New Characterizations of Algebraic Regularity

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Abstract
In this paper, we give new characterizations of algebraic regularity by using differential forms and difference quotients.

1 Introduction

In [2], we introduced the algebraic regularity on the quaternions by using a new generalization of Cauchy-Riemann system, and characterized the new generalization of Cauchy-Riemann system by using Fueter-operators. In this paper, we characterize the algebraic regularity on the quaternions by using differential forms and difference quotients.

Throughout this paper, we let $\mathcal{H} = \mathcal{R}e_1 \oplus \mathcal{R}e_2 \oplus \mathcal{R}e_3 \oplus \mathcal{R}e_4$ be the quaternions discovered by W. R. Hamilton in 1843, where $\mathcal{R}$ is the real number field, $e_1$ is the identity of the real division associative algebra $\mathcal{H}$, and the multiplication among the remaining three elements in the $\mathcal{R}$-basis $\{e_1, e_2, e_3, e_4\}$ is defined by

$$e_2^2 = e_3^2 = e_4^2 = -e_1, \quad e_ie_j = -e_je_i = (-1)^{i+j+1}e_{9-i-j},$$

where $2 \leq i < j \leq 4$.

Recall that the algebraic regularity on the quaternions is defined in the following way:

**Definition 1.1** Let $U$ be an open subset of $\mathcal{H}$. We say that a quaternion-valued function $f : U \to \mathcal{H}$ is algebraic regular at $c = \sum_{i=1}^{4} c_i e_i \in U$ if $f$ has two properties given below.
(i) There exist two $C^1$ real-valued functions $f_0 : \mathbb{R}^4 \to \mathbb{R}$ and $f_1 : \mathbb{R}^4 \to \mathbb{R}$ such that

$$f(x) = f_1(x_1, x_2, x_3, x_4)e_1 + \sum_{k=2}^{4} x_k f_0(x_1, x_2, x_3, x_4)e_k$$  \hspace{1cm} (1)$$

for all $x = \sum_{i=1}^{4} x_i e_i \in U$.

(ii) The following equations hold at $(c_1, c_2, c_3, c_4) \in \mathbb{R}^4$:

$$\frac{\partial f_1}{\partial x_1} = f_0 + x_2 \frac{\partial f_0}{\partial x_2} + x_3 \frac{\partial f_0}{\partial x_3} + x_4 \frac{\partial f_0}{\partial x_4},$$  \hspace{1cm} (2)$$

$$\frac{\partial f_1}{\partial x_i} = -x_i \frac{\partial f_0}{\partial x_1}, \quad x_i \frac{\partial f_0}{\partial x_j} = x_j \frac{\partial f_0}{\partial x_i},$$  \hspace{1cm} (3)$$

where $2 \leq i, j \leq 4$ and $i \neq j$.

We say that $f : U \to \mathbb{H}$ is an algebraic regular function on $U$ if $f$ is algebraic regular at every point of $U$.

2 Characterizing Algebraic Regularity by Differential Forms

Let $U$ be an open subset of the quaternion $\mathbb{H}$. A quaternion-valued function $f : U \to \mathbb{H}$ has the form

$$f(x) = f \left( \sum_{k=1}^{4} x_k e_k \right) = \sum_{k=1}^{4} f_k(x_1, x_2, x_3, x_4)e_k,$$  \hspace{1cm} (4)$$

where $x = \sum_{k=1}^{4} x_k e_k \in U$, $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and $f_k(x_1, x_2, x_3, x_4)$ is a real-valued function of four real variables $x_1, x_2, x_3$ and $x_4$. The quaternion-valued function $f : U \to \mathbb{H}$ given by (4) is said to be smooth (or $C^1$) if the real-valued function $f_k(x_1, x_2, x_3, x_4)$ is smooth (or $C^1$) for $1 \leq k \leq 4$. The alternate notations for the real-value function $f_k(x_1, x_2, x_3, x_4)$ are given as follows

$$f_k(x_1, x_2, x_3, x_4) = f_k(x) = f_k(x_1 e_1 + x_i e_i + x_j e_j + x_{9-i-j} e_{9-i-j}),$$

where $x = \sum_{k=1}^{4} x_k e_k$ and $2 \leq i \neq j \leq 4$.

For $1 \leq i \leq 4$, we define $dx^i \in Hom_{\mathbb{R}}(\mathbb{H}, \mathbb{R})$ by

$$(dx^i)(q_1 e_1 + q_2 e_2 + q_3 e_3 + q_4 e_4) := q_i,$$
where \( q_i \in \mathbb{R} \) for \( 1 \leq i \leq 4 \). Clearly, \( \{dx^1, dx^2, dx^3, dx^4\} \) is a basis for the real vector space \( \text{Hom}_\mathbb{R}(\mathcal{H}, \mathbb{R}) \).

A **quaternion-valued** \( m \)-form on an open subset \( U \) of \( \mathcal{H} \) is an expression of the form

\[
\alpha = \sum_{1 \leq i_1, \ldots, i_m \leq 4} f_{i_1, \ldots, i_m}(x) \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m}, \tag{5}
\]

where \( f_{i_1, \ldots, i_m}(x) : U \to \mathcal{H} \) is a smooth quaternion-valued function on \( U \), and \( dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m} \) is the ordinary alternating \( \mathbb{R} \)-multilinear map from \( \mathcal{H} \times \cdots \times \mathcal{H} \) to \( \mathbb{R} \). By (5), a quaternion-valued \( 0 \)-form on an open subset \( U \) of \( \mathcal{H} \) is a quaternion-valued function defined on \( U \). The smooth quaternion-valued functions \( f_{i_1, \ldots, i_m}(x) \) are called the **coefficients** of \( \alpha \). We say that \( \alpha \) is a **real-valued** \( m \)-form if all of its coefficients are smooth real-valued functions.

The **exterior product** of a quaternion-valued \( m \)-form \( \alpha \) given by (5) and a quaternion-valued \( n \)-form given by

\[
\beta = \sum_{1 \leq j_1, \ldots, j_n \leq 4} g_{j_1, \ldots, j_n}(x) \, dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_n}, \tag{6}
\]

is defined to be the quaternion-valued \( (m+n) \)-form \( \alpha \wedge \beta \) which is given by:

\[
\sum_{1 \leq i_1, \ldots, i_m \leq 4} \sum_{1 \leq j_1, \ldots, j_n \leq 4} f_{i_1, \ldots, i_m}(x) g_{j_1, \ldots, j_n}(x) \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_n}.
\]

**Proposition 2.1** Let \( \alpha \) and \( \beta \) be quaternion-valued \( m \)-form \( \alpha \) given by (5) and a quaternion-valued \( n \)-form given by (6). If either \( \alpha \) or \( \beta \) is a real-valued form, then \( \alpha \wedge \beta = (-1)^{mn} \beta \wedge \alpha \).

If \( f : U \to U \) is a quaternion-valued 0-form given by \( f(x) = \sum_{i=1}^{4} f_i(x) e_i \) with the real-valued functions \( f_1(x), f_2(x), f_3(x) \) and \( f_4(x) \), we define the **differential** \( df \) of \( f \) to be the quaternion-valued 1-form on \( U \) given by

\[
\begin{align*}
df = & \frac{\partial f}{\partial x_1} \, dx^1 + \frac{\partial f}{\partial x_2} \, dx^2 + \frac{\partial f}{\partial x_3} \, dx^3 + \frac{\partial f}{\partial x_4} \, dx^4, \tag{7}
\end{align*}
\]

where \( \frac{\partial f}{\partial x_i} := \sum_{j=1}^{4} \frac{\partial f_j}{\partial x_i} e_j \) is a quaternion-valued function for \( 1 \leq i \leq 4 \).
Proposition 2.2 If $f$ and $g$ are quaternion-valued $0$-forms on an open subset $U$ of $\mathbb{H}$, then
\begin{align*}
    d(f + g) &= df + dg, \quad \text{(8)} \\
    d(fg) &= f \wedge (dg) + (df) \wedge g, \quad \text{(9)} \\
    d(qf) &= q(df), \quad d(fq) = (df) q \quad \text{for } q \in \mathbb{H}. \quad \text{(10)}
\end{align*}

If $\alpha$ is a quaternion-valued $m$-form given by (5), we define $d\alpha$ is the quaternion-valued $(m + 1)$-form given by
\begin{equation}
    d\alpha := \sum_{1 \leq i_1, \ldots, i_m \leq 4} d(f_{i_1, \ldots, i_m}) \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m}. \quad \text{(11)}
\end{equation}

The operator $d$ is called the **exterior differentiation**. The next proposition gives the basic properties of the exterior differentiation.

**Proposition 2.3** (i) If $q \in \mathbb{H}$ and $\alpha$, $\beta$ are quaternion-valued $m$-forms, then
\[ d(\alpha + \beta) = d(\alpha) + d(\beta), \quad d(q\alpha) = q \, d(\alpha) \quad \text{and} \quad d(\alpha q) = d(\alpha) q. \]

(ii) If $\alpha$ is a quaternion-valued $m$-form and $\beta$ is a quaternion-valued $n$-form, then
\[ d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^m \alpha \wedge (d\beta). \]

(iii) If $\alpha$ is a quaternion-valued $m$-form, then
\[ d(d\alpha) = 0. \]

Following [3], we use $Dq$ to denote the following quaternion-valued 3-form:
\begin{align*}
    Dq &:= e_1 dx^2 \wedge dx^3 \wedge dx^4 - e_2 dx^1 \wedge dx^3 \wedge dx^4 + e_3 dx^1 \wedge dx^2 \wedge dx^4 - e_4 dx^1 \wedge dx^2 \wedge dx^3 \\
    \text{or} \\
    Dq &:= e_1 dx^2 \wedge dx^3 \wedge dx^4 + \sum_{2 \leq i < j \leq 4} (-1)^{i+j} e_{9-i-j} dx^1 \wedge dx^i \wedge dx^j. \quad \text{(12)}
\end{align*}

Also, recall from [3] that left Fueter operator operators $D_L := \sum_{i=1}^{4} e_i \frac{\partial}{\partial x_i}$ and the right Fueter operator $D_R := \sum_{i=1}^{4} \left( \frac{\partial}{\partial x_i} \right) e_i$ are defined by
\begin{align*}
    D_L(f) := e_1 \left( \frac{\partial f}{\partial x_1} \right) + e_2 \left( \frac{\partial f}{\partial x_2} \right) + e_3 \left( \frac{\partial f}{\partial x_3} \right) + e_4 \left( \frac{\partial f}{\partial x_4} \right), \\
    D_R(f) := \left( \frac{\partial f}{\partial x_1} \right) e_1 + \left( \frac{\partial f}{\partial x_2} \right) e_2 + \left( \frac{\partial f}{\partial x_3} \right) e_3 + \left( \frac{\partial f}{\partial x_4} \right) e_4,
\end{align*}

where
\[ \frac{\partial f}{\partial x_i} := \left( \frac{\partial f}{\partial x_i} \right) e_1 + \left( \frac{\partial f}{\partial x_i} \right) e_2 + \left( \frac{\partial f}{\partial x_i} \right) e_3 + \left( \frac{\partial f}{\partial x_i} \right) e_4 \quad \text{for } 1 \leq i \leq 4. \]
Proposition 2.4 If $f : U \to \mathcal{H}$ is a $C^1$ quaternion-valued function defined on an open subset of $\mathcal{H}$, then

$$D q \land df = -D_1(f) v \quad \text{and} \quad df \land D q = D_r(f) v,$$

where $v = dx^1 \land dx^2 \land dx^3 \land dx^4$ is the volume form.

We now introduce two more real-valued 3-forms $D_0 q$ and $D_1 q$ as follows:

$$D_0 q := (x_2 + x_3 + x_4) dx^2 \land dx^3 \land dx^4 - \sum_{2 \leq i < j \leq 4} (-1)^{i+j} x_{9-i-j} dx^1 \land dx^i \land dx^j; \quad (14)$$

$$D_1 q := dx^2 \land dx^3 \land dx^4 + \sum_{2 \leq i < j \leq 4} (-1)^{i+j} dx^1 \land dx^i \land dx^j; \quad (15)$$

The basic properties of the two real-valued 3-forms above are given in the following

Proposition 2.5 Let $U$ be an open subset of $\mathcal{H}$. If $f$ is a $C^1$ quaternion-valued function defined on $U$, then the following equations hold on $U$:

$$D_0 q \land df = -df \land D_0 q = \left( - (x_2 + x_3 + x_4) \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + x_4 \frac{\partial f}{\partial x_4} \right) v, \quad (16)$$

$$D_1 q \land df = -df \land D_1 q = - \left( \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial x_4} \right) v, \quad (17)$$

where $v = dx^1 \land dx^2 \land dx^3 \land dx^4$ is the volume form.

Using Proposition 2.5, we have

Proposition 2.6 Let $U$ be an open subset of $\mathcal{H}$. If $f : U \to \mathcal{H}$ is a function given by

$$f(x) = f_1(x_1, x_2, x_3, x_4)e_1 + \sum_{k=2}^{4} x_k f_0(x_1, x_2, x_3, x_4)e_k,$$ 

where $x = \sum_{i=1}^{4} x_i e_i \in U$ with $x_1, x_2, x_3, x_4 \in \mathcal{R}$, $f_1$ and $f_0$ are $C^1$ functions, then the following are equivalent:

(i) $f$ is algebraic regular on $U$;
(ii) Both the equation
\[ D q \wedge df + 2 D_0 q \wedge df_0 + 2 D_1 q \wedge df_1 = 0 \] (18)
and the equation
\[ df \wedge D q + 2 df_0 \wedge D_0 q + 2 df_1 \wedge D_1 q = 0 \] (19)
hold on \( U \).

3 Characterizing Algebraic Regularity by Difference Quotients

Let \( f(x) = f_1(x_1, x_2, x_3, x_4)e_1 + \sum_{k=2}^{4} x_k f_0(x_1, x_2, x_3, x_4)e_k \) be a quaternion-valued function defined on an open subset \( U \) of \( \mathbb{H} \), where \( x = \sum_{i=1}^{4} x_i e_i \) with \( x_1, x_2, x_3, x_4 \in \mathbb{R} \) for \( 1 \leq i \leq 4 \). For each \( c = \sum_{i=1}^{4} c_i e_i \) with \( c_1, c_2, c_3, c_4 \in \mathbb{R} \), we define six pure quaternion-valued functions on \( U \) as follows:

\[
\begin{align*}
f_{9-i-j}^c(x) &= [c_i f_0(c_1 e_1 + x_i e_i + c_j e_j + c_{9-i-j} e_{9-i-j}) + \\
&+ c_j f_0(c_1 e_1 + c_i e_i + x_j e_j + c_{9-i-j} e_{9-i-j})] e_{9-i-j} + \\
&- [(1)^{i+j} c_i f_0(x_1, c_2, c_3, c_4) + \\
&+ c_{9-i-j} f_0(c_1 e_1 + c_i e_i + x_j e_j + c_{9-i-j} e_{9-i-j})] e_j + \\
&+ [(1)^{i+j} c_j f_0(x_1, c_2, c_3, c_4) + \\
&- c_{9-i-j} f_0(c_1 e_1 + x_i e_i + c_j e_j + c_{9-i-j} e_{9-i-j})] e_i, \quad (20)
\end{align*}
\]

\[
\begin{align*}
f_{9-i-j}^{-c}(x) &= [c_i f_0(c_1 e_1 + x_i e_i + c_j e_j + c_{9-i-j} e_{9-i-j}) + \\
&+ c_j f_0(c_1 e_1 + c_i e_i + x_j e_j + c_{9-i-j} e_{9-i-j})] e_{9-i-j} + \\
&+ [(1)^{i+j} c_i f_0(x_1, c_2, c_3, c_4) + \\
&- c_{9-i-j} f_0(c_1 e_1 + c_i e_i + x_j e_j + c_{9-i-j} e_{9-i-j})] e_j + \\
&- [(1)^{i+j} c_j f_0(x_1, c_2, c_3, c_4) + \\
&+ c_{9-i-j} f_0(c_1 e_1 + x_i e_i + c_j e_j + c_{9-i-j} e_{9-i-j})] e_i, \quad (21)
\end{align*}
\]

where \( 2 \leq i < j \leq 4 \).
In the proposition below, we characterize the algebraic regularity by the limits of a new kind of difference quotients which use the six pure quaternion-valued functions $f_i^c(x)$ and $f_i^e(x)$ with $i = 2, 3$ and 4.

**Proposition 3.1** Let $f : U \to \mathcal{H}$ be a quaternion-valued function defined by

$$f(x) = f_1(x_1, x_2, x_3, x_4)e_1 + \sum_{i=2}^{4} x_i f_0(x_1, x_2, x_3, x_4)e_i,$$

where $U$ is an open subset of $\mathcal{H}$, $x = \sum_{k=1}^{4} x_k e_k \in U$ with $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Let

$$c = \sum_{k=1}^{4} c_k e_k \in U$$
and

$$\Delta q = \sum_{k=1}^{4} (\Delta q)_k e_k,$$

where $c_k, (\Delta q)_k \in \mathbb{R}$ for $1 \leq k \leq 4$. If $f_1(x_1, x_2, x_3, x_4)$ and $f_0(x_1, x_2, x_3, x_4)$ are $C^1$ functions, then the following are equivalent:

(i) $f$ is algebraic regular at $x = c$;

(ii) $\lim_{\Delta q \to 0} \frac{(\Delta q)^{-1}}{\Delta q} \left\{ f(c + \Delta q) - f(c) + \sum_{i=2}^{4} \left[ f_i^e \left( c + (\Delta q)_i \sum_{k=1}^{4} e_k \right) - f_i^c (c) \right] \right\}$ exists;

(iii) $\lim_{\Delta q \to 0} \left\{ f(c + \Delta q) - f(c) + \sum_{i=2}^{4} \left[ f_i^c \left( c + (\Delta q)_i \sum_{k=1}^{4} c_k \right) - f_i^e (c) \right] \right\} (\Delta q)^{-1}$ exists.

Moreover, if one of the three coditions above holds, then both the limit in (ii) and the limit in (iii) equal to $\frac{\partial f}{\partial x_1}(c)$.

Based on Proposition 3.1, we call $\frac{\partial f}{\partial x_1}$ the **quaternion derivative** of an algebraic regular function $f$ on an open subset $U$ of $\mathcal{H}$. It is easy to check that if $f$ is an algebraic regular function on an open subset $U$ of $\mathcal{H}$, then its quaternion derivative $\frac{\partial f}{\partial x_1}$ is also an algebraic regular function on the open subset $U$.

**References**

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