Semiclassical Energy Levels of Sine–Gordon Model on a Strip with Dirichlet Boundary Conditions

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Abstract

We derive analytic expressions of the semiclassical energy levels of Sine–Gordon model in a strip geometry with Dirichlet boundary condition at both edges. They are obtained by initially selecting the classical backgrounds relative to the vacuum or to the kink sectors, and then solving the Schrödinger equations (of Lamé type) associated to the stability condition. Explicit formulas are presented for the classical solutions of both the vacuum and kink states and for the energy levels at arbitrary values of the size of the system. Their ultraviolet and infrared limits are also discussed.

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1 Introduction

Since their introduction in the seminal works [1, 2], semiclassical methods have proved to be efficient tools for analysing non-perturbative effects in a large class of quantum field theories. Based on this approach, there have been recently new developments concerning form factors at a finite volume [3], non–integrable models [4] and energy levels of a quantum field theory on a cylinder geometry [5]. As we show in the following, the analysis done in [5] also admits an interesting generalization to a quantum field theory defined on a strip of width $R$, with certain boundary conditions at its edges. The example discussed here is the Sine–Gordon model, defined by the Lagrangian

$$ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{\beta^2} (1 - \cos \beta \phi) , $$ (1.1)

subjected to the Dirichlet boundary conditions (D.b.c.)

$$ \phi(0,t) = \phi_0 + \frac{2\pi}{\beta} n_0 , \quad \phi(R,t) = \phi_R + \frac{2\pi}{\beta} n_R , \quad \forall t $$ (1.2)

with $0 \leq \phi_{0,R} < \frac{2\pi}{\beta}$ and $n_{0,R} \in \mathbb{Z}$. The topological charge of this model is conserved also in the presence of boundaries and it can be conveniently defined as

$$ Q = \frac{\beta}{2\pi} \left\{ \int_0^R \partial_x \phi \, dx - (\phi_R - \phi_0) \right\} = n_R - n_0 . $$ (1.3)

Hence the space of states is split in topological sectors with $Q = 0, \pm 1, \pm 2...$, and within a given $Q$-sector the states are characterized by their energies only.

It is worth mentioning that, in recent years, this problem (and variations thereof) has attracted the attention of several groups: the case of half–plane geometry, for instance, has been discussed by bootstrap methods in [6, 7, 8] and by semiclassical ones in [9, 10, 11] whereas the thermodynamics of different cases in a strip geometry has been studied in a series of publications (see [12–18]).

The semiclassical quantization presented here adds new pieces of information on this subject and it may be seen as complementary to the aforementioned studies: for the static solutions, it basically consists of identifying, in the limit $\beta \to 0$, a proper classical background $\phi_{cl}(x)$ for the given sector of the theory in exam, and then expressing the semiclassical energy levels as

$$ E_{\{k_n\}} = \mathcal{E}_{cl} + \sum_n \left( k_n + \frac{1}{2} \right) \omega_n , \quad k_n \in \mathbb{N} , $$ (1.4)

where $\mathcal{E}_{cl}$ is the classical energy of the solution whereas the frequencies $\omega_n$ are the eigenvalues of the so-called “stability equation” [11]

$$ \left[ -\frac{d^2}{dx^2} + V''(\phi_{cl}) \right] \eta_n(x) = \omega_n^2 \eta_n(x) . $$ (1.5)

For the Sine–Gordon model with periodic boundary conditions, alias in a cylinder geometry, this program has been completed in [5]. Given the similarity of the outcoming formulas with the
ones appearing in \[5\], in the sequel we will often refer to that paper for the main mathematical
definitions as well as for the discussion of some technical details. There is though a conceptual
difference between the periodic example and the one studied here: in the periodic case, in fact,
the vacuum sector is trivial at the semiclassical level (it simply corresponds to the constant
classical solution) and therefore the semiclassical quantization provides non-perturbative results
just starting from the one-kink sector. Contrarily, on the strip with Dirichlet b.c., the vacuum
sector itself is represented by a non-trivial classical solution and its quantization is even slightly
more elaborated than the one of the kink sectors.

2 Static classical solutions

In the static case, the Euler–Lagrange equation of motion associated to (1.1) is equivalent to
the first order differential equation

\[
\frac{1}{2} \left( \frac{\partial \phi_{cl}}{\partial x} \right)^2 = \frac{m^2}{\beta^2} \left( 1 - \cos \beta \phi_{cl} + A \right),
\]

which admits three kinds of solution, depending on the sign of the constant \(A\). The simplest
corresponds to \(A = 0\) and it describes the standard kink in infinite volume:

\[
\phi_{cl}^0(x) = \frac{4}{\beta} \arctan e^{m(x-x_0)}.
\]

In this paper, we will be concerned with the solutions relative to the case \(A \neq 0\), which can be
expressed in terms of Jacobi elliptic functions\(^1\) [19]. In particular, for \(A > 0\) we have

\[
\phi_{cl}^+(x) = \frac{\pi}{\beta} + \frac{2}{\beta} \text{am} \left( \frac{m(x-x_0)}{k}, k \right), \quad k^2 = \frac{2}{2+A},
\]

which has the monotonic and unbounded behaviour in terms of the real variable \(u^+ = \frac{m(x-x_0)}{k}\)
shown in Fig.1. For \(-2 < A < 0\), the solution is given instead by

\[
\phi_{cl}^-(x) = \frac{2}{\beta} \arccos \left[ k \text{ sn} (m(x-x_0), k) \right], \quad k^2 = 1 + \frac{A}{2},
\]

and it oscillates in the real variable \(u^- = m(x-x_0)\) between the \(k\)-dependent values \(\tilde{\phi}\) and
\(\frac{2\pi}{\beta} - \tilde{\phi}\) (see Fig.1).

The SG model with the Dirichlet b.c. (1.2) can be classically described by using the two
building functions \(\phi_{cl}^+(x)\) and \(\phi_{cl}^-(x)\), thanks to their free parameters \(x_0\) and \(k\), which can be
fixed in terms of \(\phi_0, \phi_R\) and \(R\). However, in order to simplify the notation, in writing down our

\[^1\text{See \[5\] and references therein for the definitions and the basic properties of complete elliptic integrals K}(k)\]

\(\text{and E}(k), \text{Jacobi and Weierstrass functions. In the following we will also use the incomplete elliptic integrals}
\]

\[
F(\varphi, k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \quad E(\varphi, k) = \int_0^\varphi d\alpha \sqrt{1 - k^2 \sin^2 \alpha},
\]

which reduce to the complete ones at \(\varphi = \pi/2\).
solutions we will rather use $R$ and $x_0$, both considered as functions of $\phi_0$, $\phi_R$ and $k$ (as a matter of fact, $k$ can be recovered by inverting the elliptic integrals which enter the corresponding expression of $R$).

As shown below, both types of solutions $\phi^+_{cl}(x)$ and $\phi^-_{cl}(x)$ are needed, in general, to define the classical background in the vacuum sector whereas only one of them, $\phi^+_{cl}(x)$, is employed for implementing the Dirichlet b.c. in the kink sector.

### 3 The vacuum sector: $Q = 0$

To discuss the vacuum sector, it is sufficient to restrict the attention to the case\(^{\footnote{All other cases can be described in a similar way, defining properly $x_0$ and $R$, and by using antikinks when necessary.}}\) $n_0 = n_R = 0$, $\phi_0 < \phi_R$ and $|\cos \frac{\beta}{2} \phi_0| > |\cos \frac{\beta}{2} \phi_R|$. It is also convenient to introduce the compact notation

$$c_{0,R} \equiv \cos \frac{\beta}{2} \phi_{0,R}. \quad (3.1)$$

In order to write down explicitly the classical background corresponding to the vacuum state with Dirichlet b.c., it is necessary to introduce preliminarily two particular values $R_1$ and $R_2$ of the width $R$ of the strip, which mark a change in the nature of the solution. They are given by

$$mR_1 = \arctanh (c_0) - \arctanh (c_R),$$
$$mR_2 = K(\tilde{k}) - F \left( \arcsin \frac{\beta_{\phi}}{\tilde{k}}, \tilde{k} \right), \quad \tilde{k} = |c_0|.$$ 

With these definitions, the classical vacuum solution, as a function of $x \in [0, R]$, has the following behaviour in the three regimes of $R$:

$$\phi_{cl}^{\text{vac}}(x) = \begin{cases} 
\phi_{cl}^{(1)}(x) & \text{for } 0 < R < R_1 \\
\phi_{cl}^{(2)}(x) & \text{for } R_1 < R < R_2 \\
\phi_{cl}^{(3)}(x) & \text{for } R_2 < R < \infty 
\end{cases} \quad (3.2)$$
where

\[
\phi_{cl}^{(1)}(x) = \phi_{cl}^{(+)}(x) \quad \text{with} \quad \begin{cases} 
mx_0 = -k \left( \frac{\beta}{2} \phi_0 - \frac{\pi}{2} \right), k \\
mR = k \left( \frac{\beta}{2} \phi_R - \frac{\pi}{2} \right) - F \left( \frac{\beta}{2} \phi_R - \frac{\pi}{2}, k \right) 
\end{cases} \quad 0 < k < 1
\]

\[
\phi_{cl}^{(2)}(x) = \phi_{cl}^{(-)}(x) \quad \text{with} \quad \begin{cases} 
mx_0 = -2K(k) + F \left( \arcsin \frac{\phi_0}{k}, k \right) \\
mR = F \left( \arcsin \frac{\phi_R}{k}, k \right) - F \left( \arcsin \frac{\phi_0}{k}, k \right) 
\end{cases} \quad \tilde{k} < k < 1
\]

\[
\phi_{cl}^{(3)}(x) = \phi_{cl}^{(-)}(x) \quad \text{with} \quad \begin{cases} 
mx_0 = -F \left( \arcsin \frac{\phi_0}{k}, k \right) \\
mR = 2K(k) - F \left( \arcsin \frac{\phi_R}{k}, k \right) - F \left( \arcsin \frac{\phi_0}{k}, k \right) 
\end{cases} \quad \tilde{k} < k < 1
\]

It is easy to check that at the particular values \(R_1\) and \(R_2\), the different definitions of the background nicely coincide. Fig. 2 shows the classical solution at some values of \(R\), one for each of the three regimes\(^3\).

**Figure 2:** Classical solution at some value of \(R\), in the case \(\beta \phi_0 = 1\) and \(\beta \phi_R = 2\).

The classical energy of the background is expressed as

\[
E_{cl}^{\text{vac}}(R) = \begin{cases} 
\mathcal{E}_{cl}^{(1)}(R) & \text{for} \quad 0 < R < R_1 \\
\mathcal{E}_{cl}^{(2)}(R) & \text{for} \quad R_1 < R < R_2 \\
\mathcal{E}_{cl}^{(3)}(R) & \text{for} \quad R_2 < R < \infty
\end{cases}
\] (3.3)

where

\[
\mathcal{E}_{cl}^{(1)}(R) = \frac{2m}{\beta^2} \left\{ \left( 1 - \frac{1}{k^2} \right) mR + \frac{2}{k} \left[ E \left( \frac{\beta}{2} \phi_R - \frac{\pi}{2}, k \right) - E \left( \frac{\beta}{2} \phi_0 - \frac{\pi}{2}, k \right) \right] \right\},
\]

\[
\mathcal{E}_{cl}^{(2)}(R) = \frac{2m}{\beta^2} \left\{ (k^2 - 1)mR + 2 \left[ E \left( \arcsin \frac{\phi_0}{k}, k \right) - E \left( \arcsin \frac{\phi_R}{k}, k \right) \right] \right\},
\]

\[
\mathcal{E}_{cl}^{(3)}(R) = \frac{2m}{\beta^2} \left\{ (k^2 - 1)mR + 2 \left[ 2E(k) - E \left( \arcsin \frac{\phi_0}{k}, k \right) - E \left( \arcsin \frac{\phi_R}{k}, k \right) \right] \right\}
\]

\(^3\)We have chosen for the plot the specific values \(\beta \phi_0 = 1\) and \(\beta \phi_R = 2\), for which \(mR_1 = 0.76\) and \(mR_2 = 1.49\).

The same values will be considered in all other pictures since their qualitative features do not sensibly depend on these parameters, except for few particular values of \(\phi_0, R\) discussed separately.
and it is plotted in Fig. 3. As expected, the quantity (3.3) has a smooth behaviour at $R_1$ and $R_2$, which correspond to the minimum and the point of zero curvature of this function, respectively. The non monotonic behaviour of the classical energy gives an intuitive motivation for the classical background being differently defined in the three regimes of $R$.

![Figure 3: Classical energy (3.3) for $\beta\phi_0 = 1$ and $\beta\phi_R = 2$.](image)

Furthermore, the classical energy can be easily expanded in the ultraviolet (UV) or infrared (IR) limit, i.e. for small or large values of $mR$, which correspond to $k \to 0$ in the regime $0 < R < R_1$ or to $k \to 1$ in the regime $R_2 < R < \infty$, respectively.

In fact, expanding the elliptic integrals in (3.3) (see [20] for the relative formulas), and comparing the result order by order with the small-$k$ expansion of $mR$ defined in the first regime of (3.2)

$$mR = k \frac{\beta}{2}(\phi_R - \phi_0) \left[ 1 + \frac{k^2}{4} \left( 1 + \frac{\sin \beta \phi_R - \sin \beta \phi_0}{\beta(\phi_R - \phi_0)} \right) + \cdots \right],$$  

(3.4)

one obtains the small-$mR$ behaviour

$$\mathcal{E}_{cl}^{(1)}(R) = \frac{1}{2R}(\phi_R - \phi_0)^2 + R \frac{m^2}{\beta^2} \left[ 1 - \frac{\sin \beta \phi_R - \sin \beta \phi_0}{\beta(\phi_R - \phi_0)} \right] + \cdots .$$  

(3.5)

Later we will comment on the meaning of this result in the UV analysis of the ground state energy. On the other hand, comparing the expansion for $k \to 1$ of $\mathcal{E}_{cl}^{(3)}(R)$ in the third regime with

$$mR = -\log \left\{ \frac{1 - k^2}{16} \tan \frac{\beta}{4} \phi_0 \tan \frac{\beta}{4} \phi_R \right\} + \cdots ,$$  

(3.6)

one obtains the large-$mR$ behaviour

$$\mathcal{E}_{cl}^{(3)}(R) = \frac{4m}{\beta^2} \left( 2 - \cos \frac{\beta}{2} \phi_R - \cos \frac{\beta}{2} \phi_0 \right) - \frac{32m}{\beta^2} \tan \frac{\beta}{4} \phi_0 \tan \frac{\beta}{4} \phi_R e^{-mR} + \cdots .$$  

(3.7)

The first term of this expression is the classical limit of the boundary energy of the vacuum sector [12], since it is the term that needs to be subtracted by choosing to normalise the energy to zero at $R \to \infty$. 

5
The classical description of the vacuum sector can be completed by mentioning the existence of two particular cases in which the three different regimes of $R$ are not needed. The first is given by $\phi_0 = \phi_R$, for which the whole range of $R$ is described by $\phi_{cl}^{(3)}(x)$ in (3.2), since $mR_0 = 0$ in this situation. The second case, defined by $\phi_0$ arbitrary and $\phi_R = 0$, can be instead described by the antikink $\phi_{cl}^{(1)}(x) = \phi_{cl}^{(1)}(-x)$ alone, since $mR_1 = \infty$ for these values of the boundary parameters (note that $x_0$ and $R$ have to be defined as opposite to the ones in (3.2)). As a consequence, these two cases display a monotonic behaviour of the classical energy, whose UV and IR asymptotics, respectively, require a separate derivation, which can be performed by simply adapting the above procedure.

Finally, it is also worth discussing an interesting feature which emerges in the IR limit of the classical solution (3.2). As it can be seen from Fig. 2, by increasing $R$ the static background is more and more localised closely to the constant value $\phi(x) \equiv 0$ and this guarantees the finiteness of the classical energy in the $R \to \infty$ limit, given by the first term in (3.7)\(^4\). However, if the IR limit is performed directly on the classical solution, we obtain one of the static backgrounds\(^5\) studied in [11].

\[ \phi_{cl}^{(3)}(x) \to \frac{2}{R \to \infty} \arccos \left[ \tanh (m(x - x_0^\infty)) \right], \quad \text{with} \quad x_0^\infty = -\arctanh(c_0). \]

The last expression tends to zero as $x \to \infty$ and consequently has classical energy $\mathcal{E}_{cl} = \frac{4m}{\beta^2} \left( 1 - \cos \frac{\beta}{2} \phi_0 \right)$. This phenomenon can be easily understood by noting that the minimum of $\phi_{cl}^{(3)}(x)$ (which goes to zero in the IR limit), is placed at $m\bar{x} = mx_0 + K(k)$ (see Fig. 1) and this point tends itself to infinity as $k \to 1$. Hence, the information about the specific value of $\phi_R$ is lost when $R \to \infty$, i.e. only the states with $\phi_R = 0$ survive in the IR limit.

4 Semiclassical quantization on the strip

We will now perform the semiclassical quantization in the vacuum sector, around the background (3.2). Depending on the value of $mR$, the stability equation (1.5) takes the form

\[ \left\{ \frac{d^2}{d\bar{x}^2} + k^2 (\tilde{\omega}^2 + 1) - 2k^2 \text{sn}^2(\bar{x} - \bar{x}_0, k) \right\} \eta_{\omega}^{(1)}(\bar{x}) = 0, \quad \text{with} \quad \bar{x} = \frac{mx}{k}, \quad \tilde{\omega} = \frac{\omega}{m}, \quad (4.1) \]

when $0 < R < R_1$, and

\[ \left\{ \frac{d^2}{d\bar{x}^2} + \omega^2 + 1 - 2k^2 \text{sn}^2(\bar{x} - \bar{x}_0, k) \right\} \eta_{\omega}^{(2,3)}(\bar{x}) = 0, \quad \text{with} \quad \bar{x} = mx, \quad \tilde{\omega} = \frac{\omega}{m}, \quad (4.2) \]

when $R_1 < R < R_2$ and $R_2 < R < \infty$.

Equations (4.1) and (4.2) can be cast in the Lamé form with $N = 1$, which has been fully discussed in [5]. The only differences with the periodic case are the presence of a non-trivial center of mass $x_0$ and the larger number of parameters entering the expression of the size $R$ of the

\(^4\)When $|c_0| < |c_R|$, the same qualitative phenomenon occurs, but the constant value is $\phi(x) \equiv \frac{2\pi}{\beta}$ in this case.

\(^5\)Obviously, the same function is obtained as $\lim_{R \to \infty} \phi_{cl}^{(1)}(x)$, in the case $\phi_R = 0$ mentioned above.
system: these make more complicated the so-called “quantization condition” that determines the discrete eigenvalues, although they do not alter the general procedure to derive it.

The boundary conditions \[ \eta_\omega(0) = \eta_\omega(R) = 0 , \] select in this case the following eigenvalues, all with multiplicity one, \[
\omega_n^{\text{vac}}(R) = \begin{cases} 
\omega_n^{(1)}(R) & \text{for } 0 < R < R_1 \\
\omega_n^{(2)}(R) & \text{for } R_1 < R < R_2 \\
\omega_n^{(3)}(R) & \text{for } R_2 < R < \infty 
\end{cases},
\] where
\[
\omega_n^{(1)}(R) = m \frac{k^2}{2 - k^2} \left( \frac{3}{2} - \mathcal{P}(iy_n) \right), \\
\omega_n^{(2,3)}(R) = m \frac{k^2 - 1}{2} \left( \frac{3}{2} - \mathcal{P}(iy_n) \right),
\] and the \( y_n \)'s are defined through the “quantization condition”
\[
2 R_1 \zeta(iy_n) + i \log \left[ \frac{\sigma(-\bar{x}_0 + iK + iy_n) \sigma(\bar{R} - \bar{x}_0 + iK' - iy_n)}{\sigma(-\bar{x}_0 + iK' - iy_n) \sigma(\bar{R} - \bar{x}_0 + iK + iy_n)} \right] = 2n\pi, \quad n = 1, 2, \ldots
\] This equation comes from the consistency condition associated to the boundary values
\[
\begin{cases} 
D_+ \eta_a(0) + D_- \eta_{-a}(0) = 0 \\
D_+ \eta_a(R) + D_- \eta_{-a}(R) = 0 
\end{cases}
\] where \( \eta_{\pm a} \) are the two linearly independent solutions of the Lamé equation which are used to build the general solution \( \eta(x) = D_+ \eta_a(x) + D_- \eta_{-a}(x) \) (see \[5\] and \[21\] for details).

As it can be seen directly from \[14\], the frequencies \[14\] are nothing else but the energies of the excited states with respect to the ground state \( E_0^{\text{vac}}(R) \). They can be easily determined from the above equations and their behaviour, as functions of \( R \), is shown in Fig. 4.

As in the periodic case \[5\], a more explicit expression for the energy levels \[14\] can be obtained by expanding them for small or large values of \( mR \). The UV expansion, for instance, can be performed extracting from \[15\] a small-\( k \) expansion for \( y_n \), inserting it in \[14\], and finally comparing the result order by order with \[31\]. Exploiting the several properties of Weierstrass functions which follow from their relation with \( \theta \)-functions (see for instance \[21\]), one gets
\[
y_n = \arctanh \frac{f}{2n\pi} + \frac{k^2}{4} \left\{ \arctanh \frac{f}{2n\pi} + s \frac{2n\pi(4n^2\pi^2 - 3f^2)}{(4n^2\pi^2 - f^2)^2} \right\} + \cdots,
\] and
\[
\omega_n^{(1)} = \frac{m}{k} \frac{2n\pi}{f} \left\{ 1 - \frac{k^2}{4} \left[ 1 + s \frac{2fs}{4n^2\pi^2 - f^2} \right] + \cdots \right\}.
\]
where we have introduced the compact notation $f \equiv \beta(\phi_R - \phi_0)$, $s \equiv (\sin \beta \phi_R - \sin \beta \phi_0)$. This leads to the UV expansion

$$\omega_n^{(1)}(R) = \frac{n \pi}{R} + m^2 R \frac{s}{f} \frac{2n \pi}{4n^2 \pi^2 - f^2} + \cdots$$  \hspace{1cm} (4.6)

In order to complete the above analysis and obtain the reference value of the energy levels, i.e. the ground state energy $E_{0}^{\text{vac}}(R)$ of the vacuum sector, we need the classical energy $E_{cl}^{\text{vac}}(R)$ and the sum on the stability frequencies given in (4.4), i.e.

$$E_{0}^{\text{vac}}(R) = E_{cl}^{\text{vac}}(R) + \frac{1}{2} \sum_{n=1}^{\infty} \omega_n^{\text{vac}}(R) .$$  \hspace{1cm} (4.7)

The above series is divergent and its regularization has to be performed by subtracting to it a mass counterterm and the divergent term coming from the infinite volume limit – a procedure that is conceptually analogous to the one discussed in [5] for the periodic case and therefore it is not repeated here. Furthermore, as already mentioned, equation (4.7) can be made more explicit by expanding it for small or large values of $mR$. Here, for simplicity, we limit ourselves to the discussion of the leading $1/R$ term in the UV expansion since it does not receive contributions from the counterterm and therefore it can be simply regularised by using the Riemann $\zeta$–function prescription (see [5] for a detailed discussion). The higher terms, instead, require a technically more complicated regularization, although equivalent to the one presented in [5].

The UV behaviour of the ground state energy is dominated by

$$E_{0}^{\text{vac}}(R) = \frac{\pi}{R} \left[ \frac{1}{2\pi} (\phi_R - \phi_0)^2 - \frac{1}{24} \right] + \cdots$$  \hspace{1cm} (4.8)

where the coefficient $-1/24$ comes from the regularization of the leading term in the series of frequencies (4.6), while the first term simply comes from the expansion of the classical energy (3.5). It is easy to see that the above expression correctly reproduces the expected ground state
energy for the gaussian Conformal Field Theory (CFT) on a strip of width $R$ with Dirichlet boundary conditions [22, 23].

Finally, it is simple to check that also the excited energy levels display the correct UV behaviour, being expressed as

$$E_{\{k_n\}}^{\text{vac}}(R) = \frac{\pi}{R} \left[ \frac{1}{2\pi} (\phi_R - \phi_0)^2 + \sum_n k_n n - \frac{1}{24} \right] + \cdots$$

(4.9)

5 The kink sector: $Q = 1$

In discussing the kink sector we can restrict to $n_0 = 0$, $n_R = 1$, since all other cases, as well as the antikink sector with $Q = -1$, are described by straightforward generalizations of the following formulas.

The classical solution can be now expressed only in terms of the function $\phi^{(+)}_{cl}(x)$ as

$$\phi^{\text{kink}}_{cl}(x) = \phi^{(+)}_{cl}(x) \quad \text{with} \quad \begin{cases} m x_0 = -k F\left(\frac{\beta}{2} \phi_0 - \frac{\pi}{2}, k\right) \\ mR = k \left[ 2K(k) + F\left(\frac{\beta}{2} \phi_R - \frac{\pi}{2}, k\right) - F\left(\frac{\beta}{2} \phi_0 - \frac{\pi}{2}, k\right) \right] \\ 0 < k < 1 \end{cases}$$

(5.1)

since in this case the whole range $0 < mR < \infty$ is spanned by varying $k$ in $[0, 1]$. This can be intuitively understood by looking at the behaviour of (5.1) in Fig. 5.

As a consequence, the classical energy and the stability frequencies of this sector can be obtained from $E^{(1)}_{cl}$ and $\omega^{(1)}_n$ of the vacuum (given respectively in eq. (3.3) and (4.4)), by simply replacing $\phi_R \rightarrow \phi_R + \frac{2\pi}{\beta}$. The leading UV behaviour of the energy levels in this sector, given by

$$E_{\{k_n\}}^{\text{kink}}(R) = \frac{\pi}{R} \left[ \frac{1}{2\pi} \left( (\phi_R - \phi_0) + \frac{2\pi}{\beta} Q \right)^2 + \sum_n k_n n - \frac{1}{24} \right] + \cdots$$

(5.2)

with $Q = 1$, correctly matches the CFT prediction.
The only result which cannot be directly extracted from the vacuum sector analysis is the IR asymptotic behaviour of the classical energy, since now the $k \to 1$ limit has to be performed on $\mathcal{E}_{cl}^{(1)}$. We have in this case

$$mR = -\log \left\{ \frac{1 - k^2}{16} \tan \frac{\beta}{4} \phi_0 \right\} + \cdots ,$$

which leads to

$$\mathcal{E}_{cl}^{(1)}(R) = \frac{4m}{\beta^2} \left( 2 - \cos \frac{\beta}{2} \phi_R + \cos \frac{\beta}{2} \phi_0 \right) + \frac{32m}{\beta^2} \tan \frac{\beta}{4} \phi_R e^{-mR} + \cdots .$$

Analogously to the vacuum sector, the first term of this expression is related to the classical limit of the boundary energy in the one–kink sector. Notice that, differently from the vacuum case, where the asymptotic IR value of the classical energy was approached from below (see (3.7)), the coefficient of the exponential correction has now positive sign, in agreement with the monotonic behaviour of the classical energy shown in Fig. 6.

![Figure 6: Classical energy in the $Q = 1$ kink sector for $\beta \phi_0 = 1$ and $\beta \phi_R = 2$.](image)

When $R \to \infty$, a mechanism analogous to the one discussed for the vacuum also takes place here: the classical energy is finite for any value of $\phi_R$, but since $\phi_{cl}^{kink}(x)$ assumes the value $\frac{2\pi}{\beta}$ at $m\bar{x} = mx_0 + kK(k)$ (see Fig.5), a point which tends to infinity as $k \to 1$, only the states with $\phi_R = 0$ survive in this limit.

It is worth noticing that $\phi_{cl}^{(+)}(x)$ can be also used to satisfy, at finite values of $R$, Dirichlet b.c. in sectors with arbitrary topological charge (see Fig.7), giving rise to the correct UV behaviour [5.2] with $Q = n_R - n_0$. However, since $\phi_{cl}^{(+)}(x)$ always assumes the value $\frac{2\pi}{\beta}(n_0 + 1)$ at $m\bar{x} = mx_0 + kK(k)$, which is once again the point going to infinity when $k \to 1$, in the IR limit it can only correspond to $Q = 1$. This result seems natural though, since in infinite volume, static classical solutions can only describe those sectors of the theory with $Q = 0, \pm 1$, while time–dependent ones are needed for higher values of $Q$. Hence, in the topological sectors
with $|Q| > 1$ the space of states will contain, at classical level, the time–dependent backgrounds, defined for any value of $R$ (which are not discussed here), plus the static ones of the form $\phi^{(+)}_{cl}(x)$, which however disappear from the spectrum as $R \to \infty$.

![Graph](image1.png)  
![Graph](image2.png)

Figure 7: Classical solution in the $Q = 3$ sector ($n_0 = 0, n_R = 3$) at some values of $R$, in the case $\beta \phi_0 = 1$ and $\beta \phi_R = 2$.

6 Conclusions

In this paper we have presented the semiclassical energy levels of a quantum field theory on a strip geometry. Our analysis builds on and extends the semiclassical quantization of a field theory on a finite geometry introduced in [3]. The example discussed here is the Sine–Gordon model with Dirichlet boundary conditions at both edges of the strip. The semiclassical approach provides analytic and non–perturbative expressions for the energy levels, valid for arbitrary values of the size $R$ of the system, which permit to link the IR data on the half-line with the UV conformal data of boundary CFT at $c = 1$.

In comparison with a cylinder geometry, an interesting new feature of the quantum field theory defined on a strip consists in a non–trivial (and non–perturbative) semiclassical description of its vacuum sector. Therefore, we have discussed in detail the classical solutions and energy levels in the $Q = 0$ case, together with the $Q = 1$ that can also be described by static backgrounds. It should be mentioned, however, that the semiclassical methods [4] are not restricted to static backgrounds only. As in infinite volume, a complete description of the theory in all sectors requires also the study of non–perturbative time–dependent solutions.

Finally, it is worth noticing that the method used here has natural and direct extension to other quantum field theories with various kinds of boundary conditions.

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Note added. In completing this work, it has appeared the paper [24] which partially overlaps with ours.

References

[1] R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 10 (1974) 4130; Phys. Rev. D 11 (1975) 3424.
[2] J. Goldstone and R. Jackiw, Phys. Rev. D 11 (1975) 1486.
[3] G. Mussardo, V. Riva and G. Sotkov, Nucl. Phys. B 670 (2003) 464.
[4] G. Mussardo, V. Riva and G. Sotkov, Nucl. Phys. B 687 (2004) 189.
[5] G. Mussardo, V. Riva and G. Sotkov, *Semiclassical Scaling Functions of Sine–Gordon Model*, hep-th/0405139.
[6] S. Ghoshal and A.B. Zamolodchikov, Int. J. Mod. Phys. A9 (1994) 3841, Erratum-ibid. A9 (1994) 4353.
[7] P. Mattsson and P. Dorey, J. Phys. A33 (2000) 9065.
[8] Z. Bajnok, L. Palla, G. Takaes and G.Z. Toth, Nucl. Phys. B 622 (2002) 548.
[9] H. Saleur, S. Skorik and N.P. Warner, Nucl. Phys. B 441 (1995) 421.
[10] E. Corrigan and G.W. Delius, J. Phys. A 32 (1999) 8601; E. Corrigan and A. Taormina, J. Phys. A 33 (2000) 8739.
[11] M. Kormos and L. Palla, J. Phys. A35 (2002) 5471.
[12] A. LeClair, G. Mussardo, H. Saleur and S. Skorik, Nucl. Phys. B 453 (1995) 581.
[13] S. Skorik and H. Saleur, J. Phys. A 28 (1995) 6605.
[14] J.S. Caux, H. Saleur and F. Siano, Phys. Rev. Lett. 88 (2002) 106402; Nucl. Phys. B 672 (2003) 411.
[15] T. Lee and C. Rim, Nucl. Phys. B 672 (2003) 487.
[16] C. Rim, *Boundary massive Sine-Gordon model at the free Fermi limit and RG flow of Casimir energy*, hep-th/0405162.
[17] C. Ahn and R.I. Nepomechie, Nucl. Phys. B 676 (2004) 637.
C. Ahn, M. Bellacosa and F. Ravanini, *Excited states NLIE for Sine-Gordon model in a strip with Dirichlet boundary conditions*, hep-th/0312176.

K. Takayama and M. Oka, Nucl. Phys. A 551 (1993) 637.

I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, Academic Press, New York (1980).

E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge, Cambridge University Press, 1927.

H.W.J. Blote, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742; J.L. Cardy, Nucl. Phys. B 270 (1986) 186; Nucl. Phys. B 275 (1986) 200; *Conformal invariance and statistical mechanics*, Les Houches 1988, North Holland, Amsterdam.

H. Saleur, *Lectures on nonperturbative field theory and quantum impurity problems*, Les Houches 1998, North Holland, Amsterdam, cond-mat/9812110.

Z. Bajnok, L. Palla and G. Takacs, *(Semi)classical analysis of sine-Gordon theory on a strip*, hep-th/0406149.