Constructions of torsion-free countable, amenable, weakly mixing groups

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Abstract

In this note, we construct torsion-free countable, amenable, weakly mixing groups, which answer a question of V. Bergelson. Some results related to verbal subgroups and crystallographic groups are also presented.

Keywords: weakly mixing group, WM group, minimally almost periodic group, variety of groups, verbal subgroup, torsion-free group, wreath product, orderable group

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1 Introduction

Weak mixing of a group action on a measure space is a property stronger than ergodicity and it plays an important role in the modern theory of dynamical systems (see for instance [Gl03], [BG04] and the references there). For the actions of cyclic groups, it was introduced by Koopman and von Neumann in [KvN32]. Later von Neumann [vN34] introduced the class of so-called ‘minimally almost periodic groups’, which can be characterized by the

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property that any ergodic measure preserving action of such a group on a finite measure space is in fact weakly mixing. At present, it is customary to call such groups weakly mixing or WM groups for short. Whereas at the beginning of the development of subject, the locally compact groups mostly were involved, in recent investigations, abstract groups started playing an important role. The case, where a group is amenable, attracted special attention in the paper of Bergelson and Furstenberg [BF09], establishing a relation between the WM property and the Ramsey theory (see also recent [BCRZ14]). For a finitely generated group, to be a WM group is the same as to have no nontrivial finite quotients. For infinitely generated amenable groups property WM is equivalent to have no nontrivial finite quotients or abelian quotients (see Proposition 2.1 below). Thus locally finite simple groups, like the group $\text{Alt}_\text{fin}(\mathbb{N})$ of finitary even permutations of $\mathbb{N}$ for instance, are WM. These groups are torsion groups.

A few years ago V. Bergelson, in a private discussion with the first author, raised the following question:

**Question 1.** Does there exist a countable, torsion-free, amenable, WM group?

We give a positive answer to this question providing examples satisfying some additional properties.

This is done in two ways. First, we follow the ideas of B. Neumann [Ne49], B. Neumann and H. Neumann [NN59], later developed by P. Hall [H74] and other researchers. This leads, for instance, to the example of a countable orderable (and hence torsion-free) locally solvable (and hence amenable) WM group (Corollary 3.2). Additional tools allow to construct simple groups that answer Question 1.

As an alternative, we use groups of the type $F'/N'$ ($N' = [N, N]$ is the commutator of $N$), where $F$ is a free group and $N < F$ is a normal subgroup. Groups of the type $F/N'$ and more generally of type $F/V(N)$ (where $V(N)$ is some verbal subgroup of $N$) and their subgroups were studied intensively in the 60s of the last century by many researchers (from [M39] to [Sh65] and much more) mostly with the purpose of study varieties of groups ([N67]). They also play a role in study of orderable groups, as it can be seen from [KK74] and the literature cited there. We show that groups of the type $F'/N'$ lead to examples of WM groups under the condition that $F/N$ is an amenable WM group.

The principal difference between these two constructions is the following. The first constructions is an embedding construction. It is quite flexible for embedding of any countable locally solvable or amenable, or elementary amenable, or sub-exponentially amenable, or torsion-free, or locally indicable, or orderable, or right orderable group in a countable group of the same type but with additional properties. On the contrary the subgroups of the groups given by the second construction are rather special. They can be regarded as generalizations of torsion-free crystallographic groups (see Proposition 5.1), in particular, any non-free subgroup $H$ of the group $F/N'$ has a nontrivial free abelian normal subgroup, and moreover, $H$ must have non-trivial intersection with $N/N'$ (see Proposition 4.5).

In the study of amenable groups, an important role is assigned to the splitting of the class $AG$ of amenable groups into disjoint union of the class $EG$ of elementary amenable groups and of class $AG \setminus EG$ of non-elementary amenable groups. Another interesting
subclass of $AG$ is class $SG$ of subexponentially amenable groups, which together with $EG$, $SG \setminus EG$ and $AG \setminus SG$ gives splitting of $AG$ into three subclasses. We provide examples of groups answering Question 1 that belongs to these classes. A group property stronger than to be torsion-free is the property to be orderable. We provide examples with various orderability properties. Unfortunately all our examples are infinitely generated, and it is interesting question if the answer to Bergelson question can be done within the class of finitely generated groups. Observe that a right orderable amenable group can not be finitely generated as it is indicable (can be mapped onto $\mathbb{Z}$) by the result of Morris [Mo06]. An interesting open question related to the above discussion is:

**Question 2.** Does there exist a finitely generated torsion-free, amenable, simple group?

A part of our note contains some results concerning verbal subgroups (this is related to the second construction of WM groups) and a construction of crystallographic groups, which is also based on the use of groups of type $F' / N'$.

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### 2 Preliminaries

Since our note lies between group theory and ergodic theory, we write more details and give more definitions than it would require for a paper in one field.

A group $G$ is called WM (weakly mixing, or minimally almost periodic) if it is non-trivial and one of the following equivalent conditions holds:

$\alpha$) $G$ has no non-trivial finite-dimensional unitary representations.

$\beta$) $G$ does not admit non-constant almost periodic functions.

$\gamma$) Any ergodic measure preserving action of $G$ on a finite measure space is in fact weakly mixing.

Equivalences $\alpha \sim \beta$ and $\alpha \sim \gamma$ are proven in [vN34] and [Sc84] respectively.

First we provide an alternative characterization of WM groups in the presence of amenability or finite generation.

**Proposition 2.1.** Let $G$ be a group.

1. If $G$ is WM then $G$ has no non-trivial finite or abelian quotients.

2. If $G$ is non-trivial and finitely generated and has no non-trivial finite quotients then $G$ is WM.
3. If $G$ is non-trivial and does not have non-cyclic free subgroups (in particular if $G$ is amenable) then group $G$ is WM if and only if it does not have non-trivial finite or abelian quotients.

Proof. 1. Any non-trivial finite or abelian group admits non-trivial finite dimensional unitary representations. If $G$ admits such a group as a quotient, then such a representation can be pulled back to $G$ showing that $G$ is not WM.

2. Suppose $G$ is finitely generated and is not WM. Then there is a non-trivial finite dimensional unitary representation $\pi : G \to U(n)$. Note $\pi(G)$ is a non-trivial finitely generated linear group. Mal’cev proved $\text{[Ma40]}$ that all such groups are residually finite. In particular, $\pi(G)$ has a non-trivial finite quotient and therefore, $G$ also has a non-trivial finite quotient, a contradiction.

3. One direction follows immediately from the point 1. Suppose now that $G$ is non-trivial, has no free subgroups on two generators, does not have non-trivial finite or abelian quotients, and is not WM. Then there is a non-trivial finite dimensional unitary representation $\pi : G \to U(n)$. Note that $\pi(G)$ is virtually solvable by the Tits alternative $\text{[Ti72]}$. If $\pi(G)$ has a proper subgroup of finite index then, pulling back, $G$ also has a proper subgroup of finite index and therefore $G$ has a non-trivial finite quotient. If $\pi(G)$ does not have a proper subgroup of finite index then $\pi(G)$ is solvable, and hence has a non-trivial abelian quotient, which implies that $G$ has a non-trivial abelian quotient, a contradiction.

$\square$

Corollary 2.2. Let $H$ be Higman’s finitely presented group $H$ without finite quotients constructed in $\text{[H51]}$. Let $H/N$ by a quotient of $H$ by some maximal normal subgroup $N$ (it is a simple finitely generated group). Then both $H$ and $H/N$ are WM groups.

Recall that a group is called locally finite if every its finitely generated subgroup is finite, and similarly, for locally solvable groups. Such groups are amenable (and even elementary amenable, see the definition below).

Corollary 2.3. Infinite simple locally finite groups are WM.

There are many examples of simple locally finite groups (the simplest example $\text{Alt}_\infty(\mathbb{N})$ was presented in the introduction). But all these groups are torsion. As was already indicated, the question about existence of finitely generated torsion-free, amenable, simple group remains open. The following question is open too.

Question 3. Does there exist a finitely generated infinite, torsion, amenable, simple group?

Recall that a group $G$ is amenable if it has invariant mean, which is equivalent to have a finitely additive left invariant measure $\mu$ with values in $[0, 1]$ defined on the algebra of all subsets of $G$ and normalized by the condition $\mu(G) = 1$. Finite, abelian groups, and groups of subexponential growth are amenable, and the class $\text{AG}$ of amenable groups is closed with respect to operations: (i) taking subgroup, (ii) taking quotient, (iii) extension, (iv) direct
limit (the latter operation can be replaced by directed unions). Class EG of elementary amenable groups is the smallest class of groups containing finite and abelian groups and closed with respect to the operations (i)-(iv) (it was introduced by M.Day in [Da57]). The class SG of subexponentially amenable groups is the smallest class of groups containing all finitely generated groups of subexponential growth and closed with respect to the operations (i)-(iv) (it was introduced in [Gr98]). The inclusions EG ⊂ SG ⊂ AG hold and are proper [Gr98, BV05]. We will say that the type of amenability of amenable group is $T_1$, $T_2$ or $T_3$ depending to what set $EG$, $SG \setminus EG$ or $T_3 = AG \setminus SG$, respectively, the group belongs. A classical source for information about amenable groups is [Gre69], more recent source of information is the survey [CGH99] and monograph [CC10].

Recall that a group $G$ is called orderable group if it has linear order (called also total order) that is invariant with respect to left and right multiplication. It is called a right (left) orderable group if it has linear order invariant with respect to right (left) multiplication. To be orderable is a stronger condition than to be right orderable, the latter is equivalent to be left orderable and is stronger than to be torsion free. We will deal in the first construction with restricted and unrestricted wreath products of groups. Observe that although a restricted wreath product of (right) orderable groups is (right) orderable, for unrestricted products, this is not correct in the class of orderable groups (but it is correct in the class of right orderable groups [MR77, Theorem 7.3.2]). Nevertheless there is a way to set a bi-invariant order on some special subgroups of an unrestricted wreath product (see proof of part (d) of lemma 3.5). The books [KK74, MR77] are good sources of information about orderable groups.

3 The first construction of torsion free WM groups

Our first construction gives a possibility to embed a group into a simple group in the way that many group properties (first of all properties to be torsion free and amenable) can be preserved. This construction uses ideas of [Ne49, NN59] and [H74] and some of our statements are simplified versions of statement that can be found in these papers. We present the proof for the reader’s convenience. We begin with the simplest way of getting examples that answer in affirmative Question 1. The corresponding groups are elementary amenable as they are locally solvable groups.

Recall that a group $G$ is called perfect if $G = G'$, where $G' = [G,G]$ is a commutator subgroup. The derived series $G^{(s)}$ is defined inductively by the rule $G^{(0)} = G, G^{(1)} = G' = [G,G]$ and $G^{(s+1)} = [G^{(s)},G^{(s)}]$. A group is called locally indicable if every its finitely generated nontrivial subgroup is indicable, i.e. can be mapped homomorphically onto an infinite cyclic groups.

**Theorem 3.1.** Let $\mathcal{P}$ be any combination of the following group properties: to be amenable or elementary amenable, or sub-exponentially amenable, or torsion free, or locally indicable, or orderable, or right orderable group. Assume that $G$ is a countable group with the property $\mathcal{P}$. Then $G$ embeds in a countable perfect group with the property $\mathcal{P}$. 
Corollary 3.2. There is an infinite countable, locally solvable, orderable, perfect, WM group.

Remark 1. Observe that the group property to be orderable implies the property to be locally indicable (Corollary 2, section 2.2 in [KK74]). Also right orderable amenable groups are locally indicable [Mo06]. In fact, local indicability of involved groups can be seen directly from the construction if we start with indicable group and proceed as in the proof of Lemma 3.5.

The next result provides more advanced examples.

Theorem 3.3. Every countable group $A$ with the property $\mathcal{P}$ introduced in Theorem 3.1 embeds in a countable simple group $H$ with the property $\mathcal{P}$.

Corollary 3.4. There exists a countable amenable, simple, orderable, WM group. Moreover, such examples exist in the class $EG$ of elementary amenable groups, in the class $SG \setminus EG$ of sub-exponentially amenable but not elementary amenable groups, and in the class $AG \setminus SG$ of amenable but not subexponentially amenable groups.

The next lemma is the key argument in proving theorem 3.1 and it also serves as the starting point for the construction leading to the simple groups mentioned in Theorem 3.3.

Lemma 3.5. Let $G$ be a countable group. There exists a countable group $H$ and an embedding of $G$ into $H$ such that $G \leq [H, H]$, and the following properties hold.

a) If $G$ is torsion-free, then $H$ is torsion-free, and if $G$ is locally indicable, then $H$ is locally indicable.

b) If $G$ is solvable of derived length $s$, then $H$ is solvable of derived length $\leq s + 1$.

c) If $G$ is amenable, then $H$ is amenable of the same type amenability as $G$ is.

d) If $G$ is (right) ordered, then $H$ is also (right) orderable with an order extending the order on $G$.

Proof. Take $H_0$ to be the unrestricted wreath product of $G$ and $\mathbb{Z}$, $H_0 = G^\mathbb{Z} \rtimes \mathbb{Z}$.

For $g \in G$, define $\delta_g \in G^\mathbb{Z}$ by the rule $\delta_g(0) = g$, $\delta_g(n) = 1$ if $n \neq 0$. Let $G \to H_0$ be the inclusion $g \mapsto \delta_g$.

Define also $f_g(n) = g$ if $n \leq 0$ and $f_g(n) = 1$ if $n > 0$. Let $\sigma$, be the generator of the active group $\mathbb{Z}$ acting on $G^\mathbb{Z}$ as the shift to the left, that is for $n \in \mathbb{Z}$, $\sigma(f)(n) = f(n + 1)$. Then $[f_g, \sigma] = f_g \sigma f_g^{-1} \sigma^{-1} = f_g \sigma(f_g^{-1}) = \delta_g$. Now take $H$ to be the subgroup of $H_0$ generated by $\sigma$ and all $f_g$, $g \in G$. It is clear that $G \leq [H, H]$.

a) $G$ is torsion free if and only if $H_0$ is torsion free and thus $H \leq H_0$ is torsion free. If $H$ is not a subgroup of the base group $G^\mathbb{Z}$ of wreath product then $H$ surjects onto nontrivial subgroup of $\mathbb{Z}$. If $H \leq G^\mathbb{Z}$ and $H$ is finitely generated and nontrivial then its projection on some factor $G$ in the product $G^\mathbb{Z}$ is finitely generated and nontrivial and therefore $H$ can be mapped onto $\mathbb{Z}$.
b) Suppose $G^{(s)} = 1$. Since $H_0 \leq G^Z$ we have $H_0^{(s+1)} = 1$, thus $H^{(s+1)} = 1$.

c) Suppose a subgroup $U \leq H \cap G^Z$ is generated by a finite number of elements of the form $f_g^s, i \in \mathbb{Z}$. Since $f_g : Z \to G$ has only two different values, the set $Z$ is a disjoint union of finitely many subsets $Z_i = Z_i(U)$ such that every function from the generating set of $U$, and therefore every function from $U$, is constant on each $Z_i$. Therefore $U$ is embeddable into a product of finitely many copies of $G$.

Hence $U$ is amenable, and so is $H \cap G^Z$ as an ascending union of such finitely generated subgroups. Finally, the group $H$ is amenable being an extension of $H \cap G^Z$ by a cyclic group. If $G$ is elementary amenable or belongs to one of the classes $SG \setminus EG$, $AG \setminus SG$, then the product of finitely many copies of $G$ is also elementary amenable or belongs to $SG \setminus EG$ or to $AG \setminus SG$, respectively. As $H \cap G^Z$ surjects onto $G$, we conclude that $H$ is in the same class of amenability as $G$.

d) Assume that a right or two-sided order $<$ on $G$ is given. Recall that such order is determined by a semigroup (called also a cone) of positive elements (that is elements $> 1$). To extend the order $<$ to $H$, we will define a cone $U$ of positive elements in $H$. Namely $(f, m) \in U$, where $f \in G^Z$, $m \in \mathbb{Z}$, $(f, m) \in H$, if $m > 0$ or $m = 0, f \neq 1$, and the last nontrivial value of the function $f$ is positive in $G$ (i.e., $f(i) > 1$ in $G$ for some $i$, and $f(j) = 1$ for all $j > i$). The last nontrivial value of $f$ exists as every function from $H^Z$ has trivial values $f(k)$ starting with some integer $k_0$.) The set $U$ of such positive pairs is a subsemigroup of $H$, and it is easy to see that $U \cup U^{-1} = H \setminus \{1\}$ and $U \cap U^{-1} = \emptyset$ for $U^{-1} = \{u^{-1} \mid u \in U\}$. It follows that $U$ is the cone of positive elements defining the total right order on $H$ by the rule: $h_1 > h_2$ iff $h_1 h_2^{-1} \in U$. This order extends the right order on $G$. The cone $U$ is invariant under conjugations by all $h \in H$ (it suffices to check for $h = (1, n)$ and $h = (f, 0)$) if the order on $G$ is two-sided. So $U$ defines a two-sided order on $H$ in this case, as desired.

\[\square\]

**Proof of Theorem 3.1.** Construct a sequence $G_0 < G_1 < \ldots$, where $G_0 = G$ and $G_{i+1}$ is obtained from $G_i$ according to Lemma 3.5 Let $G$ be the union of $\{G_i\}$. If $g \in G$ then $g \in G_j \leq G_{j+1} \leq G'$ for some $j$. Thus $G' = G$. The property $\mathcal{P}$ holds for every $G_i$ by the inductive application of Lemma 3.5. Hence the group $G$ is a $\mathcal{P}$-group too.

\[\square\]

**Proof of Corollary 3.2.** Let us start with $G_0 = \mathbb{Z}$ with the canonical order or with any orderable countable, solvable group (for instance, finitely generated free solvable group of derived length $s \geq 1$). Then every group $G_i$ from the above construction is solvable by Lemma 3.5, and so the union $G = \bigcup_{i=0}^{\infty} G_i$ is locally solvable. It is orderable and perfect by Theorem 3.1.

Since $G = G'$, $G$ cannot have nontrivial abelian quotients, and since $G$ is locally solvable, any finite quotient $K$ of $G$ is solvable which together with $K = K'$ implies that $K = \{1\}$.

\[\square\]
Proof of Theorem 3.3. Now we are going to use a restricted wreath product of groups \( A \) and \( B \), which is denoted by \( A \wr B \). We will identify the group \( A \) with the subgroup of \( A \wr B = (\times_B A) \rtimes B \), \( a \mapsto \delta_a \in \times_B A \), \( \delta_a(1) = a \), \( \delta_a(b) = 1 \) if \( b \neq 1 \). We will use the following

Lemma 3.6. Let \( G = A \wr B \), \( N \) a normal subgroup of \( G \) containing a nontrivial element \( b \) from \( B \). Then \( N \) contains \( A' \).

Proof. Let \( x, y \in A \). Then \( b \neq 1 \) implies that \( x \) and \( byb^{-1} \) commute. Also, \( y = byb^{-1} \) mod \( N \). Thus \( xy = yx \) mod \( N \), or equivalently \( [x, y] \in N \).

Given a group \( A \), we denote by \( \bar{A} \) the group containing \( A \) and constructed according to Lemma 3.5. Denote by \( A_i \), \( i \geq 0 \) the copies of \( \bar{A} \). Define \( W_0 = A_0, W_{i+1} = W_i \wr A_{i+1} \). Identify \( W_i \) with a subgroup of \( W_{i+1} \) as suggested above. Define \( W = \cup_{i=0}^\infty W_i \).

The isomorphisms \( \phi_i : A_i \to A_{i+1} \) \( (i = 0, 1, \ldots) \) induce an endomorphism \( \phi \) of \( W \). (Here we use the property that if \( X \leq Y \) then \( X \) and \( Z \) generate a subgroup in \( Y \wr Z \) canonically isomorphic to \( X \wr Z \).)

Therefore we can define the descending HNN extension \( C \) of \( W \) using an extra generator \( t \), where \( t\omega t^{-1} = \phi(\omega) \) for every \( \omega \in W \).

The normal closure \( N \) of any \( 1 \neq a \in A_0 \) in \( C \) must contain \( A'_0 \). Indeed, \( N \) contains \( b = tat^{-1} \in A_1 \) and so it contains \( A'_0 \) by Lemma 3.6. Recall that \( A_0 \) is a copy of \( \bar{A} \), and so by Lemma 3.6 we can derive that the normal closure of any nontrivial \( a \in A \) in \( C \) contains \( A \), where \( A \) is identified with a subgroup of \( A_0 \) and so with a subgroup of \( C \).

Let us denote the operation of getting the group \( C \) from \( A \) by \( \theta \), so \( \theta = \theta(A) \). We then have the ascending series \( \theta(A) < \theta(\theta(A)) < \ldots \). Let \( H = \cup_{i=0}^\infty \theta^i(A) \). Then \( H \) is a simple group. Indeed, take nontrivial \( a \in \theta^i(A) \). Then the normal closure \( N \) of \( a \) in \( \theta^{i+1}(A) \) and hence in \( H \) contains \( \theta^i(A) \) as in the previous paragraph. Similarly, since \( a \in \theta^{i+1}(A) \), it contains \( \theta^{i+1}(A) \) as well, and so on. Thus \( N = H \). It remains to check that the group \( H \) has the property \( \mathcal{P} \).

Note that the groups \( A_i \) \( (i = 0, 1, \ldots) \) inherit this property by Lemma 3.5. The property \( \mathcal{P} \) holds for the group \( W \) as well. Indeed, the property to be locally indicable is closed under subgroups, Cartesian products, group extensions and direct unions. Also each of the groups \( W_i \) is (right) orderable if \( A_0 \) is, and the order on \( W_{i+1} \) extends the order on \( W_i \) since these properties hold for a wreath product of (right) orderable groups (see Proposition 4 in section 1.1 of \([\text{KK74}]\) or define the order as we did in the proof of Lemma 3.5). So \( W \) is orderable if \( A \) is.

The group \( C \) is a semidirect product of the group \( \bar{W} = \cup_{i=0}^\infty t^{-i} W t^i \) and the infinite cyclic group \( \langle t \rangle \). We have by induction that each of the groups \( t^{-i} W t^i \) has an order extending the order on its subgroup \( t^{-i+1} W t^{i-1} \) by the property of the endomorphism \( \phi \). Hence the group \( \bar{W} \) has an order extending the order on \( A \). Finally the order on \( C \) extending this order is given by the rule \( t^m w > 1 \) for \( w \in \bar{W} \) if \( m > 0 \) or \( m = 0, w > 1 \). The reader can easily check that we have a positive cone indeed. The order is two-sided if one starts with a two-sided order on \( A \) and is one-sided if one starts with a one-sided order on \( A \). (Use that the endomorphism \( \phi \) preserves the order to check that the cone is invariant under conjugation in the former case.)
So the group $\theta(A)$ has an order extending the order of $A$ if $A$ is (right) orderable. Then this order similarly extends to the orders of $\theta(\theta(A)), \ldots, H$. The other parts of the property $P$ extend from $A_0$ to $H$ by the standard argument. The theorem is proved.

**Proof of Corollary 3.4.** If we apply Theorem 3.3 to $A = 1$, then we will have an elementary amenable orderable group $H$. Since it is infinite and simple, it is a $WM$ group by Proposition 2.1 (3).

Let now $\mathcal{G}$ be any of 3-generated 2-groups of intermediate (between polynomial and exponential) growth constructed by the first author in [Gr84]. It belongs to $SG \setminus EG$ by the well known fact that groups of subexponential growth are amenable and the result from Ch80 showing that there is no groups of intermediate growth in the class $EG$. We present it in the form $F/N$, where $F$ is a free group of rank 3. It is known that $\mathcal{G}$ is a residually (finite 2-group), and therefore the intersection of all the derived subgroups $G^{(i)}$ is trivial. Hence the group $A = F/N''$ is orderable (see Corollary 2 on page 109 of [KK74]). We have $A \in SG$ since the class $SG$ is closed under extensions and $A \notin EG$, since the homomorphic image $G$ is not in $EG$. Then by Theorem 3.3, $H \in SG$ but $H \notin EG$ since $A$ is a subgroup of $H$. Hence $H$ is the required example.

Finally, we will use the Basilica group $\mathcal{B}$ that was constructed in [GZ02]. It is 3-generated residually finite-2 group, amenable [BV05] but not subexponentially amenable [GZ02]. Therefore if we take $G = \mathcal{B}$ in the argument of the previous paragraph, then we obtain the desired example $H \in AG \setminus SG$.

**Remark 2.** It is worth to note that the Basilica group $\mathcal{B}$ is right orderable itself. To explain this we should use some facts from [GZ02] and the terminology from [BGS03]. Proposition 2 and Lemma 7 from [GZ02] show that $\mathcal{B}$ is weakly regular branch over its commutator subgroup $\mathcal{B}'$, and the relation $\mathcal{B}' = (\mathcal{B}' \times \mathcal{B}') \rtimes <c>$ holds, where $c$ is one of the generators, and therefore $\mathcal{B}'/(\mathcal{B}' \times \mathcal{B}')$ is isomorphic to an infinite cyclic group while $\mathcal{B}'/\mathcal{B}' \simeq \mathbb{Z}^2$. It follows that $\mathcal{B}$ contains a descending sequence of normal subgroups $\{H_n\}$, $n = 0, 1, 2, \ldots$ with trivial intersection, where $H_0 = \mathcal{B}, H_1 = \mathcal{B}', H_n$ is isomorphic to the direct product of $2^{n-1}$ copies of $\mathcal{B}'$ and $H_0/H_1 \simeq \mathbb{Z}^2, H_n/H_{n+1} \simeq \mathbb{Z}^{2^{n-1}}, n \geq 1$. As the quotients $H_n/H_{n+1}$ are torsion-free abelian, by the result of Zaiceva (Proposition 1, Section 5.4 in [KK74]), $\mathcal{B}$ is right orderable.

Observe that so far it is not known if the group $\mathcal{B}$ and the torsion-free group of intermediate growth $\mathcal{G}$ constructed by the first author in [Gr85] (and later observed to be right orderable [GM93]) are orderable.

**Question 4.** Does there exist an orderable group of intermediate growth?

## 4 The second approach to WM groups

The following lemma is well known (see [H55]) in the case when $G$ is a free group. The same proof works in the following version of it.
Lemma 4.1. Let $G$ be a group such that for any subgroup $H \leq G$ the abelianization $H/H'$ is a torsion-free group. Then for any normal subgroup $N \triangleleft G$, $G/N'$ is a torsion-free group.

Proof. Let $a \in G$ and $H = \langle a, N \rangle = \langle a \rangle N$. It suffices to show that $H/N'$ is torsion-free. Consider the exact sequence

$$1 \to H'/N' \to H/N' \to H/H' \to 1.$$ 

We have that $H' \leq N$, since $H = \langle a \rangle N$. So $H'/N' \leq N/N'$, and hence $H'/N'$ is free abelian. So $H/N'$ is an extension of a torsion-free group by a torsion-free group, and thus, it is torsion-free.

Further we need the following result, which is interesting by its own right. To the reader not familiar with the notion of variety of groups, we suggest, instead of arbitrary variety, to think on the variety of abelian groups, replacing in the statement and the proof the notation of verbal subgroup $\mathcal{V}(G)$ by the notation of derived subgroup $G'$ (only this special case will be used later).

Recall that if we have a set of words in a countable group alphabet, then the corresponding variety is the class of all groups which have these words $w$ as left-hand sides of the identical relations $w = 1$ (or laws). A proper variety is a variety which is not equal to the class of all groups. If $\mathcal{V}$ is a variety and $G$ is a group, $\mathcal{V}(G)$ means the subgroup of $G$ generated by all values of the words from the corresponding set in $G$ ($\mathcal{V}(G)$ is called a verbal subgroup of $G$). Note that $\mathcal{V}(G)$ is normal and moreover fully characteristic in $G$ and $G/\mathcal{V}(G) \in \mathcal{V}$.

We prove the following theorem:

Theorem 4.2. Let $F$ be a non-cyclic free group and $N$ a normal subgroup of $F$. Let $\mathcal{V}$ be a proper variety of groups. Then the group $\mathcal{V}(F)/\mathcal{V}(N)$ has a non-trivial quotient in $\mathcal{V}$ if and only if $F/N$ has.

Proof. Define the variety $\mathcal{V}^2$ by the rule $\mathcal{V}^2(G) = \mathcal{V}(\mathcal{V}(G))$.

Suppose $\mathcal{V}(F)/\mathcal{V}(N)$ surjects onto a non-trivial group $Q \in \mathcal{V}$. It follows that

$$\mathcal{V}(\mathcal{V}(F)/\mathcal{V}(N)) = \mathcal{V}^2(\mathcal{V}(N))/\mathcal{V}(N) \neq \mathcal{V}(F)/\mathcal{V}(N).$$

So $\mathcal{V}^2(F)/\mathcal{V}(N)$ is properly contained in $\mathcal{V}(F)$. Thus we have that $F/\mathcal{V}^2(F)\mathcal{V}(N) \notin \mathcal{V}$.

To obtain a contradiction, suppose $F/N$ has no non-trivial $\mathcal{V}$ quotients. Note that $F/N$ factors onto $F/\mathcal{V}^2(F)N \in \mathcal{V}$. It follows that $F/\mathcal{V}^2(F)N = 1$, so $F = \mathcal{V}^2(F)N$. Therefore we have the surjection

$$N/\mathcal{V}(N) \to \mathcal{V}^2(F)N/\mathcal{V}^2(F)\mathcal{V}(N) = F/\mathcal{V}^2(F)\mathcal{V}(N).$$

Since $N/\mathcal{V}(N) \in \mathcal{V}$, we must have $F/\mathcal{V}^2(F)\mathcal{V}(N) \in \mathcal{V}$, a contradiction.

Assume now that $F/N$ has a non-trivial quotient $G \in \mathcal{V}$; that is we have a normal subgroup $M \supseteq N$ with $F/M = G$. Then $H = \mathcal{V}(F)/\mathcal{V}(M) \in \mathcal{V}$ since $\mathcal{V}(F) \leq M$. The group $H$ is non-trivial by Theorem 43.41 in [N67], since $M \neq F$ and the variety $\mathcal{V}$ is proper. Since $\mathcal{V}(N) \leq \mathcal{V}(M)$, it follows that $H$ is a quotient of $\mathcal{V}(F)/\mathcal{V}(N)$ as desired.

Note that only the second part of the proof uses that $F$ is non-cyclic and free (and that $\mathcal{V}$ is a proper variety).
Theorem 4.3. Let $F$ be a non-abelian free group of at most countable rank and $N \triangleleft F$ a normal subgroup.

1. If $F/N$ is amenable, then $F'/N'$ is torsion-free amenable of the same type of amenability as $F/N$.

2. If $F/N$ is amenable WM group then $F'/N'$ is a countable torsion-free, amenable, WM group.

Proof. Consider the exact sequence

$$1 \to N/N' \to F/N' \to F/N \to 1.$$ \hfill (1)

$F'/N'$ is a subgroup of the extension of the amenable group $N/N'$ by the abelian group $F/N$, and so it is also amenable if $F/N$ is amenable. It follows from Lemma 4.1 that the group $F'/N'$ is torsion-free. If $F/N$ is elementary amenable, then $F/N'$ and hence $F'/N'$ are elementary amenable. If $F/N$ belongs to the class $SG \setminus EG$ then $F/N'$ also belongs to this class and hence $F'/N'$ belongs to $SG \setminus EG$ as $F'/N'/F'/N' = F/F'$ is abelian. Finally, if $F/N$ belongs to the class $AG \setminus SG$ then the same argument shows that $F'/N' \in AG \setminus SG$. This proves the first statement.

To prove the second statement, we assume that $F/N$ is an amenable WM group. By the part (1), of this theorem and by Proposition 2.1 (3), it suffices to prove that $F'/N'$ does not admit any non-trivial finite or abelian quotients. Note that abelian groups form a proper variety, so by Theorem 4.2 if $F'/N'$ has a non-trivial abelian quotient, then $F/N$ has a non-trivial abelian quotient, a contradiction with Proposition 2.1 (1).

Suppose $F'/N'$ has a non-trivial finite quotient $H$. We may assume in addition that $H$ is simple and non-abelian. The subgroup $F' \cap N/N' \leq N/N'$ is normal and abelian, and so $H$ is in fact a quotient of $F'/F' \cap N$, which is isomorphic to $F'N/N$, the commutator subgroup of $F/N$. Since $F/N$ is WM, $F/N = (F/N)'$, so $H$ is a factor of $F/N$, a contradiction with Proposition 2.1 (1).

To prove the second statement, it remains to show that $F'/N'$ is non-trivial. Suppose instead that $F' = N'$. Then $N$ contains $F'$ so $F/N$ is abelian. Since it is also a WM group, we obtain a contradiction with Proposition 2.1 (1).

Remark 3. Suppose that $F$ is a non-abelian free group, and $N$ a normal subgroup in $F$. If $F/N$ has a non-trivial finite quotient, then $F'/N'$ also has a non-trivial finite quotient. Indeed we then have that there is a normal subgroup $N \leq R < F$ such that $F/R$ is finite. It follows that $F/R'$ is virtually free abelian group, and hence is residually finite. Thus $F'/R' \leq F/R'$ is also residually finite. Moreover $F'/R'$ is non-trivial by the Auslander-Lyndon result ([AL55, Corollary 1.2]), since $F \neq R$. It is remains to observe that $F'/R'$ is a homomorphic image of $F'/N'$ since $N' \leq R'$.

The same conclusion is true if in the above statement, one replaces the variety of abelian groups by any proper variety $\mathcal{V}$ (i.e. to replace the commutator subgroup $N'$ by $\mathcal{V}(N)$ (just use P.Neumann’s theorem 43.41 from [N67] instead of Auslander-Lyndon’s theorem). Also,
the class of finite groups can be replaced by any star class defined by K. Gruenberg in [G57] if this class is closed under homomorphic images. Gruenberg’s star property of an abstract class Π of groups is defined as follows:

Class Π of groups has the star property if

1. Π is closed under taking subgroups and direct products of two groups from Π,
2. If A is a normal subgroup of B, A is residually Π-group and B/A ∈ Π, then B is a residually Π-group.

For instance, the classes of all finite groups, finite p-groups, solvable groups are star classes. Some results about star classes and residual properties of groups of the form $F/V(\langle N, N \rangle)$ were obtained by Baumslag, Dunwoody and Andreev-Olshanskii and they are published in [B63, D65, AO68].

By Theorem 4.3, in order to construct a countable torsion-free, amenable, WM group, it suffices to construct a countable amenable, WM group $G$: simply present $G$ as $G = F/N$, then $F'/N'$ answers Bergelson question. There are some examples below.

Example 1. Let $\text{Alt}_{\text{fin}}(\mathbb{N})$ be the group of all finitely-supported even permutations of the natural numbers. This group is locally finite and therefore amenable. Because it is also simple, Corollary 2.3 implies that it is a WM group. So if $\text{Alt}_{\text{fin}}(\mathbb{N}) = F/N$ then by Theorem 4.3, $F'/N'$ is a countable torsion-free, amenable, WM group.

Example 2. Let $T$ be a minimal homeomorphism of the Cantor set $C$, i.e. a homeomorphism such that the orbit $\{T^ix \mid i \in \mathbb{Z}\}$ is dense in $C$ for every $x \in C$. Define its full topological group $[[T]]$ as the group of those homeomorphisms $g$ of $C$, such that there exists a closed and open partition $C = \bigcup_{s=1}^{n} C_s$ with the property that the restriction of $g$ to any $C_s$ coincides with some power $T^{k_s(g)}$ of $T$, where $k_s(g)$ is some integer (see [M06] or [JM13]). Let $[[T]]'$ be the commutator subgroup. By [M06], $[[T]]'$ is a countably infinite simple amenable group, which is finitely generated in the case $(T, C)$ is a subshift over a finite alphabet (see [LM95]). So if $F/N = [[T]]'$, then Theorem 4.3 implies that $F'/N'$ is a countable torsion-free, amenable, WM group.

Proposition 4.4. The group from the first example is an elementary amenable group, while the group from the second example is amenable but not elementary amenable.

Proof. Let $G = F/N$ be an amenable group. If $G$ is an elementary amenable group then $F'/N'$ is also an elementary amenable group. Indeed by (1) we have that $F'/N'$ is an extension of the abelian group $(F' \cap N)/N'$ by the subgroup $F'N/N \leq F/N$. Note that $\text{Alt}_{\text{fin}}(\mathbb{N})$ is elementary amenable (since it is a direct limit of finite groups), this shows that the group in the example 1 is elementary amenable.

In the second example, $G = F/N$ is a simple non-abelian group and thus, we have $F'N/N = F/N$. Hence $F'/N'$ factors onto $G$. Because $G$ is not elementary amenable, this implies that $F'/N'$ is also not elementary amenable.

The next statement gives important information about the subgroups of $F/[N, N]$. 

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Proposition 4.5. Let $H$ be a non-free subgroup of $F/[N,N]$. Then the intersection $H \cap N/[N,N]$ is nontrivial and therefore $H$ has a nontrivial normal free abelian subgroup.

Proof. Let $M = N/[N,N]$ and assume that $H \cap M$ is trivial. Then $HM$ is a semidirect product, and we have an exact sequence

$$1 \rightarrow M \rightarrow HM \rightarrow H \rightarrow 1$$

But $HM$ is a subgroup of $F/[N,N]$ of the form $P/[N,N]$ for some $P, N \leq P < F$, so we have an exact sequence

$$1 \rightarrow M \rightarrow P/[N,N] \rightarrow H \rightarrow 1$$

with $P$ free and $N < P$. Let $\gamma : P \rightarrow H$ be defined as $\gamma = \beta \alpha$, where $\alpha : P \rightarrow P/[N,N]$ is the canonical projection. Observe that

$$H \cong P/[N,N]/N/[N,N] \cong P/N$$

and therefore $\text{Ker} \gamma = N$.

We are going to show that for any $H$–module $A$ the second cohomology group $H^2(H, A)$ vanishes. This will imply that $H$ has cohomological dimension 1 and hence by Stallings-Swan famous result [S68, S69], $H$ would be a free group, a contradiction.

So assume that for some groups $A$ and $G$ with $A$ abelian, we have a short exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$$

Then there are homomorphisms $\psi : F \rightarrow A$ and $\varphi : P \rightarrow G$ making the diagram

commutative. Indeed, let $B = \{b_1, b_2, \ldots \}$ be a basis of $P$. Then $\{\gamma(b_j)\}$ generate $H$. For each $j$ fix a preimage $g_j \in \pi^{-1}(\gamma(b_j))$ and define $\varphi(b_j) = g_j$. This defines $\varphi$. Similarly one can construct $\psi$ using that $P$ is a free group.

Since $A$ is abelian, $\psi([N,N]) = 1$. Therefore there is a homomorphism $\xi : M \rightarrow A$ making the following diagram commutative:

$$1 \rightarrow M \rightarrow P/[N,N] \rightarrow H \rightarrow 1$$

Now if $\mu : H \rightarrow P/N'$ is a splitting homomorphism for the top row, i.e. $\beta \mu = id$, then $\varphi \mu$ splits the bottom row, as required. \qed
5 Concluding remarks

Our next observation is not related to WM groups, but it is based on the use of groups of the type $F'/N'$ and therefore we decided to include it in this note.

A crystallographic group $G$ is a discrete group of isometries of $n$-dimensional Euclidean space with a bounded fundamental domain. By a theorem of Bieberbach, it can be reformulated purely in terms of group theory, which we will use as the definition, because it is more suitable to our goals.

**Definition 1.** A group $G$, which contains a normal free abelian subgroup $N$ of finite rank having finite index in $G$, and such that the centralizer $C_G(N)$ coincides with $N$, is called *crystallographic*.

Recall that $C_G(N)$, the centralizer of $N$ in $G$, is defined as the group of those $g \in G$ which commute with every element of $N$. If $N$ is abelian, then clearly $N \leq C_G(N)$, so it is the reverse inclusion that matters.

**Proposition 5.1.** If $F$ is a finitely generated free group and $|F/N| < \infty$, then all the subgroups of the group $F/N'$ (for example, our group $F'/N'$) are crystallographic.

This proposition immediately follows from the following

**Lemma 5.2.** Let $G$ be a finitely generated torsion free group, which is virtually abelian. Then $G$ is crystallographic.

**Proof.** It follows from our assumptions that there exists a maximal normal abelian subgroup $H$ having finite index in $G$. Since $G$ is finitely generated and torsion free, $H$ is a free abelian of finite rank. Suppose $C_G(H) \neq H$. The center of $C_G(H)$ has finite index in $C_G(H)$ since it contains $H$. Therefore by well known Shur’s Theorem $C_G(H)'$ is finite and thus, it is trivial since $G$ is torsion free. Thus, $C_G(H)$ is abelian contrary to the choice of $H$. \hfill $\square$

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