Weighted function spaces and Dunkl transform

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Abstract
We introduce first weighted function spaces on $\mathbb{R}^d$ using the Dunkl convolution that we call Besov-Dunkl spaces. We provide characterizations of these spaces by decomposition of functions. Next we obtain in the real line and in radial case on $\mathbb{R}^d$ weighted $L^p$-estimates of the Dunkl transform of a function in terms of an integral modulus of continuity which gives a quantitative form of the Riemann-Lebesgue lemma. Finally, we show in both cases that the Dunkl transform of a function is in $L^1$ when this function belongs to a suitable Besov-Dunkl space.

Keywords : Dunkl operators, Dunkl transform, Dunkl translations, Dunkl convolution, Besov-Dunkl spaces.

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1 Introduction
Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^d$. It started twenty years ago with Dunkl’s seminal work [9] and was further developed by several

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We consider Dunkl operators $T_i, 1 \leq i \leq d$, on $\mathbb{R}^d$, associated to an arbitrary finite reflection group $G$ and a non negative multiplicity function $k$. The Dunkl kernel $E_k$ has been introduced by C.F. Dunkl in [10]. This kernel is used to define the Dunkl transform $F_k$. K. Trimèche has introduced in [22] the Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, on the space of infinitely differentiable functions on $\mathbb{R}^d$. At the moment an explicit formula for the Dunkl translation $\tau_x(f)$ of a function $f$ is unknown in general. However, such formula is known in two cases: when the function $f$ is radial and when $G = \mathbb{Z}_d^d$, (see next section). In particular, the boundedness of $\tau_x$ is established in these cases. As a result one obtains a formula for the convolution $*$. There are many ways to define the Besov spaces (see [4, 5, 20]) and the Besov spaces for the Dunkl operators (see [1, 3]). Let $\Phi(\mathbb{R}^d)$ be the set of all sequences $(\varphi_j)_{j \in \mathbb{N}}$ of functions in $S(\mathbb{R}^d)$ which are radial. For $\beta > 0$, $1 \leq p, q \leq +\infty$ and $(\varphi_j)_{j \in \mathbb{N}} \in \Phi(\mathbb{R}^d)$, we define the weighted Besov-Dunkl space denoted by $BD_{p,q}^{\beta,k}$ as the subspace of functions $f \in L^p_k(\mathbb{R}^d)$ satisfying

\[
(\sum_{j \in \mathbb{N}} (2^{j\beta} \| \varphi_j \ast_k f \|_{p,k})^q)^{\frac{1}{q}} < +\infty \quad if \quad q < +\infty
\]

and

\[
\sup_{j \in \mathbb{N}} 2^{j\beta} \| \varphi_j \ast_k f \|_{p,k} < +\infty \quad if \quad q = +\infty,
\]

where $L^p_k(\mathbb{R}^d)$ is the space $L^p(\mathbb{R}^d, w_k(x)dx)$, with $w_k$ the weight function given by

\[
w_k(x) = \prod_{\xi \in \mathbb{R}^d} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d,
\]
$R_+^*$ being a positive root system (see next section).

For $1 \leq p \leq 2$, we introduce the class $\mathcal{M}_p(\mathbb{R}^d)$ of sequences $(g_j)_{j \in \mathbb{N}}$ of functions in $L^p_k(\mathbb{R}^d)$ such that $\mathrm{supp} \mathcal{F}_k(g_0) \subset B(0,2)$ and $\mathrm{supp} \mathcal{F}_k(g_j) \subset A_j$ for $j \in \mathbb{N}\{0\}$.

The modulus of continuity $\omega_p(f)$ of an $L^p$-function $f$, $1 \leq p \leq 2$ is defined:

- On the real line by:
  $$\omega_p(f)(t) = \|\tau_t(f) + \tau_{-t}(f) - 2f\|_{p,k}, \quad t \geq 0,$$
  which is known as the modulus of continuity of second order of $f$.
- In radial case on $\mathbb{R}^d$ by:
  $$\omega_p(f)(t) = \int_{S^{d-1}} \|\tau_{tu}(f) - f\|_{p,k} d\sigma(u), \quad t \geq 0,$$
  where $f$ is a radial $L^p$-function and $S^{d-1}$ the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$.

In this paper, we provide first characterizations of the Besov-Dunkl spaces $\mathcal{BD}^{\beta,k}_{p,q}$ by decomposition of functions using the class $\mathcal{M}_p(\mathbb{R}^d)$ for $\beta > 0$, $1 \leq p \leq 2$ and $1 \leq q < +\infty$. This extend to the Dunkl operators on $\mathbb{R}^d$ some results obtained for the classical case in [20]. Next we obtain weighted $L^p$-estimates of the Dunkl transform of a function in terms of an integral modulus of continuity. These results are carried out on the real line and in radial case on $\mathbb{R}^d$. In both cases these are expressed as a gauge on the size of in terms of an integral modulus of continuity of $f$. As consequence, we give a quantitative form of the Riemann-Lebesgue lemma. Finally, we show that the Dunkl transform $\mathcal{F}_k(f)$ of a function $f$ is in $L^1_k(\mathbb{R}^d)$ when $f$ belongs to a suitable Besov-Dunkl space.

The contents of this paper are as follows.

In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.

In section 3, we characterize the Besov-Dunkl spaces by decomposition of functions.

In section 4, we obtain inequalities for the Dunkl transform $\mathcal{F}_k(f)$ of $f$, both on the real line and in radial case on $\mathbb{R}^d$. As consequence, we give a quantitative form of the Riemann-Lebesgue lemma. We show Finally further results
of integrability of $F_k(f)$ when $f$ satisfies a suitable condition.

Along this paper we denote by $\langle ., . \rangle$ the usual Euclidean inner product in $\mathbb{R}^d$ as well as its extension to $\mathbb{C}^d \times \mathbb{C}^d$, we write for $x \in \mathbb{R}^d, \|x\| = \sqrt{x,x}$ and we use $c$ to denote a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$.
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives.
- $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ of compactly supported functions.

2 Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the articles [8, 9, 18] or to the surveys [16].

Let $G$ be a finite reflection group on $\mathbb{R}^d$, associated with a root system $R$. For $\alpha \in R$, we denote by $H_\alpha$ the hyperplane orthogonal to $\alpha$. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$, we fix a positive subsystem $R_+ = \{ \alpha \in R : \langle \alpha, \beta \rangle > 0 \}$. We denote by $k$ a nonnegative multiplicity function defined on $R$ with the property that $k$ is $G$-invariant. We associate with $k$ the index

$$\gamma = \gamma(R) = \sum_{\xi \in R_+} k(\xi) \geq 0,$$

and the weight function $w_k$ defined by

$$w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.$$

$w_k$ is $G$-invariant and homogeneous of degree $2\gamma$.

Further, we introduce the Mehta-type constant $c_k$ by

$$c_k = \left( \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} w_k(x) dx \right)^{-1}.$$

For every $1 \leq p \leq +\infty$, we denote by $L_k^p(\mathbb{R}^d)$ the space $L^p(\mathbb{R}^d, w_k(x) dx)$, $L_k^p(\mathbb{R}^d)^{rad}$ the subspace of those $f \in L_k^p(\mathbb{R}^d)$ that are radial and we use $\| \|_{p,k}$.
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as a shorthand for \[ \| \|_{L^p_k(\mathbb{R}^d)}. \]

By using the homogeneity of \( w_k \), it is shown in [14] that for \( f \in L^1_k(\mathbb{R}^d)^{rad} \), there exists a function \( F \) on \([0, +\infty)\) such that \( f(x) = F(\|x\|) \), for all \( x \in \mathbb{R}^d \). The function \( F \) is integrable with respect to the measure \( r^{2\gamma + d - 1}dr \) on \([0, +\infty)\) and we have

\[
\int_{\mathbb{R}^d} f(x) w_k(x) dx = \int_0^{+\infty} \left( \int_{S^{d-1}} w_k(ry)d\sigma(y) \right) F(r) r^{d-1} dr = d_k \int_0^{+\infty} F(r) r^{2\gamma + d - 1} dr,
\]

where \( S^{d-1} \) is the unit sphere on \( \mathbb{R}^d \) with the normalized surface measure \( d\sigma \) and

\[
d_k = \int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma + \frac{d}{2} - 1}\Gamma(\gamma + \frac{d}{2})}.
\]

The Dunkl operators \( T_j, \ 1 \leq j \leq d, \) on \( \mathbb{R}^d \) associated with the reflection group \( G \) and the multiplicity function \( k \) are the first-order differential-difference operators given by

\[
T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in \mathcal{E}(\mathbb{R}^d), \quad x \in \mathbb{R}^d,
\]

where \( \sigma_\alpha \) is the reflection on the hyperplane \( H_\alpha \) and \( \alpha_j = \langle \alpha, e_j \rangle, \ (e_1, \ldots, e_d) \) being the canonical basis of \( \mathbb{R}^d \).

In the case \( k = 0 \), the \( T_j \) reduce to the corresponding partial derivatives.

For \( y \in \mathbb{R}^d \), the system

\[
\begin{cases}
T_j u(x, y) = y_j u(x, y), & 1 \leq j \leq d, \\
u(0, y) = 1.
\end{cases}
\]

admits a unique analytic solution on \( \mathbb{R}^d \), denoted by \( E_k(x, y) \) and called the Dunkl kernel. This kernel has a unique holomorphic extension to \( \mathbb{C}^d \times \mathbb{C}^d \).

M. Rösler has proved in [15] the following integral representation for the Dunkl kernel

\[
E_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_y^k(x), \quad x \in \mathbb{R}^d, \quad z \in \mathbb{C}^d,
\]
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where $\mu_k^x$ is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$. We have for $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$, $E_k(z, z') = E_k(z', z)$, $E_k(\lambda z, z') = E_k(z, \lambda z')$ and $|E_k(x, iy)| \leq 1$ for $x, y \in \mathbb{R}^d$. It was shown in [14, 17] that

$$\int_{S^{d-1}} E_k(ix, z) w_k(z) d\sigma(z) = d_k j_{\gamma + \frac{d}{2} - 1}(\|x\|), \quad x \in \mathbb{R}^d,$$

where $j_{\gamma + \frac{d}{2} - 1}$ is the normalized Bessel function of the first kind and order $\gamma + \frac{d}{2} - 1$.

The Dunkl transform $F_k$ is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$F_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) w_k(y) dy, \quad x \in \mathbb{R}^d.$$

We list some known properties of this transform:

i) The Dunkl transform of a function $f \in L^1_k(\mathbb{R}^d)$ has the following basic property

$$\|F_k(f)\|_{\infty,k} \leq \|f\|_{1,k}.$$

ii) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under the Dunkl transform $F_k$.

iii) When both $f$ and $F_k(f)$ are in $L^1_k(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} F_k(f)(y) E_k(ix, y) w_k(y) dy, \quad x \in \mathbb{R}^d.$$

iv) (Plancherel’s theorem) The Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

By i), Plancherel’s theorem and the Marcinkiewicz interpolation theorem (see [19]), we get for $f \in L^p_k(\mathbb{R}^d)$ with $1 \leq p \leq 2$ and $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|F_k(f)\|_{p',k} \leq c \|f\|_{p,k}. \quad (2.4)$$

The Dunkl transform of a function in $L^1_k(\mathbb{R}^d)^{rad}$ is also radial and could be expressed via the Hankel transform. More precisely, according to [14], we have the following results:

$$F_k(f)(x) = \int_0^{+\infty} \left( \int_{S^{d-1}} E_k(-ix, y) w_k(y) d\sigma(y) \right) F(r) r^{2\gamma + d - 1} dr$$

$$= \mathcal{H}_{\gamma + \frac{d}{2} - 1}(F)(\|x\|), \quad x \in \mathbb{R}^d, \quad (2.5)$$
where $F$ is the function defined on $[0, +\infty)$ by $F(||x||) = f(x)$, $x \in \mathbb{R}^d$ and $\mathcal{H}_{\gamma + \frac{d}{2} - 1}$ is the Hankel transform of order $\gamma + \frac{d}{2} - 1$.

For $\varphi \in \mathcal{S}(\mathbb{R}^d)^{\text{rad}}$ and $x \in \mathbb{R}^d$, we have $F_k^{-1}(\varphi)(x) = F_k(\varphi)(-x) = F_k(\varphi)(x)$.

K. Trimèche has introduced in [22] the Dunkl translation operators $\tau_x$, $x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have

$$F_k(\tau_x(f))(y) = E_k(ix, y)F_k(f)(y).$$

(2.6)

Notice that for all $x, y \in \mathbb{R}^d$, $\tau_x(f)(y) = \tau_y(f)(x)$ and for fixed $x \in \mathbb{R}^d$

$$\tau_x \text{ is a continuous linear mapping from } \mathcal{E}(\mathbb{R}^d) \text{ into } \mathcal{E}(\mathbb{R}^d).$$

(2.7)

As an operator on $L^2_k(\mathbb{R}^d)$, $\tau_x$ is bounded. A priori it is not at all clear whether the translation operator can be defined for $L^p$- functions with $p$ different from 2. However, according to ([18], Theorem 3.7), the operator $\tau_x$ can be extended to $L^p(\mathbb{R}^d)^{\text{rad}}$, $1 \leq p \leq 2$ and we have

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k} \text{ for } f \in L^p_k(\mathbb{R}^d)^{\text{rad}}.$$  

(2.8)

The Dunkl convolution product $*_k$ of two functions $f$ and $g$ in $L^2_k(\mathbb{R}^d)$ is given by

$$(f *_k g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y)g(y)w_k(y)dy, \quad x \in \mathbb{R}^d. $$

The Dunkl convolution product is commutative and for $f, g \in \mathcal{D}(\mathbb{R}^d)$ we have

$$F_k(f *_k g) = F_k(f)F_k(g).$$

(2.9)

It was shown in ([18], Theorem 4.1) that when $g$ is a bounded function in $L^1_k(\mathbb{R}^d)^{\text{rad}}$, then

$$(f *_k g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x(g)(-y)w_k(y)dy, \quad x \in \mathbb{R}^d, $$

(2.10)

initially defined on the intersection of $L^1_k(\mathbb{R}^d)$ and $L^2_k(\mathbb{R}^d)$ extends to $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq +\infty$ as a bounded operator. In particular,

$$\|f *_k g\|_{p,k} \leq \|f\|_{p,k}\|g\|_{1,k}. $$

(2.11)
In the case $d = 1$, $G = \mathbb{Z}_2 = \{id, -id\}$ the corresponding reflection group acting on $\mathbb{R}$ and $\gamma = k(\alpha) = \alpha + \frac{1}{2} > 0$, the Dunkl operator on the real line is defined by

$$T_1(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha + 1}{x} \left[\frac{f(x) - f(-x)}{2}\right], \quad f \in \mathcal{E}(\mathbb{R}).$$

For $\lambda \in \mathbb{C}$, the Dunkl kernel is given by

$$E_k(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha + 1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R} \quad (2.12)$$

We have

$$w_k(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha + 1)}. \quad (2.13)$$

For all $x \in \mathbb{R}$, the Dunkl translation operator $\tau_x$ extends to $L^p_k(\mathbb{R}), p \geq 1$ and we have for $f \in L^p_k(\mathbb{R})$

$$\|\tau_x(f)\|_{p,k} \leq 3\|f\|_{p,k}. \quad (2.14)$$

## 3 Characterization by decomposition

In this section, we characterize the Besov-Dunkl spaces by decomposition of functions. Before, we start with some useful remarks and propositions.

**Remark 3.1** By ([3], Proposition 1 and 2), it follows that for $\beta > 0$ and $1 \leq p, q \leq +\infty$, the Besov-Dunkl space $\mathcal{BD}^{\beta,k}_{p,q}$ is independent of the choice of the sequence $(\varphi_j)_{j \in \mathbb{N}}$ in $\Phi(\mathbb{R}^d)$. This space coincide on $L^p_k(\mathbb{R}^d)$ with the homogeneous Besov-Dunkl space.

**Remark 3.2** It was shown in ([3], Proposition 3) that for $1 \leq p, q < +\infty$ and $\beta > 0$, we have the density of $S(\mathbb{R}^d)$ in $\mathcal{BD}^{\beta,k}_{p,q}$.

Now, in order to prove the following propositions, we recall that for $1 \leq p \leq 2$, $\mathcal{M}_p(\mathbb{R}^d)$ is the class of sequences $(g_j)_{j \in \mathbb{N}}$ of functions in $L^p_k(\mathbb{R}^d)$ such that supp $\mathcal{F}_k(g_0) \subset B(0, 2)$ and supp $\mathcal{F}_k(g_j) \subset A_j = \{x \in \mathbb{R}^d ; 2^{j-1} \leq \|x\| \leq 2^{j+1}\}$ for $j \in \mathbb{N} \setminus \{0\}$. 
Proposition 3.1 Let $\beta > 0$, $1 \leq p \leq 2$, $1 \leq q < +\infty$ and $(\varphi_j)_{j \in \mathbb{N}} \in \Phi(\mathbb{R}^d)$. If $f \in BD_{p,q}^{\beta,k}$ then one has

i) $\forall n, m \in \mathbb{N}$, $f_{n,m} = \sum_{s=n}^{n+m} \varphi_s * k f \in BD_{p,q}^{\beta,k} \cap \mathcal{E}(\mathbb{R}^d)$.

ii) $(\varphi_j * k f)_{j \in \mathbb{N}} \in \mathcal{M}_p(\mathbb{R}^d)$ and $f = \sum_{j=0}^{+\infty} \varphi_j * k f$, in $L^p_k(\mathbb{R}^d)$.

Proof. Let $(\varphi_j)_{j \in \mathbb{N}} \in \Phi(\mathbb{R}^d)$ and $f \in BD_{p,q}^{\beta,k}$ with $\beta > 0$, $1 \leq p \leq 2$ and $1 \leq q < +\infty$.

i) Take for $n, m \in \mathbb{N}$, $f_{n,m} = \sum_{s=n}^{n+m} \varphi_s * k f$. By the triangle inequality, (2.11) and the property (ii) for $(\varphi_j)_{j \in \mathbb{N}}$, we have $f_{n,m} \in L^p_k(\mathbb{R}^d)$. Now, using the properties (i), (ii), (iii) for $(\varphi_j)_{j \in \mathbb{N}}$, the triangle inequality and (2.11) again, we get

$$\sum_{j=0}^{+\infty} (2^{j\beta} \| \varphi_j * k f_{n,m} \|_{p,k})^q = \sum_{j=0}^{n+m+1} (2^{j\beta} \| \varphi_j * k f_{n,m} \|_{p,k})^q \leq c \sum_{j=0}^{n+m+1} (2^{j\beta} \| \varphi_j * k f \|_{p,k})^q < +\infty.$$ We conclude from (2.7) and (2.10) that $f_{n,m} \in \mathcal{E}(\mathbb{R}^d)$.

ii) Using (2.9) and (2.11) and the property (ii) for $(\varphi_j)_{j \in \mathbb{N}}$, it’s clear that $(\varphi_j * k f)_{j \in \mathbb{N}}$ is in $\mathcal{M}_p(\mathbb{R}^d)$. For $r, s \in \mathbb{N}$ with $r < s$, we have from the Hölder inequality that

$$\| \sum_{j=r}^{s} \varphi_j * k f \|_{p,k} \leq \sum_{j=r}^{s} \| \varphi_j * k f \|_{p,k} \leq \left( \sum_{j=r}^{s} 2^{-j\beta q'} \right)^{\frac{1}{q'}} \left( \sum_{j=r}^{s} (2^{j\beta} \| \varphi_j * k f \|_{p,k})^q \right)^{\frac{1}{q}},$$

where $q'$ is the conjugate of $q$. Then the series $\sum_{j=0}^{+\infty} \varphi_j * k f$ converges in $L^p_k(\mathbb{R}^d)$ and the map $f \mapsto \sum_{j=0}^{+\infty} \varphi_j * k f$ is continuous from $BD_{p,q}^{\beta,k}$ into $L^p_k(\mathbb{R}^d)$.****
Take $\psi \in S(\mathbb{R}^d)$, the series $\sum_{j=0}^{+\infty} \varphi_j * k \psi$ converges, in particular in $L_k^2(\mathbb{R}^d)$, then by the Plancherel theorem and (2.9), we deduce that $\sum_{j=0}^{+\infty} \mathcal{F}_k(\varphi_j) \mathcal{F}_k(\psi)$ converges in $L_k^2(\mathbb{R}^d)$. Since $\sum_{j=0}^{+\infty} \mathcal{F}_k(\varphi_j)(x) = 1$, $x \in \mathbb{R}^d$, we conclude that $\sum_{j=0}^{+\infty} \mathcal{F}_k(\varphi_j) \mathcal{F}_k(\psi) = \mathcal{F}_k(\psi)$, which gives $\sum_{j=0}^{+\infty} \varphi_j * k \psi = \psi$. Using the fact that $S(\mathbb{R}^d)$ is dense in $\mathcal{B}D_{p,q}^{\beta,k}$ (see Remark 3.2) and $\mathcal{B}D_{p,q}^{\beta,k}$ is continuously included in $L_k^p(\mathbb{R}^d)$, we get

$$f = \sum_{j=0}^{+\infty} \varphi_j * k f,$$

which proves the results ii). This completes the proof. □

For $f \in \mathcal{B}D_{p,q}^{\beta,k}$, we put $\|f\|_{\mathcal{B}D_{p,q}^{\beta,k}} = \left( \sum_{j \in \mathbb{N}} (2^j \beta \|\varphi_j * k f\|_{p,k})^q \right)^{\frac{1}{q}}$.

**Proposition 3.2** Let $\beta > 0$, $1 \leq p \leq 2$ and $1 \leq q < +\infty$. Then

1) For all compact set $K$ in $\mathbb{R}^d \setminus \{0\}$, the norms $\| \cdot \|_{p,k}$ and $\| \cdot \|_{\mathcal{B}D_{p,q}^{\beta,k}}$ are equivalent on the set $F_p(K)$ of all functions $f \in L_k^p(\mathbb{R}^d)$ such that $\text{supp} \mathcal{F}_k(f) \subseteq K$.

2) The subspace $F_p$ of functions $f \in L_k^p(\mathbb{R}^d)$ such that $\text{supp} \mathcal{F}_k(f)$ is a compact set in $\mathbb{R}^d \setminus \{0\}$ is dense in $\mathcal{B}D_{p,q}^{\beta,k}$.

**Proof.** 1) If $K$ is a compact set in $\mathbb{R}^d \setminus \{0\}$, then there exists a finite set $I \subset \mathbb{N}$ such that $K \subseteq \bigcup_{j \in I} A_j$ where we take $A_0 = \{x \in \mathbb{R}^d; \|x\| \leq 2\}$ when $j = 0$. This gives for $f \in F_p(K)$ and $(\varphi_j)_{j \in \mathbb{N}}$ in $\Phi(\mathbb{R}^d)$ that $\varphi_j * k f = 0$, if $j > 1 + \max I$. Hence we obtain

$$\sum_{j=0}^{+\infty} (2^j \beta \|\varphi_j * k f\|_{p,k})^q \leq \sum_{j \in I} 2^{j\beta q} \|\varphi_j\|_{1,k}^q \|f\|_{p,k}^q \leq c \left( \sum_{j \in I} 2^{j\beta q} \|f\|_{p,k}^q \right) \leq c(K) \|f\|_{p,k}^q.$$
Conversely, if \((\varphi_j)_{j \in \mathbb{N}} \in \Phi(\mathbb{R}^d)\) and \(f \in \mathcal{BD}^{\beta,k}_{p,q}\) with \(\text{supp} \mathcal{F}_k(f) \subseteq K\), then using (3.1) we have \(f = \sum_{j=0}^{+\infty} \varphi_j * k f = \sum_{j \in I} \varphi_j * k f\). Using Hölder’s inequality, it yields

\[
\|f\|_{p,k} \leq \sum_{j \in I} \|\varphi_j * k f\|_{p,k} \leq \left( \sum_{j \in I} 2^{-j \beta q'} \right)^{\frac{1}{q'}} \left( \sum_{j \in \mathbb{N}} (2^j \|\varphi_j * k f\|_{p,k})^q \right)^{\frac{1}{q}}.
\]

where \(q'\) is the conjugate of \(q\).

2) Assume \(f \in \mathcal{BD}^{\beta,k}_{p,q}\) and \((\varphi_j)_{j \in \mathbb{N}} \in \Phi(\mathbb{R}^d)\), then we put \(f_n = \sum_{s=0}^{n} \varphi_s * k f\) for \(n \in \mathbb{N} \setminus \{0\}\). It’s clear that \(f_n \in F_p\). From (Proposition 3.1, i)), we have \(f_n \in \mathcal{BD}^{\beta,k}_{p,q}\), hence using the properties (i) and (iii) for \((\varphi_j)_{j \in \mathbb{N}}\), we obtain

\[
\sum_{j=0}^{+\infty} 2^{j \beta q} \|\varphi_j * k (f - f_n)\|_{p,k}^q \leq c \sum_{|j| \geq n} 2^{j \beta q} \|\varphi_j * k f\|_{p,k}^q.
\]

Since \(f \in \mathcal{BD}^{\beta,k}_{p,q}\), then we deduce that \(\lim_{n \to +\infty} f_n = f\) in \(\mathcal{BD}^{\beta,k}_{p,q}\). □

Let \(1 \leq p \leq 2\) and \(1 \leq q < +\infty\). In order to prove the following theorem, we denote by

\[
\|g_j\|^* = \left( \sum_{j \in \mathbb{N}} (2^{j \beta} \|g_j\|_{p,k})^q \right)^{\frac{1}{q}},
\]

for any sequence \((g_j)_{j \in \mathbb{N}}\) of functions in \(L^p_k(\mathbb{R}^d)\).

**Theorem 3.1** If \(\beta > 0\), \(1 \leq p \leq 2\) and \(1 \leq q < +\infty\), then

\[
\mathcal{BD}^{\beta,k}_{p,q} = \left\{ f \in L^p_k(\mathbb{R}^d) : \exists (g_j)_{j \in \mathbb{N}} \in \mathcal{M}_p(\mathbb{R}^d) \text{ such that } f = \sum_{j=0}^{+\infty} g_j \right\},
\]

\[
\text{in } L^p_k(\mathbb{R}^d) \text{ and } \|g_j\|^{*} < +\infty. \tag{3.2}
\]

**Proof.** (Proposition 3.1, ii)) shows that every \(f \in \mathcal{BD}^{\beta,k}_{p,q}\) can be represented by the right-hand side of (3.2). Conversely, if \(f \in L^p_k(\mathbb{R}^d)\) is given by

\[
f = \sum_{j=0}^{+\infty} g_j \quad \text{in } L^p_k(\mathbb{R}^d), \quad (g_j)_{j \in \mathbb{N}} \in \mathcal{M}_p(\mathbb{R}^d) \quad \text{and} \quad \|g_j\|^{*} < +\infty,
\]
then we have for \((\varphi_j)_{j \in \mathbb{N}} \in \Phi(\mathbb{R}^d),\)
\[
\varphi_j * k f = \varphi_j * k (g_{j-1} + g_j + g_{j+1}), \quad j \in \mathbb{N}
\]
where we put for convenience \(g_{-1} = 0\). Then by (2.11) and Hölder’s inequality for \(j \in \mathbb{N}\), we obtain
\[
\| \varphi_j * k f \|^q_{p,k} \leq c 3^{q-1} \sum_{s=j-1}^{j+1} \| g_s \|^q_{p,k}.
\]
Thus summing over \(j\) with weights \(2^j\) and using the triangle inequality, we get
\[
\sum_{j \in \mathbb{N}} (2^j \| \varphi_j * k f \|_{p,k})^q \leq c \sum_{j \in \mathbb{N}} (2^j \| g_j \|_{p,k})^q < +\infty.
\]
Thus we obtain the result. 

\[\square\]

**Remark 3.3** Put \(\mathcal{A}\) the set of all functions \(\phi \in \mathcal{S}(\mathbb{R}^d)^{rad}\) such that
\[
\text{supp } \mathcal{F}_k(\phi) \subset \left\{ x \in \mathbb{R}^d; 1 \leq \|x\| \leq 2 \right\}
\]
and denote by \(C^{\phi,\beta,k}_{p,q}\), \(\beta > 0\), \(1 \leq p,q \leq +\infty\), the subspace of functions \(f \in L^p_k(\mathbb{R}^d)\) satisfying
\[
\left( \int_0^{+\infty} \left( \frac{\| f * k \phi_t \|_{p,k}}{t^\beta} \right)^q \frac{dt}{t} \right)^\frac{1}{q} < +\infty \quad \text{if} \quad q < +\infty
\]
(usual modification when \(q = +\infty\)), where \(\phi_t(x) = \frac{1}{2(\gamma+\frac{d}{2})}\phi(\frac{x}{t})\), for all \(t \in (0, +\infty)\) and \(x \in \mathbb{R}^d\). Then from ([3], Theorem 2), we have for \(\beta > 0\), \(1 \leq p,q \leq +\infty\) and all \(\phi \in \mathcal{A}\)
\[
\mathcal{BD}^{\beta,k}_{p,q} \subset C^{\phi,\beta,k}_{p,q}.
\]

In the case \(d = 1\), \(G = \mathbb{Z}_2\), \(\gamma = k(\alpha) = \alpha + \frac{1}{2} > 0\) and
\[
T_1(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha + 1}{x} \left[ \frac{f(x) - f(-x)}{2} \right], \quad f \in \mathcal{E}(\mathbb{R}),
\]
we can characterize the Besov-Dunkl spaces by differences using the modulus of continuity of second order of \(f\). Put \(\mathcal{H}\) the set of all functions \(\phi \in \mathcal{S}_*(\mathbb{R})\)
such that \( \int_0^{+\infty} \phi(x)d\mu_\alpha(x) = 0 \) with \( d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} \, dx \) and \( S_*({\mathbb R}) \) the space of even Schwartz functions on \( \mathbb R \). Then we can assert from ([1], Theorem 3.6) that for \( 0 < \beta < 1, \, 1 < p < +\infty, \, 1 \leq q \leq +\infty \) and all \( \phi \in \mathcal{H} \), that

\[
C_{p,q}^{\phi,\beta,k} = BD_{\alpha,\beta}^{p,q},
\]

where \( BD_{\alpha,\beta}^{p,q} \) is the subspace of functions \( f \in L^p(\mu_\alpha) \) satisfying

\[
\left( \int_0^{+\infty} \left( \frac{\omega_p(f)(t)}{x^\beta} \right)^q \frac{dx}{x} \right)^{\frac{1}{q}} < +\infty \quad \text{if} \quad q < +\infty
\]

and

\[
\sup_{x \in (0, +\infty)} \frac{\omega_p(f)(t)}{x^\beta} < +\infty \quad \text{if} \quad q = +\infty.
\]

Note that when \( d = 1 \), we have \( A \subset H \), then from (3.3) and (3.4), we obtain for \( 0 < \beta < 1, \, 1 < p < +\infty, \, 1 \leq q \leq +\infty \) and all \( \phi \in A \)

\[
BD_{\alpha,\beta}^{\beta,k} \subset BD_{\alpha,\beta}^{p,q}.
\]

From (3.4), it’s clear that the space \( C_{p,q}^{\phi,\beta,k} \) is independent of the specific selection of \( \phi \) in \( \mathcal{H} \). In particular, if we take for example the function \( \phi \) defined on \( \mathbb R \) by

\[
\phi(x) = -x\varphi'(x) - 2(\alpha+1)\varphi(x),
\]

where \( \varphi \) is the Gaussian function, \( \varphi(x) = e^{-x^2/2} \), then we can see that \( \phi \in \mathcal{H} \).

The dilation \( \phi_t \) of \( \phi \) gives \( \phi_t(x) = t^\frac{\beta}{2}\varphi_t(x) \), for \( x \in \mathbb R \) and \( t \in (0, +\infty) \).

From (3.4), it yields that for \( 0 < \beta < 1, \, 1 < p < +\infty \) and \( 1 \leq q \leq +\infty \),

\[
f \in BD_{\alpha,\beta}^{p,q} \iff \int_0^{+\infty} \left( \frac{\|t^\frac{\beta}{2}(f *_k \varphi_t)\|_{p,k}}{t^\beta} \right)^q \frac{dt}{t} < +\infty,
\]

(usual modification when \( q = +\infty \)).
4 Moduli of continuity and Dunkl transform

In this section, we obtain inequalities for the Dunkl transform $F_k(f)$ of $L^p$-function $f$, $1 \leq p \leq 2$, both on the real line and in radial case on $\mathbb{R}^d$. As consequence, we give a quantitative form of the Riemann-Lebesgue lemma and further results of integrability for the Dunkl transform.

Throughout this section, we denote by $p'$ the conjugate of $p$. According to (2.8) and (2.14), we recall that for $1 \leq p \leq 2$, $\omega_p(f)$ is the modulus of continuity of $f$ and is given:

- On the real line by:
  $$\omega_p(f)(t) = \|\tau_t(f) + \tau_{-t}(f) - 2f\|_{p,k}, \quad t \geq 0, \quad f \in L^p_k(\mathbb{R}).$$

- In radial case on $\mathbb{R}^d$ by:
  $$\omega_p(f)(t) = \int_{S^{d-1}} \|\tau_{tu}(f) - f\|_{p,k}d\sigma(u), \quad t \geq 0, \quad f \in L^p_k(\mathbb{R}^d)^{rad}.$$

**Lemma 4.1** (see [6, 7]) Let $\alpha > -\frac{1}{2}$. Then there exist positive constants $c_{1,\alpha}$ and $c_{2,\alpha}$ such that

$$c_{1,\alpha}\min\{1, (\lambda t)^2\} \leq 1 - j_\alpha(\lambda t) \leq c_{2,\alpha}\min\{1, (\lambda t)^2\}, \quad t, \lambda \in \mathbb{R}. \quad (4.1)$$

**Theorem 4.1** Let $1 \leq p \leq 2$ and $f \in L^p_k(\mathbb{R})$. Then there exists a positive constant $c$ such that for any $t \in (0, +\infty)$, one has

$$\left(\int_{\mathbb{R}} \min\{1, |x|^2p'\} |F_k(f)(x)|^{p'} w_k(x)dx\right)^{\frac{1}{p'}} \leq c \omega_p(f)(t), \quad \text{if } 1 < p \leq 2,$$

$$\text{ess sup}_{x \in \mathbb{R}} \left[\min\{1, (tx)^2\} |F_k(f)(x)|\right] \leq c \omega_1(f)(t), \quad \text{if } p = 1.$$

**Proof.** For $f \in L^p_k(\mathbb{R})$, we have by (2.6)

$$F_k(\tau_t(f) + \tau_{-t}(f) - 2f)(x) = [E_k(itx) + E_k(-itx) - 2]F_k(f)(x),$$

for $t \in (0, +\infty)$ and a.e $x \in \mathbb{R}$. Applying (2.4), we get

$$\|F_k(\tau_t(f) + \tau_{-t}(f) - 2f)\|_{p',k}$$

$$= \left(\int_{\mathbb{R}} |F_k(f)(x)|^{p'}E_k(itx) + E_k(-itx) - 2|^{p'} w_k(x)dx\right)^{\frac{1}{p'}}$$

$$\leq c \omega_p(f)(t). \quad (4.2)$$
From (2.12), it yields
\[ |E_k(itx) + E_k(-itx) - 2| \geq 2|j_\alpha(tx) - 1| \] (4.3)
then using (4.1) and (4.3) in (4.2), we obtain the result. Here when \( p = 1 \), we make the usual modification. \( \square \)

Remark 4.1 Note that if \( p = 2 \), then by Plancherel’s theorem and (4.1), there exist positive constants \( c_1, c_2 \) such that
\[ c_1 \omega_2(f)(t) \leq \left( \int_{\mathbb{R}} \min\{1, (tx)^4\} |\mathcal{F}_k(f)(x)|^2 w_k(x) dx \right)^{1/2} \leq c_2 \omega_2(f)(t). \]

As consequence immediate of the theorem 4.1, we obtain the following quantitative form of the Riemann-Lebesgue lemma.

Corollary 4.1 Let \( 1 \leq p \leq 2 \) and \( f \in L^p_k(\mathbb{R}) \). Then there exists a positive constant \( c \) such that for any \( t \in (0, +\infty) \), one has
\[ \left( \int_{|x| > \frac{1}{t}} |\mathcal{F}_k(f)(x)|^{p'} w_k(x) dx \right)^{\frac{1}{p'}} \leq c \omega_p(f)(t), \quad \text{if } 1 < p \leq 2, \]
\[ \operatorname{ess} \sup_{|x| > \frac{1}{t}} |\mathcal{F}_k(f)(x)| \leq c \omega_1(f)(t), \quad \text{if } p = 1. \]

Theorem 4.2 Let \( \alpha > -\frac{1}{2} \), \( \beta > 2(\alpha + 1) \), \( A > 0 \) and \( f \in L^1_k(\mathbb{R}) \). If \( f \) satisfies
\[ \sup_{t \in (0, +\infty)} \frac{\omega_1(f)(t)}{t^\beta} < A, \] (4.4)
then
\[ \mathcal{F}_k(f) \in L^1_k(\mathbb{R}). \]

Proof. From the theorem 4.1 and (4.4), we obtain
\[ \operatorname{ess} \sup_{|x| \leq \frac{1}{t}} (tx)^2 |\mathcal{F}_k(f)(x)| \leq c \omega_1(f)(t) \leq ct^\beta \] (4.5)
By Hölder’s inequality, (4.5) and (2.13), we have
\[ \int_{|x| \leq \frac{1}{t}} |x| |\mathcal{F}_k(f)(x)| w_k(x) dx \leq \operatorname{ess} \sup_{|x| \leq \frac{1}{t}} x^2 |\mathcal{F}_k(f)(x)| \int_{|x| \leq \frac{1}{t}} |x|^{-1} w_k(x) dx \]
\[ \leq ct^{\beta-2} \int_0^\frac{1}{t} x^{2\alpha} dx \leq ct^{\beta-2(\alpha+1)-1}. \]
Integrating with respect to $t$ over $(0, 1)$ and applying Fubini’s theorem, we obtain
\[
\int_{|x| \geq 1} |F_k(f)(x)|w_k(x)dx \leq c \int_0^1 t^\beta\frac{2(\alpha+1)-1}{1-\gamma} dt < +\infty.
\]
Since $L^\infty([-1, 1], w_k(x)dx) \subset L^1([-1, 1], w_k(x)dx)$, we deduce that $F_k(f)$ is in $L^1_k(\mathbb{R})$.

**Theorem 4.3** Let $1 \leq p \leq 2$ and $f \in L^p_k(\mathbb{R}^d)^{rad}$. Then there exists a positive constant $c$ such that for any $t \in (0, +\infty)$, one has
\[
\left( \int_{\mathbb{R}^d} \min\{1, (t||x||)^{2p'}\} |F_k(f)(x)|^{p'}w_k(x)dx \right)^{\frac{1}{p'}} \leq c \omega_p(f)(t), \quad \text{if } 1 \leq p \leq 2,
\]
\[
es\sup_{x \in \mathbb{R}^d} \left[ \min\{1, (t||x||)^2\} |F_k(f)(x)| \right] \leq c \omega_1(f)(t), \quad \text{if } p = 1.
\]

**Proof.** Let $f \in L^p_k(\mathbb{R}^d)^{rad}$, we can write from (2.6)
\[
F_k(\tau_{itu}(f) - f)(x) = F_k(f)(x)[E_k(itu, x) - 1],
\]
for $u \in S^{d-1}$, $t \in (0, +\infty)$ and a.e $x \in \mathbb{R}^d$. Applying (2.4), we get
\[
\|F_k(\tau_{itu}(f) - f))\|_{p,k} = \left( \int_{\mathbb{R}^d} |F_k(f)(x)|^{p'} |E_k(itu, x) - 1|^{p'} w_k(x)dx \right)^{\frac{1}{p'}} \leq c \|\tau_{itu}(f) - f\|_{p,k}.
\]
According to (2.1) and (2.5), we have
\[
\int_{\mathbb{R}^d} |F_k(f)(x)|^{p'} |E_k(itu, x) - 1|^{p'} w_k(x)dx
\]
\[
= d_k \int_0^1 \left| H_{\gamma+\frac{d}{2}-1}(F)(r) \right|^{p'} \left( \int_{S^{d-1}} |E_k(itru, z) - 1|^{p'} w_k(z) d\sigma(z) \right)^{\frac{1}{p'}} dr,
\]
where $F$ is a function on $(0, +\infty)$ such that $F(||x||) = f(x)$, for all $x \in \mathbb{R}^d$. On the other hand, by (2.2), (2.3) and Hölder’s inequality, we get
\[
d_k |j_{\gamma+\frac{d}{2}-1}(rt) - 1|
\]
\[
= \left| \int_{S^{d-1}} [E_k(itru, z) - 1] w_k(z) d\sigma(z) \right|
\]
\[
\leq \left( \int_{S^{d-1}} w_k(z) d\sigma(z) \right)^{\frac{1}{p'}} \left( \int_{S^{d-1}} |E_k(itru, z) - 1|^{p'} w_k(z) d\sigma(z) \right)^{\frac{1}{p'}}
\]
\[
\leq d_k^{\frac{1}{p'}} \left( \int_{S^{d-1}} |E_k(itru, z) - 1|^{p'} w_k(z) d\sigma(z) \right)^{\frac{1}{p'}},
\]
hence we obtain,

$$|j_{\gamma + \frac{d}{2} - 1}(rt) - 1|^{p'} \leq c \int_{S_{d-1}} |E_k(itru, z) - 1|^{p'} w_k(z) d\sigma(z). \quad (4.7)$$

From (4.6) and (4.7), it follows that

$$\int_0^{+\infty} |H_{\gamma + \frac{d}{2} - 1}(F)(r)|^{p'} |j_{\gamma + \frac{d}{2} - 1}(rt) - 1|^{p'} r^{2\gamma + d - 1} dr \leq c \left\| \tau_{tu}(f) - f \right\|_{p,k}^{p'}.$$ 

So using (4.1), (2.1) and (2.5), we obtain

$$\int_{\mathbb{R}^d} \min\{1, (t\|x\|)^{2p'}\} |\mathcal{F}_k(f)(x)|^{p'} w_k(x) dx \leq c \left\| \tau_{tu}(f) - f \right\|_{p,k}^{p'}$$

and we deduce our result. When $p = 1$, we make the usual modification. □

**Remark 4.2** Note that if $p = 2$, by Plancherel’s theorem and (4.1), there exist positive constants $c_1, c_2$ such that

$$c_1 \omega_2(f)(t) \leq \left( \int_{\mathbb{R}^d} \min\{1, (t\|x\|)^4\} |\mathcal{F}_k(f)(x)|^2 w_k(x) dx \right)^{1/2} \leq c_2 \omega_2(f)(t).$$

As a consequence of the theorem 4.3, we obtain the following quantitative form of the Riemann-Lebesgue lemma.

**Corollary 4.2** Let $1 \leq p \leq 2$ and $f \in L_k^p(\mathbb{R}^d)^{rad}$. Then there exists a positive constant $c$ such that for any $t \in (0, +\infty)$, one has

$$\left( \int_{\|x\| > \frac{1}{t}} |\mathcal{F}_k(f)(x)|^{p'} w_k(x) dx \right)^{\frac{1}{p'}} \leq c \omega_p(f)(t), \quad \text{if } 1 < p \leq 2,$$

$$\text{ess sup}_{\|x\| > \frac{1}{t}} |\mathcal{F}_k(f)(x)| \leq c \omega_1(f)(t), \quad \text{if } p = 1.$$ 

**Theorem 4.4** Let $\beta > 2(\gamma + \frac{d}{2})$, $A > 0$ and $f \in L_k^1(\mathbb{R}^d)^{rad}$. If $f$ satisfies

$$\sup_{t \in (0, +\infty)} \frac{\omega_1(f)(t)}{t^\beta} < A,$$ 

then

$$\mathcal{F}_k(f) \in L_k^1(\mathbb{R}^d)^{rad}. \quad (4.8)$$
**Proof.** From the theorem 4.3 and (4.8), we obtain

\[
\text{ess sup}_{\|x\| \leq \frac{1}{t}} (t\|x\|)^2 |\mathcal{F}_k(f)(x)| \leq c \omega_1(f)(t) \leq c t^\beta \tag{4.9}
\]

By Hölder’s inequality, (4.9) and (2.1), we have

\[
\int_{\|x\| \leq \frac{1}{t}} \|x\| |\mathcal{F}_k(f)(x)| w_k(x) dx \\
\leq \text{ess sup}_{\|x\| \leq \frac{1}{t}} \|x\|^2 |\mathcal{F}_k(f)(x)| \int_{\|x\| \leq \frac{1}{t}} \|x\|^{-1} w_k(x) dx \\
\leq c t^{\beta - 2} \int_0^{\frac{1}{t}} r^{2\gamma + d - 2} dr \leq c t^{\beta - 2(\gamma + \frac{d}{2}) - 1}.
\]

Integrating with respect to \( t \) over (0,1) and applying Fubini’s theorem, we obtain

\[
\int_{\|x\| \geq 1} |\mathcal{F}_k(f)(x)| w_k(x) dx \leq c \int_0^1 t^{\beta - 2(\gamma + \frac{d}{2}) - 1} dt < +\infty.
\]

Since \( L^\infty_k(B(0,1), w_k(x)dx) \subset L^1_k(B(0,1), w_k(x)dx) \), we deduce that \( \mathcal{F}_k(f) \) is in \( L^1_k(\mathbb{R}^d) \).

\[\square\]

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