Onsager’s algebra and partially orthogonal polynomials

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Abstract:
The energy eigenvalues of the superintegrable chiral Potts model are determined by the zeros of special polynomials which define finite representations of Onsager’s algebra. The polynomials determining the low-sector eigenvalues have been given by Baxter in 1988. In the $Z_3$-case they satisfy 4-term recursion relations and so cannot form orthogonal sequences. However, we show that they are closely related to Jacobi polynomials and satisfy a special ”partial orthogonality” with respect to a Jacobi weight function.

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1 Introduction

F.Y.Wu and Y.K.Wang \[1\] were the first to consider the Potts model with chiral interaction terms. Their interest in this generalization arose from duality considerations, but the idea proved to be very fruitful in many respects: Ostlund \[2\] and Huse \[3\] proposed the chiral Potts (CP) model for phenomenological applications: it allows to describe incommensurate phases using nearest neighbor interactions only. We give a few references \[4, 5, 6, 7\] from which the subsequent development can be traced, and turn directly to the superintegrable chiral $Z_N$ Potts quantum chain \[8\]. This is a particularly interesting model, because it provides some of the rare representations known for Onsager’s algebra \[9\] and in this sense generalizes the Ising quantum chain (for $N = 2$ it is the Ising model). Integrability by Onsager’s algebra entails that all eigenvalues of the hamiltonian are determined by the zeros of certain polynomials, which for the chiral Potts model were first derived by Baxter \[10\]. Although the definition of Baxter’s polynomials looks very simple, the properties of these polynomials turn out to be quite non-trivial and interesting \[11\]. The main part of this note deals with the properties of these polynomials. They satisfy $N + 1$-term recursion relations, therefore for $N > 2$ they cannot form orthogonal sequences. However, as found recently \[11\], several properties which characterize orthogonal polynomials are almost true for Baxter’s polynomials (e.g. the zero separation property is true except for one extreme zero).

We first recall the definitions of the superintegrable CP-hamiltonian and Onsager’s algebra and then, following B.Davies \[12\], we sketch how the formula for the energy eigenvalues emerges. We consider Baxter’s polynomials and their recursion relations. Equivalent polynomials with their zeros in $(-1, +1)$ for $N = 3$ are written in terms of a determinant. Their
expansion in terms of Jacobi polynomials gives the surprising result that many of the expansion coefficients vanish, leading to the notion of "partial orthogonality".

The Hamiltonian defining the $Z_N$-superintegrable chiral Potts quantum chain $[4,8]$ is:

$$\mathcal{H} = -\sum_{j=1}^{N} \sum_{l=1}^{N-1} \frac{2}{1 - \omega^{-l}} (X_j^l + k Z_j^l Z_j^{N-l}).$$

(1)

Here $\omega = e^{2\pi i/N}$ and $Z_j$ and $X_j$ are $Z_N$-spin operators acting in the vector spaces $\mathbb{C}^N$ at the sites $j = 1, 2, \ldots, L$ ($L$ is the chain length). The operators obey $Z_j X_j = X_j Z_j \omega^{k_j}; \quad Z_j^N = X_j^N = 1$ and we assume $X_{L+1} = X_1$ (periodic b.c). A convenient representation is $(X_j)_{l,m} = \delta_{l,m+1}$ mod $N$ and $(Z_j)_{l,m} = \delta_{l,m} \omega^m$. For $N \geq 3$ the complex coefficients make the chain Hamiltonian parity non-invariant. For $N = 2$ we get the Ising quantum chain. For fixed $N$ there is only one parameter, the temperature variable $k$. Incommensurate phases arise due to ground state level crossings. $\mathcal{H}$ commutes with the $Z_N$-charge $\hat{Q} = \prod_{j=1}^{L} Z_j$. We write the eigenvalues of $\hat{Q}$ as $\omega^Q$. $Q = 0, 1, \ldots, N-1$ labels the charge sectors of $\mathcal{H}$.

We split $\mathcal{H}$ into two operators writing $\mathcal{H} = -\frac{1}{N} N (A_0 + k A_1)$. A remarkable property of $\mathcal{H}$ is that $A_0$ and $A_1$ satisfy $[8]$ the Dolan-Grady $[13]$ relations

$$[A_0, [A_0, [A_0, A_1]]] = 16 [A_0, A_1]; \quad [A_1, [A_1, [A_1, A_0]]] = 16 [A_1, A_0],$$

which are the conditions $[14]$ for $A_0$ and $A_1$ to generate Onsager’s algebra $A$, which is formed $[1]$ from elements $A_m, G_l, m \in \mathbb{Z}, l \in \mathbb{N}$, $l \geq m$, satisfying

$$[A_1, A_m] = 4 G_{l-m}; \quad [G_l, A_m] = 2 A_{m+l} - 2 A_{m-l}; \quad [G_l, G_m] = 0.$$

(2)

From $[2]$, there is a set of commuting operators which includes $\mathcal{H}$:

$$Q_m = \frac{1}{2} (A_m + A_{-m} + k (A_{m+1} + A_{-m+1})); \quad [Q_l, Q_m] = 0; \quad Q_0 = \mathcal{H}.$$  

To obtain finite dimensional representations of $A$ we require the $A_m$ (and analogously the $G_l$ $[12,13]$) to satisfy a finite difference equation: $\sum_{k=-n}^{n} \alpha_k A_{k-l} = 0$. This is solved introducing the polynomial (the main object of the present paper):

$$F(z) = \sum_{k=-n}^{n} \alpha_k z^{k+n}$$

(3)

(from $A$ the $\alpha_k$ are either even or odd in $k$). Now the $A_m$ and $G_m$ can be expressed in terms of the zeros $z_j$ of $F(z)$ and the set of operators $E_j^\pm, H_j$:

$$A_m = 2 \sum_{j=1}^{n} (z_j^m E_j^+ + z_j^{-m} E_j^-); \quad G_m = \sum_{j=1}^{n} (z_j^m - z_j^{-m}) H_j.$$  

(4)

From $A$ these operators obey $sl(2,C)$-commutation rules:

$$[E_j^+, E_j^-] = \delta_{jk} H_k; \quad [H_j, E_k^\pm] = \pm2 \delta_{jk} E_k^\pm.$$
So $\mathcal{A}$ is isomorphic to a subalgebra of the loop algebra of a sum of $sl(2,C)$ algebras. From the first of eqs.\(1\) we can express $\mathcal{H}$ in terms of the $z_j$ and the operators $E_j^\pm$. Writing $E_j^\pm = J_{x,j} \pm i J_{y,j}$, then in a representation $Z(n,s)$ characterized by the polynomial zeros $z_1, \ldots, z_n$ and a spin-s representation $J_j^{(s)}$ of all the $J_j$, we get:

\[
\begin{align*}
(A_0 + kA_1)_{Z(n,s)} &= 2 \sum_{j=1}^{n} \left\{ (2 + k(z_j + z_j^{-1})) J_{x,j}^{(s)} + i(z_j - z_j^{-1}) J_{y,j}^{(s)} \right\} \\
&= 4 \sum_{j=1}^{n} \sqrt{1 + 2k c_j + k^2} J_{x,j}^{(s)} 
\end{align*}
\]

where $J_{x,j}^\prime$ is a rotated $SU(2)$-operator, and $c_j = \cos \theta_j = \frac{1}{2}(z_j + z_j^{-1})$.

For the CP-hamiltonians \(2\) the spin representation turns out to be $s = \frac{1}{2}$. Accordingly, all eigenvalues of \(2\) have the form

\[
E = -N \left( a + b k + 2 \sum_{j=1}^{n} m_j \sqrt{1 + 2k \cos \theta_j + k^2} \right), \quad m_j = \pm \frac{1}{2}.
\]

\(a\) and \(b\) are non-zero if the trace of $A_0$ and $A_1$ is non-zero.

2 Baxter’s polynomials

No direct way is known to find the polynomials $\mathcal{F}(z)$ from the hamiltonian $\mathcal{H}$. However, the invention of the two-dimensional integrable CP model \(4\), which contains $\mathcal{H}$ as a special logarithmic derivative, and functional relations for its transfer matrix have enabled Baxter \(4\) to obtain the polynomials for the simplest sector of $\mathcal{H}$, which at high-temperatures contains the ground state (the polynomials corresponding to all other sectors have been obtained subsequently in \(4,13,19\)). Here we shall consider only the simplest case. Baxter \(4\) finds that in terms of the variable $t$ or $s \equiv t^N = (c - 1)/(c + 1)$ (recall $c = \cos \theta$ of \(4\)) these polynomials take the form

\[
P_Q^{(L)}(s) = \frac{1}{N} \sum_{j=0}^{N-1} \left( \frac{1 - t^N}{1 - \omega^j t} \right)^L (\omega^j t)^{-\sigma_{Q,L}}; \quad \sigma_{Q,L} = (N-1)(L + Q) \mod N.
\]

Here $Q$ denotes the $\mathbb{Z}_N$-charge sector. For $\mathbb{Z}_3$ eq.\(4\) is written more explicitly:

\[
P_Q^{(L)}(s) = \frac{t^{-\sigma_{Q,L}}}{3} \left\{ ((t^2 + t + 1)^{L} + \omega^{Q}(t^2 + \omega^2 t + \omega)^{L} + \omega^{-Q}(t^2 + \omega t + \omega^2)^{L} \right\}.
\]

Due to their $\mathbb{Z}_N$-invariance $t \to \omega t$, the $P_Q^{(L)}$ depend only on $s = t^N$. The degree of the $P_Q^{(L)}(s)$ in the variable $s$ is $b_{L,Q} = [(N-1)L - Q]/N$ where $[x]$ denotes the integer part of $x$. Considering sequences of these polynomials for fixed $Q$ and $L \in \mathbb{N}$, we notice that the dimensions $b_{L,Q}$ do not always increase by one when increasing $L$ by one: at every $N$th step the dimension stays the same: e.g. the dimensions of the $P_Q^{(L)}$ for $L \mod N = 0$ and
We start with the observation that the coefficients of the expansion of \((1 + t + t^2 + \ldots + t^{N-1})^L\) contain a lot of information, we now show how to obtain these.

Our main concern in this paper is to learn about the properties of the \(\Pi^{(L)}(c)\) or \(P^{(L)}(s)\), e.g. whether these can be arranged into orthogonal sequences etc. We will find that the \(\Pi^{(L)}(c)\) are polynomials with quite remarkable properties. A number of special features of the \(\Pi^{(L)}\) have been discussed recently [11]. Here we give some more detailed results for the \(\mathbb{Z}_3\)-case. As the recursion relations for the \(\Pi^{(L)}\) contain a lot of information, we now show how to obtain these.

### 3 Recursion relations

We start with the observation that the coefficients of the \(P^{(L)}(s)\) can be obtained from the expansion of \((1 + t + t^2 + \ldots + t^{N-1})^L\), simply by taking every \(N\)-th term of the expansion, starting with the coefficient of \(t^{(N-1)L-Q}\). More precisely, we claim that we can define the \(P^{(L)}\) by the decomposition

\[
(1 + t + t^2 + \ldots + t^{N-1})^L = \left(\frac{1 - t^N}{1 - t}\right)^L = \sum_{Q=0}^{N-1} t^{\sigma_{L,Q}} P^{(L)}(s)
\]

demanding the \(P^{(L)}\) to depend on \(s = t^N\) only. Proof: Insert (3) into (3) to get:

\[
\left(\frac{1 - t^N}{1 - t}\right)^L = (1 - t^N)^L \frac{1}{N} \sum_{Q=0}^{N-1} \sum_{j=0}^{N-1} \frac{\omega^j \omega^{(L+Q)}}{(1 - \omega^j t)^L}.
\]

Interchanging the \(Q\)- and \(j\)-summations we see that the \(Q\)-summation gives zero for \(j \neq 0\), leaving only the \(j = 0\) term, which is (8).

Eq. (8) can now be used to obtain recursion relations: Write

\[
(1 + t + t^2 + \ldots + t^{N-1}) \sum_{Q=0}^{N-1} t^{\sigma_{L,Q}} P_{Q}^{(L)}(s) = \sum_{Q=0}^{N-1} t^{\sigma_{L+1,Q}} P_{Q}^{(L+1)}(s).
\]

Comparing powers of \(t\), e.g. for \(L \mod N = 0\) this gives

\[
\begin{pmatrix}
P_0^{(L+1)} \\
P_1^{(L+1)} \\
P_2^{(L+1)} \\
\vdots \\
P_{N-1}^{(L+1)}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & s & 1 & \ldots & 1 \\
1 & s & s & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & s & s & \ldots & s
\end{pmatrix}
\begin{pmatrix}
P_0^{(L)} \\
P_1^{(L)} \\
P_2^{(L)} \\
\vdots \\
P_{N-1}^{(L)}
\end{pmatrix},
\]

(10)
For } L \mod N = k \text{ replace } P_Q \to P_{Q-k} \text{ cyclically in both column vectors, keeping the same square matrix. Recursion relations not coupling } P_Q \text{ with different } Q \text{ follow by the } N \text{-fold application of these relations, leading to } N + 1 \text{-term recursion formulae. These can be transcribed into the corresponding formulae for the } \Pi_Q^{(L)}. \text{ For the Ising case } \mathbb{Z}_2 \text{ these are of the Chebyshev type}

\[ \Pi_Q^{(L+4)} - 4c\Pi_Q^{(L+2)} + 4\Pi_Q^{(L)} = 0, \]  

and so for } N = 2 \text{ the } \Pi_Q^{(L)} \text{ form orthogonal sequences. However, for } \mathbb{Z}_3 \text{ we have } (11) \text{ (valid for all } L \geq 0 \text{ and all } Q): 

\[ \Pi_Q^{(L+9)} - 3(9c^2 - 5)\Pi_Q^{(L+6)} + 48 \Pi_Q^{(L+3)} - 64 \Pi_Q^{(L)} = 0. \]  

These } \Pi_Q^{(L)} \text{ form } 9 \text{ sequences, each labeled by } (L_0, Q), \text{ where } Q = 0, 1, 2 \text{ and } L = 3j + L_0 \text{ where } L_0 = 0, 1, 2, \ldots, j = 0, 1, 2, \ldots. \text{ The degrees of the polynomials appearing in this relation increase by two from the right to the left, but since the recursion is four-term, not three-term, these are not orthogonal sequences } (12). \text{ However, like } (11) \text{ also } (12) \text{ are of the most simple type: all coefficients are independent of } L.

\section{Expansion in terms of Jacobi polynomials}

As the zeros of our } \Pi_Q^{(L)} \text{ are confined to and dense in the interval } (-1, +1) \text{ we call this the basic interval like for orthogonal polynomials. Trying to determine (numerically) a weight function by the ansatz } \int_{-1}^{1}(1 + c)^{\alpha}(1 - c)^{\beta}P_Q^{(L)}(c) = 0 \text{ for } k < b_{Q,L} \text{ fitting } \alpha \text{ and } \beta, \text{ we find (in the following we concentrate on the } \mathbb{Z}_3\text{-case) that there is an approximate solution very close to } \alpha = -\beta = \frac{1}{2}, \text{ but } \alpha \text{ and } \beta \text{ come out to be slightly } L \text{ and } k\text{-dependent, in contrast to what is needed for orthogonality. However, for } L \to \infty \text{ and small fixed } k, \text{ the solutions converge towards } \alpha = -\beta = \frac{1}{2}. \text{ So the } \Pi_Q^{(L)} \text{ seem to be close to Jacobi polynomials } P_k^{(\frac{1}{2}, -\frac{1}{2})}, \text{ but can we formulate an exact relation valid for finite } L? \text{ Is there an exact property of the } \Pi_Q^{(L)} \text{ which replaces orthogonality?}

Numerical calculations (11) \text{ gave the surprising result that seemingly complicated } \Pi_Q^{(L)} \text{ can be written as a combination of just very few Jacobi polynomials, e.g.}

\[ \Pi_1^{(12)} = 3^{11}c^7 + 3^{10}c^6 - 5 \cdot 3^{10}c^3 - 80919c^2 + 27459c - 16839 = \]

\[ \frac{6^8}{728} \left( \frac{63}{22} P_7^{\frac{1}{2}, -\frac{1}{2}} - P_5^{\frac{1}{2}, -\frac{1}{2}} \right). \]  

For polynomials } \pi(c) \text{ of degree } n \text{ we use}

\[ \pi(c) = \sum_{k=0}^{n} \pi_k P_k^{\frac{1}{2}, -\frac{1}{2}}(c) \]  

\[ \text{(13)} \]

\footnote{If we consider } Q = L \mod 3, \text{ then we have only polynomials in } c^2, \text{ and the degrees } b_{L,Q} \text{ are consecutive in powers of } z = c^2 \text{ ("simple sets of polynomials") with integer coefficients.}

\footnote{For higher } N \text{ we get similar } N + 1 \text{-term relations, e.g. for } \mathbb{Z}_4: \]

\[ \Pi_Q^{(L+16)} - 4(64c^3 - 56c)\Pi_Q^{(L+12)} - 128(14c^4 - 17)\Pi_Q^{(L+8)} - 2048c\Pi_Q^{(L+4)} + 4096 \Pi_Q^{(L)} = 0. \]

\footnote{For Jacobi series as a generalization of Taylor series, see Ch.7 of Carlson (23). We use the standard normalization of Jacobi polynomials, see e.g. Rainville (22).}
Table 1: Examples of $\mathbb{Z}_3$–polynomials $\Pi_{Q}^{(L)}(c)$ and their Jacobi-components defined in [13].

| $L$ | $\Pi_{Q}^{(L)}(c)$ | $2^{-[2L/3]} \Pi_{Q}^{(L)}(c)$ |
|-----|---------------------|-------------------------------|
| 3   | $9c + 3$            | $[0, \frac{9}{7}]$             |
| 4   | $27c^2 - 18c - 5$   | $[3, -\frac{27}{4}, \frac{9}{7}]$ |
| 5   | $81c^3 + 27c^2 - 57c - 11$ | $[0, -\frac{9}{2}, 0, \frac{81}{20}]$ |
| 6   | $3^5c^3 + 81c^2 - 135c - 21$ | $[0, 0, 0, \frac{243}{40}]$ |
| 7   | $3^6c^4 - 2 \cdot 3^5c^3 - 540c^2 + 270c + 43$ | $[3, -\frac{40}{7}, \frac{171}{14}, -\frac{729}{40}, \frac{729}{70}]$ |
| 8   | $3^7c^5 + 3^6c^4 - 2754c^3 - 702c^2 + 711c + 85$ | $[0, 0, 0, -\frac{27}{4}, 0, \frac{243}{20}]$ |
| 9   | $3^8c^5 + 3^7c^4 - 7290c^3 - 1782c^2 + 1593c + 171$ | $[0, 0, 0, -\frac{81}{40}, 0, \frac{729}{56}]$ |
| 10  | $3^9c^6 - 2 \cdot 3^8c^5 - 25515c^4 + 14580c^3 + 7965c^2 - 3186c - 341$ | $[3, -\frac{27}{4}, \frac{135}{14}, -\frac{243}{40}, \frac{9477}{385}, -\frac{3^3}{56}, \frac{3^4}{308}]$ |

and define a scalar product with Baxter’s variable $t = ((1 - c)/(1 + c))^{1/3}$ (see (3)) as the weight function (here we will not need to specify the normalization):

$$\langle \pi^{(1)}|\pi^{(2)}\rangle = \int_{-1}^{+1} dc \left(\frac{1 - c}{1 + c}\right)^{1/3} \pi^{(1)}(c)\pi^{(2)}(c) = -\int_{-\infty}^{0} ds \frac{6s}{(1 - s)^2} \pi^{(1)}(c(s)) \pi^{(2)}(c(s)).$$

The second part of this definition shows that it makes sense also if we prefer to use polynomials in the variable $s$, and that it preserves the original $\mathbb{Z}_3$–symmetry.

Since the $\Pi_{Q}^{(L)}$ satisfy the recursion relations (12) they can be written as determinants of band matrices with a bottom line specifying the initial conditions (which are the lowest $L$ polynomials). Omitting the bottom line, we define $j \times j$–band matrices and their determinants $R_j = \det |\text{band}(\{[64, 48, 3p, 1, 0], j\})|$, e.g.

$$R_0 = \begin{vmatrix} 3p & 1 & 0 & 0 & 0 & 0 \\ 48 & 3p & 1 & 0 & 0 & 0 \\ 64 & 48 & 3p & 1 & 0 & 0 \\ 0 & 64 & 48 & 3p & 1 & 0 \\ 0 & 0 & 64 & 48 & 3p & 1 \\ 0 & 0 & 0 & 64 & 48 & 3p \end{vmatrix}.$$

where $p = 9c^2 - 5$, so that the polynomials $R_j$ depend only on $c^2$. Now we get the nine sequences of the $\Pi_{Q}^{(L)}$ as linear combinations of the $R_j$ which satisfy appropriate initial conditions. Abbreviating $Q_j = R_j + 8R_{j-1}$ these are found to be:
\[ \Pi_0^{(3j)} = \frac{1}{3} (Q_j - 8Q_{j-1} + 16Q_{j-2}); \quad \Pi_1^{(3j)} = 9cR_{j-1} \pm 3Q_{j-1}; \]
\[ \Pi_0^{(3j+1)} = \pm 18cR_{j-1} + Q_j + 2Q_{j-1}; \quad \Pi_1^{(3j+1)} = Q_j - 4Q_{j-1}; \]
\[ \Pi_0^{(3j+2)} = 3c(R_j - 4R_{j-1}) \pm (Q_j - 4Q_{j-1}); \quad \Pi_2^{(3j+2)} = 3Q_j. \]

For \( j = 1, 2 \) use \( R_0 = 1; \ R_{-1} = R_{-2} = 0 \). From (13) we see that only two of the 9 sequences are independent. There are relations like e.g. \( \Pi_0^{(3j+1)} - \Pi_1^{(3j+1)} = 2\Pi_1^{(3j)} \).

To get the Jacobi-expansion of the \( \Pi_Q^{(L)} \), we only need to expand \( R_j \) and \( cR_j \). Using Jacobi-components defined in (13), from explicit calculation (for \( j \leq 36 \)), we find that for \( k < j \) (only there) the \( j \)-dependence of the \( (R_j)_k \) obeys the simple rule:
\[ (R_j)_k = (-3)^k k! (2k + 1) \left\{ (-8)^j \sigma_k + 4^j \tau_k \right\}; \]
\[ \tau_k = \frac{1}{3 \prod_{n=0}^{k-1} (3n + 1)}; \quad \sigma_k = \frac{2(-3)^m}{3 \prod_{n=m}^{b(m)} (2n + 1)}; \quad \sigma_k = \frac{2(-3)^{m-2}}{\prod_{n=m}^{b(m)} (2n)}. \]

It follows that the \( \sigma_k \) do not contribute to the \( k < j \)-components of \( Q_j \), and, using the recursion relations for the \( P_{k}^{j} (c) \), we conclude that for \( k \leq j \) we have
\[ \frac{1}{j} (Q_j)_k = -(cR_j)_k = \frac{1}{j} (c^2 R_{j+1})_k = 4^{j} (-3)^k k! (2k + 1) \tau_k. \]

It further follows that
\[ \Pi_0^{(3j+3)} = (\Pi_1^{(3j)})_k = (\Pi_0^{(3j+1)})_k = (\Pi_1^{(3j+1)})_k = (\Pi_0^{(3j+2)})_k = 0 \quad \text{for} \quad k < j. \]

All \( k < j \) components of \( \Pi_2^{(3j)} \), \( \Pi_2^{(3j+1)} \) and \( \Pi_2^{(3j+2)} \) are proportional to \( (Q_j)_k \). We get zero overlap between a polynomial \( \Pi_Q^{(L)} \) of degree \( b_{L,Q} \) with all polynomials \( \Pi_Q^{(L')} \) which have at least \( b_{L,Q} \) vanishing low-\( k \) components (these can only be polynomials which have \( b_{L',Q'} > 2b_{L,Q} \)). One of many such relations is e.g.
\[ \langle Q_0^{(3j)} | Q_0^{(3j')} \rangle = 0 \quad \text{for} \quad Q = Q' = 1 \quad \text{and} \quad 2j \leq j' - 1. \]

This property may be called "partial orthogonality".

Further rules, this time valid for all \( k \), regard the vanishing of many components for particular linear combinations: Defining
\[ Q_+^{(j)} \equiv \Pi_1^{(3j)} = 9cR_{j-1} + 3Q_{j-1} = [\alpha_0^{(+)}, \alpha_1^{(+)}, \ldots, \alpha_{2j-1}^{(+)}]; \]
\[ Q_-^{(j)} \equiv \Pi_0^{(3j)} = Q_j + 9cR_{j-1} - Q_{j-1} = [\alpha_0^{(-)}, \alpha_1^{(-)}, \ldots, \alpha_{2j-1}^{(-)}], \]
we have checked up to \( j = 30 \) that \( \alpha_k^{(\pm)} = 0 \) for \( k < j \) and that all even (odd) \( k \) Jacobi components of \( Q_+^{(j)} \) (\( Q_-^{(j)} \)) vanish. So we conjecture for all \( j, j' \)
\[ \langle Q_+^{(j)} | Q_-^{(j')} \rangle = 0. \]

One can check some special cases of these results in Table 1. It has been found numerically that a similar partial orthogonality appears also for the \( \mathbb{Z}_N \)-Baxter polynomials with \( N = 4, 5, 6 \). For even values of \( N \) further relations emerge, but these will not be discussed here.
5 Conclusion

The polynomials which play a central role for the calculation of the energy eigenvalues of the superintegrable $Z_3$–chiral Potts model are found to be related to Jacobi polynomials in a very peculiar way. Many integrals giving the Jacobi-coefficients of Baxter’s polynomials are found to vanish. These observations should have a deeper group-theoretical background, but the underlying symmetry is not yet clear to us. By reducing the formulation of the problem to some basic facts, the present analysis tries to prepare the ground for clarifying the symmetry involved.

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