**LINEAR-QUADRATIC STOCHASTIC STACKELBERG DIFFERENTIAL GAMES FOR JUMP-DIFFUSION SYSTEMS**

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**Abstract.** This paper considers linear-quadratic (LQ) stochastic leader-follower Stackelberg differential games for jump-diffusion systems with random coefficients. We first solve the LQ problem of the follower using the stochastic maximum principle and obtain the state-feedback representation of the open-loop solution in terms of the integro-stochastic Riccati differential equation (ISRDE), where the state-feedback type control is shown to be optimal via the completion of squares method. Next, we establish the stochastic maximum principle for the LQ problem of the leader using the variational method. However, to obtain the state-feedback representation of the open-loop solution for the leader, there is a technical challenge due to the jump process. We consider two different cases, in which the state-feedback type control for the leader in terms of the ISRDE can be characterized by generalizing the Four-Step Scheme. We finally show that the state-feedback representation of the open-loop optimal solutions for the leader and the follower constitutes the Stackelberg equilibrium. Note that the LQ problem of the leader is new and nontrivial due to the coupled FBSDE constraint induced by the rational behavior of the follower.

**Key words.** leader-follower Stackelberg game, LQ control for jump diffusions, forward-backward stochastic differential equation with jump diffusions, stochastic Riccati differential equation.

**AMS subject classifications.** 91A65, 93E20, 49K45, 49N10

1. **Introduction.** We first state the notation used in this paper. The precise problem formulation and the detailed literature review are then followed.

1.1. **Notation.** Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. For $x, y \in \mathbb{R}^n$, $x^T$ denotes the transpose of $x$, $\langle x, y \rangle$ is the inner product, and $|x| := \langle x, x \rangle^{1/2}$. Let $\mathbb{S}_n$ be the set of $n \times n$ symmetric matrices. Let $|x|^2_\mathbb{S} := x^T S x$ for $s \in \mathbb{R}^n$ and $S \in \mathbb{S}_n$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the natural filtration $\mathbb{F} := \{\mathcal{F}_s, 0 \leq s \leq t\}$ generated by the following two mutually independent stochastic processes and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$:

- an one dimensional standard Brownian motion $B$ defined on $[0, T]$;
- an $E$-marked right continuous Poisson random measure (process) $N$ defined on $E \times [0, T]$, where $E := E \setminus \{0\}$ with $\bar{E} \subset \mathbb{R}$ is a Borel subset of $\mathbb{R}$ equipped with its Borel $\sigma$-field $\mathcal{B}(E)$. The intensity measure of $N$ is denoted by $\lambda'(de, dt) := \lambda(de)dt$, satisfying $\lambda(E) < \infty$, where $\{\tilde{N}(A, (0, t]) := (N - \lambda')(A, (0, t])\}_{t \in [0, T]}$ is an associated compensated $\mathcal{F}_t$-martingale random (Poisson) measure of $N$ for any $A \in \mathcal{B}(E)$. Here, $\lambda$ is an $\sigma$-finite Lévy measure on $(E, \mathcal{B}(E))$, which satisfies $\int_E (1 + |e|^2)\lambda(de) < \infty$.

We introduce the following spaces $[1]$ for $t \in [0, T]$:

- $C^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$: the space of $\mathcal{F}_t$-adapted $\mathbb{R}^n$-valued stochastic processes, which is càdlàg and satisfies $\|x\|_{C^2} := \mathbb{E}[\sup_{s \in [t, T]} |x(s)|^2]^{1/2} < \infty$;
- $L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$: the space of $\mathcal{F}_t$-adapted $\mathbb{R}^n$-valued stochastic processes, satisf-

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1If the Poisson process $N$ has jumps of unite size $(E = \{1\})$, then $\{\tilde{N}(0, t) := (N - \lambda')(0, t]\}_{t \in (0, T]}$ is the compensated Poisson process, where $\lambda'(dt) = \lambda dt$ and $\lambda > 0$ is the intensity of $N$ [1, 15, 17].
The objective functional to be minimized by the leader is given by

$$J_1(a; u_1, u_2) = E \left[ \int_t^T \left( |x(s)|_L^2 + |u_1(s)|_{H_1}^2 + |x(T)|_{M_1}^2 \right) ds \right],$$

and the objective functional of the follower is as follows

$$J_2(a; u_1, u_2) = E \left[ \int_t^T \left( |x(s)|_{L_2}^2 + |u_2(s)|_{H_2}^2 + |x(T)|_{M_2}^2 \right) ds \right].$$

The assumption of the one-dimensional $B$ and $\widetilde{N}$ in (1.1) is only for notational convenience, and we can easily extend the results of this paper to the multi-dimensional case.
Assumption 2. \( Q_i : \Omega \times [0, T] \rightarrow \mathbb{S}^n \) and \( R_i : \Omega \times [0, T] \rightarrow \mathbb{S}^m, \ i = 1, 2, \) are \( \mathcal{F}_t \)-predictable stochastic processes (random coefficients of (1.2) and (1.3)), which are continuous in \( t \in [0,T] \) and uniformly bounded in a.e \( (\omega, t) \in \Omega \times [0,T] \). Also, \( M_i : \Omega \to \mathbb{S}^n, \ i = 1, 2, \) are \( \mathcal{F}_T \)-measurable random matrices, which are uniformly bounded in a.e. \( \omega \in \Omega \).

Note that in Assumption 2, \( Q_i, R_i, \) and \( M_i, \ i = 1, 2, \) are not needed to be positive (semi)definite matrices.

The interaction between the leader and the follower of the LQ Stackelberg game of this paper can be stated as follows. The leader chooses and announces her (or his) optimal solution to the follower by considering the rational reaction of the follower. The follower then determines his (or her) optimal solution by responding to the optimal solution of the leader. Then the above problem can be referred to as the linear-quadratic (LQ) stochastic Stackelberg differential game for jump-diffusion systems with random coefficients.

Under this setting, the problem can be solved in a reverse way [22, 6, 2]. Specifically, the main objective of the follower is to minimize (1.3) subject to (1.1) for any control of the leader \( u_1 \in \mathcal{U}_1 \), i.e.,

\[
(LQ-F) \quad J_2(a; u_1, \overline{u}_2[a, u_1]) = \inf_{u_2 \in \mathcal{U}_2} J_2(a; u_1, u_2), \ \forall u_1 \in \mathcal{U}_1.
\]

We note that from (1.4), \( \overline{u}_2 \) is an optimal strategy dependent on \( (a, u_1) \in \mathbb{R}^n \times \mathcal{U}_1 \), i.e., \( \overline{u}_2 : \mathbb{R}^n \times \mathcal{U}_1 \to \mathcal{U}_2 \). Then given the optimal solution of (LQ-F), the problem of the leader can be stated as follows:

\[
(LQ-L) \quad J_1(a; \overline{u}_1, \overline{u}_2[a, \overline{u}_1]) = \inf_{u_1 \in \mathcal{U}_1} J_1(a; u_1, \overline{u}_2[a, u_1]).
\]

When the pair \( (\overline{u}_1, \overline{u}_2[a, \overline{u}_1]) \in \mathcal{U}_1 \times \mathcal{U}_2 \) in (1.4) and (1.5) exists, we say that the pair \( (\overline{u}_1, \overline{u}_2[a, \overline{u}_1]) \) constitutes an (adapted) open-loop type Stackelberg equilibrium for the leader and the follower in the Stackelberg game [2, 22, 6, 14].

Based on the Stackelberg game formulated in (LQ-F) and (LQ-L), the main results of this paper can be summarized as follows:

In Section 2, we solve (LQ-F) in (1.4). In particular, using the stochastic maximum principle for jump-diffusion systems [19], we obtain an open-loop type optimal solution for (LQ-F) in terms of the forward-backward SDE (FBSDE) with jump diffusions and random coefficients, which explicitly depends on \( (a, u_1) \in \mathbb{R}^n \times \mathcal{U}_1 \). Since the open-loop type optimal solution is not implementable in practical situations, we obtain its state-feedback representation in terms of the integro-stochastic Riccati differential equation (ISRDE) by applying the Four-Step Scheme. We then show that the corresponding state-feedback type control is the optimal solution for (LQ-F) via the completion of squares method (see Theorem 2.1).

We solve (LQ-L) in (1.5) in Section 3. Note that from (LQ-F), (LQ-L) is the LQ stochastic optimal control problem for FBSDEs with jump diffusions and random coefficients, where the FBSDE, induced from (LQ-F), characterizes the rational reaction behavior of the follower [2, 22, 6]. We first obtain the stochastic maximum principle for (LQ-L) using the variational approach. Then by the stochastic maximum principle, the open-loop optimal solution for (LQ-L) is obtained in terms of the coupled FBSDEs with jump diffusions and random coefficients (see Lemma 3.1).

The state-feedback representation of the open-loop optimal solution of (LQ-L) in terms of the ISRDE is obtained by generalizing the Four-Step Scheme. Unfortunately, there is a technical limitation when extending the Four-Step Scheme to the general
jump-diffusion model in (1.1). A detailed discussion on the technical restriction is given in Section 4. Hence, we consider two different cases: (i) when the Poisson process $N$ has jumps of unit size (see Remark 1) and (ii) when the jump part in (1.1) does not depend on the control of the follower ($G_2 = 0$). Note that the Four-Step Schemes in both cases are much more involved than that for (LQ-F) and that for the case of SDEs in a Brownian setting without jumps studied in [22] due to the presence of the coupling terms by the Brownian motion and the Poisson process (see Theorems 3.2 and 3.4). Moreover, the ISRDEs in (LQ-L) (see (3.25) and (3.35)) are nonsymmetric and highly nonlinear, whereas the SRDE in [22, Theorem 3.3] is symmetric. When (LQ-F) and (LQ-L) are solvable, the corresponding open-loop optimal solutions of (LQ-F) and (LQ-L) constitute the (open-loop type) Stackelberg equilibrium, and they admit the state-feedback representation (see Corollaries 3.3 and 3.5).

1.3. Literature Review. A leader-follower Stackelberg differential game (in the deterministic case) was first studied by H. Von Stackelberg in [20]. Since then, (deterministic and stochastic) Stackelberg differential games and their applications have been studied extensively in the literature, see [2, 16, 3, 7, 22, 18, 6, 4, 10, 13, 21, 11, 24, 14] and the references therein.

Specifically, a complete solution to the LQ stochastic Stackelberg differential game (with random coefficients) was obtained by J. Yong in [22]. In [22], the open-loop type Stackelberg equilibrium and its state-feedback representation in terms of the SRDE were obtained via the maximum principle and the Four-Step Scheme. A general stochastic maximum principle of Stackelberg differential games was established in [6] for both (adapted) open-loop and closed-loop information structures. Stochastic Stackelberg differential games for backward SDEs (BSDEs) (with deterministic coefficients) were studied in [24]. The authors in [10] considered Stackelberg games for FBSDEs, and [21] studied the delay case with deterministic coefficients. Mean-field type stochastic Stackelberg differential games were considered in [11, 14].

We note that the references mentioned above considered the case of SDEs in a Brownian setting without jumps. To the best of our knowledge, a class of (LQ or nonlinear) stochastic Stackelberg differential games for jump-diffusion systems has not been studied in the existing literature. Our paper can be viewed as extensions of [22] to the case of (linear) jump-diffusion systems. That is, the main results of this paper (see Section 1.2) are reduced to those of [22] when $F = G_1 = G_2 = 0$ in (1.1). These generalizations turn out to be not straightforward, since the jump-diffusion part induces an additional technical challenge to obtain the stochastic maximum principle of (LQ-L) and the state-feedback representation of the open-loop optimal solutions of (LQ-F) and (LQ-L) via the Four-Step Scheme.

The organization of the paper is as follows. We solve (LQ-F) and (LQ-L) in Sections 2 and 3, respectively. In Section 4, we discuss the technical restriction of the Four-step scheme in (LQ-L) for the general jump-diffusion model in (1.1), and some special cases and possible extensions of this paper.

2. LQ Optimal Control for the Follower. Suppose that $(\pi, \pi_2)$ is the optimal solution of (LQ-F). We introduce the adjoint equation:

\begin{equation}
\begin{cases}
\frac{dp(s)}{ds} = -\left[ A(s)^T p(s^-) + C(s)^T q(s) + Q_2(s)\pi(s^-) \\
+ \int_{E} F(s,e)^T r(s,e)\lambda(de) \right] ds + q(s)dB(s) + \int_{E} r(s,e)\widetilde{N}(de, ds), \quad s \in [t, T) \\
p(T) = M_2 x(T).
\end{cases}
\end{equation}
Note that (2.1) is the (linear) backward SDE (BSDE) with jump diffusions and random coefficients. There is a unique solution of (2.1) with \((p, q, r) \in C^2_b(t, T; \mathbb{R}^n) \times L^2_b(t, T; \mathbb{R}^n) \times L^2_{p,b}(t, T; \mathbb{R}^n)\) [19, Lemma 2.4] (see also [5, Theorem 2.1]).

Based on the stochastic maximum principle in [19, Theorem 2.1], \(\pi_2\) satisfies the following first-order optimality condition:

\[
B_2(s)^T p(s-) + D_2(s)^T q(s) + \int_E G_2(s, e)^T r(s, e) \lambda(de) + R_2(s) \bar{u}_2(s) = 0.
\]

We now consider the following transformation in the Four-Step Scheme:

\[
p(s) = P(s) \pi(s) + \phi(s),
\]

where \(P \in \mathbb{S}^n\) with \(P(T) = M_2\) and \(\phi \in \mathbb{R}^n\) with \(\phi(T) = 0\). Assume that \(P\) and \(\phi\) are of the following form:

\[
\begin{aligned}
dP(s) &= \Lambda_1(s)ds + L(s)dB(s) + \int_E Z(s, e) \check{N}(de, ds), \ s \in [t, T) \\
d\phi(s) &= \Lambda_2(s)ds + \theta(s)dB(s) + \int_E \psi(s, e) \check{N}(de, ds), \ s \in [t, T),
\end{aligned}
\]

where \(L, Z \in \mathbb{S}^n\) and \(\theta, \psi \in \mathbb{R}^n\). Explicit expressions of (2.4) are obtained below.

By applying Itô’s formula for general Lévy-type stochastic integrals to (2.3) and using (2.4), we have

\[
dp(s) = \left[\Lambda_1(s)ds + L(s)dB(s) + \int_E Z(s, e) \check{N}(de, ds)\right] \pi(s-) \\
+ P(s-) \left[ A(s) \pi(s-) + B_1(s)u_1(s) + B_2(s) \pi_2(s) \right] ds \\
+ P(s-) \left[ C(s) \pi(s-) + D_1(s)u_1(s) + D_2(s) \pi_2(s) \right] dB(s) \\
+ \int_E P(s-) \left[ F(s, e) \pi(s-) + G_1(s, e)u_1(s) + G_2(s, e) \pi_2(s) \right] \check{N}(de, ds) \\
+ L(s) \left[ C(s) \pi(s-) + D_1(s)u_1(s) + D_2(s) \pi_2(s) \right] ds \\
+ \int_E Z(s, e) \left[ F(s, e) \pi(s-) + G_1(s, e)u_1(s) + G_2(s, e) \pi_2(s) \right] \lambda(de) ds \\
+ \int_E Z(s, e) \left[ F(s, e) \pi(s-) + G_1(s, e)u_1(s) + G_2(s, e) \pi_2(s) \right] \check{N}(de, ds) \\
+ \Lambda_2(s)ds + \theta(s)dB(s) + \int_E \psi(s, e) \check{N}(de, ds) \\
= - A(s)^T p(s-) + C(s)^T q(s) + Q_2(s) \pi(s-) \\
+ \int_E F(s, e)^T r(s, e) \lambda(de) ds + q(s)dB(s) + \int_E r(s, e) \check{N}(de, ds).
\]

Note the coefficients of \(B\) and \(\check{N}\) in (2.5). Then \(q\) and \(r\) can be written as

\[
\begin{aligned}
q(s) &= L(s) \pi(s-) + P(s-) \left[ C(s) \pi(s-) + D_1(s)u_1(s) + D_2(s) \pi_2(s) \right] + \theta(s) \\
r(s, e) &= \left( Z(s, e) \pi(s-) + P(s-) \left[ F(s, e) \pi(s-) + G_1(s, e)u_1(s) + G_2(s, e) \pi_2(s) \right] \\
&\quad + Z(s, e) \left[ F(s, e) \pi(s-) + G_1(s, e)u_1(s) + G_2(s, e) \pi_2(s) \right] + \psi(s, e).
\end{aligned}
\]

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Substituting (2.3) and (2.6) into (2.2) yields

\[
\pi_2(s) = -\hat{R}_2(s)^{-1}\hat{S}_2(s)^\top \pi(s-1) - \hat{R}_2(s)^{-1}\left(B_2(s)^\top \phi(s-) + D_2(s)^\top \theta(s) + \int_E G_2(s,e)^\top \psi(s,e)\lambda(de) + \hat{S}_1(s)u_1(s)\right),
\]

provided that \(\hat{R}_2(s)\) is invertible, where

\[
\begin{aligned}
\hat{R}_2(s) &:= D_2(s)^\top P(s-)D_2(s) + \int_E G_2(s,e)^\top \left(P(s-) + Z(s,e)G_2(s,e)\lambda(de)\right)G_2(s,e)\lambda(de) \\
\hat{S}_2(s) &:= \left(B_2(s)^\top P(s-) + D_2(s)^\top L(s) + D_2(s)^\top P(s-)C(s)\right) \\
&\quad + \int_E \left(G_2(s,e), Z(s,e) + P(s-)F(s,e) + Z(s,e)F(s,e)\right)\lambda(de) \right)^\top \\
\hat{S}_1(s) &:= D_2(s)^\top P(s-)D_1(s) \\
&\quad + \int_E G_2(s,e), P(s-)G_1(s,e) + R(s,e)G_1(s,e)\lambda(de).
\end{aligned}
\]

Note that (2.7) is the optimal control with the state-feedback representation, which depends on \(u_1 \in \mathcal{U}_1\). We can easily see that \(\pi_2 \in \mathcal{U}_2\).

By substituting (2.7) into (1.1), we have

\[
\begin{aligned}
d\pi(s) &= \left[\hat{A}(s)\pi(s-) + \hat{B}_2(s)\phi(s-) + \hat{H}_2(s)\theta(s) + \int_E \hat{K}_2(s,e)\psi(s,e)\lambda(de) + \hat{B}_1(s)u_1(s)\right]ds \\
&\quad + \left[\hat{C}(s)\pi(s-) + \hat{H}_2(s)^\top \phi(s-) + \hat{H}_2(s)\theta(s) + \int_E \hat{K}_2(s,e)\psi(s,e)\lambda(de) + \hat{D}_1(s)u_1(s)\right]dB(s) \\
&\quad + \int_E \left[\hat{F}(s,e)\pi(s-) + \hat{K}_2(s,e)^\top \phi(s-) + \hat{K}_2(s,e)^\top \theta(s) + \int_E \hat{K}_2(s,e,e')\psi(s,e')\lambda(de') + \hat{G}_1(s,e)u_1(s)\right]dN(de,ds), \\
x(t) &= a,
\end{aligned}
\]

where

\[
\begin{aligned}
\hat{A}(s) &:= A(s) - B_2(s)\hat{R}_2(s)^{-1}\hat{S}_2(s)^\top - B_2(s)\hat{R}_2(s)^{-1}B_2(s)^\top \\
\hat{H}_2(s) &:= -B_2(s)\hat{R}_2(s)^{-1}D_2(s)^\top - B_2(s)\hat{R}_2(s)^{-1}G_2(s,e)^\top \\
\hat{B}_1(s) &:= B_1(s) - B_2(s)\hat{R}_2(s)^{-1}\hat{S}_1(s) \\
\hat{C}(s) &:= C(s) - D_2(s)\hat{R}_2(s)^{-1}\hat{S}_2(s)^\top \\
\hat{K}_2(s,e) &:= -D_2(s)\hat{R}_2(s)^{-1}D_2(s)^\top - D_2(s)\hat{R}_2(s)^{-1}G_2(s,e)^\top \\
\hat{D}_1(s) &:= D_1(s) - D_2(s)\hat{R}_2(s)^{-1}\hat{S}_1(s) \\
\hat{F}(s,e) &:= F(s,e) - G_2(s,e)\hat{R}_2(s)^{-1}\hat{S}_2(s)^\top \\
\hat{K}_2(s,e,e') &:= -G_2(s,e)\hat{R}_2(s)^{-1}G_2(s,e')^\top \\
\hat{G}_1(s,e) &:= G_1(s,e) - G_2(s,e)\hat{R}_2(s)^{-1}\hat{S}_1(s).
\end{aligned}
\]

Note that (2.9) is the rational behavior of the follower under the (state-feedback type) optimal control in (2.7).

Substituting (2.3), (2.7) and (2.6) into (2.5), we can show that \(P\) in (2.4) has to
satisfy the following integro-stochastic Riccati differential equation (ISRDE):

\begin{equation}
\begin{aligned}
\frac{dP(s)}{ds} &= -\left[A(s)^\top P(s-) + P(s-)A(s) + Q_2(s) + L(s)C(s) \right. \\
&\quad \left. + C(s)^\top L(s) + C(s)^\top P(s-)C(s) + \int_E [Z(s,e)F(s,e) + F(s,e)^\top Z(s,e)]\lambda(de) \right] \\
&\quad + (s,e)^\top P(s-)F(s,e) + F(s,e)^\top Z(s,e)F(s,e)\lambda(de) \\
&\quad - \hat{S}_2(s)\hat{R}_2(s)^{-1}\hat{S}_2(s)^\top \int_E dB(s) + \int_E Z(s,e)\tilde{N}(de,ds), \ s \in [t,T] \\
P(T) &= M_2,
\end{aligned}
\end{equation}

and \( \phi \) in (2.4) is the following BSDE with jumps and random coefficients:

\begin{equation}
\begin{aligned}
d\phi(s) &= -\left[\hat{A}(s)^\top \phi(s-) + \hat{C}(s)^\top \theta(s) + \int_E \hat{F}(s,e)^\top \psi(s,e)\lambda(de) + \hat{H}_1(s)^\top u_1(s) \right. \\
&\quad \left. + \int_E \hat{G}_1(s,e)\lambda(de) \right] ds + \theta(s)dB(s) + \int_E \psi(s,e)\tilde{N}(de,ds), \ s \in [t,T] \\
\phi(T) &= 0.
\end{aligned}
\end{equation}

with \( \hat{H}_1 \) and \( \hat{G}_1 \) defined by

\begin{equation}
\begin{aligned}
\hat{H}_1(s) &= (C(s)^\top P(s-)D_1(s) + P(s-)B_1(s) \\
&\quad + L(s)D_1(s) - \hat{S}_2(s-)\hat{R}_2(s-)^{-1}\hat{S}_1(s))\top \\
\hat{G}_1(s,e) &= (F(s,e)^\top P(s-)G_1(s,e) \\
&\quad + F(s,e)^\top Z(s,e)G_1(s,e) + Z(s,e)G_1(s,e))^\top.
\end{aligned}
\end{equation}

In summary, we have the following result:

**Theorem 2.1.** Assume that Assumptions 1 and 2 hold. Suppose that \( (P,L,Z) \in \mathcal{C}^2(t,T;\mathbb{S}^n) \times \mathcal{L}^2(t,T;\mathbb{S}^n) \times \mathcal{L}^2_{\mathbb{P}}(t,T;\mathbb{R}^n) \) is the solution of the ISRDE in (2.11), and \( (\phi,\theta,\psi) \in \mathcal{C}^2(t,T;\mathbb{R}^n) \times \mathcal{L}^2(t,T;\mathbb{R}^n) \times \mathcal{L}^2_{\mathbb{P}}(t,T;\mathbb{R}^n) \) is the solution of the BSDE in (2.12). Assume that \( \hat{R}_2 \) is uniformly positive definite for a.e. \( (\omega,s) \in \Omega \times [0,T], \) where \( \hat{R}_2 \) is defined in (2.8). Then the state-feedback representation of the optimal control for \( \textbf{LQ-F} \) can be written as

\begin{equation}
\begin{aligned}
\pi_2(s) &= -\hat{R}_2(s)^{-1}\hat{S}_2(s)^\top x(s-) - \hat{R}_2(s)^{-1}\left[\hat{B}_2(s)^\top \phi(s-) \\
&\quad + D_2(s)^\top \theta(s) + \int_E G_2(s,e)^\top \psi(s,e)\lambda(de) + \hat{S}_1(s)u_1(s) \right].
\end{aligned}
\end{equation}

Moreover, the corresponding optimal cost of \( \textbf{LQ-F} \) under (2.14) is given by

\begin{equation}
\begin{aligned}
J_2(a;u_1,\pi_2) &= \inf_{u_2 \in \mathcal{U}_2} J_2(a;u_1,u_2) \\
&= \mathbb{E} \left[ |a|^2_{P(0)} + 2\langle a,\phi(0) \rangle \right. \\
&\quad + \int_t^T |u_1(s)|^2_{D_1(s)^\top P(s-)D_1(s)} ds \\
&\quad + \int_t^T \int_E |u_1(s)|^2_{G_1(s,e)^\top P(s-)G_1(s,e) + G_1(s,e)^\top Z(s,e)G_1(s,e)} \lambda(de) ds \\
&\quad + \int_t^T \left( u_1(s),B_1(s)^\top \phi(s-) + D_1(s)^\top \theta(s) \right) ds \\
&\quad + 2 \int_t^T \int_E \left( u_1(s),G_1(s,e)^\top \psi(s,e) \right) \lambda(de) ds \right].
\end{aligned}
\end{equation}
Proof. For a given $u_1 \in U_1$, let $\bar{\sigma}$ be the state process controlled by $\bar{\nu}_2$ in (2.14). Then $\bar{\sigma}$ is equivalent to the state process in (2.9), and in view of Assumptions 1 and 2, for any $u_1 \in U_1$, (2.9) admits a unique càdlàg solution in $C^2(t,T;\mathbb{R}^n)$ [1, 15, 12]. Since $(\phi,\theta,\psi)$ in (2.12) is a linear BSDE, it admits a unique solution of $(\phi,\theta,\psi) \in C^0([t,T;\mathbb{R}^n] \times \mathcal{L}^p_{\mathcal{F}}([t,T;\mathbb{R}^n] \times \mathcal{L}^p_{\mathcal{F}}([t,T;\mathbb{R}^n]))$ [19, Lemma 2.4]. Furthermore, for a fixed $(a,u_1) \in \mathbb{R}^n \times U_1$, it holds that $\bar{\nu}_2 \in U_2$.

For any $u_2 \in U_2$, we apply Itô’s formula to $d(x(s), P(s)x(s)) + 2d(x(s), \phi(s))$, where $x$ is the SDE in (1.1), $P$ is the ISRDE in (2.11), and $\phi$ is the BSDE in (2.12). Then by integrating it from 0 to $T$ and completing the integrand with respect to $u_2$, we can show that $J_2$ can equivalently written as follows:

$$(2.16) \quad J_2(a; u_1, u_2) = \mathbb{E} \left[ |a|^2_{P(0)} + 2\langle a, \phi(0) \rangle \right. \\
+ \int_t^T \left[ u_2(s) + \tilde{R}_2(s)^{-1} \tilde{S}_2(s)^\top x(s) + \tilde{R}_2(s)^{-1} \left( B_2(s)^\top \phi(s) \\
+ D_2(s)^\top \theta(s) + \int_E G_2(s,e)^\top \psi(s,e) \lambda(de) + \tilde{S}_1(s)u_1(s) \right) \right]^2 \\
\left. ds \right] \\
+ \int_t^T \int_E |u_1(s)|^2_{D_1(s)^\top P(s^-)D_1(s)} ds \\
+ \int_t^T \int_E |u_1(s)|^2_{G_1(s,e)^\top P(s^-)G_1(s,e)+G_1(s,e)^\top Z(s,e)G_1(s,e)} \lambda(de) ds \\
+ 2\int_t^T \langle u_1(s), B_1(s)^\top \phi(s^-) + D_1(s)^\top \theta(s) \rangle ds \\
+ 2\int_t^T \int_E \langle u_1(s), G_1(s,e)^\top \psi(s,e) \rangle \lambda(de) ds \right] \]

Since $\tilde{R}_2 > 0$ for a.e. $(\omega,s) \in \Omega \times [0,T]$, for a given $u_1 \in U_1$, we have

$$(2.17) \quad J_2(a; u_1, u_2) \geq J_2(a; u_1, \bar{\nu}_2), \quad \forall u_2 \in U_2.$$ 

This shows that (2.14) is the optimal control of (LQ-F), and (2.9) is the corresponding optimal state trajectory. From (2.16) and (2.17), we can easily see that (2.15) is the optimal cost of (LQ-F). This completes the proof.

3. LQ Optimal Control for the Leader. This section addresses (LQ-L) in (1.5). Note that the constraint of (LQ-L) is (2.9) and (2.12), which characterize the rational behavior of the follower under (2.14). That is, (LQ-L) can be rewritten as

$$(3.1) \quad (LQ-L) \quad J_1(a; \bar{u}_1, \bar{u}_2) = \inf_{u_1 \in U_1} J_1(a; u_1, \bar{u}_2), \text{ subject to (2.9) and (2.12)}. $$

We can easily see that (LQ-L) is the stochastic optimal control problem for FBSDEs with jump diffusions and random coefficients.

We first state the stochastic maximum principle for (LQ-L):

**Lemma 3.1.** Suppose that Assumptions 1 and 2 hold. Let $\bar{u}_1 \in U_1$, where $\bar{\sigma}$ is the corresponding state trajectory. Let $(\bar{\nu}, \beta) \in C^2(t,T; \mathbb{R}^n \times \mathbb{R}^n)$, $(\phi, \theta, \psi) \in C^2(t,T; \mathbb{R}^n) \times C^2(t,T; \mathbb{R}^n)$, and $(\phi, \theta, \psi) \in C^2(t,T; \mathbb{R}^n) \times C^2(t,T; \mathbb{R}^n)$.
be the solution of the following coupled FBSDEs:

\[
\begin{aligned}
&\mathcal{L}_2^q(t, T; \mathbb{R}^n) \times \mathcal{L}_2^q(t, T; \mathbb{R}^n) \text{ and } (\alpha, \eta, \gamma) \in \mathcal{C}^2_2(t, T; \mathbb{R}^n) \times \mathcal{L}_2^q(t, T; \mathbb{R}^n) \times \mathcal{L}_{2,p}^q(t, T; \mathbb{R}^n) \\
\text{(3.2)}
&\begin{cases}
\d x(t) = \left[ \tilde{A}(t)x(t) + \tilde{B}_2(t)x(t) + \tilde{H}_2(t)\theta(t) + \int_E \tilde{K}_2(s,e)\psi(s,e)\lambda(de) + \int_E \tilde{D}_1(s)\mu_1(s)ds \\
+ \int_E \tilde{C}(t, \theta)\phi(s) + \tilde{H}_2(t)\theta(t) + \int_E \tilde{K}_2(s,e)\psi(s,e)\lambda(de) + \int_E \tilde{D}_1(s)\mu_1(s)ds \right] \d t \\
+ \int_0^T \int_E \tilde{F}(s, \theta)\phi(s) + \tilde{H}_2(t)\theta(t) + \int_E \tilde{K}_2(s,e)\psi(s,e)\lambda(de) + \tilde{D}_1(s)\mu_1(s) \d s \\
+ \int_0^T \int_E \tilde{R}(s, e)\psi(s, e')\lambda(de') + \tilde{D}_1(s)\mu_1(s) \d s, \quad s \in (t, T) \\
\end{cases}
\end{aligned}
\]

For \( u_1' \in \mathcal{U}_1 \), let \((x', \beta') \in \mathcal{C}^2_2(t, T; \mathbb{R}^n \times \mathbb{R}^n), (\phi', \theta', \psi') \in \mathcal{C}^2_2(t, T; \mathbb{R}^n \times \mathbb{R}^n) \times \mathcal{L}_2^q(t, T; \mathbb{R}^n) \times \mathcal{L}_{2,p}^q(t, T; \mathbb{R}^n) \) and \((\alpha', \eta', \gamma') \in \mathcal{C}^2_2(t, T; \mathbb{R}^n) \times \mathcal{L}_2^q(t, T; \mathbb{R}^n) \times \mathcal{L}_{2,p}^q(t, T; \mathbb{R}^n) \) be the coupled FBSDEs in (3.2), where the initial condition holds \((x'(t), \beta'(t), \phi'(t), \alpha'(t)) = (0, 0, 0, M_1x'(T)). Assume that the following holds:

\[
\begin{aligned}
\text{(3.3)}
&\mathbb{E} \left[ \int_0^T \left\langle u_1'(s), R_1(s)u_1'(s) \right\rangle + \left\langle u_1'(s), B_1(s)^\top \alpha'(s) + D_1(s)^\top \eta'(s) \right\rangle \right. \\
&\left. + \int_E G_1(s)^\top \gamma'(s, e)\lambda(de) + \tilde{H}_1(s)\beta'(s) + \int_E \tilde{K}_1(s, e)\beta'(s)\lambda(de) \right] ds \geq 0.
\end{aligned}
\]

Then \( \pi_1 \in \mathcal{U}_1 \) is the optimal control for \((\textbf{LQ-L})\) if and only if the following first-order optimality condition holds:

\[
\begin{aligned}
\text{(3.4)}
&R_1(s)\pi_1(s) + B_1(s)^\top \alpha(s) + D_1(s)^\top \eta(s) \\
&+ \int_E G_1(s)^\top \gamma(s, e)\lambda(de) + \tilde{H}_1(s)\beta(s) + \int_E \tilde{K}_1(s, e)\beta(s)\lambda(de) = 0.
\end{aligned}
\]

**Proof.** We note that \((\pi, \phi, \theta, \psi) \in \mathcal{C}^2_2(t, T; \mathbb{R}^n \times \mathcal{L}_2^q(t, T; \mathbb{R}^n) \times \mathcal{L}_{2,p}^q(t, T; \mathbb{R}^n) \times \mathcal{L}_{2,p}^q(t, T; \mathbb{R}^n))\) admits a unique solution in view of Theorem 2.1. \( \beta \) is the forward SDE with jump diffusions and random coefficients, and from Assumptions 1 and 2, it admits a unique solution in \( \mathcal{C}^2_2(t, T; \mathbb{R}^n) \). Moreover, \((\alpha, \eta, \gamma) \) is a linear BSDE with jump diffusions and random coefficients, which admits a unique solution in \( \mathcal{C}^2_2(t, T; \mathbb{R}^n) \times \mathcal{L}_2^q(t, T; \mathbb{R}^n) \times \mathcal{L}_{2,p}^q(t, T; \mathbb{R}^n) \) [19, Lemma 2.4] (see also [5, Theorem 2.1]).
By applying Itô’s formula, (note that \(\hat{B}_2, \tilde{H}_2\) and \(\overline{R}_2\) in (2.10) are symmetric)

\[
(3.5) \quad d(x'(s), \alpha(s)) = \left[\hat{A}(s)x'(s) + \hat{B}_2(s)\phi'(s) + \tilde{H}_2(s)\theta'(s)
  + \int_E \hat{K}_2(s, e)\psi'(s, e)\lambda(de) + \hat{B}_1(s)u'_1(s)\right]^T \alpha(s)ds
  - x'(s)^T \left[\hat{A}(s)\alpha(s) + \hat{C}(s)\eta(s)
  + \int_E \tilde{F}(s, e)^T \gamma(s, e)\lambda(de) + Q_1(s)\overline{x}(s)\right]ds
  + \left[\hat{C}(s)x'(s) + \tilde{H}_2(s)^T \phi'(s) + \tilde{H}_2(s)\theta'(s)
  + \int_E \hat{K}_2(s, e)\psi'(s, e)\lambda(de) + \tilde{D}_1(s)u'_1(s)\right]^T \eta(s)ds
  + \int_E \left[\tilde{F}(s, e)x'(s) + \tilde{K}_2(s, e)^T \phi(s) + \tilde{K}_2(s, e)^T \theta(s)
  + \int_E \overline{K}_2(s, e, e')\psi'(s, e')\lambda(de') + \tilde{G}_1(s, e)u'_1(s)\right]^T \gamma(s, e)\lambda(de)
  + \left[\cdots\right]dB(s) + \left[\cdots\right]\tilde{N}(de, ds),
\]

and

\[
(3.6) \quad d(\phi'(s), \beta(s)) = -\left[\hat{A}(s)^T \phi'(s) + \hat{C}(s)^T \theta'(s)
  + \int_E \tilde{F}(s, e)^T \psi'(s, e)\lambda(de)\right]
  - \hat{H}_1(s)^T u'_1(s) + \int_E \hat{K}_1(s, e)^T u'_1(s)\lambda(de)
  - \phi'(s)^T \left[\hat{A}(s)\beta(s) + \hat{B}_2(s)\alpha(s) + \tilde{H}_2(s)\eta(s)
  + \int_E \hat{K}_2(s, e)\gamma(s, e)\lambda(de)\right]ds
  + \theta'(s)^T \left[\hat{C}(s)\beta(s) + \tilde{H}_2(s)^T \alpha(s) + \tilde{H}_2(s)\eta'(s)
  + \int_E \hat{K}_2(s, e)\gamma(s, e)\lambda(de)\right]ds
  + \int_E \left[\tilde{F}(s, e)^T \beta(s) + \tilde{K}_2(s, e)^T \alpha(s) + \tilde{K}_2(s, e)^T \eta(s)
  + \int_E \overline{K}_2(s, e, e')\gamma(s, e')\lambda(de')\right]\lambda(de)
  + \left[\cdots\right]dB(s) + \left[\cdots\right]\tilde{N}(de, ds).
\]

Using (3.5) and (3.6), we have

\[
(3.7) \quad E[\langle x'(T), M_1\overline{x}(T) \rangle] = E\left[\langle x'(T), \alpha(T) \rangle - \langle x'(0), \alpha(0) \rangle - \langle \phi'(0), \beta(0) \rangle + \langle \phi'(T), \beta(T) \rangle\right]
  = E\left[\int_T \left[\langle x'(s), Q_1(s)\overline{x}(s) \rangle + \langle u'_1(s), \tilde{B}_1(s)^T \alpha(s) \rangle
  + \langle \tilde{D}_1(s)^T \eta(s) \rangle + \int_E \tilde{G}_1(s)^T \gamma(s, e)\lambda(de)
  + \langle \tilde{H}_1(s)^T \beta(s) \rangle + \int_E \tilde{K}_1(s, e)\beta(s)\lambda(de) \right]ds\right].
\]
Similarly, we have

\begin{equation}
J_1(a; \mathbf{\pi}_1, \mathbf{\pi}_2) = E\left[\langle \mathbf{X}(0), \alpha(0) \rangle + \int_t^T \langle \mathbf{\pi}_1(s), R_1(s)\mathbf{\pi}_1(s) \rangle \right.
\end{equation}

\begin{equation}
+ \left\langle \mathbf{\pi}_1(s), \mathbf{\tilde{B}}_1(s)^T \alpha(s) + \mathbf{\tilde{D}}_1(s)^T \eta(s) + \int_E \mathbf{\tilde{G}}_1(s)^T \gamma(s, e) \lambda(de) \right. \right.
\end{equation}

\begin{equation}
+ \left. \mathbf{\tilde{H}}_1(s)\beta(s) + \int_E \mathbf{\tilde{K}}_1(s, e)\beta(s) \lambda(de) \right] ds \right],
\end{equation}

and

\begin{equation}
J_1(0; u'_1, \mathbf{\pi}_2)
\end{equation}

\begin{equation}
= E\left[\int_t^T \left[ |x'(s)|^2_{Q_1(s)} + |u'_1(s)|^2_{H_1(s)} \right] ds + |x'(T)|^2_{M_1} \right]
\end{equation}

\begin{equation}
= E\left[\int_t^T \left\langle u'_1(s), R_1(s)u'_1(s) + \mathbf{\tilde{B}}_1(s)^T \alpha'(s) + \mathbf{\tilde{D}}_1(s)^T \eta'(s) \right.ight.
\end{equation}

\begin{equation}
+ \left. \int_E \mathbf{\tilde{G}}_1(s)^T \gamma'(s, e) \lambda(de) + \mathbf{\tilde{H}}_1(s)\beta'(s) + \int_E \mathbf{\tilde{K}}_1(s, e)\beta'(s) \lambda(de) \right] ds \right].
\end{equation}

Then from (3.7)-(3.9), for \( \kappa \in \mathbb{R} \),

\begin{equation}
J(a; \mathbf{\pi}_1 + \kappa u'_1, \mathbf{\pi}_2) - J(a; \mathbf{\pi}_1, \mathbf{\pi}_2)
\end{equation}

\begin{equation}
= 2\kappa E\left[\int_t^T \left\langle u'_1(s), R_1(s)\mathbf{\pi}_1(s) + \mathbf{\tilde{B}}_1(s)^T \alpha(s) + \mathbf{\tilde{D}}_1(s)^T \eta(s) \right.ight.
\end{equation}

\begin{equation}
+ \left. \int_E \mathbf{\tilde{G}}_1(s)^T \gamma(s, e) \lambda(de) + \mathbf{\tilde{H}}_1(s)\beta(s) + \int_E \mathbf{\tilde{K}}_1(s, e)\beta(s) \lambda(de) \right] ds \right]
\end{equation}

\begin{equation}
+ \kappa^2 E\left[\int_t^T \left\langle u'_1(s), R_1(s)u'_1(s) + \mathbf{\tilde{B}}_1(s)^T \alpha'(s) + \mathbf{\tilde{D}}_1(s)^T \eta'(s) \right.ight.
\end{equation}

\begin{equation}
+ \left. \int_E \mathbf{\tilde{G}}_1(s)^T \gamma'(s, e) \lambda(de) + \mathbf{\tilde{H}}_1(s)\beta'(s) + \int_E \mathbf{\tilde{K}}_1(s, e)\beta'(s) \lambda(de) \right] ds \right] \geq 0,
\end{equation}

which implies that under (3.3), \( \mathbf{\pi}_1 \in U_1 \) is the optimal control for (LQ-L) if and only if the first-order optimality condition in (3.4) holds. This completes the proof. \( \square \)

Remark 2. From (3.9), we can see that (3.3) holds when \( Q_1, R_1 \) and \( M_1 \) are uniformly positive (semi)definite for a.e. \((\omega, s) \in \Omega \times [0, T]\).

Below, we obtain the state-feedback representation of (3.4) for two different cases. Note that (3.4) depends on the coupled FBSDEs in (3.2).

Let

\begin{equation}
\mathcal{X}(s) := \begin{bmatrix} \mathbf{X}(s) \\ \beta(s) \end{bmatrix}, \quad \mathcal{Y}(s) := \begin{bmatrix} \alpha(s) \\ \phi(s) \end{bmatrix}, \quad \mathcal{Z}(s) := \begin{bmatrix} \eta(s) \\ \theta(s) \end{bmatrix}, \quad \mathcal{K}(s, e) := \begin{bmatrix} \gamma(s, e) \\ \psi(s, e) \end{bmatrix},
\end{equation}
where \( \overline{X} := X(t) = \begin{bmatrix} a \\ 0 \end{bmatrix} \) and we define (see (2.8), (2.10) and (2.13))

\[
(3.10) \quad \begin{cases} 
\mathcal{A}(s) := \begin{bmatrix} \tilde{A}(s) & 0 \\ \tilde{A}(s) \end{bmatrix}, \quad \mathcal{B}_2(s) := \begin{bmatrix} 0 \\ \tilde{B}_2(s) \end{bmatrix}, \\
\mathcal{H}(s) := \begin{bmatrix} 0 & \tilde{H}_2(s) \\ \tilde{H}_2(s) & 0 \end{bmatrix}, \\
\mathbb{H}(s) := \begin{bmatrix} 0 & \tilde{H}_2(s) \\ \tilde{H}_2(s) & 0 \end{bmatrix}, \\
\mathbb{K}(s,e) := \begin{bmatrix} \tilde{K}_2(s,e) & 0 \\ 0 & \tilde{K}_2(s,e) \end{bmatrix}, \\
\mathbb{M}_1(s,e) := \begin{bmatrix} \tilde{K}_2(s,e) & 0 \\ 0 & \tilde{K}_2(s,e) \end{bmatrix}, \\
\mathcal{H}_1(s,e) := \begin{bmatrix} 0 \\ \tilde{H}(s) \end{bmatrix}, \\
\mathbb{K}_1(s,e) := \begin{bmatrix} 0 \\ \tilde{K}_1(s,e) \end{bmatrix}, \\
\mathcal{M}_1 := \begin{bmatrix} M_1 \\ 0 \end{bmatrix}, \\
\end{cases}
\]

Then the coupled FBSDEs in (3.2) can be written as follows:

\[
(3.11) \quad \begin{cases} 
\mathcal{A}(s) = (\mathcal{A}(s)X(s) + \mathcal{B}_2(s)Y(s) + \mathcal{H}(s)Z(s)) \\
\quad + \int_E \mathbb{K}(s,e)\mathcal{K}(s,e)\lambda(de) + \mathbb{H}(s)\mathcal{M}_1(s) ds \\
\quad + \left[ \mathcal{C}(s)X(s) + \mathcal{H}(s)^\top Y(s) + \mathcal{H}(s)Z(s) \\
\quad + \int_E \mathbb{K}(s,e)\mathcal{K}(s,e)\lambda(de) + \mathbb{H}(s)\mathcal{M}_1(s) \right] dB(s) \\
\quad + \int_E \mathbb{F}(s,e)\mathcal{A}(s) + \mathbb{K}(s,e)\mathcal{Y}(s) + \mathbb{K}(s,e)^\top Z(s) \\
\quad + \int_E \mathbb{M}_1(s) \mathcal{H}(s, s')\lambda(de') + \mathcal{G}_1(s,e)\mathcal{M}_1(s) \end{cases}
\]

where the optimality condition in (3.4) becomes

\[
(3.12) \quad R_1(s)\mathcal{M}_1(s) + \mathbb{B}_1(s)^\top \mathcal{Y}(s) + \mathcal{D}_1(s)^\top \mathcal{Z}(s) + \int_E \mathcal{G}_1(s,e)^\top \mathcal{K}(s,e)\lambda(de) \\
\quad + \mathcal{H}_1(s)\mathcal{X}(s) + \int_E \mathbb{K}_1(s,e)\mathcal{Y}(s) - \mathcal{H}(s, s')\lambda(de') = 0.
\]

We consider the following transformation in the Four-Step Scheme:

\[
(3.13) \quad \mathcal{Y}(s) = \mathcal{P}(s)\mathcal{X}(s),
\]

where \( \mathcal{P} \) takes the following form:

\[
(3.14) \quad \begin{cases} 
d\mathcal{P}(s) = \Lambda_3(s)ds + \Psi(s)dB(s) + \int_E \Theta(s,e)\tilde{N}(de,ds), \quad s \in [t, T) \\
\mathcal{P}(T) = \mathcal{M}_1.
\end{cases}
\]
Note that $\mathcal{P}, \Psi$ and $\Theta$ are $\mathbb{R}^{2n \times 2n}$-valued processes. Let $(s$ is suppressed)

$$
\mathcal{P} = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}, \quad \mathcal{P}_{11} \text{ is an } \mathbb{R}^{n \times n} \text{-dimensional process.}
$$

By applying Itô’s formula to (3.13) and using (3.14), we have

\begin{equation}
\text{(3.15) } \quad d\mathcal{Y}(s) = -\left[ \mathcal{L}(s)\mathcal{Y}(s) + Q(s)\mathcal{X}(s) + C(s)^\top Z(s) + \int_E \mathcal{F}(s,e)\mathcal{K}(s,e)\lambda(de) + \int_E \mathcal{K}(s,e)^\top \mathcal{P}(s,e)\lambda(de) \right] ds
\end{equation}

To obtain the state-feedback representation of (3.12), we consider the following two different cases:

(i) The Poisson process $N$ has jumps of unit size ($E = \{1\}$);

(ii) The follower’s control is not included in the jump part of (1.1) ($G_2 = 0$).

**Remark 3.** A detailed discussion on these two assumptions is given in Section 4.

3.1. Case I: $N$ has jumps of unit size. Let us assume that
**Assumption 3.** The Poisson process $N$ has jumps of unit size, i.e., $E = \{1\}$.

Under Assumption 3 and from Remark 1, (3.15) is given by$^3$

\[(3.16) \quad d\mathcal{Y}(s) = \left[ A(s)^T \mathcal{P}(s) \mathcal{X}(s) - \mathcal{Q}(s) \mathcal{X}(s) + C(s)^T \mathcal{Z}(s) + \mathbb{H}_1(s)^T \mathbb{m}_1(s) ight. \\
+ \lambda \mathbb{F}(s)^T \mathbb{G}(s) + \lambda \mathbb{K}_1(s)^T \mathbb{m}_1(s) \right] ds + \mathcal{Z}(s) dB(s) + \mathbb{K}(s) d\tilde{N}(s) \\
= \left[ A_3(s) ds + \mathbb{F}(s) dB(s) + \Theta(s) d\tilde{N}(s) \right] \mathcal{X}(s) - \\
+ \mathcal{P}(s) \left[ A(s) \mathcal{X}(s) + \mathbb{B}_2(s) \mathcal{P}(s) \mathcal{X}(s) + \mathbb{H}(s) \mathcal{Z}(s) ight. \\
+ \lambda \mathbb{K}(s) \mathbb{K}(s) + \mathbb{D}_1(s) \mathbb{m}_1(s) \right] ds \\
+ \mathcal{P}(s) \left[ C(s) \mathcal{X}(s) + \mathbb{H}(s)^T \mathcal{P}(s) \mathcal{X}(s) + \mathbb{H}(s) \mathcal{Z}(s) ight. \\
+ \lambda \mathbb{K}(s) \mathbb{K}(s) + \mathbb{D}_1(s) \mathbb{m}_1(s) \right] ds \\
+ \Theta(s) \left[ \mathbb{F}(s) \mathcal{X}(s) + \mathbb{H}(s)^T \mathcal{P}(s) \mathcal{X}(s) + \mathbb{H}(s)^T \mathcal{Z}(s) ight. \\
+ \lambda \mathbb{K}(s) \mathbb{K}(s) + \mathbb{G}_1(s) \mathbb{m}_1(s) \right] ds \\
+ \left( \mathcal{P}(s) + \Theta(s) \right) \left[ \mathbb{F}(s) \mathcal{X}(s) + \mathbb{H}(s)^T \mathcal{P}(s) \mathcal{X}(s) + \mathbb{H}(s)^T \mathcal{Z}(s) ight. \\
+ \lambda \mathbb{K}(s) \mathbb{K}(s) + \mathbb{G}_1(s) \mathbb{m}_1(s) \right] d\tilde{N}(s).
\]

Let us define $(s$ is suppressed)

\[(3.17) \quad \begin{cases} 
\mathcal{A}_{11} := I - \mathcal{P}(s)^T \mathbb{H}, & \mathcal{A}_{12} := \lambda \mathcal{P}(s)^T \mathbb{K} \\
\mathcal{A}_{21} := -(\mathcal{P}(s) + \Theta) \mathbb{K}, & \mathcal{A}_{22} := I - \lambda (\mathcal{P}(s) + \Theta) \mathbb{K} \\
\mathcal{B}_{11} := \mathcal{P}(s) \mathbb{C} + \mathcal{P}(s)^T \mathbb{H}^T \mathcal{P}(s) + \Psi, & \mathcal{B}_{12} := \mathcal{P}(s) \mathbb{D}_1 \\
\mathcal{B}_{21} := (\mathcal{P}(s) + \Theta) (\mathbb{F} + \mathbb{H}^T \mathcal{P}(s)) + \Theta, & \mathcal{B}_{22} := (\mathcal{P}(s) + \Theta) \mathbb{G}_1.
\end{cases}
\]

Then from (3.16), we can see that

\[(3.18) \quad \mathcal{A} \begin{bmatrix} \mathcal{Z} \\ \mathcal{K} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{Z} \\ \mathcal{K} \end{bmatrix} = \begin{bmatrix} \mathcal{B}_{11} \mathcal{X}(s) + \mathcal{B}_{12} \mathbb{m}_1 \\ \mathcal{B}_{21} \mathcal{X}(s) + \mathcal{B}_{22} \mathbb{m}_1 \end{bmatrix},
\]

which, together with the block matrix inversion lemma [8, page 18] (assuming its invertibility), implies

\[(3.19) \quad \begin{bmatrix} \mathcal{Z} \\ \mathcal{K} \end{bmatrix} = \mathcal{A}^{-1} \begin{bmatrix} \mathcal{B}_{11} \mathcal{X}(s) + \mathcal{B}_{12} \mathbb{m}_1 \\ \mathcal{B}_{21} \mathcal{X}(s) + \mathcal{B}_{22} \mathbb{m}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11}^{-1} & \mathcal{A}_{12}^{-1} \\ \mathcal{A}_{21}^{-1} & \mathcal{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{B}_{11} \mathcal{X}(s) + \mathcal{B}_{12} \mathbb{m}_1 \\ \mathcal{B}_{21} \mathcal{X}(s) + \mathcal{B}_{22} \mathbb{m}_1 \end{bmatrix},
\]

$^3$Note that under Assumption 3, $\int_{\mathbb{R}} g(s,e) \lambda(\text{de}) ds = g(s) \lambda ds$ for $g \in C_0^2([t,T;\mathbb{R}^n])$, where $\lambda > 0$ is the intensity of $N$ [1, 17].
where

\[
\begin{aligned}
\mathcal{A}_{11} & := (\mathcal{A}_{11} - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21})^{-1}, \\
\mathcal{A}_{12} & := \mathcal{A}_{11}^{-1}\mathcal{A}_{12}(\mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12} - \mathcal{A}_{22})^{-1}, \\
\mathcal{A}_{21} & := \mathcal{A}_{22}^{-1}\mathcal{A}_{21}(\mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21} - \mathcal{A}_{11})^{-1}, \\
\mathcal{A}_{22} & := (\mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12})^{-1}.
\end{aligned}
\]

Substituting (3.19) and (3.13) into the optimality condition in (3.12) yields

\[
R_1\mathbf{\pi}(s) + \mathbb{E}_T^1\mathcal{P}(s-)\mathcal{X}(s-) + \mathbb{H}_1\mathcal{X}(s-) + \lambda\mathbb{K}_1\mathcal{X}(s-)
\]

\[
+ \left(\mathbb{D}^T_1(\mathcal{A}_{11}\mathbb{B}_{11} + \mathcal{A}_{12}\mathbb{B}_{21}) + \lambda\mathbb{G}_1^T(\mathcal{A}_{21}\mathbb{B}_{11} + \mathcal{A}_{22}\mathbb{B}_{21})\right)\mathcal{X}(s-)
\]

\[
+ \left(\mathbb{D}^T_1(\mathcal{A}_{11}\mathbb{B}_{12} + \mathcal{A}_{12}\mathbb{B}_{22}) + \lambda\mathbb{G}_1^T(\mathcal{A}_{21}\mathbb{B}_{12} + \mathcal{A}_{22}\mathbb{B}_{22})\right)\mathbf{\pi}(s) = 0,
\]

and we have

\[
\mathbf{\pi}(s) = -R_1^{-1}\left(\mathbb{E}_T^1\mathcal{P}(s-) + \mathbb{H}_1 + \lambda\mathbb{K}_1
\right.
\]

\[
+ \left(\mathbb{D}^T_1(\mathcal{A}_{11}\mathbb{B}_{11} + \mathcal{A}_{12}\mathbb{B}_{21}) + \lambda\mathbb{G}_1^T(\mathcal{A}_{21}\mathbb{B}_{11} + \mathcal{A}_{22}\mathbb{B}_{21})\right)\mathcal{X}(s-)
\]

\[
= -R_1^{-1}(s)\mathcal{H}_1(s)\mathcal{X}(s-),
\]

provided that $R_1$ is invertible, where (s is suppressed)

\[
R_1 := R_1 + \left(\mathbb{D}^T_1(\mathcal{A}_{11}\mathbb{B}_{12} + \mathcal{A}_{12}\mathbb{B}_{22}) + \lambda\mathbb{G}_1^T(\mathcal{A}_{21}\mathbb{B}_{12} + \mathcal{A}_{22}\mathbb{B}_{22})\right).
\]

By substituting (3.21) into (3.19), we have (s is suppressed)

\[
\begin{bmatrix}
\mathcal{Z} \\
\mathcal{K}
\end{bmatrix} = \begin{bmatrix}
\mathcal{F}_{11} - \mathcal{A}_{12}\mathcal{R}_{11}^{-1}\mathcal{H}_1 \\
\mathcal{F}_{21} - \mathcal{A}_{22}\mathcal{R}_{11}^{-1}\mathcal{H}_1
\end{bmatrix} \mathcal{X}(s-),
\]

where

\[
\begin{aligned}
\mathcal{F}_{11} & := \mathcal{A}_{11}\mathbb{B}_{11} + \mathcal{A}_{12}\mathbb{B}_{21}, \\
\mathcal{F}_{12} & := \mathcal{A}_{11}\mathbb{B}_{12} + \mathcal{A}_{12}\mathbb{B}_{22}, \\
\mathcal{F}_{21} & := \mathcal{A}_{21}\mathbb{B}_{11} + \mathcal{A}_{22}\mathbb{B}_{21}, \\
\mathcal{F}_{22} & := \mathcal{A}_{21}\mathbb{B}_{12} + \mathcal{A}_{22}\mathbb{B}_{22}.
\end{aligned}
\]

We substitute (3.23) and (3.21) into (3.16). Then combining (3.14) with the above invertibility conditions (see (3.19) and (3.22)) and using the notation in (3.10), (3.17), (3.20) and (3.24), the ISRDE in (3.14) can be written as

\[
\begin{aligned}
\mathrm{d}\mathcal{P}(s) &= -\left[\mathcal{A}^T\mathcal{P}(s-) + \mathcal{P}(s-}\mathcal{A} + \mathcal{Q} + \mathcal{P}(s-}\mathbb{B}\mathcal{P}(s-)
\right.
\]

\[
+ \left(\mathcal{C}^T\mathcal{C} + \mathcal{Y}^T\mathcal{Y}^T\mathcal{P}(s-) + \lambda\mathcal{Q}^T\mathcal{K}^T\mathcal{P}(s-)
\right.
\]

\[
+ \left(\mathcal{C}^T + \mathcal{P}(s-)\mathcal{H}_1 + \mathcal{Y}^T\mathcal{H}_1 + \lambda\mathcal{Q}^T\mathcal{K}_1^T\mathcal{H}_1\right)(\mathcal{F}_{11} - \mathcal{A}_{12}\mathcal{R}_{11}^{-1}\mathcal{H}_1)
\]

\[
+ \left(\mathcal{C}^T + \mathcal{P}(s-)\mathcal{K}_1 + \mathcal{Y}^T\mathcal{K}_1 + \lambda\mathcal{Q}^T\mathcal{K}_1^T\mathcal{K}_1\right)(\mathcal{F}_{21} - \mathcal{A}_{22}\mathcal{R}_{11}^{-1}\mathcal{H}_1)
\]

\[
+ \left(-\mathcal{H}_1^T + \lambda\mathcal{K}_1^T + \mathcal{P}(s-)\mathcal{B} + \mathcal{Y}^T\mathcal{D}_1 + \lambda\mathcal{Q}^T\mathcal{G}_1\mathcal{R}_{11}^{-1}\mathcal{H}_1\right)\mathrm{d}s
\]

\[
\left. + \mathcal{Y}(s)\mathrm{d}B(s) + \Theta(s)\mathrm{d}\tilde{N}(s), \ s \in [t, T]\right]
\]

\[
\mathcal{P}(T) = \mathbb{M}_1,
\]

\[
\det(\mathcal{A}_{11}(s)) \neq 0, \ \forall s \in [t, T]
\]

\[
\det(\mathcal{A}_{22}(s) - \mathcal{A}_{21}(s)\mathcal{A}_{11}(s)^{-1}\mathcal{A}_{12}(s)) \neq 0, \ \forall s \in [t, T]
\]

\[
\det(\mathcal{R}_1(s)) \neq 0, \ \forall s \in [t, T].
\]

---

4. Under Assumption 3, (3.12) becomes $R_1(s)\mathbf{\pi}(s) + \mathbb{E}_1(s)^T\mathcal{Y}(s-) + \mathbb{D}_1(s)^T\mathcal{Z}(s) + \lambda\mathbb{G}_1(s)^T\mathcal{K}(s) + \mathbb{H}_1(s)\mathcal{X}(s-) + \lambda\mathbb{K}_1(s)\mathcal{X}(s-) = 0$.

5. Note that the block matrix $\mathcal{A}$ defined in (3.18) is invertible if $\mathcal{A}_{11}$ and $\mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12}$ are invertible [8, page 18].
Finally, we substitute \((3.21), (3.13)\) and \((3.23)\) into \(X\) in \((3.11)\). Then

\[
\begin{align*}
\dot{X}(s) &= \hat{A}(s)X(s^-)ds + \hat{C}(s)X(s^-)dB(s) + \hat{F}(s)X(s^-)d\hat{N}(s), \quad s \in (t, T] \\
X(t) &= \mathcal{Y},
\end{align*}
\]

where \((s \text{ is suppressed})\)

\[
\begin{align*}
\hat{A} := A + B_2P(s^-) + \hat{H}(\mathcal{F}_{11} - \mathcal{F}_{22}R_1^{-1}H_1) \\
+ \lambda \hat{K}(\mathcal{F}_{21} - \mathcal{F}_{22}R_1^{-1}H_1) - B_1R_1^{-1}H_1, \\
\hat{C} := C + \hat{H}^\top P(s^-) + \hat{H}(\mathcal{F}_{11} - \mathcal{F}_{22}R_1^{-1}H_1) \\
+ \lambda \hat{K}(\mathcal{F}_{21} - \mathcal{F}_{22}R_1^{-1}H_1) - D_1R_1^{-1}H_1, \\
\hat{F} := F + \hat{K}^\top P(s^-) + \hat{K}^\top(\mathcal{F}_{11} - \mathcal{F}_{22}R_1^{-1}H_1) \\
+ \lambda \hat{K}(\mathcal{F}_{21} - \mathcal{F}_{22}R_1^{-1}H_1) - G_1R_1^{-1}H_1.
\end{align*}
\]

In summary, we have the following result:

**Theorem 3.2.** Suppose that Assumptions 1-3 hold. Assume that \((\mathcal{P}, \Psi, \Theta) \in \mathcal{C}^2_\mathcal{F}(t, T; \mathbb{R}^{2n \times 2n}) \times \mathcal{L}^2_\mathcal{F}(t, T; \mathbb{R}^{2n \times 2n}) \times \mathcal{L}^2_\mathcal{F}_{22}(t, T; \mathbb{R}^{2n \times 2n})\) is the solution of the ISRDE in \((3.25)\), and \(X\) is the solution of \((3.26)\). Define the transformations in \((3.13)\) and \((3.23)\), and consider the control in \((3.21)\). Then \((3.11)\) and \((3.12)\) hold. In addition, suppose that \((3.3)\) holds. Then the state-feedback type control in \((3.21)\) is the optimal control for \((LQ-L)\), and the associated optimal cost is given by

\[(3.27) \quad J_1(a; \overline{u}_1, \overline{u}_2) = \inf_{u_1 \in U_1} J_1(a; u_1, \overline{u}_2) = \langle a, P_{11}(t)a \rangle.\]

**Proof.** The statement that \((3.11)-(3.12)\) are equivalent to \((3.13), (3.23)\) and \((3.21)\) follows from the preceding analysis. Furthermore, Lemma 3.1 implies that under \((3.3)\), the state-feedback type control in \((3.21)\) is the optimal control for \((LQ-L)\).

We now prove \((3.27)\). By applying Itô’s formula to \((3.11)\) (see \((3.8)\)),

\[
J_1(a; \overline{u}_1, \overline{u}_2) = \mathbb{E} \left[ \langle X(t), Y(t) \rangle + \int_t^T \left\langle \overline{u}_1(s), R_1(s)\overline{u}_1(s) + B_1(s)^\top Y(s^-) \right\rangle ds \right] \]

\[
+ D_1(s)^\top Z(s) + \lambda G_1(s)^\top K(s) + H_1(s)Y(s^-) + \lambda \kappa_1(s)Y(s^-) \right\rangle ds \right] = \langle a, P_{11}(t)a \rangle,
\]

where the second equality follows from \((3.13)\), the first-order optimality condition in \((3.12)\), and the initial condition \(\mathcal{Y}\). This completes the proof of the theorem. \(\square\)

Under Assumptions 1-3, and using \((2.14)\) and \((3.21)\), we consider

\[
\begin{align*}
\overline{u}_1(s) &= -R_1(s)^{-1}H_1(s) \left[ x(s^-) \beta(s^-) \right] \\
\overline{u}_2(s) &= -\hat{R}_2(s)^{-1}\hat{S}_2(s)^\top x(s^-) - \hat{R}_2(s)^{-1}(B_2(s)^\top \phi(s^-) \\
+ D_2(s)^\top \theta(s) + \lambda G_2(s)^\top \psi(s) - \hat{S}_1(s)R_1(s)^{-1}H_1(s) \left[ x(s^-) \beta(s^-) \right].
\end{align*}
\]

Note that \(\overline{u}_2\) in \((3.28)\) is the state-feedback type optimal control of the follower when \(u_1 \equiv \overline{u}_1\). This corresponds to the situation when the leader announces \(\overline{u}_1\) to the follower in the Stackelberg game.
Corollary 3.3. Suppose that the assumptions of Theorems 2.1 and 3.2 hold. Then \((\pi_1, \pi_2) \in U_1 \times U_2\) in (3.28) constitutes the state-feedback representation of the open-loop Stackelberg equilibrium for the leader and the follower.

3.2. Case II: The jump part in (1.1) does not depend on \(u_2\). We assume that the control of the follower, \(u_2\), is not included in the jump part of (1.1), i.e.,

Assumption 4. \(G_2 = 0\).

Remark 4. Assumption 4 implies that \(\tilde{K}_2 = \bar{K}_2 = \underline{K} = \bar{K} = 0\)

The jump part in \((3.29)\) becomes

\[
\begin{align*}
\frac{d\mathcal{Y}}{ds} &= -\left[\mathcal{A}(s)^\top \mathcal{P}(s-)\mathcal{X}(s-) + \mathcal{Q}(s)\mathcal{X}(s-) + \mathcal{C}(s)^\top \mathcal{Z}(s) + \mathcal{H}_1(s)^\top \mathcal{\pi}_1(s)\right] \\
&\quad \quad + \int_{E} \mathcal{F}(s,e)^\top \mathcal{K}(s,e)\lambda(de) + \int_{E} \mathcal{K}_1(s,e)^\top \mathcal{\pi}_1(s)\lambda(de) \, ds \\
&\quad \quad + \mathcal{Z}(s)dB(s) + \int_{E} \mathcal{K}(s,e)\tilde{N}(de, ds) \\
&= \left[\mathcal{A}(s)ds + \mathcal{\Psi}(s)dB(s) + \int_{E} \Theta(s,e)\tilde{N}(de, ds)\right]\mathcal{X}(s-) \\
&\quad \quad + \mathcal{P}(s-)\left[\mathcal{A}(s)\mathcal{X}(s-) + \mathcal{B}_2(s)\mathcal{P}(s-)\mathcal{X}(s-) + \mathcal{H}(s)\mathcal{Z}(s) + \mathcal{H}_1(s)\mathcal{\pi}_1(s)\right] \, ds \\
&\quad \quad + \mathcal{P}(s-)\left[\mathcal{C}(s)\mathcal{X}(s-) + \mathcal{H}(s)^\top \mathcal{P}(s-)\mathcal{X}(s-) + \mathcal{H}(s)\mathcal{Z}(s) + \mathcal{D}_1(s)\mathcal{\pi}_1(s)\right] \, dB(s) \\
&\quad \quad + \mathcal{\Psi}(s)^\top \mathcal{C}(s)\mathcal{X}(s-) + \mathcal{H}(s)^\top \mathcal{P}(s-)\mathcal{X}(s-) + \mathcal{H}(s)\mathcal{Z}(s) + \mathcal{D}_1(s)\mathcal{\pi}_1(s) \, ds \\
&\quad \quad + \int_{E} \Theta(s,e)^\top \left[\mathcal{F}(s,e)\mathcal{X}(s-) + \mathcal{G}_1(s)\mathcal{\pi}_1(s)\right] \lambda(de) \, ds \\
&\quad \quad + \int_{E} \left(\mathcal{P}(s-) + \Theta(s,e)\right)^\top \left[\mathcal{F}(s,e)\mathcal{X}(s-) + \mathcal{G}_1(s)\mathcal{\pi}_1(s)\right] \tilde{N}(de, ds).
\end{align*}
\]

With the invertibility of \((I - \mathcal{P}(s-)\mathcal{H})\), \((s \text{ is suppressed})\)

\[
\begin{align*}
\mathcal{Z}(s) &= (I - \mathcal{P}(s-)\mathcal{H})^{-1}\left((\mathcal{P}(s-)(\mathcal{C} + \mathcal{P}(s-)\mathcal{H})\mathcal{P}(s-) + \mathcal{\Psi})\mathcal{X} + \mathcal{P}(s-)\mathcal{D}_1\mathcal{\pi}_1\right) \\
\mathcal{K}(s,e) &= (\Theta(s,e) + (\mathcal{P}(s-) + \Theta(s,e))\mathcal{F})\mathcal{X} + (\mathcal{P}(s-) + \Theta(s,e))\mathcal{G}_1\mathcal{\pi}_1.
\end{align*}
\]

By substituting (3.30) into (3.12), we have

\[
\mathcal{\pi}_1(s) = -\mathcal{\tilde{R}}_1(s)^{-1}\left(\mathcal{B}_1^\top \mathcal{P}(s-) + \mathcal{H}_1 + \int_{E} \mathcal{K}_1(s,e)\lambda(de)\right) \\
&\quad \quad + \mathcal{D}_1^\top (I - \mathcal{P}(s-)\mathcal{H})^{-1}(\mathcal{P}(s-)\mathcal{C} + \mathcal{P}(s-)\mathcal{H}\mathcal{P}(s-) + \mathcal{\Psi}) \\
&\quad \quad + \int_{E} \mathcal{G}_1(s,e)^\top (\Theta(s,e) + (\mathcal{P}(s-) + \Theta(s,e))\mathcal{F}(s,e))\lambda(de))\mathcal{X}(s-) \\
&= -\mathcal{\tilde{R}}_1(s)^{-1}\mathcal{\tilde{B}}_1(s)\mathcal{X}(s-),
\]
provided that \( \hat{R}_1 \) is invertible, where \((s \text{ is suppressed})\)

\[
\hat{R}_1 := R_1 + D_1^T(I - P(s)\tilde{H})^{-1}P(s)D_1 + \int_{E} G_1(s,e)^T(P(s) + \Theta(s,e))G_1(s,e)\lambda(de).
\]

Then substituting (3.31) into (3.30) yields

\[
Z(s) = \tilde{F}_1(s)X(s) -, \quad \mathcal{K}(s,e) = \tilde{F}_2(s,e)X(s),
\]

where \((s \text{ is suppressed})\)

\[
\begin{align*}
\tilde{F}_1 &:= (I - P(s)\tilde{H})^{-1}((P(s)C + P(s)\tilde{H}P(s) + \Psi) - P(s)D_1\hat{R}_1^{-1}\hat{B}_1) \quad \text{(3.34)} \\
\tilde{F}_2 &:= (\Theta(s,e) + (P(s) + \Theta(s,e))F - (P(s) + \Theta(s,e))G_1\hat{R}_1^{-1}\hat{B}_1) \quad \text{into (3.32)}
\end{align*}
\]

We substitute (3.33) and (3.31) into \( (3.29) \). Then, together with the invertibility conditions in (3.33) and (3.32) and the notation in (3.10) and (3.34), the ISRDE in (3.14) has to be as follows \((s \text{ is suppressed})\):

\[
\begin{align*}
dP(s) &= - \begin{bmatrix} \hat{A}^T P(s) + P(s)\hat{A} + Q + P(s)\mathcal{B}_2 P(s) \\
+ \Psi^T C + \Psi\tilde{H}^T P(s) + \int_{E} \Theta(s,e)^T F(s,e)\lambda(de) \\
+ (C^T + P(s)\tilde{H} + \Psi\tilde{H})\tilde{F}_1 + \int_{E} F(s,e)^T \tilde{F}_2(s,e)\lambda(de) \\
- \left( \tilde{H}^T + \int_{E} K_1(s,e)\lambda(de) + P(s)D_1 \right) \\
+ \Psi T D_1 + \int_{E} \Theta(s,e)^T \lambda(de)G_1\hat{R}_1^{-1}\hat{B}_1 \end{bmatrix} ds \\
&= \int_{E} \hat{A}(s,e)X(s)\lambda(de,ds), \quad s \in [t,T] \\
&= \int_{E} \Theta(s,e)\lambda(de,ds), \quad s \in [t,T] \\
&\text{det}(I - P(s)\tilde{H}(s)) \neq 0, \quad \forall s \in [t,T] \\
&\text{det}(\hat{R}_1(s)) \neq 0, \quad \forall s \in [t,T].
\end{align*}
\]

Applying (3.31), (3.13) and (3.33) to \( X \) in (3.11) yields

\[
\begin{align*}
dX(s) &= \hat{A}(s)X(s)ds + \hat{C}(s)X(s)dB(s) \\
&+ \int_{E} \hat{F}(s,e)X(s)\lambda(de,ds), \quad s \in [t,T] \\
X(t) &= \bar{X},
\end{align*}
\]

where \((s \text{ is suppressed})\)

\[
\begin{align*}
\hat{A} &:= \hat{A} + \mathcal{B}_2 P(s) + \tilde{H}\tilde{F}_1 - \mathcal{B}_1\hat{R}_1^{-1}\hat{B}_1 \\
\hat{C} &:= \hat{C} + \tilde{H}^T P(s) + \tilde{H}\tilde{F}_1 - D_1\hat{R}_1^{-1}\hat{H}_1, \quad \hat{F} := F - G_1\hat{R}_1^{-1}\hat{B}_1.
\end{align*}
\]

**Theorem 3.4.** Suppose that Assumptions 1, 2 and 4 hold. Let \((P, \Psi, \Theta) \in C^1_0(t, T; \mathbb{R}^{2n \times 2n}) \times L^2_0(t, T; \mathbb{R}^{2n \times 2n}) \times L^2_F(t, T; \mathbb{R}^{2n \times 2n})\) be the solution of the ISRDE in (3.35), and \( X \) the solution of (3.36). Define the transformations in (3.13) and (3.33), and consider the control in (3.31). Then (3.11) and (3.12) hold. In addition, suppose that (3.3) holds. Then the state-feedback type control in (3.31) is the optimal control for \( (LQ-L) \), and the associated optimal cost is given by

\[
J_1(a; \pi_1, \pi_2) = \inf_{u \in U_1} J_1(a; u_1, \pi_2) = \langle a, P_{11}(t)a \rangle.
\]
Using (2.14) and (3.31), we introduce

$$
\begin{align*}
\tau_1(s) &= -\tilde{R}_1(s)^{-1}\tilde{B}_1(s)\begin{bmatrix} x(s) \\ \beta(s) \end{bmatrix} \\
\tau_2(s) &= -\tilde{R}_2(s)^{-1}\tilde{S}_2(s)^\top x(s) - \tilde{R}_2(s)^{-1}\left( B_2(s)^\top \phi(s) \right) \\
&\quad + D_2(s)^\top \theta(s) - \tilde{S}_1(s)\tilde{R}_1(s)^{-1}\tilde{B}_1(s)\begin{bmatrix} x(s) \\ \beta(s) \end{bmatrix}.
\end{align*}
$$

(3.37)

**Corollary 3.5.** Suppose that the assumptions of Theorems 2.1 and 3.4 hold. Then (\(\tau_1, \tau_2\)) \(\in U_1 \times U_2\) in (3.37) constitutes the state-feedback representation of the open-loop Stackelberg equilibrium for the leader and the follower.

4. **Concluding Remarks.** We believe that unlike the SRDE in [22, (3.8)], the ISRDEs of the leader in (3.25) and (3.35) are not symmetric due to the nonsymmetric coupling nature of \(Z\) and \(K\) in (3.23) and (3.33). In fact, we can observe that \(R_1\) in (3.22) and \(\int E \xi(s, e)^\top \tilde{T}_2(s, e)\lambda(de)\) in (3.35) are not symmetric. Note that the ISRDE of the follower in (2.11) is symmetric, and its scalar version is similar to the ISRDE of the LQ control problem for jump-diffusion systems in [9].

Until now, it is hard to consider the general case (without Assumption 3 or Assumption 4) to obtain the state-feedback type control of (LQ-L). Specifically, without Assumption 3 or Assumption 4, it is necessary to use (3.15) to obtain the expression of \(Z\) and \(K\). Then from (3.15), the following holds:

$$
\begin{align*}
(I - \mathcal{P}(s-)^{\hat{H}})Z(s) &= \mathcal{P}(s-) \left[ C\mathcal{X} + \hat{H}^\top \mathcal{P}\mathcal{X} + D(s)p_1 \right] + \Psi \mathcal{X} \\
K(s, e) &= \mathcal{P}(s-) \left[ F\mathcal{X} + \hat{K}^\top \mathcal{P}(s-)\lambda + \Theta(s, e)Z(s) \right] \\
&= \mathcal{P}(s-) \left[ F\mathcal{X} + \hat{K}^\top \mathcal{P}(s-)\lambda + G(s, e)p_1 + \Theta(s, e)\mathcal{X} \right] + \Theta(s, e)\mathcal{X}.
\end{align*}
$$

(4.1)

Due to the integral terms \(\int E \hat{K}(s, e)\mathcal{K}(s, e)\lambda(de)\) and \(\int E \hat{K}(s, e, e')\mathcal{K}(s, e')\lambda(de')\), and the cross-coupling structure of \(Z\) and \(K\), there is a technical challenge to find the explicit expression of \(Z\) and \(K\) in (4.1). Note that in (4.1), Assumption 3 implies that \(\int E \hat{K}(s, e)\mathcal{K}(s, e)\lambda(de) = \lambda \hat{K}(s)\mathcal{K}(s)\) and \(\int E \hat{K}(s, e, e')\mathcal{K}(s, e')\lambda(de') = \lambda \hat{K}(s)\mathcal{K}(s)\). Moreover, we observe that Assumption 4 implies \(\int E \hat{K}(s, e)\mathcal{K}(s, e)\lambda(de) = 0\) and \(\int E \hat{K}(s, e, e')\mathcal{K}(s, e')\lambda(de') = 0\). Hence, in both cases, we are able to find the explicit expressions of \(Z\) and \(K\), which are given in (3.23) and (3.33).

When there are no jumps (1.1), i.e., \(F = G_1 = G_2 = 0\), we can easily verify that Theorems 2.1, 3.2 and 3.4 (and Corollaries 3.3 and 3.5) are reduced to the case of SDEs in a Brownian setting without jumps studied in [22, Theorems 2.3 and 3.3]. Moreover, when all the coefficients in (1.1)-(1.3) are deterministic, \(L = Z = \theta = \psi = 0\) in (2.11) and (2.12), and \(\Psi = \Theta = 0\) in (3.25) and (3.35). In this case, the ISRDEs for the leader and the follower become deterministic integro-Riccati differential equations.

There are two interesting potential future research problems of this paper. One is the mean-field type problem, in which case the expected values of \(x, u_1\) and \(u_2\), i.e., \(E[x(s)], E[u_1(s)]\) and \(E[u_2(s)]\), are included (1.1)-(1.3). This problem can be viewed as a generalization of [11] to the jump-diffusion model. Another potential problem to consider is the Markov regime-switching jump-diffusion system, for which an additional Markov jump parameter is included in (1.1)-(1.3). In this problem, we need to apply (and generalize) the stochastic maximum principle in [23].
REFERENCES

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge, 2nd ed., 2009.
[2] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, SIAM, 2nd ed., 1999.
[3] T. Başar and H. Selbuz, *Closed-loop Stackelberg strategies with applications in the optimal control of multilevel systems*, IEEE Transactions on Automatic Control, 24 (1979), pp. 166–179.
[4] T. Başar and R. Srikant, *A Stackelberg network game with a large number of followers*, Journal of Optimization Theory and Applications, 115 (2002), pp. 479–490.
[5] G. Barles, R. Buckdahn, and E. Pardoux, *Backward stochastic differential equations and integral-partial differential equations*, Stochastics and Stochastics Reports, 60 (1997), pp. 57–83.
[6] A. Bensoussan, S. Chen, and S. P. Sethi, *The maximum principle for global solutions of stochastic Stackelberg differential games*, SIAM Journal on Control and Optimization, 53 (2015), pp. 1956–1981.
[7] G. Freiling, G. Jank, and R. Lee, *Existence and uniqueness of open-loop Stackelberg equilibria in linear-quadratic differential games*, Journal of Optimization Theory and Applications, 110 (2001), pp. 515–544.
[8] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge, 2nd ed., 2013.
[9] Y. Hu and B. Øksendal, *Partial information linear quadratic control for jump diffusions*, SIAM Journal on Control and Optimization, 47 (2008), pp. 1744–1761.
[10] N. Li and Z. Yu, *Forward-backward stochastic differential equations and linear-quadratic generalized Stackelberg games*, SIAM Journal on Control and Optimization, 56 (2018), pp. 4148–4180.
[11] Y. Lin, X. Jiang, and W. Zhang, *An open-loop Stackelberg strategy for the linear quadratic mean-field stochastic differential game*, IEEE Transactions on Automatic Control, 64 (2019), pp. 97–110.
[12] J. Moon, *Backward reachability approach to state-constrained stochastic optimal control problems for jump diffusion systems*. https://arxiv.org/pdf/2006.05577.pdf, 2020.
[13] J. Moon and T. Başar, *Linear quadratic mean field Stackelberg differential games*, Automatica, 97 (2018), pp. 200–213.
[14] J. Moon and H. J. Yang, *Linear-quadratic time-inconsistent mean-field type Stackelberg differential games: Time-consistent open-loop solutions*, IEEE Transactions on Automatic Control, (2020). accepted (https://arxiv.org/pdf/1911.04110.pdf).
[15] B. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Springer, 2nd ed., 2006.
[16] G. P. Papavassilopoulos and J. B. Cruz, *Nonclassical control problems and Stackelberg games*, IEEE Transactions on Automatic Control, 24 (1979), pp. 155–166.
[17] N. Privault, *Notes on stochastic finance*. https://www.ntu.edu.sg/home/nprivault/index.html, 2020.
[18] J. Shi, G. Wang, and J. Xiong, *Leader-follower stochastic differential game with asymmetric information and applications*, Automatica, 63 (2016), pp. 60–73.
[19] S. Tang and X. Li, *Necessary conditions for optimal control of stochastic systems with random jumps*, SIAM Journal on Control and Optimization, 50 (2012), pp. 964–990.
[20] H. Von Stackelberg, *The Theory of Market Economy*, Oxford University Press, 1952.
[21] J. Xu, J. Shi, and H. Zhang, *A leader-follower stochastic linear quadratic differential game with time delay*, Science China, 61 (2018), pp. 1–13.
[22] J. Yong, *A leader-follower stochastic linear quadratic differential game*, SIAM Journal on Control and Optimization, 41 (2002), pp. 1015–1041.
[23] X. Zheng, R. J. Elliott, and T. K. Siu, *A stochastic maximum principle of a Markov regime-switching jump-diffusion model and its application to finance*, SIAM Journal on Control and Optimization, 50 (2012), pp. 964–990.
[24] Y. Zhong and J. Shi, *A Stackelberg game of backward stochastic differential equations with applications*, Dynamic Games and Applications, (2019). https://doi.org/10.1007/s13235-019-00341-z.