LOCALIZATION TRANSITION IN DISORDERED PINNING MODELS.
EFFECT OF RANDOMNESS ON THE CRITICAL PROPERTIES.

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ABSTRACT. These notes are devoted to the statistical mechanics of directed polymers interacting with one-dimensional spatial defects. We are interested in particular in the situation where frozen disorder is present. These polymer models undergo a localization/delocalization transition. There is a large (bio)-physics literature on the subject since these systems describe, for instance, the statistics of thermally created loops in DNA double strands and the interaction between (1 + 1)-dimensional interfaces and disordered walls. In these cases the transition corresponds, respectively, to the DNA denaturation transition and to the wetting transition. More abstractly, one may see these models as random and inhomogeneous perturbations of renewal processes.

The last few years have witnessed a great progress in the mathematical understanding of the equilibrium properties of these systems. In particular, many rigorous results about the location of the critical point, about critical exponents and path properties of the polymer in the two thermodynamic phases (localized and delocalized) are now available.

Here, we will focus on some aspects of this topic - in particular, on the non-perturbative effects of disorder. The mathematical tools employed range from renewal theory to large deviations and, interestingly, show tight connections with techniques developed recently in the mathematical study of mean field spin glasses.

2000 Mathematics Subject Classification: 60K35, 82B44, 82B41, 60K05

Keywords: Pinning and Wetting Models, Localization transition, Harris Criterion, Critical Exponents, Correlation Lengths, Renewal Theory, Interpolation and Replica Coupling
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1. Introduction and motivations

Consider a Markov chain \( \{ S_n \}_{n \in \mathbb{N}} \) on some state space \( \Omega \), say, \( \Omega = \mathbb{Z}^d \). We can unfold \( S \) along the discrete time axis, i.e., we can consider the sequence \( \{(n, S_n)\}_{n \in \mathbb{N}} \) and interpret it as the configuration of a directed polymer in the space \( \mathbb{N} \times \Omega \). In the examples which motivate our analysis, the discrete time is actually better interpreted as one of the space coordinates. The “directed” character of this polymer just refers to the fact that the first coordinate, \( n \), is always increasing. In particular, the polymer can have no self-intersections. Some assumptions on the law of the Markov chain will be made in Section 2, where the model is defined precisely. Now let 0 be a specific point in \( \Omega \), and assume that the polymer receives a reward \( \epsilon \) (or a penalty, if \( \epsilon < 0 \)) whenever \( S_n = 0 \), i.e., whenever it touches the defect line \( \mathbb{N} \times \{0\} \). In other words, the probability of a configuration of \( \{S_1, S_2, \ldots, S_N\} \) is modified by an exponential, Boltzmann-type factor

\[
\exp \left( \epsilon \sum_{n=1}^{N} 1_{\{S_n=0\}} \right).
\]

It is clear that if \( \epsilon > 0 \) contacts with the defect line are enhanced with respect to the \( \epsilon = 0 \) (or free) case, and that the opposite is true for \( \epsilon < 0 \). One can intuitively expect that the in the thermodynamic limit \( N \to \infty \) a phase transition occurs: for \( \epsilon > \epsilon_c \) the polymer stays close to the defect line essentially for every \( n \), while for \( \epsilon < \epsilon_c \) it is repelled by it and touches it only at a few places. This is indeed roughly speaking what happens, and the transition is given the name of localization/delocalization transition. We warn the reader that it is not true in general that the critical value is \( \epsilon_c = 0 \): if the Markov chain is transient, then \( \epsilon_c > 0 \), i.e., a strictly positive reward is needed to pin the polymer to the defect line (cf. Section 2.6).

A more interesting situation is that where the constant repulsion/attraction \( \epsilon \) is replaced by a local, site-dependent repulsion/attraction \( \epsilon_n \). One can for instance consider the situation where \( \epsilon_n \) varies periodically in \( n \), but we will rather concentrate on the case where \( \epsilon_n \) are independent and identically distributed (IID) random variables. We will see that, again, the transition exists when, say, the average \( \epsilon \) of \( \epsilon_n \) is varied. However, in this case the mechanism is much more subtle. This is reflected for instance in the counter-intuitive fact that \( \epsilon_c \) may be negative: a globally repulsive defect line can attract the polymer! Presence of disorder opens the way to a large number of exciting questions, among which we will roughly speaking select the following one: how are the critical point and the critical exponents influenced by disorder?

There are several reasons to study disordered pinning models:

- there is a vast physics and bio-physics literature on the subject, with intriguing (but often contradictory) theoretical predictions and numerical/experimental observations. See also Section 2.6;
- they are interesting generalizations of classical renewal sequences. From this point of view they raise new questions and challenges, like the problem of the speed of convergence to equilibrium for the renewal probability in absence of translation invariance (cf. in particular Section 6);
- finally (and this is my main motivation) they are genuinely quenched-disordered systems where randomness has deep, non-perturbative effects. With respect to other systems like disordered ferromagnets or spin glasses, moreover, disordered pinning models have the advantage that their homogeneous counterparts are under full mathematical control. These models, therefore, turn out to be an ideal testing
ground for theoretical physics arguments like the Harris criterion and renormal-
ization group analysis.

It is also quite encouraging, from the point of view of mathematical physics, that rigorous
methods have been able not only to confirm predictions made by theoretical physicists,
but in some cases also to resolve controversies (it is the case for instance of the results in
Section 5.6 which disprove some claims appeared previously in the physical literature).

1.1. A side remark on literature and on the scope of these notes. A excellent
recent introductory work on pinning models with quenched disorder (among other topics)
is the book [22] by Giambattista Giacomin. In order to avoid the risk of producing a
résumé of it, we have focussed on aspects which are not (or are only tangentially) touched
in [22]. On the other hand, we will say very little about “polymer path properties”,
to which Chapters 7 and 8 of [22] are devoted. A certain degree of overlap is however
inevitable, especially in the introductory sections 2 and 4; results taken from [22] will be
often stated without proofs (unless they are essential in the logic of these notes).

We would also like to mention that some of the results of these notes apply also to a
model much related to disordered pinning, namely random heteropolymers (or copolymers)
at selective interfaces. It is the case, for instance, of the results of Sections 5.6 and 6. We
have chosen to deal only with the pinning model for compactness of presentation, but we
invite readers interested in the heteropolymer problem to look, for instance, at [11], [37],
[22] and references therein.

2. The model and its free energy

2.1. The basic renewal process (“the free polymer”). Our starting point will be
a renewal $\tau$ on the integers, $\tau := \{\tau_i\}_{i=0,1,2,\ldots}$, where $\tau_0 = 0$ and $\{\tau_i - \tau_{i-1}\}_{i\geq 1}$ are IID
positive and integer-valued random variables. The law of the renewal will be denoted by $P,$
and the corresponding expectation by $E.$ In terms of the “directed polymer picture”
of the introduction, $P$ is the law of the set $\tau$ of the points where the polymer touches the
defect line, in absence of interaction: $\tau = \{n : S_n = 0\}$ (cf. also Section 2.6). We assume
that $(\tau_i - \tau_{i-1})$ or, equivalently, $\tau_1$ is $P$-almost surely finite: if
\[ K(n) := P(\tau_1 = n), \]  
(2.1)
this amounts to requiring $\sum_{n\in\mathbb{N}} K(n) = 1.$ This, of course, implies that the renewal is
recurrent: $P$-almost surely, $\tau$ contains infinitely many points. A second assumption is
that $K(.)$ has a power-like tail. More precisely, we require that
\[ K(n) = \frac{L(n)}{n^{1+\alpha}} \text{ for every } n \in \mathbb{N}, \]  
(2.2)
for some $\alpha \geq 0$ and a slowly varying function $L(.)$. We recall that a function $(0, \infty) \ni
x \mapsto L(x) \in (0,\infty)$ is said to be slowly varying at infinity if [8]
\[ \lim_{x \to \infty} \frac{L(rx)}{L(x)} = 1 \]  
(2.3)
for every $r > 0.$ In particular, a slowly varying function diverges or vanishes at infinity
slower than any power. The interested reader may look at [8] for properties and many
interesting applications of slow variation. Of course, every positive function $L(.)$ having a
non-zero limit at infinity is slowly varying. Less trivial examples are $L(x) = (\log(1 + x))^\gamma$
for $\gamma \in \mathbb{R}.$
Observe that the normalization condition \( \sum_{n \in \mathbb{N}} K(n) = 1 \) implies that, if \( \alpha = 0 \), \( L(.) \) must tend to zero at infinity (cf. also Section 2.6 below for an example).

It is important to remark that typical configurations of \( \tau \) are very different according to whether \( \alpha \) is larger or smaller than 1. Indeed the average distance between two successive points,

\[
E(\tau_i - \tau_{i-1}) = \sum_{n \in \mathbb{N}} nK(n),
\]

is finite for \( \alpha > 1 \) and infinite for \( \alpha < 1 \). In standard terminology, \( \tau \) is positively recurrent (i.e., \( \tau \) occupies a finite fraction of \( \mathbb{N} \)) for \( \alpha > 1 \) and null-recurrent for \( \alpha < 1 \) (the density of \( \tau \) in \( \mathbb{N} \) is zero). This is a simple consequence of the classical renewal theorem \([6, \text{Chap. I, Th. 2.2}]\), which states that

\[
\lim_{n \to \infty} P(n \in \tau) = \frac{1}{\sum_{n \in \mathbb{N}} nK(n)}.
\]

The distinction \( \alpha \gtrless 1 \) plays an important role, especially in the behavior of the homogeneous pinning model (cf. Section 4). Later on we will see the emergence of an even more important threshold value: \( \alpha_c = 1/2 \).

**Remark 2.1.** For \( \alpha = 1 \), the question whether the renewal is positively or null recurrent is determined by the behavior at infinity of \( L(.) \): from (2.5) we see that \( \tau \) is finitely recurrent iff \( \sum_n L(n)/n < \infty \). For instance, one has null recurrence if \( L(.) \) has a positive limit at infinity.

### 2.2. The model in presence of interaction.

Now we want to introduce an interaction which favors the occurrence of a renewal at some points and inhibits it at others. To this purpose, let \( \omega \) (referred to as quenched randomness or random charges) be a sequence \( \{\omega_n\}_{n \in \mathbb{N}} \) of IID random variables with law \( P \). The basic assumption on \( \omega_n \), apart from the fact of being IID, is that \( E\omega_1 = 0 \) and \( E\omega_1^2 = 1 \). These are rather conventions than assumptions, since by varying the parameters \( \beta \) and \( h \) in Eq. (2.6) below one can effectively tune average and variance of the charges. To be specific, in these notes we will consider only two (important) examples: the Gaussian case \( \omega_1 \overset{d}{=} \mathcal{N}(0,1) \) and the bounded case, \( |\omega_1| \leq C < \infty \). Many results are expected (or proven) to hold in wider generality and a few remarks in this direction are scattered throughout the notes.

We are now ready to define the free energy of our model: given \( h \in \mathbb{R}, \beta \geq 0 \) and \( N \in \mathbb{N} \) let

\[
F^\omega_N(\beta,h) := \frac{1}{N} \log Z_{N,\omega}(\beta,h) := \frac{1}{N} \log E(e^{\sum_{n=1}^N (\beta\omega_n + h)\delta_n \delta_N}),
\]

where for notational simplicity we put \( \delta_n := 1_{\{n \in \tau\}} \), \( 1_A \) being the indicator function of a set \( A \). The quenched average of the free energy, or quenched free energy for short, is defined as

\[
F_N(\beta,h) := \mathbb{E}F^\omega_N(\beta,h).
\]

Note that the factor \( \delta_N \) in (2.6) corresponds to imposing the boundary condition \( N \in \tau \) (the boundary condition 0 \( \in \tau \) at the left border is implicit in the law \( P \)). One could equivalently work with free boundary conditions at \( N \) (i.e., replace \( \delta_N \) by 1). The infinite-volume free energy would not change, but some technical steps in the proofs of some results would be (slightly) more involved.
We need also a notation for the Boltzmann-Gibbs average: given a realization \( \omega \) of the randomness and a system size \( N \), for a \( \mathcal{P} \)-measurable function \( f(.) \) set

\[
E_{N,\omega}^{\beta,h}(f) := \frac{E\left(f(\tau) e^{\sum_{n=1}^{N}(\beta \omega_n + h) \delta_n} \right)}{Z_{N,\omega}(\beta,h)}
\]  

(2.8)

2.3. **Existence and non-negativity of the free energy.** As usual in statistical mechanics, one is (mostly) interested in the thermodynamic limit (i.e., the limit \( N \to \infty \)).

A classical question concerns the existence of the thermodynamic limit of the free energy, and its dependence on the realization of the randomness \( \omega \). In the context of the models we are considering, the answer is well established:

**Theorem 2.2.** [22, Th. 4.1] If \( E|\omega_1| < \infty \), the limit

\[
F(\beta, h) := \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\omega}(\beta,h)
\]

(2.9)

exists for every \( \beta \geq 0, h \in \mathbb{R} \) and it is \( \mathbb{P}(d\omega) \)-almost surely independent of \( \omega \).

Of course, the limit does depend in general on the law \( \mathbb{P} \) of the disorder.

Note that the only assumption on disorder, apart from the IID character of the charges, is finiteness of the first moment, so that existence and self-averaging of the infinite-volume free energy holds in much wider generality than in the cases of Gaussian or bounded disorder we are considering here.

Some properties of the free energy come essentially for free: in particular, \( F(\beta,h) \) is convex in \((\beta,h)\), non-decreasing in \( h \), continuous everywhere and differentiable almost everywhere as a consequence of convexity. Another easy fact is that the sequence \( \{NF_N(\beta,h)\}_{N \in \mathbb{N}} \) is super-additive: for every \( N, M \in \mathbb{N} \), one has \( (N + M)F_{N+M}(\beta,h) \geq NF_N(\beta,h) + MF_M(\beta,h) \). This is easily proven:

\[
(N + M)F_{N+M}(\beta, h) = \mathbb{E} \log \mathbb{E} \left( e^{\sum_{n=1}^{N+M}(\beta \omega_n + h) \delta_n} \delta_{N+M} \right)
\]

(2.10)

\[
\geq \mathbb{E} \log \mathbb{E} \left( e^{\sum_{n=1}^{N}(\beta \omega_n + h) \delta_n} \delta_N e^{\sum_{n=N+1}^{N+M}(\beta \omega_n + h) \delta_n} \delta_{N+M} \right)
\]

\[
= NF_N(\beta,h) + MF_M(\beta,h),
\]

where in the last step we used invariance of \( \mathbb{P} \) with respect to left shifts and the renewal property of \( \mathcal{P} \). It is a standard fact that super-additivity implies

\[
F(\beta, h) \geq F_N(\beta,h) \text{ for every } N \in \mathbb{N}.
\]

(2.11)

2.4. **Contact fraction and critical point.** As we already mentioned, the interest in this class of models is mainly due to the fact that they show a so-called localization-delocalization transition. This is best understood in view of the elementary bound \( F(\beta,h) \geq 0 \). This positivity property is immediate to prove:

\[
F_N(\beta,h) \geq \frac{1}{N} \mathbb{E} \log \mathbb{E} \left( e^{\sum_{n=1}^{N}(\beta \omega_n + h) \delta_n} \mathbf{1}_{\{\tau_1 = N\}} \right) = \frac{h}{N} + \frac{1}{N} \log K(N)
\]

(2.12)

and the claimed non-negativity in the limit follows from [22]. Recalling that \( F(\beta,h) \) is non-decreasing in \( h \), for a given \( \beta \) the localization/delocalization critical point is defined to be

\[
h_c(\beta) := \sup\{h : F(\beta,h) = 0\}
\]

(2.13)

and the function \( \beta \to h_c(\beta) \) is referred to as the critical line. The region of parameters

\[
\mathcal{L} := \{(\beta,h) : \beta \geq 0, h > h_c(\beta)\}
\]
and
\[ \mathcal{D} := \{ (\beta, h) : \beta \geq 0, h \leq h_c(\beta) \} \]
are referred to as localized and delocalized phases, respectively. Since level sets of a convex function are convex, \( \mathcal{L} \) is a convex set and the function \( h_c(\cdot) : [0, \infty) \ni \beta \rightarrow h_c(\beta) \) is concave. The reason for the names “localized” and “delocalized” can be understood looking at the so-called contact fraction \( \ell_N \), defined through
\[ \ell_N := \frac{\left| \tau \cap \{1, \ldots, N\} \right|}{N} \] (2.14)
and taking values between 0 and 1 (as usual, \( |A| \) denotes the cardinality of a set \( A \)). It is immediate to check that
\[ \partial_h F_N^\omega(\beta, h) = E_{N,\omega}^{\beta,h}(\ell_N) \] (2.15)
and, by standard arguments based on convexity, this equality survives in the thermodynamic limit whenever the free energy is differentiable:
\[ \lim_{N \to \infty} E_{N,\omega}^{\beta,h}(\ell_N) \overset{a.s.}{=} \partial_h F(\beta, h) \]
for every \( h \) such that \( \partial_h^+ F(\beta, h) = \partial_h^- F(\beta, h) \). (2.16)

We have already mentioned that differentiability holds for Lebesgue-almost every value of \( h \). However, much more than this is true: as it was proven in [25], differentiability (actually, infinite differentiability) in \( h \) holds whenever \( h > h_c(\beta) \). We can therefore conclude the following: for \( h < h_c(\beta) \) (or for \( h \leq h_c(\beta) \) if \( F(\beta, h) \) is differentiable at \( h_c(\beta) \)) the thermal average of the contact fraction tends for to zero for \( N \to \infty \) (almost surely in the disorder), while for \( h > h_c(\beta) \) it tends to \( \partial_h F(\beta, h) > 0 \). The average contact fraction plays the role of an order parameter, like the spontaneous magnetization in the Ising model, which is zero above the critical temperature and positive below it.

Actually, much more refined statements about the behavior of the contact fraction in the two phases are available. In particular:

- for statements concerning the localized phase we refer to [25]. There, it is proven that, roughly speaking, not only typical configurations \( \tau \) have a number \( N \ell_N \sim N \partial_h F(\beta, h) \)
of points, but also that these points are rather uniformly distributed in \( \{1, \ldots, N\} \): long gaps between them are exponentially suppressed, and the largest gap is of order \( \log N \) (cf. Theorem 6.3 below);
- for \( h < h_c(\beta) \) we refer to [24] and [22, Ch. 8], where it is proven that \( \ell_N \) is typically at most of order \( \log N / N \).

In this sense, if one goes back to the pictorial image of \( \tau \) as the set of points of polymer-defect contact, one sees that the definition of (de)-localization in terms of free energy, as given above, does indeed correspond to the intuitive idea in terms of path properties: in \( \mathcal{L} \) the polymer stays at distance \( O(1) \) from the defect, while in \( \mathcal{D} \) it wanders away from it and touches it only a small (at most \( \log N \)) number of times.

The reader should remark that we have made no conclusive statement about the behavior of the contact fraction at \( h_c(\beta) \), since we have not attacked yet the very important question of the regularity of the free energy at the critical point. This will be the subject of Sections 4 and 5.
2.5. Quenched versus annealed free energy. Inequality (2.12) is a good example of how selecting a particular subset of configurations (in that case, those for which \( \tau_1 = N \)) provides useful free energy lower bounds. For more refined results in this direction we refer to [5] and [22, Sec. 5.2]. There, this technique is employed to prove that \( h_c(\beta) \) is strictly decreasing as a function of \( \beta \) which implies in particular that, since \( h_c(\cdot) \) is concave, \( h_c(\beta) \) tends to \(-\infty \) for \( \beta \to \infty \). This corresponds to the apriori non-intuitive fact that, as mentioned in the introduction, even if the charges are on average repulsive the defect line can pin the polymer. This is purely an effect of spatial inhomogeneities due to disorder: for \( \beta \) large, it is convenient for the polymer to touch the defect line in correspondence of attractive charges, where it gets a reward \( \beta \omega_n + h \gg 1 \), while the entropic cost of avoiding the repulsive charges is independent of \( \beta \). Free energy lower bounds were obtained also in the study of a different model, the heteropolymer at a selective interface, in [10].

Free energy upper bounds are on the other hand more subtle to get. An immediate one can be however obtained by a simple application of Jensen’s inequality:

\[
F_N(\beta, h) \leq \frac{1}{N} \log \mathbb{E} Z_{N,\omega}(\beta, h) = \frac{1}{N} \log \mathbb{E} \left( e^{\sum_{n=1}^{N}(h + \log M(\beta))a_n}\delta_N \right) = F_N(0, h + \log M(\beta)) =: F_N^a(\beta, h),
\]

where \( M(\beta) := \mathbb{E} e^{\beta a_1} \). In particular, \( \log M(\beta) = \beta^2/2 \) in the case of Gaussian disorder. \( F^a(\beta, h) := F(0, \beta + \log M(\beta)) \) is referred to as annealed free energy, and we see that it is just the free energy of the homogeneous system (with the same choice of \( K(\cdot) \)) computed for a shifted value of \( h \). The physical interpretation of the annealed free energy is clear: since configurations of \( \omega \) and \( \tau \) are averaged on the same footing, it corresponds to a system where impurities can thermalize on the same time-scales as the “polymer degrees of freedom” (i.e., \( \tau \)). This is not the physical situation one wishes to study (quenched disorder corresponds rather to impurities which are frozen, or which can evolve only on time-scales which are so long that they can be considered as infinite from the experimental point of view). All the same, the information provided by (2.17) is not at all empty. Define first of all the \textit{annealed critical point} as

\[
h_c^a(\beta) := \sup \{ h : F^a(\beta, h) = 0 \} = h_c(0) - \log M(\beta). \tag{2.18}
\]

Thanks to (2.17) and (2.13), one has immediately

\[
h_c(\beta) \geq h_c(0) - \log M(\beta), \tag{2.19}
\]

a bound which, as will be discussed in Section 5.3, is optimal for \( \alpha < 1/2 \) and \( \beta \) small.

2.6. Back to examples and motivations. Typical examples of renewal sequences satisfying (2.1) are the following. Let \( \{S_n\}_{n \geq 0} \) be the simple random walk (SRW) on \( \mathbb{Z} \), with law \( P^{SRW} \) and \( S_0 := 0 \), i.e., \( \{S_n - S_{n-1}\}_{n \in \mathbb{N}} \) are IID symmetric random variables with values in \{\(-1,+1\}\}. Then, it is known that [19] \( \tau := \{ n \in \mathbb{N} : S_{2n} = 0 \} \) is a null-recurrent renewal sequence such that the law of \( \tau_1 \) satisfies (2.2) with \( \alpha = 1/2 \) and \( L(\cdot) \) asymptotically constant. The reason why one looks only at even values of \( n \) in the definition of \( \tau \) in this case is due just to the periodicity of the SRW. If instead one takes the SRW on \( \mathbb{Z}^2 \), then \( \tau \) (defined exactly as above) is always a null-recurrent renewal but in this case \( \alpha = 0 \) and \( L(n) \sim c/(\log n)^2 \) [34]. Note that in this case, the presence of the slowly varying function \( L(\cdot) \) is essential in making \( K(\cdot) \) summable.

What happens in the case of the SRW on \( \mathbb{Z}^d \) when \( d \geq 3 \)? This example does not fall directly into the class we are considering since this process is transient, and therefore the set \( \tau \) of its returns to zero is a transient renewal sequence. However this is not too
bad. Indeed, suppose more generally that one is given $K(.)$ which satisfies (2.2) but such that $\Sigma := \sum_{n \in \mathbb{N}} K(n) < 1$, i.e., $K(.)$ is a sub-probability on $\mathbb{N}$. Then, one may define $\hat{K}(n) := K(n)/\Sigma$ which is obviously a probability. It is easy to realize from Eq. (2.8) that the Gibbs measure (and free energy) of the model defined starting from $K(.)$ is the same as that obtained starting from $\hat{K}(.)$, provided that $h$ is replaced by $h + \log \Sigma$. The case where $\tau$ are the zeros of the SRW on $\mathbb{Z}^d$ with $d \geq 3$ can then be included in our discussion: Eq. (2.2) holds with $\alpha = d/2 - 1$ and $L(.)$ asymptotically constant. In the following we will therefore always assume, without loss of generality, that $\tau$ is recurrent.

We conclude this section by listing a couple of examples of (bio)-physical situations where disordered pinning models are relevant:

- **Wetting of (1+1)-dimensional disordered substrates** [17] [21]. Consider a two-dimensional system at a first order phase transition, e.g., the 2d-Ising model at zero magnetic field and $T < T_c$, or a liquid-gas system on the coexistence line. Assume that the system is enclosed in a square box with boundary conditions imposing one of the two phases along the bottom side of the box and the other phase along the other three sides. For instance, for the Ising model one can impose + boundary conditions (b.c.) along the bottom side and − b.c. along the other ones; for the liquid-gas model, one imposes that the bottom of the box is in contact with liquid and that side and top walls are in contact with gas. Then, there is necessarily an interface joining the two bottom corners of the box and separating the two phases. At very low temperature, it is customary to describe this interface as a one-dimensional symmetric random walk (not necessarily the SRW) conditioned to be non-negative, the non-negativity constraint reflecting the fact that the interface cannot exit the box. The directed character of the random walk implies in particular that one is neglecting the occurrence of bubbles or overhangs in the interface. An interesting situation occurs when the bottom wall is “dirty” and at each point has a random interaction with the interface: at some points the wall prefers to be in contact with the gas (or − phase), and therefore tries to pin the interface, while at other points it prefers contact with the liquid (or + phase) and repels the interface. Of course, this non-homogeneous interaction is encoded in the charges $\omega_n$. In this context, the (de)-localization transition is called wetting transition. This denomination is clear if we think of the liquid-gas model: the localized phase corresponds to an interface which remains at finite distance from the wall (the wall is dry), while in the delocalized phase there are few interface-wall contacts and the height of the liquid layer on the wall diverges in the thermodynamic limit: the wall is wet. It is known that, in great generality [19], the law of the first return to zero of a one-dimensional random walk conditioned to be non-negative is of the form (2.2) with $\alpha = 1/2$ and $L(.)$ asymptotically constant (this process is transient but this fact is not so relevant, in view of the discussion at the beginning of the present section).

- **Formation of loops under thermal excitation and denaturation of DNA molecules** in the Poland-Scheraga (PS) approximation [15]. Neglecting its helical structure, the DNA molecule is essentially a double strand of complementary units, called “bases”. Upon heating, the bonds which keep base pairs together can break and the two strands can partly or entirely separate (cf. figure below). This separation, or denaturation, can be described in the context of our disordered pinning models. The set $\tau$ represents the set of bases whose bond is not broken. In the localized
phase $\tau$ contains $O(N)$ points ($N$ being interpreted here as the total DNA length), i.e., corresponds to the phase where the two strands are still essentially tightly bound. In the delocalized (or denaturated) phase, on the contrary, only few base pairs are bound. In formulating the PS model, one usually takes a value $\alpha \simeq 2.12$ (cf. [35] for a justification of this choice) and (in our notations, which are not necessarily those of the literature on the PS model)

$$L(n) = \sigma \quad \text{for} \quad n \geq 2,$$

where $\sigma$ (the cooperativity parameter) is a small number, usually of the order $10^{-5}$, while $L(1)$ is fixed by the normalization condition $\sum_{n\in\mathbb{N}} K(n) = 1$. Quenched disorder corresponds here to the fact that bases of the different types are placed inhomogeneously along the DNA chain. We refer to [22, Section 1.4] for a very clear introduction to the denaturation problem and the Poland-Scheraga model. Here we wish to emphasize only that the renewal process $\tau$ described by such a $K(.)$ is not in general the set of returns of a Markov chain, as it happens for instance in the case of the wetting model described above.

3. THE QUESTIONS WE ARE INTERESTED IN

The main questions which will be considered in these notes are the following:

1. When is the annealed bound (2.17) a good one, i.e., when are quenched and annealed systems similar? We will see that quenched and annealed free energies never coincide, except in the (trivial) case where the annealed free energy is zero (i.e., the annealed model is delocalized). However, this does not mean that the solution of the annealed system gives no information about the quenched one. For instance we will show that, for $\alpha < 1/2$ and weak enough disorder, the quenched critical point coincides with the annealed one. This will be discussed in Section 5.3.

2. What is the order of the transition? Critical exponents (in particular, the specific heat exponent, cf. next section) can be exactly computed for the homogeneous model. The Harris criterion predicts that for small $\beta$ critical exponents are those of the $\beta = 0$ (or annealed) model if $\alpha < 1/2$, and are different if $\alpha > 1/2$. This is the question of disorder relevance, discussed in Sections 5.3 and 5.6.

3. Truncated correlations functions are known to decay exponentially at large distance, in the localized phase. What is the behavior of the correlation length when the transition is approached? We will see that, due to the presence of quenched
disorder, one can actually define two different correlation lengths. In specific cases, we will identify these correlation lengths and give bounds on the critical exponents which govern their divergence at $h_c(\beta)$.

4. The homogeneous model

In absence of disorder ($\beta = 0$) the model is under full mathematical control; in particular, critical point and the order of the transition can be computed exactly. In this section, we collect a number of known results, referring to [22, Chapter 2] for their proofs.

The basic point is that the free energy $F(0, h)$ is determined as follows [26, Appendix A]: if the equation

$$\sum_{n \in \mathbb{N}} e^{-bn} K(n) = e^{-h}$$

has a positive solution $b = b(h) > 0$ then $F(0, h) = b(h)$. Otherwise, $F(0, h) = 0$. From this (recall the normalization condition $\sum_{n \in \mathbb{N}} K(n) = 1$), one finds immediately that $h_c(0) = 0$. The behavior of the free energy in the neighborhood of $h_c(0)$ can be also obtained from (4.1). Care has to be taken since a naive expansion of left- and right-hand sides of (4.1) for $b$ and $h$ small does not work in general. However, this analysis can be performed without much difficulty and one can prove the following:

**Theorem 4.1.** [22, Th. 2.1]

1. If $\alpha = 0$, $F(0, h)$ vanishes faster than any power of $h$ for $h \searrow 0$.
2. If $0 < \alpha < 1$ then for $h > 0$

$$F(0, h) = h^{1/\alpha} \hat{L}(1/h),$$

where $\hat{L}(\cdot)$ is the slowly varying function

$$\hat{L}(1/h) = \left( \frac{\alpha}{\Gamma(1 - \alpha)} \right)^{1/\alpha} h^{-1/\alpha} R_{\alpha}(h)$$

and $R_{\alpha}(\cdot)$ is asymptotically equivalent to the inverse of the map $x \to x^{\alpha} L(1/x)$.
3. If $\alpha = 1$ and $\sum_{n \in \mathbb{N}} nK(n) = \infty$ then $F(0, h) = h \hat{L}(1/h)$ for some slowly varying function $\hat{L}(\cdot)$ which vanishes at infinity.
4. If $\sum_{n \in \mathbb{N}} nK(n) < \infty$ (in particular, if $\alpha > 1$)

$$F(0, h) \overset{h \searrow 0}{\sim} \frac{h}{\sum_{n \in \mathbb{N}} nK(n)}.$$

In particular, note that in the situation (4), i.e., if $\tau$ is positively recurrent under $P$, the transition is of first order: the free energy is not differentiable at $h_c(0) = 0$, i.e., the average contact fraction has a finite jump in the thermodynamic limit. This is analogous to what happens for the Ising model in dimension $d \geq 2$: if $T < T_c$ and one varies the magnetic field $H$ from $0^{-}$ to $0^{+}$, the spontaneous magnetization has a positive jump and the free energy is not differentiable. The transition is, on the other hand, continuous (at least of second order) if $P$ is the law of a null-recurrent renewal $\tau$ and it becomes smoother as $\alpha$ decreases. In thermodynamical language, one can say that the delocalization transition is of $k^{th}$ order ($F(\beta, \cdot)$ is of class $C^{k-1}$ but not of class $C^{k}$) for $\alpha \in (1/k, 1/k - 1)$ and of infinite order for $\alpha = 0$.

---

1 In order to decide between $k^{th}$ and $(k + 1)^{th}$ order for $\alpha = 1/k$ one needs to look also at the slowly varying function $L(\cdot)$, as is already clear from points (3) and (4) in the case of $k = 1$. In any case, the precise statement is that of Theorem 4.1.
In the physics literature one introduces usually the specific heat critical exponent $\nu$ as\footnote{the symbol $\nu$ for the specific heat exponent is not standard in the literature, but we have already used the letter $\alpha$ for another purpose. The same remark applies to the symbols we use for other critical exponents.}

$$\nu = 2 - \lim_{h \to h_c(\beta)} \frac{\log F(\beta, h)}{\log(h - h_c(\beta))}$$  \hfill (4.5)

(provided the limit exists) and of course $\nu$ can depend on $\beta$. From Theorem 4.1 we see that, in absence of disorder,

$$\nu(\beta = 0) = 2 - \max(1, 1/\alpha).$$  \hfill (4.6)

In particular, note that $\nu(\beta = 0) > 0$ as soon as $\alpha > 1/2$ (this observation will become interesting in the light of the results of Section 5.6).

5. Relevance or irrelevance of disorder?

We have just seen that the phase transition of the homogeneous pinning model can be of any given order - from first to infinite - depending on the choice of $K(.)$ in (2.2) and, in particular, on the value of $\alpha$. In this section we discuss the effect of disorder on the transition and we are primarily interested in the question of disorder relevance. There are actually two distinct (but inter-related) aspects in this question:

Q1 does an arbitrarily small quantity of disorder change the critical exponent $\nu$ (i.e., the order of the transition)?

Q2 does the quenched critical point differ from the annealed one for very weak disorder?

One expects the answer to both questions to be “no” if $\alpha < 1/2$ and “yes” if $\alpha > 1/2$, while the case $\alpha = \alpha_c = 1/2$ is more subtle and not clear even heuristically \cite{17, 21} (see, however, Theorem 5.5).

The plan is the following: we will first of all (Section 5.1) make a non-rigorous computation, in the spirit of the Harris approach \cite{33}, which shows why the watershed value for $\alpha$, distinguishing between relevance and irrelevance, is expected to be $\alpha_c = 1/2$, i.e., the value for which the critical exponent $\nu$ vanishes for the homogeneous model (cf. (4.6)). Next, in Section 5.2 we prove an upper bound for the free energy which strictly improves the annealed bound (2.17). In the proof of this bound we introduce the technique of interpolation, by now classical in spin glass theory but sort of new in this context. We would like to emphasize that interpolation (and replica coupling, cf. Section 5.5) techniques have proven recently to be extremely powerful in the analysis of mean field spin glass models, cf. for instance \cite{32}, \cite{1}, \cite{39}, while their relevance in the domain of disordered pinning model had not been realized clearly so far.

As a byproduct, our new upper bound partially justifies the heuristic expansion of Section 5.1. The question of relevance is taken up more seriously in Sections 5.3 to 5.6. In the former we will see, among other results, that answers to both Q1 and Q2 are actually “no” for $\alpha < \alpha_c$. In the latter, on the other hand, we show that critical exponents are modified by disorder for $\alpha > \alpha_c$: in particular, we will see that $\nu \leq 0$ whenever $\beta > 0$.

In the whole of Section 5 we consider only the case of Gaussian disorder. This allows for technically simpler proofs, but results can be generalized for instance to the bounded disorder case.
5.1. **Harris criterion and the emergence of** $\alpha_c = 1/2$. Let us note for clarity that, putting together the discussion of Section 4 and Eq. (2.18), in the Gaussian case the annealed critical point equals $h_\alpha^0(\beta) = -\beta^2/2$. The first step of our heuristic argument is rigorous and, actually, an immediate identity:

$$F_N(\beta, h) = F^0_N(\beta, h) + \frac{1}{N} \mathbb{E} \log \left( e^{\sum_{n=1}^N (\beta \omega_n - \beta^2/2) \delta_n} \right)_{N, h-h_\alpha^0(\beta)},$$  
(5.1)

where $\langle \cdot \rangle_{N,h} := \mathbb{E}_{N,0,\cdot}^0(\cdot)$ is just the Boltzmann average for the homogeneous system (cf. Eq. 2.8). Identity (5.1) can be rewritten in a more suggestive way if we recall the last equality in (2.17) and we let $h = h_\alpha^0(\beta) + \Delta$ with $\Delta \geq 0$:

$$F_N(\beta, h_\alpha^0(\beta) + \Delta) = F_N(0, \Delta) + R_{N,\Delta}(\beta) := F_N(0, \Delta) + \frac{1}{N} \mathbb{E} \log \left( e^{\sum_{n=1}^N (\beta \omega_n - \beta^2/2) \delta_n} \right)_{N, \Delta},$$  
(5.2)

Irrelevance of disorder amounts to the fact that, for $\beta$ sufficiently small, the “error term” $R_{N,\Delta}(\beta)$ is negligible with respect to the “main term” $F_N(0, \Delta)$. As we will see, the question is subtle since we are interested in both $\Delta$ and $\beta$ small, and the two limits do not in general commute. For the moment, let us proceed without worrying about rigor and let us expand naively $R_{N,\Delta}(\beta)$ for $\beta$ small and $\Delta, N$ fixed:

$$\left( e^{\sum_{n=1}^N (\beta \omega_n - \beta^2/2) \delta_n} \right)_{N, \Delta} = 1 + \sum_{n=1}^N (\beta \omega_n - \beta^2/2) \langle \delta_n \rangle_{N, \Delta} + \frac{\beta^2}{2} \sum_{n,m=1}^N \omega_n \omega_m \langle \delta_n \delta_m \rangle_{N, \Delta} + O(\beta^3).$$  
(5.3)

Expanding the logarithm and using the fact that $\mathbb{E} \omega_n = 0$ and $\mathbb{E}(\omega_n \omega_m) = 1_{\{n=m\}}$ one has, always formally,

$$R_{N,\Delta}(\beta) = -\frac{\beta^2}{2 N} \sum_{n=1}^N \left( \langle \delta_n \rangle_{N, \Delta} \right)^2 + O(\beta^3).$$  
(5.4)

In the limit $N \to \infty$ one has by definition of the homogeneous model

$$\lim_{N \to \infty} \langle \ell_N \rangle_{N, \Delta} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \langle \delta_n \rangle_{N, \Delta} = \partial_\Delta F(0, \Delta).$$

Since $\langle \delta_n \rangle_{N, \Delta}$ should not depend on $n$ as soon as $1 \ll n \ll N$, one can expect (actually, this can be proven without much difficulty) that

$$\lim_{N \to \infty, n/N \to m \in (0,1)} \langle \delta_n \rangle_{N, \Delta} = \partial_\Delta F(0, \Delta).$$  
(5.5)

In conclusion, we find

$$F(\beta, h_\alpha^0(\beta) + \Delta) = F(0, \Delta) - \frac{\beta^2}{2} (\partial_\Delta F(0, \Delta))^2 + O(\beta^3).$$  
(5.6)

Even without trying (for the moment) to justify this expansion or to look more closely at the $\Delta$-dependence of the error term $O(\beta^3)$, we can extract something important from Eq. (5.6). We know from Theorem 1.1 that, for $\alpha < 1$ and $\Delta > 0$ small, $F(0, \Delta) \simeq \Delta^{1/\alpha}$ which implies (cf. the proof of Eq. (5.19) for details) that $\partial_\Delta F(0, \Delta) \simeq \Delta^{1/\alpha - 1}$. Then we see immediately that, indeed, for $\alpha < 1/2$

$$\frac{\beta^2}{2} (\partial_\Delta F(0, \Delta))^2 \ll F(0, \Delta)$$  
(5.7)

if $\Delta$ and $\beta$ are small. In terms of the Harris criterion, disordered is said to be irrelevant in this case and one can hope that the expansion can be actually carried on at higher orders.
For $1/2 < \alpha < 1$, however, this is false: even if $\beta$ is small, choosing $\Delta$ sufficiently close to zero the left-hand side of (5.7) is much larger than the right-hand side. This means that “disorder is relevant” and the small-disorder expansion breaks down immediately. The same holds for $\alpha \geq 1$, when $F(0, \Delta) \simeq \Delta$ and $\partial_\Delta F(0, \Delta) \sim \text{const}$. The threshold value $\alpha_c = 1/2$ is clearly a “marginal case” where relevance or irrelevance of disorder cannot be decided (even on heuristic grounds) by a naive expansion in $\beta$.

The rest of this section will be devoted to give rigorous bases to this suggestive picture. As a byproduct we will learn something interesting for the case $1 < \alpha < 1$: while disorder is relevant and changes the exponent $\nu$, it modifies the transition only “very close” to the critical point (cf. Theorem 5.3).

5.2. A rigorous approach: interpolation and an improvement upon annealing. In Section 2.5 we saw that a simple application of Jensen’s inequality implies $F(\beta, h) \leq F^\alpha(\beta, h)$. Here we wish to show that this inequality is strict as soon as disorder is present ($\beta > 0$) and the annealed system is localized. Moreover, we will partly justify the small-$\beta$ expansion of Section 5.1 for $\alpha < 1/2$, showing that it provides an upper bound for the quenched free energy.

More precisely:

**Theorem 5.1.** \cite{43} Th. 2.6 For every $\beta > 0$, $\alpha \geq 0$ and $\Delta > 0$

$$F(\beta, h^\alpha_c(\beta) + \Delta) \leq \inf_{0 \leq q \leq \Delta/\beta^2} \left( \frac{\beta^2 q^2}{2} + F(0, \Delta - \beta^2 q) \right) < F(0, \Delta) = F^\alpha(\beta, h). \tag{5.8}$$

In particular, if $0 \leq \alpha < 1/2$ there exist constants $\beta_0 > 0$, $\Delta_0 > 0$ such that

$$F(\beta, h^\alpha_c(\beta) + \Delta) \leq F(0, \Delta) - \frac{\beta^2}{2} (\partial_\Delta F(0, \Delta))^2 (1 + O(\beta^2)) \tag{5.9}$$

for $\beta \leq \beta_0$, $\Delta \leq \Delta_0$, where $O(\beta^2)$ is does not depend on $\Delta$. On the other hand, if $\beta = 0$ or $\Delta \leq 0$, then $F(\beta, h^\alpha_c(\beta) + \Delta) = F^\alpha(\beta, h^\alpha_c(\beta) + \Delta)$.

About the possibility of pushing the upper bound (5.9) to order higher than $\beta^2$ see Remark 3.1 in \cite{43}. It is obvious that (5.9) cannot hold for $\alpha > 1/2$ since, as already observed after Eq. (5.7), the right-hand side is negative for $\Delta$ sufficiently small.

Readers familiar with mean field spin glass models will remark a certain similarity between the variational bound (5.8) and the “replica symmetric” variational bound \cite{30} for the free energy of the Sherrington-Kirkpatrick model. However, we do not see a natural way to generalize (5.8) to include “replica symmetry breaking” in analogy with \cite{29} \cite{1}.

**Proof of Theorem 5.1.** The proof is rather instructive because it allows us to introduce the technique of “interpolation”, which will play a major role in the next subsection. We start from identity (5.2) and, for $\Delta > 0$, $q \in \mathbb{R}$ and $0 \leq t \leq 1$, we define

$$R_{N, \Delta}(t, \beta, q) := \frac{1}{N} \mathbb{E} \log \left( e^{\sum_{n=1}^N |\beta \sqrt{\omega_n - t \beta^2/2 + \beta^2 q (t-1) | \delta_n}} \right)_{\Delta, N}. \tag{5.10}$$

In spin glass language, this would be called an “interpolating free energy”, since by varying the parameter $t$ it relates in a smooth way the quantity we wish to estimate at $t = 1$,

$$R_{N, \Delta}(t = 1, \beta, q) = R_{N, \Delta}(\beta) \tag{5.11}$$

to something easy at $t = 0$:

$$R_{N, \Delta}(t = 0, \beta, q) = F_N(0, \Delta - \beta^2 q) - F_N(0, \Delta). \tag{5.12}$$
A priori, there is no reason why \( R_{N,\Delta}(t, \beta, q) \) should be any easier to compute for \( 0 < t < 1 \) than for \( t = 1 \). What helps us is that the \( t \)-derivative of \( R_{N,\Delta}(t, \beta, q) \) can be bounded above by throwing away a (complicated) term which, luckily, has a negative sign. To see this we need first of all manageable notations and we will set

\[
\langle g(\tau) \rangle_{N,\Delta,t} := \frac{\langle g(\tau)e^{\sum_{n=1}^{N}[\beta\sqrt{\omega_n-t\beta^2/2+\beta^2q(t-1)]\delta_n]} \rangle_{\Delta,N}}{\langle e^{\sum_{n=1}^{N}[\beta\sqrt{\omega_n-t\beta^2/2+\beta^2q(t-1)]\delta_n]} \rangle_{N,\Delta}}
\]

(5.13)

for every measurable function \( g(\tau) \). We find then

\[
\frac{d}{dt} R_{N,\Delta}(t, \beta, q) = \frac{\beta^2}{N} \left( -\frac{1}{2} + q \right) \sum_{m=1}^{N} \mathbb{E} \langle \delta_m \rangle_{N,\Delta,t} + \frac{\beta}{2\sqrt{tN}} \sum_{m=1}^{N} \mathbb{E} \omega_m \langle \delta_m \rangle_{N,\Delta,t}.
\]

(5.14)

The last term of (5.14) can be rewritten using the Gaussian integration by parts formula

\[
\mathbb{E}(\omega f(\omega)) = \mathbb{E}f'(\omega),
\]

(5.15)

which holds (if \( \omega \) is a Gaussian random variable \( \mathcal{N}(0,1) \)) for every differentiable function \( f(.) \) such that \( \lim_{|x| \to \infty} \exp(-x^2/2)f(x) = 0 \). In our case, the function \( f \) is of course \( (\delta_m)_{N,\Delta,t} \) and one finds

\[
\frac{\beta}{2\sqrt{tN}} \sum_{m=1}^{N} \mathbb{E} \omega_m \langle \delta_m \rangle_{N,\Delta,t} = \frac{\beta^2}{2N} \left( \sum_{m=1}^{N} \mathbb{E} \langle \delta_m \rangle_{N,\Delta,t} \right) - \left( \langle \delta_m \rangle_{N,\Delta,t} \right)^2.
\]

(5.16)

The positive term comes from the differentiation of the numerator of \( \langle \delta_m \rangle_{N,\Delta,t} \) (recall the definition (5.13)) and the negative one from the denominator, and we used the obvious \( \delta_m = (\delta_m)^2 \). Putting together Eqs. (5.14) and (5.16) one has therefore

\[
\frac{d}{dt} R_{N,\Delta}(t, \beta, q) = \frac{\beta^2q^2}{2} - \frac{\beta^2}{2N} \sum_{n=1}^{N} \mathbb{E} \left( \langle \delta_n \rangle_{N,\Delta,t} - q \right)^2 \leq \frac{\beta^2q^2}{2}.
\]

(5.17)

At this point we are done: we integrate on \( t \) between 0 and 1 inequality (5.17), we recall the boundary conditions (5.12) and (5.11) and we get

\[
R_{N,\Delta}(\beta) \leq F_N(0, \Delta - \beta^2q) - F_N(0, \Delta) + \frac{\beta^2q^2}{2},
\]

(5.18)

Together with Eq. (5.1), taking \( N \to \infty \) limit and minimizing over \( q \) proves (5.8). Let us remark that minimizing over \( q \in \mathbb{R} \) or on \( 0 \leq q \leq \Delta/\beta^2 \) is clearly equivalent. The strict inequality in (5.8) is just due to the fact that the derivative with respect to \( q \) of the quantity to be minimized, computed at \( q = 0 \), is negative.

The expansion (5.9) is just a consequence of (5.8). Remark first of all that, at the lowest order in \( \beta \), the minimizer in (5.8) is \( q = q_\Delta := \partial_\Delta F(0, \Delta) \). Then, from identity (4.14) one finds that there exist slowly varying functions \( L^{(i)}(\cdot), i = 1, 2 \) such that for \( \alpha < 1/2 \) and \( \Delta > 0 \)

\[
\partial_\Delta F(0, \Delta) = \Delta^{(1-\alpha)/\alpha} L^{(1)}(1/\Delta), \quad \partial_\Delta^2 F(0, \Delta) = \Delta^{(1-2\alpha)/\alpha} L^{(2)}(1/\Delta).
\]

(5.19)

Let us show for instance the first equality. Differentiating both sides of (4.14) with respect to \( \Delta \) one finds

\[
\partial_\Delta F(0, \Delta) = \frac{e^{-\Delta}}{\sum_{n \in \mathbb{N}} n^{-\alpha} L(n) \exp(-F(0, \Delta)n)}
\]

(5.20)
Using Theorems A.1 and A.2 one has then, for \( \Delta \to 0 \) (i.e., for \( F(0, \Delta) \to 0 \))

\[
\partial_{\Delta} F(0, \Delta) \overset{\Delta \to 0}{\longrightarrow} \frac{\Gamma(2 - \alpha)L(1/F(0, \Delta))}{(1 - \alpha)F(0, \Delta)^{1-\alpha}} \tag{5.21}
\]

which, together with \((5.12)\), proves the first equality in \((5.19)\) for a suitable \( L^{(1)}(.) \). Note, by the way, that thanks to \((5.19)\) one has \( q_{\Delta} < \Delta/\beta^2 \) for \( \Delta, \beta \) sufficiently small (and \( \alpha < 1/2 \), of course). Another consequence of \((5.19)\) is that \( \partial_{\Delta}^2 F(0, \Delta) \) is bounded above by a finite constant \( C \) for, say, \( \Delta \leq 1 \). Then, a Taylor expansion gives

\[
F(0, \Delta - \beta^2 q_{\Delta}) \leq F(0, \Delta) - \beta^2 (\partial_{\Delta} F(0, \Delta))^2 + C \beta^4 (\partial_{\Delta} F(0, \Delta))^2,
\]

whence Eq. \((5.9)\).

Finally, the statement for \( \beta = 0 \) or \( \Delta \leq 0 \) is trivial: for \( \beta = 0 \) there is no disorder to distinguish between quenched an annealed free energies, and for \( \Delta \leq 0 \) one has \( F^a(\beta, h^a_0(\beta) + \Delta) = 0 \) which, together with \((2.17)\) and \( F(\beta, h) \geq 0 \), implies the statement.

\[\square\]

5.3. Irrelevance of disorder for \( \alpha < 1/2 \) via replica coupling. We want to say first of all that, if \( 0 < \alpha < 1/2 \) and \( \beta \) is sufficiently small (i.e., if disorder is sufficiently weak), then \( h_c(\beta) = h^a_c(\beta) \). Recalling that \( F^a(\beta, h^a_c(\beta) + \Delta) = F(0, \Delta) \), this follows immediately from

**Theorem 5.2.** [1, 43] Assume that either \( 0 < \alpha < 1/2 \) or that

\[
\alpha = 1/2 \quad \text{and} \quad \sum_{n \in \mathbb{N}} n^{-1} L(n)^{-2} < \infty. \tag{5.22}
\]

Then, for every \( \epsilon > 0 \) there exist \( \beta_0(\epsilon) > 0 \) and \( \Delta_0(\epsilon) > 0 \) such that, for every \( \beta \leq \beta_0(\epsilon) \) and \( 0 < \Delta < \Delta_0(\epsilon) \), one has

\[
(1 - \epsilon) F(0, \Delta) \leq F(\beta, h^a_c(\beta) + \Delta) \leq F(0, \Delta). \tag{5.23}
\]

Observe that this implies in particular that, under the assumptions of the theorem, the exponent \( \nu \) equals \( 2 - 1/\alpha \) as in the homogeneous case. Indeed note that, for \( \Delta \) small,

\[
\frac{\log(1 - \epsilon) + \log F(0, \Delta)}{\log \Delta} \geq \frac{\log F(\beta, h_c(\beta) + \Delta)}{\log \Delta} \geq \frac{\log F(0, \Delta)}{\log \Delta} \tag{5.24}
\]

and the statement follows taking the limit \( \Delta \to 0 \) from definition \((4.5)\) of the specific heat exponent.

We will see in Section 5.6 that the same cannot hold for \( \alpha > 1/2 \): in that case, \( \nu \) is necessarily non-positive in for the quenched system presence of disorder, while it is positive for the annealed system. One could therefore think that quenched and annealed behaviors are completely different. This is however not completely true. Indeed, the next theorem shows that \( F(\beta, h) \) and \( F^a(\beta, h) \) are very close, provided that \( 1/2 \leq \alpha < 1 \) if one is not too close to the critical point. More precisely one has

**Theorem 5.3.** Assume that \( 1/2 < \alpha < 1 \). There exists a slowly varying function \( \tilde{L}(.) \) and, for every \( \epsilon > 0 \), constants \( a_1(\epsilon) < \infty \) and \( \Delta_0(\epsilon) > 0 \) such that, if

\[
a_1(\epsilon)\beta^{2a/(2\alpha-1)} \tilde{L}(1/\beta) \leq \Delta \leq \Delta_0(\epsilon), \tag{5.25}
\]

the inequalities \((5.23)\) hold.
To see more clearly what this says on the relation between quenched and annealed critical points, forget about the slowly varying functions; then, Theorem 5.3 implies
\[ 0 \leq h_c(\beta) - h_c^\alpha(\beta) \lesssim \beta^{2\alpha/(2\alpha - 1)}. \]

Since \( 2\alpha/(2\alpha - 1) > 2 \), this shows in particular that
\[ \lim_{\beta \searrow 0} h_c(\beta) = 1. \]

**Remark 5.4.** Theorem 5.3 was proven in [4, Th. 3] and then in [43, Th. 2.2]. The two results differ only in the form of the slowly varying function \( \tilde{L}(\cdot) \). In general, the function \( \tilde{L}(\cdot) \) which pops out from the proof in [43, Th. 2.2] is larger (i.e., worse) than that of [4, Th. 3].

Finally, we consider the “marginal case” \( \alpha = \alpha_c = 1/2 \) and \( \sum_n (L(n))^{-2} n^{-1} = \infty \). This is the case, for instance, if \( P \) is the law of the returns of a one-dimensional symmetric random walk, where \( L(\cdot) \) is asymptotically constant, as mentioned in Section 2.6 As we mentioned, this case is still debated even in the physical literature. The “most likely” scenario [17] is that disorder is “marginally relevant” in this case: \( h_c(\beta) \neq h_c^\alpha(\beta) \) for every positive \( \beta \), but the two critical points are equal at every order in a weak-disorder perturbation theory. Other works, e.g. [21], claim on the other hand that disorder is irrelevant in this situation.

What one can prove for the moment is the following:

**Theorem 5.5.** [4, 43] Assume that \( \alpha = 1/2 \) and \( \sum_n (L(n))^{-2} n^{-1} = \infty \). Let \( \ell(\cdot) \) be the slowly varying function (diverging at infinity) defined by
\[ \sum_{n=1}^{N} \frac{1}{n L(n)^2} \xrightarrow{N \to \infty} \ell(N). \]

For every \( \epsilon > 0 \) there exist constants \( a_2(\epsilon) < \infty \) and \( \Delta_0(\epsilon) > 0 \) such that, if \( 0 < \Delta \leq \Delta_0(\epsilon) \) and if the condition
\[ \frac{1}{\beta^2} \geq a_2(\epsilon) \ell \left( \frac{a_2(\epsilon) \log F(0, \Delta)}{F(0, \Delta)} \right) \]
(5.28)
is verified, then Eq. (5.23) holds.

**Remark 5.6.** To be precise, in the statement of [4, Th. 4] the condition (5.28) is replaced by a different one (essentially, the factor \( \log F(0, \Delta) \) in the argument of \( \ell(\cdot) \) does not appear). In this sense, the condition (5.28) under which we prove here (5.23) is not the best possible one. However, for many “reasonable” and physically interesting choices of \( L(\cdot) \) in (2.2), Theorem 5.5 and Theorem 4 of [4] are equivalent. In particular, if \( P \) is the law of the returns to zero of the simple random walk \( \{S_n\}_{n \geq 0} \) in one dimension, i.e. \( \tau = \{n \geq 0 : S_{2n} = 0\} \), in which case \( L(\cdot) \) and \( \tilde{L}(\cdot) \) are asymptotically constant and \( \ell(N) \sim a_3 \log N \), one sees easily that (5.23) is verified as soon as
\[ \Delta \geq a_4(\epsilon) e^{-\alpha(\epsilon) \beta^2}, \]
(5.29)
which is the same condition given in [4].

Note, by the way, that in this case the difference \( h_c(\beta) - h_c^\alpha(\beta) \) vanishes faster than any power of \( \beta \), for \( \beta \searrow 0 \). This confirms the fact that, even if the two critical points can be different, they cannot be distinguished perturbatively.
5.4. Some open problems. The results of previous section, while giving rigorous bases to predictions based on the Harris criterion, leave various intriguing gaps in our comprehension of the matter. Let us list a few of them, in random order:

- Let $\alpha < 1/2$. Does there exist a $\beta_c < \infty$ such that $h_c(\beta) \neq h_{\omega}^a(\beta)$ for $\beta > \beta_c$? If yes, how smooth is $h_c(\beta)$ at $\beta_c$? Does $\nu$ equal $2 - 1/\alpha$ also for $\beta$ large?
- Again, let $\alpha < 1/2$ and look at Eq. (5.9). Is it true that $F(\beta, h_{\omega}^a(\beta) + \Delta) \geq F(0, \Delta) - \frac{\beta^2}{2} (\partial_\Delta F(0, \Delta))^2 (1 + O(\beta^2))$?
- Under the assumptions of Theorems 5.3 or 5.5, does there exist positive values of $\beta$ for which quenched and annealed critical points coincide? It is sort of reasonable to conjecture that the answer is “no”, at least for $\alpha > 1/2$.

The reader might be tempted to think that such questions should be easy to answer numerically. If so, he should have a look at Ref. [12] where one gets an idea (in the context of random heteropolymers at selective interfaces) of why numerical tests become extremely hard in the neighborhood of the critical curve.

Remark 5.7. Between the time these notes were written and the time they were published, the above open problems have been to a large extent solved. In particular:

- in Ref. [44] it was proven that for every $\alpha > 0$, if $\beta$ is large enough and, say, $\omega$ is Gaussian, then $h_c(\beta) \neq h_{\omega}^a(\beta)$.
- The question posed in open problem (2) has been answered positively in Ref. [28], although in a slightly weaker sense.
- In Ref. [16] it was proven that as soon as $\alpha > 1/2$ and $\beta > 0$ one has $h_c(\beta) \neq h_{\omega}^a(\beta)$.

5.5. Proof of Theorems 5.2-5.5. We follow the approach of [43] which, with respect to that of [4], has the advantage of technical simplicity and of being closely related to the interpolation ideas of Section 5.2. On the other hand, we encourage the reader to look also at the methods developed in [4], which have the bonus of extending in a natural way beyond the Gaussian case and of giving in some cases sharper results (cf. Remarks 5.4 and 5.6 above).

A natural idea to show that quenched and annealed systems have (approximately) the same free energy is to apply the second moment method: one computes $\mathbb{E}(Z_N(\beta, h))$ and $\mathbb{E}((Z_N(\beta, h))^2)$ and if it happens that the ratio

$$\frac{\mathbb{E}(Z_N(\beta, h))^2}{\mathbb{E}((Z_N(\beta, h))^2)} \quad (5.30)$$

remains positive for $N \to \infty$, or at least it vanishes slower than exponentially, it is not difficult to deduce that $F(\beta, h) = F^q(\beta, h)$. This approach has turned out to be very powerful for instance in controlling the high-temperature phase of the Sherrington-Kirkpatrick mean field model in absence of magnetic field [40 Ch. 2.2]. However, this simple idea does not work in our case and the ratio (5.30) vanishes exponentially for every $\beta, \Delta > 0$. This is not surprising after all, since we already know from Theorem 5.1 that quenched and annealed free energy do not coincide. There are two possible ways out of this problem. One is to perform the second moment method not on the system of size $N$ but on a smaller system whose size $N(\Delta)$ remains finite as long as $\Delta$ is positive and fixed, and diverges only for $\Delta \to 0$. If $N(\Delta)$ is chosen to be the correlation length of the annealed system, one can see that on this scale the ratio (5.30) stays positive, so that $F_{N(\Delta)}(\beta, h_{\omega}^a(\beta) + \Delta) \approx F_N(0, \Delta)$. One is then left with the delicate problem of gluing together many blocks of size $N(\Delta)$.
to obtain an estimate of the type $F(\beta, h_n^\beta(\beta) + \Delta) \geq (1 - \epsilon) F(0, \Delta)$ for the full free energy. This is, in very rough words, the approach of Ref. [4]. The other possibility, which we are going to present, is to abandon the second moment idea in favor of a generalization of the replica coupling method [31, 33]. This method was introduced in [31] in the context of mean field spin glasses and gives a very efficient control of the Sherrington-Kirkpatrick model at high temperature ($\beta$ small), i.e., for weak disorder, which is the same situation we are after here.

The two methods are in reality not orthogonal: they share the idea that the important object to look at is the intersection of two independent renewals $\tau^{(1)}, \tau^{(2)}$. To see why this quantity arises naturally, let us compute the second moment of the partition function. If $\tau^{(1)}, \tau^{(2)}$ are independent renewal processes with product law $P^\otimes 2(\cdot)$, recalling the definition $\Delta = h + \beta^2/2$, one can write

$$E[(Z_{N,\omega}(\beta, h))^2] = E^\otimes 2 \left( e^{\sum_{n=1}^{N}(\beta \omega_n + h) \{ 1_{\{\tau^{(1)}\}} + 1_{\{\tau^{(2)}\}} \} 1_{\{N \in \tau^{(1)}\}} 1_{\{N \in \tau^{(2)}\}} } \right)$$

(5.31)

Considering also that

$$[E Z_{N,\omega}(\beta, h)]^2 = E^\otimes 2 \left( e^{\Delta |\tau^{(1)}\cap\{1,\ldots,N\}|+|\tau^{(2)}\cap\{1,\ldots,N\}|} 1_{\{N \in \tau^{(1)}\}} 1_{\{N \in \tau^{(2)}\}} \right)$$

one sees that the ratio (5.30) depends on the typical number of points that $\tau^{(1)}$ and $\tau^{(2)}$ have in common up to time $N$. One sees also why this ratio has to vanish exponentially $N \to \infty$: as long as $\Delta > 0$ the renewals $\tau^{(i)}$, with law modified by the factor $\exp(\Delta |\tau^{(1)}\cap\{1,\ldots,N\}|)$, are finitely recurrent and therefore will have a number of intersections in $\{1,\ldots,N\}$ which grows proportionally to $N$.

**Proof of Theorem 5.2.** The second inequality in (5.23) is just Eq. (2.17). As for the first one, let $\Delta > 0$ and recall identity (5.2). Define, in analogy with (5.10),

$$R_{N,\Delta}(t, \beta) := \frac{1}{N} E \log \left( e^{\sum_{n=1}^{N}(\beta \sqrt{\omega_n - t} \delta_n / 2) \Delta_{N}} \right)$$

(5.32)

for $0 \leq t \leq 1$ (to the purpose of Theorem 5.2 we do not need the variational parameter $q$) where the measure $\langle \cdot \rangle_{N,\Delta}$ was defined after Eq. (5.11). Observe that

$$R_{N,\Delta}(0, \beta) = 0$$

(5.33)

while

$$R_{N,\Delta}(1, \beta) = R_{N,\Delta}(\beta).$$

(5.34)

As for the $t$-derivative one finds (just take (5.17) and put $q = 0$):

$$\frac{d}{dt} R_{N,\Delta}(t, \beta) = -\frac{\beta^2}{2N} \sum_{m=1}^{N} E \left\{ \left( \frac{\frac{\delta_m e^{\sum_{n=1}^{N}(\beta \sqrt{\omega_n - t} \delta_n / 2) \Delta_{N}}}{\Delta_{N}}}{\frac{\delta_m e^{\sum_{n=1}^{N}(\beta \sqrt{\omega_n - t} \delta_n / 2) \Delta_{N}}}{\Delta_{N}}} \right)^2 \right\}. \quad (5.35)$$

Recall definition (5.13) (specialized to the case $q = 0$) of the random measure $\langle \cdot \rangle_{N,\Delta,t}$ and let $\langle \cdot \rangle_{N,\Delta,t}^\otimes 2$ be the product measure acting on the pair $(\tau^{(1)}, \tau^{(2)})$, while $\delta_n^{(i)} := 1_{\{n \in \tau^{(i)}\}}$. Note that the two replicas $\tau^{(i)}, i = 1, 2$ are subject to the same realization $\omega$ of disorder.
Then, one can rewrite
\[ \frac{d}{dt} R_{N,\Delta}(t, \lambda, \beta) = -\frac{\beta^2}{2N} \sum_{m=1}^{N} \left\langle \delta_{m}^{(1)} \delta_{m}^{(2)} e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta} \]

(5.37)

Since we need a lower bound for \( R_{N,\Delta}(\beta) \) to prove the first inequality in Eq. (5.23), the fact that this quantity is non-positive seems to go in the wrong direction. Let us not lose faith and let us define, for \( \lambda \geq 0 \),
\[ R_{N,\Delta}^{(2)}(t, \lambda, \beta) := \frac{1}{2N} \log \left\langle e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta} \]

(5.38)

while the factor 2 in the denominator guarantees that
\[ R_{N,\Delta}^{(2)}(t, 0, \beta) = R_{N,\Delta}(t, \beta). \]

(5.39)

Again via integration by parts (the computation is conceptually as easy as the one which led to Eq. (5.17)),
\[ \frac{d}{dt} R_{N,\Delta}^{(2)}(t, \lambda, \beta) = -\frac{\beta^2}{2N} \sum_{m=1}^{N} \left\langle \delta_{m}^{(1)} \delta_{m}^{(2)} e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta} \]

(5.40)

\[ \leq \frac{\beta^2}{2N} \sum_{m=1}^{N} \left\langle \delta_{m}^{(1)} \delta_{m}^{(2)} e^{H_N(t, \lambda, \beta; \tau^{(1)}, \tau^{(2)})} \right\rangle_{N,\Delta} \]

This can be rewritten as
\[ \frac{d}{dt} R_{N,\Delta}^{(2)}(t, \lambda - t, \beta) \leq 0 \]

which implies that, for every \( 0 \leq t \leq 1 \) and \( \lambda \),
\[ R_{N,\Delta}^{(2)}(t, \lambda, \beta) \leq R_{N,\Delta}^{(2)}(0, \lambda + t, \beta). \]

(5.41)

Going back to Eqs. (5.35) and the last equality in (5.40) and using the fact that for every convex function \( \psi(.) \) one has \( x \psi'(0) \leq \psi(x) - \psi(0) \) one finds
\[ \frac{d}{dt} (-R_{N,\Delta}(t, \beta)) = \frac{d}{d\lambda} R_{N,\Delta}^{(2)}(t, \lambda, \beta) \bigg|_{\lambda=0} \leq \frac{R_{N,\Delta}^{(2)}(t, 2 - t, \beta) - R_{N,\Delta}^{(2)}(t, 0, \beta)}{2 - t} \]

(5.42)
Finally, using monotonicity of \( R_{N,\Delta}^{(2)}(t,\lambda,\beta) \) with respect to \( \lambda \) and (5.39), one obtains the bound
\[
\frac{d}{dt} (-R_{N,\Delta}(t,\beta)) \leq R_{N,\Delta}^{(2)}(0,2,\beta) + (-R_{N,\Delta}(t,\beta)),
\]
where we used (5.41) and the fact that \( 2-t \geq 1 \) (of course, we could have chosen \( 1+\eta-t \) instead of \( 2-t \) for some \( \eta > 0 \) in (5.42) and the estimates would be modified in a straightforward way). We can now integrate with respect to \( t \) between 0 and 1 this differential inequality (or use Gronwall’s Lemma, if you prefer) and, recalling Eqs. (5.34) and (5.33), we obtain
\[
-(e-1)R_{N,\Delta}^{(2)}(0,2,\beta) \leq R_{N,\Delta}(\beta) \leq 0.
\]

Before we proceed, we would like to summarize what we did so far. To prove Theorem 5.2 we need the lower bound \( \lim_{N \to \infty} R_{N,\Delta}(\beta) \geq -eF(0,\Delta) \) but, as in Section 5.2, it seems that the interpolation method gives rather upper bounds on \( R_{N,\Delta}(\beta) \). Then, through the replica coupling trick we transferred this problem into the problem of proving an upper bound for a quantity, \( R_{N,\Delta}^{(2)}(t,\lambda,\beta) \), which is analogous to \( R_{N,\Delta}(\beta) \), except that it involves two interacting copies of the system. Moreover, by throwing away a (complicated, but with a definite sign) term in Eq. (5.40), we reduced to the problem of bounding from above \( R_{N,\Delta}^{(2)}(t,\lambda,\beta) \) computed at \( t = 0 \). In other words, we replaced the task of estimating from below \( R_{N,\Delta}(\beta) \) with that of estimating from above a quantity which involves no quenched disorder, and which is therefore easier to analyze. While this procedure might look a bit magic, the basic underlying idea is the following. \( R_{N,\Delta}^{(2)}(t,\lambda,\beta) \) is obviously non-decreasing as a function of \( \lambda \). Suppose however that, for some \( \lambda > 0 \), \( R_{N,\Delta}^{(2)}(t,\lambda,\beta) \) is not very different from the value it has at \( \lambda = 0 \) (of course, proving this amounts to proving an upper bound on \( R_{N,\Delta}^{(2)}(t,\lambda,\beta) \).) Then, looking at the definition (5.37), this means that the cardinality of the intersection \( \tau^{(1)} \cap \tau^{(2)} \cap \{1,\ldots,N\} \) is typically not large and this, through Eqs. (5.34), (5.31) and (5.36) implies a lower bound on \( R_{N,\Delta}(\beta) \).

Let us now restart from (5.44) and note that
\[
R_{N,\Delta}^{(2)}(0,2,\beta) = -F_N(0,\Delta) + \frac{1}{2N} \log E^{\otimes 2}\left(e^{2\varphi^2 \sum_{n=1}^{N} \delta_n^{(1)} \delta_n^{(2)} + \Delta \sum_{n=1}^{N} (\delta_n^{(1)} + \delta_n^{(2)})^2} \right)
\leq -F_N(0,\Delta) + \frac{F_N(0,q\Delta)}{q} + \frac{1}{2Np} \log E^{\otimes 2}\left(e^{2p\beta^2 \sum_{n=1}^{N} \delta_n^{(1)} \delta_n^{(2)}} \right)
\]
where we used Hölder’s inequality and the positive numbers \( p \) and \( q \) (satisfying \( 1/p + 1/q = 1 \) are to be determined. Taking the thermodynamic limit,
\[
\lim_{N \to \infty} R_{N,\Delta}^{(2)}(0,2,\beta) \leq \lim_{N \to \infty} \frac{1}{2Np} \log E^{\otimes 2}\left(e^{2p\beta^2 \sum_{n=1}^{N} \delta_n^{(1)} \delta_n^{(2)}} \right) + F(0,\Delta) \left( \frac{1}{q} \frac{F(0,q\Delta)}{F(0,\Delta)} - 1 \right).
\]
But we know from the expression (4.2) of the free energy of the homogeneous system and from the property (2.3) of slow variation that, for every \( q > 0 \),
\[
\lim_{\Delta \to 0^+} \frac{F(0,q\Delta)}{F(0,\Delta)} = q^{1/\alpha}.
\]
Therefore, taking \( q = q(\epsilon) \) sufficiently close to (but strictly larger than) 1 and \( \Delta_0(\epsilon) > 0 \) sufficiently small one has, uniformly on \( \beta \geq 0 \) and on \( 0 < \Delta \leq \Delta_0(\epsilon) \),
\[
\lim_{N \to \infty} R_{N, \Delta}^{(2)}(0, 2, \beta) \leq \frac{\epsilon}{e - 1} F(0, \Delta) + \lim_{N \to \infty} \frac{1}{2Np(\epsilon)} \log E^{\otimes 2} \left( e^{2p(\epsilon) \beta^2 \sum_{n=1}^{N} \delta_n^{(1)} \delta_n^{(2)}} \right) \tag{5.48}
\]
Of course, \( p(\epsilon) := q(\epsilon)/(q(\epsilon) - 1) < \infty \) as long as \( \epsilon > 0 \). Note that, in view of (5.44), Theorem 5.2 would be proved if the second term in the right-hand side of (5.48) were zero. Up to now, we have not used yet the assumption that \( \alpha < 1/2 \) or that (5.22) holds, but now the right moment has come. The way this assumption enters the game is that it guarantees that the renewal \( \tau^{(1)} \cap \tau^{(2)} \) is transient under the law \( P^{\otimes 2} \). Indeed,
\[
E^{\otimes 2} \left( \sum_{n \geq 1} 1_{n \in \tau^{(1)} \cap \tau^{(2)}} \right) = \sum_{n \geq 1} P(n \in \tau)^2 < \infty \tag{5.49}
\]
since, as proven in [18],
\[
P(n \in \tau)^{n \to \infty} \sim \frac{C_\alpha}{L(n)n^{1-\alpha}} := \frac{\alpha \sin(\pi \alpha)}{\pi} \frac{1}{L(n)n^{1-\alpha}}. \tag{5.50}
\]
Actually, Eq. (5.50) holds more generally for \( 0 < \alpha < 1 \) and we will need it to prove Theorems 5.3 and 5.5.

Transience and renewal properties of the process of \( \tau^{(1)} \cap \tau^{(2)} \) implies that
\[
P^{\otimes 2}(|\tau^{(1)} \cap \tau^{(2)}| \geq k) \leq (1 - c)^k, \tag{5.51}
\]
for some \( 0 < c < 1 \): after each “renewal epoch”, i.e., each point of \( \tau^{(1)} \cap \tau^{(2)} \), the intersection renewal has a positive probability \( c \) of jumping to infinity. Therefore, there exists \( \beta_1 > 0 \) such that
\[
\sup_N E^{\otimes 2} \left( e^{2p(\epsilon) \beta^2 \sum_{n=1}^{N} \delta_n^{(1)} \delta_n^{(2)}} \right) < \infty \tag{5.52}
\]
for every \( \beta^2 p(\epsilon) \leq \beta_1^2 \). Together with (5.48) and (5.2), this implies
\[
F(\beta, h_0^\alpha(\beta) + \Delta) \geq (1 - \epsilon) F(0, \Delta) \tag{5.53}
\]
as soon as \( \beta^2 \leq \beta_0^2(\epsilon) := \beta_1^2/p(\epsilon) \), and therefore the validity of Theorem 5.2. \( \square \)

Proof of Theorem 5.3. In what follows we assume that \( \Delta \) is sufficiently small so that \( F(0, \Delta) \ll 1 \). For simplicity of exposition, we assume also that \( L(.) \) tends to a positive constant \( L(\infty) \) at infinity (for the general case, which is not significantly more difficult, cf. [43]).

If we try to repeat the proof of Theorem 5.3 in this case, what goes wrong is that the intersection \( \tau^{(1)} \cap \tau^{(2)} \) is now recurrent, so that (5.52) does not hold any more. The natural idea is then not to let \( N \) tend to infinity at \( \Delta \) fixed, but rather to work on a system of size \( N(\Delta) \), which diverges only when \( \Delta \to 0 \), i.e., when the annealed critical point is approached. In particular, we let
\[
N = N(\Delta) := c \log F(0, \Delta)/F(0, \Delta) \quad \text{with} \quad c > 0
\]
large to be fixed later. Note also that this choice of \( N(\Delta) \) is quite similar to that made in [4], where one applies the second moment method on a system of size \( c/F(0, \Delta) \) with \( c \) large. This choice has a clear physical meaning: indeed, we will see in Section 6 that the correlation functions of the annealed system decay exponentially on distances of order \( 1/F(0, \Delta) \) (the logarithmic factor in our definition of \( N(\Delta) \) should be seen as a technical necessity).
By the superadditivity property (2.11) we have, in analogy with (5.1),
\[ F(\beta, -\beta^2/2 + \Delta) \geq F_N(\Delta) + R_N(\Delta, \Delta(\beta)). \]  
(5.54)

To prove Theorem 5.3 we need to show that the first term in the right-hand side of (5.54) is essentially \( F(0, \Delta) \), while the second is not smaller than \(-\epsilon F(0, \Delta)\), in the range of parameters determined by condition (5.25). The first fact is easy: as follows from Proposition 2.7 of [25], there exists \( a_6 \in (0, \infty) \) (depending only on the law \( K(\cdot) \) of the renewal) such that
\[ F_N(0, \Delta) \geq F(0, \Delta) - a_6 \frac{\log N}{N}. \]  
(5.55)

for every \( N \). Choosing \( c = c(\epsilon) \) large enough, Eq. (5.55) implies that
\[ F_N(0, \Delta) \geq (1 - \epsilon)F(0, \Delta). \]  
(5.56)

As for \( R_N(\Delta, \Delta(\beta)) \), we have from Eqs. (5.44) and (5.45)
\[ \frac{R_N(\Delta, \Delta(\beta))}{e - 1} \geq -F(0, \Delta) \left( \frac{1}{q} \frac{F(0, q\Delta)}{F(0, \Delta)} - 1 \right) - \epsilon F(0, \Delta) \]
\[ - \frac{1}{2N(\Delta)p} \log \mathbb{E}^\otimes 2 \left( e^{2p(\epsilon)} \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)} \right), \]
(5.57)

where we used Eqs. (5.56) and (2.11) to bound \(-1/q)F_N(\Delta)(0, q\Delta) + F_N(\Delta)(0, \Delta)\) from below. Choosing again \( q = q(\epsilon) \) we obtain, for \( \Delta \leq \Delta_0(\epsilon) \),
\[ \frac{R_N(\Delta, \Delta(\beta))}{e - 1} \geq -2\epsilon F(0, \Delta) - \frac{1}{2N(\Delta)p(\epsilon)} \log \mathbb{E}^\otimes 2 \left( e^{2p(\epsilon)} \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)} \right), \]  
(5.58)

It was proven in [4] Lemma 3] and [43 Section 3.1] that if \( 1/2 < \alpha < 1 \) there exists \( a_7 = \in (0, \infty), \) which depends in particular on \( L(\infty), \) such that for every integers \( N \) and \( k \)
\[ \mathbb{P}^\otimes 2 \left( |\tau^{(1)} \cap \tau^{(2)} \cap \{1, \ldots, N\}| \geq k \right) \leq \left( 1 - \frac{a_7}{N^{2\alpha - 1}} \right)^k, \]  
(5.59)

which should be compared with (5.51), valid for \( \alpha < 1/2. \) Thanks to the geometric bound (5.59) we have
\[ \mathbb{E}^\otimes 2 \left( e^{2p(\epsilon)} \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)} \right) = \sum_{k \geq 0} \mathbb{P}^\otimes 2 \left( \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)} = k \right) e^{2p(\epsilon)} \]
\[ \leq \left( 1 - e^{2\beta^2 p(\epsilon)} \left( 1 - \frac{a_7}{N(\Delta)^{2\alpha - 1}} \right) \right)^{-1}, \]
(5.60)

whenever
\[ e^{2\beta^2 p(\epsilon)} \left( 1 - \frac{a_7}{N(\Delta)^{2\alpha - 1}} \right) < 1 \]
and this is of course the case if
\[ e^{2\beta^2 p(\epsilon)} \left( 1 - \frac{a_7}{N(\Delta)^{2\alpha - 1}} \right) \leq \left( 1 - \frac{a_7}{2N(\Delta)^{2\alpha - 1}} \right). \]  
(5.61)

At this point, using the definition of \( N(\Delta) \) and point (2) of Theorem 4.1, it is not difficult to see that there exists a positive constant \( a_8(\epsilon) \) such that (5.61) holds if
\[ \beta^2 p(\epsilon) \leq a_8(\epsilon) \frac{\Delta^{(2\alpha - 1)/\alpha}}{\log F(0, \Delta)^{2\alpha - 1}}. \]  
(5.62)
Condition (5.62) is equivalent to the first inequality in (5.25), for a suitable choice of $a_1(\epsilon)$ and $\tilde{L}(\cdot)$. As a consequence, for $N(\Delta)$ sufficiently large (i.e., for $\Delta$ sufficiently small)\[\frac{1}{2N(\Delta)p(\epsilon)} \log E^{\otimes 2} \left( e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \frac{F(0, \Delta)}{2c(\epsilon)p(\epsilon) \log F(0, \Delta)} \log \left( \frac{2N(\Delta)^{2\alpha-1}}{a_7} \right)\] (5.63)

Recalling Eq. (4.2) one sees that, if $c(\epsilon)$ is chosen large enough,
\[\frac{1}{2N(\Delta)p(\epsilon)} \log E^{\otimes 2} \left( e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \epsilon F(0, \Delta).\] (5.64)
Together with Eqs. (5.54), (5.56) and (5.58), this concludes the proof of the theorem. \(\Box\)

**Proof of Theorem 5.5** The proof is almost identical to that of Theorem 5.3 and up to Eq. (5.55) no changes are needed. One has however to be careful with the geometric bound (5.59): in this case, it is not sufficient to replace $\alpha$ by 1/2, since the behavior at infinity of the slowly varying function $L(\cdot)$ in (2.2) is here essential. The correct bound in this case is (cf. [4, Lemma 3] and [43, Sec. 3.1])
\[P^{\otimes 2} \left( \sum_{n=1}^{N} \delta_n^{(1)} \delta_n^{(2)} \geq k \right) \leq \left( 1 - \frac{a_9}{\ell(N)} \right)^k.\] (5.65)
for every $N$, for some $a_9 > 0$. We recall that $\ell(\cdot)$ is the slowly varying function, diverging at infinity, defined by (5.27). In analogy with Eq. (5.60) one obtains then
\[E^{\otimes 2} \left( e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \left( 1 - e^{2p(\epsilon) \left( 1 - \frac{a_9}{\ell(N(\Delta))} \right)} \right)^{-1}\] (5.66)
whenever the right-hand side is positive. Choosing $a_2(\epsilon)$ large enough one sees that if condition (5.28) is fulfilled then
\[e^{2\beta^2 p(\epsilon) \left( 1 - \frac{a_9}{\ell(N(\Delta))} \right)} \leq \left( 1 - \frac{a_9}{2\ell(N(\Delta))} \right)\] (5.67)
and, in analogy with (5.63),
\[\frac{1}{2N(\Delta)p(\epsilon)} \log E^{\otimes 2} \left( e^{2p(\epsilon)\beta^2 \sum_{n=1}^{N(\Delta)} \delta_n^{(1)} \delta_n^{(2)}} \right) \leq \frac{F(0, \Delta)}{2c(\epsilon)p(\epsilon) \log F(0, \Delta)} \log \left( \frac{2\ell(N(\Delta))}{a_9} \right)\] (5.68)
From this estimate, for $c(\epsilon)$ sufficiently large one obtains again (5.64) and as a consequence the statement of Theorem 5.5. \(\Box\)

5.6. **Smoothing effect of disorder (relevance for $\alpha > 1/2$).** Section 5.3 was devoted to showing that, for $\alpha < \alpha_c$, (weak) disorder is irrelevant, in that it does not change the specific heat exponent $\nu$ and in that the transition point coincides with the annealed one as long as $\beta$ is small. We saw also that for $\alpha_c \leq \alpha < 1$ quenched and annealed free energies and critical points are very close (Theorems 5.3 and 5.5). This might leave the reader with the doubt that disorder might be irrelevant in this situation too. The purpose of the present section is to show that this is not the case.

We start by recalling that via Theorem 4.1 and (2.17) we know that $F(\beta, h_c^q(\beta) + \Delta) \lesssim \Delta^{\max(1/\alpha, 1)}$. This bound is however quite poor: if we go back to (5.8) and we choose $q = \Delta/\beta^2$ we obtain
\[F(\beta, h_c^q(\beta) + \Delta) \leq \frac{\Delta^2}{2\beta^2}\] (5.69)
which is better, for \( \Delta \) small and \( \alpha > 1/2 \). The point is however that, since one expects that \( h_c^a(\beta) \neq h_c(\beta) \) in this situation, (5.69) does not say anything about the critical behavior of the quenched system; for this, we would need rather an upper bound on \( F(\beta, h_c(\beta) + \Delta) \).

This is just the content of the following result, which we state in the case of Gaussian disorder:

**Theorem 5.8.** \([26, 27]\) For every \( \beta > 0 \), \( \alpha > 0 \) and \( \Delta > 0 \) one has

\[
F(\beta, h_c(\beta) + \Delta) \leq \frac{(1 + \alpha)}{2\beta^2} \Delta^2.
\] (5.70)

**Remark 5.9.** Theorem 5.8 actually holds beyond the Gaussian case; for instance, in the case of bounded variables \( \omega_n \). In this case the statement has to be modified in that the factor 2 in that the denominator in the right-hand side of (5.70) is replaced by \( c := c(\mathbb{P}) \), a constant which only depends on the disorder distribution \( \mathbb{P} \), and the results holds only provided \( \Delta \) is sufficiently small: \( \Delta \leq \Delta_0(\mathbb{P}) \), see [26].

**Remark 5.10.** An obvious implication of Theorem 5.8 is that \( \nu \leq 0 \) as soon as \( \beta > 0 \). In this sense, this result is much reminiscent of what was proven in [13, 14] about the specific heat exponent for the nearest-neighbor disordered Ising ferromagnet.

In particular, Theorem 5.8 shows that the specific heat exponent is modified by an arbitrary amount of disorder if \( \alpha > \alpha_c \); the phase transition is smoothed by randomness if \( \alpha > \alpha_c \) and becomes at least of second order (the effect is particularly dramatic for \( \alpha > 1 \), where the transition is of first order for \( \beta = 0 \)).

It is also interesting to compare Theorem 5.8 with the celebrated result by M. Aizenman and J. Wehr [2] which states that first order phase transition in spin systems with discrete spin-flip symmetry are smoothed by disorder as long as the spatial dimension verifies \( d \leq 2 \), while the same holds for \( d \leq 4 \) if the symmetry is continuous.

A less obvious consequence of Theorem 5.8 is the following:

**Theorem 5.11.** \([41]\) Let \( \beta > 0 \) and \( 0 \leq \alpha < \infty \). There exists \( c > 0 \) such that

\[
\lim_{N \to \infty} \mathbb{E}_{N, \omega} \mathbb{P}_{N, \omega}^{\beta, h_c(\beta)} \left( |\tau \cap \{1, \ldots, N\}| \geq c N^{2/3} \log N \right) = 0.
\] (5.71)

Moreover, under the assumptions of Theorem 5.2, for \( \beta \) sufficiently small

\[
\lim_{N \to \infty} \mathbb{E}_{N, \omega} \mathbb{P}_{N, \omega}^{\beta, h_c(\beta)} \left( |\tau \cap \{1, \ldots, N\}| \geq c N^{2\alpha/(1+\alpha)} \log N \right) = 0.
\] (5.72)

This result should be read as follows. The fact that the transition is at least of second order in presence of disorder implies already that the Gibbs average of the contact fraction defined by (2.14) tends to zero in the thermodynamic limit at the critical point. The additional information provided by Theorem 5.11 are finite-\( N \) estimates on the size of \( \tau \cap \{1, \ldots, N\} \) at criticality. Whether the exponent \( 2/3 \) in Eq. (5.71) is optimal or not is an intriguing open question.

Theorem 5.11 was proven in [41] (together with more refined finite-size estimates on \( \mathbb{P}_{N, \omega}^{\beta, h_c(\beta)}(|\tau \cap \{1, \ldots, N\}|) \) for \( h - h_c(\beta) \) going to zero with \( N \)), apart from Eq. (5.72) which is a consequence of [41] Th. 3.1 plus Theorem 5.2 (cf. also Remark 3.2 in [41]).

---

3 Theorem 3.1 in [41] is formulated in the case of bounded random variables \( \omega_n \), but it generalizes immediately to the Gaussian because the basic ingredient one needs is the concentration inequality [41] Eq. (5.2)], which holds in the case of Gaussian randomness as well.
Proof of Theorem 5.8 (sketch) For a fully detailed proof we refer to [26]. In the case of Gaussian disorder a simpler proof is hinted at in [27] and fully developed in [22] Section 5.4.

Here we give just a sketchy idea of why the transition cannot be of first order when \( \beta > 0 \). Assume by contradiction that
\[
F(\beta, h_c(\beta) + \Delta) \sim c\Delta \quad \text{for} \quad \Delta \to 0^+,
\]
and consider the system at the critical point \((\beta, h_c(\beta))\). Divide the system of size \( N \) into \( N/M \) blocks \( B_i \) of size \( M \), with the idea that \( 1 \ll M \ll N \). For a given realization of \( \omega \) mark the blocks where the empirical average of \( \omega \), i.e., \((1/M)\sum_{n \in B_i} \omega_n\) equals approximately \( \Delta/\beta \). By standard large deviation estimates, there are typically \( N_{\text{marked}} := (N/M)e^{-M\Delta^2/(2\beta^2)} \) such blocks, the typical distance between two successive ones being \( D_{\text{typ}} := Me^{M\Delta^2/(2\beta^2)} \). It is a standard fact that if we take \( M \) IID standard Gaussian variables and we condition on their empirical average to be \( \delta \), for \( M \) large they (roughly speaking) distributed like IID Gaussian variables of variance 1 and average \( \delta \). Therefore, in a marked block the system sees effective thermodynamic parameters \((\beta_{eff}, h_{eff}) := (\beta, h_c(\beta) + \Delta)\). Now we want to show that the assumption (5.73) leads to the (obviously false) conclusion that \( F(\beta, h_c(\beta)) > 0 \). Indeed, let \( S_\omega \) be the set of \( \tau \) configurations such that:

- there are no points of \( \tau \) in unmarked blocks
- the boundaries of all marked blocks belong to \( \tau \).

Note that \( S_\omega \) depends on disorder through the location and the number of marked blocks, and that there is no restriction on \( \tau \) inside marked blocks. One has the obvious bound
\[
F_N(\beta, h_c(\beta)) \geq \frac{1}{N}E \log E \left( e^{\sum_{n=1}^N (\beta\omega_n + h)\delta_n} 1_{\{\tau \in S_\omega\} \delta_N} \right). \tag{5.74}
\]
But due to the definition of the set \( S_\omega \), the restricted free energy in the right-hand side of (5.74) gets (for \( M \) large) a contribution \( N_{\text{marked}} \times (M/N)F(\beta, h_c(\beta) + \Delta) \) from marked blocks, and an entropic term \( N_{\text{marked}}/N \times \log K(D_{\text{typ}}) \) from the excursions between marked blocks. Summing the two contributions, recalling the asymptotic behavior (2.2) of \( K(.) \), the expression of \( N_{\text{marked}} \) and \( D_{\text{typ}} \) and taking the \( N \to \infty \) limit at \( M \) large but fixed one obtains then
\[
F(\beta, h_c(\beta) + \Delta) \geq e^{-M\Delta^2/(2\beta^2)} \left( F(\beta, h_c(\beta) + \Delta) - (1 + \alpha)\frac{\Delta^2}{2\beta^2} \right). \tag{5.75}
\]
Since the left-hand side of (5.75) is zero, for \( \Delta \) small and \( \beta > 0 \) this inequality is clearly in contradiction with the assumption (5.73) that the transition if of first order (actually, even with the assumption \( F(\beta, h_c(\beta) + \Delta) \sim c\Delta^y \) with \( y < 2 \)).

6. Correlation lengths and their critical behavior

From certain points of view, the localized region \( \mathcal{L} \) is analogous to the high-temperature phase of a spin system. Indeed, in this region one can prove typical high-temperature results like the following: free energy fluctuations are Gaussian on the scale \( 1/\sqrt{N} \) [3, 23], the infinite-volume Gibbs measure is almost-surely unique and ergodic [9], the free energy is infinitely differentiable, finite-size corrections to the infinite volume free energy are of order \( O(1/N) \), and truncated correlation functions decay exponentially with distance [24]. In this section we concentrate on the last point, which turns out to be more subtle than expected, in particular when one approaches the critical line.
In this section we assume that the random variables $\omega_n$ are bounded, because the results we mention have been proved in the literature under this assumption. They should however reasonably extend to more general situations, for instance to the Gaussian case.

In the following, $P_{\infty,\omega}^{\beta,h}(\cdot)$ will denote the infinite-volume Gibbs measure, defined as follows: first of all we modify definitions (2.6) and (2.8) replacing

$$\sum_{n=1}^{N}(\beta \omega_n + h)\delta_n$$

by

$$\left\lfloor \frac{N}{2} \right\rfloor \sum_{n=-\left\lfloor \frac{N}{2} \right\rfloor}^{\left\lfloor \frac{N}{2} \right\rfloor}(\beta \omega_n + h)\delta_n,$$

where $\{\omega_n\}_{n \in \mathbb{Z}}$ are IID random variables, and then for a local observable $f$, i.e., a function of $\tau$ which depends only on $\tau \cap I$ with $I$ a finite subset of $\mathbb{Z}$, we let

$$E_{\infty,\omega}^{\beta,h}(f) := \lim_{N \to \infty} E_{N,\omega}^{\beta,h}(f). \quad (6.1)$$

Existence of the limit, in the localized phase, for almost every disorder realization is proven in [25] (cf. also [9], where a DLR-like point of view is adopted).

The definition of the correlation length $\xi$ contains always some degree of arbitrariness, but conventional wisdom on universality states that the critical properties of $\xi$, close to a second-order phase transition, are insensitive to the precise definition. There is however a subtlety: in the case of disordered systems there are two possible definitions of correlation lengths, which have no reason to have the same critical behavior. Remaining for definiteness in the framework of our disordered pinning models, one can first of all define a (disorder-dependent) two-point function as

$$C_{\omega}(k,\ell) := P_{\infty,\omega}^{\beta,h}(k \in \tau | \ell \in \tau) - P_{\infty,\omega}^{\beta,h}(k \in \tau). \quad (6.2)$$

In words, $C_{\omega}(k,\ell)$ quantifies how much the occurrence of $\ell \in \tau$ influences the occurrence the event $k \in \tau$. It is then natural to define a correlation length $\xi$ as

$$\frac{1}{\xi} := - \lim_{k \to \infty} \frac{1}{k} \log |C_{\omega}(k,0)|, \quad (6.3)$$

provided the limit exists. Note that $\xi$ depends on $(\beta,h)$ and, in principle, on $\omega$. One can however define a different correlation length, $\xi^{av}$, as

$$\frac{1}{\xi^{av}} := - \lim_{k \to \infty} \frac{1}{k} \log E|C_{\omega}(k,0)|. \quad (6.4)$$

In other words, $\xi$ (respectively, $\xi^{av}$) is the length over which the two-point function (respectively, the averaged two-point function) decays exponentially. For simplicity, we will call $\xi$ the typical (or quenched) correlation length, and $\xi^{av}$ the average correlation length, although it is important to keep in mind that $\xi^{av}$ is not the disorder-average of $\xi$ (indeed, in Section 6.3 we will see an example where $\xi$ is almost-surely constant but $\xi \neq \xi^{av}$).

It is interesting that in the case of the one-dimensional quantum Ising chain with random transverse field studied in [20], the two correlation lengths are believed, on the basis of a renormalization group analysis, to diverge at criticality with two different critical exponents.

---

4 One might give a different definition of the infinite-volume Gibbs measure, considering the original system (2.8) defined in $\{1, \ldots, N\}$ and taking the $N \to \infty$ limit of the average of local functions of $\tau \cap I$, with $I$ a finite subset of $N$. In other words, with the first procedure, Eq. (6.1), we are looking at the system in a window which is situated in the bulk, very far away from both boundaries. On the other hand, the second procedure is relevant if one wants to study the system in the vicinity of one of the two boundaries (and very far away from the other one).
A simple application of Jensen’s inequality shows that $\xi_{av} \geq \xi$. This inequality can be interpreted on the basis of the following intuitive argument. Divide all possible disorder realizations into sets $A_m$ where the empirical average of $\omega$ in the region $\{1, \ldots, k\}$ is approximately $m$. Of course, for $m \neq 0$ $A_m$ is a large deviation-like event of probability $\simeq \exp(-km^2/2)$.

Conditionally on $A_m$, the system sees a defect line which is more attractive (if $m > 0$) or more repulsive (if $m < 0$) than it should and therefore it is more localized (resp. more delocalized) in this region than in the rest of the system. Therefore, conditionally on $A_m$, we can expect that $C_\omega(k,0)$ behaves like $\exp(-k/\xi(\beta, h + \beta m))$. In other words, we can argue that (looking only at the exponential behavior)

$$E C_\omega(k,0) \simeq \int dm \, e^{-km^2/2} e^{-k/\xi(\beta, h + \beta m)} \simeq e^{k \max_m \{-m^2/2 - 1/\xi(\beta, h + \beta m)\}}$$

(6.5)

for $k$ large. Since $\xi$ should diverge when the critical point is approached, it is reasonably decreasing in $h$ so that the value of $m$ which realizes the maximum is strictly negative. On the other hand, when we take the limit without disorder average as in $6.3$, the events $A_m$ with $m \neq 0$ cannot contribute, i.e., almost surely they do not occur for $k$ large enough, as follows from the Borel-Cantelli lemma.

### 6.1. Correlation length of the homogeneous model.

In the homogeneous case, $\beta = 0$, the infinite-volume Gibbs measure can be explicitly described (cf. [22 Th. 2.3]): under $P^0_{\infty}(\cdot)$, $\tau$ is a homogeneous\(^5\) positively recurrent (for $h > h_c(0) = 0$) renewal on $\mathbb{Z}$ such that

$$P^0_{\infty}(\inf\{k > 0 : k \in \tau\} = n|0 \in \tau) = K(n)e^{-F(0,h)n}e^h =: \tilde{K}_h(n)$$

(6.6)

and

$$P^0_{\infty}(n \in \tau) = \frac{1}{\sum_{m \in \mathbb{N}} m \tilde{K}_h(m)}.$$  

Note that $\tilde{K}_h(\cdot)$ is a probability on $\mathbb{N}$ (cf. Eq. (4.1) and the discussion after it) with an exponential tail. What we are interested in is the precise large-$n$ behavior of

$$P^0_{\infty}(n \in \tau|0 \in \tau) - \frac{1}{\sum_{m \in \mathbb{N}} m \tilde{K}_h(m)},$$

i.e., a refinement of the renewal theorem (which simply states that this quantity tends to zero for $n \to \infty$).

Let us for a moment widen our scope and consider a homogeneous, positively recurrent renewal, with law $\tilde{P}$, such that the law of the distance between two successive points, denoted by $\tilde{K}(\cdot)$, has exponential tail: say,

$$\lim_{n \to \infty} \frac{1}{n} \log \tilde{K}(n) = -z < 0.$$  

(6.7)

We do not require for the moment that $\tilde{K}(\cdot)$ is given by (6.6) with $K(\cdot)$ in the class (2.2).

It is known (cf. for instance [6, Chapter VII.2] and [38]) that, under condition (6.7), there exist $r > 0$ and $C < \infty$ such that

$$|\tilde{P}(n \in \tau|0 \in \tau) - \frac{1}{\sum_{m \in \mathbb{N}} m \tilde{K}(m)}| \leq Ce^{-rn}.$$  

(6.8)

\(^5\)That is, its law is invariant under translation on $\mathbb{Z}$. For instance, $P^0_{\infty}(n, m \in \tau) = P^0_{\infty}(n+k, m+k \in \tau)$ for every $k \in \mathbb{Z}$. 
However, the relation between \( z \) and the largest possible \( r \) in Eq. (6.8), call it \( r_{\text{max}} \), is not known in general. A lot of effort has been put by the queuing theory community in investigating this point, and in various special cases it has been proven that \( r_{\text{max}} \geq z \) (see for instance [7], where power series methods are employed and explicit upper bounds on the prefactor \( C \) are given). In even more special cases, for instance when \( \tilde{\mathcal{P}} \) is the law of the return times to a particular state of a Markov chain with some stochastic ordering properties, the optimal result \( r_{\text{max}} = z \) is proved (for details, see [36, 41], which are based on coupling techniques). However, the equality \( r_{\text{max}} = z \) cannot be expected in general.

In particular, if \( \tilde{K}(\cdot) \) is a geometric distribution,

\[
\tilde{K}(n) = e^{-nc} e^c - 1
\]

with \( c > 0 \), then one sees easily that the left-hand side of (6.8) vanishes for every \( n \in \mathbb{N} \) so that \( r_{\text{max}} = \infty \), while \( z = c \). On the other hand, if for instance \( \tilde{\mathcal{P}} \) is the law of the return times to a particular state of a Markov chain with some stochastic ordering properties, the optimal result \( r_{\text{max}} = z \) is proved (for details, see [36, 41], which are based on coupling techniques). However, the equality \( r_{\text{max}} = z \) cannot be expected in general.

In view of this situation, it is highly non-trivial that, restricting to our original class of renewals, the following holds:

**Theorem 6.1.** [23] Let \( \tilde{K}_h(\cdot) \) be given by (6.6) with \( K(\cdot) \) satisfying (2.2) for some \( \alpha > 0 \) and slowly varying \( L(\cdot) \). Then, there exists \( h_0 > 0 \) such that, for every \( 0 < h < h_0 \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \left| \mathbf{P}_{\infty}^{0,h}(n \in \tau | 0 \in \tau) - \frac{1}{\sum_{m \in \mathbb{N}} m \tilde{K}_h(m)} \right| = -F(0,h) \quad (6.9)
\]

and, more precisely,

\[
\mathbf{P}_{\infty}^{0,h}(n \in \tau | 0 \in \tau) - \frac{1}{\sum_{m \in \mathbb{N}} m \tilde{K}_h(m)} \sim^* \frac{Q(n) e^{-F(0,h)n}}{4[\sinh(h/2)]^2} \quad (6.10)
\]

with \( Q(\cdot) \) such that \( \sum_{j=1}^{n} Q(j) n^{-\infty} L(n)/(\alpha n^\alpha) \).

It is important to emphasize that, even under assumption (6.6), this result would be false without the restriction of \( h \) small.

In the light of (6.9), it is quite natural to expect (and in some case this can be proven, see Section 6.3) that in presence of disorder \( \xi \) is still proportional to the inverse of the free energy, at least close to the critical point. But then, what about \( \xi^{av} \)?

### 6.2. \( \mu \) versus \( F \)

To answer this question, we abandon for a while the correlation length and we discuss the relation between free energy and another quantity which, due to lack of a standard name, we will call simply \( \mu \). This was first introduced, to my knowledge, in [3] in the context of random heteropolymers:

\[
\mu(\beta, h) = -\lim_{N \to \infty} \frac{1}{N} \log \mathbf{E} \left[ \frac{1}{Z_N(\beta, h)} \right] \quad (6.11)
\]

Existence of the limit in our context is easily proven by super-additivity of \( \log Z_N(\beta, h) \) (see [25 Th. 2.5]). An argument similar to (2.12) gives immediately \( \mu \geq 0 \) while a simple application of Jensen’s inequality shows that \( \mu(\beta, h) \leq F(\beta, h) \). However, much more than this is true:
Theorem 6.2. \cite{41} For every $\beta > 0$ there exists $0 < c_3(\beta), c_4(\beta) < \infty$ such that
\[
0 < c_3(\beta) \frac{F(\beta, h)^2}{\partial_h F(\beta, h)} < \mu(\beta, h) < F(\beta, h)
\] (6.12)
if $0 < h - h_c(\beta) \leq c_4(\beta)$.

In particular, the bounds in (6.12) show that also $\mu$ vanishes continuously at the critical point, like the free energy. If we call $\eta_F$ and $\eta_{\mu}$ the critical exponents associated to the vanishing of $F$ and $\mu$ for $h \to h_c(\beta)^+$, Theorem 6.2 implies the following bounds:
\[
(2 \leq) \eta_F \leq \eta_{\mu} \leq \eta_F + 1,
\] (6.13)
the inequality in parentheses being valid for $\beta > 0$ thanks to Theorem 5.8. Just to give a flavor of why $\mu$ is relevant in the description of the system let us cite the following result. Define first of all $\Delta_N$ as the largest gap between points of $\tau$ in the system of length $N$:
\[
\Delta_N := \max_{1 \leq i < j \leq N} \{|i - j| : i \in \tau, j \in \tau, \{i + 1, \ldots, j - 1\} \cap \tau = \emptyset\}.
\] (6.14)

Then,

Theorem 6.3. \cite{25} Let $(\beta, h) \in \mathcal{L}$. For every $\epsilon > 0$,
\[
\lim_{N \to \infty} P_{\beta,h}^{N,\omega} \left( \frac{1 - \epsilon}{\mu(\beta, h)} \leq \frac{\Delta_N}{\log N} \leq \frac{1 + \epsilon}{\mu(\beta, h)} \right) = 1 \text{ in probability.}
\] (6.15)

6.3. Correlation lengths and free energy. To my knowledge, the only case where $\xi$ and $\xi_{av}$ can be fully characterized even in presence of disorder is the one where $K(\cdot)$ is the law of the first return to zero of the one-dimensional SRW conditioned to be non-negative. In other words, let $\{S_n\}_{n=0,1,\ldots}$ be the SRW on $\mathbb{Z}$ started at $S_0 = 0$ and let $P^{SRW}(\cdot)$ denote its law. We define $K^{SRW,+(n)} := P^{SRW}(\inf\{k > 0 : S_k = 0\} = 2n|S_i \geq 0 \ \forall i)$. Go back to Section 2.6 for a motivation of this example as a model of wetting of a $(1+1)$-dimensional substrate. In this case, one has the following

Theorem 6.4. \cite{41} Let $K(\cdot) = K^{SRW,+(\cdot)}$ and $\ell \in \mathbb{Z}$. For every $\beta \geq 0$ and $h > h_c(\beta)$,
\[
\frac{1}{\xi_{av}} = -\lim_{k \to \infty} \frac{1}{k} \log \mathbb{E} C_{\omega}(\ell + k, k) = \mu(\beta, h)
\] (6.16)
and, $\mathbb{P}(d\omega)$-a.s.,
\[
\frac{1}{\xi} = -\lim_{k \to \infty} \frac{1}{k} \log C_{\omega}(\ell + k, k) = F(\beta, h).
\] (6.17)

With respect to Theorem 6.1 this result is much less sharp in that it catches only the exponential behavior of the two-point function. However, note that in Theorem 6.4 $h - h_c(\beta)$ is not required to be small as in Theorem 6.1. Note also that in Eqs. (6.16), (6.17) we have not taken the absolute value of $C_{\omega}(\ell + k, k)$: this is because, in this particular case, one can prove that this quantity is non-negative \cite{41}. Finally observe that, in view of (6.12), the two correlation lengths are different. It would be extremely interesting to know whether the two associated critical exponents $\eta_F, \eta_{\mu}$ coincide or not.

Remark 6.5. Theorem 6.4 does not coincide exactly with \cite{41} Th. 3.5], e.g., because in the latter $P_{\infty,\omega}(\cdot)$ is the infinite-volume Gibbs measure obtained from the system defined in $\{1, \ldots, N\}$ letting $N \to \infty$ (cf. footnote 4). However, the proof of \cite{41} extends without difficulties to the result we stated above. We remark also that the theorem holds as well in the case where $K(n) = K^{SRW}(n) := P^{SRW}(\inf\{k > 0 : S_k = 0\} = 2n)$, i.e., the law
of the first return to zero of the unconditioned SRW. This follows from the discussion in Section 2.6 and from the fact that $K^{SRW}(n) = 2K^{SRW,+}(n)$.

**Proof of Theorem 6.4 (sketch).** The proof of Theorem 6.4 is based on a coupling argument. For simplicity let $P^+(\cdot) := P^{SRW}(\cdot|S \geq 0)$. One can then rewrite the two-point function (6.2) as

$$
C_\omega(k,\ell) = \lim_{N \to \infty} \frac{1}{Z_{N,\omega}(\beta, h)^2} \times E^{+,\omega} \left[ \sum_{n=-N/2}^{N/2} e^{(\beta h_n+\mu)} (1 \{ s^{(1)}_{h_n}=0 \} + 1 \{ s^{(2)}_{h_n}=0 \}) \left( 1 \{ S^{(1)}_k=0 \} - 1 \{ S^{(2)}_k=0 \} \right) | S^{(1)}_\ell = 0 \right],
$$

where $S^{(1)}, S^{(2)}$ are independent with law $P^+$. Since the SRW conditioned to be non-negative is a Markov chain, the expectation in the right-hand side clearly vanishes if we condition on the event that there exists $\ell < i < k$ such that $S^{(1)}_i = S^{(2)}_i$. But (and here we use explicitly the condition $S_i \geq 0$ and that two SRW trajectories which cross each other do necessarily intersect), if the complementary event happens then either $S^{(1)}$ or $S^{(2)}$ has no zeros in the interval $\{ \ell + 1, \ldots, k - 1 \}$. As a consequence, one obtains

$$
\mathbb{E} C_\omega(k, 0) \leq 2 \mathbb{E} P^{\beta,h}_{\infty,\omega}(\tau \cap \{ 1, \ldots, k - 1 \} = \emptyset)
$$

and it is not difficult to deduce from (6.11) that this probability vanishes like $\exp(-k \mu(\beta, h))$ for $k \to \infty$. For the opposite bound and for the proof of (6.17) we refer to [11].

In the general case where $P$ is not necessarily the law of the returns of the SRW (or, in general, of any Markov chain), the available results on correlation lengths in presence of disorder are much less sharp and, above all, only correlation length upper bounds are known. At present, the best one can prove in general about average correlation length is the following:

**Theorem 6.6.** [12] Let $\epsilon > 0$ and $(\beta, h) \in \mathcal{L}$. There exists $C_1 := C_1(\epsilon, \beta, h) > 0$ such that, for every $k \in \mathbb{N}$,

$$
\mathbb{E} |C_\omega(\ell + k, \ell)| \leq \frac{1}{C_1 \mu(\beta, h)^{1+C_1}} \exp\left(-k C_1 \mu(\beta, h)^{1+\epsilon}\right).
$$

The constant $C_1(\epsilon, \beta, h)$ does not vanish at the critical line: for every bounded subset $B \subset \mathcal{L}$ one has $\inf_{(\beta, h) \in B} C_1(\epsilon, \beta, h) \geq C_1(B, \epsilon) > 0$.

**Remark 6.7.** The necessity of introducing $\epsilon > 0$ (i.e., of weakening the upper bound with respect to the expected one) is probably of technical nature, as appears from the fact that for $\beta = 0$ Theorem 6.6 does not reproduce the sharp results (6.9) which hold for the homogeneous case.

Observe that Theorem 6.6 is more than just an upper bound on $\xi^{av}$. Indeed, thanks to the bound on the prefactor in front of the exponential, Eq. (6.19) says that the exponential decay, with rate at least of order $\mu^{1+\epsilon}$, starts as soon as $k \gg \mu^{-1-\epsilon} |\log \mu|$. This observation reinforces the meaning of Eq. (6.19) as an upper bound of order $\mu^{-1}$ on the correlation length of disorder-averaged correlations functions.

About the typical correlation length the following can be proven:

**Theorem 6.8.** [42] Let $\epsilon > 0$ and $(\beta, h) \in \mathcal{L}$. One has for every $k \in \mathbb{N}$

$$
|C_\omega(k, 0)| \leq C_2(\omega) \exp\left(-k C_1 F(\beta, h)^{1+\epsilon}\right),
$$

where $C_2(\omega)$ depends only on $\omega$.

**Proof of Theorem 6.8 (sketch).** For simplicity, let $P^+(\cdot) := P^{SRW}(\cdot|S \geq 0)$. One can then rewrite the two-point function (6.2) as

$$
C_\omega(k,\ell) = \lim_{N \to \infty} \frac{1}{Z_{N,\omega}(\beta, h)^2} \times E^{+,\omega} \left[ \sum_{n=-N/2}^{N/2} e^{(\beta h_n+\mu)} (1 \{ s^{(1)}_{h_n}=0 \} + 1 \{ s^{(2)}_{h_n}=0 \}) \left( 1 \{ S^{(1)}_k=0 \} - 1 \{ S^{(2)}_k=0 \} \right) | S^{(1)}_\ell = 0 \right],
$$

where $S^{(1)}, S^{(2)}$ are independent with law $P^+$. Since the SRW conditioned to be non-negative is a Markov chain, the expectation in the right-hand side clearly vanishes if we condition on the event that there exists $\ell < i < k$ such that $S^{(1)}_i = S^{(2)}_i$. But (and here we use explicitly the condition $S_i \geq 0$ and that two SRW trajectories which cross each other do necessarily intersect), if the complementary event happens then either $S^{(1)}$ or $S^{(2)}$ has no zeros in the interval $\{ \ell + 1, \ldots, k - 1 \}$. As a consequence, one obtains

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\mathbb{E} C_\omega(k, 0) \leq 2 \mathbb{E} P^{\beta,h}_{\infty,\omega}(\tau \cap \{ 1, \ldots, k - 1 \} = \emptyset)
$$

and it is not difficult to deduce from (6.11) that this probability vanishes like $\exp(-k \mu(\beta, h))$ for $k \to \infty$. For the opposite bound and for the proof of (6.17) we refer to [11].

In the general case where $P$ is not necessarily the law of the returns of the SRW (or, in general, of any Markov chain), the available results on correlation lengths in presence of disorder are much less sharp and, above all, only correlation length upper bounds are known. At present, the best one can prove in general about average correlation length is the following:

**Theorem 6.6.** [12] Let $\epsilon > 0$ and $(\beta, h) \in \mathcal{L}$. There exists $C_1 := C_1(\epsilon, \beta, h) > 0$ such that, for every $k \in \mathbb{N}$,

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\mathbb{E} |C_\omega(\ell + k, \ell)| \leq \frac{1}{C_1 \mu(\beta, h)^{1+C_1}} \exp\left(-k C_1 \mu(\beta, h)^{1+\epsilon}\right).
$$

The constant $C_1(\epsilon, \beta, h)$ does not vanish at the critical line: for every bounded subset $B \subset \mathcal{L}$ one has $\inf_{(\beta, h) \in B} C_1(\epsilon, \beta, h) \geq C_1(B, \epsilon) > 0$.

**Remark 6.7.** The necessity of introducing $\epsilon > 0$ (i.e., of weakening the upper bound with respect to the expected one) is probably of technical nature, as appears from the fact that for $\beta = 0$ Theorem 6.6 does not reproduce the sharp results (6.9) which hold for the homogeneous case.

Observe that Theorem 6.6 is more than just an upper bound on $\xi^{av}$. Indeed, thanks to the bound on the prefactor in front of the exponential, Eq. (6.19) says that the exponential decay, with rate at least of order $\mu^{1+\epsilon}$, starts as soon as $k \gg \mu^{-1-\epsilon} |\log \mu|$. This observation reinforces the meaning of Eq. (6.19) as an upper bound of order $\mu^{-1}$ on the correlation length of disorder-averaged correlations functions.

About the typical correlation length the following can be proven:

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$$
|C_\omega(k, 0)| \leq C_2(\omega) \exp\left(-k C_1 F(\beta, h)^{1+\epsilon}\right),
$$

where $C_2(\omega)$ depends only on $\omega$.
where $C_1$ is as in Theorem 6.6, while $C_2(\omega) := C_2(\omega, \epsilon, \beta, h)$ is an almost surely finite random variable.

The proof of Theorems 6.6 and 6.8 relies on a rather involved coupling/comparison argument. In simple (and imprecise) words, one first approximates $K(\cdot)$ with a new law $\tilde{K}(\cdot)$ which is the law of the returns to zero of a Markov process with continuous trajectories (defined in terms of a Bessel process), and at that point the coupling argument of last section can be applied. We refer to [42] for full details.

**Appendix A. Two Tauberian results**

For completeness, we include without proof two Tauberian theorems (i.e., results about the relation between the asymptotic behavior of a function and of its Laplace transform) which we used in Section 5.5. Given a function $Q : \mathbb{N} \rightarrow \mathbb{R}$, we define for $s \in \mathbb{R}$

$$\hat{Q}(s) := \sum_{n \in \mathbb{N}} e^{-ns}Q(n)$$

whenever the sum converges.

We begin with a (quite intuitive) fact:

**Theorem A.1.** [8, Proposition 1.5.8] If $\ell(\cdot)$ is slowly varying and $\gamma > -1$ then

$$\sum_{n=1}^{N} n^{\gamma} \ell(n) \xrightarrow{N \to \infty} \frac{N^{\gamma+1}}{\gamma+1} \ell(N). \quad (A.1)$$

Next we state Karamata’s Tauberian theorem [8, Th. 1.7.1] which for our purposes may be formulated as follows:

**Theorem A.2.** Assume that $Q(n) \geq 0$ for every $n \in \mathbb{N}$, that $\ell(\cdot)$ is slowly varying and that $\rho \geq 0$. The following are equivalent:

$$\hat{Q}(s) \xrightarrow{s \to 0} \frac{\ell(1/s)}{s^\rho} \quad (A.2)$$

and

$$\sum_{n=1}^{N} Q(n) \xrightarrow{N \to \infty} N^\rho \frac{\ell(N)}{\Gamma(1+\rho)}. \quad (A.3)$$

Recall that the function $\Gamma(z)$ can be defined, for $z > 0$, as

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt.$$ 

Finally, a theorem relating the Laplace transform of a law on the half-line to its integrated tail (cf. [8, Corollary 8.1.7]):

**Theorem A.3.** Let $X$ be an integer-valued random variables with law $P$ and $Q(n) := P(X = n)$, $\ell(\cdot)$ a slowly varying function and $0 \leq \alpha < 1$. The following are equivalent:

$$1 - \hat{Q}(s) \xrightarrow{s \to 0} s^\alpha \ell(1/s) \quad (A.4)$$

and

$$P(X > n) = \sum_{j > n} Q(j) \xrightarrow{n \to \infty} \frac{\ell(n)}{n^\alpha \Gamma(1-\alpha)}. \quad (A.5)$$
ACKNOWLEDGMENTS

I would like to thank Roman Kotecký for organizing the Prague Summer School on Mathematical Statistical Mechanics and for inviting me to give a course. Learning and teaching there was an extremely stimulating experience.

Some of the results described in these notes are based on joint work with Giambattista Giacomin, to whom I am grateful for introducing me to this subject, for countless motivating conversations, and also for communicating to me the results of [23] prior to publication.

This work was supported in part by the GIP-ANR project JC05_42461 (POLINTBIO) and my presence at the school was made possible thanks to the support from the ESF-program “Phase Transitions and Fluctuation Phenomena for Random Dynamics in Spatially Extended Systems”.

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