Stability of intersections of graphs in the plane and the van Kampen obstruction *

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Abstract

A map $\varphi : K \to \mathbb{R}^2$ of a graph $K$ is approximable by embeddings, if for each $\varepsilon > 0$ there is an $\varepsilon$-close to $\varphi$ embedding $f : K \to \mathbb{R}^2$. Analogous notions were studied in computer science under the names of cluster planarity and weak simplicity. In this survey we present criteria for approximability by embeddings (P. Minc, 1997, M. Skopenkov, 2003) and their algorithmic corollaries. We introduce the van Kampen (or Hanani-Tutte) obstruction for approximability by embeddings and discuss its completeness. We discuss analogous problems of moving graphs in the plane apart (cf. S. Spież and H. Toruńczyk, 1991) and finding closest embeddings (H. Edelsbrunner). We present higher dimensional van Kampen obstruction, its completeness result and algorithmic corollary (D. Repovš and A. Skopenkov, 1998).

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1 Definition, examples and observations

A map \( \varphi : K \to \mathbb{R}^2 \) of a graph \( K \) is **approximable by embeddings**, if for each \( \varepsilon > 0 \) there is an \( \varepsilon \)-close to \( \varphi \) embedding \( f : K \to \mathbb{R}^2 \).

We mostly consider the case when \( \varphi \) is either a path or a cycle, i.e. either \( K \cong I = [0,1] \) or \( K \cong S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \).

This notion also appeared in computer science, in studies of planarity of compacta and in dynamical systems. We give later some illustrating examples. A related notion of a **projected embedding**, **level-planarity and monotone drawings** appeared in computer science and in differential topology, see [ARS, FPSS, Fu14, PT], [Sk, §3.5], [Fu16, Theorem 1] and references therein.

The following examples (see also [RS, Fig. 5]) show that this notion is interesting in itself. See elementary introduction in [DNSS, L].

![Figure 1: Transversal intersection and the standard 2-winding are not approximable by embeddings](image1)

![Figure 2: Paths not approximable by embeddings and having no transversal intersections](image2)

**Example 1.1** (Sieklucki, 1969 [Si69]). (a) Any map \( I \to I \subset \mathbb{R}^2 \) or \( T \to I \subset \mathbb{R}^2 \) is approximable by embeddings (where \( T \) is triod).
The $\phi$ determines a path in the graph where $x$ which is not planar. But if $\phi$ Vertices a $G$ graph $K$ intersections, is it then approximable by broken lines of $i$ embeddings if and only if for each $k$.

The above examples show that approximability by embeddings is non-trivial to check, even for paths.

One can show that there is no Kuratowski-type criterion.

A map from a graph to a point in the plane is approximable by embeddings if and only if the graph is planar.

There exists an algorithm of checking whether a given simplicial map is approximable by embeddings (A. Skopenkov, 1994, M. Skopenkov, 2003 [Sk94, Sk03']).

Problem 1.2. If a broken line of $k$ edges is approximable by broken lines without self-intersections, is it then approximable by broken lines of $k$ edges without self-intersections?

Let us state more convenient to apply criteria for approximability by embeddings of a simplicial path or cycle in the plane. These criteria assert that, in some sense, transversal self-intersections are the main obstructions to approximability by embeddings. The criteria are stated in terms of the derivative which we define later.

2 Criteria for approximability by embeddings of paths

Theorem 2.1 (P. Minc, 1997, M. Skopenkov, 2003 [Mi97, Sk03']); might be reproved in [CBPP]). (I) A broken line (i.e. PL map) $\phi : I \to \mathbb{R}^2$ with $k$ vertices is approximable by embeddings if and only if for each $i = 0, \ldots, k$ the $i$-th derivative $\phi^{(i)}$ does not have transversal self-intersections.

(S) A closed broken line $\varphi : S^1 \to \mathbb{R}^2$ with $k$ vertices is approximable by embeddings if and only if for each $i = 0, \ldots, k$ the $i$-th derivative $\varphi^{(i)}$ neither contains transversal self-intersections nor is the standard winding of degree $d \notin \{-1, 0, 1\}$.

Note that there are no analogous criteria for simplicial maps $\varphi : K \to \mathbb{R}^2$ of a connected graph $K \neq I, S^1 \{Sk03\}$.

First let us define the derivative (= the line graph) $G'$ of a graph $G$. The vertex set of the graph $G'$ is the edge set of $G$. For an edge $a \subset G$ denote by $a' \subset G'$ the corresponding vertex. Vertices $a'$ and $b'$ of $G'$ are joined by an edge if and only if the edges $a$ and $b$ are adjacent in $G$.

Now let $\varphi$ be a path in the graph $G$ given by the sequence of vertices $x_1, \ldots, x_k \in G$, where $x_i$ and $x_{i+1}$ are joined by an edge. Then $(x_1x_2)', \ldots, (x_{k-1}x_k)'$ is a sequence of vertices of $G'$. In this sequence replace each segment $(x_ix_{i+1})', (x_{i+1}x_{i+2})', \ldots, (x_{j-1}x_j)'$ such that $(x_ix_{i+1})' = (x_{i+1}x_{i+2})' = \cdots = (x_{j-1}x_j)'$ by a single vertex. The obtained sequence of vertices determines a path in the graph $G'$. This path $\varphi'$ is called the pre-derivative of the path $\varphi$.

A 5-od (the cone over 5 points) is a planar graph whose derivative is the Kuratowsky graph, which is not planar. But if $G \subset \mathbb{R}^2$ and the path $\varphi$ does not have transversal self-intersections, then the image of the map $\varphi'$ is a subgraph $G'' \subset G'$ naturally embedded in the plane [Sk03'].

The derivative of the path $\varphi$ is defined by changing $G'$ to $G'' \subset \mathbb{R}^2$ and $\varphi'$ to its onto restriction $\varphi' : I \to G''$.

Define the $k$-th derivative $\varphi^{(k)}$ inductively.

For a cycle $\varphi$ the definition of the derivative cycle $\varphi'$ is analogous.
Example 2.2. (a) \( \varphi' = \varphi \) for the standard \( d \)-winding \( \varphi : S^1 \to S^1 \) with \( d \neq 0 \).

(b) \( \varphi' \) is an embedding for any Euler path or cycle \( \varphi \) having no transversal self-intersections.

(c) Let \( \varphi : S^1 \to G \) be a cycle of \( k \) vertices in a graph \( G \). Then either the domain of \( \varphi^{(k)} \) is empty or \( \varphi^{(k)} \) is a standard winding of a non-zero degree. This degree can be considered as a generalization of the degree of a map \( S^1 \to S^1 \).

Corollary 2.3 (P. F. Cortese, G. Di Battista, M. Patrignani, M. Pizzonia, 2009 [CBPP], cf. [CEX, AAFT]). There are polynomial algorithms for

- checking approximability of paths or cycles by embeddings.
- finding the degree of the winding \( \varphi^{(k)} \) for a cycle \( \varphi : S^1 \to G \) of \( k \) vertices in a graph \( G \).

See also [Fu] on approximability by embeddings having a fixed isotopy class.

3 Relation to planarity of compacta

Example 3.1 (the 2-adic van Danzig solenoid). Take a solid torus \( T_1 \subset \mathbb{R}^3 \). Let \( T_2 \subset T_1 \) be a solid torus going twice along the axis of the torus \( T_1 \). Analogously, take \( T_3 \subset T_2 \) going twice along the axis of \( T_2 \). Continuing in the similar way, we obtain an infinite sequence of solid tori \( T_1 \supset T_2 \supset T_3 \supset \ldots \). The intersection of all tori \( T_i \) is called the 2-adic van Danzig solenoid.

The inverse limit of an infinite sequence \( K_1 \xrightarrow{\varphi_1} K_2 \xrightarrow{\varphi_2} K_3 \xrightarrow{\varphi_3} \ldots \) of graphs in \( \mathbb{R}^3 \) and simplicial maps between them is the compactum

\[
K = \{ (x_1, x_2, \ldots) \in l_2 : x_i \in K_i \text{ and } \varphi_{i+1}x_{i+1} = x_i \}.
\]

E.g. for the van Danzig solenoid one can take all \( K_i = S^1 \) and all \( \varphi_i \) to be 2-windings. Any 1-dimensional compactum can be represented as an inverse limit as above.

Clearly, \( K \) is planar if for each \( i \) and each embedding \( f_i : K_i \to \mathbb{R}^2 \) the composition \( f_i \circ \varphi_i \) is approximable by embeddings.

4 Examples that appeared in dynamical systems

A map \( f : K \to M \) is said to be embeddable in \( X \) if there exists an embedding \( \psi : M \to X \) for which \( \psi \circ f \) is approximable by embeddings.

Let \( K \) and \( M \) be wedges of \( k \) and \( m \) circles, respectively, and suppose that \( f \) is represented by \( k \) words of \( m \) letters.
Example 4.1. (Smale) The map $S^1 \vee S^1 \to S^1 \vee S^1$, defined by $a \mapsto aba$ and $b \mapsto ab$ is embeddable into torus but not into plane.

(Wada–Plykin) The map $S^1 \vee S^1 \vee S^1 \to S^1 \vee S^1 \vee S^1$, defined by $a \mapsto aca^{-1}$, $b \mapsto bab^{-1}$ and $c \mapsto b$ is embeddable into plane.

(Zhirov) The map $S^1 \vee S^1 \vee S^1 \vee S^1 \to S^1 \vee S^1 \vee S^1 \vee S^1$, defined by $a \mapsto ac$, $b \mapsto ad$, $c \mapsto bac$ and $d \mapsto c$ is embeddable into pretzel but not into torus.

5 Moving graphs in the plane apart

A map $\varphi : K \sqcup L \to \mathbb{R}^2$ of the disjoint union of graphs $K$ and $L$ is approximable by maps with disjoint images, or disjoinable, if for each $\varepsilon > 0$ there is an $\varepsilon$-close to $\varphi$ map $f : K \sqcup L \to \mathbb{R}^2$ such that $f(K) \cap f(L) = \emptyset$.

Clearly,

- if $K = L = I$ and $\varphi(K) = \varphi(L) = I \times 0 \subset \mathbb{R}^2$, then $\varphi$ is disjoinable.
- transversal intersection is not disjoinable.
- there is a non-disjoinable map $\varphi : I \sqcup I \to \mathbb{R}^2$ having no transversal intersections.

Problem 5.1. Is a broken line $\varphi : I \to \mathbb{R}^2$ approximable by embeddings if for each pair $I_1, I_2 \subset I$ such that $I_1 \cap I_2 = \emptyset$ the pair of of sub-broken-lines $\varphi|_{I_1 \sqcup I_2}$ is disjoinable?

This might follow from the above Minc criterion.

Analogously to approximability by embeddings one can prove that there exists an algorithm of checking disjoinability of simplicial maps.

Problem 5.2. Find a ‘quick’ algorithm of checking disjoinability of simplicial maps, at least for $K = L = I$, i.e., for broken lines.

Example 5.3 (M. Skopenkov, 2003). There exists a pair of broken lines $\varphi, \psi : I \to \mathbb{R}^2$ (on the figure below a pair of paths $f, g$, close to $\varphi, \psi$, is shown), which is not disjoinable but for which the pair $\varphi', \psi'$ of derivatives is disjoinable.

![Figure 4: A pair of non-disjoinable paths with disjoinable derivatives](image)

6 The Hanani-Tutte obstruction

The Hanani-Tutte obstruction for planarity of graphs appeared in works of Hanani and Tutte in 1934, 1970.
The Hanani-Tutte obstruction for approximability of a broken line $\varphi : I \to \mathbb{R}^2$ by embeddings is that for each $\varepsilon > 0$ there exist a subdivision $J$ of $I$ and a general position broken line $f : J \to \mathbb{R}^2$ which is $\varepsilon$-close to $\varphi$ and such that $|f(e_1) \cap f(e_2)|$ is even for each disjoint edges $e_1, e_2$ of $I$. Note that

- $f(e_1)$ and $f(e_2)$ are broken lines, not necessarily segments.
- for small enough $\varepsilon > 0$ we may assume that $f(e_1) \cap f(e_2) = \emptyset$ whenever $\varphi(e_1) \cap \varphi(e_2) = \emptyset$.

The integer Hanani-Tutte obstruction is defined analogously, the only change is that instead of $|f(e_1) \cap f(e_2)|$ is even' we orient edges $e_1, e_2$ and require that the number of their intersection points with signs is non-zero.

7 The van Kampen obstruction

The van Kampen obstruction is a reformulation of the Hanani-Tutte obstruction an equivalent form more convenient to construct algorithms. The van Kampen obstruction for embeddability of $n$-dimensional complex in $\mathbb{R}^{2n}$ was invented by van Kampen around 1932.

Let us define the mod 2 van Kampen obstruction to approximability by embeddings of broken lines. (This construction is more visual than that for of embeddability of graphs or complexes.)

Let $\varphi : I \to \mathbb{R}^2$ be a simplicial path. Denote by $x_1, \ldots, x_k$ the vertices of $I$ in the order along $I$, and denote the edge $x_i x_{i+1}$ by $i$. Let $I^* = \bigcup_{i<j} i \times j$ be the simplicial deleted product of given subdivision of $I$.

Paint red the edges $x_i \times j, j \times x_i$, and the cells $i \times j$ of $I^*$ such that $\varphi x_i \cap \varphi j = \emptyset, \varphi i \cap \varphi j = \emptyset$, and denote by $I^*\varphi$ the red set.

![Figure 5: The Van Kampen obstruction for the 'H-path' (fig. 2.a)](image)

Take a general position map $f : I \to \mathbb{R}^2$, sufficently close to $\varphi$. To each cell $i \times j$ of ”the table” $I^*$ put the number $v_f(i \times j) = |f_i \cap f_j| \pmod{2}$.

Cut $I^*$ along the red edges, and let $C_1, C_2, \ldots, C_n$ be all the obtained components such that $\partial C_k \cap \partial I^* \subset I^*\varphi$.

Denote $v_f(C_k) = \sum_{i \times j \subset C_k} v_f(i \times j)$. The van Kampen obstruction (with $\mathbb{Z}_2$-coefficients) for approximability by embeddings is the vector

$$v(\varphi) = (v_f(C_1), v_f(C_2), \ldots, v_f(C_n)) .$$

One can easily show that $v(\varphi)$ does not depend on the choice of $f$.

Thus $v(\varphi) = 0$ is a necessary condition for approximability by embeddings.

It is easy to check that $v(\varphi) \neq 0$ for a broken line $\varphi : I \to \mathbb{R}^2$ containing a transversal self-intersection.
8 Completeness of the van Kampen obstruction

Corollary 8.1 (M. Skopenkov, 2003 [Sk03']). A broken line \( \varphi : I \to \mathbb{R}^2 \) is approximable by embeddings if and only if the van Kampen obstruction \( v(\varphi) \) is zero.

This criterion, although more difficult to state, might be used to construct a faster algorithm than the ‘derivative’ criterion above.

The van Kampen obstruction \( v(\varphi) \) for approximability by embeddings of a simplicial map \( \varphi : K \to \mathbb{R}^2 \) of any graph \( K \) is defined analogously (in the lecture or in [RS, §1]).

The analogue of this Corollary for closed broken line is false. The standard 3-winding is a counterexample [RS, Fig. 5a]. Using ‘integration’ one can construct a counterexample with the image a triod [FKMP, Fig. 18].

Problem 8.2. Is analogue of the Corollary true for the broken line replaced by a simplicial map \( \varphi : K \to \mathbb{R}^2 \) of any tree \( K \)?

This is true when \( \varphi(K) \subset \mathbb{R} \) [Fu14, Fu16].

Theorem 8.3 (M. Skopenkov, 2003 [Sk03']). Let \( K \) be a graph with \( k \) vertices and without vertices of degree \( > 3 \). A simplicial map \( \varphi : K \to S^1 \subset \mathbb{R}^2 \) is approximable by embeddings if and only if the van Kampen obstruction \( v(\varphi) = 0 \) and \( \varphi^{(k)} \) does not contain standard windings of odd degree \( d \neq 1 \).

The derivative is defined analogously.

Problem 8.4. Is analogue of the Theorem true for an arbitrary graph \( K \) and \( S^1 \) replaced by an arbitrary graph \( G \)?

A positive answer implies a positive answer to the previous problem.

One can define the van Kampen obstruction for disjoinability. However, the above example of M. Skopenkov shows that its triviality is not sufficient for disjoinability, even for \( K = L = I \), i.e., for broken lines.

9 The deleted product

To formulate some results and conjectures we need the important notion of the deleted product.

The deleted product \( \tilde{G} \) of a graph (or a topological space) \( G \) is the product of \( G \) with itself, minus the diagonal:

\[ \tilde{G} := \{ (x, y) \in G \times G \mid x \neq y \}. \]

![Diagram of the deleted product and the Gauss map](image)

Figure 6: The deleted product and the Gauss map

This is the configuration space of ordered pairs of distinct points of \( G \).

Now suppose that \( f : G \to \mathbb{R}^m \) is an embedding (e.g. \( m = 2 \)). Then the map \( \tilde{f} : \tilde{G} \to S^{m-1} \) is well-defined by the Gauss formula

\[ \tilde{f}(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}. \]

This map is equivariant with respect to the ‘exchanging factors’ involution \( (x, y) \to (y, x) \) on \( \tilde{G} \) and the antipodal involution on \( S^{m-1} \). Thus the existence of an equivariant map \( \tilde{G} \to S^{m-1} \) is a necessary condition for the embeddability of \( G \) in \( \mathbb{R}^m \).
10 How close is a map to an embedding?

Let \( \varphi : G \to \mathbb{R}^2 \) be a map from a (finite) graph to the plane.

Denote by \( a(\varphi) \) the infimum of those \( a \geq 0 \) for which there is a map \( f : G \to \mathbb{R}^2 \) which is \( a \)-close to \( \varphi \) and has no self-intersections (i.e. is an embedding).

**Problem 10.1** (H. Edelsbrunner). Estimate \( a(\varphi) \), and construct an embedding \( f \) realizing the estimation from above.

Denote by \( b(\varphi) \) the infimum of those \( b \geq 0 \) for which there is an equivariant map \( \tilde{G} \to S^1 \) whose restriction to the set

\[
\{ (x, y) \in G \times G : |\varphi(x) - \varphi(y) | > b \}
\]

is equivariantly homotopic to the map given by the formula

\[
\tilde{\varphi}(x, y) := \frac{\varphi(x) - \varphi(y)}{||\varphi(x) - \varphi(y)||}.
\]

The number \( b(\varphi) \) can be constructively estimated using the van Kampen obstruction.

Clearly, \( a(\varphi) \geq b(\varphi)/2 \).

**Problem 10.2** (appeared in a discussion with H. Edelsbrunner). Is there a number \( C \) such that \( a(\varphi) \leq C b(\varphi) \) for each broken line \( \varphi : [0,1] \to \mathbb{R}^2 \)? (So \( C \) does not depend of the number of vertices of the broken line.)

One can try to obtain an affirmative solution of the above problem (and to construct a map realizing the estimation from above) using proof of a criterion for approximability by embeddings [Sk03', Corollary 1.4].

The answer for a cycle \( \varphi : S^1 \to \mathbb{R}^2 \) is presumably ‘no’. However, one can possibly formulate a more elaborate conjecture using derivation of graphs [Sk03', Theorems 1.3.5 and 1.5].

The answer for a higher-dimensional analogue of the Problem is presumably ‘yes’. One can use the proof of a criterion for approximability by embeddings [RS]. That proof presumably gives a construction of a map realizing the estimation from above.

11 Higher-dimensional generalization

A map \( f : K \to M \) between polyhedra (=bodies of simplicial complexes) is said to be **embeddable** in \( \mathbb{R}^m \) if there exists an embedding \( \psi : M \to \mathbb{R}^m \) for which \( \psi \circ f \) is approximable by embeddings.

**Theorem 11.1.** (a) For each \( n \) every map \( f : I^n \to I^n \) is embeddable in \( \mathbb{R}^{2n} \) (Sieklucki, 1969 [Si69]).

(b) For each \( n > 1 \) every map \( f : T^n \to T^n \) between \( n \)-dimensional tori is embeddable in \( \mathbb{R}^{2n} \) (Keesling-Wilson, 1985 [KW]).

(c) For each \( n > 1 \) every map \( f : S^n \to S^n \) is embeddable in \( \mathbb{R}^{2n} \) (Akhmetiev 1996, Melikhov, 2004 [Ak96, Me04]).

The proof of (a) is obvious: one just need to take the graph of \( f \) in \( I^n \times I^n \subset \mathbb{R}^{2n} \) and compress it to the first factor. The proofs of (b,c) are much more complicated.

**Corollary 11.2.** For each fixed \( m, n \) such that \( 2m \geq 3n + 3 \) approximability by embeddings of simplicial maps \( K \to \mathbb{R}^m \) from \( n \)-complexes is decidable in polynomial time.

This follows by the 1998 algebraic (=combinatorial) criterion for approximability by embeddings ([RS], see below), and from the 2013 result of M. Čadek-M. Krčál-L. Vokřínek on algorithmic solvability of the corresponding algebraic problem [CKV].

For each fixed \( m, n \) such that \( 2m < 3n + 3 \) approximability by embeddings is NP hard by Matoušek-Tancer-Wagner, 2008 [MTW]. Cf. [AAFT, §6].
12 Criteria for approximability by embeddings

Let $K,L$ be $n$-dimensional complexes. The van Kampen obstructions for embeddability of $K$ into $\mathbb{R}^{2n}$, for approximability by embeddings of a simplicial map $\varphi : K \to \mathbb{R}^{2n}$, and for approximability of a simplicial map $\varphi : K \sqcup L \to \mathbb{R}^{2n}$ by maps with disjoint images, are defined analogously (in the lecture or in [Sk08, §4], [RS, §1]).

**Theorem 12.1** (E.R. Van Kampen, A. Shapiro, W.T. Wu, 1932-57). For $n \neq 2$ an $n$-complex is embeddable into $\mathbb{R}^{2n}$ if and only if the van Kampen obstruction is zero.

The analogue for $n = 2$ is wrong by M. Freedman, V. Krushkal and P. Teichner, 1994 [FKT] (see a simpler proof in [AMSW, §3]).

**Theorem 12.2** (D. Repovš and A. Skopenkov, 1998 [RS]). For $n > 2$ a simplicial map $K \to \mathbb{R}^{2n}$ of an $n$-complex is approximable by embeddings if and only if the van Kampen obstruction is zero.

The analogue for $n = 1, 2$ is wrong.

The deleted product obstruction is defined in §9.

**Theorem 12.3** (C. Weber, 1967). Assume that $2m \geq 3n + 3$ and $K$ is an $n$-complex. There is an embedding $K \to \mathbb{R}^m$ of if and only if there is an equivariant map $\tilde{K} \to S^{m-1}$.

**Theorem 12.4** (D. Repovš and A. Skopenkov, 1998 [RS]). Assume that $2m \geq 3n + 3$ and $\varphi : K \to \mathbb{R}^m$ is a simplicial map of an $n$-complex $K$. The map $\varphi$ is approximable by embeddings if and only if there is an equivariant map $\tilde{K} \to S^{m-1}$ whose restriction to the set $\{ (x, y) \in K \times K : \varphi x \neq \varphi y \}$ is equivariantly homotopic to the map given by the formula $\tilde{\varphi}(x, y) = \frac{\varphi(x) - \varphi(y)}{|\varphi(x) - \varphi(y)|}$.

Analogous simpler result for disjoinability was proved by S. Spiež and H. Toruńczyk in 1991 [ST].

13 Generalizations to $r$-fold points

**Problem 13.1** (Gromov, 2010 [Gr10]). Is it correct that if $r$ is not a prime power, then for each compact subset $K$ of $\mathbb{R}^m$ for some $m$, having Lebesgue dimension $\dim K = (r - 1)n$, there is a continuous map $K \to \mathbb{R}^{nr}$ each of whose point preimages contains less than $r$ points?

The answer is ‘yes’ for polyhedra $K$ and each $n \geq 2$ [MW, AMSW]. We conjecture that the answer is ‘no’ in general.

Like embeddability of compacta is being related to approximability of maps from polyhedra by embeddings, Gromov’s problem is related to approximability of maps from polyhedra by maps without $r$-fold points.

Let $K$ be a finite simplicial complex. A map $f : K \to \mathbb{R}^m$ is an **almost $r$-embedding** if $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint simplices of $K$.

The van Kampen obstructions $v(K)$ for almost $r$-embeddability of an $(n(r - 1))$-complex $K$ to $\mathbb{R}^{nr}$ is defined analogously to above (in the lecture or in [MW]).

**Theorem 13.2** (I. Mabillard and U. Wagner, 2015, S. Avvakumov, I. Mabillard, A. Skopenkov and U. Wagner, 2015 [MW, AMSW]). Suppose that $n, r \geq 2$, $n + r \geq 5$. An $n(r - 1)$-complex is almost $r$-embeddable in $\mathbb{R}^{nr}$ if and only if the van Kampen obstruction is zero.

**Problem 13.3.** Are analogous results true for approximability by almost $r$-embeddings, or by maps $f : K_1 \sqcup \ldots \sqcup K_r \to \mathbb{R}^{nr}$ such that $f(K_1) \cap \ldots \cap f(K_r) = \emptyset$?
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