Background Field Method in Thermo Field Dynamics Theory for Wave Propagation in Unmagnetized Spinor QED Plasmas

Shan Wu and Ji-ying Zhang
Institute for Interdisciplinary Research, Jianghan University, Wuhan 430056, People’s Republic of China

In this work, a many body relativistic quantum field theory for the collective modes of spinor quantum electrodynamics (QED) plasmas is developed. We introduce the thermo field dynamics into the QED plasma study. The nontrivial background field method is used to take account of the non-perturbativity of background charged plasma particles and radiation field. It is an extension of “Furry picture” which is first introduced by Yuan Shi, et al. [Phys. Rev. A 94, 012124 (2016)] in their scalar QED plasma study. However, their wave function method in evaluating the background field of ideal system is hard to extend to the general many body cases. We propose a classical limit method that most perturbative high energy and quantum many body aspects can be included in a practical way. As an example, the wave propagation in unmagnetized electron-positron plasma is discussed. In the low energy limit case, the well known wave dispersion relations for non-relativistic degenerate plasma are recovered. In addition, mass increase of plasma particles due to the relativity, effective charge decrease due to the vacuum polarization, finite light velocity influence on the dispersion relation, and temperature influence on plasma system are discussed. Besides, new phenomenons including the zero sound of the electron-positron pair plasma and the particle production induced by the plasma oscillation are first reported. At last, the high energy limit case is studied.

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I. INTRODUCTION

Classical theory is not suitable for plasma system in many cases. For the typical particle number density \( n \sim 10^{22} \text{ cm}^{-3} \) in condensed matter physics, the corresponding Fermi energy \( \varepsilon_F = \hbar^2 (3\pi^2 n)^{2/3}/2m \gtrsim k_B T \), where \( m \) is the electron mass, \( k_B \) is the Boltzmann constant, \( \hbar \) is the Plank constant \( h \) divided by \( 2\pi \), and \( T \sim 0 - 10^2 K \) is the usual temperature in experiments. The degenerate effect of electron plasma, such as metal surface plasma, in solid state material must be taken into account. In inertial confinement fusion, the deuterium and tritium reaction target with the density \( 10^3 \text{ g cm}^{-3} \) and the electron temperature \( 10 \text{ keV} \) can be considered as in partial relativistic degenerate state [1].

For the intense laser-plasma experiments, electron-positron production is important [2]. The quantum electrodynamics (QED) effects cannot be ignored. Other examples come from astrophysical environments [3]. In the process of gravitational collapse to a black hole, both Dirac process [4] and Breit-Wheeler process [5] are at work. The electron-positron plasma is created from the “blackholic energy” [6]. In addition, the photon energy is of order of and even much more than the rest electron mass in the cases of X-ray pulsar and gamma-ray bursts. What’s more, relativistic degenerate plasma is believed existing in the massive white dwarf [7]. Typical high energy and quantum many body effects appear in these extreme environments.

From the theoretical point of view, a many-body relativistic quantum theory is needed. In this paper, we focus on the quantum field theory (QFT) description of the collective motion of plasmas. There are already many theoretical efforts on this subject. It starts with the pioneering works of Klimontovich and Silin [8] and Bohm and Pines [9] who studied the dispersion relation of non-relativistic degenerate electron plasma oscillations by using Wigner distribution and density matrix method, respectively. Recently, one could observe a lot of activity in the literature regarding the non-relativistic quantum plasma by using so called “quantum hydrodynamics” theory [10] which is, however, still confusion [11,12]. Those non-relativistic theories are not satisfied for the present purpose because they can not deal with the relativistic and high energy problems such as high velocity moving particles with increased mass, pair production, etc. Relativistic quantum hydrodynamic theories for Bose plasma [13] and Fermi plasma [14] have also been developed. However these theories are constructed based on one particle Klein-Gordon equation and Dirac equation that they can not be considered as the fully quantized theories. In addition, the many body effect is not taken into account. Other schemes such as one particle relativistic wave function method has the same defects [15]. A partial solution is to absorb the nonlinear vacuum phenomena, such as vacuum polarization and photon-photon scattering, into the effective Lagrangian. It modifies Maxwell equations with additional nonlinear terms. The study of quantum theory is transformed to solving these modified classical equations. It is started with the famous work by Heisenberg and Euler [16]. Many efforts are devoted into this semi-classical method which are summarized in the review paper by Marklund and Shukla [17]. Besides, plasma response theories are developed to incorporate some QED effects [18,19]. However, a general theory on this subject must be a fully quantized
many body one which includes both the high energy and the many body effects. The most suitable scheme appeals to finite temperature relativistic quantum field theory. In the realm of dense nuclear matter, by using these theories, some works have paid attentions to several aspects of QED plasmas to some degree.

However, there is still lack of a systematic many body relativistic quantum field theory treatment for QED plasmas. Particularly, none of the following points is considered in the above theories. One is the presence of strong field which makes the problem much more complexity. It stimulates the mechanism of QED cascade which leads to enormous number of photons participating in the motion of process. The other is, as will been seen in the Sec. IV, that the plasma oscillation gives rise to the increase of particle number. The usual Green function analyse for plasma collective mode is inappropriate. Both of these points can be grouped into the nonperturbativity of backgrounds. The standard perturbative QED is of no use. An alternative nonperturbative formalism is known as “Furry picture” in which electromagnetic vector potential is decomposed into the classical background field and quantum fluctuation field. The effective quantum field model of fluctuation field is obtained. The multi-photon process is taken into account through the classical field. Based on this considering, the trivial electromagnetic background field case is studied by Raicher, et al. Yuan, et al. introduce the nontrivial background fields in the study of wave propagation in scalar QED plasmas. In their paper, both the electromagnetic field and the scalar field are decomposed into the background fields and the quantum fluctuation fields. It allows one to take account not only of the nonperturbativity of radiation field, but also of the nonperturbativity of the plasma particle background. Its validity is proved by the fact that this is the only theory capable of recovering all linear wave modes well known in classical plasma theories. The critical point of this theory is to evaluate the nontrivial background fields. They derive it by calculating the wave function of ideal system background. However, the method is hardly extended to the nonzero temperature or non-ideal system.

In the present paper, we extend the nontrivial Furry decomposition method to the spinor QED plasmas. It allows one to take account of the many body physical effects and of all the relativistic quantum corrections. We propose a classical limit method. It enable us to derive the nontrivial background fields including the thermal and non-ideal effects in a general and practical way. We introduce the thermo field dynamics (TFD) into the QED plasma study. It is quite a suitable temperature quantum field theory for developing our scheme. There is a high degree of parallelism between TFD and usual QFT. The Furry decomposition in this theoretic scheme will be convenient. More over, this method overcomes the difficult diagram analysis in the real or imaginary-time Green function method due to the change of particle number. It will be specified in detail in the Sec. IV. For readability, the formalism in this paper is constructed in the electron-positron pair plasma case which widely exists in the astronomical and laser-plasma environments. The wave dispersions of classical and non-relativistic degenerate quantum plasma are recovered from our relativistic spinor QED plasma theory. In addition to these well-known results, the relativistic correction to the mass of plasma particles, effective charge decrease due to the vacuum polarization and finite temperature influence on plasma system are discovered. Besides, finite light velocity influence on dispersion relation is also found which is similar to the case of the scalar QED plasma. We first report the zero sound of the electron-positron pair plasma and the particle production induced by the plasma oscillation. The ultra-relativistic case is studied on the other hand. Extension of the study to magnetized plasmas will be reported separately.

The rest of this paper is organized as follows. In Sec. II, we review some points of TFD needed in this study. The general relativistic quantum theory of wave propagation of the spinor QED plasma is developed based on TFD. In Sec. III, we calculate the polarization tensor by using the classical limit method for unmagnetized electron-positron pair plasma. In Sec. IV, we derive a series of collective modes for this plasma. The above mentioned relativistic quantum corrections are obtained. Summery and remarks are made in Sec. V.

We use the Heaviside-Lorentz units with the speed of light $c = 1$ and Plank constant $\hbar = 1$. In addition, we set the Boltzmann constant $k_B = 1$. The metric signature is $(+,−,−,−)$.

II. TFD WITH BACK GROUND FIELDS

A. Total Lagrangian density

TFD has an almost parallel theoretical structure to that of usual QFT. It is a very powerful many body quantum field theory for describing equilibrium and nonequilibrium statistical system. To clarify our key point clearly, we first very briefly review some ideas and results of TFD useful for the present paper. Considering a quantum system described by a Fock space $\mathcal{F}$, the key point of TFD is to introduce an ancillary Fock space $\tilde{\mathcal{F}}$ which has the same structure of $\mathcal{F}$. The total space for the system considered in TFD is $\mathcal{F} \otimes \tilde{\mathcal{F}}$. A fictitious operator set $\tilde{\mathcal{A}} = \{\tilde{A}\}$ corresponding to $\tilde{\mathcal{F}}$ is introduced in addition to the physical set $\mathcal{A} = \{A\}$ corresponding to $\mathcal{F}$. Conventionally, for the sake of notational convenience, we denote

\begin{align}
A \otimes 1 \Rightarrow A, \\
1 \otimes \tilde{A} \Rightarrow \tilde{A}
\end{align}

\begin{align}
|\Omega\rangle \otimes |\tilde{\Omega}\rangle \Rightarrow |\Omega\rangle.
\end{align}

Here, $|\Omega\rangle$ is the vacuum state vector. In TFD, the operators are governed by the following axioms:

\begin{align}
\tilde{A} = A,
\end{align}

where $\tilde{A}$ acts on the fictitious space, $A$ acts on the physical space, and $\otimes$ denotes the direct product of two spaces.
\[(AB)^{\sim} = A\hat{B}, \quad (4)\]
\[(c_1A + c_2B)^{\sim} = c_1^{\ast}\hat{A} + c_2^{\ast}\hat{B}, \quad (5)\]
for any \(c_1, c_2 \in \mathbb{C}\). The equal time commutation and anti-commutation relations for operators of boson and fermion are
\[\{A, \hat{B}\} = 0, \quad (6)\]
\[\{A, \hat{B}\} = 0, \quad (7)\]
respectively. In statistical equilibrium state case, the thermal average of an arbitrary operator \(A\) can be expressed as the expectation value with respect to the temperature-dependent vacuum \(|\Omega(\beta)\rangle\):
\[\langle \Omega(\beta)|A|\Omega(\beta)\rangle = \frac{tr(Ae^{-\beta(H-\mu N)})}{tr(e^{-\beta(H-\mu N)})}, \quad (8)\]
where \(H\) is the total Hamiltonian of the system, \(\mu\) is the chemical potential, \(N\) is the particle number operator, \(\beta = 1/T\) and \(T\) is the temperature. \(|\Omega(\beta)\rangle\) is determined by thermal state condition 32
\[A(x)|\Omega(\beta)\rangle = \sigma_A e^{-\frac{\beta}{2} \mu N_A} \hat{A}(t - \frac{i\beta}{2}, x)|\Omega(\beta)\rangle, \quad (9)\]
Here, \(N_A\) is the result of fermion particle number minus antiparticle number 34, and
\[\sigma_A = \begin{cases} 1 & \text{for bosonic } A, \\ i & \text{for fermionic } A. \end{cases} \quad (10)\]

The theory is restricted in the realm of QED. In addition, all the particles such as muon and tauon other than those of electron, positron and photon are excluded. In TFD theory, many body system is governed by total Lagrangian density defined by \(\hat{L} = L - L\). Further specification of \(L\) is included in the following discussion of our theory. We use the QED electron-positron plasma model to develop our scheme. It can be extended naturally to other spinor QED plasmas. Our start point is the standard Lagrangian density of spinor QED in usual quantum field theory,
\[\hat{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2, \quad (11)\]
where \(\psi\) is the four components Dirac spinor field, \(\bar{\psi} = \psi^\dagger\gamma^0\), \(\not{D} = \gamma^\mu D_\mu\) (\(D_\mu = \partial_\mu + ieA_\mu\) is the covariant derivative operator, \(\gamma^\mu\) are 4 x 4 the gamma matrices and \(A_\mu\) are the electromagnetic vector potential), \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic field strength tensor. Considering \(\int d^4x\partial_\mu\bar{\psi}^T\gamma^\mu\psi^* = -\int d^4x\bar{\psi}^T\gamma^\mu\partial_\mu\psi^*\), \((\gamma^\dagger)^\dagger = -\gamma^\dagger\) and \(\gamma^0\gamma^i = -\gamma^i\gamma^0\), the total Lagrangian density is derived as
\[\hat{\mathcal{L}} = \sum_{\alpha = 1, 2} \varepsilon_\alpha P_\alpha [\bar{\psi}^{(\alpha)}(i\not{D}^{(\alpha)} - m)\psi^{(\alpha)} - \frac{1}{4}(F_{\mu\nu}^{(\alpha)})^2], \quad (12)\]
where \(P_\alpha\) is the ordering operator defined as 35
\[P_\alpha (A^{(\alpha)}B^{(\alpha)} \ldots C^{(\alpha)}) = \begin{cases} A^{(\alpha)}B^{(\alpha)} \ldots C^{(\alpha)} & \alpha = 1, \\ C^{(\alpha)} \ldots B^{(\alpha)}A^{(\alpha)} & \alpha = 2 \end{cases}, \quad (13)\]
and
\[\varepsilon_\alpha = \begin{cases} 1 & \text{for } \alpha = 1, \\ -1 & \text{for } \alpha = 2. \end{cases} \quad (14)\]
Here, the field operator is also doubled to satisfy the condition Eq. 33 and Eq. 35, and is defined as
\[\psi^{(\alpha)}(x) = \begin{pmatrix} \tilde{\psi}(x) \\ i\bar{\psi}(x)^{\dagger} \end{pmatrix}, \quad (15)\]
\[A^{(\alpha)}_\mu(x) = \begin{pmatrix} A_\mu(x) \\ \bar{A}_\mu(x) \end{pmatrix}, \quad (16)\]
\[B^{(\alpha)}_{\mu\nu}(x) = \begin{pmatrix} B_{\mu\nu}(x) \\ \bar{B}_{\mu\nu}(x) \end{pmatrix}. \quad (17)\]
Here, we use \(\psi^{(\alpha)}\) and \(A^{(\alpha)}\) to denote the double field quantities that \(\phi^{(1)}(x) = \psi(x) + \not{1}_{4\times 1}, \psi^{(2)}(x) = \psi^{(1)}(x)^T + i\bar{\psi}(x)^{\dagger}T, A^{(1)}_\mu(x) = A_\mu(x) + \not{1}_{4\times 1}, A^{(2)}_\mu(x) = A^{(1)}_\mu(x) + \not{1}_{4\times 1}\). The notations \(\psi^{(1)}(x), \psi^{(2)}(x), A^{(1)}_\mu(x), A^{(2)}_\mu(x), B^{(1)}_{\mu\nu}, \text{and } B^{(2)}_{\mu\nu}\) can also be understood as the specific components of the double field quantities, i.e. \(\psi(x), i\bar{\psi}(x)^{\dagger}T, A_\mu(x), \bar{A}_\mu(x)\). The total Lagrangian Eq. 12 can also be written as
\[\hat{\mathcal{L}} = P_\alpha \{\bar{\psi}^{(\alpha)}(i\not{D} - m)\not{1}_{4\times 4} \otimes \sigma^3)|\psi^{(\alpha)}\rangle\}
- \frac{1}{4}(F_{\mu\nu}^{(\alpha)})^T \sigma^3 F_{\mu\nu}^{(\alpha)}\}, \quad (18)\]
where \(\sigma^3\) is the third component of Pauli matrices \(\sigma^i\), \(\psi^{(\alpha)}\) and \(F_{\mu\nu}^{(\alpha)}\) are understood as Eq. 15 and Eq. 17 in this case.

The next procedure of TFD is to quantize the total Lagrangian, i.e., Eq. 12 or Eq. 15. The path integral quantization is used in the following sections. It is convenient for the background field method.

**B. Background field**

Due to the possible enormous number of particles participating, the standard perturbative QED method is of no use. The alternative is a nonperturbative QED formalism known as "Furry picture" in whose attitude the
background field is considered as classical. From Yuan Shi et al.'s perspective [27], it is proper for decomposing both Dirac field and vector potential field into classical background and quantum fluctuations

$$\psi^{(a)} \rightarrow \tilde{\psi}^{(a)} + \psi^{(a)}, \quad A^\alpha_\mu \rightarrow A^\alpha_\mu + A^\alpha_\mu,$$  \hspace{1cm} (19)

where $\psi_0^{(a)}$ and $A_\mu^{(a)}$ are the right side of arrow symbols are classical Dirac background and vector potential background fields, respectively. Besides, $\psi^{(a)}$ and $A^\alpha_\mu$ denote the quantum fluctuation fields of Dirac particles and vector potential particles, respectively. We first derive the effective Lagrangian of the quantum fluctuation fields. Then we explain why the perturbation method is feasible in this nontrivial background field scheme even when the nonlinear QED effects such as QED cascade appears.

Considering Eq. (14) and that the physical and fictitious field sets having the same algebraic rules, by using variational method, the doubled background fields satisfy Maxwell-Dirac equations

$$iD^\alpha - m)\psi^{(a)} = 0,$$  \hspace{1cm} (20)

$$\partial_\mu F^{(a)\mu\nu} = e\tilde{\psi}^{(a)} \gamma_\nu \psi^{(a)}.$$  \hspace{1cm} (21)

Here, $F_\mu^\nu = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha$ are the background electromagnetic field strength tensor and the corresponding fictitious one, respectively. Insert Eq. (19)–Eq. (21) into Eq. (12), we get the total action of quantum fluctuation fields, up to an useless constant $\frac{1}{4}(F_\mu^\nu)^2$, as

$$\tilde{S} = \sum_{\alpha=1,2} \epsilon_\alpha P_\alpha \int d^4 x [\tilde{\psi}^{(a)}(iD^\alpha - m)\psi^{(a)} - \frac{1}{4}(F_\mu^\nu)^2 - e\tilde{\psi}^{(a)} A^{(a)} \psi^{(a)} - e\tilde{\psi}^{(a)} A^{(a)} \psi^{(a)} - e\tilde{\psi}^{(a)} A^{(a)} \psi^{(a)}],$$  \hspace{1cm} (22)

where $F^{(a)\mu\nu} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)}$ is the fluctuation field strength tensor. The first two terms in the integral constitute free Lagrangian density for fluctuation field and the last three terms constitute interaction Lagrangian density.

The traditional perturbative QED theory is of no use in the case where the phenomenons such as QED cascade appears. It is because the enormous states corresponding to a large number of particles participating in the initial and end states of S-matrix which involves a big number of Feynman diagrams. The contribution to the S-matrix of the $\epsilon^n$ order from a big number of Feynman diagrams may even exceed that of the $\epsilon^{n-1}$ order. It will be seen more clearly from the perspective of path integral. There, the evolution operator of QED can be written as

$$\langle \tilde{\psi}(\textbf{x}) | \psi(\textbf{x}) \rangle = e^{-iH_{\text{TT}} \langle \tilde{\psi}(\textbf{x}) | \psi(\textbf{x}) \rangle} A_\mu(x),$$

where the functions $\tilde{\psi}$, $\psi$, and $A$ over which we integrate are constrained to the specific configurations $\tilde{\psi}(\textbf{x})$, $\psi(\textbf{x})$, $A_\mu(x)$ in the state vector $\langle \tilde{\psi}(\textbf{x}) | \psi(\textbf{x}) \rangle A_\mu(x)$ at $x' = 0$ and $\langle \tilde{\psi}(\textbf{x}) | \psi(\textbf{x}) \rangle A_\mu(x)$ at $x' = T$. $\hat{L}[\psi, \psi, A]$ is denoted by Eq. (11). Only the fields in the region nearby the classical field $\psi_0$, $\psi_0$, and $A_\mu$ contribute to the integral. As will be seen in I. Sec. III, in the ideal many body case, $\psi_0(x) \sim \sqrt{n_0(t, x)}$, where $n_0(t, x)$ is the density number. The classical electromagnetic potential $A_\mu$ are also macroscopic quantities. It makes the interaction action big enough. One can not ignore the higher order contribution of $\exp(i \int d^4x L_{\text{int}})$. Alternatively, in the nontrivial background field method, the decomposition of the fields, i.e. Eq. (11), makes the integral mainly contributed from the fluctuation fields around zero. The perturbation method is feasible for this effective field theory. Intuitive speaking, the infinitely many particles is reduced by incorporating the effects of background fields directly into the Lagrangian. However, in this work, we just focus on the small amplitude wave propagation. The nonlinear QED effects is excluded.

The interaction is related to some kind of currents. In fact, the effective Lagrangian is invariant under the following transformation

$$\psi^{(a)}(x) \rightarrow (e^{i\alpha(x)} - 1)\psi^{(a)}(x) + e^{i\alpha(x)} \psi^{(a)}(x),$$  \hspace{1cm} (24)

$$\psi^{(a)}(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x),$$  \hspace{1cm} (25)

where $\alpha(x)$ is any function of $x$. This new symmetry is a nature result of the local $U(1)$ symmetry of standard spinor QED Lagrangian. It implies a conservative current

$$\mathcal{J}^{\alpha} = \bar{\psi}^{(a)} \gamma^\mu \gamma^\nu \psi^{(a)} + \bar{\psi}^{(a)} \gamma^\mu \psi^{(a)} + \bar{\psi}^{(a)} \gamma^\mu \psi^{(a)},$$  \hspace{1cm} (26)

satisfying

$$\partial_\mu \mathcal{J}^{(a)\mu} = 0.$$

Here, we define the vacuum current $\mathcal{J}^{\mu}_{\text{vac}}$ and the background current $\mathcal{J}^{\mu}_{\text{bac}}$ as

$$\mathcal{J}^{(a)\mu}_{\text{vac}} = \bar{\psi}^{(a)} \gamma^\mu \psi^{(a)}$$  \hspace{1cm} (28)

and

$$\mathcal{J}^{(a)\mu}_{\text{bac}} = \bar{\psi}^{(a)} \gamma^\mu \psi^{(a)} + \bar{\psi}^{(a)} \gamma^\mu \psi^{(a)},$$  \hspace{1cm} (29)

respectively. Then, the interaction Lagrangian can be written as

$$\hat{L}_{\text{int}} = \mathcal{J}^{(a)\mu} A^\mu_{\text{bac}} = (\mathcal{J}^{(a)\mu}_{\text{vac}} + \mathcal{J}^{(a)\mu}_{\text{bac}}) A^\mu_{\text{bac}},$$  \hspace{1cm} (30)

which involves the coupling of fluctuation vector field $A^\mu_{\text{bac}}$ and both of the background field $\psi_0^{(a)}$ and the fluctuation Dirac spinor filed $\psi^{(a)}$.

In addition, the Feynman rules in momentum space of this effective field theory in TFD are listed as follows

$$a \rightarrow p, \quad \omega \rightarrow b$$  \hspace{1cm} (31)

$$a \rightarrow k, \quad \omega \rightarrow b$$  \hspace{1cm} (32)
The first three equations are usual Feynmann rules for QED in TFD. The rest ones denote other two new rules of interactions for our effective theory in TFD. Here,

$$\delta^{\rho\tau\sigma} = \begin{cases} 1 & \text{for } \rho = \tau = \sigma = 1, \\ 0 & \text{other wise.} \end{cases}$$

Further specifications of propagators and vertexes will be given in the following subsections.

C. Effective action of fluctuation vector boson

We first calculate the effective action of the fluctuation vector boson. Its role in evaluating the wave propagation will be given in the subsequent section.

To proceed further, we make the followed specifications. For the plasmas assumed in equilibrium state, the temperature-dependent vacuum state vector $|\Omega(\beta)\rangle$ for the Hamiltonian corresponding to original Lagrangian, i.e., Eq. (11), is introduced in TFD as

$$|\Omega(\beta)\rangle = e^{-iG}|\Omega'\rangle = e^{-i(G_\psi \otimes G_A)}|\Omega'\rangle,$$

where $G$ is the generator of the unitary transformation operator, $G_\psi$ and $G_A$ are the corresponding ones of Dirac fermions and vector bosons, respectively. The corresponding temperature-dependent field operators are

$$e^{iG_A[A_\mu^{(\alpha)}(x) + A_\mu^{(\alpha)}(x)]} e^{-iG_A}$$

where $W_B(\beta, x)$ and $W_F(\beta, x)$ are the two $2 \times 2$ transformation matrices which are determined by statistical properties of the system of interesting. From the above equations, $e^{-iG_A}$ and $e^{-iG_\psi}$ can be considered as the transformation operators only concerning on the field operators of $A_\mu^{(\alpha)}(x)$ and $\psi^{(\alpha)}(x)$, respectively. It leads to the thermal transformation

$$|\Omega(\beta)\rangle = e^{-iG}|\Omega\rangle,$$

where $|\Omega\rangle$ is the vacuum state vector of the effective theory denoting by Eq. (22). In most cases, photons are not in equilibrium state. For simplicity, we consider the single frequency wave propagation case which is common seen in the studies of linear wave propagation in classical plasmas and non-relativistic degenerate plasmas. Then the state of incident light is denoted as $|k\rangle |k\rangle \cdots |k\rangle$. It indicates a group of single photons propagating in plasmas. Subsequently, the temperature dependent propagator of photon is similar to that in the usual QFT. The transformation matrix $U_B = 1$. Subsequently, for simplicity, a 2-point Green’s function for fluctuation Dirac particle is derived as

$$
\langle \Omega(\beta)| T \psi^{(\rho)}(x) \bar{\psi}^{(\sigma)}(y) \rangle \Omega(\beta) \rangle = \sum_{\lambda, \tau = 1, 2} W^{(\rho\lambda)}(x) \Omega(T \psi^{(\lambda)}(x) \bar{\psi}^{(\tau)}(y)) \Omega(W^{-1}(y))^{(\tau\sigma)} = \lim_{T \to \infty + i\epsilon} \sum_{\lambda, \tau = 1, 2} W^{(\rho\lambda)}(x) \int D\bar{\psi}(n) D\psi^{(n)} e^{i \int_{0}^{T} d^{4}x \bar{\psi}(\psi^{(n)} A^{(n)}) \bar{\psi}(\psi^{(n)} A^{(n)}) \psi^{(\lambda)}(\psi^{(n)} A^{(n)}) \psi^{(\tau)}(y) (W^{-1}(\beta, y))^{(\tau\sigma)}.$$

It shows that the temperature dependent Green’s function can be evaluated by performing a unitary transformation on the doubled usual Green’s function which is derived from the functional integral over the doubled field functions variables $\bar{\psi}^{(\alpha)}$, $\psi^{(\alpha)}$ and $A^{(\alpha)}$. By using the standard functional integral formulas over Grassmann variables, i.e.,

$$(\prod_{i} \int d\theta^{i}_{k} d\bar{\theta}^{i}_{k}) \exp(-\theta^{i}_{k} B_{ij} \bar{\theta}^{j}_{k}) = \det B,$$

$$(\prod_{i} \int d\theta^{i}_{k} d\bar{\theta}^{i}_{k}) \exp(-\theta^{i}_{k} B_{ij} \bar{\theta}^{j}_{k}) = (\det B)(B^{-1})^{kl}$$

and

$$(\prod_{i} \int d\theta^{i}_{k} d\bar{\theta}^{i}_{k}) \exp(-\theta^{i}_{k} B_{ij} \bar{\theta}^{j}_{k}) = (\det B)(B^{-1})^{kl} B_{mn} - B_{kn} B_{ml}^{-1},$$

replacing the action,
Eq. 22, by its temperature dependent form, the effective action $\Gamma_A$ can be obtained, to the second order approximation, as

$$e^{i\Gamma_A} = \int D\psi^{(\alpha)} D\bar{\psi}^{(\alpha)} \exp(i \int d^4x \mathcal{L}[\bar{\psi}^{(\alpha)}, \psi^{(\alpha)}, A^{(\alpha)}]) = \det(i\mathcal{D}^{(\alpha)} - m) e^{i P_\alpha \int d^4x (\bar{A}^{(\alpha)}_\mu x^\mu |(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)\sigma^\nu)|A^{(\alpha)}_\mu \{1 - ie \int d^4x tr[(\gamma^\mu \otimes \sigma^3) S_{T_F}(x - \bar{x})] A^{(\alpha)}_\mu (x)

- \frac{e^2}{2} P_\alpha \int d^4x d^4y (A^{(\alpha)}_\mu (x))^T \Pi^{\mu\nu}_T (x, y) A^{(\alpha)}_\nu (y)\},$$

where $S_{T_F}(x - y)$ is the temperature dependent Fermion propagator, $\Pi^{\mu\nu}_T (x, y)$ is the temperature dependent polarization tensor for fluctuation vector boson. Details of these two quantities will be stated below. It is more convenient to compute in momentum space.

By using Eq. 39, the thermal propagator for fermion in TFD is

$$S_{T_F}(p; A) = \int d^4x e^{ip(x - y)} S_F(x - y) = \int d^4x e^{ip(x - y)} (0|\bar{T}\psi^{(\alpha)}(x)\bar{\psi}^{(\alpha)}(y)|0\rangle = \int d^4x e^{ip(x - y)} (\int D\bar{\psi}^{(\alpha)} D\psi^{(\alpha)}

\times e^{i \int d^4x \mathcal{L}_0[\bar{\psi}^{(\alpha)}]} - 1W_F(\beta, x) \int D\bar{\psi}^{(\alpha)} D\psi^{(\alpha)}

\times e^{i \int d^4x \mathcal{L}_0[\bar{\psi}^{(\alpha)}]} \langle \bar{\psi}^{(\alpha)}(x)\bar{\psi}^{(\alpha)}(y)\rangle W_F^{-1}(\beta, y),$$

where $\mathcal{L}_0[\bar{\psi}^{(\alpha)}] = P_\alpha \{\bar{\psi}^{(\alpha)}[i\mathcal{D} - m 1_{4\times 4} \otimes \sigma^3]\psi^{(\alpha)}\}$ is the free Lagrangian of Dirac spinor field in TFD, $|0\rangle$ is the temperature dependent vacuum state vector for free theory. From Eq. 43, we have

$$S_{T_F}(p; A) = U_F(\beta, p) \left( \begin{array}{cc} S_F(p; A) & 0 \\ 0 & \bar{S}_F(p; A) \end{array} \right) U_F^{-1}(\beta, p),$$

where, $S_F(p; A)$ is the Feynman propagator in the external field $A^\mu$, and $\bar{S}_F(p; A)$ is the one evaluated in $\bar{F}$. In the case of $eA^\mu \ll m$, by direct computation, we have

$$S_F(p; A) = \frac{i}{p^0 - m - i\varepsilon} + O(\varepsilon),$$

$$\bar{S}_F(p; A) = \frac{i}{p^0 - m + i\varepsilon} + O(\varepsilon),$$

It is worth to point out that

$$W_F(\beta, x) = U_F(\beta, i\partial_{x\mu}),$$

which is easily checked by comparing the Eq. 11 with the Eq. 13. For the inverse matrix $W_F^{-1}(\beta, x)$ in the approximation, as

$$W_F^{-1}(\beta, y) = U_F^{-1}(\beta, \frac{\partial}{\partial y_\mu}) + O(\varepsilon),$$

where $\partial$ denotes the action to left by the partial differential operator. In general, the action direction depends on the position of the object acted by the differential operator. The thermo transformation matrix has the similar properties. If the particles are in Fermi distribution, the unitary transformation matrix is

$$U_F(\beta, p^0) = \left( \begin{array}{cc} \cos \theta_{p^0}(\beta) & \sin \theta_{p^0}(\beta) \\ -\sin \theta_{p^0}(\beta) & \cos \theta_{p^0}(\beta) \end{array} \right).$$

Here,

$$\cos \theta_{p^0}(\beta) = \frac{\theta(p^0)}{\sqrt{e^{\beta(p^0)} + 1}},$$

$$\sin \theta_{p^0}(\beta) = \frac{e^{-\frac{1}{2} \beta(p^0)} \theta(p^0)}{\sqrt{e^{\beta(p^0)} + 1}},$$

where $\mu$ is the chemical potential of Dirac particle system and $\theta_{p^0}$ is the step function

$$\theta(p^0) = \begin{cases} 1 & p^0 \geq 0, \\
0 & p^0 < 0. \end{cases}$$

After functional integral in Eq. 22, the polarization tensor is

$$i\Pi^{\mu\nu}_T (x, y) = -ie^2 \{tr[-\Gamma^\mu S_{T_F}(x, y)\Gamma^\nu S_{T_F}(y, x)

-\Gamma^\mu S_{T_F}(x, x)\Gamma^\nu S_{T_F}(y, y)]

+\bar{\psi}_0^{(\alpha)}(x)\Gamma^\mu S_{T_F}(x, y)\Gamma^\nu \psi_0^{(\alpha)}(y)

+\bar{\psi}_0^{(\alpha)}(y)\Gamma^\mu S_{T_F}(x, y)\Gamma^\nu \psi_0^{(\alpha)}(x)\},$$

where

$$\Gamma^\mu = \left( \begin{array}{cc} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{array} \right).$$

It is worth to point out that the temperature dependent polarization tensor in this background field method can be written, in a common known form, as

$$i\Pi^{\mu\nu}_{T2,\mu\nu} (x, y) = -e^2 \langle 0 | T \{ \bar{\psi}_0^{(\alpha)}(x)\Gamma^\mu S_{T_F}(x, y)\Gamma^\nu \psi_0^{(\alpha)}(y)\} | 0 \rangle.$$
which can be checked by inserting Eq. [26] into the above equation. Then, to the second order approximation, Eq. [56] becomes

\[ i\Pi^\mu_2(x, y) = -\frac{e^2}{Z} W_F(\beta, x) \int D\bar{\psi}(x) D\psi(y) W_{\beta}(\beta, y) \]

\[ \times e^{i\int d^4x \bar{\psi} \mathcal{J}^{\mu}(x) \mathcal{J}^{\nu}(y) W_{\beta}(\beta, y)} \]

\[ \approx \langle 0(\beta) | T \mathcal{J}_{\text{bac}}^{\mu}(x) \mathcal{J}_{\text{bac}}^{\nu}(y) | 0(\beta) \rangle \]

\[ + \langle 0(\beta) | T \mathcal{J}_{\text{vac}}^{\mu}(x) \mathcal{J}_{\text{vac}}^{\nu}(y) | 0(\beta) \rangle, \quad (56) \]

where

\[ \langle 0(\beta) | T \mathcal{J}_{\text{bac}}^{\mu}(x) \mathcal{J}_{\text{bac}}^{\nu}(y) | 0(\beta) \rangle = -\frac{e^2}{Z_0} W_F(\beta, x) \int D\bar{\psi}(\alpha) D\psi(\alpha) e^{i\int d^4x \bar{\psi} \mathcal{J}^{\nu}(x) W_{\beta}(\beta, y)} \]

\[ \times \mathcal{J}_{\text{bac}}^{\mu}(x) \mathcal{J}_{\text{bac}}^{\nu}(y) W_{\beta}(\beta, y), \quad (57) \]

\[ \langle 0(\beta) | T \mathcal{J}_{\text{vac}}^{\mu}(x) \mathcal{J}_{\text{vac}}^{\nu}(y) | 0(\beta) \rangle = -\frac{e^2}{Z_0} W_F(\beta, x) \int D\bar{\psi}(\alpha) D\psi(\alpha) e^{i\int d^4x \bar{\psi} \mathcal{J}^{\nu}(x) W_{\beta}(\beta, y)} \]

\[ \times \mathcal{J}_{\text{vac}}^{\mu}(x) \mathcal{J}_{\text{vac}}^{\nu}(y) W_{\beta}(\beta, y), \quad (58) \]

\[ Z = \int D\bar{\psi}(\alpha) D\psi(\alpha) D\mathcal{A}(\alpha) e^{i\int d^4x \bar{\psi} \mathcal{J}^{\nu}(x) \mathcal{A}(\alpha)}, \quad (59) \]

and

\[ Z_0 = \int D\bar{\psi}(\alpha) D\psi(\alpha) e^{i\int d^4x \bar{\psi} \mathcal{J}^{\nu}(x) \mathcal{J}^{\mu}(y)}, \quad (60) \]

Two disconnected circles represented by the first term on the right side in the last line of Eq. [56] are attached to two fluctuation vector boson propagators as,

\[ e^{2tr[\Gamma^{\mu}(\alpha) (x, y) \Gamma^{\nu}(\beta)(y, y)]]. \quad (61) \]

It represents multiplication of two temperature dependent vacuum expectation values of \( \langle 0(\beta) | A_{\mu} | 0(\beta) \rangle \). According to Lorentz’s symmetry, this field expectation will be vanished. We can therefore write

\[ i\Pi^\mu_{T,2}(x, y) = i\Pi^\mu_{T,\text{bac},2}(x, y) + i\Pi^\mu_{T,\text{vac},2}(x, y), \quad (62) \]

where

\[ i\Pi^\mu_{T,\text{bac},2}(x, y) = -e^2 \langle 0(\beta) | T \mathcal{J}_{\text{bac}}^{\mu}(x) \mathcal{J}_{\text{bac}}^{\nu}(y) | 0(\beta) \rangle \]

is the background polarization tensor to the second order approximation, and

\[ i\Pi^\mu_{T,\text{vac},2}(x, y) = -e^2 \langle 0(\beta) | T \mathcal{J}_{\text{vac}}^{\mu}(x) \mathcal{J}_{\text{vac}}^{\nu}(y) | 0(\beta) \rangle \]

is the vacuum polarization tensor to the same approximation.

If the plasma is uniform, the polarization tensor \( \Pi^\mu(x - y) \) depends only on \( x - y \). The Fourier transformation of it can be simplified as follows

\[ \Pi^\mu_T(x, x) = \int \frac{d^4k d^4k'}{(2\pi)^8} e^{-ikx - ik'y} \Pi^\mu_T(k, k') \]

\[ = \Pi^\mu_T(x - y) = \int \frac{d^4k d^4k'}{(2\pi)^8} e^{-i(k - y)\Pi^\mu_T(k)} \quad (65) \]

Therefore

\[ \Pi^\mu_T(k, k') = (2\pi)^4 \delta(4)(k + k') \Pi^\mu_T(k). \quad (66) \]

Noting that the second term in the brace in Eq. [42] is linear in \( A^{(\alpha)} \). It is responsible for the emission, absorption and scattering of fluctuation vector bosons that will not be concerned at present. We will omit it in the following calculations of wave propagation for simplicity. Then, from Fourier transformation of Eq. [42], the effective action of fluctuation gauge boson, up to an unnecessary constant \(-\ln det(\mathcal{Z}^{(\alpha)} - m)\), is

\[ \Gamma_4 \rightarrow \sum_{\alpha=1,2} \varepsilon_\alpha P_{\alpha} \int \frac{d^4k}{(2\pi)^4} A^{(\alpha)}(x) \delta^4(k)\psi^{(\mu)} \psi^{(\nu)} + \Pi^\mu_T(k)A^{(\alpha)}(-k) \]

\[ + \Pi^\mu_{T,2}(k)A^{(\alpha)}(-k). \quad (67) \]

where

\[ A^{(\mu)}(k) = \int d^4xe^{ikx} A^{(\alpha)}(x). \quad (68) \]

Here, we use Feynmann gauge for free propagator of \( A^{(\mu)} \).

D. Collective excitation

The most appropriate concept is the collective mode. From Chin[20], it can be characterized as the poles of the propagator of effective fluctuation vector boson. Our theory is suitable for clarifying this point in a quantitative way. Then, the concept of collective excitation will be constructed based on the fluctuation vector boson.

The interactions of electromagnetic and charged particle interact with each other. The motion of photon propagation is inseparable from the plasma density fluctuation. In general, each of them can not be individually regarded as the whole collective motion of the system. However, one can consider the process as the fluctuation vector boson propagating in the effective medium determined by Eq. [47]. It indicates that the effective Hamiltonian of the boson-medium system is just that of the whole “real” plasma system, up to a constant ground state energy, from the perspective of our effective theory. In other words, the fluctuation vector boson can be considered as the quanta of collective motion of the system. They share the same concept of collective excitation and the same spectrum. From the standard view point of QFT, the pole of propagator determines the dispersion relation for the fluctuation vector boson, and so does for
the collective mode. It explains the claim of Chin as mentioned at the beginning of this subsection. Then, from Eq. (67), the dispersion relation is

$$\det([-k^2 g^{\mu\nu} + k^\mu k^\nu + \Pi^{\mu\nu}_{2}(k)] \otimes \sigma^3) = 0,$$

(69)

It can also be understood from the classical point of view. One can derive a classical equation of motion of $A^{(\alpha)}$, i.e., $([-k^2 g^{\mu\nu} + k^\mu k^\nu + \Pi^{\mu\nu}_{2}(k)]\sigma^3)A^{(\alpha)} = 0$. Eq. (69) is just the necessary condition for the existence of nontrivial solution.

In the long-wavelength limit, the thermal wavelength is much larger than that of fluctuation vector bosons. The charged particle density fluctuation is obvious. It dominates the motion of plasma system if the electromagnetic wave is not too strong to stimulate new particle production. The collective mode can be just regarded as the oscillation of plasmas. It coincides with the view point of conventional studies. In the opposite case, the oscillation of electromagnetic field dominates the system. Intuitively, the motion of photon propagation and of the charged particle background is decoupled. More complicated is that, as will be seen in Sec. IV, more charged particles are produced in this case. Subsequently, strictly speaking, the oscillation of plasma background is not a typical character of the whole system. In general, the photon can be regarded as partially coupled with the charged particles. The degree of the coupling can be understood with the help of Eq. (62). It shows that the effective medium can be considered as a mixture of vacuum and plasma background. The two terms on the right side of Eq. (62) represent the coupling of the effective vector boson and these two objects, respectively. As will be seen in Sec. IV, $\Pi^{\mu\nu}_{bac,2}(x, y)$ dominates in the long-wavelength limit, and $\Pi^{\mu\nu}_{vac,2}(x, y)$ does in the opposite case.

### III. Calculation of Polarization Tensor

Polarization tensor is the key for evaluating the dispersion relations. The calculations of background and vacuum polarization tensor will be given in this section respectively.

#### A. Calculation of $\Pi^{\mu\nu}_{bac,2}(k)$

Inserting Eq. (21) and (71) into (65), considering (66) and (69), the background polarization tensor in momentum space becomes

$$i\Pi^{\mu\nu}_{T2, bac}(k) (2\pi)^4 \delta^{(4)}(0) = -e^2 U_F(p^0) \langle P_{\alpha} \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0^{(a)}(q) [\Gamma^{\mu} S_{TF}(q-k)\Gamma^{\nu} + \Gamma^{\nu} S_{TF}(q+k)\Gamma^{\mu}] \psi_0^{(a)}(q) \rangle U^{-1}_F(p^0)$$

$$= \frac{\bar{\psi}_0^{(a)}(q)}{4} \left( \gamma^{\mu} g^{\nu\rho} + \gamma^{\nu} g^{\mu\rho} \right) \frac{\psi_0^{(a)}(q)}{4} + \frac{\bar{\psi}_0^{(a)}(q)}{4} \left( \gamma^{\mu} g^{\nu\rho} + \gamma^{\nu} g^{\mu\rho} \right) \frac{\psi_0^{(a)}(q)}{4} \right)$$

(70)

The appearance of the delta function $(2\pi)^4 \delta^{(4)}(0)$ in the left hand side of this equation will be explained at the end of this subsection below Eq. (145). Considering the case where the electromagnetic field is not too strong to stimulate particle production such as Schwinger pair, i.e., $eA^{\mu}/m < 1$. $eA$ can be regarded as the next to leading order correction to the Eq. (141). By using the well-known gamma matrix properties of $[\gamma^{\mu}, \gamma^{\nu}] = i S^{\mu\nu}$ and $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$, considering Eq. (141), Eq. (149) and Eq. (70), to the order of $e^2$, the background polarization tensor, i.e., Eq. (71), becomes

$$i\Pi^{\mu\nu}_{T2, bac}(k) (2\pi)^4 \delta^{(4)}(0) = -2ie^2 \int \frac{d^4q}{(2\pi)^4} \{\bar{\psi}_0(q) \left( [(q-k)^2 - m^2][(q+k)^2 - m^2] \right) ^{-1} \gamma^{\mu} [(q^2 + k^2 - m^2)q^{\nu} - wq \cdot kk^{\nu}]$$

$$+ 2q \cdot k (\bar{\psi}_0(q) \left( [(q-k)^2 - m^2][(q+k)^2 - m^2] \right) ^{-1} \gamma^{\mu} [(q^2 + k^2 - m^2)q^{\nu} - wq \cdot kk^{\nu}]$$

$$+ 4imq \cdot k S^{\mu\nu} + 2i S^{\mu\rho} \gamma^{\mu} [(q^2 + k^2 - m^2)k_{\rho} + 2q \cdot k q_{\rho}] \psi_0(q)$$

$$+ \bar{\psi}_0(q) \left( [(q-k)^2 - m^2][(q+k)^2 - m^2] \right) ^{-1} \gamma^{\mu} [(q^2 + k^2 - m^2)q^{\nu} - wq \cdot kk^{\nu}] + 2q \cdot k (\bar{\psi}_0(q) \left( [(q-k)^2 - m^2][(q+k)^2 - m^2] \right) ^{-1} \gamma^{\mu} [(q^2 + k^2 - m^2)q^{\nu} - wq \cdot kk^{\nu}]$$

$$+ 4imq \cdot k S^{\mu\nu} + 2i S^{\mu\rho} \gamma^{\mu} [(q^2 + k^2 - m^2)k_{\rho} + 2q \cdot k q_{\rho}] \{\psi_0(q)\}^T \}.$$  

(71)
The last three lines of the integrand are just the repeat of the first three lines except for \( \bar{\psi}_0(q) \) being replaced by \( \bar{\psi}_0(q)T \). The calculation of this integral is divided into two steps. We will elaborate on each of them as follows.

1. Classical limit method

Four quantities, i.e., \( \bar{\psi}_0(q)\psi_0(q) \), \( \bar{\psi}_0(q)\gamma^\mu\psi_0(q) \), \( \bar{\psi}_0(q)S^\mu\nu\psi_0(q) \) and \( \bar{\psi}_0(q)S^\rho\sigma^\mu\nu\psi_0(q) \), in the Eq. (77) must be determined first. With the help of Eq. (8), the key observation for evaluating these quantities are

\[
\langle \bar{\psi}^{(\alpha)}(x)\psi^{(\alpha)}(x) \rangle = \langle \Omega(\beta)|\bar{\psi}^{(\alpha)}(x)\psi^{(\alpha)}(x)|\Omega(\beta) \rangle, \tag{72}
\]

\[
\langle \bar{\psi}^{(\alpha)}(x)\Gamma^\mu\psi^{(\alpha)}(x) \rangle = \langle \Omega(\beta)|\bar{\psi}^{(\alpha)}(x)\Gamma^\mu\psi^{(\alpha)}(x)|\Omega(\beta) \rangle, \tag{73}
\]

\[
\langle \bar{\psi}^{(\alpha)}(x)(S^{\mu\nu} \otimes I_{2 \times 2})\psi^{(\alpha)}(x) \rangle = \langle \Omega(\beta)|\bar{\psi}^{(\alpha)}(x)(S^{\mu\nu} \otimes I_{2 \times 2})\psi^{(\alpha)}(x)|\Omega(\beta) \rangle, \tag{74}
\]

\[
\langle \bar{\psi}^{(\alpha)}(x)S^{\rho\sigma}\Gamma^\mu\psi^{(\alpha)}(x) \rangle = \langle \Omega(\beta)|\bar{\psi}^{(\alpha)}(x)S^{\rho\sigma}\Gamma^\mu\psi^{(\alpha)}(x)|\Omega(\beta) \rangle. \tag{75}
\]

Here \( \langle \rangle \) denotes the statistical average. We denote \( \Psi = \psi_0 + \psi \) in this subsection as the spinor field for electron and positron. The classical limit of these statistical average can be easily derived by replacing the field operators by the corresponding classical fields. For example

\[
\langle \bar{\psi}^{(\alpha)}(x)\psi^{(\alpha)}(x) \rangle \sim \bar{\psi}^{(\alpha)}_0(x)\psi^{(\alpha)}_0(x), \tag{76}
\]

where \( \sim \) denotes “classical limit”. The sense of the term of “classical limit” is obvious if we note

\[
\langle \bar{\psi}^{(\alpha)}(x)\psi^{(\alpha)}(x) \rangle = \lim_{x^0 \to x^0 + \hbar^{-1}} \int D\bar{\psi}^{(\alpha)}D\psi^{(\alpha)}e^{i\int d^4xL} - 1 \times \int D\bar{\psi}^{(\alpha)}D\psi^{(\alpha)}e^{i\int d^4xL} \bar{\psi}^{(\alpha)}(x') + \psi^{(\alpha)}(x') \times [\bar{\psi}^{(\alpha)}_0(x) + \psi^{(\alpha)}_0(x)] = \bar{\psi}^{(\alpha)}_0(x)\psi^{(\alpha)}_0(x) + (\bar{\psi}^{(\alpha)}(x)\psi^{(\alpha)}(x)). \tag{77}
\]

It reduces to the Eq. (76) if we neglect the “quantum fluctuation” term \( \langle \psi(x)\psi(x) \rangle \). For the same reason,

\[
\langle \bar{\psi}^{(\alpha)}(x)\Gamma^\mu\psi^{(\alpha)}(x) \rangle \sim \bar{\psi}^{(\alpha)}_0(x)\Gamma^\mu\psi^{(\alpha)}_0(x), \tag{78}
\]

\[
\langle \bar{\psi}^{(\alpha)}(x)(S^{\mu\nu} \otimes I_{2 \times 2})\psi^{(\alpha)}(x) \rangle \sim \bar{\psi}^{(\alpha)}_0(x)(S^{\mu\nu} \otimes I_{2 \times 2})\psi^{(\alpha)}_0(x), \tag{79}
\]

\[
(\bar{\psi}^{(\alpha)}(x)S^{\rho\sigma}\Gamma^\mu\psi^{(\alpha)}(x)) \sim \bar{\psi}^{(\alpha)}_0(x)S^{\rho\sigma}\Gamma^\mu\psi^{(\alpha)}_0(x). \tag{80}
\]

In fact, the classical fields \( \psi_0(x) \) and \( A_{\mu}(x) \) are not uniquely determined by Maxwell-Dirac equations Eq. (20) and the Eq. (21) if the initial and boundary conditions are unknown. The division of fields into classical background fields and quantum fluctuations is artificial in a sense. From the perspective of path integral quantization, if the background fields are chosen not to be the solutions for the system of interesting, the fluctuation fields contributing to Eq. (77) can not be considered as just quantum one. For example, one can always choose trivial solutions of the Eq. (20) and Eq. (21), \( \psi \) will always be of order of \( \sqrt{n_0} \), as will be shown in this subsection. The term of “classical limit” is meaningless. Further, it invalids the perturbation calculations. In fact, one can not ignore the higher order terms in the expansion of the effective action in Eq. (12) if \( \psi \) and \( A_{\mu} \) are macroscopic quantities.

Next, we expand the classical fields \( \psi_0(x) \) and \( \bar{\psi}_0(x) \) as

\[
\psi^{(\alpha)}_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left[ \frac{\bar{c}^{(s)}(t, p)}{i\hbar} \right] u^s(p)e^{-ip \cdot x} + \left( \frac{\bar{d}^{(s)}(t, p)^*}{i\hbar} \right) v^s(p)e^{ip \cdot x}, \tag{81}
\]

\[
\bar{\psi}^{(\alpha)}_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left[ \frac{\bar{c}^{(s)}(t, p)^*}{i\hbar} \right] \bar{u}^s(p)e^{-ip \cdot x} + \left( \frac{\bar{d}^{(s)}(t, p)}{i\hbar} \right) \bar{v}^s(p)e^{ip \cdot x}, \tag{82}
\]

where \( u^s(p) \) and \( v^s(p) \) are two momentum dependent Dirac spinors

\[
u^s(p) = \left( \frac{\sqrt{p \cdot \sigma\xi^s}}{\sqrt{p \cdot \sigma\eta^s}}, \frac{\sqrt{p \cdot \sigma\eta^s}}{\sqrt{p \cdot \sigma\xi^s}} \right), \tag{83}
\]

\[
u^s(p) = \left( \frac{\sqrt{p \cdot \sigma\eta^s}}{\sqrt{p \cdot \sigma\xi^s}}, \frac{\sqrt{p \cdot \sigma\xi^s}}{\sqrt{p \cdot \sigma\eta^s}} \right), \tag{84}
\]

with \( \sigma = (1, \sigma) \), and \( \xi^s \) and \( \eta^s \) are two two-components spinors. For the purpose of consistency, the coefficients \( \bar{c}^{(s)}(t, p) \), \( \bar{d}^{(s)}(t, p) \), \( \bar{d}^{(s)}(t, p) \), and \( \bar{d}^{(s)}(t, p) \) must be defined carefully. First, \( \bar{c}^{(s)}(t, p) \) and \( \bar{d}^{(s)}(t, p) \) are Grassmann numbers that satisfy the relations

\[
\{\bar{c}^{(s)}(t, p), \bar{c}^{(t', p')}\} = 0, \quad \{\bar{c}^{(s)}(t, p), \bar{c}^{(t', p')}^*\} = 0,
\]

\[
\{\bar{d}^{(s)}(t, p), \bar{d}^{(t', p')}\} = 0, \quad \{\bar{d}^{(s)}(t, p), \bar{d}^{(t', p')}^*\} = 0, \tag{85}
\]

\[
\{\bar{c}^{(s)}(t, p), \bar{d}^{(t', p')}\} = 0, \quad \{\bar{c}^{(s)}(t, p), \bar{d}^{(t', p')}^*\} = 0.
\]

The corresponding coefficients \( \bar{c}^{(s)}(t, p) \) and \( \bar{d}^{(s)}(t, p) \) are also Grassmann numbers satisfying the same anti-commutation relations. In addition, we define

\[
\{\bar{c}^{(s)}(t, p), \bar{c}^{(t', p')}\} = 0, \tag{86}
\]

where
\[ [\tilde{d}^*(t, p), \tilde{d}(t, p)] = 0, \quad (87) \]
\[ [\tilde{c}^*(t, p), \tilde{d}^*(t, p)] = 0. \quad (88) \]

A remark on the tilde symbol on these coefficients is that \( \tilde{c}^*(t, p) \) and \( \tilde{\bar{c}}^*(t, p) \) are defined as the two independent coefficients in the Fourier transformation of \( \psi_0^{(\alpha)}(x) \). It is unlike the case where \( \hat{A} \) is considered as the tilde-conjugation operation of operator \( A \), as introduced at the beginning of Sec. II. It is the same for the coefficients \( \tilde{d}^*(t, p) \) and \( \tilde{\bar{d}}^*(t, p) \). However, the definitions of expansions, i.e., Eq. (81) and Eq. (82), lead to algorithm similar to the Eq. (51). Subsequently, we can also regard the coefficients \( \tilde{c}^*(t, p) \) and \( \tilde{d}^*(t, p) \) as the “tilde-conjugation” of the coefficients \( c^*(t, p) \) and \( d^*(t, p) \) in calculation.

By using Eq. (81) to Eq. (85), we have
\[
\int d^3 x \psi_0^\dagger(x) \psi_0(x) = \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1,2} (|c^*(t, p)|^2 - |d^*(t, p)|^2).
\]
\[ (89) \]
\[
\int d^3 x \tilde{\bar{c}}^*(t, p) \tilde{c}^*(t, p) = \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1,2} (|\tilde{c}^*(t, p)|^2 - |\tilde{d}^*(t, p)|^2).
\]
\[ (90) \]

From the relation
\[
\langle \Omega(\beta) | \int d^3 x \Psi^\dagger(x) \Psi(x) | \Omega(\beta) \rangle
= \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1,2} \langle \Omega(\beta) | a^s_+ a^s_0 - b^s_1 b^s_0 | \Omega(\beta) \rangle,
\]
we have
\[
\sum_{s=1,2} \langle \Omega(\beta) | a^s_+ a^s_0 | \Omega(\beta) \rangle \sim \sum_{s=1,2} |\tilde{c}^*(t, p)|^2
= N^{(+)}_0(t, p),
\]
\[ (91) \]
\[
\sum_{s=1,2} \langle \Omega(\beta) | b^s_1 b^s_0 | \Omega(\beta) \rangle \sim \sum_{s=1,2} |\tilde{d}^*(t, p)|^2
= N^{(-)}_0(t, p),
\]
\[ (92) \]

where \( N^{(+)}_0(t, p) \) and \( N^{(-)}_0(t, p) \) are the average particle number density of momentum \( p \) at time \( t \) for particles and anti-particles, respectively. In the electron-positron plasma case, they correspond to the number density of electrons and positrons. Note that
\[
\psi_0(x) = \tilde{\psi}_0(x)
\]
\[ (93) \]
which is derived immediately from the relation
\[
\langle F | \psi_0(x) + \psi(x) | F' \rangle = \langle \tilde{F} \tilde{\psi}_0(x) + \tilde{\psi}(x) | F' \rangle.
\]
\[ (94) \]

Here \( |F\rangle \) and \( |F'\rangle \) are two state vectors in the Fock space \( \mathcal{F} \). \( |\tilde{F}\rangle \) and \( |\tilde{F}'\rangle \) are the corresponding ones in \( \tilde{\mathcal{F}} \). Then, perform the tilde-conjugation of Eq. (92) and Eq. (93), we have
\[
\sum_{s=1,2} \langle \Omega(\beta) | a^s_+ a^s_0 | \Omega(\beta) \rangle \sim \sum_{s=1,2} |\tilde{c}^*(t, p)|^2
= N^{(+)}_0(t, p),
\]
\[ (95) \]
\[
\sum_{s=1,2} \langle \Omega(\beta) | b^s_1 b^s_0 | \Omega(\beta) \rangle \sim \sum_{s=1,2} |\tilde{d}^*(t, p)|^2
= N^{(-)}_0(t, p).
\]
\[ (96) \]

We list the Fourier transformation of \( \psi_0^{(\alpha)}(x) \) and \( \tilde{\psi}_0^{(\alpha)}(x) \) which is useful for evaluating the four quantities mentioned at the beginning of this subsection.
\[
\psi_0^{(\alpha)}(q) = \int d^4 x e^{iq x} \psi_0(x)
= \frac{1}{\sqrt{2E_q}} \sum_{s=1,2} \left( \begin{array}{c}
\tilde{c}^s(q^0 - E_q, \mathbf{q}) \\
\tilde{d}^s(q^0 - E_q, \mathbf{q})
\end{array} \right) \tilde{u}^s(q)
+ \left( \begin{array}{c}
\tilde{d}^s(q^0 + E_q, -\mathbf{q}) \\
\tilde{c}^s(q^0 + E_q, -\mathbf{q})
\end{array} \right) \tilde{v}^s(-q),
\]
\[ (97) \]
\[
\tilde{\psi}_0^{(\alpha)}(q) = \int d^4 x e^{iq x} \tilde{\psi}_0(x)
= \frac{1}{\sqrt{2E_q}} \sum_{s=1,2} \left( \begin{array}{c}
\tilde{c}^s(q^0 - E_q, \mathbf{q})^* \\
\tilde{d}^s(q^0 - E_q, \mathbf{q})^*
\end{array} \right) \tilde{u}^s(q)
+ \left( \begin{array}{c}
\tilde{d}^s(q^0 + E_q, -\mathbf{q})^* \\
\tilde{c}^s(q^0 + E_q, -\mathbf{q})^*
\end{array} \right) \tilde{v}^s(-q),
\]
\[ (98) \]

where
\[
\tilde{c}^s(q^0 - E_q, \mathbf{q}) = \int dt e^{i(q^0 - E_q)t} \tilde{c}^s(t, q)
= \int dt e^{i(q^0 - E_q)t} \tilde{c}^s(t, q)^*,
\]
\[ (99) \]
\[
d^s(q^0 + E_q, -\mathbf{q}) = \int dt e^{i(q^0 - E_q)t} d^s(t, -\mathbf{q})
= \int dt e^{i(q^0 - E_q)t} d^s(t, -\mathbf{q}),
\]
\[ (100) \]
\[
\tilde{\bar{c}}^s(q^0 - E_q, \mathbf{q}) = \int dt e^{i(q^0 - E_q)t} \tilde{c}^s(t, q)^*
\]
\[ (101) \]
\[
d^s(q^0 + E_q, -\mathbf{q}) = \int dt e^{i(q^0 - E_q)t} d^s(t, -\mathbf{q}).
\]
\[ (102) \]

a. Calculation of \( \tilde{\psi}_0(q) \psi_0(q) \) : From the energy-momentum tensor \( T^{\mu\nu} \) corresponding to the Lagrangian of spinor QED, we have
\[
\tilde{\psi}_0 \gamma^\mu \partial^\nu \psi_0 = T^{\mu\nu} - (T_{em})^{\mu\nu} + \epsilon_{\mu\nu} \gamma^\mu \psi_0 A^\nu,
\]
\[ (103) \]

where \( (T_{em})^{\mu\nu} = F^{\mu\rho} F_{\rho\nu} - \frac{1}{4} (F_{\mu\nu})^2 g^{\mu\nu} \) is the symmetric energy-momentum tensor of free electromagnetic field, and the third part on the right side of the equation relates to the interaction of photons and charged particles. Considering the Eq. (20), we have
\[
\tilde{\psi}_0(x) \psi_0(x) = \frac{1}{m} \left( [T_0^0(x) - (T_{em})^0_0(x)] \right)
\]
From the classical limit method

\[ \langle \Omega(\beta) \rangle \frac{1}{m} [(T^0_0) - (T_{em})_0^0(x)] - (T^i_i(x) - (T_{em})_i^i(x)) \]

\[ \sim \frac{1}{m} [(T^0_0(x) - (T_{em})_0^0(x)) - (T^i_i(x) - (T_{em})_i^i(x))] \]

\[ = \bar{\psi}_0(x) \psi_0(x), \]  

(106)

where \( E \) and \( E_{em} \) are the average energy of the whole system and electromagnetic field, respectively. It is better to rewrite it in momentum representation as

\[ \int \frac{d^4p}{(4\pi)^4} \bar{\psi}_0(p) \psi_0(p) = \frac{1}{m} \int dt [E_D(t) - \sum_i P^i_D(t)]. \]  

(107)

Here,

\[ E_D(t) = E(t) - E_{em}(t) \]  

(108)

and

\[ P^i_D(t) = P^i(t) - P^i_{em}(t) \]  

(109)

where \( E_{em}(t) \) and \( P_{em}(t) \) are the total energy and momentum of free propagating electromagnetic field, while \( E_D(t) \) and \( P_D(t) \) denote the rest ones. In the reference frame of mass point of charged particle system, \( P^i_D(t) = 0 \). Subsequently,

\[ \int \frac{d^4p}{(4\pi)^4} \bar{\psi}_0(p) \psi_0(p) = \frac{1}{m} \int dt E_D(t). \]  

(110)

b. Calculation of \( \bar{\psi}_0(q) \gamma^\mu \psi_0(q) \):

\[ \int d^4x \langle \Omega(\beta) \rangle [\bar{\psi}(x) \gamma^\mu \psi(x)] \Omega(\beta) \]

\[ = \int \frac{d^4q}{(2\pi)^4} \langle \Omega(\beta) \rangle [\bar{\psi}(q) \gamma^\mu \psi(q)] \Omega(\beta) \]

\[ \sim \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(q) \gamma^\mu \psi_0(q) \]

\[ = \int dt j_0^\mu(t, q). \]  

(111)

c. Calculation of \( \bar{\psi}_0(q) S^{\nu \rho \gamma^\mu} \psi_0(q) \): It will be discussed in eight cases as follows.

(1) For the case of \( \mu = 0, \nu = 0, \rho = 0 \),

\[ \int d^4x \bar{\psi}_0(x) S^{\nu \rho \gamma^\mu} \psi_0(x) \]

\[ = \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(q) S^{00 \gamma^\mu} \psi_0(q) \]

\[ = 0. \]  

(112)

(2) For the case of \( \mu = 0, \nu = 0, \rho = i \),

\[ \int d^4x \langle \Omega(\beta) \rangle [\bar{\psi}(x) S^{\nu \rho \gamma^\mu} \psi(x)] \Omega(\beta) \]

\[ \sim \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(q) S^{00 \gamma^\mu} \psi_0(q) \]

\[ = \int dt \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(t, q) S^{00 \gamma^\mu} \psi_0(t, q) \]

\[ = -i \frac{1}{2} \int dt \int \frac{d^4q}{(2\pi)^4} \bar{j}_0^\mu(t, q). \]  

(113)

(3) For the case of \( \mu = 0, \nu = j, \rho = 0 \),

\[ \int d^4x \langle \Omega(\beta) \rangle [\bar{\psi}(x) S^{\nu \rho \gamma^\mu} \psi(x)] \Omega(\beta) \]

\[ \sim \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(q) S^{0j0 \gamma^\mu} \psi_0(q) \]

\[ = \int dt \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(t, q) S^{0j0 \gamma^\mu} \psi_0(t, q) \]

\[ = \frac{i}{2} \int dt \int \frac{d^4q}{(2\pi)^4} \bar{j}_0^\mu(t, q). \]  

(114)

(4) For the case of \( \mu = 0, \nu = j, \rho = i \),

\[ \int d^4x \langle \Omega(\beta) \rangle [\bar{\psi}(x) S^{\nu \rho \gamma^\mu} \psi(x)] \Omega(\beta) \]

\[ \sim \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(q) S^{ij0 \gamma^\mu} \psi_0(q) \]

\[ = \int dt \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(t, q) S^{ij0 \gamma^\mu} \psi_0(t, q) \]

\[ = \frac{1}{2} \int dt \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(t, q) S^{ij0 \gamma^\mu} \psi_0(t, q) \]

\[ = \frac{1}{2} \int dt \int \frac{d^4q}{(2\pi)^4} \bar{j}_0^\mu(t, q). \]  

(115)

(5) For the case of \( \mu = l, \nu = 0, \rho = 0 \),

\[ \int d^4x \bar{\psi}_0(x) S^{\nu \rho \gamma^\mu} \psi_0(x) \]

\[ = \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(q) S^{0l0 \gamma^\mu} \psi_0(q) \]

\[ = 0. \]  

(116)

(6) For the case of \( \mu = l, \nu = 0, \rho = i \),

\[ \int d^4x \langle \Omega(\beta) \rangle [\bar{\psi}(x) S^{\nu \rho \gamma^\mu} \psi(x)] \Omega(\beta) \]

\[ \sim \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(q) S^{0l0 \gamma^\mu} \psi_0(q) \]

\[ = \int dt \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_0(t, q) S^{0l0 \gamma^\mu} \psi_0(t, q) \]

\[ = -i \frac{1}{2} \int dt \int \frac{d^4q}{(2\pi)^4} \{ \delta^{il} \bar{j}_0^0(t, q) \}

\[ + \epsilon^{ilj} \overline{\left[ \bar{\psi}_0(t, q) \right]} \}

\[ = \int \frac{d^4q}{(2\pi)^4} \bar{j}_0^\mu(t, q). \]  

(117)
\[ \begin{align*}
&= \int dt \int \frac{d^3q}{(2\pi)^3} \bar{\psi}_0(t, q) S^{0\rho}_{\gamma\gamma} \psi(t, q) \\
&= \frac{i}{2} \int dt \int \frac{d^3q}{(2\pi)^3} \{ \delta^{\beta\gamma} \bar{j}_{0\beta}^R(t, q) \\
&\quad + \varepsilon^{\beta\gamma} [\bar{j}_{0\beta}^R(t, q) - j_{0\beta}^L(t, q)] \}. \tag{118}
\end{align*} \]

(8) For the case of \( \mu = l, \nu = j, \rho = i \),
\[ \int d^4x \Omega(\beta) \bar{\psi}(x) S^{0\rho}_{\gamma\gamma} \psi(x) |\Omega(\beta) \]
\[ \sim \int \frac{d^3q}{(2\pi)^3} \bar{\psi}_0(q) S^{3i}_{\gamma\gamma} \psi_0(q) \]
\[ = \int dt \int \frac{d^3q}{(2\pi)^3} \bar{\psi}_0(t, q) S^{3i}_{\gamma\gamma} \psi_0(t, q) \]
\[ = \frac{i}{2} \int dt \int \frac{d^3q}{(2\pi)^3} \{ \varepsilon^{3i} [N_{0R}(t, q) - N_{0L}(t, q)] \\
&\quad + i \delta^{3i} [j_{0R}^i(t, q) - j_{0L}^i(t, q)] \\
&\quad - i \delta^{3i} [\bar{j}_{0R}^i(t, q) - \bar{j}_{0L}^i(t, q)] \}. \tag{119}
\]

In the above eight equations, i.e. Eq. (112)-Eq. (115), \( j_{0L}(t, q) \) and \( j_{0R}(t, q) \) are the average left handed and right handed currents in momentum space at time \( t \), respectively.

\[ j_{0L}(t, q) = \int d^3xe^iq \cdot \bar{\psi}_0(x) \gamma_1 - \gamma_2 \psi_0(x), \tag{120} \]
\[ j_{0R}(t, q) = \int d^3xe^iq \cdot \bar{\psi}_0(x) \gamma_1 + \gamma_2 \psi_0(x), \tag{121} \]
where \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). \( N_{0L}(t, q) \) and \( N_{0R}(t, q) \) are the average number densities of left handed particles and right handed particles in momentum space at time \( t \),
\[ N_{0L}(t, q) = \int d^3xe^iq \cdot \bar{\psi}_{0L}^\dagger(x) \psi_{0L}(x), \tag{122} \]
\[ N_{0R}(t, q) = \int d^3xe^iq \cdot \bar{\psi}_{0R}^\dagger(x) \psi_{0R}(x), \tag{123} \]
where
\[ \psi_{0L}(x) = \frac{1 - \gamma^5}{2} \psi_0(x), \tag{124} \]
\[ \psi_{0R}(x) = \frac{1 + \gamma^5}{2} \psi_0(x), \tag{125} \]

d. Calculation of \( \bar{\psi}_0(q) S^{\mu\nu} \psi_0(q) \) : It will be included in the calculation of background polarization tensor, i.e., Eq. (114). We just mention here that the quantity relates to the spin of the system if \( \mu = i \) and \( \nu = j \),
\[ \bar{\psi}_0(q) S^{ij} \psi_0(q) = \frac{1}{2} \bar{\psi}_0(q) \varepsilon^{ijk} \Sigma^k \psi_0(q), \tag{126} \]
where
\[ \Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{127} \]

We make the following remarks. First, although there is no classical electromagnetic field in the Eq. (71). They may still have impact on the polarization tensor through the terms responsible for \( A_i(x) \). For instance, the classical electric field drives the motion of the electrons and positrons to form a current, thus affecting \( \Pi_{0\nu,2}(k) \). Next, from the Eq. (144) below, the terms involving \( S^{\mu\nu} \) are vanished. It indicates that the system is not affected by spin directly, to the approximation of \( e^2 \) order. Third, in this approximation, the following five quantities directly determine the background polarization tensor. They are total energy of Dirac particles, the distributions of left and right handed particle densities and left and right handed current densities.

2. Assumptions

For the sake of simplicity, we make the following six assumptions. First, we consider only the small amplitude oscillation in the present paper. Next, the plasma is assumed in static state, which is a common assumption in the usual study for the wave propagation in plasma. It indicates
\[ N(t, q) = N(q), \tag{128} \]
and
\[ j(t, q) = 0. \tag{129} \]

Third, we study the neutral plasma system. It is a suitable assumption for most plasmas which implies
\[ N^{(+)}(q) = N^{(-)}(q) = \frac{1}{2} N(q), \tag{130} \]
and
\[ j^{0}_0(q) = 0. \tag{131} \]

From Eq. (129)-Eq. (131), Eq. (111) reduces to
\[ \bar{\psi}_0(q) \gamma^0 \psi_0(q) = 0. \tag{132} \]

From Eq. (86), Eq. (97) and Eq. (130), we have
\[ \bar{c}^1(t, q) = \frac{1}{2} e^{i\phi_1(t)} \sqrt{N(q)}, \tag{133} \]
\[ \bar{c}^2(t, q) = \frac{1}{2} e^{i\phi_2(t)} \sqrt{N(q)}, \tag{134} \]
\[ \bar{d}^1(t, q) = \frac{1}{2} e^{i\chi_1(t)} \sqrt{N(q)}, \tag{135} \]
\[ \bar{d}^2(t, q) = \frac{1}{2} e^{i\chi_2(t)} \sqrt{N(q)}, \tag{136} \]
where \( \phi_1(t), \phi_2(t), \chi_1(t) \) and \( \chi_2(t) \) are four functions of \( t \) which are constrained by a specific condition. However, one can set them to be zeros. The reason will be
explained in the later subsection below Eq.(145). Then, according to Eq.(100) and Eq.(102), we have

$$e^s(q^0 - E_q, q) = \pi \delta(q^0 - E_q) \sqrt{N(q)},$$  \hspace{1cm} (137)

$$d^s(q^0 + E_q, q) = \pi \delta(q^0 + E_q) \sqrt{N(q)}.$$  \hspace{1cm} (138)

Fourth, we assume the equal number of left and right handed charged Dirac particles in this plasma, i.e.,

$$N_{0L}(q) = N_{0R}(q) = \frac{1}{2} N(q).$$  \hspace{1cm} (139)

The reason is that \( \psi_L \) and \( \psi_R \) play the same roles in the system. There is no reason to think that one of them is more special. It can be easily seen in the high-energy limit. The Dirac equation reduces to the Weyl equations for left handed Weyl spinor \( \psi_L \) and right handed Weyl spinor \( \psi_R \), in the massless limit. These spinors are decoupled. The corresponding Weyl fermions obey the same distributions, obviously.

For the same reason, we have the fifth assumption that

$$j^\mu_{0L}(q) = j^\mu_{0R}(q).$$  \hspace{1cm} (140)

Considering Eq.(129)–Eq.(132), Eq.(139) and Eq.(140), Eq.(112)–Eq.(119) reduce to

$$\tilde{\psi}_0(q) S^{\mu\nu} \gamma^\mu \psi_0(q) = 0.$$  \hspace{1cm} (141)

Last, for simplicity, the Dirac plasma particles are assumed being in ideal gas state. It is a naturel assumption that if the motion of plasma were subject to classical mechanics in many cases. In the degenerate case, the assumption is satisfied if the Fermi energy is much larger than that of Coulomb interaction between the electrons and positrons, i.e., \( \varepsilon_F \gg e^2/4\pi a \) with \( a = (1/n)^{1/3} \), where \( \varepsilon_F \) is the Fermi energy. For the non-relativistic approximation, \( n \gg (\varepsilon_F^{2/3})^3 \sim 10^{10} \text{cm}^{-3} \). Subsequently, Eq.(110) reduces to

$$\tilde{\psi}_0(q) \psi_0(q) = \frac{E_q}{m} \sum_{\nu = 1, 2} (|c^s(q^0 - E_q, q)|^2$$

$$+ |d^s(q^0 + E_q, -q)|^2).$$  \hspace{1cm} (142)

From Eq.(137), Eq.(138) and Eq.(142), we have

$$\int \frac{d^4 q}{(2\pi)^4} \tilde{\psi}_0(q) f(q) \psi_0(q)$$

$$= 2\pi \delta(0) \int \frac{d^3 q}{(2\pi)^3} \frac{E_q}{m} N_0(q)^+ f(q)|q^0 = E_q$$

$$+ N_0(q)^- f(q)|q^0 = -E_q),$$  \hspace{1cm} (143)

where \( f(q) \) is an arbitrary function of \( q \).

3. Background polarization tensor

From the above analysis, the background polarization tensor Eq.(71) can be reduced to

$$i \Pi^{\mu\nu}_{Tbac,2}(k)(2\pi)^4 \delta^{(4)}(0) = -\frac{4ie^2}{V} \int \frac{d^4 q}{(2\pi)^4} \frac{E_q}{m} N_0(q)^+ (k^2 + q^2 - m^2) g^{\mu\nu} + 4i q \cdot k S^{\mu\nu}$$

$$+ 4i q \cdot k S^{\mu\nu}$$

$$\frac{N^-(k^2 + q^2 - m^2)|q^0 = E_q}{[(k + q)^2 - m^2][(k - q)^2 - m^2]} - \frac{N^-(k^2 + q^2 - m^2)|q^0 = -E_q}{[(k + q)^2 - m^2][(k - q)^2 - m^2]}$$

$$= -\frac{4ie^2}{V} \int \frac{d^3 q}{(2\pi)^3} \frac{E_q N_0(q)^+}{[(k^2)^2 - 4(k \cdot q)^2}},$$  \hspace{1cm} (144)

where

$$V = \int d^3 x = (2\pi)^3 \delta^{(3)}(0)$$  \hspace{1cm} (145)

is the volume of space the plasma occupied. The two terms involving \( S^{\mu\nu} \) cancel each other out by substituting \( p \) by \(-p\) in the second term in the brace in the integral. We make two remarks on the background polarization tensor of Eq.(144). First, A heuristic view of the appearance of \((2\pi)^4 \delta^{(4)}(0)\) in the background polarization tensor is that it comes from the S-matrix elements. Imagine the physical process for the scattering of a fluctuation vector boson by the effective medium. It is well known that the S-matrix can always be written as

$$S_{fi} = \text{out}(k|S|k)_{in}$$

$$= \mathbb{1}_{fi} + i M_{fi}(2\pi)^4 \delta^{(4)}(0)$$

$$\approx \mathbb{1}_{fi} + i \varepsilon_{fi}(k) E_{fi} \Pi^\mu\nu(k)(2\pi)^4 \delta^{(4)}(0).$$  \hspace{1cm} (146)

The delta function comes from the integral of vertex in Feynman diagram representing the momentum conservation. It also explains why \( \phi_1(t), \phi_2(t), \chi_1(t) \) and \( \chi_2(t) \) in Eq.(133)–Eq.(136) can be chosen as zero. In fact, there is no need for these constants to be unique determined. They are just being chosen for ensuring the appearance of \((2\pi)^4 \delta^{(4)}(0)\) in the background polarization tensor. Sec-
ond, it looks like that Eq. (144) violates the Ward identity \( \kappa_\mu \Pi^\mu_\nu (k) = 0 \) which asks for \( \Pi^\mu_\nu (k) \propto (k^2 g^{\mu \nu} - k^\mu k^\nu) \). The answer is that from the Eq. (70), there is no external on-shell fermion which is necessary for the validity of Ward identity. However the gauge symmetry is conserved for this theory.

For simplicity, we choose a coordinate system such that \( k^\mu = (\omega, 0, 0, |k|) \). From previous assumptions, the charged plasma particles in the absence of external field obey Fermi-Dirac distribution as

\[
N_0^{(+)}(q) = N_0^{(-)}(q) = \frac{1}{e^{\beta (\epsilon_q - m)} + 1},
\]

where \( \epsilon_q = \sqrt{q^2 + m^2} \) is the mass-energy relation for ideal gas particles. The background polarization tensor will be discussed in two cases.

In the low energy approximation, the momentum of charged particles is much less than the rest mass, i.e.,

\[
|q| \ll m.
\]

Substituting Eq. (147) for Eq. (144), to the order of \( |q|^4/m^4 \), we have

\[
i \Pi^\mu_\nu_{Tbac,2}(k) = -ig^{\rho \sigma} \frac{e^2}{2\pi^2} \left( \frac{\omega^2 - |k|^2}{\omega^2 - (|k|^2)^2} - 4m^2\omega^2 \right) \\
x \int_0^\infty dq |q| e^{\beta(q^2/2m^2) + 1} |q|^2 \\
\left( \frac{\omega^2 - |k|^2)^2 + 4m^2\omega^2 - 8m^2|k|^2}{2m^2(\omega^2 - |k|^2)^2 - 4m^2\omega^2} \right) |q|^4.
\]

(149)

Eq. (149) involves the integral of the form \( \int_0^\infty dx f(x) e^{\beta (\mu - \epsilon)} + 1 \)^{-1}, where \( f(\epsilon) \) is the function such that the integral converges. In the case of \( \mu/T \gg 1 \), the integral can be evaluated by using the formula

\[
\int_0^\infty e^{\beta (\mu - \epsilon)} d\epsilon = \int_0^\mu f(\epsilon) + 2T^2 f'(\mu) \int_0^\infty dx \frac{x}{e^x + 1} + \cdots.
\]

(150)

Then, to the order of \( (T/\epsilon_F)^2 \), we can derive from the integral Eq. (149) that

\[
i \Pi^\mu_\nu_{Tbac,2}(k) = -\frac{4ig^{\rho \sigma}m^2(\omega^2 - |k|^2)}{(\omega^2 - (|k|^2)^2 - 4m^2\omega^2) [2m^2(\omega^2 - |k|^2)^2 - 4m^2\omega^2]}
\]

\[
\times \left( \frac{\omega^2 - |k|^2)^2 + 4m^2\omega^2 - 8m^2|k|^2}{(\omega^2 - |k|^2)^2 - 4m^2\omega^2} \right) \frac{3}{10m^2(3\pi^2 n_0)^2} + \frac{9T^2}{2(3\pi^2 n_0)^2} \zeta(2)
\]

\[
\times \left( \frac{\omega^2 - |k|^2)^2 + 4m^2\omega^2 - 8m^2|k|^2}{(\omega^2 - |k|^2)^2 - 4m^2\omega^2} \right),
\]

(155)

where \( \omega_p^2 = e^2n_0/m \) is the plasma frequency.

In the relativistic limit,

\[
|q| \gg m, \quad |k| \gg m,
\]

the background polarization tensor, i.e., Eq. (144) becomes

\[
i \Pi^\mu_\nu_{Tbac,2}(k) = \frac{ig^{\rho \sigma}m^2(\omega^2 - |k|^2)}{(\omega^2 - (|k|^2)^2 - 4m^2\omega^2) [2m^2(\omega^2 - |k|^2)^2 - 4m^2\omega^2]}
\]

\[
\times \left( \frac{\omega^2 - |k|^2)^2 + 4m^2\omega^2 - 8m^2|k|^2}{(\omega^2 - |k|^2)^2 - 4m^2\omega^2} \right)
\]

\[
\times \left( \frac{\omega^2 - |k|^2)^2 + 4m^2\omega^2 - 8m^2|k|^2}{(\omega^2 - |k|^2)^2 - 4m^2\omega^2} \right).
\]

(157)

\[i \Pi^\mu_\nu_{T_{2,vac}}(k) = \frac{i}{\epsilon_F^2} \frac{1}{\sqrt{(\omega - \epsilon_F)^2 - 4m^2}} \]

can be reduced to

\[i \Pi^\mu_\nu_{T_{2,vac}}(k) = \frac{i}{\epsilon_F^2} \frac{1}{\sqrt{(\omega - \epsilon_F)^2 - 4m^2}} \]

(158)
The integral is well treated in any formal textbook on QFT. Follow the standard method, by using the dimensional regularization for one-loop integral, imposing renormalization condition that the fluctuation vector boson is massless in vacuum by introducing the corresponding counterterm, one can derive

\[
i \Pi_{\mu \nu}^{\mu \nu}(k) = -\frac{2\alpha}{\pi} (k^2 g^{\mu \nu} - k^\mu k^\nu) \int_0^1 dx (1-x) \ln \frac{m^2}{m^2 - x} - \frac{6(k^4 - 2m^2k^2 - 8m^4)}{k^3 \sqrt{4m^2 - k^2}} \arctan \frac{k}{\sqrt{4m^2 - k^2}},
\]

where \( k \) denotes the modulus of \( k^\mu \) in Minkowsky space. The condition of \( k^2 > 4m^2 \) indicates the decay process of the boson propagation.

IV. DISPERSION RELATIONS

We denote the temperature dependent background polarization tensor \( i \Pi_{T\text{vac},2}^{\mu \nu}(k) \) and vacuum polarization tensor \( i \Pi_{T\text{vac},2}^{\mu \nu}(k) \) by \( -ig^{\mu \nu}B(\omega, k) \) and \( ig^{\mu \nu} - k^\mu k^\nu \Pi_{T\text{vac},2}(k) \), respectively. Then, the tensor \( -k^2 g^{\mu \nu} + k^\mu k^\nu + \Pi_{T\text{vac},2}^{\mu \nu}(k) \) can be written as

\[
\begin{pmatrix}
|k|^2[1+i\Pi_{T\text{vac},2}^{\mu \nu}(k)] - B(\omega, |k|) & 0 & 0 & \omega|k|[1+2i\Pi_{T\text{vac},2}^{\mu \nu}(k)] \\
0 & k^2[1+i\Pi_{T\text{vac},2}^{\mu \nu}(k)] + B(\omega, |k|) & 0 & 0 \\
0 & 0 & k^2[1+i\Pi_{T\text{vac},2}^{\mu \nu}(k)] + B(\omega, |k|) & 0 \\
\omega|k|[1+2i\Pi_{T\text{vac},2}^{\mu \nu}(k)] & 0 & 0 & \omega^2[1-i\Pi_{T\text{vac},2}^{\mu \nu}(k)] + 2B(\omega, |k|)
\end{pmatrix}.
\]

Subsequently, Eq. \([155]\) reduces to

\[
B(\omega, |k|) = 0,
\]

\[
(\omega^2 - |k|^2)[1 + i\Pi_{T\text{vac},2}^{\mu \nu}(k)] + 2B(\omega, |k|) = 0.
\]

Dispersion relations for the collective modes can be obtained from these two equations, in principle. However, it asks for exact calculation of the integral Eq. \([141]\) which is extremely difficult and is of less interest for the present work. In this paper, the wave dispersion relations in low energy and high energy limit are discussed. It must be emphasized again that Eq. \([161]\) and Eq. \([162]\) give the dispersion relations for the fluctuation vector bosons propagating in the plasma. The transformation to the collective mode excitation spectrum of plasma will be discussed in this section.

A. Low energy limit

1. Longitudinal wave dispersion relation

To the second order approximation, Eq. \([155]\), Eq. \([159]\) and Eq. \([162]\) gives the longitudinal wave dispersion relation as

\[
\omega^2 = \omega_p^2[1 + \frac{\omega_p^2}{4m^2}(1 - \frac{\alpha}{\pi} - \frac{3q_F^2}{10m^2} - \frac{\pi^2 m^2 T^2}{2q_F^2}) - \frac{3q_F^2}{10m^2} - \frac{\pi^2 m^2 T^2}{2q_F^2} - \frac{\alpha}{\pi} (1 + \frac{3\omega_p^2}{4m^2})]
\]

\[
\omega^2 = \frac{\omega_p^2}{2m^2} (\frac{3\pi^2 m^2 T^2}{2q_F^2} - \frac{\omega_p^2}{2m^2} - \frac{3\pi^2 m^2 T^2}{2q_F^2}) \approx 10m^2, (163)
\]

Here, we use the approximations of \( |q| \ll m, \omega_p \ll m \) and \( |k|^2 \ll \omega_p^2 \). The last one indicates the long wavelength approximation.

The leading order approximation of the Eq. \([163]\) gives

\[
\omega^2 = \frac{2\pi^2}{2m^2}.
\]
which is just the well known plasma frequency. In the zero temperature case, to the next to leading order approximation, the longitudinal dispersion relation is

\[ \omega^2 = \omega_p^2 (1 + \frac{\omega_p^2}{4m^2} - \frac{3q^2}{10m^2} - \frac{\alpha}{9\pi}) + \left( \frac{3q^2}{5m^2} - \frac{\omega_p^2}{2m^2} \right) |k|^2 + \frac{|k|^4}{4m^2}. \]  

(165)

We explain each term as follows. The last three terms in the first parentheses are the corrections to the plasma frequency. They correspond to the corrections to the number, mass and charge of plasma particles, respectively. The term of \( \omega_p^2/4m^2 \) is explained as the frequency due to the charged particles production induced by plasma oscillation. Comparing Eq. (164) with the formula

\[ \omega_p^2 = \frac{e^2 n}{m} = \omega_p^2 (1 + \frac{\omega_p^2}{4m^2}), \]  

(166)

with \( n = n_0 + \delta n \), we have the increased particle number density as

\[ \delta n = \frac{n_0 \omega_p^2}{4m^2} = \frac{e^2 n_0^2}{4m^2}. \]  

(167)

The proportion of produced particles is shown in Fig. 1. We restrict the plasma density below \( 10^{30}\) cm\(^{-3} \). It satisfies \( \omega_p \ll m \) and the non-relativistic approximation that most particles with velocities well below the light speed. The result indicates more particles produced as the increase of the plasma oscillation frequency. They are just electron-positron pairs in the electric neutrality case. The new positrons constitute at least 0.1% charged particles at the plasma density \( 10^{30}\) cm\(^{-3} \). It is reasonably expected an increase in proportion of the produced pairs with the density increasing if we extend the calculation to the relativistic scope. Further studies on this subject is interesting. In fact, the crust of neutron star with typical density \( 10^9\) g·cm\(^{-3} \) \( \sim 10^{11}\) g·cm\(^{-3} \) indicates that the number density of degenerate electrons is about \( 10^{25}\) cm\(^{-3} \) \( \sim 10^{31}\) cm\(^{-3} \) in the fully ionized case (It is believed that the crust is constituted by iron elements). In addition, white dwarfs also have extremely high density that below the regime of neutron dripp, i.e., the mass density less than \( 10^{11}\) g·cm\(^{-3} \). Subsequently, amount of electron positron pair will be produced by plasma oscillation in those extreme astrophysical environments. The effect has not been reported before to our knowledge. It is not found even in the QFT method study of scalar QED plasmas by Shi, et al., for they evaluate the Green function from a fixed particle number wave function. We point out that there is no similar term in the usual quantum many body studies where one need to evaluate the poles of the total propagation \( D_{\mu\nu}(k) \) of photon. It can be expressed as the sum of a set of ring diagrams

\[ D_{\mu\nu}(k) = \mu \nu \nu \nu 0 \nu + \mu \nu \nu \nu \nu + \cdots. \]  

(168)

Constant Dirac particle number is assumed in the calculation of free relativistic many-body Green’s function. Eq. (168) indicates that all the virtual electron-positron pair will be annihilated. Especially, there is no particle produced on shell. However, considering the high energy effect, a complete diagram analysis must include various end states containing fermions produced which is ignored in the Eq. (168). It is difficult to sum over those complex diagrams. Alternatively, in the background field method in the present paper, the particle production can be considered as being included in the quantum fluctuation. It gives no constrain on the total particle number of the system in evaluating the Green function of fluctuation fermion. The term of \(-3q^2/10m^2\) comes from the relativistic correction to mass. It can also be derived by substituting the mass \( m \) in \( \omega_p^2 = e^2 n_0/m \) by \( \langle 1 - v_T^2 \rangle \approx m(1 - \frac{v_T^2}{2}) \approx \sqrt{1 - v_T^2} \) if \( v_T \ll 1 \). Here, \( v_T \) is the average velocity square. For the highly degenerate limit case at \( T = 0 \), \( v_T^2 = 3q^2/5m^2 \). It indicates that the plasma frequency becomes \( \omega_p^2 \approx (1 - 3q^2/10m^2)e^2 n_0/m \) if we just consider correction due to the mass increasing. There is no such term in the scalar QED plasmas, for the average velocity of bosons at zero temperature is vanished. The appearance of \(-\alpha/9\pi\) is due to the vacuum polarization which leads to the effect of charge screening. It subsequently reduces the plasma frequency. The term of \( 3q^2|k|^2/5m^2 \) in the second parentheses of Eq. (165) is common seen in the dispersion relation of non-relativistic degenerate plasmas. The term of \(-\omega_p^2|k|^2/2m^2 \) is also found in the case of scalar QED plasmas. The minus sign indicates a local minimum in the dispersion relation. This phenomenon is referred to by the Ref. [31] as negative dispersion due to the finite speed of light, and is attributed to a retardation effect in Bose plasmas. This explain can be extended to Fermi plasmas also. Neglect the effects of relativistic and vacuum polarization corrections, Eq. (165) reduces to the well-known longitudinal excitation spectrum for non-relativistic degenerate plasmas, at zero temperature, as

\[ \omega^2 = \omega_p^2 + \frac{3q^2}{5m^2}|k|^2 + \frac{|k|^4}{4m^2}. \]  

(169)
For the $T \neq 0$ case, as shown in the Eq. \ref{163}, the plasma frequency decrease with the increasing temperature. Similar trend is found in the classical plasmas \cite{40}.

2. Transverse wave dispersion relation

Another solution of Eq.\ref{162} gives

$$\omega^2 = |k|^2. \quad (170)$$

It is just the dispersion relation of free propagation of the fluctuation vector boson. As mentioned in the previous section, the physical process can also be understood as a scattering process of the fluctuation vector boson by the effective medium. From the view point of scattering theory, statistical speaking, in addition to the scattered fluctuation vector bosons, there are some other incident ones simply miss the target with the dispersion relation Eq. \ref{170}. However, it is just a purely mathematical scheme to divide the photons into background field and fluctuation vector bosons. One can not separate them physically. The photon do not propagate below the plasma frequency $\omega_p$. If we focus on the dispersion relation of photon, the plasma frequency can be considered as the rest mass of photon. It is first pointed out by Anderson \cite{41}. The excess energy is transferred to the fluctuation vector boson, and equals to the momentum of this boson in the reference frame of mass point of charged particle system (see the paragraph below Eq. \ref{109}). Then according to mass-energy relation,

$$\omega^2 = \omega_p^2 + |k|^2. \quad (171)$$

Here, $\omega$ is denoted as the frequency of photon. If the relativistic quantum and statistical effects are included, according to Eq.\ref{163} by setting $|k| = 0$, the transverse wave dispersion relation can be derived as

$$\omega^2 = \omega_p^2 + \frac{\omega^2}{4m^2}(1 - \frac{3q_F^2}{10m^2} - \frac{\pi^2 m^2 T^2}{2q_F^4})$$

$$- \frac{3q_F^2}{10m^2} - \frac{\omega_p^2}{2m^2} - \frac{\pi^2 m^2 k_B T^2}{2q_F^4} + |k|^2. \quad (172)$$

Here, the quantity $\omega_p^2[\cdots]$ is the square of cut-off frequency.

3. Acoustic wave dispersion relation

The solution of Eq.\ref{161} in the condition of $mT \ll q_F^2$, i.e. $\mu \gg T$, is

$$\omega^2 = \frac{3q_F^2}{5m^2}|k|^2 + \frac{|k|^4}{4m^2}. \quad (173)$$

Here $\sqrt{3q_F}/\sqrt{5m} = \sqrt{3}v_F/\sqrt{5}$ is not the acoustic velocity of ideal degenerate Fermi particles, i.e., $v_F/\sqrt{3}$. From Landau \cite{42}, it indicates that Eq.\ref{173} is just the dispersion relation of zero sound. In fact, the condition that the temperature is much less than Fermi energy indicates that collisions are unimportant, and thermodynamic equilibrium is not established in each volume element in the time scale of $1/\omega$. The ordinary hydrodynamic sound wave doesn’t propagate. $|k|^4/4m^2$ is the second order correction to the dispersion relation of the zero sound. The velocity of zero sound is the same as that of the longitudinal wave. It is interesting that similar effect is found in the classical electron-positron plasma. The velocity is the same for the hydrodynamic sound wave and the longitudinal wave in that plasma \cite{43}.

B. High energy limit

Inserting the Eq.\ref{157} into the Eq.\ref{162}, in the high energy limit $|k| \gg m$, we get

$$\omega^2 \to |k|^2 + \frac{4om^2}{3\pi} + \frac{m\omega^2}{2\mu} + i\infty. \quad (174)$$

The infinite real part of the frequency $\omega$ is due to the massive positive and negative Dirac particles creation in the high energy limit. In addition, the infinite imaginary part of $\omega$ indicates meaningless blowing up wave and rapid decay wave. This decay can be explained as the high energy fluctuation vector boson transforming to spinor particles and thereafter stops propagating. Both of these phenomena can be also found in the scalar QED plasmas \cite{27}.

V. SUMMERY AND REMARKS

We develop a fully quantized relativistic theory of spinor QED plasmas at finite temperature by introducing the TFD method. By decomposing the Dirac field and electromagnetic potential into the nontrivial background fields and the quantum fluctuation fields, then by integrating over the fluctuation Dirac field, we derive an effective temperature field theory of the fluctuation vector boson. The calculation of the wave dispersion relations is converted to that of the poles of dressed propagator of this boson. The background field of plasma charged particles is essential for the present study. We develop the classical limit method to derive it. To the order of $e^2$, the dispersion relations of the longitudinal, transverse and sound waves are derived in the both of low-energy and high-energy limits. The lowest order approximation gives the standard plasma oscillation expression, i.e., Eq.\ref{163}. The second order approximation gives the well-known results of longitudinal and transverse wave dispersions, i.e., Eq.\ref{165} and \ref{171}, for non-relativistic degenerate plasmas. In addition, the zero sound in the electron-positron plasma is first reported. Further corrections appear in the next order approximation. As shown in Eq.\ref{163} and Eq.\ref{172}, these corrections include the mass increase due to the nonzero velocities of plasma particles, effective charge decrease due to the vacuum polarization, negative
dispersion relation due to the finite light velocity, and temperature influence on the system. Besides, we find the effect of particle production due to the plasma oscillation which has not been reported before to our knowledge. In the high-energy limit, the wave propagating stops caused by the strong vacuum polarization.

Next, we give the following remarks: (1) Although we focus on the small amplitude vibrations, many typical strong field QED plasma processes can be treated as well in the framework of our theory. More precisely, in the high intensity radiation case, two typical characteristic quantities are the classical parameter

\[ \xi = \frac{|e| \sqrt{-A^2}}{m} \]  

and quantum parameter

\[ \chi = \frac{|e| \sqrt{(\tau^\mu \tau^\nu)^2}}{m^3} \]  

The presence of intense laser field corresponding to \( \xi \gg 1 \) and \( \chi \geq 1 \) stimulates QED cascade [21-24]. The perturbation method, i.e., Eq. (111) and Eq. (122), being used in this paper can be even extended to such cases where the the multi photon processes are present. The key observation is that one can not just replace \( N_0(t, p) \), \( N_{UL}(t, p) \), \( N_{OR}(t, p) \), \( j_{\mu L}^0(t, p) \), \( j_{\mu R}^0(t, p) \) and \( j_{\nu L}^0(t, p) \) by the static approximate average values just as we did in the Sec. III. The strong variation of the corresponding quantities must be taken into account. It is the core idea of the nontrivial “Furry picture” that the effects due to the strong radiation can be included in the background fields \( A_\mu \) and \( \psi_0 \). In the case of \( eA^\mu/m \gtrsim 1 \), \( eA^\mu \) can not be regarded just as small perturbation. It looks like that one can replace the free Dirac propagator \( i/(p - m + i\epsilon) \) by \( S_F^\mu(p) \), and \( i/(p - m - i\epsilon) \) by \( S_F^\mu(p) \) in the Eq. (70). However, the Schwinger pair [11] appears in this case which is a non-perturbative phenomenon that it can not be treated in our perturbative scheme. It restricts the theory with the electromagnetic field below to the critical field strength \( F_{cr}^\mu \sim e^2/m \sim 1.3 \times 10^{16} \text{V/cm} \sim 4.4 \times 10^{13} \text{G} \). Nonperturbative field theory is needed for the super intense radiation cases [12]. (2) Comparing to Yuan Shi, et al. [27], our classical limit method provides a scheme for calculating the many body system in nonzero temperature case. In addition, it gives conceptual clarity that each term involving the background field corresponds to a classical physical quantity, as shown in Eq. (111)-Eq. (119). (3) We assume that the background plasma charged particles obey Fermi-Dirac distribution in the present paper. However, it can be extended to the non-ideal case. In fact, considering Eq. (110), if we regard \( E_D \) as the energy of non-ideal system and replace the temperature dependent propagator for ideal Dirac particles by that for non-ideal system, the Eq. (111) will be the background polarization tensor for uniform non-ideal relativistic quantum electro-positron plasmas. Further efforts can be devoted into this subject which may exhibits majority interesting phenomenons. (4) Unlike the usual quantum field theory for many body systems, our theory could deal with the nonuniform plasmas. In fact, we make the assumptions of Eq. (129), Eq. (130), Eq. (131), Eq. (132), Eq. (133) and Eq. (140) which mean the equal numbers of electrons and positrons, and of left and right handed charged Dirac particles. In addition, it also indicates equality of left and right handed current. It makes the discussion more easily. However, those restrictions are not necessary. It indicates that the nature of spinor QED plasma also depends on those additional factors including distributions of spin and chirality. Studies on these subjects will reveal more interesting effects. (5) We want to point out that the theory works not only for the electron-positron pair plasma. The extension to other QED plasmas is straightforward. Moreover, the appearing of other particles such as neutron [10], muon, etc. in plasmas involve modifying Lagrangian. However the theoretic method would be still similar. (6) In the end, we claim again that as emphasized by Yuan Shi, et al. [27], it is necessary to develop a fully quantized theory for plasmas by using QFT. Quantum many body effects can be included without any confusion. Powerful relativistic quantum many body technologies can be used in plasma study.

VI. ACKNOWLEDGEMENT

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