEMENTARY DATA

Table B.1 shows a list of files containing supplementary material containing the data shown in Fig. 5 and Figs. A.1-A.3 as well as IDL procedures for reading reading and plotting specified models in the form of Fig. 4. Table B.2 shows a file fragment from one of the data files. The IDL procedures are self-documented with instructions for use.
Figure A.1. As Fig. 5 for $M / M_\odot = 0.56$. 
Figure A.2. As Fig. 5 for $M / M_\odot = 0.76$. 
Figure A.3. As Fig. 5 for $M/M_\odot = 0.66$ showing extensions to lower and higher luminosity and effective temperature.
Figure A.4. As Fig. 5 for mixture N3 (iron and nickel abundances enhanced ×2).
Table B.1. Files containing Supplementary Data available online.

| File            | Description                                               |
|-----------------|-----------------------------------------------------------|
| surf_read.pro   | IDL procedure to read specified pulsation model           |
| surf_plot.pro   | IDL procedure to plot specified pulsation model           |
| surf_m56.log    | Surface properties for pulsation models at 0.56 M⊙, mix N1 |
| surf_m66.log    | Surface properties for pulsation models at 0.66 M⊙, mix N1 |
| surf_m76.log    | Surface properties for pulsation models at 0.76 M⊙, mix N1 |
| surf_m66N3.log  | Surface properties for pulsation models at 0.66 M⊙, mix N3 |

Table B.2. Fragment from one of the supplementary online files containing data used in the construction of Figs. 5 and Figs. A.1-A.3 and describing the variation of velocity in km s\(^{-1}\), radius (per cent) and luminosity (per cent) with pulsation phase. Each data file contains all models for a given mass and mixture; each model is demarcated by a header line containing its label and other parameters, as shown. An IDL procedure to read these files is provided with the online material.

```
MODEL log10(<L>/Lsun) log10(<Teff>/K) M/Msun log10(<R>/Rsun) P/d phase RV/(km/s) 100*(R-<R>)/<R> 100*(L-<L>)/<L>

963666N1  2.958  4.360  0.660  0.280  0.1083 -0.1000 -43.37  0.654 -2.463
-0.0975 -44.02  0.575 -1.627
-0.0953 -44.15  0.512 -0.990
...
```