ON THE DEEP-WATER AND SHALLOW-WATER LIMITS OF THE INTERMEDIATE LONG WAVE EQUATION FROM A STATISTICAL VIEWPOINT

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Abstract. We study convergence problems for the intermediate long wave equation (ILW), with the depth parameter \( \delta > 0 \), in the deep-water limit \( (\delta \to \infty) \) and the shallow-water limit \( (\delta \to 0) \) from a statistical point of view. In particular, we establish convergence of invariant Gibbs dynamics for ILW in both the deep-water and shallow-water limits. For this purpose, we first construct the Gibbs measures for ILW, \( 0 < \delta < \infty \). As they are supported on distributions, a renormalization is required. With the Wick renormalization, we carry out the construction of the Gibbs measures for ILW. We then prove that the Gibbs measures for ILW converge in total variation to that for the Benjamin-Ono equation (BO) in the deep-water limit \( (\delta \to \infty) \). In the shallow-water regime, after applying a scaling transformation, we prove that, as \( \delta \to 0 \), the Gibbs measures for the scaled ILW converge weakly to that for the Korteweg-de Vries equation (KdV). We point out that this second result is of particular interest since the Gibbs measures for the scaled ILW and KdV are mutually singular (whereas the Gibbs measures for ILW and BO are equivalent).

In terms of dynamics, we use a compactness argument to construct invariant Gibbs dynamics for ILW (without uniqueness). Furthermore, we show that, by extracting a sequence \( \delta_m \), this invariant Gibbs dynamics for ILW converges in total variation to that for BO in the deep-water limit \( (\delta_m \to \infty) \) and to that for KdV (after the scaling) in the shallow-water limit \( (\delta_m \to 0) \), respectively.

Lastly, we point out that our results also apply to the generalized ILW equation in the defocusing case, converging to the generalized BO in the deep-water limit and to the generalized KdV in the shallow-water limit. In the non-defocusing case, however, our results can not be extended to a nonlinearity with a higher power due to the non-normalizability of the corresponding Gibbs measures.


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\end{abstract}
1. Introduction

1.1. Intermediate long wave equation. In this paper, we study the intermediate long wave equation (ILW) on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases}
\partial_t u - G_\delta \partial_x^2 u = \partial_x (u^2) \\
u|_{t=0} = u_0,
\end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{T}. \quad (1.1)$$

The equation (1.1), also known as the finite-depth fluid equation, models the internal wave propagation of the interface in a stratified fluid of finite depth $\delta > 0$, and the unknown $u : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ denotes the amplitude of the internal wave at the interface. See also Remark 1.1.

The dispersion operator $G_\delta$ characterizes the phase speed and it is defined as the following Fourier multiplier operator:

$$\widehat{G_\delta f}(n) = -i \left( \coth(\delta n) - \frac{1}{\delta n} \right) \hat{f}(n), \quad n \in \mathbb{Z},$$

where $\coth$ denotes the usual hyperbolic cotangent function:

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}, \quad x \in \mathbb{R} \setminus \{0\}$$

with the convention $\coth(\delta n) - \frac{1}{\delta n} = 0$ for $n = 0$; see (1.15). See (1.6) below for our convention of the (spatial) Fourier transform. ILW (1.1) is an important physical model, providing a natural connection between the deep-water regime (= the Benjamin-Ono regime) and the shallow-water regime (= the KdV regime). As such, it has been studied extensively from both the applied and theoretical points of view. See, for example, a recent book [49, Chapter 3] by Klein and Saut for an overview of the subject and the references therein. See also a survey [79]. These two references indicate that the rigorous mathematical study of ILW is still widely open. In particular, one of the fundamental, but challenging questions is the convergence properties of ILW in the deep-water limit (as the depth parameter $\delta$ tending to $\infty$) and in the shallow-water limit (as $\delta \to 0$). In this paper, we make the first study on this convergence issue of ILW from a statistical viewpoint.
Remark 1.1. In [50], the equation for the motion of the internal wave in a finite depth fluid was derived with two depth parameters $\delta_j$, $j = 1, 2$, where $\delta_1$ and $\delta_2$ represent the depths of the upper and lower fluids, respectively, and is given by

$$\partial_t u - c_1 G_{\delta_1} \partial_x^2 u - c_2 G_{\delta_2} \partial_x^2 u = \partial_x (u^2). \quad (1.3)$$

See (25a)-(25b) and (35a)-(35b) in [50]. In [50, VI Summary], the authors proposed a special case of interest when the internal wave is located halfway between the upper and lower fluid boundaries, namely, $\delta_1 = \delta_2$. In this case, by setting $\delta = \delta_1 + \delta_2 = 2\delta_1$, the equation (1.3) reduces to the ILW equation (1.1) (up to some inessential multiplicative constants). We also point out that by taking $\delta_1 \to 0$ while keeping $\delta_2$ fixed (or by taking $\delta_2 \to 0$ while keeping $\delta_1$ fixed), we also see that the equation (1.3) reduces to the ILW equation (1.1).

1.2. Deep-water and shallow-water limits of the generalized ILW. In the following, we consider the generalized intermediate long wave equation (gILW) on $\mathbb{T}$:

$$\begin{cases}
\partial_t u - G_{\delta} \partial_x^2 u = \partial_x (u^k) \\
u|_{t=0} = u_0,
\end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \quad (1.4)$$

where $k \geq 2$ is an integer. When $k = 2$, the equation (1.4) corresponds to ILW (1.1), while, when $k = 3$, it is known as the modified ILW equation. The equation (1.4) can be written in the following Hamiltonian formulation:

$$\partial_t u = \partial_x \frac{dE_\delta(u)}{du},$$

where $E_\delta(u)$ is the Hamiltonian (= energy) given by

$$E_\delta(u) = \frac{1}{2} \int_T u G_{\delta} \partial_x u dx + \frac{1}{k + 1} \int_T u^{k+1} dx. \quad (1.5)$$

In particular, $E_\delta(u)$ is conserved under the dynamics of (1.4). Moreover, it is easy to check that the following two quantities are conserved under the gILW dynamics:

- mean: $\int_T u dx$
- mass: $M(u) = \int_T u^2 dx$. 

We also point out that ILW ($k = 2$) is known to be completely integrable. We, however, do not make use of the completely integrable structure in the following. See Remarks 1.11 and 1.12.

For simplicity of the presentation, we impose the mean-zero condition on the initial condition $u_0$, namely, $\int_T u_0 dx = 0$, in the remaining part of the paper. In view of the conservation of the (spatial) mean, this implies that the solution $u(t)$ has mean zero as long as it exists.

In other words, defining the Fourier coefficient $\hat{f}(n)$ by

$$\hat{f}(n) = \mathcal{F}(f)(n) = \int_T f(x)e^{-inx} dx, \quad (1.6)$$

we will work with real-valued functions of the form\(^1\)

$$f(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \hat{f}(n)e_n(x), \quad (1.7)$$

where $e_n(x) = e^{inx}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

---

\(^1\)Hereafter, we may drop the harmless factor $2\pi$. 
Our main goal is to study the deep-water limit ($\delta \to \infty$) and the shallow-water limit ($\delta \to 0$) of solutions to gILW (1.4) from a statistical viewpoint. In particular, we study the convergence problem with rough and random initial data, more precisely, with the Gibbs measure initial data. In the next subsection, we provide a detailed discussion on (the construction and convergence of) Gibbs measures. In the deterministic setting, the convergence problem of gILW in both the deep-water and shallow-water limits has been studied in [1, 41, 43, 61, 55, 56], providing rigorous mathematical support of the numerical study performed in [50]. We point out that the recent work [55, 56] by the first author is the only convergence result of gILW on the circle $T$.

In the following, let us briefly go over the formal derivation of the limiting equation in each of the deep-water and shallow-water limits. With a slight abuse of notation, we set
\[
\hat{G}_\delta(n) = -i \left( \coth(\delta n) - \frac{1}{\delta n} \right). \tag{1.8}
\]

**Deep-water limit ($\delta \to \infty$).**

In this case, an elementary computation shows that
\[
\lim_{\delta \to \infty} \hat{G}_\delta(n) = -i \text{sgn}(n) \tag{1.9}
\]
for any $n \in \mathbb{Z}$. Indeed, defining $q_\delta(n)$ by
\[
q_\delta(n) = |n| + \frac{1}{\delta} - n \coth(\delta n), \tag{1.10}
\]
one may easily verify that
\[
0 \leq q_\delta(n) = q_\delta(-n) \leq \frac{2}{\delta} \tag{1.11}
\]
for any $n \in \mathbb{Z}$; see Lemma 4.1 in [1]. In fact, (1.11) holds with the right-hand side replaced by $\frac{1}{\delta}$; see Remark 2.2 below.

The limit (1.9) indicates that, in the deep-water limit, namely, as $\delta \to \infty$, the gILW equation (1.4) converges to the following generalized Benjamin-Ono equation (gBO) on $T$:
\[
\partial_t u - \mathcal{H}(\partial_x^2 u) = \partial_x(u^k), \tag{1.12}
\]
where $\mathcal{H}$ is the Hilbert transform defined by
\[
\mathcal{H}f(n) = -i \text{sgn}(n) \hat{f}(n). \tag{1.13}
\]

Formally speaking, by recasting (1.4) as
\[
\partial_t u - \mathcal{H}(\partial_x^2 u) + Q_\delta \partial_x u = \partial_x(u^k), \tag{1.13}
\]
where $Q_\delta = (\mathcal{H} - \hat{G}_\delta)\partial_x$ is defined as a Fourier multiplier operator with symbol $q_\delta$ in (1.10). Then, the bound (1.11) shows that $Q_\delta$ tends to 0 in a suitable sense, thus yielding the formal convergence of (1.13) (and hence of (1.4)) to gBO (1.12) as $\delta \to \infty$. In proving rigorous convergence, one indeed needs to show that $Q_\delta \partial_x$ tends to 0 in a suitable sense (instead of $Q_\delta$), and thus, in view of the bound (1.11), it indicates that in the deep-water regime $\delta \gg 1$, long waves (with relatively small frequencies $|n| \ll \delta$) “well approximate” long waves of infinitely deep water ($\delta = \infty$).

\[\text{While it is not needed in the mean-zero case, we may set } q_\delta(0) = 0 \text{ by continuity.}\]
• **Shallow-water limit** \((\delta \to 0)\).

A direct computation shows that, for \(n \in \mathbb{Z}^*\), we have

\[
\mathcal{G}_\delta \partial_x^2 u(n) = i \left( \coth(\delta n) - \frac{1}{\delta n} \right) n^2 \hat{u}(n) = i \frac{\delta}{3} n^3 \hat{u}(n) + o(1),
\]

as \(\delta \to 0\). The identity (1.14) follows from the following identity with \(x = \delta n\):

\[
\coth(x) - \frac{1}{x} = \frac{x(e^{2x} - 1) - (e^{2x} - 2x - 1)}{x(e^{2x} - 1)} = \frac{x}{3} + o(1),
\]

as \(x \to 0\), which can be verified by using the Taylor expansion: \(e^{2x} = 1 + 2x + \sum_{k=2}^{\infty} (2x)^k/k!\).

The identity (1.14) shows that, the dispersion in (1.4) disappears as \(\delta \to 0\), formally yielding the inviscid Burgers equation in the limit (when \(k = 2\)). In order to circumvent this issue, we introduce the following scaling transformation for each \(\delta > 0\), [1]:

\[
v(t, x) = \left( \frac{3}{\delta} \right)^{\frac{1}{k-1}} u\left( \frac{3}{\delta} t, x \right),
\]

which leads to the following scaled gILW equation:

\[
\partial_t v - \frac{3}{\delta} \mathcal{G}_\delta \partial_x^2 v = \partial_x (v^k).
\]

Namely, \(v\) is a solution to the scaled gILW (1.17) (with the scaled initial data) if and only if \(u\) is a solution to the original gILW (1.1). Note that the scaled gILW (1.17) is a Hamiltonian PDE with the Hamiltonian:

\[
E_\delta(v) = \frac{3}{2\delta} \int_{\mathbb{T}} v \mathcal{G}_\delta \partial_x v dx + \frac{1}{k+1} \int_{\mathbb{T}} v^{k+1} dx,
\]

which differs from the Hamiltonian \(E_\delta(u)\) in (1.5) by a **divergent** multiplicative constant in the kinetic part (= the quadratic part) of the Hamiltonian. In view of (1.14), the scaled gILW (1.17) formally converges to the following generalized KdV equation (gKdV) on \(\mathbb{T}\):

\[
\partial_t v + \partial_x^3 v = \partial_x (v^k).
\]

From the physical point of view, the scaling transformation (1.16) is a very natural operation to perform, when \(k = 2\). The ILW equation (1.1) describes the motion of the fluid interface in a stratified fluid of depth \(\delta > 0\), where \(u\) denotes the amplitude of the internal wave at the interface. As \(\delta \to 0\), the entire fluid depth tends to 0 and, in particular, the amplitude of the internal wave at the interface is \(O(\delta)\), which also tends to 0, in the physical model. Hence, if we want to observe any meaningful limiting behavior, we need to magnify the fluid motion by a factor \(\sim \frac{1}{\delta}\), which is exactly what the scaling transformation (1.16) does when \(k = 2\). We also point out that studying the convergence problem for the scaled ILW (1.17) (with \(k = 2\)) with \(O(1)\) initial data means that we are indeed studying the original ILW (1.1) with \(O(\delta)\) initial data, which is consistent with the physical viewpoint explained above.

As mentioned above, in the deterministic setting, the convergence problem of the gILW dynamics (and the scaled gILW dynamics, respectively) to the gBO dynamics (and to the gKdV dynamics, respectively) has been studied in [1, 41, 43, 55, 56, 23]. These works studied

\[\text{The limiting behavior (1.15) also follows from the Taylor expansion of the hyperbolic cotangent function.}\]
the convergence issue from a microscopic viewpoint in the sense that convergence was established for each fixed initial data \( u|_{t=0} = u_0 \) to gILW (1.4) (or each fixed initial data \( v|_{t=0} = v_0 \) to the scaled gILW (1.17)). In the present work, we study the convergence problem from a macroscopic viewpoint. Namely, rather than considering the limiting behavior of a single trajectory, we study the limiting behavior of solutions as a statistical ensemble. Such an approach is of fundamental importance in statistical mechanics, where one replaces “the study of the microscopic dynamical trajectory of an individual macroscopic system by the study of appropriate ensembles or probability measures on the phase space of the system” [53]. In the present work, we in particular study convergence of the dynamics at the Gibbs equilibrium for the gILW equation (1.17) in both the deep-water and shallow-water limits. From the physical point of view, it is quite natural to study the fluid motion as a statistical ensemble, since one is often interested in a prediction of typical behavior of the fluid. From the theoretical point of view, it is an interesting and challenging question to study convergence of invariant Gibbs dynamics associated with the gILW equation (1.17), in particular due to the low regularity of the support of the Gibbs measures.

Our strategy for establishing convergence of invariant Gibbs dynamics for the (scaled) gILW consists of the following three steps. For simplicity, we only discuss the deep-water limit in the following, where we treat the original gILW (1.4) (rather than the scaled gILW (1.17) relevant in the shallow-water limit), unless we need to make a specific point in the shallow-water limit.

In the following, we will restrict our attention to (i) \( k = 2 \), corresponding to ILW (1.1), and (ii) \( k \in 2\mathbb{N} + 1 \) in (1.4), corresponding to the defocusing case. This restriction comes from the Gibbs measure construction. See Remark 1.7 for a discussion on the general focusing case, namely, either for (iii) even \( k \geq 4 \) or (iv) \( k \in 2\mathbb{N} + 1 \) with the focusing sign:

\[
\partial_t u - \mathcal{G}_\delta \partial_x^2 u = -\partial_x(u^k). 
\] (1.20)

• **Step 1: Construction and convergence of the Gibbs measures.**

For each finite \( \delta > 0 \), we first construct a Gibbs measure \( \rho_\delta \) for gILW (1.4) with the Hamiltonians \( E_\delta(u) \) in (1.5), formally written as

\[
\rho_\delta(du) = Z_\delta^{-1} e^{-E_\delta(u)} du = Z_\delta^{-1} e^{-\int_T u^{k+1} dx} \frac{1}{2} \int_T u \mathcal{G}_\delta \partial_x u dx \, du. 
\] (1.21)

The expression (1.21) is merely formal and we aim to construct \( \rho_\delta \) as a weighted Gaussian measure with the base Gaussian measure given by

\[
\mu_\delta(du) = Z_\delta^{-1} e^{-\int_T u \mathcal{G}_\delta \partial_x u dx} \, du. 
\] (1.22)

See the next subsection for a precise definition of \( \mu_\delta \). For each \( \delta > 0 \), the Gaussian measure \( \mu_\delta \) is supported on distributions \( \mathcal{D}'(\mathbb{T}) \) of negative regularity and thus the potential energy \( \int_T u^{k+1} dx \) in (1.21) is divergent. In order to overcome this issue, we introduce a renormalization on the potential energy, just as in the construction of the \( \Phi_{2}^{k+1} \)-measures [80, 38, 28, 74].

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4Strictly speaking, the case (iii) even \( k \geq 4 \) is non-defocusing, not focusing. For simplicity, however, we may refer to the non-defocusing case as focusing in the remaining part of the paper.

5Henceforth, constants such as \( Z_\delta \) denote various normalizing constants, which may be different line by line.
When $k = 2$, the potential energy is not sign-definite, causing a further problem. By following the work [57], we overcome this issue by introducing a Wick-ordered $L^2$-cutoff. See Subsection 1.3 for a further discussion.

Once the Gibbs measure $\rho_\delta$ is constructed for each $\delta > 0$, we then proceed to prove convergence of the Gibbs measures $\rho_\delta$ for $gILW$ (1.17) to the Gibbs measure $\rho_{BO}$ for $gBO$ (1.12) in the deep-water limit $(\delta \to \infty)$. This step involves establishing the $L^p$-integrability bound on the densities, \textit{uniformly in} $\delta \gg 1$. We point out that, for each $\delta \gg 1$, the base Gaussian measure $\mu_\delta$ is different and thus an extra care is needed in discussing what we mean by the “density”. See Section 3 for further details.

In order to study the shallow-water limit, we need to consider the scaled $gILW$ (1.17) with the Hamiltonian $E_\delta(v)$ in (1.18). This leads to the construction of the following Gibbs measure:

$$
\tilde{\rho}_\delta(dv) = Z_\delta^{-1} e^{-E_\delta(v)} dv = Z_\delta^{-1} e^{-\frac{3}{2} \delta} \int_{\mathbb{T}^d} v G_\delta \partial_x v dx dv.
$$

(1.23)

For each fixed $\delta > 0$, we construct the Gibbs measure $\tilde{\rho}_\delta$ as a weighted Gaussian measure with the base Gaussian measure $\tilde{\mu}_\delta$ given by

$$
\tilde{\mu}_\delta(dv) = Z_\delta^{-1} e^{-\frac{3}{2} \delta} \int_{\mathbb{T}^d} v G_\delta \partial_x v dx dv.
$$

(1.24)

The construction of the Gibbs measure $\tilde{\rho}_\delta$, $\delta > 0$, follows exactly the same lines as that for the Gibbs measure $\rho_\delta$ in (1.21). There is, however, a crucial difference in the shallow-water limit in establishing convergence of the Gibbs measures $\tilde{\rho}_\delta$, $\delta \ll 1$, for the scaled $gILW$ (1.17) to the Gibbs measure $\rho_{KdV}$ for $gKdV$ (1.19). More precisely, it turns out that the Gibbs measures $\rho_\delta$, $\delta > 0$, for the scaled $gILW$ (1.17) and $\rho_{KdV}$ for $gKdV$ (1.19) are mutually singular and the mode of convergence of $\tilde{\rho}_\delta$ to $\rho_{KdV}$ is weaker (than that in the deep-water limit).

This first step is one of the main novelties of the paper, where we establish a uniform bound on the densities (with respect to the underlying probability measure $P$).

• **Step 2: Construction of invariant Gibbs dynamics for the (scaled) gILW.**

In this second step, we construct dynamics for $gILW$ (1.4) at the Gibbs equilibrium constructed in Step 1. This step follows the compactness argument introduced by Burq, Thomann, and Tzvetkov [18] in the context of dispersive PDEs. See [4, 27] for the first instance of this argument in the context of fluid. See also [74, 69]. Due to the use of the compactness argument, the dynamics constructed in this step lacks a uniqueness statement.

• **Step 3: Convergence of the (scaled) gILW dynamics at the Gibbs equilibrium.**

This last step essentially follows from the previous two steps together with the triangle inequality. In Step 2, we construct limiting Gibbs dynamics as a limit of the frequency-truncated dynamics (via the compactness argument mentioned above). In this last step, we characterize the convergence established in Step 2 in the Lévy-Prokhorov metric and conclude the desired convergence of the dynamics at the Gibbs equilibrium for the (scaled) gILW to that for $gBO$ (or to $gKdV$) via a diagonal argument. The use of the Lévy-Prokhorov metric in this context is new as far as our knowledge is concerned.
Remark 1.2. There is also a slightly different formulation for the ILW equation; see [1, p. 211]. In this formulation, the generalized ILW equation on $\mathbb{T}$ reads as

$$
\partial_t u - \left(1 + \frac{1}{\delta}\right) G_\delta \partial_x^2 u = \partial_x (u^k)
$$

with the Hamiltonian $\bar{E}_\delta(u)$ given by

$$
\bar{E}_\delta(u) = \frac{\delta + 1}{2\delta} \int_{\mathbb{T}} u G_\delta \partial_x u dx + \frac{1}{k+1} \int_{\mathbb{T}} u^{k+1} dx.
$$

In taking $\delta \to \infty$, we formally have

$$
\partial_t u - G_\delta \partial_x^2 u = \partial_x (u^k) + O(\delta^{-1}),
$$

which indicates that the same convergence result holds for this version (1.25) of gILW in the deep-water limit. On the other hand, in the shallow-water regime, in view of (1.14), the equation (1.25) can be formally written as

$$
\partial_t u - \frac{1}{\delta} G_\delta \partial_x^2 u = \partial_x (u^k) + O(\delta),
$$

which indicates convergence of (1.25) to the following gKdV:

$$
\partial_t u + \frac{1}{3} \partial_x^3 u = \partial_x (u^k)
$$

without any scaling transformation. Indeed, in the shallow-water limit, a slight modification of our argument shows that an analogue of our main result holds for the version (1.25) converging to gKdV (1.26) in the shallow-water limit. On the one hand, the formulation (1.25) may seem to be a convenient model since it does not require a scaling transformation in the shallow-water limit. On the other hand, it does not seem to reflect the physical behavior in the shallow-water regime (where the entire depth and thus the amplitude $u$ are $O(\delta)$).

1.3. Construction and convergence of Gibbs measures. Consider a finite-dimensional Hamiltonian flow on $\mathbb{R}^{2n}$:

$$
\partial_t p_j = \frac{\partial H}{\partial q_j} \quad \text{and} \quad \partial_t q_j = -\frac{\partial H}{\partial p_j}
$$

with Hamiltonian

$$
H(p, q) = H(p_1, \ldots, p_n, q_1, \ldots, q_n).
$$

The classical Liouville’s theorem states that the Lebesgue measure $dp dq = \prod_{j=1}^n dp_j dq_j$ on $\mathbb{R}^{2n}$ is invariant under the dynamics (1.27). Then, together with the conservation of the Hamiltonian $H(p, q)$, we see that the Gibbs measure $Z^{-1} e^{-H(p, q)} dp dq$ is invariant under the dynamics of (1.27). By drawing an analogy, we may hope to construct invariant Gibbs dynamics for Hamiltonian PDEs. This program was initiated by the seminal works by Lebowitz, Rose, and Speer [53] and Bourgain [13, 14], leading to the construction of invariant Gibbs dynamics as well as probabilistic well-posedness. See also [34, 92, 58]. This subject has been increasingly more popular over the last fifteen years; see, for example, survey papers [65, 8].

Our first main goal is to construct Gibbs measures for gILW (1.4) (and the scaled gILW (1.17)). For this purpose, let us first go over the known results in the limiting cases $\delta = 0$ and $\delta = \infty$. 
• Construction of Gibbs measures for gKdV on \( \mathbb{T} \).

This corresponds to the shallow-water limit (\( \delta = 0 \)) in our problem. Consider gKdV (1.19) posed on the circle with the Hamiltonian

\[
E_0(v) = \frac{1}{2} \int_\mathbb{T} (\partial_x v)^2 dx + \frac{1}{k+1} \int_\mathbb{T} v^{k+1} dx,
\]

which, in view of (1.14), is a formal limit of \( E_\delta(v) \) in (1.18) as \( \delta \to 0 \). The Gibbs measure \( \rho_{\text{KdV}} \) for gKdV is formally given by

\[
\rho_{\text{KdV}}(dv) = Z_0^{-1} e^{-E_0(v)} dv
\]

\[
= Z_0^{-1} e^{-\frac{1}{k+1} \int_\mathbb{T} v^{k+1} dx} e^{-\frac{1}{2} \int_\mathbb{T} (\partial_x v)^2 dx} dv.
\]

The Gibbs measure \( \rho_{\text{KdV}} \) can be constructed as a weighted Gaussian measure with the base Gaussian measure given by the periodic Wiener measure \( \tilde{\mu}_0 \) (restricted to mean-zero functions):

\[
\tilde{\mu}_0(dv) = Z_0^{-1} e^{-\frac{1}{2} \int_\mathbb{T} (\partial_x v)^2 dx} dv.
\]

More precisely, the periodic Wiener measure \( \tilde{\mu}_0 \) is defined as the induced probability measure under the map,

\[
\omega \in \Omega \mapsto X_{\text{KdV}}(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|n|} e_n,
\]

where \( e_n(x) = e^{inx} \) and \( \{g_n\}_{n \in \mathbb{Z}^*} \) is a sequence of independent standard complex-valued Gaussian random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) conditioned that \( g_{-n} = \overline{g_n} \), \( n \in \mathbb{Z}^* \). Indeed, by Plancherel’s theorem (see (1.6) and (1.7) for our convention of the Fourier transform), we have

\[
\int_\mathbb{T} (\partial_x v)^2 dx = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} n^2 |\hat{v}(n)|^2 = \frac{1}{\pi} \sum_{n \in \mathbb{N}} n^2 |\hat{v}(n)|^2,
\]

where the second equality follows from the fact that \( v \) is real-valued, i.e. \( \hat{v}(-n) = \overline{\hat{v}(n)} \). This shows that we formally have

\[
e^{-\frac{1}{2} \int_\mathbb{T} (\partial_x v)^2 dx} dv \sim \prod_{n \in \mathbb{N}} e^{-\frac{1}{2\pi} n^2 |\hat{v}(n)|^2} d\hat{v}(n) \sim \prod_{n \in \mathbb{N}} e^{-\frac{1}{4\pi} |g_n|^2} d\mu_g
\]

\[
\sim \left( \prod_{n \in \mathbb{N}} e^{-\frac{1}{2\pi} (\text{Re} g_n)^2} d\text{Re} g_n \right) \left( \prod_{n \in \mathbb{N}} e^{-\frac{1}{2\pi} (\text{Im} g_n)^2} d\text{Im} g_n \right)
\]

in the limiting sense with the identification \( \hat{v}(n) = \frac{g_n}{|n|} \). This shows that \( \text{Re} g_n \) and \( \text{Im} g_n \) are given by mean-zero Gaussian random variables with variance \( \pi \). Hence, \( g_n = \text{Re} g_n + i \text{Im} g_n \) has variance \( 2\pi \).

It is easy to show that the support of \( \tilde{\mu}_0 \) is contained \( H^{\frac{1}{2}+\varepsilon}_c(\mathbb{T}) \setminus H^{\frac{1}{2}}_c(\mathbb{T}) \) for any \( \varepsilon > 0 \). By Khintchine’s inequality, one may also show that the support of \( \tilde{\mu}_0 \) is indeed contained in \( W^{\frac{1}{2}-\varepsilon,\infty}_c(\mathbb{T}) \). See, for example, [7] for a further discussion on the regularity of the Brownian loop \( X_{\text{KdV}} \) in (1.30). Hence, in the defocusing case, namely, when \( k \in 2\mathbb{N} + 1 \), the density

\[
e^{-\frac{1}{k+1} \int_\mathbb{T} v^{k+1} dx}
\]

in (1.28) with respect to \( \mu_0 \) satisfies \( 0 < e^{-\frac{1}{k+1} \int_\mathbb{T} v^{k+1} dx} \leq 1 \), almost surely,

\footnote{Note that \( X_{\text{KdV}} \) is nothing but the Brownian loop on \( \mathbb{T} \) (with the zero spatial mean).
\footnote{By convention, we assume that \( g_n \) has mean 0 and variance 2\( \pi \), \( n \in \mathbb{Z}^* \). See (1.31) below.}
which is in particular integrable with respect to \( \tilde{\mu}_0 \). This shows that the Gibbs measure \( \rho_{KdV} \) can be realized as a weighted \( \tilde{\mu}_0 \):

\[
\rho_{KdV}(dv) = Z_0^{-1} e^{-\frac{1}{k+1} \int_x v^{k+1} dx} d\tilde{\mu}_0(v) \tag{1.32}
\]

in this case.

In the focusing case, namely, when \( k \in 2\mathbb{N} \) or when the potential energy \( \frac{1}{k+1} \int_T v^{k+1} dx \) in (1.32) comes with the + sign, the situation is completely different, since, in this case, the density is no longer integrable with respect to the base Gaussian measure \( \tilde{\mu}_0 \). In the seminal work [53], Lebowitz, Rose, and Speer proposed to consider the Gibbs measure with an \( L^2 \)-cutoff:

\[
\rho_{KdV}(dv) = Z_0^{-1} 1_{\{\int_x v^2 dx \leq K\}} e^{-\frac{1}{k+1} \int_T v^{k+1} dx} d\tilde{\mu}_0(v) \tag{1.33}
\]

for \( k \in 2\mathbb{N} \) in the non-defocusing case, and more generally in the focusing case:

\[
\rho_{KdV}(dv) = Z_0^{-1} 1_{\{\int_x |v|^2 dx \leq K\}} e^{-\frac{1}{k+1} \int_T |v|^{k+1} dx} d\tilde{\mu}_0(v) \tag{1.34}
\]

for any real number \( k > 1 \). In [53] [13], it was shown that, when \( k < 5 \), the Gibbs measures \( \rho_{KdV} \) in (1.33) and (1.34) can be constructed as a probability measure for any \( K > 0 \), while it is not normalizable for any cutoff size when \( k > 5 \). The situation at the critical case \( k = 5 \) (for (1.34)) is more subtle. Note that the critical value \( k = 5 \) corresponds to the smallest power of the nonlinearity, where the focusing gKdV (namely, (1.19) with the + on the nonlinearity) on the real line possesses finite-time blowup solutions [57] [59]. The Gibbs measure construction when \( k = 5 \) remained a challenging open problem for thirty years and it was completed only recently in the work [72] by Sosoe, Tolomeo, and the second author; when \( k = 5 \), the focusing Gibbs measure in (1.34) can be constructed if and only if the cutoff size \( K \) is less than or equal to the mass of the so-called ground state on the real line. See [72] for a further discussion on this issue.

As we see below, in the non-defocusing case, only the \( k = 2 \) case is relevant to us. In this case, the Gibbs measure for KdV relevant to us is given by:

\[
\rho_{KdV}(dv) = Z_0^{-1} \chi_K \left( \int_T v^2 dx - 2\pi \sigma_{KdV} \right) e^{-\frac{1}{4} \int_T v^2 dx} d\tilde{\mu}_0(v), \tag{1.35}
\]

where \( \chi_K : \mathbb{R} \to [0,1] \) is a continuous function such that \( \chi_K(x) = 1 \) for \( |x| \leq K \) and \( \chi_K(x) = 0 \) for \( |x| \geq 2K \).

See Theorem 1.3 below. Here, \( \sigma_{KdV} \) denotes the variance of \( X_{KdV}(x) \) in (1.30) given by

\[
\sigma_{KdV} = \mathbb{E}[X_{KdV}^2(x)] = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^*} \frac{2\pi^2}{n^2} = \frac{\pi}{6}, \tag{1.36}
\]

which is independent of \( x \in T \) due to the translation invariant nature of the problem.

\*Construction of Gibbs measures for gBO on \( T \).\*

Next, we go over the (non-)construction of the Gibbs measures associated with gBO (1.12),

\footnote{From a PDE point of view, this criticality corresponds to the so-called \( L^2 \)-criticality (or mass-criticality), while, from the viewpoint of mathematical physics, this criticality corresponds to the phase transitions for (non-)normalizability of the focusing Gibbs measure. Here, the phases transitions are two-fold: normalizability for \( k < 5 \) and non-normalizability for \( k \geq 5 \). Also, when \( k = 5 \), normalizability below or at the critical mass and non-normalizability above the critical mass.}

\footnote{Hereafter, we use a continuous cutoff function \( \chi_K \) as in [87].}
which corresponds to the deep-water limit \((\delta = \infty)\) in our problem. The Hamiltonian for gBO (1.12) is given by

\[
E_\infty(u) = \frac{1}{2} \int_T u\partial_x u \, dx + \frac{1}{k+1} \int_T u^{k+1} \, dx,
\]

which, in view of (1.8), is a formal limit of \(E_\delta(u)\) in (1.5) as \(\delta \to \infty\). Here, \(\mathcal{H}\) denotes the Hilbert transform. Then, the Gibbs measure \(\rho_{BO}\) for gBO is formally given by

\[
\rho_{BO}(du) = Z_\infty^{-1} e^{-E_\infty(u)} \, du
\]

As in the gKdV case, we first introduce the base Gaussian measure \(\mu_\infty\) by

\[
\mu_\infty(du) = Z_\infty^{-1} e^{-\frac{1}{4} \int_T u^{k+1} \, dx} \, du.
\]

More precisely, the Gaussian measure \(\mu_\infty\) is defined as the induced probability measure under the map:

\[
\omega \in \Omega \mapsto X_{BO}(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} g_n(\omega) e_n,
\]

where \(\{g_n\}_{n \in \mathbb{Z}}\) as in (1.30). In this case, the support of \(\mu_\infty\) is contained in \(H^{-\varepsilon}(T) \setminus L^2(T)\) for any \(\varepsilon > 0\); see (3.2) below. Namely, a typical element \(u\) in the support of \(\mu_\infty\) is merely a distribution and thus the potential energy is divergent in this case.

Let us first consider the defocusing case \(k \in 2\mathbb{N} + 1\). Noting that the Gaussian measure \(\mu_\infty\) is logarithmically correlated, by introducing a Wick renormalized power \(W(u^{k+1})\) (see (1.44) below), Nelson’s estimate allows us to define the Gibbs measure \(\rho_{BO}\):

\[
\rho_{BO}(du) = Z_\infty^{-1} e^{-\frac{1}{4} \int_T W(u^{k+1}) \, dx} \, du
\]

as a limit of the frequency-truncated version, just as in the construction of the \(\Phi_{2}^{k+1}\)-measure [28, 74]; see Theorem 1.3 below. See Subsection 2.2 for a precise definition of the Wick power \(W(u^{k+1})\).

Let us now turn to the focusing case. When \(k = 2\), Tzvetkov [87] constructed the Gibbs measure for the Benjamin-Ono equation (BO) by introducing a Wick-ordered \(L^2\)-cutoff:

\[
\rho_{BO}(du) = Z_\infty^{-1} \chi_K \left( \int_T W(u^2) \, dx \right) e^{-\frac{1}{4} \int_T u^2 \, dx} \, du.
\]

See [71] for an alternative, concise proof. Note that under the mean-zero assumption, there is no need to introduce a renormalization in this case. See Remark 1.4.

In [71], Seong, Tolomeo, and the second author showed that the Gibbs measure for the focusing modified BO (with \(k = 3\)):

\[
\rho_{BO}(du) = Z_\infty^{-1} \chi_K \left( \int_T W(u^2) \, dx \right) e^{-\frac{1}{4} \int_T W(u^{k+1}) \, dx} \, du
\]

is not normalizable. Their argument can also be adapted to show that the focusing Gibbs measure is not normalizable for any \(k \geq 3\). We mention the work [17] by Brydges and Slade on a similar non-normalizability result (but with a completely different proof) in the context of the focusing \(\Phi_2^4\)-measure. See also Remark 1.7 below.
Lastly, we point out that, due to the use of the Wick renormalization, we can only consider integer values for $k$ in this case ($\delta = \infty$) and also in the intermediate case $0 < \delta < \infty$ which we will discuss next.

- **Construction of Gibbs measures for gILW on $\mathbb{T}$**.

  We finally discuss the construction of the Gibbs measure for the (scaled) gILW. Let us first consider the unscaled gILW (1.4) with the Hamiltonian $E_\delta(u)$ in (1.5). Fix $0 < \delta < \infty$.

  Our first goal is to construct the Gibbs measure $\rho_\delta$ of the form (1.21). Let $\mu_\delta$ be the base Gaussian measure of the form (1.22), which is nothing but the induced probability measure under the map:

  \[
  \omega \in \Omega \mapsto X_\delta(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{|K_\delta(n)|^{1/2}} e_n.
  \]  

  Here, $\{g_n\}_{n \in \mathbb{Z}^*}$ is as in (1.30) and $K_\delta(n)$ is given by

  \[
  K_\delta(n) := in\hat{G}_\delta(n) = n\coth(\delta n) - \frac{1}{\delta}
  \]  

  with $\hat{G}_\delta(n)$ as in (1.8). For each $n \in \mathbb{Z}^*$, we have $K_\delta(n) > 0$ and moreover, it follows from (1.10) and (1.11) that

  \[
  K_\delta(n) = |n| + O\left(\frac{1}{\delta}\right).
  \]

  See also Lemma 2.1 and Remark 2.2. This asymptotics allows us to show that, for any given $0 < \delta < \infty$, the Gaussian measures $\mu_\delta$ in (1.22) and $\mu_\infty$ in (1.37) are equivalent.\(^\text{10}\)

  Given $N \in \mathbb{N}$, let $P_N$ be the Dirichlet projection onto the frequencies $\{|n| \leq N\}$ and set $X_{\delta,N} := P_N X_\delta$. Note that, for each fixed $\delta > 0$ and $x \in \mathbb{T}$, the random variable $X_{\delta,N}(x)$ is a real-valued, mean-zero Gaussian random variable with variance

  \[
  \sigma_{\delta,N} := \mathbb{E}[X_{\delta,N}(x)] = \frac{1}{4\pi^2} \sum_{0 < |n| \leq N} \frac{2\pi}{K_\delta(n)} \sim \delta \log(N + 1).
  \]

  Given an integer $k \geq 2$, we define the Wick ordered monomial $\mathcal{W}(X_{\delta,N}^k) = \mathcal{W}_{\delta,N}(X_{\delta,N}^k)$ by setting

  \[
  \mathcal{W}(X_{\delta,N}^k) = H_k(X_{\delta,N}; \sigma_{\delta,N}),
  \]

  where $H_k(x; \sigma)$ is the Hermite polynomial of degree $k$; see Subsection 2.2. Then, $\mathcal{W}(X_{\delta,N}^k)$ converges, in $L^p(\Omega)$ for any finite $p \geq 1$ and also almost surely, to a limit, denoted by $\mathcal{W}(X_{\delta}^k)$, in $H^{-\varepsilon}(\mathbb{T})$ for any $\varepsilon > 0$; see Proposition 3.4. In particular, the truncated renormalized potential energy $\int_{\mathbb{T}} \mathcal{W}(X_{\delta,N}^{k+1}) dx$ converges, in $L^p(\Omega)$ for any finite $p \geq 1$ and also almost surely, to a limit denoted by $\int_{\mathbb{T}} \mathcal{W}(X_{\delta}^{k+1}) dx$.

\(^{10}\)Namely, mutually absolutely continuous.
With \( u_N = P_Nu \), we define the truncated Gibbs measure \( \rho_{\delta,N} \) by\(^{11}\)
\[
\rho_{\delta,N}(du) = Z_{\delta,N}^{-1}e^{-\frac{1}{\pi+1} \int_1^\infty W(u_N^{k+1})(x)dx} \, d\mu_\delta(u).
\]
(1.45)
We also define the truncated density \( G_{\delta,N}(u) \) by
\[
G_{\delta,N}(u) = e^{-\frac{1}{\pi+1} \int_1^\infty W(u^{k+1})(x)dx} = e^{-\frac{1}{\pi+1} \int_1^\infty H_{k+1}(u_N; \sigma_{\delta,N})dx}.
\]
(1.46)
In view of the convergence of the truncated renormalized potential energy mentioned above, we see that the truncated density \( G_{\delta,N} \) converges to the limiting density
\[
G_\delta(u) = e^{-\frac{1}{\pi+1} \int_1^\infty W(u^{k+1})dx}
\]
in probability with respect to \( \mu_\delta \), as \( N \to \infty \). We now state the construction of the limiting Gibbs measure \( \rho_\delta \) and its convergence property in the deep-water limit.

**Theorem 1.3.** Let \( k \in 2\mathbb{N} + 1 \). Then, the following statements hold.
(i) Let \( 0 < \delta \leq \infty \). Then, for any finite \( p \geq 1 \), we have
\[
\lim_{N \to \infty} G_{\delta,N}(u) = G_\delta(u) \quad \text{in } L^p(\mu_\delta).
\]
(1.47)
As a consequence, the truncated Gibbs measure \( \rho_{\delta,N} \) in (1.45) converges, in the sense of (1.47), to the limiting Gibbs measure \( \rho_\delta \) given by
\[
\rho_\delta(du) = Z_{\delta,N}^{-1}G_\delta(u) \, d\mu_\delta(u)
\]
\[
= Z_{\delta,N}^{-1}e^{-\frac{1}{\pi+1} \int_1^\infty W(u^{k+1})dx} \, d\mu_\delta(u).
\]
(1.48)
In particular, \( \rho_{\delta,N} \) converges to \( \rho_\delta \) in total variation. The resulting Gibbs measure \( \rho_\delta \) and the base Gaussian measure \( \mu_\delta \) are equivalent.

For \( 2 \leq \delta \leq \infty \), the rate of convergence (1.47) is uniform and thus the rate of convergence in total variation of \( \rho_{\delta,N} \) to \( \rho_\delta \) as \( N \to \infty \) is uniform for \( 2 \leq \delta \leq \infty \).
(ii) (deep-water limit of the Gibbs measures). Let \( 0 < \delta < \infty \). Then, the Gibbs measures \( \rho_\delta \) for \( gILW \) (1.13) and \( \rho_{BO} = \rho_\infty \) for \( gBO \) (1.12) constructed in Part (i) are equivalent. Moreover, \( \rho_\delta \) converges to \( \rho_{BO} \) in total variation, as \( \delta \to \infty \).

Furthermore, when \( k = 2 \), by replacing the truncated Gibbs measure \( \rho_{\delta,N} \) in (1.45) by the truncated Gibbs measure with a Wick-ordered \( L^2 \)-cutoff:
\[
\rho_{\delta,N}(du) = Z_{\delta,N}^{-1}e^{-\frac{1}{\pi+1} \int_1^\infty W(u_N^2)(x)dx} \, d\mu_\delta(u),
\]
(1.49)
the statements (i) and (ii) hold true for any fixed \( K > 0 \). Namely, for each \( 0 < \delta \leq \infty \), the truncated Gibbs measure \( \rho_{\delta,N} \) in (1.49) converges to the limiting Gibbs measure:
\[
\rho_\delta(du) = Z_{\delta}^{-1}e^{-\frac{1}{\pi+1} \int_1^\infty W(u^2)dx} \, d\mu_\delta(u)
\]
(1.50)
in the sense of the \( L^p(\mu_\delta) \)-convergence of the truncated densities as in (1.47). Moreover, the resulting Gibbs measure \( \rho_\delta \) in (1.50) and the base Gaussian measure endowed with the Wick-ordered \( L^2 \)-cutoff \( \chi_K(\int_1^\infty W(u^2)dx) \, d\mu_\delta(u) \) are equivalent. For \( 2 \leq \delta \leq \infty \), the rate of convergence in total variation of \( \rho_{\delta,N} \) to \( \rho_\delta \) as \( N \to \infty \) is uniform for \( 2 \leq \delta \leq \infty \).

\(^{11}\)Here, with a slight abuse of notation, we use the notation \( W(u_N^{k+1}) \) to mean \( H_{k+1}(u_N; \sigma_{\delta,N}) \). In the following, we use the notation \( W(u_N^{k+1}) \) with the understanding that there is the underlying Gaussian measure \( \mu_\delta \).
For $0 < \delta < \infty$, the Gibbs measures $\rho_\delta$ in (1.50) for ILW (1.1) and $\rho_{BO}$ in (1.39) for BO are equivalent and, as $\delta \to \infty$, the Gibbs measure $\rho_\delta$ converges to $\rho_{BO}$ in total variation.

Theorem 1.3 provides the first result on the construction of the Gibbs measures for the (generalized) ILW equation and also for the defocusing gBO equation (for $k \geq 3$).

Given a parameter-dependent Hamiltonian dynamics, it is of significant physical interest to study convergence of the associated Gibbs measures, which may be viewed as the first step toward studying convergence of dynamics at the Gibbs equilibrium. Theorem 1.3 (and Theorem 1.5) is the first such result for the (generalized) ILW equation, appearing in the study of fluids. We also mention a series of recent breakthrough results on the convergence of the Gibbs measures for quantum many-body systems to that for the nonlinear Schrödinger equation, led by two groups [54] (Lewin, Nam, Rougerie) and [35] (Fröhlich, Knowles, Schlein, and Sohinger). See these papers for the references therein. While these works establish only the convergence of the Gibbs measures, we also establish convergence of the corresponding dynamics; see Theorems 1.8 and 1.10 below.

Fix $k \in 2\mathbb{N} + 1$. For each fixed $0 < \delta \leq \infty$, the construction of the Gibbs measure (Theorem 1.3 (i)) follows from a standard application of Nelson’s estimate. The main novelty is Part (ii) of Theorem 1.3. In order to prove convergence of $\rho_\delta$ in the deep-water limit, we need to estimate the truncated densities $G_{\delta,N}(u)$, uniformly in both $\delta \gg 1$ and $N \in \mathbb{N}$. One subtle point is that for different values of $\delta \gg 1$, the base Gaussian measures $\mu_\delta$ are different. In order to overcome this issue, we indeed estimate $G_{\delta,N}(X_\delta)$ in $L^p(\Omega)$, uniformly in both $\delta \gg 1$ and $N \in \mathbb{N}$. Namely, we need to directly work with the probability measure $\mathbb{P}$ on $\Omega$. See Section 3 for details. We point out that this uniform bound on the truncated densities in $\delta \gg 1$ and $N \in \mathbb{N}$ also plays an important role in the dynamical part, which we discuss in the next subsection. Another key ingredient in establishing convergence of the Gibbs measures is ‘strong’ convergence of the base Gaussian measures $\mu_\delta$ (namely, convergence in the Kullback-Leibler divergence defined in (2.37); see Proposition 3.1).

When $k = 2$, the problem is no longer defocusing and thus Nelson’s argument is not directly applicable. While we could adapt the argument by Tzvetkov [87] for the BO equation, we instead use the variational approach as in the work [71] by Seong, Tolomeo, and the second author, which provides a slightly simpler argument.

**Remark 1.4.** We point out that, when $k = 2$, there is no need for a renormalization. Indeed, recalling that $H_2(x; \sigma) = x^3 - 3\sigma x$, under the mean-zero condition, we have

$$\int_{\mathbb{T}} W(u_N^3)dx = \int_{\mathbb{T}} u_N^3 dx - 3\sigma_\delta,N \int_{\mathbb{T}} u_N dx \geq \int_{\mathbb{T}} u_N^3 dx,$$

showing that a renormalization is not necessary in the $k = 2$ case. The same comment applies to Theorem 1.5 in the shallow-water limit.

Next, we consider the scaled gILW (1.17) with the Hamiltonian $\mathcal{E}_\delta(v)$ in (1.18). Let $k \in 2\mathbb{N} + 1$. For each fixed finite $\delta > 0$, the construction of the Gibbs measure $\bar{\rho}_\delta$ in (1.23) follows exactly the same lines as above. Define the base Gaussian measure $\bar{\mu}_\delta$ in (1.24) as the induced probability measure under the map:

$$\omega \in \Omega \mapsto \bar{X}_\delta(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|L_\delta(n)|^2} e_n,$$  \tag{1.51}
ON THE DEEP-WATER AND SHALLOW-WATER LIMITS OF THE ILW EQUATION

where \( \{g_n\}_{n \in \mathbb{Z}} \) is as in (1.30) and \( L_{\delta}(n) \) is given by

\[
L_{\delta}(n) := \frac{3}{\delta} K_{\delta}(n) = \frac{3}{\delta} \tilde{G}_{\delta}(n) = \frac{3}{\delta} \left( n \coth(\delta n) - \frac{1}{\delta} \right).
\]  

(1.52)

From (1.41), (1.51), and (1.52), we have

\[
\tilde{X}_{\delta} = \sqrt{\frac{\delta}{3}} X_{\delta}
\]

(1.53)

for any \( 0 < \delta < \infty \). Hence, by setting \( \tilde{X}_{\delta,N} = P_N \tilde{X}_{\delta} \), it follows from (1.43) that

\[
\tilde{\sigma}_{\delta,N} := \mathbb{E} \left[ \tilde{X}_{\delta,N}^2(x) \right] = \frac{1}{4\pi^2} \sum_{0 < |n| \leq N} \frac{2\pi}{L_{\delta}(n)}
\]

\[
= \frac{\delta}{3} \sigma_{\delta,N} \sim \delta \log(N + 1),
\]

(1.54)

where \( \sigma_{\delta,N} \) is as in (1.43).

Given \( N \in \mathbb{N} \), we define the truncated density \( \tilde{G}_{\delta,N}(u) \) by

\[
\tilde{G}_{\delta,N}(v) = e^{-\frac{1}{\delta} \int_{-T}^{T} W(v^{(k+1)}_N) dx},
\]

where \( v_N = P_N v \) and

\[
W(v^{(k+1)}_N) = H_{k+1}(v_N; \tilde{\sigma}_{\delta,N}).
\]

(1.55)

Then, we define the truncated Gibbs measure \( \tilde{\rho}_{\delta,N} \) by

\[
\tilde{\rho}_{\delta,N}(dv) = Z_{-1}^{-1} e^{-\frac{1}{\delta} \int_{-T}^{T} W(v^{(k+1)}_N) dx} d\tilde{\mu}_{\delta}(v).
\]

(1.56)

We now state our main result on convergence of the Gibbs measures in the shallow-water limit. Due to the use of the Wick renormalization for \( \delta > 0 \), we need to consider a “renormalized” power even in the shallow-water limit (\( \delta = 0 \)):

\[
\rho_{KdV}(dv) = Z_0^{-1} e^{-\frac{1}{\ell} \int_{-T}^{T} W(v^{(k+1)}_N) dx} d\tilde{\mu}_{0}(v),
\]

(1.57)

associated with the following gKdV:

\[
\partial_t v + \partial_x^3 v = \partial_x W(v^k).
\]

(1.58)

Here, \( W(v^\ell) \) is given by

\[
W(v^\ell) = H_\ell(v; \sigma_{KdV}),
\]

(1.59)

where \( \sigma_{KdV} \) is as in (1.36). In particular, when \( \delta = 0 \), \( W(v^\ell) \) is nothing but the usual Hermite polynomial of degree \( \ell \) with the finite variance parameter \( \sigma_{KdV} \), which is well defined without any limiting procedure.

**Theorem 1.5.** Let \( k \in 2\mathbb{N} + 1 \). Then, the following statements hold.

(i) Let \( 0 < \delta < \infty \). Then, for any finite \( p \geq 1 \), we have

\[
\lim_{N \to \infty} \tilde{G}_{\delta,N}(v) = \tilde{G}_{\delta}(v) := e^{-\frac{1}{\delta} \int_{-T}^{T} W(v^{(k+1)}) dx} \text{ in } L^p(\tilde{\mu}_{\delta}).
\]

(1.60)
As a consequence, the truncated Gibbs measure \( \tilde{\rho}_{\delta,N} \) in (1.56) converges, in the sense of (1.60), to the limiting Gibbs measure \( \tilde{\rho}_\delta \) given by

\[
\tilde{\rho}_\delta(dv) = Z^{-1}_\delta \tilde{G}_\delta(v) d\tilde{\mu}_\delta(v) = Z^{-1}_\delta e^{-\frac{1}{4} \int_T W(v^2) dx} d\tilde{\mu}_\delta(v).
\] (1.61)

In particular, \( \tilde{\rho}_{\delta,N} \) converges to \( \tilde{\rho}_\delta \) in total variation. The resulting Gibbs measure \( \tilde{\rho}_\delta \) and the base Gaussian measure \( \tilde{\mu}_\delta \) are equivalent.

For \( 0 < \delta \leq 1 \), the rate of convergence (1.60) is uniform and thus the rate of convergence in total variation of \( \tilde{\rho}_{\delta,N} \) to \( \tilde{\rho}_\delta \) as \( N \to \infty \) is uniform for \( 0 < \delta \leq 1 \).

(ii) (Shallow-water limit of the Gibbs measures). Let \( 0 < \delta < \infty \). Then, the Gibbs measures \( \tilde{\rho}_\delta \) for the scaled gILW (1.17) constructed in Part (i) and \( \rho_{KdV} \) in (1.57) for gKdV (1.58) are mutually singular. As \( \delta \to 0 \), however, \( \tilde{\rho}_\delta \) converges weakly to \( \rho_{KdV} \).

Furthermore, when \( k = 2 \), by replacing the truncated Gibbs measure \( \tilde{\rho}_{\delta,N} \) in (1.56) by the truncated Gibbs measure with a Wick-ordered \( L^2 \)-cutoff:

\[
\tilde{\rho}_{\delta,N}(dv) = \frac{1}{Z^{-1}_\delta \chi_K} \left( \int_T W(v^2) dx \right) e^{-\frac{1}{4} \int_T v^3 dx} d\tilde{\mu}_\delta(v),
\] (1.62)

the statements (i) and (ii) hold true for any fixed \( K > 0 \). Namely, for each \( 0 < \delta < \infty \), the truncated Gibbs measure \( \tilde{\rho}_{\delta,N} \) in (1.62) converges to the limiting Gibbs measure:

\[
\tilde{\rho}_\delta(dv) = \frac{1}{Z^{-1}_\delta \chi_K} \left( \int_T W(v^2) dx \right) e^{-\frac{1}{4} \int_T v^3 dx} d\tilde{\mu}_\delta(v)
\] (1.63)

in the sense of the \( L^p(\tilde{\mu}_\delta) \)-convergence of the truncated densities as in (1.60). Moreover, the resulting Gibbs measure \( \tilde{\rho}_\delta \) in (1.63) and the base Gaussian measure endowed with the Wick-ordered \( L^2 \)-cutoff \( \chi_K \left( \int_T W(v^2) dx \right) d\tilde{\mu}_\delta(v) \) are equivalent. For \( 0 < \delta \leq 1 \), the rate of convergence in total variation of \( \tilde{\rho}_{\delta,N} \) in (1.62) to \( \tilde{\rho}_\delta \) in (1.63) as \( N \to \infty \) is uniform for \( 0 < \delta \leq 1 \).

For \( 0 < \delta < \infty \), the Gibbs measures \( \tilde{\rho}_\delta \) in (1.63) for the scaled ILW (1.17) (with \( k = 2 \)) and \( \rho_{KdV} \) in (1.35) for KdV (with an \( L^2 \)-cutoff) are mutually singular. As \( \delta \to 0 \), however, the Gibbs measure \( \tilde{\rho}_\delta \) converges weakly to \( \rho_{KdV} \) in (1.35).

As compared to the deep-water limit (\( \delta \to \infty \)) studied in Theorem 1.3, we have an interesting phenomenon in this shallow-water limit (\( \delta \to 0 \)). This is due to the fact that, while \( L_\delta(n) \sim_\delta |n| \) for each \( \delta > 0 \), we have

\[
\lim_{\delta \to 0} L_\delta(n) = n^2
\]

for each \( n \in \mathbb{Z}^* \). See Lemma 2.3. This causes \( \tilde{\mu}_\delta \), \( \delta > 0 \), in (1.24) and the limiting Gaussian measure \( \tilde{\mu}_0 \) in (1.29) to be mutually singular. (For each finite \( \delta > 0 \), the Gaussian measure \( \tilde{\mu}_\delta \) is supported on \( H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T}) \), \( \varepsilon > 0 \), whereas \( \tilde{\mu}_0 \) is supported on \( H^{\frac{1}{2} - \varepsilon}(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}) \), \( \varepsilon > 0 \).) In view of the equivalence of the Gibbs measures and the base Gaussian measures, the first claim in Theorem 1.5(ii) essentially follows from this observation. Due to this mutual singularity, the mode of convergence of the Gibbs measures \( \tilde{\rho}_\delta \) to \( \rho_{KdV} \) in the shallow-water limit is much weaker as compared to that in the deep-water limit stated in Theorem 1.3(i). See Section 4 for details.
Remark 1.6. Let $k \in 2\mathbb{N} + 1$. Then, the Gibbs measure $\rho_{\text{KdV}}$ in (1.57) for the gKdV equation is a well-defined probability measure on $H^{\frac{1}{2} - \epsilon}_2(\mathbb{T}) \setminus H^{\frac{1}{2}}_2(\mathbb{T})$, $\epsilon > 0$. In view of (1.59) with (1.36), we have $0 < e^{-\frac{1}{k+1}} \int_{\mathbb{T}} W(u^{k+1})dx = e^{-\frac{1}{k+1}} \int_{\mathbb{T}} H_{k+1}(v; \sigma_{\text{KdV}})dx \lesssim 1$ on $H^{\frac{1}{2} - \epsilon}_2(\mathbb{T})$, which is clearly integrable with respect to the base Gaussian measure $\tilde{\mu}_0$ in (1.29).

Remark 1.7. (i) As mentioned above, in [71], Seong, Tolomeo, and the second author proved non-normalizability of the Gibbs measure (1.40) (with $k = 3$) for the focusing modified BO (for any cutoff size $K > 0$ on the Wick-ordered $L^2$-cutoff). For each fixed $\delta > 0$, the same argument allows us to prove non-normalizability of the Gibbs measure (with $k = 3$):

$$\rho_\delta(du) = Z_\delta^{-1} \chi_K \left( \int_{\mathbb{T}} W(u^2)dx \right) e^{-\frac{1}{k+1}} \int_{\mathbb{T}} W(u^{k+1})dx d\mu_\delta(u) \tag{1.64}$$

for the focusing modified ILW equation (1.20) with $k = 3$. A straightforward modification of the argument in [71] also yields non-normalizability of the focusing $\delta$ Gibbs measures (1.64) for any $k \geq 3$ and $0 < \delta \leq \infty$. For any $k \geq 3$ and $\delta > 0$, the same non-normalizability result also applies to the Gibbs measure:

$$\tilde{\rho}_\delta(v) = Z_\delta^{-1} \chi_K \left( \int_{\mathbb{T}} W(v^2)dx \right) e^{-\frac{1}{k+1}} \int_{\mathbb{T}} W(v^{k+1})dx d\tilde{\mu}_\delta(v) \tag{1.65}$$

for the focusing scaled gILW (namely, (1.17) with the $-$ sign on the nonlinearity).

(ii) In the shallow-water limit ($\delta = 0$), the Gibbs measure $\rho_{\text{KdV}}$ for the focusing gKdV (with an appropriate $L^2$-cutoff) exists up to the $L^2$-critical case ($k = 5$). For each $\delta > 0$, however, the Gibbs measure for the focusing scaled gILW, $\delta > 0$, is not normalizable and thus it is not possible to study the convergence problem for the Gibbs measures (as well as dynamics at the Gibbs equilibrium) in this case. One possible approach may be to study convergence of the truncated Gibbs measure $\tilde{\rho}_{\delta,N}$ in (1.56) (with a Wick-ordered $L^2$-cutoff) for the frequency-truncated scaled gILW to the Gibbs measure $\rho_{\text{KdV}}$ in (1.57) for the focusing gKdV (1.58), by taking $N \to \infty$ and $\delta \to 0$ in a related manner. The associated dynamical convergence problem may be of interest as well.

1.4. Dynamical problem. Our next goal is to study the associated dynamical problems. More precisely, our goal is to construct dynamics for the (scaled) gILW at the Gibbs equilibrium and then to show that the invariant Gibbs dynamics for the (scaled) gILW converges to that for gBO in the deep-water limit (and for gKdV in the shallow-water limit, respectively) in some appropriate sense. In the following, for the sake of the presentation, we refer to the study of the original (unscaled) gILW equation (and the gBO equation) for $0 < \delta < \infty$ as the deep-water regime, and the study of the scaled gILW equation for $0 \leq \delta < \infty$ (and the gKdV equation) as the shallow-water regime.

Let us first consider the deep-water regime. In Theorem 1.3, we constructed the Gibbs measure $\rho_\delta$ in (1.48) associated with the following renormalized Hamiltonian:

$$E_\delta(u) = \frac{1}{2} \int_{\mathbb{T}} u \mathcal{G}_\delta \partial_x u dx + \frac{1}{k+1} \int_{\mathbb{T}} W(u^{k+1})dx,$$  

\text{Recall our convention that by focusing, we also include the non-defocusing case, namely, (1.64) with $k \in 2\mathbb{N}$.}
when $k \in 2\mathbb{N}+1$. The corresponding Hamiltonian dynamics is formally given by the following renormalized gILW:

$$
\partial_t u - G_\delta \partial_x^2 u = \partial_x W(u^k),
$$

which needs to be interpreted in a suitable limiting sense. When $k = 2$, the measure construction does not require any renormalization (see Remark 1.4) and thus we study ILW (1.1) as the corresponding dynamical problem. As mentioned above, our first main goal is to construct dynamics at the Gibbs equilibrium. It is, however, a rather challenging problem to construct strong solutions to these equations with the Gibbsian initial data, even in a probabilistic sense. This is mainly due to the low regularity (namely, $H^{-\epsilon}(T) \setminus L^2(T)$, $\epsilon > 0$) of the Gibbsian initial data when $\delta > 0$. In fact, for $0 < \delta \leq \infty$, the only known case is for the Benjamin-Ono equation ($k = 2$ with $\delta = \infty$) by Deng [30], where he established deterministic local well-posedness result in a space, containing the support for the Gibbs measure, by a rather intricate argument and then used Bourgain’s invariant measure argument [13] to construct global-in-time dynamics at the Gibbs equilibrium. By invariance, we mean that (with $\delta = \infty$ in the BO case)

$$
\rho_\delta(\Phi_\delta^t(A)) = \rho_\delta(A)
$$

for any measurable set $A \subset H^{-\epsilon}(T)$ with some small $\epsilon > 0$ and any $t \in \mathbb{R}$, where $\Phi_\delta^t$ denotes the solution map:

$$
\Phi_\delta^t : u_0 \in H^{-\epsilon}(T) \mapsto u(t) = \Phi_\delta^t u_0 \in H^{-\epsilon}(T),
$$

satisfying the flow property

$$
\Phi_\delta^t(t_1 + t_2) = \Phi_\delta^t(t_1) \circ \Phi_\delta^t(t_2)
$$

for any $t_1, t_2 \in \mathbb{R}$. Here, we used $H^{-\epsilon}(T)$ for simplicity but it may be another Banach space, containing the support of the Gibbs measure (as in [30]). We also mention a recent work [37] on sharp global well-posedness of BO in almost critical spaces $H^s(T)$, $s > -\frac{1}{2}$, based on the complete integrability of the equation. When $0 < \delta < \infty$, the construction of strong solutions with the Gibbsian initial data is widely open even for $k = 2$. When $k \geq 3$, the difficulty of the problem increases significantly and nothing is known up to date for the renormalized gBO (with the Gibbs measure initial data):

$$
\partial_t u - H \partial_x^2 u = \partial_x W(u^k).
$$

For example, when $k = 3$ corresponding to the (renormalized) modified BO equation (mBO), the best known (deterministic) well-posedness result for mBO is in $H^{\frac{1}{2}}(T)$ [40], while the scaling-critical space is $L^2(T)$ and the support of the Gibbs measure is contained in $H^{-\epsilon}(T) \setminus L^2(T)$. When $0 < \delta < \infty$, we expect that the problem is much harder due to a rather complicated, non-algebraic nature of the dispersion symbol (see (1.2)).

In this paper, we do not aim to construct strong solutions. By a compactness argument, we instead construct global-in-time dynamics of weak solutions at the Gibbs equilibrium (without uniqueness), including the gBO case ($\delta = \infty$). In the deep-water limit, we also show that there exists a sequence $\{\delta_m\}_{m \in \mathbb{N}}$ of the depth parameters, tending to $\infty$, and solutions, at the Gibbs equilibrium, to the renormalized gILW (1.66) with $\delta = \delta_m$, converging almost surely to solutions, at the Gibbs equilibrium, to the renormalized gBO (1.69).
Theorem 1.8 (deep-water regime). Let $k \in 2\mathbb{N} + 1$. Then, the following statements hold.

(i) Let $0 < \delta \leq \infty$. Then, there exists a set $\Sigma_\delta$ of full measure with respect to the Gibbs measure $\rho_\delta$ in (1.48) constructed in Theorem 1.3 such that for every $u_0 \in \Sigma_\delta$, there exists a global-in-time solution $u \in C(\mathbb{R}; H^s(\mathbb{T}))$, $s < 0$, to the renormalized gILW equation (1.66) (and to the renormalized gBO equation (1.69) when $\delta = \infty$) with (mean-zero) initial data $u|_{t=0} = u_0$. Moreover, for any $t \in \mathbb{R}$, the law of the solution $u(t)$ at time $t$ is given by the Gibbs measure $\rho_\delta$.

(ii) There exists an increasing sequence $\{\delta_m\}_{m \in \mathbb{N}} \subset \mathbb{N}$ tending to $\infty$ such that the following holds.

- For each $m \in \mathbb{N}$, there exists a (random) global-in-time solution $u_{\delta_m} \in C(\mathbb{R}; H^s(\mathbb{T}))$, $s < 0$, to the renormalized gILW equation (1.66), with the depth parameter $\delta = \delta_m$, with the Gibbsian initial data distributed by the Gibbs measure $\rho_{\delta_m}$. Moreover, for any $t \in \mathbb{R}$, the law of the solution $u_{\delta_m}(t)$ at time $t$ is given by the Gibbs measure $\rho_{\delta_m}$.

- As $m \to \infty$, $u_{\delta_m}$ converges almost surely in $C(\mathbb{R}; H^s(\mathbb{T}))$ to a (random) solution $u$ to the renormalized gBO equation (1.69). Moreover, for any $t \in \mathbb{R}$, the law of the limiting solution $u(t)$ at time $t$ is given by the Gibbs measure $\rho_{BO} = \rho_\infty$ in (1.48).

When $k = 2$, the statements (i) and (ii) hold true without any renormalization (but with the Gibbs measures $\rho_{\delta_m}$ in (1.50) and $\rho_{BO}$ in (1.39)).

While our construction yields only weak solutions without uniqueness, Theorem 1.8 (and Theorem 1.10) is the first result on the construction of solutions with the Gibbsian initial data for both the (generalized) ILW equation ($k \geq 2$) and the gBO equation ($k \geq 3$). Furthermore, Theorem 1.8 presents the first convergence result for the (generalized) ILW equation from a statistical viewpoint. In Theorem 1.10 below, we state an analogous result in the shallow-water regime.

In proving Theorem 1.8 we employ the compactness approach used in [18, 74, 69], which in turn was motivated by the works of Albeverio and Cruzeiro [4] and Da Prato and Debussche [27] in the study of fluids. Our strategy is to start with the frequency-truncated dynamics (say, when $k \in 2\mathbb{N} + 1$):

$$\partial_t u_{\delta,N} - G_\delta \partial^2_x u_{\delta,N} = \partial_x P_N W((P_N u_{\delta,N})^k),$$  

(1.70)

which preserves the truncated Gibbs measure $\rho_{\delta,N}$ in (1.45). By exploiting the invariance of the truncated Gibbs measures $\rho_{\delta,N}$, we establish tightness (= compactness) of the push-forward measures $\nu_{\delta,N}$ (on space-time distributions) of the truncated Gibbs measures under the truncated dynamics (1.70), which implies convergence in law (up to a subsequence) of $\{u_{\delta,N}\}_{N \in \mathbb{N}}$. Then, for each fixed $\delta \gg 1$, the Skorokhod representation theorem (Lemma 2.15) allows us to prove almost sure convergence of the solution $u_{\delta,N}$ to (1.70) (after changes of underlying probability spaces) to a limit $u$, which satisfies the renormalized gILW (1.66) in the distributional sense. This part follows from exactly the same argument as those in [18, 74, 69]. Due to the use of the compactness, we only obtain global existence of a solution $u$ to (1.66) without uniqueness. The main ingredient in this step is the uniform bound on the truncated densities $\{G_{\delta,N}\}_{N \in \mathbb{N}}$ in (1.46). Here, we only need the uniformity in $N$ for each fixed $0 < \delta < \infty$, and it is with respect to the base Gaussian measure $\mu_\delta$ in (1.22).
A new ingredient in showing convergence of the gILW dynamics \([1.66]\) to the gBO dynamics \([1.12]\) is the uniform integrability of the truncated densities in both \(\delta \gg 1\) and \(N \in \mathbb{N}\) established in Theorem 1.3. As mentioned above, for different values of \(\delta \gg 1\), the base Gaussian measures \(\mu_\delta\) are different and thus we need to work directly with the underlying probability measure \(\mathbb{P}\) on \(\Omega\). This shows tightness of the probability measures \(\{\nu_{\delta,N}\}_{\delta \geq 1, N \in \mathbb{N}}\) constructed in the first step, in both \(\delta \gg 1\) and \(N \in \mathbb{N}\). Then, by the triangle inequality for the Lévy-Prokhorov metric, which characterizes weak convergence of probability measures on a separable metric space, and a diagonal argument together with the Skorokhod representation theorem (Lemma 2.15), we extract a sequence \(\{\delta_m\}_{m \in \mathbb{N}}\), tending to \(\infty\), such that the corresponding random variables \(u_{\delta_m}\) converges almost surely to a limit \(u\). In showing that \(u_{\delta_m}\) indeed satisfies the renormalized gILW \([1.66]\), we also need to apply the Skorokhod representation theorem for each \(m \in \mathbb{N}\).

**Remark 1.9.** Our notion of solutions constructed in Theorem 1.8 (and Theorem 1.10) basically corresponds to that of martingale solutions studied in the field of stochastic PDEs. See, for example, [29].

Next, we state our dynamical result in the shallow-water regime. In this case, we study the following renormalized scaled gILW:

\[
\partial_t v - \frac{3}{\delta} \mathcal{G}_\delta \partial_x^2 v = \partial_x \mathcal{W}(v^k),
\]

generated by the renormalized Hamiltonian:

\[
\mathcal{E}_\delta(v) = \frac{3}{2\delta} \int_T v \mathcal{G}_\delta \partial_x v dx + \frac{1}{k+1} \int_T \mathcal{W}(v^{k+1}) dx.
\]

As in the deep-water regime, we construct dynamics for \([1.71]\) as a limit of the frequency-truncated dynamics:

\[
\partial_t v_{\delta,N} - \frac{3}{\delta} \mathcal{G}_\delta \partial_x^2 v_{\delta,N} = \partial_x \mathcal{P}_N \mathcal{W}(\mathcal{P}_N v_{\delta,N}^k).
\]

**Theorem 1.10** (shallow-water regime). Let \(k \in 2\mathbb{N} + 1\). Then, the following statements hold.

(i) Let \(0 < \delta < \infty\). Then, there exists a set \(\hat{\Sigma}_\delta\) of full measure with respect to the Gibbs measure \(\bar{\rho}_\delta\) in \([1.61]\) constructed in Theorem 1.5 such that for every \(v_0 \in \hat{\Sigma}_\delta\), there exists a global-in-time solution \(v \in C(\mathbb{R}; H^s(\mathbb{T}))\), \(s < 0\), to the renormalized scaled gILW equation \([1.71]\) with (mean-zero) initial data \(v|_{t=0} = v_0\). Moreover, for any \(t \in \mathbb{R}\), the law of the solution \(v(t)\) at time \(t\) is given by the Gibbs measure \(\bar{\rho}_\delta\).

(ii) There exists a decreasing sequence \(\{\delta_m\}_{m \in \mathbb{N}} \subset \mathbb{R}_+\) tending to \(0\) such that the following holds.

- For each \(m \in \mathbb{N}\), there exists a (random) global-in-time solution \(v_{\delta_m} \in C(\mathbb{R}; H^s(\mathbb{T}))\), \(s < 0\), to the renormalized scaled gILW equation \([1.71]\), with the depth parameter \(\delta = \delta_m\), with the Gibbsian initial data distributed by the Gibbs measure \(\bar{\rho}_{\delta_m}\). Moreover, for any \(t \in \mathbb{R}\), the law of the solution \(v_{\delta_m}(t)\) at time \(t\) is given by the Gibbs measure \(\bar{\rho}_{\delta_m}\).

- As \(m \to \infty\), \(v_{\delta_m}\) converges almost surely in \(C(\mathbb{R}; H^s(\mathbb{T}))\) to a (random) solution \(v\) to the gKdV equation \([1.58]\). Moreover, for any \(t \in \mathbb{R}\), the law of the limiting solution \(v(t)\) at time \(t\) is given by the Gibbs measure \(\bar{\rho}_{\text{KdV}}\) in \([1.57]\).
When \( k = 2 \), the statements (i) and (ii) hold true without any renormalization (but with the Gibbs measures \( \tilde{\rho}_\delta \) in (1.63) and \( \rho_{\text{KdV}} \) in (1.35)).

With the uniform integrability on the truncated densities in both \( 0 < \delta \leq 1 \) and \( N \in \mathbb{N} \) (established in Theorem 1.3), Theorem 1.10 follows from exactly the same argument in the proof of Theorem 1.8 and hence we omit details.

**Remark 1.11.** Theorems 1.8 and 1.10 yield the construction and convergence of weak solutions. Due to the use of a compactness argument, we do not have any uniqueness statement. While these solutions are distributional solutions, they do not satisfy the Duhamel formulation, which is the usual notion for strong solutions in the study of dispersive PDEs. Furthermore, due to the lack of uniqueness, these solutions do not enjoy the flow property and thus do not satisfy the invariance property as stated in (1.68). This is the reason why we have a weaker invariance property in Theorems 1.8 and 1.10 which is, for example, not sufficient to imply the Poincaré recurrence property. See [81] for a further discussion. We also expect that a suitable uniqueness statement would allow us to show convergence of the entire family \( \{u_\delta\}_{\delta \geq 1} \) in the deep-water limit \( (\delta \to \infty) \) (and \( \{u_\delta\}_{0 < \delta \ll 1} \) in the shallow-water limit \( (\delta \to 0) \)).

Therefore, it would be of significant interest to study probabilistic construction of strong solutions to the (scaled) ILW equation with the Gibbsian initial data. As mentioned above, the \( k \geq 3 \) case seems to be out of reach at this point. Even as for the \( k = 2 \) case, the problem is very challenging. For example, in studying low regularity well-posedness of the BO equation, the gauge transform [82] plays a crucial role. For the ILW equation, however, existence of such a gauge transform is unknown; see [49, p. 128].

When \( k = 2 \), another possible approach would be to exploit the complete integrability of the ILW equation. In the case of the BO equation, there are recent breakthrough works [37, 47] on sharp global well-posedness in \( H^s(\mathbb{T}) \), \( s > -\frac{5}{2} \). Even with the complete integrability, however, the low regularity well-posedness of the ILW equation seems to be very challenging. See [23, 22] for recent developments in this direction.

Lastly, let us point out that, as for the gKdV equation (1.19) (and also (1.58)), there is a good well-posedness theory with the Gibbsian initial data; see [12, 77, 21]. In particular, in a recent work [21], Chapouto and Kishimoto completed the program initiated by Bourgain [13] on the construction of invariant Gibbs dynamics for the (defocusing) gKdV (1.19) for any \( k \in 2\mathbb{N} + 1 \).

**Remark 1.12.** When \( k = 2 \), the ILW equation is known to be completely integrable with an infinite sequence of conservation laws of increasing regularities. In this work, we study the construction and convergence of the Gibbs measures associated with the Hamiltonians and the corresponding dynamical problem. For the ILW equation, it is also possible to study the construction of invariant measures associated with the higher order conservation laws. See [39, 88, 89, 90, 31] for such construction of invariant measures associated with the higher

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13*The solution map to the frequency-truncated equation such as (1.70) enjoys the flow property, and thus a suitable uniqueness statement would imply the flow property for the limiting dynamics.*

14*In view of the absolute continuity of the Gibbs measure with respect to the base Gaussian measure, it suffices to study probabilistic local well-posedness with the Gaussian initial data \( X_\delta \) in (1.41) (or \( \tilde{X}_\delta \) in (1.51)) in the spirit of [14, 25, 77].*
order conservation laws in the context of the KdV and BO equations. Once such construction is done, it would be of strong interest to study the related convergence problem. We plan to address this issue in a forthcoming work [24]. These invariant measures will be supported on smooth(er) functions and thus this problem is of importance even from the physical point of view.

We point out that, after the completion of the current paper, there have been very recent progresses on low-regularity well-posedness and convergence issues for the ILW equation (1.1), at the $L^2$-level [44, 23] and in negative Sobolev spaces [22].

Remark 1.13. In this work, we focus our attention to the circle $\mathbb{T}$. From the physical point of view, it seems natural to study the problem on the real line. The difficulty of this problem comes from not only the roughness of the support but also the integrability of typical functions. See [15, 68] for the construction of invariant Gibbs dynamics in the context of the nonlinear Schrödinger equations on the real line. See also [48]. In the focusing case (including the $k = 2$ case), however, we expect a triviality result (namely, a large-torus limit of the periodic Gibbs measures is “trivial” such as the Dirac delta measure on the trivial function (= the zero function) or a Gaussian measure); see [78, 84] for such triviality results and further discussions in the context of the Gibbs measures associated with the focusing nonlinear Schrödinger equations on the real line.

Remark 1.14. There are recent works [36, 94, 95] on convergence of stochastic dynamics at the Gibbs equilibrium. One key difference between our work and these works is that, in [36, 94, 95], a single Gibbs measure remains invariant for the entire one-parameter family of dynamics, whereas, in our work, the Gibbs measure (and even the base Gaussian measure) varies as the depth parameter $\delta$ changes, requiring us to first establish the convergence at the level of the Gibbs measures.

Organization of the paper. In Section 2, after introducing some notations, we study basic properties of the variance parameters $K_\delta(n)$ in (1.42) and $L_\delta(n)$ in (1.52). We then go over some tools from stochastic analysis and different modes of convergence for probability measures and random variables. In Section 3, we study the construction and convergence of the Gibbs measures in the deep-water regime and present the proof of Theorem 1.3. In Section 4, we study the corresponding problem in the shallow-water regime (Theorem 1.5). In Section 5, we then study the dynamical problem and present the proof of Theorem 1.8.

2. Preliminaries

Notations. By $A \lesssim B$, we mean $A \leq CB$ for some constant $C > 0$. We use $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$, if there is some small $c > 0$, such that $A \leq cB$. We may use subscripts to denote dependence on external parameters; for example, $A \lesssim_\delta B$ means $A \leq C(\delta)B$.

Throughout this paper, we fix a rich enough probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which all the random objects are defined. The realization $\omega \in \Omega$ is often omitted in the writing. For a random variable $X$, we denote by $\mathcal{L}(X)$ the law of $X$. 
We set \( e_n(x) = e^{inx}, \ n \in \mathbb{Z} \) and \( \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} \). Given \( N \in \mathbb{N} \), let \( P_N \) be the Dirichlet projection onto the frequencies \( \{|n| \leq N\} \) defined by

\[
P_Nf(x) = \frac{1}{2\pi} \sum_{|n| \leq N} \hat{f}(n)e_n(x).
\]

Let \( s \in \mathbb{R} \) and \( 1 \leq p \leq \infty \). We define the \( L^2 \)-based Sobolev space \( H^s(T) \) by the norm:

\[
\|f\|_{H^s} = \|\langle n \rangle^s \hat{f}(n)\|_{\ell^2}.
\]

We also define the \( L^p \)-based Sobolev space \( W^{s,p}(T) \) by the norm:

\[
\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p} = \|F^{-1}[\langle n \rangle^s \hat{f}(n)]\|_{L^p},
\]

where \( F^{-1} \) denotes the inverse Fourier transform. When \( p = 2 \), we have \( H^s(T) = W^{s,2}(T) \).

We use short-hand notations such as \( L^q_T H^s(x) \) and \( L^p_\Omega H^s(x) \) for \( L^q([-T,T]; H^s(T)) \) and \( L^p(\Omega; H^s(T)) \), respectively.

In the following, we only work with real-valued functions on \( T \) or on \( \mathbb{R} \times T \).

### 2.1. On the variance parameters.

In this subsection, we establish elementary lemmas on the variance parameters \( K_\delta(n) \) in (1.42) and \( L_\delta(n) \) in (1.52) for the Gaussian Fourier series \( X_\delta \) in (1.41) and \( \tilde{X}_\delta \) in (1.51), respectively.

**Lemma 2.1.** Let \( K_\delta(n) \) be as in (1.42). Then, for any \( \delta > 0 \), we have

\[
\max(0, |n| - \frac{1}{\delta}) \leq K_\delta(n) = n \coth(\delta n) - \frac{1}{\delta} \leq |n|,
\]

where the above inequalities are strict for \( n \neq 0 \). In particular, we have

\[
K_\delta(n) \sim_\delta |n|
\]

for any \( n \in \mathbb{Z}^* \). Furthermore, for each fixed \( n \in \mathbb{Z}^* \), \( K_\delta(n) \) is strictly increasing in \( \delta \geq 1 \) and converges to \( |n| \) as \( \delta \to \infty \).

The bound (1.11) implies that, for \( \delta \geq 2 \), we have

\[
K_\delta(n) \geq |n| - \frac{1}{2} \sim |n|
\]

for any \( n \in \mathbb{Z}^* \).

**Proof.** For \( x \in \mathbb{R} \setminus \{0\} \), define \( \mathfrak{h} \) by

\[
\mathfrak{h}(x) = 1 - x \coth(x) + |x| = 1 + |x| - x \frac{e^x + e^{-x}}{e^x - e^{-x}}
\]

such that

\[
K_\delta(n) = |n| - \frac{1}{\delta} \mathfrak{h}(\delta n).
\]

In view of (1.15), we set \( \mathfrak{h}(0) = 0 \) such that \( \mathfrak{h} \) is continuous. We claim that

\[
0 < \mathfrak{h}(x) < \min(1, |x|)
\]

(2.5)
for any $x \in \mathbb{R} \setminus \{0\}$. Indeed, we first note that $h$ is an even function. For $x > 0$, a direct computation shows

$$h(x) = 1 + x - x \frac{e^{2x} + 1}{e^{2x} - 1} = 1 - \frac{2x}{e^{2x} - 1} \in (0, 1), \quad (2.6)$$

$$h(x) - x = 1 - x \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} < 0,$$

from which the claim (2.5) follows. Then, the bound (2.4) follows from (2.1) and (2.5). The equivalence (2.2) is a direct consequence of (2.1) and the fact that $K_{\delta}(n) > 0$ for $n \in \mathbb{Z}^*$. 

Fix $n \in \mathbb{N}$. By writing $K_{\delta}(n) = |n| - \frac{h(\delta n)}{\delta n} n$, the claimed strict monotonicity of $K_{\delta}(n)$ in $\delta \geq 1$ follows once we show that $\frac{h(x)}{x}$ is strictly decreasing and its limit as $x \to \infty$ is 0. A direct computation shows that

$$\frac{d}{dx} \left( \frac{h(x)}{x} \right) = -\frac{e^{4x} - 2e^{2x} - 4x^2 e^{2x} + 1}{x^2(e^{2x} - 1)^2} < 0$$

for $x \geq 1$. Namely, $K_{\delta}(n)$ is increasing for $\delta \geq 1$. From (2.6), we have $\frac{h(x)}{x} = \frac{1}{x} - \frac{2}{e^{2x} - 1}$, from which we conclude $\lim_{x \to \infty} \frac{h(x)}{x} = 0$. This concludes the proof of Lemma 2.1. \hfill \square

**Remark 2.2.** Note that we have $q_{\delta}(n) = \delta^{-1} h(\delta n)$, where $q_{\delta}(n)$ is as in (1.10). Then, (2.5) in Lemma 2.1 yields (1.11) with the right-hand side replaced by $\frac{1}{\delta}$. 

Next, we state basic properties of $L_{\delta}(n)$ defined in (1.52). Given $\delta > 0$, it follows from $L_{\delta}(n) = \frac{3}{\delta} K_{\delta}(n)$ and Lemma 2.1 that

$$L_{\delta}(n) \sim_{\delta} |n| \quad (2.7)$$

for any $n \in \mathbb{Z}^*$.

**Lemma 2.3.** The following statements hold.

(i) $0 < L_{\delta}(n) < n^2$ for any $\delta > 0$ and $n \in \mathbb{Z}^*$.

(ii) For each $n \in \mathbb{Z}^*$, $L_{\delta}(n)$ increases to $n^2$ as $\delta \to 0$.

(iii) We have

$$L_{\delta}(n) \gtrsim \begin{cases} n^2, & \text{if } \delta |n| \lesssim 1, \\ |n|, & \text{if } \delta |n| \gg 1 \text{ and } \delta \lesssim 1. \end{cases}$$

In particular, the following uniform bound holds:

$$\inf_{0 < \delta \lesssim 1} L_{\delta}(n) \gtrsim |n| \quad (2.8)$$

for any $n \in \mathbb{Z}^*$.

(iv) Define $h(n, \delta)$ by

$$L_{\delta}(n) = n^2 - h(n, \delta)n^2. \quad (2.9)$$

Then, we have

$$\sum_{n \in \mathbb{Z}} h^2(n, \delta) = \infty \quad (2.10)$$

for any $\delta > 0$. 

Proof. From (1.32), we have \( L_\delta(n) = \frac{3}{\delta} K_\delta(n) \). Hence, from Lemma 2.1, we have \( L_\delta(n) > 0 \) for any \( n \in \mathbb{Z}^* \). On the other hand, from the Mittag-Leffler expansion \([3], (11) \) on p. 189], we have

\[
\frac{\pi z \coth(\pi z)}{i} = 1 + \sum_{k=1}^{\infty} \frac{2z^2}{k^2 + z^2}
\]

for \( z \in \mathbb{C} \setminus i\mathbb{Z} \). Then, from (1.52) and (2.11), we have

\[
L_\delta(n) = 6n^2 \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2 + \delta^2 n^2}
\]

\[
= \frac{6n^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} - 6n^2 \sum_{k=1}^{\infty} \left( \frac{1}{k^2 \pi^2} - \frac{1}{k^2 \pi^2 + \delta^2 n^2} \right)
\]

\[
= n^2 - n^2 \sum_{k=1}^{\infty} \frac{6\delta^2 n^2}{k^2 \pi^2 (k^2 \pi^2 + \delta^2 n^2)}
\]

for any \( \delta > 0 \) and \( n \in \mathbb{Z} \). Hence, we conclude that \( L_\delta(n) < n^2 \) for any \( n \in \mathbb{Z}^* \). This proves the claim (i).

From (2.9) and (2.12), we have

\[
h(n, \delta) = 6\delta^2 \sum_{k=1}^{\infty} \frac{n^2}{k^2 \pi^2 (k^2 \pi^2 + \delta^2 n^2)},
\]

which tends to 0 as \( \delta \to 0 \). We also note that the expression after the first equality in (2.12) shows that \( L_\delta(n) \) is monotonic in \( \delta \). This yields the claim (ii). From (2.13), we see that, as \( n \to \infty \), \( h(n, \delta) \not\to 0 \) (for each fixed \( \delta > 0 \)), which yields (2.10). This proves the claim (iv).

Lastly, we prove (iii). Suppose \( \delta |n| \lesssim 1 \). Then, from (2.12), we have

\[
L_\delta(n) = 6n^2 \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2 + \delta^2 n^2} \gtrsim n^2 \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} \gtrsim n^2.
\]

Now, suppose \( \delta |n| \gg 1 \) and \( \delta \lesssim 1 \). Then, from (2.12) and a Riemann sum approximation, we have

\[
L_\delta(n) \gtrsim \sum_{k=1}^{\infty} \frac{|n|}{k^2 \pi^2 + \delta^2 n^2} \gtrsim \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{1}{\pi^2 (\frac{k}{\delta |n|})^2 + 1} \gtrsim 1.
\]

Note that the implicit constants in (2.14) and (2.15) are independent of \( \delta \). This proves the claim (iii).

\[\Box\]

2.2. Tools from stochastic analysis. In the following, we review some basic facts on the Hermite polynomials and the Wiener chaos estimate. See, for example, [52, 63].

We define the \( k \)th Hermite polynomials \( H_k(x; \sigma) \) with variance \( \sigma \) via the following generating function:

\[
e^{tx - \frac{1}{2} \sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma)
\]
for \( t, x \in \mathbb{R} \) and \( \sigma > 0 \). When \( \sigma = 1 \), we set \( H_k(x) = H_k(x; 1) \). Then, we have
\[
H_k(x; \sigma) = \sigma^k \frac{1}{\sqrt{2^k \pi}} e^{-x^2/(2\sigma^2)}.
\]

It is well known that \( \{H_k/\sqrt{k!}\}_{k \in \mathbb{N} \cup \{0\}} \) form an orthonormal basis of \( L^2(\mathbb{R}; \frac{1}{\sqrt{2 \pi}} e^{-x^2/2} dx) \). In the following, we list the first few Hermite polynomials for readers’ convenience:
\[
\begin{align*}
H_0(x; \sigma) &= 1, \\
H_1(x; \sigma) &= x, \\
H_2(x; \sigma) &= x^2 - \sigma, \\
H_3(x; \sigma) &= x^3 - 3\sigma x, \\
H_4(x; \sigma) &= x^4 - 6\sigma x^2 + 3\sigma^2.
\end{align*}
\]

From (2.16), we obtain the following recursion relation:
\[
\frac{\partial}{\partial x} H_k(x; \sigma) = kH_{k-1}(x; \sigma)
\]
for any \( k \in \mathbb{N} \), and the following identity:
\[
H_k(x + y; \sigma) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_\ell(y; \sigma).
\]

Let \( \{g_n\}_{n \in \mathbb{Z}} \) be an independent family of standard complex-valued Gaussian random variables conditioned that \( g_n = g_{-n} \). We first recall the following bound:
\[
\sup_{n \in \mathbb{Z}} |g_n| \leq C_{\varepsilon, \omega} < \infty
\]
almost surely for some random constant \( C_{\varepsilon, \omega} > 0 \); see Lemma 3.4 in [25]. See also Appendix 3.4 in [64].

We define a real-valued, mean-zero Gaussian white noise \( W \) on \( T \) by
\[
W(x; \omega) = \sum_{n \in \mathbb{Z}} g_n(\omega)e^{-i n x}.
\]

Next, we introduce the isonormal Gaussian process \( \{W_f : f \in L^2(T)\} \) associated to the Gaussian white noise \( W \).

**Definition 2.4.** The isonormal Gaussian process \( \{W_f : f \in L^2(T)\} \) is a real-valued, mean-zero Gaussian process indexed by the real separable Hilbert space \( L^2(T) \) such that
\[
\mathbb{E}[W_f W_g] = \langle f, g \rangle_{L^2_T}
\]
for \( f, g \in L^2(T) \). Moreover, we can realize \( W_f \) as follows:
\[
f \in L^2(T) \mapsto W_f = \langle f, W \rangle_{L^2_T} = \sum_{n \in \mathbb{Z}} \hat{f}(n) g_n(\omega),
\]
where \( W \) is as in (2.20).

**Remark 2.5.** The action (2.21) on \( f \) by the white noise is referred to as the white noise functional in [74, 69]. Note that \( W_f \) is basically the ‘periodic’ Wiener integral on \( T \).
In the following, we denote $L^2(\Omega, \sigma\{W\}, \mathbb{P})$ the space of real-valued, square-integrable random variables that are measurable with respect to $W$. We present below a fundamental result in Gaussian analysis, providing us an orthogonal decomposition of this $L^2$-probability space. Let $\Gamma^W_k$ be the $L^2(\Omega)$-completion of the linear span of the set $\{H_k(W_f) : f \in L^2(\mathbb{T}); \|f\|_{L^2} = 1\}$. We call $\Gamma^W_k$ the $k$th Wiener chaos associated to $W$. The following Wiener-Ito chaos decomposition holds:

$$L^2(\Omega, \sigma\{W\}, \mathbb{P}) = \bigoplus_{k=0}^{\infty} \Gamma^W_k. \quad (2.22)$$

The orthogonal decomposition (2.22) indicates that random variables belonging to Wiener chaoses of different orders are uncorrelated (namely, $L^2(\Omega)$-orthogonal). See also the following particular case that we will often use in our computations.

**Lemma 2.6.** Let $Y_1, Y_2$ be two real-valued, mean-zero, and jointly Gaussian random variables with variances $\sigma_1 = \mathbb{E}[Y_1^2] > 0$ and $\sigma_2 = \mathbb{E}[Y_2^2] > 0$. Then, for $k, m \in \mathbb{N} \cup \{0\}$, we have

$$\mathbb{E}\left[H_k(Y_1; \sigma_1)H_m(Y_2; \sigma_2)\right] = 1_{k=m} \cdot k!(\mathbb{E}[Y_1Y_2])^k. \quad (2.23)$$

For example, with $f, h \in L^2(\mathbb{T})$, the random variables $Y_1 = W_f$ and $Y_2 = W_h$, with $\sigma_1 = \|f\|_{L^2}^2$ and $\sigma_2 = \|h\|_{L^2}^2$ satisfy the identity (2.23).

Next, we state the Wiener chaos estimate, which is a consequence of Nelson’s hypercontractivity \[62\]. See, for example, \[80\, \text{Theorem I.22}\]. See also \[83\, \text{Proposition 2.4}\].

**Lemma 2.7** (Wiener chaos estimate). Let $\{g_n\}_{n \in \mathbb{Z}}$ be an independent family of standard complex-valued Gaussian random variables conditioned that $g_n = \overline{g}_{-n}$. Given $k \in \mathbb{N}$, let $\{Q_j\}_{j \in \mathbb{N}}$ be a sequence of polynomials in $g = \{g_n\}_{n \in \mathbb{Z}}$ of degrees at most $k$ such that $\sum_{j \in \mathbb{N}} Q_j(g) \in \mathbb{R}$, almost surely. Then, for any finite $p \geq 2$, we have

$$\left\|\sum_{j \in \mathbb{N}} Q_j(g)\right\|_{L^p(\Omega)} \leq (p - 1)^{\frac{k}{2}} \left\|\sum_{j \in \mathbb{N}} Q_j(g)\right\|_{L^2(\Omega)}.$$

Lastly, we provide a brief discussion on the Wick renormalization.

• **Wick renormalization.** Let $\{\beta_k, k \in \mathbb{N}\}$ be i.i.d. real-valued standard Gaussian random variables, which can be built from the Gaussian white noise $W$ in (2.20). Consider the polynomial $Q(x_1, \ldots, x_n)$ with $n$ variables. We denote its degree by $\deg(Q)$. Then, the random variable $Q(\beta_1, \ldots, \beta_n)$ belongs to the sum of the first $\deg(Q)$ Wiener chaoses, that is,

$$Q(\beta_1, \ldots, \beta_n) \in \bigoplus_{k \leq \deg(Q)} \Gamma^W_k.$$

One can find a unique polynomial $P$ with the same degree and the same coefficient on the leading order term such that $P(\beta_1, \ldots, \beta_n) \in \Gamma^W_{\deg(Q)}$, that is, $P(\beta_1, \ldots, \beta_n)$ is the projection of $Q(\beta_1, \ldots, \beta_n)$ onto $\Gamma^W_{\deg(Q)}$. We call such a polynomial $P$ as the Wick-ordered version of $Q$, and we write $P = W(Q)$.

**Example 2.8.** (i) Consider the polynomial $Q(x_1, \ldots, x_n) = x_1^{k_1} \cdots x_n^{k_n}$. Then, we have

$$W(Q)(x_1, \ldots, x_n) = \prod_{j=1}^{n} H_{k_j}(x_j),$$
where $H_k$ is the $k$th Hermite polynomial with variance $\sigma = 1$.

(ii) Given $N \in \mathbb{N}$, consider the following truncated random Fourier series $X_{\delta,N}$:

$$X_{\delta,N}(x, \omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|K_\delta(n)|^2} e_n. \tag{2.24}$$

Note that $X_{\delta,N} = P_N X_\delta$, where $X_\delta$ is as in (1.41). For each $x \in \mathbb{T}$, $X_{\delta,N}(x)$ is a real-valued, mean-zero Gaussian random variable with variance $\sigma_{\delta,N}$ in (1.43). Then, the Wick-ordered version of $X_k^{\delta,N}, k \in \mathbb{N}$, is given by $W(X_k^{\delta,N}) = H_k(X_{\delta,N} ; \sigma_{\delta,N})$. Compare this with (1.44).

2.3. Various modes of convergence for probability measures and random variables.

We conclude this section by going over various modes of convergence for probability measures and random variables. See, for example, [76, Chapter 3] and [85, Chapter 2] for further discussions. See also [32].

- **Convergence in probability and the Ky-Fan distance.**

  Let $X$ and $Y$ be two real-valued random variables defined on a common probability space $\Omega$. Then, the **Ky-Fan distance** between $X$ and $Y$ is defined by

$$d_{KF}(X, Y) = \mathbb{E}[1 \land |X - Y|],$$

where $a \land b := \min(a, b)$. It is known that the Ky-Fan distance characterizes convergence in probability. Namely, a sequence $\{Z_n\}_{n \in \mathbb{N}}$ of random variables converges in probability to some limit $Z$ if and only if $d_{KF}(Z_n, Z) \to 0$ as $n \to \infty$.

The usual continuous mapping theorem [9, Problem 5.17 on p. 83] states that if a sequence $\{Z_n\}_{n \in \mathbb{N}}$ converges to a limit $Z$ in probability, then, given a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, $\{\phi(Z_n)\}_{n \in \mathbb{N}}$ converges to $\phi(Z)$ in probability. For our purpose, we need to extend this continuous mapping theorem for **uniform** convergence in probability.

**Lemma 2.9** (uniform continuous mapping theorem). Let $\mathcal{J} \subset [0, \infty]$ be an index set. Suppose that $\{Z_{\delta,n}\}_{n \in \mathbb{N}}$ converges in probability to a limit $Z_\delta$ uniformly in $\delta \in \mathcal{J}$, as $n \to \infty$ in the following sense:

$$\lim_{n \to \infty} \sup_{\delta \in \mathcal{J}} d_{KF}(Z_{\delta,n}, Z_\delta) = 0 \tag{2.25}$$

or equivalently, for any $\eta > 0$,

$$\lim_{n \to \infty} \sup_{\delta \in \mathcal{J}} \mathbb{P}(|Z_{\delta,n} - Z_\delta| > \eta) = 0. \tag{2.26}$$

Suppose that the family of random variables $\{Z_\delta\}_{\delta \in \mathcal{J}}$ is tight, meaning that for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathbb{R}$ such that

$$\sup_{\delta \in \mathcal{J}} \mathbb{P}(Z_\delta \in K_\varepsilon^c) \leq \varepsilon. \tag{2.27}$$

Then, given any continuous function $\phi : \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{n \to \infty} \sup_{\delta \in \mathcal{J}} d_{KF}(\phi(Z_{\delta,n}), \phi(Z_\delta)) = 0. \tag{2.28}$$

Note that the tightness assumption on $\{Z_\delta\}_{\delta \in \mathcal{J}}$ is crucial.
Proof. Let us first show the equivalence of (2.25) and (2.26). Let 0 < η < 1. Then, by Markov’s inequality and (2.25), we have

$$\mathbb{P}(|Z_{\delta,n} - Z_\delta| > \eta) \leq \mathbb{E}\left[ 1_{\{|Z_{\delta,n} - Z_\delta| > \eta\}} \frac{|Z_{\delta,n} - Z_\delta|}{\eta} \right] \leq \frac{1}{\eta} d_{KF}(Z_{\delta,n}, Z_\delta)$$

and

$$d_{KF}(Z_{\delta,n}, Z_\delta) = \mathbb{E}\left[ |Z_{\delta,n} - Z_\delta| \land 1 \right] \leq \eta + \mathbb{P}(|Z_{\delta,n} - Z_\delta| \geq \eta).$$

This proves the equivalence of (2.25) and (2.26).

We now prove (2.28). Fix \( \beta > 0 \). In view of (2.27), there exists \( \eta = \eta(\beta) \geq 1 \) such that

$$\sup_{\delta \in \mathcal{J}} \mathbb{P}(|Z_\delta| > \eta) \leq \beta. \quad (2.29)$$

Since \( \phi \) is continuous, it is uniformly continuous on \([-2\eta, 2\eta]\). In particular, there exists small \( \varepsilon = \varepsilon(\phi, \beta) > 0 \) such that

$$|\phi(x) - \phi(y)| \leq \beta, \text{ whenever } x, y \in [-2\eta, 2\eta] \text{ with } |x - y| \leq \varepsilon. \quad (2.30)$$

Without loss of generality, we assume that \( \varepsilon \leq \eta \). Note that these parameters \( \varepsilon, \beta, \) and \( \eta \) do not depend on \( n \in \mathbb{N} \).

From (2.26), we have

$$\lim_{n \to \infty} \sup_{\delta \in \mathcal{J}} \mathbb{E}\left[ (|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1) 1_{\{|Z_{\delta,n} - Z_\delta| > \varepsilon\}} \right]$$

$$\leq \lim_{n \to \infty} \sup_{\delta \in \mathcal{J}} \mathbb{P}(|Z_{\delta,n} - Z_\delta| > \varepsilon) = 0. \quad (2.31)$$

On the other hand, from (2.30) and (2.29), we have

$$\mathbb{E}\left[ (|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1) 1_{\{|Z_{\delta,n} - Z_\delta| \leq \varepsilon\}} \right]$$

$$\leq \mathbb{E}\left[ (|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1) 1_{\{|Z_{\delta,n} - Z_\delta| \leq \varepsilon, |Z_{\delta,n} - Z_\delta| \leq \eta\}} \right]$$

$$+ \mathbb{E}\left[ (|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1) 1_{\{|Z_\delta| > \eta\}} \right] \leq 2\beta. \quad (2.32)$$

Since \( \beta > 0 \) is arbitrary, (2.28) follows from (2.31) and (2.32). \( \square \)

• **Convergence in total variation and the Hellinger distance.**

Let \( \mu \) and \( \nu \) be two probability measures on a measurable space \((E, \mathcal{E})\), the total variation distance \( d_{TV} \) of \( \mu \) and \( \nu \) is given by

$$d_{TV}(\mu, \nu) := \sup \{ |\mu(A) - \nu(A)| : A \in \mathcal{E} \}. \quad (2.33)$$

This metric induces a much stronger topology than the one induced by the weak convergence.\(^{15}\)

Next, we recall the notion of the Hellinger integral [26, 29]. Let \( \mu \) and \( \nu \) be two probability measures on a measurable space \((E, \mathcal{E})\). Note that both \( \mu \) and \( \nu \) are absolutely continuous

\(^{15}\)For example, let \( \mu_N \) denote the law of the random variable \( \frac{1}{\sqrt{N}}(Y_1 + \ldots + Y_N) \), where \( Y_i, i \in \mathbb{N} \), are i.i.d. random variables with \( \mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2} \). Then, the classical central limit theorem asserts that \( \mu_N \) converges weakly to the standard Gaussian measure on \( \mathbb{R} \), while due to the discrete nature of \( \mu_N \), its total variation distance from the standard Gaussian measure is always one.
with respect to the probability measure $\lambda = \frac{1}{2}(\mu + \nu)$. Then, the Hellinger integral of $\mu$ and $\nu$ is defined by

$$H(\mu, \nu) = \int_E \sqrt{\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}} \, d\lambda.$$  \hfill (2.34)

In fact, the definition (2.34) is independent of the choice of a probability measure $\lambda$ such that $\mu, \nu \ll \lambda$. When $\mu$ and $\nu$ are equivalent (i.e. mutually absolutely continuous), we can write $H(\mu, \nu)$ as

$$H(\mu, \nu) = \int_E \sqrt{\frac{d\nu}{d\mu}} \, d\mu.$$  \hfill (2.35)

Note that $0 \leq H(\mu, \nu) \leq 1$. The Hellinger integral provides a criterion for singularity (and equivalence) of two probability measures. It is known [29, Proposition 2.20] that $H(\mu, \nu) = 0$ if and only if $\mu$ and $\nu$ are mutually singular. Thus, for $\mu$ and $\nu$ to be equivalent, we must have $H(\mu, \nu) > 0$. In fact, when $\mu$ and $\nu$ are product measures on $(\mathbb{R}^\infty, \mathcal{B}_\mathbb{R}^\infty)$, the condition $H(\mu, \nu) > 0$ is also sufficient (Kakutani’s theorem). See Theorem 2.7 in [26].

**Lemma 2.10.** Let $\{\mu_n\}_{n \in \mathbb{N}}$ and $\{\nu_n\}_{n \in \mathbb{N}}$ be two sequences of probability measures on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ such that $\mu_n$ and $\nu_n$ are equivalent for any $n \in \mathbb{N}$. Let $\mu = \bigotimes_{n \in \mathbb{N}} \mu_n$ and $\nu = \bigotimes_{n \in \mathbb{N}} \nu_n$. Then, we have $H(\mu, \nu) = \prod_{n \in \mathbb{N}} H(\mu_n, \nu_n)$ and

- $H(\mu, \nu) > 0$ if and only if $\mu$ and $\nu$ are equivalent. In this case, we have $\frac{d\mu}{d\nu} = \prod_{n \in \mathbb{N}} \frac{d\mu_n}{d\nu_n}$.

- $H(\mu, \nu) = 0$ if and only if $\mu$ and $\nu$ are mutually singular.

With the notations as above, we introduce the **Hellinger distance** $d_H$ of $\mu$ and $\nu$ by setting

$$d_H(\mu, \nu) = \left( \frac{1}{2} \int_E \left( \sqrt{\frac{d\mu}{d\lambda}} - \sqrt{\frac{d\nu}{d\lambda}} \right)^2 \, d\lambda \right)^{\frac{1}{2}}$$  \hfill (2.36)

where $H(\mu, \nu)$ is the Hellinger integral defined in (2.35). It is clear that $0 \leq d_H(\mu, \nu) \leq 1$. We state Le Cam’s inequality, relating the total variation distance and Hellinger distance; see Lemma 2.3 in [85].

**Lemma 2.11.** Let $d_{TV}$ and $d_H$ be as in (2.33) and (2.36), respectively. Then, we have

$$(d_H(\mu, \nu))^2 \leq d_{TV}(\mu, \nu) \leq \sqrt{2} \cdot d_H(\mu, \nu)$$

for any probability measures $\mu$ and $\nu$ on a measurable space $(E, \mathcal{E})$. In particular, a sequence $\{\mu_k\}_{k \in \mathbb{N}}$ of probability measures on $(E, \mathcal{E})$ converges to some limit $\mu$ in total variation if and only if it converges to the same limit in the Hellinger distance.

---

16 Note a slightly different multiplicative constant in the definition of the Hellinger distance in [85].
In the remaining part of the paper, we do not make use of the Hellinger distance. We, however, decided to introduce it here due to its connection to the total variation distance and also to the fact that Hellinger integral plays an important role in the proof of Lemma 3.2. See also Remark 3.3 (iii).

- **Kullback-Leibler divergence (= relative entropy).**

We now define the Kullback-Leibler divergence $d_{KL}(\mu, \nu)$ between $\mu$ and $\nu$ by setting

$$d_{KL}(\mu, \nu) = \begin{cases} \int_E \log \frac{d\mu}{d\nu} d\mu, & \text{if } \mu \ll \nu, \\ \infty, & \text{otherwise}, \end{cases}$$

which is nothing but the relative entropy of $\mu$ with respect to $\nu$. While the total variation distances and the Hellinger distance are metrics, the Kullback-Leibler divergence is not a metric. For example, $d_{KL}(\cdot, \cdot)$ is not symmetric, and moreover, the symmetrized version $d_{KL}(\mu, \nu) + d_{KL}(\mu, \nu)$ is not a metric, either. If $\mu$ and $\nu$ are product measures of the form $\mu = \otimes_{n \in \mathbb{N}} \mu_n$ and $\nu = \otimes_{n \in \mathbb{N}} \nu_n$, then we have

$$d_{KL}(\mu, \nu) = \sum_{n \in \mathbb{N}} d_{KL}(\mu_n, \nu_n).$$

(2.38)

The following lemma shows that convergence in the Kullback-Leibler divergence (or in relative entropy) implies convergence in total variation and in the Hellinger distance. See Lemmas 2.4 and 2.5 in [85] for the proof.

**Lemma 2.12.** Let $d_{TV}$, $d_H$, and $d_{KL}$ be as in (2.33), (2.36), and (2.37), respectively. Then, we have

$$d_H(\mu, \nu) \leq \frac{\sqrt{d_{KL}(\mu, \nu)}}{\sqrt{2}}$$

(2.39)

and

$$d_{TV}(\mu, \nu) \leq \frac{\sqrt{d_{KL}(\mu, \nu)}}{\sqrt{2}}.$$

(2.40)

The second inequality (2.40) is known as Pinsker’s inequality and it is slightly stronger than $d_{TV}(\mu, \nu) \leq \sqrt{d_{KL}(\mu, \nu)}$, which follows from Lemma 2.11 and (2.39).

- **Weak convergence the Lévy-Prokhorov metric.**

Finally, let us introduce the Lévy-Prokhorov metric for probability measures on a separable metric space $(M, d)$. Given $\varepsilon > 0$, we define an $\varepsilon$-neighborhood of a measurable subset $A \subset M$ by

$$A^\varepsilon := \{ z \in M : d(z, x) < \varepsilon \text{ for some } x \in M \}.$$

Given two probability measures $\mu$ and $\nu$ on $M$, their Lévy-Prokhorov distance $d_{LP}(\mu, \nu)$ is defined by

$$d_{LP}(\mu, \nu) := \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all measurable } A \subset M \}.$$

(2.41)

Note that the Lévy-Prokhorov metric is indeed a metric on the space of probability measures on $M$. It is known that the Lévy-Prokhorov metric induces the same topology as the topology for weak convergence. Together with this property, we only need one additional property of
the Lévy-Prokhorov metric in this paper, that is, the triangle inequality; see (5.33) below. See [10, 32] and [6, Section 30.3] for a further discussion.

Lastly, we recall the Prokhorov theorem and the Skorokhod representation theorem.

**Definition 2.13.** Let $J$ be any nonempty index set. A family $\{\rho_i\}_{i \in J}$ of probability measures on a metric space $M$ is said to be tight if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset M$ such that $\sup_{i \in J} \rho_i(K_\varepsilon^c) \leq \varepsilon$. We say that $\{\rho_i\}_{i \in J}$ is relatively compact, if every sequence in $\{\rho_i\}_{i \in J}$ contains a weakly convergent subsequence.

Note that the index set $J$ does not need to be countable. We now recall the following Prokhorov theorem from [10, 6].

**Lemma 2.14 (Prokhorov theorem).** If a sequence of probability measures on a metric space $M$ is tight, then it is relatively compact. If in addition, $M$ is separable and complete, then relative compactness is equivalent to tightness.

Lastly, we recall the following Skorokhod representation theorem from [6, Chapter 31].

**Lemma 2.15 (Skorokhod representation theorem).** Let $M$ be a complete separable metric space (i.e. a Polish space). Suppose that probability measures $\{\rho_n\}_{n \in \mathbb{N}}$ on $M$ converges weakly to a probability measure $\rho$ as $n \to \infty$. Then, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and random variables $X_n, X : \tilde{\Omega} \to M$ such that$
L(X_n) = \rho_n \quad \text{and} \quad L(X) = \rho,$
and $X_n$ converges $\tilde{\mathbb{P}}$-almost surely to $X$ as $n \to \infty$.

3. **Gibbs measures in the deep-water regime**

In this section, we go over the construction of the Gibbs measures for the gILW equation (1.4), including the gBO case ($\delta = \infty$), and prove convergence of the Gibbs measures in the deep-water limit (as $\delta \to \infty$). As mentioned in Section 1, we construct the Gibbs measure as a weighted Gaussian measure, where the base Gaussian measure is given by $\mu_\delta$ in (1.22) with the understanding that it is given by $\mu_\infty$ in (1.37) when $\delta = \infty$. For $0 < \delta < \infty$, let $K_\delta(n)$ be as in (1.42). We extend the definition of $K_\delta(n)$ to $\delta = \infty$ by setting

$$K_\infty(n) = |n|,$$

which is consistent with Lemma 2.1. Then, a typical element under the Gaussian measure $\mu_\delta$ in (1.22) (and $\mu_\infty$ in (1.37)) is given by $X_\delta$ in (1.41) when $0 < \delta < \infty$ and $X_\infty := X_{\text{BO}}$ in (1.38) when $\delta = \infty$. It is easy to see that, given $0 < \delta \leq \infty$, $X_\delta \in H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T})$ for any $\varepsilon > 0$, almost surely. Indeed, from Lemma 2.1, we have $K_\delta(n) \sim_\delta \mathbb{N}$. Hence, with $X_{\delta,N} = X_\delta$ in (2.21), it follows from Lemma 2.7 that there exists $C_\delta > 0$ such that, for any finite $p \geq 1$,

$$\|X_{\delta,N}\|_{L^p H^{-\varepsilon}_x} \leq p^{1/2} \|\nabla^{-\varepsilon} X_{\delta,N}(x)\|_{L^2 L^2} \leq C_\delta p^{1/2} \left( \sum_{0 < |n| \leq N} \frac{1}{|n|^{1+2\varepsilon}} \right)^{1/2} \sim C_\delta p^{1/2},$$

uniformly in $N \in \mathbb{N}$, provided that $\varepsilon > 0$. A similar computation together with the Borel-Cantelli lemma shows that $X_{\delta,N}$ converges, in $L^p(\Omega)$ and almost surely, to the limit $X_\delta$ in
$H^{-\varepsilon}(T)$ for any $\varepsilon > 0$. The fact that $X_\delta \notin L^2(T)$ almost surely follows from Lemma B.1 in [19].

In Subsection 3.1 we first study various properties of the base Gaussian measures $\mu_\delta$. See Proposition 3.1. By restricting our attention to the defocusing case $(k \in 2\mathbb{N} + 1)$, we then go over the construction of the Gibbs measures in Subsection 3.2. In Subsection 3.3 we continue to study the defocusing case and establish convergence in total variation of the Gibbs measure $\rho_\delta$ to $\rho_{BO}$ in the deep-water limit ($\delta \to \infty$). Finally, in Subsection 3.4 we present the proof of Theorem 1.3 when $k = 2$.

### 3.1. Equivalence of the base Gaussian measures.

**Proposition 3.1.** (i) Let $X_\delta$ and $X_{BO}$ be as in (1.41) and (1.38), respectively. Then, given any $\varepsilon > 0$ and finite $p \geq 1$, $X_\delta$ converges to $X_{BO}$ in $L^p(\Omega; H^{-\varepsilon}(T))$ and in $H^{-\varepsilon}(T)$ almost surely, as $\delta \to \infty$. In particular, the Gaussian measure $\mu_\delta$ in (1.22) converges weakly to the Gaussian measure $\mu_\infty$ in (1.37), as $\delta \to \infty$.

(ii) For any $0 < \delta < \infty$, the Gaussian measures $\mu_\delta$ and $\mu_\infty$ are equivalent.

(iii) As $\delta \to \infty$, the Gaussian measure $\mu_\delta$ converges to $\mu_\infty$ in the Kullback-Leibler divergence defined in (2.37). In particular, $\mu_\delta$ converges to $\mu_\infty$ in total variation.

Part (iii) of Proposition 3.1 plays an essential role in establishing convergence in total variation of the Gibbs measure $\rho_\delta$ to $\rho_{BO}$ in the deep-water limit ($\delta \to \infty$).

In proving Part (ii) of Proposition 3.1 we resort to the following Kakutani’s theorem [45] in the Gaussian setting (or the Feldman-Hájek theorem [33, 42]; see also [26, Theorem 2.9]). See, for example, [20, 73, 75, 39], where Kakutani’s theorem was used in the study of dispersive PDEs. In particular, see also Proposition B.1 in [20].

**Lemma 3.2.** Let $\{A_n\}_{n \in \mathbb{Z}^*}$ and $\{B_n\}_{n \in \mathbb{Z}^*}$ be two sequences of independent, real-valued, mean-zero Gaussian random variables with $\mathbb{E}[A_n^2] = a_n > 0$ and $\mathbb{E}[B_n^2] = b_n > 0$ for all $n \in \mathbb{Z}^*$. Then, the laws of the sequences $\{A_n\}_{n \in \mathbb{Z}^*}$ and $\{B_n\}_{n \in \mathbb{Z}^*}$ are equivalent if and only if

$$\sum_{n \in \mathbb{Z}^*} \left(\frac{a_n}{b_n} - 1\right)^2 < \infty. \quad (3.3)$$

If they are not equivalent, then they are singular.

We first present a short proof of Lemma 3.2 based on Lemma 2.10. See also the proof of Theorem 2.9 in [26].

**Proof of Lemma 3.2.** Given $n \in \mathbb{Z}^*$, let $\mu_n$ and $\nu_n$ denote the laws of $A_n$ and $B_n$, respectively, and set $\mu = \bigotimes_{n \in \mathbb{Z}^*} \mu_n$ and $\nu = \bigotimes_{n \in \mathbb{Z}^*} \nu_n$. Namely, $\mu$ are $\nu$ are the laws of the sequences $\{A_n\}_{n \in \mathbb{Z}^*}$ and $\{B_n\}_{n \in \mathbb{Z}^*}$, respectively. The Hellinger integral $H(\mu, \nu)$ defined in (2.35) is
given by an infinite product:

\[
H(\mu, \nu) = \prod_{n \in \mathbb{Z}^*} H(\mu_n, \nu_n) = \prod_{n \in \mathbb{Z}^*} \int_\mathbb{R} \frac{d\mu_n}{d\nu_n} d\nu_n
\]

\[
= \prod_{n \in \mathbb{Z}^*} \int_\mathbb{R} \frac{1}{\sqrt{2\pi}(a_n b_n)^{\frac{1}{2}}} e^{-\frac{1}{4}(\frac{1}{a_n} + \frac{1}{b_n}) x^2} dx
\]

\[
= \prod_{n \in \mathbb{Z}^*} \frac{\sqrt{2}(a_n b_n)^{\frac{1}{4}}}{\sqrt{a_n + b_n}}.
\]

Thus, we have

\[
(H(\mu, \nu))^4 = \prod_{n \in \mathbb{Z}^*} \frac{4a_n b_n}{(a_n + b_n)^2} = \prod_{n \in \mathbb{Z}^*} \left(1 - \frac{(a_n - b_n)^2}{(a_n + b_n)^2}\right).
\]

Hence, \(H(\mu, \nu) > 0\) if and only if

\[
\sum_{n \in \mathbb{Z}^*} \frac{(a_n - b_n)^2}{(a_n + b_n)^2} = \sum_{n \in \mathbb{Z}^*} \frac{(a_n - b_n)^2}{(b_n + 1)^2} < \infty.
\] (3.4)

Note that the condition (3.4) is equivalent to the condition (3.3), since if one of the sums in (3.3) or (3.4) converges, then \(\frac{a_n}{b_n}\) must tend to 1 as \(n \to \infty\), which implies the other sum also converges. Then, the desired conclusion follows from Lemma 2.10. \(\square\)

We now present the proof of Proposition 3.1.

Proof of Proposition 3.1 (i) Let \(\varepsilon > 0\) and fix finite \(p \geq 1\). Then, it follows from Lemma 2.7, (1.38), (1.41), and \(\sqrt{a} - \sqrt{b} \leq \sqrt{a - b}\) for any \(a \geq b \geq 0\) together with (2.1) in Lemma 2.1 that

\[
\|X_\delta - X_{BO}\|_{L_p^p H^{-\varepsilon}} \lesssim \|\langle \nabla \rangle^{-\varepsilon} (X_\delta - X_{BO})(x)\|_{L_p^p L_2^2}
\]

\[
\lesssim \left( \sum_{n \in \mathbb{Z}^*} \langle n \rangle^{2\varepsilon} \left( \frac{1}{K_\delta(n)} - \frac{1}{|n|^{\frac{1}{2}}} \right)^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( \sum_{n \in \mathbb{Z}^*} \frac{1}{\langle n \rangle^{2\varepsilon}} \frac{|n| - K_\delta(n)}{|n| K_\delta(n)} \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( \frac{1}{\delta} \sum_{n \in \mathbb{Z}^*} \frac{1}{\langle n \rangle^{2+2\varepsilon}} \right)^{\frac{1}{2}} \lesssim \frac{1}{\delta^{1+\varepsilon}} \to 0,
\] (3.5)

as \(\delta \to \infty\). See also (2.3) for the penultimate step in (3.5).

As for the almost sure convergence, we repeat a computation analogous to (3.5) but with (2.19) in place of \(E[|g_n|^2] \sim 1\). Then, together with Lemma 2.1 (for \(\delta \geq 2\)), we have

\[
\|X_\delta(\omega) - X_{BO}(\omega)\|_{H^{-\varepsilon}}^2 \lesssim \frac{C_{\varepsilon}, \omega}{\delta} \sum_{n \in \mathbb{Z}^*} \frac{\langle n \rangle^{2\varepsilon}}{|n|^{1+2\varepsilon}} (|n| - \frac{1}{4}) \to 0,
\] (3.6)

as \(\delta \to \infty\), provided that \(0 < \varepsilon_0 < \varepsilon\). Recalling that \(\mu_\delta\) and \(\mu_\infty\) are the laws of \(X_\delta\) and \(X_{BO}\), we conclude weak convergence of \(\mu_\delta\) to \(\mu_\infty\). This proves (i).
(ii) Rewrite \(X_\delta\) in (1.41) (and in (1.38) when \(\delta = \infty\) with the understanding (3.1)) as
\[
X_\delta(\omega) = \sum_{n \in \mathbb{N}} \left( \frac{\text{Re} g_n}{\pi K_\delta^1(n)} \cos(nx) - \frac{\text{Im} g_n}{\pi K_\delta^1(n)} \sin(nx) \right).
\]

For \(n \in \mathbb{Z}^*\), set
\[
A_n = \frac{\text{Re} g_n}{\pi |n|^\frac{1}{2}} \quad \text{and} \quad A_{-n} = -\frac{\text{Im} g_n}{\pi |n|^\frac{1}{2}},
\]
and
\[
B_n = \frac{\text{Re} g_n}{\pi |n|^\frac{1}{2}} \quad \text{and} \quad B_{-n} = -\frac{\text{Im} g_n}{\pi |n|^\frac{1}{2}} \quad \text{for} \ \delta = \infty
\]
with \(a_{\pm n} = \mathbb{E}[A_{\pm n}^2] = \frac{1}{\pi K_\delta(n)} \quad \text{and} \quad b_{\pm n} = \mathbb{E}[B_{\pm n}^2] = \frac{1}{\pi |n|}\). Then, from Lemma 2.1, we have
\[
\sum_{n \in \mathbb{Z}^*} \left( \frac{a_n}{b_n} - 1 \right)^2 = \sum_{n \in \mathbb{Z}^*} \frac{(|n| - K_\delta(n))^2}{K_\delta^2(n)} \lesssim C_\delta \sum_{n \in \mathbb{Z}^*} \frac{1}{n^2} < \infty.
\]

Therefore, the claimed equivalence of \(\mu_\delta\) and \(\mu_\infty\) follows from Kakutani’s theorem (Lemma 3.2).

(iii) In this part, we prove that \(\mu_\delta\) converges to \(\mu_\infty\) in the Kullback-Leibler divergence defined in (2.37). Once this is achieved, convergence in total variation follows from Pinsker’s inequality (2.40) in Lemma 2.12.

Let us first write \(\mu_\delta, 0 < \delta \leq \infty\), as the product of Gaussian measures on \(\mathbb{R}\) (see also (1.31)):
\[
d\mu_\delta = \bigotimes_{n \in \mathbb{N}} K_\delta^\frac{1}{2}(n) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\pi} K_\delta(n)(\text{Re} \tilde{u}(n))^2} d\text{Re} \tilde{u}(n)
\times \bigotimes_{n \in \mathbb{N}} K_\delta^\frac{1}{2}(n) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\pi} K_\delta(n)(\text{Im} \tilde{u}(n))^2} d\text{Im} \tilde{u}(n)
\]
with the identification (3.1) when \(\delta = \infty\). With \(x = (x_1, x_2) \in \mathbb{R}^2\), we then have
\[
d\mu_\delta = \bigotimes_{n \in \mathbb{N}} K_\delta(n) \frac{1}{2\pi^2} e^{-\frac{1}{2\pi} K_\delta(n)|x|^2} dx =: \bigotimes_{n \in \mathbb{N}} K_\delta(n) \frac{1}{2\pi^2} d\mu_\delta^n. \quad (3.7)
\]

Then, the Radon-Nikodym derivative \(\frac{d\mu_\delta}{d\mu_\infty}\) is given by
\[
\frac{d\mu_\delta}{d\mu_\infty} = \frac{K_\delta(n)}{n} e^{\frac{1}{2\pi} (n - K_\delta(n))|x|^2}. \quad (3.8)
\]
See (??). Then, from Part (ii), (2.37), and (2.38) with (3.7) and (3.8), we have

\[
d_{\text{KL}}(\mu_\delta, \mu_\infty) = \sum_{n \in \mathbb{N}} d_{\text{KL}}(\mu_\delta^n, \mu_\infty^n)
\]

\[
= \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^2} \left( \log \frac{K_\delta(n)}{n} + \frac{1}{2\pi} (n - K_\delta(n)) |x|^2 \right) \frac{K_\delta(n)}{2\pi^2} e^{-\frac{1}{2\pi} K_\delta(n)|x|^2} \, dx
\]

\[
= \sum_{n \in \mathbb{N}} \left( \log \frac{K_\delta(n)}{n} + \frac{1}{2\pi} (n - K_\delta(n)) \int_{\mathbb{R}^2} \frac{K_\delta(n)}{2\pi^2} |x|^2 e^{-\frac{1}{2\pi} K_\delta(n)|x|^2} \, dx \right)
\]

\[
= \sum_{n \in \mathbb{N}} \phi \left( \frac{n}{K_\delta(n)} \right),
\]

where \( \phi(t) := t - 1 - \log t \). Note that \( \phi(1) = 0 \) and \( \phi'(t) > 0 \) for \( t > 1 \). Then, it follows from Lemma 2.1 that for each fixed \( n \in \mathbb{N} \), we have

\[
\phi \left( \frac{n}{K_\delta(n)} \right) \text{ decreases to } \phi(1) = 0,
\]

as \( \delta \to \infty \), since \( \frac{n}{K_\delta(n)} \) decreases to 1 as \( \delta \to \infty \). Hence, if the right-hand side of (3.9) is finite for some \( \delta \gg 1 \), then the observation (3.10) allows us to apply the dominated convergence theorem and conclude

\[
\lim_{\delta \to \infty} d_{\text{KL}}(\mu_\delta, \mu_\infty) = \lim_{\delta \to \infty} \sum_{n \in \mathbb{N}} \phi \left( \frac{n}{K_\delta(n)} \right) = \sum_{n \in \mathbb{N}} \lim_{\delta \to \infty} \phi \left( \frac{n}{K_\delta(n)} \right) = 0,
\]

yielding the desired convergence in the Kullback-Leibler divergence.

It remains to check that the right-hand side of (3.9) is finite for some \( \delta \gg 1 \). In fact, we show that the right-hand side of (3.9) is finite for any \( \delta > 0 \). By a direct computation, we have \( \phi(t) \leq (t - 1)^2 \) for \( t \geq 1 \). Then, from Lemma 2.1 we have

\[
\sum_{n \in \mathbb{N}} \phi \left( \frac{n}{K_\delta(n)} \right) \leq \sum_{n \in \mathbb{N}} \left( \frac{n - K_\delta(n)}{K_\delta(n)} \right)^2 \leq C_\delta \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty
\]

for any \( \delta > 0 \). This concludes the proof of Proposition 3.1. \( \square \)

**Remark 3.3.** (i) By using the Wiener chaos estimate (Lemma 2.7), Chebyshev’s inequality, and the Borel-Cantelli lemma, one can easily upgrade the convergence of \( X_\delta \) to \( X_{BO} \) to that in \( L^2(\Omega; W^{-\varepsilon,\infty}(\mathbb{T})) \) and in \( W^{-\varepsilon,\infty}(\mathbb{T}) \) almost surely,

(ii) From (3.5), we see that the difference \( X_\delta - X_{BO} \) lives in \( H^{\frac{1}{2}-\varepsilon}(\mathbb{T}) \)\(^{17} \) although neither \( X_\delta \) nor \( X_{BO} \) belongs to \( L^2(\mathbb{T}) \).

(iii) In order to prove convergence of \( \mu_\delta \) to \( \mu_\infty \) in total variation, it is indeed possible to directly show that \( \mu_\delta \) converges to \( \mu_\infty \) in the Hellinger distance \( d_H \) defined in (2.36) and invoke Lemma 2.11.

\(^{17}\text{In fact, in } W^{\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}) \text{ if we use the Wiener chaos estimate (Lemma 2.7).} \)

3.2. **Construction of the Gibbs measure for the defocusing gILW equation.** In this subsection, we present the construction of the Gibbs measure for the gILW equation (1.4), $0 < \delta \leq \infty$ with the understanding that the $\delta = \infty$ case corresponds to the gBO (1.12), in the defocusing case: $k \in 2\mathbb{N} + 1$. We treat the $k = 2$ case, corresponding to the ILW equation (1.1), in Subsection 3.4. Our basic strategy is to follow the argument presented in [74] on the construction of the complex $\Phi^{k+1}_\delta$-measures, by utilizing the Wiener chaos estimate (Lemma 2.7) and Nelson’s estimate. In order to establish convergence of the Gibbs measures in the deep-water limit ($\delta \to \infty$), however, we need to establish an $L^p(\Omega)$-integrability of the (truncated) densities, uniformly in both the frequency-truncation parameter $N \in \mathbb{N}$ and the depth parameter $\delta \gg 1$. See Proposition 3.6. This uniform bound also plays a crucial role in the dynamical part presented in Section 5.

Fix the depth parameter $0 < \delta \leq \infty$. Given $N \in \mathbb{N}$, let $X_{\delta,N} = P_N X_\delta$, where $X_\delta$ is defined in (1.41):

$$X_{\delta,N}(\omega) := P_N X_\delta(\omega) = \frac{1}{2\pi} \sum_{0 < |n| \leq N} g_n(\omega) e_n$$

with the identification (3.1) when $\delta = \infty$. When $\delta = \infty$, we also set

$$X_{\delta,\infty,N} = P_N X_\infty = P_N X_{\delta,\infty}$$

where $X_{\delta,\infty}$ is as in (138). Given $k \in \mathbb{N}$, let $\mathcal{W}(X_{\delta,N}^k) = H_k(X_{\delta,N}; \sigma_{\delta,N})$ denotes the Wick power defined in (1.44), where $\sigma_{\delta,N}$ is as in (1.43). Then, the truncated Gibbs measure $\rho_{\delta,N}$ in (1.45) can be written as

$$\rho_{\delta,N}(A) = Z_{\delta,N}^{-1} \int_{H^{-\varepsilon}} 1_{\{u \in A\}} e^{-\frac{1}{\kappa_N} \int_{-\varepsilon}^\varepsilon f_x \mathcal{W}(u_N^{k+1}) dx} d\mu_\delta(u)$$

$$= Z_{\delta,N}^{-1} \int_{\Omega} 1_{\{X_{\delta}(\omega) \in A\}} e^{-\frac{1}{\kappa_N} \int_{-\varepsilon}^\varepsilon f_x \mathcal{W}(X_{\delta,N}^{k+1}(\omega)) dx} d\mathcal{P}(\omega)$$

(3.11)

for any measurable set $A \subset H^{-\varepsilon}(\mathbb{T})$ with some small $\varepsilon > 0$. where $u_N = P_N u$. In the following, we freely interchange the representations in terms of $X_\delta$ and in terms of $u$ distributed by $\mu_\delta$, when there is no confusion.

Let us first construct the limiting Wick power $\mathcal{W}(X_\delta^k)$ and the related stochastic objects.

**Proposition 3.4.** Let $k \in \mathbb{N}$ and $0 < \delta \leq \infty$. Given $N \in \mathbb{N}$, let $\mathcal{W}(X_{\delta,N}^k)$ be as in (1.44). Then, given any finite $p \geq 1$, the sequence $\{\mathcal{W}(X_{\delta,N}^k)\}_{N \in \mathbb{N}}$ is Cauchy in $L^p(\Omega; W^{s,\infty}(\mathbb{T}))$, $s < 0$, thus converging to a limit, denoted by $\mathcal{W}(X_\delta^k)$. This convergence of $\mathcal{W}(X_{\delta,N}^k)$ to $\mathcal{W}(X_\delta^k)$ also holds almost surely in $W^{s,\infty}(\mathbb{T})$. Furthermore, given any finite $p \geq 1$, we have

$$\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|\mathcal{W}(X_{\delta,N}^k)\|_{L^p(\Omega)} \leq C$$

(3.12)

and

$$\sup_{2 \leq \delta \leq \infty} \|\mathcal{W}(X_{\delta,M}^k) - \mathcal{W}(X_{\delta,N}^k)\|_{L^p(\Omega)} \rightarrow 0$$

(3.13)

for any $M \geq N$, tending to $\infty$. In particular, the rate of convergence is uniform in $2 \leq \delta \leq \infty$.

As a corollary, the following two statements hold.
(i) Let $0 < \delta \leq \infty$. Given $N \in \mathbb{N}$, let $R_{\delta,N}(u) = R_{\delta,N}(u;k+1)$ denote the truncated potential energy defined by

$$R_{\delta,N}(u) := \frac{1}{k+1} \int_\mathbb{T} \mathcal{W}(\mathcal{P}_N u^{k+1}) dx = \frac{1}{k+1} \int_\mathbb{T} H_{k+1}(\mathcal{P}_N u; \sigma_{\delta,N}) dx,$$

where $\sigma_{\delta,N}$ is as in (1.43) with the identification (3.1) when $\delta = \infty$; see $\sigma_{\infty,N}$ in (3.51). Then, given any finite $p \geq 1$, the sequence $\{R_{\delta,N}(u)\}_{N \in \mathbb{N}}$ converges to the limit:

$$R_{\delta}(u) = \frac{1}{k+1} \int_\mathbb{T} \mathcal{W}(u^{k+1}) dx = \lim_{N \to \infty} \frac{1}{k+1} \int_\mathbb{T} \mathcal{W}(\mathcal{P}_N u^{k+1}) dx$$

in $L^p(d\mu_\delta)$, as $N \to \infty$. Furthermore, there exists $\theta > 0$ such that given any finite $p \geq 1$, we have

$$\sup_{N \in \mathbb{N} \cup \{\infty\}} \sup_{2 \leq \delta \leq \infty} \|R_{\delta,N}(u)\|_{L^p(d\mu_\delta)} < \infty,$$

with $R_{\delta,\infty}(u) = R_{\delta}(u)$, and

$$\|R_{\delta,M}(u) - R_{\delta,N}(u)\|_{L^p(d\mu_\delta)} \leq \frac{C_{k,\delta} p^\frac{k+1}{p+1}}{N^\theta}$$

for any $M \geq N \geq 1$. For $2 \leq \delta \leq \infty$, we can choose the constant $C_{k,\delta}$ in (3.17) to be independent of $\delta$ and hence the rate of convergence of $R_{\delta,N}(u)$ to the limit $R_{\delta}(u)$ is uniform in $2 \leq \delta \leq \infty$.

(ii) Let $0 < \delta \leq \infty$. Given $N \in \mathbb{N}$, let $F_N(u) = F_N(u;k)$ be the truncated renormalized nonlinearity in (1.70) given by

$$F_N(u) := \partial_x \mathcal{P}_N \mathcal{W}(\mathcal{P}_N u^k) = \partial_x \mathcal{P}_N H_k(\mathcal{P}_N u; \sigma_{\delta,N}),$$

where $\sigma_{\delta,N}$ is as in (1.43) with the identification (3.1) when $\delta = \infty$; see $\sigma_{\infty,N}$ in (3.51). Then, given any finite $p \geq 1$, the sequence $\{F_N(u)\}_{N \in \mathbb{N}}$ is Cauchy in $L^p(d\mu_\delta;L^s(\mathbb{T}))$, $s < -1$, thus converging to a limit denoted by $F(u) = \partial_x W(u^k)$. Furthermore, given any finite $p \geq 1$, we have

$$\sup_{N \in \mathbb{N} \cup \{\infty\}} \sup_{2 \leq \delta \leq \infty} \|F_N(u)\|_{L^p(d\mu_\delta)} < \infty,$$

with $F_\infty(u) = F(u)$, and

$$\sup_{2 \leq \delta \leq \infty} \|F_M(u) - F_N(u)\|_{L^p(d\mu_\delta)} \to 0$$

for any $M \geq N$, tending to $\infty$. In particular, the rate of convergence of $F_N(u)$ to the limit $F(u)$ is uniform in $2 \leq \delta \leq \infty$.

Remark 3.5. In the proof of Proposition 3.4, we use (2.3) to obtain a lower bound on $K_\delta(n)$, uniformly in $2 \leq \delta \leq \infty$, for any fixed $n \in \mathbb{Z}^*$. The lower bound $\delta = 2$ is by no means sharp. For example, in view of the strict monotonicity of $K_\delta(n)$ in $\delta \geq 1$ (for fixed $n \in \mathbb{Z}^*$) and the fact that $K_\delta(n) \neq 0$ for $n \in \mathbb{Z}^*$ as stated in Lemma 2.1, a slight modification of the proof of Proposition 3.4 yields the uniform (in $\delta$) bounds for $1 \leq \delta \leq \infty$. Since our main interest is to take the limit $\delta \to \infty$, we do not attempt to optimize a lower bound for $\delta$. The same comment applies to the subsequent results presented in this section and hence to Theorem 1.3.
Proof of Proposition 3.4. Given $N \in \mathbb{N}$ and $x, y \in \mathbb{T}$, we define $\gamma_N = \gamma_N(\delta)$ by setting

$$
\gamma_N(x - y) := \mathbb{E}[X_{\delta,N}(x)X_{\delta,N}(y)] = \frac{1}{2\pi} \sum_{0 < |n| \leq N} \frac{e_n(x - y)}{K_\delta(n)}.
$$

(3.21)

Note that we have

$$
\gamma_N(x - y) = \mathbb{E}[X_{\delta,N}(x)X_{\delta,M}(y)]
$$

for any $M \geq N \geq 1$. In the following, for simplicity of notation, we set $u_N = P_N u$ and suppress the $\delta$-dependence in $\gamma_N = \gamma_N(\delta)$.

Let us first make a preliminary computation. Given $n, m \in \mathbb{Z}^*$, we have

$$
\mathbb{E} \left[ F(H_k(X_{\delta,N}; \sigma_{\delta,N}))(n) F(H_k(X_{\delta,N}; \sigma_{\delta,N}))(m) \right]
$$

$$
= \int_{\mathbb{T}^2} \mathbb{E} \left[ H_k(X_{\delta,N}(x); \sigma_{\delta,N})H_k(X_{\delta,N}(y); \sigma_{\delta,N}) \right] e^{-n+m}(x)e_{-m}(y)dydx.
$$

(3.22)

From Lemma 2.6 with (3.21) we have

$$
\mathbb{E} \left[ H_k(X_{\delta,N}(x); \sigma_{\delta,N})H_k(X_{\delta,N}(y); \sigma_{\delta,N}) \right] = k! \gamma_N^k (y-x).
$$

(3.23)

Then, from (3.22), (3.23), a change of variables $z = y - x$, and integrating in $x$, we have

$$
\mathbb{E} \left[ F(H_k(X_{\delta,N}; \sigma_{\delta,N}))(n) F(H_k(X_{\delta,N}; \sigma_{\delta,N}))(m) \right]
$$

$$
= k! \int_{\mathbb{T}} \left( \int_{\mathbb{T}} e^{-n+m}(x)dx \right) \gamma_N(z)\gamma_N(z)dz
$$

$$
= 2\pi k! \mathbf{1}_{n=m} \cdot \int_{\mathbb{T}} \gamma_N(z)\gamma_N(z)dz.
$$

(3.24)

Fix small $\varepsilon > 0$. Then, by Sobolev’s inequality with finite $r \gg 1$ such that $r\varepsilon > 1$, we have

$$
\|\mathcal{W}(u_N^k)\|_{W^{r,\infty}} \lesssim \|\mathcal{W}(u_N^k)\|_{W^{r+\varepsilon,r}}.
$$

(3.25)

Let $p \geq r$. Then, by (3.25), Minkowski’s inequality (with $p \geq r \gg 1$), the Wiener chaos estimate (Lemma 2.7, (3.24)), and the boundedness of the torus $\mathbb{T}$, we have

$$
\|\|\mathcal{W}(u_N^k)\|_{W^{r,\infty}}\|_{L^p(\mu_N)} \lesssim p^\frac{k}{2} \|\|\|\mathcal{W}(u_N^k)\|_{L^{2}(\mu_N)}\|_{L^r_k}
$$

$$
= \frac{p^\frac{k}{2}}{2\pi} \|\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s+\varepsilon} \|\|\|\mathcal{W}(u_N^k)\|_{L^{2}(\mu_N)}\|_{L^r_k}
$$

$$
= C_k p^\frac{k}{2} \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s+\varepsilon} \int_{\mathbb{T}} \gamma_N^k(z)\gamma_N(z)dz \right)^\frac{1}{2}.
$$

(3.26)

From (3.21) with $\gamma_N^k(z) = \gamma_N^k(-z)$, we have

$$
\int_{\mathbb{T}} \gamma_N^k(z)\gamma_N(z)dz = \frac{1}{(2\pi)^{k-1}} \sum_{0 < |n| \leq N} \frac{1_{n=n_1+\ldots+n_k}}{\prod_{j=1}^{k} K_\delta(n_j)}.
$$

(3.27)
Hence, from (3.26) and (3.27), and Lemma 2.1, we obtain

\[
\|\|W(u_N^k)\|_{W_z^{s, \infty}}\|_{L^p(d\mu_\delta)} \leq C_{k, \delta} p^{\frac{1}{2}} \left( \sum_{0 < |n_j| \leq N \atop j=1,\ldots,k} \frac{1}{\prod_{j=1}^k \langle n_j \rangle} (n_1 + \cdots + n_k)^{2(s+\epsilon)} \right)^{\frac{1}{2}}
\]

\[
\leq C_{k, \delta} p^{\frac{1}{2}} \left( \sum_{n_1,\ldots,n_k \in \mathbb{Z}^k} \frac{1}{\prod_{j=1}^k \langle n_j \rangle} (n_1 + \cdots + n_k)^{2(s+\epsilon)} \right)^{\frac{1}{2}} < \infty,
\]

uniformly in \( N \in \mathbb{N} \), provided that \( s + \epsilon < 0 \). This last condition can be guaranteed for \( s < 0 \) by taking \( \epsilon > 0 \) sufficiently small. In view of (2.3), the bound (3.29) holds uniformly in \( 2 \leq \delta \leq \infty \) (namely, the constant \( C_{k, \delta} \) can be chosen to be independent of \( 2 \leq \delta \leq \infty \)). This proves (3.12).

Let \( M \geq N \geq 1 \) and \( p \geq 2 \). Proceeding as above, we have

\[
\|\|W(u_M^k) - W(u_N^k)\|_{W_z^{s, \infty}}\|_{L^p(d\mu_\delta)} \leq C_k p^{\frac{1}{2}} \left( \sum_{n} \langle n \rangle^{2(s+\epsilon)} \int_{\mathbb{T}} (\gamma_M(z) - \gamma_N(z)) e_n(z) dz \right)^{\frac{1}{2}}
\]

\[
\leq C_{k, \delta} p^{\frac{1}{2}} \left( \sum_{0 < |n_j| \leq M \atop j=1,\ldots,k} \frac{1}{\prod_{j=1}^k \langle n_j \rangle} (n_1 + \cdots + n_k)^{2(s+\epsilon)} \right)^{\frac{1}{2}}
\]

\[
- \sum_{0 < |n_j| \leq N \atop j=1,\ldots,k} \frac{1}{\prod_{j=1}^k \langle n_j \rangle} (n_1 + \cdots + n_k)^{2(s+\epsilon)} \right)^{\frac{1}{2}}
\]

\[
\leq C_{k, \delta} p^{\frac{1}{2}} \left( \sum_{0 < |n_j| \leq M \atop j=1,\ldots,k} \frac{1}{\prod_{j=1}^k \langle n_j \rangle} (n_1 + \cdots + n_k)^{2(s+\epsilon)} \right)^{\frac{1}{2}}
\]

\[
\leq C_{k, \delta} p^{\frac{1}{2}} N^{\max(s, -\frac{1}{2}) + 2\epsilon}
\]

for any \( \epsilon > 0 \), provided that \( s < 0 \). By choosing \( 0 < 2\epsilon < \min \left( -s, \frac{1}{2} \right) \), we then obtain

\[
\|\|W(u_M^k) - W(u_N^k)\|_{W_z^{s, \infty}}\|_{L^p(d\mu_\delta)} \to 0,
\]

as \( N \to \infty \). In view of (2.3), the bound (3.29) holds uniformly in \( 2 \leq \delta \leq \infty \) and thus the convergence in (3.30) holds uniformly in \( 2 \leq \delta \leq \infty \), yielding (3.13).

By applying Chebyshev’s inequality (see also Lemma 4.5 in [87]), to (3.29) (with \( M = \infty \)) and summing over in \( N \in \mathbb{N} \) we have

\[
\sum_{N=1}^{\infty} \mathbb{P} \left( \|W(u_N^k)\|_{W_z^{s, \infty}} > \frac{1}{j} \right) \leq \sum_{N=1}^{\infty} e^{-cN^{\frac{1}{2}\left(\max(s, -\frac{1}{2}) + 2\epsilon\right) j^{-\frac{1}{2}}}} \lesssim e^{-c'j^{-\frac{1}{2}}} \leq \infty.
\]
Therefore, we conclude from the Borel-Cantelli lemma that there exists \( \Omega_j \) with \( \mathbb{P}(\Omega_j) = 1 \) such that for each \( \omega \in \Omega_j \), there exists \( N_j = N_j(\omega) \in \mathbb{N} \) such that

\[
\| \mathcal{W}(u^k)(\omega) - \mathcal{W}(u^k_N)(\omega) \|_{W^{s,\infty}} < \frac{1}{j}
\]

for any \( N \geq N_j \). By setting \( \Sigma = \bigcap_{j=1}^{\infty} \Omega_j \), we have \( \mathbb{P}(\Sigma) = 1 \). Hence, we conclude that \( \mathcal{W}(u^k_N) \) converges almost surely to \( \mathcal{W}(u^k) \) in \( W^{s,\infty}(\mathbb{T}) \).

Next, we study the densities for the truncated Gibbs measures \( \rho_{\delta,N} \) in \( (3.11) \). As mentioned above, we restrict our attention to the defocusing case in this subsection. Namely, we fix \( k \in 2\mathbb{N} + 1 \). See Subsection 3.3 for the \( k = 2 \) case. Given \( 0 < \delta \leq \infty \) and \( N \in \mathbb{N} \), let \( G_{\delta,N}(u) \) be the truncated density defined in \( (1.46) \). Our main goal is to establish an \( L^p \)-integrability of the truncated density \( G_{\delta,N}(u) \) for the following two purposes:

- In order to construct the limiting Gibbs measure \( \rho_{\delta} \) for each fixed \( 0 < \delta \leq \infty \) (Theorem 1.3(i)), we establish such an \( L^p \)-integrability of the truncated density, uniformly in \( N \in \mathbb{N} \) but for each fixed \( 0 < \delta \leq \infty \).
- In order to prove convergence of the Gibbs measures in the deep-water limit (Theorem 1.3(ii)), we establish an \( L^p \)-integrability of the truncated density, uniformly in both \( N \in \mathbb{N} \) and \( \delta \gg 1 \).

Here, we need to study the \( L^p \)-integrability of \( G_{\delta,N}(u) \) with respect to the Gaussian measure \( \mu_{\delta} \) in \( (1.22) \), which is different for different values of \( \delta \). In order to establish a uniform (in \( \delta \)) bound, it is therefore more convenient to work with the Gaussian process \( X_{\delta} \) and the underlying probability measure \( \mathbb{P} \) on \( \Omega \).

Given \( 0 < \delta \leq \infty \) and \( N \in \mathbb{N} \), we define \( G_{\delta,N}(X_{\delta}) = G_{\delta,N}(X_{\delta}; k + 1) \) by

\[
G_{\delta,N}(X_{\delta}) = e^{-R_{\delta,N}(X_{\delta})} = e^{-\frac{1}{k+1} \int_{\mathbb{T}} W(X_{\delta}^{k+1}) dx},
\]

where \( R_{\delta,N}(X_{\delta}) = R_{\delta,N}(X_{\delta}; k + 1) \) is the truncated potential energy defined in \( (3.14) \).
Proposition 3.6. Let $k \in 2\mathbb{N} + 1$ and fix finite $p \geq 1$. Given any $0 < \delta \leq \infty$, we have
\[ \sup_{N \in \mathbb{N}} \|G_{\delta,N}(X_{\delta})\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|G_{\delta,N}(u)\|_{L^p(\mu_\delta)} \leq C_{p,k,\delta} < \infty. \] (3.32)

In addition, the following uniform bound holds for $2 \leq \delta \leq \infty$:
\[ \sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}(X_{\delta})\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}(u)\|_{L^p(\mu_\delta)} \leq C_{p,k,\delta} < \infty. \] (3.33)

Define $G_{\delta}(X_{\delta}) = G_{\delta,\infty}(X_{\delta})$ by
\[ G_{\delta}(X_{\delta}) = e^{-R_{\delta}(X_{\delta})} \]
with $R_{\delta}(X_{\delta})$ as in (3.15). Then, $G_{\delta,N}(X_{\delta})$ converges to $G_{\delta}(X_{\delta})$ in $L^p(\Omega)$. Namely, we have
\[ \lim_{N \to \infty} \|G_{\delta,N}(X_{\delta}) - G_{\delta}(X_{\delta})\|_{L^p(\Omega)} = 0. \] (3.34)

Furthermore, the convergence is uniform in $2 \leq \delta \leq \infty$:
\[ \lim_{N \to \infty} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}(X_{\delta}) - G_{\delta}(X_{\delta})\|_{L^p(\Omega)} = 0. \] (3.35)

As a consequence, the uniform bounds (3.32) and (3.33) hold even if we replace the supremum in $N \in \mathbb{N}$ by the supremum in $N \in \mathbb{N} \cup \{\infty\}$.

Theorem 1.3(i) follows as a directly corollary to Proposition 3.6, allowing us to define the limiting Gibbs measure $\rho_{\delta}$ in (1.48). Fix $0 < \delta \leq \infty$. Then, (3.34) with $p = 1$ implies that the partition function $Z_{\delta,N} = \|G_{\delta,N}(u)\|_{L^1(\mu_\delta)}$ of the truncated Gibbs measure $\rho_{\delta,N}$ in (1.45) converges to the partition function $Z_{\delta} = \|G_{\delta}(u)\|_{L^1(\mu_\delta)}$ of the Gibbs measure $\rho_{\delta}$ in (1.48). Let $B_{H^{-\varepsilon}}$ denote the collection of Borel sets in $H^{-\varepsilon}(T)$. Then, once again from (3.34), we have
\[ \lim_{N \to \infty} \sup_{A \in B_{H^{-\varepsilon}}} |\rho_{\delta,N}(A) - \rho_{\delta}(A)| = 0. \]

This proves convergence in total variation of $\rho_{\delta,N}$ to $\rho_{\delta}$. By using (3.35) in place of (3.34), a slight modification of the argument above yields uniform convergence in total variation of $\rho_{\delta,N}$ to $\rho_{\delta}$ for $2 \leq \delta \leq \infty$. See (3.35) below. We omit details.

We now present the proof of Proposition 3.6.

Proof of Proposition 3.6. We break the proof into two steps.

- **Step 1:** We first prove the uniform $L^p$-bounds (3.32) and (3.33). Given $k \in 2\mathbb{N} + 1$, the Hermite polynomial $H_{k+1}$ has a global minimum; there exists finite $a_{k+1} > 0$ such that $H_{k+1}(x) \geq -a_{k+1}$ for any $x \in \mathbb{R}$. It follows from (2.17) that
\[ H_{k+1}(x;\sigma) \geq -\sigma^{\frac{k+1}{2}}a_{k+1} \] (3.37)
for any $x \in \mathbb{R}$ and $\sigma > 0$. Hence, from (3.14) and (3.37) with (1.43), we have

$$-R_{\delta,N}(X_\delta) = -\frac{1}{k+1} \int_{T} H_{k+1}(X_{\delta,N};\sigma_{\delta,N})dx$$

$$\leq \frac{2\pi}{k+1} \sigma_{\delta,N} a_{k+1} \leq A_{k,\delta}(\log(N+1))^{\frac{k+1}{2}}$$

(3.38)

for some $A_{k,\delta} > 0$, uniformly in $N \in \mathbb{N}$. The bound (3.38) is exactly where the defocusing nature of the equation plays a crucial role.

**Remark 3.7.** Recall the uniform lower bound (2.3) for $2 \leq \delta \leq \infty$ (with the identification (3.1) when $\delta = \infty$). In view of (1.43), we can then choose $A_{k,\delta}$ to be independent of $2 \leq \delta \leq \infty$ (and $N \in \mathbb{N}$) as in the proof of Proposition 3.4. Similarly, by restricting our attention to $2 \leq \delta \leq \infty$, we can choose the constant $c_{k,\delta}$ in (3.39) below to be independent of $2 \leq \delta \leq \infty$. As a result, the constants in $B_{k,\delta,p}$ and $C_{2}(k,\delta,p)$ in (3.43) below can be chosen to be independent of $2 \leq \delta \leq \infty$.

By applying Proposition 3.4 (i) and Chebyshev’s inequality (see also Lemma 4.5 in [87]), we have, for some $C_1 > 0$ and $c_{k,\delta} > 0$,

$$\mathbb{P}\left(\left|pR_{\delta,M}(X_\delta) - R_{\delta,N}(X_\delta)\right| > \lambda\right) \leq C_1 e^{-c_{k,\delta}p^{\frac{2}{k+1}}N^{\frac{2p}{k+1}}\lambda^{\frac{2}{k+1}}}$$

(3.39)

for any $M \geq N \geq 1$ and any $p, \lambda > 0$.

By writing

$$\|G_{\delta,N}(X_\delta)\|_{L^p(\Omega)} = \int_{0}^{\infty} \mathbb{P}\left(e^{-pR_{\delta,N}(X_\delta)} > \alpha\right)d\alpha$$

$$\leq 1 + \int_{1}^{\infty} \mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \log \alpha\right)d\alpha,$$

we see that the desired bound (3.32) follows once we show that there exist $C_2 = C_2(k,\delta,p) > 0$ and $\beta > 0$ such that

$$\mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \log \alpha\right) \leq C_2 \alpha^{-(1+\beta)}$$

(3.40)

for any $\alpha > 1$ and $N \in \mathbb{N}$. We prove (3.40) via a standard application of the so-called Nelson’s estimate. Namely, given $\alpha > 1$, we choose $N_0 = N_0(\alpha) > 0$ and establish (3.40) for $N \geq N_0$ and $N < N_0$ in two different ways.

Given $\lambda := \log \alpha > 0$, we choose $N_0 > 0$ by setting

$$\lambda = 2pA_{k,\delta}(\log(N_0+1))^{\frac{k+1}{2}}. \quad (3.41)$$

Then, from (3.38) and (3.41), we have

$$-pR_{\delta,N_0}(X_\delta) \leq pA_{k,\delta}(\log(N_0+1))^{\frac{k+1}{2}} = \frac{1}{2} \lambda.$$ \quad (3.42)
Hence, from (3.42) and (3.39), we have
\[
\mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \lambda\right) \leq \mathbb{P}\left(-p(R_{\delta,N}(X_\delta) - R_{\delta,N_0}(X_\delta)) > \frac{1}{2}\lambda\right)
\]
\[
\leq \mathbb{P}\left(p\left|R_{\delta,N}(X_\delta) - R_{\delta,N_0}(X_\delta)\right| > \frac{1}{2}\lambda\right)
\]
\[
\leq C_1 e^{-c'_{k,\delta} p - \frac{2}{k+1} \lambda \frac{2}{k+1} N_0 \frac{1}{k+1} \lambda \frac{2}{k+1}}
\]
\[
\leq C_1 e^{-c'_{k,\delta} p - \frac{2}{k+1} \lambda \frac{2}{k+1} (e^{B_{k,\delta,p}\lambda \frac{2}{k+1} N_0} - 1)}
\]
\[
\leq C_2(k, \delta, p) e^{-(1+\beta)\lambda}
\]
for any $N \geq N_0$. On the other hand, for $N < N_0$, it follows from (3.38) and (3.41) that
\[
-pR_{\delta,N}(X_\delta) \leq pA_{k,\delta}(\log(N+1))^\frac{k+1}{2} < \frac{1}{2}\lambda
\]
and thus we have
\[
\mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \lambda\right) = 0.
\]  
(3.44)
Putting (3.43) and (3.44) together, we conclude that (3.40) holds for any $\alpha > 1$ and $N \in \mathbb{N}$.

Therefore, we obtain
\[
\|G_{\delta,N}(X_\delta)\|_{L^p(\Omega)}^p \leq C_3(k, \delta, p) < \infty
\]  
(3.45)
for any $N \in \mathbb{N}$.

For $2 \leq \delta \leq \infty$, it follows from Remark 3.7 that the constant $C_3(k, \delta, p)$ in (3.45) can be chosen to be independent of $2 \leq \delta \leq \infty$, thus yielding (3.33).

**Step 2:** Next, we show the (uniform) $L^p$-convergence of the truncated densities.

Fix $0 < \delta \leq \infty$. The $L^p$-convergence (3.34) of the truncated density $G_{\delta,N}(X_\delta)$ follows from the uniform bound (3.32) and a standard argument (see [86, Remark 3.8]). More precisely, as a consequence of Proposition 3.4 (i) and the continuous mapping theorem, we see that $G_{\delta,N}(X_\delta) = e^{-R_{\delta,N}(X_\delta)}$ converges in probability to the limit $G_{\delta}(X_\delta) = e^{-R_{\delta}(X_\delta)}$. Then, the $L^p$-convergence (3.34) follows from the uniform bound (3.32) and this softer convergence in probability. While we omit details of the argument in this case, we present details of an analogous argument in establishing the uniform $L^p$-convergence (3.35) in the following.

In the following, we present the proof of (3.35) and thus restrict our attention to $2 \leq \delta \leq \infty$. Proposition 3.4 (i), the continuity of the exponential function, and the uniform continuous mapping theorem (Lemma 2.9) \(^{18}\) we see that $G_{\delta,N}(X_\delta)$ converges in probability to $G_{\delta}(X_\delta)$ as $N \to \infty$, uniformly in $2 \leq \delta \leq \infty$. Then, by setting
\[
A_{\delta,N,\varepsilon} = \left\{ |G_{\delta,N}(X_\delta) - G_{\delta}(X_\delta)| \leq \varepsilon \right\},
\]  
(3.46)
we have
\[
\sup_{2 \leq \delta \leq \infty} \mathbb{P}(A_{\delta,N,\varepsilon}^c) \to 0,
\]  
(3.47)
\(^{18}\)Here, we use the tightness of $\{R_{\delta}(X_\delta)\}_{2 \leq \delta \leq \infty}$, coming from (3.10), to verify the hypothesis (2.27) in Lemma 2.9.
as $N \to \infty$. Then, from $(3.46)$, Cauchy-Schwarz’s inequality, the uniform (in $\delta$ and $N$, including $N = \infty$) bound $(3.33)$, and $(3.47)$, we obtain

$$
\sup_{2 \leq \delta \leq \infty} \|G_\delta(X_\delta) - G_{\delta,N}(X_\delta)\|_{L^p(\Omega)}
\leq \sup_{2 \leq \delta \leq \infty} \|(G_\delta(X_\delta) - G_{\delta,N}(X_\delta)) \cdot 1_{A_{\delta,N,\varepsilon}}\|_{L^p(\Omega)}
+ \sup_{2 \leq \delta \leq \infty} \|(G_\delta(X_\delta) - G_{\delta,N}(X_\delta)) \cdot 1_{A_{\delta,N,\varepsilon}}\|_{L^p(\Omega)}
\leq \varepsilon + \sup_{2 \leq \delta \leq \infty} \|G_\delta(X_\delta) - G_{\delta,N}(X_\delta)\|_{L^{2p}(\Omega)} \cdot \sup_{2 \leq \delta \leq \infty} \mathbb{P}(A_{\delta,N,\varepsilon})^{\frac{1}{2p}}
\leq 2\varepsilon
$$

for sufficiently large $N \gg 1$. This proves the uniform (in $\delta$) $L^p$-convergence $(3.35)$. This concludes the proof of Proposition 3.6. $\square$

### 3.3. Convergence of the Gibbs measures in the deep-water limit. In this subsection, we present the proof of Theorem 1.3 (ii). Once again, we restrict our attention to the defocusing case: $k \in 2\mathbb{N} + 1$. The construction of the Gibbs measures in the previous subsection shows that, for each $0 < \delta \leq \infty$, the Gibbs measure $\rho_\delta$ and the base Gaussian measure $\mu_\delta$ are equivalent. On the other hand, from Proposition 3.1, we know that the Gaussian measures $\mu_\delta$ are all equivalent for $0 < \delta \leq \infty$. Therefore, we conclude that the Gibbs measure $\rho_\delta$, $0 < \delta < \infty$, for the defocusing gILW equation $(1.4)$ and the Gibbs measure $\rho_{BO} = \rho_\infty$ for the defocusing gBO equation $(1.12)$ are equivalent. This proves the first claim in Theorem 1.3 (ii).

Hence, it remains to show that the Gibbs measure $\rho_\delta$ converges to $\rho_{BO}$ in total variation, as $\delta \to \infty$.

Before proceeding to the proof of convergence in total variation of $\rho_\delta$ to $\rho_{BO}$, let us first present the following $L^p$-convergence of the (truncated) densities. For $0 < \delta \leq \infty$, let $X_\delta$ and $X_{BO} = X_\infty$ be as in $(1.41)$ and $(1.38)$, respectively, and let $R_\delta(X_\delta)$ (and $G_\delta(X_\delta)$, respectively) be the limit of $R_{\delta,N}(X_\delta)$ constructed in Proposition 3.4 (and of $G_{\delta,N}(X_\delta)$ constructed in Proposition 3.6, respectively).

**Lemma 3.8.** Let $k \in 2\mathbb{N} + 1$ and $1 \leq p < \infty$. Then, given $N \in \mathbb{N}$, we have

$$
\lim_{\delta \to \infty} \|G_{\delta,N}(X_\delta) - G_{\infty,N}(X_{BO})\|_{L^p(\Omega)} = 0.
$$

As a corollary, we have

$$
\lim_{\delta \to \infty} \|G_\delta(X_\delta) - G_\infty(X_{BO})\|_{L^p(\Omega)} = 0.
$$

In particular, the partition function $Z_\delta$ of the Gibbs measure $\rho_\delta$ in $(1.48)$ converges to the partition function $Z_{BO} = Z_\infty$ of the Gibbs measure $\rho_{BO} = \rho_\infty$, as $\delta \to \infty$.

**Remark 3.9.** In view of the argument presented in $(3.36)$, one may be tempted to conclude directly from $(3.49)$ in Lemma 3.8 that $\rho_\delta$ converges to $\rho_{BO} = \rho_\infty$ in total variation as $\delta \to \infty$. However, this is not possible. This is due to the fact that the base Gaussian measures $\mu_\delta$ and $\mu_\infty$ are different. If we were to mimic the argument in $(3.36)$, the integral in the third step
of (3.36) would be replaced by

\[
\int_{\Omega} \left( 1_{\{X_{\delta} \in A\}} G_{\delta,N}(X_{\delta}) - 1_{\{X_{BO} \in A\}} G_{\infty}(X_{BO}) \right) dP = \int_{\Omega} 1_{\{X_{\delta} \in A\}} \left( G_{\delta,N}(X_{\delta}) - G_{\infty}(X_{BO}) \right) dP \\
+ \int_{\Omega} \left( 1_{\{X_{\delta} \in A\}} - 1_{\{X_{BO} \in A\}} \right) G_{\infty}(X_{BO}) dP. 
\] (3.50)

While we can apply (3.49) in Lemma 3.8 to control the first term on the right-hand side of (3.50), we cannot handle the second term as it is. Note that the difference

\[
1_{\{X_{\delta} \in A\}} - 1_{\{X_{BO} \in A\}} 
\]

with respect to the \(P\)-integration (and taking the supremum in \(A \in B_{H^{-}}\)) is closely related to the convergence in total variation of \(\mu_{\delta}\) to \(\mu_{\infty}\) proven in Proposition 3.1 (iii), which plays a crucial role in the proof of convergence in total variation of \(\rho_{\delta}\) to \(\rho_{BO}\) presented below.

Proof of Lemma 3.8. Fix \(N \in \mathbb{N}\). From (1.43) and Lemma 2.1, we have

\[
\sigma_{\delta,N} = \frac{1}{2\pi} \sum_{0 < |n| \leq N} K_{\delta}(n) \rightarrow \frac{1}{2\pi} \sum_{|n| \leq N} 1_{|n|} =: \sigma_{\infty,N}, \tag{3.51}
\]
as \(\delta \rightarrow \infty\). It also follows from the definitions (1.38), (1.41), and Lemma 2.1 that, for any \(x \in T\) and \(\omega \in \Omega\),

\[
|X_{\delta,N}(x; \omega)| \lesssim \sum_{0 < |n| \leq N} \frac{|g_{n}(\omega)|}{K_{\delta}(n)} \leq C_{N,\omega} < \infty
\]
for any \(2 \leq \delta \leq \infty\), \(x \in T\), and \(\omega \in \Omega\). Then, by the dominated convergence theorem applied to the integration in \(x \in T\), we have

\[
R_{\delta,N}(X_{\delta}(\omega)) = \frac{1}{k + 1} \int_{T} H_{k}(X_{\delta,N}(x; \omega); \sigma_{\delta,N}) dx \rightarrow R_{\infty,N}(X_{BO}(\omega)) \tag{3.52}
\]
as \(\delta \rightarrow \infty\), for any \(\omega \in \Omega\). As a consequence, we see that \(G_{\delta,N}(X_{\delta}(\omega))\) converges to \(G_{\infty,N}(X_{BO}(\omega))\) as \(\delta \rightarrow \infty\), for any \(\omega \in \Omega\). Moreover, from the uniform (in \(\omega\)) bound (3.38), we conclude that \(G_{\delta,N}(X_{\delta})\) converges to \(G_{\infty,N}(X_{BO})\) in \(L^{p}(\Omega)\), as \(\delta \rightarrow \infty\). This proves (3.48).

By the triangle inequality, we have

\[
\|G_{\delta}(X_{\delta}) - G_{\infty}(X_{BO})\|_{L^{p}(\Omega)} \\
\leq \|G_{\delta}(X_{\delta}) - G_{\delta,N}(X_{\delta})\|_{L^{p}(\Omega)} + \|G_{\delta,N}(X_{\delta}) - G_{\infty,N}(X_{BO})\|_{L^{p}(\Omega)} \\
+ \|G_{\infty,N}(X_{BO}) - G_{\infty}(X_{BO})\|_{L^{p}(\Omega)}. 
\] (3.53)

\(\text{Here, we use the convention that } g_{n}(\omega) \in \mathbb{C} \text{ for every } \omega \in \Omega \text{ and } n \in \mathbb{Z}^{+}.\)
Then, by first applying (3.48) above and then (3.35) in Proposition 3.6 to (3.53) (namely, we first take \( \delta \to \infty \) and then \( N \to \infty \)), we obtain
\[
\lim_{\delta \to \infty} \| G(\delta, X_{\delta}) - G_{\infty}(X_{BO}) \|_{L^p(\Omega)} \\
\leq 2 \lim_{N \to \infty} \left( \sup_{2 \leq \delta \leq \infty} \| G(\delta, X_{\delta}) - G_{\delta,N}(X_{\delta}) \|_{L^p(\Omega)} \\
+ \lim_{\delta \to \infty} \| G_{\delta,N}(X_{\delta}) - G_{\infty,N}(X_{BO}) \|_{L^p(\Omega)} \right) \\
= 0.
\]

This proves (3.49). \( \square \)

We are now ready to show that the Gibbs measure \( \rho_{\delta} \) in (1.48) converges to \( \rho_{BO} = \rho_{\infty} \) in total variation as \( \delta \to \infty \). By the triangle inequality, we have
\[
d_{TV}(\rho_{\delta}, \rho_{BO}) \leq d_{TV}(\rho_{\delta}, \rho_{\delta,N}) + d_{TV}(\rho_{\delta,N}, \rho_{\infty,N}) + d_{TV}(\rho_{\infty,N}, \rho_{BO}) \tag{3.54}
\]
for any \( N \in \mathbb{N} \). From Theorem 1.3 (i) (see also Proposition 3.6), we have
\[
\lim_{N \to \infty} \sup_{2 \leq \delta \leq \infty} d_{TV}(\rho_{\delta,N}, \rho_{\delta}) = 0. \tag{3.55}
\]
Hence, it suffices to prove
\[
\lim_{\delta \to \infty} d_{TV}(\rho_{\delta,N}, \rho_{\infty,N}) = 0 \tag{3.56}
\]
for any \( N \in \mathbb{N} \). Indeed, by applying (3.55) and (3.56) to (3.51) (namely, by first taking \( \delta \to \infty \) and then \( N \to \infty \)), we obtain
\[
\lim_{\delta \to \infty} d_{TV}(\rho_{\delta}, \rho_{BO}) \leq \lim_{N \to \infty} \left( \sup_{2 \leq \delta \leq \infty} d_{TV}(\rho_{\delta,N}, \rho_{\delta}) + \lim_{\delta \to \infty} d_{TV}(\rho_{\delta,N}, \rho_{\infty,N}) \right) \\
= 0.
\]

In the following, we prove (3.56) for any fixed \( N \in \mathbb{N} \). Fix \( N \in \mathbb{N} \). Then, Lemma 3.8 with \( p = 1 \) implies that the partition function \( Z_{\delta,N} = \| G_{\delta,N}(u) \|_{L^1(d\mu_\delta)} \) of the truncated Gibbs measure \( \rho_{\delta,N} \) in (1.45) converges to the partition function \( Z_{\infty,N} = \| G_{\infty,N}(u) \|_{L^1(d\mu_\infty)} \) of the truncated Gibbs measure \( \rho_{\infty,N} \) for the gBO equation as \( \delta \to \infty \). Then, from Proposition 3.1 (ii), we have
\[
\lim_{\delta \to \infty} \sup_{A \in B_{H^{-\epsilon}}} | \rho_{\delta,N}(A) - \rho_{\infty,N}(A) | \\
= \lim_{\delta \to \infty} \sup_{A \in B_{H^{-\epsilon}}} \left| \frac{Z_{\delta,N}}{Z_{\infty,N}} \rho_{\delta,N}(A) - \rho_{\infty,N}(A) \right| \\
\leq Z_{\infty,N}^{-1} \lim_{\delta \to \infty} \sup_{A \in B_{H^{-\epsilon}}} \left| \int_{H^{-\epsilon}} 1_A(u) \left( G_{\delta,N}(u) \frac{d\mu_\delta}{d\mu_\infty}(u) - G_{\infty,N}(u) \right) d\mu_\infty(u) \right| \tag{3.57} \\
\leq Z_{\infty,N}^{-1} \lim_{\delta \to \infty} \int_{H^{-\epsilon}} | G_{\delta,N}(u) - G_{\infty,N}(u) | d\mu_\infty(u) \\
+ Z_{\infty,N}^{-1} \lim_{\delta \to \infty} \int_{H^{-\epsilon}} | G_{\delta,N}(u) \left( \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 \right) | d\mu_\infty(u).
From (1.46) and (3.51), we see that $G_{\delta,N}(u)$ converges to $G_{\infty,N}(u)\mu_\infty$-almost surely, as $\delta \to \infty$. Moreover, it follows from (3.38) and Remark 3.7 that
\[
|G_{\delta,N}(u) - G_{\infty,N}(u)| \leq C_N < \infty
\]
for any $2 \leq \delta \leq \infty$. Hence, by the dominated convergence theorem, we obtain
\[
\lim_{\delta \to \infty} \int_{H^-} |G_{\delta,N}(u) - G_{\infty,N}(u)| \, d\mu_\infty(u) = 0.
\]
By Scheffé’s theorem (Lemma 2.1 in [85]; see also Proposition 1.2.7 in [51]), we have
\[
d_{TV}(\mu_\delta, \mu_\infty) = \frac{1}{2} \int_{H^-} \left| \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 \right| \, d\mu_\infty(u).
\]
Then, it follows from the convergence in total variation of $\mu_\delta$ to $\mu_\infty$ as $\delta \to \infty$ (Proposition 3.1 (iii)), (3.60), and the uniform (in $\delta$) bound (3.38) (and Remark 3.7) for $2 \leq \delta \leq \infty$ that
\[
\lim_{\delta \to \infty} \int_{H^-} G_{\delta,N}(u) \left| \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 \right| \, d\mu_\infty(u) \\
\leq C_N \lim_{\delta \to \infty} \int_{H^-} \left| \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 \right| \, d\mu_\infty(u) \\
= 2C_N \lim_{\delta \to \infty} d_{TV}(\mu_\delta, \mu_\infty) \\
= 0.
\]
Therefore, from (3.57), (3.59), and (3.61), we conclude (3.56) and hence convergence in total variation of $\rho_\delta$ to $\rho_{BO}$ as $\delta \to \infty$. This concludes the proof of Theorem 1.3 when $k \in 2\mathbb{N} + 1$.

3.4. Gibbs measures for the ILW equation: variational approach. We conclude this section by presenting the proof of Theorem 1.3 for the $k = 2$ case, corresponding to the ILW equation (1.1). In this case, the problem is no longer defocusing and thus we need to consider the (truncated) Gibbs measures with a Wick-ordered $L^2$-cutoff of the form (1.49) and (1.50). As pointed out in Remark 1.4, there is no need for a renormalization on the potential energy under the current (spatial) mean-zero condition.

Fix $K > 0$ in the remaining part of this section. Given $0 < \delta \leq \infty$ and $N \in \mathbb{N}$, define the truncated density $G^K_{\delta,N}(u)$ by
\[
G^K_{\delta,N}(u) = \chi_K \left( \int_{\mathbb{T}} W(u^2_N) \, dx \right) e^{-\frac{1}{\delta} \int_{\mathbb{T}} u^3_N \, dx} \\
= \chi_K \left( \int_{\mathbb{T}} H_2(u_N; \sigma_{\delta,N}) \, dx \right) e^{-\frac{1}{4} \int_{\mathbb{T}} u^3_N \, dx},
\]
where $u_N = P_N u$ and $\chi_K : \mathbb{R} \to [0,1]$ is a continuous function such that $\chi_K(x) = 1$ for $|x| \leq K$ and $\chi_K(x) = 0$ for $|x| \geq 2K$.

In view of the discussion in Subsections 3.2 and 3.3 Theorem 1.3 for $k = 2$ follows once we prove the following uniform bounds.

**Proposition 3.10.** Fix finite $p \geq 1$ and $K > 0$. Then, given any $0 < \delta \leq \infty$, we have
\[
\sup_{N \in \mathbb{N}} \|G^K_{\delta,N}(X_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|G^K_{\delta,N}(u)\|_{L^p(d\mu_\delta)} \leq C_{p,\delta,K} < \infty.
\]
In addition, the following uniform bound holds for $2 \leq \delta \leq \infty$:

$$\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}^K(X_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}^K(u)\|_{L^p(d\mu_\delta)} \leq C_{p,K} < \infty.$$ 

Let us first discuss how to conclude Theorem 1.3 when $k = 2$, by assuming Proposition 3.10. Define the limiting density $G_{\delta}^K(u)$ by

$$G_{\delta}^K(u) = \chi_K \left( \int_{\mathbb{T}} W(u^2) dx \right) e^{-\frac{1}{2} \int_{\mathbb{T}} u^3 dx}.$$  

Note that Proposition 3.4(i) guarantees that $\int_{\mathbb{T}} W(u^2) dx$ and $\int_{\mathbb{T}} u^3 dx = \int_{\mathbb{T}} W(u^3) dx$ in (3.63) exist as the limits in $L^p(\mu_\delta)$ of the truncated versions. Hence, the truncated density $G_{\delta,N}^K(u)$ converges in measure to the limiting density $G_{\delta}^K(u)$ in (3.63). Hence, once we prove Proposition 3.10 we can repeat the argument in Step 2 in the proof of Proposition 3.6 to show the following convergence results.

**Corollary 3.11.** Let $0 < \delta \leq \infty$ and $1 \leq p < \infty$. Then, $G_{\delta,N}^K(X_\delta)$ converges to $G_{\delta}^K(X_\delta)$ in $L^p(\Omega)$ as $N \to \infty$. Namely, we have

$$\lim_{N \to \infty} \|G_{\delta,N}^K(X_\delta) - G_{\delta}^K(X_\delta)\|_{L^p(\Omega)} = 0.$$ 

Furthermore, the convergence is uniform in $2 \leq \delta \leq \infty$:

$$\lim_{N \to \infty} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}^K(X_\delta) - G_{\delta}^K(X_\delta)\|_{L^p(\Omega)} = 0.$$ 

This proves an analogue of Theorem 1.3(i) when $k = 2$. The equivalence of the Gibbs measure $\rho_\delta$ in (1.50), $0 < \delta < \infty$, and $\rho_{BO}$ in (1.39) follows from (i) the equivalence of the Gibbs measure $\rho_\delta$ and the Gaussian measure with the Wick-ordered $L^2$-cutoff:

$$\chi_K \left( \int_{\mathbb{T}} W(u^2) dx \right) d\mu_\delta(u)$$

(including $\delta = \infty$ with the understanding that $\rho_\infty = \rho_{BO}$), and (ii) the equivalence of the base Gaussian measure $\mu_\delta$, $0 < \delta < \infty$ (Proposition 3.1(ii)).

Finally, we discuss convergence of the Gibbs measure $\rho_\delta$ to $\rho_{BO}$ in the deep-water limit ($\delta \to \infty$). In the defocusing case discussed in the previous subsection, the bound (3.38) provided the uniform (in $2 \leq \delta \leq \infty$ and $\omega \in \Omega$) bound on the truncated density $G_{\delta,N}(u)$; see the discussion after (3.52). See also (3.58) and (3.61). In the current non-defocusing case, however, the bound (3.38) is not available to us. Nonetheless, in view of (1.44) and (1.43) with Lemma 2.1 the Wick-ordered $L^2$-cutoff in (3.62) with (1.44) implies

$$\left| \int_{\mathbb{T}} u_N^2 dx \right| \leq \sigma_{\delta,N} + 2K \leq C_{N,K} < \infty,$$ 

for any $2 \leq \delta \leq \infty$ and $N \in \mathbb{N}$, where $C_{N,K}$ is independent of $2 \leq \delta \leq \infty$. Then, by Sobolev’s inequality with (3.64), we have

$$\left| \int_{\mathbb{T}} u_N^3 dx \right| \lesssim \|u_N\|_{H^{\frac{1}{2}}}^3 \leq N\|C_N^3\|_{H^{\frac{1}{2}}}.$$ 

which provides a bound on the truncated density $G_{\delta,N}^K(u)$ in (3.62), uniformly in $2 \leq \delta \leq \infty$. With this bound on $G_{\delta,N}^K(u)$, we can repeat the argument presented in Subsection 3.3 to conclude the desired convergence in total variation of $\rho_\delta$ to $\rho_{BO}$ as $\delta \to \infty$.

In the remaining part of this section, we present the proof of Proposition 3.10. Given $0 < \delta \leq \infty$ and $N \in \mathbb{N}$, set

$$R_{\delta,N}(u) = \frac{1}{3} \int_{\mathbb{T}} u_N^3 \, dx + A \left| \int_{\mathbb{T}} W(u_N^2) \, dx \right|^2,$$

(3.66)

where $W(u_N^2) = W_{\delta,N}(u_N^2) = H_2(u_N; \sigma_{\delta,N})$. Then, as in [71], we consider the following truncated density:

$$G_{\delta,N}^K(u) = e^{-R_{\delta,N}(u)} = e^{-\frac{1}{3} \int_{\mathbb{T}} u_N^3 \, dx - A \int_{\mathbb{T}} W(u_N^2) \, dx}$$

(3.67)

for some suitable $A > 0$. Noting that $\chi_K(x) \leq \exp \left( -A|x|^\gamma \right) \exp(A2^K \gamma)$ (3.68) for any $K, A, \gamma > 0$, we have

$$G_{\delta,N}^K(u) \leq C_{A,K} \cdot G_{\delta,N}^K(u).$$

Hence, Proposition 3.10 follows once we prove the following uniform bounds on $G_{\delta,N}^K(u)$.

**Proposition 3.12.** Fix finite $p \geq 1$. Then, there exists $A_0 = A_0(p) > 0$ such that

$$\sup_{N \in \mathbb{N}} \|G_{\delta,N}^K(X_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|G_{\delta,N}^K(u)\|_{L^p(d\mu_\delta)} \leq C_{p,\delta,K,A} < \infty$$

(3.69)

for any $0 < \delta \leq \infty$, $K > 0$, and $A \geq A_0$. In addition, the following uniform bound holds for $2 \leq \delta \leq \infty$:

$$\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}^K(X_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|G_{\delta,N}^K(u)\|_{L^p(d\mu_\delta)} \leq C_{p,K,A} < \infty$$

(3.70)

for any $K > 0$ and $A \geq A_0$.

As mentioned in the introduction, we employ the variational approach, introduced by Barashkov and Gubinelli [5], to prove Proposition 3.12. In particular, we follow closely the argument in [71], where the $\delta = \infty$ case was treated via the variational approach. See also [39, 70, 66, 16, 67] for recent works on dispersive PDEs, where the variational approach played a crucial role.

Let us first introduce some notations. Let $W(t)$ be a cylindrical Brownian motion in $L^2_0(\mathbb{T}) = \mathbf{P}_{\neq 0}L^2(\mathbb{T})$ of mean-zero functions on $\mathbb{T}$, where $\mathbf{P}_{\neq 0}$ denotes the projection onto the non-zero frequencies. Namely, we have

$$W(t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} B_n(t)e_n,$$

(3.71)
where \( \{B_n\}_{n \in \mathbb{Z}^*} \) is a sequence of mutually independent complex-valued Brownian motions such that \( \overline{B_n} = B_{-n}, n \in \mathbb{Z}^* \). Then, we define a centered Gaussian process \( Y_{\delta}(t) \) by
\[
Y_{\delta}(t) = (G_{\delta}\partial_x)^{-\frac{1}{2}}W(t),
\]
(3.72)
where \( (G_{\delta}\partial_x)^{-\frac{1}{2}} \) is the Fourier multiplier operator with the multiplier \( K_{\delta}(n)^{-\frac{1}{2}} \) with \( K_{\delta}(n) \) as in (1.42). In view of (1.11), we have \( L(Y_{\delta}(1)) = \mu_{\delta} \). Given \( N \in \mathbb{N} \), we set \( Y_{\delta,N} = P_N Y_{\delta} \). Then, from (1.43), we have
\[
E[Y_{\delta,N}(1)]^2 = \sigma_{\delta, N} \sim \delta \log(N + 1).
\]

Next, we recall the Boué-Dupuis variational formula. Let \( H_n \) denote the collection of drifts, which are progressively measurable processes belonging to \( L^2([0, 1]; L^2_0(T)) \), \( P \)-almost surely. We now state the Boué-Dupuis variational formula [11, 91]. See, in particular, Theorem 7 in [91].

**Lemma 3.13.** Given \( 0 < \delta \leq \infty \), let \( Y_{\delta} \) be as in (3.72). Fix \( N \in \mathbb{N} \). Suppose that \( F : C^\infty(T) \to \mathbb{R} \) is measurable such that \( E[|F(Y_{\delta,N}(1))|^p] < \infty \) and \( E[|e^{-F(Y_{\delta,N}(1))}|^q] < \infty \) for some \( 1 < p, q < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, we have
\[
-\log E[e^{-F(Y_{\delta,N}(1))}] = \inf_{\theta \in H_n} E[F(Y_{\delta,N}(1) + P_N I_\delta(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|^2_{L^2} dt],
\]
(3.73)
where \( I_\delta(\theta) \) is defined by
\[
I_\delta(\theta)(t) = \int_0^t (G_{\delta}\partial_x)^{-\frac{1}{2}}\theta(t')dt'
\]
and the expectation \( E = E_P \) is an expectation with respect to the underlying probability measure \( P \).

**Remark 3.14.** (i) As far as the proof of Proposition 3.12 is concerned, we only need to work with \( Y_{\delta,N} \) evaluated at time \( t = 1 \). As such, we could have stated Lemma 3.13 with \( X_{\delta,N} \) in place of \( Y_{\delta,N}(1) \), thus allowing us to avoid introducing \( W(t) \) in (3.71) and \( Y_{\delta}(t) \) in (3.72). We, however, did not do so since the natural setting of the Boué-Dupuis formula is as stated above. For example, (3.73) allows us to choose a \( \delta \)-dependent drift \( \theta \), which is crucial in showing non-normalizability of the focusing Gibbs measures \( \rho_\delta \). See [71].

(ii) In view of the discussion above, in order to prove Proposition 3.12, it is possible to work with a slightly different and weaker variational formula stated in [39, Proposition 4.4], where an expectation is taken with respect to a shifted measure.

In the following, we prove Proposition 3.12 by applying Lemma 3.13 to \( G_{\delta,N}^K(u) \) in (3.67). Before proceeding to the proof of Proposition 3.12, let us state a preliminary lemma on the pathwise regularity bounds of \( Y_{\delta,N}(1) \) and \( I_\delta(\theta)(1) \).

**Lemma 3.15.** (i) Let \( \varepsilon > 0 \) and fix finite \( p \geq 1 \). Then, given any \( 0 < \delta \leq \infty \), we have
\[
E\left[ \|Y_{\delta,N}(1)\|^p_{W^{-\varepsilon, \infty}} + \|W(Y_{\delta,N}^2(1))\|^p_{W^{-\varepsilon, \infty}} \right.
\]
\[
+ \left. \|W(Y_{\delta,N}^3(1))\|^p_{W^{-\varepsilon, \infty}} \right] \leq C_{\varepsilon, p, \delta} < \infty,
\]
(3.74)
\[^{20}\text{By convention, we normalize } B_n \text{ such that } \text{Var}(B_n(t)) = 2\pi t.\]
uniformly in $N \in \mathbb{N}$. Furthermore, by restricting our attention to $2 \leq \delta \leq \infty$, we can choose the constant $C_{\varepsilon, p, \delta}$ in (3.74) to be independent of $\delta$.

(ii) Let $0 < \delta \leq \infty$. For any $\theta \in \mathbb{H}_a$, we have

$$
\|I_\delta(\theta)(1)\|_{H^{1/2}}^2 \leq C_\delta \int_0^1 \|\theta(t)\|_{L^2_x}^2 dt,
$$

(3.75)

where the constant $C_\delta > 0$ can be chosen to be independent of $2 \leq \delta \leq \infty$.

Proof. By noting that $L(Y_{\delta,N}(1)) = L(X_{\delta,N})$, we see that Part (i) follows from Proposition 3.4. As for the bound (3.75), it follows from Minkowski’s and Cauchy-Schwarz’s inequalities and the lower bound (2.2) of $K_{\delta}(n)$ that

$$
\|I_\delta(\theta)(1)\|_{H^{1/2}}^2 = \left\| \langle \nabla \rangle^{1/2} \int_0^1 (G_{\delta} \partial_x)^{-\frac{1}{2}} \theta(t') dt' \right\|_{L^2_x}^2 \leq C_\delta \int_0^1 \|\theta(t)\|_{L^2_x}^2 dt \leq C_\delta \left( \int_0^1 \|\theta(t)\|_{L^2_x}^2 dt \right)^{1/2}.
$$

When $2 \leq \delta \leq \infty$, the lower bound (2.3) allows us to choose the constant $C_\delta$ to be independent of $2 \leq \delta \leq \infty$. □

Fix $0 < \delta \leq \infty$ and finite $p \geq 1$. We first prove the bound (3.69). In view of the Boué-Dupuis formula (Lemma 3.13), it suffices to establish a lower bound on

$$
M_{\delta,N}(\theta) = \mathbb{E} \left[ pR_{\delta,N}(Y_{\delta}(1) + I_\delta(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2_x}^2 dt \right],
$$

(3.76)

uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{H}_a$. We set

$$
Y_{\delta,N} = \mathbb{P}_N Y_{\delta} = \mathbb{P}_N Y_{\delta}(1) \quad \text{and} \quad \Theta_{\delta,N} = \mathbb{P}_N \Theta_\delta = \mathbb{P}_N I_\delta(\theta)(1).
$$

From (3.66) and (2.1), we have

$$
R_{\delta,N}(Y_\delta + \Theta_\delta) = \frac{1}{3} \int_\mathbb{T} \mathcal{W}(Y_{\delta,N}^3) dx + \int_\mathbb{T} \mathcal{W}(Y_{\delta,N}^2) \Theta_{\delta,N} dx + \int_\mathbb{T} Y_{\delta,N} \Theta_{\delta,N}^2 dx,
$$

(3.77)

+ $\frac{1}{3} \int_\mathbb{T} \Theta_{\delta,N}^2 dx + A \left\{ \int_\mathbb{T} \left( \mathcal{W}(Y_{\delta,N}^2) + 2Y_{\delta,N} \Theta_{\delta,N} + \Theta_{\delta,N}^2 \right) dx \right\}^2,
$$

where the first term on the right-hand side vanishes under the expectation. Hence, from (3.76) and (3.77), we have

$$
M_{\delta,N}(\theta) = \mathbb{E} \left[ p \int_\mathbb{T} \mathcal{W}(Y_{\delta,N}^2) \Theta_{\delta,N} dx + p \int_\mathbb{T} Y_{\delta,N} \Theta_{\delta,N}^2 dx + \frac{P}{3} \int_\mathbb{T} \Theta_{\delta,N}^3 dx + A \left\{ \int_\mathbb{T} \left( \mathcal{W}(Y_{\delta,N}^2) + 2Y_{\delta,N} \Theta_{\delta,N} + \Theta_{\delta,N}^2 \right) dx \right\}^2 + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2_x}^2 dt \right].
$$

(3.78)

We now recall the following lemma from [71, Lemma 4.1], where the $p = 1$ case was treated. See also Lemma 5.8 in [66].
Lemma 3.16. (i) There exist small $\varepsilon > 0$ and a constant $c = c(p) > 0$ and $C_0 > 0$ such that
\[
p \left| \int_T \mathcal{W}(Y_{\delta,N}^2) \Theta_{\delta,N} dx \right| \leq c \|\mathcal{W}(Y_{\delta,N}^2)\|_{W^{-\varepsilon,\infty}}^2 + \frac{1}{100} \|\Theta_{\delta,N}\|_{H^{1/2}}^2,
\]
\[
p \left| \int_T Y_{\delta,N} \Theta_{\delta,N}^2 dx \right| \leq c \|\Theta_{\delta,N}\|_{W^{-\varepsilon,\infty}}^2 + \frac{1}{100} \left( \|\Theta_{\delta,N}\|_{H^{1/2}}^2 + \|\Theta_{\delta,N}\|_{L^2}^2 \right),
\]
\[
p \left| \frac{1}{3} \int T \Theta_{\delta,N}^2 dx \right| \leq \frac{1}{100} \|\Theta_{\delta,N}\|_{H^{1/2}}^2 + C_0 p^2 \|\Theta_{\delta,N}\|_{L^2}^2,
\]
uniformly in $N \in \mathbb{N}$ and $0 < \delta \leq \infty$.

(ii) Let $A > 0$. Given any small $\varepsilon > 0$, there exists $c = c(\varepsilon, p, A) > 0$ such that
\[
Ap \left\{ \int_T \left( \mathcal{W}(Y_{\delta,N}^2) + 2Y_{\delta,N} \Theta_{\delta,N} + \Theta_{\delta,N}^2 \right) dx \right\}^2 \geq \frac{Ap}{4} \|\Theta_{\delta,N}\|_{L^2}^2 - \frac{1}{100} \|\Theta_{\delta,N}\|_{H^{1/2}}^2 - c \left\{ \|Y_{\delta,N}\|_{W^{-\varepsilon,\infty}}^2 + \left( \int_T \mathcal{W}(Y_{\delta,N}^2)dx \right)^2 \right\},
\]
uniformly in $N \in \mathbb{N}$ and $0 < \delta \leq \infty$.

As in [71], we establish a pathwise lower bound on $\mathcal{M}_{\delta,N}(\theta)$ in (3.78), uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{H}_a$, by making use of the positive terms:
\[
\mathcal{U}_{\delta,N}(\theta) = E \left[ \frac{Ap}{4} \|\Theta_{\delta,N}\|_{L^2}^4 + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2}^2 dt \right],
\]
coming from (3.78) and (3.79). From (3.78) and (3.80) together with Lemmas 3.16 and 3.15 we obtain
\[
\inf_{N \in \mathbb{N}} \inf_{\theta \in \mathbb{H}_a} \mathcal{M}_{\delta,N}(\theta) \geq \inf_{N \in \mathbb{N}} \inf_{\theta \in \mathbb{H}_a} \left\{ - C_{p,\delta,A} + \frac{1}{10} \mathcal{U}_{\delta,N}(\theta) \right\} \geq -C_{p,\delta,A} > -\infty,
\]
provided that $A = A(p) \gg 1$ is sufficiently large. Hence, the uniform (in $N$) bound (3.69) follows from Lemma 3.13 with (3.67) and (3.81).

Next, we restrict our attention to $2 \leq \delta \leq \infty$. In this case, the constant $C_{\varepsilon,p,\delta}$ in (3.74) of Lemma 3.15 is independent of $\delta$ and, as a result, we see that the constant $C_{p,\delta,A}$ in (3.81) is also independent of $2 \leq \delta \leq \infty$. Therefore, the second bound (3.70) follows from Lemma 3.13 with (3.67) and (3.81). This concludes the proof of Proposition 3.12 and hence of Theorem 1.3 when $k = 2$.

4. Gibbs measures in the shallow-water regime

In this section, we present the proof of Theorem 1.5. Namely, we go over the construction and convergence in the shallow-water limit ($\delta \to 0$) of the Gibbs measure $\tilde{\rho}_\delta$ associated with the scaled gILW equation (1.17). For each fixed $0 < \delta < \infty$, the scaling transformation (1.16) simply introduces a constant factor, depending on $\delta$. Hence, the regularity properties of the support of the base Gaussian measures $\mu_\delta$ in (1.22) for the unscaled problem and $\tilde{\mu}_\delta$ in (1.24) for the scaled problem are the same for each fixed $0 < \delta < \infty$, and thus we can repeat the argument in Section 2 to construct the Gibbs measure $\tilde{\rho}_\delta$ supported on $H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T})$, $\varepsilon > 0$, yielding Theorem 1.5(i) for each fixed $0 < \delta < \infty$. The main difference in this shallow-water regime appears in establishing uniform (in $\delta$) bounds and convergence as $\delta \to 0$. This is due to
the singularity of the base Gaussian measures $\tilde{\mu}_\delta$, $0 < \delta < \infty$, supported on $H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T})$, and $\tilde{\mu}_0$ in (1.29) supported on $H^{1/2-\varepsilon}(\mathbb{T}) \setminus H^{1/2}(\mathbb{T})$; see Proposition 4.1.

In Subsection 4.1 we first study the singularity and convergence properties of the base Gaussian measures. Then, we briefly go over the construction and convergence of the Gibbs measure $\tilde{\rho}_\delta$ for the defocusing case $(2N + 1)$ in Subsections 4.2 and 4.3. In Subsection 4.4 we discuss the variational approach to treat the $k = 2$ case.

4.1. Singularity of the base Gaussian measures. Given $0 < \delta < \infty$, let $\tilde{\mu}_\delta$ be as in (1.24) and let $\tilde{\mu}_0$ be as in (1.29). Then, a typical element under $\tilde{\mu}_\delta$ (and under $\tilde{\mu}_0$, respectively) is given by the Gaussian Fourier series $\tilde{X}_\delta$ in (1.31) (and by $X_{\text{KdV}}$ in (1.30), respectively). Given $N \in \mathbb{N}$, set

$$\tilde{X}_{\delta,N} = P_N \tilde{X}_{\delta,N} \quad \text{and} \quad X_{\text{KdV},N} = P_N X_{\text{KdV}}.$$  \hspace{1cm} (4.1)

Then, in view of (1.53), we see that, for each $0 < \delta < \infty$, $\tilde{X}_{\delta,N}$ converges in $L^p(\Omega)$ for any finite $p \geq 1$ and almost surely to the limit $\tilde{X}_\delta$ in $H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T})$, $\varepsilon > 0$, as $N \to \infty$. On the other hand, it is well known [13, 69] that $X_{\text{KdV},N}$ converges, in $L^p(\Omega)$ and almost surely, to the limit $X_{\text{KdV}}$ in $H^{1/2-\varepsilon}(\mathbb{T}) \setminus H^{1/2}(\mathbb{T})$, $\varepsilon > 0$, as $N \to \infty$.

Proposition 4.1. (i) Given any $\varepsilon > 0$ and finite $p \geq 1$, $\tilde{X}_\delta$ converges to $X_{\text{KdV}}$ in $L^p(\Omega; H^{-\varepsilon}(\mathbb{T}))$ and almost surely in $H^{-\varepsilon}(\mathbb{T})$, as $\delta \to 0$. In particular, the Gaussian measure $\tilde{\mu}_\delta$ converges weakly to the Gaussian measure $\tilde{\mu}_0$, as $\delta \to 0$.

(ii) Let $\varepsilon > 0$. Then, for any $0 < \delta < \infty$, the Gaussian measures $\tilde{\mu}_\delta$ and $\tilde{\mu}_0$ are singular as probability measures $H^{-\varepsilon}(\mathbb{T})$.

In Section 3 the convergence in total variation of $\mu_\delta$ to $\mu_\infty$ played an essential role in establishing the convergence in total variation of $\rho_\delta$ to $\rho_{\text{BO}}$. Proposition 4.1 only provides weak convergence of the base Gaussian measures $\tilde{\mu}_\delta$ to $\tilde{\mu}_0$, and the singularity between the base Gaussian measures suggests that we do not expect any stronger mode of convergence (such as convergence in total variation). As a result, we only expect weak convergence of the associated Gibbs measures $\tilde{\rho}_\delta$ to $\rho_{\text{KdV}}$ in (1.57) in the shallow-water limit ($\delta \to 0$).

Proof of Proposition 4.1. Let $\varepsilon > 0$. From (1.30), (1.51), and Lemma 2.7 we have

$$\|\tilde{X}_\delta - X_{\text{KdV}}\|_{L^p_{\varepsilon,H_\varepsilon}} \lesssim_p \|\langle \nabla \rangle^{-\varepsilon}(\tilde{X}_\delta - X_{\text{KdV}})(x)\|_{L^2_\varepsilon L^2_\varepsilon} \sim \left(\sum_{n \in \mathbb{Z}^*} \frac{1}{\langle n \rangle^{2\varepsilon}} \left(\frac{1}{L_0^2(n)} - \frac{1}{|n|}\right)^2\right)^{1/2}. \hspace{1cm} (4.2)$$

It follows from (2.8) in Lemma 2.3 that the summand is bounded by $\langle n \rangle^{-1-2\varepsilon}$ uniformly in $0 < \delta \lesssim 1$, which is summable in $n \in \mathbb{Z}^*$. Moreover, from Lemma 2.3 (ii), we see that, for each $n \in \mathbb{Z}^*$, the summand tends to 0 as $\delta \to 0$. Hence, by the dominated convergence theorem, we conclude that $\tilde{X}_\delta$ converges to $X_{\text{KdV}}$ in $L^p(\Omega; H^{-\varepsilon}(\mathbb{T}))$. As for almost sure convergence, we repeat a computation analogous to (4.2) but with (2.19) in place of $\mathbb{E}[|g_n|^2] \sim 1$. We omit details. See (3.6) for an analogous argument in the unscaled case.

Next we prove part (ii) by using Lemma 3.2. From (1.51) and (1.30), we have

$$\tilde{X}_\delta(\omega) = \sum_{n \in \mathbb{N}} \left(\frac{\text{Re} g_n}{\pi L_0^2(n)} \cos(nx) - \frac{\text{Im} g_n}{\pi L_0^2(n)} \sin(nx)\right).$$
for $0 \leq \delta < \infty$ with the understanding that $\widetilde{X}_0 = X_{KdV}$ and $L_0(n) = n^2$. For $n \in \mathbb{Z}^*$, set

$$A_n = \frac{\text{Re} \ g_n}{\pi L_\delta(n)} \quad \text{and} \quad A_{-n} = -\frac{\text{Im} \ g_n}{\pi L_\delta(n)},$$

and

$$B_n = \frac{\text{Re} \ g_n}{\pi |n|} \quad \text{and} \quad B_{-n} = -\frac{\text{Im} \ g_n}{\pi |n|},$$

with $a_{\pm n} = \mathbb{E}[A_{\pm n}^2] = \frac{1}{\pi L_\delta(n)}$ and $b_{\pm n} = \mathbb{E}[B_{\pm n}^2] = \frac{1}{\pi n^2}$. Then, from Lemma 2.3(iv), we have

$$\sum_{n \in \mathbb{Z}^*} \left(\frac{b_n}{a_n} - 1\right)^2 = \sum_{n \in \mathbb{Z}^*} \frac{(n^2 - L_\delta(n))^2}{n^4} = \sum_{n \in \mathbb{Z}^*} h^2(n, \delta) = \infty$$

for any $\delta > 0$, where $h(n, \delta)$ is as in (2.9). Therefore, we conclude from Kakutani’s theorem (Lemma 3.2) that, for any $0 < \delta < \infty$, the Gaussian measures $\widetilde{\mu}_\delta$ and $\widetilde{\mu}_0$ are mutually singular. \hfill \Box

4.2. Construction of the Gibbs measures for the defocusing scaled gILW equation.

In this subsection, we briefly go over the construction of the Gibbs measure $\widetilde{\rho}_\delta$, $0 < \delta < \infty$, for the scaled gILW equation (1.17) in the defocusing case $k = 2N + 1$. (Theorem 1.5(i)). We treat the $k = 2$ case in Subsection 4.4.

Fix the depth parameter $0 < \delta < \infty$. Given $N \in \mathbb{N}$, let $\widetilde{X}_{\delta,N} = P_N X_\delta$, where $X_\delta$ is defined in (1.51). Given $k \in \mathbb{N}$, let

$$\mathcal{W}(\widetilde{X}_k_{\delta,N}) = H_k(\widetilde{X}_{\delta,N}; \widetilde{\sigma}_{\delta,N})$$

(4.3)

denote the Wick power defined in (1.53)\textsuperscript{21} where $\widetilde{\sigma}_{\delta,N}$ is as in (1.54). Then, the truncated Gibbs measure $\widetilde{\rho}_{\delta,N}$ in (1.56) can be written as

$$\widetilde{\rho}_{\delta,N}(A) = Z_{\delta,N}^{-1} \int_{H^{-1}} 1_{\{v \in A\}} e^{-\frac{|v|_{k+1}}{1+t}} f_t \mathcal{W}(v_N^{k+1}) dx \, d\widetilde{\mu}_\delta(v)$$

$$= Z_{\delta,N}^{-1} \int_{H^{-1}} 1_{\{\widetilde{X}_\delta(\omega) \in A\}} e^{-\frac{|\widetilde{X}_{\delta,N}(\omega)|_{k+1}}{1+t}} f_t \mathcal{W}(\widetilde{X}_{\delta,N}^{k+1}(\omega)) dx \, d\widetilde{P}(\omega),$$

where $v_N = P_N v$. By repeating the proof of Proposition 3.4 in the unscaled setting, we obtain the following result.

**Proposition 4.2.** Let $k \in \mathbb{N}$ and $0 < \delta < \infty$. Given $N \in \mathbb{N}$, let $\mathcal{W}(\widetilde{X}_{\delta,N}^k)$ be as in (4.3). Then, given any finite $p \geq 1$, the sequence $\{\mathcal{W}(\widetilde{X}_{\delta,N}^k)\}_{N \in \mathbb{N}}$ is Cauchy in $L^p(\Omega; W^{s,\infty}(\mathbb{T}))$, $s < 0$, thus converging to a limit denoted by $\mathcal{W}(\widetilde{X}_\delta^k)$. This convergence of $\mathcal{W}(\widetilde{X}_{\delta,N}^k)$ to $\mathcal{W}(\widetilde{X}_\delta^k)$ also holds almost surely in $W^{s,\infty}(\mathbb{T})$. Furthermore, given any finite $p \geq 1$, we have

$$\sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \|\mathcal{W}(\widetilde{X}_{\delta,N}^k)\|_{W_x^{s,\infty}} \|_{L^p(\Omega)} < \infty$$

and

$$\sup_{0 < \delta \leq 1} \|\mathcal{W}(\widetilde{X}_{\delta,M}^k) - \mathcal{W}(\widetilde{X}_{\delta,N}^k)\|_{L^p(\Omega)} \rightarrow 0$$

for any $M \geq N$, tending to $\infty$. In particular, the rate of convergence is uniform in $0 < \delta \leq 1$.

\textsuperscript{21}As in Section 3 we freely interchange the representations in terms of $X_\delta$ and in terms of $v$ distributed by $\widetilde{\nu}_\delta$, when there is no confusion.
As a corollary, the following two statements hold.

(i) Let $0 < \delta < \infty$. Given $N \in \mathbb{N}$, let $\widetilde{R}_\delta,N(v) = \widetilde{R}_\delta,N(v; k + 1)$ denote the truncated potential energy defined by

$$
\widetilde{R}_\delta,N(v) := \frac{1}{k + 1} \int_\mathcal{T} W((\mathbf{P}_N v)^k) dx = \frac{1}{k + 1} \int_\mathcal{T} H_{k + 1}(\mathbf{P}_N v; \bar{\sigma}_\delta,N) dx,
$$

where $\bar{\sigma}_\delta,N$ is as in (1.54). Then, given any finite $p \geq 1$, the sequence $\{\widetilde{R}_\delta,N(v)\}_{N \in \mathbb{N}}$ converges to the limit:

$$
\widetilde{R}_\delta(v) = \frac{1}{k + 1} \int_\mathcal{T} W(v^{k + 1}) dx = \lim_{N \to \infty} \frac{1}{k + 1} \int_\mathcal{T} W((\mathbf{P}_N v)^{k + 1}) dx
$$

in $L^p(d\mu_\delta)$, as $N \to \infty$. Furthermore, there exists $\theta > 0$ such that given any finite $p \geq 1$, we have

$$
\sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \| \widetilde{R}_\delta,N(v) \|_{L^p(d\mu_\delta)} < \infty,
$$

with $\widetilde{R}_{\delta,\infty}(v) = \widetilde{R}_\delta(v)$, and

$$
\| \widetilde{R}_{\delta,M}(v) - \widetilde{R}_\delta,N(v) \|_{L^p(d\mu_\delta)} \leq C_{k,\delta} \frac{p^{k + 1}}{N^\theta}
$$

for any $M \geq N \geq 1$. For $0 < \delta \leq 1$, we can choose the constant $C_{k,\delta}$ in (4.7) to be independent of $\delta$ and hence the rate of convergence of $\widetilde{R}_\delta,N(v)$ to the limit $\widetilde{R}_\delta(v)$ is uniform in $0 < \delta \leq 1$.

(ii) Let $0 < \delta < \infty$. Given $N \in \mathbb{N}$, let $\widetilde{F}_N(u) = \widetilde{F}_N(u; k)$ be the truncated renormalized nonlinearity in (1.72) given by

$$
\widetilde{F}_N(v) := \partial_x \mathbf{P}_N W((\mathbf{P}_N v)^k) = \partial_x \mathbf{P}_N H_k(\mathbf{P}_N v; \bar{\sigma}_\delta,N),
$$

where $\bar{\sigma}_\delta,N$ is as in (1.54). Then, given any finite $p \geq 1$, the sequence $\{\widetilde{F}_N(v)\}_{N \in \mathbb{N}}$ is Cauchy in $L^p(d\mu_\delta; H^s(\mathcal{T}))$, $s < -1$, thus converging to a limit denoted by $\widetilde{F}(v) = \partial_x W(v^k)$. Furthermore, given any finite $p \geq 1$, we have

$$
\sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \| \| \widetilde{F}_N(v) \|_{H^s_\delta} \|_{L^p(d\mu_\delta)} < \infty
$$

and

$$
\sup_{0 < \delta \leq 1} \| \| \widetilde{F}_M(v) - \widetilde{F}_N(v) \|_{H^s_\delta} \|_{L^p(d\mu_\delta)} \to 0
$$

for any $M \geq N$, tending to $\infty$. In particular, the rate of convergence of $\widetilde{F}_N(v)$ to the limit $\widetilde{F}(v)$ is uniform in $0 < \delta \leq 1$.

Proof. Proposition 1.2 follows from a straightforward modification of the proof of Proposition 3.3. The only notable difference is that instead of using the bounds (2.2) and (2.3) for $K_\delta(n)$, we need to use the bounds (2.7) and (2.8) for $L_\delta(n)$. We omit details. \[ \square \]

Given $0 < \delta < \infty$ and $N \in \mathbb{N}$, we define $\widetilde{G}_\delta,N(\tilde{X}_\delta) = \widetilde{G}_\delta,N(\tilde{X}_\delta; k + 1)$ by

$$
\widetilde{G}_\delta,N(\tilde{X}_\delta) = e^{-\tilde{R}_\delta,N(\tilde{X}_\delta)} = e^{-\frac{\delta}{N+1} \int_\mathcal{T} W(\tilde{X}_\delta^{k+1}) dx},
$$

where $\tilde{R}_\delta,N(\tilde{X}_\delta) = \tilde{R}_\delta,N(\tilde{X}_\delta; k + 1)$ is the truncated potential energy defined in (4.4). Then, a slight modification of the proof of Proposition 3.6 yields the following proposition.
Proposition 4.3. Let $k \in 2\mathbb{N} + 1$ and fix finite $p \geq 1$. Given any $0 < \delta < \infty$, we have

$$\sup_{N \in \mathbb{N}} \left\| \tilde{G}_{\delta,N}(\tilde{X}_\delta) \right\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \left\| \tilde{G}_{\delta,N}(v) \right\|_{L^p(\tilde{d}\tilde{\mu}_\delta)} \leq C_{p,k,\delta} < \infty.$$ 

In addition, the following uniform bound holds for $0 < \delta \leq 1$:

$$\sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \left\| \tilde{G}_{\delta,N}(\tilde{X}_\delta) \right\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \left\| \tilde{G}_{\delta,N}(v) \right\|_{L^p(\tilde{d}\tilde{\mu}_\delta)} \leq C_{p,k} < \infty. \quad (4.8)$$

Define $\tilde{G}_\delta(\tilde{X}_\delta)$ by

$$\tilde{G}_\delta(\tilde{X}_\delta) = e^{-\tilde{R}_\delta(\tilde{X}_\delta)}$$

with $\tilde{R}_\delta(\tilde{X}_\delta)$ as in (4.5). Then, $\tilde{G}_{\delta,N}(\tilde{X}_\delta)$ converges to $\tilde{G}_\delta(\tilde{X}_\delta)$ in $L^p(\Omega)$. Namely, we have

$$\lim_{N \to \infty} \left\| \tilde{G}_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_\delta(\tilde{X}_\delta) \right\|_{L^p(\Omega)} = 0. \quad (4.9)$$

Furthermore, the convergence is uniform in $0 < \delta \leq 1$:

$$\lim_{N \to \infty} \sup_{0 < \delta \leq 1} \left\| \tilde{G}_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_\delta(\tilde{X}_\delta) \right\|_{L^p(\Omega)} = 0. \quad (4.10)$$

As a consequence, the uniform bounds (4.8) and (4.9) hold even if we replace the supremum in $N \in \mathbb{N}$ by the supremum in $N \in \mathbb{N} \cup \{\infty\}$.

Theorem 1.5(i) follows as a direct corollary to Proposition 4.3 allowing us to define the limiting Gibbs measure $\tilde{\rho}_\delta$ in (1.61). See the discussion right after Proposition 3.6.

For $0 < \delta < \infty$, the Gibbs measure $\tilde{\rho}_\delta$ is equivalent to the base Gaussian measure $\tilde{\mu}_\delta$. Similarly, the Gibbs measure $\rho_{KdV}$ in (1.57) is equivalent to the base Gaussian measure $\tilde{\mu}_0$. Recalling from Proposition 4.1 that the base Gaussian measures $\tilde{\mu}_\delta$, $0 < \delta < \infty$, and $\tilde{\mu}_0$ are mutually singular, we conclude that the Gibbs measures $\tilde{\rho}_\delta$ in (1.61) and $\rho_{KdV}$ in (1.57) are mutually singular. This proves the first claim in Theorem 1.5(ii).

**Proof of Proposition 4.3.** From (3.37) with (1.54), we have

$$-\tilde{R}_{\delta,N}(\tilde{X}_\delta) = -\frac{1}{k+1} \int_0^\infty H_{k+1}(\tilde{X}_{\delta,N}; \tilde{\sigma}_{\delta,N}) \, dx$$

$$\leq \frac{2\pi}{k+1} \tilde{\sigma}_{\delta,N} a_{k+1} \leq \tilde{A}_{k,\delta} (\log(N+1))^\frac{k+1}{2}$$

for some $\tilde{A}_{k,\delta} > 0$, uniformly in $N \in \mathbb{N}$. Then, we can simply repeat the proof of Proposition 3.6 using Proposition 4.2 in place of Proposition 3.4. For $0 < \delta \leq 1$, it follows from (1.51) and Lemma 2.3 that the constant $\tilde{A}_{k,\delta}$ in (4.10) can be chosen to be independent of $0 < \delta \leq 1$. Similarly, by restricting our attention to $0 < \delta \leq 1$, we can choose the constant $c_{k,\delta}$ in an analogue of (3.39) in the current setting to be independent of $0 < \delta \leq 1$ since the constant $C_{k,\delta}$ in (4.7) is independent of $0 < \delta \leq 1$. Moreover, in applying Lemma 2.9 in Step 2 of the proof of Proposition 3.6 we need the uniform bound (4.9), replacing (3.16). This observation yields the uniform bounds (4.8) and (4.9). \[\square\]

**Remark 4.4.** Given $N$, define $\sigma_{KdV,N}$ by

$$\sigma_{KdV,N} = E[X_{KdV,N}^2(x)] = \frac{1}{4\pi^2} \sum_{0 < |n| \leq N} \frac{2\pi}{n^2}, \quad (4.11)$$
which is uniformly bounded in $N \in \mathbb{N}$. Here, $X_{KdV,N}$ is as in (4.1). We then extend the
definition of $L_\delta(n)$ and $\tilde{G}_{\delta,N}$ to the $\delta = 0$ case by setting $L_0(n) = n^2$ and

$$\tilde{G}_{0,N}(X_{KdV}) = e^{-\frac{1}{k+1} \int_\Omega f(x) W(X_{KdV, N}) dx},$$

(4.12)

where $W(X_{KdV, N}) = H_{k+1}(X_{KdV, N}; \sigma_{KdV, N})$. We also set

$$\tilde{G}_{0}(X_{KdV}) = e^{-\frac{1}{k+1} \int_\Omega f(x) W(X_{KdV}) dx},$$

(4.13)

where $W(X_{KdV}) = H_{k+1}(X_{KdV}; \sigma_{KdV})$ as in (1.59). Then, by setting $\tilde{X}_0 = X_{KdV}$, Proposition 4.3 extends to $\delta = 0$. In particular, the uniform bounds (4.8) and (4.9) hold for $0 \leq \delta \leq 1$.

4.3. Convergence of the Gibbs measures in the shallow-water limit. It remains to prove that the Gibbs measure $\tilde{\rho}_\delta$ converges weakly to $\rho_{KdV}$ as $\delta \to 0$. We first state an analogue of Lemma 3.8.

**Lemma 4.5.** Let $k \in 2\mathbb{N} + 1$ and $1 \leq p < \infty$. Then, given $N \in \mathbb{N}$, we have

$$\lim_{\delta \to 0} \|\tilde{G}_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_{0,N}(X_{KdV})\|_{L^p(\Omega)} = 0.$$

As a corollary, we have

$$\lim_{\delta \to \infty} \|\tilde{G}_\delta(\tilde{X}_\delta) - \tilde{G}_0(X_{KdV})\|_{L^p(\Omega)} = 0.$$

In particular, the partition function $Z_\delta$ of the Gibbs measure $\tilde{\rho}_\delta$ in (1.61) converges to the

partition function $Z_{KdV} = Z_0$ of the Gibbs measure $\rho_{KdV} = \tilde{\rho}_0$ in (1.58), as $\delta \to 0$.

**Proof.** From Lemma 2.3, we see that $\tilde{\sigma}_{\delta,N}$ in (1.54) converges to $\sigma_{KdV,N}$ in (1.11) as $\delta \to 0$. With this observation, we can simply repeat the proof of Lemma 3.8. We omit details. \(\square\)

We are now ready to prove weak convergence of $\tilde{\rho}_\delta$ to $\rho_{KdV}$ in the shallow-water limit ($\delta \to 0$). Fix small $\varepsilon > 0$. Let $A$ be any Borel subset of $H^{-\varepsilon}(\mathbb{T})$ with $\tilde{\mu}_0(\partial A) = 0$, where $\partial A$ denotes the boundary of the set $A$. Our goal is to show that

$$\tilde{\rho}_\delta(A) - \rho_{KdV}(A) \to 0$$

(4.14)

as $\delta \to 0$, which, together with the portmanteau theorem, yields the desired weak convergence.

By the triangle inequality, we have

$$|\tilde{\rho}_\delta(A) - \rho_{KdV}(A)| \leq |\tilde{\rho}_\delta(A) - \tilde{\rho}_{\delta,N}(A)| + |\tilde{\rho}_{\delta,N}(A) - \rho_{KdV,N}(A)| + |\rho_{KdV,N}(A) - \rho_{KdV}(A)|,$$

(4.15)

where $\rho_{KdV,N}$ denotes the truncated Gibbs measure for $\delta = 0$ given by

$$\rho_{KdV,N}(A) = Z_{0,N}^{-1} \int_{H^{\varepsilon}} 1_{\{v \in A\}} e^{-\frac{1}{k+1} \int_\Omega f(x) W(v_{N+1}^\delta) dx} d\tilde{\mu}_0(v)$$

$$= Z_{0,N}^{-1} \int_\Omega 1_{\{X_{KdV}(\omega) \in A\}} \tilde{G}_{0,N}(X_{KdV}) d\mathbb{P}(\omega)$$

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for any measurable set $A \subset H^{-\varepsilon}(\mathbb{T})$. From Proposition 4.3 and Remark 4.4 we have
\[
\lim_{N \to \infty} \sup_{0 \leq \delta \leq 1} |\tilde{\rho}\delta(A) - \tilde{\rho}\delta,N(A)| = \lim_{N \to \infty} \sup_{0 \leq \delta \leq 1} \|\overline{G}\delta,N(\bar{X}\delta) - \overline{G}\delta(\bar{X}\delta)\|_{L^1(\Omega)}
\]
(4.16)
with the identification $\bar{X}\delta = X_{KdV}$, $\tilde{\rho}\delta = \rho_{KdV}$, and $\tilde{\rho}\delta,N = \rho_{KdV,N}$, where $\overline{G}_{0,N}(\bar{X}_0)$ and $\overline{G}_0(\bar{X}_0)$ are as in (4.12) and (4.13), respectively. Hence, in view of (4.14), (4.15), and (4.16), it suffices to prove
\[
\lim_{\delta \to 0} |\tilde{\rho}\delta,N(A) - \rho_{KdV,N}(A)| = \lim_{\delta \to 0} \left|Z^{-1}_{\delta,N}[\overline{G}\delta,N(\bar{X}\delta)1_A(\bar{X}\delta)] - Z^{-1}_{0,N}[\overline{G}_0,N(X_{KdV})1_A(X_{KdV})]\right|
\]
(4.17)
for some $N \in \mathbb{N}$.

First, note that it suffices to show that
\[
\mathbb{E}[\overline{G}\delta,N(\bar{X}\delta)1_A(\bar{X}\delta)] - \mathbb{E}[\overline{G}_0,N(X_{KdV})1_A(X_{KdV})] \to 0
\]
(4.18)
as $\delta \to 0$ since, by taking $A = H^{-\varepsilon}(\mathbb{T})$, (4.18) implies $Z_{\delta,N} \to Z_{0,N}$ as $\delta \to 0$.

By the triangle inequality, we have
\[
|\mathbb{E}[\overline{G}\delta,N(\bar{X}\delta)1_A(\bar{X}\delta)] - \mathbb{E}[\overline{G}_0,N(X_{KdV})1_A(X_{KdV})]| \leq \mathbb{E}[|\overline{G}\delta,N(\bar{X}\delta) - \overline{G}_0,N(X_{KdV})|]
\]
(4.19)
+ $\mathbb{E}[\overline{G}_0,N(X_{KdV})1_A(\bar{X}\delta) - 1_A(X_{KdV})]$.

From Lemma 4.5 we have
\[
\mathbb{E}[|\overline{G}\delta,N(\bar{X}\delta) - \overline{G}_0,N(X_{KdV})|] \to 0,
\]
(4.20)
as $\delta \to \infty$. As for the second term on the right-hand side of (4.19), we first note that $\sigma_{KdV,N}$ defined in (4.11) is uniformly bounded in $N \in \mathbb{N}$. Then, together with (3.37) and (4.12), we conclude that
\[
0 < \overline{G}_{0,N}(X_{KdV}(\omega)) \lesssim 1,
\]
(4.21)
uniformly in $\omega \in \Omega$ and $N \in \mathbb{N}$. Hence, from (4.21) and $\tilde{\rho}_0(\partial A) = 0$ (which implies $\mathbb{E}[1_{\partial A}(X_{KdV})] = 0$), we have
\[
\mathbb{E}[\overline{G}_{0,N}(X_{KdV})1_A(\bar{X}\delta) - 1_A(X_{KdV})] \lesssim \mathbb{E}[|1_A(\bar{X}\delta) - 1_A(X_{KdV})|]
\]
(4.22)
+ $\mathbb{E}[1_{\text{int}A}(X_{KdV}) \cdot |1_A(\bar{X}\delta) - 1_A(X_{KdV})|]$
+ $\mathbb{E}[1_{\text{int}\mathcal{A}^c}(X_{KdV}) \cdot |1_A(\bar{X}\delta) - 1_A(X_{KdV})|]$
where $\text{int}A$ denotes the interior of $A$ given by $\text{int}A = A \setminus \partial A$. From Proposition 4.1(i) and the openness of $\text{int}A$ and $\text{int}\mathcal{A}^c$, the integrands of the terms on the right-hand side of (4.22) tend to 0 as $\delta \to 0$. Hence, by the bounded convergence theorem, we conclude that
\[
\mathbb{E}[\overline{G}_{0,N}(X_{KdV})1_A(\bar{X}\delta) - 1_A(X_{KdV})] \to 0,
\]
(4.23)
as δ → 0. Therefore, putting (1.19), (1.20), and (1.23) together, we conclude (4.18), which in turn implies (4.17). Finally, from (4.15), (4.16), and (4.17), we conclude (4.14), namely, weak convergence of ρδ to ρKdV as δ → 0. This concludes the proof of Theorem 1.5 when k ∈ 2N + 1.

4.4. Gibbs measures for the scaled ILW equation: variational approach. We conclude this section by briefly going over the proof of Theorem 1.5 when k = 2, based on the variational approach as in Subsection 3.4 The major part of the argument follows exactly as in Subsection 3.4 and thus we only describe necessary definitions and steps.

Fix K > 0 in the remaining part of this section. Given 0 ≤ δ < ∞ and N ∈ N, define the truncated density $\tilde{G}_{δ,N}^K(v)$ by

$$
\tilde{G}_{δ,N}^K(v) = \chi_K \left( \int T W(v_N^2)dx \right) e^{-\frac{1}{3} \int T v_N^3 dx} = \chi_K \left( \int T H_2(v_N; \tilde{\sigma}_{δ,N}) dx \right) e^{-\frac{1}{3} \int T v_N^3 dx},
$$

where $v_N = P_N v$, $\tilde{\sigma}_{δ,N}$ is as in (1.51) when 0 < δ < ∞, and $\tilde{\sigma}_{0,N} = \sigma_{KdV,N}$. As in the unscaled case discussed in Subsection 3.4, Theorem 1.5 for k = 2 follows once we prove the following uniform bounds.

**Proposition 4.6.** Fix finite $p ≥ 1$ and $K > 0$. Then, given any 0 ≤ δ < ∞, we have

$$
\sup_{N \in \mathbb{N}} \| \tilde{G}_{δ,N}^K(\tilde{X}_δ) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \| \tilde{G}_{δ,N}^K(v) \|_{L^p(d\tilde{\mu}_δ)} ≤ C_{p,δ,K} < \infty.
$$

In addition, the following uniform bound holds for 0 ≤ δ ≤ 1:

$$
\sup_{N \in \mathbb{N}} \sup_{0 ≤ δ ≤ 1} \| \tilde{G}_{δ,N}^K(\tilde{X}_δ) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{0 ≤ δ ≤ 1} \| \tilde{G}_{δ,N}^K(v) \|_{L^p(d\tilde{\mu}_δ)} ≤ C_{p,K} < \infty.
$$

Once we have Proposition 4.6, we can argue exactly as in Subsection 3.4 to conclude Theorem 1.5. In particular, (3.64) and (3.65) provide a bound on the truncated density $\tilde{G}_{δ,N}^K$, uniformly in 0 ≤ δ ≤ 1, replacing the defocusing bound (4.10). We omit details.

In order to prove Proposition 4.6 we consider the truncated density with a taming by a power of the Wick-ordered $L^2$-norm as in Subsection 3.4. Given 0 ≤ δ < ∞ and N ∈ N, set

$$
\tilde{R}_{δ,N}(v) = \frac{1}{3} \int T v_N^3 dx + A \left| \int T W(v_N^2) dx \right|^2,
$$

where $W(v_N^2) = W_{δ,N}(v_N^2) = H_2(v_N; \tilde{\sigma}_{δ,N})$. Then, we also define the truncated density with a taming by a power of the Wick-ordered $L^2$-norm:

$$
\tilde{G}_{δ,N}^K(v) = e^{-\tilde{R}_{δ,N}(v)} = e^{-\frac{1}{3} \int T v_N^3 dx - A | \int T W(v_N^2) dx |^2}
$$

for some suitable $A > 0$. Then, from (3.68), we have

$$
\tilde{G}_{δ,N}^K(v) ≤ C_{A,K} \cdot \tilde{G}_{δ,N}^K(v)
$$

and, hence, Proposition 4.6 follows once we prove the following uniform bounds.
Proposition 4.7. Fix finite \( p \geq 1 \). Then, there exists \( A_0 = A_0(p) > 0 \) such that

\[
\sup_{N \in \mathbb{N}} \| \mathcal{G}_{\delta,N}^{K}(\widetilde{X}_\delta) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \| \mathcal{G}_{\delta,N}^{K}(v) \|_{L^p(d\mu)} \leq C_{p,K,A} < \infty
\]

for any \( 0 \leq \delta < \infty \), \( K > 0 \), and \( A \geq A_0 \). In addition, the following uniform bound holds for \( 0 \leq \delta \leq 1 \):

\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq \delta \leq 1} \| \mathcal{G}_{\delta,N}^{K}(\widetilde{X}_\delta) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{0 \leq \delta \leq 1} \| \mathcal{G}_{\delta,N}^{K}(v) \|_{L^p(d\mu)} \leq C_{p,K,A} < \infty
\]

for any \( K > 0 \) and \( A \geq A_0 \).

In order to set up the variational formulation, let us introduce some notations. Define \( \widetilde{Y}_\delta(t) \) by

\[
\widetilde{Y}_\delta(t) = \left( \frac{3}{\delta} \mathcal{G}_\delta \partial_x \right)^{-\frac{1}{2}} W(t),
\]

where \( W(t) \) is as in (3.71) and \( \left( \frac{3}{\delta} \mathcal{G}_\delta \partial_x \right)^{-\frac{1}{2}} \) is the Fourier multiplier operator with the multiplier \( \left( L_\delta(n) \right)^{-\frac{1}{2}} \) with \( L_\delta(n) \) as in (1.52). In view of (1.51), we have \( \mathcal{L}(\widetilde{Y}_\delta(1)) = \mu_\delta \). Given \( N \in \mathbb{N} \), we set \( \widetilde{Y}_{\delta,N} = P_N \widetilde{Y}_\delta \). The variational formulation in the current problem is given by the following lemma.

Lemma 4.8. Given \( 0 \leq \delta < \infty \), let \( \widetilde{Y}_\delta \) be as in (1.24). Fix \( N \in \mathbb{N} \). Suppose that \( F : C^\infty(\mathbb{T}) \rightarrow \mathbb{R} \) is measurable such that \( \mathbb{E} \left[ |F(\widetilde{Y}_\delta,N(1))|^p \right] < \infty \) and \( \mathbb{E} \left[ |e^{-F(\widetilde{Y}_\delta,N(1))}|^q \right] < \infty \) for some \( 1 < p, q < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, we have

\[
- \log \mathbb{E} \left[ e^{-F(\widetilde{Y}_\delta,N(1))} \right] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[ F(\widetilde{Y}_\delta,N(1) + P_N \tilde{I}_\delta(\theta)(1)) + \frac{1}{2} \int_0^1 \| \theta(t) \|_{L^2}^2 dt \right],
\]

where \( \tilde{I}_\delta(\theta) \) is defined by

\[
\tilde{I}_\delta(\theta)(t) = \int_0^t \left( \frac{3}{\delta} \mathcal{G}_\delta \partial_x \right)^{-\frac{1}{2}} \theta(t') dt'.
\]

With Lemma 4.8 in hand, we can proceed as in Subsection 3.4 to prove Proposition 4.7 by using Lemma 3.16 and the following lemma.

Lemma 4.9. (i) Let \( \varepsilon > 0 \) and fix finite \( p \geq 1 \). Then, given any \( 0 \leq \delta < \infty \), we have

\[
\mathbb{E} \left[ \| \widetilde{Y}_\delta,N(1) \|_{W^{-\varepsilon,\infty}}^p + \| \mathcal{W}(\widetilde{Y}_\delta^3,N(1)) \|_{W^{-\varepsilon,\infty}}^p \right] \leq C_{\varepsilon,p,\delta} < \infty,
\]

uniformly in \( N \in \mathbb{N} \). Furthermore, by restricting our attention to \( 0 \leq \delta \leq 1 \), we can choose the constant \( C_{\varepsilon,p,\delta} \) in (4.25) to be independent of \( \delta \).

(ii) Let \( 0 \leq \delta < \infty \). For any \( \theta \in \mathbb{H}_a \), we have

\[
\| \tilde{I}_\delta(\theta)(1) \|_{H^\frac{1}{2}}^2 \lesssim \int_0^1 \| \theta(t) \|_{L^2}^2 dt,
\]

where \( \mathbb{H}_a \) denotes the collection of drifts, which are progressively measurable processes belonging to \( L^2([0,1];L^2_0(\mathbb{T})) \), \( \mathbb{P} \)-almost surely, as in Subsection 3.4.
The proof of Lemma 4.9 follows exactly as in the proof of Lemma 3.15, using the lower bounds (2.7) and (2.8) of $L_\delta(n)$ (in place of (2.2) and (2.3)). We omit details.

We conclude this section by recalling Proposition 4.7 implies Proposition 4.6, which in turn implies Theorem 1.5 for $k = 2$.

5. Dynamical problem

In this section, we study the dynamical problem associated with the Gibbs measures constructions in the previous sections. In the following, we only consider the deep-water regime $0 < \delta \leq \infty$ (namely, we work on the unscaled problem (1.1)) and present the proof of Theorem 1.8 since Theorem 1.10 in the shallow-water regime ($0 \leq \delta < \infty$) follows from a similar argument. Our main strategy is to use a compactness argument as in [18, 74, 69]. In fact, as mentioned in Section 1, the proof of Theorem 1.8 (i) follows from exactly the same argument as that presented in [74, Section 5]. As for the dynamical convergence result in Theorem 1.8 (ii), we can repeat the same argument but with one key additional ingredient: the uniform (in $\delta$ and $N$) integrability of the (truncated) densities (Proposition 3.6). For conciseness of the presentation, we restrict our attention to $2 \leq \delta \leq \infty$ in the following and discuss the proof of Theorem 1.8. For each fixed $0 < \delta < 2$, the same argument (without uniformity in $\delta$) applies to yield Theorem 1.8 (i).

In the remaining part of this section, fix $k \in 2\mathbb{N} + 1$ and $s < 0$. The $k = 2$ case follows from exactly the same argument by replacing the truncated Gibbs measure $\rho_{\delta,N}$ in (1.45) and the Gibbs measure $\rho_\delta$ in (1.48) by $\rho_{\delta,N}$ in (1.49) and $\rho_\delta$ in (1.50), respectively, and thus we omit details. In Subsection 5.1, we first study the truncated gILW equation (1.70) and construct global-in-time invariant Gibbs dynamics associated with the truncated Gibbs measure $\rho_{\delta,N}$ in (1.45) for each $N \in \mathbb{N}$ and $2 \leq \delta \leq \infty$; see Lemma 5.1 below. This allows us to construct a probability measure $\nu_{\delta,N} = \rho_{\delta,N} \circ \Phi_{\delta,N}^{-1}$ on space-time functions as the pushforward of the truncated Gibbs measure $\rho_{\delta,N}$ under the solution map $\Phi_{\delta,N}$ for the truncated gILW equation (1.70). Then, by using the uniform (in $\delta$ and $N$) bound on the (truncated) densities (Proposition 3.6), we prove that $\{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}}$ is tight (Proposition 5.2). The main new point in this work is that we prove tightness not only in the frequency cutoff parameter $N \in \mathbb{N}$ but also in the depth parameter $2 \leq \delta \leq \infty$. In Subsection 5.2, we then present the proof of Theorem 1.8 by constructing the limiting dynamics. For each fixed $2 \leq \delta \leq \infty$, we can simply repeat the argument in [18, 74, 69], based on the Skorokhod representation theorem (Lemma 2.15), and construct the limiting invariant Gibbs dynamics (without uniqueness) as $N \to \infty$, yielding Theorem 1.8 (i). As for proving Theorem 1.8 (ii), by exploiting the tightness of $\{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}}$, we use a diagonal argument together with the triangle inequality for the Lévy-Prokhorov metric, characterizing weak convergence, to show that there exists a sequence $\{\delta_m\}_{m \in \mathbb{N}}$, tending to $\infty$, such that $u_{\delta_m}$ converges almost surely to some limit $u$ in $C(\mathbb{R}; H^s(T))$. Here, in order to have the claimed almost sure convergence of $u_{\delta_m}$ to $u$, we apply the Skorokhod representation theorem (Lemma 2.15). Furthermore, in order to show that $u_{\delta_m}$, $m \in \mathbb{N}$, satisfies the renormalized gILW equation (1.66), we need to apply the Skorokhod representation theorem (Lemma 2.15) infinitely many times (i.e. once for each $m \in \mathbb{N}$).
5.1. **Pushforward of the truncated Gibbs measure.** Given $2 \leq \delta \leq \infty$ and $N \in \mathbb{N}$, consider the truncated gILW equation (1.70):

$$
\partial_t u_{\delta,N} - G_\delta \partial_x^2 u_{\delta,N} = F_N(u_{\delta,N})
= \partial_x P_N W((P_N u_{\delta,N})^k)
= \partial_x P_N H_k(P_N u_{\delta,N}; \sigma_{\delta,N}),
$$

where $\sigma_{\delta,N}$ is as in (1.43) and $F_N$ is as in (3.18). We first prove global well-posedness of (5.1) and invariance of the truncated Gibbs measure $\rho_{\delta,N}$ defined in (1.45).

**Lemma 5.1.** Let $2 \leq \delta \leq \infty$, $N \in \mathbb{N}$, and $s < 0$. Then, the truncated gILW equation (5.1) is globally well-posed in $H^s(\mathbb{T})$. Moreover, the truncated Gibbs measure $\rho_{\delta,N}$ is invariant under the dynamics of (5.1).

**Proof.** The proof of this lemma follows from that of Lemma 5.1 in [74] and thus we will be brief here. We first decompose (5.1) into two parts:

$$
u_{\delta,N} = u^\text{low}_{\delta,N} + u^\text{high}_{\delta,N} = P_N u_{\delta,N} + P^\perp_N u_{\delta,N},
$$

where $P^\perp_N = \text{Id} - P_N$. Then, $u^\text{low}_{\delta,N}$ and $u^\text{high}_{\delta,N}$ satisfy the following equations:

(i) nonlinear dynamics on the low-frequency part $\{0 < |n| \leq N\}$:

$$
\partial_t u^\text{low}_{\delta,N} - G_\delta \partial_x^2 u^\text{low}_{\delta,N} = \partial_x P_N H_k(u^\text{low}_{\delta,N}; \sigma_{\delta,N}).
$$

(ii) linear dynamics on the high frequency part $\{|n| > N\}$:

$$
\partial_t u^\text{high}_{\delta,N} - G_\delta \partial_x^2 u^\text{high}_{\delta,N} = 0.
$$

We now view the equations (5.3) and (5.4) on the Fourier side. As a decoupled system of linear equation (for each frequency $|n| > N$), (5.4) is globally well-posed. As for (5.3), it is a system of finitely many ODEs with a Lipschitz vector field and thus by the Cauchy-Lipschitz theorem, it is locally well-posed. Furthermore, a direct computation shows that the $L^2$-norm of $u^\text{low}_{\delta,N}$ is conserved under the flow of (5.3), which yields global well-posedness of (5.3). Putting together, we conclude that (5.1) is globally well-posed.

Next, we prove invariance of the truncated Gibbs measure $\rho_{\delta,N}$. We first write $\rho_{\delta,N}$ in (1.45) as

$$
\rho_{\delta,N} = \rho^\text{low}_{\delta,N} \otimes \rho^\text{high}_{\delta,N},
$$

where $\rho^\text{low}_{\delta,N}$ and $\rho^\text{high}_{\delta,N}$ are given as follows:

(i) the low-frequency component $\rho^\text{low}_{\delta,N}$ is the finite-dimensional Gibbs measure on $P_N H^s(\mathbb{T})$, defined by

$$
d\rho^\text{low}_{\delta,N}(u) = Z_{\delta,N}^{-1} e^{-\frac{1}{\kappa + 1} \int_{\mathbb{T}} H_k(u; \sigma_{\delta,N}) dx} d\mu^\text{low}_{\delta,N}(u),
$$

where $\mu^\text{low}_{\delta,N} = (P_N)_* \mu_\delta$ is the pushforward image measure under $P_N$ of the base Gaussian measure $\mu_\delta$ in (1.22). Namely, $\mu^\text{low}_{\delta,N}$ is the induced probability measure under the map $\omega \in \Omega \mapsto X_{\delta,N}(\omega) = P_N X_\delta(\omega)$, where $X_\delta$ is as in (1.31).
In view of (5.2) and (5.5), we conclude invariance of the truncated Gibbs measure 

\[ \rho_{\delta,N} \]

By the classical Liouville theorem and the conservation of the (truncated) Hamiltonian for (5.3), we see that the Gibbs measure \( \rho_{\delta} \) is invariant under the linear dynamics (5.4) acts as a rotation on the Fourier coefficient at each frequency \( |n| > N \), preserving the Gaussian measure at each frequency \( |n| > N \) in (5.6). As a result, the Gaussian measure \( \rho_{\delta,N}^{|\text{high}} = (P_N^H)^* \mu_{\delta} \) is invariant under the linear dynamics (5.4). In view of (5.2) and (5.5), we conclude invariance of the truncated Gibbs measure \( \rho_{\delta,N} \) under the flow of the truncated gILW equation (5.1).

As a consequence of Lemma 5.1, we can define the solution map \( \Phi_{\delta,N} : H^s(\mathbb{T}) \to C(\mathbb{R}; H^s(\mathbb{T})) \) associated to (5.1). More precisely, for \( t \in \mathbb{R} \), we define \( \Phi_{\delta,N}(t) : H^s(\mathbb{T}) \to H^s(\mathbb{T}) \) by

\[ \phi \in H^s(\mathbb{T}) \mapsto \Phi_{\delta,N}(t)(\phi) = u_{\delta,N}(t), \]

where \( u_{\delta,N} \) is the global-in-time solution to the truncated gILW equation (5.1) with initial data \( u_{\delta,N}(0) = \phi \).

Next, we introduce the pushforward image measure \( \nu_{\delta,N} \) of the truncated Gibbs measure \( \rho_{\delta,N} \) under the solution map \( \Phi_{\delta,N} \):

\[ \nu_{\delta,N} = \rho_{\delta,N} \circ \Phi_{\delta,N}^{-1}. \]

Here, we view \( \nu_{\delta,N} \) as a probability measure on \( C(\mathbb{R}; H^s(\mathbb{T})) \) endowed with the compact-open topology, induced by the following metric:

\[ \text{dist}(u,v) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|u - v\|_{\text{C}([-j,j]; H^s)}}{1 + \|u - v\|_{\text{C}([-j,j]; H^s)}}. \]

Recall that, under this topology, a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset C(\mathbb{R}; H^s(\mathbb{T})) \) converges if and only if it converges uniformly on \([-K,K]\) for each finite \( K > 0 \). We also recall that the metric space \( (C(\mathbb{R}; H^s(\mathbb{T})), \text{dist}) \) is complete and separable. Then, it follows from the local Lipschitz continuity of \( \Phi_{\delta,N} \) that \( \Phi_{\delta,N} \) is continuous from \( H^s(\mathbb{T}) \) into \( C(\mathbb{R}; H^s(\mathbb{T})) \), which shows that \( \nu_{\delta,N} \) is a well-defined probability measure on \( C(\mathbb{R}; H^s(\mathbb{T})) \) endowed with the compact-open topology. Note that we have

\[ \int_{C(\mathbb{R}; H^s)} F(u) \, d\nu_{\delta,N}(u) = \int_{H^s} F(\Phi_{\delta,N}(\phi)) \, d\rho_{\delta,N}(\phi) \]

for any bounded measurable function \( F : C(\mathbb{R}; H^s(\mathbb{T})) \to \mathbb{R} \).

Our main goal in this subsection is to prove the following tightness result on \( \{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}} \). We point out that tightness holds not only over \( N \in \mathbb{N} \) but also over \( 2 \leq \delta \leq \infty \), which is the key new feature of this proposition.

**Proposition 5.2.** Let \( s < 0 \). Then, the family \( \{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}} \) of probability measures on \( C(\mathbb{R}; H^s(\mathbb{T})) \) is tight, and hence is relatively compact.

\[ \text{Recall that the space of continuous functions from a separable metric space } X \text{ to another separable metric space } Y \text{ with the compact-open topology is separable; see [60]. See also the paper [46, Corollary 3.3].} \]
Before proceeding to the proof of Proposition 5.2, we state two auxiliary lemmas. The first lemma establishes uniform (in \(\delta\) and \(N\)) space-time bounds on the solutions to the truncated gILW equation \([5.1]\). We postpone its proof to the end of this subsection.

Given \(1 \leq p \leq \infty\) and \(s \in \mathbb{R}\), we define the space \(W_T^{1,p}H^s_x = W^{1,p}([-T,T]; H^s(\mathbb{T}))\) by the norm:

\[
\|u\|_{W_T^{1,p}H^s_x} = \|u\|_{L^p_TH^s_x} + \|\partial_t u\|_{L^p_TH^s_x}.
\]

**Lemma 5.3.** Let \(s < 0\), and fix finite \(p \geq 1\). Then, there exists \(C_p > 0\) such that

\[
\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|u\|_{L^p_TH^s_x} \leq C_p T^{\frac{1}{p}},
\]

\[
\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|u\|_{W_T^{1,p}H^s_x} \leq C_p T^{\frac{1}{p}},
\]

The following interpolation lemma allows us to control the Hölder regularity (in time) by the two quantities controlled in Lemma 5.3 above. For \(\alpha \in (0,1)\) and \(s \in \mathbb{R}\), define the space \(C^\alpha_TH^s_x = C^\alpha([-T,T]; H^s(\mathbb{T}))\) by the norm

\[
\|u\|_{C^\alpha_TH^s_x} = \sup_{t_1, t_2 \in [-T,T]} \frac{|u(t_1) - u(t_2)|}{|t_1 - t_2|^\alpha} + \|u\|_{L^\infty_TH^s_x}.
\]

**Lemma 5.4** ([18, Lemma 3.3]). Let \(T > 0\) and \(1 \leq p \leq \infty\). Suppose that \(u \in L^p_TH^s_x\) and \(\partial_t u \in L^p_TH^{s+2}_x\) for some \(s_2 \leq s_1\). Then, for \(\delta > p^{-1}(s_1 - s_2)\), we have

\[
\|u\|_{L^\infty_TH^{s_1-\delta}_x} \lesssim \|u\|_{L^p_TH^{s_1}}^{1-p} \|u\|_{W_T^{1,p}H^{s_2}}^p.
\]

Moreover, there exist \(\alpha > 0\) and \(\theta \in [0,1]\) such that for all \(t_1, t_2 \in [-T,T]\), we have

\[
\|u(t_2) - u(t_1)\|_{H^{s_1-\alpha}} \lesssim |t_2 - t_1|^{\alpha} \|u\|_{L^p_TH^{s_1}}^{1-\theta} \|u\|_{W_T^{1,p}H^{s_2}}^\theta.
\]

As a consequence, we have

\[
\|u\|_{C^\alpha_TH^{s_1-\delta}_x} \lesssim \|u\|_{L^p_TH^{s_1}} + \|u\|_{W_T^{1,p}H^{s_2}}.
\]

**Proof.** As for \([5.13]\) and \([5.14]\), see the proof of Lemma 3.3 in [18]. The bound \([5.15]\) follows from \([5.12]\), \([5.13]\), and \([5.14]\) with Young’s inequality. \(\square\)

We now present the proof of Proposition 5.2.

**Proof of Proposition 5.2**. Let \(s < s_1 < s_2 < 0\) and \(\alpha \in (0,1)\). By the Arzelà-Ascoli theorem, the embedding \(C^\alpha([-T,T]; H^{s_1}(\mathbb{T})) \subset C([-T,T]; H^s(\mathbb{T}))\) is compact for each \(T > 0\). From Lemma 5.4 (with large \(p > 1\)) and Lemma 5.3, we have

\[
\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|u\|_{C^\alpha_TH^s_x} \leq C_p T^{\frac{1}{p}} - C_\delta T^{\frac{1}{p}}.
\]

Given \(j \in \mathbb{N}\) and \(\varepsilon \in (0,1)\), define \(K_\varepsilon\) by setting

\[
K_\varepsilon := \{ u \in C(\mathbb{R}; H^s(\mathbb{T})) : \|u\|_{C^\alpha_TH^s_x} \leq C_\delta T^{\frac{1}{p}} \text{ for all } j \in \mathbb{N}\},
\]

(5.17)
where \( T_j = 2^j \). Then, by Chebyshev’s inequality and (5.16), we have
\[
\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \nu_{\delta,N}(K_{\varepsilon}^c) \leq \sum_{j=1}^{\infty} \nu_{\delta,N}(\|u\|_{L^p_{\partial \delta,N} H_{\varepsilon}^s} > C_0 \|f\|_T \frac{1}{j^{1 + \frac{1}{p}}}) \\
\leq C_0^{-p} \varepsilon \sum_{j=1}^{\infty} T_j^{-p-1} \|\|u\|_{C^p_{T_j} W^{s+1} H_{\varepsilon}^s}\|_{L^p(d\rho_{\delta,N})} \\
\leq \left( C_0^{-p} C_p \sum_{j=1}^{\infty} T_j^{-p} \right) \varepsilon < \varepsilon,
\]
where the last step follows from choosing \( C_0 > 0 \) sufficiently large in the definition (5.17) of \( K_{\varepsilon} \).

It remains to show that \( K_{\varepsilon} \) is compact in \( C(\mathbb{R}; H^s(\mathbb{T})), \text{dist}) \), namely, endowed with the compact-open topology. While the proof of this fact was presented in the proof of Proposition 5.4 in [74], we present the argument for readers’ convenience. Let \( \{u_n\}_{n \in \mathbb{N}} \subset K_{\varepsilon} \). It follows from (5.17) that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( C^\alpha([-T_j, T_j]; H^s(\mathbb{T})) \) for each \( j \in \mathbb{N} \) and hence is compact in \( C([-T_j, T_j]; H^s(\mathbb{T})) \) for each \( j \in \mathbb{N} \). Then, by a diagonal argument, we can extract a subsequence \( \{u_{n_t}\}_{t \in \mathbb{N}} \) that is convergent in \( C([-T_j, T_j]; H^s(\mathbb{T})) \) for each \( j \in \mathbb{N} \). Hence, \( \{u_{n_t}\}_{t \in \mathbb{N}} \) is convergent in \( C(\mathbb{R}; H^s(\mathbb{T})), \text{dist}) \). This proves that \( K_{\varepsilon} \) is relatively compact in \( C(\mathbb{R}; H^s(\mathbb{T})), \text{dist}) \). It is clear that \( K_{\varepsilon} \) is closed as well, and hence we conclude the proof. \( \square \)

We conclude this subsection by presenting the proof of Lemma 5.3.

**Proof of Lemma 5.3.** The proof essentially follows the same lines in the proof of Lemma 5.5 in [74]. From (5.9), the invariance of \( \partial \delta,N \) under the truncated gILW dynamics (5.1), Cauchy-Schwarz’s inequality, Proposition 3.4 (see 3.12 with \( k = 1 \)), and Proposition 3.6 (see 3.33), we have
\[
\|\|u\|_{L^p_{\partial \delta,N} H_{\varepsilon}^s}\|_{L^p(d\rho_{\delta,N})} = \|\|\Phi_{\delta,N}(t)\|_{L^p_{\partial \delta,N} H_{\varepsilon}^s}\|_{L^p(d\rho_{\delta,N})} \\
= \|\|\Phi_{\delta,N}(t)\|_{L^p(d\rho_{\delta,N}) H_{\varepsilon}^s}\|_{L^p_T} \\
\lesssim T^{\frac{3}{5}} \|\|\phi\|_{L^p(d\rho_{\delta,N}) H_{\varepsilon}^s} \\
\lesssim T^{\frac{3}{5}} \|\|u\|_{L^p_{\partial \delta,N} H_{\varepsilon}^s}\|_{L^p(d\rho_{\delta,N})} \\
\lesssim T^{\frac{1}{5}},
\]
uniformly in \( N \in \mathbb{N} \) and \( 2 \leq \delta \leq \infty \). This proves (5.10).

Next, we prove the second bound (5.11). By writing \( \partial \delta \partial x = (\partial \delta \partial x) \partial x \), it follows from (1.42) and Lemma 2.1 that
\[
\sup_{2 \leq \delta \leq \infty} \|\|\partial \delta \partial x\|\|_{H^{s-2}} \leq \|f\|_{H^s}.
\]
Proposition 5.5. Let existence result for the gILW equation (5.20) with the Gibbs ian initial data (Theorem 1.8 (i)).

In [18, 74, 69] (see, in particular, Subsection 5.3 in [74]), we obtain the following global

\[ \tilde{\nu} \]

as

\[ \{ \tilde{\nu} \} \]

denoted by (5.20).

We thus first work with fixed \( 2 \leq \delta \leq \infty \) and construct invariant Gibbs dynamics to the renormalized gBO equation (1.69):

\[ \partial_t u_\delta - \mathcal{G}_\delta \partial_x^2 u_\delta = F(u_\delta) \]

\[ = \partial_x \mathcal{W}(u_\delta^B) \] (5.20)

with the understanding that it corresponds to the renormalized gBO equation (1.69) when \( \delta = \infty \), where \( F(u) \) is the limit of \( F_N(u) \) in (3.18) constructed in Proposition 3.4 (ii). In view of Proposition 5.2, the family \( \{ \nu_\delta \} \) is tight. Hence, by the Prokhorov theorem (Lemma 2.14), there exists a subsequence \( \{ \nu_{\delta,N_j} \} \) converging weakly to some limit denoted by \( \nu_\delta \). Namely, we have

\[ d_{LP}(\nu_{\delta,N_j}, \nu_\delta) \to 0 \] (5.21)

as \( j \to \infty \), where \( d_{LP} \) denotes the Lévy-Prokhorov metric defined in (2.41).

By the Skorokhod representation theorem (Lemma 2.15), there exist some probability space \( (\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta) \) and \( C(\mathbb{R}; H^s(T)) \)-valued random variables \( u_{\delta,N_j} \) and \( u_\delta \), such that

\[ \mathcal{L}(u_{\delta,N_j}) = \nu_{\delta,N_j} \quad \text{and} \quad \mathcal{L}(u_\delta) = \nu_\delta, \] (5.22)

and \( u_{\delta,N_j} \) converges \( \mathbb{P}_\delta \)-almost surely to \( u_\delta \) in \( C(\mathbb{R}; H^s(T)) \) as \( j \to \infty \). By repeating the argument in [18, 74, 69] (see, in particular, Subsection 5.3 in [74]), we obtain the following global existence result for the gILW equation (5.20) with the Gibbsian initial data (Theorem 1.8 (i)).

**Proposition 5.5.** Let \( u_{\delta,N_j} \), \( j \in \mathbb{N} \), and \( u_\delta \) be as above. Then, \( u_{\delta,N_j} \) and \( u_\delta \) are global-in-time distributional solutions to the truncated gILW equation (5.1) and the renormalized gILW equation (5.20), respectively. Moreover, we have

\[ \mathcal{L}(u_{\delta,N_j}(t)) = \rho_{\delta,N_j} \quad \text{and} \quad \mathcal{L}(u_\delta(t)) = \rho_\delta \] (5.23)

for any \( t \in \mathbb{R} \).

**Proof.** While the proof of Proposition 5.5 follows exactly the same lines in Subsection 5.3 of [74], we present details (with some modifications from [74]) for readers’ convenience. We also point out that Proposition 5.5 will be applied iteratively in the proof of Theorem 1.8 (ii) presented below.

---

23The space \( \mathcal{M} = C(\mathbb{R}; H^s(T)) \) endowed with the compact-open topology is complete and separable, and thus \( \mathcal{P}(\mathcal{M}) \) is the set of all the probability measures on \( \mathcal{M} \) is complete; see, for example, [10] Theorem 6.8 on p. 73.
Fix $t \in \mathbb{R}$. Let $R_t : C(\mathbb{R}; H^s(\mathbb{T})) \to H^s(\mathbb{T})$ be the evaluation map defined by $R_t(v) = v(t)$. Note that $R_t$ is a continuous function. Then, from (5.20) and the invariance of the truncated Gibbs measure $\rho_{\delta,N}$ (Lemma 5.1), we have

$$\nu_{\delta,N} \circ R_t^{-1} = \rho_{\delta,N} \circ \Phi_{\delta,N}^{-1} \circ R_t^{-1} = \rho_{\delta,N} \circ (R_t \circ \Phi_{\delta,N})^{-1}$$

$$= (R_t \circ \Phi_{\delta,N})_* \rho_{\delta,N} = (\Phi_{\delta,N}(t))_* \rho_{\delta,N}$$

(5.24)

$$= \rho_{\delta,N}.$$  

Then, it follows from (5.22) and (5.24) that

$$\mathcal{L}(u_{\delta,N_j}(t)) = \nu_{\delta,N_j} \circ R_t^{-1} = \rho_{\delta,N_j}. \quad (5.25)$$

By the construction, $u_{\delta,N_j}$ converges to $u_\delta$ in $C(\mathbb{R}; H^s(\mathbb{T}))$ almost surely with respect to $\mathbb{P}_\delta$. Thus, we have

$$u_{\delta,N_j}(t) = R_t(u_{\delta,N_j}) \to u_\delta(t) = R_t(u_\delta)$$

almost surely as $j \to \infty$, which in particular implies $u_{\delta,N_j}(t)$ converges in law to $u_\delta(t)$ as $j \to \infty$. Namely, $\mathcal{L}(u_{\delta,N_j}(t))$ converges weakly to $\mathcal{L}(u_\delta(t))$ as $j \to \infty$. On the other hand, recall from Theorem 1.3 (i) that $\rho_{\delta,N_j}$ converges to $\rho_\delta$ in total variation as $j \to \infty$, which in particular implies that $\rho_{\delta,N_j}$ converges weakly to $\rho_\delta$. Hence, in view of (5.25) and the uniqueness of the limit, we conclude $\mathcal{L}(u_\delta(t)) = \rho_\delta$. This proves (5.23).

Next, we show that the random variable $u_{\delta,N_j}$ is indeed a global-in-time distributional solution to (5.1). Given a test function $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{T}) = C_c^\infty(\mathbb{R} \times \mathbb{T})$, define $V_{\varphi,j} : C(\mathbb{R}; H^s(\mathbb{R})) \to \mathbb{R}$ by

$$V_{\varphi,j}(u) = |\langle \varphi, \partial_t u - \mathcal{G}_\delta \partial_x^2 u - F_{N_j}(u) \rangle|, \quad (5.26)$$

where $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{D}_t \times \mathcal{D}'_x$ pairing. It is easy to see that $V_{\varphi,j}$ is continuous. In view of the separability of $\mathcal{D}(\mathbb{R} \times \mathbb{T})$, let $\{\varphi_m\}_{m \in \mathbb{N}}$ be a countable dense subset of $\mathcal{D}(\mathbb{R} \times \mathbb{T})$. Then, in view of (5.9), (5.26), and the definition (5.7) of $\Phi_{\delta,N_j}$, we have

$$\|V_{\varphi_m,j}\|_{L^1(\omega_{\delta,N_j})} = \int_{H^s} |V_{\varphi_m,j}(\Phi_{\delta,N_j}(\phi))| d\rho_{\delta,N_j}(\phi) = 0 \quad (5.27)$$

for any $m \in \mathbb{N}$. Namely, there exists a set $\Sigma_m \subset C(\mathbb{R}; H^s(\mathbb{T}))$ such that $\nu_{\delta,N_j}(\Sigma_m) = 1$ and $V_{\varphi_m,j}(u) = 0$ for any $u \in \Sigma_m$. Now, set $\Sigma = \bigcap_{m \in \mathbb{N}} \Sigma_m$. Then, we have $\nu_{\delta,N_j}(\Sigma) = 1$ and, moreover, $V_{\varphi,j}(u) = 0$ for any $u \in \Sigma$ and $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{T})$, where the latter claim follows from (5.27) and the density of $\{\varphi_m\}_{m \in \mathbb{N}}$.

Finally, we prove that the random variable $u_\delta$ is a global-in-time distributional solution to (5.20). It follows from the almost sure convergence of $u_{\delta,N_j}$ to $u_\delta$ in $C(\mathbb{R}; H^s(\mathbb{T}))$ that

$$\partial_t u_{\delta,N_j} - \mathcal{G}_\delta \partial_x^2 u_{\delta,N_j} \to \partial_t u_\delta - \mathcal{G}_\delta \partial_x^2 u_\delta$$

(5.28)

in $\mathcal{D}'(\mathbb{R} \times \mathbb{T})$, $\mathbb{P}_\delta$-almost surely, as $j \to \infty$.

Next, we show almost sure convergence of the truncated nonlinearity $F_{N_j}(u_{\delta,N_j})$ to $F(u_\delta) = \partial_x W(u_\delta)$. Given $M \in \mathbb{N}$, write

$$F_{N_j}(u_{\delta,N_j}) - F(u_\delta) = (F_{N_j}(u_{\delta,N_j}) - F_M(u_{\delta,N_j})) + (F_M(u_{\delta,N_j}) - F_M(u_\delta)) + (F_M(u_\delta) - F(u_\delta)). \quad (5.29)$$
Noting that \( u \in C(\mathbb{R}; H^s(\mathbb{T})) \mapsto F_M(u) \in C(\mathbb{R}; H^{s-1}(\mathbb{T})) \) is continuous, it follows from the almost sure convergence of \( u_{\delta,N_j} \) to \( u_\delta \) in \( C(\mathbb{R}; H^s(\mathbb{T})) \) that

\[
F_M(u_{\delta,N_j}) \to F_M(u_\delta)
\]

in \( C(\mathbb{R}; H^{s-1}(\mathbb{T})) \), \( \tilde{\mathbb{P}}_\delta \)-almost surely, as \( j \to \infty \). As for the first term on the right-hand side of (5.29), for fixed \( T > 0 \), it follows from (5.9), the invariance of the truncated Gibbs measure \( \rho_{\delta,N} \), and Proposition 3.4(iii) that

\[
\lim_{j \to \infty} \| F_{N_j}(u_{\delta,N_j}) - F_{N_j}(u_{\delta,N_j}) \|_{L^2_m H_s^{s-1}} = 0
\]

where the implicit constants are independent of \( N_j \). By applying Proposition 3.4(ii), we conclude that the first term on the right-hand side of (5.29) converges to 0 in \( L^2(\tilde{\Omega}_\delta; L^2([-T,T]; H^{s-1}(\mathbb{T}))) \) as \( j,M \to \infty \). Hence, by extracting a subsequence, the first term on the right-hand side of (5.29) converges to 0 in \( L^2([-T,T]; H^{s-1}(\mathbb{T})) \), \( \tilde{\mathbb{P}}_\delta \)-almost surely, as \( j,M \to \infty \). A similar argument shows that, by extracting a subsequence, the third term on the right-hand side of (5.29) converges to 0 in \( L^2([-T,T]; H^{s-1}(\mathbb{T})) \), \( \tilde{\mathbb{P}}_\delta \)-almost surely, as \( M \to \infty \).

Putting all together with (5.29), we conclude that, up to a subsequence, \( F_{N_j}(u_{\delta,N_j}) \) converges to \( F(u_\delta) \) in \( L^2([-T,T]; H^{s-1}(\mathbb{T})) \), \( \tilde{\mathbb{P}}_\delta \)-almost surely, as \( j \to \infty \). Since the choice of \( T > 0 \) was arbitrary, we can apply this argument for \( T_m = 2^m, m \in \mathbb{N} \). Thus, with \( m = 1 \), there exists a subsequence \( F_{N_{j_1}}(u_{\delta,N_{j_1}}) \) and a set \( \Sigma_1 \) of full \( \tilde{\mathbb{P}}_\delta \)-probability such that \( F_{N_{j_1}}(u_{\delta,N_{j_1}})(\omega) \) converges to \( F(u_\delta)(\omega) \) in \( L^2([-T_1,T_1]; H^{s-1}(\mathbb{T})) \) for each \( \omega \in \Sigma_1 \) as \( j_1 \to \infty \). For each \( m \geq 2 \), we can extract a further subsequence \( F_{N_{j_m}}(u_{\delta,N_{j_m}}) \) of \( F_{N_{j_m-1}}(u_{\delta,N_{j_m-1}}) \) and a subset \( \Sigma_m \subset \Sigma_{m-1} \) of full \( \tilde{\mathbb{P}}_\delta \)-probability such that \( F_{N_{j_m}}(u_{\delta,N_{j_m}})(\omega) \) converges to \( F(u_\delta)(\omega) \) in \( L^2([-T_m,T_m]; H^{s-1}(\mathbb{T})) \) for each \( \omega \in \Sigma_m \) as \( j_m \to \infty \). By a diagonal argument, we conclude that, passing to a subsequence, we have \( F_{N_j}(u_{\delta,N_j}) \) converges to \( F(u_\delta) \) in \( L^2_{\text{loc}} H^{s-1}(\mathbb{T}) \), \( \tilde{\mathbb{P}}_\delta \)-almost surely, which in particular implies that this subsequence converges to \( F(u_\delta) \) in \( D'(\mathbb{R} \times \mathbb{T}) \), \( \tilde{\mathbb{P}}_\delta \)-almost surely. Therefore, together with (5.28), we conclude that \( u_\delta \) is a global-in-time distributional solution to (5.20).

Finally, we present the proof of Theorem 1.8(ii). In the discussion at the beginning of this subsection, we used Proposition 5.2 and the Prokhorov theorem (Lemma 2.14) to conclude that, for each fixed \( 2 \leq \delta \leq \infty \), there exists a sequence \( N_j \to \infty \) such that (5.21) holds. In the following, we iteratively apply this argument for integers \( \delta \geq 2 \) and apply a diagonal argument.
(i) Let $\delta = 2$. Then, it follows from Proposition 5.2 that the family $\{\nu_{2,N}\}_{N \in \mathbb{N}}$ is tight. Hence, by the Prokhorov theorem (Lemma 2.14), there exists a weakly convergent subsequence $\{\nu_{2,N_j(2)}\}_{j \in \mathbb{N}}$. Namely, there exists a probability measure $\nu_2$ on $C(\mathbb{R}; H^s(\mathbb{T}))$ such that $d_{LP}(\nu_{2,N_j(2)}, \nu_2) \to 0$ as $j \to \infty$.

(ii) For $\delta = 3$, we apply the same argument to $\{\nu_{3,N_j(2)}\}_{j \in \mathbb{N}}$ to conclude that there exists a weakly convergent subsequence $\{\nu_{3,N_j(3)}\}_{j \in \mathbb{N}}$ with $\{N_j(3)\}_{j \in \mathbb{N}} \subset \{N_j(2)\}_{j \in \mathbb{N}}$. Namely, there exists a probability measure $\nu_3$ on $C(\mathbb{R}; H^s(\mathbb{T}))$ such that $d_{LP}(\nu_{3,N_j(3)}, \nu_3) \to 0$ as $j \to \infty$.

(iii) We iterate this procedure for each integer $\delta \geq 4$ and construct a weakly convergent subsequence $\{\nu_{\delta,N_j(\delta)}\}_{j \in \mathbb{N}}$ with $\{N_j(\delta)\}_{j \in \mathbb{N}} \subset \{N_j(\delta-1)\}_{j \in \mathbb{N}}$. Namely, there exists a probability measure $\nu_\delta$ on $C(\mathbb{R}; H^s(\mathbb{T}))$ such that

$$d_{LP}(\nu_{\delta,N_j(\delta)}, \nu_\delta) \to 0,$$

as $j \to \infty$.

(iv) Let $\mathbb{N}_{\geq 2} = \mathbb{N} \cap [2, \infty)$. We take a diagonal sequence $\{\nu_{\delta,N_j(\delta)}\}_{\delta \in \mathbb{N}_{\geq 2}}$, where $j(\delta)$ is chosen such that $j(\delta)$ is increasing in $\delta$ and

$$d_{LP}(\nu_{\delta,N_j(\delta)}, \nu_\delta) \leq \frac{1}{\delta}.$$  

By Proposition 5.2 and the Prokhorov theorem (Lemma 2.14), the family $\{\nu_{\delta,N_j(\delta)}\}_{\delta \in \mathbb{N}_{\geq 2}}$ is tight and thus admits a weakly convergent subsequence $\{\nu_{\delta_m,N_j(\delta_m)}\}_{m \in \mathbb{N}}$ to some limit, which we denote by $\nu_\infty$. Namely, we have

$$d_{LP}(\nu_{\delta_m,N_j(\delta_m)}, \nu_\infty) \to 0,$$

as $m \to \infty$. By the triangle inequality for the Lévy-Prokhorov metric $d_{LP}$ with (5.31) and (5.32), we have

$$d_{LP}(\nu_{\delta_m}, \nu_\infty) \leq d_{LP}(\nu_{\delta_m}, \nu_{\delta_m,N_j(\delta_m)}) + d_{LP}(\nu_{\delta_m,N_j(\delta_m)}, \nu_\infty)$$

$$\leq \frac{1}{\delta_m} + d_{LP}(\nu_{\delta_m,N_j(\delta_m)}, \nu_\infty)$$

$$\to 0,$$

as $m \to \infty$ (and hence $\delta_m \to \infty$). Hence, $\nu_{\delta_m}$ converges weakly to $\nu_\infty$ as $m \to \infty$.

By the Skorokhod representation theorem (Lemma 2.15), there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and $C(\mathbb{R}; H^s(\mathbb{T}))$-valued random variables $u_{\delta_m}$ and $u$ such that

$$\mathcal{L}(u_{\delta_m}) = \nu_{\delta_m} \quad \text{and} \quad \mathcal{L}(u) = \nu_\infty$$

and $u_{\delta_m}$ converges $\bar{\mathbb{P}}$-almost surely to $u$ in $C(\mathbb{R}; H^s(\mathbb{T}))$ as $m \to \infty$.

Next, we show that $u_{\delta_m}$ is a global-in-time distributional solution to the renormalized gILW equation (5.20) (with $\delta = \delta_m$). It follows from (5.30) and the Skorokhod representation
theorem (Lemma 2.15) that there exist a probability space \((\tilde{\Omega}_m, \tilde{F}_m, \tilde{\mathbb{P}}_m)\) and \(C(\mathbb{R}; H^s(\mathbb{T}))\)-valued random variables \(\tilde{u}_{\delta_m, N_j^{(m)}}\) and \(\tilde{u}_{\delta_m}\) such that
\[
\mathcal{L}(\tilde{u}_{\delta_m, N_j^{(m)}}) = \nu_{\delta_m, N_j^{(m)}} \quad \text{and} \quad \mathcal{L}(\tilde{u}_{\delta_m}) = \nu_{\delta_m}
\]
(5.35)
and \(\tilde{u}_{\delta_m, N_j^{(m)}}\) converges \(\tilde{\mathbb{P}}_m\)-almost surely to \(\tilde{u}_{\delta_m}\) in \(C(\mathbb{R}; H^s(\mathbb{T}))\) as \(j \to \infty\). Arguing as in the proof of Proposition 5.5, we see that \(\tilde{u}_{\delta_m}\) is a global-in-time distributional solution to the renormalized gILW equation (5.20). Hence, from (5.34) and (5.35), we conclude that \(u_{\delta_m}\) is a global-in-time distributional solution to the renormalized gILW equation (5.20).

It remains to show that \(u\) satisfies the renormalized gBO equation (1.69) in the distributional sense. The almost sure convergence of \(u_{\delta_m}\) to \(u\) implies that
\[
\partial_t u_{\delta_m} - \mathcal{G}_\delta \partial_x^2 u_{\delta_m} \to \partial_t u - \mathcal{H} \partial_x^2 u
\]
(5.36)
in \(\mathcal{D}'(\mathbb{R} \times \mathbb{T})\) as \(m \to \infty\). Next, we discuss convergence of the nonlinearity. Let \(F(u_\delta) = \partial_x W(u_\delta^2)\) be as in Proposition 3.4(ii). Given \(M \in \mathbb{N}\), write
\[
F(u_{\delta_m}) - F(u) = (F(u_{\delta_m}) - F_M(u_{\delta_m})) + (F_M(u_{\delta_m}) - F_M(u))
\]
(5.37)
From the continuity of \(F_M\) and the almost sure convergence of \(u_{\delta_m}\) to \(u\), we see that the second term on the right-hand side of (5.37) tends to 0 in \(C(\mathbb{R}; H^{s-1}(\mathbb{T}))\), \(\tilde{\mathbb{P}}\)-almost surely, as \(m \to \infty\). As for the first and third terms on the right-hand side of (5.37), we need to exploit the uniform (in \(\delta\) and \(N\)) bounds, which is the main difference from the proof of Proposition 5.5 presented above. Let \(T > 0\). Then, from (5.9) and the invariance of the truncated Gibbs measure \(\rho_{\delta,N}\), we have
\[
\|F(u_{\delta_m}) - F_M(u_{\delta_m})\|_{L^2(\mathbb{R}; H^{s-1})} \leq T\frac{1}{\gamma} \|G_{\delta_m}\|_{L^4(\mathbb{T})} \|F(\phi) - F_M(\phi)\|_{L^4(\mathbb{R}; H^{s-1})}
\]
(5.38)
with the understanding that \(u_{\delta_m} = u\) when \(m = \infty\). From Proposition 3.6 we have
\[
\sup_{m \in \mathbb{N} \cup \{\infty\}} \frac{1}{\gamma} Z_{\delta_m} + \sup_{m \in \mathbb{N} \cup \{\infty\}} \|G_{\delta_m}\|_{L^4(\mathbb{T})} \lesssim 1.
\]
(5.39)
Then, from (5.38), (5.39), and Proposition 3.4(ii) (see (3.20) with \((M,N) = (\infty,N)\)), we conclude that the first and third terms on the right-hand side of (5.37) converge to 0 in \(L^2(\mathbb{R}_\delta; L^2([0,T]; H^{s-1}(\mathbb{T})))\) as \(M \to \infty\). Then, by first taking \(m \to \infty\) and then \(M \to \infty\) in (5.37), we conclude that, by extracting a subsequence, \(F(u_{\delta_m})\) converges to \(F(u)\) in \(L^2([0,T]; H^{s-1}(\mathbb{T}))\), \(\tilde{\mathbb{P}}\)-almost surely, as \(m \to \infty\). By repeating the argument at the end of the proof of Proposition 5.5, we see that up to a further subsequence, \(F(u_{\delta_m})\) converges to \(F(u)\) in \(\mathcal{D}'(\mathbb{R} \times \mathbb{T})\), \(\tilde{\mathbb{P}}\)-almost surely, as \(m \to \infty\). Therefore, together with (5.36), we conclude that \(u\) is a global-in-time distributional solution to the renormalized gBO equation (5.20). This concludes the proof of Theorem 1.8(ii).
Remark 5.6. In this paper, we considered probability measures on $H^{-\varepsilon}(\mathbb{T})$ for fixed small $\varepsilon > 0$. In the following, we briefly explain how to remove the dependence on $\varepsilon$. First, we set $H^0(\mathbb{T}) := \bigcap_{s>0} H^{-s}(\mathbb{T}) = \bigcap_{j\in\mathbb{N}} H^{-s_j}(\mathbb{T})$, with $s_j = \frac{1}{j}$. Then, we equip $H^0(\mathbb{T})$ with the following distance:

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|f - g\|_{H^{-s_j}}}{1 + \|f - g\|_{H^{-s_j}}}.$$ 

By definition, we have $d(f_n, f) \to 0$ if and only if $f_n$ converges to $f$ in $H^{-s_j}(\mathbb{T})$ for each $j \in \mathbb{N}$. Let $D$ be the set of smooth functions $Q \in C^\infty(\mathbb{T})$ of the form

$$Q(x) = \sum_{|n|\leq N} q_n e_n(x),$$

with $q_n \in \mathbb{Q}$ and $N \in \mathbb{N}$. Then, $D$ is a countable dense subset of $H^{-s_j}(\mathbb{T})$ for any $j \in \mathbb{N}$. Let $f \in H^0(\mathbb{T})$. Then, for each $j \in \mathbb{N}$, there exists $Q_{j,N} \in D$ such that

$$\|Q_{j,N} - f\|_{H^{-s_j}} \leq 2^{-N}.$$ 

Now, set $Q_N = Q_{N,N} \in D$, $N \in \mathbb{N}$. Then, given $\varepsilon > 0$, by choosing $N \geq \frac{1}{\varepsilon}$, we have

$$\|Q_N - f\|_{H^{-\varepsilon}} \leq \|Q_N - f\|_{H^{-1/\varepsilon}} = \|Q_{N,N} - f\|_{H^{-1/\varepsilon}} \leq 2^{-N}.$$ 

Hence, we have

$$d(Q_N, f) \leq \sum_{j=1}^{N} 2^{-j} \|Q_N - f\|_{H^{-1/\varepsilon}} + \sum_{j=N+1}^{\infty} 2^{-j} \leq 2^{-N} + 2^{-N} \to 0,$$

as $N \to \infty$. In other words, we just proved that $D$ is also a countable dense subset of $H^0(\mathbb{T})$ with respect to the metric $d$. Hence, from [16], we see that $C(\mathbb{R}; H^0(\mathbb{T}))$ is separable. This allows us to repeat the entire paper by replacing $C(\mathbb{R}; H^{-\varepsilon}(\mathbb{T}))$ with $C(\mathbb{R}; H^0(\mathbb{T}))$.

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\[ C(\mathbb{R}; H^0(\mathbb{T})) = \bigcap_{j=1}^{\infty} C(\mathbb{R}; H^{-\frac{1}{j}}(\mathbb{T})). \]
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