Landau-Bloch constants for functions in $\alpha$-Bloch spaces and Hardy spaces

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Abstract. In this paper, we obtain a sharp distortion theorem for a class of functions in $\alpha$-Bloch spaces, and as an application of it, we establish the corresponding Landau's theorem. These results generalize the corresponding results of Bonk, Minda and Yanagihara, and Liu, respectively. We also prove the existence of Landau-Bloch constant for a class of functions in Hardy spaces and the obtained result is a generalization of the corresponding result of Chen and Gauthier.

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1. Introduction and main results

One of the long standing open problems of determining the precise value of the schlicht Landau-Bloch constant has attracted the attention of many authors [1, 3, 6, 12, 16, 17, 18, 19, 20]. For holomorphic functions of several complex variables, Landau-Bloch constant does not exist (cf. [11, 22]) unless one considers the class of functions under certain constraints, see the works of Fitzgerald and Gong [8], Graham and Varolin [9], Liu [15], and Chen and Gauthier [5]. The existence of the Landau-Bloch constant for the class of holomorphic quasiregular mappings and their related classes were investigated by Bochner [2], Hahn [10], Harris [11], Takahashi [21] and Wu [22].

In this paper, we obtain a sharp distortion theorem (see Theorem 1.2) for a class of holomorphic functions in $\alpha$-Bloch spaces. As an application of Theorem 1.2, we establish the Landau theorem for this class (see Theorem 1.3). Theorems 1.2 and 1.3 were proved under certain assumptions, but the proofs are similar to those in [5].
1.3 generalize the corresponding results of Bonk, et. al [4] and Liu [15], respectively. In Theorem 1.4 we present the existence of the Landau-Bloch constant for a well-known class of holomorphic functions in $H^p$ spaces (cf. [7, 25, 26]). Moreover Theorem 1.4 is a generalization of the corresponding result of Chen, et. al. [5].

In order to state our results, we need to introduce some basic notations. Throughout the discussion, for $b \in \mathbb{C}$, we let $D(b, r) = \{ z \in \mathbb{C} : |z - b| < r \}$. We use $D$ to denote the open unit disk $D(0, 1)$. Also we let $\mathbb{C}^n = \{ z = (z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C} \}$, and for $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$,

$$B_n(w, r) = \left\{ z \in \mathbb{C}^n : |z - w| = \sqrt{n \sum_{k=1}^{n} |z_k - w_k|^2} < r \right\}$$

and $B^n = B^n(0, 1)$. The class of all holomorphic functions from $B^n$ into $\mathbb{C}^n$ is denoted by $\mathcal{H}(B^n)$ (cf. [26]). Here and in the following, we always treat $z \in \mathbb{C}^n$ as a column vector, that is, $n \times 1$ column matrix

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$ 

Thus, for any $f = (f_1, \ldots, f_n) \in \mathcal{H}(B^n)$, we denote by $\partial f / \partial z_k$ the column vector formed by $\partial f_1 / \partial z_k, \ldots, \partial f_n / \partial z_k$ and by

$$f' = \left( \frac{\partial f_1}{\partial z_1}, \ldots, \frac{\partial f_n}{\partial z_n} \right) := \left( \frac{\partial f_i}{\partial z_j} \right)_{n \times n}$$

we mean the matrix formed by these column vectors, namely the $n \times n$-matrix with $(i, j)$-th entry as $\partial f_i / \partial z_j$.

For an $n \times n$ matrix $A$, the operator norm is defined by

$$|A| = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \max \{|A\theta| : \theta \in \partial B^n\}.$$

Let $B_\alpha$ denote the $\alpha$-Bloch space which consists of all $f \in \mathcal{H}(B^n)$ such that (see [24])

$$\|f\|_\alpha + |f(0)| < \infty,$$

where $\alpha > 0$ and $\|f\|_\alpha$ denotes the $\alpha$-Bloch semi-norm of $f$ defined by

$$\|f\|_\alpha = \sup_{z \in B^n} (1 - |z|^2)^\alpha |f'(z)|.$$

Recently, many authors investigated the properties of $\alpha$-Bloch functions in $B^n$, see [13, 14, 23, 25, 26].

The following lemma is easy to derive and so we omit its proof.

Lemma 1.1. For $x \in [0, 1]$, let

$$\varphi(x) = x(1 - x^2) \sqrt{\frac{\alpha(n+1)}{2} \sqrt{\alpha(1+n) + 1}} \left( \frac{\alpha(n+1) + 1}{\alpha(n+1)} \right)^{\frac{\alpha(n+1)}{2}}$$
Theorem 1.3. Suppose that $f$ is schlicht ball of $\alpha$-Bloch spaces and Hardy spaces

and

$$a_0 = \frac{1}{\sqrt{a_0(1 + n) + 1}}.$$

Then $\varphi$ is increasing in $[0, a_0]$, decreasing in $[a_0, 1]$ and $\varphi(a_0) = 1$.

As in [15], the semi-norm $\|f\|_{0, \alpha}$ of $f \in \mathcal{H}(\mathbb{B}^n)$ is given by

$$\|f\|_{0, \alpha} = \sup_{z \in \mathbb{B}^n} \{(1 - |z|^2)^{-\frac{m(\lambda)}{2}} |\det f'(z)|^{\frac{1}{n}}\}.$$

We now state our main results and their proofs will be given in Section 3.

**Theorem 1.2.** Suppose that $f \in \mathcal{H}(\mathbb{B}^n)$ such that $\|f\|_{0, \alpha} = 1$ and $\det f'(0) = \lambda \in (0, 1)$. Then for any $z$ with $|z| \leq \frac{a_0 + m(\lambda)}{1 + a_0 m(\lambda)}$, we have

$$|\det f'(z)| \geq \Re(\det f'(z)) \geq \frac{\lambda(m(\lambda) - |z|)}{m(\lambda)(1 - m(\lambda)|z|)^{\alpha(n+1)+1}},$$

where $m(\lambda)$ is the unique real root of the equation $\varphi(x) = \lambda$ in the interval $[0, a_0]$ and, $\varphi$ and $a_0$ are defined as in Lemma 1.1.

Moreover, for any $z$ with $|z| \leq \frac{a_0 - m(\lambda)}{1 - a_0 m(\lambda)}$, we have

$$|\det f'(z)| \leq \frac{\lambda(m(\lambda) + |z|)}{m(\lambda)(1 + m(\lambda)|z|)^{\alpha(n+1)+1}}.$$

The estimates (1.1) and (1.2) are sharp.

We remark that when $\lambda = 1$ and $\alpha = 1$, the inequality (1.1) coincides with [13, Theorem 5]. In particular, when $\alpha = n = 1$, Theorem 1.2 coincides with [4, Theorem 2].

For $f \in \mathcal{H}(\mathbb{B}^n)$, a schlicht ball of $f$ centered at $f(w_1)$ is a ball with center $f(w_1)$ such that $f$ maps an open subset of $\mathbb{B}^n$ containing $w_1$ biholomorphically onto this ball. For a point $w_1 \in \mathbb{B}^n$, let $r(w_1, f)$ denote the radius of the largest schlicht ball of $f$ centered at $f(w_1)$.

**Theorem 1.3.** Suppose that $f \in \mathcal{B}_n$ such that $\|f\|_{\alpha} \leq K$ for some constant $K \geq 1$, $\|f\|_{0, \alpha} = 1$ and $\det f'(0) = \lambda \in (0, 1)$. Then $f(\mathbb{B}^n)$ contains a schlicht ball of radius

$$r(0, f) \geq \frac{\lambda K^{1-n}}{m(\lambda)} \int_0^{m(\lambda)} \frac{(1 - t^2)^{\alpha(n-1)} (m(\lambda) - t)}{(1 - m(\lambda)t)^{\alpha(n+1)+1}} dt,$$

where $m(\lambda)$ is the same as in Theorem 1.2.

We remark that when $\lambda = \alpha = 1$, Theorem 1.3 coincides with [15, Theorem 6].

Finally, for $0 < p < +\infty$, the Hardy space $H^p$ consists of $f \in \mathcal{H}(\mathbb{B}^n)$ such that

$$\|f\|_p = \sup_{0 < r < 1} \left\{ \int_{\partial \mathbb{B}^n} |f(r\zeta)|^p \, d\sigma(\zeta) \right\}^{\frac{1}{p}} < +\infty,$$

where $d\sigma$ denotes the normalized surface measure.
Now we state our final result which gives the existence of Landau-Bloch constant for a class of holomorphic functions in Hardy spaces.

**Theorem 1.4.** Suppose that \( f \in H^p \) satisfies \( \|f\|_p \leq K_0 \) for some constant \( K_0 > 0 \), \( f(0) = 0 \) and \( |\det f'(0)| = \lambda_0 > 0 \), where \( K_0 \geq \lambda_0 \). Then \( f \) is univalent in \( \mathbb{B}^n(0, \rho_1(r_0)) \) with

\[
\rho_1(r_0) = \max_{0 < r < 1} \rho_1(r),
\]

where

\[
\rho_1(r) = \frac{\lambda_0}{mK_0^n}\left[p^{n+1}(1 - r^2)^{n^2}\right], \quad r_0 = \sqrt{\frac{p(n+1)}{p(n+1)+2n^2}}
\]

and

\[
m = \frac{\sqrt{2(7 + \sqrt{17})}}{4\sqrt{5} - \sqrt{17}} \approx 4.19956.
\]

Moreover, the range \( f(\mathbb{B}^n(0, \rho_1)) \) contains a univalent ball \( \mathbb{B}^n(0, R) \) with

\[
R = \frac{\lambda_0^2}{2mK_0^{2n-1}}\sqrt{\frac{p(n+1)}{p(n+1)+2n^2}} \left(\frac{2n^2}{p(n+1)+2n^2}\right)^{\frac{2n^2-n}{p(n+1)}}.
\]

We remark that Theorem 1.4 is a generalization of [5, Theorem 2].

**2. Preliminaries and Lemmas**

We begin to recall some basic facts about the hyperbolic geometry in \( \mathbb{D} \). Let

\[
\rho(z, w) = \frac{1}{2} \log \left( \frac{1 + \frac{z-w}{1 - \overline{z}w}}{1 - \frac{z-w}{1 - \overline{z}w}} \right) = \arctanh \left( \frac{|z-w|}{1-\overline{z}w} \right)
\]

denote the hyperbolic distance between \( z \) and \( w \) in \( \mathbb{D} \). Throughout this article, we denote the hyperbolic disk (resp. circle) by \( \mathbb{D}_h(b, r) = \{ z : \rho(b, z) < r \} \) (resp. \( \mathbb{S}_h(b, r) = \{ z : \rho(b, z) = r \} \)). Hyperbolic disks and circles in \( \mathbb{D} \) are actually Euclidean disks and circles, respectively, with possibly different centers and radii. For \( b \) and \( z \in \mathbb{D} \), we can easily see that

\[
\rho(b, z) = r \iff \left| \frac{z-b}{1-\overline{b}z} \right| = \tanh(r) \iff \frac{|1-\overline{b}z|^2}{1-|z|^2} = \frac{1-|b|^2}{1-\tanh^2(r)}. \tag{2.1}
\]

For the proof of our main results we need several lemmas.

**Lemma A.** ([13 Lemma 4]) Let \( A \) be an \( n \times n \) complex matrix with \( |A| > 0 \). Then for any unit vector \( \theta \in \partial \mathbb{B}^n \), the inequality

\[
|A\theta| \geq \frac{|\det A|}{|A|^{n-1}}
\]

holds.
Lemma B. ([5, Lemma 3]) Let \( f \) be a holomorphic function of \( \mathbb{B}^n \) into \( \mathbb{C}^n \) such that \( |f(z)| \leq M \) for \( z \in \mathbb{B}^n \), where \( M \) is a positive constant. Then
\[
|f'(z)| \leq \frac{M}{1-|z|^2} \quad \text{and} \quad |\det f'(z)| \leq \frac{M^n}{(1-|z|^2)^{\frac{n}{2}}}
\]

Lemma C. ([5, Lemma 4]) Let \( A \) be a holomorphic function of \( \mathbb{B}^n(0, r) \) into the space of all \( n \times n \) complex matrices. If \( A(0) = 0 \) and \( |A(z)| \leq M \) for \( z \in \mathbb{B}^n(0, r) \), then
\[
|A(z)| \leq \frac{M}{r} |z|.
\]

Definition 2.1. Let \( \Phi \) and \( \Psi \) be holomorphic in \( D \) with \( \Phi(0) = \Psi(0) \) and \( D^* \) be an open disk in \( D \) with \( 0 \in D^* \). We say \( \Phi \) is subordinate to \( \Psi \) in \( D^* \) relative to the origin, written \( \Phi \prec \Psi \), if there is a holomorphic function \( \omega \) defined in \( D^* \) with \( \omega(D^*) \subset D^* \), \( \omega(0) = 0 \) and \( \Psi \circ \omega = \Phi \). If \( D_h \) is any hyperbolic disk (relative to hyperbolic geometry in \( D^* \)) with center 0, then \( \Phi \prec \Psi \) implies \( \Phi(D_h) \subset \Psi(D_h) \) since the function \( \omega \) must map \( D_h \) into itself (see [4, page 248]).

3. Proofs

Proof of Theorem 1.2. For the proof of the inequality (1.1), we denote the inverse of the restriction of \( \phi \) to the interval \([0, a_0]\) by \( m : [0, 1] \to [0, a_0] \), where \( \phi \) is the same as in Lemma 1.1. It is not difficult to see that \( m \) is increasing with \( m(0) = 0 \), \( m(1) = a_0 \) and there is an unique \( m(\lambda) \in [0, a_0] \) such that \( \phi(m(\lambda)) = \lambda \).

For convenience, we set \( a = m(\lambda) \) and for a fixed \( \zeta \in \partial \mathbb{B}^n \), we define a function \( T \) on \( D \) by
\[
T(u) = (1 - au)^{\alpha(1+n)} \det f'(\zeta u), \quad u \in D.
\]

Then \( T \) is a holomorphic function in \( D \) and \( T(0) = \lambda \). Now, for each \( u \in S_h(a, \arctanh a_0) \), we have
\[
|T(u)| = \frac{|1 - au|^\alpha(n+1)}{(1 - |u|^2)^{\frac{\alpha(n+1)}{2}}} |\det f'(\zeta u)|
\]
\[
\leq \frac{|1 - au|^\alpha(n+1)}{(1 - |u|^2)^{\frac{\alpha(n+1)}{2}}}
\]
\[
= \left( \frac{1 - a^2}{1 - \frac{1}{(1 - |u|^2)^\alpha}} \right)^{\frac{\alpha(n+1)}{2}}
\]
\[
= (1 - a^2)^{\frac{\alpha(n+1)}{2}} \left[ \frac{\alpha(n+1) + 1}{\alpha(n+1)} \right]^{\frac{\alpha(n+1)}{2}},
\]
which implies that \( T \) maps \( \mathbb{D}_h(a, \text{arctanh } a_0) \) into
\[
\left\{ z : |z| < (1 - a^2)^{\frac{\alpha(n+1)}{2}} \left[\frac{\alpha(n+1) + 1}{\alpha(n+1)}\right]^{\frac{\alpha(n+1)}{2}} \right\}.
\]

Next, we define
\[
G_\lambda(u) = \frac{\lambda(a - u)}{a(1 - au)}.
\]

By (2.1), we observe that
\[
\rho(a, u) = \text{arctanh } a_0 \iff \frac{\lambda}{a} \left|\frac{u - a}{1 - au}\right| = \frac{a_0 \lambda}{a}.
\]

On the other hand, \( \varphi(a) = \lambda \) implies
\[
(1 - a^2)^{\frac{\alpha(n+1)}{2}} \left[\frac{\alpha(n+1) + 1}{\alpha(n+1)}\right]^{\frac{\alpha(n+1)}{2}} = \frac{\lambda}{a \sqrt{\alpha(n+1) + 1}}.
\]

These observations together with (3.1) and (3.2) show that \( G_\lambda \) is a Möbius transformation which is univalent and maps
\[
\mathbb{D}_h(a, \text{arctanh } a_0)
\]
on to the closed disk
\[
\mathbb{D}(0, r_0),
\]
where
\[
r_0 = \frac{\lambda a_0}{a} = \frac{\lambda}{a \sqrt{\alpha(n+1) + 1}}.
\]

Also, \( T(0) = \lambda = G_\lambda(0) \) and we obtain that \( T \prec G_\lambda \) on \( \mathbb{D}_h(a, \text{arctanh } a_0) \), where \( \prec \) denotes the subordination (see Definition 2.1).

Now, we fix \( u \in \left(0, \frac{a_0 + a}{1 + a_0 a}\right] \) and let \( \delta_u = \{ z : |z| = u \} \). Obviously,
\[
\delta_u \subset \mathbb{D}_h(a, \text{arctanh } a_0).
\]

Since \( T \prec G_\lambda \) on \( \mathbb{D}_h(a, \text{arctanh } a_0) \), \( T \) maps the circle \( \delta_u \) into the closed disk bounded by the circle \( G_\lambda(\delta_u) \). We find that \( G_\lambda(\delta_u) \) is a hyperbolic circle with hyperbolic center \( G_\lambda(0) = \lambda \) and symmetric about the real axis \( \mathbb{R} \). Also, it is easy to see that \( G_\lambda \) is decreasing on \( \mathbb{D}_h(a, \text{arctanh } a_0) \cap \mathbb{R} \). Hence, \( G_\lambda(u) \) satisfies
\[
G_\lambda(u) = \min \{ \Re(G_\lambda(z)) : z \in \delta_u \},
\]
which in turn implies that
\[
\Re T(u) \geq \min \{|G_\lambda(z)| : z \in \delta_u\} = G_\lambda(u)
\]
whence
\[
|\det f'(\zeta u)| \geq \Re(\det f'(\zeta u)) \geq \frac{\lambda(a - |u|)}{a(1 - a|u|)^{\alpha(n+1)+1}}.
\]

For each \( z \) with \( |z| \leq \frac{a_0 + a}{1 + a_0 a} \), by taking \( u = |z| \) and \( \zeta = z/|z| \), we have
\[
|\det f'(z)| \geq \Re(\det f'(z)) \geq \frac{\lambda(a - |z|)}{a(1 - a|z|)^{\alpha(n+1)+1}}.
\]

The proof of (1.1) is finished.
Now we shall prove (1.2). For each \( y \in \left(0, a_0 - a \right) \), we set
\[
\delta_{-y} = \{ z : |z| = y \}.
\]
Then
\[
\delta_{-y} \subset D_h(a, \text{arctanh} a_0).
\]
Since \( T \prec G \) on \( D_h(a, \text{arctanh} a_0) \), we see that \( T \) maps the circle \( \delta_{-y} \) into the closed disk bounded by the circle \( G_\lambda(\delta_{-y}) \). It is not difficult to see that \( G_\lambda(\delta_{-y}) \) is a hyperbolic circle with hyperbolic center \( G_\lambda(0) = \lambda \) and symmetric about the real axis \( \mathbb{R} \). We see that \( G_\lambda \) is decreasing on \( D_h(a, \text{arctanh} a_0) \cap \mathbb{R} \). Then
\[
G_\lambda(-y) = \max\{|G_\lambda(u)| : u \in \delta_{-y}\},
\]
which yields
\[
|\det f'(-y\zeta)| \leq \frac{\lambda(a + y)}{a(1 + ay)^{\alpha(n + 1) + 1}}.
\]
For each \( z \) with \( |z| \leq \frac{a_0 - a}{1 - a_0 a} \), by taking \( y = |z| \) and \( \zeta = -z/|z| \), we conclude that
\[
|\det f'(z)| \leq \frac{\lambda(a + |z|)}{a(1 + a|z|)^{\alpha(n + 1) + 1}}.
\]
The proof of (1.2) is finished.

Moreover, the estimates (1.1) and (1.2) are sharp. The extremal function is
\[
f(z) = \left( \int_0^{z_1} \frac{\lambda(a - \xi)}{a(1 - a\xi)^{\alpha(n + 1) + 1}} d\xi \right).
\]
The proof of this theorem is complete. \( \square \)

**Proof of Theorem 1.3.** Let \( r(0, f) \) be the supremum of the positive numbers \( r \) such that there exists a domain \( \Omega \subset \mathbb{B}^n \) containing 0 which is mapped biholomorphically onto the ball \( \mathbb{B}^n(0, r) \). Then, since \( f \) is locally biholomorphic, it follows from the monodromy theorem that there is a point \( w_0 \) on the boundary of the ball \( \mathbb{B}^n(0, r(0, f)) \) such that the arc \( \gamma = f^{-1}[0, w_0] \) originates from the origin and tends to \( \partial \mathbb{B}^n \) or to a point \( z_0 \in \mathbb{B}^n \) with \( |z_0| \geq m(\lambda) \) and \( \det f'(z_0) = 0 \) as \( w \to w_0 \). Let \( \Gamma = [0, w_0] \) denote the segment from 0 to \( w_0 \). Because \( \|f\|_\alpha \) is finite by hypothesis, it follows from the definition of \( B_\alpha \) that
\[
|f'(z)| \leq \frac{\|f\|_\alpha}{(1 - |z|^2)^\alpha} \text{ for } z \in \mathbb{B}^n.
\]
Therefore, by Lemma A and (1.1), we obtain that
\[ r(0, f) = \int_\gamma |dw| = \int_\gamma |f'(z) \frac{dz}{|dz|}| \cdot |dz| \]
\[ \geq \int_\gamma |\operatorname{det} f'(z)| |dz| \]
\[ \geq \frac{\lambda K^{1-n}}{m(\lambda)} \int_0^{m(\lambda)} \frac{(1-t^2)^{\alpha(n-1)}(m(\lambda) - t)}{(1-m(\lambda)t)^{\alpha(n+1)+1}} \, dt \]
and the proof of this theorem is complete. \( \square \)

**Proof of Theorem 1.3.** For a fixed \( r \in (0, 1) \), let \( F(z) = r^{-1}f(rz) \) for \( z \in \mathbb{B}^n \). For \( \rho^* \in (0, 1) \), it is easy to see that if \( F \) is univalent in \( \mathbb{B}^n(0, \rho^*) \), then \( f \) is univalent in \( \mathbb{B}^n(0, r\rho^*) \). On the other hand, for \( R^* \in (0, 1) \), if \( F(\mathbb{B}^n(0, \rho^*)) \) contains an univalent ball \( \mathbb{B}^n(0, R^*) \), then \( f(\mathbb{B}^n(0, r\rho^*)) \) contains an univalent ball \( \mathbb{B}^n(0, rR^*) \).

Now, we begin to prove the univalency of \( F \) in \( \mathbb{B}^n(0, \rho_0(r)) \). By [29] Theorem 4.17, we find that
\[ |F(z)| \leq \frac{\|f\|_p}{r(1-r^2)^{\alpha/n}} \leq M_0(r), \]
where \( M_0(r) = K_0/(r(1-r^2)^{\alpha/n}) \). For each \( z \in \mathbb{B}^n \), Lemma B yields that
\[ |F'(z) - F'(0)| \leq M_0(\lambda) \left( 1 + \frac{1}{1-|z|^2} \right) = M_0(r)(2 - |z|^2) \]
\[ = \frac{M_0(r)(2 - |z|^2)}{1 - |z|^2}. \]
Let \( W_1(r) = (2 - r^2)/(r(1-r^2)) \) for \( r \in (0, 1) \). It is easy to see that
\[ W_1(r_1) = \min_{r \in (0,1)} \{ W_1(r) \}, \]
where \( r_1 = \sqrt{\frac{5 - \sqrt{17}}{2}} \approx 0.662 \). We denote \( W_1(r_1) \) by \( m. \) Then, we can easily verify that
\[ m := W_1(r_1) = \frac{\sqrt{2}(7 + \sqrt{17})}{4\sqrt{5 - \sqrt{17}}} \approx 4.199556. \]
For \( |z| \leq r_1 \), by Lemma C, we have
\[ |F'(z) - F'(0)| \leq mM_0(r)|z|. \]
On the other hand, for any \( \theta \in \partial \mathbb{B}^n \), we infer from Lemma A that
\[ |F'(0)\theta| \geq \frac{\lambda_0}{|F'(0)|^{n-1}} \geq \frac{\lambda_0}{M_0^{n-1}(r)} \]
In order to prove the univalency of \( F \) in \( \mathbb{B}^n(0, \rho_0(r)) \) with \( \rho_0(r) = \frac{\lambda_0}{mM_0^{n-1}(r)} < r_1 \), we choose two distinct points \( z', z'' \in \mathbb{B}^n(0, \rho_0(r)) \) and let \( [z', z''] \) denote the segment from \( z' \) to \( z'' \) with the endpoints \( z' \) and \( z'' \). Set
\[ dz = \begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix} \text{ and } d\zeta = \begin{pmatrix} d\zeta_1 \\ \vdots \\ d\zeta_n \end{pmatrix}. \]
Then we have

\[ |F(z') - F(z'')| \geq \left| \int_{[z', z'']} F'(0) \, dz \right| - \left| \int_{[z', z'']} (F'(z) - F'(0)) \, dz \right| \]

\[ > |z' - z''| \left\{ \frac{\lambda_0}{M_0^{n-1}(r)} - mM_0(r)\rho_0(r) \right\} \]

\[ = 0 \]

which shows that \( F \) is univalent in \( \mathbb{B}^n(0, \rho_0(r)) \). Then \( f \) is univalent in \( \mathbb{B}^n(0, \rho_1(r)) \), where \( \rho_1(r) = r\rho_0(r) \).

By calculations, we have

\[ \max_{0 < r < 1} \rho_1(r) = \frac{\lambda_0}{mK_0} \max_{0 < r < 1} \left[ r^{n+1}(1 - r^2) \frac{2^2}{p} \right] = \rho_1(r_0), \]

where

\[ r_0 = \sqrt{\frac{p(n + 1)}{p(n + 1) + 2n^2}}. \]

Now, for \( z \) with \( |z| = \rho_0(r_0) = \frac{\lambda_0}{mK_0(r_0)} \), we have

\[ |F(z) - F(0)| \geq \left| \int_{[0, \rho_0(r_0)]} F'(0) \, dz \right| - \left| \int_{[0, \rho_0(r_0)]} (F'(z) - F'(0)) \, dz \right| \]

\[ \geq \frac{\lambda_0\rho_0(r_0)}{M_0^{n-1}(r_0)} - mM_0(r_0) \int_0^{\rho_0(r_0)} \rho d\rho \]

\[ = \rho_0(r_0) \left\{ \frac{\lambda_0}{M_0^{n-1}(r_0)} - \frac{mM_0(r_0)\rho_0(r_0)}{2} \right\} \]

\[ = \frac{\lambda_0^2}{2mM_0^{2n-1}(r_0)}. \]

Hence \( F(\mathbb{B}^n(0, \rho_0(r_0))) \) contains a univalent ball \( \mathbb{B}^n(0, R_0) \) with

\[ R_0 = \frac{\lambda_0^2}{2mM_0^{2n-1}(r_0)}, \]

which implies \( f(\mathbb{B}^n(0, \rho_1(r_0))) \) contains a univalent ball \( \mathbb{B}^n(0, R) \) with

\[ R = rR_0 = \frac{\lambda_0^2}{2mK_0^{2n-1}} \sqrt{\frac{p(n+1)}{p(n+1)+2n^2} \left( \frac{2n^2}{p(n+1)+2n^2} \right)^{\frac{2n^2-n}{p}}}. \]

\[
\text{References}
\]

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