NEW RESIDUAL-BASED A POSTERIORI ERROR ESTIMATORS FOR LOWEST-ORDER RAVIART-THOMAS ELEMENT APPROXIMATION TO CONVECTION-DIFFUSION-REACTION EQUATIONS

SHAOHONG DU† AND XIAOPING XIE‡

Abstract. A new technique of residual-type a posteriori error analysis is developed for the lowest-order Raviart-Thomas mixed finite element discretizations of convection-diffusion-reaction equations in two- or three-dimension. Both centered mixed scheme and upwind-weighted mixed scheme are considered. The a posteriori error estimators, derived for the stress variable error plus scalar displacement error in $L^2$-norm, can be directly computed with the solutions of the mixed schemes without any additional cost, and are robust with respect to the coefficients in the equations. Local efficiency dependent on local variations in coefficients is obtained without any saturation assumption, and holds from the cases where convection or reaction is not present to convection- or reaction-dominated problems. The main tools of analysis are the postprocessed approximation of scalar displacement, abstract error estimates, and the property of modified Oswald interpolation. Numerical experiments are reported to support our theoretical results and to show the competitive behavior of the proposed posteriori error estimates.

Key words. convection-diffusion-reaction equation, centered mixed scheme, upwind-weighted mixed scheme, postprocessed approximation, a posteriori error estimators

AMS subject classifications. 65N15, 65N30, 76S05

1. Introduction. Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2$ or 3. We consider the following homogeneous Dirichlet boundary value problem for the convection-diffusion-reaction equations:

$$
\begin{aligned}
\begin{cases}
-\nabla \cdot (S \nabla p) + \nabla \cdot (pw) + rp &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{aligned}
$$

where $S \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ denotes an inhomogeneous and anisotropic diffusion-dispersion tensor, $w$ is a (dominating) velocity field, $r$ a reaction function, $f$ a source term. The choice of boundary conditions is made for ease of presentation, since similar results are valid for other boundary conditions. This type of equations arise in many chemical and biological settings. For instance, in hydrology these equations govern the transport and degradation of adsorbing contaminants and microbe-nutrient systems in groundwater.

Reliable and efficient a posteriori error estimators are an indispensable tool for adaptive algorithms. For second-order elliptic problems without convection term, the theory of a posteriori error estimation has reached a degree of maturity for finite elements of conforming, nonconforming and mixed types; see [1-9, 11-14, 18, 20, 22-23, 27, 31-33] and the references therein. For convection-diffusion-(reaction) problems, on the contrary, the theory is still under development.

The mathematical analysis of robustness of a-posteriori estimators for the convection-diffusion-reaction equations was first addressed by Verfürth [35] in the singular perturbation case, namely $S = \varepsilon I$ with $I$ the identical matrix and $0 < \varepsilon \ll 1$. The proposed estimators for the standard Galerkin approximation and the SUPG discretization give global upper and local lower bounds on the error measured in the energy norm, and are robust when the Péclet
number becomes small. In [36] Verfürth improved the results of [35] in the sense that the derived estimates are fully robust with respect to convection dominance and uniform with respect to the size of the zero-order reaction term. Sangalli [39] developed an a posteriori estimator for the residual-free bubbles methods applied to convection-diffusion problems. Later he presented a residual-based a posteriori estimator for the one-dimensional convection-diffusion-reaction model problem [31]. In [22] Kunert carried out a posteriori error estimation for the SUPG approach to a singularly perturbed convection-diffusion problem on anisotropic meshes. One may also refer to [25, 26] for a posteriori error estimation in the framework of finite volume approximations.

For the convection-diffusion-reaction model (1.1), following an idea of postprocessing in [24] Vohralík [37] established residual a posteriori error estimates for lowest-order Raviart-Thomas mixed finite element discretizations on simplicial meshes. Global upper bounds and local lower bounds for the postprocessed approximation error, $p - \tilde{p}_h$, in the energy norm were derived with $\tilde{p}_h$ the postprocessed approximation to the finite element solution $p_h$, and the local efficiency of the estimators was shown to depend only on local variations in the coefficients and on the local Péclet number. Moreover, the developed general framework allows for asymptotic exactness and full robustness with respect to inhomogeneities and anisotropies.

In this paper, we develop a new technique for residual-based a posteriori estimation of the lowest-order Raviart-Thomas mixed finite element schemes (centered mixed scheme and upwind-mixed scheme) over both the stress error, $u - u_h$, and the displacement error, $p - p_h$, of the mixed finite element solutions $(u_h, p_h)$ for the problem (1.1) with $u := -S \nabla p$. The derived reliability results are robust with respect to the coefficients. Local efficiency depends only on local variations in the coefficients is obtained without any saturation assumption, holds for the convection or reaction dominated equations. Compared with the standard analysis to the diffusion equations, our analysis avoids, by using the postprocessed approximation $\tilde{p}_h$ as a transition, Helmholtz decomposition of stress variables and dual arguments of displacement error in $L^2$-norm, and then does not need any weak regularity assumption on the diffusion-dispersion tensor. We note that although being employed in our analysis, the postprocessed displacement approximation and its modified Oswald interpolation are not involved in our estimators.

The rest of this paper is organized as follows. In Section 2 we give notations, assumptions of data, and the weak problem. We introduce in Section 3 the mixed finite element schemes (include the centered and upwind-weighted mixed scheme) and the post-processed techniques. Section 4 includes the main results. Section 5 collects some preliminary results and remarks. Section 6 and 7 analyze respectively the a posteriori error estimates and the local efficiency. Finally, we present several numerical examples in Section 8 to test our estimators.

2. Notations, assumptions and weak problem. For a domain $A \subset \mathbb{R}^d$, we denote by $L^2(A)$ and $L^2(A) := (L^2(A))^d$ the spaces of square-integrable functions, by $\langle \cdot, \cdot \rangle_A$ the $L^2(A)$ or $L^2(A)$ inner product, by $|\cdot|_A$ the associated norm, and by $|A|$ the Lebesgue measure of $A$. Let $H^k(A)$ be the usual Sobolev space consisting of functions defined on $A$ with all derivatives of order up to $k$ square-integrable; $H^1_0(A) := \{ v \in H^1(A) : v|_{\partial A} = 0 \}$, $H(\text{div}, A) := \{ v \in L^2(A)^d : \text{div } v \in L^2(A) \}$. $\cdot >_{\partial A}$ denotes $d-1$-dimensional inner product on $\partial A$ for the duality paring between $H^{-1/2}(\partial A)$ and $H^{1/2}(\partial A)$.

Let $T_h$ be a shape regular triangulation in the sense of [15] which satisfies the angle condition, namely there exists a constant $c_0$ such that for all $K \in T_h$ with $h_K := \text{diam}(K)$,

$$c_0^{-1} h_K^d \leq |K| \leq c_0 h_K^d.$$

Let $C_Q, c_Q$ be positive constants dependent only on a quantity $Q$, and $c_i$ ($i = 1, 2, \cdots$) positive constants determined only by the shape regularity parameter, $c_0$, of $T_h$. We denote by
by

\[E_h \] the set of element sides in \( T_h \), by \( E^\text{int}_h \) and \( E^\text{ext}_h \) the sets of all interior and exterior sides of \( T_h \), respectively. For \( K \in T_h \), denote by \( E_K \) the set of sides of \( K \), especially by \( E^\text{ext}_K \) the set of the boundary sides of \( K \). Furthermore, we denote by \( \omega_\sigma \) and \( \tilde{\omega}_\sigma \) the union of all elements in \( T_h \) sharing a side \( \sigma \) and the union of all elements sharing at least one point of \( \sigma \), respectively. For an element \( K \in T_h \), the set \( \tilde{\omega}_K \) is defined analogously. We also use the "broken Sobolev space" \( H^1(\bigcup T_h) := \{ \varphi \in L^2(\Omega) : \varphi|_K \in H^1(K), \forall K \in T_h \} \), and denote by \( [v]|_\sigma := (v|_K)|_\sigma - (v|_L)|_\sigma \) the jumps of \( v \in H^1(\bigcup T_h) \) over an interior side \( \sigma := K \cap L \) of diameter \( h_\sigma := \text{diam}(\sigma) \), shared by the two neighboring (closed) elements \( K, L \in T_h \).

Especially, \( [v]|_\sigma := (v|_K)|_\sigma \) if \( \sigma \in \partial \Omega \).

We consider \( d = 2, 3 \) simultaneously and let \( m := 1 \) if \( d = 2 \) and \( m := 3 \) if \( d = 3 \). The Curl of a function \( \psi \in H^1(\Omega)^m \) is defined by

\[
\text{Curl}\psi := (-\partial \psi/\partial x_2, \partial \psi/\partial x_1) \quad \text{if} \quad d = 2 \quad \text{and} \quad \text{Curl}\psi := \nabla \times \psi \quad \text{if} \quad d = 3,
\]

where \( \times \) denotes the usual vector product of two vectors in \( \mathbb{R}^3 \). Given a unit normal vector \( \mathbf{n} = (n_1, n_2) \) along the side \( \sigma \), we define the tangential component of a vector \( \mathbf{v} \in \mathbb{R}^d \) by

\[
\gamma_{\kappa \sigma} (\mathbf{v}) := \begin{cases} 
\mathbf{v} \cdot (-n_2, n_1) & \text{if} \quad d = 2, \\
\mathbf{v} \times \mathbf{n} & \text{if} \quad d = 3.
\end{cases}
\]

We note that throughout the paper, the local versions of differential operators \( \nabla, \text{curl} \) are understood in the distribution sense, namely, \( \text{curl}_h : H^1(\bigcup T_h)^d \to L^2(\Omega) \) and \( \nabla_h : H^1(\bigcup T_h) \to L^2(\Omega)^d \) are defined with \( \text{curl}_h \varphi|_K := \text{curl}(\varphi|_K) \) and \( \nabla_h \psi|_K := \nabla(\psi|_K) \).

We need in our analysis the following inequalities. Poincaré inequality and Friedrichs inequalities \( [10,27] \): for \( K \in T_h \) and \( \varphi \in H^1(K) \),

\[
||\varphi - \varphi_K||_K^2 \leq C_{P,d} h^2_K ||\nabla \varphi||_K^2, \quad (2.1)
\]

\[
(\varphi_K - \varphi)^2 \leq \frac{3dh^2_K}{K} ||\nabla \varphi||_K^2, \quad ||\varphi - \varphi_\sigma||_K^2 \leq 3dh^2_K ||\nabla \varphi||_K^2. \quad (2.2)
\]

Here \( \varphi_K := (1, \varphi/K)/|K| \) and \( \varphi_\sigma := \langle 1, \varphi >_\sigma /|\sigma| \) denote the integrable means of \( \varphi \) over \( K \) and over \( \sigma \in \varepsilon_h \), respectively. The constant \( C_{P,d} \) can be evaluated as \( d/\pi \) for a simplex by using its convexity.

Following \( [37] \), we suppose that there exists an original triangulation \( T_0 \) of \( \Omega \) such that data of the problem \( (2.1) \) are given in the following way.

**Assumptions of data** :

(1) \( S_K := S|_K \) is a constant, symmetric, and uniformly positive definite tensor such that \( c_{S,K} \mathbf{v} \cdot \mathbf{v} \leq S_K \mathbf{v} \cdot \mathbf{v} \leq C_{S,K} \mathbf{v} \cdot \mathbf{v} \) holds for all \( \mathbf{v} \in \mathbb{R}^d \) and all \( K \in T_0 \) with \( c_{S,K}, C_{S,K} > 0 \);

(2) \( \mathbf{w} \) is \( \mathbb{R} T_0(\Omega) \) (cf. Section 3 below) such that \( ||\mathbf{w}|_K|| \leq C_{\mathbf{w},K} \) holds for all \( K \in T_0 \) with \( C_{\mathbf{w},K} \geq 0 \);

(3) \( r_K := r|_K \) is a constant for all \( K \in T_0 \);

(4) \( c_{\mathbf{w},K} := ||\nabla \mathbf{w}|_K + r_K|| \) and \( C_{\mathbf{w},r,K} := ||\nabla \mathbf{w}|_K + r_K|| \) for all \( K \in T_0 \);

(5) \( f|_K \) is a polynomial for each \( K \in T_0 \);

(6) If \( c_{\mathbf{w},r,K} = 0 \), then \( C_{\mathbf{w},r,K} = 0 \).

As pointed out in \( [37] \), all the assumptions are made for the sake of simplicity and are usually satisfied in practice. If data do not satisfy these assumptions, we may employ the interpolation or projection of data with additional occurrence of data oscillation.
Finally we show the weak problem of the model (1.1): Find $p \in H^1_0(\Omega)$ such that
\[
B(p, \varphi) = (f, \varphi) \quad \text{for all } \varphi \in H^1_0(\Omega).
\] (2.3)

Here the bilinear form
\[
B(p, \varphi) := \sum_{K \in T_h} \left\{ (S \nabla p, \nabla \varphi)_K + (\nabla \cdot (pw), \varphi)_K + (rp, \varphi)_K \right\}, p, \varphi \in H^1(\bigcup T_h),
\]
and $T_h$ is a refinement of $T_0$. We define as following an energy (semi) norm corresponding to the bilinear form $B$:
\[
||| \varphi |||_h^2 := \sum_{K \in T_h} ||| \varphi |||^2_K, ||| \varphi |||^2_K := (S \nabla \varphi, \nabla \varphi)_K + c_{w, r, K} \| \varphi \|_K^2, \varphi \in H^1(\bigcup T_h).
\]

We note that the weak problem (2.3) admits a unique solution under the Assumptions (D1)-(D6) [37].

3. Mixed finite element schemes and postprocessing. Since it is of interest in many applications, the stress variable $u := -S \nabla p$ are usually approximated by using the mixed finite elements for the problem (1.1). We introduce in this section the centered and upwind-weighted mixed finite element schemes, and show the postprocessed techniques presented by Vohralík in [37].

We define the lowest order Raviart-Thomas finite element and piecewise constant space respectively as following:
\[
RT_0(T_h) := \left\{ q_h \in H(\text{div}, \Omega) : \forall K \in T_h, \exists a \in \mathbb{R}^d, \exists b \in \mathbb{R}, \text{ such that } q_h(x) = a + bx \text{ for all } x \in K. \right\};
\]
\[
P_0(T_h) := \{ v_h \in L^\infty(\Omega) : \forall K \in T_h, v_h|_K \in P_0(K) \}.
\]
Here $n$ is the unit outer normal vector along $\sigma \in \varepsilon_h$, and $P_0(K)$ denotes the set of constant functions on each $K \in T_h$. We note that $\nabla \cdot (RT_0(T_h)) \subset P_0(T_h)$.

The centered mixed finite element scheme [17][37] reads as: Find $(u_h, p_h) \in RT_0(T_h) \times P_0(T_h)$ such that
\[
(S^{-1} u_h, v_h)_\Omega - (p_h, \nabla \cdot v_h)_\Omega = 0 \quad \text{for all } v_h \in RT_0(T_h),
\] (3.1)
\[
(\nabla \cdot u_h, \varphi_h)_\Omega - (S^{-1} u_h, w_h)_\Omega + ((r + \nabla \cdot w) p_h, \varphi_h)_\Omega = (f, \varphi_h)_\Omega \quad \text{for all } \varphi_h \in P_0(T_h).
\] (3.2)

The upwind-weighted mixed finite element scheme [16][37] reads as: Find $(u_h, p_h) \in RT_0(T_h) \times P_0(T_h)$ such that
\[
(S^{-1} u_h, v_h)_\Omega - (p_h, \nabla \cdot v_h)_\Omega = 0 \quad \text{for all } v_h \in RT_0(T_h),
\] (3.3)
\[
(\nabla \cdot u_h, \varphi_h)_\Omega + \sum_{K \in T_h} \sum_{\sigma \in T_K} \hat{p}_\sigma w_{\sigma K, \sigma} \varphi_K + (r p_h, \varphi_h)_\Omega = (f, \varphi_h)_\Omega \quad \text{for all } \varphi_h \in P_0(T_h),
\] (3.4)
where \( w_{K,\sigma} := \langle 1, \mathbf{w} \cdot \mathbf{n} \rangle \) for \( \sigma \in \mathcal{E}_K \), with \( \mathbf{n} \) the unit normal vector of \( \sigma \), outward to \( K \), \( \varphi_K = (1, \varphi_h)_K/|K| = \varphi_h|_K \) for all \( K \in \mathcal{T}_h \), and \( \tilde{p}_\sigma \) is the weighted upwind value given by

\[
\tilde{p}_\sigma := \begin{cases} 
(1 - \nu_\sigma)p_K + \nu_\sigma p_L & \text{if } w_{K,\sigma} \geq 0, \\
(1 - \nu_\sigma)p_L + \nu_\sigma p_K & \text{if } w_{K,\sigma} < 0 
\end{cases}
\]  

(3.5)

when \( \sigma \) is an interior side sharing by elements \( K \) and \( L \), and by

\[
\tilde{p}_\sigma := \begin{cases} 
(1 - \nu_\sigma)p_K & \text{if } w_{K,\sigma} \geq 0, \\
\nu_\sigma p_K & \text{if } w_{K,\sigma} < 0 
\end{cases}
\]  

(3.6)

when \( \sigma \) is a boundary side included in \( \varepsilon_K \). Here \( p_K \) and \( p_L \) denotes respectively the restrictions of \( p_h \) over \( K \) and \( L \), \( \nu_\sigma \in [0, 1/2] \) denotes the coefficient of the amount of upstream weighting which may be chosen as \[37\]

\[
\nu_\sigma := \begin{cases} 
\min\{c_{S,\sigma}, \frac{|\sigma|}{h_{\sigma|\mathcal{E}_K|}} \} & \text{if } w_{K,\sigma} \neq 0 \text{ and } \sigma \in \varepsilon_h^{\text{int}}, \\
0 & \text{if } \sigma \in \varepsilon_h^{\text{ext}} \text{ and } w_{K,\sigma} > 0, \\
& \text{if } w_{K,\sigma} = 0 \text{ or } \sigma \in \varepsilon_h^{\text{ext}} \text{ and } w_{K,\sigma} < 0. 
\end{cases}
\]  

(3.7)

where \( c_{S,\sigma} \) is the harmonic average of \( c_{S,K} \) and \( c_{S,L} \) if \( \sigma \in \partial K \cap \partial L \) and \( c_{S,K} \) otherwise.

We now introduce the postprocessed technique in \[37\], where a postprocessed approximation \( \tilde{p}_h \) to the displacement \( p \) is constructed which links \( p_h \) and \( u_h \) on each simplex in the following way:

\[
-S_K \nabla \tilde{p}_h|_K = u_h \quad \text{for all } K \in \mathcal{T}_h,
\]  

(3.8)

\[
\frac{1}{|K|} \int_K \tilde{p}_h dx = p_K \quad \text{for all } K \in \mathcal{T}_h.
\]  

(3.9)

We refer to \[37\] for the existence of \( \tilde{p}_h \). We note that the new quantity \( \tilde{p}_h \in W_0(\mathcal{T}_h) \) but \( \notin H^1(\Omega) \) (see LEMMA 6.1 in \[37\]), where

\[
W_0(\mathcal{T}_h) := \{ \varphi \in L^2(\Omega) : \varphi|_K \in H^1(\mathcal{T}_h) \text{ for all } K \in \mathcal{T}_h, \varphi|_K \text{ is continuous on } \partial K \},
\]  

(4.1)

4. Main results. With the stress variable \( \mathbf{u} = -S \nabla p \), we define the global and local errors, \( \mathcal{E} \) and \( \mathcal{E}_K \), of the stress and displacement variables as

\[
\mathcal{E} := \left\{ \sum_{K \in \mathcal{T}_h} \mathcal{E}^2_K \right\}^{1/2}, \quad \mathcal{E}^2_K := ||S^{-1/2}(\mathbf{u} - \mathbf{u}_h)||^2_K + c_{w,r,K}||p - p_h||^2_K.
\]  

(4.2)

Denote respectively by \( \eta_{D,K} \) and \( \eta_{R,K} \) the elementwise displacement and residual estimator with

\[
\eta_{D,K}^2 := c_{w,r,K} h_K^2 ||S^{-1/2}(\mathbf{u} - \mathbf{u}_h)||^2_K,
\]  

(4.3)

\[
\eta_{R,K}^2 := \alpha_K^2 ||f - \nabla \cdot \mathbf{u}_h + (S^{-1}\mathbf{u}_h) \cdot \mathbf{w} - (r + \nabla \cdot \mathbf{w})p_h||^2_K + \beta_K^2 ||S^{-1}\mathbf{u}_h||^2_K.
\]  

(4.4)

Here the residual weight factors

\[
\alpha_K := \min\{\frac{h_K}{\sqrt{c_{S,K}}}, \frac{1}{\sqrt{c_{w,r,K}}}\}, \quad \beta_K := C_{w,r,K} h \alpha_K.
\]  

(4.5)
Note that in (4.4), if \( c_{w,r,K} = 0 \), \( \alpha_K \) should be understood as \( h_K / \sqrt{c_{S,K}} \).

Let \( \nu_\sigma \) be given in (3.7) for each side \( \sigma \in \varepsilon_h \). We denote

\[
\hat{\nu}_\sigma := \begin{cases} 
(1/2 - \nu_\sigma)(p_K - p_L) & \text{if } w_{K,\sigma} \geq 0, \\
(1/2 - \nu_\sigma)(p_L - p_K) & \text{if } w_{K,\sigma} < 0
\end{cases}
\]  

(4.5)

when \( \sigma \) is an interior side sharing by elements \( K \) and \( L \), and

\[
\hat{\nu}_\sigma := \begin{cases} 
-\nu_\sigma p_K & \text{if } w_{K,\sigma} \geq 0, \\
-(1 - \nu_\sigma)p_K & \text{if } w_{K,\sigma} < 0
\end{cases}
\]  

(4.6)

when \( \sigma \) is a boundary side included in \( \varepsilon_K \). We thus define an elementwise upwind estimator \( \eta_{w,K} \) by

\[
\eta_{w,K}^2 := \frac{h_K}{c_{S,K}} \sum_{\sigma \in \varepsilon_K} \left( (\mathbf{w} \cdot \mathbf{n})|_\sigma \right)^2 \left( \|\hat{\nu}_\sigma\|_\sigma^2 + h_\sigma |S^{-1}\mathbf{u}_h\|_\sigma^2 \right).
\]  

(4.7)

In order to reflect the change of the maximum eigenvalue of the coefficients matrix \( S \) over the patch \( \tilde{\sigma}_K \) of a side \( \sigma \in \varepsilon_h \), we introduce a quantity

\[
\Lambda_\sigma := \max_{K:K\cap\sigma \neq \emptyset} \{C_{S,K}\}.
\]

Similarly, the change of one variation \( c_{w,r,K} \) of the coefficients over the patch \( \tilde{\sigma}_K \) of an element \( K \in T_h \) is described by the quantity

\[
\Lambda_{w,r,K} := \max_{K':K'\cap\tilde{\sigma}_K \neq \emptyset} \{c_{w,r,K}\}.
\]

Thus we define \( \eta_{NC,K} \) as the elementwise nonconforming estimator by

\[
\eta_{NC,K}^2 := \Lambda_{w,r,K} h_K^2 |S^{-1}\mathbf{u}_h|_K^2 + \sum_{\sigma \in \varepsilon_K} \delta_\sigma \Lambda_\sigma h_\sigma |[\gamma_t (S^{-1}\mathbf{u}_h)]|_\sigma^2,
\]  

(4.8)

where \( \delta_\sigma = 1/2 \) if \( \sigma \in \varepsilon_h^{int} \), \( \delta_\sigma = 1 \) if \( \sigma \in \varepsilon_h^{ext} \).

Since the convection occurs in the equations, we need to define two numbers \( \Lambda_{\nabla,w,K} \) and \( \Lambda_{w,\sigma} \) similar to Péclet numbers describing the convection-dominated. To this end, for each \( K \in T_h \) we denote

\[
C_{\nabla,w,K} := |\nabla \cdot \mathbf{w}|_K, \quad \Lambda_{\nabla,w,K} := \max_{K':K'\cap\tilde{\sigma}_K \neq \emptyset} \left\{ \frac{C_{\nabla,w,K'}}{\sqrt{c_{w,r,K}}} \right\},
\]

and for each \( \sigma \in \varepsilon_h \) we set \( \Lambda_{w,\sigma} := \min \{\lambda_{w,\sigma}, p_{w,\sigma}\} \) with

\[
\lambda_{w,\sigma} := \max_{K:K\cap\sigma \neq \emptyset} \left\{ \frac{C_{w,K}}{\sqrt{c_{w,r,K}}} \right\}, \quad p_{w,\sigma} := \max_{K:K\cap\sigma \neq \emptyset} \left\{ \frac{h_K C_{w,K}}{\sqrt{c_{S,K}}} \right\}.
\]

We then define \( \eta_{C,K} \) as an elementwise convection estimator by

\[
\eta_{C,K}^2 := \Lambda_{\nabla,w,K} h_K^2 |S^{-1}\mathbf{u}_h|_K^2 + \sum_{\sigma \in \varepsilon_K} \delta_\sigma \Lambda_{w,\sigma}^2 h_\sigma |[\gamma_t (S^{-1}\mathbf{u}_h)]|_\sigma^2.
\]  

(4.9)

We now state a posteriori error estimates for the global error of stress and displacement.

**Theorem 4.1.** (Global error estimate for the centered mixed scheme) Let \( p \in H_h^1(\Omega) \) be the weak solution of the problem (2.2), \( \mathbf{u} = -S\nabla p \) be the continuous stress vector, \( \mathbf{u}_h, p_h \)
be the solution of the centered mixed scheme (3.1)-(3.2). Let $\mathcal{E}$ be the error of the stress and displacement in the weighted norm defined in (4.1), $\eta_{D,K}$, $\eta_{R,K}$, $\eta_{NC,K}$, and $\eta_{C,K}$ are the corresponding elementwise displacement estimator, residual estimator, convection estimator, and nonconforming estimator, defined in (4.2)-(4.3) and (4.8)-(4.9), respectively. Then it holds

$$
\mathcal{E} \leq c_1 \left\{ \sum_{K \in T_h} (\eta_{D,K}^2 + \eta_{R,K}^2 + \eta_{C,K}^2 + \eta_{NC,K}^2) \right\}^{1/2}. 
$$

\textbf{Theorem 4.2.} (Global error estimate for the upwind-weighted scheme) Let $p \in H_0^1(\Omega)$ be the weak solution of the problem (2.2), $u = -S\nabla p$ be the continuous stress vector, $(u_h, p_h)$ be the solution of the upwind-weighted mixed scheme (3.3)-(3.4). Let $\mathcal{E}$ be the error of the stress and displacement in the weighted norm defined in (4.7), $\eta_{D,K}$, $\eta_{R,K}$, $\eta_{U,K}$, $\eta_{NC,K}$, and $\eta_{C,K}$ are the corresponding elementwise displacement estimator, residual estimator, upwind estimator, convection estimator, and nonconforming estimator, defined in (4.2)-(4.3) and (4.7)-(4.9), respectively. Then it holds

$$
\mathcal{E} \leq c_2 \left\{ \sum_{K \in T_h} (\eta_{D,K}^2 + \eta_{R,K}^2 + \eta_{C,K}^2 + \eta_{NC,K}^2 + \eta_{U,K}^2) \right\}^{1/2}. 
$$

\textbf{Remark 4.1.} We note that the constants $c_1$ in (4.10) and $c_2$ in (4.11) only depend on the spatial dimension and the shape regularity parameter of the triangulation $T_h$, and are independent of the coefficients $S, w, r$. In this sense, the proposed estimators are robust with respect to all the coefficients.

\textbf{Remark 4.2.} In [12] Carstensen presented a posteriori error estimates of the Raviart-Thomas, Brezzi-Douglas-Morini, Brezzi-Douglas-Fortin-Marini elements ($M_h, L_h$) for the diffusion equations (the case $w = r = 0$ in the model (1.1)). In his estimators, the term $\min \{ h(S^{-1}u_h - \nabla h v_h) \}$ is included. In practice one may substitute it with the term $\min_{v_h \in M_h} \{ h(S^{-1}u_h - \nabla h v_h) \}$. Then the lowest order Raviart-Thomas element, it holds $\nabla h v_h = 0$, then $\max \{ h(S^{-1}u_h - \nabla h v_h) \}$ is reduced to $\max \{ hS^{-1}u_h \}$, which shows that occurrence of $\max \{ hS^{-1}u_h \}$ is reasonable in the a posteriori error estimators $\eta_{D,K}$ defined in (4.2). In addition, we note that the postprocessing (3.3) can remove the term $\max \{ h\text{curl}(S^{-1}u_h) \}$, which is also contained in Carstensen’s estimators.

The global error estimates above show that the a posteriori indicator over each element consists of a series of estimators. Thus, the local efficiency of each component ensures the local efficiency of the a posteriori indicator over an element. Here, we point out the local efficiency in the sense that its converse estimate holds up to a different multiplicative constant.

\textbf{Theorem 4.3.} (Local efficiency for the displacement and residual estimators) For $K \in T_h$, let $\eta_{D,K}$ and $\eta_{R,K}$ denote the elementwise displacement and residual estimators defined in (4.2) and (4.3), respectively. Then it holds

$$
(\eta_{D,K}^2 + \eta_{R,K}^2)^{1/2} \leq c_3 \alpha_{*,K} \mathcal{E}_K 
$$

with

$$
\alpha_{*,K} := \max \left\{ \sqrt{\frac{c_S K}{c_S K}} + \frac{h_K c_{w,K}}{c_{w,K} K}, \frac{h_K c_{w,K}}{\sqrt{c_{w,K} K}}, \right\}
\max \left\{ \frac{h_K^2 c_{w,K}}{c_{w,K} K}, \frac{h_K c_{w,K}}{\sqrt{c_{w,K} K}} \right\} + \max \left\{ \frac{h_K c_{w,K}}{\sqrt{c_{w,K} K}}, 1 \right\}.
$$
THEOREM 4.4. (Local efficiency for the nonconforming and convection estimators) Let \( \eta_{NC,K} \) and \( \eta_{C,K} \) be the elementwise nonconforming and convection estimators defined in (4.8) and (4.9), respectively. Then it holds
\[
\{\eta_{NC,K}^2 + \eta_{C,K}^2\}^{1/2} \leq c_4 \left\{ \beta_{s,K}^2 \varepsilon_K^2 + \sum_{\sigma \in \mathcal{E}_K} c_{\omega,\sigma}^2 (\Lambda_{\sigma} + \Lambda_{w,\sigma})^{1/2} (u - u_h)_\sigma^2 \right\}^{1/2},
\] (4.13)
where
\[
\beta_{s,K} = \left( \Lambda_{w,K} + \Lambda_{w,K}^2 \right) \max\{h_K^2/c_{S,K}, 1/c_{w,r,K}\},
\]
and \( \Lambda_{w,r,K}, \Lambda_{w,r,K}^2 \) are the same as in (4.8). We finally need the following quantities for the local efficiency of the upwind estimator over an element, where \( \nu_{\sigma} \) is given in (5.7) for each side \( \sigma \in \mathcal{E}_h \).
\[
\lambda_{\sigma} := \begin{cases} 
\frac{||w||_{L_2(\mathcal{E}_{K,L})}}{\sqrt{\mathcal{E}_{S,K}}} \left( \frac{1}{2} - \nu_{\sigma} \right) \max \left( \frac{1}{\sqrt{\mathcal{E}_{S,K}}}, \frac{1}{\sqrt{\mathcal{E}_{S,K}}} \right) + \max \left( \frac{h_{\mathcal{E}_{S,K}}}{\sqrt{\mathcal{E}_{S,K}}}, \frac{h_{\mathcal{E}_{S,K}}}{\sqrt{\mathcal{E}_{S,K}}} \right) & \text{if } \sigma = \tilde{K} \cap \tilde{L}, \\
\frac{||w||_{L_2(\mathcal{E}_{S,K})}}{\sqrt{\mathcal{E}_{S,K}}} \left( 1 - \nu_{\sigma} \right) \frac{1}{\sqrt{\mathcal{E}_{S,K}}} + \frac{h_{\mathcal{E}_{S,K}}}{\sqrt{\mathcal{E}_{S,K}}} & \text{if } \sigma \in \mathcal{E}_{K}^{ext},
\end{cases}
\]
\[
\rho_{\sigma} := \begin{cases} 
\frac{||w||_{L_2(\mathcal{E}_{K,L})}}{\sqrt{\mathcal{E}_{S,K}}} \left( \frac{1}{2} - \nu_{\sigma} \right) |\sigma|^{-\frac{1}{2}} + \frac{1}{\sqrt{\mathcal{E}_{S,K}}} \max \left( \frac{1}{\sqrt{\mathcal{E}_{S,K}}}, \frac{1}{\sqrt{\mathcal{E}_{S,K}}} \right) & \text{if } \sigma = \tilde{K} \cap \tilde{L}, \\
\frac{||w||_{L_2(\mathcal{E}_{S,K})}}{\sqrt{\mathcal{E}_{S,K}}} \left( 1 - \nu_{\sigma} \right) |\sigma|^{-\frac{1}{2}} + \frac{1}{\sqrt{\mathcal{E}_{S,K}}} \frac{1}{\sqrt{\mathcal{E}_{S,K}}} & \text{if } \sigma \in \mathcal{E}_{K}^{ext},
\end{cases}
\]
and
\[
\mathcal{E}_{D,\omega,\sigma} := \begin{cases} 
(c_{w,r,K} \|p - p_h\|_{L_{\mathcal{E}_{S,K}}}^2 + c_{w,r,L} \|p - p_h\|_{L_{\mathcal{E}_{S,K}}}^2)^{1/2} & \text{if } \sigma = \tilde{K} \cap \tilde{L}, \\
\sqrt{c_{w,r,K} \|p - p_h\|_{L_{\mathcal{E}_{S,K}}}^2} & \text{if } \sigma \in \mathcal{E}_{K}^{ext}.
\end{cases}
\]

THEOREM 4.5. (Local efficiency for the upwind estimator) Let \( \eta_{U,K} \) be the elementwise upwind estimator defined in (4.7). Then, it holds
\[
\eta_{U,K} \leq c_5 \sum_{\sigma \in \mathcal{E}_K} \left( \lambda_{\sigma} \|S^{-1/2}(u - u_h)\|_{\omega,\sigma} + \rho_{\sigma} \mathcal{E}_{D,\omega,\sigma} \right).
\] (4.14)

5. Preliminary results and remarks. In this section, firstly we show the abstract error estimates developed by Vohralík in [37], and then make some remarks on Vohralík’s a posteriori error estimators. To this end, for any \( \varphi \in H_0^1(\Omega) \) we define
\[
T_R(\varphi) := \sum_{K \in \mathcal{T}_h} (f + \nabla \cdot (S\nabla \hat{p}_h) - \nabla \cdot (\hat{p}_h w) - \tau \hat{p}_h, \varphi - \varphi_K),
\] (5.1)
\[
T_C(\varphi, s) := \sum_{K \in \mathcal{T}_h} (\nabla \cdot (\hat{p}_h - s) w) - 1/2(\hat{p}_h - s) \nabla \cdot w, \varphi)_K.
\] (5.2)
\[ T_U(\varphi) := \sum_{K \in T_h} \sum_{\sigma \in K} \langle (\tilde{\varphi}_\sigma - \tilde{\varphi}_h) \mathbf{w} \cdot \mathbf{n}, \varphi_K \rangle >_\sigma, \quad (5.3) \]

where \( \varphi_K \) is the mean of \( \varphi \) over \( K \), \( s \in H^1_0(\Omega) \) is arbitrarily given, \( \tilde{\varphi}_h \) is the postprocessed approximation solution given by (3.8)-(3.9), and \( \tilde{\varphi}_\sigma \) is the weighted upwind value defined in (3.5)-(3.6).

**Lemma 5.1.** (Abstract error estimates by Vohralík) Let \( p \in H^1_0(\Omega) \) be the weak solution of the problem (2.3), and let \( s \in H^1_0(\Omega) \) be arbitrary. Then it holds

\[ |||p - \tilde{p}_h|||_\Omega \leq |||\tilde{p}_h - s|||_\Omega + \sup_{\varphi \in H^1_0(\Omega), ||\varphi||_\Omega = 1} \{ T_R(\varphi) + T_C(\varphi, s) \} \quad (5.4) \]

if \( \tilde{p}_h \) is the postprocessed solution, given by (3.8)-(3.9), of the centered mixed finite element scheme (3.1)-(3.2), and holds

\[ |||p - \tilde{p}_h|||_\Omega \leq |||\tilde{p}_h - s|||_\Omega + \sup_{\varphi \in H^1_0(\Omega), ||\varphi||_\Omega = 1} \{ T_R(\varphi) + T_C(\varphi, s) + T_U(\varphi) \} \quad (5.5) \]

if \( \tilde{p}_h \) is the postprocessed solution, given by (3.8)-(3.9), of the upwind-weighted mixed finite element scheme (3.3)-(3.4).

**Remark 5.1.** In Vohralík’s work [37], the modified Oswald interpolation, \( I_{MO}(\tilde{p}_h) \in H^1_0(\Omega) \), of \( \tilde{p}_h \) is introduced to replace \( s \) in the abstract error estimates (5.4)-(5.5) so as to obtain computable estimates of the terms.

We now state our abstract error estimates for the global error of stress and displacement in the weighted norm.

**Lemma 5.2.** (Abstract error estimates for the global error) Let \( p \in H^1_0(\Omega) \) denote the weak solution of the problem (2.3), and \( s \in H^1_0(\Omega) \) be arbitrary. Let \( \mathcal{E} \) be the global error defined in (4.1) and \( \eta_{D,K} \) be the elementwise displacement estimator defined in (4.2). Then it holds

\[ \mathcal{E} \leq \sqrt{2} \{ |||\tilde{p}_h - s|||_\Omega + \sup_{\varphi \in H^1_0(\Omega), ||\varphi||_\Omega = 1} (T_R(\varphi) + T_C(\varphi, s)) + \left( \sum_{K \in T_h} \eta_{D,K}^2 \right)^{1/2} \} \quad (5.6) \]

if \( \tilde{p}_h \) is the postprocessed solution, given by (3.8)-(3.9), of the centered mixed finite element scheme (3.1)-(3.2), and holds

\[ \mathcal{E} \leq \sqrt{2} \{ |||\tilde{p}_h - s|||_\Omega + \sup_{\varphi \in H^1_0(\Omega), ||\varphi||_\Omega = 1} (T_R(\varphi) + T_C(\varphi, s) + T_U(\varphi)) + \left( \sum_{K \in T_h} \eta_{D,K}^2 \right)^{1/2} \} \quad (5.7) \]

if \( \tilde{p}_h \) is the postprocessed solution, given by (3.8)-(3.9), of the upwind-weighted mixed finite element scheme (3.3)-(3.4).

**Proof.** By the postprocessed formulations (3.8)-(3.9) and the generalized Friedrichs inequality (2.2), we have

\[ ||p - p_h||_K \leq ||p - \tilde{p}_h||_K + ||\tilde{p}_h - p_h||_K \leq ||p - \tilde{p}_h||_K + h_K ||\nabla \tilde{p}_h||_K \]

\[ = ||p - \tilde{p}_h||_K + h_K ||S^{-1}u_h||_K \quad \text{for all } K \in T_h. \quad (5.8) \]

On the other hand, it holds

\[ ||S^{-1/2}(u - u_h)||_K^2 = ||S^{1/2}\nabla(p - \tilde{p}_h)||_K^2 \quad \text{for all } K \in T_h. \quad (5.9) \]
Summing \((5.9)\) and \((5.8)\) with a multiplier \(c_{w,r,K}^{1/2}\) over all \(K \in T_h\) yields
\[
E \leq \sqrt{2} (\| p - \tilde{p}_h \| + \| \sum_{K \in T_h} c_{w,r,K} h_K^2 \| S^{-1} u_h \|_K^2 \})^{1/2}).
\]

(5.10)

The desired results \((5.6)-(5.7)\) then follows from LEMMA 5.1.

**LEMMA 5.3.** For any \(K \in T_h\) and \(\varphi \in H^1(K)\), it holds
\[
\| \varphi - \varphi_K \|_K \leq c_6 \alpha_K \| \varphi \|_K,
\]
where \(\varphi_K\) denotes the mean of \(\varphi\) over \(K\), and \(\alpha_K\) is defined as in \((4.4)\).

**Proof.** From \((4.4)\), it holds \(\alpha_K = h_K c_{s,K}^{-1/2}\) when \(h_K c_{s,K}^{-1/2} \leq c_{w,r,K}^{-1/2}\). By Bramble-Hilbert lemma we have
\[
\| \varphi - \varphi_K \|_K \leq c_7 h_K \| \nabla \varphi \|_K \leq c_7 h_K c_{s,K}^{-1/2} \| S^{1/2} \nabla \varphi \|_K
\]
\[= c_7 \alpha_K \| S^{1/2} \varphi \|_K \leq c_7 \alpha_K \| \varphi \|_K.\]

(5.12)

On the other hand, when \(h_K c_{s,K}^{-1/2} > c_{w,r,K}^{-1/2}\), it holds \(\alpha_K = c_{w,r,K}^{-1/2}\). By the property of \(L^2\)-projection we get
\[
\| \varphi - \varphi_K \|_K \leq \| \varphi \|_K = c_{w,r,K}^{1/2} \| \varphi \|_K
\]
\[= \alpha_K c_{w,r,K}^{1/2} \| \varphi \|_K \leq \alpha_K \| \varphi \|_K.\]

(5.13)

The assertion \((5.11)\) follows from \((5.12)-(5.13)\) with \(c_6 := \max\{c_7, 1\}\).

**6. A posteriori error analysis.** We devote this section to computable estimates of \(T_R(\varphi), T_U(\varphi)\) and \(T_C(\varphi, s)\) defined in \((5.1), (5.3)\) and \((5.2)\), respectively, with the help of \(u_h\) and \(p_h\). Moreover, we derive an estimate of \(\| \tilde{p}_h - s \|\) by substituting \(s\) with the modified Oswald interpolation \(I_{MO}(\tilde{p}_h)\) (see \((37)\)), and by using the postprocessing technique as a transition. Finally, we give the proof of THEOREM 4.1-4.2.

**LEMMA 6.1.** (Residual estimator) Let \(T_R(\varphi)\) be defined as in \((5.1)\) with \(\| \varphi \|_\Omega = 1, \) and \(\eta_{R,K}\) be defined as in \((4.3)\). Then it holds
\[
T_R(\varphi) \leq c_8 \left( \sum_{K \in T_h} \eta_{R,K}^2 \right)^{1/2}.
\]

(6.1)
Proof. A combination of Assumption (D4), Lemma \(5.3\) Friedrichs inequality \(2.2\), and the postprocessing \(3.8\) - \(3.9\), yields

\[
T_K(\varphi) = \sum_{K \in T_h} (f - \nabla \cdot (S \nabla \hat{p}_h) - \nabla \cdot (\hat{p}_h w) - r \hat{p}_h \varphi - \varphi_K)_K
\]

\[
= \sum_{K \in T_h} (f - \nabla \cdot u_h + (S^{-1} u_h) \cdot w - (r + \nabla \cdot w)p_h \varphi - \varphi_K)_K
\]

\[
+ \sum_{K \in T_h} ((r + \nabla \cdot w)(p_h - \hat{p}_h), \varphi - \varphi_K)_K
\]

\[
\leq c_8 \left( \sum_{K \in T_h} \alpha_K \|f - \nabla \cdot u_h + (S^{-1} u_h) \cdot w - (r + \nabla \cdot w)p_h \|_K \|\varphi\|_K
\right)
\]

\[
+ \sum_{K \in T_h} C_{w,r,K} \|\nabla \hat{p}_h\|_K \alpha_K \|\varphi\|_K \}
\]

\[
\leq c_8 \left( \sum_{K \in T_h} \alpha_K \|f - \nabla \cdot u_h + (S^{-1} u_h) \cdot w - (r + \nabla \cdot w)p_h \|_K \|\varphi\|_K
\right)
\]

\[
+ \sum_{K \in T_h} \beta_K \|S^{-1} u_h\|_K \|\varphi\|_K \right).
\]

Then the desired result \(6.1\) follows with \(\|\varphi\|_\Omega = 1\). \(\blacksquare\)

Lemma 6.2. (Upwind estimator) Let \(T_U(\varphi)\) be defined as in \(5.3\) with \(\|\varphi\|_\Omega = 1\), and \(\eta_{U,K}\) be defined as in \(4.7\). Then it holds

\[
T_U(\varphi) \leq c_9 \left( \sum_{K \in T_h} \eta_{U,K}^2 \right)^{1/2}.
\]

Proof. We denote by \(\hat{p}_\sigma\) the mean of \(\hat{p}_h\) over \(\sigma \in e_h\), i.e., \(\hat{p}_\sigma := \frac{1}{2}, \hat{p}_h > \sigma / |\sigma|\). The definitions of \(T_U(\varphi)\) and \(w_{K,\sigma}\), together with Assumption \(D2\) of the velocity field \(w\), imply

\[
T_U(\varphi) = \sum_{K \in T_h} \sum_{\sigma \in e_K} (\hat{p}_\sigma - \hat{p}_\sigma) w_{K,\sigma} \varphi_K.
\]

For an element \(K \in T_h\), it holds \(\sigma \in e_K \cap e_L\) or \(\sigma \in e_{K}^{\text{ext}}\). For the former case, recalling \(p_K = p_h|_K, p_L = p_h|_L\), from the postprocessing \(3.9\) we obtain

\[
\hat{p}_\sigma - \hat{p}_\sigma = \hat{p}_\sigma - \frac{1}{2}(p_K + p_L) + \frac{1}{2}(p_K - \hat{p}_\sigma) + \frac{1}{2}(p_L - \hat{p}_\sigma)
\]

\[
= \hat{p}_\sigma - \frac{1}{2}(p_K + p_L) + \frac{1}{2} \left( \frac{1}{|K|} \int_K \hat{p}_h dx - \frac{1}{|\sigma|} \int_\sigma \hat{p}_h ds \right)
\]

\[
+ \frac{1}{2} \left( \frac{1}{|L|} \int_L \hat{p}_h dx - \frac{1}{|\sigma|} \int_\sigma \hat{p}_h ds \right).
\]

For the latter case, we similarly have

\[
\hat{p}_\sigma - \hat{p}_\sigma = \hat{p}_\sigma - \hat{p}_K + \left( \frac{1}{|K|} \int_K \hat{p}_h dx - \frac{1}{|\sigma|} \int_\sigma \hat{p}_h ds \right).
\]

For convenience, in what follows we denote

\[
\hat{p}_{\omega \sigma} := \frac{1}{2} \left( \frac{1}{|K|} \int_K \hat{p}_h dx - \frac{1}{|\sigma|} \int_\sigma \hat{p}_h ds \right) + \frac{1}{2} \left( \frac{1}{|L|} \int_L \hat{p}_h dx - \frac{1}{|\sigma|} \int_\sigma \hat{p}_h ds \right)
\]
when $\sigma \in \varepsilon_K \cap \varepsilon_L$, and

$$\hat{\omega}_\sigma := \frac{1}{|K|} \int_K \hat{\omega}_h \, dx - \frac{1}{|\sigma|} \int_\sigma \hat{\omega}_h \, ds$$

when $\sigma \in \varepsilon^\text{ext}_K$.

In light of the definitions of $\hat{\omega}_\sigma$ and $\hat{\omega}_h$ in (3.5)–(3.6) and (4.5)–(4.6), and from (6.4)–(6.6) we have

$$T_U(\varphi) = \sum_{K \in T_h} \sum_{\sigma \in \varepsilon_K} (\hat{\omega}_\sigma + \hat{\omega}_h) w_{K,\sigma} \varphi_K. \quad (6.7)$$

Since $\varphi \in H^1_0(\Omega)$, and $\hat{\omega}_\sigma, \hat{\omega}_h$ are constants over a side $\sigma \in \varepsilon_h$, it holds

$$\sum_{K \in T_h} \sum_{\sigma \in \varepsilon_K} \hat{\omega}_\sigma w_{K,\sigma} \varphi_K = \sum_{K \in T_h} \sum_{\sigma \in \varepsilon_K} \int_\sigma \hat{\omega}_\sigma w \cdot n(\varphi_K - \varphi), \quad (6.8)$$

$$\sum_{K \in T_h} \sum_{\sigma \in \varepsilon_K} \hat{\omega}_h w_{K,\sigma} \varphi_K = \sum_{K \in T_h} \sum_{\sigma \in \varepsilon_K} \int_\sigma \hat{\omega}_h w \cdot n(\varphi_K - \varphi). \quad (6.9)$$

From Friedrichs inequality (2.2) and the postprocessing (3.8) we have

$$|\hat{\omega}_h| \leq c_1 h^{1-d/2} ||S^{-1} u_h||_{\omega_h}. \quad (6.10)$$

The trace inequality (see LEMMA 3.1 in [35]) and local shape regularity of elements indicate

$$||\varphi_K - \varphi||_{\sigma} \leq c_3 (h^{1/2} ||\varphi - \varphi_K||_K + ||\varphi - \varphi_K||_K^{1/2} ||\nabla(\varphi - \varphi_K)||_K^{1/2}) \leq c_6 h^{1/2} ||\nabla \varphi||_K \leq c_6 h^{1/2} c_{S,K}^{1/2} ||S^{1/2} \nabla \varphi||_K. \quad (6.11)$$

A combination of (6.10)–(6.11) then yields

$$\sum_{K \in T_h} \sum_{\sigma \in \varepsilon_K} \int_\sigma \hat{\omega}_h w \cdot n(\varphi_K - \varphi) \leq c_1 \sum_{K \in T_h} \left\{ \sum_{\sigma \in \varepsilon_K} |(w \cdot n)|_{\sigma} |h^{1/2} ||S^{-1} u_h||_{\omega_h} \right\} h^{1/2} c_{S,K}^{1/2} ||\varphi||_{K}. \quad (6.12)$$

Similarly we can obtain

$$\sum_{K \in T_h} \sum_{\sigma \in \varepsilon_K} \int_\sigma \hat{\omega}_h w \cdot n(\varphi_K - \varphi) \leq c_2 \sum_{K \in T_h} \left\{ \sum_{\sigma \in \varepsilon_K} |(w \cdot n)|_{\sigma} ||\hat{\omega}_h||_{\sigma} h^{1/2} c_{S,K}^{1/2} ||\varphi||_{K}. \quad (6.13)$$

Finally, the desired result (6.3) follows from (6.7)–(6.9) and (6.12)–(6.13) with $c_5 := \max(c_1, c_2)$ and $||\varphi||_{\Omega} = 1$.

For the first term, $||\hat{\omega}_h - s||_{\Omega}$, in the right side of the abstract error estimate (5.6) or (5.7), we follow [37] to take $s := I_{MO}(\hat{\omega}_h)$ in the sequel, where $I_{MO}(\hat{\omega}_h)$ is the modified Oswald interpolation of $\hat{\omega}_h$. Recall an estimate on the modified Oswald interpolation [20],

$$||\nabla(\varphi - I_{MO}(\varphi_h))||_{K}^2 \leq c_4 \sum_{\sigma : \sigma \cap K \neq \emptyset} h^{1-d/2} ||\varphi_h||_{\sigma}^2, \quad \varphi_h \in P_d(T_h) \cap W_0(T_h), \quad (6.14)$$
where \( I_{MO}(\varphi_h) \in \mathcal{P}_d(T_h) \cap H_0^1(\Omega) \) is the modified Oswald interpolation of \( \varphi_h, \mathcal{P}_d(T_h) \) 
\((d = 2\ or\ 3)\ denotes\ the\ set\ of\ polynomials\ of\ degree\ at\ most\ d\ on\ each\ simplex,\ \sigma \cap K \neq \emptyset\ when\ \sigma\ contains\ a\ vertex\ of\ \tilde{K}\).

By definition we have
\[
|||\tilde{p}_h - s|||_\Omega = \left\{ \sum_{K \in T_h} (S\nabla(\tilde{p}_h - s), \nabla(\tilde{p}_h - s))_K + \sum_{K \in T_h} c_{w,r,K}|||\tilde{p}_h - s|||_K^2 \right\}^{1/2}.
\]

**LEMMA 6.3.** Let \( \gamma_{t_\sigma}(\cdot) \) be defined as in Section 2.1, and \( s := I_{MO}(\tilde{p}_h) \). Then it holds
\[
\left\{ \sum_{K \in T_h} |||S^{1/2}\nabla(\tilde{p}_h - s)|||_K^2 \right\}^{1/2} \leq c_{14} \sum_{\sigma \in \varepsilon_h} \Lambda_\sigma h_\sigma |||[\gamma_{t_\sigma}(S^{-1}u_h)]|||_\sigma^2, \quad (6.15)
\]
where \( \Lambda_\sigma \) is given in Section 4, and \( t_\sigma \) denotes the unit tangent vector along \( \sigma \).

**Proof.** From the estimate (6.14) we have
\[
|||\nabla(\tilde{p}_h - s)|||_K^2 \leq c_{13} \sum_{\sigma, \sigma \cap K \neq \emptyset} h_\sigma^{-1}|||\tilde{p}_h|||_\sigma^2, \quad \text{for all } K \in T_h, \quad (6.16)
\]
where \( \sigma \cap K \neq \emptyset \) when \( \sigma \) contains a vertex of \( \tilde{K} \).

Since the mean of \( \tilde{p}_h \) over interior side is continuous and its mean on exterior side vanishes, i.e., \( \int_{\sigma}[\tilde{p}_h]ds = 0 \) for all \( \sigma \in \varepsilon_h \), by Poincaré inequality it holds
\[
|||\tilde{p}_h|||_\sigma = |||\tilde{p}_h||| - \int_{\sigma}[\tilde{p}_h]||| \leq c_{15} h_\sigma |||[\gamma_{t_\sigma}(\nabla([\tilde{p}_h]))]|||_\sigma, \quad (6.17)
\]
The postprocessing (3.8) indicates
\[
\gamma_{t_\sigma}(\nabla([\tilde{p}_h])) = -[\gamma_{t_\sigma}(S^{-1}u_h)], \quad \text{for all } \sigma \in \varepsilon_h. \quad (6.18)
\]
A combination of (6.16)-(6.18) yields
\[
|||S^{1/2}\nabla(\tilde{p}_h - s)|||_K^2 \leq c_{13} c_{15} C_{S,K} \sum_{\sigma, \sigma \cap K \neq \emptyset} h_\sigma |||[\gamma_{t_\sigma}(S^{-1}u_h)]|||_\sigma^2. \quad (6.19)
\]
Summing (6.19) over each element \( K \), noticing that the number of summation over a side \( \sigma \in \varepsilon_h \) is bounded by a positive constant \( c_{17} \), and combining the definition of \( \Lambda_\sigma \), we obtain
\[
\sum_{K \in T_h} |||S^{1/2}\nabla(\tilde{p}_h - s)|||_K^2 \leq c_{13} c_{15} \sum_{K \in T_h} C_{S,K} \sum_{\sigma, \sigma \cap K \neq \emptyset} h_\sigma |||[\gamma_{t_\sigma}(S^{-1}u_h)]|||_\sigma^2 \leq c_{13} c_{15} c_{16} \sum_{\sigma \in \varepsilon_h} \Lambda_\sigma h_\sigma |||[\gamma_{t_\sigma}(S^{-1}u_h)]|||_\sigma^2. \quad (6.20)
\]
The desired result (6.15) with \( c_{14} := c_{13} c_{15} c_{16} \) follows from (6.20). \( \square \)

**REMARK 6.1.** The node with respect to which the quasi-monotone condition is violated is called singular node (cf. [28]). We can derive an alternative form of (6.20) as following:
\[
\sum_{K \in T_h} |||S^{1/2}\nabla(\tilde{p}_h - s)|||_K^2 \leq c_{14} \sum_{K \in T_h} c_K^2. \quad (6.20)
\]
Combining the definition of $\xi^2_K$ with a side $\sigma$, the following corollary is a combined result of LEMMAs 6.3-6.4.

**Lemma 6.4.** Let $\Lambda_{w,r,K}$ be the same as in (4.8) and $s := I_{MO}(\tilde{\nu}_h)$. Then it holds

$$\sum_{K \in T_h} c_{w,r,K} ||\tilde{\nu}_h - s||_K^2 \leq c_{17} \sum_{K \in T_h} \Lambda_{w,r,K} h_K^2 ||S^{-1}u_h||_K^2. \quad (6.21)$$

**Proof.** Following the line of the proof of THEOREM 2.2 in [20], we obtain

$$||\tilde{\nu}_h - s||_K^2 \leq c_{18} \sum_{\sigma, \sigma \cap K \neq \emptyset} h_{\sigma}^2 ||\tilde{\nu}_h||_{\sigma}^2. \quad (6.22)$$

Let $\tilde{\nu}_\sigma := \frac{\sum_{\sigma \cap K \neq \emptyset} \tilde{\nu}_h}{|\sigma|}$ denote the mean of the postprocessed scalar variable $\tilde{\nu}_h$ over a side $\sigma \in \mathcal{E}_h$. From the trace theory and generalized Friedrichs inequality (2.2), we obtain

$$||\tilde{\nu}_\sigma||_{\sigma} \leq c_{19} h_{\sigma}^{1/2} ||\nabla \tilde{\nu}_h||_{\omega_\sigma}. \quad (6.23)$$

A combination of (6.22), (6.23) and the postprocessing (3.8) yields that

$$||\tilde{\nu}_h - s||_K^2 \leq c_{20} \sum_{\sigma, \sigma \cap K \neq \emptyset} h_{\sigma}^2 ||S^{-1}u_h||_{\omega_\sigma}^2. \quad (6.24)$$

Summing (6.24) over each element $K$, noticing that the mesh is local quasi-uniform, and combining the definition of $\Lambda_{w,r,K}$, we finally get

$$\sum_{K \in T_h} c_{w,r,K} ||\tilde{\nu}_h - s||_K^2 \leq c_{20} \sum_{K \in T_h} c_{w,r,K} \sum_{\sigma, \sigma \cap K \neq \emptyset} h_{\sigma}^2 ||S^{-1}u_h||_{\omega_\sigma}^2 \leq c_{17} \sum_{K \in T_h} \Lambda_{w,r,K} h_K^2 ||S^{-1}u_h||_K^2.$$

**Remark 6.2.** (Alternative form) For a side $\sigma \in \mathcal{E}_h$, we denote $\Lambda_{w,r,\sigma} := \max_{K \cap \sigma \neq \emptyset} \{c_{w,r,K}\}$. A combination of (6.17), (6.18) and (6.22) yields

$$||\tilde{\nu}_h - s||_K^2 \leq c_{21} \sum_{\sigma, \sigma \cap K \neq \emptyset} h_{\sigma}^2 ||\gamma_{t_\sigma}(S^{-1}u_h)||_{\sigma}^2, \quad (6.25)$$

which leads to an alternative form of the estimate (6.21),

$$\sum_{K \in T_h} c_{w,r,K} ||\tilde{\nu}_h - s||_K^{1/2} \leq c_{22} \sum_{\sigma \in \mathcal{E}_h} \Lambda_{w,r,\sigma} h_{\sigma}^2 ||\gamma_{t_\sigma}(S^{-1}u_h)||_{\sigma}^2 \bigg\}^{1/2}. \quad (6.26)$$

This inequality shows that the term $\{ \sum_{K \in T_h} c_{w,r,K} ||\tilde{\nu}_h - s||_K^{1/2} \}^{1/2}$ can be absorbed into $\{ \sum_{K \in T_h} ||S^{1/2}\nabla(\tilde{\nu}_h - s)||_K^{1/2} \}$ when $\Lambda_{w,r,\sigma} h_{\sigma} \leq \Lambda_{\sigma}$.

The following corollary is a combined result of LEMMAs 6.3, 6.4.

**Corollary 6.5.** Let $\eta_{NC,K}$ be defined as in (4.8) and $s := I_{MO}(\tilde{\nu}_h)$. Then it holds

$$||\tilde{\nu}_h - s||_{\Omega} \leq c_{23} \sum_{K \in T_h} \eta_{NC,K}^{2} \bigg\}^{1/2}. \quad (6.26)$$
LEMMA 6.6. (Convection estimator.) Let $T_C(\varphi, s)$ be defined as in (6.2) with $|||\varphi|||_\Omega = 1$ and $s := I_{MO}(\tilde{p}_h)$, and $\eta_{C,K}$ be defined as in (6.9). Then it holds

$$ T_C(\varphi, s) \leq c_{24} \left( \sum_{K \in T_h} \eta_{C,K}^2 \right)^{1/2}. \tag{6.27} $$

Proof. By triangle inequality and Hölder inequality we obtain

$$ T_C(\varphi, s) \leq \sum_{K \in T_h} \left( C_{w,K} \| \nabla (\tilde{p}_h - s) \|_K \|\varphi\|_K + \frac{1}{2} C_{\nabla w,K} \| \tilde{p}_h - s \|_K \|\varphi\|_K \right) $$

$$ \leq \left\{ \sum_{K \in T_h} \left( \frac{C_{w,K}^2}{c_{S,K} C_{w,K}} \| S^{1/2} \nabla (\tilde{p}_h - s) \|_K^2 + \frac{C_{\nabla w,K}^2}{4c_{w,K}} \| \tilde{p}_h - s \|_K^2 \right) \right\}^{1/2}. \tag{6.28} $$

Apply (6.19) and (6.24) to the inequality (6.28), and combine the definitions of $\lambda_{w,\sigma}$ and $\Lambda_{\nabla w,K}$, we then arrive at

$$ T_C(\varphi, s) \leq c_{25} \left\{ \sum_{K \in T_h} \sum_{\sigma : \sigma \cap K \neq \emptyset} C_{w,K} C_{S,K} h_{\sigma} \| \nabla (S^{-1} u_h) \|_{\sigma}^2 \right\}^{1/2} $$

$$ + \frac{C_{\nabla w,K}^2}{4c_{w,K}} \sum_{\sigma : \sigma \cap K \neq \emptyset} h_{\sigma}^2 \| S^{-1} u_h \|_{\omega_{\sigma}}^2 \right\}^{1/2}. $$

Since the modified Oswald interpolation $s = I_{MO}(\tilde{p}_h)$ preserves the mean of $\tilde{p}_h$ on the side, and $w \cdot n$ is constant over a side, it holds

$$ (\nabla \cdot ((\tilde{p}_h - s) w), \varphi_K) = < (\tilde{p}_h - s) w \cdot n, \varphi_K >_K = 0, $$

where $\varphi_K$ is the mean of $\varphi$ over $K$. Write $v := \tilde{p}_h - s$, then we have

$$ (\nabla \cdot (v w) - 1/2 v \nabla \cdot w, \varphi_K) = (\nabla v \cdot w, \varphi - \varphi_K) + (1/2 v \nabla \cdot w, \varphi_K) + (v \nabla \cdot w, \varphi_K). \tag{6.30} $$

A combination of (6.30), (2.1), (6.19), (6.24) and Hölder inequality yields

$$ T_C(\varphi, s) \leq \sum_{K \in T_h} \left( \frac{h_K C_{w,K}}{\sqrt{c_{S,K}}} \| S^{1/2} \nabla (\tilde{p}_h - s) \|_K + \frac{3C_{\nabla w,K}}{2 \sqrt{c_{w,r,K}} \| \tilde{p}_h - s \|_K} \|\varphi\|_K \right) $$

$$ \leq c_{26} \left\{ \sum_{K \in T_h} \frac{h_K^2 C_{w,K}^2}{c_{S,K}} \sum_{\sigma : \sigma \cap K \neq \emptyset} h_{\sigma} \| \nabla (S^{-1} u_h) \|_{\sigma}^2 \right\}^{1/2} $$

$$ + \frac{C_{\nabla w,K}^2}{c_{w,K}} \sum_{\sigma : \sigma \cap K \neq \emptyset} h_{\sigma}^2 \| S^{-1} u_h \|_{\omega_{\sigma}}^2 \right\}^{1/2}. \tag{6.31} $$

This estimate, together with the definitions of $p_{w,\sigma}$ and $\Lambda_{\nabla w,K}$, indicates $T_C(\varphi, s)$ from (6.31)

$$ T_C(\varphi, s) \leq c_{26} \left\{ \sum_{\sigma \in \epsilon_h} p_{w,\sigma} h_{\sigma} \| \nabla (S^{-1} u_h) \|_{\sigma}^2 \right\}^{1/2} + \left\{ \sum_{K \in T_h} \Lambda_{\nabla w,K}^2 h_K^2 \| S^{-1} u_h \|_K^2 \right\}^{1/2}. \tag{6.32} $$
The desired result (6.27) follows from (6.29) and (6.32) with \( c_{24} = \max\{c_{25}, c_{26}\} \).

**Proof of THEOREM 4.1** For the centered mixed scheme, the desired result (4.10) follows from LEMMA 5.2, LEMMA 6.1, Corollary 6.5, LEMMA 6.6 with the positive constant \( c_2 = 2\sqrt{2}\max\{1, c_8, c_{23}, c_{24}\} \). For the upwind-weighted mixed scheme, the assertion (4.11) follows from LEMMA 6.1, Corollary 6.5, LEMMA 6.6 and LEMMA 5.2 with \( c_2 = \sqrt{10}\max\{1, c_8, c_9, c_{23}, c_{24}\} \).

**REMARK 6.3.** (Two approaches in a posteriori error analysis) There are usually two approaches in literature in the a posteriori error analysis. One is directly based on the solution of the discretization scheme, the other one is based on the postprocessed approximation. Seemingly, these two approaches are fully different. Our analysis establishes a link between them, i.e. a posteriori error estimates based on the discretization solution can be derived with the help of the postprocessing technique. In doing so, one can avoid the use of Helmholtz decomposition of the stress variable which is required in traditional a posteriori error analysis for mixed finite elements.

**REMARK 6.4.** (Pure diffusion problem) When \( \mathbf{w} = r = 0 \), the model (1.1) is reduced to a pure diffusion problem. In this case, the fact that \( -\nabla \cdot (S_K \nabla p_h) = \nabla \cdot u_h \) indicates

\[
\eta_{D,K} = 0, \eta_{C,K} = 0, \eta_{NC,K}^2 = \frac{h_K^2}{c_{S,K}} ||f - f_h||^2_K, \eta_{NC,K}^2 = \sum_{\sigma \in \mathcal{E}_K} \delta_\sigma \Lambda_\sigma h_\sigma ||\gamma_{t_\sigma}(S^{-1}u_h)||^2_\sigma.
\]

Thus, the a posteriori error estimate (4.10) is reduced to

\[
\mathcal{E} \leq c_{27} \left\{ \sum_{K \in \mathcal{T}_h} \left( \frac{h_K^2}{c_{S,K}} ||f - f_h||^2_K + \sum_{\sigma \in \mathcal{E}_K} \delta_\sigma \Lambda_\sigma h_\sigma ||\gamma_{t_\sigma}(S^{-1}u_h)||^2_\sigma \right) \right\}^{1/2}
\]

with \( \mathcal{E} = \{ \sum_{K \in \mathcal{T}_h} ||S^{-1/2}(\mathbf{u} - \mathbf{u}_h)||^2_K \}^{1/2} \). In addition, Remark 6.1 implies an alternative estimate

\[
\mathcal{E} \leq c_{27} \left\{ \sum_{K \in \mathcal{T}_h} \left( \frac{h_K^2}{c_{S,K}} ||f - f_h||^2_K + \xi_K^2 \right) \right\}^{1/2}.
\]

Note that being an oscillation term, the first term in the right side of (6.34) or (6.35) may not be computed in practice.

**REMARK 6.5.** (A posteriori error estimate of divergence of the stress variable.) The continuous weak formulation of (1.1) reads as: Find \((\mathbf{u}, p) \in H(\text{div}, \Omega) \times L^2(\Omega)\) such that

\[
(S^{-1}\mathbf{u}, \mathbf{v})_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega = 0 \quad \text{for all} \ \mathbf{v} \in H(\text{div}, \Omega),
\]

\[
(\nabla \cdot \mathbf{u}, \varphi)_\Omega - (S^{-1}\mathbf{u} \cdot \mathbf{w}, \varphi)_\Omega + ((r + \nabla \cdot \mathbf{w})p, \varphi)_\Omega = (f, \varphi)_\Omega \quad \text{for all} \ \varphi \in L^2(\Omega). \quad (6.36)
\]

Notice that (6.36) can be equivalently written as: For each \( K \in \mathcal{T}_h \)

\[
(\nabla \cdot \mathbf{u}, \varphi)_K - (S^{-1}\mathbf{u} \cdot \mathbf{w}, \varphi)_K + ((r + \nabla \cdot \mathbf{w})p, \varphi)_K = (f, \varphi)_K \quad \text{for all} \ \varphi \in L^2(K). \quad (6.37)
\]

Meanwhile, the centered mixed finite element scheme (3.2) can be equivalently written as: For every \( K \in \mathcal{T}_h \)

\[
(\nabla \cdot \mathbf{u}_h, \varphi)_K - (S^{-1}\mathbf{u}_h \cdot \mathbf{w}, \varphi)_K + ((r + \nabla \cdot \mathbf{w})p_h, \varphi)_K = (f, \varphi)_K \quad \text{for all} \ \varphi \in P_0(K). \quad (6.38)
\]
Let $\overline{R}_K$ denote the mean of the elementwise residual

$$R_K := f - \nabla \cdot \mathbf{u}_h + (S^{-1}\mathbf{u}_h) \cdot \mathbf{w} - (r + \nabla \cdot \mathbf{w})p_h$$

over $K \in \mathcal{T}_h$, and set $0 \leq \ell \leq 1$. We define the data oscillation $\text{osc}_h$ as

$$\text{osc}_h := \{ \sum_{K \in \mathcal{T}_h} h_K^2 ||R_K - \overline{R}_K||_K^2 \}^{1/2}.$$ 

For any $\varphi \in L^2(K)$, let $\varphi_K$ denote the mean of $\varphi$ over $K \in \mathcal{T}_h$, then a combination of \((6.37)\) and \((6.38)\) yields

$$(\nabla \cdot (\mathbf{u} - \mathbf{u}_h), \varphi)_K = (\nabla \cdot \mathbf{u}, \varphi)_K - (\nabla \cdot \mathbf{u}_h, \varphi - \varphi_K)_K - (\nabla \cdot \mathbf{u}_h, \varphi_K)_K$$

$$= (R_K, \varphi - \varphi_K)_K + (S^{-1}(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{w}, \varphi)_K - ((r + \nabla \cdot \mathbf{w})(p - p_h), \varphi)_K$$

$$= (R_K - \overline{R}_K, \varphi - \varphi_K)_K + (S^{-1}(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{w}, \varphi)_K - ((r + \nabla \cdot \mathbf{w})(p - p_h), \varphi)_K$$

$$\leq (||R_K - \overline{R}_K||_K + ||S^{-1}(\mathbf{u} - \mathbf{u}_h)||_K ||\mathbf{w}||_{L^\infty(K)} + C_{\mathbf{w},r,K}||p - p_h||_K)||\varphi||_K.$$ 

This means that

$$||\nabla \cdot (\mathbf{u} - \mathbf{u}_h)||_K = \sup_{\varphi \in L^2(K), \varphi \neq 0} \frac{(\nabla \cdot (\mathbf{u} - \mathbf{u}_h), \varphi)_K}{||\varphi||_{L^2(K)}}$$

$$\leq ||R_K - \overline{R}_K||_K + \frac{||\mathbf{w}||_{L^\infty(K)}}{\sqrt{\max_{K \in \mathcal{T}_h}}} ||S^{-1/2}((\mathbf{u} - \mathbf{u}_h)||_K + C_{\mathbf{w},r,K}||p - p_h||_K.$$ 

\((6.39)\)

From \((6.39)\) and \((4.10)\) we obtain the following a posteriori error estimate of the divergence of the stress variable for the centered mixed finite element scheme:

$$||h' \nabla \cdot (\mathbf{u} - \mathbf{u}_h)|| \leq 28 \{ \sum_{K \in \mathcal{T}_h} (\eta^2_{R,K} + \eta^2_{C,K} + \eta^2_{NC,K} + \eta^2_{D,K}) \}^{1/2} \beta_c + \text{osc}_h,$$

where the constant $\beta_c := \max_{K \in \mathcal{T}_h} \max \{ \frac{||\mathbf{w}||_{L^\infty(K)}}{\sqrt{\max_{K \in \mathcal{T}_h}}} h'_K, \frac{C_{\mathbf{w},r,K} h'_K}{\sqrt{C_{\mathbf{w},r,K}}} \}.

7. Analysis of local efficiency. Using standard arguments we easily derive lemmas \(7.1\) \& \(7.2\).

**Lemma 7.1.** Denote $v := f - \nabla \cdot \mathbf{u}_h + (S^{-1}\mathbf{u}_h) \cdot \mathbf{w} - (r + \nabla \cdot \mathbf{w})p_h$, and let $\mathcal{E}_K$ be the local error for the stress and displacement defined in \((4.1)\). Under Assumption \((D5)\) for $f$, it holds

$$h_K ||v||_K \leq c_{29} \max \{ \sqrt{C_{S,K}} + \frac{C_{w,K} h_K}{\sqrt{C_{S,K}}}, \frac{C_{w,r,K} h_K}{\sqrt{C_{w,r,K}}} \} \mathcal{E}_K.$$ 

\((7.1)\)

**Lemma 7.2.** It holds

$$h_{\sigma}^{1/2} ||\gamma_s (S^{-1}(\mathbf{u}_h))||_{\sigma} \leq c_{30} c_{\omega,s} ||S^{-1/2}(\mathbf{u} - \mathbf{u}_h)||_{\omega,s}.$$ 

\((7.2)\)

**Lemma 7.3.** It holds

$$h_K ||S^{-1}\mathbf{u}_h||_K \leq c_{31} \max \{ \frac{h_K}{\sqrt{C_{S,K}}}, \frac{1}{\sqrt{C_{w,r,K}}} \} \mathcal{E}_K.$$ 

\((7.3)\)
follows.

A combination of (7.4), (7.5) and inverse inequality imply

\[ ||S^{-1}u_h||_K^2 = ||S^{-1}u_h + \nabla p_h||_K^2 \leq c_{32}||\psi_K^{1/2}(S^{-1}u_h + \nabla p_h)||_K^2 = c_{32}(\psi_K S^{-1}u_h, S^{-1}u_h + \nabla p_h)_K \]

Integration by parts implies

\[ (\psi_K S^{-1}u_h, S^{-1}u + \nabla p_h)_K = (\psi_K S^{-1}u_h, \nabla (p_h - p))_K = -((\nabla \cdot (\psi_K S^{-1}u_h), p_h - p))_K. \]

A combination of (7.4), (7.5) and inverse inequality imply

\[ ||S^{-1}u_h||_K^2 \leq c_{32}\left\{ \frac{1}{\sqrt{\epsilon_{S,K}}}||S^{-1/2}(u - u_h)||_K + h_K^{-1}||p - p_h||_K \right\} ||S^{-1}u_h||_K. \]

The desired result (7.3) then follows with \( c_{31} := \sqrt{2}c_{32}. \)

**Lemma 7.4.** It holds

\[ h_K^{1/2}||\tilde{p}||_\sigma \leq \begin{cases} c_{33}|\sigma|^{-1/2}(1/2 - \nu_\sigma) (h_K||S^{-1}u_h||_K + h_L||S^{-1}u_h||_L) & \text{if } \sigma = \bar{K} \cap \bar{L}, \\
\quad c_{33}|\sigma|^{-1/2}(1 - \nu_\sigma) h_K||S^{-1}u_h||_K & \text{if } \sigma \in \epsilon_K \cap \epsilon^{ext}_K. \end{cases} \]

**Proof.** If \( \sigma = \bar{K} \cap \bar{L}, \) from \( \int_\sigma \tilde{p}_h|_K ds = \int_\sigma \tilde{p}_h|_L ds \) we have

\[ \int_\sigma |\tilde{p}_\sigma| = \begin{cases} \int_\sigma (1/2 - \nu_\sigma) \left( (p_h - \tilde{p}_h)|_K - (p_h - \tilde{p}_h)|_L \right) & \text{if } p_K \geq p_L, \\
\quad \int_\sigma (1/2 - \nu_\sigma) \left( (p_h - \tilde{p}_h)|_L - (p_h - \tilde{p}_h)|_K \right) & \text{if } p_K < p_L. \end{cases} \]

This relation, together with trace theorem and the postprocessing (3.9), indicates

\[ \int_\sigma |\tilde{p}_\sigma| \leq (1/2 - \nu_\sigma)(||p_h - \tilde{p}_h||_{\partial K} + ||p_h - \tilde{p}_h||_{\partial L}) \leq c_{33}(1/2 - \nu_\sigma)(h_K^{1/2}||\nabla \tilde{p}_h||_K + h_L^{1/2}||\nabla \tilde{p}_h||_L), \]

which, together with the local shape regularity of elements and the postprocessing (3.8), implies

\[ h_K^{1/2}||\tilde{p}||_\sigma \leq h_K^{1/2}||\tilde{p}||_\sigma^{1/2} = h_K^{1/2}||\sigma||^{-1/2} \int_\sigma |\tilde{p}_\sigma| ds \leq c_{33}|\sigma|^{-1/2}(1/2 - \nu_\sigma) \left( h_K||S^{-1}u_h||_K + h_L||S^{-1}u_h||_L \right). \]

If \( \sigma \in \epsilon_K \cap \epsilon^{ext}_K, \) from \( \int_\sigma \tilde{p}_h|_K ds = 0 \) and \( \nu_\sigma \leq 1/2 \) the second assertion of the lemma follows. □
Proof of Theorem 4.3 From the definition of $\alpha_{*,K}$ in this theorem, Lemma 7.3 shows
\[ \eta_{D,K} \leq c_{34}\alpha_{*,K}\mathcal{E}_K. \] (7.7)
Denote $v := f - \nabla \cdot u_h + (S^{-1}u_h) \cdot w - (r + \nabla \cdot w)p_h$, and then it holds
\[ \eta_{R,K} \leq \alpha_K ||v||_K + \beta_K ||S^{-1}u_h||_K \leq \frac{h_K}{\varepsilon S,K} ||v||_K + C_{w,r,K}h_K \frac{h_K}{\varepsilon S,K} ||S^{-1}u_h||_K. \] (7.8)
A combination of (7.8), Lemma 7.1 and Lemma 7.3 leads to
\[ \eta_{R,K} \leq c_{35}\alpha_{*,K}\mathcal{E}_K. \] (7.9)
The desired result (4.12) follows from (7.7) and (7.9) with $c_3 = \sqrt{2}\max\{c_{34},c_{35}\}$.

Proof of Theorem 4.4 Notice that
\[ \eta_{NC,K} \leq \sqrt{\Lambda_{w,r,K}h_K} ||S^{-1}u_h||_K + \sum_{\sigma \in \varepsilon K} \Lambda_{\sigma}^{1/2} h^{1/2}_{\sigma} ||\gamma_{\tau_{\sigma}}(S^{-1}u_h)||_{\sigma} \] (7.10)
and
\[ \eta_{C,K} \leq \Lambda_{\nabla \cdot w,K}h_K ||S^{-1}u_h||_K + \sum_{\sigma \in \varepsilon K} \Lambda_{w,\sigma} h^{1/2}_{\sigma} ||\gamma_{\tau_{\sigma}}(S^{-1}u_h)||_{\sigma}. \] (7.11)
From the definitions of $\beta_{*,K}$ and $c_{\omega,\sigma}$ in this theorem, we respectively apply Lemmas 7.2, 7.3 to the above two inequalities so as to obtain
\[ \eta_{NC,K} \leq c_{36}\{\beta_{*,K}\mathcal{E}_K + \sum_{\sigma \in \varepsilon K} c_{\omega,\sigma} \Lambda_{\sigma}^{1/2} ||S^{-1}(u - u_h)||_{\omega,\sigma}\} \] (7.12)
and
\[ \eta_{C,K} \leq c_{37}\{\beta_{*,K}\mathcal{E}_K + \sum_{\sigma \in \varepsilon K} \Lambda_{w,\sigma} c_{\omega,\sigma} ||S^{-1}(u - u_h)||_{\omega,\sigma}\}. \] (7.13)
The assertion (4.13) follows from (7.12) and (7.13) by taking $c_4 := \sqrt{2(d + 1)}\max\{c_{36},c_{37}\}$.

Proof of Theorem 4.5 The local shape regularity of elements implies
\[ \eta_{U,K} \leq \frac{c_{38}}{\sqrt{\varepsilon S,K}} \sum_{\sigma \in \varepsilon K} ||(w \cdot n)_{\sigma}||_{\sigma} (h^{1/2}_{\sigma} ||\tilde{p}_{\sigma}||_{\sigma} + h_{\sigma} ||S^{-1}u_h||_{\omega,\sigma}). \] (7.14)
Then the desired estimate (4.14) follows from (7.14), Lemma 7.4 and the definitions of the constants $\lambda_{\sigma}, \rho_{\sigma}$, and $\mathcal{E}_{D,\omega,\sigma}$.

8. Numerical experiments. In this section, we test our proposed posteriori error estimators on three model problems.

8.1. Model problem with singularity at the origin. We consider the problem (1.1) in an L-shape domain $\Omega = \{(-1, 1) \times (0, 1)\} \cup \{(-1, 0) \times (-1, 0)\}$ with $w = r = 0$ and $f = 0$. The exact solution is given by
\[ p(\rho, \theta) = \rho^{2/3} \sin(2\theta/3), \]
where $\rho, \theta$ are the polar coordinates.

It is well known that this model possesses singularity at the origin. The original mesh consists of 6 right-angled triangles. We employ the centered mixed scheme described in section 3.1 to compute the approximation solution, mark elements in terms of Dörfler marking with the marking parameter $\theta = 0.5$, and then use the "longest edge" refinement to recover an admissible mesh. Specially, the uniform refinement means that all elements should be marked. We note that in the given case, the residual estimators $\eta_{R,K}$ vanish over all $K \in T_h$.

We see in the first figure of Fig 8.1 with 1635 elements that the refinement concentrates around the origin, which means the predicted error estimator captures well the singularity of the solution. The second graph of Fig 8.1 reports the estimated and actual errors of the numerical solutions on uniformly and adaptively refined meshes. It can be seen that one can substantially reduce the number of unknowns necessary to obtain the prescribed accuracy by using the a posteriori error estimates and adaptively refined meshes, and that the error of the flux in $L^2$ norm uniformly reduces with a fixed factor on two successive meshes, and that the adaptive mixed finite element method is a contraction with respect to the energy error.

8.2. Model problem with inhomogeneous diffusion tensor[18, 29, 37]. We consider the problem (1.1) in a square domain $\Omega = (-1, 1) \times (-1, 1)$ with $w = r = 0$ and $f = 0$, where $\Omega$ is divided into four subdomains $\Omega_i$ ($i = 1, 2, 3, 4$) corresponding to the axis quadrants (in the counterclockwise direction), and the diffusion-dispersion tensor $S$ is piecewise constant matrix with $S = s_i I$ in $\Omega_i$. We suppose the exact solution of this model has the form

$$p(r, \theta) = r^\alpha(a_i \sin(\alpha \theta) + b_i \cos(\alpha \theta))$$

in each $\Omega_i$ with Dirichlet boundary conditions. Here $r, \theta$ are the polar coordinates in $\Omega$, $a_i$ and $b_i$ are constants depending on $\Omega_i$, and $\alpha$ is a parameter. We note that the stress solution $u = -S \nabla p$ is not continuous across the interfaces, and only its normal component is continuous. It finally exhibits a strong singularity at the origin. We consider two sets of coefficients in the following table:

| Case 1          | Case 2          |
|-----------------|-----------------|
| $s_1 = s_3 = s_5, s_2 = s_4 = 1$ | $s_1 = s_3 = 100, s_2 = s_4 = 1$ |
| $\alpha = 0.53544965$ | $\alpha = 0.12690207$ |
| $a_1 = 0.47421560, b_1 = 1.00000000$ | $a_1 = 0.10000000, b_1 = 1.00000000$ |
| $a_2 = -0.74535599, b_2 = 2.33333333$ | $a_2 = -9.60396040, b_2 = 2.96039604$ |
| $a_3 = -0.94117559, b_3 = 0.55555555$ | $a_3 = -0.48035487, b_3 = -0.88275659$ |
| $a_4 = -2.40170264, b_4 = -0.48148148$ | $a_4 = 7.70156488, b_4 = -6.45646175$ |
A NEW POSTERIORI ERROR ESTIMATORS FOR CONVECTION-DIFFUSION EQUATIONS

The origin mesh consists of 8 right-angled triangles. We use the centered scheme compute the approximation solution, and mark elements in terms of Dörfler marking with the marking parameter \( \theta = 0.7 \) in the first case and \( \theta = 0.94 \) in the second case. We note that the elementwise estimators \( \xi_K \) are used as the a posteriori error indicators, since the residual estimators \( \eta_{R,K} \) vanish over \( K \in \mathcal{T}_h \).

In Table 8.1 we show for Case 1 some results of the actual error \( E_k \), the a posteriori indicator \( \eta_k \), the experimental convergence rate, \( EOC_{E} \), of \( E_k \), and the experimental convergence rate, \( EOC_{\eta} \), of \( \eta_k \), where

\[
EOC_{E} := \frac{\log(E_{k-1}/E_k)}{\log(\text{DOF}_k/\text{DOF}_{k-1})}, \quad EOC_{\eta} := \frac{\log(\eta_{k-1}/\eta_k)}{\log(\text{DOF}_k/\text{DOF}_{k-1})},
\]

and \( \text{DOF}_k \) denotes the number of elements with respect to the \( k \)--th iteration. We can see that the convergence rates \( EOC_{E} \) and \( EOC_{\eta} \) are close to 0.5 as the iteration number \( k = 15 \), which means the optimal decay of the actual error and a posteriori error indicator \( \eta_k \) is almost attained after 15 iterations with optimal meshes.

**Table 8.1**

Results of actual error \( E_k \), a posteriori indicator \( \eta_k \), and their convergence rates \( EOC_{E} \) and \( EOC_{\eta} \): Case 1

| \( k \) | \( \text{DOF}_k \) | \( E_k \) | \( \eta_k \) | \( EOC_{E} \) | \( EOC_{\eta} \) |
|---|---|---|---|---|---|
| 1 | 8 | 1.3665 | 5.0938 | – | – |
| 2 | 20 | 1.1346 | 3.4700 | 0.2030 | 0.4189 |
| 3 | 44 | 0.8682 | 2.9300 | 0.3394 | 0.2145 |
| 4 | 89 | 0.6672 | 2.5032 | 0.3738 | 0.2235 |
| 5 | 171 | 0.4953 | 2.0907 | 0.4562 | 0.2757 |
| 6 | 354 | 0.3708 | 1.7170 | 0.3979 | 0.2706 |
| 7 | 760 | 0.2751 | 1.5639 | 0.3907 | 0.1222 |
| 8 | 1368 | 0.2163 | 1.3529 | 0.4091 | 0.2466 |
| 9 | 2235 | 0.1776 | 1.1115 | 0.4016 | 0.4004 |
| 10 | 4025 | 0.1381 | 0.8958 | 0.4276 | 0.3667 |
| 11 | 7165 | 0.1106 | 0.7111 | 0.3851 | 0.4004 |
| 12 | 13188 | 0.0871 | 0.5566 | 0.3915 | 0.4015 |
| 13 | 24445 | 0.0671 | 0.4368 | 0.4227 | 0.3927 |
| 14 | 43785 | 0.0510 | 0.3365 | 0.4707 | 0.4476 |
| 15 | 76770 | 0.0387 | 0.2581 | 0.4915 | 0.4724 |

Fig 8.2 shows an adaptively refined mesh with 4763 elements and the estimated and actual errors against the number of elements in adaptively refined meshes for Case 1. Fig 8.3 shows an adaptively refined mesh with 1093 elements and the actual error against the number of elements in adaptively refined meshes for Case 2.

From the first figures of Fig 8.2-8.3, we can see that the refinement again concentrates around the origin, which means the adaptive mixed finite element method detects the region of rapid variation. In the second graphs of Fig 8.1-8.3 each includes an optimal convergence line, which shows in both cases, the energy error performs a trend of descend with an optimal order convergent rate. Simultaneously, from the second graphs of Fig 8.1-8.3, we also see that the proposed estimators are efficient with respect to the strongly discontinuously coefficients.
We note that the energy error is approximated with a 7-point quadrature formula in each triangle.

![Image of mesh with 4763 triangles and estimated and actual error against number of elements in adaptively refined meshes: Case 1.]

**FIG 8.2.** A mesh with 4763 triangles (left) and the estimated and actual error against the number of elements in adaptively refined meshes (right): Case 1.

![Image of mesh with 1093 triangles and actual error against number of elements in adaptively refined mesh: Case 2.]

**FIG 8.3.** A mesh with 1093 triangles (left) and the actual error against the number of elements in adaptively refined mesh (right): Case 2.

### 8.3. Convection-dominated model problem [37]

Let $S = \varepsilon I$, $w = (0, 1)$, $r = 1$ and $\Omega = (0, 1) \times (0, 1)$ in the model (1.1). We consider four cases: $\varepsilon = 0.1, 0.01, 0.001, 0.0001$. Neumann boundary conditions on the upper side, Dirichlet boundary conditions elsewhere, and the source term $f$ are chosen such that the exact solution has the form

$$p(x, y) = 0.5(1 - \tanh\left(\frac{0.5 - x}{a}\right))$$

with $a$ a positive constant. This solution is, in fact, one-dimensional and possesses an internal layer of width $a$ which we shall set, respectively, equal to $0.1, 0.05, 0.02, 0.001$.

We still start computations from an origin mesh which consists of 8 right-angled triangles, and refine it either uniformly (up to five refinements) or adaptively.

In Fig 8.4 with $\varepsilon = 0.01$, $a = 0.05$ and Fig 8.5 with $\varepsilon = 0.001$, $a = 0.05$, we can see that the refinement concentrates at an internal layer of width $a = 0.05$, and is away from the center of the shock. Both the convection-dominated regime on coarse grids and diffusion-dominated regime obtain the progressive refinement. The effect is still rather good even if the approximation to displacement is piecewise constant.
Fig 8.4. A mesh with 12943 triangles (left) and the approximate displacement (piecewise constant) on the corresponding adaptively refined mesh (right) for $\varepsilon = 0.01$ and $a=0.05$.

Fig 8.5. A mesh with 16951 triangles (left) and approximate displacement (piecewise constant) on the corresponding adaptively refined mesh (right) for $\varepsilon = 0.001$ and $a=0.05$.

Fig 8.6 shows the mesh with 39184 triangles (left) and postprocessing approximation to the scalar displacement on the corresponding adaptively refined mesh (right) in case: $\varepsilon = 0.0001$ and width $a = 0.001$. Here the value of the postprocessing approximation on each node is taken as the algorithmic mean of the values of the displacement finite element solution on all the elements sharing the vertex. The reason for the postprocessing is that the displacement finite element solution is not continuous on each vertex of the triangulation. We again see that the refinement focuses around layer of width $a = 0.001$, this indicates that the estimators actually capture interior layers and resolve them in convection-dominated regions. In addition, the postprocessing approximation to the scalar displacement obtains a satisfactory result.

In Fig 8.7 with $\varepsilon = 0.1$, $a = 0.02$ (left), the estimated and actual errors are plotted against the number of elements in uniformly and adaptively refined meshes. Again, we see that one can substantially reduce the unknowns necessary to attain the prescribed precision by using the proposed estimators and adaptively refined grids. The second graph of Fig 8.7 shows the actual error against the number of elements in adaptively refined meshes for different $\varepsilon$ in case $a = 0.1$, and also concludes a line with optimal convergence $-1/2$. In addition, we also see that the almost same error decay occurs in cases: $\varepsilon = 0.01$ and $\varepsilon = 0.001$. 
FIG 8.6. A mesh with 39189 triangles (left) and postprocessing approximate displacement on the corresponding adaptively refined mesh (right) for $\varepsilon = 0.0001$ and $a=0.001$.

FIG 8.7. Estimated and actual error against the number of elements in uniformly and adaptively refined meshes for $\varepsilon = 0.1$, $a = 0.02$ (left) and actual error against the number of elements in adaptively refined meshes for different $\varepsilon$ for $a = 0.1$ (right).

REFERENCES

[1] B. Achchab, A. Agouzal, J. Baranger, and J.F. Maitre, Estimateur d’erreur a posteriori hiérarchique. Application aux éléments finis mixtes, Numer. Math., 80 (1998), 159-179.
[2] M. Ainsworth, A synthesis of a posteriori error estimation techniques for conforming, nonconforming and discontinuous Galerkin finite element methods, in Recent Advances in Adaptive Computation, Contemp. Math. 383, AMS, Providence, RI, 2005, 1-14.
[3] M. Ainsworth, Robust a posteriori error estimation for nonconforming finite element approximation, SIAM J. Numer. Anal., 42 (2005), 2320-2341.
[4] M. Ainsworth, J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis. Wiley, New York, 2000.
[5] A. Alonso, Error estimators for a mixed method, Numer. Math., 74 (1996), 385-395.
[6] I. Babuška and W.C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal., 15 (1978), 736-754.
[7] I. Babuška, T. Strouboulis, The finite element method and its reliability. Clarendon Press, Oxford, 2001.
[8] W. Bangerth, R. Rannacher, Adaptive Finite Element Methods for Differential Equations. Birkhäuser, Basel, 2003.
[9] C. Bernardi and R. Verfürth, Adaptive finite element methods for elliptic equations with non-smooth coefficients, Numer. Math., 85 (2000), 579-608.
[10] M. Bebendorf, A note on the Poincaré inequality for convex domains, Z. Anal. Anwend., 22 (2003), 751-756.
[11] D. Braess and R. Verfürth, A posteriori error estimators for the Raviart-Thomas element, SIAM J. Numer. Anal., 33 (1996), 2431-2444.
[12] C. Carstensen, A posteriori error estimate for the mixed finite method, Math. Comp., 66:218 (1997), 465-476.
[13] C. Carstensen and S. Bartels, Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. Part one: low order conforming, nonconforming, and mixed FEM, Math. Comp., 71:239 (2002), 945-969.

[14] C. Carstensen, J. Hu and A. Orlando, Framework for the a posteriori error analysis of nonconforming finite elements, SIAM J. Numer. Anal., 45:1 (2007), 68-82.

[15] P.G. Ciarlet, The finite element method for elliptic problems. Stud. Math. Appl. 4, North-Holland, Amsterdam, 1978.

[16] C. Dawson, Analysis of an upwind-mixed finite element method for nonlinear contaminant transport equations, SIAM J. Numer. Anal., 35 (1998), 1709-1724.

[17] J.R. Douglas and J.E. Roberts, Global estimates for mixed methods for second elliptic equations, Math. Comp., 44 (1985), 39-52.

[18] G.T. Eigestad and R.A. Klausen, On the convergence of the multi-point flux O-method: Numerical experiments for discontinuous permeability, Numer. Methods Partial Differential Equations., 21 (2005), 1079-1098.

[19] G. Kanschat, F.T. Suttmeier. A posteriori error estimates for nonconforming finite element schemes. Calcolo., 36:3 (1999), 129-141.

[20] O.A. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, SIAM J. Numer. Anal., 41 (2003), 2374-2399.

[21] R. Kirby, Residual a posteriori error estimates for the mixed finite element method, Comput. Geosci., 7 (2003), 197-214.

[22] G. Kunert, A posteriori error estimation for convection dominated problems on anisotropic meshes, Math. Methods Appl. Sci., 26 (2003), 589-617.

[23] G. Kunert, A posteriori $H^1$ error estimation for a singularly perturbed reaction diffusion problem on anisotropic meshes, IMA J. Numer. Anal., 25(2005), 408-428.

[24] C. Lovadina and R. Stenberg. Energy norm a posteriori error estimates for mixed finite element methods, Math. Comp., 75 (2006), 1659-1674.

[25] M. Ohlberger, A posteriori error estimates for vertex centered finite volume approximations of convection-diffusion-reaction equations, M2AN Math. Model. Numer. Anal., 35 (2001), 355-387.

[26] M. Ohlberger, A posteriori error estimate for finite volume approximations to singularly perturbed nonlinear convection-diffusion equations, Numer. Math., 87 (2001), 737-761.

[27] L. E. Payne and H. F. Weinberger, An optimal Poincaré inequality for convex domains, Arch. Ration. Mech. Anal., 5 (1960), 286-292.

[28] M. Petzoldt, A posteriori error estimators for elliptic equations with discontinuous coefficients, Adv. Comput. Math., 16 (2002), 47-75.

[29] B. Rivière and M.F. Wheeler, A posteriori error estimates for a discontinuous Galerkin method applied to elliptic problems, Comput. Math. Appl., 46 (2003), 141-163.

[30] G. Sangalli, A robust a posteriori estimator for the residual-free bubbles method applied to advection-diffusion problems, Numer. Math., 89 (2001), 379-399.

[31] G. Sangalli, Robust a-posteriori estimator for advection-diffusion-reaction problems, Math. Comp., 77:261 (2008), 41-70.

[32] B.J. Wohlmuth and R.H.W.Hoppe, A Comparison of a Posteriori Error Estimators for Mixed Finite Element Discretizations by Raviart-Thomas element. Math.Comput., 68 (1999), 1347-1378.

[33] R. Verfürth, A review of posteriori error estimation and adaptive mesh-refinement techniques, Teubner Wiley, Stuttgart, 1996.

[34] R. Verfürth, Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation, Numer. Math., 78 (1998), 479-493.

[35] R. Verfürth, A posteriori error estimators for convection-diffusion equations, Numer. Math., 80 (1998), 641-663.

[36] R. Verfürth, Robust a posteriori error estimates for stationary convection-diffusion equations, SIAM J. Numer. Anal., 43 (2005), 1766-1782.

[37] M. Vohralík, A posteriori error estimates for lowest-order mixed finite element discretizations of convection-diffusion-reaction equations, SIAM J. Numer. Anal., 45:4 (2007), 1570-1599.