Exploring Viable Algorithmic Options for Learning from Demonstration (LfD):
A Parameterized Complexity Approach

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Abstract: The key to reconciling the polynomial-time intractability of many machine learning tasks in the worst case with the surprising solvability of these tasks by heuristic algorithms in practice seems to be exploiting restrictions on real-world data sets. One approach to investigating such restrictions is to analyze why heuristics perform well under restrictions. A complementary approach would be to systematically determine under which sets of restrictions efficient and reliable machine learning algorithms do and do not exist. In this paper, we show how such a systematic exploration of algorithmic options can be done using parameterized complexity analysis. As an illustrative example, we give the first parameterized complexity analysis of batch and incremental policy inference under Learning from Demonstration (LfD). Relative to a basic model of LfD, we show that none of our problems can be solved efficiently either in general or relative to a number of (often simultaneous) restrictions on environments, demonstrations, and policies. We also give the first known restrictions under which efficient solvability is possible and discuss the implications of our solvability and unsolvability results for both our basic model of LfD and more complex models of LfD used in practice.

1 Introduction

In an ideal world, one wants algorithms for machine learning tasks that are both efficient and reliable, in the sense that the algorithms quickly compute the correct outputs for all possible inputs of interest. An apparent paradox of machine learning research is that while many machine learning tasks are \(NP\)-hard in the worst case and hence cannot be solved both efficiently and reliably in general, these tasks are solvable amazingly well in practice using heuristic algorithms [1, page 1]. The resolution of this paradox is that machine learning tasks encountered in practice are
characterized by restrictions on input data sets that allow heuristics to perform far better than suggested by worst case analyses [1, page 2]. One approach to exploiting these restrictions pioneered by Moitra and others is to rigorously analyze existing heuristics operating relative to such restrictions to explain the good performance of those heuristics in practice. This in turn often suggests fundamentally new ways of solving machine learning tasks.

A complementary approach would be to characterize those combinations of restrictions for which efficient and reliable algorithms for a given machine learning task do and do not exist. This can be done using techniques from the theory of parameterized computational complexity [2, 3, 4]. If these techniques are applied systematically to all possible subsets of a given set of plausible restrictions for the task of interest, the resulting overview of algorithmic options for that task relative to those restrictions would be most useful in both deriving the best possible solutions for given task instances (by allowing lookup of the most appropriate algorithms relative to restrictions characterizing those instances) and productively directing research on new efficient algorithms for that task (by highlighting those restrictions under which such algorithms can exist).

In this paper, we will show how parameterized complexity analysis can be used to systematically explore algorithmic options for fast and reliable machine learning. We will first give an overview of parameterized complexity analysis (Section 2). Such analyses are then demonstrated via the first parameterized complexity analysis of a classic machine learning task, learning from demonstration (LfD) [5, 6] (Section 3). Our analysis is done relative a basic model of LfD (formalized in Section 3.1) based on that given in [5], in which discrete feature-based positive and negative demonstrations are used to infer time-independent policies specified as single-state transducers. We show that neither batch nor incremental LfD can be done efficiently in general (Section 3.2) or under many (but not all) subsets of a given set of plausible restrictions on environments, demonstrations, and policies (Section 3.3). To illustrate how parameterized complexity analyses are performed, proofs of selected results are given in the main text; all remaining proofs are given in an appendix. Finally, after discussing the implications of our results for both our basic model of LfD and LfD in practice (Section 4), we give our conclusions and some promising directions for future research (Section 5).

2 Parameterized Complexity Analysis

In this section, we first review of how classical types of computational complexity analysis such as the theory of NP-completeness [7] are used to show that problems are not efficiently solvable in general. We then give an overview of the analogous mechanisms
from parameterized complexity theory \cite{2,3,4} used to show that problems are not efficiently solvable under restrictions. Finally, we show how parameterized complexity analysis can be used to systematically explore efficient and reliable algorithmic options for solving a problem under restrictions and give several useful rules of thumb for minimizing the effort involved in carrying out such analyses.

Both classical and parameterized complexity analyses are based on the same notion of a computational problem expressed as a relation between inputs and their associated outputs. Given an input instance, a search problem asks for the associated output itself, e.g.,

**Dominating set** (search version)

*Input:* An undirected graph $G = (V, E)$.

*Output:* A minimum dominating set of $G$, i.e., a subset $V' \subseteq V$ of the smallest possible size such that for all $v \in V$, either $v \in V'$ or there is at least one $v' \in V'$ such that $(v, v') \in E$.

In classical complexity analysis, an algorithm for a problem is efficient if that algorithm runs in polynomial time—that is, the algorithm’s running time is always upper-bounded by $n^c$ where $n$ is the size of the input and $c$ is a constant. A problem which has a polynomial-time algorithm is polynomial-time tractable. Polynomial-time algorithms are preferable because their runtimes grow much more slowly than algorithms with non-polynomial runtimes, e.g., $2^n$, as input size increases and hence allow the solution of much larger inputs in practical amounts of time.

One shows that a problem $\Pi'$ is not polynomial-time tractable by giving a reduction from a problem $\Pi$ that is either not polynomial-time tractable or not polynomial-time tractable unless a widely-believed conjecture such as $P \neq NP$ \cite{8} is false. A polynomial-time reduction from $\Pi$ to $\Pi'$ \cite{7} is essentially a polynomial-time algorithm for transforming instances of $\Pi$ into instances of $\Pi'$ such that any polynomial-time algorithm for $\Pi'$ can be used in conjunction with this instance transformation algorithm to create a polynomial-time algorithm for $\Pi$. Polynomial-time intractable problems are isolated from appropriate classes of problems using the notions of hardness and completeness. Relative to a class of $C$ of problems, if every problem in $C$ reduces to $\Pi$ then $\Pi$ is said to be $C$-hard; if $\Pi$ is also in $C$ then $\Pi$ is $C$-complete. For technical reasons, these reductions are typically done between decision versions of problems for which the output is the answer to a yes/no question, e.g.,

**Dominating set** (decision version)

*Input:* An undirected graph $G = (V, E)$ and a positive integer $k$.

*Question:* Does $G$ contain a dominating set of size $k$?

Let such a decision version of a problem $\Pi$ be denoted by $\Pi_D$. This focus on decision problems does not cause difficulties in practice because if a decision version $\Pi_D$ of
a search problem $\Pi$ is defined such that any algorithm for $\Pi$ can be used to solve $\Pi_D$, then the polynomial-time intractability of $\Pi_D$ also implies the polynomial-time intractability of $\Pi$. Such is the case for the decision and search versions of DOMINATING SET defined above. In the case of $NP$-hard decision problems, this intractability holds unless $P = NP$. This is encoded in the following useful lemma.

**Lemma 1** If $X_D$ is $NP$-hard then $X$ is not solvable in polynomial time unless $P = NP$.

Parameterized problems differ from the classical search and decision problems defined above in that each parameterized problem has an associated set of one or more parameters, where a *parameter* of a problem is an aspect of that problem’s input or output. Example input and output parameters of DOMINATING SET $D$ are the maximum degree $d$ of any vertex in the given graph $G$ and the size $k$ of the requested dominating set. Given a set $K$ of parameters relative to a problem $\Pi$, let $\langle K \rangle$-$\Pi$ denote $\Pi$ parameterized relative to $K$. For example, two parameterized problems associated with DOMINATING SET $D$ are $\langle k \rangle$-DOMINATING SET $D$ and $\langle d, k \rangle$-DOMINATING SET $D$.

A restriction on a problem is phrased in terms of restrictions on the value the corresponding parameter, and algorithm efficiency under restrictions is phrased in terms of fixed-parameter tractability. A problem $\Pi$ is *fixed-parameter (fp-)*tractable relative a set of parameters $K = \{k_1, k_2, \ldots, k_m\}$ [2, 3], i.e., $\langle K \rangle$-$\Pi$ is fp-tractable, if there is an algorithm for $\Pi$ whose running time is upper-bounded by $f(K)n^c$ for some function $f()$ where $n$ is the problem input size and $c$ is a constant. Fixed-parameter tractability generalizes polynomial-time solvability by allowing problems to be effectively solvable in polynomial time when the values of the parameters in $K$ are small, e.g., $k_1, k_2 \leq 4$, and $f()$ is well-behaved, e.g., $1.2^{k_1+k_2}$, such that the value of $f(K)$ is a small constant. Hence, if a polynomial-time intractable problem $\Pi$ is fp-tractable relative to a well-behaved $f()$ for a parameter-set $K$ then $\Pi$ can be efficiently solved even for large inputs in which the values of the parameters in $K$ are small.

One shows that a parameterized problem $\Pi'$ is not fixed-parameter tractable by giving a parameterized reduction from a parameterized problem $\Pi$ that is either not fixed-parameter tractable or not fixed-parameter tractable unless a widely-believed conjecture such as $FPT \neq W[1]$ [2, 3] is false. A parameterized reduction from $\langle K \rangle$-$\Pi$ to $\langle K' \rangle$-$\Pi'$ [2] allows the instance transformation algorithm to run in fp-time relative to $K$ and requires for each $k' \in K'$ that there is a function $g_{k'}()$ such that $k' = g_{k'}(K)$. Such an instance transformation algorithm can be used in conjunction with any fixed-parameter algorithm for $\langle K' \rangle$-$\Pi'$ to create a fixed-parameter algorithm for $\langle K \rangle$-$\Pi$. Hardness and completeness for parameterized reductions is typically done
relative to classes in the \(W\)-hierarchy = \(\{W[1], W[2], \ldots, W[P], \ldots\}\) \cite{2, 3}. Once again, for technical reasons, reductions are typically done between decision versions of parameterized problems, and as any algorithm for a search version of a parameterized problem can solve the appropriately-defined decision version, we have the following parameterized analogue of Lemma 1.

**Lemma 2** Given a parameter-set \(K\) for problem \(X\), if \(\langle K \rangle \cdot X_D\) is \(W[1]\)-hard then \(\langle K \rangle \cdot X\) is not fp-tractable unless \(FPT = W[1]\).

In certain situations, one can get a more powerful result.

**Lemma 3** \cite[Lemma 2.1.35]{2} Given a parameter-set \(K\) for problem \(X\), if \(X_D\) is \(NP\)-hard when the value of every parameter \(k \in K\) is fixed to a constant value, then \(\langle K \rangle \cdot X\) is not fp-tractable unless \(P = NP\).

We can now finally talk about how the results of a parameterized complexity analysis for a problem can be used to derive an intractability map \cite{9}, which corresponds to the desired systematic overview of algorithmic options for solving that problem described in Section \(\[\]\). Given a set \(P\) of parameters of a problem \(\Pi\), an intractability map describes the parameterized complexity status of \(\Pi\) relative to each of the \(2^{|P|} - 1\) non-empty subsets of \(P\). The choice of \(P\) depends on how one wants to use the map. If one wishes to use the map as a probe to examine the effects of various parameters on the computational complexity of our problem of interest (as we do in our parameterized complexity analysis of learning from demonstration in Section \(\[\]\)), \(P\) should consist of parameters (which need not all be of small value in practice) characterizing all aspects of the input and output of that problem. If on the other hand one wishes to use the map as a guide to either developing algorithms or selecting the most appropriate algorithms for input instances of that problem that are encountered in practice, \(P\) should consist purely of aspects of the problem that are known to be small in at least some of these instances.

It is important to note that the initial algorithms used to construct an intractability map need not have practical runtimes—at this stage in analysis, one need only establish the fact and not the best possible degree of fp-tractability. The best possible fixed-parameter algorithms are developed subsequently as needed. There are a number of techniques for deriving fixed-parameter algorithms \cite{10, 11, 12}, and it has been observed multiple times within the parameterized complexity community that once fp-tractability is established, these techniques are applied by different groups of researchers in “FPT Races” to produce increasingly (and, on occasion, spectacularly) more efficient algorithms \cite{13, 14}.

An example derivation of an intractability map for a hypothetical problem \(\Pi\) with parameter-set \(\{A, B, C, D\}\) is given in Figure \(\[\]\). Part (a) of this figure describes a set
of parameterized intractability (R1, R2) and tractability (R3) results for Π; note that tractability results are highlighted by boldfacing. Each column in this table describes a result which holds relative to the parameter-set consisting of all parameters with a @-symbol in that column. If in addition a result holds when a particular parameter has a constant value $c$, that is indicated by $c$ replacing @ for that parameter in that result’s column. Part (b) gives the intractability map associated with the results in part (a). Each cell in this map denotes the parameterized status of Π (X for fp-intractability, √ for fp-tractability) relative to by the union of the sets of parameters labelling that cell’s column and row. The “raw” results from the table in part (a) are denoted by superscripted entries ($X^{R1}$, $X^{R2}$, $\sqrt{R3}$) and all other X and √ results in the map follow from these observations:

**Lemma 4** [9, Lemma 2.1.30] If problem Π is fp-tractable relative to parameter-set $K$ then Π is fp-tractable for any parameter-set $K'$ such that $K \subseteq K'$.

**Lemma 5** [9, Lemma 2.1.31] If problem Π is fp-intractable relative to parameter-set $K$ then Π is fp-intractable for any parameter-set $K'$ such that $K' \subseteq K$.

The remaining ???-entries correspond to parameter-sets whose parameterized complexity is not specified or implied in the given results. As there are ???-entries in the map in part (b), this map is a partial intractability map.

The effort involved in constructing an intractability map can be reduced (in some cases, dramatically) by applying the following two rules of thumb. First, prove the initial polynomial-time intractability of the problem of interest using reductions from
problems like DOMINATING SET whose parameterized versions are known to be fp-
intractable; this often allows such reductions to be re-used in the subsequent derivation of parameterized results. Second, in order to exploit Lemmas 4 and 5 to maximum effect when filling in the intractability map, fp-intractability results should be proved relative to the largest possible sets of parameters and fp-tractability results should be proved relative to the smallest possible sets of parameters. Both of these rules are used to good effect in the parameterized complexity analysis of learning from demonstration given in the next section.

3 Case Study: Learning from demonstration (LfD)

Learning from demonstration (LfD) [5, 6] is a popular approach for deriving policies that specify what action should be performed next given the current state of the environment. In LfD, a policy is derived from a set of one or more demonstrations, each of which is a sequence of one or more of environment-state/actions pairs. LfD can be used by itself or as a generator of initial policies that are optimized by techniques like reinforcement learning [16, Sections 5.1 and 5.2].

A number of algorithms and systems implementing LfD have been proposed over the last 35 years (see Section 59.2 of [6] and Section 4 of [16]). Some of these systems operate in “batch mode” [5, Page 471], i.e., a policy is derived from a given set of demonstrations, while others are incremental [6, Section 59.3.2], i.e., a policy derived relative to a set of previously-encountered demonstrations (which may or may not still be available) is modified to take into account a new demonstration. Fast learning relative to few demonstrations is often desirable [5, Page 475] and in some situations necessary [16, Page 5]. However, it is not known if existing (or indeed any) LfD algorithms can perform fast and reliable learning or, if not, under which restrictions such learning is possible.

In this section, we shall illustrate how the classical and parameterized complexity analysis techniques described in Section 2 can be applied to answer these questions relative to basic formalizations of batch and incremental LfD relative to memoryless reactive policies given in Section 3.1. We prove that all of these problems are polynomial-time intractable (and inapproximable) in general (Section 3.2) and remain fixed-parameter intractable under a number of (often simultaneous) restrictions (Section 3.3). The implications of all of our results for both for the basic conception of LfD examined here and LfD in practice are then discussed in Section 4.
3.1 Formalizing LfD

In order to perform our computational complexity analyses, we must first formalize the following entities and properties associated with learning from demonstration:

- Sensed environmental features and environmental states;
- Demonstrations of activities to be learned;
- Policies describing actions taken by robots in particular situations;
- What it means for a policy to correctly describe and hence be consistent with a given demonstration; and
- What it means for a policy \( p \) to be behaviorally equivalent to and hence consistent with another policy \( p' \) that has been derived from \( p \).

We will then formalize problems corresponding to batch and incremental versions of LfD. As part of our formalization process, we shall also discuss how our formalizations compare with those given to date in the literature.

We first formalize the basic entities associated with LfD:

- **Sensed environmental features and environmental states**: Let \( F = \{f_1, f_2, \ldots, f_{|F|}\} \) be a set of features that a robot can sense in its environment. A state \( s \in 2^F \) of the environment is represented by the subset of sensed features that characterize that state. Our features can be viewed as special cases of both Boolean-predicate and other multi-valued features [16, Section 3] in which the presence of a feature in a state corresponds to that feature being true or having a particular feature-value, respectively. An example feature-set \( F = \{f_1, f_2, f_3, f_4\} \) is given in part (a) of Figure 2.

- **Demonstrations**: A demonstration \( d = (type, ((s_1, a_1), (s_2, a_2), \ldots, (s_{|d|}, a_{|d|})) \) consists of a demonstration-type \( type \in \{pos, neg\} \) and a sequence of one or more environment-state / action pairs whose actions are drawn from an action-set \( A \). If \( d \) is of type pos, \( d \) is a positive demonstration; otherwise, \( d \) is a negative demonstration. As such, our demonstrations are based on state-spaces and actions that are discrete [5, Section 4.1]. Our positive demonstrations are as standardly defined for discrete actions [5]; however, our negative demonstrations are special cases of the negative demonstrations in [17, 18] as our negative demonstrations forbid all state / action pairs in those demonstrations rather than specific state / action pairs in a demonstration sequence. A set of demonstrations \( D = \{d_1, d_2, d_3, d_4\} \) based on feature-set \( F \) and action-set \( A \) given in parts (a) and (b) of Figure 2 respectively, is given in part (c) of Figure 2.
\[ f_1 = \text{weather is raining} \quad a_1 = \text{wear a raincoat} \]
\[ f_2 = \text{weather is cold} \quad a_2 = \text{wear a sweater} \]
\[ f_3 = \text{weather is sunny} \quad a_3 = \text{wear sunglasses} \]
\[ f_4 = \text{weather is windy} \quad a_4 = \text{wear a wingsuit} \]

(a) \quad (b)

\[ d_1 = (\text{pos}, \langle \{f_1\}, a_1 \rangle, \langle \{f_2\}, a_2 \rangle, \langle \{f_3\}, a_3 \rangle, \langle \{f_4\}, a_4 \rangle) \]
\[ d_2 = (\text{neg}, \langle \{f_1\}, a_2 \rangle, \langle \{f_3\}, a_1 \rangle) \]
\[ d_3 = (\text{pos}, \langle \{f_1, f_4\}, a_1 \rangle, \langle \{f_2, f_4\}, a_4 \rangle, \langle \{f_3\}, a_2 \rangle) \]
\[ d_4 = (\text{pos}, \langle \{f_2\}, a_2 \rangle, \langle \{f_3\}, a_3 \rangle) \]

(c)

\[ p_1 : T_1 = \{ \langle \{f_1\}, a_1 \rangle, \langle \{f_2\}, a_2 \rangle, \langle \{f_3\}, a_3 \rangle, \langle \{f_4\}, a_4 \rangle \} \]
\[ p_2 : T_2 = \{ \langle \{f_1\}, a_1 \rangle, \langle \{f_2\}, a_4 \rangle, \langle \{f_3\}, a_2 \rangle \} \]

(d)

Figure 2: Examples of basicLfD entities in our model. a) A set \( F = \{ f_1, f_2, f_3, f_4 \} \) of sensed environmental features. b) a set \( A = \{ a_1, a_2, a_3, a_4 \} \) of actions. c) A set \( D = \{ d_1, d_2, d_3, d_4 \} \) of demonstrations. d) Two policies \( p_1 \) and \( p_2 \) based on transition-sets \( t_1 \) and \( t_2 \), respectively.
• **Policies**: We will consider here the simplest possible type of reactive mapping-function policy [5, Section 4.1] stated in terms of a single-state transducer consisting of a state \( q \) and a transition-set \( T = ((F^t_1, a_1), (F^t_2, a_2), \ldots, (F^t_{|T|}, a_{|T|})) \) where the action \( a_i \) in each transition is drawn from an action-set \( A \). Given an environment-state \( s \in 2^F \), we say that a transition \( (F^t, a) \) triggers on \( s \) if \( F^t \subseteq s \). Our transition-triggering feature-sets are special cases of transition-triggering patterns encoded as Boolean formulas over environmental features [19, e.g.,] in which our feature-sets correspond to patterns composed of AND-ed sets of features. As our policies are not dependent on time, they are stationary (autonomous) [16, Section 2]. Two example policies \( p_1 \) and \( p_2 \) based on the feature-set \( F \) and action-set \( A \) given in parts (a) and (b) of Figure 2, respectively, are given in part (d) of Figure 2.

The behaviour of our policies should also be simple. To this end, we adopt a conservative notion of generalization to previously unobserved environmental states, in that a policy \( p \) is not defined and hence does not produce an action for any state \( s \) for which there is no transition \( (F^t, a) \) in \( p \) such that \( F^t \subseteq s \).

We also adopt a conservative notion of policy determinism, in that a policy is only guaranteed to be deterministic and produce a single action relative to states encoded in a demonstration-set \( D \). More formally, the transition-set \( T \) in any \( p \) relative to a demonstration-set \( D \) is such that for each state \( s \) in a demonstration in \( D \), all transitions in \( T \) that trigger on \( s \) produce the same action \( a \). Such a policy \( p \) is said to be valid relative to \( D \).

In some situations it will be useful to derive one policy from another. Given two policies \( p \) and \( p' \), we say that \( p' \) is derivable from \( p \) by at most \( c \) changes if at most \( c \) modifications drawn from the set \{substitute new transition feature-set, substitute new transition action, delete transition\} are required to transform \( p \) into \( p' \).

We now formalize the following properties associated with LfD:

• **Policy-demonstration consistency**: Given a policy \( p \) and a demonstration \( d \), \( p \) is consistent with \( d \) if, starting from state \( q \) and environment-state \( s_1 \), either

1. for each of the state-action-pairs \((s, a)\) in \( d \), \( p \) produces \( a \) when run on \( s \) (if \( d \) is a positive demonstration) or
2. for each of the state-action-pairs \((s, a)\) in \( d \), \( p \) does not produce \( a \) when run on \( s \) (if \( d \) is a negative demonstration).

A policy \( p \) is in turn consistent with a demonstration-set \( D \) if \( p \) is consistent with each demonstration \( d \in D \). Consistency of a policy with a positive demonstration is as standardly defined [5, 16]. Consistency of a policy with a negative
demonstration is a special case of such consistency as defined in [17, 18] and is in line with our special-case definition of negative demonstrations given above.

• **Policy-policy consistency**: This has not to our knowledge been previously defined in the literature; however, the following version will be of use in defining and analyzing incremental LfD when previously-learned demonstrations are not available. Given two policies \( p \) and \( p' \) and a demonstration \( d \), \( p \) is consistent with \( p' \) modulo \( d \) if

1. for each \( s \) that is the triggering feature-set of some transition in \( p' \) such that \( s \not\subseteq s' \) for some state / action pair \( (s', a) \) in \( d \), \( p \) and \( p' \) produce the same action when run on \( s \); and
2. for each \( s \) in some state / action pair \( (s, a) \) in \( d \), either
   
   (a) \( p \) produces action \( a \) when run on \( s \) (if \( d \) is a positive demonstration) or
   
   (b) \( p \) either (1) produces action \( a' \) when run on \( s \) if \( p' \) produces action \( a' \neq a \) when run on \( s \) or (2) \( p \) does not produce an action for \( s \) (if \( d \) is a negative demonstration).

The focus above exclusively on the feature-sets in the transitions of \( p' \) may initially seem strange. However, as any triggering feature-set of a transition \( t \) in \( p \) that mimics the behaviour of a transition \( t' \) in \( p' \) must have a triggering feature-set that is equal to or a subset of the triggering feature-set in \( t' \), any such consistent \( p \) will (modulo the behaviors requested or forbidden by \( d \)) replicate the behaviour of \( p' \).

Relative to these property formalizations, the following statements are true for the examples of LfD entities given in Figure 2:

• \( p_1 \) is valid for \( \{d_1, d_2\} \): This is so because for each environment-state \( s \) in \( d_1 \) and \( d_2 \), \( p_1 \) produces at most (and in all cases exactly) one action.

• \( p_1 \) is not valid for \( \{d_3\} \): This is so because \( p_1 \) produces action-sets \( \{a_1, a_4\} \) and \( \{a_2, a_4\} \) for environment-states \( \{f_1, f_4\} \) and \( \{f_2, f_4\} \) in \( d_3 \), respectively.

• \( p_2 \) is valid for \( \{d_1, d_2\} \): This is so because for each environment-state \( s \) in \( d_1 \) and \( d_2 \), \( p_2 \) produces at most one action.

• \( p_2 \) is valid for \( \{d_3\} \): This is so because for each environment-state \( s \) in \( d_3 \), \( p_2 \) produces at most one action.
• *p₂ is derivable from p₁ by at most 3 changes:* If the transitions of p₁ in T₁ are numbered 1–4 as they appear in T₁, this is so by substituting transition action a₄ for a₂ in t₂, substituting transition-action a₂ for a₃ in t₃, and deleting transition t₄.

• *p₁ is not derivable from p₂ by any number of changes:* This is so because p₁ has more transitions than p₂ and adding transitions is not an allowed change.

• *p₁ is consistent with {d₁, d₂}:* This is because p₁ produces all requested actions when run on the environment-states in d₁ and none of the specified actions when run on the environment-states in d₂.

• *p₁ is not consistent with {d₃}:* This is because p₁ produces two-action instead of one-action sets for the environment-states {f₁, f₄} and {f₂, f₄} in d₃ and the wrong action for environment-state {f₃} in d₃.

• *p₂ is not consistent with {d₁, d₂}:* This is so because p₂ produces the wrong actions for environment-states {f₂}, {f₃}, and {f₄} in d₁ (with the last of these actually producing no action at all). Note that p₂ is, however, consistent with {d₂}.

• *p₂ is consistent with {d₃}:* This is because p₂ produces all requested actions when run on the environment-states in d₃.

• *p₁ is consistent with p₂ modulo {d₄}:* This is so because p₁ and p₂ produce the same actions for all transition trigger-sets in p₂ that are not subsets (proper or otherwise) of environment-states in d₄ (namely, {f₁}) and for all other environment-states in d₄ (namely {f₂} and {f₃}), p₁ produces the action requested in d₄.

• *p₂ is not consistent with p₁ modulo {d₄}:* This is so because p₂ does not produce the same action as p₁ when run on the transition trigger-set {f₄} in p₁.

Given all of the above, we can formalize the LfD problems that we will analyze in the remainder of this paper:
**Batch Learning from Demonstration (LfDBat)**

*Input:* A set $D$ of demonstrations based on a feature-set $F$ and an action-set $A$ and positive non-zero integers $t$ and $f_t$.

*Output:* A policy $p$ valid for and consistent with $D$ such that there are at most $t$ transitions in $p$ and each transition is triggered by a set of at most $f_t$ features, if such a $p$ exists, and special symbol $\bot$ otherwise.

**Incremental Learning from Demonstration with History (LfDIncHist)**

*Input:* A set $D$ of demonstrations based on a feature-set $F$ and an action-set $A$, a policy $p$ that is valid for and consistent with $D$, a demonstration $d_{new}$ based on $F$ and $A$ such that $d_{new} \not\in D$, and positive non-zero integers $t$, $f_t$, and $c$.

*Output:* A policy $p'$ derivable from $p$ by at most $c$ changes that is valid for and consistent with $D \cup \{d_{new}\}$ such that there are at most $t$ transitions in $p'$ and each transition is triggered by a set of at most $f_t$ features, if such a $p'$ exists, and special symbol $\bot$ otherwise.

**Incremental Learning from Demonstration without History (LfDIncNoHist)**

*Input:* A policy $p$ based on a feature-set $F$ and an action-set $A$, a demonstration $d_{new}$ based on $F$ and $A$, and positive non-zero integers $t$, $f_t$, and $c$.

*Output:* A policy $p'$ derivable from $p$ by at most $c$ changes that is consistent with $p$ modulo $d_{new}$ such that there are at most $t$ transitions in $p'$ and each transition in $p'$ is triggered by a set of at most $f_t$ features, if such a $p'$ exists, and special symbol $\bot$ otherwise.

Let $LfDIncHist_{pos}$ and $LfDIncHist_{neg}$ ($LfDIncNoHist_{pos}$ and $LfDIncNoHist_{neg}$) denote the versions of $LfDIncHist$ ($LfDIncNoHist$) in which $d_{new}$ is a positive and negative demonstration, respectively; furthermore, let $LfDBat_D$, $LfDIncHist_D^{pos}$, $LfDIncHist_D^{neg}$, $LfDIncNoHist_D^{pos}$, and $LfDIncNoHist_D^{neg}$ denote the decision versions of the problem above which ask if the requested policy exists. Some readers may be disconcerted that we have incorporated explicit limits on the size and structure of the requested policies. This is useful in practice for applications in which $LfD$ must be done with limited computer memory [16, Page 5]. This is also useful in allowing us to investigate the effects of various aspects of policy size and structure on the computational difficulty of $LfD$.

### 3.2 $LfD$ is Polynomial-time Intractable

Let us now revisit our first question of interest—namely, are there efficient algorithms for any of the $LfD$ problems defined in Section 3.1 that are guaranteed to always produce their requested policies? We will answer this question using polynomial-time
reductions from the problem DOMINATING SET\_D defined in Section 2. The following definitions, assumptions, and known results will be useful below. For each vertex \( v \in V \) in an instance of DOMINATING SET\_D, let the complete neighbourhood \( N_G(v) \) of \( v \) be the set composed of \( v \) and the set of all vertices in \( G \) that are adjacent to \( v \) by a single edge, i.e., \( v \cup \{ u \mid u \in V \text{ and } (u, v) \in E \} \). We assume for each instance of DOMINATING SET\_D an arbitrary ordering on the vertices of \( V \) such that \( V = \{ v_1, v_2, \ldots, v_{|V|} \} \). Let DOMINATING SET\_D\^{PD3} denote the version of DOMINATING SET\_D in which the given graph \( G \) is planar and each vertex in \( G \) has degree at most 3. Both DOMINATING SET\_D and DOMINATING SET\_D\^{PD3} are NP-hard [7, Problem GT2].

**Lemma 6** DOMINATING SET\_D polynomial-time reduces to LfDBat\_D such that in the constructed instance LfDBat\_D, \(|A| = \#d = f_t = 1\) and \( t \) is a function of \( k \) in the given instance of DOMINATING SET\_D.

**Proof:** Given an instance \( \langle G = (V, E), k \rangle \) of DOMINATING SET\_D, construct an instance \( \langle D, t, f_t \rangle \) of LfDBat\_D as follows: Let \( F = \{ f_1, f_2, \ldots, f_{|V|} \} \), \( A = \{ a \} \), \( D = \{ (pos, ((s_1, a))),(pos, ((s_2, a))),\ldots,(pos, ((s_{|V|}, a))) \} \) where \( s_i \) is the set consisting of the features in \( F \) corresponding to the complete neighbourhood of \( v_i \) in \( G \), \( t = k \), and \( f_t = 1 \). Observe that this construction can be done in time polynomial in the size of the given instance of DOMINATING SET\_D.

We shall prove the correctness of this reduction in two parts. First, suppose that there is a subset \( V' = \{ v'_1, v'_2, \ldots, v'_l \} \subseteq V \), \( l \leq k \), that is a dominating set in \( G \). Construct a policy \( p \) with a transition \( \{ f_i \}, a \) for each \( v'_i \in V' \). Observe that the number of transitions in \( p \) is at most \( t \) and that \( p \) is valid for \( D \) (as all transitions produce the same action). Moreover, as \( V' \) is a dominating set in \( G \) and the state in each demonstration in \( D \) corresponds to the complete neighbourhood of one of the vertices in \( G \), \( p \) will produce the correct action for every demonstration in \( D \) and hence is consistent with \( D \).

Conversely, suppose there is a policy \( p \) consistent with \( D \) such that there are at most \( t \) transitions in \( p \) and each transition is triggered by a set of at most \( f_t \) features. As \( f_t = 1 \), the states in the demonstrations in \( D \) correspond to the complete neighborhoods of the vertices in \( G \), and \( p \) is consistent with \( D \), the set of features labeling the transitions in \( p \) corresponds to a dominating set in \( G \). Moreover, as \( t = k \), this dominating set is of size at most \( k \).

To complete the proof, observe that in the constructed instance of LfDBat\_D, \(|A| = \#d = f_t = 1\) and \( t = k \).

**Lemma 7** DOMINATING SET\_D polynomial-time reduces to LfDIncHist\_D\^{pos} such that in the constructed instance LfDIncHist\_D\^{pos}, \(|A| = \#d = 2\), \( f_t = 1 \), and \( t \) and \( c \) are functions of \( k \) in the given instance of DOMINATING SET\_D.
Proof: Given an instance \( \langle G = (V, E), k \rangle \) of DOMINATING SET\(_D\), construct an instance \( \langle D, p, d_{\text{new}}, c, t, f_i \rangle \) of LfDIncHist\(_D^{\text{pos}}\) as follows: Let \( F = \{f_1, f_2, \ldots, f_{|V|}; f_x, f_y\} \), \( A = \{a_1, a_2\} \), and \( D = \{\langle \text{pos}, \langle (s_1, a_1) \rangle \}, \langle \text{pos}, \langle (s_2, a_1) \rangle \}, \ldots, \langle \text{pos}, \langle (s_{|V|}, a_1) \rangle \} \} \) where \( s_i \) is the set consisting of \( f_x \) and the features in \( F \) corresponding to the complete neighbourhood of \( v_i \) in \( G \). Let \( p \) have \( t = k + 1 \) transitions, where the first transition is \( \langle \{f_1\}, a_1 \rangle \) and the remaining \( k \) transitions have the form \( \langle \{f_i\}, a_1 \rangle \) where \( f_i \) is the feature corresponding to a randomly selected vertex in \( G \). Finally, let \( d_{\text{new}} = \langle \langle \{f_x, f_y\}, a_2 \rangle \rangle \), \( c = k + 2 \), and \( f_t = 1 \). Note that \( p \) is valid for \( D \) (as all transitions produce the same action) and consistent with \( D \) (as the first transition in \( T \) will always generate the correct action \( a_1 \) for each demonstration in \( D \)). Observe that this construction can be done in time polynomial in the size of the given instance of DOMINATING SET\(_D\).

We shall prove the correctness of this reduction in two parts. First, suppose that there is a subset \( V' = \{v'_1, v'_2, \ldots, v'_l\} \subseteq V, l \leq k \), that is a dominating set in \( G \). Construct a policy \( p' \) with \( l + 1 \) transitions in which the first transition is \( \langle \{f_2\}, a_2 \rangle \) and the subsequent \( l \) transitions have the form \( \langle \{f_i\}, a_1 \rangle \) for each \( v'_i \in V' \). Observe that \( p' \) can be derived from \( p \) by at most \( c = k + 2 \) changes to \( p \) (namely, change the feature-set and action of the first transition and the feature-sets of the next \( l \) transitions as necessary and delete the final \( k - l \) transitions) and that \( p' \) is valid for \( D \cup \{d_{\text{new}}\} \) (as each state in the demonstrations in \( D \cup \{d_{\text{new}}\} \) cause \( p \) to produce at most one action). As \( V' \) is a dominating set in \( G \) and the state in each demonstration in \( D \) corresponds to the complete neighbourhood of one of the vertices in \( G \), \( p' \) will produce the correct action for every demonstration in \( D \) and hence is consistent with \( D \). Moreover, the first transition in \( p' \) produces the correct action for \( d_{\text{new}} \), which means that \( p' \) is consistent with \( D \cup \{d_{\text{new}}\} \).

Conversely, suppose there is a policy \( p' \) derivable from \( p \) by at most \( c \) changes that is valid for and consistent with \( D \cup \{d_{\text{new}}\} \) and has \( l \leq t = k + 1 \) transitions, each of which is triggered by a set of at most \( f_t \) features. One of these transitions must produce action \( a_2 \) in order for \( p' \) to be consistent with \( d_{\text{new}} \); moreover, this transition must also trigger on feature-set \( \{f_y\} \) (as triggering on \( \{f_x\} \), the only other option to accommodate \( d_{\text{new}} \), would cause \( p' \) to produce the wrong action for all demonstrations in \( D \)). As \( f_y \) does not occur in any state in \( D \), the remaining \( l - 1 \) transitions in \( p' \) must produce action \( a_1 \) for all states in \( D \) for \( p' \) to be consistent with \( D \). As \( f_t = 1 \) and the states in the demonstrations in \( D \) correspond to the complete neighborhoods of the vertices in \( G \), the set of features triggering the final \( l - 1 \) transitions in \( p' \) must correspond to a dominating set of size at most \( k \) in \( G \).

To complete the proof, observe that in the constructed instance of LfDIncHist\(_D^{\text{pos}}\), \(|A| = \#d = 2, f_t = 1, c = k + 2, \text{ and } t = k + 1.\)
The remaining reductions from DOMINATING SET to LfDIncHist$^{neg}_D$, LfDIncNoHist$^{pos}_D$, and LfDIncNoHist$^{neg}_D$ are given in Lemmas 11, 13, and 15 in the appendix.

**Result A:** LfDBat, LfDIncHist$^{pos}_D$, LfDIncHist$^{neg}_D$, LfDIncNoHist$^{pos}_D$, LfDIncNoHist$^{neg}_D$, linebreak and LfDIncNoHist$^{neg}_D$ are not polynomial-time tractable unless $P = NP$.

**Proof:** The $NP$-hardness of LfDBat$_D$, LfDIncHist$^{pos}_D$, LfDIncHist$^{neg}_D$, LfDIncNoHist$^{pos}_D$, and LfDIncNoHist$^{neg}_D$ follows from the $NP$-hardness of DOMINATING SET$_D$ and the reductions in Lemmas 6, 7, 11, 13, and 15. The result then follows from Lemma 1.

Given that the $P \neq NP$ conjecture is widely believed to be true [8, 7], this establishes that the most common types of LfD cannot be done both efficiently and correctly for all inputs.

### 3.2.1 LfD is Also Polynomial-time Inapproximable

Though it is not commonly known outside computational complexity circles, $NP$-hardness results such as those underlying Result A also imply various types of polynomial-time inapproximability. A polynomial-time approximation algorithm is an algorithm that runs in polynomial time in an approximately correct (but acceptable) manner for all inputs. There are a number of ways in which an algorithm can operate in an approximately correct manner. Three of the most popular ways are as follows:

1. **Frequently Correct (Deterministic) [20]:** Such an algorithm runs in polynomial time and gives correct solutions for all but a very small number of inputs. In particular, if the number of inputs for each input-size $n$ on which the algorithm gives the wrong or no answer (denoted by the function $err(n)$) is sufficiently small (e.g., $err(n) = c$ for some constant $c$), such algorithms may be acceptable.

2. **Frequently Correct (Probabilistic) [21]:** Such an algorithm (which is typically probabilistic) runs in polynomial time and gives correct solutions with high probability. In particular, if the probability of correctness is $\geq 2/3$ (and hence can be boosted by additional computations running in polynomial time to be correct with probability arbitrarily close to 1 [22, Section 5.2]), such algorithms may be acceptable.

3. **Approximately Optimal [23]:** Such an algorithm $A$ runs in polynomial time and gives a solution $A(x)$ for an input $x$ whose value $v(A(x))$ is guaranteed to be within a multiplicative factor $f(|x|)$ of the value $v_{OPT}(x)$ of an optimal solution for $x$, i.e., $|v_{OPT}(x) - v(A(x))| \leq f(|x|) \times v_{OPT}(x)$ for any input $x$ for
some function $f()$. A problem with such an algorithm is said to be polynomial-time $f(|x|)$-approximable. In particular, if $f(|x|)$ is a constant very close to 0 (meaning that the algorithm is always guaranteed to give a solution that is either optimal or very close to optimal), such algorithms may be acceptable.

It turns out that none of our LfD problems have such algorithms.

**Result B**: If LfDBat, LfDIncHist$^\text{pos}$, LfDIncHist$^\text{neg}$, LfDIncNoHist$^\text{pos}$, or LfDIncNoHist$^\text{neg}$ is solvable by a polynomial-time algorithm with a polynomial error frequency (i.e., $\text{err}(n)$ is upper bounded by a polynomial of $n$) then $P = NP$.

**Proof**: That the existence of such an algorithm for the decision versions of any of our LfD problems implies $P = NP$ follows from the $NP$-hardness of these problems (which is established in the proof of Result A) and Corollary 2.2. in [20]. The result then follows from the fact that any such algorithm for any of our LfD problems can be used to solve the decision version of that problem.

The following holds relative to both the $P \neq NP$ and $P = BPP$ conjectures, the latter of which is also widely believed to be true [21, 22].

**Result C**: If $P = BPP$ and LfDBat, LfDIncHist$^\text{pos}$, LfDIncHist$^\text{neg}$, LfDIncNoHist$^\text{pos}$, or LfDIncNoHist$^\text{neg}$ is polynomial-time solvable by a probabilistic algorithm which operates correctly with probability $\geq 2/3$ then $P = NP$.

**Proof**: It is widely believed that $P = BPP$ [22, Section 5.2] where $BPP$ is considered the most inclusive class of decision problems that can be efficiently solved using probabilistic methods (in particular, methods whose probability of correctness is $\geq 2/3$ and can thus be efficiently boosted to be arbitrarily close to one). Hence, if any of LfDBat, LfDIncHist$^\text{pos}$, LfDIncHist$^\text{neg}$, LfDIncNoHist$^\text{pos}$, or LfDIncNoHist$^\text{neg}$ has a probabilistic polynomial-time algorithm which operates correctly with probability $\geq 2/3$ then by the observation on which Lemma 1 is based, their corresponding decision versions also have such algorithms and are by definition in $BPP$. However, if $BPP = P$ and we know that all these decision versions are $NP$-hard by the proof of Result A, this would then imply by the definition of $NP$-hardness that $P = NP$, completing the result.

Certain inapproximability results follow not so much from $NP$-hardness as approximability characteristics of the particular problems used to establish $NP$-hardness. For any of our LfD problems with name $X$, let $X_{OPT}$ be the version of $X$ that returns the policy $p$ with the smallest possible value of $t$ (with this value being $\infty$ if there is
no such $p$).

**Result D:** For any of our LfD problems with name $X$, if $X_{OPT}$ is polynomial-time $c$-approximable for any constant $c > 0$ then $P = NP$.

**Proof:** Let $DOMINATING \text{ set}_{OPT}$ be the version of $DOMINATING \text{ set}$ which returns the size of the smallest dominating set in $G$—that is, the search version of $DOMINATING \text{ set}$ defined in Section 2. Observe that in the reductions in the proof of Result A, the size $k$ of a dominating set in $G$ in the given instance of $DOMINATING \text{ set}$ is always a linear function of $t$ in the constructed instance of $X_D$ (either $k = t$ (Lemmas 6, 11 and 13) or $k = t - 1$ (Lemmas 7 and 13)). In the first case, this means that a polynomial-time $c$-approximation algorithm for $X_{OPT}$ for any constant $c$ implies the existence of a polynomial-time $c$-approximation algorithm for $DOMINATING \text{ set}_{OPT}$ In the second case, this means that the existence of a polynomial-time $c$-approximation algorithm for $X_{OPT}$ for any constant $c$ implies the existence of a polynomial-time $2c$-approximation algorithm for $DOMINATING \text{ set}_{OPT}$ (as $c \times t = c \times (k + 1) \leq 2c \times k$ for $k \geq 1$). However, if $DOMINATING \text{ set}_{OPT}$ has a polynomial-time $c$-approximation algorithm for any constant $c > 0$ then $P = NP$ [25], completing the proof. 

Results B-D are not directly relevant to the goals of this paper, as approximation algorithms by definition are not reliable in the sense defined in Section 1. However, these results are still of interest for other reasons. For example, Result D suggests that it is very difficult to efficiently obtain even approximately minimum-size policies using LfD, which gives additional motivation for the parameterized analyses in the next section.

### 3.3 What Makes LfD Fixed-parameter Tractable?

We now turn to the question of what restrictions make the LfD problems defined in Section 3.1 tractable, which we rephrase as what combinations of parameters make our problems fixed-parameter tractable. The parameters examined in this paper are shown in Table 1 and can be broken into three groups:

1. Restrictions on environments ($|F|, |A|$);
2. Restrictions on demonstrations ($#d, |d|, f_{cap}$); and
3. Restrictions on policies ($t, f_t, c$).

We consider first what parameters do not yield fp-tractability. We will do this by exploiting the polynomial-time reductions from $DOMINATING \text{ set}_D$ in Section 3.2.
Table 1: Parameters for learning from demonstration problems.

| Parameter | Description                                      | Applicability |
|-----------|--------------------------------------------------|---------------|
| $F$       | # environmental description features             | All           |
| $|A|$      | # actions that can be taken in an environment    | All           |
| $\#d$     | # given demonstrations                           | All           |
| $|d|$      | max # environment-state action pairs in a demonstration | All           |
| $f_{cap}$ | max # features in a demonstration environment-state | All           |
| $t$       | max # transitions in a policy transducer         | All           |
| $f_{t}$   | max # features in a transition triggering pattern | All           |
| $c$       | Max # changes to transform one policy into another | LfDInc*       |

which also turn out to be parameterized reductions from $\langle k \rangle$-DOMINATING SET$_D$. In addition to the definitions, assumptions, and known results given at the beginning of Section 3.2, the fact that $\langle k \rangle$-DOMINATING SET$_D$ is $W[2]$-hard \cite{[2]} will be useful, as will the following consequence of our definitions of policies and consistency in Section 3.3.

**Lemma 8** Any policy $p$ is consistent with a demonstration-set $D$ consisting of $m \geq 1$ positive single state / pair demonstrations if and only if $p$ is consistent with a demonstration-set $D'$ consisting of a single positive demonstration in which all $m$ single state / action pairs in the demonstrations in $D$ have been placed in an arbitrary order.

**Lemma 9** DOMINATING SET$_D^{PD3}$ polynomial-time reduces to LfDBat$_D$ such that in the constructed instance LfDBat$_D$, $|A| = \#d = f_t = 1$, $f_{cap} = 4$, and $t$ is a function of $k$ in the given instance of DOMINATING SET$_D^{PD3}$.

**Proof:** As DOMINATING SET$_D^{PD3}$ is a special case of DOMINATING SET$_D$, the reduction in Lemma 6 from DOMINATING SET$_D$ to LfDBat$_D$ is also a reduction from DOMINATING SET$_D^{PD3}$ to LfDBat$_D$ that constructs instances of LfDBat$_D$ such that $|A| = \#d = f_t = 1$ and $t$ is a function of $k$ in the given instance of DOMINATING SET$_D^{PD3}$. To complete the proof, note that as the degree of each vertex in graph $G$ in the given instance of DOMINATING SET$_D^{PD3}$ is at most 3, the size of each com-
plete vertex neighbourhood is of size at most 4, which means that $f_{eap} = 4$ in each constructed instance of LfDBat$_D$.

**Result E:** LfDBat is not fp-tractable relative to the following parameter-sets:

a) $\{|A|, \#d, t, f_t\}$ when $|A| = \#d = f_t = 1$ (unless $FPT = W[1]$)

b) $\{|A|, |d|, t, f_t\}$ when $|A| = |d| = f_t = 1$ (unless $FPT = W[1]$)

c) $\{|A|, \#d, f_{eap}, f_t\}$ when $|A| = \#d = f_t = 1$ and $f_{eap} = 4$ (unless $P = NP$)

d) $\{|A|, |d|, f_{eap}, f_t\}$ when $|A| = |d| = f_t = 1$ and $f_{eap} = 4$ (unless $P = NP$)

**Proof:**

**Proof of part (a):** Follows from the $W[2]$-hardness of $\langle k \rangle$-Dominating set$_D$, the reduction in Lemma 6, the inclusion of $W[1]$ in $W[2]$, and the conjecture $FPT \neq W[1]$.

**Proof of part (b):** Follows from part (a) and Lemma 8.

**Proof of part (c):** Follows from the $NP$-hardness of Dominating set$_D^D3$, the reduction in Lemma 9 and Lemma 3.

**Proof of part (d):** Follows from part (c) and Lemma 8.

The following analogous results hold for our remaining LfD problems, and their proofs are given in the appendix.

**Result F:** LfDIncHist$^{pos}$ is not fp-tractable relative to the following parameter-sets:

a) $\{|A|, \#d, t, f_t, c\}$ when $|A| = \#d = 2$ and $f_t = 1$ (unless $FPT = W[1]$)

b) $\{|A|, |d|, t, f_t, c\}$ when $|A| = |d| = 2$ and $f_t = 1$ (unless $FPT = W[1]$)

c) $\{|A|, \#d, f_{eap}, f_t\}$ when $|A| = \#d = 2, f_{eap} = 4$, and $f_t = 1$ (unless $P = NP$)

d) $\{|A|, |d|, f_{eap}, f_t\}$ when $|A| = |d| = 2, f_{eap} = 4$, and $f_t = 1$ (unless $P = NP$)

**Result G:** LfDIncHist$^{neg}$ is not fp-tractable relative to the following parameter-sets:

a) $\{|A|, \#d, t, f_t, c\}$ when $|A| = \#d = 2$ and $f_t = 1$ (unless $FPT = W[1]$)

b) $\{|A|, |d|, t, f_t, c\}$ when $|A| = |d| = 2$ and $f_t = 1$ (unless $FPT = W[1]$)

c) $\{|A|, \#d, f_{eap}, f_t\}$ when $|A| = \#d = 2, f_{eap} = 4$, and $f_t = 1$ (unless $P = NP$)
Result H: LfDIncNoHist\textsuperscript{pos} is not fp-tractable relative to the following parameter-sets:

a) \{|A|, \#d, t, f_t\} when \(|A| = \#d = f_t = 1\) (unless \(FPT = W[1]\))

b) \{|A|, |d|, t, f_t\} when \(|A| = |d| = f_t = 1\) (unless \(FPT = W[1]\))

c) \{|A|, \#d, feap, f_t\} when \(|A| = \#d = f_t = 1\) and \(feap = 4\) (unless \(P = NP\))

d) \{|A|, |d|, feap, f_t\} when \(|A| = |d| = f_t = 1\) and \(feap = 4\) (unless \(P = NP\))

Result I: LfDIncNoHist\textsuperscript{neg} is not fp-tractable relative to the following parameter-sets:

a) \{|A|, \#d, t, f_t\} when \(|A| = \#d = f_t = 1\) (unless \(FPT = W[1]\))

b) \{|A|, |d|, t, f_t\} when \(|A| = |d| = f_t = 1\) (unless \(FPT = W[1]\))

c) \{|A|, \#d, feap, f_t\} when \(|A| = \#d = f_t = 1\) and \(feap = 4\) (unless \(P = NP\))

d) \{|A|, |d|, feap, f_t\} when \(|A| = |d| = f_t = 1\) and \(feap = 4\) (unless \(P = NP\))

Given that the \(P \neq NP\) and \(FPT \neq W[1]\) conjectures are widely believed to be true \([2, 3, 8, 7]\), these results show that LfD cannot be done efficiently under a number of restrictions. These results are more powerful than they first appear courtesy of Lemma 4 which establishes in conjunction with these results that none of the parameters considered here except \(|F|\) can be either individually or in many combinations be restricted to yield tractability for any of our LfD problems. Moreover, this intractability frequently holds when these parameters are restricted to very small constant values (see Tables 2 and 3 for details).

Despite this, there are combinations of parameters relative to which our problems are fp-tractable.

Result J: LfDBat, LfDIncHist\textsuperscript{pos}, LfDIncHist\textsuperscript{neg}, LfDIncNoHist\textsuperscript{pos}, and LfDIncNoHist\textsuperscript{neg} are all fp-tractable relative to parameter-set \{|F|, |A|\}.
Table 2: A detailed summary of our parameterized complexity results. a) Results for LfDBat. b) Results for LfDIncHist$^{pos}$. c) Results for LfDIncHist$^{neg}$.
Proof: Consider the following algorithm for LfDBat: Generate all possible policies with \( t \) transitions triggered by feature-sets with at most \( f_t \) features and for each such policy \( p \), determine if \( p \) is valid for and consistent with \( D \). There are \( t \leq (2^{|F|} - 1)|A| \) possible transitions and at most \( \sum_{i=1}^{t} \binom{(2^{|F|} - 1)|A|}{i} \) ways of choosing at most \( t \) transitions to form a policy. Each such policy can be checked in low-order polynomial time to see if each transition is triggered by a set of at most \( f_t \) features; moreover, it can also be verified in low-order polynomial time if each such \( f_t \)-limited policy is valid for and consistent with \( D \) by running each environment-state in \( D \) against all transitions in \( p \) to see if the action associated with that state is generated by all transitions in \( p \) that are triggered by that state. This algorithm thus runs in time upper-bounded by some function of \(|F|\) and \(|A|\) times some polynomial of the instance size, which completes the proof for LfDBat.

The algorithm above for LfDBat can be used to find possible policies that are valid for and consistent with \( D \cup \{d_{\text{new}}\} \) in LfDIncHist\textsuperscript{pos} and LfDIncHist\textsuperscript{neg}, with the addition of a step that checks each such policy \( p \) to see if it is also derivable from \( p' \) by at most \( c \) changes. There are \( t' \leq (2^{|F|} - 1) \) transitions in \( p' \) and hence at most \( t' \) transitions can be deleted from \( p' \) to leave \( t \) transitions in \( p \). As there are at most \( t \leq 2^{|F|} - 1 \) transitions in \( p \), there are at most \( t \) transitions whose feature-sets or actions can be substituted; moreover, there are \((2^{|F|} - 1) + |A|\) choices of such substitutions. The number of choices of at most \( c \) transition deletions and substitutions that can be made to \( p' \) to create \( p \) is thus at most

\[
\sum_{i=1}^{c} \sum_{j=1}^{i} \binom{t'}{j} \binom{t}{i-j} \big( (2^{|F|} - 1) + |A| \big) \leq \sum_{i=1}^{c} \sum_{j=1}^{i} t'^j t^{i-j} \big( (2^{|F|} - 1) + |A| \big)^{i-j} \\
\leq \sum_{i=1}^{c} \sum_{j=1}^{i} t'^h t^i \big( (2^{|F|} - 1) + |A| \big)^i \\
\leq \sum_{i=1}^{c} \sum_{j=1}^{i} t'^c t^c \big( (2^{|F|} - 1) + |A| \big)^c \\
= \sum_{i=1}^{c} t'^c t^c \big( (2^{|F|} - 1) + |A| \big)^c \\
= c^2 (t'^c t^c \big( (2^{|F|} - 1) + |A| \big)^c)
\]

Each of these choices of deletions and substitutions can be applied in time polynomial in \( c \) to see if they do in fact transform \( p' \) into \( p \). As \( c \leq (2^{|F|} - 1) + (2^{|F|} - 1) \), this
algorithm thus runs in time upper-bounded by some function of $|F|$ and $|A|$ times some polynomial of the instance size, which completes the proof for $LfDIncHist^{pos}$ and $LfDIncHist^{neg}$.

Finally, the algorithms above for $LfDIncHist^{pos}$ and $LfDIncHist^{neg}$ can be used to find possible policies $p'$ that are consistent with the given $p$ modulo $d_{new}$ if one eliminates the step checking if $p'$ is consistent with $D \cup \{d_{new}\}$ and adds a step that checks each such policy $p'$ is consistent with $p$ modulo $d_{new}$. As this new step only involves straightforward low-order polynomial operations on the transitions in $p$ and $p'$ and the environment-state / action pairs in $d_{new}$, this step can be done in low-order polynomial time. Hence, the revised algorithm runs in time upper-bounded by some function of $|F|$ and $|A|$ times some polynomial of the instance size, which completes the proof for $LfDIncNoHist^{pos}$ and $LfDIncNoHist^{neg}$ as well as the proof as a whole.

\[ \] Result K: $LfDBat$, $LfDIncHist^{pos}$, and $LfDIncHist^{neg}$ are all fp-tractable relative to parameter-set \{$.d, |d|, f_{eap}$\}.

**Proof:** Follows from the algorithms given in the proof of Result H and the observation that, as both $p$ and $p'$ can only use environment features in $D$ and $d_{new}$, $|F| \leq \#d \times |d| \times f_{eap}$.

Again, these results are more powerful than they first appear courtesy of Lemma 5, which establishes in conjunction with these results that any set of parameters including both $|F|$ and $|A|$ or all of $|A|$, $\#d$, $|d|$ and $f_{eap}$ can be restricted to yield tractability for all of our LfD problems and all of $LfDBat$, $LfDIncHist^{pos}$, and $LfDIncHist^{neg}$, respectively.

The intractability maps for our LfD problems relative to the results given above are large, and hence we here only give the intractability map for LfDBat (Table 4). Though this map is partial, it has 70 out of 127 parameterized result cells filled, and this was accomplished courtesy of Lemmas 4 and 5 using only 4 fp-intractability and 2 fp-tractability results. This very nicely demonstrates how the rules of thumb for performing parameterized complexity analyses at the end of Section 2 can help minimize the effort of performing these analyses.

\section{Discussion}

What do our results mean for LfD relative to the basic model investigated here? In addition to proving that batch LfD is not polynomial-time solvable in general, we have also shown that incremental LfD is not polynomial-time solvable in general when previously-encountered demonstrations either are or are not available. This
Table 3: A detailed summary of our parameterized complexity results (Cont’d). d) Results for LfDIncNoHist$^{pos}$. e) Results for LfDIncNoHist$^{neg}$.
intractability holds whether we require that requested policies are always output correctly (Result A) or that requested policies are always output correctly for inputs under a large number of both individual and simultaneous restrictions (Results E–I). These results suggest that it may be much more computationally difficult than is often realized to not only do LfD by itself but also to use LfD as an initial generator of policies that are subsequently optimized by other techniques [16, Sections 5.1 and 5.2]. The incremental LfD results are particularly sobering, as certain applications require that LfD be done very quickly with limited memory [16, Page 5], and incremental approaches that do not require all previously-encountered demonstrations to remain available seem the best hope for achieving this. That general solvability is not possible using efficient probabilistic algorithms (Result C) may also be problematic, given the increasing popularity of statistical-inference-based approaches to LfD, e.g., [26, 27].

That being said, the various applications mentioned above may yet be guaranteed to run efficiently by exploiting various restrictions. Our results to date suggest that it is most important to restrict the environment. This can be done either directly by restricting the available features and actions in the environment (\{|F|, |A|\}; Result J) or indirectly by restricting the structure of the given demonstrations (\{|A|, #d, |d|, f_{cap}\}; Result K). The latter is particularly exciting, as it shows that LfD can be done efficiently relative to few given demonstrations as long as these demonstrations are also

| Table 4: Current Intractability Map for LfDBat. |
|---|---|---|---|---|---|---|---|---|
| —— | #d | |d| | #d | |d| | #d | f_{cap} | #d | |d| | f_{cap} | #d | |d| | f_{cap} |
| — | NPh | X | X | X | ?? | X | X | ??? | ??? |
| t | X | X | X | ??? | ??? | ??? | ??? | ??? | ??? |
| f_t | X | X | X | X | ??? | X | X | ??? | ??? |
| | | | | | | | | | |
| t, f_t | X | X | X | ??? | ??? | ??? | ??? | ??? | ??? |
| t | X | X | X | ??? | ??? | ??? | ??? | ??? | ??? |
| f_t, | F | ??? | ??? | ??? | ??? | ??? | ??? | ??? | ??? |
| f_t | X | X | X | ??? | ??? | X | X | ??? | ??? |
| | | | | | | | | | |
| t, f_t | X | X | X | ??? | ??? | ??? | ??? | ??? | ??? |
| t, | F | ??? | ??? | ??? | ??? | ??? | ??? | ??? | ??? |
| f_t, | A | ??? | ??? | ??? | ??? | ??? | ??? | ??? | ??? |
| f_t | X | X | X | ??? | ??? | ??? | ??? | ??? | ??? |
| t, f_t, | F, | A | ??? | ??? | ??? | ??? | ??? | ??? | ??? |
| t, f_t, | F, | A | ??? | ??? | ??? | ??? | ??? | ??? | ??? |

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small in the sense of having few environment-state / action pairs which invoke few features and actions. Given that it is desirable that LfD be done efficiently with few demonstrations, i.e., when \#d is restricted [5, Page 475], it would be very useful if fp-tractability held relative to a small subset of \{|A|, \#d, |d|, f_{eap}\} that includes \#d (in the best case, \#d by itself). Parts (c) and (d) of Results E, F, and G rule out the possibility of fast LfD relative to many such subsets, including \{|\#d|\}. Just how many (if any) of the parameters in \{|A|, \#d, |d|, f_{eap}\} can be removed while retaining fp-tractability is thus a very important open question.

This highlights the fact that any conclusions drawn from our parameterized results must for now be tentative, as we have not yet characterized the parameterized complexity of all possible subsets of the parameters given in Table 1. For example, the extent of our knowledge of the parameterized status of LfDBat relative to the parameters examined in this paper is brought home by the partial intractability map in Table 4. Tractability may lurk in the uncharacterized subsets in this map and the intractability maps for our other LfD problems. For example, tractability may hold relative to certain sets of restrictions on policies (with sets including the number c of allowable policy-transformation changes relative to problems LfDIncNoHist^{pos} and LfDIncNoHist^{neg} being particularly tantalizing). Tractability might also be obtained using parameters not considered here. Possible candidates include parameters enforcing a high degree of similarity between time-adjacent environment-states in demonstrations (which seems to hold in real-world demonstrations) or characterizing the requested degree of compression of the information in the given demonstrations by a policy in relative rather than absolute terms, i.e., as a ratio of given demonstration-set size to policy size rather than (as is done here) just policy size alone. Additional parameters may be suggested by aspects of real-world LfD instances whose values are typically small as well as constraints invoked in cognitive systems, e.g., children learning by imitation [28, 29, 30].

All of this is most intriguing for the basic model of LfD analyzed in this paper. However, what (if anything) do our results have to say about more complex models of LfD invoked in practice? Given that real-world LfD often uses continuous rather than discrete demonstrations and infers a number of types of policies such as decision trees, hidden Markov models, and Gaussian mixture models [26, Page 789], the answer may initially seem to be “not much at all”. We agree that our analysis is not immediately applicable in these cases and are for now merely suggestive of conditions under which tractability and intractability may hold. However, they may be applicable in future in several ways:

- If our basic model of LfD is a special case of a more complex model of LfD, all of our intractability results also apply to that more complex model (as any algorithm for the more complex model would also work for our basic model,
making our basic model tractable and thus causing a contradiction).

- If this is not the case, the techniques invoked in both our tractability and intractability results may suggest ways of deriving algorithms and intractability results for more complex models.

- If even this does not hold, doing classical and parameterized complexity analyses like those presented here may still be worthwhile, in order to characterize more accurately those situations under which fast learning is and is not possible relative to these more complex models. This would enable the invocation of LfD in more situations with greater confidence.

Given this potential utility of our analysis in terms of either results, proof techniques, or analytical frameworks, these analyses should thus be seen not as an endpoint but rather the start of a (hopefully ongoing and productive) conversation between LfD researchers and computational complexity analysts.

5 Conclusions

In this paper, we have shown how parameterized complexity analysis can be used to systematically explore the algorithmic options for efficient and reliable machine learning. As an illustrative example, we gave the first parameterized complexity analysis of batch and incremental policy inference under learning from demonstration (LfD). These analyses were done relative to a basic model of LfD which uses discrete feature-based positive and negative demonstrations and time-independent policies specified as single-state transducers. Relative to this basic model, we showed that none of our LfD problems can be solved efficiently either in general or relative to a number of (often simultaneous) restrictions on environments, demonstrations, and policies. We also gave the first known restrictions under which efficient solvability is possible and discussed the implications of our solvability and unsolvability results for both our basic model of LfD and more complex models of LfD used in practice.

There are several promising directions for future research, both with respect to LfD in particular and machine learning in general. With respect to LfD, in addition to completing the characterization of the parameterized complexity of the LfD problems defined in this paper relative to the parameters given in Table 1, analyses should be extended to more complex models of LfD. This includes not only the more complex types of demonstrations and policies mentioned in Section 4 but also more complex LfD inference problems, e.g., LfD in which learners interactively receive critiques from teachers [17] or request useful positive and/or negative demonstrations [31]. More generally, given the popularity of statistical-inference approaches in machine
learning, it would be of great interest to extend our parameterized exploration of algorithmic options to include probabilistic as well as deterministic algorithms. Initial steps in this direction have already been made for other problems [32, 33, 34] and await application to problems from machine learning. All told, we believe that there is much that parameterized complexity analysis has to offer researchers in machine learning, and hope that the techniques and analyses given in this paper are a useful first step in this endeavour.

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References

[1] A. Moitra, Algorithmic Aspects of Machine Learning. Cambridge University Press, 2019.

[2] R. G. Downey and M. R. Fellows, Parameterized Complexity. Berlin: Springer, 1999.

[3] ——, Fundamentals of Parameterized Complexity. Berlin: Springer, 2013.

[4] J. Flum and M. Grohe, Parameterized Complexity Theory. Springer, Berlin, 2006.

[5] B. D. Argall, S. Chernova, M. Veloso, and B. Browning, “A survey of robot learning from demonstration,” Robotics and Autonomous Systems, vol. 57, pp. 469–483, 2009.

[6] A. Billard, S. Callinon, Rüdiger Dillmann, and S. Schall, “Robot programming by demonstration,” in Handbook of Robotics, B. Siciliano and O. Khatib, Eds. New York: Springer, 2008, ch. 59, pp. 1371–1394.

[7] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness. San Francisco, CA: W.H. Freeman, 1979.

[8] L. Fortnow, “The status of the P versus NP problem,” Communications of the ACM, vol. 52, no. 9, pp. 78–86, 2009.
[9] T. Wareham, “Systematic parameterized complexity analysis in computational phonology,” Ph.D. dissertation, University of Victoria, 1999.

[10] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, Parameterized Algorithms. Springer, 2015.

[11] F. V. Fomin, D. Lokshtanov, S. Saurabh, and M. Zehavi, Kernalization: Theory of Parameterized Preprocessing. Cambridge University Press, 2019.

[12] R. Niedermeier, Invitation to Fixed-Parameter Algorithms. Oxford University Press, 2006.

[13] C. Komusiewicz and R. Niedermeier, “New races in parameterized algorithmics,” in International Symposium on Mathematical Foundations of Computer Science, ser. Lecture Notes in Computer Science, B. Rovan, V. Sassone, and P. Widmayer, Eds., vol. 7464. Springer, 2012, pp. 19–30.

[14] U. Stege, “The impact of parameterized complexity to interdisciplinary problem solving,” in The Multivariate Algorithmic Revolution and Beyond, ser. Lecture Notes in Computer Science. Berlin: Springer, 2012, no. 7370, pp. 56–68.

[15] T. Wareham, “Designing robot teams for distributed construction, repair, and maintenance,” ACM Transactions on Autonomous and Adaptive Systems, Submitted.

[16] A. Hussein, M. M. Gaber, E. Elyan, and C. Jayne, “Imitation learning: A survey of learning methods,” ACM Computing Surveys (CSUR), vol. 50, no. 2, p. 21, 2017.

[17] B. D. Argall, B. Browning, and M. Veloso, “Learning by demonstration with critique from a human teacher,” in Proceedings of the Second ACM/IEEE International Conference on Human-Robot Interaction. IEEE, 2007, pp. 57–64.

[18] M. N. Nicolescu and M. J. Mataric, “Natural methods for robot task learning: Instructive demonstrations, generalization and practice,” in Proceedings of the Second International Joint Conference on Autonomous Agents and Multiagent Systems. ACM, 2003, pp. 241–248.

[19] B. Dufay and J.-C. Latombe, “An approach to automatic robot programming based on inductive learning,” International Journal of Robotics Research, vol. 3, no. 4, pp. 3–20, 1984.
[20] L. A. Hemaspaandra and R. Williams, “Complexity Theory Column 76: An atypical survey of typical-case heuristic algorithms,” ACM SIGACT News, vol. 43, no. 4, pp. 70–89, 2012.

[21] R. Motwani and P. Raghavan, Randomized Algorithms. Chapman & Hall/CRC, 2010.

[22] A. Wigderson, “P, NP and mathematics — A computational complexity perspective,” in Proceedings of ICM 2006: Volume I. Zurich: EMS Publishing House, 2007, pp. 665–712.

[23] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi, Complexity and Approximation: Combinatorial Optimization Problems and their Approximability Properties. Springer, 1999.

[24] A. E. F. Clementi, J. D. P. Rolim, and L. Trevisan, “The computational complexity column: Recent advances towards proving $P = BPP$,” Bulletin of the European Association for Theoretical Computer Science, vol. 64, pp. 96–103, 1998.

[25] C. Lund and M. Yannakakis, “On the hardness of approximating minimization problems,” Journal of the ACM (JACM), vol. 41, no. 5, pp. 960–981, 1994.

[26] S. P. Chatzis, D. Korkinof, and Y. Demiris, “A nonparametric Bayesian approach toward robot learning by demonstration,” Robotics and Autonomous Systems, vol. 60, no. 6, pp. 789–802, 2012.

[27] S. Choi, K. Lee, and S. Oh, “Robust learning from demonstration using leveraged Gaussian processes and sparse-constrained optimization,” in Proceedings of the 2016 IEEE International Conference on Robotics and Automation. IEEE, 2016, pp. 470–475.

[28] D. Buchsbaum, A. Gopnik, T. L. Griffiths, and P. Shafto, “Children’s imitation of causal action sequences is influenced by statistical and pedagogical evidence,” Cognition, vol. 120, no. 3, pp. 331–340, 2011.

[29] A. N. Meltzoff, “Imitation and other minds: The “like me” hypothesis,” in Perspectives on Imitation: From Neuroscience to Social Science, S. Hurley and N. Chater, Eds. Cambridge, MA: The MIT Press, 2005, vol. 2, pp. 55–77.

[30] S. Schaal, A. Ijspeert, and A. Billard, “Computational approaches to motor learning by imitation,” Philosophical Transactions of the Royal Society of London B: Biological Sciences, vol. 358, no. 1431, pp. 537–547, 2003.
[31] S. Chernova and M. Veloso, “Multi-thresholded approach to demonstration selection for interactive robot learning,” in Proceedings of the Third ACM/IEEE International Conference on Human-Robot Interaction. ACM, 2008, pp. 225–232.

[32] M. Blokpoel, J. Kwisthout, T. P. van der Weide, T. Wareham, and I. van Rooij, “A computational-level explanation of the speed of goal inference,” Journal of Mathematical Psychology, vol. 57, no. 3/4, pp. 117–133, 2013.

[33] J. Kwisthout, “Tree-width and the computational complexity of MAP approximations in Bayesian networks,” Journal of Artificial Intelligence Research, vol. 53, pp. 699–720, 2015.

[34] J. A. Montoya and M. Müller, “Parameterized random complexity,” Theory of Computing Systems, vol. 52, no. 2, pp. 221–270, 2013.

Appendix A: Proofs of Results

In this appendix, we give proofs of various results stated in the main text that were not given in the main text.

**Lemma 10** Dominating set\(^{PD3}\) polynomial-time reduces to LfDIncHist\(_D^{pos}\) such that in the constructed instance LfDIncHist\(_D^{pos}\), \(|A| = \#d = 2\), \(f_t = 1\), \(f_\text{cap} = 4\), and \(t \) and \(c\) are functions of \(k\) in the given instance of Dominating set\(^{PD3}\).

**Proof:** As Dominating set\(_D^{PD3}\) is a special case of Dominating set\(_D\), the reduction in Lemma 7 from Dominating set\(_D\) to LfDIncHist\(_D^{pos}\) is also a reduction from Dominating set\(^{PD3}\) to LfDIncHist\(_D^{pos}\) that constructs instances of LfDIncHist\(_D^{pos}\) such that \(|A| = \#d = 2\), \(f_t = 1\) and \(c\) and \(t\) are functions of \(k\) in the given instance of Dominating set\(^{PD3}\). To complete the proof, note that as the degree of each vertex in graph \(G\) in the given instance of Dominating set\(^{PD3}\) is at most 3, the size of each complete vertex neighbourhood is of size at most 4, which means that \(f_\text{cap} = 4\) in each constructed instance of LfDIncHist\(_D^{pos}\). \(\blacksquare\)

**Lemma 11** Dominating set\(_D\) polynomial-time reduces to LfDIncHist\(_D^{neg}\) such that in the constructed instance LfDIncHist\(_D^{neg}\), \(|A| = f_t = 1\), \(\#d = 2\), and \(t\) and \(c\) are functions of \(k\) in the given instance of Dominating set\(_D\).
Proof: Given an instance $\langle G = (V, E), k \rangle$ of DOMINATING SET$_D$, construct an instance $\langle D, p, d_{new}, c, t, f_i \rangle$ of LfDIncHist$_D^{neg}$ as follows: Let $F = \{f_1, f_2, \ldots, f_{|V|}, f_x\}$, $A = \{a\}$, and $D$ be as in the reduction in the proof of Lemma 7. Let $p$ have $t = k$ transitions, where the first transition is $(\{f_x\}, a)$ and the remaining $k - 1$ transitions have the form $(\{f_i\}, a)$ where $f_i$ is the feature corresponding to a randomly selected vertex in $G$. Finally, let $d_{new} = (neg, (\{\{f_x\}, a\}))$, $c = k$ and $f_i = 1$. Note that $p$ is valid for $D$ (as all transitions produce the same action) and consistent with $D$ (as the first transition in $T$ will always generate the correct action $a$ for each demonstration in $D$). Observe that this construction can be done in time polynomial in the size of the given instance of DOMINATING SET$_D$.

We shall prove the correctness of this reduction in two parts. First, suppose that there is a subset $V' = \{v'_1, v'_2, \ldots, v'_l\} \subseteq V$, $l \leq k$, that is a dominating set in $G$. Construct a policy $p'$ with $l$ transitions which have the form $(\{f_i\}, a)$ for each $v'_i \in V'$. Observe that $p'$ can be derived from $p$ by at most $c = k$ changes to $p$ (namely, change the triggering feature-sets of the first $l$ transitions as necessary and delete the final $k - l$ transitions) and that $p'$ is valid for $D \cup \{d_{new}\}$ (as all transitions produce the same action). As $V'$ is a dominating set in $G$ and the state in each demonstration in $D$ corresponds to the complete neighbourhood of one of the vertices in $G$, $p'$ will produce the correct action for every demonstration in $D$ and hence is consistent with $D$. Moreover, $p'$ cannot produce the action forbidden by $d_{new}$ for state $\{f_x\}$, which means that $p'$ is consistent with $D \cup \{d_{new}\}$.

Conversely, suppose there is a policy $p'$ derivable from $p$ by at most $c$ changes that is valid for and consistent with $D \cup \{d_{new}\}$ and has $l \leq t = k$ transitions, each of which is triggered by a set of at most $f_i$ features. None of these transitions can produce action $a$ on state $\{f_x\}$ in order for $p'$ to be consistent with $d_{new}$; moreover, the $l$ transitions in $p'$ must produce action $a$ for all states in $D$ in order for $p'$ to be consistent with $D$. As $f_i = 1$ and the states in the demonstrations in $D$ correspond to the complete neighborhoods of the vertices in $G$, the set of features triggering these $l$ transitions in $p$ must correspond to a dominating set of size at most $k$ in $G$.

To complete the proof, observe that in the constructed instance of LfDIncHist$_D^{neg}$, $|A| = f_t = 1$, $\#d = 2$, and $c = t = k$.

Lemma 12 DOMINATING SET$_D^{PD3}$ polynomial-time reduces to LfDIncHist$_D^{neg}$ such that in the constructed instance LfDIncHist$_D^{neg}$, $|A| = f_t = 1$, $\#d = 2$, $f_{cap} = 4$, and $t$ and $c$ are functions of $k$ in the given instance of DOMINATING SET$_D^{PD3}$.

Proof: As DOMINATING SET$_D^{PD3}$ is a special case of DOMINATING SET$_D$, the reduction in Lemma 11 from DOMINATING SET$_D$ to LfDIncHist$_D^{neg}$ is also a reduction from DOMINATING SET$_D^{PD3}$ to LfDIncHist$_D^{neg}$ that constructs instances of LfDIncHist$_D^{neg}$ such that $|A| = f_t = 1$, $\#d = 2$, and $c$ and $t$ are functions of $k$ in the given instance of
DOMINATING SET\textsuperscript{PD3}. To complete the proof, note that as the degree of each vertex in graph G in the given instance of DOMINATING SET\textsuperscript{PD3} is at most 3, the size of each complete vertex neighbourhood is of size at most 4, which means that \( f_{\text{cap}} = 4 \) in each constructed instance of LfDIncHist\textsuperscript{neg}.

**Lemma 13** DOMINATING SET\textsubscript{D} polynomial-time reduces to LfDIncNoHist\textsuperscript{pos} such that in the constructed instance LfDIncNoHist\textsuperscript{pos}, \(|A| = \#d = f_t = 1, \) and \( t \) is a function of \( k \) in the given instance of DOMINATING SET\textsubscript{D}.

**Proof:** Given an instance \( \langle G = (V, E), k \rangle \) of DOMINATING SET\textsubscript{D}, construct an instance \( \langle p, d_{\text{new}}, c, t, f_t \rangle \) of LfDIncNoHist\textsuperscript{pos} as follows: Let \( F = \{f_1, f_2, \ldots, f_{|V|}, f_x\} \), \( A = \{a\}, \) \( p \) have \(|V| + 1 \) transitions such that the \( i \)th, \( 1 \leq i \leq |V| \), transition has a triggering feature-set consisting of the features in \( F \) corresponding to the complete neighbourhood of \( v_i \) in \( G \) and action \( a \) and the final transition has a triggering feature-set that is an arbitrary subset of the features of \( F \) that is not the same as the triggering feature-set of any previous transition and action \( a, d_{\text{new}} = (\text{pos}, \langle \{f_x\}, a \rangle), \) \( c = |V| + 1, t = k + 1, \) and \( f_t = 1. \) Observe that this construction can be done in time polynomial in the size of the given instance of DOMINATING SET\textsubscript{D}.

We shall prove the correctness of this reduction in two parts. First, suppose that there is a subset \( V' = \{v'_1, v'_2, \ldots, v'_l\} \subseteq V, l \leq k, \) that is a dominating set in \( G. \) Construct a policy \( p' \) with \( l + 1 \) transitions in which the first transition is \( \langle \{f_x\}, a \rangle \) and the subsequent \( l \) transitions have the form \( \langle \{f_i\}, a \rangle \) for each \( v_i' \in V'. \) Observe that \( p' \) can be derived from \( p \) by at most \( c \) changes to \( p \) (namely, change the feature-sets of the first \( l + 1 \) transitions as necessary and delete the final \(|V| + 1 - (l + 1) \) transitions). As \( V' \) is a dominating set in \( G \) and the triggering feature-set in each transition in \( p \) corresponds to the complete neighbourhood of one of the vertices in \( G, \) \( p' \) will produce the correct action for the triggering feature-set associated with each transition in \( p; \) moreover, the first transition in \( p' \) produces the correct action for \( d_{\text{new}}, \) which means that \( p' \) is consistent with \( p \) modulo \( d_{\text{new}}. \)

Conversely, suppose there is a policy \( p' \) derivable from \( p \) by at most \( c \) changes that is consistent with \( p \) modulo \( d_{\text{new}} \) and has \( l \leq t = k + 1 \) transitions, each of which is triggered by a set of at most \( f_t \) features. One of these transitions must produce action \( a \) relative to \( f_x \) in order for \( p' \) to be consistent with \( d_{\text{new}}; \) moreover, as \( f_x \) does not occur in the triggering feature-set of any transition in \( p, \) the remaining \( l - 1 \) transitions in \( p' \) must produce action \( a \) for all of these triggering feature-sets in order for \( p' \) to be consistent with \( p. \) As \( f_t = 1 \) and the triggering feature-sets in the transitions in \( p \) correspond to the complete neighborhoods of the vertices in \( G, \) the set of features triggering the final \( l - 1 \) transitions in \( p' \) must correspond to a dominating set of size at most \( k \) in \( G. \)

To complete the proof, observe that in the constructed instance of LfDIncNoHist\textsuperscript{pos}, \(|A| = \#d = f_t = 1, \) and \( t = k + 1. \)
**Lemma 14** DOMINATING SET$^{PD3}_{D}$ polynomial-time reduces to LfDIncNoHist$^{pos}_{D}$ such that in the constructed instance LfDIncNoHist$^{pos}_{D}$, $|A| = \#d = f_i = 1$, $f_{cap} = 4$, and $t$ is a function of $k$ in the given instance of DOMINATING SET$^{PD3}_{D}$.

**Proof:** As DOMINATING SET$^{PD3}_{D}$ is a special case of DOMINATING SET$_D$, the reduction in Lemma 13 from DOMINATING SET$_D$ to LfDIncNoHist$^{pos}_{D}$ is also a reduction from DOMINATING SET$^{PD3}_{D}$ to LfDIncNoHist$^{pos}_{D}$ that constructs instances of LfDIncNoHist$^{pos}_{D}$ such that $|A| = \#d = f_i = 1$ and $t$ is a function of $k$ in the given instance of DOMINATING SET$^{PD3}_{D}$. To complete the proof, note that as the degree of each vertex in graph $G$ in the given instance of DOMINATING SET$^{PD3}_{D}$ is at most 3, the size of each complete vertex neighbourhood is of size at most 4, which means that $f_{cap} = 4$ in each constructed instance of LfDIncNoHist$^{pos}_{D}$. \[\square\]

**Lemma 15** DOMINATING SET$_D$ polynomial-time reduces to LfDIncNoHist$^{neg}_{D}$ such that in the constructed instance LfDIncNoHist$^{neg}_{D}$, $|A| = \#d = f_i = 1$ and $t$ is a function of $k$ in the given instance of DOMINATING SET$_D$.

**Proof:** Given an instance $\langle G = (V, E), k \rangle$ of DOMINATING SET$_D$, construct an instance $\langle p, d_{new}, c, t, f_t \rangle$ of LfDIncNoHist$^{neg}_{D}$ as follows: Let $F = \{f_1, f_2, \ldots, f_{|V|}, f_x\}$, $A = \{a\}$, $p$ have $|V|$ transitions such that the $i$th, $1 \leq i \leq |V|$, transition has a triggering feature-set consisting of the features in $F$ corresponding to the complete neighbourhood of $v_i$ in $G$ and action $a$, $d_{new} = (neg, \langle \{f_x\}, a \rangle)$, $c = |V|$, $t = k$, and $f_i = 1$. Observe that this construction can be done in time polynomial in the size of the given instance of DOMINATING SET$_D$.

We shall prove the correctness of this reduction in two parts. First, suppose that there is a subset $V' = \{v'_1, v'_2, \ldots, v'_l\} \subseteq V$, $l \leq k$, that is a dominating set in $G$. Construct a policy $p'$ with $l$ transitions in which the $i$th, $1 \leq i \leq l$, transition has the form $\langle \{f_i\}, a \rangle$ for each $v'_i \in V'$. Observe that $p'$ can be derived from $p$ by at most $c$ changes to $p$ (namely, change feature-sets of the first $l$ transitions as necessary and delete the final $|V| - l$ transitions). As $V'$ is a dominating set in $G$ and the triggering feature-set in each transition in $p$ corresponds to the complete neighbourhood of one of the vertices in $G$, $p'$ will produce the correct action for the triggering feature-sets associated with each transition in $p$; moreover, $p'$ cannot produce action $a$ for $d_{new}$, which means that $p'$ is consistent with $p$ modulo $d_{new}$.

Conversely, suppose there is a policy $p'$ derivable from $p$ by at most $c$ changes that is consistent with $p$ modulo $d_{new}$ and has $l \leq t = k$ transitions, each of which is triggered by a set of at most $f_t$ features. The $l$ transitions in $p'$ must produce action $a$ for all of the triggering feature-sets in $p$ in order for $p'$ to be consistent with $p$. As $f_t = 1$ and the triggering feature-sets in the transitions in $p$ correspond to the complete neighborhoods of the vertices in $G$, the set of features triggering the $l$ transitions in $p'$ must correspond to a dominating set of size at most $k$ in $G$. \[35\]
To complete the proof, observe that in the constructed instance of LfDIncNoHist$^D_{\text{pos}}$, $|A| = \#d = f_t = 1$ and $t = k$.

Lemma 16 DOMINATING SET$^D_{\text{PD3}}$ polynomial-time reduces to LfDIncNoHist$^D_{\text{neg}}$ such that in the constructed instance LfDIncNoHist$^D_{\text{pos}}$, $|A| = \#d = f_t = 1$, $f_{eap} = 4$, and $t$ is a function of $k$ in the given instance of DOMINATING SET$^D_{\text{PD3}}$.

Proof: As DOMINATING SET$^D_{\text{PD3}}$ is a special case of DOMINATING SET$^D$, the reduction in Lemma 15 from DOMINATING SET$^D$ to LfDIncNoHist$^D_{\text{neg}}$ is also a reduction from DOMINATING SET$^D_{\text{PD3}}$ to LfDIncNoHist$^D_{\text{neg}}$ that constructs instances of LfDIncNoHist$^D_{\text{neg}}$ such that $|A| = \#d = f_t = 1$ and $t$ is a function of $k$ in the given instance of DOMINATING SET$^D_{\text{PD3}}$. To complete the proof, note that as the degree of each vertex in graph $G$ in the given instance of DOMINATING SET$^D_{\text{PD3}}$ is at most 3, the size of each complete vertex neighbourhood is of size at most 4, which means that $f_{eap} = 4$ in each constructed instance of LfDIncNoHist$^D_{\text{neg}}$.

Result F: LfDIncHist$^D_{\text{pos}}$ is not fp-tractable relative to the following parameter-sets:

a) $\{|A|, \#d, t, f_t, c\}$ when $|A| = \#d = 2$ and $f_t = 1$ (unless $FPT = W[1]$)
b) $\{|A|, |d|, t, f_t, c\}$ when $|A| = |d| = 2$ and $f_t = 1$ (unless $FPT = W[1]$)
c) $\{|A|, \#d, f_{eap}, f_t\}$ when $|A| = \#d = 2$, $f_{eap} = 4$, and $f_t = 1$ (unless $P = NP$)
d) $\{|A|, |d|, f_{eap}, f_t\}$ when $|A| = |d| = 2$, $f_{eap} = 4$, and $f_t = 1$ (unless $P = NP$)

Proof:
Proof of part (a): Follows from the $W[2]$-hardness of $\langle k \rangle$-DOMINATING SET$^D$, the reduction in Lemma 15, the inclusion of $W[1]$ in $W[2]$, and the conjecture $FPT \neq W[1]$.
Proof of part (b): Follows from part (a) and Lemma 8.
Proof of part (c): Follows from the NP-hardness of DOMINATING SET$^D_{\text{PD3}}$, the reduction in Lemma 10 and Lemma 3.
Proof of part (d): Follows from part (c) and Lemma 8.

Result G: LfDIncHist$^D_{\text{neg}}$ is not fp-tractable relative to the following parameter-sets:

a) $\{|A|, \#d, t, f_t, c\}$ when $|A| = \#d = 2$ and $f_t = 1$ (unless $FPT = W[1]$)
b) $\{|A|, |d|, t, f_t, c\}$ when $|A| = |d| = 2$ and $f_t = 1$ (unless $FPT = W[1]$)
c) $\{|A|, \#d, f_{eap}, f_t\}$ when $|A| = \#d = 2$, $f_{eap} = 4$, and $f_t = 1$ (unless $P = NP$)
d) \{ |A|, |d|, f_{eap}, f_t \} when |A| = |d| = 2, f_{eap} = 4, and f_t = 1 (unless \( P = NP \))

Proof:

Proof of part (a): Follows from the \( W[2]- \)hardness of \( \langle k \rangle \)-DOMINATING SET\(_P\), the reduction in Lemma 11, the inclusion of \( W[1] \) in \( W[2] \), and the conjecture \( FPT \neq W[1] \).

Proof of part (b): Follows from part (a) and Lemma 8

Proof of part (c): Follows from the \( NP \)-hardness of DOMINATING SET\(_P^{PD3} \), the reduction in Lemma 12 and Lemma 3

Proof of part (d): Follows from part (c) and Lemma 8

Result H: LfDIncNoHist\(^{pos} \) is not fp-tractable relative to the following parameter-sets:

- a) \{ |A|, \#d, t, f_t \} when |A| = \#d = f_t = 1 (unless \( FPT = W[1] \))
- b) \{ |A|, |d|, t, f_t \} when |A| = |d| = f_t = 1 (unless \( FPT = W[1] \))
- c) \{ |A|, \#d, f_{eap}, f_t \} when |A| = \#d = f_t = 1 and \( f_{eap} = 4 \) (unless \( P = NP \))
- d) \{ |A|, |d|, f_{eap}, f_t \} when |A| = |d| = f_t = 1 and \( f_{eap} = 4 \) (unless \( P = NP \))

Proof:

Proof of part (a): Follows from the \( W[2]- \)hardness of \( \langle k \rangle \)-DOMINATING SET\(_P\), the reduction in Lemma 13, the inclusion of \( W[1] \) in \( W[2] \), and the conjecture \( FPT \neq W[1] \).

Proof of part (b): Follows from part (a) and Lemma 8

Proof of part (c): Follows from the \( NP \)-hardness of DOMINATING SET\(_P^{PD3} \), the reduction in Lemma 14 and Lemma 3

Proof of part (d): Follows from part (c) and Lemma 8

Result I: LfDIncNoHist\(^{neg} \) is not fp-tractable relative to the following parameter-sets:

- a) \{ |A|, \#d, t, f_t \} when |A| = \#d = f_t = 1 (unless \( FPT = W[1] \))
- b) \{ |A|, |d|, t, f_t \} when |A| = |d| = f_t = 1 (unless \( FPT = W[1] \))
- c) \{ |A|, \#d, f_{eap}, f_t \} when |A| = \#d = f_t = 1 and \( f_{eap} = 4 \) (unless \( P = NP \))
- d) \{ |A|, |d|, f_{eap}, f_t \} when |A| = |d| = f_t = 1 and \( f_{eap} = 4 \) (unless \( P = NP \))
Proof:
Proof of part (a): Follows from the $W[2]$-hardness of $\langle k \rangle$-DOMINATING SET$_D$, the reduction in Lemma 15, the inclusion of $W[1]$ in $W[2]$, and the conjecture $FPT \neq W[1]$.
Proof of part (b): Follows from part (a) and Lemma 8.
Proof of part (c): Follows from the $NP$-hardness of DOMINATING SET$_{D^3}$, the reduction in Lemma 16, and Lemma 3.
Proof of part (d): Follows from part (c) and Lemma 8.