SIMULTANEOUS RUIN PROBABILITY FOR TWO-DIMENSIONAL BROWNIAN AND LÉVY RISK MODELS

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Abstract: The ruin probability in the classical Brownian risk model can be explicitly calculated for both finite and infinite-time horizon. This is not the case for the simultaneous ruin probability in two-dimensional Brownian risk model. Resorting on asymptotic theory, we derive in this contribution approximations of both simultaneous ruin probability and simultaneous ruin time for the two-dimensional Brownian risk model when the initial capital increases to infinity. Given the interest in proportional reinsurance, we consider in some details the case where the correlation is 1. This model is tractable allowing for explicit formulas for the simultaneous ruin probability for linearly dependent spectrally positive Lévy processes. Examples include perturbed Brownian and gamma Lévy processes.

Key Words: two-dimensional Brownian risk model; Brownian motion; simultaneous ruin probability; simultaneous ruin time; ruin time approximation

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1. TWO-DIMENSIONAL BROWNIAN RISK MODEL

The classical Brownian risk model (BRM) of an insurance portfolio

\[ R_1(t) = u + c_1 t - \sigma_1 W_1(t), \quad t \geq 0, \]

with \( W_1 \) a standard Brownian motion, \( \sigma_1 > 0 \), the initial capital \( u > 0 \) and the premium rate \( c_1 > 0 \), is a key benchmark model in risk theory; see e.g., [1].

The ruin probability in the time horizon \([0, T]\) for some finite positive \( T \) is given by (see e.g., [2])

\[
\tilde{\psi}_T(u) := \mathbb{P} \left\{ \inf_{t \in [0, T]} R_1(t) < 0 \right\} = \mathbb{P} \left\{ \sup_{t \in [0, T]} (\sigma_1 W_1(t) - c_1 t) > u \right\} = \Phi \left( \frac{-u}{\sigma_1 \sqrt{T}} + \frac{c_1 \sqrt{T}}{\sigma_1} \right) + e^{-2c_1 u/\sigma_1} \Phi \left( \frac{-u}{\sigma_1 \sqrt{T}} - \frac{c_1 \sqrt{T}}{\sigma_1} \right)
\]

for any \( u \geq 0 \), with \( \Phi \) the distribution function of an \( N(0,1) \) random variable.

In the infinite-time horizon, i.e., for \( T = \infty \), the corresponding ruin probability for this risk model is

\[
\tilde{\psi}_\infty(u) := \mathbb{P} \left\{ \inf_{t \geq 0} R_1(t) < 0 \right\} = e^{-2c_1 u/\sigma_1^2}.
\]

Since in practice an insurance company runs multiple portfolios simultaneously, it is of interest to calculate the simultaneous ruin probability for the classical benchmark BRM. For notational simplicity, we shall consider only the two dimensional setup, where for the second portfolio we consider the risk process

\[ R_2(t) = v + c_2 t - \sigma_2 W_2(t), \quad t \geq 0, \]
with \( W_2 \) another standard Brownian motion, \( v \) the initial capital and \( c_2 > 0 \). Hereafter \((W_1(t), W_2(t)), t \geq 0 \) are assumed to be jointly Gaussian with the same law as

\[
(B_1(t), \rho B_1(t) + \rho^* B_2(t)), \quad t \geq 0, \quad \rho^* = \sqrt{1 - \rho^2}, \quad \rho \in (-1, 1],
\]

where \( B_1, B_2 \) are two independent standard Brownian motions. Thus the correlation between \( W_1(t) \) and \( W_2(t) \) is \( \rho \) for \( t > 0 \). The special case \( \rho = 1 \) will be discussed separately in Section 3. In this bivariate risk model, tractable expressions for the simultaneous ruin probability are not available for both finite and infinity-time horizon. Here we are concerned with the study of the ruin probability in finite-time, which from practical point of view is more natural.

In the 2-dimensional BRM the probability of simultaneous ruin of both portfolios in the time period \([0,T]\) is given by

\[
P\{\exists t \in [0,T] : R_1(t) < 0, R_2(t) < 0\} = P\{\exists t \in [0,T] : \sigma_1 W_1(t) - c_1 t > u, \sigma_2 W_2(t) - c_2 t > v\},
\]

which by self-similarity (time-scaling property) of Brownian motion reduces to

\[
P\left\{ \exists t \in [0,1] : W_1(t) - \frac{c_1 \sqrt{T}}{\sigma_1} t > \frac{u}{\sigma_1 \sqrt{T}}, W_2(t) - \frac{c_2 \sqrt{T}}{\sigma_2} t > \frac{v}{\sigma_2 \sqrt{T}} \right\}.
\]

Consequently, in order to simplify the presentation, we shall consider in the following \( T = 1, \sigma_1 = \sigma_2 = 1 \) and define for any \( u, v \) non-negative the simultaneous ruin probability as

\[
\psi(u,v) := P\{\exists t \in [0,1] : W_1(t) - c_1 t > u, W_2(t) - c_2 t > v\}.
\]

The main findings of this contribution concern the approximation of

\[
\psi(u,au) = P\{\exists t \in [0,1] : W_1(t) - c_1 t > u, W_2(t) - c_2 t > au\}
\]
as \( u \to \infty \), for any given constant \( a \in (-\infty, 1] \). Note that there is no restriction to consider only \( a \leq 1 \) and in our model it is possible to deal also with \( a, c_1, c_2 \) being negative. This reflects the fact that depending on the correlations between two portfolios, the need for initial capital \( u \) and \( v \) can be different.

Clearly, the simplest possible model is when \( W_1 \) and \( W_2 \) are independent. Even in this model, it is not possible to calculate \( \psi(u,au) \) explicitly. Since for independent Gaussian processes the main tools of asymptotic theory of those processes are still available, the asymptotic behaviour of \( \psi(u,au) \) can be established by modifying the classical approach (i.e., using Gordon inequality, see [4, Prop. 3.6], instead of the well-known Slepian inequality, see e.g., [3, 4, 5]).

If \( W_1, W_2 \) are jointly Gaussian and dependent, then the calculation of the simultaneous ruin probability is much more difficult to deal with, since there is no substitute for Gordon inequality and the current methodology cannot cover the approximation of extremes of vector-valued dependent risk processes, see also discussion in Section 2. In order to understand the asymptotic behaviour of the simultaneous ruin probability as the initial capital \( u \) tends to infinity, we present next a sharp bounds for \( \psi(u,v) \), which also give some insights on the asymptotic approximation of the simultaneous ruin probability when \( u \) tends to infinity.

First, observe that for any \( u, c_1, c_2 \) we have a simple upper bound

\[
\psi(u,au) \leq \min \left( P\left\{ \sup_{t \in [0,1]} (W_1(t) - c_1 t) > u \right\}, P\left\{ \sup_{t \in [0,1]} (W_2(t) - c_2 t) > au \right\} \right) =: g(u,au).
\]
In view of (1) the upper bound $g(u, au)$ can be calculated explicitly. However, if $a \in (\rho, 1]$ this upper bound is too rough as the next result shows. Throughout in the following $I(\cdot)$ is the indicator function and $\rho^* = \sqrt{1 - \rho^2} \in [0, 1]$. Further $\Psi = 1 - \Phi$ with $\Phi$ the standard normal distribution on $\mathbb{R}$.

**Proposition 1.1.** For $c_1, c_2 \in \mathbb{R}$, $(u, v) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$ and $\rho \in (-1, 1]$ we have

$$
(4) \mathbb{P}\{W_1(1) > u + c_1, W_2(1) > v + c_2\} \leq \psi(u, v) \leq \frac{\mathbb{P}\{W_1(1) > u + c_1, W_2(1) > v + c_2\}}{\mathbb{P}\{W_1(1) > \max(c_1, 0), W_2(1) > \max(c_2, 0)\}}.
$$

The main result of this contribution, given in Theorem 2.1 below, shows that a precise asymptotic approximation of $\psi(u, au)$, as $u \to \infty$, can be obtained by using more advanced techniques. Theorem 2.1 presents interesting insight on the simultaneous probability of ruin given the correlation $\rho$ that governs the risk processes $R_1$ and $R_2$. For this case, we have that if the proportion of initial capitals between first and second risk process is larger than the correlation function, that is $\rho \geq a$, then the ruin probability is much smaller.

Related results for the infinite-time horizon are obtained in [6–12]. The first three papers consider the case that $\rho = 1$. In [9] the case $\rho \in (-1, 1)$ is dealt with and [11] extends [7, 8] to the $d$-dimensional setup of non-degenerated risk processes of Sparre-Andersen type.

The asymptotic behaviour of the ruin probability in finite-time horizon, when $u \to \infty$, compared with the results of [9], is completely different. In particular, the leading term in the asymptotics for the finite-time horizon is $e^{-q_{a, \rho}u^2/2}$ with

$$
(5) q_{a, \rho} = \frac{1 - 2a\rho + a^2}{1 - \rho^2} - \mathbb{I}(a > \rho) + \mathbb{I}(a \leq \rho).
$$

Note that if $a \in (\rho, 1)$, then $q_{a, \rho} > 1$. In the infinite-time horizon, the leading term in the asymptotic of simultaneous ruin probability equals $e^{-c_{a, \rho}u}$ for some positive $c_{a, \rho}$; see [4, 12].

In the literature two-dimensional risk models are mainly concerned with heavy-tailed setup, see e.g., [13–16] and the references therein. The light-tailed assumption is different; see [17] for some explanations and the difficulties in the light-tailed settings.

Brief organisation of the rest of the paper. In the next section we give short discussions of our results including the case $\rho = 1$ and the approximation of the conditional ruin time. All the proofs are displayed in Section 4.

2. Main result

Let $\varphi_{\rho}$ stands for the joint probability density function (pdf) of $(W_1(1), W_2(1))$ and $\sim$ means asymptotic equivalence of two functions when the argument $u$ tends to infinity. For $a \in (\rho, 1]$ let the constant $C_{a, \rho} \in (0, \infty)$ be given by

$$
(6) C_{a, \rho} = \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists t \geq 0 : \begin{array}{c} W_1(t) - t > x \\ W_2(t) - at > y \end{array} \right\} e^{\lambda_1 x + \lambda_2 y} \, dx \, dy,
$$

where

$$
(7) \lambda_1 = \frac{1 - a\rho}{1 - \rho^2}, \quad \lambda_2 = \frac{a - \rho}{1 - \rho^2}
$$

are both positive.
Theorem 2.1. Let $c_1, c_2$ be two given constants and let $\rho \in (-1, 1)$.

i) If $a \in (\rho, 1]$, then as $u \to \infty$

\begin{equation}
\psi(u, au) \sim C_{a, \rho} u^{-2} \varphi_{\rho}(u + c_1, au + c_2).
\end{equation}

ii) If $a \leq \rho$, then we have as $u \to \infty$

\begin{equation}
\psi(u, au) \sim 2\sqrt{2\pi(1 - \rho^2)} \Phi^*(c_1 \rho - c_2) e^{\frac{(c_2 - c_1)^2}{2(1 - \rho^2)}} u^{-1} \varphi_{\rho}(u + c_1, \rho u + c_2),
\end{equation}

where $\Phi^*(c_1 \rho - c_2) = 1$ if $a < \rho$ and $\Phi^*$ is the df of $\sqrt{1 - \rho^2} W_1(1)$ when $a = \rho$.

iii) Note that $C_{a, \rho}$ is not a type of Pickands constant in these setting, see [17] for the multivariate version of those constants and [13, 14, 19].

\begin{remark}
\begin{enumerate}
\item In view of [20] Theorem 4.1 and Theorem 2.1 we have that for $a \in (\rho, 1]$

\begin{equation}
\psi(u, au) \sim C_{a, \rho} \lambda_1 \lambda_2 \mathbb{P}\left\{W_1(1) > u + c_1, W_2(1) > au + c_2\right\}, \quad u \to \infty,
\end{equation}

with $\lambda_1, \lambda_2$ defined in [7], whereas if $a \leq \rho$, then

\begin{equation}
\psi(u, au) \sim 2 \mathbb{P}\left\{W_1(1) > u + c_1, W_2(1) > au + c_2\right\} \sim 2 \Phi^*(c_1 \rho - c_2) \mathbb{P}\left\{W_1(1) > u + c_1\right\}, \quad u \to \infty.
\end{equation}

Moreover, combination of Theorem 2.1 with Proposition 1.1 gives the following upper bound

\begin{equation}
C_{a, \rho} \leq \frac{1}{\lambda_1 \lambda_2} \mathbb{P}\left\{W_1(1) > \max(0, c_1), W_2(1) > \max(0, c_2)\right\}.
\end{equation}

\item From the above results, for any $a \in (\rho, 1]$ and $b \leq \rho$ we have

\begin{equation}
\lim_{u \to \infty} \frac{\psi(u, au)}{\psi(u, bu)} = 0.
\end{equation}

In particular, if $\rho = 0$, the above holds for any $a \in (0, 1], b \leq 0$.
\end{enumerate}
\end{remark}

Theorem 2.1 enables us to analyze the simultaneous ruin time $\tau_{\text{sim}}(u)$ on $[0, 1]$ defined by

\begin{equation}
\tau_{\text{sim}}(u) = \inf\{t \in [0, 1] : W_1(t) - c_1 t > u, W_2(t) - c_2 t > au\}.
\end{equation}

Our result below shows that $u^2(1 - \tau_{\text{sim}}(u))$ conditioned that $\tau_{\text{sim}}(u) \leq 1$, converges as $u \to \infty$, to an exponentially distributed random variable.

Theorem 2.2. If $a \leq 1, \rho \in (-1, 1)$ and $x \geq 0$, then with $q_{a, \rho}$ defined in [5] we have

\begin{equation}
\lim_{u \to \infty} \mathbb{P}\left\{u^2(1 - \tau_{\text{sim}}(u)) \leq x | \tau_{\text{sim}}(u) \leq 1\right\} = 1 - \exp\left(-q_{a, \rho} x / 2\right).
\end{equation}

Note that if $a > \rho$, then $q_{a, \rho} > 1$ and $q_{a, \rho} = 1$ for $a \leq \rho$.

3. Proportional portfolios with one-sided Lévy risk processes

In this section we consider the case when the insurance companies share the same portfolio of claims, with some proportion $r_1, r_2 > 0$, respectively and the portfolio is modeled by a Lévy process. This is typical for proportional reinsurance treaties. We refer to, e.g., [6, 8, 21] for the analysis of this model for infinite-time ruin problem in Brownian and Lévy setup. Following recent results of Michna [22], we shall derive exact distribution of the corresponding ruin probability for the claim process modeled by a
spectrally one-sided Lévy process $Z$ with absolutely continuous one-dimensional distributions. Since for positive $r_1, r_2$

$$
P \{ \exists t \in [0, T] : r_1 Z(t) - c_1 t > x, r_2 Z(t) - c_2 t > y \} = P \left\{ \exists t \in [0, T] : Z(t) - \frac{c_1}{r_1} t > \frac{x}{r_1}, Z(t) - \frac{c_2}{r_2} t > \frac{y}{r_2} \right\},$$

in the rest of this section, with no loss of generality, we suppose that $r_1 = r_2 = 1$. Thus the aim of this section is to obtain exact (non-asymptotic) expressions for the simultaneous ruin probability on finite time horizon $[0, T]$ defined by

$$\psi_Z(x, y) = P \{ \exists t \in [0, T] : Z(t) - c_1 t > x, Z(t) - c_2 t > y \}.$$ 

Below we exclude the degenerated scenario $c_1 = c_2$ and by the symmetry of the considered problem we assume that $c_1 > c_2$. Utilising the findings in Michna [22] we shall derive an explicit formula for $\psi_Z(x, y)$ both for spectrally positive and spectrally negative $Z$, which is the main result of this section.

Suppose first that $Z$ is spectrally positive. For $T, u$ positive and arbitrary constant $c$ set

$$\mathcal{L}(c, T, u) := P \{ Z(T) - cT > u \} - \int_0^T \mathbb{E} \left\{ \min(0, Z(T-s) - c(T-s)) \right\} f(u + cs, s) \, ds,$$

where $f(u, t)$ is the density function of $Z(t)$. We note that in the light of [22], for $u \geq 0$,

$$P \left\{ \sup_{t \in [0, T]} (Z(t) - ct) > u \right\} = \mathcal{L}(c, T, u).$$

**Theorem 3.1.** Let $Z$ be a spectrally positive Lévy process with cádlág sample paths and $P \{ Z(0) = 0 \} = 1$. Suppose that $Z(t), t > 0$ has density function $f(u, t)$ and let $c_1, c_2$ be two given constants such that $\delta := c_1 - c_2 > 0$.

i) If $x \geq y \geq 0$, then

$$\psi_Z(x, y) = \mathcal{L}(c_1, T, x).$$

ii) If $0 \leq x < y < x + \delta T$, then setting $\xi = (y - x)/\delta$ we have

$$\psi_Z(x, y) = \mathcal{L}(c_2, \xi, y) + \int_0^\infty \mathcal{L}(c_1, T - \xi, z) f(y + c_2 \xi - z, \xi) \, dz$$

$$- \int_0^\infty z \mathcal{L}(c_1, T - \xi, z) \, dz \int_0^\xi \frac{f(y + c_2 s, s)}{\xi - s} f(c_2(\xi - s) - z, \xi - s) \, ds.$$

iii) If $y \geq x + \delta T$ and $x \geq 0$, then (11) holds substituting $c_1, x$ by $c_2, y$, respectively.

Next, let us suppose that the Lévy process $Z$ is spectrally negative. In view of [22] [Thm 5] we obtain the following result.

**Theorem 3.2.** Let $Z$ be a spectrally negative Lévy process with cádlág sample paths and $P \{ Z(0) = 0 \} = 1$. Suppose that $Z(t), t > 0$ has density function $p(u, t)$ and let $c_1, c_2$ be two given constants such that $\delta := c_1 - c_2 > 0$.

i) If $x \geq y \geq 0$, then

$$\psi_Z(x, y) = x \int_0^T \frac{p(x + c_1 s, s)}{s} \, ds.$$

ii) If $0 \leq x < y < x + \delta T$, then setting $\xi = (y - x)/\delta$ we have

$$\psi_Z(x, y) = y \int_0^\xi \frac{p(y + c_2 s, s)}{s} \, ds + \int_0^\infty z p(-z + y + c_2 \xi, \xi) \, dz \int_0^{T-\xi} \frac{p(z + c_1 s, s)}{s} \, ds$$

$$- \int_0^\infty z \mathcal{L}(c_1, T - \xi, z) \, dz \int_0^\xi \frac{p(y + c_2 s, s)}{\xi - s} f(c_2(\xi - s) - z, \xi - s) \, ds.$$
- \int_0^\infty zdz \int_0^{T-\xi} \frac{p(z+c_1s,s)}{s} ds
\cdot \int_0^\xi \frac{p(-z+c_2t,t)}{\xi-s} p(u+c_2(\xi-t),\xi-t) dt
\]

iii) If \( y \geq x + \delta T \) and \( x \geq 0 \), then (11) holds substituting \( c_1, x \) by \( c_2, y \), respectively.

In the rest of this section we apply Theorem 3.1 to important Lévy risk models.

Example 3.3. If \( Z(t), t \geq 0 \) is a standard Brownian motion, then Theorem 3.1 is satisfied with \( f(u,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \) and
\[
\mathcal{L}(c,T,u) = \Phi(-uT^{-1/2} - c\sqrt{T}) + e^{-2uc\Phi(-uT^{-1/2} + c\sqrt{T})}.
\]

Example 3.4. Let \( Z \) be a gamma Lévy process with parameter \( \lambda > 0 \) where the density function of \( Z(t), t > 0 \) is given by
\[
f(u,t) = \frac{\lambda^t}{\Gamma(t)} u^{t-1} e^{-\lambda u} u \geq 0.
\]
Then Theorem 3.1 holds with
\[
\mathcal{L}(c,T,u) = \frac{\lambda^T}{\Gamma(T)} \int_u^{\infty} z^{T-1} e^{-\lambda z} dz
+ \lambda^T e^{-\lambda u} \int_0^T ds \int_0^{c(T-s)} \frac{(u+cs)^{-\lambda s} e^{-c\lambda s}}{\Gamma(s)\Gamma(T-s+1)} (c(T-s)-z) z^{-\lambda s} e^{-\lambda z} dz
\]
for \( c, T, u \) positive.

Example 3.5. Suppose that \( Z = Z_{\alpha,1,1} \) is an \( \alpha \)-stable Lévy process with \( 1 < \alpha < 2 \), \( \beta = 1 \) (i.e., skewed to the right) and scale parameter \( \sigma = 1 \); see, e.g., Samorodnitsky and Taqqu [23]. Then
\[
f(u,t) = \frac{1}{\pi t^{1/\alpha}} \int_0^\infty e^{-\alpha^\alpha} \cos \left( ut^{\alpha-1}/\alpha - x^\alpha \tan \frac{\pi \alpha}{2} \right) dx
\]
and Theorem 3.1 is satisfied with
\[
\mathcal{L}(c,T,u) = \frac{1}{\pi t^{1/\alpha}} \int_0^\infty dz \int_0^\infty e^{-\alpha^\alpha} \cos \left( (z+cT)x^{\alpha-1}/\alpha - x^\alpha \tan \frac{\pi \alpha}{2} \right) dx
- \frac{1}{\pi} \int_0^T E\left\{ \min(0, Z_{\alpha,1,1}(T-s) - c(T-s)) \right\} ds \int_0^\infty e^{-\alpha^\alpha} \cos \left( (u+cs)x^{\alpha-1}/\alpha - x^\alpha \tan \frac{\pi \alpha}{2} \right) dx
\]
for \( T > 0, c \in \mathbb{R} \) and \( u > 0 \), where
\[
E\left\{ \min(0, Z_{\alpha,1,1}(T-s) - c(T-s)) \right\} = \frac{1}{\pi s^{1/\alpha}} \int_0^\infty dz \int_0^\infty e^{-\alpha^\alpha} \cos \left( (z+cs)x^{\alpha-1}/\alpha - x^\alpha \tan \frac{\pi \alpha}{2} \right) dx.
\]

Example 3.6. Consider gamma Lévy risk process perturbed by Brownian motion, i.e. suppose that \( Z(t) = Z_1(t) + \sigma Z_2(t) \), where \( Z_1(t), t \geq 0 \) is a gamma Lévy process, as defined in Example 3.4. \( Z_2(t), t \geq 0 \) is a standard Brownian motion independent of \( Z_1 \) and \( \sigma > 0 \). Then Theorem 3.1 holds with
\[
f(u,t) = \frac{\lambda^t}{\Gamma(t)} e^{-\frac{(u+y)^2}{2\sigma^2}} y^{t-1} dy
\]
and
\[
\mathcal{L}(c,T,u) = \frac{\lambda^T}{\Gamma(T)\sigma \sqrt{2\pi T}} \int_u^\infty dz \int_0^\infty e^{-\frac{(z+cT-y)^2}{2\sigma^2}} y^{T-1} dy
- \frac{1}{\sigma \sqrt{2\pi}} \int_0^T E\left\{ \min(0, Z(T-s) - c(T-s)) \right\} ds \int_0^\infty e^{-\frac{(u+c(T-s)+y)^2}{2\sigma^2}} y^{s-1} dy
\]
for $T > 0$, $c \in \mathbb{R}$ and $u > 0$, where

$$
\mathbb{E} \{ \min(0, Z(s) - cs) \} = \frac{\lambda^s}{\Gamma(s) \sigma \sqrt{2\pi s}} \int_{-\infty}^{0} zdz \int_{0}^{\infty} e^{-\frac{(s+cz-y)^2}{2\sigma^2s}} \lambda y^{-1} dy .
$$

4. Proofs

First recall that in our notation $B_1, B_2$ are two independent standard Brownian motions and $(W_1, W_2)$ has law given by (3) for some $\rho \in (-1, 1)$. In order to shorten the notation, in the following we set $W_i^*(t) = W_i(t) - c_i t, i = 1, 2$, with $c_1, c_2$ two given constants (not necessarily positive). We shall write $\Psi_\rho$ for the tail distribution function of $(W_1(1), W_2(1))$ and $\varphi_\rho$ for its pdf.

4.1. Proof of Proposition 1.1. The proof of the lower bound is immediate. For the proof of the upper bound we follow the same idea as in the proof of [14][Thm 1.1]. We shall use the standard notation for vectors which are denoted in bold. Let $W(t) = (W_1(t), W_2(t))$, $c = (c_1, c_2)$. For $u = (u_1, u_2) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$ define next $B_u := \{(x, y) \in \mathbb{R}^2 : (x = u_1 \wedge y \geq u_2) \vee (x \geq u_1 \wedge y = u_2)\}$, the boundary of the set $\{(x, y) \in \mathbb{R}^2 : x \geq u_1, y \geq u_2\}$, and

$$
\tau(u) := \inf\{t \in [0, 1] : W_1(t) - c_1 t \geq u_1, W_2(t) - c_2 t \geq u_2\}.
$$

Observe that

$$
\mathbb{P} \{W_1(1) - c_1 \geq u_1, W_2(1) - c_2 \geq u_2\} \\
= \int_{0}^{1} \mathbb{P} \{\tau(u) \in dt\} \int_{B_u} \mathbb{P} \{W(t) - ct \in dx | \tau(u) = t\} \mathbb{P} \{W(1-t) - c(1-t) \geq u - x\},
$$

where we used the Strong Markov property and the fact that $W(1) - W(t)$ has the same law as $W(1-t)$ for any $t \in [0, 1]$. Since for $x_1 \geq u_1, x_2 \geq u_2$ and any $t \in [0, 1]$ we have

$$
\mathbb{P} \{W(1-t) - c(1-t) \geq u - x\} \geq \mathbb{P} \{W(1-t) \geq \tilde{c}(1-t)\} \\
= \mathbb{P} \{W(1) \geq \tilde{c}\sqrt{1-t}\} \\
\geq \mathbb{P} \{W(1) \geq \tilde{c}\},
$$

where $\tilde{c} = (\max(c_1, 0), \max(c_2, 0))$, the proof is complete.

4.2. Proof of Theorem 2.1. Let in the following $\delta(u, T) = 1 - Tu^{-2}$, for $T, u > 0$. Before proceeding to the proof of Theorem 2.1 we present two lemmas: Lemma 4.1 that provides a sharp upper bound for

$$
m(u, T) := \mathbb{P} \{\exists t \in [0, \delta(u, T)] : W_1(t) - c_1 t > u, W_2(t) - c_2 t > au\}
$$

and Lemma 4.2 which gives precise asymptotics of

$$
M(u, T) := \mathbb{P} \{\exists t \in [\delta(u, T), 1] : W_1(t) - c_1 t > u, W_2(t) - c_2 t > au\}
$$
as $u \to \infty$.

Lemma 4.1. For any $T > 0$, $a \in (-\infty, 1]$ and sufficiently large $u$

$$
m(u, T) \leq e^{-T/s} \frac{\mathbb{P} \{W_1(1) \geq u + c_1, W_2(1) \geq au + c_2\}}{\mathbb{P} \{W_1(1) > \max(c_1, 0), W_2(1) > \max(c_2, 0)\}}.
$$
Proof of Lemma 4.1. For notation simplicity, we suppress the argument $u$ writing only $\delta(T)$ instead of $\delta(u, T)$ in the following. By the self-similarity of Brownian motion, combined with Proposition 4.1. for any $u > 0$

$$m(u, T) = \mathbb{P}\{\exists t \in [0, 1]: W_1(t) - c_1 \delta^{1/2}(T)t > \delta^{-1/2}(T)u, W_2(t) - c_2 \delta^{1/2}(T)t > \delta^{-1/2}(T)au\} \leq \frac{\mathbb{P}\{W_1(\delta(T)) \geq u + c_1 \delta(T), W_2(\delta(T)) \geq au + c_2 \delta(T)\}}{\mathbb{P}\{W_1(1) > \max(c_1, 0), W_2(1) > \max(c_2, 0)\}}.$$

Since, for sufficiently large $u$ (set below $\nu = \delta^{-1/2}(T), \overline{\nu} = \delta^{-1/2}(T/2)$ and recall that both $\nu$ and $\overline{\nu}$ depend on $u$)

$$\nu u + c_1/\nu \geq \overline{\nu}(u + c_1), \quad \nu au + c_2/\nu \geq \overline{\nu}(au + c_2),$$

then we have

$$\mathbb{P}\{W_1(1) \geq \nu u + c_1/\nu, W_2(1) \geq \nu au + c_2/\nu\} \leq \mathbb{P}\{W_1(1) \geq \overline{\nu}(u + c_1), W_2(1) \geq \overline{\nu}(au + c_2)\} = \overline{\nu}^2 \int_{u+c_1}^{\infty} \int_{au+c_2}^{\infty} \varphi_{\rho}(\overline{\nu}x, \overline{\nu}y) \, dx \, dy$$

for sufficiently large $u$. Taking into account that

$$\varphi_{\rho}(\overline{\nu}x, \overline{\nu}y) = \frac{1}{2\pi(1 - \rho^2)} \exp\left(-\frac{\overline{\nu}^2}{2(1 - \rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

$$\leq \frac{1}{2\pi(1 - \rho^2)} \exp\left(-\frac{1 + Tu^{-2}/2}{2(1 - \rho^2)}((\rho x - y)^2 + (1 - \rho^2)x^2)\right)$$

$$\leq \varphi_{\rho}(x, y) \exp\left(-\frac{Tu^{-2}}{4}x^2\right)$$

we get, for sufficiently large $u$ that

$$\overline{\nu}^2 \int_{u+c_1}^{\infty} \int_{au+c_2}^{\infty} \varphi_{\rho}(\overline{\nu}x, \overline{\nu}y) \, dx \, dy \leq \overline{\nu}^2 \int_{u+c_1}^{\infty} \int_{au+c_2}^{\infty} \varphi_{\rho}(x, y) \exp\left(-\frac{Tu^{-2}}{4}x^2\right) \, dx \, dy$$

$$\leq \exp\left(-\frac{T}{8}\right) \mathbb{P}\{W_1(1) \geq u + c_1, W_2(1) \geq au + c_2\}.$$

This completes the proof. □

Lemma 4.2. i) For any $a \in (\rho, 1]$ and any $T > 0$ we have

$$(13) \quad M(u, T) \sim u^{-2} \varphi_{\rho}(u + c_1, au + c_2)I(T), \quad u \to \infty,$$

where

$$I(T) := \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists t \in [0, T]: \begin{array}{l} W_1(t) - t > x \vspace{1mm} \quad W_2(t) - at > y \end{array}\right\} e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \in (0, \infty).$$

ii) For any $a \leq \rho, T > 0$ with $\rho \in (-1, 1)$ we have

$$M(u, T) \sim u^{-1} \varphi_{\rho}(u + c_1, \rho u + c_2)I(T), \quad u \to \infty,$$

where

$$I(T) := \int_{\mathbb{R}^2} \mathbb{P}\left\{\sup_{t \in [0, T]} (W_1(t) - t) > x\right\} \left[I(a < \rho) + I(y < 0, a = \rho)\right] e^{-\frac{y^2 - 2\rho(c_2 + c_1\rho)}{2(1 - \rho^2)}} \, dx \, dy.$$
Proof of Lemma 4.2 For any $x, y$ and
$$u_x = u + c_1 - x/u, \quad u_y = au + c_2 - y/u$$
we have (recall that the pdf of $(W_1(1), W_2(1))$ is denoted $\varphi_p$)
$$\varphi_p(u_x, u_y) =: \varphi_p(u + c_1, au + c_2)\psi_u(x, y) \sim \varphi_p(u + c_1, au + c_2)e^{\lambda_1 x + \lambda_2 y}, \quad u \to \infty,$$
where $\lambda_i$’s are given in (7). Set below
$$u_{x,y} := u_y - pu_x = (a - p)u - (y - px)/u + c_2 - pc_1$$
and let $B_1, B_2$ be two independent standard Brownian motions.
For any $u > 0$ set further $\bar{t}_u = 1 - t/u^2$ and
$$\bar{x}_u = 1 - x/u^2, \quad x_u = x/u^2, \quad A(u) = u^{-2}\varphi_p(u + c_1, au + c_2).$$
i) We have (recall $W_i^*(t) = W_i(t) - c_i t$)
$$M(u, T) = u^{-2}\int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists \in [u, T], 1 : W_1^*(t) > u, W_2^*(t) > au \mid W_1(1) = u_x, W_2(1) = u_y \right\} \varphi_p(u_x, u_y) dx dy$$
$$= A(u) \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists \in [0, T] : B_1(\bar{t}_u) - c_1 \bar{t}_u > u, \quad \rho B_1(\bar{t}_u) + \rho^* B_2(\bar{t}_u) - c_2 \bar{t}_u > au \mid B_1(1) = u_x, B_2(1) = u_{x,y} \right\} \psi_u(x, y) dx dy$$
$$= A(u) \int_{\mathbb{R}^2} h_u(T, x, y) \psi_u(x, y) dx dy,$$
where $\rho^* = \sqrt{1 - \rho^2}$. For notational simplicity we define further
$$B_{u,1}(t) + \bar{t}_u u_x := B_1(\bar{t}_u)(1 = u_x), \quad t \geq 0,$$
$$B_{u,2}(t) + \bar{t}_u u_{x,y}/\rho^* := B_2(\bar{t}_u)(\rho^* B_2(1) = u_{x,y}), \quad t \geq 0.$$
The following weak convergence holds for all $x \in \mathbb{R}$
$$u\left[B_{u,1}(t) + \bar{t}_u u_x - c_1 \bar{t}_u - u\right] \to B_1(t) - t - x, \quad t \in [0, T]$$
as $u \to \infty$. The above implies the weak convergence as $u \to \infty$
$$u\rho^* B_{u,2}(t) + u\left[\bar{t}_u u_{x,y} - (c_2 - pc_1) \bar{t}_u - u(a - p)\right] \to \rho^* B_2(t) - (a - p)t - (y - px), \quad t \in [0, T]$$
for any $x, y \in \mathbb{R}$. The following function
$$h(T, x, y) = \mathbb{P}\left\{ \sup_{t \in [0,T]} \min\left(B_1(t) - t - x, B_{12}(t) - y\right) > 0 \right\}$$
is non-increasing in both $x$ and $y$ and therefore it is continuous for $x, y \in \mathbb{R}$ almost everywhere where
$$B_{12}(t) = \rho [B_1(t) - t] + \rho^*[B_2(t) - t(a - p)/\rho^*].$$
Note that by the independence of $B_1$ and $B_2$ we have that $(B_1(t) - t, B_{12}(t))$, $t \geq 0$ has the same law as
$$(W_1(t) - t, W_2(t) - at), \quad t \geq 0$$
implicating that
$$h(T, x, y) = \mathbb{P}\left\{ \sup_{t \in [0,T]} \min\left(W_1(t) - t - x, W_{2}(t) - at - y\right) > 0 \right\} > 0.$$
For any \((x, y)\) continuity point of \(h(T, x, y)\), since \(B_1, B_2\) are independent it follows by continuous mapping theorem that

\[
\begin{align*}
    h_u(T, x, y) & := \mathbb{P} \left\{ \exists t \in [0, T] : \quad B_{u,1}(t) + \bar{t}_u u_x - c_1 \bar{t}_u > u \right. \\
    & \left. \quad \rho[B_{u,1}(t) + \bar{t}_u u_x] + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}] - c_2 \bar{t}_u > au \right\} \\
    & = \mathbb{P} \left\{ \sup_{t \in [0, T]} \min \left( u(B_{u,1}(t) + \bar{t}_u (u_x - c_1) - u), u(\rho[B_{u,1}(t) + \bar{t}_u u_x] \right. \\
    & \left. \quad + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}] - c_2 \bar{t}_u - au) > 0 \right) \right\} \\
    & \to h(T, x, y), \quad u \to \infty.
\end{align*}
\]

The above convergence holds for almost all \(x, y \in \mathbb{R}\), consequently using the dominated convergence theorem, we have

\[
M(u, T) = A(u) \int_{\mathbb{R}^2} h_u(T, x, y) \psi_u(x, y) \, dx \, dy
\]

\[
\sim A(u) \int_{\mathbb{R}^2} h(T, x, y) e^{\lambda_1 x + \lambda_2 y} \, dx \, dy, \quad u \to \infty.
\]

The application of the dominated convergence theorem can be justified as follows. First note that for all \(u\) large and some \(\varepsilon > 0\) we have

\[
\psi_u(x, y) \leq e^{\lambda_1 \varepsilon x + \lambda_2 \varepsilon y}, \quad x, y \in \mathbb{R},
\]

where \(\lambda_i \varepsilon(x) = \lambda_i + \text{sign}(x) \varepsilon\). Moreover, using that for sufficiently large \(u\) and \(s, t \in [0, T]\) we have

\[
u^2 \mathbb{E} \left\{ (B_{u,i}(s) - B_{u,i}(t))^2 \right\} \leq \text{Const}|t-s|\text{ for some Const} > 0\]

and the application of Piterbarg inequality (see, e.g., [Thm 8.1]) implies that for \(x, y \geq 0\) and some constant \(C_1\)

\[
h_u(T, x, y) = \mathbb{P} \left\{ \sup_{t \in [0, T]} \min \left( u(B_{u,1}(t) + \bar{t}_u (u_x - c_1) - u), u(\rho[B_{u,1}(t) + \bar{t}_u u_x] \right. \\
    & \left. \quad + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}] - c_2 \bar{t}_u - au) > 0 \right) \right\} \\
    \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} \left( u(B_{u,1}(t) + \bar{t}_u (u_x - c_1) - u) + u(\rho[B_{u,1}(t) + \bar{t}_u u_x] \right. \\
    & \left. \quad + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}] - c_2 \bar{t}_u - au) > 0 \right) \right\} \\
    \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} \left( uB_{u,1}(t) + u\rho B_{u,1}(t) + u\rho^* B_{u,2}(t) - (t(a + 1) + x + y - \varepsilon) \right) > 0 \right\} \\
    \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} \left( uB_{u,1}(t) + u\rho B_{u,1}(t) + u\rho^* B_{u,2}(t) \right) > x + y - C_1 \right\} \\
    \leq \bar{C} e^{-C(x+y)^2} \leq \bar{C} e^{-C(x^2+y^2)},
\]

for \(x \geq 0, y \leq 0\)

\[
h_u(T, x, y) \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} u(B_{u,1}(t) + \bar{t}_u (u_x - c_1) - u) > 0 \right\} \leq \bar{C} e^{-Cx^2}
\]

and for \(x \leq 0, y \geq 0\)

\[
h_u(T, x, y) \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} u(\rho[B_{u,1}(t) + \bar{t}_u u_x] + \rho^*[B_{u,2}(t) + \bar{t}_u u_{x,y}] - c_2 \bar{t}_u - au) > 0 \right\} \leq \bar{C} e^{-Cy^2}
\]
for some $C, \bar{C} > 0$. Hence we have
\[
\int_{x \geq 0, y \geq 0} h_u(T, x, y) \psi_u(x, y) \, dx \, dy \leq \bar{C} \int_{0}^{\infty} e^{(\lambda_1 + \varepsilon)x} e^{-Cx^2} \, dx \int_{0}^{\infty} e^{(\lambda_2 + \varepsilon)y} e^{-Cy^2} \, dy < \infty
\]
\[
\int_{x \geq 0, y \leq 0} h_u(T, x, y) \psi_u(x, y) \, dx \, dy \leq \bar{C} \int_{-\infty}^{0} e^{(\lambda_2 - \varepsilon)y} \, dy \int_{0}^{\infty} e^{(\lambda_1 + \varepsilon)x} e^{-Cx^2} \, dx < \infty
\]
\[
\int_{x \leq 0, y \geq 0} h_u(T, x, y) \psi_u(x, y) \, dx \, dy \leq \bar{C} \int_{-\infty}^{0} e^{(\lambda_1 - \varepsilon)x} \, dx \int_{0}^{\infty} e^{(\lambda_2 + \varepsilon)y} e^{-Cy^2} \, dy < \infty
\]
\[
\int_{x \leq 0, y \leq 0} h_u(T, x, y) \psi_u(x, y) \, dx \, dy \leq \int_{-\infty}^{0} e^{(\lambda_1 - \varepsilon)x} \, dx \int_{-\infty}^{0} e^{(\lambda_2 - \varepsilon)y} \, dy < \infty,
\]
which confirms the validity of the dominated convergence theorem.

ii) Next, when $a \leq \rho$ we shall apply a different transformation, namely
\[
u_x = u + c_1 - x/u, \quad \nu_y = \rho u + c_2 - y.
\]

With this notation we have
\[
u_{x,y} := \nu_y - \rho \nu_x = \rho x/u - y + c_2 - \rho c_1
\]
and
\[
\varphi_\rho(u_x, u_y) := \psi_u(x, y) e^{x - \frac{y^2}{2(1-\rho^2)}} \sim \varphi_\rho(u + c_1, \rho u + c_2) e^{x - \frac{y^2 - 2u(c_2 - c_1 \rho)}{2(1-\rho^2)}}, \quad u \to \infty.
\]

For any $x, y \in \mathbb{R}$ we have thus
\[
\lim_{u \to \infty} h_u(T, x, y) = \mathbb{P} \left\{ \sup_{t \in [0, T]} (W_1(t) - t) > x \right\} \left[ \mathbb{I}(a < \rho) + \mathbb{I}(y < 0, a = \rho) \right] + \mathbb{P} \left\{ \sup_{t \in [0, T]} \min(W_1(t) - t - x, W_2(t) - at) > 0 \right\} \mathbb{I}(y = 0, a = \rho)
\]
\[
= : h(T, x, y).
\]

Setting $A(u) := u^{-1} \varphi_\rho(u + c_1, \rho u + c_2)$ we have further
\[
M(u, T) =
\]
\[
u - 1 \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists t \in [\delta(u, T), 1]: W_1^*(t) > u, W_2^*(t) > au \mid W_1(1) = u_x, W_2(1) = u_y \right\} \varphi_\rho(u_x, u_y) \, dx \, dy
\]
\[
= u^{-1} \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists t \in [0, T]: \frac{B_1(t) - c_1 t_u}{\rho B_1(t) + \rho \sigma B_2(t) - c_2 t_u > u} \mid B_1(1) = u_x \right\} \psi_u(x, y) e^{x - \frac{y^2 - 2u(c_2 - c_1 \rho)}{2(1-\rho^2)}} \, dx \, dy
\]
\[
\sim A(u) \int_{\mathbb{R}^2} h(T, x, y) e^{x - \frac{y^2 - 2u(c_2 - c_1 \rho)}{2(1-\rho^2)}} \, dx \, dy, \quad u \to \infty.
\]

The application of the dominated convergence theorem is simpler in this case and is therefore omitted.

\[
\square
\]

**Proof of Theorem 2.3** Recall first that we define $\delta(u, T) = 1 - Tu^{-2}$. In view of Lemma 4.1 combined with Proposition 1.1 we immediately obtain that
\[
\lim_{T \to \infty} \lim_{u \to \infty} \frac{\mathbb{P} \left\{ \exists t \in [0, \delta(u, T)]: W_1^*(t) > u, W_2^*(t) > au \right\}}{\psi(u, au)} = 0.
\]

Hence, using that (recall $M(u, T) := \mathbb{P} \left\{ \exists t \in [\delta(u, T), 1]: W_1^*(t) > u, W_2^*(t) > au \right\}$)
\[
M(u, T) \leq \psi(u, au) \leq \mathbb{P} \left\{ \exists t \in [0, \delta(u, T)]: W_1^*(t) > u, W_2^*(t) > au \right\} + M(u, T)
\]
we obtain
\[ \lim_{T \to \infty} \lim_{u \to \infty} \frac{M(u,T)}{\psi(u,au)} = 1. \]
Consequently, in view of Lemma 4.2 it suffices to prove that
\[ \lim_{T \to \infty} I(T) \in (0, \infty), \]
where \( I(T) \) is defined in Lemma 4.2. We derive the above one considering separately \( a \in (\rho, 1] \) and \( a \leq \rho \).

i) If \( a \in (\rho, 1] \), then we have
\[
\begin{align*}
\lim_{T \to \infty} I(T) &= \lim_{T \to \infty} \int_{\mathbb{R}^2} h(T, x, y) e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \\
&= \int_{\mathbb{R}^2} \lim_{T \to \infty} h(T, x, y) e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \in (0, \infty),
\end{align*}
\]
where \( h \) is as in the proof of Lemma 4.2, \( \lambda_1, \lambda_2 \) are positive constants defined in (7) and
\[
\tilde{h}(x, y) := \lim_{T \to \infty} h(T, x, y).
\]
We have the following upper bound
\[
\begin{align*}
\int_{\mathbb{R}^2} \tilde{h}(x, y) e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \\
&\leq \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} h([i, i+1], x, y) e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \\
&= \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} P \left\{ \exists t \in [i, (i+1)] : W_1(t) - t > x \quad W_2(t) - at > y \right\} e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \\
&\leq \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} P \left\{ \sup_{t \in [i, (i+1)]} (W_1(t) - t) > x, \sup_{t \in [i, (i+1)]} (W_2(t) - at) > y \right\} e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \\
&= \frac{1}{\lambda_1 \lambda_2} \sum_{i=0}^{\infty} \mathbb{E} \left\{ e^{\lambda_1 M_i + \lambda_2 M_i^*} \right\}.
\end{align*}
\]
Using further the independence of increments of the Brownian motion, the following equality in distribution (abbreviated as \( \overset{d}{=} \)) holds
\[
(M_i, M_i^*) \overset{d}{=} \left( \sup_{t \in [i, i+1]} (W_1(t) - t), \sup_{t \in [i, i+1]} (W_2(t) - at) \right) = (Q_1, Q_2) + \left( V_1(i) - i, V_2(i) - ai \right),
\]
with \((V_1, V_2)\) an independent copy of \((W_1, W_2)\). By the definition of \( \lambda_1 \) and \( \lambda_2 \) we have \( \lambda_1 + \lambda_2 \rho = 1 \).
Consequently, since for \( \tilde{V}_2 \) an independent copy of \( V_1 \)
\[
\lambda_1 V_1(i) + \lambda_2 V_2(i) \overset{d}{=} (\lambda_1 + \lambda_2 \rho) V_1(i) + \lambda_2 \rho^* \tilde{V}_2(i) = V_1(i) + \lambda_2 \rho^* \tilde{V}_2(i)
\]
we obtain
\[
\ln \mathbb{E} \left\{ e^{\lambda_1 M_i + \lambda_2 M_i^*} \right\} - \ln \mathbb{E} \left\{ e^{\lambda_1 Q_1 + \lambda_2 Q_2} \right\} = \begin{cases} i, & a - \rho \geq 0 \\ \frac{(a - \rho)^2}{2(1 - \rho^2)} - \frac{1 - a \rho}{1 - \rho^2} i - \frac{a - \rho}{1 - \rho^2} ai \\ \frac{i}{2(1 - \rho^2)} [2 + 2a^2 - (1 - \rho^2) - (a - \rho)^2] \end{cases}
\]
where $\kappa = \frac{1 - 2a + a^2}{1 - \rho^2} > 1$. Consequently, by the monotone convergence theorem

$$\lim_{T \to \infty} \int_{\mathbb{R}^2} h(T, x, y)e^{\lambda_1 x + \lambda_2 y} \, dx \, dy = \int_{\mathbb{R}^2} \tilde{h}(x, y)e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \in (0, \infty)$$

implying the claim.

**ii)** Next suppose that $a \leq \rho$. Again by Lemma 4.2 the proof that $\lim_{T \to \infty} I(T) \in (0, \infty)$ follows if we show that

$$\int_{\mathbb{R}} e^x \mathbb{P} \left\{ \sup_{t \geq 0} (W_1(t) - t) > x \right\} \, dx < \infty.$$ 

In view of the fact that $\mathbb{P} \left\{ \sup_{t \geq 0} (W_1(t) - t) > x \right\} = e^{-2x}$ for $x \geq 0$ we have that

$$\int_{\mathbb{R}} e^x \mathbb{P} \left\{ \sup_{t \geq 0} (W_1(t) - t) > x \right\} \, dx = 2$$

implying that for $a < \rho$

$$\int_{\mathbb{R}^2} \tilde{h}(x, y)e^{-x^2 - 2y(c_2 - c_1 \rho)} \, dx \, dy = 2 \int_{\mathbb{R}} e^{-2y(c_2 - c_1 \rho)} \, dy$$

$$= 2\sqrt{2\pi(1 - \rho^2)} e^{\frac{(c_2 - c_1 \rho)^2}{2(1 - \rho^2)}}$$

and for $a = \rho$

$$\int_{\mathbb{R} \times (-\infty, 0)} \tilde{h}(x, y)e^{-\frac{y^2 - 2y(c_2 - c_1 \rho)}{2(1 - \rho^2)}} \, dx \, dy$$

$$= 2 \int_0^\infty e^{-\frac{y^2 - 2y(c_2 - c_1 \rho)}{2(1 - \rho^2)}} \, dy$$

$$= 2e^{\frac{(c_2 - c_1 \rho)^2}{2(1 - \rho^2)}} \int_0^\infty e^{-\frac{(y - (c_1 - c_2) \rho)}{2(1 - \rho^2)}} \, dy$$

$$= 2e^{\frac{(c_2 - c_1 \rho)^2}{2(1 - \rho^2)}} \sqrt{2\pi(1 - \rho^2)} \Phi \left( \frac{(c_1 - c_2) \rho}{\sqrt{2(1 - \rho^2)}} \right),$$

where $\Phi$ is the standard normal distribution, hence the proof follows easily. \qed

### 4.3. Proof of Theorem 3.1

The proof is based on the observation that each case i), ii), and iii) makes reduction of the original problem to a simpler one, which can be solved.

In the light of [25] and [26] for $x \geq 0$ we have

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} (Z(t) - ct) > x \right\} = \mathcal{L}(c, T, x),$$

where $\mathcal{L}(c, T, x)$ is defined in (10).

The proof of case i) follows now easily since $c_1 > c_2$ implies that

$$\psi_Z(x, y) = \mathbb{P} \left\{ \sup_{t \in [0, T]} (Z(t) - c_1 t) > x \right\}$$

for any $x \geq y \geq 0$. The case iii) follows with similar arguments, therefore we prove next only the remaining claim.
ii) If $0 \leq x < y < x + T(c_1 - c_2)$, then

$$
\psi_{Z}(x, y) = \mathbb{P} \left\{ \sup_{t \in [0, T]} (Z(t) - c(t)) > y \right\},
$$

with (set below $\delta = c_1 - c_2 > 0$)

$$
c(t) = \begin{cases} 
    c_2 t & \text{if } t \in \left[0, \frac{y-x}{\delta} \right] \\
    c_1 t + x - y & \text{if } t \in (\frac{y-x}{\delta}, T]. 
\end{cases}
$$

Hence, following [22][Thm 4] we have with $\xi = (y - x)/\delta \in (0, T)$

$$
\psi_{Z}(x, y) = \mathcal{L}(c_2, \xi, y) + \int_{0}^{\infty} \mathcal{L}(c_1, T - \xi, z) f(y + c_2 - z, T) \, dz \\
- \int_{0}^{\infty} z \mathcal{L}(c_1, T - \xi, z) \, dz \int_{0}^{\xi} \frac{f(y + c_2s, s)}{\xi - s} f(c_2(\xi - s) - z, \xi - s) \, ds
$$
establishing the proof.

□

4.4. **Proof of Theorem 3.2.** The proof of Theorem 3.2 is analogous to the proof of Theorem 3.1. □

4.5. **Proof of Theorem 2.3.** We focus only on the case $c_1, c_2 \geq 0$, since other scenarios follow by similar arguments. Using that (recall $W^{*}_i(t) = W_i(t) - c_i t$), for any $u > 0$ and $x > 0$

$$
\mathbb{P} \left\{ u^2(1 - \tau_{\text{sim}}(u)) > x | \tau_{\text{sim}}(u) \leq 1 \right\} = \frac{\mathbb{P} \left\{ \exists t \in [0, 1 - \frac{x}{u^2}] : W^{*}_1(t) > u, W^{*}_2(t) > au \right\}}{\mathbb{P} \left\{ \exists t \in [0, 1] : W^{*}_1(t) > u, W^{*}_2(t) > au \right\}}
$$
in conjunction with Theorem 2.1, we are left with finding the asymptotics of

$$
\mathbb{P} \left\{ \exists t \in [0, 1 - \frac{x}{u^2}] : W^{*}_1(t) > u, W^{*}_2(t) > au \right\}
$$
as $u \to \infty$. By self-similarity of $W_1, W_2$, we have (recall that we denote $\bar{x}_u = 1 - x/u^2$)

$$
\Pi(u, x) := \mathbb{P} \left\{ \exists t \in [0, \bar{x}_u] : W^{*}_1(t) > u, W^{*}_2(t) > au \right\} \\
= \mathbb{P} \left\{ \exists t \in [0, 1] : \sqrt{\bar{x}_u}W_1(t) - c_1 t \bar{x}_u > u, \sqrt{\bar{x}_u}W_2(t) - c_2 t \bar{x}_u > au \right\}.
$$
Consequently, for all $u, x$ positive

$$
\Pi(u, x) \geq \psi \left( \frac{u}{\sqrt{\bar{x}_u}}, \frac{au}{\sqrt{\bar{x}_u}} \right),
$$
and

$$
\Pi(u, x) \leq \psi \left( \frac{u}{\sqrt{\bar{x}_u}} - \frac{c_1 x}{u^2}, \frac{au}{\sqrt{\bar{x}_u}} - \frac{c_2 x}{u^2} \right).
$$
Hence the proof follows by a direct application of Theorem 2.1 to (17) and (18).

□

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