A Bayesian Approach for the Variance of Fine Stratification

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Abstract

Fine stratification is a popular design as it permits the stratification to be carried out to the fullest possible extent. Some examples include the Current Population Survey and National Crime Victimization Survey both conducted by the U.S. Census Bureau, and the National Survey of Family Growth conducted by the University of Michigan’s Institute for Social Research. Clearly, the fine stratification survey has proved useful in many applications as its point estimator is unbiased and efficient. A common practice to estimate the variance in this context is collapsing the adjacent strata to create pseudo-strata and then estimating the variance, but the attained estimator of variance is not design-unbiased, and the bias increases as the population means of the pseudo-strata become more variant. Additionally, the estimator may suffer from a large mean squared error (MSE). In this paper, we propose a hierarchical Bayesian estimator for the variance of collapsed strata and compare the results with a nonparametric Bayes variance estimator. Additionally, we make comparisons with a kernel-based variance estimator recently proposed by Breidt et al. (2016). We show our proposed estimator is superior compared to the alternatives given in the literature such that it has a smaller frequentist MSE and bias. We verify this throughout multiple simulation studies and data analysis from the 2007-8 National Health and Nutrition Examination Survey and the 1998 Survey of Mental Health Organizations.

Keywords: Collapsed strata; hierarchical Bayes estimator; nonparametric Bayes estimator; primary sampling unit.

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1 Introduction

The process of independently selecting a very small number of units often one or two primary sampling units (PSUs) per disjoint countable subpopulations (so-called strata) is called fine stratification. Surveys such as the Survey of Income and Program Participation of the U.S. Census Bureau and the U.S. Department of Agriculture’s National Resources Inventory use a multi-stage two-per-stratum design and a stratified two-stage area sampling design, respectively. The Canadian Health Measures Survey with a three-stage sample design is also based on the small number of PSUs per stratum.

Despite the popularity of the design which allows the usual Horvitz & Thompson (1952) estimator for an interested parameter such as population mean or total to be unbiased and efficient, it is also recognized to have a number of caveats. In particular, the variance under the design for one PSU per stratum does not exist and for two or three PSUs per stratum contains a large amount of variation. A traditional method for estimating the variance of an unknown parameter under fine stratification is collapsing the adjacent strata to create pseudo-strata with more number of PSUs and then calculating the variance. This method was first introduced by Hansen et al. (1953), but it often causes an overestimation in the variance of estimator.

Hansen et al. (1953) and Isaki (1983) proposed to use some auxiliary variables well-correlated with the expected values of the mean of strata to reduce the bias of variance estimator. Mantel & Giroux (2009) proposed a new approach based on the components of variance from different stages of sampling and applied it to the Canadian Health Measures Survey. Mosaferi (2015) proposed an empirical and constrained empirical Bayes estimators for the variance of one unit per stratum sample design. The author also compared one PSU per stratum design with two PSUs per stratum design and pointed out some of the instabilities of the proposed estimators due to the method of moment parameter estimation.
Recently, Breidt et al. (2016) proposed a nonparametric alternative method that replaces a collapsed stratum estimator by a kernel-weighted stratum neighbourhood estimator, which uses the deviations from a fitted mean function to estimate the variance. The underlying assumption in their paper is stratum means (or totals) vary smoothly with an auxiliary variable. Fogarty (2018) used classical least squares theory to present an improved variance estimator for finely stratified experiments. The variance estimator is conservative in the expectation but is asymptotically no more conservative than the traditional estimator. Majority of the proposed estimators in the literature are based on some auxiliary variables well-correlated with the expected values of the mean of strata which may not be always available.

In this paper, we propose a hierarchical Bayes (HB) approach to estimate the variance of fine stratification. The advantages of the proposed estimator are (i) reducing the MSE significantly, (ii) outperforming in comparison with the alternative estimators given in the paper, (iii) not relying on the choice of super-population for generating the data in the simulation studies, (iv) easy to be implemented by the government agencies, and (v) depending on as few information as possible, which only come from the sample itself without using any external auxiliary information or the sampling frame. Additionally, we compare our results with a nonparametric Bayes approach of Lahiri & Tiwari (1991), where we modify it for the variance of fine stratification. We consider several simulation studies and data analysis to prove the superiority of our methodology.

The rest of the paper is organized as follows. In Section 2 we provide some notation and review the variance of fine stratification based on the collapsed strata method. In Section 3 we propose our set-up of HB for estimating the variance, and we explain some of its properties. In Section 4 we define a nonparametric Bayesian variance estimator. Section 5 demonstrates our methodology and its superiority throughout several simulation studies. Furtherly, we apply our method to real data sets in Section 6 where the first data comes
from 2007-8 National Health and Nutrition Examination Survey, and the second data comes from the 1998 Survey of Mental Health Organizations conducted by the U.S. Substance Abuse and Mental Health Services Administration. Some conclusions are given in Section 7. Appendices contain some details on derivations and computations.

2 Variance of Fine Stratification

In stratified sampling design, we partition the target population into disjoint finite number of strata where the units per each stratum are homogeneous and share at least one common characteristic such as location, income, education or gender. This causes the variance of an interested parameter to be small. Also, stratification allows a flexible method of sample selection per stratum such as simple random sampling with or without replacement and systematic sampling as strata are assumed to be independent. In this manuscript, we focus on the particular case of fine stratification design where the sample size per stratum is as small as one PSU or two PSUs selected without replacement. Our target parameter is the population mean, and our goal is estimating the variance of estimated mean. Often, this variance brings complications since it is not estimable when the sample size per stratum is as small as one unit. Throughout the rest of this section, we provide some notation and explain the traditional collapsed strata variance estimator.

2.1 Notation and Estimation

Consider a finite population with $N$ units $\{1, 2, ..., N\}$, where we partition it into $H$ disjoint sub-groups (strata) each with the size of $N_h$ for $h = 1, 2, ..., H$ such that $N = \sum_{h=1}^{H} N_h$. Let $y_{hj}$ denote the interested value from $j$-th individual nested in the $h$-th stratum such that $j = 1, ..., N_h$. Additionally, assume the value of $y_{hj}$ is observed without error. Randomly and without replacement, we select a sample of size $n_h$ from stratum $h$-th, where $n_h = 1, 2$
or larger, and the total sample size is \( n = \sum_{h=1}^{H} n_h \). In many applications, these units could be the primary stage units. Here, we only concentrate on a single-stage sampling and assume we have a stratified simple random sampling without replacement (see, Valliant et al. (2013) chapter 3 for the definition of this design). This design could be easily generalized to much more complicated survey designs such as multi-stage two-per-stratum design or multiple stages of selection within strata.

As noted earlier, we are interested in estimating the finite population mean denoted by \( \bar{y}_{st} = \sum_{h=1}^{H} W_h \bar{y}_h \), where \( W_h = N_h/N \) is the relative size of \( h \)-th stratum such that \( \sum_{h=1}^{H} W_h = 1 \). Note that \( \bar{y}_h \) is the finite population mean of the \( h \)-th stratum defined as \( N_h^{-1} \sum_{j=1}^{N_h} y_{hj} \). The population sampling variance of the stratified estimator is

\[
V(\bar{y}_{st}) = \sum_{h=1}^{H} W_h^2 \frac{1}{n_h} (1 - f_h) S_h^2,
\]

where \( f_h = n_h/N_h \) is the sampling fraction, and \( 1 - f_h \) is the finite population correction (FPC) factor in stratum \( h \). Also, \( S_h^2 = \sum_{j=1}^{N_h} (y_{hj} - \bar{y}_h)^2/(N_h - 1) \) is the finite population variance for the \( h \)-th stratum (see, Särndal et al. (2003) section 3.7 and Valliant et al. (2013) chapter 3).

Let \( s_h \) denote the selected sample with size \( n_h \) from stratum \( h \) such that the total sample is \( s = s_1 \cup s_2 \cup ... \cup s_H \). Also, let \( \pi_j = \mathbb{P}\{j \in s_h\} \) be the first-order inclusion probability and \( \pi_{jk} = \mathbb{P}\{j, k \in s_h\} \) be the second-order inclusion probability. The Horvitz & Thompson (1952) unbiased estimator of the finite population mean under stratified design is as follows:

\[
\bar{y}_{HT} = \frac{1}{N} \sum_{h=1}^{H} \sum_{j=1}^{n_h} \frac{y_{hj}}{\pi_j}.
\]

Under simple random sampling without replacement, \( \pi_j = n_h/N_h \) and \( \pi_{jk} = n_h(n_h - \)
1) \(N_h(N_h - 1)\). An unbiased estimator of the variance of \(\bar{y}_{HT}\) in (1) is

\[
\text{var}(\bar{y}_{HT}) = \frac{1}{N^2} \sum_{h=1}^{H} \sum_{(j,k) \in s_h} \sum \Delta_{jk} \frac{y_{hj} y_{hk}}{\pi_j \pi_k},
\]

(2)

where \(\Delta_{jk} = 1 - (\pi_j \pi_k / \pi_{jk})\) for \(j \neq k\) and \(\Delta_{jj} = 1 - \pi_j\).

### 2.2 Collapsed Strata Variance Estimator

When there exits only one PSU per stratum, the variance estimator given in equation (2) cannot be obtained as \(\pi_{jk} = 0\). Instead, a collapsed variance estimator can be used, where we create larger strata by often pairing the adjacent strata to have more PSUs per the new created pseudo-stratum. Pseudo strata can be created by using the spatial location or stratum size measure which often come from the population frame and should be done before the process of sample selection. For simplicity, assume that the total number of strata “\(H\)” is an even number. If \(\{x_h\}\) denotes the collapsing index for \(h = 1, ..., H\), then we can create successive pseudo strata by sorting strata based on \(x_h\).

Let strata \(h\) and \(l\) \((h \neq l)\) belong to the same pseudo-stratum. The collapsed strata variance estimator is

\[
\text{var}_{\text{Coll}} = \frac{1}{N^2} \sum_{(h \neq l) = 1}^{H} \frac{N_{hl}^2}{n_{hl}} (1 - f_{hl}) s_{hl}^2,
\]

(3)

where \(n_{hl}\) and \(N_{hl}\) are the merged sample size and merged population size of strata \(h\) and \(l\), respectively. Additionally, \(f_{hl} = n_{hl} / N_{hl}\) and \(s_{hl}^2 = \frac{1}{n_{hl}-1} \sum_{j=1}^{n_{hl}} (y_{hl,j} - \bar{y}_{hl})^2\). An interested reader can review chapter 3 of Särndal et al. (2003) for the collapsed strata variance estimator literature. The variance estimator given in (3) is design-biased, and its bias with respect to the design is

\[
\text{Bias}(\text{var}_{\text{Coll}}) = \frac{1}{N^2} \sum_{(h \neq l) = 1}^{H} \frac{N_{hl}^2}{n_{hl}} (1 - f_{hl}) (\bar{y}_h - \bar{y})^2,
\]
see Wolter (2007) section 2.5 and Breidt et al. (2016).

Note that the bias term is positive, and it suggests a strategy on how we can group the strata to reduce the amount of bias by putting more similar strata in pairs with respect to the characteristic of interest to minimize the difference $|\bar{y}_h - \bar{y}_l|$. If we assume $N_h = N/H$, $n_{hl} = 2$, and the FPC factor is negligible, then the bias term summarizes to $2H^{-2} \sum_{(h \neq l)=1}^{H} (\bar{y}_h - \bar{y}_l)^2$, which is of order $O(H^{-1})$ as $N \to \infty$. We expect by increasing the number of strata, the amount of bias decreases. When $H$ is an odd number, we can create a single pseudo stratum by combining three original strata (say) and the rest of pseudo strata by combining two original strata. Then, we can use the same methodology described in this paper. Rust & Kalton (1987) compared variance estimators by collapsing two, three or more strata. They showed that increasing the number of collapsed strata results into a more biased variance estimator. Hence, collapsing strata in pairs seems to be a safe strategy.

3 Hierarchical Bayesian Variance Estimator

We propose our Bayes estimator such that strata $h$ and $l$ belong to the same pseudo-stratum. Assume $L_g$ is the number of original strata which are combined in the same $g$-th pseudo stratum. In surveys, often the collection of variances $s_{hl}^2$’s is positively skewed. In order to stabilize the skewness, we use the log-transformation of them such that $\log s_{hl}^2$ follows a normal distribution. This could prevent the occurrence of heavy tails for the unknown parameter $S_{hl}^2$ in the full conditional distribution. We propose the following hierarchical model to the data:

Stage one: $\log s_{hl}^2 | S_{hl}^2, \sigma^2 \overset{\text{ind}}{\sim} \text{Normal}(\log S_{hl}^2, \sigma^2)$ for $(h \neq l) = 1, ..., H$,

Stage two: $S_{hl}^2 \overset{\text{ind}}{\sim} \text{Half-t}(L_g + 1, d)$ for $d > 0$, and
Stage three: $\sigma^{-2} \sim \text{Gamma}(L_g, \zeta)$, where

$$
\zeta = \begin{cases} 
n_{hl} & \text{if } n_{hl} \text{ is small}, \\
1 & \text{otherwise}. 
\end{cases}
$$

The Half-t distribution in stage two of the hierarchical model is a special case of folded non-standardized t-distribution with mean equals to zero and degrees of freedom $L_g + 1$. This distribution can be presented as

$$
P(S^2_{hl}) \propto \left\{1 + \frac{1}{L_g + 1} \left(\frac{S^2_{hl}}{d}\right)^2\right\}^{-\frac{L_g+2}{2}}, \quad S^2_{hl} \geq 0.
$$

Remark 1: For large enough values of $n_{hl}$, we suggest $\zeta = 1$ in stage three of the hierarchical model. This could alleviate the possibility of receiving unreasonably small values for the variance and non-converging coefficients in the hierarchical Bayes. Although, the structure of population could have some effects on this choice.

The proposed hierarchical model has the following desirable properties:

1. It does not depend on an extra auxiliary information which may come from the population or the sampling frame. It only depends on the sample information $L_g$, $n_{hl}$, and $s^2_{hl}$.

2. It can be easily implemented by the government agencies.

3. Its bias and MSE are reasonably small compared to the other methods given in the literature of fine stratification and described in this manuscript.

4. Based on simulation studies, this hierarchy is not restricted to a particular super population for generating the data. Thus, it can be used for the data generated from normal, gamma or other distributions of the exponential family. We only need to
define a correct value for $d$, which can be simply identified by a preliminary analysis with a small training data set based on bias and MSE criteria or some other desirable criteria.

**Remark 2**  The proposed hierarchical model can be naturally extended to a case when more than two strata are combined. The main change is on the value of $L_g$. Also, note that as the number of PSUs per stratum increases, the performance of collapsed variance estimator relatively improves, and the results between hierarchical Bayes variance estimator and collapsed variance estimator become more alike.

Finally, we do not recommend the consideration of $S_{hl} \sim \text{Half-t}(L_g + 1, d)$ over $S_{hl}^2 \sim \text{Half-t}(L_g + 1, d)$ in stage two of hierarchical model due to its slow run-time and computational challenges; see Table B.2 given in the Appendix B. It cannot also have significant improvements on the results.

### 3.1 Conditional Posterior Distributions

Let $S^2 = (S_1^2, ..., S_H^2)'$ and $s^2 = (s_1^2, ..., s_H^2)'$. After some simplification, the joint posterior distribution is

$$
\pi(S^2, \sigma^2 | s^2) \propto \prod_{(h \neq l) = 1}^H (\sigma^2)^{-1/2} \exp \left\{ -\left( \log s_{hl}^2 - \log S_{hl}^2 \right)^2 / 2\sigma^2 \right\} \\
\times \prod_{(h \neq l) = 1}^H \left\{ 1 + \frac{1}{L_g + 1} \left( \frac{S_{hl}^2}{d} \right)^2 \right\}^{-L_g+2} \times \left\{ (\sigma^2)^{-L_g-1} \exp \left( -\frac{\zeta}{\sigma^2} \right) \right\}.
$$

This joint posterior is proper, and it is verified in Appendix A.1. Since it is not feasible to sample directly from the joint posterior, we make an inference via a Gibbs sampler that cycles through drawing from the full conditional posterior distributions. Now, we provide these full conditional distributions:

(i) $\pi(S_{hl}^2 | s^2, \sigma^2) \propto S_{hl}^2 \times \text{log-Normal}(\log s_{hl}^2, \sigma^2) \text{Half-t}(L_g + 1, d)$, and
(ii) $[\sigma^2 | s^2, S^2] \sim \text{Inv-Gamma} \left( L_g + \frac{H}{2L_d}, \zeta + \frac{1}{2} \sum_{(h \neq l)=1}^H \{\log s^2_{hl} - \log S^2_{hl}\}^2 \right)$.

### 3.2 Variance Estimator

In the process of deriving a Bayes estimator, the objective is estimating the parameter $S^2_h$ by $\hat{\varrho}_h$ (say) under a suitable loss function. We would like the right departure of an estimator $\varrho_h$ from the true value of $S^2_h$ i.e. $\varrho_h > S^2_h$ to be more penalized compared to the left departure of $\varrho_h < S^2_h$. Thus, we suggest the conventional squared error loss function in the form of $L(\varrho, S^2) = H^{-1}(\varrho - S^2)'(\varrho - S^2)$ for $\varrho = (\varrho_1, ..., \varrho_H)'$. For other conventional losses, the Bayes estimators have complicated forms or cannot be expressed in closed forms. The optimal predictor under the risk function $R(\varrho, S^2) = \mathbb{E}_{S^2}[L(\varrho, S^2)]$ is $\hat{\varrho} = \mathbb{E}[S^2|\text{data}]$. In this manner, the mean of the conditional distribution (i) in the form of $\hat{\varrho}_{h\beta} = \mathbb{E}[S^2_{hl}|s^2, \sigma^2]$ is our optimal predictor. Observe that in our setting, the posterior distribution of $S^2_{hl}$ does not have a known closed form. Therefore in order to generate samples from it, we need a special effort. We handle this situation by a Metropolis-Hastings (MH) accept-reject within Gibbs sampler algorithm such that we consider the log-Normal($\log s^2_{hl}, \sigma^2$) as a candidate distribution for targeting $[S^2_{hl}|s^2, \sigma^2]$; see Algorithm B.1 and other computational details given in the Appendix B. The hierarchical Bayes estimator of variance is

$$var_{HB} = \frac{1}{N^2} \sum_{(h \neq l)=1}^H \frac{N^2_{hl}}{n_{hl}} (1 - f_{hl}) \hat{\varrho}^\beta_{hl}.$$  (4)

### 4 Nonparametric Bayesian Variance Estimator

Here, we assume that the strata values are independently drawn from independent super-populations with a common Dirichlet process prior. This assumption also has been made by (1973) and Lahiri & Tiwari (1991) for some nonparametric problems and variance of stratified sampling design, respectively. Let $y_h = (y_{h1}, ..., y_{hn_h})'$ denote the
vector of sample observations from stratum $h$ where they are iid with distribution $P_h$, and $P_h$’s are iid with a Dirichlet process prior with parameter $M$ defined in what follows. Let $M$ be a non-null finite measure on the space $(\mathbb{R}, \mathfrak{B})$, where $\mathbb{R}$ is a real line and $\mathfrak{B}$ is the $\sigma$-field of Borel subsets of $\mathbb{R}$. Thus, we have the following model

(i) $y_h | P_h \overset{iid}{\sim} P_h$ for $h = 1, \ldots, H$, and

(ii) $P_h \overset{iid}{\sim} \text{Dirichlet}(M)$.

Then, the posterior distribution is

$$P_h | y_h \overset{iid}{\sim} \text{Dirichlet}(M + \sum_{j=1}^{n_h} \delta_{y_{hj}}),$$

where $\delta_{y_{hj}}$ denotes the measure giving mass one to $y_{hj}$. Under the squared error loss function $L(\varrho, S^2)$, the nonparametric Bayes (NB) estimator $S_{hl}^2$ denoted by $\hat{\varrho}_{NB}$ is

$$\hat{\varrho}_{NB} = \frac{M + n_{hl}}{M + n_{hl} + 1} \left\{ \Lambda_{hl} \alpha_{hl} \sigma_0^2 + (1 - \Lambda_{hl}) \beta_{hl} \frac{n_{hl} - 1}{n_{hl}} s_{hl}^2 + \Lambda_{hl} (1 - \Lambda_{hl}) \gamma_{hl} (\bar{y}_{hl} - \mu_0)^2 \right\}, (5)$$

where $\Lambda_{hl} = M/[M + n_{hl}]$ (see Appendix A.2 for the further details of this estimator).

Expression (5) has been previously derived by Lahiri & Tiwari (1991) under stratified sampling design without the consideration of very small number of PSUs per stratum and collapsing strata. Also, the authors did not provide any numerical results regarding the performance of their estimator. Here, for the sake of completeness of comparisons, we review this estimator.

After substituting the estimators of unknown parameters $\mu_0, \sigma_0^2$, and $M$ (with further details given in Appendix A.2) into expression (5), we can obtain the empirical Bayes estimator $\hat{\varrho}_{EB-NB}^{\text{EB-NB}}$ as follows

$$\hat{\varrho}_{EB-NB}^{\text{EB-NB}} = \frac{\hat{M} + n_{hl}}{\hat{M} + n_{hl} + 1} \left\{ \hat{\Lambda}_{hl} \hat{\alpha}_{hl} \hat{\sigma}_0^2 + (1 - \hat{\Lambda}_{hl}) \hat{\beta}_{hl} \frac{n_{hl} - 1}{n_{hl}} \hat{s}_{hl}^2 + \hat{\Lambda}_{hl} (1 - \hat{\Lambda}_{hl}) \hat{\gamma}_{hl} (\bar{y}_{hl} - \hat{\mu}_0)^2 \right\}.$$
To this end, the empirical Bayes estimator of nonparametric Bayes variance is

$$
var_{\text{Dir}} = \frac{1}{N^2} \sum_{(h \neq l) = 1}^{H} \frac{N_{hl}^2}{n_{hl}} (1 - f_{hl}) \hat{\varphi}_{hl}^{\text{EB-NB}}.
$$

(6)

5 Simulation Studies

In this section, we explore the performance of our proposed methods throughout the simulated data sets. For this purpose, we consider two simulated data sets. For the first simulated data set, we assume the super-population follows a normal distribution under different scenarios. For the second simulated data set, we assume the super-population follows a gamma distribution. The aim of two simulated data sets is to confirm that the HB variance estimator is superior regardless of the choice of super-population for generating the data sets.

5.1 Gaussian Super-Population

For the simulation study of this section, we follow the set up of Breidt et al. (2016) to generate a finite population with multiple response variables. This set up was used by Salinas et al. (2020) to propose a new calibration estimator. It also has been used in a sequence of papers such as Dahlke et al. (2013), Diana & Perri (2012), Singh (2012), and Breidt & Opsomer (2008) among others. We generate a stratified finite population with $H$ strata each of them with the size of $N_h \equiv NH^{-1} = 60$ such that we have seven survey variables of interest. For stratum $h$, $x_h = hH^{-1}$. For these seven variables, the mean function is equal to

$$
m^{(\ell)}_s(x) = 2 \frac{g_\ell(x) - \min_{x \in [0,1]} g_\ell(x)}{\max_{x \in [0,1]} g_\ell(x) - \min_{x \in [0,1]} g_\ell(x)}, \quad \text{for} \quad \ell = 1, ..., 7,
$$
where line: \( g_1(x) = 1 + 2(x - 0.5) \), quad: \( g_2(x) = 1 + 2(x - 0.5)^2 \), bump: \( g_3(x) = 1 + 2(x - 0.5) + \exp(-200(x - 0.5)^2) \), jump: \( g_4(x) = \{1 + 2(x - 0.5)\}1_{\{x \leq 0.65\}} + 0.65 \{x > 0.65\} \), expo: \( g_5(x) = \exp(-8x) \), cycle1: \( g_6(x) = 2 + \sin(2\pi x) \), and cycle4: \( g_7(x) = 2 + \sin(8\pi x) \).

In order to generate the values of the \( \ell \)-th survey variable, we have

\[
y_j^{(\ell)} = m_s^{(\ell)}(x_h) + e_j, \quad \text{for} \quad j \in N_h,
\]

such that \( e_j \sim N(0, \phi^2) \). Therefore,

\[
\bar{y}_h^{(\ell)} = m_s^{(\ell)}(x_h) + \frac{1}{N_h} \sum_{j \in N_h} e_j.
\]

For the rest of parameters, we assume \( \phi = \{0.25, 0.5, 5\} \), \( H = \{50, 100, 200\} \) and the sample size to be \( n_h = 1, 2, 3, 4, 5 \) per stratum. The bandwidth \( b \) can be chosen as \( \frac{1}{H} < b < \frac{2}{H} \) for the nonparametric kernel-based variance estimator to achieve the smallest possible nonempty kernel window (see Breidt et al. (2016) for the related discussion).

Here, we assume \( b \in \{1/40, 1/8, 1/6, 1/4\} \) following the Table given in Breidt et al. (2016). See Appendix A.3 for the further details of nonparametric kernel-based variance estimator. We consider the Horvitz and Thompson estimator of the mean given in (1) and make comparisons among 4 different methods of estimating the variance as follows:

1) collapsed variance estimator denoted by \( \text{var}_{\text{Coll}} \), 2) nonparametric kernel-based variance estimator of Breidt et al. (2016) denoted by \( \text{var}_{\text{Ker}} \), 3) nonparametric Bayes variance estimator denoted by \( \text{var}_{\text{Dir}} \), and 4) hierarchical Bayes variance estimator denoted by \( \text{var}_{\text{HB}} \). For collapsing the strata, we pair the adjacent strata based on the ordered values of \( x_h \)’s.

We conduct a Monte Carlo study with the size of \( R = 1000 \), where each replication contains an MCMC run of \( 2 \times 10^5 \) iterations for the MH accept-reject within Gibbs sampler algorithm for estimating the HB variance; see Appendix B for the further details of algorithm. Our two criteria of evaluation are absolute relative bias (RB) and relative root
MSE (RRMSE) defined as follows

\[
\text{RB}(\text{var}) = \frac{|E_R(\text{var}) - V(\bar{y}_{st})|}{V(\bar{y}_{st})}, \quad \text{and}
\]

\[
\text{RRMSE}(\text{var}) = \frac{\{E_R(\text{var}) - V(\bar{y}_{st})\}^{1/2}}{V(\bar{y}_{st})},
\]

where \(\text{var}\) is the estimator of variance based on the 4 methods given in the manuscript, and the expected values are with respect to the Monte Carlo study. The results of comparisons for 1 PSU and 2 PSUs are given in Tables 1–3. Based on the results, \(\text{var}_{HB}\) has small values of RB and particularly RRMSE compared to the other estimators. Additionally, the results based on cycle4 are often larger than the other scenarios. As the number of PSUs increases, the variance estimators become more accurate, i.e. both RB and RRMSE decrease. The results for 3, 4, and 5 PSUs are given in the Appendix C.

5.1.1 Choice of An Optimal “d”

In order to understand which value of \(d\) is suitable, we can select a sample (or a training data) from the entire real data set and investigate bias and MSE (or other desirable criteria) for a chosen grid values. The value from the grid which corresponds to the smallest bias and MSE can be considered as an optimal \(d\). Here, we have used our Gaussian super-population and have done a Monte Carlo study with as small as \(R = 50\) replications. Then, we have found the values of empirical RB and RRMSE on a fixed grid for all the 7 scenarios described in section 5.1. The results for 1 PSU are given in Figure 1. We observe the optimal value of \(d\) is pretty consistent across different scenarios particularly as \(\phi\) increases. When \(\phi\) is small, the optimal value of \(d\) for the scenario cycle4 is quite different.
Table 1: Comparisons among 4 different variance estimators for a Gaussian super-

population when $\phi = 0.25$. We assume $b = 1/40$ for $\text{var}_{\text{Ker}}$.

| PSUs | Variance Criterion of Scenario |
|------|--------------------------------|
|      | Evaluation line quad bump jump expo cycle1 cycle4 |
| 1    | $\text{var}_{\text{Coll}}$ |
|      | RB 0.007 0.066 0.053 0.000 0.013 0.068 0.925 |
|      | RRMSE 0.286 0.302 0.284 0.266 0.296 0.300 1.036 |
|      | $\text{var}_{\text{Ker}}$ |
|      | RB 0.010 0.001 0.002 0.092 0.004 0.004 0.112 |
|      | RRMSE 0.284 0.264 0.263 0.303 0.265 0.263 0.289 |
|      | $\text{var}_{\text{Dir}}$ |
|      | RB 0.326 0.286 0.295 0.330 0.295 0.285 0.289 |
|      | RRMSE 0.378 0.347 0.349 0.375 0.354 0.346 0.426 |
|      | $\text{var}_{\text{HB}}$ |
|      | RB 0.001 0.022 0.016 0.002 0.013 0.022 0.001 |
|      | RRMSE 0.102 0.105 0.101 0.097 0.101 0.105 0.114 |
| 2    | $\text{var}_{\text{Coll}}$ |
|      | RB 0.000 0.047 0.035 0.002 0.040 0.048 0.622 |
|      | RRMSE 0.161 0.175 0.169 0.161 0.173 0.177 0.664 |
|      | $\text{var}_{\text{Ker}}$ |
|      | RB 0.017 0.031 0.026 0.209 0.039 0.016 0.257 |
|      | RRMSE 0.280 0.283 0.281 0.401 0.286 0.277 0.384 |
|      | $\text{var}_{\text{Dir}}$ |
|      | RB 0.195 0.157 0.167 0.194 0.163 0.156 0.306 |
|      | RRMSE 0.234 0.207 0.213 0.233 0.212 0.208 0.359 |
|      | $\text{var}_{\text{HB}}$ |
|      | RB 0.000 0.021 0.017 0.001 0.016 0.024 0.001 |
|      | RRMSE 0.084 0.087 0.087 0.084 0.085 0.088 0.055 |

5.2 HMT Population

In order to conduct the simulation study for a different super-population rather than the
Gaussian one, we use HMT \{PracTools\} in R to generate a population based on the model
$\mathbb{E}(y) = \alpha + \beta x$, $V(y) = \sigma^2 x^{3/2}$, where both $x$ and $y$ have gamma distributions (see Valliant et al. (2020) for the further details). Under this function, strata are formed to approximately
have the same total of $x$. The HMT population has been firstly proposed by Hansen et al. (1983). This function can generate an $N \times 3$ matrix where each row represents a unique
observation, and three columns are as follows: (1) “strat” for the stratum ID, (2) “x”
for the auxiliary variable $x$, and (3) “y” for the analysis variable $y$. For our purpose, we
generate $N = 2000$ observations with $H = 20$ strata. Afterwards, we order the strata by
Table 2: Comparisons among 4 different variance estimators for a Gaussian super-population when $\phi = 0.5$. We assume $b = 1/40$ for $var_{Ker}$.

| PSUs | Variance | Scenario | Evaluation | line | quad | bump | jump | expo | cycle1 | cycle4 |
|------|----------|----------|------------|------|------|------|------|------|--------|--------|
| 1    | $var_{Coll}$ | RB | 0.011 | 0.006 | 0.002 | 0.011 | 0.006 | 0.012 | 0.201 |
|      |          | RRMSE | 0.267 | 0.273 | 0.269 | 0.266 | 0.272 | 0.275 | 0.378 |
|      | $var_{Ker}$ | RB | 0.001 | 0.003 | 0.002 | 0.018 | 0.005 | 0.002 | 0.030 |
|      |          | RRMSE | 0.263 | 0.264 | 0.263 | 0.271 | 0.264 | 0.263 | 0.266 |
|      | $var_{Dir}$ | RB | 0.341 | 0.333 | 0.330 | 0.363 | 0.367 | 0.322 | 0.198 |
|      |          | RRMSE | 0.383 | 0.376 | 0.375 | 0.398 | 0.399 | 0.371 | 0.290 |
|      | $var_{HB}$  | RB | 0.005 | 0.008 | 0.010 | 0.005 | 0.008 | 0.012 | 0.002 |
|      |          | RRMSE | 0.094 | 0.093 | 0.095 | 0.093 | 0.093 | 0.096 | 0.111 |
| 2    | $var_{Coll}$ | RB | 0.005 | 0.010 | 0.003 | 0.005 | 0.007 | 0.014 | 0.140 |
|      |          | RRMSE | 0.160 | 0.163 | 0.161 | 0.160 | 0.163 | 0.163 | 0.227 |
|      | $var_{Ker}$ | RB | 0.016 | 0.021 | 0.019 | 0.049 | 0.024 | 0.018 | 0.076 |
|      |          | RRMSE | 0.280 | 0.281 | 0.280 | 0.302 | 0.282 | 0.280 | 0.291 |
|      | $var_{Dir}$ | RB | 0.199 | 0.187 | 0.192 | 0.199 | 0.189 | 0.184 | 0.082 |
|      |          | RRMSE | 0.237 | 0.228 | 0.232 | 0.237 | 0.230 | 0.226 | 0.166 |
|      | $var_{HB}$  | RB | 0.018 | 0.025 | 0.023 | 0.018 | 0.024 | 0.006 | 0.033 |
|      |          | RRMSE | 0.087 | 0.088 | 0.088 | 0.087 | 0.088 | 0.084 | 0.084 |

$x_h = \log(\sum_{j=1}^{N_h} x_{hj})$ in stratum $h$ and select 1, 2, 3, 4, and 5 PSUs per stratum without replacement.

For this model-based simulation, we replicate the process of sample selection 1000 times ($R = 1000$) and estimate the variance per each replication based on the 4 methods given in the manuscript. The results of comparisons based on RB and RRMSE are given in Table 4. Overall, as the number of PSUs increases, both RB and RRMSE decrease. We observe that $var_{HB}$ outperforms compared to the rest of other estimators. Additionally, $var_{Ker}$ is given for $b \in \{1/8, 1/6, 1/4\}$. We observe that the values of RB and RRMSE for $var_{Ker}$ are similar for different $b$’s when the values are rounded to the 3rd decimal places per each scenario of sample selection.
Table 3: Comparisons among 4 different variance estimators for a Gaussian super-
population when $\phi = 5$. We assume $b = 1/40$ for $\text{var}_{\text{Ker}}$.

| $\phi = 5$ | Variance Estimator | Criterion of Evaluation | Scenario |
|------------|-------------------|-------------------------|----------|
|            |                   | line  | quad  | bump | jump | expo | cycle1 | cycle4 |
| 1          | $\text{var}_{\text{Coll}}$ | RB   | 0.012 | 0.012 | 0.012 | 0.011 | 0.011 | 0.013 |
|            |                   | RRMSE | 0.266 | 0.267 | 0.266 | 0.267 | 0.267 | 0.266 |
|            | $\text{var}_{\text{Ker}}$ | RB   | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
|            |                   | RRMSE | 0.263 | 0.263 | 0.263 | 0.263 | 0.263 | 0.263 |
|            | $\text{var}_{\text{Dir}}$ | RB   | 0.593 | 0.590 | 0.579 | 0.594 | 0.594 | 0.584 | 0.595 |
|            |                   | RRMSE | 0.600 | 0.600 | 0.590 | 0.601 | 0.602 | 0.593 | 0.603 |
|            | $\text{var}_{\text{HB}}$ | RB   | 0.003 | 0.003 | 0.004 | 0.003 | 0.003 | 0.004 | 0.003 |
|            |                   | RRMSE | 0.092 | 0.092 | 0.092 | 0.092 | 0.092 | 0.092 | 0.091 |
| 2          | $\text{var}_{\text{Coll}}$ | RB   | 0.005 | 0.004 | 0.005 | 0.005 | 0.004 | 0.004 | 0.005 |
|            |                   | RRMSE | 0.160 | 0.160 | 0.160 | 0.160 | 0.160 | 0.160 | 0.160 |
|            | $\text{var}_{\text{Ker}}$ | RB   | 0.016 | 0.016 | 0.014 | 0.017 | 0.016 | 0.017 | 0.017 |
|            |                   | RRMSE | 0.280 | 0.280 | 0.280 | 0.281 | 0.281 | 0.280 | 0.280 |
|            | $\text{var}_{\text{Dir}}$ | RB   | 0.209 | 0.210 | 0.209 | 0.210 | 0.210 | 0.208 | 0.209 |
|            |                   | RRMSE | 0.243 | 0.244 | 0.243 | 0.244 | 0.244 | 0.242 | 0.243 |
|            | $\text{var}_{\text{HB}}$ | RB   | 0.018 | 0.018 | 0.018 | 0.018 | 0.018 | 0.019 | 0.018 |
|            |                   | RRMSE | 0.086 | 0.087 | 0.086 | 0.086 | 0.086 | 0.087 | 0.086 |

6 Data Analysis

In this section, we explore the performance of our proposed methods with real data sets. The first data set comes from National Health and Nutrition Examination Survey 2007–8, where we treat it as our complete population and conduct a design-based simulation for evaluating the variance estimators. The second data set comes from the 1998 survey of mental health organizations in the PracTools package of R, and we estimate the variances based on the available data. All the results confirm the superiority of our proposed HB variance estimator.
Figure 1: Optimal $d$ selection plots under one PSU selection design. For $\phi = 0.25$, the line that stands apart from the rest is associated to the cycle4 scenario. By increasing $\phi$, cycle4 behaves similar to the rest of scenarios.

### 6.1 NHANES 2007–8

We conduct a design-based simulation study using a dietary intake data from the National Health and Nutrition Examination Survey (NHANES) 2007–8. This data has been previously used by [Liao & Valliant (2012)](#). The dietary intake data is used to estimate the types and amounts of foods and beverages consumed during the 24 hour period prior to the interview (midnight to midnight), and to estimate intakes of energy, nutrients, and other food components from those foods and beverages. The design is approximated by the stratified selection of 1 PSU or 2 PSUs within each stratum. The size of our available data is $N = 4329$ and the total number of strata is $H = 16$.

We consider this data set as our super-population and order the strata by $x_h = \log(N_h)$ in stratum $h$. Then, we select 1 PSU or 2 PSUs without replacement per stratum and repeat this process for $R = 1000$ times. Per each iteration of $r$-th where $r = 1, \ldots, R$, we estimate the variance based on 4 approaches given in this manuscript for 3 variables as
Table 4: Comparisons among 4 different variance estimators from \textit{HMT population}. Note that the values of $\text{var}_{\text{Ker}}$ are associated with $b \in \{1/8, 1/6, 1/4\}$.

| Number of PSUs | Criterion of Evaluation | $\text{var}_{\text{Coll}}$ | $\text{var}_{\text{Ker}}$ | $\text{var}_{\text{Dir}}$ | $\text{var}_{\text{HB}}$ |
|----------------|-------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1              | RB                      | 1.057                       | \{0.060, 0.060, 0.060\}     | 0.655                       | 0.012                       |
|                | RRMSE                   | 1.613                       | \{0.545, 0.545, 0.545\}     | 1.049                       | 0.261                       |
| 2              | RB                      | 0.771                       | \{0.090, 0.090, 0.090\}     | 0.867                       | 0.034                       |
|                | RRMSE                   | 1.017                       | \{0.491, 0.491, 0.491\}     | 1.048                       | 0.155                       |
| 3              | RB                      | 0.737                       | \{0.169, 0.169, 0.169\}     | 0.809                       | 0.001                       |
|                | RRMSE                   | 0.911                       | \{0.529, 0.529, 0.529\}     | 0.939                       | 0.112                       |
| 4              | RB                      | 0.711                       | \{0.224, 0.224, 0.224\}     | 0.774                       | 0.072                       |
|                | RRMSE                   | 0.830                       | \{0.582, 0.582, 0.582\}     | 0.869                       | 0.105                       |
| 5              | RB                      | 0.700                       | \{0.309, 0.309, 0.309\}     | 0.754                       | 0.066                       |
|                | RRMSE                   | 0.795                       | \{0.657, 0.657, 0.657\}     | 0.832                       | 0.093                       |

follows: DR1TCARB (total carbohydrate in gm), DR1TSUGR (total sugar in gm), and DR1TSFAT (total saturated fatty acids in gm). Distributions of the estimated variances for these 3 variables are given in Figure 2. Observe that $\text{var}_{\text{HB}}$ is closely centered around the true variance of population. Thus, it performs better than the other estimators. Other variance estimators have a highly right-skewed distribution with a significant number of outliers.

### 6.2 1998 SMHO Data

The 1998 Survey of Mental Health Organizations (SMHO) was conducted by the U.S. Substance Abuse and Mental Health Services Administration. It collected data on mental health care organizations and general hospitals that provide mental health care services for total expenditure, full-time equivalent staff, bed count, and total caseload by type of organization. This data set is in the \texttt{PracTools} package in \texttt{R}, and its size is $N = 875$ with 12 variables. We first eliminate outpatient facilities, i.e. only organizations with BEDS$>$0.
are remained. The original data set has 16 strata, and we have to combine some of them after the elimination step because of the remained small size per some strata. Thus, we create 6 new strata.

Here, we only consider two variables \( \text{BEDS} \) (total inpatient beds) and \( \text{EXPTOTAL} \) (total expenditures) and order strata by \( x_h = \log \text{ of total BEDS in stratum } h \). We select a simple random sample without replacement (SRSWOR) and a systematic sample with the size of 1 PSU and 2 PSUs per stratum and estimate the variances based on the 4 methods defined in this manuscript. For \( \text{var}_{\text{Ker}} \), we assume the bandwidth \( b = 1/8 \). Afterwards, we look at the coefficient of variation (cv) defined as \( \bar{y}_{\text{HT}} / \sqrt{\text{var}(\bar{y}_{\text{HT}})} \). The results are given in Table 5.

We observe that there are cases where we are unable to estimate the variance based on the kernel-based and nonparametric Bayes methods. For the kernel-based variance estimator, this is due to receiving \( C_d = 0 \), and for the nonparametric Bayes variance estimator, this is due to \( \hat{M}^{-1} = 0 \); see Appendices A.2 and A.3 for the further details of these variances. Additionally, we observe that the hierarchical Bayes variance estimator has the smallest value of cv among the rest.

Table 5: 1998 SMHO coefficient of variation results. “NA” means the variance cannot be calculated.

| Number of PSUs | Sample Selection | \( \text{var}_\text{Coll} \) | \( \text{var}_\text{Ker} \) | \( \text{var}_\text{Dir} \) | \( \text{var}_\text{HB} \) |
|---------------|-----------------|------------------|------------------|------------------|------------------|
| 1 SRSWOR      | 0.435           | NA               | NA               | 1.813e-8         |
| SYSTEMATIC    | 0.125           | NA               | 0.100            | 1.648e-8         |
| 2 SRSWOR      | 0.279           | NA               | 0.239            | 1.921e-8         |
| SYSTEMATIC    | 0.621           | NA               | NA               | 6.006e-9         |
Figure 2: Distribution of variance estimators based on a Monte Carlo study for 3 selected variables from the NHANES data set. The horizontal lines stand for the true variances of population.
7 Conclusion

Fine stratification design has been widely used in the literature of survey statistics. Even though, the estimator of an interested parameter is efficient, its variance under the design either does not exist or has a huge variation. A traditional method is collapsing the adjacent strata and then estimating the variance, but this estimator suffers from large MSEs. A number of alternative variance estimators have been proposed in the literature, but they often rely on some strong auxiliary variables well-correlated with the response variable, or they have a complex form, which make them inapplicable in the real life. Here, we propose a promising solution based on a hierarchical Bayes approach for this long-standing problem. We compare our proposed estimator with a number of estimators given in the literature. Our estimator has a simple form, and it outperforms compared to the rest of other estimators given in the manuscript.

Appendix A: Technical Details of Estimators

A.1 Proper Joint Posterior Distribution

The joint posterior distribution given in section 3.1 is a proper probability density function verified as follows:

\[
\int_0^{+\infty} \int_0^{+\infty} \ldots \int_0^{+\infty} \pi(S^2, \sigma^2 | s^2) \, d\sigma^2 \, dS_1 \ldots dS_H \\
\propto \prod_{(h \neq l)=1}^{H} \int_0^{+\infty} \frac{\{1 + \frac{1}{L_g+1}(\frac{S_{hl}^2}{d})^2\} - \frac{L_g+2}{2} \left\{ \frac{1}{2} \sum_{(h \neq l)=1}^{H} (\log s_{hl}^2 - \log S_{hl}^2)^2 + \zeta \right\}^{H/4+L_g}}{dS_{hl}} dS_{hl} \\
\leq \prod_{(h \neq l)=1}^{H} \int_0^{+\infty} \left\{ 1 + \frac{1}{L_g + 1} \left( \frac{S_{hl}^2}{d} \right)^2 \right\} - \frac{L_g+2}{2} \left\{ \frac{1}{2} \sum_{(h \neq l)=1}^{H} (\log s_{hl}^2 - \log S_{hl}^2)^2 + \zeta \right\}^{H/4+L_g} \frac{dS_{hl}}{(L_g + 1) \pi d} \right\}^{1/2} < +\infty,
\]

since \( d \) is positive and is bounded from the above.
A.2 NB Variance Estimator Details

In expression (5), \(\alpha_{hl}, \beta_{hl},\) and \(\gamma_{hl}\) are adjusting terms for taking into account the FPC as follows:

\[
\alpha_{hl} = (1 - f_{hl})[N_{hl} - \Lambda_{hl}]/N_{hl} - 1], \quad \beta_{hl} = [M + n_{hl} + 1]/[N_{hl} - 1] + (1 - f_{hl})\left\{1 + [1 - \Lambda_{hl}]/[N_{hl} - 1]\right\}, \quad \gamma_{hl} = (1 - f_{hl})[M + N_{hl}]/[N_{hl} - 1],
\]
where \(\Lambda_{hl} = M/[M + n_{hl}].\) If \(N_{hl} \to \infty,\) then \((1 - f_{hl}) \to 1\) and therefore all the terms \(\{\alpha_{hl}, \beta_{hl}, \gamma_{hl}\} \to 1,\) which gives the Bayes estimator of (5) for an infinite population. In expression (5), \((M, \mu_0, \sigma_0^2)\) are unknowns and should be estimated. Indeed, \(\mu_0\) and \(\sigma_0^2\) can be estimated by a function of \(\bar{y}_{hl}\) and \(s_{hl}^2\) from all the strata, respectively, and \(M\) can be estimated by a function of mean squared error within and between strata.

Following Ghosh & Lahiri (1987), the unknown \(M\) can be estimated as follows

\[
\hat{M}^{-1} = \max \left\{0, \left(\frac{(H - 1)\text{MSB}}{(H - 3)\text{MSW}} - 1\right)(H - 1)g^{-1}\right\},
\]

(A.1)

where \(g = \sum_{(h \neq l)=1}^H n_{hl} - \left\{\sum_{(h \neq l)=1}^H n_{hl}^2/\sum_{(h \neq l)=1}^H n_{hl}\right\},\) MSW and MSB are mean squared error within and between strata, respectively. Note that \(\hat{M}^{-1} \to_P M^{-1}\) as \(H \to \infty.\) Expression (A.1) is identical to the James-Stein estimator under the assumption of equality of \(n_{hl}\)'s (see Ghosh & Meeden (1986)). The only remained unknown parameters are \(\mu_0\) and \(\sigma_0^2,\) which can be estimated by

\[
\hat{\mu}_0 = \sum_{(h \neq l)=1}^H \frac{(1 - \hat{\Lambda}_{hl})}{\sum_{(h \neq l)=1}^H (1 - \hat{\Lambda}_{hl})} \bar{y}_{hl}, \quad \text{and} \quad \hat{\sigma}_0^2 = \frac{\hat{M} + 1}{\hat{M}} \sum_{(h \neq l)=1}^H \frac{(n_{hl} - 1)}{\sum_{(h \neq l)=1}^H (n_{hl} - 1)} s_{hl}^2.
\]

One can consult Lahiri & Tiwari (1991) for the Bayes risk and asymptotic optimality of the empirical Bayes estimator \(\hat{\varrho}_{EB-NB}^{EB-NB}.\)
### A.3 Kernel-Based Variance Estimator

For the sake of completeness, we briefly review the nonparametric kernel-based variance estimator of $\text{Var}_{\text{Ker}}$ given by [Breidt et al. (2016)](Breidt et al. 2016) as follows

$$
\text{Var}_{\text{Ker}} = \frac{1}{N^2 C_d} \sum_{h=1}^{H} \left( \sum_{j \in s_h} \frac{y_{hj}}{\pi_j} - \sum_{l=1}^{H} d_l(h) \left[ \sum_{j \in s_l} \frac{y_{lj}}{\pi_j} \right] \right)^2,
$$

where the kernel weight is $d_l(h) = K[\frac{x_h - x_l}{b}] / \sum_{l=1}^{H} K[\frac{x_h - x_l}{b}]$, such that $K[.]$ is a symmetric bounded kernel function and $b$ is the bandwidth parameter. The nonrandom normalizing constant $C_d$ is

$$
C_d = \frac{1}{H} \sum_{h=1}^{H} \left( 1 - 2d_h(h) + \sum_{l=1}^{H} d_l^2(h) \right).
$$

We use the Epanechnikov kernel $K(u) = 0.75(1 - u^2)1_{\{|u| \leq 1\}}$ for the kernel weight $d_l(h)$.

### Appendix B: Details of MCMC Algorithm

In this section, we provide some computational details on the MCMC algorithm used to approximate the posterior distribution of $S_{hl}^2$ in section 3. Posterior computation is performed using a Metropolis-Hastings accept-reject within Gibbs sampler algorithm described in section 3 of the paper. The details of this algorithm are given in Algorithm B.1 in what follows. The results are based on MCMC runs of $2 \times 10^5$ iterations, discarding the first 2000 samples as burn-in and then thinning every 200 iterations. We have looked at 4 chains with 4 different starting values of parameters, but we only include the results from chain 1 throughout the manuscript. The algorithms were implemented in R and a desktop computer with 9.7GB of RAM and an i7 Intel processor was used to perform the simulations.

In all cases, standard convergence diagnostics and traceplots did not show any significant mixing issues, and there were no correlations among $S_{hl}^2$’s of different strata. MCMC
runtimes for all the simulated data sets as well as real data sets are given in Table B.2 based on hour per run. The results are for 1 PSU sample selection design. In Table B.2 we consider two cases of $S_{hi}^2 \iid \sim \text{Half-t}(L_g + 1, d)$ and $S_{hi}^2 \iid \sim \text{Half-t}(L_g + 1, d)$. We observe that the run-time for $S_{hi}^2 \iid \sim \text{Half-t}(L_g + 1, d)$ is much smaller than the other case. Thus, we do not recommend the consideration of $S_{hi}^2 \iid \sim \text{Half-t}(L_g + 1, d)$ for Stage two of the hierarchical model given in section 3.

Figures B.1–B.5 show the traceplots and posterior distributions of $S_{hi}^2$ for some of the pseudo strata based on 4 MCMC chains. We do not observe any issues regarding the convergence of the chains from the plots. However, the mixing of the chains for the NHANES data is much slower compared to the rest. Table B.1 displays the estimated MCMC standard errors for the estimation of the expected posterior after discarding the first 2000 iterations of each run as a burn-in and thinning every 200 iterations. The MCMC standard errors were computed using the function summary.mcmc from the R package CODA (Plummer et al. (2006)). The estimated standard errors are all between 0.002 and 0.068, indicating that the $\text{var}_\text{HB}$ estimates are reliable enough. By increasing the number of iterations, the time-series MCMC error could relatively decrease. The results of Gaussian super-population are only for the line scenario. For the other scenarios, the results are almost the same as the ones given in this section.

Table B.1: Time-series MCMC error for the posterior expected values under one PSU selection design. The result for the Gaussian super-population is for the line scenario under the assumption of $\phi = 0.25$.

| Gaussian Super-Population | HMT Population | NHANES | 1998 SMHO |
|---------------------------|----------------|--------|----------|
| 0.002                     | 0.040          | 0.025  | 0.015    |
| Carbohydrate              | SRSWOR         | Sugar  | SYSTEMATIC |
| 0.068                     | 0.017          |        |          |
| 0.015 (Fatty Acids)       |                |        |          |
Algorithm B.1 MH Accept-Reject within Gibbs Sampler Algorithm

1: for $t = 1, 2, \ldots$ do
2: Initialize $S_{hl}^{2*} \sim \text{log-Normal}(\log s_{hl}^2, \sigma^2(t))$.
3: Set $S_{hl}^{2(t+1)} = S_{hl}^{2(t)} + (S_{hl}^{2*} - S_{hl}^{2(t)}) \log u < \rho,$
    \[ u \sim \text{Uniform}(u; 0, 1), \text{ and} \]
    \[ \rho = \log \pi(S_{hl}^{2*}|\text{rest}) + \log q(S_{hl}^{2(t)}|S_{hl}^{2*}) - \log \pi(S_{hl}^{2(t)}|\text{rest}) - \log q(S_{hl}^{2*}|S_{hl}^{2(t)}). \]
4: Generate $\sigma^{2(t+1)} \sim \text{Inv-Gamma}(L_g + \frac{H}{2L_g}, \zeta + \frac{1}{2} \sum_{(h \neq l)}^{H} \{\log s_{hl}^2 - \log S_{hl}^{2(t+1)}\}^2)$.
5: end for
6: return Samples $\{S_{hl}^{2(t+1)}\}_{t \geq 1}$ and $\{\sigma^{2(t+1)}\}_{t \geq 1}$.

Note that in Algorithm B.1, $\pi(S_{hl}^{2*}|\text{rest})$ and $\pi(S_{hl}^{2(t)}|\text{rest})$ follow the full conditional distribution (i) given in section 3.1. Additionally, $q(S_{hl}^{2(t)}|S_{hl}^{2*})$ and $q(S_{hl}^{2*}|S_{hl}^{2(t)})$ are candidate distributions following log-Normal($S_{hl}^{2(t)}; \log s_{hl}^2, \sigma^2(t)$) and log-Normal($S_{hl}^{2*}; \log s_{hl}^2, \sigma^2(t)$), respectively.

Table B.2: MCMC runtime for the posterior expected values under one PSU selection design. The results are based on 1 MCMC chain. The result of the Gaussian super-population is for the line scenario under the assumption of $\phi = 0.25$. “$>21$ hrs” means that the code was still busy after 21 hrs.

| Modeling Super-Population | Gaussian | HMT Population | NHANES | 1998 SMHO |
|---------------------------|----------|----------------|--------|-----------|
| Hierarchical model assuming $S_{hl}^2 \overset{iid}{\sim} \text{Half-t}(L_g + 1, d)$ | 0.021 hrs | 0.009 hrs | 0.007 hrs | 0.002 hrs |
| for all 3 variables       |          |                |        |           |
| Hierarchical model assuming $S_{hl}^2 \overset{iid}{\sim} \text{Half-t}(L_g + 1, d)$ | $>21$ hrs | 3.855 hrs | 2.112 hrs | 0.331 hrs |
| for all 3 variables       |          |                |        |           |
Figure B.1: Gaussian super-population (1 PSU). Trace plots and posterior distributions of $S_{hl}^2$ based on 4 MCMC chains for pseudo strata $g = 1, 10, \text{and } 25$ for the line scenario under the assumption of $\phi = 0.25$ and $H = 50$.

Figure B.2: HMT super-population (1 PSU). Trace plots and posterior distributions of $S_{hl}^2$ based on 4 MCMC chains for pseudo strata $g = 1, 5, \text{and } 10$ when $H = 20$. 

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Figure B.3: NHANES data (1 PSU). Trace plots and posterior distributions of $S_{hl}^2$ based on 4 MCMC chains for pseudo strata $g = 1$ and 8 when $H = 16$. 
Figure B.4: 1998 SMHO data (SRSWOR – 1 PSU). Trace plots and posterior distributions of $S^2_{hl}$ based on 4 MCMC chains for pseudo strata $g = 1, 2, \text{ and } 3$ when $H = 6$.

Figure B.5: 1998 SMHO data (SYSTEMATIC – 1 PSU). Trace plots and posterior distributions of $S^2_{hl}$ based on 4 MCMC chains for pseudo strata $g = 1, 2, \text{ and } 3$ when $H = 6$. 
Appendix C: Additional Results for the Gaussian Super-Population

For presenting the results of this Appendix, we consider an MCMC run of 25,000 iterations by discarding the first 500 samples as burn-in and thinning every 100 iterations which conserves the computing time while providing sufficient precision to be informative about the performance of variances. For the Monte Carlo study, we assume $R = 1000$. For the case of PSUs = 5, we assume $\zeta = 1$ in the final stage of hierarchical model given in section 3 to prevent the non-convergency issues and NA values for the marginal distribution of $\sigma^2$. This makes the values of RRMSE (particularly) for $var_{HB}$ become larger in comparison with the case of PSUs = 4, although it is still smaller than $var_{Coll}$.

Table C.1: Comparisons among 4 different variance estimators for a Gaussian super-population when $\phi = 0.25$ and $n_{hl} \geq 3$. We assume $b = 1/40$ for $var_{Ker}$.

| $\phi = 0.25$ | Variance Estimator | Criterion of Evaluation | Scenario | PSUs |
|----------------|---------------------|-------------------------|----------|------|
|                |                     |                         |          | 3    |
|                | $var_{Coll}$        | RB                      | 0.000    |
|                | RRMSE               |                         | 0.123    |
|                |                      |                         |          |
|                | $var_{Ker}$         | RB                      | 0.015    |
|                | RRMSE               |                         | 0.285    |
|                |                      |                         |          |
|                | $var_{Dir}$         | RB                      | 0.137    |
|                | RRMSE               |                         | 0.173    |
|                |                      |                         |          |
|                | $var_{HB}$          | RB                      | 0.004    |
|                | RRMSE               |                         | 0.058    |
|                |                      |                         |          |
|                |                     |                         |          |
|                | $var_{Coll}$        | RB                      | 0.001    |
|                | RRMSE               |                         | 0.102    |
|                |                      |                         |          |
|                | $var_{Ker}$         | RB                      | 0.047    |
|                | RRMSE               |                         | 0.276    |
|                |                      |                         |          |
|                | $var_{Dir}$         | RB                      | 0.104    |
|                | RRMSE               |                         | 0.138    |
|                |                      |                         |          |
|                | $var_{HB}$          | RB                      | 0.003    |
|                | RRMSE               |                         | 0.044    |
|                |                      |                         |          |
Table C.1 (continued)

| φ = 0.25 | Variance Estimator | Criterion of Evaluation | Scenario   |
|----------|---------------------|-------------------------|------------|
| 5        | varColl             | RB                      | 0.004 0.038 0.035 0.006 0.031 0.046 0.521 |
|          |                     | RRMSE                   | 0.087 0.098 0.096 0.086 0.095 0.102 0.535 |
|          | varKer              | RB                      | 0.067 0.104 0.093 0.602 0.124 0.083 0.704 |
|          |                     | RRMSE                   | 0.295 0.308 0.302 0.721 0.316 0.301 0.771 |
|          | varDir              | RB                      | 0.081 0.049 0.052 0.079 0.056 0.042 0.393 |
|          |                     | RRMSE                   | 0.113 0.096 0.097 0.111 0.100 0.093 0.409 |
|          | varHB               | RB                      | 0.004 0.016 0.012 0.008 0.006 0.020 0.003 |
|          |                     | RRMSE                   | 0.063 0.066 0.065 0.062 0.063 0.067 0.019 |

Table C.2: Comparisons among 4 different variance estimators for a Gaussian super-population when φ = 0.5 and nhl ≥ 3. We assume b = 1/40 for varKer.

| φ = 0.5 | Variance Estimator | Criterion of Evaluation | Scenario   |
|----------|---------------------|-------------------------|------------|
| 3        | varColl             | RB                      | 0.006 0.004 0.001 0.005 0.002 0.009 0.124 |
|          |                     | RRMSE                   | 0.123 0.127 0.122 0.123 0.127 0.125 0.186 |
|          | varKer              | RB                      | 0.014 0.021 0.018 0.076 0.023 0.017 0.107 |
|          |                     | RRMSE                   | 0.285 0.287 0.285 0.325 0.287 0.285 0.308 |
|          | varDir              | RB                      | 0.142 0.133 0.136 0.141 0.135 0.129 0.030 |
|          |                     | RRMSE                   | 0.177 0.172 0.172 0.177 0.174 0.168 0.123 |
|          | varHB               | RB                      | 0.008 0.012 0.012 0.008 0.010 0.014 0.008 |
|          |                     | RRMSE                   | 0.060 0.061 0.060 0.060 0.061 0.062 0.056 |
| 4        | varColl             | RB                      | 0.003 0.008 0.005 0.004 0.006 0.012 0.125 |
|          |                     | RRMSE                   | 0.102 0.104 0.103 0.101 0.104 0.105 0.170 |
|          | varKer              | RB                      | 0.046 0.057 0.051 0.122 0.061 0.049 0.171 |
|          |                     | RRMSE                   | 0.276 0.278 0.277 0.318 0.278 0.277 0.327 |
|          | varDir              | RB                      | 0.107 0.097 0.100 0.108 0.099 0.094 0.008 |
|          |                     | RRMSE                   | 0.141 0.135 0.136 0.141 0.136 0.133 0.103 |
|          | varHB               | RB                      | 0.001 0.003 0.003 0.001 0.001 0.005 0.003 |
|          |                     | RRMSE                   | 0.044 0.045 0.044 0.044 0.045 0.045 0.043 |
| 5        | varColl             | RB                      | 0.001 0.008 0.006 0.001 0.006 0.013 0.119 |
|          |                     | RRMSE                   | 0.086 0.088 0.087 0.086 0.087 0.089 0.151 |
|          | varKer              | RB                      | 0.066 0.079 0.072 0.167 0.084 0.070 0.226 |
|          |                     | RRMSE                   | 0.294 0.298 0.296 0.355 0.299 0.296 0.372 |
|          | varDir              | RB                      | 0.085 0.076 0.078 0.085 0.079 0.072 0.025 |
|          |                     | RRMSE                   | 0.116 0.111 0.112 0.116 0.112 0.108 0.089 |
|          | varHB               | RB                      | 0.002 0.004 0.003 0.002 0.002 0.008 0.011 |
|          |                     | RRMSE                   | 0.063 0.063 0.064 0.063 0.063 0.064 0.054 |
Table C.3: Comparisons among 4 different variance estimators for a Gaussian super-population when $\phi = 5$ and $n_{hl} \geq 3$. We assume $b = 1/40$ for $\text{var}_{\text{Ker}}$.

| $\phi = 5$ | Variance Criterion of Scenario | PSUs | Estimator | Evaluation | line | quad | bump | jump | expo | cycle1 | cycle4 |
|------------|--------------------------------|------|-----------|------------|------|------|------|------|------|--------|--------|
| 3          |                                |      | $\text{var}_{\text{Coll}}$ | RB         | 0.007 | 0.007 | 0.007 | 0.007 | 0.007 | 0.006  | 0.007  |
|            |                                |      | $\text{var}_{\text{Ker}}$   | RB         | 0.014 | 0.014 | 0.014 | 0.011 | 0.014 | 0.014  | 0.015  |
|            |                                |      | $\text{var}_{\text{Dir}}$   | RB         | 0.143 | 0.143 | 0.143 | 0.143 | 0.143 | 0.142  | 0.144  |
|            |                                |      | $\text{var}_{\text{HB}}$    | RB         | 0.008 | 0.008 | 0.008 | 0.008 | 0.008 | 0.008  | 0.008  |
| 4          |                                |      | $\text{var}_{\text{Coll}}$ | RB         | 0.004 | 0.003 | 0.004 | 0.004 | 0.003 | 0.003  | 0.004  |
|            |                                |      | $\text{var}_{\text{Ker}}$   | RB         | 0.046 | 0.046 | 0.046 | 0.046 | 0.046 | 0.046  | 0.047  |
|            |                                |      | $\text{var}_{\text{Dir}}$   | RB         | 0.108 | 0.108 | 0.108 | 0.108 | 0.108 | 0.107  | 0.108  |
|            |                                |      | $\text{var}_{\text{HB}}$    | RB         | 0.002 | 0.001 | 0.001 | 0.002 | 0.001 | 0.001  | 0.001  |
| 5          |                                |      | $\text{var}_{\text{Coll}}$ | RB         | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002  | 0.003  |
|            |                                |      | $\text{var}_{\text{Ker}}$   | RB         | 0.065 | 0.066 | 0.065 | 0.061 | 0.066 | 0.065  | 0.067  |
|            |                                |      | $\text{var}_{\text{Dir}}$   | RB         | 0.086 | 0.086 | 0.086 | 0.086 | 0.086 | 0.086  | 0.086  |
|            |                                |      | $\text{var}_{\text{HB}}$    | RB         | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003  | 0.003  |
|            |                                |      | RRMSE                  |            | 0.014 | 0.014 | 0.014 | 0.011 | 0.014 | 0.014  | 0.015  |
|            |                                |      | RRMSE                  |            | 0.275 | 0.275 | 0.275 | 0.274 | 0.275 | 0.275  | 0.276  |
|            |                                |      | RRMSE                  |            | 0.108 | 0.108 | 0.108 | 0.108 | 0.108 | 0.107  | 0.108  |
|            |                                |      | RRMSE                  |            | 0.002 | 0.001 | 0.001 | 0.002 | 0.001 | 0.001  | 0.001  |
|            |                                |      | RRMSE                  |            | 0.086 | 0.086 | 0.086 | 0.086 | 0.086 | 0.085  | 0.087  |
|            |                                |      | RRMSE                  |            | 0.117 | 0.117 | 0.117 | 0.117 | 0.117 | 0.116  | 0.117  |
|            |                                |      | RRMSE                  |            | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003  | 0.003  |
|            |                                |      | RRMSE                  |            | 0.063 | 0.063 | 0.063 | 0.063 | 0.063 | 0.063  | 0.063  |
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