Recursive transfer-matrix method for second-harmonic generation in a one-dimensional nonlinear photonic crystal at arbitrary incidence angle

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Abstract. In this paper, we derive a recursive transfer-matrix method (RTMM) and use the method to analyse the problem of second-harmonic generation (SHG) in a one-dimensional nonlinear photonic crystal (1D-NPC) under oblique incidence of fundamental-frequency (FF) wave and nondepleted pump approximation. This method is very simple and useful for determining SHG conversion efficiency at various incident angles of FF wave and thicknesses of nonlinear layer. The RTMM calculation results show that a maximum SHG conversion efficiency can be produced by the strong local FF field confinement in 1D-NPC due to optimal incidence angle and nonlinear layer thickness.

1. Introduction
Photonic crystals (PCs) are artificial optical structure, which can control the various behaviors of light propagating inside the structure. The propagation of light through PCs can be differently analyzed comparing with light propagating through free-space or homogeneous material [1]. There are many numerical methods to characterize the light propagation behaviors in the periodic structures like PCs. A well-known numerical method that use to compute the output wave from PC is the transfer matrix method (TMM) [2]. TMM could be deployed to study optical properties i.e. transmittance, reflectance, absorbance, and distribution of electric field in linear and nonlinear PC structures. Furthermore, TMM was modified and developed to RTMM for handling the nonlinear SHG problems because this method is simple, fast, and accurate method [3-4].

In this work, we formulated the calculation of phase-matched SHG conversion efficiency in the 1D-NPC structure, which composed of periodically alternating of Gallium Arsenide (GaAs) and Aluminum Arsenide (AlAs) layers, by using RTMM under nondepleted pump approximation and the oblique incidence of FF and SH waves. Furthermore, the maximum SHG conversion efficiency can be achieved by optimizing the thickness of composed layer and incident angle of FF wave.

2. Rtmm for calculating shg at oblique incidence of ff in 1d-npc structure
The simple RTMM algorithm can be used to calculated the distribution of electric and magnetic field and also transmission and reflection spectra of an incident FF and generated SH plane waves with particular polarization state inside the 1D-NPC structure. Figure 1 shows a schematic of 1D-NPC structure with \( N \) layers (\( N \) is even number). The \( i^{\text{th}} \) layer is assumed to have a thicknesses \( d_i \), refractive
index $n_i^{(1,2)}$ (with superscripts 1, 2 representing FF and SH field, respectively), and $\chi_i^{(2)}$ is represented an second-order nonlinear susceptibility. In this study, the alternated layers have the same thickness and material properties, i.e., the refractive indices $n_i^{(1)}$ and $n_i^{(2)}$ correspond FF and SH, the odd layers have thicknesses $d_i$ and even layers have thicknesses $d_{i+1}$. Both TM- and TE-polarized incident waves have the arbitrary incident angle $\theta_0$ from the incident medium labelled with a subscript “0”. Meanwhile, the substrate is indicated by the subscript “s”. So, $n_i$ and $n_s$ represent the refractive indices of the incident and substrate medium, respectively, for both FF and SH waves.

Under nondepleted FF pump approximation, the spatial evolution of FF and SH waves in $i$th layer can be retrieved from solving of coupled equations for SHG as below:

\[
\frac{d^2 E_i^{(1)}}{dz^2} + (k_W^{(1)})^2 E_i^{(1)} = 0, \tag{1a}
\]

\[
\frac{d^2 E_i^{(2)}}{dz^2} + (k_W^{(2)})^2 E_i^{(2)} = -(k_W^{(2)})^2 \chi_i^{(2)} (E_i^{(1)})^2, \tag{1b}
\]

where $\chi_i^{(2)}$ is the second-order nonlinearity, $k_W^{(1)} = n_i^{(1)} \cos \theta_i^{(1)} \omega / c$ and $k_W^{(2)} = n_i^{(2)} \cos \theta_i^{(2)} (2\omega) / c$ are the $z$-component of wave-vector of FF and SH wave, and $\theta_i^{(1)}$ and $\theta_i^{(2)}$ are the propagating angle of FF and SH wave in the $i$th layer, respectively.

In case of incident FF wave with TM-polarization, the solutions corresponding to equation (1) for calculating $E_i^{(1)}$ and $E_i^{(2)}$ under oblique incidence are given as

\[
E_i^{(1)}(z) = a_i^{(1)} \cos \theta_i^{(1)} e^{-ik_W^{(1)}z} + b_i^{(1)} \cos \theta_i^{(1)} e^{-ik_W^{(1)}z}, \tag{2a}
\]

\[
E_i^{(2)}(z) = a_i^{(2)} \cos \theta_i^{(2)} e^{-ik_W^{(2)}z} + b_i^{(2)} \cos \theta_i^{(2)} e^{-ik_W^{(2)}z} + F_i^{(1)} e^{-2ik_W^{(1)}z} + F_i^{(-1)} e^{-2ik_W^{(1)}z} + F_i^{(2)}, \tag{2b}
\]

where $z_{i-1} < z < z_i$. In equation (2b), $a_i^{(1)}$, $b_i^{(1)}$, $a_i^{(2)}$ and $b_i^{(2)}$ represent the forward and backward electric field amplitude of FF and SH in $i$th layer, respectively. Meanwhile, the parameters

![Figure 1](image-url)
\[ F_i^{(+)} = -\frac{(a_i^{(2)}k_0^{(2)} \cos \theta_0^{(2)})^2 Z_i^{(2)}}{(k_i^{(2)})^2 - 4(k_i^{(1)})^2}, \quad \text{(3a)} \]
\[ F_i^{(-)} = -\frac{(b_i^{(2)}k_0^{(2)} \cos \theta_0^{(2)})^2 Z_i^{(2)}}{(k_i^{(2)})^2 - 4(k_i^{(1)})^2}, \quad \text{(3b)} \]
\[ F_i^{(2z)} = -\frac{2(k_i^{(2)} \cos \theta_0^{(2)})^2 a_i^{(2)} b_i^{(2)} Z_i^{(2)}}{(k_i^{(2)})^2}, \quad \text{(3c)} \]

where \( k_0^{(2)} = 2\alpha t^2 / c \) and \( \theta_0^{(2)} = \sin^{-1}(2k_i^{(1)} n_1^{(1)} \sin \theta_i^{(1)} / k_i^{(2)} n_0) \).

Using boundary conditions, the forward and backward electric field amplitude of SH field at a single nonlinear layer are written in matrix form as follows:
\[
\begin{pmatrix} a_i^{(2)} \\ b_i^{(2)} \end{pmatrix} = M_i^{(2)} \begin{pmatrix} a_0^{(2)} \\ b_0^{(2)} \end{pmatrix} + G_0^{-1} \begin{pmatrix} B_i H_i - N_i B_i \\ H_i F_i^{(+)} - I - N_i F_i^{(2z)} \end{pmatrix},
\]
\[
\text{where } M_i^{(2)} = G_0 N_i G_i^{-1}, \quad N_i = G_i Q_i G_i^{-1}, \quad G_0 = \begin{pmatrix} 1 & 1 \\ n_0 & -n_0 \end{pmatrix}, \quad G_i = \begin{pmatrix} 1 & 1 \\ k_i^{(2)} / k_0^{(2)} & -k_i^{(2)} / k_0^{(2)} \end{pmatrix},
\]
\[
Q_i = \begin{pmatrix} e^{ik_i^{(2)} z} & 0 \\ 0 & e^{-ik_i^{(2)} z} \end{pmatrix}, \quad B_i = \begin{pmatrix} 1 & 1 \\ 2k_i^{(2)} / k_0^{(2)} -2k_i^{(2)} / k_0^{(2)} \end{pmatrix}, \quad \text{and } H_i = \begin{pmatrix} e^{2ik_i^{(2)} z} & 0 \\ 0 & e^{-2ik_i^{(2)} z} \end{pmatrix}, \quad i = 1, 2, 3, \ldots
\]

For \( N \)-layer system, the formula of RTMM for calculating the forward and backward SH electric field amplitude can be derived by equation (4) as:
\[
\begin{pmatrix} a_{s}^{(2)} \\ b_{s}^{(2)} \end{pmatrix} = M_{s}^{(2)} \begin{pmatrix} a_0^{(2)} \\ b_0^{(2)} \end{pmatrix} + \begin{pmatrix} R_{s}^{(+)} \\ R_{s}^{(-)} \end{pmatrix},
\]
\[
\text{and } \begin{pmatrix} R_{s}^{(+)} \\ R_{s}^{(-)} \end{pmatrix} = N_s \begin{pmatrix} R_{0}^{(+)} \\ R_{0}^{(-)} \end{pmatrix} + G_0^{-1} \begin{pmatrix} B_i H_i - N_s B_i \\ H_i F_i^{(+)} - I - N_i F_i^{(2z)} \end{pmatrix},
\]
\[\text{where } \begin{pmatrix} R_{0}^{(+)} \\ R_{0}^{(-)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Using the boundary conditions \( a_i^{(2)} = 0 \) and \( b_i^{(2)} = 0 \), all of the coefficients for \( a_i^{(2)} \) and \( b_i^{(2)} \) can be computed by using the derived RTMM formula numerically. Similarly, the RTMM formula for SHG process with incident FF wave in TE-mode can be derived with the same approach as shown in TM-mode case.

3. Numerical results and discussions

In this work, 1D-PNC is composed of 50-periods of GaAs and AlAs layers. The incident FF wave, which has wavelength 1,600 nm, is consisted of both TM- and TE-polarization. The corresponding second-order susceptibilities for GaAs is \( \chi^{(2)} = 2d_{14} \sin \theta \cos \theta \) and for AlAs is zero, where the nonlinear coefficient \( d_{14} \) is 200 pm/V and \( \theta = \theta_0^{(1)} \) denotes the propagation angle of FF wave with respect to \( z \) axis of the structure. Figure 2a and 2b shows the numerical results of internal conversion efficiencies of generated SH wave with TE- and TM-polarization from 1D-PNC as function of various incident angle of FF wave (\( \theta_0 = 0^\circ - 30^\circ \)) and thickness of GaAs layer (\( d_l = 140.5 - 143.5 \text{ nm} \)), while the thickness of AlAs layer (\( d_2 \)) is fixed at 92 nm. In addition, the refractive indices for FF are \( n_1^{(1)} = 3.3608 \),
\[ n_1^{(1)} = 2.8942 \quad \text{while for SH are} \quad n_1^{(2)} = 3.6641, \quad n_2^{(2)} = 3.039 \quad \text{and for the incident and substrate medium are assumed} \quad n_0 = 1 \quad \text{and} \quad n_s = 1 \quad \text{respectively} \ [5].\]

According to figure 2(a), the maximum SH conversion efficiency in case of TE-polarization is about \(5.182 \times 10^{-4} \text{V}^2/\text{m}^2\) at optimal layer thickness of GaAs: \(d_1 = 142.7 \text{ nm} \quad \text{and optimal incident angle} \quad \theta_0 = 21^\circ.\) Meanwhile, the maximum SH conversion efficiency for TM-polarization is \(4.545 \times 10^{-4} \text{V}^2/\text{m}^2\) for optimal layer thickness of GaAs: \(d_1 = 142.5 \text{ nm} \quad \text{and optimal incident angle} \quad \theta_0 = 18^\circ\) as shown in figure 2(b). It is clear that the incident angle is important parameter to achieve the maximum SH conversion efficiencies due to the second-order susceptibility is function of \(\sin \theta_1^{(1)} \cos \theta_2^{(1)}\). Moreover, the 1D-NPC structure provides the enhancement of SHG process with local FF field-enhancement from electric field resonance due to multi-reflections of FF wave inside the structure.

![Figure 2](image-url)  
**Figure 2.** Calculated internal SHG conversion efficiencies versus the incident angle \(\theta_0 \quad (0^\circ - 30^\circ)\) of FF wave and the layer thickness of GaAs. For \(d_2 = 92 \text{ nm}\), wavelength at \(\lambda_{FF} = 1,600 \text{ nm}\) and value of \(d_{14}\) is 200 pm/V, (a) conversion efficiency of TE-polarized SH wave and (b) conversion efficiency of TM-polarized SH wave.

### 4. Conclusions

In this work, a formula of RTMM for calculating the SH waves, which generated by an oblique incident FF wave in a 50-periods one-dimensional GaAs/AlAs nonlinear photonic crystal, was derived by using concepts of TMM and boundary conditions. According to the numerical results under nondepleted pump approximation, we found that the SHG conversion efficiencies for TM- and TE-polarized SH waves are dependent on incident angle of FF and thickness of nonlinear layer. The phase-matched condition could be achieved by optimizing both incident angle and thickness of nonlinear layer.

### References

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