MODULI SPACES OF COLORED GRAPHS

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Abstract. We introduce moduli spaces of colored graphs, defined as spaces of non-degenerate metrics on certain families of edge-colored graphs. Apart from fixing the rank and number of legs these families are determined by various conditions on the coloring of their graphs. The motivation for this is to study Feynman integrals in quantum field theory using the combinatorial structure of these moduli spaces, so that the conditions defining a family of graphs reflect the Feynman rules in a physical quantum field theory such as (massive) scalar fields or quantum electrodynamics. In any case the resulting spaces decompose into various types of cell complexes with a rich and interesting combinatorial structure. We treat some examples in detail and discuss their topological properties, connectivity and homology groups, for the case of rank one graphs.

1. Introduction

The purpose of this article is to define moduli spaces of colored graphs in the spirit of [HV98] and [CHKV16] where such spaces for uncolored graphs were used to study the homology of automorphism groups of free groups. Our motivation stems from the connection between these constructions in geometric group theory and the study of Feynman integrals as pointed out in [BK15].

Let us briefly sketch these two fields and what is known so far about their relation.

The analysis of Feynman integrals as complex functions of their external parameters (e.g. momenta and masses) is a special instance of a very general problem, understanding the analytic structure of functions defined by integrals. That is, given a complex function \( f : \mathbb{C}^n \to \mathbb{C} \) with

\[
f(z) = \int_{\Gamma} g(w, z)dw,
\]

what can be deduced about of \( f \) from studying the integration contour \( \Gamma \subset \mathbb{C}^m \) and the integrand \( g : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C} \)?

There is a mathematical account for a class of well-behaved cases [Pha11], but unfortunately Feynman integrals are generally too complicated to answer this question thoroughly. They have however enough (combinatorial) structure allowing for the deduction of partial results by various methods, although some have yet to be put on rigorous mathematical footing. For example, Cutkosky’s theorem [Cut60], relating the imaginary part of a Feynman integral to a simpler integral over generalized residues (in physics terms, certain edge propagators put on mass-shell), was just recently proven in [BK15]. Along the way, Bloch and Kreimer mention the

\footnote{For what is known [ELOP66] is the classic reference for physicists while [HT66] advocates a more mathematical point of view.}
idea of studying *Outer space* and closely related spaces to gain new insights into the properties of Feynman integrals.

Outer space $CV_n$ is a topological space that arises in geometric group theory where it is used to study the automorphism group of a free group $F_n$. Inspired by ideas from Teichmüller theory, it is constructed as a space in which points are equivalence classes of marked metric graphs and it comes with a group action of $\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n)$, the group of outer automorphism of $F_n$, permuting the markings \cite{CV86}. The quotient of $CV_n$ by this action is a moduli space for connected metric graphs of rank $n$ (or genus $n$ *tropical curves* \cite{Cap11}) with no vertices of valence one or two. Moreover, it is a rational classifying space for $\text{Out}(F_n)$, hence computes the group homology $H_*(\text{Out}(F_n); \mathbb{Q})$.

There are actually many homotopy equivalent models for Outer space. For instance, $CV_n$ deformation retracts to the so-called spine of Outer space, a simplicial complex whose elements assemble naturally into cubes. This produces a cubical complex with its elements represented by pairs $(G, F)$ of (marked) graphs and spanning forests $F \subset G$. Furthermore, the group action behaves nicely on the spine of Outer space, allowing to set up a cubical chain complex to compute $H_*(\text{Out}(F_n); \mathbb{Q})$ as the (cubical) homology of the moduli space of rank $n$ graphs.

The connection to physics, as proposed by Bloch and Kreimer, is established by the fact that this cubical complex also captures the combinatorial structure of cut and reduced graphs which show up in the study of Feynman integrals as complex functions of their external parameters. More precisely, to any pair $(G, F)$ where $F$ is a spanning forest of $G$ they associate two new graphs, a *reduced graph* obtained by collapsing all edges of $G$ which do not connect different components of the spanning forest $F$ and a *cut graph* where all those edges connecting different components are put *on-shell* (see \cite{Kre16} for details and examples). This data allows to analyse a Feynman integral through the study of simpler integrals, i.e. to determine its singular loci and branch cuts together with their associated discontinuities.

On the other hand, this is the same data that describes the boundary operator in the cubical chain complex for $CV_n$, or $CV_n/\text{Out}(F_n)$ respectively. Hence, the combinatorial topology of these spaces is somehow related to the analytic structure of Feynman integrals. So far this holds at least for every cell, i.e. every pair $(G, F)$ (see \cite{BK15, Kre16}). Understanding the relations between neighboring cells is the main motivation for the present article.

In contrast to the elements in Outer space or the moduli space of graphs, physicists usually consider Feynman diagrams as graphs with additional structure. Depending on a chosen theory one needs to distinguish between different masses or particle types assigned to graph edges. Moreover, there are rules for which particles may interact, so that not all vertex types will be allowed. Such additional data can be represented, as a first approximation, by coloring the edges of a graph. Adjusting the definitions to this case we obtain moduli spaces that parametrize colored graphs. These spaces have very similar structures, such as the cubical decompositions described above. This allows to mimic the ideas of \cite{HV98} to compute their homology groups algorithmically.

The idea (or hope) is that understanding the combinatorics and topology of these moduli spaces of colored graphs will give new insight into the study of Feynman integrals. A closer look at this connection and its applications will be pursued in
future work, including a comparison to the algebraic approach via coactions that
has received quite some attention lately (see e.g. [ABDG17]). In the present paper
we study only mathematical properties of these moduli spaces of colored graphs
which are pretty interesting in their own right.

While we give rather general definitions, our concrete results are mainly focused
on the one-loop case which appears to be much simpler than all other cases. More
precisely, three versions of moduli spaces are under consideration: A space which
allows for arbitrary colorings of its elements by a fixed number of colors \( m \) and
two spaces of holocolored graphs in which each edge has a different color, one of
which has the additional feature that edges keep the information of their coloring
when shrunk to zero length. In physics terms we think of the first type as a
moduli space for all possible Feynman diagrams in a theory with \( m \) different scalar
particles while the second type models the "generic case" where only diagrams with
different masses occur (this case is much easier from an analytic perspective). The
third space of holocolored graphs with "remembering edges" is built to model a
diagrammatic account of the operator product expansion (see for example [BZ81]
or [IZ05]). Here, in contrast to the analysis of a Feynman integral by looking at
its cut and reduced graphs, the type of a new vertex formed by collapsing an edge
depends on the label of the latter. To model this effect we therefore consider a
space of graphs where edges of zero length still carry a color.

In all three cases we study the homology groups of these spaces. Our main com-
putational tool, generalizing the ideas of [HV98] to the colored case, is a cubical
chain complex that allows to compute their rational homology with computer as-
sistance. Moreover, we discuss some special cases in detail and derive a couple of
general results by hand. For the spaces of holocolored graphs we obtain the highest
non-trivial homology group by an explicit construction. For arbitrary colorings we
show that all one-loop spaces are simply connected. Thus, all homology in rank
one vanishes, as is the case for the moduli spaces of uncolored graphs. In fact, the
computer results suggest that all homology groups, except in the top dimensional
rank, are isomorphic, hence independent of the number of colors. We comment on
some partial results to prove this conjecture.

The exposition is organized as follows. Section 2 serves as a quick reminder
on graphs and cell complexes, setting up the necessary definitions and notation
for the rest of this work. In the next section, after recalling the notion of moduli
spaces of graphs in the classical sense, we define moduli spaces of colored graphs
formally. Several examples are considered in detail and we discuss their most im-
portant properties. The rest of the paper is concerned with the topology of these
moduli spaces for one-loop graphs. In Section 4 we study the case of arbitrarily
colored graphs. For this we introduce in 4.1 the cubical chain complex that com-
putes the rational homology of these spaces. Then we list and discuss the results
of these calculations and prove some general statements on homological and homo-
topical properties. Section 5 is concerned with holocolored graphs, with or without
remembering edges. We display the results of calculations in the cubical complex
as well as some direct results obtained by algebraic and combinatorial methods.

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2. Preliminaries

Let us start by introducing the basic definitions and notational conventions for the central objects in this article, graphs and cell complexes.

2.1. Graphs. Graphs are very versatile mathematical objects that show up in a variety of fields including discrete mathematics, computer science and quantum field theory. They prominently arise in the perturbative approach to the latter in form of Feynman diagrams which represent integrals contributing to probability amplitudes in high-energy particle scattering processes (see for example [IZ05]).

There are various possibilities to define graphs\(^2\) each suited for different purposes. Physicists often use a definition based on half-edges which allows for a distinction between internal and external edges as needed for Feynman diagrams.

Definition 2.1. A graph is a tuple \(G = (V,H,s,c)\) consisting of a set of vertices \(V = V(G)\), a set of half-edges \(H = H(G)\), a map \(s = s_G : H \to V\) which connects each half-edge to its source vertex and a map \(c = c_G : H \to H\) with \(c^2 = \text{id}_H\) that connects half-edges with each other; if \(h_1 \neq h_2 \in H\) are two distinct half-edges with \(c(h_1) = h_2\), the pair \((h_1, h_2)\) is called an (internal) edge\(^3\) of \(G\), otherwise \(h_1 = h_2\) is called an external edge, leg or hair. We denote the set of internal edges of \(G\) by \(E(G)\).

A subgraph \(\gamma \subset G\) is a graph such that \(V(\gamma) \subset V(G), H(\gamma) \subset H(G)\) and \(s_\gamma = s_G|_{H(\gamma)}, c_\gamma = c_G|_{H(\gamma)}\). Subgraphs can have external legs, but we will only consider so-called internal subgraphs, i.e. subgraphs without legs.

The following notations occur frequently throughout this article.

- The rank or loop number of \(G\) is denoted by \(|G|\) or \(h_3(G)\) (the first Betti number of \(G\)).
- For any vertex \(v \in V(G)\) we denote its valency by \(|v| := |s^{-1}(v)|\).
- \(G\) is called one-particle irreducible (1PI) or bridge-free or core if it is connected and still connected upon removal of any internal edge \(e \in E(G)\). In case \(G\) is not 1PI, the edges leaving the graph disconnected upon removal are called bridges or separating edges.
- A graph \(G\) is called admissible if it is 1PI and \(|v| \geq 3\) for all \(v \in V(G)\).
- A graph \(F\) is called a k-forest if it has no external edges, no loops, \(|F| = 0\), and \(k\) components. In particular, a 1-forest is called a tree. A subgraph \(F \subset G\) is called a spanning k-forest if \(F\) is a k-forest and \(V(F) = V(G)\). If \(k = 1\), then \(F\) is said to be a spanning tree.
- A graph with a single vertex, \(n\) internal edges and \(s\) legs is called a rose with \(n\) petals (and \(s\) thorns) and denoted by \(R_{n,s}\).

Sometimes, especially for topological considerations, it will be more convenient to think of graphs as one dimensional CW-complexes. In this case legs can be modeled as attached to univalent vertices or as additional labels on the vertices of \(G\) (the definition of admissibility has then to be adjusted, allowing for univalent

\(^2\)Here we always consider non-empty finite multi-graphs.

\(^3\)By using ordered pairs \((h_1, h_2)\) one obtains oriented edges; \(G\) is then called a directed graph.
vertices or for labeled bivalent vertices, respectively).

In the following we will need two operations on graphs, the removing and contracting of edges.

**Definition 2.2.** Let $G$ be a graph and $\gamma \subset G$ an (internal) subgraph.

1. $G \setminus \gamma$ denotes the graph $G$ with all edges of $\gamma$ removed, i.e. $V(G \setminus \gamma) = V(G)$ and $E(G \setminus \gamma) = E(G) \setminus E(\gamma)$.

2. For $\gamma$ connected, $G/\gamma$ denotes the graph obtained from $G$ by collapsing $\gamma$ to a single vertex, i.e. $\gamma$ is replaced by a vertex with all edges connecting $\gamma$ and $G \setminus \gamma$ attached to it. For disconnected graphs $\gamma$, this operation is defined by shrinking each connected component in this manner.

The case where the subgraph $\gamma$ is a forest will occur frequently in this text. In that case the number of loops does not change when contracting $\gamma$, $|G/\gamma| = |G|$. If in particular $\gamma$ is a spanning tree, then $G/\gamma$ is a rose with $|G|$ petals.

So far we have considered graphs as purely combinatorial objects. To obtain more structure, a graph can be endowed with a metric by means of a length function that assigns to each edge a positive real number.

**Definition 2.3.** A **metric graph** is a pair $(G, \lambda)$ where $G$ is a graph and $\lambda$ a map $\lambda : E(G) \to \mathbb{R}_{>0}$ giving each edge of $G$ a length. The **volume** of $G$ is defined as the sum of all edge lengths,

$$\text{vol}_\lambda(G) := \sum_{e \in E(G)} \lambda(e).$$

Thus, given a metric graph $G$ the distance between two points can be defined as the minimum length of a path connecting them, turning $G$ into a metric space.

If we allow for an edge $e \in E(G)$ to have length zero, we typically identify $(G, \lambda)$ with the contracted graph $(G/e, \lambda|_{E(G) \setminus \{e\}})$.

Lastly, we need a notion of maps between graphs that makes sense in both the combinatorial and topological setting.

**Definition 2.4.** By a map $f : G \to G'$ we mean a cellular map between $G$ and $G'$ viewed as CW-complexes (see Definition 2.9 below). In addition, we require $f$ to map legs to legs or basepoints to basepoints, respectively.

2.2. **Cell complexes.** All spaces we encounter in this work have a particular nice structure; they decompose into various types of cell complexes. In this section we give a short account of these types, taking a geometric approach following [Hat02] and [Swi02].

**Simplicial complexes:** First, we consider spaces that decompose into simplices.

**Definition 2.5.** A collection of simplices $K$ is called a **simplicial complex** if the following holds:

1. For all $\sigma \in K$: If $\tau \subset \sigma$, then $\tau$ is also in $K$.
2. For all $\sigma, \sigma' \in K$, either $\sigma \cap \sigma' \in K$ or $\sigma \cap \sigma' = \emptyset$.

The **dimension** of a simplicial complex is defined as the largest $n$ for which there is an $n$-simplex in $K$. 

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By gluing together the elements of a simplicial complex $K$ along their common faces one obtains a topological space $|K|$, the geometric realization of $K$. Vice versa, if a topological space $X$ is homeomorphic to the geometric realization of a simplicial complex $K$, then $K$ is called a triangulation of $X$. Being somewhat imprecise we refer then to $X$ as a simplicial complex as well.

It is often useful to relax the gluing conditions on simplices a little. For instance, the second condition in Definition 2.5 implies that in a simplicial complex no two simplices can have the same vertex set. If we want more flexibility in gluing simplices together we need the notion of a semi-simplicial complex.

**Definition 2.6.** A topological space $X$ together with a collection of continuous maps $\sigma_\alpha : \Delta^n \to X$ (with $n \in \mathbb{N}$ dependent on $\alpha$) is called a semi-simplicial or $\Delta$-complex if

1. All restrictions of $\sigma_\alpha$ to the interior of $\Delta^n$ are injective such that each $x \in X$ is in the image of exactly one such restriction.
2. For all $\sigma_\alpha : \Delta^n \to X$ the restriction to any face is a map $\sigma_\beta : \Delta^{n-1} \to X$.
3. For any $\alpha$, any $A \subset X$: $\sigma_\alpha^{-1}(A)$ is open $\iff A$ is open.

This indeed generalizes the notion of a simplicial complex: If $K = \{\sigma_\alpha\}_{\alpha \in I}$ is a simplicial complex and $X = |K|$ its geometric realization, then the natural inclusion maps $\sigma_\alpha \hookrightarrow X$ give $X$ the structure of a $\Delta$-complex. On the other hand, iteratively subdividing the simplices in a $\Delta$-complex $X$ (for instance by barycentric subdivision) produces eventually a simplicial complex structure on $X$.

**Cubical complexes:** Obviously, one may take also other types of building blocks to decompose a given space. For instance, by replacing simplices with cubes. This method does not appear very often in the literature but it has certain computational advantages, some of which are utilized in Sections 4 and 5.

Let $I$ denote the interval $[0, 1] \subset \mathbb{R}$ and define the standard $n$-cube as the product $\square^n := I^n = [0, 1] \times \ldots \times [0, 1]$ and $\square^0 := \{0\}$.

Given a topological space $X$ we want to decompose it into an union of such cubes. The direct analog to a simplicial complex is to restrictive for the purpose of this work. Therefore, we mimic the above definition of a $\Delta$-complex:

**Definition 2.7.** A (singular) $n$-cube in $X$ is a continuous map $\sigma : \square^n \to X$. If $\sigma$ is injective, the cube is called regular, otherwise degenerate.

The $i$-th (primary) faces of a cube $\sigma : \square^n \to X$ are defined as the maps

$$f^i_+ \sigma : \square^{n-1} \to X, \ (x_1, \ldots, x_n) \mapsto \sigma(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

and

$$f^i_- \sigma : \square^{n-1} \to X, \ (x_1, \ldots, x_n) \mapsto \sigma(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n).$$

**Definition 2.8.** A topological space $X$ together with a collection of maps $\sigma_\alpha : \square^n \to X$ (with $n \in \mathbb{N}$ dependent on $\alpha$) is called a cubical complex if

- All restrictions of $\sigma_\alpha$ to the interior of $\square^n$ are injective such that each $x \in X$ is in the image of exactly one such restriction.
- For every $\sigma_\alpha : \square^n \to X$ the restriction to any primary face is a map $\sigma_\beta : \square^{n-1} \to X$.
- For any $\alpha$ and any $A \subset X$: $\sigma_\alpha^{-1}(A)$ is open $\iff A$ is open.
For a more detailed treatment of cubical complexes the interested reader is referred to [Mas91].

**CW-complexes:** Last, but not least, by replacing simplices or cubes by disks $D^n := \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$ and allowing much more general gluing maps we obtain the notion of **CW-complexes.** Instead of a formal definition, we give a building recipe for these kind of spaces (as in [Hat02], for a precise definition we refer to Swi02).

**Definition 2.9.** A CW- or (regular) cell complex is a space $K$ constructed inductively as follows.

1. Start with a discrete set $K^{(0)}$, the set of 0-cells of $K$.
2. Inductively form $K^{(n)}$ from $K^{(n-1)}$ by attaching $n$-cells $D^n_\alpha$ via maps $\varphi_\alpha : S^{n-1} \approx \partial D^n_\alpha \to K^{(n-1)}$. Thus, $K^{(n)}$ is the quotient space of $K^{(n-1)} \cup \alpha D^n_\alpha$ under the relation $x \sim \varphi_\alpha(x^\prime)$ for $x^\prime \in \partial D^n_\alpha$. The $n$-cells $e^n_\alpha$ of $K$ are the homeomorphic images of $D^n_\alpha \setminus \partial D^n_\alpha$ under this quotient map.
3. $K = \cup K^{(n)}$ with the weak topology: A set $U \subset K$ is open if and only if $U \cap K^{(n)}$ is open for all $n$.

The spaces $K^{(n)}$ are referred to as the $n$-skeleta of $K$. Given two CW-complexes, a continuous map $f : K \to L$ is called **cellular** if $f(K^{(n)}) \subset L^{(n)}$ for all $n$.

Clearly, cell complexes provide a very flexible setting to deal with topological spaces. One major advantage is that for a CW-decomposition of a given space much fewer cells are needed than in the simplicial or $\Delta$ setting. For the sake of brevity, we refer for examples and applications to [Hat02, Swi02].

**Cubical Homology:** Given any of the above types of decompositions of a space $X$ there is a corresponding chain complex whose homology is isomorphic to the singular homology of $X$. Simplicial and cellular homology are well-known, but one may also use cubes to calculate $H_\ast(X)$. The chain complex $(C^n_c(X), \partial_c)$ associated to a cubical complex $(X, (\sigma_\alpha)_{\alpha \in A})$ is constructed by defining the chain groups $C^n_c(X)$ to be the free abelian groups generated by all (regular) cubes $\sigma_\alpha : \Box^n \to X$. The boundary morphism $\partial^n_c$ acts linearly on the $n$-chains and its action on a single generator $\sigma_\alpha$ is defined by

\[
\partial^n_c \sigma_\alpha := \partial^n_+ \sigma_\alpha + \partial^n_- \sigma_\alpha,
\]

where

\[
\partial^n_+\sigma_\alpha := \sum_{i=1}^n (-1)^{i-1} f^i_+ \sigma_\alpha = \sum_{i=1}^n (-1)^{i-1} \sigma_\alpha|_{I^{i-1} \times \{0\} \times I^{n-i}},
\]

\[
\partial^n_-\sigma_\alpha := \sum_{i=1}^n (-1)^{i-1} f^i_- \sigma_\alpha = \sum_{i=1}^n (-1)^{i} \sigma_\alpha|_{I^{i-1} \times \{1\} \times I^{n-i}}.
\]

This produces a chain complex whose homology is isomorphic to the singular homology of $X$. It must be remarked though that caution is required when dealing with degenerate cubes, i.e. cubes $\sigma : \Box^n \to X$ which are not injective (cf. Mas91). The treatment of this technicality is omitted here since it does not occur in the cases considered in this work.
3. Moduli spaces of graphs

Moduli spaces of (uncolored) graphs can be defined as quotients of Culler-Vogtmann Outer space $CV_n$ and generalizations thereof. Points in these spaces are tuples $(G, \lambda, g)$ where $(G, \lambda)$ is a metric graph of rank $n$ and $g$ a marking, a (homotopy class of a) homotopy equivalence between $G$ and the rose graph $R_n$ [CV86]. Roughly speaking, Outer space $CV_n$ is an enlarged moduli space of graphs that is equipped with an action of $\text{Out}(F_n)$ which acts by changing the markings. Defining a moduli space of graphs as the orbit space of this action has certain advantages, but for the sake of brevity and having the application to Feynman diagrams in mind we stick to a more direct definition. Nevertheless, our construction is heavily inspired and conceptually quite close to the case of Outer space and its generalizations. For more on the "approach from Outer space" we refer to the survey in [Vog16].

3.1. The "classical"/uncolored case. For $n > 1$ let $(G, \lambda)$ denote an admissible graph $G = (V,E)$ without legs of rank $n$ employed with a metric $\lambda$. We define an equivalence relation on the set of such metric graphs by

\[(G, \lambda) \sim (G', \lambda') \iff \exists \text{ homothety } \varphi : G \to G' \text{ such that } \lambda = \lambda' \circ \varphi.\]

This relation allows to consider metric graphs with their volume normalized to one. Moreover, if $\lambda$ vanishes on some $E' \subset E$ we identify $(G, \lambda)$ with the contracted graph $G/E'$ and $\lambda|_{E\setminus E'}$ appropriately normalized. If $\lambda$ vanishes only on forests $F \subset E(G)$, the metric is called regular, otherwise degenerate.

Definition 3.1. The moduli space of rank $n$ graphs $\mathcal{M}G_n$ is defined as

$$\mathcal{M}G_n := \{(G, \lambda) \mid G \text{ admissible with } |G| = n, \lambda \text{ a regular metric on } G\} / \sim.$$ 

As a topological space $\mathcal{M}G_n$ is best understood by first looking at space $K_n = \{(G, \lambda) \mid G \text{ admissible with } |G| = n, \lambda \text{ regular with } \text{vol}_\lambda(G) = 1\}$ and then considering the quotient with respect to $\sim$. The nice property of $K_n$ is that it decomposes into a disjoint union of open simplices in the following way.

For each graph $G$ consider the set of normalized regular metrics $\lambda$ on $G$ with $\lambda(e) > 0$ for all $e \in E$. The space of such metrics on $G$ parametrizes the interior of an $|E| - 1$ dimensional simplex $\Delta_G$. Here one may think of the edges of $G$ as being ordered; the corresponding cell in $\mathcal{M}G_n$ is given by taking the quotient with respect to the relation which forgets the ordering - if $G$ has multi-edges, this is a nontrivial operation, "folding" $\Delta_G$ onto itself (cf. Figure 1).

A face of $\Delta_G$ lies in $K_n$ if and only if the edge set of $G$ on which $\lambda$ vanishes forms a forest in $G$, i.e. when $\lambda$ is regular. On the other hand, some faces of $\Delta_G$ may be missing, namely those corresponding to degenerate metrics vanishing on subgraphs $\gamma \subset G$ with $|\gamma| > 0$. Points in these faces are said to lie at infinity.

Thus, $K_n$ decomposes into a disjoint union of open simplices, one for each admissible graph of rank $n$. These simplices are glued together using the identification of metrics as described above,

$$\Delta_G \subset \Delta_{G'} \iff \exists F \subset G' \text{ a forest with } G'/F = G.$$ 

The same holds for an algebro-geometric approach via tropical curves, see for example the survey in [Cap11].
Since an admissible graph of rank \( n \) can have at most \( 3n - 3 \) edges (all vertices trivalent), we find \( \dim K_n = \dim \mathcal{MG}_n = 3n - 4 \).

In summary, \( K_n \) is not a simplicial complex because

1. some faces of simplices are missing (those corresponding to points at infinity) and
2. in some cases the intersection of two simplices is the union of more than one of their faces.

However, \( K_n \) is not too badly behaved; it is a \( \Delta \)-complex with some of its simplices deleted.

Unfortunately, this description does not hold for \( \mathcal{MG}_n \). In the process of taking the quotient of \( K_n \) with respect to the relation \( \sim \) some simplices get folded onto themselves, see Figure [1]. After normalizing this is essentially the identification of isomorphic graphs and permuting the metrics on multi-edges. In fact, this is the only bad thing that can happen, so \( \mathcal{MG}_n \) can be described as a CW-complex with missing cells.

Another way to understand the topology of \( \mathcal{MG}_n \) is to first replace \( K_n \) by a subspace \( SK_n \) to which it deformation retracts and then take the quotient with respect to \( \sim \). The space \( SK_n \subset K_n \) is a simplicial complex and plays the same role for \( K_n \) as does the spine for Outer space. It can be defined as the geometric realization of the poset \( (G_n, \preceq) \) where

\[
G_n := \{ G | G \text{ adm.} \land |G| = n \}, \ G \preceq G' \iff \exists F \subset G \text{ a forest : } G = G'/F.
\]

Hence, a \( k \)-simplex in \( SK_n \) is represented by a chain \( G_0 \preceq \ldots \preceq G_k \) of \( k+1 \) graphs in the poset \( G_n \).

Moreover, \( SK_n \) can be further simplified by grouping its simplices into cubes. In this description a cube \((G,F)\) is specified by a graph \( G \in G_n \) and a spanning forest \( F \) of \( G \). Such a cube \((G,F)\) contains all simplices of \( SK_n \) that can be obtained by ordering the edges of \( F = (e_1, \ldots, e_k) \) and setting

\[
G_0 = G/F, \ G_1 = G/(e_1, \ldots, e_{k-1}), \ldots, G_{k-1} = G/e_1, \ G_k = G.
\]

Thus, for a forest \( F \) with \( k \) edges the corresponding cube is \( k \)-dimensional and contains \( k! \) simplices. Since an admissible graph can have at most \( 3n - 3 \) edges, we find \( \dim SK_n = 2n - 3 \) as the number of edges of a spanning tree of such a graph.

The whole construction and the decompositions described above naturally generalize to the case of graphs with \( s \) external edges. Here we think of these legs as labeled basepoints and adjust the definition of admissibility accordingly. One then considers basepointed metric graphs \((G, \{v_1, \ldots, v_s\}, \lambda)\) and declares two such points to be equivalent if and only if there is a basepoint-preserving homothety \( \varphi : (G, \{v_1, \ldots, v_s\}) \to (G', \{w_1, \ldots, w_s\}) \) such that \( \lambda' \circ \varphi = \lambda \).

**Definition 3.2.** For \( n,s \in \mathbb{N} \) the moduli space of rank \( n \) graphs with \( s \) legs \( \mathcal{MG}_{n,s} \) is defined as

\[
\mathcal{MG}_{n,s} := \left\{ (G, \{v_1, \ldots, v_s\}, \lambda) \mid G \text{ admissible with } s \text{ legs } v_1, \ldots, v_s, n \text{ loops and } \lambda : E \to \mathbb{R}_{\geq 0} \text{ regular } \right\}/\sim.
\]

\[\text{In the category of simplicial complexes such a space } K \text{ is known as a \textit{relative simplicial complex}, i.e. a pair } (X,Y) \text{ where } X \text{ is a simplicial complex and } Y \subset X \text{ a subcomplex, such that } K = X \setminus Y \text{ (cf. [Sta87]).}\]
For \( s > 0 \) these moduli spaces play the same role for a sequence of groups \( \Gamma_{n,s} \) as does \( \mathcal{M}G_n \) for \( \text{Out}(F_n) \). Here \( \Gamma_{n,s} \) is the group of (relative) homotopy classes of self-homotopy equivalences of a rank \( n \) graph fixing its \( s \) legs, say of the rose graph \( R_{n,s} \) with \( n \) petals and \( s \) thorns,

\[
\Gamma_{n,s} := \pi_0(\text{aut}(R_{n,s})) \implies \Gamma_{n,0} = \text{Out}(F_n), \Gamma_{n,1} = \text{Aut}(F_n), \ldots
\]

For a precise definition and further reading on these groups with applications in geometric group theory we refer to [CHKV16] and the survey in [Vog16].

Similar to the case of \( \mathcal{M}G_n \) the space

\[
K_{n,s} := \left\{ (G, \{v_1, \ldots, v_s\}, \lambda) \middle| G \text{ admissible with } s \text{ legs } v_1, \ldots, v_s, \right. \left. |G| = n, \lambda \text{ regular with } \text{vol}_\lambda(G) = 1 \right\}
\]

decomposes into an union of open simplices \( \Delta_G \). From the definition of admissibility and by an Euler characteristic argument we deduce \( \dim K_{n,s} = \dim \mathcal{M}G_{n,s} = 3n + s - 4 \).

There is deformation retract \( SK_{n,s} \subset K_{n,s} \), defined analogously to the case without legs, which is a simplicial (or cubical) complex of dimension \( 2n - 3 + s \). It may be used to setup a cubical chain complex that allows to compute the homology of \( \mathcal{M}G_{n,s} \) as was done in [HV98] for the case \( s = 1 \). Moreover, these moduli spaces are rational classifying spaces for the groups \( \Gamma_{n,s} \), so that these complexes actually compute the algebraically defined group homology \( H_*(\Gamma_{n,s}; \mathbb{Q}) \).

### 3.2. Moduli spaces of colored graphs

In the previous section we used edge-metrics to define topological spaces populated by (admissible) graphs. When physicists draw Feynman diagrams to represent particle scattering processes, edges may describe different kinds of particles. To encode this additional piece of data we consider now colored edges.

**Definition 3.3.** Let \( G = (V, E) \) be a graph and \( m \in \mathbb{N} \). An \textit{m-coloring} of \( G \) is a map \( c : E \to \{1, 2, \ldots, m\} \).
An $m$-coloring of a graph represents some (physical) property of the edges which can take $m$ different values. For example one might think of a Feynman diagram in a scalar field theory of three different massive particles as a graph endowed with a 3-coloring, one color for each particle in the theory.

Analogous to the constructions in the last section metric graphs endowed with a coloring can be represented by points in a moduli space of colored graphs. Here we will consider three different cases in detail: Spaces in which any $m$-coloring is allowed and spaces of holocolored graphs in which only injective colorings with a fixed set of colors are admitted, so that each edge is assigned a different color. In the latter case two types of spaces are distinguished, one using colored graphs as before while the other deals with graphs which retain the information of their coloring upon shrinking edges.

To make things precise, we start with a definition of equivalence for colored metric graphs. For this we define two relations by

$$(G, \lambda, c) \sim (G', \lambda', c') \iff \exists \varphi : G \to G' \text{ homothety with } c(e) = c' (\varphi(e)) \text{ if } \lambda(e) \neq 0$$

and

$$(G, \lambda, c) \sim_* (G', \lambda', c') \iff \exists \varphi : G \to G' \text{ homothety s.t. } c = c' \circ \varphi.$$  

Note that the second condition on $\sim$ actually allows to forget the color of an edge that is collapsed to zero length. Therefore, this is the appropriate generalization of (3.1) to the colored case and we keep using the same symbol.

**Definition 3.4.** Let $n, s \in \mathbb{N}$ and define the moduli space of $m$-colored graphs as

$$\mathcal{MCG}_{n,s} := \left\{ (G, \{v_1, \ldots, v_s\}, \lambda, c) \mid G \text{ adm. with } s \text{ legs, } |G| = n, \lambda \text{ regular, } c : E \to \{1, \ldots, m\} \right\} / \sim.$$  

**Definition 3.5.** For $n, s \in \mathbb{N}$ set $C := \{1, \ldots, 3(n-1) + s\}$ and define

- the moduli space of holocolored graphs by

$$\mathcal{MHG}_{n,s} := \left\{ (G, \{v_1, \ldots, v_s\}, \lambda, c) \mid G \text{ adm. with } s \text{ legs, } |G| = n, \lambda \text{ regular, } c : E \to C \text{ injective} \right\} / \sim.$$  

- the moduli space of holocolored graphs with remembering edges by

$$\mathcal{MRG}_{n,s} := \left\{ (G, \{v_1, \ldots, v_s\}, \lambda, c) \mid G \text{ adm. with } s \text{ legs, } |G| = n, \lambda \text{ regular, } c : E \to C \text{ injective} \right\} / \sim_*.$$  

If $n > 1$, then arguing as in the uncolored case we see that each moduli space is the quotient of a space that decomposes into a disjoint union of open simplices, one for each admissible colored graph with $n$ loops and $s$ legs. In the case of $m$- and holocoloring the face relations are again given by contracting forests, but now with the additional requirement of matching colors. In the case of remembering edges this is a bit more delicate: Here it is best to think of a point as given by either

- an admissible colored metric graph with additional labels on its vertices that keep track of the contracted edges (cf. Section 5.2), or

---

6In many actual quantum field theoretical calculations, the spin of particles adds an additional feature; half-integer spin particles come as oriented edges.
• an admissible colored combinatorial graph with all legs distinct and without
identifying edges of length zero and contracted edges, \( \lambda(e) = 0 \neq G = G/e \),
and adjust the partial order accordingly.

Either way, for \( n > 1 \) all moduli spaces contain deformation retracts that have
the structure of simplicial or cubical complexes (the case \( n = 1 \) is slightly different
and will be considered in detail in Section 4.1).

The only (and rather nontrivial) difference is that the notion of isomorphic graphs
now depends on the coloring. In particular, the folding due to the symmetry of
multi-edges does not occur if these edges are all colored differently. Therefore,
both spaces of holocolored graphs, \( \mathcal{MG}_{n,s} \) and \( \mathcal{MRG}_{n,s} \), decompose directly into
\( \Delta \)-complexes with missing faces. Their building blocks are the simplices \( \Delta(G,c) \) as
described above, one for each isomorphism class of admissible colored graphs \( (G,c) \).

On the other hand, the moduli spaces of arbitrarily colored graphs \( \mathcal{MC}G_{n,s} \) have
no such restrictions and contain thus rather “wildly folded” simplices.

In the following section we will study the topology of these three spaces in the
one-loop case. This case has two advantages: No missing faces occur and the class
of admissible graphs is very simple, allowing for direct calculations that would get
quickly out of hand for graphs of higher rank.

Before we continue let us briefly describe two other moduli spaces that should
also be of interest for the study of Feynman diagrams; a detailed analysis is reserved
for future work.

Directed graphs: A directed graph is a graph \( G \) together with an orientation of
its edges, i.e. each edge \( e \in E(G) \) has a source and target vertex, denoted by \( v_−(e) \)
and \( v_+(e) \), respectively. We abbreviate this data by \( o = (v_−, v_+) : E(G) \to V(G)^2 \).

To define the notion of equivalence for directed metric graphs we henceforth
assume that for a point \( (G, \lambda, o) \) the metric \( \lambda \) is strictly positive - otherwise we
replace \( G \) by \( G/Z \) where \( Z \subset E(G) \) is the set of edges of zero length with respect
to \( \lambda \).

Definition 3.6. Let \( \sim \) denote the generalization of the equivalence relation \((3.1)\)
to the case of directed metric graphs,

\[
(G, \lambda, o) \sim (G', \lambda', o') \iff \exists \varphi : G \to G' \text{ homothety and } \forall e \in E(G) : o(e) = (x, y) \Rightarrow o'(\varphi(e)) = (\varphi(x), \varphi(y)).
\]

Then for \( n, s \in \mathbb{N} \) the moduli space of directed graphs is defined as

\[
\mathcal{MDG}_{n,s} := \left\{ (G, \{v_1, \ldots, v_s\}, \lambda, o) \mid G \text{ adm. with } s \text{ legs, } |G| = n, \text{ directed by } o : E \to V^2, \lambda \text{ regular} \right\} / \sim.
\]

The definition of \( \sim \) is compatible with the equivalence of colored graphs, so that
the above construction naturally generalizes to moduli spaces of colored directed
graphs. Moreover, if needed we can also allow only for some edges to be oriented
and/or colored, although the notation blows up considerably in this case.

Restriction on vertex types: In a realistic quantum field theory not all types
of particle interactions are allowed which translates in the language of Feynman
diagrams to restrictions on the possible vertex types. For instance, in quantum
electrodynamics there is only one interaction vertex, a trivalent node connecting the three edge types present in this theory, an electron, a positron and a photon. Hence, a corresponding moduli space, say of 3-colored graphs (ignoring orientations; in fact electrons and positrons call for directed edges as well), should have fewer top-dimensional cells. Namely those only having vertices connecting three different edge types. In lower dimensions other vertices should be admitted though since all graphs obtained by contracting edges in allowed graphs contribute to the analysis of the corresponding Feynman integrals (or to the operator product expansion).

Thus, in this exemplary case we should define a moduli space of QED Feynman diagrams as $\mathcal{MCG}^3_{n,s} \setminus Z$ where $Z$ is the union of all top-dimensional open cells that correspond to graphs having forbidden vertices and all lower dimensional cells corresponding to graphs that cannot be obtained by shrinking edges in allowed graphs.

Again, using colored and/or directed edges is a mere technicality and can in principle done as above.

4. $m$-COLORED GRAPHS

The primary focus of this and the following sections will be the calculation of the rational homology $H_\ast(X; \mathbb{Q})$ for $X = \mathcal{MCG}^m_{1,s}$, $\mathcal{M}_H G_{1,s}$ and $\mathcal{M}_R G_{1,s}$. The main computational tool for this endeavor is a cubical chain complex that was used in [HV98] to calculate the rational homology of $\mathcal{M} n, 1$. For $n > 1$ this method generalizes straightforward to the cases $s > 1$ and colored graphs. For $n = 1$ the same ideas work although coming from a slightly different setup.

4.1. The cubical complex. In the following let $X^m_s := \mathcal{MCG}^m_{1,s}$ denote the moduli spaces of $m$-colored one-loop graphs with $s$ legs. We now describe a decomposition of $X^m_s$ into singular cubes.

For any colored graph $(G, c)$ the set of regular metrics, normalized to volume one, describes a closed (no faces at infinity) simplex of dimension $d = |E(G)| - 1$. However, as discussed in the previous section, the images of these simplices under the quotient operation with respect to $\sim$ do not assemble themselves into a simplicial or semi-simplicial complex. The structure can be saved though by performing a barycentric subdivision before taking the quotient.

For this we start with the space

$$K^m_s := \left\{(G, \{v_1, \ldots, v_s\}, \lambda, c) \mid \begin{array}{l} \text{G adm. with } s \text{ legs, } |G| = 1, \lambda \text{ regular} \\ \text{and } \text{vol}_\lambda(G) = 1, c : E \to \{1, \ldots, m\} \end{array} \right\}$$

and consider its barycentric subdivision $BK^m_s$. A $k$-simplex in $BK^m_s$ is represented by a chain $(G_0, c_0) \preceq \ldots \preceq (G_k, c_k)$ of $k + 1$ colored graphs where the partial order $\preceq$ is given by

$$(G, c) \preceq (G', c') \iff \exists F \subset G' \text{ a forest with } G = G'/F \text{ and } c = c'|_{E(G') \setminus E(F)}.$$

These simplices can be reassembled into cubes. In this picture a $k$-cube is given by a colored graph $(G, c)$ and a forest $F \subset G$ with $k = |E(F)|$. This cube, denoted by $(G, F, c)$, is the collection of all $k!$ simplices in $BK^m_s$ that are given by choosing an order $(e_1, \ldots, e_k)$ on the edges of $F$ and setting

$$G_0 = G/F, \ G_1 = G/(e_1, \ldots, e_{k-1}), \ldots, \ G_{k-1} = G/e_1, \ G_k = G$$
Figure 2. A geometric representation of $X^1_3$ as a cubical complex.

and

$$c_0 = c|_{E(G)\setminus E(F)}, \quad c_1 = c|_{E(G)\setminus \{e_1, \ldots, e_{k-1}\}}, \quad \ldots, \quad c_{k-1} = c|_{E(G)\setminus \{e_1\}}, \quad c_k = c.$$  

This decomposition of $BK^m_s$ is fine enough to survive the quotient operation with respect to $\sim$; in every cube no pair of points belongs to the same equivalence class, i.e. no folding occurs, and only whole cubes get identified with each other. Therefore, we find a similar decomposition of $X^m_s$ into cubes, one for each pair $(G, F, c)$ where $(G, c)$ is an isomorphism class of admissible colored one-loop graphs with $s$ legs and $F \subset G$ a forest.

Example 4.1. Figure 2 depicts the cubical decomposition of $X^1_3 \cong \mathcal{MG}_{1,3}$ (cf. Figure 1). Here the label of a cube $(G, F)$ is drawn as a graph with the edges of $F$ colored in red. Vertices and edges with the same label have to be identified.

4.2. Rational homology. The case without colors, or equivalently a single color $m = 1$, serves as a good starting point to understand the computation and compare results with already established knowledge. A lot is already known about the homology of $\mathcal{MG}_{n,s}$. In fact, the cases $s = 0$ and $s = 1$ encode the group homology of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$, an active area of research. For the case $s > 1$ the rational homology of $\mathcal{MG}_{n,s}$ is fully determined for $n = 1, 2$ in [CHKV16]. For $n = 1$ we have

$$H_k(\mathcal{MG}_{1,s}; \mathbb{Q}) = \begin{cases} \mathbb{Q}(i) & \text{if } k \text{ is even} \\ 0 & \text{otherwise}. \end{cases}$$  

Interest in these homology groups stems from the fact that gluing together graphs of low rank along their legs induces so-called assembly maps in homology which produce potential new homology classes in $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ for larger $n$. 
For the moduli space of $m$-colored graphs a cubical chain complex can be used to compute $H_*(\mathcal{MC}_{n,s}^m; \mathbb{Q})$, for $n > 1$ by generalizing the construction of [HV98] to the colored case and for $n = 1$ by using the cubical complex described in the previous section.

In both cases we obtain similarly defined cubical chain complex that computes the rational homology of $\mathcal{MC}_{n,s}^m$. Its chain groups $C_k(\mathcal{MC}_{n,s}^m)$ are the free abelian groups generated by all cubes $(G,F,c)$ where $|E(F)| = k$. The boundary operator $\partial$ from (2.1) can easily be generalized to act on these triples $(G,F,c)$. We define the action of $\partial^m : C_*(\mathcal{MC}_{n,s}^m) \to C_*(\mathcal{MC}_{n,s}^m)$ on a cube $(G,F,c) \in C_k(\mathcal{MC}_{n,s}^m)$ by

$$\partial_k^m(G,F,c) = \sum_{i=1}^{k} (-1)^{i-1} \left( \partial (G,F \setminus \{e_i\}) - (G/e_i,F/e_i,c_{e_i}) \right),$$

where $c_{e_i} := c|_{E(G) \setminus \{e_i\}}$ is the coloring of $G$ with the edge $e_i$ collapsed. By construction this operator squares to zero.

From now on we stick to the one-loop case, i.e the spaces $X_s^m = \mathcal{MC}_{1,s}^m$.

As in the uncolored case (cf. [HV98]) the homology groups of $X_s^m$ can be calculated by a computer program (for details see [Muh18]). Endowing graphs with the additional data of a coloring leads to an even greater growth of the number of cubes with increasing $s$. Additionally, computing the homology of $X_s^m$ by explicit calculation gets more difficult when increasing the number of colors. Thus, the maximal number of external legs $s$ to which the calculations can be performed

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c} \hline & H_0 & H_1 & H_2 & H_3 & H_4 \\ \hline X_2^5 & 2 & - & - & - & - \\ X_2^6 & 1 & 0 & - & - & - \\ X_3^4 & 1 & 0 & 6 & - & - \\ X_4^3 & 1 & 0 & 3 & 9 & - \\ X_5^2 & 1 & 0 & 6 & 0 & 84 \\ \hline \end{array} \]

\[ \begin{array}{c|c|c|c|c|c|c|c} \hline & H_0 & H_1 & H_2 & H_3 & H_4 & H_5 \\ \hline X_1^5 & 3 & - & - & - & X_1^4 & 4 & - & - & - \\ X_2^5 & 1 & 1 & - & - & X_2^4 & 1 & 3 & - & - \\ X_3^4 & 1 & 0 & 20 & - & X_3^4 & 1 & 0 & 49 & - \\ X_4^3 & 1 & 0 & 3 & 103 & 1 & 0 & 3 & 426 & - \\ \hline \end{array} \]

Table 1. The dimension of the homology groups $H_k(X_s^m; \mathbb{Q})$ for $1 \leq m \leq 7$ and various $s$.

\[ \begin{array}{c|c|c|c} \hline & H_0 & H_1 & H_2 \\ \hline X_1^5 & 5 & - & - \\ X_2^5 & 6 & - & - \\ X_3^5 & 1 & 0 & 99 \\ \hline \end{array} \]

\[ \begin{array}{c|c|c|c|c|c|c} \hline & H_0 & H_1 & H_2 & H_3 & H_4 \\ \hline X_1^6 & 7 & - & - \\ X_2^6 & 1 & 10 & - \\ X_3^6 & 1 & 0 & 176 \\ X_5^3 & 1 & 0 & 286 \\ \hline \end{array} \]
decreases with \( m \). The results for the homology dimensions for different numbers of colors are listed in Table 1, a specific choice of generators for each group can be found in [M"uh18].

4.3. Special cases. In some cases the usual suspects in the algebraic topologist’s toolbox allow for direct derivation of some results.

For \( s = 1 \) the moduli spaces \( X_1^m \) simply consists of \( m \) points, one for each rose graph \( R_{1,1} \) colored by \( c \in \{1, \ldots, m\} \). Hence, \( H_0(X_1^m; \mathbb{Z}) \cong \mathbb{Z}^m \) and trivial otherwise. Note that in every other case \( X_s^m \) is path-connected and therefore \( H_0(X_s^m; \mathbb{Z}) \cong \mathbb{Z} \). From this the ranks of all homology groups for the case \( s = 2 \) can be calculated by an Euler characteristic argument.

**Proposition 4.2.** For \( X_2^m = \text{MCG}_{1,2}^m \) we have

\[
H_1(X_2^m; \mathbb{Z}) \cong \mathbb{Z} \frac{(m-1)(m-2)}{2}.
\]

*Proof.* Instead of giving a combinatorial proof, we use a Mayer-Vietoris sequence to provide some insight into the topology at work.

Divide the space \( X_2^m \) into \( X_2^m = \overline{A} \cup \overline{B} \) with

\[
A := \{(G,F,c) \mid m \in \text{Im}(c)\},
B := \{(G,F,c) \mid c(e) \neq m \text{ for at least one } e \in E(G)\}.
\]

The subspaces \( A \) and \( B \) are depicted in Figure 3 and Figure 4, respectively. In these figures, black labels represent the labels of external legs, while red labels represent the coloring.

![Figure 3](image3.png)

**Figure 3.** The subspace \( A \subset X_2^m \) containing all cubes corresponding to graphs with at least one edge of color \( m \).

The intersection \( A \cap B \) consists of all cubes corresponding to graphs that contain an edge colored with \( m \) but with the other edge colored differently. Figure 3 illustrates this intersection. \( A \) is contractible and \( B \) deformation retracts to a subspace homeomorphic to \( X_2^{m-1} \) by shrinking all cubes containing an edge colored by \( m \) to zero length. Their intersection \( A \cap B \) consists of \( m - 1 \) disjoint lines and can be deformation retracted to \( m - 1 \) disjoint points.

The reduced Mayer-Vietoris sequence for \( X_2^m \) and its subspaces \( A \) and \( B \) reads

\[
\begin{align*}
0 & \rightarrow \tilde{H}_1(A \cap B; \mathbb{Z}) \rightarrow \tilde{H}_1(A; \mathbb{Z}) \oplus \tilde{H}_1(B; \mathbb{Z}) \rightarrow \tilde{H}_1(X_2^m; \mathbb{Z}) \\
& \rightarrow \tilde{H}_0(A \cap B; \mathbb{Z}) \rightarrow \tilde{H}_0(A; \mathbb{Z}) \oplus \tilde{H}_0(B; \mathbb{Z}) \rightarrow 0
\end{align*}
\]
Figure 4. The subspace $B \subset X^m_2$ containing all cubes corresponding to graphs with at least one edge not colored by $m$.

Figure 5. The intersection $A \cap B$ of the involved subspaces is a disjoint union of line segments.

Now $\widetilde{H}_n(A; \mathbb{Z}) = 0$ since $A$ is homotopy equivalent to a point. By the additivity axiom

\[ \widetilde{H}_1(A \cap B; \mathbb{Z}) \cong \bigoplus_{i=1}^{m-1} \widetilde{H}_1(\{\ast\}; \mathbb{Z}) = 0 \]

and

\[ H_0(A \cap B; \mathbb{Z}) \cong H_0(\bigcup_{i=1}^{m-1} \{\ast\}; \mathbb{Z}) \cong \mathbb{Z}^{m-1}. \]

Hence, $\widetilde{H}_0(A \cap B; \mathbb{Z}) \cong \mathbb{Z}^{m-2}$ and furthermore $\widetilde{H}_1(B; \mathbb{Z}) \cong \widetilde{H}_1(X^m_2; \mathbb{Z})$. Therefore, the above long exact sequence contains a short exact sequence which after application of the above isomorphisms reads

\[ 0 \rightarrow \widetilde{H}_1(X^m_2; \mathbb{Z}) \rightarrow \widetilde{H}_1(X^m_2; \mathbb{Z}) \rightarrow \mathbb{Z}^{m-2} \rightarrow 0, \]
where the first zero is \( \tilde{H}_1(A \cap B; \mathbb{Z}) \). This is a split exact sequence, so we immediately obtain
\[
\tilde{H}_1(X^m_s; \mathbb{Z}) \cong \tilde{H}_1(X^m_s; \mathbb{Z}) \oplus \mathbb{Z}^{m-2}
\]
With this the proposition follows by induction. \( \square \)

**Rational homology in the highest non-trivial dimension:** The spaces \( \mathcal{MG}^m_{n,s} \) naturally contain \( m \) copies of \( \mathcal{MG}_{n,s} \) as the subspaces corresponding to all graphs whose edges are colored identically. We use this fact by considering the subspace
\[
A := \{(G, \lambda, c) \in \mathcal{MG}^m_{n,s} \mid c \text{ is constant}\}
\]
and the long exact sequence of the pair \( (\mathcal{MG}^m_{n,s}, A) \). To simplify notation we set \( X := \mathcal{MG}^m_{n,s} \). The exact sequence then reads
\[
\ldots \rightarrow H_{k+1}(X, A; \mathbb{Q}) \rightarrow H_k(A; \mathbb{Q}) \rightarrow H_k(X; \mathbb{Q}) \rightarrow H_k(X, A; \mathbb{Q}) \rightarrow \ldots
\]
We have \( A \cong \bigcup_{n=1}^m \mathcal{MG}_{n,s} \) and therefore \( H_k(A; \mathbb{Q}) \cong H_k(\mathcal{MG}_{n,s}; \mathbb{Q})^m \) for all \( k \in \mathbb{N}_0 \). For the case \( n=0 \) this means in particular that \( H_k(A; \mathbb{Q}) = 0 \) for all odd \( k \) (see equation (4.1)). Thus, in case of an odd number of external legs \( s \) this yields a short exact sequence
\[
0 \rightarrow H_{s-1}(A; \mathbb{Q}) \rightarrow H_{s-1}(X^m_s; \mathbb{Q}) \rightarrow H_{s-1}(X^m_s, A; \mathbb{Q}) \rightarrow 0,
\]
which splits since we consider coefficients in \( \mathbb{Q} \). Therefore,
\[
H_{s-1}(X^m_s; \mathbb{Q}) \cong H_{s-1}(\mathcal{MG}_{1,s}; \mathbb{Q})^m \oplus H_{s-1}(X, A; \mathbb{Q}).
\]
This shows that a generator of \( H_{s-1}(\mathcal{MG}_{1,s}; \mathbb{Q}) \) endowed with a constant coloring is a generator of \( H_{s-1}(X^m_s, \mathbb{Q}) \) and moreover these generators are independent for different colors. Therefore, the full homology can in principle be understood by studying the relative homology groups \( H_{s-1}(X^m_s, A; \mathbb{Q}) \) only.

In particular, this and variations of this ansatz using various filtrations induced by the coloring may also be used to improve computer calculations. Furthermore, there are many interesting maps between these moduli space for different \( n, s \) and \( m \) given by attaching self-loops, adding, removing or gluing of legs and recoloring edges. The most interesting case surely is the variation of \( n \) which will be pursued in future work (cf. [CHKV16] for applications of these ideas in the uncolored case).

### 4.4. Homological stability

In the case of more than two legs \( s > 2 \) the first homology groups of \( X^m_s \) stabilize in the sense that \( H_1 \) is trivial for all numbers of legs and colors.

**Proposition 4.3.** For all \( s \geq 3 \) and \( m \geq 1 \)
\[
H_1(X^m_s, \mathbb{Z}) = 0.
\]

**Proof.** The statement actually follows from Theorem 4.6 below immediately, but we give an elementary proof providing a good look at the structure of the spaces \( X^m_s \). It relies on a careful examination of their two-skeleta (which in turn is needed for proving Theorem 4.6).

Let \( s \geq 3 \) and \( m \geq 1 \) and denote by \( X \) the two-skeleton of \( X^m_s \). It has a rather simple CW-complex structure:

- Its 0-dimensional cells \( \rho \) correspond to
  \( 1 \) all roses \( R^1_{1,s} \) with their single edge colored by \( c \in \{1, \ldots, m\}^8 \)

---

8. By abuse of notation we use here \( c \) to denote a single color instead of the coloring map.
(2) all "bananas" \( B_{U,V} \) with both edges of length \( \frac{1}{2} \) and colored by \( c \in \{1, \ldots, m\} \) and \( U \sqcup V = \{1, \ldots, s\} \) a partition of the set of legs,

- Its 1-dimensional cells \( \tau \) correspond to
  
  (1) arcs of length \( \frac{1}{2} \) parametrizing bananas \( B_{U,V} \) with two edges of unequal length, both colored by \( c \in \{1, \ldots, m\} \) and \( U \sqcup V = \{1, \ldots, s\} \),
  
  (2) arcs of unit length parametrizing bananas \( B^{a,b}_{U,V} \) with both edges colored differently by \( a,b \in \{1, \ldots, m\} \), \( a \neq b \), and \( U \sqcup V = \{1, \ldots, s\} \).

- Its 2-dimensional cells \( \sigma \) are disks parametrizing "triangle" graphs \( T^{a,b,c}_{A,B,C} \) where \( a,b,c \in \{1, \ldots, m\} \) and \( A \sqcup B \sqcup C = \{1, \ldots, s\} \).

The attaching maps are induced by edge contractions: Arcs of type (2) connect two colored roses, i.e. vertices of type (1), whereas type (1) arcs have a rose \( R_{1,s} \) and a banana \( B^{c}_{U,V} \) with edge lengths \( \frac{1}{2} \) as their endpoints. A disk \( \sigma^{a,b,c}_{A,B,C} \) is attached to three of the above described arcs, \( \tau^{a,c}_{A,B,C}, \tau^{c,b}_{A,C,B} \) and \( \tau^{a,b}_{A,C,B} \), describing the three possible banana configurations obtained from the corresponding triangle graph by contracting one of its three edges.

**Claim 1:** Let \( Y \) be the space obtained from \( X \) by contracting all \( m \) arcs of type (1) to their rose endpoint. Then \( X \cong Y \) are homotopy equivalent.

Proof: The \( m \) arcs form a disjoint union of contractible subspaces \( A_1, \ldots, A_m \) in \( X \). This allows to construct \( Y \) inductively. The claim then follows from the fact that for a contractible subcomplex \( A \subset X \) the quotient map \( \pi : X \to X/A \) is a homotopy equivalence, see e.g. [Hat02].

Thus, we can replace \( X \) by an even simpler CW-complex \( Y \) that has the following structure.

- Its 0-dimensional cells \( \rho^c \) correspond to roses \( R_{1,s} \) with their single edge colored by \( c \in \{1, \ldots, m\} \).
- Its 1-dimensional cells \( \tau^{a,b}_{U,V} \) correspond to arcs of unit length describing bananas \( B^{a,b}_{U,V} \) with both their edges colored differently by \( a,b \in \{1, \ldots, m\}, \ a \neq b \), and \( U \sqcup V = \{1, \ldots, s\} \). Their endpoints are the vertices \( \rho^c \) and \( \rho^b \).
- Its 2-dimensional cells are disks \( \sigma^{a,b,c}_{A,B,C} \) parametrizing triangle graphs \( T^{a,b,c}_{A,B,C} \) where \( a,b,c \in \{1, \ldots, m\} \) and \( A \sqcup B \sqcup C = \{1, \ldots, s\} \). If \( a \neq b \neq c \) the boundary of such a cell is given by three banana arcs as above. If two colors are equal, say \( a = b \), then this disk is attached to \( \tau^{a,b,c}_{A,B,C} \) and \( \tau^{a,c,b}_{A,C,B} \). If all three colors are equal, then \( \partial \sigma^{a,b,c}_{A,B,C} \) is the vertex \( \rho^c \).

**Claim 2:** All cycles in \( C_1(Y; \mathbb{Z}) \) are homologous to zero, i.e \( H_1(Y; \mathbb{Z}) = 0 \).

Proof: The key ingredient for showing this claim is a "reshuffling" property on the partitions \( U \sqcup V = \{1, \ldots, s\} \) which determine the leg structure of a banana graph \( B_{U,V} \). To make this precise, let \( \gamma \in C_1(Y; \mathbb{Z}) \) be a cycle of the form

\[
\gamma = \tau^{a,b}_{U,V} - \tau^{a,b}_{U,V}, \quad \text{so} \ \partial \gamma = 0.
\]
W.l.o.g. assume that $\bar{U} \cap V \neq \emptyset$ and $\bar{U} \cap U \neq \emptyset$ (otherwise we exchange the labels). The arc $\tau_{U,V}^{a,b}$ is one boundary component of the disk $\sigma_{A,B,C}^{a,a,b}$ with
\[
A = U, \\
B = \bar{U} \cap V, \\
C = (\bar{U} \setminus U) \triangle V,
\]
where $\triangle$ denotes the symmetric difference of sets. Note that $C$ is also given by $\{1,\ldots,s\} \setminus (U \cup (\bar{U} \cap V))$. Then (with a suitable choice of orientations) we compute
\[
\partial \sigma_{A,B,C}^{a,a,b} = \tau_{A,B\cup C}^{a,b} - \tau_{A\cup B,C}^{a,b} = \tau_{U,V}^{a,b} - \tau_{U\cup C,\bar{U} \cap V}^{a,b}.
\]
The other arc $\tau_{U\cup C,V \cap U}^{a,b}$ lies in the boundary of a disk $\sigma_{A,B,C}^{a,a,b}$ with
\[
\partial \sigma_{A,B,C}^{a,a,b} = \tau_{U\cap V,S,T}^{a,b} - \tau_{(U\cap V)\cup T}^{a,b},
\]
Now choose $T = \bar{U} \cap U$ which implies
\[
S = \{1,\ldots,s\} \setminus ((\bar{U} \cap V) \cup (\bar{U} \cap U)) = \{1,\ldots,s\} \setminus \bar{U} = \bar{V}.
\]
Thus,
\[
\partial \sigma_{A,B,C}^{a,a,b} = \tau_{U\cap V,S,T}^{a,b} - \tau_{(U\cap V)\cup T}^{a,b} = \tau_{U\cap V,S,T}^{a,b} - \tau_{U\cap V,S,T}^{a,b},
\]
and we conclude that there exists $\omega \in C_2(Y;\mathbb{Z})$ with $\gamma = \partial \omega$.

Now if $\gamma \in C_1(Y;\mathbb{Z})$ is a cycle consisting of three arcs $\tau_{a,b}^{a,b}$, $\tau_{b,c}^{b,c}$ and $\tau_{a,c}^{a,c}$ connecting three roses $\rho_a$, $\rho_b$ and $\rho_c$, then we can use the above construction to find a disk $\sigma_{U,V,W}^{a,b,c}$ such that
\[
\gamma = \partial \sigma_{U,V,W}^{a,b,c} + \partial \left( \sum_{i \in I} \sigma_{i}^{a,a,b} + \sum_{j \in J} \sigma_{j}^{b,b,c} + \sum_{k \in K} \sigma_{k}^{a,a,c} \right).
\]
Finally, if $\gamma \in C_1(Y;\mathbb{Z})$ is any linear combination of more than three arcs with $\partial \gamma = 0$, we decompose $\gamma$ into a sum of elementary three-piece cycles and use the previous step. □

The results in Table 1 suggest that this stability with respect to $m$ actually holds for all groups $H_i(X_m^n)$ with $i < s - 1$.  

**Conjecture 4.4.** For all $m > 1$ and $0 \leq i < s - 1$ there is an isomorphism
\[
H_i(\mathcal{MCG}_m^{1,s};\mathbb{Q}) \cong H_i(\mathcal{MCG}_1^{1,s};\mathbb{Q}) = H_i(\mathcal{MCG}_{1,s};\mathbb{Q}).
\]

Consider the cubical chain complexes used to compute the homology groups of $\mathcal{MCG}_m^{1,s}$. For any $m \in \mathbb{N}$ there is a canonical map that forgets the coloring of each graph,
\[
f^{(m)} : C_*(\mathcal{MCG}_m^{1,s}) \longrightarrow C_*(\mathcal{MCG}_{1,s}), \\
(G,F,c) \mapsto (G,F).
In the other direction there is also a map defined by equipping a graph with all possible colorings
\[ c^{(m)} : C_*(\mathcal{MG}_{n,s}) \to C_*(\mathcal{MCG}_{n,s}^m), \]
\[ (G, F) \mapsto \sum_{\text{all colorings } c} \frac{1}{m|E(G)|} (G, F, c). \]

A short calculation shows that both maps are chain maps. Let \( f^{(m)}_* \) and \( c^{(m)}_* \) denote the corresponding induced homomorphism on homology. Then the following relation holds.

**Proposition 4.5.** \( f^{(m)}_* \circ c^{(m)}_* = id_{H_*({\mathcal{MG}}_{n,s}; \mathbb{Q})} \). In particular, \( f^{(m)}_* \) is surjective and \( c^{(m)}_* \) is injective.

**Proof.** Let \( k \in \mathbb{N}_0 \). For all \([x] \in H_k(\mathcal{MG}_{n,s}; \mathbb{Q})\) represented by \( \sum_i q_i(G_i, F_i) \) with \( q_i \in \mathbb{Q} \) we have
\[ (f^{(m)}_* \circ c^{(m)}_*)([x]) = \left[ \sum_i q_i(f^{(m)}(c^{(m)}_*)(G_i, F_i)) \right] \]
and
\[ (f^{(m)}_* \circ c^{(m)}_*)(G_i, F_i) = f^{(m)}( \sum_{\text{all colorings } c} \frac{1}{m|E(G)|} (G_i, F_i, c)) \]
\[ = \sum_{\text{all colorings } c} \frac{1}{m|E(G)|} f^{(m)}(G_i, F_i, c) \]
\[ = \sum_{\text{all colorings } c} \frac{1}{m|E(G)|} (G_i, F_i) \]
\[ = (G_i, F_i). \]

This means in particular that \( \dim H_k(X^1_{m,s}; \mathbb{Q}) \leq \dim H_k(X^m_{s,m}; \mathbb{Q}) \) holds for all \( k, s \) and \( m \), proving "one half" of the conjecture.

**Homotopy type.** Now we take a look at the connectivity of the moduli spaces of \( m \)-colored one-loop graphs with \( s \) legs.

First consider the cases \( s \in \{1, 2\} \). The moduli spaces \( X^m_s \) simply consist of \( m \) points, one for each rose graph \( R_{1,1} \) colored by \( c \in \{1, \ldots, m\} \). Thus, \( \pi_0(X^m_s) = \mathbb{Z}^m \) and trivial otherwise.

The case \( s = 2 \) is also easy to describe. A CW-decomposition similar to the one described in the proof of Proposition 1.3 shows that \( X^m_2 \) is homotopy equivalent to the complete graph \( K_m \) on \( m \) vertices. This, in turn, is homotopy equivalent to a wedge of \( \frac{(m-1)(m-2)}{2} \) circles \( S^1 \). Thus, its fundamental group is isomorphic to the free group on these generators, \( \pi_1(X^m_2) \cong F_{\frac{(m-1)(m-2)}{2}} \).

Now turning to the general case, let \( s > 2, m > 1 \) and recall the description of the CW-structure of the two skeleton of \( X^m_s \) from the proof of Proposition 1.3.

**Theorem 4.6.** For \( s \geq 3 \) all the moduli spaces \( X^m_s \) are simply connected.
Remark 4.7. The theorem should be optimal in the sense that the spaces $X_m$ cannot be higher connected. This follows from the Hurewicz theorem (see e.g. [Hat02]) together with the results for $H_2(X_m)$ in Table 1. In fact, as a corollary we find $\pi_2(X_m) \cong H_2(X_m)$.

For the proof we need a general fact about CW-complexes that allows to compute their fundamental group by a rather elementary process.

Proposition 4.8 (Proposition 6.48 in [Swi02]). Let $X$ be a connected CW-complex. Let $i : X^{(1)} \to X$ denote the inclusion of its one-skeleton and choose a basepoint $x_0 \in X^{(1)}$. Then $i_* : \pi_1(X^{(1)}, x_0) \to \pi_1(X, x_0)$ is an epimorphism and $\ker i_*$ is the normal subgroup generated by the elements $[\varphi_\alpha]$ for $\varphi_\alpha : (S^1, s_0) \to (X^{(1)}, x_0)$ the attaching maps of the two-cells $e_\alpha^2$.

Proof. (of Theorem 4.6) We start by replacing $X_m$ with a simpler homotopy equivalent space as in Proposition 4.3. For this, we simply collapse the contractible subcomplex $\mathcal{Z}$ consisting of all arcs $\tau_{U,V}$ describing the one special arc between $V$ colored by a single color $c \in \{1, \ldots, m\}$ and one freely chosen arc $\tau_\mathcal{Z} := \tau_{a_0,b_0}$ with $a_0, b_0 \in \{1, \ldots, m\}, a_0 \neq b_0$ and $U \sqcup V = \{\ldots, s\}$. Note that $\mathcal{Z}$ forms a maximal tree in the one-skeleton of $X_m$. In particular, all zero-dimensional cells of $X_m$ lie in $\mathcal{Z}$. Therefore, in the process of collapsing this subcomplex all these cells get identified to a single vertex $\rho_0$.

Let us denote the resulting space by $Y$. Following the recipe for determining $\pi_1(Y)$ we start with the one-skeleton $Y^{(1)}$ which decomposes into

- a zero-dimensional cell $\rho_0$,
- one-dimensional cells $\tau_{U,V}^{a,b}$, one for every partition $\{1, \ldots, s\} = U \sqcup V$ and color pair $a, b \in \{1, \ldots, m\}$, $a \neq b$, except the one combination $U_0, V_0, a_0, b_0$ describing the one special arc $\tau_\mathcal{Z}$ chosen above.

Thus, $Y^{(1)}$ is a wedge of $k$ circles, $k$ being the number of partitions $\{1, \ldots, s\} = U \sqcup V$ times the number of distinct pairs $\{a, b\} \subset \{1, \ldots, m\}$ minus one,

$$k = \left\lfloor \frac{s}{2} \right\rfloor m(m - 1) - 1.$$

with $\left\lfloor \cdot \right\rfloor$ denoting the Stirling number of the second kind [AS64].

According to Proposition 4.8 the fundamental group of $Y$ is the free group on $k$ generators, subject to relations coming from the attaching maps of all of its two-cells. These relations are of the form

$$\sigma_{A,B,C}^{a,b,c} : \tau_{A,B\cup C}^{a,b} \cdot (\tau_{A\cup B,C}^{a,c})^{-1} = e,$$

$$\sigma_{A,B,C}^{a,b,c} : \tau_{A\cup B,C}^{a,b} \cdot (\tau_{A,B\cup C}^{a,c})^{-1} = e,$$

$$\sigma_{A,B,C}^{a,b,c} : \tau_{A\cup B,C}^{a,c} \cdot (\tau_{A,B\cup C}^{b,c})^{-1} = e.$$

We now show that they kill all generators, so that $\pi_1(Y, \rho_0) = 0$. From the first row we find for every pair of partitions $U \sqcup V = S \sqcup T = \{1, \ldots, s\}$

$$\tau_{U,V}^{a,b} = \tau_{S,T}^{a,b}$$

for all $\{a, b\} \subset \{1, \ldots, m\}$

and therefore in the case $a = a_0, b = b_0$

$$\tau_{U,V}^{a_0,b_0} = \tau_\mathcal{Z} = e$$

for all $U, V$ with $U \sqcup V = \{1, \ldots, s\}$. 

From this and the second row we deduce the relation
\[ \tau_{b^0,c}^{U,V} = \tau_{a_0,c}^{U,V} \text{ for all } c \in \{1, \ldots, m\} \text{ and } U \sqcup V = \{1, \ldots, s\} \]
while the third row implies
\[ \tau_{b^0,c}^{U,V} = (\tau_{a_0,c}^{U,V})^{-1} \text{ for all } c \in \{1, \ldots, m\} \text{ and } U \sqcup V = \{1, \ldots, s\}. \]
Hence, we conclude \( \tau_{b^0,c}^{U,V} = \tau_{a_0,c}^{U,V} = e \) which allows to rewrite the third row in the case \( a = a_0 \) as
\[ e = \tau_{b,c}^{A \sqcup B, C} \cdot \tau_{a_0,b}^{A, B \cup C} \cdot (\tau_{a_0,b}^{A,C,B})^{-1} = \tau_{b,c}^{A, B \cup C} \]
and therefore \( \tau_{b,c}^{A, B \cup C} = e \) for all \( \{a,b\} \subset \{1, \ldots, m\} \) and \( U \sqcup V = \{1, \ldots, s\}. \)

Remark 4.9. The statement does not hold for the moduli spaces of holocolored graphs. Indeed, as we will see in the next section, \( \mathcal{M}H_G^{1,s} \) and \( \mathcal{M}R_G^{1,s} \) are homeomorphic to the two-dimensional torus \( T^2 \) which has \( \mathbb{Z}^2 \) as fundamental group.

5. Holocolored graphs

In contrast to the case of arbitrarily colored graphs the spaces \( \mathcal{M}H_G^{1,s} \) and \( \mathcal{M}R_G^{1,s} \) decompose directly into a \( \Delta \)-complexes. Therefore, we may use instead of the cubical chain complex the associated semi-simplicial chain complex which has fewer generators. We thereby obtain a graph complex, somewhat in the spirit of Kontsevich’s graph homology \([Kon93, Kon94]\), that computes the homology of the moduli spaces of holocolored graphs.

5.1. The moduli spaces \( \mathcal{M}H_G^{1,s} \). Let \( \tilde{X}_s \) denote the space \( \mathcal{M}H_G^{1,s} \). In the semi-simplicial chain complex for \( \tilde{X}_s \) the \( k \)-dimensional chain groups are generated by simplices \( \Delta_{(G,c)} \), one for each isomorphism class of admissible colored graphs on \( k + 1 \) edges. As in the cubical case we abbreviate these generators by \( (G,c) \). The boundary operator of this chain complex contains the terms with shrunken edges only and reads\(^9\)
\[ \partial_k (G,c) := \sum_{i=1}^{k} (-1)^{i-1} (G_{e_i}, c_{e_i}). \]

Recall that the number of allowed colors is \( 3(n-1)+s \) which in the one loop case equals \( s \), the maximal number of internal edges of the graphs representing elements in \( \tilde{X}_s \).

Example 5.1. While for \( s = 1, 2 \) the space \( \tilde{X}_s \) is just a single point or a single edge, respectively, the space \( \tilde{X}_3 \) is homeomorphic to the two-dimensional torus \( T^2 \). Its \( \Delta \)-complex structure consists of six 2-simplices, nine 1-simplices and 3 vertices with identifications as depicted in Figure 6.

As in the previous case the homology of the moduli space of holocolored graphs can be calculated with the help of a computer program. The dimensions of the homology groups of the spaces \( \tilde{X}_s \) that could be obtained are listed in Table 2 for \( 1 \leq s \leq 5 \). A particular choice of generators for these groups can be found in [Müh18].

\(^9\)Note that this simple definition works only for the case \( n = 1 \). Otherwise it might not be possible to shrink an edge to zero length, owing to the occurrence of missing faces.
Figure 6. A geometric representation of $\tilde{X}_3$ as a $\Delta$-complex.

Table 2. The dimension of the homology groups $H_k(\tilde{X}_s; \mathbb{Q})$ for $1 \leq s \leq 5$ and $0 \leq k \leq 4$.

| $\tilde{X}_1$ | $H_0$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ |
|----------|--------|--------|--------|--------|--------|
| $\tilde{X}_2$ | 1 | - | - | - | - |
| $\tilde{X}_3$ | 1 | 0 | - | - | - |
| $\tilde{X}_4$ | 1 | 2 | 1 | - | - |
| $\tilde{X}_5$ | 1 | 0 | 36 | 3 | - |

Homology in the highest non-trivial dimension: The group $H_{s-1}(\tilde{X}_s)$ can be determined explicitly by merging the simplices in the $\Delta$-decomposition of $\tilde{X}_s$ into larger cubes.

**Proposition 5.2.** For $s \geq 3$ we have

$$H_{s-1}(\tilde{X}_s, \mathbb{Z}) \cong \mathbb{Z}^{(s-1)!}.$$ 

**Proof.** We describe a cubical chain complex that computes $H_{s-1}(\tilde{X}_s)$.

A point $(G, \lambda, c) \in \tilde{X}_s$ can be thought of as embedded in $\mathbb{R}^2$ by drawing a circle of finite radius (which is $\frac{1}{2\pi}$ for normalized graphs) to represent the union of all of its edges. Fix an arbitrary point on the circle and identify it with a leg/vertex $v_0 \in V(G)$. The length of each edge is uniquely determined by fixing $s-1$ angles...
\( \theta_1, \ldots, \theta_{s-1} \) representing the position of the legs/vertices \( v_1, \ldots, v_{s-1} \in V(G) \) on the circle with respect to the position of the distinguished vertex \( v_0 \). For convenience we choose a parametrization such that angles \( \theta_i \) reach from 0 to 1.

In this way \((G, \lambda, c)\) can be represented by a tuple \((\theta_1, \ldots, \theta_{s-1}, \sigma)\) where \( \sigma \in \Sigma_s \) encodes the coloring \( c \). This representation is not unique, since we have

\[
(\theta_1, \ldots, \theta_{s-1}, \sigma) \sim (\theta_{s-1}, \ldots, \theta_1, \sigma^-)
\]

where \( \sigma^- \) denotes the permutation \((\frac{1}{\sigma_1} \frac{2}{\sigma_2} \ldots \frac{s}{\sigma_s})\).

This defines a cubical complex in which each \( \sigma \in \Sigma_s / \mathbb{Z}_2 \) designates an \( s-1 \) dimensional cube \( w_\sigma := (\theta_1, \ldots, \theta_{s-1}, \sigma) \). On the associated chain complex the cubical boundary operator \( \partial = \partial^+ + \partial^- \) is then given by

\[
\partial^+ w_\sigma = \sum_{i=1}^{s-1} (-1)^{i-1} (\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_{s-1}, \sigma)
\]

and

\[
\partial^- w_\sigma = \sum_{i=1}^{s-1} (-1)^i (\theta_1, \ldots, \theta_{i+1}, 1, \theta_{i+1}, \ldots, \theta_{s-1}, \sigma).
\]

To describe the face relations in this complex we define a "shuffling" operator \( \tau_+ : \Sigma_s \rightarrow \Sigma_s \) by

\[
\tau_+ \sigma := \tau_+ (\frac{1}{\sigma_1} \frac{2}{\sigma_2} \ldots \frac{s}{\sigma_s}) = (\frac{1}{\sigma_2} \frac{2}{\sigma_3} \ldots \frac{s-1}{\sigma_s} \frac{s}{\sigma_1}).
\]

Letting any external leg rotate once around the graph leads to a cyclic permutation of the edges. More precisely, for any \( \sigma \in \Sigma_s / \mathbb{Z}_2 \) we have

\[
(\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_{s-1}, \tau_+ \sigma) = (\theta_1, \ldots, \theta_{i-1}, 1, \theta_{i+1}, \ldots, \theta_{s-1}, \tau_+ \sigma).
\]

This immediately yields

\[
(5.2) \quad \partial_\sigma \tau_+ \sigma = \partial^+ \partial^- w_\sigma + \partial^+ \partial^- w_\sigma = \partial^+ \partial^- w_\sigma - \partial^+ \partial^- w_\sigma.
\]

The highest non-trivial homology group \( H_{s-1}(X_s) \) is just ker \( \partial_{s-1} \) and equation \([5.2]\) sets up a linear system of equations to determine this kernel. To make this explicit, we quotient out the cyclic permutations generated by \( \tau_+ \). The quotient group is thus \((\Sigma_s / \mathbb{Z}_2) / C_s \cong \Sigma_s / (C_s \ltimes \mathbb{Z}_2) \cong \Sigma_s / D_s \) for \( D_s \) the dihedral group. It consists of \( \frac{n(s)}{2} = \frac{\left(s-1\right)!}{2} \) equivalence classes and we choose a representative \( \sigma_1, \ldots, \sigma_n(s) \) (with \( n(s) = \frac{\left(s-1\right)!}{2} \)) for each one.

The matrix representation of \( \partial_{s-1} \) with respect to the basis

\[
\{ \partial^+ \partial^- w_{\sigma_1}, \partial^+ \partial^- w_{\tau_+ \sigma_1}, \ldots, \partial^+ \partial^- w_{\tau_+^{n(s)-1} \sigma_1} \}_{i=1, \ldots, n(s)}
\]

of the target space \( C_{s-2}(X_s) \) then reads

\[
A_{\partial_{s-1}} = \begin{pmatrix}
A & A & \cdots \\
A & & \\
& & A
\end{pmatrix}.
\]
containing \( n(s) \) copies of

\[
A = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & -1 \\
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 1 \\
\end{pmatrix}.
\]

The matrix \( A \) can easily be brought into row-echelon form and reveals its rank to be \( \text{rank}(A) = s - 1 \). Thus there are \( \frac{(s-1)!}{2} \) solutions to the homogeneous system of equations defined by \( A_0 \), and we obtain

\[
H_{s-1}(\tilde{X}_{1,s}) \cong \mathbb{Z}^{\frac{(s-1)!}{2}}.
\]

\( \square \)

**Permuting colors:** Let \( \Sigma_C := \text{Perm}(C) \cong \Sigma_{3(n-1)+s} \) denote the group of permutations of the set of colors \( C \). It acts on \( \mathcal{MHG}_{n,s} \) by changing the coloring, for \( g \in \Sigma_C \) by \( g.(G, \lambda, c) := (G, \lambda, g \circ c) \).

This action respects the "relative semi-simplicial structure" by mapping open \( k \)-simplices to open \( k \)-simplices. Furthermore, it does so transitively on each set of open \( k \)-simplices in \( \mathcal{MC}_{G_{n,s}} \). The stabilizer of a point \((G, \lambda, c)\) depends only on the number of edges of its representing graph \( G \); if \( G \) has \( k \) edges we find its stabilizer as the set of permutations that act only on the colors not in the image of \( c \), \( \forall x = (G, \lambda, c) \in \mathcal{MC}_{G_{n,s}} : \Sigma_{C_x} \cong \text{Perm}(C \setminus \text{im}(c)) \cong \Sigma_{3(n-1)+s-k} \).

This shows that the orbit space of this action is the moduli space of uncolored graphs \( \mathcal{MG}_{n,s} \) and the projection \( \pi : \mathcal{MC}_{G_{n,s}} \to \mathcal{MC}_{G_{n,s}}/\Sigma_C = \mathcal{MG}_{n,s} \) is a branched covering map, i.e. a covering map outside a nowhere-dense set, in this case outside of \( B = \{(G, \lambda, c) \in \mathcal{MC}_{G_{n,s}} \mid |E(G)| < 3(n-1) + s\} \).

For \( n = 1 \) this means that if we decompose \( \tilde{X}_s \) into a simplicial complex, for instance by performing two barycentric subdivisions, then the action of \( \Sigma_C \cong \Sigma_s \) is simplicial, i.e. for every \( g \in \Sigma_C \) the map \( v \mapsto g.v \) is simplicial\(^{10}\).

In that case the projection \( \pi : \tilde{X}_s \to \tilde{X}_s/\Sigma_C = \mathcal{MG}_{1,s} \) that forgets the coloring is a simplicial branched covering map, i.e. a simplicial covering map outside the nowhere-dense set \( \tilde{X}_s^{(s-2)} \), the \((s-2)\)-skeleton of \( \tilde{X}_s \).

**The Euler characteristic of \( \tilde{X}_s \):** The number of \( k \)-simplices in the moduli space of holocolored graphs can be determined by combinatorial means.

For an admissible graph \( G \) with \( k \) internal edges, denote the set of all holocolorings (with \( s \) available colors) of \( G \) by \( \mathcal{C}_{k,s}(G) \), and the set of its non-equivalent external leg structures with \( s \) legs by \( \mathcal{S}_{k,s}(G) \).

For an one-loop holocolored graph \( G \) with \( k \) internal edges there are \( \binom{s}{k} \) ways to choose which colors the edges of \( G \) can have. Furthermore, there are \( k! \) ways to permute these edges to get different graphs, except in the case \( k = 2 \) where both

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\(^{10}\) A map between two simplicial complexes \( f : K \to K' \) is simplicial if it sends every simplex in \( K \) to a simplex in \( K' \) by a map taking vertices to vertices.
permutations of edges yield the same graph. Thus,
\begin{equation}
|\mathcal{C}_{k,s}(G)| = \binom{s}{k} \frac{k!}{1 + \delta_{k,2}} = \frac{s!}{(s-k)!(1+\delta_{k,2})}.
\end{equation}

The leg structure is uniquely determined by a partition of the $s$ legs into $k$ non-empty groups and an element of $S_k/(C_k \ltimes \mathbb{Z}_2)$ which represents their ordering. There are $\{\binom{s}{k}\}$ number of ways to choose such a partition where $\{n\}$ is the Stirling number of the second kind \cite{AS64},
\[\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} n^j.\]

For $k \geq 3$ the group $S_k/(C_k \ltimes \mathbb{Z}_2)$ has $\frac{(k-1)!}{2}$ elements. Thus, there are exactly this many ways to order the $s$ legs to yield distinct graphs. For $k = 1$ and $k = 2$ there is clearly only one way to organize the legs. Hence,
\begin{equation}
|\mathcal{S}_{k,s}(G)| = \left\{ \binom{s}{k} \frac{(k-1)!}{2} (1+\delta_{k,1} + \delta_{k,2}) \right\}.
\end{equation}

With this the number of $(k-1)$-simplices $N_{k,s}$ in $\tilde{X}_s$ can be calculated. Each such $(k-1)$-simplex belongs to a one-loop holocolored graph with $k$ edges and $s$ legs. There are $|\mathcal{C}_{k,s}(G)| \cdot |\mathcal{S}_{k,s}(G)|$ such graphs since the choices of coloring and leg structure are independent, so by (5.3) and (5.4)
\[N_{k,s} = \binom{s}{k} k! \left\{ \binom{s}{k} \frac{(k-1)!}{2} (1+\delta_{k,1}) \right\} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^s.\]

This in turn can be used to calculate the Euler characteristic of $\tilde{X}_s$,
\[\chi(\tilde{X}_s) = \sum_{k=1}^{s} (-1)^{k-1} N_{k,s} = \sum_{k=1}^{s} (-1)^{k-1} \frac{(k-1)!}{2 - \delta_{k,1}} \binom{s}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^s = \sum_{k=1}^{s} \sum_{j=0}^{k} (-1)^{j+1} \binom{s}{k} \binom{k}{j} \frac{j^s (k-1)!}{2 - \delta_{k,1}}.\]

For $1 \leq s \leq 8$ the Euler characteristic obtained by this formula can be found in Table 3.

5.2. Graphs with remembering edges. When considering Feynman graphs in the operator product expansion \cite{BZ81, LZ05}, an edge shrunk to zero length still carries the physical information assigned to it (for instance, its mass or particle type). More precisely, the vertex it gets identified with describes a new type of interaction, depending on the type of the contracted edge. Thus, from a physics
perspective it might be worthwhile to consider an alternative complex in which face relations respect this restriction.

Recall the definition of $\mathcal{MRG}_{n,s}$ where again $n$ is the number of loops, $s$ the number of external legs and the number of colors is taken to be the maximal number of internal edges that can occur, $|C| = 3(n - 1) + s$. Note that we consider the same ground set of holocolored graphs but identify points with respect to a different relation $\sim_s$ given by

$$(G, \lambda, c) \sim_s (G', \lambda', c') \iff \exists \varphi : G \to G' \text{ homothety s.t. } c = c' \circ \varphi.$$ 

In the following let $n = 1$ and denote by $\bar{X}_s$ the space $\mathcal{MRG}_{1,s}$. Up to $s = 3$ there is no difference between this space and the moduli space of holocolored graphs with "forgetful edges". A difference can only occur if there are at least two vertices to which more than one leg is connected. In particular, $H_{s-1}(\bar{X}_s) \cong H_{s-1}(\tilde{X}_s)$ which means that the results from equation (5.1) are still valid. Hence,

$$H_{s-1}(\bar{X}_s) \cong \mathbb{Q}^{\frac{(s-1)(s+1)}{2}}.$$ 

For explicit calculations of the homology of $\bar{X}_s$ the representation of a simplex has to be slightly modified to account for the new face relations induced by $\sim_s$. The color of shrunk edges is now relevant when identifying faces of different simplices. We therefore add a weight to each vertex that has more than one leg attached. This weight is the set of colors of the edges that were collapsed to that vertex (i.e. one for each additional leg).

The homology groups in the one-loop case for up to five legs that were calculated with computer assistance are listed in Table 4. An explicit choice of generators can be found in [Müh18].

| $s$ | $H_0$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ |
|-----|------|------|------|------|------|
| $X_1$ | 1 | - | - | - | - |
| $X_2$ | 1 | 0 | - | - | - |
| $X_3$ | 1 | 2 | 1 | - | - |
| $X_4$ | 1 | 0 | 18 | 3 | - |
| $X_5$ | 1 | 0 | 48 | 166 | 12 |

Table 4. The dimension of the homology groups $H_k(\bar{X}_s; \mathbb{Q})$ for $1 \leq s \leq 5$ and $0 \leq k \leq 4$.

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