Garsia–Rodemich spaces:
Local maximal functions and interpolation

by

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Abstract. We characterize the Garsia–Rodemich spaces $\text{GaRo}_X$ associated with a rearrangement invariant space via local maximal operators. Let $Q_0$ be a cube in $\mathbb{R}^n$. We show that there exists $s_0 \in (0, 1)$ such that for all $0 < s < s_0$, and for all r.i. spaces $X(Q_0)$, we have

$$\text{GaRo}_X(Q_0) = \{ f \in L^1(Q_0) : \|f\|_{\text{GaRo}_X} \simeq \|M_{s,Q_0}^# f\|_X < \infty \},$$

where $M_{s,Q_0}^#$ is the Strömberg–Jawerth–Torchinsky local maximal operator. Combined with a formula for the $K$-functional of the pair $(L^1, \text{BMO})$ obtained by Jawerth–Torchinsky, our result shows that the $\text{GaRo}_X$ spaces are interpolation spaces between $L^1$ and $\text{BMO}$. Among the applications, we prove, using real interpolation, the monotonicity under rearrangements of Garsia–Rodemich type functionals. We also give an approach to Sobolev–Morrey inequalities via Garsia–Rodemich norms, and prove necessary and sufficient conditions for $\text{GaRo}_X(Q_0) = X(Q_0)$. Using packings, we obtain a new expression for the $K$-functional of the pair $(L^1, \text{BMO})$.

1. Introduction. The starting point of this research is the celebrated John–Nirenberg Lemma which we now recall. Let $Q_0 \subset \mathbb{R}^n$ be a fixed cube $\text{(1)}$ $1 < p < \infty$. The John–Nirenberg spaces $\text{JN}_p := \text{JN}_p(Q_0)$ consist of all functions $f \in L^1(Q_0)$ such that (cf. [18], [32])

$$\|f\|_{\text{JN}_p} = \sup_{\{Q_i\}_{i \in I} \in P} \left\{ \sum_i |Q_i| \left( \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \, dx \right)^p \right\}^{1/p} < \infty,$$

$\text{(1.1)}$

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$\text{(1)}$ A “cube” in this paper will always mean a cube with sides parallel to the coordinate axes. We normalize $Q_0$ to have measure 1.

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where \( f_Q = \frac{1}{|Q|} \int_Q f \, dx \) and

\[
P := P(Q_0) = \{ \{ Q_i \}_{i \in \mathbb{N}} : Q_i \text{ are subcubes of } Q_0 \text{ with pairwise disjoint interiors} \}.
\]

To understand the correct definition for \( p = \infty \), we proceed as follows. For \( \pi = \{ Q_i \}_{i \in \mathbb{N}} \in P \), let

\[
f_{\pi}(x) = i \int_{Q_i} |f - f_{Q_i}| \chi_{Q_i}(x).
\]

Then

\[
\|f\|_{JN_p} = \sup_{\pi \in P} \|f_{\pi}\|_{L^p(Q_0)}.
\]

This justifies the definition: \( JN_\infty := JN_\infty(Q_0) \) consists of all functions \( f \in L^1(Q_0) \) such that

\[
\|f\|_{JN_\infty} = \sup_{\pi \in P} \|f_{\pi}\|_{L^\infty(Q_0)}.
\]

It follows readily that

\[
JN_\infty(Q_0) = \text{BMO}(Q_0).
\]

The John–Nirenberg Lemma \cite{18} implies the following embeddings:

\[
JN_p(Q_0) \subset \begin{cases} L(p, \infty)(Q_0), & 1 < p < \infty, \\ e^L(Q_0), & p = \infty, \end{cases}
\]

where \( L(p, \infty)(Q_0) \) is the “weak” \( L^p \) space and \( e^L(Q_0) \) is the Orlicz space of exponentially integrable functions. This result, and the spaces involved, has been the object of intensive study over the years and, in particular, the space \( \text{BMO} \) now plays a very important role in harmonic analysis. We refer to \cite{18}, \cite{7}, \cite{14}, \cite{32} for background, different proofs and extensive bibliographies.

In their paper, Garsia–Rodemich \cite{16} proposed a very original approach \textsuperscript{(2)} to (1.2). It is based on the following idea: To effectively compare \( JN_p \) with \( L(p, \infty) \), \( 1 < p < \infty \), a new class of spaces was introduced in \cite{16}. We shall say that \( f \in G_p := G_p(Q_0) \) if and only if \( f \in L^1(Q_0) \) and there exists \( C > 0 \) such that for all \( \{ Q_i \}_{i \in \mathbb{N}} \in P \) we have

\[
\sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq C \left( \sum_i |Q_i| \right)^{1/p'},
\]

where \( 1/p' = 1 - 1/p \), and we let

\[
\|f\|_{G_p} = \inf \{ C : (1.3) \text{ holds} \}.
\]

\textsuperscript{(2)} While \cite{16} contains many interesting results, and indeed has been widely quoted in the literature, their approach to (1.2) has remained largely unnoticed until very recently (cf. \cite{24}).

\textsuperscript{(3)} To describe the original results we shall use a temporary notation.
The connection between the JN\(_p\) and \(G_p\) spaces can be seen from the readily verified computation
\[
(1.4) \quad \int_Q |f(x) - f_Q| \, dx \leq \frac{1}{|Q|} \int_Q \int_Q |f(x) - f(y)| \, dx \, dy \leq 2 \int_Q |f(x) - f_Q| \, dx.
\]
Indeed, combining (1.4) with Hölder’s inequality, we find that for each \(\{Q_i\}_{i \in \mathbb{N}} \in P\) we have
\[
\sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq 2 \sum_i \int_{Q_i} |f(x) - f_{Q_i}| \, dx
\]
\[
= 2 \sum_i |Q_i|^{1/p'} \left( |Q_i|^{1/p} \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right)
\]
\[
\leq 2 \left( \sum_i |Q_i| \right)^{1/p'} \left( \sum_i \left( \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right)^p \right)^{1/p}.
\]
Consequently,
\[
\|f\|_{G_p} \leq 2 \|f\|_{\text{JN}_p}.
\]
It is remarkable that actually (cf. [16] for the one-dimensional case and [24] in general), as sets,
\[
(1.5) \quad G_p = L(p, \infty), \quad 1 < p < \infty.
\]
It is easy to see that the definition of \(G_p\) also makes sense for \(p = 1\) and \(p = \infty\).

For \(p = \infty\), letting \(p' = 1\) in (1.3) we get
\[
(1.6) \quad G_\infty = \text{BMO}.
\]
In fact, since for any cube \(Q\), we have \(\{Q\} \in P\), it follows from (1.4) that
\[
\int_Q |f(x) - f_Q| \, dx \leq \frac{1}{|Q|} \int_Q \int_Q |f(x) - f(y)| \, dx \, dy \leq \|f\|_{G_\infty} |Q|,
\]
yielding
\[
\|f\|_{\text{BMO}} \leq \|f\|_{G_\infty}.
\]
On the other hand, if \(f \in \text{BMO}\), then once again using (1.4) we see that for any \(\pi \in P\),
\[
\sum_{Q \in \pi} \frac{1}{|Q|} \int_Q \int_Q |f(x) - f(y)| \, dx \, dy \leq 2 \sum_{Q \in \pi} \frac{|Q|}{|Q|} \int_Q \int_Q |f(x) - f_Q| \, dx \, dy
\]
\[
\leq 2 \|f\|_{\text{BMO}} \sum_{Q \in \pi} |Q|.
\]
Consequently,
\[
\|f\|_{G_\infty} \leq 2 \|f\|_{\text{BMO}}.
\]
For $p = 1$, we let $p' = \infty$, and
\[
\|f\|_{G_1} := \sup_{\pi \in P} \sum_{Q \in \pi} \frac{1}{|Q|} \int_{Q} \int_{Q} |f(x) - f(y)| \, dx \, dy.
\]
Then (4) (modulo constants) $G_1 = L^1$. Indeed, it is easy to see that $\|f\|_{G_1} \leq 2\|f\|_{L^1}$.

Conversely, let $\widetilde{L}^1$ be the space $L^1$ modulo constants. Then since $\{Q_0\} \in P$, we have
\[
\|f\|_{\widetilde{L}^1} \leq \|f - f_{Q_0}\|_{L^1} \leq \frac{1}{|Q_0|} \int_{Q_0} \int_{Q_0} |f(x) - f(y)| \, dx \, dy \quad \text{(by (1.4))}
\]
\[
\leq \|f\|_{G_1}.
\]
For $p = \infty$, the method of proof of (1.5) that was given in [24] also yields (1.7)
\[
G_\infty \subset L(\infty, \infty),
\]
where $L(\infty, \infty)$ is the Bennett–DeVore–Sharpley space
\[
L(\infty, \infty) = \left\{ f \in L^1(Q_0) : \|f\|_{L(\infty, \infty)} = \sup_t (f**(t) - f^*(t)) < \infty \right\}
\]
(here, $f^*$ is the decreasing rearrangement of $f$ and $f**(t) := \frac{1}{t} \int_0^t f^*(s) \, ds$). Together, (1.6) and (1.7) therefore provide the improvement of the John–Nirenberg inequality obtained by Bennett–DeVore–Sharpley [4] [5], namely
\[
\text{BMO} \subset L(\infty, \infty).
\]

In [25] it was shown how the Garsia–Rodemich spaces fit in the theory of Sobolev embeddings, and in [26] the Garsia–Rodemich characterization of the weak $L^p$ spaces was used to provide a streamlined proof of the embedding theorem for the Bourgain–Brezis–Mironescu space $B$ (cf. [8]),
\[
B \subset L(n', \infty).
\]

In short, the Garsia–Rodemich spaces provide a framework that can be used to study a number of classical problems in analysis. It was then natural to consider the problem of extending the Garsia–Rodemich construction. In particular, in view of the characterization of $L(p, \infty)$ provided by (1.5), we

(4) When comparing $G_p$ spaces with other function spaces we must take into account that for any constant $c$, $\|f - c\|_{G_p} = \|f\|_{C_p}$.

(5) It is well known and easy to see that if $f \in L(\infty, \infty)$ then $f \in e^L$. Indeed, since $(f**(s))' = \frac{f^*(s) - f**(s)}{s}$, we have
\[
f**(t) - f**(1) = \int_1^t (f**(s) - f^*(s)) \frac{ds}{s} \leq \|f\|_{L(\infty, \infty)} \log \frac{1}{t}, \quad 0 < t < 1.
\]
Moreover, as shown by Bennett–DeVore–Sharpley [4], $L(\infty, \infty)$ is the rearrangement invariant hull of BMO.
ask: what other rearrangement invariant spaces can be characterized via a suitable extension of the Garsia–Rodemich conditions? In this direction the following generalization of the condition (1.3) was proposed in [26].

Let \( X := X(Q_0) \) be a rearrangement invariant space; for a given integrable function \( f \) we consider the class \( \Gamma_f \) of integrable functions \( \gamma \) such that for all \( \{Q_i\}_{i \in \mathbb{N}} \in P \),

\[
\sum_{i \in \mathbb{N}} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq \sum_{i \in I} \int_{Q_i} \gamma(x) \, dx.
\]

(1.8)

To describe the corresponding enlarged class of spaces associated with these conditions, it will be convenient to replace our temporary notation for the \( G \)-spaces as follows. We let

\[
\text{GaRo}_{X} := \text{GaRo}_{X}(Q_0) = \{ f : \|f\|_{\text{GaRo}_{X}} < \infty \},
\]

where \(^{(6)}\)

\[
\|f\|_{\text{GaRo}_{X}} = \inf \{ \|\gamma\|_{X} : \gamma \in \Gamma_f \}.
\]

(1.9)

It is shown in [26, Corollary 1 and Remark 2] that, in this new notation, we have

\[
\text{GaRo}_{L(p,\infty)} = G_p, \quad 1 < p < \infty.
\]

Moreover, at the end points it is easy to verify that

\[
\text{GaRo}_{L(\infty,\infty)} = G_\infty = \text{BMO} \quad \text{and} \quad \text{GaRo}_{L^1} = G_1 = L^1.
\]

Thus, (1.5) and (1.7) now read

\[
\text{GaRo}_{L(p,\infty)} = L(p,\infty), \quad 1 < p < \infty, \quad \text{GaRo}_{L(\infty,\infty)} \subset L(\infty,\infty).
\]

More generally, the following generalization holds for any r.i. space \( X \) (cf. [26]),

\[
(1.10) \quad \text{GaRo}_{X} = X \quad \text{if} \quad 0 < \alpha_X \leq \beta_X < 1,
\]

where \( \alpha_X, \beta_X \) are the Boyd indices of \( X \) (cf. Section 2.3).

The characterization (1.10) is very satisfactory since it captures all the main results at the level of \( L^p \) spaces, \( 1 < p < \infty \). However, the methods of [26] are unsuitable to understand what happens when the Boyd indices are 0 or 1. In fact, the analysis of the end point cases of (1.10) seems to require new ideas.

In this paper we obtain a new characterization of the Garsia–Rodemich spaces via the Strömberg–Jawerth–Torchinsky local maximal operators (cf. \(^{(6)}\) We let \( \|f\|_{\text{GaRo}_{X}} = \infty \) if \( \Gamma_f = \emptyset \); moreover we shall use the convention \( \|\gamma\|_{X} = \infty \) if \( \gamma \notin X \).
Let \( s \in (0, 1) \). Then

\[
M_{s,Q_0}^# f(x) := \sup_{Q_0 \supset Q \ni x} \inf_{c \in \mathbb{R}} \{ \alpha \geq 0 : |\{y \in Q : |f(y) - c| > \alpha\}| < s|Q| \}, \quad x \in Q_0,
\]

where the supremum is taken over all cubes \( Q \) contained in \( Q_0 \) such that \( x \in Q \). One of our main results (Theorem 3.1 below) states that there exists \( s_0 \in (0, 1) \) such that, for all \( 0 < s < s_0 \), and all r.i. spaces \( X \),

\[
\|f\|_{\text{GaRo}_X} \simeq \|M_{s,Q_0}^# f\|_X,
\]

where the implied constants are independent of \( f \).

This result not only allows us to study the limiting cases of \((1.10)\) but at the same time provides a connection between \( \text{GaRo}_X \) spaces and classical harmonic analysis. In particular, in Theorem 4.1 we show a significant improvement over \((1.10)\):

\[
\alpha_X > 0 \implies \text{GaRo}_X = X.
\]

In fact, for a large class of r.i. spaces of fundamental type (cf. Definition 4.2), \((1.12)\) is best possible. In Corollary 4.6 we prove that for every r.i. space \( X \) of fundamental type

\[
\text{GaRo}_X = X \iff \text{GaRo}_X \subset X \iff \alpha_X > 0.
\]

Another consequence of \((1.11)\) is that the \( \text{GaRo}_X \) spaces are real interpolation spaces between \( L^1 \) and BMO. For example, this can be seen as a consequence of \((1.11)\) and the formula for the \( K \)-functional of the pair \((L^1, \text{BMO})\) obtained by Jawerth–Torchinsky [17],

\[
K(t, f; L^1, \text{BMO}) \simeq \int_0^t (M_{s,Q_0}^# f)^*(u) \, du, \quad t > 0.
\]

The characterization \((1.11)\) connects \( \text{GaRo}_X \) spaces with classical harmonic analysis. Let

\[
f_{Q_0}^#(x) = \sup_{Q_0 \supset Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx, \quad x \in Q_0,
\]

and for a r.i. space \( X \) define

\[
X^# = \{ f : f_{Q_0}^# \in X \}
\]

with

\[
\|f\|_{X^#} = \|f_{Q_0}^#\|_X.
\]

\((7)\) The expression \( F \lesssim G \) means that \( F \leq c \cdot G \) for some constant \( c > 0 \) independent of all or part of the arguments of \( F \) and \( G \). If \( F \lesssim G \) and \( G \lesssim F \) we write \( F \simeq G \).
We will show (cf. Theorem 6.2) that
\[(1.17) \quad \|f\|_{X^\#} \simeq \|f\|_{\text{GaRo}_X} \quad \text{if and only if} \quad \beta_X < 1.\]

Moreover, we will consider generalized Fefferman–Stein inequalities of the form \((8)\)
\[
\inf_{c, \text{constant}} \|f - c\|_X \leq C \|f\|_{X^\#},
\]
and prove that this inequality holds if \(\alpha_X > 0\) (cf. Theorem 6.1) \((9)\).

It is of interest that the conditions on the indices that appear in the results described above are connected with considerations arising from interpolation theory. For example, to compare \(\text{GaRo}_X\) and \(X^\#\) one needs to understand the relationship between the sharp maximal operator \(f_{Q_0}^\#\) and the local maximal operator \(M_{s,Q_0}^\# f\), and one way to achieve this is via the formula for the \(K\)-functional for the pair \((L^1, \text{BMO})\) provided by \((1.14)\), and the formula obtained by Bennett–Sharpley (cf. Example 2.2 below),
\[(1.18) \quad (f_{Q_0}^\#)^*(t) \simeq \frac{1}{t} \int_0^t (M_{s,Q_0}^\# f)^*(u) \, du =: (M_{s,Q_0}^\# f)^{**}(t).\]

From \((1.14)\) and \((1.18)\) we see that the relationship between the sharp maximal operator and the local maximal operator is analogous to the classical relationship between \(f^*\) and \(f^{**}\). Moreover, if we write \(\text{GaRo}_{L(p, \infty)} = L(p, \infty) = (L^1, \text{BMO})_{1/p', \infty}\) we see that as \(p \to \infty\) we approach the space \(\text{BMO}\), thus we expect to lose “rearrangement invariance”, and this may help to explain the requirement \(\alpha_X > 0\), to be able to attain results of the form \(\text{GaRo}_X = X\).

The connection between Garsia–Rodemich spaces and interpolation goes deeper. In fact, the ideas associated with the construction of Garsia–Rodemich spaces lead us to find a new formula for the \(K\)-functional associated with the pair \((L^1, \text{BMO})\), using packings (cf. Section 7 below), which we believe should be of interest when comparing pointwise averages, as one often does in the theory of weighted norm inequalities. As a concrete application of this circle of ideas we show how one can use interpolation methods to prove the monotonicity under rearrangements of certain Garsia–Rodemich type functionals (our approach should be compared with the one in \([16]\)).

Finally, returning to some of the original results of Garsia and his collaborators, we give a simple proof of a Sobolev–Morrey embedding in Section 8.

We refer the reader to Section 2 and to the monographs \([6]\), \([22]\), \([19]\), \([10]\) and \([32]\) for background information and notation.

\((8)\) The classical inequalities of Fefferman–Stein \([15]\) correspond to \(X = L^p, 1 < p < \infty\).

\((9)\) For a different approach to Fefferman–Stein inequalities in the more general setting of Banach function spaces we refer to Lerner \([21]\).
2. Background information

2.1. Rearrangements. Let \((\Omega, \mu)\) be a resonant Borel probability space (cf. [6, p. 64]). For a measurable function \(f : \Omega \to \mathbb{R}\), the distribution function of \(f\) is given by
\[
\lambda_f(t) := \mu\{x \in \Omega : |f(x)| > t\}, \quad t > 0.
\]
The decreasing rearrangement \(f^*\) of a measurable function \(f\) is the right-continuous non-increasing function, mapping \((0, 1]\) into \([0, \infty)\), which is equimeasurable with \(f\), i.e.,
\[
\lambda_f(t) = |\{s \in [0, 1] : f^*(s) > t\}|, \quad t > 0,
\]
where \(| \cdot |\) denotes the Lebesgue measure on \([0, 1]\). It can be defined by the formula
\[
f^*(s) := \inf\{t > 0 : \lambda_f(t) \leq s\}, \quad s \in (0, 1].
\]
The maximal average \(f^{**}(t)\) is defined by
\[
f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \, ds = \frac{1}{t} \sup \left\{ \int_E |f(s)| \, d\mu : \mu(E) = t \right\}, \quad 0 < t \leq 1.
\]

2.2. Rearrangement invariant spaces. We briefly recall the basic definitions and conventions we use from the theory of rearrangement invariant (r.i.) spaces, and refer the reader to the books [6], [19] and [22] for a complete treatment. In the next definition we follow [22].

Let \(X := X(\Omega)\) be a Banach function space on \((\Omega, \mu)\), which is either separable or has the Fatou property (the latter means that if \(f_n \geq 0, f_n \in X, f_n \uparrow f\), and \(\sup_{n=1,2,\ldots} \|f_n\|_X < \infty\), then \(f \in X\) and \(\|f_n\|_X \uparrow \|f\|_X\)). We shall say that \(X\) is a rearrangement invariant (r.i.) space if \(g \in X\) implies that all \(\mu\)-measurable functions \(f\) with \(f^* = g^*\) also belong to \(X\) and \(\|f\|_X = \|g\|_X\).

For any r.i. space \(X\) we have
\[
L^\infty \subset X \subset L^1,
\]
with continuous embeddings. Many of the familiar spaces we use in analysis are examples of r.i. spaces, e.g. the \(L^p\)-spaces, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces, etc.

Let \(M\) be an increasing convex function on \([0, \infty)\) such that \(M(0) = 0\). The Orlicz space \(L_M\) consists of all measurable functions \(x(\cdot)\) on \([0, 1]\) such that \(M(|x(\cdot)|/\lambda) \in L^1\) for some \(\lambda > 0\). This space is equipped with the Luxemburg norm
\[
\|x\|_{L_M} := \inf \left\{ \lambda > 0 : \int_0^1 M(|x(t)|/\lambda) \, dt \leq 1 \right\}.
\]
In particular, if \(M(u) = u^p, 1 \leq p < \infty\), we obtain the usual \(L^p\)-spaces.
Let $\varphi$ be an increasing concave function on $[0, 1]$ with $\varphi(0) = 0$. The Marcinkiewicz space $\mathcal{M}(\varphi)$ consists of all measurable functions $x(t)$ such that

$$\|x\|_{\mathcal{M}(\varphi)} := \sup_{0 < s \leq 1} \frac{\varphi(s)}{s} \int_0^s x^*(t) \, dt < \infty.$$ 

The space $L(p, \infty)$, $1 < p < \infty$, corresponds to taking $\varphi(s) = s^{-1/p}$.

Let $X(\Omega)$ be a r.i. space. Then there exists a unique r.i. space (cf. [6, p. 64]) (the representation space of $X(\Omega)$) $\bar{X} = \bar{X}(0,1)$ on $((0,1), |\cdot|)$ such that

$$\|f\|_{X(\Omega)} = \|f^*\|_{\bar{X}(0,1)}.$$ 

In what follows, if no confusion is possible we shall not distinguish between $X$ and $\bar{X}$.

The following majorization principle, usually associated to the names of Hardy–Littlewood–Pólya–Calderón (cf. [11], [22, Proposition 2.a.8]), holds for r.i. spaces: if

$$(2.2) \int_0^t f^*(s) \, ds \leq \int_0^t g^*(s) \, ds \quad \text{for all } t > 0,$$ 

then, for any r.i. space $\bar{X}$,

$$\|f^*\|_{\bar{X}} \leq \|g^*\|_{\bar{X}},$$

or equivalently

$$\|f\|_{X} \leq \|g\|_{X}.$$ 

The fundamental function of $X$ is defined by

$$\phi_X(s) = \|X[0,s]\|_{\bar{X}}, \quad 0 \leq s \leq 1.$$ 

We can assume without loss of generality that $\phi_X$ is concave (cf. [9]). For example, for an Orlicz space $L_N$ (cf. (2.1) above) we have $\phi_{L_N}(t) = 1/N^{-1}(1/t)$, and for a Marcinkiewicz space $\mathcal{M}(\varphi)$, $\phi_{\mathcal{M}(\varphi)}(t) = \varphi(t)$.

**2.3. Boyd indices and Hardy operators.** Let $X = X(\Omega)$ be an arbitrary r.i. space. Then the compression/dilation operator $\sigma_s$ on $\bar{X}$, defined by

$$\sigma_s f(t) = \begin{cases} f^*(t/s), & 0 < t < \min(1, s), \\ 0, & s \leq t \leq 1, \end{cases}$$

is bounded on $\bar{X}$, and moreover (cf. [19 §2.4])

$$(2.3) \|\sigma_s\|_{\bar{X} \to \bar{X}} \leq \max\{1, s\} \quad \text{for all } s > 0.$$ 

The Boyd indices (cf. [9]) are defined by

$$\alpha_X := \lim_{s \to 0^+} \frac{\ln \|\sigma_s\|_{\bar{X} \to \bar{X}}}{\ln s} \quad \text{and} \quad \beta_X := \lim_{s \to \infty} \frac{\ln \|\sigma_s\|_{\bar{X} \to \bar{X}}}{\ln s}.$$
For each r.i. space $X$ we have $0 \leq \alpha_X \leq \beta_X \leq 1$. For example, it follows readily that $\alpha_{L^p} = \beta_{L^p} = 1/p$ for all $1 \leq p \leq \infty$.

It is known that the Boyd indices control the boundedness of the Hardy operators, defined by

$$Pf(t) := \frac{1}{t} \int_0^t f(s) \, ds, \quad Qf(t) := \frac{1}{t} \int f(s) \, \frac{ds}{s}.$$ 

In fact, it is well known (cf. [9], [19, Theorems 2.6.6 and 2.6.8]) that

$$P \text{ is bounded on } \overline{X} \iff \beta_{\overline{X}} < 1,$$

$$Q \text{ is bounded on } \overline{X} \iff \alpha_{\overline{X}} > 0.$$ 

### 2.4. $K$-functionals and real interpolation.

Let $(A_0, A_1)$ be a compatible pair of Banach spaces. For all $f \in A_0 + A_1$ and $t > 0$, we define the Peetre $K$-functional by

$$K(t, f; A_0, A_1) := \inf \{ \|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1, f_i \in A_i, i = 0, 1 \}.$$ 

Let $\theta \in (0, 1)$ and $1 \leq q \leq \infty$. The interpolation spaces $(A_0, A_1)_{\theta, q}$ are defined by

$$(A_0, A_1)_{\theta, q} := \{ f : f \in A_0 + A_1, \|f\|_{(A_0, A_1)_{\theta, q}} < \infty \},$$

where

$$\|f\|_{(A_0, A_1)_{\theta, q}} := \begin{cases} \left\{ \int_0^\infty (s^{-\theta} K(s, f; A_0, A_1))^q \frac{ds}{s} \right\}^{1/q} & \text{if } q < \infty, \\ \sup_{s > 0} \{ s^{-\theta} K(s, f; A_0, A_1) \} & \text{if } q = \infty. \end{cases}$$

**Example 2.1** (Peetre–Oklander formula; cf. [6, (1.28), p. 298], [27]). For the pair $(L^1, L^\infty)$ the $K$-functional is given by

$$K(t, f; L^1, L^\infty) = \int_0^t f^*(u) \, du, \quad t > 0.$$

Let $M_{Q_0}$ be the Hardy–Littlewood maximal operator,

$$M_{Q_0}f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad x \in Q_0.$$ 

The maximal operator $M_{Q_0}$ is connected with $K(t, \cdot; L^1, L^\infty)$ via the Herz–Stein inequalities (cf. [6, Theorem 3.8, p. 122]):

$$M_{Q_0}f^*(t) \simeq f^{**}(t) := \frac{1}{t} \int_0^t f^*(u) \, du, \quad 0 < t \leq 1.$$ 

**Example 2.2.** For the pair $(L^1, \text{BMO})$ (we consider classes of equivalence modulo constants), we have the following formula due to Bennett–Sharpley
Garsia–Rodemich spaces

(2.8) \[ K(t, f; L^1, \text{BMO}) \simeq t(f^\#_{Q_0})^*(t), \quad 0 < t \leq 1. \]

Comparing this with the Jawerth–Torchinsky formula (1.14) we see the equivalence (1.18).

In what follows any constant appearing in inequalities and depending only on the dimension \( n \) will be referred to as absolute.

3. A new description of Garsia–Rodemich spaces. In this section we give a new characterization of Garsia–Rodemich spaces using local maximal operators. To motivate our result it will be useful to reformulate somewhat the definition of the \( \Gamma_f \) classes (cf. (1.8) above).

It follows from inequalities (1.4) that if an integrable function \( \gamma \) belongs to \( \Gamma_f \), then for all subcubes \( Q \subset Q_0 \), we have

\[
\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq \frac{1}{|Q|} \int_Q |\gamma(x)| \, dx,
\]

whence

\[ f^\#_{Q_0}(x) \leq M_{Q_0} \gamma(x), \quad x \in Q_0. \]

The idea behind our main result can now be summarized as follows: for every \( f \in L^1 \), the Strömberg–Jawerth–Torchinsky maximal function \( M^\#_{s,Q_0} f \) is an “optimal” choice of \( \gamma \) from \( \Gamma_f \).

**Theorem 3.1.** There exists \( s_0 \in (0, 1) \), depending only on dimension \( n \), such that for all \( s \in (0, s_0) \) and every r.i. space \( X \), we have

(3.1) \[ \text{GaRo}_X = \{ f \in L^1 : \|M^\#_{s,Q_0} f\|_X < \infty \}. \]

Moreover, with constants of equivalence depending on \( n \in \mathbb{N} \) and \( s \in (0, s_0) \),

(3.2) \[ \|f\|_{\text{GaRo}_X} \simeq \|M^\#_{s,Q_0} f\|_X. \]

For the proof we shall need the following

**Lemma 3.2.**

(i) For every cube \( Q \subset Q_0 \), all \( 0 < s < 1 \) and \( c \in \mathbb{R} \), and each \( f \in L^1(Q_0) \) we have

\[ |\{ y \in Q : M^\#_{s,Q} f(y) > \lambda \} | \leq \frac{4^n}{s} |\{ y \in Q : |f(y) - c| > \lambda \} | \quad \text{for all } \lambda > 0. \]

(ii) There exists \( 0 < s_0 < 1 \) such that for all \( s \in (0, s_0) \) and all \( f \in L^1(Q_0) \),

(3.3) \[ M_{Q_0}(M^\#_{s,Q_0} f)(x) \leq \frac{2 \cdot 8^n}{s} f^\#_{Q_0}(x), \quad x \in Q_0, \]

where \( M_{Q_0} \) is the Hardy–Littlewood maximal function (cf. (2.6)).
Proof. (i) Let \( Q \subset Q_0 \) be an arbitrary cube. If \( M_{s,Q}^\# f(y) > \lambda \) for \( y \in Q \), then there is a cube \( Q' \subset Q \) such that \( y \in Q' \) and for all \( c \in \mathbb{R} \),

\[
|\{ z \in Q' : |f(z) - c| > \lambda \}| > s|Q'|.
\]

Therefore, we have

\[
M_Q(\chi_{\{|f-c|>\lambda\}})(y) \geq \frac{1}{|Q'|} \int_{Q'} \chi_{\{|f-c|>\lambda\}}(z) \, dz > s
\]

(here \( M_Q \) is the Hardy–Littlewood maximal operator corresponding to the cube \( Q \)). Hence,

\[
|\{ y \in Q : M_{s,Q}^\# f(y) > \lambda \}| \leq |\{ y \in Q : M_Q(\chi_{\{|f-c|>\lambda\}})(y) > s\}|.
\]

Combining this estimate with the fact that \( M_Q \) is of weak type \((1, 1)\) (cf. [6, Theorem 3.3.3]), we see that

\[
|\{ y \in Q : M_{s,Q}^\# f(y) > \lambda \}| \leq \frac{4^n}{s} \| \chi_{\{|f-c|>\lambda\}} \|_{L^1(Q)} = \frac{4^n}{s}|\{ y \in Q : |f-c| > \lambda \}|.
\]

(ii) Let \( x \in Q_0 \) and \( Q \subset Q_0 \) be an arbitrary cube such that \( x \in Q \). Denote by \( 2Q \) the cube concentric with \( Q \) and with double side length. Clearly, there is a cube \( \hat{Q} \) such that \( Q_0 \cap 2Q \subset \hat{Q} \subset Q_0 \) and \( |\hat{Q}| \leq |2Q| \). In particular, if \( 2Q \subset Q_0 \), we take \( \hat{Q} = 2Q \). Note that \( \hat{Q} \supset Q \).

Further, for all \( y \in Q \) we have

\[
M_{s,Q_0}^\# f(y) \leq M_{s,\hat{Q}}^\# f(y) + R_{s,\hat{Q}}^\# f(y),
\]

where the operator \( R_{s,\hat{Q}}^\# \) is defined just as \( M_{s,\hat{Q}}^\# \) except that the supremum is now taken over all cubes intersecting \( Q_0 \setminus \hat{Q} \). From the preceding inequality it follows that

\[
(3.4) \quad \frac{1}{|Q|} \int_Q M_{s,Q_0}^\# f(y) \, dy \leq \frac{1}{|Q|} \int_Q M_{s,\hat{Q}}^\# f(y) \, dy + \frac{1}{|Q|} \int_Q R_{s,\hat{Q}}^\# f(y) \, dy.
\]

Applying part (i) of this lemma to \( \hat{Q} \) and using the properties of that cube, we estimate the first integral on the right-hand side:

\[
\frac{1}{|Q|} \int_Q M_{s,\hat{Q}}^\# f(y) \, dy \leq 2^n \frac{1}{|Q|} \int_Q M_{s,\hat{Q}}^\# f(y) \, dy \leq \frac{8^n}{s} \frac{1}{|\hat{Q}|} \int_{\hat{Q}} |f(y) - c| \, dy
\]

for any \( c \in \mathbb{R} \). On the other hand, since the cube \( \hat{Q} \) is fixed, for each \( \varepsilon > 0 \) we can choose a constant \( c' \) such that

\[
\frac{1}{|\hat{Q}|} \int_{\hat{Q}} |f(y) - c' - \varepsilon| \, dy \leq (1 + \varepsilon) \inf_{c' \in \mathbb{R}} \frac{1}{|\hat{Q}|} \int_{\hat{Q}} |f(y) - c'| \, dy.
\]
Combining these inequalities with the definition of $f^\#_{Q_0}(x)$, we infer that

$$(3.5) \quad \frac{1}{|Q|} \int_M M^\#_{s,Q_0} f(y) \, dy \leq (1 + \varepsilon) \frac{8^n}{s} f^\#_{Q_0}(x).$$

To estimate the second integral on the right-hand side of (3.4), we will use the following observation. For each cube $Q' \subset Q_0$, from $Q' \cap (Q_0 \setminus \bar{Q}) \neq \emptyset$ it follows that $Q' \cap (\mathbb{R}^n \setminus (2Q)) \neq \emptyset$. Therefore, there is a cube $Q'' \subset Q_0$ such that $Q'' \supset Q$ and $|Q''| \leq 3^n|Q'|$ and so from the definition of the operators $M^\#_{s,Q_0}$ and $R^\#_{s,Q_0}$ we see that

$$\sup_{y \in Q} R^\#_{s,Q_0} f(y) \leq \inf_{y \in \bar{Q}} M^\#_{s',Q_0} f(y),$$

where $s' = s3^{-n}$. Now since $x \in Q$, we obtain

$$\frac{1}{|Q|} \int R^\#_{s,2Q} f(y) \, dy \leq M^\#_{s',Q_0} f(x) \leq \frac{3^n}{s} f^\#_{Q_0}(x),$$

where the last inequality follows from Chebyshev’s inequality. Combining our findings with (3.4) and (3.5), we obtain

$$\frac{1}{|Q|} \int M^\#_{s,Q_0} f(y) \, dy \leq 2(1 + \varepsilon) \frac{8^n}{s} f^\#_{Q_0}(x).$$

Taking the supremum over all cubes $Q \subset Q_0$ such that $x \in Q$ and letting $\varepsilon \to 0$ we achieve the desired inequality (3.3). 

Proof of Theorem 3.1. Suppose that $f \in L^1$ is such that $\|M^\#_{s,Q_0} f\|_X < \infty$ for some $s \in (0, 1)$. Recall that by [20] Lemma 2.4, there exists $s_0 = s_0(n) > 0$ such that, for all $0 < s < s_0$ and for every cube $Q \subset Q_0$,

$$(3.6) \quad \int_Q |f - f_Q| \, dx \leq 8 \int_Q M^\#_{s,Q_0} f \, dx.$$

Consequently, by (1.4), $16M^\#_{s,Q_0} f \in \Gamma_f$. Thus, for each $s \in (0, s_0)$,

$$\|f\|_{GaRo_X} \leq 16\|M^\#_{s,Q_0} f\|_X.$$

Conversely, let $f \in GaRo_X$. Given $\varepsilon > 0$ we can select $\gamma \in \Gamma_f \cap \Gamma_f$ such that

$$(3.7) \quad \|\gamma\|_X \leq \|f\|_{GaRo_X} + \varepsilon.$$

From the fact that $\gamma \in \Gamma_f$ it follows that (see the observation at the beginning of this section)

$$(3.8) \quad f^\#_{Q_0}(x) \leq M_{Q_0} \gamma(x), \quad x \in Q_0.$$
Consequently, by (3.3), for all $0 < s < 1$,

$$M_{Q_0}(M_{s,Q_0}^## f)(x) \leq \frac{2 \cdot 8^n}{s} M_{Q_0} \gamma(x), \quad x \in Q_0.$$  

Taking rearrangements in (3.9), and using Herz’s rearrangement inequality for the Hardy–Littlewood maximal operator (cf. (2.7)), for each $0 < s < 1$ we can find a constant $c = c(n,s)$ such that

$$\int_0^t (M_{s,Q_0}^## f)^*(s) \, ds \leq c \int_0^t \gamma^*(s) \, ds \quad \text{for all } t > 0.$$  

Hence, using successively the Hardy–Littlewood–Pólya–Calderón majorization principle (cf. (2.2)) and inequality (3.7), we get

$$\|M_{s,Q_0}^## f\|_X \leq c \|\gamma\|_X \leq c \|f\|_{\text{GaRo}_X} + c\varepsilon.$$  

Letting $\varepsilon \to 0$ we obtain the desired converse inequality. \[\Box\]

From Theorem 3.1 and its proof, we readily obtain the following alternative description of Garsia–Rodemich spaces. Denote by $\Gamma'_f$ the set of all functions $\gamma \in L^1(Q_0)$ satisfying (3.8).

**Corollary 3.3.** Let $X$ be a r.i. space. Then the Garsia–Rodemich space $\text{GaRo}_X$ consists of all $f \in L^1(Q_0)$ for which $\Gamma'_f \cap X \neq \emptyset$. Moreover, there exists an absolute constant $c = c(n)$ such that

$$\inf\{\|\gamma\|_X : \gamma \in \Gamma'_f \cap X\} \leq \|f\|_{\text{GaRo}_X} \leq c \inf\{\|\gamma\|_X : \gamma \in \Gamma'_f \cap X\}.$$  

**4. A characterization of rearrangement invariant spaces via Garsia–Rodemich conditions.** The main result of this section is the following characterization of r.i. spaces which improves on (1.10) above.

**Theorem 4.1.** Let $X$ be a r.i. space such that $\alpha_X > 0$. Then

$$\text{GaRo}_X = X.$$  

**Proof.** Let $f \in X$. Since for all cubes $Q \subset Q_0$ we have

$$\frac{1}{|Q|} \int_{QQ} |f(x) - f(y)| \, dx \, dy \leq 2 \int_Q |f(x) - f_Q| \, dx \leq 4 \int_Q |f(x)| \, dx,$$

it follows from (1.8) that $4|f| \in \Gamma_f$. Consequently, the embedding $X \subset \text{GaRo}_X$ holds for every r.i. space $X$ and moreover

$$\|f\|_{\text{GaRo}_X} \leq 4\|f\|_X.$$  

We now show that if $\alpha_X > 0$, then $\text{GaRo}_X \subset X$. Let $f \in \text{GaRo}_X$, and let $\gamma \in \Gamma_f$. Then we have (3.8), which combined with (2.7) implies

$$(f_{Q_0}^#)^*(t) \leq (M_{Q_0}^## \gamma)^*(t) \leq \gamma^{**}(t) := \frac{1}{t} \int_0^t \gamma^*(u) \, du.$$
Thus, from (2.5) and (2.8), we get
\[(4.1) \quad K(t, f; L^1, \text{BMO}) \preceq K(t, \gamma; L^1, L^\infty),\]
where the implied constants are independent of \(f\) and \(\gamma\). Fix \(p > 1/\alpha_X\). It is well known that (cf. [6, Theorem 8.11, p. 398])
\[(L^1, L^\infty)_{\theta,p} = (L^1, \text{BMO})_{\theta,p} = L^p, \quad \theta = 1 - 1/p.\]
Therefore, by Holmstedt’s reiteration formula (cf. [6, Corollary 2.3, p. 310]), we have
\[K(t, f; L^1, L^p) \simeq t \left( \int_{t^{1/\theta}}^\infty (s^{-\theta} K(t, f; L^1, \text{BMO}))^p \frac{ds}{s} \right)^{1/p},\]
\[K(t, \gamma; L^1, L^p) \simeq t \left( \int_{t^{1/\theta}}^\infty (s^{-\theta} K(t, \gamma; L^1, L^\infty))^p \frac{ds}{s} \right)^{1/p},\]
with constants that depend only on \(p\) (and hence on \(X\)). Combining these estimates with (4.1) yields
\[K(t, f; L^1, L^p) \preceq K(t, \gamma; L^1, L^p),\]
with constants that depend only on \(X\) and \(n\). Since the pair \((L^1, L^p)\) is \(K\)-monotone (cf. [29], [12, Theorem 4])\(^{(10)}\), it follows that there exists a bounded linear operator \(T\) acting on the pair \((L^1, L^p)\) such that \(f = T\gamma\). Moreover, from the fact that \(p > 1/\alpha_X\), we can deduce that \(X\) is an interpolation space with respect to the pair \((L^1, L^p)\) (cf. [2, Theorem 2]). Consequently, by the \(K\)-monotonicity of \((L^1, L^p)\), there exists a Banach lattice \((\Phi, \|\cdot\|_\Phi)\) of Lebesgue measurable functions on \((0, \infty)\) such that the norm of \(X\) can be represented as follows (cf. [10, Theorems 4.4.5 and 4.4.38]):
\[(4.2) \quad \|x\|_X \simeq \|K(t, x; L^1, L^p)\|_\Phi \quad \text{for all } x \in X.\]
It follows that the operator \(T\) is bounded on \(X\), and consequently
\[\|f\|_X \leq c\|\gamma\|_X\]
for some constant \(c = c(n, X)\). Taking the infimum over all \(\gamma \in \Gamma_f\) yields
\[\|f\|_X \leq c\|f\|_{\text{GaRo}_X},\]
as we wished to show. \(\blacksquare\)

Theorem 4.1 has a partial converse. To state the result we introduce the class of r.i. spaces of fundamental type.

**Definition 4.2.** Let \(X = X(Q_0)\) be a r.i. space on \(Q_0\), and let \(\widetilde{X} = X(0, 1)\) be its Luxemburg representation on \((0, 1)\) (cf. Section 2.2). We shall say that \(X\)
is of fundamental type if there exists a constant $C > 0$ such that (cf. Section 2.3 above)
\[ \|\sigma_t\|_{\overline{X}\to\overline{X}} \leq C \sup_{0 < s \leq \min(1,1/t)} \frac{\phi_X(st)}{\phi_X(s)}, \quad t > 0. \]

Remark 4.3. It is easy to verify that Orlicz, Lorentz, Marcinkiewicz spaces, etc., are all of fundamental type.

Definition 4.4. A median value \((11)\) of \(f\) on \(Q\) is a number \(m_f(Q)\) such that
\[ \{|x \in Q : f(x) > m_f(Q)\}| \leq \frac{1}{2}|Q| \]
and
\[ \{|x \in Q : f(x) < m_f(Q)\}| \leq \frac{1}{2}|Q|. \]

It is well known that \(m_f(Q)\) is one of the constants \(c\) minimizing some functionals depending on the deviation \(|f - c|\). In particular (cf. \([20, \S 2, p. 2450]\)),
\[ (f - m_f(Q_0))^*(t) \leq 2 \inf_{c \in \mathbb{R}} (f - c)^*(t), \quad 0 < t \leq 1/2. \]
From this inequality one can easily deduce that for every r.i. space \(X\),
\[ (4.3) \quad \|f - m_f(Q_0)\|_X \leq 4 \inf_{c \in \mathbb{R}} \|f - c\|_X. \]

Theorem 4.5. Let \(X\) be a r.i. space of fundamental type, and let \(0 < s \leq 1/2\). If there exists a constant \(C > 0\) such that
\[ (4.4) \quad \inf_{c \in \mathbb{R}} \|f - c\|_X \leq C \|M^#_{s,Q_0}f\|_X \quad \text{for all } f \in X, \]
then \(\alpha_X > 0\).

Proof. To the contrary, suppose that \(\alpha_X = 0\). Since \(X\) is of fundamental type we can find two numerical sequences \(\{u_k\}_{k \in \mathbb{N}}, \{a_k\}_{k \in \mathbb{N}}\) contained in \((0,1)\), converging to zero, and such that
\[ (4.5) \quad \phi_X(u_k a_k) \geq \frac{1}{2} \phi_X(a_k), \quad k = 1, 2, \ldots. \]
Without loss of generality we can assume that \(Q_0 = [0,1]^n\). Moreover, if \(b > 0\), we set \(bQ_0 := [0,b]^n\). For \(a \in (0,1)\), let \(f_a(x) := n \ln(\frac{a^{1/n}}{|x| \infty}) \chi_{a^{1/n}Q_0}(x), x \neq 0\), with \(|x|_{\infty} := \max_{i=1,\ldots,n} |x_i|\) for \(x = (x_i)_{i=1}^n \in \mathbb{R}^n\). One can readily verify that there exists a constant \(D \geq 1\), depending only on the dimension \(n\) and \(s\), such that \(M^#_{s,Q_0}f_a(x) \leq D\) if \(|x| \leq Da\), and \(M^#_{s,Q_0}f_a(x) = 0\) if \(|x| > Da\). Thus, using the concavity of the fundamental function \(\phi_X\) (see Section 2.2), we get
\[ (4.6) \quad \|M^#_{s,Q_0}f_a\|_X \leq D \phi_X(Da) \leq D^2 \phi_X(a), \quad 0 < a \leq 1. \]

(11) Note that \(m_f(Q)\) is not uniquely defined.
Moreover, it can be easily checked that \( f_a^*(t) = \ln(a/t) \chi_{(0,a)}(t) \) and \( m_{f_a}(Q_0) = 0 \) if \( a \) is sufficiently small. Thus, using (4.3)–(4.6), for sufficiently large \( k \in \mathbb{N} \) we have

\[
D^2 C\phi_X(a_k) \geq C \| M^s_{s,Q_0} f_{a_k} \|_X \geq \inf_{c \in \mathbb{R}} \| f_{a_k} - c \|_X \geq \frac{1}{4} \| f_{a_k} - m_{f_{a_k}}(Q_0) \|_X \\
= \frac{1}{4} \| f_{a_k} \|_X \geq \frac{1}{4} \| \ln(a_k/t) \chi_{(0,a_k)}(t) \|_{\bar{X}} \geq \frac{1}{4} \| \ln(a_k/t) \chi_{(0,a_k u_k)}(t) \|_{\bar{X}} \geq \frac{1}{4} \ln(u_k^{-1}) \| \chi_{(a_k u_k)}(t) \|_{\bar{X}} \geq \frac{1}{4} \ln(u_k^{-1}) \phi_X(a_k).
\]

This leads to a contradiction since \( \lim_{k \to \infty} \ln(u_k^{-1}) = \infty \).

Applying Theorems 3.1, 4.1, and 4.5, we immediately obtain the following result.

**Corollary 4.6.** Let \( X \) be an r.i. space of fundamental type. Then the following conditions are equivalent:

(a) \( \text{GaRo}_X = X \);
(b) \( \text{GaRo}_X \subset X \);
(c) \( \alpha_X > 0 \).

5. **K-functionals and rearrangement inequalities.** In this section we consider some examples of interaction of Garsia–Rodemich functionals with rearrangements, which are connected with our discussion.

Our first application deals with a new proof of an inequality due to Bennett–Sharpley (cf. [6, Theorem 7.3, p. 377]).

**Example 5.1.** There exists an absolute constant \( c > 0 \) such that for all \( f \in L^1(Q_0) \),

\[
(5.1) \quad f^{**}(t) - f^*(t) \leq c(f_{Q_0}^#)^*(t), \quad 0 < t < 1/6.
\]

**Proof.** We recall the following fact from [20]: There exists an absolute constant \( c_1 \) such that for all \( f \in L^1(Q_0) \) and all \( \gamma \in \Gamma_f \),

\[
(5.2) \quad f^{**}(t) - f^*(t) \leq c_1 \gamma^{**}(t), \quad 0 < t < 1/6.
\]

On the other hand, from (3.6) we know that for sufficiently small \( s > 0 \) we have \( 16 M^s_{s,Q_0} f \in \Gamma_f \). Consequently, by (5.2),

\[
(5.3) \quad f^{**}(t) - f^*(t) \leq 16 c_1 (M^s_{s,Q_0} f)^{**}(t), \quad 0 < t < 1/6.
\]

Combining the last inequality with the fact that there exists an absolute constant \( c_2 \) such that (cf. (1.18))

\[
(M^s_{s,Q_0} f)^{**}(t) \leq c_2 (f_{Q_0}^#)^*(t),
\]

we obtain (5.1). ■

Our second result shows how the continuity of rearrangements on Garsia–Rodemich spaces can be easily established using their description obtained
in Theorem 3.1 and interpolation (compare with the methods to establish related rearrangement inequalities, developed in [16] and [3]).

**Theorem 5.2.** There exists an absolute constant $c > 0$ such that for all $f \in \text{GaRo}_X$,

$$\|f^*\|_{\text{GaRo}_X(0,1)} \leq c \|f\|_{\text{GaRo}_X(Q_0)}.$$

**Proof.** From [16] and [4] (cf. also [13]), we know that there exists an absolute constant $c_1 \geq 1$ such that

$$\|f^*\|_{\text{BMO}(0,1)} \leq c_1 \|f\|_{\text{BMO}}.$$

On the other hand, it is well known (cf. [16], [19, Theorem 2.3.1]) that for all $f, g \in L^1(Q_0)$,

$$\|f^* - g^*\|_{L^1(0,1)} \leq \|f - g\|_{L^1(Q_0)}.$$

Consequently, for every $f \in L^1(Q_0)$,

$$K(t, f^*; L^1(0,1), \text{BMO}(0,1)) = \inf \{\|f_1\|_{L^1(0,1)} + t\|f_2\|_{\text{BMO}(0,1)} : f^* = f_1 + f_2\}$$

$$\leq \inf \{\|f^* - g^*\|_{L^1(0,1)} + t\|g^*\|_{\text{BMO}(0,1)} : g \in \text{BMO}(Q_0)\}$$

$$\leq \inf \{\|f - g\|_{L^1(Q_0)} + tc_1\|g\|_{\text{BMO}(Q_0)} : g \in \text{BMO}(Q_0)\}$$

$$\leq c_1K(t, f; L^1(Q_0), \text{BMO}(Q_0))$$

$$\leq c_1K(t, f^*; L^1(Q_0), \text{BMO}(Q_0)) \quad (\text{since } K(t)/t \text{ decreases}).$$

In particular, in view of (1.14), there exists an absolute constant $c_2 > 0$ such that

$$\tag{5.3} (M_{s,(0,1)}^# f^*)^{**}(t) \leq c_2(M_{s,Q_0}^# f)^{**}(t).$$

By the Hardy–Littlewood–Pólya–Calderón principle, it follows that

$$\|M_{s,(0,1)}^# f^*\|_{X(0,1)} \leq c'\|(M_{s,Q_0}^# f)^*\|_{X(0,1)} = c'\|M_{s,Q_0}^# f\|_{X(Q_0)}.$$

Applying (3.2) we finally obtain

$$\|f^*\|_{\text{GaRo}_X(0,1)} \leq c\|f\|_{\text{GaRo}_X(Q_0)},$$

as we wished to show. ■

**Remark 5.3.** Essentially the same argument shows that if $T$ is a bounded operator on the pair $(L^1, \text{BMO})$, then $T$ is a bounded operator in the space $\text{GaRo}_X$. Indeed, for such operators we have

$$K(t, Tf; L^1, \text{BMO}) \leq cK(t, f; L^1, \text{BMO}), \quad t > 0,$$

which in view of (1.14) implies

$$\int_0^t (M_{s,Q_0}^# Tf)^*(s) \, ds \leq c \int_0^t (M_{s,Q_0}^# f)^*(s) \, ds.$$
Therefore, we get (2.2) and, as above, for any r.i. space $X$ we have
\[ \|M_{s,Q_0}^# Tf\|_X \leq c \|M_{s,Q_0}^# f\|_X. \]
The desired result now follows from Theorem 3.1.

**Remark 5.4.** As we have seen before (cf. (1.18)),
\[ (M_{s,(0,1)}^# f^*)^{**}(t) \simeq ((f^*)^{#(0,1)})^*(t) \]
and
\[ (M_{s,Q_0}^# f)^*(t) \simeq (f_{Q_0}^#)^*(t), \]
thus for a suitable constant $C > 0$, from (5.3) it follows that
\[ ((f^*)^{#(0,1)})^*(t) \leq C(f_{Q_0}^#)^*(t), \]
which should be compared with Example 5.1.

**Remark 5.5.** The $K$-functional for the pair $(L^\infty, \text{BMO})$ was computed by several authors including Janson, Jawerth–Torchinsky, Shvartsman (cf. [17], [28] and the references therein). It would be of interest to connect interpolation spaces with respect to the pair $(L^\infty, \text{BMO})$ and Garsia–Rodemich constructions.

### 6. Fefferman–Stein inequality via Garsia–Rodemich spaces.

The original Fefferman–Stein inequality (cf. [15] and also [31] and the references therein) concerns the embedding (cf. (1.15) and (1.16) above)
\[ L^p^# \subset L^p, \quad 1 < p < \infty. \]

In [31], Strömberg extended this result to an appropriate class of Orlicz spaces.

The connection between $X^#$ and $\text{GaRo}_X$ can be seen from the fact that
\[ (6.1) \]
indeed, we can easily show that $f \in X^#$ implies $2f_{Q_0}^# \in \Gamma_f$. This follows directly from (1.4) since for each $Q \subset Q_0$ we have
\[ \frac{1}{|Q|} \int_Q |f(x) - f(y)| \, dx \, dy \leq 2 \int_Q |f(x) - f_Q| \, dx = 2 \frac{|Q|}{|Q_0|} \int_Q |f(x) - f_Q| \, dx \]
\[ = 2 \int_Q \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \right) \, dy \leq 2 \int_{Q_0} f_{Q_0}^#(y) \, dy, \]
and so $\gamma := 2f_{Q_0}^#$ satisfies inequality (1.8). Consequently, (6.1) holds for all r.i. spaces $X$, and moreover
\[ \|f\|_{\text{GaRo}_X} \leq 2\|f\|_{X^#}. \]

Using the above observation, one can extend the Fefferman–Stein–Strömberg result (12) to the setting of r.i. spaces.

(12) However, note that unlike [31] we consider functions defined on a fixed cube $Q_0$. 

Theorem 6.1. If the lower Boyd index $\alpha_X$ of the r.i. space $X$ is positive, then $X^# \subset X$.

Proof. From the condition $\alpha_X > 0$ and Theorem 4.1 we infer that $\text{GaRo}_X = X$. We conclude by combining this fact with (6.1).

The next result establishes necessary and sufficient conditions for the opposite embedding $X \subset X^#$ to hold.

Theorem 6.2. Let $X$ be an r.i. space on $[0, 1]$. The following conditions are equivalent:

(i) $\beta_X < 1$;
(ii) $\text{GaRo}_X \subset X^#$;
(iii) $X \subset X^#$.

Proof. (i)$\Rightarrow$(ii). Let $f \in \text{GaRo}_X$. As we have seen above, for every $\gamma \in \Gamma_f$ we have $f_Q^\#(x) \leq M_{Q_0} \gamma(x)$. Since we are assuming that $\beta_X < 1$, the Hardy–Littlewood operator $M_{Q_0}$ is bounded on $X$. Hence,

$$\|f_Q^\#\|_X \leq \|M_{Q_0} \gamma\|_X \leq \|M_{Q_0}\|_{X \rightarrow X} \|\gamma\|_X.$$

Taking the infimum over all $\gamma \in \Gamma_f$, we get

$$\|f_Q^\#\|_X \leq \|M_{Q_0}\|_{X \rightarrow X} \|f\|_{\text{GaRo}_X},$$

whence $f \in X^#$.

(ii)$\Rightarrow$(iii). The implication is trivial since the embedding $X \subset \text{GaRo}_X$ holds for all r.i. spaces $X$ (see the beginning of the proof of Theorem 4.1).

(iii)$\Rightarrow$(i). By [6, Theorem 5.7.3] (cf. also Example 5.1 in Section 5), we have

$$f^{**}(t) - f^*(t) \leq c'(f_Q^\#)^*(t), \quad 0 < t < 1/6,$$

for some absolute constant $c'$. Therefore,

$$f^{**}(t/6) \leq f^*(t/6) + c'(f_Q^\#)^*(t/6), \quad 0 < t < 1.$$

From the latter inequality, (2.3), and our current assumption, it follows that

$$\|f^{**}\|_X \leq \|\sigma_6 f^{**}\|_X \leq \|\sigma_6 f\|_X + c'\|\sigma_6 f_Q^\#\|_X \leq 6c'(\|f\|_X + \|f_Q^\#\|_X) \leq c\|f\|_X.$$

This shows that the Hardy operator $P$ is bounded on $X$, and therefore, by (2.4), $\beta_X < 1$.

7. A packing formula for the $K$-functional of $(L^1, \text{BMO})$. The new characterization of Garsia–Rodemich spaces discussed in the introduction (cf. (1.11) above) suggested a new formula for the $K$-functional of $(L^1, \text{BMO})$ (see Section 2.4).
Remark 7.1. In order to properly interpret \((L^1, \text{BMO})\) as a compatible pair of Banach spaces, it is necessary to factor out the constant functions. Equivalently, we can restrict ourselves to functions with zero mean, i.e. \(\int_{Q_0} f(x) \, dx = 0\).

For any family of cubes \(\pi = \{Q_i\} \in P := P(Q_0)\), we define
\[
S_{\pi,\#}(f)(x) = \sum_{Q_i \in \pi} \left( \frac{1}{|Q_i|} \int_{Q_i} |f(y) - f_{Q_i}| \, dy \right) \chi_{Q_i}(x), \quad x \in Q_0,
\]
and let
\[
F_{f,\#}(t) = \sup_{\pi \in P} (S_{\pi,\#}(f))^*(t), \quad 0 < t \leq 1.
\]

Theorem 7.2. There exist absolute constants such that for all \(f \in L^1\),
\[
K(t, f; L^1, \text{BMO}) \simeq t F_{f,\#}(t), \quad 0 < t \leq 1.
\]

Proof. It is plain that
\[
F_{f,\#}(t) \leq f^{\#*}(t), \quad 0 < t \leq 1.
\]
Consequently, by (2.8) (where the implied constants depend only on the dimension), we have
\[
t F_{f,\#}(t) \preceq K(t, f; L^1, \text{BMO}), \quad 0 < t \leq 1.
\]
Thus, the desired result will follow if we show that
\[
K(t, f; L^1, \text{BMO}) \preceq t F_{f,\#}(t), \quad 0 < t \leq 1,
\]
with some absolute constant.

Given \(t \in (0, 1]\), we consider the set
\[
\Omega(t) := \{x \in Q_0 : f^{\#}(x) > f^{\#*}(t)\}.
\]
It follows that for each \(x \in \Omega(t)\) there exists a cube \(Q_x\) such that \(Q_x \subset Q_0\), \(x \in Q_x\), and
\[
\frac{1}{|Q_x|} \int_{Q_x} |f - f_{Q_x}| > f^{\#*}(t) . \quad (7.2)
\]
Note that, by the definition of \(\Omega(t)\), we have \(Q_x \subset \Omega(t)\) for every \(x \in \Omega(t)\). Consider the family \(\{Q_x\}_{x \in \Omega(t)}\) of cubes. Using a Vitali type covering lemma (cf. [30, p. 9]), we can select a subfamily \(\{Q_k\}\) of pairwise disjoint cubes (which may contain a finite number of elements) such that
\[
|\Omega(t)| = \left| \bigcup_{x \in \Omega(t)} Q_x \right| \leq 5^n \sum_k |Q_k|. \quad (7.3)
\]
Clearly \(\pi = \{Q_k\} \in P\), and moreover, by (7.2),
\[
S_{\pi,\#}(f)(x) > f^{\#*}(t) \quad \text{for all } x \in \bigcup_k Q_k.
\]
Therefore, combining (7.3) and $|\Omega(t)| \geq t$, we obtain
\[ |\{ x \in Q_0 : S_{\pi,\sharp}(f)(x) > f^\sharp(t) \}| \geq 5^{-n} |\Omega(t)| \geq 5^{-n} t. \]
Thus, by the definition of the decreasing rearrangement of a measurable function, it follows that
\[ F_{f,\sharp}(5^{-n} t) \geq S_{\pi,\sharp}(f)^*(5^{-n} t) \geq f^\sharp(t), \quad 0 < t \leq 1. \]
Equivalently,
\[ f^\sharp(5^n t) \leq F_{f,\sharp}(t), \quad 0 < t \leq 5^{-n}. \]
From the latter inequality, (2.8) and the fact that $K(t) := K(t, f; L^1, BMO)$ is an increasing function, we have
\[ K(t) \leq K(5^n t) \simeq 5^n t f^\sharp(5^n t) \leq 5^n t F_{f,\sharp}(t), \quad 0 < t \leq 5^{-n}. \]
Suppose now that $5^{-n} < t \leq 1$. Let us first remark that $K(1) \leq \|f\|_{L^1}$. Indeed, we may assume that $\int_{Q_0} f(x) \, dx = 0$ (see Remark 7.1) and therefore to compute $K(1)$ we can use the decomposition $f = f + 0$, and the assertion follows since
\[ \|f\|_{L^1} = \frac{1}{|Q_0|} \int_{Q_0} |f - f_{Q_0}| \, dx \leq \|f\|_{BMO}. \]
Let us also note that since $\pi = \{Q_0\} \in P$, we have $F_{f,\sharp}(1) \geq \|f\|_{L^1}$. Consequently, using successively the facts that $K(t)$ is increasing, $F_{f,\sharp}(t)$ is decreasing, and $5^n t > 1$, we get
\[ K(t) \leq K(1) \leq \|f\|_{L^1} \leq F_{f,\sharp}(1) \leq 5^n t F_{f,\sharp}(t). \]
Thus, inequality (7.1) holds for all $0 < t \leq 1$ with constant $c = 5^n$. ■

**Remark 7.3.** Let $p \in (0, 1)$. For any family $\pi = \{Q_i\} \in P(Q_0)$ of cubes we let
\[ S_{\pi,\sharp}^p(f)(x) := \sum_i \left( \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}|^p \right)^{1/p} \chi_{Q_i}(x), \]
\[ F_{f,\sharp}^p(t) := \sup_{\pi \in P} (S_{\pi,\sharp}^p(f))^*(t). \]
Then, by a slight modification of the proof of Theorem 7.2 we see that
\[ K(t, f; L^p, BMO) \simeq t F_{f,\sharp}^p(t), \quad 0 < t \leq 1 \]
(cf. [5, Remark 6.3]).

**8. Extensions of the Garsia–Rodemich construction.** We very briefly illustrate some of the results discussed in this paper by showing how adding a parameter to the Garsia–Rodemich construction leads to a connection with the theory of Campanato spaces and the Morrey–Sobolev theorem. We refer to [1] for more information and background.
DEFINITION 8.1. Let $\lambda \in (-n, 0]$ and $1 < p \leq \infty$. We shall say that $f \in L^1$ belongs to $\text{GaRo}_{p,\lambda}$ if there exists a constant $C > 0$ such that for all $\{Q_i\} \in P$,

\begin{equation}
\sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq C \left( \sum_i |Q_i|^{1+\lambda/n} \right)^{1/p'},
\end{equation}

where $1/p' = 1 - 1/p$, and let

$$
\|f\|_{\text{GaRo}_{p,\lambda}} := \inf \{C : \text{(8.1) holds}\}.
$$

Recall the definition of the homogeneous Campanato space $\dot{L}^{1,\lambda}$ (cf. [1, Section 2.2, p. 8]):

DEFINITION 8.2.

$$
\dot{L}^{1,\lambda} := \left\{ f : \|f\|_{\dot{L}^{1,\lambda}} := \sup_{Q \subset Q_0} |Q|^{-\lambda/n} \left( \frac{1}{|Q|} \int_{Q} |f - f_Q| \right) < \infty \right\}.
$$

THEOREM 8.3.

$$
\text{GaRo}_{\infty,\lambda} = \begin{cases} 
\dot{L}^{1,\lambda} & \text{if } \lambda \in (-n, 0), \\
\text{BMO} & \text{if } \lambda = 0.
\end{cases}
$$

Proof. Clearly, it is sufficient to consider the case when $\lambda \in (-n, 0)$.

We will repeatedly use the fact (see [1,4]) that

$$
\frac{1}{|Q|} \int_{Q} \int_{Q} |f(x) - f(y)| \, dx \, dy \simeq \int_{Q} |f(x) - f_Q| \, dx.
$$

Consequently, we can write

$$
\|f\|_{\dot{L}^{1,\lambda}} \simeq \sup_{Q \subset Q_0} |Q|^{-\lambda/n-1} \frac{1}{|Q|} \int_{Q} \int_{Q} |f(x) - f(y)| \, dx \, dy.
$$

Suppose that $f \in \text{GaRo}_{\infty,\lambda}$. Then, since for each $Q \subset Q_0$ we have $\{Q\} \in P$, we see that

$$
\frac{1}{|Q|} \int_{Q} \int_{Q} |f(x) - f(y)| \, dx \, dy \leq |Q|^\lambda n^1 \|f\|_{\text{GaRo}_{\infty,\lambda}}.
$$

Hence,

$$
\|f\|_{\dot{L}^{1,\lambda}} \leq \|f\|_{\text{GaRo}_{\infty,\lambda}}.
$$
Conversely, suppose that \( f \in \dot{L}^{1,\lambda} \) and let \( \{Q_i\} \in P \). We compute
\[
\sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy
\]
\[
= \sum_i |Q_i|^{\lambda/n+1}|Q_i|^{-\lambda/n-1} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy
\]
\[
\leq \|f\|_{\dot{L}^{1,\lambda}} \sum_i |Q_i|^{\lambda/n+1}.
\]
Consequently, \( \|f\|_{\text{GaRo}_{\infty,\lambda}} \leq \|f\|_{\dot{L}^{1,\lambda}} \). \( \blacksquare \)

The import of the Campanato spaces stems from a well known result by Campanato and Meyers (cf. [1, (2.3), p. 9]) that for \( \lambda \in (-1, 0) \),
\[
(8.2) \quad \dot{L}^{1,\lambda}(Q_0) = \text{Lip}(-\lambda)(Q_0).
\]

Let \( \alpha \in (0, 1) \) and \( p \geq 1 \). Define
\[
W^{\alpha,p} := W^{\alpha,p}(Q_0) = \left\{ f : \|f\|_{W^{\alpha,p}} = \left\{ \int_{Q_0} \int_{Q_0} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha p}} \, dx \, dy \right\}^{1/p} < \infty \right\}.
\]

Then we have the classical

**Theorem 8.4.** Let \( p > n/\alpha \). Then
\[
W^{\alpha,p} \subset \text{GaR}_{\infty,n/p-\alpha} = \dot{L}^{1,n/p-\alpha} = \text{Lip}(\alpha - n/p).
\]

**Proof.** Note that \( -1 < n/p - \alpha < 0 \). In view of Theorem [8.3] (8.1) and (8.2), for any cube \( Q \subset Q_0 \) we need to estimate from above the quantity
\[
I := |Q|^{\alpha/n-1/p-1} \frac{1}{|Q|} \int_{Q} \int_{Q} |f(x) - f(y)| \, dx \, dy.
\]
We proceed as follows:
\[
I \leq |Q|^{\alpha/n-1/p-2}|Q|^{\frac{n+\alpha p}{n p}} \int_{Q} \int_{Q} \frac{|f(x) - f(y)|}{|x - y|^{(n+\alpha p)/p}} \, dx \, dy
\]
\[
\leq |Q|^{\alpha/n-1/p-2+1/p+\alpha/n}|Q|^{2(1-1/p)} \left\{ \int_{Q} \int_{Q} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha p}} \, dx \, dy \right\}^{1/p}
\]
(by Hölder’s inequality)
\[
\leq |Q|^{2(\alpha/n-1/p)} \|f\|_{W^{\alpha,p}} \leq |Q_0|^{2(\alpha/n-1/p)} \|f\|_{W^{\alpha,p}},
\]
as we wished to prove. \( \blacksquare \)

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