Tensor description of vector field property and its application in classical continuum mechanics

Peng SHI1 *

1 Well Logging Key Laboratory, China National Logging Corporation, No.50 Zhangba five road, Xi’an, 710077, P. R. China

* Corresponding author: sp198911@outlook.com;
Abstract. In classical continuum mechanics the stress tensors are introduced to describe the stress on an arbitrary micro surface at a point like the Kirchhoff stress tensor and Cauchy stress tensor. Inspired by this idea, three second-order tensors, spherical tensor, deviatoric tensor and spin tensor, are introduced to describe the characteristics of any vector field. The study shows that the spherical tensor and spin tensor describe the divergence and curl of the vector field at a point respectively and the direction of the divergence field or curl field at one point can be determined only when the spherical tensor or spin tensor is combined with the deviatoric tensor. On this basis, the study analyzed the shear wave in elastomer and shear flow in fluid. It is concluded that the spin tensor describing local rigid body rotation in elastic theory and fluid dynamics contributes stress, which is different from the traditional understanding and shear stress reciprocity is unnecessary in classical continuum mechanics.

Keywords: Classical continuum mechanics; Vector field; Shear stress reciprocity; Elastic wave; Viscous flow
1. **INTRODUCTION**

Classical mechanics deals with the motion of an object under various forces in the absolute concept of time-space [1-3]. To describe the kinematic and dynamic characteristics of different kind of objects under the action of external forces, various branches of classical mechanics have been proposed and developed, such as particle mechanics, rigid body mechanics and continuum mechanics [4-8]. With the efforts of mechanics masters, classical mechanics has been widely applied in the field of engineering technology [9, 10]. Due to the success in describing the motion of an object with particle model, the community of mechanics generally believes that the basic laws of classical mechanics are Newton's three laws of motion or other mechanical principles related to and equivalent to them [1-11]. The study shows that the motion of a continuum in the domain of continuum mechanics may not satisfy Newton's three laws of motion.

Continuum mechanics, as an important branch of classical mechanics, concerns with the stresses in solids and fluids and their deformation or flow [7, 8]. In continuum mechanics, the macroscopic medium is considered to satisfy the continuum hypothesis, where the structures of real fluid and solid are considered to be perfectly continuous and are paid no attention to their molecular structure. In order to describe the distribution of displacement field and stress field in continuum and the relationship between stress field and displacement field, the tensors like stress tensor, strain (or strain rate) tensor and spin (or spin rate) tensor are introduced. The nonzero-volume elements constituting a continuous medium are treated as particles and their dynamics is described with Newton’s second law under the initial configuration [7, 8, 10]. On this basis, the classical continuum mechanics is established and the stress tensors like the Kirchhoff press tensor and Cauchy
stress tensor are introduced to describe the stresses at a point in initial configuration and current configuration, respectively [7, 8]. Under the assumption of particle motion of material element, the stress tensor is proved to be a symmetric tensor [7, 8, 12]. For the sake of ensuring the symmetry of stress tensor, the strain and strain rate tensors, considered to contribute the stress and described with the gradient of displacement and velocity respectively, are also second-order symmetric tensors [12, 13].

In the classical theory of continuum mechanics, there is a fatal conflict that the rotation of a material element is admitted when the deformation of a continuous material is described, but the rotation of material element is ignored when its motion is described [7, 8, 12, 13]. This conflict makes it difficult to explain why the displacement in wave equations and the velocity in Navier-Stokes equations are curl, since the displacement and velocity caused by strain and strain rate respectively should be curl free. In my opinion, the problems in classical continuum mechanics stem from the inapplicability of Newton's three laws of motion. In recent decades, some new continuum mechanics models have been proposed and the rotation of material elements has been declared [14, 15]. No researcher declares that these models violate Newton's second law of motion. This study tries to show that the stress tensors can be asymmetric in classical continuum mechanics and the definition of shear wave and shear flow goes beyond the particle motion description of material element.

2. STATE TENSOR DESCRIBING VECTOR FIELD

Due to the complexity of the relationship between stress field and displacement field in continuum, the stress tensor, strain (or strain rate) tensor and rotation (or rotation rate) tensor are introduced in continuum mechanics. Inspired by this idea, three second-order
tensors, spherical tensor, deviatoric tensor and spin tensor, are introduced to describe the characteristics of any vector field.

Assuming that $A$ represents an arbitrary vector field, two vectors infinitesimal close in distance satisfy the following relationship under linear expansion [12, 13]:

$$A(R + \delta R) = A(R) + (\alpha + \alpha' + \chi) \cdot \delta R ,$$

with

$$\alpha = \frac{1}{6} tr(\nabla A + A \nabla)I ,$$

$$\alpha' = \frac{1}{2} (\nabla A + A \nabla) - \alpha ,$$

$$\chi = \frac{1}{2} (\nabla A - A \nabla),$$

where, $\alpha$ is a spherical tensor, $\alpha'$ is a deviatoric tensor, $\chi$ is a spin tensor, $\nabla$ is the vector operator del, $R$ is the radius vector and $I$ represents unit tensor. With Equations (2) to (4), the divergence and curl of $A$ can be expressed as:

$$\nabla \cdot A = tr(\alpha + \alpha' + \chi) = tr(\alpha) ,$$

$$\nabla \times A = \varepsilon : (\alpha + \alpha' + \chi) = \varepsilon : \chi ,$$

here, $\varepsilon$ is the permutation symbol. It is obtained from Equations (5) to (6) that the divergence and curl of a vector field are included in spherical tensor and spin tensor, respectively. The deviatoric tensor contributes neither divergence nor curl, but the direction of vector field at one point is determined only when the spherical tensor and spin tensor are combined with the deviatoric tensor.

By taking the divergence of a gradient, the Laplace operator is obtained. The Laplacian of a vector field is another vector field and is expressed as:
\[ \nabla^2 A = \nabla \nabla \cdot A - \nabla \nabla \times A. \]  \tag{7}

By submitting Equations (5) and (6) into Equation (7), the Laplacian of the vector field \( A \) can be rewritten as:

\[ \nabla^2 A = \nabla \left( \text{tr} (\alpha) \right) - \nabla \times (\varepsilon : \chi). \]  \tag{8}

If a vector field is the electric field or magnetic field when the electromagnetic wave propagates in vacuum, the electric field or magnetic field is curl field [16]. The conversion between electric field and magnetic field can be described with state tensors as:

\[ \nabla \times (\varepsilon : \chi_E) = -\varepsilon : \chi_B, \]  \tag{9}
\[ \nabla \times (\varepsilon : \chi_H) = -\varepsilon : \chi_D. \]  \tag{10}

here, \( \chi_E, \chi_B, \chi_H \) and \( \chi_D \) the are spin tensors of electric field intensity \( E \), magnetic induction intensity \( B \), magnetic field intensity \( H \) and electric displacement vector \( D \), respectively. Equations (9) and (10) show that the transverse wave arises from the conversion between two curl fields in time and space domains. Below, it will be shown that the elastic theory denies that the propagation of shear wave is the conversion of curl strain field and curl acceleration field.

3. ELASTIC WAVE EQUATION DERIVATION FROM TRADITIONAL MOTION DESCRIPTION OF CONTINUUM

In classical continuum mechanics, a continuum is regarded as a set of particles. By treating a material element as particle and only considering its translation, the motion equation of material element in differential form is expressed in the following form [12]:

\[ \rho \frac{Dv}{Dt} - \nabla \cdot \sigma^s - f = 0, \]  \tag{11}
where $D/Dt$ is the material derivative, $v$ is the velocity of material element translation, $\rho$ is the mass density, $\sigma^S$ is the symmetric second-order stress tensor, $f$ is the body force which is a non-curl field. It is known from above analysis that the symmetric second-order stress tensor is the change intensity of a curl free vector field $\Sigma^S$ at one point and can be expressed by the gradient of the vector field $\Sigma^S$:

$$
\sigma^S = \nabla \Sigma^S.
$$

Then the stress $F$ can be expressed as:

$$
F = \sigma^S \cdot n = \frac{\Sigma^S (R + \delta R) - \Sigma^S (R)}{|\delta R|}.
$$

with $n$ the unit vector of outer normal of surface element. Due to the vector field $\Sigma^S$ is a curl free field, the stress field $F$ is also a curl free field.

Submitting Equation (12) into Equation (11), the motion equation of material element forming a continuum can be rewritten as:

$$
\rho \frac{Dv}{Dt} - \nabla \cdot \sigma^S - f = 0.
$$

It is obtained from Equation (14) that the translation of material element is determined by the normal stress and deviatoric stress does not contribute the translation of material element, which is different from the traditional understanding [8, 12, 13].

For elastomer with small deformation, the element translation is expressed in differential form as [12]:

$$
\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \sigma^S - f = 0,
$$

here, $\partial/\partial t$ is the time derivative, $u$ is the displacement. Its constitutive relations and strain-displacement relations in component form are expressed as [12, 17]:
\[ \sigma_{ij}^s = C_{ijkl} e_{kl}^s, \quad (16) \]
\[ e_{kl}^s = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad (17) \]
\[ C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad (18) \]

where, \( e_{kl}^s \) is strain tensor, \( C_{ijkl} \) is elastic tensor, \( \delta_{ij} \) is the Kronecker delta. \( \lambda \) and \( \mu \) are the Lamé constants. When the displacement field is only caused by deformation, the displacement field is a curl free field.

Substituting the strain-displacement relations (Equation (17)) into the constitutive relations (Equation (16)) and subsequently substituting the constitutive relations expressed with displacement in the equation of motion (Equation (15)), the displacement equation of motion can be obtained in vector notation [12, 17]:

\[ \mu \nabla^2 u + (\lambda + \mu) \nabla \nabla \cdot u + f = \rho \frac{\partial^2 u}{\partial t^2}, \quad (19) \]

which is also called the Navier equation. In the traditional derivation of elastic wave equation, the following formula is considered to be true [8, 12]:

\[ \nabla^2 u = \nabla (\nabla \cdot u) - \nabla \times \nabla \times u. \quad (20) \]

Then the Navier equation is rewritten as:

\[ (\lambda + 2\mu) \nabla \nabla \cdot u - \mu \nabla \times \nabla \times u + f = \rho \frac{\partial^2 u}{\partial t^2}. \quad (21) \]

From the analysis in above section, we know that the stress field and the displacement caused by deformation are curl free fields. Therefore Equation (20) is not true and only longitudinal wave can be derived based the traditional motion description. Since shear waves exist objectively in elastic media, the motion of material element forming a
continuum should be beyond the particle motion description and cannot be described fully by Newton's second law of motion.

4. ELASTIC WAVE EQUATION DERIVATION FROM NEW MOTION DESCRIPTION OF CONTINUUM

Though the existence of shear wave is objective in real solid media, it is hard to say whether the shear wave can be correctly described by Equation (21). Here it is assumed that Equation (21) can objectively describe the propagation of shear wave in elastic media, and the wave equation including both longitudinal wave and shear wave is derived from a new motion equation.

To theoretically obtain the elastic wave equation, a new motion equation needs to be proposed which can generally describe the motion of material elements in classical continuum mechanics domain and the constitutive relations and strain-displacement relationships of elastomer should be modified correspondingly. The study believes that the motion equation of material element forming a classical continuum can be expressed as:

\[
\rho \frac{Dv}{Dt} - \nabla \Sigma - f = 0,
\]

here, \(\Sigma\) is an arbitrary vector field. In this case, the stress tensor is an asymmetric tensor and the symmetric part and antisymmetric part of stress tensor are expressed respectively as:

\[
\sigma^s = \frac{1}{2}(\nabla \Sigma + \Sigma \nabla),
\]

\[
\sigma^t = \frac{1}{2}(\nabla \Sigma - \Sigma \nabla).
\]

(22)
Replacing the vector field $\Sigma$ with stress tensor, the motion equation of material element is rewritten as:

$$\nabla \left( \text{tr} \left( \sigma^s \right) \right) - \nabla \times \left( \varepsilon : \sigma^d \right) + f = \rho \frac{Dv}{Dt}.$$  \hspace{1cm} (25)

It is seen that under the new motion description of material element shear stress reciprocity is unnecessary in classical continuum mechanics. When stress field is curl, the stress tensor is asymmetric.

For elastomer with small deformation, the convective acceleration can be ignored and the motion equation is written as:

$$\nabla \left( \text{tr} \left( \sigma^s \right) \right) - \nabla \times \left( \varepsilon : \sigma^d \right) + f = \rho \frac{\delta^3}{v^2}.$$  \hspace{1cm} (26)

Under the new motion description, the constitutive relations need to be modified appropriately to have the deformation of an elastomer produce an asymmetrical strain tensor. The study believes that the constitutive relations and strain-displacement relationships of elastomer should be expressed as follows:

$$\sigma = C : e ,$$

$$e = \nabla u ,$$

here, $e$ is the strain including the traditional defined strain and local rigid body rotation. For isotropic elastomers, the elastic tensor should be expressed as:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl}.$$  \hspace{1cm} (29)

It is seen from Equations (27) to (28) that the stress tensor is asymmetric when the displacement field is rotational and is symmetric when the displacement field is irrotational. This means that Equations (27) to (28) contain the case in the classical theory of elasticity. In fact, the traditional expression of the elastic tensor is redundant since that the local rigid
body rotation does not contribute to the stress has been declared in the classical theory of elasticity.

Submitting Equation (27) into Equation (26), the motion equation of elastomer is expressed as:

$$
(\lambda + \mu) \nabla (tr(e^s)) - \mu \nabla \times (e^s : e^s) + f = \rho \frac{\partial^2 u}{\partial t^2},
$$

with

$$
e^s = \frac{1}{2} (\nabla u + u \nabla),
$$

$$
e^A = \frac{1}{2} (\nabla u - u \nabla),
$$

where, $e^s$ and $e^A$ are the strain tensor and rotation tensor. Via the relationship between vector field and its state tensor (Equations (7) and (8)), Equation (30) is rewritten as:

$$
(\lambda + \mu) \nabla \nabla \cdot u - \mu \nabla \times \nabla \times u + f = \rho \frac{\partial^2 u}{\partial t^2}.
$$

Equation (33) can both predict the existence of longitudinal wave and shear wave in elastomer.

In the classical theory of elasticity, though the local rigid body rotation is admitted, the traditionally defined deformation is only considered for the deformation coordination, which means that the material elements composing an elastomer can rotate freely. This is inconsistent with the facts. In admitting the rotation of material element, the study considers that the deformation coordination of an elastomer should be described as follows [18]:

$$
\nabla \times e = 0.
$$
5. DERIVATION OF NAVIER-STOKES EQUATION FROM NEW MOTION EQUATION

To theoretically derive the elastic wave equation, a new motion equation of material element forming continuum is proposed in last section. Here we verify its generality in describing the motion of material element by deriving the Navier-Stokes equation.

For viscous fluids, the stress comes from volume deformation and shear flow. When the shear flow occurs in viscous fluid, the velocity field is a curl field. Therefore, the stress field in the viscous fluid is a curl field and the stress tensor is an asymmetric tensor. Separating the stress caused by volume deformation from the stress caused by shear flow, the motion equation of material element is expressed as:

\[
\nabla (tr(\sigma)) - \nabla \times (\varepsilon \cdot d) + f = \rho \frac{Dv}{Dt}, \tag{35}
\]

with

\[
d = \sigma - \frac{1}{3} tr(\sigma) I, \tag{36}
\]

here, \(d\) is a deviatoric stress. For a Newtonian fluid, the relations between deviatoric stress and rate of deviatoric strain \(\xi\) in component form is expressed as:

\[
d_{ij} = \eta \delta_{ik} \delta_{jk} \xi_{kj}, \tag{37}
\]

where, \(\eta\) is the viscosity of fluid. The relations between rate of deviatoric strain \(\xi\) and velocity \(v\) are

\[
\xi = \nabla v - \frac{1}{3} tr(\nabla v) I. \tag{38}
\]

Submitting Equation (38) into Equation (37) and consequently submitting the relations between deviatoric stress and rate of deviatoric strain expressed with velocity into Equation
(35), the motion equation can be expressed with velocity as:

$$-\nabla p - \eta \nabla \times (\varepsilon : \nabla \nu) + f = \rho \frac{D\nu}{Dt}. \quad (39)$$

Since the following relations are hold:

$$\nabla \times (\varepsilon : \nabla \nu) = \nabla \times \nabla \times \nu = -\nabla^2 v + \nabla \nabla \cdot \nu, \quad (40)$$

Equation (39) can be rewritten as:

$$-\nabla p + \eta \nabla^2 v - \nabla \nabla \cdot \nu + f = \rho \frac{D\nu}{Dt}. \quad (41)$$

For further special case of incompressible fluid, the mass conservation reduces to $\nabla \cdot \nu = 0$ and Equation (41) reduces to:

$$-\nabla p + \eta \nabla^2 v + f = \rho \frac{D\nu}{Dt}. \quad (42)$$

Equation (42) is the same as the Navier-Stokes equation of motion for incompressible fluid. This means that the new motion equation proposed in this study is also suitable for describing the motion of material elements forming fluids.

6. DISCUSSION AND CONCLUSIONS

The study introduced three second-order tensors, spherical tensor, deviatoric tensor and spin tensor to describe the characteristics of any vector field. The divergence of vector field is described by the spherical tensor and the curl of vector field is described by spin tensor. The deviatoric tensor contributes neither divergence nor curl, but the direction of vector field at one point is determined only when the spherical tensor and spin tensor are combined with the deviatoric tensor. On this basis, it is concluded that the symmetry of stress tensor indicates that the stress field in continuum is a curl free field and the
longitudinal wave can only be derived from the traditional motion equation.

In order to obtain the wave equation that can describe both longitudinal wave and shear wave in elastomer, the study proposed a new motion equation. The new motion equation shows that the reciprocity of stress tensor is unnecessary in continuum mechanics and the stress field is a curl field when stress tensor is asymmetric. In the derivation, the constitutive relations, strain-displacement relations and deformation coordination are correspondingly modified. The new constitutive relations indicate that the local rigid body rotation contributes stress and a curl stress field related to local rigid body rotation is produced to balance the curl acceleration field when the shear wave propagates in elastomer. The study also verified the generality of the new motion equation in describing the motion of material element by deriving the Navier-Stokes equation. The result shows that the Navier-Stokes equation can be derived from the new motion equation when the relations between deviatoric stress and rate of deviatoric strain is correspondingly modified to fit the new stress tensor. It is obtained from the derivation of the Navier-Stokes equation that the viscous force field related to shear flow is a curl field and can balance the curl free force field like pressure field.

It should be pointed out that the new motion equation has made the motion of material elements forming classical continua beyond the description of particle motion. At present only two motion model, particle model and rigid body model, are proposed in classical mechanics and the latter is considered to be the collection of the motion of the former. Only from the derivation of the wave equation, the shear wave can be derived from the conservation of momentum moment by adding rotational degrees of freedom to the material element. In this case, the longitudinal and shear waves in elastomer correspond to
the translation and rotation of material element, respectively. However, if we believe that
the motion of classical continua can be described by the same equation of motion, it is not
advisable to regard the translation and rotation of material elements as independent because
the coupling between the curl field and curl free field in fluid cannot be reasonably
explained. This implies that the motion and force of continuum is beyond the current
cognition.

DECLARATION OF COMPETING INTEREST

The author declares that he has no known competing financial interests or personal
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REFERENCES

[1] Goldstein, H. Poole, C., Safko, J., 2002. Classical mechanics. Am. J. Phys. 70, 782-783.
[2] Cohen, M., 2012. Classical Mechanics: A Critical Introduction. Hindawi Publishing
Corporation, New York, USA.
[3] Feynman, R.P., Leighton, R. B., Sands, M., 1963. The Feynman Lectures on Physics –
Volume 1. Addison-Wesley Publishing Company Inc., Massachusetts, USA.
[4] Batterman, R.W., 2013. The Oxford Handbook of Philosophy of Physics. Oxford Univ.
Press, New York, USA.

[5] Fasano, A. Marmi, S., Pelloni, B., 2006. Analytical mechanics: an introduction. Oxford Univ. Press, New York, USA.

[6] Becker, R.A. 1954. Introduction to theoretical mechanics. McGraw-Hill, New York, USA.

[7] Lai, W.M., Rubin, D.H., Krempl, E., 2009. Introduction to continuum mechanics. Butterworth-Heinemann, Oxford England.

[8] Malvern, L.E., 1969. Introduction to the Mechanics of a Continuous Medium, Prentice Hall, New Jersey, USA.

[9] Truesdell, C., 1976. History of classical mechanics. Naturwissenschaften 63, 53-62.

[10] Timoshenko, S.P., 1983. History of Strength of Materials. Dover Publications, New York, USA.

[11] Newton, I., 1687. Philosophiae Naturalis Principia Mathematica.

[12] Achenbach, J. D., 1973. Wave propagation in elastic solids. Elsevier, Amsterdam, Netherlands.

[13] Batchelor, G.K., 2000. An introduction to fluid dynamics, Cambridge Univ. press, Cambridge, England.

[14] Mindlin, R.D., Tiersten, H.F., 1962, Effects of couple-stresses in linear elasticity. Arch. Ration. Mech. An. 11, 415-448.

[15] Scheibner, C., Souslov, A., Banerjee, D., Surówka, P., Irvine, W. T., Vitelli, V., 2020. Odd elasticity. Nat. Phys. 16, 475–480.

[16] Jackson, J.D., 1999. Classical electrodynamics, Wiley, New York.

[17] Graff, K.F., 1975. Wave Motion in Elastic Solids, Dover publications, New York, USA.
[18] Qiu, Z., 2020. A simple theory of asymmetric linear elasticity. *World J. Mech.* 10, 166-185.