The “measurability” of the non-minimal coupling is discussed by considering the correction to the Newtonian static potential in the semi-classical approach. The coefficient of the “gravitational Darwin term” (GDT) gets redefined by the non-minimal torsion scalar couplings. Based on a similar analysis of the GDT in the effective field theory approach to non-minimal scalar we conclude that for reasonable values of the couplings the correction is very small.
1 Introduction

The interface of (classical) gravitational and quantum realms [1, 2] has been the subject of a considerable literature for several decades that we can say that the gravitational effects on quantum systems are no more beyond our reach (see e.g. [3, 4, 5]). The theoretical analysis consisted basically in inserting the Newtonian gravitational potential into the Schrödinger equation. On the other hand, considering gravity as an affective field theory certain quantum predictions can be made (see [6, 7] and references therein). One direction to improve the analysis may be to learn how to handle relativistic field equations in a curved background space-time and study the low enough energy and small curvature regions.

A relativistic quantum mechanical system and the effects of external fields coupled to it can be studied by constructing the Foldy-Wouthuysen transformation (FWT) [8, 9]. The main advantage of the FWT transformation is that the Hamiltonian and all operators in this representation are block-diagonal. This transformation holds only in the one-particle approximation where the loop calculations are not taken into account and this description is acceptable if the external fields are very weak and the particle production processes can be neglected. However, there are very few known problems in flat space that admit an exact FWT [10, 11]. In curved space the known exact FWT are those related to Dirac [12] and spin zero particles [13, 14] coupled to a static spacetime metric. For a scalar coupled to higher derivative gravity see [15].

The coupling between the curvature and scalar field of the form $\lambda R \phi$ is the only possible local term with dimensionless coupling constant $\lambda$ [16, 17]. The values $\lambda = 0$ (minimal coupling) and $\lambda = 1/6$ (for massless scalars) are used in the literature. A general value of $\lambda \neq 0$ is the so-called non-minimal coupling and the question of which value(s) of $\lambda$ should constitute the correct coupling to gravity depends on the particular field theory used for the scalar field (see, e.g. [18] and references therein). Given the current theoretical situation it seems more of an experimental problem to identify which would be the correct $\lambda$ coupling(s) for the various scalar particles. It has been suggested that the action of the gravity-scalar theory should contain, along with the Einstein-Hilbert action, some non-minimal couplings of the scalar field with the curvature tensor, see e.g. [19].

As for alternatives to classical General Relativity, there are different options (see e.g. [20, 21, 22, 23, 24]). Here we consider the so-called Einstein-Cartan theory [23]. In this context, the torsion field does not interact minimally with scalar fields, but a non-minimal formulation allows the introduction of certain interaction terms with some coupling constants $\xi_i$. Moreover, this type of scalar-torsion interaction is a necessary condition for the renormalizability of the quantum field theory in curved space-time with torsion [23, 25].

Here we extend the results of [13, 14] to the case of a massive scalar coupled to gravity and torsion. We describe the formulation of the metric-scalar gravity with torsion and provide an on-shell action obtained after the substitution of the torsion fields into the original action by using their relevant field equations. We will show that the problem of finding the exact FWT for the metric-scalar gravity with torsion reduces to that of the non-minimal metric-scalar system and that the torsion field effects are encoded only in the $\xi_i$ dependence of the redefined non-minimal coupling $\hat{\lambda}$.

The torsion field effects on the Dirac equation in the non-relativistic limit have been
considered using the Foldy-Wouthuysen transformation and the semi-classical approach (see e.g., [26, 27, 28, 29]).

In the next section we describe the metric-scalar gravity with torsion and write the on-shell action written in terms only of the scalar field once the torsion field components are substituted into the action by making use of their equations of motion. In section 3 we write the FWT and the quasi-relativistic Hamiltonian for the system. In section 4 the Foldy’s Hamiltonian is written for a static background metric and its properties are discussed.

2 The metric-scalar gravity with torsion

Supposing that the affine connection $\tilde{\Gamma}_{\beta\gamma}^\alpha$ is not symmetric, i.e.,
\[ \tilde{\Gamma}_{\beta\gamma}^\alpha - \tilde{\Gamma}_{\gamma\beta}^\alpha = T_{\beta\gamma}^\alpha, \] (2.1)

one defines the tensor $T_{\beta\gamma}^\alpha$ called torsion.

For our purposes it is useful to decompose the torsion into its three irreducible components: i) the vector $T_\beta = T_{\beta\alpha}^\alpha$, ii) the axial vector $S^\nu = \epsilon^{\alpha\beta\mu\nu}T_{\alpha\beta\mu}$, and iii) the tensor $q_{\alpha\beta\gamma}$ satisfying $q_{\alpha\beta\alpha} = 0$ and $\epsilon^{\alpha\beta\mu\nu}q_{\alpha\beta\mu} = 0$. Then the torsion becomes
\[ T_{\alpha,\beta\gamma} = \frac{1}{3}(T_\beta g_{\alpha\gamma} - T_\gamma g_{\alpha\beta}) - \frac{1}{6}\epsilon_{\alpha\beta\gamma\mu}S^\mu + q_{\alpha\beta\gamma}. \] (2.2)

It is also useful to write the scalar curvature in terms of these irreducible components;
\[ \tilde{R} = R - 2\nabla_\alpha T^\alpha - \frac{4}{3}T^\alpha T_\alpha + \frac{1}{2}g_{\alpha\beta\gamma}q^{\alpha\beta\gamma} + \frac{1}{24}S^\alpha S_\alpha. \] (2.3)

The covariant derivative $\nabla_\alpha$ and the Riemannian curvature tensor $R_{\beta\gamma\delta}^\alpha$ are obtained from the symmetric connection $\Gamma_{\beta\gamma\delta}^\alpha = \frac{1}{2}g^{\alpha\tau}(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma})$.

The general non-minimal action for the scalar field coupled to metric and torsion is given by [23]
\[ S = \int \sqrt{-g} \frac{1}{2} \left[ g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 + \sum_{i=1}^{5} \xi_i P_i \phi^2 \right] d^4x. \] (2.4)

We have the following structures: $P_1 = R$ (the Riemannian curvature scalar), $P_2 = \nabla_\alpha T^\alpha$, $P_3 = T^\alpha T_\alpha$, $P_4 = S^\alpha S_\alpha$, $P_5 = q_{\alpha\beta\gamma}q^{\alpha\beta\gamma}$. Therefore, there are five non-minimal parameters $\xi_1, \ldots, \xi_5$. In the torsionless case the only non-minimal term $\xi_1 R$ must be considered.

Our aim is to find the semi-classical Hamiltonian of the massive scalar theory coupled to external gravity and on-shell torsion fields with the general couplings $\xi_i$ (2.4) in the context of the exact Foldy-Wouthuysen transformation [13]. Then, making use of the decomposition

\[^1\text{We consider the approach in which torsion mass is dominating over the possible kinetic terms. In the case of fermion coupled to torsion this assumption is sufficient to provide the contact spin-spin interactions [29, 23].}\]
(2.2), the equations of motion for the torsion tensor can be written as the equations of motion for the relevant components $T$, $S$, and $q_{\alpha\beta\gamma}$,

$$T_\alpha = \frac{\xi_2}{\xi_3} \frac{\nabla_\alpha \phi}{\phi}, \quad S_\alpha = q_{\alpha\beta\gamma} = 0. \quad (2.5)$$

This relation is in accordance with the result that scalar fields produce torsion only in non-minimally coupled theories. Then, the torsion is related to the gradient of the field [30, 23] (in general, spin is considered as the source of torsion, however, the emergence of torsion in other contexts has been discussed in e.g. [31]).

Substituting these expressions into (2.4) one gets the on-shell action

$$S = \int \sqrt{-g} \frac{1}{2} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \hat{m}^2 \phi^2 + \hat{\lambda} \phi^2 \hat{R} \right] d^4x, \quad (2.6)$$

where

$$\hat{m}^2 = \frac{1}{1 + \frac{\xi_2}{\xi_3}} m^2, \quad \hat{\lambda} = \frac{\xi_1}{1 + \frac{\xi_2}{\xi_3}}. \quad (2.7)$$

The effect of torsion coupled to scalar and metric fields are encoded in the $\xi_2$ and $\xi_3$ parameters dependence of the re-defined coupling $\hat{\lambda}$ and mass $\hat{m}$ parameters of the action (2.6). The mass redefinition due to the effect of torsion trace has also been reported in [32].

Notice that an attempt to directly generalize the Einstein-Cartan theory coupled to a scalar reduces the number of free parameters $\xi_i$ to just one parameter. To observe this, write the “minimal” action

$$S = \int \sqrt{-g} \frac{1}{2} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 + \lambda \bar{R} \phi^2 \right] d^4x, \quad (2.8)$$

where the expression (2.3) for the curvature in the space with torsion must be used. Therefore, comparing (2.4) and (2.8) we get

$$\xi_1 = \lambda, \quad \xi_2 = -2\lambda, \quad \xi_3 = -\frac{4}{3} \lambda, \quad \xi_4 = \frac{1}{2} \lambda, \quad \xi_5 = \frac{1}{24} \lambda \quad (2.9)$$

In particular, using the relations (2.9) for the redefined $\hat{\lambda}$ and $\hat{m}$ parameters one gets

$$\hat{m}^2 = \frac{1}{1 + 3\lambda} m^2, \quad \hat{\lambda} = \frac{\lambda}{1 + 3\lambda}. \quad (2.10)$$

Observe that the usual value in the torsionless case $\hat{\lambda} = \xi_1 = 1/6$ (for massless scalar) corresponds to $\xi_1 = \lambda = 1/3$ when torsion is considered. The shifting of the conformal value from $1/6$ to $1/3$ is due to the non-trivial transformation of torsion under conformal symmetry [23, 25, 33].
3 Exact Foldy-Wouthuysen transformation

In order to find the non-relativistic Hamiltonian of the system we will use the procedure developed in [13] for a scalar coupled non-minimally to gravity. This problem has been addressed for a real spin-0 particle coupled to the static metrics
\[ ds^2 = V^2 dt^2 - W^2 dx^2, \]  
where \( V = V(x) \) and \( W = W(x) \).

The treatment uses the properties of a pseudo-Hermitian Hamiltonian [34, 35, 36] which appears when the Klein-Gordon equation for the model (2.6) is written in the two-component Schrödinger formulation [37]
\[ i\dot{\Phi} = \mathcal{H}\Phi, \]  
with the Hamiltonian given by
\[ \mathcal{H} = \frac{\hat{m}}{2} \zeta^T - \zeta \theta, \]  
where
\[ \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \zeta = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \]  
and the operator \( \theta \) is defined by
\[ \theta \equiv \frac{F^2}{2\hat{m}} \nabla^2 \frac{1}{W} \nabla \ln(VW) \cdot \nabla - \frac{\hat{m}}{2} V^2 - \frac{\lambda}{2\hat{m}} V^2 R, \quad F^2 \equiv \frac{V^2}{W^2}. \]

Notice that the Hamiltonian satisfies the pseudo-Hermiticity property \( \mathcal{H}^\dagger = \sigma_3 \mathcal{H} \sigma_3 \), \( \sigma_3 \) being the diagonal Pauli matrix \( \text{diag}(1, -1) \). Here we simply quote the exact Foldy’s Hamiltonian [13]
\[ \mathcal{H}'' = (-2\hat{m}\theta')^{1/2} \sigma_3, \]  
where
\[ \theta' = \frac{\hat{m}}{2} V^2 - \frac{1}{2\hat{m}} F \hat{p}^2 F + \frac{1}{8\hat{m}} \nabla F \cdot \nabla F + \mathcal{D}_\lambda(V, W), \quad \theta' = f \theta f^{-1}, \]
\[ f = V^{-1/2} W^{3/2}, \quad \hat{p} = -i \nabla, \]
\[ \mathcal{D}_\lambda(V, W) = \hat{\lambda}[(1 - 2\lambda) \frac{V}{W^2} \nabla^2 V - 2 \frac{V}{W^2} \nabla V \cdot \nabla W + (1 - 4\lambda) \frac{V^2}{W^3} \nabla^2 W + 2 \frac{V^2}{W^4} (\nabla W)^2]. \]  

The Eq. (3.6) has been obtained in two steps. First, the operator \( \theta' \) is constructed demanding it to be hermitian with respect to the usual flat space measure. Second, it is a simple observation that \( \mathcal{H}^2 \) is diagonal, so, taking the square-root of this operator
and conveniently diagonalizing the 2x2 identity matrix provides (3.6). However, as pointed out in [38] a simple diagonalization procedure of the Hamiltonian may be nonequivalent to the Foldy-Wouthuysen transformation. As it is well known the FWT provides the correct physical interpretation of Klein-Gordon equation written in the form (3.2)-(3.3) [39]. The Hamiltonian (3.6) is exactly the same as the one recently proposed in Eq. (83) of Ref. [34], in which a rigorous Hilbert space construction based on the solutions of a Klein-Gordon-type field equation is considered for $t$–independent $\theta$ operator. As pointed out in [34] the Eq. (3.6) is the Foldy-Wouthuysen Hamiltonian in the Schrödinger picture of the first quantized scalar field theory. According to the discussions above we will consider $H''$ in (3.6) as the true FW Hamiltonian.

The quasirelativistic Hamiltonian and the first order terms in the $1/\hat{m}$ expansion of (3.6) becomes

$$H'' \approx \left[ \hat{m}V + \frac{1}{4\hat{m}} \left( W^{-1}p^2F + Fp^2W^{-1} \right) - \frac{1}{8mV} \nabla F \cdot \nabla F + \frac{1}{2mV} \mathcal{D}_\lambda(V, W) \right] \sigma_3 . \quad (3.8)$$

The so-called “gravitational Darwin term” (GDT) is given by $\frac{1}{2mV} \mathcal{D}_\lambda(V, W)$ [13]. A remarkable fact is that one can rewrite (3.7) as

$$\mathcal{D}_\lambda(V, W) \equiv \hat{\lambda}F \nabla^2 F + 3(\frac{1}{6} - \hat{\lambda}) \frac{F}{W} \left\{ \nabla^2 V + F \nabla^2 W \right\} . \quad (3.9)$$

Notice that the last term in (3.9) inside brackets does not contribute if $\hat{\lambda} = 1/6$ (for the “minimal” action (2.8) and according to the Eq. (2.10) and the discussion below it, this corresponds to $\xi_1 = 1/3$) providing a simple form for the Darwin term as discussed in some detail in Ref [14]. The last term in (3.9) for general $\hat{\lambda}$, of course, does not give a vanishing contribution and then provides a complicated Darwin term.

4 The semi-classical approximation for the Hamiltonian

The external gravitational field is assumed to be weak, then a Newtonian approximation will be sufficient. Thus, far from the source the solution of the Einstein equation for a point particle of mass $M$ located at $r = 0$ can be taken as

$$g_{00} \approx 1 - \frac{2MG}{r} , \quad (4.1)$$

$$g_{11} = g_{22} = g_{33} \approx -1 - \frac{2MG}{r} . \quad (4.2)$$

From (4.1) and (4.2) we get immediately

$$V \approx 1 - \frac{MG}{r} , \quad W \approx 1 + \frac{MG}{r} \quad (4.3)$$

and

$$F \approx 1 - \frac{2MG}{r} , \quad (4.4)$$
Inserting (4.3) and (4.4) into (3.8) we obtain the non-relativistic FW Hamiltonian

$$\mathcal{H}'' = \left[ \dot{\mathbf{m}} + \mathbf{m} \cdot \mathbf{g} + \frac{\mathbf{p}^2}{2\mathbf{m}} + \frac{3}{2\mathbf{m}} \mathbf{p} \cdot (\mathbf{g} \cdot \mathbf{x}) \mathbf{p} - \frac{4\pi GM}{m} \lambda_{\xi} \delta^3(\mathbf{r}) \right] \sigma_3, \quad (4.5)$$

where \( \mathbf{g} = -GM \mathbf{r} / r^3 \) and \( \lambda_{\xi} \equiv \xi_1 / \sqrt{1 - \xi_2^2} \).

The ‘mass’ \( \dot{m} \) is the redefined parameter in (2.7). It is known that a spinless particle in an external torsion field undergoes a shift of its mass [23, 32]. The first three terms in (4.5) represent the rest energy, the gravitational potential, the non-relativistic kinetic term, respectively, while the fourth term is the first relativistic correction for the gravitational potential. The last term proportional to \( \nabla \cdot \mathbf{E} \) arising from ‘zitterbewegung’ (the particle’s coordinate is ‘smeared out’ over a length \( \approx \hbar/mc \)).

In order to get (4.5) we have used the fact that for the special type of metrics (3.1) and in the approximation (4.3)-(4.4), the last term of (3.9) containing the brackets gives a negligible contribution for \( \lambda \neq \frac{1}{6} \). So, the GDT arises only from the \( \dot{\lambda} \mathbf{F} \nabla^2 \mathbf{F} \) sector of (3.9)

$$\frac{\dot{\lambda}}{2\mathbf{m}} \nabla_{\dot{\lambda}} (V, W) \approx -\frac{4\pi GM}{m} \lambda_{\xi} \delta^3(\mathbf{r}), \quad (4.6)$$

where in the denominator of the right hand side one has the scalar mass \( m \) parameter of action (2.4). For the case (2.10) the relationship \( \lambda_{\xi} = \xi / \sqrt{1 + 3\xi} \) holds.

The GDT without torsion presented in [14], in the case under consideration here gets redefined by a coefficient depending on the extra \( \xi_{2,3} \) couplings. Moreover, the GDT (4.6) is the analog to the one obtained for the torsionless case in the effective field theory approach [40] where a \( \lambda \) dependent term \( \lambda_{\xi} \lambda \) has been obtained as the next to leading correction to the Newtonian potential at tree level due to the non-minimal coupling.

On the other hand, this contribution to the static gravitational potential is similar to the one obtained in quadratic gravity. Quadratic gravity is constructed by the addition of non-linear terms to the curvature in the Einstein-Hilbert action. The coupling parameters of these terms in the Lagrangian must be determined by experiments. Experimental constraints on these parameters can be set out from the dispersive character of the bending of light in higher derivative gravity and based on the fact that rainbow effect is currently undetectable [42] or from sub-millimeter tests of the inverse square law [43, 44]. To gain insight into the nature of the term (4.6) let us write the potential which follows from quadratic gravity [45, 46]

$$V(r) = -\frac{GMm}{r} \left( 1 + \frac{1}{3} e^{-m_0 r} - \frac{4}{3} e^{-m_1 r} \right), \quad (4.7)$$

where \( m_0 = \sqrt{\frac{1}{\kappa^2(3\alpha + \beta)}} \) and \( m_1 = \sqrt{-2/(\kappa^2 \beta)} \), \( \alpha \) and \( \beta \) are dimensionless parameters, and \( \kappa^2 = 32\pi G \) is the Einstein’s constant. The \( \alpha \) and \( \beta \) are the parameters in the \( \alpha \mathbf{R}^2 + \beta R^2_{\mu\nu} \) terms of the higher order gravity.

Regarding the potentials described above, let us point out that also a Yukawa type potential has been reported for the static limit of the vector component of torsion in Eq. (2.2) in the framework of transposition invariance formulation (see [22] and references therein).
Experimental bounds on the parameter $\alpha$ and $\beta$ given in Refs. [42, 43, 44] are

$$\alpha, |\beta| < 10^{60}, \quad (4.8)$$

which improves the earlier bound $10^{74}$ of Stelle [45, 46].

In the limit $\alpha, \beta \to 0$, as is usually more appropriate for a perturbation in an effective field theory, the potential (4.7) at first order becomes [6, 7]

$$V(r) = -GMm \left( \frac{1}{r} - 128\pi^2 G(\alpha + \beta)\delta^3(\vec{r}) \right). \quad (4.9)$$

The above limiting procedure provides the low energy potential. Then the $R^2$ terms give rise to a very weak and short-ranged contribution to the Newtonian potential.

The last term contribution obtained in (4.5) due to the non-minimal couplings has the same structure as the delta function term in (4.9). Therefore, since the torsion field effect on the scalar particle manifests itself by redefining the GDT coefficient $^3$, following similar arguments used in the quadratic gravity case and in the effective field theory approach to dealing with the non-minimal coupling [40], one may conclude that it is not possible to measure the non-minimal coupling $\lambda^i_\xi$ in the region of energy/curvature where the non-relativistic approximation is valid. Then, the effective field theory of gravity is not affected by the presence of the non-minimal coupling terms $\xi_i$ in (2.4).

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\(^{3}\)Regarding this point, the authors of Ref. [41] considered propagating torsion effects on test particles and observed that the non-relativistic limit of the auto-parallel equation shows that the “force” due to the torsion potential is manifested in the same way as the gravitational one. Moreover, for static and weak potentials of the trace part of the torsion $\varphi$ and the gravitational field $h_{00}$ they showed that in the non-relativistic limit the potentials obey a Poisson partial differential equation and concluded that due to their similar effects on the test particle it is impossible to distinguish their effects unless the source and the initial condition for the torsion field is supplied; thus, the smallness of the intensity of the torsion forces, makes their detection even more difficult.
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