Mock theta functions and characters of N=3 superconformal modules

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Abstract

In this paper we study the characters of N=3 superconformal modules by using the Zwegers’ theory on modification of mock theta functions.

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1 Introduction

As it is known well, the N=3 superconformal algebra is constructed as the W-algebra associated to the affine superalgebra $\widehat{B}(1,1) = \widehat{osp}(3|2)$. In this paper, we consider the N=3 modules obtained by the quantum Hamiltonian

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reduction of $\hat{B}(1, 1)$-modules which are integrable with respect to the roots $\alpha_2$ and $\delta - \alpha_2$ and atypical with respect to $\alpha_1$. Among these modules, we consider the highest weight $\hat{B}(1, 1)$-module $L(\Lambda^{[K(m), m_2]})$ where $m$ is a positive integer and $m_2$ is a non-negative integer such that $m_2 \leq m$ and $K := -\frac{m+2}{4}$ and $\Lambda^{[K(m), m_2]} := K(m)\Lambda_0 - \frac{m_2^2}{2}\alpha_1$.

The characters of these $N=3$ modules are Appell’s functions (cf. [3]) and their modular properties are usually unclear and difficult.

The case $m = 1$ (i.e., $K = -\frac{3}{4}$) was discussed in §5 of [10].

The purpose of the present paper is to study the characters in the case $m = 2$ (i.e., $K = -1$) by using the Zwegers’ theory on mock modular forms. In this case, we will show that the space of characters is not $SL_2(\mathbb{Z})$-invariant but the $SL_2(\mathbb{Z})$-invariance is achieved by collaboration of honest characters and modified characters. Also we show that the modified characters are explicitly written by the Dedekind’s $\eta$-function and the Mumford’s theta functions $\vartheta_{ab}(\tau, z)$ ($a, b \in \{0, 1\}$). The method to compute the characters of $N=3$ modules in this paper is as follows:

- First, compute all modified characters by using the action of $SL_2(\mathbb{Z})$.
- Next, compute the Zwegers’ correction terms.

These are possible in the case $m = 2$ namely $K = -1$, and then we obtain the explicit formulas for both honest and modified characters. The results obtained in the case $m = 2$ enable us to study also the case $m = 4$.

This paper is organized as follows.

In section 2 we review the Zwegers’ modification theory and collect formulas which are necessary to work out our calculation.

In section 3 as preparation for calculation in this paper, we give formulas for (super)characters and twisted characters of integrable $\hat{B}(1, 1)$-modules.

In section 4 we compute the characters of the $N=3$ modules obtained from the quantum Hamiltonian reduction of integrable $\hat{B}(1, 1)$-modules. For the $SL_2(\mathbb{Z})$-invariance of the space of modified characters, we consider 2 kinds of twisted characters.

In section 5 we compute modified characters in the case $m = 2$ by using their modular properties.

In section 6 as continuation from section 5 we compute the Zwegers’ correction terms and obtain the explicit formulas for the honest characters in the case $m = 2$. By this calculation it turns out that the modified character, in this case, is nothing else but the sum of two honest characters.

In section 7 we compute the asymptotic behavior of (super)characters as $\tau \downarrow 0$ in the case $m = 2$ (i.e., $K = -1$). We see that the asymptotics of super-characters remain unchanged under the modification, whereas the asymptotics of characters behave in somehow strange way.

In sections 8 and 9 we discuss about relations between characters of $H(\Lambda^{[K(m), m_2]})$ and $H(\Lambda^{[K(2m), m_2]})$ and, as its application, we deduce explicit formulas for some of modified and honest characters in the case $m = 4$. 
Finally in section 10 we show some theta-relations of the Mumford’s theta functions $\vartheta_{ab}(\tau, z)$ obtained from the modification of characters.

Through the study developped in this paper, we are led to a conjecture about the relation between honest characters and modified characters. If this conjecture is true, we can expect that the Zwegers’ modification theory of mock theta functions will provide very powerful tools for the study of characters of W-algebras.

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2 Functions $\Phi^{[m,s]}$ and $\tilde{\Phi}^{[m,s]}$

We begin this section with brief and quick review on the Zwegers’ modification theory of mock theta functions which are used in this paper. For $m \in \frac{1}{2}\mathbb{N}$ and $s \in \frac{1}{2}\mathbb{Z}$, we define the functions $\Phi^{[m,s]}_i(\tau, z_1, z_2, t)$ by

$$\Phi^{[m,s]}_1(\tau, z_1, z_2, t) := e^{-2\pi int} \sum_{j \in \mathbb{Z}} \frac{e^{2\pi imj(z_1+z_2)+2\piisz_1q^{mj^2+s_j}}}{1 - e^{2\piisz_1q^j}}$$

$$\Phi^{[m,s]}_2(\tau, z_1, z_2, t) := e^{-2\pi int} \sum_{j \in \mathbb{Z}} \frac{e^{-2\pi imj(z_1+z_2)-2\piisz_2q^{mj^2+s_j}}}{1 - e^{-2\piisz_2q^j}}$$

where $q := e^{2\pi i\tau}$ ($\tau \in \mathbb{C}_+$) and $z_1, z_2, t \in \mathbb{C}$, and put

$$\Phi^{[m,s]}(\tau, z_1, z_2, t) := \Phi^{[m,s]}_1(\tau, z_1, z_2, t) - \Phi^{[m,s]}_2(\tau, z_1, z_2, t)$$

These functions appear usually in the characters of affine superalgebras and superconformal algebras, but do not have good modular properties. This situation is improved by the Zwegers’ modification, using the functions $R_{j;m}(\tau, w)$ for $(m, j) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{Z}$ defined by

$$R_{j;m}(\tau, w) := \sum_{n \equiv j \mod 2m} \left\{ \text{sgn} \left( n - \frac{1}{2} - j + 2m \right) - E \left( \left( n - 2m \frac{\text{Im}(w)}{\text{Im}(\tau)} \right) \sqrt{\frac{\text{Im}(\tau)}{m}} \right) \right\} \times e^{-\frac{2\pi iz_1}{m} + 2\pi iwn}$$

where $(\tau, w) \in \mathbb{C}_+ \times \mathbb{C}$ and $E(z) := 2 \int_0^z e^{-\pi t^2} dt$. For $m \in \frac{1}{2}\mathbb{N}$ and $s \in \frac{1}{2}\mathbb{Z}$, we put

$$\Phi^{[m,s]}_{\text{add}}(\tau, z_1, z_2, t)$$

$$:= \frac{1}{2} e^{-2\pi int} \sum_{k \in s+\mathbb{Z}} R_{k;m} \left( \tau, \frac{z_1 - z_2}{2} \right) \frac{\theta_{k;m} - \theta_{-k;m}}{s \leq k < s + 2m}$$

$$\tilde{\Phi}^{[m,s]}(\tau, z_1, z_2, t) := \Phi^{[m,s]}(\tau, z_1, z_2, t) + \Phi^{[m,s]}_{\text{add}}(\tau, z_1, z_2, t)$$
and call $\tilde{\Phi}^{[m:s]}$ the “modification” of $\Phi^{[m:s]}$ and $\Phi^{[m:s]}_{\text{add}}$ the “correction term”. In the formula (2.4a), $\theta_{k,m}$ is the Jacobi’s theta function:

$$\theta_{k,m}(\tau, z) := \sum_{j \in \mathbb{Z}} e^{2\pi i m(j + \frac{k}{m})} q^{m(j + \frac{k}{m})^2}$$ (2.5)

The properties of these functions were studied in [6], [7], [8], [9], [10] and [12]. Among them we collect here only the formulas which are necessary for our purpose in this paper.

**Lemma 2.1.** If $m \in \mathbb{N}$ and $j \in \frac{1}{2} \mathbb{Z}$, then $R_{j,m}(\tau, w \pm \frac{1}{2}) = e^{\pm \pi i j} R_{j,m}(\tau, w)$

**Lemma 2.2.** Let $m \in \frac{1}{2} \mathbb{N}$ and $s, s' \in \frac{1}{2} \mathbb{Z}$. Then

1) $\tilde{\Phi}^{[m:s]}(\tau, z_2, z_1, t) = \Phi^{[m:1-s]}(\tau, z_1, z_2, t)$.

2) $\Phi^{[m,s]}(\tau, z_2, z_1, t) = \Phi^{[m,s]}(\tau, z_1, z_2, t)$.

3) $\Phi^{[m,s]}_{\text{add}}(\tau, z_2, z_1, t) = \Phi^{[m,s]}_{1}(\tau, z_1, z_2, t)$.

4) $\Phi^{[m,s]}_{2}(\tau, z_1, z_2, t) = \Phi^{[m,s]}_{1}(\tau, -z_2, -z_1, t)$.

**Lemma 2.3.** Let $m \in \frac{1}{2} \mathbb{N}$ and $s \in \frac{1}{2} \mathbb{Z}$. Then

1) if $s \in \mathbb{Z}$, $\Phi^{[m:s]}(\tau, -\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_1}{\tau}, t) = \tau e^{\frac{2\pi i m}{\tau} z_1 z_2} \tilde{\Phi}^{[m:s]}(\tau, z_1, z_2, t)$

2) if $m + s \in \mathbb{Z}$, then
   - (i) $\Phi^{[m:s]}(\tau + 1, z_1, z_2, t) = \Phi^{[m:s]}(\tau, z_1, z_2, t)$
   - (ii) $\Phi^{[m:s]}(\tau + 1, z_1, z_2, t) = \Phi^{[m,s]}(\tau, z_1, z_2, t)$

and similar formula holds for $\Phi^{[m,s]}_i$ (i $\in \{1, 2\}$).

**Lemma 2.4.** Let $m \in \mathbb{N}$ and $s \in \frac{1}{2} \mathbb{Z}$ and $a, b \in \mathbb{Z}$ such that $a + b \in 2 \mathbb{Z}$. Then

1) $\Phi^{[m,s]}(\tau, z_1 + a, z_2 + b, t) = (-1)^{2sa} \Phi^{[m,s]}(\tau, z_1, z_2, t)$

2) In the case $s \in \mathbb{Z}$,

$$\Phi^{[m,s]}(\tau, z_1 + a, z_2 + b, t) = \Phi^{[m,s]}(\tau, z_1, z_2, t) \quad \text{for } \forall a, \forall b \in \mathbb{Z}$$

and similar formula holds for $\Phi^{[m,s]}_i$, $\Phi^{[m,s]}_{\text{add}}$ and $\tilde{\Phi}^{[m,s]}_i$ (i $\in \{1, 2\}$).
Lemma 2.5. For $m \in \frac{1}{2} \mathbb{N}$ and $s \in \frac{1}{2} \mathbb{Z}$,

$$
\Phi_{m,s}(2\tau, z_1, z_2, t)
= \frac{1}{2} \left( \Phi_{2m,2s}(\tau, \frac{z_1}{2}, \frac{z_2}{2}) + e^{-2\pi is} \Phi_{2m,2s}(\tau, \frac{z_1+1}{2}, \frac{z_2-1}{2}) \right)
$$

and similar formula holds for $\Phi_{i,s,\text{add}}$ and $\Phi_{i,s,\text{ini}}$ ($i \in \{1, 2\}$).

Lemma 2.6. Let $m \in \frac{1}{2} \mathbb{N}$, $s \in \frac{1}{2} \mathbb{Z}$ and $a, b \in \mathbb{C}$. Then

$$
\Phi_{i,s}(\tau, z_1 + a + \frac{\tau}{2}, z + b - \frac{\tau}{2}, 0) = e^{2\pi im(a-b)} \Phi_{i,s}(\tau, z_1 - a - \frac{\tau}{2}, z + b + \frac{\tau}{2}, 0)
$$

and similar formula holds for $\Phi_{i,s}^\prime$ ($i \in \{1, 2\}$).

Lemma 2.7. ($sl(2|1)$-denominator identity):

$$
\Phi_{1,s}^\prime(\tau, z_1, z_2, t) = \Phi_{1,s}(\tau, z_1, z_2, t) = -ie^{-2\pi it} \eta(\tau)^3 \theta_{11}(\tau, z_1, z_2)
$$

for $s \in \mathbb{Z}$, where $\theta_{ab}(\tau, z)$ are the Mumford’s theta functions ([11]).

We note also the following simple but useful formulas:

Lemma 2.8. Let $m \in \frac{1}{2} \mathbb{N}$, $s \in \frac{1}{2} \mathbb{Z}$ and $a \in \mathbb{N}$. Then

1) $\Phi_{1}^{[m,s]}(\tau, z_1, z_2, 0) - \Phi_{1}^{[m,s+a]}(\tau, z_1, z_2, 0)$

$$
= \sum_{k=0}^{a-1} e^{\pi i (s+k)(z_1-z_2)} q^{-\frac{(s+k)^2}{4m}} \theta_{s+k, m}(\tau, z_1 + z_2)
$$

2) $\Phi_{2}^{[m,s]}(\tau, z_1, z_2, 0) - \Phi_{2}^{[m,s+a]}(\tau, z_1, z_2, 0)$

$$
= \sum_{k=0}^{a-1} e^{\pi i (s+k)(z_1-z_2)} q^{-\frac{(s+k)^2}{4m}} \theta_{-(s+k), m}(\tau, z_1 + z_2)
$$

3) $\Phi^{[m,s]}(\tau, z_1, z_2, t) - \Phi^{[m,s+a]}(\tau, z_1, z_2, t)$

$$
= e^{-2\pi imt} \sum_{k=0}^{a-1} e^{\pi i (s+k)(z_1-z_2)} q^{-\frac{(s+k)^2}{4m}} \left[ \theta_{s+k, m} - \theta_{-(s+k), m} \right](\tau, z_1 + z_2)
$$
Proof. 1) Letting $s \to s + a$ in (2.1a), we have

$$\Phi_{1}^{[m,s+a]}(\tau, z_1, z_2, 0) = \sum_{j \in \mathbb{Z}} e^{2\pi im j (z_1 + z_2)} q^{m j^2} \frac{(e^{2\pi i z_1} q^{j})^{s+a}}{1 - e^{2\pi i z_1} q^{j}},$$

and so

$$\Phi_{1}^{[m,s]}(\tau, z_1, z_2, 0) - \Phi_{1}^{[m,s+a]}(\tau, z_1, z_2, 0) = \sum_{j \in \mathbb{Z}} \sum_{k=0}^{a-1} e^{2\pi im j (z_1 + z_2)} q^{m j^2} \frac{(e^{2\pi i z_1} q^{j})^{s}}{1 - e^{2\pi i z_1} q^{j}}$$

proving 1).

2) By Lemma 2.2, we have

$$\Phi_{2}^{[m,s]}(\tau, z_1, z_2, 0) - \Phi_{2}^{[m,s+a]}(\tau, z_1, z_2, 0) = \Phi_{1}^{[m,s]}(\tau, -z_2, -z_1, 0) - \Phi_{1}^{[m,s+a]}(\tau, -z_2, -z_1, 0)$$

$$= \sum_{k=0}^{a-1} e^{\pi i (s+k)(z_1 - z_2)} q^{-\frac{(s+k)^2}{4m}} \theta_{s+k}, m(\tau, -(z_1 + z_2))$$

proving 2). 3) follows from 1) and 2).

The following functions $\widetilde{A}_{i}^{[m]}(\tau, z)$ ($m \in \mathbb{N}$ and $i = 1 \sim 6$) play important roles to describe the characters of $N=3$ modules.

$$\widetilde{A}_{1}^{[m]}(\tau, z) := \widetilde{\Phi}^{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{4} + \frac{1}{4}, \frac{z}{2} - \frac{\tau}{4} - \frac{1}{4}, 0) \quad (2.6a)$$

$$\widetilde{A}_{2}^{[m]}(\tau, z) := \widetilde{\Phi}^{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{4} - \frac{1}{4}, \frac{z}{2} - \frac{\tau}{4} + \frac{1}{4}, 0) \quad (2.6b)$$

$$\widetilde{A}_{3}^{[m]}(\tau, z) := \widetilde{\Phi}^{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{4} + \frac{1}{2}, \frac{z}{2} - \frac{\tau}{4} - \frac{1}{2}, 0) \quad (2.6c)$$
Lemma 2.9.

We note that, by Lemma 2.6, \( \overline{A}_6^{[m]}(\tau, z) \) is written as follows:

\[
\overline{A}_6^{[m]}(\tau, z) = e^{-\pi im} \Phi^{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{4}, \frac{z}{2} - \frac{\tau}{4}, 0).
\]

The modular transformation properties of these functions are easily computed by using Lemma 2.3 to obtain the following:

**Lemma 2.9.**

1) **S-transformation** :

(i) \( \overline{A}_1^{[m]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) \) \( = \tau e^{\frac{2\pi im}{\tau} (\frac{\tau}{2}+\frac{z}{\tau})} \overline{A}_2^{[m]}(\tau, z) \)

(ii) \( \overline{A}_2^{[m]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) \) \( = \tau e^{\frac{2\pi im}{\tau} (\frac{\tau}{2}-\frac{z}{\tau})} \overline{A}_1^{[m]}(\tau, z) \)

(iii) \( \overline{A}_3^{[m]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) \) \( = \tau e^{\frac{2\pi im}{\tau} (\frac{\tau}{2}+\frac{z}{\tau})} \overline{A}_6^{[m]}(\tau, z) \)

(iv) \( \overline{A}_4^{[m]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) \) \( = \tau e^{\frac{2\pi im}{\tau} (\frac{\tau}{2}-\frac{z}{\tau})} \overline{A}_5^{[m]}(\tau, z) \)

(v) \( \overline{A}_5^{[m]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) \) \( = \tau e^{\frac{2\pi im}{\tau} (\frac{\tau}{2}+\frac{z}{\tau})} \overline{A}_4^{[m]}(\tau, z) \)

(vi) \( \overline{A}_6^{[m]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) \) \( = \tau e^{\frac{2\pi im}{\tau} (\frac{\tau}{2}-\frac{z}{\tau})} \overline{A}_3^{[m]}(\tau, z) \)

2) **T-transformation** :

(i) \( \overline{A}_1^{[m]}(\tau + 1, z) \) \( = \overline{A}_1^{[m]}(\tau, z) \)

(ii) \( \overline{A}_2^{[m]}(\tau + 1, z) \) \( = \overline{A}_4^{[m]}(\tau, z) \)

(iii) \( \overline{A}_3^{[m]}(\tau + 1, z) \) \( = \overline{A}_2^{[m]}(\tau, z) \)

(iv) \( \overline{A}_4^{[m]}(\tau + 1, z) \) \( = \overline{A}_1^{[m]}(\tau, z) \)

(v) \( \overline{A}_5^{[m]}(\tau + 1, z) \) \( = \overline{A}_5^{[m]}(\tau, z) \)

(vi) \( \overline{A}_6^{[m]}(\tau + 1, z) \) \( = e^{\pi im} \overline{A}_6^{[m]}(\tau, z) \)

Define the functions \( \overline{A}_i^{[m]}(\tau, z) \) by

\[
\overline{A}_i^{[m]}(\tau, z) := \begin{cases} 
  e^{-\frac{2\pi im}{\tau}} \overline{A}_i^{[m]}(\tau, z) & (1 \leq i \leq 4) \\
  \overline{A}_5^{[m]}(\tau, z) & (i = 5) \\
  e^{-\frac{2\pi im}{\tau}} \overline{A}_6^{[m]}(\tau, z) & (i = 6)
\end{cases}
\]
Then the transformation properties of these functions are obtained from Lemma 2.9 as follows:

Lemma 2.10.

1) S-transformation :

(i) \[ A^{[m]}_1 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{\pi i m \tau} e^{\frac{\pi i m z}{2\tau}} A^{[m]}_2 (\tau, z) \]

(ii) \[ A^{[m]}_2 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{\pi i m \tau} e^{\frac{\pi i m z}{2\tau}} A^{[m]}_1 (\tau, z) \]

(iii) \[ A^{[m]}_3 \left( -\frac{1}{z}, \frac{z}{\tau} \right) = e^{\pi i m \tau} e^{\frac{\pi i m z}{2\tau}} A^{[m]}_6 (\tau, z) \]

(iv) \[ A^{[m]}_4 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \tau e^{\pi i m \tau} A^{[m]}_5 (\tau, z) \]

(v) \[ A^{[m]}_5 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \tau e^{\pi i m \tau} A^{[m]}_4 (\tau, z) \]

(vi) \[ A^{[m]}_6 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{\pi i m \tau} e^{\frac{\pi i m z}{2\tau}} A^{[m]}_3 (\tau, z) \]

2) T-transformation :

(i) \[ A^{[m]}_1 (\tau + 1, z) = e^{-\pi i m} A^{[m]}_3 (\tau, z) \]

(ii) \[ A^{[m]}_2 (\tau + 1, z) = e^{-\pi i m} A^{[m]}_4 (\tau, z) \]

(iii) \[ A^{[m]}_3 (\tau + 1, z) = e^{-\pi i m} A^{[m]}_2 (\tau, z) \]

(iv) \[ A^{[m]}_4 (\tau + 1, z) = e^{-\pi i m} A^{[m]}_1 (\tau, z) \]

(v) \[ A^{[m]}_5 (\tau + 1, z) = A^{[m]}_5 (\tau, z) \]

(vi) \[ A^{[m]}_6 (\tau + 1, z) = e^{\pi i m} A^{[m]}_6 (\tau, z) \]

We note that, by Lemma 2.5, these functions \( A^{[m]}_j (\tau, z) \) are connected to \( \Phi^{[\tau, s]} \) by the following relations:

\[
\tilde{\Phi}^{[\tau, 0]} \left( 2\tau, z + \frac{\tau}{2}, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} + \frac{\tau}{2}, \sqrt{8} \right) = \frac{1}{2} \left\{ A^{[m]}_4 (\tau, z) + A^{[m]}_1 (\tau, z) \right\} \tag{2.8a}
\]

\[
\tilde{\Phi}^{[\tau, 1]} \left( 2\tau, z + \frac{\tau}{2}, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} + \frac{\tau}{2}, \sqrt{8} \right) = \frac{1}{2} \left\{ A^{[m]}_4 (\tau, z) - A^{[m]}_1 (\tau, z) \right\} \tag{2.8b}
\]

\[
\tilde{\Phi}^{[\tau, 0]} \left( 2\tau, z + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, \sqrt{8} \right) = \frac{1}{2} \left\{ A^{[m]}_4 (\tau, z) + A^{[m]}_3 (\tau, z) \right\} \tag{2.8c}
\]

\[
\tilde{\Phi}^{[\tau, 1]} \left( 2\tau, z + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, \sqrt{8} \right) = \frac{1}{2} \left\{ A^{[m]}_4 (\tau, z) - A^{[m]}_3 (\tau, z) \right\} \tag{2.8d}
\]
\[ \tilde{\Phi}[\frac{m}{2},0] \left( 2\tau, z - \frac{1}{2}, z + \frac{1}{2}, 0 \right) = \tilde{A}_3^m(\tau, z) \]  
\[ \tilde{\Phi}[\frac{m}{2},0] \left( 2\tau, z - \frac{1}{2}, z - \tau + \frac{1}{2}, \frac{\tau}{2} \right) = \frac{1}{2} \left\{ 1 + (-1)^{m+2s} \right\} A_6^m(\tau, z) \]  
(2.8e)

(2.8f)

3 Integrable \( \hat{osp}(3|2) \)-modules and their characters

We consider the Dynkin diagram of \( \hat{B}(1,1) = \hat{osp}(3|2) \)

\[
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\circ \quad \circ \\
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\]

with the inner product \((\mid \cdot \mid)\) such that \( (\alpha_i \mid \alpha_j)_{i,j=0,1,2} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \).

Then the dual Coxeter number of \( \hat{B}(1,1) \) is \( h^\vee = \frac{1}{2} \). Let \( \Lambda_0 \) be the element in \( \mathfrak{h}^\ast \) satisfying the conditions \( (\Lambda_0^\vee \mid \alpha_j) = \delta_{j,0} \) and \( (\Lambda_0^\vee \mid \Lambda_0) = 0 \). Let \( \delta = \alpha_0 + 2\alpha_1 + 2\alpha_2 \) be the primitive imaginary root and \( \rho = \frac{1}{2}\Lambda_0 - \frac{1}{2}\alpha_1 \) be the Weyl vector.

We put
\[ K(m) := -\frac{m+2}{4} \quad \text{so that} \quad m = -4 \left( K(m) + \frac{1}{2} \right) \]  
(3.1a)

\[ \Lambda^{[K(m),m_2]} := K(m)\Lambda_0 - \frac{m_2}{2}\alpha_1 \]  
(3.1b)

Note that the weight \( \Lambda^{[K(m),m_2]} \) is atypical with respect to \( \alpha_1 \), namely \( (\Lambda^{[K(m),m_2]} \mid \rho \mid \alpha_1) = 0 \).

Lemma 3.1. The weight \( \Lambda^{[K(m),m_2]} \) is integrable with respect to \( \alpha_2 \) and \( \delta - \alpha_2 \) if and only if \( m \) and \( m_2 \) are non-negative integers satisfying \( m_2 \leq m \).

In this paper, a \( \hat{B}(1,1) \)-module \( L(\Lambda) \) which is integrable with respect to \( \alpha_2 \) and \( \delta - \alpha_2 \) is called simply an “integrable” \( \hat{B}(1,1) \)-module, and \( \Lambda \) is called simply an “integrable” weight.

Define the coordinates on the Cartan subalgebra \( \mathfrak{h} \) of \( \hat{B}(1,1) \) by
\[ (\tau, z_1, z_2, t) := 2\pi i \left\{ -\tau \Lambda_0 - z_1(\alpha_1 + 2\alpha_2) - z_2 \alpha_1 + t\delta \right\} \]  
(3.2)

For an integrable weight \( \Lambda \) which is atypical with respect to \( \alpha_1 \), the supercharacter \( \text{ch}_\Lambda^{-} \) and the character \( \text{ch}_\Lambda^{+} \) of \( L(\Lambda) \) are obtained by the formulas
where \( t \) we have

\[ R(-) \chi^{(-)}_\Lambda := \sum_{w \in \mathcal{R}} \varepsilon(w) w \left( \frac{e^{\lambda + \rho}}{1 - e^{-\alpha_1}} \right) \]

\[ = \sum_{j \in \mathbb{Z}} t_{j \alpha_2} \left( \frac{e^{\lambda + \rho}}{1 - e^{-\alpha_1}} \right) - \sum_{j \in \mathbb{Z}} r_{\alpha_2} t_{j \alpha_2} \left( \frac{e^{\lambda + \rho}}{1 - e^{-\alpha_1}} \right) \]  

(3.3a)

\[ (R^+) \chi^{(+)}_{\Lambda (K(m),m2)}(\tau, z_1, z_2, t) := (R(-) \chi^{(-)}_\Lambda) \left( \tau, z_1 - \frac{1}{2}, z_2 - \frac{1}{2}, t \right) \]  

(3.3b)

where \( t_\alpha (\alpha \in \mathfrak{h}) \) is the linear automorphism of \( \mathfrak{h} \) defined, in [1], by

\[ t_\alpha(\lambda) := \lambda + (\lambda|\delta)\alpha - \left\{ \frac{(\alpha|\alpha)}{2} (\lambda|\delta) + (\lambda|\alpha) \right\} \delta \]  

(3.4)

and \( R^\pm \) are (super)denominator of \( \tilde{B}(1,1) \).

**Lemma 3.2.** For \( m \in \mathbb{N} \), the (super)character of integrable \( \tilde{B}(1,1) \)-module \( L(\Lambda^{(K(m),m2)}) \) is given by the following formulas:

1) \( (R^+) \chi^{(+)}_{\Lambda (K(m),m2)}(\tau, z_1, z_2, t) \)

\[ = \Phi^\tau \left( \frac{e^{\lambda + \rho}}{1 - e^{-\alpha_1}} \right) \left( 2\tau, z_1 - \frac{1}{2}, z_2 + \frac{1}{2}, t \right) \]

2) \( (R(-) \chi^{(-)}_{\Lambda (K(m),m2)}(\tau, z_1, z_2, t) \)

\[ = \Phi^\tau \left( \frac{e^{\lambda + \rho}}{1 - e^{-\alpha_1}} \right) \left( 2\tau, z_1, z_2 - \frac{1}{2}, t \right) \]

In order to have \( SL_2(\mathbb{Z}) \)-invariance of the space of characters, we need “twisted” characters. For this sake, we define the character twisted by \( \sigma_{j,k} := t_{j \alpha_1 + k(\alpha_1 + 2\alpha_2)} \) as follows:

\[ (R^+)^{tw} \chi^{(+)}_{\Lambda (K(m),m2)}(h) := (R^+) \chi^{(+)}_{\Lambda (K(m),m2)}(\sigma_{j,k}(h)) \quad (h \in \mathfrak{h}) \]  

(3.5)

where \( R^+ \) is the twisted denominator, but we will not need here to know its explicit formula. The twisted numerator \( (R^+)^{tw} \chi^{(+)}_{\Lambda (K(m),m2)}(\sigma_{j,k}(h)) \) is computed easily as follows:

**Lemma 3.3.** \( (R^+)^{tw} \chi^{(+)}_{\Lambda (K(m),m2)}(\sigma_{j,k}(h)) \)

\[ = \Phi^\tau \left( \frac{e^{\lambda + \rho}}{1 - e^{-\alpha_1}} \right) \left( 2\tau, z_1 + k\tau - \frac{1}{2}, -z_2 - j\tau + \frac{1}{2}, 2 \atop t + jz_1 + kz_2 + jk\tau \right) \]

Proof. By (3.4) we have

\[
\begin{align*}
\sigma_{j,k}(\Lambda_0) &= \Lambda_0 + j\alpha_1 + k(\alpha_1 + 2\alpha_2) - jk\delta \\
\sigma_{j,k}(\alpha_1) &= \alpha_1 - k\delta \\
\sigma_{j,k}(\alpha_1 + 2\alpha_2) &= \alpha_1 + 2\alpha_2 - j\delta.
\end{align*}
\]
Then we have
\[
\sigma_{j,k}(\tau, z_1, z_2, t) = 2\pi i \left\{-\tau \sigma_{j,k}(\Lambda_0) - z_1 \sigma_{j,k}(\alpha_1 + 2\alpha_2) - z_2 \sigma_{j,k}(\alpha_1) + t \delta\right\}
\]
\[
= 2\pi i \left\{-\tau \Lambda_0 - (z_1 + k\tau)(\alpha_1 + 2\alpha_2) - (z_2 + j\tau)\alpha_1 + (t + jz_1 + kz_2 + jk\tau)\delta\right\}
\]
\[
= (\tau, z_1 + k\tau, z_2 + j\tau, t + jz_1 + kz_2 + jk\tau),
\]
so
\[
(R^{(+)}\text{tw})\text{ch}_{\sigma_{j,k}}(\tau, z_1, z_2, t)
\]
\[
= (R^{(+)}\text{ch}_{\sigma_{j,k}}(\tau, z_1 + k\tau, z_2 + j\tau, t + jz_1 + kz_2 + jk\tau))
\]
\[
= \Phi_{\tau, z_1 + k\tau, z_2 + j\tau, t + jz_1 + kz_2 + jk\tau}(4.1)
\]
by Lemma 3.2 proving the claim. □

4 Characters of quantum reduction

We now consider the quantum Hamiltonian reduction associated to the pair
\((x = \frac{1}{2}\theta, f = e^{-\theta})\), where \(\theta := 2(\alpha_1 + \alpha_2)\) is the highest root of the finite-dimensional Lie superalgebra \(\mathfrak{g} := \text{osp}(3|2)\). The Cartan subalgebra of \(\mathfrak{g}\) is \(\mathfrak{h} := \mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2\). Taking a basis \(J_0 := -2\alpha_2\), we have
\[
2\pi i \left\{-\tau \Lambda_0 - \tau x + zJ_0 + \frac{\tau}{2}(x|x)\delta\right\}
\]
\[
= 2\pi i \left\{-\tau \Lambda_0 - \left(z + \frac{\tau}{2}\right)(\alpha_1 + 2\alpha_2) - \left(-z + \frac{\tau}{2}\right)\alpha_1 + \frac{\tau}{4}\delta\right\}
\]
\[
= (\tau, z + \frac{\tau}{2}, -z + \frac{\tau}{2}, \frac{\tau}{4})(4.1)
\]
Then the (super)character of the quantum Hamiltonian reduction \(H(\Lambda)\) of \(\hat{B}(1,1)\) module \(L(\Lambda)\) is obtained by the formula :
\[
\left(\frac{N=3}{R^{(\pm)}\text{ch}_{H(\Lambda)}}\right)(\tau, z) = (R^{(\pm)}\text{ch}_{\Lambda})(2\pi i \left\{-\tau \Lambda_0 - \tau x + zJ_0 + \frac{\tau}{2}(x|x)\delta\right\})
\]
\[
= (R^{\pm}\text{ch}_{\Lambda})(\tau, z + \frac{\tau}{2}, -z + \frac{\tau}{2}, \frac{\tau}{4})(4.2)
\]
And also similar for the twisted characters, where \(N=3 R^{(+)}\) and \(N=3 R^{(-)}\) and \(N=3 R^{(+)\text{tw}}\) are the denominator and superdenominator and twisted denominator respectively of the N=3 superconformal algebra defined, by using the Mumford’s
Proof.

1) and 2) follow from (4.2) and Lemma 3.2. 3) is shown as follows. By theta functions \( \vartheta \) functions Proposition 4.1. For twisted) characters of the \( N=3 \) module \( H(\Lambda(K(k),m_2)) \) are as follows:

\[
\begin{aligned}
\text{1) } & \quad N=3 \quad \bar{R}^{(+)}(\tau,z) := \eta(\frac{\tau}{2}) \eta(2\tau) \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \\
\text{2) } & \quad N=3 \quad \bar{R}^{(-)}(\tau,z) := \eta(\tau)^3 \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \\
\text{3) } & \quad N=3 \quad \bar{R}^{(+)}(\tau,z) := \frac{1}{\sqrt{2}} \eta(\tau)^3 \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \\
\end{aligned}
\tag{4.3}
\]

Then, by (4.2) and Lemma 3.2 and Lemma 3.3, we obtain the following :

**Proposition 4.1.** For \( m \in \mathbb{N} \) and \( m_2 \in \mathbb{Z}_{\geq 0} \) such that \( m_2 \leq m_2 \), the (super or twisted) characters of the \( N=3 \) module \( H(\Lambda(K(k),m_2)) \) are as follows:

\[
\begin{aligned}
\text{1) } & \quad \left( \bar{R}^{(+)} \right)_{\Lambda(K(m),m_2)}^{(+)tw}((\tau,z)) = \Phi_{\frac{\tau}{2}} \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right) \\
\text{2) } & \quad \left( \bar{R}^{(-)} \right)_{\Lambda(K(m),m_2)}^{(-)tw}((\tau,z)) = \Phi_{\frac{\tau}{2}} \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right) \\
\text{3) } & \quad \left( \bar{R}^{(+)} \right)_{\Lambda(K(m),m_2)}^{(+)tw}((\tau,z)) \left( \sigma_+ \right) = \Phi_{\frac{\tau}{2}} \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right) \\
\text{4) } & \quad \left( \bar{R}^{(+)} \right)_{\Lambda(K(m),m_2)}^{(+)tw}((\tau,z)) \left( \sigma_- \right) = \Phi_{\frac{\tau}{2}} \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right)
\end{aligned}
\]

4) Let \( \sigma_\pm \) be the linear automorphisms of \( \mathfrak{h} \) defined by

\[
\begin{cases}
\sigma_+ := \sigma_{\frac{\tau}{2}, \frac{\tau}{2}} = t_{\alpha_1 + \alpha_2} \\
\sigma_- := \sigma_{-\frac{\tau}{2}, -\frac{\tau}{2}} = t_{-(\alpha_1 + \alpha_2)}
\end{cases}
\]

Then

\[
\begin{aligned}
\text{(i) } & \quad \left( \bar{R}^{(+)} \right)_{\Lambda(K(m),m_2)}^{(+)tw}((\tau,z)) = \Phi_{\frac{\tau}{2}} \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right) \\
\text{(ii) } & \quad \left( \bar{R}^{(+)} \right)_{\Lambda(K(m),m_2)}^{(+)tw}((\tau,z)) = \Phi_{\frac{\tau}{2}} \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right)
\end{aligned}
\]

**Proof.** 1) and 2) follow from (4.2) and Lemma 3.2 and Lemma 3.3, 3) is shown as follows. By the twisted version of (4.2), we have

\[
\begin{aligned}
\text{(i) } & \quad \left( \bar{R}^{(+)} \right)_{\Lambda(K(m),m_2)}^{(+)tw}((\tau,z)) = \Phi_{\frac{\tau}{2}} \frac{\vartheta_{11}(\tau,z)}{\vartheta_{00}(\tau,z)} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right)
\end{aligned}
\]
The RHS of this equation (4.4) is rewritten, by using Lemma 3.3, as follows:

\[
\text{RHS of (4.4)} = \phi\left(\frac{\tau - \frac{1}{2}}{2\pi R}\right) \left(2\tau, \left(z + \frac{\tau}{2}\right) + k\tau - \frac{j}{2}, \left(-z + \frac{\tau}{2}\right) - j\tau + \frac{1}{2}\right)
\]

\[
= \phi\left(\frac{\tau - \frac{1}{2}}{2\pi R}\right) \left(2\tau, z + \left(k + \frac{1}{2}\right)\tau - \frac{1}{2}, z - \left(j + \frac{1}{2}\right)\tau + \frac{1}{2}\right)
\]

\[
= \phi\left(\frac{\tau - \frac{1}{2}}{2\pi R}\right) \left(2\tau, \frac{(j - k)z}{2} + \frac{1}{2} \left(j + \frac{1}{2}\right) \left(k + \frac{1}{2}\right)\tau\right)
\]

proven 3). 4) follows from 3) immediately.

We call these characters “honest” characters. “Modified” characters \(\overline{\text{ch}}_H(\Lambda)\) are defined by replacing \(\Phi\) with \(\Phi\) in the formulas in Proposition 4.1, namely

\[
\begin{align*}
N &= 3 \quad (R^{+}) \overline{\text{ch}}_H(\Lambda)(\tau, z) := \phi\left(\frac{\tau - \frac{1}{2}}{2\pi R}\right) \left(2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2} \frac{\tau}{8}\right) \\
N &= 3 \quad (R^{-}) \overline{\text{ch}}_H(\Lambda)(\tau, z) := \phi\left(\frac{\tau - \frac{1}{2}}{2\pi R}\right) \left(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{\tau}{8}\right) \\
N &= 3 \quad (R^{\text{tw}})(+)(\tau, z) := \phi\left(\frac{\tau - \frac{1}{2}}{2\pi R}\right) \left(2\tau, z + \frac{\tau}{2} - \frac{1}{2} \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{\tau}{2}\right) \\
N &= 3 \quad (R^{\text{tw}})(-)(\tau, z) := \phi\left(\frac{\tau - \frac{1}{2}}{2\pi R}\right) \left(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2} \frac{\tau}{2}\right)
\end{align*}
\]

By Lemma 2.2 and Lemma 2.3 and the formula (2.62), these formulas are rewritten as follows:

**Proposition 4.2.**

1) \(N^3\quad (R^{+}) \overline{\text{ch}}_H(\Lambda)(\tau, z) = \frac{1}{2} \left\{ A_2^{[m]}(\tau, z) + \pi i (m+1) A_1^{[m]}(\tau, z) \right\} \)

2) \(N^3\quad (R^{-}) \overline{\text{ch}}_H(\Lambda)(\tau, z) = \frac{1}{2} \left\{ A_4^{[m]}(\tau, z) + \pi i (m+1) A_3^{[m]}(\tau, z) \right\} \)

3) \(N^3\quad (R^{\text{tw}})(\tau, z) = \frac{1}{2} \left\{ 1 + \pi i (m+1) \right\} A_0^{[m]}(\tau, z) \)

4) \(N^3\quad (R^{\text{tw}})(\tau, z) = \frac{1}{2} \left\{ 1 + \pi i (m+1) \right\} A_2^{[m]}(\tau, z) \)

From these formulas we see that

\[
\begin{align*}
\overline{\text{ch}}_H(\Lambda)(\tau, z) &= 0 \quad \text{if} \quad m + m \in 2\mathbb{Z} \\
\overline{\text{ch}}_H(\Lambda)(\tau, z) &= 0 \quad \text{if} \quad m \in 2\mathbb{Z}
\end{align*}
\]
Then, putting
\[
p(m) := \begin{cases} 
1 & \text{if } m \in 2\mathbb{Z} \\
0 & \text{if } m \in \mathbb{Z}_{\text{odd}}
\end{cases},
\] (4.7)
we have
\[
\begin{align*}
&\left\{ \begin{array}{l}
N^3_{=3} (+)tw \tilde{CH}_{H(\Lambda(K(m), p(m)))}(\tau, z) = \tilde{\sigma}^m_6(\tau, z) \\
N^3_{=3} (-)tw \tilde{CH}_{H(\Lambda(K(m), p(m)))}(\tau, z) = \tilde{\sigma}^m_5(\tau, z)
\end{array} \right. \\
\end{align*}
\] (4.8)

Note also that
\[
\begin{align*}
&\left\{ \begin{array}{l}
\tilde{\sigma}^m_{H(\Lambda(K(m), m_2))}(\tau, z) = \tilde{\sigma}^m_{H(\Lambda(K(m), m_2))}(\tau, z) \\
\tilde{\sigma}^m_{H(\Lambda(K(m), m_2))}(\tau, z) = \tilde{\sigma}^m_{H(\Lambda(K(m), m_2))}(\tau, z)
\end{array} \right. \\
\end{align*}
\] (4.9)

So, among modified characters, we need to consider only the following 6 characters:
\[
\left\{ \begin{array}{l}
\tilde{\sigma}^m_{H(\Lambda(K(m), m_2))}(\tau, z) \\
\tilde{\sigma}^m_{H(\Lambda(K(m), m_2))}(\tau, z)
\end{array} \right. \\
\] (4.10)

Then, by Proposition 4.2 we have

**Corollary 4.1.**

1) \(N^3_{=3} (+)tw \tilde{CH}_{H(\Lambda(K(m), m_2))}(\tau, z) = \frac{1}{2} \left\{ \tilde{\sigma}^m_6(\tau, z) - \tilde{\sigma}^m_5(\tau, z) \right\} \)

2) \(N^3_{=3} (+)tw \tilde{CH}_{H(\Lambda(K(m), m_2))}(\tau, z) = \frac{1}{2} \left\{ \tilde{\sigma}^m_6(\tau, z) + \tilde{\sigma}^m_5(\tau, z) \right\} \)

3) \(N^3_{=3} (-)tw \tilde{CH}_{H(\Lambda(K(m), m_2))}(\tau, z) = \frac{1}{2} \left\{ \tilde{\sigma}^m_6(\tau, z) - \tilde{\sigma}^m_5(\tau, z) \right\} \)

4) \(N^3_{=3} (-)tw \tilde{CH}_{H(\Lambda(K(m), m_2))}(\tau, z) = \frac{1}{2} \left\{ \tilde{\sigma}^m_6(\tau, z) + \tilde{\sigma}^m_5(\tau, z) \right\} \)

5) \(N^3_{=3} (+)tw \tilde{CH}_{H(\Lambda(K(m), m_2))}(\tau, z) = \tilde{\sigma}^m_6(\tau, z) \)

6) \(N^3_{=3} (+)tw \tilde{CH}_{H(\Lambda(K(m), m_2))}(\tau, z) = \tilde{\sigma}^m_5(\tau, z) \)

The modular transformation properties of these functions are obtained by Lemma 2.10 as follows:

**Lemma 4.1.**

1) \(S\)-transformation :
\[
(1+e^{-\mu_2z})(R \, (+) \, \overline{ch}_{H(\Lambda|\Lambda(m),0)}) (\tau, z) + (e^{\mu_2z} - 1)(R \, (+) \, \overline{ch}_{H(\Lambda|\Lambda(m),0)}) (\tau, z) \]

3) \( T \)-transformation:

i) \( (R \, (+) \, \overline{ch}_{H(\Lambda|\Lambda(m),0)}) (\tau + 1, z) = e^{-\mu_2z} (R \, (-) \, \overline{ch}_{H(\Lambda|\Lambda(m),0)}) (\tau, z) \)

ii) \( (R \, (+) \, \overline{ch}_{H(\Lambda|\Lambda(m),1)}) (\tau + 1, z) = e^{-\mu_2z} (R \, (-) \, \overline{ch}_{H(\Lambda|\Lambda(m),1)}) (\tau, z) \)

iii) \( (R \, (-) \, \overline{ch}_{H(\Lambda|\Lambda(m),0)}) (\tau + 1, z) = -e^{\mu_2z} (R \, (+) \, \overline{ch}_{H(\Lambda|\Lambda(m),0)}) (\tau, z) \)

iv) \( (R \, (+) \, \overline{ch}_{H(\Lambda|\Lambda(m),1)}) (\tau + 1, z) = e^{-\mu_2z} (R \, (-) \, \overline{ch}_{H(\Lambda|\Lambda(m),1)}) (\tau, z) \)

v) \( (R \, (+) \, \overline{ch}_{H(\Lambda|\Lambda(m),p(m))}) (\tau + 1, z) = e^{\mu_2z} (R \, (-) \, \overline{ch}_{H(\Lambda|\Lambda(m),p(m))}) (\tau, z) \)
Following modular transformation properties:

Proposition 4.3.

Then by these formulas (4.11a) and (4.11b) and by Lemma 4.1, we obtain the

We note that the denominators of the N=3 SCA defined by (4.3) satisfy the
following modular transformation properties:

\[
\begin{align*}
\left(\begin{array}{c}
N=3 \\
R^{(+)}(+) + N=3 \\
R^{(-)}(+) + N=3 \\
R^{(+)}tw(+) + N=3 \\
R^{(-)}tw(+) + N=3 \\
R^{(+)}tw(+)
\end{array}\right) \left(\begin{array}{c}
\tau + 1, z \\
\tau \\
\tau \\
\tau \\
\tau \\
\tau
\end{array}\right) &= \left(\begin{array}{c}
N=3 \\
R^{(+)}(+) \\
N=3 \\
N=3 \\
N=3 \\
N=3
\end{array}\right) \left(\begin{array}{c}
\tau, z \\
\tau, z \\
\tau, z \\
\tau, z \\
\tau, z \\
\tau, z
\end{array}\right)
\end{align*}
\]

(4.11a)

(4.11b)

Then by these formulas (4.11a) and (4.11b) and by Lemma 4.1, we obtain the

Proposition 4.3.

1) S-transformation:

(i) \(\tilde{\chi}_{H[A[K(m),0]]}^{(+)}\left(\begin{array}{c}
- \frac{1}{\tau}, \frac{z}{\tau}
\end{array}\right) = - \frac{1}{2} e^{-\frac{\pi i m}{4}} e^{\frac{\pi i m z}{4}} \left\{ \right. \)

\[
(1 - e^{\frac{\pi i m}{2}}) \tilde{\chi}_{H[A[K(m),0]]}^{(+)}(\tau, z) - (e^{\frac{\pi i m}{2}} + 1) \tilde{\chi}_{H[A[K(m),0]]}^{(+)}(\tau, z) \left. \right\}
\]

(ii) \(\tilde{\chi}_{H[A[K(m),1]]}^{(+)}\left(\begin{array}{c}
- \frac{1}{\tau}, \frac{z}{\tau}
\end{array}\right) = - \frac{1}{2} e^{-\frac{\pi i m}{4}} e^{\frac{\pi i m z}{4}} \left\{ \right. \)

\[
(1 + e^{\frac{\pi i m}{2}}) \tilde{\chi}_{H[A[K(m),1]]}^{(+)}(\tau, z) + (e^{\frac{\pi i m}{2}} - 1) \tilde{\chi}_{H[A[K(m),0]]}^{(+)}(\tau, z) \left. \right\}
\]

(iii) \(\tilde{\chi}_{H[A[K(m),0]]}^{(-)}\left(\begin{array}{c}
- \frac{1}{\tau}, \frac{z}{\tau}
\end{array}\right) = - \frac{1}{2} e^{\frac{\pi i m z}{2}} \left\{ \right. \)

\[
\tilde{\chi}_{H[A[K(m),1]]}^{(+)}(\tau, z) - e^{\frac{\pi i m}{2}} \tilde{\chi}_{H[A[K(m),p(m)]]}^{(+)}(\tau, z) \left. \right\}
\]

(iv) \(\tilde{\chi}_{H[A[K(m),1]]}^{(-)}\left(\begin{array}{c}
- \frac{1}{\tau}, \frac{z}{\tau}
\end{array}\right) = - \frac{1}{2} e^{\frac{\pi i m z}{2}} \left\{ \right. \)

\[
\tilde{\chi}_{H[A[K(m),1]]}^{(+)}(\tau, z) + e^{\frac{\pi i m}{2}} \tilde{\chi}_{H[A[K(m),p(m)]]}^{(+)}(\tau, z) \left. \right\}
\]

(v) \(\tilde{\chi}_{H[A[K(m),0]]}^{(+)}(\tau, z) - \tilde{\chi}_{H[A[K(m),1]]}^{(-)}(\tau, z) = e^{\frac{\pi i m z}{2}} \left\{ \right. \)

\[
\tilde{\chi}_{H[A[K(m),0]]}^{(-)}(\tau, z) - \tilde{\chi}_{H[A[K(m),1]]}^{(-)}(\tau, z) \left. \right\}
\]
Lemma 5.1. From Proposition 4.3, satisfy the following modular transformation properties:

\[ \tau \to -\tau, \quad z \to \frac{z}{\tau} \]

\[ \hat{c}_{\bar{H}(A[K^{(m),1}])}^{-1}(\tau, z) = -e^{\pi i \frac{m - 2}{2}} \left\{ \hat{c}_{\bar{H}(A[K^{(m),0}])}(\tau, z) + \hat{c}_{\bar{H}(A[K^{(m),1}])}(\tau, z) \right\} \]

2) \( T \)-transformation :

\[ \begin{align*}
(\pi) & \quad \hat{c}_{\bar{H}(A[K^{(m),0}])}(\tau + 1, z) = e^{-\left(\frac{\pi i}{12} + \frac{\pi i}{24}\right)} \hat{c}_{\bar{H}(A[K^{(m),0}])}(\tau, z) \\
(\pi) & \quad \hat{c}_{\bar{H}(A[K^{(m),1}])}(\tau + 1, z) = e^{-\left(\frac{3\pi i}{12} + \frac{3\pi i}{24}\right)} \hat{c}_{\bar{H}(A[K^{(m),1}])}(\tau, z) \\
(\pi) & \quad \hat{c}_{\bar{H}(A[K^{(m),0}])}(\tau + 1, z) = -e^{-\left(\frac{3\pi i}{12} + \frac{3\pi i}{24}\right)} \hat{c}_{\bar{H}(A[K^{(m),0}])}(\tau, z) \\
(\pi) & \quad \hat{c}_{\bar{H}(A[K^{(m),1}])}(\tau + 1, z) = e^{-\left(\frac{3\pi i}{12} + \frac{3\pi i}{24}\right)} \hat{c}_{\bar{H}(A[K^{(m),1}])}(\tau, z) \\
(\pi) & \quad \hat{c}_{\bar{H}(A[K^{(m),0}])}(\tau + 1, z) = e^{-\left(\frac{\pi i}{12} + \frac{\pi i}{24}\right)} \hat{c}_{\bar{H}(A[K^{(m),0}])}(\tau, z) \\
(\pi) & \quad \hat{c}_{\bar{H}(A[K^{(m),1}])}(\tau + 1, z) = e^{-\left(\frac{3\pi i}{12} + \frac{3\pi i}{24}\right)} \hat{c}_{\bar{H}(A[K^{(m),1}])}(\tau, z)
\end{align*} \]

Remark 4.1. From this Proposition 4.3 we can know all of modified (super)characters if we know only one of them, except in the case \( m \in 4\mathbb{N} \).

We note that the central charge \( c(m) \) of the \( N=3 \) module \( H(A[K^{(m),m_2}]) \) is

\[ c(m) = -6K(m) - \frac{7}{2} = \frac{3m - 1}{2} \quad (4.12) \]

5 Modified characters in the case \( m = 2 \)

In this section, we deduce the explicit formulas for the modified characters in the case \( m = 2 \), namely for the \( N=3 \) modules \( H(\Lambda^{[-1,m_2]}) \) \( (m_2 = 0, 1, 2) \) of central charge \( c(2) = \frac{5}{2} \). By Proposition 4.3, we need to consider the following 6 characters in this case:

\[ \tilde{A}^{(\pm)}_{H(\Lambda^{[-1,0]})}, \tilde{A}^{(\pm)}_{H(\Lambda^{[-1,1]})}, \tilde{A}^{(\pm)}_{H(\Lambda^{[-1,2]})} \]

The numerators of these characters are written by \( A^{[2]}_j(\tau, z) \) which, by Proposition 4.3, satisfy the following modular transformation properties:

**Lemma 5.1.**

1) \( S \)-transformation :

\[ \begin{align*}
(i) & \quad A^{[2]}_1 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = i \tau e^{\frac{\pi i}{24}} A^{[2]}_1(\tau, z) \\
(ii) & \quad A^{[2]}_2 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = -i \tau e^{\frac{3\pi i}{24}} A^{[2]}_1(\tau, z) \\
(iii) & \quad A^{[2]}_3 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = -\tau e^{\frac{\pi i}{24}} A^{[2]}_1(\tau, z) \\
(iv) & \quad A^{[2]}_4 \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \tau e^{\frac{\pi i}{24}} A^{[2]}_1(\tau, z)
\end{align*} \]
\( (v) \quad A_5^2 \left( \frac{-1}{\tau}, \frac{z}{\tau} \right) = \tau e^{\frac{\pi i z^2}{2\tau}} A_4^2(\tau, z) \)

\( (vi) \quad A_6^2 \left( \frac{-1}{\tau}, \frac{z}{\tau} \right) = -\tau e^{\frac{\pi i z^2}{2\tau}} A_3^2(\tau, z) \)

2) \text{T-transformation :}

\( (i) \quad A_1^2(\tau + 1, z) = e^{-\frac{\pi i}{4}} A_3^2(\tau, z) \)

\( (ii) \quad A_2^2(\tau + 1, z) = e^{-\frac{\pi i}{4}} A_4^2(\tau, z) \)

\( (iii) \quad A_3^2(\tau + 1, z) = e^{-\frac{\pi i}{4}} A_2^2(\tau, z) \)

\( (iv) \quad A_4^2(\tau + 1, z) = e^{-\frac{\pi i}{4}} A_1^2(\tau, z) \)

\( (v) \quad A_5^2(\tau + 1, z) = A_5(\tau, z) \)

\( (vi) \quad A_6^2(\tau + 1, z) = -A_6^2(\tau, z) \)

In order to write these functions \( A_j^2(\tau, z) \) explicitly, we consider the following functions \( g_i(\pm) \) defined above:

\[
\begin{align*}
g_1^{(+)}(\tau, z) &:= \frac{\eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{10}(\tau, z)}{\vartheta_{01}(\tau, z)} \\
g_1^{(-)}(\tau, z) &:= \frac{\eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{10}(\tau, z)}{\vartheta_{00}(\tau, z)} \\
g_2^{(+)}(\tau, z) &:= \frac{\eta(\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{01}(\tau, z)}{\vartheta_{10}(\tau, z)} \\
g_2^{(-)}(\tau, z) &:= \frac{\eta(\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{01}(\tau, z)}{\vartheta_{00}(\tau, z)} \\
g_3^{(+)}(\tau, z) &:= \frac{\eta(\tau)^5}{\eta(\tau)^2 \eta(2\tau)^2} \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{00}(\tau, z)}{\vartheta_{10}(\tau, z)} \\
g_3^{(-)}(\tau, z) &:= \frac{\eta(\tau)^5}{\eta(\tau)^2 \eta(2\tau)^2} \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{00}(\tau, z)}{\vartheta_{01}(\tau, z)}
\end{align*}
\]

\textbf{Lemma 5.2.} The functions \( g_i^{(\pm)}(\tau, z) \) defined above satisfy the following transformation properties:

1) \text{S-transformation :}

\( (i) \quad g_1^{(+)} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = -\frac{1}{2} \tau e^{\frac{\pi i z^2}{\tau}} g_2^{(+)}(\tau, z) \)

\( (ii) \quad g_1^{(-)} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = -\frac{1}{2} \tau e^{\frac{\pi i z^2}{\tau}} g_2^{(-)}(\tau, z) \)
Lemma 5.3.

2) \( T \)-transformation :

(i) \( g_1^{(+)}(\tau + 1, z) = e^{\frac{2\pi i}{\tau}} g_1^{(-)}(\tau, z) \)

(ii) \( g_1^{(-)}(\tau + 1, z) = e^{2\pi i/\tau} g_1^{(+)}(\tau, z) \)

(iii) \( g_2^{(+)}(\tau + 1, z) = g_3^{(+)}(\tau, z) \)

(iv) \( g_2^{(-)}(\tau + 1, z) = e^{2\pi i/\tau} g_3^{(-)}(\tau, z) \)

(v) \( g_3^{(+)}(\tau + 1, z) = g_2^{(+)}(\tau, z) \)

(vi) \( g_3^{(-)}(\tau + 1, z) = e^{\frac{2\pi i}{\tau}} g_2^{(-)}(\tau, z) \)

Now we see that the functions \( A_j^{[2]}(\tau, z) \) and \( g_j^{(\pm)}(\tau, z) \) are related as follows:

Lemma 5.3.

1) \( i \)

(i) \( A_1^{[2]}(\tau, z) + A_2^{[2]}(\tau, z) = 2i g_1^{(-)}(\tau, z) \)

(ii) \( A_3^{[2]}(\tau, z) + A_4^{[2]}(\tau, z) = -2i g_1^{(+)}(\tau, z) \)

(iii) \( A_1^{[2]}(\tau, z) - A_2^{[2]}(\tau, z) = g_2^{(-)}(\tau, z) \)

(iv) \( A_3^{[2]}(\tau, z) - A_4^{[2]}(\tau, z) = i g_2^{(+)}(\tau, z) \)

(v) \( A_3^{[2]}(\tau, z) - A_4^{[2]}(\tau, z) = i g_3^{(-)}(\tau, z) \)

(vi) \( A_5^{[2]}(\tau, z) + A_6^{[2]}(\tau, z) = i g_3^{(+)}(\tau, z) \)

2) \( i \)

(i) \( 2 A_1^{[2]}(\tau, z) = 2i g_1^{(-)}(\tau, z) + g_2^{(-)}(\tau, z) \)

(ii) \( 2 A_2^{[2]}(\tau, z) = 2i g_1^{(-)}(\tau, z) - g_2^{(-)}(\tau, z) \)

(iii) \( 2 A_3^{[2]}(\tau, z) = i \{ -2 g_1^{(+)}(\tau, z) + g_3^{(-)}(\tau, z) \} \)

(iv) \( 2 A_4^{[2]}(\tau, z) = -i \{ 2 g_1^{(+)}(\tau, z) + g_3^{(-)}(\tau, z) \} \)

(v) \( 2 A_5^{[2]}(\tau, z) = i \{ g_3^{(+)}(\tau, z) + g_2^{(+)}(\tau, z) \} \)

(vi) \( 2 A_6^{[2]}(\tau, z) = i \{ g_3^{(+)}(\tau, z) - g_2^{(+)}(\tau, z) \} \)
Proof. 1) In order to prove (i) and (ii), we let \( m = 2 \) in the formulas (2.8a) and (2.8c):

\[
\tilde{\Phi}^{[1,0]} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right) = \frac{1}{2} \left( A_{1}^{[2]}(\tau, z) + A_{2}^{[2]}(\tau, z) \right)
\]

(5.2a)

\[
\tilde{\Phi}^{[1,0]} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} - \frac{1}{2}, \frac{\tau}{8} \right) = \frac{1}{2} \left( A_{3}^{[2]}(\tau, z) + A_{4}^{[2]}(\tau, z) \right)
\]

(5.2b)

The LHS of these equations can be computed by using Lemma 2.7 as follows:

\[
\tilde{\Phi}^{[1,0]} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right)
= -iq^{-\frac{1}{8}} \frac{\eta(\tau)^3 \vartheta_{11}(2\tau, 2z)}{\vartheta_{11}(2\tau, z + \frac{\tau}{2} - \frac{1}{2}) \vartheta_{10}(2\tau, z - \frac{\tau}{2} + \frac{1}{2})}
\]

(5.3a)

\[
\tilde{\Phi}^{[1,0]} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2} - \frac{1}{2}, \frac{\tau}{8} \right)
= -iq^{-\frac{1}{8}} \frac{\eta(\tau)^3 \vartheta_{11}(2\tau, 2z)}{\vartheta_{11}(2\tau, z + \frac{\tau}{2} - \frac{1}{2}) \vartheta_{10}(2\tau, z - \frac{\tau}{2} + \frac{1}{2})}
\]

(5.3b)

where we used the following formulas:

\[
\vartheta_{11}(2\tau, 2z) = \frac{\eta(2\tau)}{\eta(\tau)^2} \vartheta_{11}(\tau, z) \vartheta_{10}(\tau, z)
\]

(5.4a)

\[
\vartheta_{10}(2\tau, z + \frac{\tau}{2}) \vartheta_{10}(2\tau, z - \frac{\tau}{2}) = q^{-\frac{1}{2}} \frac{\eta(2\tau)^2}{\eta(\tau)} \vartheta_{00}(\tau, z)
\]

(5.4b)

\[
\vartheta_{11}(2\tau, z + \frac{\tau}{2}) \vartheta_{11}(2\tau, z - \frac{\tau}{2}) = q^{-\frac{1}{2}} \frac{\eta(2\tau)^2}{\eta(\tau)} \vartheta_{01}(\tau, z)
\]

(5.4c)

Then, by (5.2a) and (5.3a) we obtain (i), and by (5.2b) and (5.3b) we obtain (ii). Since (i) and (ii) are thus established, (iii) is obtained by applying \( S \)-transformation to (i), (iv) is obtained by applying \( S \)-transformation to (ii), (v) is obtained by applying \( T \)-transformation to (iii), (vi) is obtained by applying \( T \)-transformation to (iv), which complete proof of 1). 2) follows from 1) immediately.

Theorem 5.1. In the case \( m = 2 \), the modified characters are as follows:

1) \( \tilde{c}_{H(\Lambda_{1}, \emptyset)}^{(\tau, z)}(\tau, z) = -\frac{1}{2} \frac{\eta(z)}{\eta(2\tau) \eta(\tau)} \cdot \vartheta_{01}(\tau, z) \)
2) \( \text{ch}_{H(A[-1, 1])}^{(+)}(\tau, z) = i \frac{\eta(2\tau)}{\eta(\frac{\tau}{2})\eta(\tau)} \cdot \vartheta_{10}(\tau, z) \)

3) \( \text{ch}_{H(A[-1, 0])}^{(-)}(\tau, z) = -i \frac{\eta(\tau)}{2} \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})^2\eta(2\tau)} \cdot \vartheta_{00}(\tau, z) \)

4) \( \text{ch}_{H(A[-1, 1])}^{(\cdot)}(\tau, z) = -i \frac{\eta(\frac{\tau}{2})\eta(2\tau)}{\eta(\tau)^4} \cdot \vartheta_{10}(\tau, z) \)

5) \( \text{ch}_{H(A[-1, 1])}^{(\cdot w(\tau^3))}(\tau, z) = \frac{i}{\sqrt{2}} \left\{ \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})^2\eta(2\tau)} \vartheta_{00}(\tau, z) + \frac{\eta(\frac{\tau}{2})^2\eta(2\tau)}{\eta(\tau)^4} \vartheta_{01}(\tau, z) \right\} \)

6) \( \text{ch}_{H(A[-1, 1])}^{(\cdot w(\tau^3))}(\tau, z) = \frac{i}{\sqrt{2}} \left\{ \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})^2\eta(2\tau)} \vartheta_{00}(\tau, z) - \frac{\eta(\frac{\tau}{2})^2\eta(2\tau)}{\eta(\tau)^4} \vartheta_{01}(\tau, z) \right\} \)

**Proof.** This theorem is obtained from Corollary 4.1 and Lemma 2.10 and Lemma 5.3 and the formula 4.3 as follows:

\[
\text{ch}_{H(A[-1, 0])}^{(+)}(\tau, z) = \frac{1}{N=3} \left( \frac{R^{(-)}(\tau, z)}{N=3} \text{ch}_{H(A[-1, 0])}^{(+)}(\tau, z) \right)
\]

\[
= - \frac{1}{2} \frac{1}{\eta(\frac{\tau}{2})\eta(2\tau)} \cdot \vartheta_{00}(\tau, z) \cdot \eta(\tau)^2 \cdot \vartheta_{11}(\tau, z) \cdot \frac{\eta(\tau)}{\eta(\frac{\tau}{2})} \cdot \frac{\eta(2\tau)}{\eta(\tau)} 
\]

\[
= - \frac{1}{2} \eta(\frac{\tau}{2}) \cdot \vartheta_{10}(\tau, z)
\]

\[
\text{ch}_{H(A[-1, 1])}^{(+)}(\tau, z) = \frac{1}{N=3} \left( \frac{R^{(+)}(\tau, z)}{N=3} \text{ch}_{H(A[-1, 1])}^{(+)}(\tau, z) \right)
\]

\[
= i \frac{1}{\eta(\frac{\tau}{2})\eta(2\tau)} \cdot \vartheta_{00}(\tau, z) \cdot \eta(2\tau)^2 \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{10}(\tau, z)}{\vartheta_{00}(\tau, z)} = i \frac{\eta(2\tau)}{\eta(\frac{\tau}{2})\eta(\tau)} \cdot \vartheta_{10}(\tau, z)
\]

\[
\text{ch}_{H(A[-1, 0])}^{(-)}(\tau, z) = \frac{1}{N=3} \left( \frac{R^{(-)}(\tau, z)}{N=3} \text{ch}_{H(A[-1, 0])}^{(-)}(\tau, z) \right)
\]

\[
= - i \frac{\eta(\frac{\tau}{2})}{\eta(\tau)^3} \cdot \vartheta_{01}(\tau, z) \cdot \eta(\tau)^5 \cdot \frac{\vartheta_{11}(\tau, z) \vartheta_{00}(\tau, z)}{\vartheta_{01}(\tau, z)} = - \frac{i}{2} \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)} \cdot \vartheta_{00}(\tau, z)
\]

\[
\text{ch}_{H(A[-1, 1])}^{(-)}(\tau, z) = \frac{1}{N=3} \left( \frac{R^{(-)}(\tau, z)}{N=3} \text{ch}_{H(A[-1, 1])}^{(-)}(\tau, z) \right)
\]
Corollary 5.1. \( m = 2 \) which, of course, coincide with Proposition 4.3.

\[
\tilde{\chi}^{(+)}_{H(\Lambda^{[-1,1]})}(\tau, z) = \frac{1}{N = 3} R^{(+) tw}(\tau, z) \left\{ \eta(\tau) \chi^{(+)}_{H(\Lambda^{[-1,1]})}(\tau, z) \right\}
\]

\[
\tilde{\chi}^{(-)}_{H(\Lambda^{[-1,1]})}(\tau, z) = \frac{1}{N = 3} R^{(-) tw}(\tau, z) \left\{ \eta(\tau) \chi^{(-)}_{H(\Lambda^{[-1,1]})}(\tau, z) \right\}
\]

Thus the proof is completed. Since, by Lemma 2.7, the modified characters are the same with the honest characters for \( H(\Lambda^{[-1,1]}) \), the formulas for the characters of \( H(\Lambda^{[-1,1]}) \) are exhibited by using “\( \chi^{(-)} \)” in place of “\( \tilde{\chi}^{(-)} \)” in this theorem.

Then by modular transformation properties of \( \eta(\tau) \) and \( \vartheta_{ab}(\tau, z) \), we obtain the following transformation properties of the modified characters in the case \( m = 2 \) which, of course, coincide with Proposition 4.3.

**Corollary 5.1.**

1) **S-transformation:**

(i) \( \tilde{\chi}^{(+)}_{H(\Lambda^{[-1,1]})}\left( \frac{-1}{\tau}, \frac{z}{\tau} \right) = i \, e^\frac{\pi iz}{\tau} \tilde{\chi}^{(+)}_{H(\Lambda^{[-1,1]})}(\tau, z) \)

(ii) \( \tilde{\chi}^{(+)}_{H(\Lambda^{[-1,1]})}\left( \frac{-1}{\tau}, \frac{z}{\tau} \right) = -i \, e^\frac{\pi iz}{\tau} \tilde{\chi}^{(+)}_{H(\Lambda^{[-1,1]})}(\tau, z) \)
In this section, we compute the correction term \( \Phi_{\text{add}}^{[1, \frac{1}{4}]} \). Hence, in the case \( m = 2 \), the \( SL_2(\mathbb{Z}) \)-invariance holds for the space in which honest characters and modified characters collaborate.

6 Honest characters in the case \( m = 2 \)

In this section, we compute the correction term \( \Phi_{\text{add}}^{[1, \frac{1}{4}]} \) to obtain the honest (super)characters of \( H(\Lambda[-1,m]) \) for \( m_2 = 0, 2 \).

Lemma 6.1.

1) \( R_{5,1} \left( \frac{\tau}{2}, \frac{\tau}{4} \right) = \begin{cases} q^{\frac{\pi}{7}} & \text{if } j = \frac{1}{2} \\ 0 & \text{if } j = \frac{3}{2} \end{cases} \)
2) \( \Phi^{[1, \frac{1}{2}]}_{\text{add}}(\tau, z + \frac{\tau}{4}, z - \frac{\tau}{4}, 0) = -\frac{1}{2} q^{\frac{\tau}{16}} [\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}](\tau, 2z) \)

3) \( \Phi^{[1, \frac{1}{2}]}_{\text{add}}(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0) = i \frac{1}{2} q^{\frac{\tau}{8}} \theta_{11}(\tau, z) \)

4) \( \Phi^{[1, \frac{1}{2}]}_{\text{add}}(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, \tau) = i \frac{1}{2} \theta_{11}(\tau, z) \)

Proof. 1) Letting \( m = 1 \) and \( w = \frac{\tau}{4} \) in (2.3), we have

\[
R_{j;1}(\tau, \frac{\tau}{4}) = \sum_{n \equiv j \mod 2} \left\{ \text{sgn} \left( n + \frac{3}{2} - j \right) - E \left( \left( n - \frac{1}{2} \right) \sqrt{\text{Im}(\tau)} \right) \right\} e^{-\pi in^2 + \pi iz^2}
\]

Putting \( n = j + 2k \), this equation is rewritten as follows:

\[
R_{j;1}(\tau, \frac{\tau}{4}) = \sum_{k \in \mathbb{Z}} \left\{ \text{sgn} \left( 2k + \frac{3}{2} \right) - E \left( \left( j + 2k - \frac{1}{2} \right) \sqrt{\text{Im}(\tau)} \right) \right\} q^{\frac{1}{2}(j+2k)(1-j-2k)}
\]

\[
= \left[ \sum_{k \geq 0} - \sum_{k < 0} \right] q^{\frac{1}{2}(j+2k)(1-j-2k)}
\]

\[
- \sum_{k \in \mathbb{Z}} E \left( \left( j + 2k - \frac{1}{2} \right) \sqrt{\text{Im}(\tau)} \right) q^{\frac{1}{2}(j+2k)(1-j-2k)}
\]

Putting (I)

\[
(R_{j;1}(\tau, \frac{\tau}{4}))_{j} = \left\{ \begin{array}{ll}
q^{\frac{\tau}{16}} & \text{if } j = \frac{1}{2} \\
0 & \text{if } j = \frac{3}{2}
\end{array} \right.
\]

and that (II)

\[
(R_{j;1}(\tau, \frac{\tau}{4}))_{j} = 0
\]

since \( E(-x) = -E(x) \). Thus we have

\[
(R_{j;1}(\tau, \frac{\tau}{4}))_{j} + (R_{j;1}(\tau, \frac{\tau}{4}))_{j} = 0
\]

proving 1).

2) By the equation (2.4a) and 1), we have

\[
\Phi^{[1, \frac{1}{2}]}_{\text{add}}(\tau, z + \frac{\tau}{4}, z - \frac{\tau}{4}, 0) = -\frac{1}{2} \left( R_{\frac{1}{2}, 1}(\tau, \frac{\tau}{4}) - R_{\frac{1}{2}, 1}(\tau, \frac{\tau}{4}) \right) [\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}](\tau, 2z)
\]
\[
= -\frac{1}{2} q^{\frac{r}{\pi}} [\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}](\tau, 2z)
\]
proving 2).

3) Letting \( \tau \to 2\tau \) in 2), we have
\[
\Phi^{[1, \frac{1}{2}]} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0 \right) = -\frac{1}{2} q^{\frac{r}{\pi}} [\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}](2\tau, 2z) .
\]
Since
\[
[\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}](2\tau, 2z) = [\theta_{1, 2} - \theta_{-1, 2}](\tau, z) = -i\vartheta_{11}(\tau, z) ,
\]
we obtain 3). 4) follows from 3) immediately. \( \square \)

Lemma 6.2.

1) \( R_{j, 1} \left( \tau, \frac{\tau}{4} - \frac{1}{2} \right) = \begin{cases} -iq^{\frac{r}{\pi}} & \text{if } j = \frac{1}{2} \\ 0 & \text{if } j = \frac{3}{2} \end{cases} \)

2) \( \Phi_{\text{add}}^{[1, \frac{1}{2}]} \left( \tau, z + \frac{\tau}{4} - \frac{1}{2}, z - \frac{\tau}{4} + \frac{1}{2}, 0 \right) = \frac{i}{2} q^{\frac{r}{\pi}} [\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}](\tau, 2z) \)

3) \( \Phi_{\text{add}}^{[1, \frac{1}{2}]} \left( 2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2}, 0 \right) = \frac{1}{2} q^{\frac{r}{\pi}} \vartheta_{11}(\tau, z) \)

4) \( \Phi_{\text{add}}^{[1, \frac{1}{2}]} \left( 2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right) = \frac{1}{2} \vartheta_{11}(\tau, z) \)

Proof. 1) By Lemma [2.1] and Lemma 6.1, we have
\[
R_{j, 1} \left( \tau, \frac{\tau}{4} - \frac{1}{2} \right) = e^{-\pi ij} R_{j, 1} \left( \tau, \frac{\tau}{4} \right) = \begin{cases} -iq^{\frac{r}{\pi}} & \text{if } j = \frac{1}{2} \\ 0 & \text{if } j = \frac{3}{2} \end{cases}
\]
proving 1). Proof of 2) and 3) and 4) is obtained by similar arguments as in the proof of Lemma 6.1. \( \square \)

Using these Lemmas, we obtain \( \Phi^{[1, \frac{1}{2}]} \) and \( \Phi^{[1, \frac{3}{2}]} \) as follows:

Proposition 6.1.

1) (i) \( \Phi^{[1, \frac{1}{2}]} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, \frac{\tau}{8} \right) = -\frac{i}{2} \left\{ \eta(\tau)^{\frac{3}{2}} \eta(2\tau)^{\frac{1}{2}} \vartheta_{11}(\tau, z) \vartheta_{01}(\tau, z) \vartheta_{00}(\tau, z) + \vartheta_{11}(\tau, z) \right\} \)

(ii) \( \Phi^{[1, \frac{3}{2}]} \left( 2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{8} \right) = -\frac{1}{2} \left\{ \eta(\tau)^{\frac{3}{2}} \vartheta_{11}(\tau, z) \vartheta_{01}(\tau, z) \vartheta_{00}(\tau, z) + \vartheta_{11}(\tau, z) \right\} \)
Proof. 1) By (2.4b), we have

\[ \Phi^{1, \frac{3}{2}} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, \frac{\tau}{8} \right) = -\frac{i}{2} \left\{ g_{3}^{-}(\tau, z) - \vartheta_{11}(\tau, z) \right\} \]

\[ = -\frac{i}{2} \left\{ \frac{\eta(\tau)^{5}}{\eta(\frac{\tau}{2})^{2} \eta(2\tau)^{2}} \frac{\vartheta_{11}(\tau, z) \vartheta_{00}(\tau, z)}{\vartheta_{01}(\tau, z)} - \vartheta_{11}(\tau, z) \right\} \]

(ii) \( \Phi^{1, \frac{3}{2}} \left( 2\tau + \frac{\tau}{2}, z + \tau - \frac{\tau}{2} = \frac{1}{2}, z + \frac{\tau}{2} \right) = -\frac{1}{2} \left\{ g_{2}^{-}(\tau, z) - \vartheta_{11}(\tau, z) \right\} \]

\[ = -\frac{1}{2} \left\{ \frac{\eta(\tau)^{5}}{\eta(\tau)} \frac{\vartheta_{11}(\tau, z) \vartheta_{01}(\tau, z)}{\vartheta_{00}(\tau, z)} - \vartheta_{11}(\tau, z) \right\} \]

The RHS of these equations are rewritten, by (2.8b) and (2.8d) and Lemma 5.3 and Lemma 6.1 and Lemma 6.2 and (5.1d) and (5.1f), as follows:

RHS of (6.2a) = \(- \frac{1}{2} \left\{ \vartheta_{3}^{-}(\tau, z) - \vartheta_{11}(\tau, z) \right\} - \frac{i}{2} \vartheta_{11}(\tau, z) \)

\[ = -\frac{i}{2} g_{3}^{-}(\tau, z) - i \frac{\eta(\tau)^{5}}{\eta(\frac{\tau}{2})^{2} \eta(2\tau)^{2}} \frac{\vartheta_{11}(\tau, z) \vartheta_{00}(\tau, z)}{\vartheta_{01}(\tau, z)} - \frac{i}{2} \vartheta_{11}(\tau, z) \]

RHS of (6.2b) = \(- \frac{1}{2} \left\{ \vartheta_{3}^{-}(\tau, z) - \vartheta_{11}(\tau, z) \right\} - \frac{1}{2} \vartheta_{11}(\tau, z) \)

\[ = -\frac{1}{2} g_{2}^{-}(\tau, z) - \frac{1}{2} \vartheta_{11}(\tau, z) = -\frac{1}{2} \left\{ \frac{\eta(\tau)^{5}}{\eta(\tau)} \frac{\vartheta_{11}(\tau, z) \vartheta_{01}(\tau, z)}{\vartheta_{00}(\tau, z)} - \frac{1}{2} \vartheta_{11}(\tau, z) \right\} \]

Thus we have proved 1).

2) Letting \( m = 1, s = \frac{1}{2}, a = 1 \) and \( \tau \to 2\tau \) in Lemma 2.5, we have

\[ \left[ \Phi^{1, \frac{3}{2}} - \Phi^{1, \frac{3}{2}} \right] \left( 2\tau, z_{1}, z_{2}, t \right) = e^{-2\pi i t} e^{\frac{\pi}{4} (z_{1} - z_{2})} q^{2(-\frac{t}{4})} \left[ \theta_{\psi, 1} - \theta_{\psi, 1} \right] (2\tau, z_{1} + 2z_{2}) \]

Letting \( t = \frac{z_{1}}{8} \), we have

\[ \left[ \Phi^{1, \frac{3}{2}} - \Phi^{1, \frac{3}{2}} \right] \left( 2\tau, z_{1}, z_{2}, \frac{z_{1}}{8} \right) = e^{\frac{\pi}{4} (z_{1} - 2z_{2})} q^{-\frac{t}{4}} \left[ \theta_{\psi, 1} - \theta_{\psi, 1} \right] (2\tau, z_{1} + z_{2}) (6.3) \]
In this equation \((6.3)\), we let \[
\begin{align*}
\{ z_1 & = z + \frac{\tau}{2} \quad \text{and} \quad z_1 = z + \frac{\tau}{2} - \frac{1}{2} \\
\{ z_2 & = z - \frac{\tau}{2} \quad \text{and} \quad z_2 = z - \frac{\tau}{2} + \frac{1}{2}.
\end{align*}
\]

Then, since \[
\begin{align*}
\{ z_1 - z_2 & = \tau \quad \text{and} \quad z_1 - z_2 = \tau - 1 \\
z_1 + z_2 & = 2z \quad \text{respectively, we have}
\end{align*}
\]

\[
\begin{align*}
\left[ \Phi_1^{(a)} - \Phi_1^{(b)} \right] (2\tau, z + \frac{\tau}{2} z - \frac{\tau}{2} - \frac{\tau}{8}) & = e^{\frac{i\pi}{4} q^{-\frac{1}{2}} [\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}]} (2\tau, 2z) = -i\theta_{11}(\tau, z) \quad (6.4a) \\
\left[ \Phi_1^{(a)} - \Phi_1^{(b)} \right] (2\tau, z + \frac{\tau}{2} z - \frac{1}{2} z - \frac{\tau}{2} + \frac{1}{2} \frac{\tau}{8}) & = e^{\frac{i\pi}{4} (\tau - 1) q^{-\frac{1}{2}} [\theta_{\frac{1}{2}, 1} - \theta_{-\frac{1}{2}, 1}]} (2\tau, 2z) = -i\theta_{11}(\tau, z) \quad (6.4b)
\end{align*}
\]

by \((6.1)\). Now the claim 2) follows from 1) and \((6.4a)\) and \((6.4b)\). \(\square\)

Then we can obtain the honest characters of the \(N=3\) modules \(H(\Lambda^{[-1,0]})\) and \(H(\Lambda^{[-1,2]})\) as follows:

**Theorem 6.1.**

1. (i) \(\chi^{(+)\Lambda^{[-1,0]}}_H(\tau, z)\)

\[
= -\frac{1}{2} \left\{ \frac{\eta(\frac{\tau}{2})}{\eta(2\tau) \eta(\tau)} \vartheta_{01}(\tau, z) + \frac{1}{\eta(\frac{\tau}{2})} \vartheta_{00}(\tau, z) \right\}
\]

(ii) \(\chi^{(+)\Lambda^{[-1,2]}}_H(\tau, z)\)

\[
= -\frac{1}{2} \left\{ \frac{\eta(\frac{\tau}{2})}{\eta(2\tau) \eta(\tau)} \vartheta_{01}(\tau, z) - \frac{1}{\eta(\frac{\tau}{2})} \vartheta_{00}(\tau, z) \right\}
\]

2. (i) \(\chi^{(-)\Lambda^{[-1,0]}}_H(\tau, z)\)

\[
= -\frac{i}{2} \left\{ \frac{\eta(\tau)^2}{\eta(\tau)^2 \eta(2\tau)^2} \vartheta_{00}(\tau, z) + \frac{\eta(\frac{\tau}{2})}{\eta(\tau)^2} \vartheta_{01}(\tau, z) \right\}
\]

(ii) \(\chi^{(-)\Lambda^{[-1,2]}}_H(\tau, z)\)

\[
= -\frac{i}{2} \left\{ \frac{\eta(\tau)^2}{\eta(\tau)^2 \eta(2\tau)^2} \vartheta_{00}(\tau, z) - \frac{\eta(\frac{\tau}{2})}{\eta(\tau)^2} \vartheta_{01}(\tau, z) \right\}
\]

**Corollary 6.1.** The modified (super)characters \(\tilde{\chi}^{(\pm)}_{H(\Lambda^{[-1,0]})}\) are written by the honest (super)characters as follows:

\[
\tilde{\chi}_{H(\Lambda^{[-1,0]})}(\tau, z) = \frac{1}{2} \left\{ \chi^{(\pm)\Lambda^{[-1,0]}}_H(\tau, z) + \chi^{(\pm)\Lambda^{[-1,2]}}_H(\tau, z) \right\}
\]
Proof. This is obvious from Theorems 5.1 and 6.1.

Then, by the above Corollary 6.1 and Remark 5.1 we obtain the following:

**Corollary 6.2.** The linear space spanned by
\[ \text{ch}_{H(\Lambda[−1,i])}^{(±)}(\tau, z), \quad [\text{ch}_{H(\Lambda[−1,0])}^{(±)} + \text{ch}_{H(\Lambda[−1,i])}^{(±)}](\tau, z), \quad \text{ch}_{H(\Lambda[−1,i])}^{(+\text{tw}(\sigma\tau))}(\tau, z) \]
is $SL_2(\mathbb{Z})$-invariant. Namely, though the space of honest characters is not $SL_2(\mathbb{Z})$-invariant, it contains (non-trivial) $SL_2(\mathbb{Z})$-invariant subspace.

### 7 Asymptotics of characters in the case $m = 2$

In this section we consider the asymptotic behavior of characters of N=3 modules $H(\Lambda[−1,m\mathbb{Z}])$ as $\tau \downarrow 0$, namely $\tau = iT$ ($T > 0$) and $T \to 0$.

**Lemma 7.1.** For $a \in \mathbb{C}$, the asymptotics of $\vartheta_{ab}(\tau, a\tau)$ as $\tau \downarrow 0$ are as follows:

1) $\vartheta_{00}(\tau, a\tau) \sim (-i\tau)^{-\frac{1}{2}}$
2) $\vartheta_{01}(\tau, a\tau) \sim (-i\tau)^{-\frac{1}{2}} \cdot 2 \cos(a\pi) \ e^{-\frac{\pi}{4}\tau}$
3) $\vartheta_{10}(\tau, a\tau) \sim (-i\tau)^{-\frac{1}{2}}$
4) $\vartheta_{11}(\tau, a\tau) \sim (-i\tau)^{-\frac{1}{2}} \cdot 2i \sin(a\pi) \ e^{-\frac{\pi}{4}\tau}$

Proof. These are obtained easily from the $S$-transformation property of $\vartheta_{ab}(\tau, z)$:
\[
\vartheta_{ab}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = (-i)^a (-i\tau)^{\frac{1}{2}} e^{\frac{a\pi}{2} \tau} \vartheta_{ba}(\tau, z)
\]
and the power series expansion of $\vartheta_{ab}(\tau, z)$:
\[
\vartheta_{00}(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n z} q^{\frac{n^2}{2}}
\]
\[
\vartheta_{01}(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi i n z} q^{\frac{n^2}{2}}
\]
\[
\vartheta_{10}(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2\pi i (n+\frac{1}{2}) z} q^{\frac{1}{2}(n+\frac{1}{2})^2}
\]
\[
\vartheta_{11}(\tau, z) = i \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi i (n+\frac{1}{2}) z} q^{\frac{1}{2}(n+\frac{1}{2})^2}
\]

Then, by using
\[
\eta(\tau) \sim (-i\tau)^{\frac{1}{2}} e^{-\frac{\pi}{4}\tau}
\]
and Lemma 7.1 we obtain the asymptotics of characters of N=3 modules as follows:
**Proposition 7.1.** For \( a \in \mathbb{C} \), the asymptotics of the honest and modified (super)characters are as follows:

1) **Asymptotics of characters:**
   
   (i) \( \tilde{\text{ch}}^{(+)}_{H(A^{-1,0})}(\tau, a\tau) \sim \tau^{10} \cdot -2 \cos(a\pi) e^{-\frac{\pi}{2\tau} \varphi} \)
   
   (ii) \( \text{ch}^{(+)}_{H(A^{-1,0})}(\tau, a\tau) \sim \tau^{10} \cdot -\frac{1}{2} (-i\tau)^{\frac{3}{2}} e^{\frac{\pi}{2\tau} \varphi} \)
   
   (iii) \( \text{ch}^{(+)}_{H(A^{-1,2})}(\tau, a\tau) \sim \tau^{10} \cdot \frac{1}{2} (-i\tau)^{\frac{3}{2}} e^{\frac{\pi}{2\tau} \varphi} \)
   
   (iv) \( \text{ch}^{(+)}_{H(A^{-1,1})}(\tau, a\tau) \sim \tau^{10} \cdot \frac{i}{2} e^{\frac{\pi}{2\tau} \varphi} \)

2) **Asymptotics of super-characters:**

   (i) \( \tilde{\text{ch}}^{(-)}_{H(A^{-1,0})}(\tau, a\tau) \sim -\frac{i}{\sqrt{2}} e^{\frac{\pi}{2\tau} \varphi} \)

   (ii) \( \text{ch}^{(-)}_{H(A^{-1,m_2})}(\tau, a\tau) \sim -\frac{i}{\sqrt{2}} e^{\frac{\pi}{2\tau} \varphi} \quad (m_2 = 0, 1, 2) \)

**Proof.** First we note that (7.1) gives the following asymptotics:

\[
\begin{align*}
\frac{\eta(\frac{2}{a})}{\eta(2\tau)\eta(\tau)} & \sim \tau^{10} \cdot 2 (-i\tau)^{\frac{3}{2}} e^{-\frac{\pi}{2\tau} \varphi} \\
\frac{\eta(2\tau)}{\eta(\frac{2}{a})\eta(\tau)} & \sim \tau^{10} \cdot \frac{1}{2} (-i\tau)^{\frac{3}{2}} e^{\frac{\pi}{2\tau} \varphi} \\
\frac{\eta(\tau)^2}{\eta(\frac{2}{a})\eta(2\tau^2)} & \sim \sqrt{2} (-i\tau)^{\frac{3}{2}} e^{\frac{\pi}{2\tau} \varphi} \\
\frac{\eta(\frac{2}{a})\eta(2\tau^2)}{\eta(\tau)^4} & \sim \frac{1}{\sqrt{2}} (-i\tau)^{\frac{3}{2}} e^{\frac{\pi}{2\tau} \varphi} \\
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{\eta(\frac{2}{a})\eta(2\tau)} & \sim -i\tau e^{\frac{\pi}{2\tau} \varphi} \\
\frac{\eta(\frac{2}{a})}{\eta(\tau)^3} & \sim \frac{1}{\sqrt{2}} (-i\tau)^{\frac{3}{2}} e^{\frac{\pi}{2\tau} \varphi}
\end{align*}
\]

Then, by these formulas (7.2a) and (7.2b) and Theorem 5.1 and Lemma 7.1 we have

\[
\begin{align*}
\tilde{\text{ch}}^{(+)}_{H(A^{-1,0})}(\tau, a\tau) & = -\frac{1}{2} \cdot \frac{\eta(\frac{2}{a})}{\eta(2\tau)\eta(\tau)} \cdot \vartheta_{01}(\tau, a\tau) \\
& \sim \tau^{10} \cdot -\frac{1}{2} \cdot 2 (-i\tau)^{\frac{3}{2}} e^{-\frac{\pi}{2\tau} \varphi} \cdot (-i\tau)^{-\frac{3}{2}} \cdot 2 \cos(a\pi) e^{-\frac{\pi}{2\tau} \varphi} = -2 \cos(a\pi) e^{-\frac{\pi}{2\tau} \varphi} \\
\text{ch}^{(+)}_{H(A^{-1,1})}(\tau, a\tau) & = i \cdot \frac{\eta(2\tau)}{\eta(\frac{2}{a})\eta(\tau)} \cdot \vartheta_{10}(\tau, a\tau)
\end{align*}
\]
In this section, we consider the relations of modified characters between Proposition 8.1.

Modified characters in the case $m$ follows:

\[ \sim \frac{1}{2} (-i \tau)^\frac{1}{2} e^{\frac{\pi i}{12}} \cdot (-i \tau)^{-\frac{1}{2}} = \frac{i}{2} e^{\frac{\pi i}{12}} \]

\[ \tilde{\chi}_{H^{(-)}_{A^{-1,0}}} (\tau, a \tau) = -\frac{i}{2} \frac{\eta(\tau)^2}{\eta(2 \tau)^2} \cdot \vartheta_{00}(\tau, a \tau) \]

\[ \sim \frac{1}{2} \sqrt{2} (-i \tau)^\frac{1}{2} e^{\frac{\pi i}{12}} \cdot (-i \tau)^{-\frac{1}{2}} = -\frac{i}{\sqrt{2}} e^{\frac{\pi i}{12}} \]

\[ \chi^{(-)}_{H^{(--)}_{A^{-1,1}}} (\tau, a \tau) = -\frac{i}{2} \frac{\eta(\tau)^2}{\eta(2 \tau)^2} \cdot \vartheta_{10}(\tau, a \tau) \]

\[ \sim \frac{1}{\sqrt{2}} (-i \tau)^\frac{1}{2} e^{\frac{\pi i}{12}} \cdot (-i \tau)^{-\frac{1}{2}} = -\frac{i}{\sqrt{2}} e^{\frac{\pi i}{12}} \]

and

\[ \frac{1}{\eta(\tau)^2} \vartheta_{00}(\tau, a \tau) \sim (-i \tau) e^{\frac{\pi i}{12}} \cdot (-i \tau)^{-\frac{1}{2}} = (-i \tau)^{\frac{1}{2}} e^{\frac{\pi i}{12}} \]

\[ \frac{\eta(\tau)^2}{\eta(2 \tau)^2} \vartheta_{10}(\tau, a \tau) \sim \frac{1}{\sqrt{2}} (-i \tau) e^{\frac{\pi i}{12}} \cdot (-i \tau)^{-\frac{1}{2}} \cdot 2 \cos(a \tau) e^{-\frac{\pi i}{12}} \]

\[ = \sqrt{2} \cos(a \tau) (-i \tau)^{\frac{1}{2}} e^{-\frac{\pi i}{12}} \]

Then, by Theorem [1], the asymptotics of honest (super)characters are as follows:

\[ \chi^{(+)}_{H^{(+)_{A^{-1,0}}}} (\tau, a \tau) \sim -\frac{1}{2} (-i \tau)^{\frac{1}{2}} e^{\frac{\pi i}{12}} \]

\[ \chi^{(+)}_{H^{(+)_{A^{-1,1}}}} (\tau, a \tau) \sim \frac{1}{2} (-i \tau)^{\frac{1}{2}} e^{\frac{\pi i}{12}} \]

\[ \chi^{(-)}_{H^{(-)}_{A^{-1,m_2}}} (\tau, a \tau) \sim -\frac{i}{\sqrt{2}} e^{\frac{\pi i}{12}} \quad (m_2 = 0, 2). \]

Thus the proof of Proposition is completed. \hfill \Box

8 Modified characters in the case $m = 4$

In this section, we consider the relations of modified characters between $m$ and $2m$.

**Proposition 8.1.** \( \overset{0}{A}^{|m|}_{j}(\tau, z) \)'s and \( \overset{0}{A}^{2|m|}_{j}(\tau, z) \)'s are related to each other by the following formulas:

\[ 2 \overset{0}{A}^{m|}_{3}(2 \tau, 2z) = \overset{0}{A}^{2|m|}_{1}(\tau, z) + \overset{0}{A}^{2|m|}_{2}(\tau, z) \quad (8.1a) \]

\[ \overset{0}{A}^{m|}_{6}(\frac{\tau}{2}, z) = \overset{0}{A}^{2|m|}_{1}(\tau, z) + \overset{0}{A}^{2|m|}_{2}(\tau, z) \quad e^{-\pi \tau m} \overset{0}{A}^{2|m|}_{2}(\tau, z) \quad (8.1b) \]

\[ 2 \overset{0}{A}^{m|}_{4}(2 \tau, 2z) = \overset{0}{A}^{2|m|}_{3}(\tau, z) + \overset{0}{A}^{2|m|}_{4}(\tau, z) \quad (8.1c) \]
\[
e^{\frac{\pi im}{2}} A^m_6 \left( \frac{\tau + 1}{2}, z \right) = A^{[2m]}_3(\tau, z) + e^{-\pi im} A^{[2m]}_4(\tau, z) \quad (8.1d)
\]
\[
A^m_5 \left( \frac{\tau + 1}{2}, z \right) = A^{[2m]}_5(\tau, z) + e^{\pi im} A^{[2m]}_6(\tau, z) \quad (8.1e)
\]
\[
A^m_6 \left( -\frac{1}{2\tau} + \frac{1}{2}, \frac{z}{\tau} \right) = \tau e^{\frac{\pi im}{2}} e^{\frac{\pi im}{2}} \left[ A^{[2m]}_5(\tau, z) + A^{[2m]}_6(\tau, z) \right] \quad (8.1f)
\]

**Proof.** First look at the formula for \( A^{[m]}_2(\tau, z) \) in (2.6b) and (2.7):
\[
A^{[m]}_2(\tau, z) = \Phi_{[m,0]} \left( \tau, \frac{z}{2} + \frac{\tau - 1}{2}, \frac{z}{2} - \frac{\tau + 1}{2}, \frac{\tau - 1}{16} \right).
\]
Letting \( \tau \to \tau - 1 \), we have
\[
A^{[m]}_2(\tau - 1, z) = \Phi_{[m,0]} \left( \tau, \frac{z}{2} + \frac{\tau - 1}{2}, \frac{z}{2} - \frac{\tau + 1}{2}, \frac{\tau - 1}{16} \right) = e^{\frac{\pi im}{2}} \Phi_{[m,0]} \left( \tau, \frac{z}{2} + \frac{\tau - 1}{2}, \frac{z}{2} - \frac{\tau + 1}{2}, \frac{\tau - 1}{16} \right)
\]
Letting \( \tau \to 2\tau \) and \( z \to 2z \), we have
\[
A^{[m]}_2(2\tau - 1, 2z) = e^{\frac{\pi im}{2}} \Phi_{[m,0]} \left( 2\tau, \frac{z}{2} + \frac{\tau - 1}{2}, \frac{z}{2} - \frac{\tau + 1}{2}, \frac{\tau - 1}{8} \right) \quad (8.2a)
\]
By Lemma 2.5 and the formulas (2.6a) and (2.6b), the RHS of this equation is written as follows:
\[
\text{RHS of } (8.2a) = e^{\frac{\pi im}{2}} \cdot \frac{1}{2} \left\{ A^{[2m]}_1(\tau, z) + A^{[2m]}_2(\tau, z) \right\} \quad (8.2b)
\]
The LHS of (8.2a) is written, by using the T-transformation properties in Lemma 2.10 as follows:
\[
\text{LHS of } (8.2a) = e^{\frac{\pi im}{2}} A^{[m]}_3(2\tau, 2z) \quad (8.2c)
\]
Then by (8.2a) and (8.2b) and (8.2c), we have
\[
A^{[m]}_3(2\tau, 2z) = \frac{1}{2} \left\{ A^{[2m]}_1(\tau, z) + A^{[2m]}_2(\tau, z) \right\},
\]
proving (8.1a). Then (8.1b) is obtained by applying S-transformation to (8.1a). (8.1c) is obtained by applying T-transformation to (8.1a). (8.1d) is obtained by applying T-transformation to (8.1b). (8.1e) is obtained by applying S-transformation to (8.1c). (8.1f) is obtained by applying S-transformation to (8.1d).

Writing the formulas (8.1a) \( \sim \) (8.1c) in the case \( m = 2 \), we obtain the following:
Lemma 8.1. $\tilde{A}_j^{(4)}$’s are related to $A_j^{[4]}$’s as follows:

1) $2\tilde{A}_0^{[2]}(2\tau, 2z) = \tilde{A}_1^{[4]}(\tau, z) + A_2^{[4]}(\tau, z)$

2) $2\tilde{A}_4^{[2]}(2\tau, 2z) = A_3^{[4]}(\tau, z) + \tilde{A}_4^{[4]}(\tau, z)$

3) $\tilde{A}_5^{[2]}(\frac{\tau}{2}, z) = A_3^{[4]}(\tau, z) + \tilde{A}_6^{[4]}(\tau, z)$

4) $2\tilde{A}_0^{[4]}(2\tau, 2z) = \tilde{A}_6^{[2]}(\frac{\tau}{2}, z)$

Then by Corollary 4.1, we obtain the explicit formula for $\tilde{\chi}_{H(\Lambda(K,4),1)}^{(+)}(\tau, z)$ as follows:

Proposition 8.2.

1) $\tilde{\chi}_{H(\Lambda(K,4),1)}^{(+)}(\tau, z)$

$$= \frac{i}{2} \frac{1}{\eta(2\tau)^2} \frac{2 \eta(2\tau)^5}{\eta(2\tau)^2 \eta(4\tau)^2} \cdot \vartheta_{00}(2\tau, 2z)$$

2) $\tilde{\chi}_{H(\Lambda(K,4),1)}^{(-)}(\tau, z)$

$$= -\frac{i}{2} \frac{\eta(2\tau)}{\eta(2\tau)^2} \cdot \vartheta_{10}(\tau, z) \left\{ \begin{array}{c} \frac{\eta(2\tau)^5}{\eta(2\tau)^2 \eta(4\tau)^2} \cdot \vartheta_{00}(2\tau, 2z) + \frac{2 \eta(4\tau)^2}{\eta(2\tau)} \cdot \vartheta_{10}(2\tau, 2z) \end{array} \right\}$$

Proof. 1) We compute $\tilde{A}_2^{[4]}(\tau, z) + A_1^{[4]}(\tau, z)$ by using Lemma 8.1

$$A_2^{[4]}(\tau, z) + A_1^{[4]}(\tau, z) = 2\tilde{A}_3^{[2]}(2\tau, 2z) = i \left\{ -2g_1^{(+)}(2\tau, 2z) + g_0^{(-)}(2\tau, 2z) \right\}$$

$$= i \left\{ -2 \frac{\eta(4\tau)^2}{\eta(2\tau)} \cdot \vartheta_{11}(2\tau, 2z) \frac{\vartheta_{10}(2\tau, 2z)}{\vartheta_{00}(2\tau, 2z)} \right. \right.$$ 

$$\left. + \frac{\eta(2\tau)^5}{\eta(2\tau)^2 \eta(4\tau)^2} \cdot \vartheta_{11}(2\tau, 2z) \frac{\vartheta_{00}(2\tau, 2z)}{\vartheta_{01}(2\tau, 2z)} \right\}$$

$$= i \frac{\vartheta_{11}(2\tau, 2z)}{\vartheta_{01}(2\tau, 2z)} \left\{ -2 \frac{\eta(4\tau)^2}{\eta(2\tau)} \cdot \vartheta_{10}(2\tau, 2z) + \frac{\eta(2\tau)^5}{\eta(2\tau)^2 \eta(4\tau)^2} \cdot \vartheta_{00}(2\tau, 2z) \right\}$$

$$= i \frac{\vartheta_{11}(\tau, z)}{\vartheta_{01}(\tau, z)} \frac{\vartheta_{10}(\tau, z)}{\vartheta_{00}(\tau, z)} \left\{ -2 \frac{\eta(4\tau)^2}{\eta(2\tau)} \cdot \vartheta_{10}(2\tau, 2z) + \frac{\eta(2\tau)^5}{\eta(2\tau)^2 \eta(4\tau)^2} \cdot \vartheta_{00}(2\tau, 2z) \right\}$$

Then, by Corollary 4.1, we have

$$\tilde{\chi}_{H(\Lambda(K,4),1)}^{(+)}(\tau, z) = \frac{1}{N=3} \cdot \frac{1}{2} \cdot \left[ A_2^{[4]} + A_1^{[4]} \right](\tau, z)$$


\[ = \frac{i}{2} \left\{ \frac{\eta(2\tau)}{\eta(2\tau)} \cdot \vartheta_{01}(2\tau, z) \cdot \vartheta_{11}(2\tau, z) \cdot \vartheta_{10}(2\tau, z) \right\} \]

proving 1).

2) Next we compute \( \tilde{A}_3^{[4]}(\tau, z) \) and \( \tilde{A}_4^{[4]}(\tau, z) \) by using Lemma 8.1.

\[ \tilde{A}_3^{[4]}(\tau, z) + \tilde{A}_4^{[4]}(\tau, z) = 2 \tilde{A}_1^{[4]}(2\tau, 2z) = -i \left\{ 2g_1^{(+)}(2\tau, 2z) + g_3^{-}(2\tau, 2z) \right\} \]

Then, by Corollary 8.1, we have

\[ \tilde{\chi}_{H(\Lambda(K, 0), 1)}^{(-)}(\tau, z) = \frac{1}{\eta(2\tau)} \cdot \left\{ \frac{\eta(4\tau^2)}{\eta(4\tau)} \cdot \vartheta_{11}(4\tau, z) \cdot \vartheta_{10}(4\tau, z) \right\} \]

proving 2).

**Corollary 8.1.** When \( m = 4 \), the modified character and supercharacter \( \tilde{\chi}_{H(\Lambda(K, 0), 1)}^{(\pm)}(\tau, z) \) are holomorphic functions of \((\tau, z) \in \mathbb{C} \times \mathbb{C} \).

The modified character \( \tilde{\chi}_{H(\Lambda(K, 0), 1)}^{(±)}(\tau, z) \) cannot be obtained by the argument in this section, but further analysis via another approach suggests the following:
Conjecture 8.1. \( \tilde{c}_H^{(+)}(\Lambda_{[K^4,0]}) (\tau, z) \) will be written in the form

\[
\tilde{c}_H^{(+)}(\Lambda_{[K^4,0]}) (\tau, z) = \frac{1}{\eta^2(\tau)} \left\{ C_1(\tau) \vartheta_{01}(\tau, z)^2 + C_2(\tau) \vartheta_{10}(\tau, z)^2 \right\}
\]

(8.3)

where \( C_i(\tau) \) are holomorphic functions in \( \tau \in \mathbb{C}_+ \) satisfying \( C_i(-\frac{1}{\tau}) = -C_i(\tau) \).

9 Honest characters in the case \( m = 4 \)

For \( m \in \mathbb{N} \) and \( s \in \frac{1}{2} \mathbb{Z} \), we define the functions \( P^{[m,s]}(\tau, z) \) and \( Q^{[m,s]}(\tau, z) \) as follows:

\[
P^{[m,s]}(\tau, z) := \Phi^{[2m,s]}(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0) \quad \text{(9.1a)}
\]

\[
Q^{[m,s]}(\tau, z) := \Phi^{[2m,s]}(2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2}, 0) \quad \text{(9.1b)}
\]

Lemma 9.1. Let \( m \in \mathbb{N} \) and \( s \in \frac{1}{2} \mathbb{Z} \). Then the following formulas hold:

1) \( P^{[m,s+1]}(\tau, z) = P^{[m,s]}(\tau, z) = -q^{s+m} \left[ \theta_s, \varphi - \theta_{-s}, \varphi \right](2\tau, 2z) \)

2) \( Q^{[m,s+1]}(\tau, z) = Q^{[m,s]}(\tau, z) = -e^{-\pi i s} q^{s+m} \left[ \theta_s, \varphi - \theta_{-s}, \varphi \right](2\tau, 2z) \)

Proof. To prove these formulas we use the following formula which is obtained from Lemma 2.8:

\[
\Phi^{[2m,s]}(2\tau, z_1, z_2, 0) - \Phi^{[2m,s+1]}(2\tau, z_1, z_2, 0) = e^{\pi i s (z_1 - z_2)} q^{-\frac{s}{m}} \left[ \theta_s, \varphi - \theta_{-s}, \varphi \right](2\tau, z_1 + z_2)
\]

(9.2)

Putting \( z_1 = z + \frac{\tau}{2} \) and \( z_2 = z - \frac{\tau}{2} \), this formula (9.2) gives

\[
\Phi^{[2m,s]}(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0) - \Phi^{[2m,s+1]}(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0) = e^{\pi i s} q^{-\frac{s}{m}} \left[ \theta_s, \varphi - \theta_{-s}, \varphi \right](2\tau, 2z)
\]

proving 1).

Next, putting \( z_1 = z + \frac{\tau}{2} + \frac{1}{2} \) and \( z_2 = z - \frac{\tau}{2} + \frac{1}{2} \), the formula (9.2) gives

\[
\Phi^{[2m,s]}(2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2}, 0) - \Phi^{[2m,s+1]}(2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2}, 0) = e^{-\pi i s} q^{\frac{s}{m}} \left[ \theta_s, \varphi - \theta_{-s}, \varphi \right](2\tau, 2z)
\]

proving 2).
Lemma 9.2. Let \( m \in \mathbb{N} \) and \( s \in \frac{1}{2}\mathbb{Z} \). Then

1. \( 2P^{[m,s]}(2\tau, 2z) = P^{[2m,2s]}(\tau, z) + e^{-2\pi is}Q^{[2m,2s]}(\tau, z) \)

2. \( \begin{align*} \text{(i)} & \quad P^{[2m,2s]}(\tau, z) = P^{[m,s]}(2\tau, 2z) + P^{[m,s+\frac{1}{2}]}(2\tau, 2z) \\ \text{(ii)} & \quad Q^{[2m,2s]}(\tau, z) = e^{2\pi is}\left\{P^{[m,s]}(2\tau, 2z) - P^{[m,s+\frac{1}{2}]}(2\tau, 2z)\right\} \end{align*} \)

Proof. 1) By (9.1a) and Lemma 2.5, we have

\[ 2P^{[m,s]}(2\tau, 2z) = 2\Phi^{[m,s]}(4\tau, 2z + \tau, 2z - \tau, 0) \]

where (I) is computed by using Lemma 2.5 as follows:

\[
(I) = \Phi^{[m,2s]}(2\tau, \left( z + \frac{\tau}{2} + \frac{1}{2}\right) - 1, \left( z - \frac{\tau}{2} - \frac{1}{2}\right) + 1, 0) = \Phi^{(+)[m,2s]}(2\tau, z + \frac{\tau}{2} - \frac{1}{2}, z - \frac{\tau}{2} + \frac{1}{2}, 0) = Q^{[2m,2s]}(\tau, z)
\]

Thus we have

\[ 2P^{[m,s]}(2\tau, 2z) = P^{[2m,2s]}(\tau, z) + e^{-2\pi is}Q^{[2m,2s]}(\tau, z), \quad (9.3a) \]

proving 1). In order to prove 2), we let \( s \to s + \frac{1}{2} \) in (9.3a). Then, by using Lemma 9.1, we have

\[ 2P^{[m,s+\frac{1}{2}]}(2\tau, 2z) = P^{[2m,2s+1]}(\tau, z) + e^{-2\pi i(s+\frac{1}{2})}Q^{[2m,2s+1]}(\tau, z) \]

\[
= \left\{ \begin{align*} P^{[2m,2s]}(\tau, z) & - q^{\frac{\tau}{4} - \frac{(2is)^2}{4m}} \theta_{2s, 2m - \theta_{-2s, 2m}}(2\tau, 2z) \\ - e^{-2\pi is}Q^{[2m,2s]}(\tau, z) & - e^{2\pi is}q^{\frac{\tau}{4} - \frac{(2is)^2}{4m}} \theta_{2s, 2m - \theta_{-2s, 2m}}(2\tau, 2z) \end{align*} \right\}
\]

\[ = P^{[2m,2s]}(\tau, z) - e^{-2\pi is}Q^{[2m,2s]}(\tau, z) \quad (9.3b) \]

Then by making \((9.3a) \pm (9.3b)\), we have

\[
2P^{[m,s]}(2\tau, 2z) + 2P^{[m,s+\frac{1}{2}]}(2\tau, 2z) = 2P^{[2m,2s]}(\tau, z) \]

\[
2P^{[m,s]}(2\tau, 2z) - 2P^{[m,s+\frac{1}{2}]}(2\tau, 2z) = 2e^{-2\pi is}Q^{[2m,2s]}(\tau, z)
\]

so we have

\[
\begin{align*}
\text{either} & \quad P^{[2m,2s]}(\tau, z) = P^{[m,s]}(2\tau, 2z) + P^{[m,s+\frac{1}{2}]}(2\tau, 2z) \\
& \quad Q^{[2m,2s]}(\tau, z) = e^{2\pi is}\left\{P^{[m,s]}(2\tau, 2z) - P^{[m,s+\frac{1}{2}]}(2\tau, 2z)\right\}
\end{align*}
\]

proving 2).
In order to compute the honest characters in the case \( m = 4 \), we need to know \( P^{[4,s]}(\tau, z) \) and \( Q^{[4,s]}(\tau, z) \), which are obtained from Lemma 9.2 as follows:

**Lemma 9.3.**

1) \( P^{[2,s]}(\tau, z) \) (\( s \in \frac{1}{2} \mathbb{Z} \) such that \( \frac{1}{3} \leq s \leq \frac{3}{2} \)) are as follows:

(i) \( P^{[2,2]}(\tau, z) = -\frac{i}{2} q^{\frac{1}{2}} \left\{ g_3(\tau, z) + \theta_{11}(\tau, z) \right\} \)

\[
= -\frac{i}{2} q^{\frac{1}{2}} \left\{ \eta(\tau)^{\frac{5}{2}} \eta(\tau)^{2} \cdot \frac{\theta_{11}(\tau, z) \theta_{00}(\tau, z)}{\theta_{01}(\tau, z)} + \theta_{11}(\tau, z) \right\}
\]

(ii) \( P^{[2,1]}(\tau, z) = -iq^{\frac{1}{2}} g_1(\tau, z) = -iq^{\frac{1}{2}} \eta(2\tau)^{2} \cdot \frac{\theta_{11}(\tau, z) \theta_{10}(\tau, z)}{\theta_{01}(\tau, z)} \)

(iii) \( P^{[2,\frac{3}{2}]}(\tau, z) = -\frac{i}{2} q^{\frac{1}{2}} \left\{ g_3(\tau, z) - \theta_{11}(\tau, z) \right\} \)

\[
= -\frac{i}{2} q^{\frac{1}{2}} \left\{ \eta(\tau)^{\frac{5}{2}} \eta(\tau)^{2} \cdot \frac{\theta_{11}(\tau, z) \theta_{00}(\tau, z)}{\theta_{01}(\tau, z)} - \theta_{11}(\tau, z) \right\}
\]

2) \( P^{[4,s]}(\tau, z) \) (\( s = 1, 2 \)) are as follows:

(i) \( P^{[4,1]}(\tau, z) = -\frac{i}{2} q^{\frac{1}{2}} \left\{ g_3(\tau, z) + 2 g_1(\tau, z) + \theta_{11}(\tau, z) \right\} \)

\[
= -\frac{i}{2} q^{\frac{1}{2}} \theta_{11}(\tau, z) \left[ \eta(\tau)^{\frac{5}{2}} \eta(\tau)^{2} \cdot \frac{\theta_{00}(\tau, z)}{\theta_{01}(\tau, z)} + \frac{\eta(4\tau)^{2}}{\eta(2\tau)} \cdot \frac{\theta_{10}(\tau, z)}{\theta_{01}(\tau, z)} + 1 \right]
\]

(ii) \( P^{[4,2]}(\tau, z) = -\frac{i}{2} q^{\frac{1}{2}} \left\{ g_3(\tau, z) + 2 g_1(\tau, z) - \theta_{11}(\tau, z) \right\} \)

\[
= -\frac{i}{2} q^{\frac{1}{2}} \theta_{11}(\tau, z) \left[ \eta(\tau)^{\frac{5}{2}} \eta(\tau)^{2} \cdot \frac{\theta_{00}(\tau, z)}{\theta_{01}(\tau, z)} + \frac{\eta(4\tau)^{2}}{\eta(2\tau)} \cdot \frac{\theta_{10}(\tau, z)}{\theta_{01}(\tau, z)} - 1 \right]
\]

3) \( Q^{[4,s]}(\tau, z) \) (\( s = 1, 2 \)) are as follows:

(i) \( Q^{[4,1]}(\tau, z) = \frac{i}{2} q^{\frac{1}{2}} \left\{ g_3(\tau, z) - 2 g_1(\tau, z) + \theta_{11}(\tau, z) \right\} \)

\[
= \frac{i}{2} q^{\frac{1}{2}} \theta_{11}(\tau, z) \left[ \eta(\tau)^{\frac{5}{2}} \eta(\tau)^{2} \cdot \frac{\theta_{00}(\tau, z)}{\theta_{01}(\tau, z)} - \frac{\eta(4\tau)^{2}}{\eta(2\tau)} \cdot \frac{\theta_{10}(\tau, z)}{\theta_{01}(\tau, z)} + 1 \right]
\]

(ii) \( Q^{[4,2]}(\tau, z) = \frac{i}{2} q^{\frac{1}{2}} \left\{ g_3(\tau, z) - 2 g_1(\tau, z) - \theta_{11}(\tau, z) \right\} \)

\[
= \frac{i}{2} q^{\frac{1}{2}} \theta_{11}(\tau, z) \left[ \eta(\tau)^{\frac{5}{2}} \eta(\tau)^{2} \cdot \frac{\theta_{00}(\tau, z)}{\theta_{01}(\tau, z)} - \frac{\eta(4\tau)^{2}}{\eta(2\tau)} \cdot \frac{\theta_{10}(\tau, z)}{\theta_{01}(\tau, z)} - 1 \right]
\]
Proof. 1) follows from definition (9.1a) of $P^{[m,s]}(\tau, z)$ and Proposition 6.1 and the formula (5.3b). 2) and 3) follow from 1) and Lemma 9.2.

Then, in the case $m = 4$, we obtain some of honest characters as follows:

**Proposition 9.1.** In the case $m = 4$, the honest (super)characters of $H(\Lambda^{[K(4), m]})$ ($m_2 = 1, 3$) are given by the following formulas:

1) (i) $\text{ch}^{(+)}_{H(\Lambda^{[K(4), 1]})}(\tau, z) = \frac{i}{2} \left( \frac{1}{\eta(\tau)} \frac{1}{\eta(2\tau)} \right) \frac{\vartheta_{10}(\tau, z)}{\vartheta_{01}(\tau, z)}$

\[
\times \left\{ \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2} \vartheta_{00}(2\tau, 2z) - 2 \frac{\eta(4\tau)^2}{\eta(2\tau)} \vartheta_{10}(2\tau, 2z) + \vartheta_{01}(2\tau, 2z) \right\}
\]

(ii) $\text{ch}^{(+)}_{H(\Lambda^{[K(4), 3]})}(\tau, z) = \frac{i}{2} \left( \frac{1}{\eta(\tau)} \frac{1}{\eta(2\tau)} \right) \frac{\vartheta_{10}(\tau, z)}{\vartheta_{01}(\tau, z)}$

\[
\times \left\{ \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2} \vartheta_{00}(2\tau, 2z) - 2 \frac{\eta(4\tau)^2}{\eta(2\tau)} \vartheta_{10}(2\tau, 2z) - \vartheta_{01}(2\tau, 2z) \right\}
\]

2) (i) $\text{ch}^{(-)}_{H(\Lambda^{[K(4), 1]})}(\tau, z) = -\frac{i}{2} \left( \frac{1}{\eta(\tau)} \frac{1}{\eta(\tau)} \right) \frac{\vartheta_{10}(\tau, z)}{\vartheta_{00}(\tau, z)}$

\[
\times \left\{ \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2} \vartheta_{00}(2\tau, 2z) + 2 \frac{\eta(4\tau)^2}{\eta(2\tau)} \vartheta_{10}(2\tau, 2z) + \vartheta_{01}(2\tau, 2z) \right\}
\]

(ii) $\text{ch}^{(-)}_{H(\Lambda^{[K(4), 3]})}(\tau, z) = -\frac{i}{2} \left( \frac{1}{\eta(\tau)} \frac{1}{\eta(\tau)} \right) \frac{\vartheta_{10}(\tau, z)}{\vartheta_{00}(\tau, z)}$

\[
\times \left\{ \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2} \vartheta_{00}(2\tau, 2z) + 2 \frac{\eta(4\tau)^2}{\eta(2\tau)} \vartheta_{10}(2\tau, 2z) - \vartheta_{01}(2\tau, 2z) \right\}
\]

Proof. By Proposition 4.1 and the formulas (9.1a) and (9.1b), the numerators of the (super)characters of $N=3$ module $H(\Lambda^{[K(4), m]})$ are given by the following formulas:

\[
[N^{=3}_{R} \begin{cases} \text{ch}^{(+)}_{H(\Lambda^{[K(m), m]})}(\tau, z) & = q^{-\frac{m+1}{4}} Q^{[m, \frac{m+1}{2}]}(\tau, z) \\ \text{ch}^{(-)}_{H(\Lambda^{[K(m), m]})}(\tau, z) & = q^{-\frac{m+1}{4}} P^{[m, \frac{m+1}{2}]}(\tau, z) \end{cases}]
\]

Then this proposition follows from Lemma 9.3 and the formula (4.3) for the $N=3$ denominators.

From Proposition 8.2 and Proposition 9.1 we see the following:
Corollary 9.1. In the case \( m = 4 \), the modified character \( \widetilde{\chi}_{H(\Lambda^{(\kappa(4),1)})}^{(+)}(\tau, z) \) is a sum of honest characters, namely,

\[
\sum_{m_2 = 1, 3} \chi_{H(\Lambda^{(\kappa(4),m_2)})}^{(+)}(\tau, z) = 2 \cdot \widetilde{\chi}_{H(\Lambda^{(\kappa(4),1)})}^{(+)}(\tau, z)
\]

Below we will show that the honest characters are holomorphic functions.

Lemma 9.4. For \( m \in \frac{1}{2} \mathbb{N} \) and \( s \in \frac{1}{2} \mathbb{Z} \), we put

\[
f(\tau, z) := \Phi^{[m,s]}(2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0)
\]

Then \( f(\tau, n + p\tau) = 0 \) for \( \forall n, \forall p \in \mathbb{Z} \).

Proof. By the definition (2.2) of \( \Phi^{[m,s]} \), the function \( f(\tau, z) \) is written as follows:

\[
f(\tau, z) = f_1(\tau, z) - f_2(\tau, z)
\]

where

\[
f_1(\tau, z) = \sum_{j \in \mathbb{Z}} e^{4\pi imjz} e^{2\pi isz} q^{2mj^2 + 2sj + \frac{z}{2}} / (1 - e^{2\pi iz} q^{j + \frac{1}{2}})
\]

\[
f_2(\tau, z) = \sum_{j \in \mathbb{Z}} e^{-4\pi imjz} e^{-2\pi isz} q^{2mj^2 + 2sj + \frac{z}{2}} / (1 - e^{2\pi iz} q^{j + \frac{1}{2}})
\]

Then we have

\[
f_1(\tau, n + p\tau) = (-1)^{2s} n \sum_{j \in \mathbb{Z}} q^{2mj(j + p) + s(2j + p + \frac{1}{2})} / (1 - q^{2j + p + \frac{1}{2}}) \tag{9.4a}
\]

\[
f_2(\tau, n + p\tau) = (-1)^{2s} n \sum_{j \in \mathbb{Z}} q^{2mj(j - p) + s(2j - p + \frac{1}{2})} / (1 - q^{2j - p + \frac{1}{2}}) \tag{9.4b}
\]

Putting \( j = k + p \), this equation (9.4b) is rewritten as follows:

\[
f_2(\tau, n + p\tau) = (-1)^{2s} n \sum_{k \in \mathbb{Z}} q^{2mk(k + p) + s(2k + p + \frac{1}{2})} / (1 - q^{2k + p + \frac{1}{2}}) \tag{9.4c}
\]

Then by (9.4a) and (9.4c), one has

\[
f(\tau, n + p\tau) = f_1(\tau, n + p\tau) - f_2(\tau, n + p\tau) = 0
\]

proving lemma.

From this lemma, we have the following:
Lemma 9.5. Let \( m \) be a positive integer and \( m_2 \) be a non-negative integer such that \( m_2 \leq m \). Then
\[
\left( R^3(--) \right)_{H(\Lambda^{[K(m),m_2]})}(\tau, n + p\tau) = 0 \quad \text{for } \forall n, \forall p \in \mathbb{Z}
\]

Proof. By Proposition 4.1 we have
\[
\left( R^3(--) \right)_{H(\Lambda^{[K(m),m_2]})}(\tau, z) = q^{-\frac{m_2 + 1}{2}} \phi_{\frac{m_2 + 1}{2}} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0 \right).
\]

Then the claim follows from Lemma 9.1. \( \square \)

Proposition 9.2. Let \( m \) be a positive integer and \( m_2 \) be a non-negative integer such that \( m_2 \leq m \). Then the (super)characters \( ch_{H(\Lambda^{[K(m),m_2]})}(\tau, z) \) are holomorphic in the domain \((\tau, z) \in \mathbb{C} \times \mathbb{C}\).

Proof. First we consider the super-character. By (4.3) and Proposition 4.1 we have
\[
ch_{H(\Lambda^{[K(m),m_2]})}(\tau, z) = \eta(\frac{\tau}{2}) \cdot \frac{\eta(\tau)}{\eta(\tau)^3} \cdot \vartheta_{01}(\tau, z) \cdot q^{-\frac{m_2 + 1}{2}} \phi_{\frac{m_2 + 1}{2}} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0 \right)
\]

where
\[
\phi_{\frac{m_2 + 1}{2}} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0 \right) = \sum_{j \in \mathbb{Z}} \frac{e^{2\pi imjz + \pi i (m+1)z} q^{jm^2 + (m+1)(j+\frac{1}{2})}}{1 - e^{2\pi iz} q^{j^2 + \frac{1}{4}}} - \sum_{j \in \mathbb{Z}} \frac{e^{2\pi imjz - \pi i (m+1)z} q^{jm^2 + (m+1)(j+\frac{1}{2})}}{1 - e^{-2\pi iz} q^{j^2 + \frac{1}{4}}}
\]

From these equations, we see the following:

(i) \( \phi_{\frac{m_2 + 1}{2}} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0 \right) \) has singularities at \( e^{\pm 2\pi iz} q^{j^2 + \frac{1}{2}} = 1 \), but these singularities are cancelled by \( \vartheta_{01}(\tau, z) \).

(ii) \( \frac{1}{\vartheta_{11}(\tau, z)} \) has singularities at \( e^{\pm 2\pi iz} q^{j^2} = 1 \), but these singularities are cancelled by \( \phi_{\frac{m_2 + 1}{2}} \left( 2\tau, z + \frac{\tau}{2}, z - \frac{\tau}{2}, 0 \right) \) due to Lemma 9.6.

Thus all of singularities in the RHS of (9.5) disappear, so \( ch_{H(\Lambda^{[K(m),m_2]})}(\tau, z) \) is holomorphic. Then \( ch_{H(\Lambda^{[K(m),m_2]})}(\tau, 1, z) \) is holomorphic, since it is a scalar multiple of \( ch_{H(\Lambda^{[K(m),m_2]})}(\tau + 1, z) \). \( \square \)

From Corollary 6.1 and Corollary 9.1 we conjecture the following:

Conjecture 9.1. For \( m \in \mathbb{N} \) and \( j \in \{0,1\} \), there exists \( \mu^{[m]}(m_2) \) such that
\(\mu[m](m_2) \in \mathbb{N}\) for each \(m_2 \in \mathbb{Z}^m_j\)

\[
\sum_{m_2 \in \mathbb{Z}^m_j} \mu[m](m_2) \cdot \chi_{H(\Lambda[K(m),m_2])}^{(+)}(\tau, z)
= \left( \sum_{m_2 \in \mathbb{Z}^m_j} \mu[m](m_2) \right) \times \chi_{H(\Lambda[K(m),m_2])}^{(+)}(\tau, z)
\]

where \(\mathbb{Z}^m_j := \{m_2 \in \mathbb{Z} : 0 \leq m_2 \leq m \text{ and } m_2 - j \in 2\mathbb{Z}\}\).

**Conjecture 9.2.** For \(m \in \mathbb{N}\), modified characters are holomorphic functions of \((\tau, z) \in \mathbb{C} \times \mathbb{C}\).

We note that, if Conjecture 9.1 is true, then Conjecture 9.2 is true by Proposition 9.2. Also note that, if Conjecture 9.1 is true, then the honest characters \(\chi_{H(\Lambda[K(m),m_2])}^{(+)}(\tau, z)\) \((m_2 \in \mathbb{Z}^m_j)\) can be written explicitly by the modified character \(\overline{\chi}_{H(\Lambda[K(m),m_2])}^{(+)}(\tau, z)\), as is seen in the following Remark 9.1 in the case \(m = 4\).

**Remark 9.1.** Assume that Conjecture 9.1 is true for \(m = 4\) and \(j = 0\) with \((\mu[4](0), \mu[4](2), \mu[4](4)) = (1, 2, 1)\). Then the honest characters \(\chi_{H(\Lambda[K(4),m_2])}^{(+)}(\tau, z)\) \((m_2 \in \{0, 2, 4\})\) are expected to be as follows by using the formula (8.3) for the modified character \(\overline{\chi}_{H(\Lambda[K(4),m_2])}^{(+)}(\tau, z)\) in Conjecture 5.1:

1) \(\chi_{H(\Lambda[K(4),0])}^{(+)}(\tau, z) = \overline{\chi}_{H(\Lambda[K(4),0])}^{(+)}(\tau, z)\)

\[
+ \frac{i}{4} q^{-\frac{1}{4}} \frac{1}{\eta(\frac{\tau}{2})\eta(2\tau)} \cdot \frac{\vartheta_{00}(\tau, z)}{\vartheta_{11}(\tau, z)} \left[ -3(\theta_{\frac{1}{2},2} - \theta_{-\frac{1}{2},2}) + (\theta_{\frac{1}{2},2} - \theta_{-\frac{1}{2},2}) \right] (2\tau, 2z)
\]

2) \(\chi_{H(\Lambda[K(4),2])}^{(+)}(\tau, z) = \overline{\chi}_{H(\Lambda[K(4),2])}^{(+)}(\tau, z)\)

\[
+ \frac{i}{4} q^{-\frac{1}{4}} \frac{1}{\eta(\frac{\tau}{2})\eta(2\tau)} \cdot \frac{\vartheta_{00}(\tau, z)}{\vartheta_{11}(\tau, z)} \left[ (\theta_{\frac{1}{2},2} - \theta_{-\frac{1}{2},2}) + (\theta_{\frac{1}{2},2} - \theta_{-\frac{1}{2},2}) \right] (2\tau, 2z)
\]

3) \(\chi_{H(\Lambda[K(4),4])}^{(+)}(\tau, z) = \overline{\chi}_{H(\Lambda[K(4),4])}^{(+)}(\tau, z)\)

\[
+ \frac{i}{4} q^{-\frac{1}{4}} \frac{1}{\eta(\frac{\tau}{2})\eta(2\tau)} \cdot \frac{\vartheta_{00}(\tau, z)}{\vartheta_{11}(\tau, z)} \left[ (\theta_{\frac{1}{2},2} - \theta_{-\frac{1}{2},2}) - 3(\theta_{\frac{1}{2},2} - \theta_{-\frac{1}{2},2}) \right] (2\tau, 2z)
\]

10 \(\vartheta\)-relations

Using Lemma 2.7 and the formulas for \(A_j^{[2]}(\tau, z)\) in Lemma 5.3 we obtain the following formulas for \(\vartheta_{ab}\):
Proposition 10.1.

1) \( \frac{\partial_{00}(2\tau, z)}{\partial_{10}(2\tau, z)} + \frac{\partial_{10}(2\tau, z)}{\partial_{00}(2\tau, z)} = \frac{\eta(\tau)^6}{\eta(\tau)^2 \eta(2\tau)^4} \cdot \frac{\partial_{00}(\tau, z)}{\partial_{10}(\tau, z)} \)

2) \( \frac{\partial_{00}(2\tau, z)}{\partial_{10}(2\tau, z)} - \frac{\partial_{10}(2\tau, z)}{\partial_{00}(2\tau, z)} = \frac{\eta(\tau)^2}{\eta(2\tau)^2} \cdot \frac{\partial_{01}(\tau, z)}{\partial_{10}(\tau, z)} \)

Proof. Letting \( m = 2 \) in the formulas (2.8a) and (2.8b) and using Lemma 5.3 for the explicit formulas for \( A_{0}^{[2]}(\tau, z) \) and \( A_{0}^{[2]}(\tau, z) \), we have

\[
\Phi(1.0) \left( 2\tau, z - \frac{1}{2}, z + \frac{1}{2}, 0 \right) = A_{0}^{[2]}(\tau, z) = \frac{i}{2} \left\{ g_{3}^{(+)}(\tau, z) + g_{2}^{(+)}(\tau, z) \right\} 
\]

\[
= \frac{i}{2} \cdot \frac{\partial_{11}(\tau, z)}{\partial_{10}(\tau, z)} \left\{ \frac{\eta(\tau)^5}{\eta(\tau)^2 \eta(2\tau)^2} \cdot \partial_{00}(\tau, z) + \frac{\eta(\tau)^2}{\eta(\tau)} \cdot \partial_{01}(\tau, z) \right\} 
\]

(10.1a)

\[
\Phi(1.0) \left( 2\tau, z + \tau - \frac{1}{2}, z - \tau + \frac{1}{2}, \tau \right) = A_{0}^{[2]}(\tau, z) = \frac{i}{2} \left\{ g_{3}^{(+)}(\tau, z) - g_{2}^{(+)}(\tau, z) \right\} 
\]

(10.1b)

We compute the LHS of (10.1a) and (10.1b). By Lemma 2.7, these are rewritten as follows:

LHS of (10.1a) \( = -i \eta(2\tau)^3 \cdot \frac{\partial_{11}(2\tau, 2z)}{\partial_{11}(2\tau, z - \frac{1}{2}) \cdot \partial_{11}(2\tau, z + \frac{1}{2})} \)

\[
= i \frac{\eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\partial_{00}(2\tau, z)}{\partial_{10}(2\tau, z)} \cdot \partial_{11}(\tau, z) 
\]

(10.2a)

LHS of (10.1b) \( = -i q^{\frac{1}{2}} \eta(2\tau)^3 \cdot \frac{\partial_{11}(2\tau, 2z)}{\partial_{11}(2\tau, \tau + \frac{1}{2}) \partial_{11}(2\tau, z - \tau + \frac{1}{2})} \)

\[
= \frac{i \eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\partial_{10}(2\tau, z)}{\partial_{00}(2\tau, z)} \cdot \partial_{11}(\tau, z) 
\]

(10.2b)

where we used the formula (5.4a) and the following formulas:

\[
\partial_{11}(2\tau, z - \frac{1}{2}) \partial_{11}(2\tau, z + \frac{1}{2}) = -\partial_{10}(2\tau, z)^2 
\]

\[
\partial_{11}(2\tau, z + \tau - \frac{1}{2}) \partial_{11}(2\tau, z - \tau + \frac{1}{2}) = -q^{-\frac{1}{2}} \partial_{00}(2\tau, z)^2 
\]

and

\[
\begin{align*}
\frac{\partial_{10}(\tau, z)}{\partial_{00}(2\tau, z)} &= \frac{\eta(\tau)}{\eta(2\tau)^2} \cdot \partial_{00}(2\tau, z) \\
\frac{\partial_{10}(\tau, z)}{\partial_{00}(2\tau, z)} &= \frac{\eta(\tau)}{\eta(2\tau)^2} \cdot \partial_{10}(2\tau, z) 
\end{align*}
\]
Then we have, by (10.1a) and (10.2a),
\[
\frac{\eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{00}(2\tau, z)}{\vartheta_{10}(2\tau, z)} = \frac{1}{2} \cdot \frac{1}{\vartheta_{10}(\tau, z)} \left\{ \frac{\eta(\tau)^5}{\eta(\tau)^2\eta(2\tau)^2} \vartheta_{00}(\tau, z) + \frac{\eta(\tau^2)^2}{\eta(\tau)} \vartheta_{01}(\tau, z) \right\}
\]
and, by (10.1b) and (10.2b),
\[
\frac{\eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{10}(2\tau, z)}{\vartheta_{00}(2\tau, z)} = \frac{1}{2} \cdot \frac{1}{\vartheta_{10}(\tau, z)} \left\{ \frac{\eta(\tau)^5}{\eta(\tau)^2\eta(2\tau)^2} \vartheta_{00}(\tau, z) - \frac{\eta(\tau^2)^2}{\eta(\tau)} \vartheta_{01}(\tau, z) \right\}.
\]

Now, by making (10.3a) ± (10.3b), we have
\[
\frac{\eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{00}(2\tau, z)}{\vartheta_{10}(2\tau, z)} = \frac{\eta(\tau)^5}{\eta(\tau)^2\eta(2\tau)^2} \vartheta_{00}(\tau, z) + \frac{\eta(\tau^2)^2}{\eta(\tau)} \vartheta_{01}(\tau, z)
\]
\[
\frac{\eta(2\tau)^2}{\eta(\tau)} \cdot \frac{\vartheta_{10}(2\tau, z)}{\vartheta_{00}(2\tau, z)} = \frac{\eta(\tau)^5}{\eta(\tau)^2\eta(2\tau)^2} \vartheta_{00}(\tau, z) - \frac{\eta(\tau^2)^2}{\eta(\tau)} \vartheta_{01}(\tau, z)
\]

namely
\[
\frac{\vartheta_{00}(2\tau, z)}{\vartheta_{10}(2\tau, z)} + \frac{\vartheta_{10}(2\tau, z)}{\vartheta_{00}(2\tau, z)} = \frac{\eta(\tau)^6}{\eta(\tau)^2\eta(2\tau)^4} \cdot \frac{\vartheta_{00}(\tau, z)}{\vartheta_{10}(\tau, z)}
\]
\[
\frac{\vartheta_{00}(2\tau, z)}{\vartheta_{10}(2\tau, z)} - \frac{\vartheta_{10}(2\tau, z)}{\vartheta_{00}(2\tau, z)} = \frac{\eta(\tau^2)^2}{\eta(2\tau)^2} \cdot \frac{\vartheta_{01}(\tau, z)}{\vartheta_{10}(\tau, z)}
\]
proving proposition. \(\square\)

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