Experimental entanglement redistribution under decoherence channels

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(Dated: July 16, 2014)

When an initially entangled pair of qubits undergoes local decoherence processes, there are a number of ways in which the original entanglement can spread throughout the multipartite system consisting of the two qubits and their environments. Here we report theoretical and experimental results regarding the dynamics of the distribution of entanglement in this system. The experiment employs an all optical set-up in which the qubits are encoded in the polarization degrees of freedom of two photons, and each local decoherence channel is implemented with an interferometer that couples the polarization to the path of each photon, which acts as an environment. We monitor the dynamics and distribution of entanglement and observe the transition from bipartite to multipartite entanglement and back, and show how these transitions are intimately related to the sudden death and sudden birth of entanglement. The multipartite entanglement is further analyzed in terms of 3- and 4-partite entanglement contributions, and genuine four-qubit entanglement is observed at some points of the evolution.

Introduction: In one of his landmark papers published in 1935, Erwin Schrödinger characterized entanglement in terms that are closely related to the modern notion of information [1]. According to him, for an entangled system “best possible knowledge of a whole does not include best possible knowledge of its parts”. In the same vein, the process of decoherence can be attributed to the loss of information of an individual quantum system to the environment with which it gets entangled [2]. In the case of a composite system in an entangled state, the initial entanglement, or “information of the whole”, can spread and distribute throughout the system and environment in a number of ways. Depending on the initial state and on the particular type of interaction with the environment, this redistribution of the initial entanglement can give rise to phenomena like the Entanglement Sudden Death (ESD) [3,4], and Entanglement Sudden Birth (ESB) [5]. In this paper, we address the question of what happens with the entanglement in the cases it disappears, and how it may eventually reappear. This dynamical process leads to multipartite entanglement involving system and environment. For an initial bipartite entangled state where one subsystem interacts with the environment, it was shown [7,8] that genuine tripartite entanglement may arise in the form of GHZ [9] or W type [10] of states, including the environmental degrees of freedom.

Here, we experimentally study the situation in which two entangled qubits are coupled to their local environments, a system that can give rise to a much richer dynamics, as compared to the tripartite case [7,8], allowing for the detailed study of the entanglement sudden-death and sudden-birth processes, and the corresponding emergence of genuine multipartite entanglement involving qubits and environment. The qubits are encoded in the polarizations of two photons, while the decoherence is implemented by optically coupling the polarization to the spatial mode, which plays the role of the environment. Performing quantum state tomography of the whole, system and environmental degrees of freedom, we observe the entanglement as a function of the amount of decoherence applied to the system. Along this evolution, we observe the sudden-death and sudden-birth effects, and the redistribution of the entanglement from bipartite to tri- and four-partite forms. We also present a theory based on monogamy relations that provides entanglement quantifiers for some types of entanglement.

Monogamy inequalities and residual entanglement: In the study of the distribution of entanglement in N-qubit systems, the monogamous feature of bipartite entanglement plays a fundamental role, restricting the way bipartite entanglement can be shared among the different components of the multipartite system. This monogamy arises from the following inequality [11]

\[ C_{i|j_1,j_2,...,j_{N-1}} \geq C_{i|j_1}^2 + C_{i|j_2}^2 + \ldots + C_{i|j_{N-1}}^2, \]  

where \( C_{A|B}^2 \) is known as the tangle and represents a measure of the bipartite entanglement between systems \( A \) and \( B \). If \( A \) and \( B \) represent any two subsets of qubits then

\[ C_{A|B}^2 = \inf_{\{\phi_i\}} \sum_i 2p_i \left[ 1 - \text{tr} \left( \rho_A^i \right)^2 \right], \]  

where \( \rho_A^i = \text{tr}_B \left( \left| \phi_i \right\rangle \left\langle \phi_i | \right\rangle \right) \), and the pure states \( \left| \phi_i \right\rangle \) are the possible decompositions of \( \rho_{AB} \) as \( \sum_i p_i \left| \phi_i \right\rangle \left\langle \phi_i | \right\rangle \). Note that the infimum is taken over \{\{\phi_i\} \}. If \( A \) and \( B \) represent single qubits, so that \( A = i, B = j \), then \( C_{A|B} \) is the usual two-qubit concurrence \( C_{ij} \), given by [12]

\[ C_{ij} = \max \{0, \Gamma\}, \]  

where \( \Gamma = \sqrt{\lambda_1 - \lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \) and the \( \lambda_i \)'s are the eigenvalues (in decreasing order) of \( \rho_{ij}(\sigma_y \otimes \sigma_y)\rho_{ij}^\dagger(\sigma_y \otimes \sigma_y) \).
spontaneous parametric down-conversion \[15\]. The setup is shown in Fig. 1. The system qubits are encoded \(C\) and equals the initial entanglement \(3\)-tangle defined in \[14\].

subject to the condition that the reduced density matrix \(\rho_{ij}\) is a rank-two matrix.

Let us now analyze the monogamy relations in the context of a 4-qubit system composed of two open systems, \(S_1\) and \(S_2\), and their respective environments, \(E_1\) and \(E_2\). We assume that the initial (pure) state has the structure

\[
|\Psi(0)\rangle_{S_1S_2E_1E_2} = |\psi(0)\rangle_{S_1S_2} |1\rangle_{E_2} |0\rangle_{E_1},
\]

with \(|\psi(0)\rangle\) an entangled state, and that only local interactions occur between systems \((S_1, S_2)\) and their respective environments, so that the evolved state is

\[
|\Psi(t)\rangle = U_{S_1E_1}(t) U_{S_2E_2}(t) |\Psi(0)\rangle.
\]

With these assumptions, direct calculation shows that the reduced density matrices \(\rho_{S_1E_i}(t)\) and \(\rho_{S_2E_j}(t)\) are rank-two matrices during the whole evolution, hence we can resort to Eq. \[4\] and write the residual entanglement corresponding to the bipartition \(S_1E_1|S_2E_2\) as

\[
R_{S_1E_1|S_2E_2} = C^2_{S_1E_1|S_2E_2} - C^2_{S_2E_1|S_1E_2} - C^2_{S_1E_2|S_2E_1} + C^2_{S_1E_1|S_2E_2},
\]

where, as seen from Eq. \[4\], \(R_{S_1E_1|S_2E_2}\) is positive. The residual entanglement corresponding to the most unbalanced bipartition is defined via Eq. \[1\]

\[
R_i = C^2_{S_1E_i|S_2E_2} - C^2_{S_2E_i|S_1E_2} - C^2_{S_1E_2|S_2E_1} + C^2_{S_1E_1|S_2E_2},
\]

where \(i, j, k, l = S_1, S_2, E_1\) and \(E_2\). The residual entanglements indicate the presence of multipartite (rather than bipartite) entanglement. In particular, for a pure state of three qubits, the residual entanglement becomes independent of the choice of qubit \(i\) and coincides with the 3-tangle

\[
\tau_{ijk} = C^2_{i,jk} - C^2_{ij} - C^2_{ik},
\]

defined in \[14\].

Since the evolution of Eq. \[6\] involves only local unitary transformations, the bipartite entanglement between systems \((S_1E_1)\) and \((S_2E_2)\) remains constant. The tangle \(C^2_{S_1E_1|S_2E_2}\) is thus conserved along the evolution and equals the initial entanglement \(C^2_0\) of \(S_1E_1\).

**Experimental Setup:** A sketch of the experimental setup is shown in Fig. 1. The system qubits are encoded in the polarization of two photons, produced with type-I spontaneous parametric down-conversion \[15\]. The environment is represented by the spatial mode (path) of each photon, initially in state \(|0\rangle_E\). Ideally, the initial two-photon state can be written as

\[
|\Psi\rangle = (\alpha |0\rangle_{S_1} |0\rangle_{S_2} + \beta |1\rangle_{S_1} |1\rangle_{S_2}) |0\rangle_{E_1} |0\rangle_{E_2},
\]

where \(|0\rangle_S\) and \(|1\rangle_S\) are represented by horizontal and vertical polarization of the photons, respectively. Each one of the photons is directed to optical interferometers that implement a unitary transformation that models the interaction between the polarization \(S_i\) and the spatial mode \(E_i\). Tracing out the environment degrees of freedom, this operation corresponds to a quantum channel \[7\]. A second interferometer, wave plates and a polarizing beam splitter (PBS) are used to perform full state tomography on the polarization and path degrees of freedom simultaneously.

![Decoherence Channel Tomographic Meas.](image)

FIG. 1: Experimental Setup: An ultraviolet laser pumps a non-linear BBO crystal, in which two photons are produced via parametric down conversion. Both photons are directed to two separated nested interferometers. The first two interferometers (light red rectangles) are constructed with a beam displacer (BD) and a modified beam displacer (MBD). These implements the decoherence channels as was done in \[7\] \[8\]. The optical elements in the light green rectangles are used to make tomographic measurements on the polarization and path degrees of freedom simultaneously.
where \( p = \sin^2(2\theta_p) \). This transformation corresponds to the amplitude damping channel (AD) when the environments are traced out. The analogy with a relaxation process can be made by associating the parameter \( p \) with time, so that \( p(t = 0) = 0 \), and \( p(t \to \infty) = 1 \). The second interferometer and the polarization optics shown in the light green rectangles of Figs. 1 are used to perform complete tomography of the polarization and spatial mode using 256 different settings of the HWPs and quarter-wave plates (QWP) \([R1\ \| \] \[LG]\). That is, we perform complete 4 qubit tomography, allowing us to reconstruct the total quadripartite state of system + environment, and to completely analyze the evolution of entanglement as a function of \( p \).

**Distribution of entanglement:** We analyze the “sudden death” and “sudden birth” of entanglement, showing that what happens in fact is a redistribution of the initial entanglement between system \( S \) and environment \( E \) to completely disappear. Between the systems \( S \) and \( E \) concurrence in Eq. (3)) is null before the entanglement was calculated for several values of \( p \). \( \alpha/\beta \) and \( \sqrt{\alpha/\beta} \) are very close to zero during the residual entanglement of Eq. (7), in the Lower Bound approximation \([17]\). The areas are theoretical predictions (see the text). In the inset we show \( \Gamma \), defined in Eq. (5), where one can see that the entanglement is actually born at \( p_{ESB} \).

![FIG. 2: Experimental tangles \( C_{ij} \) versus \( p \), the evolution parameter. The red squares represent \( C_{S1S2}^2 \), which decays monotonically until \( p = 0.34 \pm 0.04 \) where the sudden death of entanglement takes place. \( C_{E1E2}^2 \) (blue circles) is null for \( p \) in the range \([0, 0.62]\) where it begins to monotonically grow, the entanglement sudden birth. Black diamonds represent the residual entanglement of Eq. (7), in the Lower Bound approximation \([17]\).](image)
swapped to the environments $E_1$ and $E_2$.

Further decomposition of the multipartite entanglement: We now go into more detail regarding the type of multipartite entanglement that appears during the evolution. An explicit decomposition of $R_{S_1E_1|S_2E_2}$ in terms of well-identified multipartite entanglements is needed to provide information about the distribution of entanglement. We first use Eqs. (5) (applied to $i = S_1, E_1$) and (7) to write

$$R_{S_1E_1} = C_{S_1E_1}^2 + 2C_{S_1E_1}^2 - (C_{S_1E_1}^2 + C_{S_2E_2}^2 + C_{E_1S_1S_2E_2}^2),$$

where we have defined $R_{S_1E_1} = R_{S_1E_1|S_2E_2} - (R_{S_1} + R_{E_1})$. Since $R_{S_1E_1}$ is defined as a linear combination of residual multipartite entanglements, it must be possible to express it in terms of non-bipartite entanglement contributions only. In order to do this we first use Eq. (6) and observe that $\langle \Psi(t) | \Psi(0) \rangle$ can be obtained by applying $U_{S_2E_2}$ to the intermediate state $U_{S_1E_1}(t) | \Psi(0) \rangle$. As is stated below Eq. (6), the reduced density matrix $\rho_{S_2E_2}$ is a rank-two matrix, so that at this intermediate stage the problem is that of a 3-qubit ($S_1$, $E_1$ and $S_2E_2$) system in a pure state, and in which only $S_1$ and $E_1$ interact. In this case one multipartite entanglement arises, corresponding precisely to the 3-tangle in Eq. (8) with the qubit $i$ being $S_1$, $j$ being $E_1$ and $k$ being the effective two-level system $(S_2E_2)$. We denote this quantity as $\tau_{S_1E_1(S_2E_2)}$.

Now, by invoking the invariance of entanglement under local operations, it follows that $\tau_{S_1E_1(S_2E_2)}$ remains unaffected when the transformation $U_{S_2E_2}$ is applied to the state $U_{S_1E_1}(t) | \Psi(0) \rangle$, so that the final state $| \Psi(t) \rangle$ has a multipartite entanglement $\tau_{S_1E_1(S_2E_2)}$ whose value is independent of $U_{S_2E_2}$. On the other hand, as seen from Eq. (14), $R_{S_1E_1}$ depends only on the reduced density matrices $\rho_{S_1E_1}$, $\rho_{E_1}$, and $\rho_{S_1}$, and consequently represents a multipartite entanglement that does not depend on the transformation $U_{S_2E_2}$ neither. Since $\tau_{S_1E_1(S_2E_2)}$ and $R_{S_1E_1}$ are both independent of $U_{S_2E_2}$, we can compute them assuming that $U_{S_2E_2} = 1$. In this case $E_2$ remains in its ground state, the system $(S_1E_1)$ remains in a pure state and thus $\tau_{S_1E_1(S_2E_2)}$ is simply given by the 3-tangle $\tau_{S_1E_1}$ that corresponds to the 3-qubit state calculated with equation (9). As for $R_{S_1E_1}$, we notice that with $U_{S_2E_2} = I$ we can apply the decomposition (10) to $C_{S_1E_1}^2 = C_{S_1}^2 + C_{E_1}^2$, and so obtained $R_{S_1E_1} = -\tau_{S_1E_1}$. From the definition of $R_{S_1E_1}$ we are led to write

$$R_{S_1E_1|S_2E_2} = R_{S_1} + R_{E_1} - \tau_{S_1E_1(S_2E_2)}.$$  

Decomposing the residual entanglements $R_i$, we can rewrite $R_{S_1E_1|S_2E_2}$ in terms of more explicit multipartite entanglement contributions. This can be accomplished considering that if $\rho_{S_2E_2}$ is a rank-two matrix, so that the subsystem $(S_2E_2)$ can be considered as single qubit, and the complete system $S_1E_1(S_2E_2)$ is in a pure state,

then we can use Eq. (9) and write

$$C_{ij|kl}^2 = C_{ij}^2 + C_{ik}^2 + C_{jk}^2,$$

where in the second line we used Eq. (1) and denoted with $\tau_{ij|kl}$ the corresponding residual entanglement. The underline distinguishes the reference qubit and stresses the fact that $\tau_{ij|kl}$ is not necessarily invariant under a permutation of indices that involves $i$, as is $\tau_{ij}$ in Eq. (8). In fact, the 3-partite entanglement $\tau_{ijkl}$ coincides with the 3-tangle $\tau_{ijkl}$ only when the subsystem $ikl$ is in a pure state, that is, only when $j$ is separable from the rest of the qubits. Comparison of Eqs. (8) and (10) leads to

$$R_i = \tau_{ijkl},$$

an expression that allows us to compute the 3-partite entanglement $\tau_{ijkl}$ of the mixed state $\rho_{ijkl}$ as

$$\tau_{ijkl} = C_{ijkl}^2 - (C_{ij}^2 + C_{ik}^2 + C_{jk}^2 - \tau_{ij} \tau_{kl}).$$

Equations (15), (17), and the analogous equation that results from the latter by performing the substitution $1 \leftrightarrow 2$, lead to

$$R_{S_1E_1|S_2E_2} = \frac{1}{2} (\tau_{S_1E_1} + \tau_{S_1E_1 + \tau_{S_1E_1} + \tau_{S_2E_2} + \tau_{S_2E_2}(S_1E_1)).$$

With these results we now analyze our experimental data regarding the residual entanglements $R_i$ which, as seen from Eq. (17), can be separated in a tripartite entanglement contribution, measured by $\tau_{ijkl}$, plus a contribution that may have information about four-partite
that for the state \( \tau \), the four-partite genuine entanglement. For this purpose, we use however, we do not have information about the generation of the experimental results corresponding to \( p \), \( \rho \), and \( \tau \) acquaintance. We observed the transition bipartite \( \leftrightarrow \) multipartite entanglement along the evolution and showed that sudden death of entanglement occurs when all the entanglement becomes completely multipartite, whereas the sudden birth of entanglement occurs when the entanglement ceases to be completely multipartite and gets redistributed also in bipartite form. We believe that this is the first experimental demonstration of entanglement sudden birth. We also presented a novel decomposition of the residual entanglement measures that allowed us to analyze our results in terms of well-identified 3- and 4-partite entanglement contributions. In addition, we used the fidelity as a witness of multipartite entanglement to demonstrate that there is an interval in the evolution in which the complete system is found in a genuine four-partite entangled state. Our results represent a significant step towards a deeper understanding of decoherence processes and the distribution of entanglement in multiqubit systems.

Acknowledgments

Financial support was provided by Brazilian agencies FAPERJ, CNPq, CAPES, and the National Institute of Science and Technology for Quantum Information. AVH acknowledges financial support from Programa Ciência sem Fronteiras (CNPq, CAPES).

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