HADAMARD TYPE VARIATION FORMULAS FOR THE EIGENVALUES OF THE $\eta$-LAPLACIAN AND APPLICATIONS

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Abstract. In this paper we consider an analytic family of Riemannian structures on a compact smooth manifold $M$ with boundary. We impose the Dirichlet condition to the $\eta$-Laplacian and show the existence of analytic curves of its eigenfunctions and eigenvalues. We derive Hadamard type variation formulas. As an application we show that for a subset of $C^r$-metrics, $1 \leq r < \infty$, all eigenvalues of $\eta$-Laplacian operator are generically simple. Moreover, we consider families of perturbations of domains in $M$ and obtain Hadamard type formulas for the eigenvalues of the $\eta$-Laplacian in this case. We also establish the generic simplicity of the eigenvalues in this context.

1. Introduction

In [6] Berger derived variation formulas for the eigenvalues of the Laplace-Beltrami operator with respect to smooth one-parameter family of Riemannian metrics $g(t)$ on a differentiable manifold $M$. Such formulas are known as Hadamard type variation formulas. In a seminal work, Uhlenbeck [22] proved important results on generic properties of the eigenvalues and eigenfunctions of the Laplace-Beltrami operator $\Delta_g$ on a compact Riemannian manifold $(M, g)$ without boundary. In order to prove her results on genericity of the eigenvalues of $\Delta_g$, Uhlenbeck used the Thom transversality theorem. In our case, we work with the $\eta$-Laplacian $L_g := \Delta_g - g(\nabla \eta, \nabla \cdot)$, $\eta \in C^\infty(M)$, and instead of applying topological theorems, we obtain our result with the Hadamard type variation formulas.

Our motivation to look at the $\eta$-Laplacian is the following. By extending the Bochner formula for the $\eta$-Laplacian, we naturally give rise to an extension of the Ricci curvature tensor. It is the so-called Bakry-Émery-Ricci tensor given by $\text{Ric}_\eta = \text{Ric} + \nabla^2 \eta$. Now, this tensor naturally appears in the study of Ricci flow [11], almost Ricci solitons [4, 9, 18], CPE metrics [5] and quasi-Einstein metrics [3, 8]. Other two very interesting applications that are related to our work are [7] and [16]. We refer to the papers of J. Lott [15] and G. Perelman [19] for geometric background of the $\eta$-Laplacian. It is natural to look for extensions of the Berger formulas for this operator. Thus, our main goal in this paper is to derive Hadamard type formulas for the $\eta$-Laplacian and to present some applications. The use of this technique is common in the works of some authors. We would like to mention the works of Albert [1], Henry [12], Pereira [17] and Soufi-Ilias [21].

We first impose the Dirichlet boundary condition and obtain Hadamard type variation formulas (cf. Proposition 2 and 3). An important intermediate step in our approach is Lemma 2 which computes the variation rate of the $\eta$-Laplacian

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with respect to the metric variation. This lemma is an extension for the \( \eta \)-laplacian of a result due to Berger, and we offer a very simple proof. At the end we apply our Hadamard type formulas to obtain the following applications. In the first application we derive generic properties of the eigenvalues with respect to variation metrics. In the second one we consider perturbations of a bounded domain \( \Omega \) (given by diffeomorphisms) in Riemannian manifold \( (M, g) \) and we derive generic properties of the eigenvalues with respect to perturbations. In this case, it will be necessary to consider a family of operators \( \eta(t) \)-Laplacian where the function \( \eta \) depends on the parameter \( t \), see (3.7). We use a such modification mainly to prove the Proposition 3. Now we are going to state our main results.

We let \( M^r \) denote the Banach space of \( C^r \) metrics on \( M \), \( r > 1 \). Recall that a subset \( \Gamma \subset M^r \) is called residual if it contains a countable intersection of open dense sets. The property of metrics in \( \Gamma \subset M^r \) will be called generic if it holds on a residual subset.

**Theorem 1.** Given a compact Riemannian manifold \( (M, g) \), there exists a residual subset \( \Gamma \subset M^r \) such that, for all \( g \in \Gamma \), the eigenvalues of the Dirichlet problem for the \( \eta \)-Laplacian \( L_g \) are simple.

Let \( \Omega \) be a bounded domain in a Riemannian manifold \( (M, g) \), and \( D^r(\Omega) \) be the set of \( f : \Omega \to M \) such that \( f \) is \( C^r \)-diffeomorphism from \( \Omega \) to \( f(\Omega) \). It is known that this set is an affine manifold of a Banach space (see [10]). Then, we will show that the following property is generic:

**Theorem 2.** Define \( \mathcal{D} \subset D^r(\Omega) \) to be the subset of \( f : \Omega \to (M, g) \) such that all eigenvalues of \( \eta \)-Laplacian operator \( L_g \) on \( C^\infty_0(\Omega) \) (with Dirichlet boundary condition on \( f(\Omega) \)) are simple. Then \( \mathcal{D} \) is residual.

A interesting case to be considered is when metric \( g \) and the function \( \eta \) are deformed simultaneously. We hope to address this issues elsewhere.

### 2. Preliminaries

Let us consider an oriented compact Riemannian manifold \( (M, g) \) with boundary \( \partial M \) and volume form \( d\nu \). It is endowed with a weighted measure of the form \( dm = e^{-\eta}d\nu \), where \( \eta : M \to \mathbb{R} \) is a smooth function. Let \( d\nu \) be the volume form induced on \( \partial M \) and \( d\mu = e^{-\eta}d\nu \) be the corresponding weighted measure on \( \partial M \). Now, we define the \( \eta \)-Laplacian operator by \( L_g = \Delta_g - g(\nabla \eta, \nabla \cdot) \) which is essentially self-adjoint on \( C^\infty_0(M) \). Observe that this allows us to use perturbation theory for linear operators [13]. To do this, we consider the set \( M^r \) of all \( C^r \)-Riemannian metrics on \( M \). Then every \( g \in M^r \) determines the sequence \( 0 = \mu_0(g) < \mu_1(g) \leq \mu_2(g) \leq \cdots \leq \mu_k(g) \leq \cdots \) of the eigenvalues of \( L_g \) counted with their multiplicities. We regard each eigenvalue \( \mu_k(g) \) as a function of \( g \) in \( M^r \). Note that, in general, the functions \( g \to \mu_k(g) \) are continuous but not differentiable (see [14]). They are differentiable when \( \mu_k \) is simple. With this notation, the divergence theorem remains valid under the form \( \int_M Lf dm = \int_{\partial M} g(\nabla f, \nu)d\mu \). Thus, the formula of integration by parts is given by

\[
\int_M \ell Lf dm = -\int_M g(\nabla \ell, \nabla f) dm + \int_{\partial M} \ell g(\nabla f, \nu)d\mu, \tag{2.1}
\]

for all \( f, \ell \in C^\infty(M) \).
We consider the inner product $\langle T, S \rangle = \text{tr}(TS^*)$ induced by $g$ on the space of $(0,2)$-tensors on $M$, where $S^*$ denotes the adjoint tensor of $S$. Clearly, we have in local coordinates

$$\langle T, S \rangle = \sum_{i,j,k,l} g^{ik} g^{jl} T_{ij} S_{kl}.$$ 

Furthermore, we have $\Delta_g f = \langle \nabla^2 f, g \rangle$ where $\nabla^2 f = \nabla df$ is the Hessian of $f$. We also recall that each $(0,2)$-tensor $T$ on $(M, g)$ can be associated to a unique $(1,1)$-tensor by $g(T(Z), Y) := T(Z, Y)$ for all $Y, Z \in \mathfrak{X}(M)$. We shall slightly abuse notation here and will also write $T$ for this $(1,1)$-tensor. So, we consider the $(0,1)$-tensor given by

$$(\text{div} T)(v)(p) = \text{tr}(w \mapsto (\nabla_w T)(v)(p)),$$

where $p \in M$ and $v \in T_p M$. Moreover, one can define a $(0,1)$-tensor $\text{div}_g T$ putting $\text{div}_g T := \text{div} T - dq \circ T$.

Before deriving our main results, we present the following one.

**Lemma 1.** Let $T$ be a symmetric $(0,2)$-tensor on a Riemannian manifold $(M, g)$. Then

$$\text{div}_g (T(\varphi Z)) = \varphi(\text{div}_g T, Z) + \varphi(\nabla Z, T) + T(\nabla \varphi, Z), \tag{2.2}$$

for each $Z \in \mathfrak{X}(M)$ and any smooth function $\varphi$ on $M$.

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on $(M, g)$. Using the properties of $\text{div}_g$ and the symmetry of $T$, for each $Z \in \mathfrak{X}(M)$ and any smooth function $\varphi$ on $M$, we compute

\[
\begin{align*}
\text{div}_g (T(\varphi Z)) &= \text{div}_g (\varphi T(Z)) = \varphi \text{div}_g (T(Z)) + g(\nabla \varphi, T(Z)) \\
&= \varphi(\text{div} T)(Z) + \varphi \sum_i g(T(\nabla e_i Z, e_i)) - \varphi g(\nabla \eta, T(Z)) + T(\nabla \varphi, Z) \\
&= \varphi(\text{div}_g T)(Z) + \varphi \sum_i g(\nabla e_i Z, T(e_i)) + T(\nabla \varphi, Z) \\
&= \varphi(\text{div}_g T)(Z) + \varphi(\nabla Z, T) + T(\nabla \varphi, Z).
\end{align*}
\]

Finally, we use the duality $\langle \text{div}_g T(Z), Z \rangle = \langle \text{div}_g T, Z \rangle$ to conclude the proof of the lemma. \hfill $\square$

Let us observe that for every $X \in \mathfrak{X}(M)$ the operator $\text{div}_g X = \text{div} X - g(\nabla \eta, X)$ has the same properties of the operator $\text{div} X$ as well as is valid $\int_M \text{div}_g X \text{d}m = \int_{\partial M} g(X, \nu) \text{d}\mu$.

In the following, we assume all manifolds to be oriented and those that are compact are assumed to have a boundary.

### 3. Hadamard Type Variation Formulas

In this section we consider a smooth variation $g(t)$ of the metric $g$, so that $(M, g(t), dm_t)$ is a Riemannian manifold with a smooth measure. Denoting by $H$ the $(0,2)$-tensor given by $H_{ij} = \frac{d}{dt} |_{t=0} g_{ij}(t)$ and writing $h = \langle H, g \rangle$, we easily get

$$\frac{d}{dt} |_{t=0} dm_t = \frac{1}{2} h \text{d}m.$$ 

From now on we shall write in local coordinates $f_i = \partial_i f$. We first prove the following lemma.
Lemma 2. Let \((M, g)\) be a Riemannian manifold and \(g(t)\) be a differentiable variation of the metric \(g\). Then, for all \(f \in C_c^\infty(M)\), we have

\[
L'f = \left\langle \frac{1}{2}dh - \text{div}_\eta(H, df) - \langle H, \nabla^2 f \rangle, \right.
\]

where \(L' := \left. \frac{d}{dt} \right|_{t=0} L_{g(t)}\).

Proof. Indeed, since \(\langle df, d\ell \rangle = g^{ij}(t)f_i \ell_j\), for any \(\ell \in C_c^\infty(M)\), and \(\left. \frac{d}{dt} \right|_{t=0} g^{ij}(t) = -g^{ik}g^{js}H_{ks}\), we have

\[
\frac{d}{dt} \left|_{t=0} \right. \langle df, d\ell \rangle = -g^{ik}g^{js}H_{ks}f_i \ell_j = -H(g^{ik}f_i \partial_s, g^{js} \partial_s) = -H(\nabla f, \nabla \ell). \tag{3.2}
\]

From the formula of integration by parts we have

\[
\int_M \ell L_{g(t)} f \, dm = - \int_M \langle df, d\ell \rangle \, dm.
\]

Hence, from equation (3.2), we have at \(t = 0\)

\[
\int_M \ell L' f \, dm + \frac{1}{2} \int_M hL f \, dm = \int_M H(\nabla f, \nabla \ell) \, dm - \frac{1}{2} \int_M h \langle df, d\ell \rangle \, dm. \tag{3.3}
\]

Now, applying Lemma 1 for \(T = H, \varphi = \ell\) and \(Z = \nabla f\) we have

\[
\text{div}_\eta(H(\nabla f)) = \ell(\text{div}_\eta(H, df)) + \ell(H, \nabla^2 f) + H(\nabla f, \nabla \ell). \tag{3.4}
\]

Moreover,

\[
\text{div}_\eta(h \nabla f) = \ell h L f + \ell(\text{div}_\eta(H, df)) + h \langle df, d\ell \rangle. \tag{3.5}
\]

Hence, plugging (3.2) and (3.5) in (3.3), we obtain

\[
\int_M \ell L' f \, dm = \int_M \ell \left( \frac{1}{2} \langle df, d\ell \rangle - \langle \text{div}_\eta(H, df) - \langle H, \nabla^2 f \rangle \right) \, dm, \tag{3.6}
\]

which concludes the proof of the lemma. \(\square\)

Next, we consider a smooth function \(\eta : I \times M \to \mathbb{R}\) and write for simplicity \(\dot{\eta} = \left. \frac{d}{dt} \right|_{t=0} \eta(t)\). Then for all \(f \in C_c^\infty(M)\) we define

\[
\bar{L}_t f := \Delta_t f - g(t)(\nabla \eta(t), \nabla f). \tag{3.7}
\]

Thus,

\[
\left. \frac{d}{dt} \right|_{t=0} \bar{L}_t f = \Delta'f - \left. \frac{d}{dt} \right|_{t=0} g^{ij}(t) \dot{\eta}_i(t) f_j = \Delta'f - \left( \left. \frac{d}{dt} \right|_{t=0} g^{ij}(t) \right) \dot{\eta}_i f_j - g^{ij} \dot{\partial}_i \left. \frac{d}{dt} \right|_{t=0} \eta(t) f_j = L'f - \langle \nabla \eta, \nabla f \rangle. \tag{3.8}
\]

The next result extends the Lemma 3.15 of Berger in [6] for the \(\eta\)-Laplacian.

Proposition 1. Let \((M, g)\) be a compact Riemannian manifold. Consider a real analytic one-parameter family of Riemannian structures \(g(t)\) on \(M\) with \(g = g(0)\). If \(\lambda\) is an eigenvalue of multiplicity \(m > 1\) for the \(\eta\)-Laplacian \(L_\eta\), then there exists \(\varepsilon > 0\), and there exist scalars \(\lambda_i\) \((i = 1, \ldots, m)\) and functions \(\phi_i\) varying analytically in \(t\), such that, for all \(|t| < \varepsilon\) the following relations hold:

1. \(L_{g(t)} \phi_i(t) = \lambda_i(t) \phi_i(t)\);
2. \(\lambda_i(0) = \lambda\);
3. \(\{\phi_i(t)\}\) is orthonormal in \(L^2(M, dm_t)\).
Proof. First, let us consider an extension $g(z)$ of $g(t)$ to a domain $D_0$ of the complex plane $\mathbb{C}$. So, we consider the operator

$$L_{g(z)} : C^\infty(M; \mathbb{C}) \to C^\infty(M; \mathbb{C}),$$

that in a local coordinate system is given by

$$L_{g(z)} f = g^{ij}(z) \left( \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}(z) \frac{\partial f}{\partial x_k} - \frac{\partial g_{ij}}{\partial x_k} \right), \tag{3.9}$$

for all $f \in C^\infty(M; \mathbb{C})$, with

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left( \frac{\partial g_{ij}}{\partial x_{\ell}} + \frac{\partial g_{j\ell}}{\partial x_{i}} - \frac{\partial g_{i\ell}}{\partial x_{j}} \right).$$

Now we observe that the domain $D = H^2(M) \cap H^1_0(M)$ of the operator $L_{g(z)}$ is independent of $z$, since $M$ is compact, any two metrics are equivalent. Besides, the application $z \mapsto L_{g(z)} f$ is holomorphic for $z \in D_0$ and for every $f \in D$. Thus, $L_{g(z)}$ is a holomorphic family of type $(A)$ in [14]. Now we need to prove that the operator $L_{g(z)}$ is self-adjoint with fixed inner product, for this purpose, for each $t$, we can construct an isometry

$$P : L^2(M, dm) \to L^2(M, dm_t)$$

taking, for each $u$, $P(u) = \frac{\sqrt{\det(g_{ij}(0))}}{\sqrt{\det(g_{ij}(t))}} u$. In fact,

$$\int_M P(u)P(v)dm_t = \int_M \frac{\sqrt{\det(g_{ij}(0))}}{\sqrt{\det(g_{ij}(t))}} uvdm_t = \int_M uvdm.$$

Thus, the operator $\tilde{L}_t := P^{-1} \circ L_t \circ P$ will have the same eigenvalues of $L_t : H^2(M, dm_t) \cap H^1_0(M, dm_t) \to L^2(M, dm_t)$. But the compactness of the $M$ implies that $\tilde{L}_t$ is self-adjoint, since

$$\int_M u\tilde{L}_t vdm \overset{(isom.)}{=} \int_M P(v)L_t P(u)dm_t = \int_M P(u)L_t P(v)dm_t \overset{(isom.)}{=} \int_M P^{-1}(u)P^{-1}L_t P(v)dm = \int_M u\tilde{L}_t vdm.$$

Under these conditions we can apply a theorem due to Rellich [20] or Theorem 3.9 in Kato [14], to obtain the result of the proposition. \hfill \Box

Now, we will derive the first Hadamard type variation formula which generalizes substantially one of Berger’s formulas [6].

**Proposition 2.** Let $(M, g)$ be a compact Riemannian manifold and $g(t)$ be a differentiable variation of the metric $g$, $\{\phi_i(t)\} \subset C^\infty(M)$ a differentiable family of functions and $\lambda(t)$ a differentiable family of real numbers such that $\lambda_i(0) = \lambda$ for each $i = 1, \ldots, m$ and for all $t$

$$\begin{cases} -\tilde{L}_t \phi_i(t) = \lambda_i(t) \phi_i(t) & \text{in } M \\ \phi_i(t) = 0 & \text{on } \partial M, \end{cases}$$

with $\langle \phi_i(t), \phi_j(t) \rangle_{L^2(M, dm_t)} = \delta_{ij}$. Then the derivative of the $t \to (\lambda_i(t) + \lambda_j(t)) \delta_{ij}$ is given by

$$(\lambda_i + \lambda_j) \delta_{ij} = \int_M \left\langle \frac{1}{2} L(\phi_i \phi_j) g - 2d\phi_i \otimes d\phi_j, H \right\rangle dm + \int_M \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle dm. \tag{3.10}$$
Proof. We first do the case when $\eta$ does not depend on $t$. Differentiating the equation $-L_{\eta(t)}\phi_i(t) = \lambda_i(t)\phi_i(t)$, we have at $t = 0$, $-L'\phi_i - L\phi_i' = \lambda'_i\phi_i + \lambda_i\phi''_i$, so

$$-\int_M (\phi_j L'\phi_i + \phi_j L\phi_i')dm = \int_M (\lambda_i\phi_j + \lambda_j\phi_i')dm = \lambda'_i \int_M \phi_j\phi_i dm - \int_M \phi_i'L\phi_j dm.$$  

Using integration by parts and the fact that $\phi_i = 0$ on $\partial M$, we obtain

$$\lambda'_i\delta_{ij} = -\int_M \phi_j L'\phi_i dm.$$

Thus, writing $s_{ij} = (\lambda'_i + \lambda'_j)$ we deduce from Lemma 2 that

$$-s_{ij}\delta_{ij} = \int_M \phi_j L'\phi_i dm + \int_M \phi_i L'\phi_j dm$$

$$= \int_M (\frac{1}{2}dh - div_{\nu}H, \phi_j d\phi_i + \phi_i d\phi_j) - \langle H, \phi_j \nabla^2 \phi_i + \phi_i \nabla^2 \phi_j \rangle dm$$

$$= \int_M (\frac{1}{2}dh, d(\phi_i, \phi_j)) dm - \int_M \phi_j ((div_{\nu}H, d\phi_i) + \langle H, \nabla^2 \phi_i \rangle) dm$$

$$- \int_M \phi_i ((div_{\nu}H, d\phi_j) + \langle H, \nabla^2 \phi_j \rangle) dm.$$

We next use Lemma 1 and again integration by parts to get

$$-s_{ij}\delta_{ij} = -\int_M \frac{h}{2}L(\phi_i, \phi_j) dm + 2\int_M H(\nabla \phi_i, \nabla \phi_j) dm,$$  

(3.11)

or equivalently

$$s_{ij}\delta_{ij} = \int_M \frac{h}{2}L(\phi_i, \phi_j) g - 2d\phi_i \otimes d\phi_j, H \rangle dm.$$

In the general case, we differentiate the equation $-\tilde{L}_t\phi_i(t) = \lambda_i(t)\phi_i(t)$ at $t = 0$, $-L'\phi_i - L\phi_i' = \lambda'_i\phi_i + \lambda_i\phi''_i$, so $-L'\phi_i - L\phi_i' = \lambda'_i\phi_i + \lambda_i\phi''_i - \langle \nabla \eta, \nabla \phi_i \rangle$. Thus, we have that

$$\lambda'_i\delta_{ij} = -\int_M \phi_j L'\phi_i dm + \int_M \phi_j \langle \nabla \eta, \nabla \phi_i \rangle dm.$$

A calculation analogous to the one above completes the proof. 

In what follows, we assume that $\Omega \subset M$ is a bounded domain with smooth boundary and prove the following general formula which generalizes a result of Soulé and Ilias [21].

Proposition 3. Let $(M, g)$ be a real analytic Riemannian manifold, $\Omega \subset M$ a bounded domain, $f_t : \Omega \rightarrow (M, g)$ a analytic family of diffeomorphisms from $\Omega$ to $\Omega_t = f_t(\Omega)$, $f_0$ is the identity map and $\lambda$ an eigenvalue of multiplicity $m > 1$. Then there exists an analytic family of $m$ functions $\{\phi_i(t)\} \in C^\infty(\Omega_t)$ with $\langle \phi_i(t), \phi_j(t) \rangle_{L^2(\Omega_t, dm)} = \delta_{ij}$ and real numbers $\lambda_i(t)$ with $\lambda_i(0) = \lambda$, such that they are solutions for the Dirichlet problem

$$\begin{cases} 
-L\phi_i(t) = \lambda_i(t)\phi_i(t) & \Omega_t \\
\phi_i(t) = 0 & \partial \Omega_t, 
\end{cases}$$

for all $t, i = 1, \ldots, m$. Moreover, the derivative of the $t \mapsto (\lambda_i(t) + \lambda_j(t))\delta_{ij}$ is given by

$$(\lambda_i + \lambda_j)\delta_{ij} = -2\int_{\partial \Omega} \langle V, \nu \rangle \frac{\partial \phi_i}{\partial \nu} \frac{\partial \phi_j}{\partial \nu} d\mu,$$  

(3.12)
Recall that, $f(t) = f_t^* g$ on $\Omega$. Thus, considering

$$L_t(\phi_i(t) \circ f_t) := \Delta_t(\phi_i(t) \circ f_t) - g(t)(\nabla(\eta \circ f_t), \nabla(\phi_i(t) \circ f_t)),$$

we obtain

$$L_t(\phi_i(t) \circ f_t)(p) = -\lambda_i(t) \phi_i(t) \circ f_t(p).$$

For $\phi_i(t) = \phi_i(t) \circ f_t$, we have $\forall t$, $(\phi_i(t), \phi_j(t))_{L^2(\Omega, dm_t)} = \delta_{ij}$ and

$$\begin{cases} -L_t \phi_i(t) = \lambda_i(t) \phi_i(t) & \text{in } \Omega \\ \phi_i(t) = 0 & \text{on } \partial \Omega. \end{cases}$$

Since $\phi_i(0) \circ f_0(p) = \phi_i(p)$ and $\eta(t) = \eta \circ f_t$, we have by Proposition 2

$$s_{ij} \delta_{ij} = \int_{\Omega} \frac{h}{2} L(\phi_i \phi_j) dm - 2 \int_{\Omega} H(\nabla \phi_i, \nabla \phi_j) dm + \int_{\Omega} \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle dm.$$

Recall that, $H = \frac{d}{dt} \bigg|_{t=0} f_t^* g = L_V g$ where $V = \frac{d}{dt} \bigg|_{t=0} f_t$. Then

$$s_{ij} \delta_{ij} = \int_{\Omega} \frac{1}{2} L(\phi_i \phi_j) (g, H) dm - 2 \int_{\Omega} \left( \frac{d}{dt} \bigg|_{t=0} f_t^* g \right) (\nabla \phi_i, \nabla \phi_j) dm$$

$$+ \int_{\Omega} \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle dm$$

$$= \int_{\Omega} L(\phi_i \phi_j) \text{div} V \ dm - 2 \int_{\Omega} \langle \nabla \phi_i, V, \nabla \phi_j \rangle \ dm$$

$$- 2 \int_{\Omega} \langle \nabla \phi_i, V, \nabla \phi_j \rangle \ dm + \int_{\Omega} \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle \ dm.$$

But,

$$\langle \nabla \phi_i, V, \nabla \phi_j \rangle = \text{div}_\eta ((V, \nabla \phi_j) \nabla \phi_i) + \lambda (V, \nabla \phi_j) \phi_i - \nabla^2 \phi_j (V, \nabla \phi_i).$$

Since $\lambda = \lambda_i(0) = \lambda_j(0)$ and $\frac{a_{ij}}{2} \delta_{ij} = a_{ij}$ we have

$$a_{ij} = -\lambda \int_{\Omega} \phi_i \phi_j \text{div} V dm + \int_{\Omega} \langle \nabla \phi_i, \nabla \phi_j \rangle \text{div} V dm - \int_{\Omega} \text{div}_\eta ((V, \nabla \phi_j) \nabla \phi_i) dm$$

$$- \lambda \int_{\Omega} \langle V, \nabla \phi_j \rangle \phi_i dm + \int_{\Omega} \nabla^2 \phi_j (V, \nabla \phi_i) dm - \int_{\Omega} \text{div}_\eta ((V, \nabla \phi_i) \nabla \phi_j) dm$$

$$- \lambda \int_{\Omega} \langle V, \nabla \phi_i \rangle \phi_j dm + \int_{\Omega} \nabla^2 \phi_i (V, \nabla \phi_j) dm + \frac{1}{2} \int_{\Omega} \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle dm$$

$$= -\lambda \int_{\Omega} \left( \phi_i \phi_j \text{div} V + \langle V, \nabla (\phi_i \phi_j) \rangle \right) dm - \int_{\Omega} \langle V, \nabla \phi_j \rangle \langle \phi_i, \nu \rangle dm$$

$$- \int_{\partial \Omega} \langle V, \nabla \phi_j \rangle \langle \nu, \phi_i \rangle dm + \int_{\partial \Omega} \langle \nabla \phi_i, \nabla \phi_j \rangle \text{div} V dm + \int_{\Omega} \nabla^2 \phi_j (V, \nabla \phi_i) dm$$

$$+ \int_{\Omega} \nabla^2 \phi_i (V, \nabla \phi_j) dm + \frac{1}{2} \int_{\Omega} \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle dm.$$
As \( \phi_i = 0 \) on \( \partial \Omega \), we have \( \nabla \phi_i = \langle \nabla \phi_i, \nu \rangle \nu = \frac{\partial \phi_i}{\partial \nu} \nu \) on \( \partial \Omega \). Moreover,

\[
\text{div}_\Omega \left( \langle \nabla \phi_i, \nabla \phi_j \rangle V \right) + \langle \nabla \phi_i, \nabla \phi_j \rangle \langle \nabla \eta, V \rangle = \text{div}(\langle \nabla \phi_i, \nabla \phi_j \rangle V)
\]

\[
= \langle \nabla \phi_i, \nabla \phi_j \rangle \text{div} V + \langle \nabla \phi_i, \nabla \phi_j \rangle, V \rangle
\]

\[
= \langle \nabla \phi_i, \nabla \phi_j \rangle \text{div} V + \nabla^2 \phi_i (V, \nabla \phi_i) + \nabla^2 \phi_j (V, \nabla \phi_j).
\]

So,

\[
a_{ij} = -\lambda \int_\Omega \text{div}(\phi_i \phi_j V) \, dm - 2 \int_{\partial \Omega} \langle V, \nu \rangle \frac{\partial \phi_i}{\partial \nu} \frac{\partial \phi_j}{\partial \nu} \, d\mu + \int_\Omega \text{div}_\Omega \left( \langle \nabla \phi_i, \nabla \phi_j \rangle V \right) \, dm + \frac{1}{2} \int_\Omega \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle \, dm.
\]

It follows that

\[
a_{ij} = -\lambda \int_\Omega \text{div}(\phi_i \phi_j V) \, dm - 2 \int_{\partial \Omega} \langle V, \nu \rangle \frac{\partial \phi_i}{\partial \nu} \frac{\partial \phi_j}{\partial \nu} \, d\mu + \int_\Omega \text{div}_\Omega \left( \langle \nabla \phi_i, \nabla \phi_j \rangle V \right) \, dm + \frac{1}{2} \int_\Omega \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle \, dm
\]

\[
= -\int_{\partial \Omega} \langle V, \nu \rangle \frac{\partial \phi_i}{\partial \nu} \frac{\partial \phi_j}{\partial \nu} \, d\mu - \lambda \int_\Omega \text{div}(\phi_i \phi_j V) \, dm + \frac{1}{2} \int_\Omega \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle \, dm.
\]

On the other hand,

\[
0 = \int_\Omega \text{div}_\eta (\phi_i \phi_j V) \, dm = \int_\Omega \text{div}(\phi_i \phi_j V) \, dm - \int_\Omega \phi_i \phi_j \langle \nabla \eta, V \rangle \, dm.
\]

Hence

\[
a_{ij} = -\int_{\partial \Omega} \langle V, \nu \rangle \frac{\partial \phi_i}{\partial \nu} \frac{\partial \phi_j}{\partial \nu} \, d\mu - \lambda \int_\Omega \text{div}(\phi_i \phi_j V) \, dm + \frac{1}{2} \int_\Omega \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle \, dm.
\] (3.13)

Since \( \eta(t, p) = \eta \circ f(t, p) \) we have

\[
\hat{\eta} = \frac{d}{dt} \bigg|_{t=0} \eta(t, p) = \frac{d}{dt} \bigg|_{t=0} (\eta \circ f)(t, p) = d\eta \bigg|_{f(0,p)} \cdot f_t(p) = d\eta \bigg|_p (V) = \langle \nabla \eta, V \rangle.
\]

We next use that \( \lambda_i(0) = \lambda_j(0) = \lambda \), \( L(\phi_i \phi_j) = \phi_i L \phi_j + \phi_j L \phi_i + 2\langle \nabla \phi_i, \nabla \phi_j \rangle \) and integration by parts to calculate

\[
\frac{1}{2} \int_\Omega \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle \, dm = -\frac{1}{2} \int_\Omega \hat{\eta} L(\phi_i \phi_j) \, dm + \frac{1}{2} \int_{\partial \Omega} \langle \nabla \eta, \nabla (\phi_i \phi_j) \rangle \, d\mu
\]

\[
= \int_\Omega \langle \nabla \eta, V \rangle (\lambda \phi_i \phi_j - \langle \nabla \phi_i, \nabla \phi_j \rangle) \, dm.
\]

This computation tells us that the last two terms in (3.13) cancel each other, which concludes the proof of the proposition. \( \square \)
4. Applications

We now turn our attention to two applications of the Hadamard type formulas we derived in the previous section. We first prove the following:

**Proposition 4.** Let \((M, g_0)\) be a compact Riemannian manifold and \(\lambda\) an eigenvalue of the \(\eta\)-Laplacian for the Dirichlet problem with multiplicity \(m > 1\). Then in every neighborhood of \(g_0\) with respect to the \(C^r\) topology, there exists \(g\) and a positive \(\epsilon_{g,r}\) such that if \(|\lambda(g) - \lambda(g_0)| < \epsilon_{g,r}\), then \(\lambda(g)\) is simple.

**Proof.** Consider \(g(t) = g + tT\), where \(T\) is any symmetric \((0, 2)\)-tensor \(T\) on \((M^n, g)\). We can choose \(t\) sufficiently small so that \(g(t)\) is a Riemannian metric and the eigenvalue \(\lambda(t)\) satisfies

\[
\begin{align*}
-L_{g(t)}\phi_i(t) &= \lambda(t)\phi_i(t) & \text{in } M \\
\phi_i(t) &= 0 & \text{on } \partial M.
\end{align*}
\]

Since \(H = \frac{d}{dt}g(t) = T\) and \(L = L_g\), by Proposition 2 we have

\[
\langle d\phi_i \otimes d\phi_j, T \rangle_{\delta_{ij}} = \int_M \left(\frac{1}{4} L(\phi_i, \phi_j)g - d\phi_i \otimes d\phi_j, T\right)_{\delta_{ij}} dm.
\]

(4.1)

Now, considering the symmetrizer \(S = \frac{d\phi_i \otimes d\phi_j + d\phi_j \otimes d\phi_i}{2}\) and using the fact that

\[
\langle d\phi_i \otimes d\phi_j, T \rangle = \langle d\phi_j \otimes d\phi_i, T \rangle
\]

we deduce the next identity

\[
\lambda'\delta_{ij} = \int_M \left(\frac{1}{4} L(\phi_i, \phi_j)g - S, T\right)_{\delta_{ij}} dm.
\]

(4.2)

If \(i \neq j\), we have

\[
\frac{1}{4} L(\phi_i, \phi_j)g = S.
\]

(4.3)

Furthermore, taking the trace in equation (4.3), we have

\[
g(\nabla \phi_i, \nabla \phi_j) = \frac{n}{4} L(\phi_i, \phi_j) = \frac{n}{4} (\phi_i L \phi_j + \phi_j L \phi_i + 2g(\nabla \phi_i, \nabla \phi_j))
\]

\[
= \frac{n}{2} (-\lambda \phi_i \phi_j + g(\nabla \phi_i, \nabla \phi_j)).
\]

(4.4)

For \(n \neq 2\) we can write

\[
\frac{n\lambda}{n-2} \phi_i \phi_j = g(\nabla \phi_i, \nabla \phi_j).
\]

(4.5)

Fixing \(p \in M\) we consider an integral curve \(\alpha\) in \(M\) such that \(\alpha(0) = p\) and \(\alpha'(s) = \nabla \phi_i(\alpha(s))\). Defining \(\beta(s) := \phi_j(\alpha(s))\), we compute

\[
\beta'(s) = \langle \nabla \phi_j(\alpha(s)), \alpha'(s) \rangle = g(\nabla \phi_j, \nabla \phi_i)(\alpha(s)) = \frac{n\lambda}{n-2} \phi_i \phi_j(\alpha(s))
\]

\[
= \frac{n\lambda}{n-2} \phi_i(\alpha(s)) \beta(s),
\]

which is a contradiction, since \(M\) is compact. For the case \(n = 2\), we have from equation (4.3) that \(\phi_i \phi_j = 0\). Then, it follows from the principle of the unique continuation that at least one of the eigenfunctions vanishes, which is again a contradiction. Therefore, we complete the proof of Proposition 4. \(\Box\)
Proposition 5. Let \((M, g)\) be a Riemannian manifold and \(\Omega\) a bounded domain in \(M\). Let \(\lambda\) be an eigenvalue of the \(\eta\)-Laplacian for the Dirichlet problem with multiplicity \(m > 1\). Then in every neighborhood of the identity with respect to the \(C^r\) topology, there exists a diffeomorphism \(f\) and a positive \(\epsilon_{f,r}\) such that if \(|\lambda(f) - \lambda| < \epsilon_{f,r}\), then \(\lambda(f)\) is simple.

Proof. Note also that in this case Proposition 3 is still valid without the analyticity assumption. Let \(\lambda\) be an eigenvalue of multiplicity \(m > 1\). Suppose that for all perturbations by a diffeomorphism of \(\Omega\) the multiplicity of \(\lambda\) cannot be reduced. Then, it follows from (3.12) that \(\partial \phi_i / \partial \nu \partial \phi_j / \partial \nu = 0\) on \(\partial \Omega\). This way, we have either \(\partial \phi_i / \partial \nu = 0\) or \(\partial \phi_j / \partial \nu = 0\) in some open set \(U\) of \(\partial \Omega\). If \(\partial \phi_i / \partial \nu = 0\) in \(U\), since \(\phi_i = 0\) on \(\partial \Omega\), it follows from the unique continuation principle [13] that \(\phi_i = 0\) on \(\Omega\), which is a contradiction. 

4.1. Proof of Theorem 1.

Proof. Let \(C_m\) be the set of metrics in \(\mathcal{M}^r\) such that the first \(m\) eigenvalues of \(L_g\) are simple. It is known that if these eigenvalues depend continuously on the metric (see [2]), then for each \(m\) the set \(C_m\) is open in \(\mathcal{M}^r\). On the other hand, it follows from Proposition 4 that the set \(C_m\) is dense in \(\mathcal{M}^r\). Since \(\mathcal{M}^r\) is a complete metric space in the \(C^r\) topology the set \(\Gamma = \cap_{m=1}^\infty C_m\) is dense, which proves this theorem. 

4.2. Proof of Theorem 2.

Proof. Since \(D^r(\Omega)\) is an affine manifold in a Banach space, similar arguments as above allow us to obtain the Theorem 2.

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