Stable CMC and index one minimal surfaces in conformally flat manifolds

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Abstract Let \( M \) be a Riemannian 3-manifold of nonnegative Ricci curvature, \( \text{Ric} \geq 0 \). We suppose that \( M \) is conformally flat and simply connected or more generally that it admits a conformal immersion into the standard 3-sphere. Let \( \Sigma \) be a compact connected and orientable surface immersed in \( M \) which is a stable constant mean curvature (CMC) surface or an index one minimal surface. We prove that \( \Sigma \) is homeomorphic either to a sphere or to a torus. Moreover, in case \( \Sigma \) is homeomorphic to a torus, then it is embedded, minimal, conformal to a flat square torus and \( \text{Ric}(N) = 0 \) where \( N \) is a unit field normal to \( \Sigma \). The result is sharp, we can perturb the standard metric on the 3-sphere in its conformal class to obtain metrics of nonnegative Ricci curvature admitting minimal tori which are stable as CMC surfaces.

As a consequence, in any 3-sphere of positive Ricci curvature which is conformally flat, the isoperimetric domains are topologically 3-balls.

Keywords Constant mean curvature surfaces, minimal surfaces, stability, isoperimetric problem.

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1. Introduction

Minimal surfaces and constant mean curvature surfaces (CMC) in Riemannian 3-manifolds are both critical points of the area functional. The former are critical for compactly supported variations and the latter for variations which furthermore keep the enclosed volume constant. They are called stable if they minimize the area up to second order for those variations. In the CMC case, the terminologies volume preserving stable and weakly stable are sometimes used to emphasize the distinction with the minimal case. Stable minimal surfaces are of fundamental importance in the general theory of minimal surfaces and compact stable CMC surfaces are equally important in studying the isoperimetric problem in Riemannian 3-manifolds since the boundary of an isoperimetric region is a stable CMC.

From the variational view point, the next interesting class of surfaces to consider in the minimal case are those having Morse index one. Actually, these surfaces have

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received much attention, Pitts [7] and Pitts and Rubinstein [8] have, for instance, constructed many examples using minimax methods.

A positivity assumption on the curvature of the ambient 3-manifold makes the stability condition more tractable and one can indeed control the topology of stable CMC and index one minimal surfaces. Actually, improving previous results, Ros [13] proved the sharp result that compact orientable stable CMC surfaces and compact orientable index one minimal surfaces in orientable 3-manifolds with nonnegative Ricci curvature have genus ≤ 3. Note that, under this assumption, it is an easy fact that a compact orientable minimal surface is stable if and only if it is totally geodesic and the Ricci curvature evaluated on a unit normal to it vanishes. Let us mention also the works by Ritoré-Ros [10] and Ritoré [9] about index one minimal surfaces in flat 3-manifolds.

In this paper, we consider these questions in 3-manifolds of nonnegative Ricci curvature which are conformally flat. More precisely, we consider 3-manifolds with nonnegative Ricci curvature which admit a conformal immersion into the standard 3-sphere \( S^3 \). Simply connected and conformally flat 3-manifolds are examples of manifolds satisfying the latter condition. Indeed they can be conformally immersed into \( S^3 \) by means of a developing map. We will prove that a connected compact and orientable surface in this kind of manifolds which is a stable CMC or a minimal surface of index one is topologically a sphere or a torus. Moreover, if such a surface is a torus, then it has to be embedded, minimal, conformal to a flat square torus and \( \text{Ric}(N) = 0 \) where \( N \) is a unit field normal to the torus and \( \text{Ric} \) denotes the Ricci curvature of the ambient manifold. The result is sharp, we are able to perturb the standard metric on \( S^3 \) in its conformal class to exhibit examples of metrics of nonnegative Ricci curvature admitting minimal tori of index one which are stable as CMC surfaces. An interesting corollary is that in a 3-sphere of positive Ricci curvature which is conformally flat, the isoperimetric regions are topological 3-balls. This answers partially a conjecture of Ros [11, 12].

2. Preliminaries

Let \((M, \langle \cdot, \cdot \rangle)\) be an orientable Riemannian 3-manifold and \( \Sigma \) an orientable immersed compact CMC surface without boundary in \( M \). We note that when the (constant) mean curvature of \( \Sigma \) is not zero then \( \Sigma \) is automatically orientable since its mean curvature field is a non vanishing global normal. Call \( N \) a unit field normal to \( \Sigma \). We consider on \( \Sigma \) the following quadratic form:

\[
Q(u, v) = \int_{\Sigma} \langle \nabla u, \nabla v \rangle - (|\sigma|^2 + \text{Ric}(N))uv \, dA = - \int_{\Sigma} uLv \, dA, \quad u, v \in C^\infty(\Sigma),
\]

where \( \nabla u \) stands for the gradient of \( u \), \( |\sigma|^2 \) is the square of the norm of second fundamental form \( \sigma \) of the immersion and \( \text{Ric}(N) \) denotes the Ricci curvature of \( M \) evaluated on the field \( N \). The linear operator \( L = \Delta + |\sigma|^2 + \text{Ric}(N) \) is the Jacobi operator of the surface, \( \Delta \) being the Laplacian on \( \Sigma \).

The stability condition in the minimal case means that the quadratic form \( Q \) is nonnegative on \( C^\infty(\Sigma) \) and in the CMC case it means that:
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(2.1) \[ Q(u, u) \geq 0, \quad \text{for any } u \in C^\infty(\Sigma) \quad \text{satisfying } \int_\Sigma u \, dA = 0. \]

See [1] for the details.

Let \( \Sigma \) be a minimal surface as above. Its index is by definition the number of negative eigenvalues, counted with multiplicities, of its Jacobi operator. So \( \Sigma \) is stable if and only if it has index zero. Taking as a test function a constant (non-zero) function, we see immediately that when the Ricci curvature is nonnegative, a minimal surface \( \Sigma \) as above, is stable as a minimal surface if and only if it is totally geodesic and \( \text{Ric}(N) = 0 \).

We will consider 3-manifolds which have nonnegative Ricci curvature and admit a conformal immersion into \( S^3 \). This includes the simply connected conformally flat manifolds of nonnegative Ricci curvature. Recall that a Riemannian manifold \( (M, g) \) of dimension \( n \) is said to be conformally flat (or locally conformally flat) if it admits a coordinate covering \( \{ U_\alpha, \phi_\alpha \} \) with \( \phi_\alpha : (U_\alpha, g) \to S^n \) conformal. When \( n \geq 3 \) and \( M \) is simply connected, it is a classical fact that this implies the existence of a conformal immersion \( (M, g) \to S^n \), the developing map, which is unique up to Möbius transformations. This is shown using Liouville’s theorem and a standard monodromy argument. We emphasize that do not assume our manifolds to be complete. Assuming completeness, Zhu [15] proved that a complete simply connected conformally flat \( n \)-manifold of nonnegative Ricci curvature is either conformally equivalent to \( R^n \) or \( S^n \), or is isometric to \( S^{n-1} \times R \). It is clear that small perturbations of the (standard) metric of \( S^n \) in its conformal class produce metrics with positive Ricci curvature. Also, Zhu [15] exhibited (rotationally symmetric) complete metrics on \( R^n \) of nonnegative Ricci curvature which are conformally flat and non flat.

In the sequel we denote by \( S^n \) the standard \( n \)-dimensional sphere. We write \( S^n \subset R^{n+1} \) to mean we identify it with the unit sphere centered at the origin in the Euclidean space \( R^{n+1} \).

To get information about stable CMC (resp. index one minimal) surfaces, one needs to choose suitable test functions and evaluate the quadratic form \( Q \) on them. A very useful source of test functions is the following

**Lemma 2.1.** [4, 6] Let \( \Sigma \) be a compact Riemannian surface admitting a conformal immersion \( \Psi : \Sigma \to S^n \) and \( \rho : \Sigma \to R \) a smooth positive function. Then there exists a conformal diffeomorphism \( T : S^n \to S^n \) such that the vector valued map \( T \circ \Psi : \Sigma \to S^n \subset R^{n+1} \) has mean value zero

\[ \int_\Sigma (T \circ \Psi) \rho \, dA = 0, \]

where \( dA \) is the volume element on \( \Sigma \).

Lemma 2.1 was widely used by several authors, in the context of stability, for meromorphic maps \( \Sigma \to S^3 \) of controlled energy. Our idea here is to use it for conformal maps into \( S^3 \) taking advantage of the existence of a conformal immersion of the ambient manifold into \( S^3 \) and utilizing the following known fact

**Proposition 2.2.** Let \( M \) be an orientable Riemannian 3-manifold and \( \Sigma \) a compact orientable surface without boundary immersed in \( M \). Denote by \( K_s \) the sectional
curvature of $M$ evaluated on the tangent plane to $\Sigma$, by $H$ the mean curvature of $\Sigma$ and by $dA$ its area element. Then the quantity

$$ (2.2) \quad \int_{\Sigma} (H^2 + K_s) dA $$

is invariant under conformal changes of the metric on $M$.

Proof. (sketch) Let $\kappa_1$ and $\kappa_2$ denote the principal curvatures of $\Sigma$. It is straightforward to check that the density $(\kappa_1 - \kappa_2)^2 dA$ is invariant under conformal changes of the metric on $M$. Integrating over $\Sigma$ and using Gauss equation and Gauss-Bonnet formula gives the invariance of (2.2). \hfill \Box

3. The result

We now state our main result. It improves a weaker one we obtained previously ([14], Theorem 3.3 (i)).

Theorem 3.1. Let $M$ be a Riemannian 3-manifold of nonnegative Ricci curvature, $\text{Ric} \geq 0$. Suppose $M$ is simply connected and conformally flat or more generally that it admits a conformal immersion into $\mathbb{S}^3$. Let $\Sigma$ be a compact orientable surface without boundary immersed in $M$ with unit normal $N$. Suppose $\Sigma$ is a stable CMC or an index one minimal surface. Then

(i) If $\Sigma$ is not connected then it is a finite union of totally geodesic surfaces and $\text{Ric} (N) = 0$. In this case $\Sigma$ is a stable minimal surface.

(ii) If $\Sigma$ is connected then it is homeomorphic to either a sphere or to a torus which it is minimal, embedded, has the conformal structure of a flat square torus and $\text{Ric} (N) = 0$.

Before proving the theorem, we note that we make no assumption about the completeness of $M$. Also, since by assumption $M$ immerses into $\mathbb{S}^3$, it is orientable and so the existence of a global unit normal $N$ to the surface $\Sigma$ is equivalent to its orientability.

Proof. We first treat the CMC case. The proof in the minimal case is similar and will be sketched below.

If $\Sigma$ is not connected then one can take as a test function a function which is a non zero constant on each connected component and with mean value zero and obtain immediately the conclusion of (i).

To prove (ii), denote by $X: \Sigma \to M$ the immersion of the CMC surface $\Sigma$ and by $F: M \to \mathbb{S}^3 \subset \mathbb{R}^4$ a conformal immersion. Set $\Psi = F \circ X$. By Lemma 2.1 applied to $\Psi$ with $\rho \equiv 1$, we can assume that

$$ \int_\Sigma \Psi \, dA = 0. $$

We therefore can use the coordinate functions of $\Psi$ as test functions. We thus have

$$ (3.1) \quad 0 \leq Q(\Psi_i, \Psi_i) = \int_\Sigma |\nabla \Psi_i|^2 - (|\sigma|^2 + \text{Ric} (N)) \Psi_i^2, \quad i = 1, \ldots, 4.$$
Summing up these inequalities and taking into account that \( \Psi \) takes its values in the unit sphere, we get:

\[
\int_{\Sigma} (|\sigma|^2 + \text{Ric}(N)) \, dA \leq \int_{\Sigma} |\nabla \Psi|^2 \, dA.
\]

Denote by \( K \) and \( K_s \) respectively the intrinsic curvature of \( \Sigma \) and the sectional curvature of \( M \) evaluated on the tangent plane to \( \Sigma \). By Gauss equation we have

\[
|\sigma|^2 = 4H^2 + 2K_s - 2K.
\]

So we rewrite (3.2) as follows:

\[
\int_{\Sigma} (4H^2 + 2K_s - 2K + \text{Ric}(N)) \, dA \leq \int_{\Sigma} |\nabla \Psi|^2 \, dA.
\]

Call \( g \) the standard metric on \( S^3 \). As \( \Psi \) is conformal, \( \int_{\Sigma} |\nabla \Psi|^2 \, dA = \int_{\Sigma} 2 \text{Jacobian}(\Psi) \, dA = 2 \text{area}(\Sigma, \Psi^* g) \). Here \( \text{area}(\Sigma, \Psi^* g) \) denotes the area of \( \Sigma \) for the metric \( \Psi^* g \) induced by the immersion \( \Psi \).

Using Gauss-Bonnet formula, we transform inequality (3.3) into:

\[
\int_{\Sigma} (2H^2 + \text{Ric}(N)) \, dA + 2 \int_{\Sigma} (H^2 + K_s) \, dA - 4\pi \chi(\Sigma) \leq 2 \text{area}(\Sigma, \Psi^* g).
\]

Denote by \( \bar{H} \) and \( d \bar{A} \), respectively the mean curvature of the immersion \( \Psi \) and the area element induced on \( \Sigma \). From Proposition 2.2, one has:

\[
\text{area}(\Sigma, \Psi^* g) \leq \int_{\Sigma} (\bar{H}^2 + 1) \, d\bar{A} = \int_{\Sigma} (H^2 + K_s) \, dA
\]

Putting this into (3.3), one obtains:

\[
\int_{\Sigma} (2H^2 + \text{Ric}(N)) \, dA \leq 4\pi \chi(\Sigma)
\]

Since \( M \) has nonnegative Ricci curvature, this shows \( \chi(\Sigma) \geq 0 \), that is, \( \Sigma \) is homeomorphic either to a sphere or to a torus.

Suppose \( \Sigma \) is topologically a torus, then the equality is reached in (3.6) and so \( H = 0 \) and \( \text{Ric}(N) = 0 \). Furthermore the equality is also reached in the intermediate inequalities and in particular in (3.5). So \( \bar{H} = 0 \), that is, \( \Psi \) is a conformal minimal immersion into \( S^3 \). Again, as equality is reached in (3.2), we have:

\[
\int_{\Sigma} |\sigma|^2 \, dA = \int_{\Sigma} |\nabla \Psi|^2 \, dA = 2 \text{area}(\Sigma, \Psi^* g)
\]

We now prove \( \Psi \) is an embedding, this will show \( X \) is an embedding too. As \( \Sigma \) is a torus, there exists a meromorphic map \( \Sigma \rightarrow S^2 \) of degree 2 which we can take, again by Lemma 2.1, such that \( \int_{\Sigma} \phi \, dA = 0 \). Taking as test functions the coordinate functions \( \phi_i \), \( i = 1, 2, 3 \), we have:

\[
\int_{\Sigma} |\nabla \phi_i|^2 \, dA \geq \int_{\Sigma} |\sigma|^2 \phi_i^2 \, dA, \quad i = 1, 2, 3.
\]

Summing up these inequalities and taking into account (3.7), we get:

\[
2 \text{area}(\Sigma, \Psi^* g) = \int_{\Sigma} |\sigma|^2 \, dA \leq \int_{\Sigma} |\nabla \phi|^2 \, dA = 16\pi.
\]

Suppose \( \Psi \) is not an embedding, then Li and Yau (6) have shown that in this case \( \text{area}(\Sigma, \Psi^* g) \geq 8\pi \). So equality is achieved in (3.9) and hence also in (3.8).
So the holomorphic map $\phi$ satisfies $Q(\phi_i, \phi_i) = 0$, for $i = 1, 2, 3$. As $\Sigma$ is stable, for any $v \in C^\infty(\Sigma)$ satisfying $\int_\Sigma v = 0$ and any $t \in \mathbb{R}$ we have $Q(\phi_i + tv, \phi_i + tv) \geq 0$ and so $Q(\phi_i, v) = 0$. It follows that each of the functions $\phi_i$ satisfies the equation

$$\Delta \phi_i + |\sigma|^2 \phi_i = c_i,$$

for some real constant $c_i$, $i = 1, 2, 3$. So $\phi$ satisfies an equation of the type:

$$\Delta \phi + |\sigma|^2 \phi = \vec{c}$$

with $\vec{c}$ a constant vector in $\mathbb{R}^3$.

On the other hand, since $\phi : \Sigma \rightarrow S^2$ is holomorphic it is harmonic and therefore satisfies the equation:

$$\Delta \phi + |\nabla \phi|^2 \phi = 0.$$  

As $\phi$ takes its values in the sphere $S^2$ and is non constant, it follows easily from (3.10) and (3.11) that necessarily $\vec{c} = 0$ and $|\sigma|^2 = |\nabla \phi|^2$. So the Jacobi operator of $\Sigma$ writes as $L = \Delta + |\nabla \phi|^2$ and the stability assumption implies that $L$ has only one negative eigenvalue. Otherwise said the holomorphic map $\phi$ has index one. However such maps do not exist on tori (cf. [13]), a contradiction. Therefore $\Psi$ is a conformal minimal embedding of a torus in $S^3$. By the recent solution by Brendle [3] to Lawson’s conjecture, we know $\Psi(\Sigma)$ is congruent to the Clifford torus. This shows $\Sigma$ is conformal to a square flat torus.

Assume now $\Sigma$ is an index one minimal surface. Let $\varphi_1 \in C^\infty(\Sigma)$ be a non-trivial first eigenfunction of the Jacobi operator $L$. The index one hypothesis means that

$$Q(u, u) \geq 0, \text{ for any } u \in C^\infty(\Sigma) \text{ satisfying } \int_\Sigma u \varphi_1 \, dA = 0.$$  

It is well known that $\varphi_1$ has no zeros and can thus be taken positive. We then apply, as above, Lemma 2.1 to the map $\Psi$ with $\rho = \varphi_1$. The rest of the proof is similar, we omit the details.

□

Remark 3.2. Possibility (i) in Theorem 3.1 happens for instance in the Riemannian product $S^2 \times \mathbb{R}$ where a finite union of slices $S^2 \times \{t\}$ is a stable minimal surface.

We end this section with an application to the isoperimetric problem. Let $M$ be a compact Riemannian 3-manifold. From Geometric Measure Theory results, one knows that for any $0 < V < \text{Vol}(M)$ there exists a regular domain $\Omega \subset M$ of volume $V$ such that $\partial \Omega$ minimizes the area among all regular domains in $M$ of volume $V$. Moreover $\partial \Omega$ is a (possibly disconnected) regular CMC surface and is stable. Solving the isoperimetric problem in $M$ consists in determining the isoperimetric regions. This is known in very few cases. See [12] for an account on the subject. Ros [11, 12] conjectured that the isoperimetric regions in the 3-sphere endowed with a metric of positive Ricci curvature are topologically 3-balls. As a consequence of Theorem 3.1 we have this positive partial answer to Ros’ conjecture

Corollary 3.3. Let $M$ be the 3-sphere endowed with a metric of positive Ricci curvature such that $M$ is conformally flat. Then the isoperimetric regions in $M$ are topologically 3-balls.
Note that by the existence of the developing map we recalled in Sect. 2, the conformal flatness of $M$ means that $M$ is conformally equivalent to the standard sphere $S^3$ (Kuiper’s theorem).

4. Examples with stable tori

We show in this section that Theorem 3.1 is optimal. We will perturb conformally the standard metric on the 3-sphere to obtain metrics of nonnegative Ricci curvature having minimal tori which are stable as CMC surfaces (and have index one as minimal surfaces). We will do this keeping the Clifford torus minimal and making it stable for the new metrics.

We start with the following parametrization (see [5]) of the unit sphere centered at the origin $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$

$$\Phi : \mathbb{R}^2 \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \rightarrow S^3$$

$$\Phi(\theta, \phi, t) = \sin \left(t + \frac{\pi}{4}\right) \left(\cos(\sqrt{2} \theta), \sin(\sqrt{2} \theta), 0, 0\right) + \cos \left(t + \frac{\pi}{4}\right) \left(0, 0, \cos(\sqrt{2} \phi), \sin(\sqrt{2} \phi)\right)$$

This parametrization covers the unit sphere with two orthogonal circles removed, namely $S^3 \setminus \{(z_1, z_2) \in \mathbb{C}^2, \quad z_1 = 0 \quad \text{or} \quad z_2 = 0\}$.

Note that $\Phi(., ., 0) : \mathbb{R}^2 \rightarrow S^3$ parametrizes the Clifford torus

$$T = \{(z_1, z_2) \in \mathbb{C}^2 : \quad |z_1|^2 = |z_2|^2 = \frac{1}{\sqrt{2}}\}.$$ and, for each $t \in (\frac{-\pi}{4}, \frac{\pi}{4})$, the map $\Phi(., ., t)$ parametrizes the surface parallel to $T$ at signed distance $t$.

The expression of the standard spherical metric in these (local) coordinates is

$$g = (1 + \sin(2t))d\theta^2 + (1 - \sin(2t))d\phi^2 + dt^2.$$ We modify conformally the metric $g$ into $\bar{g} = e^{2\varphi}g$, where $\varphi$ is a smooth function of $t$ alone to be determined. The Ricci curvature tensor $\overline{\text{Ric}}$ of $\bar{g}$ is related to the Ricci tensor, $\text{Ric}$, of $g$ as follows (cf. [2] p. 59, the sign convention for the Laplacian is opposite to ours)

$$\overline{\text{Ric}} = \text{Ric} - (\nabla d\varphi - d\varphi \otimes d\varphi) - (\Delta \varphi + |\nabla \varphi|^2)g$$

Straightforward computations, taking into account that $\text{Ric} = 2g$, give

$$\begin{aligned}
&\overline{\text{Ric}} \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}\right) = 2(1 - \varphi'') + 4 \frac{\sin(2t)}{\cos^2(2t)} \varphi' \\
&\overline{\text{Ric}} \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right) = (1 + \sin(2t))^2 \left(2 - \varphi'' + 4 \frac{\sin(2t)}{\cos^2(2t)} \varphi' - (\varphi')^2\right) - \varphi' \cos(2t) \\
&\overline{\text{Ric}} \left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\right) = (1 - \sin(2t))^2 \left(2 - \varphi'' + 4 \frac{\sin(2t)}{\cos^2(2t)} \varphi' - (\varphi')^2\right) + \varphi' \cos(2t) \\
&\overline{\text{Ric}} \left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}\right) = \overline{\text{Ric}} \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right) = \overline{\text{Ric}} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\right) = \overline{\text{Ric}} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \phi}\right) = 0.
\end{aligned}$$
Let now $r \in (0, \frac{\pi}{8})$ and take a smooth even function $\zeta : (-\frac{\pi}{4}, \frac{\pi}{4}) \rightarrow \mathbb{R}$ such that

(i) $\zeta(0) = 1$, $\zeta$ is decreasing on $(0, r)$ and $\zeta(r) = 0$
(ii) $\zeta$ is negative on $(r, 2r)$
(iii) $\zeta \equiv 0$ on $[2r, \frac{\pi}{4})$
(iv) $\int_0^{2r} \zeta(s) \, ds = 0$.

We set $\varphi(t) = \int_0^t (\int_0^s \zeta(x) \, dx) \, ds$.

Denote by $\kappa_1, \kappa_2$ the principal curvatures of a surface with unit normal $N$ for the metric $g$. Its principal curvatures for the metric $\bar{g}$ are given by

$$\bar{\kappa}_i = e^{-2\varphi}(\kappa_i + \langle \nabla \varphi, N \rangle), \quad i = 1, 2$$

For the Clifford torus a unit normal is $N = \frac{\partial}{\partial t}|_{t=0}$ and so

$$\bar{\kappa}_i = e^{-2\varphi(0)}(\kappa_i + \varphi'(0)), \quad i = 1, 2$$

We the choice of $\varphi$ as above, we have $\varphi(0) = 0$ so that the metric on the Clifford torus $\mathbb{T}$ is unchanged and $\bar{N} = N$ is a unit normal to $\mathbb{T}$ for the metric $\bar{g}$. As $\varphi'(0) = 0$ we have $\bar{\kappa}_i = \kappa_i$, $i = 1, 2$. The Clifford torus $\mathbb{T}$ is therefore still minimal for the metric $\bar{g}$ and the square of the norm of its second fundamental form $\bar{\sigma}$ is $|\bar{\sigma}|^2 = |\sigma|^2 = 2$. Since $\varphi''(0) = 1$, we have $\text{Ric}(\bar{N}, N) = 0$. So the Jacobi operator of $\mathbb{T}$ for the new metric is $L = \Delta + 2$. For the Clifford torus, the first non zero eigenvalue of $\Delta$ is $\lambda_1 = 2$. It follows that $\mathbb{T}$ is stable as a CMC and has index one as a minimal surface for the metric $\bar{g}$.

For $|t| \geq 2r$, we have $\bar{g} = C \, g$ where $C = e^{2\varphi(2r)}$ is a constant and so $\bar{g}$ extends to the whole of $S^3$. It remains to show that we can choose $r$ so that with $\varphi$ as above the Ricci curvature of $\bar{g}$ is nonnegative.

Note that $\varphi'(t) = \int_0^t (\int_0^s \zeta(s) \, ds) \geq 0$ on $[0, \frac{\pi}{8})$ (resp. $\leq 0$ on $(-\frac{\pi}{8}, 0]$) and so, for any $t \in (-\frac{\pi}{16}, \frac{\pi}{16})$, $\sin(2t)\varphi'(t) \geq 0$. Observe also that, by the choice of $\varphi$, we have $|\varphi'| \leq r$ and $\varphi'' \leq 1$ on $(-\frac{\pi}{16}, \frac{\pi}{16})$. Using this one checks easily from (4.1) that $\text{Ric} \geq 0$ if $r$ is taken small enough. It is interesting to note that, with the choice of $r$ small and $\varphi$ as above, $\text{Ric}$ vanishes only on the direction of $\frac{\partial}{\partial t}|_{t=0}$.

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