Finite presentations of centrally extended mapping class groups

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Abstract
We describe a finite presentation of $T_{g,r}$ for $g \geq 3$. Here $T_{g,r}$ is the universal central extension of the mapping class group of the surface of genus $g$ with $r$-boundaries. We also investigate the cases of $g = 2, 3$, and give an application.

Keywords  mapping class group, central extension

MSC2010 database  16S20, 37E30, 57M07

1 Introduction and main results

The mapping class group $M_{g,r}$ of the surface $\Sigma_{g,r}$ of genus $g$ with $r$-boundaries has been heavily studied in low-dimensional topology. Especially, in quantum topology, Witten [Wi] has made a prophetic discovery that the Chern-Simons quantum field theory of level $k$ (with Wilson loops) produces 3-manifold invariants and (quantum) representations $V_k$ of $M_{g,r}$ with an “anomaly”; this prophecy has since been mathematically formulated in some cases (see, e.g., [BH, GM, Koh]) and the anomaly has been interpreted as a 2-framing [Ati] or a $p_1$-structure [BH]. Because of the obstacles from by $p_1$-structures, the space $V_k$ is not always some right module of $M_{g,r}$, but that of a central extension over $M_{g,r}$.

Such central extensions are sometimes considered to be complicated in a sense. In contrast, the subjacent groups $M_{g,r}$ have been widely analysed with finite presentations (see, e.g., [FM, G2]). To see this, since $M_{g,r}$ with $g \geq 3$ is perfect (see [Kor]), we can set up the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow T_{g,r} \longrightarrow M_{g,r} \longrightarrow 0 \quad \text{(exact)},$$

associated with the group cohomology $H^2(M_{g,r}; \mathbb{Z}) \cong \mathbb{Z}$, which can be computed in a combinatorial way [Kor §6]. A 2-cocycle $\tau_g$ corresponding to a generator of the center $\mathbb{Z}$ has some difficulties, as mentioned in [Ati]; the results of Meyer [Me] and Turaev [T] indicate that the quadruple $4\tau_g$ can be geometrically described as a signature from the modular group $Sp(2g; \mathbb{Z})$; however, known formulations of $\tau_g$ are algebraically a little complicated. For example, Turaev formula of $\tau_g$ [T] is a signature 2-cocycle with a modification using a Maslov index; furthermore, in [GM], the structure of $T_{g,r}$ was described in terms of Lagrangian cobordisms.

1.1 Results; The cases of $g \geq 3$ and $r = 0, 1$.

This paper explicitly describes a finite presentation of $T_{g,r}$ for $g \geq 3$, $r \geq 0$. To begin, the presentations with $r = 0, 1$ are as follows:
Theorem 1. (I) Let \( g \geq 3 \). The central extension \( \mathcal{T}_{g,1} \) of \( \mathcal{M}_{g,1} \) in (I) has a presentation with generators \( c_0, c_1, \ldots, c_{2g+1}, \mu \) and the following relations:

(Braid relation) \[ c_i c_j = c_j c_i, \quad \text{if } I(c_i, c_j) = 0, \]  
(3-chain relation) \[ (c_1 c_2 c_3)^4 (c_0 b_0)^{-1} = \mu, \quad c_i \mu = \mu c_\bar{i}, \]  
where \( I \) means the geometric intersection number in Figure 1, and the notation \( b_0, b_1, b_2, b_3 \) is common ones in the Wajnryb's presentation [Wa]: Precisely, we have

\[
\begin{align*}
    b_0 &= (c_4 c_2 c_1 c_1 c_2 c_3 c_4)^{-1} c_0 (c_4 c_2 c_1 c_2 c_3 c_4), \\
    b_1 &= (c_4 c_5 c_3 c_4)^{-1} c_0 (c_4 c_5 c_3 c_4), \\
    b_2 &= (c_2 c_3 c_1 c_2)^{-1} b_1 (c_2 c_3 c_1 c_2), \\
    b_3 &= (c_6 c_5 c_4 c_3 c_2 c_5)^{-1} c_0 (c_6 c_5 c_4 c_3 c_2 c_5)^{-1} c_0 (c_6 c_5 c_4 c_3 c_2 c_5)^{-1} b_1 c_6 c_5 c_1 c_2^{-1} c_3^{-1} c_4^{-1}).
\end{align*}
\]

(II) Furthermore, concerning the closed surface of genus \( g \geq 3 \), the group \( \mathcal{T}_{g,0} \) can be presented as above with adding the following commutator relation:

\[
[(c_{2g} c_{2g-1} \cdots c_1 c_1 c_2 \cdots c_2), c_{2g+1}] = 1,
\]

where the bracket \([a, b]\) is the abbreviation of \( a b a^{-1} b^{-1} \).

This presentation is a lift of the Wajnryb presentation of \( \mathcal{M}_{g,r} \) [Wa]. To be precise, his presentation can be made exactly as the quotient of the above presentation by adding the relation \( \mu = 1 \); see also [FM, Theorem 5.3] for the detail. Correspondingly, the symbols \( c_i \) and \( b_i \) in \( \mathcal{M}_{g,r} \) can be interpreted as Dehn twists along the respective curves \( \gamma_i \) and \( \beta_i \) in Figure 1.

![Figure 1: Generators of \( \mathcal{M}_{g,r} \) with \( g \geq 2, r \leq 1 \), and the curves in the lantern relation with \( g \geq 3 \).](image)

1.2 Result II; Punctured cases with \( g \geq 3 \), and the genus two case.

Furthermore, we will examine the case \( g \geq 3, \ r \geq 1 \). Following [G2], we call a triple \((i, j, k) \in \{1, \ldots, 2g + r - 2\}^3\) good, if it satisfies \( i \leq j \leq k \), or \( j \leq k \leq i \) or \( k \leq i \leq j \) without \( i = j = k \). Considering the closed curves in \( \Sigma_{g,r} \) depicted in Figure 2 let us state the expression using good indices:
Theorem 2 (cf. [G2, Theorem 1]). Let \( g \geq 3 \) and \( r \geq 1 \). The central extension \( \mathcal{T}_{g,r} \) admits a presentation with generators \( b, b_1, \ldots, b_{g-1}, a_1, \ldots, a_{2g+r-2} ; \{ c_{i,j} \}_{1 \leq i,j \leq 2g+r-2, \ i \neq j} \) and \( \mu \). Here, the relations are as follows:

(i) “Handles”: \( c_{2i,2i+1} = c_{2i+1,2i} \) for all \( i \) with \( 1 \leq i \leq g - 1 \).

(ii) “Braids”: \( xy(y^{-1}I(x,y)) = yx \) holds for all \( x, y \) among the generators with \( I(x,y) \leq 1 \), where \( I(x,y) \) is the geometric intersection number of \( x, y \) according to Figure 2.

(iii) “Stars”: \( c_{i,j}c_{j,k}c_{k,i} = (a_ia_ja_k)^3\mu^{-1} \) for all good triples \( (i, j, k) \). Here, we set \( c_{1,1} = 1 \).

(iv) “Centralization”: \( [b, \mu] = [b_i, \mu] = [a_i, \mu] = [c_{i,j}, \mu] = 1 \).

In analogy to Theorem 1, this presentation is a lift of Gervais’s presentation [G2] of \( \mathcal{M}_{g,r} \). More precisely, the theorem 1 in [G2] with \( g \geq 1 \), \( r \geq 1 \) says that the group with the presentation subject to \( \mu = 1 \) is isomorphic to \( \mathcal{M}_{g,r} \).

![Figure 2: Gervais’s generators of \( \mathcal{M}_{g,r} \) with \( g \geq 2 \), \( r \geq 1 \).](image)

Finally, we focus on the case \( g = 2 \). Although \( \mathcal{T}_2 \) is not perfect, we will give a presentation of a \( \mathbb{Z} \)-central extension \( \mathcal{T}_2 \) of \( \mathcal{M}_2 \). Here, \( \mathcal{T}_2 \) is the central extension associated with a generator of \( H^2(\mathcal{M}_2; \mathbb{Z}) = H_1(\mathcal{M}_2) \cong \mathbb{Z}/10 \).

Theorem 3. The \( \mathbb{Z} \)-central extension \( \mathcal{T}_2 \) of \( \mathcal{M}_2 \) has a presentation with generators \( c_1, c_2, c_3, c_4 \) and \( c_5 \). Here, the relations are defined by the two preceding ones (2), (3) and the followings:

\[
(c_1c_2c_3)^4c_5^{-2} = c_5c_4c_3c_2c_1c_2c_3c_4c_5,
\]

\[
[c_5c_4c_3c_2c_1c_2c_3c_4c_5, c_5] = 1.
\]

Although we do not focus on the remaining case of genus one, the presentation of \( \mathcal{T}_{1,r} \) was already studied (see [GM]). In conclusion, our results are summarized to that most of the central extensions \( \mathcal{T}_{g,r} \) can be dealt with concretely as finitely presented groups.

Finally, we list several little applications in some topics. First, the results are useful for checking whether a linear representation \( \rho : \mathcal{T}_{g,r} \to GL_n(\mathbb{C}) \) is well-defined or not: For instance, if \( r = 0 \) or \( r = 1 \), according to Theorem 1 it is necessary to check that the lantern relation and the 3-chain relation are sent to \( c \cdot \text{id}_{\mathbb{C}^n} \) for some \( c \in \mathbb{C} \). Furthermore, the theorems might be
useful in normalizing (constant factors of) Kohno’s description [Koh] on 3-manifold invariants and in computing concretely Lefschetz fibration invariants in [N2], which are constructed from $\rho_{g,k}$. In addition, the presentation of $T_{g,r}$ seems compatible with the setting in the chart description [EHKT].

2 Proof of the theorems.

This section is dedicated to proving the theorems stated above. Here, we assume that the reader has basic knowledge of group cohomology and the mapping class group (see [FM, §1-5] and [Kor]).

We briefly explain an outline of the proofs. First, we review the group “$\text{As}(D_{g}^{n})$” that is introduced in the study of the TQFT [MR]; we show an isomorphism $\text{As}(D_{g}^{n}) \cong \mathbb{Z} \times T_{g,0}$ for $g \geq 3$; see Theorem 4. As a corollary, we verify that the group presented in Theorem 1 (II) is isomorphic to $\text{As}(D_{g}^{n})$ without the $\mathbb{Z}$-factor. In parallel, concerning Theorem 3, a similar discussion is applicable to the genus two case. After that, we show Theorem 1 (I) by using the universality of the central extensions and Harer stability [H]. Next, we will give a proof of Theorem 2 by induction on $r \geq 1$. Actually, the group presented in Theorem 2 for $r = 1$ is shown to be isomorphic to that in Theorem 1, and the proof for $r \geq 2$ is similarly done by using a result of Harer stability [H].

2.1 Preliminaries: the associated group

To accomplish the outline, we start by introducing terminologies and state a key proposition (Theorem 4). First, denoting $\mathcal{M}_{g,0}$ by $\mathcal{M}_g$ for short, we set the following three subsets:

$$D_g := \{ \tau_\alpha \in \mathcal{M}_g \mid \alpha \text{ is a (unoriented) simple closed curve } \gamma \text{ in } \Sigma_g \},$$  \hspace{1cm} (9)

$$D_{g}^{n} := \{ \tau_\alpha \in D_g \mid \alpha \text{ is a non-separating simple closed curve } \gamma \text{ in } \Sigma_g \},$$  \hspace{1cm} (10)

$$D_{g}^{(k)} := \{ \tau_\alpha \in D_g \mid \text{ The complement } \Sigma_g \setminus \alpha \text{ is homeomorphic to } \Sigma_{k,1} \sqcup \Sigma_{g-k,1} \},$$  \hspace{1cm} (11)

where the symbol $\tau_\alpha$ is the (positive) Dehn twist along $\alpha$. Next, for $Z = D_g$ or $Z = D_{g}^{n}$, we will analyze the group $\text{As}(Z)$, which is considered in [MR]. Here $\text{As}(Z)$ is defined to be the abstract group generated by symbols $e_z$ with $z \in Z$ subject to the relation $e_{w-1zw} = e_{w}^{-1}e_z e_w$ and is called the associated group. Note that the inclusion $D_g \hookrightarrow \mathcal{M}_g$ gives rise to a group epimorphism $\mathcal{E} : \text{As}(Z) \rightarrow \mathcal{M}_g$ by definition, and we obtain the equality

$$ge_zg^{-1} = e_{\mathcal{E}(ge_zg^{-1})} \in \text{As}(Z), \quad \text{for any } z \in Z, \ g \in \text{As}(Z),$$  \hspace{1cm} (12)

which is easily verified by induction on the word length of $g$. The reader should keep in mind this equality, since we will use (12) in several times. For example, as a result of (12), the kernel $\text{Ker}(\mathcal{E})$ is contained in the center. In summary, we have

$$0 \rightarrow \text{Ker}(\mathcal{E}) \rightarrow \text{As}(Z) \xrightarrow{\mathcal{E}} \mathcal{M}_g \rightarrow 0,$$  \hspace{1cm} (central extension).
Theorem 4. (cf. [GI] Theorem C) If \( g \geq 3 \), there are isomorphisms \( \text{As}(D_g) \cong \mathcal{T}_{g,0} \times \mathbb{Z}^{[g/2]+1} \) and \( \text{As}(D_g^{\text{ns}}) \cong \mathcal{T}_{g,0} \times \mathbb{Z} \).

Here, we shall refer to [GI] Theorem C which claimed the same statement on \( \text{As}(D_g^{\text{ns}}) \) and discussed an infinite presentation of \( \mathcal{T}_{g,r} \) from the \( p^1 \)-structure. But, his proofs used functoriality with respect to lantern relations of \( \mathcal{M}_g \) [GI] Theorems C, 4.1], and might contain a little gap with \( g = 3 \). In contrast, the proofs in this paper use neither combinatorial computation as in [GI] nor the Maslov index as in [Ati, GM, T], but only basic knowledge of group cohomology of degree 2.

In addition, let us describe the concepts of lantern relations in \( \text{As}(D_g) \). If \( g \geq 3 \), consider two elements of the form
\[
\kappa_{3\text{-chain}} := (e_{c_1} e_{c_2} e_{c_3})^4 e_{c_0}^{-1} e_{b_0}, \quad \kappa_{\text{lantern}} := e_{c_1}^{-1} e_{c_3}^{-1} e_{b_3}^{-1} e_{b_0} e_{b_2} e_{b_1} \in \text{As}(D_g),
\]
where \( b_i \) and \( c_i \) are the respective Dehn twists of the curves \( \beta_i \) and \( \gamma_i \) in Figure 1 if \( g = 2 \), we define \( \kappa_{3\text{-chain}} \) to be \( (e_{c_1} e_{c_2} e_{c_3})^4 e_{c_3}^{-2} \) and \( \kappa_{\text{lantern}} \) to be \( 1_{\text{As}(D_g)} \). As is well-known, \( E(\kappa_{3\text{-chain}}) \) and \( E(\kappa_{\text{lantern}}) \) are the identity in \( \mathcal{M}_g \), which are commonly called the 3-chain relation and the lantern relation, respectively. Furthermore, for \( k < g/2 \), set up seven curves \( \alpha_k, \beta_k, \gamma_k, \delta_k, x_k, y_k, z_k \) in \( \Sigma_g \) illustrated in Figure 3 and consider the product
\[
\mathcal{L}_k := e_{\tau_{\alpha_k}}^{-1} e_{\tau_{\beta_k}}^{-1} e_{\tau_{\gamma_k}}^{-1} e_{\tau_{x_k}}^{-1} e_{\tau_{y_k}}^{-1} e_{\tau_{z_k}} \in \text{As}(D_g).
\]
The lantern relations in \( \mathcal{M}_g \) tell us that these \( \kappa_{\text{lantern}} \) and \( \mathcal{L}_k \) lie in \( \text{Ker}(E) \).

In addition, for \( \dagger = 1, 2, \ldots, [g/2] \) or \( \dagger = \text{ns} \), we define a homomorphism \( \epsilon_{\dagger} : \text{As}(D_g) \to \mathbb{Z} \) by setting \( \epsilon_{\dagger}(e_x) = 1 \in \mathbb{Z} \) if \( x \in D_g^{(\dagger)} \) and by setting \( \epsilon_{\dagger}(e_x) = 0 \in \mathbb{Z} \) otherwise. Since the orbit decomposition of the conjugate action \( D_g \curvearrowright \mathcal{M}_g \) is presented as \( D_g = D_g^{\text{ns}} \cup \left( \bigcup_{0 < k \leq g/2} D_g^{(k)} \right) \), it follows from (11) that the sum \( \left( \bigoplus \epsilon_j \right) \oplus \epsilon_{\text{ns}} \) gives an abelianization \( H_1(\text{As}(D_g)) \cong \mathbb{Z}^{[g/2]+1} \).

![Figure 3: The k-th lantern relation \( E(\mathcal{L}_k) \) in \( \mathcal{M}_g \) with \( g > 2, k \geq 1 \), and the curve \( e_{\text{sep}} \) with \( g = 2 \).](image)

Furthermore, let us review [EN] Theorem 4.2]. Notice from (11) that, if a tuple \( z = (z_1, \ldots, z_m) \in (D_g)^m \) satisfies \( z_1 \cdots z_m = 1_{\mathcal{M}_g} \), the product \( e_{z_1} \cdots e_{z_m} \in \text{As}(D_g) \) lies in the central kernel \( \text{Ker}(E : \text{As}(D_g) \to \mathcal{M}_g) \). Furthermore, we can construct a closed oriented 4-manifold \( E_g \) from such a tuple \( z \in (D_g)^m \), which is called a Lefschetz fibration (over \( S^2 \)); see, e.g., [EN] for the definition. Inspired by [GI] and the Hopf theorem on \( H_2, \text{Endo and Nagami} \) [EN] Definition 3.3 and Proposition 3.6] constructed a homomorphism \( I_g : \text{Ker}(E) \to \mathbb{Z} \) that enjoys the following property:
Theorem 5 (EN, Theorem 4.2 and Propositions 3.10-3.12). The homomorphism $I_g$ satisfies

$$I_g(e_{z_1} \cdots e_{z_m}) = \sigma_z \in \mathbb{Z},$$

for any $m$-tuple $(z_1, \ldots, z_m) \in (D_g)^m$ with $z_1 \cdots z_m = 1, \mathcal{M}_g \in \mathcal{M}_g$. Here $m_{ns} \in \mathbb{Z}$ is the number

$$\{ j \mid z_j \in D_g^m \},$$

and $\sigma_z$ is the signature of the associated 4-manifold $E_z$.

Moreover, $I_g(\kappa_{\text{chain}}) = -6$; Furthermore, if $g \geq 3$, then $I_g(\kappa_{\text{lantern}}) = I_g(\mathcal{L}_k) = 1$.

2.2 Proofs of theorems for $g \geq 3$.

We begin by proving Theorem 4 and show Theorems 1–2. Here, we should set up the epimorphism

$$\theta_r : \mathcal{M}_{g,r} \twoheadrightarrow \mathcal{M}_{g,r-1},$$

induced by gluing a disc to the boundary component of $\Sigma_{g,1}$.

Proof of Theorem 4. For the proof, we now analyze the group $\text{As}(\mathcal{D}_g)$ with $g \geq 3$, as a special tool in quandle theory similar to [NI §32–3]. Consider a homomorphism $s_j : \mathbb{Z} \rightarrow \text{As}(\mathcal{D}_g)$ which sends $n$ to $(\mathcal{L}_j)^n$. These $s_j$ is a section of $\epsilon_j$, and the image is contained in the center $\text{Ker}(\mathcal{E})$ because of (3). Similarly, take the map $s_{ns} : \mathbb{Z} \rightarrow \text{As}(\mathcal{D}_g)$ which sends $n$ to $(e_{c_0} e_{c_1} e_{c_2} e_{c_3} e_{c_4})^{-n}$. Therefore, the semi-direct product structure associated with $s_j$ and $s_{ns}$ is trivial; hence, we have the decomposition $\text{As}(\mathcal{D}_g) \cong \tilde{\mathcal{M}}_g \times \mathbb{Z}^{[\frac{n}{2}]+1}$ for some central extension $\tilde{\mathcal{M}}_g$ of $\mathcal{M}_g$. By Künneth formula on $H_1$, this $\tilde{\mathcal{M}}_g$ is perfect.

Hence, it is sufficient to show $\tilde{\mathcal{M}}_g \cong \mathcal{T}_{g,0}$. To this end, notice from the infration-restriction exact sequence (cf. [18] below) that the kernel of $\tilde{\mathcal{M}}_g \rightarrow \mathcal{M}_g$ is a quotient of $H_2(\mathcal{M}_g)$, since $\tilde{\mathcal{M}}_g$ is perfect. Here, we should notice that the kernel contains $\mathbb{Z}$. Actually, it follows from Theorem 5 that the sum of the homomorphisms $(I_g \oplus s_{ns}) \oplus (\oplus \epsilon_j) : \text{Ker}(\mathcal{E}) \rightarrow \mathbb{Z}^{[\frac{n}{2}]+2}$ is of order 4 at most.

We will show $\tilde{\mathcal{M}}_g \cong \mathcal{T}_{g,0}$ as a result of the preceding claim. First, if $g \geq 4$, the kernel of $\tilde{\mathcal{M}}_g \rightarrow \mathcal{M}_g$ must be $H_2(\mathcal{M}_g) \cong \mathbb{Z}$; hence, the universality of the central extensions implies $\tilde{\mathcal{M}}_g \cong \mathcal{T}_{g,0}$ as desired.

Next, we address the case $g = 3$. We should refer the fact $H_2(\mathcal{M}_{3,1}) \cong H_2(\mathcal{M}_3) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ [S Corollary 4.10]; Thus, the kernel of the projection $q : \tilde{\mathcal{M}}_3 \rightarrow \mathcal{M}_3$ is either $\mathbb{Z}$ or $\mathbb{Z}/2 \oplus \mathbb{Z}$. We will show that the kernel $\text{Ker}(q)$ is $\mathbb{Z}$. For this, consider the pullback of $q : \tilde{\mathcal{M}}_3 \rightarrow \mathcal{M}_3$ and $\theta_1$, and denote it by $\tilde{\mathcal{M}}_{3,1}$. Since the induced map $\tilde{\theta}_1 : H_2(\mathcal{M}_{3,1}) \rightarrow H_2(\mathcal{M}_3)$ is known to be isomorphic [S §4], it is enough for the proof of $\text{Ker}(q) \cong \mathbb{Z}$ to show that $\mathcal{M}_3$ is not the universal central extension of $\mathcal{M}_{3,1}$. Consider the group with presentation

$$\mathcal{M}_{3,1}^{\text{pre}} := \langle c_0, c_1, \ldots, c_7 \mid \text{the relations (2), (3), and (4)} \rangle.$$

The Wajnryb presentation [Wa] implies that $\mathcal{M}_{3,1}^{\text{pre}}$ is a central $\mathbb{Z}$-extension over $\mathcal{M}_{3,1}$ and the kernel is $\mathbb{Z} = \{ \mu^m \}_{m \in \mathbb{Z}}$, and that $\mathcal{M}_{3,1}^{\text{pre}}$ is perfect by the lantern relation (5). Then, we shall notice that the canonical map $\mathcal{M}_{3,1}^{\text{pre}} \rightarrow \mathcal{M}_{3,1}$ which sends $c_i$ to $e_{c_i}$ is surjective, since every generator $e_{\tau_m}$ is conjugate to $c_0$. Hence, if $\tilde{\mathcal{M}}_{3,1}$ is the universal central extension of $\mathcal{M}_{3,1}$,
there is such no map from any central extension of $\mathcal{M}_{3,1}$ with fiber $\mathbb{Z}$. Hence, $\tilde{\mathcal{M}}_{3,1}$ is not universal. Hence, the kernel $\text{Ker}(q)$ is $\mathbb{Z}$, leading to $\tilde{\mathcal{M}}_3 \cong T_3$.

Finally, we show $\text{As}(D_g^{\text{ns}}) \cong T_{g,0} \times \mathbb{Z}$ with $g \geq 3$. Notice from (11) that $\epsilon_{\text{ns}} : \text{As}(D_g^{\text{ns}}) \rightarrow \mathbb{Z}$ is the abelianization. Similarly, the image of $\sigma_{\text{ns}} : \mathbb{Z} \rightarrow \text{As}(D_g^{\text{ns}})$ is contained in the center $\text{Ker}(E)$. Therefore, the semi-direct product structure is trivial, i.e., $\text{As}(D_g^{\text{ns}}) \cong \tilde{\mathcal{M}}_g \times \mathbb{Z}$ for a central extension $\tilde{\mathcal{M}}_g$ of $\mathcal{M}_g$. Thus, it is enough to show $\tilde{\mathcal{M}}_g \cong T_{g,0}$. For this, the map $\text{As}(D_g^{\text{ns}}) \rightarrow \text{As}(D_g)$ induced by the inclusion $D_g^{\text{ns}} \hookrightarrow D_g$ implies $\tilde{\mathcal{M}}_g \times \mathbb{Z} \rightarrow T_{g,0} \times \mathbb{Z}$ over $\mathcal{M}_g$. Note that the image of $\tilde{\mathcal{M}}_g$ is contained in $T_{g,0}$, since $\tilde{\mathcal{M}}_g$ is also perfect. Thus, the universality of $T_{g,0}$ immediately lead to the desired $\tilde{\mathcal{M}}_g \cong T_{g,0}$.

We are now in a position to prove Theorem 1. Notice from Theorem 5 that

$$I_g(\kappa_{3\text{-chain}} \kappa_{10}^{\text{lantern}}) = 4, \quad \epsilon_{\text{ns}}(\kappa_{3\text{-chain}} \kappa_{10}^{\text{lantern}}) = 0.$$  (14)

Since the cokernel of $(I_g \oplus \epsilon_{\text{ns}}) \oplus (\bigoplus \epsilon_j)$ is $\mathbb{Z}/4$ as in the proof above, this $\kappa_{3\text{-chain}} \kappa_{10}^{\text{lantern}}$ is a generator of the center $\mathbb{Z} = \text{Ker}(T_{g,0} \rightarrow \mathcal{M}_g)$ in (11).

**Proof of Theorem 2.** Let $r = 0$ or 1, let $\mathcal{G}_{g,r}$ be the group with the presentation given in Theorem 1 and let $q_r : \mathcal{G}_{g,r} \rightarrow \mathcal{M}_{g,r}$ be the quotient map by adding the relation $\mu = 1$. Noting the relation (4), the map $q_r$ is a central extension with fiber $\mathbb{Z}$.

We first show (II). Using (11), we can verify that the correspondence

$$c_i \mapsto c_i \kappa_{\text{lantern}}, \quad \mu \mapsto \kappa_{3\text{-chain}} \kappa_{10}^{\text{lantern}}$$

defines a homomorphism $\psi : \mathcal{G}_{g,0} \rightarrow \text{As}(D_g^{\text{ns}})$ over $\mathcal{M}_g$. The image is contained in $T_{g,0} \cong \text{Ker}(\epsilon_{\text{ns}} : \text{As}(D_g^{\text{ns}}) \rightarrow \mathbb{Z})$ by definition. Hence, since $T_{g,0}$ is universal modulo torsion subgroup, $\psi$ must be an isomorphism $\mathcal{G}_{g,0} \cong T_{g,0}$ as required.

Next, we show (I). From the definition of (13), we have a commutative diagram:

$$\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \text{proj.} \\
0 & \rightarrow & \mathcal{G}_{g,0} \\
\downarrow q_r & & \downarrow q_1 \\
0 & \rightarrow & \mathcal{M}_{g,0} \\
\downarrow \theta_1 & & \downarrow \theta_1 \\
0 & \rightarrow & 0
\end{array} \quad (\text{central extension})$$  (15)

From the definitions of $\mathcal{G}_{r,s}$, the left vertical map is isomorphic. It is known as the Harer stability with $g \geq 3$ (see [Kor, §6]) that the right induced map $\theta_1^* : H^2(\mathcal{M}_{g,1}; \mathbb{Z}) \rightarrow H^2(\mathcal{M}_{g,1}; \mathbb{Z})$ is an isomorphism on $\mathbb{Z}$. Since $\mathcal{G}_{g,0} \cong T_{g,0}$, the universality of the central $\mathbb{Z}$-extensions implies the desired isomorphism $\mathcal{G}_{g,1} \cong T_{g,1}$.

Now, let us turn into proving Theorem 2 for the punctured groups. Let us denote by $\mathcal{G}_{g,r}'$ the group with the presentation given in Theorem 2. Recalling the quotient $\mathcal{G}_{g,r}'/\langle \mu = 1 \rangle \cong \mathcal{M}_{g,r}$ as a result of (12). Theorem 1, we see from the relation (iv) that the projection $\mathcal{G}_{g,r}' \rightarrow \mathcal{M}_{g,r}$ is a central $\mathbb{Z}$-extension.

**Proof of Theorem 2.** As the first step of the induction on $r$, we let $g \geq 3$, $r = 1$ and will observe a diagram similar to (15). Consider the map $q_1 : \mathcal{G}_{g,1}' \rightarrow \mathcal{M}_{g,1}$ which takes each generator $\alpha$ of
$G'_{g,1}$ to the corresponding Dehn twist $\tau_\alpha$. It immediately follows from [Q2, Lemma 5] that the kernel of $q_1$ is generated by only $\mu = 1$, and that $q_1$ is a central $\mathbb{Z}$-extension from the definition of $\mu$. Furthermore, using the homomorphism $\theta_1$ in (15), consider the correspondence from $D_g$ to $\text{As}(D_g)$ defined by

$$\theta_1(\gamma) \longmapsto e_{\theta_1(\gamma)}e_{\text{lantern}}^c L_1^{c_\gamma(\theta_1(\gamma))} \ldots L_{[g/2]}^{c_\gamma(\theta_1(\gamma))}, \quad \theta_1(\mu) \longmapsto \kappa_{3\text{-chain}}^{10}.$$

(16)

Here, $\gamma$ runs over the generators in Theorem 2 and $L_k$ is the central element defined in [29]. Notice from [EN, Proposition 3.13] that the homomorphism $I_g : \text{Ker}(E) \to \mathbb{Z}$ in Theorem 3 (resp. $\epsilon_{\text{ns}}$) sends the star relation (iii) to $5-N_{i,j,k}$ (resp. $9+N_{i,j,k}$), where $N_{i,j,k}$ is the cardinality of $\{c_{i,j}, c_{j,k}, c_{k,i} | c_{x,y}$ is separating $\}$. Hence, compared with (14), the correspondence (16) defines a map $G'_{g,1} \to \text{As}(D_g)$ as a centrally extended homomorphism over $\theta_1$. Note that the image is $\text{Ker}(\oplus \epsilon_1 : \text{As}(D_g) \to \mathbb{Z}[g/2]+1 \cong T_{g,0}$ by definition. Hence, $G'_{g,1}$ must be $T_{g,1}$ by a diagram chasing similar to (15) and the universality of central $\mathbb{Z}$-extensions.

Finally, we now complete the proof with $r \geq 2$. Consider the canonical surjection $p_r : G'_{g,r} \to G'_{g,r-1}$ obtained from the presentations. Then, the quotient $p_r$ modulo $\mu = 1$ is identified with $\theta_r : M_{g,r} \to M_{g,r-1}$ in (13). In addition, consider the injection $\iota_r : M_{g,r-1} \to M_{g,r}$ induced by gluing a two holed disc to a boundary component of $\Sigma_{g,r-1}$. From the Harer stability (see [Kor]), the induced map $\iota^*_r : H^2(M_{g,r}; \mathbb{Z}) \to H^2(M_{g,r-1}; \mathbb{Z})$ is known to be an isomorphism on $\mathbb{Z}$. Furthermore, since $\theta_r \circ \iota_r = \text{id}$, the induced map $\theta^*_r : H^2(M_{g,r-1}; \mathbb{Z}) \to H^2(M_{g,r}; \mathbb{Z})$ is an isomorphism on $\mathbb{Z}$. Consequently, the surjection $p_r$ induces an isomorphism on $H^2$. Hence, since $H^2(G'_{g,r-1}; \mathbb{Z}) \cong 0$ by induction on $r$, we have $H^2(G'_{g,r}; \mathbb{Z}) \cong 0$. In conclusion, the central $\mathbb{Z}$-extension $G'_{g,r}$ is also universal up to torsion subgroup, that is, $G'_{g,r} \cong T_{g,r}$ as desired.

2.3 The case of two genus

This subsection is devoted to giving the proof with $g = 2$.

As a preliminary, we will describe the extension $T_2$ in terms of $D_2$.

For this, we begin by showing the isomorphism (17) below. Consider the homomorphism $s$ defined by setting

$$s : \mathbb{Z} \longrightarrow \text{As}(D_2); \quad n \longmapsto \left( (e_{c_1} e_{c_2})^6 e_{\text{sep}}^{-1} \right)^n,$$

Here and $e_{\text{sep}}$ are the curve described in Figure 3 respectively. Then the 2-chain relation in $M_2$ implies that this $s$ is a splitting of $\epsilon_1 : \text{As}(D_2) \to \mathbb{Z}$. Since the image of $s$ is contained in the central kernel of $\text{As}(D_2) \to M_2$, the semi-direct product arising from $\epsilon_1$ is trivial, i.e.,

$$\text{As}(D_2) \cong \mathbb{Z} \times \text{Ker}(\epsilon_1).$$

(17)

Moreover, the inclusion $\text{As}(D_2)^{\text{ns}} \subset \text{As}(D_2)$ leads to $\text{As}(D_2)^{\text{ns}} \cong \text{Ker}(\epsilon_1)$ in a similar way to the case $g \geq 3$.

Next, recall $H_1(\text{As}(D_2)^{\text{ns}}) \cong \mathbb{Z}$ mentioned above, and the basic facts $H_1(M_2) \cong \mathbb{Z}/10$ and $H_2(M_2) \cong \mathbb{Z}/2$; see [FM, Kor]. Then the inflation-restriction exact sequence for $E : \text{As}(D_2)^{\text{ns}} \to M_2$ can be written as

$$H_2(\text{As}(D_2)^{\text{ns}}) \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Ker}(E) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/10 \longrightarrow 0,$$

(exact).

(18)
Therefore, Ker(\(\mathcal{E}\)) is either \(\mathbb{Z}\) or \(\mathbb{Z} \oplus \mathbb{Z}/2\). Although the image of \(\delta^*\) is mysterious, the quotient \(\text{As}(\mathcal{D}_{ns}^2)/\text{Im}(\delta^*)\) is isomorphic to \(\mathcal{T}_2\).

**Proof of Theorem 3.** Denote the group presented in the statement by \(\mathcal{G}\). Then, the quotient of \(\mathcal{G}\) subject to \((c_1c_2c_3)^4c_5^2 = 1\) is \(\mathcal{M}_2\). Further, by (11), the quotient map \(\mathcal{G} \to \mathcal{M}_2\) is a central \(\mathbb{Z}\)-extension. A diagram chasing of (18) easily reveals that \(\kappa_3\)-chain is equal to \(e_5e_4e_3e_2e_1e_7e_6e_5e_4e_3\) contained in the center Ker(\(\mathcal{E}\))/\(\text{Im}\delta^*\). Moreover, we should notice that \(A := \{e_7e_6e_5e_4e_3e_2e_1\} \subset \text{Im}(\mathcal{D}_{ns}^2)/\text{Im}(\delta^*)\). Indeed, although \(A\) is contained in the center Ker(\(\mathcal{E}\))/\(\text{Im}\delta^*\) \(\cong \mathbb{Z}\) by (11), \(c_1(A) = 0\) and (18) imply \(A = 0\). In summary, as before, the correspondence \(c_i \mapsto e_i\) defines an epimorphism between central \(\mathbb{Z}\)-extensions \(\mathcal{G}\) and \(\text{As}(\mathcal{D}_{ns}^2)\) over \(\mathcal{M}_g\), which is an isomorphism. Hence, the above result \(\text{As}(\mathcal{D}_{ns}^2) \cong \mathcal{T}_2\) immediately leads to the conclusion. \(\Box\)

**Remark 6.** From the discussion, the kernel of \(\text{As}(\mathcal{D}_2)/\text{Im}\delta^* \to \mathcal{M}_2\) is generated by \((e_1e_2c_5)^6e_{\tau\text{sep}}^{-1}\) and \(\kappa_3\)-chain.

### 2.4 An application to Lefschetz fibrations

Finally, we conclude this paper by discussing right \(\mathcal{T}_{g,0}\)-modules and tuples \((z_1, \ldots, z_m) \in (\mathcal{D}_g)^m\) with \(z_1 \cdots z_m = 1\). Recall that the product \(e_{z_1} \cdots e_{z_m} \in \text{As}(\mathcal{D}_g)\) lies in the center Ker(\(\mathcal{E}\)). Furthermore, in the study of Lefschetz fibration invariants, it is important to verify whether the identity \(e_{z_1} \cdots e_{z_m} = \text{id}_M\) holds or not (see [N2, §3.2] for the details).

To do so in an easy way, we will show that the identity can be established by the central elements \(\kappa_3\)-chain, \(\kappa_{\text{lantern}}\) and the signature of 4-dimensional Lefschetz fibrations: (This is also a criterion for which the quantum representation is useful for some Lefschetz fibration invariants; see [N2] for details.) Precisely,

**Proposition 7.** Let us regard a right \(\mathcal{T}_{g,0}\)-module \(M\) as an \(\text{As}(\mathcal{D}_g)\)-module via the isomorphism \(\text{As}(\mathcal{D}_g) \cong \mathcal{T}_{g,0} \times \mathbb{Z}\). Denote the associated map \(\text{As}(\mathcal{D}_g) \to \text{End}(M)\) by \(\rho\).

(I) Let \(g \geq 3\). For any tuple \((z_1, \ldots, z_m) \in (\mathcal{D}_g)^m\) with \(z_1 \cdots z_m = 1\), the product \(\rho(e_{z_1} \cdots e_{z_m})\) is equal to

\[
\rho(\kappa_3\text{-chain})^{(\sigma_2 + m)/4} \rho(\kappa_{\text{lantern}})^{5\sigma_2 + 5m}/2 - mns \rho(L_1)^{n_1} \cdots \rho(L_{[\frac{g}{2}]})^{n_{[\frac{g}{2}]}} \in \text{End}(M). 
\]

Here, \(n_k := \#\{z_i \in \mathcal{D}_g^{(k)}\}\), and the notation \(\sigma_2, mns \in \mathbb{N}\) are the same as Theorem 3.

In particular, if the right hand side is the identity, then \(\rho(e_{z_1} \cdots e_{z_m}) = \text{id}_M\).

(II) If \(g = 2\), then the identity

\[
\rho(e_{z_1} \cdots e_{z_m}) = \rho((e_1e_2c_5)^6e_{\tau\text{sep}}^{-1})^{m_{ns} - m} \rho(\kappa_3\text{-chain})^{(\sigma_2 - 7m + 7mns)/6} \in \text{End}(M)
\]

holds for any tuple \(z = (z_1, \ldots, z_m) \in (\mathcal{D}_g)^m\) satisfying \(z_1 \cdots z_m = 1\).

**Proof.** It follows from the proof of Theorem 3 that the kernel of \(\text{As}(\mathcal{D}_g) \to \mathcal{M}_g\) is generated by \((\kappa_3\text{-chain})^{10}_{\text{lantern}}\) and the lantern relations \(\, \kappa_{\text{lantern}} = L_1, \ldots, L_{[\frac{g}{2}]}.\) Since \(e_{z_1} \cdots e_{z_m}\) is contained in the kernel, there exist \(N_C, N_L \in \mathbb{Z}\) for which the following holds:

\[
e_{z_1} \cdots e_{z_m} = (\kappa_3\text{-chain})^{N_C}_{\text{lantern}} \kappa_{\text{lantern}}^{N_L}(L_1)^{n_1} \cdots (L_{[\frac{g}{2}]})^{n_{[\frac{g}{2}]}} \in \text{As}(\mathcal{D}_g) 
\]
First, we show (I). Note \(m_{ns} = \varepsilon_{ns}(e_{z_1} \cdots e_{z_m}) = -N_L \in \mathbb{Z}\). Furthermore, recall from Theorem 5 that the homomorphism \(I_g\) satisfies \(I_g(\kappa_{3\text{-chain}}) = -6\) and \(I_g(\kappa_{\text{lantern}}) = I_g(\mathcal{L}_k) = -1\); The former statement in Theorem 5 yields \(\sigma_z = I_g(e_{z_1} \cdots e_{z_m}) = -6N_C + 10N_C - N_L - n_1 - \cdots - n_{[\frac{m}{2}]} = 4N_C - m\), leading to the solution \(N_C = (\sigma_z + m)/4\). Hence, applying \(\rho\) to (20) implies the desired equality (19) as claimed.

(II) Finally, we deal with the case \(g = 2\). By Remark 6, the kernel of \(A_{s}(D_2) \to M_2\) is generated by \((e_{c_1}e_{c_2})^6e^{-1}_{\tau_{\text{sep}}}\) and \(\kappa_{3\text{-chain}}\). Hence, we have \(e_{z_1} \cdots e_{z_m} = ((e_{c_1}e_{c_2})^6e^{-1}_{\tau_{\text{sep}}})^{N_1}(\kappa_{3\text{-chain}})^{N_2}\). Then, \(m - m_{ns} = \varepsilon_1(e_{z_1} \cdots e_{z_m}) = -N_1\). Noting from [EN, Proposition 3.9] that \(I_g((e_{c_1}e_{c_2})^6e^{-1}_{\tau_{\text{sep}}}) = -7\), we have \(\sigma_z = I_g(e_{z_1} \cdots e_{z_m}) = -7N_1 - 6N_2\) Consequently, the solution \(N_2 = (\sigma_z - 7m + 7m_{ns})/6\) yields the required equality. \(\square\)

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