Access to improve the muon mass and magnetic moment anomaly via the bound-muon $g$ factor

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A theoretical description of the $g$ factor of a muon bound in a nuclear potential is presented. One-loop self-energy and multi-loop vacuum polarization corrections are calculated, taking into account the interaction with the binding potential exactly. Nuclear effects on the bound-muon $g$ factor are also evaluated. We put forward the measurement of the bound-muon $g$ factor via the continuous Stern-Gerlach effect as an independent means to determine the free muons magnetic moment anomaly and mass. The scheme presented enables to increase the accuracy of the mass by more than an order of magnitude.

The physics of muons features puzzling discrepancies. The disagreement of the free muons experimental and theoretical $g$ factor by $3\sigma$ represents the largest deviation from the Standard Model observed in an electroweak quantity \( \Gamma \). Recently, high-precision spectroscopy experiments with the muonic H atom yielded a value for the proton radius which strongly disagrees with that obtained from measurements on regular H \( 2 \) (see also \( 3 \)). Therefore, experiments aiming at improved determinations of the muons properties help to clarify these issues and can be a hint for New Physics.

The fast progress in the theoretical understanding and experimental precision of the bound-electron $g$ factor (see e.g. \( 6 \) and references therein) has also enabled the most accurate determination of the mass of the electron in Penning trap $g$-factor experiments by means of the continuous Stern-Gerlach effect \( 3, 14, 17 \). In this Letter we put forward a similar method for the extraction of the mass of the muon by employing light muonic ions, by which we mean here bound systems solely consisting of a nucleus and a muon without further surrounding electrons. Since currently the mass of the muon is only known from muonium spectroscopy \( 18 \) and to a fractional standard uncertainty of \( 2.2 \times 10^{-8} \) \( 19, 20 \), alternative methods for its determination are especially desirable.

When a muonic ion is subjected to a magnetic field of strength $B$, the Larmor frequency between the bound-muon Zeeman sublevels depends on the magnetic moment $\mu$ of the muon by the formula

$$\omega_L = \frac{2\mu}{\hbar} B = \frac{g}{2 m_\mu} B,$$

with $e$ being the (positive) unit charge, and $g$ and $m_\mu$ the bound muons $g$ factor and mass, respectively. Determining the magnetic field at the location of the ion becomes possible through a measurement of the cyclotron frequency of the ion,

$$\omega_c = \frac{Q}{M} B,$$

where $Q$ and $M$ are the charge and mass of the muonic ion, respectively. Thus, $m_\mu$ can be expressed by $M$ as

$$m_\mu = \frac{g e^{\omega_L}}{2 Q \omega_c} M,$$

where the theoretical value $g_{\text{theo}}$ for the bound-muon $g$ factor is to be substituted. The quantity to be measured is the ratio of the two frequencies, $\Gamma = \omega_L/\omega_c$. For determining $m_\mu$ with a given fractional uncertainty, all the quantities $g_{\text{theo}}, \Gamma$ and $M$ have to be known at the same level of accuracy. Alternatively, Eq. \( 1 \) and \( 2 \) can be combined to yield an experimental bound-muon $g$ factor

$$g = 2 \frac{m_\mu Q}{M} \frac{\Gamma}{e}.$$  

Such a determination of $g = 2 + 2a_\mu + \Delta g_{\text{bind}}$ constitutes an alternative access to the free muons magnetic moment anomaly $a_\mu$ at a level at which $m_\mu$ is known from an independent experiment, and provided the binding contribution $\Delta g_{\text{bind}}$ can be calculated to sufficient accuracy.

In the electron mass experiments \( 6, 15, 16 \), $^{12}\text{C}^5+$ ions were employed because the atomic mass unit is defined in terms of the mass of the $^{12}\text{C}$ atom. In determining the muon mass in a similar fashion, a lighter element, namely $^4\text{He}$ is more appropriate to minimize uncertainties due to nuclear effects. In addition, the mass of $^4\text{He}$ is known to sufficient accuracy. Therefore, in the following, we present the theory of the $g$ factor of muonic $^4\text{He}^+$ and show that a 9-digit fractional accuracy is achievable, which corresponds to the same accuracy in the extracted muon mass or magnetic moment anomaly, provided the ratio of the Larmor and cyclotron frequencies can be measured with matched precision.

**Theoretical approach** – The Dirac value $g_D$ corresponds to the leading tree-level Feynman diagram with the assumption of a pointlike nucleus. It was first calculated by Breit in 1928 \( 21 \). For a Dirac particle in the $1s$ state of an ion with a charge number $Z$ it is
constant. Various effects shift the bound-muon $g$ factor: the electric and (b) magnetic loop vacuum polarization corrections, and (c) the self-energy wave function and (d) self-energy vertex correction terms. A double external line represents a muonic Coulomb-Dirac wave function, and the wave line terminated by a triangle stands for the interaction with the external magnetic field. The internal wave line represents a photon propagator, and the internal double line depicts a Coulomb-Dirac muon propagator; in vacuum polarization loops, it may also represent an electron-positron propagator.

$g_D = \frac{\alpha}{\pi} + \frac{1}{4\pi} \sqrt{1 - (Z\alpha)^2}$, where $\alpha$ is the fine-structure constant. Various effects shift the bound-muon $g$ factor from this value: Firstly, due to the finite size of the nucleus, the interaction potential between the muon and the nucleus deviates from a pure Coulomb potential on the fm scale. Therefore, the wave function of the bound muon and hence its $g$ factor deviate from the corresponding quantities computed for a pure Coulomb potential. This finite size (FS) correction to the bound-muon $g$ factor can be expressed with the nuclear root-mean-square radius $r_{\text{RMS}}$ by the approximate formula \[ \Delta g_{\text{FS}} = \frac{2}{3} \left( Z\alpha \right)^2 r_{\text{RMS}}^2 + O\left( (Z\alpha)^6 \right), \] in agreement with Ref. [23]. As one can see on this formula, the FS correction for bound muons is more than 4 orders of magnitude larger than for bound electrons. The accuracy of $\Delta g_{\text{FS}}$ is mostly limited by the uncertainty of $r_{\text{RMS}}$. The correction due to the deformation of the nuclear charge distribution was estimated using the method described in Ref. [22] and nuclear data from Ref. [24], and was taken to be less than $10^{-14}$. We also assume a negligibly small magnitude for the nuclear polarization correction Ref. [25], [26].

The leading quantum electrodynamic (QED) corrections correspond to the one-loop Feynman diagrams shown on Fig. 1. These diagrams represent the electric and magnetic loop vacuum polarization (VP) corrections [Fig. 1(a) and (b), respectively] and the self-energy (SE) wave function and vertex corrections [Fig. 1(c) and (d), respectively]. As in free-particle QED, these loop diagrams are ultraviolet (UV) divergent. The renormalization procedure used to cancel the divergences is based on the expansion of the internal fermion lines in each diagram in powers of interactions with the nuclear potential. We apply the two-time Green’s function method Ref. [27] for obtaining expressions for the individual terms.

The fermion loop in the VP electric loop diagram modifies the nuclear potential at distances on the scale of the Compton wavelength of the loop particle. In a good approximation, this can be reduced to a free fermion loop with one interaction with the nuclear field, leading to the Uehling potential $V_{\text{Ueh}}(r)$ Ref. [28]. The effect of the Uehling term was evaluated in different ways. First, the $g$ factor contribution of the first-order Uehling diagram can be calculated as

$$\Delta g_{\text{Ueh}} = -\frac{8m_{\mu}}{3} \langle a | V_{\text{Ueh}} | \delta a \rangle, \quad (5)$$

where $|a\rangle$ is the bound-muon Dirac wave function and $|\delta a\rangle$ is the wave function linearly perturbed by the magnetic interaction. For a point-like nucleus, $|\delta a\rangle$ is known analytically Ref. [29]. Since the Uehling potential does not depend on the mass of the bound particle, but only on the mass of the particle in the loop, the Uehling term can also be computed as

$$\Delta g_{\text{Ueh}} = -\frac{4}{3m_{\mu}} \langle a | \partial_r V_{\text{Ueh}} \partial_r | a \rangle, \quad (6)$$

according to the method described in Ref. [30]. In both cases, the $g$-factor contribution was obtained by numerical integration, yielding an excellent numerical agreement between the two methods. In the pointlike nuclear model, the results were also compared to the exact analytical formula Ref. [31]. We note that $Z\alpha$ expansion results derived for electronic atoms can not be straightforwardly applied to the case of muonic atoms, since they assume the loop particle to be identical to the bound particle. Furthermore, electronic VP effects would be largely overestimated by $Z\alpha$ expansion formulas, thus they need to be calculated to all orders in this parameter even at low $Z$.

The higher-order term of the electric loop VP diagram, the Wichmann-Kroll contribution, was calculated with the method of Ref. [32] and was found to be negligible. Hadronic VP corrections were estimated from the muonic Uehling term, following Ref. [33], as $\Delta g_{\text{VP}}^{\text{had}} = 0.671(15)\Delta g_{Ueh}^{\mu}$. The contribution of the Uehling potential was also evaluated in an all-order treatment by including it in the radial Dirac equation, and calculating the bound-muon wave function numerically in a B-spline representation Ref. [34], as described in Ref. [35]. This allows the extraction of the 2nd-order Uehling corrections, shown in Fig. 2 (a). Finally, the Källén-Sabry two-loop VP correction Ref. [36], illustrated in Fig. 2 (b), was evaluated employing B-splines, and the effective potential given in Ref. [37], [38].

The lowest-order term in the expansion of the magnetic loop VP diagram [Fig. 1(b)] corresponds to the diagram with the Coulomb-Dirac propagator replaced by the free
Dirac propagator. This diagram is UV divergent, and its $g$-factor contribution is canceled by charge renormalization [3]. Higher-order contributions to this diagram, such as the virtual light-by-light scattering (LBL) term, are finite. We evaluate the LBL term as it was performed in Ref. [39, 40], with the difference that we include the finite nuclear size effect in the bound muon wave function. Also, we calculate the mixed magnetic and electron loop effect by repeating the above calculation with the inclusion of the effect of the Uehling potential in the bound-muon wave function. The corresponding two-loop contribution is slightly below the uncertainty at which we aim. We note that further two-loop VP corrections evaluated very recently for electronic ions [13, 41] may also contribute, and their calculation can be extended to the case of muons in a straightforward manner.

In the calculation of the SE wave function correction [Fig. 1(c)], the muon propagator between the magnetic interaction and the SE loop can be expressed as a spectral sum over all eigenfunctions $|n\rangle$ of the Coulomb-Dirac Hamiltonian as

$$\sum_n \frac{|n\rangle \langle n|}{E_n - E_n + \text{sgn}(E_n)\epsilon_0},$$

with the $E_n$ being the eigenenergies of the $|n\rangle$ and $E_a$ being the eigenenergy of the reference state $|a\rangle$. The diagram needs to be split into the irreducible ($E_n \neq E_a$) and the reducible ($E_n = E_a$) part. The $g$-factor correction of the irreducible part can be expressed using the SE operator $\Sigma$ as

$$\Delta g^{[0]}_{\text{SE, irred}} = -\frac{8m_e}{3}\langle \delta a | \gamma^0 \Sigma | a \rangle.$$

Here, $\gamma^0$ is the time-like Dirac matrix. The irreducible part can be separated into the zero-potential contribution (free internal muon line), the one-potential contribution (free internal muon line with one interaction with the nuclear potential) and the many-potential contribution (two and more interactions with the nuclear potential). While the zero-potential and one-potential contributions are UV divergent, the many-potential contribution is finite. The zero-potential contribution can be written as

$$\Delta g^{[0]}_{\text{SE, irred, ren}} = -\frac{8m_e}{3}\langle \delta a | \gamma^0 \Sigma | a \rangle.$$

Here, $\Sigma_2(p)$ is the momentum-space SE function of the free muon and using dimensional regularization in $d = 4 - \epsilon$ dimensions, it can be expressed as [42]

$$\Sigma_2(p) = \frac{\delta m}{4\pi} \Delta_s(p - m_\mu) + \Sigma_R(p),$$

with $\delta m = \frac{3\alpha\mu_\mu}{4\pi} (\Delta_s + \frac{\Delta_\gamma}{2})$ and $\Delta_s = \frac{\alpha}{\pi} - \gamma_E - \ln m_\mu^2 + \ln 4\pi$, where $\gamma_E = 0.57721\ldots$ is Euler’s constant. The $\delta m$ term is cancelled by mass renormalization, and the $\Delta_s$ term will be cancelled by a similar term in the one-potential contribution. The renormalized zero-potential contribution is defined as

$$\Delta g^{[0]}_{\text{SE, irred, ren}} = -\frac{8m_e}{3}\langle \delta a | \gamma^0 \Sigma \Delta_R | a \rangle,$$

while the one-potential contribution is

$$\Delta g^{[1]}_{\text{SE, irred, ren}} = -\frac{8m_e}{3}\langle \delta a | \gamma^0 \Gamma^\nu_2 V | a \rangle,$$

with $V$ being the interaction potential of the nucleus. $\Gamma^\nu_2(p', p)$ ($\nu \in \{0, 1, 2, 3\}$) is the vertex function for free fermions and can be separated into a divergent and a regular part as [42]

$$\Gamma^\nu_2(p', p) = \frac{\alpha}{4\pi} \Delta_\gamma^\nu + \Gamma^\nu_2(p', p).$$

The $\Delta_s$ term in the one-potential contribution cancels the corresponding $\Delta_s$ term in the zero-potential contribution. For details of the renormalization procedure and for expressions of $\Sigma_R(p)$ and $\Gamma^\nu_2(p', p)$ see Ref. [42]. The renormalized one-potential term is then defined as

$$\Delta g^{[1]}_{\text{SE, irred, ren}} = -\frac{8m_e}{3}\langle \delta a | \gamma^0 \Gamma^\nu_2 V | a \rangle.$$

The many-potential contribution was evaluated using methods described in Ref. [42, 13]. It is straightforward to generalize the calculation of the Lamb-shift diagram to the many-potential contribution of the $g$-factor SE diagram. The integration over the virtual photon frequency required in the many-potential contribution was split into a low-energy and a high-energy part. The partial-wave expansion of the low-energy part converges rapidly and does not require any extrapolation. The high-energy term converges slower. The series was computed up to Dirac angular momentum quantum numbers $|\kappa| \approx 40$, and the remainder of the series was estimated using the Richardson extrapolation method [44].
The $g$-factor contribution of the reducible SE diagram is calculated from the energy derivative of the Lamb shift matrix element:

$$
\Delta g_{\text{red}} = g_D \frac{\partial}{\partial E} \langle a | \gamma^0 \Sigma(E) | a \rangle \bigg|_{E=E_a}.
$$

(15)

It can be again split into the zero- and many-potential contributions. While the zero-potential contribution is UV divergent, the one-potential part is finite and can therefore be included in the many-potential term.

The SE vertex correction [Fig. 1(b)] can be expressed as $\Delta E_{\text{ver}} = -e(a)\gamma_0 \mathbf{T} \cdot \mathbf{A}(a)$, with $\mathbf{T}$ being the 3-vector component of the vertex function. This expression can be split into the zero- and many-potential contributions. The zero-potential term is UV divergent, but the many-potential part does not contain UV divergences. The UV-divergent terms in the zero-potential contributions of the vertex diagram and the reducible SE diagram cancel each other. The renormalized zero-potential contribution can be calculated in momentum space, using the magnetic vector potential $\mathbf{A}(\mathbf{p}') - \mathbf{p} = -\frac{1}{2}(2\pi)^3 \mathbf{B} \times \nabla_a \delta(\mathbf{p}' - \mathbf{p})$, and can be expressed as

$$
\Delta g_{\text{ver}}^{[0]} = -2im \int \frac{d^3 p}{(2\pi)^3} \int d^3 p' \nabla_p \delta(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{p}' \times \mathbf{T}(\mathbf{p})) a(\mathbf{p}'),
$$

(16)

with $a(\mathbf{p})$ being the muon wave function in momentum space. This can be further evaluated using integration by parts. For further details see Ref. [10]. We note that our numerical results for the SE terms agree well with $\alpha$ expansion formulas [11, 49, 51].

The calculations so far have been performed in the Furry picture [62], i.e. using a static external field to describe the nucleus. The nuclear recoil contribution is the correction to the $g$ factor due to the finiteness of the nuclear mass. Formulas derived for bound electrons [55–57] are applicable also to the case of muonic ions.

Results and conclusions – The highest theoretical accuracy can be achieved in light muonic ions, as all binding corrections scale with high powers of the atomic number $Z$. Especially nuclear structural effects are suppressed, which, given the uncertainties of nuclear parameters, is necessary for a sufficiently accurate competitive determination of $a_\mu$ or $m_\mu$. Therefore, we chose the muonic ion with the lightest spinless nucleus, namely, $^4\text{He}^+$. For an illustration of our theory, Table II lists numerical results for the contributing terms.

The binding effect on the $g$ factor for this element can be calculated with a $10^{-9}$ fractional accuracy, allowing for the improvement of the muon mass, or a determination of the free muon magnetic moment anomaly by subtracting theoretical binding effects from the mea-

| Effect | Term | Numerical value | Ref. |
|--------|------|----------------|------|
| Dirac value | | 1.999 857 988 8 | [19, 21] |
| Finite nuclear size | | 0.000 000 094 6(4) | [45] |
| One-loop SE | $(Z\alpha)^0$ | 0.000 000 322 819 5 | [19, 46] |
| | all-order binding | 0.000 000 084 9(10) | |
| One-loop VP | $e\text{VP}$, Uehling | -0.000 000 479 6 | |
| | $e\text{VP}$, magnetic loop | 0.000 000 127 2(4) | |
| | $\mu\text{VP}$, Uehling | -0.000 000 000 1 | |
| | hadronic VP, Uehling | -0.000 000 000 1(1) | |
| Two-loop QED | $(Z\alpha)^0$ | 0.000 000 268 4 | [47, 48] |
| | $\text{SE-SE}$, $(Z\alpha)^2$ | -0.000 000 000 1 | [13, 49–51] |
| | $S(e\text{VP})E$, $(Z\alpha)^2$ | 0.000 000 000 4 | [47, 50] |
| | 2nd-order Uehling | -0.000 000 001 1(4) | |
| | Källén-Sabry | -0.000 000 003 5 | |
| | magnetic loop+Uehling | 0.000 000 000 3 | |
| $\geq$ Three-loop QED | $(Z\alpha)^0$ | 0.000 000 160 6 | [19, 52–54] |
| Nuclear recoil | $(\alpha)^1$, all orders in $Z\alpha$ | 0.000 000 007 2 | [55] |
| | $(\alpha)^2$, $(Z\alpha)^2$ | -0.000 000 000 7 | [56] |
| | radiative recoil | -0.000 000 000 4 | [57] |
| Weak interaction | $(Z\alpha)^0$ | 0.000 000 003 1 | [19, 58] |
| Hadronic contributions | $(Z\alpha)^0$ | 0.000 000 139 3(12) | [19, 59–61] |
| Sum | | 2.002 195 193 4(20) | |

TABLE I: Various contributions to the $g$ factor of $\mu^+\text{He}^+$. The abbreviations are: “$e\text{VP}$”/“$\mu\text{VP}$”: VP due to virtual $e^-e^+$/$\mu^+\mu^-$ pairs. The estimated uncertainty of the nuclear size effect stems from the error bar of the root-mean-square nuclear radius and the uncertainty of the nuclear charge distribution model.
sured bound-muon $g$ factor. Such experiments are challenging due to the short lifetime of the muon. Nevertheless, in light of recent advances in the creation and precision spectroscopy of light muonic atoms and Penning-trap techniques such as phase-sensitive cyclotron frequency measurements, this method may serve in near future as an independent muon mass or magnetic moment anomaly determination technique, and along with corresponding experimental developments, will improve the mass uncertainty by more than an order of magnitude.

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