Solitons and other waves on a quantum vortex filament

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The quantum form of the local induction approximation (LIA, a model approximating the motion of a thin vortex filament in superfluid) including superfluid friction effects is put into correspondence with a type of complex Ginzburg-Landau equation, in a manner analogous to the Hasimoto map taking the classical LIA into the cubic nonlinear Schrödinger equation. From this formulation, we determine the form and behavior of Stokes waves, 1-solitons, and other traveling wave solutions under normal and binormal friction. The most important of these solutions is the soliton on a quantum vortex filament, which is a natural generalization of the 1-soliton solution constructed mathematically by Hasimoto which motivated subsequent real-world experiments. We also conjecture on the possibility of chaos in such systems, and on the existence more complicated solitons such as breathers.

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The self-induced motion of a vortex filament in a superfluid is approximated by the HVBK model [1]. Applying the local induction approximation (LIA) to the non-local term, Schwarz [2] obtained a kind of quantum LIA which is given in non-dimensional form by

\[
v = \gamma \kappa t \times n + \alpha t \times (U - \gamma \kappa t \times n) - \alpha' t \times (t \times (U - \gamma \kappa t \times n)),
\]

(1)

where \( U \) is the dimensionless normal fluid velocity, \( t \) and \( n \) are the unit tangent and unit normal vectors to the vortex filament, \( \kappa \) is the dimensionless average curvature, \( \gamma = \Gamma \ln(c/\kappa a_0) \) is a dimensionless composite parameter (\( \Gamma \) is the dimensionless quantum of circulation, \( c \) is a scaling factor of order unity, \( a_0 \approx 1.3 \times 10^{-8} \text{cm} \) is the effective core radius of the vortex), \( \alpha \) and \( \alpha' \) are dimensionless friction coefficients which are small (except near the \( \lambda \)-point; for reference, the \( \lambda \)-point is the temperature \(( \approx 2.17 \text{K}, \text{at atmospheric pressure}) \) below which normal fluid Helium transitions to superfluid Helium [3]). Table 1 of Schwarz [2] shows that at temperature \( T = 1 \text{K} \) we have \( \alpha = 0.005 \) and \( \alpha' = 0.003 \), while at temperature \( T = 1.5 \text{K} \) we have \( \alpha = 0.073 \) and \( \alpha' = 0.018 \). In the limit \(( \alpha, \alpha' ) \rightarrow (0,0) \), we recover the classical Da Rios equations for the motion of a vortex filament in a classical fluid [4].

A number of studies exist on the solutions to the quantum LIA. In the \( \alpha, \alpha' \rightarrow 0 \) limit, these solutions should collapse into solutions of the classical LIA. One highly important class of solutions to the classical LIA would be the 1-soliton solution found by Hasimoto [5], by way of what is now referred to as the Hasimoto transformation, which puts the classical LIA into correspondence with the cubic NLS. While a number of solutions to the quantum LIA have been studied either numerically or analytically, the Hasimoto 1-soliton have never been extended to the quantum LIA. The purpose of this paper is to fill this important gap. Applying a method analogous to that of Hasimoto, we are able to put the quantum LIA [1] into correspondence with a type of complex Ginzburg-Landau equation (a natural complex-coefficient generalization of NLS). From this, we study Stokes waves, 1-solitons, and other traveling wave solutions. Each of these solutions generalizes known results for the classical LIA. We also conjecture on the possibility of chaos in such systems.

Differentiating with respect to the arclength variable \( s \), and performing several vector manipulations, we have that the quantum LIA [1] becomes

\[
\dot{t} = \frac{\partial}{\partial s} \left\{ (1 - \alpha' |t|^2) t \times t_s - \alpha (t \cdot t_s) t - |t|^2 t_s \right\}
\]

(2)

Taking \( t \) to be a unit vector, the equation simplifies slightly to

\[
\dot{t} = \frac{\partial}{\partial s} \left\{ (1 - \alpha') t \times t_s + \alpha t_s + \alpha t \times U - \alpha' (t \cdot U) t \right\}.
\]

(3)

This puts the quantum LIA [1] into the form of a vector conservation law.

In what follows, we shall take \( U = 0 \), for brevity of the calculations. Many studies on specific structures in the quantum LIA model have taken the normal fluid velocity to zero, as it permits one to study such structures without the influence of drift or other distorting effects on the filaments [6]. The physical applicability of such a scenario is limited to the very low temperature regime in superfluid Helium 4. On the other hand, in the case of superfluid Helium 3, the normal fluid velocity \( U \) is zero (because Helium 3 is very viscous, unlike Helium 4, so it is always at rest or in solid body rotation, but \( \alpha \) and \( \alpha' \) are not zero [5]). Similar results were recently attempted in the case of \( U \neq 0 \) [8], however the system was not solved and only the limiting reduction to \( \alpha, \alpha' = 0 \) was given. Some qualitative observations were also given at lowest order.
Let $\mathbf{b}$ denote the binormal vector, and take $\kappa$ and $\tau$ to be the curvature and torsion, respectively. Recall that $\mathbf{t}_s = \kappa \mathbf{n}, \mathbf{n}_s = -\kappa \mathbf{t} + \tau \mathbf{b}, \mathbf{b}_s = -\kappa \mathbf{n}$. After setting $\mathbf{U} = \mathbf{0}$, we can write $\mathbf{U}$ as $\mathbf{t} = (1-\alpha')(\kappa \mathbf{b})_s + \alpha \mathbf{t}_{ss}$.

Let us define the function $\psi(s,t) = \kappa(s,t) \exp(i \int_0^s \tau(s,t) \, ds)$ and also a new vector-valued function by $\mathbf{m} = (\mathbf{n} + i \mathbf{b}) \exp(i \int_0^s \tau(s,t) \, ds)$. Note that $\mathbf{m}_s = -i \mathbf{t}$ and $\mathbf{t}_s = \frac{1}{2} \left((\psi^* \mathbf{m} + \psi \mathbf{m}^*) \right)$. The quantum LIA \cite{9} therefore takes the form

$$\mathbf{t} = \frac{i}{2} (1-\alpha')(\psi \mathbf{m}^* - \psi^* \mathbf{m}) + \frac{\alpha}{2} (\psi^* \mathbf{m} + \psi \mathbf{m}^*)_s. \quad (4)$$

We seek to derive an equation for $\psi$ in analogy to that which was obtained by Hasimoto in the case of a standard fluid (i.e., $\alpha = \alpha' = 0$). On the one hand, note that

$$\mathbf{m}_s = -\dot{\psi} \mathbf{t} - \dot{\psi} \mathbf{t} = -(\dot{\psi} + \alpha |\psi|^2 \psi) \mathbf{t} + \frac{i(1-\alpha')}{2} \psi \psi^* \mathbf{m}$$

$$= -i(1-\alpha') + \alpha \psi \psi^* \mathbf{m} + \frac{i(1-\alpha')}{2} \psi \psi^* \mathbf{m}^*.$$

On the other hand, assume that we have a representation for $\mathbf{m}$ of the form

$$\mathbf{m} = a \mathbf{m} + b \mathbf{m}^* + ct. \quad (6)$$

First, observe that

$$a + a^* = \frac{1}{2}(\mathbf{m} \cdot \mathbf{m}^* + \mathbf{m}^* \cdot \mathbf{m}) = \frac{1}{2} \partial_t (\mathbf{m} \cdot \mathbf{m}^*) = 0,$$ 

therefore $a$ must take the form $a = i \phi(s,t)$, for some real-valued function $\phi$. By a similar process, $b \equiv 0$. We should also find that

$$c = -\mathbf{m} \cdot \mathbf{t} = -(i(1-\alpha') + \alpha) \psi_s. \quad (8)$$

Therefore, we have the representation

$$\mathbf{m} = i \phi(s,t) \mathbf{m} - (i(1-\alpha') + \alpha) \psi_s(s,t) \mathbf{t}. \quad (9)$$

Differentiation of this representation with respect to arclength gives

$$\mathbf{m}_s = i \phi_s \mathbf{m} - i \phi \psi \mathbf{t} - (i(1-\alpha') + \alpha) \psi_s \mathbf{t}$$

$$= \frac{1}{2} (i(1-\alpha') + \alpha) \psi_s (\psi^* \mathbf{m} + \psi \mathbf{m}^*). \quad (10)$$

Clearly, the coefficients of $\mathbf{t}, \mathbf{m}$ and $\mathbf{m}^*$ in equations \cite{5} and \cite{10} should match exactly. The $\mathbf{m}^*$ coefficients already match exactly. Setting the $\mathbf{m}$ coefficients equal, we obtain

$$\phi_s = \frac{1}{2} - \frac{\alpha'}{2} \partial_s |\psi|^2 - \frac{\alpha i}{2} (\psi^* \psi_s - \psi \psi_s^*), \quad (11)$$

hence

$$\phi(s,t) = \frac{1}{2} \frac{\alpha'}{2} - \alpha |\psi|^2 + \alpha (\text{Re}(\psi))(\text{Im}(\psi)) + A(t)$$

$$= \frac{1}{2} \frac{\alpha'}{2} - \alpha |\psi|^2 - \frac{\alpha i}{4} (\psi^2 - \psi^* 2) + A(t), \quad (12)$$

where $A(t)$ is an arbitrary function of time. Despite the appearance of $i$, this representation is real-valued, since $\psi^2 - \psi^* 2$ is purely imaginary. Matching the coefficients of $\mathbf{t}$, we obtain

$$\psi + \alpha |\psi|^2 \psi = i \phi + (i(1-\alpha') + \alpha) \psi_{ss}. \quad (13)$$

Using \cite{12}, we obtain an evolution equation for the function $\psi$:

$$\dot{\psi} = i (1-\alpha') + \alpha) \psi_{ss}$$

$$+ \left( i(1-\alpha') - \alpha \right) |\psi|^2 \psi + \frac{\alpha}{4} (\psi^2 - \psi^* 2) \psi. \quad (14)$$

Evidently, for the solutions we take interest in, the term $\psi^2 - \psi^* 2$ will be small (negligible), so we remove it. This term would need to be considered in the case of higher-order perturbations to the system (at order $\alpha^2$). Making this reasonable reduction, we obtain

$$\psi = i A(t) + (i(1-\alpha') + \alpha) \psi_{ss}$$

$$+ \left( i(1-\alpha') - \alpha \right) |\psi|^2 \psi + \frac{\alpha}{4} (\psi^2 - \psi^* 2) \psi. \quad (15)$$

Under an appropriate scaling of time $(T = (1-\alpha')t)$ and defining a function $\Psi$ such that $\psi(s,t) = \sqrt{\Psi(s,T)} \exp(i \int_0^t A(t) dt)$, we can reduce \cite{13} into

$$\Psi_T = (i + \epsilon) \Psi_{ss} + (i - 2 \epsilon) |\Psi|^2 \Psi, \quad (16)$$

where $\epsilon = \alpha/(1-\alpha') << 1$. Eq. \cite{10} is a type of complex Ginzburg-Landau equation. If we take $\alpha, \alpha' = 0$ (which corresponds to a standard fluid), then $\epsilon = 0$, and \cite{10} reduces to the cubic NLS, and therefore these results are completely consistent with those of Hasimoto for the standard fluid LIA.

A Stokes wave solution exists for the classical LIA. To recover a Stokes wave along a quantum vortex filament, we assume a solution of the form $\Psi(s,T) = P(T)$, so that

$$i P_T + (1 + 2 \epsilon) |P|^2 P = 0. \quad (17)$$

Writing $P(T) = R(T) \exp(i \Theta(T))$, we find $R_T = -2 \epsilon R^2$ and $\Theta_T = 2 \epsilon R^2$, which gives $R(T) = (1 + 4 \epsilon T)^{-1/2}$ and $\Theta(T) = (4 \epsilon)^{-1} \ln(2 + 4 \epsilon T)$. $P(T)$ then takes the form

$$P(T) = \cos \left( \frac{\ln(1 + 4 \epsilon T)}{4 \epsilon} \right) + i \sin \left( \frac{\ln(1 + 4 \epsilon T)}{4 \epsilon} \right). \quad (18)$$

Taking $\psi(s,t) = P(T) \exp(i \int_0^t A(t) dt)$ gives us the general form of a Stokes wave. In the $\epsilon = 0$ limit, we obtain
the standard Stokes wave of constant modulus.

The most interesting solution associated with the Hase- moto transformation of the LIA is likely the soliton on a vortex filament. It is natural to wonder if such a solit-
on solution is possible for the quantum LIA. In order to obtain a soliton, we shall consider a stationary solution \( \Psi(s,T) = \sqrt{2\omega}q(S)\exp(-i\omega t) \), with \( S = \sqrt{2s} \). This puts \( 10 \) into the form

\[
(1 - \epsilon i)q'' + 2(1 + 2\epsilon i)|q|^2q + q = 0 ,
\]

or, equivalently,

\[
q'' + 2\left( \frac{1 - \epsilon^2}{1 + \epsilon^2} + \frac{3\epsilon}{1 + \epsilon^2} \right)|q|^2q - \left( \frac{1}{1 + \epsilon^2} + \frac{\epsilon i}{1 + \epsilon^2} \right) q = 0 .
\]

Since \( \epsilon = O(\alpha) \) and \( \alpha << 1 \), we may neglect terms of order \( \epsilon^2 \) and higher, to obtain

\[
q'' + 2(1 + 3\epsilon)|q|^2q - (1 + \epsilon)i q = 0 .
\]

When \( \epsilon = 0 \) (corresponding to the standard fluid case), we find \( q(S) = \text{sech}(S) \), so any solution for \( \epsilon > 0 \) should reduce to this case in the \( \epsilon \to 0 \) limit. We should therefore consider a solution of the form \( q(S) = \rho(S)\exp(i\theta(S)) \). This has the interpretation that curvature is determined by the \( \epsilon = 0 \) case, while \( \epsilon > 0 \) influence the torsion of the filament. Making the relevant transformation, discarding contributions of order \( \epsilon^2 \) or higher, and splitting \( 21 \) into real and imaginary parts, we obtain

\[
\rho'' + 2\rho^3 - \rho = 0 , \quad 2\rho\theta' + \rho\theta'' + 6\rho^3 - \rho = 0 .
\]

Clearly, \( \rho(S) = \text{sech}(S) \), which is just the soliton from the standard fluid case. We then find that

\[
\theta'(S) = (C_1 + 5\tanh(S) - 2\tanh^3(S))\cosh^2(S) .
\]

This derivative blows up rapidly for all values of \( C_1 \) except for \( C_1 = 3 \). When \( C_1 = 3 \), \( \theta'(S) \to 1/2 \) as \( S \) gets large. This in turn implies that \( \theta(S) \) would grow linearly as \( S \) gets large. Therefore, we pick \( C_1 = 3 \), and upon integrating \( 22 \) we find

\[
\theta(S) = C_0 + \frac{3}{2}S + \frac{3}{4}(\cosh(2S) - \sinh(2S)) + 2\ln(\cosh(S)) .
\]

There is a far simpler, yet still rather accurate, way to represent \( \theta \) by way of an asymptotic expansion. We find that

\[
\theta(S) = C_0 + \frac{7}{4} - 2\ln(2) + \frac{S}{2} + \frac{11}{4}e^{-2S} + O(e^{-4S}) .
\]

Picking \( C_0 = 2\ln(2) - 7/4 \) to simplify the expansion,

\[
\theta(S) = \frac{S}{2} + \frac{11}{4}e^{-2S} + O(e^{-4S}) .
\]

This solution gives a linear growth in \( S \), for large enough \( S \). We therefore have that \( q(S) \) is accurately approximate by

\[
q(S) = \text{sech}(S)\exp\left( ie \left( \frac{S}{2} + \frac{11}{4}e^{-2S} \right) \right)
\]

up to order \( \epsilon^2 \). Putting this solution back into the natural variables \( s \) and \( t \), we obtain a soliton on a quantum vortex filament:

\[
\psi(s,t) = \sqrt{2\omega}\text{sech}(\sqrt{2s})\exp(i\mu(s,t)) ,
\]

\[
\mu(s,t) = \frac{\alpha}{1 - \alpha'} \left( \frac{\sqrt{\omega}}{2} + \frac{11}{4}e^{-2\sqrt{2s}} \right) - (1 - \alpha')\omega t .
\]

The solution \( 23, 24 \) constitutes a soliton along a vortex filament. The solution is stationary, with only the phase depending on time. It is, however, possible to con-
sider a traveling wave along a vortex filament. In the case, both the phase and the amplitude of \( \psi \) would vary in time. Let us define \( \Psi(s,T) = \Psi(\xi) \), where \( \xi = s - qT \). Ignoring terms of order \( \epsilon^2 \) and higher, we obtain

\[
\Psi'' + \epsilon(\mu - \nu)\Psi' + (1 + 3\epsilon)|\Psi|^2\Psi = 0 .
\]

Writing \( \hat{\Psi}(\xi) = F(\xi)\exp(i\int_0^\xi G(\nu)d\nu) \), we obtain the system

\[
F'' - FG^2 + \eta F' + \eta FG + F^3 = 0 ,
\]

\[
2F'G + FG' - \eta F' + \eta FG + 3\epsilon F^3 = 0 ,
\]

which is effectively a third-order dynamical system. The system \( 31 \) has the interesting property that it has either one or infinitely many equilibrium points, depending on the value of the wave speed, \( \eta \). If \( \eta = 3/2 \), there exist infinitely many equilibria of the form \( (F,G) = (F_0, -2F_0^2) \), where \( F_0 \in \mathbb{R} \). On the other hand, when \( \eta \neq 3/2 \), the only equilibrium is the zero equilibrium \( (F,G) = (0,0) \).

From numerical simulations, we find that there is an inter-

ingesting bursting pattern associated with the solutions to \( 31 \). For large negative values of \( \xi \), the phase and amplitude functions are reasonably well-behaved. Then, near some finite value \( \xi = \xi_0 \) (which in general depends on both \( \epsilon \) and \( \eta \)), there is a bursting behavior to the phase \( \Psi(\xi) \), near where the amplitude \( F(\xi) \) reaches its minimal value. Past \( \xi_0 \), the phase switches signs and gradually the dynamics become more tame. This behavior is demon-

strated in Fig. 1. This behavior becomes more clear when we view the system in phase space. In Fig. 2, we plot the solution to \( 31 \) in the phase space \( (F,F',G) \). The solution corresponds to a trajectory which originates infinitely far from the origin as \( \xi \to -\infty \), then approaches the origin, goes through a bursting pattern, and then leaves the origin in a similar manner to which it came.

With this, we have successfully transformed the quan-
FIG. 1: (Color Online) Plot of the solution to the dynamical system (31) when $\alpha = 0.073$, $\alpha' = 0.018$, $\eta = 0.1$. Initial conditions are $F(0) = 1$, $F'(0) = 0$ and $G(0) = 0$.

FIG. 2: (Color Online) Phase space plot of the solution to the dynamical system (31) when $\alpha = 0.073$, $\alpha' = 0.018$, $\eta = 0.1$. Initial conditions are $F(0) = 1$, $F'(0) = 0$ and $G(0) = 0$.

We considered a family of traveling waves solutions. The phase of the waves undergo a type of bursting behavior, during which they change sign (going from positive to negative). However, we did not find more complicated dynamics, such as chaos. Still, there are other possible solutions to the PDE (10), so more complex dynamics are certainly possible. Indeed, chaos has been shown to arise from related models [11]. Chaos in the quantum LIA was previously conjectured [12], but as of yet has not been shown. Note that our derivations exclude any strong effects from the normal fluid velocity vector, $U$. It is possible to include the effects of the normal fluid, although the derivations will be much more complicated and lengthy. Due to the added complexity of the resulting equation, it may be possible to demonstrate chaotic behavior in the analogous equations which account for the normal fluid flow. It is also possible that the inclusion of the term $\psi^2 - \psi^*^2$ will give more complicated dynamics in some instances.

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