Learning ReLU Networks via Alternating Minimization

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Abstract

We propose and analyze a new family of algorithms for training neural networks with ReLU activations. Our algorithms are based on the technique of alternating minimization: estimating the activation patterns of each ReLU for all given samples, interleaved with weight updates via a least-squares step. We consider three different cases of this model: (i) a single ReLU; (ii) 1-hidden layer networks with $k$ hidden ReLUs; (iii) 2-hidden layer networks. We show that under standard distributional assumptions on the input data, our algorithm provably recovers the true “ground truth” parameters in a linearly convergent fashion; furthermore, our method exhibits requires only $O(d)$ samples for the single ReLU case and $\tilde{O}(dk^2)$ samples in the 1-hidden layer case. We also extend this framework to deeper networks, and empirically demonstrate its convergence to a global minimum.

1 Introduction

Motivation

Deep neural networks have found success in a wide range of machine learning applications. However, despite significant empirical success, a rigorous algorithmic understanding of training such networks remains far less well understood.

Our focus in this paper are on a class of neural networks with rectified linear units (ReLUs) as activation functions. The method of choice to train such networks is the popular (stochastic) gradient descent. ReLU networks are computationally less expensive to train when compared to networks with tanh or sigmoid activations since they generally involve simpler gradient update steps. Due to their utility as well as amenability to analysis, several recent papers have addressed the problem of provably showing that gradient descent for ReLU networks succeeds under various assumptions [1, 2, 3, 4].

Our contributions

In this paper, we depart from the standard approach of gradient descent for learning ReLU-based neural networks. Instead, we propose a new approach based on the technique of alternating minimization. In contrast with gradient-based learning, our algorithm is parameter-free: it does not involve any tuning parameters (such as learning rate, damping factor, dropout ratio, etc.) other than setting the number of training epochs. To the best of our knowledge, such an alternating minimization approach in the context of neural network learning is novel.

We supplement our learning algorithm with a rigorous analysis for both 0- and 1-hidden layer ReLU networks. In particular, we prove that under a generative modeling assumption where there is a “ground truth” (or teacher) network and the data samples $x$ are Gaussian distributed, our algorithm exhibits linear convergence provided a sufficient number of samples. This means that the parameter estimation error reduces to $\varepsilon$ after $O(\log 1/\varepsilon)$ training epochs. For $d$-dimensional inputs, our approach requires $O(d)$ samples for 0-hidden layers and $O(dk^2 \log k)$ for 1-hidden layer networks.

In the 1-hidden layer case, our analysis works for both dense architectures [2] as well as residual networks (resnets) with skip connections [3]. We remark that our rate of convergence matches that of the algorithm provided in [2] for dense networks, and improves upon the standard SGD for resnets, as analyzed in [3].
However, in contrast with [2], this paper offers the following improvements: our algorithm is parameter-free, we require no assumption on spectral properties (such as condition number) of the network, and our analysis is far simpler. Proving convergence of our alternating minimization-based learning algorithm requires a suitable initialization. We discuss a range of initialization strategies for both standard and residual architectures.

Finally, we provide a range of numerical experiments that support our theoretical analysis. Our experiments show that our proposed algorithm provides better numerical performance than those provided in [3, 2]. While our theory is only for 0- and 1-hidden layers, we extend the algorithm to deeper ReLU networks, and show that the algorithm works well empirically (i.e., it gives zero training loss for sufficiently many samples). Establishing rigorous guarantees for ReLU networks with depth $\geq 2$ is an interesting direction for future work. Overall, our work can be viewed as a first step towards an algorithmic approach that goes beyond traditional SGD for training ReLU networks.

**Techniques** At a high level, our algorithm is based upon a simple (but key) idea that we call the “linearization” trick. For a given sample $x$ and a given estimate of the network parameter $W$, define the state of the network as the collection of binary variables that indicate whether a given ReLU is active or not. (In the literature, these have been referred to as signatures [5].) Since ReLU networks simulate piecewise linear functions, if we fix the state, then the mapping from $x$ to the label $y$ can be approximated using a linear neural network.

We can repeat this local linearization step for all samples. Overall, we obtain a system of linear equations with the weights of the network as the variables, which we can solve using either black-box direct methods (such as LU or Cholesky decompositions) or iterative solvers (such as conjugate gradients) depending on the size of the problem. Once we obtain an improved estimate of the weights, we can update the the state of the network for each sample, and repeat the above procedure by alternating between updating the state variables as well as the weights until convergence.

Two main conceptual challenges arise here. The first is to rigorously prove that our linearization trick actually converges to a global optimum in the absence of noise. (In fact, proving convergence even to a local optimum has not been shown before, to our knowledge.) To show this, we appeal to recent algorithmic advances for solving phase retrieval problems [6, 7, 8, 9, 10, 11]. In particular, in the case of Gaussian-distributed data we adapt the proof of [6] to prove linear convergence of the weights to an $\epsilon$-ball around the ground truth teacher network parameters. In contrast with [6] which requires drawing a fresh set of samples across different epochs, our method supports sample re-use and therefore models what is typically performed in practice.

Our above alternating minimization succeeds (in theory) for arbitrary networks. However, our analysis is local in nature, and only succeeds provided the network weights are initialized suitably close to the ground truth. The second conceptual challenge is to actually produce such an initialization. We discuss this in the context of several common architectural assumptions. For dense networks, we can leverage the tensor-based initialization of [2], while for residual networks, we propose a simple identity initialization that can be directly derived by inspecting the network architecture. In both cases, we perform several numerical experiments to demonstrate the benefits of our proposed methods.

**Comparison with prior work** Owing to the tremendous success of deep learning approaches in practice [12, 13], numerous theoretical contributions have emerged that study various aspects of deep learning algorithms (such as expressive power, accuracy of learning from an optimization perspective, and generalization). See Section 2 of [2] for a comprehensive overview.

Our focus in this paper is on designing provably accurate algorithms for optimizing the squared-error training loss for ReLU networks. Several recent papers have studied this from a theoretical perspective. The works of [1, 14] have proved global convergence of gradient descent for a 0-hidden layer network with a ReLU activation, given a sufficient number of samples. The case of 1-hidden layer ReLU networks is more challenging, and it is now known that the landscape of the squared-error loss function is rife with local minima; therefore, algorithmic aspects such as initialization and local/global convergence need to be carefully
handled. For random initializations, the recent papers [1, 15] provides a symmetry-breaking convergence analysis for 2-layer ReLU networks where the weight vectors of the hidden layer possesses disjoint supports. In contrast, our algorithm succeeds without any such assumptions on the weights of the network.

On the other hand, the approach in [4] constructs a special non-convex loss function which does not suffer from local minima, and whose minima correspond to the ground truth parameters of the ReLU networks. However, in contrast our metric of success is directly measured in terms of the Euclidean error between the estimated weights and the ground truth parameters.

The work perhaps most closely related to our approach is that of [2], who provide recovery guarantees for 1-hidden layer networks with a variety of activation functions, including ReLU’s. Like us, they provide a two-stage scheme: a suitable initialization, followed by gradient descent. (Such a two-stage approach has been proposed for solving several other non-convex machine learning problems [6, 16].) In contrast, our method is based on alternating minimization, and therefore is parameter free (and not sensitive to choice of learning rate). Moreover, our sample-complexity upper bounds for 1-hidden layer networks are polynomial improvements over that of [2], and we also show that our algorithm can be (empirically) extended to 2-hidden layer networks.

In addition, in contrast with [2], our algorithm (and analysis) for 1-hidden layer networks extend to residual networks with skip connections. Residual networks have shown to give empirical advantage over standard networks and are empirically easier to optimize over [17]. An added benefit of optimizing with skipped connections is that constitutes a simpler optimization landscape with fewer local minima [18]. We leverage this fact, and show that even with a random initial estimate, one is able to constrain the set of possible solutions $I + W$ to a smaller subset of the space of optimization variables, hence enabling global convergence guarantees. While preparing this paper we became aware of the work in [19], that also utilizes an identity initialization to train deep residual networks. However, their analysis only holds for linear neural networks, while our focus is on ReLU networks; moreover, they advocate using gradient descent to learn the weights of the network, while our approach is based on alternating minimization.

Finally, we remark that a similar alternating minimization framework has been utilized to study the problem of phase retrieval [6, 20, 9], with near-optimal computational and statistical guarantees. Indeed, our algorithm for learning 0-hidden layer networks is a direct extension of that of [6]. Our contribution in this paper is to show that the same alternating minimization framework generalizes to networks with 1 or more hidden layers as well, and may pave the way for further algorithmic connections between the two problems.

The rest of this paper is organized as follows. Section 2 establishes notation and mathematical basics. Section 3 introduces our alternating minimization framework for both 0- and 1-hidden layers, establishes convergence bounds, and includes a sketch of the analysis techniques. Section 4 supports our analysis via several numerical experiments.

2 Mathematical model

Notation Let us first establish some notation. Scalars and vectors are denoted by small case letters, and matrices are denoted by upper case letters, with elements of both indexed by subscripts. The indicator vector $p$ stores the missing sign information from prior to the ReLU operation. The matrix $P = \text{diag}(p)$ represents the matrix with the elements of $p$ along its diagonal and zero elsewhere. The symbol ‘$\circ$’ denotes the element-wise Hadamard product. Vectors and matrices with superscript ‘$\ast$’ denote the ground truth. Vectorization (or flattening) of a matrix $M$ is represented as vec$(M)$, producing a long vector with columns of $M$ stacked one beneath another. Small constants are represented by $\delta$ and large constants by $C$. The non-linear operation of ReLU is denoted by $\sigma(\cdot)$. The indicator function is denoted as $1_{\{Xw^\ast > 0\}} := \frac{\text{sign}(Xw^\ast) + 1}{2}$, where each entry:

$$1_{\{Xw^\ast > 0\}}_i = \begin{cases} 1 & x_i^\top w^\ast > 0 \\ 0 & x_i^\top w^\ast < 0 \end{cases},$$

for all $i \in \{1 \ldots m\}$. Boldface $\mathbf{1}$ represents a vector of ones. Similarly, $\mathbf{I}$ is the identity matrix.
In each of the ReLU network architectures considered below, we assume that the training data consists of i.i.d. samples \( \{(x_1, y_1), \ldots, (x_n, y_n)\} \), where each input \( x_i \in \mathbb{R}^d \) is constructed by sampling independently from a zero-mean \( d \)-variate Gaussian distribution with variance \( 1/n \), and the output \( y_i \) obeys the generative model:

\[
y_i = f(x_i),
\]

where \( f \) varies depending on the architecture. We denote the data matrix \( X \in \mathbb{R}^{n \times d} \), with each row representing a \( d \)-dimensional sample point, with \( n \) such samples. Scalar labels corresponding to each sample point are contained in \( y \in \mathbb{R}^n \). If skipped connections are considered, we denote the network weights by \( W^* \) and effective weights of the forward model as \( \tilde{W}^* = W^* + I \).

We devise algorithms that recover the “ground truth” parameters of the ReLU network. In the case of neural networks with 1 or more hidden layers, such a recovery can only be performed modulo some permutation of the weight vectors. Therefore, to measure the quality of the algorithm, we define a notion of distance between two weight matrices:

\[
\text{dist} (W, W') := \| W - W' \|_F = \min_{\pi' \in \Pi} \| W - \pi' W\|_F,
\]

where \( \pi' \) is any permutation of the column indices of the weight matrix, such that \( W_{\pi'} = \tilde{W} \) (column permuted version of \( W \)) and \( \Pi \) is a set of all possible permutations of the column indices of \( W \). For notational convenience, we assume without loss of generality that \( \pi \) is the identity permutation, in which case we simply can replace the \( \text{dist} (\cdot, \cdot) \) with the more familiar Frobenius norm.

**Single ReLU (0-hidden layers)** For a single neuron, i.e., 0-hidden layer network with ReLU activation, the forward model can be written as:

\[
y = \text{ReLU}(Xw^*),
\]

where \( w^* \in \mathbb{R}^{d \times 1} \) are the neuron weights, and \( \text{ReLU}(z) = \max(z, 0) \) is applied component-wise.

**1-hidden layer** For a 1-hidden layer network with \( k \) ReLUs, the forward model can be written as:

\[
f(X) = \sum_{q=1}^{k} \text{ReLU}(Xw^*_q) := \sum_{q=1}^{k} p^*_q \circ (Xw^*_q) := \sigma(XW^*)1_{k \times 1},
\]

where \( W^* := [w^*_1 \ldots w^*_k] \in \mathbb{R}^{d \times k} \), and \( p^*_q = \#\{w^*_q > 0\} := \frac{\text{sign}(Xw^*_q) + 1}{2} \quad \forall q \). We can rewrite this as a linear function of the weights \( W^* \):

\[
f(X) = [ \text{diag}(p^*_1 \ldots \text{diag}(p^*_k))X]_{n \times dk} \cdot \text{vec}(W^*)_{dk \times 1} := A^* \cdot \text{vec}(W^*).
\]

**2-hidden layers** For a 2-hidden layer network with \( k \) ReLUs in the first layer and \( k_0 \) ReLUs in the second layer, the forward model can be written as:

\[
f(X) = \sum_{r=1}^{k_0} \sigma(\tilde{X}w^*_r) := \sum_{r=1}^{k_0} \sigma(\sigma(XW^1)w^*_r) = \sigma(\sigma(XW^1)(W^2)W^3),
\]

where \( W^{1*} := [w^*_1 \ldots w^*_k] \in \mathbb{R}^{d \times k} \), and \( W^{2*} := [w^*_1 \ldots w^*_k] \in \mathbb{R}^{k \times k_0} \), \( W^{3*} = 1_{k_0 \times 1} \).

\[1\text{Since the ReLU is a homogenous function, we can assume that the second layer of weights is the all-ones vector without loss of generality.}\]
Algorithm 1 ReLU training via Alternating Minimization

Input: $X, y, T$

1: Initialize $w^0 \leftarrow 0$.
2: for $t = 0, \cdots, T - 1$ do
3: $p^t \leftarrow \mathbb{1}_{\{Xw^t > 0\}}$,
4: $w^{t+1} \leftarrow \arg\min_w \|p^t \circ (Xw) - y\|_2$.
5: end for

Output: $w^T \leftarrow w^t$.

Algorithm 2 Training 1-hidden layer ReLU network via Alternating Minimization

Input: $X, y, T, k$

1: Initialize $W^0 \leftarrow \text{INITIALIZE\_TWO\_LAYER}(X, y, k)$, s.t. dist $(W^0, W^*) \leq \delta_1 \|W^*\|_F$.
2: for $t = 0, \cdots, T - 1$ do
3: $p^t_q \leftarrow \mathbb{1}_{\{Xw^t_q > 0\}}, \ \forall q \in [k],$
4: $A^t \leftarrow [\text{diag}(p^t_1)X \cdots \text{diag}(p^t_k)X]_{n \times dk},$
5: $W^{t+1} \leftarrow \text{reshape}(\arg\min_{\text{vec}(W)} \|A^t \cdot \text{vec}(W) - y\|_2, [d, k]).$
6: end for

Output: $W^T \leftarrow W^t$.

3 Algorithms and Analysis

We now propose our alternating minimization-based framework to learn shallow networks with ReLU activations. At a high level, our algorithm rests on the following idea: given the knowledge of the correct signs of the input to each ReLU, the forward models depicted above can be linearized, i.e., they now represent linear neural networks. Therefore, the weights can be estimated as in the case of any other linear model (e.g. via least-squares).

This immediately motivates an iterative two-phase approach: alternate between (i) a sign estimation (linearization) step, followed by (ii) an update to the estimate of the weights via least-squares estimation. The first phase can be computationally achieved by a single forward pass through the network for each sample, while the second phase can be computationally achieved by any standard (stable) solver for linear systems of equations. These steps are iterated to convergence (and we elaborate on how many iterations are required below in our analysis).

The pseudocode for training 0- and 1-hidden layer networks using our above framework is provided in the form of Algorithms 1 and 2 respectively. The pseudocode for training 2-hidden layer networks is presented in the Appendix A as Algorithm 3.

3.1 Algorithmic guarantees

We now provide a theoretical analysis of our proposed algorithms. Our first theorem proves that alternating minimization for training a single ReLU exhibits linear convergence.

Theorem 1. Suppose that the coordinates of each data sample $x_i$ are drawn i.i.d. from the Gaussian distribution $\mathcal{N}(0, 1/\sqrt{n})$. Given an initialization $w^0$ satisfying dist $(w^0, w^*) \leq \delta_0 \|w^*\|_2$, for a small constant
where $l = \{1, 2, \ldots, L\}$ is the counter for each iteration of gradient descent per least-squares estimation step and gradient descent is run for a total of $L$ iterations, and $A := \mathbb{P}X := \text{diag}(\rho) \cdot X \in \mathbb{R}^{n \times d}$. The convergence parameters $\rho_1, \rho_2 < 1$ and $0 < \mu < 2$. The result follows by evaluating the sign estimation error $\|E_{\text{sgn}}\|_2^2$ in the $t^{th}$ iteration in terms of $\|w^t - w^*\|_2^2$, via Lemma 3 in Appendix B.

Our second theorem states that training 1-hidden layer ReLU network via Alternating Minimization also converges linearly to the true weight parameters.

**Theorem 2.** Given an initialization $W^0$ satisfying $\text{dist}(W^0, W^*) \leq \delta_1 \|W^*\|_F$, for $0 < \delta_1 < 1$, if we have number of training samples $n > C \cdot d \cdot k^2 \cdot \log k$, then with high probability $(1 - e^{-\gamma n})$ the itertes of Algorithm 2 satisfy:

$$\text{dist}(W^{t+1}, W^*) \leq \rho_0 \text{dist}(W^t, W^*) .$$

Here, $\gamma$ is a positive constant and $0 < \rho_0 < 1$.

**Proof sketch:** Again, we defer the full proof to Appendix A. Similar to the set of arguments for a single neuron case, we obtain bounds on the distance of the $t^{th}$ update of the weight matrix $W^t_{\pi}$ from the true weights $W^*_{\pi}$, where $\pi$ is a particular permutation of the columns of $W^t$ and $W^*$ respectively, corresponding to the minimum Euclidean distance between the two terms.

$$\|W^{t+1, l+1}_{\pi} - W^*_{\pi}\|_F \leq \|I - \mu B^T B\|_2 \|W^{t+1, l}_{\pi} - W^*_{\pi}\|_F + \|\mu B^T (B - B^*) \text{vec}(W^*_{\pi})\|_2 ,$$

(weights estm. error) (sign estm. error)

where we have used the bounds $\|I - \mu B^T B\|_2^2 \leq \rho_1 < 1$ and $\|B^T\|_2 \leq R = \sqrt{k}$, through Corollaries 1 and 2 in Appendix B and $\mu = 1/k$ is chosen as per Corollaries 1 and 2. The cumulative sign estimation error $\sum_{q=1}^{k} \|E_{\text{sgn}_q}\|_2^2$ is bounded in terms of $\|W^{t}_{\pi} - W^*_{\pi}\|_F$ via Corollary 3 in Appendix B.
3.2 Techniques for initialization

To prove convergence of Algorithms 1 and 2, we require that the initial weights $w^0$ and $W^0$ are such that they meet the constraints $\text{dist}(W^0, W^*) \leq \delta_0 \|w^*\|_2$ and $\text{dist}(w^0, w^*) \leq \delta_1 \|W^*\|_F$, respectively for (small enough) constants $\delta_0$ and $\delta_1$. We now discuss strategies that meet such a requirement. We begin with the 0-hidden layer case.

**Theorem 3.** In Algorithm 1, the initialization, $w^0 \leftarrow 0$ is a small constant distance $0 < \delta_0 < 1$ away from the true mapping $w^*$, i.e.,

$$\text{dist}(w^0, w^*) \leq \delta_0 \|w^*\|_2,$$

as long as the number of training examples 'n' satisfy the following bound,

$$n \geq Cd,$$

with probability greater than $1 - e^{-\gamma n}$ for some constant $\gamma$.

**Proof sketch:** We utilize the update rule formulated to analyze the convergence of our algorithm, and use $w^0 \leftarrow 0$ as the initialization, to obtain the following bound:

$$\|w^1 - w^*\|_2 \leq \left(\rho_1^L + \mu \frac{\rho_5}{1 + \rho_4}\right) \|0 - w^*\|_2 := \delta \|w^*\|_2,$$

where $\rho_4 < 1$ and $\rho_5 \approx 1$, and $0 < \mu < 2$.

For the 1-hidden layer case, we use the tensor initialization method proposed by Zhong et. al. in [2]. For completeness, we provide a description of this method in the Appendix A.

For residual network with skipped connections, we consider a modification of Equation 2, of the form:

$$f(X) = \sum_{q=1}^{k} \sigma(X(w_q^* + e_q^S)) := \sigma(X(W^* + I_S)W^{2*} := \sigma(X(\hat{W}^*))W^{2*},$$

where $W^* := [w_1^* \ldots w_k^*] \in \mathbb{R}^{d \times k}$, effective weights of the forward mapping $\hat{W}^* = W^* + I_S$, $W^{2*} = I_{k \times 1}$, and $I_S \in \mathbb{R}^{d \times k}$ is a sub-matrix of $I_{d \times d}$, with $k \leq d$ (locations are known) out of $d$ columns picked, and $e_q^S$ are the columns of $I_S$. Furthermore, a common assumption in the literature is that $\|W^*\|_F \leq \gamma$ [3, 21, 19]. For our algorithm, we require the assumption that $\text{dist}(\hat{W}^0, W^*) \leq \delta_1 \|W^*\|_F$. Now, suppose we initialize $\hat{W}^0 \leftarrow I_S$, where support $S$ is known. Then,

$$\text{dist}(\hat{W}^0, \hat{W}^*) = \text{dist}(I_S, W^* + I_S) = \|W^* + I_S - I_S\|_F = \|W^*\|_F.$$  

If the underlying weights corresponding to the true mapping are such that $\|W^*\|_F \leq \delta_1 \|W^* + I_S\|_F$, then the requirement for convergence of Algorithm 2 is met. That is, if there exists some $\gamma$, such that $\|W^*\|_F \leq \gamma \leq \delta_1 k/(1 - \delta_1)$, then the output of the alternating minimization algorithm $W^T$ will converge to $W^*$, via Theorem 2.

3.3 Sample complexity

Through our algorithms, we are able to establish recovery guarantees, under certain sample complexity constraints. In the case of a single neuron model, we are able to prove that $O(d)$ samples are required for successfully learning the true single neuron mapping $w^*$, using alternating minimization. This matches the results for learning the weights of a single neuron via gradient descent [14]. For single hidden layer model, we obtain a sample complexity requirement of $O(dk^2 \log k)$ for global convergence, as long as a certain (refer Theorem 2) weights initialization condition is met. This is comparable to the results in Theorem D.2. of [2],
where the authors derive that $O(dk^2\text{poly}(\log d))$ samples are required to ensure linear local convergence of vanilla gradient descent to learn the weights of a 1-hidden layer network. The sample complexity bottleneck lies in the initialization stage.

This is also the first paper that comments on the sample complexity requirement to successfully learn the weights of (one block) of a 1-hidden layer network with skipped connections, as long as the network is initialized to an identity (or subset of identity) mapping (refer Eqn. 8). In this case, the sample complexity requirement is exactly $O(dk^2\log k)$, as long as $\|W^*\|_F < \gamma$.

4 Experiments

Single ReLU We select a data matrix $X \in \mathbb{R}^{n \times d}$ with entries picked from $\mathcal{N}(0,1/d)$ distribution. We construct a (single) ReLU with weights $w^* \in \mathbb{R}^d$ picked from a Gaussian distribution $\mathcal{N}(0,1)$. Our goal is to recover $w^*$, (forward model in Eqn. 1) using our proposed approach. In our experiments, we set $w_0 \leftarrow 0$, but in our experiments we have seen that random initialization with weights $O(1/\sqrt{d})$ (which is typically prescribed in practice) yields identical phase transitions (not shown in this paper).

The dimension of the input $d$ is swept in steps of 25 from $d = 50$ to $d = 200$, and the number of training samples $n$ is swept from $n = 50$ to $n = 500$ in steps of 50. The training process was repeated for 100 trials, with each trial corresponding to a different set of random data samples $X$. Recovery is said to be successful if $\|w^T - w^*\|_2/\|w^*\|_2 < 0.01$. The probability of recovery is calculated as (number of trials in which recovery is successful)/(total number of trials). We present the results in terms of a phase transition diagram (Figure 1). As can be seen from the figure, there appears to be a linear relationship between the dimension $d$ and the number of samples required by alternating minimization for accurate parameter recovery, which is predicted by our theoretical analysis.

One-hidden layer ReLU network Again, we select the data matrix $X \in \mathbb{R}^{n \times d}$ with entries picked from $\mathcal{N}(0,1/d)$ distribution. Following the setup of [2], we pick $W^*$ such that $W^* = U\Sigma V^T$, where $U \in \mathbb{R}^{d \times k}$ and $V \in \mathbb{R}^{k \times k}$ are obtained from the reduced QR decomposition of a random Gaussian matrix. The diagonal matrix $\Sigma$ is chosen to have entries $1, 1 + \frac{\kappa-1}{\kappa-1}, 1 + \frac{2(\kappa-1)}{\kappa-1} \ldots \kappa$, with $\kappa = 2$ and varying values of $k$, constituting the forward model in Eqn. 2. The weight matrix is initialized according to the tensor initialization strategy of [2]. We fix the dimension of the input layer to $d = 100$ and vary the dimension of the hidden layer $k = 2, 3, 4, 5$. We also sweep the number of training samples $n$ was swept from 500 to 5000 in steps of 500. The training process was repeated for 40 trials. Recovery is said to be successful if $\|W^*_\pi - W^*_\pi\|_F/\|W^*_\pi\|_F < 0.01$. We also use a (heuristic) random initialization with all weights with magnitude $O(1/\sqrt{d})$ and plotted corresponding phase transition 2. It may be noted that even though
Number of samples $n$

| Probability of recovery | $K=2$ | $K=3$ | $K=4$ | $K=5$ |
|-------------------------|-------|-------|-------|-------|
| $K=2$ (random)          |       |       |       |       |
| $K=3$ (random)          |       |       |       |       |
| $K=4$ (random)          |       |       |       |       |
| $K=5$ (random)          |       |       |       |       |

Number of samples $n$

| Probability of recovery | $K=5$ | $K=10$ | $K=15$ | $K=20$ |
|-------------------------|-------|--------|--------|--------|
| $K=5$ (random)          |       |        |        |        |
| $K=10$ (random)         |       |        |        |        |
| $K=15$ (random)         |       |        |        |        |
| $K=20$ (random)         |       |        |        |        |

Figure 2: Phase transitions and sample complexity and for 1-hidden layer network with ReLU activation (left) vanilla network (2), $d = 100$, solid lines represent recovery from the initialization in [2], dotted lines represent recovery with random initialization; (right) skipped connections (7), $d = 20$, solid lines represent recovery when network is initialized with submatrix of identity $I_S$, dotted lines represent recovery with random initialization.

this gives comparatively worse performance than when the initialization from [2] is used, we observe that random initialization is capable of recovering weights $W^*$ accurately with sufficiently many samples. Further experimental results for this setup are provided in the Appendix.

Residual networks We consider a similar 1-hidden layer network as above, but now add skipped connections from a random subset of the input layer to the output. Therefore, the “effective” weights now become $\hat{W}^* = W^* + I_S$ with individual elements of $W^*$ picked from the Gaussian distribution $c \cdot \mathcal{N}(0, 1/d)$ with $c = 3$ (i.e. $\gamma = 3$) and $I_S$ being a subset of columns of the identity matrix $I_{d \times d}$, $S$ representing corresponding indices, and $\text{card}(S) = k \leq d$.

First, we let $d = k = 20$, so that effective weights $\hat{W}^* = I + W^* \in \mathbb{R}^{20 \times 20}$. We then reduce the number of hidden neurons to $k = 5, 10, 15$, respectively, by dropping or deactivating some pre-fixed neurons. If $S = S_5, S_{10}, S_{15}$ (card($S_5$) = 5, card($S_{10}$) = 10 and card($S_{15}$) = 15) are randomly permuted indices between 1 to 20, the weight matrix is initialized simply as $\hat{W}_0 \leftarrow I_S$. The number of training samples $n$ was swept from 100 to 1000 in steps of 100. We repeat over 20 trials with each trial being a random instantiation of the data matrix $X$. Recovery is said to be successful if $\frac{\|\hat{W}_\pi^T - \hat{W}_{z_\pi}^*\|_F}{\|\hat{W}_\pi^*\|_F} < 0.01$. We observe that our simple identity-based initialization provides a (strictly) better phase transition than a random initialization; see Figure 2.

In the supplementary material we demonstrate additional experiments for 2-hidden layer ReLU networks, and prove that our framework produces accurate parameter estimates.

A Appendix A

In this section we state our alternating minimization algorithm for solving 2-hidden layer networks with ReLU activation and the proofs for Theorems 1, 2 (convergence, 0-,1-hidden layers), 3 (initialization, 0-hidden layer) and a summary of the initialization approach for single hidden layer networks as proposed in [2] (initialization, 1-hidden layer).

A.1 Training 2-hidden layer network with ReLU activation

See Algorithm 3.
Algorithm 3 Training 2-hidden layer ReLU network via Alternating Minimization

Input: $X, y, T, k, k_o$

1: Initialize: $W_1^0, W_2^0 \leftarrow \text{INITIALIZETHREELAYER}(X, y, k, k_o)$:
   
   \[ \text{s.t. } \text{dist} (W_1^0, W_1^T) \leq \delta_1 \|W_1^T\|_F, \text{dist} (W_2^0, W_2^T) \leq \delta_2 \|W_2^T\|_F. \]

2: for $t = 0, \cdots, T - 1$ do
   
   3: $p_1^{1,t} \leftarrow 1_{\{Xw_1^t > 0\}}, \forall q \in [k],$
   
   4: $p_2^{2,t} \leftarrow 1_{\{p_1^{1,t} > 0\}}, \forall r \in [k],$
   
   5: $C_q^t \leftarrow \left( \sum_{r=1}^{k_0} w_2^{2,t} \text{diag}(p_r^{2,t}) \right) \cdot \text{diag}(p_q^{1,t}) \cdot X, \forall q \in [k],$
   
   6: $C_t^t \leftarrow [C_1^t \ldots C_k^t],$
   
   7: $W_1^{1,t} \leftarrow \text{reshape}(\text{argmin}_{\text{vec}(W_1)} \|C_t^t \cdot \text{vec}(W_1) - y\|_2, [n, k]),$
   
   8: $B_t^t \leftarrow \text{diag}(p_{r}^{2,t})_{m \times m} \cdot \text{diag}(p_{1}^{1,t}Xw_1^{1,t+1} \cdots \text{diag}(p_{k}^{1,t}Xw_k^{1,t+1})_{m \times k}, \forall r \in [k],$
   
   9: $B_t^t \leftarrow \left[ B_1^t \ldots B_{k_o}^t \right],$
   
   10: $W_2^{2,t} \leftarrow \text{reshape}(\text{argmin}_{\text{vec}(W_2)} \|B_t^t \cdot \text{vec}(W_2) - y\|_2, [k, k_o]),$

end for

Output: $W_1^{1,T} \leftarrow W_1^{1,t}, W_2^{2,T} \leftarrow W_2^{2,t}.$

A.2 Proofs of guarantees on global convergence

Theorem 1. Suppose that the coordinates of each data sample $x_i$ are drawn i.i.d. from the Gaussian distribution $\mathcal{N}(0, 1/\sqrt{n})$. Given an initialization $w^0$ satisfying $\text{dist} (w^0, w^*) \leq \delta_0 \|w^*\|_2$, for a small constant $0 < \delta_0 < 1$, if we have number of training samples $n > Cd$, then with high probability $(1 - e^{-\gamma n})$ the iterates of Algorithm 1 satisfy:

\[ \text{dist} (w^{t+1}, w^*) \leq \rho_0 \text{dist} (w^t, w^*). \tag{4} \]

Here, $\gamma$ is a positive constant and $0 < \rho_0 < 1$.

Proof. Note: $P^t := \text{diag}(p^t), A^t = A := P^t X.$

We analyze the $w$—update step in Line 4 of Algorithm (1):

Objective function to be minimized is:

\[ f(w) = \frac{1}{2} \|y - A^t w\|_2^2 = 2 \|y - P^t X w\|_2^2 \]

\[ \implies \nabla_w f(w) = -2A^T(y - Aw) = A^T(A - A^*)w^* + A^T(A(w - w^*)) \tag{9} \]

If $L$ iterations of gradient descent are carried out to minimize the Least Squares (LS) problem in Step 4 of
Algorithm (1), with $l$ indicating the iteration counter ranging from 1 to $L$, then the iterates of $w$ follow:
\[
\begin{align*}
    w^{t+1,l+1} &= w^{t+1,l} - \mu \nabla_w f(w) \\
    &= w^{t+1,l} - \mu (A^T (A - A^*) w^* + A^T A (w^{t+1,l} - w^*)) \\
    \Rightarrow w^{t+1,l+1} - w^* &= (I - \mu A^T A) (w^{t+1,l} - w^*) - \mu A^T (A - A^*) w^*
\end{align*}
\]
where we have used the bounds $\|I - \mu A^T A\|_2 \leq \rho_1$ and $\|A^T\|_2 \leq 1$ through Lemmas 1 and 2 in Appendix B, as long as $n > C \cdot d$ and $0 < \rho_1 < 1$, with probability greater than $1 - \frac{m}{n}$ and $0 < \mu < 2$.

We can then proceed to bound the non-linearity estimation error:
\[
E_{sgn} := (A - A^*) w^* = (\mathbb{F}^d - \mathbb{P}^*) X w^*
\]
\[
\Rightarrow \|E_{sgn}\|_2^2 = \sum_{i=1}^{m} (x_i^T w^*)^2 \cdot 1_{\{x_i^T w^* < 0\}} \leq \rho_2^2 \|w^t - w^*\|_2^2
\]
\[
\Rightarrow \|E_{sgn}\|_2 \leq \rho_2 \|w^t - w^*\|_2,
\]
where we have used Lemma 3 in Appendix B and $0 < \rho_2 < 1$, as long as $n > C \cdot d$, with probability greater than $1 - e^{-\gamma n}$, for some positive constant $\gamma$.

Hence the criterion for convergence is as follows:
\[
\|w^{t+1} - w^*\|_2 \leq \left( \rho_1^L + \frac{\mu R}{1 + \rho_1} \right) \|w^t - w^*\|_2 := \rho_0 \|w^t - w^*\|_2,
\]
where $L$ is the number of iterations of Gradient Descent used for solving the LS step in Line 4 of Algorithm 1. \hfill \Box

**Theorem 2.** Given an initialization $W^{0}$ satisfying $\text{dist} (W^{0}, W^*) \leq \delta_1 \|W^*\|_F$, for $0 < \delta_1 < 1$, if we have number of training samples $n > C \cdot d \cdot k^2 \cdot \log k$, then with high probability $\left(1 - e^{-\gamma n}\right)$ the iterates of Algorithm 2 satisfy:
\[
\text{dist} (W^{t+1}, W^*) \leq \rho_0 \text{dist} (W^t, W^*).
\]

*Here, $\gamma$ is a positive constant and $0 < \rho_0 < 1$.*

**Proof. Note:** Here $B^t = [B_1 \ldots B_k]_{\pi} = B$ and subscript $\pi$ represents a specific permutation of the indices $\{1 \ldots k\}$.

Objective function to be minimized is:
\[
\begin{align*}
    f(W) &= \frac{1}{2} \|y - [A_1 \ldots A_k]_{\pi} \cdot \text{vec}(W_\pi)\|_2^2 := \frac{1}{2} \|y - [\mathbb{F}_1^d \ldots \mathbb{F}_k^d]_{\pi} \cdot \text{vec}(W_\pi)\|_2^2 \\
    \Rightarrow \nabla_W f(W) &= -2B^T (y - B \cdot \text{vec}(W_\pi)) \\
    &= B^T (B - B^*) \cdot \text{vec}(W^*_\pi) + B^T B \cdot (\text{vec}(W_\pi) - \text{vec}(W^*_\pi))
\end{align*}
\]
The error metric accounts for permutation $\pi$ of columns of $W$.

\[
\|W^{t+1,l+1}_\pi - W^*_\pi\|_F \leq \|I - \mu B^T B\|_2 \|W^{t+1,l}_\pi - W^*_\pi\|_F + \|\mu B^T (B - B^*) \text{ vec}(W^*_\pi)\|_2 \\
\leq \rho_1 \|W^{t+1,l}_\pi - W^*_\pi\|_F + \|B (B - B^*) \cdot \text{ vec}(W^*_\pi)\|_2.
\]

(14)

In Algorithm 2, we have used the bounds $\|I - \mu B^T B\|_2 \leq \rho_1$ and $\|B (B - B^*)\|_2 \leq R$, through Corollaries 1 and 2 in Appendix B and $\varepsilon := \sum_{q=1}^k \|E_{\text{sgn},q}\|_2$ and $\mu = 1/k$ is chosen as per Corollaries 1 and 2.

We can then proceed to bound the phase error:

\[
E_{\text{sgn},q} = (A_q - A^*_q)w^*_q = (P^*_q - P_q)Xw^*_q
\]

\[
\implies \|E_{\text{sgn},q}\|_2^2 = \sum_{i=1}^m (x_i^T w^*_q)^2 \cdot 1_{\{x_i^T w^*_q(x_i^T w^*_q) < 0\}} \leq \rho_{2,q}^2 \|w^*_q - w^*_q\|_2^2
\]

\[
\implies \sum_{q=1}^k \|E_{\text{sgn},q}\|_2^2 \leq \sum_{q=1}^k \rho_{2,q}^2 \|w^*_q - w^*_q\|_2^2 \leq \max_q (\rho_{2,q}^2) \sum_{q=1}^k \|w^*_q - w^*_q\|_2^2 = \max_q (\rho_{2,q}^2) \|W^*_q - W^*_q\|_F^2
\]

(17)

\[
\implies \left(\sum_{q=1}^k \|E_{\text{sgn},q}\|_2^2 \right)^2 \leq k \sum_{q=1}^k \|E_{\text{sgn},q}\|_2^2 \leq k \max_q (\rho_{2,q}^2) \|W^*_q - W^*_q\|_F^2
\]

\[
\implies \sum_{q=1}^k \|E_{\text{sgn},q}\|_2 \leq \sqrt{k} \max_q (\rho_{2,q}) \|W^*_q - W^*_q\|_F = \rho_3 \|W^*_q - W^*_q\|_F,
\]

(18)

where we have used Corollary 3 in Appendix B, with probability greater than $1 - \eta$, where $\eta$ is small constant close to 0 as long as $n > C \cdot d \cdot k^2 \cdot \log k$.

Hence the criterion for convergence is as follows:

\[
\|W^t+1,1 - W^*_\|_F \leq \left(\rho_1^L + \frac{\mu R \rho_3}{1 + \rho_1}\right) \|W^t - W^*_\|_F
\]

(19)

where $L$ is the number of iterations of Gradient Descent used for solving the LS step in Step 5 of Algorithm (2).

\[\square\]

A.3 Initialization techniques

**Theorem 3.** In Algorithm 1, the initialization, $w^0 \leftarrow 0$ is a small constant distance $0 < \delta_0 < 1$ away from the true mapping $w^*$, i.e.,

\[
\text{dist} (w^0, w^*) \leq \delta_0 \|w^*\|_2,
\]
as long as the number of training examples \( n \) satisfy the following bound,
\[
n \geq Cd, \tag{6}
\]
with probability greater than \( 1 - e^{-\gamma n} \) for some constant \( \gamma \).

**Proof.** The convergence of the alternating minimization algorithm requires a good initial point. One can show that initializing \( w^0 \leftarrow 0 \), gives a good start to our algorithm. This can be shown as follows.

If \( w^0 \leftarrow 0 \), then \( P^0 \leftarrow I \) and subsequently \( A \leftarrow X \). Evaluating the gradient, using 9:
\[
w^{1,1+1} = w^{1,1} + \mu X^T P^* X w^* - \mu X^T P^* X w^{1,1} \\
\implies w^{1,1+1} - w^* \leq (I - \mu X^T X)(w^{1,1+1} - w^*) + \mu X^T (I - P^*) X w^* \\
\implies \|w^{1,1+1} - w^*\|_2 \leq \|I - \mu X^T X\|_2 \|w^{1,1+1} - w^*\|_2 + \mu \|X^T (I - P^*) X\|_2 \|w^*\|_2 \\
= \rho_4 \|w^{1,1+1} - w^*\|_2 + \mu \rho_5 \|0 - w^*\|_2 \\
\implies \|w^1 - w^*\|_2 \leq \left( \frac{\rho_4}{\mu} + \frac{\rho_5}{1 + \rho_4} \right) \|0 - w^*\|_2 = \delta \|w^*\|_2,
\]
where \( w^{1,0} = 0 \) and \( l \) is the iteration counter of gradient descent, total \( L \) iterations, where \( \|I - \mu X^T X\|_2 \leq \rho_4 < 1 \) and \( \|X^T (I - P^*) X\|_2 \leq \rho_5 \approx 1 \) are bounded via Lemma 2 in Appendix B.

**Initialization for single hidden layer model:** Zhong et. al. in [2] develop tensor initialization method (one of the approaches for INITIALIZE_TWOLAYER subroutine in Algorithm 2) for one-hidden-layer networks for ReLU and other activations. The basis of their scheme is to obtain an orthonormal set of vectors \( W^0 := \{w_1^0, w_2^0 \ldots w_k^0\} \), that have the same span as the true weight vectors \( W^* := \{w_1^*, w_2^* \ldots w_k^*\} \).

To do this, they evaluate an empirical estimator \( \hat{P} = \sum_{i=1}^{m} y_i (x_i x_i^T - I) \), such that \( P = \mathbb{E}[\hat{P}] = \sum_{q=1}^K (\gamma_2(w_q^*) - \gamma_0(w_q^*)) w_q^* w_q^* \) (note that in this paper, we assume that the second layer consists of weights \( v_q^* = 1 \), for \( q = \{1,2, \ldots k\} \)). To estimate the initial span \( \{w_1^0, w_2^0 \ldots w_k^0\} \), the authors contract two matrices \( \hat{P}_1 = CI + \hat{P} \) and \( \hat{P}_2 = CI - \hat{P} \) (where \( C > 2 \|\hat{P}\|_2 \)) and evaluate the top-k eigenvectors and corresponding eigenvalues (in terms of absolute value), each, of \( \hat{P}_1 \) and \( \hat{P}_2 \) respectively. They merge the top \( k_1 \) and \( k_2 \) eigenvalues of \( \hat{P}_1 \) and \( \hat{P}_2 \) respectively, such that \( k_1 + k_2 = k \). This is done by ordering all eigenvectors, of \( \hat{P}_1 \) and \( \hat{P}_2 \) in descending order and picking eigenvectors corresponding to the top-k eigenvalues from the combined pool of eigenvalues. This is followed by an orthogonalization step, such that all \( k_1 + k_2 \) eigenvectors extracted are orthogonal to each other.

If the singular values of \( W^* \) are \( \sigma_1 \geq \cdots \geq \sigma_k \), then, condition number is \( \kappa = \sigma_1 / \sigma_k \). Then, the authors claim that the procedure described above gives a good initialization:
\[
\|W^0 - W^*\|_F \leq \epsilon_0 \cdot \text{poly}(k, \kappa) \|W^*\|_F,
\]
with high probability if number of samples \( n > \epsilon_0^{-2} d \cdot \text{poly}(k, \kappa, \log d) \) (refer Theorem 5.6 of [2]).

**B Appendix B**

In this section, we state some Lemmas with or without proofs, required for the proofs of Theorems in Appendix A.

**Lemma 1.** If \( A := PX \), where \( P \) is a diagonal matrix with indicators 0 and 1 on the diagonal, and the entries of \( X \) are from \( \mathcal{N}(0, 1 / \sqrt{n}) \), then the operation of \( P \) on Gaussian matrix \( X \), extracts a sub-matrix of \( X \) such that,
\[
\|A^T\|_2 \leq 1 + 1 / \sqrt{C} \approx 1,
\]
with probability greater than \( 1 - \frac{\eta}{16} \) if \( n > Cd \cdot \log \frac{1}{\eta} \) where \( C \) is large enough and \( \eta \) is a constant.

Subsequently, \( \|A^T A\|_2 := \|X^T PX\|_2 \leq \|X^T X\|_2 \leq (1 + 1 / \sqrt{C})^2 \).

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Proof. Then spectral norm of the indicator matrix is bounded as $0 \leq \|P\|_2 \leq 1$. Subsequently,

$$\|A^\top\|_2 = \|A\|_2 \leq \|P\|_2 \|X\|_2 \leq 1 \cdot \|X\|_2 \leq 1 + 1/\sqrt{C},$$

where $\|X\|_2$ is bounded using standard results from random matrix theory [22], with probability greater than $1 - \frac{n}{16}$, if $n > Cd \cdot \log \frac{1}{\eta}$, where $C$ is a large constant and $\eta$ is a constant.

Corollary 1. For $B := [A_1 A_2 \ldots A_K] := [P_1 X \; P_2 X \ldots P_K X]$, where $P_k$’s are diagonal matrices with indicators 0 and 1 on the diagonal, and the entries of $X$ are from $\mathcal{N}(0, 1/\sqrt{n})$, then the operation of each $P_k$ on Gaussian matrix $X$, extracts a sub-matrix of $X$ such that:

$$\|B^\top\|_2 = \|B\|_2 = \|P_1 X \; P_2 X \ldots P_K X\|_2 \leq \|X \ldots (K \text{ times}) \ldots X\|_2 \leq \sqrt{K} = R,$$

as long as $n > C \cdot d \cdot \log \frac{1}{\eta}$, with probability greater than $1 - \frac{n}{16}$. Subsequently, in expression 16, we should pick $\mu$ such that $\mu < (1 + \rho_1)/\sqrt{K}$, and hence the product $\mu R \leq 1 + \rho_1$.

Lemma 2. If $A := P X$, where $P$ is a diagonal matrix with indicators 0 and 1 on the diagonal, and the entries of $X$ are from $\mathcal{N}(0, 1/\sqrt{n})$, then, the entity $\|I - \mu A^\top A\|_2$, with $0 \leq \mu \leq 1$ is bounded as:

$$\|I - \mu A^\top A\|_2 \leq \rho$$

where $0 \leq \rho \leq 1$, with probability greater than $1 - \frac{n}{16}$, if $n > Cd \cdot \log \frac{1}{\eta}$, where $C$ is a large constant and $\eta$ is a constant.

Proof. For any matrix $A$ with spectral norm $\sigma_{\min} \leq \|A\|_2 \leq \sigma_{\max}$, the entity $\|I - \mu A^\top A\|_2$, with $0 \leq \mu \leq 1$ is bounded as:

$$\|I - \mu A^\top A\|_2 \leq \max (|1 - \mu \sigma_{\min}^2|, |1 - \mu \sigma_{\max}^2|)$$

Using standard results from random matrix theory [22], the spectral norm of $X$ can be bounded as $1 - 1/\sqrt{C} \leq \|X\|_2 \leq 1 + 1/\sqrt{C}$, w.h.p, as long as $n > Cd$ and we choose $0 < \mu < 2$. We obtain the bound:

$$\|I - \mu X^\top X\|_2 \leq \left((\mu - 1) + 2\mu/\sqrt{C} + \mu/C\right) \leq \rho_0 < 1,$$

where $0 < \mu < 2$ (‘−’ for $0 < \mu < 1$ and ‘+’ for $1 < \mu < 2$) and $C$ is large enough. For smallest $\rho_0$, $\mu \to 1$. Then,

$$\|I - X^\top X\|_2 \leq 2/\sqrt{C} + 1/C < \rho_0$$

(20)

where $\rho_0 \to 0$ if $C$ is large enough. Consider a submatrix of $X$, $X_S$, such that the permutation of the rows of $X$ gives the decomposition $X = \begin{bmatrix} X_S & X_{\bar{S}} \end{bmatrix}$.

Through Lemma 1, $\|X_S\|_2 < \|X\|_2 \leq 1 + 1/\sqrt{C} - \delta < 1 + 1/\sqrt{C}$. for $\delta$ small enough and $0 < \delta < 1$. Then the bound on $\|I - X^\top P X\|_2$ is modified as:

$$\|I - X_S^\top X_S\|_2 \leq 2/\sqrt{C} + 1/C + (1 - \delta)^2 < \rho_0 + (1 - \delta)^2$$

(21)

If $C$ is large enough,

$$\|I - X^\top P X\|_2 \leq (1 - \delta)^2 = \rho < 1.$$
Corollary 2. For $B := [A_1 A_2 \ldots A_K] := [P_1 X \ P_2 X \ldots P_K X]$, where $P_k$’s are diagonal matrices with indicators 0 and 1 on the diagonal, and the entries of $X$ are from $\mathcal{N}(0, 1/\sqrt{n})$,

$$\|I - \mu B^\top B\|_2 \leq \rho$$

where $0 \leq \rho \leq 1$, with probability greater than $1 - \frac{\eta}{10}$, if $n > Cd \cdot \log \frac{1}{\eta}$, where $C$ is a large constant and $\mu < 1/K$.

Proof. From Corollary 1, $1 - 1/\sqrt{C} < \frac{\|\rho\|_2}{\sqrt{K}} < 1 + 1/\sqrt{C}$, with probability $1 - \frac{\eta}{10}$ and $n > Cd \cdot \log \frac{1}{\eta}$. Then,

$$\|I - \mu B^\top B\|_2 \leq \pm \left( (K\mu - 1) + 2K\mu/\sqrt{C} + K\mu/C \right) \leq \rho$$

if $K\mu \to 1$ or $\mu = 1/K$.

Lemma 3. As long as the initial estimate $w^0$ is a small distance away from the true weights $w^*$, dist $(w^0, w^*) \leq \delta_0 \|w^*\|_2$, and subsequently, dist $(w^t, w^*) \leq \delta_0 \|w^*\|_2$, where $w^t$ is the $t$th estimate of weight vector $w$, then the following bound holds,

$$\sum_{i=1}^{m} (x_i^\top w^*)^2 \cdot 1\{ (x_i^\top w^*) (x_i^\top w^*) \leq 0 \} \leq \rho_2 \|w^t - w^*\|_2^2,$$

with probability greater than $1 - e^{-\gamma n}$, where $\gamma$ is a positive constant, as long as $n > C \cdot d$ and $\rho_2 \approx 0.14$, if $\delta_0 = \frac{1}{10}$ and elements of $x_i, x_{ij} \sim \mathcal{N}(0, 1/\sqrt{n})$.

This lemma has been adapted from Lemma 3 of [11].

Corollary 3. As long as the initial estimate $w^0$ is a small distance away from the true weights $w^*$, dist $(w^0, w^*) \leq \delta_0 \|w^*\|_2$, and subsequently, dist $(w^t, w^*) \leq \delta_0 \|w^*\|_2$, where $w^t$ is the $t$th estimate of weight vector $w$, then the following bound holds,

$$\|E_{nl}\|_2^2 = \sum_{i=1}^{m} (x_i^\top w^*)^2 \cdot 1\{ (x_i^\top w^*) (x_i^\top w^*) \leq 0 \} \leq \frac{\rho_2}{K} \|w^t - w^*\|_2^2,$$

with probability greater than $1 - \eta'$, where $\gamma$ is a positive constant, $\eta'$ is a small constant close to 0, as long as $n > C \cdot dK^2$ and $\rho_2 \approx 0.14$, if $\delta_0 = \frac{1}{10}$ and elements of $x_i, x_{ij} \sim \mathcal{N}(0, 1/\sqrt{n})$.

Subsequently, for $K$ such neurons $w_1, w_2, \ldots, w_K$, we take a union bound over this probability, such that,

$$\left( \sum_{k=1}^{K} \|E_{nl_k}\|_2 \right)^2 \leq K \sum_{k=1}^{K} \|E_{nl_k}\|_2^2 \leq K \max_{k} (\rho_2^2, K) \|W^t - W^*_n\|_F^2 \leq \rho_3 \|W^t - W^*_n\|_F^2$$

where $\rho_3 < 0.14$, with probability greater than $1 - \eta$, to yield sample complexity $n > C \cdot d \cdot K^2 \cdot \log K$, where $\eta$ is a small constant close to 0.

Proof. This lemma has been adapted from Lemma 3 of [11] by modifying equation (58) and subsequently equation (63) as follows:

$$\|E_{nl}\|_2^2 \leq (0.13 + c\eta) \|w^t - w^*\|_2^2 \quad (22)$$

with probability greater than $1 - (1 + \frac{2}{\eta})e^{-cm^2}$ where $c$ is a small constant, and the factor $c_0 = 0.13$ can be reduced by reducing the initial distance factor $\delta_0$ (Refer Proof of Lemma 3 of [11] for further details) and the factor $c_0 \propto \delta_0$.  

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To deconstruct $\delta_1$, we have
\[
\text{dist} \left( W^{t+1}, W^* \right) \leq \delta_1 \text{dist} \left( W^t, W^* \right)
\]
\[
\sum_{q=1}^{k} \text{dist} \left( w^{t+1}_q, w^*_q \right)^2 \leq \delta_1^2 \sum_{q=1}^{k} \text{dist} \left( w^t_q, w^*_q \right)^2
\]
We effectively require for each neuron, that
\[
\text{dist} \left( w^{t+1}_q, w^*_q \right)^2 \lesssim \delta_1^2 \text{dist} \left( w^t_q, w^*_q \right)^2
\]
Therefore $\delta_1 \approx \delta_0$. We need to ensure that the value of $\rho_3 < 1$ in the expression 18. For this, we need to ensure that each of $\rho_{2,k}^2 < \frac{1}{K}$. Hence we are required to evaluate Equation 22, such that:
\[
\|E_{nl}\|_2^2 \leq \frac{1}{K} (0.13 + \alpha c) \|w^f - w^*\|_2^2
\]
for $k^{th}$ neuron. The probability of this event is $1 - (1 + \frac{2}{\epsilon})^d e^{-cm \frac{\epsilon^2}{K^2}}$ where $c$ is a small constant. We further require this condition to hold for all $K$ neurons. Hence we take a union bound, such that 23 holds with probability $1 - K (1 + \frac{2}{\epsilon})^d e^{-cm \frac{\epsilon^2}{K^2}}$ for all $K$ neurons. To evaluate sample complexity,
\[
K \left( 1 + \frac{2}{\epsilon} \right)^d e^{-cm \frac{\epsilon^2}{K^2}} < \eta
\]
\[\implies (\log K) \cdot d \cdot \log \left( 1 + \frac{2}{\epsilon} \right) - cm \frac{\epsilon^2}{K^2} < \log \eta\]
\[\implies m > C \left( K^2 \cdot (\log K) \cdot d \cdot \epsilon^{-2} \cdot \log \epsilon^{-1} + \epsilon^{-2} \cdot K^2 \cdot \log \frac{1}{\eta} \right),\]
where $C$ is a constant large enough.

C Supplementary experiments

C.1 Training loss
We traced the training losses under the “good” initialization and random $O(1/\sqrt{d})$ schemes, which is presented in Figures 3 and 4. In Figure 3 we compare the training losses, for a 1 hidden layer network (Eqn.
Figure 4: (a) Sample complexity and (b) training loss for $n = 2,000$ for 2-hidden layer network (3) with $d = 20, k = 3, k_o = 2$ and ReLU activation.

2), for two separate trials with $n = 10,000$, for cases $k = 2$ and $k = 5$, under random $O(1/\sqrt{d})$ and tensor initialization [2] schemes.

C.2 Two hidden layer ReLU model

Entries of the data matrix $X \in \mathbb{R}^{n \times d}$ are picked from $\mathcal{N}(0, 1/\sqrt{n})$ distribution. A three layer network is considered (Eqn. 3), and is assigned weights $W^{1*}$ and $W^{2*}$ with individual elements of both matrices picked from Gaussian distribution $\mathcal{N}(0, 1)$. For the sake of experimental analysis, the weight matrices are initialized as $[W_0^1, W_0^2] \leftarrow [W^{1*} + E^1, W^{2*} + E^2]$, where $\|E^1\|_2 = 0.2 \|W^{1*}\|_2$ and $\|E^2\|_2 = 0.2 \|W^{2*}\|_2$. We fix the dimension of the input layer to $d = 20$, the dimension of the first hidden layer to $k = 3$ and the dimension of the second hidden layer to $k_o = 2$. The number of training samples was set to $n = 300, 600, 900, 1, 200, 1, 500$ (Figure 4(a)) for 10 trials, and 2,000 for 2 trials (Figure 4(b)).

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