Stability of the Second Order Delay Differential Equations with a Damping Term

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Abstract

For the delay differential equations

\[ \ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0, \quad g(t) \leq t, \quad h(t) \leq t, \]

and

\[ \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + a_1(t)\dot{x}(g(t)) + b_1(t)x(h(t)) = 0 \]

explicit exponential stability conditions are obtained.

Keywords: exponential stability, nonoscillation, positive fundamental function, second order delay equations

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1 Introduction

This paper deals with the scalar linear delay differential equation of the second order with a damping term

\[ \ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0 \tag{1} \]

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and the equation which also involves nondelay terms

\[ \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + a_1(t)\dot{x}(g(t)) + b_1(t)x(h(t)) = 0. \] (2)

Such linear and nonlinear equations attract attention of many mathematicians due to their significance in applications. We mention here the monographs of Myshkis [17], Norkin [18], Ladde, Lakshmikantham and Zhang [15], Györi and Ladas [13], Erbe, Kong and Zhang [12], Burton [7], Kolmanovskiy and Nosov [14] and references therein.

In particular, Minorsky [16] in 1962 considered the problem of stabilizing the rolling of a ship by the “activated tanks method” in which ballast water is pumped from one position to another. To solve this problem he constructed several delay differential equations with damping of the form (1) and (2).

In spite of obvious importance in applications, there are only few papers on delay differential equations with damping.

In [10] the authors considered autonomous equation (2) and obtained stability results using analysis of the roots of the characteristic equation.

In [5] stability of the autonomous equation

\[ \ddot{x}(t) + ax(t) + bx(t-\tau) = 0 \] (3)

was studied using Lyapunov functions. It was demonstrated that if \( a > 0, b > 0 \) and \( b\tau < a \), then equation (3) is exponentially stable. Other results obtained by Lyapunov functions method can be found in [9, 19].

In [8] Burton and Furumochi applied fixed point theorems to equation (1) and obtained new stability results. In particular, the equation

\[ \ddot{x}(t) + \frac{1}{3} \dot{x}(t) + \frac{1}{48}x(t-16) = 0 \]

is exponentially stable, where the condition \( b\tau < a \) does not hold. Here \( b\tau = a \).

In [6, 7] some other stability conditions were obtained by the fixed point method for (1) in the case \( g(t) \equiv t, h(t) = t - \tau \).

To the best of our knowledge, there is only one paper [2] where stability of the general nonautonomous equation (1) was investigated. In [2] the authors applied the W-method [1] which is based on the application of the Bohl-Perron type theorem (Lemma 3 of the present paper).

Note also the paper [11] where nonoscillation of systems of delay differential equations was considered and on this basis several results on nonoscillation and exponential stability of second order delay differential equations were obtained.

Here we will employ the method of [2] and consider equation (2), which was not studied in [2]. Furthermore, we will use a new approach (also based on a Bohl-Perron type theorem) and obtain sharper stability results for equation (1) than in [2].

In particular, for (3) we obtain the same stability condition \( b\tau < a \) as in [5], but our results are applicable to more general nonautonomous equations as well.
2 Preliminaries

We consider the scalar second order delay differential equation (1) under the following conditions:

(a1) $a(t), b(t)$, are Lebesgue measurable and essentially bounded functions on $[0, \infty)$; 
(a2) $g : [0, \infty) \rightarrow \mathbb{R}, h : [0, \infty) \rightarrow \mathbb{R}$ are Lebesgue measurable functions, $g(t) \leq t$, $h(t) \leq t$, $t \geq 0$, $\lim_{t \to \infty} \sup (t - g(t)) < \infty$, $\lim_{t \to \infty} \sup (t - h(t)) < \infty$.

Together with (1) consider for each $t_0 \geq 0$ an initial value problem

$$
\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(h(t)) = f(t), \quad t \geq t_0, \\
x(t) = \varphi(t), \quad \dot{x}(t) = \psi(t), \quad t < t_0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = x_0'.
$$

(4)

We also assume that the following hypothesis holds

(a3) $f : [t_0, \infty) \rightarrow \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function, $\varphi : (-\infty, t_0) \rightarrow \mathbb{R}, \psi : (-\infty, t_0) \rightarrow \mathbb{R}$ are Borel measurable bounded functions.

**Definition.** A function $x : \mathbb{R} \rightarrow \mathbb{R}$ with locally absolutely continuous on $[t_0, \infty)$ derivative $\dot{x}$ is called a solution of problem (1), (5) if it satisfies equation (1) for almost every $t \in [t_0, \infty)$ and equalities (5) for $t \leq t_0$.

**Definition.** For each $s \geq 0$, the solution $X(t, s)$ of the problem

$$
\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(h(t)) = 0, \quad t \geq s, \\
x(t) = 0, \quad \dot{x}(t) = 0, \quad t < s, \quad x(s) = 0, \quad \dot{x}(s) = 1
$$

(6)

is called the fundamental function of equation (1).

We assume $X(t, s) = 0$, $0 \leq t < s$.

Let functions $x_1$ and $x_2$ be the solutions of the following equation

$$
\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(h(t)) = 0, \quad t \geq t_0, \quad x(t) = 0, \quad \dot{x}(t) = 0, \quad t < t_0,
$$

with initial values $x(t_0) = 1, \dot{x}(t_0) = 0$ for $x_1$ and $x(t_0) = 0, \dot{x}(t_0) = 1$ for $x_2$, respectively.

By definition $x_2(t) = X(t,t_0)$.

**Lemma 1** Let (a1)-(a3) hold. Then there exists one and only one solution of problem (1), (5) that can be presented in the form

$$
x(t) = x_1(t)x_0 + x_2(t)x_0' + \int_{t_0}^{t} X(t, s)f(s)ds - \int_{t_0}^{t} X(t, s)[a(s)\psi(g(s)) + b(s)\varphi(h(s))]ds,
$$

(7)

where $\varphi(h(s)) = 0$ if $h(s) > t_0$ and $\psi(g(s)) = 0$ if $g(s) > t_0$.

**Definition.** Eq. (1) is (uniformly) exponentially stable, if there exist $M > 0$, $\mu > 0$, such that the solution of problem (1), (5) has the estimate

$$
|x(t)| \leq M e^{-\mu(t-t_0)} \left[|x(t_0)| + \sup_{t<t_0} (|\varphi(t)| + |\psi(t)|)\right], \quad t \geq t_0,
$$

(8)
where $M$ and $\mu$ do not depend on $t_0$.

**Definition.** The fundamental function $X(t, s)$ of (1) has an exponential estimate if there exist positive numbers $K > 0, \lambda > 0$, such that

$$|X(t, s)| \leq K e^{-\lambda(t-s)}, \ t \geq s \geq 0. \quad (8)$$

For the linear equation (1) with bounded delays ((a2) holds) the last two definitions are equivalent.

Under (a2) the exponential stability does not depend on values of equation parameters on any finite interval.

**Remark.** All definitions and Lemma 1 can also be applied to equation (2), for which we will assume that conditions (a2)-(a3) hold and all coefficients are Lebesgue measurable essentially bounded on $[0, \infty)$ functions.

Consider the equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0, \quad (9)$$

where $a > 0, b > 0$ are positive numbers. This equation is exponentially stable. Denote by $Y(t, s)$ the fundamental function of (9).

**Lemma 2** [4] Let $a > 0, b > 0$.

1) If $a^2 > 4b$ then $\int_0^t |Y(t, s)|ds \leq \frac{1}{b}$, $\int_0^t |Y'(t, s)|ds \leq \frac{2a}{\sqrt{a^2 - 4b(a - \sqrt{a^2 - 4b})}}$.

2) If $a^2 < 4b$ then $\int_0^t |Y(t, s)|ds \leq \frac{4}{a\sqrt{4b - a^2}}$, $\int_0^t |Y'(t, s)|ds \leq \frac{2(a + \sqrt{4b - a^2})}{a\sqrt{4b - a^2}}$.

3) If $a^2 = 4b$ then $\int_0^t |Y(t, s)|ds \leq \frac{1}{b}$, $\int_0^t |Y'(t, s)|ds \leq \frac{2}{\sqrt{b}}$.

Let us introduce some functional spaces on a semi-axis. Denote by $L_\infty[t_0, \infty)$ the space of all essentially bounded on $[t_0, \infty)$ scalar functions and by $C[t_0, \infty)$ the space of all continuous bounded on $[t_0, \infty)$ scalar functions with the supremum norm.

**Lemma 3** [1] Suppose there exists $t_0 \geq 0$ such that for every $f \in L_\infty[t_0, \infty)$ both the solution $x$ of the problem

$$\ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = f(t), \ t \geq t_0, \quad x(t) = 0, \dot{x}(t) = 0, \ t \leq t_0,$$

and its derivative $\dot{x}$ belong to $C[t_0, \infty)$. Then equation (7) is exponentially stable.

**Remark.** A similar result is valid for equation (2).
Lemma 4 If $a(t) \geq 0$ is essentially bounded on $[0, \infty)$, the fundamental function $Z(t, s)$ of the equation
\[ \dot{x}(t) + a(t)x(g(t)) = 0 \] (10)
is positive: $Z(t, s) > 0, t \geq s \geq t_0 \geq 0$ and $t - g(t) \leq \delta$, then
\[ \int_{t_0 + \delta}^{t} Z(t, s)a(s)ds \leq 1 \text{ for all } t \geq t_0 + \delta. \]

Lemma 5 If $a(t) \geq \alpha > 0$ is essentially bounded in $[0, \infty)$, \(\limsup_{t \to \infty}(t - g(t)) < \infty\) and the fundamental function $Z(t, s)$ of equation (10) is positive, then (10) is exponentially stable and $Z(t, s)$ has an exponential estimate.

Lemma 6 Suppose $a(t) \geq 0$ and
\[ \int_{g(t)}^{t} a(s)ds \leq \frac{1}{e^t}, t \geq t_0 \geq 0. \]
Then the fundamental function of (10) is positive: $Z(t, s) > 0, t \geq s \geq t_0$.

3 Stability Conditions, I

In this section we consider equation (2) as a perturbation of an exponentially stable ordinary differential equation for which integral estimations of the fundamental function and its derivative are known.

We will start with the main result of [2].
Denote by $Y(t, s)$ the fundamental function of the equation
\[ \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0. \] (11)
If equation (11) is exponentially stable then there exist
\[ Y = \sup_{t > t_0} \int_{t_0}^{t} |Y(t, s)|ds < \infty, \quad Y' = \sup_{t > t_0} \int_{t_0}^{t} |Y'(t, s)|ds < \infty. \] (12)

Denote
\[ a = \left\{ \begin{array}{ll}
0, & g(t) \equiv t, \\
\sup_{t > t_0} a(t), & g(t) \not\equiv t,
\end{array} \right. \quad b = \left\{ \begin{array}{ll}
0, & h(t) \equiv t, \\
\sup_{t > t_0} b(t), & h(t) \not\equiv t,
\end{array} \right.
\]
\[ h = \max \left\{ \sup_{t > t_0}(t - g(t)), \sup_{t > t_0}(t - h(t)) \right\}. \]

Theorem A Suppose $a(t) \geq 0, b(t) \geq 0$, equation (11) is exponentially stable and for some $t_0 \geq 0$
\[ h < \frac{1}{a^2Y'' + abY' + bY''}. \]
Then (11) is exponentially stable.

Now we will apply the method of [2] to equation (2).
Theorem 1 Suppose $a(t) \geq 0, b(t) \geq 0$, equation (11) is exponentially stable and for some $t_0 \geq 0$
\[
\|a_1\|Y' + \|b_1\|Y < 1,
\]
where $\| \cdot \|$ is the norm in the space $L_\infty[t_0, \infty)$. Then (2) is exponentially stable.

Proof. Without loss of generality we can assume $t_0 = 0$. Consider the following problem
\[
\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + a_1(t)\dot{x}(g(t)) + b_1(t)x(h(t)) = f(t), \; t \geq 0,
\]
\[
x(t) = \dot{x}(t) = 0, \; t \leq 0.
\]

Let us demonstrate that for every $f \in L_\infty$ the solution $x$ of (13) and its derivative are bounded. Due to the zero initial conditions, we can assume that $h(t) = g(t) = 0, \; t < 0$.

The solution $x$ of (13) is also a solution of the following problem
\[
\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = z(t), \; t \geq 0, \; x(t) = \dot{x}(t) = 0, \; t \leq 0,
\]
with some function $z(t)$. Then
\[
x(t) = \int_0^t Y(t, s)z(s)ds, \; x'(t) = \int_0^t Y'_t(t, s)z(s)ds,
\]
where $Y(t, s)$ is the fundamental function of equation (11). Hence equation (13) is equivalent to the equation
\[
z(t) + a_1(t)\int_0^t Y'(g(t), s)z(s)ds + b_1(t)\int_0^t Y(h(t), s)z(s)ds = f(t).
\]
Equation (16) has the form $z + Hz = f$, which we consider in the space $L_\infty[0, \infty)$. We have
\[
\|H\| \leq \|a_1\|Y' + \|b_1\|Y < 1.
\]
Then for the solution of (16) we have $z \in L_\infty$.

Equalities (15) imply $\|x\| \leq Y\|z\|, \; \|x'\| \leq Y'\|z\|$.

By Lemma 3 equation (2) is exponentially stable. \qed

Consider now equation (2) with $a(t) \equiv a, b(t) \equiv b$. By Lemma 2 we have the following statement.

Corollary 1 Suppose for some $t_0 \geq 0$ one of the following conditions holds:
1) \[ a^2 > 4b, \]
\[
\frac{2a}{\sqrt{a^2 - 4b(a - \sqrt{a^2 - 4b})}}\|a_1\| + \frac{1}{b}\|b_1\| < 1,
\]
2) \[ 4b > a^2, \]
\[
\frac{2(a + \sqrt{4b - a^2})}{a\sqrt{4b - a^2}}\|a_1\| + \frac{4}{a\sqrt{4b - a^2}}\|b_1\| < 1,
\]
3) \[ a^2 = 4b, \]
\[
\frac{2}{\sqrt{b}}\|a_1\| + \frac{1}{b}\|b_1\| < 1,
\]
where $\| \cdot \|$ is the norm in the space $L_\infty[t_0, \infty)$. Then equation (2) is exponentially stable.
Example 1. Let us illustrate the exponential stability domain in Corollary 1 for two cases. If $a = 3$, $b = 2$ then $a^2 > 4b$ and the condition of 1) becomes $3|a_1| + 0.5|b_1| < 1$ for constant $a_1$, $b_1$ which corresponds to the domain inside the vertically stretched rhombus in Fig. 1. If $a = 3$, $b = 2.5$ then $a^2 < 4b$ and the condition of 2) becomes $2|a_1| + |b_1| < 0.75$ which is inside a smaller rhombus in Fig. 1.

![Diagram](image)

Figure 1: The domain inside the rhombus gives the area of parameters $a_1$, $b_1$ for the exponential stability of (2): for $a = 3$, $b = 2$ and $a = 3$, $b = 2.5$, respectively.

4 Stability Conditions II

In this section we consider equations (11) and (12) as perturbations of an exponentially stable ordinary differential equation for which only an integral estimation of its fundamental function is known.

Theorem 2 Suppose (11) is exponentially stable, (10) has a positive fundamental function $Z(t, s) > 0$, $t \geq s \geq t_0 \geq 0$, $a(t) \geq \alpha > 0$, $t - g(t) \leq \delta$, $t - h(t) \leq \tau$.

If for some $t_1 \geq t_0 + \delta$

$$Y \left[ \delta \| a \| \left( \| a \| \left\| \frac{b}{a} \right\| + \| b \| \right) + \tau \| b \| \left\| \frac{b}{a} \right\| \right] < 1,$$

(17)

where $Y$ is denoted in (12), $\| \cdot \|$ is the norm in the space $L_{\infty}[t_1, \infty)$, then equation (1) is exponentially stable.

Proof. Consider the following problem

$$\ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = f(t),$$
$$x(t) = \dot{x}(t) = 0, \ t \leq t_1.$$ 

(18)

As in the proof of Theorem 1 we can assume that $h(t) = g(t) = 0$, $t < t_1$. Let us demonstrate that for every $f \in L_{\infty}[t_1, \infty)$ the solution of problem (18) and its derivative are bounded.
The equation in problem (18) can be rewritten as
\[ \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) - a(t)\int_{g(t)}^{t} \ddot{x}(s)ds - b(t)\int_{h(t)}^{t} \dot{x}(s)ds = f(t). \] (19)

Equation (19) is equivalent to the following one
\[ x(t) - \int_{t_1}^{t} Y(t, s)a(s)\int_{g(s)}^{s} [a(\tau)\dot{x}(g(\tau)) + b(\tau)x(h(\tau))]d\tau ds - \int_{t_1}^{t} Y(t, s)b(s)\int_{h(s)}^{s} \dot{x}(\tau)d\tau ds = f_1(t), \] (20)

where \( f_1(t) = \int_{t_1}^{t} Y(t, s)f(s)ds \) and \( Y(t, s) \) is the fundamental function of equation (11). Hence \( f_1 \in L_\infty[t_1, \infty) \).

The equation in problem (18) can be rewritten in a different form
\[ \dot{x}(t) + \int_{t_1}^{t} Z(t, s)b(s)x(h(s))ds = r(t), \] (21)

where \( r(t) = \int_{t_1}^{t} Z(t, s)f(s)ds \), \( Z(t, s) \) is the fundamental function of (10). Since \( a(t) \geq 0 \) and \( Z(t, s) > 0 \), then by Lemma 3 fundamental function \( Z(t, s) \) has an exponential estimation. Hence \( r \in L_\infty[t_1, \infty) \).

From (18) and (20) we have
\[ x(t) = \int_{t_1}^{t} Y(t, s)a(s)\int_{g(s)}^{s} [a(\tau)\dot{x}(g(\tau)) + b(\tau)x(h(\tau))]d\tau ds - \int_{t_1}^{t} Y(t, s)b(s)\int_{h(s)}^{s} \dot{x}(\tau)d\tau ds = f_2(t), \] (22)

where
\[ f_2(t) = f_1(t) + \int_{t_1}^{t} Y(t, s)a(s)\int_{g(s)}^{s} f(\tau)d\tau ds. \]

Since \( \|f_2\| \leq \|f_1\| + Y\delta\|a\|\|f\| \), then \( f_2 \in L_\infty[t_1, \infty) \).

Substituting \( \dot{x} \) from (21) into (22), we obtain
\[ x(t) - \int_{t_1}^{t} Y(t, s)a(s)\int_{g(s)}^{s} [a(\tau)\int_{t_1}^{\tau} Z(g(\tau), \xi)b(\xi)x(h(\xi))d\xi]d\tau ds - \int_{t_1}^{t} Y(t, s)b(s)\int_{h(s)}^{s} \dot{x}(\tau)d\tau ds = f_3(t), \] (23)

where
\[ f_3(t) = f_2(t) - \int_{t_1}^{t} Y(t, s) \left[a(s)\int_{g(s)}^{s} r(g(\tau))d\tau - b(s)\int_{h(s)}^{s} r(\tau)d\tau\right]ds. \]

Since \( \|f_3\| \leq \|f_2\| + Y\|r\|(\|a\|\delta + \|b\|\tau) \) then \( f_3 \in L_\infty[t_1, \infty) \).

Equation (23) has the form \( x - Tx = f_3 \). Lemma 4 yields that
\[ \int_{t_1}^{g(t)} Z(g(t), s)b(s)ds \leq \sup_{t > t_1} \int_{t_1}^{t} Z(t, s)b(s)ds \leq \sup_{t > t_1} \int_{t_1}^{t} Z(t, s)a(s)\frac{b(s)}{a(s)}ds \leq \|b\| \frac{1}{\|a\|}. \]
Consider the equation with constant coefficients

\[ \ddot{x}(t) + a \dot{x}(g(t)) + b x(h(t)) = 0, \quad (24) \]

where \( a > 0, b > 0, t - g(t) \leq \delta, t - h(t) \leq \tau \).

**Corollary 2** Suppose \( a \delta \leq \frac{1}{e} \) and one of the following conditions holds:

1) \( a^2 \geq 4b, \quad 2\delta a + \frac{\tau b}{a} < 1, \)

2) \( a^2 < 4b, \quad 2\delta ab + \frac{\tau b^2}{a} < \frac{a\sqrt{4b-a^2}}{4}. \)

Then equation (24) is exponentially stable.

**Example 2.** Let us illustrate the exponential stability domain in Corollary 2 for both cases. If \( a = 3, b = 2 \) then \( a^2 > 4b \) and the condition of 1) becomes \( 6\delta + \frac{2}{3}\tau < 1 \). If \( a = 3, b = 2.5 \) then \( a^2 < 4b \) and the condition of 2) becomes \( 15\delta + \frac{25}{12}\tau < \frac{3}{4} \) or \( 20\delta + \frac{25}{9}\tau < 1 \). Here the inequality \( a \delta = 3\delta < 1/e \) should also be satisfied (the area under the horizontal line).

![Image showing a diagram with parameters \( \delta, a, \tau \) and \( \sigma \) and the area under the line \( a \sigma = 3\sigma = 1/e \) for the domain of parameters \( \tau, \sigma \) for the exponential stability of (24).

**Figure 2:** The domain inside the triangle under the horizontal line \( a \sigma = 3\sigma = 1/e \) gives the domain of parameters \( \tau, \sigma \) for the exponential stability of (24): for \( a = 3, b = 2 \) and \( a = 3, b = 2.5 \), respectively. The domain of parameters in the former case involves the domain in the latter case.

**Corollary 3** Suppose \( g(t) \equiv t, a^2 \geq 4b, \tau b < a \). Then equation (24) is exponentially stable.

**Remark.** Corollary 3 gives the same condition \( \tau b < a \) as was obtained by Burton in [5] for autonomous equation (3); however, equation (24) is not autonomous.

In Theorem 2 it was assumed that the first order equation (10) has a positive fundamental function. In the next theorem we will omit this restriction.
Theorem 3 Suppose (11) is exponentially stable, \(a(t) \geq \alpha > 0\), \(t - g(t) \leq \delta\), \(t - h(t) \leq \tau\).

If for some \(t_0 \geq 0\) the inequality \(\delta\|a\| < 1\) holds and

\[
Y \left[ \frac{(\delta\|a\|^2 + \tau\|b\|)(\|b\| + \delta\|b\|)}{1 - \delta\|a\|} + \delta\|b\| \right] < 1, \tag{25}
\]

where \(\| \cdot \|\) is the norm in the space \(L_\infty[t_0, \infty)\), then equation (7) is exponentially stable.

Proof. Without loss of generality we can assume \(t_0 = 0\) and \(h(t) = g(t) = 0\), \(t < 0\). As in the proof of Theorem 2 we will demonstrate that for every \(f \in L_\infty[0, \infty)\) the solution of (18) and its derivative are bounded.

First we will obtain an apriori estimation for the derivative of the solution of (18). Equation (18) can be rewritten in the form

\[
\ddot{x}(t) + a(t)\dot{x}(t) = f(t). \tag{26}
\]

Substituting \(\dot{x}\) from (18) into (26) we have

\[
\ddot{x}(t) + a(t)\dot{x}(t) + a(t) \int_{g(t)}^{t} [a(s)\dot{x}(g(s)) + b(s)x(h(s))] ds + b(t)x(h(t)) = r(t), \tag{27}
\]

where \(r(t) = f(t) + a(t) \int_{g(t)}^{t} f(s) ds\). Evidently \(r \in L_\infty[0, \infty)\).

Equation (27) is equivalent to the following one

\[
\dot{x}(t) + \int_{0}^{t} e^{-\int_{0}^{s} a(\xi)d\xi} \left( a(s) \int_{g(s)}^{s} [a(\tau)\dot{x}(g(\tau)) + b(\tau)x(h(\tau))] d\tau + b(s)x(h(s)) \right) ds = r_1(t), \tag{28}
\]

where \(r_1(t) = \int_{0}^{t} e^{-\int_{0}^{s} a(\xi)d\xi} r(s) ds\). Since \(a(t) \geq \alpha > 0\), then \(r_1 \in L_\infty[0, \infty)\).

Denote by \(\| \cdot \|_T\) the norm in the space \(L_\infty[0, T]\). From (28) we have

\[
\|\dot{x}\|_T \leq \delta(\|a\|\|\dot{x}\|_T + \|b\|\|x\|_T) + \left\| \frac{b}{a} \right\| \|x\|_T + \|r_1\|.
\]

Hence we obtain the following apriori estimation for \(\|\dot{x}\|_T\)

\[
\|\dot{x}\|_T \leq \frac{\delta\|b\| + \|b\|}{1 - \delta\|a\|} \|x\|_T + \frac{\|r_1\|}{1 - \delta\|a\|}. \tag{29}
\]

Equation (18) is equivalent to (20) which can be rewritten as

\[
x(t) + \int_{0}^{t} Y(t, s)a(s) \int_{g(s)}^{s} [a(\tau)\dot{x}(g(\tau)) + b(\tau)x(h(\tau))] d\tau ds - \int_{0}^{t} Y(t, s)b(s) \int_{h(s)}^{s} \dot{x}(\tau) d\tau ds = f_2(t), \tag{30}
\]

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where \( f_2(t) = f_1(t) + \int_0^t Y(t, s) a(s) \int_{g(s)} f(\tau) d\tau ds \), \( Y(t, s) \) is the fundamental function of (11). Then \( f_2 \in L_\infty[0, \infty) \) and

\[
\|x\|_T \leq Y \left[ |\delta| a(\|a\| \|\dot{x}\|_T + \|b\| \|x\|_T) + \|b\| \|\dot{x}\|_T \right] + \|f_2\|
\]

(31)

Inequalities (29) and (31) imply

\[
\|x\|_T \leq Y \left[ \frac{(|\delta| a)^2 + \|b\| (\|a\| \|\dot{x}\|_T + \|b\| \|x\|_T)}{1 - \delta \|a\|} + \delta \|b\| \right] \|x\|_T + C,
\]

where \( C \) is a positive number.

Inequality (32) has the form \( \|x\|_T \leq A \|x\|_T + C \), where \( A \) and \( C \) do not depend on \( T \) and \( 0 < A < 1 \). Hence \( x \in L_\infty[0, \infty) \). Inequality (29) implies \( \dot{x} \in L_\infty[0, \infty) \). By Lemma 3 equation (11) is exponentially stable.

Now let us apply the method of apriori estimation to equation (2).

**Theorem 4** Suppose (11) is exponentially stable, \( a(t) \geq \alpha > 0, t - g(t) \leq \delta, t - h(t) \leq \tau \).

If for some \( t_0 \geq 0 \) the inequality \( \frac{a_1}{a} < 1 \) holds and

\[
Y \left[ \frac{\|a_1\| (\|b\| \|a\| + \|b\|)}{1 - \|a_1\|} + \|b_1\| \right] < 1,
\]

(33)

where \( \| \cdot \| \) is the norm in the space \( L_\infty[0, \infty) \), \( Y \) is defined in (12), then equation (2) is exponentially stable.

**Proof.** Without loss of generality we can assume \( t_0 = 0 \). Let us demonstrate that for every \( f \in L_\infty[0, \infty) \) both the solution \( x \) of (13) and its derivative are bounded.

Equation (13) is equivalent to the following one

\[
\dot{x}(t) = -\int_0^t e^{-\int_s^t a(\tau) d\tau} [a_1(s) \dot{x}(g(s)) + b_1(s)x(h(s)) + b(s)x(s)] ds + r(t),
\]

(34)

where \( r(t) = \int_0^t e^{-\int_s^t a(\tau) d\tau} f(s) ds \). Evidently \( r \in L_\infty[0, \infty) \). Hence

\[
\|\dot{x}\|_T \leq \frac{a_1}{a} \|\dot{x}\|_T + \left( \|b\| \|a\| + \|b_1\| \right) \|x\|_T + \|r\|,
\]

where \( \| \cdot \|_T \) is the norm in \( L_\infty[0, T] \). Then

\[
\|\dot{x}\|_T \leq \frac{\|b\| + \|b_1\|}{1 - \frac{a_1}{a}} \|x\|_T + \frac{\|r\|}{1 - \frac{a_1}{a}}.
\]

(35)
Equation (13) is also equivalent to
\[
x(t) = -\int_0^t Y(t, s) [a_1(s)\dot{x}(g(s)) + b_1(s)x(h(s))] \, ds + f_1(t),
\]
where \(f_1(t) = \int_0^t Y(t, s)f(s)ds\), \(Y(t, s)\) is the fundamental function of (11). The first inequality in (12) implies \(f_1 \in L_{\infty}[0, \infty)\). Thus
\[
\|x\|_T \leq Y(\|a_1\|\|\dot{x}\|_T + \|b_1\|\|x\|_T) + \|f_1\|,
\]
which together with inequality (35) yields
\[
\|x\|_T \leq Y \left[ \|a_1\| \left( \frac{\|\dot{x}\|}{\underline{a}} + \frac{\|X_1\|}{\underline{a}} \right) + \|b_1\| \right] \|x\|_T + C.
\]
Here \(C\) is a positive number which does not depend on \(x\) and \(T\).

As in the proof of Theorem 3, we obtain that \(x\) and \(\dot{x}\) are bounded functions. Then by Lemma 3 equation (2) is exponentially stable. \(\square\)

**Remark.** If \(a_1(t) \equiv 0\) then the statements of Theorems 1 and 4 coincide.

**Corollary 4** Suppose \(a(t) \equiv a > 0\), \(b(t) \equiv b > 0\), \(a^2 \geq 4b\), \(t - g(t) \leq \delta\), \(t - h(t) \leq \tau\) and for some \(t_0 \geq 0\) we have \(a > \|a_1\|\), \(\|a_1\|(b + \|b_1\|) < (a - \|a_1\|)(b - \|b_1\|)\), where \(\|\cdot\|\) is the norm in the space \(L_{\infty}[t_0, \infty)\). Then equation (2) is exponentially stable.

**Example 3.** Let us illustrate the exponential stability domain in Corollary 4. If \(a = 3\), \(b = 2\) then \(a^2 > 4b\) and the condition becomes \(4|a_1| + 3|b_1| < 6\) for constant \(a_1, b_1\) (the domain inside the rhombus in Fig. 3) and bounded delays.

![Figure 3](image-url)

**Figure 3:** The domain inside a rhombus gives the area of parameters \(a_1, b_1\) for the exponential stability of (2) for \(a = 3\), \(b = 2\).

**Corollary 5** Suppose \(a_1(t) \equiv 0\), \(a(t) \equiv a > 0\), \(b(t) \equiv b > 0\), \(a^2 \geq 4b\), \(t - h(t) \leq \tau\) and for some \(t_0 \geq 0\) we have \(\|b_1\| < b\), where \(\|\cdot\|\) is the norm in the space \(L_{\infty}[t_0, \infty)\). Then equation (2) is exponentially stable.
5 Explicit Stability Conditions

In all previous results we have assumed that the ordinary differential equation (11) is exponentially stable and an estimation on the integral of its fundamental function is known. All these conditions can hold, as the following lemma demonstrates.

Lemma 7 Suppose for some \( t_0 \geq 0 \)
\[
a = \inf\limits_{t \geq t_0} a(t) > 0, \quad b = \inf\limits_{t \geq t_0} b(t) > 0, \quad B = \sup\limits_{t \geq t_0} b(t), \quad a^2 \geq 4B. \tag{37}
\]
Then the fundamental function of (11) is positive: \( Y(t, s) > 0, t > s \geq t_0. \)
Moreover, (11) is exponentially stable and for its fundamental function we have
\[
\int_{t_0}^t Y(t, s)b(s)ds \leq 1. \tag{38}
\]

As an application of Lemma 7, we will obtain new explicit stability conditions for equations (1) and (2).

Theorem 5 Suppose there exists \( t_0 \geq 0 \) such that condition (37) holds,
\[
\int_{g(t)}^t a(s)ds \leq \frac{1}{e}, \quad t \geq t_0, \tag{39}
\]
and \( t - g(t) \leq \delta, \quad t - h(t) \leq \tau \) for \( t \geq t_0. \)
If for some \( t_1 \geq t_0 + \delta \)
\[
\delta \left\| \frac{a}{b} \right\| \left( \left\| a \right\| \left\| \frac{b}{a} \right\| + \left\| b \right\| \right) + \tau \left| \frac{b}{a} \right| < 1, \tag{40}
\]
where \( \left\| \cdot \right\| \) is the norm in the space \( L_{\infty}[t_1, \infty) \), then equation (1) is exponentially stable.

Proof. Let us demonstrate that for every \( f \in L_{\infty}[t_1, \infty) \) the solution \( x \) of problem (18) and its derivative are bounded. The equation in problem (18) is equivalent to (23), which has the form \( x - Hx = f_3 \). Condition (39) and Lemma 6 imply that the fundamental function of equation (10) satisfies \( Z(t, s) > 0, t \geq t_0 \geq 0 \). Lemmas 4 and 6 yield that
\[
|(Hx)(t)| \leq \int_{t_1}^t Y(t, s)b(s)\left\{ \frac{a(s)}{b(s)} \int_{g(s)}^{g(t)} Z(g(\zeta), t, \xi)a(\zeta)\frac{b(\zeta)}{a(\zeta)}d\zeta + b(\zeta) \right\} d\zeta ds \| x \|
\]
\[
+ \int_{t_1}^t Y(t, s)b(s)\left[ \int_{g(s)}^{s} d\zeta \int_{t_1}^{s} Z(\zeta, \xi)a(\xi)\frac{b(\xi)}{a(\xi)}d\xi \right] ds \| x \|
\]
\[
\leq \delta \left\| \frac{a}{b} \right\| \left( \left\| a \right\| \left\| \frac{b}{a} \right\| + \left\| b \right\| \right) + \tau \left| \frac{b}{a} \right| \| x \|.
\]

Inequality (40) implies \( \| H \| < 1 \), hence the solution \( x \) of (23) and therefore of (1) is a bounded function. Equality (21) yields that \( \dot{x} \) is a bounded function. By Lemma 3 equation (1) is exponentially stable. \( \square \)
Corollary 6 Suppose there exists $t_0 \geq 0$ such that condition (37) holds, $g(t) \equiv 0$, $t - h(t) \leq \tau, t \geq t_0$. If $\| \frac{a_1}{b} \| < 1$, where $\| \cdot \|$ is the norm in the space $L_{\infty}[t_0, \infty)$, then equation (1) is exponentially stable.

Let us proceed to stability conditions for equation (2).

Theorem 6 Suppose there exists $t_0 \geq 0$ such that condition (37) holds, $t - g(t) \leq \delta, t - h(t) \leq \tau, t \geq t_0$. If $\| \frac{a_1}{b} \| < 1$ and

$$\left\| \frac{a_1}{b} \right\| \left( \left\| \frac{b_1}{b} \right\| + \left\| \frac{b_1}{b} \right\| \right) + \left( \left\| \frac{b_1}{b} \right\| + \left\| \frac{b_1}{b} \right\| \right) \| \frac{a_1}{b} \| < 1,$$

where $\| \cdot \|$ is the norm in the space $L_{\infty}[t_0, \infty)$, then equation (2) is exponentially stable.

Proof. We follow the proof of Theorem 4. From (35) and (36) we have

$$\| x \|_T \leq \int_0^T Y(t, s)b(s)ds \left( \left\| \frac{a_1}{b} \right\| \| \dot{x} \|_T + \left\| \frac{b_1}{b} \right\| \| x \|_T \right) + \| f_1 \| \leq \left( \left\| \frac{a_1}{b} \right\| \left\| \frac{b_1}{b} \right\| + \left\| \frac{b_1}{b} \right\| \right) \| x \|_T + C.$$

The end of the proof is similar to the proof of Theorem 4. \hfill \Box

Corollary 7 Suppose there exists $t_0 \geq 0$ such that condition (37) holds, $a_1(t) \equiv 0, t - h(t) \leq \tau, t \geq t_0$. If $\| \frac{b_1}{b} \| < 1$, where $\| \cdot \|$ is the norm in the space $L_{\infty}[t_0, \infty)$, then equation (2) is exponentially stable.

References

[1] N. V. Azbelev and P. M. Simonov, Stability of Differential Equations with Aftereffect. Stability and Control: Theory, Methods and Applications, 20. Taylor & Francis, London, 2003.

[2] D. Bainov and A. Domoshnitsky, Stability of a second-order differential equation with retarded argument, Dynamics and Stability of Systems 9 (1994), no. 2, 145–151.

[3] L. Berezansky and E. Braverman, On exponential stability of linear differential equations with several delays, J. Math. Anal. Appl. 324 (2006), no. 2, 1336–1355.

[4] L. Berezansky, E. Braverman and A. Domoshnitsky, Positive solutions and stability of a linear ordinary differential equation of the second order with damping term, arXiv:0807.2227v1 [math.DS] (July 14, 2008); to appear in Funct. Differ. Equ., 2009.
[5] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Academic Press, *Mathematics in Science and Engineering* **178**, 1985.

[6] T. A. Burton, Stability by Fixed Point Theory for Functional Differential Equations. Dover Publications, Mineola, New York, 2006.

[7] T. A. Burton, Fixed points, stability, and exact linearization, *Nonlinear Anal.* **61** (2005), 857–870.

[8] T. A. Burton and T. Furumochi, Asymptotic behavior of solutions of functional differential equations by fixed point theorems, *Dynam. Systems Appl.* **11** (2002), no. 4, 499–519.

[9] T. A. Burton and L. Hatvani, Asymptotic stability of second order ordinary, functional, and partial differential equations, *J. Math. Anal. Appl.* **176** (1993), 261–281.

[10] B. Cahlon and D. Schmidt, Stability criteria for certain second-order delay differential equations with mixed coefficients, *J. Comput. Appl. Math.* **170** (1994), no. 1, 79–102.

[11] A. Domoshnitsky, Componentwise applicability of Chaplygin’s theorem to a system of linear differential equations with delay, *Differential Equations* **26** (1991), no. 10, 1254–1259.

[12] L.N. Erbe, Q. Kong and B.G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, Basel, 1995.

[13] I. Győri and G. Ladas, Oscillation Theory of Delay Differential Equations, Clarendon Press, Oxford, 1991.

[14] V.B. Kolmanovskiy and V.R. Nosov, Stability of Functional Differential Equations. Academic Press, 1986.

[15] G.S. Ladde, V. Lakshmikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Argument, Marcel Dekker, New York, Basel, 1987.

[16] N. Minorski, Nonlinear Oscillations, Van Nostrand, New York, 1962.

[17] A.D. Myshkis, Linear Differential Equations with Retarded Argument, Nauka, Moscow, 1972 (in Russian).

[18] S.B. Norkin, Differential Equations of the Second Order with Retarded Argument, Translation of Mathematical Monographs, AMS, V. 31, Providence, R.I., 1972.

[19] B. Zhang, On the retarded Liénard equation, *Proc. Amer. Math. Soc.* **115** (1992), no.3, 779–785.