Abstract

We study a coupled thermo-diffusion system that accounts for the dynamics of hot colloids in periodically heterogeneous media. Our model describes the joint evolution of temperature and colloidal concentrations in a saturated porous structure, where the Smoluchowski interactions are responsible for aggregation and fragmentation processes in the presence of Soret-Dufour type effects. Additionally, we allow for deposition and depletion on internal micro-surfaces. In this work, we derive corrector estimates quantifying the rate of convergence of the periodic homogenization limit process performed in [24] via two-scale convergence arguments. The major technical difficulties in the proof are linked to the estimates between nonlinear processes of aggregation and deposition and to the convergence arguments of the a priori information of the oscillating weak solutions and cell functions in high dimensions. Essentially, we circumvent the arisen difficulties by a suitable use of the energy method and of fine integral estimates controlling interactions at the level of micro-surfaces.

1 Introduction

Diffusion and heat conduction, taken separately, are well understood processes at a large variety of space scales. However, as soon as diffusion interplays with the conduction of heat, it appears that the structure of the model equations is not so clear as one would expect, especially if one wants to describe settings away from the somewhat better understood thermodynamic equilibrium, where statistical mechanics is the main investigation tool.

Driven by possible applications in the context of efficient drug-delivery and in the design of intelligent packaging materials, we wish to understand mathematically the upscaling of the following basic thermo-diffusion scenario: We look at a population of colloidal particles (monomers) driven by a flux linearly combining Fick and Fourier contributions. We assume that monomers undergo a Smoluchowski-like dynamics producing populations of i-mers that finally meet and travel through a transversal porous membrane. The microscopic boundaries (at the level of the membrane pores) are active in the sense that they host adsorption and desorption of clusters of colloidal particles.

The starting PDE model is formulated in [24] by Krekel and his co-authors. Their thermo-diffusion system is posed in perforated media with uniform periodicity inside the domain. As main outcome, they prove both the global weak solvability of the model as well as the periodic homogenization limit. As byproduct, they also obtain the precise structure of the effective transport parameters. Now, is the moment to: Justify the two-scale asymptotics by proving corrector/error estimates for the homogenization limit for periodic arrangements of membrane pores/microstructures.

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In our context, the structure of the corrector estimate for the involved concentrations and temperature fields we wish to prove is

\[
\|\theta^\varepsilon - \theta^\varepsilon_0\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + \|u^\varepsilon - u^\varepsilon_0\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 \\
+ \|\nabla (\theta^\varepsilon - \theta^\varepsilon_1)\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + \|\nabla (u^\varepsilon - u^\varepsilon_1)\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + \|v^\varepsilon_0\|_{L^2((0,T) \times \Gamma^\varepsilon)}^2 \leq C \varepsilon, 
\]

where \(C > 0\) is a generic constant independent of the choice of the scale parameter \(\varepsilon > 0\).

To obtain this corrector estimate, our strategy is to use an energy-like method and macroscopic reconstructions (cf. e.g. [9], but also [10]). This technique basically relies on the choice of test functions able to capture in suitable norms the difference between the micro-and macro-concentrations as well as micro- and macro-temperatures and their transport fluxes. Careful attention needs to be payed to the regularity of the limit solutions as well as of the cell functions involved in the asymptotic procedure; see e.g. [22, 15]. A similar approach has been followed by Eck et al. (cf. e.g. [8, 9]) concerning the upscaling of the phase field model in high contrast regimes. Besides handling new nonlinear terms, the novel aspect in our context is the handling of the errors produced in the upscaling due to micro-surfaces. A similar analysis can be carried over the settings in [4, 34, 36, 14], e.g.

Besides the energy-like approach used here for a periodic homogenization case, powerful contributions can be obtained using variants of the bulk and boundary unfolding operators: see, for instance, [18, 31, 15, 27]. Using somewhat more regularity, high-order corrector estimates can be obtained for semi-linear elliptic systems via an iteration method that uses explicitly the expected structure of the two-scale asymptotic expansion; compare [22, 21]. Settings involving locally-periodic microstructures can be treated as in [28], e.g., while the random case is in most of the cases out of reach; see [23, 33] for some details in this direction.

Having available corrector estimates like (1.1) allows in principle the construction of convergence proofs as well as a priori error estimate for MsFEM applied to problems in perforated media like in [7], for instance.

This paper is structured as follows: Section 2 is devoted to the presentation of the Smoluchowski-Soret-Dufour model posed in a perforated domain. In this section, we also list a couple of preliminary results about the two-scale convergence and compactness arguments and about the weak solvability of both the microscopic and limit models (recalling from [24]). Our main result is Theorem 12 as presented in Section 3. We then introduce the derivation of the difference system resulting from the microscopic problem and the "macroscopic reconstructed" system. On top of that, we prepare in this part a few helpful integral estimates. The proof of Theorem 12 is provided in Section 4. We conclude the paper with the remarks from Section 5.

2 Setting of the problem

2.1 The coupled thermo-diffusion model

2.1.1 A geometrical interpretation of porous medium

Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^d \) \((d \in \{2,3\})\) with \( \partial \Omega \in C^{0,1} \). Without loss of generality, we reduce ourselves to consider \( \Omega \) as the parallelepiped \((0,a_1) \times \ldots \times (0,a_d)\) with \(a_i > 0, i \in \{1, \ldots, d\}\). Let \( Y \) be the representative unit cell defined by

\[
Y := \left\{ \sum_{i=1}^{d} \lambda_i \bar{e}_i : 0 < \lambda_i < 1 \right\},
\]

where \( \bar{e}_i \) is the \( i \)th unit vector in \( \mathbb{R}^d \).
Let \( Y_0 \) be an open subset of \( Y \) with a Lipschitz boundary \( \Gamma = \partial Y_0 \) which is divided into two disjoint closed parts \( \Gamma_N \) and \( \Gamma_R \) with \( \Gamma_N \cap \Gamma_R = \emptyset \). Let \( Z \in \mathbb{R}^d \) be a hypercube. Then for \( X \subset Z \) we denote by \( X^k \) the shifted subset

\[
X^k := X + \sum_{i=1}^d k_i \vec{e}_i,
\]

where \( k = (k_1, ..., k_d) \in \mathbb{Z}^d \) is a vector of indices.

Assume that a scale factor \( \varepsilon > 0 \) is given. The pore skeleton is then defined as the union of \( \varepsilon Y_0^k \) the \( \varepsilon \)-homothetic sets of \( Y_0^k \), i.e.

\[
\Omega^\varepsilon_0 := \bigcup_{k \in \mathbb{Z}^d} \{ \varepsilon Y_0^k : Y_0^k \subset \Omega \}.
\]

Thus, the total pore space we have in mind is \( \Omega^\varepsilon = \Omega \setminus \Omega^\varepsilon_0 \).

Set \( Y_1 := Y \setminus Y_0 \). The unit cell \( Y \) is made of two parts including the gas phase \( Y_1 \) and the solid phase \( Y_0 \). We denote the total pore surface of the skeleton by \( \Gamma^\varepsilon := \partial \Omega^\varepsilon_0 \). The pore surface \( \Gamma^\varepsilon \) consists of two parts satisfying \( \Gamma^\varepsilon = \Gamma^\varepsilon_N \cup \Gamma^\varepsilon_R \) where \( \Gamma^\varepsilon_N \) and \( \Gamma^\varepsilon_R \) are disjoint closed sets possessing a nonzero \( (d-1) \)-dimensional measure. The Neumann boundary \( \Gamma^\varepsilon_N \) indicates the insulation for the heat flow, whilst at \( \Gamma^\varepsilon_R \) we allow for a flux of mass through a Robin-type condition. The union of the cell regions \( \varepsilon Y_1^k \) (without the solid grains \( \varepsilon Y_0^k \)) represents the total available space for thermo-diffusion.

In Figure 2.1 and Figure 2.2, we show a admissible 2d domain with microstructures. We let throughout the paper \( n := (n_1, ..., n_d) \) be the unit outward normal vector on the boundary \( \partial \Omega^\varepsilon \). The representation of the periodic geometries is inspired from [20, 22, 34] and references cited therein, but other possibilities exist as well. The practical problem usually delimitates the freedom in choosing the precise structure of \( Y_0 \); see Figure 2.2 for a couple of options.

Figure 2.1: An admissible 2d perforated domain.

Figure 2.2: Possible choices for \( Y_0 \). The choice of (a) fits to the geometry described in Figure 2.1.
2.1.2 Model description

Before describing the microscopic problem (which we refer to as \((P^\varepsilon)\)), we define some useful notation. For \(\delta > 0\), let \(\nabla^\delta\) be the so-called mollified gradient

\[
\nabla^\delta f (x) := \nabla \left[ \int_{B(x, \delta)} J_\delta (x - y) f (y) \, dy \right],
\]

where \(J_\delta\) is a mollifier (see e.g. [12]) and \(B (x, \delta)\) is the ball centered in \(x \in \Omega\) with radius \(\delta\). The radius \(\delta\) is assumed to be an \(\varepsilon\)-independent constant.

We denote by \(x \in \Omega^\varepsilon\) the macroscopic variable and by \(y = x/\varepsilon\) the microscopic variable representing fast variations at the microscopic geometry. With this convention, we write

\[
\kappa^\varepsilon (x) = \kappa \left( \frac{x}{\varepsilon} \right) = \kappa (y).
\]

The same convention applies to all the other oscillating coefficients involved our problem.

We denote by \(\mathcal{A}^\varepsilon_T\) the second-order elliptic operator in divergence form with rapidly oscillating coefficients, i.e.

\[
\mathcal{A}^\varepsilon_T := \nabla \cdot \left( - \mathbb{T} \left( \frac{x}{\varepsilon} \right) \nabla \right) = \frac{\partial}{\partial x_i} \left[ - \tau^{\alpha\beta}_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right]. \tag{2.1}
\]

Concerning the structure of \(\mathcal{A}^\varepsilon_T\), we assume that for all \(y \in Y\), \(\mathbb{T} (y) = \left( \tau^{\alpha\beta}_{ij} (y) \right) : \mathbb{R}^d \to \mathbb{R}^{m^2 \times d^2}\) for \(1 \leq i, j \leq d, 1 \leq \alpha, \beta \leq m\) is a second-order tensor that depends on the position vector \(y\) and satisfies a uniform (in \(\varepsilon\)) ellipticity condition. Depending on the situation, we have either \(\mathbb{T}\) is the tensor \(\kappa\) (heat conductivity) or the tensor \(d_i\) (diffusion coefficients). Note that \(m \geq 1\) denotes the number of base equations in the system.

In this framework, we consider that maximum \(N > 2\) colloidal species are involved in the thermo-diffusion process. We denote by \((\theta^\varepsilon, u^\varepsilon_i, v^\varepsilon_i)\) for \(i \in \{1, \ldots, N\}\) the triplet of real-valued solutions of our thermo-diffusion model, i.e. a system of coupled ordinary differential equations with semi-linear parabolic equations for the evolution of temperature and colloidal concentrations. Denote by \(u^\varepsilon := (u^\varepsilon_1, ..., u^\varepsilon_N)\) the vector of all active colloidal concentrations \(u^\varepsilon_i\). We assume that these species obey the population balance equation as postulated by Smoluchowski in \([37]\), i.e.

\[
R_i(s) := \frac{1}{2} \sum_{k+j=i} \beta_{kj} s_k s_j - \sum_{j=1}^N \beta_{ij} s_i s_j, \quad \text{(with } R_i : \mathbb{R}^N \to \mathbb{R}, i \in \{1, \ldots, N\})
\]

theoretically representing a quadratic-like rate of change of \(s_i\). The presence of coagulation coefficients \(\beta_{ij} > 0\) accounts for the rate aggregation and fragmentation between populations of particles of size \(i\) and \(j\). For further modeling details, we refer the reader to \([11, 16, 17]\) and \([25]\), e.g.

We denote the parabolic cylinders as \(Q^\varepsilon_T := (0, T) \times \Omega^\varepsilon\) and \(Q_T := (0, T) \times \Omega\). Now, we detail the structure of our microscopic problem \((P^\varepsilon)\). For \(i \in \{1, \ldots, N\}\), we consider the following coupled thermo-diffusion system:

\[
\partial_t \theta^\varepsilon + \mathcal{A}^\varepsilon_k \theta^\varepsilon = \tau^\varepsilon \sum_{i=1}^N \nabla^\delta u^\varepsilon_i \cdot \nabla \theta^\varepsilon \quad \text{in } Q^\varepsilon_T, \tag{2.2}
\]

\[
\partial_t u^\varepsilon_i + \mathcal{A}^\varepsilon_{di} u^\varepsilon_i = \rho^\varepsilon_i \nabla^\delta \theta^\varepsilon \cdot \nabla u^\varepsilon_i + R_i (u^\varepsilon_i) \quad \text{in } Q^\varepsilon_T, \tag{2.3}
\]

\[
\partial_t v^\varepsilon_i = a^\varepsilon_i u^\varepsilon_i - b^\varepsilon_i v^\varepsilon_i \quad \text{on } (0, T) \times \Gamma^\varepsilon, \tag{2.4}
\]

subject to the boundary conditions

\[
- \kappa^\varepsilon \nabla \theta^\varepsilon \cdot n = 0 \quad \text{on } (0, T) \times \Gamma^\varepsilon_N. \tag{2.5}
\]
\[-\kappa^\varepsilon \nabla \theta^\varepsilon \cdot n = \varepsilon g_0^\varepsilon \theta^\varepsilon \quad \text{on} \quad (0, T) \times \Gamma_R^\varepsilon, \tag{2.6}\]
\[-\kappa^\varepsilon \nabla \theta^\varepsilon \cdot n = 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \tag{2.7}\]
\[-d_i^\varepsilon \nabla u_i^\varepsilon \cdot n = \varepsilon (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \quad \text{on} \quad (0, T) \times \Gamma^\varepsilon, \tag{2.8}\]
\[-d_i^\varepsilon \nabla u_i^\varepsilon \cdot n = 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \tag{2.9}\]

and the initial data
\[\theta^\varepsilon(0, x) = \theta_0^\varepsilon(x) \quad \text{for} \quad x \in \Omega^\varepsilon, \tag{2.10}\]
\[u_i^\varepsilon(0, x) = u_i^\varepsilon(0) \quad \text{for} \quad x \in \Omega^\varepsilon, \tag{2.11}\]
\[v_i^\varepsilon(0, x) = v_i^\varepsilon(0) \quad \text{for} \quad x \in \Gamma^\varepsilon. \tag{2.12}\]

(2.2)-(2.12) form our microscopic problem \((P^\varepsilon)\).

| \(\kappa^\varepsilon\) | heat conductivity (tensor) |
| \(\tau^\varepsilon\) | Soret coefficient (tensor) |
| \(g_0^\varepsilon\) | heat absorption (scalar) |
| \(d_i^\varepsilon\) | diffusion coefficients (tensor) |
| \(\rho_i^\varepsilon\) | Dufour coefficients (tensor) |
| \(a_i^\varepsilon, b_i^\varepsilon\) | deposition rate coefficients (scalars) |

Table 1: Physical parameters of the microscopic problem \((P^\varepsilon)\).

**Remark 1.** Our thermo-diffusion system is made of \(N + 1\) equations where the short-hand explanation for physical parameters in this model can be found in Table 1. Physically, equation (2.2) describes the changes of the temperature \(\theta^\varepsilon\) in \(\Omega^\varepsilon\) according to a heat conduction equation with a production term depending on \(\nabla \delta \theta^\varepsilon\), whilst the colloidal concentration \(u_i^\varepsilon\) is assumed to satisfy \(N\) reaction-diffusion like equations given by (2.3) with a chemical reaction term depending on \(\nabla \delta \theta^\varepsilon\). This type of special right-hand sides is mimicking the so-called Soret and Dufour effects. In (2.8), \(v_i^\varepsilon\) denotes the mass of the deposited species on the boundary of the pore skeleton \(\Gamma^\varepsilon\). These quantities are also supposed to satisfy the following ordinary differential equations (2.4).

We make use of the following assumptions:

- \((A_1)\) The coefficients \(\kappa^\varepsilon, \tau^\varepsilon, d_i^\varepsilon, \rho_i^\varepsilon \in [H^1(\Omega^\varepsilon)]^d \cap [L^\infty(\Omega^\varepsilon)]^d\), \(g_0^\varepsilon \in L^\infty(\Gamma_R^\varepsilon)\) and \(a_i^\varepsilon, b_i^\varepsilon \in L^\infty(\Gamma^\varepsilon)\) are \(Y\)-periodic. Also, there exist positive constants \(\kappa_{\min}, \kappa_{\max}, \tau_{\min}, \tau_{\max}, d_{\min}, d_{\max}, \rho_{\min}, \rho_{\max}, a_{\min}, a_{\max}, b_{\min}, b_{\max}\) such that \(\kappa_{\min} \leq \kappa_{jk} \leq \kappa_{\max}, \tau_{\min} \leq \tau_{jk} \leq \tau_{\max}, d_{\min} \leq d_{jk} \leq d_{\max}, \rho_{\min} \leq \rho_{jk} \leq \rho_{\max}, a_{\min} \leq a_{jk} \leq a_{\max}, b_{\min} \leq b_{jk} \leq b_{\max}\) for \(i \in \{1, \ldots, N\}\) and \(j, k \in \{1, \ldots, d\}\). Furthermore, there also exist positive constants \(\alpha_i\) for \(i \in \{0, \ldots, N\}\) such that

\[\kappa_{jk}(y) \xi_j \xi_k \geq \alpha_0 |\xi|^2 \quad \text{and} \quad d_{jk}^i(y) \xi_j \xi_k \geq \alpha_i |\xi|^2 \quad \text{for any} \quad \xi \in \mathbb{R}^d, i \in \{1, \ldots, N\}, j \quad \text{and} \quad k \in \{1, \ldots, d\}\]

to guarantee the ellipticity of the operators \(A_k^\varepsilon\) and \(A_{d_k}^\varepsilon\).

- \((A_2)\) The initial conditions satisfy \(\theta^\varepsilon(0) \in L^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon), u_i^\varepsilon(0) \in L^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon), v_i^\varepsilon(0) \in L^\infty(\Gamma^\varepsilon)\) for \(i \in \{1, \ldots, N\}\), such that we can find \(C_0 > 0\) satisfying

\[
\|\theta^\varepsilon(0)\|_{H^1(\Omega^\varepsilon)} + \sum_{i=1}^N \left(\|u_i^\varepsilon(0)\|_{H^1(\Omega^\varepsilon)} + \|v_i^\varepsilon(0)\|_{L^\infty(\Gamma^\varepsilon)}\right) \leq C_0,
\]

where \(C_0\) is independent of the choice of \(\varepsilon\).

**Remark 2.** By the definitions of \(\kappa, \tau, d_i, \rho_i\) and \((A_1)\), there exist positive constants that bound from below and above these coefficients on \(Y\) for each choice of \(\varepsilon\).
Unless otherwise specified, all the constants $C$ are independent of the homogenization parameter $\varepsilon$, but the precise values may differ from line to line or even within a single chain of estimates. Throughout this paper, we use the superscript $\varepsilon$ to emphasize the dependence on the heterogeneity of the material characterized by the homogenization parameter $\varepsilon$. In the sequel, we use $dS_\varepsilon$ as a shorthand for $ndS$, where $S_\varepsilon$ can be viewed as a common notation for a boundary of any surface. Moreover, the notation $|\cdot|$ for a domain indicates in this work the volume of that domain.

### 2.2 Preliminary results

In this subsection, we present the definition of two-scale convergence as well as its compactness arguments (cf. [2 30]) together with the fact already known concerning the weak solvability and periodic homogenization of $(P^\varepsilon)$.

**Definition 3. Two-scale convergence**

Let $(u^\varepsilon)$ be a sequence of functions in $L^2 (0, T; L^2 (\Omega))$ with $\Omega$ being an open set in $\mathbb{R}^d$, then it two-scale converges to a unique function $u^0 \in L^2 ((0, T) \times \Omega \times Y)$, denoted by $u^\varepsilon \rightharpoonup^2 u^0$, if for any $\varphi \in C_0^\infty ((0, T) \times \Omega; C_0^\infty (Y))$ we have

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u^\varepsilon (t, x) \varphi (t, x, x/\varepsilon) \, dx \, dt = \int_0^T \int_\Omega \int_Y u^0 (t, x, y) \varphi (t, x, y) \, dy \, dx \, dt.$$ 

**Theorem 4. Two-scale compactness**

- Let $(u^\varepsilon)$ be a bounded sequence in $L^2 ((0, T) \times \Omega)$. Then there exists a function $u^0 \in L^2 ((0, T) \times \Omega \times Y)$ such that, up to a subsequence, $u^\varepsilon$ two-scale converges to $u^0$.

- Let $(u^\varepsilon)$ be a bounded sequence in $L^2 (0, T; H^1 (\Omega))$, then up to a subsequence, we have the two-scale convergence in gradient $\nabla u^\varepsilon \rightharpoonup^2 \nabla_x u^0 + \nabla_y u^1$ for $u^0 \in L^2 (0, T; H^1 (\Omega))$ and $u^1 \in L^2 ((0, T) \times \Omega; H^1_\# (Y) / \mathbb{R})$.

**Remark 5.** The concepts of two-scale convergence and compactness for $\varepsilon$-periodic hypersurfaces were originally introduced in [29 31] and have been used in [14 24]. For brevity, let $(u^\varepsilon)$ be a sequence of functions in $L^2 (0, T; L^2 (\Gamma^\varepsilon))$. We say $u^\varepsilon$ two-scale converges to a limit $u^0$ in $L^2 ((0, T) \times \Omega \times \Gamma)$ with $\Gamma = \partial \Omega$ if for any $\varphi \in C_0^\infty (\Omega \times \partial \Omega)$ we have

$$\lim_{\varepsilon \to 0} \int_{\Gamma^\varepsilon} \varepsilon u^\varepsilon (t, x) \varphi (t, x, x/\varepsilon) \, d\sigma (y) \, dx \, dt = \int_0^T \int_\Omega \int_\Gamma u^0 (t, x, y) \varphi (t, x, y) \, d\sigma (y) \, dx \, dt.$$ 

Thereby, we obtain the two-scale compactness on surfaces that for each bounded sequence $(u^\varepsilon)$ in $L^2 (0, T; L^2 (\Gamma^\varepsilon))$, one can extract a subsequence which two-scale converges to $u^0 \in L^2 ((0, T) \times \Omega \times \Gamma)$. Furthermore, if $(u^\varepsilon)$ is bounded in $H^1 (\Omega^\varepsilon)$, then two-scale converges to a limit function $u^0 \in L^\infty ((0, T) \times \Omega \times \Gamma)$.

It is important to note that, for our choice of $Y_0$, the interior extension from $H^1 (\Omega^\varepsilon)$ into $H^1 (\Omega)$ exists with extension constants independent of $\varepsilon$ (see [20 Lemma 5] and [6 Theorem 2.10]).

**Definition 6. The weak formulation of $(P^\varepsilon)$**

For $i \in \{1, ..., N\}$, the triplet $(\theta^\varepsilon, u^\varepsilon_i, v^\varepsilon_i)$ satisfying

$$\theta^\varepsilon, u^\varepsilon_i \in H^1 (0, T; L^2 (\Omega^\varepsilon)) \cap L^\infty (0, T; H^1 (\Omega^\varepsilon)) \cap L^\infty ((0, T) \times \Omega^\varepsilon),$$

$$v^\varepsilon_i \in H^1 (0, T; L^2 (\Gamma^\varepsilon)) \cap L^\infty ((0, T) \times \Gamma^\varepsilon).$$
is a weak solution to \((P^\varepsilon)\) provided that

\[
\begin{aligned}
\int_{\Omega^\varepsilon} \partial_t \theta^\varepsilon \varphi dx + \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla \theta^\varepsilon \cdot \nabla \varphi dx + \varepsilon \int_{\Gamma_R^\varepsilon} g_0 \theta^\varepsilon \varphi dS^\varepsilon &= \int_{\Omega^\varepsilon} \varepsilon \sum_{i=1}^N \nabla^\delta u^\varepsilon_i \cdot \nabla \theta^\varepsilon \varphi dx, \\
\int_{\Omega^\varepsilon} \partial_t u^\varepsilon_i \phi_i dx + \int_{\Omega^\varepsilon} d^\varepsilon_i \nabla u^\varepsilon_i \cdot \nabla \phi_i dx + \varepsilon \int_{\Gamma^\varepsilon} (a^\varepsilon_i u^\varepsilon_i - b^\varepsilon_i v^\varepsilon_i) \phi_i dS^\varepsilon &= \int_{\Omega^\varepsilon} R^\varepsilon_i (u^\varepsilon) \phi_i dx + \int_{\Omega^\varepsilon} \rho^\varepsilon_i \nabla^\delta \theta^\varepsilon \cdot \nabla u^\varepsilon_i \phi_i dx, \\
\varepsilon \int_{\Gamma^\varepsilon} \partial_t v^\varepsilon_i \psi_i dS^\varepsilon &= \varepsilon \int_{\Gamma^\varepsilon} (a^\varepsilon_i u^\varepsilon_i - b^\varepsilon_i v^\varepsilon_i) \psi_i dS^\varepsilon,
\end{aligned}
\]

for all \((\varphi, \phi_i, \psi_i) \in H^1(\Omega^\varepsilon) \times H^1(\Omega^\varepsilon) \times L^2(\Gamma^\varepsilon)\).

**Theorem 7. Well-posedness and Positivity of solution**

Assume \((A_1)-(A_2)\) and \(i \in \{1, \ldots, N\}\). The microscopic problem \((P^\varepsilon)\) admits a unique solution \((\theta^\varepsilon, u^\varepsilon_i, v^\varepsilon_i)\) in the sense of Definition \(\overline{\theta}\) belonging to

\[
K(T, M) := \{z \in L^2((0,T) \times \Omega^\varepsilon) : |z| \leq M \text{ a.e. in } (0,T) \times \Omega^\varepsilon\}
\]

for some \(M > 0\). Additionally,

\[
\begin{align*}
\theta^\varepsilon, u_i^\varepsilon &\in H^1(0,T; L^2(\Omega^\varepsilon)) \cap L^\infty(0,T; H^1(\Omega^\varepsilon)) \cap L^\infty((0,T) \times \Omega^\varepsilon), \\
v_i^\varepsilon &\in H^1(0,T; L^2(\Gamma^\varepsilon)) \cap L^\infty((0,T) \times \Gamma^\varepsilon).
\end{align*}
\]

Furthermore, this triplet \((\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)\) is positive and the following energy estimates hold

\[
\begin{align*}
\kappa_{\min} \|\nabla \theta^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + \int_0^t \|\partial_t \theta^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} dt &\leq C, \\
\|\nabla u_i^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + \int_0^T \left(\|\partial_t u_i^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + \|\partial_t v_i^\varepsilon(t)\|^2_{L^2(\Gamma^\varepsilon)}\right) dt &\leq C \text{ for a.e. } t \in (0,T).
\end{align*}
\]

We denote by \((P^0)\) the strong formulation of the macroscopic (limit) problem. We introduce below the limit problem whose precise structure has been obtained via a two-scale convergence procedure in \cite{24}.

**Theorem 8. Strong formulation of the macroscopic problem – \((P^0)\)**

Assume \((A_1)-(A_2)\). For \(i \in \{1, \ldots, N\}\), the triplet \((\theta^0, u_i^0, v_i^0)\) of limit solutions \((\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)\) to \((P^\varepsilon)\) in the sense of Definition \(\overline{\theta}\) satisfies the following macroscopic system

\[
\begin{align}
\partial_t \theta^0 + \nabla \cdot (\mathbb{K} \nabla \theta^0) + g_0 |\Gamma_R| \theta^0 &= \sum_{i=1}^N (\mathbb{T}^i \nabla^\delta u_i^0) \cdot \nabla \theta^0 \text{ in } Q_T, \\
\partial_t u_i^0 + \nabla \cdot (\mathbb{D}^i \nabla u_i^0) + A_i u_i^0 - B_i v_i^0 &= (\mathbb{F}^i \nabla u_i^0) \cdot \nabla \theta^0 + R_i (u^0) \text{ in } Q_T,
\end{align}
\]

subject to the boundary conditions

\[
\begin{align}
- \mathbb{K} \nabla \theta^0 \cdot n &= 0 \text{ on } (0,T) \times \partial \Omega, \\
- \mathbb{D}^i \nabla u_i^0 \cdot n &= 0 \text{ on } (0,T) \times \partial \Omega,
\end{align}
\]

(2.17)
and associated with the ordinary differential equations
\[ \partial_t v_i^0 = A_i u_i^0 - B_i v_i^0 \quad \text{in } Q_T, \]
(2.18)
where we have denoted by \( K = K_0 \mathbb{I} + (K_{ij})_{ij}, T^i = T_0^i \mathbb{I} + (T_{jk}^i)_{jk}, D^i = D_i \mathbb{I} + D_0^i, F^i = F_i \mathbb{I} + F_0^i \) for \( j, k \in \{1, ..., d\} \) with \( \mathbb{I} \) standing for the identity matrix and the quantities \( K_0, K_{ij}, T_0^i, T_{jk}^i, D_i, D_0^i, F_i, F_0^i, A_i, B_i \)
being effective constants corresponding, respectively, to the oscillating coefficients and defined in \( (2.23) \) \( - \) \( (2.27) \).

Furthermore, the initial conditions are provided by
\[ \begin{align*}
\theta^0 (t = 0) &= \theta^{0,0} \quad \text{in } \Omega, \\
u_i^0 (t = 0) &= u_i^{0,0} \quad \text{in } \Omega, \\
v_i^0 (t = 0) &= v_i^{0,0} \quad \text{on } \Gamma.
\end{align*} \]
(2.19) \( - \) (2.21)

**Theorem 9.** The weak formulation of \( (P^0) \)

Assume \((A_1)-(A_2)\) and take \( i \in \{1, ..., N\} \), the triplet \((\theta^0, u_i^0, v_i^0)\) satisfying
\[ \theta^0, u_i^0 \in H^1 (0, T; L^2 (\Omega)) \cap L^2 (0, T; H^1 (\Omega)) \cap L^\infty ((0, T) \times \Omega), \]
\[ v_i^0 \in H^1 (0, T; L^2 (\Omega)) \cap L^\infty ((0, T) \times \Omega), \]
is a weak solution to \( (P^0) \) provided that
\[ \begin{align*}
\int_\Omega \partial_t \theta^0 \varphi dx + \int_\Omega K \nabla \theta^0 \cdot \nabla \varphi dx + \int_\Omega g_0 |\Gamma_R| |Y_1| \theta^0 \varphi dx &= \int_\Omega \sum_{i=1}^N (T^i \delta^u_i u_i^0) \cdot \nabla \theta^0 \varphi dx, \\
\int_\Omega \partial_i u_i^0 \phi_i dx + \int_\Omega D^i \nabla u_i^0 \cdot \nabla \phi_i dx + \int_\Omega \left( A_i u_i^0 - B_i v_i^0 \right) \phi_i dx &= \int_\Omega (F^i \delta^u_i u_i^0) \cdot \nabla \theta^0 \phi_i dx + \int_\Omega R_i (\varphi^0) \phi_i dx, \\
\int_\Omega \partial_i v_i^0 \psi_i dx &= \int_\Omega \left( A_i u_i^0 - B_i v_i^0 \right) \psi_i dx,
\end{align*} \]
(2.22)
hold for all \( (\varphi, \phi, \psi) \in C^\infty (\Omega) \times C^\infty (\Omega) \times C^\infty (\Omega) \).

For \( i \in \{1, ..., N\} \) and \( j, k \in \{1, ..., d\} \), the effective constants in Theorem 8 are defined, as follows:
\[ \begin{align*}
K_0 &:= \frac{1}{|Y_1|} \int_{Y_1} \kappa(y) dy, \quad K_{ij} := \frac{1}{|Y_1|} \int_{Y_1} \kappa(y) \frac{\partial \theta^0_j}{\partial y_i} dy, \\
T_0^i &:= \frac{1}{|Y_1|} \int_{Y_1} \tau_i(y) dy, \quad T_{jk}^i := \frac{1}{|Y_1|} \int_{Y_1} \tau_i(y) \frac{\partial \theta^0_j}{\partial y_k} dy, \\
D_i &:= \frac{1}{|Y_1|} \int_{Y_1} d_i(y) dy, \quad D_0^i := \left( \frac{1}{|Y_1|} \int_{Y_1} d_i(y) \frac{\partial u_i^0}{\partial y_k} dy \right)_{jk}, \\
F_i &:= \frac{1}{|Y_1|} \int_{Y_1} \rho_i(y) dy, \quad F^i := \left( \frac{1}{|Y_1|} \int_{Y_1} \rho_i(y) \frac{\partial u_i^0}{\partial y_k} dy \right)_{jk}, \\
A_i &:= \frac{1}{|Y_1|} \int_{\partial Y_0} a_i dy, \quad B_i := \frac{1}{|Y_1|} \int_{\partial Y_0} b_i dy.
\end{align*} \]
(2.23) \( - \) (2.27)
Hereby, the functions $\bar{\theta}$ and $\bar{u}_i$ linearly formulate the limit functions $\theta^1$ and $u^1_i$ by $\theta^1 := \bar{\theta} \cdot \nabla_x \theta^0 = \sum_{j=1}^d \partial_x_j \theta^0 \bar{\theta}^j$ and $u^1_i := \bar{u}_i \cdot \nabla_x u^0_i = \sum_{j=1}^d \partial_x_j u^0_i \bar{u}^j_i$ for $i \in \{1, \ldots, N\}$. Moreover, they solve, respectively, the cell problems introduced in the following Theorem.

**Theorem 10. The cell problems**

Assume $(A_1)$ holds. The limit functions $\theta^1$ and $u^1_i$ defined as above solve the following cell problems:

\[
\begin{aligned}
\nabla_y \cdot (-\kappa (y) \nabla_y \bar{\theta}^j(x, y)) &= \nabla_y \cdot (\kappa n_j) & \text{in } Y_1, \\
-\kappa (y) \nabla_y \bar{\theta}^j \cdot n &= \kappa n_j & \text{on } \partial Y_0, \\
\bar{\theta}^j &\text{ is } Y\text{-periodic},
\end{aligned}
\]

where $n_j$ is the $j$th unit vector of $\mathbb{R}^d$ and $i \in \{1, \ldots, N\}, j \in \{1, \ldots, d\}$. Furthermore,

(i) If $\kappa, d_i \in [H^1 (Y_1)]^d$ are Lipschitz continuous, the system (2.28)–(2.29) admits a unique solution $(\bar{\theta}^j, \bar{u}^j_i) \in H^2_{\text{loc}} (Y_1) \times H^2_{\text{loc}} (Y_1);

(ii) If $k, d_i \in [H^1 (Y_1)]^d \cap [H^{-\frac{1}{2} + s} (\partial Y_0)]^d$ for every $s \in (-\frac{1}{2}, \frac{1}{2})$ are Lipschitz continuous, the system (2.28)–(2.29) admits a unique solution $(\bar{\theta}^j, \bar{u}^j_i) \in H^{1+s} (Y_1) \times H^{1+s} (Y_1).

The weak solvability of the cell problems (2.28) and (2.29) shall be further discussed in the proof of our main result – Theorem 12. To derive our corrector estimates, we need a number of elementary inequalities.

- For all $1 \leq p \leq \infty$, the following estimates hold:
  \[
  \|\nabla^\delta f \cdot g\|_{L^p (\Omega^\varepsilon)} \leq C_\delta \|f\|_{L^\infty (\Omega^\varepsilon)} \|g\|_{L^p (\Omega^\varepsilon)} \quad \text{for } f \in L^\infty (\Omega^\varepsilon), \quad g \in [L^p (\Omega^\varepsilon)]^d, \tag{2.30}
  \]
  \[
  \|\nabla^\delta f\|_{L^p (\Omega^\varepsilon)} \leq C_\delta \|f\|_{L^2 (\Omega^\varepsilon)} \quad \text{for } f \in L^2 (\Omega^\varepsilon), \tag{2.31}
  \]
  where $C > 0$ depends only on $\delta$. See [24], e.g., for a proof of (2.30) and (2.31).

- To estimate the correctors for both the temperature $\theta^\varepsilon$ and colloidal concentrations $u^\varepsilon_i$, we consider the real-valued cut-off function $m^\varepsilon \in C^1_0 (\Omega)$ satisfying $0 \leq m^\varepsilon \leq 1$, $\varepsilon |\nabla m^\varepsilon| \leq C$, and $m^\varepsilon = 1$ on $\{x \in \Omega : \text{dist} (x, \Gamma) \geq \varepsilon\}$. Furthermore, one can prove that
  \[
  \|1 - m^\varepsilon\|_{L^2 (\Omega^\varepsilon)} \leq C \varepsilon^{1/2}, \quad \varepsilon \|\nabla m^\varepsilon\|_{L^2 (\Omega^\varepsilon)} \leq C \varepsilon^{1/2}. \tag{2.32}
  \]

- (A Young-type inequality) Let $\delta > 0$ and $a, b \geq 0$ be arbitrarily real numbers and take $q, q' > 1$ real constants that are Hölder conjugates of each other. Then the following inequality holds
  \[
  ab \leq \frac{1}{q} \delta^a a^q + \frac{1}{q'} \delta^{-q} b^{q'}. \tag{2.33}
  \]
• (Trace inequality for $\varepsilon$-dependent hypersurfaces $\Gamma^\varepsilon$) Let $\Gamma^\varepsilon$ be as in Subsection 2.1.1. For $\varphi^\varepsilon \in H^1(\Omega^\varepsilon)$, there exists a constant $C > 0$ (independent of $\varepsilon$) such that
\[
\varepsilon \|\varphi^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 \leq C \left( \|\varphi^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \varepsilon^2 \|\nabla \varphi^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \right).
\] (2.34)

The proof of (2.34) can be found in [20, Lemma 3].

Theorem 11. Existence and uniqueness results for $(P^0)$ Assume $(A_1)$-$\mathcal{A}_2$. For $i \in \{1, \ldots, N\}$, the macroscopic problem $(P^0)$ admits a unique (local) weak solution in $L^2((0, T) \times \Omega)$.

Proof. Due to the homogenization limit results in [24, Lemma 4.3], the existence of the triplet $(\theta^0, u_i^0, v_i^0)$ in Theorem 9 is guaranteed. The contraction of these functions in a closed subspace of $[L^2((0, T) \times \Omega)]^{N+2}$ can be proved concisely by a linearization argument. The proof can be sketched as follows: We define
\[
K_1(M, T) := \{ z \in L^2((0, T) \times \Omega) : |z| \leq M \text{ a.e. in } Q_T \}.
\]
For $i \in \{1, \ldots, N\}$, let $\theta_i^{0,1}, u_i^{0,1}, v_i^{0,1} \in K_1(M_i, T_i)$ and $\theta_i^{0,2}, u_i^{0,2}, v_i^{0,2} \in K_1(M_2, T_2)$ be two pairs of (weak) solutions of the macro system. By choosing $T = \min\{T_1, T_2\}$ and $M = \max\{M_1, M_2\}$ and suitable test functions $\varphi, \phi_i, \psi_i$ in (2.22), we get $d(\theta_i^0) := \theta_i^{0,1} - \theta_i^{0,2}$, $d(u_i^0) := u_i^{0,1} - u_i^{0,2}$, $d(v_i^0) := v_i^{0,1} - v_i^{0,2} \in K_1(M, T)$, which satisfy the following equalities:
\[
\frac{1}{2} \partial_t \| d(\theta_i^0) \|_{L^2(\Omega)}^2 + K \| \nabla d(\theta_i^0) \|_{L^2(\Omega)}^2 + g_0 \left[ \frac{\Gamma R}{|Y_1|} \right] \| d(\theta_i^0) \|_{L^2(\Omega)}^2
\]
\[
= \int_{\Omega} \sum_{i=1}^N \left( \left( \mathbb{T}^i \nabla \delta u_i^{0,1} \cdot \nabla \theta_i^{0,1} - \mathbb{T}^i \nabla \delta u_i^{0,2} \cdot \nabla \theta_i^{0,2} \right) d(\theta_i^0) \right) dx, \quad (2.35)
\]
\[
\frac{1}{2} \partial_t \| d(u_i^0) \|_{L^2(\Omega)}^2 + \mathbb{D}^i \| \nabla d(u_i^0) \|_{L^2(\Omega)}^2 + A_i \| d(u_i^0) \|_{L^2(\Omega)}^2 - \int_{\Omega} B_i d(v_i^0) d(u_i^0) dx
\]
\[
= \int_{\Omega} \left( (\mathbb{F}^i \nabla u_i^{0,1} ) \cdot \nabla \theta_i^{0,1} - (\mathbb{F}^i \nabla u_i^{0,2} ) \cdot \nabla \theta_i^{0,2} \right) d(u_i^0) dx
\]
\[
+ \int_{\Omega} \left( R_i (u_i^{0,1}) - R_i (u_i^{0,2}) \right) d(u_i^0) dx, \quad (2.36)
\]
\[
\frac{1}{2} \partial_t \| d(v_i^0) \|_{L^2(\Omega)}^2 + B_i \| d(v_i^0) \|_{L^2(\Omega)}^2 = \int_{\Omega} A_i d(u_i^0) d(v_i^0) dx.
\]
Then, with the help of the estimates (2.30)-(2.31) and the Young-type inequality (2.33) under a suitable choice of a pair $(\delta, q, q')$ to get rid of the gradient norms $\| \nabla d(\theta_i^0) \|_{L^2(\Omega)}^2$ and $\| \nabla d(u_i^0) \|_{L^2(\Omega)}^2$ on the left-hand side of (2.35)-(2.36), one can find a constant $C(M, \delta) > 0$ such that for all $i \in \{1, \ldots, N\}$
\[
\partial_t \| d(\theta_i^0) \|_{L^2(\Omega)}^2 + \partial_t \| d(u_i^0) \|_{L^2(\Omega)}^2 + \partial_t \| d(v_i^0) \|_{L^2(\Omega)}^2
\]
\[
\leq C(M, \delta) \left( \| d(\theta_i^0) \|_{L^2(\Omega)}^2 + \| d(u_i^0) \|_{L^2(\Omega)}^2 + \| d(v_i^0) \|_{L^2(\Omega)}^2 + 1 \right). \quad (2.37)
\]

Hereby, we apply the Gronwall inequality to (2.37) and then integrate the resulting estimate over $(0, T)$ to obtain that
\[
\| d(\theta_i^0) \|_{L^2((0, T) \times \Omega)}^2 + \| d(u_i^0) \|_{L^2((0, T) \times \Omega)}^2 + \| d(v_i^0) \|_{L^2((0, T) \times \Omega)}^2 \leq T^2 C(M, \delta) \exp(TC(M, \delta)) \quad (2.38)
\]

Since $T^2 C(M, \delta) \exp(TC(M, \delta)) \to 0$ as $T \to 0$, we can construct an approximation scheme $(\theta_i^{0,n}, u_i^{0,n}, v_i^{0,n})$ for $n \in \mathbb{N}$ for the macro system in which the involved nonlinear terms are linearized. With a small enough $T_0$ such that $T_0^2 C(M, \delta) \exp(T_0C(M, \delta)) < 1$, we claim that $\{\theta_i^{0,n}\}_{n \in \mathbb{N}}$, $\{u_i^{0,n}\}_{n \in \mathbb{N}}$ and $\{v_i^{0,n}\}_{n \in \mathbb{N}}$ are the Cauchy sequences in $K_1(M, T_0)$ by (2.38). Thus, the local existence and uniqueness of solutions in $[L^2((0, T) \times \Omega)]^{N+2}$ to $(P^0)$ is guaranteed.
3 Main result

The main result of this paper is stated in the next Theorem whose applicability is delimited by the assumptions (A1)-(A2) and the extra regularity assumptions shall also be provided therein. Note that the involved macro reconstructions $\theta_0^\varepsilon, u_i^\varepsilon, v_i^\varepsilon$ for $i \in \{1, ..., N\}$ shall be defined right in the next Subsection.

**Theorem 12.** Assume (A1)-(A2). Let $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ and $(\theta^0, u_i^0, v_i^0)$ for $i \in \{1, ..., N\}$ be weak solutions to $(P^\varepsilon)$ and $(P^0)$ in the sense of Definition \[9\] and Theorem \[9\] respectively. Let $\bar{\theta}, \bar{u}_i$ be the cell functions solving the cell problems (2.28)-(2.29) and satisfying
\[
\bar{\theta}, \bar{u}_i \in L^\infty(\Omega^\varepsilon; W^{1+s,2}_0(Y_1)) \cap H^1(\Omega^\varepsilon; W^{s,2}_0(Y_1)) \quad \text{for } s > d/2.
\]

For every $t \in (0, T]$, we also assume that $\theta^0(t, \cdot), u_i^0(t, \cdot) \in W^{1,\infty}(\Omega^\varepsilon) \cap H^2(\Omega^\varepsilon)$ for $i \in \{1, ..., N\}$. On top of that, we assume the initial homogenization limit is of the rate
\[
\|\theta^{\varepsilon,0} - \theta^{0,0}\|_{L^2(\Omega^\varepsilon)}^2 + \sum_{i=1}^N \|u_i^{\varepsilon,0} - u_i^{0,0}\|_{L^2(\Omega^\varepsilon)}^2 + \sum_{i=1}^N \|v_i^{\varepsilon,0} - v_i^{0,0}\|_{L^2(\Gamma^\varepsilon)}^2 \leq \varepsilon^\gamma,
\]
for some $\gamma \in \mathbb{R}_+$. Then the following corrector estimate holds
\[
\|\theta^\varepsilon - \theta^0\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + \sum_{i=1}^N \|u_i^\varepsilon - u_i^0\|_{L^2((0,T) \times \Omega^\varepsilon)}^2
\]
\[
+ \|\nabla (\theta^\varepsilon - \theta_i^0)\|_{L^2((0,T);L^2(\Omega^\varepsilon)^d)}^2 + \sum_{i=1}^N \|\nabla (u_i^\varepsilon - u_i^0)\|_{L^2((0,T);L^2(\Omega^\varepsilon)^d)}^2 \leq C \max \{\varepsilon, \varepsilon^\gamma\},
\]
where $C$ is a generic positive constant that is independent of $\varepsilon$.

Furthermore, if $\gamma \geq 1$, then we obtain
\[
\varepsilon \sum_{i=1}^N \|v_i^\varepsilon - v_i^0\|_{L^2((0,T) \times \Gamma^\varepsilon)}^2 \leq C\varepsilon.
\]

3.1 Macroscopic reconstruction

To derive correctors estimates for our problem, we use the concept of the macroscopic reconstruction. We borrow this terminology from Eck[9], but note that it is also connected to similar concepts in the \textit{a posteriori} numerical analysis of PDEs (see e.g. [26]). It turns out that we derive operators that could bring us the link between the strong formulations $(P^\varepsilon)$ and $(P^0)$. For a.e. $t \in [0, T]$ and $x \in \Omega^\varepsilon$ we provide that
\[
\theta_0^\varepsilon(t, x) := \theta^0(t, x),
\]
\[
u_i^0(t, x) := u_i^0(t, x),
\]
\[
\nu_i^0(t, x) := v_i^0(t, x).
\]

Henceforward, we obtain the system of macroscopic reconstruction whose expression is similar to the strong formulations $(P^0)$, but acting on $x \in \Omega^\varepsilon$. We accordingly subtract this system from the macroscopic system $(P^\varepsilon)$ equation-by-equation and gain the difference system over $\Omega^\varepsilon$. Then we proceed to the correctors justification by the following choice of test functions:
\[
\varphi(t, x) := \theta^\varepsilon(t, x) - \left(\theta_0^\varepsilon(t, x) + \varepsilon m^\varepsilon(x) \bar{\theta}\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla_x \theta^0(t, x)\right),
\]
\[
\phi_i(t, x) := u_i^\varepsilon(t, x) - \left(u_i^0(t, x) + \varepsilon m^\varepsilon(x) \bar{u}_i\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla_x u_i^0(t, x)\right),
\]
where $m_\varepsilon$ is a cut-off function with the properties (2.32).

Multiplying the difference system by the test functions $\varphi, \phi_i \in H^1(\Omega^\varepsilon)$ and integrating the resulting equations over $\Omega^\varepsilon$, we obtain the system, denoted by (3.1), as follows:

$$
\int_{\Omega^\varepsilon} \partial_t (\theta^\varepsilon - \theta^0_0) \varphi dx + \int_{\Omega^\varepsilon} (\kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta^0_0) \cdot \nabla \varphi dx + \varepsilon \int_{\Gamma^\varepsilon R} g_0 \theta^\varepsilon \varphi dS_\varepsilon

- g_0 \frac{|\Gamma_R|}{|Y_1|} \int_{\Omega^\varepsilon} \theta^\varepsilon \varphi dx = \int_{\Omega^\varepsilon} \left( \tau^\varepsilon \sum_{i=1}^N \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon - \sum_{i=1}^N (T^i \nabla^\delta u_{i,0}^\varepsilon) \cdot \nabla \theta^0_0 \right) \varphi dx,
$$

$$
\int_{\Omega^\varepsilon} \partial_t (u_i^\varepsilon - u_{i,0}^\varepsilon) \phi_i dx + \int_{\Omega^\varepsilon} (d_i^\varepsilon \nabla u_i^\varepsilon - D_i^\varepsilon \nabla u_{i,0}^\varepsilon) \cdot \nabla \phi_i dx + \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \phi_i dS_\varepsilon

- \int_{\Omega^\varepsilon} (A_i u_{i,0}^\varepsilon - B_i v_{i,0}^\varepsilon) \phi_i dx = \int_{\Omega^\varepsilon} (\rho_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon - (\mathbb{F}^i \nabla u_{i,0}^\varepsilon) \cdot \nabla^\delta \theta^0_0) \phi_i dx

+ \int_{\Omega^\varepsilon} (R_i (u^\varepsilon) - R_i (u_{i,0}^\varepsilon)) \phi_i dx,
$$

According to the system (3.1), we denote the following terms:

$$
I_1 := \int_{\Omega^\varepsilon} \partial_t (\theta^\varepsilon - \theta^0_0) \varphi dx,
$$

$$
I_2 := \int_{\Omega^\varepsilon} (\kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta^0_0) \cdot \nabla \varphi dx,
$$

$$
I_3 := \varepsilon \int_{\Gamma^\varepsilon R} g_0 \theta^\varepsilon \varphi dS_\varepsilon - g_0 \frac{|\Gamma_R|}{|Y_1|} \int_{\Omega^\varepsilon} \theta^\varepsilon \varphi dx,
$$

$$
I_4 := \int_{\Omega^\varepsilon} \left( \tau^\varepsilon \sum_{i=1}^N \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon - \sum_{i=1}^N (T^i \nabla^\delta u_{i,0}^\varepsilon) \cdot \nabla \theta^0_0 \right) \varphi dx,
$$

$$
J^1_i := \int_{\Omega^\varepsilon} \partial_t (u_i^\varepsilon - u_{i,0}^\varepsilon) \phi_i dx,
$$

$$
J^2_i := \int_{\Omega^\varepsilon} (d_i^\varepsilon \nabla u_i^\varepsilon - D_i^\varepsilon \nabla u_{i,0}^\varepsilon) \cdot \nabla \phi_i dx,
$$

$$
J^3_i := \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \phi_i dS_\varepsilon - \int_{\Omega^\varepsilon} (A_i u_{i,0}^\varepsilon - B_i v_{i,0}^\varepsilon) \phi_i dx,
$$

$$
J^4_i := \int_{\Omega^\varepsilon} (\rho_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon - (\mathbb{F}^i \nabla u_{i,0}^\varepsilon) \cdot \nabla^\delta \theta^0_0) \phi_i dx + \int_{\Omega^\varepsilon} (R_i (u^\varepsilon) - R_i (u_{i,0}^\varepsilon)) \phi_i dx.
$$

We introduce, in the same spirit as for (3.1) and (3.2), another macroscopic reconstruction $\theta^\varepsilon_1(t, x)$ and $u^\varepsilon_{i,1}(t, x)$ defined as follows:

$$
\theta^\varepsilon_1(t, x) := \theta^0_0(t, x) + \varepsilon \tilde{\theta} \left( x, \frac{x}{\varepsilon} \right) \cdot \nabla x \theta^0(t, x),
$$

$$
u^\varepsilon_{i,1}(t, x) := u^\varepsilon_{i,0}(t, x) + \varepsilon \tilde{u}_i \left( x, \frac{x}{\varepsilon} \right) \cdot \nabla x u^0_i(t, x),
$$

where $\tilde{\theta}$ and $\tilde{u}_i$ are the cell functions introduced in Theorem 10.

By definition (3.1)-(3.2), the macroscopic reconstruction $\theta^\varepsilon_0(t, x)$ and $u^\varepsilon_{i,0}(t, x)$ are interchangeable, respectively, in notation with the limit functions $\theta^0(t, x)$ and $u^0_i(t, x)$ in Theorem 12.
3.2 Integral estimates

Remark 13. From Lemma 14, one can apply directly the $L^2$-estimate between the space-dependent physical parameters of the microscopic problem (e.g. $\kappa^\varepsilon$, $\tau^\varepsilon$) and their averages, even if the parameters in discussion are actually tensors. To this end, these estimates are controlled as $\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}$, where $p^\varepsilon$ refers to the oscillating coefficient and $\bar{p}$ denotes its average.

Lemma 14. Let $Y_1$ as defined in Subsection 2.1.1. Let $p^\varepsilon(x) := p(x^\varepsilon)$ belong to $H^1(\Omega^\varepsilon)$ satisfying

$$\bar{p} := \frac{1}{|Y_1|} \int_{Y_1} p(y) \, dy.$$  

Then the following estimate holds

$$\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2} \|p^\varepsilon\|_{H^1(\Omega^\varepsilon)}.$$  

Proof. We consider the periodic geometry described in Figure 2.1 in Subsection 2.1.1. For a fixed test function $\phi \in H^1(\Omega^\varepsilon)$, we see that

$$\int_{\Omega^\varepsilon} (p^\varepsilon - \bar{p}) \phi \, dx = \sum_{k \in \mathbb{Z}^d} \int_{eY_1^k} (p^\varepsilon - \bar{p}) \phi \, dx$$

$$\leq C \int_{eY_1} (p^\varepsilon - \bar{p}) \phi \, dx.$$  

By changing the variable $x = \varepsilon y$, the relations

$$\int_{\varepsilon Y_1} p\left(\frac{x}{\varepsilon}\right) \phi(x) \, dx = \varepsilon^d \int_{Y_1} p(y) \phi(\varepsilon y) \, dy,$$

$$\int_{\varepsilon Y_1} p(y) \phi(x) \, dy \, dx = \varepsilon^d \int_{Y_1} \int_{Y_1} p(y) \phi(\varepsilon z) \, dz \, dy,$$

enable us to write:

$$\int_{\varepsilon Y_1} (p^\varepsilon - \bar{p}) \phi \, dx = \varepsilon^d |Y_1|^{-1} \int_{Y_1} \int_{Y_1} (p(y) \phi(\varepsilon y) - p(y) \phi(\varepsilon z)) \, dz \, dy. \quad (3.14)$$  

Thanks to the representation

$$\phi(\varepsilon y) - \phi(\varepsilon z) = \varepsilon \int_0^1 \nabla \phi(t\varepsilon y + (1 - t)\varepsilon z) \cdot (y - z) \, dt,$$

with $\xi = ty + (1-t)z$ and $\eta = y - z$, we note that (3.14) can be bounded from above by

$$\left| \int_{\varepsilon Y_1} (p^\varepsilon - \bar{p}) \phi \, dx \right| \leq \varepsilon^{d+1} |Y_1|^{-1} \left( \int_{Y_1} \int_{Y_2} |\nabla \phi(\varepsilon \xi) \cdot \eta|^2 \, d\eta \, d\xi \right)^{1/2} \left( \int_{Y_1} \int_{Y_1} |p(y)|^2 \, dy \, dz \right)^{1/2}. \quad (3.15)$$  

In (3.15), we have denoted $Y_2 := \{y - z \mid \text{for } y, z \in Y_1\}$. Also, (3.15) leads to

$$\int_{\Omega^\varepsilon} (p^\varepsilon - \bar{p}) \phi \, dx \leq C\varepsilon \|p^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\nabla \phi\|_{L^2(\Omega^\varepsilon)},$$

and with $\phi = p^\varepsilon - \bar{p}$ and (2.33), (3.15) becomes $\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon \left( \|p^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \|\nabla p^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \right)$ and hence, we finally get

$$\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2} \|p^\varepsilon\|_{H^1(\Omega^\varepsilon)}.$$

This completes the proof of the lemma. \qed
Due to the no-flux boundary condition (2.5), we define the function space
\[ H^1(\Gamma^\varepsilon_N) := \{ v \in H^1(\Gamma^\varepsilon) \mid -\kappa^\varepsilon \nabla v^\varepsilon \cdot n = 0 \text{ on } \Gamma^\varepsilon_N \}, \]
which is a closed subspace of \( H^1(\Gamma^\varepsilon) \). This plays a role inside Lemma 15.

**Lemma 15.** Let \( \theta^\varepsilon \in L^2(0, T; H^1(\Gamma^\varepsilon_N)) \) and \( \theta^0 \in L^2(0, T; H^1(\Omega^\varepsilon)) \). For any
\[ f_1 \in C([0, T]; H^1_+(\Omega^\varepsilon) \cap L^\infty_+ (\Omega^\varepsilon)), \]
\[ f_2 \in C([0, T]; H^1_+ (\Gamma^\varepsilon) \cap L^\infty_+ (\Gamma^\varepsilon)), \]
suppose that there exists \( f_3 \in C[0, T] \) such that
\[ \int_{\Omega^\varepsilon} f_1 \theta^0 dx = \int_{\Gamma^\varepsilon_R} f_2 \theta^\varepsilon dS^\varepsilon + \varepsilon f_3. \]

Then, it exists a \( C > 0 \) such that
\[ \left| \int_{\Omega^\varepsilon} f_1 \theta^0 \varphi dx - \varepsilon \int_{\Gamma^\varepsilon_R} (f_2 \theta^\varepsilon + \varepsilon f_3) \varphi dS^\varepsilon \right| \leq \varepsilon C \| \varphi \|_{H^1(\Omega^\varepsilon)}, \]
for any \( \varphi \in H^1(\Omega^\varepsilon) \).

**Proof.** We adapt Lemma 5.2 from [28] to our context. The proof of the lemma is based on the following auxiliary problem: Given \( f_1, f_2, \theta^\varepsilon, \theta^0 \) as above and \( \bar{f} \in C[0, T] \), find \( \Psi \) such that
\[ \begin{cases} 
\Delta_y \Psi (\cdot, x, y) \big|_{y=x}^\varepsilon = f_1 \theta^0 & \text{for } x \in \Omega^\varepsilon, \\
\nabla_y \Psi (\cdot, x, y) \cdot n = f_2 \theta^\varepsilon + \varepsilon \bar{f} & \text{for } (x, y) \in \Gamma^\varepsilon_R, \\
\nabla_y \Psi \cdot n = 0 & \text{at } \Gamma^\varepsilon_N. 
\end{cases} \tag{3.16} \]

By [32, Lemma 2.1] and also [5], the problem (3.16) has a (weak) \( Y \)-periodic solution
\[ \Psi (\cdot, x, y) \big|_{y=x}^\varepsilon \in L^2 (0, T; H^1(\Omega^\varepsilon)) \]
satisfying the integral equality
\[ \int_{\Omega^\varepsilon} f_1 \theta^0 dx = \int_{\Gamma^\varepsilon} (f_2 \theta^\varepsilon + \varepsilon \bar{f}) dS^\varepsilon = \int_{\Gamma^\varepsilon_R} f_2 \theta^\varepsilon S^\varepsilon + \varepsilon f_3, \]
with \( f_3 \) being \( \| \Gamma^\varepsilon_R \|^{-1} \bar{f} \). Moreover, that solution is unique up to an additive constant.

Multiplying the first equation in (3.16) by \( \varphi \in H^1(\Omega^\varepsilon) \) and then integrating the resulting equation over \( \Omega^\varepsilon \), we arrive at
\[ \left| \int_{\Omega^\varepsilon} f_1 \theta^0 \varphi dx - \varepsilon \int_{\Gamma^\varepsilon_R} (f_2 \theta^\varepsilon + \varepsilon \bar{f}) \varphi dS^\varepsilon \right| = \left| \int_{\Omega^\varepsilon} \Delta_y \Psi (\cdot, x, y) \big|_{y=x}^\varepsilon \varphi dx - \varepsilon \int_{\Gamma^\varepsilon_R} f_2 \theta^\varepsilon \varphi dS^\varepsilon - \varepsilon^2 \int_{\Gamma^\varepsilon_R} \bar{f} \varphi dS^\varepsilon \right|. \]
\[
\int_{\Omega} \varepsilon \left( \nabla_x \left[ \nabla_y \Psi (\cdot, x, y) \right]_{y=\frac{x}{\varepsilon}} - \nabla_x \nabla_y \Psi (\cdot, x, y) \right) \varphi dx - \varepsilon \int_{\Gamma_R} f_2 \theta^\varepsilon \varphi dS_{\varepsilon} - \varepsilon^2 |\Gamma_R|^{-1} \int_{\Gamma_R} f_3 \varphi dS_{\varepsilon}
\]
\[
= \left[ \varepsilon \int_{\Omega} \nabla_y \Psi (\cdot, x, y)_{y=\frac{x}{\varepsilon}} \cdot n \varphi dS_{\varepsilon} - \varepsilon \int_{\Omega} \nabla_y \Psi (\cdot, x, y)_{y=\frac{x}{\varepsilon}} \nabla_x \varphi dx \right] - \varepsilon \int_{\Omega} \nabla_x \nabla_y \Psi (\cdot, x, y)_{y=\frac{x}{\varepsilon}} \varphi dx - \varepsilon \int_{\Gamma_R} f_2 \theta^\varepsilon \varphi dS_{\varepsilon} - \varepsilon^2 |\Gamma_R|^{-1} \int_{\Gamma_R} f_3 \varphi dS_{\varepsilon}
\]
\[
= \left[ \int_{\Omega} f_t \theta^0 \varphi dx - \varepsilon \int_{\Gamma_R} \left( f_2 \theta^\varepsilon + \varepsilon \tilde{f} \right) \varphi dS_{\varepsilon} \right] \leq 
\]
\[
C \varepsilon \| \varphi \|_{H^1(\Omega)}.
\]

This completes the proof of the lemma.

4 Proof of Theorem 12

The proof of Theorem 12 relies on a fine control of the \( \varepsilon \)-dependence needed to estimate each term in (3.6)-(3.13). At first, the term \( \mathcal{I}_1 \) can be rewritten as:

\[
\int_{\Omega} \partial_t \left( \theta^\varepsilon - \theta^0 \right) \left( \theta^\varepsilon - \theta^0 - \varepsilon m^\varepsilon \tilde{\theta} \left( x, \frac{x}{\varepsilon} \right) \cdot \nabla_x \theta^0 \right) = \frac{1}{2} \frac{d}{dt} \left\| \theta^\varepsilon \left( t \right) - \theta^0 \left( t \right) \right\|^2_{L^2(\Omega)} - \varepsilon \int_{\Omega} \partial_t \left( \theta^\varepsilon - \theta^0 \right) m^\varepsilon \tilde{\theta} \left( x, \frac{x}{\varepsilon} \right) \cdot \nabla_x \theta^0 dx.
\]

Similarly, we proceed to estimate \( \mathcal{J}_1^i \) as follows:

\[
\int_{\Omega} \partial_t \left( u_i^\varepsilon - u_i^0 \right) \left( u_i^\varepsilon - u_i^0 - \varepsilon m^\varepsilon \tilde{u}_i \left( x, \frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0 \right) = \frac{1}{2} \frac{d}{dt} \left\| u_i^\varepsilon \left( t \right) - u_i^0 \left( t \right) \right\|^2_{L^2(\Omega)} - \varepsilon \int_{\Omega} \partial_t \left( u_i^\varepsilon - u_i^0 \right) m^\varepsilon \tilde{u}_i \left( x, \frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0 dx.
\]

Using the decomposition

\[
\kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta^0 = \kappa^\varepsilon \nabla \left( \theta^\varepsilon - \theta^0 \right) + \kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta^0,
\]

the term \( \mathcal{I}_2 \) thus becomes

\[
\mathcal{I}_2 = \int_{\Omega} \kappa^\varepsilon \nabla \left( \theta^\varepsilon - \theta^0 \right) \cdot \nabla \varphi dx + \int_{\Omega} \left( \kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta^0 \right) \cdot \nabla \varphi dx.
\]
Concerning the first term on the right-hand side of (4.3), we get

\[
\int_{\Omega^c} \kappa^\varepsilon \nabla (\theta^\varepsilon - \theta^\varepsilon_0) \cdot \nabla \varphi \, dx \ge \frac{K_{\min}}{2} \| \nabla (\theta^\varepsilon - \theta^\varepsilon_0) (t) \|_{L^2(\Omega^c)}^2 - C \varepsilon^2 \| \nabla ((1 - m^\varepsilon) \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0 (t)) \|_{L^2(\Omega^c)}^2.
\]

It is worth pointing out that the cell problems (2.28) and (2.29) require more regularity on the heat conductivity \( \kappa \) and the diffusion coefficient \( d_i \), namely we need \( \kappa, d_i \in H^1(\bar{Y}_1) \). On the other side, since these cell problems are elliptic problems on a non-convex polygon, it is well-known that the cell functions \( \bar{\theta} \) and \( \bar{u}_i \) usually do not belong to \( H^2(\bar{Y}_1) \) in \( y \) no matter how smooth the right-hand sides of (2.28) and (2.29) are (cf. [19]). Due to the extra regularity on \( \kappa \) and \( d_i \) leading to their Lipschitz property in space and due to the Lipschitz boundary of the microstructure, the solutions can be at most in \( H^2_{\text{loc}}(\bar{Y}_1) \) (see, e.g. [19] Theorem 2.2.2.3)). Notably, that result will not change even if the microstructure boundary is very smooth as in this case. We also emphasize that when investigating problems on domains without holes, the cell problems are then considered in the unit cell \( Y \) and by the convexity of that cell, one obtains the regularity of the cell functions up to \( H^3(Y) \).

It follows from [35] Theorem 4] that the cell problems (2.28)-(2.29) admit a unique solution \((\bar{\theta}, \bar{u}_i) \in H^{1+s}_{\#}(Y_1) \times H^{1+r}_{\#}(Y_1)\) for some \( s, r \in (-\frac{1}{2}, \frac{1}{2}) \). Essentially, this hinders us when dealing with the term \( \varepsilon \| \nabla ((1 - m^\varepsilon) \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0 (t)) \|_{L^2(\Omega^c)} \). In fact, we need \( \bar{\theta} \in L^\infty(\Omega^c; \bar{Y}_1) \), whereas its maximal regularity only gives \( L^\infty(\Omega^c; \bar{Y}_1) \) (a similar situation holds for \( \bar{u}_i \)). Recall the Sobolev embedding \( W^{j+s, p}(Y_1) \subset C^j(\bar{Y}_1) \) for \( s > d \) (cf. [11]). Our Hilbertian framework, i.e. \( p = 2, j = 1 \), requires \( s > d/2 \geq 1/2 \) which leads to the impossibility of getting \( C^1_{\#}(Y_1) \) from \( H^{1+s}_{\#}(Y_1) \). Obviously, one of the possibilities is to work with the domain without holes in 1D, i.e. \( d = 1 \) and \( s = 1 \). The fact that \((\bar{\theta}, \bar{u}_i) \in [L^\infty(\Omega^c; W^{1+s, 2}(Y_1))]^2 \) for \( s > d/2 \) is strictly needed to obtain \( (\bar{\theta}, \bar{u}_i) \in [L^\infty(\Omega^c; C^1_{\#}(\bar{Y}_1))]^2 \).

Then, with the assumption \( \theta^0(t, \cdot) \in W^{1, \infty}(\Omega^c) \cap H^2(\Omega^c) \) and the extra regularity \( \bar{\theta} \in H^1(\Omega^c; W^{1, 2}_{\#}(Y_1)) \) providing \( \bar{\theta} \in H^1(\Omega^c; C^1_{\#}(\bar{Y}_1)) \), we estimate that

\[
\varepsilon \| \nabla ((1 - m^\varepsilon) \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0 (t)) \|_{L^2(\Omega^c)} \leq \varepsilon \| \nabla m^\varepsilon \|_{L^2(\Omega^c)} \| \bar{\theta} \|_{L^\infty(\Omega^c; \bar{Y}_1)} \| \theta^0 (t) \|_{W^{1, \infty}(\Omega^c)} + \varepsilon \| \nabla_x \bar{\theta} \|_{L^2(\Omega^c; \bar{Y}_1)} \| \theta^0 (t) \|_{W^{1, \infty}(\Omega^c)} + \| 1 - m^\varepsilon \|_{L^2(\Omega^c)} \| \nabla_y \bar{\theta} \|_{L^\infty(\Omega^c; \bar{Y}_1)} \| \theta^0 (t) \|_{W^{1, \infty}(\Omega^c)} + \varepsilon \| \bar{\theta} \|_{L^\infty(\Omega^c; \bar{Y}_1)} \| \theta^0 (t) \|_{H^2(\Omega^c)} \leq C (\varepsilon + \varepsilon^{1/2}),
\]

where we use the inequalities (2.32) together with the fact that \( \nabla = \nabla_x + \varepsilon^{-1} \nabla_y \).

Observe that

\[
\nabla \theta^\varepsilon = \nabla_x \theta^0 + (\nabla_y \bar{\theta})^\varepsilon \nabla_x \theta^0 + \varepsilon \bar{\theta}^\varepsilon \nabla_x \theta^0 = (\bar{\theta}^\varepsilon \nabla_x \theta^0 + \varepsilon \nabla \theta^\varepsilon \nabla_x \theta^0) + \varepsilon (\nabla_x \bar{\theta})^\varepsilon \nabla_x \theta^0.
\]

(4.4)

Hence, we get

\[
\kappa^\varepsilon \nabla \theta^\varepsilon - \kappa \nabla \theta^0 = \kappa^\varepsilon (\nabla \theta^0 + (\nabla_y \bar{\theta})^\varepsilon \nabla_x \theta^0) - \kappa \nabla \theta^0 + \kappa \varepsilon (\bar{\theta}^\varepsilon \nabla_x \theta^0 + (\nabla_x \bar{\theta})^\varepsilon \nabla_x \theta^0).
\]

(4.5)

We note that the \( L^2 \)-norm of the second term on the right-hand side of (4.5) is bounded from above by

\[
\varepsilon \| \kappa^\varepsilon (\bar{\theta}^\varepsilon \nabla_x \theta^0 + (\nabla_x \bar{\theta})^\varepsilon \nabla_x \theta^0) \|_{L^2(\Omega^c)} \leq C \varepsilon \| \bar{\theta} \|_{L^\infty(\Omega^c; \bar{Y}_1)} \| \theta^0 \|_{H^2(\Omega^c)} + C \varepsilon \| \nabla_x \bar{\theta} \|_{L^2(\Omega^c; \bar{Y}_1)} \| \theta^0 \|_{H^{1, \infty}(\Omega^c)}.
\]
Let us handle now the following quantity $\kappa^\varepsilon \left( \nabla \theta^0 + (\nabla y \tilde{\theta})^\varepsilon \nabla x \theta^0 \right) - \mathbb{K} \nabla \theta^0$. In fact, recall that $\mathcal{G} := \kappa (I + \nabla y \tilde{\theta}) - \mathbb{K}$ is divergence-free with respect to $y \in Y_1$ due to the structure of the cell problems in Theorem 10. Moreover, we know that its average also vanishes, i.e.

$$\int_{Y_1} \mathcal{G} \, dy = 0,$$

by virtue of the definition of the homogenized heat conductivity $\mathbb{K}$ in Theorem 8.

As a consequence, $\mathcal{G}$ possesses a vector potential $V$ and this vector potential is skew-symmetric such that $\mathcal{G} = \nabla_y V$. In general, the selection of the vector potential is non-unique. However, we can choose $V$ to solve the Poisson equation $\Delta_y V = \eta(x, y) \nabla_y \mathcal{G}$ for some function $\eta$ just depending on the dimensions. Using this equation together with the periodic boundary conditions at $\partial Y_0$ and the vanishing cell average, we can determine this vector potential $V$ uniquely. Now, we formulate the quantity $\mathcal{G}^\varepsilon \nabla \theta^0 = \kappa^\varepsilon \left( \nabla \theta^0 + (\nabla y \tilde{\theta})^\varepsilon \nabla x \theta^0 \right) - \mathbb{K} \nabla \theta^0$ in terms of this vector potential. Using the relation that $\nabla_y = \varepsilon \nabla - \varepsilon \nabla V$, we have

$$\mathcal{G}^\varepsilon \nabla \theta^0 = \varepsilon \nabla \cdot \left( V^\varepsilon \nabla \theta^0 \right) - \varepsilon V^\varepsilon \Delta \theta^0 - \varepsilon (\nabla_x V)^\varepsilon \nabla \theta^0. \quad (4.6)$$

Due to the skew-symmetry of $V$ (and also that of $V^\varepsilon$), the first term on the right-hand side of (4.6) is divergence-free, indicating the boundedness in $L^2(\Omega^\varepsilon)$ with the order of $O(\varepsilon)$. In addition, combining $\tilde{\theta} \in L^\infty(\Omega^\varepsilon; W^{1+s,2}_\#(Y_1)) \cap H^1(\Omega^\varepsilon; W^{s,2}_\#(Y_1))$ with the above Poisson equation $\Delta_y V = \eta(x, y) \nabla_y \mathcal{G}$ yields

$$\|V\|_{W^{1+s,2}(Y_1)} \leq C \|\mathcal{G}\|_{W^{s,2}(Y_1)}.$$

By the compact embedding $W^{s,2}(Y_1) \subset C(Y_1)$ for $s > d/2 \geq 1$, we thus get

$$V \in L^\infty \left( \Omega^\varepsilon; C_\# \left( Y_1 \right) \right) \cap H^1 \left( \Omega^\varepsilon; C_\# \left( Y_1 \right) \right).$$

As a consequence, the boundedness in $L^2(\Omega^\varepsilon)$ of the second and third terms on the right-hand side of (4.6) is given by

$$\varepsilon \|V^\varepsilon \Delta \theta^0 + (\nabla_x V)^\varepsilon \nabla \theta^0\|_{L^2(\Omega^\varepsilon)} \leq \varepsilon \|V\|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \|\theta^0\|_{H^2(\Omega^\varepsilon)} + \varepsilon \|V\|_{H^1(\Omega^\varepsilon; C(Y_1))} \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)}.$$

Therefore, with the help of the Hölder inequality, we note that

$$\int_{\Omega^\varepsilon} (\kappa^\varepsilon \nabla \theta^0 - \mathbb{K} \nabla \theta^0) \cdot \nabla \varphi \, dx \leq C \varepsilon,$$

which completes the estimates for $\mathcal{I}_2$.

Consequently, we can write

$$\mathcal{I}_2 \geq C \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{L^2(\Omega^\varepsilon)}^2 - C \left( \varepsilon^2 + \varepsilon \right). \quad (4.7)$$

Similarly, estimating the term $\mathcal{J}_2'$ leads to

$$\mathcal{J}_2' \geq C \|\nabla (u^\varepsilon - u_{1,1}^\varepsilon)(t)\|_{L^2(\Omega^\varepsilon)}^2 - C \left( \varepsilon^2 + \varepsilon \right). \quad (4.8)$$

Concerning the estimate of the term $\mathcal{I}_3$, we note the following: Thanks to the compatibility constraint (Theorem 15) with the choice $\varphi = \theta^\varepsilon - \theta^0$, we get that

$$\mathcal{I}_3 \leq C \varepsilon \|\varphi\|_{H^1(\Omega^\varepsilon)}$$

$$\leq C \varepsilon \left( \|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)} + \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{L^2(\Omega^\varepsilon)} + \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{L^2(\Omega^\varepsilon)} \right)$$

$$\leq C \varepsilon \left( \|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)} + \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{L^2(\Omega^\varepsilon)} + C(1 + \varepsilon) \right), \quad (4.9)$$
where we use again the difference relation (4.4) and get the following bound from above

\[
\| \nabla (\theta^\varepsilon_1 - \theta^0) \|_{L^2(\Omega^\varepsilon)} \leq \| \nabla \bar{\theta} \|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \| \theta^0 \|_{W^{1,\infty}(\Omega^\varepsilon)} \\
+ \varepsilon \left( \| \bar{\theta} \|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \| \theta^0 \|_{H^2(\Omega^\varepsilon)} + \| \nabla x \bar{\theta} \|_{L^2(\Omega^\varepsilon; C(Y_1))} \| \theta^0 \|_{W^{1,\infty}(\Omega^\varepsilon)} \right).
\]

Similarly, the term \( J_3^i \) is bounded from above by

\[
J_3^i \leq C \varepsilon \left( \| u^\varepsilon_i - u^0_i \|_{L^2(\Omega^\varepsilon)} + \| \nabla (u^\varepsilon_i - u^0_i) \|_{L^2(\Omega^\varepsilon)} \right)^d + C(1 + \varepsilon). \tag{4.10}
\]

Note the elementary decomposition:

\[
\tau^\varepsilon \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon - (T^i \nabla^\varepsilon u^\varepsilon_i) \cdot \nabla \theta^0 = \tau^\varepsilon - T^i \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon \\
+ T^i \left( \nabla^\varepsilon u^\varepsilon_i - \nabla^\varepsilon u^0_i \right) \cdot \nabla \theta^0 + \nabla \theta^0 \cdot \nabla \theta^0.
\]

Multiplying the above equation by the test function \( \varphi \), we arrive at

\[
(\tau^\varepsilon \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon - (T^i \nabla^\varepsilon u^\varepsilon_i) \cdot \nabla \theta^0) \varphi = \tau^\varepsilon - T^i \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon (\theta^\varepsilon - \theta^0) \\
- \varepsilon (\tau^\varepsilon - T^i) \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon m^\varepsilon \bar{\theta} \cdot \nabla x \theta^0 \\
+ T^i \left( \nabla^\varepsilon u^\varepsilon_i - \nabla^\varepsilon u^0_i \right) \cdot \nabla \theta^0 (\theta^\varepsilon - \theta^0) \\
- \varepsilon T^i \left( \nabla \theta^\varepsilon - \nabla \theta^0 \right) \cdot \nabla \theta^0 (\theta^\varepsilon - \theta^0) \\
- \varepsilon T^i \left( \nabla \theta^\varepsilon - \nabla \theta^0 \right) \cdot \nabla \theta^0 (\theta^\varepsilon - \theta^0)
\]

\[
\quad = \sum_{k=1}^6 I^k_4.
\]

To be able to estimate \( I_4 \), we need to ensure the boundedness of each of the terms \( \int_{\Omega^\varepsilon} I^k_4 \) for \( k_i \in \{1, \ldots, 6 \} \) and \( i \in \{1, \ldots, N\} \). We obtain:

\[
\int_{\Omega^\varepsilon} |I^k_4| \, dx \leq \varepsilon \| \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \left( \| \tau^\varepsilon - T^i \| m^\varepsilon \bar{\theta} \left( \frac{x}{\varepsilon} \right) \cdot \nabla x \theta^0 \right) \leq \varepsilon \| u^\varepsilon_i \|_{L^\infty(\Omega^\varepsilon)} \| \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \| \bar{\theta} \|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \| \theta^0 \|_{W^{1,\infty}(\Omega^\varepsilon)} \| \tau^\varepsilon - T^i \|_{L^2(\Omega^\varepsilon)}, \tag{4.11}
\]

and

\[
\int_{\Omega^\varepsilon} |I^4_4| \, dx \leq \frac{\varepsilon}{2} \| \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \left( \| \tau^\varepsilon - T^i \| \| m^\varepsilon \bar{\theta} \left( \frac{x}{\varepsilon} \right) \cdot \nabla x \theta^0 \|_{L^2(\Omega^\varepsilon)} \right) \leq \frac{\varepsilon}{2} \| \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \leq \varepsilon \| u^\varepsilon_i - u^0_i \|_{L^2(\Omega^\varepsilon)} \| \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \| \bar{\theta} \|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \| \theta^0 \|_{W^{1,\infty}(\Omega^\varepsilon)} \leq C \varepsilon. \tag{4.12}
\]

Furthermore, we estimate

\[
\int_{\Omega^\varepsilon} |I^4_4| \, dx \leq \| \tau^\varepsilon - T^i \|_{L^2(\Omega^\varepsilon)} \| \nabla^\varepsilon u^\varepsilon_i \cdot \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \| \theta^\varepsilon - \theta^0 \|_{L^\infty(\Omega^\varepsilon)} \leq C \varepsilon \| \tau^\varepsilon - T^i \|_{L^2(\Omega^\varepsilon)} \| u^\varepsilon_i \|_{L^\infty(\Omega^\varepsilon)} \| \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \| \bar{\theta} \|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \| \theta^0 \|_{W^{1,\infty}(\Omega^\varepsilon)} \leq C \varepsilon. \tag{4.13}
\]

and by Young’s inequality, it yields

\[
\int_{\Omega^\varepsilon} |I^3_4| \, dx \leq \frac{\| \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \| u^\varepsilon_i - u^0_i \|_{L^2(\Omega^\varepsilon)} \| \bar{\theta} \|_{L^\infty(\Omega^\varepsilon)} \| \theta^0 \|_{W^{1,\infty}(\Omega^\varepsilon)}}{2} + \frac{1}{2} \| \theta^\varepsilon - \theta^0 \|_{L^2(\Omega^\varepsilon)} \leq \frac{\| \nabla \theta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \| u^\varepsilon_i - u^0_i \|_{L^2(\Omega^\varepsilon)} \| \bar{\theta} \|_{L^\infty(\Omega^\varepsilon)} \| \theta^0 \|_{W^{1,\infty}(\Omega^\varepsilon)}}{2} + \frac{1}{2} \| \theta^\varepsilon - \theta^0 \|_{L^2(\Omega^\varepsilon)} \leq C \varepsilon. \tag{4.14}
\]
and
\[
\int_{\Omega^c} |T_3^\varepsilon| \, dx \leq \frac{|T_4|^2}{2} \left\| \left( \nabla \theta^\varepsilon - \nabla \theta^0 \right) \cdot \nabla^d u^0_i \right\|_{L^2(\Omega^c)}^2 + \frac{1}{2} \left\| \theta^\varepsilon - \theta^0 \right\|_{L^2(\Omega^c)}^2
\]
\[
\leq \frac{|T_4|^2}{2} C_\delta \left\| u^0_i \right\|_{L^2(\Omega^c)}^2 \left\| \nabla \theta^\varepsilon - \nabla \theta^0 \right\|_{L^2(\Omega^c)}^2 + \frac{1}{2} \left\| \theta^\varepsilon - \theta^0 \right\|_{L^2(\Omega^c)}^2, \quad (4.15)
\]
\[
\int_{\Omega^c} |T_4^\varepsilon| \, dx \leq \frac{\varepsilon |T_4|^2}{2} \left\| u^0_i \right\|_{L^2(\Omega^c)}^2 \left\| \nabla \theta^\varepsilon - \nabla \theta^0 \right\|_{L^2(\Omega^c)}^2 + \varepsilon \left\| \bar{\theta} \right\|_{L^\infty(\Omega^c; C(\bar{Y}_1))} \left\| \theta^0 \right\|_{W^{1,\infty}(\Omega^c)}^2. \quad (4.16)
\]

Remark that the first integral in $J^i_4$ can be estimated similarly. On top of that, observe that we can find constants $C_{Ri} > 0$ (independent of $\varepsilon$) such that
\[
\left\| R_i \left( u^\varepsilon \right) - R_i \left( u^0 \right) \right\|_{L^2(\Omega^c)} \leq C_{Ri} \sum_{j=1}^N \left\| u_j^\varepsilon - u_j^0 \right\|_{L^2(\Omega^c)} \quad \text{for } i \in \{1, \ldots, N\},
\]
in which the constants $C_{Ri}$ depend on the $L^\infty$-bounds of the concentrations $u^\varepsilon, u^0$ as discussed in [22 Section 5].

The estimate on the second integral of $J^i_4$ can be computed directly. Note that for $i \in \{1, \ldots, N\}$, we have:
\[
\left( R_i \left( u^\varepsilon \right) - R_i \left( u^0 \right) \right) \phi_i = \left( R_i \left( u^\varepsilon \right) - R_i \left( u^0 \right) \right) \left( u_i^\varepsilon - u_i^0 \right)
\]
\[
= -\varepsilon \left( R_i \left( u^\varepsilon \right) - R_i \left( u^0 \right) \right) m^\varepsilon \vec{u}_i \left( \frac{x_j}{\varepsilon} \right) \cdot \nabla u_i^0.
\]

This gives
\[
\int_{\Omega^c} \left( R_i \left( u^\varepsilon \right) - R_i \left( u^0 \right) \right) \phi_i \, dx \leq C_{Ri} \sum_{j=1}^N \left\| u_j^\varepsilon - u_j^0 \right\|_{L^2(\Omega^c)} \left( \left\| u_j^\varepsilon - u_j^0 \right\|_{L^2(\Omega^c)} + \right.
\]
\[
\left. + \varepsilon \left\| \bar{u}_i \right\|_{L^\infty(\Omega^c; C(\bar{Y}_1))} \left\| u_i^0 \right\|_{W^{1,\infty}(\Omega^c)} \right). \quad (4.17)
\]

Collecting the estimates (4.7), (4.8), (4.9), (4.10), (4.11)-(4.16) and (4.17), we obtain:
\[
\left\| \nabla \left( \theta^\varepsilon - \theta^0 \right) \left( t \right) \right\|_{L^2(\Omega^c)}^2 + \sum_{i=1}^N \left\| \nabla \left( u_i^\varepsilon - u_i^0 \right) \left( t \right) \right\|_{L^2(\Omega^c)}^2
\]
\[
\leq C (\varepsilon^2 + \varepsilon) + C \varepsilon \left( \left\| \theta^\varepsilon \left( t \right) - \theta^0 \left( t \right) \right\|_{L^2(\Omega^c)} + \left\| \nabla \left( \theta^\varepsilon - \theta^0 \right) \left( t \right) \right\|_{L^2(\Omega^c)} + C \left( 1 + \varepsilon \right) \right)
\]
\[
+ C \varepsilon \sum_{i=1}^N \left( \left\| u_i^\varepsilon \left( t \right) - u_i^0 \left( t \right) \right\|_{L^2(\Omega^c)} + \left\| \nabla \left( u_i^\varepsilon - u_i^0 \right) \left( t \right) \right\|_{L^2(\Omega^c)} + C \left( 1 + \varepsilon \right) \right)
\]
\[
+ C \varepsilon \left( \sum_{i=1}^N \left\| u_i^\varepsilon \left( t \right) - u_i^0 \left( t \right) \right\|_{L^2(\Omega^c)} + \left\| \theta^\varepsilon \left( t \right) - \theta^0 \left( t \right) \right\|_{L^2(\Omega^c)} \right)
\]
\[
+ C \left( \sum_{i=1}^N \left\| u_i^\varepsilon \left( t \right) - u_i^0 \left( t \right) \right\|_{L^2(\Omega^c)} + \left\| \theta^\varepsilon \left( t \right) - \theta^0 \left( t \right) \right\|_{L^2(\Omega^c)} \right)
\]
\[
+ C \varepsilon \left( \left\| \nabla \left( \theta^\varepsilon - \theta^0 \right) \left( t \right) \right\|_{L^2(\Omega^c)}^2 + \sum_{i=1}^N \left\| \nabla \left( u_i^\varepsilon - u_i^0 \right) \left( t \right) \right\|_{L^2(\Omega^c)}^2 \right) + C \varepsilon.
\]
Notably, Theorem 14 provides us that the $L^2$-error estimates between the Soret and Dufour coefficients and their homogenized (averaged) versions, i.e. $\| \tau^\varepsilon - \tau^0 \|_{L^2(\Omega^\varepsilon)}$ and $\| \rho^\varepsilon - \rho^0 \|_{L^2(\Omega^\varepsilon)}$ are of the order $\mathcal{O}(\varepsilon^{1/2})$. It thus yields that

\[
\| \nabla (\theta^\varepsilon - \theta^0) (t) \|^2_{L^2(\Omega^\varepsilon)} + \sum_{i=1}^{N} \| \nabla (u_i^\varepsilon - u_i^0) (t) \|^2_{L^2(\Omega^\varepsilon)} \\
\leq C \left( \varepsilon^2 + \varepsilon \right) + C \varepsilon \left( \| \theta^\varepsilon (t) - \theta^0 (t) \|_{L^2(\Omega^\varepsilon)} + \| \nabla (\theta^\varepsilon - \theta^0) (t) \|_{L^2(\Omega^\varepsilon)} \right) \\
+ C \varepsilon \sum_{i=1}^{N} \left( \| u_i^\varepsilon (t) - u_i^0 (t) \|_{L^2(\Omega^\varepsilon)} + \| \nabla (u_i^\varepsilon - u_i^0,1) (t) \|_{L^2(\Omega^\varepsilon)} \right) \\
+ C \varepsilon^{1/2} \left( \| \theta^\varepsilon (t) - \theta^0 (t) \|_{L^2(\Omega^\varepsilon)} + \sum_{i=1}^{N} \| u_i^\varepsilon (t) - u_i^0 (t) \|_{L^2(\Omega^\varepsilon)} \right) \\
+ C \left( \sum_{i=1}^{N} \| u_i^\varepsilon (t) - u_i^0 (t) \|^2_{L^2(\Omega^\varepsilon)} + \| \theta^\varepsilon (t) - \theta^0 (t) \|^2_{L^2(\Omega^\varepsilon)} \right) \\
+ C \varepsilon \left( \| \nabla (\theta^\varepsilon - \theta^0) (t) \|^2_{L^2(\Omega^\varepsilon)} + \sum_{i=1}^{N} \| \nabla (u_i^\varepsilon - u_i^0,1) (t) \|^2_{L^2(\Omega^\varepsilon)} \right). 
\]

(4.18)

It now remains to estimate the second term on the right-hand side of (4.11)–(4.12). In fact, integrating by parts gives

\[
\int_0^t \int_{\Omega^\varepsilon} m^\varepsilon \partial_t (u_i^\varepsilon - u_i^0) \bar{u}_i \left( \frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0 (s, x) \, dx \, ds = \int_{\Omega^\varepsilon} m^\varepsilon (u_i^\varepsilon - u_i^0) \bar{u}_i \left( \frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0 (s, x) \, dx \bigg|_{s=0}^{s=t} \\
- \int_0^t \int_{\Omega^\varepsilon} m^\varepsilon (u_i^\varepsilon - u_i^0) \bar{u}_i \left( \frac{x}{\varepsilon} \right) \cdot \nabla_x \partial_t u_i^0 (s, x) \, dx \, ds.
\]

We then observe that

\[
\varepsilon \left| \int_{\Omega^\varepsilon} m^\varepsilon \left[ (u_i^\varepsilon - u_i^0) - (u_i^\varepsilon (0) - u_i^0 (0)) \right] \bar{u}_i \cdot \nabla_x u_i^0 (t, x) \, dx \right| \\
\leq C \varepsilon \left( \| u_i^\varepsilon (t) - u_i^0 (t) \|_{L^2(\Omega^\varepsilon)} + \| u_i^\varepsilon,0 - u_i^0,0 \|_{L^2(\Omega^\varepsilon)} \right),
\]

and hence,

\[
\varepsilon \left| \int_{\Omega^\varepsilon} m^\varepsilon \left[ (\theta^\varepsilon - \theta^0) - (\theta^\varepsilon (0) - \theta^0 (0)) \right] \bar{\theta} \cdot \nabla_x \theta^0 (t, x) \, dx \right| \\
\leq C \varepsilon \left( \| \theta^\varepsilon (t) - \theta^0 (t) \|_{L^2(\Omega^\varepsilon)} + \| \theta^\varepsilon,0 - \theta^0,0 \|_{L^2(\Omega^\varepsilon)} \right).
\]

For all $t \in (0, T]$, we set

\[
w_1 (t) = \| \theta^\varepsilon (t) - \theta^0 (t) \|^2_{L^2(\Omega^\varepsilon)} + \sum_{i=1}^{N} \| u_i^\varepsilon (t) - u_i^0 (t) \|^2_{L^2(\Omega^\varepsilon)},
\]

\[
w_2 (t) = \| \nabla (\theta^\varepsilon - \theta^0) (t) \|^2_{L^2(\Omega^\varepsilon)},
\]

\[
w_0 = \| \theta^\varepsilon,0 - \theta^0,0 \|^2_{L^2(\Omega^\varepsilon)} + \sum_{i=1}^{N} \| u_i^\varepsilon,0 - u_i^0,0 \|^2_{L^2(\Omega^\varepsilon)}.
\]
Then, when integrating (4.18) and (4.1)-(4.2) from 0 to \( t \), we are led to the following Gronwall-like estimate

\[
w_1(t) + \int_0^t w_2(s) \, ds \leq C \left( \varepsilon^2 + \varepsilon + (1 + \varepsilon) \, w_0 + \varepsilon \int_0^t w_1(s) \, ds \right),
\]

which can be rewritten as

\[
w_1(t) + \int_0^t w_2(s) \, ds \leq C \left( \varepsilon + (1 + \varepsilon) \, w_0 \right) e^{C \varepsilon t} \quad \text{for} \ t \in [0,T]. \tag{4.19}
\]

Finally, we turn our attention to the corrector estimate for \( v^*_i \). For \( i \in \{1,\ldots,N\} \) we consider the equation for the reconstruction \( v^*_i = v^i_0 \), obtained from (2.18), with the test function \( \psi_i \in L^2(\Gamma^\varepsilon) \) and integrate the resulting equation over \( \Gamma^\varepsilon \) to get

\[
\varepsilon \int_{\Gamma^\varepsilon} \partial_t v^i_0 \psi_i dS_\varepsilon = \varepsilon \int_{\Gamma^\varepsilon} (A_i u^i_0 - B_i v^0_0) \psi_i dS_\varepsilon. \tag{4.20}
\]

Then, we find the difference equation for the micro concentration \( v^*_i \) and the reconstruction \( v^i_0 \) by subtracting the third equation of (2.13) and (4.20), provided that

\[
\varepsilon \int_{\Gamma^\varepsilon} \partial_t (v^*_i - v^i_0) \psi_i dS_\varepsilon = \varepsilon \int_{\Gamma^\varepsilon} (a^\varepsilon_i u^i_0 - A_i u^0_0) \psi_i dS_\varepsilon - \varepsilon \int_{\Gamma^\varepsilon} (b^\varepsilon_i v^*_i - B_i v^0_0) \psi_i dS_\varepsilon
\]

\[
- \varepsilon \int_{\Gamma^\varepsilon} (a^\varepsilon_i - A_i) (u^i_0 - v^0_0) \psi_i dS_\varepsilon.
\]

Hence, we choose \( \psi_i = v^*_i - v^0_i \) to obtain the following estimate

\[
\frac{\varepsilon}{2} \frac{d}{dt} \left\| v^*_i - v^0_i \right\|^2_{L^2(\Gamma^\varepsilon)} \leq C \varepsilon \left( \left\| u^i_0 - v^0_0 \right\|^2_{L^2(\Gamma^\varepsilon)} + \left\| v^*_i - v^0_i \right\|^2_{L^2(\Gamma^\varepsilon)} \right)
\]

\[
+ \varepsilon \int_{\Gamma^\varepsilon} \left| a^\varepsilon_i - A_i \right| \left| u^i_0 - v^0_0 \right| \left| v^*_i - v^0_i \right| dS_\varepsilon + \varepsilon \int_{\Gamma^\varepsilon} \left| b^\varepsilon_i - B_i \right| \left| v^0_0 \right| \left| v^*_i - v^0_i \right| dS_\varepsilon. \tag{4.21}
\]

Since \( \Omega^\varepsilon \) is a Lipschitz domain, we recall the trace embedding \( H^1(\Omega^\varepsilon) \subset L^q(\partial\Omega^\varepsilon) \) which holds for \( 1 \leq q \leq 2^*_\partial\Omega^\varepsilon = 2(d-1)/(d-2) \) if \( d \geq 3 \), and \( 2^*_\partial\Omega^\varepsilon = \infty \) if \( d = 2 \) (cf. [13]). Therefore, when the two-dimensional case is concentrated, we conclude to estimate (4.21), as follows:

\[
\frac{\varepsilon}{2} \frac{d}{dt} \sum_{i=1}^N \left\| v^*_i - v^0_i \right\|^2_{L^2(\Gamma^\varepsilon)} \leq C \varepsilon \left( \sum_{i=1}^N \left\| u^i_0 - u^0_0 \right\|^2_{L^2(\Gamma^\varepsilon)} + \sum_{i=1}^N \left\| v^*_i - v^0_i \right\|^2_{L^2(\Gamma^\varepsilon)} \right)
\]

\[
+ C \varepsilon \left( \sum_{i=1}^N \left| a^\varepsilon_i - A_i \right|^2_{L^2(\Gamma^\varepsilon)} + \sum_{i=1}^N \left| b^\varepsilon_i - B_i \right|^2_{L^2(\Gamma^\varepsilon)} \right).
\]

Observe that using the trace inequality (2.34) for the difference norms \( \left\| a^\varepsilon_i - A_i \right\|_{L^2(\Gamma^\varepsilon)} \) and \( \left\| u^i_0 - v^0_0 \right\|_{L^2(\Gamma^\varepsilon)} \), together with Lemma 14 and (4.19) gives

\[
\frac{\varepsilon}{2} \frac{d}{dt} \sum_{i=1}^N \left\| v^*_i - v^0_i \right\|^2_{L^2(\Gamma^\varepsilon)} \leq C \varepsilon \max \{ \varepsilon, \varepsilon^\gamma \} + C \varepsilon \sum_{i=1}^N \left\| v^*_i - v^0_i \right\|^2_{L^2(\Gamma^\varepsilon)}. \tag{4.22}
\]

Note herein that the gradient norms are ignored when applying the trace inequality to the differences. It is simply because that they are of the order \( O(\varepsilon^2) \) by their own regularity.
Henceforward, we apply the Gronwall inequality to (4.22) and obtain
\[
\varepsilon \sum_{i=1}^{N} \| v_i^\varepsilon - v_i^0 \|_{L_2(\Gamma^\varepsilon)}^2 \leq C \max \{ \varepsilon, \varepsilon^\gamma \} e^{C\varepsilon t}.
\]

In the same manner, if \( d \geq 3 \) is applied, we can bound the absolute differences \( |a_i^\varepsilon - A_i| \) and \( |b_i^\varepsilon - B_i| \) in (4.21) from above by a constant \( C \) independent of \( \varepsilon \) (by (A1)) and then get back the estimate (4.22). This completes the proof of Theorem 12.

5 Conclusions

In this work, we have presented corrector estimates for the homogenization limit for a thermo-diffusion system with Smoluchowski interactions coupled with a system of differential equations, posed in a perforated domain. This type of error-control justifies the formal homogenization asymptotics obtained in [25] and completes the convergence result in [24] by giving convergence rates. This is done using the concept of macroscopic reconstruction together with fine integral estimates on the solution and oscillating coefficients. Our working technique can be applied to a larger class of coupled nonlinear systems of partial differential equations posed in perforated media.

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