Recurrent Trajectories and Finite Critical Trajectories of Quadratic Differentials on the Riemann Sphere

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Abstract

In this paper, the focus will be on both the existence and non-existence respectively of finite critical trajectories and recurrent trajectories of a quadratic differential on the Riemann sphere. We show the connection between these two items. More precisely, we collect some criterions for the non-existence of recurrent trajectories. The first criterion is in the same vein of Jenkins Three-pole Theorem, while the second one is in relation with the so-called Level function of quadratic differentials.

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1 Introduction

One of the most frequent problems once investigating the finite critical trajectories of a given quadratic differential on the Riemann sphere $\mathbb{C}$ is the existence of the infamous recurrent trajectories. Jenkins Three-pole Theorem paves the way in a particular case. In this paper, our main emphasis is on the connection between the existence of these two kinds of trajectories, which appears to be permanent. More precisely, we gather two criterions for the non-existence of recurrent trajectories. The first criterion is
based on a proof of Jenkins Three-pole Theorem. The second one is derived from the so-called Level function of quadratic differentials as determined by Y.Baryshnikov and B.Shapiro.

A quadratic differential on the Riemann sphere \( \hat{\mathbb{C}} \) is a 2-form \( \varpi_\varphi(z) = \varphi(z) \, dz^2 \) where \( \varphi \) is a rational function on \( \mathbb{C} \). In a first plan, we give some immediate and brief observations from the theory of quadratic differentials. For more details, we refer the reader to [3],[4]...

**Critical points** of \( \varpi_\varphi \) are its zero’s and poles in \( \hat{\mathbb{C}} \). Zeros and simple poles are called **finite critical points**, while poles of order 2 or greater are called **infinite critical points**. All other points of \( \hat{\mathbb{C}} \) are called **regular points**.

Horizontal trajectories (or just trajectories) of the quadratic differential \( \varpi_\varphi \) are the zero loci of the equation

\[
\Im \int z \sqrt{\varphi(t)} \, dt = \text{const},
\]

or equivalently

\( \varpi_\varphi = \varphi(z) \, dz^2 > 0 \).

The **vertical** (or, orthogonal) trajectories are obtained by replacing \( \Im \) by \( \Re \) in equation (1). Horizontal and vertical trajectories of the quadratic differential \( \varpi_\varphi \) produce two pairwise orthogonal foliations of the Riemann sphere \( \hat{\mathbb{C}} \).

A trajectory passing through a critical point of \( \varpi_\varphi \) is called **critical trajectory**. In particular, if it starts and ends at a finite critical point, it is called **finite critical trajectory** or **short trajectory**, otherwise, we call it an **infinite critical trajectory**. The closure of the set of finite and infinite critical trajectories, that we denote by \( \Gamma_\varphi \), is called the **critical graph**.

The local and global structure of the trajectories are studied in [4, Theorem3.5],[3]. In the large, any trajectory is either a closed curve containing no critical points, or, is an arc connecting two critical points, or, is an arc that has no limit along at least one of its rays. The structure of the set \( \hat{\mathbb{C}} \setminus \Gamma_\varphi \) depends on the local and global behaviors of trajectories. It consists of a finite number of domains called the **domain configurations** of \( \varpi_\varphi \). The following Theorem gives the possible structures of domain configurations (see [4, Theorem3.5],[3]):

**Theorem 1** There are five kinds of domain configurations, :

- Half-plane domain: It is swept by trajectories converging to a pole of order 3 in its two ends, and along consecutive critical directions. Its conformally mapped to a half plane.
• Strip domain: \textit{It is swept by trajectories which both ends tend to poles of order 2. Its conformally mapped to a strip domain.}

• Ring domain: \textit{It is swept by closed trajectories. Its conformally mapped to an annulus.}

• Circle domain: \textit{It is swept by closed trajectories and contains exactly one double pole. Its conformally mapped to the a circle.}

• Dense domain: \textit{It is swept by recurrent critical trajectory, i.e., the interior of its closure is non-empty.}

As it is mentioned in the title, we are interested in the fifth kind of domain configuration. The dense domain is either the whole complex plane, or, is bordered by finite critical trajectories. Figure 1 illustrates two quadratics differentials with recurrent trajectories. In the first one, the whole complex plane is a set domain, while in the second, the dense domain is bounded by a closed finite critical trajectory. See [7], [8].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Recurrent trajectory through the regular point $z = 1$ of the quadratic differential $-dz^2/(z - 0.5) (z + 0.5) (z - 1 - i) (z + 1 + i)$, (left). $-zd^2/(z - 0.5) (z - 1 - i) (z - 2 + i)$ (right)}
\end{figure}

The quadratic differential $\varpi_\varphi$ defines a $\varphi$-metric on the Riemann sphere with the differential element $\sqrt{|\varphi(z)|} |dz|$. If $\gamma$ is a rectifiable arc in $\widehat{\mathbb{C}}$, then its $\varphi$-length is defined by

$$|\gamma|_\varphi = \int_\gamma \sqrt{|\varphi(z)|} |dz|.$$  

A trajectory of $\varpi_\varphi$ is finite if, and only if its $\varphi$-length is finite, otherwise it is infinite.
A domain in $\hat{\mathbb{C}}$ bounded only by segments of horizontal and/or vertical trajectories of $\varpi_\phi$ (and their endpoints) is called $\varpi_\phi$-polygon. A useful tool in the investigation of the critical graph of a quadratic differential is:

**Lemma 2 (Teichmüller)** Let $\Omega$ be a $\varpi_\phi$-polygon, and let $z_j$ be the critical points on the boundary $\partial\Omega$ of $\Omega$, and let $t_j$ be the corresponding interior angles with vertices at $z_j$, respectively. Then

$$\sum_j \left(1 - \frac{(n_j + 2)t_j}{2\pi}\right) = 2 + \sum_i m_i,$$

where $n_j$ are the multiplicities of $z_j = 1$, and $m_i$ are the multiplicities of critical points inside $\Omega$.

### 2 Recurrent trajectories

Suppose that the quadratic differential $\varpi_\phi$ has an infinite critical point $p$. From the local behavior of the trajectories, there exists a neighborhood $U$ of $p$, such that, any trajectory in $U$, either is a closed curve encircling $p$, if $p$ is a double pole with negative residue, or, it diverges to $p$ following a certain direction depending on the order of $p$. In other word, there is absence of recurrent trajectory in $U$. If $\varpi_\phi$ has a recurrent trajectory, then its respective dense domain cannot be the whole complex plane, and then, $\varpi_\phi$ should have finite critical trajectories (Theorem 1). These facts show the following:

**Proposition 3** Assume that the quadratic differential $\varpi_\phi$ satisfies:

(i) it has at least an infinite critical point,

(ii) it has no finite critical trajectory.

Then, $\varpi_\phi$ has no recurrent trajectory.

In general, the non-existence of recurrent trajectories is not guaranteed. The most well-known result is the so-called Jenkins Three-pole Theorem, which asserts that

**Theorem 4 (Jenkins Three-pole Theorem)** A quadratic differential on the Riemann sphere with at most three distinct poles cannot have recurrent trajectories.
Two different proofs of this theorem can be found in [4] and [3, Theorem 15.2]. But the proof given in the second reference seems to get a more precise result. What comes next is a refinement of the mentioned Theorem:

**Proposition 5** No recurrent trajectories of a quadratic differential on the Riemann sphere with at most three critical points with odd multiplicities.

**Proof.** We follow the idea of the proof of [3, Theorem 15.2]. Let \( \gamma(t), t \in [0,1[ \) be a ray of a recurrent trajectory of a quadratic differential \( \varpi \) on the Riemann sphere with at most three critical points with odd multiplicities. Set \( t_0 \in [0,1[ \) such that \( \gamma(t), t \geq t_0 \) does not pass through any critical point. Let \( \gamma^\perp \) be an interval of the orthogonal trajectory starting at the point \( z_0 = \gamma(t_0) \). By hypothesis, \( \gamma(t), t \geq t_0 \) cuts \( \gamma^\perp \) infinitely many times. Let \( z_1 = \gamma(t_1), t_1 > t_0 \) be the first intersection between \( \gamma(t) \) and \( \gamma^\perp \). We consider the Jordan curve \( \delta \) composed by the two sub-arcs \( \gamma_0 \) and \( \gamma_0^\perp \) of \( \gamma(t), t \geq t_0 \) and \( \gamma^\perp \) between the points \( z_0 \) and \( z_1 \), see Figure 2. As we can take the next intersection between \( \gamma(t), t > t_1 \) and \( \gamma^\perp \), we may assume that the interior of \( \delta \) at \( z_1 \) equals \( \frac{3\pi}{2} \). Let \( z_2 = \gamma(t_2), t_2 > t_1 \) be the next intersection of \( \gamma(t), t > t_1 \) and \( \gamma^\perp \). Clearly, \( z_2 \in \gamma_0^\perp \), the sub-arc \( \gamma([t_1,t_2]) \) splits a component \( \Omega \) of \( \hat{\mathbb{C}} \setminus \delta \) into two sub-domains \( \Omega_1 \) and \( \Omega_2 \), with

\[
\partial\Omega_1 = \gamma([t_0,t_2]) \cup \gamma_1^\perp, \\
\partial\Omega_2 = \gamma([t_1,t_2]) \cup \gamma_2^\perp,
\]

where \( \gamma_1^\perp \) and \( \gamma_2^\perp \) are the sub-arcs of \( \gamma_0^\perp \) joining \( z_2 \) respectively to \( z_0 \) and \( z_1 \).

| \( \partial\Omega_1 \) | \( z_0, z_1, z_2 \) | \( \pi/2, \pi, \pi/2 \) |
| \( \partial\Omega_2 \) | \( z_1, z_2 \) | \( \pi/2, \pi/2 \) |

Applying Lemma 2 to the \( \varpi \)-polygons \( \partial\Omega_1 \) and \( \partial\Omega_2 \), we get

\[
1 = 2 + \sum m_i \text{ for } \partial\Omega_i, i = 1,2.
\]

We conclude that each of the domains \( \Omega_1 \) and \( \Omega_2 \) must contain at least a critical point with odd multiplicity. Now, let \( z_3 = \gamma(t_3), t_3 > t_2 \) be the next intersection of \( \gamma(t), t > t_2 \) and \( \gamma^\perp \). It is obvious that the union of \( \gamma([t_2,t_3]) \)
and the sub-arc of $\gamma^+_{0}$ between $z_2$ and $z_3$ splits the exterior of $\Omega$ into two sub-domains $F_1$ and $F_2$.

|          | vertices | respective interior angles       |
|----------|----------|----------------------------------|
| $\partial F_1$ | $z_0, z_1, z_2, z_3$ | $3\pi/2, \pi/2, \pi/2, \pi/2$ |
| $\partial F_2$ | $z_2, z_3$ | $\pi/2, \pi/2$                  |

Once more, applying Lemma 2 to the $\omega_{\varphi}$-polygons $\partial F_1$ and $\partial F_2$, we reach the same conclusion that each of the domains $F_1$ and $F_2$ must contain at least a critical point with odd multiplicity. We just proved that the quadratic differential should have at least four critical points with odd multiplicities; a contradiction.

![Figure 2](image)

**Figure 2:**

**Example 6** A particular case is when all the multiplicities of the critical points of a quadratic differential are even. A lemniscate of a rational function $r(z) = \frac{p(z)}{q(z)}$, where $p$ and $q$ are two co-prime polynomials, is defined for $c > 0$ by

$$\left\{ z \in \hat{C} : |r(z)| = c \right\}.$$  

It is a real algebraic curve of degree $2 \max(\deg p, \deg q)$; indeed, its defining equation can be seen as

$$p(x, y)p(x, y) - c^2 q(x, y)q(x, y) = 0,$$

with $z = x + iy$. For more details, see [10]. These sets can be determined immediately (equation (1)) as trajectories of the quadratic differential

$$- \left( \frac{r'(z)}{r(z)} \right)^2 dz^2 = - \left( \frac{p'(z) q(z) - p(z) q'(z)}{p(z) q(z)} \right)^2 dz^2.$$  

From the equality

\[
\frac{p'(z)q(z) - p(z)q'(z)}{p(z)q(z)} = \sum_{p(a)q(a) = 0} \frac{m_a}{z - a},
\]

where \(m_a \in \mathbb{N}^*\) are the multiplicity of the zero's of \(p(z)q(z)\), we deduce that the finite critical points are the zero’s of \(\sum_{p(a)q(a) = 0} \frac{m_a}{z - a}\); the zero’s of \(p(z)q(z)\) (and possibly \(\infty\)) are the only infinite critical points of \(-\left(\frac{r'(z)}{r(z)}\right)^2 \, dz^2\), and they are all double poles with negative residues. By Proposition 5, this quadratic differential has no recurrent trajectories. Its critical graph can be found easily since the local and global structures of the trajectories are well-known. The set of closed trajectories of \(-\left(\frac{r'(z)}{r(z)}\right)^2 \, dz^2\) cover the whole complex plane minus a zero Lebesgue set, this kind of quadratic differential is called Strebel differential.

A more elaborated exploitation of the proof of Proposition 5 allows us to show the following:

**Proposition 7** Let \(\varphi(z) = \frac{p(z)}{(q(z))^2}\), where \(p(z)\) and \(q(z)\) are two co-prime polynomials. We suppose that:

(a) for each \(i = 1, \ldots, n\), the zero’s \(a_{2i-1}\) and \(a_{2i}\) of \(p(z)\) are connected by a short trajectory of \(\varphi\),

(b) each pair of zero’s \(a_{2i-1}\) and \(a_{2i}\) have the same parity of multiplicities,

Then, the quadratic differential \(\varphi\) has no recurrent trajectory.

Here is a starting point which is an idea of Y. Baryshnikov and B. Shapiro, it gives a necessary and sufficient conditions on the existence of recurrent trajectories of the quadratic differential \(\varphi\):

**Lemma 8** Assume that there exists a function \(f : \mathbb{C} \setminus \mathcal{I} \to \mathbb{R}\) (called level function of \(\varphi\)) such that:

(i) \(f\) is continuous and piecewise smooth on \(\mathbb{C} \setminus \mathcal{I}\);

(ii) \(f\) is constant on the trajectories of \(\varphi\);
(iii) \( f \) is non-constant on any open subset of \( \mathbb{C} \).

Then, the quadratic differential \( \varpi \phi \) has no recurrent trajectory. Conversely, if \( \varpi \phi \) has no recurrent trajectory, then such a function exists.

**Proof.** If \( \varpi \phi \) has no recurrent trajectory, \( \Gamma \phi \) splits \( \hat{\mathbb{C}} \) into at most the first four domain configurations defined previously in the introduction: half-plane domains, ring domains, circle domains, and strip domains. On each of these domains we can construct a function that is continuous, constant on the trajectories, but not on any open set, and which is vanishing on the boundary of the domain. Gluing together these functions along delivers the desired continuous function.

If \( \varpi \phi \) has a recurrent trajectory \( \gamma \), by continuity, the function \( f \) will be constant on its closure, which violates (iii) of.

This leads us to an interesting particular case when \( \phi(z) = \frac{p(z)}{(q(z))^r} \), where \( p(z) \) and \( q(z) \) are two co-prime polynomials:

**Proposition 9** We suppose that:

(a) for each \( i = 1, \ldots, n \), the zero’s \( a_{2i-1} \) and \( a_{2i} \) of \( p(z) \) are connected by a short trajectory of \( \varpi \phi \),

(b) the residue of the rational function \( \phi(z) \) in every zero of \( q(z) \) is purely imaginary number.

Then, the quadratic differential \( \varpi \phi \) has no recurrent trajectory.

**Proof.** For \( i = 1, \ldots, n \), let \( \gamma_i \) be the short trajectory (taken with an orientation) of \( \varpi \phi \) connecting \( a_{2i-1} \) and \( a_{2i} \). We define a single-valued function \( \sqrt{p(z)} \) in \( \mathbb{C} \setminus \bigcup_{i=1}^{n} \gamma_i \). For \( t \in \bigcup_{i=1}^{n} \gamma_i \), we denote by \( \left( \sqrt{p(t)} \right)_+ \) and \( \left( \sqrt{p(t)} \right)_- \) the limits of \( \sqrt{p(z)} \) from the + and − sides, respectively. (As usual, the +-side of an oriented curve lies to the left and the −-side lies to the right, if one traverses the curve according to its orientation). We have then

\[
\frac{\left( \sqrt{p(t)} \right)_+}{q(t)} = - \frac{\left( \sqrt{p(t)} \right)_-}{q(t)}, \quad t \in \gamma_i, \ i = 1, \ldots, n.
\]
Since $\gamma_i$ is a short trajectory of $\varpi_\varphi$,
\[
\Im \left( \int_{a_{2i-1}}^t \frac{\sqrt{p(s)}}{q(s)} \, ds \right) = 0, \ t \in \gamma_i, \ i = 1, \ldots, n.
\]
It follows that, if $\gamma$ is a closed Jordan curve encircling $\gamma_i$ (or parts of it) and none of the poles of $\varphi$,
\[
\Im \int_{\gamma} \sqrt{p(z)} \, q(z) \, dz = \pm 2 \Im \int_{\gamma_i} \frac{\sqrt{p(z)}}{q(z)} \, dz = 0.
\]
We consider the multi-valued function
\[
f : z \mapsto \int_{a_{1}}^{z} \frac{\sqrt{p(s)}}{q(s)} \, ds
\]
defined in $\mathbb{C} \setminus \mathcal{I}$. For $z \in \mathbb{C} \setminus (\cup_{i=1}^{n} \gamma_i \cup \mathcal{I})$, let $\beta_1$ and $\beta_2$ be two oriented Jordan arcs connecting $a_{1}$ and $z$ in $\mathbb{C} \setminus (\cup_{i=1}^{n} \gamma_i \cup \mathcal{I})$. If we denote by $\mathcal{J}$ the subset of poles of $\varphi$ encircled by the Jordan closed curve $\beta_1 - \beta_2$, then we have
\[
\int_{\beta_1 - \beta_2} \frac{\sqrt{p(s)}}{q(s)} \, ds = \pm 2 \pi i \sum_{a \in \mathcal{J}} \text{res}_f(a).
\]
By condition (a), we conclude that
\[
\Im \int_{\beta_1} \frac{\sqrt{p(s)}}{q(s)} \, ds = \Im \int_{\beta_2} \frac{\sqrt{p(s)}}{q(s)} \, ds,
\]
and the function $\Im f$ is well defined in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$. Obviously, it satisfies (i) and (ii) of Lemma 8, the point (iii) follows from the harmonicity of $\Im f$ in $\mathbb{C} \setminus (\cup_{i=1}^{n} \gamma_i) \cup \{\text{poles of } \varphi\}$.

As a consequence, if $p$ and $q$ have simple and real zero’s respectively $a_1 < b_1 < a_2 < b_2 < \cdots < a_{2n} < b_m$, such that $p(b_i) < 0$ for $i = 1, \ldots, m$. Then the quadratic differential $-\frac{p(z)}{(q(z))^2} \, dz^2$ has no recurrent trajectory. Indeed, for any $i = 1, \ldots, n$, the segment $[a_{2i-1}, a_{2i}]$ is a short trajectory of the quadratic differential $-\frac{p(z)}{(q(z))^2} \, dz^2$, and the result follows from Proposition 9.

**Example 10** Let $p(z), q(z)$, and $r(z)$ be three complex polynomials, $p$ and $q$ are co-prime. If the algebraic equation
\[
p(z) C^2(z) + q(z) C(z) + r(z) = 0
\]
admits solution as Cauchy transform of some compactly supported Borel signed measure \( \mu \) in the \( \mathbb{C} \)-plane, then, by Plemelj-Sokhotsky Lemma, this measure lives in the short trajectories of the quadratic differential

\[
- \frac{q^2(z) - 4p(z)r(z)}{p^2(z)} dz^2,
\]

and it is given by

\[
d\mu(z) = \frac{1}{2\pi i} \frac{\sqrt{q^2(z) - 4p(z)r(z)}}{p(z)} dz,
\]

see [11] or [5]. The information by Proposition 9 that the above quadratic differential cannot have a recurrent trajectory is essential in the finding of short trajectories if exist.

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