TORUS ACTIONS, LOCALIZATION AND INDUCED REPRESENTATIONS ON COHOMOLOGY

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Abstract. The purpose of this note is to describe the action of a (say finite) group $W$ on the cohomology algebra of a projective variety $X$ over $\mathbb{C}$ in the following setting: there exists an equivariantly formal action on $X$ by an algebraic torus $S$ such that $W$ acts on $H^*_S(X)$ as a graded algebra and $\mathbb{C}[\text{Lie}(S)]$-module and on $H^*(X^S)$ as a graded $\mathbb{C}$-algebra and, in addition, the cohomology restriction map $H^*_S(X) \to H^*_S(X^S)$ is $W$-equivariant. A sufficient condition for an action of a group $W$ on $H^*(X^S)$ to define an action on $H^*(X)$ which admits a lift to $H^*_S(X)$ satisfying the above conditions was given in [CK]. In this note we show that under the above conditions, the actions of $W$ on $H^*_S(X)$ and $H^*(X^S)$ are equivalent. In particular, if $W$ acts on the fixed point set $X^S$, then the action of $W$ on $H^*(X)$ is a sum of induced representations, one for component of $X^S$. This gives a simple proof of the Alvis-Lusztig-Treumann theorem which describes Springer’s Weyl group action on the cohomology of a Springer variety in a flag variety of type $A$. A slight rewording of the hypotheses allows one to state the above result for arbitrary actions of a algebraic torus.

1. Introduction

Let $X$ be a projective variety, and let $S$ be an algebraic torus acting on $X$. Suppose this torus action is equivariantly formal. In [CK], the author and Kiumars Kaveh gave a sufficient condition for a (finite) group $W$ which acts on the cohomology algebra $H^*(X^S)$ of the fixed point set $X^S$ to also act on cohomology algebra $H^*(X)$ of $X$. Here all cohomology will be over $\mathbb{C}$, so coefficients are suppressed. The goal of this note is to continue to develop this theme. In particular, we will give a sufficient condition for representations of $W$ on $H^*(X^S)$ and $H^*(X)$ to be equivalent. A key step is to show that $H^*(X^S)$ has a filtration whose associated graded is isomorphic with $H^*(X)$. The proof of this result uses the evaluation mapping due to Puppe. An interesting corollary is that if $(S, X)$ and $(S, Y)$ are a pair of equivariantly formal torus actions, then an $S$-equivariant map $\phi : X \to Y$ which induces a surjection $\phi^* : H^*(Y) \to H^*(X)$ also induces a surjection on the cohomology of fixed point sets: that is, $\phi$ induces a surjection $H^*(Y^S) \to H^*(X^S)$. Using an assumption of surjectivity, we will give a short proof of the Alvis-Lusztig and Treumann theorem [AL, Treu] which shows a large class of Weyl group representations of the Weyl group on the cohomology of a Springer fibre are certain induced representations.

2. The main result

Let us first describe the general set up and recall the main result from [CK]. Suppose $Y$ is a complex projective variety with vanishing odd Betti numbers and $H^*(Y)$ is its cohomology algebra over $\mathbb{C}$. Suppose that $Y$ admits an algebraic torus action $(S, Y)$. Then it follows from the Borel fixed point theorem that the fixed point locus $Y^S$ is nonempty.
Recall that the equivariant cohomology of an arbitrary space $Y$ over $\mathbb{C}$ is defined to be $H^*_\mathcal{W}(Y) = H^*(Y_\mathcal{W})$ where $Y_\mathcal{W}$ is the Borel space $(Y \times E)/\mathcal{W}$, $E$ being a contractible space with a free $\mathcal{W}$-action (cf. [Bri]). Furthermore, $H^*(E/\mathcal{W})$ has a natural identification with the algebra $\mathbb{C}[s]$, where $s = \text{Lie}(\mathcal{W})$, which is assumed to be graded in even degrees.

Recall also that the forgetful map $H^*_\mathcal{S}(Y) \to H^*_\mathcal{W}(Y)$ is the cohomology restriction map induced by the inclusion of $\mathcal{W}$ into $\mathcal{S}$ along a fibre. A key fact which we will use throughout is the well known localization theorem (cf. [Bri]): the inclusion $i : Y^\mathcal{S} \to Y$ induces an injection $i^*_\mathcal{S} : H^*_\mathcal{S}(Y) \to H^*_\mathcal{W}(Y^\mathcal{S})$, and, furthermore, $i^*_\mathcal{S}$ becomes an isomorphism after localizing $H^*_\mathcal{W}(Y^\mathcal{S})$ at finitely many elements $f_1, \ldots, f_r \in \mathbb{C}[s]$. Note that by the Kunneth formula, $H^*_\mathcal{S}(Y^\mathcal{S}) = \mathbb{C}^s \otimes H^*_\mathcal{W}(Y^\mathcal{S})$.

Now let $\mathcal{W}$ be a finite group which acts topologically on $Y$, and suppose $\mathcal{W}$ commutes with the action of $\mathcal{S}$; that is, $\mathcal{W} \times \mathcal{S}$ acts on $Y$. Then $\mathcal{W}$ acts on $H^*_\mathcal{S}(Y)$ and $H^*_\mathcal{S}(Y^\mathcal{S})$ by $\mathbb{C}[s]$-module isomorphisms. Assume also that $X$ is an $\mathcal{S}$-stable subvariety of $Y$, and note that $X^\mathcal{S}$ is necessarily nonempty too. We are not assuming, however, that $\mathcal{W}$ acts on $X$ or $X^\mathcal{S}$. The main result in [CK] is the following:

**Theorem 2.1.** Suppose the cohomology restriction map $H^*_\mathcal{W}(Y) \to H^*_\mathcal{W}(X)$ is surjective: hence $X$ has trivial odd cohomology. Assume also that $H^*_\mathcal{W}(X^\mathcal{S})$ is a graded $\mathcal{W}$-algebra. Then there exist graded $\mathbb{C}$-algebra representations of $\mathcal{W}$ on the equivariant cohomology algebras $H^*_\mathcal{S}(X)$ and $H^*_\mathcal{S}(X^\mathcal{S})$ via $\mathbb{C}[s]$-module isomorphisms such that the localization map $i^*_\mathcal{S} : H^*_\mathcal{S}(X) \to H^*_\mathcal{S}(X^\mathcal{S})$ is $\mathcal{W}$-equivariant. In particular, the forgetful map induces a graded $\mathbb{C}$-algebra representation of $\mathcal{W}$ on $H^*_\mathcal{W}(X)$.

A nice consequence of this theorem is that it gives an elementary construction of Springer’s representation [Spr2, Spr3] of the Weyl group of $SL(n, \mathbb{C})$ on $H^*(B_x)$ for the Springer variety $B_x$ of complete flags in $\mathbb{C}^n$ fixed by an arbitrary nilpotent $x \in \text{sl}(n, \mathbb{C})$. See Section 6 for more details. In this note we treat the question of how, in the setting described above, one can determine the irreducible decomposition of $\mathcal{W}$ on $H^*_\mathcal{W}(X)$.

The next theorem is a statement of the main result of this note.

**Theorem 2.2.** Assume $X$ has vanishing odd cohomology. Suppose a finite group $\mathcal{W}$ acts on both of the $\mathbb{C}$-algebras $H^*_\mathcal{S}(X)$ and $H^*_\mathcal{S}(X^\mathcal{S})$ via graded $\mathbb{C}[s]$-module isomorphisms and that the cohomology restriction map $H^*_\mathcal{S}(X) \to H^*_\mathcal{S}(X^\mathcal{S})$ is $\mathcal{W}$-equivariant. Then $\mathcal{W}$ acts linearly on both $H^*_\mathcal{W}(X)$ and $H^*_\mathcal{W}(X^\mathcal{S})$ via the forgetful maps, and these actions are equivalent: that is, $H^*_\mathcal{W}(X) \cong H^*_\mathcal{W}(X^\mathcal{S})$ as $\mathcal{W}$-modules. More precisely, $H^*_\mathcal{W}(X^\mathcal{S})$ has a $\mathcal{W}$-stable filtration $F_{-1} = \{0\} \subset F_0 \subset F_1 \subset \cdots \subset F_m \subset F_{m+1} \subset \cdots \subset H^*_\mathcal{W}(X^\mathcal{S})$ such that $F_i F_j \subset F_{i+j}$ and $Gr_F H^*_\mathcal{W}(X^\mathcal{S}) \cong H^*_\mathcal{W}(X)$ via a $\mathcal{W}$-equivariant isomorphism of graded $\mathbb{C}$-algebras.

Of course, the assumption that $\mathcal{W}$ acts on both $H^*_\mathcal{S}(X^\mathcal{S})$ and $H^*_\mathcal{S}(X)$ is part of the conclusion of Theorem 2.1. In the case the action of $\mathcal{W}$ on $H^*_\mathcal{W}(X)$ is induced by an action of $\mathcal{W}$ on $X^\mathcal{S}$ and $X^\mathcal{S}$ is finite, we get the following more explicit description:

**Corollary 2.3.** Assume $X^\mathcal{S}$ is finite and $\mathcal{W}$ acts on it. If this action of $\mathcal{W}$ is simply transitive (resp. transitive), then $H^*_\mathcal{W}(X)$ is the regular representation of $\mathcal{W}$ (resp. the induced representation $\text{Ind}_{\mathcal{W}_z}(\mathbb{C})$ associated to the trivial one dimensional representation of the isotropy group $\mathcal{W}_z$ for any $z \in X^\mathcal{S}$). In general, the representation of $\mathcal{W}$ on $H^*_\mathcal{W}(X)$ is either the regular representation or a direct sum of induced representations corresponding to the orbits of $\mathcal{W}$ on $X^\mathcal{S}$.

**Example 2.4.** Let $G$ denote a semisimple complex algebraic group, and suppose $B$ and $T$ denote, respectively, a Borel subgroup of $G$ and a maximal torus of $G$ contained in $B$. 

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The Weyl group $W = N_G(T)/T$ acts smoothly on the flag variety $B = G/B$ of $G$ (on the right) by viewing $G/B$ as $K/H$. Since this action is simply transitively on $\mathcal{B}^T$, the corollary implies the action $(W, H^*(B))$ on cohomology is the regular representation. This is of course well known.

3. GKM-VARIETIES AND $W$-ACTIONS ON THE MOMENTUM GRAPH

Recall that one calls $X$ a GKM-variety if the torus action $(S, X)$ is equivariantly formal (for example, $X$ has vanishing odd cohomology), and $X$ has only finitely many $S$-stable curves [GKM]. Let $E(X)$ denote the set of these $S$-stable curves. It follows immediately from [Car1] that $X^S$ is finite since every fixed point lies on an $S$-stable curve (in fact, on at least dim $X$ such curves), and every $S$-stable curve contains exactly two fixed points.

Recall that the momentum graph of $(S, X)$ is the graph $\mathfrak{M}$ with vertex set $X^S$ and edge set $E(X)$, where two vertices $x$ and $y$ are on an edge $C$ if and only if $C^S = \{x, y\}$. By fixing a $G_m$-action $\lambda : \mathbb{C}^* \rightarrow S$ on $X$ with fixed point set $X^S$, one may orient each edge $C = S \cdot z$ of $\mathfrak{M}$ by defining the source of $C$ to be $\lim_{t \to 0} \lambda(t)z$ and sink of $C$ to be $\lim_{t \to \infty} \lambda(t)z$. If $x$ is the source of $C$ and $y$ is its sink, we will write $C = C(x, y)$. Every $S$-stable curve $C$ in $X$ is smooth, hence is determined by its $S$-stable tangent line in $T_x(X)$ on which $S$ acts by a character $\chi$ of $S$ such that $<\chi, \lambda> > 0$. This determines a unique weight $\alpha_C$ of degree two in $\ast \subset \mathbb{C}[s]$. The edge $C$ of $\mathfrak{M}$ is labelled be this weight.

Since $H^*_S(X^S) = \mathbb{C}[s] \otimes H^0(X^S)$ and $H^0(X^S) = \mathbb{C}^{X^S}$, $H^*_S(X^S)$ is naturally identified with $(\mathbb{C}[s])^{X^S}$, the algebra of $\mathbb{C}[s]$-valued functions on $X^S$. By the fundamental theorem of [GKM], the image of the restriction map $i^*_S : H^*_S(X) \rightarrow H^*_S(X^S)$ is

$$\{ f \in (\mathbb{C}[s])^{X^S} \mid f(x) - f(y) \in \alpha_C \mathbb{C}[s] \text{ if } C = C(x, y) \}.$$  

We will say that $W$ acts on $\mathfrak{M}$ if $W$ acts on $X^S$ and acts linearly on $\ast$ so that if $x, y \in X^S$ lie on an edge $C(x, y)$ with weight $\alpha$, then $w \cdot x$ and $w \cdot y$ lie on an edge $C(w \cdot x, w \cdot y)$ with weight $w(\alpha)$. Then $W$ acts in two ways on $H^*_S(X^S)$: on the right by

$$f \cdot w(x) = f(w \cdot x),$$

and on the left by

$$w \cdot f(x) = w \cdot (f(w^{-1} \cdot x)),$$

The right action of $W$ commutes with the $\mathbb{C}[s]$-module structure of $H^*_S(X^S)$ while the left action commutes with a twist.

**Lemma 3.1.** Suppose $X$ is a GKM-variety with respect to the torus action $(S, X)$, let $\mathfrak{M}$ be its momentum graph, and suppose $W$ acts on $\mathfrak{M}$. Then the image the localization map $i^*_S : H^*_S(X) \rightarrow H^*_S(X^S)$ is stable under both the left and right actions of $W$ on $H^*_S(X^S)$, and these actions descend via the forgetful map to $H^*(X)$.

**Proof.** Suppose $f \in H^*_S(X^S)$ lies in the image of $i^*_S$, and consider an edge $C = C(x, y) \in \mathfrak{M}$. Then $f(x) - f(y) \in \alpha_C \mathbb{C}[s]$. Thus,

$$f \cdot w(x) - f \cdot w(y) = f(w \cdot x) - f(w \cdot y) \in w(\alpha) \mathbb{C}[s],$$

where $\beta \in \ast$ is the weight of $C(w \cdot x, w \cdot y)$. Hence the image of $i^*_S$ is stable under the right action. For the left action, the same reasoning gives

$$w \cdot f(x) - w \cdot f(y) = w \cdot (f(w^{-1} \cdot x)) - w \cdot (f(w^{-1} \cdot y)) \in w(w^{-1}(\alpha)) \mathbb{C}[s] = \alpha \mathbb{C}[s].$$

The final assertion is obvious. \qed

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Example 3.2 ([Carl]). Let $G$, $B$, $T$ and $W$ be as in Example 2.4. An important example of a GKM-variety is $G/B$ with its canonical $T$-action or, more generally, any $T$-invariant subvariety of $G/B$. The fixed point set $(G/B)^T = \{ wB \mid w \in W \}$ is naturally identified with $W$ via $w = wB$, and every $T$-stable curve $C$ in $G/B$ has exactly two fixed points which can be written $x$ and $y = rx$, where $r$ is a reflection in $W$ and $x < y$ in the Bruhat order on $W$. Thus the momentum graph of $G/B$ is identified with the Bruhat graph of $W$, namely the graph with vertex set $W$, where two vertices $x$, $y$ are joined by a directed edge $[x,y]$ from $x$ to $y$ if $x^{-1}y$ is a reflection and the length of $y$ is greater than the length of $x$. The directed edge $[x,y]$ corresponds to a $T$-stable curve $C(x,y)$. Namely, if $x^{-1}y = r_\alpha$, where $\alpha > 0$, and $U_\alpha$ is the one dimensional unipotent subgroup of $G$ corresponding to $\alpha$, then $C(x,y) = \overline{U_\alpha xB}$ is a $T$-stable curve joining $xB$ and $yB$. The curve $w\overline{U_\alpha xB}$ is also $T$-stable and joins $wxB$ and $wyB$. Hence the action of $W$ on itself by left translation induces an action of $W$ on the momentum graph of $G/B$ by $w \cdot [x,y] = [w \cdot x, w \cdot y]$.

The right action of $W$ on $H^*_T(G/B)$ is well known to be induced from the (topological) action of $W$ on $G/B$. It’s important properties are proven in [KK, Km, Tym]. Borel’s classical picture of the cohomology $H^*(G/B)$ of $G/B$ as the coinvariant algebra $\mathbb{C}[t]/I_W^T$ of $W$ implies the right action of $W$ on $H^*(G/B)$ is the regular representation of $W$. Here $I_W^T$ is the ideal generated by the $W$-invariants vanishing at 0. As remarked, this is also a direct consequence of Theorem 2.2. The left action of $W$ on $H^*_T(G/B)$ is defined in the unpublished paper [Knu], and its properties were studied by Tymoczko. In fact, the left action of $W$ is trivial on $H^*(G/B)$ ([Tym, Prop. 4.2]). Tymoczko also posed question of describing the left action of $W$ on the cohomology of a regular semisimple Hessenberg variety. We will describe these varieties next.

Remark 3.3. To each $B$-submodule $\mathfrak{h}$ of $\mathfrak{g}$ and each semi-simple element $t \in t$, there is a smooth $T$-variety $X_\mathfrak{h}$ in $G/B$ called a Hessenberg variety such that $(X_\mathfrak{h})^T = W$ ([DPS]). By definition, $X_\mathfrak{h} = \{ gB \mid g^{-1}t \in \mathfrak{h} \}$.

The above remarks imply that $X_\mathfrak{h}$ admits a left $W$-action. Interestingly, it is no longer the case that the action of $W$ on $H^*(X_\mathfrak{h})$ is trivial. In fact, it was recently shown that $H^*(X_\mathfrak{h})^W$ is the image of $H^*(G/B)$ under the cohomology restriction map (cf. [AHHM]). Since $(G/B)^T = (X_\mathfrak{h})^T$ it cannot be the case that the cohomology of $G/B$ surjects on that of $X_\mathfrak{h}$ unless they coincide. When $\Delta$ is the set of simple roots and $\mathfrak{h}/\mathfrak{b} = \sum_{\alpha \in \Delta} \mathfrak{b} - \alpha$, $X_\mathfrak{h}$ is the toric variety associated with the Weyl fan composed of the Weyl chambers. In this case, the representation of $W$ on $H^*(X_\mathfrak{h})$ was described by Procesi in [Proc]. The question of what this $W$-representation is in general was originally posed by Tymoczko (loc-cit). In [SW], Shareshian and Wachs conjectured a combinatorial formula for the character of the left $W$-action on $H^*(X_\mathfrak{h})$ involving quasisymmetric functions and indifference graphs which was proved by Brosnan and Chow in [BC] and later by Guay-Paquet [GP].

Summarizing the information about the right action from Theorem 2.2 and Corollary 2.3, we get

Proposition 3.4. If $X$ is a GKM-variety with respect to an algebraic torus $S$ and a (finite) group $W$ acts on the moment graph of $(S,X)$, then $W$ acts on $H^*_S(X)$ on the right by $\mathbb{C}[S]$-automorphisms. The representation $(W, H^*(X))$ obtained via the forgetful map is equivalent to the representation of $W$ on $H^0(X^S)$ which is completely determined as a sum of induced representations by the orbit structure of $W$ on $X^S$ as in Corollary 2.3.
4. The Evaluation mapping

Suppose as usual that $X$ has an algebraic torus action $(S, X)$ and vanishing odd cohomology. In this section we will obtain a filtration of $H^*(X^S)$ whose associated graded algebra is isomorphic with $H^*(X)$. This generalizes the main theorem in [CKP]. For simplicity, we will assume $S = \mathbb{G}_m$. The generalization to arbitrary $S$ is straightforward.

**Theorem 4.1.** Assume $X$ has vanishing odd cohomology, and let $S$ be a one dimensional algebraic torus acting on $X$. Then $H^*(X^S)$ admits a filtration

$$(1) \quad F_{-1} = \{0\} \subset F_0 \subset F_1 \subset \cdots \subset F_m \subset F_{m+1} \subset \cdots \subset H^*(X^S)$$

satisfying $F_i F_j \subset F_{i+j}$ with the property that there exists a graded $\mathbb{C}$-algebra isomorphism

$$(2) \quad \text{Gr}_FH^*(X^S) = \bigoplus_{i \geq 0} F_i/F_{i-1} \cong \bigoplus_{i \geq 0} H^{2i}(X) = H^*(X).$$

**Proof.** Since the cohomology $H^*(X)$ is trivial in odd degrees, it follows that the action of $S$ is equivariantly formal. Thus $H^*_S(X)$ is a free $\mathbb{C}[s]$-module of rank $r = \dim H^0(X^S)$. Let $f_1, \ldots, f_r \in \mathbb{C}[s]$ be elements such that the restriction $i_S^* : H^*_S(X) \to H^*_S(X^S)$ is an isomorphism after localization at these elements. We will say $a \in \mathfrak{s}$ is regular if $f_i(a) \neq 0$ for $i = 1, \ldots, r$. By equivariant formality, we have the following well known exact sequence

$$(3) \quad 0 \longrightarrow \mathbb{C}[s]^+ H^*_S(X) \longrightarrow H^*_S(X) \longrightarrow H^*(X) \longrightarrow 0,$$

where the map $H^*_S(X) \longrightarrow H^*(X)$ is induced by inclusion of $X$ in $X_S$ and $\mathbb{C}[s]^+$ is the augmentation ideal (i.e. the maximal ideal of $0 \in \mathfrak{s}$). If $V$ is a $\mathbb{C}$-vector space and $a \in \mathfrak{s}$, let $V_a$ be the $\mathbb{C}[s]$-module defined by putting $f \cdot v = f(a)v$ for any $f \in \mathbb{C}[s]$ and $v \in V$. Similarly, if $M$ is an $\mathbb{C}[s]$-module, put $M[a] = M \otimes_{\mathbb{C}[s]} \mathbb{C}_a$. When $M$ is free of rank $r$, then $\dim_{\mathbb{C}} M[a] = r$. Now, the exact sequence (3) implies

$$H^*_S(X)[a] \cong H^*(X)[a] = H^*(X).$$

Furthermore, the Localization Theorem implies that

$$(4) \quad i_S^* : H^*_S(X)[a] \to H^*_S(X^S)[a]$$

is an isomorphism of $\mathbb{C}[s]$-algebras for any regular $a \in \mathfrak{s}$, hence also an isomorphism of (ungraded) $\mathbb{C}$-algebras. For any $i \geq 0$, put

$$ F_i = \sum_{j \leq i} H^*_S(X).$$

This defines an increasing filtration of $H^*_S(X)$ such that $F_i F_j \subset F_{i+j}$ for $i, j \geq 0$. Moreover, defining $F_i[a]$ to be the image of $F_i$ in $H^*_S(X)[a]$, we obtain, for any regular $a$, a filtration $F(i, a)$ of $H^*(X^S)$ as follows. Let $e_a : H^*_S(X^S)[a] \to H^*(X^S)$ be the unique $\mathbb{C}$-algebra isomorphism such that $e_a(f \otimes \phi) = f(a) \phi$. Now put

$$ F(i, a) = e_a(i_S^*(F_i[a])).$$

Then define the filtration (1) of $H^*(X^S)$ to be this filtration. Returning to the filtration $F_i[a]$ of $H^*_S(X)[a]$, we obtain from the forgetful map in (3) a filtration $G(i, a) \subset G(i + 1, a)$ of $H^*(X) \equiv H^*(X)[a]$ such that

$$ \bigoplus_{i \geq 0} G(i, a)/G(i - 1, a) \cong \bigoplus_{i \geq 0} F(i, a)/F(i - 1, a)$$
is a graded \( \mathbb{C} \)-algebra isomorphism via (4). It remains to establish the graded ring isomorphism
\[
\bigoplus_{i \geq 0} G(i, a)/G(i - 1, a) \cong H^*(X).
\]
But \( G(i, a)/G(i - 1, a) \cong H^{2i}(X)[a] \), and this gives rise to the natural multiplication
\[
G(i, a)/G(i - 1, a) \otimes G(j, a)/G(j - 1, a) \to H^{2i}(X)[a] \otimes H^{2j}(X)[a] \to H^{2(i+j)}(X)[a]
\]
via the ring structure of \( H^*(X)[a] \). But \( H^*(X) = H^*(X)[a] \) by definition, so the proof of Theorem 4.1 is finished.

It is clear that the filtration is preserved by equivariant maps. This has an interesting and useful consequence. Suppose \( X \) is an \( S \)-stable subvariety of \( Y \) such that the cohomology restriction map \( i^* : H^*(Y) \to H^*(X) \) is surjective and suppose also that \( Y \) has vanishing odd cohomology. Let \( E \) and \( F \) be the filtrations of \( H^*(Y^S) \) and \( H^*(X^S) \) defined in Theorem 4.1. Then, if \( \rho : X^S \to Y^S \) is the inclusion, \( \rho^*(E_i) \subset F_i \) for all \( i \). This gives the commutative diagram
\[
\begin{array}{ccc}
\text{Gr}_E H^*(Y^S) & \longrightarrow & H^*(Y) \\
\rho^* & | & | \\
\text{Gr}_F H^*(X^S) & \longrightarrow & H^*(X),
\end{array}
\]
where \( \rho \) is the graded \( \mathbb{C} \)-algebra morphism induced by \( \rho^* \). Since the horizontal maps are isomorphisms and \( i^* \) is surjective, it follows that \( \rho^*(E_i) = F_i \) for all \( i \). Since \( \bigcup F_i = H^*(Y^S) \), it follows that \( \rho^* \) is surjective. Thus we have proved the following, which may be of some interest independently.

**Theorem 4.2.** Assume that \( X \) is an \( S \)-stable subvariety of the \( S \)-variety \( Y \) such that the cohomology restriction map \( i^* : H^*(Y) \to H^*(X) \) is surjective. Suppose also that the odd cohomology of \( Y \) vanishes. Then the cohomology restriction map \( \rho^* : H^*(Y^S) \to H^*(X^S) \) is also surjective.

We will give a somewhat surprising application of this theorem in Section 6.

**Remark 4.3.** The fact that the evaluation map and localization can be used to obtain a filtration of \( H^*(X^S) \) whose associated graded is \( H^*(X) \) is a remark of Puppe (cf. [P1, P2]), who was interested in obtaining results about \( H^*(X^S) \) from \( H^*(X) \) rather than the reverse. Puppe’s results appeared at almost the same time as a paper of the author and David Lieberman which asserts the existence of the filtration in Theorem 4.1 when \( X \) is smooth and the torus action \((S, X)\) is replaced by a holomorphic vector field with simple zeros (cf. [CL, CKP]). The connection between these results was finally noticed at an Oberwolfach workshop in 2006 and resulted in the paper [CKP].
equivalent to the representation of \( W \) on \( \text{Gr}_FW^*(X^S) = \bigoplus_{i \geq 0} F_i/F_{i-1} \). This follows from the following lemma.

**Lemma 5.1.** Let \( W \) be a finite group acting linearly on a \( \mathbb{C} \)-vector space \( V \) having an \( W \)-invariant filtration

\[
V_0 = \{0\} \subset V_1 \subset \cdots V_{m-1} \subset V_m = V.
\]

Then the induced representation of \( W \) on \( \text{Gr } V = \bigoplus_{i \geq 0} V_i/V_{i-1} \) is equivalent to the given representation of \( W \) on \( V \).

**Proof.** Since \( W \) is finite, every \( W \) invariant subspace of \( V \) has an \( W \)-invariant complement. Applying this fact to \( F_i \subset F_{i+1} \) for each \( i \), one gets a \( \mathbb{C} \)-linear \( W \)-equivariant isomorphism between \( V \) and \( \text{Gr } V \). \( \square \)

To prove Corollary 5.3 suppose \( X^S = \{x_1, \ldots, x_r\} \) and \( W \) acts transitively. Let \( K \) be the isotropy group of \( x_1 \). By the Orbit Stabilizer Theorem, \( |W/K| = r \). Since \( H^0(X^S) = \mathbb{C}^{|X^S|} \), a basis of \( H^0(X^S) \) is given by the functions \( \delta_1, \ldots, \delta_r \) on \( X^S \) defined by the conditions \( \delta_i(x_j) = \delta_{ij} \). By definition, the subspace \( Y = \mathbb{C}\delta_1 \) is a \( K \)-stable line. Moreover, if \( W/K = \{w_1K, \ldots, w_rK\} \), then \( w_i \cdot \delta_1 = \delta_1 \) provided \( w_i \cdot x_1 = x_i \). For \( w_i \cdot \delta_1(x_i) = \delta_1((w_i)^{-1}, x_i) = \delta_1(x_1) = 1 \) while if \( i \neq j \), \( (w_i)^{-1} \cdot x_j \neq x_i \), so \( w_i \cdot \delta_1(x_j) = 0 \). Consequently,

\[
H^0(X^S) = \sum_{\sigma \in W/K} \sigma Y.
\]

If \( K = \{e\} \), then \((W, H^0(X^S))\) is the regular representation. Otherwise, the representation of \( W \) on \( H^0(X^S) \) is induced as claimed. The other assertion of the Corollary is straightforward. \( \square \)

6. **Remarks on Springer varieties**

Let \( G, B, T \) and \( W \) be as in Example 2.4. Recall that the flag variety \( B = G/B \) of \( G \) parameterizes the set of all Borel subgroups of \( G \), or, equivalently, the set of all Borel subalgebras of \( \mathfrak{g} \). Let \( \mathcal{N} \subset \mathfrak{g} \) denote the nilpotent cone: that is, the set of all nilpotent elements of \( \mathfrak{g} \). For any \( x \in \mathcal{N} \), the set \( \mathcal{B}_x \) of all Borel subalgebras of \( \mathfrak{g} \) containing \( x \) is a closed connected subvariety of \( B \) called the Springer variety associated to \( x \). Springer varieties are also called a Springer fibres since every \( \mathcal{B}_x \) is the fibre over \( x \in \mathcal{N} \) of Springer’s resolution of \( \mathcal{N} \) \([Spr1]\). Although \( W \) acts on \( B \), it does not in general act on \( \mathcal{B}_x \). However, a celebrated result of Springer \([Spr2, Spr3]\) says that \( W \) admits a graded \( \mathbb{C} \)-algebra representation on \( H^*(\mathcal{B}_x) \) for any \( x \in \mathcal{N} \). In Springer’s original construction, \( W \) acts on \( \ell \)-adic cohomology, but there are also constructions of Springer’s representations on classical cohomology via intersection cohomology and homotopy theory \([\text{BBM}, \text{KL}, \text{Ross}, \text{Sloth}, \text{Treu}]\).

We will be concerned, primarily, with Springer fibres \( \mathcal{B}_x \) for Levi nilpotents \( x \) that is, nilpotents that lie in a Levi subalgebra of \( \mathfrak{g} \). We will use the letter \( I \) to denote a Levi subalgebra of \( \mathfrak{g} \), and \( L \) to denote its corresponding Levi subgroup of \( G \). Without loss of generality, we may assume the parabolic subgroup associated to \( L \) is a subgroup of \( B \) and that the centre of \( L \) is a torus \( S \subset T \). If \( x \in I \) is nilpotent, then \( \mathcal{B}_{L,x} \) will denote the Springer fibre associated to \( x \) in the flag variety \( L/L \cap B \) of \( L \). The components of the fixed point set \( \mathcal{B}_S \) are identified with the flag varieties \( L/(L \cap wBw^{-1}) \) as \( w \) ranges over a set of representatives of \( W/W_L \). A nilpotent \( x \in I \) is called parabolic if it is also regular in \( I \) in the sense that it lies in a unique Borel subalgebra of \( I \). As mentioned in the introduction, if \( x \in I \) is parabolic and the cohomology restriction map \( i^* : H^*(B) \to H^*(\mathcal{B}_x) \) is surjective, then Springer’s action of \( W \) lifts from \( H^*(\mathcal{B}_x) \) to \( H^*_S(\mathcal{B}_x) \), where \( S \subset T \) is the center of the
Levi subgroup $L$ of $G$ with Lie algebra $\mathfrak{l}$. Let $W_L$ be the Weyl group of $L$ with respect to $T$. Now, the surjectivity result Theorem \ref{thm:main} has the following slightly surprising consequence:

**Theorem 6.1.** Let $x \in \mathfrak{l}$ be a Levi nilpotent such that $i^*: H^*(\mathcal{B}) \to H^*(\mathcal{B}_x)$ is surjective. Then the cohomology restriction map $H^*(\mathcal{B}_L) \to H^*(\mathcal{B}_{L,x})$ is also surjective. Furthermore, if $H$ is a subtorus of $T$ containing $\mathcal{S}$ which acts on $\mathcal{B}_{L,x}$ such that $(\mathcal{B}_{L,x})^H = (\mathcal{B}_{L,x})^S$, then $W_L$ lifts to $H^*_H(\mathcal{B}_{L,x})$.

**Proof.** This follows immediately from Theorems \ref{thm:main} and \ref{thm:coho}.

The theorem of Alvis–Lusztig and Treumann (\cite{AL, Lus, Treu}) says the following over $\mathbb{C}$:

**Theorem 6.2.** Suppose $x$ is a nilpotent element of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. Then the $W$-action on $H^*(\mathcal{B}_x)$ is equivalent to the induced representation $\text{Ind}_{W_L}^W(H^*(\mathcal{B}_{x,L}))$, where $L$ is the Levi subgroup of $G$ corresponding to $\mathfrak{l}$, $W_L \subset W$ is its Weyl group, and $\mathcal{B}_{x,L}$ is the Springer variety in the flag variety of $L$ corresponding to $x$.

We will give a geometric proof under the assumption that the cohomology restriction map $i^*: H^*(\mathcal{B}) \to H^*(\mathcal{B}_x)$ is surjective. Now the Weyl group $W_L = N_L(T)/T$ of $L$ acts on the flag variety $L_w = L/(L \cap wBu^{-1})$ for each $w \in W$. Moreover, the connected components of $\mathcal{B}^S$ are permuted by $W$ and each one is isomorphic to some $L_w$. Furthermore, the components are fixed by $W_L$, so the set of components of $\mathcal{B}^S$ is in one to one correspondence with $W/W_L$. Now let $w_1, \ldots, w_k \in W$ denote a complete set of representatives for $W/W_L$. Denoting $L_w \cap \mathcal{B}_x$ by $L_{w,x}$, we have

$$(\mathcal{B}_x)^S = \bigcup_{1 \leq i \leq k} L_{w_i,x}.$$ 

Now the action of $W$ on $H^*(\mathcal{B}^S)$ is induced by the action of $W$ on $\mathcal{B}^S$, and, by Theorem \ref{thm:induced}, we have a filtration of $H^*((\mathcal{B}_x)^S)$ such that $\text{Gr}H^*((\mathcal{B}_x)^S) \cong H^*(\mathcal{B}_x)$.

From the diagram \ref{fig:coho} with $Y = \mathcal{B}$ and $X = \mathcal{B}_x$, it follows that one can construct inductively a graded action of $W$ on $H^*((\mathcal{B}_x)^S)$ so that all the morphisms in the diagram are $W$-equivariant. Applying the surjectivity assumption together with Theorems \ref{thm:main} and \ref{thm:coho} it follows that since

$$H^*((\mathcal{B}_x)^S) = \bigoplus H^*(L_{w_i,x}),$$

then $H^*((\mathcal{B}_x)^S) \cong \text{Ind}_{W_L}^W(H^*(\mathcal{B}_{x,L}))$ where $\mathcal{B}_{x,L}$ denotes $L_{e,x}$. \hfill \Box

If $x$ is a regular nilpotent in the Levi subalgebra $\mathfrak{l}$, then the fixed point set $(\mathcal{B}_x)^S$ is finite, and in fact $W$ acts on this finite set. Thus we obtain a more concrete picture.

**Corollary 6.3.** Suppose $x$ is a regular nilpotent in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. Assume also that the cohomology restriction map $H^*(\mathcal{B}) \to H^*(\mathcal{B}_x)$ is surjective. Then Springer’s representation of $W$ on $H^*(\mathcal{B}_x)$ is $\text{Ind}_{W_L}^W(\mathbb{C})$. That is, it is induced by the trivial representation of the Weyl group $W_L$ of $L$.

**Remark 6.4.** It is well known that every nilpotent in $\mathfrak{sl}(n, \mathbb{C})$ is regular in a Levi subalgebra $\mathfrak{l}$ and, moreover, the cohomology restriction map $H^*(\mathcal{B}) \to H^*(\mathcal{B}_x)$ is surjective (cf. \cite{Spalt}). The above proofs of Theorem \ref{thm:main} are therefore valid for type $A$. The determination of when the cohomology restriction map $H^*(\mathcal{B}) \to H^*(\mathcal{B}_x)$ is surjective is an interesting open question. A necessary condition for surjectivity of the cohomology restriction map in the middle dimension is that the component group $A(x) = Z_G(x)^0/Z_G(x)$ is trivial. The conjugacy classes of these groups have been computed in \cite{Somm}. It has also been recently shown that if $G$ is simply laced, then $H^2(\mathcal{B}) \to H^2(\mathcal{B}_x)$ is always surjective \cite{CVX}.
Springer’s Weyl group action in type $A$ has an interesting history which we will briefly recall here. Let us assume $G = GL(n, \mathbb{C})$ and $B$ and $T$ are the subgroups of upper triangular matrices and the diagonal matrices in $G$ respectively. Here we may take $W$ to be the $n \times n$ permutation matrices $P(n)$ acting explicitly on $B$ on the left. Suppose $x$ is an upper triangular nilpotent element of $\mathfrak{gl}(n, \mathbb{C})$. Then $B$ is the set of complete flags in $\mathbb{C}^n$, and $B_x$ consists of all flags

$$V_0 = \{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$$

such that $x(V_i) \subset V_{i-1}$ for all $i \geq 1$. By the above remark, Theorem 6.2 says that Springer’s representation always has the form $\text{Ind}_{W_{L_x}}^W(\mathbb{C})$. Here $W_L = P(\mu_1) \times \cdots \times P(\mu_m)$, where $\mu_1 \geq \cdots \geq \mu_m$ is the partition of $n$ dual to the partition corresponding to a regular nilpotent in $I$. This fact was originally conjectured by Kraft in [Kraft] and subsequently verified in a beautiful paper of DeConcini and Procesi in [DP]. In fact, Kraft made two conjectures. First, suppose $y \in \mathfrak{gl}(n, \mathbb{C})$ is nilpotent and $C_y$ is the closure of its $G$-conjugacy class. Let $C_y \cap \mathfrak{t}$ denote the schematic intersection of $C_y$ and the Lie algebra $\mathfrak{t}$ of $T$. Kraft first conjectured that if $P$ is a parabolic subgroup of $GL(n, \mathbb{C})$ containing $B$ and $y$ is a Richardson element in $P$ (that is, the $P$-conjugacy class of $y$ is dense in the nilradical of $P$), then the natural representation of $W$ on $A(C_y \cap \mathfrak{t})$ is $\text{Ind}_{W_{L_y}}^W(\mathbb{C})$. He then conjectured that if $x$ is a regular nilpotent in $I$, then $A(C_y \cap \mathfrak{t})$ and $H^*(B_x)$ are isomorphic both as $S_n$-modules and graded $\mathbb{C}$-algebras. As mentioned above, this was verified by DeConcini and Procesi loc.cit. In [Car2], using torus actions, the Kraft-DeConcini-Procesi isomorphism was extended to certain pairs $(x, y)$ of nilpotents in an arbitrary semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ where $x$ is a regular nilpotent in a Levi subalgebra $I$ and $y$ is a Richardson element in the nilradical of the parabolic subalgebra associated to $I$. This requires the surjectivity assumption for $H^*(B_x)$ and the further assumption that the stabilizer in $G$ of $y$ be connected.

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References

[AHHM] Abe, H.; Harada, M.; Horiguchi, T.; Masuda, M. The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A. [arXiv:1404.1217] 3.

[AL] Alvis, D.; Lusztig, G. On Springer’s correspondence for simple groups of type $E_n$ ($n = 6, 7, 8$). Math. Proc. Camb. Phil. Soc. 92, pp. 6572, 1982. MR662961 (83k:20040)

[BBM] Borho, W.; Brylinski, J.-L.; MacPherson, R. Springer’s Weyl group representations through characteristic classes of cone bundles. Math. Ann. 278 (1987), no. 1-4, 273289.

[Br] Brion, M. Equivariant cohomology and equivariant intersection theory. Notes by Alvaro Rittatore. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514.

[BC] Brosnan, P.; Chow, T. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. (English summary) Sm. Lothar. Combin. 78B (2017), Art. 3, 12 pp.

[Car1] Carrell, J. B. The Bruhat Graph of a Coxeter Group, a Conjecture of Deodhar, and Rational Smoothness of Schubert Varieties. Proc. Symp. Pure Math. 56, 5361 (1994)

[Car2] Carrell, J. B. Orbits of the Weyl group and a theorem of DeConcini and Procesi. Compositio Math. 60 (1986), no. 1, 45–52.

[Car3] Carrell, J. B. Torus actions and cohomology. The adjoint representation and the adjoint action, 83–158. Encyclopaedia Math. Sci., 131, Springer, Berlin, 2002.

[CK] Carrell, J. B.; Kaveh, K. Springer’s Weyl group representation via localization, Can. Math. Bull.

[CKP] Carrell, J.; Kaveh, K.; Puppe, V. Vector fields, torus actions and equivariant cohomology. Pacific J. Math. 232 (2007), 61–76.

[CL] Carrell, J. B.; Lieberman, D. A. Vector fields and Chern numbers, Math. Ann. 225 (1977), 263273.

[CVX] Chen, T.-H.; Vilonen, K.; Xue, T. On the cohomology of Fano varieties and the Springer correspondence. With an appendix by Dennis Stanton. Adv. Math. 318 (2017), 515533.
[DLP] De Concini, C.; Lusztig, G.; Procesi, C. Homology of the zero-set of a nilpotent vector field on a flag manifold. J. Amer. Math. Soc. 1 (1988), no. 1, 15–34.

[DP] De Concini, C.; Procesi, C. Symmetric functions, conjugacy classes, and the flag variety. Invent. Math., 64 (1981) 203–219.

[DPS] De Mari, F.; Procesi, C.; Shayman, M. Hessenberg varieties. Trans. Amer. Math. Soc. 332 (1992), no. 2, 529534.

[GKM] Goresky, M.; Kottwitz, R.; MacPherson, R. Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. 131 (1998), no. 1, 253–283.

[GMcP] Goresky, M.; MacPherson, R. On the spectrum of the equivariant cohomology ring. Canad. J. Math. 62 (2010), no. 2, 262–283.

[G-P] Guay-Paquet, M. A modular relation for the chromatic symmetric functions of $(3 + 1)$-free posets. 2013. arXiv:1306.2400.

[HS] Hotta, R.; Springer, T. A. A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups. Invent. Math. 41 (1977) 113–127.

[KL] Kazhdan, D.; Lusztig, G. A topological approach to Springer’s representations. Adv. in Math. 38 (1980), 222–228.

[Knu] Knutson, A. A Schubert calculus recurrence from the noncomplex $W$-action on $G/B$, arXiv:math.CO/0306304.

[Kraft] Kraft, H. P. Conjugacy classes and Weyl group representations, Young tableaux and Schur functions in algebra and geometry (Torun, 1980), 191205, Astérisque, 8788, Soc. Math. France, Paris, 1981.

[KK] Kostant, B.; Kumar, S. $T$-equivariant $K$-theory of generalized flag varieties. J. Differential Geom. 32 (1990), no. 2, 549603.

[KP] Kumar, S.; Procesi, C. An algebro-geometric realization of equivariant cohomology of some Springer fibers. Journal of Algebra 368 (2012), 70–74.

[Lus] Lusztig, G. An induction theorem for Springer’s representations. Representation theory of algebraic groups and quantum groups, 253259, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.

[Proc] Procesi, C. The toric variety associated to Weyl chambers. Mots, 153161, Lang. Raison. Calc., Herm, Paris, 1990.

[P1] Puppe, V. Cohomology of fixed point sets and deformation of algebras. Manuscripta Math. 23 (1977/78), no. 4, 343–354.

[P2] Puppe, V. Deformations of algebras and cohomology of fixed point sets. Manuscripta Math. 30 (1979/80), no. 2, 119–136.

[Ross] Rossmann, W. Picard-Lefschetz theory for the coadjoint quotient of a semisimple Lie algebra. (English summary) Invent. Math. 121 (1995), 531–578.

[SW] Shareshian, J.; Wachs, M. L. Chromatic quasisymmetric functions and Hessenberg varieties. Configuration Spaces. CRM Series, Springer, 2012.

[Skl] Slodowy, P. Four lectures on simple groups and singularities. Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, 11. Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 1980. ii+64 pp.

[Treu] Treumann, D. A topological approach to induction theorems in Springer theory. Represent. Theory 13 (2009), 8–18.

[Tym] Tymoczko, J. S. Permutation actions on equivariant cohomology of flag varieties. Toric topology, 365384, Contemp. Math., 460, Amer. Math. Soc., Providence, RI, 2008.
