Abstract

Novel method of calculating Nuclear Statistical Equilibrium is presented. Basic equations are carefully solved using arbitrary precision arithmetic. Special interpolation procedure is then used to retrieve all abundances using tabulated results for neutrons and protons, together with basic nuclear data. Proton and neutron abundance tables, basic nuclear data and partition functions for nuclides used in calculations are provided. Simple interpolation algorithm using pre-calculated p and n abundances tabulated as a functions of kT, ρ and Ye is outlined. Unique properties of this method are: (1) ability to pick-up out of NSE selected nuclei only (2) computational time scaling linearly with number of re-calculated abundances (3) relatively small amount of stored data: only two large tables (4) slightly faster than solving NSE equations using traditional Newton-Raphson methods for small networks (few tens of species); superior for huge (800-3000) networks (5) do not require initial guess; works well on random input (6) can tailored to specific application (7) ability to use third-party NSE solvers to obtain fully compatible tables (8) encapsulation of the NSE code for bug-free calculations.

Range of applications for this approach is possible: coverage test of traditional NSE Newton-Raphson codes, generating starting values, code-to-code verification and possible replacement of the old legacy procedures in supernova simulations.

Contents

1 Brief introduction ........................................... 2
2 NSE ....................................................... 3
   2.1 Basic equations .............................................. 3
   2.2 Limitations of the Newton-Raphson NSE solvers .......... 5
1. Brief introduction

Main goal of the article is to provide new method of computing the Nuclear Statistical Equilibrium (NSE) abundances of the nuclear species. Vast range of conditions can be analyzed: \( Y_e = 0.0 \ldots 1.0 \), \( \rho = 10^2 \ldots 10^{13} \text{ g/cm}^3 \) and \( T = 2 \times 10^9 \ldots 10^{11} \text{ K} \) covering almost any astrophysical situation imaginable. While this article do not concentrate on particular target object, these results are useful for study of pre-supernova stars after Si burning [11], thermonuclear supernovae [12], core-collapse [3], and protoneutron stars [1].

We calculate NSE abundances using reliable arbitrary precision arithmetic approach. Tables of pre-calculated proton \((X_p)\) and neutron \((X_n)\) abundances as functions of the thermodynamic conditions defined by the triad: \( \rho, T, Y_e \) are
stored. Recovering of the remaining several hundred abundances from these two tables is non-trivial task. Detailed description of the working procedure used to calculate NSE abundances is provided. Algorithm is fast thanks to use of pre-tabulated $X_{p,n}$. It has unique ability to pick up out of NSE ensemble only species of interest and other features. Computational time scales linearly with number of required nuclides.

2. NSE

2.1. Basic equations

Well-known equations for the ensemble of $N_{iso} + 1$ nuclei in thermal equilibrium [2, 14] are:

$$\sum_{k=0}^{N_{iso}} X_k = 1$$ (1a)

$$\frac{1}{A_k} \sum_{k=0}^{N_{iso}} Z_k X_k = Y_e$$ (1b)

where abundance $X_k$ for $k$-th nuclei with atomic number $Z_k$ and mass number $A_k$ is:

$$X_k = \frac{1}{2} G_k(T) \left( \frac{1}{2} \rho N_A \lambda^3 \right)^{A_k-1} A_k^{5/2} X_n^{A_k-Z_k} X_p^{Z_k} e^{Q_k/kT}.$$ (2)

Temperature dependent partition function for $k$-th nuclei is given by:

$$G_k(T) = \sum_{i=0}^{i_{max}} (2J_{ik} + 1)e^{-\frac{E_{ik}}{kT}}$$ (3)

where summation is over all known excited states (numbered by the index $i$) of the $k$-th nucleus; $J_{ik}$ and $E_{ik}$ are the spin and the excitation energy, respectively. $Q_k$ is the binding energy, $\rho, T$ - density and temperature of the plasma, $N_A$ is the Avogadro number and $k$ - Boltzmann constant. Thermal de’Broglie wavelength used in eq. (2) is:

$$\lambda = \frac{h}{\sqrt{2\pi m_H kT}}$$ (4)

where $m_H$ is the mass of the hydrogen atom and $h$ denotes Planck’s constant.

Partition function has been calculated directly from nuclear database using [3]. Missing spins were assumed to be equal zero. For uncertain lowest possible was used. Results are in good agreement with data used in [4, 5], cf. Fig. 2.
Figure 1: Nuclides included in NSE calculations. Much larger (800 nuclear species) networks were also tested.
2.2. Limitations of the Newton-Raphson NSE solvers

NSE equations, from mathematical point of view, form the system of two large high order polynomial equations (polynomial system) for unknown proton $X_p$ and neutron $X_n$ abundances. System is solved numerically using two-dimensional Newton-Raphson technique. Due to large integer powers and other factors this approach is prone to numerous convergence problems. While in "normal" situation (typical thermodynamic parameters, good initial guess, standard selection of species) convergence of codes using machine floating point arithmetic is amazingly fast, failures are inevitable. Limited numerical precision might be problematic issue. This forces programmers to include multi-level fail-safe procedures. They are by many orders of magnitude slower, and not guaranteed to converge. Careful programming with proper handling of round-off errors is required to get correct results, leading to additional complications. Due to problems with numerical precision and unpredictable iteration numbers, rapidly growing with number of species and for low temperatures, procedures are long, complicated and hard to parallelize.

Moreover, even if we are interested in abundance of single nucleus entire

\footnote{In principle polynomial system might be reduced using Groebner basis methods, especially over rational field. In practice ensemble including protons, neutrons, $^4$He and single heavy nuclei can be solved, but additional components cause Groebner basis algorithms to fail in sense of computational time: no result is returned in a period of several hours.}
system has to be solved. Such a situation is typical for neutrino spectrum calculations, as usually much more nuclear species are included in NSE than those with known neutrino emission rates. Usually very few of them contribute at non-negligible level, e.g. $p$, $^{56}\text{Ni}$ and $^{55}\text{Co}$ for $\nu_e$ emission at $Y_e = 0.5$. Large part of $kT-\rho-Y_e$ space is completely dominated by processes involving neutrons and protons only. In the course of the research we have faced this problem. In recent article NSE ensemble included 800 nuclides while FFN tables used include only 189 of them. Interpolation of the pre-calculated results has been found to be optimal solution. Similar problem arise in core-collapse supernova simulations. Depending on temperature, NSE or full reaction network is solved. Again, NSE can be computed for larger ensemble, but due to limitations of network ODE solvers only fraction of species is traced.

2.3. Interpolation algorithm

To handle results of the NSE calculations efficiently, interpolation seems to be wrong solution. Naively, one might try to interpolate stored proton $X_p$ and neutron $X_n$ abundances obtained from Eqns. (1), and get $X_k$ from (2). Unfortunately, this does not work. Even a very small inaccuracy in $X_n$ or $X_p$ produces enormous error in $X_k$ due to large ($\sim A$) integer powers in (2). Another ,,brute force" method is tabulation of every $X_k$. This might be useful if a few out of NSE species are of interest. This is also the fastest approach. However, for larger number of species amount of stored data becomes very large: tens or hundreds of tables like Table 7 instead of two. Fortunately, we found a compromise, which successfully combines both ideas. Inability to get accurate abundances using interpolated $X_n, X_p$ does not include grid points, as they can be stored with accuracy equal to the machine precision, or even better if required. First, we calculate abundance of selected species $X_k$ at grid points neighboring given ($\rho, T, Y_e$) point. Next, we interpolate using computed $X_k$’s. Only proton $X_p$ and neutron $X_n$ abundances need to be tabulated, but more (using formula (2) at 8 corners of a cube) computational time is required compared to interpolation of stored $X_k$ values for all nuclei. Additionally, partition function $G_k(T)$, atomic and mass numbers $Z_k, A_k$ and binding energy $Q_k$ has to be stored for all nuclei to use (2). Using (tri)linear interpolation eqns. (1) are fulfilled automatically up to original solving accuracy.

We still have to solve (1) to generate $X_p$ and $X_n$ tables. Any method e.g. existing codes [10], pre-calculated results or a web service [8] may be used in this purpose. Because efficiency and speed of the code is not of primary importance if one use interpolating scheme, Eqns. (1a, 1b) has been solved numerically using MATHEMATICA code. Entero code [9] has approx. 100 lines including database loading, writing C headers, and solving (1) with arbitrary precision. Code is slow compared to FORTRAN equivalents, a price paid for arbitrary precision. This is not an important issue, as all we want is to generate

\[ \delta(X_{N}^{X}X_{p}^{Z}) \sim A2^AX, \]

where $\delta X$ is typical relative error of $X_n, X_p$ and $A$ is mass number. For $A \sim 60$ amplification of relative error might be as large as $10^{18}$ (!) for $X_n \sim X_p \sim 0.5$. Entire code has approx. 100 lines including database loading, writing C headers, and solving (1) with arbitrary precision. Code is slow compared to FORTRAN equivalents, a price paid for arbitrary precision. This is not an important issue, as all we want is to generate
Table 1: Minimum number of nuclides required to compute all abundances above $X_{\text{min}}$.

| $X_{\text{min}}$ | Z | A | Last included nuclide |
|------------------|---|---|-----------------------|
| $10^{-1}$        | 28| 56| 56Ni                  |
| $10^{-2}$        | 28| 57| 57Ni                  |
| $10^{-4}$        | 29| 59| 59Cu                  |
| $10^{-5}$        | 30| 60| 60Zn                  |
| $10^{-6}$        | 30| 61| 61Zn                  |
| $10^{-7}$        | 30| 63| 63Zn                  |
| $10^{-8}$        | 31| 63| 63Ga                  |
| $10^{-9}$        | 31| 65| 65Ga                  |
| $10^{-10}$       | 32| 66| 66Ge                  |
| $10^{-12}$       | 32| 68| 68Ge                  |
| $10^{-20}$       | 36| 75| 75Kr                  |
| $10^{-30}$       | 41| 87| 87Nb                  |

used, including excited states and spins. This let us to calculate temperature dependent partition function. Measured excited states were used if present in database, otherwise neglected. Third party partition functions can be used as well. No Coulomb and screening corrections were applied.

Proton and neutron abundances are then tabulated as a functions of temperature, density and electron fraction. NSE results are checked against available codes/results [16, 8, 13] with good agreement.

2.4. Brief discussion of NSE results

Determination of NSE abundances is crucial for many applications, including nucleosynthesis, neutrino emission, nuclear energy generation and equation of state. Therefore we have made some tests to verify results and accuracy estimates. Despite known physical issues (temperature-dependent partition function, Coulomb corrections [13], screening [7]) one of the most important factors is number and selection of species included in equations (1). Even a single one important nuclei missing in NSE ensemble may lead to radically different results. While inclusion of some species seems obvious (p, n, $^4$He, $^{56}$Ni, iron group) further selection is more or less arbitrary.

To quantify problem I tried to answer the following question: what is the maximum required atomic (Z) and mass number (A) to get solution including all species with abundance larger than prescribed $X_{\text{min}}$. Results are presented in Table 1 and Fig. 3. For example, from Table 1 if we do not want to miss any of species with abundance above e.g. $10^{-6}$, we need at least nuclides up to $^{61}$Zn. Nuclei in Fig. 3 are ordered according to [6]: approximate Z and A are included as a tick marks for a top axis. This estimate gives an upper limit for number of required nuclei. To get true minimal number of nuclides required to

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footnotes:
1. We do it once, in parallel if required. Later we use interpolators, which are very fast, even compared to codes using hardware floats.
get all species above assumed accuracy one have to consider all subsets for entire considered \( kT - \rho - Y_e \) space. Number of subsets, given by the Bell number \( B_{N,\text{iso}} \) is very large. Therefore, rigorous selection of species is impossible for large sets, and the safest thing to do is to use estimates given by Table 1 or consider all nuclei available [8]. In practice however, other factors decide, e.g. limited computational resources in supernova simulations.

From Fig. 3 we can conclude that the most primitive NSE including p and n only is not useful, maybe except for very high temperatures, cf. Fig. 4. Inclusion of the alpha particle extends applications to lower temperatures but usually p and n abundances are wrong by few orders of magnitude. To get correct abundances of p and n for lower temperatures entire iron peak has to be included. \( X_p \) and \( X_n \) are rock-stable if all nuclei below \( Z=28, A=56 \) are included. This number might be seriously reduced if we focus on narrow \( Y_e \) range and exclude low mass (\( A=3..16 \)) elements. Anyway, results in Table 1 indicate, that no more than 1000 nuclides [8], it does not change results significantly.
2.5. NSE viewgraphs

We discuss some properties of the NSE state for completeness. For a very high temperatures above $kT \simeq 0.5$ MeV in Fig. 4 ($T_9 \simeq 5.8$) no bound nuclei exist and we have a mixture of free neutrons and protons (Fig. 4). In this case solution of system (1) is:

$$X_n = 1 - Y_e, \quad X_p = Y_e.$$ 

If temperature decrease helium is being "synthesized" like in Big Bang nucleosynthesis. If temperature drops further more below $kT \simeq 0.35$ MeV ($T \simeq 4 \times 10^9$ K) and thermodynamic conditions are maintained for long enough, heavy most bound nuclei are preferred. Finally, cold catalyzed matter state is a pure (for $Y_e = 0.45$) $^{58}$Fe, cf. Fig. 4 for $Y_e = 0.5$ it is $^{56}$Ni. This is appealing physical picture. Note extremely strong $Y_e$ dependence of the NSE state (Fig. 5) for $0.35 < Y_e < 0.55$. $Y_e$ dependence for large temperatures is trivial: smooth balance between p, n and $\alpha$ abundances. The most interesting is the temperature range where heavy nuclei dominate. Note that, for higher densities, temperature threshold for heavy nuclei formation moves to higher temperatures, see eq. (2) and footnote 4.

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4 Actually, if we forget temperature dependent partition function, according to eq. (2) solution depends on factor proportional to $\rho^2/kT^3$. 

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Figure 5: NSE abundance versus electron fraction $Y_e$.

The striking feature of Fig. 5 is a rapid variation of the abundances within range of $Y_e = 0.35 \cdots 0.55$, cf. Fig. 6. NSE prefer nucleus with individual $Y_e^{(k)} = Z_k / A_k$ as close as possible to $Y_e$ for entire thermodynamic ensemble. For example, double magic nuclei $^{78}$Ni with largest known neutron excess $^5$ (lowest $Y_e = 28/78 \simeq 0.36$) dominates for $Y_e < 0.365$ (not included in example network) until neutrons (with $Y_e = 0$) take a lead. For opposite side, $Y_e \gg 0.5$, protons are dominant $^6$.

Note, that for $Y_e = 0$ exact solution of NSE equations is $X_n = 1$; for $Y_e = 1$ we get $X_p = 1$. Abundances for intermediate values of $Y_e$ continuously approach these values for $Y_e \to 0$, and $Y_e \to 1$, cf. Fig. 5. Other abundances rapidly approach zero.

Rapid abundance variation has strong imprint on neutrino emission. For example, known for large electron capture rate $^{55}$Co has non-negligible abundance only in narrow range of $Y_e = 0.47 \cdots 0.5$.

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$^5$Neutron excess is equivalent to $Y_e$: $\eta = 1 - 2Y_e$.

$^6$Normally, for $Y_e \gg 0.5$ protons dominate. But if $^3$Li would exist, it should take the role of hydrogen under NSE conditions if density is high enough. This species is still present in nuclide databases with atomic mass 3.030775 and binding energy 2.2676 MeV, despite experimental detection $^{17}$ has never been confirmed $^{15}$; see also comments in ESNDF data at http://ie.lbl.gov/ensdf/.
3. Proton and neutron NSE abundance tables: explanation and examples of use

Dataset described and presented in the article is meant to be simple example of the methods used. It is tailored to test against 32 isotope NSE solver used by Garching group, based on serial code of Hix & Thielemann [4, 5]. In real application user should use larger tables for bigger networks available online or generate (request from author) user-defined dataset tailored to specific application.

Here we provide tables of the proton, $X_p$ and neutron, $X_n$ abundances, together with nuclear data required to calculate all remaining abundances, $X_k$. Additionally, list of nuclides is required, including:

1. atomic and mass numbers
2. masses and binding energies
3. spins and excited states or, equivalently, tabulated temperature-dependent partition function

In Table 7 we have provided: in the first column temperature $kT$ in MeV, second column include base 10 logarithm of the density in g/cm$^3$, and third column electron fraction $Y_e$, i.e. number of electrons divided by number of baryons. Fourth column include proton abundance under NSE defined by the $kT$, $\rho$ and $Y_e$, and fifth column contain neutron abundance.
### Table 2: Symbols used in nuclide table, main Table 5 begins on page 10

| Col | Symbol | Description                  |
|-----|--------|------------------------------|
| 1   | No     | position                     |
| 2   | Symbol | Standard element symbol      |
| 3   | A      | Mass number                  |
| 4   | N      | Neutron number               |
| 5   | Z      | Atomic number                |
| 6   | Q      | Binding energy per nucleon [MeV] |
| 7   | $J_0$  | ground state spin (0 if not known) |

### Table 3: Symbols used in temperature dependent partition function table, main Table 5 begins on page 17

**NOTE:** without ground state partition function 2$J_0 + 1$, included in Table 5.

Total partition function is a sum of 2$J_0 + 1$ and function tabulated below. Results are truncated below $10^{-6}$.

| Col | Symbol | Description                  |
|-----|--------|------------------------------|
| 1   | No     | position                     |
| 2   | Symbol | Standard element symbol      |
| 3   | kT=0.2 | Partition function for $kT = 0.2$ MeV |
| 4   | kT=0.4 | Partition function for $kT = 0.4$ MeV |
| 5   | kT=0.6 | Partition function for $kT = 0.6$ MeV |
| 6   | kT=0.8 | Partition function for $kT = 0.8$ MeV |
| 7   | kT=1.0 | Partition function for $kT = 1.0$ MeV |

### Table 4: Symbols used in NSE proton and neutron abundance table, main Table 7 begins on page 18

| Col | Symbol | Description                  |
|-----|--------|------------------------------|
| 1   | kT     | Temperature [MeV]            |
| 2   | lg $\rho$ | base 10 logarithm of the density [g/cm$^3$] |
| 3   | $Y_e$  | number of electrons per baryon |
| 4   | $X_p$  | abundance of free protons    |
| 5   | $X_n$  | abundance of free neutrons   |
To calculate all NSE abundance we need basic nuclear data, presented in Table 5 and partition functions, from Table 5.

Using approach presented here, main computational cost is the partition function, so use of tabulated version instead of eq. (3) is important.

Detailed description of the algorithm is presented below. Goal is to calculate abundance $X_k$ of species $k$ for given temperature $T$, density $\rho$ and $Y_e$:

$$X_k = NSE(T, \rho, Y_e, k).$$

1. from tables of the proton(neutron) abundance we pick up points surrounding requested $T, \rho, Y_e$; in case of e.g. trilinear interpolation these points are 8 corners of a cuboid – requested point must be inside or at the edge of the cuboid
2. for all these points we calculate abundance $X_k$ from (2)
3. now we have machine-precision accurate abundances $X_i$ at 8 corners of the cuboid
4. interpolate (trilinear interpolation in the example) to get $X_i$ at desired point

We point out again, that we interpolate $X_k$ NOT $X_p$ or $X_n$. $X_k$ must be calculated exactly at grid points. Example implementation of the algorithm is included in libnse library [10].

3.1. Implementation notes

Article deals with interpolation of the functions of three variables. Despite progress in computer hardware, available memory amount in particular, it is hard to find sophisticated 3D interpolators. Therefore trilinear, or mixed bilinear on $T - \rho$ plane and staircase for $Y_e$, interpolations were used.

A lot of computational time is spend on $X_p^Z X_n^{A-Z}$. Large integer power of the floating-point can be computed nearly optimal using double-exponentiation algorithm, usually included in standard math libraries. Minor improvements for range of interest can be achieved using optimal chain of powers, C++ template programming or other technique devoid of if instruction.

We still recommend caution with integer powers of floating point numbers. For example standard math.h from C does not include integer powers, GNU Gsl only up to 9, while cmath.h from C++ standard library does. This cause large variation of the computational time.

Higher-order interpolation might possibly help to fit procedure into CPU cache memory thanks to reduced amount of data. However we are also in danger of overfitting resulting in catastrophic errors, e.g. negative abundances. If amount of memory is not an issue, linear interpolation is recommended.

Partition function is evaluated using linear interpolation.

4. Additional numerical data

Printed tables and results described in the article are meant to be simple examples of the proposed method. Extended versions of the tables, custom
datasets and numerical library can be downloaded from http://ribes.if.uj.edu.pl/libnse/ or requested from the author.

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5. NSE proton and neutron abundance table

Table 5: Nuclei included in NSE and required nuclear data.

| No | Symbol | Z  | N  | A  | Q          | $J_0$ |
|----|--------|----|----|----|------------|------|
| 1  | $^1\text{n}$ | 0  | 1  | 1  | 0          | 1/2  |
| 2  | $^1\text{H}$  | 1  | 0  | 1  | 0          | 1/2  |
| 3  | $^4\text{He}$ | 2  | 2  | 4  | 7.0739150  | 0    |
| 4  | $^{12}\text{C}$ | 6  | 6  | 12 | 7.6801440  | 0    |
| 5  | $^{16}\text{O}$ | 8  | 8  | 16 | 7.9762060  | 0    |
| 6  | $^{20}\text{Ne}$ | 10 | 10 | 20 | 8.0322400  | 0    |
| 7  | $^{24}\text{Mg}$ | 12 | 12 | 24 | 8.2607090  | 0    |
| 8  | $^{28}\text{Si}$ | 14 | 14 | 28 | 8.4477440  | 0    |
| 9  | $^{32}\text{S}$  | 16 | 16 | 32 | 8.4931340  | 0    |
| 10 | $^{36}\text{Ar}$ | 18 | 18 | 36 | 8.5199900  | 0    |
| 11 | $^{40}\text{Ca}$ | 20 | 20 | 40 | 8.5513010  | 0    |
| 12 | $^{44}\text{Ti}$ | 22 | 22 | 44 | 8.5335180  | 0    |
| 13 | $^{50}\text{Ti}$ | 22 | 28 | 50 | 8.7556180  | 0    |
| 14 | $^{48}\text{Cr}$ | 24 | 24 | 48 | 8.5722100  | 0    |
| 15 | $^{54}\text{Cr}$ | 24 | 30 | 54 | 8.7779140  | 0    |
| 16 | $^{55}\text{Cr}$ | 24 | 31 | 55 | 8.7318840  | 3/2  |
| 17 | $^{54}\text{Mn}$ | 25 | 29 | 54 | 8.7379230  | 3    |
| 18 | $^{55}\text{Mn}$ | 25 | 30 | 55 | 8.7649880  | 5/2  |
| 19 | $^{56}\text{Mn}$ | 25 | 31 | 56 | 8.7383000  | 3    |
| 20 | $^{52}\text{Fe}$ | 26 | 26 | 52 | 8.6095980  | 0    |
| 21 | $^{54}\text{Fe}$ | 26 | 28 | 54 | 8.7363440  | 0    |
| 22 | $^{55}\text{Fe}$ | 26 | 29 | 55 | 8.7465600  | 3/2  |
| 23 | $^{56}\text{Fe}$ | 26 | 30 | 56 | 8.7903230  | 0    |
| 24 | $^{57}\text{Fe}$ | 26 | 31 | 57 | 8.7702490  | 1/2  |
| 25 | $^{58}\text{Fe}$ | 26 | 32 | 58 | 8.7922210  | 0    |
| 26 | $^{55}\text{Co}$ | 27 | 28 | 55 | 8.6695750  | 7/2  |
| 27 | $^{56}\text{Co}$ | 27 | 29 | 56 | 8.6948170  | 4    |
| 28 | $^{56}\text{Ni}$ | 28 | 28 | 56 | 8.6427090  | 0    |
| 29 | $^{57}\text{Ni}$ | 28 | 29 | 57 | 8.6709010  | 3/2  |
| 30 | $^{58}\text{Ni}$ | 28 | 30 | 58 | 8.7320410  | 0    |
| 31 | $^{60}\text{Ni}$ | 28 | 32 | 60 | 8.7807570  | 0    |
| 32 | $^{60}\text{Zn}$ | 30 | 30 | 60 | 8.5832730  | 0    |
Table 6: Temperature dependent partition function

| No | Name  | 0.20 | 0.40 | 0.60 | 0.80 | 1.00 |
|----|-------|------|------|------|------|------|
| 3  | $^4$He | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 4  | $^{12}$C | 0.00 | 0.00 | 0.00 | 0.02 | 0.06 |
| 5  | $^{16}$O | 0.00 | 0.00 | 0.00 | 0.00 | 0.03 |
| 6  | $^{20}$Ne | 0.00 | 0.08 | 0.34 | 0.72 | 1.21 |
| 7  | $^{24}$Mg | 0.00 | 0.16 | 0.53 | 1.00 | 1.58 |
| 8  | $^{28}$Si | 0.00 | 0.06 | 0.26 | 0.58 | 1.00 |
| 9  | $^{32}$S | 0.00 | 0.02 | 0.14 | 0.44 | 1.02 |
| 10 | $^{36}$Ar | 0.00 | 0.04 | 0.21 | 0.61 | 1.37 |
| 11 | $^{40}$Ca | 0.00 | 0.00 | 0.04 | 0.24 | 0.87 |
| 12 | $^{44}$Ti | 0.02 | 0.38 | 1.32 | 3.09 | 5.90 |
| 13 | $^{50}$Ti | 0.00 | 0.12 | 0.62 | 1.79 | 4.04 |
| 14 | $^{48}$Cr | 0.12 | 0.85 | 1.98 | 3.60 | 5.95 |
| 15 | $^{54}$Cr | 0.08 | 0.74 | 2.09 | 4.58 | 8.62 |
| 16 | $^{55}$Cr | 1.39 | 5.48 | 11.30 | 18.60 | 27.00 |
| 17 | $^{54}$Mn | 10.90 | 22.30 | 36.70 | 54.70 | 75.20 |
| 18 | $^{55}$Mn | 4.36 | 7.81 | 14.00 | 24.00 | 37.40 |
| 19 | $^{56}$Mn | 15.80 | 33.00 | 50.80 | 70.20 | 90.80 |
| 20 | $^{52}$Fe | 0.07 | 0.63 | 1.49 | 2.70 | 4.46 |
| 21 | $^{54}$Fe | 0.00 | 0.19 | 1.11 | 3.43 | 7.57 |
| 22 | $^{56}$Fe | 0.33 | 2.12 | 6.25 | 13.20 | 22.50 |
| 23 | $^{58}$Fe | 0.07 | 0.68 | 2.01 | 4.78 | 9.60 |
| 24 | $^{57}$Fe | 7.67 | 12.80 | 20.20 | 30.50 | 43.60 |
| 25 | $^{58}$Fe | 0.09 | 0.87 | 2.81 | 6.71 | 13.10 |
| 26 | $^{55}$Co | 0.00 | 0.07 | 0.74 | 2.86 | 7.02 |
| 27 | $^{56}$Co | 4.07 | 10.60 | 18.50 | 27.90 | 38.50 |
| 28 | $^{56}$Ni | 0.00 | 0.00 | 0.08 | 0.32 | 0.86 |
| 29 | $^{57}$Ni | 0.14 | 1.04 | 2.46 | 4.63 | 7.87 |
| 30 | $^{58}$Ni | 0.00 | 0.17 | 0.95 | 2.93 | 6.68 |
| 31 | $^{60}$Ni | 0.00 | 0.25 | 1.31 | 3.89 | 8.48 |
| 32 | $^{60}$Zn | 0.03 | 0.46 | 1.36 | 2.84 | 5.03 |
Table 7: Proton and neutron abundance tables.

| kT  | lg ρ | Y_e | X_p                      | X_n                      |
|-----|------|-----|--------------------------|--------------------------|
| 0.20| 6    | 0.350 | 1.4733451757175688e-32   | 1.9792286428615166e-01   |
| 0.20| 8    | 0.350 | 4.659459569570535e-37    | 1.979172869601915e-01    |
| 0.20| 10   | 0.350 | 1.4734610247732251e-41   | 2.003978128597258e-01    |
| 0.40| 6    | 0.350 | 2.50356829594687e-14     | 1.982260532795905e-01    |
| 0.40| 8    | 0.350 | 7.9451430714268810e-19   | 1.9791672869601915e-01   |
| 0.40| 10   | 0.350 | 1.9792286428615166e-01   |                       |
| 0.60| 6    | 0.350 | 4.659459569570535e-37    | 1.9792286428615166e-01   |
| 0.60| 8    | 0.350 | 4.659459569570535e-37    | 1.9792286428615166e-01   |
| 0.60| 10   | 0.350 | 1.4734610247732251e-41   | 2.003978128597258e-01    |
| 0.80| 6    | 0.350 | 4.659459569570535e-37    | 1.9792286428615166e-01   |
| 0.80| 8    | 0.350 | 4.659459569570535e-37    | 1.9792286428615166e-01   |
| 0.80| 10   | 0.350 | 1.4734610247732251e-41   | 2.003978128597258e-01    |
| 1.00| 6    | 0.350 | 4.659459569570535e-37    | 1.9792286428615166e-01   |

18
Table 7: Proton and neutron abundance tables [continued].

| kT | lg ρ | Ye   | X_p  | X_n  |
|----|------|------|------|------|
| 1.00 | 8    | 0.450 | 2.4074257595157450e-01 | 3.4074257595157448e-01 |
| 1.00 | 10   | 0.450 | 2.6369076822919937e-03 | 2.3110858387951739e-02 |
| 0.20 | 6    | 0.500 | 2.1027565926622363e-06 | 3.3591508816189998e-23 |
| 0.20 | 8    | 0.500 | 2.9902933564115093e-07 | 3.9833138940328108e-26 |
| 0.20 | 10   | 0.500 | 2.1502284101490003e-08 | 4.5645516281556141e-29 |
| 0.40 | 6    | 0.500 | 2.4656257836791810e-02 | 1.1948909248170071e-07 |
| 0.40 | 8    | 0.500 | 2.9898277397867462e-03 | 1.2462253458017922e-10 |
| 0.40 | 10   | 0.500 | 2.7224923203959757e-04 | 1.6226416771168784e-13 |
| 0.60 | 6    | 0.500 | 5.105593212499610e-02 | 5.105593212499276e-02 |
| 0.60 | 8    | 0.500 | 2.7482483740179343e-02 | 6.7168188134800034e-05 |
| 0.60 | 10   | 0.500 | 6.4480794989162076e-03 | 3.9682763842094195e-08 |
| 0.80 | 6    | 0.500 | 4.9402975852718162e-01 | 4.9402975852718162e-01 |
| 0.80 | 8    | 0.500 | 4.6130285406449010e-02 | 4.613028479868531e-02 |
| 0.80 | 10   | 0.500 | 2.1479785529572089e-02 | 3.4028673055878073e-05 |
| 1.00 | 6    | 0.500 | 4.9999805645135403e-01 | 4.9999805645135403e-01 |
| 1.00 | 8    | 0.500 | 2.8751054543310173e-01 | 2.8751054543310173e-01 |
| 1.00 | 10   | 0.500 | 3.0258682045056728e-02 | 2.9362486076929912e-03 |