Entanglement Dynamics for Two Spins in an Optical Cavity —Field Interaction Induced Decoherence and Coherence Revival

XUE-MIN BAI,1 CHUN-PING GAO,1 JUN-QI LI,1,2 NI LIU,1,3 J.-Q. LIANG,1,*

1Institute of Theoretical Physics, Shanxi University, Taiyuan, Shanxi 030006, China
2ljqsxu@163.com
3liuni2011520@sxu.edu.cn
*jqliang@sxu.edu.cn

Abstract: We in this paper study quantum correlations for two neutral spin-particles coupled with a single-mode optical cavity through the usual magnetic interaction. Two-spin entangled states for both antiparallel and parallel spin-polarizations are generated under the photon coherent-state assumption. Based on the quantum master equation we derive the time-dependent quantum correlation of Clauser-Horne-Shimony-Holt (CHSH) type explicitly in comparison with the well known entanglement-measure concurrence. In the two-spin singlet state, which is recognized as one eigenstate of the system, the CHSH correlation and concurrence remain in their maximum values invariant with time and independent of the average photon-numbers either. The correlation varies periodically with time in the general entangled-states for the low average photon-numbers. When the photon number increases to a certain value the oscillation becomes random and the correlations are suppressed below the Bell bound indicating the decoherence of the entangled states. In the high photon-number limit the coherence revivals periodically such that the CHSH correlation approaches the upper bound value at particular time points associated with the cavity-field period.

© 2022 Optical Society of America

OCIS codes: (270.0270) Quantum optics; (270.5580) Quantum electrodynamics; (270.5585 ) Quantum information and processing.

References and links
1. M. A. Nielsen, and I. L. Chuang, “Quantum Computation and Quantum Information” (New York: Cambridge University Press, 2000).
2. C. Branciard, N. Brunner, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, A. Ling and V. Scarani, “Testing quantum correlations versus single-particle properties within Leggett’s model and beyond”, Nature Phys. 4, 681 (2008).
3. S. Gröblacher, T. Paterek, R. Kaltenbaek, Č. Brukner, M. Zukowski, M. Aspelmeyer and A. Zeilinger, “An experimental test of non-local realism”, Nature 446, 871 (2007).
4. T. Paterek, A. Fedrizzi, S. Gröblacher, T. Jennewein, M. Zukowski, M. Aspelmeyer, and A. Zeilinger, “Experimental test of nonlocal realistic theories without the rotational symmetry assumption”, Phys. Rev. Lett. 99, 210406 (2007).
5. S. Pirrino, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, “Random numbers certified by Bell’s theorem”, Nature 464, 1021 (2010).
6. R. Rabelo, M. Ho, D. Cavalcanti, N. Brunner, and V. Scarani, “Device-independent certification of entangled measurements”, Phys. Rev. Lett. 107, 050502 (2011).
7. G. Jaeger, “Entanglement, Information, and the Interpretation of Quantum Mechanics”, (Springer, Heidelberg-Berlin-New York, 2010).
8. C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, and V. Scarani, “Experimental falsification of Leggett’s nonlocal variable model”, Phys. Rev. Lett. 99, 210407 (2007).
9. M. D. Eisaman, E. A. Goldschmidt, J. Chen, J. Fan, and A. Migdall, “Experimental test of nonlocal realism using a fiber-based source of polarization-entangled photon pairs”, Phys. Rev. A 77, 032339 (2008).
10. M. Paternostro, and H. Jeong, “Testing nonlocal realism with entangled coherent states”, Phys. Rev. A 81, 032115 (2010).
11. C.-W. Lee, M. Paternostro, and H. Jeong, “Faithful test of nonlocal realism with entangled coherent states”, Phys. Rev. A 83, 022102 (2011).
12. M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, “Information causality as a physical principle”, Nature 461, 1101 (2009).
13. G. Wehri, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, “Violation of Bell’s inequality under strict Einstein locality conditions”, Phys. Rev. Lett. 81, 5039 (1998).
14. A. Aspect, “Violation of Bell’s inequality under strict Einstein locality conditions”, Nature 398, 189 (1999).
15. W. Titel, J. Brendel, B. Gisin, T. Herzog, H. Zbinden, and N. Gisin, “Experimental demonstration of quantum correlations over more than 10 km”, Phys. Rev. A 57, 3229 (1998).
16. M. A. Rowe, D. Kiepinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, “Experimental violation of a Bell’s inequality with efficient detection”, Nature (London) 409, 791 (2001).
17. L. F. Wei, Y. X. Liu, and F. Nori, “Testing Bell’s inequality in a constantly coupled Josephson circuit by effective single-qubit operations”, Phys. Rev. B 72, 104516 (2005).
18. P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, “New high-intensity source of polarization-entangled photon pairs”, Phys. Rev. Lett. 75, 4337 (1995).
19. H. Sakai, T. Saito, T. Ikeda, K. Itoh, T. Kawabata, H. Kuboki, Y. Maeda, N. Matsui, C. Rangacharyulu, M. Sasano, Y. Satou, K. Sekiguchi, K. Suda, A. Tamii, T. Uesaka, and K. Yako, “Spin correlations of strongly interacting massive fermion pairs as a test of Bell’s inequality”, Phys. Rev. Lett. 97, 150405 (2006).
20. A. C. Dada, J. Leach, G. S. Buller, M. J. Padgett, and E. Andersson, “Experimental high-dimensional two-photon entanglement and violations of generalized Bell inequalities”, Nature Physics 7, 677 (2011).
21. J. Barrett, D. Collins, L. Hardy, A. Kent, and S. Popescu, “Quantum nonlocality, Bell inequalities, and the memory loophole”, Phys. Rev. A 66, 042111 (2002).
22. Y. Zhang, S. Glancy, and E. Knill, “Asymptotically optimal data analysis for rejecting local realism”, Phys. Rev. A 84, 062118 (2011).
23. P. Pandy, A. Misra, and I. Chakrabarty, “Complementarity between tripartite quantum correlation and bipartite Bell-inequality violation in three-qubit states”, Phys. Rev. A 94, 052126 (2016).
24. A. A. Semenov and W. Vogel, “Entanglement transfer through the turbulent atmosphere”, Phys. Rev. A 81, 023835 (2010).
25. M. O. Gumberidze, A. A. Semenov, D. Vasylyev, and W. Vogel, “Bell nonlocality in the turbulent atmosphere”, Phys. Rev. A 94, 053801 (2016).
26. L. Mazzola, B. Bellomo, R. L. Franco, and G. Compagno, “Bell nonlocality in the turbulent atmosphere”, Phys. Rev. A 81, 052116 (2010).
27. A. G. Kofman, and A. N. Korotkov, “Bell-inequality violation versus entanglement in the presence of local decoherence”, Phys. Rev. A 77, 052329 (2008).
28. J. Q. Li, and J. Q. Liang, “Disentanglement and Bell nonlocality in a classical dephasing environment”, Phys. Lett. A 374, 1975 (2010).
29. F. Altintas, and R. J. Eryigit, “Dynamics of entanglement and Bell nonlocality for two stochastic qubits with dipole–dipole interaction”, J. Phys. A: Math. Theor. 43, 415306 (2010).
30. L. Derkacz, and L. Jakobczyk, “Clauser-Horne-Shimony-Holt violation and the entropy-concurrence plane”, Phys. Rev. A 72, 042321 (2005).
31. R. F. Werner, “Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model”, Phys. Rev. A 40, 4277 (1989).
32. J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, “Proposed experiment to test local hidden-variable theories”, Phys. Rev. Lett. 23, 880 (1969).
33. P. Bierhorst, “A robust mathematical model for a loophole-free Clauser–Horne experiment”, J. Phys. A 48, 195302 (2015).
34. A. J. Leggett, “Nonlocal hidden-variable theories and quantum mechanics: An incompatibility theorem”, Found. Phys. 33, 1469 (2003).
35. B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau and R. Hanson, “Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres”, Nature 526, 682 (2015).
36. M. T. Quintino, T. Vértesi, and N. Brunner, “Joint measurability, einstein-podolsky-rosen steering, and bell nonlocality”, Phys. Rev. Lett. 113, 160402 (2014).
37. M. Żukowski, A. Dutta, and Z. Yin, “Geometric Bell-like inequalities for steering”, Phys. Rev. A 91, 032107 (2015).
38. A. Roy, S. S. Bhattacharya, A. Mukherjee, and M. Banik, “Optimal quantum violation of Clauser-Horne-Shimony-Holt like steering inequality”, J. Phys. A 48, 145302 (2015).
39. X. L. Zhen, Q. Yang, M. Yang, and Z. L. Cao, “Bell-Nonlocality Dynamics of Three Remote Atoms in Tavis—Cummings and Jaynes—Cummings Models”, Commun. Theor. Phys. 62, 795 (2014).
40. R. X. Chen, C. Hu, and L. Miao, “Dynamics of Bell-nonlocality of two-mode squeezed vacuum fields interacting with atoms”, Opt. Commun. 284, 2955 (2011).
41. S. B. Li, and J. B. Xu, “Entanglement, Bell violation, and phase decoherence of two atoms inside an optical cavity”, Phys. Rev. A 72, 022332 (2005).
42. J.-B. Xu, and S.-B. Li, “Control of the entanglement of two atoms in an optical cavity via white noise”, New J. Phys. 7, 72 (2005).
43. T. M. Stace, G. J. Milburn, and C. H. W. Barnes, “Entangled two-photon source using biexciton emission of an asymmetric quantum dot in a cavity”, Phys. Rev. B 67, 085317 (2003).
The non-locality is one of the most peculiar characteristics of quantum mechanics without classical correspondence. As a typical example of non-locality the two-particle entangled-state has become the essential ingredients in quantum information and computation [1–3], although...
it was originally considered by Einstein, Podolsky, and Rosen to question the completeness of quantum mechanics. In general, an entangled state for a composite system is not able to be factorized into a direct product of the individual states of subsystems. Bell’s inequality (BI) derived in terms of classical statistics is a criteria of the local correlation for the classical bipartite-model. It actually provides a quantitative test of the entangled states, which are fundamentally different from the classical world [4–11], while the underlying physics is obscure [12]. The BI and its violation in agreement with quantum mechanical predictions [13–20] have attracted considerable attentions both theoretically and experimentally in recent years [21–31]. Indeed, the violation of BI has been verified undoubtedly in various systems, for example, with entangled photons [13–15], trapped ions [16] and switchable Josephson qubits [17]. Soon after the Bell’s pioneer work the original BI was modified to various new forms [32–38], among which a more suitable inequality for the quantitative test was formulated by Clauser-Horne-Shimony-Holt (CHSH). The entanglement and BI have been also investigated in optical cavity with identical atoms [39–46]. Recently a loop-free experiment on the violation of BI was reported [47, 48] using the spins of a nitrogen-vacancy defect in diamond [49–54]. Nonetheless, most of these current studies are focused on the measurement of static entangled states. It is of fundamental importance to explore the dynamic behavior of quantum correlation for the bipartite entangled states and the related violation of BI in an interacting system. The environment induced decoherence is considered as one of the most serious obstacle to realize the quantum operation. The dynamic evolution of quantum correlations for a two-qubit system has been studied extensively in various dissipative-environments [55–60]. Motivated by the experimental test of BI in spin systems [47, 48, 61, 62] we in the present paper study a dynamic model for two spins interacting with a cavity-field to see the field-interaction effect on the entanglement. We demonstrate that general entangled states indeed can be generated and then derive the quantum correlation in terms of quantum probability statistics under the assumption of measuring outcome-independence [63]. The quantum CHSH-correlation denoted by \( P_{CHSH} \) results generally in the violation of BI, \( P_{CHSH} < 2 \), for the entangled states, and the maximum correlation-value is found as \( P_{\text{max}}^{CHSH} = 2\sqrt{2} \). The dynamic evolution of the maximum quantum CHSH-correlation \( P_{\text{max}}^{CHSH} \) is evaluated explicitly with the help of quantum master equation and the result is compared with the entanglement measure, concurrence. We reveal the field-interaction induced decoherence and the coherence revival, which is characterized by the violation of BI.

The paper is organized as follows: we present the model Hamiltonian and stationary solutions in Sec.II. The general entangled states are generated in the spin cavity system. In Sec. III, we derive the quantum CHSH-correlation in terms of quantum statistics for the two-spin entangled states. The quantum dynamics of density operator is formulated in Sec.IV. The time-evolutions of \( P_{\text{max}}^{CHSH} \) and concurrence are presented in Sec.V. Finally, we in Sec.VI summarize and discuss the results.

2. Stationary solutions and entangled states

Following the recent experiment for the violation of BI with nitrogen-vacancy defects in diamond [47–50] we assume that the two spins are subject to a one-mode optical cavity of frequency \( \omega \). The usual Zeeman energy for two spins in the magnetic component of quantized light field gives rise to the Hamiltonian

\[
H = \omega a^\dagger a + ig \sum_{i=1}^{2} (a \sigma_i^+ - a^\dagger \sigma_i^-),
\]

with the convention \( \hbar = 1 \) throughout the paper. The Pauli matrices \( \sigma_i^x, \sigma_i^y, \sigma_i^z = \sigma_i^+ \pm i \sigma_i^- \) are defined to describe the spin operators. \( a (a^\dagger) \) is the annihilation (creation) operator of the single-mode cavity field.
We are looking for the stationary solutions of Hamiltonian Eq. (1) in the optical coherent state $|\alpha\rangle$ under the semiclassical approximation. Taking the average in the coherent state $|\alpha\rangle$, in which the complex eigenvalue of the photon annihilation operator, that $a|\alpha\rangle = \alpha |\alpha\rangle$, is parameterized as $\alpha = \gamma e^{i\phi}$, we have an effective Hamiltonian with spin operator only

$$H_{sp}(\alpha) = \langle \alpha | H | \alpha \rangle = \omega \gamma^2 + \sum_{i=1}^{2} i \gamma g [e^{i\phi} \sigma_i^+ - e^{-i\phi} \sigma_i^-].$$

(2)

$\gamma^2 = \langle \alpha | a^\dagger a | \alpha \rangle$ denotes the average photon-number, which plays a important role in the entanglement dynamics. The spin Hamiltonian $H_{sp}(\alpha)$ can be diagonalized in the two-qubit bases $|e_1\rangle = |+, +\rangle$, $|e_2\rangle = |+, -\rangle$, $|e_3\rangle = |-, +\rangle$, $|e_4\rangle = |-, -\rangle$ with the energy eigenvalues given by

$$\begin{align*}
\epsilon_0(\gamma) &= \omega \gamma^2 - 2g\gamma, \\
\epsilon_1(\gamma) &= \omega \gamma^2 + 2g\gamma, \\
\epsilon_2(\gamma) &= \omega \gamma^2,
\end{align*}$$

(3)

which are functions of parameter $\gamma$. The corresponding eigenstates are seen to be

$$\begin{align*}
|\psi_0\rangle &= \frac{1}{2} [e^{-i\phi} |-, -\rangle - e^{i\phi} |+, +\rangle - i(|+, -\rangle + |-, +\rangle)], \\
|\psi_3\rangle &= \frac{1}{2} [e^{-i\phi} |-, -\rangle - e^{i\phi} |+, +\rangle + i(|+, -\rangle + |-, +\rangle)], \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}} (e^{i\phi} |+, +\rangle + e^{-i\phi} |-, -\rangle), \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}} (|+, -\rangle - |-, +\rangle).
\end{align*}$$

(4)

The two-spin entangled states indeed can be generated in the optical cavity under the semi-classical field approximation. $|\psi_1\rangle$ is the well studied two-spin singlet state in relation with the nonlocality and BI. While $|\psi_2\rangle$, which is a degenerate state of $|\psi_1\rangle$, is the entangled state with parallel spin polarization. The stability of the states $|\psi_0\rangle$, $|\psi_1\rangle$, and $|\psi_2\rangle$ can be examined with variational method regarding $\gamma$ as a variational parameter of energy function. For example, the solution $\gamma = 0$ of the energy extremum equation $\partial \epsilon_1(\gamma) / \partial \gamma = 0$ is obviously the energy minimum of the energy function $\epsilon_1(\gamma)$. Thus $|\psi_1\rangle$, and $|\psi_2\rangle$ are stable states of normal phase with zero average photon-number. While $|\psi_0\rangle$ is the ground state of superradiant phase [64–66]. We solve in the present paper the dynamic equation of motion for the given initial entangled states and then study the time-evolution of measuring outcome-correlations. Particular attention is paid on the field interaction induced decoherence and coherence revival. For the sake of simplicity we consider the dynamics associated with the initial states $|\psi_1\rangle$, and $|\psi_2\rangle$.

3. Measuring outcome correlation and Bell’s inequality

BI derived from classical statistics with the assumption of locality, was one of the first criteria to detect the quantum entanglement. Recently the BIs and their violations were revisited in terms of quantum probability statistics with the help of state density-operator [63], which has advantage to formulate the various forms of inequality and the violation in a unified manner. We consider a general two-spin entangled state with antiparallel spin-polarization

$$|\psi\rangle = \sin \xi e^{i\eta} |+, -\rangle + \cos \xi e^{-i\eta} |-, +\rangle,$$

(5)

where $\xi$, $\eta$ are two arbitrary angle-parameters. The density operator can be separated into the local and non-local parts

$$\rho_{\psi} = \rho_{\psi}^{lc} + \rho_{\psi}^{nlc}.$$ 

(6)
The local part
\[ \rho_{lc}^{\psi} = \sin^2 \xi \langle +, + \rangle \langle +, + \rangle + \cos^2 \xi \langle -, + \rangle \langle -, + \rangle, \] (7)
which is the classical probability state, gives rise to BIs. While the non-local part
\[ \rho_{nlc}^{\psi} = \sin \xi \cos \xi e^{-i2\eta} \langle +, - \rangle \langle -, + \rangle + \sin \xi \cos \xi e^{i2\eta} \langle - , - \rangle \langle +, - \rangle, \] (8)
which comes from the quantum interference between two components of the entangled state leads to the violation of BIs. Following Bell the measurements of two spins are performed independently along arbitrary directions, say \( \mathbf{a} \) and \( \mathbf{b} \), respectively. The measuring outcomes fall into the eigenvalues of projection spin-operators
\[ \sigma \cdot \mathbf{a} | \pm \rangle \] and
\[ \sigma \cdot \mathbf{b} | \pm \rangle = \pm | \pm \rangle, \] according to the quantum measurement theory. By solving the above eigenequations we obtain the explicit form of eigenstates
\[ | + \rangle_n = \cos \left( \frac{\theta_n}{2} \right) | + \rangle + \sin \left( \frac{\theta_n}{2} \right) e^{i\phi_n} | - \rangle \] and
\[ | - \rangle_n = \sin \left( \frac{\theta_n}{2} \right) | + \rangle - \cos \left( \frac{\theta_n}{2} \right) e^{i\phi_n} | - \rangle, \]
which are known as the spin coherent states [64–66] of north- and south-pole gauges. Here, \( \mathbf{n} = (\sin \theta_n \cos \Phi_n, \sin \theta_n \sin \Phi_n, \cos \theta_n) \) with \( \mathbf{n} = \mathbf{a}, \mathbf{b} \) is a unit vector parameterized by the polar and azimuthal angles \( (\theta_n, \Phi_n) \). The outcome-independent base vectors of two-spin measurements are the product eigenstates of operators \( \sigma \cdot \mathbf{a} \) and \( \sigma \cdot \mathbf{b} \) labeled arbitrarily as
\[ |1\rangle = |\mathbf{a} + \mathbf{b}\rangle, |2\rangle = |\mathbf{a} - \mathbf{b}\rangle, |3\rangle = |\mathbf{a} + \mathbf{b}\rangle, |4\rangle = |\mathbf{a} - \mathbf{b}\rangle. \] (9)
The measuring outcome correlation-probability for two spins respectively along directions \( \mathbf{a} \) and \( \mathbf{b} \) is thus obtained in terms of the quantum statistical-average under the outcome-independent base vectors Eq. (9)
\[ P(a, b) = \text{Tr}[\sigma \cdot \mathbf{a}(\sigma \cdot \mathbf{b})\rho_{\psi}]. \] (10)
Using the separated density operators Eq. (6) the measuring outcome correlation-probability Eq. (10) can be also split into the local and non-local parts
\[ P(a, b) = P_{lc}(a, b) + P_{nlc}(a, b). \]
The local part Eq. (7) gives rise to the Bell correlation [63] of two-direction \( (a, b) \) measurements
\[ P_{lc}(a, b) = - \cos \theta_a \cos \theta_b. \] (11)
Substituting the the Bell correlation Eq. (11) into the CHSH-correlation-probability of four-direction measurements defined as
\[ P_{CHSH} = |P(a, b) + P(a, c) + P(d, b) - P(d, c)|, \] (12)
we recover the well known CHSH-inequality
\[ P_{CHSH}^{lc} = |\cos \theta_a (\cos \theta_b + \cos \theta_c) + \cos \theta_d (\cos \theta_b - \cos \theta_c)| \leq 2, \] (13)
which was derived originally by CHSH from classical statistics with the assumption of locality. The CHSH-inequality Eq. (13) is valid for arbitrary entangled states given by Eq. (5) considering the local part Eq. (7) only. Including the non-local part Eq. (8) the quantum correlation for the two-direction measurements is simply a scalar product of the two unit-vectors \( \mathbf{a} \) and \( \mathbf{b} \)

\[
P(a, b) = -\mathbf{a} \cdot \mathbf{b},
\]

which is valid, however, under the condition \( \xi = 3\pi/4, \eta = n\pi \) with \( n \) being a integer. Thus the arbitrary entangled state Eq. (5) reduces to the two-spin singlet state \( |\psi_1\rangle \) in Eq. (4). With the quantum correlation Eq. (14) the quantum CHSH-probability Eq. (12) becomes [63, 67]

\[
P_{CHSH} = |\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) + \mathbf{d} \cdot (\mathbf{b} - \mathbf{c})|.
\]

When the unit vector \( \mathbf{b} \) is perpendicular to \( \mathbf{c} \), and \( \mathbf{a}, \mathbf{d} \) are respectively parallel to \( (\mathbf{b} + \mathbf{c}), (\mathbf{b} - \mathbf{c}) \), we obtain the maximum quantum CHSH-probability as

\[
P_{CHSH}^{\text{max}} = 2\sqrt{2},
\]

which indicates the maximum violation of CHSH-inequality Eq. (13). Our formalism shows a direct relation between the decoherence and CHSH-inequality, which can serve a criteria of coherence for the two parts of entangled state. Most recently it was proved that for the entangled state of parallel spin polarization

\[
|\psi\rangle = \sin \xi e^{i\eta} |+, +\rangle + \cos \xi e^{-i\eta} |-, -\rangle,
\]

the Bell correlation of local part is simply [67]

\[
P_L(a, b) = \cos \theta_a \cos \theta_b,
\]

which is different from that of antiparallel spin-polarization Eq. (11) only by a sign change. The CHSH-inequality for the local part Eq. (13) remains not changed. The quantum CHSH-correlation-probability Eq. (16) is valid for the entangled state under the condition \( \xi = \pi/4, \eta = n\pi \), namely, \( |\psi\rangle = 1/2(|+, +\rangle |-, -\rangle) \).

We are going to study the dynamics of CHSH-probability to reveal the field interaction induced effect on the entanglement. The dynamic evolution of \( P_{CHSH} \) can be obtained from the time dependent density-operator \( \rho(t) \) in terms of the quantum probability statistics presented in this paper. For speciality we consider only the maximum quantum CHSH-probability Eq. (16), which has been shown to evaluate in a clever way [68, 69] directly from the state-density operator itself.

4. Quantum dynamics with photon coherent state

The dynamic evolution of two-qubit entangled states in the optical cavity can be evaluated from the Schrödinger equation \( i\partial / \partial t |\psi\rangle = H |\psi\rangle \). To this end we begin with the interaction picture \( i\partial / \partial t |\psi_i\rangle = H_i |\psi_i\rangle \), where \( |\psi_i\rangle = R(t)|\psi\rangle \) and the unitary operator is

\[
R(t) = e^{i\omega a^\dagger a t}.
\]

The interaction-picture Hamiltonian is seen to be

\[
H_i = R(t)HR^\dagger(t) - iR(t)\frac{\partial}{\partial t}R^\dagger(t)
\]

\[
= ig \sum_{i=1}^{2} (ae^{-i\omega t} \sigma_i^+ - a^\dagger e^{i\omega t} \sigma_i^-).
\]

\[
4. Quantum dynamics with photon coherent state

The dynamic evolution of two-qubit entangled states in the optical cavity can be evaluated from the Schrödinger equation \( i\partial / \partial t |\psi\rangle = H |\psi\rangle \). To this end we begin with the interaction picture \( i\partial / \partial t |\psi_i\rangle = H_i |\psi_i\rangle \), where \( |\psi_i\rangle = R(t)|\psi\rangle \) and the unitary operator is

\[
R(t) = e^{i\omega a^\dagger a t}.
\]

The interaction-picture Hamiltonian is seen to be

\[
H_i = R(t)HR^\dagger(t) - iR(t)\frac{\partial}{\partial t}R^\dagger(t)
\]

\[
= ig \sum_{i=1}^{2} (ae^{-i\omega t} \sigma_i^+ - a^\dagger e^{i\omega t} \sigma_i^-).
\]

\[
4. Quantum dynamics with photon coherent state

The dynamic evolution of two-qubit entangled states in the optical cavity can be evaluated from the Schrödinger equation \( i\partial / \partial t |\psi\rangle = H |\psi\rangle \). To this end we begin with the interaction picture \( i\partial / \partial t |\psi_i\rangle = H_i |\psi_i\rangle \), where \( |\psi_i\rangle = R(t)|\psi\rangle \) and the unitary operator is

\[
R(t) = e^{i\omega a^\dagger a t}.
\]

The interaction-picture Hamiltonian is seen to be

\[
H_i = R(t)HR^\dagger(t) - iR(t)\frac{\partial}{\partial t}R^\dagger(t)
\]

\[
= ig \sum_{i=1}^{2} (ae^{-i\omega t} \sigma_i^+ - a^\dagger e^{i\omega t} \sigma_i^-).
\]
The entail time-evolution operator of the system reads as

$$U(t) = R^\dagger(t)U_i(t),$$  \hspace{1cm} (19)

with $U_i(t) = \exp(-iH_i t)$ being the time evolution operator for the interaction-picture Hamiltonian Eq. (1). In the two-qubit bases $|e_i\rangle \ (i = 1, 2, 3, 4)$, time-evolution operator possesses a matrix form [70],

$$U_i(t) = \begin{pmatrix}
1 - 2a^2 S & a \sin(\sqrt{S}a) a^\dagger & a \sin(\sqrt{S}a) \sqrt{\frac{S}{2}} & 2a \sin(\sqrt{S}a) \\
-a \sin(\sqrt{S}a) a^\dagger & \cos^2(\sqrt{S}a) - \sin^2(\sqrt{S}a) & 0 & 0 \\
-a \sin(\sqrt{S}a) \sqrt{\frac{S}{2}} & 0 & \cos(\sqrt{S}a) & 0 \\
2a \sin(\sqrt{S}a) \sqrt{\frac{S}{2}} & 0 & 0 & 1 - 2a \sin(\sqrt{S}a) a^\dagger
\end{pmatrix},$$  \hspace{1cm} (20)

where $S = 2a^2 a + 1$.

From the quantum master equation

$$i \frac{d\rho(t)}{dt} = [H, \rho(t)],$$  \hspace{1cm} (21)

the density operator at time $t$ is found as $\rho(t) = U(t)\rho(0)U^\dagger(t)$, where the initial density operator $\rho(0) = \rho_\phi(0)\rho_\gamma(0)$, is the product of the spin part $\rho_\phi(0) = |\psi(0)\rangle \langle \psi(0)|$ and the cavity-field part $\rho_\gamma(0) = |\alpha\rangle \langle \alpha|$ with $|\alpha\rangle$ being the photon coherent state [64]. We are interested in the field-interaction effect on the quantum measuring-outcome correlation and thus assume the initial state being normalized two-spin entangled-state Eq. (5). The matrix representation of the density operator Eq. (5) in the two-qubit bases is

$$\rho_\phi(0) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \sin^2 \xi & \sin \xi \cos \xi e^{i2\eta} & 0 \\
0 & \sin \xi \cos \xi e^{-i2\eta} & \cos^2 \xi & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.  \hspace{1cm} (22)$$

Using the explicit form of photon coherent state in the Fock space, $|\alpha\rangle = \sum_n \alpha^n / \sqrt{n!} \exp\{-|\alpha|^2/2\}|n\rangle$, we take the trace of density operator over the photon number state $|n\rangle$ to obtain the reduced density operator that

$$\rho_\gamma(t) = \sum_{n=0}^\infty \rho_n(t), \quad \rho_n(t) = \langle n|\rho(t)|n\rangle,$$

with

$$\rho_n(t) = \sum_{j,l=0}^\infty \langle n|U(t)|j\rangle \rho_\phi(0) \langle l|U^\dagger(t)|n\rangle \frac{\gamma^{j+l} e^{i(j-l)\phi}}{\sqrt{j!l!}} e^{-\gamma^2}.$$  \hspace{1cm} (23)

The reduced density matrix is written in the two-qubit bases explicitly as

$$\rho_\gamma(t) = \sum_{n=0}^\infty \begin{pmatrix}
(\rho_n)_{11} & (\rho_n)_{12} & (\rho_n)_{13} & (\rho_n)_{14} \\
(\rho_n)_{12} & (\rho_n)_{22} & (\rho_n)_{23} & (\rho_n)_{24} \\
(\rho_n)_{13} & (\rho_n)_{23} & (\rho_n)_{33} & (\rho_n)_{34} \\
(\rho_n)_{14} & (\rho_n)_{24} & (\rho_n)_{34} & (\rho_n)_{44}
\end{pmatrix}.  \hspace{1cm} (24)$$
The diagonal elements of the density matrix $\rho_n(t)$ are seen to be

\[
\begin{align*}
(\rho_n)_{11} &= \langle e_1 | \rho_n | e_1 \rangle = \frac{e^{-\gamma^2}}{n!} C_1^2 (\sin 2\xi \cos 2\eta + 1), \\
(\rho_n)_{22} &= \frac{e^{-\gamma^2}}{n!} (C_2^2 \sin^2 \xi + C_3^2 \cos^2 \xi + C_2 C_3 \sin 2\xi \cos 2\eta), \\
(\rho_n)_{33} &= \frac{e^{-\gamma^2}}{n!} (C_2^2 \cos^2 \xi + C_3^2 \sin^2 \xi + C_2 C_3 \sin 2\xi \cos 2\eta), \\
(\rho_n)_{44} &= \frac{e^{-\gamma^2}}{n!} C_4^2 (\sin 2\xi \cos 2\eta + 1),
\end{align*}
\]

and the off-diagonal elements are

\[
\begin{align*}
(\rho_n)_{12} &= \frac{e^{-\gamma^2}}{2n!} C_1 e^{-2i\eta} (C_2 \sin \xi + C_3 e^{2i\eta} \cos \xi) (e^{2i\eta} \sin \xi + \cos \xi), \\
(\rho_n)_{13} &= \frac{e^{-\gamma^2}}{n!} C_1 e^{-2i\eta} (C_2 e^{2i\eta} \cos \xi + C_3 \sin \xi) (e^{2i\eta} \sin \xi + \cos \xi), \\
(\rho_n)_{14} &= \frac{e^{-\gamma^2}}{n!} C_1 C_4 (\sin 2\xi \cos 2\eta + 1), \\
(\rho_n)_{23} &= \frac{e^{-\gamma^2}}{2n!} \left( C_2^2 e^{2i\eta} \sin 2\xi + 2C_2 C_3 + C_2^2 \sin 2\xi \right), \\
(\rho_n)_{24} &= \frac{e^{-\gamma^2}}{n!} C_4 e^{-2i\eta} (C_2 e^{2i\eta} \sin \xi + C_3 \cos \xi) (e^{2i\eta} \cos \xi + \sin \xi), \\
(\rho_n)_{34} &= \frac{e^{-\gamma^2}}{n!} C_4 e^{-2i\eta} (C_2 \cos \xi + C_3 e^{2i\eta} \sin \xi) (e^{2i\eta} \cos \xi + \sin \xi).
\end{align*}
\]

We have $(\rho_n)_{ij} = (\rho_n)_{ji}^*$, since the state-density operator $\rho_n(t)$ is Hermitian. The time functions are given by

\[
\begin{align*}
C_1 &= \gamma^{n+1} \sin(gt \sqrt{4n + 6}) / \sqrt{4n + 6}, \\
C_2 &= \gamma^n \cos^2(gt \sqrt{2n + 1} / 2), \\
C_3 &= -\gamma^n \sin^2(gt \sqrt{2n + 1} / 2), \\
C_4 &= -\gamma^{n-1} n \sin(gt \sqrt{4n - 2}) / \sqrt{4n - 2}.
\end{align*}
\]

For the entangled initial-state of parallel spin polarization Eq. (17) the density operator becomes

\[
\rho_{\psi}(0) = \begin{pmatrix}
\sin^2 \xi & 0 & 0 & \sin \xi \cos \xi e^{2i\eta} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\sin \xi \cos \xi e^{-2i\eta} & 0 & 0 & \cos^2 \xi
\end{pmatrix}.
\]
We now study the entanglement dynamics in terms of the time-evolution of density matrix with the time functions being

\[ (\rho_n)_{11} = \frac{e^{-\gamma^2}}{n!} (D_2^2 \sin^2 \xi + D_2^2 \cos^2 \xi + D_1 D_2 \sin 2\xi \cos 2\eta), \]

\[ (\rho_n)_{22} = \frac{e^{-\gamma^2}}{n!} (D_3^2 \sin^2 \xi + D_3^2 \cos^2 \xi + D_3 D_4 \sin 2\xi \cos 2\eta), \]

\[ (\rho_n)_{44} = \frac{e^{-\gamma^2}}{n!} (D_5^2 \sin^2 \xi + D_5^2 \cos^2 \xi + D_5 D_6 \sin 2\xi \cos 2\eta), \]

\[ (\rho_n)_{33} = (\rho_n)_{22}, \]

and the off-diagonal elements read

\[ (\rho_n)_{12} = \frac{e^{-\gamma^2}}{n!} (D_4 \cos \xi + e^{-2i\eta} D_6 \sin \xi)(D_2 \cos \xi + e^{2i\eta} D_1 \sin \xi), \]

\[ (\rho_n)_{14} = \frac{e^{-\gamma^2}}{n!} (D_6 \cos \xi + e^{-2i\eta} D_5 \sin \xi)(D_2 \cos \xi + e^{2i\eta} D_1 \sin \xi), \]

\[ (\rho_n)_{24} = \frac{e^{-\gamma^2}}{n!} (D_6 \cos \xi + e^{-2i\eta} D_5 \sin \xi)(D_4 \cos \xi + e^{2i\eta} D_3 \sin \xi), \]

\[ (\rho_n)_{13} = (\rho_n)_{12}, \]

\[ (\rho_n)_{23} = (\rho_n)_{22}, \]

\[ (\rho_n)_{34} = (\rho_n)_{24}, \]

with the time functions being

\[ D_1 = \gamma^n (1 - \frac{2(n+1) \sin^2 (gt \sqrt{\frac{2n+3}{2}})}{2n + 3}), \]

\[ D_2 = \gamma^{n+2} \frac{2 \sin^2 (gt \sqrt{\frac{2n+3}{2}})}{2n + 3}, \]

\[ D_3 = -\gamma^{n-1} \frac{n \sin (gt \sqrt{4n + 2})}{\sqrt{4n + 2}}, \]

\[ D_4 = \gamma^{n+1} \frac{\sin (gt \sqrt{4n + 2})}{\sqrt{4n + 2}}, \]

\[ D_5 = \gamma^{n-2} \frac{2(n-1) \sin^2 (gt \sqrt{\frac{2n-1}{2}})}{2n - 1}, \]

\[ D_6 = \gamma^n (1 - \frac{2n \sin^2 (gt \sqrt{\frac{2n-1}{2}})}{2n - 1}). \]

In the following we are going to investigate measuring outcome-correlations of two spins in terms of the density operator \( \rho_r(t) \) to see the field-interaction induced effect on the entanglement. The dynamic evolution of quantum CHSH-correlation-particularly \( \rho_{CHSH}^{max} \) is evaluated in comparison with the entanglement measure, concurrence.

5. **Time evolution of quantum correlation**

We now study the entanglement dynamics in terms of the time-evolution of density matrix \( \rho_r(t) \), from which the time-dependent maximum quantum CHSH-correlation-probability is derived.
To this end we define [68, 69] a matrix relating to $\rho_r(t)$ as

$$T_{\rho_r(t)} = \rho_r(t)\sigma \otimes \sigma.$$  

A symmetric matrix is given by

$$U_{\rho_r(t)} = T^T_{\rho_r(t)}T_{\rho_r(t)},$$  

where $T^T$ is the transposition of $T$. The maximum quantum CHSH-correlation-probability can be found [68, 69] from the eigenvalues of the symmetric matrix $U_{\rho_r(t)}$

$$P_{CHSH}^{max}(t) = 2\sqrt{m(\rho_r)},$$  

where $m(\rho_r(t)) = \max_{j<k}(u_j + u_k)$ with $u_j$ ($j = 1, 2, 3$) being the eigenvalues of $U_{\rho_r(t)}$ defined in Eq. (26). In the absence of the cavity field we have $P_{CHSH}^{max} = 2\sqrt{m(\rho_0)} = 2\sqrt{2}$, which coincides with Eq. (16). The $P_{CHSH}^{max}(t)$-value can be used to indicate the entanglement that when $P_{CHSH}^{max}(t) < 2$, the entanglement collapses due to the field-interaction induced quantum decoherence.

It is interesting to compare the $P_{CHSH}^{max}(t)$ with the concurrence, which is a well known measure of the entanglement [71] for a two-qubit system. The instantaneous concurrence is defined by [71]

$$C(t) = \max\{0, \Lambda(t)\},$$  

where $\Lambda(t) = \lambda_2(t) - \lambda_3(t) - \lambda_4(t)$. $\lambda_i(t)$ denotes the square root of the instantaneous eigenvalue of the matrix $\rho_r(t)(\sigma_x \otimes \sigma_y)\rho^*_r(t)(\sigma_x \otimes \sigma_y)$ for $i = 1, 2, 3, 4$ in the decreasing order of eigenvalue magnitudes. $\rho^*_r(t)$ is the complex conjugate of the two-qubit density matrix $\rho_r(t)$.

The maximum CHSH-correlation-probability $P_{CHSH}^{max}(t)$ is compared with concurrence $C(t)$, so that to establish a dynamic relation between the violation of CHSH inequality and the entanglement measure. In the photon coherent-state the average photon-number $\gamma^2$, which, we will see, affects greatly the entanglement dynamics.

### 5.1. Entangled states of antiparallel spin polarizations

The time-variation curves of maximum CHSH-correlation-probability $P_{CHSH}^{max}(t)$ (a) and the concurrence $C(t)$ (b) are displayed in Figs. 1-2, for initial entangled states of antiparallel spin-polarizations Eq. (5) with various average photon-numbers $\gamma^2 = 0.01$ (1), 1 (2), 15 (3), 150 (4). The time $t$ is measured in the optic-field period $T = 2\pi/\omega$ and the atom-filed coupling $g = 1$ is in the unit of field frequency $\omega$. The time-variation curves of $P_{CHSH}^{max}(t)$ and $C(t)$ have one to one correspondence with the upper bound value $P_{CHSH}^{max} = 2\sqrt{2}$, which corresponds to the concurrence $C = 1$ indicating the maximum entanglement. The dynamics is sensitive to the angle parameters $\xi$ and $\eta$ of the entangled states. For $\eta = 0$ the two coefficients of the entangled states are real and the time-evolution curves are plotted for the parameters $\xi = \pi/6$ (green solid line), $\pi/4$ (blue dot and dash line), and $3\pi/4$ (red dash line) in Fig. 1. It is remarkable to see a fact that both the maximum CHSH-correlation-probability and concurrence remain in the maximum values $P_{CHSH}^{max}(t) = 2\sqrt{2}$, $C(t) = 1$ not varying with time when $\xi = 3\pi/4$. In this case the initial state becomes an eigenstate of the system $|\psi_1\rangle$ i.e. the two-spin singlet state as shown in Eq. (4), which is independent of the cavity-field parameters $\gamma$ and $\phi$. The measuring outcome of two-spin correlation operator is invariant in the Hamiltonian eigenstate $|\psi_1\rangle$, since the density operator commutes with the Hamiltonian and thus is a conserved quantity. If the initial state is not the eigenstate with $\xi = \pi/6, \pi/4$, $P_{CHSH}^{max}(t)$ oscillates with time below the quantum upper-bound $2\sqrt{2}$. The oscillation behavior depends on the average photon numbers. In the lower value of average photon-number $\gamma^2 = 0.01$, $P_{CHSH}^{max}(t)$ is a periodic oscillation around the classical upper-bound value of $P_{CHSH}^{lc} = 2$ seen in Eq. (13). The oscillation becomes random for $\gamma^2$ tending to
Fig. 1. Color online) The maximum quantum CHSH-probability $P^{\text{max}}_{\text{CHSH}}(t)$ (a) and concurrence $C(t)$ (b) as functions of time $t$ measured in the period $T$ of optic field for the entangled-state of antiparallel spin-polarization with angle parameters $\eta = 0$ and $\xi = \pi/6$ (green solid line), $\pi/4$ (blue dot and dash line) $\pi/3$ (red dash line) with average photon-number $\gamma^2 = 0.01$ (1), 1 (2), 15 (3), 150 (4). $P^{\text{max}}_{\text{CHSH}}(t) = 2\sqrt{2}$, $C(t) = 1$ not varying with time for the spin singlet $|\Psi_1\rangle$.

Fig. 2. Color online) Time-variation curves of $P^{\text{max}}_{\text{CHSH}}(t)$ (a) and $C(t)$ (b) for the entangled-state of antiparallel spin-polarization with angle parameters $\eta = \pi/4$ and $\xi = \pi/6$ (green solid line), $\pi/4$ (blue dot and dash line) $\pi/3$ (pink dash line) with average photon-number $\gamma^2 = 0.01$ (1), 1 (2), 15 (3), 150 (4). $P^{\text{max}}_{\text{CHSH}}(t) = 2\sqrt{2}$, $C(t) = 1$ not varying with time for the spin singlet $|\Psi_1\rangle$. 
1. When the average photon-number increases to $\gamma^2 = 15$, the entanglement collapses seen from Fig. 1(b3), since we have $P_{\text{CHSH}}^\text{max}(t) < 2$, namely the CHSH inequality Eq. (13) is satisfied. In other words the quantum interference part $\rho_{\psi}^\text{nic}$ disappears due to the field-interaction induced decoherence. In the large photon-number limit the coherence revives periodically characterized by the sharp-peak values, which approaches the quantum upper-bound $2\sqrt{\eta}$ (1) seen from Fig. 1(a4,b4). The discrete peak-positions of $P_{\text{CHSH}}^\text{max}(t)$ curves are located precisely at $t = n5T$ with $n$ being a integer. When the superposition coefficients of entangled states become complex with non-vanishing $\eta$, the periodic coherence-revival basically remains as shown in Fig. 2 for $\xi = \pi/4$ and various phase angles $\eta$ for small or large photon-number limit. It is interesting to see a fact that the CHSH inequality is nearly always violated independent of average photon numbers for the state with angle parameter $\eta = \pi/3$ (pink dash line), although the $P_{\text{CHSH}}^\text{max}(t)$ value is suppressed from the quantum upper-bound. The field interaction does not lead to the decoherence for this state. While the entanglement collapses for the states with $\eta = \pi/4$, $\pi/6$ at the average photon number $\gamma^2 = 15$ (b3). The sharp peak-positions of $P_{\text{CHSH}}^\text{max}(t)$ curves become $t = (2n)5T$ in the large photon-number limit with $\gamma^2 = 150$ (a4). While The peak-height almost shrinks to the classical bound 2 at the time points $t = (2n + 1)5T$. The concurrence plots are displayed in (b1) to (b4) showing the same behaviors as $P_{\text{CHSH}}^\text{max}(t)$.

5.2. Entangled states for parallel spin polarizations

The BI was originally formulated and well studied based on the two-spin singlet state $|\psi_1\rangle$. It was proved long ago [72] and verified recently by means of quantum statistics [63] that the BI and its violation are actually valid for arbitrary two-spin entangled states with antiparallel polarization [63,72] given by Eq. (5). Most recently it was demonstrated that the original formula of BI has to be slightly modified for the two-spin entangled state of parallel spin polarization, while the CHSH inequality remains not changed [67]. Now we turn our attention to the dynamics of the maximum quantum-CHSH-probability $P_{\text{CHSH}}^\text{max}(t)$ and the concurrence $C(t)$ for the initial entangled states with parallel spin-polarizations given by Eq. (17). The $P_{\text{CHSH}}^\text{max}(t)$ and $C(t)$ curves are displayed respectively in upper (a) and lower (b) panels in Fig. 3 for the average photon-numbers $\gamma^2 = 0.01$ (1), 5 (2), 15 (3), 150 (4) with the state parameters $\eta = 0$, $\xi = \pi/3$ (green solid-line), $\pi/4$ (blue dot and dash line), $3\pi/4$ (red dash-line). Again the time evolution becomes random when the average photon number reaches $\gamma^2 = 1$. When the average photon-number increases to $\gamma^2 = 15$, the entanglement collapses only for the initial state with $\xi = 3\pi/4$. The CHSH inequality is, however, violated for the states with $\xi = \pi/6$, $\pi/4$ since $P_{\text{CHSH}}^\text{max}(t) > 2$ in these cases. $P_{\text{CHSH}}^\text{max}(t)$ remains in the upper quantum bound value with small oscillation for the state $\xi = \pi/4$ (blue dot-and dash-line), while the average value of $P_{\text{CHSH}}^\text{max}(t)$ for the state $\xi = \pi/6$ (green solid line) is slightly lower than the upper bound. In the large photon-number limit $\gamma^2 = 150$ (a4), the decoherence for the initial state $\xi = 3\pi/4$ revivals periodically at the time points $t = n5T$. Correspondingly the time-variation curves of $P_{\text{CHSH}}^\text{max}(t)$ and $C(t)$ are displayed in Fig. 4 for the initial states with complex superposition coefficients, in which the angle parameters are chosen as $\xi = \pi/4$, $\eta = \pi/6$ (green solid-line), $\pi/4$ (blue dot- and dash-line), $\pi/3$ (pink dash-line) respectively. At the average photon-number $\gamma^2 = 15$, the entanglement collapses for the initial states with $\eta = \pi/4$, $\pi/3$ (b3) and revivals at the time points of $t = (n/2)5T$ in the large photon-number limit $\gamma^2 = 150$ (b4). However, the peak-height of $P_{\text{CHSH}}^\text{max}(t)$ can approach the upper quantum bound only at $t = (2n)5T$. The $P_{\text{CHSH}}^\text{max}(t)$-curve varies with time for the state $\eta = \pi/6$ above the classical bound and so that the coherence remains.
Fig. 3. Color online) The maximum CHSH correlation-probability $P_{CHSH}^{max}(t)$ (upper panel) concurrence $C(t)$ (lower panel) as functions of time $t$ in the initial entangled states of parallel spin-polarizations with angle parameters $\eta = 0$, $\xi = \pi/3$, $\pi/4$, and $3\pi/4$ for the average photon-number $\gamma^2 = 0.01$ (1), 1 (2), 15 (3), 150 (4).

Fig. 4. Color online) The time variation of $P_{CHSH}^{max}(t)$ (upper panel) and concurrence $C(t)$ (lower panel) in the entangled states of parallel spin-polarizations with the angle parameters $\xi = \pi/4$, $\eta = \pi/6$, $\pi/4$, and $\pi/3$ for the average photon-number $\gamma^2 = 0.01$ (1), 1 (2), 15 (3), 150 (4).
6. Conclusion and discussion

We propose a theoretical model for two spins in a single-mode optical cavity with the ordinary magnetic interaction. Stationary entangled-states are realized in terms of variational method under the condition of optical coherent-state. The well studied two-spin singlet state is indeed generated in this simple model. Besides we also obtain the entangled state with parallel spin polarizations. The BI and its violation are reformulated in a unified way by means of quantum probability statistics with the help of state density operator. This formulation establishes a direct relation between decoherence and BI, which is obtained by neglecting the quantum interference part of density operator. Thus the BI indeed can serve as a criteria of decoherence. The BI in the CHSH form is also valid for the general two-spin entangled states of parallel spin polarizations so that the entanglement dynamics can be investigated for arbitrary entangled states with both antiparallel and parallel spin polarizations. Based on the time-evolution of state-density operator derived with the quantum master equation the dynamics of maximum CHSH-correlation-probability \( P_{\text{max}}^{\text{CHSH}}(t) \) is evaluated in comparison with the concurrence \( C(t) \). It is interesting to see a fact that the maximum quantum CHSH-probability as well as the concurrence does not vary with time in the spin singlet state. This is because that the density operator, which commutes with Hamiltonian, is a conserved quantity, and the measuring outcome is invariant with time in this state. In the general entangled states with antiparallel spin polarizations, the time-evolutions of \( P_{\text{max}}^{\text{CHSH}}(t) \) and \( C(t) \) are periodic in the weak field regimen with the average photon number in the order of \( \gamma^2 = 0.01 \). The spin flip due to the field interaction leads to the time oscillation of measuring outcome correlation. When the field intensity increases to the order of \( \gamma^2 = 1 \) the time evolution becomes random. The entanglement collapses at the average photon number \( \gamma^2 = 15 \) due to the field induced decoherence, which is characterized by the fulfilling of CHSH-inequality. The coherence revivals periodically in the large photon-number limit \( \gamma^2 = 150 \), such that the sharp peaks of \( P_{\text{max}}^{\text{CHSH}}(t) \) curves approach the upper quantum-bound \( 2\sqrt{2} \). The peak positions are located precisely at \( t = n5T \). For the complex superposition-coefficients of the entangled states the complete decoherence appears for the states \( \eta = \pi/6, \pi/4 \). Coherence revivals at the time points \( t = (2n)5T \), but not at \( t = (2n + 1)5T \), where the peak heights shrink to the classical bound value 2. The entanglement dynamics is similar but different in details for the states with parallel spin-polarizations. The precise period of coherence revival associated with the cavity-field frequency is an interesting observation in the large photon-number limit.

Acknowledgments

This work was supported by the Natural Science Foundation of China (Grants Nos. 11105087, 61275210, 11275118, 11404198) and Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (STIP) (Grant No. 2014102).