MAGNETIC HELICITY AND SUBSOLUTIONS IN IDEAL MHD

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Abstract. We show that ideal 2D MHD does not possess weak solutions (or even subsolutions) with compact support in time and non-trivial magnetic field. We also show that the $\Lambda$-convex hull of ideal MHD has empty interior in both 2D and 3D; this is seen by finding suitable $\Lambda$-convex functions. As a consequence we show that mean-square magnetic potential is conserved in 2D by subsolutions and weak limits of solutions in the physically natural energy space $L^\infty_t L^2_x$, and in 3D we show the conservation of magnetic helicity by $L^3$-integrable subsolutions and weak limits of solutions. However, in 3D the $\Lambda$-convex hull is shown to be large enough that nontrivial smooth, compactly supported strict subsolutions exist.

1. Introduction

Magnetohydrodynamics (MHD in short) couples Maxwell’s equations with hydrodynamics to study the macroscopic behaviour of electrically conducting fluids such as plasmas and liquid metals (see [Dav] and [ST]). On the $n$-dimensional torus $T^n = [0,1]^n$ the Cauchy problem for ideal (nonviscous) MHD consists of the equations

\begin{equation}
\partial_t u + \text{div}(u \otimes u - b \otimes b) + \nabla \Pi = 0,
\end{equation}

\begin{equation}
\partial_t b + \text{div}(b \otimes u - u \otimes b) = 0,
\end{equation}

\begin{equation}
\text{div} u = \text{div} b = 0,
\end{equation}

\begin{equation}
u(\cdot, 0) = u_0, b(\cdot, 0) = b_0,
\end{equation}

\begin{equation}
\int_{T^n} u(x,t) \, dx = \int_{T^n} b(x,t) \, dx = 0 \quad \text{for almost every } t \in [0,T],
\end{equation}

where $T > 0$, $u \in L^2_{\text{loc}}(T^n \times [0,T]; \mathbb{R}^n)$ is the velocity field, $b \in L^2_{\text{loc}}(T^n \times [0,T]; \mathbb{R}^n)$ is the magnetic field, $\Pi \in L^1_{\text{loc}}(T^n \times [0,T])$ is the total pressure and the initial datas $u_0, b_0 \in L^2(T^n; \mathbb{R}^n)$ are divergence-free. Equations (1.1)-(1.4) are understood in the sense of distributions, that is,

\begin{align*}
\int_0^T \int_{T^n} [u \cdot \partial_t \varphi + (u \otimes u - b \otimes b) : \nabla \varphi] + \int_{T^n} u_0 \cdot \varphi(\cdot, 0) &= 0, \\
\int_0^T \int_{T^n} [b \cdot \partial_t \varphi + (b \otimes u - u \otimes b) : \nabla \varphi] + \int_{T^n} b_0 \cdot \varphi(\cdot, 0) &= 0,
\end{align*}

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\[
\int_0^T \int_{\mathbb{T}^n} (u \cdot \nabla)\varphi = \int_0^T \int_{\mathbb{T}^n} (b \cdot \nabla)\varphi = 0
\]

for all \( \varphi \in C_c^\infty(\mathbb{T}^n \times [0, T]; \mathbb{R}^n) \). An analogous definition, without condition (1.5), is given in \( \mathbb{R}^n \) with \( u, b \in L^2_{\text{loc}}(\mathbb{R}^n \times [0, T]; \mathbb{R}^n) \) and test functions \( \varphi \in C_c^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^n) \). The Euler equations are a special case of ideal MHD where \( b \equiv 0 \).

Starting from the pioneering work of De Lellis and Székelyhidi on the Euler equations in [DLS09], the Tartar framework has been used to show the existence of solutions with compact support in time for many equations of hydrodynamics. Such pathological weak solutions were already known to exist in the case of Euler equations (see \cite{Sch, Shn}) but the method of De Lellis and Székelyhidi is very robust and extends even to subsolutions and weak limits of solutions. See 

\cite{Dan, DS, DLS09, DLS10, DLS13, DLS14, DLS16, Eyi, Ise13, Ise16, Ise17, IO, Ons and Shv1}).

In this work we instead use the Tartar framework to show conservation of integral quantities under weak assumptions. In the Tartar framework the ideal MHD equations (1.1)–(1.3) are decoupled into a set of linear partial differential equations and a pointwise constraint: the equations are

\begin{align*}
\text{(1.6)} & \quad \text{div} \, u = \text{div} \, b = 0, \\
\text{(1.7)} & \quad \partial_t u + \text{div} \, S = 0, \\
\text{(1.8)} & \quad \partial_t b + \text{div} \, A = 0,
\end{align*}

where \( S \) takes values in the set \( S^{n \times n} \) of symmetric matrices and \( A \) takes values in the set \( A^{n \times n} \) of antisymmetric matrices and the constraint set is

\[
K := \{(u, b, S, A) \in \mathbb{R}^n \times \mathbb{R}^n \times S^{n \times n} \times A^{n \times n} : S = u \otimes u - b \otimes b + \Pi I, \, \Pi \in \mathbb{R}, \\
A = b \otimes u - u \otimes b \}.
\]

The wave cone \( \Lambda \) is, loosely speaking, the set of directions in which 1-dimensional oscillating waves satisfy equations (1.3)–(1.8), and the \( \Lambda \)-convex hull \( K^\Lambda \) consists of those points that cannot be separated from \( K \) by functions that are convex in the directions of \( \Lambda \) (see \S 3.2 for the exact definitions). A solution of (1.5)–(1.8) that takes values in \( K^\Lambda \) at a.e. \((x, t) \in \mathbb{T}^n \times [0, T]\) is called a subsolution.

It is a classical fact that smooth solutions of (1.1)–(1.4) conserve magnetic helicity in 3D and mean-square magnetic potential in 2D. We show that the conservation is in fact very robust and extends even to subsolutions and weak limits of solutions. This phenomenon is seen here as a reflection of the shape of the \( \Lambda \)-convex hull of the constraint set \( K \). As a corollary, in 2D, there exist no weak solutions of the MHD equations (1.1)–(1.5) that have compact support in time and non-trivial magnetic field \( b \).

In 2D we consider weak solutions of (1.1)–(1.5) where \( u \) and \( b \) belong to the natural energy space \( L^\infty_t L^2_x(\mathbb{T}^2 \times [0, T]; \mathbb{R}^2) \). By redefining \( u \) and \( b \) in a set of times of measure zero we may then assume that \( u, b \in C_w([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2)) \) (where we denote \( v \in C_w([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2)) \) when \( v \in L^\infty_t L^2_x(\mathbb{T}^2 \times [0, T]; \mathbb{R}^2) \) and \( t_j \rightarrow t \) implies \( v(\cdot, t_j) \rightharpoonup v(\cdot, t) \) in \( L^2(\mathbb{T}^2; \mathbb{R}^2) \) for every \( t \in [0, T] \); this can be seen by a modification of \cite{Gal} Lemmas 2.2 and 2.4). In our result \( \Psi \in C_w([0, T]; W^{1,2}(\mathbb{T}^2)) \)}
is the unique stream function of \( b \) that satisfies \( \int_{\mathbb{T}^2} \Psi(x, t) \, dx = 0 \) for every \( t \in [0, T] \) (see Lemma 2.1). The result is new also for weak solutions of (1.1)–(1.3).

**Theorem 1.1.** If \( u, b \in C_w([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2)) \), \( S \in L^1_{loc}(\mathbb{T}^2 \times [0, T]; \mathcal{A}^{2 \times 2}) \) and \( A \in L^1_{loc}((\mathbb{T}^2 \times [0, T]; \mathbb{A}^{2 \times 2}) \) form a solution of (1.4)–(1.8) such that \((u, b, S, A)(x, t) \in K^A \) a.e. \((x, t) \in \mathbb{T}^2 \times [0, T]\), then the mean-square magnetic potential \( \int_{\mathbb{T}^2} |\Psi(x, t)|^2 \, dx \) is constant in \( t \).

Theorem 1.1 is a reflection of the existence of a suitable \( \Lambda \)-convex function which shows that if \((u, b, S, A)(x, t) \in K^A \) a.e., then \( A = b \otimes u - u \otimes b \) and so (1.2) is satisfied (see [4]). Another main ingredient of the proof is the fact that the evolution of \( \Psi \) can be described in terms of the Jacobian of a two-dimensional map: \( \partial_t \Psi - J_{(\Psi, \Phi)} = 0 \), where \( \Phi \) is the stream function of \( u \) (see Lemma 2.2). Thus we can use the Hardy space theory of Jacobians: first, if \( u \) and \( b \) are smooth, we use an integration by parts to compute

\[
\partial_t \frac{1}{2} \int_{\mathbb{T}^2} |\Psi(x, t)|^2 \, dx = \int_{\mathbb{T}^2} \Psi(x, t) J_{(\Psi, \Phi)}(x, t) \, dx = -\int_{\mathbb{T}^2} \Phi(x, t) J_{(\Psi, \Phi)}(x, t) \, dx = 0;
\]

in the general case where \( u, b \in C_w([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2)) \) we use Sobolev embedding to get \( \Psi \in L^\infty_{\text{loc}}(\mathbb{T}^2 \times [0, T]) \) and the \( \mathcal{H}^1 \) regularity theory of Coifman, Lions, Meyer and Sermes from [CLMS] to get \( J_{(\Psi, \Phi)} \in L^\infty_{\text{loc}}(\mathbb{T}^2 \times [0, T]) \), and Fefferman’s classical \( \mathcal{H}^1 \)–BMO duality result from [FS] is then used to provide the necessary approximation argument. The proof is presented in [12] where we also show that the mean-square magnetic potential is conserved by weak limits of solutions (see Remark 4.5). Theorem 1.1 implies Corollary 1.2 by using the Poincaré inequality at every \( t \in [0, T] \) to estimate \( \int_{\mathbb{T}^2} |b(x, t)|^2 \, dx = \int_{\mathbb{T}^2} |\nabla \Psi(x, t)|^2 \, dx \geq C \int_{\mathbb{T}^2} |\Psi(x, t)|^2 \, dx \).

**Corollary 1.2.** Suppose that \( u, b \in C_w([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2)) \) satisfy (1.2)–(1.6). Then either \( b \equiv 0 \) or there exists \( C > 0 \) such that \( \int_{\mathbb{T}^2} |b(x, t)|^2 \, dx \geq C \) for every \( t \in [0, T] \).

Corollary 1.2 rules out convex integration solutions that are in the energy space and compactly supported in time (aside from Euler solutions for which \( b \equiv 0 \)). Note also that by Corollary 1.2 it is not possible to construct weak solutions which dissipate magnetic energy to zero – in fact, one can easily replace \( T \) by \( \infty \) in Theorem 1.1 and Corollary 1.2.

In 3D, in contrast to Corollary 1.2 Bronzi, Lopes Filho and Nussenzveig Lopes used in [BLFNL] convex integration in the Tartar framework to show that there exist infinitely many bounded weak solutions of (1.1)–(1.3) that are of the symmetry reduced form

\[
(1.9) \quad u(x_1, x_2, x_3, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0), \quad b(x_1, x_2, x_3, t) = (0, 0, b_3(x_1, x_2, t)),
\]

compactly supported in time and with \( u, b \not\equiv 0 \). The solutions were obtained cleverly via two-dimensional Euler equations with a passive tracer (see [6.5]). Note, however, that the solutions are independent of the \( x_3 \) variable. In particular, the construction of [BLFNL] does not give solutions of MHD that are compactly supported in space.

We put the results of [BLFNL] in context. In 3D we may in (1.8) identify the antisymmetric matrix \( A = [a_{ij}]_{i,j=1}^3 \) with \( a := (a_{23}, a_{31}, a_{12}) \in \mathbb{R}^3 \) and write (1.8) as

\[
(1.10) \quad \partial_t b + \nabla \times a = 0
\]
C integration solutions and the uniqueness of the Cauchy problem. The general problem whether there exists a dichotomy between the existence of convex b determines a unique with the relaxation of MHD. The third one is whether, in 2D, every Cauchy data explicit computation of the \( \Lambda \)-convex hulls and the question whether they coincide of the convex integration approach works in this setting. The second one is an \( \Lambda \)-affine quantity a \( \cdot \) b vanishes in \( \Lambda \). This implies that \( \Lambda \) has empty interior, which would normally make the use of convex integration very difficult (for a situation where a non-linear pointwise constraint is successfully understood see [MS99]). Under the restrictions (1.9) the constraint a \( \cdot \) b = 0 does not cause trouble precisely because a and b take values in orthogonal subspaces of \( \mathbb{R}^3 \) (see (5.5).

We use the Tartar framework to show the following 3D analogue of Theorem 1.3 where \( \Psi \) is a vector potential of b, i.e. \( \nabla \times \Psi = b \) (see Lemma 2.5).

**Theorem 1.3.** Suppose that \( u, b \in L^3(\mathbb{T}^3 \times ]0, T[; \mathbb{R}^3) \), \( S \in L^1_{\text{loc}}(\mathbb{T}^3 \times ]0, T[; \mathbb{S}^{3 \times 3}) \) and a \( \in L^{3/2}(\mathbb{T}^3 \times ]0, T[; \mathbb{R}^3) \) form a solution of (1.5)–(1.7), (1.10) that takes values in \( \Lambda \). Then the magnetic helicity \( \int_{\mathbb{T}^3} \Psi(x, t) \cdot b(x, t) \, dx \) is constant a.e. in t.

For solutions of the linearized MHD equations (1.5)–(1.7), (1.10) on \( \mathbb{T}^3 \), the time evolution of magnetic helicity is given by

\[
(1.11) \quad \partial_t \int_{\mathbb{T}^3} \Psi(x, t) \cdot b(x, t) \, dx = -2 \int_{\mathbb{T}^3} a(x, t) \cdot b(x, t) \, dx,
\]

and Theorem 1.3 follows from the fact that a \( \cdot \) b vanishes in \( \Lambda \). The details of the proof are presented in 6.2.

As a quadratic \( \Lambda \)-affine quantity a \( \cdot \) b is weakly continuous (see Lemma 5.3), and so, using (1.11), we also show in Theorem 5.3 that magnetic helicity is conserved by weak limits of \( L^3 \) solutions of 3D MHD. The same principle is behind the conservation of mean-square magnetic potential by weak limits of solutions in 2D; there the quadratic \( \Lambda \)-affine quantity is b \( \times \) u = \( J_{(\Psi, \Psi)} \).

Despite the remarkable robustness of magnetic helicity conservation, compactly supported convex integration solutions cannot, at this point, be ruled out in 3D MHD. In fact, the \( \Lambda \)-convex hull K \( ^\Lambda \) turns out to have non-empty relative interior (relative to the constraint a \( \cdot \) b = 0). This result, recorded in the following theorem, is the technically most difficult part of the paper and requires careful analysis of the interplay between K and \( \Lambda \).

**Theorem 1.4.** In 3D MHD, \( \operatorname{int}(K^\Lambda) \neq \emptyset \). However, the point \( (0, 0, 0, 0) \) belongs to the relative interior of \( K^\Lambda \) in the set \( \{(u, b, S, a): b \cdot a = 0\} \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^{3 \times 3} \times \mathbb{R}^3 \).

Theorem 1.4 suggests defining strict subsolutions in analogy to incompressible Euler equations and many other equations of fluid dynamics (see 6.1 for the precise definition). The following 3D result is in stark contrast to Theorem 1.1.

**Theorem 1.5.** In \( \mathbb{R}^3 \) the ideal MHD equations have strict subsolutions \( (u, b, S, a) \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^{3 \times 3} \times \mathbb{R}^3) \) with u, b \( \neq \) 0.

Three interesting problems arise from our work. The first one is whether the emptiness of the interior of the 3D \( \Lambda \)-convex hull can be overcome and some variant of the convex integration approach works in this setting. The second one is an explicit computation of the \( \Lambda \)-convex hulls and the question whether they coincide with the relaxation of MHD. The third one is whether, in 2D, every Cauchy data determines a unique b in the energy space. The last question is a special case of the general problem whether there exists a dichotomy between the existence of convex integration solutions and the uniqueness of the Cauchy problem.
2. Background

In this section we give results on stream functions and vector and scalar potentials, and we present proofs in cases where they are difficult to find in the literature. We also discuss previously known results on conserved integral quantities in MHD.

2.1. Stream functions in 2D. In the following standard lemma we find stream functions for solutions of the ideal MHD equations on the torus $\mathbb{T}^2$. The lemma concerns time-dependent mappings in Bochner spaces, and for more information on Bochner spaces we refer to [HVNVW].

**Lemma 2.1.** If $v \in C_v([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2))$ satisfies $\text{div} \, v = 0$ and $\int_{T^2} v(x, t) \, dx = 0$ for every $t \in [0, T]$, then there exists a unique function $\Theta \in C_v([0, T]; W^{1, 2}(\mathbb{T}^2)) \cap C([0, T]; L^2(\mathbb{T}^2))$ with $-\nabla^2 \Theta := (\partial_2 \Theta, -\partial_1 \Theta) = v$ and $\int_{T^2} \Theta(x, t) \, dx = 0$ for every $t \in [0, T]$.

**Sketch of proof.** At every $t \in [0, T]$ we get $\text{div} \, v(\cdot, t) = 0$ (by integrating $v$ against suitable test functions of the form $\varphi_1(x)\varphi_2(t)$). The existence and uniqueness of $\Theta(\cdot, t)$ is proven by standard Fourier analysis (similar to the proof of Lemma 2.5), and Poincaré inequality gives $||\Theta(\cdot, t)||_{W^{1, 2}(\mathbb{T}^2)} \lesssim ||v(\cdot, t)||_{L^2}$. The mapping $t \mapsto \Theta(\cdot, t) \colon [0, T] \to W^{1, 2}(\mathbb{T}^2)$ is strongly measurable by the linearity and boundedness of the operator that maps $v(\cdot, t)$ to $\Theta(\cdot, t)$ and the strong measurability of $t \mapsto v(\cdot, t)$. Now a standard application of the Rellich-Kondrachov Theorem gives $\Theta \in C([0, T]; L^2(\mathbb{T}^2))$.

One of the main ideas behind Theorem 1.1 is that the time evolution of the stream function of $b$ is governed by a Jacobian determinant which is an $\mathcal{H}^1$ integrable quantity.

**Lemma 2.2.** Suppose that $u, b \in C_v([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2))$ satisfy (1.2) and that $\Phi, \Psi \in C_v([0, T]; W^{1, 2}(\mathbb{T}^2))$ are the stream functions of $u$ and $b$ given by Lemma 2.1. Then

$$\partial_t \Psi - J(\Psi, \Phi) = 0$$

in $\mathbb{T}^2 \times [0, T]$.

Note that Lemma 2.2 does not require (1.1) as an assumption. Before presenting the proof of Lemma 2.2 we fix, for the rest of this article, a mollifier $\chi \in C_c^\infty(\mathbb{T}^2 \times \mathbb{R})$ of the tensor product form $\chi(x, t) = \chi_x(t)\chi_t(x)$, where $\int_{\mathbb{T}^2} \chi_x(x) \, dx = \int_{-\infty}^{\infty} \chi_t(t) \, dt = 1$. We assume that $\chi$ is even and supp($\chi$) $\subset \mathbb{T}^2 \times ]-1, 1[$. When $\delta > 0$, we define $\chi^\delta(x, t) := \delta^{-3}\chi(x/\delta, t/\delta)$. We also denote, e.g., $\Psi_\delta := \Psi * \chi^\delta$, where $\Psi$ is the stream function of $b$. Note that for every $\delta > 0$ and every $t \in [\delta, T - \delta]$,

$$\int_{T^2} \Psi_\delta(x, t) \, dx = \int_{t-\delta}^{t+\delta} \int_{T^2} \Psi(y, s) \int_{\mathbb{T}^2} \chi^\delta_x(x - y) \, dx \, dy \, ds = 0.$$

**Proof.** The second MHD equation (1.2) can be written as

$$-\nabla^2 \partial_t \Psi + \text{div}((\nabla^\perp \Psi \otimes \nabla^\perp \Phi - \nabla^\perp \Phi \otimes \nabla^\perp \Psi) = 0.$$

Note that $-\nabla^2 J(\Psi, \Phi) = \text{div}((\nabla^\perp \Psi \otimes \nabla^\perp \Phi - \nabla^\perp \Phi \otimes \nabla^\perp \Psi)$, which implies that

$$\partial_t \Psi - J(\Psi, \Phi) = : g \in \mathcal{D}'(\mathbb{T}^2 \times [0, T])$$

satisfies $\nabla g = 0$. By (2.2) and a similar formula for $J(\Psi, \Phi)$ we therefore get $g_\delta = 0$ in $\mathbb{T}^2 \times [\delta, T - \delta]$ for every $\delta \in [0, T/2]$, and so $g = 0$. $\square$
Remark 2.3. As pointed out to the authors by László Székelyhidi Jr., in the 2D Euler equations the vorticity satisfies an identity similar to (2.1): when \( \phi \) is the stream function and \( \omega := \nabla \times u \) is the vorticity of the velocity \( u \), we have \( \partial_t \omega - J(\omega, \phi) = 0 \) (see [VN, p. 442]).

2.2. Vector and scalar potentials in 3D. We will present 3D analogues of Lemmas 2.1 and 2.2, and we first recall the Helmholtz-Hodge decomposition in suitable Bochner spaces can be proved by using Lemma 2.4 at a.e. time \( t \).

Lemma 2.4. Suppose \( 1 < p < \infty \). Then every \( v \in L^p(T^3, \mathbb{R}^3) \) with \( \int_{T^3} v(x) \, dx = 0 \) can be written uniquely as

\[
v = u + \nabla g,
\]

where \( u \in L^p(T^3, \mathbb{R}^3) \) satisfies \( \text{div} u = 0 \), \( \int_{T^3} u(x) \, dx = 0 \) and \( \int_{T^3} |u(x)|^p \, dx \lesssim_p \int_{T^3} |v(x)|^p \, dx \) whereas \( g \in W^{1, p}(T^3) \) satisfies \( \int_{T^3} g(x) \, dx = 0 \) and \( \int_{T^3} |\nabla g(x)|^p \, dx \lesssim_p \int_{T^3} |v(x)|^p \, dx \).

For time-dependent mappings in \( L^p(T^3 \times [0, T]; \mathbb{R}^3) \), a Helmholtz-Weyl decomposition in suitable Bochner spaces can be proved by using Lemma 2.4 at a.e. time \( t \); strong measurability follows from the fact that the operators \( v \mapsto u \) and \( v \mapsto \nabla g \) of Lemma 2.4 are linear. However, the particular form of the Helmholtz-Weyl decomposition needed here appears difficult to find in the literature, and we therefore sketch a proof. Lemma 2.5 also provides information on the divergence-free component that is crucial in §5.2.

Lemma 2.5. Let \( 1 < p < \infty \). If \( v \in L^p(T^3 \times [0, T]; \mathbb{R}^3) \) satisfies \( \int_{T^3} v(x,t) \, dx = 0 \) a.e. \( t \in ]0, T[ \), then \( v \) can be uniquely written as

\[
v = \nabla \times \Theta + \nabla g,
\]

where \( \Theta \in L^p_{L^2} W^{1, p}(T^3 \times [0, T]; \mathbb{R}^3) \) satisfies \( \int_{T^3} \Theta(x,t) \, dx = 0 \) a.e. \( t \in ]0, T[ \) and \( \text{div} \Theta = 0 \) whereas \( g \in L^p_{L^2} W^{1, p}(T^3 \times [0, T[ \) satisfies \( \int_{T^3} g(x,t) \, dx = 0 \) a.e. \( t \in ]0, T[ \). Furthermore,

\[
\|\Theta\|_{L^p_{L^2} W^{1, p} \times [0, T[} + \|g\|_{L^p_{L^2} W^{1, p} \times [0, T[} \lesssim_p \|v\|_{L^p}.
\]

If \( \text{div} v = 0 \), then \( g = 0 \), and if \( \nabla \times v = 0 \), then \( \Theta = 0 \). As in 2D, we fix for the rest of the article an even mollifier \( \chi \in C_0^\infty(T^3 \times \mathbb{R}) \) of the tensor product form \( \chi(x,t) = \chi^2(x) \chi^4(t) \), where \( \int_{T^3} \chi^2(x) \, dx = \int_{-\infty}^{\infty} \chi^4(t) \, dt = 1 \) and \( \text{supp}(\chi) \subset T^3 \times ]-1, 1[ \). When \( \delta > 0 \), we denote \( \chi^\delta(x,t) := \delta^{-4} \chi(x/\delta, t/\delta) \).

Proof. We get \( \Theta \) and \( g \) as limits of smooth mappings. Let \( 0 < \delta < \epsilon < T - \epsilon \). We denote

\[
v_\delta(x,t) := v \ast \chi^\delta := \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{c_k^\delta(t) \cdot k}{|k|^2} k e^{2\pi i k \cdot x} = \chi^\delta(x, t) + \nabla g^\delta(x, t)
\]

for every \( x \in T^3 \).
Now $T^\delta(x, t) := \sum_{k \in \mathbb{Z} \setminus \{0\}} |2\pi k|^{-2} a_k(x)e^{2\pi ik \cdot x}$ is a solution of the Poisson equation $-\Delta T^\delta(x, t) = u^\delta(x, t)$ and furthermore $\text{div } T^\delta(x, t) = 0$. Thus
\[
\nabla \times (\nabla \times T^\delta)(x, t) = \nabla \text{div } T^\delta(x, t) - \Delta T^\delta(x, t) = u^\delta(x, t).
\]
We set $\Theta^\delta(x, t) := \nabla \times T^\delta(x, t)$ so that $u^\delta(x, t) = \nabla \times \Theta^\delta(x, t)$. The natural norm bound $\|\partial_i \partial_j T^\delta(x, t)\|_{L^p(T^3)} \lesssim_p \|\Delta T^\delta\|_{L^p(T^3)}$ for all $i, j \in \{1, 2, 3\}$ (see RRS Theorem B.7), combined with Lemma 2.4 and the Poincaré inequality, yields
\[
\|\Theta^\delta(x, t)\|_{W^{1, p}(T^3)} + \|g^\delta(x, t)\|_{W^{1, p}(T^3)} \lesssim_p \|u_3^\delta(x, t)\|_{L^p(T^3)}.
\]
Now a decomposition $v = \nabla \times \Theta + \nabla g$ in $T^3 \times [\epsilon, T - \epsilon]$ is found via standard limiting arguments.

Uniqueness of $g$ in $T^3 \times [\epsilon, T - \epsilon]$ follows from Lemma 2.4. For the uniqueness of $\Theta$ suppose $\hat{\Theta}$ is another vector potential that satisfies the conditions of Lemma 2.5 in $T^3 \times [\epsilon, T - \epsilon]$. Then $\Delta(\Theta - \hat{\Theta}) = -\nabla \times (\nabla \times (\Theta - \hat{\Theta})) = 0$ and $\int_{T^3}(\Theta(x, t) - \hat{\Theta}(x, t)) \, dx = 0$ a.e. $t \in [\epsilon, T - \epsilon]$, which leads to $\Theta = \hat{\Theta}$ since the periodic extension of $(\Theta - \hat{\Theta})$ is bounded and harmonic. This gives the unique Hodge decomposition in $T^3 \times [0, T]$ with the desired norm bounds.

In order to finish the proof of the lemma suppose that $\text{div } v = 0$. Then $-\Delta g = \text{div}(v - \nabla \times \Theta) = 0$ and $\int_{T^3} g(x, t) \, dx = 0$ for a.e. $t \in [0, T]$, which implies that $g = 0$. Similarly, $\nabla \times v = 0$ leads to $-\Delta \Theta = \nabla \times (\nabla \times \Theta) = \nabla \times (v - \nabla g) = 0$, yielding $\Theta = 0$. □

Lemma 2.6 implies the following Poincaré-type lemma with norm bounds for solutions of linearized 3D MHD.

**Lemma 2.6.** Suppose $b \in L^3(T^3 \times [0, T]; \mathbb{R}^3)$ and $a \in L^{3/2}(T^3 \times [0, T]; \mathbb{R}^3)$ satisfy
\[
\text{div } b = 0, \quad \partial_t b + \nabla \times a = 0.
\]
Then there exist unique $\Psi \in L^3_x W^{1,3}_t(T^3 \times [0, T]; \mathbb{R}^3)$ and $g \in L^3_x W^{1,3/2}_t(T^3 \times [0, T])$ such that
\[
b = \nabla \times \Psi \quad \text{and} \quad \partial_t \Psi + a - \int_{T^3} a(y, \cdot) \, dy = \nabla g
\]
with $\int_{T^3} \Psi(x, t) \, dx = 0$ and $\int_{T^3} g(x, t) \, dx = 0$ for a.e. $t \in [0, T]$ and $\text{div } \Psi = 0$.
Furthermore,
\[
\|\Psi\|_{L^3_x W^{1,3}_t} \lesssim \|b\|_{L^3} \quad \text{and} \quad \|\partial_t \Psi\|_{L^{3/2}_x W^{1,3/2}_t} \lesssim \|a\|_{L^{3/2}_x}.
\]

**Proof.** The only part of Lemma 2.6 that does not follow immediately from Lemma 2.5 is the claim that in the decomposition $a - \int_{T^3} a(y, \cdot) \, dy = \nabla \times \Theta + \nabla g$ we have $\nabla \times \Theta = -\partial_t \Psi$. In order to show this let $0 < \epsilon < T/2$. Whenever $0 < \delta < \epsilon$, we write $a_\delta - (\int_{T^3} a(y, \cdot) \, dy)_\delta = \nabla \times \Theta_\delta + \nabla g_\delta$. On the other hand, from the equation $\nabla \times (\partial_t \Psi + a - \int_{T^3} a(y, \cdot) \, dy) = 0$ and Lemma 2.5 we get $\partial_t \Psi_\delta + a_\delta - (\int_{T^3} a(y, \cdot) \, dy)_\delta = \nabla \hat{g}$. By Lemma 2.5, $\text{div } \hat{g} = 0$, and so $\text{div } \partial_t \Psi_\delta = 0$. Since $\int_{T^3} \partial_t \Psi_\delta(x, t) \, dx = 0$ for every $t \in [\epsilon, T - \epsilon]$, the uniqueness of the Helmholtz-Weyl decomposition in Lemma 2.4 implies that $\nabla \times \Theta_\delta = -\partial_t \Psi_\delta$ and $g_\delta = \hat{g}$. Thus $\nabla \times \Theta = \lim_{\delta \searrow 0} \nabla \times \Theta_\delta = \lim_{\delta \searrow 0} -\partial_t \Psi_\delta = -\partial_t \Psi$ in $\mathcal{D}'(T^3 \times [\epsilon, T - \epsilon]; \mathbb{R}^3)$, which proves the claim. □
2.3. Classically conserved quantities of ideal MHD. We define three classically conserved quantities of ideal 3D MHD on the torus $\mathbb{T}^3$; analogous definitions, under suitable assumptions, are available in $\mathbb{R}^3$.

**Definition 2.7.** Suppose that $u, b \in C^\infty(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)$ and $\Pi \in C^\infty(\mathbb{T}^3 \times [0, T])$ satisfy the ideal MHD equations and that $\Psi \in C^\infty(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)$ satisfies $\nabla \times \Psi = b$ and $\int_{\mathbb{T}^3} \Psi(x, t) \, dx = 0$ for every $t \in [0, T]$. The total energy, magnetic helicity and cross helicity of $(u, b, \Pi)$ are defined as

$$
\frac{1}{2} \int_{\mathbb{T}^3} (|u(x, t)|^2 + |b(x, t)|^2) \, dx,
$$

$$
\int_{\mathbb{T}^3} \Psi(x, t) \cdot b(x, t) \, dx,
$$

$$
\int_{\mathbb{T}^3} u(x, t) \cdot b(x, t) \, dx.
$$

All three quantities defined above are conserved in time by smooth solutions. For results on total energy and cross helicity conservation for weak solutions we refer to [CKS], [KL] and [Yu]. Conservation of the magnetic helicity was shown in [CKS] for $u \in C([0, T]; B^2_{3,\infty}(\mathbb{T}^3; \mathbb{R}^3))$ and $b \in C([0, T]; B^2_{3,\infty}(\mathbb{T}^3; \mathbb{R}^3))$ with $\alpha_1 + 2\alpha_2 > 0$. In [KL], Kang and Lee showed magnetic helicity conservation under the assumption that $u, b \in C_w([0, T]; L^2(\mathbb{R}^3; \mathbb{R}^3)) \cap L^2_1 L^3_3(\mathbb{R}^3 \times [0, T]; \mathbb{R}^3)$, although the assumptions on $\Psi$ are not made completely explicit. Theorems 1.3 and 5.5 generalise magnetic helicity conservation to subsolutions and weak limits of $L^3$ solutions. In 2D, magnetic helicity has the following natural counterpart.

**Definition 2.8.** Suppose $u, b \in C^\infty(\mathbb{T}^2 \times [0, T]; \mathbb{R}^2)$ and $\Pi \in C^\infty(\mathbb{T}^2 \times [0, T])$ is the stream function $\Psi$ of $b$ with $\int_{\mathbb{T}^2} \Psi(x, t) \, dx = 0$ for every $t \in [0, T]$. The mean-square magnetic potential of $(u, b, \Pi)$ is defined as $\int_{\mathbb{T}^2} |\Psi(x, t)|^2 \, dx$.

In [CKS], the conservation of the mean-square magnetic potential is shown for $u$ and $\Psi$ in the Besov spaces $C([0, T]; B^2_{3,\infty}(\mathbb{T}^2; \mathbb{R}^2))$ and $C([0, T]; B^2_{3,\infty}(\mathbb{T}^2; \mathbb{R}^2))$ respectively, where $\alpha_1 + 2\alpha_2 > 1$. In Theorem 1.4 we prove conservation under the assumption that $u, b \in C_w([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2))$.

3. Ideal MHD in the Tartar framework

This article is devoted to studying the MHD equations in the Tartar framework, and in this section we recall many of the relevant definitions. We also compute the wave cone in both 2D and 3D. The computation of the wave cone and Theorem 6.3 are also partial results towards the existence of compactly supported convex integration solutions of 3D MHD. For more information on the Tartar framework see [Tar79], [Tar83] and in the context of fluid dynamics see [CFG], [DLS12], [Sze].

3.1. Linearization of MHD in Elsässer variables. In order to facilitate the ensuing computations we use the Elsässer (characteristic) variables $z^\pm := u \pm b$ to rewrite the MHD equations in terms of $(z^+, z^-, \Pi)$ in the symmetric form

$$
\begin{align*}
\partial_z z^+ + \text{div}(z^+ \otimes z^- + \Pi I) &= 0, \\
\partial_z z^- + \text{div}(z^- \otimes z^+ + \Pi I) &= 0, \\
\text{div} z^\pm &= 0
\end{align*}
$$

(3.1)
We also define normalized versions of $K$ in this formalism we denote $M := S + A : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times n}$ so that $S = (M + M^T)/2$ and $A = (M - M^T)/2$.

**Definition 3.1.** The linear partial differential operator $\mathcal{L} : \mathcal{D}'(\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times n}) \to \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R})$ is defined by

$$
\mathcal{L}(z^+, z^-, M) := \begin{bmatrix}
\partial_t z^+ + \text{div } M \\
\partial_t z^- + \text{div } M^T \\
\text{div } z^+ \\
\text{div } z^-
\end{bmatrix}.
$$

On the torus $\mathbb{T}^n$ we add to $\mathcal{L}$ the two components $t \mapsto \int_{\mathbb{T}^n} z^+(x,t) \, dx$.

We decouple (3.1) into the linear equation $\mathcal{L}(z^+, z^-, M) = 0$ and the pointwise constraint that $(z^+, z^-, M)$ takes values in the set

$$K := \{(z^+, z^-, M) : M = z^+ \otimes z^- + \Pi I, \Pi \in \mathbb{R}\}.$$

We also define normalized versions of $K$ by setting

$$K_{r,s} := \{(z^+, z^-, M) : |z^+| = r, |z^-| = s, M = z^+ \otimes z^- + \Pi I, |\Pi| \leq rs\}$$

for all $r, s > 0$.

**Remark 3.2.** The use of Eötvös variables is natural in the context of convex integration for MHD. Indeed, it is still an open problem whether in $\mathbb{R}^3$ there exist compactly supported weak solutions of MHD which do not conserve cross helicity. This suggests, in analogy to the work done on many other equations of fluid dynamics, an attempt to prescribe the total energy and the cross helicity (densities) at every time $t$. One would achieve this if one could prescribe $|z^+|$ and $|z^-|$, as $((|u|^2 + |b|^2)/2 = (|z^+|^2 + |z^-|^2)/4$ and $u \cdot b = (|z^+|^2 - |z^-|^2)/4$. In contrast, $u \cdot b$ obviously cannot be written in terms of $|u|$ and $|b|$.

### 3.2. The wave cone and the lamination convex hull

**Plane waves** are one-dimensional oscillations $(x,t) \mapsto h((x,t)-(\xi,c))(\alpha,\beta,M)$ with $(\alpha,\beta,M) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$, $\xi, c \in (\mathbb{R}^n \times \mathbb{R}) \setminus \{0\}$ and $h : \mathbb{R} \to \mathbb{R}$, and the wave cone, defined below, gives the set of directions of plane waves satisfying the linearized MHD equation $\mathcal{L}(z^+, z^-, M) = 0$. Here and in the sequel we will often use the isomorphism

$$(\alpha,\beta,M) \mapsto \begin{bmatrix}
M \\
\beta^T \\
\alpha \\
0
\end{bmatrix} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \to \mathbb{R}_0^{(n+1) \times (n+1)},$$

where $\mathbb{R}_0^{(n+1) \times (n+1)} := \{A \in \mathbb{R}^{(n+1) \times (n+1)} : a_{n+1,n+1} = 0\}$. The linearized MHD equation $\mathcal{L}(z^+, z^-, M) = 0$ can be written in divergence form as

$$\text{div}_{x,t} \begin{bmatrix}
M \\
(z^-)^T \\
0
\end{bmatrix} = \text{div}_{x,t} \begin{bmatrix}
M^T \\
(z^+)^T \\
0
\end{bmatrix} = 0.$$

**Definition 3.3.** The **wave cone** of ideal MHD is

$$\Lambda_0 = \left\{ V \in \mathbb{R}^{4 \times 4}_0 : \exists \begin{bmatrix}
\xi \\
c
\end{bmatrix}_c \in \mathbb{R}^{n+1} \setminus \{0\} \text{ such that } V \begin{bmatrix}
\xi \\
c
\end{bmatrix} = V^T \begin{bmatrix}
\xi \\
c
\end{bmatrix} = 0 \right\}.$$

We also denote

$$\Lambda = \left\{ V \in \mathbb{R}^{4 \times 4}_0 : \exists \begin{bmatrix}
\xi \\
c
\end{bmatrix}_c \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R} \text{ such that } V \begin{bmatrix}
\xi \\
c
\end{bmatrix} = V^T \begin{bmatrix}
\xi \\
c
\end{bmatrix} = 0 \right\}.$$
We will use the subset $\Lambda$ of the wave cone $\Lambda_0$ as it is much easier to use in convex integration. For convex integration it is important to gain enough information about the lamination convex hull of $K$, which we next define. Given a set $Y \subset \mathbb{R}_0^{(n+1) \times (n+1)}$ we denote $Y^{0,\Lambda} := Y$ and define inductively

$$Y^{N+1,\Lambda} := Y^{N,\Lambda} \cup \{\lambda V + (1 - \lambda)W : \lambda \in [0, 1], \ V, W \in Y^{N,\Lambda}, \ V - W \in \Lambda\}$$

for all $N \in \mathbb{N}_0$.

**Definition 3.4.** When $Y \subset \mathbb{R}_0^{(n+1) \times (n+1)}$, the lamination convex hull of $Y$ (with respect to $\Lambda$) is

$$Y^{lc,\Lambda} := \bigcup_{N \geq 0} Y^{N,\Lambda}.$$

In addition to the lamination convex hull, another, potentially larger, hull is used in convex integration theory. In order to define it recall the following

**Definition 3.5.** A function $f : \mathbb{R}_0^{(n+1) \times (n+1)} \to \mathbb{R}$ is said to be $\Lambda$-convex if the function $t \mapsto f(V + tW) : \mathbb{R} \to \mathbb{R}$ is convex for every $V \in \mathbb{R}_0^{(n+1) \times (n+1)}$ and every $W \in \Lambda$.

While the lamination convex hull is defined by taking convex combinations, the $\Lambda$-convex hull of $Y \subset \mathbb{R}_0^{(n+1) \times (n+1)}$ is defined as the set of points that cannot be separated from $Y$ by $\Lambda$-convex functions.

**Definition 3.6.** When $Y \subset \mathbb{R}_0^{(n+1) \times (n+1)}$, the $\Lambda$-convex hull $Y^\Lambda$ consists of points $W \in \mathbb{R}_0^{(n+1) \times (n+1)}$ with the following property: if $f : \mathbb{R}_0^{(n+1) \times (n+1)} \to \mathbb{R}$ is $\Lambda$-convex and $f|_Y \leq 0$, then $f(W) \leq 0$.

3.3. Computation of the wave cone in 3D MHD. Recall that $(\alpha, \beta, M) \in \Lambda$ if and only if there exists $(\xi, c) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ such that

$$\begin{cases}
M\xi + c\alpha = 0, \\
M^T\xi + c\beta = 0, \\
\alpha \cdot \xi = 0, \\
\beta \cdot \xi = 0.
\end{cases}$$

(3.4)

We split the computation of $\Lambda$ into a few special cases. When $\alpha, \beta \in \mathbb{R}^3$ with $\alpha \times \beta \neq 0$, the nine tensor products $\alpha \otimes \alpha$, $\alpha \otimes \beta$, $\alpha \otimes (\alpha \times \beta)$, $\beta \otimes \alpha$, $\beta \otimes \beta$, $\beta \otimes (\alpha \times \beta)$, $(\alpha \times \beta) \otimes \alpha$, $(\alpha \times \beta) \otimes \beta$, $(\alpha \times \beta) \otimes (\alpha \times \beta)$ form a basis of $\mathbb{R}^{3 \times 3}$, and we write elements of $\mathbb{R}^{3 \times 3}$ in the form $M = c_{11} \alpha \otimes \alpha + \cdots + c_{33} (\alpha \times \beta) \otimes (\alpha \times \beta)$.

**Lemma 3.7.** When $\alpha \times \beta \neq 0$, the triple $(\alpha, \beta, M) \in \Lambda$ if and only if

$$M = c_{11} \alpha \otimes \alpha + c_{12} \alpha \otimes \beta + c_{13} [\alpha \otimes (\alpha \times \beta) + (\alpha \times \beta) \otimes \beta] + c_{21} \beta \otimes \alpha + c_{22} \beta \otimes \beta$$

for some $c_{11}, \ldots, c_{22} \in \mathbb{R}$.

**Proof.** Suppose (3.4) are satisfied. Then $\xi$ is necessarily a nonzero multiple of $\alpha \times \beta$, and we may assume that $\xi = \alpha \times \beta$. By using the fact that $\alpha \cdot \alpha \times \beta = \beta \cdot \alpha \times \beta = 0$ we get

$$M(\alpha \times \beta)$$

$$= c_{13} [\alpha \otimes (\alpha \times \beta)] (\alpha \times \beta) + c_{23} [\beta \otimes (\alpha \times \beta)] (\alpha \times \beta)$$

$$+ c_{33} [(\alpha \times \beta) \otimes (\alpha \times \beta)] (\alpha \times \beta)$$

$$= c_{13} |\alpha \times \beta|^2 \alpha + c_{23} |\alpha \times \beta|^2 \beta + c_{33} |\alpha \times \beta|^2 \alpha \times \beta$$
and
\[ M^T(\alpha \times \beta) = c_{31}(\alpha \times \beta) + c_{32}(\alpha \times \beta) + c_{33}(\alpha \times \beta) \]
\[ = c_{31}(\alpha \times \beta)\alpha + c_{32}(\alpha \times \beta)\beta + c_{33}(\alpha \times \beta)\beta. \]
Now (3.3) implies that \(c_{13} = c_{32}\) and \(c_{23} = c_{31} = c_{33} = 0\) but the coefficients \(c_{11}, c_{12}, c_{21}\) and \(c_{22}\) are free. For the converse choose \((\xi, c) = (\alpha \times \beta, -c_{13}(\alpha \times \beta)^2)\) in (3.4). □

**Lemma 3.8.** Suppose \(\alpha \neq 0\) and \(k \in \mathbb{R}\). We have \((\alpha, k\alpha, M) \in \Lambda\) if and only if \(M\) is of the form
\[
M = c_{11}\alpha \otimes \alpha + c_{12}\alpha \otimes \gamma + \cdots + c_{33}(\alpha \times \gamma) \otimes (\alpha \times \gamma).
\]
for some \(\gamma \neq 0\) with \(\gamma \perp \alpha\) and \(c_{11}, \ldots, c_{33} \in \mathbb{R}\). In particular, \((\alpha, k\alpha, f \otimes \alpha + \alpha \otimes g) \in \Lambda\) for every \(f, g \in \mathbb{R}^3\).

**Proof.** Let \(0 \neq \xi \perp \alpha\). Denote \(\gamma = \xi\) and write
\[
M = c_{11}\alpha \otimes \alpha + c_{12}\alpha \otimes \gamma + \cdots + c_{33}(\alpha \times \gamma) \otimes (\alpha \times \gamma).
\]
Now (3.3) is equivalent to
\[
M\gamma + c\alpha = c_{12}|\gamma|^2\alpha + c_{22}|\gamma|^2\gamma + c_{32}\gamma|\gamma|^2\alpha + c\alpha = 0,
\]
\[
M^T\gamma + k\alpha = c_{21}|\gamma|^2\alpha + c_{22}|\gamma|^2\gamma + c_{33}\gamma|\gamma|^2\alpha + k\alpha = 0,
\]
which proves the first claim.

In order to demonstrate the second claim we fix \(f, g \in \mathbb{R}^3\). If \(\alpha \times (f - kg) \neq 0\), we set \(\gamma := \alpha \times (f - kg)/|\alpha \times (f - kg)|\). Writing \(f = \langle f, \alpha/|\alpha|\rangle\alpha/|\alpha| + \langle f, \gamma\rangle\gamma + \langle f, \alpha/|\alpha| \times \gamma\rangle\alpha/|\alpha| \times \gamma\) and similarly for \(g\) we note that \(\langle f, \gamma\rangle = k\langle \gamma, g\rangle\) and so \(f \otimes \alpha + \alpha \otimes g\) is of the form (3.3), where \(c_{12} = \langle \gamma, g\rangle\). If, on the other hand, \(\alpha \times f = k\alpha \times g \neq 0\), we set \(\gamma = \alpha \times f/|\alpha \times f|\). As above, \(f \otimes \alpha + \alpha \otimes g\) is of the form (3.3) with \(c_{12} = 0\). Next, if \(\alpha \times f = k\alpha \times g = 0\) and \(k \neq 0\), then \(f \otimes \alpha + \alpha \otimes g\) is simply a multiple of \(\alpha \otimes \alpha\). Finally, if \(\alpha \times f = 0\) and \(k = 0\), we choose any unit vector \(\gamma \perp \alpha\); (3.3) is satisfied with \(c_{12} = (\gamma, g)\). □

All that is left is the characterization of the case \(\alpha = 0\). When \(\alpha = 0\) and \(\beta \neq 0\), we have a similar situation as in the preceding lemma:

**Lemma 3.9.** When \(\beta \neq 0\), we have \((0, \beta, M) \in \Lambda\) if and only if
\[
M = c_{12}\gamma \otimes \beta + c_{22}\beta \otimes \beta + c_{23}(\gamma \times \beta) \otimes \beta + c_{33}(\gamma \times \beta) \otimes (\gamma \times \beta)
\]
for some \(\gamma \neq 0\) with \(\gamma \perp \beta\) and \(c_{12}, \ldots, c_{32} \in \mathbb{R}\). In particular, \((0, \beta, f \otimes \beta + \beta \otimes g) \in \Lambda\) for all \(f, g \in \mathbb{R}^3\).

**Proof.** For the first claim let us denote \(0 \neq \xi = \gamma \perp \beta\). We write \(M = c_{11}\gamma \otimes \gamma + c_{12}\gamma \otimes \beta + \cdots + c_{32}(\gamma \times \beta) \otimes \beta + c_{33}(\gamma \times \beta) \otimes (\gamma \times \beta)\). Now (3.3) is equivalent to
\[
M\gamma = c_{11}|\gamma|^2\gamma + c_{21}|\gamma|^2\beta + c_{31}|\gamma|^2\gamma \times \beta = 0,
\]
\[
M^T\gamma + c\beta = c_{11}|\gamma|^2\gamma + c_{22}|\gamma|^2\gamma + c_{33}|\gamma|^2\gamma \times \beta + c\beta = 0,
\]
proving the first claim of the lemma. The second claim is proved as in the case \(k = 0\) of Lemma 3.8. □
Whenever $\alpha \neq 0$ or $\beta \neq 0$, the conditions $(\alpha, \beta, M) \in \Lambda$ and $(\alpha, \beta, M) \in \Lambda_0$ are in fact equivalent. However, when $\alpha = \beta = 0$, belonging to $\Lambda$ is a much more restrictive condition.

**Lemma 3.10.** If $M \in \mathbb{R}^{3 \times 3}$, we have $(0,0,M) \in \Lambda$ if and only if there exist orthonormal vectors $f_1, f_2 \in \mathbb{R}^3$ and coefficients $c_{11}, c_{12}, c_{21}, c_{22} \in \mathbb{R}$ such that

\[
M = \sum_{i,j=1}^{2} c_{ij} f_i \otimes f_j.
\]

Proof. If $(0,0,M) \in \Lambda$, choose $(\xi, c) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ such that $M\xi = M^T\xi = 0$. Form an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathbb{R}^3$, where $f_3 = \xi/|\xi|$, and write $M = \sum_{i,j=1}^{2} c_{ij} f_i \otimes f_j$. Then $M\xi = \sum_{i=1}^{2} c_{i3} f_i = 0$ and $M^T\xi = \sum_{j=1}^{2} c_{3j} f_j = 0$ imply that $M = \sum_{i,j=1}^{2} c_{ij} f_i \otimes f_j$. The converse is proved by setting $\xi = f_1 \times f_2$ and choosing any $c \in \mathbb{R}$. \(\square\)

3.4. **The wave cone in 2D MHD.** In 2D MHD the $\Lambda$-convex hull $K^\Lambda$ has empty interior. However, $K^\Lambda$ is strictly larger than $K$ itself; these results are shown in Proposition 4.1. We first compute the wave cone in this section.

Recall that the subset of the wave cone that we use is

\[
\Lambda = \left\{ V \in \mathbb{R}^{3 \times 3} : \exists \begin{bmatrix} \xi \\ c \end{bmatrix} \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R} \text{ such that } V \begin{bmatrix} \xi \\ c \end{bmatrix} = V^T \begin{bmatrix} \xi \\ c \end{bmatrix} = 0 \right\}.
\]

If $\alpha \in \mathbb{R}^2 \setminus \{0\}$, we may write any $M \in \mathbb{R}^{2 \times 2}$ as a linear combination of four basis elements of $\mathbb{R}^{2 \times 2}$: $M = c_{11}\alpha \otimes \alpha + c_{12}\alpha \otimes \alpha^\perp + c_{21}\alpha^\perp \otimes \alpha + c_{22}\alpha^\perp \otimes \alpha^\perp$, where $\alpha^\perp = (-\alpha_2, \alpha_1)$. Now a necessary condition for $(\alpha, \beta, M) \in \Lambda$ is that $0 \neq \xi \perp \{\alpha, \beta\}$ so that $\beta = k\alpha$ with $k \in \mathbb{R}$. Lemma 3.11 is proved by an easy adaptation of the proofs of Lemmas 3.8 and 3.10.

**Lemma 3.11.** When $\alpha \neq 0$ and $k \in \mathbb{R}$, the triple $(\alpha, k\alpha, M) \in \Lambda$ if and only if

\[
M = c_{11}\alpha \otimes \alpha + c_{12}\alpha \otimes \alpha^\perp + k\alpha^\perp \otimes \alpha,
\]

where $c_{11}, c_{12} \in \mathbb{R}$. When $\beta \neq 0$, we have $(0, \beta, M) \in \Lambda$ if and only if

\[
M = c_{11}\beta \otimes \beta + c_{21}\beta^\perp \otimes \beta,
\]

where $c_{11}, c_{12} \in \mathbb{R}$. Furthermore, $(0,0,M) \in \Lambda$ if and only if $M = \gamma \otimes \gamma$ for some $\gamma \in \mathbb{R}^2$.

4. **Proof of Theorem 1.1**

Theorem 1.1 is proved in this section. In Proposition 3.11, we find a suitable $\Lambda$-convex function which gives us crucial information about the shape of the $\Lambda$-convex hull $K^\Lambda$, and in Proposition 4.1 we finish the proof of Theorem 1.1.

4.1. **Emptiness of the interior of the hull in 2D.** The constraint set $K$ for 2D ideal MHD was defined in (3.3). The following result on the $\Lambda$-convex hull of $K$ was the initial motivation behind Theorem 1.1 and Corollary 1.2.

**Proposition 4.1.** Whenever $(\alpha, \beta, \alpha \otimes \beta + N) \in K^\Lambda$, the matrix $N$ is symmetric. In particular, $K^\Lambda$ has empty interior. However, $K^\Lambda \setminus K \neq \emptyset$.

Proof. The idea of the proof of the emptiness statement is to construct a $\Lambda$-convex function $f$ such that $f$ vanishes on $K$ but $f(\alpha, \beta, M) > 0$ whenever $M - \alpha \otimes \beta$ is non-symmetric.
We define

\begin{equation}
 f(\alpha, \beta, M) := |(m_{12} - \alpha_1 \beta_2) - (m_{21} - \alpha_2 \beta_1)|^2
\end{equation}

intending to show that when \( V := (\alpha, \beta, M) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \) and \( W := (\gamma, k_\gamma, N) \in \Lambda \), the function

\[ t \mapsto g(t) := f(V + tW) : \mathbb{R} \to \mathbb{R} \]

is convex. (A similar proof works for triples of the form \( (0, \gamma, N) \in \Lambda \).)

With \( V \) and \( W \) fixed and \( t \in \mathbb{R} \), we get

\[
g(t) - g(0) = |(m_{12} + tm_{12} - (\alpha_1 + t\gamma_1)(\beta_2 + t\gamma_2)) - (m_{21} + tm_{21} - (\alpha_2 + t\gamma_2)(\beta_1 + t\gamma_1))|^2 - |(m_{12} - \alpha_1 \beta_2) - (m_{21} - \alpha_2 \beta_1)|^2
\]

\[ = ct + |dt|^2, \]

since the terms \( \pm t^2k\gamma_1\gamma_2 \) cancel out. Thus \( g''(t) = 2d^2 \geq 0 \) for all \( t \), so that \( g \) is convex.

We then show that \( K^\Lambda \setminus K \neq \emptyset \). When \( \alpha, \beta, \gamma, \delta \in \mathbb{R}^2 \), we have \( (\alpha, \beta, \alpha \otimes \beta) - (\gamma, \delta, \gamma \otimes \delta) \in \Lambda \) precisely when \( \alpha, \beta, \gamma, \delta \) lie on the same line (see Lemma 6.6 and its proof for an analogous statement in 3D). Now \( \alpha^\pm = (\pm 3, 1) \) and \( \beta^\pm = (\pm 2, 1) \) belong to the horizontal line through \((0, 1)\) and \((\alpha^\pm, \beta^\pm, \alpha^\pm \otimes \beta^\pm) \in K \), yet

\[
\frac{(\alpha^+, \beta^+ \otimes \beta^+)}{2} + \frac{(\alpha^-, \beta^- \otimes \beta^-)}{2} = \left( 0, 1, 0, 1, \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \right) \notin K.
\]

The claim \( K^\Lambda \setminus K \neq \emptyset \) also follows from the fact that the \( \Lambda \)-convex hull for Euler equations is non-trivial. Indeed, when \( z^+ = z^- \) and \( M = 0 \) (see (3.2)) reduce to the linearized Euler equations.

When written in terms of \((u, b, S, A)\), the \( \Lambda \)-convex function in (4.1) is of the form \( f(u, b, S, A) := 4|a_{12} - b \times u|^2 \) and Proposition 11 obtains the following form.

Corollary 4.2. If \((u, b, S, A) \in K^\Lambda \), then \( A = b \otimes u - u \otimes b \).

By Corollary 4.2, the assumptions of Theorem 1.1 imply that (1.2) holds. Thus Theorem 1.1 will be proved once we show that the mean-square magnetic potential is conserved by solutions of (1.2) - (1.5).

4.2. Conservation of the mean-square magnetic potential. The aim of this section is to finish the proof of Theorem 1.1, our proof is reminiscent of that of [LV] Theorem 1.4. We use the \( H^1 \) regularity theory of Coifman, Lions, Meyer and Semmes from [CLMS], more precisely the following adaptation of the classical Wente inequality to the torus \( T^2 \) (see [FMS] Theorem A.1).

Lemma 4.3. When \((f_1, f_2, f_3) \in W^{1,2}(T^2, \mathbb{R}^3)\), we have

\[
\int_{T^2} f_1(x)J(f_2, f_3)(x)\,dx \lesssim \| f_1 \|_{BMO(T^2)} \| J(f_2, f_3) \|_{H^1(T^2)} \lesssim \| \nabla f_1 \|_{L^2(T^2)} \| \nabla f_2 \|_{L^2(T^2)} \| \nabla f_3 \|_{L^2(T^2)}.
\]
The left-hand side of (1.2) can be understood in terms of $H^1$–BMO duality, but we will in fact use estimate (1.2) only in cases where the left-hand side is Lebesgue integrable. For the proof of Theorem 1.1, we fix a mollifier $\chi \in C_\infty^\infty(T^2 \times \mathbb{R})$ as in 2.1. Since $\chi$ is even, we have $\int_{\mathbb{T}}^T \int_{\mathbb{T}} f(x, t) g(x, t) \, dx \, dt = \int_{\mathbb{T}}^T \int_{\mathbb{T}} f_\delta(x, t) g(x, t) \, dx \, dt$ for all $f \in L^1(T^2 \times [0, T])$ and $g \in L^\infty(T^2 \times [0, T])$ whenever $0 < \delta < \epsilon < T - \epsilon$.

**Proof of Theorem 1.1.** Since $\Psi \in C([0, T]; L^2(T^2))$, it suffices to show that

$$(4.3) \quad \int_0^T \partial_t \eta(t) \int_{\mathbb{T}^2} |\Psi(x, t)|^2 \, dx \, dt = 0$$

for every $\eta \in C_\infty^\infty([0, T])$. We fix $\eta$ and choose $\epsilon > 0$ such that $\text{supp}(\eta) \subset [2\epsilon, T - 2\epsilon]$. Now $\|\Psi - \Psi_\delta\|_{L^2(T^2 \times (\epsilon, T - \epsilon))} \to 0$ and $\|\nabla \Psi - \nabla \Psi_\delta\|_{L^2(T^2 \times (\epsilon, T - \epsilon))} \to 0$ as $\delta \searrow 0$, and so (2.1) yields

$$\int_{\mathbb{T}}^T \partial_t \eta(t) \int_{\mathbb{T}^2} |\Psi(x, t)|^2 \, dx \, dt = \lim_{\delta \searrow 0} \int_{\mathbb{T}}^T \partial_t \eta(t) \int_{\mathbb{T}^2} |\Psi_\delta(x, t)|^2 \, dx \, dt$$

$$= 2 \lim_{\delta \searrow 0} \int_{\mathbb{T}}^T \eta(t) \int_{\mathbb{T}^2} \Psi_\delta(x, t)[J_{\Psi, \phi}]_\delta(x, t) \, dx \, dt.$$ 

When $\delta > 0$ is small, we write

$$[J_{\Psi, \phi}]_\delta = [J_{\Psi - \Psi_\delta, \phi}]_\delta + ([J_{\Psi_\delta, \phi}]_\delta - J_{\Psi_\delta, \phi}) + J_{\Psi_\delta, \phi}$$

and estimate the resulting integrals separately. First, we use Lemma 4.3 Hölder’s inequality and Young’s inequality to estimate

$$\left| \int_\epsilon^T \eta(t) \int_{\mathbb{T}^2} \Psi_\delta(x, t)[J_{\Psi - \Psi_\delta, \phi}]_\delta(x, t) \, dx \, dt \right|$$

$$\leq \int_{\mathbb{T}}^T \|\eta \nabla \Psi_\delta\|_{L^2(T^2 \times [\epsilon, T - \epsilon])} \|\nabla (\Psi - \Psi_\delta)\|_{L^2(T^2)} \|\nabla \Phi\|_{L^2(T^2)} \, dt$$

$$\leq \|\eta \nabla \Psi_\delta\|_{L^2(T^2 \times [\epsilon, T - \epsilon])} \|\nabla (\Psi - \Psi_\delta)\|_{L^2(T^2 \times [\epsilon, T - \epsilon])} \|\nabla \Phi\|_{L^\infty L^2(T^2 \times [\epsilon, T - \epsilon])} \to 0$$

as $\delta \searrow 0$. Similarly,

$$\left| \int_\epsilon^T \eta(t) \int_{\mathbb{T}^2} \Psi_\delta(x, t)[J_{\Psi_\delta, \phi}]_\delta(x, t) \, dx \, dt \right|$$

$$\leq \int_{\mathbb{T}}^T \|\eta \nabla \Psi_\delta\|_{L^2(T^2 \times [\epsilon, T - \epsilon])} \|\nabla \Psi_\delta\|_{L^2(T^2)} \|\nabla \Phi\|_{L^\infty L^2(T^2 \times [\epsilon, T - \epsilon])} \to 0$$

as $\delta \searrow 0$. Similarly,
and we get
\[ \| [\eta \nabla \Psi]_0 - \eta \nabla \Psi_0 \|_{L^2(T^2 \times \epsilon, T-\epsilon)} \]
\[ \leq \| [\eta \nabla \Psi_0 - \eta \nabla \Psi]_0 \|_{L^2(T^2 \times \epsilon, T-\epsilon)} + \| [\| \eta \nabla \Psi \|_0 - \eta \nabla \Psi \|_{L^2(T^2 \times \epsilon, T-\epsilon)} \| \to 0 \]
as \delta \searrow 0. Finally,
\[ \int_0^{T-\epsilon} \eta(t) \int_{T_2} \Psi_\delta(x, t) \Phi(\Psi_\delta, \Psi)(x, t) \, dx \, dt = - \int_0^{T-\epsilon} \eta(t) \int_{T_2} \Phi(x, t) \Phi(\Psi_\delta, \Psi)(x, t) \, dx \, dt = 0 \]
for every \( \delta > 0 \), finishing the proof of (4.3).

It is natural to ask whether an analogue of Theorem 1.1 holds in the whole space \( \mathbb{R}^2 \). However, square integrable divergence-free vector fields do not in general have a square integrable stream function in \( \mathbb{R}^2 \), a fact that has sometimes been overlooked in the literature. The following simple proposition quantifies this phenomenon in terms of Baire category and shows that the natural analogue of Lemma 2.1 in \( \mathbb{R}^2 \) is false.

Proposition 4.4. The set \( \{ \nabla \Phi : \Phi \in W^{1,2}(\mathbb{R}^2) \} \) is of the first Baire category in \( \{ v \in L^2(\mathbb{R}^2)^2 : \text{div} \, v = 0 \} \).

Proof. We write \( \{ \nabla \Phi : \Phi \in W^{1,2}(\mathbb{R}^2) \} = \bigcup_{k=1}^{\infty} \{ \nabla \Phi : \| \Phi \|_{W^{1,2}} \leq k \} \) intending to prove that each of the closed sets \( \{ \nabla \Phi : \| \Phi \|_{W^{1,2}} \leq k \} \) has empty interior. Fix \( k \in \mathbb{N} \). By the linearity of \( \nabla \), it suffices to show that \( \{ \nabla \Phi : \| \Phi \|_{W^{1,2}} \leq k \} \) does not contain a ball centered at the origin.

Choose \( \Theta \in W^{1,2}(\mathbb{R}^2) \) such that \( \Theta + C \notin W^{1,2}(\mathbb{R}^2) \) for every \( C \in \mathbb{R} \), and denote \( v := \nabla \Theta \in L^2(\mathbb{R}^2)^2 \). Seeking contradiction, suppose there exist \( \Psi \in W^{1,2}(\mathbb{R}^2) \) and \( c \neq 0 \) such that \( \| \Psi \|_{W^{1,2}} \leq k \) and \( \nabla \Phi = cv \). Then \( \nabla \Phi \rangle \Phi \rangle \Psi = 0 \) and so for some \( C \in \mathbb{R} \) we have \( \Theta + C = \Psi/c \in W^{1,2}(\mathbb{R}^2) \), which yields the desired contradiction. \( \square \)

Remark 4.5. Mean-square magnetic potential is also conserved by weak \( L^2 \) limits of sequences of solutions that are bounded in \( L^2 \times L^2_2(T^2 \times [0, T]; \mathbb{R}^2) \); this boils down to the fact that the induction equation \( \partial_b b + \text{div}(b \otimes u - u \otimes b) = 0 \) is weakly compact. Indeed, suppose \( b^j, u^j \in C_w([0, T]; L^2(T^2; \mathbb{R}^2)) \) solve (4.2) for every \( j \in \mathbb{N} \) with \( \sup_{j \in \mathbb{N}} \| u^j \|_{L^2(T^2; \mathbb{R}^2)} + \| b^j \|_{L^2(T^2; \mathbb{R}^2)} < \infty \) and assume \( b^j \rightharpoonup b, u^j \rightharpoonup u \) in \( L^2_2(T^2 \times [0, T]; \mathbb{R}^2) \). Up to passing to a subsequence, the initial data \( b^0 \) have a weak limit \( b_0 \) in \( L^2(T^2; \mathbb{R}^2) \). Furthermore, by Lemma 2.2, \( \sup_{j \in \mathbb{N}} \| \partial_b u^j \|_{L^2(T^2; \mathbb{R}^2)} < \infty \), giving \( \Psi^j \rightharpoonup \Psi \) in \( L^2_2(T^2 \times [0, T]) \) by the Aubin-Lions-Simon Lemma (for a version that suffices for us see e.g. Lemma 7.1). Now \( b^j \times u^j = J(\Psi^j, \Psi^j) \rightharpoonup J(\Psi, \Psi) = b \times u \) in \( \mathcal{D}'(T^2 \times [0, T]) \), yielding \( \partial_b b + \text{div}(b \otimes u - u \otimes b) = 0 \) with \( b(\cdot, 0) = b_0 \). Furthermore, \( b, u \in L^\infty(T^2 \times [0, T]; \mathbb{R}^2) \), and so again, by a modification of Lemmas 2.2 and 2.4, \( b \in C_w([0, T]; L^2(T^2; \mathbb{R}^2)) \). Now \( b \) and \( u \) essentially satisfy the assumptions of Theorem 1.1 (the weak continuity of \( u \) in time is not needed for the conclusion to hold) and so they conserve mean-square magnetic potential in time.
5. Proof of Theorem 1.3

Theorem 1.3 gives an analogue of Theorem 1.1 in 3D, and the proof is presented in this section. In §5.1 we find a \( \Lambda \)-affine function which shows that the \( \Lambda \)-convex hull \( K^\Lambda \) has empty interior, and the proof of Theorem 1.3 is finished in §5.2. Theorem 1.3 does not, however, immediately rule out compactly supported convex integration solutions of 3D MHD with \( b \neq 0 \).

5.1. Emptiness of the interior of the hull in 3D. The main aim of this subsection is the construction of a suitable \( \Lambda \)-affine function. The function gets a much more intuitive form in the formalism with velocity field \( u \) and magnetic field \( b \) rather than the Elsässer variables \( z^\pm \). Recall from the Introduction that the linearized MHD equations can be written in terms of \( (u, b, S, A) \) as

\[
\text{div}_x, t \begin{bmatrix} S \\ u^T \end{bmatrix} = 0, \quad \text{div}_x, t \begin{bmatrix} A \\ -b^T \end{bmatrix} = 0,
\]

so that linearized MHD is divided into the 'symmetric part' satisfied by \( u \) and the 'antisymmetric part' satisfied by \( b \).

Furthermore, we may identify

\[
A = \begin{bmatrix} 0 & a_{12} & -a_{31} \\ -a_{12} & 0 & a_{23} \\ a_{31} & -a_{23} & 0 \end{bmatrix} \equiv (a_{23}, a_{31}, a_{12}) := a \in \mathbb{R}^3.
\]

In this identification we have \( A^T = \xi \cdot a \) for all \( \xi \in \mathbb{R}^3 \) and \( b \otimes u - u \otimes b \simeq b \times u \) for all \( b, u \in \mathbb{R}^3 \). The evolution equation of \( b \) thus obtains the more intuitive form

\[
(5.1) \quad \partial_t b + \nabla \times a = 0.
\]

We write \( K \) in terms of \( u, b, S \) and \( a \) as

\[
K = \{(z^+, z^-, M) : M = z^+ \otimes z^- + \Pi I, \; \Pi \in \mathbb{R} \}
\subseteq \{(u, b, S, a) : S = u \otimes u - b \otimes b + \Pi I, \; a = b \times u, \; \Pi \in \mathbb{R} \}.
\]

The wave cone conditions \( M^\xi + cz^+ = 0, \; M^T \xi + c z^- = 0, \; z^\pm \cdot \xi = 0 \) (see (3.4)) translate into

\[
(5.2) \quad S^\xi + cu = 0, \quad \xi \times a + cb = 0, \quad u \cdot \xi = b \cdot \xi = 0.
\]

Now the sought \( \Lambda \)-affine function finds a particularly simple form.

**Theorem 5.1.** If \( (u, b, S, a) \in \Lambda \) or \( (u, b, S, a) \in K^\Lambda \), then \( a \cdot b = 0 \).

**Proof.** We define

\[
Q(u, b, S, a) := a \cdot b,
\]

note that \( Q|_K = 0 \) and set out to show that \( Q \) is \( \Lambda \)-affine. Since \( Q \) is quadratic, it suffices to show that \( a \cdot b = 0 \) for all \( (u, b, S, a) \in \Lambda \).

Let \( (u, b, S, a) \in \Lambda \) and choose \( (\xi, c) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R} \) such that (5.2) holds. Thus, writing \( b = (b_1, b_2, b_3) \), we have

\[
(5.3) \quad \xi_2 a_3 - \xi_3 a_2 + cb_1 = 0, \quad \xi_3 a_1 - \xi_1 a_3 + cb_2 = 0, \quad \xi_1 a_2 - \xi_2 a_1 + cb_3 = 0.
\]

By multiplying the three equations of (5.3) by \( a_1, a_2 \) and \( a_3 \) respectively and taking the sum of the left-hand sides we get \( c(a_1 b_1 + a_2 b_2 + a_3 b_3) = 0 \), that is, \( cb \cdot a = 0 \), which finishes the proof when \( c \neq 0 \).
Suppose then $c = 0$. Let $\xi_1 \neq 0$, the cases $\xi_2 \neq 0$ and $\xi_3 \neq 0$ being similar because of symmetry. By assumption, $b \cdot \xi = 0$, and so, by (5.3),

$$0 = \frac{a_1}{\xi_1}(\xi_1 b_1 + \xi_2 b_2 + \xi_3 b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3 = b \cdot a.$$ 

\[ \square \]

**Remark 5.2.** Note that the variable $a$ is the electric field, $b \times u$ is the Maxwell-Faraday equation and the pointwise constraint $a = b \times u$ is ideal Ohm’s law.

### 5.2. Conservation of magnetic helicity by subsolutions and weak limits of solutions.

The conservation of magnetic helicity by weak solutions $u, b \in L^1_t L^3_x(T^3 \times [0, T]; \mathbb{R}^3)$ of (1.1)–(1.3) was proved in [KL], and we present below a more elementary proof of the result along the lines of [IV, Theorem 1.4], while generalizing the result to subsolutions. Recall the even mollifier $\chi \in C_c^\infty(T^3 \times \mathbb{R})$ defined in (2.2).

**Proof of Theorem 1.3.** Suppose $u, b \in L^3_t(L^3_x \times [0, T]; \mathbb{R}^3)$, $S \in L^3_{t, loc}(T^3 \times [0, T]; S^{1 \times 3})$ and $a \in L^{3/2}_t(L^3_x \times [0, T]; \mathbb{R}^3)$ form a solution of (1.5)–(1.7) and (1.10) that takes values in $K^A$ a.e. Let $\eta \in C_c^\infty([0, T])$, so that for $\epsilon > 0$ small enough, $\text{supp}(\eta) \subset [\epsilon, T - \epsilon]$. By using Lemma 2.6 and integrating by parts a few times we get

$$\int_t^{T-\epsilon} \eta(t) \int_{T^3} \Psi(x, t) \cdot b(x, t) \, dx \, dt$$

$$= \lim_{\delta \downarrow 0} \int_t^{T-\epsilon} \eta(t) \int_{T^3} \Psi_\delta(x, t) \cdot b_\delta(x, t) \, dx \, dt$$

$$= \lim_{\delta \downarrow 0} \int_t^{T-\epsilon} \eta(t) \int_{T^3} \left( a_\delta(x, t) - \int_{T^3} a_\delta(y, t) \, dy - \nabla g_\delta(x, t) \right) \cdot b_\delta(x, t) \, dx \, dt$$

$$+ \int_t^{T-\epsilon} \eta(t) \int_{T^3} \Psi_\delta(x, t) \cdot \nabla \times a_\delta(x, t) \, dx \, dt$$

$$= 2 \lim_{\delta \downarrow 0} \int_t^{T-\epsilon} \eta(t) \int_{T^3} a_\delta(x, t) \cdot b_\delta(x, t) \, dx \, dt$$

$$= 2 \int_t^{T-\epsilon} \eta(t) \int_{T^3} a(x, t) \cdot b(x, t) \, dx \, dt = 0,$$

since $a \cdot b = 0$. 

We present a variant of Theorem 1.3 which says that magnetic helicity is also conserved by weak limits of solutions. The proof is based on the observation that as a $\Lambda$-affine function, $Q(u, b, S, a) := a \cdot b$ is weakly continuous for solutions of linearized 3D MHD.

**Lemma 5.3.** Suppose $b^1, w^1 \in L^3_t(T^3 \times [0, T]; \mathbb{R}^3)$, $S^1 \in L^1_{t, loc}(T^3 \times [0, T]; S^{3 \times 3})$ and $a^1 \in L^{3/2}_t(T^3 \times [0, T]; \mathbb{R}^3)$ satisfy the linearized MHD equations (1.5)–(1.7)–(1.10). Assume $b^1 \rightarrow b$ and $w^1 \rightarrow u$ in $L^3_t(T^3 \times [0, T]; \mathbb{R}^3)$ and $a^1 \rightarrow a$ in $L^{3/2}_t(T^3 \times [0, T]; \mathbb{R}^3)$. Then $a^1 \cdot b^1 \rightarrow a \cdot b$ in $\mathcal{D}'(T^3 \times [0, T])$.

By a theorem of Tartar, every quadratic $\Lambda$-affine function is weakly continuous (see [Tar79, Corollary 13]), but in [Tar79] the assumptions on the $L^p$ exponents are different (here $p = 3$ for $b$ and $p = 3/2$ for $a$) and the proof uses techniques of Fourier analysis that do not transfer immediately to our setting. We use instead the potentials provided by Lemma 2.6.
Remark 5.4. The expression \((a \cup b)\) (which we do not relabel); we show that there exists a further subsequence converging to \(a \cdot b\) in \(D'(\mathbb{T}^3 \times [0, T])\). We use Lemma 2.6 to write \(b^j = \nabla \times \Psi^j\) and \(a^j - \int_{\mathbb{T}^3} a^j(y, \cdot)\, dy = -\partial_t \Psi^j + \nabla g^j\). We use the weak compactness of \(L^p\) spaces in order to pass to a further subsequence and get \(\Psi^j \rightharpoonup \Psi\) in \(L^3(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\), \(D\Psi^j \rightharpoonup M\) in \(L^3(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\) and \(\partial_t \Psi^j \rightharpoonup f\) in \(L^{3/2}(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\), and clearly \(M = D\Psi\) and \(f = \partial_t \Psi\). By the Aubin-Lions compactness lemma (see e.g. \([\text{Rou, Lemma 7.7}]\)), we conclude that \(\Psi^j \rightharpoonup \Psi\) in \(L^3(\mathbb{T}^3 \times ]0, T[; \mathbb{R}^3)\) whenever \(0 < \epsilon < T\). We write \(a = a^j \uparrow a\) and \(b = b^j \uparrow b\) in \(D'(\mathbb{T}^3 \times [0, T])\).

Proof of Lemma 5.3. Fix any subsequence of \(a^j \cdot b^j\) (which we do not relabel); we show that there exists a further subsequence converging to \(a \cdot b\) in \(D'(\mathbb{T}^3 \times [0, T])\). We use Lemma 2.6 to write \(b^j = \nabla \times \Psi^j\) and \(a^j - \int_{\mathbb{T}^3} a^j(y, \cdot)\, dy = -\partial_t \Psi^j + \nabla g^j\). We use the weak compactness of \(L^p\) spaces in order to pass to a further subsequence and get \(\Psi^j \rightharpoonup \Psi\) in \(L^3(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\), \(D\Psi^j \rightharpoonup M\) in \(L^3(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\) and \(\partial_t \Psi^j \rightharpoonup f\) in \(L^{3/2}(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\), and clearly \(M = D\Psi\) and \(f = \partial_t \Psi\). By the Aubin-Lions compactness lemma (see e.g. \([\text{Rou, Lemma 7.7}]\)), we conclude that \(\Psi^j \rightharpoonup \Psi\) in \(L^3(\mathbb{T}^3 \times ]0, T[; \mathbb{R}^3)\) whenever \(0 < \epsilon < T\). We write \(a = a^j \uparrow a\) and \(b = b^j \uparrow b\) in \(D'(\mathbb{T}^3 \times [0, T])\).

Remark 5.4. The expression \((-\partial_t \Psi^j + \nabla g^j) \cdot \nabla \times \Psi^j\) appearing in (5.4) is, up to a constant, a familiar compensated compactness quantity called the \(P\)faffian of the antisymmetrized Jacobian matrix \(2^{-1}(D_{x \cdot t} - D_{x \cdot t}'')(\Psi, g): \mathbb{R}^3 \times ]0, T[ \to \mathcal{A}^{1/4})\).

Theorem 5.5. Suppose \(b^j, u^j \in L^3(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\) and \(\Pi^j \in L^1_{\text{loc}}(\mathbb{T}^3 \times [0, T])\) satisfy the MHD equations \((1.4) = (1.3)\) and condition \((1.5)\). Suppose \(b^j \uparrow b\) and \(u^j \uparrow u\) in \(L^3(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\) and that \(\Pi^j \rightharpoonup \Pi\) in \(L^1_{\text{loc}}(\mathbb{T}^3 \times [0, T])\). Then \((b, u, \Pi)\) conserves magnetic helicity a.e. \(t \in ]0, T[\).

Proof. We denote \(a^j := b^j \times u^j \in L^{3/2}(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\) and \(S^j := u^j \otimes u^j - b^j \otimes b^j \in L^{3/2}(\mathbb{T}^3 \times [0, T]; S^{3 \times 3})\). Thus \((u^j, b^j, S^j, a^j)\) satisfy the linearized MHD equations \((1.5) - (1.7)\) and \((1.10)\). By passing to a subsequence, \(a^j \rightharpoonup a\) in \(L^{3/2}(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)\). If \(\eta \in C_c^\infty([0, T])\), then by Lemma 5.3 and the fact that \(a^j \cdot b^j = 0\) for every \(j \in \mathbb{N}\),
\[
\int_0^T \partial_t \eta(t) \int_{\mathbb{T}^3} \Psi(x, t) \cdot b(x, t)\, dx = 2 \int_0^T \eta(t) \int_{\mathbb{T}^3} a(x, t) \cdot b(x, t)\, dx dt = 0.
\]

Theorem 5.5 can also be proved by noting, as in the proof of Lemma 5.3, that \(\Psi^j \rightharpoonup \Psi\) in \(L^3\). However, Lemma 5.3 is of independent interest and sheds light on
the question whether the relaxation of 3D MHD coincides with the Λ-convex hull $K^Λ$ by showing that the relaxation respects the constraint $a \cdot b = 0$.

6. THEOREMS 1.4, 1.5 AND ESTIMATES ON $K^Λ$

The aim of this section is to show the existence of non-trivial compactly supported strict subsolutions of 3D MHD. In §6.1 we give the definition of strict subsolutions and formulate in Theorem 6.3 a strengthening of Theorem 1.4 which says that the origin is in the relative interior of the lamination convex hull $K^{lc,Λ}$ (relative to the constraint given in Theorem 5.1), the proof is presented in §6.2 §6.3 Theorem 1.5 is obtained as a rather straightforward corollary at the end of §6.1.

We also briefly discuss the work of Bronzi, Lopes Filho and Nussenzweig Lopes on convex integration in 3D MHD in §6.1.

6.1. Definition of strict subsolutions and proof of Theorem 1.5. Recall that in terms of the variables $u$, $b$, $S$ and $a$ we have $K_{r,s} = \{(u, b, S, a) : |u + b| = r, |u - b| = s, |u| \leq rs, S = u \otimes u - b \otimes b + II, a = b \times u\}$. Then $K_{r,s}$ is denoted by $U_{r,s}$.

Definition 6.1. Let $r, s > 0$. The relative interior of $K_{r,s}^{lc,Λ}$ in \{(u, b, S, a) ∈ $R^3 \times R^3 \times S^{3 \times 3} \times R^3 : a \cdot b = 0\}$ is denoted by $U_{r,s}$.

It will turn out in Theorem 6.3 that $U_{r,s} ≠ \emptyset$. This motivates the following definition in analogy to, among others, the definition of subsolutions of Euler equations in [DLS12] p. 350.

Definition 6.2. Let $r, s > 0$. The mappings $u, b ∈ L^2_{loc}(R^3, R^3), S ∈ L^1_{loc}(R^3, S^{3 \times 3})$ and $a ∈ L^1_{loc}(R^3, R^3)$ form a strict subsolution of (1.1)–(1.3) if $(u, b, S, a)$ satisfies (1.6)–(1.7), (1.10) and $(u, b, S, a)(x, t) ∈ U_{r,s}$ for almost every $(x, t) ∈ R^3 \times R$.

The existence of strict subsolutions plays a pivotal part in the construction of convex integration solutions of equations of fluid dynamics in the Tartar framework. For the existence of strict subsolutions it is of course mandatory that the set $U_{r,s}$ be non-empty. We record the following strengthening of Theorem 1.4.

Theorem 6.3. Let $r, s > 0$. Then $0 \in U_{r,s}$.

Sections 6.2–6.4 are devoted to the proof of Theorem 6.3. Assuming Theorem 6.3 we now prove Theorem 1.5 via the existence of compactly supported strict subsolutions of Euler equations and the following simple result.

Lemma 6.4. Let $E ∈ C^∞_c(R^3 \times R)$ and $η = (η', η_4) ∈ R^3 \times R$. Then

\begin{equation}
\begin{align*}
b := \nabla E \times η', & \quad a := -(∂_t E)η' + η_4 \nabla E
\end{align*}
\end{equation}

satisfy the conditions

\begin{align*}
div b = 0, \\
∂_t b + ∇ \times a = 0, \\
b \cdot a = 0.
\end{align*}

Proof of Theorem 1.5. Let $r, s > 0$ and choose a solution $(u, S) ∈ C^∞(R^3 \times R; R^3 \times S^{3 \times 3})$ of the linearized Euler equations $∂_t u + div S = 0$ and $div u = 0$ with $u ≠ 0$ (see [DLS09] Lemma 4.4). Then choose $E ∈ C^∞_c(R^3 \times R)$ and $η = (η', η_4) ∈ R^3 \times R$ with $∇ E \times η' ≠ 0$ and define $b, a ∈ C^∞_c(R^3 \times R; R^3)$ by (6.1). Now $(u, b, S, a) ∈ C^∞_c(R^3 \times R; R^3)$
\( C_c^\infty(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{R}^3) \), and after possibly multiplying by a small non-zero constant, Theorem 6.3 gives \((u, b, S, a)(x, t) \in \mathcal{U}_{r,s}\) for every \((x, t) \in \mathbb{R}^3 \times \mathbb{R}_.\) □

6.2. A lemma on the relative interior of the \(\Lambda\)-convex hull. We begin the proof of Theorem 6.3 by formulating in Lemma 6.5 a slightly stronger result which is also of independent interest. The lemma is stated in terms of Els"asser variables, and we next discuss the relevant definitions in this formalism.

We first write the \(\Lambda\)-affine quantity \(a \cdot b\) of Theorem 6.3 in terms of Els"asser variables. When \(\alpha, \beta \in \mathbb{R}^3\) and \(\alpha \neq \beta\), we denote \(f_1 := (\alpha - \beta)/|\alpha - \beta| = b/|b|\) and suppose that \(f_1, f_2, f_3\) form an orthonormal basis of \(\mathbb{R}^3\) with \(f_1 \times f_2 = f_3\). If we write a general \(3 \times 3\) matrix as \(N := \sum_{i,j=1}^{3} c_{ij} f_i \otimes f_j\), then \(a \cdot b = |b|(c_{23} - c_{32})/2\). Indeed,

\[
c_{23} - c_{32} = N f_3 \cdot f_2 - N f_2 \cdot f_3 = (N - N^T) f_3 \cdot f_2 = 2(A + u \otimes b - b \otimes u) f_3 \cdot f_2 = f_3 \times 2(a + u \times b) \cdot f_2 = 2(a + u \times b) \cdot f_2 \times f_3 = 2a \cdot \frac{b}{|b|} = 0.
\]

Theorem 6.3 is a rather direct consequence of the following lemma once one sets \(\tau = 0\), as we show after presenting the lemma.

**Lemma 6.5.** If \(r, s > 0\) and \(0 \leq \tau < 1\), then there exists a constant \(c_{\tau,r,s} > 0\) with the following property: if

(i) \(\alpha, \beta \in \mathbb{R}^3\) and \(\Pi \in \mathbb{R}\),

(ii) \(\{f_1, f_2, f_3\}\) is an orthonormal basis of \(\mathbb{R}^3\), where \(f_1 = (\alpha - \beta)/|\alpha - \beta|\) if \(\alpha \neq \beta\),

(iii) \(N = \sum_{i,j=1}^{3} c_{ij} f_i \otimes f_j\) with \(c_{23} = c_{32}\),

(iv) \(\max(|\alpha| - \tau r, |\beta| - \tau s, |N|, |\Pi| - \tau^2 rs) < c_{\tau,r,s}\),

then

\[(\alpha, \beta, \alpha \otimes \beta + \Pi I + N) \in K_{r,s}^{\ell c, \Lambda}.
\]

We indicate why Lemma 6.5 implies Theorem 6.3. Suppose that \((u, b, S, a) \cong (\alpha, \beta, \alpha \otimes \beta + \Pi I + N)\) is close to the origin and \(a \cdot b = 0\). If \(\alpha - \beta \neq 0\), then we write \(N = \sum_{i,j=1}^{3} c_{ij} f_i \otimes f_j\) and all the assumptions of Lemma 6.5 are satisfied for \(\tau = 0\) so that \((u, b, S, a) \in K_{r,s}^{\ell c, \Lambda}\). If on the other hand \(\alpha = \beta\), then there exists an orthonormal basis \(\{f_1, f_2, f_3\}\) of \(\mathbb{R}^3\) such that condition (iii) of Lemma 6.5 holds. Indeed, condition (iii) reads as \(c_{23} - c_{32} = (N - N^T) f_3 \cdot f_2 = 0\), and writing \(N - N^T \cong z \in \mathbb{R}^3\) we have \((N - N^T) f_3 = z \times f_3\), so that we get (iii) by choosing \(f_3 = z/|z|\) if \(z \neq 0\) and \(f_3 = (1, 0, 0)\) if \(z = 0\). Thus the assumptions of Lemma 6.5 are satisfied and \((u, b, S, a) \in K_{r,s}^{\ell c, \Lambda}\). Lemma 6.5 is proved in the following two subsections.

6.3. The case with \(c_{23} = 0\). We prove Lemma 6.5 by gradually weakening the assumptions on \(N = \sum_{i,j=1}^{3} c_{ij} f_i \otimes f_j\) in condition (iii). In this subsection we cover matrices \(N\) with \(c_{23} = 0\). First, in Lemma 6.6 we give a geometric characterization of the pairs \(V,W \in K\) such that \(V - W \in \Lambda\). By iterating the idea we handle, in
Lemma 6.8 the case where $N$ is a rank-one matrix, and in Lemma 6.9 a more elaborate use of $\Lambda$-convex combinations gives every $N$ with $c_{23} = 0$. The significantly harder case $c_{23} \neq 0$ is proved in §6.4.

**Lemma 6.6.** Suppose $\alpha, \beta, \gamma, \delta \in \mathbb{R}^3$ and $\Pi_1, \Pi_2 \in \mathbb{R}$. Then the following conditions are equivalent.

(i) $(\alpha, \beta, \alpha \otimes \beta + \Pi_1 I) - (\gamma, \delta, \gamma \otimes \delta + \Pi_2 I) \in \Lambda$,

(ii) the four points $\alpha, \beta, \gamma$ and $\delta$ lie on the same hyperplane, that is, in a set of the form \( \{x \in \mathbb{R}^3 : \langle x, \xi \rangle + c = 0 \} \) with $\xi \neq 0$ and $c \in \mathbb{R}$. Furthermore, $\Pi_1 = \Pi_2$.

**Proof.** Suppose $\alpha, \beta, \gamma, \delta \in \{x \in \mathbb{R}^3 : \langle x, \xi \rangle + c = 0 \}$, where $\xi \in \mathbb{R}^3 \setminus \{0\}$, and $\Pi_1 = \Pi_2$. Then

\[
\langle \alpha - \gamma, \xi \rangle = \langle \beta - \delta, \xi \rangle = 0,
\]

(6.2) $(\alpha \otimes \beta - \gamma \otimes \delta + \Pi_1 I - \Pi_2 I)\xi = \langle \beta, \xi \rangle\alpha - \langle \delta, \xi \rangle\gamma = \langle \delta, \xi \rangle(\alpha - \gamma) = -c(\alpha - \gamma),$

(6.3) $(\alpha \otimes \beta - \gamma \otimes \delta + \Pi_1 I - \Pi_2 I)\xi = \langle \alpha, \xi \rangle\beta - \langle \gamma, \xi \rangle\delta = \langle \gamma, \xi \rangle(\beta - \delta) = -c(\beta - \delta),$

and so $(\alpha, \beta, \alpha \otimes \beta + \Pi_1 I) - (\gamma, \delta, \gamma \otimes \delta + \Pi_2 I)$ satisfies condition (6.4).

Conversely, if $(\alpha, \beta, \alpha \otimes \beta + \Pi_1 I) - (\gamma, \delta, \gamma \otimes \delta + \Pi_2 I)$ satisfies (6.4) with $(\xi, c) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$, then (6.2) follows immediately, and next $(\alpha \otimes \beta - \gamma \otimes \delta + (\Pi_1 - \Pi_2) I)\xi + \alpha c = 0$ implies $\Pi_1 = \Pi_2$, which in turn implies (6.3)–(6.4), and so $\alpha, \beta, \gamma, \delta \in \{x \in \mathbb{R}^3 : \langle x, \xi \rangle + c = 0 \}$.

The following lemma covers the case $N = 0$ and gives us freedom in the selection of the points $\alpha, \beta \in \mathbb{R}^3$ in ensuing arguments.

**Lemma 6.7.** If $|\alpha| \leq r$, $|\beta| \leq s$ and $|\Pi| \leq rs$, then $(\alpha, \beta, \alpha \otimes \beta + \Pi I) \in K_{r,s}^{2,\Lambda}$.

**Proof.** We prove the case $0 < |\alpha| \leq r$, $0 < |\beta| \leq s$, the remaining cases being similar. By Lemma 6.3, the difference of the triples $(\alpha \otimes \beta + \Pi I)$ and $(\alpha \otimes \beta - \Pi I)$ is in $K_{r,s}^{2,\Lambda}$. Then the following condition holds: $(\alpha \otimes \beta + \Pi I) - (\Pi I) \in K_{r,s}^{2,\Lambda}$.

We next prove the case of rank-one matrices $N = c_{ij} f_i \otimes f_j$ with $\{i, j\} \neq \{2, 3\}$ – in fact, we prove a slightly more general statement. Recall that $f_1 = (\alpha - \beta) / |\alpha - \beta|$ when $\alpha \neq \beta$.

**Lemma 6.8.** There exists a constant $c_{r,s}^{2,\Lambda} \in ]0, 1]$ with the following property. If $|\Pi| \leq rs$, max\{\begin{small}$|\alpha| - \tau r$, $|\beta| - \tau s$, $|\alpha|$
\end{small} \} \leq c_{r,s}^{1,\Lambda} and $e \in S^2$, then

$(\alpha, \beta, \alpha \otimes \beta + ae \otimes e + \Pi I), (\alpha, \beta, \alpha \otimes \beta + a f_1 \otimes e + \Pi I), (\alpha, \beta, \alpha \otimes \beta + ae \otimes f_1 + \Pi I) \in K_{r,s}^{3,\Lambda}$.

**Proof.** We first prove the case $N = ae \otimes e$. Choose $c_{r,s}^{2,\Lambda} := (1 - \tau)^2 r s / (r + s + 1)^2$. If max\{\begin{small}$|\alpha| - \tau r$, $|\beta| - \tau s$, $|\alpha|$
\end{small} \} \leq c_{r,s}^{1,\Lambda} and $e \in S^2$, write $a = k c_{r,s}^{2,\Lambda}$, where $|k| \leq 1$. Then choose $b = k (1 - \tau) r / 2 (r + s + 1)$ and $c = d (1 - \tau) s / 2 (r + s + 1)$ so that $bc = a$. Now $|\alpha \pm be| \leq r$ and $|\beta \pm ce| \leq s$, and so Lemma 6.7 gives

$V^\pm := (\alpha \pm be, \beta \pm ce, (\alpha \pm be) \otimes (\beta \pm ce) + \Pi I) \in K_{r,s}^{2,\Lambda}.$
The points $\alpha \pm be, \beta \pm ce \in \mathbb{R}^3$ all belong to the same hyperplane (or straight) $\beta + \text{span}\{f_1, e\}$, and thus, by Lemma 6.6 \((V^+ + V^-)/2 = (\alpha, \beta, \alpha \otimes \beta + ae \otimes e + IIJ) \in K_{r,s}^A\)

The case $N = af_1 \otimes e$ is proved by setting

$$V^\pm := (\alpha \pm b f_1, \beta \pm ce, (\alpha \pm b f_1) \otimes (\beta \pm ce) + IIJ) \in K_{r,s}^A$$

and repeating the argument above. The case $N = ae \otimes f_1$ is similar. \qed

With the conclusion of Lemma 6.5 demonstrated for rank-one matrices $N$, we now prove the general case where $c_{23} \neq 0$. We present the result in an equivalent form that is easier to use in [6, 8].

**Lemma 6.9.** There exists $c''_{t,r,s} \in [0, 1]$ with the following property. Whenever $g_1, \ldots, g_5 \in S^2$, $d_1, \ldots, d_5 \in \mathbb{R}$, max\{||\alpha| - r\tau|, ||\beta| - \tau s|, |d_1|, \ldots, |d_5|\} \leq c''_{t,r,s}$ and $IIJ \leq rs$, we have

$$(\alpha, \beta, \alpha \otimes \beta + d_1 g_1 \otimes g_1 + d_2 g_2 \otimes g_2 + d_3 g_3 \otimes g_3 + d_4 f_1 \otimes g_4 + d_5 g_5 \otimes f_1 + IIJ) \in K_{r,s}^A.$$ 

**Proof.** First assume $d_3 = d_4 = d_5 = 0$ and max\{||\alpha| - r\tau|, ||\beta| - \tau s|, |d_1|, |d_2|\} \leq (c''_{t,r,s})^2/2$. Writing $b = \sqrt{|d_2|}$ and $c = \text{sgn}(d_2) \sqrt{|d_2|}$, Lemma 6.8 gives

$$V^\pm := (\alpha \pm b g_2, \beta \pm c g_2, (\alpha \pm b g_2) \otimes (\beta \pm c g_2) + d_1 g_1 \otimes g_1 + IIJ) \in K_{r,s}^A$$

and Lemma 6.8 gives $V^+ - V^- \in \Lambda$, so that $(\alpha, \beta, \alpha \otimes \beta + d_1 g_1 \otimes g_1 + d_2 g_2 \otimes g_2 + IIJ) = (V^+ + V^-)/2 \in K_{r,s}^A$.

We next consider the general case. Set $c''_{t,r,s} := (c''_{t,r,s})^2/16$ and suppose now that we have max\{||\alpha| - r\tau|, ||\beta| - \tau s|, |d_1|, \ldots, |d_5|\} \leq c''_{t,r,s}$. By choosing $d_6 \neq 0$ with $|d_6| \leq c''_{t,r,s}$ and repeatedly using Lemma 6.8 as above,

$V := (\alpha, \beta - d_6 f_1, \alpha \otimes (\beta - d_6 f_1) + d_1 g_1 \otimes g_1 + d_2 g_2 \otimes g_2 + d_3 g_3 \otimes g_3 + 2d_4 f_1 \otimes g_4 + IIJ),$

$W := (\alpha, \beta - d_6 f_1, \alpha \otimes (\beta - d_6 f_1) + d_1 g_1 \otimes g_1 + d_2 g_2 \otimes g_2 + d_3 g_3 \otimes g_3 + 2d_5 g_5 \otimes f_1 + IIJ)$

belong to $K_{r,s}^A$. (Here we used the fact that $\alpha - (\beta \pm d_6 f_1)$ are parallel to $f_1$.) Lemma 6.9 gives

$$V - W = \left(0, 2d_6 f_1, \left(\alpha - \frac{d_5}{d_6} g_3\right) \otimes 2d_6 f_1 + 2d_6 f_1 \otimes \frac{d_2}{d_6} g_4\right) \in \Lambda,$$

and the claim follows by taking the average of $V$ and $W$. \qed

**6.4. Completion of the proof of Lemma 6.5.** One of the difficulties in proving the case $N = \sum_{i,j=1}^3 c_{ij} f_i \otimes f_j$ with $c_{23} = c_{32} \neq 0$ is the fact that the basis \{\text{f}_1, f_2, f_3\} of $\mathbb{R}^3$ depends on $\alpha$ and $\beta$. We wish to use Lemma 6.9 to write

$$\left(\alpha, \beta, \alpha \otimes \beta + \sum_{i,j=1}^3 c_{ij} f_i \otimes f_j + IIJ\right)$$

$$= \lambda V_1 + \mu V_2$$

$$:= \lambda \left(\alpha + \mu f_1, \beta + \mu g, (\alpha + \mu f) \otimes (\beta + \mu g) + \sum_{i,j=1}^3 c''_{ij} f'_i \otimes f'_j + IIJ\right)$$

$$+ \mu \left(\alpha - \lambda f_1, \beta - \lambda g, (\alpha - \lambda f) \otimes (\beta - \lambda g) + \sum_{i,j=1}^3 c''_{ij} f''_i \otimes f''_j + IIJ\right).$$
where \( f'_1 \) and \( \alpha + \mu f - (\beta + \mu g) \) are parallel, \( c'_{32} = c'_{32} = 0 \), similar conditions hold for \( V_2 \) and furthermore \( 0 < \lambda < 1, \lambda + \mu = 1 \) and \( V_1 - V_2 \in \Lambda \). This leads, by necessity, to much more complicated computations than the ones done in \[6.3\].

In particular, when \( \alpha \neq \beta \), natural directions of the form \( (f, g) = (c f_i, d f_j) \) with \( c, d \in \mathbb{R} \) and \( i, j \in \{1, 2, 3\} \) always lead to \( c_{23} = c_{32} = 0 \) unless \( c, d \neq 0 \) and \( \{i, j\} = \{2, 3\} \). With this specific choice of \( (f, g) \) we are, however, eventually able to achieve \( c_{23} = c_{32} \neq 0 \). We finish the proof of Lemma \[6.3\] in two steps, starting with the following lemma.

**Lemma 6.10.** There exists a constant \( e''_{\tau, r, s} \in]0, 1[ \) such that if at most one of \( c_{12}, c_{13}, c_{21}, c_{31} \) nonzero and max\{\( |\alpha - \tau r|, |\beta - \tau s|, |c_{ij}|, |\Pi| \)\} \( \leq e''_{\tau, r, s} \), then we have \( (\alpha, \beta, \alpha \otimes \beta + \sum_{i,j=1}^{3} c_{ij} f_i \otimes f_j + \Pi) \in K^7_{r,s} \).

**Proof.** We will find a constant \( e''_{\tau, r, s} \) for the case where \( c_{13} = c_{21} = c_{31} = 0 \); the modifications required for the other three cases are rather obvious. We will write \( N = 2N'/3 + N''/3 \), where \( N' := \sum_{i,j=1}^{3} c'_{ij} f_i \otimes f_j \) satisfies \( c'_{ij} = c_{ij} \) for all \( \{i, j\} \notin \{(1, 2), (1, 3)\} \) and the same condition holds for \( N'' := \sum_{i,j=1}^{3} c''_{ij} f_i \otimes f_j \), so that \( 0, N' - N'' \in \Lambda \). In turn, we will write

\[
(\alpha, \beta, \alpha \otimes \beta + N' + \Pi) = \frac{1}{2} (V + W),
\]

and similarly for \( N'' \), where \( V, W \in K^7_{r,s} \) satisfy \( V - W \in \Lambda \), fall under the scope of Lemma \[6.9\] and will be specified below.

We initially work under the full generality allowed by Lemma \[6.9\]. Recall that \( \alpha - \beta = |\alpha - \beta| f_1 \). By Lemma \[6.9\] whenever all the coefficients of \( V \) and \( W \) defined below are small enough, we have

\[
V = (\alpha + c f_2, \beta + d f_3, (\alpha + c f_2) \otimes (\beta + d f_3) + v_{f_1 \otimes f_1} f_1 \otimes f_1 + v_{f_2 \otimes f_2} f_2 \otimes f_2 + v_{f_3 \otimes f_3} f_3 \otimes f_3 + (\alpha - \beta) f_1 + c f_2 - d f_3) \otimes (a_1 f_1 + a_2 f_2 + a_3 f_3) + (b_1 f_1 + b_2 f_2 + b_3 f_3) \otimes (|\alpha - \beta| f_1 + c f_2 - d f_3) + \Pi) \in K^7_{r,s},
\]

\[
W = (\alpha - c f_2, \beta - d f_3, (\alpha - c f_2) \otimes (\beta - d f_3) + w_{f_1 \otimes f_1} f_1 \otimes f_1 + w_{f_2 \otimes f_2} f_2 \otimes f_2 + w_{f_3 \otimes f_3} f_3 \otimes f_3 + (\alpha - \beta) f_1 - c f_2 + d f_3) \otimes (c_1 f_1 + c_2 f_2 + c_3 f_3) + (d_1 f_1 + d_2 f_2 + d_3 f_3) \otimes (|\alpha - \beta| f_1 - c f_2 + d f_3) + \Pi) \in K^7_{r,s}.
\]

We will choose \( c, d \neq 0 \), and so, by Lemma \[6.7\] and the fact that

\[
\alpha \otimes 2df_3 + 2c f_2 \otimes \beta = \beta \otimes 2df_3 + 2c f_2 \otimes \beta + |\alpha - \beta| f_1 \otimes 2df_3,
\]

the condition \( V - W =: (2c f_2, 2d f_3, d f_1 \otimes f_1 f_1 \otimes f_1 + d f_1 \otimes f_2 f_1 \otimes f_2 + \cdots + d f_3 \otimes f_3 f_3 \otimes f_3) \in \Lambda \) reads as

\[
\begin{align*}
(6.5) \quad & d_{f_1 \otimes f_1} = |\alpha - \beta| (a_1 + b_1 - c_1 - d_1) + v_{f_1 \otimes f_1} - w_{f_1 \otimes f_1} = 0, \\
(6.6) \quad & d_{f_1 \otimes f_2} = |\alpha - \beta| (a_2 - c_2) + c(b_1 + d_1) = 0, \\
(6.7) \quad & d_{f_1 \otimes f_3} = |\alpha - \beta| (b_3 - d_3) - d(a_1 + c_1) = 0, \\
(6.8) \quad & d \cdot d_{f_2 \otimes f_1} = cd(a_1 + c_1) + |\alpha - \beta| d(b_2 - d_2) + 2cd(\beta, f_1) \\
(6.9) \quad & = c \cdot |\alpha - \beta| (a_3 - c_3 + 2d) - cd(b_1 + d_1) + 2cd(\beta, f_1) = c \cdot d_{f_1 \otimes f_3}
\end{align*}
\]
(recall that the relative lengths of the vectors $2c f_2$ and $2df_3$ matter in condition \((0.8) - (0.9)\).)

On the other hand, the sought equation \((\alpha, \beta, \alpha \otimes \beta + N' + II) = (V + W)/2\) can be written as

\[
2 \sum_{i,j=1}^3 c_{ij} f_i \otimes f_j + 2(c'_1 - c_1) f_1 \otimes f_2 + 2(c'_2 - c_2) f_2 \otimes f_3 \\
= [c(a_3 - c_3) + d(d_2 - b_2) + 2cd] f_2 \otimes f_3 \\
+ [d(c_2 - a_2) + c(b_3 - d_3)] f_3 \otimes f_2 \\
+ [c(b_1 - d_1) + |\alpha - \beta|(a_2 + c_2)] f_1 \otimes f_2 \\
+ [c(a_1 - c_1) + |\alpha - \beta|(b_2 + d_2)] f_2 \otimes f_1 \\
+ [-d(b_1 - d_1) + |\alpha - \beta|(a_3 + c_3)] f_1 \otimes f_3 \\
+ [-d(a_1 - c_1) + |\alpha - \beta|(b_3 + d_3)] f_3 \otimes f_1 \\
+ |\alpha - \beta|(a_1 + b_1 + c_1 + d_1) + v f_1 \otimes f_1 f_1 + f_1 \\
+ [c(a_2 + b_2 - c_2 - d_2) + v f_2 \otimes f_2 + w f_2 \otimes f_3] f_2 \otimes f_2 \\
+ [d(-a_3 - b_3 + c_3 + d_3) + v f_3 \otimes f_3 + w f_3 \otimes f_3] f_3 \otimes f_3.
\]

Despite conditions \((0.8) - (0.9)\) there is some freedom in the choice of the coefficients. We choose

\[
V = (\alpha + c f_2, \beta + d f_3, (\alpha + c f_2) \otimes (\beta + d f_3) \\
+ (c_{11} - q |\alpha - \beta| - 2p |\alpha - \beta|^2) f_1 \otimes f_1 \\
+ (c_{22} + q |\alpha - \beta| - 2p |\alpha - \beta|^2) f_2 \otimes f_2 + (c_{33} + (3p - 1)d^2) f_3 \otimes f_3 \\
+ (|\alpha - \beta| f_1 + c f_2 - d f_3) \otimes (p |\alpha - \beta| f_1 + c f_2 - d f_3) + (2p - 1) f_3) \\
+ (|p |\alpha - \beta| + q) f_1 + d f_3 \otimes ((|\alpha - \beta| f_1 + c f_2 - d f_3) + III) \\
\in K^{7, A}_{r,s},
\]

\[
W = (\alpha - c f_2, \beta - d f_3, (\alpha - c f_2) \otimes (\beta - d f_3) \\
+ (c_{11} + q |\alpha - \beta| - 2p |\alpha - \beta|^2) f_1 \otimes f_1 \\
+ (c_{22} + q |\alpha - \beta| - 2p |\alpha - \beta|^2) f_2 \otimes f_2 + (c_{33} + (3p - 1)d^2) f_3 \otimes f_3 \\
+ (|\alpha - \beta| f_1 - c f_2 + d f_3) \otimes (p |\alpha - \beta| f_1 + c f_2 - d f_3) + (2p - 1) f_3) \\
+ (|p |\alpha - \beta| - q) f_1 - d f_3 \otimes ((|\alpha - \beta| f_1 - c f_2 + d f_3) + III) \\
\in K^{7, A}_{r,s},
\]

where the coefficients \(c, d, p, q \in \mathbb{R}\) will be specified at the end of the proof. The wave cone conditions \((0.8) - (0.9)\) are satisfied and thus

\[
\frac{V + W}{2} = (\alpha, \beta, \alpha \otimes \beta + c_{11} f_1 \otimes f_1 + c q f_1 \otimes f_2 - d q f_1 \otimes f_3 + c_{22} f_2 \otimes f_2 \\
+ 2 c d f_3 \otimes f_3 + f_3 \otimes f_3) + c_{33} f_3 \otimes f_3 + III \\
=: (\alpha, \beta, \alpha \otimes \beta + N' + III) \in K^{8, A}_{r,s}.
\]

We repeat the argument with \(c\) and \(d\) replaced by \(-c/2\) and \(-2d\) to form

\[
N'' = c_{11} f_1 \otimes f_1 - \frac{c}{2} q f_1 \otimes f_2 + 2 d q f_1 \otimes f_3 + c_{22} f_2 \otimes f_2 + 2 p c d f_3 \otimes f_3 + c_{33} f_3 \otimes f_3,
\]
and then \((0,0,N' - N'') = (0,0,(3cq/2)f_1 \otimes f_2 - 3dqf_1 \otimes f_3) \in \Lambda\) implies
\[
\frac{2}{3}(\alpha, \beta, \alpha \otimes \beta + N' + \Pi I) + \frac{1}{3}(\alpha, \beta, \alpha \otimes \beta + N'' + \Pi I)
= (\alpha, \beta, \alpha \otimes \beta + c_{ij}f_i \otimes f_1 + \frac{1}{2}qf_1 \otimes f_2 + c_{22}f_2 \otimes f_2
+ 2pcd(f_2 \otimes f_3 + f_3 \otimes f_2) + c_{33}f_3 \otimes f_3 + \Pi I) \in K_{r,s}^9, \Lambda.
\]

We are now ready to specify all the parameters. Set \(c_{r,s}'' := (c'_{r,s})^4/1000(r + s + 1)^8\) and denote \(c_{12} = kc_{r,s}''\) with \(|k| \leq 1\) and \(c_{23} = lc_{r,s}''\) with \(|l| \leq 1\). Choose 
\[c = (c''_{r,s})^{1/2}, d = (c''_{r,s})^{1/4}, p = l(c''_{r,s})^{1/4}/2\]
and \(q = 2k(c''_{r,s})^{1/2}\), so that \(cq/2 = c_{12}, 2pcd = c_{23}\). This proves the case \(c_{13} = c_{21} = c_{31} = 0\). When forming \(N'' \in R^{3 \times 3}\), if we replace \(c\) and \(d\) by \(-2c\) and \(-d/2\) (instead of \(-c/2\) and \(-d/2\) as above) and modifying \(p\) and \(q\) accordingly, we get the case \(c_{13} \neq 0\). The cases \(c_{21} \neq 0\) and \(c_{31} \neq 0\) are similar by symmetry.

We finish the proof of Lemma 6.5 by reducing the general case \(N = \sum_{i,j=1}^{3} c_{ij}f_i \otimes f_j\) with \(c_{23} = c_{32}\) to the situation of Lemma 6.10.

Completion of the proof of Lemma 6.5 Suppose \((\alpha, \beta, \alpha \otimes \beta + \sum_{i,j=1}^{3} c_{ij}f_i \otimes f_j + \Pi I)\) satisfies the assumptions of Lemma 6.5 where \(c_{r,s}'' := c_{r,s}''/8\). We write
\[\sum_{i,j=1}^{3} c_{ij}f_i \otimes f_j =: N' + c_{12}f_1 \otimes f_2 + c_{13}f_1 \otimes f_3 + c_{21}f_2 \otimes f_1 + c_{31}f_3 \otimes f_1\]
with the intention of using Lemma 6.10 and convex combinations in directions of the form \((0,0,M) \in \Lambda\) to prove that \((\alpha, \beta, \alpha \otimes \beta + \sum_{i,j=1}^{3} c_{ij}f_i \otimes f_j + \Pi I) \in K_{r,s}^{12,\Lambda}\). We first form a \(\Lambda\)-convex combination via
\[
c_{12}f_1 \otimes f_2 + c_{13}f_1 \otimes f_3 + c_{21}f_2 \otimes f_1 + c_{31}f_3 \otimes f_1
= \frac{1}{2}(2c_{12}f_1 \otimes f_2 + 2c_{21}f_2 \otimes f_1 + 2c_{31}f_3 \otimes f_1)
+ \frac{1}{2}(2c_{13}f_1 \otimes f_3 + 2c_{23}f_3 \otimes f_1 + 2c_{33}f_3 \otimes f_1)
= \frac{1}{2}\left(\frac{1}{2}(4c_{12}f_1 \otimes f_2 + 4c_{23}f_3 \otimes f_1) + \frac{1}{2}(2c_{21}f_2 \otimes f_1 + 2c_{33}f_3 \otimes f_1)\right)
+ \frac{1}{2}\left(\frac{1}{2}(4c_{13}f_1 \otimes f_3 + 4c_{21}f_2 \otimes f_1) + \frac{1}{2}(2c_{23}f_3 \otimes f_1 + 2c_{33}f_3 \otimes f_1)\right).
\]
Note that \((\alpha, \beta, \alpha \otimes \beta + N' + c_{12}f_1 \otimes f_2 + c_{13}f_1 \otimes f_3 + c_{21}f_2 \otimes f_1 + c_{31}f_3 \otimes f_1 + \Pi I)\) is, furthermore, a \(\Lambda\)-convex combination of \((\alpha, \beta, \alpha \otimes \beta + N' + 4c_{12}f_1 \otimes f_2 + 4c_{13}f_1 \otimes f_3 + 4c_{21}f_2 \otimes f_1 + 4c_{31}f_3 \otimes f_1 + \Pi I) \in K_{r,s}^{9,\Lambda}\) and \((\alpha, \beta, \alpha \otimes \beta + N' + 2c_{12}f_1 \otimes f_2 + 2c_{13}f_1 \otimes f_3 + 2c_{21}f_2 \otimes f_1 + 2c_{31}f_3 \otimes f_1 + \Pi I) \in K_{r,s}^{9,\Lambda}\), and a similar argument works on \(c_{21}f_2 \otimes f_1 + 2c_{31}f_3 \otimes f_1\).

For the matrix \(4c_{12}f_1 \otimes f_2 + c_{31}f_3 \otimes f_1\) we get
\[
(\alpha, \beta, \alpha \otimes \beta + N' + 4c_{12}f_1 \otimes f_2 + c_{31}f_3 \otimes f_1 + \Pi I) = \frac{V + W}{2} \in K_{r,s}^{10,\Lambda}
\]
where, for a small enough \(d \in R \setminus \{0\}\), we have
\[
V := (\alpha, \beta + df_1, \alpha \otimes (\beta + df_1) + N' + 8c_{12}f_1 \otimes f_2 + \Pi I) \in K_{r,s}^{9,\Lambda},
\]
\[
W := (\alpha, \beta - df_1, \alpha \otimes (\beta - df_1) + N' + 2c_{31}f_3 \otimes f_1 + \Pi I) \in K_{r,s}^{9,\Lambda}
\]
by Lemma 6.10 and \(V - W \in \Lambda\) by Lemma 6.5. Analogous reasoning shows that \((\alpha, \beta, \alpha \otimes \beta + N' + 4c_{13}f_1 \otimes f_3 + c_{21}f_2 \otimes f_1 + \Pi I) \in K_{r,s}^{10,\Lambda}\), and as a result, \((\alpha, \beta, \alpha \otimes \beta + \sum_{i,j=1}^{3} c_{ij}f_i \otimes f_j + \Pi I) \in K_{r,s}^{12,\Lambda}.\) \qed
We have now proved Lemma 6.5 and, thereby, Theorems 1.4, 1.5 and 6.3. We finish the article by making some brief remarks on the convex integration solutions of [BLFNL].

6.5. Discussion of the solutions of Bronzi & al. As mentioned in the Introduction, Bronzi, Lopes Filho and Nussenzweig Lopes showed in [BLFNL] the existence of bounded weak solutions of 3D MHD that are compactly supported in time. In this subsection we briefly discuss their construction.

Bronzi & al. studied two-dimensional incompressible Euler equations with a passive tracer,

\begin{align}
\partial_t v + \text{div}(v \otimes v) + \nabla p &= 0, \\
\partial_t b + \text{div}(bv) &= 0, \\
\text{div } v &= 0,
\end{align}

where \( v: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) is the velocity field, \( b: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) is the tracer and \( p: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) is the pressure. Their main result, proved by adapting arguments of [DLS09] on Euler equations, reads as follows:

**Theorem 6.11 ([BLFNL]).** Given a bounded domain \( \Omega \subset \mathbb{R}^2 \times \mathbb{R} \), there exists a weak solution \( (v, b) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R}) \) of \( (6.10) - (6.12) \) with the following properties:

(i) \( |v(x, t)| = 1 \) and \( |b(x, t)| = 1 \) for almost every \( (x, t) \in \Omega \),

(ii) \( v(x, t) = 0, b(x, t) = 0 \) and \( p(x, t) = 0 \) for almost every \( (x, t) \in (\mathbb{R}^2 \times \mathbb{R}) \setminus \Omega \).

Bronzi & al. obtained bounded weak solutions of 3D MHD from Theorem 6.11 as follows: when \( u, b: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \) are of the symmetry reduced forms

\begin{align}
u(x_1, x_2, x_3, t) := (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0), \quad b(x_1, x_2, x_3, t) = (0, 0, b_3(x_1, x_2, t)),
\end{align}

the 3D MHD equations \( (1.1) - (1.3) \) are reduced to \( (6.10) - (6.12) \), and so Theorem 6.11 yields solutions of \( (1.1) - (1.3) \). However, as mentioned in the Introduction, solutions \( u \) and \( b \) obtained in this way are not compactly supported in space.

The main obstacle to finding compactly supported solutions of 3D MHD is the non-linear constraint \( a \cdot b = 0 \). By using Lemma 6.4 one may show that a single localized 1D wave can take values in \( U_{r,s} \), but the localization makes it difficult to superimpose waves while satisfying the condition \( a \cdot b = 0 \). In [BLFNL] this issue did not arise simply because Bronzi & al. imposed the symmetry restrictions \( (6.13) \) which lead (in this article’s notation) to \( a \) and \( b \) taking values in orthogonal subspaces of \( \mathbb{R}^3 \). It seems that going beyond the symmetry reduced solutions of [BLFNL] and attaining compact support in space will require very careful analysis.

Note also that the symmetry restrictions \( (6.13) \) ensure that the cross helicity \( \int_{\mathbb{R}^3} u(x, t) \cdot b(x, t) \) vanishes identically for the solutions of [BLFNL]. It is therefore still an open question whether cross helicity is conserved by weak solutions of 3D MHD.

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