Supplementary Information to
*Fixed-effects inference and tests of correlation for longitudinal functional data*

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### S.1 ADDITIONAL SIMULATION ON INITIAL ESTIMATES OF FIXED EFFECTS AND ITERATIVE ESTIMATION

We have done a small simulation to evaluate the effects of initial estimate of fixed effects on the proposed model estimation method. We consider two initial estimates: (1) The ind_obs method, which was the one proposed in the paper and the initial estimate of fixed effects is obtained by fitting model equation (4) assuming all observations are independent; (2) The ind_func method, which assumes that all functions are independent and then obtain initial estimate of fixed effects by fitting model equation (4). For the latter, the marginal eigenfunctions \( \phi_k(\cdot) \) are obtained by the face function in which the mean function has no covariate. When the initial estimates are based on the ind_obs method, we also iterate between estimates of fixed effects and covariance estimation twice (denoted by "ind_obs & iterative") to evaluate if additional iterations will lead to improved estimate of fixed effects. Note that the proposed method only iterates twice and the refined and final estimate of fixed effects is obtained after fitting the covariance function. For the above, we consider unspecified covariance data with \( n = 100 \) and \( n_i \sim \text{Unif}[3, 6] \).

Table S1 displays the results for estimating coefficient functions. We see that the \( \sqrt{\text{MISE}} \) from the three methods are comparable, which shows that the final model estimation is not sensitive to changes in the initial estimate. In addition, additional iterations between fixed effects estimation and covariance estimation do not seem to improve fixed effects estimation.

| Initial          | \( \beta_0(s) \) \( \sqrt{\text{MISE}} \) | IAW | IAC | \( \beta_1(s) \) \( \sqrt{\text{MISE}} \) | IAW | IAC | \( \beta_2(s) \) \( \sqrt{\text{MISE}} \) | IAW | IAC |
|------------------|-------------------------------------|-----|-----|-------------------------------------|-----|-----|-------------------------------------|-----|-----|
| ind_func         | 0.078                               | 0.62| 1.00| 0.073                               | 0.31| 0.95| 0.049                               | 0.22| 0.95|
| ind_obs          | 0.078                               | 0.62| 1.00| 0.073                               | 0.31| 0.95| 0.049                               | 0.22| 0.95|
| ind_obs & iterative| 0.078                           | 0.62| 1.00| 0.072                               | 0.31| 0.95| 0.049                               | 0.22| 0.95|

### S.2 ESTIMATION ALGORITHM WITH INDEPENDENT AND EXCHANGEABLE LONGITUDINAL COVARIANCE

Algorithm S1 shows the estimation algorithm. For the working independence model, \( W_{ij}(s) = \sum_{k=1}^{K} \xi_{ijk} \phi_k(s) \), where \( \xi_{ijk} \) are uncorrelated across \( i \) and \( j \) for each \( k \) and \( \text{Var}(\xi_{ij}) = \sigma_{\xi k}^{2} \). With a diagonal covariance matrix \( \Sigma = \text{diag}(\sigma_{\xi 1}^{2}, \ldots, \sigma_{\xi K}^{2}) \) for \( \xi_{ij} \), the mixed model representation in equation (4) under independence is

\[
y_i = X_i\alpha + (I_{n_i} \otimes \Phi)\xi_i + \epsilon_i.
\]

For the exchangeable model, we have \( W_{ij}(s) = \sum_{k=1}^{K} (\xi_{ij} + \zeta_{ij}) \phi_k(s) \), where \( \xi_{ij} \) is uncorrelated with all \( \zeta_{ijk} \) and \( \zeta_{ijk} \) are mutually uncorrelated across \( i, j \) and \( k \). Let \( \xi_i = [\xi_{i1}, \ldots, \xi_{iK}]^{T} \). We have \( W_{ij} = \Phi(\xi_i + \zeta_{ij}) \), and \( \xi_i \) has a diagonal covariance matrix \( \Sigma_{0} = \text{diag}(\sigma_{\xi 0,1}^{2}, \ldots, \sigma_{\xi 0,K}^{2}) \). Then the mixed model representation in equation (4) under exchangeability is

\[
y_i = X_i\alpha + (I_{n_i} \otimes \Phi)(I_{n_i} \otimes \xi_i + \epsilon_i) + \epsilon_i.
\]

For the exchangeable model, we have \( W_{ij}(s) = \sum_{k=1}^{K} (\xi_{ij} + \zeta_{ij}) \phi_k(s) \), where \( \xi_{ij} \) is uncorrelated with all \( \zeta_{ijk} \) and \( \zeta_{ijk} \) are mutually uncorrelated across \( i, j \) and \( k \). Let \( \xi_i = [\xi_{i1}, \ldots, \xi_{iK}]^{T} \). We have \( W_{ij} = \Phi(\xi_i + \zeta_{ij}) \), and \( \xi_i \) has a diagonal covariance matrix \( \Sigma_{0} = \text{diag}(\sigma_{\xi 0,1}^{2}, \ldots, \sigma_{\xi 0,K}^{2}) \). Then the mixed model representation in equation (4) under exchangeability is

\[
y_i = X_i\alpha + (I_{n_i} \otimes \Phi)(I_{n_i} \otimes \xi_i + \epsilon_i) + \epsilon_i.
\]
### Algorithm S1: Estimate fixed effects with independent or exchangeable covariance

1. Estimate the initial fixed effects as $\mu_{ij}^0(s)$ by existed methods assuming that all observations $y_{ij}(s)$ are mutually independent, and subtract the initial mean functions from observed data to get residuals as $r_{ij}(s) = y_{ij}(s) - \mu_{ij}^0(s)$.
2. Apply the face algorithm to $r_{ij}(s)$ to extract eigenfunctions $\phi_k(s)$.
3. With covariates $x_{ijp}$ and estimated eigenfunctions $\phi_k(s)$, fit the mixed model using the *gam* or *bam* function to obtain the final fixed effects as $\mu_{ij}(s)$.

### S.3 | EXTRA PLOTS IN SIMULATION

The three plots in Figure S1 show the estimated coefficient functions when the data ($n = 100$ and $n_i \sim \text{Unif}[3, 6]$) are generated from independent, exchangeable, or unspecified covariances, respectively.

### S.4 | POSITIVE DEPENDENCY AMONG TEST STATISTICS

Based on 5000 datasets generated from the null hypothesis in simulation, we test the correlation between test statistics for layers $k = 1, 2$ with $H_0 : \rho = 0$ vs. $H_A : \rho > 0$. Table S2 presents values of correlations and corresponding p-values. It shows that the test statistics from different layers in the simulation are slightly positive dependent.

|                  | exchangeable  | independent |
|------------------|---------------|-------------|
| $n = 100$        | $n = 200$     | $n = 100$   | $n = 200$   |
| $n_i \sim \text{Unif}[3, 6]$ | 0.15 (0.00) | 0.15 (0.00) | 0.24 (0.00) | 0.12 (0.00) |
| $n_i \sim \text{Unif}[8, 12]$ | 0.20 (0.00) | 0.14 (0.00) | 0.05 (0.00) | 0.02 (0.13) |

### S.5 | EXTRA PLOTS OF NHANES

Figure S2 presents the estimated covariance functions within each day and cross days. Physical activity demonstrates large variation in the morning. Moreover, there is a negative correlation between morning activities and midnight activities. If individuals are energetic at midnight, they tend to have less activity in the morning.

The top three estimated eigenfunctions are shown in Figure S3 which explain 58.53% of the total variability. The first eigenfunction has large values during the late morning (8AM - 11AM), suggesting that study participants with positive scores on this component will have more activity in the morning. The second eigenfunction has a sharp peak during the early morning and is negative at night, suggesting that study participants with positive scores on this component will have intense activity during the early morning and less activity at night. Study participants with positive scores on the third component have more activity during working hours (9AM - 5PM) and less activity at the other time of a day.

### S.6 | ADDITIONAL SIMULATION

In this section we present the performance of inferential approaches for the longitudinal functional models via more complex simulation settings.
FIGURE S1 Estimated coefficient functions when the data are generated with \( n = 100 \) and \( \eta_i \sim \text{Unif}[3, 6] \). The columns from left to right display the estimates using the ind_obs method, the independent model, the exchangeable model and the unspecified model, respectively. For each panel, the solid red line represents the true coefficient function, the gray lines are the estimates from 500 replications, and the dashed black line represents the mean function of the 500 gray lines.

### S.6.1 Simulation setting

Data are generated from the model \( y_{ij}(s) = \mu_{ij}(s) + W_{ij}(s) + e_{ij}(s), s \in [0, 1] \). There are four covariates and the functional mean is \( \mu_{ij}(s) = \beta_0(s) + x_{ij1}\beta_1(s) + x_{ij2}\beta_2(s) + x_{ij3}\beta_3(s) + x_{ij4}\beta_4(s) \), where \( x_{ij1} \sim \text{iid } N(0, 1), x_{ij2} \sim \text{iid } N(0, 2), x_{ij3} \sim \text{iid } N(0, 1) \) and \( x_{ij4} = dT_{ij} + e_{ij} \), where \( e_{ij} \sim \text{N}(\rho e_{(i-1)j}, 1) \) with \( e_{i0} = 0 \). We let \( a = 1 \) and \( \rho = 0.7 \). The coefficient functions are \( \beta_0(s) = 1 + \sqrt{3}\cos(3\pi s), \beta_1(s) = \sqrt{5}(6s^2 - 6s + 1), \beta_2(s) = 2 + \sin(\pi s), \beta_3(s) = 2 + \cos(3\pi s) \) and \( \beta_4(s) = 1 + \sin(\pi s) + \cos(\pi s) \).
**FIGURE S2** Estimated covariance functions captured by the exchangeable covariance model. Left panel: covariance within each day. Right panel: covariance across days.

**FIGURE S3** Estimated eigenfunctions of the physical activity data among NHANES population. The proportion of variability explained in each principal component is shown on the title of each panel.

Using the orthonormal functions
\[
\phi_1(s) = \sqrt{2} \sin(2\pi s), \quad \phi_2(s) = \sqrt{2} \cos(2\pi s), \quad \phi_3(s) = \sqrt{2} \sin(4\pi s) \quad \text{and} \quad \phi_4(s) = \sqrt{2} \cos(4\pi s),
\]

\(W_{ij}(\cdot)\) is generated from one of the three models below:

1. **Independent covariance**: 
   \[
   W_{ij}(s) = \sum_{k=1}^{4} \xi_{ik} \phi_k(s).
   \]

2. **Exchangeable covariance**: 
   \[
   W_{ij}(s) = \sum_{k=1}^{4} (\xi_{ik} + \xi_{ik}) \phi_k(s).
   \]

3. **Unspecified covariance**: 
   \[
   W_{ij}(s) = \sum_{k=1}^{4} \left\{ \sum_{l=1}^{2} \eta_{ik} \psi_{k} (T_{ij}) + \xi_{ijk} \right\} \phi_k(s).
   \]

For the independent covariance,
\[
\xi_{ij1} \sim N(0, 6) \quad \xi_{ij2} \sim N(0, 4) \quad \xi_{ij3} \sim N(0, 3) \quad \xi_{ij4} \sim N(0, 2)
\]

For the exchangeable covariance,
\[
\xi_{ij1} \sim N(0, 4) \quad \xi_{ij2} \sim N(0, 2.5) \quad \xi_{ij3} \sim N(0, 2) \quad \xi_{ij4} \sim N(0, 1.5)
\]
\[
\zeta_{ij1} \sim N(0, 2) \quad \zeta_{ij2} \sim N(0, 1.5) \quad \zeta_{ij3} \sim N(0, 1) \quad \zeta_{ij4} \sim N(0, 0.5)
\]
The eigenfunctions for the unspecified covariance are

\[
\begin{align*}
\psi_{11}(T) &= \sqrt{2} \cos(4\pi T) \\
\psi_{12}(T) &= \sqrt{2} \sin(4\pi T) \\
\psi_{21}(T) &= \sqrt{2} \cos(4\pi T) \\
\psi_{22}(T) &= \sqrt{2} \sin(4\pi T) \\
\psi_{31}(T) &= \sqrt{2} \cos(2\pi T) \\
\psi_{32}(T) &= \sqrt{2} \sin(2\pi T) \\
\psi_{41}(T) &= \sqrt{2} \cos(2\pi T) \\
\psi_{42}(T) &= \sqrt{2} \sin(2\pi T)
\end{align*}
\]

We generate the random terms as

\[
\begin{align*}
\eta_{11} &\sim N(0, 3) & \eta_{21} &\sim N(0, 2) & \eta_{31} &\sim N(0, 1.5) & \eta_{41} &\sim N(0, 1) \\
\eta_{12} &\sim N(0, 2) & \eta_{22} &\sim N(0, 1.5) & \eta_{32} &\sim N(0, 1) & \eta_{42} &\sim N(0, 0.5) \\
\zeta_{ij1} &\sim N(0, 0.1) & \zeta_{ij2} &\sim N(0, 0.5) & \zeta_{ij3} &\sim N(0, 0.5) & \zeta_{ij4} &\sim N(0, 0.5)
\end{align*}
\]

The functional arguments \{s_1, \ldots, s_n\} form a grid of 101 equidistant points in the unit interval. The mutually independent random errors \(e_{ij}(s)\) are generated from \(N(0, 3)\).

For simulation, we used the same factorial design in Section 4.1: (a) the number of subjects is either 100 or 200; (b) the number of visits \(n_i\) are generated from either \(\text{Unif}[3, \ldots, 6]\) or \(\text{Unif}[8, \ldots, 12]\); and (c) the covariance model is either independent, exchangeable, or unspecified. There are 12 model conditions and we simulate 100 datasets for each condition.

For tests of longitudinal covariances, data are generated similarly. However, under the alternative we add terms to ensure that the null model is misspecified. Specifically, we consider

1. Deviation from independent covariance: \(\xi_{ijk} \rightarrow \xi_{ijk} + \Delta z_{ik}(T_{ij})\).
2. Deviation from exchangeable covariance: \(\xi_{ijk} + \zeta_{ijk} \rightarrow \xi_{ijk} + \zeta_{ijk} + \Delta z_{ik}(T_{ij})\).

The scalar \(\Delta\) controls the magnitude of the deviation from the null model. The scores \(\xi_{ijk}\) and \(\zeta_{ijk}\) are independent from the non-linear random function \(z_{ik}(\cdot)\), where \(z_{ik}(T) = \sum_{\ell=1}^{2n} \eta_{ik\ell} \psi_{k\ell}(T)\). The terms \(\eta_{ik\ell}\) and \(\psi_{k\ell}(T)\) are specified as before.

### S.6.2 Simulation results for estimation

The values for parameters, such as PVE and the number of knots, are the same as those in Section 4.2. The same evaluation criteria are used to assess estimation accuracy. Tables S4 and S5 show the simulation results for estimating the coefficient functions when the data generating mechanism uses independent, exchangeable, or unspecified covariances, respectively. Conclusions for MISE and confidence bands are consistent with those in Section 4.3. Compared to the simple setting in Section 4, the improvement of estimation accuracy for the correct model becomes more pronounced under a complex setting.

Figure S4 and S5 display the estimated coefficient functions when the data \((n = 100\text{ and }n_i \sim \text{Unif}[3, 6])\) are generated from independent, exchangeable, or unspecified covariances, respectively. The columns from left to right display the estimates using the ind_obs method, the independent model, the exchangeable model and the unspecified model, respectively. Within each panel, solid red curves denote the true coefficient functions, and dashed black curves represent the mean function of estimates from all replications. The dashed black curves align well with the true coefficient functions, indicating that the point estimator is unbiased. For data with exchangeable and unspecified covariance structures, we observe larger variability from the ind_obs method than the estimates from the correct model. Such higher accuracy of the correct model is also shown in Table S4 and S5.

For example, in Table S5 for data with \(n = 100\text{ and }n_i \sim \text{Unif}[8, 12]\), the \(\sqrt{\text{MISE}(\hat{\beta}_s(s))}\) is 0.042 for the method which accounted for unspecified correlation, while it is 0.088 for the ind_obs method. Moreover, the unspecified method gives a much smaller IAW with a value of 0.16 than that from the bootstrap method with a value of 0.38, while their similar coverage rates are similar.

### S.6.3 Simulation results for longitudinal covariance tests

The empirical type I error rate (estimated size) for nominal levels \(\alpha = 0.05\) and 0.10 based on 5000 simulated datasets is reported in Table S6. For the joint test, we use the Bonferroni correction to account for the multiplicity of hypotheses tests. Since tests can be slightly conservative for exchangeable covariance with a larger number of visits per subject, we also calculated the size under Benjamini-Hochberg (BH) correction. However, it doesn’t outperform the Bonferroni correction. Though Benjamini-Hochberg provides less stringent control of type I errors compared to the Bonferroni correction, it should not make much difference when there are few tests. In the article, we do not consider many layers of \(\phi_s(s)\). As only a small number of multiple comparisons, the Bonferroni correction is recommended.

Figure S7 presents the power curves at the \(\alpha = 0.05\) level with 1000 simulated datasets. In all settings, the proposed tests maintain proper size and have the power to detect alternative longitudinal covariance both when the null hypothesis assumes independent covariance and exchangeable covariance.
TABLE S3 \(\sqrt{MISE}\), IAC for 95% pointwise confidence interval and IAW for coefficient functions based on independent covariance data using five methods across 100 simulations. Time (standard error) with second as a unit is the run time per simulation averaged by 100 replications.

| Method         | \(n = 100\) & \(n_i \sim \text{Unif}[3, 6]\) | \(n = 100\) & \(n_i \sim \text{Unif}[8, 12]\) | \(n = 100\) & \(n_i \sim \text{Unif}[3, 6]\) | \(n = 200\) & \(n_i \sim \text{Unif}[8, 12]\) |
|----------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| \(\beta_{i(s)}\) | \(\sqrt{MISE}\) | IAC | \(\sqrt{MISE}\) | IAC | \(\sqrt{MISE}\) | IAC | \(\sqrt{MISE}\) | IAC | \(\sqrt{MISE}\) | IAC | \(\sqrt{MISE}\) | IAC |
| **Ind_obs**    | 0 (0.0) | 0.184 | 0.32 | 0.57 | 0.178 | 0.21 | 0.40 | 0.123 | 0.14 | 0.40 | 0.183 | 0.22 | 0.42 | 0.119 | 0.14 | 0.41 |
| **Bootstrap**  | 118 (0.4) | 0.184 | 0.77 | 0.95 | 0.178 | 0.71 | 0.94 | 0.123 | 0.51 | 0.94 | 0.183 | 0.72 | 0.93 | 0.119 | 0.50 | 0.95 |
| **Independent** | 22 (0.3) | 0.175 | 0.98 | 0.96 | 0.139 | 0.64 | 0.98 | 0.081 | 0.38 | 0.97 | 0.184 | 0.69 | 0.93 | 0.097 | 0.38 | 0.94 |
| **Exchangeable** | 60 (1.0) | 0.175 | 1.00 | 0.96 | 0.138 | 0.65 | 0.98 | 0.080 | 0.38 | 0.97 | 0.185 | 0.69 | 0.92 | 0.097 | 0.38 | 0.94 |
| **Unspecified** | 32 (0.3) | 0.175 | 0.98 | 0.96 | 0.139 | 0.64 | 0.98 | 0.081 | 0.38 | 0.97 | 0.184 | 0.69 | 0.93 | 0.097 | 0.38 | 0.94 |
| **Ind_obs**    | 1 (0.0) | 0.124 | 0.22 | 0.58 | 0.116 | 0.15 | 0.45 | 0.077 | 0.09 | 0.45 | 0.126 | 0.15 | 0.42 | 0.075 | 0.10 | 0.45 |
| **Bootstrap**  | 239 (1.2) | 0.124 | 0.52 | 0.94 | 0.116 | 0.49 | 0.94 | 0.077 | 0.34 | 0.95 | 0.126 | 0.48 | 0.92 | 0.075 | 0.33 | 0.95 |
| **Independent** | 226 (1.8) | 0.120 | 0.66 | 0.96 | 0.098 | 0.45 | 0.96 | 0.058 | 0.27 | 0.96 | 0.128 | 0.47 | 0.92 | 0.063 | 0.28 | 0.95 |
| **Exchangeable** | 444 (6.0) | 0.120 | 0.68 | 0.96 | 0.097 | 0.46 | 0.97 | 0.057 | 0.27 | 0.97 | 0.128 | 0.47 | 0.92 | 0.063 | 0.28 | 0.95 |
| **Unspecified** | 236 (1.8) | 0.120 | 0.66 | 0.96 | 0.098 | 0.45 | 0.96 | 0.058 | 0.27 | 0.96 | 0.128 | 0.47 | 0.92 | 0.063 | 0.28 | 0.95 |
| **Ind_obs**    | 1 (0.0) | 0.137 | 0.23 | 0.56 | 0.123 | 0.15 | 0.41 | 0.079 | 0.10 | 0.44 | 0.126 | 0.16 | 0.42 | 0.079 | 0.10 | 0.44 |
| **Bootstrap**  | 220 (1.1) | 0.137 | 0.54 | 0.93 | 0.123 | 0.50 | 0.94 | 0.079 | 0.35 | 0.96 | 0.126 | 0.51 | 0.94 | 0.079 | 0.35 | 0.94 |
| **Independent** | 167 (1.3) | 0.134 | 0.69 | 0.95 | 0.102 | 0.46 | 0.96 | 0.056 | 0.28 | 0.98 | 0.128 | 0.50 | 0.94 | 0.068 | 0.29 | 0.94 |
| **Exchangeable** | 457 (6.4) | 0.134 | 0.71 | 0.95 | 0.102 | 0.47 | 0.97 | 0.056 | 0.28 | 0.98 | 0.128 | 0.50 | 0.94 | 0.068 | 0.29 | 0.94 |
| **Unspecified** | 186 (1.3) | 0.134 | 0.69 | 0.95 | 0.102 | 0.46 | 0.96 | 0.056 | 0.28 | 0.98 | 0.128 | 0.50 | 0.94 | 0.068 | 0.29 | 0.94 |
| **Ind_obs**    | 2 (0.0) | 0.085 | 0.16 | 0.62 | 0.080 | 0.10 | 0.47 | 0.059 | 0.07 | 0.40 | 0.084 | 0.11 | 0.49 | 0.060 | 0.07 | 0.39 |
| **Bootstrap**  | 464 (2.0) | 0.085 | 0.36 | 0.95 | 0.080 | 0.34 | 0.94 | 0.059 | 0.24 | 0.94 | 0.084 | 0.34 | 0.93 | 0.060 | 0.23 | 0.94 |
| **Independent** | 1797 (10.8) | 0.086 | 0.47 | 0.95 | 0.072 | 0.32 | 0.96 | 0.044 | 0.20 | 0.97 | 0.082 | 0.34 | 0.94 | 0.055 | 0.21 | 0.92 |
| **Exchangeable** | 3333 (50.5) | 0.086 | 0.48 | 0.95 | 0.072 | 0.33 | 0.96 | 0.044 | 0.21 | 0.97 | 0.081 | 0.34 | 0.94 | 0.055 | 0.21 | 0.92 |
| **Unspecified** | 1822 (10.6) | 0.086 | 0.47 | 0.95 | 0.072 | 0.32 | 0.96 | 0.044 | 0.20 | 0.97 | 0.082 | 0.34 | 0.94 | 0.055 | 0.21 | 0.92 |

FIGURE S4 Estimated coefficient functions when the data are generated from independent covariance with \(n = 100\) and \(n_i \sim \text{Unif}[3, 6]\). The columns from left to right display the estimates using the ind_obs method, the independent model, the exchangeable model and the unspecified model, respectively. For each panel, the solid red line represents the true coefficient function, the gray lines are the estimates from 100 replications, and the dashed black line represents the mean function of the 100 gray lines.
**TABLE S4** √MISE, IAC for 95% pointwise confidence interval and IAW for coefficient functions based on exchangeable covariance data using five methods across 100 simulations. Time (standard error) with second as a unit is the run time per simulation averaged by 100 replications.

| Method             | Time | √MISE | IAC | √MISE | IAC | √MISE | IAC | √MISE | IAC | √MISE | IAC | √MISE | IAC | √MISE | IAC |
|--------------------|------|-------|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|-----|
| ind_obs            | 0.00 | 0.339 | 0.35 | 0.311 | 0.24 | 0.213 | 0.15 | 0.170 | 0.22 | 0.176 | 0.14 | 0.176 | 0.14 | 0.29  |
| bootstrap          | 0.00 | 0.339 | 0.35 | 0.311 | 0.24 | 0.213 | 0.15 | 0.170 | 0.22 | 0.176 | 0.14 | 0.176 | 0.14 | 0.29  |
| independent        | 0.00 | 0.339 | 0.35 | 0.311 | 0.24 | 0.213 | 0.15 | 0.170 | 0.22 | 0.176 | 0.14 | 0.176 | 0.14 | 0.29  |
| exchangeable       | 0.00 | 0.339 | 0.35 | 0.311 | 0.24 | 0.213 | 0.15 | 0.170 | 0.22 | 0.176 | 0.14 | 0.176 | 0.14 | 0.29  |
| unspecified        | 0.00 | 0.339 | 0.35 | 0.311 | 0.24 | 0.213 | 0.15 | 0.170 | 0.22 | 0.176 | 0.14 | 0.176 | 0.14 | 0.29  |
| n = 100 & n_i ~ Unif[3, 6] |      |       |     |       |     |       |     |       |     |       |     |       |     |       |     |
| n = 100 & n_i ~ Unif[8, 12] |     |       |     |       |     |       |     |       |     |       |     |       |     |       |     |

**FIGURE S5** Estimated coefficient functions when the data are generated from exchangeable covariance with n = 100 and n_i ~ Unif[3, 6]. For each panel, the solid red line represents the true coefficient function, the gray lines are the estimates from 100 replications, and the dashed black line represents the mean function of the 100 gray lines.
TABLE S5 √MISE, IAC for 95% pointwise confidence interval and IAW for coefficient functions based on unspecified covariance data using five methods across 100 simulations. Time (standard error) with second as a unit is the run time per simulation averaged by 100 replications

| Method       | Time  | √MISE | IAW | IAC | √MISE | IAW | IAC | √MISE | IAW | IAC | √MISE | IAW | IAC |
|--------------|-------|-------|-----|-----|-------|-----|-----|-------|-----|-----|-------|-----|-----|
|              |       |       |     |     |       |     |     |       |     |     |       |     |     |
| n = 100 & n_i ~ Unif[3, 6] |       |       |     |     |       |     |     |       |     |     |       |     |     |
| ind_obs      | 0.00  | 0.183 | 0.32| 0.58| 0.162 | 0.21| 0.46| 0.111 | 0.14| 0.45| 0.178 | 0.22| 0.45|
| bootstrap    | 118   | 0.183 | 0.75| 0.94| 0.162 | 0.70| 0.95| 0.111 | 0.49| 0.95| 0.178 | 0.72| 0.94|
| independent  | 22    | 0.181 | 0.98| 0.95| 0.126 | 0.64| 0.97| 0.072 | 0.37| 0.97| 0.177 | 0.69| 0.93|
| exchangeable | 55    | 0.181 | 0.99| 0.95| 0.125 | 0.65| 0.98| 0.071 | 0.38| 0.98| 0.177 | 0.69| 0.93|
| unspecified  | 87    | 0.126 | 0.82| 0.99| 0.092 | 0.42| 0.97| 0.058 | 0.26| 0.96| 0.113 | 0.46| 0.94|
| n = 100 & n_i ~ Unif[8, 12] |       |       |     |     |       |     |     |       |     |     |       |     |     |
| ind_obs      | 1.00  | 0.114 | 0.22| 0.63| 0.110 | 0.15| 0.47| 0.069 | 0.10| 0.48| 0.109 | 0.15| 0.48|
| bootstrap    | 244   | 0.114 | 0.48| 0.95| 0.110 | 0.45| 0.93| 0.069 | 0.31| 0.95| 0.109 | 0.48| 0.96|
| independent  | 233   | 0.112 | 0.66| 0.97| 0.094 | 0.45| 0.96| 0.049 | 0.27| 0.99| 0.111 | 0.47| 0.95|
| exchangeable | 392   | 0.112 | 0.67| 0.97| 0.094 | 0.45| 0.97| 0.049 | 0.27| 0.99| 0.111 | 0.47| 0.95|
| unspecified  | 568   | 0.069 | 0.47| 1.00| 0.062 | 0.25| 0.94| 0.034 | 0.15| 0.97| 0.059 | 0.26| 0.96|
| n = 200 & n_i ~ Unif[3, 6] |       |       |     |     |       |     |     |       |     |     |       |     |     |
| ind_obs      | 1.00  | 0.134 | 0.23| 0.57| 0.114 | 0.15| 0.46| 0.081 | 0.10| 0.43| 0.125 | 0.16| 0.43|
| bootstrap    | 221   | 0.134 | 0.53| 0.94| 0.114 | 0.49| 0.94| 0.081 | 0.34| 0.95| 0.125 | 0.51| 0.95|
| independent  | 169   | 0.130 | 0.70| 0.96| 0.097 | 0.47| 0.97| 0.055 | 0.29| 0.99| 0.127 | 0.50| 0.94|
| exchangeable | 444   | 0.130 | 0.70| 0.96| 0.097 | 0.47| 0.97| 0.055 | 0.29| 0.99| 0.127 | 0.50| 0.94|
| unspecified  | 663   | 0.081 | 0.55| 1.00| 0.068 | 0.29| 0.95| 0.038 | 0.18| 0.98| 0.076 | 0.31| 0.94|
| n = 200 & n_i ~ Unif[8, 12] |       |       |     |     |       |     |     |       |     |     |       |     |     |
| ind_obs      | 2.00  | 0.085 | 0.16| 0.61| 0.074 | 0.10| 0.47| 0.056 | 0.07| 0.43| 0.082 | 0.11| 0.46|
| bootstrap    | 467   | 0.085 | 0.34| 0.95| 0.074 | 0.31| 0.95| 0.056 | 0.22| 0.93| 0.082 | 0.34| 0.95|
| independent  | 1820  | 0.082 | 0.47| 0.95| 0.066 | 0.33| 0.97| 0.043 | 0.20| 0.97| 0.081 | 0.34| 0.94|
| exchangeable | 2943  | 0.082 | 0.47| 0.95| 0.066 | 0.33| 0.98| 0.043 | 0.20| 0.97| 0.081 | 0.34| 0.94|
| unspecified  | 4466  | 0.047 | 0.33| 1.00| 0.041 | 0.17| 0.95| 0.025 | 0.11| 0.95| 0.044 | 0.18| 0.95|

FIGURE S6 Estimated coefficient functions when the data are generated from unspecified covariance with n = 100 and n_i ~ Unif[3, 6]. For each panel, the solid red line represents the true coefficient function, the gray lines are the estimates from 100 replications, and the dashed black line represents the mean function of the 100 gray lines.
TABLE S6 Empirical type I error of covariance tests at the nominal $\alpha = 0.05$ and 0.10 levels based on 5000 datasets, by sample size ($n$) and observations per subject ($n_i$). Bonferroni refers to the joint test by Bonferroni correction and BH refers to the joint test by Benjamini-Hochberg correction.

| Null                  | $n$ | $\alpha$ | 1st layer | 2nd layer | 3rd layer | 4th layer | Bonferroni | BH       |
|----------------------|-----|----------|-----------|-----------|-----------|-----------|------------|----------|
| exchangeable         | 100 | 0.05     | 0.050     | 0.048     | 0.048     | 0.051     | 0.051      | 0.047    |
|                      |     | 0.10     | 0.102     | 0.097     | 0.102     | 0.101     | 0.098      | 0.103    |
| $n_i \sim \text{Unif}[3, 6]$ | 200 | 0.05     | 0.050     | 0.050     | 0.048     | 0.043     | 0.050      | 0.048    |
|                      |     | 0.10     | 0.098     | 0.106     | 0.098     | 0.090     | 0.098      | 0.101    |
| exchangeable         | 100 | 0.05     | 0.041     | 0.038     | 0.044     | 0.040     | 0.034      | 0.033    |
|                      |     | 0.10     | 0.090     | 0.091     | 0.091     | 0.088     | 0.073      | 0.078    |
| $n_i \sim \text{Unif}[8, 12]$ | 200 | 0.05     | 0.046     | 0.045     | 0.039     | 0.041     | 0.039      | 0.037    |
|                      |     | 0.10     | 0.093     | 0.096     | 0.093     | 0.098     | 0.078      | 0.083    |
| independent          | 100 | 0.05     | 0.052     | 0.052     | 0.053     | 0.051     | 0.058      | 0.054    |
|                      |     | 0.10     | 0.101     | 0.102     | 0.104     | 0.100     | 0.104      | 0.107    |
| $n_i \sim \text{Unif}[3, 6]$ | 200 | 0.05     | 0.053     | 0.052     | 0.051     | 0.059     | 0.055      | 0.052    |
|                      |     | 0.10     | 0.103     | 0.097     | 0.103     | 0.104     | 0.104      | 0.108    |
| independent          | 100 | 0.05     | 0.052     | 0.049     | 0.052     | 0.048     | 0.053      | 0.051    |
|                      |     | 0.10     | 0.107     | 0.103     | 0.102     | 0.103     | 0.100      | 0.107    |
| $n_i \sim \text{Unif}[8, 12]$ | 200 | 0.05     | 0.050     | 0.051     | 0.054     | 0.053     | 0.051      | 0.048    |
|                      |     | 0.10     | 0.101     | 0.102     | 0.098     | 0.105     | 0.099      | 0.103    |

FIGURE S7 Power curves (type I error $\alpha = 0.05$) for individual and joint tests of covariances, under deviation from the null. Shown are: $n_i \sim \text{Unif}[3, 6]$ (solid line) and $n_i \sim \text{Unif}[8, 12]$ (dashed line), for $n = 100$ (gray) and $n = 200$ (black). Bonferroni refers to the joint test by Bonferroni correction and BH refers to the joint test by Benjamini-Hochberg correction.