CONTINUOUS-TIME MEAN-VARIANCE ASSET-LIABILITY
MANAGEMENT WITH STOCHASTIC INTEREST RATES
AND INFLATION RISKS

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ABSTRACT. This paper investigates a continuous-time Markowitz mean-variance asset-liability management (ALM) problem under stochastic interest rates and inflation risks. We assume that the company can invest in $n+1$ assets: one risk-free bond and $n$ risky stocks. The risky stock’s price is governed by a geometric Brownian motion (GBM), and the uncontrollable liability follows a Brownian motion with drift, respectively. The correlation between the risky assets and the liability is considered. The objective is to minimize the risk (measured by variance) of the terminal wealth subject to a given expected terminal wealth level. By applying the Lagrange multiplier method and stochastic control approach, we derive the associated Hamilton-Jacobi-Bellman (HJB) equation, which can be converted into six partial differential equations (PDEs). The closed-form solutions for these six PDEs are derived by using the homogenization approach and the variable transformation technique. Then the closed-form expressions for the efficient strategy and efficient frontier are obtained. In addition, a numerical example is presented to illustrate the results.

1. Introduction. Mean-variance portfolio selection has become the foundation of modern finance theory since the pioneering work of Markowitz [18], where he considered a one-period economy and formulated the portfolio selection problem as a static mean-variance optimization problem. Since then, continuing efforts have been devoted to extend the one-period model to dynamic multi-period or continuous-time models. Among these efforts, breakthrough works on mean-variance analysis

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have been done by Li and Ng [15] and Zhou and Li [31], where they extended the single period mean-variance model to multi-period and continuous-time versions, respectively, the explicit expressions of the efficient investment strategies and mean-variance efficient frontier are derived by employing an embedding technique and the stochastic linear quadratic control approach. Thereafter, many scholars focus on dynamic mean-variance portfolio selection problems. For more detailed discussion, readers are referred to Celikyurt and Özekici [3], Costa and Araujo [8], Cui et al. [9] and references therein for the multi-period case, and Lim and Zhou [16], Zhou and Yin [32], Wang and Forsyth [24], Shen [22], Lv et al. [17] and references therein for the continuous-time case.

Nowadays, in risk management and insurance, it is widely recognized that the ALM problem is of both theoretical interest and practical importance. In ALM, there are two main concerns, one is to maximize the terminal surplus which is the difference between asset value and liability value under the framework of mean-variance criterion (see Chiu and Li [6], Xie [26], Chen et al. [5], Chiu and Wong [7], Yu [29], Chang [4], etc.). The other is to maximize the expected utility of the terminal asset-liability ratio (see Hoevenaars et al. [12], Giamouridis et al. [11], Pan and Xiao [20]). However, most of the above-mentioned literatures ignore inflation risk. It is reasonable to take the risks of inflation rate into account when an investment project has a long time horizon such as a pension fund. Zhang [30] considered the problem of maximizing the expected utility of terminal real wealth including an inflation-linked bond. Siu [23] added the impact of (macro)-economic conditions to the settings in Zhang [30]. Some other works on maximizing the CRRA utility function include Brennan and Xia [2], Munk et al. [19] for one risky asset and Bensoussan [1] for multiple risky assets and partial information. Yao et al. [27] studied a Markowitz mean-variance defined contribution pension fund management with inflation risk, and obtained closed-form solutions of the efficient strategy and efficient frontier by using the dynamic programming approach. Recently, under considering the stochastic interest rates and inflation risks, Pan and Xiao [21] derived both the efficient strategy and the mean-variance efficient frontier by using the dynamic programming approach. However, they do not consider the correlation between the risky asset and the liability.

In this paper, based on the work of Xie et al. [25], we incorporate the stochastic interest rates and inflation risks of market into a dynamic ALM problem with multiple risky assets. In additional, we consider the correlation between the liability and the stocks. This actually relaxes the assumption of liability in the work of Chiu and Li [6], where both the risky assets’ prices and the liability are restricted to be governed by the same Brownian motions. In order to solve the optimal control problem, we first apply the Lagrange multiplier method to transform the original problem into a standard stochastic optimal control problem and establish the corresponding extended HJB equation. Then we obtain the explicit expressions of the optimal investment strategy by solving the extended HJB equation. Furthermore, by using the Lagrange dual theory, we derive the efficient investment strategy and efficient frontier of this mean-variance ALM problem. Finally, for comparison, we discuss a special case without liability and provide a numerical example to illustrate our results.

The remainder of this paper is organized as follows. Section 2 provides the formulation of the problem. In Section 3, the associated nonlinear HJB equation is obtained and converted into six PDEs. Section 4 gives the integral-form analytical
solutions for these six PDEs. The closed-form expressions for the efficient strategy and the mean-variance efficient frontier are obtained in Section 5. In Section 6, we compare our results with a special case. Section 7 gives some numerical results, and we conclude in Section 8.

2. Model formulation. Let \((\Omega, \mathcal{F}, \mathcal{P}, \mathbb{F})\) be a complete filtered probability space. Assume that the filtration \(\mathbb{F} = \{F_t\}_{t \in [0,T]}\) is generated by an \((n+1)\)-dimensional standard Brownian motion \((W_0(t), W_1(t), \ldots, W_n(t))\mid t \in [0,T]\) for a positive integer \(n\), where the positive number \(T\) is a fixed and finite time horizon, \(F_0 = \{\emptyset, \Omega\}, \mathcal{F}_T = \mathcal{F}\), and the superscript "\(^\top\)" represents the transpose of a vector or a matrix.

We denote by \(\mathcal{C}([0,T]; \mathbb{R}^{n \times k})\) the class of \(\mathbb{R}^{n \times k}\)-valued continuous bounded deterministic functions on \([0,T]\), and by \(\mathcal{L}^2_k([0,T]; \mathbb{R}^m)\) the set of all \(\mathbb{R}^m\)-valued and progressively measurable stochastic processes \(f(t)\) which are adapted to \(\{\mathcal{F}_t\}_{t \in [0,T]}\) such that \(E\left[\int_0^T \|f(t)\|^2 dt\right] < +\infty\).

We consider a financial market composed of one bond and \(n\) stocks. The bond price \(P_0(t)\) is described by the following ordinary differential equation (ODE):

\[
\begin{cases}
    dP_0(t) = r(t)P_0(t) dt, & t \in [0,T], \\
    P_0(0) = p_0 > 0,
\end{cases}
\]

where \(r(t)\) is the risk-free interest rate and follows the Hull-White interest rate model (see Hull and White [13])

\[
\begin{cases}
    dr(t) = \theta(t) - r(t) \, dt + \sigma_r(t) \, dW(t), & t \in [0,T], \\
    r(0) = r_0 > 0,
\end{cases}
\]

where \(\theta(t) \in \mathcal{C}([0,T]; \mathbb{R})\) denotes the long run mean of interest rate; \(a\) is a positive constant which denotes the mean reversion rate of interest rate; \(\sigma_r(t) = (\sigma_{r1}(t), \ldots, \sigma_{rn}(t)) \in \mathcal{C}([0,T]; \mathbb{R}^{1 \times n})\) denotes the volatility of interest rate, and \(W(t) = (W_1(t), \ldots, W_n(t))\).

The price of the \(i\)th stock, defined by \(P_i(t)\), satisfies the following stochastic differential equation (SDE):

\[
\begin{cases}
    dP_i(t) = \mu_i(t)P_i(t) dt + \sigma_i(t)P_i(t) dW(t), & t \in [0,T], \\
    P_i(0) = p_i, & i = 1, \ldots, n,
\end{cases}
\]

where \(\mu_i(t) \in \mathcal{C}([0,T]; \mathbb{R})\) and \(\sigma_i(t) = (\sigma_{i1}(t), \ldots, \sigma_{in}(t)) \in \mathcal{C}([0,T]; \mathbb{R}^{1 \times n})\) are the appreciation rate and volatility of the \(i\)th stock, respectively.

Consider a company with an initial asset \(x_0 > 0\). In a finite time horizon \(T > 0\), the above \(n+1\) assets can be traded continuously without transaction costs or any restrictions on lending, borrowing of bonds and short sale of stocks. Let \(\pi(t) = (\pi_1(t), \ldots, \pi_n(t))\) \(^\top\) be an asset portfolio with \(\pi_i(t)\) representing the proportion of money invested in the \(i\)th stock, then, the company’s asset \(X(t)\) evolves as

\[
\begin{cases}
    dX(t) = X(t) \left[1 - \sum_{i=1}^{n} \pi_i(t)\right] \frac{dP_0(t)}{P_0(t)} + X(t) \sum_{i=1}^{n} \pi_i(t) \frac{dP_i(t)}{P_i(t)}, \\
    X(0) = x_0 > 0,
\end{cases}
\]

where \(b(t) = (\mu_1(t) - r(t), \ldots, \mu_n(t) - r(t))\) \(^\top\) \(\in \mathcal{C}([0,T]; \mathbb{R}^n)\), \(\sigma(t) = [\sigma_{ij}(t)] \in \mathcal{C}([0,T]; \mathbb{R}^{n \times n})\).
Following Xie [26], we denote by $L(t)$ the company’s accumulative liability at time $t$, and assume that $L(t)$ satisfies the following SDE:

$$
\begin{align*}
\left\{ \begin{array}{l}
    dL(t) = L(t) [\alpha(t) dt + \beta(t) d\rho(t)], \quad t \in [0, T], \\
    L(0) = L_0,
\end{array} \right.
\end{align*}
$$

where $\alpha(t), \beta(t) \in C([0, T]; \mathbb{R})$, $\{\rho(t) : t \in [0, T]\}$ is a one-dimensional standard Brownian motion defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$.

In general, the dynamic evolution of the liability is not always independent with the risky assets prices. We denote by $\rho_j(t)$ the correlation coefficient between $B(t)$ and $W_j(t)$ for $j = 1, \ldots, n$, and let $\rho(t) = (\rho_1(t), \ldots, \rho_n(t))$ be the correlation coefficient vector. It follows from the formula (2.6) given in Koo [14] that the diffusion term of the liability, $B(t)$, can be expressed as the linear combination of $W_1(t), \ldots, W_n(t)$ and $W_0(t)$:

$$
B(t) = \rho(t)W(t) + \sqrt{1 - \rho(t)\rho'(t)}W_0(t),
$$

where $\rho(t)\rho'(t) \leq 1$ for all $t \in [0, T]$. When $\rho(t)\rho'(t) < 1$, the risk from the liability cannot be eliminated by trading financial assets.

By (6), we have

$$
\left\{ \begin{array}{l}
    dL(t) = L(t) \left[ \alpha(t) dt + \beta(t)\rho(t)d\rho(t) + \beta(t)\sqrt{1 - \rho(t)\rho'(t)}dW_0(t) \right], \\
    L(0) = L_0.
\end{array} \right.
$$

Eq. (7) includes two special cases. Firstly, if $B(t)$ does not correlate with $W_j(t)$ for $j = 1, \ldots, n$, that is, $\rho_j(t) = 0$ for $j = 1, \ldots, n$, then $B(t)$ is equal to $W_0(t)$. Secondly, if the stocks price and the liability are driven by the same source of randomness, that is, $\rho(t) = 1$, then $B(t) = W(t)$, this means that the risk risen from the liability can be hedged completely by the stock.

As is well known, the consumer price index (CPI) is often used to measure the inflation level, so we use CPI to represent the inflation index. As adopted in much literature (see Brennan and Xia [2] and Munk et al. [19]), we suppose that the inflation index $I(t)$ satisfies the following SDE:

$$
\left\{ \begin{array}{l}
    dI(t) = I(t) [\sigma(t) dt + \eta(t)d\rho(t)], \quad t \in [0, T], \\
    I(0) = I_0,
\end{array} \right.
$$

where $\sigma(t) \in C([0, T]; \mathbb{R})$ is the instantaneous expected inflation rate, and $\eta(t) = (\eta_1(t), \ldots, \eta_n(t)) \in C([0, T]; \mathbb{R}^{1 \times n})$ is the volatility of the price index. It should be noted that the stock prices, and the inflation index could be correlated with each other, since they are all affected by the common stochastic factor $W(t)$.

Throughout this paper we assume that for $\forall t \in [0, T]$, $\sigma(t)$ satisfies the nondegenerate conditions: $\sigma(t)\sigma'(t) > \epsilon I_n$, where $\epsilon > 0$ and $I_n$ is the identity matrix.

In ALM, we are concerned with the surplus which is the difference between asset value and liability value, we denote the surplus process of the company by $Y(t) = X(t) - L(t)$. Subtracting (7) from (4), we obtain the SDE for the surplus process as follows:

$$
\left\{ \begin{array}{l}
    dY(t) = \left[ (\sigma(t) + \dot{\sigma}(t)b(t)) (Y(t) + L(t)) - \alpha(t)L(t) \right] dt \\
    + \left[ \dot{\sigma}(t)\sigma(t) (Y(t) + L(t)) - \beta(t)\rho(t)L(t) \right] dW(t) \\
    - \beta(t)\sqrt{1 - \rho(t)\rho'(t)}L(t)dW_0(t), \\
    Y(0) = y_0 - L_0.
\end{array} \right.
$$
Let $\tilde{Y}(\cdot)$ denote the real surplus process excluding the impact of the inflation, i.e. $\tilde{Y}(\cdot) = \frac{Y(\cdot)}{I(\cdot)}$, then, by (8) and (9), and using Itô’s formula, we have

$$
\begin{aligned}
d\tilde{Y}(t) &= \left[ (r(t) + \pi'(t)\tilde{b}(t) + \delta(t)) \left( \tilde{Y}(t) + \tilde{L}(t) \right) - \tilde{\alpha}(t)\tilde{L}(t) \right] dt \\
&\quad + \left[ (\pi'(t)\sigma(t) - \eta(t)) \left( \tilde{Y}(t) + \tilde{L}(t) \right) - (\beta(t)\rho(t) - \eta(t)) \tilde{L}(t) \right] dW(t) \\
&\quad - \beta(t)\sqrt{1 - \rho(t)\rho'(t)}\tilde{L}(t) dW_0(t), \\
\tilde{Y}(0) &= \frac{x_0 - L_0}{I_0} = y_0,
\end{aligned}
$$

where $\tilde{L}(\cdot) = \frac{L(\cdot)}{I(\cdot)}$ is the real liability process satisfying

$$
\begin{aligned}
d\tilde{L}(t) &= \tilde{L}(t) \left[ \tilde{\alpha}(t) dt + (\beta(t)\rho(t) - \eta(t)) dW(t) + \beta(t)\sqrt{1 - \rho(t)\rho'(t)} dW_0(t) \right], \\
\tilde{L}(0) = \frac{L_0}{I_0} = \tilde{l}_0,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{b}(t) &= b(t) - \sigma(t)\eta'(t), \\
\tilde{\delta}(t) &= \eta(t)\eta'(t) - v(t), \\
\tilde{\alpha}(t) &= \alpha(t) - v(t) - [\beta(t)\rho(t) - \eta(t)]\eta'(t).
\end{aligned}
$$

**Definition 2.1.** An asset portfolio $\pi = \{\pi(t); t \in [0, T]\}$ is called admissible if $\pi(t) \in L^2_{\mathbb{F}}([0, T]; \mathbb{R}^n)$ and the SDE (10) has a unique solution $\tilde{Y}(t)$ corresponding to $\pi(t)$. In this case, we refer to $(\tilde{Y}(t), \pi(t))$ an admissible pair. The set of all admissible portfolios is denoted by $\Pi$.

The company’s objective is to find an admissible asset portfolio $\pi(t)$, among all the admissible portfolios whose expected terminal surplus is $E[\tilde{Y}(T)] = d$ for some given $d \in \mathbb{R}$, such that the risk measured by the variance of the terminal surplus

$$
\text{Var} [\tilde{Y}(T)] = E [\tilde{Y}(T) - E[\tilde{Y}(T)]]^2 = E \left[ \tilde{Y}(T) - d \right]^2
$$

is minimized. Specifically, we have the following formulation.

**Definition 2.2.** The mean-variance portfolio selection in ALM is the following constrained stochastic optimization problem, parameterized by $d \in \mathbb{R}$:

$$
\begin{aligned}
\min_{\pi \in \Pi} \text{Var}[\tilde{Y}(T)] = E[\tilde{Y}(T) - d]^2, \\
\text{subject to} \\
E[\tilde{Y}(T)] = d, \\
(\tilde{r}(t), \tilde{Y}(t), \tilde{L}(t), \pi(t)) \text{satisfies (2), (10) and (11)}, \\
\pi(t) \in \Pi.
\end{aligned}
$$

Let $d_{\sigma_{\text{min}}}$ be the expected terminal real surplus which has the minimum variance over all admissible strategies. Corresponding to $d > d_{\sigma_{\text{min}}}$, the optimal solution $\pi(t) = \{\pi(t); t \in [0, T]\}$ of problem (10) is called an efficient strategy, and the corresponding pair $(\text{Var}[\tilde{Y}(T)], d)$ is called an efficient point. The set of all efficient points is called the efficient frontier.

3. **Transformation of the problem.** In this section, the Lagrange method is applied to handle the constraint $E[\tilde{Y}(T)] = d$. By introducing a multiplier $\lambda$, our problem turns out to be an unconstrained problem:

$$
\begin{aligned}
\min \ E \left[ \tilde{Y}(T) - d \right]^2 + 2\lambda \left\{ E \left[ \tilde{Y}(T) \right] - d \right\}.
\end{aligned}
$$
Denoting \( u = d - \lambda \), we have
\[ E [\tilde{Y}(T) - d]^2 + 2\lambda \{ E [\tilde{Y}(T)] - d \} = E [\tilde{Y}(T) - u]^2 - \lambda^2. \]
Note that \( \lambda^2 \) is a fixed value. The problem (14) is equivalent to
\[ \min_{\pi \in \Pi} E [\tilde{Y}(T) - u]^2. \quad (15) \]
Following the framework of dynamic programming, we consider the truncated problem (15) starting from time \( t \), where \( t \in [0, T] \), with initial states \( r(t) = r, \tilde{Y}(t) = y \) and \( \tilde{L}(t) = l \). That is, the dynamics of \( r(s), \tilde{Y}(s) \) and \( \tilde{L}(s) \) for \( s \in [t, T] \) can be written as
\[
\begin{cases}
  dr(s) = a[\theta(s) - r(s)] dt + \sigma_r(s) dW(s), \\
  r(t) = r,
\end{cases}
\quad (2')
\]
and
\[
\begin{cases}
  d\tilde{Y}(s) = [(\pi'(s)\tilde{b}(s) + \tilde{\delta}(s)) (\tilde{Y}(s) + \tilde{L}(s)) - \tilde{\alpha}(s)\tilde{L}(s)] ds \\
  + [(\pi'(s)\sigma_r(s) - \eta(s)) (\tilde{Y}(s) + \tilde{L}(s)) - (\beta(s)\rho_r(s) - \eta(s)) \tilde{L}(s)] dW(s) \\
  + \beta(s)\sqrt{1 - \rho_r(s)^2}\tilde{L}(s) dW_0(s), \\
  \tilde{Y}(t) = y,
\end{cases}
\quad (10')
\]
and
\[
\begin{cases}
  d\tilde{L}(s) = \tilde{L}(s) [\tilde{\alpha}(s)ds + (\beta(s)\rho_r(s) - \eta(s)) dW(s) - \beta(s)\sqrt{1 - \rho_r(s)^2}dW_0(s)], \\
  \tilde{L}(t) = l.
\end{cases}
\quad (11')
\]
Correspondingly, the value function is defined as follows:
\[ V(t, r, y, l) = \inf_{\pi \in \Pi} E \left[ (\tilde{Y}(T) - u)^2 \mid r(t) = r, \tilde{Y}(t) = y, \tilde{L}(t) = l \right] \quad (16) \]
with the boundary condition \( V(T, r, y, l) = (y - u)^2 \). Refering to Yao et al. [27], we know that when \( t = 0 \), \( V(0, 0, y_0, l_0) \) is the value function for problem (15). By using the principle of dynamic programming (see Fleming and Soner [10]), the HJB equation for \( V = V(t, r, y, l) \) is
\[
\begin{align*}
  &V_t + a[\theta(t) - r] V_r + \frac{1}{2} \sigma_r(t)\sigma_r(t)^2 V_{rr} + \tilde{\alpha}(t) V_l \\
  &+ \frac{1}{2} \left[ \eta(t)\eta'(t) y'^2 + 2\beta(t)\rho_r(t)\eta'(t) \right] V_{yy} + \left[ (r + \delta(t)) (y + l) - \tilde{\alpha}(t) l \right] V_y \\
  &+ \frac{1}{2} \left[ \beta(t)\sigma_r(t)\eta'(t) - \sigma_r(t)\eta''(t) \right] V_{ry} - \beta(t)\rho_r(t)\eta'(t) y + \beta(t)\rho_r(t)\eta'(t) l V_{ry} \\
  &+ \eta(t)\eta'(t) y' \beta(t)\rho_r(t)\eta'(t) g + \beta(t)\rho_r(t)\eta'(t) l V_{yl} \\
  &+ \inf_{\pi(t)} \left\{ \pi(t) \tilde{b}(t)(y + l) V_y + \frac{1}{2} \left[ \pi(t)\sigma(t)\sigma(t)^2 \pi(t)(y + l)^2 \right] V_{yy} \\
  &- 2\pi(t)\sigma(t)\eta'(t)(y + l) y - 2\pi'(t)\beta(t)\sigma(t)\eta'(t) \rho_r(t)(y + l) \right\} V_{yy} \\
  &+ \pi'(t)\sigma(t)\eta'(t)(y + l) V_{ry} + \pi(t)\sigma(t) [\beta(t)\rho_r(t) - \eta'(t)] (y + l) V_{yl} = 0,
\end{align*}
\]
(17)
where \( V_t, V_r, V_l, V_y, V_{rr}, V_{rl}, V_{ry}, V_{yy}, V_{yl} \) denote the partial derivatives of first and second orders with respect to \( t, r, l, y \), respectively.

The first-order condition for \( \pi(t) \) in (17) gives the corresponding optimal strategy:
\[ \pi^*(t) = -\frac{(\sigma(t)\sigma(t)^2)^{-1}}{y + l} \left[ b(t) V_y + \sigma(t) [\beta(t)\rho_r(t) - \eta'(t)] V_{yy} + \sigma(t)\sigma(t)\eta'(t) V_{yl} \right. \\
\]
\[ - \sigma(t)\eta'(t) y - \beta(t)\sigma(t)\eta'(t) l \right]. \quad (18) \]
Plugging (18) into (17) and simplifying the equation, we have

\[
\begin{align*}
V_t + a \theta(t) - r V_r + \frac{1}{2} \sigma_r(t) \sigma_r(t) V_{rr} + \bar{a}(t) V_t + \frac{1}{2} \varphi_1(t) l^2 V_{ll} & \\
+ \left[ \left( r + \delta(t) \right) \left( y + l \right) - \varphi_2(t) \right] V_y + \varphi_3(t) y V_y + \frac{1}{2} \varphi_4(t) y^2 V_{yy} & \\
+ \varphi_5(t) y V_{yl} + \frac{1}{2} \varphi_6(t) y^2 V_{yy} + \varphi_7(t) V_{yl} - \varphi_8(t) y V_{yy} - \varphi_9(t) l V_{yl} & \\
+ \varphi_{10}(t) y V_{yl} + \varphi_{11}(t) l^2 V_{yl} - \frac{1}{2} \varphi_{12}(t) V_{ll}^2 - \varphi_{13}(t) V_{yl} V_{yy} & \\
- \varphi_{14}(t) l V_{yl} - \frac{1}{2} \varphi_{15}(t) V_{yy}^2 - \varphi_{16}(t) V_{yl} V_{yy} - \frac{1}{2} \varphi_{17}(t) l^2 V_{yy}^2 = 0,
\end{align*}
\]

where

\[
\begin{align*}
\varphi_1(t) &= \beta^2(t) - 2 \beta(t) \rho(t) \eta'(t) + \eta(t) \eta'(t), \\
\varphi_2(t) &= \bar{a}(t) - \beta(t) \bar{b}'(t) \sigma(t) \sigma'(t) + \sigma(t) \rho'(t), \\
\varphi_3(t) &= \tilde{b}'(t) \sigma(t) \sigma'(t) - \sigma(t) \eta'(t), \\
\varphi_4(t) &= \eta(t) \eta'(t) - \eta(t) \sigma(t) \sigma'(t) - \sigma(t) \eta'(t), \\
\varphi_5(t) &= \beta(t) \rho(t) \eta'(t) - \beta(t) \eta(t) \eta'(t) \sigma(t) \sigma'(t) - \sigma(t) \rho'(t), \\
\varphi_6(t) &= \beta^2(t) - \beta^2(t) \rho(t) \sigma'(t) (\sigma(t) \sigma'(t))^{-1} \sigma(t) \rho'(t), \\
\varphi_7(t) &= \sigma_r(t) \left[ \beta(t) \rho(t) - \eta(t) \right], \\
\varphi_8(t) &= \sigma_r(t) \eta'(t) - \sigma(t) \sigma'(t) \eta(t) \sigma'(t) - \sigma(t) \eta'(t), \\
\varphi_9(t) &= \beta(t) \sigma_r(t) \rho'(t) - \beta(t) \sigma_r(t) \sigma'(t) \sigma(t) \sigma'(t)^{-1} \sigma(t) \rho'(t), \\
\varphi_{10}(t) &= \eta(t) \eta'(t) - \eta(t) \eta'(t) - \beta(t) \rho(t) \eta'(t) + \beta(t) \rho(t) - \eta(t) \sigma(t) \sigma'(t) \\
&\quad \times \sigma(t) \sigma'(t)^{-1} \sigma(t) \rho'(t), \\
\varphi_{11}(t) &= \beta(t) \rho(t) \eta'(t) - \beta^2(t) + \beta(t) \beta(t) \rho(t) - \eta(t) \sigma(t) \sigma'(t) \\
&\quad \times \sigma(t) \sigma'(t)^{-1} \sigma(t) \rho'(t), \\
\varphi_{12}(t) &= \tilde{b}'(t) \sigma(t) \sigma'(t) - \tilde{b}'(t), \\
\varphi_{13}(t) &= \tilde{b}'(t) \sigma(t) \sigma'(t) - \tilde{b}'(t) \sigma_r(t) \sigma'(t), \\
\varphi_{14}(t) &= \tilde{b}'(t) \sigma(t) \sigma'(t) - \tilde{b}'(t) \sigma(t) \sigma'(t) \sigma'(t) - \tilde{b}'(t) \sigma(t) \sigma'(t) \sigma'(t) \\
&\quad \times \sigma(t) \sigma'(t)^{-1} \sigma(t) \rho'(t), \\
\varphi_{15}(t) &= \sigma_r(t) \sigma'(t) \sigma(t) \sigma'(t) - \sigma(t) \sigma'(t)^{-1} \sigma(t) \sigma'(t), \\
\varphi_{16}(t) &= \beta(t) \rho(t) - \eta(t) \sigma(t) \sigma'(t) \sigma'(t) - \sigma(t) \sigma'(t) \\
&\quad \times \sigma(t) \sigma'(t)^{-1} \sigma(t) \rho'(t), \\
\varphi_{17}(t) &= \beta(t) \rho(t) - \eta(t) \sigma(t) \sigma'(t) \sigma'(t) - \sigma(t) \sigma'(t) \sigma(t) \sigma'(t) - \beta(t) \rho(t) - \eta(t) \sigma(t) \sigma'(t) \\
&\quad \times \sigma(t) \sigma'(t)^{-1} \sigma(t) \rho'(t). 
\end{align*}
\]

Eq. (19) is a complicated nonlinear PDE which is hard to solve. Conjecturing that the solution of \( V(t, r, y, l) \) has the following form:

\[
V(t, r, y, l) = f(t, r) y^2 + g(t, r) y + h(t, r) l^2 + p(t, r) l + q(t, r) y l + z(t, r),
\]

where \( f(t, r), g(t, r), h(t, r), p(t, r), q(t, r) \) and \( z(t, r) \) are certain deterministic binary functions that need to be determined. Then, by (20), we have

\[
\begin{align*}
V_t = \frac{\partial f(t, r)}{\partial r} y^2 + \frac{\partial g(t, r)}{\partial r} y + \frac{\partial h(t, r)}{\partial r} l^2 + \frac{\partial p(t, r)}{\partial r} l + \frac{\partial q(t, r)}{\partial r} y l + \frac{\partial z(t, r)}{\partial r}, \\
V_r = \frac{\partial f(t, r)}{\partial r} y^2 + \frac{\partial g(t, r)}{\partial r} y + \frac{\partial h(t, r)}{\partial r} l^2 + \frac{\partial p(t, r)}{\partial r} l + \frac{\partial q(t, r)}{\partial r} y l + \frac{\partial z(t, r)}{\partial r}, \\
V_y = 2 f(t, r) y + g(t, r) + q(t, r) l, \\
V_{rr} = 2 \frac{\partial^2 f(t, r)}{\partial r^2} y^2 + 2 \frac{\partial^2 g(t, r)}{\partial r^2} y + 2 \frac{\partial^2 h(t, r)}{\partial r^2} l^2 + 2 \frac{\partial^2 p(t, r)}{\partial r^2} l + 2 \frac{\partial^2 q(t, r)}{\partial r^2} y l + 2 \frac{\partial^2 z(t, r)}{\partial r^2}, \\
V_{rl} = 2 \frac{\partial^2 f(t, r)}{\partial r \partial l} y^2 + 2 \frac{\partial^2 g(t, r)}{\partial r \partial l} y + 2 \frac{\partial^2 h(t, r)}{\partial r \partial l} l^2 + 2 \frac{\partial^2 p(t, r)}{\partial r \partial l} l + 2 \frac{\partial^2 q(t, r)}{\partial r \partial l} y l + 2 \frac{\partial^2 z(t, r)}{\partial r \partial l}, \\
V_{ry} = 2 \frac{\partial^2 f(t, r)}{\partial r \partial y} y + 2 \frac{\partial^2 g(t, r)}{\partial r \partial y} y + 2 \frac{\partial^2 h(t, r)}{\partial r \partial y} l^2 + 2 \frac{\partial^2 p(t, r)}{\partial r \partial y} l + 2 \frac{\partial^2 q(t, r)}{\partial r \partial y} y l + 2 \frac{\partial^2 z(t, r)}{\partial r \partial y}, \\
V_{ll} = 2 \frac{\partial^2 f(t, r)}{\partial l^2} y^2 + 2 \frac{\partial^2 g(t, r)}{\partial l^2} y + 2 \frac{\partial^2 h(t, r)}{\partial l^2} l^2 + 2 \frac{\partial^2 p(t, r)}{\partial l^2} l + 2 \frac{\partial^2 q(t, r)}{\partial l^2} y l + 2 \frac{\partial^2 z(t, r)}{\partial l^2}, \\
V_{yl} = 2 \frac{\partial^2 f(t, r)}{\partial y \partial l} y + 2 \frac{\partial^2 g(t, r)}{\partial y \partial l} y + 2 \frac{\partial^2 h(t, r)}{\partial y \partial l} l^2 + 2 \frac{\partial^2 p(t, r)}{\partial y \partial l} l + 2 \frac{\partial^2 q(t, r)}{\partial y \partial l} y l + 2 \frac{\partial^2 z(t, r)}{\partial y \partial l}, \\
V_{yy} = 2 \frac{\partial^2 f(t, r)}{\partial y^2} y^2 + 2 \frac{\partial^2 g(t, r)}{\partial y^2} y + 2 \frac{\partial^2 h(t, r)}{\partial y^2} l^2 + 2 \frac{\partial^2 p(t, r)}{\partial y^2} l + 2 \frac{\partial^2 q(t, r)}{\partial y^2} y l + 2 \frac{\partial^2 z(t, r)}{\partial y^2}.
\end{align*}
\]
\[
\frac{\partial f(t,r)}{\partial t} + \left[ a\theta(t) - ar - 2(\varphi_8(t) + \varphi_{13}(t)) \right] \frac{\partial f(t,r)}{\partial r} + \frac{\varphi_{13}(t)}{2} \frac{\partial^2 f(t,r)}{\partial r^2} + \frac{\varphi_{11}(t)}{f(t,r)} \frac{\partial f(t,r)}{\partial r} = 0, \quad (22)
\]
\[
T = t, r, \quad \frac{\partial}{\partial t} = t, r, \quad \frac{\partial^2}{\partial r^2} = t, r, \quad \frac{\partial}{\partial r} = t, r.
\]
\[
\frac{\partial g(t,r)}{\partial t} + \left[ a\theta(t) - ar - \varphi_8(t) - \varphi_{13}(t) - \varphi_{15}(t) \right] \frac{\partial g(t,r)}{\partial r} t, r, \quad + \varphi_{13}(t) \frac{\partial^2 g(t,r)}{\partial r^2} t, r, \quad + \varphi_{11}(t) \frac{\partial g(t,r)}{\partial r} t, r = 0,
\]
g(T, r) = -2u,
\]
\[
\frac{\partial h(t,r)}{\partial t} + \left[ a\theta(t) - ar + 2\varphi_7(t) \right] \frac{\partial h(t,r)}{\partial r} t, r, \quad + \frac{\varphi_{13}(t)}{2} \frac{\partial^2 h(t,r)}{\partial r^2} t, r, \quad + \varphi_{11}(t) \frac{\partial h(t,r)}{\partial r} t, r = 0,
\]
h(T, r) = 0,
\]
\[
\frac{\partial p(t,r)}{\partial t} + \left[ a\theta(t) - ar + \varphi_7(t) \right] \frac{\partial p(t,r)}{\partial r} t, r, \quad + \frac{\varphi_{13}(t)}{2} \frac{\partial^2 p(t,r)}{\partial r^2} t, r, \quad + \varphi_{11}(t) \frac{\partial p(t,r)}{\partial r} t, r = 0,
\]
p(T, r) = 0,
\]
\[
\frac{\partial q(t,r)}{\partial t} + \left[ a\theta(t) - ar + \varphi_7(t) - \varphi_8(t) - \varphi_{13}(t) - \varphi_{15}(t) \right] \frac{\partial q(t,r)}{\partial r} t, r, \quad + \frac{\varphi_{13}(t)}{2} \frac{\partial^2 q(t,r)}{\partial r^2} t, r, \quad + \varphi_{11}(t) \frac{\partial q(t,r)}{\partial r} t, r = 0,
\]
q(T, r) = 0,
\]
\[
\begin{align*}
\frac{\partial z(t,r)}{\partial t} + \left[ a\theta(t) - ar \right] \frac{\partial z(t,r)}{\partial r} + \frac{\varphi_{13}(t)}{2} \frac{\partial^2 z(t,r)}{\partial r^2} + \frac{\varphi_{11}(t)}{2} \frac{\partial z(t,r)}{\partial r} &= 0, \quad (26) \\
&= 0, \quad (27)
\end{align*}
\]
\[
z(T, r) = u^2.
\]
4. Solving the associated PDEs. Now, we are going to solve the six associated PDEs \((22)-(27)\). We start with Eq. \((22)\). Referring to Yao et al. \((27)\), we conjecture that the form of the solution is
\[
f(t, r) = e^{A_1(t)r + A_2(t)}, \quad (28)
\]
for two undetermined functions \(A_1(t)\) and \(A_2(t)\) with the boundary conditions \(A_1(T) = 0\) and \(A_2(T) = 0\). Differentiating Eq. \((28)\), we have
\[
\frac{\partial f(t,r)}{\partial t} = (A_1(t)r + \dot{A}_2(t)) f(t,r), \quad \frac{\partial f(t,r)}{\partial r} = A_1(t) f(t,r), \quad \frac{\partial^2 f(t,r)}{\partial r^2} = A_1^2(t) f(t,r), \quad (29)
\]
where the first derivatives are defined as \(\dot{A}_1(t) = \frac{dA_1(t)}{dt}\) and \(\dot{A}_2(t) = \frac{dA_2(t)}{dt}\). The first derivative for other functions in this paper will be defined in a similar way.
Then the solution of (31) can be expressed in terms of following ODEs:
\[
\begin{align*}
\dot{A}_1(t) &- aA_1(t)r + 2r = 0, \\
A_1(T) & = 0,
\end{align*}
\]
and
\[
\begin{align*}
\dot{A}_2(t) &+ [a\theta(t) - 2(\varphi_8(t) + \varphi_{13}(t))] A_1(t) + \left[ \frac{\sigma_r(s)\sigma_r'(s)}{2} - \varphi_{15}(t) \right] A_1^2(t) \\
&+ [2\delta(t) + 2\varphi_3(t) + \varphi_4(t) - \varphi_{12}(t)] = 0, \\
A_2(T) & = 0.
\end{align*}
\]
Eq. (30) can be solved as follows
\[
A_1(t) = \frac{2 - 2e^{a(t-T)}}{a}.
\]
Then the solution of (31) can be expressed in terms of $A_1(t)$:
\[
A_2(t) = \int_t^T \left\{ [a\theta(s) - 2(\varphi_8(s) + \varphi_{13}(s))] A_1(s) + \left[ \frac{\sigma_r(s)\sigma_r'(s)}{2} - \varphi_{15}(s) \right] A_1^2(s) \\
+ [2\delta(s) + 2\varphi_3(s) + \varphi_4(s) - \varphi_{12}(s)] \right\} ds.
\]
In the following, we proceed with the solution for (23). Plugging (28), which is the expression for $f(t, r)$, into (23) gives
\[
\frac{\partial g(t, r)}{\partial r} + [a\theta(t) - ar - \varphi_8(t) - \varphi_{13}(t) - \varphi_{15}(t)A_1(t)] \frac{\partial g(t, r)}{\partial t} + \frac{\sigma_r(t)\sigma_r'(t)}{2} \frac{\partial^2 g(t, r)}{\partial r^2} \\
+ r + \delta(t) + \varphi_3(t) - \varphi_{12}(t) - \varphi_{13}(t)A_1(t) g(t, r) = 0.
\]
Then we only need to solve Eq. (34). Suppose that $g(t, r)$ has the following form:
\[
g(t, r) = -2ue^{A_3(t)r + A_4(t)}.
\]
Plugging (35) into (34), we have the following ODEs:
\[
\begin{align*}
\dot{A}_3(t) &- aA_3(t)r + r = 0, \\
A_3(T) & = 0,
\end{align*}
\]
and
\[
\begin{align*}
\dot{A}_4(t) + [a\theta(t) - \varphi_8 - \varphi_{13}(t) - \varphi_{15}(t)A_1(t)] A_3(t) + \frac{\sigma_r(t)\sigma_r'(t)}{2} A_3^2(t) \\
+ [\delta(t) + \varphi_3(t) - \varphi_{12}(t) - \varphi_{13}(t)A_1(t)] = 0, \\
A_4(T) & = 0.
\end{align*}
\]
Eq. (36) can be solved easily:
\[
A_3(t) = \frac{1 - e^{a(t-T)}}{a}.
\]
Then, by ODE (37), the solution of $A_4(t)$ can be expressed in terms of $A_3(t)$:
\[
A_4(t) = \int_t^T \left\{ [a\theta(s) - \varphi_8(s) - \varphi_{13}(s) - \varphi_{15}(s)A_1(s)] A_3(s) + \frac{\sigma_r(s)\sigma_r'(s)}{2} A_3^2(s) \\
+ [\delta(s) + \varphi_3(s) - \varphi_{12}(s) - \varphi_{13}(s)A_1(s)] \right\} ds.
\]
Comparing (32) and (38), we obtain the following result.

**Proposition 1.** For any $t \in [0, T]$, we have $A_1(t) = 2A_3(t)$. 
Having addressed the solutions for (22) and (23), we turn to solve Eq. (27). Plugging the expressions for \( f(t, r) \) and \( g(t, r) \) into PDE (27), by Proposition 1, we have

\[
\begin{align*}
\frac{\partial z(t, r)}{\partial t} + [a\theta(t) - ar] \frac{\partial z(t, r)}{\partial r} + \frac{\sigma_q(t) \sigma'_q(t)}{2} \frac{\partial^2 z(t, r)}{\partial r^2} \\
- u^2 \left[ \varphi_{12}(t) + 2\varphi_{13}(t)A_3(t) + \varphi_{15}(t)A_3^2(t) \right] e^{2A_4(t) - A_2(t)} = 0,
\end{align*}
\]

(40)

Note that the expression for \( \left[ \varphi_{12}(t) + 2\varphi_{13}(t)A_3(t) + \varphi_{15}(t)A_3^2(t) \right] e^{2A_4(t) - A_2(t)} \) depends only on \( t \) and is independent of \( r \). Furthermore, by the boundary condition \( z(T, r) = u^2 \), we guess that \( z(t, r) \) also depends only on \( t \) and is independent of \( r \).

In this case, \( z(t, r) \) can be denoted as \( z(t) \), and \( \frac{\partial z(t, r)}{\partial r} = 0 \). Then, PDE (40) can be simplified as

\[
\begin{align*}
\left\{ \frac{\partial z(t)}{\partial t} - u^2 \left[ \varphi_{12}(t) + 2\varphi_{13}(t)A_3(t) + \varphi_{15}(t)A_3^2(t) \right] e^{2A_4(t) - A_2(t)} = 0,
\end{align*}
\]

(41)

By a simple calculation, the solution of (41) is

\[
z(t) = u^2 \left\{ 1 - \int_t^T \left[ \varphi_{12}(s) + 2\varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s) \right] e^{2A_4(s) - A_2(s)} ds \right\}. \tag{42}
\]

In what follows, we adopt the variable transformation technique to derive the solutions of Eqs. (24), (25) and (26). We first solve Eq. (27). Plugging the expression for \( f(t, r) \) into (26), we have

\[
\begin{align*}
\frac{\partial q(t, r)}{\partial t} + [\kappa_q(t) - ar] \frac{\partial q(t, r)}{\partial r} + \frac{\sigma_q(t) \sigma'_q(t)}{2} \frac{\partial^2 q(t, r)}{\partial r^2} \\
+ [r + \phi_q(t)] q(t, r) + 2 [r + \psi_q(t)] e^{A_1(t)r + A_2(t)} = 0,
\end{align*}
\]

(43)

where

\[
\begin{align*}
\kappa_q(t) &= a\theta(t) + \varphi_7(t) - \varphi_6(t) - \varphi_{13}(t) - \varphi_{15}(t)A_1(t), \\
\phi_q(t) &= \delta(t) + \delta(t) + \varphi_3(t) + \varphi_1(t) - \varphi_2(t) - \varphi_4(t) - \varphi_3(t) - \varphi_5(t), \\
\psi_q(t) &= \delta(t) - \varphi_2(t) + \varphi_5(t) - \varphi_9(t)A_1(t).
\end{align*}
\]

Note that Eq. (43) is no longer a homogeneous PDE due to the presence of the term \( 2 [r + \psi_q(t)] e^{A_1(t)r + A_2(t)} \), hence the analytical solution cannot be derived by using the same method as in solving (34). However, we will show in the next proposition that the solution of (43) can be derived by solving the following associated homogeneous PDE with a parameter \( \tau (\tau \leq T) \):

\[
\begin{align*}
\frac{\partial v(t, r; \tau)}{\partial t} + [\kappa_{q}(t) - ar] \frac{\partial v(t, r; \tau)}{\partial r} + \frac{\sigma_{q}(t) \sigma'_{q}(t)}{2} \frac{\partial^2 v(t, r; \tau)}{\partial r^2} \\
+ [r + \phi_{q}(t)] v(t, r; \tau) + 2 [r + \psi_{q}(t)] e^{A_1(\tau)r + A_2(\tau)} = 0,
\end{align*}
\]

(44)

By using the homogenization technique in PDE theory, we have the following result.

**Proposition 2.** Let \( v(t, r; \tau) \) be the solution of PDE (44). Then the solution of (43) can be expressed as

\[
q(t, r) = \int_t^T v(t, r; \tau) d\tau.
\]

(45)

**Proof.** The proof is similar to Proposition 2 of Yao et al.[27], we omit it here. \( \square \)
Proposition 2 shows that the solution of PDE (43) can be expressed in terms of the solution for (44). Now we start solving PDE (44).

By the terminal condition \( v(\tau; r; \tau) \), we can attempt to find a solution of the form

\[
v(t; r; \tau) = 2 [A_5(t; \tau) r + A_6(t; \tau)] e^{A_7(t; \tau) r},
\]

where \( A_5(t; \tau) \), \( A_6(t; \tau) \) and \( A_7(t; \tau) \) are three undetermined functions with the boundary conditions \( A_5(\tau; \tau) = e^{A_2(\tau)} \), \( A_6(\tau; \tau) = \psi_0(\tau) e^{A_2(\tau)} \) and \( A_7(\tau; \tau) = A_1(\tau) \). Substituting (46) into (44) and after some calculations, we obtain

\[
\begin{aligned}
&\dot{A}_5(t; \tau) + \left[ \frac{\sigma_s(t) \sigma_r(t)}{2} A_5^2(t; \tau) + \phi_q(t) - a + \kappa_q(t) A_7(t; \tau) \right] A_5(t; \tau) = 0, \\
&\dot{A}_6(t; \tau) + \left[ \kappa_q(t) + \sigma_s(t) \sigma_r(t) A_7(t; \tau) \right] A_5(t; \tau) \\
&\quad + \left[ \phi_q(t) + \kappa_q(t) A_7(t; \tau) + \frac{\sigma_s(t) \sigma_r(t)}{2} A_5^2(t; \tau) \right] A_6(t; \tau) = 0,
\end{aligned}
\]

Solving ODE (47), we have

\[
A_7(t; \tau) = \frac{1 - 2 e^{a(t-T)} + e^{a(t-\tau)}}{a}.
\]

Then, by ODEs (48) and (49), \( A_5(t; \tau) \) and \( A_6(t; \tau) \) can be expressed in terms of \( A_7(t; \tau) \):

\[
\begin{aligned}
A_5(t; \tau) &= e^{A_2(\tau)} + \int_t^\tau \left[ \frac{\sigma_s(s) \sigma_r(s)}{2} A_5^2(s; \tau) + \phi_q(s) - a + \kappa_q(s) A_7(s; \tau) \right] ds, \\
A_6(t; \tau) &= \left\{ e^{-a \tau} \int_t^\tau \left[ \kappa_q(s) + \sigma_s(s) \sigma_r(s) A_7(s; \tau) \right] e^{as} ds + \psi_0(t) \right\} \\
&\quad \times e^{A_2(\tau)} + \int_t^\tau \left[ \phi_q(s) + \kappa_q(s) A_7(s; \tau) + \frac{\sigma_s(s) \sigma_r(s)}{2} A_5^2(s; \tau) \right] ds.
\end{aligned}
\]

By Proposition 2, the solution of PDE (43) or PDE (26) is

\[
q(t, r) = \int_t^\tau v(t, r; \tau) d\tau = 2 \int_t^\tau [A_5(t; \tau) r + A_6(t; \tau)] e^{A_7(t; \tau) r} d\tau,
\]

where \( A_5(t; \tau) \), \( A_6(t; \tau) \) and \( A_7(t; \tau) \) are defined in (51), (52) and (50). Now we have two PDEs left: (24) and (25). Let

\[
\begin{aligned}
F_1(t, r) &= \varphi_0(f(t, r)) [r + \delta(t) - \varphi_2(t) + \varphi_{11}(t)] q(t, r) - \varphi_{15}(t) 4f(t, r) \left( \frac{\partial q(t, r)}{\partial r} \right)^2 \\
&\quad - \left[ \varphi_9(t) + (\varphi_{13}(t) + \varphi_{16}(t)) \right] \frac{q(t, r)}{2f(t, r)} \frac{\partial q(t, r)}{\partial r} \\
&\quad - \frac{\varphi_{12}(t) + 2\varphi_{14}(t) + \varphi_{17}(t)}{4f(t, r)} q^2(t, r), \\
F_2(t, r) &= [r + \delta(t) - \varphi_{12}(t)] g(t, r) - \left[ \frac{\varphi_{13}(t) + \varphi_{16}(t)}{2f(t, r)} A_3(t) + \varphi_{12}(t) + \varphi_{14}(t) \right] \\
&\quad \times q(t, r) g(t, r) - \frac{\varphi_{13}(t) + \varphi_{15}(t) A_5(t) \partial q(t, r)}{2f(t, r)} g(t, r).
\end{aligned}
\]
Then we rewrite PDEs (24) and (25) as
\[
\begin{align*}
\begin{cases}
\frac{\partial h(t,r)}{\partial t} + [a\theta(t) - ar + 2\varphi_7(t)] \frac{\partial h(t,r)}{\partial r} + \frac{\sigma_r(t)\sigma_r(t)}{2} \frac{\partial^2 h(t,r)}{\partial r^2} \\
+ [2\alpha(t) + \varphi_1(t)] h(t,r) + F_1(t,r) = 0,
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
\frac{\partial p(t,r)}{\partial t} + [a\theta(t) - ar + \varphi_7(t)] \frac{\partial p(t,r)}{\partial r} + \frac{\sigma_r(t)\sigma_r(t)}{2} \frac{\partial^2 p(t,r)}{\partial r^2} \\
+ \alpha(t)p(t,r) + F_2(t,r) = 0,
\end{cases}
\end{align*}
\]
respectively. With the same structure, it is obvious that PDEs (54) and (55) can be solved in similar ways. We only show the steps to solve (54) in the following.

Proposition 3. The solution of PDE (54) can be expressed as
\[
h(t,r) = \int_t^T \omega(t,r;\tau)d\tau, \quad (56)
\]
where \(\omega(t,r;\tau)\) is the solution of the following homogeneous parabolic equation with a parameter \(\tau (\tau \leq T)\):
\[
\begin{align*}
\begin{cases}
\frac{\partial \omega(t,r;\tau)}{\partial t} + [a\theta(t) - ar + 2\varphi_7(t)] \frac{\partial \omega(t,r;\tau)}{\partial r} + \frac{\sigma_r(t)\sigma_r(t)}{2} \frac{\partial^2 \omega(t,r;\tau)}{\partial r^2} \\
+ [2\alpha(t) + \varphi_1(t)] \omega(t,r;\tau) = 0, \\
\omega(t,r;\tau) = F_1(\tau,r),
\end{cases}
\end{align*}
\]
To derive the closed-form solution of PDE (57), we let
\[
\omega(t,r;\tau) = \Upsilon(\varepsilon,\varrho;\tau)e^{\int_t^\tau [2\alpha(s) + \varphi_1(s)]ds}, \quad (58)
\]
where
\[
\varepsilon = \frac{1}{2} \int_t^\tau \sigma_r(s)\sigma_r(s)e^{2\alpha(s-r)}ds, \quad \varrho = re^{\alpha(t-r)} + \int_t^\tau [a\theta(s) + 2\varphi_7(s)]e^{\alpha(s-r)}ds.
\]
Plugging (58) into (57) yields
\[
\begin{align*}
\begin{cases}
\frac{\partial \Upsilon(\varepsilon,\varrho;\tau)}{\partial \varepsilon} = \frac{\partial^2 \omega(t,r;\tau)}{\partial \varrho^2}, \\
\Upsilon(0,0;\tau) = F_1(\tau,0),
\end{cases}
\end{align*}
\]
which is a heat equation that has been studied extensively in the literature. By classical PDE theory, the solution of PDE (59) is
\[
\Upsilon(\varepsilon,\varrho;\tau) = \int_{-\infty}^{+\infty} F_1(\tau,\zeta) e^{-\frac{(\varrho-\zeta)^2}{4\varrho^2}}d\zeta. \quad (60)
\]
By Eq. (56), we have
\[
\omega(t,r;\tau) = \int_{-\infty}^{+\infty} F_1(\tau,\zeta)e^{-\frac{(\varrho-\zeta)^2}{4\varrho^2}}d\zeta \\
\times e^{\int_t^\tau [2\alpha(s) + \varphi_1(s)]ds} \quad (61)
\]
Then it follows from Proposition 3 that the solution of PDE (54) or (24) can be expressed as
\[ h(t, r) = \int_t^T \omega(t, r; \tau) d\tau \]
\[ = \int_t^T e^{\int_t^\tau [2\alpha(\zeta) + \varphi_1(\zeta)]d\zeta} \int_{-\infty}^{+\infty} F_1(\tau, \zeta)e^{-\frac{\left\{r e^{A(t-\tau)} + \left[\int_t^\tau [\alpha(s) + 2\varphi_1(s)]e^{A(s-\tau)}ds - \zeta\right] \right\}^2}{2 \int_t^\tau \sigma_r(s)\sigma_r(s)e^{2\alpha(s-\tau)}ds}} \frac{d\zeta}{d\tau}. \] (62)

Similarly, we can derive the solution for PDE (55) or (25) as follows:
\[ p(t, r) = \int_t^T e^{\int_t^\tau \bar{a}(\zeta)d\zeta} \int_{-\infty}^{+\infty} F_2(\tau, \zeta)e^{-\frac{\left\{r e^{A(t-\tau)} + \left[\int_t^\tau \alpha(s) + \varphi_1(s)\right]e^{A(s-\tau)}ds - \zeta\right] \right\}^2}{2 \int_t^\tau \sigma_r(s)\sigma_r(s)e^{2\alpha(s-\tau)}ds}} \frac{d\zeta}{d\tau}. \] (63)

So far, we have the solutions for PDEs (22)–(27). Based on the expression for \( V(t, r, y, l) \) in (20), we have
\[ V(t, r, y, l) = f(t, r)y^2 + g(t, r)y + h(t, r)l^2 + p(t, r)l + q(t, r)yl + z(t), \] (64)
where \( f(t, r), g(t, r), h(t, r), p(t, r), q(t, r) \) and \( z(t, r) \) are given by (28), (35), (62), (63), (53) and (42). It is obvious to verify that \( V_{yy} = 2f(t, r) = 2e^{-A_1(t)r + A_2(t)} > 0 \), which means that the optimization problem (16) does have the optimal solution. Substituting (21) into (18) and noticing that \( \frac{\partial f(t, r)}{\partial r} = f(t, r)A_1(t), \frac{\partial g(t, r)}{\partial r} = g(t, r)A_3(t) \), the optimal strategy for problem (15) can be derived as
\[ \pi^*(t) = -\frac{(\sigma(t)\sigma'(t))^{-1}}{2(y + l)} \left\{ \bar{h}(t) \left[ 2y + e^{-A_1(t)r - A_2(t)}q(t, r)l - 2ue^{-A_3(t)r + A_4(t)} - A_2(t) \right] \\
+ \beta(t)\sigma(t)\sigma'(t)l \left[ e^{-A_1(t)r - A_2(t)}q(t, r) - 2 \right] + \sigma(t)\sigma'(t) \right\} \\
\times \left\{ 2A_1(t)y - 2uA_3(t)e^{-A_3(t)r - A_2(t)}r - 2A_2(t) + e^{-A_1(t)r - A_2(t)} \right\} \\
\times \left. \frac{\partial q(t, r)}{\partial r} \right) - \sigma(t)\eta'(t) \left[ 2y + e^{-A_1(t)r - A_2(t)}q(t, r)l \right]. \] (65)

To sum up, we have the following theorem.

**Theorem 4.1.** For any \( t \in [0, T] \), the optimal value function \( V(t, r, y, l) \) of the optimization problem (15) is given by (64), and the corresponding optimal solution is given by (65).

The following theorem verifies that the investment strategy and value function given by Theorem 4.1 are indeed optimal.

**Theorem 4.2.** (Verification theorem) If \( J(t, r, y, l) \in C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}) \) is a candidate solution of the HJB equation (17) and satisfies the boundary condition \( (y - u)^2 \), then for all admissible strategies \( \pi(t) \in \Pi \), one has \( V(t, r, y, l) \geq J(t, r, y, l) \). Furthermore, if \( \pi^*(t) \) satisfies
\[ \pi^*(t) \in \arg\inf_{\pi \in \Pi} E \left[ (\bar{Y}(T) - u)^2 | r(t) = r, \bar{Y}(t) = y, \bar{L}(t) = l \right], \] (66)
then one has \( V(t, r, y, l) = J(t, r, y, l) \), and \( \pi^*(t) \) is the optimal investment strategy of problem (15).

Proof. The proof is similar to Theorem 4 of Chang [4] and Theorem 2 of Pan and Xiao [21], we omit it here. \(\square\)

5. Efficient portfolio and efficient frontier. In this section, we proceed to derive the solution for the original mean-variance problem (13). As in Section 3, when \( t = 0 \), \( V(0, r_0, y_0, l_0) \) is the optimal value for problem (15), and \( G(r_0, y_0, l_0, \lambda) \) defined by \( V(0, r_0, y_0, l_0) - \lambda^2 \) is the optimal value for problem (14). According to Theorem 4.1, we have

\[
G(r_0, y_0, l_0, \lambda) = V(r_0, y_0, l_0, \lambda) - \lambda^2 = f(0, r_0)y_0^2 + g(0, r_0)y_0 + h(0, r_0)l_0^2 + p(0, r_0)l_0 + z(0) - \lambda^2. \tag{67}
\]

Noticing the expressions of \( f(t, r), g(t, r), h(t, r), p(t, r), q(t, r) \) and \( z(t) \) described by (28), (35), (62), (63), (53) and (42), we find that only \( g(t, r), p(t, r) \) and \( z(t) \) have the parameter \( \lambda = d - u \). So, substituting \( g(t, r), p(t, r) \) and \( z(t) \) into (67), we obtain

\[
G(r_0, y_0, l_0, \lambda) = \int_0^T \left[ \varphi_{12}(s) + 2 \varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s) \right] e^{2A_4(s) - A_2(s)}ds 
\]

\[
+ 2A_4(s) - A_2(s)ds \right] - 2d \left[ e^{A_3(0)r_0 + A_4(0)y_0} + \bar{p}(0, r_0)l_0 \right],
\]

\[
-2A_4(s) - A_2(s)ds \right] + \left\{ f(0, r_0)y_0^2 + g(0, r_0)y_0 + h(0, r_0)l_0^2 + p(0, r_0)l_0 + z(0) - \lambda^2. \tag{68}
\]

where

\[
\bar{p}(0, r_0) = \int_0^T e^{\bar{F}_2(T, r)} e^{\int_0^T \sigma_s(s) \sigma_r(s) e^{2a(s - \tau) - \lambda^2}ds} d\zeta,
\]

\[
\bar{F}_2(t, r) = \left[ r + \delta(t) - \varphi_{12}(t) \right] e^{A_3(t)r + A_4(t)} - \left\{ \left[ \varphi_{13}(t) + \varphi_{16}(t) \right] A_3(t) + \left[ \varphi_{12}(t) + \varphi_{14}(t) \right] \right\} q(t, r) + \varphi_{13}(t) + \varphi_{15}(t)A_3(t) \frac{\partial q(t, r)}{\partial r}
\]

\[
\times q(t, r) + \frac{\varphi_{13}(t) + \varphi_{15}(t)A_3(t)}{2} \frac{\partial q(t, r)}{\partial r} \right\} e^{-A_3(t)r + A_4(t) - A_2(t)}.
\]

Note that (68) is a quadratic function with respect to \( \lambda \), which implies that (68) may exist a finite maximum value while the existence of this finite maximum value depends on the coefficient of \( \lambda^2 \). For this purpose, we give the following proposition.

Proposition 4. If \( \sigma(t)\sigma_r(t)A_3(t) + \bar{b}'(t) \neq 0 \), then we have

\[
\int_0^T \left[ \varphi_{12}(s) + 2 \varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s) \right] e^{2A_4(s) - A_2(s)}ds > 0. \tag{69}
\]
Proof. From the property of exponential function, we know that \( e^{2A_3(s) - A_2(s)} > 0 \), so we only have to prove that \( \varphi_{12}(s) + 2\varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s) > 0 \), by the expression of \( \varphi_{12}(t), \varphi_{13}(t) \) and \( \varphi_{15}(t) \), we have
\[
\varphi_{12}(t) + 2\varphi_{13}(t)A_3(t) + \varphi_{15}(t)A_3^2(t) = \left[ \sigma(t)\sigma'(t)A_3(t) + \bar{b}'(t) \right] (\sigma(t)\sigma'(t))^{-1} \\
\times \left[ \sigma(t)\sigma'(t)A_3(t) + \bar{b}'(t) \right]' > 0.
\]
Evidently, it follows the conclusion and this completes the proof.

By Proposition 4, the optimal value of \( G(r_0, y_0, l_0, \lambda) \) can be achieved when \( \lambda \) is given by
\[
\lambda^* = d + \frac{e^{A_3(0)r_0 + A_4(0)y_0} + \bar{p}(0, r_0)l_0 - d}{\int_0^T [\varphi_{12}(s) + 2\varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s)] e^{2A_3(s) - A_2(s)}ds}.
\]
(70)
Substituting (70) into (65), the optimal strategy of problem (14) is given by
\[
\pi^*(t) \\
= \frac{\left( \sigma(t)\sigma'(t) \right)^{-1}}{2(y + l)} \left\{ \bar{b}(t) \left[ 2y + e^{-A_1(t)r - A_2(t)q(t,r)l} - 2u^* e^{-A_3(t)r + A_4(t)-A_1(t)} \right] \\
+ \beta(t)\sigma(t)\sigma'(t)l \left[ e^{-A_1(t)r - A_2(t)q(t,r)l} - 2 \right] + \sigma(t)\sigma'(t) \\
\times \left[ 2A_1(t)y - 2u^* A_3(t)e^{-A_1(t)r + A_4(t)-A_2(t)} + e^{-A_1(t)r} - A_2(t) \right] \\
\times \frac{\partial q(t,r)}{\partial r} \right\} - \sigma(t)\eta'(t) \left[ 2y + e^{-A_1(t)r - A_2(t)q(t,r)l} \right].
\]
(71)
where \( u^* = d - \lambda^* \). Moreover, we can obtain the optimal value of problem (14), namely,
\[
\text{Var} \left[ Y^*(T) \right] \\
= f(0, r_0) y_0^2 + b(0, r_0) l_0^2 + q(0, r_0) y_0 l_0 \\
- \left[ e^{A_3(0)r_0 + A_4(0)y_0} + \bar{p}(0, r_0)l_0 \right]^2 \\
- \left[ 1 - \int_0^T [\varphi_{12}(s) + 2\varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s)] e^{2A_3(s) - A_2(s)}ds \right] \\
\times \left[ 1 - \int_0^T \left[ \varphi_{12}(s) + 2\varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s) \right] e^{2A_3(s) - A_2(s)}ds \right] \\
\times \left[ d - \frac{e^{A_3(0)r_0 + A_4(0)y_0} + \bar{p}(0, r_0)l_0}{1 - \int_0^T [\varphi_{12}(s) + 2\varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s)] e^{2A_3(s) - A_2(s)}ds} \right]^2.
\]
(72)
Set
\[
d^* = \frac{e^{A_3(0)r_0 + A_4(0)y_0} + \bar{p}(0, r_0)l_0}{1 - \int_0^T [\varphi_{12}(s) + 2\varphi_{13}(s)A_3(s) + \varphi_{15}(s)A_3^2(s)] e^{2A_3(s) - A_2(s)}ds}.
\]
Here we need to emphasize that rational investors should not select the expected terminal wealth less than \( d^* \), namely, \( d \geq d^* \).

In conclusion, we have the following theorem about the efficient strategy and efficient frontier of problem (13).
where the liability is ignored. It turns out to be a pure asset allocation problem respectively given by Eqs. (71) and (72), where \( d \geq d^* \).

6. Special case without liability. In this section, we consider a special case where the liability is ignored. It turns out to be a pure asset allocation problem as considered in Yao et al. [28]. In this case, all the results obtained in previous sections can be substantially simplified.

According to the liability process \( L(t) \) in (5), we set
\[
\alpha(t) = 0, \quad \beta(t) = 0, \quad \rho(t) = \begin{pmatrix} 0_n \end{pmatrix},
\]
where \( \begin{pmatrix} 0_n \end{pmatrix} \) is a zero column vector of order \( n \). Then, the Eq. (19) becomes
\[
\begin{align*}
V_t + [a \delta(t) - ar] V_r + \frac{1}{2} \sigma_r(t) \sigma_r(t)V_{rr} + [r + \delta(t) + \varphi_3(t)] xV_x + \frac{1}{2} \varphi_4(t) x^2 V_{xx} - \varphi_8(t) xV_{xx} - \frac{1}{2} \varphi_12(t) V^2_{xx} - \varphi_13(t) \frac{V^2_x}{V_{xx}} = 0,
\end{align*}
\]
where
\[
\begin{align*}
\varphi_3(t) &= \ddot{b}(t) \left( \sigma(t) \sigma(t) \right)^{-1} \sigma(t) \eta(t), \\
\varphi_4(t) &= \eta(t) \eta(t) - \eta(t) \sigma(t) \sigma(t) \left( \sigma(t) \sigma(t) \right)^{-1} \sigma(t) \eta(t), \\
\varphi_8(t) &= \sigma_r(t) \eta(t) - \sigma_r(t) \sigma(t) \sigma(t) \left( \sigma(t) \sigma(t) \right)^{-1} \sigma(t) \eta(t), \\
\varphi_{12}(t) &= \ddot{b}(t) \left( \sigma(t) \sigma(t) \right)^{-1} \dot{b}(t), \\
\varphi_{13}(t) &= \ddot{b}(t) \left( \sigma(t) \sigma(t) \right)^{-1} \left( \sigma(t) \sigma(t) \right)^{-1} \sigma(t) \sigma(t) \\
\varphi_{15}(t) &= \sigma_r(t) \sigma(t) \left( \sigma(t) \sigma(t) \right)^{-1} \sigma(t) \sigma(t).
\end{align*}
\]
In addition, the value function \( V(t, x, r) = f(t, r)x^2 + g(t, r)x + z(t, r) \) with \( f(t, r), g(t, r) \) and \( z(t, r) \) are given by (28), (35) and (42).

As a result, the optimal portfolio and the mean-variance efficient frontier of problem (13) is reduced to the following result.

**Theorem 6.1.** For a given expected terminal wealth \( d \), the efficient portfolio of problem (13) is
\[
\pi^*(t) = - \frac{\left( \sigma(t) \sigma(t) \right)^{-1} \left( \ddot{b}(t) \left( x - u^* e^{-A_3(t)^x + A_4(t)^x} - A_2(t) \right) + \sigma(t) \sigma(t) \right)}{x} \times \left[ A_1(t) x - u^* A_3(t) e^{-A_3(t)^x + A_4(t)^x - A_2(t)} - \sigma(t) \eta(t) x \right],
\]
where
\[
\begin{align*}
u^* &= \frac{d - e^{A_3(0)^x + A_4(0)^x}}{\int_0^T \left( \varphi_{12}(s) + 2 \varphi_{13}(s) A_3(s) + \varphi_{15}(s) A_3^2(s) \right) e^{2 A_4(s) - A_2(s)} ds},
\end{align*}
\]
Furthermore, the efficient frontier is
\[
\begin{align*}
Var[X^*(T)] &= f(0, r_0)x_0^2 \left( \int_0^T \left( \varphi_{12}(s) + 2 \varphi_{13}(s) A_3(s) + \varphi_{15}(s) A_3^2(s) \right) e^{2 A_4(s) - A_2(s)} ds \right)^2 \\
&\quad + \frac{1}{T} \int_0^T \left( \varphi_{12}(s) + 2 \varphi_{13}(s) A_3(s) + \varphi_{15}(s) A_3^2(s) \right) e^{2 A_4(s) - A_2(s)} ds \\
&\times \left[ d - \frac{e^{A_3(0)^x + A_4(0)^x}}{\int_0^T \left( \varphi_{12}(s) + 2 \varphi_{13}(s) A_3(s) + \varphi_{15}(s) A_3^2(s) \right) e^{2 A_4(s) - A_2(s)} ds} \right]^2.
\end{align*}
\]
7. Numerical examples. In this section, we give a numerical example to illustrate the effects of some main parameters on the efficient frontier for our mean-variance ALM problem.

Consider a company whose initial asset and liability are \( x_0 = 1.2, l_0 = 1 \). The company plans to invest in one bond and two stocks during the time horizon \( T = 1 \) (year). For convenience but without loss of generality, we suppose that the involved Brownian motions are \( W_0(t), W_1(t), W_2(t) \), and all the coefficients \( \theta(t), \sigma_r(t), \mu_1(t), \sigma_1(t), \alpha(t), \beta(t), \rho(t), v(t), \eta(t) \) are constant over time \( t \in [0, T] \). Throughout this section, unless otherwise stated, all the related parameters for our model are chosen similarly as in Chang [4], Chiu and Li [6], Pan and Xiao [21], which are specified as follows:

\[
\begin{align*}
    a &= 0.85, \quad \theta(t) = 0.015, \quad \sigma_r(t) = (0.18, 0), \quad r_0 = 0.01, \\
    \mu_1(t) &= 0.1, \quad \mu_2(t) = 0.2, \quad \sigma_1(t) = (0.1, 0.08), \quad \sigma_2(t) = (0.3, 0.2), \\
    \alpha(t) &= 0.08, \quad \beta(t) = 0.3, \quad \rho_1(t) = 0.65, \quad \rho_2(t) = 0.10, \\
    v(t) &= 0.04, \quad \eta(t) = (0.4, 0), \quad I_0 = 0.03.
\end{align*}
\]

Plugging the above parameter values into the related formulas in previous sections, we obtain the impacts of initial value of interest rate \( r_0 \), the volatility of interest rate \( \sigma_r(t) \), the correlation coefficient \( \rho(t) \), the initial inflation level \( I_0 \), the expected rate of inflation index \( v(t) \), the volatility of inflation index \( \eta(t) \) on the efficient frontier, which are plotted in FIGUREs. 1–6.

![Figure 1. Impact of \( r_0 \) on the efficient frontier](image)

FIGURE 1 contributes to the evolution of efficient frontier with respect to the initial value of interest rate \( r_0 \). We find that the efficient frontier moves upwards as \( r_0 \) increases. This means that the expected terminal wealth has a positive relationship with the initial value of interest rate if we acquire the same variance of terminal value. From FIGURE 2, we can see that the volatility of interest rate affects the efficient frontier. It is well known when the value of \( \sigma_r(t) \) increases while keeping
everything else unchanged, the risk of interest rate increases. To hedge the risk of interest rate, the company needs to invest more money in the stock, which makes the efficient frontier moves downwards.

In FIGURE 3, it can be seen that whenever $\rho(t)$ takes negative or positive values, the efficient frontier is sensitive to the change of $\rho$’s values. Since $Y(t)$ is the
difference between asset and liability, which share the same set of Brownian motions except $W_0$, the variance of each stock varies depending on the long or short position in the optimal strategy. When $\pi_i$ and $\rho_i$ have different signs, the variance is amplified and the efficient frontier moves to the right comparing with the case of $\rho = 0$. Otherwise, when $\pi_i$ and $\rho_i$ have the same sign, the variance becomes smaller and the efficient frontier moves to the left.
From FIGURE 4, it can be seen that the efficient frontier moves downwards as the initial value of inflation level $I_0$. The reason is that in the case of other parameters unchanged, the higher $I_0$ means the higher inflation risk, as a result, more money is invested in the risky assets that leads to take more risk for a given expected terminal wealth. This further shows that under the influence of inflation, the expected terminal wealth is reduced.

In FIGURE 5, we can see that when the value of inflation rate $v(t)$ increases, the corresponding variance with a fixed level of expected terminal wealth becomes higher. The reason is that, since $v(t)$ becomes larger, the inflation moves quickly, which can add to the randomness brought by the Brownian motion $W(t)$, and the accumulated risk increases with it. Similar to FIGURE 2, the volatility of inflation index greatly affects the effective frontier in FIGURE 6, and the company will have more risks for the same level of the expected terminal wealth.

8. **Conclusion.** In this paper, we investigate a continuous-time ALM problem with stochastic interest rate and inflation risks under mean-variance criterion. The financial market consists of one bond and $n$ stocks whose prices are modeled by GBMs. The inflation index which represents the consumer price index depends on the same Brownian motions as the stock prices. The uncontrollable liability of the company are modeled by another Brownian motion with drift which is correlated with the stock prices. By applying the stochastic control approach and the variable transformation technique, the explicit expressions for efficient strategies and efficient frontier are obtained. Finally, we provide a numerical example to illustrate how the key parameters of the model affect the efficient frontier. The results show that these parameters greatly affect the investment decisions, which in turn illustrate the effectiveness of our model.

There are some possible extensions that deserve to be investigated further in future study. The first one is the problem with the regime-switching jump-diffusion
model, which may be more realistic and interesting. The second one is the problem
with optimal asset-liability ratio management, which may appeal more focus by
investment institutions or individual investors. Third, market frictions such as a
spread between borrowing and lending rate, fixed or proportional transaction cost
are important issues worth investigating. ALM is an important risk management
tool; it is also interesting to see practical applications of the model and methodology.
Using more versatile stochastic models such as regime-switching jump-diffusion
to study ALM, the optimal control problems become more complicated and closed-
form solutions are virtually impossible. Numerical methods will provide an alter-
native.

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