NONSINGULAR H-TENSOR AND ITS CRITERIA

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Abstract. H-tensor is a new developed concept in tensor analysis and it is an extension of H-matrix and M-tensor. Based on the spectral theory of non-negative tensors, several equivalent conditions of nonsingular H-tensors are established in the literature. However, these conditions can not be used as a criteria to identify nonsingular H-tensors as they are hard to verify. In this paper, based on the diagonal product dominance and S diagonal product dominance of a tensor, we establish some new implementable criteria in identifying nonsingular H-tensors. The positive definiteness of nonsingular H-tensors with positive diagonal entries is also discussed in this paper. The obtained results extend the corresponding conclusions for nonsingular H-matrices and improve the existing results for nonsingular H-tensors.

1. Introduction. A high order tensor is a multi-way array whose entries are addressed via multiple indices in the following form [4, 16]

$$\mathcal{A} = (A_{i_1i_2\cdots i_m}), \ A_{i_1i_2\cdots i_m} \in \mathbb{R}, \ i_j = 1, 2, \cdots, n_j, \ j = 1, 2, \cdots, m.$$  

If the dimensions of tensor $\mathcal{A}$ in all directions are equal, i.e., $n_1 = n_2 = \cdots = n_m$, then $\mathcal{A}$ is called a square tensor, otherwise it is called a rectangular tensor. For square tensor $\mathcal{A}$, if all the entries $A_{i_1i_2\cdots i_m}$ are invariant under any permutation of their indices $\{i_1i_2\cdots i_m\}$, then tensor $\mathcal{A}$ is called symmetric [15].

Tensors have wide applications in signal and image processing, continuum physics, higher-order statistics, blind source separation and exploratory multi-way data analysis [4, 16, 22, 25]. Hence, tensor analysis and computing have received much attention of researchers in recent decade and the research branches include spectral hypergraph theory [13, 29], best low-rank approximation to tensors [15, 17, 37], positive definiteness identification of even order homogenous polynomials [12, 18, 24, 31], (sparse) tensor decomposition [1, 7, 16, 26, 30], tensor completing [9, 19, 21] and so on [3, 31, 32, 33].

Generally, tensor is a higher-order extension of matrix, and hence many concepts and related properties for matrices such as determinant, eigenvalue and singular...
value theory can be extended to higher order tensors by exploring their multilinear algebra properties [4, 12, 7, 28]. Besides, the matrices with some special structures such as symmetric matrices, nonnegative matrices, Z-matrices, \(B(B_0)\)-matrices, \(P(P_0)\)-matrices \(M(M_0)\)-matrices and \(H\)-matrices can also be extended to higher order tensors and these are becoming a hot topic of recent research on tensors, see e.g., [2, 18, 23, 29, 34, 35] and references there in.

It is well known that \(M\)-tensor is defined based on \(Z\)-tensor and its algebra properties can be explored by virtue of the spectral theory of nonnegative tensors [36]. A typical feature of a nonsingular \(M\)-tensor is that the minimal value of the real parts of all eigenvalues of \(M\)-tensor is positive and tensor’s diagonal dominance can guarantee that a \(Z\)-tensor is a nonsingular \(M\)-tensor [18, 36]. Motivated by equivalent definitions of \(M\)-matrices, some new equivalent definitions of \(M\)-tensors are established and their properties are further explored via semi-positiveness and monotonicity in [8]. From the semi-positiveness of nonsingular \(M\)-tensors, the diagonal entries of a nonsingular \(M\)-tensor are all positive.

A closely related conception with \(M\)-tensor is \(H\)-tensor which is defined on the comparison tensor [8, 14, 18]. Certainly, \(H\)-tensor is also an extension of \(H\)-matrix and it contains \(M\)-tensor as special cases [36]. Some equivalent conditions for \(H\)-tensors are presented in [8] and some criteria in identifying nonsingular \(H\)-tensors are provided in [18]. However, just as \(H\)-matrices, these equivalent conditions are not appropriate to be used as criteria in identifying \(H\)-tensors as these conditions are hard to verify. This stimulates us to establish some new “checkable” criteria to fulfill this work and this constitutes the motivation of this paper. In this paper, we establish some new implementable criteria in identifying nonsingular \(H\)-tensors by exploring the diagonal product dominance of tensors. We further discuss the heredity of the principal subtensor of nonsingular \(H\)-tensors and the positive definiteness of nonsingular \(H\)-tensors with positive diagonal entries. The obtained results extend the corresponding results for nonsingular \(H\)-matrices [5, 6] and improve the existing results for nonsingular \(H\)-tensors [8, 18, 36].

To end this section, we give the notation needed in this paper. We use bold and calligraphic letters, say, \(\mathbf{A}, \mathbf{B}, \mathbf{M}, \mathbf{I}\), to denote higher order tensors, and use \(\mathbf{A}_{i_1i_2\cdots i_m}\) to denote the \(i_1i_2\cdots i_m\)-th entry of tensor \(\mathbf{A}\); we use bold uppercase letters, say, \(\mathbf{A}, \mathbf{B}\), to denote matrices, and use \(\mathbf{A}_{ij}\) to denote the \(ij\)-th entry of matrix \(\mathbf{A}\); we use bold and lower case letters, say, \(\mathbf{x}, \mathbf{y}\), to refer to vectors and use \(\mathbf{x}_i\) to denote the \(i\)-th entry of vector \(\mathbf{x}\). We use lower case letters and Greek alphabet, say \(c, \lambda\), to refer to scalars. We write \(\mathbf{A} \geq \mathbf{0}\) or \(\mathbf{x} \geq \mathbf{0}\) to mean that every entry of tensor \(\mathbf{A}\) or vector \(\mathbf{x}\) is nonnegative, and we write \(\mathbf{A} > \mathbf{0}\) or \(\mathbf{x} > \mathbf{0}\) to mean that every entry of \(\mathbf{A}\) or \(\mathbf{x}\) is positive. We write \(|\mathbf{A}|\) to denote the tensor whose each entry is the modulus of corresponding entry in tensor \(\mathbf{A}\), and write \(N\) to denote set \(\{1, 2, \cdots, n\}\). We use \(\mathbf{e}\) to denote the vector of all ones and use \(\mathbf{e}_i\) to denote the \(i\)-th unit vector. We use \(\text{Re}(\cdot)\) to denote the real part of a complex number and use \(\text{diag}(\mathbf{x})\) to denote the diagonal matrix with diagonal entries \(x_i, i = 1, 2, \cdots, n\).

Usually, tensors considered in this paper are square tensors of order \(m\) and dimension \(n\). The product of tensor \(\mathbf{A}\) and matrix \(\mathbf{X} \in \mathbb{R}^{n \times n}\) on mode-\(k\) [16] is defined as
\[
(\mathbf{A} \times_k \mathbf{X})_{i_1i_2\cdots i_m} = \sum_{i_k = 1}^n \mathbf{A}_{i_1i_2\cdots i_k \cdots i_m} \mathbf{X}_{i_kj_k}
\]
and we denote
\[
\mathbf{A} \mathbf{X}^{m-1} = \mathbf{A} \times_2 \mathbf{X} \times_3 \mathbf{X} \cdots \times_m \mathbf{X}.
\]
For tensor $\mathbf{A}$ and vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ with entries

$$(\mathbf{A}\mathbf{x}^{m-1})_i = \sum_{i_1, i_2, \cdots, i_m=1}^{n, n, \cdots, n} A_{i_1 i_2 i_3 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \cdots, n,$$

and $\mathbf{A}\mathbf{x}^m$ is a scalar with

$$\mathbf{A}\mathbf{x}^m = \sum_{i_1, i_2, \cdots, i_m=1}^{n, n, \cdots, n} A_{i_1 i_2 i_3 \cdots i_m} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_m}.$$

We use $\mathbf{I}$ to denote $m$-th order $n$-dimensional identity tensor with entries

$$I_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise}, \end{cases}$$

and define the following $m$-order Kronecker delta

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases}$$

The remainder of this paper is organized as follows. Some preliminaries about $M$-tensors, $H$-tensors and their properties are presented in Section 2. Based on the tensors diagonal product dominance and $S$ diagonal product dominance, some new implementable criteria in identifying nonsingular $H$-tensors are established in Section 3. Further properties of nonsingular $H$-tensors with positive diagonal entries are investigated in Section 4. Some conclusions are drawn in the last section.

2. $M$-tensors, $H$-tensors and their properties. We first present some definitions developed in tensor analysis [2, 20, 28] and then introduce some kinds of specially structured tensors.

For a real $m$-th order $n$-dimensional tensor $\mathbf{A}$ and a scalar $\lambda \in \mathbb{C}$, if there exists nonzero vector $\mathbf{x} \in \mathbb{C}^n$ such that

$$\mathbf{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathbf{x}^{[m-1]} \in \mathbb{C}^n$ with $(\mathbf{x}^{[m-1]})_i = x_i^{m-1}, i = 1, 2, \cdots, n$, then $\lambda$ is said to be an eigenvalue of tensor $\mathbf{A}$ and $\mathbf{x}$ an eigenvector associated with eigenvalue $\lambda$. In particular, If $\mathbf{x}$ is real, then $\lambda$ is also real, and we say $(\lambda, \mathbf{x})$ is an $H$-eigenpair of tensor $\mathbf{A}$. The largest modulus of eigenvalue of tensor $\mathbf{A}$ is called the spectral radius of tensor $\mathbf{A}$ and we denote it by $\rho(\mathbf{A})$.

Motivated by the characteristics of nonsingular matrices, we say a square tensor is nonsingular if its all eigenvalues are nonzero. The following conclusion from [28] shows the linearity of the eigenvalue on the linear combination a square tensor and the identity tensor.

**Lemma 2.1.** For any square tensor $\mathbf{A}$, scalars $k$ and $b$, $\lambda \in \mathbb{R}$ is an eigenvalue of tensor $\mathbf{A}$ if and only if $k\lambda + b$ is an eigenvalue of tensor $k\mathbf{A} + b\mathbf{I}$. Furthermore, they have the same eigenvectors.

When $m$ is even and $\mathbf{A}$ is symmetric, tensor $\mathbf{A}$ is called positive (semi-)definite if $(\mathbf{A}\mathbf{x}^m \geq 0, \mathbf{Ax}^m > 0$ for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$. By virtue of $H$-eigenvalue, we have the following equivalent condition for positive (semi-)definiteness of a symmetric tensor (Theorem 5, [28]).

**Theorem 2.1.** An even order square symmetric tensor is positive (semi-)definite if and only if all of its $H$-eigenvalues are (nonnegative) positive.
The following concept plays an important role in spectral analysis of nonnegative tensors [2].

**Definition 2.1.** An \(m\)-th order \(n\)-dimensional tensor \(A\) is called reducible, if there exists a non-empty proper index subset \(I \subset N\) such that

\[
A_{i_1i_2\cdots i_m} = 0, \quad \forall \ i_1 \in J, \ i_2, i_3, \cdots, i_m \in \bar{J} = N \setminus J.
\]

Otherwise, tensor \(A\) is called irreducible.

The following specially structured tensors are extended from matrices [8, 36].

**Definition 2.2.** Tensor \(A\) is said to be a \(Z\)-tensor if it can be written as

\[
A = cI - B,
\]

where \(c > 0\) and \(B\) is a nonnegative tensor. Furthermore, if \(c \geq \rho(B)\), then \(A\) is said to be an \(M\)-tensor, and if \(c > \rho(B)\), then \(A\) is said to be a nonsingular \(M\)-tensor.

It is easy to see that all the off diagonal entries of a \(Z\)-tensor are non-positive. From Lemma 2.1, scalar \(c\) can be any scalar not less than \(\max_{1 \leq i \leq n} |A_{ii\cdots i}|\). The following conclusion gives two equivalent conditions for a \(Z\)-tensor to be a nonsingular \(M\)-tensor [8, 36].

**Proposition 2.1.** Let \(A\) be a \(Z\)-tensor. Then it is a nonsingular \(M\)-tensor if and only if one of the following conditions holds.

1. The real part of any eigenvalue of tensor \(A\) is positive;
2. There exists positive vector \(x \in \mathbb{R}^n\) such that \(Ax^{n-1} > 0\).

Inspired by comparison matrices and \(H\)-matrices [10], comparison tensors and \(H\)-tensors are introduced in [8].

**Definition 2.3.** For \(m\)-th order \(n\)-dimensional tensor \(A\), its comparison tensor, denoted by \(M_A\), is defined as

\[
(M_A)_{i_1i_2\cdots i_m} = \begin{cases} 
|A_{i_1i_2\cdots i_m}|, & \text{if } i_1 = i_2 = \cdots = i_m, \\
-|A_{i_1i_2\cdots i_m}|, & \text{otherwise}.
\end{cases}
\]

Clearly, for any square tensor \(A\), its comparison tensor is a \(Z\)-tensor. Thus, tensor \(A\) can be characterized via \(M\)-tensor and hence we have the following definition.

**Definition 2.4.** If comparison tensor \(M_A\) of tensor \(A\) is an \(M\)-tensor, then tensor \(A\) is called an \(H\)-tensor, and if comparison tensor \(M_A\) is a nonsingular \(M\)-tensor, then tensor \(A\) is called a nonsingular \(H\)-tensor.

From the relationship of \(M\)-tensors and \(H\)-tensors, Proposition 2.1 can be used to characterize \(H\)-tensors. However, its conditions are hard to verify and here we introduce the following popular condition in tensor analysis [28, 34, 36].

**Definition 2.5.** Tensor \(A\) is called diagonally dominant if

\[
|A_{ii\cdots i}| \geq \sum_{i_2, \cdots, i_m \neq ii\cdots i} |A_{i_2i_3\cdots i_m}|, \quad \forall \ i = 1, 2, \cdots, n,
\]

and tensor \(A\) is called strictly diagonally dominant if all the inequalities hold with strict inequality.

By virtue of the diagonal dominance, Theorem 3.15 in [36] provides two criteria in identifying nonsingular \(M\)-tensors. Based on the relationship of \(M\)-tensors and \(H\)-tensors, these criteria can be applied to nonsingular \(H\)-tensors and hence we obtain the following conclusions (also see Lemmas 7,8 in [18]).
Theorem 2.2. If square tensor \( A \) is strictly diagonally dominant or it is irreducible and diagonally dominant with at least one strict inequality holding in (2.1), then it is a nonsingular \( H \)-tensor.

To explore criteria with weaker conditions for nonsingular \( H \)-tensor, we give the following definition [8].

Definition 2.6. Tensor \( A \) is said to be generalized strictly diagonally dominant if there exists positive diagonal matrix \( D \) such that \( AD^{m-1} \) is strictly diagonally dominant.

Based on this definition, we have the following equivalent condition of nonsingular \( H \)-tensors [8]. To make the paper self-contained, we give its proof.

Proposition 2.2. Tensor \( A \) is a nonsingular \( H \)-tensor if and only if \( A \) is generalized strictly diagonally dominant.

Proof. Necessity. If \( A \) is a nonsingular \( H \)-tensor, then its comparison tensor \( M_A \) is a nonsingular \( M \)-tensor. From Proposition 2.1, we know that there exists vector \( x > 0 \) such that \( M_A x^{m-1} > 0 \). Let \( D = \text{diag}(x) \), then

\[
M_A x^{m-1} = M_A (D e)^{m-1} = (M_A D^{m-1}) e^{m-1} > 0.
\]

This means that tensor \( M_A D^{m-1} \) is strictly diagonally dominant and hence tensor \( AD^{m-1} \) is strictly diagonally dominant. Thus, \( A \) is generalized diagonally dominant.

Sufficiency. If tensor \( A \) is generalized diagonally dominant, then there exists positive diagonal matrix \( D \) such that tensor \( AD^{m-1} \) is strictly diagonally dominant. Hence,

\[
(M_A D^{m-1}) e^{m-1} = M_A (D e)^{m-1} > 0.
\]

Let \( x = D e \). Then \( x > 0 \) and \( M_A x^{m-1} > 0 \). Using Proposition 2.1 again, we know that \( M_A \) is a nonsingular \( M \)-tensor and hence \( A \) is a nonsingular \( H \)-tensor.

From the proposition, we can readily obtain the following conclusion.

Corollary 2.1. For square tensor \( A \), if there exists a positively diagonal matrix \( D \in \mathbb{R}^{n \times n} \) such that \( AD^{m-1} \) is a nonsingular \( H \)-tensor, then \( A \) is a nonsingular \( H \)-tensor.

Proof. Suppose \( AD^{m-1} \) is a nonsingular \( H \)-tensor. Then from Proposition 2.2, \( AD^{m-1} \) is generalized strictly diagonally dominant. Thus, there exists positive diagonal matrix \( G \in \mathbb{R}^{n \times n} \) such that \( (AD^{m-1}) G^{m-1} \), i.e., \( A(DG)^{m-1} \) is strictly diagonally dominant. Clearly, \( DG \) is also a positive diagonal matrix. Using Proposition 2.2 again, we conclude that \( A \) is a nonsingular \( H \)-tensor.

Combining Proposition 2.2 with Theorem 2.2, we have the following conclusion.

Proposition 2.3. If tensor \( A \) is irreducible and diagonally dominant with at least one strict inequality holding in (2.1), then it is generalized diagonally dominant.

3. Criteria for nonsingular \( H \)-tensors. Under the diagonal dominance assumption, Zhang and Li et al. [18, 36] establish some sufficient conditions for a square tensor to be a nonsingular \( M \)-tensor which can be applied to nonsingular \( H \)-tensors, i.e. Theorem 2.2. In this section, we will weaken these conditions from two aspects, one is based on the diagonal product dominance of tensors motivated by the sufficient conditions for nonsingular \( H \)-matrices established in [27], and the other is
based on the $S$ diagonal product dominance of tensor inspired by criteria for non-singular $H$-matrices [5, 6].

First, we have the following conclusion which has some relevance with Lemma 10 in [18].

**Theorem 3.1.** For $m$-th order $n$-dimensional tensor $\mathcal{A}$, if

$$|A_{i_1\cdots i_m}|^{m-1}|A_{j_1\cdots j_m}| > r_i^{m-1}(\mathcal{A})r_j(\mathcal{A}), \quad \forall i, j \in N, i \neq j,$$

where $r_i(\mathcal{A}) = \sum_{i_2i_3\cdots i_m \neq ii\cdots i} |A_{i_2i_3\cdots i_m}|$, then $\mathcal{A}$ is a nonsingular $H$-tensor.

**Proof.** For simplicity, we assume that the comparison tensor of tensor $\mathcal{A}$ is itself, i.e., $\mathcal{M}_A = \mathcal{A}$. Then, $A_{ii\cdots i} > 0$ from the assumption and $A_{i_1i_2\cdots i_m} \leq 0$ if $\delta_{i_1i_2\cdots i_m} = 0$. Hence, $\mathcal{A}$ is a $Z$-tensor and we only need to show it is a nonsingular $M$-tensor. From Proposition 2.1, we only need to show that the real part of any eigenvalue of tensor $\mathcal{A}$ is positive under the assumption.

Let $(\lambda, \mathbf{x})$ be an eigenpair of tensor $\mathcal{A}$. Then

$$\quad (\lambda - A_{i_1\cdots i})\mathbf{x}^{m-1} = \sum_{i_2i_3\cdots i_m \neq ii\cdots i} A_{i_2i_3\cdots i_m}\mathbf{x}_{i_2}\mathbf{x}_{i_3}\cdots \mathbf{x}_{i_m}, \quad \forall i \in N. \quad (3.2)$$

Since $\mathbf{x} \neq 0$, we can take the largest two entries in magnitude of eigenvector $\mathbf{x}$, say $\mathbf{x}_{i_0}$ and $\mathbf{x}_{j_0}$. Then

$$|x_{i_0}| \geq |x_{j_0}| \geq |x_i|, \quad \forall i \in N \text{ and } i \neq i_0, j_0. \quad (3.3)$$

If $\mathbf{x}_{j_0} = 0$, then $\mathbf{x} = \mathbf{x}_{i_0}e^{i_0}$. From equation (3.2), we deduce that $\lambda = A_{i_0i_0\cdots i_0} > 0$ and the conclusion follows.

Now, we assume that $\mathbf{x}_{j_0} \neq 0$. Consider equation (3.2) for $i = i_0$. Taking the model of both sides of the equation and dividing both sides by $|x_{i_0}^{m-1}|$, then we obtain from (3.3) that

$$\quad |\lambda - A_{i_0i_0\cdots i_0}| = |\sum_{i_2i_3\cdots i_m \neq ii\cdots i_0} A_{i_0i_2i_3\cdots i_m}\mathbf{x}_{i_2}\mathbf{x}_{i_3}\cdots \mathbf{x}_{i_m}x_{i_0}^{-m}| \leq \sum_{i_2i_3\cdots i_m \neq ii\cdots i_0} |A_{i_0i_2i_3\cdots i_m}| |x_{i_2}| |x_{i_3}| \cdots |x_{i_m}| x_{i_0}^{-m} \leq r_{i_0}(\mathcal{A})|x_{i_0}|^{-m} \quad (3.4)$$

Now, consider equation (3.2) for $i = j_0$. Taking the model of both sides of (3.2) and dividing both sides by $|x_{j_0}^{m-1}|$, then one has from (3.3) that

$$\quad |\lambda - A_{j_0j_0\cdots j_0}| = |\sum_{i_2i_3\cdots i_m \neq j_0j_0\cdots j_0} A_{j_0i_2i_3\cdots i_m}\mathbf{x}_{i_2}\mathbf{x}_{i_3}\cdots \mathbf{x}_{i_m}x_{j_0}^{-m}| \leq \sum_{i_2i_3\cdots i_m \neq j_0j_0\cdots j_0} |A_{j_0i_2i_3\cdots i_m}| |x_{i_2}| |x_{i_3}| \cdots |x_{i_m}| x_{j_0}^{-m} \leq r_{j_0}(\mathcal{A})|x_{j_0}|^{-m} \quad (3.5)$$

Combining (3.4) with (3.5) yields,

$$|\lambda - A_{i_0i_0\cdots i_0}|^{m-1}|\lambda - A_{j_0j_0\cdots j_0}| \leq r_{i_0}^{m-1}(\mathcal{A})r_{j_0}(\mathcal{A}).$$

Using the facts that

$$|\text{Re}(\lambda) - A_{i_0i_0\cdots i_0}|^{m-1}|\text{Re}(\lambda) - A_{j_0j_0\cdots j_0}| \leq |\lambda - A_{i_0i_0\cdots i_0}|^{m-1}|\lambda - A_{j_0j_0\cdots j_0}|$$
and $A_{i_0i_0\cdots i_0} > 0, A_{j_0j_0\cdots j_0} > 0$, we conclude that $\text{Re}(\lambda) > 0$. Hence $A$ is a nonsingular $M$-tensor from Proposition 2.1.

Certainly, a strictly diagonally dominant tensor satisfies the assumption of Theorem 3.1. However, the converse does not necessarily hold and even it is not diagonally dominant as shown from the following example. Thus, the conclusion improves the result established in [36].

Consider 3-order 2-dimensional tensor $A$ with entries

\[
A_{111} = 4, \quad A_{112} = -1, \quad A_{121} = 0, \quad A_{122} = -1.
\]

For this tensor, $r_1(A) = 2$ and $r_2(A) = 4$. The tensor is not diagonally dominant as $A_{222} = 3 < r_2(A)$. However,

\[
A_{222}^2 = 4^2 \times 3 = 48 > r_1^2(A)r_2(A) = 4 \times 4 = 16
\]

and

\[
A_{111}A_{222} = 4 \times 3^2 = 36 > r_1(A)r_2^2(A) = 2 \times 16 = 32.
\]

From Theorem 3.1, we conclude that tensor $A$ is a nonsingular $H$-tensor.

Now, we turn to consider another kind of tensor diagonal product dominance.

Let $S$ be a subset of $N$ and $\bar{S} = N \setminus S$. Then we define the following multiple index sets

\[
\Lambda = \{i_2i_3\cdots i_m \mid i_k \in S \text{ for any } k = 2, 3, \cdots, m\},
\]

\[
\bar{\Lambda} = \{i_2i_3\cdots i_m \mid i_k \in \bar{S} \text{ for some } k = 2, 3, \cdots, m\}.
\]

Based on the above sets, we split the sum $r_i(A)$ of tensor $A = (A_{i_1i_2\cdots i_m})$ into two parts

\[
r_i^\Lambda(A) = \sum_{i_2i_3\cdots i_m \notin \Lambda} |A_{i_2i_3\cdots i_m}|, \quad r_i^\bar{\Lambda}(A) = \sum_{i_2i_3\cdots i_m \in \bar{\Lambda}} |A_{i_2i_3\cdots i_m}|.
\]

For tensor $A = (A_{i_1i_2\cdots i_m})$ and a partition $(S, \bar{S})$ of $N$, denote

\[
h_i(A) = \frac{|A_{i_1i_2\cdots i_m}| - r_i^\Lambda(A)}{r_i^\Lambda(A)}, \quad i \in S; \quad H_i(A) = \frac{r_i^\bar{\Lambda}(A)}{|A_{i_1i_2\cdots i_m}| - r_i^\bar{\Lambda}(A)}, \quad i \in \bar{S}.
\]

Then we have the following conclusion.

**Theorem 3.2.** For tensor $A = (A_{i_1i_2\cdots i_m})$, if there exists a partition $(S, \bar{S})$ of the index set $N$ such that

\[
|A_{i_1i_2\cdots i_m}| - r_i(A) > 0, \quad i \in S; \quad |A_{i_1i_2\cdots i_m}| - r_i^\bar{\Lambda}(A) > 0, \quad i \in \bar{S},
\]

\[
\min_{i \in S} h_i(A) > \max_{i \in \bar{S}} H_i(A),
\]

then $A$ is generalized strictly diagonally dominant and hence it is a nonsingular $H$-tensor.

**Proof.** For simplicity, we assume that the comparison tensor of $A$ is itself.

From the first inequality of (3.6), one has $\min_{i \in S} h_i(A) > 1$. Hence we may define the following positive diagonal matrix $D$ with diagonal entries

\[
D_{ii} = \begin{cases} 1, & \text{if } i \in S, \\ d, & \text{if } i \in \bar{S}, \end{cases}
\]

where $d > 1$ is such that

\[
\min_{i \in S} h_i(A) > d^{m-1} > \max_{i \in \bar{S}} \{\max_{i \in S} H_i(A), 1\}.
\]
Certainly,
\[
\min_{i \in S} h_i(\mathcal{A}) > d^{m-1} > \max_{i \in \bar{S}} H_i(\mathcal{A}).
\]

Now, consider tensor \( \mathcal{B} = \mathcal{A} \mathcal{D}^{m-1} \). It is easy to see that for any \( i \in N \),
\[
r_i^\lambda(\mathcal{B}) = \sum_{\{i_2, \ldots, i_m \} \in \bar{\Lambda}} |\mathcal{B}_{i_2i_3 \ldots i_m}| = \sum_{\{i_2, \ldots, i_m \} \in \bar{\Lambda}} |\mathcal{A}_{i_2i_3 \ldots i_m}| = r_i^\lambda(\mathcal{A}),
\]
and
\[
r_i^{\bar{\lambda}}(\mathcal{B}) = \sum_{\{i_2, \ldots, i_m \} \in \Lambda} |\mathcal{B}_{i_2i_3 \ldots i_m}| \leq d^{m-1}r_i^{\bar{\lambda}}(\mathcal{A})
\]

Thus for \( i \in S \), if \( r_i^{\bar{\lambda}}(\mathcal{A}) > 0 \), then
\[
r_i(\mathcal{B}) = r_i^\lambda(\mathcal{B}) + r_i^{\bar{\lambda}}(\mathcal{B}) \leq r_i^\lambda(\mathcal{A}) + d^{m-1}r_i^{\bar{\lambda}}(\mathcal{A}) < r_i^\lambda(\mathcal{A}) + h_i r_i^{\bar{\lambda}}(\mathcal{A}) = r_i^\lambda(\mathcal{A}) + \frac{A_{ii \ldots i} - r_i^\lambda(\mathcal{A})}{r_i^{\bar{\lambda}}(\mathcal{A})} r_i^{\bar{\lambda}}(\mathcal{A}) = A_{ii \ldots i} = A_{ii \ldots i}^{(1)},
\]
and if \( r_i^{\bar{\lambda}}(\mathcal{A}) = 0 \), then from the first inequality of (3.6),
\[
r_i(\mathcal{B}) = r_i^\lambda(\mathcal{B}) + r_i^{\bar{\lambda}}(\mathcal{B}) \leq r_i^\lambda(\mathcal{A}) + d^{m-1}r_i^{\bar{\lambda}}(\mathcal{A}) = r_i^\lambda(\mathcal{A}) < A_{ii \ldots i} = A_{ii \ldots i}^{(1)}.
\]

For \( i \in \bar{S} \), from the second inequality of (3.6), one has
\[
B_{ii \ldots i} - r_i(\mathcal{B}) = d^{m-1}A_{ii \ldots i} - r_i^{\lambda}(\mathcal{B}) - r_i^{\bar{\lambda}}(\mathcal{B}) \geq d^{m-1}A_{ii \ldots i} - r_i^{\lambda}(\mathcal{A}) - d^{m-1}r_i^{\bar{\lambda}}(\mathcal{A}) = d^{m-1}(A_{ii \ldots i} - r_i^{\lambda}(\mathcal{A})) - r_i^{\bar{\lambda}}(\mathcal{A}) > H_i(A_{ii \ldots i} - r_i^{\lambda}(\mathcal{A})) - r_i^{\bar{\lambda}}(\mathcal{A}) = \frac{r_i^{\lambda}(\mathcal{A})}{A_{ii \ldots i} - r_i^{\lambda}(\mathcal{A})}(A_{ii \ldots i} - r_i^{\lambda}(\mathcal{A})) - r_i^{\bar{\lambda}}(\mathcal{A}) = 0.
\]

This means that tensor \( \mathcal{A} \mathcal{D}^{m-1} \) is strictly diagonally dominant, and \( \mathcal{A} \) is a nonsingular \( M \)-tensor by Proposition 2.2.

If we take \( S = \{i\} \) and let \( \bar{S} \) be its supplement set in \( N \), then Theorem 3.2 reduces to Lemma 12 in [18]. Now, we give an analysis to the assumption in Theorem 3.2. If tensor \( \mathcal{A} \) is strictly diagonally dominant, then for any partition of \( N \), it holds that
\[
A_{ii \ldots i} > r_i(\mathcal{A}) = r_i^\lambda(\mathcal{A}) + r_i^{\bar{\lambda}}(\mathcal{A}), \quad i \in N.
\]

Hence
\[
\min_{i \in S} h_i(\mathcal{A}) > \max_{i \in \bar{S}} H_i(\mathcal{A}).
\]

That is, a strictly diagonally dominant tensor satisfies the condition of Theorem 3.2. This means that Theorem 3.2 improves Theorem 2.2.

If the concerned tensor is irreducible, then the assumption in the theorem can be relaxed. To proceed, we need the following convention for \( x, y \in \mathbb{R} \),
\[
\frac{x}{y} = \begin{cases} +\infty, & \text{if } x > 0, y = 0; \\ 1, & \text{if } x = 0, y = 0. \end{cases}
\]
Theorem 3.3. For irreducible tensor $\mathcal{A}$, if there exists a partition $(\mathcal{S}, \bar{\mathcal{S}})$ of the index set $\mathbb{N}$ such that

$$|A_{ii...i}| - r_i(\mathcal{A}) \geq 0, \quad i \in \mathcal{S}; \quad |A_{ii...i}| - \bar{r}_i(\mathcal{A}) \geq 0, \quad i \in \bar{\mathcal{S}},$$

$$\min_{i \in \mathcal{S}} h_i(\mathcal{A}) \geq \max_{i \in \mathcal{S}} H_i(\mathcal{A})$$

and there exists index $i_0 \in \mathcal{S}$ such that

$$|A_{i_0i_0...i_0}| - r_{i_0}(\mathcal{A}) > 0, \quad h_{i_0}(\mathcal{A}) > \max_{i \in \mathcal{S}} H_i(\mathcal{A}),$$

then $\mathcal{A}$ is a nonsingular $H$-tensor.

Proof. Similar to the proof of Theorem 3.2, we assume that the comparison tensor of $\mathcal{A}$ is itself. Then from the first inequality of (3.7) and the convention $0^1$, one has $\min_{i \in \mathcal{S}} h_i(\mathcal{A}) \geq 1$. Further, from (3.8) and (3.9), there exists $d \geq 1$ such that

$$\min_{i \in \mathcal{S}} h_i(\mathcal{A}) \geq d^{m-1} \geq \max_{i \in \mathcal{S}} H_i(\mathcal{A})$$

and $h_{i_0}(\mathcal{A}) > d^{m-1}$.

Define positive diagonal matrix $\mathbf{D}$ with diagonal entries

$$\mathbf{D}_{ii} = \begin{cases} 1, & \text{if } i \in \mathcal{S}, \\ d, & \text{if } i \in \bar{\mathcal{S}}, \end{cases}$$

and $\mathcal{B} = \mathcal{A} \mathbf{D}^{m-1}$. Then $\mathcal{B}$ remains irreducible as $\mathbf{D}$ is positively diagonal. Just as in the proof of Theorem 3.2, for any $i \in \mathbb{N}$, it holds that

$$r_i^\Lambda(\mathcal{B}) = r_i^\Lambda(\mathcal{A}), \quad r_i^\Lambda(\mathcal{B}) \leq d^{m-1} r_i^\Lambda(\mathcal{A})$$

and

$$\mathcal{B}_{ii...i} = \mathcal{A}_{ii...i}, \quad i \in \mathcal{S}; \quad \mathcal{A}^{(1)}_{ii...i} = d^{m-1} \mathcal{A}_{ii...i}, \quad i \in \bar{\mathcal{S}}.$$
For \( i \in \bar{S} \), if \( A_{i_1 \cdots i} - r_i^\Lambda(A) > 0 \) then

\[
B_{i_1 \cdots i} - r_i(B) = d^{m-1}A_{i_1 \cdots i} - r_i^\Lambda(B) - r_i^\Lambda(B) \\
\geq d^{m-1}A_{i_1 \cdots i} - r_i^\Lambda(B) - d^{m-1}r_i^\Lambda(A) \\
= d^{m-1}(A_{i_1 \cdots i} - r_i^\Lambda(A)) - r_i^\Lambda(A) \\
\geq H_i(A)(A_{i_1 \cdots i} - r_i^\Lambda(A)) - r_i^\Lambda(A) \\
= \frac{r_i^\Lambda(A)}{\Lambda_{i_1 \cdots i} - r_i^\Lambda(A)}(A_{i_1 \cdots i} - r_i^\Lambda(A)) - r_i^\Lambda(A) = 0,
\]

and if \( A_{i_1 \cdots i} - r_i^\Lambda(A) = 0 \), then from \( h_{i_0}(A) > \max_{i \in \bar{S}} H_i(A) \), one has \( H_i(A) < \infty \).

Thus, \( r_i^\Lambda(A) = 0 \) and hence

\[
B_{i_1 \cdots i} - r_i(B) = d^{m-1}A_{i_1 \cdots i} - r_i^\Lambda(B) - r_i^\Lambda(B) \\
\geq d^{m-1}A_{i_1 \cdots i} - r_i^\Lambda(B) - d^{m-1}r_i^\Lambda(A) \\
= d^{m-1}(A_{i_1 \cdots i} - r_i^\Lambda(A)) - r_i^\Lambda(A) = 0.
\]

Thus, \( AD^{m-1} \) is diagonally dominant with at least one strict inequality. Taking the irreducibility of tensor \( AD^{m-1} \) into consideration, we know that \( AD^{m-1} \) is generalized diagonally dominant by Proposition 2.3. Recalling Proposition 2.2, we know that \( A \) is nonsingular \( H \)-tensor.

It can easily be verified that an irreducible tensor which is diagonally dominant with one strict inequality holding in (2.1) satisfies the condition of the theorem. This means that the assumption in Theorem 3.3 is weaker than the condition of the irreducibility and the diagonal dominance with one strict inequality holding in (2.1) of a tensor. Thus, the conclusion improves the second part of Theorem 2.2.

4. Principal subtensor and nonsingular \( H \)-tensor with positive diagonal entries. In this section, we first explore the heredity of the principal subtensor of nonsingular \( H \)-tensors and then investigate the positive definiteness of nonsingular \( H \)-tensor with positive diagonal entries.

For nonnegative tensors \( A \) and \( B \), it holds that \( \rho(A) \geq \rho(B) \) provided that \( A \geq B \geq 0 \) [34]. This can be strengthened as follows.

**Lemma 4.1.** For nonnegative tensors \( A \) and \( B \), if \( A \geq B \), then \( \rho(A) \geq \rho(B) \). Furthermore, the spectral radius of any principal subtensor of a nonnegative tensor is not larger than that of this tensor.

**Proof.** For any nonzero vector \( x \geq 0 \), from \( A \geq B \geq 0 \), we know that

\[
(Ax^{m-1})_i \geq (Bx^{m-1})_i, \quad i \in N.
\]

Thus, from Theorem 5.3 in [34], one has

\[
\rho(A) = \max_{x \geq 0, x \neq 0} \min_{x_i > 0} \frac{(Ax^{m-1})_i}{x_i^{m-1}} \geq \max_{x \geq 0, x \neq 0} \min_{x_i > 0} \frac{(Bx^{m-1})_i}{x_i^{m-1}} = \rho(B).
\]

The first conclusion follows.

Let \( A_J \) be a principal subtensor of tensor \( A \) whose entries are indexed by subset \( J \) of \( N \), i.e.,

\[
A_J = (A_{i_1 i_2 \cdots i_m}), \quad i_j \in J, \quad j = 1, 2, \cdots, m.
\]

Define the following tensor with same dimensions as tensor \( A \),

\[
B_{i_1 i_2 \cdots i_m} = \begin{cases} A_{i_1 i_2 \cdots i_m}, & \text{for } i_j \in J, j = 1, 2, \cdots, m, \\ 0, & \text{otherwise}. \end{cases}
\]
Certainly, \( Ax^{m-1} \geq Bx^{m-1} \) for any nonnegative vector \( x \in \mathbb{R}^n \) and a similar argument to the proof of the first assertion yields that \( \rho(A) \geq \rho(B) \).

On the other way, for principal subtensor \( A_J \), its any eigenvalue with associated eigenvector \( x_J \) is also an eigenvalue of tensor \( B \) with associated eigenvector \( \begin{pmatrix} x_J \\ 0 \end{pmatrix} \), and vise versa. Hence, \( \rho(A_J) = \rho(B) \) from the definition of spectral radius and the second conclusion follows.

From Lemma 4.1, we can readily obtain the following conclusions.

**Theorem 4.1.** For any nonsingular \( H \)-tensor, its any principal subtensor is also a nonsingular \( H \)-tensor.

**Theorem 4.2.** Let \( A \) and \( B \) be \( m \)-th order \( n \)-dimensional tensors such that
\[
|A_{i_1i_2\cdots i_m}| = \begin{cases} 
\geq |B_{i_1i_2\cdots i_m}|, & \text{if } \delta_{i_1i_2\cdots i_m} = 1; \\
\leq |B_{i_1i_2\cdots i_m}|, & \text{if } \delta_{i_1i_2\cdots i_m} = 0.
\end{cases}
\]

Then \( A \) is a nonsingular \( H \)-tensor if \( B \) is a nonsingular \( H \)-tensor.

Now, we consider nonsingular \( H \)-tensors with positive diagonal entries. To proceed, we need the following conclusion [34].

**Lemma 4.2.** Let \( A \) and \( B \) be two square tensors of same order and same dimension. If \( A \geq |B| \), then \( \rho(A) \geq \rho(B) \).

The following conclusion shows that nonsingular \( H \)-tensors with positive diagonal entries have similar properties to nonsingular \( M \)-tensors.

**Theorem 4.3.** Let \( A \) be a nonsingular \( H \)-tensor with positive diagonal entries. Then for its any eigenvalue \( \lambda \), it holds that \( \Re(\lambda) > 0 \).

**Proof.** Since \( A \) is a nonsingular \( H \)-tensor, then its compare tensor \( M_A \) is a nonsingular \( M \)-tensor, and there exist a nonnegative tensor \( B \) and a positive number \( c > \rho(B) \) such that
\[
M_A = cI - B.
\]
(4.1)

Since all the diagonal entries of tensor \( A \) are positive, tensor \( A \) can be written as
\[
A = cI - C,
\]
where tensor \( C \) satisfies that \( B = |C| \). From Lemma 4.2, one has
\[
\rho(B) \geq \rho(C).
\]
Then from the second inequality in (4.1), one has \( c > \rho(C) \).

On the other hand, from Lemma 2.1, \( \lambda \) is an eigenvalue of tensor \( A \) if and only if \( (c - \lambda) \) is an eigenvalue of tensor \( cI - A \), i.e., tensor \( C \). From \( \rho(C) \leq \rho(B) < c \), we know that \( |c - \lambda| < c \), and hence \( \Re(\lambda) > 0 \). 

The following example shows that the converse of the theorem does not hold, i.e., for tensor \( A \) with positive diagonal entries, if the real part of its any eigenvalue is positive, then \( A \) is not necessarily a nonsingular \( H \)-tensor. This shows the difference between \( H \)-tensor with positive diagonal entries and \( M \)-tensor.

Consider 3-order 2-dimensional tensor \( A \) with
\[
A_{111} = 1, \ A_{112} = 2, \ A_{212} = -2, \ A_{222} = 1
\]
and all other entries are zeros. From the definition of eigenvalue, its any eigenpair \( (\lambda, x) \) satisfies that
\[
\begin{cases}
\begin{align*}
x_1^2 + 2x_1x_2 &= \lambda x_1^2, \\
-2x_1x_2 + x_2^2 &= \lambda x_2^2.
\end{align*}
\end{cases}
\]
A straightforward computing gives the spectral of tensor $\mathcal{A}$, i.e., its all eigenvalues: $\lambda_\mathcal{A} = 1, 1+2i, 1-2i$. Clearly, the real part of each eigenvalue of tensor $\mathcal{A}$ is positive. The eigenvalues of the comparison tensor of tensor $\mathcal{A}$ can similarly be computed: $\lambda_{\mathcal{MA}} = -1, 1, 3$. From (1) of Proposition 2.1, we know that $\mathcal{A}$ is not a nonsingular $H$-tensor.

From Theorem 4.3, we conclude that all $H$-eigenvalues of a nonsingular $H$-tensor with positive diagonal entries are positive. Recalling Theorem 2.1, we have the following conclusion for even order symmetric tensor (see Theorem 8 in [18]).

**Theorem 4.4.** For any symmetric nonsingular $H$-tensor of even order, if its all diagonal entries are positive, then it is positive definite.

For general nonsingular $H$-tensor with positive diagonal entries, it can be characterized via semi-positiveness.

**Theorem 4.5.** Let $\mathcal{A}$ be a nonsingular $H$-tensor with positive diagonal entries. Then there exists positive vector $x \in \mathbb{R}^n$ such that $\mathcal{A}x^{m-1} > 0$.

**Proof.** Suppose tensor $\mathcal{A}$ satisfies the assumption. Then its comparison tensor $\mathcal{MA}$ is a nonsingular $M$-tensor and from Proposition 2.2, there exists positive diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $\mathcal{MA}D^{m-1}$ is strictly diagonally dominant. Hence

$$(\mathcal{MA}D^{m-1})e^{m-1} = \mathcal{MA}(De)^{m-1} > 0.$$  

Using the fact that $\mathcal{A} \geq \mathcal{MA}$ and $De > 0$, we further have

$$\mathcal{A}(De)^{m-1} \geq \mathcal{MA}(De)^{m-1} > 0.$$  

Let $x = De$. The desired conclusion follows. 

We end this section by presenting a counter example to show that the converse of the theorem does not hold.

Consider 4-order 2-dimensional symmetric tensor $\mathcal{A}$ such that

$$\mathcal{A}_{1111} = 1, \quad \mathcal{A}_{1112} = 2, \quad \mathcal{A}_{1121} = 2, \quad \mathcal{A}_{1211} = 2, \quad \mathcal{A}_{2111} = 2, \quad \mathcal{A}_{2222} = 1$$

and all other entries are zero. A straightforward computing gives

$$\mathcal{A}e^3 = \begin{pmatrix} 7 \\ 3 \end{pmatrix} > 0.$$  

However, for $x = (1; -1) \in \mathbb{R}^2$,

$$\mathcal{A}x^4 = -6 < 0.$$  

This means that tensor $\mathcal{A}$ is not positive definite. By Theorem 4.4, we know that it is not a nonsingular $H$-tenor.

5. **Conclusion.** In this paper, we considered the nonsingular $H$-tensors by establishing its equivalence with generalized diagonal dominance, and investigated nonsingular $H$-tensors with positive diagonal entries based on the real part of tensor eigenvalue and tensor semi-positiveness. We also established some new implementable criteria in identifying nonsingular $H$-tensors based on the strict diagonal product dominance and $S$ diagonal product dominance. The obtained results improve the existing results and extend the corresponding conclusions for matrices.

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