On stability of a stationary solution to the Hotelling migration equation

T N Gogoleva, I N Shchepina, M V Polovinkina and S A Rabeeakh

1Department of economic theory and world economy, Voronezh State University, Holzunov str., 42-v, build 5-a, Voronezh, Russian Federation
2Department of information technologies and mathematical methods in Economics, Voronezh State University, Holzunov str., 42-v, build 5-a, Voronezh, Russian Federation
3Department of higher mathematics and information technologies, Voronezh State University of Engineering Technologies, Revolution Av., 19, Voronezh, Russian Federation
4Department of mathematical and applied analysis, Voronezh State University, Universitetskaya pl., 1, Voronezh, Russian Federation
E-mail: polovinkin@yandex.ru

Abstract. We prove a sufficient condition for stability of a stationary solution to a nonlinear partial differential equation describing migration processes.

1. Introduction
The ability to support growing populations within existing economic systems and environments has been one of the main concerns of societies throughout history. Population growth and migration processes have been the subject of research by demographers, ecologists, economists and mathematicians. Questions and challenges raised by complex demographic processes are often addressed practically and conceptually through the use of mathematical models.

One of the first researchers into population dynamics was Thomas Malthus. Malthus observed in an essay written in 1798 that the growth of the human population was fundamentally different from the growth of the food supply to feed that population. He wrote that the human population was growing geometrically (i.e. exponentially) while the food supply was growing arithmetically (i.e. linearly). He concluded that left unchecked, it would only be a matter of time before the world’s population would be too large to feed itself. Malthus assumed that the rate, at which the population grew, was directly proportional to its current size. If the population at time $t$ is denoted by $p(t)$, then the assumption of natural growth can be written symbolically as (see [1])

$$\frac{dp}{dt} = kp,$$

where $k$ is a positive constant. Equation (1) is called the Malthusian model. The solution of equation (1) is the exponential function

$$p(t) = p_0e^{kt},$$

where $p_0$ is the initial population. The solution predicts
population explosion if \( k > 0 \),
- population extinction if \( k < 0 \),
- no change if \( k = 0 \).

The Malthusian model is commonly called the natural growth model or the exponential growth model. This model may be useful in situations in which the time scale of observation is small enough to make it acceptable to assume that \( k > 0 \) remains nearly constant, resources appear to be unlimited, and \( p_0 \) is small.

Later Pierre François Verhulst (1838) replaced the constant relative growth rate \( k \) in (1) by the relative growth rate

\[
k \left( 1 - \frac{p}{M} \right),
\]

that decreases linearly as a function of \( p \). The dimensionless factor \( k(1-p/M) \) serves to diminish the relative growth rate from \( k \) down to zero as the population increases from its initial level \( p_0 \) to \( M \). The constant \( M \) represents the maximum sustainable population beyond which \( p \) cannot increase. The resulting model (see [1]),

\[
\frac{dp}{dt} = kp \left( 1 - \frac{p}{M} \right),
\]

is called the logistic growth model or the Verhulst model. The Verhulst model assumes that the growth rate declines from a value \( k \), when conditions are very favorable, to the value 0, when the population has increased to the maximum value \( M \) that the environment can support. The solution of equation (2) is

\[
p(t) = \frac{M p_0}{p_0 + (M - p_0)e^{-kt}}.
\]

The logistic model predicts rapid initial growth for \( 0 < p_0 < M \), then a decrease in growth rate as time passes so that the size of the population approaches a limit. This behavior is in agreement with the observed behavior of many populations, and for this reason, the logistic model is often used as a means of describing population size.

The Verhulst model and the Malthusian model didn’t take into account population migration. Harold Hotelling (1921) added to the Verhulst equation diffusion term describing migration. As a result, the equation took the form (see [2])

\[
\frac{\partial p}{\partial t} = A(s - p)p + B \Delta p,
\]

where \( p = p(x_1, x_2, t) \) is the population density,

\[
\Delta = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}
\]

is the Laplace operator,

\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right)
\]

is the Hamilton operator (symbolic vector), \( A \) is the population growth rate, \( B \) is the migration rate, \( s \) is the coefficient of the saturated population density, \( t \) is the time parameter, \( x_1, x_2 \) are the geographical coordinates.

The Hotelling model describes both population growth and migration processes. Population growth is modelled as a logistic process. Migration processes are described using Fourier’s Law of Heat Conduction. Hotelling introduced the notion of saturated population density. If
a real population density is higher than a saturated one, the population decreases, if a real population density is lower than a saturated one, the population increases. The spatial diffusion was explained by the fact that population growth declines output per capita, and people move from more populated places to less populated ones.

A significant weakness of the Hotelling model is that livelihood stocks are assumed to be equal to a given constant, not depending on time and population (labour force). Therefore, this model is more suitable for animal populations, as evidenced through its successful application in ecology in 30 years after creation.

Men, unlike animals, can produce their own livelihood, moreover, as a result of technical progress, this production becomes increasingly effective. For biological populations, the quantity of food stocks also changes over time, but these changes, resulting from external factors rather than from activity of populations, are more of a cyclical nature than a long-term one. And furthermore, when certain stocks are already established, there is no fixed relationship between them and an unsaturated population for which a share of population change is exactly equal to zero.

In this regard, T. Puu proposed to replace the growth term \( p(1 - p) \) by the term

\[
q = \alpha(\beta p^2 - p^3),
\]

where \( q \) is the production function with a scale effect, increasing at the beginning and decreasing at the end:

\[
q = \alpha(\beta p^2 - p^3).
\]

The coefficient \( \alpha \) determines technological efficiency, and \( \beta \) represents a threshold scale of production. The transition of scale effect from increasing to decreasing occurs when \( p = 1/2 \beta \). The output reaches a maximum at \( p = 2/3 \beta \) and reduced to zero at \( p = \beta \). The coefficient \( \gamma \) represents an acceptable per capita income, leading to stationarity. One can choose time and population units in such a way that the growth rate and the stationary population in the Hotelling model will be equal to one.

After introducing the explicit production function, T. Puu stated that it is also required to modify the original diffusion term in the Hotelling model. According to his reasoning, the population spreads from places with higher density to places with lower density, because the reduction of income from per capita production has a negative impact on population size. Based on these arguments, T. Puu suggested as relation, describing the growth and the diffusion of human population, the following equation:

\[
\frac{\partial p}{\partial t} = p(1 + \alpha(\beta p^2 - p^3) - \gamma p) + \frac{1}{6} \Delta(\alpha(3\beta p^2 - 2p^3)).
\]

(4)

T. Puu [2] considered a problem of stability of a stationary solution (denoted by \( \pi \)) to equation (3) and showed that a condition

\[
\pi > s/2
\]

(5)

is sufficient for the stability.

For equation (4), T. Puu also considered the question of stability of a stationary solution (again denoted by \( \pi \)). He proved that the condition

\[
\mu \leq 0,
\]

(6)

where

\[
\mu = 1 + \alpha(3\beta \pi^2 - 4\pi^3) - 2\gamma \pi,
\]

(7)
is sufficient for the stability of the stationary solution to equation (4).

V. Z. Meshkov, I. P. Polovinkin and M. E. Semenov [3] replaced this condition by a weaker one. They showed that the condition

\[ \pi > \frac{s}{2} - \frac{B}{2A d^2}, \]

(8)

where \( d \) is diameter of a domain, is sufficient for the stability too.

These authors also improved the condition (6). They proved that the condition

\[ \mu < \frac{\alpha \pi (\beta - \pi)}{d^2}, \]

where \( \mu \) is defined by the formula (7), is sufficient for the stability of the stationary solution to equation (4).

Let’s select units of measurement in such a way that all of exogenous constants (the population growth rate, the migration rate and the coefficient of saturated population density) become equal to one. Then equation (3) takes the form

\[ \frac{\partial p}{\partial t} = (1 - p)p + \Delta p. \]

(9)

Below we consider another nonlinear partial differential equation that models a physical process and at the same time biological and social processes. T. Puu [2] addressed the issue of sufficient condition of stability for stationary solutions to this equation. That condition was somewhat overstated. The purpose of this paper is to show that the stability of the stationary solution will be achieved with less stringent restrictions. We will prove the above-mentioned assertion, following [3] and taking into account the geometry of the area in which the equation is considered.

2. Materials and methods

In this paper we use classical methods of mathematical analysis and theory of partial differential equations.

3. Model description

Let \( \Omega \) be a bounded domain in the plane of variables \( x_1 \) and \( x_2 \) with a piecewise smooth boundary \( \Gamma \) and a diameter \( d \). Let us consider the Hotelling equation in \( \Omega \) [2].

\[ \frac{\partial p}{\partial t} = (1 - p)p + \Delta p + (\vec{u} + p\vec{v}) \nabla p, \]

(10)

where \( \vec{u}, \vec{v} \) are two constant vectors, determining the intensity and direction of autonomous components movements. We consider, that all of variables are chosen to be dimensionless.

4. A stationary solution of the equation and its stability

In accordance with general positions of systems theory we define a stationary solution to equation (10) as a function \( \pi(x_1, x_2) \), which satisfies equation (10) and does not depend on time variable \( t \), that is

\[ \frac{\partial \pi}{\partial t} = 0. \]

In another words the stationary solution of equation (10) is a solution to the equation

\[ (1 - \pi) \pi + \Delta \pi + (\vec{u} + \pi \vec{v}) \nabla \pi = 0. \]

(11)
Further, as usual, we consider the stationary solution to be stable if after any small deviation from it, the solution, received as a result of the deviation, has to approach to this stationary solution.

"Smallness" of deviation will mean for us as follows. We will multiply the equation by a nonnegative integer power of a deviation and integrate over $\Omega$. Then we will try to judge the stationary solution stability by a sign of the derivative with respect to variable $t$ from the weighted integral of the deviation. If the deviation is small enough, terms, involved in a certain sum, containing degrees of deviation greater than 2, will not have influence on the sign of the sum.

5. Conclusion of the sufficient condition for the stability of the stationary solution

Let

$$z = p - \pi$$

be a small deviation from the stationary solution. Then

$$p = \pi + z.$$ (12)

Substituting the expression (12) into equation (10), we obtain

$$\frac{\partial z}{\partial t} = (1 - \pi - z)(\pi + z) + \Delta\pi + \Delta z + (\bar{u} + (\pi + z) \bar{v}) (\nabla\pi + \nabla z)$$

or after opening the brackets,

$$\frac{\partial z}{\partial t} = (1 - \pi) \pi + \Delta\pi + (\bar{u} + \pi \bar{v}) \nabla\pi + (1 - 2\pi) z +$$

$$+ \Delta z + (\bar{u} + \pi \bar{v} + z\bar{v}) \nabla z - z^2 + z\bar{v} \nabla\pi.$$ 

With respect to the fact that $\pi$ is a stationary solution and satisfies (11) we obtain

$$\frac{\partial z}{\partial t} = (1 - 2\pi) z + \Delta z + (\bar{u} + \pi \bar{v} + z\bar{v}) \nabla z + z\bar{v} \nabla\pi - z^2.$$ (13)

In order to research the stability of a stationary solution let’s multiply equation (13) by $z$ and integrate over the domain $\Omega$, where we consider equation (10). After that we get

$$\int \int_{\Omega} z \frac{\partial z}{\partial t} \ dx = \int \int_{\Omega} (1 - 2\pi) z^2 \ dx +$$

$$+ \int \int_{\Omega} z \Delta z \ dx + \int \int_{\Omega} z (\bar{u} + \pi \bar{v} + z\bar{v}) \nabla z \ dx +$$

$$+ \int \int_{\Omega} z^2 \bar{v} \nabla\pi \ dx - \int \int_{\Omega} z^3 \ dx.$$ (14)

Denote $dx = dx_1 dx_2$. Let’s consider separately every term on the right hand side. We have to proceed from the fact that solution $p$ and stationary solution $\pi$ take the same values on the
boundary $\Gamma = \partial \Omega$. Therefore, the deviation $z$ vanishes on the boundary $\Gamma$. In accordance with the Green formula [4] for two functions $f \in C^1(\Omega)$ and $g \in C^2(\Omega) \cap C^1(\Omega)$, one has

$$
\iint_{\Omega} f \Delta g dx = - \iint_{\Omega} \nabla f \nabla g dx + \int_{\Gamma} f \frac{\partial g}{\partial \nu} ds,
$$

where $\nu$ is a unit external normal vector to $\Gamma$. Here $ds$ is an arc element of the boundary $\Gamma = \partial \Omega$. Substituting $f = z, g = z$, we obtain

$$
\iint_{\Omega} z \Delta z dx = - \iint_{\Omega} (\nabla z)^2 dx + \int_{\Gamma} z \frac{\partial z}{\partial \nu} ds.
$$

The deviation $z$ vanishes on the boundary $\Gamma$ as we noted. Hence every integral of the deviation and its powers over $\Gamma$ is equal to zero. That’s why the last term on the right hand side, i.e. the integral over $\Gamma$ is equal to zero too and we have

$$
\iint_{\Omega} z \Delta z dx = - \iint_{\Omega} (\nabla z)^2 dx.
$$

Furthermore we have

$$
z \vec{u} \nabla z = \frac{1}{2} \sum_i \frac{\partial}{\partial x_i} (u_i z^2) = \frac{1}{2} \text{div} (z^2 \vec{u}).
$$

In accordance with the Gauss-Ostrogradskiy formula, one has

$$
\iint_{\Omega} \text{div} \vec{F} dx = \int_{\Gamma} \vec{F} \cdot \vec{\nu} ds,
$$

where

$$\vec{F} = (F_1(x_1, x_2), F_2(x_1, x_2)), \quad F_j(x_1, x_2) \in C^1(\Omega) \cap C(\Omega), \quad j = 1, 2,$$

$$\text{div} \vec{F} = \nabla \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}.$$ 

With respect to the fact that $z$ vanishes at $\Gamma$, we obtain

$$
\iint_{\Omega} \vec{u} \ z \Delta z dx = \frac{1}{2} \iint_{\Omega} \text{div} (z^2 \vec{u}) dx =
$$

$$= \frac{1}{2} \int_{\Gamma} z^2 \vec{u} \cdot \vec{\nu} ds = 0.
$$

For the integral of the product $z^2 \vec{u} \nabla z$, we get in the same way

$$
\iint_{\Omega} z^2 \vec{u} \nabla z dx = \frac{1}{3} \iint_{\Omega} \text{div} (z^3 \vec{u}) dx =
$$

$$= \frac{1}{3} \int_{\Gamma} z^3 \vec{u} \cdot \vec{\nu} ds = 0.
$$

In addition,
\[ z \, \pi \, \vec{v} \, \nabla z = \pi \sum_i v_i \, z \frac{\partial z}{\partial x_i} = \]
\[ = \frac{1}{2} \pi \sum_i \frac{\partial}{\partial x_i} (v_i \, z^2) = \]
\[ = \frac{1}{2} \sum_i \frac{\partial}{\partial x_i} (v_i \, \pi \, z^2) - \frac{1}{2} \sum_i v_i \, z^2 \frac{\partial \pi}{\partial x_i} = \]
\[ = \frac{1}{2} \text{div}(\pi z^2 \vec{v}) - \frac{1}{2} z^2 \vec{v} \, \nabla \pi. \]

Integrating this equality, we obtain
\[ \iint_\Omega z \, \pi \, \vec{v} \, \nabla z \, dx = \]
\[ = \frac{1}{2} \iint_\Omega \text{div}(\pi z^2 \vec{v}) \, dx - \frac{1}{2} \iint_\Omega z^2 \vec{v} \, \nabla \pi \, dx. \]

Now, using the Ostrogradskiy-Gauss formula, we obtain
\[ \iint_\Omega z \, \pi \, \vec{v} \, \nabla z \, dx = \frac{1}{2} \iint_\Gamma \pi z^2 \vec{v} \, ds - \frac{1}{2} \iint_\Omega z^2 \vec{v} \, \nabla \pi \, dx. \]

The above expression contains an integral of the product with the factor \( z^2 \) over boundary \( \Gamma \).

It is equal to zero, as previously mentioned. So we have
\[ \iint_\Omega z \, \pi \, \vec{v} \, \nabla z \, dx = -\frac{1}{2} \iint_\Omega z^2 \vec{v} \, \nabla \pi \, dx. \]

Thus, multiplying equation (13) by the deviation \( z \) and integrating over the domain \( \Omega \), we finally obtain an equality
\[ \frac{1}{2} \frac{\partial}{\partial t} \iint_\Omega z^2 \, dx = \iint_\Omega \left( 1 - 2 \pi - \frac{1}{2} \vec{v} \, \nabla \pi \right) \, z^2 \, dx - \]
\[ - \iint_\Omega (\nabla z)^2 \, dx - \iint_\Omega z^3 \, dx. \]

We are interested in a sign of this expression. If it is negative, then the stationary solution is stable [2].

So far we have made our computations in accordance with T. Puu [2]. When T. Puu obtains a similar expression he notes that

i) the absolute value of
\[ \iint_\Omega z^3 \, dx \]

is too small in comparison with absolute value of
\[ \iint_\Omega \left( 1 - 2 \pi - \frac{1}{2} \vec{v} \, \nabla \pi \right) \, z^2 \, dx - \iint_\Omega (\nabla z)^2 \, dx \]

(16)
so that its sign doesn’t influence on the sign of (15) (moreover he ignores terms containing powers of \( z \) more that 2-nd before integrating) and we respect this fact too;

ii) the term

\[-\iint_\Omega (\nabla z)^2 \, dx\]

is negative and we respect this fact too. Furthermore T. Puu noted that if

\[1 - 2\pi - \frac{1}{2} \vec{v} \nabla \pi < 0,\]  

(17)

then the left hand side of (15) is negative and the stationary solution is stable. Certainly that’s true.

Another approach was suggested by V.Z. Meshkov in [3]. In accordance with it we use the Poincare-Steklov-Friedrichs inequality (see [5], [6], [7], [8]). By this inequality we get

\[\iint_\Omega |\nabla z^2| \, dx \geq \frac{1}{d^2} \iint_\Omega z^2 \, dx.\]

Hence we obtain

\[\frac{1}{2} \frac{\partial}{\partial t} \iint_\Omega z^2 \, dx \leq \iint_\Omega \left(1 - 2\pi - \frac{1}{2} \vec{v} \nabla \pi - \frac{1}{d^2}\right) z^2 \, dx - \iint_\Omega z^3 \, dx.\]

We assumed that the deviation \( z \) is small enough. More exactly let \( \varepsilon > 0 \) and

\[|z| = |z^3/z^2| < \varepsilon.\]

If

\[\varepsilon = \frac{1}{2} \left|1 - 2\pi - \frac{1}{2} \vec{v} \nabla \pi - \frac{1}{d^2}\right|,\]

then

\[|z^3| < \frac{1}{2} \left|1 - 2\pi - \frac{1}{2} \vec{v} \nabla \pi - \frac{1}{d^2}\right| z^2.\]

Integrating this inequality we obtain

\[\left|\iint_\Omega z^3 \, dx\right| < \frac{1}{2} \left|\iint_\Omega \left(1 - 2\pi - \frac{1}{2} \vec{v} \nabla \pi - \frac{1}{d^2}\right) z^2 \, dx\right|,\]

so that the sign of integral

\[\iint_\Omega z^3 \, dx\]

doesn’t influence on the sign of the sum.

Finally we have proved the theorem about a sufficient condition for the stability of a stationary solution to the Hotelling equation.
Theorem. Let $\pi = \pi(x_1, x_2)$ be a stationary solution to the Hotelling equation. Let the inequality

$$1 - \frac{1}{d^2} - 2\pi - \frac{1}{2} \vec{v} \cdot \nabla \pi < 0$$

(18)

holds. Then $\pi$ is stable.

Condition (18), which we obtained as a sufficient condition for the stability of the stationary solution to the Hotelling equation, is softer than condition (17) by T. Puu [2]. When $\vec{u} = 0, \vec{v} = 0$, condition (18), which is (8) for the stability of the stationary solution, was obtained in [3]. Let’s note that our condition (18) improves T. Puu’s condition (17), essentially, when the diameter $d$ of the domain $\Omega$ is small. If it is large, then condition (18) is not far from (17).

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