On the dual–vector field condensation
in the dual Monopole Nambu–Jona–Lasinio model with dual
Dirac strings

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Abstract
The condensation of a dual–vector field is investigated in the dual Monopole Nambu–Jona–Lasinio model with dual Dirac strings. The condensate of a dual–vector field is calculated as a functional of a shape of a dual Dirac string. The obtained result is compared with the gluon condensate calculated in a QCD sum rules approach and on lattice.
1 Introduction

Effective Lagrangian of the dual magnetic Monopole Nambu–Jona–Lasinio model. In our recent publications [1–4] we have shown that the dual Monopole Nambu–Jona–Lasinio model (MNJL) with dual Dirac strings is a perfect continuum space–time analogy of Compact Quantum Electrodynamics (CQED) [5]. As has been shown in Ref.[5] CQED possesses the same non–perturbative phenomena as low–energy QCD. The MNJL model is based on a Lagrangian, invariant under magnetic $U(1)$ symmetry, with massless magnetic monopoles self–coupled through a local four–monopole interaction [1,2]:

$$L(x) = \bar{\chi}(x)i\gamma^\mu\partial_\mu\chi(x) + G[\bar{\chi}(x)\chi(x)]^2 - G_1[\bar{\chi}(x)\gamma_\mu\chi(x)][\bar{\chi}(x)\gamma^\mu\chi(x)],$$  (1.1)

where $\chi(x)$ is a massless magnetic monopole field, $G$ and $G_1$ are positive phenomenological constants. Below we will show that we have to choose $G_1 = G/4$ for the self consistency of the theory in the one loop approximation [3,4].

The magnetic monopole condensation accompanies the creation of $\bar{\chi}\chi$ collective excitations with the quantum numbers of a scalar Higgs meson field $\rho$ and a dual–vector field $C_\mu$.

For the derivation of an effective Lagrangian the $\rho$ and $C_\mu$ fields are introduced as cyclic variables.

$$L(x) = \bar{\chi}(x)i\gamma^\mu\partial_\mu\chi(x) - V(x),$$  (1.2)

where $V(x)$ is defined

$$-V(x) = \bar{\chi}(x)(-g\gamma^\mu C_\mu(x) - \kappa \rho(x))\chi(x) - \frac{\kappa^2}{4G}\rho^2(x) + \frac{g^2}{4G_1}C_\mu(x)C^\mu(x).$$  (1.3)

Now we can show that the vacuum expectation value (v.e.v) of the $\rho$ field does not vanish. For this aim we have to derive the equation of motion of the $\rho$ field by varying the Lagrangian Eq.(1.1) with respect to the $\rho$ field:

$$\frac{\partial L(x)}{\partial \rho(x)} = -\kappa \bar{\chi}(x)\chi(x) - \frac{\kappa^2}{4G}\rho(x) = 0.$$  (1.4)

This leads to

$$\rho(x) = -\frac{2G}{\kappa} \bar{\chi}(x)\chi(x).$$  (1.5)

Taking the v.e.v. of both sides of Eq.(1.4) we get

$$\langle \rho(x) \rangle = -\frac{2G}{\kappa} \langle \bar{\chi}(x)\chi(x) \rangle = -\frac{2G}{\kappa} \langle \bar{\chi}(0)\chi(0) \rangle,$$  (1.6)

where $\langle \bar{\chi}(0)\chi(0) \rangle$ is the magnetic monopole condensate. Thus, the non–zero value of the v.e.v. of the $\rho$ field is related to the monopole condensation. In order to deal with a physical scalar field, the $\sigma$–field, we should follow the standard procedure and subtract $\langle \rho(x) \rangle$. This gives $\sigma(x) = \rho(x) - \langle \rho(x) \rangle$. The v.e.v. of the $\rho$–field can be expressed in terms of the mass of the magnetic monopole $M$ in the superconducting phase [1–4], $\langle \rho(x) \rangle = M/\kappa$, where $M$ is proportional to $\langle \bar{\chi}(0)\chi(0) \rangle$ [1–4]

$$M = -2G \langle \bar{\chi}(0)\chi(0) \rangle.$$  (1.7)

This is the gap–equation testifying the appearance of massive magnetic monopoles in the superconducting phase, where $\langle \bar{\chi}(0)\chi(0) \rangle \neq 0$. As has been shown in Refs.[1,2] it also leads to the
suppression of direct transitions between the physical scalar field $\sigma$ and the non-perturbative vacuum.

In terms of the $\sigma$–field the Lagrangian Eq.(1.2) reads

$$\mathcal{L}(x) = \bar{\chi}(x) (i \gamma^\mu \partial_\mu - M) \chi(x) - \tilde{\mathcal{V}}(x),$$

where now $\tilde{\mathcal{V}}(x)$ reads

$$-\tilde{\mathcal{V}}(x) = \bar{\chi}(x) (-g \gamma^\mu C_\mu(x) - \kappa \sigma(x)) \chi(x) - \frac{\kappa^2}{4G} \rho^2(x) + \frac{g^2}{4G_1} C_\mu(x) C^\mu(x).$$

Integrating out the magnetic monopole fields we arrive at the effective Lagrangian

$$\mathcal{L}_{\text{eff}}(x) = \tilde{\mathcal{L}}_{\text{eff}} - \frac{\kappa^2}{4G} \rho^2(x) + \frac{g^2}{4G_1} C_\mu(x) C^\mu(x)$$

with $\tilde{\mathcal{L}}(x)_{\text{eff}}$ defined as

$$\tilde{\mathcal{L}}_{\text{eff}}(x) = -i \left[ x \left( \frac{1}{\det(i\hat{\beta} - M + \Phi)} \right)^{\eta} \right].$$

Here we have denoted $\Phi = -g \gamma^\mu C_\mu - \kappa \sigma$, and $\sigma = \rho - M/\kappa$.

The effective Lagrangian $\tilde{\mathcal{L}}_{\text{eff}}(x)$ can be represented by an infinite series

$$\tilde{\mathcal{L}}_{\text{eff}}(x) = \sum_{n=1}^{\infty} \frac{i}{n} \text{tr}_L \left( x \left( \frac{1}{M - i\hat{\beta}} \right) \chi \right) = \sum_{n=1}^{\infty} \tilde{\mathcal{L}}^{(n)}_{\text{eff}}(x).$$

The index $L$ means the evaluation of the trace over the Lorentz indices. The effective Lagrangian $\tilde{\mathcal{L}}^{(n)}_{\text{eff}}(x)$ is given by

$$\tilde{\mathcal{L}}^{(n)}_{\text{eff}}(x) = \int \frac{d^4x d^4k \chi(x, k_1, \ldots, k_n)}{(2\pi)^4} \left( i k_1 \cdot x \cdots i k_n \cdot x \left( -\frac{1}{n} \frac{1}{16\pi^2} \right) \int \frac{d^4k}{\pi^2} \right. \left. \times \text{tr}_L \left\{ \frac{1}{M - k} \Phi(x_1) \frac{1}{M - k - k_1} \Phi(x_2) \cdots \frac{1}{M - k - k_1 - \ldots - k_{n-1}} \Phi(x) \right\} \right).$$

at $k_1 + k_2 + \ldots + k_n = 0$. The r.h.s. of Eq.(1.13) describes the one–massive–monopole loop diagram with $n$–vertices. The monopole–loop diagrams with two vertices ($n = 2$) determine the kinetic term of the $\sigma$–field and give the contribution to the kinetic term of the $C_\mu$–field, while the diagrams with ($n \geq 3$) describe the vertices of interactions of the $\sigma$ and $C_\mu$ fields. In accordance with the prescription given in [1,2] the effective Lagrangian $\tilde{\mathcal{L}}_{\text{eff}}(x)$ should be defined by the set of divergent one–massive–monopole–loop diagrams with $n = 1, 2, 3$ and 4 vertices. The evaluation of these diagrams gives

$$\mathcal{L}_{\text{eff}}(x) = \frac{1}{2} \kappa^2 J_2(M) \partial_\mu \sigma(x) \partial^\mu \sigma(x) - M \left[ \frac{\kappa}{2G} - \frac{\kappa}{4\pi^2} J_1(M) \right] \sigma(x) +$$

$$+ \frac{1}{2} \left[ -\frac{\kappa^2}{2G} + \frac{\kappa^2}{4\pi^2} J_1(M) - 4M^2 \frac{\kappa^2}{8\pi^2} J_2(M) \right] \sigma^2(x).$$
\[-2M \kappa \frac{\kappa^2}{8 \pi^2} J_2(M) \sigma^3(x) - \frac{1}{2} \kappa^2 \frac{\kappa^2}{8 \pi^2} J_2(M) \sigma^4(x) - \frac{g^2}{48 \pi^2} J_2(M) C_{\mu\nu}(x) C(x)^{\mu\nu} + \left[ \frac{g^2}{4G_1} - \frac{g^2}{16 \pi^2} [J_1(M) + M^2 J_2(M)] \right] C_{\mu}(x) C^{\mu}(x), \tag{1.14} \]

where we have defined $C^{\mu\nu}(x) = \partial^\mu C^\nu(x) - \partial^\nu C^\mu(x)$. Then, $J_1(M)$ and $J_2(M)$ are the following quadratically and logarithmically divergent integrals

\[
J_1(M) = \int \frac{d^4 k}{\pi^2} \left( \frac{1}{(M^2 - k^2)^2} \right) = \Lambda^2 - M^2 \elln \left(1 + \frac{\Lambda^2}{M^2}\right) - \frac{\Lambda^2}{M^2 + \Lambda^2},
\]

\[
J_2(M) = \int \frac{d^4 k}{\pi^2} \left( \frac{1}{(M^2 - k^2)^2} \right) = \elln \left(1 + \frac{\Lambda^2}{M^2}\right) - \frac{\Lambda^2}{M^2 + \Lambda^2}. \tag{1.15} \]

In order to get correct factors of the $\sigma$ and $C_{\mu}$ field kinetic terms we have to set [1,2]

\[
\frac{g^2}{12 \pi^2} J_2(M) = 1, \quad \frac{\kappa^2}{8 \pi^2} J_2(M) = 1. \tag{1.16} \]

So, the coupling constants are connected by the relation $\kappa^2 = 2g^2/3$ [1,2].

The effective Lagrangian Eq. (1.14) contains a term linear in the $\sigma$–field. This part of the effective Lagrangian leads to direct transitions $\sigma \to$ vacuum. In the case of a physical $\sigma$–field such transitions should be suppressed. In order to suppress these transitions we have to impose the constraint [1–4]

\[
\frac{1}{G} - \frac{1}{2 \pi^2} J_1(M) = 0, \tag{1.17} \]

where $J_1(M)$ can be connected with the monopole condensate [1–4]

\[
< \bar{\chi}(0) \chi(0) >= - \frac{1}{4\pi^2} MJ_1(M). \tag{1.18} \]

Inserting Eq. (1.18) into Eq. (1.17) we arrive at the gap–equation (1.7)

The coefficient in front of the last term in Eq. (1.14) defines the mass of the $C_{\mu}$ field:

\[
M_C^2 = \frac{g^2}{2G_1} - \frac{g^2}{8 \pi^2} [J_1(M) + M^2 J_2(M)]. \tag{1.19} \]

Piling up the gap–equation (1.7) and the constraint (1.16) we recast the effective Lagrangian (1.14) into the form

\[
\mathcal{L}_{\text{eff}}(x) = -\frac{1}{4} C_{\mu\nu}(x) C^{\mu\nu}(x) + \frac{1}{2} M_C^2 C_{\mu}(x) C^{\mu}(x) + \frac{1}{2} \partial_{\mu} \sigma(x) \partial^{\mu} \sigma(x) - \frac{1}{2} M_{\sigma}^2 \sigma^2(x) \left[1 + \kappa \frac{\sigma(x)}{M_{\sigma}}\right]^2 + \frac{1}{2} \partial_{\mu} \sigma(x) \partial^{\mu} \sigma(x) - \frac{1}{2} M_{\sigma}^2 \sigma^2(x) + \mathcal{L}_{\text{int}}[\sigma(x)], \tag{1.20} \]
where \( M_\sigma = 2 M \) is the mass of the \( \sigma \)-field and \( \mathcal{L}_{\text{int}}[\sigma(x)] \) describes the self–interactions of the \( \sigma \)-field

\[
\mathcal{L}_{\text{int}}[\sigma(x)] = -\kappa M_\sigma \sigma^3(x) - \frac{1}{2} \kappa^2 \sigma^4(x).
\]  

(1.21)

As has been shown in Ref.[3,4] the gap–equation, in turn, can be derived in the one–monopole loop approximation by using only the Lagrangian Eq.(1.1). When comparing these two gap–equations we get the relation \( G_1 = G/4 \) that reduces the number of input parameters [3,4].

**Wave function of the non–perturbative vacuum.** The MNJL model as well as the NJL model [6] and the BCS theory of superconductivity [7] possesses a non–trivial non–perturbative vacuum with a wave function [2]

\[
|0\rangle^{(M)} = \prod_{\vec{p}, \lambda = \pm 1} \left[ \sqrt{\frac{1+\beta_{\vec{p}}^2}{2}} + \lambda \sqrt{\frac{1-\beta_{\vec{p}}^2}{2}} a^{(0)\dagger}(\vec{p}, \lambda) b^{(0)\dagger}(\vec{p}, \lambda) \right] |0\rangle,
\]  

(1.22)

where \( \beta_{\vec{p}} = \vec{p}/E_{\vec{p}} = \vec{p}/\sqrt{\vec{p}^2 + M^2} \) is the velocity of a massive monopole with the mass \( M \), and \( a^{(0)\dagger}(\vec{p}, \lambda) \) (or \( b^{(0)\dagger}(\vec{p}, \lambda) \)) denotes the creation operator of a massless monopole ( or anti–monopole) with a momentum \( \vec{p} \) and helicity \( \lambda \); \( |0\rangle = |0\rangle^{(0)} \) is the wave function of the perturbative vacuum of the non–condensed phase. The wave function \( |0\rangle^{(M)} \) of the non–perturbative vacuum is distinctly invariant under magnetic \( U(1) \) symmetry. The former implies that in the condensed phase the magnetic \( U(1) \) symmetry is not broken [2]. This is the main peculiarity of the MNJL model with respect to the dual Higgs model with dual Dirac strings [8–10], where the magnetic \( U(1) \) symmetry is broken spontaneously in the superconducting phase.

The wave function of the non–perturbative vacuum \( |0\rangle^{(M)} \) is invariant under parity transformations \( \mathcal{P} \), \( \mathcal{P}|0\rangle^{(M)} = |0\rangle^{(M)} \).

In order to show that the magnetic monopole condensate \( \langle \bar{\chi}(0)\chi(0) \rangle \) has a distinct meaning of the order parameter we, following the BCS theory of superconductivity [7], can introduce two operators possessing different properties under parity transformations

\[
\mathcal{O}_+ = 2 \sum_{\vec{p}} \sum_{\lambda = \pm 1} \lambda b^{(0)\dagger}(\vec{p}, \lambda) a^{(0)\dagger}(\vec{p}, \lambda), \quad \mathcal{P}\mathcal{O}_+\mathcal{P}^\dagger = + \mathcal{O}_+,
\]

\[
\mathcal{O}_- = 2 \sum_{\vec{p}} \sum_{\lambda = \pm 1} b^{(0)\dagger}(\vec{p}, \lambda) a^{(0)\dagger}(\vec{p}, \lambda), \quad \mathcal{P}\mathcal{O}_-\mathcal{P}^\dagger = - \mathcal{O}_-. \quad (1.23)
\]

As has been shown in Ref.[2] the v.e.v of the \( \mathcal{O}_+ \) operator per unit volume coincides with the magnetic monopole condensate \( \langle \bar{\chi}(0)\chi(0) \rangle \):

\[
\langle \mathcal{O}_+ \rangle = \frac{1}{V}^{(M)}|0\rangle\langle 0|^{(M)} = -\frac{1}{V} \sum_{\vec{p}} \sum_{\lambda = \pm 1} \lambda^2 \sqrt{1-\beta_{\vec{p}}^2} = -4M \sum_{\vec{p}} \frac{1}{2E_{\vec{p}}V} = -4M \int \frac{d^3p}{2E_{\vec{p}}} = -4M \int \frac{d^4p}{(2\pi)^4i} \frac{1}{M^2 - \vec{p}^2 - i 0} = -\frac{M}{4\pi^2} J_1(M) = \langle \bar{\chi}(0)\chi(0) \rangle,
\]  

(1.24)

where \( V \) is a normalization volume. In turn, one can show that \( \langle \mathcal{O}_- \rangle = 0 \). This confirms the parity conservation in the MNJL model.

**Dual Dirac strings.** Dual Dirac strings are included in the MNJL model in the form of a dual electric field strength \( \mathcal{E}^{\mu\nu}(x) \) defined by [1–4]

\[
\mathcal{E}^{\mu\nu}(x) = Q \int \int d\tau d\sigma \left( \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\nu}{\partial \tau} \frac{\partial X^\mu}{\partial \sigma} \right) \delta^{(4)}(x - X), \quad (1.25)
\]
where \( X^\mu = X^\mu(\tau, \sigma) \) represents the position of a point on the world sheet swept by the string. The sheet is parameterized by internal coordinates \(-\infty < \tau < \infty\) and \(0 \leq \sigma \leq \pi\), so that \( X^\mu(\tau, 0) = X^\mu_Q(\tau) \) and \( X^\mu(\tau, \pi) = X^\mu_{\bar{Q}}(\tau) \) represent the world lines of an anti-quark and a quark \([1–4,8–10]\).

The effective Lagrangian of the dual Higgs field \( \sigma(x) \) and vector \( C^\mu(x) \) fields is then defined

\[
\mathcal{L}_{\text{eff}}(x) = \frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} M_C^2 C_\mu(x) C^\mu(x) + \\
\frac{1}{2} \partial_\mu \sigma(x) \partial^\mu \sigma(x) - \frac{1}{2} M_\sigma^2 \sigma^2(x) \left[ 1 + \kappa \frac{\sigma(x)}{M_\sigma} \right]^2,
\]

where

\[
= \frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} M_C^2 C_\mu(x) C^\mu(x) + \\
\frac{1}{2} \partial_\mu \sigma(x) \partial^\mu \sigma(x) - \frac{1}{2} M_\sigma^2 \sigma^2(x) + \mathcal{L}_{\text{int}}[\sigma(x)],
\]

(1.26)

where the field strength \( F^{\mu\nu}(x) \) reads \([1–4,8–10]\): \( F^{\mu\nu}(x) = \mathcal{E}^{\mu\nu}(x) - *C^{\mu\nu}(x) \) and \( *C^{\mu\nu}(x) \) is the dual version of \( C^{\mu\nu}(x) \), \( *C^{\mu\nu}(x) = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} C_{\alpha\beta}(x) (\varepsilon^{0123} = 1) \).

**Quarks and anti-quarks.** In the MNJL model a quark and an anti-quarks are point-like particles with masses \( m_q = m_{\bar{q}} = m \), electric charges \( Q_q = -Q_{\bar{q}} = Q \), and trajectories \( X_q^\mu(\tau) \) and \( X_{\bar{q}}^\mu(\tau) \), respectively, attached to the ends of a dual Dirac string. They are described by the Lagrangian

\[
\mathcal{L}_{\text{free quark}}(x) = - \sum_{i=q,\bar{q}} m_i \int d\tau \left( \frac{d X_i^\mu(\tau)}{d\tau} \frac{d X_{\bar{i}}^\mu(\tau)}{d\tau} g^{\mu\nu} \right)^{1/2} \delta^{(4)}(x - X_i(\tau)).
\]

(1.27)

Substituting Eq.(1.27) in Eq.(1.26) we arrive at the total effective Lagrangian of the MNJL model with dual Dirac strings, quarks and anti-quarks

\[
\mathcal{L}_{\text{eff}}(x) = \frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} M_C^2 C_\mu(x) C^\mu(x) + \\
\frac{1}{2} \partial_\mu \sigma(x) \partial^\mu \sigma(x) - \frac{1}{2} M_\sigma^2 \sigma^2(x) \left[ 1 + \kappa \frac{\sigma(x)}{M_\sigma} \right]^2,
\]

(1.28)

This effective Lagrangian should be used for the evaluation of different observables defined in terms of vacuum expectation values related to the magnetic monopole Green functions \([1–4]\).

**Magnetic monopole Green functions.** The \( n \)–point magnetic monopole Green function can be defined as the vacuum expectation value of the time ordered product of the massless magnetic monopole densities Refs.\([1–4,8–10]\):

\[
G(x_1, \ldots, x_n) = \langle 0| T(\bar{\chi}(x_1) \Gamma_1 \chi(x_1) \ldots \bar{\chi}(x_n) \Gamma_n \chi(x_n))|0\rangle_{\text{conn}},
\]

(1.29)

where \( \Gamma_i(i = 1, \ldots, n) \) are the Dirac matrices. As has been shown in Ref.\([11]\) the vacuum expectation value Eq.(1.29) can be represented in terms of the vacuum expectation values of the densities of the massive magnetic monopole fields \( \chi_M(x) \) coupled to the fields of the collective
excitations $\sigma$ and $C_\mu$

$$G(x_1, \ldots, x_n) = \langle 0| T(\bar{\chi}(x_1)\Gamma_1\chi(x_1) \cdots \bar{\chi}(x_n)\Gamma_n\chi(x_n))|0\rangle_{\text{conn.}} =$$

$$= (M) \langle 0| T(\bar{\chi}_M(x_1)\Gamma_1\chi_M(x_1) \cdots \bar{\chi}_M(x_n)\Gamma_n\chi_M(x_n))$$

$$\times \exp i \int d^4x \{ -g\bar{\chi}_M(x)\gamma^\nu\chi_M(x)C_\nu(x) - \kappa\bar{\chi}_M(x)\chi_M(x)\sigma(x) + \mathcal{L}_{\text{int}}[\sigma(x)] \} |0\rangle_{\text{conn.}}^{(M)}.$$  

(1.30)

Here $|0\rangle_{\text{conn.}}^{(M)}$ is the wave function of the non–perturbative vacuum of the MNJL model in the condensed phase and $|0\rangle$ the wave function of the perturbative vacuum of the non–condensed phase.

The self–interactions $\mathcal{L}_{\text{int}}[\sigma(x)]$ provide $\sigma$–field loop contributions and can be dropped out in the tree $\sigma$–field approximation Refs. [1–4]. The tree $\sigma$–field approximation can be justified keeping massive magnetic monopoles very heavy, i.e. $M \gg M_C$. This corresponds to the London limit $M_\sigma = 2M \gg M_C$ in the dual Higgs model with dual Dirac strings [8–10]. The inequality $M_\sigma \gg M_C$ means also that in the MNJL model we deal with Dual Superconductivity of type II [12]. In the tree $\sigma$–field approximation the r.h.s. of Eq. (1.30) can be recast into the form

$$G(x_1, \ldots, x_n) = < 0| T(\bar{\chi}(x_1)\Gamma_1\chi(x_1) \cdots \bar{\chi}(x_n)\Gamma_n\chi(x_n))|0 >_{\text{conn.}} =$$

$$= (M) < 0| T(\bar{\chi}_M(x_1)\Gamma_1\chi_M(x_1) \cdots \bar{\chi}_M(x_n)\Gamma_n\chi_M(x_n))$$

$$\exp i \int d^4x \{ -g\bar{\chi}_M(x)\gamma^\nu\chi_M(x)C_\nu(x) - \kappa\bar{\chi}_M(x)\chi_M(x)\sigma(x) \} |0 >_{\text{conn.}}^{(M)}.$$  

(1.31)

For the subsequent investigation it is convenient to represent the r.h.s. of Eq. (1.31) in terms of the generating functional of the monopole Green functions [1–4]

$$G(x_1, \ldots, x_n) = \prod_{i=1}^n \frac{\delta}{\delta \eta(x_i)} \frac{\delta}{\delta \bar{\eta}(x_i)} Z[\eta, \bar{\eta}] \bigg|_{\eta=\bar{\eta}=0} ,$$  

(1.32)

where $\bar{\eta}(\eta)$ are the external sources of the massive monopole (antimonopole) fields, and $Z[\eta, \bar{\eta}]$ is the generating functional of the monopole Green functions defined by

$$Z[\eta, \bar{\eta}] = \frac{1}{Z} \int D\chi_M D\bar{\chi}_M DC_\mu D\bar{C}_\mu \exp i \int d^4x \left[ \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2} M_C^2 C_\mu(x) C^\mu(x) + \frac{1}{2} \bar{\eta}_\mu \bar{C}_\mu(x) - \frac{1}{2} M_\sigma^2 \sigma^2(x) + \bar{\chi}_M(x)i \sigma^\mu \partial_\mu - M - g \gamma^\mu C_\mu(x) - \kappa \sigma(x) \right] \chi_M(x)$$

$$+ \bar{\eta}(x) + \bar{\chi}_M(x) \eta(x) + \mathcal{L}_{\text{free quark}}(x) \bigg] .$$  

(1.33)

The normalization factor $Z$ is defined by the condition $Z[0,0] = 1$.

The paper is organized as follows. In Sect. 2 we discuss the gluon condensate in non–perturbative QCD both in the QCD sum rules approach and in lattice simulations. In Sect. 3 we calculate the dual–vector field condensate and compare the obtained result with the gluon condensate of non–perturbative QCD. In the Conclusion we discuss the obtained results.

## 2 Gluon condensate of non–perturbative QCD

For the first time, the condensate of the gluon fields

$$\left\langle \frac{g^2}{4\pi^2} G^a_{\mu\nu}(0) G^{a\mu\nu}(0) \right\rangle \neq 0,$$  

(2.1)
where \( g_s \) and \( G_{\mu\nu}^a(0) \) \((a = 1, \ldots, 8)\) are the quark–gluon coupling constant and the gluon field strength, has been introduced as a phenomenological parameter characterizing quantitatively non–trivial properties of a non–perturbative vacuum of QCD in the QCD sum rules approach [13]. The gluon condensate breaks the dilatation invariance at a scale of energy transferred of order \( \Lambda_D \sim 4 \text{ GeV} \) [14]. This gives a signal to the breaking of chiral symmetry which becomes broken spontaneously at a scale of energy transferred of order \( \Lambda_\chi \sim 1 \text{ GeV} \) [14]. As has been shown in Ref.[15] the contribution of the gluon condensate to the condensate of light quarks \( u, d \) and \( s \) makes up \( 1/3 \) of the meanvalue of the quark condensate \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle = -0.253 \text{ GeV} \) [15].

The numerical value of the gluon condensate

\[
\left\langle \frac{g_s^2}{4\pi^2} G_{\mu\nu}^a(0) G_{\rho\sigma}^a(0) \right\rangle = (0.331 \text{ GeV})^4 \quad (2.2)
\]

obtained in Ref.[13] by studying the charmonium channel has been obviously underestimated [16–18]. The correct value of the gluon condensate increased by a factor of 4 has been obtained in Ref.[19]. The meanvalue of the gluon condensate is equal to [19]

\[
\left\langle \frac{g_s^2}{4\pi^2} G_{\mu\nu}^a(0) G_{\rho\sigma}^a(0) \right\rangle = (0.458 \text{ GeV})^4. \quad (2.3)
\]

In the dilute–instanton gas approximation the gluon condensate has been defined as [13]

\[
\left\langle \frac{g_s^2}{4\pi^2} G_{\mu\nu}^a(0) G_{\rho\sigma}^a(0) \right\rangle_{\text{inst.+anti–inst.}} = 16 \int_0^{\rho_c} \frac{d\rho}{\rho^5} d(\rho), \quad (2.4)
\]

where \( d(\rho) \) is the instanton density function. For the \( SU(N) \) gauge group the instanton density function is defined by [20]

\[
d(\rho) = \frac{C_1 e^{-C_2 N}}{(N-1)! (N-2)!} \left[ \frac{8\pi^2}{g_s^2(\rho)} \right]^{2N} e^{-8\pi^2/g_s^2(\rho)}. \quad (2.5)
\]

The coefficients \( C_1 \) and \( C_2 \) are given by [20]

\[
C_1 = \frac{2 e^{5/6}}{\pi^2} = 0.466, \quad \quad C_2 = \frac{5}{3} \ell_n 2 - \frac{11}{36} + \frac{1}{3} (\ell_n (2\pi) + \gamma) + \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{\ell_n n}{n^2} = 1.679, \quad (2.6)
\]

where \( \gamma = 0.5277\ldots \) is Euler’s constant.

In the absence of quark contributions and in the one–loop approximation the running coupling constant \( g(\rho) \) is determined [20]

\[
g_s^2(\rho) = \frac{g_s^2(\rho_0)}{1 + \frac{11}{3} N \frac{g_s^2(\rho_0)}{8\pi^2} \frac{\ell_n \rho}{\rho_0}}, \quad (2.7)
\]

where \( \rho_0 = 1/\Lambda_U \) and \( \Lambda_U \) is the ultra–violet cut–off. Then, the integral over \( \rho \) defining the gluon condensate Eq.(2.4) is infrared divergent and the parameter \( \rho_c = 1/\Lambda_R \) plays the role of the infrared cut–off [13]. According to estimates of Ref.[13] the infrared cut–off should of order \( \Lambda_R \sim 200 \text{ MeV} \).
Thus, in the absence of quark contributions and in the one–loop approximation the instanton density function \(d(\rho)\) reads [20]

\[
d(\rho) = \frac{C_1 e^{-C_2 N}}{(N-1)!/(N-2)!} \left( \frac{8\pi^2}{g_s^2(\Lambda_U)} - \frac{11}{3} N \ell n(\rho \Lambda_U) \right)^{2N} e^{-8\pi^2/g_s^2(\Lambda_U)}.
\] (2.8)

For QCD when \(N = 3\) we obtain

\[
d(\rho) = 1.513 10^{-3} (\rho \Lambda_U)^{11} \left[ \frac{8\pi^2}{g_s^2(\Lambda_U)} - 11 \ell n(\rho \Lambda_U) \right]^6 e^{-8\pi^2/g_s^2(\Lambda_U)}.
\] (2.9)

Substituting Eq.(2.9) in Eq.(2.4) one can see that the integral over \(\rho\) is concentrated near the infrared cut–off \(\rho_c = 1/\Lambda_R\). Therefore, the result of the integration can be given by the expression [13,20]

\[
\langle g_s^2 4\pi^2 G_{a\mu\nu}(0) G^{a\mu\nu}(0) \rangle_{\text{inst.+anti–inst.}} \approx 1.513 10^{-3} \frac{16}{7} \Lambda^4_R \left[ \frac{8\pi^2}{g_s^2(\Lambda_R)} \right]^6 e^{-8\pi^2/g_s^2(\Lambda_R)}
\]
\[
\approx 3.458 10^{-3} \Lambda^4_R \left[ \frac{8\pi^2}{g_s^2(\Lambda_R)} \right]^6 e^{-8\pi^2/g_s^2(\Lambda_R)}.
\] (2.10)

According to lattice formulation of QCD the gluon condensate can be represented in the following general form [21]

\[
\frac{1}{\Lambda^4_U} \langle g_s^2 4\pi^2 G_{a\mu\nu}(0) G^{a\mu\nu}(0) \rangle = W_{40} + W_{04} \frac{\Lambda^4_{QCD}}{\Lambda^4_U} + W_{22} \frac{\Lambda^2_{QCD}}{\Lambda^2_U} + \ldots,
\] (2.11)

where \(W_{mn}\) are the numerical coefficients and \(\Lambda_{QCD}\) enters to the definition of the running coupling constant [19,22]

\[
\alpha_s(\rho) = \frac{g_s^2(\rho)}{4\pi} = \frac{4\pi}{b_0} \frac{1}{\ell n(\rho \Lambda_{QCD})},
\] (2.12)

where \(b_0 = (11/3)N\) [22]. The typical value of \(\Lambda_{QCD}\) is \(\Lambda_{QCD} = 100 \div 300\) MeV [22].

The coefficient \(W_{40}\) describes a perturbative contribution to the gluon condensate, whilst the coefficients \(W_{04}\) and \(W_{22}\) are fully non–perturbative. The presence of the term proportional to \(\Lambda^2_{QCD}/\Lambda^2_U\) differs the expression for the gluon condensate obtained within the dilute–instanton gas approximation from the gluon condensate calculated on lattice. The appearance of this term is related to power corrections [13,21], whereas the expression Eq.(2.10) corresponds to the leading order contribution in power expansion [13]. The gluon condensate given by Eqs.(2.10) and (2.11) we would compare with the condensate of the dual–vector field \(C_\mu\) which we calculate below in the MNJL model.

### 3 Dual–vector field condensate in the MNJL model

In the MNJL model we defined the condensate of the dual–vector field \(C_\mu\) by analogy with the magnetic monopole condensate [2–4]

\[
\langle \frac{g_s^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \rangle = \frac{1}{Z} \int \int \int D\chi D\bar{\chi} DC^\mu D\sigma \left[ \frac{g_s^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right]
\]
\[
\exp i \int d^4z \left[ \frac{1}{2} C_\nu(z) \left( \Box + M_\xi^2 \right) C^\nu(z) + C^\nu(z) \partial^{\mu*} E_{\mu\nu}(z) \right. \\
+ \frac{1}{2} \sigma(z) \left( \Box + M_\sigma^2 \right) \sigma(z) + \mathcal{L}_{\text{int}}[\sigma(z)] \\
- g \bar{\chi}_M(z) \gamma_\nu \chi_M(z) C^\nu(z) - \kappa \bar{\chi}_M(z) \chi_M(z) \sigma(z) + \bar{\chi}_M(z) (i \gamma_\nu \partial^\nu - M) \chi_M(z) \biggr], 
\]

where \( Z \) is the normalization constant

\[
Z = \int \frac{\mathcal{D}x \mathcal{D}\bar{x} \mathcal{D}c \mathcal{D}\bar{c}}{Z} \exp i \int d^4z \left[ \frac{1}{2} C_\nu(z) \left( \Box + M_\xi^2 \right) C^\nu(z) \\
+ C^\nu(z) \partial^{\mu*} E_{\mu\nu}(z) + \frac{1}{2} \sigma(z) \left( \Box + M_\sigma^2 \right) \sigma(z) + \mathcal{L}_{\text{int}}[\sigma(z)] \\
- g \bar{\chi}_M(z) \gamma_\nu \chi_M(z) C^\nu(z) - \kappa \bar{\chi}_M(z) \chi_M(z) \sigma(z) + \bar{\chi}_M(z) (i \gamma_\nu \partial^\nu - M) \chi_M(z) \biggr]. 
\]

Recall that we are working in the London limit, \( M_\sigma = 2M \gg M_C \). One can show that in this limit the contribution of the \( \sigma \) field exchanges to the dual–vector field condensate are insignificant and can be neglected with respect to the contributions of the dual–vector field exchanges. As a result the integral over the \( \sigma \) field can be absorbed by the normalization constant \( Z \)

\[
\left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \frac{1}{Z} \int d^4z \left[ \frac{g^2}{4\pi^2} C_\nu(x) C^{\mu\nu}(x) \right]
\]

\[
\exp i \int d^4z \left[ \frac{1}{2} C_\nu(z) \left( \Box + M_\xi^2 \right) C^\nu(z) + C^\nu(z) \partial^{\mu*} E_{\mu\nu}(z) - g \bar{\chi}_M(z) \gamma_\nu \chi_M(z) C^\nu(z) \\
+ \bar{\chi}_M(z) (i \gamma_\nu \partial^\nu - M) \chi_M(z) \biggr]. 
\]

The integration over the dual–vector field \( C_\mu \) we perform around the Abrikosov flux line. For this aim we represent the dual–vector field \( C_\mu \) as follows [3,4,9,10]

\[
C_\nu(x) = C_\nu[\mathcal{E}(x)] + c_\nu(x).
\]

The field \( C_\mu[\mathcal{E}(x)] \) is the Abrikosov flux line induced by a dual Dirac string obeying the equation [1–4,8–10]

\[
\left( \Box + M_C^2 \right) C_\nu[\mathcal{E}(x)] = -\partial^{\mu*} E_{\mu\nu}(x) 
\]

possessing the solution

\[
C_\mu[\mathcal{E}(x)] = - \int d^4x' \Delta \left( x - x', M_C \right) \partial^{\mu*} E_{\mu\nu}(x'),
\]

where \( \Delta \left( x - x', M_C \right) \) is the Green function

\[
\Delta \left( x - x', M_C \right) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i k \cdot (x - x')}}{M_C^2 - k^2 - i0}. 
\]

The field \( c_\mu(x) \) stands for the quantum fluctuations of the dual–vector field around the Abrikosov flux line [2–4,9–10]. Denoting \( \bar{F}_{\mu\nu}[\mathcal{E}(x)] = C_{\mu\nu}[\mathcal{E}(x)] \) and subtracting a trivial contribution proportional to \( \bar{F}_{\mu\nu}[\mathcal{E}(x)] \bar{F}^{\mu\nu}[\mathcal{E}(x)] \) we obtain

\[
\delta \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle - \frac{g^2}{4\pi^2} \bar{F}_{\mu\nu}[\mathcal{E}(x)] \bar{F}^{\mu\nu}[\mathcal{E}(x)] = \frac{1}{Z} \frac{g^2}{2\pi^2} \\
\times \int \mathcal{D}x \mathcal{D}\bar{x} \mathcal{D}c \mathcal{D}\bar{c} \left\{ \partial^\mu c^\nu(x) \left( \partial_\mu c_\nu(x) - \partial_\nu c_\mu(x) \right) + \bar{F}^{\mu\nu}[\mathcal{E}(x)] \left( \partial_\mu c_\nu(x) - \partial_\nu c_\mu(x) \right) \right\} \\
\exp i \int d^4z \left[ \frac{1}{2} c_\nu(z) \left( \Box + M_\xi^2 \right) c^\nu(z) - g \bar{\chi}_M(z) \gamma_\nu \chi_M(z) c^\nu(z) \\
- g \bar{\chi}_M(z) \gamma_\nu \chi_M(z) C^{\nu}(\mathcal{E}(z)) + \bar{\chi}_M(z) (i \gamma^\nu \partial_\nu - M) \chi_M(z) \biggr]. 
\]
For the integration over the $c_\mu$-field we suggest consider the auxiliary path integral

$$
\mathcal{J}(\bar{\chi}_M, \chi_M, C^\nu[\mathcal{E}]) = \int Dc^\mu \{ \partial_\mu c_\nu(x) (\partial_\mu c_\nu(x) - \partial_\nu c_\mu(x)) + \bar{F}^{\mu\nu}[\mathcal{E}(x)] (\partial_\mu c_\nu(x) - \partial_\nu c_\mu(x)) \}
\exp i \int d^4z \left[ \frac{1}{2} c_\nu(z) (\Box + M_\nu^2) e^\nu(z) - e^\nu(z) J_\nu(z) \right],
$$

(3.9)

where $J_\nu(z)$ is a conserving current, $\partial^\nu J_\nu(z) = 0$, defined by

$$
J_\nu(z) = g \bar{\chi}_M(z) \gamma_\nu \chi_M(z) + j_\nu(z)
$$

(3.10)

and $j_\nu(z)$ is an external source which should be put zero finally.

The r.h.s. of Eq. (3.9) can be represented in the form of functional derivatives with respect to $j_\nu(x)$

$$
\mathcal{J}(\bar{\chi}_M, \chi_M, C^\nu[\mathcal{E}]) = \left\{ \frac{\partial}{\partial x^\mu} \right\} \frac{\delta}{\delta j_\nu(x)} \left( \frac{\partial}{\partial x^\mu} \frac{\delta}{\delta j_\nu(x)} - \frac{\partial}{\partial x^\nu} \frac{\delta}{\delta j_\mu(x)} \right) + i \bar{F}^{\mu\nu}[\mathcal{E}(x)] \left( \frac{\partial}{\partial x^\mu} \frac{\delta}{\delta j_\nu(x)} - \frac{\partial}{\partial x^\nu} \frac{\delta}{\delta j_\mu(x)} \right) \right\}
\times \int Dc^\mu \exp i \int d^4z \left[ \frac{1}{2} c_\nu(z) (\Box + M_\nu^2) e^\nu(z) - e^\nu(z) J_\nu(z) \right]_{j_\nu=0}.
$$

(3.11)

At $J_\nu(z) = 0$ the path integral over $c^\mu$ is normalized to unity. The integral over $c^\mu$ is a Gaussian and the result of the integration reads

$$
\int Dc^\mu \exp i \int d^4z \left[ \frac{1}{2} c_\nu(z) (\Box + M_\nu^2) e^\nu(z) - e^\nu(z) J_\nu(z) \right] = \exp \left[ -i \frac{1}{2} \int d^4z \int d^4z' J_\nu(z) \Delta(z - z', M_C) J^\nu(z') \right],
$$

(3.12)

where the Green function $\Delta(z - z', M_C)$ is given by Eq. (3.7).

The functional derivatives with respect to $j_\nu(x)$ are equal to

$$
\frac{\partial}{\partial x^\mu} \frac{\delta}{\delta j_\nu(x)} \exp \left[ -i \frac{1}{2} \int d^4z \int d^4z' J_\nu(z) \Delta(z - z', M_C) J^\nu(z') \right] = \left[ -\int d^4z \Box_x \Delta(x - z, M_C) J_\nu(z) \int d^4z' \Delta(x - z', M_C) J^\nu(z') \right]
$$
where we have taken into account that \( \partial \mu J_\mu(z) = 0 \).

The functional \( \mathcal{J}(\bar{\chi}_M, \chi_M, C^{\mu}[\mathcal{E}]) \) is then defined by

\[
\mathcal{J}(\bar{\chi}_M, \chi_M, C^{\mu}[\mathcal{E}]) = \\
= \left\{ g^2 \int d^4 z \Box \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \int d^4 z' \Delta(x - z', M_C) \bar{\chi}_M(z') \chi_M(z') \right\} \\
+ g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) \bar{\chi}_M(z') \chi_M(z') \\
- g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) \bar{\chi}_M(z') \chi_M(z') \\
+ 2 g \tilde{F}^{\mu \nu}[\mathcal{E}(x)] \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \right\} \\
\times \exp \left\{ -ig^2 \frac{1}{2} \int d^4 z d^4 z' \left[ \bar{\chi}_M(z) \chi_M(z') \right] \Delta(z - z', M_C) \left[ \bar{\chi}_M(z') \chi_M(z') \right] \right\}. 
\tag{3.14}
\]

Substituting the functional \( \mathcal{J}(\bar{\chi}_M, \chi_M, C^{\mu}[\mathcal{E}]) \) in the integrand of the r.h.s. of Eq. (3.8) we express the dual–vector field condensate as a path integral over the massive magnetic monopole fields.

\[
\delta \left\{ \frac{g^2}{4\pi^2} C_{\mu \nu}(x) C^{\mu \nu}(x) \right\} = \frac{1}{Z} \frac{g^2}{2\pi^2} \int \int \int D\chi D\bar{\chi} \\
\left\{ g^2 \int d^4 z \Box \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \int d^4 z' \Delta(x - z', M_C) \bar{\chi}_M(z') \chi_M(z') \right\} \\
+ g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) \bar{\chi}_M(z') \chi_M(z') \\
- g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) \bar{\chi}_M(z') \chi_M(z') \\
+ 2 g \tilde{F}^{\mu \nu}[\mathcal{E}(x)] \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) \bar{\chi}_M(z) \chi_M(z) \right\} \\
\times \exp i \int d^4 z \left\{ \bar{\chi}_M(z) (i \gamma_\nu \partial_\nu - M) \chi_M(z) - g \bar{\chi}_M(z) \gamma_\nu \chi_M(z) C^{\mu}[\mathcal{E}(z)] \\
- ig^2 \frac{1}{2} \int d^4 z d^4 z' \left[ \bar{\chi}_M(z) \chi_M(z') \right] \Delta(z - z', M_C) \left[ \bar{\chi}_M(z') \chi_M(z') \right] \right\}. 
\tag{3.15}
\]
monopole interactions. This reduces the r.h.s. of Eq. (3.15) to the form \([3,4]\)

\[
\delta \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \frac{1}{Z} \frac{g^2}{2\pi^2} \int \int D\chi D\bar{\chi}
\]

\[
\left\{ g^2 \int d^4 z \Box x \Delta(x - z, M_C) [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] \int d^4 z' \Delta(x - z', M_C) [\bar{\chi}_M(z')\gamma''\chi_M(z')]
\right.
\]

\[
+ g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) [\bar{\chi}_M(z')\gamma''\chi_M(z')]
\]

\[
- g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) [\bar{\chi}_M(z')\gamma''\chi_M(z')]
\]

\[
\times \exp \int d^4 z \left\{ \bar{\chi}_M(z) (i \gamma'' \partial_\nu - M) \chi_M(z) - g \bar{\chi}_M(z)\gamma_\nu\chi_M(z) C''[\epsilon(z)]
\right.
\]

\[
- \frac{g^2}{2M_C^2} [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] [\bar{\chi}_M(z)\gamma''\chi_M(z)] \right\}
\]

\( (3.16) \)

For the integration over the massive magnetic monopole fields it is convenient to decompose the r.h.s. of Eq. (3.16) conventionally into two parts

\[
\delta \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle_{\text{non-string}}
\]

\[
+ \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle_{\text{string}},
\]

where the terms are given by

\[
\left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle_{\text{non-string}} = \frac{1}{Z} \frac{g^2}{2\pi^2} \int D\chi D\bar{\chi}
\]

\[
\left\{ g^2 \int d^4 z \Box x \Delta(x - z, M_C) [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] \int d^4 z' \Delta(x - z', M_C) [\bar{\chi}_M(z')\gamma''\chi_M(z')]
\right.
\]

\[
+ g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) [\bar{\chi}_M(z')\gamma''\chi_M(z')]
\]

\[
- g^2 \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] \int d^4 z' \frac{\partial}{\partial x_\mu} \Delta(x - z', M_C) [\bar{\chi}_M(z')\gamma''\chi_M(z')]
\]

\[
\times \exp \int d^4 z \left\{ \bar{\chi}_M(z) (i \gamma'' \partial_\nu - M) \chi_M(z) - g \bar{\chi}_M(z)\gamma_\nu\chi_M(z) C''[\epsilon(z)]
\right.
\]

\[
- \frac{g^2}{2M_C^2} [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] [\bar{\chi}_M(z)\gamma''\chi_M(z)] \right\}
\]

\( (3.18) \)

and

\[
\left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle_{\text{string}} = \frac{1}{Z} \frac{g^2}{2\pi^2} \bar{F}^{\mu\nu}[\epsilon(x)] \int \int D\chi D\bar{\chi}
\]

\[
\times \int d^4 z \frac{\partial}{\partial x_\mu} \Delta(x - z, M_C) g [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)]
\]

\[
\times \exp \int d^4 z \left\{ \bar{\chi}_M(z) (i \gamma'' \partial_\nu - M) \chi_M(z) - g \bar{\chi}_M(z)\gamma_\nu\chi_M(z) C''[\epsilon(z)]
\right.
\]

\[
- \frac{g^2}{2M_C^2} [\bar{\chi}_M(z)\gamma_\nu\chi_M(z)] [\bar{\chi}_M(z)\gamma''\chi_M(z)] \right\}
\]

\( (3.19) \)
As we would show below the non–string part of the dual–vector field condensate does not depend on a dual Dirac string, whereas the string part does.

The integration over massive monopole fields is analogous to that carried out in Refs.[3–4], where the r.h.s. of Eq. (3.19) has been approximated by massive monopole–loop diagrams by keeping only leading divergent contributions.

In terms of the massive magnetic monopole–loop diagrams the non–string part of the dual–vector field condensate is determined by

\[ \left< \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right>_{\text{non-string}} = \left( \frac{g^2}{16\pi^2} \right)^2 \int \frac{d^4q}{2\pi^4 i} \frac{q^\mu q^\nu}{(M_C^2 - q^2 - i0)^2} \]

\[ \times \left[ \int \frac{d^4k_1}{\pi^2 i} \text{tr} \left\{ \frac{1}{M - \hat{k}_1 + g \hat{C}[\mathcal{E}(z)]} \frac{1}{M - \hat{k}_1 + g \hat{C}[\mathcal{E}(z)]} \right\} + \sum_{n=1}^{\infty} \left( \frac{g^2}{M_C^2} \right)^n \left( \frac{1}{16\pi^2} \right)^n \right] \]

\[ \times \left[ \int \frac{d^4k_2}{\pi^2 i} \text{tr} \left\{ \frac{1}{M - \hat{k}_2 + g \hat{C}[\mathcal{E}(z)]} \frac{1}{M - \hat{k}_2 + g \hat{C}[\mathcal{E}(z)]} \right\} \right] \]

\[ \times \left[ \int \frac{d^4k_n}{\pi^2 i} \text{tr} \left\{ \frac{1}{M - \hat{k}_n + g \hat{C}[\mathcal{E}(z)]} \frac{1}{M - \hat{k}_n + g \hat{C}[\mathcal{E}(z)]} \right\} \right] \]

\[ \times \left[ \int \frac{d^4k_{n+1}}{\pi^2 i} \text{tr} \left\{ \frac{1}{M - \hat{k}_{n+1} + g \hat{C}[\mathcal{E}(z)]} \frac{1}{M - \hat{k}_{n+1} + g \hat{C}[\mathcal{E}(z)]} \right\} \right]. \quad (3.20) \]

Here we have taken into account that the contributions of the first two terms in the non–string part cancel each other after integration over virtual momenta.

Integrating over \( k \) and \( k_i (i = 1, \ldots, n + 1) \) we obtain [3,4]

\[ \left< \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right>_{\text{non-string}} = \left( \frac{g^2}{16\pi^2} \right)^2 \int \frac{d^4q}{2\pi^4 i} \frac{q^\mu q^\nu}{(M_C^2 - q^2 - i0)^2} \]

\[ 2 g_{\mu\nu} [J_1(M) + M^2 J_2(M)] \sum_{n=1}^{\infty} \left( \frac{1}{M_C^2} \right)^n \left[ J_1(M) + M^2 J_2(M) \right] \]

\[ = \frac{g^2}{32\pi^2} [J_1(M_C) - M_C^2 J_2(M_C)] \frac{g^2}{8\pi^2} [J_1(M) + M^2 J_2(M)] \]

\[ = 1 - \frac{1}{M_C^2} \frac{g^2}{8\pi^2} [J_1(M) + M^2 J_2(M)] \]

where \( J_1(M_C) \) and \( J_2(M_C) \) are the quadratically and logarithmically divergent integrals defined by Eq.(1.13) with \( M \) replaced by \( M_C \). Since the cut–off \( \Lambda \) is much greater than \( M_C \), we can replace \( [J_1(M_C) - M_C^2 J_2(M_C)] \) by \( \Lambda^2 \). The non–string part of the dual–vector field condensate is then given by

\[ \frac{1}{\Lambda^4} \left< \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right>_{\text{non-string}} = \frac{1}{\Lambda^2 32\pi^4} \frac{g^2}{8\pi^2} [J_1(M) + M^2 J_2(M)] \]

(3.22)

We have represented the non–string part of the dual–vector field condensate in the form convenient for the comparison with the lattice calculations where we have set \( \Lambda_U = \Lambda \).
The string part of the dual–vector field condensate is determined by the following expression in terms of the massive magnetic monopole–loop diagrams [3,4]

\[
\left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle_{\text{string}} = \frac{g^2}{\pi^2} F^{\mu\nu}[\mathcal{E}(x)] \int d^4z \frac{\partial}{\partial x^\mu} \Delta(x - z, M_C) \\
\left( - \frac{g}{16\pi^2} \right) \int \frac{d^4k}{\pi^2} \text{tr} \left\{ \gamma^\mu \frac{1}{M - k + g \hat{C}[\mathcal{E}(z)]} \right\} + \sum_{n=1}^{\infty} \left[ \frac{g^2}{M_C^2} \right]^n \left( \frac{1}{16\pi^2} \right)^n \\
\times \int \frac{d^4k_1}{\pi^2} \text{tr} \left\{ \gamma^\mu \frac{1}{M - k_1 + g \hat{C}[\mathcal{E}(z)]} \right\} \gamma^{\alpha_1} \frac{1}{M - k_1 + g \hat{C}[\mathcal{E}(z)]} \\
\times \int \frac{d^4k_2}{\pi^2} \text{tr} \left\{ \gamma^{\alpha_1} \frac{1}{M - k_2 + g \hat{C}[\mathcal{E}(z)]} \right\} \gamma^{\alpha_2} \frac{1}{M - k_2 + g \hat{C}[\mathcal{E}(z)]} \\
\times \int \frac{d^4k_{n-1}}{\pi^2} \text{tr} \left\{ \gamma^{\alpha_{n-1}} \frac{1}{M - k_{n-1} + g \hat{C}[\mathcal{E}(z)]} \right\} \gamma^{\alpha_n} \frac{1}{M - k_{n-1} + g \hat{C}[\mathcal{E}(z)]} \\
\times \left( - \frac{g}{16\pi^2} \right) \int \frac{d^4k_n}{\pi^2} \text{tr} \left\{ \gamma^{\alpha_n} \frac{1}{M - k_n + g \hat{C}[\mathcal{E}(z)]} \right\} \right].
\] (3.23)

Keeping only leading divergent contributions [2–4] we obtain

\[
\left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle_{\text{string}} = \frac{g^2}{\pi^2} F^{\mu\nu}[\mathcal{E}(x)] \int d^4z \frac{\partial}{\partial x^\mu} \Delta(x - z, M_C) C_\nu[\mathcal{E}(z)] \\
\times \frac{g^2}{8\pi^2} \frac{[J_1(M) + M^2 J_2(M)]}{1 - \frac{g^2}{M_C^2} \frac{8\pi^2}{8\pi^2} [J_1(M) + M^2 J_2(M)]}.
\] (3.24)

Integrating by parts over \( z \) we arrive at the expression

\[
\left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle_{\text{string}} =
\frac{g^2}{8\pi^2} \frac{[J_1(M) + M^2 J_2(M)]}{1 - \frac{g^2}{M_C^2} \frac{8\pi^2}{8\pi^2} [J_1(M) + M^2 J_2(M)]} \int d^4z \frac{g^2}{2\pi^2} F^{\mu\nu}[\mathcal{E}(x)] \Delta(x - z, M_C) \bar{F}_{\mu\nu}[\mathcal{E}(z)].
\] (3.25)

Collecting all pieces we obtain the condensate of the dual–vector field

\[
\frac{1}{\Lambda^4} \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \frac{1}{\Lambda^4} \frac{g^2}{4\pi^2} F^{\mu\nu}[\mathcal{E}(x)] \bar{F}_{\mu\nu}[\mathcal{E}(x)] + \frac{g^2}{8\pi^2} \frac{[J_1(M) + M^2 J_2(M)]}{1 - \frac{g^2}{M_C^2} \frac{8\pi^2}{8\pi^2} [J_1(M) + M^2 J_2(M)]} \\
\times \left( \frac{1}{\Lambda^4} \frac{g^2}{32\pi^4} + \frac{1}{\Lambda^4} \int d^4z \frac{g^2}{2\pi^2} F^{\mu\nu}[\mathcal{E}(x)] \Delta(x - z, M_C) \bar{F}_{\mu\nu}[\mathcal{E}(z)] \right),
\] (3.26)

where the first term comes from the tree–approximation defined by the Abrikosov flux line, whereas the second one is fully caused by quantum field fluctuations around the Abrikosov flux line. Below we show that the term caused by quantum field fluctuations dominates in the dual–vector field condensate.
By using Eqs. (1.19) - (1.17) and the relation \( G_1 = G/4 \) we can recast the r.h.s. of Eq. (3.26) into the form

\[
\frac{1}{\Lambda^4} \left( \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right) = \frac{1}{\Lambda^4} \left( \frac{g^2}{4\pi^2} F^{\mu\nu}[\mathcal{E}(x)] F_{\mu\nu}[\mathcal{E}(x)] \right) + \frac{1 - \frac{g^2}{3} \langle \bar{\chi}\chi \rangle}{M^3} \left( 1 + \frac{2g^2}{M^2} \langle \bar{\chi}\chi \rangle \right) 
\]

\[
\times \left( \frac{M^2}{\Lambda^2} \frac{3g^2}{128\pi^4} + \frac{M^2}{\Lambda^4} \int d^4z \frac{3g^2}{8\pi^2} F^{\mu\nu}[\mathcal{E}(x)] \Delta(x - z, M_C) F_{\mu\nu}[\mathcal{E}(z)] \right) \quad (3.27)
\]

The dependence of the dual–vector field condensate on the shape of a dual Dirac string enters through the term \( \bar{F}^{\mu\nu}[\mathcal{E}(x)] F_{\mu\nu}[\mathcal{E}(x)] \).

Since the confinement regime can be described by static infinitely long dual Dirac strings, we would make an estimate of the numerical value of the dual–vector field condensate in the approximation of the infinitely long dual Dirac string strained along the \( z \)–axis. In this approximation the electric field of a static dual Dirac string is given by \[8\]

\[
\vec{E}(\vec{r}) = e_z Q \delta^{(2)}(\vec{r}),
\]

where \( \vec{r} \) is the radius–vector in the plane perpendicular to the \( z \)–axis. The dual–vector potential possesses only the azimuthal component and reads \[8\]

\[
C_\alpha(r) = - \frac{Q M_C}{2\pi} K_1(M_C r),
\]

where \( K_1(M_C r) \) is the McDonald function. The dual electric field induced by an infinitely long dual Dirac string amounts to

\[
\vec{E}(r) = \text{rot} \vec{C}(\vec{r}) = -e_z \frac{Q M_C^2}{2\pi} K_0(M_C r).
\]

Substituting the dual electric field Eq. (3.30) in the dual–vector field condensate Eq. (3.26) we obtain

\[
\frac{1}{\Lambda^4} \left( \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right) = \frac{1}{\Lambda^4} \frac{g^2}{2\pi^2} \frac{Q^2 M_C^4}{4\pi^2} K_0^2(M_C r) + \frac{1 - \frac{g^2}{3} \langle \bar{\chi}\chi \rangle}{M^3} \left( 1 + \frac{2g^2}{M^2} \langle \bar{\chi}\chi \rangle \right) 
\]

\[
\times \left( \frac{M^2}{\Lambda^2} \frac{3g^2}{128\pi^4} + \frac{M^2 M_C^2}{\Lambda^4} \frac{3g^2}{8\pi^2} \frac{Q^2}{4\pi^2} M_C r K_0(M_C r) K_1(M_C r) \right) \quad (3.31)
\]

By using the Dirac quantization condition \( g Q = 2\pi \) we bring up the r.h.s. of Eq. (3.32) to the form

\[
\frac{1}{\Lambda^4} \left( \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right) = \frac{M_C^4}{\Lambda^4} \frac{1}{2\pi^2} \frac{Q^2 M_C^4}{4\pi^2} K_0^2(M_C r) + \frac{1 - \frac{g^2}{3} \langle \bar{\chi}\chi \rangle}{M^3} \left( 1 + \frac{2g^2}{M^2} \langle \bar{\chi}\chi \rangle \right) 
\]

\[
\times \left( \frac{M^2}{\Lambda^2} \frac{3g^2}{128\pi^4} + \frac{M^2 M_C^2}{\Lambda^4} \frac{3g^2}{8\pi^2} M_C r K_0(M_C r) K_1(M_C r) \right) \quad (3.32)
\]
At $r \to \infty$ the dual–vector field condensate is defined by only the non–string part and reads

$$\frac{1}{\Lambda^4} \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \frac{M^2}{\Lambda^2} \frac{3g^2}{128\pi^4} \left[ 1 - \frac{g^2}{3} \frac{\langle \bar{\chi}\chi \rangle}{M^3} \right] + \frac{1 + 2g^2}{M^2} \frac{\langle \bar{\chi}\chi \rangle}{M^2}. \quad (3.33)$$

One can show that this value is positive.

At $r \to 0$ the dual–vector field condensate is infinite. However, as has been noted in Refs.[2–4] in the MNJL model as well as in the dual Higgs model with dual Dirac strings [8–10] the minimal transversal distances are restricted by inequality $r \geq 1/M_\sigma = 1/2M$. Thus, at $r = 1/2M$ we get the maximal value of the dual–vector field condensate, since the contributions of the dual Dirac string described by the terms proportional to $K_0^2(MCr)$ and $r K_0(MCr) K_1(MCr)$ are positive.

$$\frac{1}{\Lambda^4} \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \frac{M_C^4}{\Lambda^4} \frac{1}{2\pi^2} \ln^2 \left( \frac{M^2}{M_C^2} \right) + \frac{1}{1 + \frac{2g^2}{M_C^2}} \left( \frac{M^2}{M_C^2} \right) \left[ 1 - \frac{g^2}{3} \frac{\langle \bar{\chi}\chi \rangle}{M^3} \right] \left( \frac{M^2}{M_C^2} \right) \ln \left( \frac{M^2}{M_C^2} \right). \quad (3.34)$$

We should emphasize that unlike the magnetic monopole condensate the dual–vector field condensate does not vanish in the close vicinity of a dual Dirac string [3,4].

4 Conclusion

The evaluation of the dual–vector field condensate in the MNJL model has confirmed our statement we have pointed out in Refs.[2–4,9,10] that quantum fluctuations of the fields of monopole–(anti)monopole collective excitations around the Abrikosov flux lines induced by dual Dirac strings in the condensed phase play an important role for the understanding of the confinement mechanism. We have shown that these fluctuations determine fully the dual–vector field condensate independent on the shape of a dual Dirac string and non–vanishing at large distances.

In the case of the neglect of quantum field fluctuations this part of the dual–vector field condensate does not appear and the condensate is defined by

$$\frac{1}{\Lambda^4} \left\langle \frac{g^2}{4\pi^2} C_{\mu\nu}(x) C^{\mu\nu}(x) \right\rangle = \frac{M_C^4}{\Lambda^4} \frac{1}{2\pi^2} \ln^2 \left( \frac{M^2}{M_C^2} \right) + \frac{1}{1 + \frac{2g^2}{M_C^2}} \left( \frac{M^2}{M_C^2} \right) \ln \left( \frac{M^2}{M_C^2} \right).$$

for the infinitely long static dual Dirac string strained along the $z$–axis.

Unlike Eq.(3.32) the dual–vector field condensate given by Eq.(4.1) vanishes at large distances in the plane transversal to a dual Dirac string. This result should contradict to the gluon condensate having a non–vanishing part at large distances.

When matching our expression for the dual–vector field condensate given by Eq.(3.32) with that calculated on lattice we would like to emphasize the absence of the perturbative term $W_{40}$ and the presence of the term proportional to $M^2/\Lambda^2$ having a non–perturbative nature. The absence of the perturbative contribution to the dual–vector field condensate is rather clear. Indeed, the evaluation of the dual–vector field condensate is carried out in the non–perturbative (condensed) phase of the MNJL model with a non–perturbative dual–superconducting vacuum filled with dual Dirac strings. Thereby, the perturbative contributions cannot appear in principle.

Then, non–perturbative contributions having the structure $M_C^4/\Lambda^4$ and $M_C^6 M^2/\Lambda^4$ are determined fully by the dual Dirac string. They vanish at large distances. At short distances
restricted from below by inequality \( r \geq 1/M_\sigma = 1/2M \) the contributions of these terms are positive. Due to this they increase the value of the non-string part of the dual–vector field condensate in the vicinity of a dual Dirac string. This result is opposite to that we have obtained for the magnetic monopole condensate. Unlike the dual–vector field condensate the magnetic monopole condensate vanishes in the close vicinity of a dual Dirac string [2–4].

Unfortunately, we could not derive with one–to–one correspondence the contribution to the dual–vector field condensate analogous to that caused by a dilute–instanton vacuum in the gluon condensate. In the MNJL model according to the Dirac quantization this contribution should be of order \((M_C^4/\Lambda^4) O(g^6)\) or \((M_C^4/\Lambda^4) O(g^6)\). Such contributions can be found among other terms in the part of the dual–vector field condensate depending on the shape of a dual Dirac string. However, these terms vanish at large distances and do not give a dominant contribution to the dual–vector field condensate. Thus, we can conclude that the non–perturbative dual–superconducting vacuum of the MNJL model possesses partly the properties of a dilute–instanton gas vacuum but this part of the wave function of the non–perturbative vacuum does not play a dominant role.

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