Local asymptotically optimal test in ARCH model

Lounis tewfik
Laboratoire de mathématiques Nicolas Oresme, CNRS UMR 6139 Université de Caen
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Abstract. This work is an extension in Arch models of the theorem of S.Y. Hwang and I.V. Basawa (2001) which was used before in nonlinear time series contiguous to $AR(1)$ processes. Our results are established under some general assumptions and stationarity and ergodicity conditions. Local asymptotic normality (LAN) for the log likelihood ratio was established. An optimal test was constructed when the parameter is unspecified. The method is based on the introducing of a new estimator.

Keywords and phrases: Local asymptotic normality, Contiguity, efficiency, identifiable models, Le Cam’s third lemma, discrete estimate, modified estimator, time series models, ARCH models.

1 Introduction

The study of the chronicles emanating from economic, biological, financial, hydrological, biomedical data or others make use of relevant mathematical models, namely the time series models that allow to model this type of problems provided that this framework takes into account several criteria, such as, for instance, the dependance of the observations, or the mean and the variance which are functions that depend on time. This often leads us to choose a class of well adapted models to aggregate these differences best. The chosen class will be that of stochastic models which will be detailed in the following. Let $\{Y_i, X_i\}$ be a sequence of stationary and ergodic random vectors with finite second and third moment such that for all $i \in \mathbb{Z}$, $Y_i$ a univariate random variable and $X_i$ a $d$-variate random vector. We consider the class of stochastic models

$$Y_i = T(Z_i) + V(Z_i) \epsilon_i, \quad i \in \mathbb{Z},$$

where the random vectors $Z_i = (Y_{i-1}, Y_{i-2}, \ldots, Y_{i-s}, X_{i-1}, X_{i-2}, \ldots, X_{i-\varrho})$, for given non negative integers $s$ and $\varrho$, the $\epsilon_i$’s are centred iid random variables with unit variance and density function $f$, such that for all $i \in \mathbb{Z}$, $\epsilon_i$ is independent of $\mathcal{F}_i = \sigma(Z_j, j \leq i)$, the real-valued functions $T(\cdot)$ and $V(\cdot)$ are unknown.

In this paper we study the problem of testing of the couple of functions $(T(\cdot), V(\cdot))$ in a class of parametric functions. Another words, let

$$\mathcal{M} = \{(m(\rho, \cdot), \sigma(\theta, \cdot)), \ (\rho^\top, \theta^\top)^\top \in \Theta_1 \times \Theta_2\},$$

where for all set $A$, $\text{int}(A)$ denotes the interior of the set $A$ and the script $\top$ denotes the transpose. $\ell$ and $p$ are two positive integers, and each one of the two functions $m(\rho, \cdot)$ and $\sigma(\theta, \cdot)$ has a known form such that $\sigma(\theta, \cdot) > 0$. For a sample of length $n$, we derive a test of $H_0 [(T(\cdot), V(\cdot)) \in \mathcal{M}]$ against $H_1 [(T(\cdot), V(\cdot)) \notin \mathcal{M}]$, one can remark that the null hypothesis $H_0$ is equivalent to :

$$H_0[(T(\cdot), V(\cdot))] = \left(m(\rho_0, \cdot), \sigma(\theta_0, \cdot)\right),$$

for some $(\rho_0^\top, \theta_0^\top)^\top \in \Theta_1 \times \Theta_2$ while the alternative hypothesis $H_1$ is equivalent to

$$H_1[(T(\cdot), V(\cdot))] \neq \left(m(\rho_0, \cdot), \sigma(\theta_0, \cdot)\right).$$
When we choose the alternative hypothesis like this
For all integers \( n \geq 1 \) the alternative hypothesis \( H_1^{(n)} \) is define by the following equality
\[
H_1^{(n)}[(T(\cdot), V(\cdot)) = \left( m(\rho_0, \cdot) + h n^{-\frac{1}{2}} G(\cdot), \sigma(\theta_0, \cdot) + h' n^{-\frac{1}{2}} S(\cdot) \right).
\]
\( G \) and \( S \) are two specified functions with values in \( \mathbb{R} \), \((h, h') \in K_1 \times K_2 \) where \( K_1 \) et \( K_2 \) are two compacts of \( \mathbb{R} \) and \( hh' \neq 0 \).
Under the null hypothesis \((H_0)\), the time series model (1.1)
\[
Y_i = m(\rho_0, Z_i) + \sigma(\theta_0, Z_i) \epsilon_i.
\]
(1.2)
And under the alternative hypothesis \( H_1^{(n)} \), the time series model (1.1) begin
\[
Y_i = m(\rho_0, Z_i) + h n^{-\frac{1}{2}} G(Z_i) + \left( \sigma(\theta_0, Z_i) + h' n^{-\frac{1}{2}} S(Z_i) \right) \epsilon_i,
\]
(1.3)
Let \( f_0 \) and \( f_{n, h, h'} \) denote the density function of the random variable \( Y_i \) corresponding to the time series model (1.2) and (1.3) respectively, and let \( f_{n,0} \) and \( f_{n,h,h'} \) denote the density function of the random vector \((Y_1, \ldots Y_n)\) corresponding to the time series model (1.2) and (1.3) respectively. Different specifications of \( m(\rho_0, \cdot) \) and \( \sigma(\theta_0, \cdot) \) show that (1.2) embodies a large class of time series models, for instance, we name AR, ARMA, SETAR, SETAR-ARCH and \( \beta \)-ARCH.
We consider the problem of testing the null hypothesis \((H_0)\) against the alternative hypothesis \( (H_1^{(n)}) \) such that
\[
(H_0) : m(\rho, Z_i) = m(\rho_0, Z_i) \quad \text{and} \quad \sigma(\theta, Z_i) = \sigma(\theta_0, Z_i),
\]
and,
\[
(H_1^{(n)}) : m(\rho, Z_i) = m(\rho_0, Z_i) + h n^{-\frac{1}{2}} G(Z_i) \quad \text{and} \quad \sigma(\theta, Z_i) = \sigma(\theta_0, Z_i) + h' n^{-\frac{1}{2}} S(Z_i).
\]
We use the Neyman-Pearson test statistic based on the log-likelihood ratio \( \Lambda_{n,h,h'} \) which is defined by the following equality
\[
\Lambda_{n,h,h'} = \log \left( \frac{f_{n,h,h'}}{f_{n,0}} \right) = \sum_{i=1}^{n} \log(g_{n,i,h,h'}).
\]
(1.4)
Our aim is to establish the normality of the test. Based on (Hwang and Basawa, 2001, Theorem 1) and under some hypothesis and conditions and to a constant close, the log-likelihood ratio (1.4) is asymptotically equivalent to a sequence of random variables which is called the central sequence, therefore we obtain an optimal test in the case where the parameter \((\rho_0, \theta_0)\) is specified. In a general case, the parameter \((\rho_0, \theta_0)\) is unknown, so the propriety of the optimality of the test is not asserted. In order to estimate this parameter, we use locally discrete estimates, this kind of estimates was introduced by Le Cam (1960), and used by Bickel (1982) and Kreiss (1987).
The advantage of discrete estimates is the Lemma (4.4) of (Kreiss (1987). This Lemma was among the fundamental tool used by several authors to complete their research works, we can name the articles of Hallin and Puri (1994), Benghabrit and Hallin (1998, 1996) and Cassart et al. (2008).
When we consider the difference between the two expressions of the central sequence and an estimated central sequence, sometimes it is possible to prove the optimality of the test. In our case and after the difference between the two central sequences, we get asymptotically a non-degenerate term. In order to solve this very problem and on the basis of the discrete estimates, we introduce a new estimator, the principle is to absorb the error of the difference between the estimated central sequence and the central sequence with the unknown parameter by modifying one component of the discrete estimate, this method is presented in (Lounis, 2012, Section 1). Consequently, under some assumptions, the optimality of the constructed test is proved.
The paper is organized as follows
In the forthcoming (2), we establish some general assumptions and results which are used in order to construct the test when the parameter is assumed known, the local asymptotic normality is established, an optimal test is constructed and it’s asymptotic power is derived. In section (3), supplementary assumptions are given, the discrete estimates were introduced and applied for the central sequences. In section (4), we prove the optimality of the test when the parameter is unknown, the proof is based on the modified estimate which is defined in the work of Lounis (2012). Section (5) concerned the generalization of our results in \( \mathbb{Z} \). In section (6), we conduct a simulations in order to investigate the performance of the proposed test. All mathematical developments are relegated to the Section (7).
2 The construction of the test when the parameter is known

Many results and assumptions are stated in the next subsection in order to construct our test in the case when the parameter of the study time series model is specified.

2.1 Main results and assumption

Throughout we assume that \( i \in \mathbb{N} \). An extension on \( \mathbb{Z} \) will be made at the end of this paper. Consider the time series models

\[
Y_i = m(\rho_0, Z_i) + \sigma(\theta_0, Z_i) \epsilon_i,
\]

and,

\[
Y_i = m(\rho_0, Z_i) + h \bar{n}^{-\frac{1}{2}} G(Z_i) + \left( \sigma(\theta_0, Z_i) + h' \bar{n}^{-\frac{1}{2}} S(Z_i) \right) \epsilon_i.
\]

In order to establish the principle of local asymptotic normality (LAN) for the log-likelihood ratio \( A_{n,h,h'} \), we use (Hwang and Basawa, 2001, Theorem (1)), so we check the three conditions noted (C.1), (C.2) and (C.3) such that:

For a fixed step \((h, h')\) in \( K_1 \times K_2 \) where \( hh' \neq 0 \), we have

(C.1) \( \max_{1 \leq i \leq n} |g_{n,i,h,h'} - 1| = o_P(1) \).

(C.2) There exist a positive constant \( \tau_{h,h'}^2 \) such that

\[
\sum_{i=1}^{n} (g_{n,i,h,h'} - 1)^2 = \tau_{h,h'}^2 + o_P(1).
\]

(C.3) There exist a \( \mathcal{F}_n \) measurable random variable \( V_{n,h,h'} \) such that

\[
\sum_{i=1}^{n} (g_{n,i,h,h'} - 1) = V_{n,h,h'} + o_P(1).
\]

In order to establish our results, we need the following assumptions and notations.

For all \( x \in \mathbb{R} \), let

\[
M_f(x) = \frac{f(x)}{f(x)}.
\]

We assume that the function \( x \mapsto M_f(x) \) is differentiable, we denote by \( \dot{M}_f \) the derivative function of \( M_f \).

Consider the function \( F \) defined by

\[
F(x; a, b) = \frac{1}{b} f \left( \frac{x - a}{b} \right), \quad \text{where } |a| < \infty \text{ and } 0 < b < \infty.
\]

We assume that the following assumptions are satisfied:

\begin{itemize}
  \item [(A_1)] \( (A_{1,1}) \) : There exist a measurable positive function \( \varphi \), a real \( p > 1 \) such that 
    \[
    E(\varphi^p(\epsilon_0)) < +\infty \text{ and a strictly positive real } \zeta, \text{ where } \zeta > \max(|a|, |b - 1|) \text{ such that }
    \frac{\partial^2 F(x; a, b)}{\partial a \partial b^k} \leq \varphi(x),
    \]
    \( j \) and \( k \) are two positive integers such that \( j + k = 2 \).
  \item [(A_{1,2})] \text{ There exist a positive functions } V_1 \text{ and } V_2 \text{ such that }
    \[
    \left| \frac{\partial^2 F(x; a, b)}{\partial a^j \partial b^k} - \frac{\partial^2 F(x; a', b)}{\partial a^j \partial b^k} \right| \leq V_1(x; a^*, b)|a - a'| \text{ and }
    \]
    \[
    \left| \frac{\partial^2 F(x; a, b)}{\partial a^j \partial b^k} - \frac{\partial^2 F(x; a, b')}{\partial a^j \partial b^k} \right| \leq V_2(x; a, b^*)|b - b'|,
    \]
\end{itemize}
where \((a^*, b^*) \in [a, a'] \times [b, b']\), \(j\) and \(k\) are two positive integers such that \(j + k = 2\).

There exist a measurable positive function \(\phi\) such that \(E(\phi(\epsilon_0)) < +\infty\) and a strictly positive real \(\zeta'\), where \(\zeta' > \max(|\alpha|, |\beta - 1|)\) such that
\[
\frac{|V_i(x; \alpha, \beta)}{f(x)} \leq \phi(x), \quad i = 1, 2.
\]

\((A_2)\) There exist \(\lambda > 0\) such that:
\[
\begin{align*}
(A_{2.1}) & \quad E\{M_f(\epsilon_0)\} = 0. \\
(A_{2.2}) & \quad E\{\epsilon_0 M_f(\epsilon_0)\} = -1. \\
(A_{2.3}) & \quad E\{\tilde{M}_f(\epsilon_0) + M_f^2(\epsilon_0)\} = 0. \\
(A_{2.4}) & \quad E\{\epsilon_0(M_f(\epsilon_0) + M_f^2(\epsilon_0))\} = 0. \\
(A_{2.5}) & \quad E\{\epsilon_0^2(M_f(\epsilon_0) + M_f^2(\epsilon_0))\} = 2.
\end{align*}
\]

2.2 Optimal test when the parameter is known

In this subsection, we proceed to construct the test in the case when the parameter \(K\) under the hypothesis \(H_0\) is assumed known, \(\tilde{\theta} = \theta_0\), where \(\epsilon_0\) is a measurable positive function. Theorem 2.1 under the previous assumptions and conditions, we have the following Theorem:

\[
\text{Theorem 2.1} \quad \text{Under the hypothesis } (H_0), \text{ we have}
\]
\[
\begin{align*}
\Lambda_{n,h,h'} &= V_{n,h,h'} - \frac{\tau^2_{h,h'}}{2} + o_P(1), \quad \text{where } V_{n,h,h'} \xrightarrow{p} N(0, \tau^2_{h,h'}), \\
\tau^2_{h,h'} &= h^2 I_0 E \left( \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right)^2 + h^2 (I_2 - 1) E \left( \frac{S(Z_0)}{\sigma(\theta_0, Z_0)} \right)^2 + 2h^2 (I_1) E \left( \frac{G(Z_0)S(Z_0)}{\sigma^2(\theta_0, Z_0)} \right), \\
\end{align*}
\]

and
\[
I_j = E\{\epsilon_0^j M_f^2(\epsilon_0)\}, \quad j \in \{0, 1, 2\}.
\]

2.3 Efficiency and power of the test

In order to test the null hypothesis \((H_0)\) against the alternative hypothesis \((H_1^\ast)\) and for a fixed step \((h, h')\) in \(K_1 \times K_2\), we use the Neumyuan-Pearson statistics \(T_{n,h,h'}\) defined by
\[
T_{n,h,h'} = I \left\{ \frac{V_{n,h,h'}}{\tau_{h,h'}} \geq Z(u) \right\},
\]

where \(a^*, b^* \in [a, a'] \times [b, b']\), \(j\) and \(k\) are two positive integers such that \(j + k = 2\).
Where $Z(u)$ is the quantile with order $1-u$ of the standard normal distribution ($\Phi(Z(u)) = 1-u$).

We can deduce from the equality (2.1) that $(H_0)$ and $(H_1^\tau)$ are contiguous see for instance (Droesbeke and Fine, 1996, Corollary (4.3)). Under $(H_1^\tau)$ and from Le Cam’s third’s lemma Hall and Mathiason (1990), we shall prove that the random variable $V_{n,h,h'}$ converges in distribution to $\mathcal{N}(\tau_{n,h,h'}, \sigma^2)$ as $n \to +\infty$, therefore we obtain under the assumptions of the Theorem (2.1) the following statement:

**Theorem 2.2** The statistics test is asymptotically optimal with a power function equal to $1-\Phi(Z(1-u)-\tau_{n,h,h'})$.

3 Estimation of the parameters and the link between the random local sequences.

In practice the parameter $(\rho_0, \theta_0)$ is unknown, so we can’t assert the optimality of the test. For estimating the unknown parameter, we use the discrete estimates. Firstly, we begin by introducing the local random sequences $\rho_n$ and $\theta_n$ of the parameters $\rho_0$ and $\theta_0$ respectively, secondly we establish the difference between the central sequences $V_{n,h,h'}(\rho_0, \theta_0)$ and $V_{n,h,h'}(\rho_n, \theta_n)$, where $V_{n,h,h'}(\rho_n, \theta_n)$ is the central sequence obtained after replacing the parameter $(\rho_0, \theta_0)$ by the parameter $(\rho_n, \theta_n)$ in the expression of $V_{n,h,h'}$, finally, and based on of (Kreiss, 1987, Lemma (4.4)), we introduce the discrete estimates. This kind of estimator was introduced by Le Cam (1960), and applied by (Hallin and Puri (1994)), (Benghabrit and Hallin (1998, 1996) and (Cassart et al. (2008))).

We need in this work to remind some definitions and notations, and we assume some supplementary assumptions. The core of proof of the optimality of the test is based on the instrumental Proposition (3.1) which will be stated and proved later.

**Notations and definitions**

Throughout, $\| \cdot \|_p$ and $\| \cdot \|_\ell$ are the euclidian norms in $\mathbb{R}^\ell$ and $\mathbb{R}^p$ respectively. We define the local sequences $\rho_n$ and $\theta_n$ of the parameters $\rho_0$ and $\theta_0$ respectively by the following equalities

$$
\rho_n = \rho_0 + n^{-\frac{1}{2}}u^{(n)}, \quad \theta_n = \theta_0 + n^{-\frac{1}{2}}v^{(n)},
$$

such that

$$
(u^{(n)})^T = (u_1^{(n)}, \ldots, u_\ell^{(n)}), \quad (v^{(n)})^T = (v_1^{(n)}, \ldots, v_p^{(n)}), \quad (\tau^{(n)})^T = (u^{(n)}^T, v^{(n)}^T)^T,
$$

and $\sup_n |(\tau^{(n)})^T (\tau^{(n)})| < +\infty$.

For all $n \geq 1$, we denote by

$$
r_n = \|\rho_n - \rho_0\|_\ell \quad \text{and} \quad r_n' = \|\theta_n - \theta_0\|_p.
$$

For all integers $i$, we define the residual $\epsilon_i$ by the following equation

$$
\epsilon_i = \frac{Y_i - m(\rho_0, Z_i)}{\sigma(\theta_0, Z_i)}. \tag{3.1}
$$

By replacing in (3.1) the parameters $\rho_0$ and $\theta_0$ by the local sequences $\rho_n$ and $\theta_n$ respectively, we obtained the expression of the natural estimate of the residuals $\epsilon_i$ defined in the following equation

$$
\tilde{\epsilon}_{i,n} = \frac{Y_i - m(\rho_0 + n^{-\frac{1}{2}}u^{(n)}, Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}}v^{(n)}, Z_i)}. \tag{3.2}
$$

Let

$$
r_{f,h,n}(\rho_0, \theta_0) = -n^{-\frac{1}{2}} \sum_{i=1}^n hM_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \tag{3.3}
$$

and

$$
q_{f,h,n}(\rho_0, \theta_0) = -n^{-\frac{1}{2}} \sum_{i=1}^n h'(1+\epsilon_i M_f(\epsilon_i)) \frac{S(Z_i)}{\sigma(\theta_0, Z_i)}. \tag{3.4}
$$
Clearly, we have:

\[ V_{n,h,h'}(\rho_0, \theta_0) = r_{f,h,n}(\rho_0, \theta_0) + q_{f,h',n}(\rho_0, \theta_0). \]  

By replacing in (3.4), \( \epsilon_i \) and \( \theta_0 \) by \( \tilde{\epsilon}_{i,n} \) and \( \theta_n \) respectively, we get the following equalities

\[ r_{f,h,n}(\rho_n, \theta_n) = -n^{-\frac{2}{3}} \sum_{i=1}^{n} h M_f(\tilde{\epsilon}_{i,n}) \frac{G(Z_i)}{\sigma(\theta_0 + n^{-\frac{2}{3}} v(n), Z_i)}, \]  

\[ q_{f,h',n}(\rho_n, \theta_n) = -n^{-\frac{2}{3}} \sum_{i=1}^{n} h'(1 + \tilde{\epsilon}_{i,n} M_f(\tilde{\epsilon}_{i,n})) \frac{S(Z_i)}{\sigma(\theta_0 + n^{-\frac{2}{3}} v(n), Z_i)}, \]  

\[ V_{n,h,h'}(\rho_n, \theta_n) = r_{f,h,n}(\rho_n, \theta_n) + q_{f,h',n}(\rho_n, \theta_n). \]  

**Assumptions**

We suppose that the conditions \((A)_1-(A)_4\) remains satisfied and we assume that for all fixed \( x \), the functions \( \rho \rightarrow m(\rho, x) \) and \( \theta \rightarrow \sigma(\theta, x) \) are twice differentiable, we denote by

\[ \partial m(\rho, \cdot)^\top = \left( \frac{\partial m(\rho, \cdot)}{\partial \rho_1}, \ldots, \frac{\partial m(\rho, \cdot)}{\partial \rho_t} \right), \quad \partial \sigma(\theta, \cdot)^\top = \left( \frac{\partial \sigma(\theta, \cdot)}{\partial \theta_1}, \ldots, \frac{\partial \sigma(\theta, \cdot)}{\partial \theta_p} \right), \]

\[ \partial^2 m(\rho, \cdot) = \left( \frac{\partial^2 m(\rho, \cdot)}{\partial \rho_i \partial \rho_j} \right)_{1 \leq i, j \leq t}, \quad \text{and} \quad \partial^2 \sigma(\theta, \cdot) = \left( \frac{\partial^2 \sigma(\theta, \cdot)}{\partial \theta_i \partial \theta_j} \right)_{1 \leq i, j \leq p}. \]

\( \partial^2 m(\rho, \cdot) \) and \( \partial^2 \sigma(\theta, \cdot) \) are the hessian matrix of \( m(\rho, \cdot) \) in \( \rho \) and \( \sigma(\theta, \cdot) \) in \( \theta \) respectively.

We assume that the function \( x \mapsto M_f(x) \) is twice differentiable with a bounded second derivative, \( \tilde{M}_f \) is the second derivative of \( M_f \) (in this case we assume that the function \( f \) has a third derivative). We define the function \( N_f \) by

\[ N_f : x \mapsto N_f(x) = 1 + x M_f(x). \]

Note that the function \( N_f \) is twice differentiable with

\[ \tilde{N}_f(x) = M_f(x) + x \tilde{M}_f(x), \quad \text{and} \quad \hat{N}_f(x) = 2 \tilde{M}_f(x) + x \tilde{M}_f(x). \]  

\( \tilde{N}_f \) and \( \hat{N}_f \) are respectively the derivative and the second derivative of \( N_f \), we suppose that \( \tilde{N}_f \) is bounded. According to the notations of the previous subsection, we assume that the following conditions are satisfied:

\begin{itemize}
  \item \((A_4)_1\) For all \( n \geq 1 \), there exist two closed balls \( \overline{B}_{1,n} = \overline{B}_{1,n}(\rho_0, r_{1,n}) \subset \text{int}(\Theta_1) \) and \( \overline{B}_{2,n} = \overline{B}_{2,n}(\theta_0, r_{2,n}) \subset \text{int}(\Theta_2) \) where \( r_{1,n} \geq r_n \) and \( r_{2,n} \geq r_n' \) and a positive function \( N_{1,n} \), such that \( E\left( \sup_{n \geq 1} N_{1,n}(Z_0) \right) < \infty \), where \( \mu > 0 \), such that, for all fixed \( x \), we have
    \[ \max \left( \sup_{(\rho, \theta) \in \overline{B}_{1,n} \times \overline{B}_{2,n}} \frac{\max_{1 \leq i \leq t} |\partial m(\rho, x)|}{\sigma(\theta, x)}, \sup_{(u, \theta) \in \overline{B}_{2,n} \times \overline{B}_{2,n}} \frac{\max_{1 \leq j \leq p} |\partial \sigma(u, x)|}{\sigma(\theta, x)} \right) \leq N_{1,n}(x). \]
  \item \((A_4)_2\) For all \( n \geq 1 \), there exist two closed balls \( \overline{B}_{1,n} = \overline{B}_{1,n}(\rho_0, r'_{1,n}) \subset \text{int}(\Theta_1) \) and \( \overline{B}_{2,n} = \overline{B}_{2,n}(\theta_0, r'_{2,n}) \subset \text{int}(\Theta_2) \) where \( r'_{1,n} \geq r_n \) and \( r'_{2,n} \geq r'_n \), and a positive function \( N_{2,n} \), such that \( E\left( \sup_{n \geq 1} N_{2,n}(Z_0) \right) < \infty \), where \( \mu' > 0 \), such that, for all fixed \( x \), we have
    \[ \max \left( \sup_{(\rho, \theta) \in \overline{B}_{1,n} \times \overline{B}_{2,n}} \frac{\max_{1 \leq i \leq t} |\partial m(\rho, x)|}{\sigma(\theta, x)}, \sup_{(u, \theta) \in \overline{B}_{2,n} \times \overline{B}_{2,n}} \frac{\max_{1 \leq j \leq p} |\partial \sigma(u, x)|}{\sigma(\theta, x)} \right) \leq N_{2,n}(x). \]
\end{itemize}
For all $n \geq 1$, there exist two closed balls $B_{1,n}^{(3)} = B_{1,n}^{(3)}(\rho_0, r_n^{(3)}) \subset \text{int}(\Theta_1)$ and $B_{2,n}^{(3)} = B_{2,n}^{(3)}(\theta_0, r_n^{(3)}) \subset \text{int}(\Theta_2)$ where $r_n^{(3)} \geq r_n$ and $r_n^{(3)} \geq r_n'$ and a positive function $N_{3,n}$ such that

\[
E\left(\sup_{n \geq 1} N_{3,n}(Z_0)\right)^{\mu_3+1} < \infty, \text{ where } \mu_3 > 0, \text{ such that, for all fixed } x, \text{ we have }
\max \left( \sup_{(\rho, \eta) \in B_{1,n}^{(3)} \times B_{3,n}^{(3)}} \max_{1 \leq i \leq \ell} \frac{\partial m(\rho, \eta)}{\partial \rho_i} \sigma(\theta, x), \sup_{(u, \eta) \in B_{2,n}^{(3)} \times B_{3,n}^{(3)}} \max_{1 \leq j \leq p} \frac{\partial^2 m(u, \eta)}{\partial \rho_j \partial \rho_0} \sigma(\theta, x) \right) \leq N_{3,n}(x).
\]

(A4.4)

For all $n \geq 1$, there exist two closed balls $B_{1,n}^{(4)} = B_{1,n}^{(4)}(\rho_0, r_n^{(4)}) \subset \text{int}(\Theta_1)$ and $B_{2,n}^{(4)} = B_{2,n}^{(4)}(\theta_0, r_n^{(4)}) \subset \text{int}(\Theta_2)$ where $r_n^{(4)} \geq r_n$ and $r_n \geq r_n'$, and a positive function $N_{4,n}$ such that

\[
E\left(\sup_{n \geq 1} N_{4,n}(Z_0)^{\mu_4+1}\right) < \infty, \text{ where } \mu_4 > 0, \text{ such that, for all fixed } x, \text{ we have }
\max \left( \sup_{(\rho, \eta) \in B_{1,n}^{(4)} \times B_{3,n}^{(4)}} \max_{1 \leq i \leq \ell} \frac{\partial^2 m(\rho, \eta)}{\partial \rho_i \partial \rho_0} \sigma(\theta, x), \sup_{(u, \eta) \in B_{2,n}^{(4)} \times B_{3,n}^{(4)}} \max_{1 \leq j \leq p} \frac{\partial^2 m(u, \eta)}{\partial \rho_j \partial \rho_0} \sigma(\theta, x) \right) \leq N_{4,n}(x).
\]

Remark 3.1 Several families of distribution assumed the condition " $\hat{M}_f$ is bounded ", we can for example cite the case where $f$ is a standard normal distribution, then we have $|M_f(\epsilon_0)| = 1$ and $|M_f(\epsilon_0)| = 0$. When $f$ is the student distribution with a degree of freedom greater than 3, it is easy to prove with using simple calculation that the functions $x \mapsto M_f(x)$, $x \mapsto \hat{M}_f(x)$ and $x \mapsto x\hat{M}_f(x)$ are bounded (see Appendix).

Locally asymptotic discrete estimates

The great advantage of discrete estimates is (Kreiss, 1987, Lemma (4.4)) who goes back to Le Cam and is also used by (Bickel (1982)), (Linton (1993)), (Hallin and Puri (1994)), (Benghabrit and Hallin (1998, 1996)) and (Cassart et al. (2008)). The parameters $\rho_0$ and $\theta_0$ are unknown, in order to estimate these parameters, we introduce the discrete estimates $\hat{\rho}_n$ and $\hat{\theta}_n$ of $\rho_0$ and $\theta_0$ respectively, such that these two conditions (D1) and (D2) are satisfied:

(D1) : $\hat{\rho}_n$ is $\sqrt{n}$ consistent, i.e for all $\epsilon > 0$, there exist $\eta_1(\rho_0, \epsilon)$ and $\eta_1(\rho_0, \epsilon)$ such that under $(H_0)$, we have $\forall n \geq n_1(\rho_0, \epsilon), \mathbb{P}(\sqrt{n}\|\hat{\rho}_n - \rho_0\| > \eta_1) \leq \epsilon$.

$\hat{\theta}_n$ is $\sqrt{n}$ consistent, i.e for all $\epsilon > 0$, there exist $\eta_2(\theta_0, \epsilon)$ and $\eta_2(\theta_0, \epsilon)$ such that under $(H_0)$, we have $\forall n \geq n_2(\theta_0, \epsilon), \mathbb{P}(\sqrt{n}\|\hat{\theta}_n - \theta_0\| > \eta_2) \leq \epsilon$.

(D2) : $\hat{\rho}_n$, $\hat{\theta}_n$ are locally discrete, i.e for all fixed value $c > 0$ and under $(H_0)$ and as $n \to +\infty$, the number of possible values of $\hat{\rho}_n$ in $B_1 = \{u \in \mathbb{R}^n, \sqrt{n}\|u - \rho_0\| \leq c\}$ and $\hat{\theta}_n$ in $B_2 = \{v \in \mathbb{R}^n, \sqrt{n}\|v - \theta_0\| \leq c\}$ is bounded.

Note that the condition (D1) concerned the appropriate rate of convergence in probability of the estimates, this condition is satisfied by a several estimates such as the maximum likelihood estimates, the Yule-Walker estimates, the M-estimates and the least square estimates.

We now may state the fundamental proposition which is the the core of the proof of the optimality.

Proposition 3.1 For $(j, k) \in \{1, \ldots, \ell\} \times \{1, \ldots, p\}$, let

\[
K^T = (K_1, \ldots, K_\ell), \quad K'^T = (K'_1, \ldots, K'_\ell)
\]

\[
K_j = \mathbb{E} \frac{\partial m(\rho_0, Z_0)}{\partial \rho_j} \hat{M}_f(\epsilon_0) G(Z_0) \frac{\sigma(\theta_0, Z_0)}{\sigma(\hat{\theta}_0, Z_0)}, \quad K'_j = \mathbb{E} \frac{\partial m(\rho_0, Z_0)}{\partial \rho_j} \hat{N}_f(\epsilon_0) S(Z_0) \frac{\sigma(\theta_0, Z_0)}{\sigma(\hat{\theta}_0, Z_0)}.
\]

\[
J^T = (J_1, \ldots, J_\ell), \quad J'^T = (J'_1, \ldots, J'_\ell)
\]

\[
J_k = \mathbb{E} \frac{\partial \sigma(\rho_0, Z_0)}{\partial \rho_k} \epsilon_0 \hat{M}_f(\epsilon_0) G(Z_0) \frac{\sigma(\theta_0, Z_0)}{\sigma(\hat{\theta}_0, Z_0)}, \quad J'_k = \mathbb{E} \frac{\partial \sigma(\rho_0, Z_0)}{\partial \rho_k} \epsilon_0 \hat{N}_f(\epsilon_0) S(Z_0) \frac{\sigma(\theta_0, Z_0)}{\sigma(\hat{\theta}_0, Z_0)}.
\]
Then, we have the following equalities
\[
\begin{align*}
  r_{f,h,n}(\rho_n, \theta_n) - r_{f,h,n}(\rho_0, \theta_0) &= h(u^{(n)})^T K^T + h(v^{(n)})^T J^T + o_P(1), \\
  q_{f,h',n}(\rho_n, \theta_n) - q_{f,h',n}(\rho_0, \theta_0) &= h'(u^{(n)})^T K'^T + h'(v^{(n)})^T J'^T + o_P(1), \\
  \mathcal{V}_{n,h,h'}(\rho_n, \theta_n) - \mathcal{V}_{n,h,h'}(\rho_0, \theta_0) &= (u^{(n)})^T (h K^T + h' K'^T) + (v^{(n)})^T (h J^T + h' J'^T) + o_P(1).
\end{align*}
\]

(3.12) (3.13) (3.14)

Remark 3.2 The condition “\(\hat{N}_f\) is bounded” is satisfied by a large class of distribution functions.

Based on the remark (3.1) and the equality (3.9), we can deduce that, when \(f\) is the density function of the standard normal distribution, we have \(|\hat{N}_f(\epsilon_0)| = 2\), and when \(f\) is the density of the student distribution with freedom greater than 3, \(\hat{N}_f\) is bounded (see appendix).

Using the estimator \(\hat{\rho}_n\) and \(\hat{\theta}_n\) of \(\rho_0\) and \(\theta_0\) respectively and such that the conditions \((D_1)\) and \((D_2)\) are satisfied, with the replacing of the local sequences \(\rho_n\) and \(\theta_n\) by \(\hat{\rho}_n\) and \(\hat{\theta}_n\) in (3.12), (3.13) and (3.14) respectively, and under the assumptions of proposition (3.1), we obtain the following statement:

Proposition 3.2
\[
\begin{align*}
  r_{f,h,n}(\rho_n, \theta_n) - r_{f,h,n}(\rho_0, \theta_0) &= \sqrt{n}(\hat{\rho}_n - \rho_0)^T h K^T + \sqrt{n}(\hat{\theta}_n - \theta_0)^T h J^T + o_P(1), \\
  q_{f,h',n}(\rho_n, \theta_n) - q_{f,h',n}(\rho_0, \theta_0) &= \sqrt{n}(\hat{\rho}_n - \rho_0)^T h' K'^T + \sqrt{n}(\hat{\theta}_n - \theta_0)^T h' J'^T + o_P(1), \\
  \mathcal{V}_{n,h,h'}(\rho_n, \theta_n) - \mathcal{V}_{n,h,h'}(\rho_0, \theta_0) &= \sqrt{n}(\hat{\rho}_n - \rho_0)^T (h K^T + h' K'^T) + \sqrt{n}(\hat{\theta}_n - \theta_0)^T (h J^T + h' J'^T) + o_P(1), \\
  &= D_{h,h'}(n) + o_P(1).
\end{align*}
\]

This last result, is a fundamental tool used later for the proof of optimality of the test.

Consider again the equalities (3.17), we remark that
\[
\left(\hat{\rho}_n, \hat{\theta}_n\right) = \left(\rho_0 + n^{-\frac{1}{2}} \sqrt{n}(\hat{\rho}_n - \rho_0), \theta_0 + n^{-\frac{1}{2}} \sqrt{n}(\hat{\theta}_n - \theta_0)\right),
\]

with a probability close to 1, the condition \((D_1)\) gives the following condition
\[
\sup_n \left\{\left(\sqrt{n}(\hat{\rho}_n - \rho_0), \sqrt{n}(\hat{\theta}_n - \theta_0)\right)^T \left(\sqrt{n}(\hat{\rho}_n - \rho_0), \sqrt{n}(\hat{\theta}_n - \theta_0)\right)\right\} < +\infty.
\]

Since \(\sqrt{n}(\hat{\rho}_n - \rho_0) = o_P(1)\) and \(\sqrt{n}(\hat{\theta}_n - \theta_0) = o_P(1)\), we concluded in a particular case corresponding to the equalities \(K = K' = J = J = 0\), that the central sequences \(\mathcal{V}_{n,h,h'}(\rho_n, \theta_n)\) and \(\mathcal{V}_{n,h,h'}(\rho_0, \theta_0)\) are equivalent, in a general case the right both side of the last previous equality is not \(o_P(1)\) as \(n \to \infty\), so it is not possible to assert the optimality of the constructed test, in order to solve this problem, we need to introduce another estimator which is defined and described in the work of (Lounis, 2012, Section 1).

4 Optimal test

Throughout, we denote by \(\Omega' = (\rho_n, \theta_n)\) the discrete estimate of the unspecified parameter \(\Omega = (\rho_0, \theta_0)\), with the use of the results of Lounis (2012), we shall construct another \(\sqrt{n}\)-consistency estimate \(\Omega_n\) of the parameter \(\Omega\). According to the notations of (Lounis, 2012, Section (1)), we call this estimate the modified discrete estimator which is denoted by M.D.E, under a supplementary assumptions, we shall prove in the next subsection that with the use of the M.D.E., it is possible to construct an optimal test based on the Neyman-Pearson statistics.

We now may proceed to the proof of the optimality of the test, we need that the conditions (P.0) (or (P’.0)) and (P.1) (or (P’.1) ) are fulfilled, such that :

1. \((P.0) : \frac{\partial \mathcal{V}_{n,h,h'}(\Omega_n)}{\partial \rho_n} \neq 0,\)
2. (P.0) : \( \frac{\partial V_{n,h,h'}(\Omega_n)}{\partial \rho_n} \neq 0 \),

3. (P.1) : \( \frac{1}{\sqrt{n}} \frac{\partial V_{n,h,h'}(\Omega_n)}{\partial \rho_n} \xrightarrow{P} c_1 \) as \( n \to \infty \),

4. (P'.1) : \( \frac{1}{\sqrt{n}} \frac{\partial V_{n,h,h'}(\Omega_n)}{\partial \rho_n} \xrightarrow{P} c_2 \) as \( n \to \infty \), where \( c_1 \) and \( c_2 \) are two constantes, such that \( c_1 \neq 0 \) and \( c_2 \neq 0 \).

Remark 4.1 – The assumptions (P.0), (P’.0), (P.1) and (P’.1) are fixed in (Lounis, 2012, Section 1) in order to prove the existence and the \( \sqrt{n} \)-consistency of the modified estimator.

– Sufficient condition was stated for univariate time series model, for more details see (Lounis, 2012, Lemma 3.1). A generalization of this result concerned the AR(\( m \)) model is presented in the following subsection:

About a sufficient condition in \( AR(m) \) model

Consider the following \( AR(m) \) model:

\[
Y_i = \sum_{j=1}^{m} (\rho_j Y_{i-j}) + \epsilon_i, \quad \text{where} \quad \sum_{j=1}^{m} |\rho_j| < 1. \tag{4.1}
\]

It will assumed that the model (4.1) is stationary and ergodic with finite second and fourth moments, in this case, and according to the previous notations, we have

\[
m(\rho_0, Z_i) = \sum_{j=1}^{m} (\rho_j Y_{i-j}), \quad \sigma(\theta, Z_i) = 1 \quad \text{and} \quad \Omega^\top = (\rho_1, \ldots, \rho_m)'. \tag{4.2}
\]

We denote by \( \hat{\rho}_n = (\hat{\rho}_{n,1}, \ldots, \hat{\rho}_{n,m})' \) the estimator of the unknown parameter \( \rho = (\rho_1, \ldots, \rho_m)' \).

Another estimator was introduced in (Lounis, 2012, Section 1), its consistency is satisfied under the following statement :

(C.1) \[
\frac{1}{\sqrt{n}} \frac{\partial V_n(\hat{\rho}_n)}{\partial \rho_n} \xrightarrow{P} c_1 \quad \text{as} \quad n \to \infty,
\]

where \( c_1 \) is some constant no equal to 0.

Observe that, in practice, it is difficult to check this last condition, therefore it is possible to give an equivalent condition which is easier to establish. According to the previous notations and assumptions, we have the following statement :

Lemma 4.1 \( \epsilon_i \) are i.i.d. standard normal distribution with function density \( f \), Under \( H_0 \), we have

\[
\frac{1}{\sqrt{n}} \frac{\partial V_n(\hat{\rho}_n)}{\partial \rho_j} = \frac{1}{\sqrt{n}} \frac{\partial V_n(\rho_0)}{\partial \rho_j} + o_p(1).
\]

Consequence

This lemma enables us to get an equivalent condition for the consistency of the modified estimator of the unknown parameter in \( AR(m) \) model, the use of the estimator of the unknown parameter in the stated condition (C.1) remains difficult, more precisely, it is possible to calculate this limit with the unknown parameter. In this case, the great advantage is that the result depends only on the observations, under the condition of ergodicity and stationarity of \( AR(m) \) model, it is easy to prove that \( \frac{1}{\sqrt{n}} \frac{\partial V_n(\rho_0)}{\partial \rho_j} \xrightarrow{P} \mathbb{E}(Y_{i-j}G(Z_j)) \).

In short, we shall replace in this case, the condition (C’.1) by the condition :
Optimality

We assume that the conditions (A.1) - (A.4) are satisfied, now it is obvious from the previous results that we can state the following theorem:

**Theorem 4.1** Under LAN and the conditions (P.0), (or (P′.0),) and (P.1) or ((P′.1)), the asymptotic power of $\bar{T}_n$ under $H_1^0$ is equal to

$$1 - \Phi(Z(\alpha) - \hat{\sigma}^2).$$

Furthermore, $\bar{T}_n$ is asymptotically optimal.

**5 Generalization in Z**

Our results are established for $i \in \mathbb{N}$, doing an extension for $i \in \mathbb{Z}$, then, we process the case where $i \in \mathbb{Z}^-$. Consider the following random variables $\mathcal{Y}$, $Z$ and $\varepsilon$, such that, for all $i \in \mathbb{Z}^-$, we have

$$\mathcal{Y}_{-i} = Y_i, \quad Z_{-i} = Z_i, \quad \text{and} \quad \varepsilon_{-i} = \varepsilon_i.$$

Clearly, $i' = -i \in \mathbb{N}$, therefore we obtain

$$\mathcal{Y}_{i'} = T(Z_{i'}) + V(Z_{i'})\varepsilon_{i'}, \quad \text{where} \quad i' \in \mathbb{N}.$$

The last time series model is similar to the model (1), by following the same previous reasoning in the case corresponding to the model (1), we shall construct a test $T'_{n,h,h'}$, which is defined by the following equality

$$T'_{n,h,h'} = I \left\{ \frac{\mathcal{V}_{n,h,h'}(u)}{Z(u)} \geq Z(u) \right\},$$

where

$$\tau_{h,h'}^2 = h^2 I_0' \mathbb{E} \left( G(Z_0) \sigma(\theta_0, Z_0) \right)^2 + h^2 (I_2' - 1) \mathbb{E} \left( S(Z_0) \sigma(\theta_0, Z_0) \right)^2 + 2hh'(I_1') \mathbb{E} \left( \frac{G(Z_0)S(Z_0)}{\sigma(\theta_0, Z_0)} \right),$$

$$U_{n,i',h,h'} = -n^{-\frac{k}{2}} \left\{ hM_f(\varepsilon_{i'}) \frac{G(Z_{i'})}{\sigma(\theta_0, Z_{i'})} + h'(M_f(\varepsilon_{i'})\varepsilon_{i'} + 1) \frac{S(Z_{i'})}{\sigma(\theta_0, Z_{i'})} \right\},$$

$$I_j' = \mathbb{E} \left( \varepsilon_{i'}^j M_j^2(\varepsilon_{i'}) \right), \quad j \in \{0, 1, 2\} \quad \text{and} \quad \mathcal{V}'_{n,h,h'} = \sum_{i' = 1}^{n} U_{n,i',h,h'}.$$

**6 Simulations**

In order to investigate the performance of the proposed test, we conduct simulations, the considering time series models are AR(1) and AR(2). We give simultaneously the power functions with the true parameter, the estimated parameter and the estimated parameter by the M.D.E. respectively. The power relative for each test estimated upon $m = 1000$ replicates, all those representations use the discretized form of the modified estimate. We devote a big importance about the choice of the functions $G$ and $S$ to aim to satisfied the stated conditions. In a sequel, we assume that : $\varepsilon_i$'s are centred iid and $\varepsilon_0 \overset{D}{\rightarrow} \mathcal{N}(0, 1)$, in this case, we have

$$\mathbb{E}(\varepsilon_i) = 0, \quad \mathbb{E}(\varepsilon_i^2) = 1, \quad \mathbb{E}(\varepsilon_i^4) = 3.$$
Example 1:
Nonlinear time series contiguous to AR(1) processes

Consider the $s$th order (nonlinear) time series

$$Y_i = \rho_0 Y_{i-1} + \alpha G(Y(i-1)) + \epsilon_i, \quad |\rho_0| < 1. \quad (6.1)$$

It will be assumed that the time series model (6.1) is stationary and ergodic with finite second moments. Consider again the problem of testing the null hypothesis ($H_0$) : $\alpha = 0$ (linearity of the AR(1) model) against the alternative hypothesis ($H^a_0$) : $\alpha = n^{-\frac{1}{2}}$ (nonlinearity of the AR(1) model). The purpose of this subsection is to treat this problem of the testing when $h = h' = 1$, in this case, we have, for all integers $i$, the following equalities:

$$D_{h,h'}(n) = D_n = -\left(\sqrt{n}(\hat{\rho}_n - \rho_0)^T(K^T + K'^T) + \sqrt{n}(\hat{\theta}_n - \theta_0)^T(J^T + J'^T)\right), \quad (6.2)$$

$$m(\rho_0, Z_i) = \rho_0 Y_{i-1}, \quad \sigma(\theta, Z_i) = 1, \quad M_f(\epsilon_i) = -\epsilon_i, \quad \hat{M}_f(\epsilon_i) = -1, \quad N_f(\epsilon_i) = 1 - \epsilon_i^2,$

$$\hat{N}_f(\epsilon_i) = -2\epsilon_i, \quad \Omega = \rho_0, \quad \Omega_n = \hat{\rho}_n \quad \text{and} \quad Z_i = \left(Y_{i-1}, Y_{i-2}, \cdots, Y_{i-s}, X_i, X_{i-1}, \cdots, X_{i-q}\right).$$

We choose $G : (x_1, x_2, \cdots, x_s, x_{s+1}, x_{s+2}, \cdots, x_{s+q}) \rightarrow \frac{6a}{1 + x_1^2}$, note that this choice of the functions $G$ and $S$ enables us to obey the conditions $(A_{3.1})$ and $(A_{3.2})$.

The parameter $\rho_0$ is estimated by the least square estimate $\hat{\rho}_n = \frac{\sum_{i=1}^{n} Y_i Y_{i-1}}{\sum_{i=1}^{n} Y_i^2}$ and the residual $\epsilon_i$ is estimated by $\epsilon_{i,n} = Y_i - \hat{\rho}_n Y_{i-1}$. We have $\hat{\rho}_{n,h,h'}(\Omega) = \frac{6a}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_i Y_{i-1}}{1 + Y_i^2}$. Then, from the equalities $(3.10), (3.11)$, the ergodicity and the stationarity of model (6.1), it follows that:

$$\frac{1}{\sqrt{n}} \hat{V}_{n,h,h'}(\Omega) = \frac{-6a}{n} \sum_{i=1}^{n} \frac{Y_i Y_{i-1}}{1 + Y_i^2}, \quad \frac{1}{\sqrt{n}} \hat{V}_{n,h,h'}(\Omega_n) = \frac{-6a}{n} \sum_{i=1}^{n} \frac{Y_i Y_{i-1}}{1 + Y_i^2},$$

$$J = J' = K = 0 \quad \text{and} \quad K = -6aE\left[\frac{Y_{i-1}}{1 + Y_{i-1}^2}\right].$$

We denote by $\text{discrete}(\hat{\rho}_n)$ the discretization of the least square estimator L.S.E. $\hat{\rho}_n$. Note that from the ergodicity and the stationarity of the model (6.1), it follows that the random variable $\frac{1}{\sqrt{n}} \hat{V}_{n,h,h'}(\Omega) \overset{a.s.}{\rightarrow} -6aE\left[\frac{Y_{i-1}}{1 + Y_{i-1}^2}\right]$ as $n \rightarrow \infty$. With the use of (3.17) combined with equality (3.10), it follows that:

$$\hat{V}_{n,h,h'} - \hat{V}_{n,h,h'} = -\sqrt{n}(\text{discrete}(\hat{\rho}_n) - \rho_0)6aE\left[\frac{Y_{i-1}}{1 + Y_{i-1}^2}\right] + o_p(1). \quad (6.3)$$

Under the conditions $(P.0)$ and $(P.1)$, we have the $\sqrt{n}$-consistency of the modified estimated M.D.E. which is noted $\bar{\rho}_n$, with:

$$\hat{\rho}_n = \frac{D_n}{\frac{D_{h,h'}(\Omega_n)}{\rho_0}} + (\text{discrete}(\hat{\rho}_n)),$$

where the quantity $D_n$ is defined in the equality (6.2), it result that:

$$\bar{\rho}_n = \frac{\sqrt{n}(\text{discrete}(\hat{\rho}_n) - \rho_0)6aE\left[\frac{Y_{i-1}}{1 + Y_{i-1}^2}\right]}{\hat{V}_n(\hat{\rho}_n)} + (\text{discrete}(\hat{\rho}_n)). \quad (6.4)$$
For a fixed $\alpha = 0.05$, the test proposed is $T_n = \left\{ \frac{V_n(\hat{\rho}_n)}{\tau(\hat{\rho}_n)} \geq Z(\alpha) \right\}$, with the subsisting the parameter $\rho_0$ in the expressions of the proposed test and the power function $1 - \Phi(Z(\alpha) - \tau^2(\rho_0))$, by it’s modified estimate $\hat{\rho}_n$ defined by the equality (6.4), it result from the theorem (4.1) that the statistic test $\hat{T}_n$ is asymptotically equivalent to $T_n$ and it’s power is equal to $1 - \Phi(Z(\alpha) - \tau^2(\hat{\rho}_n))$.

The true value of the parameter $\rho_0$ is fixed at 0.1 and the sample sizes are $n = 30, 40, 60$ and 80. We obtain the following representations :

We remark that, the power function with true value and the empirical power function with the M.D.E. are close as the value $n$ is large.

Example 2:

An extension to ARCH processes

Consider the following time series model with conditional heteroscedasticity

\[
Y_i = \rho_0 Y_{i-1} + \alpha G(Y(i - 1)) + \sqrt{1 + \beta B(Y(i - 1))} \epsilon_i, \quad i \in \mathbb{Z}.
\] (6.5)

It is assumed that the model (6.5) is ergodic and stationary. We conduct our simulation with the same method as the previous case, we define the functions $G$ and $S$ by :

\[
G : (x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+q}) \rightarrow \frac{5a}{1 + x_1^2} \quad \text{and} \quad S = G.
\]

Therefore, we obtain the following equalities :

\[
\begin{align*}
\frac{1}{\sqrt{n}} \frac{\partial V_{n,h,h'}}{\partial \rho} (\Omega) &= \frac{-5a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2} + \frac{-10a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2} \epsilon_i, \\
\frac{1}{\sqrt{n}} \frac{\partial V_{n,h,h'}}{\partial \rho} (\Omega_n) &= \frac{-5a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2} + \frac{-10a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2} (Y_i - \hat{\rho}_n Y_{i-1}), \\
J &= J' = K' = 0, \\
K &= -5a \mathbb{E} \left[ \frac{Y_{i-1}}{1 + Y_{i-1}^2} \right].
\end{align*}
\]

Then we obtain :

\[
\hat{V}_{n,h,h'} - V_{n,h,h'} = -\sqrt{n}(\hat{\rho}_n - \rho_0) h 5a \mathbb{E} \left[ \frac{Y_{i-1}}{1 + Y_{i-1}^2} \right] + o_P(1).
\]

For a fixed $\alpha = 0.05$, the test proposed is $T_n = \left\{ \frac{V_n(\hat{\rho}_n)}{\tau(\hat{\rho}_n)} \geq Z(\alpha) \right\}$, with the subsisting the parameter $\rho_0$ by it’s estimator $\hat{\rho}_n$ in the expressions of the proposed test and the power function $1 - \Phi(Z(\alpha) - \tau^2(\rho_0))$, we obtain from theorem (4.1) an optimal equivalent test $T_n$ with a power $1 - \Phi(Z(\alpha) - \tau^2(\hat{\rho}_n))$, the true value of the parameter $\rho_0$ is fixed at 0.1 and the sample sizes are $n = 30, 40, 50$ and 80. We obtain the following representations :
Example 3: AR(2) model

Consider the following AR(2) model:

\[ Y_i = \rho_1 Y_{i-1} + \rho_2 Y_{i-2} + \epsilon_i, \quad \text{where} \quad |\rho_1| + |\rho_2| < 1. \]  

(6.6)

It will be assumed that the model (6.6) is stationary and ergodic, in this case, we have

\[ m(\rho, Z_i) = \rho_1 Y_{i-1} + \rho_2 Y_{i-2}, \quad \sigma(\theta, Z_i) = 1 \quad \text{and} \quad \Omega^\top = (\rho_1, \rho_2). \]  

(6.7)

We choose

\[ S, G : (x_1, x_2, \ldots, x_s, x_{s+1}, x_{s+2}, \ldots, x_{s+q}) \longrightarrow \frac{8a}{1 + x_1^2 + x_s^2}, \quad \text{where} \quad a \neq 0, \quad \text{clearly, we obtain} \]  

\[ S(Z_0) = G(Z_0) = \frac{8a}{1 + Y_{-1}^2 + Y_{-2}^2}. \]

Note that the choice of the functions \( G \) et \( S \) enables us to obey the conditions \((A_{3,1})\) and \((A_{3,2})\). We denote by \( \Omega_n = (\rho_{1,n}, \rho_{2,n}) \) the least square estimate of the parameter \( \Omega^\top = (\rho_1, \rho_2) \) such that:

\[ \Omega_n = [X^\top X]^{-1} X^\top Y, \]  

(6.8)

\[ Y = \begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{pmatrix}, \quad X = \begin{pmatrix} Y_0 & Y_{-1} \\ \cdot & \cdot \\ \cdot & \cdot \\ Y_{n-1} & Y_{n-2} \end{pmatrix} \quad \text{and} \quad X^\top = \begin{pmatrix} Y_0 & \cdots & Y_{n-1} \\ Y_{-1} & \cdots & Y_{n-2} \end{pmatrix}. \]

Recall that for each \( i \), the residual \( \epsilon_i \) is estimated by the following random variable

\[ \hat{\epsilon}_{i,n} = Y_i - \rho_{1,n} Y_{i-1} - \rho_{2,n} Y_{i-2}. \]  

(6.9)

We have:

\[ \frac{\partial V_{n,h,h'}}{\partial \rho_1}(\Omega) = -\frac{8a}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} - \frac{16a}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} (Y_i - \rho_1 Y_{i-1} - \rho_2 Y_{i-2}), \]

\[ \frac{\partial V_{n,h,h'}}{\partial \rho_2}(\Omega) = -\frac{8a}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} - \frac{16a}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} (Y_i - \rho_1 Y_{i-1} - \rho_2 Y_{i-2}). \]

We obtain:

\[ \frac{1}{\sqrt{n}} \frac{\partial V_{n,h,h'}}{\partial \rho_1}(\Omega) = -\frac{8a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} - \frac{16a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \epsilon_i, \]  

(6.10)

\[ \frac{1}{\sqrt{n}} \frac{\partial V_{n,h,h'}}{\partial \rho_2}(\Omega) = -\frac{8a}{n} \sum_{i=1}^{n} \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} - \frac{16a}{n} \sum_{i=1}^{n} \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \epsilon_i, \]  

(6.11)
then:

\[
\frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_1} (\Omega_n) = -\frac{8a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} - \frac{16a}{n} \sum_{i=1}^{n} \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \\hat{\epsilon}_{i,n}, \tag{6.12}
\]

\[
\frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_2} (\Omega_n) = -\frac{8a}{n} \sum_{i=1}^{n} \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} - \frac{16a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \\hat{\epsilon}_{i,n}. \tag{6.13}
\]

**Correction with respect to the first parameter \( \rho_1 \):**

The combination of the equalities (6.6) with (6.9) enables us to deduce that

\[
\hat{\epsilon}_{i,n} - \epsilon_i = -Y_{i-1}(\rho_{n,1} - \rho_1) - Y_{i-2}(\rho_{n,2} - \rho_2). \tag{6.14}
\]

From the difference between the equalities (6.12) and (6.10) combined with (6.14), it follows that

\[
\left| \frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_1} (\Omega_n) - \frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_1} (\Omega) \right| = \frac{16a}{n} \sum_{i=1}^{n} \left| \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right| (\hat{\epsilon}_{i,n} - \epsilon_i),
\]

\[
\leq \left| \rho_{n,1} - \rho_1 \right| \times \frac{16a}{n} \sum_{i=1}^{n} \frac{Y_{i-1}^2}{1 + Y_{i-1}^2 + Y_{i-2}^2} + \left| \rho_{n,2} - \rho_2 \right| \times \frac{16a}{n} \sum_{i=1}^{n} \frac{Y_{i-2}^2}{1 + Y_{i-1}^2 + Y_{i-2}^2}. \tag{6.15}
\]

Remark that:

\[
\frac{1}{1 + Y_{i-1}^2 + Y_{i-2}^2} \leq 1 \quad \text{then} \quad \frac{Y_{i-1}^2}{1 + Y_{i-1}^2 + Y_{i-2}^2} \leq Y_{i-1}^2 \quad \text{this implies that} \quad \left| \sum_{i=1}^{n} \frac{Y_{i-1}^2}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right| \leq \sum_{i=1}^{n} Y_{i-1}^2. \tag{6.16}
\]

We can also remark that:

\[
\left| \sum_{i=1}^{n} \frac{Y_{i-1}Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right| \leq \left| \sum_{i=1}^{n} Y_{i-1}Y_{i-2} \right| \leq \frac{1}{2} \sum_{i=1}^{n} (Y_{i-1}^2 + Y_{i-2}^2). \tag{6.17}
\]

From the ergodicity, the stationarity and since the model is with finite second moments, it follows the convergence almost surely of the random variables \( \frac{1}{n} \sum_{i=1}^{n} Y_{i-1}^2 \) and \( \frac{1}{n} \sum_{i=1}^{n} (Y_{i-1}^2 + Y_{i-2}^2) \) to constants \( a_1 \) and \( a_2 \) respectively. The couples \( \left( \rho_{n,1} - \rho_1, \frac{16a}{n} \sum_{i=1}^{n} Y_{i-1}^2 \right) \) and \( \left( \rho_{n,2} - \rho_2, \frac{16a}{n} \sum_{i=1}^{n} (Y_{i-1}^2 + Y_{i-2}^2) \right) \) converge in probability to \( (0, 16a_1) \) and \( (0, 8a_2) \) respectively, it follows from the continuous mapping theorem (see for instance *van der Vaart (1998)*) applied on the product and the sum of the functions that

\[
\left| \rho_{n,1} - \rho_1 \right| \times \frac{16a}{n} \sum_{i=1}^{n} \left| \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right| + \left| \rho_{n,2} - \rho_2 \right| \times \frac{16a}{n} \sum_{i=1}^{n} \left| \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right| \overset{P}{\longrightarrow} 0.
\]

In connection with (6.15), it follows that, asymptotically, the quantities \( \frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_1} (\Omega_n) \) and \( \frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_1} (\Omega) \) have the same limit (in probability sense). The random variables \( \frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_1} (\Omega_n) \) and \( \frac{1}{\sqrt{n}} \frac{\partial Y_{n,h,h'}}{\partial \rho_2} (\Omega) \) converge to the constants \( -8a \mathbb{E} \left[ \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right] \) and \( -8a \mathbb{E} \left[ \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right] \) respectively. From the equalities (3.10) and (3.11), it follows that:

\[
K^T = -8a \left( \mathbb{E} \left[ \frac{Y_{i-1}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right], \mathbb{E} \left[ \frac{Y_{i-2}}{1 + Y_{i-1}^2 + Y_{i-2}^2} \right] \right), \tag{6.19}
\]

\[
J = J' = 0, \quad \text{and} \quad K^{T,c} = (0, 0). \tag{6.20}
\]
In sequel, we denote by \( \hat{\Omega}_{1,n} \) the modified estimate obtained after modifying the first component \( \rho_{1,n} \), under the assumptions \((P.0)\) and \((P.1)\), we have the following equalities:

\[
\hat{\rho}_{n,1} = \frac{D_{h,h'}(n)}{\partial y_{\hat{\rho}_{n,1}}(\Omega_n)} + \hat{\rho}_{n,1} \quad \text{and} \quad \hat{\rho}_{n,2} = \hat{\rho}_{n,2}, \quad \text{with} \quad D_{h,h'}(n) = D_n = -\sqrt{n}(\hat{\Omega}_n - \Omega). (\hat{K} + \hat{K}').
\]

For a fixed \( \alpha = 0.05 \), the true value of the parameter \((\rho_1, \rho_2)^T\) is fixed at \((0.2, 0.2)^T\) and the sample sizes are \( n = 30, 40, 50, \) and \( 80 \).

We represent simultaneously the power test with a true parameter \( \rho \) respect to the first parameter by its least square estimator L.S.E \( \hat{\rho}_n \) and the empirical power test which is obtained with the subsisting the true value \( \rho_0 \) by it’s estimate M.D.E. The correction of the estimation is made with respect to the first parameter \( \rho_1 \). Throughout, we denote by \( \text{discrete}(\Omega_n) \) the descritized form of the estimator \( \Omega_n \), we obtain then

\[
\hat{\rho}_{n,1} = \frac{D_{h,h'}(n)}{\partial y_{\text{discrete}(\Omega_n)}} + \hat{\rho}_{n,1} \quad \text{and} \quad \hat{\rho}_{n,2} = \text{discrete}(\hat{\rho}_{n,2}),
\]

with \( D_{h,h'}(n) = D_n = -\sqrt{n}(\text{discrete}(\Omega_n) - \Omega). (\hat{K} + \hat{K}') \).

By the replacing of the parameter \( \Omega \) by it’s estimator \( \hat{\Omega}_n \) in the expression 2.3, we obtain the following sequence of the test \( T_{1,n,h,h'} \), such that:

\[
T_{1,n,h,h'} = I\left\{ \frac{Y_{n,h,h'}}{\hat{\tau}_{1,h,h'}} \geq Z(u) \right\}, \quad (6.21)
\]

where

\[
\hat{\tau}_{1,h,h'}^2 = 64a^2E\left( \frac{1}{1 + Y_{-1}^2 + Y_{-2}^2} \right)^2 \left[ h^2 \hat{I}_{n,0} + h^2(\hat{I}_{n,1})^2 + 2hh'(\hat{I}_{n,1}) \right], \quad (6.22)
\]

and

\[
\hat{I}_{n,j} = E\left( \hat{\tau}_{n,0}^{j+2} \right) = E\left( (Y_0 - \rho_{n,1}Y_{-1} - \rho_{n,2}Y_{-2})^{j+2} \right), \quad (6.23)\]

\[
\hat{I}_{n,j} = E\left( \hat{\tau}_{n,0}^{j+2} \right) = E\left( (Y_0 - \rho_{n,1}Y_{-1} - \text{discrete}(\hat{\rho}_{n,2})Y_{-2})^{j+2} \right), \quad (6.24)\]

we give the representations of the power functions in terms to the value of the constant \( a \), the first representation (blue color) corresponded to the power function with the true value of the parameter,
the second corresponded (green color) to the power function with the least square estimator of the parameter and the third (red color) corresponded to the power function with the modified estimator M.D.E, then we obtain:

**Correction with respect to \( \rho_2 \)**

With the same reasoning as the previous case and with the use of the estimate \( \Omega_{n,2} \), we obtain following sequence of the test \( T_{2,n,h,h'} \), such that:

\[
T_{n,h,h'} = I\left\{ \frac{\bar{\Omega}_{n,h,h'}}{\tau_{2,h,h'}} \geq Z(u) \right\},
\]

where

\[
\hat{\rho}_{n,2} = \frac{D_{h,h'}(n)}{\bar{\rho}_{n,2}} + \hat{\rho}_{n,1}, \quad \bar{\rho}_{n,1} = \hat{\rho}_{n,1},
\]

\[
\tau_{1,h,h'} = a^2 E\left( \frac{1}{1 + Y_{-1}^2 + Y_{-2}^2} \right)^2 \left[ h^2 I_{n,0} + h^2 (I_{n,2} - 1) + 2hh' I_{n,1} \right],
\]

\[
I_{n,j} = E\left( e_{n,0}^{j+2} \right) = E\left( (Y_0 - \hat{\rho}_{n,1} Y_{-1} - \hat{\rho}_{n,2} Y_{-2})^{j+2} \right),
\]

\( j = 0, 1, 2 \).

Therefore, we obtain the following representations:

**7 Proofs of the results**

Throughout we fixe the step \((h,h')\) in the compact set \( K_1 \times K_2 \), where \( hh' \neq 0 \). \( o_P(1) \in \mathbb{R} \xrightarrow{P} 0 \) as \( n \to \infty \).

For some demonstrations, we need to prove the following lemma:

**Lemma 7.1** Let \( a \) and \( b \) are two positive reals and \( \xi \) a real greater than 2, then we have

\[
(a + b)^\xi \leq 2^{\xi-1} (a^\xi + b^\xi).
\]
Proof of the Lemma 7.1.

The function $d : x \mapsto x^\xi$ is twice differentiable on $\mathbb{R}$, the second derivative function $\dd d : \xi (\xi - 1) x^{\xi-2}$ is positive on $\mathbb{R}^+$, therefore $d : x \mapsto x^\xi$ is a convex function on $\mathbb{R}^+$, then:

$\forall (a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\forall (\lambda_1, \lambda_2) \in [0, 1] \times [0, 1]$ with $\lambda_1 + \lambda_2 = 1$, we have $(\lambda_1 a + \lambda_2 b)^\xi \leq \lambda_1 a^\xi + \lambda_2 b^\xi$.

By choosing $\lambda_1 = \lambda_2 = \frac{1}{2}$, we obtain the result.

Proof of the theorem 2.1

We check the three conditions (C.1), (C.2) and (C.3) of (Hwang and Basawa, 2001, Theorem 1).

Verification of the condition (C.1)

Under $(H_0)$, and for $i \in \{1, \ldots, n\}$ and we have:

$$
|g_{n,i,h,h'} - 1| = \left| \frac{f_{h,h'}(Y_i)}{f_0(Y_i)} - 1 \right| = \left| \frac{\left( \frac{\epsilon_i - \alpha_{n,i,h}}{\beta_{n,i,h'}} \right) - f(\epsilon_i)}{f(\epsilon_i)} \right|,
$$

where

$$
\alpha_{n,i,h} = h n^{-\frac{1}{2}} \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \quad \text{and} \quad \beta_{n,i,h'} = 1 + h' n^{-\frac{1}{2}} \frac{S(Z_i)}{\sigma(\theta_0, Z_i)}.
$$

Observe that

$$
|g_{n,i,h,h'} - 1| = \left| F(\epsilon_i; \alpha_{n,i,h}, \beta_{n,i,h'}) - F(\epsilon_i; 0, 1) \right| \left| \frac{1}{f(\epsilon_i)} \right|.
$$

By Taylor expansion of the function $F(\epsilon_i; \cdot, \cdot)$ around $(0, 1)$, we obtain

$$
|g_{n,i,h,h'} - 1| = \left| \frac{\partial F(\epsilon_i; 0, 1)}{f(\epsilon_i)} \alpha_{n,i,h} + \frac{\partial F(\epsilon_i; 0, 1)}{f(\epsilon_i)} \beta_{n,i,h'} (\beta_{n,i,h'} - 1) + \frac{R_{n,i,h,h'}}{f(\epsilon_i)} \right|,
$$

and,

$$
R_{n,i,h,h'} = \frac{1}{2} \left[ \alpha_{n,i,h}, \beta_{n,i,h'} - 1 \right] \partial^2 A_{n,i,h,h'}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h'}^*) \left[ \alpha_{n,i,h}, \beta_{n,i,h'} - 1 \right]^T,
$$

where, $\alpha_{n,i,h}^*, \beta_{n,i,h'}^* \in [0, \alpha_{n,i,h}] \times [1, \beta_{n,i,h'}]$, and, $\partial^2 A_{n,i,h,h'}(\cdot; \alpha_{n,i,h}^*, \beta_{n,i,h'}^*)$ is the hessian matrix of the function $F$ in $(\cdot; \alpha_{n,i,h}^*, \beta_{n,i,h'}^*)$.

Let

$$
U_{n,i,h,h'} = \frac{\partial F(\epsilon_i; 0, 1)}{f(\epsilon_i)} \alpha_{n,i,h} + \frac{\partial F(\epsilon_i; 0, 1)}{f(\epsilon_i)} \beta_{n,i,h'} (\beta_{n,i,h'} - 1) \quad \text{and} \quad R_{n,i,h,h'}^* = \frac{R_{n,i,h,h'}}{f(\epsilon_i)}.
$$

We have

$$
\frac{\partial F(\epsilon_i; 0, 1)}{\partial a} = -\dot{f}(\epsilon_i), \quad \text{and}, \quad \frac{\partial F(\epsilon_i; 0, 1)}{\partial b} = -\left( f(\epsilon_i) + \epsilon_i \dot{f}(\epsilon_i) \right).
$$

Then

$$
U_{n,i,h,h'} = -n^{-\frac{1}{2}} \left\{ h M f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} + h' (M f(\epsilon_i) \epsilon_i + 1) \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right\}.
$$

We have:

\begin{align*}
\text{(7.1)}
\end{align*}
\[ g_{n,i,h,h'} - 1 = U_{n,i,h,h'} + R^*_{n,i,h,h'} \]

From (A1.1), there exist \( p > 1 \), a strictly positive real \( \zeta \), where \( \zeta > \max(\{\alpha^*_{n,i,h}, |\beta^*_{n,h'} - 1|\}) \) and a positive measurable function \( \varphi \) with \( E(\varphi(\theta_0)) < +\infty \) such that

\[
|R^*_{n,i,h,h'}| \leq \frac{1}{2} \left\{ \alpha^2_{n,i,h} + (\beta_{n,i,h} - 1)^2 + 2\alpha_{n,i,h}(\beta_{n,i,h} - 1) \right\} \varphi(\epsilon_i)
\]

\[
\leq \frac{1}{2} \left\{ (\alpha^2_{n,i,h} + (\beta_{n,i,h} - 1)^2) \varphi(\epsilon_i) \right\}
\]

\[
\leq \frac{1}{2n} \left\{ \frac{hG(Z_i) + h'S(Z_i)}{\sigma(\theta_0, Z_i)} \right\}^2 \varphi(\epsilon_i)
\]

\[
\leq \frac{\delta}{n} \left\{ \frac{G(Z_i) + S(Z_i)}{\sigma(\theta_0, Z_i)} \right\}^2 \varphi(\epsilon_i), \quad (7.2)
\]

where, \( \delta = \max(\delta_1^2, \delta_2^2) \), and \( \delta_1 \) and \( \delta_2 \) are the diameters of the compact sets \( K_1 \) and \( K_2 \) respectively. Let \( \nu > 1 \), by Markov’s inequality, we have for all \( \gamma > 0 \)

\[
P\left(|R^*_{n,i,h,h'}| > \gamma \right) = P\left(|R^*_{n,i,h,h'}|^\nu > \gamma^\nu \right) \leq \frac{1}{\gamma^\nu} E[R^*_{n,i,h,h'}|^\nu].
\]

Then by the inequality (7.2), we obtain

\[
P\left(|R^*_{n,i,h,h'}| > \gamma \right) \leq \frac{1}{\gamma^\nu n^\nu} E\left\{ \left[ \frac{G(Z_i) + S(Z_i)}{\sigma(\theta_0, Z_i)} \right]^{2\nu} \varphi(\epsilon_i) \right\}.
\]

It follows from the lemma (7.1), that

\[
\left( \frac{G(Z_i) + S(Z_i)}{\sigma(\theta_0, Z_i)} \right)^{2\nu} \leq \left( \frac{|G(Z_i)| + |S(Z_i)|}{\sigma(\theta_0, Z_i)} \right)^{2\nu}
\]

\[
\leq 2^{2\nu-1} \left\{ \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^{2\nu} + \left| \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right|^{2\nu} \right\}.
\]

Therefore by the stationarity, we have

\[
P\left(\max_{i \in \{1, \ldots, n\}} |R^*_{n,i,h,h'}| > \gamma \right) \leq 2^{2\nu-1} \frac{\delta^\nu}{\gamma^\nu n^\nu} \left\{ \sum_{i=1}^{n} E\left| \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right|^{2\nu} + \sum_{i=1}^{n} E\left| \frac{S(Z_0)}{\sigma(\theta_0, Z_0)} \right|^{2\nu} \right\}
\]

\[
\leq K \left\{ \sum_{i=1}^{n} E\left| \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right|^{\lambda+2} + \sum_{i=1}^{n} E\left| \frac{S(Z_0)}{\sigma(\theta_0, Z_0)} \right|^{\lambda+2} \right\}.
\]

We have \( 2\nu > 2 \), then there exist \( \lambda > 0 \), such that \( 2\nu = \lambda + 2 \), we obtain

\[
P\left(\max_{i \in \{1, \ldots, n\}} |R^*_{n,i,h,h'}| > \gamma \right) \leq \sum_{i=1}^{n} E\left| \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right|^{\lambda+2} + \sum_{i=1}^{n} E\left| \frac{S(Z_0)}{\sigma(\theta_0, Z_0)} \right|^{\lambda+2}.
\]

It follows from (A3.1) and (A3.2) that
\[ P \left( \max_{i \in \{1, \ldots, n\}} |P_{n,i,h,h}'| > \gamma \right) \to 0 \text{ as } n \to +\infty. \]

So we have
\[ \max_{i \in \{1, \ldots, n\}} |P_{n,i,h,h}'| = o_P(1). \quad (7.3) \]

Now we have to show that
\[ \max_{i \in \{1, \ldots, n\}} |U_{n,i,h,h'}| = o_P(1). \]

Remark that
\[ P \left( \max_{i \in \{1, \ldots, n\}} |U_{n,i,h,h'}| > \gamma \right) \leq \sum_{i=1}^{n} P \left( |U_{n,i,h,h'}|^{2\nu} > \gamma^{2\nu} \right). \]

It follows from Markov’s inequality that, for all \( \gamma > 0 \), we have
\[ P \left( \max_{i \in \{1, \ldots, n\}} |U_{n,i,h,h'}| > \gamma \right) \leq \frac{1}{\gamma^{2\nu}} \sum_{i=1}^{n} E \left| U_{n,i,h,h'} \right|^{2\nu}. \quad (7.4) \]

From the lemma (7.1), we can deduce that
\[ E \left| U_{n,i,h,h'} \right|^{2\nu} \leq n^{-\nu} \delta^{2\nu-1} \left\{ E \left( \left| M_f(\epsilon_i) \right|^{2\nu} \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right) + E \left( \left| M_f(\epsilon_i) \epsilon_i + 1 \right|^{2\nu} \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right) \right\}. \]

Combined this in connection with (7.4), it results that
\[ P \left( \max_{i \in \{1, \ldots, n\}} |U_{n,i,h,h'}| > \gamma \right) \leq \frac{\delta^{2\nu-1} \lambda^{2\nu} - 1}{n^{\nu-1} \gamma^{2\nu}} \left\{ E \left( \sum_{i=1}^{n} \left| M_f(\epsilon_i) \right|^{2\nu} \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right) \right\}
+ \frac{\delta^{2\nu-1} \lambda^{2\nu} - 1}{n^{\nu-1} \gamma^{2\nu}} \left\{ E \left( \sum_{i=1}^{n} \left| M_f(\epsilon_i) \epsilon_i + 1 \right|^{2\nu} \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right) \right\}
\leq \frac{\lambda^{2\nu} - 1}{n^{\frac{\nu}{2}} \gamma^{2\nu}} \left\{ E \left( \sum_{i=1}^{n} \left| M_f(\epsilon_i) \right|^{\lambda^{2\nu}} \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right) \right\}
+ \frac{\lambda^{2\nu} - 1}{n^{\frac{\nu}{2}} \gamma^{2\nu}} \left\{ E \left( \sum_{i=1}^{n} \left| M_f(\epsilon_i) \epsilon_i + 1 \right|^{\lambda^{2\nu}} \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right) \right\}. \]

We can remark after using the lemma (7.1) that
\[ \left| M_f(\epsilon_i) \epsilon_i + 1 \right|^{\lambda^{2\nu}} \leq 2^{\lambda^{2\nu} + 1} \left| M_f(\epsilon_i) \epsilon_i \right|^{\lambda^{2\nu}} + 2^{\lambda + 1}. \quad (7.5) \]

It follows from (A.3.1), (A.3.2), (A.3.3), (A.3.4) and the stationarity of the model that
\[ \max_{i \in \{1, \ldots, n\}} |U_{n,i,h,h'}| = o_P(1). \quad (7.6) \]

We deduce from the equalities (7.3) and (7.6) that the condition (CC1) is satisfied.
Verification of the condition \((C.2)\)

We have

\[
\sum_{i=1}^{n} (g_{n,i,h,h'} - 1)^2 = \sum_{i=1}^{n} U_{n,i,h,h'}^2 + \sum_{i=1}^{n} (R_{n,i,h,h'}^*)^2 + 2 \sum_{i=1}^{n} U_{n,i,h,h'} R_{n,i,h,h'}^*.
\]

Using the inequality (7.2) followed by a simple majoration, we obtain

\[
\sum_{i=1}^{n} (R_{n,i,h,h'}^*)^2 \leq \max_{i \in \{1,\ldots,n\}} \left| R_{n,i,h,h'}^* \right| \sum_{i=1}^{n} \left| R_{n,i,h,h'}^* \right|
\leq \max_{i \in \{1,\ldots,n\}} \left| R_{n,i,h,h'}^* \right| \left\{ \frac{\delta}{2n} \sum_{i=1}^{n} \left| \frac{G(Z_i) + S(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \varphi(\epsilon_i) \right\}
\leq \max_{i \in \{1,\ldots,n\}} \left| R_{n,i,h,h'}^* \right| \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \varphi(\epsilon_i) \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 + \frac{\delta}{n} \sum_{i=1}^{n} \varphi(\epsilon_i) \left| \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\}.
\]

Let

\[
A_{n,i,\delta} = \frac{\delta}{n} \sum_{i=1}^{n} \varphi(\epsilon_i) \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2, \quad \text{and} \quad B_{n,i,\delta} = \frac{\delta}{n} \sum_{i=1}^{n} \varphi(\epsilon_i) \left| \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2.
\]

We consider the set of the events \(\Omega_1\) such that \(\Omega_1 = \{\omega, \varphi(\epsilon_i) \leq 1\}\), it is clear that on the complementary \(\Omega_1^c\) of the set \(\Omega_1\), we have, for all real \(p > 1\), \(\varphi(\epsilon_i) \leq \varphi^p(\epsilon_i)\) (In this case we choose a value \(p\) which is corresponded to the condition \((A_{1,1})\)), therefore:

\[
\begin{align*}
|A_{n,i,\delta}| &\leq \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \varphi(\epsilon_i) I_{\Omega_1} \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\} + \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \varphi(\epsilon_i) I_{\Omega_1^c} \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\} \\
&\leq \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\} + \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \varphi^p(\epsilon_i) \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\},
\end{align*}
\]

where \(I(\cdot)\) denotes the indicator function.

Let

\[
A_{n,i,\delta}^* = \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\} + \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \varphi^p(\epsilon_i) \left| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\}.
\]

From the ergodic theorem and \((A_{1,1})\) and since the second moments of the model are finite, it results that the random variable \(A_{n,i,\delta}^*\) converges a.s. to some constant \(c_1\) as \(n \to +\infty\).

Let

\[
B_{n,i,\delta}^* = \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \left| \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\} + \left\{ \frac{\delta}{n} \sum_{i=1}^{n} \varphi^p(\epsilon_i) \left| \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right|^2 \right\}.
\]

With a same reasoning as \(A_{n,i,\delta}^*\), we can show that the random variable \(B_{n,i,\delta}^*\) converges a.s. to some constant \(c_2\) as \(n \to +\infty\), therefore the random variable \(A_{n,i,\delta}^* + B_{n,i,\delta}^*\) converges to \(c = c_1 + c_2\) a.s. as \(n \to +\infty\).

The random vector \(\left( A_{n,i,\delta}^* + B_{n,i,\delta}^* \right) \), \(\max_{i \in \{1,\ldots,n\}} |R_{n,i,h,h'}^*|\) converges in probability to \((c,0)\). Since the function, \((x,y) \mapsto xy\) is continuous, it results from continuous mapping theorem (van der Vaart (1998)) that

\[
\max_{i \in \{1,\ldots,n\}} |R_{n,i,h,h'}^*| (A_{n,i,\delta}^* + B_{n,i,\delta}^*) \xrightarrow{P} 0 \quad \text{a.s.} \quad n \to \infty,
\]

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which implies

\[ \sum_{i=1}^{n} (R_{n,i,h,h'}^*)^2 = o_P(1). \quad (7.7) \]

We have

\[ \sum_{i=1}^{n} U_{n,i,h,h'} R_{n,i,h,h'}^* \leq \sum_{i=1}^{n} \left| U_{n,i,h,h'} \right| R_{n,i,h,h'}^* \]

\[ \leq \max_{i \in \{1, \ldots, n\}} \left| U_{n,i,h,h'} \right| \sum_{i=1}^{n} R_{n,i,h,h'}^* \]

\[ \leq \max_{i \in \{1, \ldots, n\}} \left| U_{n,i,h,h'} \right| (A_{n,i,\delta}^* + B_{n,i,\delta}^*). \]

Using the same arguments as in the last case and (7.6), we can show that,

\[ \sum_{i=1}^{n} U_{n,i,h,h'} R_{n,i,h,h'}^* = o_P(1). \quad (7.8) \]

We have

\[ \sum_{i=1}^{n} U_{n,i,h,h'}^2 = \sum_{i=1}^{n} \left\{ -n^{-\frac{1}{2}} \left[ hM_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} + h'(M_f(\epsilon_i)\epsilon_i + 1) \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right] \right\}^2 \]

\[ = \frac{1}{n} \sum_{i=1}^{n} \left( hM_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right)^2 + \left( h'(M_f(\epsilon_i)\epsilon_i + 1) \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right)^2 \]

\[ + 2hh' \frac{1}{n} \left\{ \sum_{i=1}^{n} M_f(\epsilon_i)[M_f(\epsilon_i)\epsilon_i + 1] \frac{G(Z_i)S(Z_i)}{\sigma(\theta_0, Z_i)} \right\} . \]

Note that

\[ E \left[ \frac{G(Z_i)S(Z_i)}{\sigma^2(\theta_0, Z_i)} \right] \leq E \left[ \frac{|G(Z_i)S(Z_i)|}{\sigma^2(\theta_0, Z_i)} \right] \]

\[ \leq \frac{1}{2} E \left[ \frac{G^2(Z_i)}{\sigma^2(\theta_0, Z_i)} \right] + \frac{1}{2} E \left[ \frac{S^2(Z_i)}{\sigma^2(\theta_0, Z_i)} \right] < +\infty. \]

It follows from the ergodicity and stationarity of the model, that the random variable \( \sum_{i=1}^{n} U_{n,i,h,h'}^2 \) converges a.s. to a positive constant \( \tau_{h,h'}^2 \) as \( n \to +\infty \), where

\[ \tau_{h,h'}^2 = h^2 E \left[ \left( hM_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right)^2 \right] + h'^2 E \left[ \left( 1 + \epsilon_i M_f(\epsilon_i) \right)^2 \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 \]

\[ + 2hh' E \left[ \epsilon_i M_f^2(\epsilon_i) + M_f(\epsilon_i) \frac{G(Z_i)S(Z_i)}{\sigma(\theta_0, Z_i)} \right]. \]

(7.9)

Let \( I_j = E \left( \epsilon_0^j M_f^2(\epsilon_0) \right) \) and \( K_j = E \left( \epsilon_0^j M_f(\epsilon_0) \right) \), \( j \in \{0, 1, 2\} \). It results from (A1.2) and (A2.2), that

\[ \tau_{h,h'}^2 = h^2 I_0 E \left( \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right)^2 + h'^2 (I_2 + 2K_1 + 1) E \left( \frac{S(Z_0)}{\sigma(\theta_0, Z_0)} \right)^2 + 2hh' (I_1 + K_0) E \left( \frac{G(Z_0)S(Z_0)}{\sigma^2(\theta_0, Z_0)} \right) \]

\[ = h^2 I_0 E \left( \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right)^2 + h'^2 (I_2 - 1) E \left( \frac{S(Z_0)}{\sigma(\theta_0, Z_0)} \right)^2 + 2hh' (I_1) E \left( \frac{G(Z_0)S(Z_0)}{\sigma^2(\theta_0, Z_0)} \right). \]

(7.10)

It follows from (7.7), (7.8) and (7.10), that the condition (C.2) is satisfied.
Verification of the condition (C.3)

Let
\[ V_{n,h,h'}(\rho_0, \theta_0) = \sum_{i=1}^{n} U_{n,i,h,h'}. \]

From (A2.1) and (A2.2), \( U_{n,i,h,h'} \) is a \( F_n \) centred martingale. In order to prove that the random variable \( V_{n,h,h'}(\rho_0, \theta_0) \) converges in distribution to \( \mathcal{N}(0, \sigma_{h,h'}^2) \) as \( n \to +\infty \), we use (Hall and Heyde, 1980, Theorem 3.2., Corollaries 3.1., and 3.2,) therefore we check the following conditions:

- (i) **Linderberg condition**: for all \( \gamma > 0 \),
\[ \sum_{i=1}^{n} \mathbb{E} \left( U_{n,i,h,h'}^2 I_{\{U_{n,h,h'}|>\gamma\}} / F_{i-1} \right) \overset{P}{\to} 0 \mbox{ as } n \to \infty. \]

- (ii) **Conditionally variance**: \( \sum_{i=1}^{n} \mathbb{E}(U_{n,i,h,h'}^2 / F_{i-1}) \overset{P}{\to} \eta^2 \mbox{ as } n \to \infty. \)

- (iii) **Measurability**: The random variable \( \eta \) is measurable on the field \( F_{i-1} \).

**Verification of the Linderberg condition**

By the conditionally Hölder’s inequality, there exist \( \nu > 1 \) and \( p > 1 \), \( \frac{1}{p} + \frac{1}{\nu} = 1 \) such that:
\[
\mathbb{E}\left(U_{n,i,h,h'}^2 I\{U_{n,i,h,h'}|>\gamma\} / F_{i-1}\right) \leq \left\{ \mathbb{E}(|U_{n,i,h,h'}|^{2\nu} / F_{i-1}) \right\}^{\frac{1}{\nu}} \left\{ \mathbb{P}(|U_{n,i,h,h'}| > \gamma / F_{i-1}) \right\}^{\frac{1}{p}} \\
\leq \left\{ \mathbb{E}(|U_{n,i,h,h'}|^{2+\lambda} / F_{i-1}) \right\}^{\frac{1}{\nu(p+1)}} \left\{ \mathbb{P}(|U_{n,i,h,h'}| > \gamma / F_{i-1}) \right\}^{\frac{1}{p}} \\
\leq \left\{ \mathbb{E}(|U_{n,i,h,h'}|^{2+\lambda} / F_{i-1}) \right\}^{\frac{1}{\nu(p+1)}} \left\{ \mathbb{P}(|U_{n,i,h,h'}|^{2+\lambda} > \gamma^{2+\lambda} / F_{i-1}) \right\}^{\frac{1}{p}},
\]

where \( \nu = 1 + \frac{\lambda}{2} \) and \( \lambda > 0 \). Note that from the lemma (7.1), it follows:
\[ |\epsilon_i M_f(\epsilon_i) + 1|^{2+\lambda} \leq 2^{1+\lambda}(|\epsilon_i M_f(\epsilon_i)|^{2+\lambda} + 1). \]

By (A3.4), we have
\[ \mathbb{E}\left|\epsilon_i M_f(\epsilon_i) + 1\right|^{2+\lambda} \leq 2^{1+\lambda}\mathbb{E}(|\epsilon_i M_f(\epsilon_i)|^{2+\lambda}) + 2^{1+\lambda} < +\infty, \]

It follows from Markov’s conditionally inequality that
\[
\mathbb{E}\left(U_{n,i,h,h'}^2 I\{U_{n,i,h,h'}|>\gamma\} / F_{i-1}\right) \leq \gamma^{-\frac{(2+\lambda)}{p}} \left\{ \mathbb{E}(|U_{n,i,h,h'}|^{2+\lambda} / F_{i-1}) \right\}^{\frac{1}{p}} \times \\
\times \left\{ \mathbb{E}(|U_{n,i,h,h'}|^{2+\lambda} / F_{i-1}) \right\} \frac{1}{p} \times \\
\leq \gamma^{-\frac{(2+\lambda)}{p}} \left\{ \mathbb{E}(|U_{n,i,h,h'}|^{2+\lambda} / F_{i-1}) \right\}.
\]
It results from The lemma (7.1) followed by the properties of the conditionally expectation that
\[
\sum_{i=1}^{n} \mathbb{E}(U_{n,i,h,h'}^{2}I_{\{U_{n,h,h'} > \gamma\}}) / \mathcal{F}_{i-1} \leq 2^{(1+\lambda)} \gamma^{-\frac{(2+\lambda)}{p}} \sum_{i=1}^{n} \mathbb{E}\left[\left|hM_{f}(\epsilon_{i}) - \frac{G(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right|^{2+\lambda} / \mathcal{F}_{i-1}\right] \\
+ n^{-\frac{(1+\lambda)}{p}} \sum_{i=1}^{n} \mathbb{E}\left|h'(\epsilon_{i}M_{f}(\epsilon_{i}) + 1) - \frac{S(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right|^{2+\lambda} / \mathcal{F}_{i-1} \right]\]
\[
\leq 2^{(1+\lambda)} \delta^{1+\frac{\lambda}{2}} \gamma^{-\frac{(2+\lambda)}{p}} n^{-\frac{(1+\lambda)}{p}} \left\{ \sum_{i=1}^{n} \left|\frac{G(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right|^{2+\lambda} \mathbb{E} \left|\epsilon_{0}M_{f}(\epsilon_{0})\right|^{2+\lambda} \right\} \\
+ n \sum_{i=1}^{n} \left|\frac{S(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right|^{2+\lambda} \mathbb{E}\left|\epsilon_{0}M_{f}(\epsilon_{0}) + 1\right|^{2+\lambda}\right\}
\leq K^{2(1+\lambda)} \delta^{1+\frac{\lambda}{2}} \gamma^{-\frac{(2+\lambda)}{p}} n^{-\frac{\lambda}{2}} \times \\
\times \left\{ \frac{n}{n} \sum_{i=1}^{n} \left|\frac{G(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right|^{2+\lambda} + \frac{1}{n} \sum_{i=1}^{n} \left|\frac{S(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right|^{2+\lambda} \right\}. \tag{7.12}
\]

Using the inequality (7.12) and from the ergodicity, the stationarity, \((A_{3,1})\) and \((A_{3,2})\), it results that
\[
\sum_{i=1}^{n} \mathbb{E}(U_{n,i,h,h'}^{2}I_{\{U_{n,h,h'} > \gamma\}}) / \mathcal{F}_{i-1} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.
\]
Which implies that the Linderberg condition is satisfied.

Conditionally variance
\[
\sum_{i=1}^{n} \mathbb{E}\left(U_{n,i,h,h'}^{2} / \mathcal{F}_{i-1}\right) = \frac{1}{n} \left\{ \sum_{i=1}^{n} \mathbb{E}\left(\left[hM_{f}(\epsilon_{i}) - \frac{G(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right]^{2} / \mathcal{F}_{i-1}\right) \right\} \\
+ \sum_{i=1}^{n} \mathbb{E}\left(\left|h'(\epsilon_{i}M_{f}(\epsilon_{i}) + 1) - \frac{S(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right|^{2} / \mathcal{F}_{i-1}\right) \\
+ 2hh' \sum_{i=1}^{n} \mathbb{E}\left(\left[M_{f}(\epsilon_{i})[M_{f}(\epsilon_{i})\epsilon_{i} + 1] - \frac{G(Z_{i})S(Z_{i})}{\sigma(\theta_{0}, Z_{i})}\right] / \mathcal{F}_{i-1}\right) \right\}.
\]

Using the properties of the conditionally expectation, and since the random variables \(\epsilon_{i}\) are independent of \(\mathcal{F}_{i} = \sigma(Z_{j}, j \leq i)\) and after the application of the ergodic theorem, it follows the convergence of
\[
\sum_{i=1}^{n} \mathbb{E}(U_{n,i,h,h'}^{2} / \mathcal{F}_{i-1}) \xrightarrow{\text{a.s.}} \frac{\eta^{2}}{\mathcal{T}_{h,h'}} \text{as} \ n \to \infty \quad \text{(so in Probability)}.
\]

**Measurability :**
The random variable \(\eta\) is a constant, so it is measurable on \(\mathcal{F}_{i-1}\), therefore we obtain the measurability. In summary, by collecting the conditions \((i)\), \((ii)\) and \((iii)\), we deduce that the random variable \(V_{n,h,h'}(\rho_{0}, \theta_{0})\) converges in distribution to \(\mathcal{N}(0, \mathcal{T}_{h,h'})\) as \(n \to +\infty\). It remains to prove that \(\sum_{i=1}^{n} R_{n,i,h,h'}^{*} = o_{P}(1)\), where
\[
R_{n,i,h,h'}^{*} = \frac{1}{2f(\epsilon_{i})}[\alpha_{n,h,h'}(\beta_{n,i,h'} - 1)]^{\top} \partial^{2} A_{n,i,h,h'}(\epsilon_{i}; \alpha_{n,i,h,h'}^{*}, \beta_{n,i,h,h'}^{*})[\alpha_{n,h,h'}(\beta_{n,i,h'} - 1)]^{\top},
\]
and,
\[
\partial^{2} A_{n,i,h,h'}(\epsilon_{i}; \alpha_{n,i,h,h'}^{*}, \beta_{n,i,h,h'}^{*}) = \begin{pmatrix}
D_{1,1}(\epsilon_{i}; \alpha_{n,i,h,h'}^{*}, \beta_{n,i,h,h'}^{*}) & D_{1,2}(\epsilon_{i}; \alpha_{n,i,h,h'}^{*}, \beta_{n,i,h,h'}^{*}) \\
D_{2,1}(\epsilon_{i}; \alpha_{n,i,h,h'}^{*}, \beta_{n,i,h,h'}^{*}) & D_{2,2}(\epsilon_{i}; \alpha_{n,i,h,h'}^{*}, \beta_{n,i,h,h'}^{*})
\end{pmatrix}.
\]
We have
\[
\sum_{i=1}^{n} R_{n,i,h,h'} = \sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} - 2f(\epsilon_i) \alpha_{n,i,h}^2 + \sum_{i=1}^{n} \frac{D_{2,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} (\beta_{n,i,h'} - 1)^2 \]
\[
+ \sum_{i=1}^{n} \alpha_{n,i,h}(\beta_{n,i,h'} - 1) \frac{D_{1,2}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} + \sum_{i=1}^{n} \alpha_{n,i,h}(\beta_{n,i,h} - 1) \frac{D_{2,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)}.
\]

We have
\[
\sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} \alpha_{n,i,h}^2 = \frac{h^2}{n} \sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2.
\]
We have the following decomposition
\[
\frac{h^2}{n} \sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 = \frac{h^2}{n} \sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 + \frac{h^2}{n} \sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}')}{2f(\epsilon_i)} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2.
\]

From (A.1), there exist a positive function \(V_1\), a strictly positive real \(\epsilon'\), which
\[\epsilon' > \max(|\alpha_{n,i,h}^*, |\beta_{n,i,h}^* - 1|)\] and a measurable positive function \(\phi\) such that \(E(\phi(\epsilon_0)) < +\infty\) such that
\[
\left| D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}') - D_{1,1}(\epsilon_i; 0, 1) \right| \leq |\alpha_{n,i,h}^*| V_1(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}'),
\]
where \(\alpha_{n,i,h}^* \in [0, \alpha_{n,i,h}^*].\)

For all integers \(n \geq 1\), we have
\[
E \left\{ \frac{h^2}{n} \sum_{i=1}^{n} \left| D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}') - D_{1,1}(\epsilon_i; 0, 1) \right| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right\} \leq h^2 \frac{2n}{n} \sum_{i=1}^{n} E \left\{ \phi(\epsilon_i) |\alpha_{n,i,h}^*| \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 \right\}.
\]
Since \(\alpha_{n,i,h}^*\) is in the interval \([0, \alpha_{n,i,h}]\), therefore there exist a random sequence of parameter \((\theta_n)_{n \geq 1}\) with values in \([0, 1]\) such that
\[\alpha_{n,i,h}^* = \theta_n \alpha_{n,i,h}^*.
\]
Then we obtain
\[
E \left\{ \frac{h^2}{n} \sum_{i=1}^{n} \left| D_{1,1}(\epsilon_i; \alpha_{n,i,h}', \beta_{n,i,h}') - D_{1,1}(\epsilon_i; 0, 1) \right| \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right\} \leq h^2 \frac{2n}{n} \sum_{i=1}^{n} \theta_n E \left\{ \phi(\epsilon_0) |\alpha_{n,i,h}^*| \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 \right\} \leq K \theta^3 n^{-\frac{1}{2}} \sum_{i=1}^{n} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^3.
\]
By Markov’s inequality, for all $\gamma > 0$, we have
\[
P \left( \frac{h^2}{n} \sum_{i=1}^{n} \left| D_{1.1}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.1}(\epsilon_i; 0, 1) \right| \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 > \gamma \right) \leq \frac{1}{\gamma} K h^2 n^{-2} \mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^3 \right\}.
\]

From the ergodicity and the stationarity of the model and since $n^{-\frac{3}{2}} \rightarrow 0$, it results that
\[
\frac{h^2}{n} \sum_{i=1}^{n} \left| D_{1.1}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.1}(\epsilon_i; 0, 1) \right| \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.
\]

Finally, we get
\[
R_{n,i,h,h'}^{(1)} = \sum_{i=1}^{n} \left| D_{1.1}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.1}(\epsilon_i; 0, 1) \right| \alpha_{n,i,h}^2 = o_P(1). \quad (7.13)
\]

By following the same previous reasoning in the last case, we shall prove that
\[
R_{n,i,h,h'}^{(2)} = \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \beta_{n,i,h'} - 1)^2 = o_P(1), \quad (7.14)
\]
\[
R_{n,i,h,h'}^{(3)} = \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \alpha_{n,i,h}^2 = o_P(1), \quad (7.15)
\]
\[
R_{n,i,h,h'}^{(4)} = \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \beta_{n,i,h'} - 1)^2 = o_P(1), \quad (7.16)
\]
\[
R_{n,i,h,h'}^{(5)} = \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \alpha_{n,i,h}^2 = o_P(1), \quad (7.17)
\]
\[
R_{n,i,h,h'}^{(6)} = \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \beta_{n,i,h'} - 1)^2 = o_P(1), \quad (7.18)
\]

Let
\[
R_{n,i,h,h'}^{(7)} = \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \alpha_{n,i,h}(\beta_{n,i,h'} - 1), \quad (7.17)
\]
\[
R_{n,i,h,h'}^{(8)} = \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \alpha_{n,i,h}(\beta_{n,i,h'} - 1). \quad (7.18)
\]

From the following inequality
\[
\left| \alpha_{n,i,h}(\beta_{n,i,h'} - 1) \right| \leq \frac{1}{2} \left[ \alpha_{n,i,h}^2 + (\beta_{n,i,h'} - 1)^2 \right],
\]

It results that
\[
|R_{n,i,h,h'}^{(7)}| \leq \frac{1}{2} \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| \alpha_{n,i,h}^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \left| D_{1.2}(\epsilon_i; \alpha_{n,i,h}^*, \beta_{n,i,h}^*) - D_{1.2}(\epsilon_i; 0, 1) \right| (\beta_{n,i,h'} - 1)^2.
\]
We have
\[
\sum_{i=1}^{n} R_{n,i,h,h'}^* = \sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} \alpha_{n,i,h}^2 - \frac{D_{1,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} \alpha_{n,i,h}^2 \\
+ \sum_{i=1}^{n} \frac{D_{2,2}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} (\beta_{n,i,h'} - 1)^2 \\
+ \sum_{i=1}^{n} \frac{D_{1,2}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} \alpha_{n,i,h} (\beta_{n,i,h'} - 1) \\
+ \sum_{i=1}^{n} \frac{D_{2,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} \alpha_{n,i,h} (\beta_{n,i,h'} - 1) \\
+ L_{n,i,h,h'}
\]
with
\[
L_{n,i,h,h'} = \sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} \alpha_{n,i,h}^2 + \sum_{i=1}^{n} \frac{D_{2,2}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} (\beta_{n,i,h'} - 1)^2 \\
+ \sum_{i=1}^{n} \alpha_{n,i,h} (\beta_{n,i,h'} - 1) \frac{D_{1,2}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} + \sum_{i=1}^{n} \alpha_{n,i,h} (\beta_{n,i,h'} - 1) \frac{D_{2,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)}
\]
From the equalities (7.13), (7.14), (7.19) and (7.20), it results that
\[
\sum_{i=1}^{n} R_{n,i,h,h'}^* \leq R_{n,i,h,h'}^{(1)} + R_{n,i,h,h'}^{(2)} + R_{n,i,h,h'}^{(7)} + R_{n,i,h,h'}^{(8)} + L_{n,i,h,h'}
\]
We have
\[
D_{1,1}(\epsilon_i; 0, 1) = \bar{f}(\epsilon_i), \quad D_{1,2}(\epsilon_i; 0, 1) = D_{2,1}(\epsilon_i; 0, 1) = \bar{f}(\epsilon_i) + \epsilon_i \bar{f}(\epsilon_i)
\]
and
\[
D_{2,2}(\epsilon_i; 0, 1) = 2\epsilon_i \bar{f}(\epsilon_i) + \epsilon_i^2 \bar{f}(\epsilon_i)
\]
By simple calculation, it is easy to prove that :
\[
\frac{\bar{f}(x)}{f(x)} = \dot{M}(x) + M_f^2(x).
\]
- By (A2.3) combined with the ergodicity and the stationarity of the model, it results that :
\[
\sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} \alpha_{n,i,h}^2 \xrightarrow{a.s.} \frac{h^2}{n} \sum_{i=1}^{n} \frac{\bar{f}(\epsilon_i)}{2f(\epsilon_i)} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 \xrightarrow{a.s.} h^2 E \left\{ \frac{\bar{f}(\epsilon_i)}{2f(\epsilon_i)} \left[ \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 \right\}, \text{ as } n \to \infty.
\]
then
\[
\sum_{i=1}^{n} \frac{D_{1,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} \alpha_{n,i,h}^2 \xrightarrow{a.s.} \frac{h^2}{2} E \left\{ \hat{M}(\epsilon_0) + M_f^2(\epsilon_0) \right\} E \left[ \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right]^2 = 0 \text{ as } n \to \infty.
\]
By (A2.2) and (A2.3) combined with the ergodicity and the stationarity of the model, it results that:

$$\frac{h^2}{n} \sum_{i=1}^{n} \frac{D_{2,2}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} (\beta_{n,i,h'} - 1)^2$$

$$= \frac{h^2}{n} \sum_{i=1}^{n} \left\{ \epsilon_i M_f(\epsilon_i) + \frac{1}{2} \epsilon_i^2 \left( \hat{M}_f(\epsilon_i) + M_f^2(\epsilon_i) \right) \right\} \left[ \frac{S(Z_i)}{\sigma(\theta_0, Z_i)} \right]^2 \overset{a.s.}{\longrightarrow} 0, \text{ as } n \to \infty.$$ 

It follows by (A2.1) and (A2.4) and the ergodicity and the stationarity of the model that:

$$\sum_{i=1}^{n} \alpha_{n,i,h}(\beta_{n,i,h'} - 1) \frac{D_{1,2}(\epsilon_i; 0, 1)}{2f(\epsilon_i)} + \sum_{i=1}^{n} \alpha_{n,i,h}(\beta_{n,i,h'} - 1) \frac{D_{2,1}(\epsilon_i; 0, 1)}{2f(\epsilon_i)}$$

$$= h h' \sum_{i=1}^{n} \frac{G(Z_i)S(Z_i)}{\sigma^2(\theta_0, Z_i)} \left[ M_f(\epsilon_i) + \epsilon_i (\hat{M}_f(\epsilon_i) + M_f^2(\epsilon_i)) \right]$$

$$\overset{a.s.}{\longrightarrow} hh' \mathbb{E} \left[ \frac{G(Z_i)S(Z_i)}{\sigma^2(\theta_0, Z_i)} \left[ M_f(\epsilon_0) + \epsilon_0 (\hat{M}_f(\epsilon_0) + M_f^2(\epsilon_0)) \right] \right] = 0 \text{ as } n \to \infty.$$ 

Consequently, the random variable $L_{n,i,h,h'} \overset{a.s.}{\longrightarrow} 0$ as $n \to +\infty$.

The random vector $\left(R_{n,i,h,h'}^{(1)} + R_{n,i,h,h'}^{(2)} + R_{n,i,h,h'}^{(7)} + R_{n,i,h,h'}^{(8)}, L_{n,i,h,h'} \right)$ $\overset{P}{\longrightarrow} (0, 0)$ as $n \to +\infty$. Since the function $(x, y) \mapsto x + y$ is continuous, it results that

$$\sum_{i=1}^{n} R_{n,i,h,h'}^* = o_P(1).$$

**Conclusion**

The conditions (C.1), (C.2) and (C.3) are established, from the (Hwang and Basawa, 2001, Theorem 1), it follows, under the hypothesis $(H_0)$, that:

$$\Lambda_{n,h,h'} = V_{n,h,h'}(\rho_0, \theta_0) - \frac{\rho_0^2 \sigma(h,h')^2}{2} + o_P(1). \quad (7.21)$$

**Proof of the Theorem 2.2**

The proof is similar as the proof of (Hwang and Basawa, 2001, Theorem 3).

**Proof of the Proposition 3.1**

Based on the equations (3.1) and (3.2), we have

$$\sigma(\theta_0 + n^{-\frac{1}{2}} u^{(n)}, Z_i) \tilde{e}_{i,n} - \sigma(\theta_0, Z_i) \epsilon_i = - \left( m(\rho_0 + n^{-\frac{1}{2}} u^{(n)}, Z_i) - m(\rho_0, Z_i) \right),$$

Then

$$\tilde{e}_{i,n} - \epsilon_i = \frac{m(\rho_0 + n^{-\frac{1}{2}} u^{(n)}, Z_i) - m(\rho_0, Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} u^{(n)}, Z_i)} \frac{\sigma(\theta_0 + n^{-\frac{1}{2}} u^{(n)}, Z_i) - \sigma(\theta_0, Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} u^{(n)}, Z_i)} \epsilon_i,$$
By Taylor’s expansion with order 1 of the functions $\rho \rightarrow m(\rho, \cdot)$ and $\theta \rightarrow \sigma(\theta, \cdot)$ around $\rho_0$ and $\theta_0$ respectively, we obtain the following equalities

\[
\begin{align*}
m(\rho_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i) - m(\rho_0, Z_i) &= n^{-\frac{1}{2}} (v^{(n)})^\top \partial m(\tilde{\rho}_n, Z_i)^\top, \\
\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i) - \sigma(\theta_0, Z_i) &= n^{-\frac{1}{2}} (v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top,
\end{align*}
\]

(7.22)  
(7.23)

\[
\tilde{e}_{i,n} - \epsilon_i = -\frac{n^{-\frac{1}{2}} (v^{(n)})^\top \partial m(\tilde{\rho}_n, Z_i)^\top}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} - \frac{n^{-\frac{1}{2}} (v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} \epsilon_i.
\]

(7.24)

The parameters $\tilde{\rho}_n$ and $\tilde{\theta}_n$ are between $\rho_0$ and $\rho_n$ and $\theta_0$ and $\theta_n$ respectively. By Taylor’s expansion with order 2 of the function $u \mapsto M_f(u)$ around $\epsilon_i$ combined with the equality (7.23), we obtain

\[
\frac{M_f(\tilde{e}_{i,n})}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} - \frac{M_f(\epsilon_i)}{\sigma(\theta_0, Z_i)} = \frac{\sigma(\theta_0, Z_i) M_f(\tilde{e}_{i,n}) - \sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i) M_f(\epsilon_i)}{\sigma(\theta_0, Z_i) \sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)}
\]

\[
= \frac{\sigma(\theta_0, Z_i) M_f(\tilde{e}_{i,n}) - \sigma(\theta_0, Z_i) + n^{-\frac{1}{2}} (v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top M_f(\epsilon_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)}
\]

\[
= \frac{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i) M_f(\epsilon_i) - n^{-\frac{1}{2}} (v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top M_f(\epsilon_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)}
\]

\[
= \frac{M_f(\tilde{e}_{i,n}) - M_f(\epsilon_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} - \frac{n^{-\frac{1}{2}} (v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} M_f(\epsilon_i)
\]

\[
= \frac{(\tilde{e}_{i,n} - \epsilon_i) M_f(\epsilon_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} + \frac{2 \sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i) (\tilde{e}_{i,n} - \epsilon_i)^2 M_f(\tilde{e}_{i,n})}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i) \sigma(\theta_0, Z_i)} M_f(\epsilon_i),
\]

(7.25)

where $\tilde{e}_{i,n}$ is between $\epsilon_i$ et $\tilde{e}_{i,n}$. By a difference between the equalities (3.6) and (3.3), it follows that

\[
\tilde{r}_{f,h,n} - r_{f,h,n} = -n^{-\frac{1}{2}} h \sum_{i=1}^{n} \left[ \frac{M_f(\tilde{e}_{i,n})}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} - \frac{M_f(\epsilon_i)}{\sigma(\theta_0, Z_i)} \right] G(Z_i).
\]

Using the equality (7.25), we obtain:

\[
\tilde{r}_{f,h,n} - r_{f,h,n} = I_{n,h,1} + I_{n,h,2} + I_{n,h,3}.
\]

With

\[
I_{n,h,1} = -n^{-\frac{1}{2}} h \sum_{i=1}^{n} \frac{\tilde{e}_{i,n} - \epsilon_i) M_f(\epsilon_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} G(Z_i),
\]

(7.26)

\[
I_{n,h,2} = -n^{-\frac{1}{2}} h \sum_{i=1}^{n} \frac{(\tilde{e}_{i,n} - \epsilon_i)^2 M_f(\tilde{e}_{i,n}) G(Z_i)}{2 \sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)}
\]

(7.27)

and

\[
I_{n,h,3} = \frac{1}{n} h \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top M_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i) \sigma(\theta_0, Z_i)}
\]

(7.28)

Now we proceed to evaluate the terms $I_{n,h,1}$, $I_{n,h,2}$ and $I_{n,h,3}$; all the limits are calculated under the hypothesis $(H_0)$. 

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Evaluation of the term $I_{n,h,1}$

We have

$$I_{n,h,1} = -n^{-\frac{1}{2}}h \sum_{i=1}^{n} \frac{(\bar{\epsilon}_{i,n} - \epsilon_i) \bar{M}_f(\epsilon_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} G(Z_i)$$

$$+ n^{-\frac{1}{2}}h \sum_{i=1}^{n} \frac{(\bar{\epsilon}_{i,n} - \epsilon_i) \bar{M}_f(\epsilon_i)}{\sigma(\theta_0, Z_i)} G(Z_i)$$

$$- n^{-\frac{1}{2}}h \sum_{i=1}^{n} \frac{1}{\sigma(\theta_0, Z_i)} G(Z_i)$$

$$= n^{-\frac{1}{2}}h \sum_{i=1}^{n} \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} G(Z_i)$$

From the equality (7.30), we have

$$I_{n,h,1} = I_{n,h,1}^{(1)} + I_{n,h,1}^{(2)}.$$

Using the equality (7.24), we obtain

$$I_{n,h,1}^{(1)} = -n^{-\frac{1}{2}}h \sum_{i=1}^{n} A(\theta_0, \tilde{\rho}_n, \tilde{n}, Z_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} + n^{-\frac{1}{2}}h \sum_{i=1}^{n} B(\theta_0, \tilde{n}, Z_i) \epsilon_i \bar{M}_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)},$$

with

$$A(\theta_0, \tilde{\rho}_n, \tilde{n}, Z_i) = \frac{(v(n)^T \partial \sigma(\tilde{n}, Z_i)^T (u(n)^T \partial m(\tilde{\rho}_n, Z_i)^T)}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} \sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i),$$

$$B(\theta_0, \tilde{n}, Z_i) = -\left[\frac{(v(n)^T \partial \sigma(\tilde{n}, Z_i)^T)}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)}\right]^2.$$

The parameters $\tilde{\rho}_n$ and $\tilde{n}$ are into the convex segments $[\rho_0, \rho_n]$ of $\mathbb{R}^\ell$ and $[\theta_0, \theta_n]$ of $\mathbb{R}^p$ respectively, then there exist for all integers $n$, a sequence $(s_n, t_n)$ with values in $[0, 1] \times [0, 1]$, such that

$$\tilde{\rho}_n = s_n \rho_0 + (1 - s_n) \rho_n \quad \text{and} \quad \tilde{n} = t_n \theta_0 + (1 - t_n) \theta_n.$$

It result that

$$||\tilde{\rho}_n - \rho_0||_\ell \leq (1 - s_n)||\rho_n - \rho_0||_\ell \leq ||\rho_n - \rho_0||_\ell,$$

and

$$||\tilde{n} - \theta_0||_p \leq (1 - t_n)||\theta_n - \theta_0||_p \leq ||\theta_n - \theta_0||_p.$$
By applying Cauchy-Schwartz’s inequality on each term of the product (7.29) and doing a majoration, we obtain

\[ |A(\theta_0, \tilde{\rho}_n, \tilde{\theta}_n, Z_i)| \leq \left\{ \frac{\|(v^{(n)})\|_p \|\partial \sigma(\tilde{\theta}_n, Z_i)\|_p \cdot \|(u^{(n)})\|_Q \|\partial m(\tilde{\rho}_n, Z_i)\|_Q}{\sigma(\theta_0 + n^{-\frac{2}{5}}v^{(n)}, Z_i)} \right\} \]

\[ \leq \frac{1}{2} \left\{ \left[ \frac{\|(u^{(n)})\|_Q \|\partial m(\tilde{\rho}_n, Z_i)\|_Q}{\sigma(\theta_0 + n^{-\frac{2}{5}}v^{(n)}, Z_i)} \right]^2 + \left[ \frac{\|(u^{(n)})\|_Q \|\partial m(\tilde{\rho}_n, Z_i)\|_Q}{\sigma(\theta_0 + n^{-\frac{2}{5}}v^{(n)}, Z_i)} \right]^2 \right\}. \]

It results that

\[ \left| \frac{1}{n} \sum_{i=1}^{n} A(\theta_0, \tilde{\rho}_n, \tilde{\theta}_n, Z_i) \, \hat{M}_f(\epsilon_i) \, \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right| \]

\[ \leq \frac{1}{2n} \sum_{i=1}^{n} \left[ \frac{\|(v^{(n)})\|_p \|\partial \sigma(\tilde{\theta}_n, Z_i)\|_p}{\sigma(\theta_0 + n^{-\frac{2}{5}}v^{(n)}, Z_i)} \right]^2 \left| \hat{M}_f(\epsilon_i) \, \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right| \]

\[ + \frac{1}{2n} \sum_{i=1}^{n} \left[ \frac{\|(u^{(n)})\|_Q \|\partial m(\tilde{\rho}_n, Z_i)\|_Q}{\sigma(\theta_0 + n^{-\frac{2}{5}}v^{(n)}, Z_i)} \right]^2 \left| \hat{M}_f(\epsilon_i) \, \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|. \] (7.33)

Since that for all \( x \), we have

\[ \|\partial m(\rho, x)\|_Q \leq \sqrt{\ell} \max_{1 \leq j \leq \ell} \left| \frac{\partial m(\rho, x)}{\partial \rho_j} \right|, \]

and

\[ \|\partial \sigma(\theta, x)\|_p \leq \sqrt{p} \max_{1 \leq j \leq p} \left| \frac{\partial \sigma(\theta, x)}{\partial \theta_j} \right|. \] (7.34) (7.35)

Therefore, it follows from the inequalities (7.31), (7.32), (7.33) and the conditions \((A_{4.1})\), that

There exist two closed balls \( B_{1,n} = B_{1,n}(\rho_0, r_{1,n}) \subset \text{int}(\Theta_1) \) and \( B_{2,n} = B_{2,n}(\theta_0, r_{2,n}) \subset \text{int}(\Theta_2) \) where \( r_{1,n} \geq r_n \) and \( r_{2,n} \geq r'_n \) and a positive function \( N_{1,n} \), such that

\[ E\left( \sup_{n \geq 1} \left[ N_{1,n}(Z_0) \right] \right)^{\mu + 2} < \infty \], where \( \mu > 0 \), such that

\[ \left| \frac{1}{n} \sum_{i=1}^{n} A(\theta_0, \tilde{\rho}_n, \tilde{\theta}_n, Z_i) \, \hat{M}_f(\epsilon_i) \, \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right| \leq \max(\ell, p). \sup_n \left[ (\sigma^{(n)})^\top (\sigma^{(n)}) \right] \]

\[ \times \left| \frac{1}{n} \sum_{i=1}^{n} \left( \sup_{n \geq 1} \left[ N_{1,n}(Z_i) \right] \right)^{\mu + 2} \right| \left| \hat{M}_f(\epsilon_i) \, \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right|. \] (7.36)

Note that the quantity

\[ E\left| \hat{M}_f(\epsilon_0) \, \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \left( \sup_{n \geq 1} \left[ N_{1,n}(Z_0) \right] \right)^{2} \right| < +\infty, \]

In fact, by Hölder’s inequality, we have

\[ E\left| \hat{M}_f(\epsilon_0) \, \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \left( \sup_{n \geq 1} \left[ N_{1,n}(Z_0) \right] \right)^{2} \right| \]

\[ \leq \left\{ E\left| \hat{M}_f(\epsilon_0) \, \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right|^{\lambda + 2} \right\} \left\{ E\left( \sup_{n \geq 1} \left[ N_{1,n}(Z_0) \right] \right)^{2t} \right\} \]

\[ \leq \left\{ E\left| \hat{M}_f(\epsilon_0) \right|^{\lambda + 2} \right\} \left\{ E\left( \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \right)^{\lambda + 2} \right\} \left\{ E\left( \sup_{n \geq 1} \left[ N_{1,n}(Z_0) \right] \right)^{2t} \right\} \]

Since that \( \frac{1}{\lambda + 2} + \frac{1}{2t} = 1 \), then \( t = 1 + \frac{1}{\lambda + 2} \) then \( 2t = 2 + \mu \), the conditions \((A_{3.1})\), \((A_{3.5})\) and \((A_{4.1})\) enable us to conclude that

\[ E\left| \hat{M}_f(\epsilon_0) \, \frac{G(Z_0)}{\sigma(\theta_0, Z_0)} \left( \sup_{n \geq 1} \left[ N_{1,n}(Z_0) \right] \right)^{2} \right| < +\infty. \]

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It follows from the stationarity and the ergodicity of model that the random variable
\[ \frac{1}{n} \sum_{i=1}^{n} |M_f(\varepsilon_i) \frac{G(Z)}{\sigma(\theta_0, Z_i)} (\sup_{n \geq 1} [N_1, n(Z_0)])^2| \] converges a.s. to the constant
\[ E[|M_f(\varepsilon_0) \frac{G(Z)}{\sigma(\theta_0, Z_0)} (\sup_{n \geq 1} [N_1, n(Z_0)])^2|] \] as \( n \to +\infty \). From (3.1) and the inequality (7.36), it results that:
\begin{equation}
\begin{aligned}
n^{-\frac{1}{2}} \frac{1}{n} \sum_{i=1}^{n} A(\theta_0, \tilde{\theta}_n, Z_i) M_f(\varepsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} &= o_P(1). \\
(7.37)
\end{aligned}
\end{equation}

By following the same previous reasoning in the last case and changing \( (A_{3.5}) \) by \( (A_{3.6}) \), we shall prove that
\[ |B(\theta_0, \tilde{\theta}_n, Z_i)| \leq \left[ \frac{\|v(n)\|}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right]^2, \quad \text{and} \quad n^{-\frac{1}{2}} h \frac{1}{n} \sum_{i=1}^{n} B(\theta_0, \tilde{\theta}_n, Z_i) \varepsilon_i M_f(\varepsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} = o_P(1).
\]

From the equalities (7.37) and (7.38), we deduce that
\begin{equation}
\begin{aligned}
I_{n,h,1}^{(1)} &= o_P(1). \\
I_{n,h,1}^{(2)} &= -n^{-\frac{1}{2}} h \frac{1}{n} \sum_{i=1}^{n} (\tilde{\varepsilon}_i - \varepsilon_i) M_f(\varepsilon_i) G(Z_i) \\
&= h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial m(\tilde{\rho}_n, Z_i)^\top M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} - h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} + h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)}.
\end{aligned}
\end{equation}

We have the following decomposition:
\begin{equation}
\begin{aligned}
h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial m(\tilde{\rho}_n, Z_i)^\top M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} &= h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial m(\tilde{\rho}_n, Z_i)^\top M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} + h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} + h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} \\
&= I_{n,h,1}^{(2,1)} + I_{n,h,1}^{(2,2)}, \quad \text{where}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
I_{n,h,1}^{(2,1)} &= h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial m(\tilde{\rho}_n, Z_i)^\top M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} - h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} + h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)}, \\
I_{n,h,1}^{(2,2)} &= h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial m(\tilde{\rho}_n, Z_i)^\top M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} - h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} + h \frac{1}{n} \sum_{i=1}^{n} \frac{(u(n))^\top \partial \varepsilon_i M_f(\varepsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)}.
\end{aligned}
\end{equation}
And
\[
\frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top \epsilon_i \hat{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} = \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top \epsilon_i \hat{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} - \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\hat{\theta}_n, Z_i)^\top \epsilon_i \hat{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} + \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top \epsilon_i \hat{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)}.
\]

We have then
\[
I_{n,h,1}^{(2,1)} = I_{n,h,1}^{(2,2)} + I_{n,h,1}^{(2,3)} + I_{n,h,1}^{(2,4)}.
\]

We evaluate the terms \(I_{n,h,1}^{(2,1)}\), \(I_{n,h,1}^{(2,2)}\), \(I_{n,h,1}^{(2,3)}\) and \(I_{n,h,1}^{(2,4)}\). From (7.23) the expression \(I_{n,h,1}^{(2,1)}\) can also be written
\[
I_{n,h,1}^{(2,1)} = -n^{-\frac{1}{2}} h \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top (u^{(n)})^\top \partial m(\tilde{\rho}_n, Z_i)^\top G(Z_i)}{\sigma(\theta_0, Z_i)} \hat{M}_f(\epsilon_i).
\]

By Cauchy-Schwarz's inequality, we obtain
\[
\left| \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top (u^{(n)})^\top \partial m(\tilde{\rho}_n, Z_i)^\top}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} \right| \leq \frac{\|v^{(n)}\|_p \|\partial \sigma(\tilde{\theta}_n, Z_i)\|_p}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} \frac{\|u^{(n)}\|_\ell \|\partial m(\tilde{\rho}_n, Z_i)\|_\ell}{\sigma(\theta_0, Z_i)}
\]
\[
\leq \frac{1}{2} \left\{ \frac{\|v^{(n)}\|_p \|\partial \sigma(\tilde{\theta}_n, Z_i)\|_p}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} \right\}^2 + \frac{1}{2} \left\{ \frac{\|u^{(n)}\|_\ell \|\partial m(\tilde{\rho}_n, Z_i)\|_\ell}{\sigma(\theta_0, Z_i)} \right\}^2 \leq \frac{1}{2} \sup_n \|\tau^{(n)}\| \tau^{(n)}) \left\{ \left\{ \frac{\|\partial \sigma(\tilde{\theta}_n, Z_i)\|_p}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} \right\}^2 + \left\{ \frac{\|\partial m(\tilde{\rho}_n, Z_i)\|_\ell}{\sigma(\theta_0, Z_i)} \right\}^2 \right\}.
\]

Then, we obtain
\[
|I_{n,h,1}^{(2,1)}| \leq n^{-\frac{1}{2}} h \sup_n \left| \tau^{(n)} \right| \left| \tau^{(n)} \right| \frac{1}{n} \sum_{i=1}^{n} \frac{G(Z_i) \hat{M}_f(\epsilon_i)}{\sigma(\theta_0, Z_i)} |\sup_{n \geq 1} |N_{i,n}(Z)| | \right|^2.
\]

(7.40)
Using the inequality (7.31), (7.32), (7.34), (7.35), (7.40) and from (A.1), (A.4), (A.5), (3.1) and the ergodic theorem, it results that
\[ I_{n,h,1}^{(2,1)} = o_p(1). \]
(7.41)

With a same reasoning and changing (A.5) by (A.6), we shall prove that:
\[ I_{n,h,1}^{(2,3)} = o_p(1). \]
(7.42)

It remains to evaluate the terms \( I_{n,h,1}^{(2,2)} \) and \( I_{n,h,1}^{(2,4)} \).

\[ I_{n,h,1}^{(2,2)} = h \frac{1}{n} \sum_{i=1}^{n} \left( \frac{(u(n))^{T} \partial m(\tilde{\rho}_n, Z_i)^{T} \tilde{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} \right). \]

By Taylor’s expansion with order 2 of the functions \( \rho \rightarrow m(\rho, \cdot) \) and \( \theta \rightarrow \sigma(\theta, \cdot) \) around \( \rho_0 \) and \( \theta_0 \) respectively, we obtain the following equalities
\[ n^{-\frac{1}{2}} (u(n))^{T} \partial m(\tilde{\rho}_n, Z_i)^{T} = n^{-\frac{1}{2}} (u(n))^{T} \partial m(\rho_0, Z_i)^{T} + \frac{1}{2} n^{-\frac{1}{2}} (u(n))^{T} \partial^2 m(\tilde{\rho}_n, Z_i)n^{-\frac{1}{2}} (u(n)), \]
\[ n^{-\frac{1}{2}} (v(n))^{T} \partial \sigma(\tilde{\theta}_n, Z_i)^{T} = n^{-\frac{1}{2}} (v(n))^{T} \partial \sigma(\theta_0, Z_i)^{T} + \frac{1}{2} n^{-\frac{1}{2}} (v(n))^{T} \partial^2 \sigma(\tilde{\theta}_n, Z_i)n^{-\frac{1}{2}} (v(n)), \]
where \( \tilde{\rho}_n \) and \( \tilde{\theta}_n \) are between \( \rho_0 \) and \( \rho_n \) and \( \theta_0 \) and \( \theta_n \) respectively. From (7.43), the expression \( I_{n,h,1}^{(2,2)} \) can also be written
\[ I_{n,h,1}^{(2,2)} = h \frac{1}{n} \sum_{i=1}^{n} \left( \frac{(u(n))^{T} \partial m(\rho_0, Z_i)^{T} \tilde{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} \right) + \frac{h}{2} n^{-\frac{1}{2}} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{(u(n))^{T} \partial^2 m(\tilde{\rho}_n, Z_i)(u(n)) \tilde{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} \right). \]

We consider the following term
\[ I_{n,h,1}^{(2,2,2)} = \frac{h}{2} n^{-\frac{1}{2}} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{(u(n))^{T} \partial^2 m(\tilde{\rho}_n, Z_i)(u(n)) \tilde{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} \right). \]
(7.45)

For all integers \( i \), we have
\[ (u(n))^{T} \partial^2 m(\tilde{\rho}_n, Z_i)(u(n)) = \sum_{k=1}^{\ell} \frac{\partial^2 m(\tilde{\rho}_n, Z_i)}{\partial \rho_k^2}(u_k(n))^{2} + \sum_{1 \leq k, j \leq \ell, k \neq j} \frac{\partial^2 m(\tilde{\rho}_n, Z_i)}{\partial \rho_k \partial \rho_j}(u_k(n))(u_j(n)). \]
(7.46)

We have the following inequalities
\[ (u_k(n))(u_j(n)) \leq |u_k(n))(u_j(n))| \]
\[ \leq \frac{1}{2}[(u_k(n))^{2} + (u_j(n))^{2}] \]
\[ \leq \frac{1}{2}||u(n)||^2. \]
(7.47)
Using the inequality (7.47), we obtain

\[
(u^{(n)})^\top \partial^2 m(\tilde{\rho}_n, Z_i)(u^{(n)}) \leq \max_{1 \leq i, j \leq \ell} \left| \frac{\partial^2 m(\tilde{\rho}_n, Z_i)}{\partial \rho_k \partial \rho_j} \right| \left[ \ell \| u^{(n)} \|_2^2 + \frac{\ell (\ell - 1)}{2} \| u^{(n)} \|_2^2 \right] \\
\leq \max_{1 \leq k, j \leq \ell} \left| \frac{\partial^2 m(\tilde{\rho}_n, Z_i)}{\partial \rho_k \partial \rho_j} \right| \left[ \ell + \frac{\ell (\ell - 1)}{2} \| u^{(n)} \|_2^2 \right] \\
\leq \max_{1 \leq k, j \leq \ell} \left| \frac{\partial^2 m(\tilde{\rho}_n, Z_i)}{\partial \rho_k \partial \rho_j} \right| \left[ \frac{\ell^2 + \ell}{2} \right] \sup_n \| (\tau^{(n)})^\top (\tau^{(n)}) \|.
\]  

(7.48)

With a same reasoning as (7.31) and (7.32), we shall prove that

\[
\| \tilde{\rho}_n - \rho_0 \|_\ell \leq \| \rho_n - \rho_0 \|_\ell.
\]

(7.49)

\[
\| \tilde{\theta}_n - \theta_0 \|_p \leq \| \theta_n - \theta_0 \|_p.
\]

(7.50)

The inequality (7.48) associated with (7.49), (7.50), (A.3.1), (A.3.5), (A.4.1), (3.1) and the ergodicity and the stationarity of the model implies that, when \( n \to +\infty \), we obtain

\[
I_{n,h,1}^{(2,2,2)} = o_p(1).
\]

(7.51)

It remains to treat the term \( I_{n,h,1}^{(2,2,1)} \), such that

\[
I_{n,h,1}^{(2,2,1)} = h \frac{1}{n} \sum_{i=1}^{n} \frac{(u^{(n)})^\top \partial m(\rho_0, Z_i)^\top \hat{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)}.
\]

We have for all integers \( i \)

\[
(u^{(n)})^\top \partial m(\rho_0, Z_i)^\top = \sum_{j=1}^{\ell} u_j^{(n)} \frac{\partial m(\rho_0, Z_i)}{\partial \rho_j}.
\]

We obtain

\[
I_{n,h,1}^{(2,2,1)} = h \frac{1}{n} \sum_{i=1}^{n} \frac{(u^{(n)})^\top \partial m(\rho_0, Z_i)^\top \hat{M}_f(\epsilon_i) G(Z_i)}{\sigma(\theta_0, Z_i)} = h \frac{1}{n} \sum_{i=1}^{n} \frac{u_j^{(n)} \frac{\partial m(\rho_0, Z_i)}{\partial \rho_j}}{\sigma(\theta_0, Z_i)} \hat{M}_f(\epsilon_i) G(Z_i) + \ldots + h \frac{1}{n} \sum_{i=1}^{n} \frac{u_j^{(n)} \frac{\partial m(\rho_0, Z_i)}{\partial \rho_j}}{\sigma(\theta_0, Z_i)} \hat{M}_f(\epsilon_i) G(Z_i).
\]

It follows from (A.3.1), (A.3.5) and (A.4.3) that

For all \( j \in \{1, \ldots, \ell \} \) and as \( n \to +\infty \), we have

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{\partial m(\rho_0, Z_i)}{\partial \rho_j} \hat{M}_f(\epsilon_i) G(Z_i) \to_a E \left[ \frac{\partial m(\rho_0, Z_0)}{\partial \rho_j} \hat{M}_f(\epsilon_0) G(Z_0) \right] = K_j.
\]

Therefore, there exist for all \( j \in \{1, \ldots, \ell \} \) a random variable \( E_{j,n} \), where \( E_{j,n} \) converges a.s to 0 as \( n \to +\infty \), such that

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{\partial m(\rho_0, Z_i)}{\partial \rho_j} \hat{M}_f(\epsilon_i) G(Z_i) = K_j + E_{j,n}.
\]

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We have
\[ u_j^{(n)} \leq \|u^{(n)}\|_\ell \leq [\sup_n (\tau^{(n)})^\top (\tau^{(n)})] < +\infty. \]
Therefore
\[ u_j^{(n)} E_{j,n} = o_P(1). \]
It results that
\[ I_{n,h,1}^{(2,2,1)} = h (u^{(n)})^\top K^\top + o_P(1), \quad (7.52) \]
with
\[
K^\top = (K_1,\cdots,K_\ell), \\
K_j = \mathbb{E} \left[ \frac{\partial m(\theta_0,Z_0)}{\partial \theta_j} M_f(\epsilon_0) G(Z_0) \right], \\
j \in \{1,\ldots,\ell\}.
\]
It follows from the equalities (7.51) and (7.52) that
\[ I_{n,h,1}^{(2,2)} = h (u^{(n)})^\top K^\top + o_P(1). \quad (7.53) \]
It remain to process the term \( I_{n,h,1}^{(2,4)} \).
With a similar method, we shall give a similar inequality as (7.48), therefore we obtain
\[
(u^{(n)})^\top \partial^2 \sigma(\tilde{\theta}_n, Z_i)(u^{(n)}) \leq \max_{1 \leq k,j \leq p} \left[ \frac{\partial^2 \sigma(\tilde{\theta}_n, Z_i)}{\partial \theta_k \partial \theta_j} \right] \left[ \frac{p^2 + p}{2} \right] \sup_n (\tau^{(n)})^\top (\tau^{(n)})
\]
\[. \quad (7.54) \]
By changing (7.43), (7.48) and (A3b) by (7.44), (7.54) and (A3a) respectively and using the same reasoning as the term \( I_{n,h,1}^{(2,2)} \), we obtain the following equation:
\[ I_{n,h,1}^{(2,4)} = h (u^{(n)})^\top J^\top + o_P(1), \quad (7.55) \]
where \( J^\top = (J_1,\cdots,J_p) \),
\[ J_k = \mathbb{E} \left[ \frac{\partial \sigma(\theta_0, Z_0)}{\partial \theta_k} \epsilon_0 M_f(\epsilon_0) G(Z_0) \right], \quad (7.56) \]
In summary, we have
\[
I_{n,h,1} = I_{n,h,1}^{(1)} + I_{n,h,1}^{(2)}, \\
I_{n,h,1}^{(2)} = I_{n,h,1}^{(2,1)} + I_{n,h,1}^{(2,2)} + I_{n,h,1}^{(2,3)} + I_{n,h,1}^{(2,4)}.
\]
It follows from the equalities (7.39), (7.41), (7.42), (7.53) and (7.55), that:
\[ I_{n,h,1} = h (u^{(n)})^\top K^\top + h (u^{(n)})^\top J^\top + o_P(1). \quad (7.57) \]
Evaluation of the term $I_{n,h,3}$

From the equality (7.44), we obtain

$$I_{n,h,3} = \frac{1}{n} \sum_{i=1}^{n} (v^{(n)})^\top \frac{\partial \sigma(\theta_n, Z_i)^\top}{\sigma(\theta_n, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_n, Z_i)}$$

where $I_{n,h,3}^{(1)} = \frac{1}{n} \sum_{i=1}^{n} (v^{(n)})^\top \frac{\partial \sigma(\theta, Z_i)^\top}{\sigma(\theta, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta, Z_i)}$, and $I_{n,h,3}^{(2)} = \frac{n^{-\frac{1}{2}}}{2} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial^2 \sigma(\tilde{\theta}_n, Z_i)(v^{(n)})}{\sigma(\theta_n, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_n, Z_i)}.$

We have

$$|I_{n,h,3}^{(2)}| \leq \frac{n^{-\frac{1}{2}}}{2} |h| \frac{1}{n} \sum_{i=1}^{n} \frac{|(v^{(n)})^\top \partial^2 \sigma(\tilde{\theta}_n, Z_i)(v^{(n)})|}{\sigma(\theta_n, Z_i)} |M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_n, Z_i)}|.$$ 

It follows from (3.1), (7.49), (7.50), (7.54), (A3.1), (A3.3), (A4.4) and the ergodicity and the stationarity that, when $n \to +\infty$, we obtain

$$I_{n,h,3}^{(2)} = o_P(1).$$

With the use of Taylor's expansion with order 1 of the function $\sigma(\theta, \cdot)$ around $\theta_0$, we have

$$I_{n,h,3}^{(1)} = \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top}{\sigma(\theta_0, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top}{\sigma(\theta, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}$$

$$- \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top}{\sigma(\theta_0, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta, Z_i)}$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top}{\sigma(\theta_0, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} - \frac{1}{\sigma(\theta_0, Z_i)} \right] (v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top}{\sigma(\theta_0, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta, Z_i)}$$

$$= -n^{-\frac{1}{2}} \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top}{\sigma(\theta_0 + n^{-\frac{1}{2}} v^{(n)}, Z_i)} (v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \frac{(v^{(n)})^\top \partial \sigma(\theta_0, Z_i)^\top}{\sigma(\theta_0, Z_i)} M_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}$$

$$= I_{n,h,3}^{(1,1)} + I_{n,h,3}^{(1,2)}.$$
By Cauchy-Schwartz’s inequality followed by the use of (3.1), (7.31), (7.32), (A_{3.1}), (A_{3.3}), (A_{4.1}) and the ergodicity and the stationarity of the model, we shall to prove that

\[
I_{n,h,3}^{(1,1)} = -n^{-\frac{1}{2}} \frac{1}{n} \sum_{i=1}^{n} \frac{(\bar{y}(n))^\top \partial \varphi(\tilde{\theta}_i, Z_i)^\top (\bar{y}(n))^\top \partial \varphi(\tilde{\theta}_0, Z_i)^\top}{\sigma(\tilde{\theta}_0, Z_i)^\top} M_f(\varepsilon_i) \frac{G(Z_i)}{\sigma(\tilde{\theta}_0, Z_i)}
= o_P(1).
\] (7.60)

It remains to evaluate the term \(I_{n,h,3}^{(1,2)}\), where

\[
I_{n,h,3}^{(1,2)} = \frac{1}{n} \sum_{i=1}^{n} (\bar{y}(n))^\top \partial \varphi(\tilde{\theta}_0, Z_i)^\top M_f(\varepsilon_i) \frac{G(Z_i)}{\sigma(\tilde{\theta}_0, Z_i)}
\]

Using the same reasoning applied on the term \(I_{n,h,3}^{(2,2)}\) with changing the condition (A_{3.5}) by (A_{3.3}) and using (A_{2.1}), we shall prove that

\[
I_{n,h,3}^{(1,2)} = \frac{1}{n} h (\bar{y}(n))^\top Q + o_P(1)
= o_P(1),
\] (7.61)
such that for all \(j \in \{1, \ldots, p\}\), we have

\[
Q^\top = (Q_1, \ldots, Q_p),
\]

\[
Q_j = E \left[ \frac{\partial \varphi(\tilde{\theta}_0, Z_0)}{\partial \theta_j} M_f(\varepsilon_0) G(Z_0) \right] = 0.
\]

In summary

From the equalities (7.59), (7.60) and (7.61), we deduce that

\[
I_{n,h,3} = o_P(1).
\] (7.62)

Evaluation of the term \(I_{n,h,2}\)

We have

\[
I_{n,h,2} = -n^{-\frac{1}{2}} \frac{1}{2} \sum_{i=1}^{n} (\tilde{y}_{i,n} - \varepsilon_i)^2 \tilde{M}_f(\tilde{\varepsilon}_i) \frac{G(Z_i)}{\sigma(\tilde{\theta}_0 + n^{-\frac{1}{2}} \bar{y}(n), Z_i)}
= -n^{-\frac{1}{2}} \frac{1}{2} \sum_{i=1}^{n} (\tilde{y}_{i,n} - \varepsilon_i)^2 \tilde{M}_f(\tilde{\varepsilon}_i, Z_i)
= -n^{-\frac{1}{2}} \frac{1}{2} \sum_{i=1}^{n} (\tilde{y}_{i,n} - \varepsilon_i)^2 \tilde{M}_f(\tilde{\varepsilon}_i, Z_i)
= -n^{-\frac{1}{2}} \frac{1}{2} \sum_{i=1}^{n} (\tilde{y}_{i,n} - \varepsilon_i)^2 \tilde{M}_f(\tilde{\varepsilon}_i, Z_i)
= -n^{-\frac{1}{2}} \frac{1}{2} \sum_{i=1}^{n} (\tilde{y}_{i,n} - \varepsilon_i)^2 \tilde{M}_f(\tilde{\varepsilon}_i, Z_i)
= I_{n,h,2}^{(1)} + I_{n,h,2}^{(2)}.
\]

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where
\[
I^{(1)}_{n,h,2} = -n^{-\frac{1}{2}} \frac{h}{2} \sum_{i=1}^{n} (\tilde{\epsilon}_{i,n} - \epsilon_i)^2 \tilde{M}_f(\epsilon_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}
\]
\[
I^{(2)}_{n,h,2} = -n^{-\frac{1}{2}} \frac{h}{2} \sum_{i=1}^{n} (\tilde{\epsilon}_{i,n} - \epsilon_i)^2 \tilde{M}_f(\epsilon_i) G(Z_i) \left[ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} - \frac{1}{\sigma(\theta_0, Z_i)} \right].
\]

From (7.24) and after majoration and the use of the Cauchy Schwartz’s inequality, it results that:
\[
(\tilde{\epsilon}_{i,n} - \epsilon_i)^2 \leq \frac{2}{n} \left\{ \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right) \sum_{i=1}^{n} \left( \frac{(v^{(n)})^\top \partial m(\tilde{\rho}_n, Z_i)^\top}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 \right\} + \frac{2}{n} \left\{ \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right) \sum_{i=1}^{n} \left( \frac{(v^{(n)})^\top \partial \sigma(\tilde{\theta}_n, Z_i)^\top}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 \right\} \epsilon_i^2. \tag{7.63}
\]

Then
\[
|I^{(1)}_{n,h,2}| \leq n^{-\frac{1}{2}} |h| \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 |\tilde{M}_f(\tilde{\epsilon}_i)| \right\} G(Z_i) \sigma(\theta_0, Z_i)
\]
\[
+ n^{-\frac{1}{2}} |h| \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 |\tilde{M}_f(\tilde{\epsilon}_i)| \right\} G(Z_i) \sigma(\theta_0, Z_i)
\]
\[
\leq n^{-\frac{1}{2}} |h| \left[ \sup_{n}(\tau(n)^\top \tau(n)) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 |\tilde{M}_f(\tilde{\epsilon}_i)| \right\} G(Z_i) \sigma(\theta_0, Z_i) \right]
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 \right\} \epsilon_i^2 |\tilde{M}_f(\tilde{\epsilon}_i)| |\sigma(\theta_0, Z_i)| G(Z_i).
\]

Since the second derivative $\tilde{M}_f$ is bounded, then there exist a positive real $\vartheta$ such that $\forall x \in \mathbb{R}$, we have
\[
|\tilde{M}_f(x)| \leq \vartheta. \tag{7.64}
\]

It follows that
\[
|I^{(1)}_{n,h,2}| \leq \vartheta n^{-\frac{1}{2}} |h| \left[ \sup_{n}(\tau(n)^\top \tau(n)) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 \right\} G(Z_i) \sigma(\theta_0, Z_i) \right]
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \right)^2 \right\} \epsilon_i^2 \left( \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \right).
\]

From (3.1), (7.31), (7.32), (A.3.1), (A.3.7), (A.4.1) and the ergodic theorem, it follows asymptotically that
\[
I^{(1)}_{n,h,2} = o_p(1). \tag{7.65}
\]

It remains to evaluate the term $I^{(2)}_{n,h,2}$.

By Taylor’s expansion with order 1 of the function $\sigma(\theta, \cdot)$ around $\theta_0$, the expression $I^{(2)}_{n,h,2}$ can also be written
\[
I^{(2)}_{n,h,2} = \frac{1}{n} h \sum_{i=1}^{n} \left( \frac{1}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \partial \sigma(\tilde{\theta}_n, Z_i)^\top \right) (\tilde{\epsilon}_{i,n} - \epsilon_i)^2 \tilde{M}_f(\tilde{\epsilon}_i) \frac{G(Z_i)}{\sigma(\theta_0, Z_i)}.
\]
From the inequalities (7.63) and (7.64) followed by Cauchy -Schwartz inequality and a simple majoration, we obtain

\[ |I_{n,h,2}^{(2)}| \leq \frac{1}{n} \vartheta |h| |\sup_n \| r^{(n)} | (r^{(n)})^\top \| \| \frac{1}{n} \sum_{i=1}^n \left\{ \frac{\| \partial m(\tilde{\theta}_n, Z_i) \|_p}{\sigma(\theta_0 + n^{-\frac{1}{2}}, Z_i)} \right\}^2 \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \frac{\| \partial \sigma(\tilde{\theta}_n, Z_i) \|}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} | \]

\[ + \frac{1}{n} \sum_{i=1}^n \left\{ \frac{\| \partial \sigma(\tilde{\theta}_n, Z_i) \|_p}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} \right\}^2 \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \frac{\| \partial \sigma(\tilde{\theta}_n, Z_i) \|}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} \]

\[ \leq \frac{1}{n} \vartheta |h| |\sup_n \| r^{(n)} | (r^{(n)})^\top \| \| \frac{1}{n} \sum_{i=1}^n \left\{ \frac{\| \partial m(\tilde{\theta}_n, Z_i) \|_p}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} \right\}^3 \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \]

\[ + \frac{1}{n} \sum_{i=1}^n \left\{ \frac{\| \partial \sigma(\tilde{\theta}_n, Z_i) \|_p}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} \right\}^3 \frac{G(Z_i)}{\sigma(\theta_0, Z_i)} \].

From (7.31), (7.32), (3.1), (A3.1), (A3.7), (A4.2), and the ergodicity of the model, it follows that

\[ I_{n,h,2}^{(2)} = o_P(1). \] (7.66)

From the equalities (7.65) and (7.66), we deduce that

\[ I_{n,h,2} = o_P(1). \] (7.67)

In summary, we have the following equalities

\[ \tilde{r}_{f,h,n} - r_{f,h,n} = I_{n,h,1} + I_{n,h,2} + I_{n,h,3}, \]

\[ I_{n,h,1} = h(u(n))^\top K^\top + h(v(n))^\top J^\top + o_P(1), \]

\[ I_{n,h,2} = o_P(1), \]

\[ I_{n,h,3} = o_P(1). \]

We deduce that

\[ \tilde{r}_{f,h,n} - r_{f,h,n} = h(u(n))^\top K^\top + h(v(n))^\top J^\top + o_P(1). \] (7.68)

In order to evaluate the term \( \tilde{q}_{f,h',n} - q_{f,h',n} \), we consider the difference between the equations (3.7) et (3.4), then we obtain

\[ \tilde{q}_{f,h',n} - q_{f,h',n} = -n^{-\frac{1}{2}} h' \sum_{i=1}^n \frac{N_f(\tilde{\epsilon}_{i,n})}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} - \frac{N_f(\tilde{\epsilon}_i)}{\sigma(\theta_0, Z_i)} S(Z_i). \]

Using the same reasoning that (7.25), it results that

\[ \frac{N_f(\tilde{\epsilon}_{i,n})}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} - \frac{N_f(\tilde{\epsilon}_i)}{\sigma(\theta_0, Z_i)} = \frac{N_f(\tilde{\epsilon}_{i,n}) - N_f(\tilde{\epsilon}_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} - \frac{n^{-\frac{1}{2}}(v(n))^\top \partial \sigma(\tilde{\theta}_n, Z_i)\partial \sigma(\tilde{\theta}_n, Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} \]

\[ = \frac{\tilde{\epsilon}_{i,n} - \epsilon_i - \epsilon_i^2}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} + \frac{\tilde{\epsilon}_i}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} - \frac{n^{-\frac{1}{2}}(v(n))^\top \partial \sigma(\tilde{\theta}_n, Z_i)\partial \sigma(\tilde{\theta}_n, Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}}v(n), Z_i)} \]

\[ \frac{N_f(\tilde{\epsilon}_i)}{\sigma(\theta_0, Z_i)}. \] (7.69)
Hence

\[ \tilde{q}_{f,h',n} - q_{f,h',n} = I'_{n,h',1} + I'_{n,h',2} + I'_{n,h',3}, \]

with

\[
I'_{n,h',1} = -n^{-\frac{1}{2}} h' \sum_{i=1}^{n} \frac{\left( \hat{\epsilon}_{i,n} - \epsilon_{i} \right) \tilde{N}_f(\epsilon_{i})}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} S(Z_i), \tag{7.70}
\]

\[
I'_{n,h',2} = -n^{-\frac{1}{2}} h' \sum_{i=1}^{n} \frac{\left( \hat{\epsilon}_{i,n} - \epsilon_{i} \right)^2}{2 \sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i)} \tilde{N}_f(\hat{\epsilon}_{i}) S(Z_i), \tag{7.71}
\]

\[
I'_{n,h',3} = \frac{1}{n} h' \sum_{i=1}^{n} \frac{(v(n))^\top \partial \sigma(\hat{\theta}_{n}, Z_i)\top N_f(\epsilon_{i}) S(Z_i)}{\sigma(\theta_0 + n^{-\frac{1}{2}} v(n), Z_i) \sigma(\theta_0, Z_i)} N_f(\epsilon_{i}) S(Z_i). \tag{7.72}
\]

**Evaluation of the term** \( I'_{n,h',1} \)

Firstly, from (3.9), we remark that

\[ |\tilde{N}_f(x)| \leq |M_f(x)| + |x \tilde{M}_f(x)|. \tag{7.73} \]

It results from the application of the Lemma (7.1) on the inequality (7.73) and the use of the conditions \((A_{3.3})\) and \((A_{3.6})\), that

- \((A_{3.5})\):
  
  There exist \( \lambda > 0 \) such that : \( \mathbb{E}|\tilde{N}_f(\epsilon_0)|^{\lambda+2} < +\infty \).

We have from the equality (3.9), the following equality

\[ x \tilde{N}_f(x) = x M_f(x) + x^2 \tilde{M}_f(x). \]

By applying on this last equality the Lemma (7.1) combined with the conditions \((A_{3.4})\) and \((A_{3.8})\), we deduce that

\((A_{3.6})\): There existe \( \lambda > 0 \), such that : \( \mathbb{E}\left|\epsilon_0 \tilde{N}_f(\epsilon_0)\right|^{\lambda+2} < +\infty \). By changing respectively \((A_{3.1})\), \((A_{3.5})\) and \((A_{3.6})\) by \((A_{3.2})\), \((A_{3.3})\) and \((A_{3.6})\) and with applying on the expression \( I'_{n,h',1} \) the same previous reasoning applied on the expression \( I_{n,h,1} \), we shall prove that

\[ I'_{n,h,1} = h' (u(n))^{\top} K^{\top} + h'(v(n))^{\top} J^{\top} + o_P(1), \]

such that

\[ K^{\top} = (K'_1, \cdots, K'_n), \]

\[ K'_\ell = \mathbb{E} \left[ \frac{\partial m(\psi_0, Z_0)}{\partial \psi_\ell} \tilde{N}_f(\epsilon_0) S(Z_0) \right], \]

\[ J^{\top} = (J'_1, \cdots, J'_p), \]

\[ J'_k = \mathbb{E} \left[ \frac{\partial \sigma(\theta_0, Z_0)}{\partial \theta_k} \epsilon_0 \tilde{N}_f(\epsilon_0) S(Z_0) \right]. \]

**Evaluation of the term** \( I'_{n,h',2} \)

In this case, the condition \((7.64)\) is replaced by the following condition :

\[ |\tilde{N}_f(x)| \leq \psi', \tag{7.74} \]

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where $\theta'$ is strictly positive real.
By changing $(A_{3.1})$ by $(A_{3.2})$ and with applying on the expression $I'_{n,h',2}$ the same previous reasoning applied on the expression $I_{n,h,2}$, we shall prove that

$$I'_{n,h',2} = o_P(1). \quad (7.75)$$

**Evaluation of the $I'_{n,h',3}$**

From the definition of the function $N_f$, and using the condition $(A_{2.2})$, we obtain the following condition : $(A_{2.1}) : \mathbb{E} \{ N_f(\epsilon_0) \} = 0$. By changing respectively $(A_{2.1})$ by $(A_{2.1}')$ and with applying on the expression $I'_{n,h',3}$ the same previous reasoning applied on the expression $I_{n,h,3}$, we shall prove that

$$I'_{n,h',3} = o_P(1). \quad (7.76)$$

**In summary** It follows from the equalities (7.74), (7.75) and (7.76)

$$\tilde{q}_{f,h',n} - q_{f,h',n} = h'(u^{(n)})^\top K'^\top + h'(v^{(n)})^\top J'^\top + o_P(1). \quad (7.77)$$

Hence the proposition is established.

**Proof of the Proposition 3.2**

The proof of proposition 3.2 is a consequence of the works of Le Cam (1960) and Kreiss (1987). The interested reader can refer to in (Kreiss, 1987, Lemma (4.4)) for more details.

**Proof of the Theorem**

Consider again the equality

$$\hat{V}_{n,h,h'} - V_{n,h,h'} = \sqrt{n}(\hat{\theta}_n - \theta_0)^\top (h K'^\top + h' K'^\top) + \sqrt{n}(\hat{\theta}_n - \theta_0)^\top (h J'^\top + h' J'^\top) + o_P(1). \quad (7.78)$$

and let

$$D_{n,h,h'} = - \left( \sqrt{n}(\hat{\theta}_n - \theta_0)^\top (h K'^\top + h' K'^\top) + \sqrt{n}(\hat{\theta}_n - \theta_0)^\top (h J'^\top + h' J'^\top) \right),$$

clearly, $|D_{n,h,h'}| = O_P(1)$, in fact by applying the Cauchy Schwartz inequality combined with the triangle inequality, it follows that :

$$|D_{n,h,h'}| \leq \sqrt{n}||\hat{\theta}_n - \theta_0||_p h K'^\top + h' K'^\top ||_p + \sqrt{n}||\hat{\theta}_n - \theta_0||_p ||h J'^\top + h' J'^\top||_p.$$ 

Since the estimates $\rho_n$ and $\theta_n$ are consistent, it follows that $D_{n,h,h'} = O_P(1)$, therefore the equality (7.78) can also rewritten

$$\hat{V}_{n,h,h'} - V_{n,h,h'} = -D_{n,h,h'} + o_P(1). \quad (7.79)$$

From the assumption (P.0), there exists another estimate $\Omega_n = \Omega_n^{(1)}$ of the unknown parameter $\Omega$ such that

$$V_{n,h,h'}(\Omega_n) = V_{n,h,h'} + o_P(1). \quad (7.80)$$

Under a additional assumptions (P.1), $\Omega_n$ is $\sqrt{n}$-root consistent, see (Lounis, 2012, Subsection 1.2)

The equality (7.80), enables us to deduce that, with $o_P(1)$ close, the replacing in the expression (2.3)
of the test of the central sequence $V_{n,h,h'}(\Omega)\) by the estimate central sequence $V_{n,h,h'}(\Omega_{n})$ has no effect.

From the continuity of the function $\tau^2(\cdot,\cdot)$ and the convergence in probability of the random sequence $\Omega_{n}$ to the unknown parameter $\Omega$, it follows that under the hypothesis $H_0$ and under contiguous alternatives, we get

$$I\left\{\frac{V_{n,h,h'}(\Omega_{n})}{\tau_{n,h,h'}(\hat{\rho}_n,\hat{\theta}_n)} \geq Z(u)\right\} = I\left\{\frac{V_{n,h,h'}(\Omega)}{\tau_{n,h,h'}(\rho,\theta)} \geq Z(u)\right\} + o_P(1).$$

The two sequences of tests $\hat{T}_n = I\left\{\frac{V_{n,h,h'}(\Omega_{n})}{\tau_{n,h,h'}(\hat{\rho}_n,\hat{\theta}_n)} \geq Z(u)\right\}$ and $T_n = I\left\{\frac{V_{n,h,h'}(\Omega)}{\tau_{n,h,h'}(\rho,\theta)} \geq Z(u)\right\}$ are locally and asymptotically equivalent, hence the optimality of the test. The asymptotic power of this test is equal to $1 - \Phi(Z(\alpha) - \tau^2(\hat{\rho}_n))$, see (Hwang and Basawa, 2001, Theorem 3).

**Remark 7.1** We can also get the optimality of the test when we replace the estimate $(\Omega_{n})$ by the $(\Omega_{n})$ in this previous proof.

**Conclusion 7.1** On a basis of the discrete estimates and for each step $n$, we have modified one component of our estimate in order to absorb the error, this new estimate was constructed on the tangent space of the discrete estimate in each step $n$, so the introduction of this kind of estimate has enabled us to get the optimality of the test which is based on the Neyman-Pearson statistic when we replace in the expression of this statistic the unknown parameter by the M.D.E.

In practise, we shall obtain a good M.D.E. when the errors $\|\hat{\rho}_n - \rho_0\|_i$ and $\|\hat{\theta}_n - \theta_0\|_p$ are best estimated, in this case, we shall used the bootstrap methods.

**Proof of the Lemma (4.1)**

For the AR(m) model, the expression of the central sequence is given by:

$$V_n(\rho_0) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_f(\epsilon_i) G(Y(i-1)) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} N_f(\epsilon_i) S(Y(i-1)), \quad \text{where} \quad N_f(\epsilon_i) = 1 + \epsilon_i M_f(\epsilon_i).$$

In order to evaluate the difference between the two partial derivatives central sequences, we calculate the derivative with respect to the component $\rho_j$, then we obtain:

For each integer $i \in \{1, \ldots, m\}$ we have:

$$\frac{\partial \epsilon_i}{\partial \rho_j} = -Y_{i-j}, \quad \hat{M}_f(\epsilon_i) = -1, \quad \text{and} \quad \hat{N}_f(\epsilon_i) = -2\epsilon_i.$$

With a simple calculation, we shall prove that:

$$\frac{1}{\sqrt{n}} \frac{\partial V_n(\rho)}{\partial \rho_j} = -\frac{1}{n} \sum_{i=1}^{n} \frac{\partial (\epsilon_i)}{\partial \rho_j} \hat{M}_f(\epsilon_i) G(Z_i) - \frac{1}{n} \sum_{i=1}^{n} \frac{\partial (\epsilon_i)}{\partial \rho_j} \hat{N}_f(\epsilon_i) S(Z_i),$$

and,

$$\frac{1}{\sqrt{n}} \frac{\partial V_n(\hat{\rho}_n)}{\partial \rho_j} = -\frac{1}{n} \sum_{i=1}^{n} \frac{\partial (\hat{\epsilon}_i,n)}{\partial \rho_j} \hat{M}_f(\hat{\epsilon}_i,n) G(Z_i) - \frac{1}{n} \sum_{i=1}^{n} \frac{\partial (\hat{\epsilon}_i,n)}{\partial \rho_j} \hat{N}_f(\hat{\epsilon}_i,n) S(Z_i).$$

(7.81) (7.82) (7.83)
From the difference between the equalities (7.83) and (7.81), it follows that:

\[
\frac{1}{\sqrt{n}} \frac{\partial V_n(\rho)}{\partial \rho_j} - \frac{1}{\sqrt{n}} \frac{\partial V_n(\hat{\rho}_n)}{\partial \rho_j} = \frac{-1}{n} \sum_{i=1}^{n} \left( \frac{\partial (\hat{\epsilon}_{i,n})}{\partial \rho_j} \hat{M}_f(\hat{\epsilon}_{i,n}) - \frac{\partial (\epsilon_i)}{\partial \rho_j} \hat{M}_f(\epsilon_i) \right) G(Z_i)
\]

\[
-\frac{1}{n} \sum_{i=1}^{n} \left( \frac{\partial (\hat{\epsilon}_{i,n})}{\partial \rho_j} \hat{N}_f(\hat{\epsilon}_{i,n}) - \frac{\partial (\epsilon_i)}{\partial \rho_j} \hat{N}_f(\epsilon_i) \right) S(Z_i),
\]

\[
= -\frac{2}{n} \sum_{i=1}^{n} (\hat{\epsilon}_{i,n} - \epsilon_i) Y_{i-j} S(Z_i).
\]

(7.84)

From the equalities the previous equalities, it follows that:

\[
\frac{1}{\sqrt{n}} \frac{\partial V_n(\rho)}{\partial \rho_j} - \frac{1}{\sqrt{n}} \frac{\partial V_n(\hat{\rho}_n)}{\partial \rho_j} = (\hat{\rho}_{1,n} - \rho_1) \times \frac{-2}{n} \sum_{i=1}^{n} Y_{i-1} Y_{i-j} S(Z_i)
\]

\[
+ \cdots + (\hat{\rho}_{m,n} - \rho_m) \times \frac{-2}{n} \sum_{i=1}^{n} Y_{i-m} Y_{i-j} S(Z_i).
\]

(7.85)

(7.86)

For all integers \(i\) and \(j\), we have the following equalities:

\[
|Y_{i-m} Y_{i-j} S(Z_i)| \leq \frac{1}{2} \left[ |Y_{i-m}|^2 + |S(Z_i)|^2 \right] \leq \frac{1}{4} \left[ |Y_{i-m}|^4 + |Y_{i-j}|^4 \right] + \frac{1}{2} |S(Z_i)|^2.
\]

(7.87)

By applying this last equalities on the next previous equality, it results that:

\[
\left| \frac{1}{\sqrt{n}} \frac{\partial V_n(\rho)}{\partial \rho_j} - \frac{1}{\sqrt{n}} \frac{\partial V_n(\hat{\rho}_n)}{\partial \rho_j} \right| \leq (\hat{\rho}_{1,n} - \rho_1) \times \left[ \frac{1}{2n} \sum_{i=1}^{n} Y_{i-1}^4 + \frac{1}{2n} \sum_{i=1}^{n} Y_{i-j}^4 + \frac{1}{n} \sum_{i=1}^{n} S^2(Z_i) \right]
\]

\[
+ \cdots + (\hat{\rho}_{m,n} - \rho_m) \times \left[ \frac{1}{2n} \sum_{i=1}^{n} Y_{i-m}^4 + \frac{1}{2n} \sum_{i=1}^{n} Y_{i-j}^4 + \frac{1}{n} \sum_{i=1}^{n} S^2(Z_i) \right].
\]

(7.88)

Recall that the estimator \(\hat{\rho}_n = (\hat{\rho}_{1,n}, \ldots, \hat{\rho}_{m,n})'\) is consistent, it follows that, for each integer \(k \in \{1, \ldots, m\}\), the quantity \(\hat{\rho}_{k,n} - \rho_k \xrightarrow{p} 0\) as \(n \rightarrow \infty\), remark that this convergence in probability is one consequence of the continuous mapping theorem, see for instance van der Vaart (1998). Since the model is ergodic with finite second and fourth moments, we obtain under \(H_0\):

\[
\frac{1}{\sqrt{n}} \frac{\partial V_n(\hat{\rho}_n)}{\partial \rho_j} = \frac{1}{\sqrt{n}} \frac{\partial V_n(\rho_0)}{\partial \rho_j} + o_p(1).
\]

### Appendix

We prove the results which are stated in the remark 3.1, more precisely when \(f\) is density of a student distribution with a degree of freedom \(l\) greater than 3, the functions \(x \mapsto M_f(x), x \mapsto \hat{M}_f(x)\) and \(x \mapsto x \hat{M}_f(x)\) are bounded.

We have

\[
f(x) = C_l (1 + \frac{x^2}{l})^{\frac{1}{l+1}},
\]

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where $C_l = \frac{\Gamma(l+\frac{1}{2})}{\sqrt{\pi \Gamma(l+1/2)}}$, and $\Gamma$ is the gamma function. Then we have

\[
M_f(x) = -\frac{l+1}{l} \frac{x}{(1 + \frac{x^2}{l})},
\]

\[
\dot{M}_f(x) = -\frac{l+1}{l} \left[ \frac{2x^2}{(1 + \frac{x^2}{l})^2} - \frac{1}{1 + \frac{x^2}{l}} \right],
\]

\[
\ddot{M}_f(x) = -\frac{l+1}{l} \left[ \frac{8x^3}{l} \frac{1}{(1 + \frac{x^2}{l})^3} - \frac{4x}{l} \frac{1}{(1 + \frac{x^2}{l})^2} - \frac{2x}{l} \right].
\]

We have

\[
|\dot{M}_f(x)| \leq \frac{l+1}{l} \left[ \frac{2x^2}{(1 + \frac{x^2}{l})^2} + \frac{1}{1 + \frac{x^2}{l}} \right].
\]

We can remark that

\[
\frac{2x^2}{(1 + \frac{x^2}{l})^2} = \frac{2x}{l} \frac{1}{(1 + \frac{x^2}{l})} \frac{2x}{l}.
\]

Since $\frac{2x}{l} \leq (1 + \frac{x^2}{l})$ and $1 \leq (1 + \frac{x^2}{l})$, it results that

\[
|\dot{M}_f(x)| \leq \frac{3(l+1)}{2l}.
\]

(7.89)

We have

\[
|\ddot{M}_f(x)| \leq \frac{l+1}{l} \left[ \left| \frac{8x^3}{l} \frac{1}{(1 + \frac{x^2}{l})^3} \right| + \left| \frac{4x}{l} \frac{1}{(1 + \frac{x^2}{l})^2} \right| + \left| \frac{2x}{l} \right| \right].
\]

We can remark that

\[
\frac{8x^3}{l} \frac{1}{(1 + \frac{x^2}{l})^3} = \frac{\sqrt{l}}{l} \frac{2x}{l} \frac{1}{(1 + \frac{x^2}{l})} \frac{2x}{l} \frac{1}{l}.
\]

\[
\frac{4x}{l} \frac{1}{(1 + \frac{x^2}{l})^2} = \frac{2}{\sqrt{l}} \frac{2x}{l} \frac{1}{(1 + \frac{x^2}{l})}.
\]

\[
\frac{2x}{l} \frac{1}{(1 + \frac{x^2}{l})^2} = \frac{\sqrt{l}}{l} \frac{1}{(1 + \frac{x^2}{l})}.
\]

It results that

\[
|\ddot{M}_f(x)| \leq \frac{(l+1)(4\sqrt{l})}{l^2}.
\]

(7.90)

It remains to show that the function $x \mapsto x\ddot{M}_f(x)$ is bounded. In fact, we have

\[
x\ddot{M}_f(x) = -\frac{l+1}{l} \left[ \frac{8x^4}{l} \frac{1}{(1 + \frac{x^2}{l})^3} - \frac{4x^2}{l} \frac{1}{(1 + \frac{x^2}{l})^2} - \frac{2x^2}{l} \right].
\]
We have

\[ \frac{8x^4}{l^2} \left( 1 + \frac{x^2}{l^2} \right)^3 = 8 \frac{x^2}{l} \left( 1 + \frac{x^2}{l^2} \right)^2 \left( 1 + \frac{x^2}{l^2} \right). \]

\[ \frac{2x^2}{l^2} \left( 1 + \frac{x^2}{l^2} \right)^2 = 4 \frac{x^2}{l} \frac{1}{1 + \frac{x^2}{l^2}}. \]

\[ \frac{2x^2}{l^2} \left( 1 + \frac{x^2}{l^2} \right) = 2 \frac{x^2}{l} \frac{1}{1 + \frac{x^2}{l^2}}. \]

Since \( \frac{x^2}{l^2} \leq (1 + \frac{x^2}{l^2}) \leq (1 + \frac{x^2}{l^2})^2 \), it result that

\[ |x \tilde{N}_f(x)| \leq \frac{14(l + 1)}{l}. \] (7.91)

Using the equality (3.9) and from the equalities (7.89) and (7.91), it results that the second derivative \( \tilde{N}_f \) is bounded. Obviously, this previous results remain satisfied when the value of the degree of freedom is smaller than 3.

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