Multiple orthogonal polynomials associated with branched continued fractions for ratios of hypergeometric series

Hélder Lima*

Department of Mathematics, KU Leuven, Celestijnenlaan 200B box 2400, 3001 Leuven, Belgium

To appear in Advances in Applied Mathematics

Abstract

The main objects of the investigation presented in this paper are branched-continued-fraction representations of ratios of contiguous hypergeometric series and type II multiple orthogonal polynomials on the step-line with respect to linear functionals or measures whose moments are ratios of products of Pochhammer symbols. This is an interesting case study of the recently found connection between multiple orthogonal polynomials and branched continued fractions that gives a clear example of how this connection leads to considerable advances on both topics.

We start by obtaining new results about generating polynomials of lattice paths and total positivity of matrices and giving new contributions to the general theory of the connection between multiple orthogonal polynomials and branched continued fractions with emphasis on its application to the analysis of multiple orthogonal polynomials. Then, we construct new branched continued fractions for ratios of contiguous hypergeometric series. We give conditions for positivity of the coefficients of these branched continued fractions and we show that the ratios of products of Pochhammer symbols are generating polynomials of lattice paths for a special case of the branched continued fractions under study. Next, we introduce a family of type II multiple orthogonal polynomials on the step-line associated with those branched continued fractions. We present a formula as terminating hypergeometric series for these polynomials, we study their differential properties, and we find an explicit recurrence relation satisfied by them. Finally, we focus the analysis of the multiple orthogonal polynomials to the cases where the corresponding branched-continued-fraction coefficients are all positive. In those cases, the orthogonality conditions can be written using measures on the positive real line involving Meijer G-functions and we obtain results about the location of the zeros and the asymptotic behaviour of the polynomials.

Keywords: Multiple orthogonal polynomials, branched continued fractions, hypergeometric series, Pochhammer symbols, production matrices, lower-Hessenberg matrices.

Mathematics Subject Classification 2000: Primary: 11J70, 33C45, 42C05; Secondary: 05A10, 15B99, 30B70, 30E05, 33C20.

*Email address: helder.lima@kuleuven.be
## Contents

1. **Introduction and motivation**  
   - 3

2. **Background**  
   - 5
     - 2.1 Hypergeometric series  
       - 5
     - 2.2 Multiple orthogonal polynomials  
       - 6
     - 2.3 Lattice paths and branched continued fractions  
       - 7
     - 2.4 Production matrices  
       - 10
     - 2.5 Total positivity and oscillation matrices  
       - 10

3. **Results about generalised and modified $m$-Stieltjes-Rogers polynomials**  
   - 11

4. **Multiple orthogonal polynomials and branched continued fractions**  
   - 15
     - 4.1 Connection via production matrices  
       - 16
     - 4.2 Positive branched-continued-fraction coefficients, orthogonality measures, and zeros  
       - 19

5. **Branched continued fractions for ratios of hypergeometric series**  
   - 21
     - 5.1 Construction of the branched continued fractions  
       - 21
     - 5.2 Conditions for positivity of the coefficients  
       - 27
     - 5.3 Modified $m$-Stieltjes-Rogers-polynomials when $a_{r+1} = 1$  
       - 28
     - 5.4 A limiting type of ratios  
       - 29

6. **A general class of hypergeometric multiple orthogonal polynomials**  
   - 30
     - 6.1 Explicit expressions as terminating hypergeometric series  
       - 30
     - 6.2 Differential properties  
       - 35
     - 6.3 Recurrence relation  
       - 36

7. **Generalised $m$-Stieltjes-Rogers polynomials**  
   - 37

8. **Multiple orthogonal polynomials with respect to Meijer G-functions**  
   - 39
1 Introduction and motivation

This paper gives a detailed investigation of a case study of the connection between two different corners of Mathematics: multiple orthogonal polynomials, studied by the special-functions community, and branched continued fractions, introduced by the enumerative-combinatorics community to solve total-positivity problems. This connection was introduced and analysed in the recent paper [37].

The main objects of the investigation presented here are branched-continued-fraction representations for ratios of contiguous hypergeometric series $r+1F_s$ and multiple orthogonal polynomials with respect to $m = \max(r,s)$ linear functionals or measures whose moments are ratios of Pochhammer symbols

$$\left(\frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n}\right)_{n \in \mathbb{N}} \text{ with } (r,s) \in \mathbb{N}^2 \setminus \{(0,0)\}. \quad (1.1)$$

The Pochhammer symbol $(z)_n$, also known as the rising factorial, is defined by

$$(z)_0 = 1 \quad \text{and} \quad (z)_n := z(z+1)\cdots(z+n-1) \quad \text{for } n \geq 1. \quad (1.2)$$

These branched continued fractions and multiple orthogonal polynomials are linked because the ordinary generating function of the moment sequence in (1.1) is

$$\sum_{n=0}^{\infty} \left(\frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n}\right) t^n = r+1F_s \left(\frac{a_1, \cdots, a_r, 1}{b_1, \cdots, b_s} \bigg| t\right). \quad (1.3)$$

The generating function (1.3) is the special case $a_{r+1} = 1$ of ratios of contiguous hypergeometric series for which branched-continued-fraction representations were introduced in [33] and are generalised here.

There are several already known particular cases of multiple orthogonal polynomials with respect to measures whose moment sequences are of the form (1.1). The cases where $m = 1$ are $(r,s)$ equal to $(1,0)$, $(1,1)$, and $(0,1)$, and they correspond to the well-known Laguerre, Jacobi, and Bessel classical orthogonal polynomials, respectively. When $s \leq r = 2$, we obtain multiple orthogonal polynomials with respect to Nikishin systems of 2 measures, supported on the whole positive real line if $s < r$ (i.e., $s \in \{0,1\}$) or on the interval $(0,1)$ if $s = r$ (i.e., $s = 2$). The multiple orthogonal polynomials corresponding to the cases $(r,s)$ equal to $(2,0)$, $(2,1)$, and $(2,2)$ were investigated in [3, 9, 41], [24], and [25], respectively; their orthogonality measures involve modified Bessel functions of the second kind, confluent hypergeometric functions of the second kind, and Gauss’ hypergeometric function, respectively. Finally, when $s = 0$ and $r$ is an arbitrary positive integer, the moments in (1.1) reduce to products of Pochhammer symbols, which are associated with
multiple orthogonal polynomials with respect to $r$ measures on the positive real line involving Meijer G-functions $G_{r,0}^{0,r}$ introduced in [23] to investigate singular values of products of Ginibre random matrices. The special cases $r = 1$ and $r = 2$ of the latter polynomials are, respectively, the classical Laguerre polynomials and the multiple orthogonal polynomials investigated in [3, 9, 41].

When $m = 1$, the branched continued fractions appearing in this paper reduce to classical continued fractions. If $(r,s) = (1,1)$, we recover Gauss’ continued fraction for ratios of contiguous $2F_1$; if $(r,s)$ is equal to $(1,0)$ or $(0,1)$, we obtain known continued-fraction representations for ratios of contiguous $2F_0$ and $1F_1$, respectively, which can be obtained as limiting cases of Gauss’ continued fraction. These continued fractions are connected to the Jacobi, Laguerre, and Bessel polynomials, respectively. These are particular instances of the well-known relation between continued fractions and orthogonal polynomials (see [6, Ch. III.4] and [42]).

We end this introductory section with an outline of the paper.

In Section 2, we give some background on the main topics involved. In particular, we give a brief introduction to multiple orthogonal polynomials and to branched continued fractions.

In Section 3, we present new results about generalised and modified $m$-Stieltjes-Rogers polynomials (which are generating polynomials of lattice paths, more precisely partial $m$-Dyck paths) and total positivity. These results were motivated by their applications to the study of multiple orthogonal polynomials, which are made clear in Section 4, but they are worthy of interest on their own.

In Section 4, we revisit the connection between multiple orthogonal polynomials and branched continued fractions introduced in [37] and give new contributions to the study of this connection, with emphasis on its application to the analysis of multiple orthogonal polynomials.

In Section 5, we construct new branched continued fractions for ratios of contiguous hypergeometric series, which generalise the branched continued fractions introduced in [33, §14]. In addition, we give conditions for non-negativity of the coefficients of our branched continued fractions and we show that the modified $m$-Stieltjes-Rogers polynomials linked to the special case $a_{r+1} = 1$ of our branched continued fractions are ratios of products of Pochhammer symbols of the form in (1.1) multiplied by a binomial coefficient.

In Section 6, we analyse the type II multiple orthogonal polynomials on the step-line with respect to linear functionals whose moments are the ratios of products of Pochhammer symbols shown to be modified $m$-Stieltjes-Rogers polynomials in Section 5. We obtain explicit expressions for these multiple orthogonal polynomials as terminating hypergeometric series, we derive differential properties satisfied by them, and we use their connection with the branched continued fractions constructed in Section 5 to obtain expressions for their recurrence coefficients as combinations of branched-continued-fraction coefficients.

In Section 7, we present an explicit formula for the generalised $m$-Stieltjes-Rogers polynomials corresponding to the special case $a_{r+1} = 1$ of the branched continued fractions constructed in Section 5.

In Section 8, we focus the analysis of the multiple orthogonal polynomials introduced in Section 6 to the cases where the corresponding branched-continued-fraction coefficients are all positive. In those cases, the
linear functionals of orthogonality are induced by measures on the positive real line whose densities are
Meijer G-functions, we have positivity of the recurrence coefficients, we obtain results about the location of
the zeros and the asymptotic behaviour of the multiple orthogonal polynomials, and we show that special
cases of these polynomials include polynomial sequences with constant recurrence coefficients and particu-
lar instances of the Jacobi-Piñeiro polynomials.

We finish the paper with some final remarks about applications and future directions of investigation related
with the work presented here.

In summary, this paper brings to light new contributions to the general theory of lattice paths and branched
continued fractions (Section 3) and to the general theory of multiple orthogonal polynomials via their con-
nection with branched continued fractions (Section 4) as well as new results about its main objects of inves-
tigation: branched continued fractions for ratios of hypergeometric series (Sections 5 and 7) and a new class
of hypergeometric multiple orthogonal polynomials (Sections 6 and 8).

2 Background

In this section we present some definitions and basic results about hypergeometric series, multiple orthogo-
nal polynomials, lattice paths and branched continued fractions, production matrices, and total positivity.

2.1 Hypergeometric series

For \( p, q \in \mathbb{N} \), the (generalised) hypergeometric series (see \([1, 14, 26]\)) is defined by

\[
pFq\left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_q} \bigg| \frac{z}{\alpha} \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.
\]  

(2.1)

We treat this expression as a formal power series. Convergence of non-terminating hypergeometric series
plays no role here, except in Proposition 8.3.

The hypergeometric series (2.1) is a solution of the differential equation

\[
\left[ \prod_{j=1}^{q} \left( z \frac{d}{dz} + b_j \right) \frac{d}{dz} - \prod_{i=1}^{p} \left( z \frac{d}{dz} + a_i \right) \right] F(z) = 0.
\]  

(2.2)

The derivative of a hypergeometric series is equal to a shift in the parameters up to multiplication by a
constant:

\[
\frac{d}{dz} pFq\left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_q} \bigg| \frac{z}{\alpha} \right) = \frac{a_1 \cdots a_p}{b_1 \cdots b_q} pFq\left( \frac{a_1+1, \ldots, a_p+1}{b_1+1, \ldots, b_q+1} \bigg| \frac{z}{\alpha} \right).
\]  

(2.3)

The hypergeometric series satisfies the confluent relation

\[
\lim_{|\alpha| \to \infty} p+1Fq\left( \frac{a_1, \ldots, a_p, \alpha}{b_1, \ldots, b_q} \bigg| \frac{z}{\alpha} \right) = pFq\left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_q} \bigg| \frac{z}{\alpha} \right).
\]  

(2.4)
2.2 Multiple orthogonal polynomials

Multiple orthogonal polynomials (see [20, Ch. 23], [31, Ch. 4], and [40, §3]) are a generalization of conventional orthogonal polynomials [6, 20, 38] in which the polynomials satisfy orthogonality relations with respect to several measures rather than just one. We revisit and algebraise the theory of multiple orthogonal polynomials, defining the orthogonality conditions via linear functionals in the ring $R[x]$ of polynomials in one variable with coefficients in a commutative ring $R$.

A linear functional $u$ defined on $R[x]$ is a linear map $u : R[x] \to R$. The dual space of $R[x]$ is the vector space consisting of all linear functionals on $R[x]$. The action of a linear functional $u$ on a polynomial $f \in R[x]$ is denoted by $u[f]$. The moment of order $n$ of a linear functional $u$ is equal to $u[x^n]$. By linearity, every linear functional $u$ is uniquely determined by its moments or, alternatively, by the values of $u$ at the elements of any basis of $R[x]$.

There are two types of multiple orthogonal polynomials: type I and type II. Both type I and type II polynomials satisfy orthogonality conditions with respect to a vector of $d$ linear functionals for a positive integer $d$ and depend on a multi-index $\vec{n} = (n_0, \cdots, n_{d-1}) \in \mathbb{N}^d$ of norm $|\vec{n}| = n_0 + \cdots + n_{d-1}$, and both reduce to standard orthogonal polynomials when the number of orthogonality functionals, $d$, is 1.

The type I multiple orthogonal polynomials for $\vec{n} = (n_0, \cdots, n_{d-1}) \in \mathbb{N}^d$ with respect to a vector of $d$ linear functionals $(u_0, \cdots, u_{d-1})$ are given by a vector of $r$ polynomials $(A_{\vec{n},0}, \cdots, A_{\vec{n},d-1})$, with $\deg A_{\vec{n},j} \leq n_j - 1$ for each $j \in \{0, \cdots, d-1\}$, satisfying the orthogonality and normalisation conditions

$$\sum_{j=0}^{d-1} u_j \left[ x^k A_{\vec{n},j} \right] = \begin{cases} 0 & \text{if } 0 \leq k \leq |\vec{n}| - 2, \\ 1 & \text{if } k = |\vec{n}| - 1. \end{cases} \quad (2.5)$$

The type II multiple orthogonal polynomial with respect to a vector of $d$ linear functionals $(u_0, \cdots, u_{d-1})$ for $\vec{n} = (n_0, \cdots, n_{d-1}) \in \mathbb{N}^d$ consists of a monic polynomial $P_{\vec{n}}$ of degree $|\vec{n}| = n_0 + \cdots + n_{d-1}$, the norm of $\vec{n}$, which satisfies, for each $j \in \{0, \cdots, d-1\}$, the orthogonality conditions

$$u_j \left[ x^k P_{\vec{n}} \right] = 0 \quad \text{if } k \in \{0, \cdots, n_j - 1\}. \quad (2.6)$$

Here we focus only on type II multiple orthogonal polynomials.

The orthogonality conditions for multiple (and standard) orthogonal polynomials are usually defined with respect to vectors of measures on the real line or the complex plane instead of linear functionals. To obtain the type II orthogonality conditions with respect to vectors of measures instead of linear functionals on a commutative ring $R$, consider $R \in \{\mathbb{R}, \mathbb{C}\}$ and linear functionals defined by the integrals with respect to the orthogonality measures, that is, consider linear functionals defined via a measure $\mu$ by

$$u[f] = \int f(x) d\mu(x) \quad \text{for any } f \in R[x]. \quad (2.7)$$

We are interested in the type II multiple orthogonal polynomials for multi-indices on the so-called step-line. A multi-index $(n_0, \cdots, n_{d-1}) \in \mathbb{N}^d$ is on the step-line if $n_0 \geq n_1 \geq \cdots \geq n_{d-1} \geq n_0 - 1$. For a fixed $d \in \mathbb{Z}^+$,
there is a unique multi-index of norm \( n \) on the step-line of \( \mathbb{N}^d \), for each \( n \in \mathbb{N} \). Therefore, the type II multiple orthogonal polynomials on the step-line form a sequence with exactly one polynomial of degree \( n \), for each \( n \in \mathbb{N} \), equal to the length of the corresponding multi-index, and we can replace the multi-index by its length without any ambiguity. The type II multiple orthogonal polynomials on the step-line of \( \mathbb{N}^d \) are often referred to as \( d \)-orthogonal polynomials, as introduced in [27], where \( d \) is the number of orthogonality functionals. This means that a polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\) is \( d \)-orthogonal with respect to \((u_0, \cdots, u_{d-1})\) if \( P_n(x) \) is the type II multiple orthogonal polynomial for the multi-index on the step-line of \( \mathbb{N}^d \) with norm \( n \). Applying (2.6) to the multi-indices on the step-line, \((P_n(x))_{n \in \mathbb{N}}\) satisfies

\[
\int x^k P_n(x) d\mu_j(x) = \begin{cases} N_n & \text{if } n = dk + j \\ 0 & \text{if } n \geq dk + j + 1 \end{cases} \quad \text{for } j \in \{0, \cdots, d - 1\}. \tag{2.8}
\]

According to [27, Th. 2.1], a polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\) is \( d \)-orthogonal if and only if it satisfies a \((d + 1)\)-order recurrence relation of the form

\[
P_{n+1}(x) = xP_n(x) - \sum_{k=0}^{d} \gamma_{n-k}^{[k]} P_{n-k}(x). \tag{2.9}
\]

The recurrence coefficients in (2.9) are collected in the infinite \((d + 2)\)-banded unit-lower-Hessenberg matrix

\[
H = (h_{i,j})_{i,j \in \mathbb{N}} = \begin{bmatrix}
\gamma_0^{[0]} & 1 \\
\gamma_0^{[1]} & \gamma_1^{[0]} & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\gamma_0^{[d]} & \cdots & \gamma_{d-1}^{[1]} & \gamma_d^{[0]} & 1 \\
\gamma_1^{[d]} & \cdots & \gamma_d^{[1]} & \gamma_{d+1}^{[0]} & 1
\end{bmatrix}. \tag{2.10}
\]

The \( d \)-orthogonal polynomials \((P_n(x))_{n \in \mathbb{N}}\) are the characteristic polynomials of the truncated finite matrices formed by the first \( n \) rows and columns of \( H \) (see [10, §2.2]). We revisit this property in Proposition 4.1.

### 2.3 Lattice paths and branched continued fractions

The branched continued fractions appearing in this paper (other types of branched continued fractions exist in the literature) were introduced in [33] and were further explored in [32]. We follow their definitions.

For a positive integer \( m \), a \( m \)-Dyck path is a path in the lattice \( \mathbb{N} \times \mathbb{N} \), starting and ending on the horizontal axis, using steps \((1,1)\), called rises, and \((1,-m)\), called \( m \)-falls (see Fig.1 for an example of a 2-Dyck path). When \( m = 1 \), the 1-Dyck paths are simply known as Dyck paths.

More generally, we consider \( m \)-Dyck paths at level \( k \), which are paths in \( \mathbb{N} \times \mathbb{N}_{\geq k} \) using steps \((1,1)\) and \((1,-m)\) and starting and ending at height \( k \), and partial \( m \)-Dyck paths, which are paths in \( \mathbb{N} \times \mathbb{N} \) using steps \((1,1)\) and \((1,-m)\) and allowed to start and end anywhere in \( \mathbb{N} \times \mathbb{N} \).
Observe that the length of a $m$-Dyck path, as well as a $m$-Dyck path at level $k$, is always a multiple of $m + 1$. For an infinite sequence of indeterminates $\alpha = (\alpha_i + m)_{i \in \mathbb{N}}$, the $m$-Stieltjes-Rogers polynomial $S_n^{(m)}(\alpha)$, with $n \in \mathbb{N}$, is the generating polynomial for $m$-Dyck paths of length $(m + 1)n$, with each rise having weight 1 and each $m$-fall from height $i$ having weight $\alpha_i$. They are an extension of the Stieltjes-Rogers polynomials introduced by Flajolet in [17], which correspond to the case $m = 1$.

Let $f_0(t)$ be the generating function for $m$-Dyck paths with the weights specified above, considered as a formal power series in $t$, that is,

$$f_0(t) = \sum_{n=0}^{\infty} S_n^{(m)}(\alpha) t^n. \tag{2.11}$$

More generally, let $f_k(t)$ be the generating function for $m$-Dyck paths at level $k$ with the same weights. Observe that $f_k(t)$ is $f_0(t)$ with each $\alpha_i$ replaced by $\alpha_i + k$. The sequence $(f_k(t))_{k \in \mathbb{N}}$ satisfies the functional equations (see [33, Eqs. 2.26-2.27])

$$f_k(t) = 1 + \alpha_{k+m} t \prod_{j=0}^{m} f_{k+j}(t) \quad \text{and} \quad f_k(t) = \frac{1}{1 - \alpha_{k+m} t \prod_{j=1}^{m} f_{k+j}(t)} \tag{2.12}$$

Successively iterating the former we find that

$$f_k(t) = \frac{1}{1 - \alpha_{k+m} t \prod_{i_1=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1} t \prod_{i_2=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2} t \prod_{i_3=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3} t \prod_{i_4=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3+i_4} t \prod_{i_5=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3+i_4+i_5} t \prod_{i_6=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3+i_4+i_5+i_6} t \prod_{i_7=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3+i_4+i_5+i_6+i_7} t \prod_{i_8=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3+i_4+i_5+i_6+i_7+i_8} t \prod_{i_9=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3+i_4+i_5+i_6+i_7+i_8+i_9} t \prod_{i_{10}=1}^{m} \frac{1}{1 - \alpha_{k+m+i_1+i_2+i_3+i_4+i_5+i_6+i_7+i_8+i_9+i_{10}} \cdots}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}} \tag{2.13}$$

In particular, a representation for $f_0(t)$ is obtained by taking $k = 0$ in (2.13). We call the right-hand side of (2.13) a Stieltjes-type $m$-branched continued fraction, or a $m$-branched S-fraction for short.
When $m = 1$, (2.13) reduces to a classical Stieltjes continued fraction or S-fraction

$$f_k(t) = \frac{1}{\alpha_{k+1} t - \frac{\alpha_{k+2} t}{1 - \ldots}}. \quad (2.14)$$

The representation for the generating function of the classical Dyck paths with each rise having weight 1 and each $m$-fall from height $i$ having weight $\alpha_i$ as the continued fraction (2.14) was shown in [17].

We are interested in generalisations of the $m$-Stieltjes-Rogers polynomials. Observe that every vertex $(x, y)$ of a partial $m$-Dyck path starting at $(0, 0)$, and in particular its final vertex, satisfies $x \equiv y \mod (m+1)$. For an infinite sequence of indeterminates $\alpha = (\alpha_i)_{i \geq m}$ and $n, k, j \in \mathbb{N}$, the generalised $m$-Stieltjes-Rogers polynomials of type $j$, $S_{n,k}^{(m;j)}(\alpha)$, are defined in [11] as the generating polynomials for partial $m$-Dyck paths from $(0,0)$ to $((m+1)n + j, (m+1)k + j)$ in which each rise gets weight 1 and each $m$-fall from height $i$ gets weight $\alpha_i$. The fundamental generalised $m$-Stieltjes-Rogers polynomials are the ones of type $0 \leq j \leq m$, because we can reduce the generalised $m$-Stieltjes-Rogers polynomials of higher type $j$ to the fundamental types via the trivial relation $S_{n,k}^{(m;j+\ell(m+1))}(\alpha) = S_{n+\ell,k+\ell}^{(m;j)}(\alpha)$.

Here we are interested in the cases where either $j = 0$ or $k = 0$. When $j = 0$, we have the generalised $m$-
Stieltjes-Rogers polynomials, $S_{n,k}^{(m)}(\alpha)$, generating partial $m$-Dyck paths from $(0,0)$ to $((m+1)n, (m+1)k)$; when $k = 0$, we have the modified $m$-Stieltjes-Rogers polynomials of type $j$, $S_{n}^{(m;j)}(\alpha)$, counting partial $m$-Dyck paths from $(0,0)$ to $((m+1)n + k, k)$. Based on [33, Prop. 2.3], we know that the generating function of the modified $m$-Stieltjes-Rogers polynomials of type $j$, for any $j \in \mathbb{N}$, is

$$f_0(t) \cdots f_j(t) = \sum_{n=0}^{\infty} S_{n}^{(m;j)}(\alpha) t^n. \quad (2.15)$$

We consider the matrices $S_{n,k}^{(m;j)} = \left( S_{n,k}^{(m;j)}(\alpha) \right)_{n,k \in \mathbb{N}}$ of the generalised $m$-Stieltjes-Rogers polynomials of type $j$, for $j \in \mathbb{N}$. When $j = 0$, we denote this matrix by $S_{n}^{(m)}$, that is, $S_{n}^{(m)} = \left( S_{n,k}^{(m)}(\alpha) \right)_{n,k \in \mathbb{N}}$ is the matrix of the generalised $m$-Stieltjes-Rogers polynomials. Moreover, we introduce the matrix of modified $m$-Stieltjes-
Rogers polynomials $\hat{S}_{n}^{(m;j)} = \left( \hat{S}_{n,k}^{(m;j)}(\alpha) \right)_{n,j \in \mathbb{N}}$. Note that the $j$th-column of $\hat{S}_{n}^{(m)}$ is the $0$th-column of $S_{n}^{(m;j)}$. The classical case $m = 1$ of this observation corresponds to [36, Prop. 9.2].

It is clear that $S_{n,0}^{(m;j)}(\alpha) = S_{n}^{(m)}(\alpha)$ for all $n \in \mathbb{N}$, so the $0$th-columns of the matrices $S_{n}^{(m)}$ and $\hat{S}_{n}^{(m)}$ display the ordinary $m$-Stieltjes-Rogers polynomials. Furthermore, for any $j \in \mathbb{N}$, $S_{n,n}^{(m;j)}(\alpha) = 1$ for all $n \in \mathbb{N}$ and $S_{n,k}^{(m;j)}(\alpha) = 0$ whenever $k > n$, so the generalised $m$-Stieltjes-Rogers polynomials of type $j$ form a unit-lower-triangular matrix. In particular, $S_{n}^{(m)}$ is a unit-lower-triangular matrix. The same is not true for $\hat{S}_{n}^{(m)}$. For instance, $\hat{S}_{0}^{(m;k)}(\alpha) = 1$ for all $k \in \mathbb{N}$, because the only partial $m$-Dyck path from $(0,0)$ to $(k,k)$ is formed by $k$ consecutive rises, so all entries in the $0$th-row of $\hat{S}_{n}^{(m)}$ are equal to 1.
2.4 Production matrices

Let \( \Pi = (\pi_{i,j})_{i,j \in \mathbb{N}} \) be an infinite matrix with entries in a commutative ring \( \mathcal{R} \). If all the powers of \( \Pi \) are well-defined, we can define an infinite matrix \( A = (a_{n,k})_{n,k \in \mathbb{N}} \) by \( a_{n,k} = (\Pi^n)_{0,k} \). In particular, \( a_{0,0} = 1 \) and \( a_{0,k} = 0 \), if \( k \geq 1 \). We call \( \Pi \) the production matrix and \( A \) the output matrix. The method of production matrices was introduced in \([12, 13]\).

Production matrices play a fundamental role in the connection between multiple orthogonal polynomials and branched continued fractions, which we explore in detail in Section 4. This is a consequence of both the production matrix of the generalised \( m \)-Stieltjes-Rogers polynomials \( (s_{n,k}^{(m)}(\alpha))_{n,k \in \mathbb{N}} \) (see explicit formulas for this production matrix in \((3.8)-(3.9))\) and the matrix encoding the recurrence relation of a \( m \)-orthogonal polynomial sequence being \((m+2)\)-banded unit-lower-Hessenberg matrices.

Therefore, we are interested in production matrices that are unit-lower-Hessenberg. A unit-lower-Hessenberg matrix \( \Pi \) is always row-finite (i.e., \( \Pi \) has only finitely many nonzero entries in each row), so all the powers of \( \Pi \) are well defined and we can construct the output matrix of \( \Pi \), which is a unit-lower-triangular matrix. Conversely, the production matrix of a unit-lower-triangular matrix, which always exists and is unique, is a unit-lower-Hessenberg matrix.

2.5 Total positivity and oscillation matrices

We say that a matrix with real entries is *totally positive* and *strictly totally positive* if all its minors are, respectively, nonnegative and positive. This terminology is the same used in \([33]\) and \([35]\). However, we warn the reader that other references on total positivity, including \([18]\) and \([15]\), use the terms totally nonnegative and totally positive matrices for what we define here as totally positive and strictly totally positive matrices, respectively.

Oscillation matrices are a class of matrices intermediary between totally positive and strictly totally positive matrices. A \((n \times n)\)-matrix \( A \) with real entries is an oscillation matrix if \( A \) is totally positive and some power of \( A \) is strictly totally positive. We are interested in oscillation matrices because they share the nice spectral properties of strictly totally positive matrices: according to the Gantmacher-Krein theorem (see \([18, \text{Ths. II-6, II-14}])\), the eigenvalues of an oscillation matrix are all simple, real, and positive, and interlace with the eigenvalues of the submatrices obtained by removing either its first or last column and row.

If we consider matrices with entries in a partially ordered commutative ring \( \mathcal{R} \), we can still define a matrix to be totally positive if all its minors are nonnegative. The definition of a partially ordered commutative ring \( \mathcal{R} \) here is the same as in \([33, \S 9]\): the nonnegative elements form a subset \( P \subset \mathcal{R} \) such that \( 0, 1 \in P \), \( a, b \in P \) implies \( a + b, ab \in P \), and \( P \cup (-P) = \{0\} \); for \( a, b \in \mathcal{R} \) we write \( a \leq b \) if \( b - a \in P \). Moreover, we say that a matrix with entries in a ring of polynomials \( \mathcal{R}[\mathbf{x}] \), where \( \mathcal{R} \) is again a partially ordered commutative ring and \( \mathbf{x} \) is a (finite or infinite) set of indeterminates, is *coefficient-wise totally positive* if that matrix is totally positive in \( \mathcal{R}[\mathbf{x}] \) equipped with the coefficient-wise partial order: a polynomial in \( \mathcal{R}[\mathbf{x}] \) is nonnegative if all its coefficients are nonnegative.
The theory of production matrices is connected to the study of total positivity because, if $\Pi$ is a totally positive matrix that is either row-finite or column-finite, with entries in a partially ordered commutative ring, then its output matrix is also a totally positive matrix (see [33, Th. 9.4]).

### 3 Results about generalised and modified $m$-Stieltjes-Rogers polynomials

In this section, we find a relation between the matrices of generalised and modified $m$-Stieltjes-Rogers polynomials (Proposition 3.1) and show that the matrix relating them is coefficient-wise totally positive (Proposition 3.2). As a result, we conclude that the matrix of modified $m$-Stieltjes-Rogers polynomials is also coefficient-wise totally positive (Theorem 3.3). Furthermore, we give explicit expressions for the entries of the production matrix of the generalised $m$-Stieltjes-Rogers polynomials (Proposition 3.5).

We start by proving the following two propositions:

**Proposition 3.1.** For $m \in \mathbb{Z}^+$, let $S^{(m)} = \left(\binom{m}{n,k} (\alpha)\right)_{n,k \in \mathbb{N}}$ and $\hat{S}^{(m)} = \left(S^{(m;j)} (\alpha)\right)_{n,j \in \mathbb{N}}$ be the matrices of generalised and modified $m$-Stieltjes-Rogers polynomials, respectively, with weights $\alpha = (\alpha_{k+m})_{k \in \mathbb{N}}$. Then,

$$\hat{S}^{(m)} = S^{(m)} \Lambda^{(m)},$$

where $\Lambda^{(m)} = \left(\lambda_{i,j}^{(m)}\right)_{i,j \in \mathbb{N}}$ is the upper-triangular matrix such that $\lambda_{i,j}^{(m)}$ is the generating polynomial of the partial $m$-Dyck paths from $(0,m+1) \ell_i$ to $(j,0)$. Therefore, $\lambda_{0,j}^{(m)} = 1$ for any $j \in \mathbb{N}$ and

$$\lambda_{i,j}^{(m)} = \sum_{\substack{j \geq \ell_1 \geq \cdots \geq \ell_i \geq 1}} \prod_{k=1}^{i} \alpha_{m+\ell_k} = \sum_{\ell_1 = i}^{j} \alpha_{m+\ell_1} \sum_{\ell_2 = \ell_1}^{\ell_2} \alpha_{2m+\ell_2} \cdots \sum_{\ell_i = \ell_{i-1}}^{\ell_i} \alpha_{m+\ell_i} \quad \text{when } 1 \leq i \leq j. \quad (3.2)$$

**Proposition 3.2.** For any $m \in \mathbb{Z}^+$, the upper-triangular matrix $\Lambda^{(m)}$ defined in Proposition 3.1 is totally positive in the polynomial ring $\mathbb{Z}[\alpha]$ equipped with the coefficient-wise partial order.

Observe that (3.1) gives the unique LU-factorisation of $\hat{S}^{(m)}$ where $L = S^{(m)}$ is a unit-lower-triangular matrix. Moreover, according to [33, Th. 9.12], the $j$th-column of $\hat{S}^{(m)}$, $\left(S^{(m;j)} (\alpha)\right)_{n \in \mathbb{N}}$, is a coefficient-wise Hankel-totally positive sequence (or, equivalently, a Stieltjes moment sequence). Therefore, Proposition 3.1 gives an answer to the remark in [37, §4].

Note that, when $j = i$, (3.2) reduces to $\lambda_{i,i}^{(m)} = \prod_{k=1}^{i} \alpha_{m+i}$ for any $i \in \mathbb{N}$. As a result, if the sequence $\left(\alpha_{k+m}\right)_{n \in \mathbb{N}}$ does not have any zeroes or divisors of zero, all the diagonal entries of $\Lambda^{(m)}$ are different from zero.

The upper-triangular matrix defined in Proposition 3.1 is

$$\Lambda^{(m)} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ \alpha_{m+1} & \alpha_{m+1} + \alpha_{m+2} & \alpha_{m+1} + \alpha_{m+2} + \alpha_{m+3} & \cdots \\ \alpha_{m+2} \alpha_{2m+2} & \alpha_{m+2} \alpha_{2m+2} + \alpha_{m+3} (\alpha_{2m+2} + \alpha_{2m+3}) & \cdots \\ \alpha_{m+3} \alpha_{2m+3} \alpha_{3m+3} & \cdots \\ \cdots \end{bmatrix}. \quad (3.3)$$
In particular, for the classical case \( m = 1 \),
\[
\Lambda^{(1)} = \begin{bmatrix}
1 & 1 & 1 & 1 & \\
\alpha_2 & \alpha_2 + \alpha_3 & \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 & \\
\alpha_3 \alpha_4 & \alpha_4 (\alpha_3 + \alpha_4 + \alpha_5) & \alpha_4 (\alpha_3 + \alpha_4 + \alpha_5) + \alpha_5 (\alpha_4 + \alpha_5 + \alpha_6) & \alpha_4 \alpha_5 \alpha_6 (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) & \\
\alpha_4 \alpha_5 \alpha_6 & \alpha_4 \alpha_5 \alpha_6 (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) & \alpha_5 \alpha_6 \alpha_7 \alpha_8 & \alpha_5 \alpha_6 \alpha_7 \alpha_8 & \\
& & & & \\
\end{bmatrix}.
\tag{3.4}
\]

The matrix of generalised \( m \)-Stieltjes-Rogers polynomials \( S^{(m)} = \left( S^{(m)}_{n,k} (\alpha) \right)_{n,k \in \mathbb{N}} \) is totally positive in the polynomial ring \( \mathbb{Z}[\alpha] \) equipped with the coefficient-wise partial order [33, Th. 9.8]. Moreover, the product of totally positive matrices (when well-defined) is also totally positive, due to the Cauchy-Binet formula (see, for instance, [35, §1.1]). Therefore, combining Propositions 3.1 and 3.2, we obtain the following result.

**Theorem 3.3.** For any \( m \in \mathbb{Z}^+ \), the matrix of modified \( m \)-Stieltjes-Rogers polynomials \( \hat{S}^{(m)} = \left( \hat{S}^{(m);j}_{n,i} (\alpha) \right)_{n,j \in \mathbb{N}} \) is totally positive in the polynomial ring \( \mathbb{Z}[\alpha] \) equipped with the coefficient-wise partial order.

**Proof of Proposition 3.1.** Firstly, we prove that (3.1) holds. Observe that, when \( i > j \), there are no partial \( m \)-Dyck paths from \( (0, (m+1)i) \) to \( (j, j) \), because the lowest point you can go to from \( (0, (m+1)i) \) in \( j \) steps is \( (j, (m+1)i - mj) \) and \( i > j \) implies that \( (m+1)i - mj > j \). Therefore, \( \hat{\lambda}^{(m)}_{i,j} (\alpha) = 0 \) for \( i > j \), \( \Lambda^{(m)} \) is indeed an upper-triangular matrix, and (3.1) is equivalent to

\[
S^{(m);j}_{n,i} (\alpha) = \sum_{i=0}^{j} S^{(m);i}_{n,i} (\alpha) \hat{\lambda}^{(m)}_{i,j} (\alpha) \quad \text{for all } n, j \in \mathbb{N}. \tag{3.5}
\]

Let \( n, j \in \mathbb{N} \). Each partial \( m \)-Dyck path from \( (0, 0) \) to \( ((m+1)n + j, j) \) contains a unique point \( ((m+1)n + i, (m+1)i) \), with \( 0 \leq i \leq j \), and it can be split in two at that point. Hence, we obtain (3.5), with \( \hat{\lambda}^{(m)}_{i,j} (\alpha) \) equal to the generating polynomial of the partial \( m \)-Dyck paths from \( ((m+1)n, (m+1)i) \) to \( ((m+1)n + j, j) \).

Furthermore, the weights of the \( m \)-falls in any partial \( m \)-Dyck path are invariable with horizontal translations, because they only depend on the height from where the \( m \)-fall occurs. Hence, \( \hat{\lambda}^{(m)}_{i,j} (\alpha) \) does not depend on \( n \) and it is equal to the generating polynomial of the partial \( m \)-Dyck paths from \( (0, (m+1)i) \) to \( (j, j) \).

Now we find the explicit expressions for \( \hat{\lambda}^{(m)}_{i,j} (\alpha) \), with \( i \leq j \). Observe that it takes \( i \) \( m \)-falls and \( j - i \) rises to go from \( (0, (m+1)i) \) to \( (j, j) \), with the \( k^{th} \) \( m \)-fall counting from the end occuring from height \( km + j - \sigma_k \), where \( \sigma_k \) is the number of rises happening after this \( m \)-fall, so \( 0 \leq \sigma_1 \leq \cdots \leq \sigma_i \leq j - i \). Therefore, it is clear that if \( i = 0 \) then \( \hat{\lambda}^{(m)}_{0,j} (\alpha) = 1 \), while for \( i \geq 1 \), we define \( \ell_k = j - \sigma_k \) for \( 1 \leq k \leq i \) to obtain (3.2). □

To prove the total positivity of \( \Lambda^{(m)} \), we use a similar argument to the total-positivity combinatorial proofs using the Lindström-Gessel-Viennot lemma in [33, §9.4]. For that purpose, we need to first recall some relevant definitions and results, connecting total positivity with walks on graphs.

Let \( G = (V, \bar{E}) \) be a directed graph with vertex set \( V \) and edge set \( \bar{E} \), and let \( w = (w_{i,j})_{(i,j) \in \bar{E}} \) be a set of commuting indeterminates associated to the edges of \( G \), to which we refer to as the *edge weights*. For
If \( i, j \in V \), a walk from \( i \) to \( j \) (of length \( n \in \mathbb{N} \)) is a sequence \( \gamma = (\gamma_0, \ldots, \gamma_n) \in V^{n+1} \) such that \( \gamma_0 = i \), \( \gamma_n = j \), and \( (\gamma_k, \gamma_{k+1}) \in \bar{E} \) for all \( 0 \leq k \leq n - 1 \). The weight of a walk \( \gamma = (\gamma_0, \ldots, \gamma_n) \), which we denote by \( W(\gamma) \), is the product of its edge weights, that is, \( W(\gamma) = w_{\gamma_0,\gamma_1} \cdots w_{\gamma_{n-1},\gamma_n} \), with \( W(\gamma) = 1 \) if \( \gamma \) is a walk of length 0.

We define the walk matrix \( B = (b_{i,j})_{i,j \in V} \) with entries \( b_{i,j} \in \mathbb{Z}[[w]] \) equal to the sum of the weights of all the walks from \( i \) to \( j \).

For \( r \in \mathbb{Z}^{+} \), let \( i = (i_1, \ldots, i_r) \), \( j = (j_1, \ldots, j_r) \in V^{r} \) be two ordered \( r \)-tuples of distinct vertices of \( G \). We say that the pair \((i,j)\) is nonpermutable if the set of vertex-distinct walk systems \( \gamma = (\gamma_1, \ldots, \gamma_r) \) satisfying \( \gamma_k : i_k \to j_{\sigma(k)} \) is empty whenever \( \sigma \) is not the identity permutation. Now let \( I \) and \( J \) be (not necessarily finite) subsets of \( V \), equipped with total orders \( <_{I} \) and \( <_{J} \), respectively. We say that the pair \((I, <_{I}), (J, <_{J})\) is fully nonpermutable if each pair \((i,j)\) of increasing \( r \)-tuples \( i = (i_1, \ldots, i_r) \) in \( (I, <_{I}) \) and \( j = (j_1, \ldots, j_r) \) in \( (J, <_{J}) \) is nonpermutable for any \( r \geq 1 \). Here we use a fully nonpermutable pair to prove the total positivity of \( \Lambda^{(m)} \) via the following lemma.

**Lemma 3.4.** (cf. [33, Cor. 9.17]) For an acyclic graph \( G = (V, \bar{E}) \), let \( (I, <_{I}) \) and \((J, <_{J})\) be totally ordered subsets of \( V \) such that the pair \(( (I, <_{I}), (J, <_{J}) \) is fully nonpermutable. Then, the submatrix \( B_{I,J} \), with rows and columns ordered accordingly with \( <_{I} \) and \( <_{J} \), of the walk matrix of \( G \) is totally positive with respect to the coefficient-wise order on \( \mathbb{Z}[[w]] \).

**Proof of Proposition 3.2.** Recall that \( \lambda_{i,j}^{(m)} \) is the generating polynomial of the partial \( m \)-Dyck paths from \((0,(m+1)i)\) to \((j,j)\). Observe that every vertex \((x,y)\) of a partial \( m \)-Dyck path starting from \((0,(m+1)i)\) and ending at \((j,j)\) satisfies \( y \geq x \) and \( x \equiv y \mod (m+1) \). Therefore, we can define \( \lambda_{i,j}^{(m)}(\alpha) \) as the generating polynomial of the paths from \((0,(m+1)i)\) to \((j,j)\) in the directed graph \( G_m = \left( V_m, \bar{E}_m \right) \) with vertex set

\[
V_m = \{(x,y) \in \mathbb{N} \times \mathbb{N} : y \geq x \text{ and } x \equiv y \mod (m+1)\}
\]

and edge set

\[
\bar{E}_m = \{(x_1,y_1),(x_2,y_2) \in V_m \times V_m : x_2 - x_1 = 1 \text{ and } y_2 - y_1 \in \{1,-m\}\}.
\]

This means that \( \Lambda^{(m)} \) is the submatrix of the walk matrix of \( G_m \) corresponding to paths with source vertices \( I = \{i_n = (0,(m+1)n) : n \in \mathbb{N}\} \) and sink vertices \( J = \{j_n = (n,n) : n \in \mathbb{N}\} \), totally ordered by \((0,(m+1)n)) < (0,(m+1)n') \) and \((n,n) < (n',n')\) if \( n < n' \) (see Fig.2). Moreover, observe that \( G_m \) is a graph embedded in the plane and the vertices of \( I \cup J \) lie on the boundary of \( G_m \) in the order “first \( I \) in reverse order, then \( J \) in order”, so the pair \((I,J)\) is clearly fully nonpermutable. Therefore, because \( G_m \) is acyclic, \( \Lambda^{(m)} \) is totally positive as a consequence of Lemma 3.4. \( \Box \)
Based on [33, Prop. 8.2], the production matrix of the generalised \( m \)-Stieltjes-Rogers polynomials \( S_{n,k}^{(m)}(\alpha) \) is a \((m + 2)\)-banded unit-lower-Hessenberg matrix admitting a decomposition in \( m + 1 \) bidiagonal matrices:

\[
H^{(m)} = \prod_{i=0}^{m-1} L \left( (\alpha_{k(m+1)+i})_{k \in \mathbb{Z}^+} \right) \cdot U \left( (\alpha_{k(m+1)+m})_{k \in \mathbb{N}} \right),
\]

where

- \( L \left( (l_k)_{k \in \mathbb{Z}^+} \right) \) is the lower-bidiagonal infinite matrix with entries \( L_{k,k} = 1 \) and \( L_{k+1,k} = l_{k+1} \) for all \( k \in \mathbb{N} \),
- \( U \left( (u_k)_{k \in \mathbb{N}} \right) \) is the upper-bidiagonal infinite matrix with entries \( U_{k,k+1} = 1 \) and \( U_{k,k} = u_k \) for all \( k \in \mathbb{N} \).

In the following result, we give explicit formulas for the entries of this production matrix.

**Proposition 3.5.** For \( m \in \mathbb{Z}^+ \) and a sequence of indeterminates \( \alpha = (\alpha_{k+m})_{k \in \mathbb{N}} \), the production matrix of the generalised \( m \)-Stieltjes-Rogers polynomials \( S_{n,k}^{(m)}(\alpha) \) is the \((m + 2)\)-banded unit-lower-Hessenberg matrix \( H^{(m)} = \left( h_{i,n}^{(m)}(\alpha) \right)_{i,n \in \mathbb{N}} \) whose entries are \( h_{i,n}^{(m)}(\alpha) = 0 \) if \( i \leq n - 2 \) or \( i \geq n + m + 1 \), \( h_{n-1,n}^{(m)}(\alpha) = 1 \) for all \( n \geq 1 \), and, setting \( \alpha_j = 0 \) for \( 0 \leq j \leq m - 1 \),

\[
h_{n+k,n}^{(m)}(\alpha) = \sum_{m \geq \ell_0 \geq \cdots \geq \ell_k \geq 0} \prod_{j=0}^{k} \alpha_{(m+1)(n+j)+\ell_j} \text{ for all } n \in \mathbb{N} \text{ and } 0 \leq k \leq m.
\]

The cases \( m = 1 \) and \( m = 2 \) of Equations (3.8)-(3.9) are explicitly written in [33, Eqs. 7.7-7.8].

**Proof.** We prove this result by induction on \( m \in \mathbb{Z}^+ \).
When \( m = 1 \), (3.8) reduces to \( H^{(1)} = L\left((\alpha_{2k})_{k \in \mathbb{Z}^+}\right) U\left((\alpha_{2k+1})_{k \in \mathbb{N}}\right) \). As a result,

\[
h_{n,n+1}^{(1)}(\alpha) = 1, \quad h_{n,n}^{(1)}(\alpha) = \alpha_{2n} + \alpha_{2n+1}, \quad \text{and} \quad h_{n+1,n}^{(1)}(\alpha) = \alpha_{2n+1} \alpha_{2n+2} \quad \text{for all} \ n \in \mathbb{N},
\]

(3.10)

and all other entries of \( H^{(1)} \) are equal to zero. Therefore, Proposition 3.5 holds for \( m = 1 \).

Now we suppose that Proposition 3.5 holds for some \( m \in \mathbb{Z}^+ \) and we show that it also holds for \( m + 1 \).

Recalling again (3.8),

\[
H^{(m+1)} = \prod_{i=0}^{m} L\left((\alpha_{k(m+2)+i})_{k \in \mathbb{Z}^+}\right) \cdot U\left((\alpha_{k(m+2)+(m+1)})_{k \in \mathbb{N}}\right) = L\left((\alpha_{k(m+2)})_{k \in \mathbb{Z}^+}\right) \cdot L^{(m)},
\]

(3.11)

where

\[
\hat{H}^{(m)} = \prod_{i=0}^{m} L\left((\alpha_{k(m+2)+i})_{k \in \mathbb{Z}^+}\right) \cdot U\left((\alpha_{k(m+2)+(m+1)})_{k \in \mathbb{N}}\right).
\]

(3.12)

Writing \( \hat{H}^{(m)} = \left(\hat{h}_{i,n}^{(m)}\right)_{i,n \in \mathbb{N}} \), we have

\[
\hat{h}_{n+k,n}^{(m)}(\alpha) = \hat{h}_{n+k,n}^{(m)}(\alpha) + \alpha_{(m+2)(n+k)} \hat{h}_{n+k-1,n}^{(m)}(\alpha).
\]

(3.13)

Using the induction hypothesis, \( \hat{H}^{(m)} \) is a \((m + 2)\)-banded unit-lower-Hessenberg matrix such that

\[
\hat{h}_{n+k,n}^{(m)}(\alpha) = \sum_{m \geq \ell_0 \geq \cdots \geq \ell_k \geq 0} \prod_{j=0}^{k} \alpha_{(m+2)(n+j) + (\ell_j+1)} \quad \text{for} \ n \in \mathbb{N} \text{ and } 0 \leq k \leq m.
\]

(3.14)

Hence, (3.13) implies that \( h_{i,n}^{(m+1)}(\alpha) = 0 \) if \( i \leq n - 2 \) or \( i \geq n + m + 2 \), \( h_{n-1,n}^{(m+1)}(\alpha) = 1 \) for any \( n \geq 1 \), and

\[
h_{n+k,n}^{(m+1)}(\alpha) = \sum_{m \geq \ell_0 \geq \cdots \geq \ell_k \geq 0} \prod_{j=0}^{k} \alpha_{(m+2)(n+j) + (\ell_j+1)} + \alpha_{(m+2)(n+k)} \sum_{m \geq \ell_0 \geq \cdots \geq \ell_{k-1} \geq 0} \prod_{j=0}^{k-1} \alpha_{(m+2)(n+j) + (\ell_j+1)},
\]

(3.15)

for any \( n \in \mathbb{N} \) and \( 0 \leq k \leq m + 1 \), with the first sum being an empty sum (thus, equal to 0) when \( k = m + 1 \).

Setting \( \ell_j = \ell_j + 1 \) in both sums and \( \lambda_k = 0 \) in the second sum, we obtain

\[
h_{n+k,n}^{(m+1)}(\alpha) = \sum_{m+1 \geq \ell_0 \geq \cdots \geq \ell_k \geq 0} \prod_{j=0}^{k} \alpha_{(m+2)(n+j) + \lambda_j}.
\]

(3.16)

Therefore, Proposition 3.5 holds for \( m + 1 \), which concludes our proof. \( \square \)

4 Multiple orthogonal polynomials and branched continued fractions

The connection between multiple orthogonal polynomials and branched continued fractions was recently introduced and analysed in [37]. Here we revisit and further explore this connection and its applications in the study of multiple orthogonal polynomials.
Firstly, we define a \( m \)-orthogonal polynomial sequence such that the moments of its dual sequence are generalised \( m \)-Stieltjes-Rogers polynomials (Theorem 4.2), we show that this polynomial sequence is \( m \)-orthogonal with respect to functionals whose moments are modified \( m \)-Stieltjes-Rogers polynomials (Proposition 4.4), and we give explicit formulas for the recurrence coefficients of this \( m \)-orthogonal polynomial sequence (Theorem 4.5). Then, we assume the positivity of the branched-continued-fraction coefficients and, under that assumption, we show that the orthogonality conditions can be written via positive measures on the positive real line (Corollary 4.6) and that the zeros of the \( m \)-orthogonal polynomials are all simple, real, and positive and the zeros of consecutive polynomials interlace (Theorem 4.8). Finally, we give an upper bound for the zeros of \( m \)-orthogonal polynomials using the asymptotic behaviour of their recurrence coefficients (Theorem 4.9).

### 4.1 Connection via production matrices

Here it is useful to introduce matrices representing sequences of linear functionals and polynomial sequences. Precisely, we call the moment matrix of a sequence of linear functionals \((u_k)_{k \in \mathbb{N}}\) to the matrix \( A = (a_{n,k})_{n,k \in \mathbb{N}} \) such that \( a_{n,k} = u_k [x^n] \) and the coefficient matrix of a polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\) to the matrix \( B = (b_{n,k})_{n,k \in \mathbb{N}} \) such that \( P_n(x) = \sum_{k=0}^{n} b_{n,k} x^k \). A sequence of linear functionals \((u_k)_{k \in \mathbb{N}}\) is the dual sequence of a polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\) if \( u_k [P_n] = \delta_{k,n} \). Using their representing matrices, this is equivalent to say that \((u_k)_{k \in \mathbb{N}}\) is the dual sequence of \((P_n(x))_{n \in \mathbb{N}}\) if and only if the moment matrix of \((u_k)_{k \in \mathbb{N}}\) is the inverse of the coefficient matrix of \((P_n(x))_{n \in \mathbb{N}}\).

The theory of production matrices is instrumental to the study of the connection between multiple orthogonal polynomials and branched continued fractions. The key result linking polynomial sequences with production matrices is the following.

**Proposition 4.1.** [37, Prop. 3.2 & 3.4] Given a monic polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\), there exists an unique unit-lower-Hessenberg matrix \( H = (h_{n,k})_{n,k \in \mathbb{N}} \) such that

\[
P_{n+1}(x) = x P_n(x) - \sum_{k=0}^{n} h_{n,k} P_k(x).
\]  

Conversely, given any unit-lower-Hessenberg matrix \( H = (h_{n,k})_{n,k \in \mathbb{N}} \), the recurrence relation (4.1) with the initial condition \( P_0(x) = 1 \) determines an unique polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\). Moreover, for \( n \geq 1 \), \( P_n(x) \) is the characteristic polynomial of the \((n \times n)\)-matrix \( H_n \) formed by the first \( n \) rows and columns of \( H \), which means that, if we denote the \((n \times n)\)-identity matrix by \( I_n \), then

\[
P_n(x) = \det(x I_n - H_n) \quad \text{for any } n \geq 1.
\]  

Furthermore, the coefficient matrix of the sequence \((P_n(x))_{n \in \mathbb{N}}\) is the inverse of the output matrix of \( H \). Therefore, \( H \) is the production matrix of the moment matrix of the dual sequence of \((P_n(x))_{n \in \mathbb{N}}\).

See the remarks after [37, Prop. 3.4] to find some references for (4.2). In the case where \( H \) is a \((d + 2)\)-banded unit-lower-Hessenberg matrix, and consequently \((P_n(x))_{n \in \mathbb{N}}\) is a \( d \)-orthogonal polynomial sequence, (4.2) was mentioned in the introduction, at the end of Subsection 2.2.
If we consider the previous proposition with the unit-lower-Hessenberg matrix H being the production matrix of the generalised m-Stieltjes-Rogers polynomials \( \left( S^{(m)}_{n,k}(\alpha) \right)_{n,k \in \mathbb{N}} \), we obtain the following result connecting lattice paths and branched continued fractions with multiple orthogonal polynomials.

**Theorem 4.2.** For \( m \in \mathbb{Z}^+ \) and a sequence \( \alpha = (\alpha_{k+m})_{k \in \mathbb{N}} \) in a commutative ring R, let \( H = (h_{n,k})_{n,k \in \mathbb{N}} \) be the \((m+2)\)-banded unit-lower-Hessenberg production matrix of the generalised m-Stieltjes-Rogers polynomials \( \left( S^{(m)}_{n,k}(\alpha) \right)_{n,k \in \mathbb{N}} \) and \( (P_n(x))_{n \in \mathbb{N}} \) be the polynomial sequence satisfying the recurrence relation

\[
P_{n+1}(x) = xP_n(x) - \sum_{j=0}^{\min(n,m)} h_{n,n-j} P_{n-j}(x), \tag{4.3}
\]

with the initial condition \( P_0(x) = 1 \). Then, the dual sequence of \( (P_n(x))_{n \in \mathbb{N}} \) is \( (u_k)_{k \in \mathbb{N}} \) defined by

\[
u_k[x^n] = S^{(m)}_{n,k}(\alpha) \quad \text{for all } k, n \in \mathbb{N}, \tag{4.4}
\]

and, if \( h_{n+m,n} \neq 0 \) for all \( n \in \mathbb{N} \), \( (P_n(x))_{n \in \mathbb{N}} \) is \( m \)-orthogonal with respect to \( (u_0, \ldots, u_{m-1}) \).

*Proof.* Because \( H \) is a \((m+2)\)-banded unit-lower-Hessenberg matrix, (4.1) reduces to (4.3). As a result, the polynomial sequence \( (P_n(x))_{n \in \mathbb{N}} \) is \( m \)-orthogonal with respect to the first \( m \) elements of its dual sequence. Furthermore, due to Proposition 4.1, we know that, because \( H \) is the production matrix of \( \left( S^{(m)}_{n,k}(\alpha) \right)_{n,k \in \mathbb{N}} \), the dual sequence \( (u_k)_{k \in \mathbb{N}} \) of \( (P_n(x))_{n \in \mathbb{N}} \) satisfies (4.4).

We know from the latter theorem that the production matrix of the generalised m-Stieltjes-Rogers polynomials \( \left( S^{(m)}_{n,k}(\alpha) \right)_{n,k \in \mathbb{N}} \) determines a \( m \)-orthogonal polynomial sequence with respect to linear functionals whose moments are the same generalised m-Stieltjes-Rogers polynomials. The following result gives a relation between the sequences of linear functionals whose moment matrices are formed by the generalised and modified m-Stieltjes-Rogers polynomials as a corollary of Proposition 3.1.

**Lemma 4.3.** For \( m \in \mathbb{Z}^+ \) and a sequence \( \alpha = (\alpha_{k+m})_{k \in \mathbb{N}} \) in a commutative ring R, let \( \Lambda = (\lambda_{i,j}(\alpha))_{i,j \in \mathbb{N}} \) be the upper-triangular matrix defined in Proposition 3.1 and let \( (u_k)_{k \in \mathbb{N}} \) and \( (v_k)_{k \in \mathbb{N}} \) be the sequences of linear functionals defined by

\[
u_k[x^n] = S^{(m)}_{n,k}(\alpha) \quad \text{and} \quad v_k[x^n] = S^{(m,k)}_{n}(\alpha) \quad \text{for all } n, k \in \mathbb{N}. \tag{4.5}
\]

Then, the sequences \( (u_k)_{k \in \mathbb{N}} \) and \( (v_k)_{k \in \mathbb{N}} \) are related by

\[
v_k = \sum_{i=0}^{k} \lambda_{i,k}(\alpha) u_i \quad \text{for all } k \in \mathbb{N}. \tag{4.6}
\]

Recall that if the sequence \( \alpha = (\alpha_{k+m})_{k \in \mathbb{N}} \) has no zeroes or divisors of zero, then \( \lambda_{i,i}(\alpha) \neq 0 \) for all \( i \in \mathbb{N} \). Therefore, the type II multiple orthogonal polynomials on the step-line with respect to the first elements of the sequences of linear functionals \( (u_k)_{k \in \mathbb{N}} \) and \( (v_k)_{k \in \mathbb{N}} \) coincide, as explained in the following result.

17
Proposition 4.4. For $m \in \mathbb{Z}^+$ and a sequence $\alpha = (\alpha_{k+m})_{k \in \mathbb{N}}$ without any zeroes or divisors of zero in a commutative ring $R$, let $(u_k)_{k \in \mathbb{N}}$ and $(v_k)_{k \in \mathbb{N}}$ be the sequences of linear functionals defined by (4.5). Then, for any $d \in \mathbb{Z}^+$, a polynomial sequence is $d$-orthogonal with respect to $(v_0, \cdots, v_{d-1})$ if and only if it is $d$-orthogonal with respect to $(u_0, \cdots, u_{d-1})$. In particular, the $m$-orthogonal polynomial sequence $(P_n(x))_{n \in \mathbb{N}}$ with respect to $(v_0, \cdots, v_{m-1})$ exists and it satisfies the recurrence relation (4.3), where $H = (h_{n,k})_{n,k \in \mathbb{N}}$ is the production matrix of the generalised $m$-Stieltjes-Rogers polynomials $\left(S^{(m)}_{n,k}(\alpha)\right)_{n,k \in \mathbb{N}}$.

We know from Theorem 4.2 that the coefficients of the recurrence relation satisfied by the $m$-orthogonal polynomial sequence $(P_n(x))_{n \in \mathbb{N}}$ are the nontrivial entries of the production matrix of the corresponding generalised $m$-Stieltjes-Rogers polynomials. In the following result we give explicit expressions for the recurrence coefficients as a consequence of Proposition 3.5.

Theorem 4.5. For $m \in \mathbb{Z}^+$ and a sequence $\alpha = (\alpha_{k+m})_{k \in \mathbb{N}}$ without any zeroes or divisors of zero in a commutative ring $R$, let $(P_n(x))_{n \in \mathbb{N}}$ be the monic $m$-orthogonal polynomial sequence with respect to the vectors of linear functionals $(u_0, \cdots, u_{m-1})$ and $(v_0, \cdots, v_{m-1})$ such that

$$u_k[x^n] = S^{(m)}_{n,k}(\alpha) \quad \text{and} \quad v_k[x^n] = S^{(m,k)}_n(\alpha) \quad \text{for all } n \in \mathbb{N} \text{ and } 0 \leq k \leq m - 1. \quad (4.7)$$

Then, $(P_n(x))_{n \in \mathbb{N}}$ satisfies the recurrence relation

$$P_{n+1}(x) = xP_n(x) - \sum_{k=0}^{\min(m,n)} \gamma^{[k]}_{n-k} P_{n-k}(x), \quad (4.8)$$

with initial condition $P_0(x) = 1$ and coefficients

$$\gamma^{[k]}_n = \sum_{m \geq \ell_0 > \cdots > \ell_k \geq 0} \alpha_{(m+1)(n+j)+\ell_j} \quad \text{for any } n \in \mathbb{N} \text{ and } 0 \leq k \leq m, \quad (4.9)$$

where $\alpha_j = 0$ for $0 \leq j \leq m - 1$.

Note that the number of summands in (4.9) is equal to $\binom{m+1}{k+1}$, because each summand corresponds to one way of choosing $k + 1$ elements from $\{0, \cdots, m\}$ to be $\ell_0, \cdots, \ell_k$.

Because the sequence $(\alpha_{k+m})_{n \in \mathbb{N}}$ does not have any zeroes or divisors of zero,

$$\gamma^{[m]}_n = \prod_{j=0}^{m} \alpha_{(m+1)n+m(j+1)} \neq 0 \quad \text{for all } n \in \mathbb{N}. \quad (4.10)$$

For the classical case $m = 1$, Theorem 4.5 states that the orthogonal polynomials $(P_n(x))_{n \in \mathbb{N}}$ with respect to the linear functional $u$ defined by $u[x^n] = S^{(m)}_n(\alpha)$ for all $n \in \mathbb{N}$ satisfy the second-order recurrence relation

$$P_{n+1}(x) = (x - \alpha_{2n} - \alpha_{2n+1})P_n(x) - \alpha_{2n+1}\alpha_{2n+2}P_{n-1}(x). \quad (4.11)$$

Analogously, for any $m \geq 2$, Theorem 4.5 gives an explicit $(m+1)$-order recurrence relation satisfied by the $m$-orthogonal polynomials $(P_n(x))_{n \in \mathbb{N}}$ with respect to the functionals whose moments are given in (4.7).
4.2 Positive branched-continued-fraction coefficients, orthogonality measures, and zeros

In the study of multiple orthogonal polynomials, we are usually interested in considering orthogonality measures instead of linear functionals. Here we are particularly interested in positive orthogonality measures over $\mathbb{R}^+$. However, we know that $(S_{n,k}^{(m)}(\alpha))_{n\in\mathbb{N}}$ cannot be a moment sequence of a positive measure over $\mathbb{R}^+$ for any $k \geq 1$, because $s_{n,k}^{(m)}(\alpha) = 0$ for $n < k$, and, in particular, the moment of order zero is 0. Combining Proposition 4.4 with [33, Th. 9.12], we find that the modified $m$-Stieltjes-Rogers polynomials solve this problem when the coefficients $\alpha_k$ are all positive.

**Corollary 4.6.** For $m \in \mathbb{Z}^+$ and a sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ of positive real numbers, $(S_{n,k}^{(m)}(\alpha))_{n\in\mathbb{N}}$ is the moment sequence of a positive measure $\mu_k$ on $\mathbb{R}^+$ for any $0 \leq k \leq m$ (cf. [33, Th. 9.12]). Furthermore, as a consequence of Proposition 4.4, the $m$-orthogonal polynomials with respect to the vector of measures $(\mu_0, \cdots, \mu_{m-1})$ exist and satisfy the recurrence relation (4.3).

When the branched-continued-fraction coefficients $\alpha_k+m$ are all positive, it is clear from (4.9) that the recurrence coefficients of the $m$-orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}}$ are also all positive. Moreover, the positivity of the branched-continued-fraction coefficients also leads to nice properties about the location of the zeros of $P_n(x)$, as detailed in Theorem 4.8, which is a consequence of the following lemma.

**Lemma 4.7.** For $m \in \mathbb{Z}^+$, let $(P_n(x))_{n\in\mathbb{N}}$ be a $m$-orthogonal polynomial sequence and $H = (h_{n,k})_{n,k\in\mathbb{N}}$ be the $(m+2)$-banded unit-lower-Hessenberg such that $(P_n(x))_{n\in\mathbb{N}}$ satisfies the recurrence relation (4.3). If the matrices $H_n$ formed by the first $n$ rows and columns of $H$ are oscillation matrices for all $n \in \mathbb{Z}^+$, then the zeros of $P_n(x)$ are all simple, real, and positive, and the zeros of consecutive polynomials interlace.

**Proof.** Recalling (4.2), $P_n(x)$ is the characteristic polynomial of $H_n$ for any $n \geq 1$, so the zeros of $P_n(x)$ are the eigenvalues of $H_n$. Moreover, due to the Gantmacher-Krein theorem (see [18, Ths. II-6 & II-14]), the eigenvalues of an oscillation matrix are all simple, real, and positive, and interlace with the eigenvalues of the submatrices obtained by removing either its first or last column and row. In addition, observe that if we remove the last column and row of $H_n$, with $n \geq 2$, we obtain $H_{n-1}$. Therefore, the lemma holds.

A bidiagonal matrix is totally positive if and only if all its entries are nonnegative (see [33, Lemma 9.1]). Moreover, the product of matrices preserves total positivity. Therefore, if $\alpha_{k+m} \geq 0$ for all $k \in \mathbb{N}$, the production matrix of the generalised $m$-Stieltjes-Rogers polynomials $(S_{n,k}^{(m)}(\alpha))_{n,k\in\mathbb{N}}$ is totally positive. Furthermore, based on [35, Th. 5.2], a $(n \times n)$-matrix of real numbers is an oscillation matrix if and only if it is totally positive, nonsingular, and all the entries lying in its subdiagonal and its supradiagonal are positive. Therefore, if $\alpha_{k+m} > 0$ for any $k \in \mathbb{N}$, the $(n \times n)$-matrices $H_n$ formed by its first $n$ rows and columns are oscillation matrices for all $n \geq 1$. Hence, combining Proposition 4.4 and Lemma 4.7, we obtain the following result.

**Theorem 4.8.** For $m \in \mathbb{Z}^+$ and a sequence of positive real constants $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, let $(P_n(x))_{n\in\mathbb{N}}$ be the $m$-orthogonal polynomial sequence with respect to the vector of linear functionals $(\mu_0, \cdots, \mu_{m-1})$ and to the vector of measures $(\mu_0, \cdots, \mu_{m-1})$ supported on a subset of $\mathbb{R}^+$ such that

$$u_k[x^n] = S_{n,k}^{(m)}(\alpha) \quad \text{and} \quad \int x^n d\mu_k(x) = S_{n,k}^{(m)}(\alpha) \quad \text{for all} \ n \in \mathbb{N} \ \text{and} \ 0 \leq k \leq m-1. \tag{4.12}$$
Therefore, based on [19, Cor. 6.1.8], we have

\[ \text{maximum of the absolute values of the eigenvalues of } H_n \]

Then, each zero of \( P_n(x) \) is an eigenvalue of the matrix \( H_n \) and

\[ \text{for } m \in \mathbb{Z}^+, \text{ let } x^{(n)}_n \text{ be a } m\text{-orthogonal polynomial sequence satisfying a recurrence relation of the form } (4.8) \text{ such that } |a_n^{(k)}| \in \mathbb{R} \text{ for any } n \in \mathbb{N} \text{ and } 0 \leq k \leq m, \text{ with } |a_n^{[m]}| > 0, \text{ and suppose there exist real constants } |b_n^{[m]}| > 0 \text{ and } |b_n^{[k]}| \geq 0 \text{ for } 0 \leq k \leq m - 1 \text{ and a non-decreasing unbounded positive sequence } (f_n)_{n \in \mathbb{N}} \text{ such that} \]

\[ |a_n^{[k]}| \leq \gamma_n^{[k]} |f_n|^{k+1} + o \left( f_n \right) \text{ as } n \to +\infty. \]

(4.13)

For any \( n \in \mathbb{Z}^+ \), we denote by \( x^{(n)}_n \) the largest zero in absolute value of \( P_n(x) \). Then,

\[ |x^{(n)}_n| \leq \min_{t \in \mathbb{R}^+} \left( t + \sum_{k=0}^{r} \frac{\gamma_n^{[k]}}{t^k} \right) f_n + o \left( f_n \right) \text{ as } n \to +\infty. \]

(4.14)

The particular case \( r = 2 \) of the latter theorem corresponds to [24, Th. 3.5] and the following proof is a generalisation of the proof therein.

**Proof.** Let \( n \in \mathbb{Z}^+ \) and \( H_n \) be the \((m + 2)\)-banded lower Hessenberg matrix such that \( P_n(x) = \det (xI_n - H_n) \).

Then, each zero of \( P_n \) is an eigenvalue of the matrix \( H_n \) and \( |x^{(n)}_n| \) is equal to the spectral radius of \( H_n \), the maximum of the absolute values of the eigenvalues of \( H_n \).

Therefore, based on [19, Cor. 6.1.8], we have

\[ |x^{(n)}_n| \leq \min_{t_0, \ldots, t_{n-1} \in \mathbb{R}^+} \max_{i \in \{0, \ldots, n-1\}} \left\{ \sum_{j=0}^{n-1} \frac{t_j}{t_i} |(H_n)_{i,j}| \right\}. \]

(4.15)

Recalling the values of the entries of \( H_n \) from (2.10) with \( d = m \), the latter implies that

\[ |x^{(n)}_n| \leq \min_{t_0, \ldots, t_{n-1} \in \mathbb{R}^+} \max_{i \in \{0, \ldots, n-1\}} \left\{ \frac{t_i+1}{t_i} + \sum_{k=0}^{m} |\gamma_n^{[k]}| |t_i-k| \right\}, \]

(4.16)

with \( t_j = 0 \) if \( j = n \) or \( j < 0 \).

In particular, we can set \( t_j = t^{j} \prod_{l=1}^{j} f_l > 0 \) for \( 0 \leq j \leq n-1 \) and \( t \in \mathbb{R}^+ \), to find that

\[ |x^{(n)}_n| \leq \min_{t \in \mathbb{R}^+} \max_{i \in \{0, \ldots, n-1\}} \left\{ tf_{i+1} + \sum_{k=0}^{m} |\gamma_n^{[k]}| t^{k-1} \prod_{l=0}^{k-1} f_{i-l} \right\}. \]

(4.17)

Furthermore, recalling (4.13), we derive that

\[ |x^{(n)}_n| \leq \min_{t \in \mathbb{R}^+} \max_{i \in \{0, \ldots, n-1\}} \left\{ tf_{i+1} + \sum_{k=0}^{m} |\gamma_n^{[k]}| t^{k-1} \prod_{l=0}^{k-1} f_{i-l} + o \left( f_{i+1} \right) \right\}. \]

(4.18)
Therefore, due to the sequence \((f_n)_{n \in \mathbb{N}}\) being non-decreasing,

\[
\left| x_n^{(n)} \right| \leq \min_{r \in \mathbb{R}^+} \max_{i \in \{0, \ldots, n-1\}} \left\{ \left( t + \sum_{k=0}^{m} r^k \right) f_{i+1} + o(f_{i+1}) \right\},
\]

(4.19)

and, because \((f_n)_{n \in \mathbb{N}}\) is unbounded, we can conclude that (4.14) holds.

\[
\square
\]

5 Branched continued fractions for ratios of hypergeometric series

Branched-continued-fraction representations for three types of ratios of contiguous hypergeometric series,

\[
\begin{align*}
&\frac{r+1 F_3 \left( \begin{array}{c} a_1, \ldots, a_r+1 \\ b_1, \ldots, b_s \end{array} \bigg| t \right) - r+1 F_3 \left( \begin{array}{c} a_1, \ldots, a_r+1 - 1 \\ b_1, \ldots, b_{s-1}, b_s - 1 \end{array} \bigg| t \right)}{r+1 F_3 \left( \begin{array}{c} a_1, \ldots, a_r+1 - 1 \\ b_1, \ldots, b_{s-1}, b_s - 1 \end{array} \bigg| t \right)}, \\
&\frac{r F_3 \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \bigg| t \right) - r F_3 \left( \begin{array}{c} a_1, \ldots, a_r - 1 \\ b_1, \ldots, b_{s-1}, b_s - 1 \end{array} \bigg| t \right)}{r F_3 \left( \begin{array}{c} a_1, \ldots, a_r - 1 \\ b_1, \ldots, b_{s-1}, b_s - 1 \end{array} \bigg| t \right)},
\end{align*}
\]

(5.1)

were introduced in [33, §14], where they are referred to as, respectively, the first, second, and third ratios of contiguous hypergeometric series.

The main result of this section is Theorem 5.5, where we present new branched continued fractions that include the first and second ratios of contiguous hypergeometric series in [33] as particular cases. In Proposition 5.6, we find necessary and sufficient conditions for non-negativity and positivity of the coefficients of the branched continued fractions in Theorem 5.5 when \(r \geq s\) and all the indeterminates are real and positive. In Corollary 5.7, we give explicit expressions for the modified \(m\)-Stieltjes-Rogers polynomials corresponding to the branched continued fractions in Theorem 5.5 when \(a_{r+1} = 1\); they are ratios of products of Pochhammer symbols up to multiplication by a binomial coefficient. At the end of the section, we explain how an extension of the third ratio in (5.1) can be obtained as a limiting case of the branched continued fractions introduced in Theorem 5.5.

Throughout this section, we work on the commutative ring \(R = \mathbb{Q}(b_1, \ldots, b_s) [a_1, \ldots, a_{r+1}]\) of polynomials in the indeterminates \(a_1, \ldots, a_{r+1}\) whose coefficients are rational functions in the indeterminates \(b_1, \ldots, b_s\).

To simplify the notation, we denote by \([k]_n\), with \(k, n \in \mathbb{Z}\) and \(n \geq 1\), the unique element of \(\{1, \ldots, n\}\) congruent with \(k\) modulo \(n\), that is, \([k]_n = [(k - 1) \mod n] + 1\).

5.1 Construction of the branched continued fractions

To construct branched continued fractions, we use the Euler-Gauss recurrence method for \(m\)-S-fractions introduced in [33], which is a generalisation of the Euler-Gauss method for classical S-fractions (see [36]).

**Lemma 5.1.** (cf. [33, Prop. 2.3]) For \(m \in \mathbb{Z}^+\), let \((a_{i+m})_{i \in \mathbb{N}}\) be a sequence in a commutative ring \(R\) and let \((f_k(t))_{k \in \mathbb{N}}\) and \((g_k(t))_{k \geq -1}\) be two sequences of formal power series related by \(f_k(t) = g_k(t)/g_{k-1}(t)\) for all
Lemma 5.2. \([33, \text{Lemma 14.1}]\) The hypergeometric series \(pF_q\) satisfies the following three-term contiguous relations:

\[
pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right | t \right) - \frac{p \prod_{k=1, k \neq i}^{p} a_k}{(b_j - 1) \prod_{l=1}^{q} b_l} \frac{1}{t} pF_q \left( \begin{array}{c} a_1 + 1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_p + 1 \\ b_1, \ldots, b_q + 1 \end{array} \right | t \right),
\]

\[
(5.3)
\]

\[
pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right | t \right) - \frac{p \prod_{k=1, k \neq i}^{p} a_k}{(b_j - 1) \prod_{l=1}^{q} b_l} \frac{1}{t} pF_q \left( \begin{array}{c} a_1 + 1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_p + 1 \\ b_1 + 1, \ldots, b_q + 1 \end{array} \right | t \right),
\]

\[
(5.4)
\]

\[
pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right | t \right) - \frac{p \prod_{k=1}^{p} a_k}{(b_j - 1) \prod_{l=1}^{q} b_l} \frac{1}{t} pF_q \left( \begin{array}{c} a_1, \ldots, a_p + 1 \\ b_1, \ldots, b_q + 1 \end{array} \right | t \right),
\]

\[
(5.5)
\]

As in \([33]\), we start from the case \(r = s = m\). In that case, our branched continued fraction coincides with the one introduced in \([33, \text{Th. 14.2}]\) for the first ratio in (5.1), which we revisit in the following result.

Theorem 5.3. (cf. \([33, \text{Th. 14.2}]\)) For \(m \geq 1\), let

\[
g_k(t) = m+1F_m \left( \begin{array}{c} a_1^{(k)}, \ldots, a_{m+1}^{(k)} \\ b_1^{(k)}, \ldots, b_m^{(k)} \end{array} \right | t \right),
\]

\[
(5.6)
\]

for any \(k \geq -1\), with

\[
a_i^{(k)} = a_i + \left\lfloor \frac{k + 1 - i}{m + 1} \right\rfloor \text{ and } b_j^{(k)} = b_j + \left\lfloor \frac{k + 1 - j}{m} \right\rfloor.
\]

(5.7)
Then, the ratios of contiguous hypergeometric series \( f_k(t) = \frac{g_k(t)}{g_{k-1}(t)} \) admit the \( m \)-branched-continued-fraction representation (2.13) with coefficients

\[
\alpha_{k+m} = \frac{(b'_k - a'_k) \prod_{i=1, i \neq k}^{m+1} a_i^{(k)} }{(b'_k - 1) \prod_{i=1}^{m} b'^{(k)}_i} \quad \text{for any } k \in \mathbb{N},
\]

where

\[
a'_k = a_{[k]}^{(k)} = a_{[k], m+1} + \left[ \frac{k}{m+1} \right] \quad \text{and} \quad b'_k = b_{[k]}^{(k)} = b_{[k], m} + \left[ \frac{k}{m} \right].
\]

Note that, for any \( k \in \mathbb{N}, 1 \leq i \leq m+1, \) and \( 1 \leq j \leq m, \) the coefficients defined in (5.9) satisfy

\[
a'_{k+1-i} = a_{[k+1-i], m+1}^{(k-i)} + \left[ \frac{k + 1 - i}{m+1} \right] = a_{[k+1-i], m+1}^{(k)} \quad \text{and} \quad b'_{k+1-j} = b_{[k+1-j], m}^{(k+1-j)} + \left[ \frac{k + 1 - j}{m} \right] = b_{[k+1-j], m}^{(k)}
\]

Therefore,

\[
\{ a_{1}^{(k)}, \ldots, a_{m}^{(k)} \} = \{ a'_{k}, \ldots, a'_{k-m} \} \quad \text{and} \quad \{ b_{1}^{(k)}, \ldots, b_{m}^{(k)} \} = \{ b'_{k}, \ldots, b'_{k-m+1} \}.
\]

Moreover, \( b_{k}' - 1 = b_{k-m}' \) for all \( k \in \mathbb{N}. \) As a result, we can simplify (5.6) and (5.8) to obtain the following alternative version of [33, Th. 14.2].

**Corollary 5.4.** For \( m \geq 1, \) let

\[
g_k(t) = F_m^{(a_1', \ldots, a_{k-m}', b_1', \ldots, b_{k-m+1}')}(t) \quad \text{for any } k \geq -1,
\]

with \( a'_k \) and \( b'_k \) defined by (5.9). Then, the ratios of contiguous hypergeometric series \( f_k(t) = \frac{g_k(t)}{g_{k-1}(t)} \) admit the \( m \)-branched-continued-fraction representation (2.13) with coefficients

\[
\alpha_{k+m} = \frac{(b'_k - a'_k) \prod_{i=1}^{m} a'_k}{\prod_{i=0}^{m} b'_k} \quad \text{for any } k \in \mathbb{N}.
\]

Alternatively, we could set \( \alpha_m = a_1 \cdots a_m \) and let \( \alpha_{k+m} \) be defined by (5.8) or (5.13) for any \( k \geq 1. \) This choice of coefficients \( (\alpha_{k+m})_{k \in \mathbb{N}} \) gives the branched-continued-fraction for the second ratio of contiguous \( m+1 \) \( F_m \)-hypergeometric series obtained in [33, Th. 14.5]. This is a particular instance of a more generic observation: if \( (g_k(t))_{k \geq -1} \) is a sequence of functions satisfying the recurrence relation (5.2), then changing the value of \( \alpha_m \) changes \( g_{-1}(t) \), without changing \( g_k(t) \) for \( k \in \mathbb{N}. \) More generally, changing the values of \( \alpha_m, \ldots, \alpha_{m+n} \) with \( n \in \mathbb{N} \) changes \( g_{-1}(t), \ldots, g_{n-1}(t), \) without changing \( g_k(t) \) for \( k \geq n. \)

We focus now on the cases with \( r \neq s. \) When \( r > s, \) the branched continued fractions in [33] are obtained from the case \( r = s \) replacing \( t \) by \( b_1 \cdots b_{r-s} t, \) taking \( b_1, \ldots, b_{r-s} \to \infty, \) and relabelling \( b_1 \to b_{1-(r-s)}; \) when
where \( s > r \) they are obtained from the case \( r = s \) replacing \( t \) by \((a_1 \cdots a_{s-r})^{-1} t\), taking \( a_1, \ldots, a_{s-r} \to \infty\), and relabelling \( a_i \to a_{i-(s-r)}\). Here we generalise this process, also starting from the case \( r = s \), but considering that the indeterminates which we take to infinity do not need to be \( b_1, \ldots, b_{r-s} \) (if \( r > s \)) or \( a_1, \ldots, a_{s-r} \) (if \( s > r \)), but can instead be any \( r-s \) indeterminates among \( b_1, \ldots, b_s \) (if \( r > s \)) or any \( s-r \) indeterminates among \( a_1, \ldots, a_r \) (if \( s > r \)). For this purpose, when \( r > s \), we choose \( 1 \leq \lambda_1 < \cdots < \lambda_s \leq r \), define \( B = \prod_{j \in J} b_j \) with \( J := \{1, \ldots, r\} \setminus \{\lambda_1, \ldots, \lambda_s\} \neq \emptyset\), and construct new branched continued fractions by replacing \( t \) by \( Bt \), taking \( b_j \to \infty \) for all \( j \in J \), and relabelling \( b_{\lambda_i} \to b_j \) for \( 1 \leq j \leq s \). Similarly, when \( r < s \), we choose \( 1 \leq \sigma_1 < \cdots < \sigma_r \leq s \), define \( A = \prod_{i \in I} a_i \) with \( I := \{1, \ldots, s\} \setminus \{\sigma_1, \ldots, \sigma_r\} \neq \emptyset\), and construct new branched continued fractions by replacing \( t \) by \( A^{-1} t \), taking \( a_i \to \infty \) for all \( i \in I \), and relabelling \( a_{\sigma_i} \to \tilde{a}_i \) for \( 1 \leq i \leq r \).

Taking these limits in the branched continued fractions from Theorem 5.3 and Corollary 5.4, we obtain the following result.

**Theorem 5.5.** For \( r,s \in \mathbb{N} \) such that \( m = \max(r,s) \geq 1 \), let \( 1 \leq \lambda_1 < \cdots < \lambda_s \leq r \) and \( \Lambda = \{\lambda_1, \ldots, \lambda_s\} \) if \( r \geq s \) or let \( 1 \leq \sigma_1 < \cdots < \sigma_r \leq s \), \( \sigma_{r+1} = s+1 \), and \( \Sigma = \{\sigma_1, \ldots, \sigma_r, s+1\} \) if \( r < s \), and define

\[
g_k(t) = r+1F_s \left( \begin{array}{c} a_1^{(k)}, \ldots, a_{r+1}^{(k)} \\ b_1^{(k)}, \ldots, b_s^{(k)} \end{array} \right) t \quad \text{for any} \quad k \geq -1,
\]

where

\[
a_i^{(k)} = a_i + \left\lfloor \frac{k+1-i}{r+1} \right\rfloor \quad \text{and} \quad b_j^{(k)} = b_j + \left\lfloor \frac{k+1 \lambda_j}{r} \right\rfloor \quad \text{if} \quad r \geq s,
\]

or

\[
a_i^{(k)} = a_i + \left\lfloor \frac{k+1-\sigma_i}{s+1} \right\rfloor \quad \text{and} \quad b_j^{(k)} = b_j + \left\lfloor \frac{k+1-j}{s} \right\rfloor \quad \text{if} \quad r \leq s.
\]

Then, the ratios of contiguous hypergeometric series \( \left( f_k(t) = \frac{g_k(t)}{g_{k-1}(t)} \right) \) admit a \( m \)-branched-continued-fraction representation of the form (2.13) with coefficients \( (\alpha_{k+m})_{k \in \mathbb{N}} \) defined as follows:

- if \( r \geq s \),

\[
\alpha_{k+r} = \begin{cases} \frac{r+1}{\prod_{i=1, i \neq [k]+1}^{r} a_i^{(k)}} \prod_{i=1}^{r} a_i' \\ \frac{\prod_{j=1}^{s} b_j^{(k)}}{\prod_{i=1, i \neq [k], i \in \Lambda}^{r} b_i^{(k)}} \\ \frac{(b_i^{(k)} - a_i^{(k)})}{\prod_{j=1}^{r} b_j^{(k)}} \frac{r+1}{\prod_{i=1, i \neq [k]+1}^{r} a_i^{(k)}} = \frac{(b_i' - a_i')}{\prod_{i=1, i \neq [k], i \in \Lambda}^{r} b_i'} \\ \frac{(b_i^{(k)} - 1)}{\prod_{j=1}^{r} b_j^{(k)}} \frac{r+1}{\prod_{i=1, i \neq [k]+1}^{r} a_i^{(k)}} \end{cases} \quad \text{if} \quad [k]_r \notin \Lambda,
\]

with

\[
a_i' = a_i^{(k)} = a_{[k]+1} + \left\lfloor \frac{k}{r+1} \right\rfloor \quad \text{and} \quad b_i' = b_i^{(k)} = b_i + \left\lfloor \frac{k}{r} \right\rfloor \quad \text{if} \quad [k]_r = \lambda_i;
\]

and

\[
a_i^{(k)} = a_i^{(k)} = a_i = \frac{s}{r+1} \quad \text{and} \quad b_i = b_i^{(k)} = b_i + \frac{s}{r} \quad \text{if} \quad [k]_r = \lambda_i\; (5.18)
\]

24
Furthermore, we can always choose if we get a branched-continued-fraction for the first or second ratio of contiguous hypergeometric series when the continued-fraction coefficients are, respectively, giving well-known classical-continued-fraction representations of the form (2.14) for the ratios of contiguous hypergeometric series

\[ 2F_1\left(a_1 + \left[ \frac{k}{s+1} \right], a_2 + \left[ \frac{k-1}{2} \right] \middle| t \right), \quad 2F_0\left(a_1 + \left[ \frac{k}{s+1} \right], a_2 + \left[ \frac{k-1}{2} \right] \middle| - \right), \quad \#F_1\left(a + \left[ \frac{k-1}{2} \right] \middle| t \right), \quad \#F_0\left(a + \left[ \frac{k-1}{2} \right] \middle| - \right), \quad \text{and} \quad \#F_1\left(a + \left[ \frac{k+2}{2} \right] \middle| t \right), \quad \#F_0\left(a + \left[ \frac{k+2}{2} \right] \middle| - \right) \]

(5.21)

where the continued-fraction coefficients are, respectively,

\[ \alpha_{2k+1} = \frac{(a_1 + k)(b - a_2 + k)}{(b + 2k + 1)(b + 2k)} \quad \text{and} \quad \alpha_{2k+2} = \frac{(a_2 + k)(b - a_1 + k)}{(b + 2k + 1)(b + 2k + 1)} \quad \text{for } k \in \mathbb{N}, \]

(5.22)

(5.23)

(5.24)

For any \( r, s \in \mathbb{N} \) not both equal to 0, Theorem 5.5 gives \( \binom{\max(r,s)}{\min(r,s)} \) distinct branched continued fractions, including as particular cases the representations obtained in [33, §14] for the first and second ratios of contiguous \( r+1F_s \)-hypergeometric series in (5.1). In Section 6, we investigate multiple orthogonal polynomials corresponding to each of these branched continued fractions (with \( a_{r+1} = 1 \)).

When \( r \geq s \), setting \( \lambda_j = r - s + j \) for all \( 1 \leq j \leq s \) in Theorem 5.5 gives the branched-continued-fraction for the first ratio of contiguous \( r+1F_s \)-hypergeometric series obtained in [33, Th. 14.3] if \( \lambda_s \) is \( r \) or the branched-continued-fraction for the first ratio of contiguous \( r+1F_s \)-hypergeometric series obtained in [33, Th. 14.6] if \( \lambda_s \leq r \). When \( r \leq s \), setting \( \sigma_i = s - r + i \) for all \( 1 \leq i \leq r \) in Theorem 5.5 gives the branched-continued-fraction for the first ratio of contiguous \( r+1F_s \)-hypergeometric series obtained in [33, Th. 14.3].

Furthermore, we can always choose if we get a branched-continued-fraction for the first or second ratio of contiguous \( r+1F_s \)-hypergeometric series by keeping the values of \( \alpha_{k+m} \) for \( k \geq 1 \) defined in Theorem 5.5 and
changing \( \alpha_m \) and, consequently, changing \( g_i(t) \) without \( \alpha_i \) for any \( k \in \mathbb{N} \). Precisely, we get the first and second ratios in (5.1) if we set, respectively,

\[
\alpha_m = \frac{a_1 \cdots a_r (b_s - a_{r+1})}{b_1 \cdots b_s (b_s - 1)} \quad \text{or} \quad \alpha_m = \frac{a_1 \cdots a_r}{b_1 \cdots b_s}. \tag{5.25}
\]

The expressions on the left- and right-hand sides of (5.17) and (5.19) are obtained by taking the limits described before the statement of Theorem 5.5 in (5.8) or in (5.13), respectively. Alternatively, the expressions on the right-hand side can be obtained from the expressions on left-hand side analogously to how we derived (5.13) from (5.8). We now give an alternative proof of Theorem 5.5, using the relations in Lemma 5.2.

**Second proof of Theorem 5.5.** It is sufficient to show that the sequence \((g_k(t))_{k \geq 1}\) defined in (5.14) satisfies the recurrence relation

\[
g_k(t) - g_{k-1}(t) = \alpha_{k+m} t \ g_{k+m}(t) \quad \text{for all} \ k \in \mathbb{N}, \tag{5.26}
\]

involving the coefficients \((\alpha_{k+m})_{k\in\mathbb{N}}\) defined by (5.17) if \( r \geq s \) or by (5.19) if \( r \leq s \). Then, using Lemma 5.1, \( f_k(t) = g_k(t)/g_{k-1}(t) \) is the generating function of the \( m \)-Dyck paths at height \( k \) for any \( k \in \mathbb{N} \) and admits the \( m \)-branched-continued-fraction representation (2.13).

To prove that (5.26) holds, we use the relations in Lemma 5.2. We start by considering the case \( r \geq s \). Then, for any \( k \in \mathbb{N}, \)

- if \( i = [k]_{r+1} \) and \( a_i = a_{i-1} \) if \( i \neq [k]_{r+1}, \)
- \( b_j = b_{j-1} + 1 \) if \( \lambda_j = [k]_{r} \) and \( b_j = b_{j-1} \) if \( \lambda_j \neq [k]_{r}. \)

If \([k]_{r} \notin \Lambda = \{\lambda_1, \cdots, \lambda_s\}\), we use (5.3) to find that

\[
g_k(t) - g_{k-1}(t) = r+1F_s \left( \begin{array}{c}
\frac{a_1^{(k)}}{b_1^{(k)}}, \cdots, \frac{a_{r-1}^{(k)}}{b_{r-1}^{(k)}} \end{array} \bigg| t \right) - r+1F_s \left( \begin{array}{c}
\frac{a_1^{(k)} - 1}{b_1^{(k)}}, \cdots, \frac{a_{r-1}^{(k)} - 1}{b_{r-1}^{(k)}} \end{array} \bigg| t \right)
\]

\[
= \frac{\prod_{i=1, i \neq [k]_{r+1}}^{r} a_i^{(k)}}{\prod_{j=1}^{s} b_j^{(k)}} \ r+1F_s \left( \begin{array}{c}
\frac{a_1^{(k)} + 1}{b_1^{(k)}}, \cdots, \frac{a_{r-1}^{(k)} + 1}{b_{r-1}^{(k)}} \end{array} \bigg| t \right)
\]

\[
= \alpha_{k+r} t \ g_{k+r}(t). \tag{5.27}
\]

Otherwise, \([k]_{r} = \lambda_{\ell} \in \Lambda \) with \( \ell \in \{1, \cdots, s\} \), and, using (5.4), we get

\[
g_k(t) - g_{k-1}(t) = r+1F_s \left( \begin{array}{c}
\frac{a_1^{(k)}}{b_1^{(k)}}, \cdots, \frac{a_{r-1}^{(k)}}{b_{r-1}^{(k)}} \end{array} \bigg| t \right) - r+1F_s \left( \begin{array}{c}
\frac{a_1^{(k)} - 1}{b_1^{(k)}}, \cdots, \frac{a_{r-1}^{(k)} - 1}{b_{r-1}^{(k)}} \end{array} \bigg| t \right)
\]

\[
= \frac{\prod_{i=1, i \neq [k]_{r+1}}^{r} a_i^{(k)}}{\prod_{j=1}^{s} b_j^{(k)}} \ r+1F_s \left( \begin{array}{c}
\frac{a_1^{(k)} + 1}{b_1^{(k)}}, \cdots, \frac{a_{r-1}^{(k)} + 1}{b_{r-1}^{(k)}} \end{array} \bigg| t \right)
\]

\[
= \alpha_{k+r} t \ g_{k+r}(t). \tag{5.28}
\]
Next, we consider the case $r \leq s$. Then, for any $k \in \mathbb{N}$,

- $a_i^{(k)} = a_i^{(k-1)} + 1$ if $\sigma_i = [k]_{j+1}$ and $a_i^{(k)} = a_i^{(k-1)}$ if $\sigma_i \neq [k]_{j+1}$,
- $b_j^{(k)} = b_j^{(k-1)} + 1$ if $j = [k]_s$ and $b_j^{(k)} = b_j^{(k-1)}$ if $j \neq [k]_s$.

If $[k]_{j+1} \notin \Sigma = \{\sigma_1, \cdots, \sigma_r, s + 1\}$, we use (5.5) to find that

$$g_k(t) - g_{k-1}(t) = r + 1 F_3 \begin{pmatrix} a_1^{(k)}, \cdots, a_{r+1}^{(k)} \\ b_1^{(k)}, \cdots, b_s^{(k)} \end{pmatrix} t - r + 1 F_3 \begin{pmatrix} a_1^{(k)}, \cdots, a_{r+1}^{(k)} \\ b_1^{(k)}, \cdots, b_s^{(k)} \end{pmatrix} t$$

$$= - \frac{\prod_{i=1}^{r+1} a_i^{(k)}}{(b_{[k]_s} - 1) \prod_{j=1}^{r+1} b_j^{(k)}} r + 1 F_3 \begin{pmatrix} a_1^{(k)} + 1, \cdots, a_r^{(k)} + 1 \\ b_1^{(k)} + 1, \cdots, b_s^{(k)} + 1 \end{pmatrix} t$$

$$= \alpha_{k+s} t g_{k+s}(t).$$

Otherwise, $[k]_{j+1} = \sigma_\ell \in \Sigma$ with $\ell \in \{1, \cdots, r+1\} (\sigma_{r+1} = s + 1)$, and, using (5.4), we get

$$g_k(t) - g_{k-1}(t) = r + 1 F_3 \begin{pmatrix} a_1^{(k)}, \cdots, a_{r+1}^{(k)} \\ b_1^{(k)}, \cdots, b_s^{(k)} \end{pmatrix} t - r + 1 F_3 \begin{pmatrix} a_1^{(k)}, \cdots, a_{r+1}^{(k)} \\ b_1^{(k)}, \cdots, b_s^{(k)} \end{pmatrix} t$$

$$= \frac{(b_{[k]_s} - a_{\ell}^{(k)}) \prod_{i=1, i \neq \ell}^{r} a_i^{(k)}}{(b_{[k]_s} - 1) \prod_{j=1}^{r+1} b_j^{(k)}} r + 1 F_3 \begin{pmatrix} a_1^{(k)} + 1, \cdots, a_{r+1}^{(k)} + 1 \\ b_1^{(k)} + 1, \cdots, b_s^{(k)} + 1 \end{pmatrix} t$$

$$= \alpha_{k+s} t g_{k+s}(t).$$

(5.30)

\[\square\]

### 5.2 Conditions for positivity of the coefficients

The following result gives necessary and sufficient conditions for the non-negativity and for positivity of all coefficients of the branched continued fractions introduced in Theorem 5.5. In Section 8, we revisit in more detail these conditions when $a_{r+1} = 1$.

**Proposition 5.6.** For $r, s \in \mathbb{N}$ such that $s \leq r \neq 0$, let $1 \leq \lambda_1 < \cdots < \lambda_s \leq r$ and $a_1, \cdots, a_{r+1}, b_1, \cdots, b_s \in \mathbb{R}^+$. Then, the coefficients in the sequence $(\alpha_{k+r})_{k \in \mathbb{N}}$ defined by (5.17) are all nonnegative if and only if

$$b_j \geq a_i - \left[ \frac{i - \lambda_j}{r} \right] = \begin{cases} a_i & \text{if } i \leq \lambda_j \\ a_i - 1 & \text{if } i > \lambda_j + 1 \end{cases} \quad \text{for all } 1 \leq i \leq r+1 \text{ and } 1 \leq j \leq s,$$  

(5.31)

and

$$\frac{b_s - a_{r+1}}{b_s - 1} \geq 0 \quad \text{when } \lambda_s = r.$$  

(5.32)

Furthermore, the coefficients in $(\alpha_{k+r})_{k \in \mathbb{N}}$ are all positive if and only if all the inequalities above are strict.
When $r < s$, we cannot have non-negativity of all coefficients in $(\alpha_{k+r})_{k \in \mathbb{N}}$ due to the minus sign in (5.19).

**Proof.** For any $k \in \mathbb{N}$, $a_i^k \geq 0$ for $i \in \{1, \ldots, r+1\}$ and $b_j^k \geq b_j$ for $j \in \{1, \ldots, r\}$, so $\alpha_{k+r} > 0$ whenever $[k]_r \notin \{\lambda_1, \ldots, \lambda_s\}$. Otherwise, we consider separately the cases $k = 0$ and $k \geq 1$. If $k = 0$, $[k]_r \in \{\lambda_1, \ldots, \lambda_s\}$ implies that $[k]_r = r = \lambda_j$, so $\alpha_k \geq 0$ if and only if (5.32) holds. If $k \geq 1$ and $[k]_r = \lambda_j$ for some $j \in \{1, \ldots, s\}$, then $\alpha_{k+r} \geq 0$ if and only if $b_j^k \geq a_{[k]_r+1}^k$, because $b_j^k \geq b_j + 1$ implies $b_j^k - 1 \in \mathbb{R}^+$ for any $k \geq 1$. We will show now that $b_j^k \geq a_{[k]_r+1}^k$ for all $k \geq 1$ if and only if (5.31) holds.

If $k = r(r+1) + \ell$, with $\ell \geq 1$, and $[k]_r = [\ell]_r = \lambda_j$, we have

$$b_j^k - a_{[k]_r+1}^k = (b_j^k + (r+1)) - a_{[k]_r+1}^k = b_j^k - a_{[k]_r+1}^k + 1. \quad (5.33)$$

Hence, $b_j^k - a_{[k]_r+1}^k \geq 0$ holds for all $k \geq 1$ such that $[k]_r = \lambda_j$ for some $j \in \{1, \ldots, s\}$ if and only if that holds for all $1 \leq k \leq r(r+1)$. Note that each $k \in \{1, \ldots, r(r+1)\}$ can be uniquely written in the form

$$k = qr + p = q(r+1) + (p-q) \quad \text{with} \quad q \in \{0, \ldots, r\} \quad \text{and} \quad p \in \{1, \ldots, r\}. \quad (5.34)$$

Therefore, $\alpha_{k+r} \geq 0$ for all $k \geq 1$ if and only if $b_j^k - a_{[k]_r+1}^k \geq 0$ for all $k \in \{1, \ldots, r(r+1)\}$ of the form (5.34) with $p = [k]_r = \lambda_j$ with $j \in \{1, \ldots, s\}$. For $k$ of the form in (5.34), we get

$$b_j^k = b_j + q + 1 \quad \text{and} \quad a_{[k]_r+1}^k = \begin{cases} a_{\lambda_j-q} + (q+1) & \text{if } 0 \leq q < \lambda_j - 1, \\ a_{\lambda_j-q+r+1} + q & \text{if } \lambda_j - 1 \leq q \leq r. \end{cases} \quad (5.35)$$

If $0 \leq q < \lambda_j - 1$, we set $i = \lambda_j - q \in \{1, \ldots, \lambda_j\}$ and we have $b_j^k - a_{[k]_r+1}^k \geq 0$ if and only if $b_j \geq a_i$. Otherwise, $\lambda_j \leq q \leq r$, we set $i = \lambda_j - q + r + 1 \in \{\lambda_j + 1, \ldots, r+1\}$, and we get $b_j^k - a_{[k]_r+1}^k \geq 0$ if and only if $b_j \geq a_i - 1$. Therefore, we conclude that $b_j^k \geq a_{[k]_r+1}^k$ for all $k \geq 1$ if and only if (5.31) holds.

### 5.3 Modified $m$-Stieltjes-Rogers-polynomials when $a_{r+1} = 1$

The following result focus on the particular case $a_{r+1} = 1$ of Theorem 5.5. In that case, we can find explicit expressions for the modified $m$-Stieltjes-Rogers-polynomials $(s_n^{(m,k)}(\alpha))_{n,k \in \mathbb{N}}$ and they reduce to ratios of products of Pochhammer symbols when $k \leq m$, which correspond to the moments of the linear functionals of orthogonality for the multiple orthogonal polynomials studied in Section 6.

**Corollary 5.7.** For $r, s \in \mathbb{N}$ with $m = \max(r, s) \geq 1$, let $(\alpha_{k+m})_{k \in \mathbb{N}}$ be defined, as in Theorem 5.5, by (5.17) if $r \geq s$ or by (5.19) if $r \leq s$, and suppose that $a_{r+1} = 1$. Then, the generating function of the modified $m$-Stieltjes-Rogers-polynomials of type $k$, $(s_n^{(m,k)}(\alpha))_{n \in \mathbb{N}}$, with $k \in \mathbb{N}$, is $g_k(t)$ defined in (5.14), and

$$s_n^{(m,k)}(\alpha) = \binom{n + \lceil \frac{k-m}{m+1} \rceil}{n} \binom{a_1^{(k)}}{b_1^{(k)}} \cdots \binom{a_r^{(k)}}{b_s^{(k)}} \quad \text{for any } n, k \in \mathbb{N}. \quad (5.36)$$
In particular,

\[ S_n^{(m,k)}(\alpha) = \left( \frac{a_1^{(k)}}{b_1^{(k)}} \right)_n \cdots \left( \frac{a_r^{(k)}}{b_r^{(k)}} \right)_n \] for \( n \in \mathbb{N} \) and \( 0 \leq k \leq m \). \hspace{1cm} (5.37)

**Proof.** If we take \( a_{r+1} = 1 \) in (5.14), then \( a_r^{(-1)} = a_{r+1} - 1 = 0 \) and, as a result, \( g_{-1}(t) = 1 \). Therefore, applying Lemma 5.1 to the recurrence relation (5.26), we find that the generating function of the modified \( m \)-Stieltjes-Rogers polynomials of type \( k \) is \( g_k(t) = f_0(t) \cdots f_k(t) \). Furthermore,

\[ a_r^{(k)} = 1 + \left\lfloor \frac{k-m}{m+1} \right\rfloor \implies \left( \frac{a_r^{(k)}}{n!} \right)_n = \left( n + \left\lfloor \frac{k-m}{m+1} \right\rfloor \right)_n. \hspace{1cm} (5.38) \]

Therefore,

\[ \sum_{n=0}^{\infty} S_n^{(m,k)}(\alpha) t^n = r+1 F_s \left( a_1^{(k)}, \ldots, a_r^{(k)}, 1 + \left\lfloor \frac{k-m}{m+1} \right\rfloor \right)_n \left( b_1^{(k)}, \ldots, b_s^{(k)} \right)_n. \hspace{1cm} (5.39) \]

As a result, we obtain (5.36). When \( 0 \leq k \leq m \), \( \left\lfloor \frac{k-m}{m+1} \right\rfloor = 0 \) and (5.36) reduces to (5.37). \( \square \)

### 5.4 A limiting type of ratios

The limiting case \( a_{r+1} \to \infty \) of the branched continued fractions introduced in Theorem 5.5 gives new branched-continued-fraction representations for the third ratio of hypergeometric series \( r F_s \) in (5.1). Let \( (g_k(t))_{k \geq -1} \) and \( (\alpha_k)_k \) be defined by (5.14) and (5.17)-(5.19), respectively, as in Theorem 5.5, and define

\[ \hat{g}_k(t) = \lim_{a_{r+1} \to \infty} g_k \left( \frac{t}{a_{r+1}} \right) = \frac{F_s \left( a_1^{(k)}, \ldots, a_r^{(k)} \right)_n}{b_1^{(k)}, \ldots, b_s^{(k)} \right)_n}. \hspace{1cm} (5.40) \]

Then, the ratios of contiguous hypergeometric series \( \hat{f}_k(t) = \frac{\hat{g}_k(t)}{\hat{g}_{k-1}(t)} \) admit a \( m \)-branched-continued-fraction representation with coefficients \( (\alpha_{k+m})_{k \in \mathbb{N}} \) defined by

\[ \alpha_{k+m} = \lim_{a_{r+1} \to \infty} \frac{\alpha_{k+m}}{a_{r+1}}. \hspace{1cm} (5.41) \]

This construction generalises the branched continued fractions obtained in [33, §14] for the third ratio of contiguous hypergeometric series in (5.1), analogously to how Theorem 5.5 generalises the branched continued fractions obtained in [33, §14] for the first and second ratios of contiguous hypergeometric series \( r+1F_s \). In fact, this construction reduces to parts (a) and (b) of [33, Th. 14.12] when \( \lambda_j = r - s + j \) for all \( 1 \leq j \leq s \) with \( r \geq s \) and when \( \sigma_i = s - r + i \) for all \( 1 \leq i \leq r \) with \( r \leq s \), respectively.

However, we do not explicitly write the coefficients of these branched continued fractions here, because they are very similar to the coefficients in Theorem 5.5. Moreover, Theorem 5.5 already gives branched-continued-fraction representations for ratios of contiguous hypergeometric series \( r F_s \) when \( r \geq 1 \). To find branched-continued-fraction representations for ratios of contiguous \( q F_s \) with \( s \geq 1 \), see [33, Th. 14.8].

29
The functions $g_0$ and $g_{-1}$ in Theorem 5.5, and consequently their ratio $f_0$, are invariant under permutations of $(a_1, \cdots, a_r)$ (but not $a_{r+1}$) and $(b_1, \cdots, b_{s-1})$ (but not necessarily $b_s$). The branched-continued-fraction coefficients $(\alpha_{k+m})_{k \in \mathbb{N}}$ defined by (5.17)-(5.19) are not invariant under these permutations. However, when $a_{r+1} = 1$, their production matrix is invariant under permutations of $(a_1, \cdots, a_r)$, but not of any of the $b_j$, as well as under different choices of $(\sigma_1, \cdots, \sigma_r)$ when $r < s$, but not of $(\lambda_1, \cdots, \lambda_s)$ when $r > s$. We will prove this statement at the end of the next section, as a consequence of the corresponding multiple orthogonal polynomials having the same symmetries.

6 A general class of hypergeometric multiple orthogonal polynomials

In this section, we investigate the type II multiple orthogonal polynomials on the step-line with respect to the linear functionals $(u_0, \cdots, u_{m-1})$, with $m \in \mathbb{Z}^+$, whose moments are the modified $m$-Stieltjes-Rogers polynomials in (5.37), equal to ratios of products of Pochhammer symbols. These multiple orthogonal polynomials have coefficients in the ring $R = \mathbb{Q}(b_1, \cdots, b_s)[a_1, \cdots, a_r]$ of polynomials in the indeterminates $a_1, \cdots, a_r$ whose coefficients are rational functions in the indeterminates $b_1, \cdots, b_s$.

Firstly, we find explicit expressions as terminating hypergeometric series for the multiple orthogonal polynomials under study (Theorem 6.1). Then, we derive differential properties satisfied by the polynomials (Propositions 6.4 and 6.5). Finally, we use the connection of these multiple orthogonal polynomials with the branched continued fractions introduced in Theorem 5.5 to obtain expressions for the recurrence coefficients of the polynomials as combinations of the branched-continued-fraction coefficients (Theorem 6.6).

6.1 Explicit expressions as terminating hypergeometric series

The main goal of this section is to prove the following result.

Theorem 6.1. For $r, s \in \mathbb{N}$ with $m = \max(r, s) \geq 1$, let:

1. $1 \leq \lambda_1 < \cdots < \lambda_s \leq r$ when $r \geq s$ and $1 \leq \sigma_1 < \cdots < \sigma_s \leq s$ when $r \leq s$;
2. $R = \mathbb{Q}(b_1, \cdots, b_s)[a_1, \cdots, a_r]$ for $a_1, \cdots, a_r, b_1, \cdots, b_s$ such that, for all $1 \leq i \leq r$, $1 \leq j \leq s$ and $n \in \mathbb{N}$,
   - $a_i + n, b_j + n \neq 0$,
   - $b_j - a_i + n \neq 0$ if $i \leq \lambda_j$ and $b_j - a_i + 1 + n \neq 0$ if $i \geq \lambda_j + 1$ when $r \geq s$,
   - $b_j - a_i + n \neq 0$ if $\sigma_i \leq j$ and $b_j - a_i + 1 + n \neq 0$ if $\sigma_i > j + 1$ when $r \leq s$;
3. $(v_0, \cdots, v_{m-1})$ the vector of linear functionals acting on $R[x]$ with moments

$$
\nu_k [x^n] = \left( \begin{array}{c}
\alpha_1^{(k)}
\vdots
\alpha_r^{(k)}
\end{array} \right)_n \left( \begin{array}{c}
\beta_1^{(k)}
\vdots
\beta_s^{(k)}
\end{array} \right)_n \quad \text{for } n \in \mathbb{N} \text{ and } 0 \leq k \leq m - 1,
$$

(6.1)

where the $\alpha_i^{(k)}$ and $\beta_j^{(k)}$ are defined by (5.15)-(5.16), which means that, for any $0 \leq k \leq m - 1$,

$$
a_i^{(k)} = \begin{cases} 
a_i + 1 & \text{if } 1 \leq i \leq k \\
a_i & \text{if } k + 1 \leq i \leq r \end{cases} \quad \text{and} \quad b_j^{(k)} = \begin{cases} 
b_j + 1 & \text{if } 1 \leq \lambda_j \leq k \\
b_j & \text{if } k + 1 \leq \lambda_j \leq r \end{cases} \quad \text{when } r \geq s,
$$

(6.2)
By definition of the hypergeometric series, the limiting relations hold for the corresponding orthogonality functionals.

Then, the $m$-orthogonal polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\) with respect to \((u_0, \cdots, u_{m-1})\) is given by

\[
P_n(x) = \frac{(-1)^n (a_1)_n \cdots (a_r)_n}{(b_1^{(n-1)})_n \cdots (b_s^{(n-1)})_n} \mathbf{^sF_r} \left( \begin{array}{c} -n, b_1^{(n-1)}, \cdots, b_s^{(n-1)} \\ a_1, \cdots, a_r \end{array} \right| x \right),
\]

with

\[
b_j^{(n-1)} = b_j + \left[ \frac{n - \lambda_j}{r} \right] \text{ when } r \geq s \quad \text{or} \quad b_j^{(n-1)} = b_j + \left[ \frac{n - j}{s} \right] \text{ when } r \leq s.
\]

When \(r = s = m\), we have

\[
P_n(x) = \frac{(-1)^n (a_1)_n \cdots (a_m)_n}{(b_1^{(n-1)})_n \cdots (b_m^{(n-1)})_n} \mathbf{^mF_m} \left( \begin{array}{c} -n, b_1^{(n-1)}, \cdots, b_m^{(n-1)} \\ a_1, \cdots, a_m \end{array} \right| x \right) \quad \text{with } b_j^{(n-1)} = b_j + \left[ \frac{n - j}{m} \right].
\]

The polynomials in (6.4) with \(r \neq s\) can be obtained as limiting cases of (6.6).

When we want to highlight the parameters in the polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\) defined by (6.4), we denote \(P_n(x)\) by \(P_n^{[r,s],(\lambda_1, \cdots, \lambda_s)}(x)\) for \(r \geq s\) or by \(P_n^{[r,s]}(x)\) for \(r \leq s\). Note that \((P_n(x))_{n \in \mathbb{N}}\) defined by (6.4) with \(r \leq s\) does not depend on the choice of \((\sigma_1, \cdots, \sigma_r)\), although the same is not true for the corresponding linear functionals.

If \(s < r\), let \(J := \{1, \cdots, r\} \setminus \{\lambda_1, \cdots, \lambda_s\} \neq \emptyset\), \(B = \prod_{j \in J} b_j\), and \(\hat{b}_j = b_{\lambda_j}\) for \(1 \leq j \leq s\). Then,

\[
P_n^{[r,s],(\lambda_1, \cdots, \lambda_s)}(x) = \lim_{j \rightarrow \infty} \mathbf{^mF_m}^{[r,r]} \left( \begin{array}{c} a_1, \cdots, a_r \\ \hat{b}_1, \cdots, \hat{b}_s \end{array} \right| \frac{x}{B} b_{\lambda_1, \cdots, \lambda_s} \right).
\]

Analogously, if \(r < s\), let \(I := \{1, \cdots, s\} \setminus \{\sigma_1, \cdots, \sigma_r\} \neq \emptyset\), \(A = \prod_{i \in I} a_i\), and \(\hat{a}_i = a_{\sigma_i}\) for \(1 \leq i \leq r\). Then,

\[
P_n^{[r,s]}(x) = \lim_{a_i \rightarrow \infty} A^{-n} P_n^{[s,s]} \left( Ax \begin{array}{c} a_1, \cdots, a_s \\ b_{\lambda_1, \cdots, \lambda_s} \end{array} \right).
\]

Similar limiting relations hold for the corresponding orthogonality functionals.

By definition of the hypergeometric series, the \(m\)-orthogonal polynomials \((P_n(x))_{n \in \mathbb{N}}\) can be written as

\[
P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a_1 + n - k)_k \cdots (a_r + n - k)_k}{(b_1^{(n-1)} + n - k)_k \cdots (b_s^{(n-1)} + n - k)_k} x^{n-k}.
\]
Therefore, the reciprocal polynomials \(\left(x^n P_n \left(\frac{1}{x}\right)\right)_{n \in \mathbb{N}}\), which are the polynomials with the same coefficients as \((P_n(x))_{n \in \mathbb{N}}\) in reverse order, are

\[
x^n P_n \left(\frac{1}{x}\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a_1 + n - k) \cdots (a_r + n - k)}{(b_1^{(n-1)} + n - k) \cdots (b_s^{(n-1)} + n - k)} x^k.
\]  
(6.10)

Using again the definition of the hypergeometric series, the latter is equivalent to

\[
x^n P_n \left(\frac{1}{x}\right) = \pFq{r+1}{p+q}{-n, -n - a_1 + 1, \ldots, -n - a_r + 1}{-n - b_1^{(n-1)} + 1, \ldots, -n - b_s^{(n-1)} + 1}{-1}^{r+s} x.
\]  
(6.11)

The latter formula can also be obtained as a particular case of the more general formula for the reciprocal of a hypergeometric polynomial:

\[
x^p_{p+1} F_q \left(-n, c_1, \ldots, c_p \atop d_1, \ldots, d_q\right) \left(\frac{1}{x}\right) = \frac{(-1)^n (c_1) \cdots (c_p) \left(c_1 \cdots (c_p) \right)_n}{(d_1) \cdots (d_q) \left(d_1 \cdots (d_q) \right)_n} \pFq{q+1}{p+q}{-n, -n - d_1 + 1, \ldots, -n - d_q + 1}{-n - c_1 + 1, \ldots, -n - c_p + 1}{-1}^{p+q} x.
\]  
(6.12)

The latter result can be found in [16]. Therein it is assumed that \(c_1 = n + 1\) and \(d_1 = \frac{1}{2}\), but the proof is the same in the general case.

Recalling the orthogonality conditions in (2.8), Theorem 6.1 holds if and only if \((P_n(x))_{n \in \mathbb{N}}\) defined by (6.4) satisfies the orthogonality conditions

\[
v_{\ell} \left[x^k P_n \right] = \begin{cases} N_n \neq 0 & \text{if } n = m k + \ell, \\ 0 & \text{if } n \geq m k + \ell + 1, \end{cases} \quad \text{for all } k, n \in \mathbb{N} \text{ and } \ell \in \{0, \ldots, m - 1\}.
\]  
(6.13)

To show that (6.13) holds, we use the following Lemmas 6.2 and 6.3.

**Lemma 6.2.** For \(p, q, r, s \in \mathbb{N}\), let:

- \(R := \mathbb{Q} (b_1, \ldots, b_s, c_1, \ldots, c_p) [a_1, \ldots, a_r, d_1, \ldots, d_q]\),
- \(v : R[x] \to R\) the linear functional defined by

\[
v [x^n] = \frac{(a_1) \cdots (a_r)}{(b_1) \cdots (b_s)} \quad \text{for any } n \in \mathbb{N},
\]  
(6.14)

- \((P_n(x))_{n \in \mathbb{N}}\) the polynomial sequence in \(R[x]\) whose elements are

\[
P_n(x) = \frac{(-1)^n (d_1) \cdots (d_q) \left(d_1 \cdots (d_q) \right)_n}{(c_1) \cdots (c_p) \left(c_1 \cdots (c_p) \right)_n} \pFq{q+1}{p+q}{-n, c_1, \ldots, c_p}{d_1, \ldots, d_q}{x}.
\]  
(6.15)

Then, for any \(k, n \in \mathbb{N}\),

\[
v [x^k P_n] = \frac{(-1)^n (d_1) \cdots (d_q) \left(a_1\right)_k \cdots (a_r)_k}{(c_1) \cdots (c_p) \left(b_1\right)_k \cdots (b_s)_k} \pFq{p+q+1}{r+s}{-n, c_1, \ldots, c_p, a_1 + k, \ldots, a_r + k}{d_1, \ldots, d_q, b_1 + k, \ldots, b_s + k}{1}.
\]  
(6.16)
Our aim is to prove that (6.13) holds. Using Lemma 6.2, we have

\[ v \left[ x^k P_n \right] = \frac{(1)^n (d_1) \cdots (d_q) n^n}{(c_1) \cdots (c_p) n^n} \sum_{j=0}^{n} \frac{(c_1)_j \cdots (c_p)_j}{j! (d_1)_j \cdots (d_q)_j} v \left[ x^j \right]. \quad (6.17) \]

Applying (6.14) to the latter, we get

\[ v \left[ x^k P_n \right] = \frac{(1)^n (d_1) \cdots (d_q) n^n}{(c_1) \cdots (c_p) n^n} \sum_{j=0}^{n} \frac{(c_1)_j \cdots (c_p)_j}{j! (d_1)_j \cdots (d_q)_j} v \left[ x^j (a_1 k + j) \cdots (a_r k + j) \right], \quad (6.18) \]

which is equivalent to

\[ v \left[ x^k P_n \right] = \frac{(1)^n (d_1) \cdots (d_q) n^n (a_1 k + j) \cdots (a_r k + j)}{(c_1) \cdots (c_p) n^n (b_1 k + j) \cdots (b_s k + j)} \sum_{j=0}^{n} \frac{(c_1)_j \cdots (c_p)_j}{j! (d_1)_j \cdots (d_q)_j (b_1 k + j) \cdots (b_s k + j)} v \left[ x^j \right]. \quad (6.19) \]

Therefore, by definition of the generalised hypergeometric series, (6.16) holds.

**Lemma 6.3.** Let \( p, n, m_1, \ldots, m_p \in \mathbb{N} \) such that \( m := \sum_{i=1}^{p} m_i \leq n \) and let \( R = \mathbb{Q} (c_1, \cdots, c_p, d) \). Then,

\[ p+1F_p \left( \begin{array}{c} -n, c_1 + m_1, \cdots, c_p + m_p \\ c_1, \cdots, c_p \end{array} \bigg| 1 \right) = \begin{cases} 0 & \text{if } m \leq n - 1, \\ \frac{(1)^n n!}{\prod_{i=1}^{p} (c_i)} & \text{if } m = n; \end{cases} \quad (6.20) \]

and

\[ p+2F_{p+1} \left( \begin{array}{c} -n, d, c_1 + m_1, \cdots, c_p + m_p \\ d+1, c_1, \cdots, c_p \end{array} \bigg| 1 \right) = \frac{n! \prod_{i=1}^{p} (c_i - d)}{\prod_{i=1}^{p} (c_i)}, \quad (6.21) \]

Formula (6.21) was deduced in [28] and (6.20) can be obtained by taking the limit \( d \to +\infty \) in (6.21).

**Proof of Theorem 6.1.** Our aim is to prove that (6.13) holds. Using Lemma 6.2, we have

\[ v_{\ell} \left[ x^k P_n \right] = \frac{(1)^n \prod_{i=1}^{r} (a_i)_n \prod_{i=1}^{r} (a_i^{(\ell)})^k}{\prod_{j=1}^{s} (b_j^{(n-1)}) \prod_{j=1}^{s} (b_j^{(n-1)})^k} r_{s+1}F_{r+s} \left( \begin{array}{c} -n, a_1^{(\ell)} + k, \cdots, a_r^{(\ell)} + k, b_1^{(n-1)}, \cdots, b_s^{(n-1)} \\ a_1, \cdots, a_r, b_1^{(\ell)} + k, \cdots, b_s^{(\ell)} + k \end{array} \bigg| 1 \right). \quad (6.22) \]

for any \( k, n \in \mathbb{N} \) and \( \ell \in \{0, \cdots, m - 1\} \). To shorten the notation throughout this proof, we define

\[ F_{\ell}(n, k) = r_{s+1}F_{r+s} \left( \begin{array}{c} -n, a_1^{(\ell)} + k, \cdots, a_r^{(\ell)} + k, b_1^{(n-1)}, \cdots, b_s^{(n-1)} \\ a_1, \cdots, a_r, b_1^{(\ell)} + k, \cdots, b_s^{(\ell)} + k \end{array} \bigg| 1 \right). \quad (6.23) \]
Suppose that \( n \geq mk + \ell + 1 \). Then,
\[
b^{(n-1)}_j = b_j + \left\lfloor \frac{n-\lambda_j}{m} \right\rfloor \geq b_j + k + \left\lfloor \frac{\ell+1-\lambda_j}{m} \right\rfloor = b^{(\ell)}_j + k \quad \text{for each } 1 \leq j \leq m.
\] (6.24)

Therefore, \( a^{(\ell)}_i + k - a_i \) and \( b^{(n-1)}_j - b^{(\ell)}_j - k \) are nonnegative integers for any \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \). Furthermore,
\[
\sum_{i=1}^r \left(a^{(\ell)}_i + k - a_i\right) \leq rk + \ell \leq mk + \ell
\] (6.25)

and
\[
\sum_{j=1}^s \left(b^{(n-1)}_j - b^{(\ell)}_j - k\right) = \sum_{j=1}^s \left(\left\lfloor \frac{n-\lambda_j}{m} \right\rfloor - \left\lfloor \frac{\ell+1-\lambda_j}{m} \right\rfloor - k\right)
\leq \sum_{i=1}^m \left(\left\lfloor \frac{n-i}{m} \right\rfloor - \left\lfloor \frac{\ell+1-i}{m} \right\rfloor - k\right) = n - 1 - mk - \ell.
\] (6.26)

As a result,
\[
\sum_{i=1}^r \left(a^{(\ell)}_i + k - a_i\right) + \sum_{j=1}^s \left(b^{(n-1)}_j - b^{(\ell)}_j - k\right) \leq n - 1.
\] (6.27)

Hence, using (6.20), we find that \( F_{\ell}(n,k) = 0 \) and, consequently, \( v_{\ell} [x^k P_n] = 0 \) whenever \( n \geq mk + \ell + 1 \).

Suppose now that \( n = mk + \ell \). Then,
\[
b^{(n-1)}_j = b_j + \left\lfloor \frac{n-\lambda_j}{m} \right\rfloor = b_j + k + \left\lfloor \frac{\ell-\lambda_j}{m} \right\rfloor = \begin{cases} b_j + k + 1 & \text{if } 1 \leq \lambda_j \leq \ell - 1, \\ b_j + k & \text{if } \ell \leq \lambda_j \leq m \end{cases}
\] for \( 1 \leq \ell \leq m - 1 \), (6.28)

and
\[
b^{(mk-1)}_j = b_j + \left\lfloor \frac{mk-j}{m} \right\rfloor = \begin{cases} b_j + k & \text{if } 1 \leq \lambda_j \leq m - 1, \\ b_j + k - 1 & \text{if } \lambda_j = m \end{cases}
\] for \( \ell = 0 \). (6.29)

Therefore, there are two cases to consider separately: either \([\ell]_m \not\subseteq \{\lambda_1, \ldots, \lambda_s\}\) or \([\ell]_m \subseteq \{\lambda_1, \ldots, \lambda_s\}\).

We define \( \eta(\ell) = \max\{0 \leq j \leq s | \lambda_j \leq \ell\} \) and \( \zeta(\ell) = \max\{0 \leq i \leq r | \sigma_i \leq \ell\} \) with \( \lambda_0 = \sigma_0 = 0 \). Note that \( \eta(\ell) \leq \ell \) with \( \eta(\ell) = \ell \) when \( r \leq s \), and \( \zeta(\ell) \leq \ell \) with \( \zeta(\ell) = \ell \) when \( r \geq s \).

If \([\ell]_m \not\subseteq \{\lambda_1, \ldots, \lambda_s\}\), then, recalling (6.20),
\[
F_{\ell}(n,k) = r+1F_r\left(\begin{array}{c} -n, a^{(\ell)}_1 + k, \ldots, a^{(\ell)}_r + k \\ a_1, \ldots, a_r \end{array} \right)_1 \frac{(-1)^n n!}{\prod_{i=1}^{\eta(\ell)} (a_i)_{k+1} \prod_{i=\zeta(\ell)+1}^r (a_i)_k}. \] (6.30)

Therefore,
\[
v_{\ell} [x^k P_n] = \frac{n! \prod_{i=1}^{\zeta(\ell)} (a_i + 1)_{n-1} \prod_{i=\zeta(\ell)+1}^r (a_i)_n}{\prod_{j=1}^s \left(b^{(n-1)}_j\right)_{\eta(\ell)} \prod_{j=1}^s \left(b^{(\ell)}_j\right)_k} \neq 0. \] (6.31)
If \([\ell]_m = \{\lambda_1, \ldots, \lambda_s\}\), then \([\ell]_m = \lambda_{\eta(\ell)}\). As a result, using (6.21) and defining \(b_0 = b_m - 1\), we have

\[
F_{\ell}(n, k) = r + 2F_{r+1}\left(\begin{array}{c}
-n, a_1^{(\ell)}, \ldots, a_r^{(\ell)}, b_{\eta(\ell)} + k + 1 \\
a_1, \ldots, a_r, b_{\eta(\ell)} + k + 1
\end{array}\right) = \frac{n!}{(b_{\eta(\ell)} + k + 1)_n} \prod_{i=1}^{\zeta(\ell)} (a_i - b_{\eta(\ell)} - k)_k \prod_{i=\zeta(\ell)+1}^{r} (a_i)_k,
\]

which means that

\[
F_{\ell}(n, k) = (-1)^{k+\zeta(\ell)} n! \prod_{i=1}^{\zeta(\ell)} (b_{\eta(\ell)} - a_i)_{k+1} \prod_{i=\zeta(\ell)+1}^{r} (b_{\eta(\ell)} - a_i + 1)_k
\]

As a result,

\[
\nu_{\ell} \left[ x^k P_n \right] = \frac{(-1)^{(m-r)k + \ell - \zeta(\ell)}}{(b_{\eta(\ell)} + k)_n \prod_{j=1}^{\xi(\ell)} (b_{\eta(\ell)} + 1)_{n+k} \prod_{j=\zeta(\ell)+1}^{s} (b_j)_{n+k}} \neq 0.
\]

Hence, we have proved that (6.13) always holds. \(\square\)

### 6.2 Differential properties

In the following result, we write the \(m\)-orthogonal polynomials defined by (6.4) as solutions to an ordinary differential equation, as a consequence of their explicit representation as hypergeometric series.

**Proposition 6.4.** For \(r, s \in \mathbb{N}\) and \(m = \max(r, s) \geq 1\), let \((P_n(x))_{n\in\mathbb{N}}\) be the \(m\)-orthogonal polynomial sequence defined by (6.4). Then, \(P_n(x)\) satisfies the \((m+1)\)-order differential equation

\[
\left[ \prod_{i=1}^{r} \left( x \frac{d}{dx} + a_i \right) \right] \frac{d}{dx} P_n(x) = \left[ \prod_{j=1}^{s} \left( x \frac{d}{dx} + b_j^{(n-1)} \right) \right] \left( x \frac{d}{dx} - n \right) P_n(x),
\]

which can be written in the form

\[
m \prod_{j=1}^{s} b_j^{(n-1)} P_n(x) + \sum_{k=0}^{m} x^k \psi_n^{[k]}(x) \frac{d^{k+1}}{dx^{k+1}} (P_n(x)) = 0,
\]

for some polynomials \(\psi_n^{[k]}\), \(0 \leq k \leq m\), of degree not greater than 1. In particular, the coefficient of the highest-order derivative, \(x^m \psi_n^{[m]}\), is equal to \(x^m\) when \(s \neq r\) and to \(x^m(1-x)\) when \(s = r\).

**Proof.** Applying the differential equation (2.2) to (6.4), we derive (6.35). Expanding both sides of (6.35),

\[
\left[ \prod_{i=1}^{r} \left( x \frac{d}{dx} + a_i \right) \right] \frac{d}{dx} P_n(x) = \sum_{k=0}^{r} \eta^{[k]} x^k \frac{d^{k+1}}{dx^{k+1}} P_n(x)
\]
and

\[
\left[ \prod_{j=1}^{s} \left( x \frac{d}{dx} + b_j^{(n-1)} \right) \right] \left( x \frac{d}{dx} - n \right) P_n(x) = -n \prod_{j=1}^{s} b_j^{(n-1)} P_n(x) + \sum_{k=0}^{s} \xi_n^{[k]} x^{k+1} \frac{d^{k+1}}{dx^{k+1}} P_n(x), \tag{6.38}
\]

with \( \eta^{[r]} = 1 = \frac{r}{\psi_0} \). Therefore, defining \( \psi_n^{[k]} = \eta^{[k]} - \frac{r}{\psi_0} x \) for \( 0 \leq k \leq m \), we obtain (6.36).

Furthermore, applying the differentiation formula (2.3) to the \( m \)-orthogonal polynomials defined by (6.4), we find that the differentiation operator acts on them as a shift on the parameters. Therefore, these polynomials satisfy Hahn’s property, that is, their sequence of derivatives is also \( m \)-orthogonal.

**Proposition 6.5.** For \( r, s \in \mathbb{N} \) and \( m = \max(r, s) \geq 1 \), let \( (P_n(x))_{n \in \mathbb{N}} \) be the \( m \)-orthogonal polynomial sequence defined by (6.4). Then, when \( r \geq s \), we have

\[
\frac{d}{dx} P_n^{[(r,s), (\lambda_1, \ldots, \lambda_s)]} \left( x \middle| a_1, \ldots, a_r \atop b_1, \ldots, b_s \right) = \begin{cases} n P_n^{[(r,s), (\lambda_2-1, \ldots, \lambda_r-1, r), \lambda_1]} \left( x \middle| a_1+1, \ldots, a_r+1 \atop b_2+1, \ldots, b_s+1, b_1+2 \right) & \text{if } \lambda_1 = 1, \\ n P_n^{[(r,s), (\lambda_1-1, \ldots, \lambda_s-1), \lambda_r]} \left( x \middle| a_1+1, \ldots, a_r+1 \atop b_1+1, \ldots, b_s+1 \right) & \text{if } \lambda_1 \geq 2, \end{cases}
\]

and, when \( r \leq s \),

\[
\frac{d}{dx} P_n^{[(r,s)]} \left( x \middle| a_1, \ldots, a_r \atop b_1, \ldots, b_s \right) = n P_n^{[(r,s)]} \left( x \middle| a_1+1, \ldots, a_r+1 \atop b_2+1, \ldots, b_s+1, b_1+2 \right). \tag{6.40}
\]

### 6.3 Recurrence relation

We end this section showing explicit expressions for the coefficients of the recurrence relation satisfied by the \( m \)-orthogonal polynomials defined by (6.4). Recall that the parameters \( a_i^{(k)} \) and \( b_j^{(k)} \) in (6.1) are defined by (5.15)-(5.16) with \( a_{r+1} = 1 \). Therefore, based on Theorem 4.5, we obtain the following result.

**Theorem 6.6.** For \( r, s \in \mathbb{N} \) and \( m = \max(r, s) \geq 1 \), let \( (P_n(x))_{n \in \mathbb{N}} \) be the \( m \)-orthogonal polynomial sequence in \( R = \mathbb{Q} (b_1, \ldots, b_s) [a_1, \ldots, a_r] \) defined by (6.4) and let \( H^{(m)} = (h_{n,k})_{n,k \in \mathbb{N}} \) be the production matrix of the generalised \( m \)-Stieltjes-Rogers polynomials \( S_n^{(m)} (\alpha) \), where \( \alpha = (\alpha_{k+m})_{k \in \mathbb{N}} \) is defined by (5.17)-(5.19) with \( a_{r+1} = 1 \). Then, \( (P_n(x))_{n \in \mathbb{N}} \) satisfies the \( (m+1) \)-order recurrence relation

\[
P_{n+1}(x) = x P_n(x) - \sum_{k=0}^{m} \gamma_{n-k}^{[k]} P_{n-k}(x), \tag{6.41}
\]

where, setting \( \alpha_j = 0 \) for \( 0 \leq j \leq m-1 \),

\[
\gamma_{n}^{[k]} = h_{n+k,n} = \sum_{m_0 \geq \ell_0 \geq \cdots \geq \ell_k \geq 0} \prod_{j=0}^{k} \alpha_{(m+1)(n+j)+\ell_j} \text{ for all } n \in \mathbb{N} \text{ and } 0 \leq k \leq m. \tag{6.42}
\]
A consequence of this theorem is that the production matrix $H^{(m)}$ of the generalised $m$-Stieltjes-Rogers polynomials $S_{n,k}^{(m)}(\alpha)$, where $\alpha = (\alpha_{k+m})_{k \in \mathbb{N}}$ is defined by (5.17)-(5.19) with $a_{r+1} = 1$, is determined by the polynomial sequence $(P_n(x))_{n \in \mathbb{N}}$ defined in Theorem 6.1. Recalling (6.4), $(P_n(x))_{n \in \mathbb{N}}$ is invariant under permutations of $(a_1, \ldots, a_r)$ as well as under different choices of $(\sigma_1, \ldots, \sigma_r)$ when $r < s$. However, $(P_n(x))_{n \in \mathbb{N}}$ is not invariant under permutations of $(b_1, \ldots, b_s)$ or under different choices of $(\lambda_1, \ldots, \lambda_s)$ when $r > s$.

Therefore, as mentioned at the end of Section 5, the production matrix $H^{(m)}$ is invariant under permutations of $(a_1, \ldots, a_r)$ (but not of $(b_1, \ldots, b_s)$) and under different choices of $(\sigma_1, \ldots, \sigma_r)$ (but not of $(\lambda_1, \ldots, \lambda_s)$), in spite of these permutations and choices corresponding to different branched-continued-fraction coefficients. As a result, permutations of $(a_1, \ldots, a_r)$ and different choices of $(\sigma_1, \ldots, \sigma_r)$ lead to different decompositions in bidiagonal matrices of the same production matrix $H^{(m)}$. However, taking $k = (m + 1)n$, with $n \in \mathbb{N}$, in (5.17)-(5.19), we check that $a_{(m+1)n+m}$ is invariant under these permutations and choices, so the LU-factorisation of $H^{(m)}$ given by its different decompositions in bidiagonal matrices is the same (which had to be the case because a matrix LU-factorisation with L unit-lower-triangular is unique).

7 Generalised $m$-Stieltjes-Rogers polynomials

In this section, we present an explicit formula for the generalised $m$-Stieltjes-Rogers polynomials corresponding to the branched continued fractions introduced in Theorem 5.5 with $a_{r+1} = 1$, which are the moments of the dual sequence of the $m$-orthogonal polynomials defined in Theorem 6.1.

Theorem 7.1. For $r, s \in \mathbb{N}$ and $m = \max(r, s) \geq 1$, let $(\alpha_{k+m})_{k \in \mathbb{N}}$ in $R = \mathbb{Q}(b_1, \ldots, b_s)[a_1, \ldots, a_r]$ defined by (5.17)-(5.19) with $a_{r+1} = 1$. The generalised $m$-Stieltjes-Rogers polynomials with weights $(\alpha_{k+m})_{k \in \mathbb{N}}$ are

$$S_{n,k}^{(m)}(\alpha) = \binom{n}{k} \frac{\prod_{i=1}^{r} (a_i + k)_{n-k}}{\prod_{j=1}^{s} (b_j + k)_{n-k}} \text{ with } b_j^{(k)} = b_j + \left\lfloor \frac{k+1 - \lambda_j}{m} \right\rfloor. \tag{7.1}$$

The case $s = 0$ and $m = r = 2$ of this theorem corresponds to [37, Prop. 7.1].

Proof. Let $S = (s_{n,k})_{n,k \in \mathbb{N}}$ and $T = (t_{n,k})_{n,k \in \mathbb{N}}$ be the unit-lower-triangular matrices with entries

$$s_{n,k} = \binom{n}{k} \frac{\prod_{i=1}^{r} (a_i + k)_{n-k}}{\prod_{j=1}^{s} (b_j + k)_{n-k}} \quad \text{and} \quad t_{n,k} = (-1)^{n-k} \binom{n}{k} \frac{\prod_{i=1}^{r} (a_i + k)_{n-k}}{\prod_{j=1}^{s} (b_j^{(n-1)} + k)_{n-k}} \quad \text{when } n \geq k. \tag{7.2}$$

Reverting the order of summation in (6.9), we rewrite the polynomials defined in Theorem 6.1 as

$$P_n(x) = \sum_{k=0}^{n} t_{n,k} x^k \quad \text{for all } n \in \mathbb{N}. \tag{7.3}$$

Hence, $T$ is the coefficient matrix of $(P_n(x))_{n \in \mathbb{N}}$. Moreover, based on Proposition 4.1, $S^{(m)} = \left( S_{n,k}^{(m)}(\alpha) \right)_{n,k \in \mathbb{N}}$ is the moment matrix of the dual sequence of $(P_n(x))_{n \in \mathbb{N}}$. Therefore, $S^{(m)}$ is the inverse matrix of $T$, so
Furthermore, for any row-generating polynomials of the matrix $S^{(m)} = S$ if and only if
\[
(TS)_{n,k} = \sum_{l=k}^{n} t_{n,l} s_{l,k} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases} \tag{7.4}
\]

It is clear that $(TS)_{n,k} = 0$ whenever $n < k$ and $(TS)_{n,n} = t_{n,n} s_{n,n} = 1$ for any $n \in \mathbb{N}$. For $n > k$,
\[
\sum_{l=k}^{n} t_{n,l} s_{l,k} = \sum_{l=k}^{n} (-1)^{n-l} \binom{n}{l} \frac{\prod_{i=1}^{r} (a_i+k)_{l-k}(a_i+l)_{n-l}}{\prod_{j=1}^{s} (b_j^{(n-1)}+k)_{l-k} (b_j^{(n-1)}+l)_{n-l}}. \tag{7.5}
\]

Furthermore, for any $k < l < n$, we have
\[
\binom{n}{l} \binom{l-k}{k} = \binom{n-k}{l-k}, \quad (a_i+k)_{l-k}(a_i+l)_{n-l} = (a_i+k)_{n-k}, \quad \text{and} \quad (b_j^{(n-1)}+l)_{n-l} = \frac{\left(\frac{b_j^{(n-1)}+k}{b_j^{(n-1)}+l}\right)_{n-k}}{l-k}. \tag{7.6}
\]

Applying the formulas above to (7.5) and making the change of variable $\ell = l - k$, we find that
\[
\sum_{l=k}^{n} t_{n,l} s_{l,k} = (-1)^{n-k} \binom{n}{k} \frac{\prod_{i=1}^{r} (a_i+k)_{n-k}}{\prod_{j=1}^{s} (b_j^{(n-1)}+k)_{n-k}} \sum_{\ell=0}^{n-k} (-1)^{\ell} \binom{n-k}{\ell} \prod_{j=1}^{\ell} \left(\frac{b_j^{(n-1)}+k}{b_j^{(n-1)}+\ell}\right), \tag{7.7}
\]
which is equivalent to
\[
\sum_{l=k}^{n} t_{n,l} s_{l,k} = (-1)^{n-k} \binom{n}{k} \frac{\prod_{i=1}^{r} (a_i+k)_{n-k}}{\prod_{j=1}^{s} (b_j^{(n-1)}+k)_{n-k}} s+1\text{F}_s \left(\begin{array}{c} -(n-k), b_1^{(n-1)}+k, \ldots, b_s^{(n-1)}+k \\ b_1^{(k)}+k, \ldots, b_s^{(k)}+k \end{array} \right). \tag{7.8}
\]

When $n > k$, $b_j^{(n-1)} - b_j^{(k)}$ is a nonnegative integer for any $1 \leq j \leq s$
\[
\sum_{j=1}^{s} (b_j^{(n-1)} - b_j^{(k)}) = \sum_{j=1}^{s} \left(\left\lfloor \frac{n-\lambda_j}{m} \right\rfloor - \left\lfloor \frac{k+1-\lambda_j}{m} \right\rfloor \right) \leq \sum_{i=1}^{m} \left(\left\lfloor \frac{n-i}{m} \right\rfloor - \left\lfloor \frac{k+1-i}{m} \right\rfloor \right) = n - k - 1. \tag{7.9}
\]

Therefore, recalling (6.20), we have
\[
s+1\text{F}_s \left(\begin{array}{c} -(n-k), b_1^{(n-1)}+k, \ldots, b_s^{(n-1)}+k \\ b_1^{(k)}+k, \ldots, b_s^{(k)}+k \end{array} \right) = 0 \quad \text{for } n > k. \tag{7.10}
\]
As a result, we conclude that (7.4) and, consequently, (7.1) hold.

The row-generating polynomials of the matrix $S^{(m)} = \left(S^{(m)}_{n,k}(\alpha)\right)_{n,k \in \mathbb{N}}$ determined by (7.1) are
\[
\sum_{k=0}^{n} S^{(m)}_{n,k}(\alpha) x^k = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{i=1}^{r} (a_i+k)_{n-k}}{\prod_{j=1}^{s} (b_j^{(k)}+k)_{n-k}} x^k. \tag{7.11}
\]

38
When \( s = 0 \), \( S^{(m)} \) is its own unsigned inverse, which means that the inverse of \( S^{(m)} \) is \( \left( (-1)^{n-k} S_{n,k}^{(m)}(\alpha) \right)_{n,k \in \mathbb{N}} \), and the row-generating polynomials of \( S^{(m)} \) are

\[
\sum_{k=0}^{n} \binom{n}{k} \prod_{i=1}^{m} (a_i + k)_{n-k} x^k = (-1)^n P_n(-x).
\] (7.12)

That is not the case when \( s \geq 1 \), because then the entries of \( S^{(m)} \) have terms \( b_j^{(k)} \), while the entries of its inverse \( T = (t_{n,k})_{n,k \in \mathbb{N}} \) have terms \( b_j^{(n-1)} \) instead. In that case, it is not clear what are the row-generating polynomials (7.11), but we can determine the column-generating series of \( S^{(m)} \):

\[
\sum_{n=k}^{\infty} S_{n,k}^{(m)}(\alpha) x^{n-k} = \sum_{n=k}^{\infty} \binom{n}{k} \prod_{i=1}^{r} (a_i + k)_{n-k} x^{n-k} = \binom{r}{s} \prod_{j=1}^{s} \left( b_j^{(k)} \right)_{n-k} x^{n-k} = F_{s+1}(a_1 + k, \ldots, a_r + k, 1 + k \mid b_1^{(k)} + k, \ldots, b_s^{(k)} + k \mid x).
\] (7.13)

8 Multiple orthogonal polynomials with respect to Meijer G-functions

In this section, we focus the analysis of the multiple orthogonal polynomials introduced in Section 6 to the cases where the branched-continued-fraction coefficients defined by (5.17) with \( a_{r+1} = 1 \) are all positive. The positivity of the branched-continued-fraction coefficients implies that the zeros of the corresponding multiple orthogonal polynomials are all simple, real, and positive, the zeros of consecutive polynomials interlace, and the recurrence coefficients are all positive. Furthermore, it is a sufficient (but far from necessary) condition for the linear functionals of orthogonality in Theorem 6.1 to be induced by measures on the positive real line whose densities are Meijer G-functions (see Theorem 8.1).

We find the asymptotic behaviour of the recurrence coefficients (Proposition 8.2) and we present a Mehler-Heine-type asymptotic formula near the origin (Proposition 8.3). Then, we use these two results to find an upper bound for the largest zero as well as the asymptotic behaviour of the zeros near the origin (Theorem 8.4). Then, we focus our analysis on two special cases with \( s = r \): a \( r \)-orthogonal polynomial sequence with constant recurrence coefficients and particular instances of the Jacobi-Piñeiro polynomials (see Subsections 8.3 and 8.4, respectively). Finally, we use the connection with the Jacobi-Piñeiro polynomials to find the asymptotic zero distribution of the polynomials under analysis here when \( s = r \) (Theorem 8.6).

8.1 Positivity of branched-continued-fraction coefficients and Meijer G-functions

When \( a_{r+1} = 1 \), the condition (5.32) is trivial and the same is true for the condition (5.31) with \( i = r + 1 \), because it reduces to \( b_j \geq 0 \) for all \( 1 \leq j \leq s \). Therefore, the coefficients in \( (\alpha_k)_{k \in \mathbb{N}} \) defined by (5.17), with \( a_{r+1} = 1 \) and \( a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{R}^+ \), are all positive if and only if

\[
b_j > a_i - \left[ \frac{i - \lambda_j}{r} \right] = \begin{cases} a_i & \text{if } i \leq \lambda_j \\ a_i - 1 & \text{if } i \geq \lambda_j + 1 \end{cases} \quad \text{for all } 1 \leq i \leq r \text{ and } 1 \leq j \leq s.
\] (8.1)
Recalling Remark 4.6, these conditions imply that the sequence of modified \( r \)-Stieltjes-Rogers polynomials \( S^{r,k}_n(a) \) \( n \in \mathbb{N} \), which accordingly to Corollary 5.7 is the moment sequence in (6.1), is a Stieltjes moment sequence for any \( 0 \leq k \leq r \). Furthermore, these conditions imply that \( b_j > a_{\lambda_j} \) and, consequently, \( b_j^{(k)} > a_{\lambda_j}^{(k)} \) for any \( 1 \leq j \leq s \) and \( 0 \leq k \leq r-1 \). Therefore, the moment sequence in (6.1) with \( r \geq s \) and \( a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{R}^+ \) satisfying (8.1) is the entrywise product of \( r \) Stieltjes moment sequences

\[
m_n^{(k)} = \left( \frac{a_1^{(k)}}{b_1^{(k)}} \right)_n \cdots \left( \frac{a_r^{(k)}}{b_r^{(k)}} \right)_n = \prod_{j=1}^s \left( \frac{a_j^{(k)}}{b_j^{(k)}} \right)_n \prod_{i=1}^r \left( \frac{a_i^{(k)}}{b_i^{(k)}} \right)_n \quad \text{with } I = \{1, \ldots, r\} \setminus \{\lambda_1, \ldots, \lambda_s\}. \tag{8.2}
\]

This decomposition gives an alternative proof that the conditions in (8.1) are sufficient for \( (m_n^{(k)})_{n \in \mathbb{N}} \) to be a Stieltjes moment sequence for any \( 0 \leq k \leq r-1 \), because an entrywise product of Stieltjes moment sequences is also a Stieltjes moment sequence. However, these conditions are not necessary; see [22, §2] for more information about sharper sufficient conditions for \( m_n^{(k)} \) to be a Stieltjes moment sequence.

The Meijer G-function \( G_{s,r}^{r,0} \) (see [14, 26] for more details) is defined by the Mellin-Barnes type integral

\[
G_{s,r}^{r,0} \left( \begin{array}{c} b_1, \ldots, b_s \\ a_1, \ldots, a_r \end{array} \bigg| x \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(a_1+u) \cdots \Gamma(a_r+u)}{\Gamma(b_1+u) \cdots \Gamma(b_s+u)} x^{-u} du, \quad c > \min_{1 \leq i \leq r} \{\Re(a_i)\}. \tag{8.3}
\]

When it exists, the Mellin transform of the Meijer G-function \( G(x) \) is equal to the ratio of gamma functions in the integrand on the right-hand side of (8.3). For \( r, s \in \mathbb{N} \) with \( r \geq 1 \) and \( s \geq r \), let \( a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{C} \). Then, based on [5, Eq. 2.24.2.1],

\[
\int_0^\infty G_{s,r}^{r,0} \left( \begin{array}{c} b_1, \ldots, b_s \\ a_1, \ldots, a_r \end{array} \bigg| x \right) x^z dx = \frac{\Gamma(a_1+z) \cdots \Gamma(a_r+z)}{\Gamma(b_1+z) \cdots \Gamma(b_s+z)}, \quad \text{for any } z \in \mathbb{C} \text{ such that } \Re(z) > \min_{1 \leq i \leq r} \{\Re(a_i)\}. \tag{8.4}
\]

In particular, if \( \Re(a_i) > 0 \) for all \( 1 \leq i \leq r \), we have

\[
\frac{\Gamma(b_1) \cdots \Gamma(b_s)}{\Gamma(a_1) \cdots \Gamma(a_r)} \int_0^\infty G_{s,r}^{r,0} \left( \begin{array}{c} b_1, \ldots, b_s \\ a_1, \ldots, a_r \end{array} \bigg| x \right) x^{n-1} dx = \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \quad \text{for all } n \in \mathbb{N}. \tag{8.5}
\]

We assume now that \( s = r \) and \( a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{R}^+ \) satisfy (8.1). In particular, \( b_i > a_i \) for all \( 1 \leq i \leq r \), which implies that \( \sum_{i=1}^r (b_i - a_i) > 0 \). Therefore, based on [21, Lemma 1],

\[
G_{s,r}^{r,0} \left( \begin{array}{c} b_1, \ldots, b_r \\ a_1, \ldots, a_r \end{array} \bigg| x \right) = 0 \quad \text{for all } x > 1. \tag{8.6}
\]

As a result, the integration in (8.5) is over the interval \((0, 1)\) instead of the whole positive real line.

Considering (8.5)-(8.6), the following result is the special case of Theorem 6.1 obtained when the parameters satisfy the conditions for positivity of the corresponding branched-continued-fraction coefficients.

40
Theorem 8.1. For \( r, s \in \mathbb{N} \) such that \( s \leq r \neq 0 \), let \( 1 \leq \lambda_1 < \cdots < \lambda_s \leq r, a_1, \cdots, a_r, b_1, \cdots, b_s \in \mathbb{R}^+ \) satisfying (8.1), and \((\mu_0, \cdots, \mu_{r-1})\) the vector of measures supported on the whole positive real line if \( s < r \) or on the interval \((0, 1)\) if \( s = r \) with densities

\[
d\mu_k(x) = G_{s,r} \left( \frac{b_1^{(k)} \cdots b_s^{(k)}}{a_1^{(k)} \cdots a_r^{(k)}} \right) x \right| \frac{dx}{x} = G_{s,r} \left( \frac{b_1^{(k)} - 1, \cdots, b_s^{(k)} - 1}{a_1^{(k)} - 1, \cdots, a_r^{(k)} - 1} \right) x \right| \frac{dx}{x} \quad \text{for } k \in \{0, \cdots, r-1\}, \quad (8.7)
\]

where

\[
a_i^{(k)} = \begin{cases} a_i + 1 & \text{if } 1 \leq i \leq k \\ a_i & \text{if } k + 1 \leq i \leq r \end{cases} \quad \text{and} \quad b_j^{(k)} = \begin{cases} b_j + 1 & \text{if } 1 \leq \lambda_j \leq k \\ b_j & \text{if } k + 1 \leq \lambda_j \leq r. \end{cases} \quad (8.8)
\]

Then, the \( r \)-orthogonal polynomial sequence \((P_n(x))_{n \in \mathbb{N}}\) with respect to \((\mu_0, \cdots, \mu_{r-1})\) is given by

\[
P_n(x) = \frac{(-1)^n (a_1 \cdots a_r)^n}{b_1^{(n-1)} \cdots b_s^{(n-1)}} s+1 \mathbb{F}_r \left( \begin{array}{ll} -n, b_1^{(n-1)}, \cdots, b_s^{(n-1)} \\ a_1, \cdots, a_r \end{array} \right) x \right| \frac{dx}{x} \quad \text{with } b_j^{(n-1)} = b_j + \left[ \frac{n-\lambda_j}{r} \right]. \quad (8.9)
\]

See [26, Eq. 5.4.4] for the equality of the two densities involving Meijer G-functions in (8.7).

Recall that when \( a_1, \cdots, a_r, b_1, \cdots, b_s \in \mathbb{R}^+ \) satisfy the conditions (8.1), the coefficients \( \alpha = (\alpha_{k+m})_{k \in \mathbb{N}} \) defined by (5.17) with \( a_{r+1} = 1 \) are all positive. Therefore, based on Theorem 4.8, the zeros of the corresponding \( r \)-orthogonal polynomials \((P_n(x))_{n \in \mathbb{N}}\) determined by (8.9) are all simple, real, and positive, and the zeros of consecutive polynomials interlace. Furthermore, when \( s = r \), the orthogonality measures are all supported on the interval \((0, 1)\), so it is natural to conjecture that the zeros of \((P_n(x))_{n \in \mathbb{N}}\) are all located on that interval. This conjecture is clearly true when \( r = 1 \) and \((P_n(x))_{n \in \mathbb{N}}\) are the Jacobi polynomials orthogonal with respect to the positive measure \( \mu \) supported on the interval \((0, 1)\) with density \( d\mu(x) = x^{a-1} (1-x)^{b-a-1} \) for \( a, b \in \mathbb{R}^+ \) such that \( a < b \); it is also true for \( r = 2 \), because the orthogonality measures form a Nikishin system on the interval \((0, 1)\) (see [25, Th. 1]). In fact, we show that this conjecture is true for any positive integer \( r \), as we give the corresponding asymptotic zero distribution in Theorem 8.6.

When \( r = 1 \) and \( s \leq r \) (that is, \( s \in \{0, 1\} \)), the conditions (8.1) for positivity of the branched-continued-fraction coefficients defined by (5.17) with \( a_{r+1} = 1 \) correspond to the necessary and sufficient conditions for positivity of the orthogonality measure of the corresponding Laguerre and Jacobi polynomials, respectively. Moreover, when \( r = 2 \) and \( s \leq r \) (that is, \( s \in \{0, 1, 2\} \)), the conditions in (8.1) correspond to the conditions for the orthogonality measures defined by (8.7)-(8.8) to form a Nikishin system. It would be interesting to find out whether the conditions in (8.1) also imply that the orthogonality measures defined by (8.7)-(8.8) form a Nikishin system when \( r \geq 3 \).

### 8.2 Recurrence coefficients and location of the zeros

Throughout the rest of this section, we use the notation \([k]_r\), with \( k, r \in \mathbb{Z} \) and \( r \geq 1 \), for the unique element of \( \{1, \cdots, r\} \) congruent with \( k \) mod \( r \), as we have done in Section 5. When \( r \geq s \) and \( a_1, \cdots, a_r, b_1, \cdots, b_s \in \mathbb{R}^+ \)}
satisfy (8.1), the coefficients defined by (5.17) have asymptotic behaviour

\[ \alpha_{k+r} \sim \begin{cases} \frac{r^s}{(r+1)^s} k^{-s} & \text{if } [k]_r \notin \{ \lambda_1, \cdots, \lambda_s \}, \\ \frac{r^s}{(r+1)^{s+1}} k^{-s} & \text{if } [k]_r \in \{ \lambda_1, \cdots, \lambda_s \}, \end{cases} \quad \text{as } k \to \infty, \quad (8.10) \]

which implies that

\[ \alpha_{(r+1)(n+j)+\ell} \sim \begin{cases} \left( \frac{r}{r+1} \right)^s n^{-s} & \text{if } [n+j+\ell]_r \notin \{ \lambda_1, \cdots, \lambda_s \}, \\ \frac{r^s}{(r+1)^{s+1}} n^{-s} & \text{if } [n+j+\ell]_r \in \{ \lambda_1, \cdots, \lambda_s \}, \end{cases} \quad \text{as } n \to \infty, \quad (8.11) \]

Therefore, recalling Theorem 6.6, we obtain the following result.

**Proposition 8.2.** For \( r, s \in \mathbb{N} \) such that \( s \leq r \neq 0 \), \( 1 \leq \lambda_1 < \cdots < \lambda_s \leq r \), and \( a_1, \cdots, a_r, b_1, \cdots, b_s \in \mathbb{R}^+ \) satisfying (8.1), let \( (P_n(x))_{n \in \mathbb{N}} \) be the \( r \)-orthogonal polynomial sequence defined by (8.9). Then, \( (P_n(x))_{n \in \mathbb{N}} \) satisfies the recurrence relation

\[ P_{n+1}(x) = x P_n(x) - \sum_{k=0}^{r} \gamma_{n-k}^{[k]} P_{n-k}(x), \quad (8.12) \]

where the recurrence coefficients are given by (6.42), are all positive, and have asymptotic behaviour

\[ \gamma_{n}^{[k]} \sim C_{[n]}^{[k]} n^{(r-s)(k+1)} \quad \text{as } n \to \infty, \quad (8.13) \]

where, for any \( 1 \leq j \leq r \),

\[ C_{j}^{[k]} = \left( \frac{r}{r+1} \right)^{s(k+1)} \sum_{\ell_0, \cdots, \ell_k \geq 0} (r+1)^{-f_j(\ell_0, \cdots, \ell_k)}, \quad (8.14) \]

with

\[ f_j(\ell_0, \cdots, \ell_k) = \left| \{ 0 \leq i \leq k \mid [j+\ell_i]_r \in \{ \lambda_1, \cdots, \lambda_s \} \} \right|. \quad (8.15) \]

When \( s = 0 \), \( \{ \lambda_1, \cdots, \lambda_s \} = \emptyset \) and we have (cf. [23, Lemma 4.3])

\[ \gamma_{n}^{[k]} \sim \left( \frac{r+1}{k+1} \right)^{n^{r(k+1)}} \quad \text{as } n \to \infty, \quad \text{for any } 0 \leq k \leq r. \quad (8.16) \]

When \( s = r \), \( \{ \lambda_1, \cdots, \lambda_s \} = \{ 1, \cdots, r \} \) and, as a result, we find that

\[ \gamma_{n}^{[k]} \to \left( \frac{r+1}{k+1} \right)^{k+1} \left( \frac{r^r}{(r+1)^{r+1}} \right)^{k+1} \quad \text{as } n \to \infty, \quad \text{for any } 0 \leq k \leq r. \quad (8.17) \]

When \( 0 < s < r \), the recurrence coefficients have an asymptotic behaviour of period \( r \). The simplest example of this periodic asymptotic behaviour is the case \( (r, s) = (2, 1) \), for which the recurrence coefficients have an asymptotic behaviour of period 2 given in [24, Th. 3.4].
Combining the periodic asymptotic behaviour of the recurrence coefficients in (8.13) with Theorem 4.9, we obtain an upper bound for the largest zeros of the \( r \)-orthogonal polynomials \( (P_n(x))_{n \in \mathbb{N}} \) defined by (8.9). Moreover, we can relate the asymptotic behaviour of the zeros near the origin with the location of the zeros of the hypergeometric function \( _0F_r(−;a_1, \ldots, a_r|−z) \), which are in infinite number and are all real and positive (see [39, §4]), as a consequence of the following Mehler-Heine-type asymptotic formula.

**Proposition 8.3.** For \( r, s \in \mathbb{N} \) such that \( s \leq r \neq 0 \), \( 1 \leq \lambda_1 < \cdots < \lambda_s \leq r \), and \( a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{R}^+ \) satisfying (8.1), let \( (P_n(x))_{n \in \mathbb{N}} \) be the \( r \)-orthogonal polynomial sequence defined by (8.9). Then,

\[
\lim_{n \to \infty} (−1)^n \frac{b_1^{(n-1)} \cdots b_s^{(n-1)}}{(a_1)_n \cdots (a_r)_n} P_n \left( \frac{z}{n^s+1} \right) = _0F_r \left( \begin{array}{c} -n; b_1^{(n-1)}, \ldots, b_s^{(n-1)} \\ a_1, \ldots, a_r \end{array} \left| \frac{z}{n^s+1} \right. \right). \tag{8.18}
\]

uniformly on compact subsets of \( \mathbb{C} \).

**Proof.** Recalling (8.9),

\[
(−1)^n \frac{b_1^{(n-1)} \cdots b_s^{(n-1)}}{(a_1)_n \cdots (a_r)_n} P_n \left( \frac{z}{n^s+1} \right) = s+1 _0F_r \left( \begin{array}{c} -n, b_1^{(n-1)}, \ldots, b_s^{(n-1)} \\ a_1, \ldots, a_r \end{array} \left| \frac{z}{n^s+1} \right. \right). \tag{8.19}
\]

Furthermore, observe that \( b_j^{(n-1)} \sim \frac{n}{r} \) for each \( 1 \leq j \leq s \). Hence, successively applying the first confluent relation in (2.4) to the formula above, we obtain (8.18).

Our results on the location of the zeros of \( (P_n(x))_{n \in \mathbb{N}} \) are summarised in the following theorem.

**Theorem 8.4.** For \( r, s \in \mathbb{N} \) such that \( s \leq r \neq 0 \), \( 1 \leq \lambda_1 < \cdots < \lambda_s \leq r \), and \( a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{R}^+ \) satisfying (8.1), let \( (P_n(x))_{n \in \mathbb{N}} \) be the \( r \)-orthogonal polynomial sequence defined by (8.9). Then:

- all the zeros of \( P_n(x) \) are simple, real, and positive, and the zeros of consecutive polynomials interlace;
- if we denote the zeros of \( P_n(x) \) and \( _0F_r(−;a_1, \ldots, a_r|−z) \) in increasing order, by \( \left( x_k^{(n)} \right)_{k=1}^\infty \) and \( (f_k)_{k=1}^\infty \), respectively, we have
  \[
  \lim_{n \to \infty} n^{r+1} x_k^{(n)} = r^s f_k \quad \text{for all } k \geq 1; \tag{8.20}
  \]
- there exists a constant \( K(r,s) \in \mathbb{R}^+ \) such that the largest zero of \( P_n(x) \) satisfies
  \[
  x_n^{(n)} < K(r,s) n^{-r} + o\left( n^{-r} \right) \quad \text{as } n \to \infty. \tag{8.21}
  \]

When \( s = r \), (8.21) is equivalent to say that the zeros of \( (P_n(x))_{n \in \mathbb{N}} \) are all located on a bounded interval \( (0, K(r,r)) \). Based on [2, Th. 1.1], this is a corollary of the boundedness of the recurrence coefficients in (8.17). In fact, we show in Theorem 8.6 that this bounded interval is \( (0, 1) \).

When \( s = 0 \), we can find a simple expression for the constant \( K(r,0) \) in (8.21). Based on Theorem 4.9 and recalling the asymptotic behaviour of the recurrence coefficients given in (8.16), we have

\[
 x_n^{(n)} < \min_{t \in \mathbb{R}^+} \left( t + \sum_{k=0}^r \frac{r+1}{(k+1)t^{-k}} \right) n^r + o\left( n^r \right) = \min_{t \in \mathbb{R}^+} \left( \frac{(t+1)^{r+1}}{t^r} \right) n^r + o\left( n^r \right). \tag{8.22}
\]
The minimum appearing in the latter formula is obtained when \( t = r \). Therefore, we find that, when \( s = 0 \), the largest zero of \( P_n(x) \) defined by (8.9) satisfies
\[
x_n^{(s)} < \frac{(r+1)^{r+1}}{r^r} n^r + o(n^r) \quad \text{as } n \to \infty.
\] (8.23)

The asymptotic zero distribution of \( P_n(n' x) \) on the interval \( 0, \frac{(r+1)^{r+1}}{r^r} \) is given in [29, Th. 3.2].

When \( 0 < s < r \), the asymptotic behaviour of the recurrence coefficients is more convoluted and, as a consequence, the constant \( K(r,s) \) in the upper bound for the largest zero becomes more complicated to compute and less sharp. For instance, see [24, Cor. 3.6] for an upper bound for the largest zero of the 2-orthogonal polynomials corresponding to the case \( (r,s) = (2,1) \).

### 8.3 A \( r \)-orthogonal polynomial sequence with constant recurrence coefficients

Here we prove that, for \( s = r \) and a particular choice of parameters \( a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{R}^+ \), the \( r \)-orthogonal polynomials given by (8.9) satisfy a recurrence relation with constant coefficients as follows. This is equivalent to say that each diagonal of the corresponding unit-lower-Hessenberg matrix is constant, i.e. that matrix is Toeplitz. We give explicit formulas for these \( r \)-orthogonal polynomials and for densities and moments of their orthogonality measures.

**Theorem 8.5.** For \( r \in \mathbb{Z}^+ \), let \( (P_n(x))_{n \in \mathbb{N}} \) be the polynomial sequence defined by (8.9) with \( s = r \),
\[
a_i = 1 + \frac{i}{r+1} = \frac{r+1+i}{r+1}, \quad \text{and} \quad b_i = 1 + \frac{i+1}{r} = \frac{r+1+i}{r} \quad \text{for each } 1 \leq i \leq r.
\] (8.24)

Then:

(a) \( (P_n(x))_{n \in \mathbb{N}} \) satisfies the recurrence relation with constant coefficients
\[
P_{n+1}(x) = x P_n(x) - \sum_{k=0}^{\min(n,r)} \binom{r+1}{k+1} \binom{r}{(r+1)^{r+1}}^{k+1} P_{n-k}(x).
\] (8.25)

(b) \( (P_n(x))_{n \in \mathbb{N}} \) can be explicitly written by
\[
P_n(x) = \binom{n+r}{r} \left( \frac{-r^r}{(r+1)^{r+1}} \right)^n r^{r+1} F_r \left( \binom{-n, \frac{r+1}{r}, \ldots, \frac{n+2}{r}}{\frac{r+2}{r+1}, \ldots, \frac{2r+1}{r+1}} ; x \right) \quad \text{for any } n \in \mathbb{N}.
\] (8.26)

(c) \( (P_n(x))_{n \in \mathbb{N}} \) is \( r \)-orthogonal with respect to the vector of measures \( (\mu_0, \ldots, \mu_{r-1}) \) supported on \( (0,1) \) with densities
\[
d\mu_k(x) = G^r_{r,r} \left( \binom{k+j+1}{r+1}_{i=1} \frac{r}{i} \right) x \quad \text{for } 0 \leq k \leq r-1.
\] (8.27)

and moments
\[
\int_0^1 x^k d\mu_k(x) = \binom{r}{(r+1)^{r+1}}^n \frac{1}{n+1} \binom{(r+1)(n+1)+k}{n} \quad \text{for any } 0 \leq k \leq r-1 \text{ and } n \in \mathbb{N}.
\] (8.28)
Observe that:

- the parameters in (8.24) satisfy the conditions in (8.1) for positivity of the branched-continued-fraction coefficients,
- the hypergeometric function in (8.26) is \((-\frac{1}{2})\)-balanced because
  \[
  r \sum_{i=1}^r \left(1 + \frac{i}{r+1}\right) = r + \frac{1}{r+1} \sum_{i=1}^r i = \frac{3r}{2} \quad \text{and} \quad -n + \frac{1}{r+1} \sum_{i=1}^r \left(1 + \frac{n+i}{r}\right) = r + \frac{1}{r} \sum_{i=1}^r i = \frac{3r+1}{2},
  \]
- the moments in (8.28) are related to the Fuss-Catalan numbers.

For \(r = 1\), the polynomials defined in Theorem 8.5 are orthogonal with respect to the measure \(\mu_0\) on \((0, 1)\) with density \(\sqrt{x(1-x)}dx\) and correspond to the Chebyshev polynomials of the second kind \((U_n(x))_{n \in \mathbb{N}}\), up to a linear transformation of the variable (see [20, Eqs. 4.5.15, 4.5.22]):

\[
U_n(x) = (n + 1) \, {}_2F_1\left(\frac{-n, n+2}{3}; \frac{1-x}{2}\right) = (-4)^n \, P_n\left(\frac{1-x}{2}\right).
\]

For \(r = 2\), the polynomials defined in Theorem 8.5 reduce to the 2-orthogonal polynomial sequence with constant recurrence coefficients introduced in [25, §4.5]. The densities of the corresponding orthogonality measures \((\mu_0, \mu_1)\) are rational functions given in [25, Eq. 126]. It is natural to ask whether the Meijer G-functions in (8.27) can also be expressed as rational functions for \(r \geq 3\).

**Proof.** Firstly, we prove (a). Recalling Theorem 6.6, \((P_n(x))_{n \in \mathbb{N}}\) satisfies the recurrence relation

\[
P_{n+1}(x) = xP_n(x) - \sum_{k=0}^r \gamma_n^{[k]} P_{n-k}(x),
\]

with coefficients

\[
\gamma_n^{[k]} = \sum_{r \geq \ell_0 > \cdots > \ell_k \geq 0} \prod_{j=0}^k \alpha_{(r+1)(n+j)+\ell_j} \quad \text{for all } n \in \mathbb{N} \text{ and } 0 \leq k \leq r,
\]

where \(\alpha_j = 0\) for \(0 \leq j \leq r - 1\) and, recalling (5.13),

\[
\alpha_k+r = \frac{(b'_k-a'_k) \prod_{i=1}^r a'_{k-i}}{\prod_{i=0}^r b'_{k-i}} \quad \text{for any } k \in \mathbb{N},
\]

with

\[
a'_k = a_{[k],r+1} + \left\lceil \frac{k}{r+1} \right\rceil \quad \text{and} \quad b'_k = b_{[k],r} + \left\lceil \frac{k}{r} \right\rceil.
\]

To prove (a) is equivalent to show that

\[
\gamma_n^{[k]} = \binom{r+1}{k+1} \left(\frac{r^r}{(r+1)^{r+1}}\right)^{k+1} \quad \text{for all } n \in \mathbb{N} \text{ and } 0 \leq k \leq r.
\]

Taking the parameters in (8.24), we have
We can now compute the values of $\alpha_{k+r}$ for $k \in \mathbb{N}$.

Firstly, we take $k = (r+1)n$ with $n \in \mathbb{N}$ to compute $\alpha_{(r+1)n+r}$. Then,

\[
\begin{align*}
\bullet \quad b'_{(r+1)n} - a'_{(r+1)n} &= \frac{(r+1)n + 2r + 1}{r} - (n + 1) = \frac{n + r + 1}{r}, \\
\prod_{i=0}^{r} b'_i &= \prod_{j=0}^{r} b'_{(r+1)(n-1)+j} = \prod_{j=0}^{r} \left( \frac{(r+1)(n+1)+j}{r} \right) = \frac{(r+1)(n+1)}{r+1}, \\
\prod_{i=1}^{r} d'_{i} &= \prod_{j=0}^{r} d'_{(r+1)(n-1)+j} = \prod_{j=0}^{r} \left( \frac{(r+1)(n+1)+j}{r+1} \right) = \frac{(r+1)(n+1)}{r+1}.
\end{align*}
\]

Therefore, recalling (8.33), we have

\[
\alpha_{(r+1)n+r} = \frac{\left( b'_{(r+1)n} - a'_{(r+1)n} \right) \prod_{i=1}^{r} d'_{i}}{\prod_{i=0}^{r} b'_i} = \frac{r'(n+r+1)}{(r+1)^{r+1}(n+1)} \quad \text{for all } n \in \mathbb{N}.
\]

Next, we compute $\alpha_{(r+1)n+j}$, with $n \geq 1$ and $0 \leq j \leq r-1$. We take $k = (r+1)(n-1) + (j+1)$, so that $k+r = (r+1)n+j$. Note that $1 \leq j+1 \leq r$, so $(r+1) \nmid k$. Therefore,

\[
\begin{align*}
\bullet \quad b'_k - a'_k &= \frac{k + 1}{r} - \frac{k}{r+1} = \frac{k + r + 1}{r(r+1)} = \frac{(r+1)n+(j+1)}{r(r+1)}, \\
\prod_{i=0}^{r} (b'_i)^{k-i} &= \prod_{i=0}^{r} \left( k - i + 2r + 1 \right) = \prod_{i=0}^{r} \left( k + r + 1 + i \right) = \frac{(k + r + 1)^{r+1}}{r^{r+1}} = \frac{(r+1)n+(j+1)^{r+1}}{r^{r+1}}, \\
\prod_{i=1}^{r} (d'_i)^{k-i} &= \prod_{i=0}^{r} \left( (r+1)(n+1)+(j-i) \right) = n \prod_{i=0}^{r} \left( (r+1)(n+1)+(j-i) \right) = \frac{n((r+1)n+j+2)}{(r+1)^{r+1}(n+1)}.
\end{align*}
\]

Hence, recalling again (8.33), we find that

\[
\alpha_{(r+1)n+j} = \frac{r' n}{(r+1)^{r+1}(n+1)} \quad \text{for all } n \geq 1 \text{ and } 0 \leq j \leq r-1.
\]

This formula is also valid for $n = 0$, because then it reduces to $\alpha_j = 0$ for all $0 \leq j \leq r-1$.

Now we can compute $\gamma_{n}^{[k]}$ for any $n \in \mathbb{N}$ and $0 \leq k \leq r$ by inputting (8.38) and (8.40) in (8.32). If $\ell_0 = r$,

\[
\prod_{j=0}^{k} \alpha_{(r+1)(n+j)+\ell_j} = \prod_{j=0}^{k} \frac{r'(n+j)}{(r+1)^{r+1}(n+1)} = \left( \frac{r'}{(r+1)^{r+1}} \right)^{k+1} \frac{n+r+1}{n+k+1},
\]

and, if $\ell_0 \leq r-1$,

\[
\prod_{j=0}^{k} \alpha_{(r+1)(n+j)+\ell_j} = \prod_{j=0}^{k} \frac{r'(n+j)}{(r+1)^{r+1}(n+j+1)} = \left( \frac{r'}{(r+1)^{r+1}} \right)^{k+1} \frac{n}{n+k+1}.
\]
In (8.32), there are \( \binom{k}{r} \) summands with \( \ell_0 = r \) and \( \binom{k}{r+1} \) summands with \( \ell_0 \leq r - 1 \), corresponding, respectively, to choosing \( k \) elements of \( \{0, \ldots, r\} \) to be \( \ell_1, \ldots, \ell_r \) and to choosing \( k + 1 \) elements of \( \{0, \ldots, r - 1\} \) to be \( \ell_0, \ldots, \ell_{r-1} \). Therefore, we have

\[
\gamma_{n}^{[k]} = \left( \frac{r^r}{r+1} \right)^{k+1} \left( \frac{r}{k} \right)^{n + r + 1} + \left( \frac{r}{k+1} \right)^{n + k + 1} = \left( \frac{r^r}{r+1} \right)^{k+1} \left( \frac{r+1}{k+1} \right), \tag{8.43}
\]

which means that (8.35) and, consequently, (8.25) hold.

Now we prove (b). Combining (8.9) with \( s = r \) and (5.11) with \( m = r \), we get

\[
P_n(x) = \left( \frac{-1}{a_1 \cdots a_r} \right)^n \frac{(a_1 \cdots a_r)_{n} \cdots (a_r)_{n}}{a_1 \cdots a_r} \cdot \frac{r}{r+1} \sum_{k=0}^{r} \frac{\Gamma(1)}{\Gamma(k+1)} \left( \frac{n}{r+1} \right)_k x^k.
\]

Therefore, recalling (8.24) and (8.36a), we have

\[
P_n(x) = \left( \frac{-1}{a_1 \cdots a_r} \right)^n \frac{(a_1 \cdots a_r)_{n} \cdots (a_r)_{n}}{a_1 \cdots a_r} \cdot \frac{r}{r+1} \sum_{k=0}^{r} \frac{\Gamma(1)}{\Gamma(k+1)} \left( \frac{n}{r+1} \right)_k x^k.
\]

Furthermore,

\[
\prod_{j=1}^{r} \left( \frac{n + r + i}{r} \right)_n = \prod_{j=1}^{r} \frac{n + r + i}{r + 1} = r^{-n} \prod_{k=r+1}^{r(n+1)} (n+k) = \frac{(n + r + 1)_n}{r^n},
\]

and

\[
\prod_{j=1}^{r} \left( \frac{r + i + 1}{r + 1} \right)_n = \prod_{j=1}^{r} \frac{r + i + 1}{r + 1} = r^{-n} \prod_{k=r+2}^{r(n+1)} (r + i + 1)_k = \frac{(r+1)_{n+r}}{r! (r+1)^{n+r}}.
\]

Hence,

\[
\frac{(-1)^n (r+1)_{n} \cdots (r+1)_{n}}{r! (r+1)^{n+r} (n + r + 1)_n} = \frac{n + r}{r} \left( \frac{n + r}{r+1} \right)^{n} = \left( \frac{n + r}{r} \right)^{n} \left( \frac{r}{r+1} \right)^{n}.
\]

and (8.45) implies (8.26).

Finally, we prove (c). Considering (8.7)-(8.8) with the parameters in (8.24), we obtain (8.27) and we find that, for any \( 0 \leq k \leq r - 1 \) and \( n \in \mathbb{N} \),

\[
\int_{0}^{1} x^k \text{d} \mu_k(x) = \prod_{i=1}^{r+1} \frac{r+1+k+i}{r+1} \frac{(n+1)!}{\prod_{i=1}^{r} \frac{r+1+k+i}{r+1}}.
\]

Analogously to (8.46)-(8.47), we have

\[
\prod_{i=1}^{r} \left( \frac{r + k + 2}{r} \right)_n = \left( \frac{r + k + 2}{r} \right)^{n}, \quad \text{and} \quad \prod_{i=1}^{r+1} \left( \frac{r + k + 2}{r + 1} \right)_n = \left( \frac{r + k + 2}{r + 1} \right)^{n}.
\]

Therefore,

\[
\int_{0}^{1} x^k \text{d} \mu_k(x) = \left( \frac{r^r}{(r+1)^{r+1}} \right)^n \frac{(n+1)!}{\prod_{i=1}^{r} \frac{r+1+k+i}{r+1}}.
\]

which is equivalent to (8.28).
8.4 Connection with Jacobi-Piñeiro polynomials and asymptotic zero distribution

Here we show that, for \( s = r \) and parameters \( a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{R}^+ \) satisfying a certain set of relations, the \( r \)-orthogonal polynomials given by (8.9) correspond to Jacobi-Piñeiro polynomials on the step-line.

For \( a_1, \ldots, a_r \in \mathbb{R}^+ \), let \( b_r = a_1 + 1 \) and \( b_i = a_{i+1} \) for \( 1 \leq i \leq r - 1 \). Note that these parameters satisfy the conditions in (8.1) if and only if \( a_1 < \cdots < a_r < a_1 + 1 \). Then, for any \( 1 \leq j \leq r - 1 \):

- \( b_i^{(j)} = b_i + 1 = a_i + 1 = a_{i+1}^{(j)} \) for \( 1 \leq i \leq j - 1 \), (8.52a)
- \( b_j^{(j)} = b_j + 1 = a_j + 1 = a_{j+1}^{(j)} + 1 \), (8.52b)
- \( b_i^{(j)} = b_i = a_{i+1} = a_{i+1}^{(j)} \) for \( j + 1 \leq i \leq r - 1 \), and (8.52c)
- \( b_r^{(j)} = b_r = a_1 + 1 = a_1^{(j)} \). (8.52d)

Hence, the orthogonality measures in (8.7), with \( s = r \) and the choice of parameters above, reduce to

\[
\mathrm{d}\mu_j(x) = \frac{\Gamma(a_j+1)}{\Gamma(a_{j+1})} \frac{1}{x} G_{1,1}^{1,0} \left( \begin{array}{c} a_j+1+1 \\ a_{j+1} \end{array} \right) \frac{1}{x} \mathrm{d}x = a_{j+1} x^{a_{j+1}-1} \mathrm{d}x \quad \text{for all } 0 \leq j \leq r - 1. \quad (8.53)
\]

The multiple orthogonal polynomials with respect to \((\mu_0, \ldots, \mu_{r-1})\) determined by (8.53) are a particular case of the Jacobi-Piñeiro polynomials originally introduced by Piñeiro in [34]. In fact, the formula (8.9) with \( b_r = a_1 + 1 \) and \( b_i = a_{i+1} \) for \( 1 \leq i \leq r - 1 \) reduces to the explicit formula for the Jacobi-Piñeiro polynomials given in [20, Eq. 23.3.5], with \( \beta = 0 \) and the multi-index \( \vec{n} = (n_1, \ldots, n_r) \) on the step-line. Furthermore, it is clear from [20, Eq. 23.3.5] that the Jacobi-Piñeiro polynomials with \( \beta \neq 0 \) cannot be a particular case of the polynomials given by (8.9).

This connection with the Jacobi-Piñeiro polynomials suggests that our polynomials share the asymptotic behaviour of the recurrence coefficients, the ratio asymptotics, and the asymptotic zero distribution with the Jacobi-Piñeiro polynomials. We show that this is true and, as a result, that we can obtain the asymptotic zero distribution of our polynomials from [30, Th. 1.1].

The asymptotic zero distribution \( \nu \) of \((P_n(x))_{n \in \mathbb{N}}\) is the limit (if it exists) for the normalised zero counting measure of \( P_n(x) \) in the sense of the weak convergence of measures, that is,

\[
\int f \mathrm{d}\nu(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left( x_k^{(n)} \right), \quad \text{where } x_1^{(n)}, \ldots, x_n^{(n)} \text{ are the zeros of } P_n(x), \quad (8.54)
\]

for all bounded and continuous functions \( f \) on \((0, 1)\).

If the zeros of \((P_n(x))_{n \in \mathbb{N}}\) are all real and simple and the zeros of consecutive polynomials interlace and the limit of the ratio of two consecutive polynomials, which we refer to as the ratio asymptotics of \((P_n(x))_{n \in \mathbb{N}}\), exists and converges uniformly on compact subsets of \( \mathbb{C} \setminus (0, 1) \), the Stieltjes transform of the asymptotic zero distribution \( \nu \) is (see [30, §4] for more details)

\[
\int \frac{\mathrm{d}\nu(t)}{x-t} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{x-x_k^{(n)}} = \lim_{n \to \infty} \frac{P'_n(x)}{nP_n(x)} = \frac{F'(z)}{F(z)} \quad \text{with } F(z) = \lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)}. \quad (8.55)
\]
Let \( a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{R}^+ \) satisfying (8.1) with \( s = r \) and \( \lambda_j = j \) for all \( 1 \leq j \leq r \) and \( (P_n(x))_{n \in \mathbb{N}} \) be the \( r \)-orthogonal polynomials defined by (8.9). Then, the asymptotic behaviour of the recurrence coefficients of \( (P_n(x))_{n \in \mathbb{N}} \) is given by (8.17) and it does not depend on the choice of parameters. As a result, the ratio asymptotics and the asymptotic zero distribution do not depend on the parameters either, because the ratio asymptotics is determined by the asymptotic behaviour of the recurrence coefficients (see [2, Lemma 3.2]) and, recalling (8.55), the asymptotic zero distribution is determined by the ratio asymptotics.

On the other hand, the ratio asymptotics and the asymptotic zero distribution of the Jacobi-Piñeiro polynomials on the step-line were obtained in [30, Th. 1.1]. In particular, the results in [30, Th. 1.1] hold for our polynomials \( (P_n(x))_{n \in \mathbb{N}} \) when \( b_r = a_1 + 1 \) and \( b_i = a_i + 1 \) for \( 1 \leq i \leq r - 1 \), because we have seen that these polynomials are a particular case of the Jacobi-Piñeiro polynomials on the step-line. Consequently, those results also hold for any \( a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{R}^+ \) satisfying (8.1). Therefore, we will show that the ratio asymptotics and the asymptotic zero distribution of our polynomials are the same as for the Jacobi-Piñeiro polynomials for multi-indices near the diagonal.

Using [2, Remark 3.1] (with \( F_i(x) = F(x) \) for all \( i \in \mathbb{N} \)) and the asymptotic behaviour (8.17) of the recurrence coefficients, we find that the ratio asymptotics \( F(z) \) defined in (8.55) satisfies the algebraic equation

\[
F(x) = x - \sum_{k=0}^{r} \binom{r+1}{k+1} \left( \frac{r^k}{(r+1)^{k+1}} \right)^{k+1} (F(x))^{-k} \leftrightarrow \left( F(x) + \frac{r^k}{(r+1)^{k+1}} \right)^{r+1} = x(F(x))^r. \tag{8.56}
\]

This algebraic equation is equivalent to [30, Eq. 2.11], with \( F(x) = z - p \) and \( p = \left( \frac{r}{r+1} \right)^r \). Therefore, we can replicate the method in [30, §3-4] to show that the ratio asymptotics and the asymptotic zero distribution of \( (P_n(x))_{n \in \mathbb{N}} \) are the same as for the Jacobi-Piñeiro polynomials for multi-indices near the diagonal. The asymptotic zero distribution is presented in the following result.

**Theorem 8.6.** (cf. [30, Th. 1.1]) For \( r \in \mathbb{Z}^+ \), let \( (P_n(x))_{n \in \mathbb{N}} \) be the \( r \)-orthogonal polynomial sequence defined by (8.9) with \( s = r \) and \( a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{R}^+ \) satisfying (8.1) and denote by \( x_1^{(n)}, \ldots, x_n^{(n)} \) the zeros of \( P_n(x) \) for \( n \geq 1 \). Then, the asymptotic zero distribution \( \nu_r \) of \( P_n(x) \),

\[
\int_0^1 f(t) \nu_r(t) \, dt = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(x_k^{(n)}\right), \tag{8.57}
\]

is supported on \([0, 1]\) with density

\[
\nu_r(x) = \frac{(r + 1) \sin \phi \sin(r \phi) \sin((r + 1) \phi)}{\phi \cos \phi + r^2 \sin^2((r + 1) \phi)}, \tag{8.58}
\]

after the change of variables

\[
x = \frac{r \sin \phi \sin((r + 1) \phi)}{(r + 1) \sin \phi \sin(r \phi)} \quad \text{with} \quad \phi \in \left(0, \frac{\pi}{r + 1}\right). \tag{8.59}
\]

This asymptotic zero distribution is related to the Fuss-Catalan numbers (see [30, §3] for more details). The asymptotic zero distribution of \( (P_n(x))_{n \in \mathbb{N}} \) when \( s = 0 \) presented in [29, Th. 3.2] is also related to the Fuss-Catalan numbers. It would be of interest to find the asymptotic zero distribution of \( (P_n(x))_{n \in \mathbb{N}} \) when \( 0 < s < r \) and check if it is also related to the Fuss-Catalan numbers.
For $r = 1$, Theorem 8.6 recovers the arcsin density of the asymptotic zero distribution of the Jacobi polynomials on the interval $(0, 1)$:

$$ v_1(x) = \frac{1}{\pi \sqrt{x(1-x)}} \quad \text{for } x \in (0, 1). \quad (8.60) $$

For $r = 2$, Theorem 8.6 reduces to [25, Th. 10 (b)] and the density of the corresponding asymptotic zero distribution was found in [8, Th. 2.1]:

$$ v_2(x) = \frac{\sqrt{3}}{4\pi} \left( \frac{1 + \sqrt{1-x}}{x^{\frac{1}{2}}} + \frac{1 - \sqrt{1-x}}{x^{\frac{1}{2}}} \right) \quad \text{for } x \in (0, 1). \quad (8.61) $$

Reciprocally, if the ratio asymptotics $F(z)$ defined in (8.55) exists and converges uniformly for a $r$-orthogonal polynomial sequence $(P_n(x))_{n \in \mathbb{N}}$, then the asymptotic behaviour of the recurrence coefficients is uniquely determined by this limit (see the proof of [2, Lemma 3.1] for an algorithm to compute the asymptotic behaviour of the recurrence coefficients from the ratio asymptotics). Therefore, the asymptotic behaviour of the recurrence coefficients for the Jacobi-Piñeiro polynomials on the step-line is the same as for the $r$-orthogonal polynomial sequence $(P_n(x))_{n \in \mathbb{N}}$ in Theorem 8.6 and it is given by (8.17).

Final remarks

The investigation presented in this paper provides an excellent example on how the study of the connection between multiple orthogonal polynomials and branched continued fractions leads to considerable advances on both topics. We will produce analogous investigations for other examples of this connection in future work, giving rise to new multiple orthogonal polynomials and branched continued fractions.

The focus in this paper was only on the type II multiple orthogonal polynomials. In forthcoming work, we will investigate the corresponding type I multiple orthogonal polynomials and study the connection between type I multiple orthogonal polynomials and branched continued fractions in a more general setting.

We are also interested in exploring the applications of the multiple orthogonal polynomials studied here to other fields of Mathematics. Particular cases of these polynomials have direct applications in the analysis of singular values of products of random matrices (see [23]), are known to be random walk polynomials (see [4]), and are connected to the study of rational solutions of Painlevé equations (see [7]). We expect that these applications can be extended for the general class of multiple orthogonal polynomials investigated in this paper and we intend to explore those extensions in future work.

Acknowledgements: I am very grateful to Alan Sokal for many enlightening discussions about all the topics of the investigation presented here, especially about the topics concerning lattice paths, branched continued fractions, and total positivity, for numerous pertinent suggestions that considerably improved this paper, and for kindly sharing draft versions of [37] and [11], to Ana Loureiro for illuminating discussions on several results presented here, particularly related to multiple orthogonal polynomials, and to LMS and EPSRC for their support of this work through grants ECF-1920-18 and EP/W522454/1, respectively.
References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia Math. Appl.* Cambridge University Press, 1999.

[2] A.I. Aptekarev, V.A. Kalyagin, G. López Lagomasino, and I.A. Rocha. On the limit behavior of recurrence coefficients for multiple orthogonal polynomials. *J. Approx. Theory*, 13:779–811, 2011.

[3] Y. Ben Cheikh and K. Douak. On two-orthogonal polynomials related to the Bateman’s $J_{\nu}^m$-function. *Methods Appl. Anal.*, 7(4):641–662, 2000.

[4] A. Branquinho, A. Foulquié-Moreno, M. Mañas, and J. E. Fernández-Díaz. Hypergeometric multiple orthogonal polynomials and random walks. Preprint available at arXiv:2107.00770, 2021.

[5] Yu. A. Brychkov, O. I. Marichev, and A. P. Prudnikov. *Integrals and Series Vol. 3. More Special Functions*. Gordon and Breach Science Publishers, New York, 1990.

[6] Theodore S Chihara. *An Introduction to Orthogonal Polynomials*, volume 13 of *Mathematics and its Applications*. Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[7] P. A. Clarkson and E. L. Mansfield. The second Painlevé equation, its hierarchy and associated special polynomials. *Nonlinearity*, 16:R1–R26, 2003.

[8] E. Coussement, J. Coussement, and W. Van Assche. Asymptotic zero distribution for a class of multiple orthogonal polynomials. *Trans. Amer. Math. Soc.*, 360(10):5571–5588, 2008.

[9] E. Coussement and W. Van Assche. Some properties of multiple orthogonal polynomials associated with Macdonald functions. *J. Comput. Appl. Math.*, 133:253–261, 2001.

[10] J. Coussement and W. Van Assche. Gaussian quadrature for multiple orthogonal polynomials. *J. Comput. Appl. Math.*, 178:131–145, 2005.

[11] B. Deb, M. Pétréolle, A. Sokal, and B. Zhu. Lattice paths and branched continued fractions III: Generalizations of the Laguerre, rook and Lah polynomials. In preparation, 2022.

[12] E. Deutsch, L. Ferrari, and S. Rinaldi. Production matrices. *Adv. Appl. Math.*, 34(1):101–122, 2005.

[13] E. Deutsch, L. Ferrari, and S. Rinaldi. Production matrices and Riordan arrays. *Ann. Comb.*, 13(1):65–85, 2009.

[14] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.17 of 2017-12-22. F.W.J. Olver, A.B.Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller and B.V. Saunders, eds.

[15] S. M. Fallat and C. R. Johnson. *Totally Nonnegative Matrices*, volume 35 of *Princeton Ser. Appl. Math*. Princeton University Press, 2011.

[16] M.C.F. Fasenmyer. Some generalized hypergeometric polynomials. *Bull. Amer. Math. Soc.*, 53(8):806–812, 1947.
[17] P. Flajolet. Combinatorial aspects of continued fractions. *Discrete Math.*, 32(2):125–161, 1980.

[18] F.R. Gantmacher and M.G. Krein. *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems (Revised English Edition).* AMS Chelsea Publishing, Providence, R.I., 2002.

[19] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, 1985.

[20] M.E.H. Ismail. *Classical and Quantum Orthogonal Polynomials in One Variable.* Cambridge University Press, 2005.

[21] D.B. Karp and E.G. Prilepkina. Hypergeometric functions as generalized Stieltjes transforms. *J. Math. Anal. Appl.*, 393(2):348–359, 2012.

[22] D.B. Karp and E.G. Prilepkina. Completely monotonic gamma ratio and infinitely divisible H-function of Fox. *Comput. Methods Funct. Theory*, 16:135–153, 2016.

[23] A.B.J. Kuijlaars and L. Zhang. Singular values of products of Ginibre random matrices, multiple orthogonal polynomials and hard edge scaling limits. *Comm. Math. Phys.*, 2(332):759–781, 2014.

[24] H. Lima and A. Loureiro. Multiple orthogonal polynomials associated with confluent hypergeometric functions. *J. Approx. Theory*, 92, 2020.

[25] H. Lima and A. Loureiro. Multiple orthogonal polynomials with respect to Gauss’ hypergeometric function. *Stud. Appl. Math.*, 148(1):154–185, 2022.

[26] Y. L. Luke. *The special functions and their approximations, Vol.I*, volume 53 of *Math. Sci. Eng.* Academic Press, New York-London, 1969.

[27] P. Maroni. L’orthogonalité et les récurrences de polynômes d’ordre supérieur à deux. *Ann. Fac. Sci. Toulouse*, 10(1):105–139, 1989.

[28] B.M. Minton. Generalized hypergeometric functions at unit argument. *J. Math. Phys.*, 12:1375–1376, 1970.

[29] T. Neuschel. Plancherel-Rotach formulae for average characteristic polynomials of products of Ginibre random matrices and the Fuss-Catalan distribution. *Random Matrices Theory Appl.*, 3(1), 2014.

[30] T. Neuschel and W. Van Assche. Asymptotic zero distribution of Jacobi-Piñeiro and multiple Laguerre polynomials. *J. Approx. Theory*, 205:114–132, 2016.

[31] E.M. Nikishin and V.N. Sorokin. *Rational Approximations and Orthogonality*, volume 92 of *Transl. Math. Monogr.* Amer. Math. Soc., Providence, R.I., 1991.

[32] M. Pétrotolle and A. Sokal. Lattice paths and branched continued fractions II: Multivariate Lah polynomials and Lah symmetric functions. *Eur. J. Combinatorics*, 92(103235), 2021.

[33] M. Pétrotolle, A. Sokal, and B. Zhu. Lattice paths and branched continued fractions: An infinite sequence of generalizations of the Stieltjes-Rogers and Thron-Rogers polynomials, with coefficient-wise Hankel-total positivity. *To appear in Mem. Amer. Math. Soc.*, 2021. Preprint available at arXiv:1807.03271.
[34] L.R. Piñeiro. On simultaneous approximations for a collection of Markov functions. *Vestnik Mosk. Univ., Ser. I*, 2:67–70, 1987.

[35] A. Pinkus. *Totally Positive Matrices*, volume 181 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 2010.

[36] A. Sokal. A simple algorithm for expanding a power series as a continued fraction. Preprint available at arXiv:2206.15434, 2022.

[37] A. Sokal. Multiple orthogonal polynomials, $d$-orthogonal polynomials, production matrices and branched continued fractions. Preprint available at arXiv:2204.11528, 2022.

[38] G. Szegö. *Orthogonal Polynomials*, volume 23 of *Amer. Math. Soc. Colloq. Publ., Vol. XXIII*. Amer. Math. Soc., Providence, R.I., Fourth edition, 1975.

[39] W. Van Assche. Mehler-Heine asymptotics for multiple orthogonal polynomials. *Proc. Amer. Math. Soc.*, 145:303–314, 2017.

[40] W. Van Assche. Orthogonal and multiple orthogonal polynomials, random matrices, and Painlevé equations. In *AIMS-Volkswagen Stiftung Workshops*, pages 629–683. Springer, 2018.

[41] W. Van Assche and S. Yakubovich. Multiple orthogonal polynomials associated with Macdonald functions. *Integral Transforms Spec. Funct.*, 9(3):229–244, 2000.

[42] J. Zeng. Combinatorics of orthogonal polynomials and their moments. In H.S. Cohl and M.E.H. Ismail, editors, *Lectures on Orthogonal Polynomials and Special Functions*, volume 464, pages 280–334. Cambridge University Press, 2021.