CHARACTER VALUES AND HOCHSCHILD HOMOLOGY

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Abstract. We present a conjecture (and a proof for $G = SL(2)$) generalizing a result of J. Arthur which expresses a character value of a cuspidal representation of a $p$-adic group as a weighted orbital integral of its matrix coefficient. It also generalizes a conjecture by the second author proved by Schneider-Stuhler and (independently) the first author. The latter statement expresses an elliptic character value as an orbital integral of a pseudo-matrix coefficient defined via the Chern character map taking value in zeroth Hochschild homology of the Hecke algebra. The present conjecture generalizes the construction of pseudo-matrix coefficient using compactly supported Hochschild homology, as well as a modification of the category of smooth representations, the so called compactified category of smooth $G$-modules. This newly defined ”compactified pseudo-matrix coefficient” lies in a certain space $K$ on which the weighted orbital integral is a conjugation invariant linear functional, our conjecture states that evaluating a weighted orbital integral on the compactified pseudo-matrix coefficient one recovers the corresponding character value of the representation.

We also discuss the properties of the averaging map from $K$ to the space of invariant distributions, partly building on works of Waldspurger and Beuzart-Plessis.

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1. Introduction

Let $G$ be a reductive group over a local nonArchimedean field $F$.

The goal of the article is to present an algebraic expression for a character of an admissible representation of $G$ on a compact element.
The statement is presented as a conjecture (see Conjecture 4.3) for a general reductive group, it is proved in the paper for $G = SL(2)$. We also describe a modification of the category $Sm = Sm(G)$ of finitely generated smooth representations, the so called compactified category of smooth $G$-modules, which plays a key role in our algebraic description of character values and may have an independent interest.

To describe the context for these constructions recall a conjecture of [14] proved in [19] and [7].

Let $\mathcal{H} = \cup_K \mathcal{H}_K$ be the Hecke algebra of locally constant compactly supported $\mathbb{C}$-valued measures. Thus $Sm(G)$ is identified with the category of finitely generated nondegenerate $\mathcal{H}$ modules [10].

Let $C(\mathcal{H}) = \mathcal{H}/[\mathcal{H}, \mathcal{H}] = \mathcal{H}_G = HH_0(\mathcal{H}) = HH_0(Sm)$ be the cocenter of $\mathcal{H}$. Here $\mathcal{H}_G$ denotes coinvariants with respect to the conjugation action, while $HH_*$ stands for Hochschild homology, and its second appearance refers to the notion of Hochschild homology of an abelian category.

Since $\mathcal{H}$ is Noetherian and has finite homological dimension, there is a well defined Chern character (also called the Hattori-Stallings or Dennis trace) map $ch : K^0(Sm) \to C(\mathcal{H})$ (we will abbreviate $ch([M])$ to $ch(M)$). It has been conjectured in [14] and proven in [19], [7] that for an elliptic regular semisimple element $g \in G$ and an admissible representation $\rho$ we have

$$\chi_\rho(g) = O_\rho(ch(\rho)), \tag{1}$$

where $O_\rho$ denotes the orbital integral. Here we use that $O_\rho : \mathcal{H} \to \mathbb{C}$ being conjugation invariant factors through $C(\mathcal{H})$.

If $\rho$ is a cuspidal irreducible representation then (assuming that $G$ has a compact center) a matrix coefficient $m_\rho \in \mathcal{H}$ is a representative of the class $ch(\rho) \in C(\mathcal{H})$. Thus in this case (1) reduces to an earlier result of Arthur [2]. However, the latter applies also to nonelliptic regular semisimple elements: for such an element $g$ and a cuspidal irreducible representation $\rho$ Arthur has proved that

$$\chi_\rho(g) = WO_\rho(m_\rho), \tag{2}$$

where $WO_\rho$ denotes the weighted orbital integral. Our Conjecture 4.3 provides a generalization of (1) to all regular semisimple compact elements $g$, which for a cuspidal representation $\rho$ reduces to (2).

The first step in this direction is a generalization of the map $ch : K^0(Sm) \to C(\mathcal{H})$. For our present purposes we need to modify both the source and the target of this map. We replace the target $C(\mathcal{H}) = \mathcal{H}_G$ by $K_G$ where $K \subset \mathcal{H}$ is a subspace invariant under the conjugation action of $G$, the so called space of "weightless" functions.\footnote{This adjective reflects the fact that weighted orbital integrals restricted to this space are independent of the choices involved in choosing the weight function on an orbit. This space has appeared in the literature (see [6], [24] and references therein) where it was called the space of strongly cuspidal function. We refrain from using this terminology since we use the term "cuspidal function" in the sense of [14] where it refers to a function acting by zero in any parabolically induced representation, thus in our terminology $K$ contains the space of cuspidal functions. The term "cuspidal function" was used in a different sense in [6], [24] etc., so that $K$ is contained in the set of cuspidal functions in the sense of loc. cit.}

Definition and some properties of $K$ are discussed in section 2. The key property is that $WO_\rho|_K$ is a $G$-invariant functional for any regular semisimple element $g \in G$; furthermore, there is a well defined averaging map $Av$ from $K$ to the space of invariant generalized functions on $G$ and for $\phi \in K$ the value of $WO_\rho(\phi)$ coincides with $Av(\phi)(g)$ (the latter is well defined since $Av(\phi)$ is in fact a locally constant function on the set of regular elements in $G$). We also provide a conjecture with a proof for $PGL(2, F)$, $char(F) = 0$ describing the image of the averaging map from $K$ to the space of invariant distributions.
To describe the source of the map generalizing $ch$ we need some new ingredients. One of them is the so called compactified category of smooth (finitely generated) representations $Sm$.

The abelian category $\overline{Sm}$ is defined in section 2. Recall that according to Bernstein [9], $Sm$ can be identified with the category of coherent sheaves of modules over a certain sheaf of algebras over a scheme which is an infinite union of affine algebraic varieties, the spectrum $\mathfrak{Z}$ of the Bernstein center of $G$. The category $\overline{Sm}$ can be described as the category of coherent sheaves of modules over a certain coherent sheaf of algebras over a (componentwise) compactification of $\mathfrak{Z}$. An admissible module $\rho$ can also be viewed as an object in $\overline{Sm}$, so we can apply the Chern character to the class of $\rho$ obtaining $\bar{ch}(\rho) \in HH_0(\overline{Sm})$.

We also need another invariant of the category $Sm$, namely the compactly supported Hochschild homology $HH_c(Sm)$ which is the derived global sections with compact support in the sense of [12] of localized Hochschild homology $R\text{Hom}_{\mathcal{H}\otimes \mathcal{H}^\text{op}}(\mathcal{H}, \mathcal{H})$.

We have natural maps $HH_c(Sm) \to HH_*(\overline{Sm}) \to HH_*(Sm)$; for an admissible module $\rho$ we have its compactly supported Chern character $\bar{ch}(\rho) \in HH_0(Sm)$, so that $ch(\rho)$ and $ch(\rho)$ equal the images of $ch(\rho)$ under the corresponding maps.

The first statement in the main conjecture (a theorem for $SL(2)$) provides a natural isomorphism

$$K_G^c \cong \text{Im}(HH^c_0(Sm) \to HH_0(\overline{Sm})),$$

where $K^c \subset K$ is the subspace of measures supported on compact elements. By the previous paragraph, $\bar{ch}(\rho)$ belongs to that image, thus we obtain a homological construction of an element in $K_G^c$ from an admissible representation, the so called "compactified pseudo-matrix coefficient" of the representation.

The second main statement (proved for $SL(2)$) asserts that for a compact regular element $g$ and an admissible representation $\rho$ we have $WO_\rho(\bar{ch}(\rho)) = \chi_\rho(g)$. Notice that for a noncompact regular element $g$ the value of $\chi_\rho(g)$ coincides with a character value of the Jacquet functor applied to $\rho$ [11], in particular it vanishes for a cuspidal module.

We view Conjecture 4.3 as an algebraic statement underlying some aspects of Arthur’s local trace formula [11], while equality 11 underlies the elliptic part of the local trace formula (see also Remark 3.16 below). We plan to develop this theme in a future publication.

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2. **Weightless functions and invariant distributions**

Let $\mathcal{H} = \mathcal{H}(G)$ be the Hecke algebra of compactly supported locally constant measures on $G$, the convolution product on $\mathcal{H}(G)$ will be denoted by $. For an open subsemigroup $S \subset G$ we let $\mathcal{H}(S) \subset \mathcal{H}(G)$ denote the subalgebra of measures supported on $S$. We denote by $D$ the space of generalized functions on $G$ (that is the space of linear functionals on $\mathcal{H}(G)$) and by $D^G \subset D$ the subspace of invariant generalized functions. Let $\mathcal{H}\text{cusp}$, $D^G_{\text{cusp}}$ be the cuspidal part of $\mathcal{H}$, $D^G$, i.e. $\mathcal{H}\text{cusp}$ consists of functions acting by zero in any parabolically induced representation and
\( \mathcal{D}_{\text{cusp}}^G \) consists of distributions vanishing on the orthogonal complement of \( \mathcal{H}_{\text{cusp}} \) (cf. footnote 1 above). Until the end of the section we assume for simplicity of notation that \( G \) has compact center.

Then averaging with respect to conjugations yields a well defined map \( H_0(G, \mathcal{H}_{\text{cusp}}) \to \mathcal{D}_{\text{cusp}}^G \) given by \( f \mapsto \int_X \frac{\,df}{\mathcal{H}} \) for a Haar measure \( \mathcal{H} \) on \( G \).

In this section we define a larger subspace \( K \) in \( \mathcal{H} \) on which the averaging map is still well defined and conjecture that the map \( \tau \) defines an embedding \( H_0(G, K) \to \mathcal{D}_G \). Moreover, we propose a conjectural description of the image of \( \tau \). We prove this conjecture in the case when \( G = \text{PGL}(2) \).

2.1. The conjecture. We start with some notation. Let \( \mathcal{O} \subset F \) be the ring of integers and \( \pi \) be a generator of the maximal ideal \( m \) of \( \mathcal{O} \). Let \( \text{val} : F^* \to \mathbb{Z} \) be the valuation such that \( \text{val}(\pi) = 1 \). We define \( \|x\| = q^{-\text{val}(x)}, x \in F^* \) where \( q = |\mathcal{O}/m| \).

For any smooth \( F \)-variety \( X \) we denote by \( S(X) \) the space of locally constant measures on \( X \) with compact support. In the case when \( X \) is a homogeneous \( G \)-variety with a \( G \)-invariant measure \( dx \) the map \( f \to f/\mathcal{H} \) identifies \( S(X) \) with the space of compactly supported locally constant functions on \( X \). In this case we will not distinguish between functions and measures on \( X \). Also, we will work with spaces \( Y_P = (G/U_P \times G/U_P)/L \) for a parabolic subgroup \( P = LU_P \subset G \). In this case \( S(Y_P) \) will denote the space of integral kernels of operators \( S(G/U_P) \to S(G/U_P) \), i.e. locally constant compactly supported sections of the \( G \)-equivariant locally constant sheaf \( pr_1^*(\mu) \), where \( pr_1 : Y_P \to G/P \) is the first projection.

In particular, we fix a Haar measure \( \mathcal{H} \) on \( G \) and identify the space \( \mathcal{S}(G) = \mathcal{H}(G) \) with the space of compactly supported locally constant functions on \( G \). Then \( \Delta \) is identified with the space of distributions, we also get the \( L^2 \) pairing \( (\ , \ , ) \) on \( \mathcal{H}(G) \).

Let \( \mathfrak{g} \) be the Lie algebra of \( G \). We will assume that \( \mathfrak{g} \) has a finite number of nilpotent conjugacy classes and that there exists a \( G \)-equivariant \( F \)-analytic bijection \( \phi \) between a neighbourhood of \( 0 \) in \( \mathfrak{g} \) and a neighbourhood of \( e \) in \( G \). We also assume that for a semisimple element \( s \in G \) the Lie algebra \( \mathfrak{g} \) of its centralizer \( Z_G(s) \) admits a \( Z_G(s) \)-invariant complement. These assumptions are well known to hold if \( \text{char}(F) = 0 \) or if \( \text{char}(F) > N \) for some \( N \) depending on the rank of \( G \), see [15 §1.8] for more precise information.

**Definition 2.1.**
- We denote by \( G_{Z_G(s),s} \) the space of germs of \( \text{Ad} \)-invariant distributions near \( 0 \) on \( \mathfrak{g} \) which are restrictions of linear combinations of the Fourier transforms of invariant measures on a nilpotent orbits. Using the bijection \( \phi \) we consider \( G_{Z_G(s),s} \) as a space of germs of \( \text{Ad} \)-invariant distributions on \( G \) near the identity.
- For a semisimple element \( s \in G \) we denote by \( G_{Z_G(s),s} \) the space of germs of distributions on \( Z_G(s) \) at \( s \) obtained from the space \( G_{Z_G(s),s} \) by the shift by \( s \).
- Let \( s \in G \) be a semisimple element, \( X_s = G/Z_G(s) \), \( r : G \to X_s \) be the natural projection and \( \gamma : X_s \to G \) be a continuous section. We denote by \( dz \) a \( G \)-invariant measure on \( X_s \).
- We denote by \( \hat{\kappa} : Z_G(s) \times X_s \to G \) the map given by \( (z, x) \to \gamma(x)sz\gamma(x)^{-1} \).

Let \( \mathfrak{z} \subset \mathfrak{g} \) be the Lie algebra of \( Z_G(s) \). By assumption there exists a complementary \( \mathfrak{z} \)-invariant subspace \( W \subset \mathfrak{g} \) and the map \( \kappa_0 : \mathfrak{z} \oplus W \to \mathfrak{g}_s \), \( (z, w) \to z + \text{Ad}(s)w \) is a bijection. Therefore there exists an open neighborhood \( R \subset Z_G(s) \) of \( e \) such that the restriction \( \kappa \) of \( \hat{\kappa} \) on \( R \times X_s \) is an open embedding.

**Definition 2.2.** For any \( \psi \in G_{Z_G(s),s} \) we choose a representative \( \hat{\psi} \in \mathcal{D}_{Z_G(s)}(Z_G(s)) \) of \( \psi \).
- For a function \( f \in \mathcal{S}(G) \) and \( z \in R \subset X_s \) we define \( f_z \in \mathcal{S}(Z_G(s)) \) by \( f_z := f(\kappa(z, x)) \).
We define a function $\tilde{f}$ on $R$ by $\tilde{f}(z) := \tilde{\psi}(f_z)$.

We define a distribution $\psi(\tilde{\psi})$ on $G$ by $\psi(f) := \int_{X_s} \tilde{f}(z)$.

We denote by $\langle \psi(\tilde{\psi}) \rangle$ the germ of the distribution $\psi$ at $s$.

It is clear that for any two choices $\tilde{\psi}$, $\tilde{\psi}'$ of representatives of $\tilde{\psi}$ the difference $\psi(\tilde{\psi}) - \psi(\tilde{\psi}')$ vanishes on a $G$-invariant open neighborhood of $s$. Therefore the germ $\langle \tilde{\psi} \rangle$ does not depend on a choice of a representative of $\tilde{\psi}$.

**Definition 2.3.**

- We denote by $G_s$ the space of germs at $s$ of $Ad$-invariant distributions of the form $[\psi]$, $\psi \in G_{Z_G(s),s}$.
- We denote by $\mathcal{F} \subset \mathcal{D}^G$ the subspace of distributions $\alpha$ such that
  a) there exists a compact subset $C$ in $G$ such that $\text{supp}(\alpha) \subset G(C)$;  
  b) for any semisimple $s \in G$ the germ of $\alpha$ at $s$ belongs to $G_s$.

**Remark 2.4.** If $\text{char}(F) = 0$ then $\mathcal{F}$ admits an equivalent description as the space of invariant distributions $\alpha$ satisfying the following requirements:

- a) there exists a compact subset $\mathcal{C}$ in $G$ such that $\text{supp}(\alpha) \subset G(\mathcal{C})$;
- b) there exists a compact open subgroup $K \subset G$ such that for every element $z$ in the Bernstein center satisfying $\delta_K * z = 0$ we have $\alpha * z = 0$.

Equivalence of the two definitions of $\mathcal{F}$ follows from [13, Theorem 16.2].

**Definition 2.5.**

- We define the space $\mathcal{K}(G)$ of *weightless* functions as the subspace in $S(G)$ of functions $f$ such that $\int_{u \in U_Q} f(lu)du = 0$, $l \in L$ for all proper parabolic subgroups $Q = LU_Q \subset G$.
- For a closed conjugation invariant subset $X$ of $G$ we define the space $\mathcal{K}(X) = K_X \subset S(X)$ as the subspace of functions $f$ such that
  $\int_{u \in U_Q} f(lu)du = 0,$
  for all proper parabolic subgroups $Q = LU_Q \subset G$ and $l \in L$ such that $lU_Q \subset X$.

**Remark 2.6.** For $f \in \mathcal{H}$ and a parabolic $P = LU_P \subset G$ let $\mathcal{A}_P(f) \in S(Y_P)$ denote its orishperic transform, i.e. the integral kernel of the action of $f$ on $S(G/U_P)$. Let $\Delta_{Y_P} \subset Y_P$ be the preimage of diagonal under the projection $Y_P \to (G/P)^2$. Then $\Delta_{Y_P} \cong (G/U_P \times L)/L$, where $L$ acts on the first factor by right translations and on the second one by conjugation. It is easy to see that for $f \in \mathcal{H}$ we have $f \in \mathcal{K}$ iff for any parabolic subgroup $P \subsetneq G$ we have $\mathcal{A}_P(f)|_{\Delta_{Y_P}} = 0$.

The following result is due to J.-L. Waldspurger (see [24, Lemma 9]).

**Proposition 2.7.** For $f \in \mathcal{H}(G)$ the following are equivalent:

a) $f \in \mathcal{K}$.

b) For any $h \in \mathcal{H}(G)$ the function $g \mapsto \langle f, h \rangle$ has compact support.

For any $f \in S(G)$ we define distribution $\tilde{f}$ by:

$$\langle \tilde{f}, h \rangle := \int_{g \in G} \langle f, h \rangle dg.$$

For future reference we mention the following.

**Lemma 2.8.** For $f \in \mathcal{K}$ the distribution $\tilde{f}|_{G^{rs}}$ (where $G^{rs}$ is the open set of regular semisimple elements) is a locally constant function. For $g \in G^{rs}$ we have

$$\tilde{f}(g) = WO_g(f),$$

where $WO_g$ denotes the *weighted orbital integral*.

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2We thank Raphaël Beuzart-Plessis for pointing this out to us.
Proof follows from the definition and basic properties of the weighted orbital integral, see e.g. [3] §I.11.

The group $G$ acts on $K$ by conjugation. It is clear that the map $f \rightarrow \tilde{f}$ factors through a map $\tau : H_0(G, K) \rightarrow D^G$. For any $f \in K$ we denote by $[f]$ its image in $H_0(G, K)$.

Conjecture 2.9. a) $\tilde{f} \in \mathcal{E}$ for $f \in K$.

b) The map $\tau$ defines an isomorphism between $H_0(G, K)$ and $\mathcal{E}$.

c) $\dim(H_0(G, K(\Omega))) = 1$ for any regular semisimple conjugacy class $\Omega \subset G$.

Remark 2.10. One can check that $\tilde{f}$ satisfies the conditions of Remark 2.4 thus if $\text{char}(F) = 0$ then part (a) of the conjecture follows from Harish-Chandra’s Theorem [13, Theorem 16.2]; see [23, Corollary 5.9], [6, Proposition 5.6.1] for details.

Remark 2.11. It is clear that part c) follows from a) and b).

Remark 2.12. If the centralizer of an element $g \in \Omega$ is an anisotropic (compact) torus then statement (c) clearly follows from uniqueness (up to scaling) of a Haar measure on $G$. In the case when that centralizer has split rank one the statement is checked in the next subsection.

2.2. Almost elliptic orbits. To simplify the wording we assume in this subsection that the center of $G$ is compact. A regular semisimple element $g \in G$ will be called almost elliptic if the split rank of its centralizer is at most one. We now prove Conjecture 2.9(c) in the case when $\Omega$ consists of almost elliptic elements.

Fix $g \in \Omega$ and let $T$ be the centralizer of $G$, thus $\Omega \cong G/T$. In view of Remark 2.12 it suffices to consider the case when the split rank of $T$ equals one; we also assume without loss of generality that $G$ is almost simple.

There are exactly two parabolic subgroups $P, P' \subset G$ containing $T$. Let $U, U'$ be their unipotent radicals.

Consider the complex

\begin{equation}
0 \rightarrow K_\Omega \rightarrow S(G/T) \rightarrow S(G/TU) \oplus S(G/TU') \rightarrow \mathbb{C} \rightarrow 0;
\end{equation}

here $S$ stands for the space of locally constant compactly supported measures as before, the third arrow send $\phi$ to $(pr_*(\phi), pr'_*(\phi))$, where $pr : G/T \rightarrow G/TU$, $pr' : G/T \rightarrow G/TU'$ are the projections and the fourth arrow sends $(\phi, \phi')$ to $\int \phi - \int \phi'$.

Lemma 2.13. The complex 3 is exact.

Proof. Exactness at all the terms except for $S(G/TU) \oplus S(G/TU')$ is clear. Suppose that $(\psi, \psi') : S(G/TU) \oplus S(G/TU') \rightarrow \mathbb{C}$ is a linear functional vanishing on the image of $S(G/T)$. Let $\tilde{\psi} : S(G) \rightarrow \mathbb{C}$ be the composition of the direct image map $S(G) \rightarrow S(G/TU)$ with $\psi$. Then $\tilde{\psi}$ is a right $TU$ invariant generalized function on $G$. On the other hand, $-\psi$ is equal to the composition of the direct image map $S(G) \rightarrow S(G/TU')$ with $\psi'$, which shows that $\tilde{\psi}$ is also right $TU'$ invariant. Since $G$ is assumed to be almost simple, $U$ and $U'$ together generate $G$, thus we see that $\tilde{\psi}$ is right $G$ invariant. It follows that $\tilde{\psi}$ is proportional to the functional $\phi \rightarrow \int \phi$, hence the functional $(\psi, \psi')$ factors through the differential in 3, which yields exactness of 3.

We can now finish the proof of Conjecture 2.9(c) in the present case. Breaking 3 into short exact sequences we get

\begin{align*}
0 & \rightarrow K_\Omega \rightarrow S(G/T) \rightarrow M \rightarrow 0, \\
0 & \rightarrow M \rightarrow S(G/TU) \oplus S(G/TU') \rightarrow \mathbb{C} \rightarrow 0.
\end{align*}

Considering the corresponding long exact sequences on homology we see that it suffices to check that the map $\mathbb{C} = H_0(G, S(G/T)) \rightarrow H_0(G, M)$ is nonzero while
the map $H_1(G, S(G/T)) \to H_1(G, M)$ has one-dimensional cokernel. The former statement is clear since the composition $H_0(G, S(G/T)) \to H_0(G, M) \to H_0(G, S(G/TU))$ is nonzero. To check the latter recall that $H_1(G, \mathbb{C}) = 0$ for $i > 0$ since the resolution of $\mathbb{C}$ provided by the simplicial complex for computation of homology of the Bruhat-Tits building $\mathcal{B}$ shows that $H_i(G, \mathbb{C}) \cong H_i(\mathcal{B}/G)$, while $\mathcal{B}/G$ is a product of simplices. Thus

$$H_1(G, M) \cong H_1(G, S(G/TU) \oplus S(G/TU')) \cong H_1(T, \mathbb{C}) \oplus H_1(T, \mathbb{C}).$$

Since $S(G/TU) \cong H_1(T, \mathbb{C})$ we see that

$$\text{CoKer} (H_1(G, S(G/T)) \to H_1(G, M)) \cong H_1(T, \mathbb{C}),$$

which is one-dimensional.

\[\Box\]

2.3. The case of $\text{PGL}(2)$. To simplify the argument we assume in this subsection that $\text{char}(F) = 0$.

**Theorem 2.14.** Conjecture 2.9 is true for $G = \text{PGL}(2)$.

The rest of the subsection is devoted to the proof of the Theorem.

**Claim 2.15.** We have $\mathcal{K}_{\text{cusp}} \subset \mathcal{K}$.

**Proposition 2.16.** The map $\tau : H_0(G, \mathcal{K}) \to D^G$ is an embedding.

**Proof.** We need more notation.

**Definition 2.17.**

- For $\epsilon > 0$ we define $G^\epsilon = \{g \in G | |p(g)| \leq \epsilon\}$, where $p(g) = \frac{\text{tr}(\tilde{g})}{\det(g)} - 4$; here $\tilde{g} \in GL(2, F)$ is a representative of $g$.
- We let $G_s, G_e, N, \tilde{N}$ denote, respectively, the sets of regular semisimple split, regular semisimple elliptic, regular unipotent and all unipotent elements.
- We set
  $$\mathcal{K}_u = \mathcal{K}(\tilde{N}) := \{f \in S(\tilde{N}) | \int_{u \in U} f(u)du = 0 \forall B = TU \subset G\},$$
  $$\mathcal{K}_0 := \{f \in \mathcal{K}_u | f(\epsilon) = 0\},$$
  where $B$ runs over the set of Borel subgroups in $G$.
- For $f \in \mathcal{K}$ we denote by $\kappa(f) \in \mathcal{K}_u$ the restriction of $f$ to $\tilde{N}$ and by $[\kappa(f)]$ the image of $\kappa(f)$ in $H_0(G, \mathcal{K}_u)$.

We start with the following geometric statement.

**Lemma 2.18.** Let $f \in S(G)$ be such that $f|_{\tilde{N}} \in \mathcal{K}_u$. Then there exists $\epsilon > 0$ such that $f|_{G^\epsilon} \in \mathcal{K}$.

**Proof.** Recall notations of Remark 2.6. We have $\Delta_{Y_B} = G/B \times T$, where $T$ is the (abstract) Cartan subgroup of $G$. It is easy to see that condition $f|_{\tilde{N}} \in \mathcal{K}_u$ is equivalent to vanishing of the restriction of $A_B(f)$ to $G/B \times \{1\} \subset \Delta_{Y_B}$. Also, condition $f|_{G^\epsilon} \in \mathcal{K}$ is equivalent to vanishing of $A_B(f)$ on $G/B \times T_\epsilon \subset \Delta_{Y_B}$, where $T_\epsilon = G_\epsilon \cap T$ (here we abuse notation by identifying the abstract Cartan subgroup $T$ with an arbitrarily chosen Cartan subgroup). Since $A_B(f)$ is locally constant for $f \in \mathcal{H}$, the statement follows from compactness of $G/B$. \[\Box\]

**Corollary 2.19.** For any $f \in \mathcal{K}$ such that $[\kappa(f)] = 0$ there exists $f' \in \mathcal{K}$ with the same image in $H_0(G, \mathcal{K}_u)$ and such that $f'|_{\tilde{N}} = 0$.

**Proof.** Since $[f|_{\tilde{N}}] = 0$ we can write the restriction of $f$ to $\tilde{N}$ as a finite sum $\sum_i (f_i - \bar{f}_i), \bar{f}_i \in \mathcal{K}_0^\epsilon, g_i \in G$. As follows from the Lemma 2.18 we can choose $f_i \in \mathcal{K}$ such that $\bar{f}_i = f_i|_{\tilde{N}}$. Then the function $f' := f - \sum (f_i^{\bar{g}_i} - f_i)$ satisfies the conditions of Corollary. \[\Box\]
Proposition 2.20. The space $H_0(G, K_u)$ is two dimensional.

Proof. We first show that $\dim(H_0(G, K_0)) = 1$.

Let $\mathcal{B}$ be the variety of Borel subgroups, $p : N \to \mathcal{B}$ the map which associates to $u \in N$ the Borel subgroup containing $u$. By definition we have an exact sequence

$$0 \to K_0 \to \mathcal{S}(N) \to \mathcal{S}(\mathcal{B}) \to 0$$

and therefore an exact sequence

$$H_1(G, \mathcal{S}(N)) \to H_1(G, \mathcal{S}(\mathcal{B})) \to H_0(G, K_0) \to H_0(G, \mathcal{S}(N)) \to H_0(G, \mathcal{S}(\mathcal{B})).$$

Lemma 2.21. $H_1(G, \mathcal{S}(N)) = 0$.

Proof. Fix a Borel subgroup $B = TU$. We can write $U$ as a union of open compact subgroups $U_1 \subset ... \subset U_n \subset ...$. Therefore $\mathcal{S}(N) = \mathcal{S}(G/U)$ is the direct limit of $\mathcal{S}(G/U_n)$. Since the functor $M \mapsto H_1(G, M)$ commutes with direct limits it is sufficient to show that $H_1(G, \mathcal{S}(G/U_n)) = \{0\}$. Since $U_n \subset G$ is a compact subgroup the space $H_1(G, \mathcal{S}(G/U_n))$ is a direct summand of $\mathcal{S}(G)$. Since $H_1(G, \mathcal{S}(G)) = 0$ the Lemma is proven.

Since $G$ acts transitively on $N$ and on $\mathcal{B}$ we have $H_0(G, \mathcal{S}(N)) \cong H_0(G, \mathcal{S}(\mathcal{B})) = \mathbb{C}$. Since $H_1(G, \mathcal{S}(N)) = 0$ we see that the map $H_1(G, \mathcal{S}(\mathcal{B})) \to H_0(G, K_0)$ is an isomorphism. Since $\mathcal{B} = G/B$ we have:

$$H_1(G, \mathcal{S}(\mathcal{B})) = H_1(B, \mathbb{C}) = H_1(T, \mathbb{C}) = \mathbb{C}.$$

So $\dim(H_0(G, K_0)) = 1$.

To conclude the argument, recall the short exact sequence $0 \to K_0 \to K_u \to \mathbb{C} \to 0$.

Since $H_1(G, \mathbb{C}) = 0$ we have an exact sequence:

$$0 \to H_0(G, K_0) \to H_0(G, K_u) \to \mathbb{C} \to 0$$

So $\dim(H_0(G, K_u)) = 2$. □

Let $r : K_{\text{cusp}} \to K_u$ be the restriction and $[r] : K_{\text{cusp}} \to H_0(G, K_u)$ be the composition of $r$ and projection $K_u \to H_0(G, K_u)$.

Lemma 2.22. The map $[r]$ is onto.

Proof. Recall the map $\tau : H_0(G, K) \to \mathcal{D}^G$, $[f] \mapsto \hat{f}$. Let $\mathcal{D}^G$ be the space of germs of invariant distributions at $e$ and $\tilde{\tau} : H_0(G, K) \to \mathcal{D}^G$ be the composition of $\tau$ with the restriction map.

Corollary 2.13 implies that the map $\tilde{\tau}$ vanishes on the kernel of the map $H_0(G, K) \to H_0(G, K_u)$. Thus it suffices to show that $\tilde{\tau}|_{K_{\text{cusp}}}$ has rank at least two, i.e. that there exist irreducible cuspidal representations $\rho_1$, $\rho_2$, such that their characters restricted to any $G$-invariant open neighborhood of identity are not proportional. This is easily done by inspecting the character tables, see e.g. [20] §2.6. □

Corollary 2.23. a) For any $f \in \mathcal{K}$ there exists $f_{\text{cusp}} \in K_{\text{cusp}}$ such that $[\kappa(f)] = [\kappa(f_{cusp})]$.

b) For any $f_0 \in K_u$ there exists $f \in K_{\text{cusp}}$ such that $[f_0] = [\kappa(f)]$. □

Let $s \in G$ be a regular split semisimple element and $\Omega \subset G$ be the conjugacy class of $s$.

Proposition 2.24. $\dim(H_0(G, K_{\Omega})) = 1$.

Proof. Let $T = Z_G(t)$ be the split torus and $B, B' \subset G$ be Borel subgroups containing $T$. Since $\Omega = G/T$ we have maps $r : \Omega \to G/B$ and $r' : \Omega \to G/B'$ and therefore morphisms $r_* : \mathcal{S}(\Omega) \to \mathcal{S}(G/B)$ and $r'_* : \mathcal{S}(\Omega) \to \mathcal{S}(G/B')$.

As a special case of Lemma 2.13 we get:
Lemma 2.25. The sequence
\[ 0 \to K_{\Omega} \to S(\Omega) \to S(G/B) \oplus S(G/B') \to C \to 0, \]
where the last map \( l \) is given by \((\nu, \nu') \mapsto \int \nu - \int \nu'\), is exact. \( \square \)

Let \( L := \ker(l) \). We have an exact sequence
\[ 0 \to L \to S(G/B) \oplus S(G/B') \to C \to 0. \]

Using that \( G \) has homological dimension one, we get that the corresponding long exact sequence of homology contains the following fragment:
\[ 0 \to H_1(G, L) \to H_1(B, C) \oplus H_1(B', C) \to H_1(G, C). \]
It is easy to see that
\[ \dim(H_1(B, C)) = \dim(H_1(B', C)) = \dim(H_1(T, C)) = 1 \]
Since the quotient \( G/[G, G] \) is finite we see that \( H_1(G, C) = 0 \). Therefore \( \dim(H_1(G, L)) = 2 \).

On the other hand we have an exact sequence
\[ 0 \to K_{\Omega} \to S(\Omega) \to L \to 0 \]
and therefore an exact sequence
\[ H_1(G, S(\Omega)) \to H_1(G, L) \to H_0(G, K_{\Omega}) \to H_0(G, S(\Omega)). \]
Since \( H_1(G, S(\Omega)) = H_1(T, C) \) we see \( \dim(H_1(G, S(\Omega))) = 1 \).

Lemma 2.26. The map \( a : H_1(G, S(\Omega)) \to H_1(G, L) \) is an embedding.

Proof. It is sufficient to show that the map \( H_1(G, S(\Omega)) \to H_1(G, S(G/B)) \) induced by the composition \( p_\ast \circ a : L \to S(G/B) \) is an embedding.

Since \( H_1(G, S(\Omega)) = H_1(T, C) \), \( H_1(G, S(G/B)) = H_1(B, C) \) and \( H_{>0}(U, C) = 0 \), we see that this map is an isomorphism. \( \square \)

We can now finish the proof of Proposition 2.24. Since \( G \) acts transitively on \( G/B \) the map \( \mu \to \int \mu \) defines an isomorphism \( H_0(G, S(G/B)) \to C \). On the other hand, since \( \int \nu = 0 \) for any \( \nu \in K_{\Omega} \) the map \( H_0(G, K_{\Omega}) \to H_0(G, S(\Omega)) \) equals zero. So \( \dim(H_0(G, K_{\Omega})) = 1 \). \( \square \)

Recall that \( G_s \subset G \) is the subset of regular split semisimple elements, let \( K_{\ast} \subset K \) be the subspace of functions in \( K \) supported on \( G_s \). We fix a Cartan subgroup \( T \), the Weyl group \( W = \mathbb{Z}/2\mathbb{Z} \) acts on \( T \) and on \( G/T \) in the usual way. Then the map
\[ (T - \{ e \}) \times G/T \to G_s, (s, g) \to gsg^{-1} \]
induces an isomorphism
\[ S(G_s) \to (S(T - \{ e \}) \otimes S(G/T))^W. \]

Corollary 2.27. a) For \( f \in K_{\ast} \) the distribution \( \hat{f} \) is locally constant on \( G_s \).

b) The map \( f \mapsto (t \mapsto \hat{f}(t)) \) induces an isomorphism
\[ H_0(G, K_{\ast}) \to S(T - \{ e \})^W. \]

Proof. The isomorphism \( (5) \) is clearly compatible with the averaging map \( f \mapsto \hat{f} \), which implies a). Likewise, restriction to an orbit is compatible with averaging, thus in view of \( (5) \) it suffices to show that for a fixed orbit \( \Omega \subset G_s \) the map \( t \mapsto \hat{f}(t) \), \( t \in \Omega \) induces an isomorphism \( H_0(G, K(\Omega)) \to C \). By Proposition 2.24 it suffices to see that this map is nonzero. This follows, for example, from the fact that the character of a cuspidal representation does not necessarily vanish on \( G_s \), while the character of an irreducible cuspidal representation is obtained by averaging from its matrix coefficient. \( \square \)
Now we can finish the proof of Proposition 2.16. Let \( f \in K \) be such that \( \hat{f} = 0 \). It follows from Corollary 2.19 and Lemma 2.18 that we can find \( f' \) with the same image in \( H_0(G, K) \) such that \( f' = f_s + f_e \) where \( f_s \) is supported on regular split semisimple elements and \( f_e \) on regular elliptic elements. The condition \( \hat{f} = 0 \) implies that \( \hat{f'} = 0 \), hence \( \hat{f}_s = 0 \) and \( \hat{f}_e = 0 \). It is easy to see that the condition \( \hat{f}_e = 0 \) implies that \( [f_e] = 0 \). So we may assume that the support of \( f \) is contained in the subset \( G_\alpha \subset \hat{G} \) of regular split semisimple elements. Now Proposition 2.16 follows from Corollary 2.27.

Let \( \hat{D}_e \) be the space of germs of distributions at \( e \) and \( D_e \subset \hat{D}_e \) be the subspaces spanned by germs of characters of irreducible representations.

**Lemma 2.28.** The space \( D_e \) is 2-dimensional. It is spanned by germs of characters of irreducible cuspidal representations.

**Proof.** The second statement is a special case of a theorem of Harish-Chandra [13]. The first one also follows from loc. cit., as it shown there that more generally the space \( D_e \) has a basis indexed by unipotent orbits.

Recall that \( E \subset D \) is the subspace of distributions \( \alpha \) satisfying the following three conditions:

a) There exists a compact subset \( C \) in \( G \) such that \( supp(\alpha) \subset C^G \).

b) The restriction of \( \alpha \) on \( G - \{ e \} \) is given by a locally constant function.

c) The germ of \( \alpha \) at \( e \) belongs to \( D_e \).

**Lemma 2.29.** \( \tau(K) \subset E \).

**Proof.** Fix \( f \in K \). It is clear that the distribution \( \hat{f} \) satisfies condition a).

To prove that \( \hat{f} \) satisfies condition b) we have to show that for any semisimple element \( s \in G - \{ e \} \) there exists an open neighborhood \( R \subset G \) of \( s \) such that the restriction \( \hat{f}|_R \) is a constant. If \( s \) is split then this follows from Corollary 2.27(a), if \( s \) is elliptic the proof is similar.

To prove that \( \hat{f} \) satisfies condition c) we observe that Corollary 2.23 implies existence of \( f_{cusp} \in K_{cusp} \) such that \( [\kappa(f)] = [\kappa(f_{cusp})] \).

It is easy to see that when \( f \) is a matrix coefficient of an irreducible cuspidal representation \( \rho \) then \( \hat{f} \) is proportional to the character of \( \rho \). Thus condition c) is satisfied by \( \alpha_{cusp} = f_{cusp} \). However, by Lemma 2.18 and Corollary 2.19 the germs of \( \alpha \) and \( \alpha_{cusp} \) at \( e \) coincide.

**Proposition 2.30.** \( \tau(K) = E \).

**Proof.** It remains to show that every \( \alpha \in E \) is in the image of \( \tau \). Lemma 2.28 shows that there exists \( \beta \in E \) which is a linear combination of characters of cuspidal representations such \( \alpha - \beta \) vanishes on an open neighborhood of \( e \). Thus we have \( \alpha - \beta = \alpha_s + \alpha_e \), where \( \alpha_s \) is supported on \( G_s \), while \( \alpha_e \) is supported on \( G_e \).

Now \( \alpha_s \) is in the image of \( \tau \) by Corollary 2.24 while \( \alpha_e \) is in the image of \( \tau \) by a similar argument. Also, \( \beta \) is in the image of \( \tau \) since the character of an irreducible cuspidal representation \( \rho \) equals \( \hat{f} \) where \( f \) is a matrix coefficient of \( \rho \).

Proposition 2.30 and therefore Theorem 2.14 are proven.

3. The compactified category of smooth modules

### 3.1. Definition of the compactified category

For a parabolic \( P = LU \) let \( L^0 \subset L \) be the subgroup generated by compact subgroups; thus \( L^0 \) is the kernel of the unramified characters of \( L \). Set \( \Lambda_P = L/L^0 \).

Let \( \Lambda_P \) be the group of \( F \)-rational characters of \( L \) and \( \Lambda_P^\vee \) be the subset of \( P \)-dominant weights, i.e. weights which are (non-strictly) dominant with respect to

Let \( \Lambda_P \) be the group of \( F \)-rational characters of \( L \) and \( \Lambda_P^\vee \) be the subset of \( P \)-dominant weights, i.e. weights which are (non-strictly) dominant with respect to
any (not necessarily $F$-rational) Borel subgroup $B \subset P$. We have a nondegenerate pairing between the lattices $\Lambda_P$ and $\Lambda_P$ given by:

$\langle xL^0, \lambda \rangle = val_P(\lambda(x))$.

Let $\Lambda^+_P$ be the subsemigroup defined by:

$\Lambda^+_P = \{ x \in \Lambda_P \mid \langle x, \lambda \rangle \geq 0 \ \forall \lambda \in \Lambda^+_P \}$,

and let $L^+_P \subset L$ be the preimage of $\Lambda^+_P$ under the projection $L \rightarrow \Lambda_P$.

For a pair of parabolics $P \supset Q$ let $L^+_P \subset L_Q$ denote the image of $L^+_P \cap Q$ in $L_Q = Q/U_Q$. It is easy to see that $L^+_Q \supset L^+_P$.

For an open submonoid $M \subset G$ we let $Sm(M)$ denote the category of nondegenerate finitely generated smooth $M$-modules; this is easily seen to be equivalent to the category of finitely generated smooth $M$-modules.

For parabolic subgroups $P \supset Q$ we have the "Jacquet" functor $J^+_Q : Sm(L^+_P) \rightarrow Sm(L^+_Q)$, $M \mapsto M_{U_Q}$, where $U_Q$ is image of $U_Q$ in $L_P = P/U_P$.

To simplify the wording in the following definition we fix a minimal parabolic $P_0$, then by a standard parabolic we mean a subgroup $P$ containing $P_0$.

**Definition 3.1.** The compactified category of smooth $G$-modules $\overline{Sm}^{nc} = \overline{Sm}^{nc}(G)$ is the category whose object is a collection $(M_P)$ indexed by standard parabolic subgroups $P = L_P U_P$, where $M_P$ is a smooth module over $L^+_P$, together with isomorphisms

\[
J^+_Q(M_P) \cong H(L^+_P) \otimes H(L^+_Q) M_Q
\]

fixed for every pair of standard parabolic subgroups $P = L_P U_P \supset Q = L_Q U_Q$; here $H$ denotes the algebra of locally constant compactly supported distributions. The isomorphisms are required to satisfy the associativity identity for each triple of parabolics $P_1 \supset P_2 \supset P_3$.

An object in the compactified category is called coherent if the module $M_P$ is finitely generated for all $P$.

We let $\overline{Sm} = \overline{Sm}(G) \subset \overline{Sm}^{nc}$ denote the full subcategory of coherent objects.

It is easy to see that $\overline{Sm}(G)$ is an abelian category, the functor sending $(M_P) \in \overline{Sm}$ to $M_G$ identifies $Sm(G)$ with a Serre quotient of $\overline{Sm}$.

We also have an adjoint functor $Sm(G) \rightarrow \overline{Sm}^{nc}$. This functor sends admissible modules but not general finitely generated modules to $\overline{Sm}$.

**Example 3.2.** Let $G = SL(2)$. In this case the category $\overline{Sm}$ admits the following more direct description. A component of the spectrum $\mathfrak{Z}$ of Bernstein center is in this case either a point or an affine curve, thus $\mathfrak{Z}$ admits a canonical (componentwise) compactification $\overline{\mathfrak{Z}}$. Notice that $\partial \mathfrak{Z} = \mathfrak{Z} \setminus \mathfrak{Z}$ is identified with the set $(O^*)^+$ of characters of $T_0 = O^x$. Let $T^+ = O \setminus \{0\} \subset F^x = T$, thus $T^+ \cong O^x \times \mathbb{Z}_{\geq 0}$. Set $\mathfrak{Z}^+ := Spec(H(T^+)) \cong (O^*)^+ \times A^1$. Notice that we have a natural map $\pi : \mathfrak{Z}^+ \rightarrow \mathfrak{Z}$ inducing an isomorphism $(O^*)^+ \times \{0\} \rightarrow \partial \mathfrak{Z}$. Moreover, $\pi$ is etale at $(O^*)^+ \times \{0\}$.

The full Hecke algebra $H$ defines a quasicoherent sheaf of algebras on $\mathfrak{Z}$, we now describe its extension to a quasicoherent sheaf of algebras $\overline{H}$ on $\overline{\mathfrak{Z}}$. The latter depends on the choice of a maximal open compact subgroup $K_0 = SL(2, O)$. Fixing this choice we set $H^+ = End_{T^+}(S(G/U^+))$, where $(G/U)^+ = O^2 \setminus \{0\} \subset F^2 \setminus \{0\} \subset G/U$. We also let $\hat{H} = End_T(S(G/U))$. It is clear that $\hat{H}$ defines a quasicoherent sheaf on $\mathfrak{Z}$ whose restriction to the open subset $\mathfrak{Z} := \mathfrak{Z} \setminus (O^*)^+ \times \{0\}$ is the quasicoherent sheaf defined by $\hat{H}$.

The action of $H$ on $S(G/U)$ defines a homomorphism $\pi^*(H) \rightarrow \hat{H}$ which is an isomorphism on a Zariski neighborhood of $\partial \mathfrak{Z}^+ = \mathfrak{Z}^+ \setminus \mathfrak{Z}$. Thus we get a well defined quasicoherent sheaf of algebras $\overline{H}$ on $\overline{\mathfrak{Z}}$ such that $\overline{H}|_\mathfrak{Z} = H$ and the induced
map $\pi^*(\mathcal{H})|_\mathfrak{g} \to \mathcal{H}$ extends to a map $\pi^*(\mathcal{H}) \to \mathcal{H}^+$ which is an isomorphism on a neighborhood of $\partial \mathfrak{g}$. It is clear that $K_0^0$ acts on $\mathcal{H}$ and for an open subgroup $K \subset K_0$ the subsheaf $\mathcal{H}_K$ of $K^2$ invariants is a coherent sheaf of algebras.

We leave it to the reader to show that although $\mathcal{H}$ depends on an auxiliary choice, different choices lead to algebras which are canonically Morita equivalent. Thus we can consider the category of sheaves of nondegenerate $\mathcal{H}_K$-modules which can be checked to be canonically equivalent to $Sm$. If the subgroup $K \subset K_0$ is nice in the sense of [9] then for every component $X$ of $\mathfrak{g}$ either the coherent sheaf of algebras $\mathcal{H}_K|_X$ is zero or the corresponding summand in $Sm$ (respectively, $Sm^c$) is canonically equivalent to the category of coherent (respectively, quasicoherent) sheaves of $\mathcal{H}_K|_X$-modules.

3.2. Compactified center and a spectral description of the compactified category. Let $Z = Z(G)$ be the Bernstein center of $G$ and $\mathfrak{g} = Spec(Z)$ be its spectrum. By the main result of [9] (the set of closed points of) $\mathfrak{g}$ is in bijection with the set $Cusp(G)$ of cuspidal data, i.e. the set of $G$-conjugacy classes of pairs $(L, \rho)$, where $L \subset G$ is a Levi subgroup and $\rho$ is a cuspidal irreducible representation of $L$.

3.2.1. Compactified center. Let $\mathfrak{z}$ denote its compactification described as follows. We have a canonical isomorphism $\mathfrak{g} = \mathfrak{g}/W$ where $W$ is the Weyl group, and $\mathfrak{z}$ parametrizes pairs $(L, \rho)$ where $L$ is a Levi subgroup containing a fixed maximally split Cartan $T$ and $\rho$ is a cuspidal representation of $L$. The complex torus $L^T = X(L)$ acts on the union $\mathfrak{z}_L$ of components corresponding to a given Levi subgroup $L \supset T$; here $X(L)$ stands for the group of unramified characters of $L$ acting on the set of representations by twisting. Notice that $L^T$ is a torus with $X^*(L^T) = L/L^0$. The action is transitive on each component and the stabilizer of each point is finite. The space $X^*(L^T)_R = X_*(Z(L))_R$ (where $X_*$ stands for the lattice of $F$-rational cocharacters) contains hyperplanes corresponding to the roots of $Z(L)$ in $\mathfrak{g}$; the fan formed by these hyperplanes defines an equivariant compactification $\mathfrak{z}^c_T$ of $L^T$. We set

$$\mathfrak{z} = \mathfrak{z}/W, \quad \mathfrak{z} = \bigcup_L L^T \times \mathfrak{z}_L,$$

where the right hand side makes sense because the action of $W$ on $\mathfrak{z}$ extends to the compactification, here we use the notation $X \times^H Y = (X \times Y)/H$. Notice that every component of $\mathfrak{z}_L$ is of the form $L^T/A$ for a finite subgroup $A \subset L^T$, thus the corresponding component of $\mathfrak{z}^c_T = \mathfrak{z}_L$ is identified with $\mathfrak{z}^c_T/A$.

For a parabolic $P = LU$ let $Z^0(L) \subset Z(L)$, $Z^+(L) \subset Z(L)$ be the subalgebras consisting of distributions supported on $L^0$ and $L^+_P$ respectively, set also $\mathfrak{z}^0(L) = Spec(Z^0(L))$, $\mathfrak{z}^+(L) = Spec(Z^+(L))$.

It is clear that

$$\mathfrak{z}^0(L) = \mathfrak{z}(L)/X(L),$$

where $X(L)$ is the group of unramified characters of $L$.

Proposition 3.3. a) $\mathfrak{z}$ admits a canonical stratification indexed by conjugacy classes of parabolic subgroups, where the stratum $\mathfrak{z}_P$ corresponding to the class of a parabolic $P$ is identified with $\mathfrak{z}^0(L)$.

b) The embedding $\mathfrak{z}^0(L) \to \mathfrak{z}$ canonically extends to a map $\mathfrak{z}^+(L) \to \mathfrak{z}$ which is etale on a Zariski neighborhood of $\mathfrak{z}_P \cong \mathfrak{z}^0(L)$.

Given two parabolics $P \subset Q$ we have a canonical map $c_P^Q : \mathfrak{z}^+(L_P) \to \mathfrak{z}^+(L_Q)$ which is compatible with maps to $\mathfrak{z}$.

Moreover, for three parabolics $P_1 \subset P_2 \subset P_3$ we have

$$c_{P_3}^{P_2} = c_{P_2}^{P_1}c_{P_1}^{P_3}.$$
Proof. Let \( \overline{3}_L = \overline{\mathcal{T}^L} \times \overline{\mathcal{T}} \overline{Z}_L \).

It is a standard fact that \( \overline{\mathcal{T}} \overline{T} \overline{Q} \) are in bijection with parabolic subgroups containing \( L \), so that the orbit \( \overline{\mathcal{T}} \overline{T} \overline{Q} \) corresponding to a parabolic \( Q = MU_Q \) is identified with \( \mathcal{X}(L)/\mathcal{X}(M) \). The stratification of \( \overline{\mathcal{T}} \overline{T} \) by \( \overline{\mathcal{T}} \overline{T} \) orbits induces a stratification on \( \overline{3}_L \), the stratum corresponding to a parabolic \( Q \) will be denoted by \( \overline{3}_L \{Q\} \).

Fix a conjugacy class \( P \) of parabolic subgroups and set \( \overline{3}_L(P) = \bigcup_{Q \in P} \overline{3}_L \{Q\} \).

Let \( \overline{3}_P = \bigcup_{L} \overline{3}_L(P) \).

It is clear that \( \overline{3}_P \) is a stratification of \( \overline{3} \) and each stratum is \( W \)-invariant. Thus \( \overline{3}_P := \overline{3}_P/W \) are strata of a stratification of \( \overline{3} \).

The map \( Q \mapsto \overline{3}_L \{Q\} \) is easily seen to be \( W \)-equivariant, it follows that for a parabolic \( P = LU \in P \) we have

\[
\overline{3}_P \cong \bigcup_{M,T \subseteq M \subseteq L} \overline{3}_M \{P\}/W_L.
\]

The above isomorphism \( \overline{\mathcal{T}} \overline{T} \overline{Q} \cong \mathcal{X}(L)/\mathcal{X}(M) \) shows that \( \overline{3}_M \{P\} \cong \overline{3}_M(L)/\mathcal{X}(L) \). Passing to the union over \( M \) and taking quotient by the action of \( W_L \) (which commutes with the action of \( \mathcal{X}(L) \)) we get \( \overline{3}_P \cong \overline{3}(L)/\mathcal{X}(L) \) which yields (a) in view of (7).

To check (b) observe that for parabolic subgroups \( Q = MU_Q \supset P = LU_P \supset T \) the cone \( \mathbb{R}^{\geq 0} \Lambda^+_Q \) belongs to the fan defining the toric variety \( \overline{\mathcal{T}} \overline{T} \). Let \( V_L \{Q\} \) be the corresponding affine open subset in \( \overline{\mathcal{T}} \overline{T} \) and \( V_L \{Q\} \) be the corresponding open affine in \( \overline{3}_L \). Thus \( V_L \{Q\} \) is a Zariski open neighborhood of \( \overline{3}_L \{Q\} \).

It is easy to see that \( V_L \{Q\} \) is \( W_M \) invariant and \( V_L \{Q\}/W_M \cong \overline{3}(L) = \overline{3}(L)/\mathcal{X}(L) \). Since \( \overline{3} = \overline{3}/W \), claim (b) follows from the fact that the stabilizer of any point \( x \in \overline{3} \{Q\} \) is contained in \( W_M \).

c) follows by inspection.

In order to relate \( \overline{3} \) to \( \overline{3} \) we will need the following general concept. Let \( X \) be an algebraic variety. By a **quasicoherent enrichment** of a category \( \mathcal{C} \) over \( X \) we will mean assigning to objects \( M, N \in \mathcal{C} \) an object \( \underline{\text{Hom}}(M, N) \in QCoh(X) \) together with an isomorphism \( \underline{\text{Hom}}(M, N) = \Gamma(\underline{\text{Hom}}(M, N)) \) and maps \( \underline{\text{Hom}}(M_1, M_2) \otimes_{\mathcal{O}_X} \underline{\text{Hom}}(M_2, M_3) \rightarrow \underline{\text{Hom}}(M_1, M_3) \) satisfying the associativity constraint and compatible with the composition of morphisms in \( \mathcal{C} \). If the quasicoherent sheaf \( \underline{\text{Hom}}(M, N) \) is actually coherent for all \( M, N \in \mathcal{C} \) we say that the enrichment is coherent.

**Proposition 3.4.** The category \( \overline{3} \) (respectively, \( \overline{3}^{qc}(G) \)) admits a natural lifts to a category coherent (respectively, quasicoherent) enrichment over \( \overline{3} \).

**Corollary 3.5.** The categories \( \overline{3}, \overline{3}^{qc} \) split as a direct sum indexed by components of \( \overline{3} \).

Before proceeding to prove the Proposition we state a general elementary Lemma.

**Lemma 3.6.** Let \( X = \bigsqcup X_i \) be a scheme with a fixed stratification (i.e. the closure of \( X_i \) coincides with \( \bigsqcup_{j \leq i} X_j \) for some partial order \( \leq \) on the set \( I \) of strata). Set \( U_i = \bigsqcup_{j \geq i} X_j \), this is an open subset of \( X \). Suppose that for each \( i \) we are given a map \( u_i : Y_i \rightarrow U_i \), such that

i) \( X_i \times_X Y_i \supseteq X_i \)

ii) \( u_i \) is etale over a Zariski neighborhood of \( X_i \).
iii) For $j \leq i$ set $Y_{ji} = Y_j \times_X U_i$. Then the map $Y_{ji} \to U_i$ factors through a map $u_{ji} : Y_j \to Y_i$. Moreover, for $k < j < i$ the map $Y_{ki} \to U_j \supset U_i$ factors through a map $u_{kji} : Y_{ki} \to Y_{ji}$.

Let $Y_d = \bigsqcup_{i_1 < \ldots < i_d} Y_{i_1i_2 \ldots i_d}$. Then the diagram $Y_d \ni Y_2 \ni Y_1 \to X$ satisfies descent for quasicoherent sheaves, i.e. $Q\text{Coh}(X)$ is equivalent to the category of quasicoherent sheaves on $Y_1$ with isomorphisms of the two pull-backs to $Y_2$ whose pull-backs to $Y_3$ satisfy the natural compatibility.

Remark 3.7. To fix ideas let us first prove the Lemma for sheaves in the analytic topology assuming we work over the base field $\mathbb{C}$. Then it is easy to see that we can find an open subset $Y^\alpha_i \subset Y_i$ for each $i$ so that $Y^\alpha_i$ maps isomorphically to a neighborhood of $X_i$ in $X$. Moreover, we can arrange it so that the images of $Y^\alpha_j$ and $Y^\alpha_i$ have a nonempty intersection only if $i < j$ or $j < i$. Replacing $Y_i$ by $Y^\alpha_i$ does not affect the category of gluing data; however, $(Y^\alpha_i)$ is just an open covering of $X$, so the claim is clear.

3.2.2. Proof of Lemma 3.6. Let $X_0$ be a closed stratum. Running an inductive argument, we can assume the theorem is known for the stratified space $X \setminus X_0$. Then we are reduced to proving the claim in the situation when the stratification consists of two strata $X = X_0 \bigsqcup X_1$. Replacing $X_0$ by its open subset containing $X_0$ clearly does not affect the category of descent data, so we can assume without loss of generality that $Y_1 \to X$ is etale. By a standard argument the claim reduces to exactness of the complex of sheaves on $X$:

$$0 \to \mathcal{O} \to (Y_0 \to X)_*(\mathcal{O}) \oplus (Y_1 \to X)_*(\mathcal{O}) \to (Y_{01} \to X)_*(\mathcal{O}) \to 0.$$ 

The complex is clearly exact over $X_1$, so it is enough to show that local cohomology of this complex with support on $X_1$ vanishes. This reduces to showing that $\alpha_1'(\mathcal{O}) \rightarrowtail \beta'(\mathcal{O})$, where $\beta : X_0 \to X$, $\alpha_1 : X_0 \to Y_0$. This follows from conditions (i), (ii).

3.2.3. Proof of Proposition 3.4. Proposition 3.3 implies that $X = X_3$ with the stratification of Proposition 3.3(a), and $Y_1 = \mathfrak{F}(L^+)_{\mathbb{C}^*}$ satisfy the conditions of Lemma 3.6. It is easy to see from the definition of $\mathfrak{sm}$ that the collection of quasi-coherent sheaves $\Hom(M_P, N_P)$ provides gluing data described in Lemma 3.6. Also, for $M, N \in \mathfrak{sm}$ the module $\Hom(M_P, N_P)$ is finitely generated, so the resulting quasicoherent sheaf is in fact coherent. 

3.3. The spectral description of $\mathfrak{sm}$. Recall that for every component $X \subset X$ the choice of a sufficiently small nice (in the sense of [9]) open compact subgroup $K \subset G$ defines a coherent sheaf of algebras $A = \mathcal{A}_X(K)$ on $X$ with an equivalence between $\mathcal{A}_X(K)$ modules and the corresponding summands in smooth $G$-modules.

Let $\mathfrak{B}$ be the (reductive) Bruhat-Tits building of $G$. We fix a special vertex $x \in \mathfrak{B}$ and let $K_x \subset G$ denote the corresponding maximal compact subgroup. We also fix a maximally split Cartan subgroup $T$, such that the corresponding apartment $A_T \subset \mathfrak{B}$ contains $x$. Thus $A_T$ is an affine space with underlying vector space $V = X_+T \otimes \mathbb{R}$. Let $V_+ \subset A_T$ be a Weyl cone with vertex at $x$ and $B$ the corresponding minimal parabolic subgroup.

It follows from the Iwasawa decomposition (see e.g. [22, §3.3.2]) that we have a natural bijection

$$\mathfrak{P}' \setminus G/K_x \cong \check{\mathcal{A}}_P,$$

where $\mathfrak{P}'$ denotes the commutator group. Let $G^+(x, P)$ be the union of cosets corresponding to elements in $\check{\mathcal{A}}_P \subset \check{\mathcal{A}}_P$.

Let $(U_P \setminus G)^+(x)$ be the image of $G^+(x, P)$ in $U_P \setminus G$ and $\mathcal{S}^+((U_P \setminus G)(x) \subset \mathcal{S}(U_P \setminus G)$ be the subspace of functions whose support is contained in $(U_P \setminus G)^+(x)$. 

\[ (8) \]
Proposition 3.8. a) The left action of $L$ on $S(U_P \backslash G)$ restrict to an $L^+_P$ action on $S^+(U_P \backslash G)(x)^K$.

b) Let $K \subset K_x$ be an open subgroup. The $H(L^+_P)$-module $S^+(U_P \backslash G)(x)^K$ is finitely generated and projective.

c) There exists a unique object $P(x)^K \in \overline{Sm}_X$ such that $P(x)^K = S^+(U_P \backslash G)(x)^K$ while the isomorphism $\Phi$ for $P = G$ comes from the natural arrow $S^+(U_Q \backslash G)(x)^K \to S(U_Q \backslash G)(x)^K = JD_G^q(S(G)^K)$.

d) For every open subgroup $K \subset K_x$ the object $P(x)^K$ is locally projective, i.e. the functor $\text{Hom}(P(x)^K, -)$ is exact.

e) For every component $X$ there is an open subgroup $S \subset K_x$ such that for every open subgroup $K \subset S$ the object $P^K$ is local generator of the corresponding summand $\overline{Sm}_X \subset \overline{Sm}$.

Proof. a) Bijection $\Phi$ intertwines the left action of $L$ on $P^\lambda \backslash G/K_x$ with the action of $\Lambda_P = L/L^0$ on itself by translations, this implies part a). The space $S^+(U_P \backslash G)(x)^K$ splits as a direct sum indexed by $P^\lambda \backslash G/K$, each summand is isomorphic to the space $S(L^+_P)^{KL}$ for some open compact subgroup $K_L \subset K$, this implies b).

Notice that given modules $M_P$ and isomorphisms $\Phi$ for $P = G$ as in Definition 3.1, the rest of the isomorphisms $\Phi$ satisfying the requirements of the Definition are defined uniquely (if they exist) provided that each module $M_P$ is torsion free as a module over $Z^+(L_P)$. This implies uniqueness in c). Existence follows from the fact that $G^\lambda(x, P) \subset G^\lambda(x, Q)$ for parabolic subgroups $P \subset Q$.

Statement d) follows from b).

Finally e) follows from the corresponding statement about $H_K$ established in 9.

Set $\overline{A}_X = \overline{A}_X(x, K) := \overline{\text{Hom}}(P(x)^K, P(x)^K)|_X$, this is a sheaf of algebras on $X$.

Corollary 3.9. Given a component $X \subset \overline{X}$ for any small enough open subgroup $K \subset K_x$ we have a canonical equivalence between the category of (quasi)coherent sheaves of $\overline{A}_X(x, K)$-modules and the summand in $\overline{Sm}^{-}$ (respectively, $\overline{Sm}^+$) corresponding to $X$.

3.4. Compactified category and filtered modules. For a simple root $\alpha$ let $P_\alpha$ be the corresponding maximal proper parabolic and $\overline{F}_\alpha$ be the closure of the corresponding stratum in $\overline{X}$. It is easy to see that $\overline{F}_\alpha$ is a divisor. For a weight $\lambda$ set $D_\lambda = \sum_\alpha \langle \alpha, \lambda \rangle |\overline{F}_\alpha|$ where the sum runs over the set of simple roots, let also $O(\lambda) = C_{\overline{X}}(D_\lambda)$ and $\mathcal{F}(\lambda) = \mathcal{F} \otimes_{O(\overline{X})} O(\lambda)$.

A coherent sheaf $\mathcal{F}$ on $\overline{X}$ is determined by the graded module $\oplus_\lambda \Gamma(\mathcal{F}(\lambda)), \overline{O}(\lambda)$.

If $\mathcal{F}$ is torsion free then the natural map $\mathcal{F}(\mu) \to \mathcal{F}(\lambda + \mu), \lambda \in \Lambda^+$ is injective and $\bigcup \mathcal{F}(\lambda) = j_+ j^+(\mathcal{F})$ where $j : \overline{X} \to \overline{X}$ is the embedding. Thus the category of torsion free coherent sheaves $\text{Coh}_{tf}(\overline{X})$ admits a full embedding into the category of $O(\overline{X})$-modules equipped with a filtration indexed by $\Lambda^+$ compatible with the natural filtration on the ring $O(\overline{X})$: to a sheaf $\mathcal{F}$ it assigns the module $\Gamma(j_+^*(\mathcal{F}))$ with the filtration by the subspaces $\Gamma(\mathcal{F}(\lambda))$.

Applying it to the sheaf of rings $\overline{A}_X(x, K)$ we get a filtration on the Hecke algebra $F^{\text{spec}}_{\Lambda^+}(H_K)$.

Thus we obtain the following:

Proposition 3.10. We have a full embedding from the category $\overline{Sm}_K^{tf}$ of torsion free objects in $\overline{Sm}_K$ to the category of modules over $H_K$ equipped with a filtration compatible with the filtration $F^{\text{spec}}$ on $H_K$. 

It is clear that the associated graded of the filtered module in the image of such an embedding is finitely generated; recall that a filtration with this property is called a good filtration.

We also have the left adjoint functor $\text{Loc}$ from the category of $\mathcal{H}_K$ modules with a good filtration to $\text{Spec}m_K$.

3.4.1. A geometric description of the filtration. We now provide a more explicit description of the filtration $F^{\text{spec}}_{\leq \lambda}(\mathcal{H}_K)$.

Recall that the two sided cosets of $K_x$ in $G$ are indexed by $X^+_B$ (see e.g. [22 §3.3.3]), for $\lambda \in X^+_B$ let $G_{\lambda}$ denote the corresponding coset. Let $F^{\text{geom}}_{\leq \lambda}(\mathcal{H}_K)$ be the space of functions whose support is contained in $\bigcup \mu \leq \lambda_{\mu}$ for some generators $h_{\mu}$.

Proposition 3.11. For every open compact $K \subset K_x$ there exists $\lambda_0 \in X^+_B$ such that:

$$(9) \quad F^{\text{spec}}_{\leq \lambda_0}(\mathcal{H}_K) = \{ h \in \mathcal{H}_K \mid \forall \mu \leq \lambda_0 + X^+_B : \ h \ast F^{\text{geom}}_{\leq \mu}(\mathcal{H}_K) \subset F^{\text{geom}}_{\leq \mu + \lambda_0}(\mathcal{H}_K) \}.$$ 

Proof. It follows from the definition that

$$(10) \quad h \in F^{\text{spec}}_{\leq \lambda_0}(\mathcal{H}_K) \iff \forall P, \mu : \mathcal{S}((U_P \setminus \mathcal{G})_{\leq \mu}) \ast h \subset \mathcal{S}((U_P \setminus \mathcal{G})_{\leq \lambda_0 + \mu}),$$

where $\lambda_P$ is the image of $\lambda$ in $\Lambda_P$ and $(U_P \setminus \mathcal{G})_{\leq \mu} = \bigcup \left( (U_P \setminus \mathcal{G})_{\nu} \right)_{\nu \leq \mu}$ for the standard partial order $\leq$ on $\Lambda_P$; here $(U_P \setminus \mathcal{G})_{\nu}$ is the image of the coset corresponding to $\nu$ under bijection (8). Let $X_P = (G/U_P \times G/U_P)/L$. We have $K_x \setminus X_P/K_x \cong \Lambda_P$, so we need to check the condition in the right hand side of (10) is equivalent to:

$$(11) \quad \forall \mu \leq \lambda_0 : \ h \ast \mathcal{S}(\mathcal{X}_P) \subset \mathcal{S}(\mathcal{X}_P \setminus \lambda_0 + \mu),$$

where $\mathcal{X}_P = \bigcup \mathcal{X}_P \setminus \nu \leq \mu$. This is true by considering the projection $X \to G/P^-$ with fiber $G/U_P$. Assume now that $h$ lies in the set in the right hand side of (10). Applying the map $B_1$ of [8, Definition 5.3] and using [8 Lemma 5.5], we see that $h$ also satisfies (11). This shows that the right hand side of (10) is contained in the left hand side. We proceed to check the opposite inclusion. The Rees ring $\oplus \mathcal{F}^{\text{spec}}_{\leq \lambda}(\mathcal{H}_K)$ is finite over its center which is a finitely generated commutative ring, hence the Rees ring is finitely generated. Thus it suffices to check that for a finite set of generators $h_i \in F^{\text{spec}}_{\leq \lambda}(\mathcal{H}_K)$ we have:

$$(h_i \ast F^{\text{geom}}_{\leq \mu}(\mathcal{H}_K) \subset F^{\text{geom}}_{\leq \mu + \lambda_0}(\mathcal{H}_K) \forall \mu \leq \lambda_0 + \Lambda^+$$

for some $\lambda_0 \in \Lambda^+$. Existence of such a $\lambda_0$ for a given $h_i$ follows from [8 Lemma 5.5]. Since we consider a finite set of $h_i$, there exists $\lambda_0$ which satisfies the requirement for all $h_i$. \qed

Remark 3.12. It easily follows from the construction that $\text{gr}(F^{\text{spec}})$ is a Noetherian ring.

Notice that the geometric filtration on $\mathcal{H}$ is also compatible with the algebra structure; however, its associated graded is neither Noetherian, nor finitely generated in general. This is closely related to the fact that the intertwining operator acting on the space of functions over $(G/U)(O/\pi^n O)$ is not an isomorphism for $n > 1$ (here $(G/U)$ is a scheme over $\mathcal{O}$ coming from the O-group scheme with generic fiber $G$ corresponding to $x \in \mathcal{O}$).

Example 3.13. In view of Proposition [5.11] the geometric filtration $F^{\text{geom}}$ on $\mathcal{H}$ makes it into a filtered module over the filtered algebra $(\mathcal{H}, F^{\text{spec}})$. Applying the functor $\text{Loc}$ to that filtered module we get an object in $\text{Spec}m$. We denote it by $\mathcal{H}$ and call it the intertwining object in $\text{Spec}m$. 

Lemma 3.14. Assume that $G = SL(2)$.

The intertwining object $\mathcal{H}$ is equivalently described by $\mathcal{H}_G = \mathcal{H}$,

$$\mathcal{H}_B = \{ f \in \mathcal{S}(G/U) \mid supp(I^{-1}(f)) \subset (G/U_\cdot)^+ \},$$

where $I^{-1}$ denotes the inverse intertwining operator taking values in functions of bounded support.

Proof. Without loss of generality we can assume that $x$ is the standard vertex, so that $K_x = SL(2,O)$, we will write $K_0$ instead of $K_x$.

We need to check that for $h \in \mathcal{H}_K$ and large $\lambda \in \Lambda$ we have:

$$h \in F_{\leq \lambda}^{\text{geom}} \iff supp(I_p^{-1}A_p(h)) \subset (X_p)_{\leq \lambda},$$

where $X_{\leq \lambda}$ is the union of two-sided $K_0$ cosets corresponding to weights $\mu \leq \lambda$ and $A : \mathcal{H} \to \mathcal{S}(Y_p)$. By [8] Theorem 7.6] $I^{-1}A = B^\star$ (notations of loc. cit.). It follows from [8] Lemma 5.5] that for large $\lambda$ we have

$$supp(h) \subset G_{\leq \lambda} \iff supp(B^\star(f)) \subset X_{\leq \lambda}.$$ 

The claim follows. \hfill \Box

Remark 3.15. A similar statement can also be checked for an arbitrary reductive group $G$ based on a generalization of [8] Theorem 7.6] to an arbitrary parabolic subgroup.

Remark 3.16. We expect the local trace formula [11] to be closely related to the computation of Chern character of $\mathcal{H}$ taking values in the appropriate Hochschild homology group.

4. Hochschild homology and character values

Recall the notion of Hochschild homology of an abelian category [18], [16]. For a cochain sheaf of algebras $A$ over a quasi-projective algebraic variety $X$ over a field we have $HH_* (X; A) \cong HH_* (A - \text{mod})$, where $A - \text{mod}$ is the abelian category of coherent sheaves of $A$-modules and $HH_*(X; A)$ is defined as the derived global section of a naturally defined object $R\text{Hom}_{\mathcal{A} \otimes A^{op}}(A, A)$ in the derived category of sheaves on $X$, here the isomorphism is shown in [17].

One can also define the compactly supported Hochschild homology $HH^c_\star (X; A)$ as the derived global sections with compact support in the sense of [12] of $R\text{Hom}_{\mathcal{A} \otimes A^{op}}(A, A)$. If $X$ is projective then we also have

$$HH^c_\star (X; A) \cong HH_* (X; A) \cong HH_* (A - \text{mod}),$$

where the first isomorphism is clear since $R\Gamma = R\Gamma_\epsilon$ for a projective scheme. Also, for an open subscheme $U \subset X$ we have a natural push forward map $HH^c_\star (U; A|_X) \to HH^c_\star (X; A)$.

Assume that the sheaf of algebras $A$ (locally) has finite homological dimension. Then we have the Chern character map $ch : K^0(A - \text{mod}) \to HH_0(A - \text{mod})$, see e.g. [17] §4.2] for the definition (it is called the Euler class map in loc. cit.).

Example 4.1. For future reference we spell out this general construction in some simple special cases. We leave the proofs of these standard facts to the interested reader.

1. Assume $X$ is affine, so that $A - \text{mod}$ is the category of finitely generated modules over a Noetherian ring which we also denote by $A$. Then $HH_*(X; A) = HH_*(A)$ is computed by the bar complex of $A$, in particular $HH_0(X; A) = HH_0(A) = A/[A, A]$. A finitely generated projective module $M$ is isomorphic to $A^{\oplus n}e$ for an idempotent $e \in \text{Mat}_n(A)$ for some $n$. Then $ch(M) = \sum(e_{ii} \mod [A, A])$.

---

3Here by a bounded set in $G/U_\cdot = F^2 \setminus \{0\}$ we mean a subset with compact closure in $F^2$. 
(2) Assume that \( X = U_1 \cup U_2 \) for affine open subsets \( U_1, U_2 \). Let \( A_i = \Gamma(U_i, A) \), \( i = 1, 2 \) and \( A_{12} = \Gamma(U_1 \cap U_2, A) \). Then \( HH_*(X; A) \) is computed by the complex

\[
\text{Cone}(\text{Bar}(A_1) \oplus \text{Bar}(A_2) \to \text{Bar}(A_{12})) [-1].
\]

For a locally projective module coherent sheaf \( M \) of \( A \)-modules we can find integers \( n, m \) and idempotents \( e^1 \in \text{Mat}_n(A_1) \), \( e^2 \in \text{Mat}_m(A_2) \) together with isomorphisms \( \Gamma(U_1, M) \cong A_1^{\otimes n} e^1 \), \( \Gamma(U_2, M) \cong A_2^{\otimes m} e^2 \) and matrices \( a \in \text{Mat}_m(A_{12}) \), \( b \in \text{Mat}_n(A_{21}) \) which induce the inverse isomorphisms between \( A_1^{\otimes n} e^1 \) and \( A_2^{\otimes m} e^2 \) coming from the identification of both with \( \Gamma(U_1 \cap U_2, M) \). In this setting \( ch(M) \) is represented by the cocycle \((\sum e^1_{ii}, \sum e^2_{jj}, \sum a_{ij} \otimes b_{ji}) \in A_1 \oplus A_2 \oplus A_{12}^{\otimes 2}\).

(3) Let \( U \subset X \) be an affine open subset such that its complement \( Z \) is also affine. Let \( A_1 = \Gamma(U, A) \), and let \( A_1, A_{12} \) be the algebras of sections of \( A \) on the formal neighborhood and the punctured formal neighborhood of \( Z \) respectively. Then the statements of part (b) continue to hold mutatis mutandis.

Definition 4.2. In the above setting, for an object \( M \in A - \text{mod} \) with proper support we have the Chern character \( ch^c(M) \in HH^0_0(X; A) \) as follows. Fix a proper subscheme \( Z \subset X \) containing support of cohomology of \( M \). Then \([17, \S 5.7]\) yields \( ch_Z(M) \in R\Gamma_Z(X, R\text{Hom}_{A^{\otimes c}}(A, A)) \). We define \( ch^c(M) \) to be the image of \( ch_Z(M) \) under the canonical map \( R\Gamma_Z(X, F) \to R\Gamma_c(X, F) \), \( F = R\text{Hom}_{A^{\otimes c}}(A, A) \).

It is immediate to check that \( ch^c(M) \) is independent of the auxiliary choice of \( Z \).

In particular, we have Chern character map \( \bar{ch} : K^0(\text{Sm}) \to HH_0(\text{Sm}) \) and \( ch^c : K^0(\text{Adm}) \to HH_0^c(\text{Sm}) \), where \( \text{Adm} \subset \text{Sm} \) is the subcategory of admissible modules.

Let \( K^c \subset K, K^{nc} \subset K \) be the subspace of measures supported on compact (respectively, noncompact) elements.

Conjecture 4.3. a) We have a canonical isomorphism:

\[
K^c_G \cong \text{Im}(HH^0_0(Sm) \to HH_0(\text{Sm})).
\]

b) For \( \rho \in \text{Adm} \) and \( g \in G \) we have \( \text{WO}_g(\bar{ch}(\rho)) \) if \( g \) is compact regular semisimple. (Here we use that \( ch(\rho) \) is the image of \( ch^c(\rho) \) thus it belongs to \( \text{Im}(HH^0_0(Sm) \to HH_0(\text{Sm})) \) which we identify with \( K^c_G \) by a).

The proof of Conjecture for \( G = SL(2) \) is presented in the next section; we plan to present the proof in the general case in a later publication.

5. \( SL(2) \) Calculations

Theorem 5.1. Conjecture 4.3 holds for \( G = SL(2) \).

The rest of the section is devoted to the proof of the Theorem. From now on set \( G = SL(2) \).

5.1. Explicit complexes for Hochschild homology. Applying the general construction of section 4.3 with \( \sigma = \{ x \} \) where \( x \) is the vertex with stabilizer \( G(O) \) we arrive at a sheaf of algebras \( A \) on \( \mathbb{F} \) such that a direct summand in \( \mathbb{S} \) is canonically identified with the category of coherent sheaves of \( A \)-modules. It is easily seen to coincide with the sheaf \( H_K \) introduced in Example 3.2. We keep notations of that Example.

We also let \( \mathcal{H}^+_K \) denote its sections on the formal neighborhood \( \hat{\mathcal{P}} \) of \( \partial \mathcal{P} = \mathbb{F} \setminus \mathcal{P} \) and \( \mathcal{H}_K \) the sections on the punctured formal neighborhood of \( \partial \mathcal{P} \). We set \( \mathcal{H}^+ = \mathcal{H}^+_K \).
compute, respectively, \( R_G/U \) matrices in sections of \( S \). We also let \( X = (G/U \times G/U^-)/T \) be the set of rank one matrices in \( Mat_2(F) \) and \( X_+ = X \cap Mat_2(O) \).

Then we have \( H^+_K = \text{End}_{\text{mod}}(S(G/U|^+)) \otimes \mathbb{C}[t] \mathbb{C}(t) \), where \( S_K = \mathbb{C}(G/U) \otimes \mathbb{C}[t] \mathbb{C}(t) \). We have \( \hat{H}_K = \text{End}_{\text{mod}}(S_K(G/U)) \), where \( S_K(G/U) \) is the space of functions on \( G/U \) with bounded support.

Also notice that \( \hat{H} \) (though not \( \hat{H}^+ \)) carries a \( G \times G \) action.

Lemma 5.2. a) \( HH_*(\overline{Sm}) \) is computed by the complex

\[
(12) \quad \text{Bar}(\overline{Sm}) := \text{cone}[\text{Bar}(\hat{H}) \oplus \text{Bar}(\hat{H}^+)] \to \text{Bar}(\hat{H})[1].
\]

b) \( HH_i^c(Sm) \) is computed by the complex

\[
(13) \quad \text{Bar}^c(Sm) := \text{cone}[\text{Bar}(\hat{H}) \to \text{Bar}(\hat{H})][1].
\]

c) \( HH_i^c(Sm) \) is canonically isomorphic to derived \( G \) coinvariants in the two-term complex

\[
(14) \quad \mathcal{H} \to \hat{\mathcal{H}}
\]

(placed in degrees 0,1).

Proof. For a complex \( F \) of coherent sheaves on \( \overline{Y} \) the complexes

\[
\text{cone}[\Gamma(\mathcal{F}) \oplus \hat{\Gamma}^+(\mathcal{F}) \to \hat{\Gamma}(\mathcal{F})][1],
\]

compute, respectively, \( R\Gamma(\mathcal{F}) \) and \( R\Gamma^c(\mathcal{F}), \) where \( \hat{\Gamma}^+(\mathcal{F}) \) and \( \hat{\Gamma}(\mathcal{F}) \) denote, respectively, sections of \( \mathcal{F} \) on the formal neighborhood and on the punctured formal neighborhood of \( \partial \overline{Y} \). Applying this to a complex representing \( R\text{Hom}_{\mathcal{F}_K}^{\mathcal{F}_K} \mathcal{F}_K \) and observing that we have canonical quasi-isomorphisms \( \hat{\Gamma}^+(\mathcal{F}) \to \text{Bar}(\hat{H}^+), \)

\( \hat{\Gamma}(\mathcal{F}) \to \text{Bar}(\hat{H}) \) we get statements (a,b).

Statement (c) follows from isomorphisms \( HH_*(M) \cong H_*(G, M), HH_i^c(N) \cong HH_{i+1}(N) \) for an \( \hat{H} \)-bimodule \( M \) and an \( \hat{H} \)-bimodule \( N \).

5.2. Calculation of \( HH_0 \). In this subsection we prove part (a) of the Theorem.

Proposition 5.3. a) We have a short exact sequence:

\[
(15) \quad 0 \to \mathcal{K}_G \to HH_0^c(Sm) \to \overline{S(T)/S(T)} \to 0.
\]

b) We have a natural isomorphism \( \mathcal{K}_G \cong \text{Im}(HH_0^c(Sm) \to HH_0^c(\overline{Sm})) \).

We start the proof with the following

Lemma 5.4. \( G \) acts trivially on the cokernel of the map \( H \to S(Y_\Delta) \).

Proof follows from Lemma 2.25 \( \square \)

Corollary 5.5. We have:

\[
H_1(G, \text{CoKer}(H \to S(Y_\Delta))) = 0, \quad H_1(G, \text{CoKer}(H \to S(Y_\Delta))) = S(T)/S(T),
\]

where \( S(T) \) is the Tate completion of \( S(T) \) (functions on the torus).

Proof of Proposition 5.3. By Lemma 5.2 (c), \( HH_0^c(Sm) \) is identified with the derived \( G \)-coinvariants in the complex \( \mathcal{H} \to \overline{S(Y)} \) which is clearly isomorphic to \( \mathcal{L}^2 \). We have a short exact sequence

\[
0 \to S(Y_0) \to S(Y) \to S(Y_\Delta) \to 0,
\]
where \( Y_0 = Y \setminus Y_\Delta \) is the open set of non-colinear pairs of vectors. Thus the action of \( G \) on \( Y_0 \) is almost free, i.e. the stabilizer of every point in \( \{ \pm 1 \} \), and the orbits of \( G \) on \( Y_0 \) are indexed by the finite set \( F^\times/(F^\times)^2 \). This shows that \( H_i(G, S(Y_0)) = 0 \) and \( H_0(G, S(Y_0)) \) is finite dimensional. Denoting as above \( \hat{S}(Y) = S(Y) \otimes_{C[z, \varpi^{-1}]} \mathbb{C}((\varpi)) \) and similarly for \( \hat{S}(Y_0) \) we see \( H_i(G, S(Y_0)) = 0 \) for all \( i \). Thus derived \( G \) coinvariants of the complex \( \mathcal{E} \) are canonically isomorphic to derived \( G \)-coinvariants of the quotient complex \( \mathcal{E} = \mathcal{H} \to \hat{S}(Y_\Delta) \). Notice that \( H^0(\mathcal{E}) = \mathcal{K}, \ H^1(\mathcal{E}) = \text{CoKer}(\mathcal{H} \to \hat{S}(Y_\Delta)) \) and \( \mathcal{E} \) is quasi-isomorphic to the complex with zero differential since the category of \( G \)-modules has homological dimension one. Thus part (a) of the Proposition follows from Corollary 5.3.

Lemma 5.2 shows that \( \text{Bar}^e(\text{Sm}) \) is canonically quasiisomorphic to \( \mathbb{C} \otimes_G \mathcal{E} \). One the other hand, we have a short exact sequence of complexes

\[
0 \to \text{Bar}^e(\text{Sm}) \to \overline{\text{Bar}^e(\text{Sm})} \to \text{Bar}(\mathcal{H}(T^+)) \to 0,
\]

which yields a long exact sequence on cohomology:

\[
\cdots \to HH_{i+1}(\mathcal{H}(T^+)) \to HH_i(\text{Sm}) \to HH_i(\overline{\text{Sm}}) \to \cdots
\]

We are interested in \( i = 0 \). We have \( HH_1(\mathcal{H}(T^+)) \cong \mathcal{H}(T^+) \). To finish the proof we need another

Lemma 5.6. a) Let us identify

\[
HH_1(\mathcal{H}(T)) = H_1(G, \hat{S}(Y_\Delta)) = \hat{S}(T).
\]

Then the image of the natural map \( HH_1(\mathcal{H}(T^+)) \to HH_1(\mathcal{H}(T)) \) coincides with \( \varpi \hat{S}(T^+) \), i.e. with the space of functions supported on \( \varpi T^+ \), where \( \varpi \in F \) is a uniformizer.

b) In view of (15), part (a) yields a map \( \varpi \hat{S}(T^+) \cap S(T) \to \mathcal{K}_G \subset HH_0(\text{Sm}) \).

This map induces an isomorphism onto \( (\mathcal{K}_G)_G \) where \( \mathcal{K}_G \subset \mathcal{K} \) is the space of measures supported on noncompact elements.

Proof of the Lemma. By Hochschild-Kostant-Rosenberg, we can identify Hochschild homology with forms. It is easy to see that elements in \( S(T^0) \subset \hat{S}(T) \) (where \( T^0 \) is the maximal compact subgroup in the torus \( T \) locally on each component of Bernstein center are proportional to \( \frac{dz}{z} \) for a local coordinate coming from a global coordinate on \( \mathbb{C}^\times \). On the other hand, the image of \( HH_1(\mathcal{H}(T^+)) \) equals \( \mathbb{C}[z]dz \), this proves (a).

To prove (b) observe that the map in question can be described as follows. We have a short exact sequence of \( G \)-modules

\[
0 \to \mathcal{K} \to \mathcal{H} \to I \to 0,
\]

where \( I = \text{Im}(\mathcal{H} \to S(Y_\Delta)) \) which yields the Bockstein homomorphism \( \phi : H_1(G, I) \to H_0(G, K) \). By Corollary 5.3 we have \( H_1(G, I) \cong H_1(G, S(Y_\Delta)) = H_1(G, \mu_{p^1}) \otimes S(T) = \hat{S}(T) \). In view of (a) it suffices to check that \( \phi : S(\varpi T^+) \to H_0(G, K^{nc}) \).

We have a direct sum decomposition \( \mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}^{nc} \) compatible with the decomposition \( S(Y_\Delta) = S(Y^{\Delta}_c) \oplus S(Y^{\Delta nc}) \), where \( Y^\Delta_c = \mathbb{P}^1 \times O^\times \) and \( Y^\Delta nc = Y_\Delta \setminus Y^\Delta_c \). Let us further decompose \( Y^{\Delta nc} = Y^{\Delta +} \cup Y^{\Delta -} \), where \( Y^{\Delta +} = \mathbb{P}^1 \times (\varpi O \setminus \{ 0 \}) \), \( Y^{\Delta -} = \mathbb{P}^1 \times (F \setminus O) \).

The image of \( \phi \) clearly coincides with the image of \( H_1(G, S(Y^{\Delta +}_c)) \) under the map coming from the class in \( \text{Ext}^1(S(Y^{\Delta}_c), K) \) induced from the above short exact sequence. It is clear that it is contained in \( H_0(G, K^{nc}) \). It coincides with \( H_0(G, K^{nc}) \) for the following reason. Consider a regular orbit \( O = G/T \), then the image \( K_O \) of \( K \) in \( S(O) \) is the kernel of the pushforward map \( S(G/T) \to \mu_{G/B} \oplus \mu_{G/B} \). Then
$\langle K_O \rangle_G$ is one dimensional and the Bockstein map $H_1(G, \mu_{G/B}) \to H_0(G, K_O)$ is nonzero.

The Lemma clearly implies statement (b) of Proposition. □

5.3. Cocycles for Chern character and Euler characteristic. Let now $M$ be a locally projective object of $\mathcal{S}m$.

Thus there exist idempotents $c \in \text{Mat}_n(\mathcal{H})$, $e_+ \in \text{Mat}_m(\mathcal{H}^+)$ and $a, b \in \text{Mat}_{n,m}(\mathcal{H})$, such that $ab = e$, $ba = e_+$ with isomorphisms $M|_3 \cong H^n e$, $M|_3 \cong \mathcal{H}^+ e_+$, so that $a, b$ induces the two inverse isomorphisms between the restrictions of $H^n e$ and $\mathcal{H}^+ e_+$ to the punctured formal neighborhood of $\partial 3$ coming from the identification with the restriction of $M$.

Lemma 5.7. In the above notation, $\check{c}h(M)$ is represented by the cocycle for $\text{Bar}(\mathcal{S}m)$

\[ c = (c_H, e_+, c), \]

where

\[ c = \sum_i c_{ii} \in \mathcal{H}, \quad c_+ = \sum_j (e_+)_{jj} \in \mathcal{H}^+, \quad c = \sum_{i,j} a_{ij} \otimes b_{ji} \in \mathcal{H} \otimes \mathcal{H}. \]

Proof. This is a special case of Example [14](c). □

To state the next result we need the following notation. Fix $g \in G(O)$ normalizing $K$ thus $g$ acts by conjugation on the sheaf of algebras $\mathcal{A}$ and on global sections of an object in $\mathcal{A} - \text{mod}$.

We introduce a zero-cochain $\tau_g$ for the dual complex $\text{Bar}(\mathcal{S}m)^*$, $\tau_g = (\tau_H(g), \tau_+(g), \hat{\tau}(g))$, where $\tau_H(g) = WO_g \in \mathcal{H}^*$ (the weighted orbital integral), $\tau_+(g) = 0 \in \mathcal{H}_+^e$ and $\hat{\tau}(g) : \alpha \otimes \beta \mapsto \text{Tr}((\alpha, \Pi) \beta \circ g, \mathcal{S}(X))$, where $\alpha, \beta \in \mathcal{H}$ act on $\mathcal{S}(X)$ by left multiplication, $\Pi$ is the operator of multiplication by the characteristic function of $X_+$ acting on $\mathcal{S}(X)$ and $g$ is acting on the right.

Proposition 5.8. a) The cochain $\tau_g$ is a cocycle. It satisfies:

\[ \text{Tr}(g, \text{RHom}(\mathcal{H}^*, \mathcal{M})) = \langle \tau_g, \check{c}h(M) \rangle, \]

for $M \in \mathcal{S}m$, where $\mathcal{H}^*$ is the intertwining bimodule introduced at the end of section 3.

b) For $\phi \in K_G^*$ it is $\text{Im}(HH_0^c(Sm) \to HH_0(\mathcal{S}m))$ we have:

\[ \langle \phi, \tau_g \rangle = WO_g(\phi). \]

c) The image of $\tau_g$ in $HH_0^c(\mathcal{S}m)$ is independent of the auxiliary choices in the definition of $\tau_g$.

Proof. Part (c) clearly follows from (b).

We deduce (b) from Lemma[14] Let us present the map $K_G^* \to HH_0(\text{Bar}^c(Sm))$ by an explicit cocycle. Fix $f \in K \cap \mathcal{H}_K$. Consider the decomposition of $\mathcal{S}(G/U)^K$ as a direct sum of spaces of functions on preimages of $K$ orbits in $G/B$. Then the action of $f$ on this space has zero block diagonal components with respect to this decomposition. Also, the image of $f$ in $\mathcal{H} = \mathcal{S}(Y)$ is contained in $\mathcal{S}(Y_0)$, while $\mathcal{S}(\mathcal{Y}_0)_G = 0$ by the proof of Proposition[14]. Thus $f = \sum h_i - g_i h_i, g_i \in G, h_i \in \mathcal{S}(Y_0)$. It is easy to check that the image of $f$ in $HH_0(\text{Bar}^c(Sm))$ is represented by $(f, \sum g_i \delta_K \otimes h_i g^{-1}) \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. It remains to check that for $h_i$ as above we have

\[ \text{Tr}([g_i, \Pi] h_i g_i^{-1} \circ g, \mathcal{S}(X)) = 0. \]

We claim that the following stronger statement holds: the operator $[g_i, \Pi] h_i g_i^{-1}$ acting on $\mathcal{S}(G/U)$ has zero diagonal blocks with respect to the above block decomposition. To see this pick an open compact subgroup $K'$ such that $h_i \in \mathcal{H}_{K'}$, for all $i$. Then $g_i^{-1}$ sends a summand of the decomposition corresponding to a
$K'$-orbit $U \subset G/B$ to a summand corresponding to the $g_i^{-1} K'$ orbits $g_i^{-1}(U)$, the endomorphism $h_i$ acts by an operator with zero diagonal blocks with respect to the decomposition corresponding to $g_i^{-1} K'$ orbits, while $[g_i, \Pi]$ sends the summand corresponding to an orbit $g_i^{-1}(U)$ to the summand corresponding to the orbit $U$ (this is clear since $\Pi$ preserves all the direct sum decompositions above). Equality \(\text{(10)}\) and hence part (b) of the Proposition follows.

The proof of part (a) occupies the next subsection.

5.4. Tate cocycle and weighted orbital integral.

5.4.1. The Tate cocycle. A general reference for the following material is \([5]\) or \([4]\), especially \([4]\), §4.2.13, p 142; §7.13.18, p 344, developing the theme started in \([21]\).

Let $V$ be a Tate vector space, then the Lie algebra $\text{End}(V)$ of continuous endomorphism of $V$ has a canonical central extension, which we will denote by $\tilde{\text{End}}(V)$.

Let $\text{End}^b(V) \subset \text{End}(V)$ denote the subspace of endomorphisms with bounded image, and $\text{End}^d(V) \subset \text{End}(V)$ be the subspace of endomorphisms having open kernel (here $b$ stands for bounded, and $d$ for discrete). Then $\text{End}^b(V)$, $\text{End}^d(V)$ are two-sided ideals in the associative algebra $\text{End}(V)$, and hence ideals in the Lie algebra $\text{End}(V)$. Further, the trace functional is defined on the intersection $\text{End}^{bd} = \text{End}^b(V) \cap \text{End}^d(V)$, $tr : \text{End}^{bd} \to k$; it satisfies
\[
tr([E, E^{bd}]) = 0; \quad tr([E^b, E^d]) = 0
\] (17)
for $E^b \in \text{End}^b(V)$, $E^d \in \text{End}^d(V)$, $E^{bd} \in \text{End}^{bd}(V)$, $E \in \text{End}(V)$.

The central extension $\tilde{\text{End}}(V) \to \text{End}(V)$ is specified by the requirement that it is trivialized on the ideals $\text{End}^b(V)$, $\text{End}^d(V)$, i.e. the embeddings $\text{End}^b(V) \hookrightarrow \text{End}(V)$, $\text{End}^d(V) \hookrightarrow \text{End}(V)$ are lifted to (fixed) homomorphisms $s_b : \text{End}^b(V) \hookrightarrow \tilde{\text{End}}(V)$, $s_d : \text{End}^d(V) \hookrightarrow \tilde{\text{End}}(V)$, where $s_b$, $s_d$ intertwine the adjoint action of $\text{End}(V)$; and for $E^{bd} \in \text{End}^{bd}(V)$ we have
\[
s_b(E^{bd}) - s_d(E^{bd}) = tr(E^{bd}) \cdot c,
\] (18)
where $c \in \tilde{\text{End}}(V)$ is the generator of the kernel of the projection $\tilde{\text{End}}(V) \to \text{End}(V)$.

Suppose now that a decomposition of $V$
\[
V = V^+ \oplus V^-
\] (19)
into a sum of a bounded open and a discrete subspaces is given, and let $\Pi$ denote the projection to the bounded open $V^+$ along the discrete $V^-$. Then for any $E \in \text{End}(V)$ in the right-hand side of
\[
E = E \cdot \Pi + E \cdot (Id - \Pi)
\]
the first summand lies in $\text{End}^b(V)$, and the second one in $\text{End}^d(V)$. Thus we get a splitting $s = s_{\Pi} : \text{End}(V) \to \tilde{\text{End}}(V)$,
\[
s_{\Pi}(E) = s_b(E \cdot \Pi) + s_d(E \cdot (Id - \Pi)).
\] (20)

It is also easy to see that for $E$ preserving $V_-$ (respectively, $V_+$) the element $s_{\Pi}(E)$ is independent of the choice of the complement $V_-$ (respectively, $V_+$). Thus we get a canonical splitting $s_{V^+}$ (respectively, $s_{V^-}$) of the central extension on the subalgebras $\text{End}_{V_+}(V)$, $\text{End}_{V_-}(V)$ consisting of endomorphisms preserving $V_+$ (respectively, $V_-$).

Denote by $C(E_1, E_2)$ the corresponding 2-cocycle of $\text{End}(V)$, i.e. a bi-linear functional, such that
\[
[s(E_1), s(E_2)] = s([E_1, E_2]) + C(E_1, E_2) \cdot c.
\]
Then we have

\[(21) \quad C(E_1, E_2) = \text{Tr}(E_1 \circ \Pi \circ E_2 \circ (Id - \Pi) - E_2 \circ \Pi \circ E_1 \circ (Id - \Pi)).\]

Finally, suppose that another discrete cobounded space \( W \subset V \) is fixed. The splitting \( s_W \) of the central extension on \( \text{End}_W(V) \) yields a linear functional \( \sigma_W = s_W - s_{\Pi|\text{End}_W(V)} \) on \( \text{End}_W(V) \).

**Example 5.9.** We have

\[(22) \quad \sigma_W(Id) = \dim(V^+ \cap W) - \co \dim(V^+ + W). \]

Also, suppose that \( F \) is an automorphism of \( V \) such that either \( F(V^+) \subset V^+ \), or \( V^+ \subset F(V^+) \); set \( d_{V^+}(F) = -\dim(V^+/F(V^+)) \) in the former and \( d_{V^+}(F) = \dim(F(V^+)/V^+) \) in the latter case. Then

\[(23) \quad C(F, F^{-1}) = d_{V^+}(F). \]

Both equalities follow directly from the definitions.

Consider now the complex

\[(24) \quad \text{Cone}(\text{Bar}(\text{End}_{V^+}(V)) \oplus \text{Bar}(\text{End}_W(V)) \to \text{Bar}(\text{End}(V)))[-1] \]

and define a zero-cochain for the dual complex \( \epsilon = (0, \sigma_W, C) \).

It is easy to check that the zero-cochain \( \epsilon \) is in fact a cocycle whose cohomology class does not depend on the choice of \( V^- \) for a fixed \( V^+ \).

**5.4.2. Weighted orbital integral and Tate cocycle.** We now apply this in the following example. Consider the Tate space

\[(25) \quad V = \hat{\mathcal{H}} \cong \hat{\mathcal{S}}(Y) \cong \hat{\mathcal{S}}(X) \]

where \( X \) was defined in the second paragraph of §5.1 the first isomorphism was discussed above and the second one is induced by the inverse intertwining operator. Let \( W \) be the image of \( \mathcal{H} \) in the space \( (25) \) and \( \hat{\mathcal{S}}(X)^* \) be the space of functions supported on \( X_+ \) while \( \hat{\mathcal{S}}(X)^{-} \) is the space of functions supported on the complement of \( X_+ \).

The group \( K_0 \) acts on all these spaces. Fix a representation \( \rho \) of \( K_0 \) and set \( V_\rho = \text{Hom}_{K_0, \rho}(\mathcal{H}, V) \), \( W_\rho = \text{Hom}_{K_0, \rho}(\mathcal{H}) \) and \( V_\rho^\pm = \text{Hom}_{K_0, \rho}(\mathcal{S}(X)^\pm) \).

The complex \( (24) \) maps naturally to the complex \( (24) \) constructed from \( V = V_\rho \) \( V^\pm = V_\rho^\pm \) and \( W = W_\rho \), let \( a_\rho \) denote that map.

**Lemma 5.10.** We have \( a_\rho^*(\epsilon) = \int_{K_0} \tau_g \text{Tr}(g, \rho)dg \).

**Proof.** Equality of components in \( (\mathcal{H}^{\otimes 2})^* \) follows by inspection.

It remains to check equality of components in \( \mathcal{H}^* \). Recall (see e.g. §3 §I.11) that \( WO_\rho(f) = \phi(0) \), where \( \phi \) is a linear function on dominant weights such that for large \( \lambda \) we have

\[ \phi(\lambda) = \text{Tr}(f \circ \Pi_\lambda \circ g, \mathcal{H}). \]

Here \( f \) acts on \( \mathcal{H} \) by convolution on the left, \( g \) acts by right translation and \( \Pi_\lambda \) is the characteristic function of \( G_{\leq \lambda} \) which is the union of two sided \( K_0 = G(O) \) cosets corresponding to \( \mu \leq \lambda \).

It follows that for a locally constant function \( \psi \) on \( K_0 \) supported on regular semisimple elements there exists an affine linear function \( \phi_\psi(\lambda) \), such that

\[ \phi_\psi(\lambda) = \text{Tr}(f \circ \Pi_\lambda \circ \psi, \mathcal{H}) \quad \text{for} \ \lambda \gg 0, \]

\[ \int_{K_0} WO_\rho(f) \psi(g)dg = \phi_\psi(0), \]

where \( \psi \) acts by convolution on the right.
Comparing this with the definition of Tate cocycle $\epsilon$ and of the intertwining bimodule $\mathcal{P}$ (see Example 3.13) and using Lemma 3.14, we see that for $\psi$ as above:

$$\int_{K_0} \text{WO}_g(f) \psi(g) dg = \sigma_H(f \otimes \psi),$$

where $f \otimes \psi$ acts on the Tate space (26) via its natural $H$-bimodule structure. Both sides of the last equality are continuous in $\psi$ with respect to the $L^1$ norm: this is clear for the right hand side and it follows from Theorem 2.14 for the left hand side. Thus validity of the equality for $\psi$ supported on regular semisimple elements implies its validity for all $\psi$. □

5.4.3. Sheaves of algebras on curves. We now apply the above construction in the following setting. Let $C$ be a smooth complete curve with a finite collection of points $x = \{x_1, \ldots, x_n\}$. Let $\hat{C}_x$, $\hat{C}_x^0$ be the formal neighborhood and the punctured formal neighborhood of $x$ respectively. Let $\mathcal{V}$ be a torsion free coherent sheaf on $C$, and set $V = \Gamma(\hat{C}_x, \mathcal{V})$, $W = \Gamma(C \setminus x, \mathcal{V})$, $V^+ = \Gamma(\hat{C}_x, \mathcal{V})$. Fixing formal coordinates $z_i$ at $x_i$ and trivializations of $\mathcal{V}$ on the formal neighborhood of $x_i$ we get an isomorphism $V \cong \mathbb{C}((z))^N$ sending $V_+$ to $\mathbb{C}[[z]]^N$, thus we get a splitting $\mathcal{V} = V^+ \oplus V^-$ where $V^- \cong z^{-1} \mathbb{C}[[z]]^N$.

Let $\mathcal{A}$ be a torsion free coherent sheaf of algebras on $C$ with a right action on $\mathcal{V}$; we assume also that $\mathcal{A}$ has finite homological dimension (i.e. that the algebra of sections of $\mathcal{A}$ on an affine open set has this property).

Let $A = \Gamma(C \setminus x, \mathcal{A})$, $\hat{A}^+ = \Gamma(\hat{C}_x, \mathcal{A})$ and $\hat{A} = \Gamma(\hat{C}_x^0, \mathcal{A})$. Then the complex

$$(26) \quad \text{Cone}(\text{Bar}(A) \oplus \text{Bar}(\hat{A}^+) \to \text{Bar}(\hat{A})) [-1]$$

computes Hochschild homology of the category $\mathcal{A} - \text{mod}$. On the other hand, it maps naturally to the complex (24), let $\alpha$ denote this map.

Lemma 5.11. Suppose that $\mathcal{V}$ is a locally projective sheaf of right $\mathcal{A}$-modules. Then for a coherent sheaf $\mathcal{M}$ of (left) $\mathcal{A}$-modules we have:

$$<\text{ch}(\mathcal{M}), \alpha^*(\epsilon)> = \chi(V \otimes_A \mathcal{M}),$$

where $\chi$ denotes Euler characteristic.

Proof. Both sides of (27) do not change if we replace $\mathcal{A}$ by $\text{End}_{O_C}(\mathcal{V})^{op}$ and $\mathcal{M}$ by $\text{End}_{O_C}(\mathcal{V})^{op} \otimes_A \mathcal{M}$. Thus we can assume without loss of generality that $\mathcal{A} = \text{End}_{O_C}(\mathcal{V})^{op}$. In that case we have a canonical equivalence of categories $\text{Coh}(C) \cong \mathcal{A} - \text{mod}$, $\mathcal{F} \mapsto \mathcal{F} \otimes_{O_C} \mathcal{V}^*$.

Furthermore, the operator $[\mathcal{F}] \mapsto [\mathcal{F} \otimes \mathcal{V}]$ induces an automorphism of the rational Grothendieck group $K^0(\text{Coh}(C)) \otimes \mathbb{Q}$, thus it suffices to prove (27) for $\mathcal{A} = \text{End}_{O_C}(\mathcal{V})^{op}$, $\mathcal{M} = \mathcal{F} \otimes \mathcal{V}^*$, where $\mathcal{F} = \mathcal{V} \otimes \mathcal{F'}$, thus $\mathcal{F} = \mathcal{F'} \otimes_{O_C} \mathcal{A}$. Now, both sides of (27) do not change if we replace $\mathcal{A}$ by $O_C$ and $\mathcal{M}$ by $\mathcal{F'}$; the locally projective module $\mathcal{V}$ is replaced by the corresponding coherent sheaf $\mathcal{V}^{op}_{O_C}$. Thus we have reduced to the case $\mathcal{A} = O_C$.

It is easy to see that both sides of (27) do not change if $\mathcal{V}$ and $\mathcal{F}$. A locally free sheaf on a curve admits a filtration whose associated graded is a sum of line bundles; also, since $C$ is smooth the Grothendieck group $K^0(\text{Coh}(C))$ is generated by line bundles. Thus we can assume without loss of generality that both $\mathcal{M}$ and $\mathcal{V}$ are line bundles.

If $\mathcal{M}$ is a trivial line bundle then $\text{ch}(\mathcal{M})$ is represented by the cocycle $(1, 1, 1 \otimes 1)$, so (27) follows from (22). Notice also that both sides of (27) for a fixed $\mathcal{V}$ depend only on the degree of the line bundle $\mathcal{M}$ (here we assume without loss of generality that the curve $C$ is irreducible): this is clear for the right hand side, while for the

\[\text{This assumption is likely unnecessary but it allows one to simplify the statements and the proofs.}\]
left hand side it follows from the fact that a regular function on an abelian variety is constant. Thus it suffices to consider the case when \( M = \mathcal{O}(nx), \; x \in \mathfrak{x} \). In this case \( ch(M) \) is represented by the cocycle \((1, 1, f \otimes f^{-1})\) where \( f \) is a function on the formal punctured neighborhood of \( \mathfrak{x} \) having order \( n \) at \( x \) and order 0 at other points. In this case \((27)\) follows from \((22)\) and \((23)\).

**Corollary 5.12.** a) Suppose that \( V \cong eA \) is a direct summand in a free rank one module for an idempotent \( e \in \Gamma(C, A) \). Then for a coherent sheaf \( M \) of \( A \)-modules we have:

\[
\langle ch(M), e^*(\epsilon) \rangle = \chi(eM),
\]

where \( \chi \) denotes the Euler characteristic.

b) Suppose that \( M \) has finite support contained in \( C \setminus \mathfrak{x} \). Then the equality in (a) holds under a weaker assumption that \( V|_{C \setminus \mathfrak{x}} \cong e\mathcal{A}|_{C \setminus \mathfrak{x}} \).

5.4.4. **Proof of Proposition 5.3 (a).** It suffices to check that we get an equality upon averaging against a character of any representation \( \psi \) of \( K_0 = G(O) \).

In view of Corollary 5.12(b) it suffices to check that the cochain \( \int_{K_0} \tau_g Tr(g, \psi) dg \) is obtained as in Corollary 5.12 for an appropriate choice of trivialization for \( V \) on the formal neighborhood of \( \mathfrak{x} \), where \( C \subset \mathfrak{G} \) is the union of one dimensional components containing representations with nonzero \( K \)-invariant vectors, \( \mathfrak{x} = (\mathfrak{G} \setminus \mathfrak{3}) \cap C, A = \overline{\mathcal{A}}, \) \( z_i \) are the natural local coordinates and \( \mathcal{V} = Hom_{K_0}(\psi, \overline{\mathcal{A}}) \). Applying Lemma 5.10 we reduce to showing that \( a^*_0(\epsilon) \) has the required form.

It follows from Lemma 5.11 that the space \( V_+ \) of sections of \( \overline{\mathcal{A}} \) on the formal neighborhood of \( \mathfrak{G} \setminus \mathfrak{3} \) is identified with \( \mathcal{S}(X_+) \) (notation introduced in the second paragraph of §(5.11)): moreover, we can choose a trivialization of the sheaf \( \overline{\mathcal{A}} \) on the formal neighborhood of \( \mathfrak{G} \setminus \mathfrak{3} \) so that constant sections correspond to functions supported on the set \( X(O) = X_+ \setminus \{tX_+\} \). Then the space \( V_- \) is identified with \( \mathcal{S}(X \setminus X_+)^{K \times K} \). The desired equality now follows by inspecting the definitions.

5.5. **Proof of part (b) of Theorem 5.1.** For \( g \in K_0 \) the formula follows from Proposition 5.8. The general case of a compact element follows by conjugating with an element of \( GL(2, F) \).

**REFERENCES**

[1] J. Arthur, *A local trace formula*, Inst. Hautes Études Sci. Publ. Math. 73 (1991), 5–96.
[2] J. Arthur, *The characters of supercuspidal representations as weighted orbital integrals*, Proc. Indian Acad. Sci. Math. Sci. 97 (1987), no. 1-3, 3–19 (1988).
[3] A. Beilinson, *An introduction to the trace formula*, Harmonic analysis, the trace formula, and Shimura varieties, 1–263, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.
[4] A. Beilinson, V. Drinfeld, *Quantization of Hitchin’s Integrable System and Hecke Eigensheaves*, unpublished manuscript available at: http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf
[5] A. Beilinson, B. Feigin, B. Mazur, *Notes on Conformal Field Theory*, unpublished manuscript available at: http://www.math.stonybrook.edu/~kirillov/manuscripts.html
[6] R. Bezrukavnikov, *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the archimedean case*, preprint arXiv:1506.01452
[7] R. Bezrukavnikov, *Homological properties of representations of p-adic groups related to geometry of the group at infinity*, Ph.D. thesis, preprint arXiv:math/0406223
[8] R. Bezrukavnikov, D. Kazhdan, *Geometry of second adjointness for p-adic groups*, With an appendix by R. Bezrukavnikov, D. Kazhdan and Y. Varshavsky, Represent. Theory 19 (2015), 299–332.
[9] J. Bernstein, *Le "centre" de Bernstein*, Edited by P. Deligne. Travaux en Cours, “Representations of reductive groups over a local field”, 1–32, Hermann, Paris, 1984.
[10] J. Bernstein, A. Zelevinsky, *Representations of the group GL(n, F), where F is a local non-Archimedean field*, (Russian) Uspekhi Mat. Nauk 31 (1976), no. 3(189), 5–70.
[11] W. Casselman, *Characters and Jacquet modules*, Math. Ann. 230 (1977), no. 2, 101–105.
[12] P. Deligne, Appendix to: R. Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. Lecture Notes in Mathematics, No. 20 Springer-Verlag, Berlin-New York 1966.

[13] Harish-Chandra *Admissible invariant distributions on reductive p-adic groups*, with a preface and notes by Stephen DeBacker and Paul J. Sally, Jr., University Lecture Series, 16. American Mathematical Society, Providence, RI, 1999.

[14] D. Kazhdan *Cuspidal geometry of p-adic groups*, J. Analyse Math. 47 (1986), 1–36.

[15] D. Kazhdan, Y. Varshavsky, *Endoscopic decomposition of certain depth zero representations*, Studies in Lie theory, 223–301, Progr. Math., 243, Birkhäuser Boston, Boston, MA, 2006.

[16] B. Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra 136 (1999), no. 1, 1–56.

[17] B. Keller, *On the cyclic homology of ringed spaces and schemes*, Doc. Math. 3 (1998), 231–259.

[18] R. McCarthy, *The cyclic homology of an exact category*, J. Pure Appl. Algebra 93 (1994), no. 3, 251–296.

[19] R. Schneider, U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. 85 (1997), 97–191.

[20] A. Silberger, *PGL2 over the p-adics: its representations, spherical functions, and Fourier analysis*, Lecture Notes in Mathematics, Vol. 166 Springer-Verlag, Berlin-New York 1970, 204 pp.

[21] J. Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. (4) 1 (1968) 149–159.

[22] J. Tits, *Reductive groups over local fields*, in: ”Automorphic forms, representations and L-functions” (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 29–69, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[23] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad*, Compositio Math. 146 (2010) no. 5, 1180–1290.

[24] J.-L. Waldspurger, *Caractères de représentations de niveau 0*, arXiv 1602.01440.