THE IMPROVED ISOPERIMETRIC INEQUALITY AND THE WIGNER CAUSTIC OF PLANAR OVALS

MICHAL ZWIERZYŃSKI

Abstract. The classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$ states that for a simple closed curve $M$ of the length $L_M$, enclosing a region of the area $A_M$, one gets

$$L_M^2 \geq 4\pi A_M.$$  

In this paper we present the improved isoperimetric inequality, which states that if $M$ is a closed regular simple convex curve, then

$$L_M^2 \geq 4\pi A_M + 8\pi \bigg|\tilde{A}_{E_{\frac{1}{2}}(M)}\bigg|,$$

where $\tilde{A}_{E_{\frac{1}{2}}(M)}$ is an oriented area of the Wigner caustic of $M$, and the equality holds if and only if $M$ is a curve of constant width.

Furthermore we also present a stability property of the improved isoperimetric inequality (near equality implies curve nearly of constant width).

The Wigner caustic is an example of an affine $\lambda$-equidistant (for $\lambda = \frac{1}{2}$) and the improved isoperimetric inequality is a consequence of certain bounds of oriented areas of affine equidistants.

1. Introduction

The classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$ states that:

Theorem 1.1. (Isoperimetric inequality) Let $M$ be a simple closed curve of the length $L_M$, enclosing a region of the area $A_M$, then

$$L_M^2 \geq 4\pi A_M,$$

and the equality (1.1) holds if and only if $M$ is a circle.

This fact was already known in ancient Greece. The first mathematical proof was given in the nineteenth century by Steiner [25]. After that, there have been many new proofs, generalizations, and applications of this famous theorem, see for instance [3, 12, 16, 19, 20, 21, 23, 25], and the literature therein. In 1902 Hurtwiz [19] and later Gao [12] showed the reverse isoperimetric inequality.

Theorem 1.2. (Reverse isoperimetric inequality) Let $K$ be a strictly convex domain whose support function $p$ has the property that $p''$ exists and is absolutely continuous, and let $\tilde{A}$ denote the oriented area of the evolute of the boundary curve of $K$.

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Let $L_K$ be the perimeter of $K$ and $A_K$ be the area of $K$. Then
\[ L_M^2 \leq 4\pi A_M + \pi|\tilde{A}|, \]
(1.2)
Equality holds if and only if $p(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$.

In this paper we present bounds of oriented areas of affine equidistants and thanks to it we will prove the improved isoperimetric inequality, which states that if $M$ is a closed regular simple convex curve, then
\[ L_M^2 \geq 4\pi A_M + 8\pi \left| \tilde{A}_{E_{\frac{1}{2}}(M)} \right|, \]
where $\tilde{A}_{E_{\frac{1}{2}}(M)}$ is an oriented area of the Wigner caustic of $M$, and the equality holds if and only if $M$ is a curve of constant width. This is very interesting that the absolute value of the oriented area of the Wigner caustic improves the classical isoperimetric inequality and also gives the exact link between the area and the length of constant width curves.

The family of affine $\lambda$ - equidistants arises as the counterpart of parallels or offsets in Euclidean geometry. An affine equidistant for us is the set of points of chords connecting points on $M$ where tangent lines to $M$ are parallel, which divide the chord segments between the base points with a fixed ratio $\lambda$, also called the affine time. When in affine $\lambda$-equidistants the ratio $\lambda$ is equal to $\frac{1}{2}$ then this set is also known as the Wigner caustic. The Wigner caustic of a smooth convex closed curve on affine symplectic plane was first introduced by Berry, in his celebrated 1977 paper [1] on the semiclassical limit of Wigner’s phase-space representation of quantum states. They are many papers considering affine equidistants, see for instance [4, 5, 6, 7, 8, 9, 13, 14, 22, 26], and the literature therein. The Wigner caustic is also known as the area evolute, see [4, 13].

2. Geometric quantities, affine equidistants and Fourier series

Let $M$ be a smooth planar curve, i.e. the image of the $C^\infty$ smooth map from an interval to $\mathbb{R}^2$. A smooth curve is closed if it is the image of a $C^\infty$ smooth map from $S^1$ to $\mathbb{R}^2$. A smooth curve is regular if its velocity does not vanish. A regular closed curve is convex if its signed curvature has a constant sign. An oval is a smooth closed convex curve which is simple, i.e. it has no selfintersections. In our case it is enough to consider $C^2$ - smooth curves.

**Definition 2.1.** A pair $a, b \in M$ ($a \neq b$) is called the parallel pair if tangent lines to $M$ at points $a, b$ are parallel.

**Definition 2.2.** A chord passing through a pair $a, b \in M$ is the line:
\[ l(a, b) = \{ \lambda a + (1 - \lambda)b \mid \lambda \in \mathbb{R} \}. \]

**Definition 2.3.** An affine $\lambda$-equidistant is the following set.
\[ E_\lambda(M) = \{ \lambda a + (1 - \lambda)b \mid a, b \text{ is a parallel pair of } M \}. \]

The set $E_{\frac{1}{2}}(M)$ will be called the Wigner caustic of $M$.

Note that, for any given $\lambda \in \mathbb{R}$ we have an equality $E_\lambda(M) = E_{1-\lambda}(M)$. Thus, the case $\lambda = \frac{1}{2}$ is special. In particular we have also equalities $E_0(M) = E_1(M) = M$. 


It is well known that if $M$ is a generic oval, then $E_\lambda(M)$ are smooth closed curves with cusp singularities only \[1, 15\], the number of cusps of $E_1(M)$ is odd and not smaller than 3 \[1, 13\] and the number of cusps of $E_\lambda(M)$ for a generic value of $\lambda \neq \frac{1}{2}$ is even \[9\].

**Definition 2.4.** An oval is said to have constant width if the distance between every pair of parallel tangent lines is constant. This constant is called the width of the curve.

![Figure 1](image1.png)

**Figure 1.** An oval $M$ and (i) $E_\frac{1}{2}(M)$, (ii) $E_\frac{3}{2}(M)$.

Let us recall some basic facts about plane ovals which will be used later. The details can be found in the classical literature \[10, 18\].

Let $M$ be a positively oriented oval. Take a point $O$ inside $M$ as the origin of our frame. Let $p$ be the oriented perpendicular distance from $O$ to the tangent line at a point on $M$, and $\theta$ the oriented angle from the positive $x_1$-axis to this perpendicular ray. Clearly, $p$ is a single-valued periodic function of $\theta$ with period $2\pi$ and the parameterization of $M$ in terms of $\theta$ and $p(\theta)$ is as follows

\[\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) = (p(\theta) \cos \theta - p'(\theta) \sin \theta, p(\theta) \sin \theta + p'(\theta) \cos \theta).\]

The couple $(\theta, p(\theta))$ is usually called the polar tangential coordinate on $M$, and $p(\theta)$ its Minkowski’s support function.

Then, the curvature $\kappa$ of $M$ is in the following form

\[\kappa(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0,\]

or equivalently, the radius of a curvature $\rho$ of $M$ is given by

\[\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta).\]

Let $L_M$ and $A_M$ be the length of $M$ and the area it bounds, respectively. Then one can get that

\[L_M = \int_M ds = \int_0^{2\pi} p(\theta) d\theta = \int_0^{2\pi} p(\theta) d\theta,\]
and

\[ A_M = \frac{1}{2} \int_M p(\theta) \, ds \]

\[ = \frac{1}{2} \int_0^{2\pi} p(\theta) [p(\theta) + p''(\theta)] \, d\theta = \frac{1}{2} \int_0^{2\pi} [p^2(\theta) - p'^2(\theta)] \, d\theta. \]

(2.3) and (2.4) are known as Cauchy’s formula and Blaschke’s formula, respectively.

Since the Minkowski support function of \( M \) is smooth bounded and \( 2\pi \)-periodic, its Fourier series is in the form

\[ p(\theta) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\theta + b_n \sin n\theta \right). \]

(2.5)

Differentiation of (2.5) with respect to \( \theta \) gives

\[ p'(\theta) = \sum_{n=1}^{\infty} n \left( -a_n \sin n\theta + b_n \cos n\theta \right). \]

(2.6)

By (2.5), (2.6) and the Parseval equality one can express \( L_M \) and \( A_M \) in terms of Fourier coefficients of \( p(\theta) \) in the following way.

\[ L_M = 2\pi a_0. \]

(2.7)

\[ A_M = \pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1)(a_n^2 + b_n^2). \]

(2.8)

One can notice that \( \gamma(\theta), \gamma(\theta + \pi) \) is a parallel pair of \( M \), hence \( \gamma_\lambda \) - the parameterization of \( E_\lambda(M) \) is as follows

\[ \gamma_\lambda(\theta) = \left( \gamma_{\lambda,1}(\theta), \gamma_{\lambda,2}(\theta) \right) \]

\[ = \lambda \gamma(\theta) + (1 - \lambda) \gamma(\theta + \pi) \]

\[ = (P_\lambda(\theta) \cos \theta - P'_\lambda \sin \theta, P_\lambda(\theta) \sin \theta + P'_\lambda(\theta) \cos \theta), \]

where \( P_\lambda(\theta) = \lambda p(\theta) - (1 - \lambda)p(\theta + \pi), \theta \in [0, 2\pi]. \) Furthermore if \( \lambda = \frac{1}{2} \), then the map \( M \ni \gamma(\theta) \mapsto \gamma_{\frac{1}{2}}(\theta) \in E_{\frac{1}{2}}(M) \) for \( \theta \in [0, 2\pi] \) is the double covering of the Wigner caustic of \( M \).

\section{3. Oriented Areas of Equidistants and the Improved Isoperimetric Inequality}

Let \( L_{E_\lambda(M)}, A_{E_\lambda(M)} \) denote the length of \( E_\lambda(M) \) and the oriented area of \( E_\lambda(M) \), respectively.

Similarly like in [9] we can show the following proposition.

\textbf{Proposition 3.1.} [9] Let \( M \) be an oval. Then

(i) if \( \lambda \neq \frac{1}{2} \), then \( L_{E_\lambda(M)} \leq \left( |\lambda| + |1 - \lambda| \right) L_M. \)

In particular if \( \lambda \in \left( 0, \frac{1}{2} \right) \cup \left( \frac{1}{2}, 1 \right) \), then \( L_{E_1(M)} \leq L_M. \)

(ii) \( 2L_{E_{\frac{1}{2}}(M)} \leq L_M. \)
Proof. The parameterization of $M$ and $E_\lambda(M)$ is like in (2.1) and (2.9), respectively. Then

$$\begin{align*}
L_{E_\lambda(M)} &= \int_0^{2\pi} |\gamma'_\lambda(\theta)|d\theta \\
&= \int_0^{2\pi} |\lambda\gamma' + (1 - \lambda)\gamma' + \pi)|d\theta \\
&\leq |\lambda|\int_0^{2\pi} |\gamma'(\theta)|d\theta + |1 - \lambda|\int_0^{2\pi} |\gamma'(\theta + \pi)|d\theta \\
&= (|\lambda| + |1 - \lambda|)L_M.
\end{align*}$$

If $\lambda = \frac{1}{2}$, then the map $M \ni \gamma(\theta) \mapsto \gamma_\frac{1}{2}(\theta) \in E_\frac{1}{2}(M)$ for $\theta \in [0, 2\pi]$ is the double covering of the Wigner caustic of $M$. Thus $2L_{E_\frac{1}{2}(M)} \leq L_M$. \hfill $\Box$

**Theorem 3.2.** Let $M$ be a positively oriented oval of the length $L_M$, enclosing the region of the area $A_M$. Let $\tilde{A}_{E_\lambda(M)}$ denote an oriented area of $E_\lambda(M)$. Then

(i) if $\lambda \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, then

$$A_M - \frac{\lambda(1 - \lambda)}{\pi}L_M^2 \leq \tilde{A}_{E_\lambda(M)} \leq (2\lambda - 1)^2A_M. \tag{3.1}$$

(ii) if $\lambda = \frac{1}{2}$, then

$$A_M - \frac{L_M^2}{4\pi} \leq 2\tilde{A}_{E_\frac{1}{2}(M)} \leq 0. \tag{3.2}$$

(iii) if $\lambda \in (-\infty, 0) \cup (1, \infty)$, then

$$\frac{(2\lambda - 1)^2A_M \leq \tilde{A}_{E_\lambda(M)} \leq A_M - \frac{\lambda(1 - \lambda)}{\pi}L_M^2. \tag{3.3}$$

(iv) for all $\lambda \neq 0, \frac{1}{2}, 1$ $M$ is a curve of constant width if and only if

$$\tilde{A}_{E_\lambda(M)} = A_M - \frac{\lambda(1 - \lambda)}{\pi}L_M^2. \tag{3.4}$$

(v) $M$ is a curve of constant width if and only if

$$2\tilde{A}_{E_\frac{1}{2}(M)} = A_M - \frac{L_M^2}{4\pi}. \tag{3.5}$$
Proof. Let (2.9) be the parameterization of $E_\lambda(M)$. Then the oriented area of $E_\lambda(M)$ is equal to

$$
\bar{A}_{E_\lambda(M)} = \frac{1}{2} \int_{E_\lambda(M)} \gamma_{\lambda,1}d\gamma_{\lambda,2} - \gamma_{\lambda,2}d\gamma_{\lambda,1}
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left[ (P_\lambda(\theta) \cos \theta - P'_\lambda(\theta) \sin \theta) (P_\lambda(\theta) + P''_\lambda(\theta)) \cos \theta \\
+ (P_\lambda(\theta) \sin \theta + P'_\lambda \cos \theta) (P_\lambda(\theta) + P''_\lambda(\theta)) \sin \theta \right] d\theta
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left[ P^2_\lambda(\theta) + P_\lambda(\theta)P''_\lambda(\theta) \right] d\theta
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left[ P^2_\lambda(\theta) - P'^2_\lambda(\theta) \right] d\theta
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left[ (\lambda p(\theta) - (1 - \lambda)p(\theta + \pi))^2 - (\lambda p'(\theta) - (1 - \lambda)p'(\theta + \pi))^2 \right] d\theta
$$

$$
= \lambda^2 \cdot \frac{1}{2} \int_0^{2\pi} [p^2(\theta) - p'^2(\theta)] d\theta + (1 - \lambda)^2 \cdot \frac{1}{2} \int_0^{2\pi} [p^2(\theta + \pi) - p'^2(\theta + \pi)] d\theta
$$

$$
- 2\lambda(1 - \lambda) \cdot \frac{1}{2} \int_0^{2\pi} [p(\theta)p(\theta + \pi) - p'(\theta)p'(\theta + \pi)] d\theta
$$

$$
= (2\lambda^2 - 2\lambda + 1) A_M - 2\lambda(1 - \lambda) \cdot \frac{1}{2} \int_0^{2\pi} [p(\theta)p(\theta + \pi) - p'(\theta)p'(\theta + \pi)] d\theta.
$$

Let $\Psi_M = \frac{1}{2} \int_0^{2\pi} [p(\theta)p(\theta + \pi) - p'(\theta)p'(\theta + \pi)] d\theta$, then the oriented area of $E_\lambda(M)$ is in the following form.

$$
(3.6) \quad \bar{A}_{E_\lambda(M)} = (2\lambda^2 - 2\lambda + 1) A_M - 2\lambda(1 - \lambda) \Psi_M.
$$

Let us find formula for $\Psi_M$ in terms of coefficients of Fourier series of the Minkowski support function $p(\theta)$. By (2.5) and (2.6) there are also equalities:

$$
p(\theta + \pi) = a_0 + \sum_{n=1}^{\infty} (-1)^n (a_n \cos(n\theta) + b_n \sin(n\theta)),
$$

$$
p'(\theta + \pi) = \sum_{n=1}^{\infty} (-1)^n n (-a_n \sin(n\theta) + b_n \cos(n\theta)),
$$

$$
(3.7) \quad \Psi_M = \pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (-1)^n (n^2 - 1)(a_n^2 + b_n^2).
$$

Then bounds of $\Psi_M$ are as follows.

$$
(3.8) \quad \pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) \leq \Psi_M \leq \pi a_0^2 + \frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1)(a_n^2 + b_n^2).
$$

By (2.7) and (2.8) one can rewrite (3.8) as

$$
(3.9) \quad A_M \leq \Psi_M \leq \frac{L_M^2}{2\pi} - A_M
$$
Then using (3.9) to (3.6) and because the map \( M \ni \gamma(\theta) \mapsto \gamma_2(\theta) \in E_{1/2}(M) \) for \( \theta \in [0, 2\pi] \) is the double covering of the Wigner caustic of \( M \), one can obtain (3.1), (3.2), (3.3).

To prove (3.4) and (3.5) let us notice that \( M \) is a curve of constant width if and only if coefficients \( a_{2n}, b_{2n} \) for \( n \geq 1 \) in the Fourier series of \( p(\theta) \) are all equal to zero \([10, 16]\), and then the formula (3.7) of \( \Psi_M \) is as follows:

\[
\Psi_M = \pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (-1)^n (n^2 - 1)(a_n^2 + b_n^2)
\]

\[
= \pi a_0^2 + \frac{\pi}{2} \sum_{n=2, n \text{ is odd}}^{\infty} (n^2 - 1)(a_n^2 + b_n^2)
\]

\[
= 2\pi a_0^2 - A_M
\]

\[
= \frac{L_M^2}{2\pi} - A_M.
\]

\[ \square \]

A simple consequence of the above theorem is the following remark and also the main result, the improved isoperimetric inequality for planar ovals.

**Remark 3.3.** Let \( M \) be an positively oriented oval. Then Theorem 3.2 gives us that \( \bar{A}_{E_{1/2}(M)} \leq 0 \), which leads to the fact that the Wigner caustic of \( M \) has a reversed orientation against that of the original curve \( M \).

**Theorem 3.4.** *(The improved isoperimetric inequality)* If \( M \) is an oval of the length \( L_M \), enclosing a region of the area \( A_M \), then

\[
L_M^2 \geq 4\pi A_M + 8\pi |\bar{A}_{E_{1/2}(M)}|,
\]

where \( \bar{A}_{E_{1/2}(M)} \) is an oriented area of the Wigner caustic of \( M \), and equality (3.10) holds if and only if \( M \) is a curve of constant width.

**Remark 3.5.** The improved isoperimetric inequality becomes isoperimetric inequality if and only if \( \bar{A}_{E_{1/2}(M)} = 0 \), that means when \( M \) has the center of symmetry.

**Theorem 3.6.** *(Barbier’s Theorem)*\([2]\) Let \( M \) be a curve of constant width \( w \). Then the length of \( M \) is equal to \( \pi w \).

By Barbier’s Theorem and Theorem 3.4 one can get the following corollary.

**Corollary 3.7.** Let \( M \) be an oval of constant width \( w \), enclosing region with the area \( A_M \). Then

\[
A_M = \frac{\pi w^2}{4} - 2 |\bar{A}_{E_{1/2}(M)}|,
\]

where \( \bar{A}_{E_{1/2}(M)} \) is an oriented area of the Wigner caustic of \( M \).

One can check that curve \( M \) for which \( p(\theta) = \cos 3\theta + 11 \) (see Fig. 2) is an oval of constant width \( w = 22 \), enclosing the area \( A_M = 117\pi \), and a signed area of the Wigner caustic of \( M \) is equal to \( A_{E_{1/2}(M)} = -2\pi \).
In Proposition 3.8 we present explicitly curves for which the Wigner caustic has exactly $2n + 1$ cusps (this number must be odd and not smaller than $3 - [1, 13]$).

**Proposition 3.8.** Let $n$ be a positive integer. Let $M_{2n+1}$ be a curve for which $p_{2n+1} (\theta) = \cos \left[ \theta (2n+1) \right] + (2n+1)^2 + 2$ is its Minkowski support function. Then $M$ is an oval of constant width and $E_2 (M_{2n+1})$ has exactly $2n + 1$ cusps.

**Proof.** $M$ is singular if and only if $p_{2n+1} (\theta) + p''_{2n+1} (\theta) = 0$, but one can check that is impossible. $M$ is curve of constant width, because $p_{2n+1} (\theta) + p_{2n+1} (\theta + \pi)$ is constant for all values of $\theta$.

The cusp of the Wigner caustic of an oval appears when curvatures of the original curve at points in parallel pair are equal [1].

One can check that equation $\kappa (\theta) = \kappa (\theta + \pi)$, where $\theta \in [0, \pi]$ holds if and only if $\theta = \frac{\pi + 2k \pi}{4n+2}$ for $k \in \{0, 1, 2, \ldots, 2n\}$. □
4. The stability of the improved isoperimetric inequality

A bounded convex subset in \( \mathbb{R}^n \) is said to be an \( n \) - dimensional convex body if it is closed and has interior points. Let \( C^n \) denote the set of all \( n \) - dimensional convex bodies.

There are many important inequalities in convex geometry and differential geometry, such as the isoperimetric inequality, the Brunn–Minkowski inequality, Aleksandrov–Fenchel inequality. The stability property of them are of great interest in geometric analysis, see [11, 17, 19, 21, 24] and the literature therein. An inequality in convex geometry can be written as

\[
\Phi(K) \geq 0,
\]

where \( \Phi : C^n \rightarrow \mathbb{R} \) is a real-valued function and \( (4.1) \) holds for all \( K \in C^n \). Let \( C^n_\Phi \) be a subset of \( C^n \) for which equality in \( (4.1) \) holds.

For example, if \( n = 2 \), and like in previous sections let \( L_{\partial K} \) denote the length of the boundary of \( K \), let \( A_{\partial K} \) denote the area enclosed by \( \partial K \) (i.e. the area of \( K \)), \( \Phi(K) = L^2_{\partial K} - 4\pi A_{\partial K} \). Then inequality \( \Phi(K) \geq 0 \) is then the classical isoperimetric inequality in \( \mathbb{R}^2 \). In this case \( C^2_\Phi \) is a set of disks.

In this section we will study stability properties associated with \( (4.1) \). We ask if \( K \) must be close to a member of \( C^n_\Phi \) whenever \( \Phi(K) \) is close to zero. Let \( d : C^n \times C^n \rightarrow \mathbb{R} \) denoted in some sense the deviation between two convex bodies. \( d \) should satisfy two following conditions:

(i) \( d(K, L) \geq 0 \) for all \( K, L \in C^n \),
(ii) \( d(K, L) = 0 \) if and only if \( K = L \).

If \( \Phi, C^n_\Phi \) and \( d \) are given, then the stability problem associated with \( (4.1) \) is as follows.

Find positive constants \( c, \alpha \) such that for each \( K \in C^n \), there exists \( N \in C^n_\Phi \) such that

\[
\Phi(K) \geq cg^\alpha(K, N).
\]

(4.2)

From this point let us assume that \( n = 2 \) and by Theorem 3.4 let

\[
\Phi(K) = L^2_{\partial K} - 4\pi A_{\partial K} - 8\pi \left| \bar{A}_{E_2}(M) \right| \geq 0.
\]

(4.3)

From Theorem 3.4 one can see that \( C^2_\Phi \) consists of bodies of constant width.

Let us recall two \( d \) measure functions.

Let \( K \) and \( N \) be two convex bodies with respective support functions \( p_{\partial K} \) and \( p_{\partial N} \). Usually to measure the deviation between \( K \) and \( N \) one can use the Hausdorff distance,

\[
d_\infty(K, N) = \max_\theta \left| p_{\partial K}(\theta) - p_{\partial N}(\theta) \right|.
\]

(4.4)

Another such measure is the measure that corresponds to the \( L_2 \)-metric in the function space. It is defined by

\[
d_2(K, N) = \left( \int_0^{2\pi} \left| p_{\partial K}(\theta) - p_{\partial N}(\theta) \right|^2 d\theta \right)^{\frac{1}{2}}.
\]

(4.5)

It is obvious that \( d_\infty(K, N) = 0 \) (or \( d_2(K, N) = 0 \)) if and only of \( K = N \).
Lemma 4.1. Let \( c_k, d_k \in \mathbb{R} \) for \( k \in \{1, 2, \ldots, n\} \). Then

\[
\max_{\theta} \left| \sum_{k=1, k \text{ odd}}^{n} (c_k \cos k\theta + d_k \sin k\theta) \right| \leq \max_{\theta} \left| \sum_{k=1}^{n} (c_k \cos k\theta + d_k \sin k\theta) \right|.
\]

Proof. Let

\[
f_{\text{odd}}(\theta) = \sum_{k=1, k \text{ odd}}^{n} (c_k \cos k\theta + d_k \sin k\theta),
\]

\[
f_{\text{even}}(\theta) = \sum_{k=1, k \text{ even}}^{n} (c_k \cos k\theta + d_k \sin k\theta),
\]

\[
f(\theta) = f_{\text{odd}}(\theta) + f_{\text{even}}(\theta).
\]

One can see that \( f_{\text{odd}} \) is bounded, \( 2\pi \)-periodic and \( f_{\text{odd}}(\theta) = -f_{\text{odd}}(\theta + \pi) \). Let \( \theta_0 \) be an argument for which \( f_{\text{odd}}(\theta_0) = \max_{\theta} f_{\text{odd}}(\theta) \), then \( f_{\text{odd}}(\theta_0 + \pi) = \min_{\theta} f_{\text{odd}}(\theta) = -f_{\text{odd}}(\theta_0) \).

Because \( f_{\text{even}} \) is \( \pi \)-periodic, then \( f_{\text{even}}(\theta_0) = f_{\text{even}}(\theta_0 + \pi) \). One can see that:

- If \( f_{\text{even}}(\theta_0) \geq 0 \), then \( \max_{\theta} |f(\theta)| = |f(\theta_0)| = f(\theta_0) \geq f_{\text{odd}}(\theta_0) = \max_{\theta} |f_{\text{odd}}(\theta)|. \)
- If \( f_{\text{even}}(\theta_0) < 0 \), then \( \max_{\theta} |f(\theta)| = |f(\theta_0 + \pi)| = -f(\theta_0 + \pi) \geq -f_{\text{odd}}(\theta_0 + \pi) = \max_{\theta} |f_{\text{odd}}(\theta)|. \)

\[\square\]

Definition 4.2. Let \( p_M \) be the Minkowski support function of a positively oriented oval \( M \) of length \( L_M \). Then

\[
p_{\text{Wigner}}(\theta) = \frac{L_M}{2\pi} + \frac{p_M(\theta) - p_M(\theta + \pi)}{2}
\]

will be the support function of a curve \( W_M \) which will be called the Wigner caustic type curve associated with \( M \).

Proposition 4.3. Let \( W_M \) be the Wigner caustic type curve associated with an oval \( M \). Then \( W_M \) has following properties:

(i) \( W_M \) is an oval of constant width,

(ii) \( L_{W_M} = L_M \),

(iii) \( E_{\frac{1}{2}}(W_M) = E_{\frac{1}{2}}(M) \),

(iv) \( A_{W_M} \geq A_M \) and the equality holds if and only if \( M \) is a curve of constant width,

(v) \( W_M = M \) if and only if \( M \) is a curve of constant width.

Proof. By (4.6), to prove that \( W_M \) we will show \( p_{\text{Wigner}}(\theta) > 0 \).

By (2.5) and (2.2) the radius of a curvature of \( W_M \) is equal to

\[
\rho_{\text{Wigner}}(\theta) = p_{\text{Wigner}}(\theta) + p''_{\text{Wigner}}(\theta)
\]

\[
= a_0 + \sum_{n=1, n \text{ odd}}^{\infty} (\cos n\theta + b_n \sin n\theta)
\]

\[
W_M(\theta) = \rho_{\text{Wigner}}(\theta) + p_M(\theta) - p_M(\theta + \pi)
\]

\[
W_M(\theta) = \frac{L_M}{2\pi} + \frac{p_M(\theta) - p_M(\theta + \pi)}{2}
\]

\[
W_M(\theta) = \frac{L_M}{2\pi} + \frac{p_M(\theta) - p_M(\theta + \pi)}{2}
\]

\[
W_M(\theta) = \frac{L_M}{2\pi} + \frac{p_M(\theta) - p_M(\theta + \pi)}{2}
\]
Because $\rho_M(\theta) = a_0 + \sum_{n=1}^{\infty} (-n^2 + 1)(a_n \cos n\theta + b_n \sin n\theta) > 0$, then also by inequality $\rho_{W_M} > 0$ holds. This is a consequence of that the range of
\[
\left| \sum_{n \text{ is odd}} (-n^2 + 1)(a_n \cos n\theta + b_n \sin n\theta) \right|
\]
is a subset of the range of
\[
\left| \sum_n (-n^2 + 1)(a_n \cos n\theta + b_n \sin n\theta) \right| - \text{see Lemma 4.1.}
\]
To check that $W_M$ is a curve of constant width let us notice that $p_{W_M}(\theta) + p_{W_M}(\theta + \pi)$ is a constant.

By (2.3) one can get
\[
L_{W_M} = \int_0^{2\pi} p_{W_M}(\theta) d\theta = \int_0^{2\pi} \left( \frac{L_M}{2\pi} + \frac{p_M(\theta) - p_M(\theta + \pi)}{2} \right) d\theta = L_M.
\]

By (2.9) the support function of the Wigner caustic of $M$ is $p(\theta) - p(\theta + \pi)$. One can check that also this is the support function of the Wigner caustic of $W_M$. Then using the improved isoperimetric inequality it is easy to show that $A_{W_M} \geq A_M$ and the equality holds if and only if $M$ is a curve of constant width.

**Theorem 4.4.** Let $K$ be strictly convex domain of area $A_{\partial K}$ and perimeter $L_{\partial K}$ and let $\tilde{A}_{E_1/2}(\partial K)$ denote the oriented area of the Wigner caustic of $\partial K$. Let $W_K$ denote the convex body for which $\partial W_K$ is the Wigner caustic type curve associated with $\partial K$. Then
\[
L_{\partial K}^2 - 4\pi A_{\partial K} - 8\pi \left| \tilde{A}_{E_1/2}(\partial K) \right| \geq 4\pi^2 d_\infty^2(K, W_K),
\]
where equality holds if and only if $\partial K$ is a curve of constant width.

**Proof.** Because of (2.5), (2.6), (2.3), (2.4), the support functions $p_{\partial K}$ and $p_{\partial W_K}$ have the following Fourier series:

\[
p_{\partial K}(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),
\]
\[
p_{\partial W_K}(\theta) = a_0 + \sum_{n=1, n \text{ is odd}} (a_n \cos n\theta + b_n \sin n\theta).
\]

Also one can get the Fourier series of $\Phi$ (see (4.3)):

\[
\Phi(K) = L_{\partial K}^2 - 4\pi A_{\partial K} - 8\pi \left| \tilde{A}_{E_1/2}(M) \right| = 2\pi^2 \left( \sum_{n=2, n \text{ is even}}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) \right).
\]

One can check that $|a_n \cos n\theta + b_n \sin n\theta| \leq \sqrt{a_n^2 + b_n^2}$ and then by (4.4) and Hölder’s inequality:
\[ d_\infty(K, W_K) = \max_\theta \left| p_{\partial K}(\theta) - p_{\partial W_K}(\theta) \right| \]
\[ = \max_\theta \left| \sum_{n=2, \text{n is even}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \right| \]
\[ \leq \max_\theta \left( \sum_{n=2, \text{n is even}}^{\infty} |a_n \cos n\theta + b_n \sin n\theta| \right) \]
\[ \leq \sum_{n=2, \text{n is even}}^{\infty} \frac{1}{\sqrt{n^2 - 1}} \cdot \sqrt{n^2 - 1} \sqrt{a_n^2 + b_n^2} \]
\[ \leq \sqrt{\frac{1}{2}} \cdot \sqrt{\Phi(K)} \cdot \frac{1}{2\pi^2}. \]

And the equality holds if and only if \( a_{2m} = b_{2m} = 0 \) for all \( m \in \mathbb{N} \), so \( \partial K \) is a curve of constant width. \( \square \)

**Theorem 4.5.** Under the same assumptions of Theorem 4.4, one gets
\[
L^2_{\partial K} - 4\pi A_{\partial K} - 8\pi \left| \tilde{A}_{E_1}(\partial K) \right| \geq 6\pi d_2^2(K, W_K),
\]
where equality holds if and only if \( \partial K \) is a curve of constant width, or the Minkowski support function of \( \partial K \) is in the form
\[ p_{\partial K}(\theta) = a_0 + a_2 \cos 2\theta + b_2 \sin 2\theta + \sum_{n=1, n \text{ is odd}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \]

**Proof.** Because of (4.10) and (4.11)
\[
d_2^2(K, W_K) = \int_0^{2\pi} \left| p_{\partial K}(\theta) - p_{\partial W_K}(\theta) \right|^2 d\theta \]
\[ = \int_0^{2\pi} \left| \sum_{n=2, \text{n is even}}^{\infty} (a_n \cos n\theta + b_n \sin \theta) \right|^2 d\theta \]
\[ = \pi \sum_{n=2, \text{n is even}}^{\infty} (a_n^2 + b_n^2) \]
\[ \leq \frac{1}{6\pi} \cdot 2\pi^2 \sum_{n=2, \text{n is even}}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) \]
\[ = \frac{1}{6\pi} \Phi(K). \]

And the equality holds if and only if \( a_{2m} = b_{2m} = 0 \) for all \( m \in \mathbb{N} \), so \( \partial K \) is a curve of constant width, or \( p_{\partial K}(\theta) = a_0 + a_2 \cos 2\theta + b_2 \sin 2\theta + \sum_{n=1, n \text{ is odd}}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \) \( \square \)
Let us consider the convex body $K$ for which \((4.13)\) is its Minkowski support function – see Fig. 4. We will check how close the right hand sides in the stability inequalities \((4.9)\) and \((4.12)\) are to be optimal.

\[
(4.13) \quad p_{\partial K}(\theta) = 10 + 2 \cos 2\theta - \frac{1}{3} \sin 3\theta - \frac{1}{4} \cos 4\theta.
\]

Then

\[
(4.14) \quad p_{\partial W_K}(\theta) = 10 - \frac{1}{3} \sin 3\theta,
\]

\[
(4.15) \quad p_{E_{\frac{1}{2}}(\partial K)}(\theta) = -\frac{1}{3} \sin 3\theta.
\]

One can check that

\[
(4.16) \quad L_{\partial K} = 20\pi,
\]

\[
(4.17) \quad A_{\partial K} = \frac{26809}{288}\pi,
\]

\[
(4.18) \quad \tilde{A}_{E_{\frac{1}{2}}(\partial K)} = \frac{2\pi}{9},
\]

\[
(4.19) \quad L_{\partial K}^2 - 4\pi A_{\partial K} - 8\pi \left| \tilde{A}_{E_{\frac{1}{2}}(\partial K)} \right| = 25.875\pi^2.
\]
\[ 4\pi^2 d_{\infty}^2(K, W_K) = 4\pi^2 \left( \max_{\theta} \left| 2\cos 2\theta - \frac{1}{4} \cos 4\theta \right| \right)^2 = 20.25\pi^2, \]

(4.20)

\[ 6\pi d_2^2(K, W_K) = 6\pi \int_0^{2\pi} \left( 2\cos 2\theta - \frac{1}{4} \cos 4\theta \right)^2 d\theta = 24.375\pi^2. \]

(4.21)

Then by (4.19), (4.20) the stability inequality (4.9) in Theorem 4.4 is in the following form:

\[ 25.875\pi^2 \geq 20.25\pi^2, \]

(4.22)

and by (4.19), (4.21) the stability inequality (4.12) in Theorem 4.5 is in the following form:

\[ 25.875\pi^2 \geq 24.375\pi^2. \]

(4.23)

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Warsaw University of Technology, Faculty of Mathematics and Information Science, Plac Politechniki 1, 00-661 Warsaw, Poland,
E-mail address: zwierzynskim@mini.pw.edu.pl