Waring–Goldbach Problem: One Square, 
Four Cubes and Higher Powers

Jinjiang Li* & Min Zhang†
Department of Mathematics, China University of Mining and Technology*
Beijing 100083, P. R. China

Abstract: Let $P_r$ denote an almost–prime with at most $r$ prime factors, counted according to multiplicity. In this paper, it is proved that, for $12 \leq b \leq 35$ and for every sufficiently large odd integer $N$, the equation

$$N = x^2 + p_1^3 + p_2^3 + p_3^3 + p_4^4 + p_5^5 + p_6^6$$

is solvable with $x$ being an almost–prime $P_{r(b)}$ and the other variables primes, where $r(b)$ is defined in the Theorem. This result constitutes an improvement upon that of Lü and Mu.

Keywords: Waring–Goldbach problem; Hardy–Littlewood Method; almost–prime; sieve method

MR(2010) Subject Classification: 11P05, 11P32, 11P55, 11N36.

1 Introduction and main result

Let $a, b$ and $N$ be positive integers and define $H_{a,b}(N)$ to be the number of solutions of the following Diophantine equation

$$N = x_1^2 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^6 + x_7^7,$$

with all the variables $x_j$ being positive integers. In 1981, Hooley [7] obtained an asymptotic formula for $H_{3,5}(N)$. In 1991, from Brüdern’s work, Lu [12] get the asymptotic formula for $H_{3,b}(N)$. In addition, by using a sort of pruning technique, Lu [12] established the asymptotic formula for $H_{4,b}(N)$ ($4 \leq b \leq 6$) and gave the lower bound estimates of the expected order of magnitude for $H_{4,b}(7 \leq b \leq 17)$, $H_{5,b}(5 \leq b \leq 9)$. 

*Corresponding author.

E-mail addresses: jinjiang.li.math@gmail.com (J. Li), min.zhang.math@gmail.com (M. Zhang).
and $H_{6,b}(N) (6 \leq b \leq 7)$. Motivated by the work of Lu [12], Dashkevich [5] obtained the asymptotic formula for $H_{6,8}(N)$. In view of the results of Hooley, Lu and A. M. Dashkevich, it is reasonable to conjecture that, for every sufficiently large odd integer $N$ the following equation

$$N = p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^3 \quad (3 \leq a \leq b)$$  \hspace{1cm} (1.1)$$

is solvable, where and below the letter $p$, with or without subscript, always denotes a prime number. But this conjecture is perhaps out of reach at present. However, it is possible to replace a variable by an almost–prime. In 2016, Lű and Mu [13] employed the sieve theory and the Hardy–Littlewood method to obtain the following approximation to the conjecture (1.1).

**Theorem 1.1 (Lű and Mu, 2016)** Let $a$ and $b$ be positive integers such that

$$\frac{5}{18} < \frac{1}{a} + \frac{1}{b} \leq \frac{1}{3}.$$ \hspace{1cm} (1.2)$$

For every sufficiently large odd integer $N$, let $R_{a,b}(N)$ denote the number of solutions of the following equation

$$N = x^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^3$$ \hspace{1cm} (1.3)$$

with $x$ being an almost–prime $\mathcal{P}_{r(a,b)}$ and the other variables primes, where $r(a,b)$ is equal to $\left(\frac{4}{3}\left(\frac{1}{a} + \frac{1}{b} - \frac{5}{18}\right)^{-1}\right)$. Then we have

$$R_{a,b}(N) \gg N^{\frac{2}{3} + \frac{1}{a} + \frac{1}{b}} \log^{-7} N.$$$$

Especially, if $a = 4$, then there holds $12 \leq b \leq 35$ from the condition (1.2), and the values of $r(4,b)$ are as follows:

$r(4,12) = 24$, $r(4,13) = 27$, $r(4,14) = 30$, $r(4,15) = 34$, $r(4,16) = 38,$

$r(4,17) = 42$, $r(4,18) = 48$, $r(4,19) = 53$, $r(4,20) = 60$, $r(4,21) = 67,$

$r(4,22) = 75$, $r(4,23) = 84$, $r(4,24) = 96$, $r(4,25) = 109$, $r(4,26) = 124,$

$r(4,27) = 144$, $r(4,28) = 168$, $r(4,29) = 198$, $r(4,30) = 240$, $r(4,31) = 297,$

$r(4,32) = 384$, $r(4,33) = 528$, $r(4,34) = 816$, $r(4,35) = 1680.$

In this paper, we shall improve the result of Lű and Mu [13] in the cases $a = 4$, $12 \leq b \leq 35$ and establish the following theorem.
Theorem 1.2 For $12 \leq b \leq 35$, let $R_b(N)$ denote the number of solutions of the following equation

$$N = x^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3$$  \hspace{1cm} (1.4)

with $x$ being an almost–prime $\mathcal{P}_{r(b)}$ and the other variables primes. Then, for sufficiently large odd integer $N$, we have

$$R_b(N) \gg N^{\frac{35}{36} + \frac{1}{b} \log^{-7} N},$$

where

- $r(12) = 6$,\; $r(13) = 7$,\; $r(14) = 7$,\; $r(15) = 7$,\; $r(16) = 8$,\; $r(17) = 8$,
- $r(18) = 8$,\; $r(19) = 8$,\; $r(20) = 9$,\; $r(21) = 9$,\; $r(22) = 9$,\; $r(23) = 10$,
- $r(24) = 10$,\; $r(25) = 10$,\; $r(26) = 11$,\; $r(27) = 11$,\; $r(28) = 11$,\; $r(29) = 12$,
- $r(30) = 12$,\; $r(31) = 13$,\; $r(32) = 13$,\; $r(33) = 14$,\; $r(34) = 15$,\; $r(35) = 17$.

The proof of our result employs the Hardy–Littlewood circle method and Iwaniec’s linear sieve method, from which we can give a lower bound estimate of $R_b(N)$, which is stronger than that of the result of Lü and Mu [13] and leads to the refinement.

2 Notation

Throughout this paper, $N$ always denotes a sufficiently large odd integer; $\mathcal{P}_r$ denote an almost–prime with at most $r$ prime factors, counted according to multiplicity; $\varepsilon$ always denotes an arbitrary small positive constant, which may not be the same at different occurrences; $\gamma$ denotes Euler’s constant; $f(x) \ll g(x)$ means that $f(x) = \mathcal{O}(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$; the letter $p$, with or without subscript, always stands for a prime number; the constants in the $\mathcal{O}$–term and $\ll$–symbol depend at most on $\varepsilon$. As usual, $\varphi(n)$, $\mu(n)$ and $\tau_k(n)$ denote Euler’s function, Möbius’ function and the $k$–dimensional divisor function, respectively. Especially, we write $\tau(n) = \tau_2(n)$. $p^f \| m$ means that $p^f | m$ but $p^{f+1} \nmid m$. We denote by $a(m)$ and $b(n)$ arithmetical functions satisfying $|a(m)| \ll 1$ and $|b(n)| \ll 1$; $(m, n)$ denotes the greatest common divisor of $m$ and $n$; $e(\alpha) = e^{2\pi i \alpha}$. We always denote by $\chi$ a Dirichlet character (mod$q$), and by $\chi^0$ the principal Dirichlet character (mod$q$). Let

$$A = 10^{100}, \quad Q_0 = \log^{20A} N, \quad Q_1 = N^{\frac{12}{36} - \frac{1}{b} + 50\varepsilon}, \quad Q_2 = N^{\frac{17}{36} + \frac{1}{b} - 50\varepsilon}, \quad D = N^{\frac{3}{16} - \frac{1}{b} - 51\varepsilon},$$

$$z = D^\frac{1}{3}, \quad U_k = \frac{1}{k} N^\frac{k}{3}, \quad U^* = \frac{1}{3} N^\frac{3}{16}, \quad F_3(\alpha) = \sum_{U^* \leq n \leq 2U^*} e(n^3 \alpha),$$
In order to prove Theorem we need the following lemmas.

3 Preliminary Lemmas

Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic, and $F'(x) \geq m > 0$, or $F'(x) \leq -m < 0$, throughout the interval $[a, b]$. Then we have

$$\left| \int_a^b e^{iF(x)} \, dx \right| \leq \frac{4}{m}.$$  

**Proof.**  See Lemma 4.2 of Titchmarsh [15].

Let $f(x)$ be a real differentiable function in the interval $[a, b]$. If $f'(x)$ is monotonic and satisfies $|f'(x)| \leq \theta < 1$. Then we have

$$\sum_{a < n < b} e^{2\pi if(n)} = \int_a^b e^{2\pi if(x)} \, dx + O(1).$$
Proof. See Lemma 4.8 of Titchmarsh [15].

Lemma 3.3 Let \( 2 \leq k_1 \leq k_2 \leq \cdots \leq k_s \) be natural numbers such that
\[
\sum_{i=j+1}^{s} \frac{1}{k_i} \leq \frac{1}{k_j}, \quad 1 \leq j \leq s-1.
\]
Then we have
\[
\int_0^1 \left| \prod_{i=1}^{s} f_{k_i}(\alpha) \right|^2 d\alpha \ll N^{\frac{1}{k_1} + \cdots + \frac{1}{k_s} + \varepsilon}.
\]
Proof. See Lemma 1 of Brüdern [1].

Lemma 3.4 For \((a, q) = 1\), we have
\begin{enumerate}
\item[(i)] \( S_j(q, a) \ll q^{1 - \frac{1}{j}} \);
\item[(ii)] \( G_j(\chi, a) \ll q^{\frac{1}{2} + \varepsilon} \).
\end{enumerate}
In particular, for \((a, p) = 1\), we have
\begin{enumerate}
\item[(iii)] \(|S_j(p, a)| \leq ((j, p - 1) - 1) \sqrt{p} \);
\item[(iv)] \(|S_j^*(p, a)| \leq ((j, p - 1) - 1) \sqrt{p} + 1 \);
\item[(v)] \( S_j^*(p^\ell, a) = 0 \) for \( \ell \geq \gamma(p) \), where
\[
\gamma(p) = \begin{cases}
\theta + 2, & \text{if } p^\theta \parallel j, \ p \neq 2 \text{ or } p = 2, \ \theta = 0, \\
\theta + 3, & \text{if } p^\theta \parallel j, \ p = 2, \ \theta > 0.
\end{cases}
\]
\end{enumerate}
Proof. For (i) and (iii)–(iv), see Theorem 4.2 and Lemma 4.3 of Vaughan [17], respectively. For (ii), see Lemma 8.5 of Hua [8] or the Problem 14 of Chapter VI of Vinogradov [18]. For (v), see Lemma 8.3 of Hua [8].

Lemma 3.5 We have
\begin{enumerate}
\item[(i)] \( \int_0^1 |F_3(\alpha)F_3^*(\alpha)|^2 d\alpha \ll N^\frac{8}{7} + \varepsilon \),
\item[(ii)] \( \int_0^1 |F_3(\alpha)F_3^*(\alpha)|^4 d\alpha \ll N^\frac{11}{7} \),
\item[(iii)] \( \int_0^1 |f_3(\alpha)f_3^*(\alpha)|^2 d\alpha \ll N^\frac{8}{7} + \varepsilon \),
\item[(iv)] \( \int_0^1 |f_3(\alpha)f_3^*(\alpha)|^4 d\alpha \ll N^\frac{11}{7} \log^8 N \).
\end{enumerate}
Proof. For (i), one can see the Theorem of Vaughan [16], and for (ii), one can see Lemma 2.4 of Cai [4]. Moreover, (iii) and (iv) follow from (i) and (ii) by considering the number of solutions of the underlying Diophantine equations, respectively.
Lemma 3.6 For $\alpha = \frac{a}{q} + \beta$, define

$$\mathfrak{N}(q, a) = \left(\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0}\right),$$  \hspace{1cm} (3.1)$$

$$\Delta_4(\alpha) = f_4(\alpha) - \frac{S_4^*(q, a)}{\varphi(q)} \sum_{U_4 < n \leq 2U_4} e(\beta n^4),$$  \hspace{1cm} (3.2)$$

$$W(\alpha) = \sum_{d \leq D} \frac{c(d)}{dq} S_2(q, ad^2)v_2(\beta),$$  \hspace{1cm} (3.3)$$

where

$$c(d) = \sum_{\substack{d = mn \cr m \leq D^{2/3} \cr n \leq D^{1/3}}} a(m)b(n) \ll \tau(d).$$

Then we have

$$\sum_{1 \leq q \leq Q_0} \sum_{\substack{a = -q \cr (a, q) = 1}} 2q \int_{\mathfrak{N}(q, a)} |W(\alpha)\Delta_4(\alpha)|^2 d\alpha \ll N^2 \log^{-100A} N$$ \hspace{1cm} (3.4)$$

and

$$\sum_{1 \leq q \leq Q_0} \sum_{\substack{a = -q \cr (a, q) = 1}} 2q \int_{\mathfrak{N}(q, a)} |W(\alpha)|^2 d\alpha \ll \log^{21A} N.$$ \hspace{1cm} (3.5)$$

**Proof.** For (3.4) and (3.5), one can refer to Lemma 2.4 and Lemma 2.5 of Li and Cai [11], respectively. □

Lemma 3.7 For $\alpha = \frac{a}{q} + \beta$, define

$$V_k(\alpha) = \frac{S_k^*(q, a)}{\varphi(q)} v_k(\beta),$$  \hspace{1cm} (3.6)$$

Then we have

$$\sum_{1 \leq q \leq Q_0} \sum_{\substack{a = -q \cr (a, q) = 1}} 2q \int_{\mathfrak{N}(q, a)} |V_4(\alpha)|^2 d\alpha \ll N^{-\frac{1}{2}} \log^{21A} N,$$ \hspace{1cm} (3.7)$$

where $\mathfrak{N}(q, a)$ is defined by (3.1).

**Proof.** See (2.12) of Li and Cai [11]. □
Lemma 3.8 For \( r = \frac{a}{q} + \beta \), define

\[
V^*_r(\alpha) = \frac{S^*_r(q,a)}{\varphi(q)} v^*_r(\beta).
\]

Then \( \alpha = \frac{a}{q} + \beta \in \mathcal{M}_0 \), we have

\[
f_k(\alpha) = V_k(\alpha) + O\left(U_k \exp(- \log^{1/3} N)\right),
\]

\[
f^*_k(\alpha) = V^*_k(\alpha) + O\left(U^*_k \exp(- \log^{1/3} N)\right),
\]

\[
g_r(\alpha) = \frac{c_r(b)V_2(\alpha)}{\log U_2} + O\left(U_2 \exp(- \log^{1/3} N)\right),
\]

where \( V_k(\alpha) \) is defined (3.6), and

\[
c_r(b) = (1 + O(\varepsilon))
\]

\[
\times \int_{r-1}^{3h-36} \int_{r-2}^{t_1-1} \frac{dt_1}{t_1} \int_{r-2}^{t_2-1} \frac{dt_2}{t_2} \cdots \int_{r-4}^{t_r-1} \frac{dt_{r-3}}{t_{r-3}} \int_{r-3}^{t_r-1} \frac{\log(t_{r-2} - 1)}{t_{r-2}} dt_{r-2}.
\]

Proof. By Siegel–Walfisz theorem and partial summation, we obtain

\[
g_r(\alpha) = \sum_{\ell \in \mathcal{A}_r \atop \ell p \equiv \ell \mod q} e\left((\frac{a}{q} + \beta)(\ell p)^2\right) \frac{\log p}{\log U_2}
\]

\[
= \sum_{h=1}^{q} e\left(\frac{ah^2}{q}\right) \sum_{\ell \in \mathcal{A}_r} \frac{1}{\log U_2} \sum_{\ell p \equiv \ell \mod q} (\log p) e(\beta(\ell p)^2)
\]

\[
= \sum_{h=1}^{q} e\left(\frac{ah^2}{q}\right) \sum_{\ell \in \mathcal{A}_r} \frac{1}{\log U_2} \int_{2U_2}^{2U_2} e(\beta(\ell \nu)^2) d\left(\sum_{p \equiv \nu \mod q} \log p\right)
\]

\[
= \frac{S^*_r(q,a)}{\varphi(q)} v_2(\beta) \sum_{\ell \in \mathcal{A}_r} \frac{1}{\ell \log U_2} + O\left(U_2 \exp(- \log^{1/3} N)\right)
\]

\[
= \frac{c_r(b)V_2(\alpha)}{\log U_2} + O\left(U_2 \exp(- \log^{1/3} N)\right).
\]
This completes the proof of (3.11). Also, (3.9) and (3.10) can be proved in similar but simpler processes.

**Lemma 3.9** For \( \alpha \in \mathfrak{m}_2 \), we have
\[
h(\alpha) \ll N^{17/72} + \frac{1}{8} - 24 \varepsilon.
\]

**Proof.** By the estimate (4.5) of Lemma 4.2 in Brüdern and Kawada [3], we deduce that
\[
h(\alpha) \ll \frac{N^{17/72} (q \log N)}{(q + N|a| - N)^{1/2}} + N^{1/4} D_4^2 \ll N^{17/72} + \frac{1}{8} - 24 \varepsilon.
\]
This completes the proof of Lemma 3.9.

## 4 Mean Value Theorems

In this section, we shall prove the mean value theorems for the proof of Theorem 1.2.

**Proposition 4.1** For \( 12 \leq b \leq 35 \), define
\[
J(N, d) = \sum_{m \in \mathcal{L}} \prod_{j=1}^{p_d} \log p_j.
\]
Then we have
\[
\sum_{m \in D^{2/3}} a(m) \sum_{n \in D^{1/3}} b(n) \left( J(N, mn) - \frac{S_{mn}(N)}{mn} J(N) \right) \ll N^{35/36} + \frac{1}{10} \log A N.
\]

**Proof.** Let
\[
K(\alpha) = h(\alpha) f_2^2(\alpha) f_3^2(\alpha) f_4(\alpha) f_6(\alpha) e(-N \alpha).
\]
By the Farey dissection (3.8), we have
\[
\sum_{m \in D^{2/3}} a(m) \sum_{n \in D^{1/3}} b(n) J(N, mn) = \int_0^1 K(\alpha) d\alpha = \left( \int_{m_0} + \int_{m_1} + \int_{m_2} + \int_{m_3} \right) K(\alpha) d\alpha.
\]
From Cauchy’s inequality, Lemma 3.3 and (iii) of Lemma 3.5, we obtain
\[
\int_0^1 |f_2^2(\alpha) f_3^2(\alpha) f_4(\alpha) f_6(\alpha)| d\alpha
\]
\[
\ll \left( \int_0^1 \left| f_3(\alpha)f_4(\alpha)f_b(\alpha) \right|^2 d\alpha \right)^\frac{1}{2} \left( \int_0^1 \left| f_3(\alpha)f_3^2(\alpha) \right|^2 d\alpha \right)^\frac{1}{2} \\
\ll (N^{\frac{1}{12} + \frac{1}{5} + \varepsilon})^{1/2} (N^{\frac{1}{12} + \frac{1}{5} + \varepsilon})^{1/2} \ll N^{\frac{1}{24} + \frac{1}{9} + \varepsilon}.
\]

(4.2)

By Lemma 3.9 and (4.2), we get

\[
\int_{m_2} K(\alpha) d\alpha \ll \sup_{\alpha \in m_2} |h(\alpha)| \int_0^1 \left| f_3^2(\alpha)f_3^2(\alpha)f_4(\alpha)f_b(\alpha) \right| d\alpha \\
\ll N^{\frac{1}{24} + \frac{1}{9} - 24\varepsilon} \cdot N^{\frac{1}{24} + \frac{1}{9} + \varepsilon} \ll N^{\frac{1}{24} + \frac{1}{9} - \varepsilon}.
\]

(4.3)

From Theorem 4.1 of Vaughan [17], for \( \alpha \in m_1 \), we have

\[
h(\alpha) = W(\alpha) + O(DQ^{\frac{1}{2} + \varepsilon}) = W(\alpha) + O(N^{\frac{35}{144} + \frac{1}{12} - 25\varepsilon}),
\]

(4.4)

where \( W(\alpha) \) is defined by (3.3). Define

\[
K_1(\alpha) = W(\alpha)f_3^2(\alpha)f_3^2(\alpha)f_4(\alpha)f_b(\alpha)e(-N\alpha).
\]

(4.5)

Then, by (4.2) and (4.4), we have

\[
\int_{m_1} K(\alpha) d\alpha = \int_{m_1} K_1(\alpha) d\alpha + O(N^{\frac{1}{24} + \frac{1}{9} - 20\varepsilon}).
\]

(4.6)

Let

\[
\mathcal{N}_0(q, a) = \left( \frac{a}{q} - \frac{1}{N^{709/945}}, \frac{a}{q} + \frac{1}{N^{709/945}} \right), \quad \mathcal{N}_0 = \bigcup_{1 \leq q \leq Q_0} \bigcup_{\substack{a=-q \lfloor a \rfloor=1}}^{2q} \mathcal{N}_0(q, a),
\]

\[
\mathcal{N}_1(q, a) = \mathcal{N}(q, a) \setminus \mathcal{N}_0(q, a), \quad \mathcal{N}_1 = \bigcup_{1 \leq q \leq Q_0} \bigcup_{\substack{a=-q \lfloor a \rfloor=1}}^{2q} \mathcal{N}_1(q, a),
\]

\[
\mathcal{N} = \bigcup_{1 \leq q \leq Q_0} \bigcup_{\substack{a=-q \lfloor a \rfloor=1}}^{2q} \mathcal{N}(q, a),
\]

where \( \mathcal{N}(q, a) \) is defined by (3.1). Then we have \( m_1 \subset \mathcal{I}_0 \subset \mathcal{N} \). By the rational approximation theorem of Dirichlet, we get

\[
\int_{m_1} K_1(\alpha) d\alpha \ll \int_{m_1 \cap \mathcal{I}_0} |K_1(\alpha)| d\alpha + \int_{m_1 \cap \mathcal{N}_1} |K_1(\alpha)| d\alpha \\
\ll \sum_{1 \leq q \leq Q_0} \sum_{\substack{a=-q \lfloor a \rfloor=1}}^{2q} \int_{m_1 \cap \mathcal{I}_0(q, a)} |K_1(\alpha)| d\alpha \\
+ \sum_{1 \leq q \leq Q_0} \sum_{\substack{a=-q \lfloor a \rfloor=1}}^{2q} \int_{m_1 \cap \mathcal{N}_1(q, a)} |K_1(\alpha)| d\alpha.
\]

(4.7)
By Lemma 3.1, we have
\[ v_k(\beta) \ll \frac{U_k}{1 + |\beta|N} \]
From the trivial inequality \((q, d^2) \leq (q, d)^2\) and above estimate, we have
\[
|W(\alpha)| \ll \sum_{d \in D} \frac{\tau(d)}{d} (q, d^2)^{1/2} (\beta)^{1/2} \ll \tau_3(q) (q, d^2)^{1/2} (\beta)^{1/2} N \ll \frac{\tau_3(q) U_2 \log^2 N}{q^{1/2}(1 + |\beta|N)}.
\]
(4.8)
Thus, for \(\alpha \in \mathcal{N}_1(q, a)\), we get
\[
W(\alpha) \ll N^{\frac{473}{1890}} \log^2 N,
\]
(4.9)
from which and (4.2) we have
\[
\sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \sum_{(a,q)=1} \int_{m_1 \cap \mathcal{N}_1(q, a)} |K_1(\alpha)|d\alpha \\
\ll N^{\frac{473}{1890}} \log^2 N \cdot \int_0^1 |f_3^2(\alpha) f_3^*2(\alpha) f_6(\alpha) f_7(\alpha)|d\alpha \ll N^{\frac{35}{36} + \frac{1}{6} - \varepsilon}.
\]
(4.10)
By Lemma 3.2, we derive
\[
f_4(\alpha) = \Delta_4(\alpha) + V_4(\alpha) + O(1).
\]
Therefore, we have
\[
\sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \sum_{(a,q)=1} \int_{m_1 \cap \mathcal{N}_0(q, a)} |K_1(\alpha)|d\alpha \\
\ll \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \sum_{(a,q)=1} \int_{m_1 \cap \mathcal{N}_0(q, a)} |W(\alpha) \Delta_4(\alpha) f_3^2(\alpha) f_3^*2(\alpha) f_b(\alpha)|d\alpha \\
+ \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \sum_{(a,q)=1} \int_{m_1 \cap \mathcal{N}_0(q, a)} |W(\alpha) V_4(\alpha) f_3^2(\alpha) f_3^*2(\alpha) f_b(\alpha)|d\alpha \\
+ \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \sum_{(a,q)=1} \int_{m_1 \cap \mathcal{N}_0(q, a)} |W(\alpha) f_3^2(\alpha) f_3^*2(\alpha) f_b(\alpha)|d\alpha \\
=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\]
(4.11)
where \(\Delta_4(\alpha)\) and \(V_4(\alpha)\) are defined by (3.2) and (3.6), respectively.
It follows from Cauchy’s inequality, (iv) of Lemma 3.5 and (3.4) that
\[ I_1 \ll \sup_{\alpha \in \mathbb{G}_0} |f_b(\alpha)| \left( \sum_{1 \leq q \leq Q_0} \sum_{a = -q}^{2q} \int_{\mathbb{R}(q,a)} |W(\alpha)\Delta_4(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_0^1 |f_3(\alpha)f_3^*(\alpha)|^4 d\alpha \right)^{1/2} \]
\[ \ll N^{1/2}(N^{1/2}\log^{-10A} N)^{1/2}(N^{13/2}\log^8 N)^{1/2} \ll N^{\frac{45}{35} + \frac{1}{2}}\log^{-10A}N. \quad (4.12) \]

From (4.8), we know that, for \( \alpha \in m_1 \), there holds
\[ \sup_{\alpha \in m_1} |W(\alpha)| \ll N^{1/2} \log^{-10A}N. \quad (4.13) \]

Therefore, by Cauchy’s inequality, (3.7), (4.13) and (iv) of Lemma 3.5, we obtain
\[ I_2 \ll \left( \sup_{\alpha \in \mathbb{G}_0} |f_b(\alpha)| \right) \left( \sup_{\alpha \in m_1} |W(\alpha)| \right) \cdot \left( \sum_{1 \leq q \leq Q_0} \sum_{a = -q}^{2q} \int_{\mathbb{R}(q,a)} |V_4(\alpha)|^2 d\alpha \right)^{1/2} \]
\[ \times \left( \int_0^1 |f_3(\alpha)f_3^*(\alpha)|^4 d\alpha \right)^{1/2} \]
\[ \ll N^{1/2} \cdot N^{1/2} \log^{-30A}N \cdot (N^{-\frac{1}{2}} \log^{21A}N)^{1/2} \cdot (N^{\frac{13}{2}} \log^8 N)^{1/2} \]
\[ \ll N^{\frac{45}{35} + \frac{1}{2}}\log^{-10A}N. \quad (4.14) \]

From Cauchy’s inequality, (3.5), and (iv) of Lemma 3.5, we derive that
\[ I_3 \ll \sup_{\alpha \in \mathbb{G}_0} |f_b(\alpha)| \left( \sum_{1 \leq q \leq Q_0} \sum_{a = -q}^{2q} \int_{\mathbb{R}(q,a)} |W(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_0^1 |f_3(\alpha)f_3^*(\alpha)|^4 d\alpha \right)^{1/2} \]
\[ \ll N^{1/2}(\log^{21A} N)^{1/2}(N^{13/2}\log^8 N)^{1/2} \ll N^{\frac{13}{20} + \frac{1}{2}}\log^{20A}N \ll N^{\frac{45}{35} + \frac{1}{2}}\log^{-10A}N. \quad (4.15) \]

Combining (4.11), (4.12), (4.14) and (4.15), we can deduce that
\[ \sum_{1 \leq q \leq Q_0} \sum_{a = -q}^{2q} \int_{m_1 \cap \mathbb{G}_0(q,a)} |K_1(\alpha)| d\alpha \ll N^{\frac{45}{35} + \frac{1}{2}}\log^{-10A}N. \quad (4.16) \]

From (4.6), (4.7), (4.10) and (4.16) we conclude that
\[ \int_{m_1} K(\alpha) d\alpha \ll N^{\frac{45}{35} + \frac{1}{2}}\log^{-10A}N. \quad (4.17) \]

Similarly, we obtain
\[ \int_{m_0} K(\alpha) d\alpha \ll N^{\frac{45}{35} + \frac{1}{2}}\log^{-10A}N. \quad (4.18) \]

For \( \alpha \in m_0 \), define
\[ K_0(\alpha) = W(\alpha)V_3^2(\alpha)V_3^2(\alpha)V_4(\alpha)V_5(\alpha)e(-N\alpha). \]
Noticing that (4.4) still holds for \( \alpha \in \mathfrak{M}_0 \), it follows from (3.9), (3.10) and (4.4) that
\[
K(\alpha) - K_0(\alpha) \ll N^{\frac{71}{56} + \frac{1}{b}} \exp \left( - \log^{1/4} N \right).
\]

By the above estimate, we derive that
\[
\int_{\mathfrak{M}_0} K(\alpha) d\alpha = \int_{\mathfrak{M}_0} K_0(\alpha) d\alpha + O \left( N^{\frac{11}{56} + \frac{1}{b}} \log^{-A} N \right). \tag{4.19}
\]

By the well–known standard technique in the Hardy–Littlewood method, we deduce that
\[
\int_{\mathfrak{M}_0} K_0(\alpha) d\alpha = \sum_{m \leq D^{2/3}} a(m) \sum_{n \leq D^{1/3}} b(n) \frac{\mathcal{H}_{mn}(N)}{mn} J(N) + O \left( N^{\frac{11}{56} + \frac{1}{b}} \log^{-A} N \right), \tag{4.20}
\]
and
\[
J(N) \asymp N^{\frac{11}{56} + \frac{1}{b}}. \tag{4.21}
\]
From (4.1), (4.3), (4.17)–(4.21), the result of Proposition 4.1 follows.

In a similar way, we have

**Proposition 4.2** For \( 12 \leq b \leq 35 \), define
\[
J_r(N, d) = \sum_{(p^2 + p^2 + p^2 + p^4 + p^4 + p_6^6 \equiv N \pmod{q}, \ell_2 \in L, \ell_2 \in \mathcal{A}, \ell \equiv 0 \pmod{d})} \left( \frac{\log p}{\log \ell_2} \prod_{j=2}^{6} \log p_j \right).
\]
Then we have
\[
\sum_{m \leq D^{2/3}} a(m) \sum_{n \leq D^{1/3}} b(n) \left( J_r(N, mn) - \frac{c_r(b) \mathcal{H}_{mn}(N)}{mn \log U_2 J(N)} \right) \ll N^{\frac{11}{56} + \frac{1}{b}} \log^{-A} N,
\]
where \( c_r(b) \) is defined by (3.12).

**5 On the function \( \omega(d) \)**

In this section, we shall investigate the function \( \omega(d) \) which is defined in (5.12) and required in the proof of the Theorem 1.2.

**Lemma 5.1** Let \( \mathcal{R}(q, N) \) and \( \mathcal{L}(q, N) \) denote the number of solutions of the following congruences
\[
u_j^3 + u_j^3 + u_j^3 + u_j^4 + u_j^6 + p_6^6 \equiv N \pmod{q}, \ 1 \leq u_j \leq q, \ (u_j, q) = 1, \ 1 \leq j \leq 6,
\]
and

\[ x^2 + u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3 + p_6^5 \equiv N(\mod q), \quad 1 \leq x, u_j \leq q, \quad (u_j, q) = 1, \quad 1 \leq j \leq 6, \]

respectively. Then we have \( \mathcal{L}(p, N) > \mathcal{R}(p, N) \). Moreover, there holds

\[ \mathcal{L}(p, N) = p^6 + O(p^5), \quad (5.1) \]

\[ \mathcal{R}(p, N) = p^5 + O(p^4). \quad (5.2) \]

**Proof.** Let \( \mathcal{L}^*(q, N) \) denote the number of solutions to the following congruence

\[ x^2 + u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3 + p_6^5 \equiv N(\mod q), \quad 1 \leq x, u_j \leq q, \quad (xu_j, q) = 1, \quad 1 \leq j \leq 6. \]

Then we have

\[
\begin{align*}
p \cdot \mathcal{L}^*(p, N) &= \sum_{a=1}^{p} S_2^*(p, a) S_3^4(p, a) S_4^*(p, a) S_5^*(p, a) e\left( -\frac{aN}{p} \right) \\
&= (p - 1)^7 + E_p, \quad (5.3)
\end{align*}
\]

where

\[
E_p = \sum_{a=1}^{p-1} S_2^*(p, a) S_3^4(p, a) S_4^*(p, a) S_5^*(p, a) e\left( -\frac{aN}{p} \right).
\]

By (iv) of Lemma 3.4, we obtain

\[
|E_p| \leq (p - 1)^2(\sqrt{p} + 1)(2\sqrt{p} + 1)^4(3\sqrt{p} + 1). \quad (5.4)
\]

It is easy to verify that \( |E_p| < (p - 1)^7 \) for \( p \geq 13 \), hence we have \( \mathcal{L}^*(p, N) > 0 \) for \( p \geq 13 \). In addition, for \( p = 2, 3, 5, 7, 11 \), we can check one by one directly by hand that \( \mathcal{L}^*(p, N) > 0 \). Therefore, we have \( \mathcal{L}^*(p, N) > 0 \) for every prime \( p \), and

\[ \mathcal{L}(p, N) = \mathcal{L}^*(p, N) + \mathcal{R}(p, N) > \mathcal{R}(p, N). \quad (5.5) \]

From (5.3) and (5.4), we deduce that

\[ \mathcal{L}^*(p, N) = p^6 + O(p^5). \quad (5.6) \]

By similar arguments that lead to (5.3) and (5.4), we have

\[ \mathcal{R}(p, N) = p^5 + O(p^4). \quad (5.7) \]

Combining (5.5)–(5.7), we get (5.1). This completes the proof of Lemma 5.1. **■**

**Lemma 5.2** The series \( \mathcal{S}(N) \) is convergent and satisfying \( \mathcal{S}(N) > 0 \).
Proof. From (i) and (ii) of Lemma 3.4, we get

\[ |A(q, N)| \ll \frac{|B(q, N)|}{q^{5/2+6\epsilon}} \ll \frac{q^{5/2+6\epsilon}(\log \log q)^5}{q^2} \ll \frac{1}{q^2}. \]

Thus, the series

\[ \mathcal{S}(N) = \sum_{q=1}^{\infty} A(q, N) \]

converges absolutely. Noting that \( A(q, N) \) is multiplicative in \( q \) and by (v) of Lemma 3.4, we have

\[ \mathcal{S}(N) = \prod_p (1 + A(p, N)). \] (5.8)

From (iii) and (iv), we know that, for \( p \geq 19 \), there holds

\[ |A(p, N)| \leq \frac{(p-1)^2 \sqrt{p}(2\sqrt{p} + 1)^4 (3\sqrt{p} + 1)}{p(p-1)^6} \leq \frac{100}{p^2}. \]

Therefore, there holds

\[ \prod_{p \geq 19} (1 + A(p, N)) \geq \prod_{p \geq 19} \left(1 - \frac{100}{p^2}\right) \geq c_1 > 0. \] (5.9)

On the other hand, it is easy to see that

\[ 1 + A(p, N) = \frac{\mathcal{G}(p, N)}{(p-1)^6}, \] (5.10)

from which and (5.5), we have \( 1 + A(p, N) > 0 \). Therefore, there holds

\[ \prod_{p < 19} (1 + A(p, N)) \geq c_2 > 0. \] (5.11)

Finally, from (5.8)–(5.11), we conclude that \( \mathcal{G}(N) > 0 \). This completes the proof of Lemma 5.2. ■

In view of Lemma 5.2, we define

\[ \omega(d) = \frac{\mathcal{G}_d(N)}{\mathcal{G}(N)}. \] (5.12)

Similar to (5.8), we have

\[ \mathcal{G}_d(N) = \prod_p (1 + A_d(p, N)) = \prod_{p|d} (1 + A_d(p, N)) \prod_{p|d} \left(1 + A_d(p, N)\right). \] (5.13)

If \( (d, q) = 1 \), then we have \( S_k(q, ad^k) = S_k(q, a) \). Moreover, if \( p|d \), then we get \( A_d(p, N) = A_p(p, N) \). Therefore, it follows from (5.8), (5.12) and (5.13) that

\[ \omega(p) = \frac{1 + A_p(p, N)}{1 + A(p, N)}, \quad \omega(d) = \prod_{p|d} \omega(p). \] (5.14)
Also, it is easy to show that
\[
1 + A_p(p, N) = \frac{p}{(p - 1)^6}R(p, N). \tag{5.15}
\]
From (5.10), (5.14) and (5.15), we deduce that
\[
\omega(p) = \frac{p \cdot R(p, N)}{\Omega(p, N)}. \tag{5.16}
\]
According to (5.1), (5.2), (5.14) and (5.16), we obtain the following lemma.

**Lemma 5.3** The function \(\omega(d)\) is multiplicative and satisfies
\[
0 \leq \omega(p) < p, \quad \omega(p) = 1 + O(p^{-1}). \tag{5.17}
\]

### 6 Proof of Theorem 1.2

In this section, let \(f(s)\) and \(F(s)\) denote the classical functions in the linear sieve theory. Then by (2.8) and (2.9) of Chapter 8 in [6], we have
\[
F(s) = \frac{2e^\gamma}{s}, \quad 1 \leq s \leq 3; \quad f(s) = \frac{2e^\gamma \log(s - 1)}{s}, \quad 2 \leq s \leq 4.
\]
In the proof of Theorem 1.2, let \(\lambda^\pm(d)\) be the lower and upper bounds for Rosser’s weights of level \(D\), hence for any positive integer \(d\) we have
\[
|\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \text{ if } d > D \text{ or } \mu(d) = 0.
\]
For further properties of Rosser’s weights we refer to Iwaniec [9]. Let
\[
\mathcal{Y}(z) = \prod_{2 < p < z} \left(1 - \frac{\omega(p)}{p}\right).
\]
Then from Lemma 5.3 and Mertens’ prime number theorem (See [14]) we obtain
\[
\mathcal{Y}(z) \asymp \frac{1}{\log N}. \tag{6.1}
\]
In order to prove Theorem 1.2, we need the following lemma:

**Lemma 6.1** Under the condition (5.17), then if \(z \leq D\), there holds
\[
\sum_{d \mid \emptyset} \frac{\lambda^-(d)\omega(d)}{d} \geq \mathcal{Y}(z) \left(f \left(\frac{\log D}{\log z}\right) + O\left(\log^{-1/3} D\right)\right), \tag{6.2}
\]
and if \(z \leq D^{1/2}\), there holds
\[
\sum_{d \mid \emptyset} \frac{\lambda^+(d)\omega(d)}{d} \leq \mathcal{Y}(z) \left(F \left(\frac{\log D}{\log z}\right) + O\left(\log^{-1/3} D\right)\right). \tag{6.3}
\]
Proof. See (12) and (13) of Lemma 3 in Iwaniec [10]. ■

Let $M(b) = \lfloor \frac{72b}{36-b} \rfloor$. From the definition of $\mathcal{M}_r$, we know that $r \leq M(b)$. Therefore, we have

$$\mathcal{R}_b(N) \geq \sum_{m \in \mathcal{L}, m(3,4) = 1} \mathcal{R}(m) \sum_{1 \leq j \leq 6} \log p_j \quad \text{for } m^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 = N,$$

where $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3$, $U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$, $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3, U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$, $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3, U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$.

Thus,

$$= \Gamma_0 - \sum_{r = r(b)+1}^{M(b)} \Gamma_r.$$ 

By the property of Rosser’s weight $\lambda^{-}(d)$ and Proposition 4.1, we get

$$\Gamma_0 \geq \frac{1}{\log U} \sum_{m \in \mathcal{L}, m(3,4) = 1} \mathcal{R}(m) \sum_{1 \leq j \leq 6} \log p_j \quad \text{for } m^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 = N,$$

where $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3$, $U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$, $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3, U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$.

Similarly,

$$\Gamma_0 \geq \frac{1}{\log U} \sum_{m \in \mathcal{L}, m(3,4) = 1} \mathcal{R}(m) \sum_{1 \leq j \leq 6} \lambda^{-}(d) \mu(d) \quad \text{for } m^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 = N,$$

where $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3$, $U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$, $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3, U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$.

Similarly,

$$\Gamma_0 \geq \frac{1}{\log U} \sum_{m \in \mathcal{L}, m(3,4) = 1} \mathcal{R}(m) \sum_{1 \leq j \leq 6} \lambda^{-}(d) \mu(d) \quad \text{for } m^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 = N,$$

where $U_3 < p_1, p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3, U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$.

By the property of Rosser’s weight $\lambda^{+}(d)$ and Proposition 4.2, we have

$$\Gamma_r \leq \sum_{(\ell p)^2 + m^2 + p_j^2 + p_j^2 + p_j^2 + p_j^2 = N} 1 \quad \text{for } \ell \leq b, \quad \ell \in \mathcal{L}, \quad (m,4) = 1,$$

where $U_3 < p_2 < 2U_3, U_3 < p_3, p_4 < 2U_3$, $U_4 < p_1, p_2 < 2U_4, U_5 < p_6 < 2U_5$. 

16
\[
\leq \frac{1}{\log W} \sum_{\ell \in \mathcal{K}, \ell' \in \mathcal{L}} \frac{\log p}{\log U_2} \prod_{j=2}^6 \log p_j \sum_{d|(m, \Psi)} \mu(d)
\]

\[
= \frac{1}{\log W} \sum_{\ell \in \mathcal{K}, \ell' \in \mathcal{L}} \left( \frac{\log p}{\log U_2} \prod_{j=2}^6 \log p_j \right) \sum_{d|(m, \Psi)} \lambda^+(d)
\]

\[
\leq \frac{1}{\log W} \sum_{d|\Psi} \sum_{\lambda^+(d) \leq 2} (\log p) \log W \sum_{d|\Psi} \lambda^+(d) \omega(d) \frac{1}{d} + O(N^{\frac{35}{36} + \frac{1}{2}} \log^{-1} A N)
\]

Define
\[
C(b) = \sum_{r=r(b)+1}^{M(b)} c_r(b).
\]

According to simple numerical calculation, we obtain

\[
C(12) < 0.681372, \ C(13) < 0.430703, \ C(14) < 0.408611, \ C(15) < 0.649606, \quad (6.7)
\]

\[
C(16) < 0.496677, \ C(17) < 0.386493, \ C(18) < 0.621141, \ C(19) < 0.651975, \quad (6.8)
\]

\[
C(20) < 0.382485, \ C(21) < 0.631281, \ C(22) < 0.599447, \ C(23) < 0.426621, \quad (6.9)
\]

\[
C(24) < 0.394069, \ C(25) < 0.644773, \ C(26) < 0.603438, \ C(27) < 0.510736, \quad (6.10)
\]

\[
C(28) < 0.615415, \ C(29) < 0.502098, \ C(30) < 0.660826, \ C(31) < 0.403155, \quad (6.11)
\]

\[
C(32) < 0.656868, \ C(33) < 0.635545, \ C(34) < 0.669316, \ C(35) < 0.547965. \quad (6.12)
\]

From (6.4)–(6.12), we deduce that

\[
\mathcal{R}_b(N) \geq (f(3) - F(3)C(b)) \left( 1 + O\left( \log^{-\frac{1}{D}} \right) \right) \frac{\mathcal{S}(N)\mathcal{J}(N)Y(z)}{\log U} + O\left( N^{\frac{35}{36} + \frac{1}{2}} \log^{-1} A N \right)
\]
\[ \geq \frac{2e}{3} (\log 2 - 0.681372) \left( 1 + O\left( \frac{1}{\log^{1/3} D} \right) \right) \frac{\mathcal{S}(N)J(N)Y(z)}{\log U} + O\left( \frac{N^{35/36 + \frac{1}{b}}}{\log^{4} N} \right) \]

\[ \gg N^{35/36 + \frac{1}{b}} \log^{-7} N, \]

which completes the proof of Theorem 1.2.

**Acknowledgement**

The authors would like to express the most sincere gratitude to Professor Wenguang Zhai for his valuable advice and constant encouragement.

**References**

[1] J. Brüdern, *Sums of squares and higher powers*, J. London Math. Soc. (2), 35(2) (1987), 233–243.

[2] J. Brüdern, *A problem in additive number theory*, Math. Proc. Cambridge Philos. Soc., 103(1) (1988), 27–33.

[3] J. Brüdern, K. Kawada, *Ternary problems in additive prime number theory*, in: Analytic Number Theory, C. Jia and K. Matsumoto (eds.), Dev. Math. 6, Kluwer, Dordrecht, 2002, 39–91.

[4] Y. C. Cai, *The Waring–Goldbach problem: one square and five cubes*, Ramanujan J., 34(1) (2014), 57–72.

[5] A. M. Dashkevich, *On representing natural numbers as a sum of mixed powers*, Math. Notes, 57(3), 254–260.

[6] H. Halberstam, H. E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[7] C. Hooley, *On a new approach to various problems of Waring’s type*, in: Recent progress in analytic number theory, Vol. 1, Academic Press, London, 1981, 127–191.

[8] L. K. Hua, *Additive Theory of Prime Numbers*, Amer. Math. Soc., Providence, Rhode Island, 1965.

[9] H. Iwaniec, *Rossers sieve*, Acta Arith., 36(2) (1980), 171–202.

[10] H. Iwaniec, *A new form of the error term in the linear sieve*, Acta Arith., 37(1) (1980), 307–320.
[11] Y. J. Li, Y. C. Cai, Waring–Goldbach problem: two squares and some higher powers, J. Number Theory, 162 (2016), 116–136.

[12] M. G. Lu, On a problem of sums of mixed powers, Acta Arith., 58(1) (1991), 89–102.

[13] X. D. Lü, Q. W. Mu, On Waring–Goldbach problem of mixed powers, J. Théor. Nombres Bordeaux, 28(2) (2016), 523–538.

[14] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie, J. Reine Angew. Math., 78 (1874), 46–62.

[15] E. C. Titchmarsh, The Theory of the Riemann Zeta–Function, 2nd edn., (Revised by D. R. Heath–Brown), Oxford University Press, Oxford, 1986.

[16] R. C. Vaughan, Sums of three cubes, Bull. London Math. Soc., 17(1) (1985), 17–20.

[17] R. C. Vaughan, The Hardy–Littlewood Method, 2nd edn., Cambridge Tracts Math., Vol. 125, Cambridge University Press, 1997.

[18] I. M. Vinogradov, Elements of Number Theory, Dover Publications, New York, 1954.