Chaos synchronization of canonically and Lie-algebraically deformed Henon-Heiles systems by active control

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Abstract

Recently, there has been provided two chaotic models based on the twist-deformation of classical Henon-Heiles system. First of them has been constructed on the well-known, canonical space-time noncommutativity, while the second one on the Lie-algebraically type of quantum space, with two spatial directions commuting to classical time. In this article, we find the direct link between mentioned above systems, by synchronization both of them in the framework of active control method. Particularly, we derive at the canonical phase-space level the corresponding active controllers as well as we perform (as an example) the numerical synchronization of analyzed models.
1 Introduction

Since Edward Lorenz proposed his widely-known "model of weather", there appeared a lot of papers dealing with so-called chaotic models, whose dynamics is described by strongly sensitive with respect initial conditions, nonlinear differential equations. The most popular of them are: Lorenz system [1], Roessler system [2], Rayleigh-Benard system [3], Henon-Heiles system [4], jerk equation [5], Duffing equation [6], Lotka-Volter system [7], Liu system [8], Chen system [9] and Sprott system [10]. A lot of them have been applied in various fields of industrial and scientific divisions, such as, for example: Physics, Chemistry, Biology, Microbiology, Economics, Electronics, Engineering, Computer Science, Secure Communications, Image Processing and Robotics.

The one of the most interesting among the above models seems to be so-called Henon-Heiles system, which has been provided in pure astrophysical context. It concerns the problem of nonlinear motion of a star around of a galactic center, where the motion is restricted to a plane. It is defined by the following Hamiltonian function

\[ H(p, x) = \frac{1}{2} \sum_{i=1}^{2} (p_i^2 + x_i^2) + x_1^2 x_2 - \frac{1}{3} x_2^3, \]

which in cartesian coordinates \( x_1 \) and \( x_2 \) describes the set of two nonlinearly coupled harmonic oscillators. In polar coordinates \( r \) and \( \theta \) it corresponds to the particle moving in noncentral potential of the form

\[ V(r, \varphi) = \frac{r^2}{2} + \frac{r^3}{3} \sin (3\varphi), \]

with \( x_1 = r \cos \varphi \) and \( x_2 = r \sin \varphi \). The above model has been inspired by observations indicating, that star moving in a weakly perturbated central potential should has apart of total energy \( E_{\text{tot}} \) constant in time, also the second conserved physical quantity \( I \). It has been demonstrated with use of so-called Poincare section method, that such a situation appears in the case of Henon-Heiles system only for the values of control parameter \( E_{\text{tot}} \) below the threshold \( E_{\text{th}} = 1/6 \). For higher energies the trajectories in phase space become chaotic and the quantity \( I \) does not exist (see e.g. [11], [12]).

Recently, there has been proposed in articles [13] and [14] two noncommutative counterparts of the above mentioned Henon-Heiles system. They have been defined respectively on the following canonically as well as Lie-algebraically deformed Galilei space-times [15]-[17][3]

\[ [t, \hat{x}_i] = 0, \quad [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \]

\[ \text{1) The quantity } I \text{ plays the role of additional constant of motion, which leads to the regular trajectories of particle.} \]

\[ \text{2) The canonically and Lie-algebraically noncommutative space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [15], [16]), for the twist-deformed quantum Galilei Hopf algebras } \mathcal{U}_\theta(G) \text{ and } \mathcal{U}_\kappa(G) \text{ respectively.} \]

\[ \text{3) It should be noted that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see references [18], [19]), apart of canonical [15]-[17] space-time noncommutativity, there also exist Lie-algebraic [17]-[22] and quadratic [17], [22]-[24] type of quantum spaces.} \]
and

\[ [t, \dot{x}_i] = 0 \ , \ \ [\dot{x}_i, \dot{x}_j] = \frac{i}{\kappa} \epsilon_{ij} \ , \quad (4) \]

with constant deformation parameters \( \theta_{ij} = -\theta_{ji} \) and \( \kappa \). Particularly, there has been provided the Hamiltonian functions of the models as well as the corresponding canonical equations of motion. Besides, it has been demonstrated that for proper values of deformation parameters \( \theta \) and \( \kappa \), and for proper values of control parameters, there appears (much more intensively) chaos in both systems. Consequently, in such a way, it has been shown the impact of the above noncommutative space-times on the basic dynamical properties of this important classical chaotic model. It should be noted, that such deformed constructions are inspired by investigations, dealing with noncommutative classical and quantum mechanics (see e.g. \[25\]-\[28\]) as well as with field theoretical systems (see e.g. \[29\]-\[31\]), in which the quantum space-time is not classical. Such models follow (particularly) from formal arguments based mainly on Quantum Gravity \[32\], \[33\] and String Theory \[34\], \[35\], indicating that space-time at Planck scale becomes noncommutative.

One of the most important problem of the chaos theory concerns so-called chaos synchronization phenomena. Since Pecora and Carroll \[36\] introduced a method to synchronize two identical chaotic systems, the chaos synchronization has received increasing attention due to great potential applications in many scientific discipline. Generally, there are known several methods of chaos synchronization, such as: OGY method \[37\], active control method \[38\], \[39\], adaptive control method \[40\], \[41\], backstepping method \[42\], \[43\], sampled-data feedback synchronization method \[44\], time-delay feedback method \[45\] and sliding mode control method \[46\], \[47\]. The mentioned methods have been applied to the synchronization of many identical as well as different chaotic models, such as, for example, Sprott, Lorenz and Roessler systems respectively \[48\], \[49\].

In this article we synchronize by active control scheme canonically deformed Henon-Heiles (master) system \[13\] with its Lie-algebraically noncommutative (slave) partner \[14\]. In this aim we establish the proper so-called active controllers with use of the Lyapunov stabilization theory \[50\]. Additionally, we illustrate the obtained results by numerical calculations performed for particular values of deformation parameters \( \theta_{ij} \) and \( \kappa \).

The paper is organized as follows. In second Section we recall chaotic canonically and Lie-algebraically deformed Henon-Heiles models proposed in articles \[13\] and \[14\] respectively. In Section 3 we remaind the basic concepts of active synchronization method, while in fourth Section we find the active controllers which synchronize both noncommutative systems. The conclusions and final remarks are discussed in the last Section.

## 2 The noncommutative Henon-Heiles models

In this Section we very shortly remaind the basic facts concerning two chaotic Henon-Heiles models defined on noncommutative Galilei space-times \[5\] and \[6\] respectively. As it was mentioned in Introduction, first of them has been provided in paper \[13\] while the second one in article \[14\].
2.1 Classical Henon-Heiles system on canonically deformed spacetime

In accordance with [13], the dynamics of the model is given by the following Hamiltonian function

\[ H(\hat{p}, \hat{x}) = \frac{1}{2} \sum_{i=1}^{2} (\hat{p}_i^2 + \hat{x}_i^2) + \hat{x}_1^2 \hat{x}_2 - \frac{1}{3} \hat{x}_2^3 , \]  
(5)

defined on the canonically deformed phase space of the form 4

\[ \{ \hat{x}_1, \hat{x}_2 \} = 2\theta , \quad \{ \hat{p}_1, \hat{p}_2 \} = \{ \hat{x}_i, \hat{p}_j \} = 0 , \]  
(6)

with constant parameter \( \theta = \theta_{12} = -\theta_{21} \). In terms of commutative canonical variables \((x_i, p_i)\) the Hamiltonian looks as follows

\[ H(p, x) = \frac{1}{2M(\theta)} (p_1^2 + p_2^2) + \frac{1}{2} M(\theta) \Omega^2(\theta) (x_1^2 + x_2^2) - S(\theta) L + \]  
(7)

where

\[ L = x_1 p_2 - x_2 p_1 , \]  
(8)

\[ 1/M(\theta) = 1 + \theta^2 , \]  
(9)

\[ \Omega(\theta) = \sqrt{(1 + \theta^2)} , \]  
(10)

and

\[ S(\theta) = \theta . \]  
(11)

Due to the form of the above energy function, the symbols \( M(\theta) \) and \( \Omega(\theta) \) denote the new, deformed mass and frequency of particle, respectively. Obviously, quantity \( L \) plays the role of the angular momentum vector, while \( S(\theta) \) can be interpreted as the present in third term of the Hamiltonian, the new \( \theta \)-dependent coefficient. It should be also noted, that two last, nonlinear members of formula (7) remain responsible for chaotic behaviour of the system, while the corresponding to \( H(p, x) \) canonical equations of motion are given by

\[ \dot{x}_1 = \left[ 1/M(\theta) \right] p_1 + S(\theta) x_2 + \]  
\[ \quad + \left[ (x_1 - \theta p_2)^2 - (x_2 + \theta p_1)^2 \right] \theta , \]  
(12)

\[ \dot{x}_2 = \left[ 1/M(\theta) \right] p_2 - S(\theta) x_1 - 2(x_2 + \theta p_1) (x_1 - \theta p_2) \theta , \]  
(13)

\[ \dot{p}_1 = -M(\theta) \Omega^2(\theta) x_1 + S(\theta) p_2 - 2(x_2 + \theta p_1) (x_1 - \theta p_2) , \]  
(14)

\[ \dot{p}_2 = -M(\theta) \Omega^2(\theta) x_2 - S(\theta) p_1 + \]  
\[ \quad - (x_1 - \theta p_2)^2 + (x_2 + \theta p_1)^2 . \]  
(15)

\[ ^4 \text{The correspondence relations are } \{ \cdot, \cdot \} = \frac{1}{i} [ \cdot, \cdot ] . \]
Of course, for deformation parameter $\theta$ approaching zero the above system becomes classical.

### 2.2 Classical Henon-Heiles system on Lie-algebraically deformed space-time

The model is defined by the Hamiltonian function (5) given on the following Lie-algebraically deformed phase space

$$
\{ \hat{x}_1, \hat{x}_2 \} = \frac{2t}{\kappa}, \quad \{ \hat{p}_i, \hat{p}_j \} = 0, \quad \{ \hat{x}_i, \hat{p}_j \} = \delta_{ij},
$$

with constant, mass-like parameter $\kappa$. In terms of commutative variables the above Hamiltonian takes the form

$$
H(p, x, t) = \frac{1}{2M(\frac{t}{\kappa})} (p_1^2 + p_2^2) + \frac{1}{2} M(\frac{t}{\kappa}) \Omega^2(\frac{t}{\kappa}) (x_1^2 + x_2^2) - S(\frac{t}{\kappa}) L + \frac{1}{3} \left( x_2 + \frac{t}{\kappa} p_1 \right)^3,
$$

where

$$
L = x_1 p_2 - x_2 p_1,
$$

$$
M(\frac{t}{\kappa}) = 1 + \left( \frac{t}{\kappa} \right)^2,
$$

$$
\Omega(\frac{t}{\kappa}) = \sqrt{1 + \left( \frac{t}{\kappa} \right)^2},
$$

and

$$
S(\frac{t}{\kappa}) = \frac{t}{\kappa}.
$$

It is worth to notice, that due to the similar form of energy functions (7) and (17), the all coefficients $M(\frac{t}{\kappa})$, $\Omega(\frac{t}{\kappa})$ as well as $S(\frac{t}{\kappa})$ can be interpreted in the same manner as their $\theta$-deformed counterparts (9)-(11). However, contrary to the previous case, the Lie-algebraically modified quantities (19)-(21) are time-dependent, and the corresponding

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5One can check that $[\kappa] = \text{kg}$.  

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5
canonical equations of motion look as follows

\[
\begin{align*}
\dot{x}_1 &= \frac{1}{M} \left( \frac{t}{\kappa} \right) p_1 + S \left( \frac{t}{\kappa} \right) x_2 + S \left( \frac{t}{\kappa} \right) x_2 + \left[ \left( x_1 - \frac{t}{\kappa} p_2 \right)^2 - \left( x_2 + \frac{t}{\kappa} p_1 \right)^2 \right] \frac{t}{\kappa} \\
\dot{x}_2 &= \frac{1}{M} \left( \frac{t}{\kappa} \right) p_2 - S \left( \frac{t}{\kappa} \right) x_1 - 2 \left[ x_2 + \frac{t}{\kappa} p_1 \right] \left[ x_1 - \frac{t}{\kappa} p_2 \right] \frac{t}{\kappa} \\
\dot{p}_1 &= -M \left( \frac{t}{\kappa} \right) \Omega^2 \left( \frac{t}{\kappa} \right) x_1 + S \left( \frac{t}{\kappa} \right) p_2 - 2 \left[ x_2 + \frac{t}{\kappa} p_1 \right] \left[ x_1 - \frac{t}{\kappa} p_2 \right] \\
\dot{p}_2 &= -M \left( \frac{t}{\kappa} \right) \Omega^2 \left( \frac{t}{\kappa} \right) x_2 - S \left( \frac{t}{\kappa} \right) p_1 + \left[ x_1 - \frac{t}{\kappa} p_2 \right]^2 + \left[ x_2 + \frac{t}{\kappa} p_1 \right]^2 .
\end{align*}
\]

(22) \quad (23) \quad (24) \quad (25)

Obviously, for deformation parameter \( \kappa \) running to infinity the above model becomes commutative.

3 **Chaos synchronization by active control - general prescription**

In this Section we remaind the general scheme of chaos synchronization of two systems by so-called active control procedure \cite{38, 39}. Let us start with the following master mode\footnote{\( \frac{d\alpha}{dt} = \dot{\alpha} \).}

\[
\dot{x} = Ax + F(x) ,
\]

(26) with \( x = [ x_1, x_2, \ldots, x_n ] \) being the state of the system, \( A \) denotes the \( n \times n \) matrix of the system parameters and \( F(x) \) plays the role of the nonlinear part of the differential equation (26). The slave model dynamics is described by

\[
\dot{y} = By + G(y) + u ,
\]

(27) with \( y = [ y_1, y_2, \ldots, y_n ] \) being the state of the system, \( B \) denoting the \( n \)-dimensional quadratic matrix of the system, \( G(y) \) playing the role of nonlinearity of the equation (27) and \( u = [ u_1, u_2, \ldots, u_n ] \) being the active controller of the slave model. Besides, it should be mentioned that for matrices \( A = B \) and functions \( F = G \) the states \( x \) and \( y \) describe two identical chaotic systems. In the case \( A \neq B \) or \( F \neq G \) they correspond to the two different chaotic models.
Let us now provide the following synchronization error vector
\[ e = y - x , \] (28)
which in accordance with (26) and (27) obeys
\[ \dot{e} = By - Ax + G(y) - F(x) + u . \] (29)

In active control method we try to find such a controller \( u \), which synchronizes the state of the master system (26) with the state of the slave system (27) for any initial condition \( x_0 = x(0) \) and \( y_0 = y(0) \). In other words, we design a controller \( u \) in such a way that for system (29) we have
\[ \lim_{t \to \infty} ||e(t)|| = 0 , \] (30)
for all initial conditions \( e_0 = e(0) \). In order to establish the synchronization (29) we use the Lyapunov stabilization theory \([50]\). It means, that if we take as a candidate Lyapunov function of the form
\[ V(e) = e^T Pe , \] (31)
with \( P \) being a positive \( n \times n \) matrix, then we wish to find the active controller \( u \) so that
\[ \dot{V}(e) = -e^T Q e , \] (32)
where \( Q \) is a positive definite \( n \times n \) matrix as well. Then the systems (26) and (27) remain synchronized.

4 Chaos synchronization of the models

The described in previous Section algorithm can be used to the synchronization of two above remained noncommutative Henon-Heiles systems. In our treatment the canonically deformed model \([13]\) plays the role of master system
\[ \dot{x}_1 = [1/M(\theta)] p_1 + S(\theta) x_2 + \\
\quad + [(x_1 - \theta p_2)^2 - (x_2 + \theta p_1)^2] \theta , \] (33)
\[ \dot{x}_2 = [1/M(\theta)] p_2 - S(\theta) x_1 - 2(x_2 + \theta p_1) (x_1 - \theta p_2) \theta , \] (34)
\[ \dot{p}_1 = -M(\theta) \Omega^2(\theta) x_1 + S(\theta) p_2 - 2(x_2 + \theta p_1) (x_1 - \theta p_2) , \] (35)
\[ \dot{p}_2 = -M(\theta) \Omega^2(\theta) x_2 - S(\theta) p_1 + \\
\quad - (x_1 - \theta p_2)^2 + (x_2 + \theta p_1)^2 . \] (36)
while its slave partner is given by Lie-algebraically noncommutative model \[14\]

\[
\dot{y}_1 = 1/M \left( t / \kappa \right) \pi_1 + S \left( t / \kappa \right) y_2 + \\
\quad + \left[ \left( y_1 - t / \pi_2 \right)^2 - \left( y_2 + t / \pi_1 \right)^2 \right] \frac{t}{\kappa} + u_{y_1}, \tag{37}
\]

\[
\dot{y}_2 = 1/M \left( t / \kappa \right) \pi_2 - S \left( t / \kappa \right) y_1 - 2 \left[ y_2 + t / \pi_1 \right] \left[ y_1 - t / \pi_2 \right] \frac{t}{\kappa} + u_{y_2}, \tag{38}
\]

\[
\dot{\pi}_1 = -M \left( t / \kappa \right) \Omega^2 \left( t / \kappa \right) y_1 + S \left( t / \kappa \right) \pi_2 - 2 \left[ y_2 + t / \pi_1 \right] \left[ y_1 - t / \pi_2 \right] + u_{\pi_1}, \tag{39}
\]

\[
\dot{\pi}_2 = -M \left( t / \kappa \right) \Omega^2 \left( t / \kappa \right) y_2 - S \left( t / \kappa \right) \pi_1 + \\
\quad - \left[ y_1 - t / \pi_2 \right]^2 + \left[ y_2 + t / \pi_1 \right]^2 + u_{\pi_2}, \tag{40}
\]

with active controllers \( u_{y_1}, u_{y_2}, u_{\pi_1} \) and \( u_{\pi_2} \) respectively.

Using the above equations of motion one can check that the dynamics of synchronization errors \( e_{y_i} = y_i - x_i \) and \( e_{\pi_i} = \pi_i - p_i \) is obtained as\[7\]

\[
\dot{e}_{y_i} = 1/M \left( t / \kappa \right) \pi_1 + S \left( t / \kappa \right) y_2 + \\
\quad + \left[ \left( y_1 - t / \pi_2 \right)^2 - \left( y_2 + t / \pi_1 \right)^2 \right] \frac{t}{\kappa} + \\
\quad - \frac{1}{M(\theta)} p_1 - S(\theta) x_2 - \left[ (x_1 - \theta p_2)^2 + (x_2 + \theta p_1)^2 \right] \theta + u_{y_1}, \tag{41}
\]

\[
\dot{e}_{y_2} = 1/M \left( t / \kappa \right) \pi_2 - S \left( t / \kappa \right) y_1 - 2 \left[ y_2 + t / \pi_1 \right] \left[ y_1 - t / \pi_2 \right] \frac{t}{\kappa} + \\
\quad - \frac{1}{M(\theta)} p_2 + S(\theta) x_1 + 2(x_2 + \theta p_1) (x_1 - \theta p_2) \theta + u_{y_2}, \tag{42}
\]

\[
\dot{e}_{\pi_1} = -M \left( t / \kappa \right) \Omega^2 \left( t / \kappa \right) y_1 + S \left( t / \kappa \right) \pi_2 - 2 \left[ y_2 + t / \pi_1 \right] \left[ y_1 - t / \pi_2 \right] + \\
\quad + M(\theta) \Omega^2(\theta) x_1 - S(\theta) p_2 + 2(x_2 + \theta p_1) (x_1 - \theta p_2) + u_{\pi_1}, \tag{43}
\]

\[7\]See also formula (29).
\[
\dot{e}_{\pi_2} = -M \left( \frac{t}{\kappa} \right) \Omega^2 \left( \frac{t}{\kappa} \right) y_2 - S \left( \frac{t}{\kappa} \right) \pi_1 + \\
- \left[ y_1 - \frac{t}{\kappa} \pi_2 \right]^2 + \left[ y_2 + \frac{t}{\kappa} \pi_1 \right]^2 + \\
+ M(\theta)\Omega^2(\theta)x_2 + S(\theta)p_1 + (x_1 - \theta p_2)^2 - (x_2 + \theta p_1)^2 + u_{\pi_2}.
\]

Besides, if we define the positive Lyapunov function by
\[
V(e) = \frac{1}{2} \left( e_{y_1}^2 + e_{y_2}^2 + e_{\pi_1}^2 + e_{\pi_2}^2 \right),
\]
then for the following choice of control functions
\[
\begin{align*}
   u_{y_1} &= \left[ 1/M(\theta) \right] p_1 + S(\theta)x_2 + [(x_1 - \theta p_2)^2 - (x_2 + \theta p_1)^2] \theta + \\
   &- 1/M \left( \frac{t}{\kappa} \right) \pi_1 - S \left( \frac{t}{\kappa} \right) y_2 + \\
   &- \left[ \left( y_1 - \frac{t}{\kappa} \pi_2 \right)^2 + \left( y_2 + \frac{t}{\kappa} \pi_1 \right)^2 \right] \frac{t}{\kappa} - e_{y_1},
\end{align*}
\]
\[
\begin{align*}
   u_{y_2} &= \left[ 1/M(\theta) \right] p_2 - S(\theta)x_1 - 2(x_2 + \theta p_1) (x_1 - \theta p_2) \theta + \\
   &- 1/M \left( \frac{t}{\kappa} \right) \pi_2 + S \left( \frac{t}{\kappa} \right) y_1 + 2 \left[ y_2 + \frac{t}{\kappa} \pi_1 \right] \left[ y_1 - \frac{t}{\kappa} \pi_2 \right] \frac{t}{\kappa} - e_{y_2},
\end{align*}
\]
\[
\begin{align*}
   u_{\pi_1} &= -M(\theta)\Omega^2(\theta)x_1 + S(\theta)p_2 - 2(x_2 + \theta p_1) (x_1 - \theta p_2) + \\
   &+ M \left( \frac{t}{\kappa} \right) \Omega^2 \left( \frac{t}{\kappa} \right) y_1 - S \left( \frac{t}{\kappa} \right) \pi_2 + 2 \left[ y_2 + \frac{t}{\kappa} \pi_1 \right] \left[ y_1 - \frac{t}{\kappa} \pi_2 \right] - e_{\pi_1},
\end{align*}
\]
\[
\begin{align*}
   u_{\pi_2} &= -M(\theta)\Omega^2(\theta)x_2 - S(\theta)p_1 - (x_1 - \theta p_2)^2 + (x_2 + \theta p_1)^2 + \\
   &+ M \left( \frac{t}{\kappa} \right) \Omega^2 \left( \frac{t}{\kappa} \right) y_2 + S \left( \frac{t}{\kappa} \right) \pi_1 + \\
   &+ \left[ y_1 - \frac{t}{\kappa} \pi_2 \right]^2 + \left[ y_2 + \frac{t}{\kappa} \pi_1 \right]^2 - e_{\pi_2},
\end{align*}
\]
we have
\[
\dot{V}(e) = -\left( e_{y_1}^2 + e_{y_2}^2 + e_{\pi_1}^2 + e_{\pi_2}^2 \right).
\]

Such a result means (see general prescription) that the canonically (see (33)-(36)) and Lie-algebraically (see (37)-(40)) Henon-Heiles systems are synchronized for all initial conditions with active controllers (46)-(49).

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8 The matrix \( P = 1 \) in the formula (31).
9 The matrix \( Q = 1 \) in the formula (32).
Let us now illustrate the above considerations by the proper numerical calculations.

First of all, we solve canonically deformed system (33)-(36) with \( \theta = 1 \) as well as we integrate the Lie-algebraically model (37)-(40) for \( \kappa = 1 \) and without active controllers \( u_{y1}, u_{y2}, u_{\pi1} \) and \( u_{\pi2} \), for two different sets of initial conditions
\[
(x_{01}, x_{02}; p_{01}, p_{02}) = (0.01, -0.01; 0, 0),
\]
and
\[
(y_{01}, y_{02}; \pi_{01}, \pi_{02}) = (0, 0; -0.02, 0.02),
\]
respectively. The results are presented on Figure 1 - one can see that there exist (in fact) the divergences between both phase space trajectories. Next, we find the solutions for the master system (33)-(36) (the \((x,p)\)-trajectory) and for its slave partner (37)-(40) with active controllers (46)-(49) (the \((y,\pi)\)-trajectory) for initial data (51) and (52) respectively. Now, we see that the corresponding phase space trajectories become synchronized - the vanishing in time error functions \( e_{y_i} \) and \( e_{\pi_i} \) are presented on Figure 2. Additionally, we repeat the above numerical procedure for two another sets of initial data: \((x_0; p_0) = (0, 0; 0, 0)\) and \((y_0; \pi_0) = (0.02, -0.02; 0.01, -0.01)\); the obtained results are presented on Figures 3 and 4 respectively.

5 Final remarks

In this article we synchronize two noncommutative Henon-Heiles models with use of active control method. Particularly, we find the proper active controllers (46)-(49) as well as we perform numerical synchronization of the systems for fixed values of deformation parameters \( \theta \) and \( \kappa \).

In our opinion the obtained result seems to be quite interesting due to the two reasons at least. Firstly, it finds the direct dynamical link between two models defined on the completely different noncommutative space-times - the canonically twisted space and the Lie-algebraically deformed space-time respectively. Such a connection suggests, that there may exist other, more fundamental (for example taken at the kinematical level) link between both, considered here systems. Secondly, it combines in quite matured way two disparate scientific fields, such as the elements of Quantum Group Theory with the techniques typical for the Classical Chaos domain.

Finally, it should be noted that the presented investigations can be extended in various ways. For example, one may consider synchronization of the noncommutative Henon-Heiles models with use of other mentioned in Introduction methods. Obviously, the works in this direction already started and are in progress.

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Figure 1: The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for canonically deformed Henon-Heiles system with initial conditions (51) (the $(x, p)$-trajectory), and for Lie-algebraically noncommutative Henon-Heiles model without correlation functions $u_{y_i}, u_{\pi_i}$ for the initial conditions (52) (the $(y, \pi)$-trajectory). The blue line corresponds to the $e_{y_1}$-error function, the orange one - to $e_{y_2}$, the green one - to $e_{\pi_1}$ and the red one - to $e_{\pi_2}$ respectively.
Figure 2: The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for canonically deformed Henon-Heiles model defined by the master system (33)-(36) with the initial conditions (51) (the $(x,p)$-trajectory), and for the slave Lie-algebraically noncommutative Henon-Heiles system (37)-(40) with the initial conditions (52) (the $(y,\pi)$-trajectory). The blue line corresponds to the $e_{y_1}$-error function, the orange one - to $e_{y_2}$, the green one - to $e_{\pi_1}$ and the red one - to $e_{\pi_2}$ respectively.
Figure 3: The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for canonically deformed Henon-Heiles model with the initial conditions $(x_0; p_0) = (0, 0; 0, 0)$ (the $(x, p)$-trajectory), and for Lie-algebraically noncommutative Henon-Heiles model without correlation functions $u_{y_i}$, $u_{\pi_i}$ for the initial conditions $(y_0, \pi_0) = (0.02, -0.02; 0.01, -0.01)$ (the $(y, \pi)$-trajectory). The blue line corresponds to the $e_{y_1}$-error function, the orange one - to $e_{y_2}$, the green one - to $e_{\pi_1}$ and the red one - to $e_{\pi_2}$ respectively.
Figure 4: The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for canonically deformed Henon-Heiles model defined by the master system (33)-(36) with the initial conditions $(x_0; p_0) = (0, 0; 0, 0)$ (the $(x, p)$-trajectory), and for the slave Lie-algebraically noncommutative Henon-Heiles system (37)-(40) with the initial conditions $(y_0, \pi_0) = (0.02, -0.02; 0.01, -0.01)$ (the $(y, \pi)$-trajectory). The blue line corresponds to the $e_{y_1}$-error function, the orange one - to $e_{y_2}$, the green one - to $e_{\pi_1}$ and the red one - to $e_{\pi_2}$ respectively.