N-STRONGLY QUASI-INVARIANT MEASURE ON DOUBLE COSET SPACES

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Abstract. Let $G$ be a locally compact group, $H$ and $K$ be two closed subgroups of $G$, and $N$ be the normalizer group of $K$ in $G$. In this paper, the existence and properties of a rho-function for the triple $(K,G,H)$ and an $N$-strongly quasi-invariant measure of double coset space $K\backslash G/H$ is investigated. In particular, it is shown that any such measure arises from a rho-function. Furthermore, the conditions under which an $N$-strongly quasi-invariant measure arises from a rho-function are studied.

1. Introduction

Let $G$ be a locally compact group and $H$ and $K$ be closed subgroups of $G$. The double coset space of $G$ by $H$ and $K$ respectively, is

$$K\backslash G/H = \{KxH; x \in G\},$$

which induced by Liu in [8].

When $K$ is trivial, a double coset $K\backslash G/H$ changes to a homogeneous space $G/H$. The existence of quasi-invariant measures on homogeneous spaces $G/H$ (with merely measurable rho-functions) was first proved by Mackey [10] under the assumption that $G$ is second countable. Bruhat [2] and Loomis [9] showed how to obtain strongly quasi-invariant measures with no countability hypotheses. This work is extended in a special case in [5]. Also, the existence of a homomorphism rho-function causes the existence of a relatively invariant measure on $G/H$ is in [11].

One may refer to [11, 6] to find more informations about homogeneous space $G/H$. When $K = H$, a double coset space $K\backslash G/H$ changes to a hypergroup in which the homogeneous space $G/H$ is a semi hypergroup [7]. It is worthwhile to note that the hypergroup plays important rules in physics.

In this paper, we construct an $N$-strongly quasi-invariant measure on $K\backslash G/H$ when $H$ and $K$ are used subgroups, not necessarily compact. Also we investigate when $K$ is a normal closed subgroup of $G$ then $K\backslash G/H$ possesses a $G$-strongly quasi-invariant measure. In addition, when $H$ is trivial we show the existence of an $N$-strongly quasi-invariant measure on the right cosets of $K$ in $G$.

It is worth mentioning that in [8] the conditions for the existence of $N$-relatively invariant measures and $N$-invariant measures are investigated.

Some preliminaries and notations about coset space $K\backslash G/H$ and related measures on it are stated in Section 2.

In Section 3, we construct a rho-function for the triple $(K,G,H)$ and introduce
an $N$-strongly quasi-invariant measure which arises from this rho-function. 
In particular, we obtain in Section 4., conditions under which an $N$-strongly quasi-invariant measure arises from a rho-function.

2. Notations and Preliminary Results

Let $G$ be a locally compact Hausdorff group and let $H$ and $K$ be closed subgroups of $G$. Throughout this paper, we denote the left Haar measures on $G$, $H$ and $K$ respectively, by $dx$, $dh$, $dk$, and their modular functions by $\Delta_G$, $\Delta_H$ and $\Delta_K$, respectively. If $S$ is a locally compact Hausdorff space, a (left) action of $G$ on $S$ is a continuous map $(x, s) \mapsto xs$ from $G \times S$ to $S$ such that (i) $s \rightarrow xs$ is a homeomorphism of $S$ for each $x \in G$, and (ii) $x(ys) = (xy)s$ for all $x, y \in G$ and $s \in S$. A space $S$ equipped with an action of $G$ is called a $G$-space. A $G$-space $S$ is called transitive if for every $s, t \in S$ there exists $x \in G$ such that $xs = t$.

The standard examples of transitive $G$-spaces are the quotient spaces $G/H$ (where $H$ is a closed subgroup of $G$), equipped with the quotient topology on which $G$ acts by left multiplication. We shall use the term homogeneous space to mean a transitive space $S$ that is isomorphic to a quotient space $G/H$. In homogeneous space $G/H$, if $\mu$ is a positive Radon measure on $G/H$, Borel set $E$ is called negligible with respect to $\mu$, if $\mu(E) = 0$. Let $\mu_x$ denote its transfer by $x \in G$, that is $\mu_x(E) = \mu(x \cdot E)$ for any Borel set $E \subseteq G/H$. $\mu$ is called strongly quasi invariant if there is a positive continuous function $\lambda$ on $G \times G/H$ such that $d\mu_x(yH) = \lambda(x, yH)d\mu(yH)$, for all $x, y \in G$. A rho-function for the pair $(G, H)$ is defined to be a positive locally integrable function $\rho$ on $G$ which satisfies

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)} \rho(x), \quad (x \in G, \ h \in H).$$

It is known that for each pair $(G, H)$ there is a strictly positive rho-function which constructs a strongly quasi-invariant measure $\mu$ on $G/H$ such that

$$\int_G f(x) \rho(x) dx = \int_G \int_H f(xh) dh d\mu(xH), \quad (2.1)$$

for all $f \in C_c(G)$, the space of all continuous functions on $G$ with compact supports. And conversely, each strongly quasi-invariant measure on $G/H$ arises from a rho-function which satisfies (2.1) for a rho-function $\rho$, and all such measures are strongly equivalent. That is to say, all strongly quasi-invariant measures on $G/H$ have the same negligible sets (see [6, 11]).

The notion of double coset space is a natural generalization of that of coset space arising by two subgroups, simultaneously. Recall that if $K \backslash G/H$ is a double coset space of $G$ by $H$ and $K$, then elements of $K \backslash G/H$ are given by $\{KxH: x \in G\}$. The canonical mapping of which, is $q : G \rightarrow K \backslash G/H$, defined by $q(x) = KxH$, which is abbreviated by $\bar{x}$, and which is surjective. The double coset space $K \backslash G/H$ equipped with the quotient topology, which is the largest topology, that makes $q$ continuous. In this topology $q$ is also an open mapping and proper—that is for each compact set $F \subseteq K \backslash G/H$ there is a compact set $E \subseteq G$ with $q(E) = F$. Based on the above mentioned case, $K \backslash G/H$ is a locally compact and Hausdorff space.

Let $N$ be the normalizer of $K$ in $G$, i.e.,

$$N = \{g \in G; \ gK = Kg\}.$$
Then, there is a naturally defined mapping
\[ \varphi : N \times K \backslash G / H \to K \backslash G / H \]
given by
\[ \varphi(n, q(x)) := KnxH. \]
It can be verified that \( \varphi \) is a well-defined, continuous, transitive action of \( N \) on \( K \backslash G / H \). Considering \( K \backslash G / H \) with this transitive action, we now denote \( \varphi(n, q(x)) \) by \( n \cdot q(x) \).

We define the mapping \( Q \) from \( C_c(G) \) to \( C_c(K \backslash G / H) \) by
\[
Q(f)(KxH) = \int_K \int_H f(k^{-1}xh)dhdk.
\]
It is evident that \( Q \) is a well-defined continuous linear map, as well as \( \text{supp}(Q(f)) \subseteq q(\text{supp} f) \). In the following, the properties of this mapping is investigated. However, we first recall that the definition of \( \text{IN} \)-group and verification of a property of it is used in the sequel.

A locally compact group \( G \) is called an \( \text{IN} \)-group if there is a compact unit neighbourhood \( U \) in \( G \) which is invariant under inner automorphism, that is, for any \( x \in G \), \( xUx^{-1} = U \). It is known that the \( \text{IN} \)-groups are unimodular.

**Lemma 2.1.** If \( K \) is also an \( \text{IN} \)-group, then \( \int_K f(k)dk = \int_K f(nkn^{-1})dk \), for all \( f \in C_c(K) \) and \( n \in N \).

**Proof.** Let for \( n \in N \), \( \lambda_n : C_c(K) \to \mathbb{C} \) be given by
\[ \lambda_n(f) = \int_K f(nkn^{-1})dk. \]
Then for every \( t \in K \), we have
\[ \lambda_n(L_t^{-1}f) = \int_K L_t^{-1}f(nkn^{-1})dk = \int_K f(tnk^{-1})dk = \int_K f(nkn^{-1})dk. \]
This shows that \( \lambda_n \) is left invariant, so it induced a left Haar measure \( \lambda_n \) on \( K \).

Therefore, there is \( c > 0 \) such that
\[ \int_K f(k)d\lambda_n(k) = c \int_K f(k)dk. \]
Since \( K \) is an \( \text{IN} \)-group, then there is a compact unit neighbourhood \( U \) in \( K \) such that \( xUx^{-1} = U \) for all \( x \in K \). Thus, \( |n^{-1}Un| \leq |U| \) for each \( n \in N \), where \( |U| \) denotes the measure of \( U \). Therefore, we can write
\[
|c - 1||U| = ||c|U| - |U|| = |\lambda_n(U) - |U|| = ||nUn^{-1}| - |U|| \leq ||U| - |U|| = 0.
\]
This implies that \( c = 1 \). \( \square \)

**Lemma 2.2.** For any compact set \( F \subseteq K \backslash G / H \) there exists \( f \in C_c^+(G) \) such that \( Qf = 1 \) on \( F \).

**Proof.** The proof is straightforward. \( \square \)
Note that for $f \in C_c(G)$ and $g \in G$, we consider $L_g f(x) = f(g^{-1}x)$ and $R_g f(x) = f(xg)$, and for each $n \in \mathbb{N}$ and $F \in C_c(K \backslash G/H)$, we define $L_g F(\tilde{x}) = F((g^{-1}x)^\circ)$ and $R_g F(\tilde{x}) = F((xg)^\circ)$. 

**Lemma 2.3.** Given the notation at the beginning of the section, the map $Q : C_c(G) \rightarrow C_c(K \backslash G/H)$ has the following properties.

(i) $Q(C_c(G)) = C_c(K \backslash G/H)$

(ii) If $K$ is also an $IN$-group, then for each $n \in \mathbb{N}$

$$Q(L_n f) = L_n Q(f), \quad f \in C_c(G).$$

**Proof.** For (i) suppose that $F \in C_c(K \backslash G/H)$. Since $q$ is proper, then there is a compact subset $D \subseteq G$ such that $q(D) = \text{supp } F$. Let $f \in C_c(G)$ be such that $f(d) > 0$ for all $d \in D$. Consider the function $f_1$ defined on $G$ by

$$f_1(x) = \begin{cases} f(x) & \text{if } Q(f)(q(x)) \neq 0 \\ 0 & \text{if } Q(f)(q(x)) = 0 \end{cases}.$$ 

Since $Q(f)(q(x)) > 0$ for $x \in q^{-1}(\text{supp } F)$, and $F(q(x)) = 0$, for $x \in G \setminus q^{-1}(\text{supp } F)$, which is an open subset in $G$, $f_1 \in C_c(G)$ and $Q(f_1) = F$.

Finally, for (ii) according to Lemma 2.1, we may state that

$$Q(L_n f)(KxH) = \int_K \int_H f(nk^{-1}xh)dhdk$$

$$= \int_H \int_K f(nk^{-1}nxh)dkdh$$

$$= \int_H \int_K f(k^{-1}nxh)dkdh$$

$$= L_n Q(f)(KxH).$$

Next theorem gives a necessary and sufficient condition for the existence of a positive Radon measure on $K \backslash G/H$.

**Theorem 2.4.** If $\mu$ is a positive Radon measure on $K \backslash G/H$, then positive Radon measure $\tilde{\mu}$ on $G$ is defined by

$$\int_G f(x)d\tilde{\mu}(x) = \int_{K \backslash G/H} Q(f)(\tilde{x})d\mu(\tilde{x}), \quad (2.2)$$

satisfying

$$\int_G f(kxh^{-1})d\tilde{\mu}(x) = \Delta_K(k)\Delta_H(h) \int_G f(x)d\mu(x). \quad (2.3)$$

Conversely, if a positive Radon measure $\tilde{\mu}$ on $G$ has the property $(2.3)$, then the equation $(2.2)$ defines a positive Radon measure $\mu$ on $K \backslash G/H$.

**Proof.** Suppose that $\mu$ is a positive Radon measure on $K \backslash G/H$, then $\tilde{\mu}$ defined by $(2.2)$ is clearly a positive Radon measure on $G$. Also, for each $h_0 \in H$, $k_0 \in K$,
Let $f \in C_c(G)$, we have
\[
\int_G f(k_0 x h_0^{-1}) d\tilde{\mu}(x) = \int_{K \backslash G / H} \int_K \int_H L_{k_0^{-1}} \circ R_{h_0^{-1}} f(k^{-1} x h) dh dk d\tilde{\mu}(x)
\]
\[
= \int_{K \backslash G / H} \int_K \int_H f(k_0 k_0^{-1} x h h_0^{-1}) dh dk d\tilde{\mu}(\tilde{x})
\]
\[
= \Delta_H(h_0) \Delta_K(k_0) \int_{K \backslash G / H} Q(f)(\tilde{x}) d\mu(\tilde{x})
\]
\[
= \Delta_H(h_0) \Delta_K(k_0) \int_{G} f(x) d\tilde{\mu}(x).
\]

Conversely, suppose that the positive Radon measure $\tilde{\mu}$ on $G$ has the property 2.3, then take
\[
\mu : C_c(K \backslash G / H) \to (0, +\infty),
\]
by
\[
\mu(Q(f)) = \int_G f(x) d\tilde{\mu}.
\]

Now we show that $\mu$ is well-defined, let $f \in C_c(G)$ such that $Q(f) = 0$. According to Lemma 2.2, there is $g$ in $C_c(G)$ such that $Q(g) \equiv 1$ on $Q(supp f)$. By using the Fubini’s Theorem, we have
\[
\int_G f(x) d\tilde{\mu} = \int_{K \backslash G / H} \int_K \int_H g(k^{-1} x h) dh dk d\tilde{\mu}(x)
\]
\[
= \int_{K \backslash G / H} \int_K \int_H f(k x h^{-1}) g(x) \Delta_K(k) \Delta_H(h) d\tilde{\mu}(x) dk
\]
\[
= \int_{G} g(x) \int_H \int_K f(k^{-1} x h) dh dk d\tilde{\mu}(x)
\]
\[
= \int_{G} g(x) Q(f)(q(x)) d\tilde{\mu}(x) = 0.
\]

It is easy to check that $\mu$ is a positive linear functional, therefore it induces a positive Radon measure $\mu$ on $K \backslash G / H$ such that,
\[
\int_{G} f(x) d\tilde{\mu}(x) = \int_{K \backslash G / H} Q(f)(\tilde{x}) d\mu(\tilde{x}).
\]

\[\square\]

**Corollary 2.5.** Considering the assumptions of Theorem 2.4, there is a correspondence between the positive Radon measure $\tilde{\mu}$ on $G$ and $\mu$ on the double coset space $G // H$, such that
\[
\int_{G} f(x) d\tilde{\mu}(x) = \int_{G / H} Q(f)(\tilde{x}) d\mu(\tilde{x})
\]
and
\[
\int_{G} f(h x h^{-1}) d\tilde{\mu}(x) = \int_{G} f(x) d\tilde{\mu}(x)
\]
for all $f \in C_c(G)$.
3. The existence of $N$-strongly quasi invariant measure

In this section, we refine and generalize the concept of strongly quasi invariant measure on double coset spaces. Moreover, we investigate the existence of $N$-strongly quasi-invariant measure on these spaces. We start our work with the following definitions.

**Definition 3.1.** Let $G$ be a locally compact group and $H$ and $K$ be closed subgroups of it. For a positive Radon measure $\mu$ on $K\backslash G/H$, assume that $\mu_n$ is its transfer by $n \in N$, that is, $\mu_n(E) = \mu(n \cdot E)$, for any Borel set $E \subseteq K\backslash G/H$. $\mu$ is called $N$-strongly quasi invariant if there is a continuous positive function $\lambda$ on $N \times K\backslash G/H$ such that for all $n \in N$, $d\mu_n(\tilde{y}) = \lambda(n, \tilde{y}) d\mu(\tilde{y})$ ($\tilde{y} \in K\backslash G/H$). We call such $\lambda$ the modular function of $\mu$.

**Remark 3.2.** Note that if $K$ is normal in $G$, then the $N$-strongly quasi-invariant measure $\mu$ is the $G$-strongly quasi invariant on $K\backslash G/H$ and if $K = \{e\}$, $\mu$ is the strongly quasi invariant measure on $G/H$.

**Definition 3.3.** Suppose that $G$ is a locally compact group and $H$ and $K$ are closed subgroups of $G$. A rho-function for the triple $(K, G, H)$ is a non-negative locally integrable function $\rho$ on $G$, which satisfies

$$\rho(kxh) = \frac{\Delta_K(k)\Delta_H(h)}{\Delta_G(h)} \rho(x).$$

In the following, it is shown that for every triple $(K, G, H)$ there exists a rho-function and an $N$-strongly quasi-invariant measure on $K\backslash G/H$, which arises from this rho-function. For this, first it is shown that for each $f \in C_c(G)$ there exists a rho-function $\rho_f$ for the triple $(K, G, H)$.

**Proposition 3.4.** Suppose that $G$ is a locally compact group and $H$ and $K$ are closed subgroups of $G$. Then for each $f \in C_c(G)$ there exists a continuous rho-function $\rho_f$ on $G$.

**Proof.** For each $f \in C_c(G)$, take

$$\rho_f(x) = \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh) dh dk.$$

It is clear that $\rho_f$ is a well-defined positive linear map and according to Fubini’s formula we have

$$\int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh) dh dk = \int_H \int_K \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh) dk dh = \int_{K \times H} \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh) dk dh.$$

First, we show that $\rho_f$ is uniformly continuous. Suppose that $V$ is a compact unit neighbourhood in $G$. Since $f \in C_c(G)$, for given $\varepsilon > 0$ there is a symmetric neighbourhood $U$ of $e$ such that $U \subseteq V$ and for each $y \in Ux$, $|f(x) - f(y)| < \varepsilon$.

Take $M = V \cdot \text{supp} f \cdot V$. If $x \in G\setminus KMH$, then $f(k^{-1}xh) = f(k^{-1}yh) = 0$ for all $k \in K$ and $h \in H$, and if $x \in KMH$, there is $k_0 \in K$ and $h_0 \in H$ such that $k_0^{-1}xh_0 \in M$. If $y \in Ux$, then we have two cases:

1. If $k^{-1}k_0^{-1}y \in \text{supp} f$, then $k^{-1}k_0^{-1}x \in M$. Therefore, $k^{-1} \in Mh_0M^{-1} \cap K$. Also, if $k^{-1}k_0^{-1}x \in \text{supp} f$, then $k^{-1} \in Mh_0M^{-1} \cap K$. 

Lemma 3.5. Let $U$ be a symmetric unit neighbourhood of $G$ with compact closure and $N$ be the normalizer of $K$ in $G$. If $U_N = U \cap N$ is taken, there exists a subset $A$ of $G$ with the following properties:

(i) For every $x \in G$, we have $KxH \cap U_Na \neq \emptyset$, for some $a \in A$.
(ii) If $M$ is a compact subset of $G$, then $\{ a \in A : KMH \cap U_Na \neq \emptyset \}$ is finite.

Proof. Let $A = \{ A \subseteq G ; \text{ for all } a \neq b \text{ in } A, a \notin KU_NbH \}$. According to Zorn’s Lemma, $A$ has a maximal element, say $A$. We claim that $A$ satisfies (i) and (ii).

(i) If $x \in A$ the claim is clear. If $x \in G \setminus A$ where such that $KxH \cap U_Na = \emptyset$, for all $a \in A$, then we could add $x$ to $A$ and make $A$ strictly larger. So (i) holds for $A$. 

This proves $\rho_f$.
(ii) Let $M$ be a compact subset of $G$ and $A_M = \{ a \in A; KMH \cap U_N a \neq \emptyset \}$. For every $a \in A_M$, $KMH \cap U_N a \neq \emptyset$ implies $KaH \cap U_N M \neq \emptyset$ and conversely.

Pick $x_a \in KaH \cap U_N M$. If $A_M$ is infinite, then $\{ x_a; a \in A_M \}$ would have a cluster point $x$, say, in the compact set $\bar{U}_N M$.

Let $V$ be a unit neighbourhood such that $VV^{-1} \subseteq U$. Then, by choosing $V_N = V \cap N$, we have $V_N V_N^{-1} \subseteq U_N$. Since the $x_a$ is a cluster at $x$, there exist distinct $a, b \in A_M$ such that $x_a, x_b \in VNx$. This implies that $x_a x_b^{-1} \in V_N V_N^{-1} \subseteq U_N$. But $x_a \in KaH$ and $x_b \in KbH$, so $x_a \in KU_N bH$ which forces $a \in KU_N bH$, in contradiction to $a \in A$. So, $A_M$ is finite and (ii) is met.

□

Next we use Lemma 3.5 and Proposition 3.4 to give a rho-function for each triple $(K, G, H)$ mentioned above, which is strictly positive on $G$.

**Proposition 3.6.** With the above notation, there exists a rho-function $\rho$ for the triple $(K, G, H)$, which is continuous and everywhere strictly positive on $G$.

**Proof.** Choose $f \in C^+_c(G)$ such that $f(c) > 0$ and $f(x) = f(x^{-1})$ for all $x \in G$.

Put $U = \{ x \in G; f(x) > 0 \}$, by choosing $U_N = U \cap N$ and according to Lemma 3.5 there is subset $A$ of $G$ with properties (i) and (ii) which are mentioned in this Lemma.

Let for every $y \in A$, $f^y(x) = f(xy^{-1})$ for $x \in G$. By using Proposition 3.4, we can define a continuous rho-function $\rho_{f^y}$ by

$$\rho_{f^y}(x) = \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h) \Delta_K(k^{-1})} f(k^{-1} x h y^{-1}) dh dk.$$ 

Now, by using the fact that $\rho_{f^y}(x) = 0$ if $x \notin KU_N y H$ and applying the Proposition 3.4, for any compact subset $M$ of $G$, we have $\rho_{f^y}$ as being zero on $M$ for all but finitely many $y \in A$. Thus $\rho = \sum_{y \in A} \rho_{f^y}$ is a continuous function on $G$. Also, it is evident that $\rho$ is a rho-function.

According to Lemma 3.5 (i), for each $x \in G$, there is $y \in A$ such that $f^y(k x h) > 0$ for some $k \in K$ and $h \in H$. Therefore, $\rho_{f^y}(x) > 0$ and hence $\rho(x) > 0$. □

Next, we use Proposition 3.4 to construct a positive measure on $K \setminus G \setminus H$.

**Theorem 3.7.** Let $\rho$ be a rho-function for the triple $(K, G, H)$. Then there exists a positive Radon measure $\mu_\rho$ on $K \setminus G \setminus H$ such that

$$\int_{K \setminus G \setminus H} Q(f)(\bar{x}) d\mu_\rho(\bar{x}) = \int_G f(x) \rho(x) dx$$

for all $f \in C_c(G)$.

**Proof.** By applying Proposition 3.6, for each triple $(K, G, H)$ we can get a rho-function $\rho$. Take the linear functional $I_\rho$ on $C_c(K \setminus G \setminus H)$ by

$$I_\rho(Q(f)) = \int_G f(x) \rho(x) dx.$$
By using Lemma 2.2, there exists \( g \in C_c(G) \) such that \( Q(g)(\bar{x}) = 1 \) on \( \text{supp}(Q(f)) \). That is, \( \int_K \int_H g(k^{-1}xh)dhdk = 1 \) for all \( x \in \text{Supp} f \) therefore we can write

\[
\int_G f(x)\rho(x)dx = \int_G f(x)\rho(x)Q(g)(\bar{x})dx \\
= \int_K \int_H \int_G f(x)\rho(x)g(k^{-1}xh)dxdhdk \\
= \int_G \int_K \int_H f(kxh^{-1})\Delta_H(h^{-1})\Delta_K(k)\rho(x)g(x)dhdxdx \\
= \int_G g(x)\rho(x)\left( \int_K \int_H f(k^{-1}xh)dhdk \right)dx.
\]

Now if \( Q(f) = 0 \), then \( \int_G f(x)\rho(x)dx = 0 \). Therefore, \( I_\rho \) is a well-defined positive linear functional on \( C_c(K\backslash G/H) \). We conclude that there exists a positive Radon measure \( \mu_\rho \) on \( K\backslash G/H \) such that

\[
\int_{K\backslash G/H} Q(f)(\bar{y})d\mu_\rho(\bar{y}) = \int_G f(x)\rho(x)dx.
\]

\( \square \)

We add the \( IN \)-group condition for closed subgroup \( K \) of \( G \) in Theorem 3.7 to achieve our result.

**Theorem 3.8.** Suppose also that \( K \) is an \( IN \)-group. Given any rho-function \( \rho \) for the triple \( (K, G, H) \), there is an \( N \)-strongly quasi-invariant measure \( \mu_\rho \) on \( K\backslash G/H \) such that

\[
\int_{K\backslash G/H} f(y)\rho(y)dy = \int_{K\backslash G/H} Q(f)(\bar{y})d\mu_\rho(\bar{y}) \\
= \int_{K\backslash G/H} \int_K \int_H f(k^{-1}yh)dhdkd\mu_\rho(\bar{y}).
\]

**Proof.** By applying Theorem 3.7, we can get a unique measure \( \mu_\rho \) on \( K\backslash G/H \), which satisfies the following:

\[
\int_G f(y)\rho(y)dy = \int_{K\backslash G/H} \int_K \int_H f(k^{-1}yh)dhdkd\mu_\rho(\bar{y}).
\]

\( \mu_\rho \) is an \( N \)-strongly quasi invariant. Indeed, let

\[
\lambda : N \times K\backslash G/H \times \rightarrow (0, +\infty)
\]

by

\[
\lambda(n, \bar{y}) = \frac{\rho(ny)}{\rho(y)}.
\]

(3.1)

By using the fact that \( K \) is an \( IN \)-group, one can prove that \( \lambda \) is well-defined.
The continuity of rho-function \( \rho \) results in the fact that \( \lambda \) is also continuous. Moreover, for each \( n \in \mathbb{N} \), we have

\[
\int_{K \backslash G/H} Q(f(\tilde{y}))d\mu_n(\tilde{y}) = \int_{K \backslash G/H} Q(L_n f)(\tilde{y})d\mu(y)
\]

\[
= \int_G L_n f(y)\cdot \rho(y)dy
\]

\[
= \int_{K \backslash G/H} Q(f, \lambda(n, .))(\tilde{y})d\mu(\tilde{y}).
\]

Therefore,

\[
\frac{d\mu_n(\tilde{y})}{d\mu(\tilde{y})} = \lambda(n, \tilde{y}).
\]

\[\square\]

**Remark 3.9.** According to Theorem 3.8, if \( K \) is also normal in \( G \), then \( K \backslash G/H \) has a \( G \)-strongly quasi-invariant measure.

In the following proposition we list some properties of \( \mu_n \).

**Proposition 3.10.** Let \( \rho \) be a rho-function for the triple \((K, G, H)\).

(i). If \( A \) is a closed subset of \( K \backslash G/H \) such that \( \rho(n) = 0 \), for all \( n \in N \backslash q^{-1}(A) \), then \( \text{Supp}\mu_n \subseteq A \).

(ii). For each \( n \in N \), \( L_n \rho \) is also a rho-function for the triple \((K, G, H)\) and \( \mu_{L_n \rho} = (\mu_n)^{-1} \).

(iii). Suppose that \( K \) is an \( IN \)-group, then if \( f \in C_c^+(G) \) and take \( \rho = \rho_f \), therefore for any \( \alpha \in C_c(K \backslash G/H) \)

\[
\int_{K \backslash G/H} \alpha(\tilde{x})d\mu_n(\tilde{x}) = \int_G \alpha(q(x)) f(x)dx.
\]

**Proof.** The proof of (i) and (ii) are straightforward. For each \( \alpha \in C_c(K \backslash G/H) \), there is \( \varphi \in C_c(G) \) such that \( Q(\varphi) = \alpha \). Therefore, we have

\[
\int_{K \backslash G/H} \alpha(\tilde{x})d\mu_n(\tilde{x}) = \int_{K \backslash G/H} \alpha(\tilde{x})d\mu_n(\tilde{x})
\]

\[
= \int_G \varphi(x)\rho(x)dx
\]

\[
= \int_G \varphi(x) \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)} f(k^{-1}xh)dhdkdx
\]

\[
= \int_G \int_{H \times K} \varphi(k^{-1}xh)f(x)d(h \times k)dx
\]

\[
= \int_G f(x) \left( \int_{H \times K} \varphi(k^{-1}xh)d(h \times k) \right)dx
\]

\[
= \int_G f(x)Q(\varphi)(q(x))dx.
\]

This proves (iii). \[\square\]
4. RHO-FUNCTION AND $N$-STRONGLY QUASI-INVARIANT MEASURE

Suppose that $G$ is a locally compact group, $H$ and $K$ are closed subgroups of $G$ and $N$ is the normalizer group of $K$ in $G$. Also, suppose that $\omega$ is a left Haar measure on $N$ with the modular function $\Delta_N$. In this section, we want to consider under which conditions an $N$-strongly quasi-invariant measure on $K \setminus G/H$ arises from a rho-function.

First, we recall that if $X$ is a locally compact Hausdorff space and $\mu$ is a positive Radon measure on $X$, then subset $B$ is called locally negligible, if for each compact subset $M$ of $X$, $\mu(B \cap M) = 0$.

Remark 4.1. In [1] has been shown that $N$ is not locally negligible if and only if $N$ is open subgroup of $G$.

Lemma 4.2. If $N$ is an open subgroup of $G$ then each $f \in C_c(N)$ may be regarded as a function in $C_c(G)$ and $Q : C_c(G)|_{C_c(N)}$ is surjective on $B = \{F \in C_c(K \setminus G/H), \text{supp} F \subseteq q(N)\}$.

Proof. Proof is straightforward. □

Our main result in this section is as follows:

Theorem 4.3. Suppose also that $K$ is an IN-group, $N$ is not locally negligible, and $H \subseteq N$. Then every $N$-strongly quasi-invariant measure $\mu$ on $K \setminus G/H$ arises from a rho-function. That is, there is a rho-function $\rho : G \to (0, +\infty)$, such that

$$\int_{K \setminus G/H} \int_K \int_H f(k^{-1} x h) dh dk d\mu(knH) = \int_G f(x) \rho(x) dx$$

for all $f \in C_c(N)$, (4.1)

and all such measures are $N$-strongly equivalent. That is to say that they have the same negligible sets on $q(N)$.

Proof. Suppose that $\mu$ is an $N$-strongly quasi-invariant measure on $K \setminus G/H$, then there is a positive continuous function $\lambda$ on $N \times (K \setminus G/H)$, such that $(d\mu_x/d\mu)(\bar{y}) = \lambda(x, \bar{y})$. It is easy to check that

$$\lambda(n_1n_2p) = \lambda(n_1, n_2p)\lambda(n_2, p).$$

According to Remark 4.1, $N$ is an open subgroup of $G$. Therefore, by applying Lemma 4.2, each function in $C_c(N)$ may be regarded as a function in $C_c(G)$. Also, Range $Q|_{C_c(N)}$ is $\{F \in C_c(K \setminus G/H); \text{supp} F \subseteq q(N)\}$. The mapping $f \mapsto \int_{K \setminus G/H} Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH)$ is a left invariant positive linear
functional on \( C_c(N) \). Indeed:

\[
\int_{K \backslash G / H} Q(L_m f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH)
= \int_{K \backslash G / H} L_m Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH)
= \int_{K \backslash G / H} Q(f)(Km^{-1}nH)\lambda(n, KH)^{-1}d\mu(KnH)
= \int_{K \backslash G / H} Q(f)(KnH)\lambda(mn, KH)^{-1}\lambda(n, KH)^{-1}\lambda(m, KnH)d\mu(KnH)
= \int_{K \backslash G / H} Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH).
\]

By uniqueness of Haar measure on \( N \), there is \( c > 0 \) such that

\[
\int_{K \backslash G / H} Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH) = c \int_N f(x)d\omega(x). \tag{4.2}
\]

Let \( \rho_1 : N \to (0, +\infty) \) be given by \( \rho_1(n) = c\lambda(n, KH) \). By replacing \( f \) by \( f \cdot \lambda(n, KH) \) in (4.2), we see that

\[
\int_{K \backslash G / H} \int_K \int_H f(k^{-1}nh)dh dk = \int_N f(n)\rho_1(n)dn \tag{4.3}
\]

Now \( \rho_1 \) can be extended on \( G \) by the following definition:

\[
\rho : G \to (0, +\infty)
\]

\[
\rho(x) = \begin{cases} 
\rho_1(x) & x \in N \\
0 & x \notin N 
\end{cases}
\]

then \( \rho \) is a positive continuous function on \( G \). Moreover, if \( h_0 \in H \) and \( k_0 \in K \), then by using \( H \subseteq N \), we can write

\[
\int_G f(x)\rho(k_0xh_0)dx = \int_N f(x)\rho(k_0xh_0)\omega(x)
= \int_G f(k_0^{-1}xh_0^{-1})\rho(x)\Delta_N(h_0^{-1})\omega(x)
= \Delta_G(h_0^{-1})\int_{K \backslash G / H} \int_K \int_H f(k_0^{-1}xh_0^{-1}kdh d\mu(\bar{x})
= \Delta_G(h_0^{-1})\Delta_H(h_0)\Delta_K(k_0)\int_G f(x)\rho(x)\omega(x)
= \frac{\Delta_K(k_0)\Delta_H(h_0)}{\Delta_G(h_0)} \int_N f(x)\rho(x)\omega(x).
\]
This being for all \( f \in C_c(N) \), \( \rho(knh_0) = \frac{\Delta_\mu(h_0) \Delta_k(k_0)}{\Delta_G(h_0)} \rho(n) \). When \( x \notin N \), the equality \( \rho(k_0 x h_0) = \frac{\Delta_\mu(h_0) \Delta_k(k_0)}{\Delta_G(h_0)} \rho(x) \) is trivial. This proves that \( \rho \) is a rho-function.

Suppose that \( \mu_1 \) and \( \mu_2 \) are \( N \)-strongly quasi-invariant measures on \( K \triangleleft G/H \) associated with rho-function \( \rho_1 \) and \( \rho_2 \) on \( G \), respectively. Then, we have

\[
\frac{\rho_1(knh)}{\rho_2(knh)} = \frac{\Delta_\mu(h) \Delta_k(k)}{\Delta_G(h)} \frac{\rho_1(n)}{\rho_2(n)}
\]

for all \( n \in N \).

Take \( \varphi : K \triangleleft G/H \to [0, +\infty) \) by

\[
\varphi(KxH) = \begin{cases} 
\frac{\rho_1(x)}{\rho_2(x)} & \text{if } x \in N \\
0 & \text{if } x \notin N
\end{cases}
\]

clearly \( \varphi \) is well-defined and continuous.

Let \( f \in C_c(G) \). Then we can write

\[
Q(f \cdot \frac{\rho_1}{\rho_2})(KnH) = \int_K \int_H f(k^{-1}nh) \frac{\rho_1(k^{-1}nh)}{\rho_2(k^{-1}nh)} d\mu dh
\]

\[
= \frac{\rho_1(n)}{\rho_2(n)} \int_K \int_H f(k^{-1}nh) d\mu dh,
\]

for all \( n \in N \).

Therefore, we have

\[
\int_{K \triangleleft G/H} Q(f)(\tilde{n}) d\mu_1(\tilde{n}) = \int_N f(n) \rho_1(n) dn
\]

\[
= \int_N f(n) \left( \frac{\rho_1(n)}{\rho_2(n)} \right) \rho_2(n) dn
\]

\[
= \int_{K \triangleleft G/H} Q(f)(\tilde{n}) \varphi(\tilde{n}) d\mu_2(\tilde{n}).
\]

Hence, \( \frac{d\mu_1}{d\mu_2}(\tilde{n}) = \varphi(\tilde{n}) \) for all \( \tilde{n} \in K \triangleleft G/H \).

Now, if \( A \subseteq q(N) \) is a negligible set with respect to \( \mu_1 \), then we have

\[
0 = \int_{K \triangleleft G/H} 1_A(\tilde{x}) d\mu_1(\tilde{x}) = \int_{K \triangleleft G/H} 1_A(\tilde{x}) \varphi(\tilde{x}) d\mu_2(\tilde{x}).
\]

Therefore, \( \int_{K \triangleleft G/H} 1_A(\tilde{x}) \varphi(\tilde{x}) d\mu_2(\tilde{x}) = 0 \). By using the fact that for each \( n \in N \) we can get \( \varphi(\tilde{n}) > 0 \) we have \( \int_{K \triangleleft G/H} 1_A(\tilde{x}) d\mu_2(\tilde{x}) = 0 \), so \( \mu_2(A) = 0 \).

Therefore, the negligible sets of \( q(N) \) with respect to \( \mu_1 \) are the same as the negligible sets, with respect to \( \mu_2 \), and we are done. \( \square \)

**Corollary 4.4.** Let \( G \) be a semidirect product of \( K \) and \( H \), respectively. Double coset space \( K \triangleleft G/H \) possesses a strongly quasi-invariant measure.

**Corollary 4.5.** If \( K \triangleleft G \), then each strongly quasi-invariant measure on \( K \triangleleft G/H \) arises from a rho-function. In other words, there exists a rho-function \( \rho \) on \( G \) such that

\[
\int_{K \triangleleft G/H} \int_K \int_H f(k^{-1}xh) d\mu dh dx = \int_G f(x) \rho(x) dx \quad \text{for all } f \in C_c(G)
\]
Proof. It is sufficient to apply Theorem 4.3 and to note the fact that $N = G$. 

**Proposition 4.6.** If $\mu$ is an $N$-strongly quasi-invariant measure on $K \backslash G/H$ which arises from a rho-function, then supp $\mu = K \backslash G/H$.

**Proof.** Suppose that $\mu$ is an $N$-strongly quasi-invariant measure on $K \backslash G/H$ which arises from a rho-function $\rho$. Therefore, we can write

$$\int_{K \backslash G/H} \int_K \int_H f(k^{-1}xh)dhdkd\mu(\tilde{x}) = \int_G f(x)\rho(x)dx \quad \text{for all } f \in C_c(G).$$

Now if supp $\mu \neq K \backslash G/H$, then there is a non-empty open subset $U$ of $K \backslash G/H$ such that $\mu(U) = 0$. By applying Urysohn’s Lemma, there is a non-zero $F \in C_c(K \backslash G/H)$ such that supp $F \subseteq U$. Also, there is a non-zero $f \in C_c(G)$ such that $Q(f) = F$. So,

$$0 = \int_{K \backslash G/H} F(\tilde{x})d\mu(\tilde{x}) = \int_{K \backslash G/H} Q(f)(\tilde{x})d\mu(\tilde{x})$$

$$= \int_{K \backslash G/H} \int_K \int_H f(k^{-1}xh)dhdkd\mu(\tilde{x})$$

$$= \int_G f(x)\rho(x)dx > 0$$

which is a contradiction. Therefore, supp $\mu = K \backslash G/H$. 

**Proposition 4.7.** Let $N$ be open in $G$, and $\mu$ be any $N$-strongly quasi-invariant measure on $K \backslash G/H$ which arises from a rho-function. Then, for a Borel subset $A \subseteq K \backslash G/H$, $A \cap q(N)$ is locally negligible if and only if $q^{-1}(A) \cap N$ is locally negligible in $G$.

**Proof.** Let $A \subseteq K \backslash G/H$ be a Borel set such that $A \cap q(N)$ be locally negligible. By intersecting $A \cap q(N)$ with an arbitrary compact subset of $K \backslash G/H$, we may assume, without loss of generality, that $A \cap q(N)$ is relatively compact, that is, $A \cap q(N)$ is compact.

Let $f \in C^+(G)$ be such that $f \neq 0$. By applying Fubini’s Theorem, we can write

$$\int_{K \backslash G/H} \int_N f(x)1_{A \cap q(N)}(x \cdot \tilde{y})d\omega(x)d\mu(\tilde{y}) = \int_N \int_{K \backslash G/H} f(x)1_{A \cap q(N)}(x \cdot \tilde{y})d\mu(\tilde{y})d\omega(x).$$

(4.4)

Suppose that $\mu(A \cap q(N)) = 0$, then $\mu(x^{-1} \cdot A \cap q(N)) = 0$ for all $x \in N$. Thus, the right hand side of (4.4) is zero, and so the left hand side. Therefore, we may state that

$$\int_N f(x)1_{A \cap q(N)}(x \cdot \tilde{y})d\omega(x) = 0$$

for almost all $\tilde{y} \in K \backslash G/H$.

Let $C$ be any compact subset of $G$ such that $C \cap N \neq \emptyset$ and $U$ a compact unit neighbourhood in $G$. Select $f \in C_c^+(G)$ so that $f(x) > 1$ for all $x \in CU^{-1} \cap N$. Since $\mu(q(U \cap N)) > 0$, there exists $y \in U \cap N$ such that

$$\int_N f(x) \cdot 1_{A \cap q(N)}(x \cdot \tilde{y})d\omega(x) = 0.$$
So,

\[ 0 = \Delta_N(y) \int_N f(x) \cdot 1_{A \cap q(N)}(q(xy)) \, d\omega(x) \]
\[ = \int_N f(xy^{-1}) 1_{q^{-1}(A) \cap N}(x) \, d\omega(x). \]

Now, for each \( x \in CU^{-1} \cap N \), we have \( f(xy^{-1}) \geq 1 \) which implies that
\[ \int_N 1_{q^{-1}(A) \cap N \cap C}(x) \, d\omega(x) = 0. \]

Thus, \( q^{-1}(A) \cap N \cap C \) is negligible set for any compact set \( C \subseteq G \), that is, \( q^{-1}(A) \cap N \) is locally negligible.

Conversely, suppose that \( q^{-1}(A) \cap N \) is locally negligible. Again, let \( f \in C^+(N) \), and \( \hat{y} \in q(N) \) be arbitrary and from now fixed, since \( q \) is onto, choose \( y \in N \) such that \( q(y) = \hat{y} \). Then, \( x \mapsto f(xy^{-1}) \) is continuous with compact support.

Then, the left hand side of (4.4) is zero, therefore the right hand side is zero as well. Hence, for almost all \( x \in N \)

\[ 0 = \int_{K \setminus G/H} f(x) \cdot 1_{A \cap q(N)}(x \cdot \hat{y}) \, d\mu(\hat{y}) \]
\[ = f(x) \cdot \mu(x^{-1} \cdot A \cap q(N)). \]

Since \( f \neq 0 \), there is \( x \in N \) so that \( \mu(x^{-1} \cdot A \cap q(N)) = 0 \) which implies that \( \mu(A \cap q(N)) = 0. \)

\[ \square \]

**Theorem 4.8.** If \( K \) is also an IN-group and \( \mu \) is an \( N \)-strongly quasi-invariant measure on \( K \setminus G/H \), then \( \tilde{\mu} \) defined by \( \tilde{\mu}(f) = \int_{K \setminus G/H} Q(f)(\hat{x}) \, d\tilde{\mu}(\hat{x}) \) has the following property:

\[ \int_G f(nx^{-1}) \, d\tilde{\mu}(x) = \Delta_H(h) \int_G f(x) \cdot \lambda(n, q(x)) \, d\tilde{\mu}(x). \quad (4.5) \]

**Proof.** Suppose that \( \mu \) is an \( N \)-strongly quasi-invariant measure. Therefore, there is the continuous positive function \( \lambda \) on \( N \times K \setminus G/H \) such that \( d\mu_n(\hat{x}) = \lambda(n, \hat{x}) \, d\mu(\hat{x}) \).
for all $n \in N$. Hence, by applying Theorem 2.4, we have
\[
\int_G f(nxh^{-1})d\tilde{\mu}(x) = \int_G L_n^{-1} \circ R_h^{-1} f(x)d\tilde{\mu}(x)
\]
\[
= \Delta_H(h) \int_G L_n^{-1} f(x)d\tilde{\mu}(x)
\]
\[
= \Delta_H(h) \int_{K\setminus G/H} L_n^{-1} Q(f)q(\bar{x})d\mu(\bar{x})
\]
\[
= \Delta_H(h) \int_{K\setminus G/H} Q(f)q(\bar{x})\lambda(n, \bar{x})d\mu(\bar{x})
\]
\[
= \Delta_H(h) \int_G \left( f \cdot \lambda(n, q(\cdot)) \right)(x)d\tilde{\mu}(x)
\]
\[\square\]

Remark 4.9. Note that if $K = \{e\}$, then we conclude that each strongly quasi-invariant measure on $G/H$ arises from a rho-function and if $H = \{e\}$, then $K \setminus G$ (the right cosets of $K$ in $G$) has $N$-strongly quasi-invariant measure by the left action and if $N$ is not locally negligible, this measure arises from a rho-function.

Remark 4.10. Take $K = H$. Now if $N$ is not locally negligible, each $N$-strongly quasi-invariant measure on $G//H$ arises from a rho-function.

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