QUASI-LINEAR STOKES PHENOMENON FOR THE HASTINGS-MCLEOD SOLUTION OF THE SECOND PAINLEVÉ EQUATION

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Abstract. Using the Riemann-Hilbert approach, we explicitly construct the asymptotic Ψ-function corresponding to the solution $y \sim \pm \sqrt{-x/2}$ as $|x| \to \infty$ to the second Painlevé equation $y_{xx} = 2y^3 + xy - \alpha$. We precisely describe the exponentially small jump in the dominant solution and the coefficient asymptotics in its power-like expansion.

1. Introduction

The second Painlevé equation,

$$(P_2) \quad y_{xx} = 2y^3 + xy - \alpha, \quad \alpha = \text{const},$$

was introduced more than a century ago in a classification of the second order ODEs $y_{xx} = R(x, y, y_x)$ with the Painlevé property [1]. To this date, equation $P_2$ has found interesting and important applications in the modern theory of nonlinear waves [2,3], plasma physics [4], bifurcation theory [5], random matrices and combinatorics [6,7], theory of semi-classical orthogonal polynomials [8] and others.

Among its various solutions, we distinguish those with a monotonic asymptotic behavior as $x \to \pm \infty$ (look for complete list of asymptotics and connection formulae to transcendent solutions of $P_2$ in [3]). For instance, if $\alpha = 0$ and $x \to +\infty$, there exists a 1-parameter family of solutions approximated with exponential accuracy by decreasing solutions of the Airy equation, $y_{xx} = xy$, see [2],

$$y \simeq a \text{Ai}(x) = \frac{a}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3} x^{3/2}} (1 + O(x^{-3/2})), \quad a = \text{const}.$$  

If in the above asymptotics $0 \leq a < 1$, then the Painlevé function is bounded and oscillates as $x \to -\infty$,

$$y \simeq b(-x)^{-1/4} \sin\left(\frac{2}{3} (-x)^{3/2} - \frac{3}{2} b^2 \ln(-x) + \phi\right),$$

with real constants $b$ and $\phi$ determined by $a$, cf. [2]. If $a > 1$, then the solution blows up at a finite point. If $a = 1$, then the solution of $P_2(\alpha = 0)$ grows like a square root as $x \to -\infty$, $y \simeq \sqrt{-x/2}$, see [3].

For the first time, the asymptotic as $|x| \to \infty$ behavior of the Painlevé transcendent in the complex domain was studied by Boutroux [10]. Generically, the Painlevé asymptotics within the sectors arg $x \in \left(\frac{\pi}{3}, \frac{\pi}{3}(n+1)\right)$, $n \in \mathbb{Z}$ is described by a modulated elliptic sine which trigonometrically degenerates along the directions arg $x = \frac{\pi}{3} n$.

Besides generic solutions, Boutroux described 1- and 0-parameter reductions of the trigonometric asymptotic solutions which admit analytic continuation from the ray arg $x = \frac{\pi}{3} n$ into an adjacent complex sector. As $x \to \infty$ in the interior of
the relevant complex sector, such solution is represented in the leading order by a
power sum of \( x \) and of a trans-series which is a sum of exponentially small terms,
\begin{equation}
(1.2) \\
y = (\text{power series}) + (\text{exponential terms}).
\end{equation}

Expansions of such kind can be obtained using a conventional perturbation analysis.

In [11, 12], it was observed that asymptotic solutions of the form \( \Omega_k \) exhibit a
quasi-linear Stokes phenomenon, i.e. a discontinuity in a minor term with respect to
\arg x. The first rigorous study of the quasi-linear Stokes phenomenon associated to
solutions \( y \sim \alpha/x \) as \( x \to \infty \) for \( P_2 \) and \( y \sim \sqrt{-x/6} \) as \( x \to \infty \) for \( P_1 \) is presented
in [13, 14], where the reader can find a discussion of other approaches to the same
problem.

Below, we study a quasi-linear Stokes phenomenon for the second Painlevé tran-
scendent with the monotonic asymptotic behavior \( y \sim \sqrt{-x/2} \) as \( x \to -\infty \). Our
main tool is the isomonodromy deformation method [15, 16, 17] in the form of the
Riemann-Hilbert problem approach via the nonlinear steepest descent method [18].

As in [13, 14], we pursue a two-fold goal, i.e. (a) we bring an exact meaning to
the formal expression (1.2), and (b) we evaluate the asymptotics of the coefficients
of the leading power series in (1.2).

2. RIEMANN-HILBERT PROBLEM FOR \( P_2 \)

The inverse problem method in the form of the Riemann-Hilbert (RH) problem
was first applied to an asymptotic study of \( P_2 \) in [10]. Further study of the RH
problem can be found in [18, 19, 20, 21, 22].

According to [10], the set of generic Painlevé functions is parameterized by two of
the Stokes multipliers of the associated linear system denoted below by the symbols
\( s_k, k = 0, 1, 2 \). As it is shown in [22], if \( s_0(1 + s_0 s_1) \neq 0 \) then the asymptotic solution
of the RH problem within the sector \( \arg x \in (\frac{2\pi}{3}, \pi) \) can be expressed in terms of
elliptic \( \theta \)-functions which in turn yields an elliptic asymptotics to the Painlevé
function itself. Assuming that the condition \( s_0 = 0 \) holds true, we arrive to an RH
problem which leads to a decreasing asymptotics \( y \sim \alpha/x \), see [13].

In the present paper, we first construct an asymptotic solution to the relevant
RH problem as \( |x| \to \infty, \arg x \in (\frac{2\pi}{3}, \pi) \) assuming that
\begin{equation}
(2.1) \\
1 + s_0 s_1 = 0.
\end{equation}

Below, we use the RH problem of Flaschka and Newell [10] modified as it is
proposed in [13]. This RH problem comes in a standard way from a collection of
properly normalized solutions of the Lax pair for \( P_2 \),
\begin{align}
(2.2a) \\
\frac{\partial \Psi}{\partial \lambda} \Psi^{-1} &= -i (4\lambda^2 + x + 2y^2) \sigma_3 - (4y \lambda + \frac{\alpha}{\lambda}) \sigma_2 - 2y_x \sigma_1, \\
(2.2b) \\
\frac{\partial \Psi}{\partial x} \Psi^{-1} &= -i \lambda \sigma_3 - y \sigma_2,
\end{align}
where \( \sigma_3 = (1 \ 0 \ 0), \sigma_2 = (0 \ -i \ 0), \sigma_1 = (0 \ 1 \ 0) \).

Let us introduce the piece-wise oriented contour \( \gamma = C \cup \rho_+ \cup \rho_- \cup \gamma_{\infty} \gamma_k \) which is
the union of the rays \( \gamma_k = \{ \lambda \in \mathbb{C} : |\lambda| > r, \ \arg \lambda = \frac{n \pi}{4} + \frac{\pi}{4}(k-1) \}, k = 0, 1, \ldots, 5 \),
oriented toward infinity, the clock-wise oriented circle \( C = \{ \lambda \in \mathbb{C} : |\lambda| = r \} \), and
of two vertical radiuses \( \rho_+ = \{ \lambda \in \mathbb{C} : |\lambda| < r, \ \arg \lambda = \frac{n \pi}{4} \} \) and \( \rho_- = \{ \lambda \in \mathbb{C} : |\lambda| < r, \ \arg \lambda = -\frac{n \pi}{4} \} \) oriented to the origin. The contour \( \gamma \) divides the complex \( \lambda \)-plane
into 8 regions \( \Omega_k, k \in \{ \text{left}, \text{right}, 0, 1, \ldots, 5 \} \). \( \Omega_{\text{left}} \) and \( \Omega_{\text{right}} \) are left and right
halves of the interior of the circle $C$ deprived the radiuses $\rho_+, \rho_-$. The regions $\Omega_k$, $k = 0, 1, \ldots, 5$, are the sectors between the rays $\gamma_k$ and $\gamma_{k-1}$ outside the circle, see Figure 2.1.

![Figure 2.1. A Riemann-Hilbert problem graph for $\Psi(\lambda)$ associated to $P_2$.](image)

Let each of the regions $\Omega_k$, $k = \text{right}, 0, 1, 2$, be a domain for a holomorphic $2 \times 2$ matrix function $\Psi_k(\lambda)$. Denote the collection of $\Psi_k(\lambda)$ by $\Psi(\lambda)$,

$$\Psi(\lambda) \big|_{\lambda \in \Omega_k} = \Psi_k(\lambda), \quad \Psi(e^{i\pi}\lambda) = \sigma_2 \Psi(\lambda) \sigma_2.$$  

Let $\Psi_+(\lambda)$ and $\Psi_-(\lambda)$ be the limits of $\Psi(\lambda)$ on $\gamma$ to the left and to the right, respectively. Let us also introduce two matrices $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

The RH problem we talk about is the following one:

1. Find a piece-wise holomorphic $2 \times 2$ matrix function $\Psi(\lambda)$ such that

$$\Psi(\lambda)e^{\theta\sigma_3} \to I, \quad \lambda \to \infty, \quad \theta = i\left(\frac{4}{3}x\lambda^3 + x\lambda\right),$$

and

$$\|\Psi_{\text{right}}(\lambda)\lambda^{-\alpha_3}\| \leq \text{const}, \quad \lambda \to 0;$$

2. on the contour $\gamma$, the jump condition holds

$$\Psi_+(\lambda) = \Psi_-(\lambda)S(\lambda),$$

where the piece-wise constant matrix $S(\lambda)$ is given by equations:

- on the rays $\gamma_k$,

$$S(\lambda)\big|_{\gamma_k} = S_k, \quad S_{2k-1} = I + s_{2k-1}\sigma_-, \quad S_{2k} = I + s_{2k}\sigma_+,$$

with the constants $s_k$ satisfying the constraints

$$s_{k+3} = -s_k, \quad s_1 - s_2 + s_3 + s_1s_2s_3 = -2\sin \pi\alpha.$$
above RH problem, the values $\alpha$ for the Remark 2.1.

\begin{equation}
(2.12)\quad E S \lambda
\end{equation}

\begin{equation}
(2.10)\quad M = \left(e^{i\pi(\alpha - \frac{1}{2})}\sigma_3 + J_{\pm}\sigma_\pm\right)(-i\sigma_2),
\end{equation}

\begin{align*}
J_+ &= 0 \quad \text{if} \quad \frac{1}{2} + \alpha \notin \mathbb{N}, \quad \text{i.e.} \quad \alpha \notin \left\{\frac{1}{2}, \frac{3}{2}, \ldots \right\}, \\
J_- &= 0 \quad \text{if} \quad \frac{1}{2} - \alpha \notin \mathbb{N}, \quad \text{i.e.} \quad \alpha \notin \left\{-\frac{1}{2}, -\frac{3}{2}, \ldots \right\};
\end{align*}

– on the circle $C$, the piece-wise constant jump matrix $S(\lambda)$ is defined by the equations

\begin{align*}
(2.11)\quad &\Psi_0(\lambda) = \Psi_{\text{right}}(\lambda)E S_0^{-1}, \quad \Psi_1(\lambda) = \Psi_{\text{right}}(\lambda)E, \quad \Psi_2(\lambda) = \Psi_{\text{right}}(\lambda)E S_1, \quad \\
&\Psi_3(\lambda) = \Psi_{\text{left}}(\lambda)\sigma_2 E \sigma_2 S_3^{-1}, \quad \Psi_4(\lambda) = \Psi_{\text{left}}(\lambda)\sigma_2 E \sigma_2, \quad \\
&\Psi_5(\lambda) = \Psi_{\text{left}}(\lambda)\sigma_2 E \sigma_2 S_4.
\end{align*}

The connection matrix $E$, the Stokes matrices $S_k$ and the monodromy matrix $M$ satisfy the cyclic relation,

\begin{equation}
(2.12)\quad E S_1 S_2 S_3 = \sigma_2 M^{-1} E \sigma_2.
\end{equation}

A solution of the RH problem (2.9), (2.10), if exists, is unique.

**Remark 2.1.** For the $\lambda$-equation associated with PII (2.2a) as well as for the above RH problem, the values $\alpha = \frac{1}{2} + n, n \in \mathbb{Z}$, are called resonant since the corresponding $\Psi$-function may have a logarithmic singularity at the origin. It is a quite common fallacy, that the RH problem is not uniquely solvable for resonant values of such parameters. As a matter of facts, monodromy data form a locally smooth complex surface (2.3), with the special points $\alpha = \frac{1}{2} + n, n \in \mathbb{Z}, s_1 = -s_2 = s_3 = (-1)^{n+1}$. To the latter, one has to attach a copy of $\mathbb{C}P^1$. A complex parameter describing the attached space $\mathbb{C}P^1$ can be interpreted as the ratio of a column entries of the connection matrix $E$ [16]. Thus neglecting the attached space $\mathbb{C}P^1$ may lead to the loss of uniqueness in the RH problem solution.

The asymptotics of $\Psi(\lambda)$ as $\lambda \to \infty$ is given by

\begin{equation}
(2.13)\quad Y(\lambda) := \Psi(\lambda)e^{\theta \sigma_3} = I + \lambda^{-1}\left(-i\mathcal{H}(\sigma_3 + \frac{y}{2}\sigma_1)\right) + \mathcal{O}(\lambda^{-2}),
\end{equation}

where

\begin{equation}
(2.14)\quad \mathcal{H} = \frac{1}{2}y_x^2 - \frac{1}{2}y^4 - \frac{1}{2}xy^2 + \alpha y
\end{equation}

is the Hamiltonian for the second Painlevé equation. Thus $y(x)$ can be extracted from the “residue” of $Y(\lambda)$ at infinity,

\begin{equation}
(2.15)\quad y = 2 \lim_{\lambda \to \infty} \lambda Y_{12}(\lambda) = 2 \lim_{\lambda \to \infty} \lambda Y_{21}(\lambda).
\end{equation}

Equation (2.13) specifies the Painlevé transcendent as the function $y = f(x, \alpha, \{s_k\})$ of the independent variable $x$, of the parameter $\alpha$ in the equation and of the Stokes multipliers $s_k$ (generically, the connection matrix $E$ can be expressed via $s_k$ using
equation (2.12) modulo arbitrary left diagonal (or triangular for half-integer $\alpha$) multiplier; at the special points of the monodromy surface (2.8) $s_1 = -s_2 = s_3 = (-1)^{n+1}$, $\alpha = \frac{1}{2} + n$, $n \in \mathbb{Z}$, the connection matrix $E$ contains a parameter $r \in \mathbb{C}P^1$ which specifies a relevant classical solution of $P_2$. Using the solution $y = f(x, \alpha, \{s_k\})$ and the symmetries of the Stokes multipliers described in [23], we obtain further solutions of $P_2$:

\begin{align}
\begin{aligned}
y &= -f(x, -\alpha, \{-s_k\}), \\
y &= e^{i\frac{2}{3}n}f(e^{i\frac{2}{3}n}x, \alpha, \{s_k+2n\}),
\end{aligned}
\end{align}

where the bar means the complex conjugation.

3. Riemann-Hilbert problem for $1 + s_0 s_1 = 0$

First of all observe that our RH problem can be transformed to an equivalent RH problem with the jump contour consisting of three branches, see Figure 3.1 (the method of transformation of Riemann-Hilbert graphs is explained in detail in [22, 13]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.1}
\caption{A transformed RH problem graph.}
\end{figure}

Let us assume now that

\begin{equation}
1 + s_0 s_1 = 0.
\end{equation}

Constraints (2.8) imply that $s_1 - s_0 = -2 \sin \pi \alpha$ as well, so that

\begin{align}
\begin{aligned}
s_1 &= e^{-i\pi \sigma (\alpha + \frac{1}{2})}, \\
s_0 &= -1/s_1 = e^{i\pi \sigma (\alpha - \frac{1}{2})}, \\
\sigma^2 &= 1,
\end{aligned}
\end{align}
while the parameter $s_2$ remains arbitrary. Using \((3.1)\), it is straightforward to check that the jump matrices in the transformed jump graph in Figure 3.1 are as follows,

\[
(S_1S_0S_1)^{-1} = \begin{pmatrix} 0 & 1/s_1 \\ -s_1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_2(S_1S_0S_1)^{-1} \sigma_2 = \begin{pmatrix} 0 & s_1 \\ -1/s_1 & 0 \end{pmatrix},
\]

\[
S_1S_0S_1S_2\sigma_2S_1^{-1}\sigma_2 = \begin{pmatrix} 0 & -1/s_1 \\ s_1 & s_1(s_1 + s_2) \end{pmatrix}, \quad \text{and} \quad \sigma_2S_1S_0S_1S_2\sigma_2S_1^{-1} = \begin{pmatrix} s_1(s_1 + s_2) & -s_1 \\ 1/s_1 & 0 \end{pmatrix}.
\]

For the connection matrix $E$ we have:

\[
(3.4a) \quad \alpha - \frac{1}{2} \notin \mathbb{Z}, \quad J_+ = J_- = 0, \quad M = e^{i\pi \alpha} \sigma_1:
\]

\[
\begin{cases}
\sigma = -1, & s_1 = -e^{i\pi(\alpha - \frac{1}{2})}; \quad ES_0^{-1}S_1^{-1} = p^{\sigma_3} \begin{pmatrix} 1 & 0 \\ s_1^2(s_1 + s_2) & 1 \end{pmatrix}, \\
\sigma = +1, & s_1 = -e^{-i\pi(\alpha - \frac{1}{2})}; \quad ES_0^{-1}S_1^{-1} = p^{-\sigma_3} \sigma_2 \begin{pmatrix} 1 & 0 \\ -s_1 & 1 \end{pmatrix},
\end{cases}
\]

where $p$ is constant;

\[
(3.4b) \quad \alpha - \frac{1}{2} = n \in \mathbb{Z}, \quad s_1 = -s_0 = (-1)^n + 1, \quad M i\sigma_2 = (-1)^n I + J_{\pm \sigma_4}:
\]

\[
\begin{cases}
\begin{array}{c}
n \in \mathbb{Z}_{<0}, \quad J_+ = 0, \quad J_- \neq 0: \quad ES_0^{-1}S_1^{-1} = p^{\sigma_3} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}, \\
n \in \mathbb{Z}_{>0}, \quad J_+ \neq 0, \quad J_- = 0: \quad ES_0^{-1}S_1^{-1} = p^{-\sigma_3} \sigma_2 \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix},
\end{array}
\end{cases}
\]

where $p$ and $q$ are constants.

If $\alpha - \frac{1}{2} = n \in \mathbb{Z}$ and $s_1 = -s_0 = (-1)^n + 1$ then the condition $J_+ = J_- = 0$ is equivalent to the equation $s_2 = (-1)^n$. On the one hand, these points have important geometrical meaning as being special points of the monodromy surface \((4.8)\). On the other hand, in this case, equation \((4.12)\) does not provide any restriction to the connection matrix $E$ which becomes arbitrary. Taking into account that, for $\alpha - \frac{1}{2} = n \in \mathbb{Z}$, the matrix $E$ is determined up to an upper or lower in dependence on the sign of $n$ triangular left multiplier, it contains one essential parameter which parameterizes a family of the (classical) Painlevé functions.

4. CASE $\alpha - \frac{1}{2} \notin \mathbb{Z}$ AND $1 + s_0s_1 = 0$

In the case of the non-special point of the monodromy surface, even for a half-integer $\alpha$, the entries of the connection matrix $E$ do not affect the Painlevé function, and thus the jump graph can be deformed to one depicted in Figure 4.1. Here,

\[
(4.1) \quad \Sigma = \begin{pmatrix} 0 & 1/s_1 \\ -s_1 & 0 \end{pmatrix}, \quad S_2S_4^{-1} = \begin{pmatrix} 1 & s_1 + s_2 \\ 0 & 1 \end{pmatrix},
\]

and other jump matrices are as above. For our convenience, we put the node points of the jump graph to the points $\lambda_{1,2} = \pm \sqrt{-x/2}$, $\arg x \in \left[\frac{2\pi}{3}, \pi\right]$. It is worth to note that the jump graph consists of the level lines $\text{Im} g(\lambda) = \text{const}$, $\text{Re} g(\lambda) = \text{const}$.
Quasi-linear Stokes phenomenon for the Hasting-McLeod solution of $P_2$

emanating from the critical points $\lambda = \pm \sqrt{-x/2}$ and $\lambda = 0$ for the function

$$g(\lambda) = i^{\frac{3}{2}} (\lambda^2 + \frac{x}{2})^{3/2}.$$  

The function $\Psi(\lambda)$ is normalized at infinity by the asymptotic condition (2.4).

Since we eliminate from the jump graph the circle around the origin, the asymptotics for $\Psi(\lambda)$ at the origin (2.5) has to be replaced by the condition

$$\|\Psi(\lambda)S_1S_0E^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}\| \leq \text{const}, \quad \lambda \to +0.$$  

The RH problem depends on the free parameter $s_2$ due to a jump across the imaginary axis. Using a quadratic change $\lambda^2 = \xi$ supplemented by a gauge transformation of the $\Psi$-function, it is possible to obtain a disjoint jump graph. Unfortunately, unlike the cases considered in \[13 \, 14\], in this case, one meets a difficulty with the normalization of the transformed RH problem which results in ambiguity in the asymptotics of the Painlevé transcendent. Thus we prefer to deal with the connected jump graph shown in Figure 4.1.

4.1. Reduced RH problem with $1 + s_0s_1 = s_1 + s_2 = 0$. Consider the non-special RH problem for $P_2$ corresponding to $1 + s_0s_1 = s_1 + s_2 = 0$. Then the non-specialty assumption implies that $\alpha - \frac{1}{3}$ is not integer. The RH problem jump graph coincides with one depicted in Figure 4.1 except for the jump across the curve lines emanating from the origin and approaching the vertical direction since now

$$S_2S_4^{-1} = \sigma_2S_4^{-1}\sigma_2 = I.$$  

The reduced RH problem is formulated as follows: find a piece-wise holomorphic function $\hat{\Psi}(\lambda)$ such that

$$\hat{\Psi}(\lambda)e^{\theta_3} \to I, \quad \lambda \to \infty;$$  

Figure 4.1. An RH problem graph for non-special points of the monodromy surface.
\( \| \hat{\Psi}(-\lambda)S_1S_0E_0^{-1}\lambda^{-\alpha_3} \| \leq \text{const}, \quad \lambda \to 0, \) where \( E_0 \) is the connection matrix \( E \) defined in (3.4) corresponding to \( s_1 + s_2 = 0 \);

\( \hat{\Psi}(\lambda) \) is discontinuous across the contour shown in Figure 4.1 (with the trivial jumps across the lines emanating from the origin and approaching the vertical directions).

**Theorem 4.1.** Let \( \text{arg } x \in [\frac{2\pi}{3}, \pi], \alpha - \frac{1}{2} \notin \mathbb{Z} \) and \( 1 + s_0s_1 = s_1 + s_2 = 0 \). Then, for large enough \( |x| \), there exists a unique solution to the reduced RH problem. If additionally the parameter \( \sigma \in \{+1, -1\} \) is defined in such a way that \( s_1 = e^{-i\pi\sigma(\alpha + \frac{1}{2})} \) then the corresponding Painlevé function has the asymptotics

\[
(4.4) \quad y(x) = y_1(x, \alpha, \sigma) = \sigma \sqrt{e^{-i\pi \frac{x}{2}}} \mathcal{O}(x^{-2/5}), \quad |x| \to \infty, \quad \text{arg } x \in [\frac{2\pi}{3}, \pi].
\]

**Proof.** Uniqueness. Let the reduced RH problem admit two solutions \( \hat{\Psi}_1(\lambda) \) and \( \hat{\Psi}_2(\lambda) \). Since all the jump matrices are unimodular, their determinants \( \det \hat{\Psi}_j(\lambda) \) are continuous across the jump contour, bounded at the origin and therefore are entire functions. Using the Liouville theorem and the normalization of \( \hat{\Psi}(\lambda) \) we conclude that \( \det \hat{\Psi}_j(\lambda) \equiv 1 \). Therefore there exists the ratio \( \chi(\lambda) = \hat{\Psi}_1(\lambda)\hat{\Psi}_2^{-1}(\lambda) \) which is continuous across the jump contour for \( \hat{\Psi}_j(\lambda) \), remains bounded as \( \lambda \to 0 \) and is normalized to the unit matrix as \( \lambda \to \infty \). Therefore, by the Liouville theorem, \( \chi(\lambda) \equiv I \).

Existence. Consider an RH problem on the curve line segment \([-\sqrt{-x/2}, 0) \cup (0, \sqrt{-x/2})\), where \( \text{arg } \sqrt{-x/2} \in [-\frac{\pi}{6}, 0] \), see Figure 4.2 (because \( \Psi \)-function can be analytically continued to \( \mathbb{C}\backslash\{0\} \), the jump contours in the RH problem can be bent in any convenient way and even “kiss” the infinity), with the quasi-permutation jump matrices:

\[
\begin{pmatrix}
\sigma_1 & \Sigma \\
\Sigma & -\sigma_1
\end{pmatrix}
\]

**Figure 4.2.** Jump graph to the RH model problem 1.
Quasi-linear Stokes phenomenon for the Hasting-McLeod solution of $P_2$.  

RH model problem 1.  

1. $\Phi_0^0(\lambda)e^{\sigma_3} \to I, \lambda \to \infty$;  
2. $\|\Phi_0^0(\lambda)S_1S_0E_0^{-1}\lambda^{-\alpha\sigma_3}\| \leq \text{const}, \lambda \to 0$;  
3. $\Phi_0^0(\lambda) = \Phi_0^0(\lambda)\sigma_2\sigma_2, \lambda \in (-\sqrt{-x/2}, 0)$,  
$\Phi_0^0(\lambda) = \Phi_0^0(\lambda)\Sigma^{-1}, \lambda \in (0, \sqrt{-x/2})$,  
$\sigma_2\Sigma\sigma_2 = \left( \begin{array}{cc} 0 & s_1 \\ -1/s_1 & 0 \end{array} \right), \Sigma^{-1} = \left( \begin{array}{cc} 0 & -1/s_1 \\ s_1 & 0 \end{array} \right)$.  

A solution to this RH problem is found explicitly in [21] in some different notations. Namely, on the complex $\lambda$-plane cut along $[-\sqrt{-x/2}, \sqrt{-x/2}]$, define the following scalar and matrix functions $\beta, Y, \delta$:  

\begin{equation} \beta(\lambda) = \left( \frac{\lambda - \sqrt{-x/2}}{\lambda + \sqrt{-x/2}} \right)^{1/4}, \end{equation}  
whose branch is fixed by the condition  

$\beta \to 1$ as $\lambda \to \infty$;  

\begin{equation} Y(\lambda) = \frac{1}{2} \begin{pmatrix} \beta + \beta^{-1} & -\sigma(\beta - \beta^{-1}) \\ -\sigma(\beta - \beta^{-1}) & \beta + \beta^{-1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \beta^{-\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \end{equation}  

\begin{equation} \delta(\lambda) = \left( \frac{\sqrt{\lambda^2 + \frac{x^2}{4}} - i\sqrt{-x/2}}{\sqrt{\lambda^2 + \frac{x^2}{4}} + i\sqrt{-x/2}} \right)^{\nu}, \quad \nu = -\frac{1}{2\pi i} \ln(i\sigma s_1) = \frac{\sigma\alpha}{2}, \end{equation}  

where the branch of the square root is fixed by its asymptotics $\sqrt{\lambda^2 + \frac{x^2}{4}} = \lambda + O(\lambda^{-1})$ as $\lambda \to +\infty$, and $x''$ is defined on the plane cut along the negative part of the real axis and its main branch is chosen.  

As easy to see,  

\begin{equation} \delta(-\lambda)\delta(\lambda) = 1. \end{equation}  

Furthermore, the functions introduced above enjoy the following jump properties:  

\begin{equation} Y_+(\lambda) = Y_-(\lambda)(-i\sigma_1), \quad \lambda \in (-\sqrt{-x/2}, \sqrt{-x/2}), \end{equation}  
\begin{equation} \delta_+(\lambda)\delta_-(\lambda) = e^{2\pi i\nu}, \quad \lambda \in (-\sqrt{-x/2}, 0), \end{equation}  
\begin{equation} \delta_+(\lambda)\delta_-(\lambda) = e^{-2\pi i\nu}, \quad \lambda \in (0, \sqrt{-x/2}), \end{equation}  
\begin{equation} g_+(\lambda) = -g_-(\lambda), \quad \lambda \in (-\sqrt{-x/2}, \sqrt{-x/2}). \end{equation}  

Therefore the function  

\begin{equation} \Phi_0^0 = Y\delta^{\sigma_3}e^{-g\sigma_3} \end{equation}  
satisfies the jump conditions of model RH problem 1.
Because of the asymptotics as \( \lambda \to \infty \)

\[
\begin{align*}
\beta(\lambda) &= 1 - \frac{1}{2\lambda} \sqrt{-x/2} + O(\lambda^{-2}), \\
Y(\lambda) &= I + \frac{1}{2\lambda} \sqrt{-\frac{x}{2}} \sigma_1 + O(\lambda^{-2}), \\
\delta(\lambda) &= 1 + O(\lambda^{-2}), \\
g(\lambda) &= i \left( \frac{1}{3} \lambda^3 + x \lambda \right) + i \frac{x^2}{6\lambda} + O(\lambda^{-3}),
\end{align*}
\]

(4.13)

we find the asymptotics as \( \lambda \to \infty \) of \( \Phi_0(\lambda) \),

\[
\Phi(\lambda) = \left( I + \frac{1}{\lambda^2} \left( -i\frac{x^2}{3} \sigma_3 + \sigma \sqrt{-\frac{x}{2}} \sigma_1 \right) + O(\lambda^{-2}) \right) e^{-\sigma_3}.
\]

(4.14)

Using asymptotics as \( \lambda \to 0 \),

\[
\begin{align*}
\beta_-(\lambda) &= e^{-ix/4} + O(\lambda), \\
Y_-(\lambda) &= \frac{1}{\sqrt{2}} \left( I + i \sigma \sigma_1 + O(\lambda) \right), \\
\delta_-(\lambda) &= e^{-i\pi(1-x)}(\lambda x - 1 + O(\lambda^2)), \\
g_-(\lambda) &= -\sqrt{2}(\sigma_3)^{3/2} + \sqrt{2}(\sigma_3)^{1/2} + O(\lambda^4),
\end{align*}
\]

(4.15)

it is easy to see that

\[
\Phi^0(\lambda) = C \left( I + O(\lambda) \right) \lambda^{-\sigma_3}
\]

(4.16)

with some constant matrix \( C \), therefore, using (3.4a), \( \| \Phi^0(\lambda) S^{-1} E_0^{-1} \lambda^{-\sigma_3} \| \leq \text{const} \). Thus the function \( \Phi^0(\lambda) \) solves model RH problem 1.

Remark 4.1. A similar quasi-permutation RH problem in an elliptic case is solved in [22]. More general quasi-permutation RH problem is solved in [24].

For the subsequent discussion it is also worth to note that

\[
\sigma_2 \Phi^0(\lambda) = \Phi(\lambda).
\]

(4.17)

We also point out that \( \Phi^0(\lambda) \) is singular at \( \lambda = \pm \sqrt{-x/2} \). To “smoothen” this singularity, let us consider the following

**RH model problem 2.** Find a piece-wise holomorphic function \( \Phi(\lambda) \) with the jumps indicated in Figure 4.3. Since we do not normalize \( \Phi(\lambda) \), it is determined up to a left multiplication in an entire matrix function. If we will use this solution in a domain different from \( \mathbb{C} \), then \( \Phi(\lambda) \) is determined up to a left multiplication in a matrix function holomorphic in this domain.

It is possible to construct a solution \( \Phi(\lambda) \) of this model problem using the classical Airy functions. The Airy function \( \text{Ai}(z) \) can be defined using the Taylor expansion

\[
\begin{align*}
\text{Ai}(z) &= \frac{1}{3^{2/3} \Gamma\left(\frac{2}{3}\right)} \sum_{k=0}^{\infty} \frac{3^k \Gamma\left(k + \frac{1}{3}\right) z^{3k}}{\Gamma\left(\frac{2}{3}\right)(3k)!} - \frac{1}{3^{1/3} \Gamma\left(\frac{1}{3}\right)} \sum_{k=0}^{\infty} \frac{3^k \Gamma\left(k + \frac{2}{3}\right) z^{3k+1}}{\Gamma\left(\frac{1}{3}\right)(3k+1)!}.
\end{align*}
\]

(4.18)
Asymptotics at infinity of this function and its derivative are as follows,

\begin{align}
\text{Ai}(z) & = \frac{z^{-1/4}}{2\sqrt{\pi}} e^{-\frac{3}{4}z^{3/2}} \left\{ \sum_{n=0}^{N} (-1)^{n} 3^{-2n} \frac{\Gamma(3n + \frac{1}{2})}{\Gamma(\frac{1}{2})(2n)!} z^{-\frac{3n}{2}} \right. + O(z^{-\frac{3}{2}(N+1)}) \right\}, \\
\text{Ai}'(z) & = -\frac{z^{1/4}}{2\sqrt{\pi}} e^{-\frac{3}{4}z^{3/2}} \left\{ \sum_{n=0}^{N} (-1)^{n+1} 3^{-2n} (3n + \frac{1}{2}) \frac{\Gamma(3n - \frac{1}{2})}{\Gamma(\frac{1}{2})(2n)!} z^{-\frac{3n}{2}} + O(z^{-\frac{3}{2}(N+1)}) \right\},
\end{align}

as \( z \to \infty \), \( \arg z \in (-\pi, \pi) \).

Introduce the matrix function \( Z_0(\lambda) \),

\begin{align}
Z_0(\lambda) = \sqrt{2\pi} e^{-i\pi/4} \left( \begin{array}{cc}
v_2(z) & v_1(z) \\
\frac{d}{dz}v_2(z) & \frac{d}{dz}v_1(z)
\end{array} \right) e^{-izi\frac{3}{2}},
\end{align}

where

\begin{align}
v_2(z) = e^{i2\pi/3} \text{Ai}(e^{i2\pi/3}z), \quad v_1(z) = \text{Ai}(z).
\end{align}

As \( |z| \to \infty \), \( \arg z \in (-\pi, \frac{\pi}{3}) \), this function has the asymptotics

\begin{align}
Z_0(z) = z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left( I + O(z^{-3/2}) \right) e^{z^{3/2}\sigma_3}.
\end{align}

Also, introduce the matrix functions

\begin{align}
Z_1(z) := Z_0(z)G_0, \quad Z_2(z) := Z_1(z)G_1, \quad Z_3(z) := Z_2(z)G_2,
\end{align}

\begin{align}
G_0 = G_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}.
\end{align}

By the properties of the Airy functions \([25, 26]\), \( Z_j(z) \) have the asymptotics \([27, 28]\) as \( z \to \infty \) and \( \arg z \in (-\pi + \frac{2\pi}{3}j, \frac{2\pi}{3} + \frac{2\pi}{3}j) \), \( j = 0, 1, 2, 3 \).
Define a piece-wise holomorphic function $Z^{RH}(z)$,

\[
Z^{RH}(z) = \begin{cases} 
Z_0(z)(is_1)^{\sigma_3/2}, & \arg z \in (-\frac{\pi}{3}, 0), \\
Z_j(z)(is_1)^{\sigma_3/2}, & \arg z \in (-\frac{2\pi}{3} + \frac{2\pi}{3}j, \frac{2\pi}{3}j), \quad j = 1, 2, \\
Z_3(z)(is_1)^{\sigma_3/2}, & \arg z \in (\frac{4\pi}{3}, \frac{5\pi}{3}).
\end{cases}
\]

By construction, $Z^{RH}(z)$ has the uniform asymptotics

\[
Z^{RH}(z) = z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\
1 & -1 \end{pmatrix} \left( I + \mathcal{O}(z^{-3/2}) \right) e^{\frac{i\pi}{4} \frac{\sigma_3}{2} (\lambda^2 + \frac{x}{2})}, \quad z \to \infty,
\]

and jump properties indicated in Figure 4.4.

\[\text{Figure 4.4. A model RH problem solvable by the Airy functions.}\]

Using the change of the independent variable

\[
z = e^{i\pi/2} 2^{3/4} (\lambda^2 + \frac{x}{2}),
\]

and the notation

\[
\zeta := \lambda - \sqrt{-x/2},
\]

we compute the “ratio” of $\Phi^0$ and $Z^{RH}$ in the annulus $c_1|x|^{-\frac{1}{4}+\epsilon} \leq |\zeta| \leq c_2|x|^{-\frac{1}{4}+\epsilon}$, where $0 < c_1 < c_2$ and $\frac{1}{4} < \epsilon < \frac{1}{2}$ are some constants,

\[
\Phi^0(\lambda)(Z^{RH}(z))^{-1} = \left( I + \mathcal{O}(x^{-1/4} \zeta^{1/2}) + \mathcal{O}(x^{-1/2} \zeta^{-2}) \right) \times 
\hat{\zeta} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\
1 & -1 \end{pmatrix} \left( \frac{1-\sigma}{2} i\sigma_1 + \frac{1+\sigma}{2} I \right) e^{i\frac{\pi}{4} \frac{\sigma_3}{2} (\frac{x}{2}) \zeta^{1/2} \sigma_3},
\]

Define the matrix function $\Phi_r(\lambda)$,

\[
\Phi_r(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\
1 & -1 \end{pmatrix} \left( \frac{1-\sigma}{2} i\sigma_1 + \frac{1+\sigma}{2} I \right) e^{i\frac{\pi}{4} \frac{\sigma_3}{2} (\frac{x}{2}) \zeta^{1/2} \sigma_3} Z^{RH}(z(\lambda)).
\]
Let us introduce the piece-wise holomorphic function \( \Phi(\lambda) \),

\[
\Phi(\lambda) = \begin{cases} 
\Phi_r(\lambda), & |\lambda - \sqrt{-x/2}| < R, \\
\sigma_2 \Phi_r(e^{-i\pi} \lambda) \sigma_2, & |\lambda + \sqrt{-x/2}| < R, \\
\Phi_0(\lambda), & |\lambda \pm \sqrt{-x/2}| > R,
\end{cases}
\]

where \( R = c|x|^{-\frac{1}{4} + \epsilon} \) for a constant \( c > 0 \) and \( \frac{1}{4} < \epsilon < \frac{1}{2} \). We look for the exact solution of the reduced RH problem in the form of the product

\[
\hat{\Psi}(\lambda) = \chi(\lambda) \Phi(\lambda).
\]

The correction function \( \chi(\lambda) \) solves the RH problem whose jump graph is shown in Figure 4.5. The jump matrix across the circle centered at \( \lambda = \sqrt{-x/2} \) is given by

![Figure 4.5. RH problem jump graph for the correction function \( \chi(\lambda) \).](image)

the ratio \( \Phi_0(\lambda) \Phi_r^{-1}(\lambda) \) which, due to \( 4.28 \), \( 4.29 \) and choosing \( \epsilon = 2/5 \), satisfies the estimate

\[
\| \Phi_0(\lambda) \Phi_r^{-1}(\lambda) - I \| \leq c|x|^{-3/10}.
\]

The jump matrices across the exterior parts of infinite contours approach the unit matrices exponentially fast,

\[
\| \Phi_0(\lambda) S_j (\Phi_0(\lambda))^{-1} - I \| \leq C|x|^{3/20} e^{-\frac{15}{4}|x|^{3/4}} |\zeta|^{3/2}.
\]

Now, the solvability of the reduced RH problem is straightforward. Indeed, consider the system of singular integral equations \( \chi_+ = I + K \chi_+ \) equivalent to the RH problem for \( \chi(\lambda) \),

\[
\chi_-(\lambda) = I + \frac{1}{2\pi i} \int_{\gamma} \chi_-(\xi) (G(\xi) - I) \frac{d\xi}{\xi - \lambda_-}.
\]
where $\gamma$ is the contour shown in Figure 1.6 and $G(\lambda) = \Phi_-(\lambda)S_j\Phi_-^{-1}(\lambda)$ is the relevant jump matrix. Using the estimates (4.32), (4.33) and the boundedness of the Cauchy operator in $L^2(\gamma)$, the singular integral operator $K$, which is a superposition of the operator of the right multiplication in $G(\lambda) - I$ and of the Cauchy operator $C_-$, satisfies the estimate, $\|K\|_{L^2(\gamma)} \leq c|x|^{-2/5}$, where the precise value of the positive constant $c$ is not important for us. Thus $K$ is a contracting operator in $L^2(\gamma)$ for large enough $|x|$. Since $KI$ is a square integrable function on $\gamma$, and observing that $\rho = \chi_- - I$ satisfies the singular integral equation $\rho = KI + K\rho$, we find $\rho \in L^2(\gamma)$ and therefore $\chi_- = I + \rho$ by iterations.

The solution found by iterations implies the asymptotics of $\chi(\lambda)$ as $\lambda \to \infty$,

\begin{equation}
\chi(\lambda) = I + \frac{1}{\lambda}O(x^{-2/5}) + O(\lambda^{-2}).
\end{equation}

Using (4.35) and (4.14) in (4.31) and taking into account the expansion (2.13), we see that the Painlevé function corresponding to the reduced RH problem has the asymptotics as $|x| \to \infty$, $\arg x \in \left[\frac{3\pi}{2}, \pi\right]$,

\begin{equation}
y(x) = \sigma\sqrt{e^{-i\pi x/2}} + O(x^{-2/5}).
\end{equation}

This completes the proof. \hfill \Box

4.2. RH problem with $1 + s_0s_1 = 0$ and $s_1 + s_2 \neq 0$. Let us go to the case of $1 + s_0s_1 = 0$ and arbitrary $s_1 + s_2 \neq 0$, see Figure 1.4. We look for the solution $\hat{\Psi}(\lambda)$ in the form of the product

\begin{equation}
\hat{\Psi}(\lambda) = \hat{\chi}(\lambda)\tilde{\Psi}(\lambda),
\end{equation}

where $\tilde{\Psi}(\lambda)$ is the solution of the reduced RH problem, i.e. with $s_1 + s_2 = 0$.

The correction function $\hat{\chi}(\lambda)$ satisfies the RH problem on two level lines $\text{Im} \ g(\lambda) = \text{const}$ emanating from the origin and approaching the vertical direction, see Figure 1.6.

The correction function $\hat{\chi}(\lambda)$ satisfies the RH problem

\begin{enumerate}
  \item $\hat{\chi}(\lambda) \to I$, $\lambda \to \infty$;
  \item $\|\hat{\chi}(\lambda)\tilde{\Psi}(\lambda)S_1S_0E^{-1}\lambda^{-\alpha\sigma_3}\| \leq \text{const}$, $\lambda \to +0$;
  \item $\hat{\chi}(\lambda) = \hat{\chi}_-(\lambda)\hat{G}(\lambda)$,
    $\hat{G}(\lambda) := \hat{G}_1(\lambda) = \Psi_-(\lambda)S_2\Psi_+^{-1}(\lambda)$, $\lambda \in (0, +i\infty)$,
    $\hat{G}(\lambda) := \hat{G}_2(\lambda) = \Psi_-(\lambda)\sigma_2S_2\sigma_3^{-1}\sigma_2\Psi_+^{-1}(\lambda)$, $\lambda \in (0, -i\infty)$,
    $\sigma_2\hat{G}_1(-\lambda)\sigma_2 = \hat{\tilde{G}}_2(\lambda)$.
\end{enumerate}

Theorem 4.2. If $\alpha - \frac{1}{3} \notin \mathbb{Z}$, $1 + s_0s_1 = 0$, $\text{arg} \ x \in \left[\frac{3\pi}{2}, \pi\right]$ and $|x|$ is large enough, then the RH problem (4.3), (4.14) is uniquely solvable. The asymptotics of the corresponding Painlevé function as $|x| \to \infty$ in the indicated sector is given by

\begin{equation}
y = y_1(x, \alpha, \sigma)-
- \frac{s_1 + s_2}{\pi} - \frac{i}{2}\sigma_3 - \frac{i}{2}\Gamma(1 + \sigma\alpha)(e^{-i\pi x})^{-\frac{3}{2}\sigma_3}e^{-\frac{2\pi}{3}(e^{-i\pi x})^{3/2}}(1 + O(x^{-1/4})),
\end{equation}

where $y_1(x, \alpha, \sigma) \sim \sigma e^{-i\pi x/2}$ is the solution of the Painlevé equation corresponding to $1 + s_0s_1 = s_1 + s_2 = 0$, while the parameter $\sigma \in \{+1, -1\}$ is defined by the use of the equation $s_1 = e^{-i\pi\sigma(\alpha + \frac{1}{3})}$.
Proof. It is enough to prove the solvability of the RH problem for \( \hat{\chi}(\lambda) \) above. In a neighborhood of the imaginary axis, \( \hat{\Psi}(\lambda) = \chi(\lambda)\Phi^0(\lambda) \), see (4.31) and (4.30). Hence, using (4.12), we have the following expression for the jump matrix across \((0, +i\infty)\),

\[
\hat{G}_1(\lambda) = I + (s_1 + s_2)\delta^2 e^{-2g}Y\sigma_+Y^{-1}\chi^{-1},
\]

(4.39a)

\[
\hat{G}_2(\lambda) = I - (s_1 + s_2)\delta^{-2}e^{2g}Y\sigma_-Y^{-1}\chi^{-1}.
\]

(4.39b)

Because the jump contour coincides with the level line \( \text{Im}(\lambda) = \text{const} \), the jump matrix \( \hat{G}_1(\lambda) \) exponentially approaches the unit matrix as \( \lambda \to +i\infty \). Similarly, \( \hat{G}_2(\lambda) - I \) decreases exponentially as \( \lambda \to -i\infty \). For the exponential \( e^{g(\lambda)} \), due to (4.15), the origin is the saddle point.

The jump matrices have algebraic singularity at \( \lambda = 0 \),

\[
\hat{G}_1(\lambda) = I + (s_1 + s_2)e^{-2\pi i\nu}(-2x)^{-2\nu}\lambda^{4\nu}e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}+2\sqrt{2}i(-x)^{1/2}}\times
\]

(4.40a)

\[
\times\frac{1}{2}(i\sigma_3 + \sigma_1)(1 + O(\lambda) + O(x^{-9/10})),
\]

\[
\hat{G}_2(\lambda) = I - (s_1 + s_2)e^{2\pi i\nu}(-2x)^{-2\nu}\lambda^{4\nu}e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}+2\sqrt{2}i(-x)^{1/2}}\times
\]

(4.40b)

\[
\times\frac{1}{2}(i\sigma_3 + \sigma_1)(1 + O(\lambda) + O(x^{-9/10})).
\]

Consider the model RH problem with the jump matrices \( \hat{G}_j(\lambda) \), \( j = 1, 2 \), of the form (4.39) but with the matrix \( \chi(\lambda)Y(\lambda) \) replaced by its asymptotics at \( \lambda = 0 \), i.e.

\[
\hat{G}_1(\lambda) = I + (s_1 + s_2)\delta^2 e^{-2g}Y\sigma_3 + \sigma_1,
\]

(4.41a)

\[
\hat{G}_2(\lambda) = I - (s_1 + s_2)\delta^{-2}e^{2g}Y\sigma_3 + \sigma_1.
\]

(4.41b)
Assume for a moment that
\begin{equation}
(4.42) \quad \Re \nu \geq 0, \quad \text{i.e., } \Re (\sigma \alpha) \geq 0.
\end{equation}
Since $\tilde{G}_j(\lambda) - I$ are constant nilpotent matrices multiplied in scalar functions, solution of the jump problem is given by the Cauchy integral,
\begin{equation}
(4.43) \quad \hat{\chi}(\lambda) = I + \frac{s_1 + s_2}{2\pi i} \left\{ \int_{0}^{+\infty} \delta^2 e^{-2g} \int_{-\infty}^{0} \delta^{-2}e^{2g} \frac{d\zeta}{\zeta - \lambda} \right\} \frac{1}{2} (i\sigma \sigma_3 + \sigma_1) =
I + \frac{s_1 + s_2}{2\pi i} \int_{0}^{+\infty} \delta^2 e^{-2g} \frac{\lambda d\zeta}{\zeta^2 - \lambda^2} (i\sigma \sigma_3 + \sigma_1).
\end{equation}
The condition $\frac{4.42}{4.12}$ ensures the convergence of the above integrals. Obviously, $\hat{\chi}(\lambda) \to I$ as $\lambda \to \infty$. As to asymptotics at the origin, the same condition $\frac{4.42}{4.12}$ ensures that $\| \hat{\chi}(\lambda)\| \leq \text{const.}$, and $\| \hat{\chi}(\lambda) \hat{\Psi}(\lambda) S_I S_0 E^{-1} \lambda^{-\alpha_3} \| \leq \text{const}.$

We will look for $\hat{\chi}(\lambda)$ in the form of the product
\begin{equation}
(4.44) \quad \hat{\chi}(\lambda) = X(\lambda)\tilde{\chi}(\lambda).
\end{equation}
For the correction function $X(\lambda)$, we have the RH problem with the jump contour shown in Figure 4.10 but with different jump matrices:
\begin{enumerate}
\item $X(\lambda) \to I$, $\lambda \to \infty$;
\item $\|X(\lambda)\| \leq \text{const}$, $\lambda \to +0$;
\item $X_+(\lambda) = X_-(\lambda)H(\lambda)$, $H(\lambda) = \hat{\chi}_-(\lambda) \tilde{G}(\lambda) \tilde{G}^{-1}(\lambda) \hat{\chi}^{-1}_-(\lambda)$.
\end{enumerate}

In the equivalent singular integral equation,
\begin{equation}
(4.45) \quad X_-(\lambda) = I + \frac{1}{2\pi i} \int_{\gamma} X_-(\zeta) \left( H(\zeta) - I \right) \frac{d\zeta}{\zeta - \lambda - },
\end{equation}
where $\gamma$ is the jump graph in Figure 4.10 and $\lambda_-$ is the right limit of $\lambda$ on the jump contour, or, in the symbolic form, $X_- = I + KX_-$, the operator $K$ is the superposition of the right multiplication in $H - I$ and of the Cauchy operator $C_-$. Using the boundedness of $C_-$ in $L^2(\gamma)$, we estimate the norm
\begin{equation}
(4.46) \quad \|K\|_{L^2(\gamma)} \leq c \|H - I\|_{L^2(\gamma)} \leq c|\arg x - \pi|^{1/2} \|\hat{\chi}_-(\lambda)\| \leq c' |x|^{-1/2},
\end{equation}
where $c$, $c'$ are positive constants whose precise value is not important for us. Taking into account the assumption $\Re \nu \geq 0 \frac{4.12}{4.12}$, the singular integral operator $K$ is contracting for large enough $|x|$, $\arg x \in \left[\frac{2\pi}{3}, \pi\right]$, and therefore equation $\frac{4.45}{4.45}$ is solvable by iterations.

To find the asymptotics of the relevant Painlevé function, it is enough to find asymptotics of $\tilde{\chi}(\lambda) \frac{4.38}{4.38}$ as $\lambda \to \infty$,
\begin{equation}
(4.47) \quad \tilde{\chi}(\lambda) = I - \frac{s_1 + s_2}{2\pi i} \int_{0}^{+\infty} \delta^2 e^{-2g} \int_{-\infty}^{0} \delta^{-2}e^{2g} \frac{d\zeta}{\zeta - \lambda} \right\} \frac{1}{2} (i\sigma \sigma_3 + \sigma_1) =
I + \lambda^{-1} h(i\sigma \sigma_3 + \sigma_1) + O(\lambda^{-3}),
\end{equation}
\begin{align*}
h &= -\frac{s_1 + s_2}{2\pi} 2^{-5\nu+\frac{7}{2}} \Gamma(2\nu+\frac{1}{3}) (-x)^{-3\nu+\frac{2}{3}} e^{-2\pi i x^{3/2}} (1 + O(\lambda^{-1/2})).
\end{align*}
Thus the asymptotics of $\Psi(\lambda)$ as $\lambda \to \infty$ is given by
\begin{equation}
\Phi(\lambda)e^{\Theta_3} = X \hat{\Psi}e^{\Theta_3} = I + \lambda^{-1}(-i\hat{h}\sigma_3 + \frac{y}{2}\sigma_1) + O(\lambda^{-2}) = \\
= I + \lambda^{-1}(-i(\hat{h}(2 - \sigma h)\sigma_3 + \frac{y_1 + 2\hat{h}_1}{2}\sigma_1) + O(\lambda^{-2}),
\end{equation}
where $\hat{h}_j = h(1 + O(x^{-2v-1/4})), j = 3, 1,$ involves the contribution of the correction function $X(\lambda)$. Comparison of two lines in (4.48) yields the asymptotics of the Painlevé function, $y = y_1 + 2\hat{h}_1$, which turns into (4.38) for $Re(\sigma\alpha) \geq 0, \alpha - \frac{1}{2} \notin \mathbb{Z}$ substituting $\nu = \sigma\alpha/2$.

Validity of the asymptotic formula (4.38) without the restriction (4.49) follows from the observation that (4.38) is invariant with respect to Bäcklund transformations. Indeed, the Schlesinger transformation of the $\Psi$-function is described by (4.38) (supplemented by tilde if necessary). The increasing degenerate Painlevé functions.

\begin{equation}
\Psi(\lambda) = R(\lambda)\Psi(\lambda), \quad R(\lambda) = I - \frac{q_\epsilon}{2\lambda}(\sigma_3 - i\epsilon\sigma_1), \quad q_\epsilon = \frac{\alpha + \frac{1}{2}}{y_x - \epsilon(y^2 + \frac{1}{2})^2}, \quad \epsilon \in \{+1, -1\},
\end{equation}
yields the $\Psi$-function, associated to a new Painlevé transcendent (4.50)
$$
\hat{y} = y + \epsilon q_\epsilon, \quad \hat{y}_x = y_x + \epsilon(y^2 - y^2), \quad \alpha = -\alpha - \epsilon,
$$
characterized however by the same Stokes multipliers. The latter with the definition of the parameter $\sigma$ via $s_1 = e^{-i\pi\sigma(\alpha + \frac{1}{2})} = \hat{s}_1 = e^{-i\pi\sigma(\alpha + \frac{1}{2})}$ implies the transformation of the parameter $\sigma$, $\hat{\sigma} = -\sigma$ for either value of $\epsilon = \pm 1$. If both $Re(\sigma\alpha) \geq 0$ and $Re(\hat{\sigma}\alpha) = Re(\sigma\alpha) + \sigma\epsilon \geq 0$, then the asymptotics of both $y$ and $\hat{y}$ is described by (4.38) (supplemented by tilde over $\sigma\alpha$, $\sigma$ and $\alpha$ if necessary). This observation constitutes the invariance of the asymptotic formula with respect to the Bäcklund transformation. Since the latter transformation is a bi-rational transformation in the space with coordinates $(y, y_x)$, the above invariance can be confirmed algebraically. Therefore, if $Re(\hat{\sigma}\alpha) < 0$ while $Re(\sigma\alpha) \geq 0$, the asymptotics of $\hat{y}$ and $y$ are described by the formula (4.38) (supplemented by tilde if necessary). The iterated use of the Bäcklund transformations completes the proof. \hfill \square

4.3. The increasing degenerate Painlevé functions. Applying the second of the symmetries (2.10) to (4.38) and changing the argument of $x$ in $2\pi$, we obtain

\begin{equation}
y = y_0(x, \alpha, \sigma) - \\
-\frac{s_2 - s_0}{\pi} - \frac{2\sigma\alpha - \frac{1}{2}}{4} \Gamma(\frac{1}{2} + \sigma\alpha)(e^{-i\pi x})^{-\frac{1}{2} + \sigma\alpha - \frac{1}{2}} e^{-\frac{i\pi x}{2}}(x^{-1/4})(1 + O(x^{-1/4})),
\end{equation}
where $y_0(x, \alpha, \sigma) = y_1(x, \alpha, \sigma) \sim \sigma \sqrt{e^{-i\pi x}/2}$ is the solution of the Painlevé equation corresponding to $1 + s_0s_1 = s_2 - s_0 = 0$, while the parameter $\sigma \in \{+1, -1\}$ is defined by the use of the equation $s_1 = e^{-i\pi\sigma(\alpha + \frac{1}{2})}$.

The solutions $y_1(x, \alpha, \sigma)$ and $y_0(x, \alpha, \sigma) = y_1(x, \alpha, \sigma)$ are meromorphic functions of $x \in \mathbb{C}$ and thus can be continued beyond the sectors indicated in Theorems 4.2 and 4.3. To find the asymptotics of the solution $y_0(x, \alpha, \sigma)$ in the interior of the
sector $\arg x \in [\frac{2\pi}{3}, \pi]$, we apply (4.35) with $s_2 = s_0$. Similarly, we find the asymptotics of the solution $y_1(x, \alpha, \sigma)$ in the interior of the sector $\arg x \in [\pi, \frac{4\pi}{3}]$ using (4.50) with $s_2 = -s_1$. Either expression implies that, if $|x| \to \infty$, $\arg x \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$,

$$y_0(x, \alpha, \sigma) = y_1(x, \alpha, \sigma) = i\sigma \frac{2^{-\frac{2\sigma-2}{\alpha}}}{\Gamma\left(\frac{1}{2} - \sigma\alpha\right)} \left(e^{-i\pi x} \right)^{-\frac{3}{2}\sigma\alpha - \frac{3}{4}} e^{-\frac{2\pi^2}{3} \left(e^{-i\pi x}\right)^{3/2} \left(1 + O(x^{-1/4})\right)},$$

where we have used the relation

$$(s_0 + s_1)\Gamma\left(\frac{1}{2} + \sigma\alpha\right) = -\frac{2i\pi\sigma}{\Gamma\left(\frac{1}{2} - \sigma\alpha\right)}.$$ 

The formula (4.52) constitutes the quasi-linear Stokes phenomenon for the increasing degenerate asymptotic solution of $P_2$.

Due to exponential decay of the difference (4.52) in the interior of the sector $\arg x \in (\frac{2\pi}{3}, \frac{4\pi}{3})$, we have solutions of $P_2$,

$$y = y_1(x, \alpha, \sigma) \simeq \sigma \sqrt{e^{-i\pi x}2}, \quad |x| \to \infty, \quad \arg x \in [\frac{2\pi}{3}, \frac{4\pi}{3}],$$

for $s_0 = -e^{i\pi\sigma(\alpha + \frac{1}{2})}$, $s_1 = e^{-i\pi\sigma(\alpha + \frac{1}{2})}$, $s_2 = -e^{-i\pi\sigma(\alpha + \frac{1}{2})}$, and

$$y = y_0(x, \alpha, \sigma) \simeq \sigma \sqrt{e^{-i\pi x}2}, \quad |x| \to \infty, \quad \arg x \in (\frac{2\pi}{3}, \frac{4\pi}{3}),$$

for $s_0 = -e^{i\pi\sigma(\alpha + \frac{1}{2})}$, $s_1 = e^{-i\pi\sigma(\alpha + \frac{1}{2})}$, $s_2 = -e^{-i\pi\sigma(\alpha + \frac{1}{2})}.$

Applying the rotational symmetry of (2.10) to $y_1$ and $y_0$, we find solutions

$$y = y_{2n+1}(x, \alpha, \sigma) := e^{i\frac{2\pi}{3} n} y_1(e^{i\frac{2\pi}{3} n} x, \alpha, \sigma) \simeq \sigma(-1)^n \sqrt{e^{-i\pi x}2}, \quad |x| \to \infty, \quad \arg x \in [\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3} - \frac{2\pi}{3} n],$$

for $s_{2n} = -e^{i\pi\sigma(\alpha + \frac{1}{2})}$, $s_{2n+1} = e^{-i\pi\sigma(\alpha + \frac{1}{2})}$, $s_{2n+2} = -e^{-i\pi\sigma(\alpha + \frac{1}{2})}$, and

$$y = y_{2n}(x, \alpha, \sigma) := e^{i\frac{2\pi}{3} n} y_0(e^{i\frac{2\pi}{3} n} x, \alpha, \sigma) \simeq \sigma(-1)^n \sqrt{e^{-i\pi x}2}, \quad |x| \to \infty, \quad \arg x \in [\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3} - \frac{2\pi}{3} n],$$

for $s_{2n} = -e^{i\pi\sigma(\alpha + \frac{1}{2})}$, $s_{2n+1} = e^{-i\pi\sigma(\alpha + \frac{1}{2})}$, $s_{2n+2} = -e^{-i\pi\sigma(\alpha + \frac{1}{2})}.$

Comparing the Stokes multipliers, we observe the symmetries

$$y_{n}(x, \alpha, \sigma) = y_{n+6}(x, \alpha, \sigma), \quad y_{2n-1}(x, \alpha, \sigma) = y_{2n}(x, \alpha, -\sigma).$$

Equation (4.52) and definitions (4.54) and (4.56) imply the differences take place,

$$y_{2n}(x, \alpha, \sigma) - y_{2n+1}(x, \alpha, \sigma) =$$

$$i\sigma \frac{2^{-\frac{2\sigma-2}{\alpha}}}{\Gamma\left(\frac{1}{2} - \sigma\alpha\right)} e^{i\frac{2\pi}{3} n} \left(e^{-i\pi e^{i\frac{2\pi}{3} n} x} \right)^{-\frac{3}{2}\sigma\alpha - \frac{3}{4}} e^{-\frac{2\pi^2}{3} \left(e^{-i\pi e^{i\frac{2\pi}{3} n} x}\right)^{3/2} \left(1 + O(x^{-1/4})\right)}, \quad |x| \to \infty, \quad \arg x \in [\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3} - \frac{2\pi}{3} n],$$

which are exponentially small in the interior of the indicated sectors.
Using the second of the equations (4.57), we establish the analytic continuation of the asymptotics as \(|x| \to \infty\),
\[
y_{2n-1}(x, \alpha, (1)^{n-1} \sigma) = y_{2n}(x, \alpha, (1)^{n} \sigma) \simeq \sigma \sqrt{e^{-i\pi x/2}},
\]
\(|x| \to \infty, \quad \arg x \in \left(\frac{2\pi}{7}(1-n), \frac{2\pi}{3}(3-n)\right)\).

5. Coefficient asymptotics for \(\alpha - \frac{1}{2} \notin \mathbb{Z}\)

An elementary investigation shows that the equation \(P_2, y_{xx} = 2y^2 + xy - \alpha\), can be satisfied by the formal series depending on \(\alpha\) and a parameter \(\sigma = \pm 1\),
\[
y_f(x, \alpha, \sigma) = \sigma \sqrt{-x/2} \sum_{n=0}^{\infty} b_n(-x)^{-3n/2} + \mathcal{O}(x^{-\infty}).
\]

Given \(\sigma\), the series \((5.1)\) is determined uniquely since its coefficients \(b_n\) are determined by the recurrence relation,
\[
b_0 = 1, \quad b_1 = \frac{\sigma \alpha}{\sqrt{2}},
\]
\[
b_{n+2} = \frac{9n^2 - 1}{8} b_n - \sum_{m=1}^{n+1} \frac{b_m b_{n+2-m}}{n+2-l} - \frac{1}{2} \sum_{l=1}^{n+2} \sum_{m=1}^{l} b_l b_m b_{n+2-l-m}.
\]

Several initial terms of the expansion are given by
\[
y_f(x, \alpha, \sigma) = \sigma \sqrt{-x/2} \left\{1 + \frac{\sigma \alpha}{\sqrt{2}(-x)^{3/2}} - \frac{1 + 6\alpha^2}{8(-x)^3} + \frac{\sigma \alpha (11 + 16\alpha^2)}{8\sqrt{2}(-x)^{9/2}} - \frac{73 + 708\alpha^2 + 420\alpha^4}{128(-x)^6} + \frac{\sigma \alpha (1021 + 2504\alpha^2 + 768\alpha^4)}{64\sqrt{2}(-x)^{15/2}} - \frac{10657 + 129918\alpha^2 + 132060\alpha^4 + 24024\alpha^6}{1024(-x)^9} + \frac{\sigma \alpha (248831 + 786304\alpha^2 + 416400\alpha^4 + 49152\alpha^6)}{512\sqrt{2}(-x)^{21/2}} + \mathcal{O}(x^{-12}) \right\}.
\]

To find the asymptotics of the coefficients \(b_n\) in \((5.1)\) as \(n \to \infty\), let us construct a sectorial analytic function \(\hat{y}(t)\),
\[
\hat{y}(t) = y_{2n-1}(e^{i\pi t^2}, \alpha, (1)^{n-1} \sigma) = y_{2n}(e^{i\pi t^2}, \alpha, (1)^{n} \sigma),
\]
\(|t| \to \infty, \quad \arg t \in \left(-\frac{3}{4}n, \frac{3}{4} - \frac{3}{4}n\right)\).

The function \(\hat{y}(x)\) has a finite number of simple poles and, by construction, has the uniform asymptotics near infinity, \(\hat{y}(t) \simeq \sigma t/\sqrt{2}\) as \(|t| \to \infty\). Using \((5.1)\), it has the following formal series expansion,
\[
\hat{y}(t) = \sigma \frac{t}{\sqrt{2}} \sum_{n=0}^{\infty} b_n t^{-3n}.
\]

Let \(y^{(N)}(x)\) be a partial sum
\[
y^{(N)}(t) = \sigma \frac{t}{\sqrt{2}} \sum_{n=0}^{N-1} b_n t^{-3n},
\]
and $v^{(N)}(t)$ be a product

$$ v^{(N)}(t) = t^{3N-2}(\hat{y}(t) - y^{(N)}(t)) = \frac{\sigma}{t^{\sqrt{2}}} \sum_{n=0}^{\infty} b_{n+N} t^{-3n}. \quad (5.7) $$

Because $t^{3N-2}y^{(N)}(t)$ is polynomial and $\hat{y}(t)/t$ is bounded as $|t| \geq \rho$, the integral of $v^{(N)}(t)$ along the circle of the radius $|t| = \rho$ containing all the pole singularities of $\hat{y}(t)$ satisfies the estimate

$$ \left| \oint_{|t| = \rho} v^{(N)}(t) \, dt \right| \leq \rho^{3N-2} \oint_{|t| = \rho} |\hat{y}(t)| \, dl \leq 2\pi \rho^{3N} \max_{|t| = \rho} |\hat{y}(t)| = C\rho^{3N}. \quad (5.8) $$

On the other hand, inflating the sectorial arcs of the circle $|t| = \rho$, we find that

$$ \oint_{|t| = \rho} v^{(N)}(t) \, dt = \oint_{|t| = R} v^{(N)}(t) \, dt + \sum_{n=-3}^{2} \oint_{|t|=R} e^{i\hat{\pi}n(t,R)} \left( v^{(N)}(t) - v^{(N)}_{-}(t) \right) \, dt. \quad (5.9) $$

Since $v^{(N)}(t) = \frac{\sigma}{t^{\sqrt{2}}} b_{N} + O(t^{-3})$, the first of the integrals in the r.h.s. of (5.9) is

$$ \oint_{|t|=R} v^{(N)}(t) \, dt = \pi i \sigma \sqrt{2} b_{N} + O(R^{-2}). \quad (5.10) $$

Last six integrals in (5.9) are computed using (5.4), (4.55), (4.56), (4.57) and (4.52),

$$ \sum_{n=-3}^{2} \oint_{|t|=R} e^{i\hat{\pi}n(t,R)} \left( v^{(N)}_{+}(t) - v^{(N)}_{-}(t) \right) \, dt = $$

$$ = 3 \int_{\rho}^{R} t^{3N-2} \left( y_{0}(e^{i\pi t^{2}, \alpha, \sigma}) - y_{1}(e^{i\pi t^{2}, \alpha, \sigma}) \right) \, dt + $$

$$ + 3(-1)^{N-1} \int_{\rho}^{R} t^{3N-2} \left( y_{0}(e^{i\pi t^{2}, \alpha, -\sigma}) - y_{1}(e^{i\pi t^{2}, \alpha, -\sigma}) \right) \, dt = $$

$$ = 3i\sigma \frac{-2^{-\frac{3}{4}\sigma - \frac{1}{4}}}{\Gamma \left( \frac{1}{2} - \sigma \alpha \right)} \int_{\rho}^{R} t^{3N-2\alpha - \frac{1}{4}} e^{-2\frac{\pi}{2\sqrt{2}} t^{3}} (1 + O(t^{-1/2})) \, dt + $$

$$ + 3(-1)^{N} i\sigma \frac{2^{\frac{3}{4}\sigma - \frac{1}{4}}}{\Gamma \left( \frac{3}{2} + \sigma \alpha \right)} \int_{\rho}^{R} t^{3N+\sigma \alpha - \frac{1}{4}} e^{-2\frac{\pi}{2\sqrt{2}} t^{3}} (1 + O(t^{-1/2})) \, dt = $$

$$ = i\sigma 2^{-\frac{3}{4}N} \sqrt{3} \left\{ 6^{-\sigma \alpha} \frac{\Gamma \left( N - \sigma \alpha - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} - \sigma \alpha \right)} \left[ 1 + O(N^{-1/6}) \right] + $$

$$ + (-1)^{N} 6^{\sigma \alpha} \frac{\Gamma \left( N + \sigma \alpha - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + \sigma \alpha \right)} \left[ 1 + O(N^{-1/6}) \right] \right\} + O(\rho^{3N} \sqrt{R^{3(N+|\text{Re}|)} - \frac{1}{2}}). $$

Thus, letting $R = \infty$, we find the asymptotics as $N \to \infty$ of the coefficient $b_{N}$ in (5.11).

$$ b_{N} = -\frac{1}{\frac{3}{2}\sqrt{2}} \left\{ 6^{-\sigma \alpha} \frac{\Gamma \left( N - \sigma \alpha - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} - \sigma \alpha \right)} \left[ 1 + O(N^{-1/6}) \right] + $$

$$ + (-1)^{N} 6^{\sigma \alpha} \frac{\Gamma \left( N + \sigma \alpha - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + \sigma \alpha \right)} \left[ 1 + O(N^{-1/6}) \right] \right\} + O(\rho^{3N}). \quad (5.12) $$
In particular, the asymptotics \[ b_{2n} \rightarrow 0 \] is consistent with the elementary observation that, for \( \alpha = 0 \), all odd coefficients vanish, \( b_{2n-1} = 0 \); for even coefficients, the leading order asymptotics reduces to the one-term expression, i.e.

\[
\alpha = 0: \quad b_{2N} = -\frac{\sqrt{2}}{\pi^{3/2} \sqrt{3}} \left( \frac{3}{2 \sqrt{2}} \right)^{2N} \Gamma(2N - \frac{1}{2}) \left( 1 + O(N^{-1/6}) \right) + O(\rho^{3N}).
\]

**Remark 5.1.** The error bound in (5.12) can be improved to \( O(x^{-3/2}) \) which implies the error bound for (5.12) \( O(N^{-1}) \).

**Remark 5.2.** It is possible to prove that the asymptotic formula (5.12) remains valid for half-integer values of \( \alpha \) as well.

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