Bifurcation from semi-trivial standing waves and ground states for a system of nonlinear Schrödinger equations

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Abstract

We consider a system of nonlinear Schrödinger equations related to the Raman amplification in a plasma. We study the orbital stability and instability of standing waves bifurcating from the semi-trivial standing wave of the system. The stability and instability of the semi-trivial standing wave at the bifurcation point are also studied. Moreover, we determine the set of the ground states completely.

1 Introduction

1.1 Motivation

In this paper, we consider the following system of nonlinear Schrödinger equations

\[
\begin{align*}
    i\partial_t u_1 &= -\Delta u_1 - \kappa |u_1| u_1 - \gamma \overline{u_1} u_2 \\
    i\partial_t u_2 &= -2\Delta u_2 - 2|u_2| u_2 - \gamma u_1^2
\end{align*}
\]

for \((t, x) \in \mathbb{R} \times \mathbb{R}^N\), where \(u_1\) and \(u_2\) are complex-valued functions of \((t, x)\), \(\kappa \in \mathbb{R}\) and \(\gamma > 0\) are constants and \(N \leq 3\). System (1.1) is a reduced system studied in [7, 8] and related to the Raman amplification in a plasma.

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Roughly speaking, the Raman amplification is an instability phenomenon taking place when an incident laser field propagates into a plasma. We refer to [5, 6] for a precise description of the phenomenon. A similar system to (1.1) also appears as an optics model with quadratic nonlinearity (see [21]).

In [7, 8], the authors studied the following three-component system

\[
\begin{align*}
  i\partial_t v_1 &= -\Delta v_1 - |v_1|^{p-1}v_1 - \gamma v_3 \overline{v_2} \\
  i\partial_t v_2 &= -\Delta v_2 - |v_2|^{p-1}v_2 - \gamma v_3 \overline{v_1} \\
  i\partial_t v_3 &= -\Delta v_3 - |v_3|^{p-1}v_3 - \gamma v_1 v_2,
\end{align*}
\]

where \(1 < p < 1 + 4/N\) and \(N \leq 3\). Let \(\omega > 0\) and let \(\varphi_\omega \in H^1(\mathbb{R}^N)\) be a unique positive radial solution of

\[
-\Delta \varphi + \omega \varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}^N.
\]

Then, \((0, 0, e^{i\omega t} \varphi_\omega)\) solves (1.2). We note that \(e^{i\omega t} \varphi_\omega\) is a standing wave solution of the single nonlinear Schrödinger equation

\[
i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,
\]

and that \(e^{i\omega t} \varphi_\omega\) is orbitally stable for (1.4) if \(1 < p < 1 + 4/N\), and it is unstable if \(1 + 4/N \leq p < 1 + 4/(N - 2)\) (see [1, 4] and also [3, Chapter 8]). In [7, 8], the authors proved the following result on the semi-trivial standing wave solution \((0, 0, e^{i\omega t} \varphi_\omega)\) of (1.2).

**Theorem 0.** ([7, 8]) Let \(N \leq 3\), \(1 < p < 1 + 4/N\), \(\omega > 0\), and let \(\varphi_\omega\) be the positive radial solution of (1.3). Then, there exists a positive constant \(\gamma^*\) such that the semi-trivial standing wave solution \((0, 0, e^{i\omega t} \varphi_\omega)\) of (1.2) is stable if \(0 < \gamma < \gamma^*\), and it is unstable if \(\gamma > \gamma^*\).

By the local bifurcation theorem by Crandall and Rabinowitz [10], it is easy to see that \(\gamma = \gamma^*\) is a bifurcation point. We are interested in the structure of the bifurcation from the semi-trivial standing wave of (1.2) and its stability property. However, this problem is difficult to study in the general case \(1 < p < 1 + 4/N\), so we consider the special case \(p = 2\). Moreover, since \(v_1\) and \(v_2\) play the same role in the proof of Theorem 0, we consider a reduced system (1.1) assuming \(v_1 = v_2\) in (1.2). We also introduce a parameter \(\kappa\) in the first equation of (1.1), which makes the structure of standing wave solutions richer as we will see below. We remark that the positive constant \(\gamma^*\) in Theorem 0 is given by

\[
\gamma^* = \inf \left\{ \frac{\|\nabla v\|^2_{L^2} + \omega \|v\|^2_{L^2}}{\int_{\mathbb{R}^N} \varphi_\omega(x) |v(x)|^2 \, dx} : v \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}.
\]
For the case $p = 2$, since $\varphi_\omega$ is the positive radial solution of
\begin{equation}
-\Delta \varphi + \omega \varphi - |\varphi|^p \varphi = 0, \quad x \in \mathbb{R}^N, \tag{1.6}
\end{equation}
we see that the infimum in (1.5) is attained at $v = \varphi_\omega$ and $\gamma^* = 1$. In the same way as the proof of Theorem 0, we can prove the following.

**Theorem 1.** Let $N \leq 3$, $\kappa \in \mathbb{R}$, $\gamma > 0$, $\omega > 0$, and let $\varphi_\omega$ be the positive radial solution of (1.6). Then, the semi-trivial standing wave solution $(0, e^{2i\omega t} \varphi_\omega)$ of (1.1) is stable if $0 < \gamma < 1$, and it is unstable if $\gamma > 1$.

We remark that the stability property of the semi-trivial standing wave of (1.1) is independent of $\kappa$ for the case $\gamma \neq 1$. On the other hand, we will see that the sign of $\kappa$ plays an important role for the case $\gamma = 1$ (see Theorems 4 and 5 below).

### 1.2 Notation and Definitions

Before we state our main results, we prepare some notation and definitions. For a complex number $z$, we denote by $\Re z$ and $\Im z$ its real and imaginary parts. Throughout this paper, we assume that $N \leq 3$. We regard $L^2(\mathbb{R}^N, \mathbb{C})$ as a real Hilbert space with the inner product
\[
(u, v)_{L^2} = \Re \int_{\mathbb{R}^N} u(x)\overline{v(x)} \, dx,
\]
and we define the inner products of real Hilbert spaces $H = L^2(\mathbb{R}^N, \mathbb{C})^2$ and $X = H^1(\mathbb{R}^N, \mathbb{C})^2$ by
\[
(\tilde{u}, \tilde{v})_H = (u_1, v_1)_{L^2} + (u_2, v_2)_{L^2}, \quad (\tilde{u}, \tilde{v})_X = (\tilde{u}, \tilde{v})_H + (\nabla \tilde{u}, \nabla \tilde{v})_H.
\]
Here and hereafter, we use the vectorial notation $\tilde{u} = (u_1, u_2)$, and it is considered to be a column vector.

The energy $E$ and the charge $Q$ are defined by
\[
E(\tilde{u}) = \frac{1}{2} \|\nabla \tilde{u}\|_H^2 - \frac{\kappa}{3} \|u_1\|_{L^3}^3 - \frac{1}{3} \|u_2\|_{L^3}^3 - \Re \int_{\mathbb{R}^N} u_1^2 u_2 \, dx,
\]
\[
Q(\tilde{u}) = \frac{1}{2} \|	ilde{u}\|_H^2, \quad \tilde{u} \in X.
\]
For $\theta \in \mathbb{R}$, we define $G(\theta)$ and $J$ by
\[
G(\theta)\tilde{u} = (e^{i\theta} u_1, e^{2i\theta} u_2), \quad J\tilde{u} = (i u_1, 2i u_2), \quad \tilde{u} \in X,
\]
and
\[ \langle G(\theta)\vec{f}, \vec{u} \rangle = \langle \vec{f}, G(-\theta)\vec{u} \rangle, \quad \langle J\vec{f}, \vec{u} \rangle = -\langle \vec{f}, J\vec{u} \rangle \]
for \( \vec{f} \in X^* \) and \( \vec{u} \in X \), where \( X^* \) is the dual space of \( X \). For \( y \in \mathbb{R}^N \), we define
\[ \tau_y \vec{u}(x) = \vec{u}(x - y), \quad \vec{u} \in X, \quad x \in \mathbb{R}^N. \]
Note that (1.1) is written as
\[ \partial_t \vec{u}(t) = -JE'(\vec{u}(t)), \]
and that \( E(G(\theta)\tau_y \vec{u}) = E(\vec{u}) \) for all \( \theta \in \mathbb{R}, \quad y \in \mathbb{R}^N \) and \( \vec{u} \in X \).

By the standard theory (see, e.g., [3, Chapter 4]), we see that the Cauchy problem for (1.1) is globally well-posed in \( X \), and the energy and the charge are conserved. For \( \omega > 0 \), we define the action \( S_\omega \) by
\[ S_\omega(\vec{v}) = E(\vec{v}) + \omega Q(\vec{v}), \quad \vec{v} \in X. \]
Note that the Euler-Lagrange equation \( S'_\omega(\vec{\phi}) = 0 \) is written as
\[ \begin{cases} -\Delta \phi_1 + \omega \phi_1 = \kappa|\phi_1|\phi_1 + \gamma \bar{\phi}_1 \phi_2 \\ -\Delta \phi_2 + \omega \phi_2 = |\phi_2|\phi_2 + (\gamma/2)\bar{\phi}_1^2 \end{cases} \tag{1.7} \]
and that if \( \vec{\phi} \in X \) satisfies \( S'_\omega(\vec{\phi}) = 0 \), then \( G(\omega t)\vec{\phi} \) is a solution of (1.1).

**Definition 1.** We say that a standing wave solution \( G(\omega t)\vec{\phi} \) of (1.1) is **stable** if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) with the following property. If \( \vec{u}_0 \in X \) satisfies \( \|\vec{u}_0 - \vec{\phi}\|_X < \delta \), then the solution \( \vec{u}(t) \) of (1.1) with \( \vec{u}(0) = \vec{u}_0 \) exists for all \( t \geq 0 \), and satisfies
\[ \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|\vec{u}(t) - G(\theta)\tau_y \vec{\phi}\|_X < \varepsilon \]
for all \( t \geq 0 \). Otherwise, \( G(\omega t)\vec{\phi} \) is called **unstable**.

In this article, we are also interested in the classification of ground states of (1.7). A ground state of (1.7) is a nontrivial solution which minimizes the action \( S_\omega' \) among all the nontrivial solutions of (1.7). The set \( G_\omega \) of the ground states for (1.7) is then defined as follows:
\[ A_\omega = \{ \vec{v} \in X : S'_\omega(\vec{v}) = 0, \quad \vec{v} \neq 0 \}, \]
\[ d(\omega) = \inf \{ S_\omega(\vec{v}) : \vec{v} \in A_\omega \}, \]
\[ G_\omega = \{ \vec{u} \in A_\omega : S_\omega(\vec{u}) = d(\omega) \}. \]
1.3 Main Results

We first look for solutions of (1.7) of the form \( \vec{\phi} = (\alpha \phi_\omega, \beta \phi_\omega) \) with \((\alpha, \beta) \in ]0, \infty[^2\), where \(\phi_\omega\) is the positive radial solution of (1.6). It is clear that if \((\alpha, \beta) \in ]0, \infty[^2\) satisfies

\[
\kappa \alpha + \gamma \beta = 1, \quad \gamma \alpha^2 + 2 \beta^2 = 2 \beta,
\]

then \((\alpha \phi_\omega, \beta \phi_\omega)\) is a solution of (1.7). For \(\kappa \in \mathbb{R}\) and \(\gamma > 0\), we define

\[
S_{\kappa, \gamma} = \{(x, y) \in ]0, \infty[^2\colon \kappa x + \gamma y = 1, \gamma x^2 + 2y^2 = 2y\}.
\]

Note that \(\gamma x^2 + 2y^2 = 2y\) is an ellipse with vertices \((x, y) = (0, 0), (0, 1), (\pm 1/\sqrt{2\gamma}, 1/2)\), and that \(S_{\kappa, \gamma} \subset \{(x, y) : 0 < y < 1\}\).

To determine the structure of the set \(S_{\kappa, \gamma}\), which is one of the crucial points of our analysis, for \(\kappa^2 \geq 2\gamma (1 - \gamma)\) we define

\[
\alpha_\pm = \frac{(2 - \gamma) \kappa \pm \gamma \sqrt{\kappa^2 + 2 \gamma (\gamma - 1)}}{2 \kappa^2 + \gamma^3},
\]

\[
\beta_\pm = \frac{\kappa^2 + \gamma^2 \pm \kappa \sqrt{\kappa^2 + 2 \gamma (\gamma - 1)}}{2 \kappa^2 + \gamma^3},
\]

\[
\alpha_0 = \frac{(2 - \gamma) \kappa}{2 \kappa^2 + \gamma^3}, \quad \beta_0 = \frac{\kappa^2 + \gamma^2}{2 \kappa^2 + \gamma^3}.
\]

We also divide the parameter domain \(D = \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\}\) into the following sets (see Figure 1).

\[
\mathcal{J}_1 = \{(\kappa, \gamma) : \kappa \leq 0, \gamma > 1\} \cup \{(\kappa, \gamma) : \kappa > 0, \gamma \geq 1\},
\]

\[
\mathcal{J}_2 = \{(\kappa, \gamma) : 0 < \gamma < 1, \kappa > \sqrt{2\gamma (1 - \gamma)}\},
\]

\[
\mathcal{J}_3 = \{(\kappa, \gamma) : 0 < \gamma < 1, \kappa = \sqrt{2\gamma (1 - \gamma)}\},
\]

\[
\mathcal{J}_0 = \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\} \setminus (\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3).
\]

Notice that the sets \(\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2\) and \(\mathcal{J}_3\) are mutually disjoint, and \(D = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3\). Note also that for \(0 < \kappa \leq 1/\sqrt{2}\), the equation \(2\gamma (1 - \gamma) = \kappa^2\) has solutions \(\gamma = \gamma_\pm := (1 \pm \sqrt{1 - 2\kappa^2})/2\). It is then possible to determine the set \(S_{\kappa, \gamma}\) in terms of \(\alpha_\pm, \beta_\pm, \alpha_0\) and \(\beta_0\). Indeed, by elementary computations, we obtain the following.

**Proposition 1.**

(0) If \((\kappa, \gamma) \in \mathcal{J}_0\), then \(S_{\kappa, \gamma}\) is empty.

(1) If \((\kappa, \gamma) \in \mathcal{J}_1\), then \(S_{\kappa, \gamma} = \{(\alpha_+, \beta_-)\}\).

(2) If \((\kappa, \gamma) \in \mathcal{J}_2\), then \(S_{\kappa, \gamma} = \{(\alpha_+, \beta_-), (\alpha_-, \beta_+)\}\).

(3) If \((\kappa, \gamma) \in \mathcal{J}_3\), then \(S_{\kappa, \gamma} = \{(\alpha_0, \beta_0)\}\).
Remark 1. (1) When $\kappa \leq 0$, $(\alpha_+, \beta_-) \to (0, 1)$ as $\gamma \to 1 + 0$. That is, the branch $\{(\alpha_+\varphi_\omega, \beta_-\varphi_\omega) : \gamma > 1\}$ of positive solutions of (1.7) bifurcates from the semi-trivial solution $(0, \varphi_\omega)$ at $\gamma = 1$.

(2) When $\kappa > 0$, $(\alpha_-, \beta_+) \to (0, 1)$ as $\gamma \to 1 - 0$. That is, the branch $\{(\alpha_-\varphi_\omega, \beta_+\varphi_\omega) : \gamma_m < \gamma < 1\}$ of positive solutions of (1.7) bifurcates from the semi-trivial solution $(0, \varphi_\omega)$ at $\gamma = 1$, where $\gamma_m = \inf\{\gamma : (\kappa, \gamma) \in S_{\kappa, \gamma}\}$, and it is given by $\gamma_m = 0$ if $\kappa > 1/\sqrt{2}$, and $\gamma_m = \gamma_+$ if $0 < \kappa \leq 1/\sqrt{2}$.

We obtain the following stability and instability results of standing waves of (1.1) associated with Proposition 1. Recall that $\varphi_\omega$ is the positive radial solution of (1.6).

**Theorem 2.** Let $N \leq 3$ and $(\kappa, \gamma) \in J_1 \cup J_2$. For any $\omega > 0$, the standing wave solution $G(\omega t)(\alpha_+\varphi_\omega, \beta_-\varphi_\omega)$ of (1.1) is stable.

**Theorem 3.** Let $N \leq 3$ and $(\kappa, \gamma) \in J_2$. For any $\omega > 0$, the standing wave solution $G(\omega t)(\alpha_-\varphi_\omega, \beta_+\varphi_\omega)$ of (1.1) is unstable.

**Remark 2.** In this paper, we do not study the stability/instability problem of $G(\omega t)(\alpha_0\varphi_\omega, \beta_0\varphi_\omega)$ for the case $(\kappa, \gamma) \in J_3$.

**Remark 3.** The result for the case $\kappa = 1$ in Theorem 3 is announced in [16] together with an outline of the proof.

We also obtain the stability and instability results of semi-trivial standing wave at the bifurcation point $\gamma = 1$. The results depend on the sign of $\kappa$.

**Theorem 4.** Let $N \leq 3$, $\kappa > 0$ and $\gamma = 1$. For any $\omega > 0$, the standing wave solution $(0, e^{2i\omega t}\varphi_\omega)$ of (1.1) is unstable.
Theorem 5. Let $N \leq 3$, $\kappa \leq 0$ and $\gamma = 1$. For any $\omega > 0$, the standing wave solution $(0, e^{2i\omega t}, \varphi_\omega)$ of (1.1) is stable.

Remark 4. The linearized operator $S''_\omega(0, \varphi_\omega)$ around the semi-trivial standing wave is independent of $\kappa$ (see (2.2) and (2.3) below). Therefore, Theorems 4 and 5 are never obtained from the linearized analysis only. The proof of Theorem 5 relies on the variational method of Shatah [18] and on the characterization of the ground states in Theorem 6 below.

Remark 5. For the case $\gamma = 1$, using the notation in Section 2, we have

$$L_R \vec{v} = (L_1 v_1, L_2 v_2)$$

and

$$L_I \vec{v} = (L_{-1} v_1, L_1 v_2),$$

and the kernel of $S''_\omega(0, \varphi_\omega)$ contains a nontrivial element $(\varphi_\omega, 0)$ other than the elements $\nabla(0, \varphi_\omega)$ and $J(0, \varphi_\omega)$ naturally coming from the symmetries of $S_\omega$ (see (2.4) below).

Next, we consider the ground state problem for (1.7). We define $\kappa_c(\gamma) = \frac{1}{2}(\gamma + 2)\sqrt{1-\gamma}$, $0 < \gamma < 1$. (1.9)

Then, $\kappa_c$ is strictly decreasing on the open interval $]0, 1[$, $\kappa_c(0) = 1$ and $\kappa_c(1) = 0$. We define a function $\gamma_c$ on $]0, 1[$ by the inverse function of $\kappa_c$. For the ground state problem, it is convenient to divide the parameter domain $D = \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\}$ into the following sets (see Figure 2).

$$K_1 = \{(\kappa, \gamma) : \kappa \leq 0, \gamma > 1\} \cup \{(\kappa, \gamma) : \kappa \geq 1, \gamma > 0\}$$

$$K_2 = \{(\kappa, \gamma) : 0 < \kappa < 1, \gamma > \gamma_c(\kappa)\},$$

$$K_3 = \{(\kappa, \gamma) : 0 < \kappa < 1, \gamma = \gamma_c(\kappa)\}.$$  

Note that $K_1$, $K_2$ and $K_3$ are mutually disjoint, and $D = K_1 \cup K_2 \cup K_3$.

Remark also that since $\sqrt{2\gamma(1-\gamma)} < \kappa_c(\gamma)$ for $0 < \gamma < 1$, we have $J_0 \subset K_2$.

Moreover, we define

$$G_0^0 = \{G(\theta)\tau_y(0, \varphi_\omega) : \theta \in \mathbb{R}, y \in \mathbb{R}^N\},$$

$$G_1^1 = \{G(\theta)\tau_y(\alpha_+ \varphi_\omega, \beta_- \varphi_\omega) : \theta \in \mathbb{R}, y \in \mathbb{R}^N\}.$$

Then, the set $G_\omega$ of the ground states for (1.7) is determined as follows.

Theorem 6. Let $N \leq 3$ and $\omega > 0$.

1. If $(\kappa, \gamma) \in K_1$, then $G_\omega = G_1^1$.
2. If $(\kappa, \gamma) \in K_2$, then $G_\omega = G_0^0$.
3. If $(\kappa, \gamma) \in K_3$, then $G_\omega = G_0^0 \cup G_1^1$. 


The rest of the paper is organized as follows. In Section 2, we study some spectral properties of the linearized operators around standing waves, which are needed in Sections 3 and 4. In Section 3, we prove Theorems 2 and 3, while Section 4 is devoted to the proof of Theorem 4. In Section 5, we study the ground state problem for (1.7), and prove Theorem 6. Finally, Theorem 5 is proved as a corollary of Theorem 6.

2 Linearized Operators

In this section, we study spectral properties of the linearized operator $S''_\omega(\Phi)$. Here and hereafter, for $\alpha \geq 0$ and $\beta > 0$, we put

$$\Phi = (\alpha \varphi_\omega, \beta \varphi_\omega), \quad \Phi_1 = (-\beta \varphi_\omega, \alpha \varphi_\omega), \quad \Phi_2 = (\alpha \varphi_\omega, 2\beta \varphi_\omega).$$

First, by direct computations, we have

$$\langle S''_\omega(\Phi) \bar{u}, \bar{u} \rangle = \langle \mathcal{L}_R \bar{u}, \bar{u} \rangle + \langle \mathcal{L}_I \Im \bar{u}, \Im \bar{u} \rangle \quad \text{(2.1)}$$

for $\bar{u} = (u_1, u_2) \in X$, where $\mathcal{R} \bar{u} = (\mathcal{R}u_1, \mathcal{R}u_2)$, $\Im \bar{u} = (\Im u_1, \Im u_2)$, and

$$\mathcal{L}_R = \begin{bmatrix} -\Delta + \omega & 0 \\ 0 & -\Delta + \omega \end{bmatrix} - \begin{bmatrix} (2\alpha + \gamma \beta) \varphi_\omega \\ \gamma \alpha \varphi_\omega \\ 2\beta \varphi_\omega \end{bmatrix}, \quad \text{(2.2)}$$

$$\mathcal{L}_I = \begin{bmatrix} -\Delta + \omega & 0 \\ 0 & -\Delta + \omega \end{bmatrix} - \begin{bmatrix} (\alpha - \gamma \beta) \varphi_\omega \\ \gamma \alpha \varphi_\omega \\ \beta \varphi_\omega \end{bmatrix}. \quad \text{(2.3)}$$
Since \( S'(G(\theta)\tau _b\Phi ) = 0 \) for \( y \in \mathbb{R}^N \) and \( \theta \in \mathbb{R} \), we see that
\[
\nabla \Phi \in \ker \mathcal{L}_R, \quad \Phi_2 \in \ker \mathcal{L}_I.
\]
For \( a \in \mathbb{R} \), we define \( \mathcal{L}_a \) by
\[
\mathcal{L}_a v = -\Delta v + \omega v - a\varphi \omega v, \quad v \in H^1(\mathbb{R}^N, \mathbb{R}).
\]
We recall some known results on \( \mathcal{L}_a \).

**Lemma 1.** Let \( N \leq 3 \) and let \( \varphi_\omega \) be the positive radial solution of (1.6).

1. \( \mathcal{L}_2 \) has one negative eigenvalue, \( \ker \mathcal{L}_2 \) is spanned by \( \{\nabla \varphi_\omega\} \), and there exists a constant \( c_1 > 0 \) such that \( \langle \mathcal{L}_2 v, v \rangle \geq c_1 \|v\|^2_{H^1} \) for all \( v \in H^1(\mathbb{R}^N, \mathbb{R}) \) satisfying \( (v, \varphi_\omega)_{L^2} = 0 \) and \( (v, \nabla \varphi_\omega)_{L^2} = 0 \).
2. \( \mathcal{L}_1 \) is non-negative, \( \ker \mathcal{L}_1 \) is spanned by \( \{\varphi_\omega\} \), and there exists \( c_2 > 0 \) such that \( \langle \mathcal{L}_1 v, v \rangle \geq c_2 \|v\|^2_{H^1} \) for all \( v \in H^1(\mathbb{R}^N, \mathbb{R}) \) satisfying \( (v, \varphi_\omega)_{L^2} = 0 \).
3. If \( a < 1 \), then \( \mathcal{L}_a \) is positive on \( H^1(\mathbb{R}^N, \mathbb{R}) \).
4. If \( 1 < a < 2 \), then \( \langle \mathcal{L}_a \varphi_\omega, \varphi_\omega \rangle < 0 \), and there exists \( c_4 > 0 \) such that \( \langle \mathcal{L}_a v, v \rangle \geq c_4 \|v\|^2_{H^1} \) for all \( v \in H^1(\mathbb{R}^N, \mathbb{R}) \) satisfying \( (v, \varphi_\omega)_{L^2} = 0 \).

**Proof.** Parts (1) and (2) are well-known (see [22]). Note that the quadratic nonlinearity in (1.6) is \( L^2 \)-subcritical if and only if \( N \leq 3 \), and that the assumption \( N \leq 3 \) is essential for (1). Parts (3) and (4) follow from (1) and (2) immediately.

In the next lemma, we give the diagonalization of \( \mathcal{L}_R \) and \( \mathcal{L}_I \).

**Lemma 2.** By orthogonal matrices
\[
A = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad B = \frac{1}{\sqrt{\alpha^2 + 4\beta^2}} \begin{bmatrix} \alpha & 2\beta \\ -2\beta & \alpha \end{bmatrix},
\]
\( \mathcal{L}_R \) and \( \mathcal{L}_I \) are diagonalized as follows:
\[
\mathcal{L}_R = A^* \begin{bmatrix} L_2 & 0 \\ 0 & L_{(2-\gamma)\beta} \end{bmatrix} A, \quad \mathcal{L}_I = B^* \begin{bmatrix} L_1 & 0 \\ 0 & L_{(1-2\gamma)\beta} \end{bmatrix} B.
\]

**Proof.** The computation is straightforward, and we omit the details.

The next three lemmas establish the coercivity properties of the operators \( \mathcal{L}_R \) and \( \mathcal{L}_I \). They represent the main results of this section, and are the key points in the proofs of Theorems 2 and 3.

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Lemma 3. If \((2-\gamma)\beta < 1\), then there exists a constant \(\delta_1 > 0\) such that \((\mathcal{L}_R\vec{v}, \vec{v}) \geq \delta_1 \|\vec{v}\|_X^2\) for all \(\vec{v} \in H^1(\mathbb{R}^N, \mathbb{R})^2\) satisfying \((\vec{v}, \Phi)_H = 0\) and \((\vec{v}, \nabla \Phi)_H = 0\).

Proof. By Lemma 2, we have \((\mathcal{L}_R\vec{v}, \vec{v}) = (L_2w_1, w_1) + (L_{(2-\gamma)\beta}w_2, w_2)\), where \(\vec{w} = A\vec{v}\). Since we have
\[
(w_1, \varphi_\omega)_{L^2} = \frac{(\vec{v}, \Phi)_H}{\sqrt{\alpha^2 + \beta^2}} = 0, \quad (w_1, \nabla \varphi_\omega)_{L^2} = \frac{(\vec{v}, \nabla \Phi)_H}{\sqrt{\alpha^2 + \beta^2}} = 0,
\]
it follows from Lemma 1 (1) that \((L_2w_1, w_1) \geq c_1 \|w_1\|_{H^1}^2\). Moreover, by the assumption \((2-\gamma)\beta < 1\) and by Lemma 1 (3), we have \((L_{(2-\gamma)\beta}w_2, w_2) \geq c_3 \|w_2\|_{H^1}^2\). This completes the proof.

Lemma 4. If \(1 \leq (2-\gamma)\beta < 2\), then there exists a constant \(\delta_2 > 0\) such that \((\mathcal{L}_R\vec{v}, \vec{v}) \geq \delta_2 \|\vec{v}\|_X^2\) for all \(\vec{v} \in H^1_{\text{rad}}(\mathbb{R}^N, \mathbb{R})^2\) satisfying \((\vec{v}, \Phi)_H = 0\) and \((\vec{v}, \Phi_1)_H = 0\), where \(\Phi_1 = (-\beta \varphi_\omega, \alpha \varphi_\omega)\).

Proof. By Lemma 2, we have \((\mathcal{L}_R\vec{v}, \vec{v}) = (L_2w_1, w_1) + (L_{(2-\gamma)\beta}w_2, w_2)\), where \(\vec{w} = A\vec{v}\). Then we have \((w_1, \varphi_\omega)_{L^2} = (\vec{v}, \Phi)_H/\sqrt{\alpha^2 + \beta^2} = 0\). Moreover, since \(\varphi_\omega\) and \(w_1\) are radially symmetric, we have \((w_1, \nabla \varphi_\omega)_{L^2} = 0\). Thus, it follows from Lemma 1 (1) that \((L_2w_1, w_1) \geq c_1 \|w_1\|_{H^1}^2\). Moreover, since \((w_2, \varphi_\omega)_{L^2} = (\vec{v}, \Phi_1)_H/\sqrt{\alpha^2 + \beta^2} = 0\), it follows from the assumption \(1 \leq (2-\gamma)\beta < 2\) and Lemma 1 (2), (4) that \((L_{(2-\gamma)\beta}w_2, w_2) \geq c_2 \|w_2\|_{H^1}^2\).

Lemma 5. There exists a constant \(\delta_3 > 0\) such that \((\mathcal{L}_I\vec{v}, \vec{v}) \geq \delta_3 \|\vec{v}\|_X^2\) for all \(\vec{v} \in H^1(\mathbb{R}^N, \mathbb{R})^2\) satisfying \((\vec{v}, \Phi_2)_H = 0\), where \(\Phi_2 = (\alpha \varphi_\omega, 2\beta \varphi_\omega)\).

Proof. By Lemma 2, we have \((\mathcal{L}_I\vec{v}, \vec{v}) = (L_1w_1, w_1) + (L_{(1-2\gamma)\beta}w_2, w_2)\), where \(\vec{w} = B\vec{v}\). Since \((w_1, \varphi_\omega)_{L^2} = (\vec{v}, \Phi_2)_H/\sqrt{\alpha^2 + 4\beta^2} = 0\), Lemma 1 (2) implies \((L_1w_1, w_1) \geq c_2 \|w_1\|_{H^1}^2\). Moreover, since \((1-2\gamma)\beta < 1\), it follows from Lemma 1 (3) that \((L_{(1-2\gamma)\beta}w_2, w_2) \geq c_3 \|w_2\|_{H^1}^2\).

The last two lemmas of this section make connections between parameters \((\kappa, \gamma)\) and the criteria used in Lemma 3, 4 and 5 on \(\beta\).

Lemma 6. Let \((\kappa, \gamma) \in J_1 \cup J_2\). Then, \((2-\gamma)\beta_- < 1\) and \((1-2\gamma)\beta_- < 1\).

Proof. We put \(D = \kappa^2 + 2\gamma(\gamma - 1)\). By the second equation of (1.8), we have \(0 < \beta_- < 1\). Thus, we have \((1-2\gamma)\beta_- < \beta_- < 1\). If \(\gamma > 1\), then \((2-\gamma)\beta_- < \beta_- < 1\). While, if \(0 < \gamma \leq 1\), then \(\kappa > 0\), \(D > 0\) and \((2-\gamma)\beta_- < (2-\gamma)(\kappa^2 + \gamma^2)/(2\kappa^2 + \gamma^3) < 1\). Note that the last inequality is equivalent to \(D > 0\).
Lemma 7. Let \((\kappa, \gamma) \in J_2\). Then, \(1 < (2 - \gamma)\beta_+ < 2\) and \((1 - 2\gamma)\beta_+ < 1\).

Proof. We put \(D = \kappa^2 + 2(\gamma - 1)\). Since \(0 < \beta_+ < 1\), we have \((2 - \gamma)\beta_+ < 2\beta_+ < 2\), and \((1 - 2\gamma)\beta_+ < \beta_+ < 1\). Next, we see that \((2 - \gamma)\beta_+ > 1\) is equivalent to \((2 - \gamma)\kappa > \gamma\sqrt{D}\). Since \(0 < \gamma < 1\) and \(\kappa > 0\), we have \(\gamma\sqrt{D} < \gamma\kappa < (2 - \gamma)\kappa\).

\[\text{Remark 6. When } (\kappa, \gamma) \in J_3, \text{ we have } D = \kappa^2 + 2(\gamma - 1) = 0, (2 - \gamma)\beta_0 = 1 \text{ and } (1 - 2\gamma)\beta_0 < 1.\]

3 Proofs of Theorems 2 and 3

In this section we prove Theorems 2 and 3 using the results of Section 2 and the following propositions. Proposition 2 follows from Theorem 3.4 of Grillakis, Shatah and Strauss [11] (see also [23] and [7, Section 3]). While, Proposition 3 follows from Theorem 1 of [16] (see also [11, 15, 19]).

Proposition 2. Let \(\tilde{\phi} \in A_\omega\). Assume that there exists a constant \(\delta > 0\) such that \(\langle S''(\tilde{\phi}) \tilde{w}, \tilde{w} \rangle \geq \delta \|\tilde{w}\|_X^2\) for all \(\tilde{w} \in X\) satisfying \((\tilde{\phi}, \tilde{w})_H = (\tilde{\phi}, \tilde{w})_H = 0\) and \((\nabla \tilde{\phi}, \tilde{w})_H = 0\). Then the standing wave solution \(G(\omega t)\tilde{\phi}\) of (1.1) is stable.

Proposition 3. Let \(\tilde{\phi} \in A_\omega\) be radially symmetric. Assume that there exist \(\tilde{\psi} \in X_{rad}\) and a constant \(\delta > 0\) such that \(\|\tilde{\psi}\|_H = 1\), \((\tilde{\psi}, \tilde{\phi})_H = (\tilde{\psi}, \tilde{\phi})_H = 0\), \((S''(\tilde{\phi}) \tilde{\psi}, \tilde{\psi}) < 0\), and \(\langle S''(\tilde{\phi}) \tilde{w}, \tilde{w} \rangle \geq \delta \|\tilde{w}\|_X^2\) for all \(\tilde{w} \in X_{rad}\) satisfying \((\tilde{\phi}, \tilde{w})_H = (\tilde{\phi}, \tilde{w})_H = 0\). Then the standing wave solution \(G(\omega t)\tilde{\phi}\) of (1.1) is unstable.

Proof of Theorem 2. For \((\kappa, \gamma) \in J_1 \cup J_2\), let \((\alpha, \beta) = (\alpha_+, \beta_-)\). Let \(\tilde{w} \in X\) satisfy \((\Phi, \tilde{w})_H = (J\Phi, \tilde{w})_H = 0\) and \((\nabla \Phi, \tilde{w})_H = 0\). By (2.1), we have
\[
\langle S''(\Phi) \tilde{w}, \tilde{w} \rangle = \langle L_R \Re \tilde{w}, \Re \tilde{w} \rangle + \langle L_I \Im \tilde{w}, \Im \tilde{w} \rangle.
\]
Since \((\Phi, \Re \tilde{w})_H = (\Phi, \Re \tilde{w})_H = 0\) and \((\nabla \Phi, \Re \tilde{w})_H = (\nabla \Phi, \Re \tilde{w})_H = 0\), it follows from Lemmas 6 and 3 that \(\langle L_R \Re \tilde{w}, \Re \tilde{w} \rangle \geq \delta_1 \|\Re \tilde{w}\|_X^2\). While, since \((\Im \tilde{w}, \Phi_2)_H = (J\Phi, \tilde{w})_H = 0\), Lemma 5 implies \(\langle L_I \Im \tilde{w}, \Im \tilde{w} \rangle \geq \delta_3 \|\Im \tilde{w}\|_X^2\).

Therefore, Theorem 2 follows from Proposition 2.
Proof of Theorem 3. For \((\kappa, \gamma) \in \mathcal{J}_2\), let \((\alpha, \beta) = (\alpha_-, \beta_+)\). We take \(\vec{\psi} = \Phi_1 / \|\Phi_1\|_H\). Then we have \(\|\vec{\psi}\|_H = 1\), \((\vec{\psi}, \Phi)_H = 0\) and \((\vec{\psi}, J\Phi)_H = 0\).

Moreover, by Lemma 7 and Lemma 1 (4), we have
\[
\langle S''_\omega (\vec{\varphi}) \vec{w}, \vec{w} \rangle = \langle L R \vec{w}, \vec{w} \rangle = \langle L (2-\gamma) \beta \varphi_\omega, \varphi_\omega \rangle / \|\varphi_\omega\|_{L^2}^2 < 0.
\]

Finally, let \(\vec{w} \in X_{rad}\) satisfy \((\vec{\phi}, \vec{w})_H = (J\Phi, \vec{w})_H = (\vec{\psi}, \vec{w})_H = 0\). Since \((\vec{\phi}, \Re \vec{w})_H = (\vec{\phi}, \vec{w})_H = 0\) and \((\vec{\phi}_1, \Re \vec{w})_H = (\vec{\phi}_1, \vec{w})_H = 0\), by Lemmas 7 and 4, we have \(\langle L_R \vec{w}, \vec{w} \rangle \geq \delta_2 \|\Re \vec{w}\|_X^2\). While, since \((\Im \vec{w}, \varphi_\omega)_H = (\vec{\psi}, \varphi_\omega)_H = 0\), by Lemma 5, we have \(\langle L_I \Im \vec{w}, \Im \vec{w} \rangle \geq \delta_3 \|\Im \vec{w}\|_X^2\). Thus, by (2.1), we have \(\langle S''_\omega (\vec{\varphi}) \vec{w}, \vec{w} \rangle \geq \delta \|\vec{w}\|_X^2\), and Theorem 3 follows from Proposition 3.

\[\square\]

4 Proof of Theorem 4

We introduce the following Proposition 4 to prove Theorem 4. It is a modification of Theorem 2 of [16]. In what follows, \(\text{sgn}(\mu)\) denotes the sign of any real \(\mu\).

Proposition 4. Let \(\vec{\varphi} \in A_\omega\) be radially symmetric. Assume that there exist \(\vec{\varphi} \in X_{rad}\) such that
\[
\begin{align*}
(i) \quad & \|\vec{\psi}\|_H = 1, \ (\vec{\psi}, \vec{\varphi})_H = 0, \ (\vec{\psi}, J\vec{\varphi})_X = 0, \ S''_\omega (\vec{\varphi}) \vec{\psi} = 0, \\
(ii) \quad & \text{there exists a positive constant } k_0 \text{ such that } \langle S''_\omega (\vec{\varphi}) \vec{w}, \vec{w} \rangle \geq k_0 \|\vec{w}\|_X^2 \\
& \text{for all } \vec{w} \in X_{rad} \text{ satisfying } (\vec{\phi}, \vec{w})_H = (J\vec{\phi}, \vec{w})_H = (\vec{\psi}, \vec{w})_H = 0, \\
(iii) \quad & \text{there exist positive constants } k_1, k_2 \text{ and } \varepsilon \text{ such that} \\
\text{sgn}(\lambda) \cdot \langle S'_\omega (\vec{\phi} + \lambda \vec{\psi} + \vec{z}), \vec{\psi} \rangle \leq -k_1 \lambda^2 + k_2 \|\vec{z}\|_X^2 + o(\lambda^2 + \|\vec{z}\|_X^2) \\
& \text{for all } \lambda \in \mathbb{R} \text{ and } \vec{z} \in X_{rad} \text{ satisfying } |\lambda| + \|\vec{z}\|_X < \varepsilon.
\end{align*}
\]

Then the standing wave solution \(G(\omega t) \vec{\varphi}\) of (1.1) is unstable.

We first prove Theorem 4 using Proposition 4.

Proof of Theorem 4. The proof consists of verifying the assumptions (i), (ii), (iii) of Proposition 4. Let \((\alpha, \beta) = (0, 1)\) and \(\Phi = (0, \varphi_\omega)\). We take
\[
\vec{\psi} = (\psi_1, \psi_2) = (\varphi_\omega, 0)/\|\varphi_\omega\|_{L^2}.
\]
Then, \(\|\vec{\psi}\|_H = 1\), \((\vec{\psi}, \Phi)_H = 0\), \((\vec{\psi}, J\Phi)_X = 0\), and
\[
S''_\omega (\vec{\varphi}) \vec{\psi} = (L_1 \varphi_\omega, 0)/\|\varphi_\omega\|_{L^2} = (0, 0).
\]
Thus, (i) is satisfied. The assumption (ii) is proved in the same way as the proof of Theorem 3. Finally, we prove (iii). Let $\lambda \in \mathbb{R}$ and $\vec{z} = (z_1, z_2) \in X_{rad}$, and put $\vec{v} = (v_1, v_2) = \lambda \vec{v} + \vec{z}$. Then, we have

$$v_1 = \lambda \psi + z_1, \quad v_2 = z_2, \quad \psi_1 = \varphi_\omega / \| \varphi_\omega \|_{L^2},$$

and

$$\| \varphi_\omega \|_{L^2} \langle S'_\omega (\Phi + \vec{v}), \vec{\psi} \rangle = \mathcal{R} \int_{\mathbb{R}^N} \{ \nabla v_1 \cdot \nabla \varphi_\omega + \omega v_1 \varphi_\omega - \kappa |v_1| v_1 \varphi_\omega - v_1 \varphi_\omega + \overline{v_2} \varphi_\omega \} dx$$

$$= \mathcal{R} \int_{\mathbb{R}^N} \{ v_1 (-\Delta \varphi_\omega + \omega \varphi_\omega - \varphi_\omega^2) - \kappa |v_1| v_1 \varphi_\omega - v_1 \overline{v_2} \varphi_\omega \} dx$$

$$= -\kappa \mathcal{R} \int_{\mathbb{R}^N} v_1 v_1 \varphi_\omega dx - \mathcal{R} \int_{\mathbb{R}^N} v_1 \overline{v_2} \varphi_\omega dx.$$

Thus, we have

$$\langle S'_\omega (\Phi + \vec{v}), \vec{\psi} \rangle = -\kappa \mathcal{R} \int_{\mathbb{R}^N} v_1 v_1 \psi_1 dx - \mathcal{R} \int_{\mathbb{R}^N} v_1 \overline{v_2} \psi_1 dx. \quad (4.1)$$

Here, we have

$$\text{sgn}(\lambda) \cdot \kappa \mathcal{R} \int_{\mathbb{R}^N} |\lambda \psi_1| |\varphi_\omega| \psi_1 dx = C_0 \lambda^2,$$

where $C_0 := \kappa \| \varphi_\omega \|_{L^3}^2 / \| \varphi_\omega \|_2^2$, and the first term of the right hand side of (4.1) is estimated as follows.

$$\left| \text{sgn}(\lambda) \cdot \kappa \mathcal{R} \int_{\mathbb{R}^N} |v_1| v_1 \psi_1 dx - C_0 \lambda^2 \right| \leq \kappa \int_{\mathbb{R}^N} |v_1| |v_1 - |\lambda \psi_1| \varphi_\omega| \psi_1 dx$$

$$\leq C \int_{\mathbb{R}^N} (|v_1| + |\lambda \psi_1|)|v_1 - \lambda \psi_1| \psi_1 dx \leq C \int_{\mathbb{R}^N} \left( |\lambda \psi_1| + |z_1| \right) |z_1| \psi_1 dx$$

$$\leq C \| \lambda \|_{L^3} \| \psi_1 \|_{L^3}^2 + C \| z_1 \|_{L^3}^2 \| \psi_1 \|_{L^3} \leq C_0 \lambda^2 / 4 + C_1 \| z_1 \|_{H^1}^2$$

for some constant $C_0$ depending on $\varphi_\omega$. Here, in the last inequality, we used the inequality of the type $2ab \leq \varepsilon^2 a^2 + b^2 / \varepsilon^2$. While, the second term of the right hand side of (4.1) is estimated as follows.

$$\left| \mathcal{R} \int_{\mathbb{R}^N} v_1 \overline{v_2} \psi_1 dx \right| \leq \| \lambda \|_{L^3} \| \overline{v_2} \|_{L^3} \| \psi_1 \|_{L^3}^2 + \| z_1 \|_{L^3} \| z_2 \|_{L^3} \| \psi_1 \|_{L^3}$$

$$\leq C_0 \lambda^2 / 4 + C_2 \| z_2 \|_{H^1}^2 + C_3 \| z_1 \|_{H^1} \| z_2 \|_{H^1}$$

for some positive constants $C_2$ and $C_3$. Thus, we have

$$\text{sgn} \lambda \cdot \langle S'_\omega (\Phi + \lambda \vec{v} + \vec{z}), \vec{\psi} \rangle \leq -C_0 \lambda^2 / 2 + C_4 \| \vec{z} \|_X^2$$

for some constant $C_4 > 0$. This completes the proof. \qed
In the rest of this section, we give the proof of Proposition 4 by modifying the proof of Theorem 2 of [16]. We define

\[ N_\varepsilon(\phi) = \{ \bar{u} \in X_{\text{rad}} : \inf_{\theta \in \mathbb{R}} \| G(\theta)\bar{u} - \bar{\phi} \|_X < \varepsilon \} \]

and the identification operator \( I : X \to X^* \) by

\[ \langle I\bar{u}, \bar{v} \rangle = \langle \bar{u}, \bar{v} \rangle_H, \quad \bar{u}, \bar{v} \in X. \]

**Lemma 8.** There exist \( \varepsilon > 0 \) and a \( C^2 \) map \( \Theta : N_\varepsilon(\phi) \to \mathbb{R}/2\pi\mathbb{Z} \) such that

\[
\|G(\Theta(\bar{u}))\bar{u} - \bar{\phi}\|_X \leq \|G(\theta)\bar{u} - \bar{\phi}\|_X,
\]

\[
(G(\Theta(\bar{u}))(\bar{u}, J\tilde{\phi})_X = 0, \quad \Theta(G(\theta)\bar{u}) = \Theta(\bar{u}) - \theta,
\]

\[
I^{-1}\Theta'(\bar{u}) = \frac{JG(-\Theta(\bar{u}))(1 - \Delta)\tilde{\phi}}{(G(\Theta(\bar{u}))\bar{u}, J^2\tilde{\phi})_X} \tag{4.2}
\]

for all \( \bar{u} \in N_\varepsilon(\phi) \) and \( \theta \in \mathbb{R}/2\pi\mathbb{Z} \).

**Proof.** See Lemma 3.2 of [11]. Note that \( \bar{\phi} \in H^3(\mathbb{R}^N)^2 \) by the elliptic regularity for (1.7).

We put \( M(\bar{u}) = G(\Theta(\bar{u}))\bar{u} \). Then we have \( M(\tilde{\phi}) = \bar{\phi} \) and \( M(G(\theta)\bar{u}) = M(\bar{u}) \) for \( \bar{u} \in N_\varepsilon(\phi) \) and \( \theta \in \mathbb{R} \). We define \( \mathcal{A} \) and \( \Lambda \) by

\[
\mathcal{A}(\bar{u}) = (M(\bar{u}), J^{-1}\tilde{\psi})_H, \quad \Lambda(\bar{u}) = (M(\bar{u}), \tilde{\psi})_H \tag{4.3}
\]

for \( \bar{u} \in N_\varepsilon(\phi) \). Then we have

\[
JI^{-1}\mathcal{A}'(\bar{u}) = G(-\Theta(\bar{u}))\tilde{\psi} - \Lambda(\bar{u})JI^{-1}\Theta'(\bar{u}), \tag{4.4}
\]

\[
0 = \frac{d}{d\theta} A(G(\theta)\bar{u})|_{\theta=0} = \langle \mathcal{A}'(\bar{u}), J\bar{u} \rangle = -\langle I\bar{u}, JI^{-1}\mathcal{A}'(\bar{u}) \rangle. \tag{4.5}
\]

We define \( \mathcal{P} \) by

\[
\mathcal{P}(\bar{u}) = \langle E'(\bar{u}), JI^{-1}\mathcal{A}'(\bar{u}) \rangle
\]

for \( \bar{u} \in N_\varepsilon(\phi) \). Note that by (4.2), (4.4) and (4.5), we have

\[
\mathcal{P}(\bar{u}) = \langle S'_\omega(\bar{u}), JI^{-1}\mathcal{A}'(\bar{u}) \rangle
\]

\[
= \langle S'_\omega(M(\bar{u})), \tilde{\psi} \rangle - \frac{\Lambda(\bar{u})}{(M(\bar{u}), J^2\tilde{\phi})_X} \langle S'_\omega(M(\bar{u})), J^2(1 - \Delta)\tilde{\phi} \rangle. \tag{4.6}
\]
Lemma 9. Let $\mathcal{I}$ be an interval of $\mathbb{R}$. Let $\bar{u} \in C(\mathcal{I}, X) \cap C^1(\mathcal{I}, X^*)$ be a solution of (1.1), and assume that $\bar{u}(t) \in \mathcal{N}_c(\bar{\phi})$ for all $t \in \mathcal{I}$. Then

$$\frac{d}{dt} A(\bar{u}(t)) = \mathcal{P}(\bar{u}(t)), \quad t \in \mathcal{I}.$$ 

Proof. See Lemma 4.6 of [11] and Lemma 2 of [16]. \qed

Lemma 10. There exist positive constants $k^*$ and $\varepsilon_0$ such that

$$E(\bar{u}) \geq E(\bar{\phi}) + k^* \text{sgn} \Lambda(\bar{u}) \cdot \mathcal{P}(\bar{u})$$

for all $\bar{u} \in \mathcal{N}_{\varepsilon_0}(\bar{\phi})$ satisfying $Q(\bar{u}) = Q(\bar{\phi})$.

Proof. We put $\bar{v} = M(\bar{u}) - \bar{\phi}$, and decompose $\bar{v}$ as

$$\bar{v} = a\bar{\phi} + bJ\bar{\phi} + c\bar{\phi} + \bar{w},$$

where $a, b, c \in \mathbb{R}$, and $\bar{w} \in X_{\text{rad}}$ satisfies $(\bar{\phi}, \bar{w})_H = (J\bar{\phi}, \bar{w})_H = (\bar{\phi}, \bar{w})_H = 0.$ Since $Q(\bar{\phi}) = Q(\bar{u}) = Q(M(\bar{u})) = Q(\bar{\phi}) + (\bar{\phi}, \bar{v})_H + Q(\bar{v})$ and $(\bar{\phi}, \bar{v})_H = a\|\bar{\phi}\|_H^2$, we have $a = O(\|\bar{v}\|_X^2)$. Moreover, by Lemma 8 and by the assumption (i) of Proposition 4, we have $(M(\bar{u}), J\bar{\phi})_X = 0$ and $(J\bar{\phi}, \bar{v})_X = 0$. Thus, we have 0 = $(\bar{v}, J\bar{\phi})_X = b\|J\bar{\phi}\|_X^2 + (\bar{w}, J\bar{\phi})_X$, $|b|\|J\bar{\phi}\|_X \leq \|\bar{w}\|_X$ and

$$\|\bar{v}\|_X \leq |c|\|\bar{\phi}\|_X + 2\|\bar{w}\|_X + O(\|\bar{v}\|_X^2). \quad (4.7)$$

Since $S'_\omega(\bar{\phi}) = 0$ and $Q(\bar{u}) = Q(\bar{\phi})$, by the Taylor expansion, we have

$$E(\bar{u}) - E(\bar{\phi}) = S'_\omega(M(\bar{u})) - S'_\omega(\bar{\phi}) = \frac{1}{2} \langle S''_\omega(\bar{\phi}) \bar{v}, \bar{v} \rangle + o(\|\bar{v}\|_X^2). \quad (4.8)$$

Here, since $a = O(\|\bar{v}\|_X^2)$ and $S''_\omega(\bar{\phi})(J\bar{\phi}) = S''_\omega(\bar{\phi})\bar{\phi} = 0$, by the assumption (ii) of Proposition 4, we have

$$E(\bar{u}) - E(\bar{\phi}) = \frac{1}{2} \langle S''_\omega(\bar{\phi}) \bar{v}, \bar{v} \rangle + o(\|\bar{v}\|_X^2)$$

$$\quad = \frac{1}{2} \langle S''_\omega(\bar{\phi}) \bar{w}, \bar{w} \rangle + o(\|\bar{v}\|_X^2) \geq \frac{k_0}{2} \|\bar{w}\|_X^2 - o(\|\bar{v}\|_X^2). \quad (4.9)$$

On the other hand, we have $c = (\bar{v}, \bar{\psi})_H = \Lambda(\bar{u}) = O(\|\bar{v}\|_X),$

$$S'_\omega(\bar{\phi} + \bar{v}) = S'_\omega(\bar{\phi}) + S'_\omega(\bar{\phi})\bar{v} + o(\|\bar{v}\|_X) = S'_\omega(\bar{\phi})\bar{w} + o(\|\bar{v}\|_X),$$

where $\Lambda(\bar{u}) \equiv 0$. Therefore, for all $\bar{u} \in \mathcal{N}_{\varepsilon_0}(\bar{\phi})$ satisfying $Q(\bar{u}) = Q(\bar{\phi})$ and $\text{sgn} \Lambda(\bar{u}) \cdot \mathcal{P}(\bar{u}) = k^* > 0$, we have $\bar{v} = a\bar{\phi} + bJ\bar{\phi} + c\bar{\phi} + \bar{w}$ with $\|\bar{w}\|_X < \varepsilon_0$, and $E(\bar{u}) - E(\bar{\phi})$ is the $\|\bar{v}\|_X$-open region of $\mathbb{R}$ as a subset of $\mathbb{R}$.
Moreover, we have
\[ (M(\tilde{u}), J^2 \tilde{\phi})_X = (\tilde{\phi}, J^2 \tilde{\phi})_X + O(\|\tilde{v}\|_X). \]
Thus, by (4.6) we have
\[ \mathcal{P}(\tilde{u}) = \langle S'_\omega(\tilde{\phi} + \tilde{v}), \tilde{\psi} \rangle + \frac{c}{\|J\tilde{\phi}\|_X^2} \langle S''_\omega(\tilde{\phi})\tilde{w}, J^2(1 - \Delta)\tilde{\phi} \rangle + o(\|\tilde{v}\|_X^2). \]

Here, by the assumption (iii) of Proposition 4, we have
\[ \text{sgn}(c) \cdot \langle S'_\omega(\tilde{\phi} + \tilde{v}), \tilde{\psi} \rangle \]
\[ \leq -k_1c^2 + k_2\|a\tilde{\phi} + bJ\tilde{\phi} + \tilde{w}\|_X^2 + o(c^2 + \|a\tilde{\phi} + bJ\tilde{\phi} + \tilde{w}\|_X^2) \]
\[ \leq -k_1c^2 + k_3\|\tilde{w}\|_X^2 + o(\|\tilde{v}\|_X^2). \]

Moreover, we have
\[ \left| \frac{c}{\|J\tilde{\phi}\|_X^2} \langle S''_\omega(\tilde{\phi})\tilde{w}, J^2(1 - \Delta)\tilde{\phi} \rangle \right| \leq k|c|\|\tilde{w}\|_X \leq \frac{k_1}{2}c^2 + k_4\|\tilde{w}\|_X^2. \]

Thus, we have
\[ -\text{sgn}\Lambda(\tilde{u}) \cdot \mathcal{P}(\tilde{u}) \geq \frac{k_1}{2}c^2 - k_5\|\tilde{w}\|_X^3 - o(\|\tilde{v}\|_X^2) \quad (4.10) \]
with some constant \( k_5 > 0 \). By (4.9) and (4.10), we have
\[ E(\tilde{u}) - E(\tilde{\phi}) - k^* \text{sgn}\Lambda(\tilde{u}) \cdot \mathcal{P}(\tilde{u}) \geq k_6c^2 + k_7\|\tilde{w}\|_X^2 - o(\|\tilde{v}\|_X^2), \quad (4.11) \]
where we put \( k^* = k_0/(4k_5) \), \( k_6 = k^*k_1/2 \) and \( k_7 = k_0/4 \). Finally, since \( \|\tilde{w}\|_X = \|M(\tilde{u}) - \psi\|_X < \varepsilon_0 \), it follows from (4.7) that the right hand side of (4.11) is non-negative, if \( \varepsilon_0 \) is sufficiently small. This completes the proof. \( \square \)

**Lemma 11.** There exist \( \lambda_1 > 0 \) and a continuous curve \( (-\lambda_1, \lambda_1) \ni \lambda \mapsto \tilde{\phi}_\lambda \in X_{\text{rad}} \) such that \( \tilde{\phi}_0 = \tilde{\phi} \) and
\[ E(\tilde{\phi}_\lambda) < E(\tilde{\phi}), \quad Q(\tilde{\phi}_\lambda) = Q(\tilde{\phi}), \quad \lambda \mathcal{P}(\tilde{\phi}_\lambda) < 0 \]
for \( 0 < |\lambda| < \lambda_1 \).

**Proof.** For \( \lambda \) close to 0, we define
\[ \tilde{\phi}_\lambda = \tilde{\phi} + \lambda \tilde{v} + \sigma(\lambda)\tilde{\phi}, \quad \sigma(\lambda) = \left(1 - \frac{Q(\tilde{\psi})}{Q(\tilde{\phi})}\lambda^2\right)^{1/2} - 1. \]

Then, we have \( Q(\tilde{\phi}_\lambda) = Q(\tilde{\phi}), \sigma(\lambda) = O(\lambda^2), \sigma'(\lambda) = O(\lambda) \) and
\[ S'_\omega(\tilde{\phi}_\lambda) - S'_\omega(\tilde{\phi}) = \int_0^\lambda \frac{d}{ds} S'_\omega(\tilde{\phi}_s) \, ds = \int_0^\lambda \langle S''_\omega(\tilde{\phi}_s), \tilde{\psi} + \sigma'(s)\tilde{\phi} \rangle \, ds. \]
Here, by the assumption (iii) of Proposition 4, we have

\[ \text{sgn}(s) \cdot \langle S'(\tilde{\phi}_s), \psi \rangle \leq -k_1 s^2 + o(s^2). \]

Moreover, since \( S'_\omega(\tilde{\phi}_s) = S'_\omega(\tilde{\phi}) + S''_\omega(s\tilde{\phi} + \sigma(s)\tilde{\phi}) + o(s) = o(s^2) \), we have \( \langle S'_\omega(\tilde{\phi}_s), \sigma'(s)\tilde{\phi} \rangle = o(s^2) \). Thus, we have \( S_\omega(\tilde{\phi}_\lambda) - S_\omega(\tilde{\phi}) \leq -k_1 |\lambda|^2/3 + o(\lambda^3) \).

Finally, by (4.10), we have \( \lambda P(\tilde{\phi}_\lambda) \leq -k_1 |\lambda|^2/2 + o(\lambda^3) \).

\[ \text{Proof of Proposition 4.} \]
Suppose that \( G(\omega t)\tilde{\phi} \) is stable. For \( \lambda \) close to 0, let \( \tilde{\phi}_\lambda \in X_\text{rad} \) be the function given in Lemma 11, and let \( \tilde{u}_\lambda(t) \) be the solution of (1.1) with \( \tilde{u}_\lambda(0) = \tilde{\phi}_\lambda \). Then, there exists \( \lambda_0 > 0 \) such that if \( |\lambda| < \lambda_0 \), then \( \tilde{u}_\lambda(t) \in N_\varepsilon(\phi) \) for all \( t \geq 0 \), where \( \varepsilon_0 \) is the positive constant given in Lemma 10. Moreover, by the definition (4.3), there exist positive constants \( C_1 \) and \( C_2 \) such that \( |A(\tilde{u})| \leq C_1 \) and \( |\Lambda(\tilde{u})| \leq C_2 \) for all \( \tilde{u} \in N_\varepsilon(\phi) \). Let \( -\lambda_0 < \lambda < 0 \) and put \( \delta_\lambda = E(\tilde{\phi}) - E(\tilde{\phi}_\lambda) \). Since \( P(\tilde{\phi}_\lambda) > 0 \) and \( t \mapsto P(\tilde{u}_\lambda(t)) \) is continuous, by Lemma 10 and by the conservation laws of \( E \) and \( Q \), we see that \( P(\tilde{u}_\lambda(t)) > 0 \) for all \( t \geq 0 \) and

\[ \delta_\lambda = E(\tilde{\phi}) - E(\tilde{u}_\lambda(t)) \leq -k^* \text{sgn} \Lambda(\tilde{u}_\lambda(t)) \cdot P(\tilde{u}_\lambda(t)) \leq k^* C_2 P(\tilde{u}_\lambda(t)) \]

for all \( t \geq 0 \). Moreover, by Lemma 9, we have

\[ \frac{d}{dt} A(\tilde{u}_\lambda(t)) = P(\tilde{u}_\lambda(t)) \geq \frac{\delta_\lambda}{k^* C_2} \]

for all \( t \geq 0 \), which implies that \( A(\tilde{u}_\lambda(t)) \to \infty \) as \( t \to \infty \). This contradicts the fact that \( |A(\tilde{u}_\lambda(t))| \leq C_1 \) for all \( t \geq 0 \). Hence, \( G(\omega t)\tilde{\phi} \) is unstable. \( \square \)

5 Ground States

5.1 Existence and Stability of Ground States

In this subsection, we briefly recall the existence and stability of ground states for (1.7). We define

\[ \|\tilde{u}\|_{X_\omega}^2 = \|\nabla \tilde{u}\|_H^2 + \omega \|\tilde{u}\|_H^2, \]

\[ V(\tilde{u}) = \kappa \|u_1\|_{L^3}^3 + \|u_2\|_{L^3}^3 + \frac{3}{2} \gamma |\Re \int_{\mathbb{R}^N} u_1^2 \overline{u}_2^2 \, dx, \]

\[ K_\omega(\tilde{u}) = \|\tilde{u}\|_{X_\omega}^2 - V(\tilde{u}) \]

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for \( \vec{u} \in X \). Then the action \( S_\omega \) associated with (1.7) is written as
\[
S_\omega(\vec{u}) = \frac{1}{2}||\vec{u}||^2_{X_\omega} - \frac{1}{3}V(\vec{u}).
\]
Remark that for \( \vec{u} \in X \) satisfying \( K_\omega(\vec{u}) = 0 \), one has
\[
S_\omega(\vec{u}) = \frac{1}{6}||\vec{u}||^2_{X_\omega}.
\] (5.1)
Moreover, we define
\[
\mu(\omega) = \inf\{S_\omega(\vec{u}) : \vec{u} \in X, \ K_\omega(\vec{u}) = 0, \ \vec{u} \neq (0,0)\},
\]
\[
\mathcal{M}_\omega = \{\vec{\phi} \in X : S_\omega(\vec{\phi}) = \mu(\omega), \ K_\omega(\vec{\phi}) = 0\}.
\]

The following Lemma 12 establishes the existence of a ground state for (1.7). Since it can be proved by the standard variational method (see [2, 13, 14, 24] and also [9, 17]), we omit the proof.

**Lemma 12.** Let \( \kappa \in \mathbb{R}, \ \gamma > 0 \) and \( \omega > 0 \). If \( \{\vec{u}_n\} \subset X \) satisfies \( S_\omega(\vec{u}_n) \rightarrow \mu(\omega) \) and \( K_\omega(\vec{u}_n) \rightarrow 0 \), then there exist a sequence \( \{y_n\} \subset \mathbb{R}^N \) and \( \vec{\phi} \in \mathcal{M}_\omega \) such that \( \{\tau_{y_n}\vec{u}_n\} \) has a subsequence that converges to \( \vec{\phi} \) strongly in \( X \). Moreover, \( \mathcal{M}_\omega = \mathcal{G}_\omega \) and \( \mu(\omega) = d(\omega) \). As a consequence, the set \( \mathcal{G}_\omega \) is not empty.

Next, we consider the stability of ground states. By the scale invariance of (1.7), we see that \( d(\omega) = \omega^{3-3N/2}d(1) \) for all \( \omega > 0 \). Since \( N \leq 3 \) and \( d(1) > 0 \), we have \( d''(\omega) > 0 \) for all \( \omega > 0 \). Using this fact and Lemma 12, the following Proposition 5 can be proved by the method of Shatah [18] (see also [9]). Since it is standard, we omit the proof.

**Proposition 5.** Let \( \kappa \in \mathbb{R} \) and \( \gamma > 0 \). For any \( \omega > 0 \), the set \( \mathcal{G}_\omega \) is stable in the following sense. For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \vec{u}_0 \in X \) satisfies \( \text{dist}(\vec{u}_0, \mathcal{G}_\omega) < \delta \), then the solution \( \vec{u}(t) \) of (1.1) with \( \vec{u}(0) = \vec{u}_0 \) exists for all \( t \geq 0 \), and satisfies \( \text{dist}(\vec{u}(t), \mathcal{G}_\omega) < \varepsilon \) for all \( t \geq 0 \), where we put
\[
\text{dist}(\vec{u}, \mathcal{G}_\omega) = \inf\{||\vec{u} - \vec{\phi}||_X : \vec{\phi} \in \mathcal{G}_\omega\}.
\]

### 5.2 Preliminaries from Elementary Geometry

In this section, we explain some basic geometric properties concerning the line and the ellipse defined by (1.8). In the proof of Theorem 6, one has to compare, for a given \( (\alpha, \beta) \in \mathcal{S}_{\kappa, \gamma} \), the quantities \( \alpha^2 + \beta^2 \) and 1. This is the purpose of Lemmas 14 and 15.
Lemma 13. Let \( \gamma > 0 \) and \( 0 < r \leq 1 \), and put

\[
B = \{(x, y) \in ]0, 1[^2 : \gamma x^2 + 2y^2 = 2y, \ x^2 + y^2 = r^2\}.
\]

(1) If \( 0 < r < 1 \), then \( B \) consists of one point.
(2) If \( r = 1 \) and \( 0 < \gamma < 1 \), then \( B = \{(2\sqrt{1-\gamma}/(2-\gamma), \gamma/(2-\gamma))\} \).
(3) If \( r = 1 \) and \( \gamma \geq 1 \), then \( B \) is empty.

Proof. First we prove (1). Let \( 0 < r < 1 \). Recall that \( \gamma x^2 + 2y^2 = 2y \) is an ellipse with vertices \((x, y) = (0, 0), (0, 1)\) and \((\pm 1/\sqrt{2\gamma}, 1/2)\), and that
\[
B \subset \{(x, y) : 0 < y < 1\}.
\]
By the equations in \( B \), we have \( g(y) := (2-\gamma)y^2 - 2y + \gamma r^2 = 0 \). Since \( g(0) = \gamma r^2 > 0 \) and \( g(1) = \gamma (r^2 - 1) < 0 \), the equation \( g(y) = 0 \) has only one solution in \([0, 1]\). This proves (1). Parts (2) and (3) are obtained by direct computations.

Lemma 14. Let \((\kappa, \gamma) \in J_1 \cup J_2\). Then, \( \alpha_+^2 + \beta_-^2 = 1 \) if and only if \((\kappa, \gamma) \in K_3\).

Proof. Assume that \((\alpha_+, \beta_-)\) satisfies \( \alpha_+^2 + \beta_-^2 = 1 \). Since \((\alpha_+, \beta_-)\) satisfies \( \gamma \alpha_+^2 + 2\beta_-^2 = 2\beta_- \), it follows from (2) and (3) of Lemma 13 that \( 0 < \gamma < 1 \) and \( (\alpha_+, \beta_-) = (2\sqrt{1-\gamma}/(2-\gamma), \gamma/(2-\gamma)) \). Substituting this into \( \kappa \alpha_+ + \gamma \beta_- = 1 \), we have \( \kappa = \kappa_\gamma(\gamma) \). Thus, \((\kappa, \gamma) \in K_3\). Conversely, it is easy to see that \( \alpha_+^2 + \beta_-^2 = 1 \) if \((\kappa, \gamma) \in K_3\).

Lemma 15. If \((\kappa, \gamma) \in K_1\), then \( \alpha_+^2 + \beta_-^2 < 1 \).

Proof. First, we remark that the function \( f(\kappa, \gamma) := \alpha_+^2 + \beta_-^2 - 1 \) is continuous in \( J_1 \cup J_2 \), and that \( K_1 \) is a connected subset of \( J_1 \cup J_2 \). By Lemma 14, \( f \) has no zeros in \( K_1 \). Thus, \( f \) has a constant sign in \( K_1 \). Finally, since \( f(0, \gamma) \to -1 \) as \( \gamma \to \infty \), we conclude that \( f(\kappa, \gamma) < 0 \) for all \((\kappa, \gamma) \in K_1\).

In the same way as Lemma 15, we see that \( \alpha_+^2 + \beta_-^2 > 1 \) for \((\kappa, \gamma) \in K_2 \cap J_2 \), but this fact is not used in what follows. The following Lemma 16 plays an important role in the proof of Lemma 17.

Lemma 16. Let \((\kappa, \gamma) \in J_1 \cup J_2\). Then, \( \gamma \alpha_+ > \kappa \beta_- \).

Proof. Since \( \gamma \), \( \alpha_+ \) and \( \beta_- \) are positive, the inequality is trivial for the case \( \kappa \leq 0 \). Let \( \kappa > 0 \) and put \( D = \kappa^2 + 2\gamma(\gamma - 1) \). Then, \( D > 0 \) and

\[
\gamma \alpha_+ > \kappa \beta_- \iff (2-\gamma)\gamma \kappa + \gamma^2 \sqrt{D} > \kappa(\kappa^2 + \gamma^2) - \kappa^2 \sqrt{D} \\
\iff (\gamma^2 + \kappa^2) \sqrt{D} > \kappa D.
\]

Since \( (\gamma^2 + \kappa^2)^2 - \kappa^2 D = \gamma^4 + 2\gamma \kappa^2 > 0 \), the last inequality holds.
We define
\[
\ell = \begin{cases} 
\alpha_k^2 + \beta_k^2 & \text{if } (\kappa, \gamma) \in \mathcal{K}_1, \\
1 & \text{if } (\kappa, \gamma) \in \mathcal{K}_2 \cup \mathcal{K}_3,
\end{cases}
\] (5.2)
and for a given \((\kappa, \gamma)\),
\[
E_1 = \{(x, y) \in \mathbb{R}^2 : \kappa x + \gamma y \geq 1\},
\]
\[
E_2 = \{(x, y) \in \mathbb{R}^2 : \gamma x^2 + 2 y^2 \geq 2 y, \ x^2 + y^2 \leq \ell\}.
\]

In Lemmas 17 and 18, we establish the structure of the set \(E_1 \cap E_2\) with respect to \((\kappa, \gamma)\).

**Lemma 17.** If \((\kappa, \gamma) \in \mathcal{K}_1 \cup \mathcal{K}_3\), then \(E_1 \cap E_2 = \{\alpha_+, \beta_-\}\).

**Proof.** Since it is clear that \{\((\alpha_+, \beta_-)\}\} \subset E_1 \cap E_2, we prove the inverse inclusion. By Lemmas 13, 14 and 15, we see that the ellipse \(\gamma x^2 + 2 y^2 = 2 y\) and the circle \(x^2 + y^2 = \alpha_k^2 + \beta_k^2\) intersect at only one point \((x, y) = (\alpha_+, \beta_-)\) in \{\((x, y) : x > 0\}\}. The normal \(y = f_1(x)\) and the tangent \(y = f_2(x)\) of the circle \(x^2 + y^2 = \alpha_k^2 + \beta_k^2\) at the point \((x, y) = (\alpha_+, \beta_-)\) are given by
\[
f_1(x) = \frac{\beta_-}{\alpha_+} (x - \alpha_+) + \beta_-, \quad f_2(x) = -\frac{\alpha_+}{\beta_-} (x - \alpha_+) + \beta_-,
\]
and we see that \(E_2 \subset E_3 := \{(x, y) : y \leq f_1(x), \ y \leq f_2(x)\}\). By Lemma 16 and by \(\kappa \alpha_+ + \gamma \beta_- = 1\), we have 
\[\frac{-\alpha_+}{\beta_-} < -\kappa/\gamma < \beta_-/\alpha_+\]. That is, the slope of the line \(\kappa x + \gamma y = 1\) is less than that of the normal \(y = f_1(x)\), and is greater than or equal to that of the tangent \(y = f_2(x)\). Recalling that \((\alpha_+, \beta_-)\) is on the line \(\kappa x + \gamma y = 1\), we see that \(E_1 \cap E_2 \subset E_1 \cap E_3 = \{\alpha_+, \beta_-\}\). This completes the proof. \(\square\)

**Lemma 18.** If \((\kappa, \gamma) \in \mathcal{K}_2\), then \(E_1 \cap E_2\) is empty.

**Proof.** First, we consider the case where \(\kappa \leq 0\) and \(0 < \gamma \leq 1\). Then, \(E_1 \subset \{(x, y) : y \geq 1/\gamma\} \subset \{(x, y) : y \geq 1\}\), and we see that \(E_1 \cap E_2\) is empty.

Next, we consider the case where \(0 < \gamma < 1\) and \(0 < \kappa < \kappa_c(\gamma)\). We fix \(\gamma \in ]0, 1[\) and denote \(E_1 = E_1(\kappa)\) for \(0 < \kappa \leq \kappa_c\). Remark that \(E_2\) is independent of \(\kappa \in ]0, \kappa_c[\). When \(\kappa = \kappa_c\), by Lemma 17, we have \(E_1(\kappa_c) \cap E_2 = \{\alpha_+, \beta_-\}\).
Moreover, when \(0 < \kappa < \kappa_c\), \(E_1(\kappa)\) is strictly smaller than \(E_1(\kappa_c)\). Thus, we see that \(E_1(\kappa) \cap E_2\) is empty if \(0 < \kappa < \kappa_c\). \(\square\)
5.3 Determination of Ground States

We are now able to determine the structure of the set $G_\omega$. We use an idea of Sirakov [20] (see also [12]). We denote

$$\|u\|_{H^1_\omega}^2 = \|\nabla u\|_{L^2}^2 + \omega \|u\|_{L^2}^2, \quad u \in H^1(\mathbb{R}^N).$$

**Lemma 19.** Let $\vec{u} = (u_1, u_2) \in \mathcal{A}_\omega$. Then we have

$$\|u_1\|_{H^1_\omega}^2 = \kappa \|u_1\|_{L^3}^3 + \gamma \int_{\mathbb{R}^N} u_1^2 u_2 \, dx,$n

$$\|u_2\|_{H^1_\omega}^2 = \|u_2\|_{L^3}^3 + \frac{\gamma}{2} \int_{\mathbb{R}^N} u_1^2 u_2 \, dx.$$

**Proof.** The first identity is obtained by multiplying the first equation of (1.7) by $u_1$ and by integrating by parts. The second identity is obtained in the same way. \qed

**Lemma 20.** For any $\omega > 0$, $6d(\omega) \leq \ell \|\varphi_\omega\|_{L^3}^3$, where $\ell$ is the number defined by (5.2).

**Proof.** Let $(\alpha \varphi_\omega, \beta \varphi_\omega) \in \mathcal{A}_\omega$. Then, we have $K_\omega(\alpha \varphi_\omega, \beta \varphi_\omega) = 0$, and so by (5.1),

$$S_\omega(\alpha \varphi_\omega, \beta \varphi_\omega) = \frac{1}{6} \|\varphi_\omega\|_{L^3}^2 = \frac{\alpha^2 + \beta^2}{6} \|\varphi_\omega\|_{H^1_\omega}^2.$$

Moreover, since $\varphi_\omega$ is a solution of (1.6), we have $\|\varphi_\omega\|_{H^1_\omega}^2 = \|\varphi_\omega\|_{L^3}^3$, and $6S_\omega(\alpha \varphi_\omega, \beta \varphi_\omega) = (\alpha^2 + \beta^2)\|\varphi_\omega\|_{L^3}^3$. Finally, by the definitions of $d(\omega)$ and $\ell$, we obtain the desired estimate. \qed

The following variational characterization of $\varphi_\omega$ is well-known (see [3, 12, 20, 24]), and we omit the proof.

**Lemma 21.** Let $\omega > 0$. Then

$$\|\varphi_\omega\|_{L^3} = \inf \left\{ \|v\|_{H^1_\omega}^2 / \|v\|_{L^3}^2 : v \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}. \quad (5.3)$$

Moreover,

$$\{ v \in H^1(\mathbb{R}^N) : \|v\|_{H^1_\omega}^2 = \|v\|_{L^3}^3 = \|\varphi_\omega\|_{L^3}^3 \} = \{ e^{i\theta} \varphi_\omega(\cdot + y) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^N \}. \quad (5.4)$$
The next lemma is linked to the key Lemmas 17 and 18.

**Lemma 22.** Let \((u_1, u_2) \in \mathcal{G}_\omega\), and put

\[
a = \|u_1\|_{L^3}/\|\varphi_\omega\|_{L^3}, \quad b = \|u_2\|_{L^3}/\|\varphi_\omega\|_{L^3}.
\]

Then, \(a \geq 0\), \(b > 0\), and \((a, b)\) satisfies

\[
a^2 \leq a^2(ka + \gamma b), \quad 2b \leq 2b^2 + \gamma a^2, \quad a^2 + b^2 \leq \ell. \tag{5.5}
\]

Moreover,

1. If \((\kappa, \gamma) \in \mathcal{K}_1\), then \((a, b) = (\alpha_+, \beta_-)\).
2. If \((\kappa, \gamma) \in \mathcal{K}_2\), then \((a, b) = (0, 1)\).
3. If \((\kappa, \gamma) \in \mathcal{K}_3\), then \((a, b) \in \{(\alpha_+, \beta_-), (0, 1)\}\).

**Proof.** We first prove (5.5). If \(u_2 = 0\), then the second equation of (1.7) implies \(u_1 = 0\). This contradicts \((u_1, u_2) \in \mathcal{A}_\omega\). Thus, \(u_2 \neq 0\) and \(b > 0\). By (5.3), Lemma 19 and the Hölder inequality, we have

\[
\|\varphi_\omega\|_{L^3}\|u_1\|_{L^3}^2 \leq \|u_1\|_{H^1_{\omega}}^3 + \gamma \int_{\mathbb{R}^N} u_1^2 u_2^2 \, dx \\
\leq \kappa \|u_1\|_{L^3}^3 + \gamma \|u_1\|_{L^3}^2 \|u_2\|_{L^3}, \tag{5.6}
\]

which provides \(a^2 \leq a^2(ka + \gamma b)\). In the same way, we have

\[
\|\varphi_\omega\|_{L^3}\|u_2\|_{L^3}^2 \leq \|u_2\|_{H^1_{\omega}}^3 = \|u_2\|_{L^3}^3 + \gamma \int_{\mathbb{R}^N} u_1^2 u_2^2 \, dx \\
\leq \|u_2\|_{L^3}^3 + \frac{\gamma}{2} \|u_1\|_{L^3}^2 \|u_2\|_{L^3}. \tag{5.7}
\]

Since \(b > 0\), this gives \(2b \leq 2b^2 + \gamma a^2\). Finally, by Lemma 20 and (5.3), we obtain

\[
\ell \|\varphi_\omega\|_{L^3}^3 \geq 6d(\omega) = 6S_{\omega}(\vec{u}) = \|\vec{u}\|_{X_{\omega}}^2 \geq \|\varphi_\omega\|_{L^3}\left(\|u_1\|_{L^3}^2 + \|u_2\|_{L^3}^2\right), \tag{5.8}
\]

which implies \(a^2 + b^2 \leq \ell\). Hence, (5.5) is proved.

We now prove (1), (2) and (3). Let \((\kappa, \gamma) \in \mathcal{K}_1\). Then, since \(\ell < 1\), by Lemma 22, we see that \(a > 0\) and \((a, b) \in E_1 \cap E_2\). Thus, (1) follows from (5.5). Next, let \((\kappa, \gamma) \in \mathcal{K}_2\). Suppose that \(a > 0\). Then, by (5.5), we have \((a, b) \in E_1 \cap E_2\). However, this contradicts Lemma 18. Thus, we have \(a = 0\) and \(b = 1\), which proves (2). Part (3) can be proved similarly. \(\square\)
Proof of Theorem 6. We consider the case \((\kappa, \gamma) \in \mathcal{K}_1\). Let \(\vec{u} \in \mathcal{G}_\omega\). By (5.6), (5.7) and Lemma 22, we see that
\[
\|u_1/\alpha_+\|_{\mathcal{H}_L^2} = \|\varphi_\omega\|_{L^3}^3 = \|u_1/\alpha_+\|_{L^3}^3, \tag{5.9}
\]
\[
\|u_2/\beta_-\|_{\mathcal{H}_L^2} = \|\varphi_\omega\|_{L^3}^3 = \|u_2/\beta_-\|_{L^3}^3, \tag{5.10}
\]
\[
\int_{\mathbb{R}^N} u_1^2 u_2^2 \, dx = \|u_1\|_{L^3}^2 \|u_2\|_{L^3}. \tag{5.11}
\]
By (5.4) and by (5.9) and (5.10), there exist \((\theta_1, y_1), (\theta_2, y_2) \in \mathbb{R} \times \mathbb{R}^N\) such that \(u_1 = e^{i\theta_1} \alpha_+ \varphi_\omega (\cdot + y_1)\) and \(u_2 = e^{i\theta_2} \beta_- \varphi_\omega (\cdot + y_2)\). Moreover, by (5.11), we see that \(2\theta_1 - \theta_2 \in 2\pi \mathbb{Z}\) and \(y_1 = y_2\). Thus, we have \(\mathcal{G}_\omega \subset \mathcal{G}_\omega^0\). Since \(\mathcal{G}_\omega\) is not empty, (1) is proved. (2) and (3) can be proved in the same way. 

Finally, Theorem 5 is obtained as a corollary of Theorem 6.

Proof of Theorem 5. Let \(\kappa \leq 0\) and \(\gamma = 1\). Then, \((\kappa, \gamma) \in \mathcal{K}_2\), and by Theorem 6, \(\mathcal{G}_\omega = \mathcal{G}_\omega^0\). Thus, Theorem 5 follows from Proposition 5. 

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