An interior boundedness result for an elliptic equation.

Samy Skander Bahoura*
Equipe d’Analyse Complexe et Géométrie.
Université Pierre et Marie Curie, 75005 Paris, France.

Abstract
We derive a local uniform boundedness result for an equation with weight having interior singularity.

Keywords: C° weight, interior singularity, a priori estimate, maximum principle.
MSC: 35J60, 35B44, 35B45, 35B50

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on open set $\Omega$ of $\mathbb{R}^2$ with a smooth boundary.

We consider the following equation:

\[
(P) \begin{cases}
-\Delta u = \frac{1}{-\log |x|/2d} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\
u = 0 & \text{in } \partial\Omega.
\end{cases}
\]

Here:

\[
0 \leq V \leq b, \quad \int_{\Omega} \frac{1}{-\log |x|/2d} e^u dx \leq C, \quad u \in W^{1,1}_0(\Omega),
\]

and,

\[
d = \text{diam}(\Omega), \quad 0 \in \Omega
\]

Equations of the previous type were studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [11][19], where one can find some existence and compactness results.

Among other results, we can see in [11] the following important Theorem:

Theorem A (Brezis-Merle [11]). If $(u_i)$ is a sequence of solutions of problem $(P)$ with $(V_i)$ satisfying $0 < a \leq V_i \leq b < +\infty$ and without the term \(\frac{1}{-\log |x|/2d}\),

then, for any compact subset $K$ of $\Omega$, it holds:

*e-mails: samybahoura@yahoo.fr, samybahoura@gmail.com

1
\[
\sup_K u_i \leq c,
\]

with \(c\) depending on \(a, b, K, \Omega\).

One can find in [11] an interior estimate if we assume \(a = 0\), but we need an assumption on the integral of \(e^{u_i}\), namely, we have:

**Theorem B** (Brezis-Merle [11]). For \((u_i)_i\) and \((V_i)_i\), two sequences of functions relative to the problem \((P)\) without the term \(\frac{1}{x} - \log \frac{|x|}{2d}\) and with,

\[
0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,
\]

then for all compact set \(K\) of \(\Omega\) it holds;

\[
\sup_K u_i \leq c,
\]

with \(c\) depending on \(b, C, K\) and \(\Omega\).

If we assume \(V\) with more regularity, we can have another type of estimates, a \(\sup + \inf\) type inequalities. It was proved by Shafrir see [18], that, if \((u_i)_i\) is a sequence of functions solutions of the previous equation without assumption on the boundary with \(V_i\) satisfying \(0 < a \leq V_i \leq b < +\infty\), then we have a \(\sup + \inf\) inequality.

Here, we have:

**Theorem** For sequences \((u_i)_i\) and \((V_i)_i\) of the Problem \((P)\), for all compact subsets \(K\) of \(\Omega\) we have:

\[
||u_i||_{L^\infty(K)} \leq c(b, C, K, \Omega),
\]

**Remark:** Remark that we have a \(C^0\) weight \(\frac{1}{x} - \log \frac{|x|}{2d}\), the solutions are not \(C^2\), but if we add some assumptions on \(V_i\) one can consider \(C^2\) solutions and \(C^2\) convergence of sequences.

On can have the regularity \(C^2\) of the solutions and the \(C^2\) convergence of the solutions if we suppose for example \(V_i \in C^{0, \epsilon}, \epsilon > 0\) and for the convergence \(V_i \rightarrow V\) in the space \(C^{0, \epsilon}, \epsilon > 0\). Indeed, one can reduce the problem to regularity and convergence of the Newtonian potential of a radial distribution \(f(x) = f(|x|) = V_i(0)e^{u_i(0)} - \log \left(\frac{|x|}{2d}\right) \eta(|x|), \text{ with } \eta \text{ a cutoff function (}\eta \equiv 1 \text{ in a neighborhood of } 0\right)\), see for example the book of Dautray-Lions, chapter 2, Laplace operator.

By a duality theorem one can prove that (see [12]):

\[
||\nabla u_i||_q \leq C_q, \ \forall 1 \leq q < 2.
\]

If we add the assumption that
then by a result of Chen-Li of "moving-plane" we have a compactness of \((u_i)\), near the boundary, see [13].

We ask the following question about inequality of type \(\sup + \inf\), as in the work of Tarantello, see [19] and Bartolucci-Tarantello, see [8]:

**Problems.** 1) Consider the Problem \((P)\) without the boundary condition (without Dirichlet condition) and assume that:

\[
0 < a \leq V \leq b < +\infty,
\]

Does exists constants \(C_1 = C_1(a, b, K, \Omega), C_2 = C_2(a, b, K, \Omega)\) such that:

\[
\sup_K u + C_1 \inf_{\Omega} u \leq C_2,
\]

for all solution \(u\) of \((P)\) ?

2) If we add the condition \(||\nabla V||_\infty \leq A\), can we have a sharp inequality:

\[
\sup_K u + \inf_{\Omega} u \leq c(a, b, A, K, \Omega)?
\]

2 Proof of the Theorem

We have:

\[
u_i \in W^{1,1}_0(\Omega), \quad \frac{1}{-\log \frac{|x|}{2d}} e^{u_i} \in L^1(\Omega).
\]

Thus, by corollary 1 of Brezis and Merle we have:

\[
e^{u_i} \in L^k(\Omega), \forall k > 2.
\]

Using the elliptic estimates and the Sobolev embedding, we have:

\[
u_i \in W^{2,1}(\Omega) \cap C^{1,\epsilon}(\Omega).
\]

By the maximum principle \(u_i \geq 0\).

Also, by a duality theorem or a result of Brezis-Strauss, we have:

\[
||\nabla u_i||_q \leq C_q, \quad 1 \leq q < 2.
\]

Since,

\[
\int_\Omega \frac{1}{-\log \frac{|x|}{2d}} V_i e^{u_i} dx \leq C,
\]

We have a convergence to a nonegative measure \(\mu\):

\[
\int_\Omega \frac{1}{-\log \frac{|x|}{2d}} V_i e^{u_i} \phi dx \to \int_\Omega \phi d\mu, \quad \forall \phi \in C_c(\Omega).
\]
We set $S$ the following set:

$$S = \{ x \in \Omega, \exists (x_i) \in \Omega, x_i \to x, u_i(x_i) \to +\infty \}.$$  

We say that $x_0$ is a regular point of $\mu$ if there function $\psi \in C_c(\Omega)$, $0 \leq \psi \leq 1$, with $\psi = 1$ in a neighborhood of $x_0$ such that:

$$\int \psi d\mu < 4\pi. \quad (1)$$

We can deduce that a point $x_0$ is non-regular if and only if $\mu(x_0) \geq 4\pi$.

A consequence of this fact is that if $x_0$ is a regular point then:

$$\exists R_0 > 0 \text{ such that one can bound } (u_i) = (u_i^+) \text{ in } L^\infty(B_{R_0}(x_0)). \quad (2)$$

We deduce (2) from corollary 4 of Brezis-Merle paper, because we have by the Gagliardo-Nirenberg-Sobolev inequality:

$$||u_i||_1 = ||u_i|| \leq c_q ||u_i||^q \leq C_q' ||\nabla u||_q \leq C_q, \quad 1 \leq q < 2.$$  

We denote by $\Sigma$ the set of non-regular points.

**Step 1:** $S = \Sigma$.

We have $S \subset \Sigma$. Let’s consider $x_0 \in \Sigma$. Then we have:

$$\forall R > 0, \lim ||u_i^+||_{L^\infty(B_R(x_0))} = +\infty. \quad (3)$$

Suppose contrary that:

$$||u_i^+||_{L^\infty(B_R(x_0))} \leq C.$$  

Then:

$$||e^{u_i^+}||_{L^\infty(B_R(x_0))} \leq C, \quad \text{and}$$

$$\int_{B_R(x_0)} \frac{1}{1 - \log \frac{|x|}{2d}} V_i e^{u_i^+} = o(1).$$

For $R$ small enough, which imply (1) for a function $\psi$ and $x_0$ will be regular, contradiction. Then we have (3). We choose $R_0 > 0$ small such that $B_{R_0}(x_0)$ contain only $x_0$ as non-regular point. $\Sigma$. Let’s $x_i \in B_R(x_0)$ such that:

$$u_i^+(x_i) = \max_{B_R(x_0)} u_i^+ \to +\infty.$$  

We have $x_i \to x_0$. Else, there exists $x_{ik} \to \bar{x} \neq x_0$ and $\bar{x} \notin \Sigma$, i.e. $\bar{x}$ is a regular point. It is impossible because we would have (2).

Since the measure is finite, if there are blow-up points, or non-regular points, $S = \Sigma$ is finite.

**Step 2:** $\Sigma = \{ \emptyset \}$.

Now: suppose contrary that there exists a non-regular point $x_0$. We choose a radius $R > 0$ such that $B_R(x_0)$ contain only $x_0$ as non-regular point. Thus outside $\Sigma$ we have local uniform boundedness of $u_i$, also in $C^1$ norm. Also, we have weak $^*$-convergence of $V_i$ to $V \geq 0$ with $V \leq b$.  

\[\text{4} \]
Let’s consider (by a variational method):

\[ z_i \in W^{1,2}_0(B_R(x_0)), \]

\[ -\Delta z_i = f_i = \frac{1}{-\log \frac{|x|}{2d}} V e^{\eta} \quad \text{in} \quad B_R(x_0), \quad \text{et} \quad z_i = 0 \quad \text{on} \quad \partial B_R(x_0). \]

By a duality theorem:

\[ z_i \in W^{1, q}_0(B_R), \quad ||\nabla z_i||_q \leq C_q. \]

By the maximum principle, \( u_i \geq z_i \) in \( B_R(x_0) \).

\[ \int -\log \frac{|x|}{2d} e^{z_i} \leq \int -\log \frac{|x|}{2d} e^{u_i} \leq C. \quad (4) \]

On the other hand, \( z_i \to z \) a.e. (uniformly on compact sets of \( B_R(x_0) - \{x_0\} \)) with \( z \) solution of:

\[ -\Delta z = \mu \quad \text{in} \quad B_R(x_0), \quad \text{et} \quad z = 0 \quad \text{on} \quad \partial B_R(x_0). \]

Also, we have up to a subsequence, \( z_i \to z \) in \( W^{1, q}_0(B_R(x_0)), 1 \leq q < 2 \) weakly, and thus \( z \in W^{1, q}_0(B_R(x_0)) \).

Then by Fatou lemma:

\[ \int -\log \frac{|x|}{2d} e^z \leq C. \quad (5) \]

As \( x_0 \in S \) is not regular point we have \( \mu(\{x_0\}) \geq 4\pi \), which imply that, \( \mu \geq 4\pi \delta_{x_0} \) and by the maximum principle in \( W^{1, 1}_0(B_R(x_0)) \) (obtained by Kato’s inequality)

\[ z(x) \geq 2 \log \frac{1}{|x - x_0|} + O(1) \text{ if } x \to x_0. \]

Because,

\[ z_1 \equiv 2 \log \frac{1}{|x - x_0|} + 2 \log R \in W^{1, s}_0(B_R(x_0)), \quad 1 \leq s < 2. \]

Thus,

\[ \frac{1}{-\log \frac{|x|}{2d}} e^z \geq \frac{C}{-|x - x_0|^2 \log \frac{|x|}{2d}}, \quad C > 0. \]

Both in the cases \( x_0 = 0 \) and \( x_0 \neq 0 \) we have:

\[ \int_{B_R(x_0)} \frac{1}{-\log \frac{|x|}{2d}} e^z = \infty. \]

But, by (5):
\[
\int \frac{1}{-\log \frac{|x|}{2\theta}} e^z \leq C.
\]
which a contradiction.

References

[1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.

[2] C. Bandle. Isoperimetric Inequalities and Applications. Pitman, 1980.

[3] Bahoura, S.S. About Brezis Merle problem with Lipschitz condition. arXiv:0705.4004.

[4] Bartolucci, D. A "sup+Cinf" inequality for Liouville-type equations with singular potentials. Math. Nachr. 284 (2011), no. 13, 1639-1651.

[5] Bartolucci, D. A "sup+Cinf" inequality for the equation \(-\Delta u = Ve^u/|x|^{2\alpha}\). Proc. Roy. Soc. Edinburgh Sect. A 140 (2010), no. 6, 1119-1139.

[6] Bartolucci, D. A sup+inf inequality for Liouville type equations with weights. J. Anal. Math. 117 (2012), 29-46.

[7] Bartolucci, D. A sup \times inf-type inequality for conformal metrics on Riemann surfaces with conical singularities. J. Math. Anal. Appl. 403 (2013), no. 2, 571-579.

[8] Bartolucci, D. Tarantello. G. The Liouville equation with singular data: a concentration-compactness principle via a local representation formula, Journal of Differential Equations 185 (2002), 161-180.

[9] L. Boccardo, T. Gallouet. Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal. 87 no 1, (1989), 149-169.

[10] H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.

[11] H. Brezis, F. Merle. Uniform estimates and Blow-up behavior for solutions of \(-\Delta u = V(x)e^u\) in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).

[12] H. Brezis, W. A. Strauss. Semi-linear second-order elliptic equations in L1. J. Math. Soc. Japan 25 (1973), 565-590.

[13] W. Chen, C. Li. A priori estimates for solutions to nonlinear elliptic equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.

[14] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \(\mathbb{R}^2\). Commun. Anal. Geom. 6, No.1, 1-19 (1998).

[15] R. Dautray, J-L. Lions. Part 2, Laplace operator.
[16] YY. Li, I. Shafrir. Blow-up analysis for solutions of $-\Delta u = V e^u$ in dimension two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.

[17] YY. Li. Harnack Type Inequality: the method of moving planes. Commun. Math. Phys. 200, 421-444 (1999).

[18] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = V e^u$. C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

[19] G. Tarantello. A Harnack inequality for Liouville-type equation with Singular sources. Indiana University Mathematics Journal. Vol 54, No 2 (2005). pp 599-615.