Population dynamics and control with imposed interbirth refractory periods

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Abstract

We consider age-structured models with an imposed refractory period between births. These models can be used to formulate alternative population control strategies to China’s one-child policy. By allowing any number of births, but with an imposed delay between births, we show how the total population can be decreased and how a relatively younger age distribution generated. This delay represents a more “continuous” form of population management for which the one-child policy is a limiting case. Such a policy approach could be more easily accepted by society. We also propose alternative birth rate functions that might result from a societal response to imposed refractory periods. Our numerical and asymptotic analyses provides an initial framework for studying demographics and how social dynamics influences population structure.

Keywords: population dynamics, demographics, McKendrick equation, population control, one-child policy

1 Introduction

Models of age-structured population dynamics are often based on the classic McKendrick equation \citep{McKendrick1926, Kermack1927} (sometimes called von Foerster equation \citep{vonFoerster1959}). These equations describe the dynamics of the mean population as a function of time $t$ and expressed as a density in age $a$. The solutions to the McKendrick equations can be partially solved using the method of characteristics and numerical approximations \citep{Perthame2007, Keyfitz1997} across many contexts. Moreover, stochastic extensions to incorporate the random times of birth and death (demographic stochasticity) have been developed \citep{Greenman2016, Chou2016, Greenman2017}.

Age-structured equations have been used to predict the evolution of human and animal populations. Using such models and ideas from control theory to frame population control strategies was vogue in the 1970s \citep{Langhaar1972, Pollard1973, Falkenburg1973, Hritonenko2010}. A historically remarkable example was it’s use in 1979 by Jian Song \citep{Song1980, Song1982}, a Chinese engineer who numerically solved the one-component McKendrick equation using birth rates associated with China in the late 1970s (see Fig. \ref{fig:one-child}). By projecting future populations associated with different birth rates (expressed by the mean number of children per woman), he found that in order to keep the population manageable ($\sim 700$ million - 1 billion) within 100 years, this control parameter would have
to be decreased to the point that each woman is allowed only one child [Song, 1980, 1982, Song et al., 1988]. This research provided the technical basis of the one-child policy in China [Greenhalgh, 2004, Bacaër, 2011].

Despite opposition from social scientists and proposed alternatives, the Chinese leadership implemented the one-child policy in 1980 based on the implications of Jian Song’s numerical solutions to the McKendrick equation. A less strident alternative to the one-child policy was proposed by social scientists and demographers: rather than imposing a maximal number of children, a minimum delay between two consecutive births could be enforced [Greenhalgh, 2004]. Such a policy would arguably have been more readily accepted by the populace and would have led to fewer unintended consequences such as a skewed sex ratio and an elder-heavy age distribution. However the effectiveness of such alternatives had not been modeled or analyzed.

Here, we extend the McKendrick age-structured model to incorporate a delay between successive births by each female. We numerically solve the equations using parameters appropriate to 1980 China and compare predictions of the graded policies with those of the one-child policy. We explore how the total population and age distribution are affected by refractory periods of different lengths. Finally, we also discuss how social behaviors that compensate for restricted births might affect the birth rates and resulting population structures.

2 Mathematical Model

When applying age-structured PDE models to two-sex populations, a simple assumption is to consider only the female population density \( f(a,t) \) (i.e., \( f(a,t)da \) represents the mean number of women of age between \( a \) and \( a + da \) at time \( t \)). Indeed, unless the populations of the sexes are very unbalanced, the female population can be considered as the “limiting quantity” that determines the number of births. In other words, the frequency of births in the total population is relatively insensitive to the male population. The McKendrick equations describing the female population density \( f(a,t) \) are formulated as

\[
\frac{\partial}{\partial t} f(t,a) + \frac{\partial}{\partial a} f(t,a) = -\mu(a)f(t,a) \\
\begin{align*}
  f(t,0) &= \xi \int_0^\infty f(t,a)\beta_0(a)da \\
  f(0,a) &= I(a),
\end{align*}
\]

where \( \mu(a) \) represents the death rate of individuals of age \( a \), \( \beta_0(a) \) is the birth rate of women of age \( a \), and \( \xi \) is the fraction of births that produce girls. Equation (1a) describes the evolution of the population through time, Eq. (1b) denotes the boundary condition at age \( a = 0 \) describing the number of girls born at time \( t \), and Eq. (1c) specifies the initial condition.

If one were also interested in the male population, it would be coupled to the female population through both a male-population-dependent birth rate \( \beta_0 \) and the number of male newborns at time \( t \): \( (1 - \xi) \int_0^\infty f(t,a)\beta_0(a)da \). An equation for the male population density \( m(a,t) \) with a “male-specific” death rate would determine its evolution in time. While the female population strongly controls the male population through births, the male population influences the female population only through its relatively weak influence on the birth rate (as long as males are not rare and birth “limiting”). Thus, we henceforth neglect male population dynamics and concentrate only on the “limiting” female population.

Now, in order to introduce a delay between consecutive births, we need to further separate the females into those who have never had a child (and thus who are not restricted in having their first child) and those who have already had a child (and who may need to wait a certain time before having another one). The population densities for each of these classes of females are defined as:
Figure 1: China’s 1980 birth and death rates Adapted from Bacaër [2011]. The area under the birth rate
\( \beta_T \equiv \int_0^\infty \beta(a) \, da \) represents the mean number of children born by each woman.

\[ f_0(t, a) \] the population density of childless females. The quantity \( f_0(t, a) \, da \) is the number of women with age between \( a \) and \( a + da \) and who have never had a child up to the current time \( t \).

\[ f(t, a, \tau) \] the population of females who have had at least one child. The quantity \( f(t, a, \tau) \, da \, d\tau \) is the number of females at time \( t \) whose age is between \( a \) and \( a + da \) and whose youngest child’s age is between \( \tau \) and \( \tau + d\tau \).

We will assume that these two populations have the same age-dependent death rate \( \mu(a) \) but may give birth at different rates \( \beta_0 \) and \( \beta \), respectively. We also define the total female population density as

\[ f_{\text{tot}}(t, a) = f_0(t, a) + \int_a^\infty f(t, a, \tau) \, d\tau \quad (2) \]

and the total number of women at time \( t \) as

\[ n(t) = \int_0^\infty f_{\text{tot}}(t, a) \, da = \int_0^\infty f_0(t, a) \, da + \int_0^\infty \int_0^\infty f(t, a, \tau) \, d\tau \, da. \quad (3) \]

The age-structured McKendrick equations for \( f_0 \) and \( f \) are:

\[ \frac{\partial}{\partial t} f_0(t, a) + \frac{\partial}{\partial a} f_0(t, a) = - (\mu(a) + \beta_0(a)) f_0(t, a) \quad (4a) \]

\[ \frac{\partial}{\partial t} f(t, a, \tau) + \frac{\partial}{\partial a} f(t, a, \tau) + \frac{\partial}{\partial \tau} f(t, a, \tau) = - (\mu(a) + \beta(a, \tau)) f(t, a, \tau) \quad (4b) \]

\[ f_0(t, 0) = \xi \left( \int_0^\infty f_0(t, a) \beta_0(a) \, da + \int_0^\infty \int_0^\infty f(t, a, \tau) \beta(a, \tau) \, d\tau \, da \right) \quad (4c) \]

\[ f(t, a, 0) = f_0(t, a) \beta_0(a) + \int_0^\infty f(t, a, \tau) \beta(a, \tau) \, d\tau \quad (4d) \]

\[ f_0(0, a) = I_0(a) \quad \text{and} \quad f(0, a, \tau) = I(a, \tau) \quad (4e) \]

Equation (4a) describes the evolution of \( f_0 \) as in the classical McKendrick equation (cf Eq. (1a)) with birth rate \( \beta_0(a) \). For \( f(t, a, \tau) \), we must introduce the new variable \( \tau \) to mark the time since the last
birth. This brings in another convection term in Eq. \((4b)\) since \(\tau\) increases alongside time \(t\) and age \(a\). The birth rate \(\beta\) for this population can depend on both the age \(a\) and the time \(\tau\) since the last birth.

Eq. \((4c)\) gives the number of girls \(f_0(t,0)\) born at time \(t\), while Eq. \((4d)\) describes \(f(t,a,0)\), the density of females with age in \((a,a+da)\) who just gave birth at time \(t\). These individuals can arise from the \(f_0\) population (females who have never had a child) or from the \(f\) population itself (females who have already had at least one child). Thus, the boundary conditions \((4c)\) and \((4d)\) couple the two populations \(f_0\) and \(f\). Finally, equations \((4e)\) simply describe the initial conditions for \(f_0\) and \(f\).

The dependence of the rate \(\beta(a, \tau)\) on both \(a\) and \(\tau\) allows us to impose a delay between births by setting \(\beta = 0\) for \(\tau\) below a certain threshold. As a first model, we will consider

\[
\beta(a, \tau) = \beta_0(a) \mathbb{I}(\tau, \delta)
\]

with the indicator function \(\mathbb{I}(\tau, \delta) = 1\) for \(\tau > \delta\) and \(\mathbb{I}(\tau, \delta) = 0\) for \(\tau \leq \delta\). We will further elaborate on this choice for the birth rate \(\beta\) in the Discussion and Conclusions section.

### 2.1 Asymptotic behavior

We first analyze the asymptotic behavior of our model. An important feature of renewal transport equations such as the McKendrick model is that as \(t \to \infty\), the total population \(n(t)\) will grow exponentially (in the absence of nonlinear regulation terms [Gurtin and MacCamy 1974]), while the normalized, age-dependent population density converges to a time-independent distribution (see [Perthame 2007, Chapter 3] and [Arino 1995]). This property is independent of the initial condition. We will assume that this steady-state asymptotic property arises in our two-component, three-variable model; i.e., the normalized \(f_0(t,a)/n(t)\) and \(f(t,a,\tau)/n(t)\) converge to stable distributions. To find these asymptotic distributions, we search for solutions of Eqs. \((4)\) of the form

\[
f_0(t,a) = h_0(a)e^{\lambda t} \quad \text{and} \quad f(t,a,\tau) = h(a,\tau)e^{\lambda t}.
\]

Here, \(h_0(a)\) and \(h(a,\tau)\) are the steady-state distributions and \(\lambda\) is the exponential rate at which the total population increases as \(t \to \infty\). The sign of \(\lambda\) determines whether the population increases or decreases at long times. Similar to the definition of \(f_{tot}\) (Equation \((2)\)), the asymptotic age distribution of the total female population is given by

\[
h_{tot}(a) = h_0(a) + \int_0^a h(a,\tau) d\tau.
\]

If we consider solutions for Eqs. \((4)\) of the forms given in Eqs. \((6)\), we find the age-structured equations become

\[
\begin{align*}
\frac{\partial}{\partial a} h_0(a) &= - (\mu(a) + \beta_0(a) + \lambda) h_0(a) \quad \text{(8a)} \\
\frac{\partial}{\partial a} h(a,\tau) + \frac{\partial}{\partial \tau} h(a,\tau) &= - (\mu(a) + \beta(a,\tau) + \lambda) h(a,\tau) \quad \text{(8b)} \\
h_0(0) &= \xi \left( \int_0^\infty \beta_0(a) h_0(a) da + \int_0^\infty \int_0^a \beta(a,\tau) h(a,\tau) d\tau da \right) \quad \text{(8c)} \\
h(a,0) &= \beta_0(a)r_0(a) + \int_0^a \beta(a,\tau) h(a,\tau) d\tau \quad \text{(8d)}
\end{align*}
\]

Solutions of this system of PDEs will be given as a triplet \((h_0, h, \lambda)\). For a given particular solution \((h_0, h, \lambda)\), one can easily check that \((Ch_0, Ch, \lambda)\), for any nonnegative real constant \(C\), is also a solution of Eqs. \((3)\). For simplicity, we will search only for the solutions associated with \(h_0(0) = 1\)
Figure 2: The asymptotic population distribution associated with the birth and death rates shown in Fig. 1. A delay of 9 months ($\delta = 0.75$) corresponding to the human gestation period was imposed. (a) Steady-state age distributions of females without children, $h_0(a)$, and all females, $h_{\text{tot}}(a)$, respectively. A monotonically decreasing $h_{\text{tot}}(a)$ indicates that there are larger numbers of younger females, consistent with an increasing population and $\lambda > 0$. (b) The full double density $h(a, \tau)$ of females of age $a$ and whose youngest child is age $\tau$. (one can still then normalize $h_0$ and $h$ in order to derive real probability distributions). Upon using this normalization and the method of characteristics, we find

$$h_0(a) = \exp \left[ - \int_0^a (\mu(a') + \beta_0(a') + \lambda) \, da' \right]$$

(9a)

$$h(a, \tau) = \left( \beta_0(a - \tau) h_0(a - \tau) + \int_0^a \beta(a - \tau, \tau') h(a - \tau, \tau') \, d\tau' \right) \times \exp \left[ - \int_0^\tau (\mu(a - \tau') + \beta(a - \tau', \tau - \tau') + \lambda) \, d\tau' \right]$$

(9b)

$$1 = \xi \left( \int_0^\infty \beta_0(a) h_0(a) \, da + \int_0^\infty \int_0^a \beta(a, \tau) h(a, \tau) \, d\tau \, da \right)$$

(9c)

Eqs. (9) represent a fixed-point problem in the variable $\lambda$. For a fixed value of $\lambda$, we can determine the functions $h_0^\lambda$ and $h^\lambda$ using Equations (9a). Consequently, the triplet $(h_0^\lambda, h^\lambda, \lambda)$ is a solution of the Eqs. (9) if and only if

$$1 = \xi \left( \int_0^\infty \beta_0(a) h_0^\lambda(a) \, da + \int_0^\infty \int_0^a \beta(a, \tau) h^\lambda(a, \tau) \, d\tau \, da \right)$$

(10)

We can numerically solve this fixed-point problem using, for instance, a Newton optimization method.

Figure 2 shows the partitioning of the steady-state prefactors into $h_0$ and $h$ using the birth rate $\beta_0$ and death rate $\mu$ shown in Figure 1. Specifically, we used the birth rate $\beta(a, \tau)$ as per Eq. 5 with $\delta = 0.75$ years = 9 months, the natural human gestation period. We also assumed an even sex ratio at birth, $\xi = 0.5$. Using these parameters, we computed the associated asymptotic growth rate $\lambda$ and the corresponding asymptotic distributions $h_0$, $h$ and $h_{\text{tot}}$. In this case, we found $\lambda \approx 0.003 > 0$ indicating an exponentially growing total population. The shape of $h_{\text{tot}}(a)$ is consistent with this growth as it is monotonically decreasing, indicating that every new generation has a larger population than the previous one.

In Figure 3 we see how increases in the refractory period $\delta$ decrease the asymptotic growth rate $\lambda$ and affect the distribution $h_{\text{tot}}(a)$. A negative overall birth rate $\lambda < 0$ (i.e., an asymptotically decaying
population) arises when $\delta \gtrsim 1.7$ yrs $\approx 20$ months. As soon as $\lambda < 0$, the distribution $h_{\text{tot}}(a)$ becomes nonmonotonic and starts to increase at small age $a$. In this regime, the maximum female population occurs at a finite age $a > 0$.

In Appendix A, we show the asymptotic total population $h_{\text{tot}}$ can be described by the solution of a McKendrick-type equation with only one subpopulation (rather than both $h_0(a)$ and $h(a)$ used in Eqs. [6]). In such a reduced model, the death rate $\mu(a)$ and the growth rate $\lambda$ are the same, but $h_{\text{tot}}$ is also renewed through an effective birth rate function $\beta_{\text{eff}}(a)$ that corresponds to the weighted birth rates of the two subpopulations in the full model. We show that this effective birth rate for the total population incorporates the interbirth delay $\delta$.

This formulation is valid only in the asymptotic case with a fixed delay $\delta$ that remains unchanged for a long period of time. For practical modeling of policies in which delays $\delta$ are used as a time-dependent control variable, such as China’s 1980 one-child policy and its subsequent modification in 2015, it is necessary to analyze the full model that delineates the two female populations.

### 2.2 Temporal evolution

As was used to predict the effects of the one-child policy, we use China’s female age distribution in 1980 (see [Bacaër 2011, Chapter 25]) as a starting point to explore how the total population is predicted to evolve under different values of the imposed delay $\delta$. In Appendix B, we propose a way to reconstruct the initial (1980) age distributions for females who have not given birth and those who have, $I_0(a)$ and $I(a, \tau)$, respectively. These initial distributions are plotted in Figure 3(a).

As with $h_0(a)$ and $h(a)$ solved in the previous section, we found analytic solutions of Eqs. (4) using the method of characteristics:
Using the fundamental rates $\beta_0(a)$, $\mu(a)$ and $\beta(a,\tau)$ as those used in the previous section for the full model (see Figure 1 and Eq. (5)), we evaluate Eqs. 11 and construct the total female population (see Eq. 3). The evolution of $n(t)$ over one century, under different interbirth delays, are plotted in Figure 4(b). At long times, the total population exhibits the asymptotic behavior predicted by the eigenvalues shown in Figure 3(a). For $\delta \gtrsim 1.7$ years, the total population will decrease exponentially. Because the effect on reducing the total population is strongest at small values imposed delay, even a delay of $\delta \sim 2$ years is sufficient to dramatically reduce population over the next 100 years.

## 3 Discussion and Conclusions

We have formulated a “continuum” of birth-control policies for population management in which the one-child policy is a limiting case. Of course, our modeling approach is relevant for any system in which the gestation time is appreciable compared to an organism’s duration of fertility. For example, animals like the Greater cane rat (*Thryonomys swinderianus*), the Pacarana (*Dinomys branickii*) and the Steenbok (*Raphicerus campestris*) have gestation times longer than 5% of their maximum longevity. For humans, it is around 0.6% [Tacutu et al., 2018]. Under an imposed policy, we analyze within our modified age-structured PDE model how imposed interbirth refractory periods affect the predicted total female population and its steady-state age distribution. For long delays $\delta$, the model approaches the one-child policy as a larger fraction of women are pushed past menopause. Nonetheless, the framing of a policy with a tunable parameter $\delta$ may have been politically and socially easier to implement.
However, our modeling, as with the rudimentary analysis of the simple McKendrick equation [Song, 1982] that supported the one-child policy, neglects many important socio-economic factors that have an important impact on the birth rates but are difficult to model mathematically. Aspects such as the evolution of the birth and death rates after the 1980s, the difficulties in enforcement of delay policies, and the reaction of the population to the policy are all factors that can qualitatively affect the evolution of the population.

Specifically, behavioral responses could modify the effects of any policy. For the one-child policy, the bias favoring male offspring has reduced the fraction of girls to $\xi \approx 0.45$ [Zeng et al., 1992, Li and Meng, 2014]. This reduction does not affect the prediction of the model except to slightly decrease the limiting female population, which simply exists in a slightly larger male population background.

Behavioral responses by females are more likely to mitigate the expected population reduction. For example, our choice for the birth rate $\beta$ for women with at least one child (Eq. (5)) assumes that there is no behavioral reaction to the policy. We have assumed that $\beta(a, \tau) = \beta_0(a)$ for $\tau > \delta$; in other words, once the imposed refractory period has passed, the reproduction behavior reverts to that of birthless females. In reality, it is more likely that women would be more eager to have a child once the delay passed in order to "catch up" and mitigate the "missed opportunity". Although the real shape of $\beta$ is not easy to estimate, we can propose a more realistic shape of $\beta$ that is still based on $\beta_0$ but that also incorporates the proportion of births which had been prevented by the delay. Consider the ratio

$$\frac{\int_{a-\tau}^\infty \beta_0(a')da'}{\int_{a-\tau}^\infty (a' - (a' - \tau), \delta)\beta_0(a')da'} = \frac{\int_{a-\tau}^\infty \beta_0(a')da'}{\int_{a-\tau+\delta}^{\infty} \beta_0(a')da'}.$$ (12)

that represents the probability of a woman who had her last child at age $a - \tau$ to have another child if there were no imposed delay, divided by her probability to have another child under imposed delay. This ratio, which is above 1, represents the "missed opportunity" induced by the delay. We use this ratio in order to build a more realistic rate function

$$\beta(a, \tau) = \varphi \left( \frac{\beta_0(a)\mathbb{1}(\tau, \delta)\int_{a-\tau}^\infty \beta_0(a')da'}{\int_{a-\tau+\delta}^{\infty} \beta_0(a')da'} \right),$$ (13)

where $\varphi$ is a regularizing function that dampens extreme values of the ratio in Eq. (12) as menopause is approached. Thus, $\varphi$ can be viewed as incorporating the physiological limitations of giving birth. For example, we can take

$$\varphi(x) = \min\{x, 12\},$$ (14)

where 12 year$^{-1}$ is a theoretically maximum conception rate.

Asymptotic and temporal results of this behavioral response model are shown in Figure 5. The asymptotic growth rate $\lambda$ (Fig. 5(a)) of the population does not decrease as steeply as that of the model without behavioral response (Fig. 3) and exhibits a plateau around $\lambda = -0.07$ year$^{-1}$. This growth-rate plateau collapses at very long delays (here, $\delta \sim 20$ years) when the model approaches the one-child policy limit. One can see this effect on the temporal evolution (Figure 13(b)) where delays between 10 and 20 years show little difference in the total population 100 years later.

Another possibility of behavioral response is shifting the birth-rate function to younger ages as females give birth at younger ages. These behavioral adjustments will also mitigate population reduction. Under this response scenario, a birth-delay policy would be more effective in jurisdictions where the intrinsic birth rate is already high at very young ages.

Our age-structured model incorporates last birth times and provides a mathematical framework with which to investigate different proposed population management policies. It allows one to test
Figure 5: Effect of the delay $\delta$ between births on the asymptotic growth rate $\lambda$ (Figure (a)) and the temporal evolution (Figure (b)) with the rate $\beta$ as described at Equation 13.

different delays and different forms of the birth-rate function that can be modified according to behavioral responses. The overall model is also amenable to a control theory formulation, allowing future analyses in such a context.

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Mathematical Appendices

A  Effective asymptotic model

We show that the asymptotic total population $h_{tot}(a)$ is a solution to the single-population McKendrick equation (the asymptotic version of Eqs. (1)). The death rate $\mu$ and the asymptotic growth rate $\lambda$ will be the same as in the full model with two female subpopulations (Eqs. (6)), but we will consider an effective birth rate $\beta_{eff}(a)$ that will depend only on the age $a$ of females.

First, note that, by definition of $h_{tot}$ (Eq. (7)), we have

\[ \frac{\partial}{\partial a} h_{tot}(a) = \frac{\partial}{\partial a} h_0(a) + \int_0^\tau \frac{\partial}{\partial a} h(a, \tau) d\tau. \] (15)

With the PDEs in Eqs. (8), we have

\[ \frac{\partial}{\partial a} h_{tot}(a) = -[(\mu(a) + \beta_0(a) + \lambda) h_0(a) + \int_0^\tau (\frac{\partial}{\partial \tau} h(a, \tau) + (\mu(a) + \beta(a, \tau) + \lambda) h(a, \tau) ) d\tau] \]

\[ = -[(\mu(a) + \lambda) h_{tot}(a) + \int_0^\tau (\frac{\partial}{\partial \tau} h(a, \tau) ) d\tau + h(a, 0)] \]

\[ = -[(\mu(a) + \lambda) h_{tot}(a) + h(a, a) - h(a, 0) + h(a, 0)]. \]

Since there are no females with children of their own age $a$, we have the boundary condition $h(a, a) = 0$ and

\[ \frac{\partial}{\partial a} h_{tot}(a) = - (\mu(a) + \lambda) h_{tot}(a), \] (16)

which shows that $h_{tot}$ follows the McKendrick PDEs.

For the renewal term, we set $h_{tot}(0) = h_0(0)$ (girls who have just been born cannot give birth). If we consider the effective birth rate

\[ \beta_{eff}(a) = \beta_0(a) h_0(a) + \int_0^a \beta(a, \tau) h(a, \tau) d\tau \]

we have

\[ h_{tot}(0) = \xi \int_0^\infty \left( \beta_0(a) h_0(a) + \int_0^a \beta(a, \tau) h(a, \tau) d\tau \right) da = \xi \int_0^\infty \beta_{eff}(a) h_{tot}(a) da, \]

so that $h_{tot}(a)$ is a solution of an asymptotic ($t \to \infty$) McKendrick equation:

\[ \frac{\partial}{\partial a} h_{tot}(a) = - (\mu(a) + \lambda) h_{tot}(a), \] (19a)

\[ h_{tot}(0) = \xi \int_0^\infty \beta_{eff}(a) h_{tot}(a) da. \] (19b)

Note that $\beta_{eff}(a)$ (Equation 17) is just the birth rate weighted according to the two female subpopulations $h_0(a)$ and $h(a)$. If we randomly select a female of age $a$ (whether or not she has had a child), her expected birth rate is $\beta_{eff}(a)$. In Fig. 6 we plot the effective birth rate $\beta_{eff}(a)$ associated with different delays $\delta$. 

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Figure 6: The effective birth rate $\beta_{\text{eff}}(a)$ as defined in Eq. (17) for different delays $\delta$ between births. For larger delays, the areas under $\beta_{\text{eff}}(a)$ (the average number of children born per woman) decreases.

**B  Reconstruction of the initial population $f_0$ and $f$**

The available data gives the total female population in China in 1980 $f_{\text{tot}}(0, a)$ [Bacaër, 2011]. We need to separate this population into the two subpopulations $f_0(0, a)$ and $f(t, a, \tau)$ in order to have the complete initial conditions for our model.

Consider the asymptotic population and define the proportion of females of age $a$ who have not given birth

$$
\phi(a) = \frac{h_0(a)}{h_{\text{tot}}(a)} = \frac{h_0(a)}{h_0(a) + \int_0^a h(a, \tau) d\tau}. \quad (20)
$$

We then use this ratio to reconstruct the population of women who have not given birth in the original (1980) population:

$$
f_0(0, a) = \phi(a) f_{\text{tot}}(0, a).
$$

In the same manner, in order to derive the proportion of women of age $a$ and whose last birth occurred in $[\tau, \tau + d\tau]$, we use the asymptotic population $h(a, \tau)$:

$$
f(0, a, \tau) = (1 - \phi(a)) h(a, \tau) f_{\text{tot}}(0, a).
$$
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