Experimental setup for the evaluation of large displacements in the inflected beams sustained to ground

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Abstract. In engineering practice, it often occurs to tackle structures with systems of forces such as to induce displacements beyond the linear response. If the material remains in the elastic field, the behaviour is generally addressed as geometric non-linearity. Taking into consideration beam structures subjected to distributed and concentrated loads, the differential equations that describe the structural behavior need to be solved with numerical methods (e.g. Runge-Kutta algorithm or F.E.M.) or analytical perturbative approaches. This work presents experimental results concerning a beam, sustained to the ground, loaded to own weight and lifted by concentrated loads at the free-end. The loads are such as to cause very large displacements. The work describes some useful tricks to carry out the tests with good precision and repeatability. Several experimental data are compared with numerical results.

1. Introduction

Beams are structural elements widely used in various sectors of mechanical and civil engineering. In most cases, the structures are subjected to loads such as to cause small displacements. However, some force systems can trigger large displacements, even if the material remains in its elastic domain. These structures with elastic geometrically nonlinear behavior are often called Highly Flexible Structures (HFS). HFS has been used in many mechanical, aircraft and widely in spatial structural systems. Generally, they are used in any structural application where it is mandatory to satisfy space limits, reduce structural weight and / or provide special mechanisms. The behavior, in this case, is nonlinear and the differential equations that describe the phenomenon must take into account the significant variation of the geometry. Looking at the equilibrium, a nonlinear differential system of equations result. The expression that describes the curvature of the element cannot be linearized and therefore the rotation angle cannot be confused with its tangent, and this creates difficulties for a closed-form integration. Recently, the nonlinear beam theory has been used in the study of compliant mechanisms, such as soft robots, medical or surgical instruments, flexible electronics and MEMS (Micro-Electro-Mechanical Systems) [1] and [2].

In the past, many famous researchers have faced the large rotation and displacements beam problem, first of all Bernoulli and Euler. L. Saalschütz, in 1880 [3] solves the differential equations of the geometrically nonlinear-beam, loaded only by concentrated force at the extremes, by expressing the curvature as a function of the curvilinear abscissa and by making use of the Legendre forms of elliptic integrals. More recently, Barten [4] and Bisshop and Drucker, [5], that solve the problem with an
approach similar to that of Saalschütz, using the first and second complete and incomplete elliptical integrals, have revisited the solution of the cantilever Elastica problem. Since then, other works based on analytical solutions (also different from the use of elliptical integrals) are found, for example, in [6], [7], [8] and [9].

An extension of the Elastica problem, is that of Heavy Elastica (i.e. when the beam is loaded by distributed loads), that unlike the previous one, does not have a closed form solution. Rohde in 1953 [10] proposes an analytical solution of the Heavy Elastica based on a perturbative approach (Taylor Series expansion). Many other authors follow the Rohde approach to face this case with different boundary conditions, but Wang C.Y. in [11] and [12] proved that the series of functions obtained through the perturbative approach does not converge for large displacements. Following this, the perturbative approach was gradually abandoned, in favor of numerical approaches [13], [14] and [15].

In this paper, we investigate experimentally the problem of the beam sustained to the ground, loaded by own weight and lifted by concentrated load and bending moment at the free-end. This problem is treated in [16] using the perturbative approach previously mentioned, in the case of non-inflectional Elastica (no inflection point), without extremal bending moment applied. Kooi and Kuipers [17], address the case with an inflection point with an extremal bending moment applied, using the finite difference method coupled with the Newton-Raphson method, and in [18] using F.E.M., in particular beam elements and gap elements.

The simplest mathematical model of the curved beam with finite displacements and rotations can be obtained assuming:

- Linear elastic, homogeneous and isotropic material;
- The deformation shear and extensional energy are neglected as much smaller of bending deformation energy (thin beam)
- Invariability of the beam section under deformation.

Downstream of these, considering a thin beam i.e. \( d(s) \cdot k(s) \ll 1 \ \forall \ s \in [0,L] \), where \( d \) is the maximum distance from the neutral axis, \( k(s) \) the curvature, \( s \) the curvilinear abscissa and \( L \) the length of the beam, the expressions of the linear bending-curvature relationship is derived (Clebsch hypothesis). Using the previous assumptions in the equilibrium equations, the integro-differential system is obtained:

\[
E(s)I(s) \left( \frac{d^2 \psi(s)}{ds^2} - \frac{d^2 \psi_{in}(s)}{ds^2} \right) + \frac{dE(s)}{ds}I(s) \left( \frac{d\psi(s)}{ds} - \frac{d\psi_{in}(s)}{ds} \right) + \nonumber \\
+ \left( F_x + \int_{s}^{L} q_x d\bar{s} \right) \sin \psi(s) + \left( F_y + \int_{s}^{L} q_y d\bar{s} \right) \cos \psi(s) = 0 \quad (1)
\]

\[
x(s) = \int_{0}^{s} \cos \psi(s) \ ds \quad (2)
\]

\[
y(s) = \int_{0}^{s} \sin \psi(s) \ ds \quad (3)
\]

Referring to the cantilever beam of figure 1, indicating with \( s \) the curvilinear abscissa of the beam, with \( \psi_{in}(s) \) the initial tangent angle at any point by respect to \( x-axis, \psi(s) \) the variation of the tangent
angle after deformation, and with $q_x$ and $q_y$ the horizontal and vertical distributed loads respectively, the generic differential equation of the static equilibrium is shown in equation (1).

The complete development of these equations is described in detail in [19].

The integration of equation (1) requires knowledge of the limit values for $\psi(s)$ e $\psi'(s)$ on specific points. Generally, these points are different (e.g. for a cantilever beam with extremal concentrated force $\psi(s) = 0$ on the constraint and $\psi'(s) = 0$ on the free-end), this implies that the boundary condition are mixed (Dirichlet and Neumann boundary condition) and this greatly increases the difficulty of a numerical solution of this problem. Downstream of this, each numerical method used to solve the differential system must be coupled with a numerical attempt (shooting) algorithm.

Unlike the extensive analytical and numerical literature present for these problems, there are very few experimental validations in this regard. In particular, the authors are not aware of any experimental investigation regarding the topic of Heavy Elastica ground-sustained, loaded by own weight and lifted by concentrated load and bending moment. For a detailed discussion of the boundary conditions for non-inflectional and inflectional cases, see [16] and [17].

The kinematic measurements of structures that undergo large displacements are not always easy to carry out; the major difficulties are due to the structure size and the necessity to realize accurate and repeatable measurements. This work aims to indicate a possible technical/experimental approach, useful for the verification and validation of the numerical method. Techniques for displacement measuring on slender, rectangular section beams, partially supported on the ground, are presented and described. The beam is subject to its own weight and lifted to an extreme, employing end loads. Two case studies are presented in the paper, focusing the discussion on the experimental precautions useful for implementing the distributed loads, for accurately determining the necessary displacements, and the search of the last point in contact with the ground. The kinematic values are processed through the mixed use of photographic techniques and laser pointers, while the loads are determined by appropriate positioning.
of load cells. The experimental values are compared with results obtained from analytical and numerical models.

2. Experimental Setup

This section describes the two types of tests performed. Both tests consist of a beam initially resting on the ground. By applying loads, the beam is partially raised above the ground.

The first test, called Case 1, consists in imposing a known displacement at the free end of a beam ground sustained, leaving free the degree of freedom of free-end rotation, so that any inflection-point (i.e. point where the curvature changes sign) does not occur. The test scheme is shown in figure 2. The imposed displacement is high, and the final configuration assumed by the beam present a partial contact on the ground. The raised vertical free end is also subjected to a considerable horizontal backward movement. The resulting lifting force is detected by a load cell, and a cinematic apparatus is able to supply its vertical direction by following the structure in its movement. The force is supplied progressively until the desired vertical displacement is reached. This condition is detected by installing a laser optical sight that signals the achievement of the desired displacement (see figure 4, a)). The force measurement thus detected is the consequence of the geometrical nonlinear response of the structure. The alignment of the vertical force is guaranteed by a system of carriages, one mobile and one fixed, see the diagram in figure 2.

![Figure 2. Scheme of Case 1.](image)

To obtain a beam with a different density, a simple trick is used. A system of little weights glued and piled on the upper side of the beam are regularly positioned. The weights employed are those used in the tire balancing and, thanks to their shape, they can be piled one over the other. This allows an increase of the distributed vertical load, without introducing a further flexural rigidity, see figure 3.
In the deformed configuration, the system will be partially lifted from the ground and a portion remains in contact with it. The residual length that remains in contact is a crucial measurement. The point of contact estimate is provided by sliding a thin calibrated sheet of paper under the beam. The measurement is obtained by sliding it under the beam, starting from the portion raised from the ground towards the portion in contact with it. Upon reaching the contact point, the movement of the sheet is prevented, thus detecting the desired point. To make this measurement as repeatable as possible, contact recognition must take place with the same force with which the sheet is pulled. This aspect was solved by using a magnet: the sheet of paper is dragged by two wires at the ends of which there are two metal elements attracted by a magnet. When the sheet fits between the beam and the ground, and the magnet attraction force is reached, the metallic heads disengage, evidencing the distance sought. Figure 5 shows a scheme of the described technique.

![Figure 3. View of the piling of four weight](image)

**Figure 3.** View of the piling of four weight

![Figure 4. Case 1: a) view of laser target; b) view of lifted beam with four tiers of weight.](image)

**Figure 4.** Case 1: a) view of laser target; b) view of lifted beam with four tiers of weight.
The second test, called Case 2, differs from the first by the presence of a further bending moment (oriented as to lower the free-end) applied in its extremal point provided by a pair of forces. The applied moment is such as to force the end point to be horizontally tangent. The configuration provides, after deformation, an inflection point. Figure 6 shows the scheme of the experiment.

The distributed vertical distributed force is increased by four piled weights. The use of a laser pointer was necessary for the evaluation of the tangent angle of the beam at the point of maximum height, which corresponds to the point where the vertical displacement is applied. This configuration involves, as already anticipated, an inflection-point on the beam line raised from the ground. The point of contact with the ground was detected with the sheet technique already described for Case 1. In figure 7 a picture
of the experiment is shown. Force $F_1$ and $F_2$ are detected by load cells. The distance between the couple of forces allows the estimate of the moment applied.

![Figure 7. Picture of experimental Case 2.](image)

3. Results

This section shows the comparison of the numerical results with the two experimental cases described in the previous sections. Both refer to a rectangular section steel beam whose dimensions are 2.95 mm x 19.95 mm; the length is 2670 mm for test Case 1 and 2400 mm for the other test Case 2. The density of the steel turns out to be 7763 kg/m$^3$ and consequently, the weight alone implies a vertical distributed load equal to 4.4819 N/m. The weights used have a size of 4x11x19 mm$^3$ and a weight of 5 g each, they are mounted with a pitch of 15 mm, as shown in figure 3. By stacking the weights, a discrete increase in the distributed load is achieved creating a distributed load law according to the stacking level $n$:

$$q_y = q_w + 3.27 \cdot n$$  \hspace{1cm} (4)

Where with $q_y$ indicates the total vertical distributed load, $q_w$ the distribution load belonging to the beam, and $n$ is the number of weights piled one over the other.

For Case 1, three elevation value are considered: 20, 40 and 60 cm. The measurements refer to the force required to apply the imposed displacement. Three tests are repeated for each elevation level, accounting of the arithmetic average. In the overall, the distributed loads applied considers 5 subcases: own weight, 1,2,3,4 rows of stacked weights. The experimental data refer to the force required $F$ to maintain the vertical displacement, and the measurement ($L_{tot} - L$) relative to the residual length of the beam in contact with the ground. For the validation of each test, an accurate vertical alignment checks of the cable that performs the pull upwards is guaranteed. The experimental results are compared with Runge-Kutta integration method solving the equations (1), (2) and (3), and the Finite Elements Method. Experimental data of Case 1 are provided in tables 1, 2, 3, 4, 5. Each table refers to a case of distributed load, in particular for table 1 the experimental setup provides no additional weights. As shown in the tables, the error on the distance from the beginning of the detachment is caught very well, with errors of max 3%.

It is interesting to remark that the comparison is slightly affected by the number of weights piled, probably because the added weights are progressively distanced by the beam axis, introducing an approximation in the numerical and analytical estimate.

For Case 2, a single distributed load equivalent to the application of 4 rows of weights distributed along the beam was used. The vertical displacement imposed is 50.8 cm and the force is 24.927 N. The resulting torque is 11.755 Nm. The vertical and reaction forces were calculated considering the effect...
caused by the weight of the portion of the beam between the two forces, and consequently the resulting torque.

Particular attention must be paid to the implementation of the contact in the R-K and FEM model. It is crucial to impose a contact-threshold value to indicate to the solver when to consider active contact. As a matter of fact, the experimental measurement was conducted with a thin paper whose presence may alter the real contact position. Considering the numerical solutions, it is easy to take into account this discrepancy. If not considered, the effect can lead to discordant results. The values shown in the table are obtained with $10^{-4} m$ of contact-threshold.

Table 6 shows the experimental and numerical results relative to testing Case 2. In this case, the kinematic parameters are highlighted, such as the maximum height of the beam, the $F_1$ force reached and the corresponding equivalent moment at the point of maximum height. The slope angle reached by the aforementioned point is also indicated. In this last case the matching reaches its worst. This is probably due to the difficulties in the measurement of the bending moment whose effect has however a deep influence, caused mainly the inflection point presence, on the beam elevation.

### Table 1. Case 1: experimental data and comparison: self-weight.

| $Y_L$ [cm] | $F$ [N] | $L_{tot}$ $L$ [cm] | $F$ [N] | $L_{tot}$ $L$ [cm] | $\alpha$ [deg] | $F$ [N] | $L_{tot}$ $L$ [cm] | $\alpha$ [deg] |
|------------|---------|---------------------|---------|---------------------|----------------|---------|---------------------|----------------|
| 20 1°      | 3.924  | 103.3               | 3.943  | 103.76              | 13.30          | 3.971  | 102.34              | 12.99          |
| 3°         | 3.972  | 103.3               | 4.954  | 71.9                |                | 4.799  | 70.72              | 21.87          |
| 40 2°      | 4.856  | 71.8                | 4.905  | 71.78              | 22.20          | 4.799  | 70.72              | 21.87          |
| 3°         | 4.905  | 71.7                | 5.395  | 39.8                |                | 5.425  | 40.93              | 29.82          |
| 60 2°      | 5.543  | 39.7                | 5.454  | 39.70              | 30.40          | 5.425  | 40.93              | 29.82          |
| 3°         | 5.444  | 39.7                |        |                     |                | 5.435  | 40.96              | 29.69          |
Table 2. Case 1: experimental data and comparison: self-weight and one tier of weight.

| Y₁ [cm] | #  | F  [N] | Lₘ₀-L [cm] | F  [N] | Lₘ₀-L [cm] | α  [deg] | F  [N] | Lₘ₀-L [cm] | α  [deg] | F  [N] | Lₘ₀-L [cm] | α  [deg] |
|---------|----|--------|------------|--------|------------|----------|--------|------------|----------|--------|------------|----------|
| 1°      | 5.886 | 120.5  | 5.906      | 120.53 | 6.005      | 122.5    | 6.013  | 122.64     | 14.98    |
| 2°      | 5.886 | 120.5  | 5.906      | 120.53 | 6.005      | 122.5    | 6.013  | 122.64     | 14.98    |
| 3°      | 5.935 | 120.6  | 5.935      | 120.6  | 5.935      | 120.6    | 5.935  | 120.6      | 5.935    |
| 1°      | 7.210 | 93.8   | 7.210      | 93.8   | 7.210      | 93.8     | 7.210  | 93.8       | 7.210    |
| 2°      | 7.259 | 94.5   | 7.220      | 94.05  | 7.297      | 95.25    | 7.308  | 95.37      | 25.16    |
| 3°      | 7.210 | 93.9   | 7.210      | 93.9   | 7.210      | 93.9     | 7.210  | 93.9       | 7.210    |
| 1°      | 8.387 | 70.8   | 8.387      | 70.8   | 8.387      | 70.8     | 8.387  | 70.8       | 8.387    |
| 2°      | 8.486 | 70.4   | 8.397      | 70.73  | 8.364      | 68.64    | 8.328  | 68.68      | 8.342    |
| 3°      | 8.338 | 71.0   | 8.338      | 71.0   | 8.338      | 71.0     | 8.338  | 71.0       | 8.338    |

Table 3. Case 1: experimental data and comparison: self-weight and two tiers of weight.

| Y₁ [cm] | #  | F  [N] | Lₘ₀-L [cm] | F  [N] | Lₘ₀-L [cm] | α  [deg] | F  [N] | Lₘ₀-L [cm] | α  [deg] | F  [N] | Lₘ₀-L [cm] | α  [deg] |
|---------|----|--------|------------|--------|------------|----------|--------|------------|----------|--------|------------|----------|
| 1°      | 7.750 | 126.4  | 7.711      | 126.83 | 7.833      | 129.82   | 7.848  | 129.93     | 16.36    |
| 2°      | 7.701 | 127.5  | 7.711      | 126.83 | 7.833      | 129.82   | 7.848  | 129.93     | 16.36    |
| 3°      | 7.701 | 126.6  | 7.701      | 126.6  | 7.701      | 126.6    | 7.701  | 126.6      | 7.701    |
| 1°      | 9.221 | 106.8  | 9.221      | 106.8  | 9.221      | 106.8    | 9.221  | 106.8      | 9.221    |
| 2°      | 9.270 | 106.8  | 9.470      | 106.73 | 9.572      | 105.41   | 9.584  | 105.59     | 27.45    |
| 3°      | 9.968 | 106.6  | 9.968      | 106.6  | 9.968      | 106.6    | 9.968  | 106.6      | 9.968    |
| 1°      | 10.987 | 78.7   | 10.987     | 78.7   | 10.987     | 78.7     | 10.987 | 78.7       | 10.987   |
| 2°      | 10.840 | 79.0   | 10.918     | 78.85  | 10.961     | 79.96    | 10.987 | 79.93      | 37.08    |
| 3°      | 10.938 | 78.9   | 10.938     | 78.9   | 10.938     | 78.9     | 10.938 | 78.9       | 10.938   |
### Table 4. Case 1: experimental data and comparison: self-weight and three tiers of weight.

| Y_L [cm] | #   | F [N] | L_total-L [cm] | F [N] | L_total-L [cm] | α [deg] | F [N] | L_total-L [cm] | α [deg] | F [N] | L_total-L [cm] | α [deg] |
|----------|-----|-------|----------------|-------|----------------|---------|-------|----------------|---------|-------|----------------|---------|
| 1°       | 20  | 9.221 | 137.8          | 9.241 | 137.73         | 17.60   | 9.562 | 138.20         | 17.43   | 9.565 | 138.24         | 17.49   |
|          | 3°  | 9.270 | 138.6          |       |                |         |       |                |         |       |                |         |
|          | 1°  | 11.330| 115.8          |       |                |         |       |                |         |       |                |         |
| 40       | 2°  | 11.232| 116.9          | 11.330| 116.50         | 29.90   | 11.694| 115.39         | 29.20   | 11.703| 115.40         | 29.29   |
|          | 3°  | 11.429| 116.9          |       |                |         |       |                |         |       |                |         |
|          | 1°  | 13.145| 96.4           |       |                |         |       |                |         |       |                |         |
| 60       | 2°  | 13.047| 96.5           | 13.126| 96.40          | 39.80   | 13.456| 94.93          | 39.21   | 13.489| 94.98          | 39.53   |
|          | 3°  | 13.194| 95.3           |       |                |         |       |                |         |       |                |         |

### Table 5. Case 1: experimental data and comparison: self-weight and four tiers of weight.

| Y_L [cm] | #   | F [N] | L_total-L [cm] | F [N] | L_total-L [cm] | α [deg] | F [N] | L_total-L [cm] | α [deg] | F [N] | L_total-L [cm] | α [deg] |
|----------|-----|-------|----------------|-------|----------------|---------|-------|----------------|---------|-------|----------------|---------|
| 1°       | 20  | 10.791| 141.8          | 10.869| 141.70         | 18.50   | 11.157| 144.79         | 18.29   | 11.164| 144.80         | 18.37   |
|          | 3°  | 10.889| 141.8          |       |                |         |       |                |         |       |                |         |
|          | 1°  | 13.293| 120.8          |       |                |         |       |                |         |       |                |         |
| 40       | 2°  | 13.194| 120.6          | 13.175| 120.63         | 30.70   | 13.730| 118.68         | 30.78   | 13.734| 118.64         | 30.82   |
|          | 3°  | 13.047| 120.5          |       |                |         |       |                |         |       |                |         |
|          | 1°  | 15.451| 104.6          |       |                |         |       |                |         |       |                |         |
| 60       | 2°  | 15.304| 104.8          | 15.352| 104.67         | 41.50   | 15.882| 102.71         | 41.24   | 15.902| 102.88         | 41.57   |
|          | 3°  | 15.304| 104.6          |       |                |         |       |                |         |       |                |         |
Table 6. Case 2: experimental data and comparison with a vertical load and a bending moment.

|                | Experimental data | Runge Kutta | FEM (1000 beam elements) |
|----------------|-------------------|-------------|--------------------------|
| Run 1 - \(\psi_L = 0\) |                   |             |                          |
| \(y_L\) [cm]   | 50.80             | 50.82       | 50.52                    |
| \(\alpha\) [°] | 0                 | -0.12       | 0.18                     |
| L_{tot} [cm]   | 37.25             | 37.20       | 37.16                    |
| F [N]          | 24.927            | 24.920      | 24.927                   |
| M [Nm]         | 11.755            | 12.526      | 12.30                    |
| Run 2 - \(\psi_L < 0\) |                   |             |                          |
| \(y_L\) [cm]   | 50.80             | 50.31       | 50.28                    |
| \(\alpha\) [°] | -6.785            | -7.10       | -6.79                    |
| L_{tot} [cm]   | 25.50             | 26.78       | 26.64                    |
| F [N]          | 26.095            | 26.375      | 26.000                   |
| M [Nm]         | 13.967            | 14.690      | 14.437                   |

4. Conclusions
In this work, the Heavy Elastica lifted by concentrated load and ground sustained problem is investigated with evidences of an experimental campaign. Results are compared with Runge-Kutta integration method and F.E.M. results. The measurements regard a beam soil-supported in the order of one fourth of the total length. The weight of the beam was discretely varied by means of stacked weights applied in a regular manner along the beam, so that as to simulate an increased self-weight. The application of the additional weights is such as to keep unchanged the beam flexibility. The kinematic quantities of the loaded configuration are detected with laser targets that allowed a reliable repeatability of the measurements. The resulting applied forces are obtained from a single load cell or a couple of them, suitably arranged. The evaluation of the distance between the part of the beam in stable contact and the initial point of detachment is a fundamental parameter, for comparison purposes. It was obtained by means of a system composed of a sheet of paper of known thickness pulled by a magnet that guarantees a repeatable detachment force. Experimental evaluations, although limited to particular cases, provide a starting point for further experiments to be undertaken, which can be conducted with a high reliability reached with low cost instrumentation.

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