$C^{k,\alpha}$ estimates of the $\bar{\partial}$ equation on product domains

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Abstract

This paper concerns a solution operator to the $\bar{\partial}$ equation on products of planar domains. We show that given a smooth data, there is a smooth solution on such domains. The Hölder estimate of the solution is also obtained at each Hölder level. Indeed, an example of Stein and Kerzman indicates solutions with respect to $L^\infty$ data on product domains do not gain regularity to be in any Hölder space. Similar examples are constructed to reveal the $\bar{\partial}$ problem on product domains has no additional gain of regularity in Hölder spaces as well. In particular, making use of an integral representation of the solutions, we show that for the $\bar{\partial}$ problem on product domains, given a $C^{k,\alpha}$ data, $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha \leq 1$, there is a Hölder solution in $C^{k,\alpha'}$ for any $0 < \alpha' < \alpha$ with the desired estimates.

1 Introduction and the main theorems

Let $D_j \subset \mathbb{C}, j = 1, \ldots, n$, be bounded domains in the complex plane, $n \geq 2$. In particular, each $\partial D_j$ consists of a finite number of rectifiable Jordan curves which do not intersect one another throughout the rest of the paper. Consider the product domain $\Omega := D_1 \times \cdots \times D_n$ in $\mathbb{C}^n$. Then $\Omega$ is a bounded pseudoconvex domain (but not convex in general) with at most Lipschitz boundary. In this paper, we prove the following theorem.

**Theorem 1.1.** Let $D_j \subset \mathbb{C}, j = 1, \ldots, n$, be bounded domains with $C^\infty$ boundary, $n \geq 2$, and $\Omega := D_1 \times \cdots \times D_n$. Assume $f = \sum_{j=1}^n f_j d\bar{z}_j \in C^\infty(\overline{\Omega})$ is a $\bar{\partial}$-closed $(0,1)$ form on $\Omega$. Then there exists a solution $u \in C^\infty(\overline{\Omega})$ to $\bar{\partial} u = f$ in $\Omega$.

The existence and regularity of the Cauchy-Riemann equations have been thoroughly studied in literature along the line of Hörmander’s $L^2$ theory. An alternative approach is to

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express solutions in integral representations. Through a series of work including Grauert-Lieb [9], Henkin [10], Kerzman [13], Henkin-Romanov [12] and Diederich-Fischer-Fornæss [4], supnorm and Hölder estimates of solutions were established for sufficiently smooth bounded domains which are strictly convex, or strongly pseudoconvex, or convex of finite type. For the first case of non-smooth domains, for instance, products of bounded planar domains, Henkin [11] derived an integral representation of a solution operator and proved the supnorm estimate on bidisc for $C^1$ data.

Recently, the $\overline{\partial}$ equations on product domains have attracted much attention. Chen-McNeal [2] studied a type of $L^p$-Sobolev estimates for product domains in $\mathbb{C}^2$ and further gave a simple example showing that Henkin’s solution operator is unbounded in $L^p, 1 \leq p < 2$. For product domains of arbitrary dimensions, Fassina-Pan [8] constructed a solution operator through one-dimensional method, from which they obtained $L^\infty$ estimates for smooth data. See also Ehsani[7], Chakrabarti-Shaw[1], Dong-Li-Treuer [5] and the references therein for investigation of the canonical solutions on product domains.

In this paper, we seek for Hölder solutions on product domains in terms of an integral representation. A natural question is, given a Hölder data on product domains, whether a Hölder solution of the same regularity level exists. In [3], solutions in some nonstandard Hölder spaces were introduced with estimates that require higher order derivatives of the data.

We should point out, unlike strictly pseudoconvex smooth domains, the $\overline{\partial}$ problem on product domains does not gain regularity. Indeed, Stein and Kerzman [13] constructed an example with $L^\infty$ data that has no Hölder solutions on bidisc. Motivated by this, one can similarly construct examples satisfying the following conditions, proving the $\overline{\partial}$ problem on product domains in general has no gain of regularity in Hölder spaces. The examples are verified at the end of Section 4.

**Example 1.2.** Let $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the bidisc. For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \alpha < 1$, consider $\overline{\partial} u = f := \overline{\partial}((z_1 - 1)^{k+\alpha} \overline{z}_2)$ on $\Delta^2, \frac{1}{2}\pi < \arg(z_1 - 1) < \frac{3}{2}\pi$. Then $f = (z_1 - 1)^{k+\alpha} d\overline{z}_2 \in C^{k,\alpha}(\Delta^2)$ is a $\overline{\partial}$-closed $(0,1)$ form. However, there does not exist a solution $u \in C^{k,\alpha'}(\Delta^2)$ to $\overline{\partial} u = f$ on $\Delta^2$ for any $\alpha' > \alpha$.

**Example 1.3.** Let $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the bidisc. For each $k \in \mathbb{Z}^+ \cup \{0\}$, consider $\overline{\partial} u = f := \overline{\partial}((z_1 - 1)^{k+1} \overline{z}_2) / \log(z_1 - 1)$ on $\Delta^2, \frac{1}{2}\pi < \arg(z_1 - 1) < \frac{3}{2}\pi$. Then $f = ((z_1 - 1)^{k+1} / \log(z_1 - 1)) d\overline{z}_2 \in C^{k,1}(\Delta^2)$ is a $\overline{\partial}$-closed $(0,1)$ form. However, there does not exist a solution $u \in C^{k+1,\alpha}(\Delta^2)$ to $\overline{\partial} u = f$ on $\Delta^2$ for any $\alpha > 0$.

Here $C^{k,\alpha}(\Omega)$ is the standard Hölder space (see Section 2 for the definition), and a $(0,1)$ form $f$ is said to be in $C^{k,\alpha}(\Omega)$ if all its coefficients are in $C^{k,\alpha}(\Omega)$. 

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Motivated by the solution formula in [8], we observe a solution operator consisting of compositions of solid and boundary Cauchy type integrals on product domains. By slicing down to planar domains and deriving sharp Hölder estimates of those Cauchy type integrals, we are able to obtain an estimate of the solution in Hölder spaces with a loss of regularity that can be made arbitrarily small. Our main theorem is stated as follows.

**Theorem 1.4.** Let $D_j \subset \mathbb{C}$, $j = 1, \ldots, n$, be bounded domains with $C^{k+1, \alpha}$ boundary, $n \geq 2$, $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha \leq 1$, and let $\Omega := D_1 \times \cdots \times D_n$. Assume $f = \sum_{j=1}^{n} f_j d\bar{z}_j \in C^{k, \alpha}(\Omega)$ is a $\bar{\partial}$-closed $(0,1)$ form on $\Omega$ (in the sense of distributions if $k = 0$). There exists a solution operator $T$ to $\bar{\partial} u = f$ such that for any $0 < \alpha' < \alpha$, $Tf \in C^{k, \alpha'}(\Omega)$, $\bar{\partial} Tf = f$ (in the sense of distributions if $k = 0$) and $\|Tf\|_{C^{k, \alpha'}(\Omega)} \leq C\|f\|_{C^{k, \alpha}(\Omega)}$, where $C$ depends only on $\Omega, k, \alpha$ and $\alpha'$.

It is desirable to know whether there exists a solution operator that can achieve the same regularity as that of data in Hölder spaces. However, we do not have answers at this point. As a direct consequence of Theorem 1.4, we obtain the following regularity theorem for smooth $(0,1)$ forms up to the boundary, from which Theorem 1.1 follows immediately.

**Theorem 1.5.** Let $D_j \subset \mathbb{C}$, $j = 1, \ldots, n$, be bounded domains with $C^\infty$ boundary, $n \geq 2$, and $\Omega := D_1 \times \cdots \times D_n$. Assume $f = \sum_{j=1}^{n} f_j d\bar{z}_j \in C^\infty(\overline{\Omega})$ is a $\bar{\partial}$-closed $(0,1)$ form on $\Omega$. There exists a solution $u \in C^\infty(\overline{\Omega})$ to $\bar{\partial} u = f$ in $\Omega$. Moreover, $\|u\|_{C^0(\Omega)} \leq C\|f\|_{C^0(\Omega)}$, and for all $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha' < \alpha \leq 1$, $\|u\|_{C^{k, \alpha'}(\overline{\Omega})} \leq C_{k,\alpha,\alpha'}\|f\|_{C^{k, \alpha}(\overline{\Omega})}$, where $C$ depends only on $\Omega$, and $C_{k,\alpha,\alpha'}$ depends only on $\Omega, k, \alpha$ and $\alpha'$.

The rest of the paper is organized as follows. In Section 2, preliminaries about solid and boundary Cauchy type integrals on the complex plane are stated with references. In Section 3, the solution operator on product domains is introduced, along with the proof of Theorem 1.4 in the case when $k \in \mathbb{Z}^+$ and Theorem 1.5. The last section is devoted to the remaining case of Theorem 1.4 when $k = 0$. A convergence result of the mollifier method in Hölder spaces is proved in Appendix.

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## 2 Notations and Cauchy-Riemann operators in $\mathbb{C}$

As a common notice, letters $k, \alpha, \alpha'$ throughout the paper are always referred to (part of) the indices of Hölder spaces. Depending on the context, $\gamma$ is either a positive integer or an
n-tuple. $u$ and $f$ represent functions, and the boldface $f$ represents a $(0,1)$ form. Unless otherwise specified, we use $C$ to represent a constant dependent only on $\Omega, k, \alpha$ and $\alpha'$, which may be of different values in different places.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, the standard Hölder space $C^{k,\alpha}(\Omega), k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$ is defined by

$$\{f \in C^k(\Omega) : \|f\|_{C^{k,\alpha}()} := \sum_{|\gamma|=0}^{k} \sup_{z \in \Omega} |D^\gamma f(z)| + \sum_{|\gamma|=k} \sup_{z,z' \in \Omega, z \neq z'} \frac{|D^\gamma f(z) - D^\gamma f(z')|}{|z - z'|^\alpha} < \infty\}.$$ 

Here $D^\gamma$ represents any $|\gamma|$-th derivative operator. When $k = 0$, we write $C^{0,\alpha}(\Omega) = C^{\alpha}(\Omega)$. Moreover, given $f \in C^{k,\alpha}(\Omega)$, denote by

$$\|f\|_{C^k()} := \sum_{|\gamma|=0}^{k} \sup_{z \in \Omega} |D^\gamma f(z)|$$

and the Hölder semi-norm by

$$H^\alpha[f] := \sup_{z,z' \in \Omega, z \neq z'} \frac{|f(z) - f(z')|}{|z - z'|^\alpha}.$$ 

Consequently, $\|f\|_{C^{k,\alpha}()} = \|f\|_{C^k()} + \sum_{|\gamma|=k} H^\alpha[D^\gamma f]$. In particular, for each $j \in \{1, \ldots, n\}$, the Hölder semi-norm with respect to $j$-th variable for each fixed $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in D_1 \times \cdots \times D_{j-1} \times D_{j+1} \times \cdots \times D_n$ is defined as follows.

$$H^\alpha_j[f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)] := \sup_{\zeta,\zeta' \in D_j, \zeta \neq \zeta'} \frac{|f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) - f(z_1, \ldots, z_{j-1}, \zeta', z_{j+1}, \ldots, z_n)|}{|\zeta - \zeta'|^\alpha}.$$ 

Apparently, the above expression is always bounded by $H^\alpha[f]$ for each $j \in \{1, \ldots, n\}$. On the other hand, the following elementary lemma for Hölder functions is observed.

**Lemma 2.1.** If there exists a constant $C$ such that for each $j \in \{1, \ldots, n\}$ and for each $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in D_1 \times \cdots \times D_{j-1} \times D_{j+1} \times \cdots \times D_n$, $f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)$ as a function of $\zeta \in D_j$ satisfies

$$H^\alpha_j[f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)] \leq C,$$

then $H^\alpha[f] \leq nC$ for the same constant $C$. 

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Proof. We use the standard triangle inequality method. Without loss of generality, assume $n = 2$ with $\Omega = D_1 \times D_2$. Indeed, for any $z = (z_1, z_2) \in D_1 \times D_2$, then $(z_1', z_2) \in D_1 \times D_2$. Hence $|f(z_1, z_2) - f(z_1', z_2)| \leq |f(z_1, z_2) - f(z_1', z_2)| + |f(z_1', z_2) - f(z_1', z_2')| \leq C|z_1 - z_1'| + C|z_2 - z_2'| \leq 2C|z - z'|.

The rest of the section is devoted to some classical results in one variable complex analysis. Let $D$ be a bounded domain in the complex plane $\mathbb{C}$ with $C^{k+1,\alpha}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$. Given a complex valued function $f \in C(D)$, we define the following two operators related to the Cauchy kernel for $z \in D$:

\[
Tf(z) := \frac{-1}{2\pi i} \int_D \frac{f(\zeta)}{\zeta - z} d\zeta; \\
Sf(z) := \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.
\]

Here the positive orientation of $\partial D$ is adopted for the contour integral such that $D$ is always to the left while traversing along the contour(s). As is well known, $T$ is the universal solution operator for the $\partial$ operator on $D$, while $S$ turns integrable functions on $\partial D$ to holomorphic functions in $D$. In the following, we state with references some properties of the two operators that will be used later.

**Theorem 2.2.** [16] Let $D$ be a bounded domain with $C^{1,\alpha}$ boundary, $f \in C(\bar{D})$ and $f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} \in L^p(D), p > 2$. Then $f = Sf + T(f_{\bar{z}})$ in $D$.

*Proof. See [16] formula 6.10 (p. 41).*

**Theorem 2.3.** [16] Let $D$ be a bounded domain with $C^{k+1,\alpha}$ boundary, and $f \in C^{k,\alpha}(D), k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$. Then $Tf \in C^{k+1,\alpha}(D)$ and $Sf \in C^{k,\alpha}(D)$. Moreover, there exists a constant $C$ dependent only on $D, k$ and $\alpha$, such that

\[
\|Tf\|_{C^{k+1,\alpha}(D)} \leq C\|f\|_{C^{k,\alpha}(D)}; \\
\|Sf\|_{C^{k,\alpha}(D)} \leq C\|f\|_{C^{k,\alpha}(D)}.
\]

*Proof. See [16] Theorem 1.32 (p. 56) for operator $T$, and Theorem 1.10 (p. 21) for operator $S).*

**Theorem 2.4.** [16] Let $D$ be a bounded domain. $Tf \in C^{\alpha}(D)$ if $f \in L^p(D), p > 2, \alpha = \frac{p-2}{p}$, and there exists a constant $C$ dependent only on $D$ and $p$, such that

\[
\|Tf\|_{C^{\alpha}(D)} \leq C\|f\|_{L^p}.
\]

Moreover, $\bar{\partial}T = id$ on $L^p(D), 1 \leq p < \infty$ in the sense of distributions.
Proof. See [16] Theorem 1.19 (p. 38) for the first part and Theorem 1.14 (p. 29) for the second part.

3 The solution operator and the Hölder norms, \( n \geq 2 \)

Let \( D_j \subset \mathbb{C}, j = 1, \ldots, n, \) be a bounded domain with \( C^{k+1,\alpha} \) boundary, \( n \geq 2, k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1, \) and \( \Omega := D_1 \times \cdots \times D_n. \) Given a function \( f \in C^{k,\alpha}(\Omega), \) define for \( z \in \Omega, \)

\[
T_j f(z) := -\frac{1}{2\pi i} \int_{D_j} \frac{f(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z_j} \, d\zeta_j \wedge \zeta_j; \\
S_j f(z) := \frac{1}{2\pi i} \int_{\partial D_j} \frac{f(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z_j} \, d\zeta_j.
\]

(1)

Theorem 2.3-2.4 immediately imply the following lemma.

**Lemma 3.1.** Let \( j \in \{1, \ldots, n\}. \) There exists a constant \( C \) dependent only on \( \Omega, k \) and \( \alpha, \) such that for any \( f \in C^{k,\alpha}(\Omega), \) \( k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1, \) and \( \gamma(\in \mathbb{Z}^+ \cup \{0\}) \leq k, \) \( T_j f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) \) and \( S_j f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) \) as functions of \( \zeta \in D_j \) satisfy

\[
\|D_j^\gamma T_j f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)\|_{C^\alpha(D_j)} \leq \begin{cases} C\|f\|_{C^\gamma(\Omega)}, & \gamma = 0 \\ C\|f\|_{C^{\gamma-1,\alpha}(\Omega)}, & \gamma \geq 1 \end{cases} \leq C\|f\|_{C^\gamma,\alpha(\Omega)}; \\
\|D_j^\gamma S_j f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)\|_{C^\alpha(D_j)} \leq C\|f\|_{C^{\gamma,\alpha}(\Omega)}.
\]

Here \( D_j^\gamma \) represents any \( \gamma \)-th derivative operator with respect to the \( j \)-th variable.

We are now ready to prove the boundedness of operators \( T_j \) and \( S_j \) in Hölder spaces, \( 0 < \alpha < 1. \)

**Proposition 3.2.** For each \( j \in \{1, \ldots, n\}, \) \( T_j \) is a bounded linear operator sending \( C^{k,\alpha}(\Omega) \) into \( C^{k,\alpha}(\Omega), \) \( k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1. \) Namely, there exists some constant \( C \) dependent only on \( \Omega, k \) and \( \alpha, \) such that for \( f \in C^{k,\alpha}(\Omega), \)

\[
\|T_j f\|_{C^{k,\alpha}(\Omega)} \leq C\|f\|_{C^{k,\alpha}(\Omega)}. \quad (2)
\]

Moreover, for any \( f \in C^{k,\alpha}(\Omega), \) \( \bar{\partial}_j T_j f = f \) in \( \Omega, \) \( 1 \leq j \leq n. \)
Proposition 3.2 is sharp for $T_j$ for some constant $C$ independent of regularity along slice of higher dimensional domains. Consider a trivial example $D^\gamma = D_1^{\gamma_1}D_2^{\gamma_2}$, $\gamma_1 + \gamma_2 \leq k$. Then $D^\gamma T_1 f = D_1^{\gamma_1}T_1(D_2^{\gamma_2} f)$. Hence by Lemma 3.1

$$\|D^\gamma T_1 f\|_{C^0(\Omega)} = \sup_{z_2 \in D_2} \|D_1^{\gamma_1}T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^0(D_1)} \leq C \|D_2^{\gamma_2} f\|_{C^{\gamma_1}(\Omega)} \leq C\|f\|_{C^{k,0}(\Omega)}.$$  

Next, we show $H^k[D^\gamma T_1 f] \leq C\|f\|_{C^{k,0}(\Omega)}$ for some constant $C$ independent of $f$ for all $|\gamma| = k$. By Lemma 3.1, for each $z_2 \in D_2$, $D^\gamma T_1 f(\cdot, z_2)$ as a function of $\zeta \in D_1$ satisfies

$$H^k[D^\gamma T_1 f(\cdot, z_2)] \leq \|D_1^{\gamma_1}T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^0(D_1)} \leq C \|D_2^{\gamma_2} f\|_{C^{\gamma_1}(\Omega)} \leq C\|f\|_{C^{k,0}(\Omega)}$$

for some constant $C$ independent of $f$ and $z_2$.

On the other hand, let $z'_2 \neq z_2 \in D_2$ and consider $F_{z_2,z'_2}(\zeta) := \frac{D_2^{\gamma_2} f(\zeta, z_2) - D_2^{\gamma_2} f(\zeta, z'_2)}{|z_2 - z'_2|^{\gamma_2}}$ on $D_1$. Since $f \in C^{k,0}(\Omega)$, it follows $F_{z_2,z'_2} \in C^{\gamma_1}(D_1)$ and $\|F_{z_2,z'_2}\|_{C^{\gamma_1}(D_1)} \leq \|f\|_{C^{k,0}(\Omega)}$. If $\gamma_1 = 0$, by Lemma 3.1,

$$\|D_1^{\gamma_1}T_1 F_{z_2,z'_2}\|_{C^0(D_1)} = \|T_1 F_{z_2,z'_2}\|_{C^0(D_1)} \leq C \|F_{z_2,z'_2}\|_{C^0(D_1)} \leq C\|f\|_{C^{k,0}(\Omega)},$$

where $C$ is independent of $f$, $z_2$ and $z'_2$. For $\gamma_1 \geq 1$, we have by Lemma 3.1

$$\|D_1^{\gamma_1}T_1 F_{z_2,z'_2}\|_{C^0(D_1)} \leq C \|F_{z_2,z'_2}\|_{C^{\gamma_1-1,0}(D_1)} \leq C \|F_{z_2,z'_2}\|_{C^{\gamma_1}(D_1)} \leq C\|f\|_{C^{k,0}(\Omega)}$$

for some constant $C$ independent of $f$, $z_2$ and $z'_2$. In sum, for each fixed $z_1 \in D_1$,

$$\frac{|D^\gamma T_1 f(z_1, z_2) - D^\gamma T_1 f(z_1, z'_2)|}{|z_2 - z'_2|^{\gamma_2}} = |D_1^{\gamma_1}T_1 F_{z_2,z'_2}(z_1)| \leq \|D_1^{\gamma_1}T_1 F_{z_2,z'_2}\|_{C^0(D_1)} \leq C\|f\|_{C^{k,0}(\Omega)},$$

where $C$ is independent of $f$, $z_1$, $z_2$ and $z'_2$. We have thus proved $H^k_2[D^\gamma T_1 f(z_1, \cdot)] \leq C\|f\|_{C^{k,0}(\Omega)}$ with $C$ independent of $f$ and $z_1$, and the proposition as a consequence of Lemma 2.1.

We note that although $T$ is a smoothing operator in dimension one, $T_j$ does not improve regularity along slice of higher dimensional domains. Consider a trivial example $f(z_1, z_2) = |z_2|^\alpha$ on $\Delta^2$. $f \in C^{\alpha}(\Delta^2)$ but $T_1 f(z_1, z_2) = |z_1| |z_2|^\alpha \notin C^{\alpha+\epsilon}(\Delta^2)$ for any $\epsilon > 0$. Therefore Proposition 3.2 is sharp for $T_j$ in Hölder spaces.
Proposition 3.3. For each \( j \in \{1, \ldots, n\} \) and \( 0 < \alpha' < \alpha < 1 \), \( S_j \) is a bounded linear operator sending \( C^{k,\alpha}(\Omega) \) into \( C^{k,\alpha'}(\Omega) \), \( k \in \mathbb{Z}^+ \cup \{0\} \). Namely, there exists some \( C \) dependent only on \( \Omega, k, \alpha \) and \( \alpha' \), such that for \( f \in C^{k,\alpha}(\Omega) \),

\[
\|S_j f\|_{C^{k,\alpha'}(\Omega)} \leq C\|f\|_{C^{k,\alpha}(\Omega)}.
\]

Moreover, for any \( f \in C^{k,\alpha}(\Omega) \) with \( k \in \mathbb{Z}^+ \), \( \partial_z S_j f = 0 \) in \( \Omega \), \( 1 \leq j \leq n \).

Proof. The last statement is clear since \( S_j f \) is holomorphic with respect to \( z_j \) variable in \( D_j \). As in the previous proposition, we only prove (3) for \( j = 1 \) and \( n = 2 \). Let \( \gamma \) be \( j \)-th Jordan curve of \( \partial D \), and of total arclength \( s_j \). Let \( \zeta_1(s) \) be a parameterization of \( \partial D_1 \) in terms of the arclength variable \( s \), such that \( \zeta_1|_{s \in \{\sum_{m=1}^{j-1} s_m, \sum_{m=1}^{j} s_m\}} \) is a \( C^{k+1,\alpha} \) parametrization of \( \Gamma_j \). For any \( (z_1, z_2) \in \Omega \), it follows by repeated integration by part,

\[
D^2 S_1 f(z_1, z_2) = \partial_1 \partial_1 y S_1 D_2^2 f(z_1, z_2) = \frac{1}{2\pi i} \partial_1 \partial_1 \int_{\partial D_1} \frac{D_2^2 f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1
\]

\[
= \frac{1}{2\pi i} \partial_1 \partial_1 \sum_{j=1}^{N_j} \int_{\partial D_1} \frac{D_2^2 f(\zeta_1, z_2)}{\zeta_1 - z_1} \cdot \partial_1 \left( \frac{1}{\zeta_1 - z_1} \right) d\zeta_1
\]

\[
= \frac{1}{2\pi i} \partial_1 \partial_1 \sum_{j=1}^{N_j} \int_{\partial D_1} \frac{D_2^2 f(\zeta_1, z_2)}{(\zeta_1 - z_1)^2} \cdot \partial_1 \left( \frac{1}{\zeta_1 - z_1} \right) d\zeta_1
\]

\[
= \frac{1}{2\pi i} \partial_1 \partial_1 \sum_{j=1}^{N_j} \int_{\partial D_1} \frac{D_2^2 f(\zeta_1, z_2)}{(\zeta_1 - z_1)^3} \cdot \partial_1 \left( \frac{1}{\zeta_1 - z_1} \right) d\zeta_1
\]

\[
\ldots
\]

\[
= \frac{1}{2\pi i} \int_{\partial D_1} \frac{\partial_1 D_2^2 f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1 =: S_1 \tilde{f}(z_1, z_2)
\]

where \( \tilde{f} = \partial_1 D_2^2 f \in C^{\alpha}(\Omega) \) with \( \|\tilde{f}\|_{C^{\alpha}(\Omega)} \leq \|f\|_{C^{k,\alpha}(\Omega)} \). (See also [16] p. 21-22.) Therefore, we only need to prove \( \|S_1 \tilde{f}\|_{C^{\alpha'}(\Omega)} \leq C\|\tilde{f}\|_{C^{\alpha}(\Omega)} \) for some constant \( C \) independent of \( \tilde{f} \).
Firstly, by Lemma 3.1, one has

\[ \|S_1 \tilde{f}\|_{C^\alpha(\Omega)} = \sup_{z_2 \in D_2} \|S_1 \tilde{f}(\cdot, z_2)\|_{C^\alpha(D_1)} \leq C \|\tilde{f}\|_{C^\alpha(\Omega)} \]

for some constant \( C \) independent of \( \tilde{f} \).

Next, we show \( H^{\alpha'}[S_1 \tilde{f}] \leq C\|\tilde{f}\|_{C^\alpha(\Omega)} \) for some constant \( C \) independent of \( \tilde{f} \). By Lemma 3.1 for each \( z_2 \in D_2 \), \( S_1 \tilde{f}(\zeta, z_2) \) as a function of \( \zeta \in D_1 \) satisfies

\[ H^{\alpha'}[S_1 \tilde{f}(\cdot, z_2)] \leq \|S_1 \tilde{f}(\cdot, z_2)\|_{C^{\alpha'}(D_1)} \leq C \|\tilde{f}\|_{C^{\alpha'}(\Omega)} \leq C \|\tilde{f}\|_{C^\alpha(\Omega)} \]

for some constant \( C \) independent of \( \tilde{f} \) and \( z_2 \).

We further show there exists a constant \( C \) independent of \( \tilde{f} \) and \( z_1 \), such that for each \( z_1 \in D_1 \), \( H^\alpha[S_1 \tilde{f}(z_1, \cdot)] \leq C\|\tilde{f}\|_{C^\alpha(\Omega)} \). First consider \( z_1 = t_1 \in \partial D_1 \). Without loss of generality, assume \( t_1 \in \Gamma_j \) and \( \zeta_1|_{s=0} = t_1 \), where \( \zeta_1 \) is the parameterization of \( \partial D_1 = \bigcup_{j=1}^N \Gamma_j \) as before. Since \( \partial D_1 \subset C^1 \), \( \partial D_1 \) satisfies the so-called chord-arc condition. In other words, for any \( \zeta_1(s), \zeta_1(s') \in \Gamma_j, j = 1, \ldots, N \), there exists a constant \( C \geq 1 \) dependent only on \( \partial D_1 \) such that

\[ |\zeta_1(s) - \zeta_1(s')| \leq \min\{s - s', s' + s_j - s\} \leq C|\zeta(s) - \zeta(s')| \]

Here \( s_j \) is the total arclength of \( \Gamma_j \). In particular, when \( 0 \leq s \leq s_1 \),

\[ |d\zeta_1| \leq C|ds| \quad \text{and} \quad |\zeta_1(s) - t_1| \geq C \min\{s, s_1 - s\} \quad (4) \]

for some constant \( C \) depending only on \( D_1 \). By Sokhotski–Plemelj Formula (see [14] for instance), the non-tangential limit of \( S_1 \tilde{f} \) at \( (t_1, z_2) \in \partial D_1 \times D_2 \) is

\[ \Phi_1 \tilde{f}(t_1, z_2) := \frac{1}{2\pi i} \int_{\partial D_1} \frac{\tilde{f}(\zeta_1, z_2)}{\zeta_1 - t_1} d\zeta_1 + \frac{1}{2} \tilde{f}(t_1, z_2). \]

Here the first term is interpreted as the Principal Value. We shall prove that for \( z_2, z'_2 \in D_2 \) with \( h := |z_2 - z'_2| \neq 0 \),

\[ |\Phi_1 \tilde{f}(t_1, z_2) - \Phi_1 \tilde{f}(t_1, z'_2)| \leq C h^{\alpha'} \|\tilde{f}\|_{C^\alpha(\Omega)} \]

for some constant \( C \) independent of \( \tilde{f}, t_1, z_2 \) and \( z'_2 \), essentially following the idea of Muskhenlishvili [14].

Let \( h_0 \) be a positive number such that \( h^{\alpha-\alpha'} \ln \frac{1}{h} \leq 1 \) for \( 0 < h \leq h_0 < \min\{1, \frac{1}{2} \} \). Then \( h_0 \) depends only on \( \alpha \) and \( \alpha' \). When \( h \geq h_0 \),

\[ |\Phi_1 \tilde{f}(t_1, z_2) - \Phi_1 \tilde{f}(t_1, z'_2)| \leq 2 \|S_1 \tilde{f}\|_{C^\alpha(\Omega)} \leq C\|\tilde{f}\|_{C^\alpha(\Omega)} \leq \frac{C}{h_0^{\alpha'}} h^{\alpha'} \|\tilde{f}\|_{C^\alpha(\Omega)} \leq C h^{\alpha'} \|\tilde{f}\|_{C^\alpha(\Omega)} \]

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for some constant $C$ independent of $\tilde{f}, t_1, z_2$ and $z'_2$.

When $h < h_0$, write

$$
\Phi_1 \tilde{f}(t_1, z_2) - \Phi_1 \tilde{f}(t_1, z'_2) = \frac{1}{2\pi i} \int_{\partial D_1} \frac{\tilde{f}(\zeta, z_2) - \tilde{f}(t_1, z_2) - \tilde{f}(\zeta, z'_2) + \tilde{f}(t_1, z'_2)}{\zeta - t_1} d\zeta_1
$$

$$+ \frac{1}{2\pi i} \int_{\partial D_1} \frac{1}{\zeta - t_1} \int \frac{\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z'_2)}{2} d\zeta_1 d\zeta_1 + \left( \frac{\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z'_2)}{\zeta - t_1} \right)
$$

$$= : I + II.$$

Here the second equality have used the fact that $\int_{\partial D_1} \frac{1}{\zeta - t_1} d\zeta_1 = \pi i$ when interpreted as the Principal Value, due to the positive orientation of $\partial D_1$. Obviously

$$|II| \leq Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$$

for some constant $C$ independent of $\tilde{f}, t_1, z_2$ and $z'_2$.

Let $l$ be the arc on $\partial D_1$ that are centered at $t_1$ with arclength $2h$. Consequently, $l \subset \Gamma_1$ due to the fact that $h \leq \frac{\pi}{2}$. Write $I$ as follows.

$$I = \frac{1}{2\pi i} \int_{\Gamma_1 \setminus l} \frac{\tilde{f}(\zeta, z_2) - \tilde{f}(t_1, z_2) - \tilde{f}(\zeta, z'_2) + \tilde{f}(t_1, z'_2)}{\zeta - t_1} d\zeta_1$$

$$+ \frac{1}{2\pi i} \int_l \left( \frac{\tilde{f}(\zeta, z_2) - \tilde{f}(t_1, z_2) - \tilde{f}(\zeta, z'_2) + \tilde{f}(t_1, z'_2)}{\zeta - t_1} \right) d\zeta_1$$

$$+ \frac{1}{2\pi i} \int_{\cup_{j=2}^N \Gamma_j} \left( \frac{\tilde{f}(\zeta, z_2) - \tilde{f}(t_1, z_2) - \tilde{f}(\zeta, z'_2) + \tilde{f}(t_1, z'_2)}{\zeta - t_1} \right) d\zeta_1$$

$$= : I_1 + I_2 + I_3.$$

For $I_3$, since $\cup_{j=2}^N \Gamma_j$ does not intersect with $\Gamma_1$ and $t_1 \in \Gamma_1$, $|\zeta_1 - t_1| \geq C$ on $\cup_{j=2}^N \Gamma_j$ for some positive $C$ dependent only on $\partial D_1$. On the other hand, the absolute value of the numerator in $I_3$ is less than $Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$. It immediately follows that

$$|I_3| \leq Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}.$$

For $I_2$, the absolute value of the numerator of the integrand is less than $C|\zeta_1 - t_1|^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$. We infer from (1) that

$$|I_2| \leq C \|\tilde{f}\|_{C^\alpha(\Omega)} \int_l \frac{1}{|\zeta_1 - t_1|^{1-\alpha}} d\zeta_1 \leq C \|\tilde{f}\|_{C^\alpha(\Omega)} \int_0^h \frac{1}{s^{1-\alpha}} ds \leq Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$$
for some constant $C$ independent of $\tilde{f}, t_1, z_2$ and $z_2'$. Now we treat with the remaining term $I_1$. Rearrange $I_1$ so it becomes

$$|I_1| \leq \frac{1}{2\pi} \int_{\gamma \setminus \Gamma} \frac{\tilde{f}(\zeta_1, z_2) - \tilde{f}(\zeta_1, z_2')}{\zeta_1 - t_1} d\zeta_1 + \frac{1}{2\pi} \int_{\gamma \setminus \Gamma} \frac{\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z_2')}{\zeta_1 - t_1} d\zeta_1.$$  

The second term of the above inequality is bounded by $C h \|\tilde{f}\|_{C^\alpha(\Omega)}$ for some constant $C$ independent of $\tilde{f}, t_1, z_2$ and $z_2'$, as in the argument for II. The first term when $h < h_0$ is bounded by

$$Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)} \int_{h}^{\infty} \frac{1}{s} ds \leq Ch^\alpha \ln \frac{1}{h} \|\tilde{f}\|_{C^\alpha(\Omega)} \leq Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}.$$  

We have thus shown there exists a constant $C$ independent of $t_1$ and $\tilde{f}$, such that for each $z_1 = t_1 \in \partial D_1$, $H_2^\alpha [\Phi_1 \tilde{f}(t_1, \cdot)] \leq C \|\tilde{f}\|_{C^\alpha(\Omega)}$. Notice that for each fixed $\zeta \in D_2$, $S_1 \tilde{f}(z_1, \zeta)$ is holomorphic as a function of $z_1 \in D_1$ and $C^\alpha$ continuous up to the boundary with boundary value equal to $\Phi_1 \tilde{f}(z_1, \zeta)$ by Plemelj–Privalov Theorem. For each fixed $z_2$ and $z_2'$ with $|z_2 - z_2'| \neq 0$, applying Maximum Modulus Theorem to the holomorphic function $\frac{S_1 \tilde{f}(z_1, z_2) - S_1 \tilde{f}(z_1, z_2')}{|z_2 - z_2'|^\alpha}$ of $z_1$ in $D_1$, we immediately obtain

$$\sup_{z_1 \in D_1} \left| \frac{S_1 \tilde{f}(z_1, z_2) - S_1 \tilde{f}(z_1, z_2')}{|z_2 - z_2'|^\alpha} \right| \leq \sup_{t_1 \in \partial D_1} \left| \frac{\Phi_1 \tilde{f}(t_1, z_2) - \Phi_1 \tilde{f}(t_1, z_2')}{|z_2 - z_2'|^\alpha} \right| = \sup_{t_1 \in \partial D_1} H_2^\alpha [\Phi_1 \tilde{f}(t_1, \cdot)] \leq C \|\tilde{f}\|_{C^\alpha(\Omega)},$$

with $C$ independent of $f, z_1, z_2$ and $z_2'$. Therefore

$$H_2^\alpha [S_1 \tilde{f}(z_1, \cdot)] \leq C \|\tilde{f}\|_{C^\alpha(\Omega)}$$

with $C$ independent of $\tilde{f}$ and $z_1$. The proof of the proposition is complete.

It is worth pointing out that $S_j$ does not send $C^{k,\alpha}(\Omega)$ into itself. Indeed, in strong contrast to the Cauchy integral operator $S$ in one dimensional case, Tumanov [15] constructed a concrete function $f \in C^\alpha(\Delta^2)$ such that $S_1 f \notin C^\alpha(\Delta^2), 0 < \alpha < 1$. In view of the example, Proposition 3.3 is optimal for $S_j$ between Hölder spaces.
\textbf{Theorem 3.4.} Let $f = \sum_{j=1}^{n} f_j d\bar{z}_j \in C^{k,\alpha}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$. Then for any $0 < \alpha' < \alpha$,

$$Tf := \sum_{j=1}^{n} \prod_{l=1}^{j-1} T_j S_l f_j = T_1 f_1 + T_2 S_1 f_2 + \cdots + T_n S_1 \cdots S_{n-1} f_n$$

(5)

is in $C^{k,\alpha'}(\Omega)$ with $\|Tf\|_{C^{k,\alpha'}(\Omega)} \leq C \|f\|_{C^{k,\alpha}(\Omega)}$. If further $f$ is $\tilde{\partial}$-closed and $k > 0$, then $\tilde{\partial}Tf = f$.

\textit{Proof.} The operator $T$ defined by (5) is well defined on $C^{k,\alpha}(\Omega)$ due to Proposition 3.2, 3.3. Choose some positive constant $\epsilon < \frac{\alpha - \alpha'}{n-1}$. Then $\alpha' + (n-1)\epsilon < \alpha \leq 1$. Applying Proposition 3.2, 3.3 repeatedly, it follows for each $j \leq n$,

$$\|\prod_{l=1}^{j-1} T_j S_l f_j\|_{C^{k,\alpha'}(\Omega)} \leq C \|\prod_{l=1}^{j-1} S_l f_j\|_{C^{k,\alpha'}(\Omega)}$$

$$\leq C \|\prod_{l=1}^{j-2} S_l f_j\|_{C^{k,\alpha'+\epsilon}(\Omega)}$$

$$\leq \cdots$$

$$\leq C \|f_j\|_{C^{k,\alpha'+(n-1)\epsilon}(\Omega)}$$

$$\leq C \|f_j\|_{C^{k,\alpha}(\Omega)}.$$ 

Therefore, $\|Tf\|_{C^{k,\alpha'}(\Omega)} \leq C \|f\|_{C^{k,\alpha}(\Omega)}$.

Furthermore for $k \in \mathbb{Z}^+$, making use of Theorem 2.2, Proposition 3.2, 3.3, the closedness of $f$ and Fubini’s Theorem, we obtain

$$\tilde{\partial}_{z_1} Tf = \tilde{\partial}_{z_2} T_1 f_1 + \tilde{\partial}_{z_2} T_2 S_1 f_2 + \cdots + \tilde{\partial}_{z_2} T_n S_1 \cdots S_{n-1} f_n = f_1;$$

$$\tilde{\partial}_{z_2} Tf = \tilde{\partial}_{z_2} T_1 f_1 + \tilde{\partial}_{z_2} T_2 S_1 f_2 + \cdots + \tilde{\partial}_{z_2} T_n S_1 \cdots S_{n-1} f_n$$

$$= T_1(\tilde{\partial}_{z_2} f_1) + S_1 f_2 = T_1(\tilde{\partial}_{z_1} f_2) + S_1 f_2 = f_2;$$

$$\cdots$$

$$\tilde{\partial}_{z_n} Tf = \tilde{\partial}_{z_n} T_1 f_1 + \tilde{\partial}_{z_n} T_2 S_1 f_2 + \cdots + \tilde{\partial}_{z_n} T_n S_1 \cdots S_{n-1} f_n$$

$$= T_1(\tilde{\partial}_{z_n} f_1) + T_2 S_1 \tilde{\partial}_{z_n} f_2 + \cdots + S_1 \cdots S_{n-1} f_n$$

$$= T_1(\tilde{\partial}_{z_1} f_n) + S_1 T_2 \tilde{\partial}_{z_1} f_n + \cdots + S_1 \cdots S_{n-1} f_n$$

$$= f_n - S_1 f_n + S_1(f_n - S_2 f_n) + \cdots + S_1 \cdots S_{n-1} f_n = f_n.$$ 

\[\square\]
In the following lemma, we show that when $f \in C^{n-1,\alpha}(\Omega)$ in particular, $T$ defined by (5) coincides with the solution operator constructed in [8]. Therefore the same supnorm estimate in [8] passes onto $T$ if the data is in addition $C^{n-1,\alpha}(\Omega)$.

**Lemma 3.5.** Let $T^*$ be the solution operator in [8] with

$$T^*f := \sum_{s=1}^{n} (-1)^{s-1} \sum_{1 \leq i_1 < \cdots < i_s \leq n} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^s f_{i_s}}{\partial \bar{z}_{i_1} \cdots \partial \bar{z}_{i_{s-1}}} \right)$$

for $f \in C^{n-1,\alpha}(\Omega)$ (not necessarily closed) and $Tf$ be as in (5). Then

$$T^*f = Tf$$

(6)

for all $f \in C^{n-1,\alpha}(\Omega)$.

**Proof.** We first show by induction that given $f \in C^{n-1,\alpha}(\Omega)$,

$$\prod_{l=1}^{n-1} S_l f = S_1 \cdots S_{n-1} f = f - \sum_{s=1}^{n-1} (-1)^{s-1} \sum_{1 \leq i_1 < \cdots < i_s \leq n-1} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^s f}{\partial \bar{z}_{i_1} \cdots \partial \bar{z}_{i_s}} \right).$$

(7)

When $n = 2$, (7) follows from Theorem 2.2. Suppose (7) holds for $n = k$, i.e.,

$$S_1 \cdots S_{k-1} f = f - \sum_{s=1}^{k-1} (-1)^{s-1} \sum_{1 \leq i_1 < \cdots < i_s \leq k-1} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^s f}{\partial \bar{z}_{i_1} \cdots \partial \bar{z}_{i_s}} \right).$$

Suppose (7) holds for $n = k$, i.e.,

$$S_1 \cdots S_k f = f - \sum_{s=1}^{k} (-1)^{s-1} \sum_{1 \leq i_1 < \cdots < i_s \leq k} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^s f}{\partial \bar{z}_{i_1} \cdots \partial \bar{z}_{i_s}} \right).$$
When $n = k + 1$, making use of Theorem 2.2 and Fubini’s Theorem repeatedly, we have

\[ S_1 \cdots S_k f = S_k(S_1 \cdots S_{k-1} f) \]

Thus, (7) is proved.

We are ready to verify (6) by induction. When $n = 2$, (6) follows directly from Theorem 2.2. Assume (6) holds for $n = k$. In other words, for any $f \in C^{k-1,\alpha}(\Omega)$,

\[ \sum_{s=1}^{k} (-1)^{s-1} \sum_{1 \leq i_1 < \cdots < i_s \leq k} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^{s-1} f_{i_s}}{\partial z_{i_1} \cdots \partial z_{i_s}} \right) = \sum_{j=1}^{k} \prod_{l=1}^{j-1} T_j S_l f_j. \]
When \( n = k + 1 \),

\[
T^*f = \sum_{s=1}^{k+1} (-1)^{s-1} \sum_{1 \leq i_1 < \ldots < i_s \leq k+1} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^{s-1} f_{i_s}}{\partial z_{i_1} \cdots \partial \bar{z}_{i_{s-1}}} \right)
\]

\[
= \sum_{s=1}^{k} (-1)^{s-1} \sum_{1 \leq i_1 < \ldots < i_s \leq k} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^{s-1} f_{i_s}}{\partial z_{i_1} \cdots \partial \bar{z}_{i_{s-1}}} \right) +
\]

\[
+ \sum_{s=1}^{k+1} (-1)^{s-1} \sum_{1 \leq i_1 < \ldots < i_s = k+1} T_{i_1} \cdots T_{i_{s-1}} T_{k+1} \left( \frac{\partial^{s-1} f_{k+1}}{\partial z_{i_1} \cdots \partial \bar{z}_{i_{s-1}}} \right)
\]

\[
= \sum \prod_{j=1}^{k} T_j S_j f_j + T_{k+1} (f_{k+1} - \sum_{s=1}^{k} (-1)^{s-1} \sum_{1 \leq i_1 < \ldots < i_s \leq k} T_{i_1} \cdots T_{i_s} \left( \frac{\partial^{s} f_{k+1}}{\partial z_{i_1} \cdots \partial \bar{z}_{i_{s}}} \right))
\]

\[
= \sum \prod_{j=1}^{k} T_j S_j f_j + \prod_{l=1}^{k} T_{k+1} S_l f_{k+1}
\]

\[
= \sum \prod_{j=1}^{k+1} T_j S_j f_j = Tf.
\]

Here the fourth equality is because of the identity (7).

**Proof of Theorem 1.5.** Observe that \( C^\infty(\Omega) \subset C^{k,\alpha}(\Omega) \) for any integer \( k \in \mathbb{Z}^+ \cup \{0\} \) and \( 0 < \alpha \leq 1 \). Theorem 1.5 follows directly from the supnorm estimate in [8] and the proof of Theorem 3.4 in view of Lemma 3.5.

4 Proof of Theorem 1.4

Assuming \( f \in C^\alpha(\Omega) \), the following proposition shows that \( Tf \) defined by (5) solves \( \bar{x} \partial u = f \) in the sense of distributions.

**Proposition 4.1.** Let \( D_j \subset \mathbb{C}, j = 1, \ldots, n \), be bounded domains with \( C^{1,\alpha} \) boundary, \( n \geq 2 \), \( 0 < \alpha \leq 1 \) and \( \Omega := D_1 \times \cdots \times D_n \). Assume \( f = \sum_{j=1}^{n} f_j d\bar{z}_j \in C^\alpha(\Omega) \) is a \( \bar{x} \)-closed \((0,1)\) form on \( \Omega \) in the sense of distributions. Then \( u := Tf \) defined in (5) is in \( C^{\alpha'}(\Omega) \) for any \( 0 < \alpha' < \alpha \) and solves \( \bar{x} \partial u = f \) in the sense of distributions.
Proof. Given \( f \in C^\alpha(\Omega) \) for \( 0 < \alpha \leq 1 \), \( Tf \in C^{\alpha'}(\Omega) \) with \( 0 < \alpha' < \alpha \) by Theorem 3.4 with \( k = 0 \). Next we show \( Tf \) solves \( \partial u = f \) in \( \Omega \) in the sense of distributions using the standard mollifier argument. For each \( j \in \{1, \ldots, n\} \), let \( \{D_j^{(l)}\}_{l=1}^{\infty} \) be a family of strictly increasing open subsets of \( D_j \) such that when \( l \geq N_0 \in \mathbb{N}, \partial D_j^{(l)} \) is \( C^{2,\alpha} \), \( \frac{1}{l+1} < \text{dist}(D_j^{(l)}, D_j^{(l+1)}) < \frac{1}{l} \), and \( F_j^{(l)} \) being a \( C^1 \) diffeomorphism between \( \partial D_j \) and \( \partial D_j^{(l)} \) satisfies \( \lim_{l \to \infty} \|F_j^{(l)} - Id\|_{C^1(\partial D_j)} = 0 \). Let \( \Omega^{(l)} = D_1^{(l)} \times \cdots \times D_n^{(l)} \) be the product of those planar domains. Denote by \( T_j^{(l)}, S_j^{(l)} \) and \( T^{(l)} \) the operators defined in (1) and (5) accordingly, with \( \Omega \) replaced by \( \Omega^{(l)} \). Then \( T^{(l)}f \in C^{\alpha'}(\Omega^{(l)}) \) for each \( 0 < \alpha' < \alpha \). Adopting the mollifier argument to \( f \in C^\alpha(\Omega) \), we obtain \( f^\epsilon \in C^{1,\alpha}(\Omega^{(l)}) \) such that for each fixed \( 0 < \alpha' < \alpha \), \( \|f^\epsilon - f\|_{C^{\alpha'}(\Omega^{(l)})} \to 0 \) (see Appendix) as \( \epsilon \to 0 \) and \( \bar{\partial}f^\epsilon = 0 \) on \( \Omega^{(l)} \).

Fix an \( \alpha'(< \alpha) \). For each \( l \), \( T^{(l)}f^\epsilon \in C^{1,\alpha'}(\Omega^{(l)}) \) when \( \epsilon \) is small and \( \bar{\partial}T^{(l)}f^\epsilon = f^\epsilon \) in \( \Omega^{(l)} \) by Theorem 3.4. Furthermore, applying Proposition 3.2 at \( k = 0 \), we have \( \|T^{(l)}f^\epsilon - T^{(l)}f\|_{C^{\alpha'}(\Omega^{(l)})} \leq C\|f^\epsilon - f\|_{C^{\alpha'}(\Omega^{(l)})} \to 0 \) as \( \epsilon \to 0 \). We thus have \( \lim_{\epsilon \to 0} T^{(l)}f^\epsilon \) exists and is equal to \( T^{(l)}f \in C^{\alpha'}(\Omega^{(l)}) \) pointwisely.

Given a testing function \( \phi \in C_\infty(\Omega) \), let \( l_0 \geq N_0 \) be such that \( K := \text{supp} \phi \subset \Omega^{(l_0 - 2)} \). Denote by \( \bar{\partial}^* := -\bar{\partial} \) the formal adjoint of \( \bar{\partial} \). For \( l \geq l_0 \),

\[
(T^{(l)}f, \bar{\partial}^* \phi)_{\Omega^{(l_0)}} = \lim_{\epsilon \to 0} (T^{(l)}f^\epsilon, \bar{\partial}^* \phi)_{\Omega^{(l_0)}} = \lim_{\epsilon \to 0} (\bar{\partial}T^{(l)}f^\epsilon, \phi)_{\Omega^{(l_0)}} = \lim_{\epsilon \to 0} (f^\epsilon, \phi)_{\Omega^{(l_0)}} = (f, \phi)_{\Omega^{(l_0)}} \tag{8}
\]

We further claim

\[
(Tf, \bar{\partial}^* \phi)_{\Omega^{(l_0)}} = \lim_{l \to \infty} (T^{(l)}f, \bar{\partial}^* \phi)_{\Omega^{(l_0)}}. \tag{9}
\]

Indeed, for each \( j \geq 1 \),

\[
-(2i)^n (2\pi i)^j (T^{(l)}S_1^{(l)} \cdots S_{j-1}^{(l)}f_j, \bar{\partial}^* \phi)_{\Omega^{(l_0)}}
\]

\[
= \int_{K} \int_{D_j^{(l)}} \cdots \int_{D_1^{(l)}} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\bar{\partial}^* \phi(z)}{(\zeta_1 - z_1) \cdots (\zeta_j - z_j)} d\zeta_1 d\zeta_j d\bar{\zeta} \wedge dz
\]

\[
= \int_{K \times D_j} \cdots \int_{D_1} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\chi_{D_j^{(l)}}(\zeta)\bar{\partial}^* \phi(z)}{(\zeta_1 - z_1) \cdots (\zeta_j - z_j)} d\zeta_1 \cdots d\zeta_j d\bar{\zeta} \wedge dz.
\]

Here \( \chi_{D_j^{(l)}} \) is the step function on \( \mathbb{C} \) such that \( \chi_{D_j^{(l)}} = 1 \) in \( D_j^{(l)} \) and 0 otherwise. Firstly, as a function of \((z, \xi_j) \in K \times D_j\),

\[
\int_{D_1^{(l)}} \cdots \int_{D_{j-1}^{(l)}} f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\chi_{D_j^{(l)}}(\zeta)\bar{\partial}^* \phi(z) (\zeta_1 - z_1) \cdots (\zeta_j - z_j).
\]

\[
\in L^1(K \times D_j).
\]

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To see this, notice that if $z \in K(\subset \Omega^{l_0-2})$ and $\zeta_k \in \partial D^{(l)}_k, l \geq l_0, k = 1, \ldots, j - 1$, then
\[
|\zeta_k - z_k| \geq \text{dist}((\Omega^{(l)})^c, \Omega^{l_0-2}) \geq \text{dist}((\Omega^{(l_0)})^c, \Omega^{l_0-2}) > \frac{1}{J^2} := \delta_0.
\]
Hence for each $(z, \zeta_j) \in K \times D_j$,
\[
|\int_{\partial D^{(l)}_j} \cdots \int_{\partial D^{(l)}_{j-1}} f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\chi_{D^{(l)}_j}(\zeta_j)\bar{\partial}^* \phi(z) \frac{d\zeta_{j-1} \cdots d\zeta_1}{(\zeta_1 - z_1) \cdots (\zeta_j - z_j)}| \leq \frac{C}{\delta_0^j |\zeta_j - z_j|}
\]
for some constant $C > 0$, which is integrable in $K \times D_j$. On the other hand, by continuity of $f_j$ and the construction of $\Omega^{(l)}$,
\[
\lim_{l \to \infty} \int_{\partial D^{(l)}_1} \cdots \int_{\partial D^{(l)}_{j-1}} f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\chi_{D^{(l)}_j}(\zeta_j)\bar{\partial}^* \phi(z) \frac{d\zeta_{j-1} \cdots d\zeta_1}{(\zeta_1 - z_1) \cdots (\zeta_j - z_j)} = \int_{\partial D_1} \cdots \int_{\partial D_{j-1}} f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\bar{\partial}^* \phi(z) \frac{d\zeta_{j-1} \cdots d\zeta_1}{(\zeta_1 - z_1) \cdots (\zeta_j - z_j)}
\]
pointwisely in $K \times D_j$. Applying Dominated Convergence Theorem, we obtain
\[
\lim_{l \to \infty} \int_{K \times D_j} \int_{\partial D_1} \cdots \int_{\partial D_{j-1}} f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\bar{\partial}^* \phi(z) \frac{d\zeta_{j-1} \cdots d\zeta_1}{(\zeta_1 - z_1) \cdots (\zeta_j - z_j)} = - (2i)^n(2\pi i)^j (T_j^{(l)} S_1^{(l)} \cdots S_{j-1}^{(l)} f_j, \bar{\partial}^* \phi)_{\Omega^{(l_0)}}
\]
Thus (9) holds by the definition (5) of $T$.
Finally, combining (8) with (9), we deduce that
\[
(\bar{\partial}Tf, \phi)_{\Omega} = (\bar{\partial}Tf, \phi)_{\Omega^{(l_0)}} = (Tf, \bar{\partial}^* \phi)_{\Omega^{(l_0)}} = \lim_{l \to \infty} (T^{(l)} f, \bar{\partial}^* \phi)_{\Omega^{(l_0)}} = (f, \phi)_{\Omega^{(l_0)}} = (f, \phi)_{\Omega}.
\]
The proof of Proposition 4.1 is complete.

Proof of Theorem 1.4. Theorem 1.4 follows directly from Theorem 5.4 and Proposition 4.1.

Finally, making use of the idea of Kerzman [13], we argue by the following examples the regularity of the solution can not be improved in Hölder spaces in general.

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Proof of Example 1.2. \( f \) is well defined in \( \Delta^2 \) and \( f = (z_1 - 1)^{k+\alpha}d\bar{z}_2 \in C^{k,\alpha}(\Delta^2) \). Assume by contradiction that there exists a solution \( u \in C^{k,\alpha}(\Delta^2) \) to \( \bar{\partial}u = f \) in \( \Delta^2 \) for some \( \alpha' \) with \( \alpha < \alpha' < 1 \). Then \( u = h + (z_1 - 1)^{k+\alpha}\bar{z}_2 \) for some holomorphic function \( h \) in \( \Delta^2 \).

Consider \( w(\xi) := \int_{|z_2| = \frac{1}{2}} u(\xi, z_2)dz_2 \) for \( \xi \in \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \). Since \( u \in C^{k,\alpha}(\Delta^2) \), we have \( w \in C^{k,\alpha'}(\Delta) \) as well. On the other hand, by Cauchy’s Theorem,

\[
w(\xi) = \int_{|z_2| = \frac{1}{2}} (\xi - 1)^{k+\alpha}\bar{z}_2dz_2 = (\xi - 1)^{k+\alpha}\int_{|z_2| = \frac{1}{2}} \frac{1}{4z_2}dz_2 = \frac{\pi i}{2}(\xi - 1)^{k+\alpha}.
\]

This is a contradiction since \( (\xi - 1)^{k+\alpha} \notin C^{k,\alpha'}(\Delta) \) for any \( \alpha' > \alpha \).

Proof of Example 1.3. \( f \) is well defined in \( \Delta^2 \) and \( f = \frac{(z_1 - 1)^{k+1}}{\log(z_1 - 1)}d\bar{z}_2 \in C^{k,1}(\Delta^2) \). As in the proof of the previous example, assume by contradiction that there exists a solution \( u \in C^{k+1,\alpha}(\Delta^2) \) to \( \bar{\partial}u = f \) in \( \Delta^2 \) for some \( \alpha > 0 \). Then \( u = h + \frac{(z_1 - 1)^{k+1}}{\log(z_1 - 1)}\bar{z}_2 \) for some holomorphic function \( h \) in \( \Delta^2 \).

Define similarly \( w(\xi) := \int_{|z_2| = \frac{1}{2}} u(\xi, z_2)dz_2 \) on \( \Delta \). Since \( u \in C^{k+1,\alpha}(\Delta^2) \), we have \( w \in C^{k+1,\alpha}(\Delta) \). On the other hand, by Cauchy’s Theorem,

\[
w(\xi) = \int_{|z_2| = \frac{1}{2}} \frac{(\xi - 1)^{k+1}}{\log(\xi - 1)}\bar{z}_2dz_2 = \frac{\pi i}{2\log(\xi - 1)}(\xi - 1)^{k+1}.
\]

This contradicts with the fact that \( \frac{(\xi - 1)^{k+1}}{\log(\xi - 1)} \notin C^{k+1,\alpha}(\Delta) \) for any \( \alpha > 0 \).

A Appendix

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( \Omega_j := \{ x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{2} \} \) when \( j \) is large, and \( \rho \) be a smooth function in \( \mathbb{R}^n \) by

\[
\rho(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right), & |x| < 1; \\ 0, & |x| \geq 1, \end{cases}
\]

where \( C \) is selected such that \( \int_{\mathbb{R}^n} \rho(y)dy = 1 \). \( \rho \) is called the standard mollifier. Let \( f \in L^1_{\text{loc}}(\Omega) \) and define for \( x \in \Omega_j \),

\[
f_j(x) := \int_{|y| \leq 1} \rho(y)f(x - \frac{y}{j})dy.
\]

Then \( f_j \in C^\infty(\Omega_j) \). The mollifier argument is a standard method dealing with weak derivatives in Sobolev spaces (See, for instance, [6] p. 717). The following theorem ought to be
well-known for Hölder spaces, however we could not locate a reference. For convenience of the reader, we include the proof below.

**Theorem A.1.** Let \( \tilde{\Omega} \subset \subset \Omega \) and \( 0 < \alpha' < \alpha \). If \( f \in C^\alpha(\Omega) \), then \( f_j \to f \) in \( C^{\alpha'}(\tilde{\Omega}) \). I.e., \( \| f_j - f \|_{C^{\alpha'}(\tilde{\Omega})} \to 0 \) as \( j \to \infty \).

**Proof.** Let \( j_0 \) be such that \( \tilde{\Omega} \subset \Omega_{j_0} \) and assume \( j \geq j_0 \). \( \| f_j - f \|_{C(\tilde{\Omega})} \to 0 \) due to the uniform continuity of \( f \) on \( \Omega \) ([6] p.718). Write \( \phi_j(x) := f_j(x) - f(x) = \int_{|y| \leq 1} \rho(y)(f(x - \frac{y}{j}) - f(x))dy \).

We next show for any \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that when \( j \geq N \),

\[
\frac{|\phi_j(x) - \phi_j(x')|}{|x - x'|^{\alpha'}} \leq \epsilon,
\]

for all \( x, x' \in \tilde{\Omega} \). Indeed, choose \( \delta_0 > 0 \) satisfying \( \| f \|_{C^\alpha(\Omega)} \delta_0^{\alpha - \alpha'} \leq \frac{\epsilon}{2} \).

When \( |x - x'| \leq \delta_0 \),

\[
\frac{|\phi_j(x) - \phi_j(x')|}{|x - x'|^{\alpha'}} \leq \int_{|y| \leq 1} \rho(y) \frac{|f(x - \frac{y}{j}) - f(x' - \frac{y}{j})|}{|x - x'|^{\alpha'}} dy + \int_{|y| \leq 1} \rho(y) \frac{|f(x) - f(x')|}{|x - x'|^{\alpha'}} dy \\
\leq 2\| f \|_{C^\alpha(\Omega)} |x - x'|^{\alpha - \alpha'} \leq \epsilon.
\]

When \( |x - x'| > \delta_0 \), choose \( N \in \mathbb{N} \) such that \( \| f \|_{C^\alpha(\Omega)} \delta_0^{\alpha' - \alpha'} N^{-\alpha} \leq \frac{\epsilon}{2} \). Then for any \( j \geq N \), \( |x - x'| > \delta_0 \), we have

\[
\frac{|\phi_j(x) - \phi_j(x')|}{|x - x'|^{\alpha'}} \leq \int_{|y| \leq 1} \rho(y) \frac{|f(x - \frac{y}{j}) - f(x)|}{|x - x'|^{\alpha'}} dy + \int_{|y| \leq 1} \rho(y) \frac{|f(x' - \frac{y}{j}) - f(x')|}{|x - x'|^{\alpha'}} dy \\
\leq 2\| f \|_{C^\alpha(\Omega)} |x - x'|^{-\alpha'} j^{-\alpha} \leq \epsilon.
\]

Given \( f \in C^\alpha(\Omega) \), although only the \( C^{\alpha'} \) convergence of the family \( \{ f_j \} \) defined by (11) for some \( \alpha' > 0 \) is needed in Proposition 11, it is worth pointing out that the \( C^\alpha \) convergence of \( \{ f_j \} \) can not be achieved in general. The following simple counter-example was provided by Liding Yao.

**Example A.2.** Let \( \Omega = (-1, 1) \subset \mathbb{R} \) and

\[
f(x) = \left\{ \begin{array}{ll} 0, & x \leq 0; \\ x^\alpha, & x > 0. \end{array} \right.
\]

Then \( f \in C^\alpha(\Omega) \). However, for any \( \tilde{\Omega} \subset \subset \Omega \) containing the origin, \( \| f_j - f \|_{C^\alpha(\tilde{\Omega})} \geq \int_0^1 \rho(y)y^\alpha dy > 0 \) for sufficiently large \( j \).
Proof. Let $j_0$ be such that $\tilde{\Omega} \subset \Omega_{j_0}$ and assume $j \geq j_0$. Write $\phi_j(x) := f_j(x) - f(x) = \int_{-1}^{1} \rho(y)(f(x - \frac{y}{j}) - f(x))dy$. For each fixed $j$, it can be verified that

$$\phi_j(-\frac{1}{j}) = \int_{-1}^{1} \rho(y)f(-\frac{1+y}{j})dy = 0$$

and

$$\phi_j(0) = \int_{-1}^{1} \rho(y)f(-\frac{y}{j})dy = \frac{1}{j} \int_{0}^{1} \rho(y)y^\alpha dy.$$

However for all $j$,

$$\|\phi_j\|_{C^\alpha(\tilde{\Omega})} \geq \frac{\phi_j(0) - \phi_j(-\frac{1}{j})}{(\frac{1}{j})^\alpha} = \int_{0}^{1} \rho(y)y^\alpha dy > 0.$$

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