A Riemann-Hilbert approach to the existence of global solutions to the Fokas-Lenells equation on the line

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Abstract

We obtain the existence of global solutions to the Cauchy problem of the Fokas-Lenells (FL) equation on the line

\[ u_{xt} + \alpha \beta^2 u - 2i\alpha \beta u_x - \alpha u_{xx} - i\alpha \beta^2 |u|^2 u_x = 0, \]
\[ u(x, t = 0) = u_0(x), \]

without the small-norm assumption on initial data \( u_0(x) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \).

Our main technical tool is the inverse scattering transform method based on the representation of a Riemann-Hilbert (RH) problem associated with the above Cauchy problem. The existence and the uniqueness of the RH problem is shown via a general vanishing lemma. The spectral problem associated with the FL equation is changed into an equivalent Zakharov-Shabat-type spectral problem to establish the RH problems on the real axis. By representing the solutions of the RH problem via the Cauchy integral protection and the reflection coefficients, the reconstruction formula is used to obtain a unique local solution of the FL equation. Further, the eigenfunctions and the reflection coefficients are shown Lipschitz continuous with respect to initial data, which provides a prior estimate of the solution to the FL equation. Based on the local solution and the uniformly prior estimate, we construct a unique global solution in \( H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \) to the FL equation.

Keywords: Fokas-Lenells equation, Riemann-Hilbert problem, Lipschitz continuous, prior estimate, global solutions.

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1 Introduction

In this paper, we study the existence of global solutions to the Cauchy problem of the Fokas-Lenells (FL) equation

\[ u_{tx} + \alpha \beta^2 u - 2i\alpha \beta u_x - \alpha u_{xx} + \sigma i \alpha \beta^2 |u|^2 u_x = 0 \]  

(1.1)

\[ u(x,t)|_{t=0} = u_0(x), \]  

(1.2)

The FL equation is an integrable generalization of the nonlinear Schrödinger (NLS) equation, which is also tightly related with the derivative NLS model. Analogous to the derivation of the Camassa-Holm equation from the KdV equation [1], the FL equation was initially derived from the NLS equation by utilizing two Hamiltonian operators [2]. In optics, the FL equation characterizes the higher-order linear and nonlinear optical effects, and has been derived as a suitable model for the femtosecond pulse propagation through single mode optical silica fiber, for which several interesting solutions have been constructed [3–5].

Prior studies on the FL equation mainly concentrate on constructing the soliton solutions and deriving the long time asymptotics. Lenells and Fokas are the first to discover the soliton solutions to the FL equation via the inverse scattering transform and the dressing methods [6, 7]. Matsuno obtained the bright and the dark soliton solutions to the FL equation via the Hirota method [8, 9]. Later, Liu et al. derived the double Wronskian solutions to the FL equation via a bilinear approach [10], Vekslerchik et al. constructed the lattice representation and the n-dark solitons of the FL equation in [11], while Wright et al. found the breather solutions of the FL equation via a dressing-Bäcklund transformation related to the Riemann-Hilbert (RH) problem formulation [12]. A kind of rogue wave solution to the FL equation were obtained by using the Darboux transformation [13]. The Fokas method was used to investigate the initial-boundary value problem for the FL equation on the half-line and a finite interval, respectively [14, 15]. An algebro-geometric method was used to obtain algebro-geometric solutions for the FL equation [16]. The explicit onesoliton of the initial value problem for the FL equation was given by the RH method [17]. The inverse scattering transform for the FL equation with nonzero boundary
conditions was further investigated by using the RH method [18]. Recently the perturbation theory was also used to obtain the exact solution to the Fokas-Lenells equation [19]. On long time asymptotics, our recent works respectively characterized the long time behaviors of defocusing and focusing FL equations [20, 21]. However, little is known on the existence of local and global solutions to the FL equation. In recent years, the global well-posedness of the periodic initial value problem for the FL equation in a Sobolev space was proved by Fokas and Himonas [22]. But to the best of our knowledge, the existence of local and global solutions to the FL equation on the line is still unknown.

In this paper, we would like to prove the existence of global solutions to the FL equation on the line from the perspective of inverse scattering transform. This technique has been applied to prove the existence of global solutions to the derivative NLS equation by Pelinovsky and Shimabukuro [23]. By establishing the Lipschitz continuity of the eigenfunctions and the scattering data, we obtain the Lipschitz continuous mapping between the initial data and the solution to the FL equation. In this way we are able to establish the existence of local and global solutions to the FL equation. Compared with the derivative NLS equation, we find that extension of this approach to the FL equation will confront some substantial difficulties, which are also different from the derivative NLS equation [23].

▷ **Spectral singularity** $k = 0$: For the FL equation, its Lax pair involves a singularity at $k = 0$, which is however the key to control the non-blow-up of the reflection coefficients during the time evolution analysis. So we need additional requirements to handle the Jost functions, scattering data and their norm estimates. The singularity $k = 0$ does not meet the premise in the Zhou’s technique [24, Theorem 1.8]. Therefore, existing techniques have to be strengthened for analyzing the more complicated FL equation. We overcome this technical difficulty of the FL equation by drawing the equivalence between the Jost function at $k = \infty$ and at $k = 0$. We obtain a key asymptotic estimate on scattering data, which sufficiently cover the singularity of reflection coefficient $r(k)$ at $k = 0$ in its time evolution. This allows us to extend the
usage of the Beals-Coifman theory and Zhou’s technique to the case of $k = 0$.

- **Two kinds of reconstruction formulas:** For the FL equation, we are only able to recover its potential $u_x(x, t)$ from the limit of the Jost function as $k \to \infty$. We should resort to the time spectral problem (2.2) to obtain the original potential $u(x, t)$ from the asymptotic expansion of the Jost function as $k \to 0$. From these results, we are able to derive the reconstruction formula for $u(x, t)$ as $k \to 0$. While we obtain other reconstruction formula for $u_x(x, t)$ as $z \to \infty$ in new $z$-plane. These two kinds of reconstruction formulas are combined together to obtain the estimates on the solution to the FL equation.

- We transform original spectral problem and RH problem from the $k$ plane to the $z$ plane by exploiting the odd-even property of the Jost functions in the column. This facilitates the further application of the Fourier transform and Cauchy integral projection to estimate the space of the solution to the RH problem and the potential.

The structure of the paper is as follows. In Section 2, we carry out inverse scattering transform on the $k$-plane. First, based on the Lax pair of the FL equation (1.1), we construct the Jost functions $\psi(x; k)$ and analyze their asymptotics of at two spectral singularities $k = 0$ and $k = \infty$. Further we construct a RH problem $N(x; k)$ associated with the Cauchy problem (1.1)-(1.2) and then prove its existence and the uniqueness via a general vanishing lemma. In Section 3, we carry out inverse scattering transform on the $z$-plane. We impose a transformation on $\psi(x; k)$ to obtain Jost function $\Psi(x; z)$ and a new RH problem for $M(x; z)$ on the $z$-plane. We further establish the Lipschitz continuous mapping from the initial data to the reflection coefficient $r_{1,2}(z)$. In Section 4, to estimate the solutions to the RH problem $M(x; z)$, we make transitions of the RH problem of $M(x; z)$ to the RH problem of $Q_{1,2}(x; k)$ which allows us to use the existing properties of the scattering data to derive the estimates on the Beals-Coifman solutions to the RH problem of $M(x; z)$. In Section 5, based on the reconstruction formulae and the Cauchy integral, subsequently we obtain estimates on the potential $u(x)$ for the FL equation on the positive and
negative half-lines respectively. In Section 6, based on the local solution and the uniformly priori estimates, we show that there exists a global solution $u(x, t) \in C([0, \infty), H^3(\mathbb{R}) \cup H^{2,1}(\mathbb{R})$ to the Cauchy problem (1.1)-(1.2) of the FL equation.

2 Inverse scattering transform on $k$-plane

In this section, we establish a RH problem associated with the Cauchy problem (1.1)-(1.2) of the FL equation and show its existence and uniqueness.

We first fix some notations used this paper. If $I$ is an interval on the real line $\mathbb{R}$ and $X$ is a Banach space, then $C(I, X)$ denotes the space of continuous functions on $I$ taking values in $X$. It is equipped with the norm

$$
\|u\|_{C(I, X)} = \sup_{x \in I} \|u(x)\|_X.
$$

We define weighted Sobolev space by

$$
L^{p,s}(\mathbb{R}) := \{u(x) \in L^p(\mathbb{R}) : \langle x \rangle^s u(x) \in L^p(\mathbb{R})\},
$$

$$
H^{k,s}(\mathbb{R}) = \{u(x) \in L^{2,s}(\mathbb{R}) : \partial_x^j u(x) \in L^{2,s}(\mathbb{R}), \quad j = 1, \cdots, k\},
$$

$$
W(\mathbb{R}) = \{r(z) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}), z^{-2} r(z) \in L^2(\mathbb{R})\},
$$

where $\langle x \rangle := \sqrt{1 + |x|^2}$.

2.1 Jost functions

The FL equation (1.1) admits a Lax pair

$$
\phi_x + i k^2 \sigma_3 \phi = k P_x \phi, \quad (2.1)
$$

$$
\phi_t + i \eta^2 \sigma_3 \phi = H \phi, \quad (2.2)
$$

where

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix},
$$

$$
\eta = \sqrt{\alpha} \left( k - \frac{\beta}{2k} \right), \quad H = \alpha k U_x + \frac{i \alpha \beta^2}{2} \sigma_3 \left( \frac{1}{k} P - P^2 \right).
$$

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The FL spectral problem (2.1) is not the Zakharov-Shabat type spectral problem due to the multiplication by $k$ in the matrix potential $kP_x$ and the time part (2.2) admits spectral singularity at $k = 0$. To control the behavior of the eigenfunction $\psi(k)$ as $k \to \infty$ and $k \to 0$ in constructing the solution $u(x, t)$ to the FL equation (1.1), we consider two different asymptotic expansions respectively at the singularities $k = 0$ and $k = \infty$. Here, we recall the existing results on constructing the RH problem in [17, 21].

**Case I: $k = \infty$**

By making a transformation

$$\psi(x, t; k) = \phi(x, t; k)e^{i(k^2x + \eta^2t)\sigma_3}, \quad (2.4)$$

the Lax pair (2.1)-(2.2) is changed to

$$\psi_x + ik^2[\sigma_3, \psi] = kP_x \psi, \quad (2.5)$$
$$\psi_t + i\eta^2[\sigma_3, \psi] = H\psi. \quad (2.6)$$

This Lax pair admits the Jost functions with asymptotics

$$\psi^\pm(x, t; k) \sim I, \quad x \to \pm \infty,$$

which satisfy Volterra integral equations

$$\psi^\pm(x, t; k) = I + k \int_{-\infty}^\infty e^{-2ik^2(x-y)\sigma_3} P_y(y) \psi^\pm(y, t; k) dy. \quad (2.7)$$

It is obvious that the integration in (2.7) involves the term $ku_x(x)$ which is not $L^2(\mathbb{R})$ bounded since $k$ may go to infinity. In our previous work, through the small $k$ and large $k$ estimates respectively, we overcome this difficulty and prove the existence and the differentiability of the Jost functions $\psi^\pm(x, t; k)$ [21].

**Proposition 1.** Let $u_0(x) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$, and introduce the notation $\psi^\pm(x, t; k) = (\psi^+_1(x, t; k), \psi^+_2(x, t; k))$ with the scripts 1 and 2 denoting the first and second columns of $\psi^\pm(x, t; k)$. Then we have
**Analyticity:** The integral equation (2.7) admits a unique solution $\psi^\pm(x, t; k)$. Moreover, $\psi_1^-(x, t; k)$, $\psi_1^+(x, t; k)$ and $a(k)$ are analytical in the domain $D_+$; and $\psi_2^+(x, t; k)$ and $\psi_2^-(x, t; k)$ are analytical in the domain $D_-$, where (see Figure 1)

$$D_+ = \{k : \text{Im}k^2 > 0\}, \quad D_- = \{k : \text{Im}k^2 < 0\}; \quad (2.8)$$

![Figure 1: The analytical domains $D_+$ and $D_-$ for $\psi^\pm(x, t; k)$](image)

**Symmetry:** $\psi^\pm(x, t; k)$ and $S(k)$ satisfy the symmetry relations

$$\psi^\pm(x, t; k) = \sigma_2\psi^\pm(x, t; k^*)\sigma_2, \quad \psi^\pm(x, t; k) = \sigma_3\psi^\pm(x, t; -k)\sigma_3, \quad (2.9)$$

**Asymptotics:** $\psi^\pm(x, t; k)$ and $S(k)$ have asymptotic properties

$$\psi^\pm(x, t; k) = e^{-i c_\pm(x)\sigma_3} + O(k^{-1}), \quad k \to \infty, \quad (2.10)$$

where

$$c_\pm(x) = \frac{1}{2}\int_{\pm\infty}^x |u_y(y, t)|^2 dy. \quad (2.11)$$

It is noteworthy that $\psi^\pm(x, t; k)$ do not approach to the identity matrix as $k \to \infty$. To normalize the first term in the asymptotics, we define the second new function $\mu^\pm(x, t; k)$ which satisfies the following relation

$$\psi^\pm(x, t; k) = e^{-i c_\pm(x)\sigma_3}\mu^\pm(x, t; k)e^{i c_\pm(x)\sigma_3}, \quad (2.12)$$
then \( \mu^{\pm}(x, t; k) \) satisfy the following Volterra-type integral

\[
\mu^{\pm}(x, t; k) = I + \int_{\pm\infty}^{x} e^{-ik^{2}(x-y)\hat{\sigma}_3} (P_{1}\mu^{\pm})(y, t; k) \, dy,
\]  

(2.13)

where \( P_{1} \) is given by

\[
P_{1} = e^{\frac{i}{2} \int_{-\infty}^{x} |u_{y}(y, t)|^{2} dy} \hat{\sigma}_3 (kP_{x} + \frac{i}{2} |u_{x}|^{2} \hat{\sigma}_3).
\]

The potential can be recovered from the limit of the Jost function \( \psi(x, t; k) \)

\[
u_{x}(x, t) = 2 \text{Im}(m(x, t)) e^{\frac{2i}{2} \int_{-\infty}^{x} |m(s, t)|^{2} ds},
\]  

(2.14)

where

\[
m(x, t) = \lim_{k \to \infty} (k\psi(x, t; k))_{12}.
\]

The formula (2.14) shows that we only recover the potential \( u_{x}(x, t) \) from the Jost function as \( k \to \infty \). We should resort to the time spectral problem (2.2) to obtain the original potential \( u(x, t) \) from the asymptotic expansion of the Jost function \( \psi(x, t, k) \) as \( k \to 0 \).

**Case II: \( k = 0 \)**

We consider the transformation

\[
\varphi(x, t, k) = \phi(x, t; k) e^{i(k^{2}x + \eta^{2}t)\sigma_{3}},
\]  

(2.15)

then the Lax pair (2.1)-(2.2) changes to

\[
\begin{align*}
\varphi_{x} + ik^{2}[\sigma_{3}, \varphi] &= kP_{x}\varphi, \\
\varphi_{t} + i\eta^{2}[\sigma_{3}, \varphi] &= H\varphi,
\end{align*}
\]  

(2.16)

(2.17)

which is integrated along \((\pm\infty, t) \to (x, t)\) and leads to two Volterra-type integrals

\[
\varphi^{\pm}(x, t; k) = I + k \int_{\pm\infty}^{x} e^{-2ik^{2}(x-y)\hat{\sigma}_3} P_{y}(y)\varphi^{\pm}(y, t; k) \, dy.
\]  

(2.18)

We seek the asymptotic expansion of \( \varphi \) in the Lax pair (2.16)-(2.17) as \( k \to 0 \) and find that

\[
\varphi(x, t; k) = I + k \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} + O(k^{2}), \quad k \to 0,
\]  

(2.19)
which implies that the solution $u(x, t)$ to the FL equation can be reconstructed by

$$u(x, t) = \lim_{k \to 0} (k^{-1} \varphi(x, t; k))_{12}. \quad (2.20)$$

By establishing the RH problem, we strive to get a reconstruction formula for $u(x, t)$.

From $(2.12)$ and $(2.15)$, we observe that the $\mu^\pm(x, t; k)$, $\psi^\pm(x, t; k)$ and $\varphi^\pm(x, t; k)$ are related to the same Lax pair $(2.1)$-$(2.2)$ and therefore satisfy the relation

$$\psi^\pm(x, t; k) = e^{-i \sigma_3} \mu^\pm(x, t; k) e^{i \sigma_3},$$

$$\mu^\pm(x, t; k) = e^{-i \sigma_3} \varphi^\pm(x, t; k) e^{-i(k^2 + \eta^2) t} \sigma_3 C^\pm(k) e^{i(k^2 + \eta^2) t} \sigma_3 e^{i \sigma_3}. \quad (2.21)$$

By letting $x \to \pm \infty$ respectively, we find

$$C^-(k) = e^{-i \sigma_3}, \quad C^+(k) = I.$$

From $(2.19)$ and $(2.21)$, the solution $u(x, t)$ to the FL equation can be expressed as

$$u(x, t)e^{2i(\varphi(x) + c)} = \lim_{k \to 0} (k^{-1} \psi^+(x, t; k))_{12}, \quad (2.22)$$

$$u(x, t)e^{-i(2c - \varphi(x)) + c} = \lim_{k \to 0} (k^{-1} \psi^-(x, t; k))_{12}. \quad (2.23)$$

**Remark 1.** In particular, we use the relation between $u(x, t)$ and $\psi(x, t; k)$ instead of $u(x, t)$ and $\mu(x, t; k)$. The benefits of this approach will be seen later in the transformation $A(x; k)$, where the linear spectral problem $(2.1)$ is in turn transformed to a spectral problem of the Zakharov-Shabat type.

In the following sections, we first consider the partial spectral problem $(2.1)$ with $t$ being a parameter, so we omit the variable $t$ as usual, for example $\psi(x, t; k)$ is just written as $\psi(x; k)$. We will discuss the time evolution of scattering data and reconstruct the potential $u(x, t)$ with $t$ in the Section 6.

### 2.2 A basic RH problem

Since $\psi^\pm(x, t; k)$ are two solutions to the spectral problem $(2.1)$, they are linearly dependent and satisfy the scattering relation

$$\psi^-(x, t; k) = \psi^+(x, t; k) e^{-ik^2 x} S(k), \quad (2.24)$$
where $S(k)$ is a scattering matrix given by

$$S(k) = \begin{pmatrix} a(k) & b(k) \\ -b(k) & a(k) \end{pmatrix}, \quad \det S(k) = 1. \quad (2.25)$$

By Proposition 1 and (2.24), it is easy to show that $S(k)$ admits the following symmetries

$$S(k) = \sigma_2 S(\bar{k}) \sigma_2, \quad S(k) = \sigma_3 S(-k) \sigma_3 \quad (2.26)$$

and asymptotics

$$S(k) = I + O(k^{-1}), \quad k \to \infty. \quad (2.27)$$

By using the relation $\det (\psi_1^+, \psi_2^+) = 1$ and the scattering relation (2.24), the scattering data $a(k)$ and $b(k)$ can be expressed in term of determinant

$$a(k) = \det \left( \psi_1^-(0; k), \psi_2^+(0; k) \right), \quad (2.28)$$

$$b(k) = \det \left( \psi_1^+(0; k), \psi_2^-(0; k) \right). \quad (2.29)$$

and we can show that

**Proposition 2.** The scattering data $a(k)$ and $b(k)$ are even and odd functions respectively and admit the following properties

- **Symmetries:**
  $$a(-k) = a(k), \quad k \in D_+, \quad b(-k) = -b(k), \quad \text{Im} k^2 = 0. \quad (2.30)$$

- **Asymptotics:**
  $$a(k) = e^{-ic} + O(k^{-1}), \quad b(k) = O(k^{-1}), \quad k \to \infty,$$
  $$a(k) = e^{-ic} \left(1 + O(k^2)\right), \quad b(k) = O(k^3), \quad k \to 0, \quad (2.31)$$

where

$$c = c_-(x) - c_+(x) = \frac{1}{2} \int_{-\infty}^{\infty} |u_x|^2 dx,$$

and $c_\pm(x)$ are defined by (2.11).
In the analysis of a RH problem by using inverse scattering transform, to avoid technical difficulty that the singularities give rise to, in general the condition that scattering data $a(k)$ admits no eigenvalues or resonances is requested.

We denote $\mathcal{G}$ as a set of initial value $u_0(x)$ such that

$$\mathcal{G} = \{ u_0(x) : u_0(x) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}), \ a(k) \neq 0 \text{ in } \mathbb{C}^- \},$$

whose reasonableness will be shown by subsequent Proposition 7.

For $u_0(x) \in \mathcal{G}$, we define the reflection coefficient

$$r(k) := \frac{b(k)}{a(k)}, \quad k \in \mathbb{R} \cup i\mathbb{R},$$

and a matrix function

$$N(x; k) := \begin{cases} e^{ic_+(x)\sigma_3} \left( \frac{\psi^-_{\downarrow}(x; k)}{a(k)}, \psi^+_{\downarrow}(x; k) \right), & k \in D^+, \\ e^{ic_+(x)\sigma_3} \left( \psi^+_{\downarrow}(x; k), \frac{\psi^-_{\downarrow}(x; k)}{a(k)} \right), & k \in D^-, \end{cases}$$

which then satisfy the following basic RH problem

**RH Problem 2.1.** Find a matrix function $N(x; k)$ with the following properties:

- **Analyticity**: $N(x; k)$ is analytical in $\mathbb{C} \setminus \mathbb{R} \cup i\mathbb{R}$.

- **Jump condition**: $N(x; k)$ satisfies the jump condition

  $$N_+(x; k) = N_-(x; k)V(x; k),$$

  where $V(x; k) = I + J(x; k)$ is the jump matrix with

  $$J(x; k) = \begin{cases} \begin{pmatrix} |r(k)|^2 & -r(k)e^{-2ik^2x} \\ r(k)e^{2ik^2x} & 0 \end{pmatrix}, & k \in \mathbb{R}, \\ \begin{pmatrix} -|r(k)|^2 & -r(k)e^{-2ik^2x} \\ r(k)e^{2ik^2x} & 0 \end{pmatrix}, & k \in i\mathbb{R}. \end{cases}$$

- **Asymptotic conditions**:

  $$N(x; k) \to I \quad \text{as} \quad |k| \to \infty.$$

From (2.36), it is easy to find that the matrix $J(x; k)$ is Hermitian for $k \in \mathbb{R}$ and the matrix $J(x; k)$ is not Hermitian for $k \in i\mathbb{R}$. 

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2.3 Solvability of the RH problem

For a given function $h(z) \in L^p(\mathbb{R})$ with $1 \leq p < \infty$, the Cauchy operator is defined by

$$
\mathcal{C}(h)(z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(s)}{s-z} \, ds, \quad z \in \mathbb{C}\backslash\mathbb{R}.
$$

(2.38)

When $z \pm i\epsilon$ approaches to a point on the real axis $z \in \mathbb{R}$ transversely from the upper and the lower half planes, the Cauchy operator $\mathcal{C}$ becomes the Plemelj projection operators defined respectively by

$$
P_{\pm}(h)(z) := \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(s)}{s - (z \pm i\epsilon)} \, ds \quad z \in \mathbb{R}.
$$

(2.39)

We list the basic properties of the Cauchy and the Plemelj projection operators in the following proposition [23, 25, 26].

**Proposition 3.** For every $h \in L^p(\mathbb{R}), 1 \leq p < \infty$, the Cauchy operator $\mathcal{C}(h)$ and the projection operator $P_{\pm}(h)$ has the following properties:

- $\mathcal{C}(h)$ is analytic in $\mathbb{C}^\pm$ and goes to zero as $|z| \to \infty$.

- If $h \in L^1(\mathbb{R})$, then in $\mathbb{C}^+ \cup \mathbb{C}^-$, the Cauchy operator admits the following asymptotic

$$
\lim_{|z| \to \infty} z\mathcal{C}(h)(z) = -\frac{1}{2\pi i} \int_{\mathbb{R}} h(s) \, ds.
$$

(2.40)

And the projection operator $P_{\pm}(h)$ has the following properties:

- $\mathcal{C}(h)$ approaches to $P_{\pm}(h)$ almost everywhere, when a point $z \in \mathbb{C}^\pm$ approaches to a point $z_0 \in \mathbb{R}$ by any non-tangential contour from $\mathbb{C}^\pm$.

- For $1 < p < \infty$, there exists a positive constant $c$ such that

$$
\left\| P_{\pm}(h) \right\|_{L^p(\mathbb{R})} \leq c \| h \|_{L^p(\mathbb{R})}.
$$

(2.41)

Now, we begin to analyse the jump matrix $V(x;k)$ defined by the reflection coefficient $r(k)$. By using (2.25) and (2.26), we can derive the following relation

$$
|a(k)|^2 + |b(k)|^2 = 1, \quad k \in \mathbb{R},
$$

(2.42)

$$
|a(k)|^2 - |b(k)|^2 = 1, \quad k \in i\mathbb{R}.
$$

(2.43)
If $a(k)$ has no singularity on $\mathbb{R} \cap i\mathbb{R}$, we obtain the following proposition which ensures if $r(k)$ is bounded and satisfies

$$1 - |r(k)|^2 = \frac{1}{|a(k)|^2} \geq c_0^2 > 0 \quad k \in i\mathbb{R},$$

(2.44)

where $c_0 := \sup_{k \in \mathbb{R}} |a(k)|$. By using (2.36), direct calculation shows that

$$\frac{1}{2} (V + V^H) = \begin{pmatrix} 1 + |r(k)|^2 & r(k)e^{-2ik^2x} \\ r(k)e^{2ik^2x} & 1 \end{pmatrix}, \quad k \in \mathbb{R},$$

(2.45)

and

$$\frac{1}{2} (V + V^H) = \begin{pmatrix} 1 - |r(k)|^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad k \in i\mathbb{R},$$

(2.46)

which implies that the jump matrix $V$ has strictly positive real part on both $\mathbb{R}$ and $i\mathbb{R}$. This property allows us to show the following result as in [23].

**Proposition 4.** For every $r(k) \in L^\infty_z(\mathbb{R})$ with $z = k^2$ satisfying (2.44), it can be proved that for every $x \in \mathbb{R}$ and every column-vector $s \in \mathbb{C}^2$, we have

$$\text{Re}(s^HVs) \geq \alpha_- s^H s, \quad k \in \mathbb{R} \cup i\mathbb{R},$$

(2.47)

and

$$\|Vs\| \leq \alpha_+ \|s\|, \quad k \in \mathbb{R} \cup i\mathbb{R},$$

(2.48)

where $\alpha_-$ and $\alpha_+$ are positive constants.

Next, we show the solvability of the RH problem 2.1 via the solvability of a Fredholm equation. For this purpose, we make a trivial factorization to the jump matrix

$$V = b_-^{-1}b_+, \quad b_- = I, \quad b_+ = V,$$

(2.49)

which leads to

$$w_- = 0, \quad w_+ = J, \quad w = w_+ + w_+ = J.$$

(2.50)

Therefore, the Cauchy operator can be given by

$$C_w f = \mathcal{P}^+ (fw_-) + \mathcal{P}^- (fw_+) = \mathcal{P}^- (fJ).$$

(2.51)
According to Beals-Coifman theory, the solution of the RH problem 2.1 can be given by

\[ N(x; k) = I + \frac{1}{2\pi i} \int_{\mathbb{R} \cup i\mathbb{R}} \frac{\varrho(x; s)J}{s - k} \, ds = I + C(\varrho J)(z), \quad (2.52) \]

where \( z = k^2 \) and \( \varrho \) satisfies the Fredholm equation

\[ \varrho - \mathcal{P}^- (\varrho J) = I, \]

which is equivalent to

\[ N_-(x; k) = I + \mathcal{P}^- (N_- J)(z), \quad z \in \mathbb{R}, \quad (2.53) \]

by using the relation \( N_- = \varrho b_- = \varrho \). Once \( N_-(x; k) \) is found from the Fredholm integral equation (2.53), \( N_+(x; k) \) can be obtained from the projection formula

\[ N_+(x; k) = I + \mathcal{P}^+ (N_- J)(z), \quad z \in \mathbb{R}. \quad (2.54) \]

To consider solvability of the equation (2.53), we let \( G_\pm = N_\pm - I \) and transform it into a equivalent form

\[ G_-(x; k) = \mathcal{P}^- (G_- J)(z) + \mathcal{P}^- (J), \quad (2.55) \]

Therefore the solvability of the RH problem 2.1 if and only if there is a solution \( G_-(x; k) \in L^2_\mathbb{R}(\mathbb{R}) \) of the Fredholm integral equation (2.53).

**Proposition 5.** For \( r(k) \in L^2_\mathbb{R}(\mathbb{R}) \cap L^\infty_\mathbb{R}(\mathbb{R}) \), then linear inhomogeneous equation (2.55) has a unique solution \( G_-(x; k) \in L^2_\mathbb{R}(\mathbb{R}) \).

**Proof.** Since the operator \( I - \mathcal{P}^- \) is a Fredholm operator with the index zero, the uniqueness of a solution to the linear integral equation (2.55) is equivalent to show that the homogeneous equation

\[ (I - \mathcal{P}^-) s = 0 \quad (2.56) \]

has a unique zero solution in \( L^2_\mathbb{R}(\mathbb{R}) \). For a given solution \( s \in L^2_\mathbb{R}(\mathbb{R}) \) of the equation (2.56), we define two analytical functions in \( \mathbb{C} \setminus \mathbb{R} \) by

\[ s_1(z) := C(sJ)(z) \quad \text{and} \quad s_2(z) := C(sJ)^H(z), \quad (2.57) \]
where the superscript $H$ denotes the Hermite conjugate. In the upper half plane $\mathbb{C}^+$, taking 0 as the center of the circle and sufficiently large $R > 0$ as the radius, we obtain a closed enclosing line $\{|z| = R, \text{Im} z > 0\} \cup (-R, R)$. As $s_1(z)$ and $s_2(z)$ are analytic functions when $z \in \mathbb{C}^+$, the Cauchy theorem implies that

$$\oint s_1(z)s_2(z)dz = 0.$$  \hspace{1cm} (2.58)

Because $s(z), J \in L^2(\mathbb{R})$, we know $s_{1,2}(z) = O\left(z^{-1}\right)$ as $|z| \to \infty$. Thus, the integral on the semi-circle goes to zero as $R \to \infty$, from which we obtain

$$\int_{\mathbb{R}} s_1(z)s_2(z)dz = \int_{\mathbb{R}} \left[\mathcal{P}^-(sJ) + sJ\right] \left[\mathcal{P}^-(sJ)\right]^H dz = 0.$$  \hspace{1cm} (2.59)

where the identity $\mathcal{P}^+ - \mathcal{P}^- = I$ is used. As $\mathcal{P}^-(sJ) = s$, we finally obtain

$$\int_{\mathbb{R}} sVs^Hdz = 0 \implies \int_{\mathbb{R}} \text{Re}(sVs^H)dz = 0,$$  \hspace{1cm} (2.59)

which yields $s = 0$. Otherwise, if $s \neq 0$, by Proposition 4, we have

$$\int_{\mathbb{R}} \text{Re}(sVs^H)dz \geq \alpha - \int_{\mathbb{R}} |s|^2dz > 0,$$

which is contradict with (2.59). Therefore, the equation (2.59) implies that $s = 0$ and the homogeneous equation (2.56) has a unique solution in $L^2(\mathbb{R})$.

\section{Inverse scattering transform on $z$-plane}

We decide to transform original spectral problem and RH problem from the $k$-plane to the $z$-plane by exploiting the odd-even property of the Jost functions in the column. This facilitates the further application of the Fourier transform and Cauchy integral projection to estimate the space of the solution to the RH problem and the potential $u$.

\subsection{A ZS type spectral problem}

To put the original RH problem 2.1 into a RH problem with a jump contour on the real axis, we introduce a transformation defined by

$$\Phi(x; z) = A(x; k)\psi(x; k)B(x, k),$$  \hspace{1cm} (3.1)
with \( z = k^2 \) and

\[
A(x; k) = \begin{pmatrix} 1 & 0 \\ -\bar{u}_x & 2ik \end{pmatrix}, \quad B(x; k) = \begin{pmatrix} 1 & 0 \\ 0 & (2ik)^{-1} \end{pmatrix},
\]

then spectral problem (2.5) is changed into a Zakharov-Shabat type spectral problem

\[
\Psi_x + iz [\sigma_3, \Psi] = \tilde{Q} \Psi,
\]

where

\[
\tilde{Q} = \frac{1}{2i} \begin{pmatrix} |u_x|^2 & u_x \\ -2i \bar{u}_xx - u_x^* |u_x|^2 & -|u_x|^2 \end{pmatrix}.
\]

It can be shown that \( \Psi^\pm(x; z) \) satisfies the following Volterra integral equations

\[
\Psi^\pm(x; z) = I + \int_{\pm\infty} e^{-iz(x-y)} \tilde{Q} \Psi^\pm(y; z)dy.
\]

Denote \( \Psi^\pm(x; z) = (\Psi^\pm_1(x; z), \Psi^\pm_2(x; z)) \) with the subscripts 1 and 2 denoting the first and second columns of \( \Psi^\pm(x; z) \), then we can show that

**Proposition 6.** Suppose that \( u(x) \in H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R}) \). For every \( z \in \mathbb{R} \), there exists unique solutions \( \Psi^\pm(\cdot; z) \in L^\infty(\mathbb{R}) \) satisfying the integral equations (3.3) which have the following properties

- **Analyticity:** for every \( x \in \mathbb{R} \), \( \Psi^-_1(x; \cdot) \) and \( \Psi^+_2(x; \cdot) \) are analytically continued in \( \mathbb{C}^+ \), whereas \( \Psi^+_1(x; \cdot) \) and \( \Psi^-_2(x; \cdot) \) are analytically continued in \( \mathbb{C}^- \).

- **Boundedness:** there exists a positive \( z \)-independent constant \( c \) such that

\[
\|\Psi^\pm(\cdot; z)\|_{L^\infty(\mathbb{R})} \leq c, \quad z \in \mathbb{C}^\pm.
\]

**Proof.** As illustrative example, we consider the boundedness and the analyticity of the Jost functions \( \Psi^-(x; z) \). From (3.3), we get the following two integral equations

\[
\Psi^-_1(x; z) := e_1 + \int_{-\infty}^x \text{diag} \begin{pmatrix} 1, e^{2iz(x-y)} \end{pmatrix} \tilde{Q} \Psi^-_1(y; z)dy,
\]

\[
\Psi^-_2(x; z) := e_2 + \int_{-\infty}^x \text{diag} \begin{pmatrix} e^{-2iz(x-y)}, 1 \end{pmatrix} \tilde{Q} \Psi^-_2(y; z)dy.
\]
It suffices to give the proof for $\Psi_1^{-1}(x;z)$. We define the integral operator $F$ by

$$(Ff)(x;z) := \int_{-\infty}^{x} \text{diag} \left(1,e^{2iz(x-y)}\right) \tilde{Q}f(y)dy,$$  \hspace{1cm} (3.7)$$

with vector $f = (f_1,f_2)^T$, then the integral equation (3.5) can be written in an operator equation

$$\Psi_1^{-1} = e_1 + F\Psi_1^{-1}.$$  \hspace{1cm} (3.8)$$

According to the Fredholm alternative, we show that the homogeneous equation

$$(I - F)\Psi_1^{-1} = 0 \iff F\Psi_1^{-1} = \Psi_1^{-1}.$$  \hspace{1cm} (3.9)$$

has a unique zero solution. The equation (3.9) implies that we can show the operator $F$ has a unique fixed point by the Banach fixed point theorem.

For vector function $f(x) = (f_1,f_2)^T \in L^\infty$, define the norm by

$$\|f\|_{L^\infty} = \|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}.$$  

We expand (3.7) in the following form

$$Ff(x;z) = (h_1,h_2)^T,$$  \hspace{1cm} (3.10)$$

where

$$h_1 = \frac{1}{2i} \int_{-\infty}^{x} (|u_y|^2 f_1 + u_y f_2)dy,$$

$$h_2 = -\frac{1}{2i} \int_{-\infty}^{x} ((2i\bar{u}_y + \bar{u}_y |u_y|^2)f_1 + |u_y|^2 f_2)e^{2iz(x-y)}dy.$$

We divide the above components of $Ff(x;z)$ into two parts to deal with. For the first and second term, we have for every $z \in \mathbb{C}^+$ and every $x_0 \in \mathbb{R}$,

$$\|h_1\|_{L^\infty} = \sup_{x \in (-\infty,x_0)} \frac{1}{2i} \int_{-\infty}^{x} (|u_y|^2 f_1 + u_y f_2)dy$$

$$\leq \frac{1}{2} (\|f_1\|_{L^\infty} \|u_x\|_{L^2}^2 + \|f_2\|_{L^\infty} \|u_x\|_{L^1})dy$$

$$\leq \frac{1}{2} (\|u_x\|_{L^2}^2 + \|u_x\|_{L^1}) \|f\|_{L^\infty}.  \hspace{1cm} (3.11)$$
Noting that \( z \in \mathbb{C}^+ \), \( |e^{2iz(x-y)}| \leq 1 \), in the similar way above, we can show that
\[
\| h_2 \|_{L^\infty} \leq \frac{1}{2} (2\|u_{xx}\|_{L^1} + \|u_x\|_{L^3}^2 + \|u_x\|_{L^2}^2) \| f(\cdot; z) \|_{L^\infty}.
\] (3.12)
Hence,
\[
\| Ff \|_{L^\infty} \leq (2\|u_{xx}\|_{L^1} + \|u_x\|_{L^3}^2 + 2\|u_{xx}\|_{L^1} + \|u_x\|_{L^1}) \| f(\cdot; z) \|_{L^\infty}.
\] which implies that \( F \) is a contraction operator if \( x_0 \in \mathbb{R} \) is chosen such that
\[
2\|u_y\|_{L^2}^2 + \|u_y\|_{L^3}^2 + 2\|u_{yy}\|_{L^1} + \|u_y\|_{L^1} < 1.
\] (3.13)
According to the Banach fixed point theorem, for \( x_0 \) and every \( z \in \mathbb{C}^+ \), there exists a unique solution \( \Psi^{-1}_1(x; z) \in L^\infty(-\infty, x_0) \) to the equation (3.9). \( \mathbb{R} \) can be covered by a finite number of intervals, in which the estimate (3.13) is satisfied. By putting the unique solutions in each subinterval together, we finally obtain the unique solution \( \Psi^{-1}_1(\cdot; z) \in L^\infty(\mathbb{R}) \) for every \( z \in \mathbb{C}^+ \).

The analyticity of \( \Psi^{-1}_1(x; \cdot) \) in \( \mathbb{C}^+ \) for every \( x \in \mathbb{R} \) follows from the absolute and the uniform convergence of the Neumann series of the analytic functions in \( k \). Define a Neumann sequence by
\[
w_0 = e_1, \quad w_{n+1}(x; z) = Fw_n = \int_{-\infty}^x F(x, y; z)w_n(y)dy.
\] (3.14)
which constitute a series
\[
w(x; z) = \sum_{n=0}^\infty w_n(x; z) = \sum_{n=0}^\infty F^n e_1.
\] (3.15)
For a matrix function \( \tilde{Q} = (\tilde{Q}_{ij})_{ij=1}^2 \), define its \( L^1 \) matrix norm by
\[
\| \tilde{Q} \|_{L^1} := \sum_{i=1}^2 \sum_{j=1}^2 \| \tilde{Q}_{ij} \|_{L^1}.
\] (3.16)
If \( u \in H^3(\mathbb{R}) \cup H^{2,1}(\mathbb{R}) \), we have \( \tilde{Q}(u) \in L^1(\mathbb{R}) \), where the matrix \( \tilde{Q}(u) \) appears in the integral kernel \( F \) given by (3.7). For (3.14), we have
\[
\| w_n \|_{L^\infty} = \| F^n e_1 \|_{L^\infty} \leq \frac{1}{n!} \| \tilde{Q}(u) \|_{L^1}^n.
\] (3.17)
Hence, the Neumann series (3.15) absolutely and uniformly converges the solution
\( \Psi_1(x; z) \) of the Volterra integral equation (3.5) for every \( x \in \mathbb{R} \) and \( z \in \mathbb{C}^+ \). Moreover
\( \Psi_1(x; z) \) is analytic in \( \mathbb{C}^+ \) for every \( x \in \mathbb{R} \) and satisfies the bound (3.4).

**Proposition 7.** The small-norm constraint
\[
2\|u_x\|_{L^2}^2 + \|u_x\|_{L^3}^3 + 2\|u_{xx}\|_{L^1} + \|u_x\|_{L^1} < 1. \tag{3.18}
\]
is a sufficient condition which guarantees that \( a(z) \) has no eigenvalues and no spectral singularity.

**Proof.** As we have already obtained \( a(k) \) is even of \( k \) in (2.30), we will use the notation
\( a(z) \), which is essentially the same as \( a(k) \). We only need to prove that under the small-norm constraint, \( a(z) > 0 \) as \( \text{Im} z > 0 \). It is easy to see that if (3.18) holds, the Volterra integral equation (3.5) has the unique solution \( \Psi_1^{-1}(x; z) \in L^\infty_x(\mathbb{R}) \), which also satisfies
\[
\|\Psi_1^{-1}(x; z) - e_1\|_{L^\infty_x(\mathbb{R})} < 1, \quad z \in \mathbb{C}^+. \tag{3.19}
\]
Recall the integral expression of \( a(z) \)
\[
a(z) = 1 + k \int_{\mathbb{R}} u_y \Psi_1^{-21}(x; k) \, dx. \tag{3.20}
\]
While by (3.1), we get
\[
\Psi_1^{-21}(x; k) = \frac{1}{2ik} \left( \bar{u}_x \Psi_1^{-11}(x; z) + \Psi_1^{-21}(x; z) \right).
\]
The equation (3.20) can be written as
\[
a(z) = 1 + \frac{1}{2i} \int_{\mathbb{R}} \left( |u_y|^2 \Psi_1^{-11}(y; z) + u_y \Psi_1^{-21}(y; z) \right) \, dy,
\]
which yields
\[
|a(z)| \geq 1 - \left| \frac{1}{2i} \int_{\mathbb{R}} \left( |u_y|^2 \Psi_1^{-11}(y; z) + u_y \Psi_1^{-21}(y; z) \right) \, dy \right|. \tag{3.21}
\]
We estimate the above integral equation by

\[ \left| \frac{1}{2i} \int_{\mathbb{R}} (|u_y|^2 \Psi_{11}^{-1}(y; z) + u_y \Psi_{21}^{-1}(y; z)) \, dy \right| \]

\[ = \frac{1}{2} \left| \int_{\mathbb{R}} (|u_y|^2 (\Psi_{11}^{-1}(y; z) - 1) + |u_y|^2 + u_y \Psi_{21}^{-1}(y; z)) \, dy \right| \]

\[ \leq \frac{1}{2} \|u_y\|_{L^2}^2 + \frac{1}{2} \|\Psi_{11}^{-1} - 1\|_{L^\infty} \|u_x\|_{L^2}^2 + \frac{1}{2} \|\Psi_{21}^{-1}\|_{L^\infty} \|u_x\|_{L^1} \]

\[ \leq \frac{1}{2} \left( 2\|u_x\|_{L^2}^2 + \|u_x\|_{L^1} + \|u\|_{L^3}^3 + 2 \|u_{xx}\|_{L^1} \right). \]

Hence if we add the small-norm constraint to the above equation to deduce that the right-hand side is small than \(1/2\), then the equation (3.21) indicates

\[ |a(z)| > 1/2, \quad \text{Im } z > 0. \]  

(3.22)

As \(a(z)\) continues to the boundary \(\mathbb{R}\), we can still get the same result on the boundary by the order preservation of the limit

\[ |a(z)| \geq 1/2, \quad \text{Im } z = 0. \]  

(3.23)

The above procedure shows that \(a(k)\) has no eigenvalues and no spectral singularity.

This proposition show that the small-norm constraint (3.18) is a sufficient condition that the assumption (2.32) is satisfied.

The following proposition will play an important role in the subsequent various estimates on the Jost function, coefficients, Cauchy integral and potentials.

**Proposition 8.** [23] If \(w \in H^1(\mathbb{R})\), then

\[ \sup_{x \in \mathbb{R}} \left\| \int_{-\infty}^x e^{2iz(x-y)} w(y) \, dy \right\|_{L^2(\mathbb{R})} \leq \sqrt{\pi} \|w\|_{L^2}, \]  

(3.24)

and

\[ \sup_{x \in \mathbb{R}} \left\| 2iz \int_{-\infty}^x e^{2iz(x-y)} w(y) \, dy + w(x) \right\|_{L^2(\mathbb{R})} \leq \sqrt{\pi} \|\partial_x w\|_{L^2}. \]  

(3.25)
Moreover, if \( w \in L^{2,1}(\mathbb{R}) \), then for every \( x_0 \in \mathbb{R}^- \), we have

\[
\sup_{x \in (-\infty, x_0)} \left\| \langle x \rangle \int_{-\infty}^{x} e^{2iz(x-y)} w(y) dy \right\|_{L^2_{\mathbb{R}}} \leq \sqrt{\pi} \|w\|_{L^{2,1}(-\infty, x_0)},
\] (3.26)

where \( \langle x \rangle := (1 + x^2)^{1/2} \).

**Proposition 9.** Under the conditions of Proposition 6, for every \( x \in \mathbb{R} \), the Jost functions \( \Psi^\pm(x; z) \) admits the following limits

\[
\lim_{|z| \to 0} \Psi^\pm(x; z) = \begin{pmatrix} 1 \\ -u_x(x) \\ 1 \end{pmatrix},
\] (3.27)

\[
\lim_{|z| \to \infty} \Psi^\pm(x; z) = e^{-ic\pm\sigma_3}.
\] (3.28)

**Proof.** We write the integral equation (3.3) in the following scalar form

\[
\Psi^-_{11}(x; z) = 1 + \frac{1}{2i} \int_{-\infty}^{x} u_y(y) \left[ \bar{u}_y(y) \Psi^-_{11}(y; z) + \Psi^-_{21}(y; z) \right] dy,
\] (3.29)

\[
\Psi^-_{21}(x; z) = -\frac{1}{2i} \int_{-\infty}^{x} e^{2iz(x-y)} \left[ (2i\bar{u}_{yy} + |u_y|^2 \bar{u}_y) \Psi^-_{11} + |u_y|^2 \Psi^-_{21} \right] dy,
\] (3.30)

\[
\Psi^-_{12}(x; z) = \frac{1}{2i} \int_{-\infty}^{x} e^{-2iz(x-y)} u_y(y) \left[ \bar{u}_y(y) \Psi^-_{12}(y; z) + \Psi^-_{22}(y; z) \right] dy,
\] (3.31)

By using (3.4), we know that \( \Psi^-_{11}(x; z), \Psi^-_{21}(x; z) \in L^\infty_x \), hence for \( u(x) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \), we have

\[
\| (2i\bar{u}_{yy} + |u_y|^2 \bar{u}) \Psi^-_{11}(y; z) + |u_y|^2 \Psi^-_{21}(y; z) \|_{L^1} \leq c(\|u_{xx}\|_{L^1} + \|u_x\|_{L^2}^2).
\]

While for \( \text{Im} z > 0 \), we have \( |e^{2iz(x-y)}| = e^{-\text{Im} z(x-y)} \to 0, \ k \to \infty \), then by using the Lebesgue dominated convergence theorem, the equation (3.30) yields

\[
\Psi^-_{21}(x; z) \to 0, \ \ z \to \infty.
\] (3.32)

Taking the limit \( z \to \infty \) in (3.29) leads to

\[
\lim_{|z| \to \infty} \Psi^-_{11}(x; z) = 1 + \frac{1}{2i} \int_{-\infty}^{x} |u_y(y)|^2 \lim_{|z| \to \infty} \Psi^-_{11}(y; z) dy,
\]
which admits a unique solution

\[ \lim_{|z| \to \infty} \Psi_{11}(x; z) = e^{-ic_-(x)}. \]  

(3.33)

Finally combining (3.32) and (3.33) gives

\[ \lim_{|z| \to \infty} \Psi^{-1}(x; z) = e^{-ic_-(x)} e_1. \]  

(3.34)

In a similar way, we can show that

\[ \lim_{|z| \to \infty} \Psi_{2}(x; z) = e^{ic_-(x)} e_2. \]

In the following proposition, we give smooth properties of the Jost functions

**Proposition 10.** If \( u(x) \in H^{2,1}(\mathbb{R}) \), then for every \( x \in \mathbb{R}^\pm \), we have

\[ \Psi^\pm(x; \cdot) - e^{-ic_\pm(x)\sigma_3} \in H^1(\mathbb{R}). \]  

(3.35)

**Proof.** Without loss of generality, we prove the statement for the Jost function \( \Psi^{-}(x; z) \). We write the integral equation (3.3) for \( \Psi^{-1}(x; z) \) into an operator equation form

\[ (I - F)\Psi^{-} = I, \]  

(3.36)

where the operator \( F \) is given by (3.7).

Subtracting the term \((I - F)e^{-ic_-(x)\sigma_3}\) from both sides of equation (3.36), we obtain an equivalent form

\[ (I - F) \left( \Psi_{-} - e^{-ic_-(x)\sigma_3} \right) = \begin{pmatrix} 0 & n \\ m & 0 \end{pmatrix}, \]  

(3.37)

where

\[ m = \int_{-\infty}^{x} e^{2iz(x-y)} w(y) dy, \quad w(x) = -\partial_x (u_x e^{ic_-(x)}), \]  

(3.38)

\[ n = \int_{-\infty}^{x} e^{-2iz(x-y)} u_y(y) e^{ic_-(x)} dy. \]
If \( u \in H^{2,1}(\mathbb{R}) \), then \( w(x) \in L^{2,1}(\mathbb{R}) \). By the estimates (3.24) and (3.26), we have \( m(x; z), n(x; z) \in L^\infty_x (\mathbb{R}; L^2_k(\mathbb{R})) \). With the Sobolev inequality \( \|u_x\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|u\|_{H^{2,1}} \), the following bound is valid for every \( x_0 \in \mathbb{R}^- \):

\[
\sup_{x \in (-\infty, x_0] } \| \langle x \rangle m(x; z) \|_{L^2_x(\mathbb{R})} \leq \sqrt{\pi} \left( \|u_{xx}\|_{L^{2,1}} + \frac{1}{2} \|u^3_x\|_{L^{2,1}} \right) \\
\leq c \left( \|u\|_{H^{2,1}} + \|u^3\|_{H^{2,1}} \right),
\]

(3.39)

where \( c \) is a positive constant.

By a similar way in deriving (3.17), for every \( f(x; z) \in L^\infty_x (\mathbb{R}; L^2_k(\mathbb{R})) \), we can obtain

\[
\| (F^n f)(x; z) \|_{L^\infty_x L^2_k} \leq \frac{1}{n!} \| \tilde{Q}(u) \|_{L^1} \|f(x; z)\|_{L^\infty_x L^2_k}.
\]

Therefore, the operator \( I - F \) is invertible on the space \( L^\infty_x (\mathbb{R}; L^2_k(\mathbb{R})) \). The bound of the inverse operator can be obtained accordingly

\[
\| (I - F)^{-1} \|_{L^\infty_x L^2_k \rightarrow L^\infty_x L^2_k} \leq \sum_{n=0}^{\infty} \frac{1}{n!} \| \tilde{Q}(u) \|_{L^1}^n = e \| \tilde{Q}(u) \|_{L^1}.
\]

(3.40)

Clearly, (3.40) is the norm in the space \( L^\infty_x ((-\infty, x_0); L^2_k(\mathbb{R})) \) for every \( x_0 \in \mathbb{R} \). By using (3.37), (3.39), and (3.40), we obtain \( \Psi^{\pm}(x; \cdot) - e^{ic\pm\sigma} \in L^2(\mathbb{R}) \) from the following estimate for every \( x_0 \in \mathbb{R}^- \)

\[
\sup_{x \in (-\infty, x_0]} \| \langle x \rangle \left( \Psi^{-}(x; z) - e^{-ic\cdot\sigma}\right) \|_{L^2(\mathbb{R})} \leq ce \| \tilde{Q}(u) \|_{L^1} \left( \|u\|_{H^{2,1}} + \|u^3\|_{H^{2,1}} \right).
\]

(3.41)

To complete the proof of (3.35), our next task is to show that \( \partial_x \Psi^{-}(x; z) \in L^\infty_x ((-\infty, x_0); L^2_k(\mathbb{R})) \) for every \( x_0 \in \mathbb{R}^- \). Column analysis is used here. According to the characteristics of the derivative with respect to \( k \), we introduce the vector

\[
q(x; z) := [\partial_x \Psi^{-}_{11}(x; z), \partial_x \Psi^{-}_{21}(x; z) - 2ix\Psi^{-}_{21}(x; z)]^T,
\]

which satisfies the following expression

\[
(I - F)q(x; z) = m_1e_1 + m_2e_2 + m_3e_2,
\]

(3.42)
where
\[
m_1(x; z) = \int_{-\infty}^{x} yu_y(y)\Psi_{21}^-(y; z)dy,
\]
\[
m_2(x; z) = \int_{-\infty}^{x} ye^{2iz(x-y)} \left(2i\bar{u}_y(y) + |u_y(y)|^2\bar{u}_y(y)\right) \left(\Psi_{11}^-(y; z) - e^{-ic_- (x)}\right)dy,
\]
\[
m_3(x; z) = \int_{-\infty}^{x} ye^{2iz(x-y)} \left(2i\bar{u}_y(y) + |u_y(y)|^2\bar{u}_y(y)\right) e^{-ic_- (x)}dy.
\]

For every \(x_0 \in \mathbb{R}^-\), using the Hölder’s inequality to each term in the integral equation (3.42), we obtain the following bounds by (3.24):
\[
\sup_{x \in (-\infty, x_0)} \|m_1(x; z)\|_{L_2^x(\mathbb{R})} \leq \|u\|_{L^1} \sup_{x \in (-\infty, x_0)} \|\langle x \rangle \Psi_{21}^-(x; z)\|_{L_2^x(\mathbb{R})},
\]
\[
\sup_{x \in (-\infty, x_0)} \|m_2(x; z)\|_{L_2^x(\mathbb{R})} \leq (\|u_{xx}\|_{L^1} + \|u_x\|_{L^1}) \sup_{x \in (-\infty, x_0)} \|\langle x \rangle \left(\Psi_{11}^-(x; z) - e^{-ic_- (x)}\right)\|_{L_2^x(\mathbb{R})},
\]
\[
\sup_{x \in (-\infty, x_0)} \|m_3(x; z)\|_{L_2^x(\mathbb{R})} \leq \sqrt{\pi} \left(2 \|u_{xx}\|_{L^2,1} + \|u_x^2\|_{L^1,1}\right).
\]

Because of the estimate (3.41), the first two inequalities have finite bounds.

Using (3.39), (3.41), and the integral equation (3.42), we summarize that \(q(x; z) \in L_x^\infty((\infty, x_0); L_z^2(\mathbb{R}))\) for every \(x_0 \in \mathbb{R}^-\). Also under the property showed in (3.41), we finally obtain \(\partial_x \Psi_{11}^+(x; z) \in L_x^\infty((\infty, x_0); L_z^2(\mathbb{R}))\) for every \(x_0 \in \mathbb{R}^-\). This completes the proof of (3.35).

\[\square\]

**Proposition 11.** If \(u \in H^{2,1}(\mathbb{R})\) and \(u \in C^2(\mathbb{R})\), then Jost functions \(\Psi_{1}^\pm(x; z)\) admits the following limits
\[
\lim_{|z| \to \infty} z(\Psi_{1}^\pm(x; z) - e^{-ic_- (x)}e_1) = \widehat{\Psi}_{11}^\pm(x)e_1 + \widehat{\Psi}_{21}^\pm(x)e_2. \tag{3.43}
\]
with
\[
\widehat{\Psi}_{11}^\pm(x) := -\frac{1}{4}e^{-ic_\pm} \int_{-\infty}^{x} \left[ u_y(y)\bar{u}_y(y) + \frac{1}{2\imath}|u_y(y)|^4 \right] dy, \tag{3.44}
\]
\[
\widehat{\Psi}_{21}^\pm(x) = \frac{1}{2\imath}\partial_x(\bar{u}_x(x)e^{ic_\pm}). \tag{3.45}
\]
If \(u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\), then for every \(x \in \mathbb{R}\), we have
\[
z(\Psi_{1}^\pm(x; z) - e^{ic_\pm}e_1) - \left(\widehat{\Psi}_{11}^\pm(x)e_1 + \widehat{\Psi}_{21}^\pm(x)e_2\right) \in L_z^2(\mathbb{R}). \tag{3.46}
\]

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Proof. For every $x \in \mathbb{R}$ and every small $\delta > 0$, we split the integral equation of $\Psi_{21}(x; z)$ for $(-\infty, x - \delta)$ and $(x - \delta, x)$, which is rewritten in the equivalent form:

$$\Psi_{21}(x; z) = \int_{-\infty}^{x-\delta} e^{2iz(x-y)} \nu(y; z) \, dy + \nu(x; z) \int_{x-\delta}^{x} e^{2iz(x-y)} \, dy$$

$$+ \int_{x-\delta}^{x} e^{2iz(x-y)}(\nu(y; z) - \nu(x; z)) \, dy \equiv I_1 + I_2 + I_3,$$

where

$$\nu(x; z) := -\frac{1}{2i} \left[ (2i \bar{u}_x(x) + |u_x(x)|^2 \bar{u}_x(x)) \Psi_{11}(x; z) + |u_x(x)|^2 \Psi_{21}(x; z) \right].$$

As $\nu(\cdot; z) \in L^1(\mathbb{R})$, we have $I_1 \to 0$ as $k \to \infty$. Meanwhile, $\nu(\cdot; z) \in L^1(\mathbb{R})$ makes $I_3 \to 0$ as $k \to \infty$. As for $I_2$, we get the specific value

$$I_2 = -\frac{1}{2iz} (1 - e^{2iz\delta}) \nu(x; z).$$

By choosing $\delta := [\text{Im}(z)]^{-1/2}$ such that $\delta \to 0$ as $\text{Im}(z) \to \infty$. Then

$$\lim_{|k| \to \infty} (z \Psi^-)_{21}(x; k) = -\frac{1}{2i} \lim_{|\xi| \to \infty} \nu(x; z)$$

$$= -\frac{1}{4} \left[ (2i \bar{u}_x(x) + |u_x(x)|^2 \bar{u}_x(x)) e^{ic\pm(x)} \right] = \frac{1}{4} \partial_x \left( \bar{u}_x(x) e^{ic\pm(x)} \right).$$

Obviously, the limit of $z\Psi_{21}(x; z)$ reveals the relation between $z\Psi^-_{21}(x; z)$ and $u_{xx}$, which yields the limit (3.43). To deal with $\Psi^-_{11}(x; z)$, (3.29) can be rewritten as the differential equation

$$\Psi_{11,x}(x; z) = \frac{1}{2i} |u_x|^2 \Psi_{11}(x; z) + \frac{1}{2i} u_x(x) \Psi_{21}(x; z).$$

Using $e^{ic_-}(x)$ as the integrating factor,

$$\partial_x \left( e^{ic_-} \Psi_{11}(x; z) \right) = \frac{1}{2i} u_x(x) e^{ic_-} \Psi_{21}(x; z).$$

We obtain another integral equation for $\Psi_{11}(x; z)$:

$$\Psi_{11}(x; z) = e^{-ic_-} + \frac{1}{2i} e^{-ic_-} \int_{-\infty}^{x} u_y(y) e^{ic_-} \Psi_{21}(x; z) \, dy,$$

(3.49)
Also, taking the limit \(|z| \to \infty\) of \(z(\Psi^-_{11}(x; z) - e^{-ic_-(x)})\), we obtain

\[
\lim_{|z| \to \infty} z(\Psi^-_{11}(x; z) - e^{-ic_-(x)}) = \hat{\Psi}^-_{11}(x),
\]

with \(\hat{\Psi}^-_{11}(x) = -\frac{i}{4} e^{-ic_-(x)} \int_{-\infty}^{\infty} \left[ u_y(y) \bar{u}_y(y) + \frac{i}{4} |u_y(y)|^4 \right] dy\). In the end, we obtain the limit of \(\Psi^-_1\) in (3.43).

It’s a natural idea to look at the space \(z(\Psi^-_1 - e^{-ic_-(x)}e_1) - \left( \hat{\Psi}^-_{11}e_1 + \hat{\Psi}^-_{21}e_2 \right)\). To prove (3.46) for \(\Psi^-_1\), under the result shown in (3.43), we subtract the right side term from the left side and obtain

\[
(I - F) \left[ z(\Psi^-_1 - e^{-ic_-(x)}e_1) - \left( \hat{\Psi}^-_{11}e_1 + \hat{\Psi}^-_{21}e_2 \right) \right] = zde_2 - (I - F) \left( \hat{\Psi}^-_{11}e_1 + \hat{\Psi}^-_{21}e_2 \right),
\]

Combining the integral equation (3.49), we obtain

\[
zm(x; z)e_2 - (I - F) \left( \hat{\Psi}^-_{11}e_1 + \hat{\Psi}^-_{21}e_2 \right) = \tilde{m}(x; z)e_2,
\]

with \(\tilde{m}(x; z) = z \int_{-\infty}^{\infty} e^{2iz(x-y)}w(y)dy + \frac{1}{2i} w(x)
\]

\[
- \frac{1}{2i} \int_{-\infty}^{\infty} e^{2iz(x-y)} \left[ (2i \bar{u}_y(y) + \bar{u}_y(y)|u_y(y)|^2) \tilde{\Psi}^-_{11}(y) + |u_y(y)|^2 \tilde{\Psi}^-_{21}(y) \right] dy,
\]

where \(w\) is defined in (3.38). By using bounds (3.24) and (3.25), we have \(\tilde{m}(x; z) \in L_x^\infty(\mathbb{R}; L_z^2(\mathbb{Z}))\) if \(u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\). Multiply both sides of this equation by \((I - F)^{-1}\) on \(L_x^\infty(\mathbb{R}; L_z^2(\mathbb{Z}))\), we improves the proof process of (3.46) for \(\Psi^-_1(x; z)\).

**Proposition 12.** If \(u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\), then for every \(x \in \mathbb{R}^\pm\), the Jost functions \(\psi^\pm(x; k)\) have the following properties

\[
\psi^\pm_{11} - e^{-ic_\pm(x)}, \quad 2ik\psi^\pm_{21} - \bar{u}_x e^{-ic_\pm(x)} \in H^1_x(\mathbb{R}), \quad \partial_x \psi^\pm_{11} \in L^\infty_x L^\infty_z,
\]

\[
2ik\psi^\pm_{22}(x; k) - e^{ic_\pm(x)}, \quad \psi^\pm_{12}(x; k) \in H^1_z(\mathbb{R}), \quad k^{-1}\psi^\pm_{21}(x; k) \in H^1_x(\mathbb{R}).
\]

**Proof.** From the first column of the transformation (3.1), we have

\[
\Psi^-_{11}(x; k) = \psi^\pm_{11}(x; k),
\]

\[
\Psi^-_{21}(x; k) = -\bar{u}_x \psi^\pm_{11}(x; k) + 2ik\psi^\pm_{21}(x; k).
\]
Recalling the result (3.35), we immediately obtain
\[
\psi_{11}^\pm (x; k) - e^{-ic\pm(x)} = \Psi_{11}^\pm (x; k) - e^{-ic\pm(x)} \in H^1_x(\mathbb{R}).
\]
Noting that \(u_x(x) \in L^\infty(\mathbb{R}), \Psi_{21}^\pm(x; k) \in H^1_x(\mathbb{R})\), we find
\[
2ik\psi_{21}^\pm(x; k) - u_x e^{-ic\pm(x)} = \Psi_{21}^\pm(x; k) + u_x(\psi_{11}^\pm(x; k) - e^{-ic\pm(x)}) \in H^1_x(\mathbb{R}).
\]
To prove \(k^{-1}\psi_{21}^\pm(x; k)\), we start with its integral equation
\[
k^{-1}\psi_{21}^\pm(x; k) = -\int_{-\infty}^{\infty} e^{2iz(x-y)}u_\gamma e^{-ic\pm} dy
\]
\[\quad - \int_{-\infty}^{\infty} e^{2iz(x-y)}u_\gamma (\psi_{11}^\pm(y; k) - e^{-ic\pm}) dy. \tag{3.56}\]
Due to \(u \in H^2(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\) and \(\psi_{11}^\pm(x; k) - e^{-ic\pm(x)} \in H^1_x(\mathbb{R})\), by using a similar way to Proposition 8, we know that the two integrals on the right-hand side belong to \(H^1_x(\mathbb{R})\), which yields the solution \(k^{-1}\psi_{21}^\pm(0; k) \in H^1_x(\mathbb{R})\).

In a similar way to the second column of the transformation (3.1), we can show the first two formulas in (3.53).

3.2 Lipschitz continuity of the scattering data

Lipschitz continuity of the Jost functions is the basis of Lipschitz continuity of the scattering data. Our first priority is to find the Lipschitz continuous map from \(u\) to \(\Psi^\pm(x; z)\). As a direct corollary of Proposition 10, we show the map
\[
H^{2,1}(\mathbb{R}) \ni u \rightarrow \left(\Psi^\pm(x; z) - e^{-ic\pm(x)}\sigma_3\right) \in L^\infty_x(\mathbb{R}^\pm; H^1_x(\mathbb{R})) \tag{3.57}
\]
is Lipschitz continuous.

Corollary 1. Suppose that \(u, \tilde{u} \in H^{2,1}(\mathbb{R})\) satisfy \(\|u\|_{H^{2,1}}, \|\tilde{u}\|_{H^{2,1}} \leq \delta\) for some \(\delta > 0\), and their corresponding Jost functions are \(\Psi^\pm(x; z)\) and \(\tilde{\Psi}^\pm(x; z)\) respectively. Then, there is a positive constant \(c\) such that for every \(x \in \mathbb{R}^\pm\), we have
\[
\left\|\Psi^\pm(x; \cdot) - e^{-ic\pm(x)}\sigma_3 - \tilde{\Psi}^\pm(x; \cdot) + e^{-ic\pm(x)}\sigma_3\right\|_{H^1} \leq c\|u - \tilde{u}\|_{H^{2,1}}. \tag{3.58}
\]
Furthermore, if $u, \tilde{u} \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ satisfy $\|u\|_{H^3 \cap H^{2,1}} \leq \tilde{u} \leq \|\tilde{u}\|_{H^3 \cap H^{2,1}} \leq \delta$, then for every $x \in \mathbb{R}$, there is a positive $c$ such that

$$\left\| \tilde{\Psi}^\pm(x; \cdot) - \tilde{\Psi}^\pm(x; \cdot) \right\|_{L^2} \leq c \|u - \tilde{u}\|_{H^3 \cap H^{2,1}},$$

(3.59)

where

$$\tilde{\Psi}^\pm(x; z) := z (\tilde{\Psi}^\pm_1 - e^{-ic}e_1) - \left( \tilde{\Psi}^\pm_1 + e^{-ic}e_1 \right).$$

Proof. As an illustrative example, we only prove (3.58) for the Jost function $\Psi_1^-(x; z)$. For every $x \in \mathbb{R}$, it is obvious that

$$\left| e^{-ic} - e^{-i\bar{c}} \right| = \left| e^{1/2} \int_{-\infty}^\infty (|u_y(y)|^2 - |\tilde{u}_y(y)|^2) dy - 1 \right| \leq 2\delta c_1 \|u_y - \tilde{u}_y\|_{L^2}. \quad (3.60)$$

Using the integral equation (3.37), we obtain

$$\left( \psi - e^{-ic} \psi_3 \right) - \left( \psi - e^{-ic} \psi_3 \psi \right) = (I - F)^{-1} (T - \tilde{T}) + (I - F)^{-1} (F - \tilde{F}) (I - F)^{-1} \tilde{T}, \quad (3.61)$$

where

$$T = \begin{pmatrix} 0 & n \\ m & 0 \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} 0 & \tilde{n} \\ \tilde{m} & 0 \end{pmatrix},$$

and $\tilde{F}, \tilde{T}$ denote the same as $F, T$ but with $u$ being replaced by $\tilde{u}$. To estimate the first term in the right-hand side of (3.61), we write

$$m(x; z) - \tilde{m}(x; z) = \int_{-\infty}^x e^{2iz(x-y)}|w(y) - \tilde{w}(y)| dy, \quad (3.62)$$

where

$$w - \tilde{w} = \left( \tilde{u}_{xx} + \frac{1}{2i} |\tilde{u}_x|^2 \tilde{u}_x \right) e^{i\bar{c}}(x) - \left( \tilde{u}_{xx} + \frac{1}{2i} |\tilde{u}_x|^2 \tilde{u}_x \right) e^{i\bar{c}}(x) \tilde{u}.$$

By using (3.60), we obtain $\|w - \tilde{w}\|_{L^2} \leq c_2 \|u - \tilde{u}\|_{H^{2,1}}$, where $c_2$ is another positive constant.

Under (3.62) and the result in Proposition 8, we obtain for every $x_0 \in \mathbb{R}$:

$$\sup_{x \in (-\infty, x_0)} \|\langle x \rangle (m(x; z) - \tilde{m}(x; z)) \|_{L^2_x(\mathbb{R})} \leq \sqrt{x_{2}} \|u - \tilde{u}\|_{H^{2,1}}. \quad (3.63)$$

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The estimate of \( n \) is analogous. This gives the estimate for the first term in (3.61).

For the second term in the right-hand side of (3.61), we use (3.7) and find \( F \) is a Lipschitz continuous operator from \( L^\infty_x(\mathbb{R}; L^2_z(\mathbb{R})) \) to \( L^\infty_x(\mathbb{R}; L^2_z(\mathbb{R})) \) which means for every \( f \in L^\infty_x(\mathbb{R}; L^2_z(\mathbb{R})) \), we have

\[
\|(F - \tilde{F})f\|_{L^\infty_x L^2_z} \leq c_3\|u - \tilde{u}\|_{H^{2,1}}\|f\|_{L^\infty_x L^2_z},
\]

where \( c_3 \) is another positive constant independent of \( f \). Combining (3.39), (3.40), (3.61), (3.63) and (3.64), we derive for every \( x_0 \in \mathbb{R}^- \):

\[
\sup_{x \in (-\infty, x_0)} \left\langle x \right| \left( \Psi^-(x; \cdot) - e^{-icx}\sigma_3 - \tilde{\Psi}^-(x; \cdot) + e^{-icx}\sigma_3(\tilde{u}) \right) \right\rangle_{L^2(\mathbb{R})} \leq c\|u - \tilde{u}\|_{H^{2,1}},
\]

which gives the proof of (3.58) for \( \Psi^- \) and \( \tilde{\Psi}^- \). The proof of the bound (3.59) is completed by recalling the same analysis to the integral equations (3.42) and (3.50).

As the spectral problem (2.1) admits no resonances shown by Proposition 7, there hence exists a positive number \( a_0 \) such that

\[
|a(k)| \geq a_0 > 0, \quad k \in \mathbb{R} \cup i\mathbb{R},
\]

then we prove some properties of the scattering data in the \( z \)-plane.

**Proposition 13.** If \( u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \), we have the following properties about \( a(z) \) and \( b(k) \)

\[
a(z) = a(k), \quad kb(k), \quad k^{-1}b(k) \in H^1_z(\mathbb{R}),
\]

with

\[
\hat{a} = 1 + \frac{1}{2i} \int_\mathbb{R} |u_y(y)|^2 e^{-icy} \, dy,
\]

and

\[
kb(k), \quad k^{-1}b(k) \in L^2_z(\mathbb{R}).
\]

**Proof.** Due to the relation (2.24), the integration of \( a(k) \) is as follows

\[
a(z) = 1 + k \int_\mathbb{R} u_y(y) \overline{\psi_{21}^-}(x; k) \, dy.
\]
It can be deformed into

\[ a(z) - \hat{a} = \frac{1}{2i} \int_{\mathbb{R}} (|u_y(y)|^2 (\Psi_{11}^+(y; z) - e^{-ic-}) + u_y(y)\Psi_{21}^-(y; z))dy. \]

According to \( u \in H^3(\mathbb{R}) \cap H^{2.5}(\mathbb{R}) \) and (3.35),

\[ \|a(z) - \hat{a}\|_{H^1_z(\mathbb{R})} \leq \sup_{x \in \mathbb{R}} \|u_x\|_{L^2_x} \sup_{x \in \mathbb{R}} \|\Psi_{11}^- - e^{-ic-}\|_{H^1_z} + \|\bar{u}_x\|_{L^1_x} \sup_{x \in \mathbb{R}} \|\Psi_{21}^-\|_{H^1_z}, \]

which yields \( a(z) - \hat{a} \in H^1_z(\mathbb{R}) \).

We then analyze the property of \( b(k) \). By using the determinant (2.29), we write

\[ b(k) = \psi_{11}^+(0; k)\psi_{21}^- (0; k) - \psi_{21}^+(0; k)\psi_{11}^- (0; k). \] (3.69)

Under the relation \( \Psi_{11}^+(x; z) = \psi_{11}^+(x; k) \) and \( \Psi_{21}^+(x; z) = -u_x\psi_{11}^+(x; k) + 2ik\psi_{21}^+(x; k) \), we obtain

\[ 2ikb(k) = \Psi_{11}^+(0; z)\Psi_{21}^- (0; z) - \Psi_{21}^+(0; z)\Psi_{11}^- (0; z). \] (3.70)

With the same way by subtracting the limiting values of \( \Psi_{11}^+(0; z) \) and \( \Psi_{21}^+(0; z) \), we have

\[ 2ikb(k) = (\Psi_{11}^+(0; z) - e^{ic+})\Psi_{21}^- (0; z) + e^{ic+}\Psi_{21}^- (0; z) - \Psi_{21}^+(0; z)(\Psi_{11}^- (0; z) - e^{ic-}) - e^{ic-}\Psi_{11}^- (0; z). \] (3.71)

Each term of (3.71) belongs to \( H^1_z(\mathbb{R}) \). We conclude that \( kb(k) \in H^1_z(\mathbb{R}) \). In addition, based on the determinant (2.29), we have another equation about \( k^{-1}b(k) \)

\[ k^{-1}b(k) = \Psi_{11}^+(0; z)k^{-1}\psi_{21}^- (0; k) - \Psi_{11}^- (0; z)k^{-1}\psi_{21}^+(0; k) \] (3.72)

Recalling that \( k^{-1}\psi_{21}^+(0; k) \) belongs to \( H^1_z(\mathbb{R}) \) by (3.56), we obtain \( k^{-1}b(k) \in H^1_z(\mathbb{R}) \).

Since \( zk^{-1}b(k) = kb(k) \in H^1_z(\mathbb{R}) \), we note that \( k^{-1}b(k) \in L^2_z(\mathbb{R}) \). In addition, to prove that \( kb(k) \in L^2_z(\mathbb{R}) \), we multiply both sides of equation (3.70) by \( z \) and put it in the form as follows

\[ 2ikzb(k) = \Psi_{11}^+(0; z)\left(z\Psi_{21}^+(0; z) - \hat{\Psi}_{21}^+(0)\right) - \Psi_{11}^- (0; z)\left(z\Psi_{21}^-(0; z) - \hat{\Psi}_{21}^-(0)\right)
+ \hat{\Psi}_{21}^- (0)\left(\Psi_{11}^+(0; z) - e^{ic+}\right) - \hat{\Psi}_{21}^+ (0)\left(\Psi_{11}^- (0; z) - e^{ic-}\right). \] (3.73)
where we have used the identity \( \hat{\Psi}_{21}(0)e^{ic_+(0)} - \hat{\Psi}_{21}^+(0)e^{ic_-(0)} = 0 \), which is obtained from limits (3.28) and (3.43). By (3.35) and (3.46), each terms in the representation (3.73) are in \( L^2_z(\mathbb{R}) \) so far. Therefore, we derive the final result that \( kb(k) \in L^2_z(\mathbb{R}) \).

**Proposition 14.** Suppose that \( u(x) \in H^3(\mathbb{R}) \cup H^{2,1}(\mathbb{R}) \), then we have
\[
z^{-1} k^{-1} b(k) \in H^1_z(\mathbb{R}).
\] (3.74)

**Proof.** From (2.7) and (2.24), the scattering data \( b(k) \) and Jost functions \( \psi^-_{11}, \psi^-_{21} \) have the following integral representation
\[
k^{-1} b(k) = -\int_{-\infty}^{\infty} \bar{u}_y(y) \psi^-_{11}(y; k)e^{-2izy} dy,
\] (3.75)
\[
\psi^-_{11}(x; k) = 1 - k \int_{-\infty}^{x} u_g(y) \psi^-_{21}(y; k) dy,
\] (3.76)
\[
\psi^-_{21}(x; k) = k \int_{-\infty}^{x} \bar{u}_y(y) \psi^-_{11}(y; k)e^{2iz(x-y)} dy.
\] (3.77)

In order to estimate the decaying property of \( b(k) \), we write (3.75) in the form
\[
k^{-1} b(k) = \int_{-\infty}^{\infty} \bar{u}_y(y) (\psi^-_{11} - 1) e^{-2izy} dy + \int_{-\infty}^{\infty} \bar{u}_y(y) e^{-2izy} dy.
\] (3.78)

Through integration by parts and the Fourier transformation, the second integral in (3.78) becomes
\[
\int_{-\infty}^{\infty} \bar{u}_y e^{-2izy} dy = \bar{u}(y)e^{-2izy}|_{-\infty}^{+\infty} + 2iz \int_{-\infty}^{\infty} \bar{u}e^{-2izy} dy = iz \bar{u}(z).
\] (3.79)

Next we make estimate on the first integrand in (3.78). Substituting (3.77) into (3.76) yields
\[
\psi^-_{11} - 1 = -z \int_{-\infty}^{x} u_g(y)e^{2izy} dy \int_{-\infty}^{y} \bar{u}_s(s)e^{-2izs} \psi^-_{11} ds := -z K\psi^-_{11},
\] (3.80)
where integral operator is defined by
\[
K f = \int_{-\infty}^{x} u_g(y)e^{2izy} dy \int_{-\infty}^{y} \bar{u}_s(s)e^{-2izs} f ds,
\]
which implies that
\[ \|K\|_{L^\infty \to L^\infty} \leq \|u_x\|_{L^2(\mathbb{R})}^2. \]

With (3.80), the first term of (3.78) becomes
\[
\int_{-\infty}^{\infty} \hat{\bar{u}}(y) (\psi_{11} - 1) e^{-2izy} dy = -z \int_{-\infty}^{\infty} \hat{\bar{u}}(y) e^{-2izy} K \psi_{11} dy
= -\frac{1}{2} z \hat{\bar{u}}(z) K \psi_{11}(z).
\]

Combining (3.78), (3.79) and (3.81), we obtain
\[
z^{-1}k^{-1}b(k) = -\frac{1}{2} \hat{\bar{u}}(z) K \psi_{11}(z) + i\hat{\bar{u}}(z).
\]

By Plancherel formula, we have
\[
\|z^{-1}k^{-1}b(k)\|_{H^1_x} \leq \|\hat{\bar{u}}(z) K \psi_{11}\|_{H^1_x} + \|i\hat{\bar{u}}(z)\|_{H^1_x}
= \|\hat{\bar{u}}(z) K \psi_{11}(x)\|_{L^{2.1}} + \|u(x)\|_{L^{2.1}}
\leq c \|K\|_{L^\infty \to L^\infty} \sup_{x \in \mathbb{R}} \|\psi_{11}\|_{L^\infty_x} \|u_x(x)\|_{L^{2.1}_x} + c \|u(x)\|_{L^{2.1}_x}.
\]

Lemma 1. Let the set \( \Omega \subseteq \mathbb{R} \), if \( f \in L^\infty(\Omega) \), \( f \in L^2(\Omega) \), then for arbitrary \( 2 \leq p < \infty \), we have
\[
\|f\|_{L^p(\Omega)} \leq c \|f\|_{L^2(\Omega)}^{2/p}.
\]

Proof. If \( f \in L^\infty(\Omega) \), then for \( 2 \leq p < \infty \), we have \( |f|^{p-2} \in L^\infty(\Omega) \). Further by Holder inequality
\[
\|f\|_{L^p(\Omega)}^p = \int_{\Omega} |f|^p dx = \int_{\Omega} |f|^2 |f|^{p-2} dx \leq c \int_{\Omega} |f|^2 dx = c \|u\|_{L^2}^2.
\]

Proposition 15. Suppose that \( u(x) \in H^3(\mathbb{R}) \cup H^{2.1}(\mathbb{R}) \), then for fixed small \( \delta > 0 \), we have
\[
z^{-2}k^{-1}b(k) \in L^2_x(-\delta, \delta), \quad z^{-2}r_{1,2} \in L^2(\mathbb{R}).
\]
Proof. The formula (3.74) in Proposition 14 implies that

\[ z^{-1}k^{-1}b(k) \in L^2_z(\mathbb{R}), \quad z^{-1}k^{-1}b(k) \in L^\infty_z(\mathbb{R}). \]  

(3.85)

which, with Lemma 1, yields

\[ z^{-1}k^{-1}b(k) \in L^p_z(\mathbb{R}), \quad 2 \leq p < \infty. \]  

(3.86)

Noting that for fixed small \( \delta > 0 \),

\[ \|z^{-1}\|_{L^1(\mathbb{R})} = \left( \int_{-\delta}^{\delta} |z|^{-1/2} \, dz \right)^2 \leq c, \]

which together with (3.86), and by Yang inequality, we then derive that

\[ \|z^{-2}k^{-1}b(k)\|_{L^2(\mathbb{R})} = \|z^{-1}(z^{-1}k^{-1}b(k))\|_{L^2(\mathbb{R})} \]
\[ \leq \|z^{-1}\|_{L^1}(\mathbb{R})^{1/5} \|z^{-1}k^{-1}b(k)\|_{L^5}(\mathbb{R})^{4/5} \|z^{-1}k^{-1}b(k)\|_{L^2(\mathbb{R})} \leq c. \]  

(3.87)

Fix small \( \delta > 0 \), let \( \Omega = \mathbb{R} \setminus (-\delta, \delta) \) and \( \chi \) denote the indicator function of the interval \( \Omega \). Since \( r_1(0; z) \in L^\infty(\mathbb{R}) \), \( |a(k)|^{-1} \leq a_0^{-1} \) shown in (3.65), then by (3.87), we have

\[ \|z^{-2}r_1(z)\|_{L^2(\mathbb{R})} \leq \|\chi z^{-2}r_1(z)\|_{L^2(\mathbb{R})} + \|(1 - \chi) \frac{1}{2ia(k)} z^{-2}k^{-1}b(k)\|_{L^2(\mathbb{R})} \]
\[ \leq 2\|r_1(z)\|_{L^\infty(\mathbb{R})} \|z^{-2}\|_{L^2(\mathbb{R})} + a_0^{-1} \|z^{-2}k^{-1}b(k)\|_{L^2(\mathbb{R})}. \]  

(3.88)

In a similar way, by using Proposition 14, we can show that

\[ \|z^{-2}r_2(z)\|_{L^2(\mathbb{R})} \leq 2\|r_2(z)\|_{L^\infty(\mathbb{R})} \|z^{-2}\|_{L^2(\mathbb{R})} + a_0^{-1} \|z^{-1}k^{-1}b(k)\|_{L^2(\mathbb{R})}. \]  

(3.89)

In conclusion, we show that the mapping from \( u \) to scattering data \( a(z) \) and \( b(k) \)

\[ H^{2,1}(\mathbb{R}) \ni u \to a(z) - 1 \in H^1_z(\mathbb{R}), \]
\[ H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \ni u \to kb(k), k^{-1}b(k) \in H^1_z(\mathbb{R}), \]
\[ H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \ni u \to kb(k), k^{-1}b(k) \in L^2_z(\mathbb{R}), \]
\[ H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \ni u \to z^{-2}kb(k), z^{-2}k^{-1}b(k) \in L^2_z(\mathbb{R}). \]  

(3.90)
is Lipschitz continuous.

If we express this property of Lipschitz continuity in detail the following corollary can be drawn:

**Corollary 2.** Let \( u, \tilde{u} \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \) satisfy \( \| u \|_{H^3 \cap H^{2,1}} \leq \delta \) for some \( \delta > 0 \). \( u \) and \( \tilde{u} \) correspond to two sets of scattering data \( a, b \) and \( \tilde{a}, \tilde{b} \), respectively. Then, there is a positive constant \( c \) such that

\[
\| a(z) - \tilde{a}(z) \|_{H^1} \leq c \| u - \tilde{u} \|_{H^3 \cap H^{2,1}},
\]

\[
\| kb(k) - k\tilde{b}(k) \|_{L^1} + \| k^{-1}b(k) - k^{-1}\tilde{b}(k) \|_{L^1} \leq c \| u - \tilde{u} \|_{H^3 \cap H^{2,1}}.
\]

### 3.3 A new RH problem on the \( z \)-plane

In order to use the theorems on classical Cauchy integral and its projection on the real axis, we change the original RH problem 2.1 with jump contour on \( \mathbb{R} \cup i\mathbb{R} \) in the \( k \)-plane into a RH problem with jump contour on \( \mathbb{R} \) in the \( z \)-plane.

We define a matrix function

\[
M(x; z) := \begin{cases} 
  e^{ic_+(x)\sigma_3} \left( \frac{\Psi^-(x; z)}{a(z)}, \Psi^+_2(x; z) \right), & z \in \mathbb{C}^+, \\
  e^{ic_+(x)\sigma_3} \left( \Psi^+_1(x; z), \frac{\Psi^-(x; z)}{a(z)} \right), & z \in \mathbb{C}^-,
\end{cases}
\]

which then satisfies the following RH problem on \( z \)-plane (see Figure 2).

**RH Problem 3.1.** Find a matrix function \( M(x; z) \) with the following properties:

- **Analyticity:** \( M(x; z) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \).

- **Jump condition:** \( M(x; z) \) satisfies the jump condition

\[
M_+(x; z) = M_-(x; z)(I + R(x; z)), \quad z \in \mathbb{R},
\]

where the jump matrix is defined by

\[
R(x; z) = \begin{pmatrix} \tilde{r}_1(z) & r_2(z) \\ r_2(z)e^{2ix} & \tilde{r}_1(z)e^{-2ix} \end{pmatrix},
\]

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\[
    r_1(z) := -\frac{b(k)}{2ika(k)}, \quad r_2(z) := \frac{2ikb(k)}{a(k)}, \quad z \in \mathbb{R},
\]
which satisfy the relations
\[
    \begin{align*}
    r_2(z) &= 4zr_1(z), \quad z \in \mathbb{R}, \\
    \bar{r}_1(z)r_2(z) &= |r(k)|^2, \quad z \in \mathbb{R}^+, \quad k \in \mathbb{R}, \\
    \bar{r}_1(z)r_2(z) &= -|r(k)|^2, \quad z \in \mathbb{R}^-, \quad k \in i\mathbb{R}.
    \end{align*}
\]

\[\triangleq \text{Asymptotic conditions:}\]
\[
    M(x; z) \to I \quad \text{as} \quad |z| \to \infty.
\]

**Proof.** The first two assertions are easy to be checked. By using (3.1) and (??), we obtain that
\[
    \Psi_\pm^2(x; z) = \frac{1}{2ik} \left( -\bar{u}_x \psi_{12}^\pm + 2ik\psi_{22}^\pm \right),
\]
which together with (2.10) gives
\[
    \lim_{|z| \to \infty} \Psi_\pm^2(x; z) = e^{-ic_\pm}e_2,
\]
which together with (3.34) leads to (3.98). \[\square\]

**Figure 2:** The left figure is the RH problem for \(N(x; k)\) on \(k\)-plane whose jump contour is \(\mathbb{R} \cap i\mathbb{R}\); The right figure is the RH problem for \(M(x; z)\) on \(z\)-plane whose jump contour is \(\mathbb{R}\).
3.4 Lipschitz continuity of reflection coefficient

Further, we hope to obtain the Lipschitz continuity between the new scattering coefficient \( r_1,2(z) \) and the initial value \( u_0(x) \), and also analyze the properties of the original scattering coefficient \( r(k) \).

**Proposition 16.** The two relations between \( u(x) \) and \( r_1,2(z) \) are listed here:

- If \( u \in H^{2,1}(\mathbb{R}) \), then \( r_1,2 \in H^1(\mathbb{R}) \).
- If \( u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \), then \( r_1,2 \in L^{2,1}(\mathbb{R}) \) and \( z^{-2}r_1,2 \in L^2(\mathbb{R}) \).

Furthermore, the mapping

\[
H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \ni u \rightarrow (r_1, r_2) \in \mathcal{W}, \tag{3.100}
\]

is Lipschitz continuous.

**Proof.** The assertion \( r_1,2 \in H^1(\mathbb{R}) \) is directly derived from Proposition 13 and 15. Noting that

\[
r_2(z) - \bar{r}_2(z) = \frac{2ikb}{a} - \frac{2ik\bar{b}}{\bar{a}}, \tag{3.101}
\]

\[
z^{-2}(r_2(z) - \bar{r}_2(z)) = \frac{2ikz^{-2}b}{a} - \frac{2ikz^{-2}\bar{b}}{\bar{a}}. \tag{3.102}
\]

By the Lipschitz continuity from \( u(x) \) to \( a(z) \) and \( b(k) \) showed in (3.90), then Lipschitz continuity of the mapping (3.100) for \( r_2(z) \) follows from the representation (3.101) and (3.102). The Lipschitz continuity for \( r_1(z) \) also can be proved in a similar way.

**Proposition 17.** If \( r_1,2(z) \in H^1_z(\mathbb{R}) \cap L^{2,1}_z(\mathbb{R}) \), then \( r(k) \in L^{2,1}_z(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).

**Proof.** As \( r_1,2(z) \in L^{2,1}(\mathbb{R}) \) and \( |r(k)|^2 = \text{sign}(z)\bar{r}_1(z)r_2(z) \) for every \( z \in \mathbb{R} \), we have

\[
\|\bar{r}(k)\|_{L^{2,1}(\mathbb{R})} \leq \|r_1(z)\|_{L^2(\mathbb{R})}\|r_2(z)\|_{L^{2,1}(\mathbb{R})} \tag{3.103}
\]

Thus, we show that \( r(k) \in L^{2,1}_z(\mathbb{R}) \).
To give the proof of $r(k) \in L^\infty_z(\mathbb{R})$, we can define $r(k)$ in two equivalently forms according to the definition of $r_{1,2}(z)$ in (3.94):

$$r(k) = \begin{cases} -2ikr_1(z) & |k| \leq 1 \\ (2ik)^{-1}r_2(z) & |k| \geq 1 \end{cases}$$  \hfill (3.104)

As $r_{1,2} \in H^1(\mathbb{R})$, we have $r_{1,2} \in L^\infty_z(\mathbb{R})$ and thereafter $r(k) \in L^\infty_z(\mathbb{R})$.

**Proposition 18.** If $r_2(z) \in H^1_z(\mathbb{R}) \cap L^{2,1}_z(\mathbb{R})$, then

$$\|kr_2(z)\|_{L^\infty_z(\mathbb{R})} \leq \|r_2\|_{H^1 \cap L^{2,1}_z(\mathbb{R})}.$$  \hfill \Box

**Proof.** For $r_2(z) \in H^1_z(\mathbb{R}) \cap L^{2,1}_z(\mathbb{R})$, we use Cauchy-Schwartz inequality to derive

$$|kr_2(z)|^2 = |zr_2^2(z)| = \left| \int_0^z (r_2(s))^2 + 2sr_2(s)r'_2(s) \, ds \right| \leq \|r_2(z)\|_{L^2_z}^2 + 2\|r'_2(z)\|_{L^2_z}\|zr_2(z)\|_{L^2_z} \leq \|r_2(z)\|_{H^1 \cap L^{2,1}_z(\mathbb{R})},$$

which implies the desired bound.  \hfill \Box

4 Estimates on solutions to the RH problem

4.1 Transitions of the RH problem

The RH problem 3.1 admits the Beals-Coifman solution

$$M_\pm(x; z) = I + \mathcal{P}^\pm(zMr(x; \cdot)R(x; \cdot))(z), \quad z \in \mathbb{R},$$  \hfill (4.1)

which can be used to estimate the columns of $M_\pm(x; z) - I$.

Next, we introduce the 2-by-2 matrix

$$T(x; z) = (M_{-,1}(x; z) - e_1, \quad M_{+,2}(x; z) - e_2),$$  \hfill (4.2)

and rewrite (4.1) into an equivalent form

$$T - \mathcal{P}^+(TR_+) - \mathcal{P}^-(TR_-) = F,$$  \hfill (4.3)

where

$$R_+(x; z) = \begin{pmatrix} \tilde{r}_1(z)e^{-2izx} \\ 0 \\ 0 \end{pmatrix}, \quad R_-(x; z) = \begin{pmatrix} 0 \\ r_2(z)e^{2izx} \\ 0 \end{pmatrix},$$  \hfill (4.4)

$$F = (\mathcal{P}^-(r_2(z)e^{2izx})e_2, \quad \mathcal{P}^+(\tilde{r}_1(z)e^{-2izx})e_1),$$  \hfill (4.5)
The estimate on $F$ can be obtained from Proposition 23. Our main task is to make a further estimate on the Cauchy integral projection in (4.3).

To analyze the derivatives of $M_{-1}(x; z)$ and $M_{+2}(x; z)$, we take the derivative of the inhomogeneous equation (4.3) in $x$, which gives

$$\partial_x T - \mathcal{P}^+ (\partial_x T) R_+ - \mathcal{P}^- (\partial_x T) R_- = \tilde{F},$$

where

$$\tilde{F} = 2i \left( e_2 \mathcal{P}^- (zr_2(z)e^{2izx}) - e_1 \mathcal{P}^+ (-z\bar{r}_1(z)e^{-2izx}) \right) + 2i \left( \begin{array}{cc} \mathcal{P}^- (zr_2(z)M_{12}^+(x; z)e^{2izx}) & -\mathcal{P}^+ (z\bar{r}_1(z) (M_{11}^+(x; z) - 1) e^{-2izx}) \\ \mathcal{P}^- (zr_2(z) (M_{21}^+ (x; z) - 1) e^{2izx}) & -\mathcal{P}^+ (z\bar{r}_1(z) M_{21}^+(x; z)e^{-2izx}) \end{array} \right).$$

Proposition 18 inspires us to start with $kr_{1,2}$ to solve the problem. Therefore, for every $k \in \mathbb{C} \setminus \{0\}$, we introduce two new matrices

$$V_1(k) := \begin{pmatrix} 1 & 0 \\ 0 & 2ik \end{pmatrix}, \quad V_2(k) := \begin{pmatrix} (2ik)^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

(4.8)

to give the original problem the form we want and $V_1$ admits

$$V_1^{-1}(k)R(x; z)V_1(k) = V_2^{-1}(k)R(x; z)V_2(k) = J(x; k), \quad z \in \mathbb{R}, \quad k \in \mathbb{R} \cup i\mathbb{R}.$$ (4.9)

It can be seen from the following analysis that the transformation of $M(x; z)$ using these two matrices can not only complete the estimation of $M(x; z)$ itself and its derivative, but also make full use of the good properties of jump matrix $J(x; k)$.

In the following RH problem, the properties of the matrix elements are characterized in the $z$ plane: If $r_{1,2} \in H_z^1(\mathbb{R}) \cap L_z^2(\mathbb{R})$, then Proposition 18 implies that $J \in L_z^1(\mathbb{R}) \cap L_z^\infty(\mathbb{R})$ and $F(x; z) \in L_z^2(\mathbb{R})$ for every $x \in \mathbb{R}$. We consider the class of solutions to the RH problem 3.1 such that for every $x \in \mathbb{R}$. Therefore, in the following subsection, we equivalently reduce the RH problem 3.1 in the $z$ plane to the RH problem related with the matrix $J(x; k)$ instead of the matrix $R(x; z)$.

**RH Problem 4.1.** Find a matrix function $Q_j(x; k)$ with the following properties:

- **Analyticity:** $Q_j(x; k)$ are analytic functions of $z$ in $\mathbb{C}^\pm$. 39
Jump condition: $Q_j(x;k)$ satisfies the jump condition

$$Q_{j,+}(x;k) = (I + J)Q_{j,-}(x;k) + D_j, \quad k \in \mathbb{R} \cup i\mathbb{R},$$ (4.10)

with

$$Q_{j,\pm}(x;k) := M_{\pm}(x;z)V_j(k) - V_j(k), \quad D_j = V_j(k)J, \quad j = 1, 2.$$ (4.11)

Parity: the columns of $Q_{j,\pm}(x;k), Q_{j,-}(x;k)J,$ and $D_j$ have the same parity in $k$.

Asymptotic conditions:

$$Q_{j,\pm}(x;k) \to 0, \quad \text{as} \quad |k| \to \infty.$$ (4.12)

**Proposition 19.** The RH problem 4.1 exists a unique solution.

*Proof.* The property is equivalent to the existence and the uniqueness of the solution to the RH problem 2.1, which is proved in Proposition 5.

We can express the solution of the RH problem 4.1 with the Cauchy integrals

$$Q_{j,\pm}(x;k) = C(Q_{j,-}(x;k)J + D_j)(z), \quad z \in \mathbb{C}^\pm.$$ (4.13)

There is a solution $Q_{j,-}(x;k) \in L_2^2(\mathbb{R})$ to the Fredholm integral equation:

$$Q_{j,-}(x;k) = \mathcal{P}^- (Q_{j,-}(x;k)J + D_j)(z), \quad z \in \mathbb{R},$$ (4.14)

Once $Q_{j,-}(x;k) \in L_2^2(\mathbb{R})$ is found, $Q_{j,+}(x;k) \in L_2^2(\mathbb{R})$ is obtained from the projection formula

$$Q_{j,+}(x;k) = \mathcal{P}^+ (Q_{j,-}(x;k)J + D_j)(z), \quad z \in \mathbb{R}.$$ (4.15)

As a result of Proposition 19, we know that $Q_{j,-}(x;k) \in L_2^2(\mathbb{R})$ exists.

To estimate the solutions to the RH problem 3.1, we prove that the operator $(I - \mathcal{P}^-)^{-1}$ in the Fredholm equation (4.14) exists in $L_2^2(\mathbb{R})$. 

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Proposition 20. If \( r(k) \in L^2_z(\mathbb{R}) \cap L^\infty_z(\mathbb{R}) \) satisfying (2.44), then \((I - \mathcal{P}^-)^{-1}\) is a bounded operator in \( L^2_z(\mathbb{R}) \). In particular, there is a positive constant \( c \) which only depends on \( \| r(k) \|_{L^\infty_z} \) so that for every row vector \( f \in L^2_z(\mathbb{R}) \), we have

\[
\left\| (I - \mathcal{P}^-)^{-1} f \right\|_{L^2_z} \leq c \| f \|_{L^2_z},
\]

(4.16)

Proof. We consider the linear inhomogeneous equation (4.14) with \( D_j \in L^2_z(\mathbb{R}) \). Based on \( \mathcal{P}^+ - \mathcal{P}^- = I \), then (4.14) and (4.15) can be rewritten into inhomogeneous equations

\[
Q_{j,-} - \mathcal{P}^- (Q_{j,-} J) = \mathcal{P}^- (D_j), \quad Q_{j,+} - \mathcal{P}^- (Q_{j,-} J) = \mathcal{P}^+(D_j). \tag{4.17}
\]

By Proposition 5, since \( \mathcal{P}^\pm(D_j) \in L^2_z(\mathbb{R}) \), there exists a unique solution to the inhomogeneous equation (4.17), so that the decomposition \( Q_j = Q_{j,+} - Q_{j,-} \) is unique. Therefore, we only need to find the estimates of \( Q_{j,+} \) and \( Q_{j,-} \) in \( L^2_z(\mathbb{R}) \).

We begin to deal with \( Q_{j,-} \) and define two analytic functions in \( \mathbb{C} \setminus \mathbb{R} \) by

\[
q_1(z) := \mathcal{C} (Q_{j,-} J) (z) \quad \text{and} \quad q_2(z) := \mathcal{C} (Q_{j,-} J + D_j)^H (z) \tag{4.18}
\]

in a similar way as in the proof of Proposition 10. By Proposition 3, we have \( q_1(z) = \mathcal{O} (z^{-1}) \), \( q_2(z) \to 0 \) as \( |z| \to \infty \), and \( q_2(z) \to 0 \) as \( |z| \to \infty \). Since \( D_j \in L^2_z(\mathbb{R}) \), \( Q_{j,-} \in L^2_z(\mathbb{R}) \), and \( J(k) \in L^2_z(\mathbb{R}) \cap L^\infty_z(\mathbb{R}) \). Therefore, the integral on closed enclosing line \( \{|z| = R, \text{Im} z > 0\} \cup (-R, R) \) goes to zero as \( R \to \infty \) by Lebesgue’s dominated convergence theorem.

Repeating the procedures in the proof of Proposition 5, we obtain

\[
0 = \oint q_1(z) q_2(z) \, dz = \int_{\mathbb{R}} [Q_{j,-} \mathcal{P}^- (D_j) + Q_{j,-} J] Q_{j,-}^H \, dz, \tag{4.19}
\]

in which we use the first inhomogeneous equation in the system (4.17). By the bound (2.47) and the bound (2.41), under the Cauchy-Schwartz inequality, there exists a positive constant \( \rho_- \) such that

\[
\rho_- \| Q_{j,-} \|^2_{L^2_z} \leq \text{Re} \int_{\mathbb{R}} Q_{j,-} (I + J) Q_{j,-}^H \, dz = \text{Re} \int_{\mathbb{R}} \mathcal{P}^- (D_j) Q_{j,-}^H \, dz \leq \| D_j \|_{L^2} \| Q_{j,+} \|_{L^2}.
\]
It is worth to note that the above estimate holds independently for the corresponding row vectors in the matrices $Q_{j,-}$ and $D_j$. As $Q_{j,-} = (I - P^-)^{-1} P^- (D_j)$, for every row-vector $f \in L^2_\mathbb{R}$ of the matrix $D_j \in L^2_\mathbb{R}$, the above inequality yields\[ \left\| (I - P^-)^{-1} P^- f \right\|_{L^2_\mathbb{R}} \leq C^- \| f \|_{L^2_\mathbb{R}}. \quad (4.20) \]

Similarly, for $Q_{j,+}$, with the bounds (2.41) (2.47) and (2.48) in Proposition 3, there are positive constants $\rho_+$ and $\rho_-$ such that
\[
\rho_- \| Q_{j,+} \|^2_{L^2_\mathbb{R}} \leq \text{Re} \int_{\mathbb{R}} Q_{j,+} (I + J)^H Q_{j,+}^H dz \leq \rho_+ \| D_j \|_{L^2_\mathbb{R}} \| Q_{j,+} \|_{L^2_\mathbb{R}}.
\]

As $Q_{j,+} = (I - P^-)^{-1} P^+ (D_j)$, for every row vector $f \in L^2_\mathbb{R}$ of the matrix $D_j \in L^2_\mathbb{R}$, the above inequality ensures\[ \left\| (I - P^-)^{-1} P^+ f \right\|_{L^2_\mathbb{R}} \leq C^- C_+ \| f \|_{L^2_\mathbb{R}}. \quad (4.21) \]

Finally, this proposition is proved via the bounds (4.20), (4.21) and the triangle inequality.

4.2 Estimates on the Beals-Coifman solutions

In this subsection, we represent each column of the RH problem 4.1 via the projection operators and give them fairly accurate estimates. Searching for analytic matrix functions $M_\pm(x; \cdot)$ in $\mathbb{C}^\pm$ for every $x \in \mathbb{R}$, we introduce the following notations for the column vectors of the matrices $M_\pm(x; z)$\[ M_\pm(x; z) = (M_{\pm,1}(x; z), M_{\pm,2}(x; z)) \quad (4.22) \]

The expression of $Q_{j,\pm}(x; k)$ is given by
\[
Q_{1,\pm}(x; k) = M_{\pm}(x; z)V_1(k) - V_1(k) = ((M_{\pm,1}(x; z) - e_1), (2ik)M_{\pm,2}(x; z) - e_2),
\]
\[
Q_{2,\pm}(x; k) = M_{\pm}(x; z)V_2(k) - V_2(k) = ((2ik)^{-1}(M_{\pm,1}(x; z) - e_1), M_{\pm,2}(x; z) - e_2),
\]

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where
\[ D_j(x; k) := V_j(k)J = R(x; z)V_j(k), \quad j = 1, 2. \tag{4.23} \]

We use
\[ (Q - J + D_j)_{i1} = (M_-V_j)_{i1} = (M_-RV_j)_{i1} = (M_-R)_{i1}, \quad i = 1, 2, \tag{4.24} \]

to obtain the expression of the first column of \( Q_j, \pm \)
\[ M_{\pm, 1}(x; z) - e_1 = \mathcal{P}^\pm (M_-(x; \cdot)R(x; \cdot))_{11}, \quad z \in \mathbb{R}, \tag{4.25} \]
\[ (2ik)^{-1}(M_{\pm, 1}(x; z) - e_1) = \mathcal{P}^\pm ((2ik^{-1})M_-(x; \cdot)R(x; \cdot))_{11}, \quad z \in \mathbb{R}, \tag{4.26} \]

and the second column of \( Q_j, \pm \)
\[ M_{2, \pm}^+(x; z) - e_2 = \mathcal{P}^\pm (M_-(x; \cdot)R(x; \cdot))_{21}, \quad z \in \mathbb{R}, \tag{4.27} \]
\[ 2ik(M_{2, \pm}^+(x; z) - e_2) = \mathcal{P}^\pm (2ikM_-(x; \cdot)R(x; \cdot))_{21}, \quad z \in \mathbb{R}, \tag{4.28} \]

**Remark 2.** From the results (4.25)-(4.28) obtained above, we find that two versions (4.25) and (4.26) may seem to be inconsistent, as well as (4.27) and (4.28) are inconsistent unless we show that (4.26) and (4.28) are redundancy. For the purpose, considering Cauchy integral projection \( \mathcal{P}^\pm (kf(k)) \) on \( k \in \mathbb{R} \cap i\mathbb{R} \), if \( f(k) \) is even function, we define its orientated integral contour is the left graph in Figure 3, and if \( f(k) \) is even function, we take its orientated integral contour is the right graph in Figure 3. In this way we can eliminate the coefficients \((2ik)^{-1}\) at both sides of the equation (4.26) and \(2ik\) at both sides of the equation (4.28).

![Figure 3](image)

Figure 3: The orientated integral contour for \( \mathcal{P}^\pm (kf(k)) \) on \( k \in \mathbb{R} \cap i\mathbb{R} \): The left graph is the integral contour for even function \( f(k) \); The right graph is the integral contour for odd function \( f(k) \).
Combining (4.25) with (4.27), we rewrite (4.22) into
\[ M_\pm(x; z) = I + \mathcal{P}^\pm (M_- (x; \cdot) R(x; \cdot)), \quad z \in \mathbb{R}, \quad (4.29) \]
for representing the solution to the RH problem 3.1 on the real line. The analytical continuation of functions \( M_\pm(x; \cdot) \) in \( \mathbb{C}^\pm \) is given by the Cauchy operators
\[ M(x; z) = I + \mathcal{C} \left( M_- (x; \cdot) R(x; \cdot) \right), \quad z \in \mathbb{C}^\pm. \quad (4.30) \]
In the following proposition, we show that \( M_\pm(x; z) \) can be estimated with \( r_{1,2}(z) \).

**Proposition 21.** Suppose \( r_{1,2}(z) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}) \) such that the inequality (2.44) is satisfied. \( M_\pm(x; z) \) has the estimate for every \( x \in \mathbb{R} \),
\[ \| M_\pm(x; \cdot) - I \|_{L^2(\mathbb{R})} \leq c \left( \| r_1 \|_{L^2(\mathbb{R})} + \| r_2 \|_{L^2(\mathbb{R})} \right), \quad (4.31) \]
with a positive constant \( c \) which only depends on \( \| r_{1,2} \|_{L^\infty(\mathbb{R})} \).

**Proof.** Under the condition in proposition 17, \( R(x; z)V_j(k) \) belongs to \( L^2(\mathbb{R}) \) for every \( x \in \mathbb{R} \) by the explicit expressions of \( D_j \) by (4.23). Combining the expression (4.25) and (4.27), we conclude that the estimate of \( R(x; z)V_j(k) \) is essential for using the Proposition 20.

Meanwhile, there exists a positive constant \( c \) which only depends on \( \| r_{1,2} \|_{L^\infty(\mathbb{R})} \) such that for every \( x \in \mathbb{R} \),
\[ \| R(x; z)V_j(k) \|_{L^2(\mathbb{R})} \leq c \left( \| r_1 \|_{L^2(\mathbb{R})} + \| r_2 \|_{L^2(\mathbb{R})} \right), \quad j = 1, 2. \quad (4.32) \]
According to
\[ \mathcal{P}^-(D_j) = \mathcal{P}^-(R(x; z)V_j(k))(k), \quad j = 1, 2, \quad (4.33) \]
and the above discussion, the integral equation (4.29) for the projection operator \( \mathcal{P}^- \) is obtained from the integral equations (4.17).

In the end, each element of \( M_-(x; z) \) satisfies the bound (4.21) for the corresponding row vectors of \( \mathcal{P}^-(D_j) \). Combining the bounds (4.16) and (4.32), we finally derive the bound (4.31). \( \square \)
Next, we begin to derive the accurate estimates on the solution to the integral equations (4.29), where the key step is to derive the estimation on the scattering coefficients \( r_1 \) and \( r_2 \) via the Fourier theory.

For a given function \( f(z) \in L^2(\mathbb{R}) \), we define Fourier transform and inverse transform

\[
\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(z) e^{-iz\xi} dz, \quad f(z) = \int_{\mathbb{R}} \hat{f}(\xi) e^{iz\xi} d\xi,
\]

then we can show the following proposition

**Proposition 22.** If \( f(z) \in L^2(\mathbb{R}) \), then we have

\[
\mathcal{P}^+ \left( f(z)e^{-2izx} \right) = \int_{2x}^{+\infty} \hat{f}(\xi)e^{iz(\xi-2x)} d\xi, \tag{4.34}
\]

\[
\mathcal{P}^- \left( f(z)e^{2izx} \right) = -\int_{2x}^{-\infty} \hat{f}(\xi)e^{-iz(\xi-2x)} d\xi, \tag{4.35}
\]

which change the Cauchy projections into the integral on the positive half-lines.

**Proof.** By using the definition of projection operator and Fourier inverse transform, we have

\[
\mathcal{P}^+(f(z)e^{-2izx}) = \frac{1}{2\pi i} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} f(s) e^{-2isz} ds \int_{\mathbb{R}} \hat{f}(\xi) e^{iz\xi} d\xi
\]

\[
= \frac{1}{2\pi i} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \hat{f}(\xi) e^{iz\xi} d\xi \right) \frac{e^{-2isz}}{s - (z + i\epsilon)} ds
\]

\[
= \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{f}(\xi) \left( \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \frac{e^{is(\xi-2s)}}{s - (z + i\epsilon)} ds \right) d\xi. \tag{4.36}
\]

Further through residue computation, we obtain

\[
\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{is(\xi-2s)}}{s - (z + i\epsilon)} ds = \lim_{\epsilon \downarrow 0} \left\{ \begin{array}{ll} e^{i(z+i\epsilon)(\xi-2s)} , & \text{if } \xi - 2s > 0 \\ 0 , & \text{if } \xi - 2s < 0 \end{array} \right.
\]

\[
= \chi(\xi - 2s)e^{iz(\xi-2x)}, \tag{4.37}
\]

with \( \chi(s) \) being a characteristic function. Substituting (4.37) into (4.36) yields

\[
\mathcal{P}^+(f(z)e^{-2izx}) = \int_{2x}^{+\infty} \hat{f}(\xi)e^{iz(\xi-2x)} d\xi. \tag{4.38}
\]
By definition, we have

\[ P^-(f(z)e^{2izx}) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_R \frac{f(s)e^{2isz}}{s - (z - i\epsilon)} ds. \]

Taking conjugation on both sides and using (4.38), we have

\[ P^-(f(z)e^{2izx}) = -\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_R \bar{f}(s)e^{-2isz} \frac{ds}{s - (z + i\epsilon)} = -P^+(\bar{f}(z)e^{-2izx}) \]

\[ = -\int_{2x}^{+\infty} \hat{f}(\xi)e^{i\xi(\xi - 2x)} d\xi. \] (4.39)

Again taking conjugation on both sides of (4.39) gives

\[ P^-(f(z)e^{2izx}) = -\int_{2x}^{\infty} \hat{f}(\xi)e^{-i\xi(\xi - 2x)} d\xi, \]

which exactly is the formula (4.35). \( \square \)

By using the above Proposition 22, we can derive a series of estimators on reflection coefficients \( r_{1,2}(z) \).

**Proposition 23.** For every \( x_0 \in \mathbb{R}^+ \) and every \( r_{1,2}(z) \in H^1(\mathbb{R}) \), we have

\[ \sup_{x \in (x_0, \infty)} \| \langle x \rangle P^+ (z^{-i} \bar{r}_1(z)e^{-2izx}) \|_{L^2_z} \leq \| z^{-i} \bar{r}_1(z) \|_{H^1_z}, \quad i = 0, 1, \] (4.40)

\[ \sup_{x \in (x_0, \infty)} \| \langle x \rangle P^- (r_2(z)e^{2izx}) \|_{L^2_z} \leq \| r_2(z) \|_{H^1_z}. \] (4.41)

In addition, we have

\[ \sup_{x \in \mathbb{R}} \| P^+ (z^{-i} \bar{r}_1(z)e^{-2izx}) \|_{L^\infty_z} \leq \frac{1}{\sqrt{2}} \| z^{-i} \bar{r}_1(z) \|_{H^1_z}, \quad i = 0, 1, \] (4.42)

\[ \sup_{x \in \mathbb{R}} \| P^- (r_2(z)e^{2izx}) \|_{L^\infty_z} \leq \frac{1}{\sqrt{2}} \| r_2(z) \|_{H^1_z}. \] (4.43)

Furthermore, if \( r_{1,2}(z) \in L^{2,1}(\mathbb{R}) \), then

\[ \sup_{x \in \mathbb{R}} \| P^+ (z \bar{r}_1(z)e^{-2izx}) \|_{L^2_z} \leq \| r_1(z) \|_{L^{2,1}_z}, \] (4.44)

\[ \sup_{x \in \mathbb{R}} \| P^- (z r_2(z)e^{2izx}) \|_{L^2_z} \leq \| r_2(z) \|_{L^{2,1}_z}. \] (4.45)
Proof. For a given function \( r(z) \in L^2(\mathbb{R}) \), then its Fourier transform \( \hat{r}(\xi) \in L^2(\mathbb{R}) \) and by Plancherel formula, we have

\[
\|r(z)\|_{L^2}^2 = 2\pi \|\hat{r}(\xi)\|_{L^2}^2. \tag{4.46}
\]

Further, a general conclusion shows \( r(z) \in H^1(\mathbb{R}) \) if and only if \( \hat{r}(\xi) \in L^{2,1}(\mathbb{R}) \). Also, \( r(z) \in L^{2,1}(\mathbb{R}) \) if and only if \( \hat{r}(\xi) \in H^1(\mathbb{R}) \). To prove (4.40), we use Proposition 22 and get

\[
\mathcal{P}^+(z^{-1}\hat{r}_1(z)e^{-2izx}) = \int_{2\pi}^{\infty} \hat{r}_1(z)(\xi)e^{iz(y-2x)}dz. \tag{4.47}
\]

By the bound (3.26) in Proposition 8, the bound (4.40) is obtained

\[
\sup_{x \in (0,\infty)} \|\langle x \rangle \int_{2\pi}^{\infty} (z^{-1}\hat{r}_1(z))(\xi)e^{iz(-2x)}dz\|_{L^2_x} \leq \sqrt{2\pi} \left\| \hat{r}_1(z)(\xi) \right\|_{L^2_{\xi}} \leq \sqrt{2\pi} \left\| z^{-1}\hat{r}_1(z) \right\|_{H^1_z}. \tag{4.48}
\]

Similarly, we derive the bound (4.42) as follows

\[
\left\| \mathcal{P}^+ (z^{-1}\hat{r}_1(z)e^{-2izx}) \right\|_{L^\infty_{2\pi}} \leq \sqrt{\pi} \left\| (z^{-1}\hat{r}_1(z))(\xi) \right\|_{L^2_{\xi}} \leq \frac{1}{\sqrt{2\pi}} \left\| z^{-1}\hat{r}_1(z) \right\|_{H^1_z}. \tag{4.49}
\]

The bounds (4.41), (4.43), (4.44) and (4.45) are obtained in the same way. \(\square\)

Our ultimate goal is to estimate the solutions \( M(x; z) \) to the RH problem 3.1. By Proposition 21, these solutions on the real line can be written in the integral Fredholm form (4.29). One remaining task is to estimate the column vectors \( M_{-1} - e_1 \) and \( M_{+2} - e_2 \). From the equation (4.25), we obtain

\[
M_{-1}(x; z) - e_1 = \mathcal{P}^- (r_2(z)e^{2izx}M_{+2}(x; z))(z), \quad z \in \mathbb{R}. \tag{4.49}
\]

Also, starting from (4.27), we know

\[
M_{+2}(x; z) - e_2 = \mathcal{P}^+ (\hat{r}_1(z)e^{-2izx}M_{-1}(x; z))(z), \quad z \in \mathbb{R}. \tag{4.50}
\]

Both the calculations take advantage of the following facts

\[
(M_{-1})_{11} = e^{i\sigma(x)\pi c_3} (M_{-1})_{11} = r_2(z)e^{2izx}M_{+21},
\]

\[
(M_{-1})_{21} = e^{i\sigma(x)\pi c_3} (M_{-1})_{21} = \hat{r}_1(z)e^{-2izx}M_{-11}.
\]

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From the explicit expression (4.5) for $F(x; z)$, we note that the first row vector of $F(x; z)$ is equal to $F(x; z)V_2(z)$ and the first row vector of $F(x; z)$ is equal to $F(x; z)V_1(z)$ which have the following representation

$$F_1(x; z) = (F(x; z)V_2)_1 = (0, \mathcal{P}^+(\bar{r}_1(z)e^{-2ixz})), \quad (4.51)$$
$$F_2(x; z) = (F(x; z)V_1)_2 = (\mathcal{P}^-(r_2(z)e^{2ixz}), 0). \quad (4.52)$$

**Proposition 24.** For every $x_0 \in \mathbb{R}^+$ and every $r_{1,2} \in H^1(\mathbb{R})$, the unique solution to the system of the integral equations (4.49) and (4.50) satisfies the estimates

$$\sup_{x \in (x_0, \infty)} \| (x) M_{-,21}(x; z) \|_{L^2_2(\mathbb{R})} \leq c \| r_2 \|_{H^1(\mathbb{R})}, \quad (4.53)$$

and

$$\sup_{x \in (x_0, \infty)} \| (x) M_{+,12}(x; z) \|_{L^2_2(\mathbb{R})} \leq c \| r_1 \|_{H^1(\mathbb{R})}, \quad (4.54)$$

where $c$ is a positive constant depending on $\| r_{1,2} \|_{L^\infty}$. Moreover, if $r_{1,2} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, we also have

$$\sup_{x \in \mathbb{R}} \| \partial_x M_{-,21}(x; z) \|_{L^2_2(\mathbb{R})} \leq c \left( \| r_1 \|_{H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})} + \| r_2 \|_{H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})} \right), \quad (4.55)$$

and

$$\sup_{x \in \mathbb{R}} \| \partial_x M_{+,12}(x; z) \|_{L^2_2(\mathbb{R})} \leq c \left( \| r_1 \|_{H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})} + \| r_2 \|_{H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})} \right), \quad (4.56)$$

where $c$ is another positive constant which depends on $\| r_{1,2} \|_{L^\infty(\mathbb{R})}$.

**Proof.** From the explicit expression (4.5), the first row vector of $T(x; z)V_2(k)$ is given by

$$(2ik)^{-1}(M_{-,11}(x; z) - 1), \quad M_{+,12}(x; z) - \bar{r}_1(z)e^{-2ixz}(M_{-,11}(x; z) - 1), \quad (4.57)$$

and the second row vector of $T(x; z)V_1(k)$ is given by

$$M_{-,21}(x; z), \quad 2ik \left( M_{+,22}(x; z) - 1 - \bar{r}_1(z)e^{-2ixz}M_{-,21}(x; z) \right). \quad (4.58)$$
Using the bound (4.16), we have for every \( x \in \mathbb{R} \),
\[
\| M_{-,21}(x; z) \|_{L^2_x} \leq c \| P^- (r_2(z)e^{2izx}) \|_{L^2_x},
\]  
(4.59)
\[
\| (2ik)^{-1} (M_{-,11}(x; z) - 1) \|_{L^2_x} \leq c \| P^+ (\tilde{r}_1(z)e^{-2izx}) \|_{L^2_x},
\]  
(4.60)
\[
\| 2ik (M_{-,22}(x; z) - 1 - \tilde{r}_1(z)e^{-2izx}M_{-,21}(x; z)) \|_{L^2_x} \leq c \| P^- (r_2(z)e^{2izx}) \|_{L^2_x},
\]  
\[
\| M_{+,12}(x; z) - \tilde{r}_1(z)e^{-2izx}M_{-,11}(x; z) - 1 \|_{L^2_x} \leq c \| P^+ (\tilde{r}_1(z)e^{-2izx}) \|_{L^2_x}.
\]  
In the end, we find
\[
\| M_{+,12}(x; z) \|_{L^2_x} \leq c \| P^+ (\tilde{r}_1(z)e^{-2izx}) \|_{L^2_x},
\]  
(4.61)
where the positive constant \( c \) still has the only dependence on \( \|r_{1,2}\|_{L^\infty} \). According to the bounds (4.40) and (4.61), we obtain the bound (4.54).

For estimating the derivative of \( M(x; z) \), under (4.6), the first row vector of \( \tilde{F}(x; z)V_2(z) \) and the second row vector of \( \tilde{F}(x; z)V_1(z) \) belongs to \( L^2_x(\mathbb{R}) \) by \( kr_{1,2}(z) \in L^\infty(\mathbb{R}) \), owing to the bounds (4.44) and (4.45) in Proposition 23, as well as the bounds (4.31) and (4.60). Combining with the previous analysis, we obtain the bounds (4.55) and (4.56). So far we’ve proved the existence of the derivative of \( M(x; z) \). \( \square \)

5 Reconstruction and estimates of the potential

Recalling reconstruction formulas obtained in (2.22), (2.23) and (3.43), we have
\[
u(x)e^{2i(c_+(x)+c)} = \lim_{k \to 0} (k^{-1}\psi^+(x; k))_{12},
\]  
(5.1)
\[
u(x)e^{-i(2c_-(x)+c)} = \lim_{k \to 0} (k^{-1}\psi^-(x; k))_{12},
\]  
(5.2)
\[
\partial_x (\tilde{u}_x(x)e^{icx}) = 2i \lim_{|z| \to \infty} (z\Psi^+(x; z))_{21},
\]  
(5.3)
which gives the relation between the potential \( \nu(x) \) and the Jost functions.

Next, we draw the parallel between the properties of the potential \( \nu \) recovered by the equations (5.1)-(5.2) with the properties of the matrices \( M_{\pm}(x; z) \), i.e., the solution to the RH problem 3.1 satisfying the integral equations (4.29). Then, using the relation
\[
\frac{\psi^+_2(x; k)}{2ik} = \Psi^+_2(x; z),
\]  
(5.4)
we get the reconstruction of \( u(x) \) and \( u_x(x) \) as follows

\[
\begin{align*}
    u(x)e^{-i(2c_{-}(x)+c)} &= 2i \lim_{z \to 0} M_{+,12}(x; z), \\
    u(x)e^{2i(c_{-}(x)+c)} &= 2i \lim_{z \to 0} M_{-,12}(x; z), \\
    \partial_x \left( \bar{u}_x(x)e^{ic_{+}(x)} \right) &= 2i e^{-ic_{+}(x)} \lim_{|z| \to \infty} z M_{\pm,21}(x; z),
\end{align*}
\]

(5.5) \hfill (5.6) \hfill (5.7)

### 5.1 Estimates on the positive half-line

We shall prove two important conclusions:

- if \( r_1, r_2 \in H^1(\mathbb{R}) \) and \( z^{-2}r_{1,2} \in L^2(\mathbb{R}) \), the reconstruction formulas (5.8) and (5.9) recover \( u \) in the class \( H^{2,1}(\mathbb{R}^+) \).

- if \( r_{1,2} \in \mathcal{H} \), then \( u \) is in the class \( H^3(\mathbb{R}^+) \cap H^{2,1}(\mathbb{R}^+) \).

Since \( r_{1,2} \in \mathcal{H} \), we have \( R(x; \cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) for every \( x \in \mathbb{R} \). Therefore, using the solution representation (4.30), we rewrite the reconstruction formulas (5.7) and (5.6) in the explicit form

\[
e^{ic_{+}(x)}\partial_x \left( \bar{u}_x(x)e^{ic_{+}(x)} \right) = -\frac{1}{\pi} \int_{\mathbb{R}} r_2(z)e^{2izx} \left[ M_{-,22}(x; z) + \bar{r}_1(z)e^{-2izx} M_{-,21}(x; z) \right] dz \\
= -\frac{1}{\pi} \int_{\mathbb{R}} r_2(z)e^{2izx} M_{+,22}(x; z) dz,
\]

and

\[
e^{2i(c_{-}(x)+c)}u(x) = \frac{1}{\pi} \int_{\mathbb{R}} z^{-1}\bar{r}_1(z)e^{-2izx} M_{+,11}(x; z) dz.
\]

(5.8) \hfill (5.9)

**Proposition 25.** Suppose \( r_{1,2} \in \mathcal{H} \) such that the inequality (2.44) is satisfied, \( u(x) \in H^3(\mathbb{R}^+) \cap H^{2,1}(\mathbb{R}^+) \) admits the estimate

\[
\|u\|_{H^3(\mathbb{R}^+) \cap H^{2,1}(\mathbb{R}^+)} \leq c(\|r_1\|_{W(\mathbb{R})} + \|r_2\|_{W(\mathbb{R})}),
\]

(5.10)

where \( c \) is a positive constant which depends on \( \|r_{1,2}\|_{H^1 \cap L^{2,1}} \).
Proof. We use reconstruction formulas (5.8) and (5.9) to obtain the estimate (5.10). The reconstruction formula (5.9) is rewritten as:

\[
e^{2i(c_-(x)+c)}u(x) = \frac{1}{\pi} \int_{\mathbb{R}} z^{-1} \tilde{r}_1(z)e^{-2i\pi x} \, dz
\]

\[
+ \frac{1}{\pi} \int_{\mathbb{R}} z^{-1} \tilde{r}_1(z)e^{-2i\pi x} (M_{-11}(x; z) - 1) \, dz := I_1(x) + I_2(x). \tag{5.11}
\]

First, \(I_1(x)\) in \(L^{2,1}(\mathbb{R})\) can be controlled by \(z^{-1} \tilde{r}_1(z)\) in \(H^1(\mathbb{R})\) since

\[
\|I_1(x)\|_{L^{2,1}} = \frac{1}{\pi} \|(z^{-1} \tilde{r}_1)(2x)\|_{L^{2,1}} = \frac{1}{\pi} \|z^{-1} \tilde{r}_1(z)\|_{H^1}. \tag{5.12}
\]

To estimate \(I_2(x)\) in \(L^{2,1}(\mathbb{R}^+)\), by using the inhomogeneous equation (4.25) and integrating by parts, we obtain

\[
I_2(x) = -\int_{\mathbb{R}} r_2(z)M_{+12}(x; z)e^{2i\pi x} \mathcal{P}^+ (z^{-1} \tilde{r}_1(z)e^{-2i\pi x}) \, dz. \tag{5.13}
\]

With the estimates (4.40) and (4.54), by Cauchy-Schwartz inequality, we have for every \(x_0 \in \mathbb{R}^+\),

\[
\sup_{x \in (x_0, \infty)} \|x^2 I_2(x)\| \leq \|r_2\|_{L^\infty} \sup_{x \in (x_0, \infty)} \|x\|_{L^2} \sup_{x \in (x_0, \infty)} \|M_{+12}(x; z)\|_{L^1_x} \times \sup_{x \in (x_0, \infty)} \|x\|_{L^2(\mathbb{R}^+)} \leq c \|r_1\|_{H^1(\mathbb{R})}^2. \tag{5.14}
\]

By combining the estimates (5.12) and (5.14) with the triangle inequality, we obtain

\[
\|u\|_{L^{2,1}(\mathbb{R}^+)} \leq c \left( \|z^{-1} r_1(z)\|_{H^1(\mathbb{R})} + \|r_1(z)\|_{H^1(\mathbb{R})}^2 \right). \tag{5.15}
\]

By using the reconstruction formula (5.8), we have

\[
e^{ic_+(x)} \partial_x \left( \bar{u}_x(x)e^{ic_+(x)} \right) = -\frac{1}{\pi} \int_{\mathbb{R}} r_2(z)e^{2i\pi x} \, dz - \frac{1}{\pi} \int_{\mathbb{R}} r_2(z)e^{2i\pi x} [M_{+22} - 1] \, dz
\]

\[
:= I_3(x) + I_4(x). \tag{5.16}
\]

Using similar procedures as above, we also derive

\[
\left\| \partial_x \left( \bar{u}_x e^{ic_+(x)} \right) \right\|_{L^{2,1}(\mathbb{R}^+)} \leq c \left( \|r_2\|_{H^1(\mathbb{R})} + \|r_2\|_{H^1(\mathbb{R})}^2 \right). \tag{5.17}
\]
Next we derivative the equation (5.16) and obtain
\[ \partial_x \left( e^{2i(e+(x)+c)}u(x) \right) = I_1'(x) + I_2'(x). \tag{5.18} \]

Direct calculation gives
\[ I_1'(x) = -\frac{2}{\pi} \int_{\mathbb{R}} \tilde{r}_1(z)e^{-2izx}dz = \tilde{r}_1(2x), \]
which implies that
\[ \|\langle x \rangle I_1'(x)\|_{L^2(\mathbb{R}^+)} = 2\pi^{-1}\|\langle x \rangle \tilde{r}_1(z)(2x)\|_{L^2(\mathbb{R})} \leq c\|r_1(z)\|_{H^1(\mathbb{R})}, \tag{5.19} \]

Differentiating \(I_2(x)\) and by using (4.25), we obtain
\[ I_2'(x) = -2i \int_{-\infty}^{\infty} \tilde{r}_1(s)e^{-2isx} \left( M_{-11}(x; z) - 1 \right) ds + \int_{-\infty}^{\infty} z^{-1}\tilde{r}_1(z)e^{-2izx} \partial_z M_{-11}(x; z) dz \]
\[ = 2i \int_{-\infty}^{\infty} \tilde{r}_1(z)M_{+12}(x; z)e^{2izx}P^+ \left( z^{-1}\tilde{r}_1(z)e^{-2izx} \right) (z) dz \]
\[ - 2i \int_{-\infty}^{\infty} z\tilde{r}_2(z)M_{+12}(x; z)e^{2izx}P^+ \left( z^{-1}\tilde{r}_1(s)e^{-2isx} \right) (z) dz \]
\[ - \int_{-\infty}^{\infty} \tilde{r}_2(z)\partial_z M_{+12}(x; z)e^{2izx}P^+ \left( z^{-1}\tilde{r}_1(z)e^{-2izx} \right) (z) dz, \]

Further with the estimation (4.55) and (4.56), it follows that
\[ \sup_{x \in (x_0, \infty)} |\langle x \rangle I_2'(x)| \]
\[ \leq 2\|r_2\|_{L^\infty} \sup_{x \in (x_0, \infty)} \|M_{+12}(x; z)\|_{L^2_x} \sup_{x \in (x_0, \infty)} \|\langle x \rangle P^+ \left( z^{-1}\tilde{r}_1(z)e^{-2izx} \right) \|_{L^2_x} \]
\[ + 2\|r_2\|_{L^{2,1}} \sup_{x \in (x_0, \infty)} \|M_{+12}(x; z)\|_{L^2_x} \sup_{x \in (x_0, \infty)} \|\langle x \rangle P^+ \left( z^{-1}\tilde{r}_1(z)e^{-2izx} \right) \|_{L^2_x} \]
\[ + \|r_2\|_{L^\infty} \sup_{x \in (x_0, \infty)} \|\langle x \rangle \partial_z M_{+12}(x; z)\|_{L^2_x} \sup_{x \in (x_0, \infty)} \|P^+ \left( z^{-1}\tilde{r}_1(z)e^{-2izx} \right) \|_{L^2_x} \]
\[ \leq c \|r_2\|_{H^1 \cap L^{2,1}} \||r_1\|_{H^1 \cap L^{2,1}} \left( \|z^{-1}r_1\|_{H^1 \cap L^{2,1}} + \|r_2\|_{H^1 \cap L^{2,1}} \right). \tag{5.20} \]

Substituting (5.15) and (5.20) into (5.18), we then obtain
\[ \|u_x\|_{L^{2,1}(\mathbb{R}^+)} \leq c \left( \|z^{-1}r_1\|_{H^1 \cap L^{2,1}} + \|r_1\|_{H^1 \cap L^{2,1}} + \|r_2\|_{H^1 \cap L^{2,1}} \right). \tag{5.21} \]

Finally, combining (5.15), (5.17) and (5.21) yields
\[ \|u\|_{H^{2,1}(\mathbb{R}^+)} \leq c \left( \|z^{-1}r_1\|_{H^1} + \|r_1\|_{H^1} + \|r_2\|_{H^1} \right), \tag{5.22} \]
where \( c \) is a positive constant that depends on \( \|r_{1,2}\|_{H^{1} \cap L^{2,1}} \).

In order to show \( u \in H^{3}(\mathbb{R}^{+}) \), we derive the equation (5.16) and obtain
\[
\partial_{x}^{2} (e^{ic_{x}(x)}u_{x}(x)) = I_{3}'(x) + I_{4}'(x),
\]
in which
\[
I_{3}'(x) = 4\pi^{2} \int_{\mathbb{R}} zr_{2}(z)e^{2izx}dz = 4\pi^{2} zr_{2}(z)(-2x),
\]
which leads to
\[
\|I_{3}'(x)\|_{L^{2}_{x}} = 4\pi^{2} \|zr_{2}(z)(-2x)\|_{L^{2}_{x}} \leq c \|r_{2}\|_{H^{1} \cap L^{2,1}}.
\]

We take the derivative of \( I_{4} \)
\[
I_{4}(x) = -4\pi^{2} \int_{-\infty}^{\infty} r_{2}(z)e^{-2izx}(M_{+,22}(x;z) - 1)dz
- 2i\pi^{2} \int_{-\infty}^{\infty} s^{-1}r_{2}(z)e^{-2izx}\partial_{x}M_{+,22}(x;z)dz.
\]
The estimates for \( I_{4}'(x) \) can also be obtained accordingly
\[
\sup_{x \in (x_{0}, \infty)} \|\langle x \rangle I_{4}'(x)\| \leq c \|r_{2}\|_{H^{1} \cap L^{2,1}} \|r_{1}\|_{H^{1} \cap L^{2,1}} \left( \|z^{-1}r_{1}\|_{H^{1} \cap L^{2,1}} + \|r_{2}\|_{H^{1} \cap L^{2,1}} \right),
\]
by which we can further show that
\[
\|I_{4}'(x)\|_{L^{2}} \leq c \left( \|z^{-1}r_{1}\|_{H^{1} \cap L^{2,1}} + \|r_{2}\|_{H^{1} \cap L^{2,1}} \right).
\]
From (5.25) and (5.26), we find that
\[
\|u_{xxx}(x)\|_{L^{2}} \leq c \left( \|z^{-1}r_{1}\|_{H^{1} \cap L^{2,1}} + \|r_{2}\|_{H^{1} \cap L^{2,1}} \right),
\]
which together with (5.22) yields the estimate
\[
\|u\|_{H^{3}(\mathbb{R}^{+}) \cap H^{2,1}(\mathbb{R}^{+})} \leq c \left( \|r_{1}\|_{W(\mathbb{R})} + \|r_{2}\|_{W(\mathbb{R})} \right).
\]

By Proposition 25, we obtain the following Proposition:
Proposition 26. Suppose $r_{1,2} \in \mathcal{W}(\mathbb{R})$ such that the inequality \((2.44)\) is satisfied, then mapping

$$
\mathcal{W}(\mathbb{R}) \ni (r_1, r_2) \mapsto u \in H^3(\mathbb{R}^+) \cap H^{2,1}(\mathbb{R}^+),
$$

\((5.28)\)
is Lipschitz continuous.

Proof. Suppose $r_{1,2}, \tilde{r}_{1,2} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ satisfying $\|r_{1,2}\|_{H^1 \cap L^{2,1}} \leq \|\tilde{r}_{1,2}\|_{H^1 \cap L^{2,1}} \leq \rho$ for some $\rho > 0$. Denote the corresponding potentials by $u$ and $\tilde{u}$ respectively. Then, there is a positive constant $c$ such that

$$
\|u - \tilde{u}\|_{H^3(\mathbb{R}^+) \cap H^{2,1}(\mathbb{R}^+)} \leq c \left( \|z^{-1}(r_1 - \tilde{r}_1)\|_{H^1 \cap L^{2,1}} + \|r_1 - \tilde{r}_1\|_{H^1 \cap L^{2,1}} + \|r_2 - \tilde{r}_2\|_{H^1 \cap L^{2,1}} \right).
$$

\((5.29)\)
The Lipschitz continuity here follows from the reconstruction formula \((5.11)\) after repeating almost the same estimates as in Proposition 25. Using similar representation for $u$ and $\tilde{u}$, the Lipschitz continuity of \((5.28)\) is ensured with the bound \((5.29)\).

5.2 Estimates on the negative half-line

In order to obtain the estimate of potential $u(x)$ on negative real half-line, we need to rewrite the RH problem 3.1 in an equivalent form. For this purpose, we introduce a scalar RH problem

$$
\begin{cases}
\delta_{+}(z) = (1 + \tilde{r}_1(z)r_2(z))\delta_{-}(z), & z \in \mathbb{R} \\
\delta_{\pm}(z) \to 1 & \text{as} \quad |z| \to \infty.
\end{cases}
$$

\((5.30)\)

Recall from \((3.96)\) and \((3.97)\) that

$$
\begin{cases}
1 + \tilde{r}_1(z)r_2(z) = 1 + |r(k)|^2 \geq 1, & k \in \mathbb{R}^+ \\
1 + \tilde{r}_1(z)r_2(z) = 1 - |r(k)|^2 \geq c_0^2 > 0, & k \in \mathbb{R}^- 
\end{cases}
$$

\((5.31)\)

where the latter inequality is due to \((2.44)\).

Proposition 27. Suppose $r_{1,2} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ such that the inequality \((2.44)\) is satisfied. The RH problem \((5.30)\) exists unique solution $\delta_{\pm}(z)$ of the form

$$
\delta(z) = \exp[C \log (1 + \tilde{r}_1r_2)], \quad z \in \mathbb{C}^\pm,
$$

\((5.32)\)
which has the limits

\[ \delta_{\pm}(z) = \exp[\mathcal{P}^\pm \log(1 + \bar{r}_1 r_2)], \quad z \in \mathbb{R}. \quad (5.33) \]

**Proof.** As \( r_{1,2} \in L^2_{\mathbb{R}}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), we obtain \( \bar{r}_1 r_2 \in L^1(\mathbb{R}) \). It follows from the representation \((3.94)\) as well as from Propositions 17 and 18 that

\[ \langle z \rangle |r(k)| \leq |r(k)| + \frac{1}{2} |k| |r_2(z)| \leq c, \quad z \in \mathbb{R}, \quad (5.34) \]

where \( c \) is a positive constant. Therefore,

\[ \log (1 + |r(z)|^2) \leq \log (1 + c^2(z)^{-2}), \quad z \in \mathbb{R}^+, \quad z \in \mathbb{R}, \quad (5.35) \]

so that \( \log (1 + \bar{r}_1 r_2) \in L^1(\mathbb{R}^+) \). In addition, with the inequality \((2.44)\), we immediately obtain

\[ |\log (1 - |r(k)|^2)| \leq -\log (1 - c^2(z)^{-2}), \quad z \in \mathbb{R}^-, \quad z \in \mathbb{R}. \quad (5.36) \]

It follows that \( \log (1 + \bar{r}_1 r_2) \in L^1(\mathbb{R}^-) \). Consequently, we have \( \log (1 + \bar{r}_1 r_2) \in L^1(\mathbb{R}) \). Also from \((5.36)\), another conclusion is \( \log (1 + \bar{r}_1 r_2) \in L^\infty(\mathbb{R}) \).

With the help of the Hölder inequality, we derive \( \log (1 + \bar{r}_1 r_2) \in L^2(\mathbb{R}) \). By Proposition 3 with \( p = 2 \), the expression \((5.32)\) defines unique analytic functions in \( \mathbb{C}^\pm \), which recover the limits \((5.33)\) and the limits when \( |z| \to \infty \): \( \lim_{|z| \to \infty} \delta_{\pm}(z) = 1 \).

In the end, with \( \mathcal{P}^+ - \mathcal{P}^- = I \), we know

\[ \delta_{\pm}(z)\delta_{-1}(z) = e^{\log(1 + \rho_1(z)r_2(z))} = 1 + \bar{r}_1(z)r_2(z), \quad z \in \mathbb{R}. \quad (5.37) \]

In a word, \( \delta_{\pm} \) given by \((5.32)\) satisfy the scalar RH problem \((5.30)\). \( \square \)

**Proposition 28.** Suppose \( r_{1,2} \in \mathcal{W}(\mathbb{R}) \) such that the inequality \((2.44)\) is satisfied, then \( \delta_{\pm}r_{1,2} \in \mathcal{W}(\mathbb{R}) \).

**Proof.** By Sokhotski-Plemelj theorem, we have the relations

\[ \mathcal{P}^\pm (h)(z) = \pm \frac{1}{2} h(z) - \frac{i}{2} \mathcal{H}(h)(z) \quad z \in \mathbb{R}. \quad (5.38) \]
We note $\mathcal{P}^+ + \mathcal{P}^- = -i\mathcal{H}$, where $\mathcal{H}$ is the Hilbert transform and have

$$\delta_+\delta_- = e^{-i\mathcal{H}\log(1+r_1r_2)}. \quad (5.39)$$

As $\log (1 + r_1r_2) \in L^2(\mathbb{R})$, we obtain $\mathcal{H}\log (1 + r_1r_2) \in L^2(\mathbb{R})$ being a real-valued function. Thus, as $|\delta_+(z)\delta_-(z)| = 1$ for almost every $z \in \mathbb{R}$, $\delta_+\delta_-r_{1,2} \in L^{2,1}(\mathbb{R})$ follows from $r_{1,2} \in L^{2,1}(\mathbb{R})$.

A remaining task is to show $\partial_z \delta_+\delta_-r_{1,2} \in L^2(\mathbb{R})$, i.e., $\partial_z \mathcal{H}\log(1 + r_1r_2) \in L^2(\mathbb{R})$. Thanks for the Parseval’s identity and the proved fact $\|\mathcal{H}f\|_{L^2} = \|f\|_{L^2}$ for every $f \in L^2(\mathbb{R})$, it is obvious that

$$\|\partial_z \mathcal{H}\log(1 + r_1r_2)\|_{L^2} = \|\partial_z \log(1 + r_1r_2)\|_{L^2}. \quad (5.40)$$

The right-hand side is bounded as $\partial_z \log(1 + r_1r_2) = \frac{\partial_z(r_1 + r_2)}{1 + r_1r_2} \in L^2(\mathbb{R})$ under the premise of this proposition. $\partial_z \delta_+\delta_-r_{1,2} \in L^2(\mathbb{R})$ is proved to be sufficient.

In the second step, we decompose the jump matrix $R(x; z)$ in an equivalent form:

$$\begin{pmatrix} \delta_-(z) & 0 \\ 0 & \delta_-^{-1}(z) \end{pmatrix} (I + R(x; z)) \begin{pmatrix} \delta_+^{-1}(z) & 0 \\ 0 & \delta_+(z) \end{pmatrix} = \begin{pmatrix} 1 & \delta_-(z)\delta_+(z)\tilde{r}_1(z)e^{-2izx} \\ \overline{\delta_+(z)}\overline{\delta_-(z)}r_2(z)e^{2izx} & 1 + r_1(z)r_2(z) \end{pmatrix}, \quad (5.41)$$

where $\delta_+^{-1}\delta_-^{-1} = \delta_-\overline{\delta_+}$ is used.

We define a new matrix

$$\hat{R}_\delta(x; z) := \begin{pmatrix} 0 & \tilde{r}_{\delta,1}(z)e^{-2izx} \\ r_{\delta,2}(z)e^{2izx} & \tilde{r}_{\delta,1}(z)r_{\delta,2}(z) \end{pmatrix}, \quad (5.42)$$

where

$$r_{\delta,j}(z) := \overline{\delta_+(z)}\overline{\delta_-(z)}r_j(z), \quad j = 1, 2. \quad (5.43)$$

By Proposition 28, we have $r_{\delta,j} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ similarly to the scattering data $r_{1,2}$.

With the functions $M_{\pm}(x; z)$ and $\delta_{\pm}(z)$, we define the functions

$$M_{\delta,\pm}(x; z) := M_{\pm}(x; z)\delta_{\pm}(z)^{-\delta_3}, \quad (5.44)$$

which satisfies the RH problem.
RH Problem 5.1. Find a matrix function $M_\delta(x; z)$ with the following properties:

- **Analyticity:** $M_{\delta, \pm}(x; z)$ are analytic functions in $\mathbb{C}^\pm$;

- **Asymptotic conditions:**
  $$M_{\delta, \pm}(x; z) \rightarrow I, \quad \text{as} \quad |z| \rightarrow 0; \quad (5.45)$$

- **Jump condition:** $M_{\delta, \pm}(x; z)$ satisfies the jump condition
  $$M_{\delta, +}(x; z) = M_{\delta, -}(x; z)(I + \hat{R}_\delta(x; z)), \quad z \in \mathbb{R}. \quad (5.46)$$

The above RH problem is transformed from the previous RH problem 3.1. As the analysis of Proposition 5 and 21, the RH problem 5.1 admits a unique solution given by

$$M_\delta(x; z) = I + C\left(M_{\delta, -}(x; \cdot)\hat{R}_\delta(x; \cdot)\right)(z), \quad z \in \mathbb{C}^\pm. \quad (5.47)$$

We denote the column vectors of $M_{\delta, \pm}$ by $M_{\delta, \pm} = [M_{\delta, \pm, 1}, M_{\delta, \pm, 2}]$. Since $r_{\delta,j} \in H^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have $\hat{R}_\delta(x; \cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $x \in \mathbb{R}$. In consequence, the reconstruction formulas (5.7) and (5.5) become

$$u(x)e^{-i(2c_-(x)+c)} = \frac{1}{\pi} \int_\mathbb{R} z^{-1}r_{\delta, 1}(z)e^{-2izx}M_{\delta, +, 11}(x; z)dz, \quad (5.48)$$

and

$$e^{ic_+(x)}\partial_x\left(\bar{u}_x(x)e^{-ic_-(x)}\right) = -\frac{1}{\pi} \int_\mathbb{R} r_{\delta, 2}(z)e^{2izx}M_{\delta, -, 22}(x; z)dz. \quad (5.49)$$

From (5.47), we can obtain the system of integral equations for vectors $M_{\delta, +, 1}$ and $M_{\delta, -, 2}$ by projecting representation

$$M_{\delta, +, 1}(x; z) = e_1 + \mathcal{P}^+\left(r_{\delta, -, 2}e^{2izx}M_{\delta, -, 2}(x; \cdot)\right), \quad (5.50)$$

$$M_{\delta, -, 2}(x; z) = e_2 + \mathcal{P}^-\left(\bar{r}_{\delta, -1}e^{-2izx}M_{\delta, +, 1}(x; \cdot)\right). \quad (5.50)$$

We formulate the above integral equation in a unified form as

$$T_\delta - \mathcal{P}^- (T_\delta R_\delta) = F_\delta, \quad (5.51)$$
where
\[
T_\delta(x; z) := (M_{\delta,+1}(x; z) - e_1, M_{\delta,-2}(x; z) - e_2) \begin{pmatrix} 1 & 0 \\ -r_{\delta,2}(z)e^{2ix} & 1 \end{pmatrix} \quad (5.52)
\]
and
\[
F_\delta(x; z) := (P^+(r_{\delta,2}(z)e^{2ix}) e_2, P^-(\bar{r}_{\delta,1}(z)e^{-2ix}) e_1). \quad (5.53)
\]

In order to estimate on the potential \(u(x)\) on negative real axis by using reconstruction formulas (5.48) and (5.49), we give similar results to Proposition 22 and Proposition 23.

**Proposition 29.** If \(f(z) \in L^2(\mathbb{R})\), then we have
\[
P^+ (f(z)e^{2ix}) = -\int_{-\infty}^{2x} \hat{f}(\xi)e^{-iz(\xi-2x)}d\xi, \quad (5.54)
\]
\[
P^- (f(z)e^{-2ix}) = \int_{-\infty}^{2x} \hat{f}(\xi)e^{iz(\xi-2x)}d\xi. \quad (5.55)
\]

**Proof.** The proof is similar with that of Proposition 22. \(\Box\)

By using above Proposition 29, we can further show the following estimates on negative half-line.

**Proposition 30.** For every \(x_0 \in \mathbb{R}^-\) and every \(r_{1,2}(z) \in H^1(\mathbb{R})\), we have
\[
\sup_{x \in (-\infty, x_0]} \| (x)P^+ (z^{-i\bar{r}_1}(z)e^{2ix}) \|_{L^2_z} \leq \| z^{-ir_1}(z) \|_{H^1}, \quad i = 0, 1, \quad (5.56)
\]
\[
\sup_{x \in (-\infty, x_0]} \| (x)P^- (r_2(z)e^{-2ix}) \|_{L^2_z} \leq \| r_2(z) \|_{H^1}. \quad (5.57)
\]

In addition, we have
\[
\sup_{x \in \mathbb{R}} \| P^+ (z^{-i\bar{r}_1}(z)e^{2ix}) \|_{L^\infty_z} \leq \frac{1}{\sqrt{2}} \| z^{-ir_1}(z) \|_{H^1}, \quad i = 0, 1, \quad (5.58)
\]
\[
\sup_{x \in \mathbb{R}} \| P^- (r_2(z)e^{-2ix}) \|_{L^\infty_z} \leq \frac{1}{\sqrt{2}} \| r_2(z) \|_{H^1}. \quad (5.59)
\]

Furthermore, if \(r_{1,2} \in L^{2,1}(\mathbb{R})\), then
\[
\sup_{x \in \mathbb{R}} \| P^+ (z\bar{r}_1(z)e^{2ix}) \|_{L^2_z} \leq \| r_1(z) \|_{L^{2,1}_z}, \quad (5.60)
\]
\[
\sup_{x \in \mathbb{R}} \| P^- (zr_2(z)e^{-2ix}) \|_{L^2_z} \leq \| r_2(z) \|_{L^{2,2}_z}. \quad (5.61)
\]
Proof. The proof is similar with that of Proposition 23.

By using Proposition 29 and Proposition 30, in same way to the Section 4, we obtain estimates of the potential $u(x)$ on the negative half-line and Lipschitz continuity.

**Proposition 31.** Let $r_{1,2} \in \mathcal{W}(\mathbb{R})$ such that the inequality (2.44) is satisfied. Then, $u(x) \in H^3(\mathbb{R}^-) \cap H^{2,1}(\mathbb{R}^-)$ satisfies the bound

$$\|u\|_{H^3(\mathbb{R}^-) \cap H^{2,1}(\mathbb{R}^-)} \leq c \left( \|z^{-1}r_1\|_{H^1 \cap L^{2,1}} + \|r_1\|_{H^1 \cap L^{2,1}} + \|r_2\|_{H^1 \cap L^{2,1}} \right),$$

where $c$ is a positive constant that depends on $\|r_{1,2}\|_{H^1 \cap L^{2,1}}$. Moreover the mapping

$$\mathcal{W}(\mathbb{R}) \ni (r_1, r_2) \mapsto u \in H^3(\mathbb{R}^-) \cap H^{2,1}(\mathbb{R}^-)$$

is Lipschitz continuous.

**Proof.** Let $r_{1,2}, \tilde{r}_{1,2} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ satisfy $\|r_{1,2}\|_{H^1 \cap L^{2,1}}, \|\tilde{r}_{1,2}\|_{H^1 \cap L^{2,1}} \leq \rho$ for some $\rho > 0$. Denote the corresponding potentials by $u$ and $\tilde{u}$ respectively. Then, there is a positive $\rho$-dependent constant $c$ such that

$$\|u - \tilde{u}\|_{H^3(\mathbb{R}^-) \cap H^{2,1}(\mathbb{R}^-)} \leq c \left( \|r_1 - \tilde{r}_1\|_{\mathcal{W}(\mathbb{R})} + \|r_2 - \tilde{r}_2\|_{\mathcal{W}(\mathbb{R})} \right).$$

6 Existence of global solutions to the FL equation

By transforming the space part of the original Lax pair, we establish a series of RH problems and the reconstruction formulas for the potential $u(x)$ where time $t$ is just considered as a parameter. We need to incorporate the time evolution to get the solution $u(x,t)$ to the FL equation.
6.1 Time evolution from reflection coefficients to RH problem

For the partial spectral problem (2.1) and the time spectral problem (2.2), we define the fundamental solutions

\[ \psi_1(x, t; k) = \psi_1(x; k)e^{-i\eta^2 t}, \]  
\[ \psi_2(x, t; k) = \psi_2(x; k)e^{i\eta^2 t}, \]  

where \( \eta = \sqrt{\alpha(k - \frac{\beta}{2k})} \). From the Volterra integral equation (2.7), the bounded Jost functions \( \psi_1(x, t; k) \) and \( \psi_2(x, t; k) \) have the same analytical property in the \( k \) plane and satisfy the same boundary conditions

\[ \psi_1(x, t; k) \to e_1, \quad \text{as} \quad x \to \pm\infty, \]  
\[ \psi_2(x, t; k) \to e_2, \quad \text{as} \quad x \to \pm\infty, \]

for every \( t \in [0, T] \). From the linear independence of two solutions to Lax pair (2.1), the columns of the Jost functions \( \psi_{\pm}(x, t; k) \) satisfy the scattering relation

\[ \psi_1(x, t; k) = a(k)\psi_1^+(x, t; k) + b(k)e^{2i\eta^2 x + 2i\eta^2 t} \psi_2^+(x, t; k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \]  

where the scattering coefficients \( a(t; k) \) and \( b(t; k) \) are dependent of \( t \). By using the time spectral problem (2.2), we can find the time evolution relations of \( a(t; k) \) and \( b(t; k) \)

\[ a(t; k) = a(k), \quad b(t; k) = b(k)e^{-2i\eta^2 t}, \]

which together with the definition (3.94), we obtain time evolution on the scattering data \( r_{1,2}(z) \) as follows

\[ r_1(t; z) = -\frac{b(k)}{2ika(k)}e^{-2i\eta^2 t} = r_1(z)e^{-2i\eta^2 t}, \]
\[ r_2(t; z) = \frac{2ikb(k)}{a(k)}e^{-2i\eta^2 t} = r_2(z)e^{-2i\eta^2 t}. \]

**Proposition 32.** If \( r_{1,2}(z) \in \mathcal{W}(\mathbb{R}) \), then for an arbitrary fixed \( T > 0 \) and every \( t \in [0, T] \),

\[ r_{1,2}(t; z) \in \mathcal{W}(\mathbb{R}). \]
Proof. By using (6.6)-(6.7), direction calculation shows that
\[ \|r_{1,2}(t; z)\|_{L^2} = \|r_{1,2}(z)\|_{L^2}, \]  
(6.9)
\[ \|z^{-2}r_{1,2}(t; z)\|_{L^2} = \|z^{-2}r_{1,2}(z)\|_{L^2}. \]  
(6.10)

Further by differentiation to (6.6)-(6.7) and taking $L^2$-norm, for $t \in [0, T]$, then by Proposition 15, we have
\[ \|\partial_z r_{1,2}(t; z)\|_{L^2} = \|\partial_z r_{1,2}(z)e^{-2i\eta t} - 2i\alpha(1 - \frac{\beta^2}{4}z^{-2})r_{1,2}(t; z)e^{-2i\eta t}\|_{L^2} \]
\[ \leq 2\alpha T \|r_{1,2}(z)\|_{L^2} + \|\partial_z r_{1,2}(z)\|_{L^2} + \frac{1}{2}\alpha\beta^2 T\|z^{-2}r_{1,2}(z)\|_{L^2}. \]  
(6.11)

In a similar way, by using Proposition 14, we can show that
\[ \|z^{-2}r_2(z)\|_{L^2(\mathbb{R})} \leq 2\|r_2(z)\|_{L^\infty}\|z^{-2}\|_{L^2(\delta, +\infty)} + a_0^{-1}\|z^{-1}k^{-1}b(k)\|_{L^2(0, \delta))}. \]  
(6.12)

Substituting (3.88) and (6.12) into (3.88) yields
\[ \|\partial_z r_{1,2}(t; z)\|_{L^2} \leq (2\alpha T + 1)\|r_{1,2}(z)\|_{H^1} + 2\alpha\beta^2 T\|r_2(z)\|_{L^\infty}\|z^{-2}\|_{L^2(\delta, +\infty)} \]
\[ + \frac{1}{2}T\alpha\beta^2 a_0^{-1}(\|z^{-2}k^{-1}b(k)\|_{L^2((0, \delta))} + \|z^{-1}k^{-1}b(k)\|_{L^2(0, \delta)}). \]  
(6.13)

Finally (6.9), (6.10) and (6.13) leads to the result (6.8).

\[ \square \]

6.2 The local solution and global solution

For $r_{1,2}(t; z) \in W(\mathbb{R})$, $t \in [0, T]$, the constraint (2.24) and the relation (3.95) is still true for every $t \in [0, T]$.

Theorem 1. For every $u_0(x) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ such that the linear equation (2.1) admits no eigenvalues or resonances, there exists a unique local solution to the Cauchy problem
\[ u(x, t) \in C([0, T], H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})), \quad t \in [0, T]. \]

Furthermore, the map
\[ H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \ni u_0(x) \mapsto u(t, x) \in C\left([0, T], H^{3}(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\right) \]  
(6.14)
is Lipschitz continuous.
Proof. The potential $u(t,x)$ is recovered from the scattering data $r_{1,2}(t;z)$ with the inverse scattering transform shown in Section 3. Combining the whole analysis in the previous section, for every $t \in [0,T)$ we finally prove that

$$
\|u(x,t)\|_{H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})} \leq c \left( \|r_1(t;z)\|_{H^1(\mathbb{R}) \cap L^{2.1}(\mathbb{R})} + \|r_2(t;z)\|_{H^1(\mathbb{R}) \cap L^{2.1}(\mathbb{R})} \right)
\leq c \left( \|r_1(z)\|_{H^1(\mathbb{R}) \cap L^{2.1}(\mathbb{R})} + \|r_2(z)\|_{H^1(\mathbb{R}) \cap L^{2.1}(\mathbb{R})} \right)
\leq c(T) \|u_0(x)\|_{H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})}.
$$

(6.15)

It is a strong indication that there exists a unique local solution

$$
u(x,t) \in C([0,T],H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})) \ \ t \in [0,T]
$$
to the Cauchy problem (1.1)-(1.2). Combining with the Lipschitz continuity from $u_0(x)$ to $r_{1,2}(t;z)$ in Proposition 16 and

$$
\|u(x,t_2) - u(x,t_1)\|_{H^3 \cap H^{2.1}} \leq c \|r_{1,2}(z)(e^{2i\eta^2 t_2} - e^{2i\eta^2 t_1})\|_{H^1 \cap H^{2.1}}
\leq c|t_2 - t_1|\|r_{1,2}(z)\|_{H^1 \cap L^{2.1}},
$$

we prove the mapping

$$H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R}) \ni u_0 \mapsto u \in C([0,T],H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R}))
$$
is Lipschitz continuous.

The following theorem show that there is a global solution in $H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})$:

**Theorem 2.** For every $u_0(x) \in H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})$ such that the linear equation (2.1) admits no eigenvalues or resonances, there exists a unique global solution to the Cauchy problem (1.1)-(1.2)

$$u(x,t) \in C([0,\infty);H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})).
$$

Furthermore, the map

$$H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R}) \ni u_0 \mapsto u \in C([0,\infty);H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R}))
$$

(6.16)
is Lipschitz continuous.
Proof. Based on Theorem 1, as \( c(T) \) depends on \( T \) and \( \| r_{1,2}(0,z) \|_{H^1 \cap L^{2,1}} \) grows at most in a polynomial order with respect to \( \| u(x,0) \|_{H^3 \cap H^{2,1}} \), we hence reach the conclusion that the local solution exists in \( C \left([0,T], H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\right) \) for an arbitrary fixed \( T \). Then the global existence of the solution can be asserted if \( T = \infty \).

Suppose the maximal time in which the local solution exists is \( T_{\text{max}} \).

If \( T_{\text{max}} = \infty \), then the local solution is global. Otherwise, the local solution exists in the finite interval \([0,T_{\text{max}}]\) or \([0,T_{\text{max}})\).

If the local solution exists in the closed interval \([0,T_{\text{max}}]\), we can use \( u(x,T_{\text{max}}) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \) as the new initial data, according to the results derived in the previous sections, there exists \( T_1 > 0 \) such that the following local solution

\[
  u_1(x,t) \in C([T_{\text{max}}, T_{\text{max}}+T_1], H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}))
\]

exists in the time interval \([T_{\text{max}}, T_{\text{max}}+T_1]\). Therefore if we reconstruct the function

\[
  \tilde{u}(x,t) = \begin{cases} 
  u(x,t), & t \in [0,T_{\text{max}}] \\
  u_1(x,t), & t \in [T_{\text{max}}, T_{\text{max}}+T_1],
  \end{cases}
\]

which then is the solution to the Cauchy problem (1.1)-(1.2) in the time interval \([0,T_{\text{max}}+T_1]\). This contradicts with the definition of \( T_{\text{max}} \).

If the local solution exists in the open interval \([0,T_{\text{max}})\), then by priori estimation in (6.15), we have the following estimation

\[
  \| u(x,t) \|_{H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})} \leq c(T_{\text{max}}) \| u(x,0) \|_{H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})}, \quad t \in [0,T_{\text{max}}),
\]

Due to the continuity of \( u(x,t) \) with respect to the time \( t \), the limit of \( u(x,t) \) when \( t \) converges to \( T_{\text{max}} \) exists. Taking the limit by \( t \to T_{\text{max}} \) in (6.18), we have

\[
  \| u_{T_{\text{max}}} \|_{H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})} \leq c(T_{\text{max}}) \| u(x,0) \|_{H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})}, \quad t \in [0,T_{\text{max}}),
\]

where \( u_{T_{\text{max}}} = \lim_{t \to T_{\text{max}}} u(x,t) \). By defining the new function

\[
  \tilde{u}(x,t) = \begin{cases} 
  u(x,t), & t \in [0,T_{\text{max}}) \\
  u_{T_{\text{max}}}, & t = T_{\text{max}},
  \end{cases}
\]

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which also is the solution to the Cauchy problem in (1.1)-(1.2) in the time interval $[0,T_{\text{max}}]$. This contradicts with the premise that $[0,T_{\text{max}})$ is the maximal open interval.

Based on the discussion above, we conclude that the global solution exists in $u(x,t) \in C ([0, \infty), H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}))$, which finally yields the proof of Theorem 2.

\[ \square \]

**Remark 3.** In our previous work [21], we ever obtained the long-time asymptotic behavior of the FL equation with generic initial data in a Sobolev space $H^{3,3}(\mathbb{R})$. In present work, we obtain the existence of global solutions to the Cauchy problem (1.1)-(1.2) of the FL equation on the line for the initial data $u_0(x) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$. Due to $H^{3,3}(\mathbb{R}) \hookrightarrow H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$, our present work ensures the strictness of our previous work on long-time asymptotic behavior of the FL equation [21]. Based on the results obtained this paper together with Backlund transformation, we will further prove the global existence for the FL equation in the case when the initial datum includes a finite number of solitons in our future work.

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**Data Availability Statements**

The data that supports the findings of this study are available within the article.

**Conflict of Interest**

The authors have no conflicts to disclose.

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