Berezinskii-Kosterlitz-Thouless transition in lattice Schwinger model with one flavor of Wilson fermion

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We have made a detailed study of the phase structure for lattice Schwinger model with one flavor of Wilson fermion on the $(m, g)$ plane. For numerical investigation, we develop a decorated tensor renormalization method for lattice gauge theories with fermions incorporating the Grassmann tensor renormalization. Our algorithm manifestly preserves rotation and reflection symmetries. We find not only a parity-broken phase but also a Berezinskii-Kosterlitz-Thouless (BKT) transition by evaluating the central charge and an expectation value of a projection operator into the parity-odd subspace. The BKT phase boundaries converge into the degenerated doubler pole $(m, g) = (-2, 0)$, while the parity-breaking transition line ends at the physical pole $(m, g) = (0, 0)$. In addition, our analysis of scaling dimensions indicates that a conformal field theory with SU(2) symmetry arises on the line of $m = -2$.

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I. INTRODUCTION

The Wilson fermion is the standard lattice regularization for the continuum Dirac fermion, avoiding the species-doubling problem. Although the chiral symmetry is explicitly broken in this formulation, the existence of massless pseudoscalar mesons is expected by Aoki’s scenario for even numbers of flavors as gapless modes accompanied by a parity-flavor-breaking transition (see a summary report [1]). On the other hand, the phase structure of the Wilson fermion with odd numbers of flavors is not fully understood yet. The condition of the Vafa-Witten theorem [2] is not fulfilled in this case and the parity symmetry itself can be spontaneously broken, that is, the flavor singlet pseudoscalar meson can be also massless. In fact, analyses based on the strong-coupling expansion show that the parity symmetry is really broken.

A possible answer was indicated by numerical simulations for three flavor lattice QCD: The redundant phase is restricted within the strong-coupling region and the correct continuum physics can be realized in a weak-coupling region [3]. However, it is still quite hard to investigate the parity- or parity-breaking transition, because Monte Carlo approaches suffer from the numerical sign problem in the broken phase for odd numbers of flavors.

As a first step toward a long-term goal to understand the phase structure of odd numbers of flavors of the Wilson fermion completely, we study the lattice Schwinger model with one flavor of the Wilson fermion. The Schwinger model, two dimensional QED, is simple but shares several properties with QCD. It was numerically verified by several methods at high level of precision that an Ising transition occurs in the strong-coupling limit [4][5]. Moreover, we showed that such transition is retained even at finite couplings and the transition line approaches the physical pole $(m, g) = (0, 0)$ where $m$ is the fermion mass and $g$ is the gauge coupling [6]. On the other hand, the situation of the lattice Schwinger model around $m = -2$ is not clear yet. In the strong coupling limit of $g = \infty$, as discussed by Salmhofer, it is mapped to a critical six-vertex model whose criticality is described by a conformal field theory (CFT) with the central charge $c = 1$ [7]. In the free limit, the point of $(m, g) = (-2, 0)$ corresponds to the degenerated doubler pole, which is governed by the CFT with $c = \frac{1}{2} + \frac{1}{2} = 1$. A natural expectation may be an existence of a $c = 1$ CFT line or region between $(m, g) = (-2, 0)$ and $(m, g) = (-2, \infty)$, which should be confirmed by a detailed investigation of the phase structure around $m = -2$ at finite couplings by using a reliable numerical method.

In order to investigate the phase structure of the lattice Schwinger model with one flavor of the Wilson fermion throughout the $(m, g)$ plane, we have improved our method proposed in Ref. [6] which was based on the tensor renormalization group (TRG) [8]. The TRG is one of the so-called tensor network algorithms which are free from the sign problem and is suitable for computation of classical partition functions and quantum path-integrals. It was originally developed mainly in the fields of condensed matter physics and quantum information science. In recent years, however, it also attracts a lot of attention in high energy physics and there have been already many applications to lattice field theories [6][9][21]. Although our previous method succeeded in clear demonstration of the Ising transition at finite couplings by adopting a Grassmann version of the TRG proposed by Gu and others [22], the gauge redundancy demanded increasing bond dimensions of tensor to keep numerical precisions at finite couplings. After our work of Refs. [6][13], Dittrich and others proposed a variation of the TRG method which gives a better control of the gauge redundancy by introducing an additional structure decorating the tensor network [14]. In this paper we develop a more efficient method to study the lattice Schwinger model with one flavor of the Wilson fermion by combining the ideas of the decorated TRG and Grassmann TRG.
gate the phase structure in the region around \( m = -2 \) at finite couplings, which was out of reach in previous studies.

It is worth noting that other tensor network methods are also gathering much attention from the field of high energy physics. They also choose the analysis of the lattice Schwinger model as a pilot study, though the Kogut-Susskind fermion formulation has been employed so far in all such works [23–36].

The layout of this papers is as follows. In Sec. II we explain the application of our decorated tensor renormalization to the lattice Schwinger model with one flavor of the Wilson fermion. Section III presents the results for numerical investigation of the phase structure. Finally we summarize our paper in Sec. IV.

II. DECORATED TENSOR RENORMALIZATION FOR THE LATTICE SCHWINGER MODEL

A. Lattice Schwinger model with one flavor of the Wilson fermion

We follow the lattice formulation of the Schwinger model given in Ref. [3], but use \( m \) and \( g \) as parameters in the model throughout this paper instead of the hopping parameter \( \kappa \) and the inverse coupling squared \( \beta \). We give the formulation again in the following to make the paper self-contained.

The Wilson-Dirac matrix is given by

\[
\bar{\psi} D[U] \psi = (m + 2) \sum_{n, \alpha} \bar{\psi}_{n, \alpha} \psi_{n, \alpha} \\
- \frac{1}{2} \sum_{n, \mu, \alpha, \beta} \psi_{n, \alpha} \left[ (1 - \gamma_\mu)_{\alpha, \beta} U_{n, \mu} \bar{\psi}_{n+\hat{\mu}, \beta} \right] \\
+ (1 + \gamma_\mu)_{\alpha, \beta} U_{n-\hat{\mu}, \mu} \psi_{n-\hat{\mu}, \alpha},
\]

where \( \psi_n \) and \( \bar{\psi}_n \) are two-component Grassmann variables at site \( n \) and \( U_{n, \mu} \) is an \( U(1) \) link variable at site \( n \) along \( \mu \) direction. \( \alpha \) and \( \beta \) denote the Dirac indices and \( \hat{\mu} \) represents an unit vector along \( \mu \) direction. The gamma matrices are

\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Following Ref. [7], we employ another basis,

\[
\chi_{n,1} = \frac{1}{\sqrt{2}} (\psi_{n,1} + \psi_{n,2}), \quad \chi_{n,2} = \frac{1}{\sqrt{2}} (\psi_{n,1} - \psi_{n,2}),
\]

\[
\bar{\chi}_{n,1} = \frac{1}{\sqrt{2}} (\bar{\psi}_{n,1} + \bar{\psi}_{n,2}), \quad \bar{\chi}_{n,2} = \frac{1}{\sqrt{2}} (\bar{\psi}_{n,1} - \bar{\psi}_{n,2}),
\]

only in the second (space) direction, while keeping \( \{ \psi_{n, \alpha} \} \) and \( \{ \bar{\psi}_{n, \alpha} \} \) in the first (time) direction. Using this trick, the hopping terms are simply rewritten as

\[
\sum_{\alpha, \beta} \bar{\psi}_{n, \alpha} (1 + \gamma_1)_{\alpha, \beta} \psi_{n-1, \beta} = 2 \bar{\psi}_{n,1} \psi_{n-1,1},
\]

\[
\sum_{\alpha, \beta} \bar{\psi}_{n, \alpha} (1 - \gamma_1)_{\alpha, \beta} \psi_{n+1, \beta} = 2 \bar{\psi}_{n+1,1} \psi_{n,1},
\]

\[
\sum_{\alpha, \beta} \bar{\psi}_{n, \alpha} (1 + \gamma_2)_{\alpha, \beta} \psi_{n-2, \beta} = 2 \bar{\psi}_{n-1,1} \psi_{n-2,1},
\]

\[
\sum_{\alpha, \beta} \bar{\psi}_{n, \alpha} (1 - \gamma_2)_{\alpha, \beta} \psi_{n+2, \beta} = 2 \bar{\psi}_{n+1,1} \psi_{n+2,1},
\]

Each Grassmann variable satisfies the anticommutation relations,

\[
[\psi_{n, \alpha}, \bar{\psi}_{m, \beta}]_+ \equiv \psi_{n, \alpha} \bar{\psi}_{m, \beta} + \bar{\psi}_{m, \beta} \psi_{n, \alpha} = 0,
\]

\[
[\bar{\psi}_{n, \alpha}, \psi_{m, \beta}]_+ = [\bar{\psi}_{n, \alpha}, \psi_{m, \beta}]_+ = 0,
\]

\[
[\chi_{n, \alpha}, \bar{\chi}_{m, \beta}]_+ = [\chi_{n, \alpha}, \chi_{m, \beta}]_+ = [\bar{\chi}_{n, \alpha}, \bar{\chi}_{m, \beta}]_+ = 0,
\]

and the integration rules,

\[
\int d\psi_{n, \alpha} 1 = \int d\bar{\psi}_{n, \alpha} 1 = 0,
\]

\[
\int d\psi_{n, \alpha} \psi_{m, \beta} = \int d\bar{\psi}_{n, \alpha} \bar{\psi}_{m, \beta} = \delta_{n,m} \delta_{\alpha, \beta},
\]

\[
\int d\chi_{n, \alpha} 1 = \int d\bar{\chi}_{n, \alpha} 1 = 0,
\]

\[
\int d\chi_{n, \alpha} \chi_{m, \beta} = \int d\bar{\chi}_{n, \alpha} \bar{\chi}_{m, \beta} = \delta_{n,m} \delta_{\alpha, \beta}.
\]

Turning to the gauge part, the \( U(1) \) plaquette action is given by

\[
S_g[U] = -\frac{1}{g^2} \sum_{\text{all plaquette}} \cos \varphi_p,
\]

\[
\varphi_p \equiv \varphi_{n+1,1} + \varphi_{n+1,2} - \varphi_{n,2},
\]

\[
\varphi_{n,1}, \varphi_{n+1,2}, \varphi_{n+2,1}, \varphi_{n,2} \in [-\pi, \pi],
\]

where \( \varphi_{n,1}, \varphi_{n+1,2}, \varphi_{n+2,1} \) and \( \varphi_{n,2} \) are phases of link variables which compose a plaquette variable \( \varphi_p \).

B. Decorated tensor network representation

We transform the partition function,

\[
Z = \int DU \det D[U] e^{-S_g[U]},
\]
into a decorated tensor network form. First, by using Eqs. [9]–[11], the fermionic part is expanded as follows:

\[
\det D[U] = \prod_{n,\alpha} \left( \int d\psi_{n,\alpha} d\bar{\psi}_{n,\alpha} \right) e^{\tilde{D}[U] \bar{\psi}} \tag{20}
\]

\[
P_0 \int \prod_n W_n \prod_{\mu} H_{n,\mu}, \tag{21}
\]

\[
W_n \equiv (m + 2)^2 + (m + 2) d\psi_{n,1} d\bar{\psi}_{n,1}
+ (m + 2) d\psi_{n,2} d\bar{\psi}_{n,2} + d\psi_{n,1} d\bar{\psi}_{n,1} d\psi_{n,2} d\bar{\psi}_{n,2}, \tag{22}
\]

\[
H_{n,1} \equiv 1 + U_{n,1}^1 \psi_{n,1} \bar{\psi}_{n,1} + U_{n,1} \psi_{n,1} \bar{\psi}_{n,2}
+ \psi_{n+1,2} \bar{\psi}_{n+1,2} \psi_{n,1}, \tag{23}
\]

\[
H_{n,2} \equiv 1 + U_{n,1}^2 \psi_{n,1} \bar{\psi}_{n,1} + U_{n,1}^2 \psi_{n,2} \bar{\psi}_{n,2}
+ \bar{\chi}_{n+1,1} \chi_{n+2,1} + \bar{\chi}_{n+2,1} \chi_{n+2,1}, \tag{24}
\]

where \(P_0\) represents a projection to terms without any Grassmann variable. Next, we perform the character expansion for the gauge part:

\[
e e^{\frac{1}{2} \cos \varphi} = \sum_{b_n = \pm \infty} \sum_{N_{cc}} \sqrt{\frac{2}{\pi g^2}} I_{b_n} \left( \frac{1}{g^2} \right), \tag{25}
\]

\[
\approx \sum_{b_n = -N_{cc}} e^{ib_n \varphi} I_{b_n} \left( \frac{1}{g^2} \right), \tag{26}
\]

with \(I_{b_n}\) the modified Bessel function. The truncation number \(N_{cc}\) is introduced in order to obtain a finite-dimensional tensor network [10]. Integrating out all link variables, we obtain

\[
\int_{-\pi}^{\pi} \frac{d\varphi_{n,1}}{2\pi} \psi_{n,1} e^{ib_n \varphi_{n,1}} \psi_{n,1} = \begin{cases}
1 + \psi_{n+1,2} \bar{\psi}_{n+1,2} \psi_{n,1}, & b_i = b_j \\
\psi_{n+1,2} \bar{\psi}_{n+1,2}, & b_i = b_j + 1 \\
\psi_{n+1,2} \bar{\psi}_{n+1,2}, & b_i = b_j - 1 \\
0, & \text{otherwise}
\end{cases} \tag{27}
\]

Integration for \(H_{n,2}\) is performed likewise. Multiplying \(W_n\) by these integrated results yields a decorated tensor network form of the partition function:

\[
Z = \sum_{b_1, b_2, b_3, \cdots} \prod_{i, j, k, \cdots} T_{n, i, j, k, \cdots}^{(b_1, b_2, b_3, b_4)} \prod_{n + 1, m, \alpha, \beta, \cdots} \psi_{n, \alpha} \bar{\psi}_{n, \beta}
\times \prod_{n + 2, i, j, k, \cdots} G_{n, \alpha, \beta}^{(b_3, b_4)} \psi_{n, \beta} \bar{\psi}_{n, \alpha}
\tag{28}
\]

where \(G_{n, \alpha, \beta}\) is a decorated tensor kernel which includes no Grassmann number and its detail is given in Appendix A. \(h_{b_1, b_2}\) is equal to the Kronecker delta \(\delta_{b_1, b_2}\) for the initial tensor network. \(R\) is a ramp function which satisfies

\[
R(x) = \begin{cases}
x, & x \geq 0 \\
0, & x < 0
\end{cases}
\tag{32}
\]

\(p(i)\) equals to zero (one) when \(i\) indicates a parity-even (-odd) state under a reflection. Note that \(\psi_{n, \alpha}\), \(\chi_{n, \alpha}\) and others are treated as independent variables. Figure 1 illustrates Eq. (28) as a network diagram.

FIG. 1. Schematic representation of the decorated tensor network of Eq. (28). Each site indicated by a filled circle represents each decorated tensor \(T_n\). \((G_{n, \mu})\) are assigned to each link. The decoration labels \(\{b_i\}\) sit on each plaquette.
FIG. 2. Schematic representation for the decompositions of (a) Eq. (42) and (b) Eq. (49). Newly-introduced Grassmann variables \( \{ \psi \_n(1), \xi \_n(1), \bar{\eta}_n(1), \bar{\xi}_n(1) \} \) are assigned to the additional dashed lines.

FIG. 3. Schematic representation of the contraction of Eq. (55). Solid lines in a closed loop, where old Grassmann variables \( \{ \tilde{\psi}_n, \psi_n, \tilde{\xi}_n, \chi_n \} \) reside, can be contracted and replaced by \( T_{n(t)}^{(1)} \).

\[
\begin{align*}
(a) & \quad \mathcal{T}_{n(t)}^{(b_1,b_2,b_3,b_4)}(i,j) \\
& \quad \xrightarrow{\text{SVD}} \mathcal{S}_{n(t)}^{(b_1,b_2)}(i,j) \quad \mathcal{S}_{n(t)^{+}}^{(t_1,b_3)}(i,j) \\
(b) & \quad \xrightarrow{\text{SVD}} \mathcal{S}_{n(t)}^{(b_1,b_2,b_3,b_4)}(i,j) \\
& \quad \xrightarrow{\text{contract}} \mathcal{T}_{n(t)}^{(b_1,b_2,b_3,b_4)}(i,j)
\end{align*}
\]

C. Decorated TRG procedure

One TRG cycle consists of two steps, decomposing each tensor in Eq. (28) by using the singular value decomposition (SVD) and contracting the decomposed tensors to obtain coarse-grained tensors. We extend an algorithm proposed in Ref. [14] to systems including fermions. Fermionic degrees of freedom are treated by following our previous approach given in Ref. [6], but some modifications are introduced to preserve rotation and reflection symmetries.

First, let us explain the decomposition part. A matrix form of the decorated tensor kernel \( T(b_1,b_2,b_3,b_4) \) is defined as follows:

\[
M^{(b_1,b_4)}(b_1;i,j),(b_2;k,l) = T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}.
\]

Then, we perform the SVD:

\[
M^{(b_3,b_4)}(b_1;i,j),(b_2;k,l) = \sum_m U^{(b_3,b_4)}_{(b_1;i,j),m} \sigma_m^{(b_3,b_4)} V^{(b_3,b_4)}_{m,(b_2;k,l)},
\]

Only the largest \( D \) singular values out of \( \{ \sigma_m^{(b_3,b_4)} \} \) are kept and others are approximated to zero. When \( b_3 = b_4 \), \( M^{(b_3,b_4)} \) is a symmetric matrix and the eigenvalue decomposition is applied rather than the SVD, that is, \( V^{(b_3,b_4)}_{m,(b_2;k,l)} = U^{(b_3,b_4)}_{(b_1;i,j),m} \) and \( \sigma_m^{(b_3,b_4)} \) can be negative. The truncation is performed in that case, taking the absolute values of \( \{ \sigma_m^{(b_3,b_4)} \} \) into consideration. In the case \( b_3 = -b_4 \), thanks to the reflection symmetry with respect to the 1’-axis (see Appendix B), \( M^{(b_3,b_4)} \) can be block-diagonalized. To derive a block-diagonal form, we introduce special indices \( O, A, B, O’, A’ \) and \( B’ \):

\[
O \in \{ (0;i,j) | i = j \},
A \in \{ (0;i,j) | i > j \} \cup \{ (b_1;i,j) | b_1 > 0 \},
B \in \{ (0;i,j) | i < j \} \cup \{ (b_1;i,j) | b_1 < 0 \},
O’ \in \{ (0;k,l) | k = l \},
A’ \in \{ (0;k,l) | k > l \} \cup \{ (b_2;k,l) | b_2 > 0 \},
B’ \in \{ (0;k,l) | k < l \} \cup \{ (b_2;k,l) | b_2 < 0 \}.
\]

Then, \( M^{(b_3,b_4)} \) is block-diagonalized as follows:

\[
\begin{pmatrix}
1 & 0 & \frac{1}{\sqrt{2}} & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} \\
0 & 1 & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} & M_{OO’} & M_{OA’} & M_{OB’} & M_{OBB’} \\
\frac{1}{\sqrt{2}} & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} & -M_{AO’} & M_{AA’} & M_{AB’} & M_{ABB’} \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \frac{1}{\sqrt{2}} & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} \\
0 & 1 & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} & M_{OO’} & M_{OA’} & M_{OB’} & M_{OBB’} \\
\frac{1}{\sqrt{2}} & \frac{(-1)^{p(i)+p(j)+b_1}}{\sqrt{2}} & -M_{AO’} & M_{AA’} & M_{AB’} & M_{ABB’} \\
\end{pmatrix}
\]

where the superscript \( (b_3,b_4) \) is omitted. When \( b_3 \) is an odd (even) number, the upper-left (lower-right) block is parity-odd under the reflection. If the index \( m \) which is introduced in the SVD, Eq. (34) is in the parity-odd...
block, \( p(m) = 1 \); otherwise \( p(m) = 0 \). By using the above result, we define a decomposition of the Grassmann-valued tensor \( T_{n}^{(b_1,b_2,b_3,b_4)} \) on even sites,

\[
T_{n}^{(b_1,b_2,b_3,b_4)} = \sum_{m,m'} \int S_{n}^{1(b_3,b_4)}_{n(1) + \hat{1};(b_1,i,j),m'} S_{n(1);(b_2:k,l),m}^{3(b_3,b_4)} G_{n(1),1,m'}^{(b_3,b_4)} \tag{42}
\]

with

\[
G_{n(1),1,m',m}^{(b_3,b_4)} \equiv h_{m',m}^{(1)(b_3,b_4)} \xi_{n(1),1}^{R(b_3-b_4) + p(m)} \eta_{n(1) + 1,2}^{R(b_4-b_3) + p(m')} \eta_{n(1),2}^{R(b_4-b_3) + p(m)}, \tag{43}
\]

\[
h_{m',m}^{(1)(b_3,b_4)} \equiv \text{sgn} \left( \sigma^{(b_3,b_4)}_{m,m'} \right) \delta_{m,m'}, \tag{44}
\]

\[
S_{n(1)+\hat{1};(b_1,i,j),m}^{1(b_3,b_4)} \equiv S_{(b_1,i,j),m}^{1(b_3,b_4)} d\xi_{n(1)+\hat{1},1}^{R(b_3-b_4) + p(m')} d\xi_{n(1)+1,2}^{R(b_4-b_3) + p(m)} d\psi_{n,1}^{\frac{1}{2} R(b_1-b_4) + p(i)} d\psi_{n,2}^{\frac{1}{2} R(b_1-b_4) + p(i)}, \tag{45}
\]

\[
S_{(b_2;k,l),m}^{3(b_3,b_4)} \equiv (-1)^{R(b_2-b_4) + R(b_1-b_4) + R(b_3-b_4)} \sqrt{\eta_{n,2}^{(b_4,b_3)}} \frac{1}{\eta_{n,1}^{(b_4,b_3)}} d\psi_{n}^{R(b_2-b_3) + p(k)} d\psi_{n,2}^{R(b_2-b_3) + p(k)}, \tag{46}
\]

\[
S_{(b_2;k,l),m}^{3(b_3,b_4)} \equiv \sqrt{\eta_{n,2}^{(b_4,b_3)}} \frac{1}{\eta_{n,1}^{(b_4,b_3)}} d\psi_{n}^{R(b_2-b_3) + p(k)} d\psi_{n,2}^{R(b_2-b_3) + p(k)}, \tag{47}
\]

where \( \{\xi_{n(1)},\eta_{n(1)}\} \) are newly-introduced Grassmann variables with \( n^{(1)} \) a coarse-grained lattice site and \( \text{sgn} \) is a sign function. This decomposition is schematically expressed in Fig. 2 (n). It should be remarked that differently from our previous approach, \( \xi_{n(1)} \) is a twocomponent Grassmann variable and that is essential to preserve rotation and reflection symmetries. In Eq. 45, the \( \pi \)-rotation symmetry explained in Appendix B is employed and it is assumed that \( \xi_{n(1)} \) transforms under rotations as \( \psi_{n} \) does. A decomposition for odd sites is defined as follows by exploiting the rotation symmetry discussed in Appendix B.

\[
T_{n}^{(b_1,b_2,b_3,b_4)} = \sum_{m,m'} \int S_{n(1)+\hat{1};(b_1,j,k),m}^{2(b_2,b_1)} S_{n(1);(b_4:l,i),m}^{4(b_2,b_1)} G_{n(1),2,m'}^{(b_2,b_1)} \tag{49}
\]

with

\[
G_{n(1),2,m',m}^{(b_2,b_1)} \equiv h_{m',m}^{(1)(b_2,b_1)} \eta_{n(1) + 1,2}^{R(b_2-b_1) + p(m')} \eta_{n(1),2}^{R(b_2-b_1) + p(m')} \eta_{n(1) + 2',1}^{R(b_2-b_1) + p(m')} \eta_{n(1),2}^{R(b_2-b_1) + p(m)}, \tag{50}
\]

\[
h_{m',m}^{(1)(b_2,b_1)} \equiv \text{sgn} \left( \sigma^{(b_2,b_1)}_{m,m'} \right) \delta_{m,m'}, \tag{51}
\]

\[
S_{n(1)+\hat{1};(b_1,j,k),m}^{2(b_2,b_1)} \equiv S_{(b_1,j,k),m}^{2(b_2,b_1)} d\eta_{n(1)+1,2}^{R(b_2-b_1) + p(m')} d\eta_{n(1) + 2',1}^{R(b_2-b_1) + p(m')} d\chi_{n,1}^{R(b_2-b_1) + p(j)} d\chi_{n,2}^{R(b_2-b_1) + p(j)}, \tag{52}
\]

\[
S_{n(1);(b_4:l,i),m}^{4(b_2,b_1)} \equiv \sqrt{\eta_{n,2}^{(b_4,b_1)}} \frac{1}{\eta_{n,1}^{(b_4,b_1)}} d\chi_{n}^{R(b_4-b_1) + p(l)} d\chi_{n,2}^{R(b_4-b_1) + p(l)}, \tag{53}
\]

\[
S_{n(1);(b_4:l,i),m}^{4(b_2,b_1)} \equiv \sqrt{\eta_{n,2}^{(b_4,b_1)}} \frac{1}{\eta_{n,1}^{(b_4,b_1)}} d\chi_{n}^{R(b_4-b_1) + p(l)} d\chi_{n,2}^{R(b_4-b_1) + p(l)}, \tag{54}
\]

where \( \{\eta_{n(1)},\eta_{n(1)}\} \) are also newly-introduced Grassmann variables and transform under rotations as \( \chi_{n} \) does. This
decomposition is schematically expressed in Fig. 2 (b). Next, we perform a contraction for $S_{n(1)}^{(1)(b_2,b_3,b_4)}$, $S_{n(1)}^{(2)(b_2,b_4)}$, $S_{n(1)}^{(3)(b_1,b_4)}$ and $S_{n(1)}^{(4)(b_2,b_4)}$ as depicted in Fig. 3. The old Grassmann variables $\{\psi_n, \bar{\psi}_n, \bar{\xi}_n, \xi_n\}$ reside on a small closed loop in the Figure and we can integrate out all of them simultaneously with the contraction. As a result, we get a coarse-grained tensor

$$\tau_{n(1);i,j,k,l}^{(1)(b_1,b_2,b_3,b_4)} = \sum_{b_5,p,q,r,s,t,s',t'} \int S_{n(1);i,j,k,l}^{(1)(b_5,b_2)} G_{n(1);i,j,k,l}^{(2)(b_5,b_4)} G_{n(1);i,j,k,l}^{(3)(b_5,b_4)} S_{n(1);i,j,k,l}^{(4)(b_5,b_1)}$$

where only the new Grassmann variables remain. It is easily confirmed that the constraints on the tensor kernel $T_{n(1)}^{(1)(b_1,b_2,b_3,b_4)}$ from the reflection symmetry are the same as those on the initial tensor given in Appendix B. The constraints from the reflection symmetry are also derived by assuming appropriate transformality of $\eta_n^{(1)}$ and $\eta_n^{(1)}$. Roles of the 1-axis (2-axis) and 2'-axis (1'-axis) are swapped, compared with the case of the initial tensor.

Further iterations can be performed in a similar manner. After $N \equiv \log_2 L^2$ iterations with $L$ the lattice length, one $N$-times coarse-grained tensor $T_{n(N)}^{(N)(b_1,b_2,b_3,b_4)}$ covers whole lattice. Therefore, the density matrix $\rho$ is represented as

$$\rho_{(b_1;i),(b_2;k)} = \sum_{j,j',k'} \int \tau_{(i,j,k,l)}^{(N)(b_1,b_2,b_3,b_4)} \rho_{(b_2;j),(b_3;k),l}^{(N)(b_1,b_3)} G_{0,j',k}^{(N)(b_1,b_3)} G_{0,j',k'}^{(N)(b_1,b_3)} = \sum_{j} (-1)^{\epsilon_{j}} T_{i,j,k}^{(N)(b_1,b_2,b_3,b_4)} \rho_{(b_2;j),(b_1;i)l}^{(N-1)(b_1,b_3)} \rho_{(b_2;j),(b_1;i)l}^{(N-1)(b_1,b_3)} \rho_{(b_2;j),(b_1;i)l}^{(N-1)(b_1,b_3)}$$

and the partition function is $Z = \text{tr} \rho$ where periodic boundary conditions are employed. The density matrix $\rho$ can be block-diagonalized because of the reflection symmetry with respect to the 1-axis, namely the parity symmetry. That enables us to compute an expectation value of a projection operator into the parity-odd subspace as follows:

$$\langle P_{\text{odd}} \rangle = \frac{\text{tr}(P_{\text{odd}} \rho)}{Z} = \frac{1}{Z} \left[ \sum_{i,j} \rho_{(0,i),(0,i)} \right] + \left[ \sum_{b_1 > 0} \left( \rho_{(b_1;i),(b_1;i)} - (-1)^{p(i)} \rho_{(b_1;i),(b_1;i)} \right) \right]$$. (57)

Taking the large volume limit, it will be one half in the parity-broken phase; otherwise zero.

### III. NUMERICAL STUDY

#### A. Strong coupling limit

Before investigation at finite couplings, we first examine the strong coupling limit. In this case, the gauge action of Eq. (16) is dropped and the conventional TRG is applied rather than the decorated TRG as in Ref. 6. Taking advantage of the symmetry-preserving algorithm, we calculate $\langle P_{\text{odd}} \rangle$ according to Eq. (57) in order to detect a parity-broken phase in an explicit manner, while we employed the analyses of the Lee-Yang/Fisher zeros and chiral susceptibility in Ref. 6. Figure 1 shows our estimate of $\langle P_{\text{odd}} \rangle$ on a $L = 20^2$ lattice with $D = 160$ as the SVD truncation parameter. It overtly indicates a parity-breaking transition around $m = -0.7$. We present the convergence behavior of $\langle P_{\text{odd}} \rangle$ with increasing $L$ around the transition point in Fig. 5. The $D$ dependence of the
An important new finding is another phase transition criticality. As demonstrated by Gu and Wen using the TRG in Ref. 37, one can evaluate it from the largest eigenvalue of the density matrix $\rho$ with the aid of CFT. Since the eigenvalues $\{\lambda_i\}$ are of a form

$$\lambda_i = e^{-f_\infty L^2 - 2\pi (\Delta_i - \pi)},$$  \hfill (58)$$

where $f_\infty$ is a non-universal constant and $\{\Delta_i\}$ are scalar.

![FIG. 6. Convergence of two phase transition points evaluated by a $\langle P_{\text{odd}} \rangle$ analysis on a $L = 2^{20}$ lattice at $g = \infty$ as a function of the SVD truncation number $D$. Dotted lines are for guiding eyes.](image)

![FIG. 4. $\langle P_{\text{odd}} \rangle$ and central charge $c$ as a function of fermion mass $m$ at $g = \infty$ evaluated with $D = 160$. $\langle P_{\text{odd}} \rangle$ is evaluated on a $L = 2^{20}$ lattice and $c$ is from $\chi^2$-fits with Eq. (59) for $L \in \{2^7, 2^8\}$ (circle) or $L \in \{2^7, 2^{10}\}$ (square). Dotted lines denote $\langle P_{\text{odd}} \rangle = 0, 0.5$ and $c = 0, 1$ to guide eyes.](image)

![FIG. 5. Convergence of $\langle P_{\text{odd}} \rangle$ at $g = \infty$ evaluated with $D = 160$ as a function of $L$. The result of $m = -0.68647$ goes to zero (symmetric phase) while that of $m = -0.68649$ reaches one half (parity-broken phase). Dotted lines denote $\langle P_{\text{odd}} \rangle = 0, 0.5$ to guide eyes.](image)

![FIG. 7. The largest eigenvalue $\lambda_0$ of the density matrix $\rho$ at $g = \infty$ evaluated with $D = 160$ as a function of $L$. Two cases of $m = -0.68648$ and $m = -1.79$ are presented. Solid curves represent fit results with Eq. (59) for $L \in \{2^7, 2^8\}$.](image)

| $m$  | $c$      | $f_\infty$ | $\sqrt{\chi^2/d.o.f}$ |
|------|----------|------------|------------------------|
| -0.68648 | 0.50272(30) | -0.566320061(79) | 4.5 |
| -1.79   | 0.99668(99)  | -0.09038590(23)  | 0.035 |
ing dimensions, we use the following equation for $\chi^2$-fit:

$$\ln \lambda_0 = \frac{\pi c}{6} - f_\infty L^2. \quad (59)$$

Note that our algorithm given in Sec. II does not preserve hermiticity of the density matrix and it is numerically broken. Because of this, we instead compute singular values as our estimate of eigenvalues. Figure 4 shows $L$ dependence of the largest eigenvalue $\lambda_0$ at $m = -1.79$, $-0.68648$, whose errors are estimated by the difference between the results with $D = 160$ and 80, and Table I summarizes the fit results, which indicate $c = 1$ for $m = -1.79$ and $c = \frac{1}{2}$ for $m = -0.68648$. The $m$ dependence of the central charge is plotted in Fig. 4. We observe a wide range of $c = 1$ for $m \lesssim -1.8$, while a clear $c = \frac{1}{2}$ peak is found around $m = -0.7$ of the Ising transition point. The existence of the critical region (not point) for $m \lesssim -1.8$ is consistent with an expected property of the BKT transition. Note that the central charge in Fig. 4 rises up more gently than $\langle P_{\text{odd}} \rangle$. One of the main reasons is that the central charge is estimated with data sets of somewhat smaller lattices in order to avoid accumulation of the SVD truncation error which makes the TRG flows go away from the critical surface. The finite size effect is clearly observed by comparing two fit results between $L \in [2^5, 2^8]$ and $L \in [2^7, 2^{10}]$. On the other hand, $\langle P_{\text{odd}} \rangle$ is a quantity related to a rather trivial fixed point which renormalization group flows finally reach away from criticality. It is less contaminated by the SVD truncation error so that we are allowed to take large $L$ limit more reliably. However, it should be noted that $\langle P_{\text{odd}} \rangle$ is much more unstable around the BKT transition point than the Ising transition point: Our estimate of the BKT transition point by the $\langle P_{\text{odd}} \rangle$ analysis has larger $D$ dependence as observed in Fig. 4.

We also measure scaling dimensions using Eq. (58):

$$\Delta_i = \frac{1}{2\pi} \ln \left( \frac{\lambda_0}{\lambda_i} \right). \quad (60)$$

In a BKT phase, $\{\Delta_i\}$ vary continuously due to the existence of an exactly marginal operator. At $m = -2$ the model is mapped to a critical six-vertex model which belongs to an universality class described by a free boson compactified on a circle with radius $R = \frac{1}{\sqrt{2}}$ (see, e.g., Ref. [38]). In this case, the scaling dimensions are known to be

$$\Delta_i = \frac{M^2}{4R^2} + R^2 N^2, \quad M, N \in \mathbb{Z}, \quad (61)$$

which means that all the lowest four nonzero scaling dimensions $\Delta_1, \Delta_2, \Delta_3$ and $\Delta_4$ are equal to one half. The top panel of Fig. 8 shows $L$ dependence of the scaling dimensions. We find sizable deviation from one half, especially for $\Delta_4$, compared to the case of $(m, g) = (-0.68648, \infty)$ in Fig. 4 where $\Delta_1 = \frac{1}{2}, \Delta_2 = 1$ and $\Delta_3 = \Delta_4 = \frac{5}{8}$ is expected for the Ising transition. This is due to the well-known fact that the existence of an exactly marginal operator in the BKT transition causes an universal $O((\ln L)^{-1})$ correction to the scaling dimensions while the central charge is affected by a smaller correction of $O((\ln L)^{-3})$ [39, 40]. Although precise determination of the scaling dimensions requires much larger lattice size, accumulation of the SVD truncation error makes it difficult (see $D$ dependence of the scaling dimensions evaluated at $L = 2^{10}$ in the bottom panel of Fig. 4).

Putting aside the logarithmic finite size corrections on
the scaling dimensions, it should be important to point out that $\Delta_1$, $\Delta_2$ and $\Delta_3$ form a triplet. The free boson compactified with $R = \frac{1}{\sqrt{2}}$ corresponds to a CFT with an SU(2) symmetry and the triplet spectrum is ascribed to it. Note that the $S = \frac{1}{2}$ Heisenberg XXX model belongs to the same universality class and its scaling dimensions obtained by numerically solving the Bethe ansatz equations show similar behavior to our results [11, 12]: triplet spectrum close to the value of one half and singlet one deviated from the value of one half. A remaining question of interest is whether or not the SU(2) symmetry arises at $m = -2$ only. We present $m$ dependence of the scaling dimensions in the BKT phase around $m = -2$ in Fig. 10 where we find that the triplet spectrum is observed only at $m = -2$.

B. Finite couplings

We employ three couplings $g = 1.0, 0.5, 0.1$ to investigate the phase structure in the finite-coupling region, where an additional parameter $N_{ce}$ is required to control the truncation of the character expansion. In our previous approach with the conventional TRG method in Ref. [6, 13], only the initial tensor size is related to the truncation number $N_{ce}$ and we chose a somewhat excessive value to safely neglect the truncation error. However, the computational cost of the decorated TRG method heavily depends on $N_{ce}$ and smaller values are preferred. Figure 11 shows our estimate of $\Delta_1$ at $(m, g) = (-2, 0.1)$ with several values of $N_{ce}$. The result looks converged for $N_{ce} \geq 3$ even on smaller lattices so that we employ $N_{ce} = 3$ in this work. Since the convergence property of the character expansion becomes better toward the strong coupling limit, the choice of $N_{ce} \geq 3$ is also valid for the cases of $g = 1.0, 0.5$. It should be noted that our choice of the SVD truncation number $D$ in the finite coupling study is much smaller than in the case of the strong coupling limit with the conventional TRG method. In the case of the decorated TRG method, however, we eventually take $D \times (2N_{ce} + 1)^2$ singular values overall with the choice of $D$ singular values in each block.

Carrying out the same analyses as the strong coupling limit, we find two phase transition points at finite couplings. Figure 12 plots the values of $m_c$ at these transition points as a function of $D$, which are determined by the $\langle P_{\text{odd}} \rangle$ analysis. Figure 13 shows $L$ dependence of the largest eigenvalue of the density matrix $\rho$ at the transition points, whose errors are estimated by the difference between the results with $D = 64$ and 32. The fit with Eq. (69) gives the central charge listed in Table 1. These results clearly show that two transitions at each coupling are the Ising and BKT types and the former moves to $m = 0$ and the latter to $m = -2$ toward the weak coupling limit.

One may expect that the SU(2) symmetry arises at $m = -2$ even in finite coupling cases. We present our estimate of the scaling dimensions $\Delta_1$, $\Delta_2$, $\Delta_3$ and $\Delta_4$ at $m = -2$ as a function of $L$ in Fig. 14. The triplet spectrum is observed in all the cases, though our results are rather contaminated by the SVD truncation error (see Fig. 15 for $D$ dependence of the scaling dimensions).

From the above investigations, we determine the phase structure of the lattice Schwinger model with one flavor of the Wilson fermion as depicted in Fig. 16. The reflection symmetry of the Wilson fermion with respect to the $m = -2$ axis is exploited in the figure. One of the characteristic features different from the Aoki phase is that only one transition line arises from $(m, g) = (0, 0)$ (and $(m, g) = (-4, 0)$). The same structure is reported in Ref. [43] where the lattice Gross-Neveu model with one flavor of the Wilson fermion was studied using a method based on the weak-coupling expansion.
FIG. 12. Convergence behavior of the Ising transition point (top) and the BKT transition point (bottom) as a function of the SVD truncation number $D$. (P_{odd}) analysis is made at $g = 1.0, 0.5, 0.1$ on a $L = 2^D$ lattice with $N_{ce} = 3$. Dotted lines are for guiding eyes.

FIG. 13. The largest eigenvalue $\lambda_0$ of the density matrix $\rho$ evaluated at $g = 1.0, 0.5, 0.1$ (from top to bottom) with $D = 64$ and $N_{ce} = 3$ as a function of $L$. Both cases of the Ising (red) and BKT (green) transitions are presented. Solid curves represent fit results with Eq. (59) for $L \in [2^2, 2^8]$.

FIG. 14. Scaling dimensions $\Delta_1, \Delta_2, \Delta_3$ and $\Delta_4$ evaluated at $(m, g) = (-2, 1.0)$ (top), $(-2, 0.5)$ (middle) and $(-2, 0.1)$ (bottom) with $D = 64$ and $N_{ce} = 3$ as a function of $L$. Dotted line denotes $\Delta_1 = 0.5$ to guide eyes.

FIG. 15. $D$ dependence of scaling dimensions $\Delta_1$ (top) and $\Delta_4$ (bottom) evaluated at $(m, g) = (-2, 0.1)$ with $N_{ce} = 3$ as a function of $L$. Dotted lines denote $\Delta_1 = \Delta_4 = 0.5$ to guide eyes.
level of precision. In recent years, however, the TRG method has been improved rapidly [44][48], These methods allow us to reach larger lattices with higher accuracy even at criticality. Further development would open a new path to precise numerical studies of the BKT transition.

It should be noted that the phase structure of the lattice Schwinger model with two or more flavors of the Wilson fermion is also of great interest. The Mermin-Wagner theorem prohibits the parity-flavor breaking for finite numbers of flavors but massless pseudoscalar mesons appear in the continuum Schwinger model. Extending our method to multi-flavor cases is straightforward in principle. However, an efficient way to control the increasing degrees of freedom due to the multi-flavor would be indispensable for practical numerical computation.

**IV. SUMMARY AND OUTLOOK**

We have performed a detailed study of the phase structure of the lattice Schwinger model with one flavor of the Wilson fermion by developing a decorated Grassmann TRG method. By implementing an algorithm which preserves the reflection symmetry, we are allowed to evaluate an expectation value of an projection operator into the parity-odd subspace, which reveals a parity-broken phase in a clear manner. We have found not only an Ising phase transition, which has been already known in our previous study, but also a BKT phase transition both in the strong coupling limit and the finite coupling region by measuring the central charge. Furthermore, our analysis of the scaling dimensions has shown that a CFT with the $SU(2)$ symmetry arises at $(m, g) = (-2, \infty)$, where the lattice Schwinger model is known to be equivalent to a critical six-vertex model, and is retained along the line of $m = -2$ in the finite coupling region of the $(m, g)$ plane.

There is a numerical difficulty of accumulation of the SVD truncation error in the study of the BKT phase. The scaling dimensions suffer from a logarithmic finite size correction, which hinders us from quantitative understanding of the properties of the BKT phase at a high level of precision. In recent years, however, the TRG method has been improved rapidly [44][48]. These methods allow us to reach larger lattices with higher accuracy even at criticality. Further development would open a new path to precise numerical studies of the BKT transition.

We thank Shinji Takeda for useful discussions and Hiroshi Ueda for his teaching us the BKT transition. Numerical calculations for the present work were mainly carried out using the Cray XC40 system at Yukawa Institute for Theoretical Physics, Kyoto University and the HOKUSAI system at Advanced Center for Computing and Communication, RIKEN. Computation of the preliminary study was performed on the HA-PACS system under the Interdisciplinary Computational Science Program in Center for Computational Sciences, University of Tsukuba. This research is supported by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) as “Exploratory Challenge on Post-K computer (Frontiers of Basic Science: Challenging the Limits)” and also in part by Grants-in-Aid for Scientific Research from MEXT (No. 15H03651).

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**Appendix A: Detail of the initial tensor**

We list all the nonzero components of the decorated tensor kernel $T^{(b_1, b_2, b_3, b_4)}$ introduced in Eq. (29). Here it is useful to classify $T^{(b_1, b_2, b_3, b_4)}$ into 19 cases according to the decoration labels $b_1, b_2, b_3$ and $b_4$. The following abbreviation is used for simplicity:

$$I_{b_1, b_2, b_3, b_4} \equiv \left[ I_{b_1} \left( \frac{1}{g} \right) I_{b_2} \left( \frac{1}{g} \right) I_{b_3} \left( \frac{1}{g} \right) I_{b_4} \left( \frac{1}{g} \right) \right]^{\frac{1}{2}}. \tag{A1}$$

For each tensor index, the value of one (two) corresponds to a parity-even (-odd) state, namely $p(1) = 0$ and $p(2) = 1$. Symmetry of the tensor is discussed in Appendix B.
1. \(b_1 - b_4 = b_3 - b_1 = b_3 - b_2 = b_2 - b_4 = 0\)

The nonzero components are as follows. The nonzero components are as follows:

\[
T_{1,1,1,1}^{(b_1,b_2,b_3,b_4)} = (m+2)^2 I_{b_1,b_2,b_3,b_4},
\]

\[
T_{2,1,2,1}^{(b_1,b_2,b_3,b_4)} = \frac{1}{2} I_{b_1,b_2,b_3,b_4},
\]

\[
T_{2,1,1,2}^{(b_1,b_2,b_3,b_4)} = \frac{1}{2} I_{b_1,b_2,b_3,b_4}.
\]

2. \(b_1 - b_4 = b_3 - b_2 = 1\) and \(b_3 - b_1 = b_2 - b_4 = 0\)

The dimension of the first and third tensor indices is one. That of the second and fourth ones is two. The nonzero components are as follows:

\[
T_{1,1,1,1}^{(b_1,b_2,b_3,b_4)} = (m+2) I_{b_1,b_2,b_3,b_4},
\]

\[
T_{1,2,1,1}^{(b_1,b_2,b_3,b_4)} = -\frac{1}{2} I_{b_1,b_2,b_3,b_4}.
\]

3. \(b_1 - b_4 = b_3 - b_2 = 0\) and \(b_3 - b_1 = b_2 - b_4 = 1\)

This case is given by a \(\frac{\pi}{2}\) rotation of the case \(A_2\)

4. \(b_1 - b_4 = b_3 - b_2 = -1\) and \(b_3 - b_1 = b_2 - b_4 = 0\)

This case is given by a \(\pi\) rotation of the case \(A_2\)

5. \(b_1 - b_4 = b_3 - b_2 = 0\) and \(b_3 - b_1 = b_2 - b_4 = -1\)

This case is given by a \(-\frac{\pi}{2}\) rotation of the case \(A_2\)

6. \(b_1 - b_4 = 1\), \(b_3 - b_1 = -1\) and \(b_3 - b_2 = b_2 - b_4 = 0\)

The dimension of the first and second tensor indices is one. That of the third and fourth ones is two. The nonzero components are as follows:

\[
T_{1,1,1,1}^{(b_1,b_2,b_3,b_4)} = \frac{m+2}{\sqrt{2}} I_{b_1,b_2,b_3,b_4},
\]

\[
T_{1,1,2,1}^{(b_1,b_2,b_3,b_4)} = T_{1,1,1,2}^{(b_1,b_2,b_3,b_4)} = -\frac{1}{\sqrt{2}} I_{b_1,b_2,b_3,b_4}.
\]

7. \(b_1 - b_4 = b_3 - b_4 = 0\) and \(b_3 - b_1 = b_3 - b_2 = 1\)

This case is given by a \(\frac{\pi}{2}\) rotation of the case \(A_6\)

8. \(b_1 - b_4 = b_3 - b_1 = 0\), \(b_3 - b_2 = -1\) and \(b_2 - b_4 = 1\)

This case is given by a \(\pi\) rotation of the case \(A_6\)

9. \(b_1 - b_4 = b_2 - b_4 = -1\) and \(b_3 - b_1 = b_3 - b_2 = 0\)

This case is given by a \(-\frac{\pi}{2}\) rotation of the case \(A_6\)

10. \(b_1 - b_4 = -1\), \(b_3 - b_1 = 1\) and \(b_3 - b_2 = b_2 - b_4 = 0\)

This case is given by a reflection with respect to the 1'-axis (see Fig. 17) of the case \(A_6\)

11. \(b_1 - b_4 = b_2 - b_4 = 0\) and \(b_3 - b_1 = b_3 - b_2 = -1\)

This case is given by a reflection with respect to the 2-axis (see Fig. 17) of the case \(A_6\)

12. \(b_1 - b_4 = b_3 - b_1 = 0\), \(b_3 - b_2 = 1\) and \(b_2 - b_4 = -1\)

This case is given by a reflection with respect to the 2'-axis (see Fig. 17) of the case \(A_6\)

13. \(b_1 - b_4 = b_2 - b_4 = 1\) and \(b_3 - b_1 = b_3 - b_2 = 0\)

This case is given by a reflection with respect to the 1-axis (see Fig. 17) of the case \(A_6\)

14. \(b_1 - b_4 = b_3 - b_1 = b_3 - b_2 = b_2 - b_4 = 1\)

The dimension of all tensor indices is one in this case. Therefore, the tensor has only one component:

\[
T_{1,1,1,1}^{(b_1,b_2,b_3,b_4)} = -\frac{1}{2} I_{b_1,b_2,b_3,b_4}.
\]

15. \(b_1 - b_4 = b_3 - b_2 = -1\) and \(b_3 - b_1 = b_2 - b_4 = 1\)

This case is given by a \(\frac{\pi}{2}\) rotation of the case \(A_{14}\)

16. \(b_1 - b_4 = b_3 - b_1 = b_3 - b_2 = b_2 - b_4 = -1\)

This case is given by a \(\pi\) rotation of the case \(A_{14}\)

17. \(b_1 - b_4 = b_3 - b_2 = 1\) and \(b_3 - b_1 = b_2 - b_4 = -1\)

This case is given by a \(-\frac{\pi}{2}\) rotation of the case \(A_{14}\)
The dimension of all tensor indices is one in this case. Therefore, the tensor has only one component:

$$T^{(b_1,b_2,b_3,b_4)}_{1,1,1,1} = I_{b_1,b_2,b_3,b_4},$$

This case is given by a $\frac{\pi}{2}$ rotation of the case $A_8$.

Appendix B: Rotation and reflection symmetries

Discrete rotation and reflection symmetries in the lattice action bring some constraints on each tensor in the decorated tensor network of Eq. [28]. First, we discuss the discrete rotation symmetry. Under a $\frac{\pi}{2}$ rotation, the Grassmann variables $\psi_n$, $\chi_n$ and the decoration labels $\{b_i\}$ transform as

$$ \begin{pmatrix} \psi_{n,1} \\ \psi_{n,2} \end{pmatrix} \rightarrow \pm \begin{pmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \begin{pmatrix} \psi_{n,1} \\ \psi_{n,2} \end{pmatrix}, \quad \chi_{n,1} \rightarrow -\chi_{n,2}, \quad \chi_{n,2} \rightarrow \chi_{n,1} + \chi_{n,1} \psi_{n,1},$$

$$ (b_1, b_2, b_3, b_4) \rightarrow (b_4, b_3, b_1, b_2),$$

where a couple of signs for the Grassmann variables stems from the fact that they transform according to a two-valued representation of the rotation group. Then, we get the following constraint by imposing the $\frac{\pi}{2}$-rotation symmetry on each tensor:

$$T^{(b_4,b_3,b_1,b_2)}_{i,j,k,l} = (-1)^{R(b_4-b_1)+R(b_2-b_3)+b_2+b_4} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}. \quad (B4)$$

Similarly, other constraints from the $\pi$- and $-\frac{\pi}{2}$-rotation symmetry are obtained:

$$T^{(b_2,b_1,b_3,b_4)}_{i,j,k,l} = (-1)^{R(b_2-b_1)+R(b_3-b_1)+R(b_2-b_3)+R(b_2-b_4)} \times (-1)^{b_2+b_3} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}, \quad (B5)$$

$$T^{(b_3,b_4,b_2,b_1)}_{i,j,k,l} = (-1)^{R(b_1-b_3)+R(b_4-b_2)+b_1+b_4} \times (-1)^{b_3+b_4} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}. \quad (B6)$$

Next, let us turn to the reflection symmetry. Under a reflection with respect to the 1-axis (see Fig. [17]), namely space inversion, $\psi_n$, $\chi_n$ and $\{b_i\}$ transform as

$$ \begin{pmatrix} \psi_{n,1} \\ \psi_{n,2} \end{pmatrix} \rightarrow \gamma_1 \begin{pmatrix} \psi_{n,1} \\ -\psi_{n,2} \end{pmatrix}, \quad \chi_{n,1} \rightarrow \chi_{n,2}, \quad (b_1, b_2, b_3, b_4) \rightarrow (-b_1, -b_3, -b_2, -b_4). \quad (B7)$$

Symmetry under these transformations brings

$$T^{(-b_4,-b_3,-b_2,-b_1)}_{i,j,k} = (-1)^{R(b_1-b_3)+R(b_2-b_3)} \times (-1)^{b_1+b_3} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l} \times (-1)^{p(i)+p(j)+p(k)+p(l)} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}. \quad (B10)$$

Transformation under a reflection with respect to the 2-axis (see Fig. [17]), namely time inversion is analogous:

$$ \begin{pmatrix} \psi_{n,1} \\ \psi_{n,2} \end{pmatrix} \rightarrow \gamma_2 \begin{pmatrix} \psi_{n,1} \\ \psi_{n,2} \end{pmatrix}, \quad \chi_{n,1} \rightarrow -\chi_{n,2}, \quad (b_1, b_2, b_3, b_4) \rightarrow (-b_3, -b_1, -b_1, -b_2), \quad (B11)$$

and a derived constraint is

$$T^{(-b_3,-b_1,-b_1,-b_2)}_{i,j,i,l} = (-1)^{R(b_1-b_3)+R(b_4-b_2)} \times (-1)^{b_1+b_3} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l} \times (-1)^{p(i)+p(j)+p(k)+p(l)} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}. \quad (B12)$$

We consider reflections with respect to the 1'- and 2'-axes also which are defined in Fig. [17]. That with respect to the 1'-axis is interpreted as a $\frac{\pi}{2}$ rotation followed by a reflection with respect to the 1-axis. The other is a $\frac{\pi}{2}$ rotation followed by a reflection with respect to the 1-axis. Thus, the following constraints are obtained:

$$T^{(-b_1,-b_2,-b_4,-b_3)}_{j,i,l,k} = (-1)^{b_1+b_3} T^{(b_1,b_2+b_3)}_{i,j,k,l} \times (-1)^{p(i)+p(j)+p(k)+p(l)} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}, \quad (B15)$$

$$T^{(-b_2,-b_1,-b_3,-b_4)}_{i,j,k,l} = (-1)^{b_1+b_3} T^{(b_1+b_2)}_{i,j,k,l} \times (-1)^{p(i)+p(j)+p(k)+p(l)} T^{(b_1,b_2,b_3,b_4)}_{i,j,k,l}. \quad (B16)$$

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FIG. 17. Definition of the 1-, 2-, 1′- and 2′-axes.

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