A PDD Decoder for Binary Linear Codes With Neural Check Polytope Projection

Yi Wei, Ming-Min Zhao, Min-Jian Zhao and Ming Lei

Abstract— Linear Programming (LP) is an important decoding technique for binary linear codes. However, the advantages of LP decoding, such as low error floor and strong theoretical guarantee, etc., come at the cost of high computational complexity and poor performance at the low signal-to-noise ratio (SNR) region. In this letter, we adopt the penalty dual decomposition (PDD) framework and propose a PDD algorithm to address the fundamental polytope based maximum likelihood (ML) decoding problem. Furthermore, we propose to integrate machine learning techniques into the most time-consuming part of the PDD decoding algorithm, i.e., check polytope projection (CPP). Inspired by the fact that a multi-layer perceptron (MLP) can theoretically approximate any nonlinear mapping function, we present a specially designed neural CPP (NCPP) algorithm to decrease the decoding latency. Simulation results demonstrate the effectiveness of the proposed algorithms.

Index Terms—Binary linear codes, check polytope projection, LDPC, machine learning, MLP, neural network.

I. INTRODUCTION

Recently, the linear programming (LP) decoder, which is based on LP relaxation of the maximum likelihood (ML) decoding problem, has attracted increasing attention for decoding binary linear codes, especially for low density parity check (LDPC) codes [1], [2]. Compared with the classical belief propagation (BP) decoder, the LP decoder has stronger theoretical guarantees on decoding performance and empirically is not observed to suffer from an error floor. However, the above advantages of the LP decoder come at the cost of two drawbacks, i.e., higher computational complexity and poorer error-correcting performance when the signal-to-noise ratio (SNR) is low.

In order to overcome the above shortcomings, the work [3] first employed the alternating direction method of multipliers (ADMM) to solve the ML decoding problem by exploiting the fundamental polytope of the parity-check (PC) constraints. In order to improve the error-rate performance in the low SNR region, the work [4] added penalty terms to the linear objective function to make pseudocodewords more costly and the resulting decoder is known as the ADMM penalized decoder. Afterwards, improvements over the ADMM penalized decoder were achieved by modifying the penalty terms [5], [6]. Based on the cascaded decomposition method [7], the work [8] adopted the penalty dual decomposition (PDD) framework [9] and developed a PDD decoder, which was shown to overperform the ADMM penalized decoder.

Y. Wei, M. M. Zhao, M. J. Zhao and M. Lei are with College of Information Science and Electronic Engineering, Zhejiang University, Hangzhou 310027, China (email: {217311333, zmбл ablack, mjzhao, lm1029}@zju.edu.cn) (Corresponding author: Ming-Min Zhao and Min-Jian Zhao.)

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Focusing on the fundamental polytope based ML decoding problem, intensive studies have been conducted to simplify the check polytope projection (CPP) operations, which is the most computationally intensive and time-consuming part in the ADMM-based decoders [10]–[13]. Compared with the original CPP algorithm which involves two sorting operations [8], the work [10] employed a cut-search algorithm to remove one sorting operation. In [11], the projection algorithm was further simplified by transforming the CPP operation into the projection onto a simplex. In [12], the authors presented an iterative CPP (ICPP) algorithm which requires no sorting operation and can substantially improve the decoding speed. The work [14] revealed the recursive structure of the parity polytope and presented an efficient projection algorithm by iteratively fixing selected components of the projection.

In this letter, we propose a novel multi-layer perceptron (MLP)-aided PDD decoder for binary linear codes. Different from the PDD decoder in [8] which considers the minimum polytope based LP formulation, we show that the PDD framework can also be applied to the fundamental polytope based LP formulation. The proposed decoder consists of two loops: in the outer loop, we update the dual variables and certain penalty parameter, while in the inner loop, we divide the primal variables into several blocks and employ the block coordinate descent (BCD) method to iteratively optimize each block variable in closed-form. Furthermore, in order to simplify the CPP operations in the proposed PDD decoder, we propose a neural CPP (NCPP) algorithm, which is obtained by integrating a simple three-layer MLP (namely CPP-net) into the ICPP algorithm in [12] to reduce the corresponding iteration number. Simulation results demonstrate that the proposed PDD decoder exhibits superior error-correcting performance and the NCPP algorithm can reduce the latency significantly.

II. PROBLEM FORMULATION

Consider a binary linear code $C$ of length $N$ specified by an $M \times N$ PC matrix $H$. Let $\mathcal{I} \triangleq \{1, \ldots, N\}$ and $\mathcal{J} \triangleq \{1, \ldots, M\}$ denote the sets of variables nodes and check nodes of $C$, respectively. Suppose $x = \{x_i = \{0, 1\}, i \in \mathcal{I}\}$ is a codeword transmitted over a memoryless binary-input symmetric-output channels, and $y$ is the received signal. Then, the received log-likelihood ratio (LLR) vector $v \in \mathbb{R}^N$ can be expressed as

$$v_i = \log \left( \frac{Pr(y_i|x_i = 0)}{Pr(y_i|x_i = 1)} \right), \quad i \in \mathcal{I}. \tag{1}$$

According to [14], the ML decoding problem can be formulated as the following optimization problem:

$$\min_{\mathbf{x}} \mathbf{v}^T \mathbf{x} \quad \text{s.t.} \quad \mathbf{x} \in C. \tag{2}$$

The work [11] proposed to relax problem [2] as follows:

$$\min_{\mathbf{x}} \mathbf{v}^T \mathbf{x} \quad \text{s.t.} \quad \mathbf{x} \in \mathcal{P} := \bigcap_{j \in \mathcal{J}} \text{conv}(\mathcal{D}_j), \tag{3}$$

where $\text{conv}(\mathcal{D}_j)$ is the convex hull of the codewords defined by the $j$-th row of the PC matrix $H$, and $\mathcal{P}$ is called the...
fundamental polynote. Let \( d_j \) denote the degree of check node \( j \). (3) can be expressed in a more compact form as
\[
\min_{x} \mathbf{v}^T x \quad \text{s.t.} \quad \mathbf{P}_j x \in \mathbb{P}_{d_j}, \quad \forall j \in \mathcal{J},
\] (4)
where \( \mathbf{P}_j \) denotes a \( d_j \times N \) selection matrix which selects the elements of \( x \) that participate in the \( j \)-th check equation. \( \mathbb{P}_{d_j} \) is the PC polytope of dimension \( d_j \), which is defined as the convex hull of all even-parity binary vectors of length \( d_j \), i.e., \( \mathbb{P}_{d_j} = \text{conv}\{e \in \{0,1\}^{d_j} | \|e\|_1 \text{ is even}\} \).

III. PDD Decoding Algorithm

In this section, we adopt the PDD framework to solve problem (3) and develop a PDD decoding algorithm, where the main idea is to introduce additional equality constraints to handle the nontrivial constraints and the discrete variable \( x \).

Firstly, we introduce auxiliary variables \( \{z_i\} \in \mathbb{R}^{d_j \times 1} \) to equivalently transform constraint \( \mathbf{P}_j x \in \mathbb{P}_{d_j}, \quad \forall j \in \mathcal{J} \) into \( \mathbf{P}_j x = z_j, \quad z_j \in \mathbb{P}_{d_j}, \quad \forall j \in \mathcal{J} \). Then, we relax the binary variables \( \{x_i\} \) to the interval \([0,1]\), and instead of using penalty functions to enforce \( x_i = 0 \) or \( 1 \), we propose to introduce auxiliary variables \( \{\hat{x}_i\} \) which satisfy \( \hat{x}_i = x_i \) and \( x_i(\hat{x}_i - 1) = 0 \). Let \( \hat{x} = [\hat{x}_1, \cdots, \hat{x}_N]^T \) and \( z = [z_1^T, \cdots, z_M^T]^T \), problem (3) can be equivalently formulated as
\[
\min_{x, \hat{x}, z} \mathbf{v}^T x + \sum_{i \in \mathcal{I}} \left( \sum_{j \in \mathcal{J}} \frac{m_{j}}{\mu_{m}} \|\mathbf{P}_j x - \hat{x}_i + \frac{y_i}{\mu_{m}}\|^2 \right)
\]
\[
+ \sum_{i \in \mathcal{I}} \left( \frac{m_{i}}{\mu_{m}} \|y_i\|^2 + \frac{\mu_{m}}{\mu_{m}} \left( \hat{x}_i - x_i + \frac{\eta_i}{\mu_{m}} \right)^2 \right),
\] (5)
\( \{y_i\} \) and \( \{\eta_i\} \) denote the dual variables associated with the constraints \( \mathbf{P}_j x = z_j \). \( x_i(\hat{x}_i - 1) = 0 \) and \( x_i = \hat{x}_i \), respectively; \( \mu_{m} \) represents the penalty parameter in the \( m \)-th outer iteration. To this end, we propose to address problem (5) by employing the BCD method in the inner iterations, and update the dual variables and the penalty parameter \( \{\mu_{m}\} \) in the outer iterations.

In (5), it can be observed that the primal variables can be divided into three blocks, i.e., \( x, \hat{x} \) and \( z \). Therefore, the BSUM iterations for problem (5) consists of the following three steps (\( k \) denotes the inner iteration index):

1) **Updating** \( x^{k+1} \) given \( \{\hat{x}^k, z^k\} \): The \( x \) subproblem is a quadratic optimization problem with a simple constraint that restricts its solution to lie in the interval \([0,1]\), which can be expressed as
\[
\min_{x} \mathbf{v}^T x + P_{\mu_{m}}(x, \hat{x}, z) \quad \text{s.t.} \quad 0 \leq x_i \leq 1, \quad \forall i \in \mathcal{I}.
\] (6)
We can observe that problem (6) can be naturally decomposed into \( N \) subproblems, i.e.,
\[
\min_{x_i} a_{i,m}^k x_i^2 + b_{i,m}^k x_i \quad \text{s.t.} \quad 0 \leq x_i \leq 1,
\] (7)
where
\( a_{i,m}^k = \frac{m_{i}}{\mu_{m}} \left( d_i + (\hat{x}_i - 1)^2 + 1 \right), \quad b_{i,m}^k = v_i + 2\alpha_i + \omega_i (\hat{x}_i - 1) + \eta_i - \mu_{m} \hat{x}_i \), and \( \alpha_i \) denotes the \( i \)-th element of the vector \( \alpha = \frac{m_{i}}{\mu_{m}} \sum_{j \in \mathcal{J}} \mathbf{P}_j^T (\frac{z_j^k - \hat{x}_i^k}{\mu_{m}}) \).

By resorting to the first-order optimality condition, the optimal solution of problem (7) can be obtained by
\[
x_i^{k+1} = \Pi_{[0,1]}(-0.5b_{i,m}^k/a_{i,m}^k),
\] (8)
where \( \Pi_{[0,1]} \) denotes the Euclidean projection operation into the interval \([0,1]\).

2) **Updating** \( z^{k+1} \) given \( \{\hat{x}^k, x^{k+1}\} \): The optimization problem of \( z_i \) can be expressed as
\[
\min_{z_i} \frac{\mu_{m}}{2} \|\mathbf{P}_j x - \hat{x}_i + \frac{y_i}{\mu_{m}}\|^2 \quad \text{s.t.} \quad z_j \in \mathbb{P}_{d_j},
\] (9)
Similar to the first step, the optimal solution of problem (9) is given by
\[
z_j^{k+1} = \Pi_{\mathbb{P}_{d_j}}(\mathbf{P}_j x^{k+1} + \frac{y_j}{\mu_{m}}),
\] (10)
where \( \Pi_{\mathbb{P}_{d_j}} \) denotes the CPP operation.

3) **Updating** \( \hat{x}^{k+1} \) given \( \{x^{k+1}, z^{k+1}\} \): The \( \hat{x} \) subproblem can be written as the following unconstrained quadratic optimization problem:
\[
\min_{\hat{x}} \sum_{i \in \mathcal{I}} \left( \frac{m_{i}}{\mu_{m}} \|y_i\|^2 + \frac{\mu_{m}}{\mu_{m}} \left( \hat{x}_i - x_i + \frac{\eta_i}{\mu_{m}} \right)^2 \right),
\]
whose optimal solution can be easily obtained by
\[
\hat{x}_i^{k+1} = \frac{4(\omega_i - \mu_{m} x_i^{k+1})}{\eta_i + \mu_{m} x_i^{k+1}}.
\] (11)
Furthermore, the dual variables can be updated by
\[ y_i^{m+1} = y_i^{m} + \mu_{m} (x_i^{k+1} - x_i^{m}), \]
\[ w_i^{m+1} = w_i^{m} + \mu_{m} (x_i^{k+1} - x_i^{m}), \]
\[ \eta_i^{m+1} = \eta_i^{m} + \mu_{m} (x_i^{k+1} - x_i^{m}). \]
To summarize, the detailed steps of the PDD decoder are listed in Algorithm 1 where \( c \) denotes a control parameter that gradually increases the penalty parameter \( \mu_{m} \) by a certain amount during each outer iteration. According to (9), the proposed PDD decoder is guaranteed to converge.

IV. NCPP Algorithm

The projection \( \Pi_{\mathbb{P}_{d}}(\cdot) \) of a real-valued vector onto the check polytope \( \mathbb{P}_{d} \) in (12) is the most time-consuming part in fundamental polytope based decoders, such as the proposed PDD decoder and the ADMM-based decoders in (3) and (4), etc. In this section, we propose a novel NCPP algorithm which can further reduce the decoding latency of the ICPP algorithm in (12). The main idea of the proposed method is to reduce the number of CPP iterations through a simple three-layer MLP (namely CFP-net) with quantized parameters. In the following, we first give a brief review of the ICPP algorithm, and then the structure of CFP-net is introduced followed by the proposed NCPP algorithm, and finally we present the detailed process of training sample generation and loss function design.

**Algorithm 1 PDD Algorithm for Problem (3)**

1. Initialize \( x^0, \{y_i^0\}, \{\eta_i^0\} \) and \( z^0 \) as all-zero vectors. Initialize all elements in \( x^0 \) to 0.5. Set the initial penalty parameter \( \mu_0 \) and control parameter \( c \). Set \( m \leftarrow 0 \).

2. repeat
3. \( k \leftarrow k + 1 \).
4. repeat
5. Update \( \{x^{k+1}, \hat{x}^k, z^{k+1}\} \) by (10), (12) and (13).
6. until some convergence condition is met.
7. \( x^m \leftarrow x^{k+1}, \hat{x}^m \leftarrow \hat{x}^{k+1}, \hat{z}^m \leftarrow z^{k+1} \) and \( z^{m+1} \leftarrow z^{k+1} \).
8. Update the dual variables by (15) and set \( \mu_{m+1} = c \mu_{m} \).
9. \( x^0 \leftarrow x^{m}, \hat{x}^0 \leftarrow \hat{x}^{m}, m \leftarrow m + 1 \).
10. until some convergence condition is met.
amount of shift is determined by the value of the direction orthogonal to the assistant hyperplane \( \hat{v} \). Structure of CPP-net \( \Theta \) of \( v \) the calculation of \( r \). In [12], an estimate \( \hat{s} \) of \( s \) was iteratively obtained by \( \hat{s} = \sum \eta^{k} \), where \( \eta^{k} \) is the incremental projection coefficient and \( \eta^{k} \Theta \) is how much \( v \) is shifted at the \( k \)-th iteration. This iterative process terminates when \( \eta^{k} \) falls below a certain threshold \( \epsilon \).

B. Structure of CPP-net

Since the assistant hyperplane \( \Theta \) is relatively easy to obtain, the main difficulty of the CPP operation \( r = \Pi_{PP_{d}}(v) \) lies in the calculation of \( s \), which can be viewed as the projection of \( v \) to \( s \), i.e., \( \Pi_{v \rightarrow s}(v) \). As a result, the CPP operation can be alternatively expressed as \( r = \Pi_{0,1}^{d} (v - \Pi_{v \rightarrow s}(v) \Theta) \). Motivated by the fact that a trained MLP with enough neurons can approximate any nonlinear mappings, we introduce a simple three-layer MLP to imitate the projection \( \Pi_{v \rightarrow s}(v) \) and output an initial estimation of \( s \) for the purpose of reducing the residual iteration number. Note that a classical MLP consists of an input layer, an output layer and several hidden layers. Each layer has multiple neurons, and each neuron can execute an activation function on the weighted sum of the outputs from the preceding layer. The activation function plays an important role in neural networks and when it is non-linear, a two-layer neural network can be proven to be a universal function approximator.

The proposed CPP-net with \( d_{j} \) inputs consists of three layers, i.e., one input layer with \( d_{j} \) neurons, one hidden layer with \( \lceil d_{j}/2 \rceil \) neurons and one output layer with only one neuron. In order to introduce non-linearity into the proposed network, both hidden and output layers should contain activation functions. Note that the widely-used ReLU activation function is not employed in the proposed CPP-net since it will force almost half of the neurons to be silenced (verified by our simulations) and limit the learning ability of CPP-net. Instead, we propose a novel activation function constructed based on the \( \sin(\cdot) \) function to improve the performance of CPP-net and with low implementation cost. We refer to this function as the SinAct(\( \cdot \)) and its definition is given by

\[
\text{SinAct}(x) = \left\{ \begin{array}{ll}
\frac{1}{2} (\sin(\frac{x}{2}) + 1), & -1 \leq x \leq 1 \\
0, & x < -1 \\
1, & x > 1
\end{array} \right.
\]

For clarity, a simple example of the proposed CPP-net when \( d_{j} = 6 \) is depicted in Fig. 1. Let \( y^{h} \in \mathbb{R}^{d_{j} \times 1} \) and \( \hat{s} \in \mathbb{R} \) denote the outputs of the hidden and output layers, respectively, then the data flow of CPP-net can be expressed as follows:

\[
y^{h} = \text{SinAct}(W_{a}v + b_{a}), \quad \hat{s} = \text{SinAct}(w^{T}_{\hat{s}}y^{h} + b_{b}),
\]

where \( \Theta \triangleq \{W_{a}, w_{b}, b_{a}, b_{b}\} \) denote the set of weights and biases, which are the learnable parameters to be trained. Therefore, the input-output mapping realized by the proposed CPP-net is defined by a chain of functions depending on \( \Theta \), i.e.,

\[
\hat{s} = F_{\text{CPP-net}}(v; \Theta) = \text{SinAct}(w_{b}(\text{SinAct}(W_{a}v + b_{a})) + b_{b}).
\]

In order to further reduce the computational complexity of CPP-net, we propose to quantize the neural weights \( \{W_{a}, w_{b}\} \) obtained by training to \( \{W_{a}, w_{b}\} \), where \( w_{Q}^{a} \in [0, \pm 2^{k}][d_{j}/2] \times d_{j}, \hspace{0.2cm} w_{Q}^{b} \in [0, \pm 2^{k}] \times [d_{j}/2] \), \( k \in \mathcal{N} \) and \( \mathcal{N} \) represents the set of natural numbers. Let \( w_{Q} \) denote an arbitrary element in \( \{w_{Q}^{a}, w_{Q}^{b}\} \), we can see that the original multiplication operations involved in the CPP-net can be simplified as follows:

- If \( w_{Q} = 0 \) or \( \pm 1 \), then no multiplication is required.
- If \( w_{Q} = \pm 2^{k}, k \in \mathcal{N} \backslash \{0\} \), then the corresponding multiplication operation can be replaced by the binary shifting operation.

Considering that the addition operations are simpler than multiplications, we choose not to quantize the biases \( \{b_{a}, b_{b}\} \), but instead finetune them with fixed \( \{W_{a}, w_{b}\} \). Note that this can compensate the performance loss caused by the quantization of \( \{W_{a}, w_{b}\} \), at least to certain extent.

C. NCPP algorithm

In this subsection, we present the proposed NCPP algorithm, which is shown in Algorithm 2. We first decide whether \( r \) can be obtained within only one CPP iteration and if not, we call the CPP-net to obtain an initial estimate \( \hat{s} \) of the difference coefficient \( s \). Then, the output of the CPP-net is fed to the subsequent CPP iterations to ensure that an accurate CPP operation can be conducted even when \( \hat{s} \) is far from \( s \). Thus, the NCPP algorithm is expected to achieve the same performance as the ICPP algorithm with lower complexity. Note that the SinAct(\( \cdot \)) function can be implemented as a look-up-table and this will not degrade the error correcting performance of the proposed decoder (ensured by steps 12-14 in Algorithm 2). When the number of quantization bits is large enough (e.g., larger than 3), the average iteration number required by the NCPP algorithm is only slightly increased (less than 0.5 in our simulations).
Algorithm 2 NCPP Algorithm

1: Input: Vector \( \mathbf{v} \) with dimension \( d_j \).
2: Output: \( \mathbf{r} = \Pi_{I_{\text{ICPP}}} (\mathbf{v}) \).
3: \( \theta_i = \text{sgn}(v_i - 0.5), \) \( i = 1, \ldots, d \).
4: if \{ \{ i : \theta_i = 1 \} \} is even then
5: \[ \eta^* = \arg \min_{|v_i| \leq 0.5} \eta_i = -\theta_i. \]
6: end if
7: \( \mu = \{ \{ i : \theta_i = 1 \} \} - 1, \) \( \mathbf{u} = \Pi_{[0,1]} (\mathbf{v}), \) \( \eta = (\theta^T \mathbf{u} - p)/d. \)
8: if \( \eta < \epsilon \) then
9: \( \mathbf{r} = \mathbf{u}. \)
10: else
11: \( \hat{s} = \mathcal{F}_{I_{\text{ICPP}}} (\mathbf{v}), \eta^0 = \hat{s}, k = 0. \)
12: repeat
13: \( \mathbf{v} = \mathbf{v} - \eta^k \theta, \) \( \mathbf{u} = \Pi_{(0,1]} (\mathbf{v}), \) \( k = k + 1, \eta^k = (\theta^T \mathbf{u} - p)/d. \)
14: until \( |\eta^k| < \epsilon \)
15: \( \mathbf{r} = \mathbf{u}. \)
16: end if

D. Training Details

1) Training Sample Generation: Generally, an MLP is trained to extract the underlying features from training samples and learn the specific patterns to perform certain tasks, such as classification, clustering and forecasting, etc. Therefore, the performance of the MLP depends critically on the quality of the training data and in our case, not surprisingly, training with training samples generated under different scenarios will lead to performance differences over the same validation set. Let \( (\mathbf{v}_p, \hat{s}_p)_{p=1}^P \) denote the labeled training sample set with size \( P \), where \( \mathbf{v}_p \) and \( \hat{s}_p \) represent the \( p \)-th feature and label, respectively. More specifically, for the considered network, \( \mathbf{v}_p \) is the input of the CNN operation, which is acquired by collecting \( \mathbf{P} \mathbf{x} + \frac{\mathbf{x}}{|\mathbf{x}|} \) in (12) when running the PDD decoding algorithm, and the label \( \hat{s}_p \) is the approximation of \( s_p \), which is obtained by running the ICPP algorithm with a predetermined iteration number \( K_p \). Since the proposed network aims to reduce the number of iterations required by the ICPP algorithm, training samples obtained by using different iteration numbers would have a critical impact on the training results. In order to investigate the characteristic of the iteration number, we illustrate its probability distribution when \( E_b/N_0 \) is set to 2 dB or 5 dB in Fig. 2 where the threshold \( \epsilon \) is fixed to 10^{-4}. We can observe that for both cases, the proportion of \( K_p = 1 \) (i.e., the ICPP algorithm converges within only one iteration) is larger than the others. Since employing CNN-net is unnecessary when \( K_p = 1 \), the training samples obtained when \( K_p = 1 \) are useless for network training and these instances should not be included in the training sample set. In addition, considering that high noise levels would prevent the proposed network from learning the underlying mapping mechanism, the training samples with \( K_p \geq 2 \) are collected under a relatively high \( E_b/N_0 = 5 \) dB is used in our simulations).

2) Loss Function: Loss function is used to measure the differences between the network output and the true label, and the performance of the network is heavily dependent on it. In general, the loss function should be carefully defined according to the specific learning task. In the following, we first investigate the convergence property of the proposed NCPP algorithm, based on which we present a novel loss function that is able to accelerate the learning process.

In Fig. 3 we illustrate the typical convergence behaviors of the proposed NCPP algorithm with different values of \( \hat{s} \), where \( s \) denotes the true difference coefficient, \( \hat{s}_L \) and \( \hat{s}_S \) denote two initial estimates of \( s \) with \( \hat{s}_L > s \), \( \hat{s}_S < s \) and \( \hat{s}_S - s = s - \hat{s}_L > \epsilon \). Note that the CNN-net can be viewed as a non-linear projector which is able to output an approximate value of the difference coefficient from the input \( \mathbf{v} \), therefore, it is able to provide a good initial point for the ICPP algorithm. The accumulated projection coefficient up to iteration \( k \), i.e., \( \hat{s}^k = \sum_{j=0}^k \eta^j \), is regarded as the performance metric. Note that the ICPP algorithm can be viewed as a special case of the proposed NCPP algorithm with \( \hat{s} = 0 \), we can observe that different values of \( \hat{s} \) lead to different numbers of iterations with the same \( \epsilon \) even when \( |\hat{s}_L - s| = |\hat{s}_S - s| \), and taking \( \hat{s}_L \) as the initial point results in a smaller iteration number. Based on this observation, we design the loss function as
\[ \mathcal{L}_{\text{CNN-net}} (\Theta) = \frac{1}{P} \sum_{p=1}^P |\hat{s}_p - s_p + \kappa (\hat{s}_p - s_p)/2| \]
Note that for the considered codes, we have tested the ADMM penalized decoders with many other penalty functions, and we finally chose the ADMM \( \ell_2 \) decoder in terms of BER performance.

V. Simulation Result

In this section, computer simulations are carried out to evaluate the error-correcting performance of the proposed PDD decoder and the decoding latency of the NCPP algorithm. The proposed network is implemented in Python using the TensorFlow library with the Adam optimizer. In the simulations, we focus on additive white Gaussian noise channel with binary phase shift keying (BPSK) modulation. The considered binary linear codes are (96, 48) MacKay 96.33.964 LDPC code \( C_1 \), (575, 288) IEEE 802.16e LDPC code \( C_2 \) and (2640, 1320) Margulis code \( C_3 \). During the training process, we collect \( 10^5 \) training samples and \( 10^4 \) validation samples with \( E_b/N_0 = 5 \) dB, \( E_b/N_0 = 4.5 \) dB and \( E_b/N_0 = 3 \) dB for \( C_1 \), \( C_2 \) and \( C_3 \) codes. The learning rate and the balance coefficient \( \kappa \) are set to \( 10^{-4} \) and 4.

We first compare the BLER performance of the proposed PDD decoder, the BP decoder (sum-product), the ADMM \( \ell_2 \) decoder in [4] and the PDD decoder in [8], as shown in Fig. 4. In all the curves, we collect at least 100 block errors for
all data points. It can be observed that our proposed PDD decoder shows better BLER performance at both low and high SNR regions for C1 code. For longer LDPC codes, i.e., the C2 and C3 codes, the proposed PDD decoder achieves a similar performance as the other counterparts when the SNR is low and outperforms them when $E_b/N_0 \geq 2$ dB and $E_b/N_0 \geq 1.4$ dB for C2 and C3 codes, respectively. Specifically, 0.3 dB, 0.1 dB and 0.08 dB performance gains over the ADMM $\ell_2$ decoder can be achieved at BLER=10^{-4} for C1, C2 and C3 codes, respectively. Besides, although the proposed PDD decoder achieves a similar BLER performance as in $[8]$, it requires less auxiliary variables and thus potentially leads to lower complexity.

Then, in TABLE I we provide the iteration numbers required by the ICPP algorithm $[12]$ and Algorithm $[2]$ when decoding C1 ($d = 6$), C2 ($d = 6$ or 7) codes and (128, 64) CCSDS code C3 ($d = 8$) $[16]$ with $E_b/N_0 = 3$ dB and $\epsilon = 10^{-6}$. Since the average (Ave) and worst case (Wor) iteration numbers required by the CPP operation both affect the decoding latency and throughput, we choose them as the performance metrics. It can be seen from TABLE I that the proposed CPP-net can reduce both the average and worst case iteration numbers and in particular, the average iteration number is reduced by nearly half.

Finally, we provide a computational complexity analysis of the ICPP and NCPP algorithms, which is based on the numbers of multiplications (Muls) and additions (Adds) required by the CPP operation. For simplicity, we take C1 code as an example and the analysis for C2 code can be similarly conducted. Note that the complexity of one ICPP iteration (step 13 in Algorithm $[2]$) involves: 1) updating $v$, which requires $d$ Muls and $d$ Adds; 2) calculating $\eta$ needs $2d$ Muls and $d$ Adds. For C1 code, we list the quantized parameters $\{W_a^Q, w_b^Q\}$ of the CPP-net as follows:

\[
W_a^Q = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix}^T, \quad w_b^Q = [-1, 1, 0]^T.
\]

(18)

Therefore, the complexity of one forward pass of the CPP-net can be expressed as 2 Muls and 7 Adds. Based on the average iteration number in TABLE I, the average numbers of Adds and Muls required by the ICPP and NCPP algorithms are listed as TABLE II. Given the fact that the CPP operations are needed in each iteration of the proposed PDD decoder or the ADMM $\ell_2$ decoder, employing Algorithm $[2]$ is able to reduce the computational complexity and decoding latency of these decoders significantly.

### TABLE I: Iteration number comparison.

| Code | Iteration Numbers | Ave | Wor |
|------|-------------------|-----|-----|
| C1   |                   |     |     |
| ICPP | 20.3675           | 72  | 25.7205 |
| NCPP | 11.0653           | 61  | 15.7024 |

### TABLE II: Computation complexity comparison.

| Code | Muls | Adds |
|------|------|------|
| C1   |      |      |
| ICPP | 366.61 | 244.41 |
| NCPP | 201.17 | 139.78 |

**VI. CONCLUSION**

In this work, we presented a novel PDD decoder with for binary linear codes. We showed that other than the minimum polytope based LP problem, the PDD framework can also be utilized to address the fundamental polytope based LP decoding problem. Furthermore, a NCPP algorithm was proposed to reduce the iteration number required by the ICPP algorithm, and it is applicable to all ADMM or PDD based decoders that involve the CPP operations. Simulation results demonstrated the superior performance of the proposed PDD decoder and the effectiveness of the NCPP algorithm for complexity and latency reduction.

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