Static teams with common information

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Abstract: We consider a static team problem in which agents observe correlated Gaussian observations and seek to minimize a quadratic cost. It is assumed that the observations can be split into two parts: common observations that are observed by all agents and local observations that are observed by individual agents. It is shown that the optimal strategies are affine and the corresponding gains can be determined by solving appropriate systems of linear equations. Two structures of optimal strategies are identified. The first may be viewed as a common-information based solution; the second may be viewed as a hierarchical control based solution. A decentralized estimation example is presented to illustrate the results.

Keywords: Stochastic control, Static teams, Multi-Agent Systems, Common Information.

1. INTRODUCTION

Decentralized decision making or team problems arise in a variety of applications including networked control systems, sensor networks, communication networks, transportation networks, and economics. In such problems, there are multiple decision makers or agents that have access to different information but aim to coordinate their actions to minimize a common cost function. A team problem is called static if the observations of agents are not affected by the actions of other agents; otherwise, the problem is called dynamic.

Static team problems were first investigated by Marschak (1955); Radner (1962); Marschak and Radner (1972), who identified necessary and sufficient conditions to determine team optimal strategies. Subsequently, there has been significant work on both static and dynamic teams. Sandell and Athans (1974) extended these results to vector valued observations; Krainak and Marcus (1982); Krainak et al. (1982) extended them to the exponential cost criteria. Ho and Chu (1972); Chu (1972) investigated linear quadratic dynamic teams with partially nested information structure and showed that they can be converted to static teams with an appropriate change of variables. Sandell and Athans (1974); Yoshikawa (1975) provided an explicit solution to the linear quadratic dynamic team with one-step delay sharing. Casalino et al. (1984) generalized these results to general partially nested teams with a common past. Yüksel (2009) generalized the results of Ho and Chu to stochastically nested information structures. Recently, Mahajan and Nayyar (2015) identified sufficient statistics for best linear controllers for linear quadratic dynamic teams with partial history sharing.

In addition to the above results for linear quadratic teams, other variations of decentralized control problems have also been considered in the literature. These include models with non-linear Markovian dynamics, $H_2/H_\infty$ models with sparsity constraints, amongst others. We refer the reader to Mahajan et al. (2012) for details.

In this paper, we investigate the following question. Before the system starts running, suppose it is possible to build an observation channel and broadcast its measurements to all agents. What is the value of such common information? We investigate this question under the assumptions that the observations are jointly Gaussian and the cost is quadratic.

To answer this question, let $J^*$ denote the optimal performance when the additional common observation channel is available and let $J^w$ denote the optimal performance when such a channel is not available. Then, the value of the additional common information is $J^w - J^*$ (i.e., it is beneficial to build the common observation channel if it costs less than $J^w - J^*$).

We develop two methods to efficiently compute the optimal strategy and the optimal performance for teams with common information. The first approach is inspired by the common information approach of Nayyar et al. (2013). We view the system from the point of view of a virtual agent that has access to the common observations. This virtual agent solves a static team problem where the means and covariances of the observations are the conditional means and covariances given the common information.

The second approach is a hierarchical control approach in which agents choose a part of their action based on their local observations and a virtual coordinator provides a linear correction to these local actions. The hierarchical control approach has interesting implications for the implementation of the optimal solution. Instead of transmitting the measurements of the common observation channel to all agents, a coordinator may only transmit a small correction term to each agent. When the common observation is high-dimensional (e.g., a video), communicating corrective terms instead of the common observations may lead to substantially smaller communication overhead.

The rest of the paper is organized as follows. In Sec. 2, we describe the model and the main results. In Sec. 3, we present an example of decentralized estimation. We prove the main results in Sec. 4 and conclude in Sec. 5.
2. STATIC TEAMS WITH COMMON INFORMATION

2.1 Model and Problem Formulation

Assume the system consists of $n$ agents that are indexed by the set $N = \{1, \ldots, n\}$. We use $N_0$ to denote the set $\{0, 1, \ldots, n\}$.

Let $(x, y_0, y_1, \ldots, y_n)$ be jointly Gaussian random variables where $x \in \mathbb{R}^d$ and $y_i \in \mathbb{R}^{d_i}$ for $i \in N_0$. Let $E[x|\cdot] = \hat{x}$, $E[y_i|\cdot] = \hat{y}_i$, for $i \in N_0$. The cost is measured by the following cost function:

$$c(x, u_1, \ldots, u_n) = \sum_{i \in N} \sum_{j \in N} u^T_i R_{ij} u_j + 2 \sum_{i \in N} u^T_i P_i x,$$

where $(R_{ij})_{i,j}$ and $(P_i)_{i \in N}$ are matrices of appropriate dimensions. For ease of notation, define $u = \text{vec}(u_1, \ldots, u_n)$, $P = \text{rows}(P_1, \ldots, P_n)$, $\Theta = \text{rows}(\Theta_1, \ldots, \Theta_n)$, $\Sigma = \text{diag}(\Sigma_1, \ldots, \Sigma_n)$.

Then, the cost (1) may be written succinctly as

$$c(x, u) = u^T R u + 2 u^T P x.$$  (2)

Assume the following:

(A1) The matrix $R$ is symmetric and positive definite.

(A2) The parameters $N$, $P$, $R$, $\Theta$, and $\Sigma$ are common knowledge to all agents.

We call the following optimization problem as static team with common information.

Problem 1. Assuming (A1) and (A2) and given the joint distribution of $(x, y_0, y_1, \ldots, y_n)$ and the cost matrices $P$ and $R$, choose decision rules $g^* = (g_1, \ldots, g_n)$ such that

$$J^* = J(g^*) = \min_{g} J(g)$$

where $J(g) = E[c(x, u)]$ and $c(\cdot, \cdot)$ is defined in (1).

2.2 Main results

Define $\hat{x}_0 = E[x|y_0] = \hat{x} + \Theta_0 \Sigma_{00}^{-1}(y_0 - \hat{y}_0)$, and for $i, j \in N$,

$$\hat{y}_i = E[y_i|y_0] = \hat{y}_i + \Sigma_{io} \Sigma_{00}^{-1}(y_0 - \hat{y}_0),$$

$$\hat{\Theta}_i = \text{cov}(x, y_i|y_0) = \Theta_i - \Theta_0 \Sigma_{00}^{-1} \Sigma_{oi},$$

$$\hat{\Sigma}_{ij} = \text{cov}(y_i, y_j|y_0) = \Sigma_{ij} - \Sigma_{io} \Sigma_{00}^{-1} \Sigma_{oj}.$$
Linear equations. Thus, the method of Theorem 1 is more

\[ u_i = F_i^*(\hat{x}_i - \bar{x}) + H_i^* \hat{x} \quad \forall i \in N. \]

(6)

(2) Alternatively, by substituting the value of \( \hat{x}_i \) in (6), the optimal control laws may be written as

\[ u_i = \hat{F}_i(\hat{x}_i - \bar{x}) + \hat{G}_i(\hat{x}_0 - \bar{x}) + H_i^{\infty} \hat{x}, \quad \forall i \in N, \]

where \( \hat{F}_i = \bar{F}_i \bar{\Theta}_i \bar{\Sigma}_{i0}^{-1} \bar{\Sigma}_{ii} \bar{\Theta}_i^{-1} \) and \( \hat{G}_i = \bar{H}_i - \bar{F}_i \bar{\Theta}_i \bar{\Sigma}_{i0}^{-1} \bar{\Sigma}_{ii} \bar{\Theta}_i^{-1} \).

(3) The corresponding gains are computed as follows. \( H_i \) is given by (5). Define \( F = \text{vec}(F_1, \ldots, F_n) \), \( \tilde{\eta} = \text{vec}(P_1 \Theta_1, \ldots, P_n \Theta_n) \), and \( \tilde{F} = [\tilde{F}_{ij}]_{i,j \in N} \) where \( \tilde{F}_{ij} = \bar{\Sigma}_{ij} \otimes R_{ij} \). Then,

\[ F = -\tilde{F}^{-1} \tilde{\eta}. \]

(8)

(4) Furthermore, the optimal cost is given by

\[ J^* = -\tilde{\eta} \tilde{F}^{-1} \tilde{\eta} - \bar{z}_0^T P R^\infty \bar{x}_0. \]

The result is obtained by substituting \( y_i - \tilde{y}_i = \bar{\Sigma}_{ii} \bar{\Theta}_i^{-1} (\hat{x}_i - \bar{x}) \) in (3).

2.3 Comparison with Radner’s results

When the common information is absent (i.e., \( y_0 \) is independent of \( (x, y_1, \ldots, y_n) \)), Problem 1 is the same as the static team problem investigated by Radner (1962), who showed the following:

**Theorem 3.** (Radner (1962)). In Problem 1, if \( y_0 \) is independent of \( (x, y_1, \ldots, y_n) \) (i.e., \( \Theta_0 = 0 \) and for \( i \in N, \Sigma_i = 0 \)), then the optimal control laws are given by

\[ u_i = L_i^o (y_i - \bar{y}_i) + H_i^o \hat{x} \]

where the gains \( \{L_i^o\}_{i \in N} \) and \( \{H_i^o\}_{i \in N} \) are computed as follows. Define \( H^o = \text{rows}(H_1^o, \ldots, H_n^o) \), \( L^o = \text{vec}(L_1^o, \ldots, L_n^o) \), \( \eta^o = \text{vec}(P_1 \Theta_1, \ldots, P_n \Theta_n) \), and \( \Gamma^o = \text{vec}(\Gamma_{ij})_{i,j \in N} \) where \( \Gamma_{ij} = \Sigma_{ij} \otimes R_{ij} \). Then,

\[ H^o = -R^o P \quad \text{and} \quad L^o = -(\Gamma^o)^{-1} \eta^o. \]

Furthermore, the optimal cost is given by

\[ J^o = -\eta^o (\Gamma^o)^{-1} \eta^o - \bar{x}^T P R^o \hat{x}. \]

Note that the gains \( \{H_i^o\}_{i \in N} \) are exactly the same \( \{H_i\}_{i \in N} \) defined in Theorem 1.

It is possible to directly use Theorem 3 to solve Problem 1. The observation of each agent is \( (y_0, y_i) \). Therefore, the optimal control law is of the form

\[ u_i = L_i^o (y_i - y_0, y_i - \bar{y}_i) + H_i^o \hat{x}. \]

Such a naïve solution requires more calculations than the solution given in Theorem 1. In particular, define \( d_u = \sum_{i \in N} d_{u_i} \), \( d_l = \sum_{i \in N} d_{l_i} \times d_{l_i} \), and \( d_t = \sum_{i \in N} d_{t_i} \times (d_{t_i} + d_{l_i}) \). To obtain the gains \( \{L_i^o\}_{i \in N} \) as argued above, we need to solve a system of \( d_u \) linear equations; to obtain the gains \( \{L_i\}_{i \in N} \) using the method of Theorem 1, we need to solve a system of \( d_t \) linear equations. Thus, the method of Theorem 1 is more efficient than the naïve method described above.

It is also possible to rewrite the result of Theorem 3 in the form of Corollary 2 as follows.

**Corollary 3.** In Problem 1, if \( y_0 \) is independent of \( (x, y_1, \ldots, y_n) \) (i.e., \( \Theta_0 = 0 \) and for \( i \in N, \Sigma_i = 0 \)), then the optimal control laws are given by

\[ u_i = F_i^o (\hat{x}_i - \bar{x}) + H_i^o \hat{x} \]

where the gains \( \{F_i^o\}_{i \in N} \) and \( \{H_i^o\}_{i \in N} \) are computed as follows. Define \( H^o = \text{rows}(H_1^o, \ldots, H_n^o) \), \( F^o = \text{vec}(F_1^o, \ldots, F_n^o) \), \( \eta^o = \text{vec}(P_1 \Theta_1, \ldots, P_n \Theta_n) \), and \( \Gamma^o = \text{vec}(\Gamma_{ij})_{i,j \in N} \) where \( \Gamma_{ij} = \Sigma_{ij} \otimes R_{ij} \). Then,

\[ H^o = -R^o P \quad \text{and} \quad F^o = -(\Gamma^o)^{-1} \eta^o. \]

Furthermore, the optimal cost is given by

\[ J^o = -\eta^o (\Gamma^o)^{-1} \eta^o - \bar{x}^T P R^o \hat{x}. \]

As before, directly using Corollary 4 to solve Problem 1 requires more calculations that the solution given by Corollary 2. In particular, define \( d_m = \sum_{i \in N} d_{m_i} \) and \( d_t = \sum_{i \in N} (d_{t_i} + d_{l_i}) \). Observe that \( d_{m_i} = d_m + N d_{t_i} \). Note that matrix \( \Sigma_o = d_0^o \times d_t \) while matrix \( \Sigma_0 = d_m \times d_m \).

3. An illustrative example: Decentralized estimation

As an illustrative example of the static team model with common information, we consider a decentralized estimation problem. Suppose there is a random variable \( x \in \mathbb{R}^d \) of interest. There are \( n \) agents. Agent \( i, i \in N \), observes \( (y_0, y_i) \) where for \( k \in N_0, y_k = C_k x + w_k \), \( w_k, w_n \in \mathbb{R}^d \) and \( C_k \) is a matrix of appropriate dimension. We assume that \( (x, w_0, \ldots, w_n) \) are independent; \( x \sim N(0, \Sigma_x) \) and \( w_i \sim N(0, \Sigma_w), \ i \in N_0 \).

After observing \( (y_0, y_i) \), agent \( i \) generates an estimate \( u_i \in \mathbb{R}^{d_k} \) of \( x \). The estimation error depends on how close the estimates are to the true state \( x \) and how close are the estimates of “neighboring” agents.

To make the notion of neighbors precise, suppose there is an undirected graph \( G \) where nodes are indexed by \( N \). For each node \( i \in N \), let \( L_i \) denote the set of neighbors of \( i \). There are “weight” matrices \( M_{ii} \) associated with each node and “weight” matrices \( M_{ij} \) associated with each edge (Note that \( M_{ij} = M_{ji} \)). It is assumed that all weight matrices are positive definite. Then, the estimation error is measured by

\[ c(x, u_1, \ldots, u_n) = \sum_{i \in N} (x - u_i)^T M_{ii} (x - u_i) \]

\[ + \sum_{i \in N, j \in N_0, i \neq j} (u_i - u_j)^T M_{ij} (u_i - u_j). \]

See Figure 1 for an example.

The above model is a special case of the static team model described in Sec 2.1. In particular, \( \bar{x} = 0 \) and for \( i, j \in N_0 \),

\[ \Theta_i = \text{cov}(x, y_i) = \Sigma_x C_i^T \]

and

\[ \Sigma_{ij} = \text{cov}(y_i, y_j) = \begin{cases} \Sigma_x C_i^T + \Sigma_w & \text{if } i = j \\ \Sigma_x C_i^T \delta_{ij} & \text{if } i \neq j \end{cases} \]

Note that \( \Sigma_{ij} \) can be succinctly written as \( C_i \Sigma_x C_j^T + \Sigma_w \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta function. Then,
Furthermore, the optimal cost is given by

\[ J^\ast = \eta \hat{\Gamma}^{-1} - \text{Tr}(\theta \Sigma_{00}^{-1} \theta_0^T R^{-1} P) + \sum_{i \in \mathcal{N}} \text{Tr}(R_i \Sigma_i), \]

where \( \eta \) and \( \hat{\Gamma} \) are defined as in Theorem 1.

Now, we numerically compute the optimal strategies and optimal performance for some specific cases. We consider a system with \( n = 4 \) nodes for three different cost functions, whose graphs are shown in Fig. 2.

We assume that
- \( d_x = d_y = d_u = 1. \)
- For \( i \in \mathcal{N} \), \( C_i = 1 \), \( \Sigma_x = 1 \), \( \Sigma_u = \sigma^2 \).
- \( \Sigma_0^0 = \sigma_0^2 \).
- For \( i \in \mathcal{N} \), \( M_{ii} = 1 \) and for \( i, j \in \mathcal{N} \), whenever the edge \((i,j)\) exists, \( M_{ij} = 1. \)

Thus, we have two parameters: the variance \( \sigma^2 \) of local observations, and the variance \( \sigma_0^2 \) of the common observations. We will evaluate the performance of the system for different choice of these parameters.

We consider two cases: without common information (i.e., when \( y_0 \) is not available to the nodes) and with common information. For the case without common information, the optimal estimate is

\[ u_i = L_i^0 y_i \quad \text{or} \quad u_i = \hat{F}_i \hat{x}_i \]

while for the case with common information, the optimal estimate is

\[ u_i = L_i y_i + G_i y_0 \quad \text{or} \quad u_i = \hat{F}_i \hat{x}_i + \hat{G}_i \hat{x}_0. \]

The corresponding optimal performances are denoted by \( J^\circ \) and \( J^\ast \).

When \( \sigma^2 = \sigma_0^2 = 1 \), the optimal solution is as follows:

**Cost (a)**
- For the case without common information \( J^\circ = 3 \)
  
  \[ u_i = \frac{1}{4} y_i, \quad \text{or} \quad u_i = \frac{1}{2} \hat{x}_i, \quad i \in \mathcal{N}. \]
- For the case with common information \( J^\ast = \frac{12}{7} \approx 1.7143 \)
  
  \[ u_i = \frac{1}{4} y_i + \frac{3}{4} y_0, \quad \text{or} \quad u_i = \frac{2}{5} \hat{x}_i + \frac{3}{5} \hat{x}_0, \quad i \in \mathcal{N}. \]
- Thus, the value of the common information channel is \( J^\circ - J^\ast = \frac{1}{7} \approx 1.2857. \)

**Cost (b)**
- For the case without common information \( J^\circ = \frac{50}{19} \approx 3.1053 \)
  
  \[ u_i = \frac{4}{19} y_i, \quad \text{or} \quad u_i = \frac{8}{29} \hat{x}_i, \quad i \in \{1, 3\}, \]
  
  \[ u_i = \frac{9}{29} y_i \quad \text{or} \quad u_i = \frac{9}{29} \hat{x}_i, \quad i \in \{2, 4\}. \]
- For the case with common information \( J^\ast = \frac{166}{95} \approx 1.7474 \)
  
  \[ u_i = \frac{11}{95} y_i + \frac{42}{95} y_0 \quad \text{or} \quad u_i = \frac{22}{95} \hat{x}_i + \frac{24}{95} \hat{x}_0, \quad i \in \{1, 3\}, \]
  
  \[ u_i = \frac{13}{95} y_i + \frac{52}{95} y_0 \quad \text{or} \quad u_i = \frac{28}{95} \hat{x}_i + \frac{52}{95} \hat{x}_0, \quad i \in \{2, 4\}. \]
and \( \hat{\gamma} \) of all agents other than \( i \) best response \( g \) strategy (\( g \)).

Thus, the value of the common information channel is \( J^* - J^0 = \frac{2k^2}{95} \approx 1.3579 \).

Cost (c)

- For the case without common information \( J^0 = \frac{16}{9} = 3.2 \) and
  \[ u_i = \frac{1}{2} y_i \quad \text{or} \quad u_i = \frac{1}{2} \hat{x}_i, \quad i \in N. \]
- For the case with common information \( J^* = \frac{16}{9} = 1.7778 \) and
  \[ u_i = \frac{2}{3} y_i + \frac{1}{3} y_j \quad \text{or} \quad u_i = \frac{1}{2} \hat{x}_i + \frac{3}{2} \hat{x}_j, \quad i \in N. \]
- Thus, the value of the common information channel is \( J^* - J^0 = \frac{64}{55} \approx 1.1422 \).

Next, we plot \( J^* \) and \( J^0 \) as a function of \( \sigma^2 \) for different values of \( \sigma^0 \). See Fig. 3 for details.

4. PROOF OF THE MAIN RESULTS

The main idea of the proof is similar to that of Radner (1962). However, instead of working with the observations \( (y_0, y_i) \), we work with the orthogonal random variables \( (\hat{y}_0, \hat{y}_i) \).

Consider agent \( i \in N \) and arbitrarily fix the strategy \( g_{-i} \) of all agents other than \( i \). A necessary condition for the strategy \( g_i \) to be globally optimal is that \( g_i \) is the best response strategy to \( g_{-i} \), i.e., for any \( y_0 \in \mathbb{R}^d, y_i \in \mathbb{R}^n \), and \( u_j \in \mathbb{R}^n \) and \( u_j = g_j(y_0, y_j) \), \( j \in N \), we have that

\[ E^{g_j} [c(x, u_i, u_{-i})|y_0, y_i] \leq E^{g_{-i}} [c(x, u_i, u_{-i})|y_0, y_i]. \]

A sufficient condition for the above to hold is

\[ \frac{\partial}{\partial u_i} E^{g_{-i}} [c(x, u_i, u_{-i})|y_0, y_i] = 0. \]

Assuming that we can interchange differentiation and expectation, we get that

\[
\text{LHS of (14)} = E^{g_{-i}} \left[ \frac{\partial}{\partial u_i} c(x, u_i, u_{-i}) \right] |y_0, y_i] = \sum_{k \in N} \sum_{j \in N} R_{ij} u_k + P_i x |y_0, y_i] = 2E^{g_{-i}} \left[ \sum_{j \in N} R_{ij} u_j + P_i x |y_0, y_i] \right].
\]

Thus, a necessary condition for strategy \( g \) to be optimal is that for all \( i \in N \) and all \( y_k \in \mathbb{R}^d, k \in N \), we have that

\[
\sum_{j \in N} R_{ij} E[u_j|y_0, y_i] + P_i E[x|y_0, y_i] = 0.
\]

Hence, a necessary condition for the strategy described in Theorem 1 to be optimal is that for all \( i \in N \) and all \( y_0 \in \mathbb{R}^d, y_i \in \mathbb{R}^n \),

\[
\sum_{j \in N} R_{ij} E[L_j(y_j - \hat{\gamma}_j) + H_j \hat{x}_0|y_0, y_i] + P_i E[x|y_0, y_i] = 0.
\]

The argument so far is similar to Radner (1962). Now, to verify (16), we exploit the orthogonal projection theorem. Recall that \( \hat{\gamma}_i = E[y_i|y_0] \) and \( (x - \hat{x}_0) \) and \( (y_i - \hat{\gamma}_i) \) are both orthogonal to \( y_0 \). Hence,

\[
E[y_j - \hat{\gamma}_i|y_0, y_i] = E[y_j - \hat{\gamma}_j|y_0] + E[y_j - \hat{\gamma}_i|y_i - \hat{\gamma}_i] = \Sigma_{ji} \Sigma_{ij}^{-1}(y_i - \hat{\gamma}_i).
\]

Thus, \( \sum_{i \in N} \Sigma_{ji} \Sigma_{ij}^{-1} \) and \( \Sigma_{ij}^{-1} \) are both orthogonal to \( y_0 \).

Substituting (17)–(19) in (16), we get that a necessary condition for the strategy described in Theorem 1 to be optimal is that for all \( i \in N \) and all \( y_0 \in \mathbb{R}^d, y_i \in \mathbb{R}^n \),

\[
\left[ \sum_{j \in N} R_{ij} L_j \Sigma_{ji} + P_i \hat{\Theta}_i \right] \Sigma_{ij}^{-1}(y_i - \hat{\gamma}_i) + \sum_{i \in N} R_{ij} H_j + P_i \hat{x}_0 = 0.
\]

For the above equation to hold for all realizations of \( (y_i - \hat{\gamma}_i) \) and \( \hat{x}_0 \), it must be the case that both terms in the square bracket are zero, i.e., for all \( i \in N \),

\[
\sum_{j \in N} R_{ij} L_j \Sigma_{ji} + P_i \hat{\Theta}_i = 0,
\]

and

\[
\sum_{i \in N} R_{ij} H_j + P_i = 0.
\]

These set of equations can be further simplified as follows. To simplify the equations for the gains \( \{H_i\}_{i \in N} \), combine (22) for all \( i \in N \) to get \( RH + P = 0 \), or equivalently, \( H = -R^{-1}P \).

To simplify the equations for the gains \( \{L_i\}_{i \in N} \), vectorize both sides of (21) and use vec\((ABC) = (CT \otimes A) \times vec(B)\) to obtain

\[
\sum_{j \in N} (\Sigma_{ij} \otimes R_{ij}) vec(L_j) + vec(P_i \hat{\Theta}_i) = 0.
\]

Substituting \( \Gamma_{ij} = \Sigma_{ij} \otimes R_{ij} \) and \( \hat{\gamma}_i = vec(P_i \hat{\Theta}_i) \), we get \( \hat{\Gamma}L + \hat{\gamma} = 0 \), or equivalently, \( L = -\hat{\Gamma}^{-1}\hat{\gamma} \).

Thus, we have proved parts 1 and 3 of the Theorem. Part 2 follows from directly substituting the result of part 1. To prove the result of part 4, observe that

\[
J^* = E[E[u^T R u + 2u^T P x|y_0]]] = 0.
\]

Now consider
Substituting (27) in (23), we get
\[ E[(y - \hat{y})^T \Sigma^{-1}(y - \hat{y})] = -\hat{\eta}^T \Sigma^{-1} \hat{\eta} - \hat{\eta}^T P \Sigma^{-1} P \hat{\eta}. \] Substituting (27) in (23), we get
\[ E[u^T R u + 2 u^T P x | y_0] = E[(y - \hat{y})^T \Sigma^{-1}(y - \hat{y})]. \] Substituting (25) and (26) in (24), we get
\[ E[(y - \hat{y})^T L^T P x | y_0] = \sum_{i \in N} \sum_{j \in N} \operatorname{Tr}(L_i \hat{\Sigma}_{ij} L_j^T R_{ij}) + 2 \sum_{i \in N} \operatorname{Tr}(L_i \hat{\Theta}_i P_i^T) \]
\[ = \sum_{i \in N} \operatorname{Tr}(L_i (\sum_{j \in N} \hat{\Sigma}_{ij} L_j^T R_{ij} + 2 \hat{\Theta}_i P_i^T)) \]
\[ = \sum_{i \in N} \operatorname{Tr}(L_i \hat{\Theta}_i P_i^T) = \sum_{i \in N} \vec{(L_i)^T} \vec{(P_i \hat{\Theta}_i)}, \]
\[ = L^T \hat{\eta} = -\hat{\eta}^T \hat{\Gamma}^{-1} \hat{\eta}, \] where (a) uses the following: for any vectors a and b and matrices A and B of appropriate dimensions,
\[ E[a^T B] = E[\operatorname{Tr}(a^T B)] = E[\operatorname{Tr}(a B^T)] = \operatorname{Tr}(AE[a^T B^T]); \]
(b) uses (21); and (c) uses the following: for any matrices A and B of appropriate dimensions
\[ \operatorname{Tr}(AB^T) = \vec{A}^T \vec{B}. \]

The second part of (24) is
\[ \hat{x}_0^T R H \hat{x}_0 + 2 \hat{x}_0^T H^T P x | y_0 \]
\[ = \hat{x}_0^T H^T R \hat{x}_0 + 2 \hat{x}_0^T H^T P x_0 = \hat{x}_0^T H^T (RH + 2P) \hat{x}_0 \]
\[ = \hat{x}_0^T H^T P x = -\hat{x}_0^T P R^{-1} P \hat{x}_0. \]

Substituting (25) and (26) in (24), we get
\[ E[u^T R u + 2 u^T P x | y_0] = \hat{\eta}^T \hat{\Gamma}^{-1} \hat{\eta} - \hat{\eta}^T P R^{-1} P \hat{x}_0. \] Substituting (27) in (23), we get
\[ J^* = -\hat{\eta}^T \hat{\Gamma}^{-1} \hat{\eta} - \hat{x}_0^T P R^{-1} P \hat{x}_0 \]
\[ = -\hat{\eta}^T \hat{\Gamma}^{-1} \hat{\eta} - \hat{x}_0^T P R^{-1} P \hat{x}_0 \]
\[ = -\hat{\eta}^T \hat{\Gamma}^{-1} \hat{\eta} - \hat{x}_0^T P R^{-1} P \hat{x}_0 \]
\[ = \hat{x}_0^T H^T P x - \operatorname{Tr}(\Theta_0 \Sigma_{y_0}^{-1} \Sigma_{y_0}^{-1} (y_0 - \hat{y}_0)). \]

This completes the proof of part 4.

5. CONCLUSION

We investigate static teams with common information and present two structures of optimal strategies. The complexity of the proposed solution methodology is significantly less than naively using the existing results for static teams.

The first structure of optimal strategies can be interpreted as follows. For the given realization of the common information, all agents compute the conditional means and covariances given the common information and compute the gains corresponding to this conditional system. The equations describing the gains depends only on the conditional covariances. Since all the random variables are Gaussian, the conditional covariances do not depend on the realization of the common information and, therefore, neither do the optimal gains.

By a simple algebraic manipulation of the structure of the optimal controller, it can also be viewed as a hierarchical controller where each agent receives a "global correction signal" that it applies to its local control action. Such an implementation is more efficient if the common information is a high-dimensional signal (e.g., video).

The solution methodology developed in this paper could be useful for dynamic team problems as well. We plan to explore that direction in the future.

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