Minimizing Regret in Dynamic Decision Problems

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February 3, 2015

Abstract

The menu-dependent nature of regret-minimization creates subtleties in applying regret-minimization to dynamic decision problems. Firstly, it is not clear whether forgone opportunities should be included in the menu. We explain commonly observed behavioral patterns as minimizing regret when forgone opportunities are present, and also show how the treatment of forgone opportunities affects behavior in the classical secretary problem. Secondly, dealing with the dynamic inconsistency of non-Bayesian preferences requires techniques such as sophistication to be used in planning. Sophistication leads to even more options for the menu. We investigate different approaches to defining the menu, and the implications of each approach. Finally, we provide conditions under which dynamic consistency is guaranteed for a regret-minimizer.

1 Introduction

Savage [28] and Anscombe and Aumann [3] showed that a decision maker maximizing expected utility with respect to a probability measure over the possible states of the world is characterized by a set of arguably desirable principles. However, as Allais [2] and Ellsberg [6] point out using compelling examples, sometimes intuitive choices are incompatible with maximizing expected utility. One reason for this incompatibility is that there is often ambiguity in the problems we face. That is, we often lack sufficient information to capture all uncertainty using a single probability measure over the possible states.
To this end, there is a rich literature offering alternative means of making decisions, usually called *ambiguity-averse preferences* (see, e.g., [1, 11, 29] for surveys). For example, we might choose to represent uncertainty using a set of possible states of the world, but using no probabilistic information at all to represent how likely each state is. With this type of representation, two popular rules for decision-making are *maximin utility* and *minimax regret*. Maximin says that you should choose the option that maximizes the worst-case payoff, while minimax regret says that you should choose the option that minimizes the *regret* you’ll feel at the end, where roughly speaking, regret is the difference between the payoff you achieved, and the payoff that you could have achieved had you known what the true state of the world was. Both maximin and minimax regret can be naturally extended for decision-making with other representations of uncertainty, for example, with a set of probability measures over the possible states, minimax regret becomes minimax expected regret (MER) [17, 32].

Real-life problems are often dynamic, with many stages where actions can be taken, and information can be learned throughout the stages as well. As such, we are interested in applying regret minimization to dynamic decision problems. To do this, we need to consider a number of issues that have been considered before for MMEU, now in the context of regret. A fundamental issue concerning regret minimization in dynamic decision problems is the definition of regret. While there is a natural way to extend the maximization of expected utility to dynamic setting, it is not at all clear how the definition of regret should be carried over to the dynamic setting. In static decision problems, the regret for each act is computed with respect to a *menu*. That is, each act is judged against the other acts in the menu. However, the menu also has a second meaning. It also defines the set of choices that the decision maker can choose from, also called the set of *feasible acts*. In a dynamic decision problem, the set of feasible acts can change between the stages. We use the term *mother decision tree* to refer to the entire decision problem from beginning to end. As more actions are taken, more and more acts in the mother decision tree become *forgone opportunities*. These are plans that were initially available to the decision maker, but are no longer available due to earlier actions of the decision maker. Should the menu also change? If so, how? Since regret intuitively captures comparison of a choice against its alternatives, it is natural for the menu to consist of at least the set of all feasible acts at the point of decision-making. But should the menu include forgone opportunities?

It may seem reasonable to say that such forgone opportunities are irrelevant and hence should not be in the menu. Indeed, the *consequentialism*
approach in economics would insist that we do not care about forgone opportunities. Existing definitions for regret in dynamic settings (see, e.g., Hayashi [17]) ignore forgone opportunities. However, introspection tells us that we sometimes do take forgone opportunities into account when we feel regret – when one is finding a new job, one might very well compare the available options to what might have been available had one chosen a different career path years ago. In Section 3 we also argue that keeping forgone opportunities in the menu is necessary in order for the decision maker to carry out ex-ante optimal plans.

To illustrate how the treatment of forgone opportunities affects the decision maker’s behavior, we examine minimax regret in the classical secretary problem [10]. We find that minimax regret with forgone opportunities stops earlier than minimax regret without forgone opportunities, which in turn stops earlier than expected utility maximization. We also look at decision patterns when forgone opportunities are weighted according to how long they have been forgone. It is quite natural for someone to regret a recently forgone opportunity more than an opportunity that has been forgone long ago. For example, regret from comparing to a career opportunity forgone ten years ago is likely to have less psychological impact than regret from comparing to a career opportunity given up last month. We can capture this temporal difference by discounting the regrets depending on the alternative that it is associated with.

In more general dynamic decision problems, we may run into the dynamic inconsistency problem. It is well-known that in dynamic decision problems, ambiguity-averse decision rules may result in dynamic inconsistency [24, 1, 30]. When we adopt the predominant view that the decision maker makes a choice according to his or her beliefs at every decision node, dynamic inconsistency refers to the situation where the plan that a decision maker favors and intends to carry out at one point in the decision problem is rejected against some other plan at a later point. In fact, there are results indicating that ambiguity-averse preferences must suffer from either dynamic inconsistency or some other anomaly [1, 7].

A standard approach to avoiding dynamic inconsistency is sophistication. A sophisticated decision maker is one that understands and plans around her future preferences [16, 21, 30, 33], instead of naively believing that her future selves will carry out the plan that she currently prefers. Most of the work on sophistication focuses on variants of the maximin rule, where the issue of menu does not arise. In the context of regret, however, sophisticated behavior lead to a new type of plans that are not feasible at a decision node: unachievable plans. These are plans that are not forgone opportunities, but
cannot be carried out by the decision maker because future choices specified by this plan involve choices that her future selves are unwilling to make.

Clearly, the largest reasonable menu to use would be the set of all plans in the mother decision tree. However, it might make sense to use a smaller menu. For example, one might want to model a decision maker who do not feel regret for forgone opportunities. If we exclude both forgone opportunities and unachievable plans, then all remaining plans are feasible. Although it is possible to consider arbitrary subsets of the largest menu, there are some natural classes of menus of interest. We examine four approaches to choosing menus, depending on how we deal with forgone opportunities and unachievable plans, and we characterize the implications of using each of these approaches of regret-minimization with sophistication. Although we use minimax regret in our examples, these results also apply to maximin, MER, and other decision rules.

There are undesirable problems with sophistication. For example, Al-Najjar and Weinstein [1] note that a sophisticated decision maker may demonstrate information aversion, that is, may strictly prefer to not receive a piece of freely available information. In the context of maximin utility, information aversion can be explained by the fact that a decision maker is trying to prevent inversion of her preferences from the increase in knowledge [30]. The decision maker may also be trying to avoid temptation. For regret minimization, there is another explanation for information aversion: learning new information may cause more regret to be experienced [21].

Although there are some psychological explanations for information aversion, some might still find this behavior unattractive. Therefore, we are also interested in understanding how to guarantee dynamic consistency. We consider the Minimax Weighted Expected Regret (MWER) [14] model, and give a necessary and sufficient condition for which a Minimax Weighted Expected Regret decision maker can always carry out an ex-ante optimal plan. Because this necessary and sufficient condition may not be easy to check, we also give a simpler sufficient condition, richness, that says that if two weighted distributions are in the set of beliefs, then some other related weighted distributions must be as well. Our approach is similar in spirit to that of Epstein and Schneider [8], who define a model called recursive multiple priors, and show that dynamic consistency is guaranteed when the decision maker’s prior beliefs form what they call a rectangular set.

The remainder of the paper is organized as follows. Section 2 discuss preliminaries. Section 3 introduces forgone opportunities. Section 4 looks at regret minimization in the secretary problem. Sections 5 and 6 discuss dynamic inconsistency and sophistication consistent planning. Section 7 gives
a behavioral analysis of using different menus. Section 8 gives conditions under which consistent planning is not required. Section 9 briefly discusses some related work.

2 Preliminaries

2.1 Static decision setting and regret

Given a set $S$ of states and a set $X$ of outcomes, an act $f$ (over $S$ and $X$) is a function mapping $S$ to $\Delta(X)$, the set of all finite-support distributions over $X$. We use $\mathcal{F}$ to denote the set of all acts. For simplicity in this paper, we take $S$ to be finite. Associated with each outcome $x \in X$ is a utility: $u(x)$ is the utility of outcome $x$. We call a tuple $(S, X, u)$ a (non-probabilistic) decision problem. To define regret, we need to assume that we are also given a set $M \subseteq \mathcal{F}$ of acts, called the menu. The reason for the menu is that, as is well known (and we demonstrate by example shortly), regret can depend on the menu. We assume that every menu $M$ has utilities bounded from above. That is, we assume that for all menus $M$, $\sup_{g \in M} u(g(s))$ is finite. This ensures that the regret of each act is well defined. For a menu $M$ and act $f \in M$, the regret of $f$ with respect to $M$ and decision problem $(S, X, u)$ in state $s$ is

$$\text{reg}_M(f, s) = \left( \sup_{g \in M} u(g(s)) \right) - u(f(s)).$$

That is, the regret of $f$ in state $s$ (relative to menu $M$) is the difference between $u(f(s))$ and the highest utility possible in state $s$ (among all the acts in $M$). The regret of $f$ with respect to $M$ and decision problem $(S, X, u)$ is the worst-case regret over all states:

$$\max_{s \in S} \text{reg}_M(f, s).$$

We denote this as $\text{reg}_M^{(S, X, u)}(f)$, and usually omit the superscript $(S, X, u)$ if it is clear from context. The minimax regret decision rule minimizes $\max_{s \in S} \text{reg}_M(f, s)$. In other words, the minimax regret choice function is

$$C_{\text{reg}_M}(M) = \arg\min_{f \in M} \max_{s \in S} \text{reg}_M(f, s).$$

The choice function may return a set of objects if there are more than one regret minimizers.
If there is a probability measure \( \Pr \) over the states, then we can consider the probabilistic decision problem \((S, X, u, \Pr)\). The expected regret of \( f \) with respect to \( M \) is

\[
\text{reg}_{M, \Pr}(f) = \sum_{s \in S} \Pr(s) \text{reg}_M(f, s).
\]

If there is a set \( \mathcal{P} \) of probability measures over the states, then we consider the \( \mathcal{P} \)-decision problem \((S, X, u, \mathcal{P})\). The maximum expected regret of \( f \in M \) with respect to \( M \) and \((S, X, u, \mathcal{P})\) is

\[
\text{reg}_{M, \mathcal{P}}(f) = \sup_{\Pr \in \mathcal{P}} \left( \sum_{s \in S} \Pr(s) \text{reg}_M(f, s) \right).
\]

The minimax expected regret (MER) decision rule minimizes \( \text{reg}_{M, \mathcal{P}}(f) \).

In an earlier paper, we introduced another representation of uncertainty, weighted set of probability measures \( [14] \). A weighted set of probability measures generalizes a set of probability measures by associating each measure in the set with a weight, intuitively corresponding to the reliability or significance of the measure in capturing the true uncertainty of the world. Minimizing weighted expected regret with respect to a weighted set of probability measures gives a variant of minimax regret, called Minimax Weighted Expected Regret (MWER).

A set \( \mathcal{P}^+ \) of weighted probability measures on a set \( S \) consists of pairs \((\Pr, \alpha_\Pr)\), where \( \alpha_\Pr \in [0,1] \) and \( \Pr \) is a probability measure on \( S \). Let \( \mathcal{P} = \{ \Pr : \exists \alpha (\Pr, \alpha) \in \mathcal{P}^+ \} \). We assume that, for each \( \Pr \in \mathcal{P}^+ \), there is exactly one \( \alpha \) such that \((\Pr, \alpha) \in \mathcal{P}^+ \). We denote this number by \( \alpha_\Pr \), and view it as the weight of \( \Pr \). We further assume for convenience that weights have been normalized so that there is at least one measure \( \Pr \in \mathcal{P} \) such that \( \alpha_\Pr = 1 \).

If beliefs are modeled by weighted probabilities \( \mathcal{P}^+ \), then we consider the \( \mathcal{P}^+ \)-decision problem \((S, X, u, \mathcal{P}^+)\). The maximum weighted expected regret of \( f \in M \) with respect to \( M \) and \((S, X, u, \mathcal{P}^+)\) is

\[
\text{reg}_{M, \mathcal{P}^+}(f) = \sup_{\Pr \in \mathcal{P}} \left( \alpha_\Pr \sum_{s \in S} \Pr(s) \text{reg}_M(f, s) \right).
\]

### 2.2 Likelihood updating

As is described in \([14]\), one way of updating weighted sets of probabilities is by using likelihood updating. Let \( E \subseteq S \) be an event. Define \( \mathcal{P}^+(E) = \)
sup\{\alpha_{Pr,E} : Pr \in \mathcal{P}\}; if \overline{\mathcal{P}}^+(E) > 0, set \alpha_{Pr,E} = \sup\{Pr' \in \mathcal{P} : Pr'|E=Pr|E\} \frac{\alpha_{Pr,Pr'(E)}}{\overline{\mathcal{P}}^+(E)}.

Note that given a measure \(Pr \in \mathcal{P}\), there may be several distinct measures \(Pr'\) in \(\mathcal{P}\) such that \(Pr' | E = Pr | E\). Thus, we take the weight of \(Pr | E\) to be the sup of the possible candidate values of \(\alpha_{Pr,E}\). By dividing by \(\overline{\mathcal{P}}^+(E)\), we guarantee that \(\alpha_{Pr,E} \in [0, 1]\), and that there is some measure \(Pr\) such that \(\alpha_{Pr,E} = 1\), as long as there is some pair \((\alpha_{Pr}, Pr) \in \mathcal{P}\) such that \(\alpha_{Pr,Pr(E)} = \overline{\mathcal{P}}^+(E)\). If \(\overline{\mathcal{P}}^+(E) > 0\), we take \(\mathcal{P}^+ | E\), the result of applying likelihood updating by \(E\) to \(\mathcal{P}^+\), to be

\{(Pr | E, \alpha_{Pr,E}) : Pr \in \mathcal{P}\}.

If \(\overline{\mathcal{P}}^+(E) = 0\), then \(\mathcal{P}^+ | E\) is undefined.

In computing \(\mathcal{P}^+ | E\), we update not just the probability measures in \(\mathcal{P}\), but also their weights. Intuitively, probability measures that are supported by the new information will get larger weights than those not supported by the new information. The new weight combines the old weight with the likelihood. Clearly, if all measures in \(\mathcal{P}\) assign the same probability to the event \(E\), then likelihood updating will give the same weight to each probability measure, resulting in measure-by-measure updating. This is not surprising, since such an observation \(E\) does not give us information about the relative likelihood of measures. Using likelihood updating is appropriate only if the measure generating the observations is assumed to be stable. This is because if the generating probability distribution changes with time, then we cannot converge to a single generating distribution. For example, if observations of heads and tails are generated by coin tosses, and a coin of possibly different bias is tossed in each round, then likelihood updating would not be appropriate.

### 2.3 Dynamic decision setting

In this section we review the standard extensive-form games framework, which we use to represent dynamic decision problems. We model a dynamic decision problem using a finite extensive form game between the decision maker and \textit{nature}. This is a game with \textit{imperfect information} and \textit{perfect recall} (see, e.g., [25]). There is some set \(S\) of states of nature, and nature makes the first move, choosing a single “true” state out of the set \(S\) of states. We assume that the decision maker has no initial information about which state nature has chosen. More formally, a \textit{decision tree} is a tuple \((H, I, u)\), where

- \(H\) is a set of histories (finite sequences of actions) such that:
the empty sequence $\langle \rangle$ is in $H$,
- for each $s \in S$, $\langle s \rangle$ is in $H$,
- if $(a^k)_{k=1,\ldots,K} \in H$ and $L < K$ then $(a^k)_{k=1,\ldots,L} \in H$;

- $\cal I$ is a partition of $H$ called an information partition, with the property that $A(h) = A(h')$ whenever $h$ and $h'$ are in the same member of $\cal I$. To capture perfect recall, we also have the condition that if $h,h'$ are in the same information set $I$, then all subhistories $h_j$ must be in the same information set as the corresponding subhistory $h'_j$;

- $u$ is a real-valued utility function on $Z$, the set of terminal histories (histories $(a^k)_{k=1,\ldots,K} \in H$ such that there is no $a^{K+1}$ such that $(a^k)_{k=1,\ldots,K+1}$ is in $H$).

Given a decision tree $(H, \cal I, u)$, the set $S$ of states is the set $\{ s : \langle s \rangle \in H \}$. We let $\cal T$ denote the set of all decision trees. We often use the terminology mother decision tree to refer to the decision tree representing the entire dynamic decision problem we are interested in (as opposed to a subtree of the original dynamic decision problem).

Since the decision maker initially considers all states in $S$ to be possible, we have $|S|$ subtrees, one for each $s \in S$, in the decision maker’s initial information set. An event is a subset of the states in $S$. At all nodes in the tree, the information set is associated with a set of states representing the event that the decision maker has “observed”.

We let $E(I)$ be the event of states considered possible at information set $I$, and we let $A(I)$ denote the set of actions available at information set $I$. We consider only pure strategies, or plans, which are mappings from information sets to actions available at each information set. Equivalently, plans are decision trees with a single branch at each information set.

A plan is equivalent to an act in the static choice setting. Therefore, in the dynamic setting, a menu is a set of plans. Since we can view a tree or subtree as a collection of plans, we can also view a tree as a menu. This equivalency is reflected in our choice of notation.

3 Forgone opportunities

All the regret-minimizing decision rules discussed in Section 2 define regret with respect to a menu. As a result, in a dynamic decision problem, we have to decide if and how forgone opportunities should affect the menu. If we use a mother decision tree to represent the entire decision problem from
beginning to end, then as more actions are taken, more and more plans become *forgone opportunities*. These are plans that were initially available to the decision maker, but are no longer available due to earlier actions of the decision maker.

The consequentialism tenet in economics \cite{24} implies that forgone opportunities should simply be ignored. However, introspection tells us that we often regret not taking an opportunity that is now forgone. In other words, even though there is no hope of choosing that forgone opportunity, we still regret it. This implies that we are including such forgone opportunities in our “menu” when we “compute” regret. In this section, we look at common behavioral patterns, including procrastination and the endowment effect. The former have not been modeled in a decision-theoretic setting. Both procrastination and the endowment effect can be explained by minimizing regret when forgone opportunities are present.

3.1 Procrastination

In this section we provide an explanation for procrastination based on forgone opportunities and regret-minimization. A student has an exam in two days. She can either start studying today, play today and then study tomorrow, or just play for both days and never study at all. There are two possible states in the world: one where the exam is difficult, and one where the exam is easy. The utilities reflect a combination of the amount of pleasure that the student derives in the next two days, and her score on the exam relative to her classmates.

Suppose that the first day of play gives the student $p_1 > 0$ utils, and the second day of play gives the student $p_2 > 0$ utils. Her exam score affects her utility only in the case where the exam is hard and she studies for both days, in which case she gets an additional $g_1$ utils for doing much better than everyone else, and in the case where the exam is hard and she never studies, in which case she loses $g_2 > 0$ utils for doing much worse than everyone else.

Figure 1 provides a graphical representation of the decision problem. For compactness, we omit nature’s node which chooses between the hard exam and the easy exam. We also draw only one of the two information sets of the student, describing the payoffs for the two states in the format of \{hard, easy\}.

Consider the case where $g_1 > p_1 + p_2$. That is, the student would be happier doing well on the test if it were hard, than playing for two days. Suppose that the student uses minimax regret to make her decision. On the first day, she observes that playing for one day and then studying on the
next day has a worst-case regret of $g_1 - p_1$, while studying on both days has a worst-case regret of $g_2 - p_1 - p_2$. Suppose that $g_2 - p_2 > g_1$. Therefore, she plays on the first day. On the next day, she compares her two available options: to study, or to play. Studying has a worst-case regret of $p_2$, while playing only has a worst-case regret of $p_2 - g_2$. Therefore, she also plays on the second day.

In this example, the student does not take forgone opportunities into account when computing regret, resulting in procrastination. If the student were to take forgone opportunities into account, then the preference reversal between the first and second day would not occur, and the student would indeed play on the first day and study on the second.

It is important to note, however, that these are relatively “rare” behavioral patterns. More “rational” individuals do not exhibit these preference reversals that result in self-destructing procrastination. The fact that behavioral anomalies involving preference reversals seem rare suggests that in many scenarios, regret-minimizers do take forgone opportunities into account when making sequential decisions.

### 3.2 The high price of ownership

Ariely [4] points out that people often overvalue items that they own. For example, a lottery for highly sought-after Duke University basketball tickets inflates students sense of value for the tickets. Students who won the tickets valued them ten times more than the students who did not receive them. Ariely proposes several psychological explanations for this behavior, for example, that ownership is such a big part of our society that we tend to focus on what we may lose rather than on what we may gain.
This is an example of the *endowment effect*, the hypothesis that individuals value what they already own more than what they do not own. Another notable example of the endowment effect include a study by Kahneman et al.\[19\], where subjects were given a mug and then offered a chance to sell it back. Kahneman et al. found that subjects typically demanded twice as much money when selling the mug than the amount that they are willing to pay for the mug.

We provide an alternative explanation of this phenomenon, based on regret. In our model, the decision maker has an opportunity to buy a good that has uncertain value. We represent this uncertainty using two states. In the first state, the item is useful, and in the second state, the item is not useful. If the good is bought, the decision maker can then resell or keep the good. The key to our model lies in the alternative opportunity that becomes forgone when the good is obtained. Here, the alternative can be thought of as not buying the good, or buying something else. By buying the good in question, the decision maker no longer has the money to buy the alternative, so the alternative becomes a forgone opportunity.

Before buying the good, the alternative opportunity is still available, and with this alternative in the menu, a regret-minimizer prefers buy the good, and then sell it. After buying the good, however, the alternative opportunity becomes forgone, and comparing only the options to sell or keep the good, the regret-minimizer prefers to keep the good. Because the sell price has not changed, this implies that the regret-minimizer has increased her valuation of the good between the time that she did not own the good, and the time that she did. This setting is depicted in Figure[2] We impose a number

\[ a, b \]
\[ c, d \]
\[ e, f \]

Figure 2: The price of ownership.
of constraints on the utilities. We assume that selling the item provides
the same amount of money to the decision maker regardless of the state.
However, in the state where the item is useful, the decision maker may feel
a sense of loss in selling the item. As a result, we have \( a \leq b \). Similarly, if
the item is useful, then not buying it at all gives the DM a sense of loss,
hence \( e \leq f \). We assume that buying and then selling a useless item incurs
a monetary loss, so \( f > b \). Moreover, if the item is useful, then the decision
maker feels worse for not buying it at all than to first buy and then sell it.
Therefore we have \( a > e \). Also, from the definition of ‘usefulness’, we have
\( c > d \), \( c > e \), and also \( f \geq d \).

In order for the preference reversal to occur in the example, we also
assume that \( 0 \leq b - d < c - a \). That is, the difference in utility between
owning and selling the item if it were useful is greater than the difference
in utility between selling and keeping the item if it were not useful. This
implies that after buying the item, between the two options of keeping and
selling the item, the decision maker would rather keep the item.

Now we check that the described behavior does occur. Consider the
initial regrets with respect to between the three options, not buying has a
regret of

\[
\max\{a - e, c - e\},
\]

while buying and keeping the item has a regret of

\[
\max\{f - d, a - c\},
\]

and buying and then selling or returning the item has a regret of

\[
\max\{f - b, c - a\}.
\]

The problem is more interesting if the decision maker would actually
buy the item in the first place with the intention to resell it; this occurs if
\( c - e > f - b \geq c - a \).

This means that the difference in utility between owning and never owning a
useful item is greater than the difference between never owning and owning
then reselling a useless item; and the latter is in turn greater than the
difference between keeping and reselling a useful item.

When these constraints are satisfied, we have a decision maker who is
initially willing to resell the item at a particular price, but later changes her
mind and opts to keep the item instead, implying a raise in her valuation of
the item. For example, we can have $a = d = 1, e = 0, b = 10, c = 15, f = 20$. We can apply this model to Ariely’s example with the basketball tickets.

Just as in the procrastination example, the “endowment effect” observed here depends on the fact that the decision maker ignores forgone opportunities while computing regret. If all forgone opportunities are included in the menu, this behavior does not occur because preferences remain constant across time.

4 The secretary problem

In this section, we consider regret-minimization in a well-known optimal stopping problem — the secretary problem [10]. The secretary problem shows that the menu attitude substantially affects the decision maker’s behavior.

Since the secretary problem and its solutions are relatively simple, we forgo formal definitions in this section. In the general monotonic utility secretary problem, there are $n$ applicants for a single open position, with one coming in at each round. The decision maker can rank the applicants that she has seen relative to one another, but does not know how the applicants seen rank against unseen applicants. At each round, the decision maker can either choose to hire the current applicant and end the decision process, or she can forgo the opportunity to ever hire this applicant, and move on to the next round. The reward for hiring an applicant depends on the rank of the hired applicant among all $n$ applicants.

More formally, at each round $j$, the decision maker observes $X_j$, the relative rank of the $j$th applicant among the first $j$ applicants observed. The reward for hiring an applicant with absolute rank $k$ is $U(k)$. For simplicity, we assume that the reward is monotonically decreasing in the absolute rank of the applicant, and the reward for not hiring anyone at all is $-\infty$.

This problem has been well studied in the context of maximizing expected utility with respect to a uniform prior distribution over all possible sequence of absolute rankings of the candidates [10]. In this case, it is known that the optimal strategy is always a threshold rule. That is, at any point in the decision problem, there exists a threshold — a number corresponding to a relative ranking — for which the decision maker should stop and hire the applicant if the applicant’s relative ranking is better than the threshold. The actual optimal thresholds are characterized by recursive formulas that are quite complicated [10].

We now consider minimax regret behavior. The same analysis can be
applied to MER or MWER, albeit with more complexity due to the many different possible beliefs. We first consider the case where forgone opportunities do not affect regret. In other words, at each round, there are only two possible choices in the “menu” used to compute regret: stopping and continuing.

At round $n$, the decision maker simply chooses to accept the final applicant, or to hire no one and receive $-\infty$. At round $j < n$, the worst-case regret of stopping is realized if the applicant in the next round is the absolute best, and the current candidate has the lowest absolute rank possible that is consistent with its relative rank. If $X_j = x_j$, then the worst-case regret for stopping is:

$$R_j^{\text{stop}}(x_j) = U(1) - U(x_j + (n - j)).$$

The worst-case regret of continuing at round $j$ is realized if the decision maker keeps seeing worse and worse applicants at all subsequent rounds. It is not hard to see that, regardless of the stopping rule used after round $j$, it is possible to stop at the absolute worst applicant. Therefore, the largest possible regret occurs if the round $j$ applicant turns out to have absolute rank equal to $x_j$. Thus, the worse-case regret of continuing at round $j$ is:

$$R_j^{\text{cont}}(x_j) = U(x_j) - U(n).$$

Suppose that the decision maker uses the rule of stopping if and only if the worst-case regret for stopping is less than the worst-case regret for continuing. That is, at round $j$, the decision maker stops if and only if

$$U(1) - U(x_j + (n - j)) < U(x_j) - U(n).$$

Like the case of maximizing expected utility, the cut-off value at each round does not depend on the history of applicants seen in the first $j - 1$ rounds. This is the manifestation of the context independence axiom in the secretary problem.

If the utility difference between consecutively ranking applicants is some constant $c$, then the optimal cut-off value can be simply expressed as

$$x_j < \frac{j + 1}{2}.$$

Table 1 compares the cut-off values of maximizing expected utility (assuming a uniform distribution on all possible sequences of absolute rankings) and minimax regret without forgone opportunities, when $n = 10$. In
this example, the cut-off value for minimax regret is always weakly larger than the cut-off value for maximizing expected utility. This means that at every stage, a minimax-regret decision maker is willing to stop at a worse candidate than her expected utility maximizing counterpart.

| Round | EU | Minimax Regret (no forgone opportunities) |
|-------|----|------------------------------------------|
| 1     | -  | -                                        |
| 2     | 1  | 1                                        |
| 3     | 1  | 1                                        |
| 4     | 2  | 2                                        |
| 5     | 2  | 2                                        |
| 6     | 2  | 3                                        |
| 7     | 2  | 3                                        |
| 8     | 3  | 4                                        |
| 9     | 4  | 4                                        |

Table 1: Cut-off values for \( n = 10 \). Each entry represents the relative ranking for which it is strictly better to stop.

Next, we include forgone opportunities in the computation of regret. At round \( n \), our assumption that hiring even the worst applicant is better than not hiring anyone at all means that at round \( n \), the regret-minimizing option is to hire the \( n \)th applicant.

At round \( j < n \), the worst-case regret of stopping is identical to the previous case, where forgone opportunities are not part of the regret computations. The worst-case regret is realized if the applicant in the next round is the absolute best, and the current applicant ranks lower than all future applicants. The worst-case regret is:

\[
R_{\text{stop}}^j(x_j) = U(1) - U(x_j + (n - j)).
\]

The worst-case regret of continuing is different when forgone opportunities can affect regret. As before, regardless of what the decision maker does after continuing, it is always possible to stop at the absolute worst applicant. The regret, however, now involves comparison against hiring any of the \( j \) forgone applicants. The worst-case regret of continuing at round \( j \) is then

\[
R_{\text{cont}}^j(x_j) = \max_{i \leq j} U(x_i) - U(n) = U(1) - U(n).
\]

As before, the optimal stopping strategy for the decision maker is to stop if and only if the worst-case regret of stopping is less than the worse-case
regret of continuing, that is, if

\[ U(1) - U(x_j + (n - j)) < U(1) - U(n), \]

i.e., \( U(x_j + (n - j)) < U(n) \).

Comparing with the previous case where forgone opportunities are ignored, we see that this decision maker is more conservative – she has a greater tendency to stop at every round. In fact, the decision maker must stop if the current round applicant is not relatively-worst amongst the applicants already seen. This is consistent with the intuition that when forgone opportunities are included in the menu, then the more opportunities are passed up, the higher the potential regret; as a result a decision maker that includes forgone opportunities in the menu are less inclined to create forgone opportunities.

Unlike the case of maximizing expected utility and minimax regret without forgone opportunities, here the cut-off value at any particular stage \( j \) may depend on the relative ranking of the first \( j \) applicants.

As we see, there are multiple reasonable solutions to the classical secretary problem, depending on whether one is an expected-utility maximizer, a regret-minimizer who does not regret forgone opportunities, or a regret-minimizer who do regret forgone opportunities. These decision makers are increasingly conservative.

### 4.1 Discounted forgone opportunities

A question that arises if we choose to include forgone opportunities in the menu is to what extent forgone opportunities should be included in the menu. That is, we may choose to include “recently forgone” opportunities in the menu, while choices that have been unavailable for a long time are excluded. At what point should we consider certain forgone opportunities to be “too far back” to matter? Moreover, should there be a smooth transition between distant forgone opportunities and recent ones? That is, should the regret associated with a forgone opportunity be weighted according to how long the opportunity has been forgone? We explore the implications of such variations in the classical secretary problem.

As we argued in the introduction, humans have a tendency to regret opportunities that have been forgone long ago and are no longer viable choices. However, these forgone opportunities might affect regret to a lesser extent than the feasible choices in the menu. Therefore, we consider a more general attitude toward forgone opportunities, where forgone opportunities
are included in the computation of regret, but are discounted by some factor related to how long the opportunity has been forgone.

At round \( j < n \), the worst-case regret of stopping is identical to the previous cases. The worst-case regret is realized if the applicant in the next round is the absolute best, and the current applicant ranks lower than all future applicants. The worst-case regret is:

\[
R^j_{\text{stop}}(x_j) = U(1) - U(x_j + (n - j)).
\]

The worst-case regret of continuing is different from the previous case. As before, regardless of what the decision maker does after continuing, it is always possible to stop at the absolute worst applicant. The regret, however, now involves a weighted comparison against hiring any of the \( j \) forgone applicants. We use the sequence of weights \( w_t \leq 1 \) to capture the fact that opportunities that have been forgone long ago may result in less regret. Intuitively, \( w_0 \) would be 1, and \( w_1 \geq w_2 \geq \ldots \geq w_j \). For example, we can have \( w_t = c^t \) for some \( 0 < c < 1 \). The worst-case regret of continuing at round \( j \) is then

\[
R^j_{\text{cont}}(x_j) = \max_{t \leq j} w_j - t U(x_t) - U(n).
\]

As before, the optimal stopping strategy for the decision maker is to stop if and only if the worse-case regret of stopping is less than the worse-case regret of continuing. That is:

\[
U(1) - U(x_j + (n - j)) < \max_{t \leq j} w_j - t U(x_t) - U(n).
\]

It is easy to derive a comparative result: if \( \bar{w} \geq \bar{w}' \) point-wise, then a decision maker who uses weights \( w' \) would have a greater tendency to stop at any round.

5 Dynamic inconsistency

The “irrational” behavior described in the procrastination example in Section 3.1 and the high price of ownership example in Section 3.2 can be understood as a change in the decision maker’s preferences as time passes. A dynamic decision problem can be reduced to a static decision problem by reducing all available plans to acts, and choosing the act that has the highest expected utility (see, e.g. Kreps [22] Chapter 10). An expected utility maximizer would choose a pure strategy that maximizes expected utility
among all pure strategies. An expected utility maximizer is also guaranteed to be *dynamically consistent*.

One way of describing dynamic consistency is that the plan considered optimal at a given point in the decision process is also optimal at any preceding point in the process, as well as any future point that is reached with positive probability. When dynamic consistency fails to hold, an ex-ante optimal plan may not be favored in the future, and hence be impossible to carry out.

The procrastination example in Section 3.1 shows that even without obtaining new information, a decision maker who uses minimax regret may not be able to execute the ex-ante optimal plan. It also explains the procrastination phenomenon as a result of dynamic inconsistency.

Although the student had intended to play on the first day and then study on the next day, she cannot commit to this plan. Had she realized this on the first day, she might have wanted to start studying on the first day, since studying on the first day has a worst-case regret of 50, which is lower than the worst-case regret of playing on both days, which is 60. The key in this example is that the change in the menu used to compute regret changed the ranking of the available options, causing dynamic inconsistency. This phenomenon is generally referred to as *changing opportunities*, and is one of the two causes of dynamic inconsistency.

In the procrastination example, if the student had included forgone opportunities in the computation of regret, then she would not be comparing only studying to playing on the second day. Instead, she would be comparing the three original choices, and her decision on the second day would be the same as that on the first day. As a result, she can carry out the original plan of playing on the first day, and then studying on the second day.

In general, if the decision maker includes forgone opportunities in the menu, and if no information is gained throughout the decision problem, then including forgone opportunities is enough to rule out dynamic inconsistency due to changing opportunities. This is because the menu, and hence the preferences over the terminal nodes, remain the same throughout the decision problem. It is particularly intriguing that with regret, doing the "irrational" thing (defying consequentialism) prevents preference reversals. We make this notion of dynamic consistency in the absence of information precise in Section 7.1.

The other cause of dynamic inconsistency is the *updating of beliefs* as the decision maker gains knowledge during the dynamic decision problem. This issue has been extensively studied (see, e.g., [7, 17, 18, 23, 24, 26, 34]), especially in the context of maxmin utility (see, e.g., [8, 9, 30]). In general,
all simple methods of updating non-Bayesian beliefs can have the potential for dynamic inconsistency. For example, while a Bayesian who updates his or her belief (a probability distribution) via conditioning is always dynamically consistent, a maxmin expected utility decision maker \cite{12} who updates her belief (a set of probability distributions) by conditioning each probability distribution on the received information is prone to dynamic inconsistency. Regret-minimizing decision makers also suffer from this problem.

6 Consistent planning

In this section, we study one of the standard methods of resolving dynamic inconsistency: sophistication. The sophistication method assumes that the decision maker is aware of, and plans around, her potentially inconsistent preferences. We focus on a specific flavor of sophistication, called consistent planning.

Using a menu-dependent decision rule, there is an important design choice when using consistent planning – what menu should the decision maker use when computing regret at each decision node. We explore the implications of the different design choices, and finally note some known shortcomings of the sophistication approach.

Sophistication circumvents the dynamic inconsistency problem by using backward induction – the agent is made to correctly anticipate her future choices. A sophisticated agent is aware of the potential for dynamic inconsistency, and thus use backward induction to determine the feasible plans, which are the plans that can actually be carried out. In the procrastination example, a sophisticated agent would know that she would not be studying on the second day. Therefore, she knows that playing on the first day and then studying on the second day is not a feasible plan.

Before defining consistent planning, we first need to define the choice behavior of the decision maker at each information set. Consider an information set $I$. Let $M$ be the menu used at $I$, let $E$ be the event encoded by the information set, and let $M'$ be the set of feasible plans at the information set. $M'$ may differ from $M$, since not all plans that the decision maker may potentially feel regret about need to be feasible at the time of making the decision. An example of an infeasible plan is one that requires the decision maker to have made a different choice at a past time step. Since that choice is already forgone, the decision maker can no longer carry out the plan requiring that choice. For simplicity, we assume that $M' \subseteq M$, i.e., that the decision maker feels regret for at least all the choices that are
actually feasible, but she might feel regret for even more plans. We assume that at each information set, the decision maker’s choice is defined by the choice function $C_{M,E}(M')$, which can be minimax regret, MER, or even some arbitrary choice function.

For example, minimax regret choice with respect to a menu $M$ which is a superset of the feasible acts $M'$ is defined as:

$$C_{reg}^{M,E}(M') = \arg\min_{f \in M'} \max_{s \in E} \text{reg}_M(f, s),$$

and MWER is similarly defined as:

$$C_{MWER}^{M,E}(M') = \arg\min_{f \in M'} \text{reg}_{M,P^+|E}(f).$$

That is, when given a choice from feasible set $M'$, and the true state is known to be some state in the set $E$, and the set $M$ is the set of plans that the decision maker potentially feels regret about ex post, the decision maker chooses a plan that minimizes the anticipated expected weighted regret.

Our definition of menu-dependent consistent planning is based on the definition of menu-independent consistent planning by Siniscalchi [30], which is in turn based on definitions of Strotz [33] and Gul and Pesendorfer [13].

Because a regret-minimizing decision maker requires a menu in order to make a choice, unlike the definitions of Siniscalchi [30] and others [13, 33], our menu-dependent version of consistent planning also depends on a menu attitude $\mu$, which is a function that specifies which menu is to be used at every information set. The input to the menu attitude function, which is associated with a mother decision tree, includes the information set $I$, and a plan for each information set below $I$. The latter is required in order to compute the set of unachievable plans at information set $I$. Since there are many possibilities for the menu at each decision node, we use a menu attitude to specify the menu at each decision node.

From the perspective at an information set $I$, two plans that only differ at the ancestors and nondescendents of $I$ are essentially the same. Therefore, we will consider such plans to be equivalent with respect to information set $I$. Let $P(I)$ denote the set of plans with the domain restricted to the information set $I$ and the descendants of $I$, and let $I(\langle s, a_1, \ldots, a_k \rangle)$ denote the information set associated with the history $\langle s, a_1, \ldots, a_k \rangle$. In particular, $I(\langle s \rangle)$ denotes the information set at the beginning of the decision problem for the decision maker.

**Definition 6.1** (Menu-Dependent Consistent Planning). *Given a decision*
tree $T$ and a menu attitude $\mu_T$, we define $CP_T$ inductively. For all information sets $I$ for which there are no subsequent information sets, define

$$CP_T(I) = C_{M,E(I)}(P(I)), \quad \text{where } M = \mu_T(I, P(I)).$$

Inductively, for all information sets $I = I((s, a_1, \ldots, a_k))$ for which $CP_T(I((s, a_1, \ldots, a_{k+1})))$ is defined for all $a_{k+1} \in A(I)$, we define

$$CP_T(I) = C_{M,E(I)}(P(I)), \quad \text{where } M = \mu_T(I, \{CP_T((s, a_1, \ldots, a_{k+1})) \}_{a_{k+1} \in A(I)}).$$

Note that in the inductive case, $\langle s, a_1, \ldots, a_k \rangle$ is always an information set. The definition of consistent planning begins at the terminal nodes, where there is no choice but to accept the outcomes. For non-terminal decision nodes $I$, once the consistent planning solution has been determined for all descendent nodes, the consistent planning solution for $I$ can be determined based on the preferences $C$ over plans. The process is repeated until the consistent planning solution for the decision tree $T$ has been determined.

6.1 The menu attitude

Regret-minimization with consistent planning is not completely defined unless we specify what menu is to be used at each decision node using a menu attitude. Clearly, the largest reasonable menu to use would include all plans available at the beginning of the decision problem, the mother decision tree. However, it might make sense to use a smaller menu. For example, one might want to model a decision maker who does not feel regret for forgone opportunities.

In general, there are two types of infeasible plans that the decision maker may or may not take into consideration when computing regret:

- plans that have been rendered infeasible by previous steps: forgone opportunities.
- plans that are not forgone but nonetheless cannot be committed to: unachievable plans.

These notions are defined formally in Section 7. Note that, by definition, the set of forgone opportunities is disjoint from the set of unachievable plans. Moreover, once the set of forgone opportunities and the set of unachievable plans are removed, all remaining plans are feasible plans that can actually
be carried out. Therefore, the largest reasonable set of plans that can be ignored is the union of the forgone opportunities and the unachievable plans.

Although it is possible to consider arbitrary subsets of all plans in the mother decision tree, some natural classes of menus are particularly interesting – those generated by systematic elimination of an entire class of infeasible plans. Thus, we examine four approaches to choosing menus, depending on how we deal with forgone opportunities and unachievable plans, and we characterize the implications of using each of these approaches, under the assumption of sophistication.

We consider four menu attitudes, that is, consistent use of menus that either include or exclude forgone opportunities, and either include or exclude unachievable plans. We denote these four menu attitudes by \((FO, UP)\), \((\neg FO, UP)\), \((FO, \neg UP)\), and \((\neg FO, \neg UP)\), respectively. These menu attitudes yield possibly different menus at each information node. We show that each menu attitude results in a different behavioral pattern.

The climate change example depicted in Figure 3 illustrates how the four different menu attitudes result in four different plans being chosen for the same decision problem. Figure 3 is a compact representation of the decision problem. There are four actions that nature can choose from: “low sensitivity and technology is not ready”, “high sensitivity and technology is not ready”, “low sensitivity and technology is ready”, and “high sensitivity and technology is ready”. In this case, the true state does not affect the available actions in each stage, so we draw only one of the four subtrees, listing out all four outcomes for each terminal node. We do not include nature’s action.

The payoffs in the climate change example depends on whether the technology for eco-friendly alternatives is ready or not, the amount of time that the governments wait before taking action, as well as the intrinsic sensitivity of the Earth. A high-sensitivity Earth punishes waiting because the impact of climate change will be severe. On the other hand, if the technology is not yet ready, the costs of implementing drastic measures will be high. In this example, ignoring forgone opportunities makes one content to wait longer than if forgone opportunities are included in regret computations. This is expected, since waiting reduces the number of alternatives that one can feel regret about.
7 Implications of the menu attitudes

In this section, we explore the implications of the different menu attitudes, that is, we try to answer the question “what does the menu attitude tell us about a decision maker?”.

The objects of choice in our setting are plans. For our purposes, we view a decision problem as a collection of plans. Since consistent planning is the process of selecting a plan (or a set of equally-good plans) given a decision tree, it is natural to capture consistent planning behavior using choice functions, as opposed to preference relations. The assumption that we can ask a decision maker to tell us the set of most-preferred plans given a decision tree is weaker than the alternative assumption that we can ask a decision maker to rank all available plans given a decision tree. In particular, as Stoye [31] observed, one may in principle never be able to observe entire menu-dependent rankings. For example, if $a, b, c$ are plans in decision tree $T$, how does one observed the preferences $a \succ_T b \succ_T c$? Clearly $a$ would be chosen by the choice function, but $b \succ_T c$ cannot be observed.

Finally, all the properties and axioms that we discuss can be translated into properties about tree-indexed preference relations, so nothing is lost by modeling preferences using choice functions indexed by menus (e.g., $C_M(M')$).

To ensure that the axioms of interest are nontrivial, we need the underlying preferences over plans to be nontrivially menu-dependent. That is, there should be a menu where the addition of an unchosen act changes the choice.

**Definition 7.1.** A family of menu-dependent preferences $C$ is nontrivially menu-dependent if there exists distinct acts $a_1, a_2, a_3$, and a menu $M$ containing $a_1, a_2$, such that $C_{M \cup \{a_3\}}(M) = a_1$ and $C_M(M) = a_2$.

It is easy to see that when there is a nonconstant utility function, regret-minimization satisfies nontrivial menu dependency.

7.1 Implications of forgone opportunities

Forgone opportunities are plans that used to be feasible, but can no longer be carried out from the current information set. Given a plan $p$, let $p/I$ denote the plan $p$ restricted to the information set $I$ and the descendants of $I$.

**Definition 7.2** (Forgone Opportunity). Given a decision tree $T$, a plan $p$ is a forgone opportunity at information set $I$ if $p \in P(I(\langle s \rangle))$ and $p/I \notin P(I)$.
As discussed in Section 5, if all forgone opportunities are included in the menu, such that the menu remains the same at every history of the decision tree, then decisions will be dynamically consistent in the absence of information about the true state.

**Definition 7.3** (No preference reversal). A family of choice functions $C_h$ exhibits no preference reversal if for all $f \in C_h(M_h) \setminus C_h(\emptyset)$, $f \in C_h(M_h)$ for all histories $h$.

**Proposition 7.4.** If forgone opportunities are included in the menu, and for all histories $h$, $E(h) = \{(s,h) : s \in S\}$, then no preference reversals occur.

**Proof.**

If forgone opportunities are never included in the computation of regret, then it is easy to see that the choices at a particular point in the decision problem will be independent of what the mother decision tree was. This can be captured by the following axiom. Let $I \sim I'$ if $I$ and $I'$ are the same set of trees, up to relabeling.

**Axiom 1** (Context Independence). For all decision trees $T$ and $T'$, if $I \sim I'$ then $CP_T(I) \sim CP_{T'}(I')$.

Context independence can also be interpreted as a form of consequentialism, which, according to Machina [24], is ‘snipping’ the decision tree at the current choice node, throwing the rest of the tree away, and calculating preferences at the current choice node by applying the original preference ordering to alternative possible continuations of the tree. In this paper, context independence is synonymous with consequentialism.

**Proposition 7.5.** A family $CP$ of choice functions resulting from consistent planning using a nontrivially menu-dependent set of preferences on plans satisfies context independence if and only if the menu attitude excludes forgone opportunities.

**Proof.** We first show that consistent planning satisfies context independence if the menu attitude excludes forgone opportunities. Let $T, T'$ be decision trees, and let $I \in T$ and $I' \in T'$ such that $I \sim I'$. If $I$ has no descendants, then, by definition of consistent planning, $CP_T(I) = CP_{T'}(I')$. If $I$ has descendants, then by induction on the structure of the tree and the fact that $I \sim I'$, we have

$$\mu_T(I, \{CP_T(\langle I, a \rangle)\}_{a \in A(I)}) = \mu_{T'}(I', \{CP_{T'}(\langle I', a \rangle)\}_{a \in A(I)})$$
Therefore \( CP_T(I) = CP_{T'}(I') \).

For the converse, we show by example that context independence can be violated if the menu attitude includes forgone opportunities. By nontrivial menu-dependence, there exists distinct acts \( a_1, a_2, a_3 \), and a menu \( M \) containing \( a_1, a_2 \), such that \( C_{M \cup \{a_3\}}(M) = a_1 \) and \( C_M(M) = a_2 \). Let \( a_1, a_2, a_3 \) and menu \( M \) be those specified in the definition of nontrivial menu-dependence. Consider two decision trees. The first tree, \( T \), has a single decision node \( I \) where the available actions are \( M \). The second tree, \( T' \), has a decision node where the decision maker can choose between an action that leads to \( I' \sim I \), and another action \( a_3 \) that leads to a terminal node. By the definition of nontrivial menu-dependence, \( CP_T(I) = a_2 \neq a_1 = CP_{T'}(I') \). Therefore, context independence is violated.

7.2 Implications of ignoring unachievable plans

In this section, we show a simple property that distinguishes consistent planning behavior by their attitude toward unachievable plans, when the underlying preferences over plans show nontrivial menu dependence.

Recall that unachievable plans are plans that are not forgone opportunities but cannot be carried out because they are not part of the consistent planning solution. More precisely:

**Definition 7.6 (Unachievable Plan).** Given a tree \( T \) and consistent planning solution \( CP_T \), a plan \( p \in P(I) \) is an unachievable plan at information set \( I \) if \( p/I' \notin CP_T(I') \) for all descendants \( I' \) of \( I \).

The property we discuss is an analog of the Independence of Irrelevant Alternatives (IIA) axiom from choice theory. The intuition is that if a consistent planning process ignores plans that cannot be carried out, then adding such plans to a decision tree will not affect the consistent planning solution. For any decision tree \( T \) and plan \( p \), let \( T \setminus \{p\} \) denote the decision tree \( T \) with the plan \( p \) removed from it.

**Axiom 2 (Independence of Unachievable Plans (IUP)).** For all \( T \in \mathcal{T} \), if \( p \notin CP_T((\langle s \rangle)) \) then \( CP_T(I(\langle s \rangle)) = CP_{T \setminus \{p\}}(I(\langle s \rangle)) \).

Preferences that are independent of unachievable plans lend themselves to iterated elimination of suboptimal plans as a way of finding the optimal plan. In certain settings, it may be difficult to rank plans or find the most preferred plan among a large menu. For instance, consider the problem of deciding on a career path. In these settings, it may be relatively easy to identify bad plans, the elimination of which simplifies the problem. Conversely,
computational benefits may motivate a decision maker to ignore unachievable plans. That is, a decision maker may choose to ignore unachievable plans because doing so simplifies the search for the preferred solution.

**Proposition 7.7.** A family of choice functions resulting from consistent planning using a nontrivially menu-dependent set of preferences on plans satisfies Independence of Unachievable Plans if and only if the menu attitude excludes unachievable plans.

**Proof.** We first show that consistent planning while ignoring unachievable plans will satisfy Independence of Unachievable Plans. Consider a tree $T$ and $p \notin CP_T(I(\langle s \rangle))$. We want to show that $CP_T(I(\langle s \rangle)) = CP_T(I(\langle s \rangle))$. This holds trivially if $p$ is not a plan in $T$. Otherwise, suppose for contradiction that $CP_T(I(\langle s \rangle)) \neq CP_T(I(\langle s \rangle))$. By assumption, $p \notin CP_T(I(\langle s \rangle))$, and $p \notin CP_T(I(\langle s \rangle))$. Therefore, there exists some $p'$ and $p'' \neq p'$, both distinct from $p$, such that $p'$ is only in $CP_T(I(\langle s \rangle))$ and $p''$ is only in $CP_T(I(\langle s \rangle))$. In order for this to occur, it must be the case that there is some information set $I$ in the tree $T$ such that $CM,M,E(I(P_T(I))) \neq CM,M,E(I(P_T(I)))$, for appropriate $M_1,M_2$. However, this means that either (1) $CM,M,E(I(P_T(I)))$ is part of $p$, which is not possible because $p$ is unachievable at $I$, or (2) $M_1 \neq M_2$, which is not possible because the menu attitude is assumed to ignore unachievable plans.

To show that IUP is violated when unachievable plans are included in the menu, we need only look to a single-stage decision problem. Since we assume that the underlying menu-dependent preferences are nontrivially menu-dependent, there exists distinct acts $a_1,a_2,a_3$, and menu $M$ containing $a_1,a_2$, such that $CM,M(a_3)(M) = a_1$ and $CM(M) = a_2$.

Thus we just let the tree be $\Gamma = M$. $CM(\Gamma) = a_2$ and $CM,M(\Gamma \cup \{a_3\}) = a_1$, violating Independence of Unachievable Plans. \qed

### 7.2.1 Commitment and procrastination

In this section, we show that procrastination can happen even to sophisticated decision makers, providing another explanation for procrastination. A sophisticated decision maker who takes unachievable plans into account when computing regret can be understood as being “sophisticated enough” to understand that her preferences may change in the future, but not sophisticated enough to completely ignore the plans that she cannot force herself to commit to. On the other hand, a sophisticated decision maker who ig-
nores unachievable plans does not feel remorse for not being able to commit to certain plans.

Consider the decision problem in Figure 4, which is a variation on the decision problem in Figure 1. The scenario is the same as before, with the only difference being that there is now a third state, where the exam is of intermediate difficulty. In this third state, studying for one day gives the student an additional 45 utils.

Assume that the student ignores forgone opportunities. As before, after playing on the first day, the student would decide to play for another day, which has a worst-case regret of 20, instead of studying, which has a worst-case regret of 25. Therefore, playing on the first day and then studying on the second day is an unachievable plan.

If the student were to ignore unachievable plans in regret computation, then studying for both days has a worst-case regret of 50, while playing on both days has a worst-case regret of 60. Therefore, the student would start studying on the first day. However, if the student includes unachievable plans in regret computation, then studying for both days has a worst-case regret of 70, while the worst-case regret of playing on both days remains at 60. Therefore, the student would play on both days instead.

This example illustrates the fact that one can often justify any arbitrary decision by making up irrelevant alternatives. Therefore, when modeling a decision problem, it is not only important to have a good set of states and outcomes and acts, but the set of available choices must also be chosen carefully. Of course, this problem applies equally well to static decision problems.

Table 2 summarizes the properties of each of the menu attitudes. Note that context independence and IUP together identify the four menu attitudes. In other words, these are a subset of the axioms characterizing the different choice behavior. This is true for all forms of regret-minimization, as well as arbitrary choice functions.

|       | Context independence | IUP |
|-------|----------------------|-----|
| $FO, UP$ | No                   | No  |
| $FO, ¬UP$ | No                   | Yes |
| $¬FO, UP$ | Yes                  | No  |
| $¬FO, ¬UP$ | Yes                  | Yes |

Table 2: Properties of the different menu attitudes
8 When is consistent planning not required?

Although sophistication and consistent planning resolve the issue of dynamic inconsistency, a sophisticated decision maker can still behave “irrationally”, as described in Section 7.2.1. Moreover, it is known that sophistication can result in an anomaly called *aversion to information* [1]. There are examples where a sophisticated agent strictly prefers not to receive free information. Some experts [1] regard this aversion to information as irrational.

In light of these issues with sophistication, we consider a different approach: instead of using consistent planning, we find conditions under which dynamic inconsistency does not occur, and hence where consistent planning is not necessary. This approach is similar to those used by Sarin and Wakker [26], Epstein and Schneider [8], Maccheroni, Marinacci, and Rustichini [23], and Klibanoff, Marinacci, and Mukerji [20]. More specifically, we examine conditions under which a naive optimal plan can always be carried out to completion. This is a weaker requirement than full dynamic consistency, since preferences over non-optimal plans need not be dynamically consistent. To prevent dynamic inconsistency due to changing menus, we use a constant menu, including both forgone opportunities and unachievable plans, throughout the decision problem.

8.1 Necessity of forgone opportunities

In this section, we argue that it is necessary to include all forgone opportunities in regret computation if we are to have any hope of ensuring dynamic consistency. Our argument is simple. Consider two similar decision problems, A and B, in Figure 5.

Note that at the node after first going L, the utilities and available choices are identical if forgone opportunities are ignored. As a result, a decision maker that ignores forgone opportunities necessarily makes the same choice in both sub-trees. However, in A, the ex-ante optimal plan is LR, while in B, the ex-ante optimal plan is LL. Therefore, it is impossible for a decision maker to ignore forgone opportunities and be dynamically consistent all the time.

8.2 Conditions for dynamic consistency

In this section, we focus on how uncertainty is resolved throughout the decision problem, rather than the set of available actions at each decision node. We assume that whenever information is revealed to the decision
maker, it is of the form \( E \) or \( \overline{E} \). That is, whenever the decision maker learns a piece of information, she either learns that the true state is in \( E \), or she learns that the true state is in \( \overline{E} \). Thus, we omit the cases where it is possible for the decision maker to learn that the true state is in \( \{s_1, s_2\} \), or otherwise it is in \( \{s_2, s_3\} \). Since we also assume that the information that the decision maker receives is always correct, throughout the course of the decision problem the set of states that the decision maker considers possible becomes smaller and smaller.

The notion of dynamic consistency for menu-independent preferences is usually captured by the following axiom on preferences, called \textit{Strong Dynamic Consistency of preference} (SDC). The following is the version of the axiom given by Tallon and Vergnaud [34].

**Axiom 3.** (SDC) For all \( E \subseteq S \), for all acts \( f, g \), if \( f \succeq_E g \) and \( f \succeq_E c \), then \( f \succeq_S g \). Moreover, if one of the conditional preferences are strict, then so is the unconditional preference.

We define a \textit{menu-dependent} version of Axiom 3 and also specify an event \( E \) for which it holds, rather than insisting that it holds for all events \( E \). This modification is useful since in a single decision problem it may be the case that only a small number of possible observations (events) are possible. Furthermore, we translate the axiom to use choice functions as our primitives.

**Axiom 4.** (SDC-M with respect to event \( E \)) For all acts \( f \), menus \( M \) containing \( f \) and \( M_1, M_2 \subseteq M_3 \subseteq M \), if \( f \in C_{M,E}(M_1) \) and \( f \in C_{M,E^c}(M_2) \), then

1. \( f \in C_{M,S}(M_3) \);

2. for all acts \( g \), if \( g \notin C_{M,E}(M_1) \) or \( g \notin C_{M,E^c}(M_2) \), then \( g \notin C_{M_3,S}(M_3) \).

When the agent has observed event \( E \subseteq S \), the agent’s actions are determined by the choice function \( C_{T,E}(M) \), where \( T \) is the menu consisting of all plans at the start of the decision problem. We also assume that the following axiom holds. Roughly, it says that if the observed event is \( E \), then the part of the plan off of \( E \) does not affect the choice.

**Axiom 5** (Conditional preference). For event \( E \), all acts \( f, g, h \), menus \( M, M' \), if \( fEg \in C_{M,E}(M') \) then \( fEh \in C_{M,E}(M') \) for all \( fEh \in M' \).

We show that as long as Axiom 4 and Axiom 5 hold for all events \( E \) in a decision problem, then no preference reversals occur.
Proposition 8.1. If for all events \( E \) associated with decision nodes in the decision tree, Axiom 4 and Axiom 5 hold, then no preference reversals occur.

Proof. Due to the structure of a decision tree, we can assume that the menu allows for arbitrary contingent decisions at each decision node. That is, if some acts \( f \) and \( g \) are in the menu \( M \) of plans, and at some decision node the DM’s knowledge is the set of states \( E \), then \( fEg \) and \( gEf \) are also in \( M \). In other words, if \( f \) is a plan and if \( g \) is a plan, and if there is a decision node where the DM learns that the state of the world is in \( E \), then the DM can choose a plan that is contingent on observing \( E \). Thus, suppose that there is some ex-ante optimal plan \( f \), such that \( f \in C_{\Gamma,S}(\Gamma) \). Further suppose for contradiction that we cannot carry out \( f \). This means that at some decision node where the information available is event \( E \), there is some other plan \( f' \) where \( f' \in C_{\Gamma,E}(M) \) and \( f \notin C_{\Gamma,E}(M) \). But this means that \( f'Ef \in C_{\Gamma,E}(M) \) and \( f \notin C_{\Gamma,E}(M) \), and \( f'Ef \in C_{M,E^c}(M') \), which would mean that \( f'Ef \in C_{\Gamma,S}(\Gamma) \) and \( f \notin C_{\Gamma,S}(\Gamma) \), i.e., \( f \) was not an ex-ante optimal plan.

It turns out that a sufficient condition for Axiom 4 to hold for a pair \( f, g \) of acts, a menu \( M \), and an event \( E \) is that the weighted regrets can be computed separately on \( E \) and \( E^c \).

Definition 8.2. The weighted regret of \( f \) with respect to \( M \) and \( \mathcal{P}^+ \) is separable under event \( E \) if

\[
\text{reg}_{M,\mathcal{P}^+}(f) = \sup_{\Pr \in \mathcal{P}^+} \alpha \Pr \left( \Pr(E)\text{reg}_{M,\mathcal{P}^+}(f) + \Pr(E^c)\text{reg}_{M,\mathcal{P}^+}(f) \right).
\]

(SEP)

It is easy to see that beliefs that are singletons, as well as beliefs that contain all of \( \Delta(S) \), satisfy SEP for all acts and all events. On the other hand, it may be difficult to determine whether a set of weighted probabilities satisfies SEP. Therefore, we also have a sufficient condition for SEP. Define a sub-probability measure \( p \) on \( S \) to be like a probability measure (i.e., a function mapping measurable subsets of \( S \) to \([0, 1]\)) such that \( p(T \cup T') = p(T) + p(T') \) for disjoint sets \( T \) and \( T' \), without the requirement that \( p(S) = 1 \). We can identify a weighted probability distribution \((\Pr, \alpha)\) with the sub-probability measure \( \alpha \Pr \). (Note that given a sub-probability measure \( p \), there is a unique pair \((\alpha, \Pr)\) such that \( \Pr = \alpha \Pr \): we simply take \( \alpha = p(S) \) and \( \Pr = p/\alpha \).) We present a sufficient condition for SEP that directly describes the set of sub-probabilities corresponding to the set of weighted probabilities.
Unlike SEP, this condition, richness, is independent of the acts that are being compared. Therefore, to verify richness, one does not need to know the payoffs in the dynamic decision problem.

**Definition 8.3.** A set $C$ of subnormal probability measures is rich with respect to event $E$ if, whenever $p_1, p_2, p_3 \in C$, $p_3(E) \frac{p_1|E|}{|p_1|} + p_3(E^c) \frac{p_2|E^c|}{|p_2|} \in$ is in the convex hull of $C$.

An example of a situation where this type of belief is reasonable is if you have collected estimated distributions from a number of experts, but you suspect that the experts were biased in their sampling. For example, consider a grocery store interested in finding out the gender distribution of its customers. Let the state space consist of states of the form \{day of week, number of males, number of females\}. Suppose expert 1 tells you that the proportion of males to females is 0.5. However, expert 2 tells you that the proportion of males to females is 0.25. You believe that expert 1 might be correct, or expert 2 might be correct. However, you also come to suspect that expert 1 might have only surveyed the grocery store during peak hours (event $E$), while expert 2 might have only surveyed the grocery store on non-peak ours (event $E^c$). Suppose you have your own belief, $p_3$, about the distribution of peak and non-peak hours. Then, you believe that it is possible that the true distribution is actually what expert 1 claims during peak hours, and what expert 2 claims during non-peak hours, while the probability of peak hours is given by $p_3$.

Subprobabilities come into play if you have different levels of confidence on the different experts. For example, perhaps you only give a weight of 0.5 to expert 1, while expert 2 has a weight of 1. This means that the combined “richness” subprobability distribution you would include in your belief would include less of expert 1’s opinion than that of expert 2’s.

Many natural sets of beliefs satisfy richness. In the simplest case, any convex set of sub-probabilities over two states is rich. In addition, a set of probability distributions described by a system of linear constraints separately on $E$ and $E^c$ also satisfy richness with respect to $E$. For example, beliefs that are described by lower and upper bounds on the probability of each state $s \in S$ are rich. Proposition 8.4 generalizes this case to general linear inequalities, where $a_i, b_j$ are real-valued vectors of length $|E|$ and $|E^c|$ respectively, and $a_i \cdot p|E$ is the dot product between $a_i$ and $p|E$.

**Proposition 8.4.** A set consisting of all probability distributions $p$ satisfying
is rich with respect to event $E$.

Proof. Consider arbitrary sub-probability distributions $p_1, p_2, \text{ and } p_3$ in the set. We want to show that for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$

$$a_i \cdot (p_3(E)p_1|E) \leq c_i, \text{ and } b_j \cdot (p_3(E^c)p_2|E) \leq d_j.$$

It suffices to note that

$$a_i \cdot (p_3(E)p_1|E) \leq a_i \cdot (p_1|E) \leq c_i \text{ and } b_j \cdot (p_3(E^c)p_2|E) \leq b_j \cdot (p_2|E) \leq d_j.$$

\[\square\]

Although SEP is a sufficient condition for Axiom 4, it is actually also a necessary condition when we are concerned with whether dynamic consistency holds under all circumstances. Thus, in a sense, SEP characterizes the set of beliefs that result in dynamically consistent MWER behavior under all circumstances. The following theorem makes this notion precise. Let $C(\mathcal{P}^+) = \{p \geq 0 : \exists c, \exists \Pr, (c, \Pr) \in \mathcal{P}^+ \text{ and } p \leq c \Pr\}$.

**Theorem 8.5.** Let $E \subseteq S$ be any event. The following are equivalent:

1. Axiom 4 and Axiom 5 hold for $C_{M,E}^{Y,U,\mathcal{P}^+}$ for all acts $f,g$ in all menus $M$.

2. Axiom 4 and Axiom 5 hold for $C_{M^*,E}^{Y,U,\mathcal{P}^+}$ for all acts $f,g$ in the menu $M^*$ (the set of all acts with nonpositive utilities).

3. For all menus $M$ and acts $f,g \in M$, the weighted regrets of $f$ and $g$ with respect to $M$ and $\mathcal{P}^+$ is separable under event $E$. 

32
4. For menu $M^*$ and acts $f, g \in M^*$, the weighted regrets of $f$ and $g$ with respect to $M$ and $P^+$ is separable under event $E$.

If $C(P^+)$ is closed, then 1-4 is implied by:

5. $C(P^+)$ is rich with respect to event $E$.

See Appendix A for the proof of Theorem 8.5.

A natural question to ask is whether the SEP condition restricts beliefs to be trivial in the sense that they always contain some large class of sub-probability measures. Here we present a simple example illustrating that richness does not necessarily force the set of sub-probability measures to be trivially large.

|      | $E$ | $E^c$ |
|------|-----|-------|
|      | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
| $p_1$ | 1 | 0 | 0 | 0 |
| $p_2$ | 0 | 0.2 | 0.2 | 0.2 |
| $p_{1,E}$ | 1 | 0 | 0 | 0 |
| $p_{2,E}$ | 0 | 0.2 | 0 | 0 |
| $p_{1,E^c}$ | 0 | 0 | 0.5 | 0.5 |
| $p_{2,E^c}$ | 0 | 0 | 0.5 | 0.5 |
| $f$ | 1 | 0 | 5 | 0 |
| $g$ | 0 | 3 | 0 | 3 |
| $p_3$ | 0 | 1 | 0 | 0 |

Table 3: Example

**Example 8.6.** Table 3 contains an initial belief (the set \{p_1, p_2\}), updated beliefs after seeing $E$ and $E^c$, respectively, and two acts, $f$ and $g$. In this example, it is not hard to see that $f \succ_E g$, $f \succ_{E^c} g$, but $f \prec g$. Now, richness requires that the sub-probability measure $p_3$ is also in the set of beliefs. It is not hard to see that when $p_3$ is included in the set, the regret of $f$ becomes higher than the regret of $g$, if $E$ is observed. Thus the dynamic consistency axiom is no longer violated.

Suppose our menu-dependent preferences satisfy Axiom 4 for all events $E$. Does this mean that we have “full dynamic consistency”? The answer is no.

When menu-independent preferences satisfy Axiom 3, we can use an induction argument, starting from the terminal nodes of the information tree,
to argue that preferences are “always consistent”. More precisely, if $E_1$ and $E_2$ are disjoint and $E = E_1 \cup E_2$, then in general we have

$$f \succeq_{E_1} g \land f \succeq_{E_2} g \Rightarrow f \succeq_E g.$$ 

To see why, let $f_{E_1} g_{E_2} h_{E^c}$ denote the act that is $f$ on $E_1$, $g$ on $E_2$, and $h$ on $E^c$. We see that

$$f \succeq_{E_1} g \land f \succeq_{E_2} g \iff f_{E_1} h_{E^c} \succeq_{E_1 \cup E^c} g_{E_1} h_{E^c} \land f_{E_2} h_{E^c} \succeq_{E_2 \cup E^c} g_{E_2} h_{E^c}.$$ 

since the part of the act that is off of the conditional event does not matter.

Applying Axiom 3, we get $f_{E} h_{E^c} \succeq g_{E} h_{E^c}$, which gives $f \succeq_{E_1} g$ if we have the axiom $f \succeq_{E_1} g \iff f_{E_1} h_{E^c} \succeq_{E_1 \cup E^c} g_{E_1} h_{E^c}$ (which, in conjunction with transitivity, implies Axiom 3).

However, we cannot say the same for menu-dependent preferences satisfying Axiom 4. A simple example exists when the menus at node $E_1$ and $E_2$ are different. That is, it is fully consistent that for some choices $f, g$, $f \succeq_{E_1} g$ and $f \succeq_{E_2} g$, but at the ancestor node, $f \prec_{E_1 \cup E_2} g$.

Finally, we would like to show that richness does not force preferences to become maximization. We demonstrate using an example that we can have a set of beliefs satisfying the richness criterion, and for which the minimax weighted expected regret preferences are incompatible with maximizing expected utility. It is not difficult to verify that the set of beliefs depicted in Table 4 is rich. MWER preferences based on beliefs $\{p_1, p_2\}$ prefer $A$ to $B$ and $A'$ to $B'$. Moreover $C$ is preferred over $D$. However there is no probability distribution over $\{s_1, s_2, s_3\}$ that can result in these preferences. Therefore, the example in Table 4 shows that richness does not force preferences to be maximization.

### 8.3 Related impossibility results

Epstein and Le Breton [7] proves that under a few simple assumptions, only Bayesian beliefs can be dynamically consistent. At first glance this impossibility result may seem to contradict our sufficient conditions for dynamic consistency.

If we use a constant menu throughout a decision problem, then we fall into Epstein and Le Breton’s framework. However, Epstein and Le Breton’s impossibility result does not apply in our case. This is because one of the conditions of Epstein and Le Breton’s theorem, $P4^c$, does not hold in our case. The $P4^c$ axiom is defined as follows:
Table 4: Example showing that richness does not force preferences to be expected-utility maximizing.

|   | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $p_1$ | 2/3 | 0 | 1/3 |
| $p_2$ | 0 | 2/3 | 1/3 |
| $A$ | 0 | 0 | $\epsilon$ |
| $B$ | 1 | 0 | 0 |
| $A'$ | 0 | 0 | $\epsilon$ |
| $B'$ | 0 | 1 | 0 |
| $C$ | 1 | 1 | 0 |
| $D$ | 0 | 0 | 1 |

Axiom 6 (Conditional weak comparative probability). For all events $T$, $A$, $B$, with $A \cup B \subseteq T$, outcomes $x^*$, $x$, $y^*$, and $y$, and acts $g$, let the notation $x^*_Ax^*Tg$ denote an act that maps states in $A$ to $x^*$, states in $T \setminus A$ to $x$, and states not in $T$ to $g$. If $x^*_Tg \succ x^*_Tg$ and $y^*_Tg \succ y^*_Tg$, then

$$x^*_Ax^*Tg \succeq x^*_Bx^*Tg \Rightarrow y^*_Ax^*Tg \succeq y^*_Bx^*Tg.$$ 

$P4^c$ implies Savage’s $P4$, and hence does not hold for minimax regret, MER, and MWER in general.

A simple counterexample for MER and MWER consists of $S = \{s_1, s_2, s_3\}$ where $A = \{s_1\}$, $B = \{s_2\}$. $u(x^*) = 20$, $u(y^*) = 10$, $u(x) = 7$, $u(y) = 4$, and $u(g) = \{(s_1, 22), (s_2, 23), (s_3, 5)\}$. Let the decision maker’s belief consist of $p_1$, where $p_1(s_1) = 0.25$ and $p_1(s_2) = 0.75$; and $p_2$, where $p_2(s_3) = 1$.

To make the set of probability measures rich, we also include $p_3$, where $p_3(s_1) = 0.25$ and $p_3(s_3) = 0.75$, and $p_4$, where $p_4(s_1) = 1/16$, $p_4(s_2) = 3/16$, and $p_4(s_3) = 1/4$. Naturally, the menu consists of $\{x^*, x, y^*, y, g\}$. The worst regret of $x^*_Tg$ is 15 with respect to $p_2$, and the worst regret of $x^*_Tg$ is 15.75 with respect to $p_1$, therefore $x^*_Tg \succ x^*_Tg$. Similarly, the worst regret of $y^*_Tg$ is 15 with respect to $p_2$, and the worst regret of $y^*_Tg$ is 18.75 with respect to $p_1$, therefore $y^*_Tg \succ y^*_Tg$. Moreover, the worst regret of $x^*_Ax^*Tg$ is 15 with respect to $p_2$, and the worst regret of $x^*_Bx^*Tg$ is 15 with respect to $p_2$, therefore $x^*_Ax^*Tg \succeq x^*_Bx^*Tg$. However, the worst regret of $y^*_Ax^*Tg$ is 17.25 with respect to $p_1$, and the worst regret of $y^*_Bx^*Tg$ is 15.75 with respect to $p_3$, therefore $y^*_Ax^*Tg \not\succeq y^*_Bx^*Tg$.

In [30], Siniscalchi shows (Proposition 1) that “conditional preference systems” that are dynamically consistent must essentially have beliefs that
are updated by Bayesian updating. However, Siniscalchi’s Proposition 1 does not apply in our case, because Proposition 1 assumes consequentialism – that the conditional preference system treats identical subtrees equally, independent of the greater decision tree within which the subtrees belong.

9 Related work

In this section we discuss some related work. Siniscalchi [30] characterizes (menu-independent) consistent planning using properties of preferences over decision trees, but the results are not directly applicable to menu-dependent preferences. Hayashi [16] studies sophisticated regret-minimization in an optimal stopping problem. In Hayashi’s definitions, forgone opportunities and unachievable plans do not factor into the regret computation, and Hayashi does not discuss other possibilities for the menu with respect to which regrets could be computed.

In terms of ensuring dynamic consistency without the need for sophistication, a related piece of work is done by Hanany and Klibanoff [15] on dynamically consistent updating of multiple priors beliefs for Maxmin Expected Utility [12]. Hanany and Klibanoff [15] propose applying Bayes’ rule to subsets of the set of priors, “where the specific subset depends on the preferences, the conditioning event, and the choice problem.”

Regret minimization has also been studied in the context of first choosing a menu, and then subsequently choosing from the chosen menu. Sarver [27] axiomatizes a form of regret-minimization in a setting where the decision maker first selects a menu then selects a choice from within the selected menu. Regret is experienced when the selected choice is inferior to some other choice in the menu. Dekel, Lipman, and Rustichini [5] also consider a similar setting, but where the states are subjective, and the decision maker is not necessarily minimizing regret. In this setting, Dekel, Lipman, and Rustichini characterize a model of decision-making where the decision maker ranks menus of lotteries with respect to a subjective state space and an aggregator over the subjective states, with the understanding that he/she will choose from the chosen menu at a later time. They explore the relationship between the amount of uncertainty in the subjective state space and the uncertainty in the preferences, as well as the relationship between preferences for flexibility/commitment and the aggregator function.
Before proving the theorem, we formalize several convenient properties in the following proposition. Let $C$ be a set of subnormal probability measures. We use $\text{ext}(C)$ to denote the extreme points of $C$, and use $\text{conv}(C)$ to denote the convex hull of $C$. Firstly, we ensure that the weighted regret of any act with respect to $P^+$ is equivalent to its regret computed with respect to the set of subnormal probability measures $C(P^+)$. Secondly, we note that taking the convex hull of a set of subnormal probability distributions does not change the computed regrets. Finally, when $C$ is convex and compact, regret can be computed with respect to just the extreme points of $C$.

**Lemma A.1.** For any utility function $u$, set of weighted probabilities $P^+$, set of subnormal probabilities $C$, act $f$ and menu $M$ containing $f$,

1. $\text{reg}_{M,P^+}(f) = \text{reg}_{M,C(P^+)}(f)$;
2. $\text{reg}_{M,C}(f) = \text{reg}_{M,\text{conv}(C)}(f)$;
3. $\text{reg}_{M,C}(f) = \text{reg}_{M,\text{ext}(C)}(f)$, if $C$ is convex and compact.

(Proof of lemma). We first show that $\text{reg}_{M,P^+}(f) = \text{reg}_{M,C(P^+)}(f)$:

$$
\text{reg}_{M,P^+}(f) = \sup_{Pr \in P} \left( \alpha Pr \sum_{s \in S} \Pr(s) \text{reg}_M(f, s) \right)
= \sup_{Pr \in P} \left( \sum_{s \in S} \alpha Pr(s) \text{reg}_M(f, s) \right)
= \text{reg}_{M,C(P^+)}(f).
$$

Next we show part 2. Since $\text{conv}(C) \supseteq C$, clearly $\text{reg}_{M,\text{conv}(C)}(f) \geq \text{reg}_{M,C}(f)$. For the other inequality, consider any $q \in \text{conv}(C)$. By definition of $\text{conv}(C)$, $q = pq_1 + (1 - p)q_2$ for $q_1, q_2$ in $C$.

$$
\sum_{s \in S} q(s) \text{reg}_M(f, s) = \sum_{s \in S} pq_1(s) + (1 - p)q_2(s) \text{reg}_M(f, s)
\leq \sup \left( \sum_{s \in S} q_1(s) \text{reg}_M(f, s), \sum_{s \in S} q_2(s) \text{reg}_M(f, s) \right).
$$

Therefore, $\text{reg}_{M,\text{conv}(C)}(f) \leq \text{reg}_{M,C}(f)$.

To see that $\text{reg}_{M,C}(f) = \text{reg}_{M,\text{ext}(C)}(f)$, note that for convex and compact $C$, the Krein-Milman Theorem says that $C$ has extreme points $\text{ext}(C)$,
and that $C$ is the closed convex hull of these extreme points. The result then follows from part 2 of this Lemma.

We also make use of the following lemma, proved in [14].

**Lemma A.2.** There exists a utility function $U$ such that for every menu $M$, there exists $\epsilon \in (0, 1]$ and constant act $l^*$ such that for all $f, g \in M$, $f \geq_M g \iff t(f) \geq_{t(M)} t(g)$, where $t$ has the form $t(f) = \epsilon f + (1 - \epsilon)l^*$ and $t(M) = \{t(f) : f \in M\}$. Moreover, there exists an act $g_{t(M)}$ such that $u(g_{t(M)}(s)) = -\sup_{f \in t(M)} u(f(s))$ for all $s \in S$.

**Proof of theorem.** $2 \Rightarrow 4$.

First we show that (SEP) for menu $M^*$ is a necessary condition for Axiom 4 to hold for menu $M^*$. In turn, we first show that

$$\text{reg}_{M^*, P^+}(f) \geq \sup_{Pr \in P^+} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M^*, P^+|E}(f) + \Pr(E^c) \text{reg}_{M^*, P^+|E^c}(f) \right)$$

(1)

is a necessary condition for Axiom 4 to hold for menu $M^*$.

Suppose for contradiction that (1) does not hold. Then there exists some event $E$ and act $g \in M^*$ such that

$$\text{reg}_{M^*, P^+}(g) < \sup_{Pr \in P^+} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M^*, P^+|E}(g) + \Pr(E^c) \text{reg}_{M^*, P^+|E^c}(g) \right).$$

We can define an act $f$ such that $f \in C_{M^*, e}(M^*)$ and $f \notin C_{M^*, e}(M^*)$ and $f \notin C_{M^*, S}(M^*)$. In particular, let $f$ be an act satisfying

$$u(f, s) = \begin{cases} \sup_{h \in M^*} u(h, s) - \text{reg}_{M^*, P^+|E}(g) \text{ for all } s \in E \\ \sup_{h \in M^*} u(h, s) - \text{reg}_{M^*, P^+|E^c}(g) \text{ for all } s \in E^c \end{cases}.$$ 

(2)

This act $f$, as defined, must be in $M^*$, since $M^*$ consists of all acts with nonpositive utilities.

With this definition of $f$, $\text{reg}_{M^*, P^+|E}(f) = \text{reg}_{M^*, P^+|E}(g)$ and $\text{reg}_{M^*, P^+|E^c}(f) = \text{reg}_{M^*, P^+|E^c}(g)$. 

38
\( \text{reg}_{M^*, \mathcal{P}^+ | E^c}(g) \). To see this, note that

\[
\text{reg}_{M^*, \mathcal{P}^+ | E}(f) = \sup_{Pr \in \mathcal{P}} \left( \alpha_{Pr[E]} \sum_{s \in E} \Pr(s | E) \text{reg}_{M^*}(f, s) \right)
\]

\[
= \sup_{Pr \in \mathcal{P}} \left( \sup_{\{Pr' \in \mathcal{P} : Pr'|E = Pr[E]\}} \frac{\alpha_{Pr'}}{\mathcal{P}'}(E) \sum_{s \in E} \Pr(s | E) \text{reg}_{M^*}(f, s) \right)
\]

\[
= \sup_{Pr \in \mathcal{P}} \left( \sup_{\{Pr' \in \mathcal{P} : Pr'|E = Pr[E]\}} \frac{\alpha_{Pr'}}{\sup\{\alpha_{Pr''} \mathcal{P}''(E) : Pr'' \in \mathcal{P}\}} \sum_{s \in E} \Pr(s | E) \text{reg}_{M^*}(f, s) \right)
\]

\[
= \sup_{Pr \in \mathcal{P}} \left( \sup_{\{Pr' \in \mathcal{P} : Pr'|E = Pr[E]\}} \frac{\alpha_{Pr'}}{\sup\{\alpha_{Pr''} \mathcal{P}''(E) : Pr'' \in \mathcal{P}\}} \text{reg}_{M^*, \mathcal{P}^+ | E}(g) \right)
\]

By the same argument but with \( E \) replaced by \( E^c \), we see that \( \text{reg}_{M^*, \mathcal{P}^+ | E^c}(f) = \text{reg}_{M^*, \mathcal{P}^+ | E^c}(g) \).

Next, we show that \( \text{reg}_{M^*, \mathcal{P}^+}(f) > \text{reg}_{M^*, \mathcal{P}^+}(g) \), violating Axiom 4.

\[
\text{reg}_{M^*, \mathcal{P}^+}(f) = \sup_{Pr \in \mathcal{P}} \left( \sum_{s \in E} \Pr(s) \text{reg}_{M^*}(f, s) \right) + \left( \sum_{s \in E^c} \Pr(s) \text{reg}_{M^*}(f, s) \right)
\]

\[
= \sup_{Pr \in \mathcal{P}} \left( \Pr(E) \sum_{s \in E} \Pr(s | E) \text{reg}_{M^*}(f, s) \right) + \Pr(E^c) \left( \sum_{s \in E^c} \Pr(s | E^c) \text{reg}_{M^*}(f, s) \right)
\]

\[
= \sup_{Pr \in \mathcal{P}} \left( \Pr(E) \text{reg}_{M^*, \mathcal{P}^+ | E}(g) + \Pr(E^c) \text{reg}_{M^*, \mathcal{P}^+ | E^c}(g) \right)
\]

By a similar argument, we can show that the opposite weak inequality is also a necessary condition. The argument is analogous to the first case, but is provided here for completeness. More precisely, we show that

\[
\text{reg}_{M^*, \mathcal{P}^+}(f) \leq \sup_{Pr \in \mathcal{P}^+} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M^*, \mathcal{P}^+ | E}(f) + \Pr(E^c) \text{reg}_{M^*, \mathcal{P}^+ | E^c}(f) \right)
\]

(3) is a necessary condition.

Suppose for contradiction that (3) does not hold. Then there exists some event \( E \) and act \( f \in M^* \) such that

\[
\text{reg}_{M^*, \mathcal{P}^+}(f) > \sup_{Pr \in \mathcal{P}^+} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M^*, \mathcal{P}^+ | E}(f) + \Pr(E^c) \text{reg}_{M^*, \mathcal{P}^+ | E^c}(f) \right)
\]

39
We define an act $g$ such that $f \in C_{M^*,E}()$ and $f \in C_{M^*,E'}()$ and $f \notin C_{M^*}()$. In particular, let $g$ be an act satisfying

$$u(g,s) = \begin{cases} \sup_{h \in M^*} u(h,s) - \text{reg}_{M^*,P^+|E}(f) & \text{for all } s \in E \\ \sup_{h \in M^*} u(h,s) - \text{reg}_{M^*,P^+|E'}(f) & \text{for all } s \in E'. \end{cases}$$

With this definition of $g$, by the same argument as in the previous case, we have $\text{reg}_{M^*,P^+|E}(f) = \text{reg}_{M^*,P^+|E}(g)$ and $\text{reg}_{M^*,P^+|E'}(f) = \text{reg}_{M^*,P^+|E'}(g)$.

Next, we show that $\text{reg}_{M^*,P^+}(f) > \text{reg}_{M^*,P^+}(g)$.

$$\text{reg}_{M^*,P^+}(g) = \sup_{Pr \in P} \alpha_{Pr} \left( \left[ \sum_{s \in E} Pr(s) \text{reg}_{M^*,P^+}(g,s) \right] + \sum_{s \in E'} Pr(s) \text{reg}_{M^*,P^+}(g,s) \right)$$

$$= \sup_{Pr \in P} \left( \Pr(E) \left[ \sum_{s \in E} Pr(s|E) \text{reg}_{M^*,P^+}(g,s) \right] + \Pr(E') \left[ \sum_{s \in E'} Pr(s|E') \text{reg}_{M^*,P^+}(g,s) \right] \right)$$

$$= \sup_{Pr \in P} \left( \Pr(E) \text{reg}_{M^*,P^+|E}(f) + \Pr(E') \text{reg}_{M^*,P^+|E'}(f) \right)$$

$$< \text{reg}_{M^*,P^+}(f).$$

$4 \Rightarrow 2$.

Now, we show straight-forwardly that (SEP) for menu $M$ is sufficient to guarantee that Axiom 4 and Axiom 5 holds for $M$. It is easy to check that Axiom 5 always holds for MWER, so we only need to check Axiom 4. Suppose that (SEP) is satisfied. If $f \succeq_{M,E} g$ and $f \succeq_{M,E'} g$, then $\text{reg}_{M,P^+|E}(f) \leq \text{reg}_{M,P^+|E}(g)$ and $\text{reg}_{M,P^+|E'}(f) \leq \text{reg}_{M,P^+|E'}(g)$. This means

$$\text{reg}_{M,P^+}(f) = \sup_{Pr \in P} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M,P^+|E}(f) + (1 - \Pr(E)) \text{reg}_{M,P^+|E'}(f) \right)$$

$$\leq \sup_{Pr \in P} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M,P^+|E}(g) + (1 - \Pr(E)) \text{reg}_{M,P^+|E'}(g) \right)$$

$$= \text{reg}_{M,P^+}(g),$$

which means $f \succeq_{M,S} g$, as required.

For the case where there is at least one strict preference (say, $f \succ_{M,E} g$), then $\text{reg}_{M,P^+|E}(f) < \text{reg}_{M,P^+|E}(g)$ and $\text{reg}_{M,P^+|E'}(f) < \text{reg}_{M,P^+|E'}(g)$. This means

$$\text{reg}_{M,P^+}(f) = \sup_{Pr \in P} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M,P^+|E}(f) + (1 - \Pr(E)) \text{reg}_{M,P^+|E'}(f) \right)$$

$$< \sup_{Pr \in P} \alpha_{Pr} \left( \Pr(E) \text{reg}_{M,P^+|E}(g) + (1 - \Pr(E)) \text{reg}_{M,P^+|E'}(g) \right)$$

$$= \text{reg}_{M,P^+}(g),$$

40
which means $f \succ_{M,S} g$, as required.

1 $\iff$ 2. Clearly, if Axiom 4 holds for all menus $M$, then it also holds for the menu $M^*$, so it remains to show that if Axiom 4 holds for menu $M^*$, then it holds for all menus $M$. So suppose $f \in C_{M,E}(M)$ and $f \in C_{M,E^c}(M)$. We want to show that $f \in C_{M,S}(M)$.

Given any act $f$ in menu $M$, let $g_M$ be an act such that $\frac{1}{2}t(f) + \frac{1}{2}t(g_M)$ is an act with constant utility 0 for some transformation $t$. The existence of such an act is guaranteed by Lemma A.2. By the independence axiom, we have

$$f \in C_{M,S}(M) \iff \frac{1}{2}t(f) + \frac{1}{2}t(g_M) \in C_{M',S}(M^*).$$

For the conditional preferences, we also have

$$f \in C_{M,E}(M) \iff \frac{1}{2}t(f) + \frac{1}{2}t(g_M) \in C_{M',E}(M^*).$$

and

$$f \in C_{M,E^c}(M) \iff \frac{1}{2}t(f) + \frac{1}{2}t(g_M) \in C_{M',E^c}(M^*).$$

Since Axiom 4 holds for menu $M^*$, we can conclude that

$$\frac{1}{2}t(f) + \frac{1}{2}t(g_M) \in C_{M',S}(M^*).$$

and hence we have $f \in C_{M,S}(M)$, as required. The case where at least one of $C_E$ and $C_{E^c}$ is a singleton follows by the same argument.

3 $\implies$ 4. Clearly, if (SEP) holds for all menus $M$, then it holds for the menu $M^*$.

1 $\implies$ 3. This case is very similar to 2 $\implies$ 4. The only difference here is that we are dealing with the set of all menus, instead of the special menu $M^*$. In fact, the proof is the same up to and including the point where we define the act $f$ by equation (2). In the previous case, we argued that the act $f$ so defined must be in the menu $M^*$. In our current case, $f$ might not be in the menu $M$ being considered. However, by adding the act $f$ to menu $M$, no ranking of acts are affected, due to the independence of never strictly optimal alternatives (INA) axiom of regret. Moreover, the constructed $f$ “exists” because there are clearly outcomes that we can define lotteries over in order to get the correct utility in each state. Therefore, in this case we argue that if condition 3 does not hold, then Axiom 4 is violated for the menu $M \cup \{f\}$.

5 $\implies$ 4.
We assume that $C(\mathcal{P}^+)$ is rich with respect to $E$, and show that, for all acts $f \in M^*$,

$$\text{reg}_{M^*, \mathcal{P}^+}(f) = \sup_{\Pr \in \mathcal{P}^+} \alpha_{\Pr} \left( \Pr(E) \text{reg}_{M^*, \mathcal{P}^+ | E}(f) + \Pr(E^c) \text{reg}_{M^*, \mathcal{P}^+ | E^c}(f) \right).$$

We first show one direction of the inequality:

$$\text{reg}_{M^*, \mathcal{P}^+}(f) = \sup_{\Pr \in \mathcal{P}^+} \alpha_{\Pr} \left( \Pr(E) \sum_{s \in E} (\Pr(s | E) \text{reg}_{M^*}(f, s)) + \Pr(E^c) \sum_{s \in E^c} (\Pr(s | E^c) \text{reg}_{M^*}(f, s)) \right)$$

$$\leq \sup_{\Pr \in \mathcal{P}^+} \alpha_{\Pr} \left( \Pr(E) \sup_{\Pr \in \mathcal{P}} \left( \sup_{\{\Pr' \in \mathcal{P} : \Pr' | E = \Pr | E\}} \sup\{\alpha_{\Pr'} \Pr'(E) : \Pr'' \in \mathcal{P}\} \sum_{s \in E} \Pr(s | E) \text{reg}_{M^*}(f, s) \right) + \Pr(E^c) \sup_{\Pr \in \mathcal{P}} \left( \sup_{\{\Pr' \in \mathcal{P} : \Pr' | E = \Pr | E\}} \sup\{\alpha_{\Pr'} \Pr'(E) : \Pr'' \in \mathcal{P}\} \sum_{s \in E} \Pr(s | E) \text{reg}_{M^*}(f, s) \right) \right)$$

$$= \sup_{\Pr \in \mathcal{P}^+} \alpha_{\Pr} \left( \Pr(E) \sum_{s \in E} \Pr(s | E) \text{reg}_{M^*}(f, s) \right) + \Pr(E^c) \sum_{s \in E^c} \Pr(s | E^c) \text{reg}_{M^*}(f, s) \right)$$

$$= \sup_{\Pr \in \mathcal{P}^+} \alpha_{\Pr} \left( \Pr(E) \text{reg}_{M^*, \mathcal{P}^+ | E}(f) + \Pr(E^c) \text{reg}_{M^*, \mathcal{P}^+ | E^c}(f) \right).$$

Next we show the other inequality direction. Let $\Pr_1, \Pr_2, \Pr_3$ be weighted probability distributions such that

$$\sup_{\Pr \in \mathcal{P}^+} \alpha_{\Pr} \left( \Pr(E) \text{reg}_{M^*, \mathcal{P}^+ | E}(f) + \Pr(E^c) \text{reg}_{M^*, \mathcal{P}^+ | E^c}(f) \right) = \alpha_{\Pr_3} \left( \Pr_3(E) \text{reg}_{M^*, \Pr_1 \mid E}(f) + \Pr_3(E^c) \text{reg}_{M^*, \Pr_2 \mid E^c}(f) \right).$$

In particular, this means that $\alpha_{\Pr_1} \Pr_1, \alpha_{\Pr_2} \Pr_2, \alpha_{\Pr_3} \Pr_3 \in C(\mathcal{P}^+)$, where $\Pr_1 | E$ and $\Pr_2 | E$ have weight 1. Then

$$\text{reg}_{M^*, \mathcal{P}^+}(f) = \sup_{\Pr \in \mathcal{P}^+} \alpha_{\Pr} \left( \Pr(E) \sum_{s \in E} (\Pr(s | E) \text{reg}_{M^*}(f, s)) + \Pr(E^c) \sum_{s \in E^c} (\Pr(s | E^c) \text{reg}_{M^*}(f, s)) \right) \geq \alpha_{\Pr_3} \left( \Pr_3(E) \text{reg}_{M^*, \Pr_1 \mid E}(f) + \Pr_3(E^c) \text{reg}_{M^*, \Pr_2 \mid E^c}(f) \right),$$

by richness, since by part 2 of Lemma A.1, we can take the convex hull of $C(\mathcal{P}^+)$ without changing any of the regret values. \qed

42
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Figure 3: Climate change example illustrating four different choices depending on which menu is used for backward induction. The four possible states, from left to right, are $lsr$, $hsr$, $lsl$, and $hsl$, respectively. The four different extreme menu attitudes result in four very different choices.
Figure 4: Procrastination example. In the first state, the exam is difficult; in the second state, the exam is easy.

Figure 5: Two decision trees.