Multivariate strong invariance principles in
Markov chain Monte Carlo

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Abstract

Strong invariance principles in Markov chain Monte Carlo are crucial to theoretically grounded output analysis. Using the wide-sense regenerative nature of the process, we obtain explicit bounds on the almost sure convergence rates for partial sums of multivariate ergodic Markov chains. Further, we present results on the existence of strong invariance principles for both polynomially and geometrically ergodic Markov chains without requiring a 1-step minorization condition. Our tight and explicit rates have a direct impact on output analysis, as it allows the verification of important conditions in the strong consistency of certain variance estimators.

1 Introduction

Markov chain Monte Carlo (MCMC) is the workhorse computational algorithm for Bayesian inference. In MCMC, given an intractable target distribu-

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tion—typically a multi-dimensional Bayesian posterior—an ergodic Markov chain is constructed such that its stationary distribution is the desired target distribution. The increasing complexity of modern data and modeling strategies warrants the need for ensuring quality assessment of MCMC output. Consistent estimation of the variance of MCMC estimators often requires the assumption of a strong invariance principle (SIP) on the underlying Markov chain (Chan, 2022; Pengel and Bierkens, 2021; Vats et al., 2018).

Utilizing wide-sense regenerative properties of ergodic Markov chains, we present conditions for when a multivariate strong invariance principle holds under polynomial and geometric ergodicity. Our rates are explicit and the tightest known in this framework. We further explain how the explicit rates have a direct consequence on practical variance estimation in MCMC.

Let \( \{X_t\}_{t \geq 1} \) be a \( d \)-dimensional, \( \pi \)-stationary stochastic process on a measurable space \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) and denote the sequence of partial sum with \( S_n = \sum_{t=1}^{n} X_t \). Let \( \| \cdot \| \) denote the Euclidean norm, \( L \) be a \( d \times d \) positive-definite matrix, and \( \kappa : \mathbb{N} \to \mathbb{R}^+ \) be an increasing function. A multivariate SIP holds if on a suitably rich probability space one can construct \( \{X_t\}_{t \geq 1} \) along with a \( d \)-dimensional Wiener process \( \{W(t) : t \geq 0\} \) so that as \( n \to \infty \),

\[
\|S_n - \mathbb{E}(S_n) - LW(n)\| = \mathcal{O}(\kappa(n)) \quad \text{with probability 1}.
\]

The rate \( \kappa(n) \) often depends on the moments and the amount of correlation in the process. For independent and identically distributed (iid) univariate processes exhibiting a moment generating function, Komlós et al. (1975, 1976) (KMT) obtained the rate \( \kappa(n) = \log(n) \); if \( X_1 \) has \( r \) moments for \( r > 3 \), they obtain rate \( \kappa(n) = n^{1/r} \). These are the tightest rates possible. For correlated sequences, Philipp and Stout (1975) collate the various known SIP rates for \( \phi \)-mixing, \( \alpha \)-mixing, non-stationary, and regenerative processes. The rates obtained are of the form \( \kappa(n) = n^{1/2-\lambda} \) where \( \lambda \) is known.

The situation is quite different in the multivariate case. Although, for iid vectors Einmahl (1989) extend the KMT result, the proof techniques
used in the univariate case do not, in general, yield explicit rates in the multidimensional one (Monrad and Philipp, 1991). For general \( \phi \)-mixing processes, Berkes and Philipp (1979); Eberlein (1986); Kuelbs and Philipp (1980) obtain a rate of \( n^{1/2-\lambda} \) for some unknown \( 0 < \lambda < 1/2 \). Making certain parametric assumptions on the process and building on the work of Wu (2007), Liu and Lin (2009); Berkes et al. (2014) obtain explicit near-optimal and optimal rates; these assumptions are appropriate in applications such as time-series, but cannot be made for the Markov chains employed in MCMC. Markov chains have thus been given special focus.

For the univariate case, Jones et al. (2006) obtain explicit rates for uniformly ergodic Markov chains. For polynomially ergodic univariate continuous-time Markov processes with \( r \) moments, Pengel and Bierkens (2021) obtain rate \( \max\{n^{1/4} \log(n), n^{1/r} \log^2(n)\} \). If the Markov chain exhibits a 1-step minorization, it is classically regenerative. Let \( p \) denote the moments of the regeneration time and let the partial sum of a regenerative tour exhibit \( 2 + \delta \) finite moments; then Csáki and Csörgő (1995) obtain rate \( n^{\max\{1/(2+\delta),1/p,1/4\}} \log(n) \). This work forms the basis of obtaining explicit rates for Markov chains with Jones et al. (2006) obtaining an expression for geometrically ergodic Markov chains and Merlevède and Rio (2015) obtaining the KMT result for geometrically ergodic bounded Markov chains.

For univariate Markov chains exhibiting an \( l \)-step minorization, Samur (2004) obtain a rate of \( n^{1/2-\lambda} \) for some unknown \( 0 < \lambda < 1/2 \). The best known result under this setting is that of Dong and Glynn (2019) and Zhu (2020), who obtain rate \( n^{\max\{1/(2+\delta),1/p,1/4\}} \log(n) \) for univariate and multivariate processes, respectively, under moment assumptions on the regeneration time.

The rates obtained by Dong and Glynn (2019) and Zhu (2020) fall short of that of Csáki and Csörgő (1995). In Section 2.2, we obtain the Csáki and Csörgő (1995) rate under a general \( l \)-step minorization of a multidimensional Markov chain. Further, we show that the moment assumptions required for this re-
result hold for geometrically and polynomially ergodic Markov chains. The
tightness and tractability of the bound has a direct impact on consistent
estimation of MCMC variances, as we highlight in Section 3.1.

In addition to the almost sure convergence results, \( l \)-step minorization
of a Harris ergodic Markov chain produces a regenerative structu re of the
underlying Markov chain. This wide-sense regeneration structure yields a re-
generative estimator with desirable asymptotic properties. A brief discussion
on the regenerative estimators is provided in Section 3.2.

2 Definitions and main result

2.1 Markov chains and wide-sense regeneration

Consider a \( \pi \)-Harris ergodic Markov chain with a one-step Markov transition
kernel

\[
P(x, A) := \Pr(X_{k+1} \in A \mid X_k = x) \quad \text{for } k \geq 1, x \in \mathcal{X}, \text{ and } A \in \mathcal{B}(\mathcal{X}).
\]

Similarly the \( n \)-step Markov transitional kernel is

\[
P^n(x, A) := \Pr(X_{k+n} \in A \mid X_k = x) \quad \text{for } k \geq 1, x \in \mathcal{X}, \text{ and } A \in \mathcal{B}(\mathcal{X}).
\]

Our SIP results will apply to Markov chains exhibiting certain rates of
convergence measured via the total variation distance:

\[
\|P^n(x, \cdot) - \pi(\cdot)\|_{TV} := \sup_{A \in \mathcal{B}(\mathcal{X})} |P^n(x, A) - \pi(A)| \leq M(x)G(n),
\]

where \( G(n) \to 0 \) as \( n \to \infty \) and \( 0 < \mathbb{E}_\pi[M(X)] < \infty \); here we use notation
\( \mathbb{E}_F \) to denote expectations when \( X_1 \sim F \). If \( G(n) = n^{-k} \) for some \( k \geq 1 \),
the Markov chain is polynomially ergodic of order \( k \). If \( G(n) = t^n \) for some
\( 0 \leq t < 1 \), the Markov chain is geometrically ergodic.
Markov chains employed in MCMC are typically \( \pi \)-Harris ergodic and are thus wide-sense regenerative (Glynn, 2011) since they exhibit an \( l \)-step minorization (Athreya and Ney, 1978). An \( l \)-step minorization for \( l \geq 1 \) holds if there exists \( h : \mathcal{X} \to [0, 1] \) with \( 0 < E_x[h(X)] < \infty \) and a probability measure \( Q(\cdot) \) such that for all \( x \in \mathcal{X}, A \in \mathcal{B}(\mathcal{X}) \)

\[
P_l(x, A) \geq h(x)Q(A).
\]

Equation (2) allows the following representation of the \( l \)-step transition kernel:

\[
P_l(x, A) = h(x)Q(A) + (1 - h(x))R(x, A),
\]

where \( R(x, \cdot) \) is the residual distribution. By virtue of (3), an alternative sampling strategy is possible using an augmented Markov chain, \( \{ (X_i^*, \delta_i) \}_{i \geq 1} \) where \( \delta \)'s are binary variables. Consider \( X_1^* \sim Q \) and \( \delta_1 \sim \text{Bernoulli}(h(X_1^*)) \) for \( i \geq 1 \). For \( i \geq 2 \), if \( \delta_i = 1 \), \( X_{i+t}^* \sim Q \), else \( X_{i+t}^* \sim R(X_i^*, \cdot) \). This alternative sampling strategy is a way of generating a probabilistically similar random process to the original Markov chain with initial kernel \( Q(\cdot) \). Every such time-point \( i \), so that \( \delta_i = 1 \) is known as a regeneration time. Denote the \( k \)th regeneration time as \( T_k \), with \( T_0 = 0 \). Let \( \tau_k := T_k - T_{k-1} \) be the time to the \( k \)th regeneration from the \( (k-1) \)th regeneration. Denote \( \mu := E_Q(\tau_1) \).

Let \( f : \mathcal{X} \to \mathbb{R}^d \) for \( d \geq 1 \) be a function whose expectation under \( \pi \) is of interest. For \( k \geq 1 \), define the sum of a tour as \( Z_k := \sum_{t=T_{k-1}+1}^{T_k} f(X_t) \), and denote \( \eta := E_Q(Z_1) \). When \( l = 1 \), the Markov chain is classically regenerative and \( (Z_k, \tau_k)_{k \geq 1} \) are iid. For the one-dimensional case, this feature is exploited by Csáki and Csörgő (1995) to arrive at an SIP using classical KMT results. These results have been adapted to MCMC by Jones et al. (2006). However, for many MCMC samplers, \( l = 1 \) is a limiting assumption. Since all Harris ergodic Markov chains satisfy an \( l \)-step minorization for some \( l \) (see Athreya and Ney, 1978, for e.g.), the assumption of an \( l \)-step minorization is no longer limiting. However, for general \( l \), Glynn (2011) discuss that
$(Z_k, \tau_k)_{k \geq 1}$ is a one-dependent stationary process, and thus the classical KMT results can no longer be utilized to establish an SIP.

### 2.2 Main result

We now present our main results establishing an SIP. Define

$$
\Sigma_Z := \text{Var}_Q \left( Z_1 - \frac{\tau_1}{\mu} \eta \right) + \text{Cov}_Q \left( Z_1 - \frac{\tau_1}{\mu} \eta, Z_2 - \frac{\tau_2}{\mu} \eta \right) + \text{Cov}_Q \left( Z_2 - \frac{\tau_2}{\mu} \eta, Z_1 - \frac{\tau_1}{\mu} \eta \right).
$$

(4)

**Theorem 1.** Let $\{X_t\}_{t \geq 1}$ be a $\pi$-Harris ergodic Markov chain and thus (2) holds. Suppose

(a) $E_Q(\tau_1^p) < \infty$ for some $p > 1$ and,

(b) for some $\delta > 0$

$$
E_\pi \left[ \left( \sum_{t=1}^{\tau_1} \left\| f(X_t) - \frac{\eta}{\mu} \right\| \right)^{2+\delta} \right] < \infty.
$$

(5)

Then, on a suitably rich probability space, one can construct $\{X_t\}_{t \geq 1}$ together with a $d$-dimensional standard Wiener process $\{W(t) : t \geq 0\}$ such that for $\beta = \max\{1/(2 + \delta), 1/2p, 1/4\}$, as $n \to \infty$

$$
\left\| \sum_{t=1}^{n} f(X_t) - nE_\pi[f(X)] - \frac{\Sigma_1^{1/2} W(n)}{\sqrt{\mu}} \right\| = O(n^\beta \log(n)) \quad \text{with probability 1}.
$$

(6)

**Proof.** See Appendix [3]

**Remark 1.** As far as we are aware Theorem [1] is the first result we know that matches the Csáki and Csörgő (1995) SIP rate for general multivari-
ate functionals. Additionally, similar to Dong and Glynn (2019), we remove the assumption of a 1-step minorization. Under the same assumptions, Dong and Glynn (2019); Zhu (2020) obtain the rate with \( \beta = \max\{1/(2 + \delta), 1/p, 1/4\} \), hence our rates are tighter.

For practical application to MCMC, it is important to assess when and for what values of \( p \) is \( \mathbb{E}_Q(\tau^p_1) < \infty \). For a 1-step minorization, Hobert et al. (2002) show that \( \tau_1 \) has a moment-generating function when \( \{X_t\}_{t \geq 1} \) is geometrically ergodic. The next two lemmas are critical to obtaining moment conditions over \( \tau \) for polynomially and geometrically ergodic Markov chains. Additionally, Lemma 3 and Lemma 4 aid in proving the moment existence of regenerative sums. Proofs of the following lemmas are in Appendix C.

**Lemma 1.** Let \( \{X_t\}_{t \geq 1} \) be a \( \pi \)-stationary polynomially ergodic Markov chain of order \( \xi > (2 + \delta)(1 + (2 + \delta)/\delta^*) \) for some \( \delta > 0 \) and \( \delta^* > 0 \). Then for all \( p \in (0, \xi) \), \( \mathbb{E}_Q[\tau^p_1] < \infty \).

**Lemma 2.** Let \( \{X_t\}_{t \geq 1} \) be a \( \pi \)-stationary geometrically ergodic Markov chain. Then \( \mathbb{E}_Q[\tau^p_1] < \infty \) for any \( p > 1 \).

**Lemma 3.** Let \( \{X_t\}_{t \geq 1} \) be a \( \pi \)-Harris ergodic Markov chain so that (2) holds. Let \( \mathbb{E}_\pi(\|f(X)\|^{p+\delta*}) < \infty \) for some \( p > 1 \) and \( \delta^* > 0 \) and \( \mathbb{E}_\pi(\tau^\phi_1) < \infty \) for \( \phi > p(p + \delta^*)/\delta^* \). Then,

\[
\mathbb{E}_\pi \left( \left( \sum_{i=1}^{\tau_1} \|f(X_i)\| \right)^p \right) < \infty.
\]

**Lemma 4.** Let \( \{X_t\}_{t \geq 1} \) be a \( \pi \)-Harris ergodic Markov chain so that (2) holds. Further, let \( \{X_n\}_{n \geq 1} \) be geometrically ergodic and \( \mathbb{E}_\pi(\|f(X)\|^{p+\delta*}) < \infty \) for some \( p > 1 \) and \( \delta^* > 0 \). Then

\[
\mathbb{E}_\pi \left( \left( \sum_{i=1}^{\tau_1} \|f(X_i)\| \right)^p \right) < \infty.
\]
The above lemmas allow the following result.

**Theorem 2.** Let \( \{X_t\}_{t \geq 1} \) be a \( \pi \)-Harris ergodic Markov chain and thus (2) holds. Suppose the chain is either

(a) polynomially ergodic of order \( \xi > (2 + \delta)(1 + (2 + \delta)/\delta^*) \) for some \( \delta > 0 \) and \( \delta^* > 0 \) and \( \mathbb{E}_\pi \left( \|f(X)\|^{2+\delta+\delta^*} \right) < \infty \); or,

(b) geometrically ergodic and \( \mathbb{E}_\pi \left( \|f(X)\|^{2+\delta+\delta^*} \right) < \infty \) for some \( \delta > 0 \) and \( \delta^* > 0 \),

then (6) holds with \( \beta = \max \{1/(2 + \delta), 1/4\} \).

**Proof.** See Appendix C.

**Remark 2.** Theorem 2 presents reasonable and verifiable conditions for the existence of an SIP. Similar conditions (slightly weaker in the case of polynomial ergodicity) are also sufficient for the existence of a CLT (see Jones, 2004). Theorem 2 marks a three-fold improvement over the existing results of Jones et al. (2006) in MCMC; (i) the assumption of a 1-step minorization is completely removed, (ii) the inclusion of an explicit result for polynomially ergodic Markov chains, and (iii) the critical extension to multivariate functionals.

**Remark 3.** For any Harris ergodic Markov chain, an \( l \)-step minorization always holds for some \( l \geq 1 \). Hence wide-sense regeneration exists for the underlying Markov chain. Consequently the one-dependent random sequences \( \{(Z_t, \tau_t)\}_{t \geq 1} \) can be constructed for any \( l \geq 2 \). Different \( l \) yields different covariance structure \( \Sigma_Z \) and different \( \mu \) in such a way that the quantity \( \Sigma_Z/\mu \) remains same. This allows for a larger class of representations of the asymptotic covariance matrix. Detecting wide-sense regenerations in real applications can be quite challenging. Nevertheless, in Section 2.1 and Appendix D, we present its theoretical framework.
3 Application to MCMC variance estimation

3.1 Batch means estimator

Having generated the process \( \{X_t\}_{t=1}^n \) through an MCMC algorithm, the samples are employed to estimate \( \mathbb{E}_\pi[f(X)] \) via the Monte Carlo average since,

\[
\hat{f}_n := \frac{1}{n} \sum_{t=1}^{n} f(X_t) \xrightarrow{a.s.} \mathbb{E}_\pi[f(X)] \quad \text{as } n \to \infty.
\]

When a Markov chain CLT holds for \( \hat{f}_n \), there exists a positive-definite matrix \( \Sigma_f \) such that as \( n \to \infty \),

\[
\sqrt{n} \left( \hat{f}_n - \mathbb{E}_\pi[f(X)] \right) \xrightarrow{d} N(0, \Sigma_f), \tag{7}
\]

where

\[
\Sigma_f = \text{Var}_\pi[f(X)] + \sum_{s=1}^\infty \left[ \text{Cov}_\pi(f(X_1), f(X_{1+s})) + \text{Cov}_\pi(f(X_1), f(X_{1+s}))^\top \right].
\]

See Jones (2004) for sufficient conditions for a Markov chain CLT to hold. We note that Theorem 2 also implies a Markov chain CLT and thus \( \Sigma_f = \Sigma_Z/\mu. \)

Estimation of \( \Sigma_f \) is widely discussed, both in the univariate case (Berg and Song, 2022; Chakraborty et al., 2022; Damerdji, 1991; Geyer, 1992; Damerdji, 1995; Jones et al., 2006; Flegal and Jones, 2010), and the multivariate case (Kosorok, 2000; Dai and Jones, 2017; Seila, 1982; Vats et al., 2018, 2019; Vats and Flegal, 2021). Estimators of \( \Sigma_f \) are employed in deciding when to stop the simulation. Thus, the MCMC simulation stops at a random time and Glynn and Whitt (1992) show that validity of the subsequent estimators require strong consistency of estimators of \( \Sigma_f \). Much effort has thus gone into ensuring that estimators of \( \Sigma_f \) are strongly consistent. One particular estimator that stands out due its computational efficiency and theoretical underpinnings, is the batch-means estimator.
An existence of a multivariate SIP is assumed for the strong consistency of the batch-means estimators (Vats et al., 2019). Vats et al. (2018) showed that if the Markov chain is polynomially ergodic, then for some unknown $0 < \lambda < 1/2$ a multivariate SIP holds with rate $n^{1/2-\lambda}$. Strong consistency also depends on choosing appropriate values for the tuning parameter, the batch size. These values depend on the rate $\lambda$, which is unknown, thus making it difficult to verify the conditions. Theorem 2 overcomes this issue considerably as we will now elucidate.

Let the Monte Carlo sample size $n = a_nb_n$ where $a_n$ denotes the number of batches and $b_n$ is the batch size. For $k = 1, \ldots, a_n$ define the mean vector of the $k^{th}$ batch as, $\bar{f}_k = (\frac{b_n^{-1} \sum_{t=(k-1)b_n+1}^{kb_n} f(X_t)}{b_n})$. Then the batch-means estimator of $\Sigma_f$ is

$$\hat{\Sigma}_{BM} = \frac{b_n}{a_n-1} \sum_{k=1}^{a_n} (\bar{f}_k - \hat{f}_n)(\bar{f}_k - \hat{f}_n)^\top.$$  

**Assumption 1.** The batch size $b_n$ is such that

(a) $b_n \to \infty$ and $n/b_n \to \infty$ as $n \to \infty$ where, $b_n$ and $n/b_n$ are monotonically increasing,

(b) there exists a constant $c \geq 1$ such that $\sum_n (b_n n^{-1})^c < \infty$.

Often $b_n = \lfloor n^\nu \rfloor$ for some $\nu > 0$ so that Assumption 1 is trivially satisfied. Common choices in the literature are $b_n = \lfloor n^{1/3} \rfloor$ and $b_n = \lfloor n^{1/2} \rfloor$. The following theorem from Vats et al. (2019) presents the conditions for strong consistency of the batch-means estimator.

**Theorem 3** (Vats et al. (2019)). Suppose $f$ is such that $E_{\pi} (\|f(X)\|^{2+\delta}) < \infty$ for some $\delta > 0$ and let the Markov chain be polynomially ergodic of order $\xi > (1 + \epsilon)(1 + 2/\delta)$ for some $\epsilon > 0$. If $b_n$ satisfies Assumption 1 and $b_n^{-1} \log(n) \kappa^2(n) \to 0$ as $n \to \infty$, then $\hat{\Sigma}_{BM} \to \Sigma$ with probability 1 as $n \to \infty$. 

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For the univariate case, Jones et al. (2006) assumed the same conditions on the batch size. For geometrically ergodic Markov chains under a 1-step minorization, Jones et al. (2006) showed a univariate SIP holds with $\kappa(n) = n^\beta \log n$, where $\beta$ is the same as Theorem 2. This implies that the batch size should be chosen such that $\nu > \max\{2/(2 + \delta), 1/2\}$. For geometrically ergodic Markov chains, this is the best known batch size condition. We note, as did Damerdji (1991, 1995), that this condition excludes common choices of $b_n$.

For the multivariate case, the only known MCMC SIP result was that of Vats et al. (2018) with $\kappa(n) = n^{1/2 - \lambda}$ for some unknown $0 < \lambda < 1/2$. This implies a choice of $\nu > 1 - 2\lambda$ but since $\lambda$ is unknown, the condition cannot be verified. On the other hand, using Theorem 2 we obtain the condition that $\nu > \max\{2/(2 + \delta), 1/2\}$. This is the same condition as the univariate case, but now only requiring polynomial ergodicity.

3.2 Regenerative estimator

Regenerations, particularly wide-sense regenerations, are notoriously difficult to identify. Nevertheless, it is natural to consider a wide-sense regenerative estimator of $E_\pi[f(X)]$ and $\Sigma_f$, even if it is for theoretical exposition. For 1-step minorization, Hobert et al. (2002); Seila (1982) present regenerative estimators, and for univariate wide-sense regenerative processes, Henderson and Glynn (2001) provide an estimator of $E_\pi[f(X)]$.

Denote the number of regenerations by $R$. By a strong law of large numbers for 1-dependent processes, $T_R/R \xrightarrow{a.s.} \mu$ as $R \to \infty$. Further as $R \to \infty$, by Lemma 5 in Appendix A

$$\tilde{f}_R := \frac{1}{T_R} \sum_{t=1}^{T_R} f(X_t) = \frac{\sum_{j=1}^{R} Z_j}{\sum_{j=1}^{R} \tau_j} \xrightarrow{a.s.} \frac{\eta}{\mu} = E_\pi[f(X)]. \tag{8}$$
By using Slutsky’s theorem,

\[ \sqrt{R} \left( \hat{f}_R - E_\pi[f(X)] \right) = \sqrt{\frac{R}{T}} \left( \frac{\sum_{j=1}^{R} Z_j}{R} - E_\pi[f(X)] \right) \]

\[ = \frac{R}{T} \frac{1}{\sqrt{R}} \left( \sum_{j=1}^{R} Z_j - T R E_\pi[f(X)] \right) \]

\[ = \frac{R}{T} \frac{1}{\sqrt{R}} \sum_{j=1}^{R} (Z_j - \tau_j E_\pi[f(X)]) \]

\[ \overset{d}{\rightarrow} N_d(0, \Sigma_f/\mu). \] (9)

Thus, if regenerations can be identified, the regenerative estimator in (8) can be used in place of \( \hat{f}_n \). Regenerative estimators of \( E_\pi[f(X)] \) from 1-step minorization have been employed by Mykland et al. (1995); Roy and Hobert (2007); Chen and Botev (2018).

For an observed Markov chain of length \( n \), the estimation of the asymptotic variance \( \Sigma_f \) in (9) is done by estimating \( \Sigma_Z \) as described in (4) and \( \mu \) separately. Define \( \bar{Z} := R^{-1} \sum_{i=1}^{R} Z_i \). We can estimate \( \mu \) and \( \Sigma_Z \) with,

\[ \hat{\mu} := \frac{1}{R} \sum_{i=1}^{R} \tau_i, \quad \text{and} \]

\[ \hat{\Sigma}_Z := \frac{1}{R} \sum_{i=1}^{R} (Z_i - \bar{Z})(Z_i - \bar{Z})^T + \frac{1}{R} \sum_{i=1}^{R-1} (Z_i - \bar{Z})(Z_{i+1} - \bar{Z})^T \]

\[ + \frac{1}{R} \sum_{i=1}^{R-1} (Z_{i+1} - \bar{Z})(Z_i - \bar{Z})^T \] (11)

Using (10) and (11), the estimator for \( \Sigma_f \) is \( \hat{\Sigma}_f = \hat{\Sigma}_Z / \hat{\mu} \).

Employing these estimators in practice has two significant challenges: (i) detecting wide-sense regenerations can be difficult since one requires knowledge of \( l \) and the minorization kernel \( Q \), and (ii) when minorizations can be established, the bounds in the minorization constant are fairly weak, yielding
prohibitively large regeneration times. In Appendix D, we present a general technique of identifying wide-sense regenerations for random scan Gibbs samplers via the Gibbs sampler of Albert and Chib (1993). As it turns out, the practical implementation fails the theoretical framework since close to no regenerations are identified even in long runs of the chain. This is predominantly due to the weak bound obtained in the minorization constant. Given the framework presented in this section, it would be worthwhile pursuing improvements on this minorization constant in the future.

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A Some preliminary results

Lemma 5. Let \( \{X_t\}_{t \geq 1} \) be a \( \pi \)-Harris ergodic Markov chain. Recall \( f, \eta \) and \( \mu \) from Section 2.1. Then,

\[
E_\pi[f(X)] = \frac{\eta}{\mu}.
\]

Proof. By Glynn (2011), \( \{(Z_k, \tau_k) : k \geq 1\} \) form a stationary 1-dependent process. By a strong law of large numbers for 1-dependent processes,

\[
\frac{1}{R} \sum_{i=1}^{R} Z_i \xrightarrow{a.s.} E_Q(Z_1) \text{ as } R \to \infty \quad \text{and,} \quad (12)
\]

\[
\frac{1}{R} \sum_{i=1}^{R} \tau_i \xrightarrow{a.s.} E_Q(\tau_1) \text{ as } R \to \infty. \quad (13)
\]
By a strong law for ergodic Markov chains and from (13),

\[
\frac{1}{T_R} \sum_{t=1}^{T_R} f(X_t) = \frac{1}{1/R} \sum_{i=1}^{R} \frac{Z_i}{\tau_i} \xrightarrow{a.s.} E_\pi[f(X)] \text{ as } R \to \infty
\]

\[
\Rightarrow \left(1/R \sum_{i=1}^{R} \tau_i \right) \frac{1}{1/R} \sum_{i=1}^{R} \frac{Z_i}{\tau_i} \xrightarrow{a.s.} E_Q(\tau_1) E_\pi[f(X)] \text{ as } R \to \infty
\]

\[
\Rightarrow \left(1/R \sum_{i=1}^{R} Z_i \right) \xrightarrow{a.s.} E_Q(\tau_1) E_\pi[f(X)] \text{ as } R \to \infty.
\] (14)

Thus, by (12) and (14),

\[
E_Q(Z_1) = E_Q(\tau_1) E_\pi[f(X)] \Rightarrow E_\pi[f(X)] = \frac{\eta}{\mu}.
\] (15)

The following lemma will be employed for the proof of Theorem and is an extension of Hobert et al. (2002, Lemma 1) to the multivariate and l-step minorization case.

**Lemma 6.** Let \( \{X_t\}_{t \geq 1} \) be a \( \pi \)-Harris ergodic Markov chain so that (2) holds. Then, for any measurable function \( \Psi : \mathcal{X}^\infty \to \mathbb{R}^d \)

\[
E_\pi \|\Psi(X_1, X_2, X_3, \ldots)\| \geq E_\pi[h(X)] E_Q \|\Psi(X_1, X_2, X_3, \ldots)\|. \tag{16}
\]

**Proof.** For all \( A \in \mathcal{B}(\mathcal{X}) \) and \( x \in \mathcal{X} \),

\[
\pi(A) = (\pi P^l)(A) = \int_{\mathcal{X}} \pi(dx) P^l(x, A) \geq Q(A) \int_{\mathcal{X}} h(x) \pi(x) = Q(A) E_\pi[h(X)].
\] (17)

By taking conditional expectation over \( X_1 \)

\[
E_\pi \|\Psi(X_1, X_2, X_3, \ldots)\| = E_\pi \left( E\{\|\Psi(X_1, X_2, X_3, \ldots)\| \mid X_1\} \right).
\] (18)

Since \( E\{\|\Psi(X_1, X_2, X_3, \ldots)\| \mid X_1\} \) is a positive function of \( X_1 \), by (17)
and (18)

$$\mathbb{E}[\|\Psi(X_1, X_2, X_3, \ldots)\| = \int_{X} \mathbb{E}[\|\Psi(X_1, X_2, X_3, \ldots)\| \mid X_1 = x] \pi(dx)$$

$$\geq \mathbb{E}[h(X)] \int_{X} \mathbb{E}(\|\Psi(X_1, \ldots)\| \mid X_1 = x) Q(dx)$$

$$= \mathbb{E}[h(X)] \mathbb{E}_Q(\|\Psi(X_1, \ldots)\|)$$

$$\Rightarrow \mathbb{E}(\|\Psi(X_1, \ldots)\| \mid X_1 = x) \geq \mathbb{E}[h(X)] \mathbb{E}_Q(\|\Psi(X_1, \ldots)\||X_1 = x).$$

\[\square\]

### B Proof of Theorem 1

**Proof.** We will start by showing some moment properties of the sequence of regeneration times in order to prove strong convergence. Denote the number of regenerations in a sample of size $n$ by:

$$\xi(n) := \sup\{k \geq 1 : T_k \leq n\} = \inf\{k \geq 1 : T_{k+1} > n\}.$$

By Glynn (2011), \{(Z_k, \tau_k): k \geq 1\} is a stationary 1-dependent process. Further, by the conditions in the theorem, $\mathbb{E}_Q(\tau_1^p) < \infty$ for $p > 1$. Define $p' = \min\{2, p\}$ for $p > 1$. Thus, $\mathbb{E}_Q(\tau_1^{p'}) < \infty$. By a Marcinkiewicz-Zygmund strong law of large numbers for 1-dependent process (see Samur, 2004, for e.g.), for $1 < p' \leq 2$,

$$\left| \sum_{i=1}^{\xi(n)} \tau_i - \xi(n)\mu \right| = |T_{\xi(n)} - \xi(n)\mu| \xrightarrow{\text{a.s.}} o(\xi(n)^{1/p'}). \tag{19}$$

With $n < T_{\xi(n)+1}$, subtracting $\xi(n)\mu$ from both side

$$n - \xi(n)\mu < T_{\xi(n)+1} - (\xi(n) + 1)\mu + \mu.$$
Now, using (19), \( T_{\xi(n)} + 1 - (\xi(n) + 1) \mu \overset{a.s.}{=} \mathcal{O}(\xi(n) + 1)^{1/p'} \) and hence \( T_{\xi(n)} + 1 - (\xi(n) + 1) \mu + \mu \overset{a.s.}{=} \mathcal{O}(\xi(n) + 1)^{1/p'} \). Thus

\[
\begin{align*}
n - \xi(n) \mu & \overset{a.s.}{=} \mathcal{O}(\xi(n) + 1)^{1/p'} \\
\Rightarrow n - \xi(n) \mu & \overset{a.s.}{=} \mathcal{O}(n^{1/p'}) \Rightarrow \xi(n) \overset{a.s.}{=} n/\mu + \mathcal{O}(n^{1/p'}) \\
\Rightarrow |\xi(n) - n/\mu| & \overset{a.s.}{=} \mathcal{O}(n^{1/p'}) \quad (20)
\end{align*}
\]

Further, by the assumption in (5) and Lemma 6,

\[
\begin{align*}
\mathbb{E}_\pi \left[ \left( \sum_{t=1}^{\tau_1} \left\| f(X_t) - \frac{\eta}{\mu} \right\| \right)^{2+\delta} \right] & < \infty \\
\Rightarrow \mathbb{E}_Q \left[ \left( \sum_{t=1}^{\tau_1} \left\| f(X_t) - \frac{\eta}{\mu} \right\| \right)^{2+\delta} \right] & < \infty \\
\Rightarrow \mathbb{E}_Q \left[ \left\| Z_1 - \frac{\tau_1 \eta}{\mu} \right\|^{2+\delta} \right] & < \infty \\
\Rightarrow \left\| \mathbb{E}_Q \left( Z_1 - \frac{\tau_1 \eta}{\mu} \right) \right\|^{2+\delta} & < \infty , \quad (22)
\end{align*}
\]

where the last implication follows from Jensen’s inequality.

Note that \( \{(X_t, \delta_t)\}_{t \geq 1} \) follows all the properties of a process generated by the alternative sampling strategy as illustrated in Section 2.1. So, \( X_1 \sim Q \) and \( (X_2, \delta_2) \ldots, (X_{\tau_1+l-1}, \delta_{\tau_1+l-1}) \) are serially generated from the initial value. Define, \( S_k := \left( \sum_{i=1}^{\tau_k} X_i, \sum_{i=\tau_k+1}^{\tau_k+l-1} X_i, \delta_{\tau_k+1}, \ldots, \delta_{\tau_k+l-1} \right)^T \). By construction \( S_k \)'s are iid vectors (see Glynn, 2011, Section 2). Consequently, \( (Z_1, \tau_1) = g(S_1) \) for some measurable function \( g(\cdot) \). Again \( X_{\tau_1+l} \sim Q \) and independent to all previous elements in the chain; define \( (Z_2, \tau_2) = g(S_1, S_2) \). Hence, for all \( k \geq 1 \) we can say \( (Z_k, \tau_k) = g(S_{k-1}, S_k) \) where \( S_0 := 0 \). Also, from (5) it directly follows that \( \mathbb{E}_\pi \left( \| f(X) \|^{2+\delta} \right) < \infty \). Thus, Condition A of Liu and Lin (2009) is satisfied. By Liu and Lin (2009, Theorem 2.1) for the stationary 1-dependent process \( \{(Z_k - \tau_k \eta/\mu) : k \geq 1\} \) and \( \{W(t) : t > 0\} \), a
\(d\)-dimensional standard Wiener process,

\[
\begin{align*}
\sum_{k=1}^{\xi(n)} Z_k - T_{\xi(n)} \frac{\eta}{\mu} - \Sigma_{Z}^{1/2} W(\xi(n)) \overset{a.s.}{=} O(\xi(n)^{1/(2+\delta)}) \\
\Rightarrow \sum_{k=1}^{\xi(n)} Z_k - T_{\xi(n)} \frac{\eta}{\mu} - \Sigma_{Z}^{1/2} W(\xi(n)) \overset{a.s.}{=} O(\xi(n)^{1/(2+\delta)}).
\end{align*}
\]

(23)

By (20) and Csörgö and Révész (2014, Proposition 1.2.1),

\[
\| W(\xi(n)) - W(n/\mu) \| \overset{a.s.}{=} O(b_n),
\]

where for positive constants \(c\) and \(c'\),

\[
b_n = \left(2cn^{1/p'} \left(\log \left(\frac{n/\mu}{n^{1/p'}}\right) + \log \log (n/\mu)\right)\right)^{1/2} \\
= \left(2cn^{1/p'} \left(\log \left(\frac{n^{1-1/p'}}{\mu}\right) + \log \log (n/\mu)\right)\right)^{1/2} \\
< c'n^{1/(2p')} (\log n) .
\]

Consequently,

\[
\| W(\xi(n)) - W(n/\mu) \| \overset{a.s.}{=} O(n^{1/(2p')} \log n).
\]

(24)

Using triangle inequality and since \(T_{\xi(n)} < n < T_{\xi(n)+1}\),

\[
Y_{\xi(n)} := \sum_{i=T_{\xi(n)}+1}^{T_{\xi(n)+1}} \left\| f(X_i) - \frac{\eta}{\mu} \right\| \\
> \left\| \sum_{i=T_{\xi(n)}+1}^{n} \left( f(X_i) - \frac{\eta}{\mu} \right) \right\|.
\]

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\[
= \left| \sum_{i=1}^{n} \left( f(X_i) - \eta \right) - \xi(n) \sum_{k=1}^{\xi(n)} \left( Z_k - \tau_k \right) \right|.
\] (25)

By construction, \( \{Y_k\}_{k \geq 1} \) are positive and identical random variables generated from sum of absolute values of correlated units sampled through wide-sense regeneration. By the integral transformation inequality and the assumption in equation-(5)

\[
\sum_{i=1}^{\infty} \Pr \left( Y_i^{2+\delta} > i \right) = \sum_{i=1}^{\infty} \Pr \left( Y_1^{2+\delta} > i \right)
< \int_{1}^{\infty} \Pr \left( Y_1^{2+\delta} > x \right) dx
< \int_{0}^{\infty} \Pr \left( Y_1^{2+\delta} > x \right) dx
= \mathbb{E}_Q[Y_1^{2+\delta}]
< \infty.
\] (26)

Consequently,

\[
\sum_{i=1}^{\infty} \Pr \left( Y_i^{2+\delta} > i \right) < \infty \Rightarrow \sum_{i=1}^{\infty} \Pr \left( Y_i > i^{\frac{1}{2+\delta}} \right) < \infty.
\] (27)

Thus by Borel-Cantelli lemma

\[
Y_n \overset{a.s.}{=} \mathcal{O} \left( n^{\frac{1}{2+\delta}} \right) \quad \text{as } n \to \infty.
\] (27)

From (25) and (27) as \( n \to \infty \)

\[
Y_{\xi(n)} \overset{a.s.}{=} \mathcal{O} \left( \xi(n)^{\frac{1}{2+\delta}} \right)
\Rightarrow \left| \sum_{i=1}^{n} \left( f(X_i) - \eta \right) - \xi(n) \sum_{k=1}^{\xi(n)} \left( Z_k - \tau_k \right) \right| \overset{a.s.}{=} \mathcal{O} \left( n^{\frac{1}{2+\delta}} \right).
\] (28)
Using the triangle inequality and (23), (24), and (28)

\[
\left\| \sum_{i=1}^{n} f(X_i) - n \frac{\eta}{\mu} - \frac{\Sigma_{z}^{1/2}}{\sqrt{\mu}} W(n) \right\| < \left\| \sum_{i=1}^{n} \left( f(X_i) - \frac{\eta}{\mu} \right) - \sum_{k=1}^{\xi(n)} \left( Z_k - \tau_k \frac{\eta}{\mu} \right) \right\|
\]

\[
+ \left\| \sum_{k=1}^{\xi(n)} Z_k - T_{\xi(n)} \frac{\eta}{\mu} - \Sigma_{z}^{1/2} W(\xi(n)) \right\|
\]

\[
+ \left\| \Sigma_{z}^{1/2} \left( W(\xi(n)) - W \left( \frac{n}{\mu} \right) \right) \right\|;
\]

\[
\Rightarrow \left\| \sum_{i=1}^{n} f(X_i) - n \frac{\eta}{\mu} - \frac{\Sigma_{z}^{1/2}}{\sqrt{\mu}} W(n) \right\| = O(n^{1/(2+\delta)}) + O(n^{1/(2+\delta)})
\]

\[
+ O(n^{1/p} \log n)
\]

Thus, with \( \beta = \max \{1/(2 + \delta), 1/2p'\} = \max \{1/(2 + \delta), 1/2p, 1/4\} \) and by lemma 5 as \( n \to \infty \)

\[
\left\| \sum_{i=1}^{n} f(X_i) - n \frac{\eta}{\mu} - \frac{\Sigma_{z}^{1/2}}{\sqrt{\mu}} W(n) \right\| \overset{a.s.}{=} O(n^\beta \log n) \quad (29)
\]

\[
\Rightarrow \left\| \sum_{i=1}^{n} f(X_i) - nE_a[f(X)] - \frac{\Sigma_{z}^{1/2}}{\sqrt{\mu}} W(n) \right\| \overset{a.s.}{=} O(n^\beta \log n).
\]

\[\Box\]

C Proof of Theorem 2

Proof of Lemma 1. \( \{X_t\}_{t \geq 1} \) is a polynomially ergodic sequence of random variables of order \( \xi \) for \( \xi > (2 + \delta)(1 + (2 + \delta)/\delta^*) \). So in (11), \( G(n) = n^{-\xi} \). Further, from Jones (2004) we have that \( \alpha(n) \leq n^{-\xi} \) for \( n \geq 1 \). Consequently,
for $p < \xi$

$$\sum_{n=1}^{\infty} n^{p-1} \alpha(n) < \sum_{n=1}^{\infty} n^{p-1} n^{-\xi} < \infty.$$ 

By Samur (2004, Proposition 3.1)

$$E_{\pi}[\tau_1^p] < \infty \text{ for } p \in (0, \xi). \quad (30)$$

From Lemma 6 we have $E_Q[\tau_1^p] < \infty$ for $p \in (0, \xi)$.

**Proof of Lemma 2** Since $\{X_t\}_{t \geq 1}$ is geometrically ergodic, $G(n) = t^n$ for some $0 < t < 1$. From Jones (2004), $\alpha(n) \leq t^n$ for $n \geq 1$. Consequently for $p > 1$,

$$\sum_{n=1}^{\infty} n^{p-1} \alpha(n) < \sum_{n=1}^{\infty} n^{p-1} t^n.$$

By a ratio test, for all $p > 1$

$$\lim_{n \to \infty} \frac{(n+1)^{p-1} t^{n+1}}{n^{p-1} t^n} = \lim_{n \to \infty} (1 + 1/n)^{p-1} t = t < 1.$$

So,

$$\sum_{n=1}^{\infty} n^{p-1} \alpha(n) < \infty \text{ for } p > 1.$$

By Samur (2004, Proposition 3.1)

$$E_{\pi}[\tau_1^p] < \infty \text{ for } p > 1. \quad (31)$$

From Lemma 6 $E_Q[\tau_1^p] < \infty$ for $p > 1$.

**Proof of Lemma 3** For $p > 1$ and using triangle inequality for $L^p$-distances on $\pi$, Hölder’s inequality, Markov’s inequality, and, infinite sum of $p$-series,
\[
= \left( \mathbb{E}_\pi \left[ \left( \sum_{i=1}^{\infty} \mathbb{P}(i \leq \tau_1) \| f(X_i) \|^p \right) \right] \right)^{1/p}
\]
\[
\leq \sum_{i=1}^{\infty} \left( \mathbb{E}_\pi \mathbb{P}(i \leq \tau_1) \| f(X_i) \|^p )^{1/p}
\]
\[
\leq \sum_{i=1}^{\infty} \left( \mathbb{E}_\pi \mathbb{P}(i \leq \tau_1) \| f(X_i) \|^p \right)^{1/p}
\]
\[
\leq \sum_{i=1}^{\infty} \left( \mathbb{E}_\pi \mathbb{P}(i \leq \tau_1) \| f(X_i) \|^p \right)^{1/p}
\]
\[
\leq \left( \mathbb{E}_\pi (\| f(X) \|^{p+\delta^*}) \right)^{1/(p+\delta^*)} \sum_{i=1}^{\infty} \left( \Pr_\pi(\tau_1 \geq i) \right)^{\delta^*/(p+\delta^*)}
\]
\[
\leq \left( \mathbb{E}_\pi (\| f(X) \|^{p+\delta^*}) \right)^{1/(p+\delta^*)} \sum_{i=1}^{\infty} \left( \Pr_\pi(\tau_1^\phi \geq i^\phi) \right)^{\delta^*/(p+\delta^*)}
\]
\[
\leq \left( \mathbb{E}_\pi (\| f(X) \|^{p+\delta^*}) \right)^{1/(p+\delta^*)} \sum_{i=1}^{\infty} \left( \frac{\mathbb{E}_\pi(\tau_1^\phi)}{i^\phi} \right)^{\delta^*/(p+\delta^*)}
\]
\[
= \left( \mathbb{E}_\pi (\| f(X) \|^{p+\delta^*}) \right)^{1/(p+\delta^*)} \left( \mathbb{E}_\pi(\tau_1^\phi) \right)^{\delta^*/(p+\delta^*)} \sum_{i=1}^{\infty} \left( \frac{1}{i^\phi} \right)^{\delta^*/(p+\delta^*)} < \infty.
\]
(32)

Proof of Lemma 4. Since \( \{ X_t \}_{t \geq 1} \) is geometrically ergodic, by (31), \( \mathbb{E}_\pi[\tau_1^q] < \infty \) for \( q > 1 \). Proceeding similarly as Lemma 3, \( \mathbb{E}_\pi ((\sum_{i=1}^{\tau_1} \| f(X_i) \|)^p) < \infty. \)

Proof of Theorem 2. (a) \( \{ X_t \}_{t \geq 1} \) is polynomially ergodic of order \( \xi \) for \( \xi > (2 + \delta)(1 + (2 + \delta)/\delta^*) \). Thus (30) holds. Then by Lemma 4
\[
\mathbb{E}_Q[\tau_1^p] < \infty \quad \text{for } p \in (0, \xi).
\]
(33)
By the assumption in the theorem, \( \mathbb{E}_\pi (\| f(X) \|^{2+\delta+\delta^*}) < \infty \) for some \( \delta > 0 \).
and $\delta^* > 0$. By (30) and Lemma 3

$$E_\pi \left[ \left( \sum_{i=1}^{\tau_1} \| f(X_i) \| \right)^{2+\delta} \right] < \infty \Rightarrow E_\pi \left[ \left( \sum_{i=1}^{\tau_1} \| f(X_i) - \eta \| \right)^{2+\delta} \right] < \infty. \quad (34)$$

By Theorem 1, (33), and (34),

$$\left\| \sum_{i=1}^{n} f(X_i) - nE_\pi[f(X)] - \frac{\Sigma Z}{\sqrt{\mu}} W(n) \right\| \xrightarrow{a.s.} O(n^\beta \log(n)) \quad (35)$$

as $n \to \infty$ where $\beta = \max\{1/(2+\delta), 1/(2p), 1/4\}$ $\forall p \in (0, \xi)$. Since $\xi > 2$ always, $\beta = \max\{1/(2+\delta), 1/4\}$.

(b) Let $\{X_t\}_{t \geq 1}$ be geometrically ergodic. So (31) holds. By Lemma 2

$$E_Q[\tau_1^p] < \infty \forall p > 1. \quad (36)$$

Since $E_\pi (\| f(X) \|^{2+\delta+\delta^*}) < \infty$ for some $\delta > 0$ and $\delta^* > 0$, by (31) and Lemma 4

$$E_\pi \left[ \left( \sum_{i=1}^{\tau_1} \| f(X_i) \| \right)^{2+\delta} \right] < \infty \Rightarrow E_\pi \left[ \left( \sum_{i=1}^{\tau_1} \| f(X_i) - \eta \| \right)^{2+\delta} \right] < \infty. \quad (37)$$

By Theorem 1, (36), and (37), with $\beta = \max\{1/(2+\delta), 1/4\}$ as $n \to \infty$

$$\left\| \sum_{i=1}^{n} f(X_i) - nE_\pi[f(X)] - \frac{\Sigma Z}{\sqrt{\mu}} W(n) \right\| \xrightarrow{a.s.} O(n^\beta \log(n)). \quad (38)$$
D Identifying wide-sense regenerations

Establishing a 1-step minorization with the corresponding small set calculations has been done for deterministic scan Gibbs samplers (see Mykland et al. (1995), Roy and Hobert (2007)). Using a random scan version of the Gibbs sampler of Albert and Chib (1993), we present a framework for identifying wide-sense regenerations.

The setup is similar as purported in Roy and Hobert (2007). For $i = 1, 2, \ldots, n$, consider $Y_i \overset{iid}{\sim} \text{Bernoulli}(\Phi(x_i^T \beta))$ where $x_i \in \mathbb{R}^p$ are given and $\beta \in \mathbb{R}^p$ is the vector of coefficients. The resulting likelihood is:

$$\Pr(Y_1 = y_1, \ldots, Y_n = y_1 \mid \beta) = \prod_{i=1}^{n} \{\Phi(x_i^T \beta)^{y_i}\{1 - \Phi(x_i^T \beta)\}^{1-y_i}\}.$$

We consider a Bayesian model with a flat prior for $\beta$ for which the posterior reduces to

$$\pi(\beta \mid y) \propto \Pr(Y_1 = y_1, \ldots, Y_n = y_1 \mid \beta) \pi(\beta) = \prod_{i=1}^{n} \{\Phi(x_i^T \beta)^{y_i}\{1 - \Phi(x_i^T \beta)\}^{1-y_i}\}.$$

Although the analytical form of $\pi(\beta \mid y)$ is not easily achievable, Albert and Chib (1993) has provide a deterministic Gibbs sampler, which we refer to as the AC sampler. One iteration of the Markov chain is the following:

- draw $z = (z_1, z_2, \ldots, z_n)^T$ such that $z_i \overset{iid}{\sim} \text{Truncated Normal}(X^T \beta_{i-1}, 1, y_i)$ for all $i = 1, 2, \ldots, n$. If $y_i = 0$, $(-\infty, 0]$ will be the truncation range and if $y_i = 1$, $(0, \infty)$ will be the truncation range.
- Draw $\beta \sim N_p ((X^T X)^{-1}X^T z, (X^T X)^{-1})$.

For the $i^{th}$ iteration the deterministic scan Markov transition kernel is

$$k_{DS}(\beta_{i+1}, z_{i+1} \mid \beta_i, z_i) = \pi(z_{i+1} \mid \beta_{i+1}, y) \pi(\beta_{i+1} \mid z_i, y)$$
The deterministic scan AC sampler is geometrically ergodic (see Roy and Hobert, 2007, Theorem 1) and regenerations can be identified using “distinguished point” technique of Mykland et al. (1995). For a distinguished point $z^* \in \mathbb{R}^n$ and a rectangular small set $D^* = [c_1, d_1] \times [c_2, d_2] \times \ldots \times [c_p, d_p]$ the minorization kernel is

$$Q(\beta_{i+1}, z_{i+1}) := \frac{1}{\epsilon} \pi(\beta_{i+1} \mid z^*, y) \pi(z_{i+1} \mid \beta_{i+1}, y) I_{D^*}(\beta_{i+1});$$

where $\epsilon = \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} \pi(\beta_{i+1} \mid z^*, y) \pi(z_{i+1} \mid \beta_{i+1}, y) I_{D^*}(\beta_{i+1}) \, dz_{i+1} d\beta_{i+1}$. For $t_j$ being the $j^{th}$ term of $t = (z_i - z) = (z_i - z) \, T \, X$; the minorization constant will be

$$s(z_i) := \frac{\epsilon \exp \left( \sum_{j=1}^{p} (c_j t_j I_{R^+}(t_j) + d_j t_j I_{R^-}(t_j)) \right)}{\exp(0.5(z_i)^T X^T (X^T X)^{-1} X(z_i) - 0.5(z^*)^T X^T (X^T X)^{-1} X(z^*))}.$$ 

Consequently, the minorization holds with

$$k_{DS}(\beta_{i+1}, z_{i+1} \mid \beta_i, z_i) \geq s(z_i) \, Q(\beta_{i+1}, z_{i+1}). \tag{39}$$

For $\Delta$ being the Dirac measure, the one-step random scan AC-sampler kernel is defined as

$$k_{RS}(\beta_{i+1}, z_{i+1} \mid \beta_i, z_i) = p\pi(\beta_{i+1} \mid z_i) \Delta_{z_i}(z_{i+1}) + (1 - p)\pi(z_{i+1} \mid \beta_{i+1}) \Delta_{\beta_i}(\beta_{i+1}).$$

Similarly, the 2-step random scan Markov transition kernel is

$$k^2_{RS}(\beta_{i+2}, z_{i+2} \mid \beta_i, z_i)$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} k_{RS}(\beta_{i+2}, z_{i+2} \mid \beta_{i+1}, z_{i+1}) k_{RS}(\beta_{i+1}, z_{i+1} \mid \beta_i, z_i) \, dz_{i+1} d\beta_{i+1}$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} (p\pi(\beta_{i+2} \mid z_{i+1}) \Delta_{z_{i+1}}(z_{i+2}) + (1 - p)\pi(z_{i+2} \mid \beta_{i+1}) \Delta_{\beta_i}(\beta_{i+2}))$$

$$\quad \times (p\pi(\beta_{i+1} \mid z_i) \Delta_{z_i}(z_{i+1}) + (1 - p)\pi(z_{i+1} \mid \beta_i) \Delta_{\beta_i}(\beta_{i+1})) \, dz_{i+1} d\beta_{i+1}$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} p^2 \pi(\beta_{i+2} \mid z_{i+1}) \pi(\beta_{i+1} \mid z_i) \Delta_{z_i}(z_{i+1}) \Delta_{z_{i+1}}(z_{i+2}) \, dz_{i+1} d\beta_{i+1}$$

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Using the 1-step minorization of the deterministic scan Gibbs sampler and
so, the following way:

\[ k^2_{RS}(\beta_{i+2}, z_{i+2} \mid \beta_i, z_i) = p^2 \pi(\beta_{i+2} \mid z_{i+2}) \Delta_{z_i}(z_{i+2}) \]

\[ + p(1-p)\pi(\beta_{i+2} \mid z_{i+2}) \pi(z_{i+2} \mid \beta_i) \]

\[ + p(1-p)\pi(z_{i+2} \mid \beta_{i+2}) \pi(\beta_{i+2} \mid z_i) \]

\[ + (1-p)^2 \pi(z_{i+2} \mid \beta_i) \Delta_{\beta_i}(\beta_{i+2}). \]

Using the 1-step minorization of the deterministic scan Gibbs sampler and
with \( s'(z_i) := p(1-p)s(z_i) \), we get

\[ k^2_{RS}(\beta_{i+2}, z_{i+2} \mid \beta_i, z_i) = p^2 \pi(\beta_{i+2} \mid z_{i+2}) \Delta_{z_i}(z_{i+2}) \]

\[ + p(1-p)\pi(\beta_{i+2} \mid z_{i+2}) \pi(z_{i+2} \mid \beta_i) \]

\[ + p(1-p)\pi(z_{i+2} \mid \beta_{i+2}) \pi(\beta_{i+2} \mid z_i) \]

\[ + (1-p)^2 \pi(z_{i+2} \mid \beta_i) \Delta_{\beta_i}(\beta_{i+2}). \]

So,

\[ k^2_{RS}(\beta_{i+2}, z_{i+2} \mid \beta_i, z_i) = p^2 \pi(\beta_{i+2} \mid z_{i+2}) \Delta_{z_i}(z_{i+2}) \]

\[ + p(1-p)\pi(\beta_{i+2} \mid z_{i+2}) \pi(z_{i+2} \mid \beta_i) \]

\[ + p(1-p)\pi(z_{i+2} \mid \beta_{i+2}) \pi(\beta_{i+2} \mid z_i) \]

\[ + (1-p)^2 \pi(z_{i+2} \mid \beta_i) \Delta_{\beta_i}(\beta_{i+2}). \]

Using the 1-step minorization of the deterministic scan Gibbs sampler and
with \( s'(z_i) := p(1-p)s(z_i) \), we get

\[ k^2_{RS}(\beta_{i+2}, z_{i+2} \mid \beta_i, z_i) = p^2 \pi(\beta_{i+2} \mid z_{i+2}) \Delta_{z_i}(z_{i+2}) \]

\[ + p(1-p)\pi(\beta_{i+2} \mid z_{i+2}) \pi(z_{i+2} \mid \beta_i) \]

\[ + p(1-p)\pi(z_{i+2} \mid \beta_{i+2}) \pi(\beta_{i+2} \mid z_i) \]

\[ + (1-p)^2 \pi(z_{i+2} \mid \beta_i) \Delta_{\beta_i}(\beta_{i+2}). \]

Following the ideas in Mykland et al. (1995) and Roy and Hobert (2007),
we can obtain the probability of a regeneration from the observed chain in
the following way:

\[ \eta_i = \Pr(\delta_i = 1 \mid (\beta_i, z_i), (\beta_{i+2}, z_{i+2})); \]

\[ = \frac{\Pr(X_{i+2} = y \mid X_i = x, \delta_i = 1) \Pr(\delta_i = 1 \mid X_i = x) \Pr(X_i = x)}{\Pr(X_{i+2} = y \mid X_i = x) \Pr(X_i = x)}; \]

25
\begin{align*}
\Pr(X_{i+2} = y \mid X_i = x, \delta_i = 1) \Pr(\delta_i = 1 \mid X_i = x) \\
= \frac{s'(z_i) Q(\beta_{i+2}, z_{i+2})}{k_{RS}^2 (\beta_{i+2}, z_{i+2} \mid \beta_i, z_i)} \\
= p(1 - p) \frac{\exp(-0.5(z_i)^T X^T (X^T X)^{-1} X(z_i)))}{\exp(-0.5(z_s)^T X^T (X^T X)^{-1} X(z_s)))} \\
\times \exp \left\{ \sum_{i=1}^p (c_i t_i I_{R^+}(t_i) + d_i t_i I_{R^-}(t_i)) \right\} ; \\
\times \pi(\beta_{i+2} \mid z^*, y) \pi(z_{i+2} \mid \beta_{i+2}, y) I_{D^*}(\beta_{i+2}) ; \\
\times \frac{1}{k_{RS}^2 (\beta_{i+2}, z_{i+2} \mid \beta_i, z_i)}. \\
\end{align*}

The \( \eta_i \) can be calculated analytically from a run of the random scan Gibbs sampler. By drawing Bernoulli samples with success probability \( \eta_i \) we identify the regenerations if the outcome of a trial is 1.

For output analysis of the observed samples, the regenerative estimator of the asymptotic variance can be exploited as discussed in Section 3.2. However, the bounds obtained in the minorization are weak enough that \( \eta_i \)s are prohibitively small, yielding close to no regenerations in large simulation lengths.

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