Optimal Algorithms for Right-Sizing Data Centers — Extended Version*

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Abstract
Electricity cost is a dominant and rapidly growing expense in data centers. Unfortunately, much of the consumed energy is wasted because servers are idle for extended periods of time. We study a capacity management problem that dynamically right-sizes a data center, matching the number of active servers with the varying demand for computing capacity. We resort to a data-center optimization problem introduced by Lin, Wierman, Andrew and Thereska [17, 19] that, over a time horizon, minimizes a combined objective function consisting of operating cost, modeled by a sequence of convex functions, and server switching cost. All prior work addresses a continuous setting in which the number of active servers, at any time, may take a fractional value.

In this paper, we investigate for the first time the discrete data-center optimization problem where the number of active servers, at any time, must be integer valued. Thereby we seek truly feasible solutions. First, we show that the offline problem can be solved in polynomial time. Our algorithm relies on a new, yet intuitive graph theoretic model of the optimization problem and performs binary search in a layered graph. Second, we study the online problem and extend the algorithm Lazy Capacity Provisioning (LCP) by Lin et al. [17, 19] to the discrete setting. We prove that LCP is 3-competitive. Moreover, we show that no deterministic online algorithm can achieve a competitive ratio smaller than 3. Hence, while LCP does not attain an optimal competitiveness in the continuous setting, it does so in the discrete problem examined here. We prove that the lower bound of 3 also holds in a problem variant with more restricted operating cost functions, introduced by Lin et al. [17].

In addition, we develop a randomized online algorithm that is 2-competitive against an oblivious adversary. It is based on the algorithm of Bansal et al. [5] (a deterministic, 2-competitive algorithm for the continuous setting) and uses randomized rounding to obtain an integral solution. Moreover, we prove that 2 is a lower bound for the competitive ratio of randomized online algorithms, so our algorithm is optimal. We prove that the lower bound still holds for the more restricted model.

Finally, we address the continuous setting and give a lower bound of 2 on the best competitiveness of online algorithms. This matches an upper bound by Bansal et al. [5]. A lower bound of 2 was also recently shown by Antoniadis and Schewior [3]. We develop an independent proof that extends to the scenario with more restricted operating cost.

1 Introduction
Energy conservation in data centers is a major concern for both operators and the environment. In the U.S., about 1.8% of the total electricity consumption is attributed to data centers [22]. In 2015,

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more than 416 TWh (terawatt hours) were used by the world’s data centers, which exceeds the total power consumption in the UK [7]. Electricity cost is a significant expense in data centers [9]: about 18–28% of their budget is invested in power [13, 8]. Remarkably, the servers of a data center are only utilized 20–40% of the time on average [4, 6]. Even worse, when idle and in active mode, they consume about half of their peak power [21]. Hence, a promising approach for energy conservation and capacity management is to transition idle servers into low-power sleep states. However, state transitions, and in particular power-up operations, also incur energy/cost. Therefore, dynamically matching the number of active servers with the varying demand for computing capacity is a challenging optimization problem. In essence, the goal is to right-size a data center over time so as to minimize energy and operation costs.

**Problem Formulation.** We investigate a basic algorithmic problem with the objective of dynamically resizing a data center. Specifically, we resort to a framework that was introduced by Lin, Wierman, Andrew and Thereska [17, 19] and further explored, for instance, in [2, 3, 5, 1, 23, 18, 20].

Consider a data center with \( m \) homogeneous servers, each of which has an active state and a sleep state. An optimization is performed over a discrete, finite time horizon consisting of time steps \( t = 1, \ldots, T \). At any time \( t \), \( 1 \leq t \leq T \), a non-negative convex cost function \( f_t(\cdot) \) models the operating cost of the data center. More precisely, \( f_t(x_t) \) is the incurred cost if \( x_t \) servers are in the active state at time \( t \), where \( 0 \leq x_t \leq m \). This operating cost captures, e.g., energy cost and service delay, for an incoming workload, depending on the number of active servers. Furthermore, at any time \( t \) there is a switching cost, taking into account that the data center may be resized by changing the number of active servers. This switching cost is equal to \( \beta(x_t - x_{t-1})^+ \), where \( \beta \) is a positive real constant and \( (x)^+ = \max(0,x) \). Here we assume that transition cost is incurred when servers are powered up from the sleep state to the active state. A cost of powering down servers may be folded into this cost. The constant \( \beta \) incorporates, e.g., the energy needed to transition a server from the sleep state to the active state, as well as delays resulting from a migration of data and connections. We assume that at the beginning and at the end of the time horizon all servers are in the sleep state, i.e. \( x_0 = x_{T+1} = 0 \). The goal is to determine a vector \( X = (x_1, \ldots, x_T) \) called **schedule**, specifying at any time the number of active servers, that minimizes

\[
\sum_{t=1}^{T} f_t(x_t) + \beta \sum_{t=1}^{T} (x_t - x_{t-1})^+.
\]  

In the offline version of this data-center optimization problem, the convex functions \( f_t, 1 \leq t \leq T \), are known in advance. In the online version, the \( f_t \) arrive over time. At time \( t \), function \( f_t \) is presented. Recall that the operating cost at time \( t \) depends for instance on the incoming workload, which becomes known only at time \( t \).

All previous work on the data-center optimization problem assumes that the server numbers \( x_t, 1 \leq t \leq T \), may take fractional values. That is, \( x_t \) may be an arbitrary real number in the range \([0, m]\). From a practical point of view this is acceptable because a data center has a large number of machines. Nonetheless, from an algorithmic and optimization perspective, the proposed algorithms do not compute feasible solutions. Important questions remain if the \( x_t \) are indeed integer valued: (1) Can optimal solutions be computed in polynomial time? (2) What is the best competitive ratio achievable by online algorithms? In this paper, we present the first study of the data-center optimization problem assuming that the \( x_t \) take integer values and, in particular, settle questions (1) and (2).

**Previous Work.** As indicated above, all prior work on the data-center optimization problem assumes that the \( x_t, 1 \leq t \leq T \), may take fractional values in \([0, m]\). First, Lin et al. [19] consider the offline problem. They develop an algorithm based on a convex program that computes optimal
solutions. Second, Lin et al. [19] study the online problem. They devise a deterministic algorithm called Lazy Capacity Provisioning (LCP) and prove that it achieves a competitive ratio of exactly 3. Algorithm LCP, at any time $t$, computes a lower bound and an upper bound on the number of active servers by considering two scenarios in which the switching cost $\beta$ is charged, either when a server is powered up or when it is powered down. The tight bound of 3 on the competitiveness of LCP also holds if the algorithm has a finite prediction window $w$, i.e. a time $t$ it knows the current as well as the next $w$ arriving functions $f_t, \ldots, f_{t+w}$. Furthermore, Lin et al. [19] perform an experimental study with two real-world traces evaluating the savings resulting from right-sizing in data centers.

Bansal et al. [5] presented a 2-competitive online algorithm and showed that no deterministic or randomized online strategy can attain a competitiveness smaller than 1.86. Recently, Antoniadis and Schewior [3] improved the lower bound to 2. Bansal et al. [5] also gave a 3-competitive memoryless algorithm and showed that this is the best competitive factor achievable by a deterministic memoryless algorithm. The data-center optimization problem is an online convex optimization problem with switching costs. Andrew et al. [1] showed that there is an algorithm with sublinear regret but that $O(1)$-competitiveness and sublinear regret cannot be achieved simultaneously. Antoniadis et al. [2] examine generalized online convex optimization, where the values $x_t$ selected by an algorithm may be points in a metric space, and relate it to convex body chasing.

Further work on energy conservation in data center includes, for instance, [14, 15]. Khuller et al. [14] introduce a machine activation problem. There exists an activation cost budget and jobs have to be scheduled on the selected, activated machines so as to minimize the makespan. They present algorithms that simultaneously approximate the budget and the makespan. A second paper by Li and Khuller [15] considers a generalization where the activation cost of a machine is a non-decreasing function of the load. In the more applied computer science literature, power management strategies and the value of sleep states have been studied extensively. The papers focus mostly on experimental evaluations. Articles that also present analytic results include [10, 11, 12].

**Our Contribution.** We conduct the first investigation of the discrete data-center optimization problem, where the values $x_t$, specifying the number of active servers at any time $t \in \{1, \ldots, T\}$, must be integer valued. Thereby, we seek truly feasible solutions.

First, in Section 2 we study the offline algorithm. We show that optimal solutions can be computed in polynomial time. Our algorithm is different from the convex optimization approach by Lin et al. [19]. We propose a new, yet natural graph-based representation of the discrete data-center optimization problem. We construct a grid-structured graph containing a vertex $v_{t,j}$, for each $t \in \{1, \ldots, T\}$ and $j \in \{0, \ldots, m\}$. Edges represent right-sizing operations, i.e. changes in the number of active servers, and are labeled with operating and switching costs. An optimal solution could be determined by a shortest path computation. However, the resulting algorithm would have a pseudo-polynomial running time. Instead, we devise an algorithm that improves solutions iteratively using binary search. In each iteration the algorithm uses only a constant number of graph layers. The resulting running time is $O(T \log m)$.

The remaining paper focuses on the online problem and develops tight bounds on the competitiveness. In Section 3 we adapt the LCP algorithm by Lin et al. [19] to the discrete data-center optimization problem. We prove that LCP is 3-competitive, as in the continuous setting. We remark that our analysis is different from that by Lin et al. [19]. Specifically, our analysis resorts to the discrete structure of the problem and identifies respective properties. The analysis by Lin et al. [19] relates to their convex optimization approach that characterizes optimal solutions in the continuous setting.

In Section 4 we develop a randomized online algorithm which is 2-competitive against an obliv-
ious adversary. It is based on the algorithm of Bansal et al. [5] that achieves a competitive ratio of 2 for the continuous setting. Our algorithm works as follows. First, it extends the given discrete problem instance to the continuous setting. Then, it calculates a 2-competitive fractional schedule by using the algorithm of Bansal et al. Finally, we round the fractional schedule randomly to obtain an integral schedule. By using the right rounding technique it can be shown that the resulting schedule is 2-competitive according to the original discrete problem instance.

In Section 5 we devise lower bounds. We prove that no deterministic online algorithm can obtain a competitive ratio smaller than 3. Hence, LCP achieves an optimal competitive factor. Interestingly, while LCP does not attain an optimal competitiveness in the continuous data-center optimization problem (where the $x_t$ may take fractional values), it does so in the discrete problem (according to deterministic algorithms). We prove that the lower bound of 3 on the best possible competitive ratio also holds for a more restricted setting, originally introduced by Lin et al. [17] in the conference publication of their paper. Specifically, the problem is to find a vector $X = (x_1, \ldots, x_T)$ that minimizes

$$
\sum_{t=1}^{T} x_t f(\lambda_t/x_t) + \beta \sum_{t=1}^{T} (x_t - x_{t-1})^+,
$$

subject to $x_t \geq \lambda_t$, for $t \in \{1, \ldots, T\}$. Here $\lambda_t$ is the incoming workload at time $t$ and $f(z)$ is a non-negative convex function representing the operating cost of a single server running with load $z \in [0, 1]$. Since $f$ is convex, it is optimal to distribute the jobs equally to all active servers, so that the operating cost at time $t$ is $x_t f(\lambda_t/x_t)$. This problem setting is more restricted in that there is only a single function $f$ modeling operating cost over the time horizon. Nonetheless it is well motivated by real data center environments.

Furthermore, in Section 5 we address the continuous data-center optimization problem and prove that no deterministic online algorithm can achieve a competitive ratio smaller than 2. The same result was shown by Antoniadis and Schewior [3]. We develop an independent proof that can again be extended to the more restricted optimization problem stated in (2), i.e. the lower bound of 2 on the best competitiveness holds in this setting as well.

In addition, we show that there is no randomized online algorithm with a competitive ratio smaller than 2, so our randomized online algorithm presented in Section 4 is optimal. The construction of the lower bound uses some results of the lower bound proof for the continuous setting. Again, we show that the lower bound holds for the more restricted model.

Finally, in Section 5 we analyze online algorithms with a finite prediction window, i.e. at time $t$ an online algorithm knows the current as well as the next $w$ arriving functions $f_t, \ldots, f_{t+w}$. We show that all our lower bounds, for both settings (continuous and discrete) and both models (general and restricted), still hold.

## 2 An optimal offline algorithm

In this section we study the offline version of the discrete data-center optimization problem. We develop an algorithm that computes optimal solutions in $O(T \log m)$ time.

### 2.1 Graph-based approach

Our algorithm works with an underlying directed, weighted graph $G = (V, E)$ that we describe first. Let $[k] := \{1, 2, \ldots, k\}$ and $[k]_0 := \{0, 1, \ldots, k\}$ with $k \in \mathbb{N}$. For each $t \in [T]$ and each $j \in [m]_0$, there is a vertex $v_{t,j}$, representing the state that exactly $j$ servers are active at time $t$. Furthermore, there are two vertices $v_{0,0}$ and $v_{T+1,0}$ for the initial and final states $x_0 = 0$ and $x_{T+1} = 0$. For each
of additional term computation, which takes $O(\epsilon)$ cost of the corresponding schedule. An optimal schedule can be determined using a shortest path with $\epsilon > t$ for $t \in G$ active at time $t$. Similarly, for $t = 1$ and each $j' \in [m]_0$, there is a directed edge from $v_0,0$ to $v_{1,j'}$ with weight $f_1(j') + \beta(j')^+$. Finally, for $t = T$ and each $j \in [m]_0$, there is a directed edge from $v_{T,j}$ to $v_{T+1,0}$ of weight 0. The structure of $G$ is depicted in Figure 1.

In the following, for each $j \in [m]_0$, vertex set $R_j = \{v_{t,j} \mid t \in [T]\}$ is called row $j$. For each $t \in [T]$, vertex set $C_t = \{v_{t,j} \mid j \in [m]_0\}$ is called column $t$.

A path between $v_0$ and $v_{T+1}$ represents a schedule. If the path visits $v_{t,j}$, then $x_t = j$ servers are active at time $t$. Note that a path visits exactly one vertex in each column $C_t$, $1 \leq t \leq T$, because the directed edges connect adjacent columns. The total length (weight) of a path is equal to the cost of the corresponding schedule. An optimal schedule can be determined using a shortest path computation, which takes $O(Tm)$ time in the particular graph $G$. However, this running time is not polynomial because the encoding length of an input instance is linear in $T$ and $\log m$, in addition to the encoding of the functions $f_i$.

In the following, we present a polynomial time algorithm that improves an initial schedule iteratively using binary search. In each iteration the algorithm constructs and uses only a constant number of rows of $G$.

### 2.2 Polynomial time algorithm

An instance of the data-center optimization problem is defined by the tuple $\mathcal{P} = (T, m, \beta, F)$ with $F = (f_1, \ldots, f_T)$. We assume that $m$ is a power of two. If this is not the case we can transform the given problem instance $\mathcal{P} = (T, m, \beta, F)$ to $\mathcal{P}' = (T, m', \beta, F')$ with $m' = 2^{\lceil \log m \rceil}$ and

$$f'_t(x) = \begin{cases} f_t(x) & x \leq m \\ x \cdot (f_t(m) + \epsilon) & \text{otherwise} \end{cases}$$

with $\epsilon > 0$. The term $x \cdot f_t(m)$ ensures that $f'_t(x)$ is a convex function, since the greatest slope of $f_t$ is $f_t(m) - f_t(m-1) \leq f_t(m)$. The inequality holds because $f_t(x) \geq 0$ for all $x \in [m]_0$. The additional term $x \cdot \epsilon$ ensures that it is adverse to use a state $x > m$, because the cost of $f_t(m)$ is always smaller.

Our algorithm uses $\log m - 1$ iterations denoted reversely by $k = K := \log m - 2$ for the first iteration and $k = 0$ for the last iteration. The states used in iteration $k$ are always multiples of $2^k$. For the first iteration we use the rows $R_0, R_{m/4}, R_{m/2}, R_{3m/4}, R_m$, so that the graph of the first iteration contains the vertices

$$V^K := \{v_{0,0}, v_{T+1,0}\} \cup \{v_{t,\xi m/4} \mid t \in [T], \xi \in \{0, 1, 2, 3, 4\}\}.$$
The optimal schedule for this simplified problem instance can be calculated in \( \mathcal{O}(T) \) time, since each column contains only five states. Given an optimal schedule \( \hat{X}^k = (\hat{x}^k_1, \ldots, \hat{x}^k_T) \) of iteration \( k \), let
\[
V_t^{k-1} := \left\{ \hat{x}^k_t + \xi \cdot 2^{k-1} \mid \xi \in \{-2, -1, 0, 1, 2\} \right\} \cap \{m\}
\]
be the states used in the \( t \)-th column of the next iteration \( k-1 \). Thus the iteration \( k-1 \) uses the vertex set
\[
V^{k-1} := \left\{ v_{0,0}, v_{T+1,0} \right\} \cup \left\{ v_{i,j} \mid t \in [T], j \in V_t^{k-1} \right\}.
\]
Note that the states with \( \xi \in \{-2, 0, 2\} \) were already used in iteration \( k \) and we just insert the intermediate states \( \xi = -1 \) and \( \xi = 1 \). If \( \hat{x}^k_t = 0 \) (or \( \hat{x}^k_t = m \), then \( \xi \in \{-2, -1\} \) (or \( \xi \in \{1, 2\} \)) leads to negative states (or to states larger than \( m \)), thus the set \( V^{k-1} \) is cut with \( \{m\} \) to ensure that we only use valid states.

The last iteration \( (k = 0) \) provides an optimal schedule for the original problem instance as shown in the next section. The runtime of the algorithm is \( \mathcal{O}(T \cdot \log m) \) and thus polynomial.

### 2.3 Correctness

To prove the correctness of the algorithm described in the previous section we have to introduce some definitions:

Given the original problem instance \( \mathcal{P} = (T, m, \beta, F) \), we define \( \mathcal{P}_k \) (with \( k \in [K]_0 := [\log m-2]_0 \)) as the data-center optimization problem where we are only allowed to use the states that are multiples of \( 2^k \). Let \( M_k := \{ n \in \{m\} \mid n \mod 2^k = 0 \} \), so \( X \) is a feasible schedule for \( \mathcal{P}_k \) if \( x_t \in M_k \) holds for all \( t \in [T] \). To express \( \mathcal{P}_k \) as a tuple, we need another tuple element called \( M \) which describes the allowed states, i.e. \( x_t \in M \) for all \( t \in [T] \). The original problem instance can be written as \( \mathcal{P} = (T, m, \beta, F, \{m\}) \) and \( \mathcal{P}_k = (T, m, \beta, F, M_k) \). Note that \( \mathcal{P}_0 = \mathcal{P} \). Let \( \hat{X}^k = (\hat{x}^k_1, \ldots, \hat{x}^k_T) \) denote an optimal schedule for \( \mathcal{P}_k \). In general, for any given problem instance \( Q = (T, m, \beta, F, M) \), let \( \Phi_k(Q) := (T, m, \beta, F, M \cap \{i \cdot 2^k \mid i \in \mathbb{N}\}) \), so \( \Phi_k(\mathcal{P}) = \mathcal{P}_k \).

Instead of using only states that are multiple of \( 2^k \) we can also scale a given problem instance \( Q = (T, m, \beta, F, M) \) as follows. Let
\[
\Psi_t(Q) := (T, m/2^l, \beta \cdot 2^l, F', M')
\]
with \( M' := \{x/2^l \mid x \in M\} \), \( F' = (f'_1, \ldots, f'_T) \) and \( f'(x)_t := f_t(x \cdot 2^l) \). Given a schedule \( X = (x_1, \ldots, x_T) \) for \( Q \) with cost \( C^Q(X) \), the corresponding schedule \( X' = (x_1/2^l, \ldots, x_T/2^l) \) for \( \Psi_t(Q) \) has exactly the same cost, i.e. \( C^Q(X) = C^{\Psi_t(Q)}(X') \). Note that the problem instance \( \Phi_k(\mathcal{P}_k) \) uses all integral states less than or equal to \( m/2^l \), so there are no gaps.

Furthermore, we introduce a continuous version of any given problem instance \( Q \) where fractional schedules are allowed. Let \( \mathcal{Q} = (T, m, \beta, F, \{0, m]\) with \( F = (f_1, \ldots, f_T) \) be the continuous extension of the problem instance \( Q = (T, m, \beta, F, M) \), where \( x_t \in [0, m] \), \( f_t : [0, m] \rightarrow \mathbb{R}_{\geq 0} \) and
\[
\tilde{f}_t(x) := \begin{cases} 
  f_t(x) & \text{if } x \in \mathbb{N} \\
  (\lfloor x \rfloor - x) f_t(\lfloor x \rfloor) + (x - \lfloor x \rfloor) f_t(\lfloor x \rfloor) & \text{else}.
\end{cases}
\]
The operating cost of the fractional states is linearly interpolated, thus \( \tilde{f}_t \) is convex for all \( t \in [T] \).

Let \( X^* = (x^*_1, \ldots, x^*_T) \in [0, m]^T \) be an optimal schedule for \( \mathcal{P} \).

The set of all optimal schedules for a given problem instance \( Q \) is denoted by \( \Omega(Q) \). Let
\[
C^Q_{[a,b]}(X) := \sum_{t=a}^{b} f_t(x_t) + \sum_{t=a+1}^{b} \beta(x_t - x_{t-1})^+
\]
be the cost during the time interval \( \{a, a+1, \ldots, b\} \).

We define \( f_0(x) := 0 \), so \( C^Q_{[0,T]}(X) = C^Q(X) \).

Now, we are able to prove the correctness of our algorithm. We begin with a simple lemma showing the relationship between the functions \( \Phi \) and \( \Psi \).
Lemma 1. The problem instances $\Phi_{k-l}(\Psi_l(P_l))$ and $\Psi_l(P_k)$ are equivalent.

Proof. We begin with $\Phi_{k-l}(\Psi_l(P_l))$ and simply apply the definitions of $P_l$, $\Psi_l$ and $\Phi_{k-l}$.

We will show that it is possible to modify $X$ such that equation (4) is fulfilled.

\begin{align*}
\Phi_{k-l}(\Psi_l(P_l)) &= \Phi_{k-l}(\Psi_l((T, m, \beta, F, \{n \in [m]_0\}))) \\
&= \Phi_{k-l}(\Psi_l((T, m, \beta, F, \{n \in [m]_0 \mid n \mod 2^l = 0\}))) \\
&= \Phi_{k-l}((T, m/2^l, \beta \cdot 2^l, F_l, \{n \in [m/2^l]_0 \mid n \mod 1 = 0\})) \\
&= (T, m/2^l, \beta \cdot 2^l, F_l, \{n \in [m/2^l]_0 \mid n \mod 2^{k-l} = 0\})
\end{align*}

Afterwards, we use the definitions of $\Psi_l$, $\Phi_k$ and $P_k$ and get $\Psi_l(P_k)$ as shown below:

\begin{align*}
(T, m/2^l, \beta \cdot 2^l, F_l, \{n \in [m/2^l]_0 \mid n \mod 2^{k-l} = 0\}) \\
&= \Psi_l((T, m, \beta, F, \{n \in [m]_0 \mid n \mod 2^k = 0\})) \\
&= \Psi_l(\Phi_k((T, m, \beta, F, \{n \in [m]_0\}))) \\
&= \Psi_l(P_k)
\end{align*}

The next technical lemma will be needed later. Informally, it demonstrates that optimal solutions of the reduced discrete problem and the above continuous problem behave similarly.

Lemma 2. Let $Y \in \Omega(P_k)$ be an optimal schedule for $P_k$ with $k \in [K]_0$. There exists an optimal solution $X^* \in \Omega(\bar{P})$ such that

$$
(y_t - y_{t-1}) \cdot (x^*_t - x^*_{t-1}) \geq 0
$$

holds for all $t \in [T]$ with $|y_t - x^*_t| \geq 2^k$ or $|y_{t-1} - x^*_{t-1}| \geq 2^k$.

Proof. Let $x^*_{t \min} := \max(\arg \min_x f_t(x))$ be the greatest state that minimizes $f_t$ and let $x^*_{t \min} := \min(\arg \min_x f_t(x))$ be the smallest state that minimizes $f_t$. Let $X^* \in \Omega(\bar{P})$ be an arbitrary optimal solution. We will show that it is possible to modify $X^*$ such that it fulfills equation (4) without increasing the cost. The modified schedule is denoted by $\tilde{X}^*$. We differ between several cases according to the relations of $y_{t-1}, y_t, x_{t-1}$ and $x_t$:

1. $x^*_{t-1} > x^*_t$
   1. $y_{t-1} \geq y_t$
      Equation (4) is fulfilled.
   2. $y_{t-1} < y_t$
      (A) $y_{t-1} \leq x^*_{t-1}$
         If $x^*_{t-1} < x^*_t$, then using $\tilde{x}^*_{t-1} := x^*_t - 1$ instead of $x^*_{t-1}$ would lead to a better solution, because $f_{t-1}$ is a convex function and the switching costs between the time slots $t-2$ and $t$ are not increased, so $x^*_{t-1} < x^*_{t-1}$ must be fulfilled. If $x^*_{t-1} > x^*_t$, then $\tilde{x}^*_t := x^*_t + 1$ would lead to a better solution for the same reason, so $x^*_{t-1} \leq x^*_t$.
      (i) $y_t \leq x^*_{t-1}$
         If $x^*_{t-1} > y_{t-1}$, then using $\tilde{y}_{t-1} := y_t \leq x^*_{t-1} \leq x^*_{t-1}$ instead of $y_{t-1}$ would lead to a better solution, so $x^*_{t-1} \leq y_{t-1}$ must be fulfilled.
         **Case 1:** $x^*_t \geq y_{t-1}$
         We set $\tilde{x}^*_{t-1} := x^*_t$, so equation (4) is fulfilled. Since $x^*_{t-1} \leq y_{t-1} \leq x^*_t < x^*_{t-1} \leq x^*_{t-1}$ the cost of $\tilde{X}^*$ is not increased.
III.  

Case 2: $x_t^* < y_{t-1}$

We set $\tilde{x}_{t-1} := y_{t-1}$ which does not increase the cost of $\tilde{X}^*$ because $x_{t-1}^* \geq y_{t-1} > x_t^*$ and $x_{t-1}^{\min} \leq y_{t-1} \leq x_{t-1}^* \leq x_{t-1}^{\min+}$. We have $y_t \geq y_{t-1} = \tilde{x}_{t-1}^* > x_t^*$. If $x_t^{\min+} < y_t$, then $\tilde{y}_t := y_{t-1}$ would lead to a better solution, so $x_t^{\min+} \geq y_t$. We set $\tilde{x}_t^* := \tilde{x}_{t-1}^*$, so equation (4) is fulfilled. Since $x_t^{\min-} \leq x_t^* < \tilde{x}_{t-1}^* \leq y_t \leq x_t^{\min+}$ the cost of $\tilde{X}^*$ is not increased.

(ii) $y_t > x_t^{*}$

Case 1: $x_t^* \leq y_{t-1}$

We have $y_t > y_{t-1} \geq x_t^* \geq x_t^{\min-}$. If $x_t^{\min+} < y_t$, then $\tilde{y}_t := y_{t-1}$ would lead to a better solution, so $x_t^{\min+} \geq y_t$. We set $\tilde{x}_t^* := x_{t-1}^*$, so equation (4) is fulfilled. Since $x_t^{\min-} \leq x_t^* < x_{t-1}^* < y_t \leq x_t^{\min+}$ the cost of $\tilde{X}^*$ is not increased.

Case 2: $x_t^* > y_{t-1}$ and $|y_{t-1} - x_{t-1}^*| \geq 2^k$

There exists a state $\tilde{y}_{t-1}$ with $y_{t-1} < \tilde{y}_{t-1} \leq x_{t-1}^*$. If $x_{t-1}^{\min-} > y_{t-1}$, then using $\tilde{y}_{t-1}$ instead of $y_{t-1}$ would lead to a better solution, so $x_{t-1}^{\min-} \leq y_{t-1}$ must be fulfilled. We set $\tilde{x}_{t-1}^* := x_t^*$, so equation (4) is fulfilled. Since $x_{t-1}^{\min-} \leq x_t^* < x_{t-1}^* \leq x_{t-1}^{\min+}$ the cost of $\tilde{X}^*$ is not increased.

Case 3: $x_t^* > y_{t-1}$ and $|y_{t-1} - x_{t-1}^*| < 2^k$

There exists a state $\tilde{y}_t$ with $x_t^* \leq \tilde{y}_t < y_t$. If $x_t^{\min+} < y_t$, then using $\tilde{y}_t$ instead of $y_t$ would lead to a better solution, so $x_t^{\min+} \geq y_t$. We set $\tilde{x}_t^* := x_{t-1}^*$, so equation (4) is fulfilled. Since $x_t^{\min-} \leq x_t^* < x_{t-1}^* < y_t \leq x_t^{\min+}$ the cost of $\tilde{X}^*$ is not increased.

(B) $y_t > x_{t-1}^*$

If $x_{t-1}^{\min-} > x_t^*$, then $\tilde{x}_t^* := x_t^* + 1$ would lead to a better solution, so $x_t^{\min-} \leq x_t^*$. If $x_t^{\min+} < y_t$, then $\tilde{y}_t := y_{t-1}$ would lead to a better solution, so $x_t^{\min+} \geq y_t$. We set $\tilde{x}_t^* := x_{t-1}^*$, so equation (4) is fulfilled. Since $x_t^{\min-} \leq x_t^* < x_{t-1}^* < y_t \leq x_t^{\min+}$ the cost of $\tilde{X}^*$ is not increased.

II. $x_{t-1}^* = x_t^*$

Equation (4) is fulfilled.

III. $x_{t-1}^* < x_t^*$

This case is symmetric to case 1.

By using Lemma 2 we can show that an optimal solution for a discrete problem instance $\mathcal{P}_k$ cannot be very far from an optimal solution of the continuous problem instance $\mathcal{P}$.

Lemma 3. Let $\hat{X}^k \in \Omega(\mathcal{P}_k)$ be an arbitrary optimal schedule for $\mathcal{P}_k$ with $k \in [K]_0$. There exists an optimal schedule $X^* \in \Omega(\mathcal{P})$ for $\mathcal{P}$ such that $|\hat{x}_t^k - x_t^*| < 2^k$ holds for all $t \in [T]$. Formally,

$$\forall k \in [K]_0 : \forall \hat{X}^k \in \Omega(\mathcal{P}_k) : \exists X^* \in \Omega(\mathcal{P}) : \forall t \in [T] : |\hat{x}_t^k - x_t^*| < 2^k.$$

Proof. To get a contradiction, we assume that there exists a $\hat{X}^k \in \Omega(\mathcal{P}_k)$ with $k \in [K]_0$ such that for all optimal schedules $X^* \in \Omega(\mathcal{P})$ there is at least one $t \in [T]$ with $|\hat{x}_t^k - x_t^*| \geq 2^k$. Let $X^* \in \Omega(\mathcal{P})$ be arbitrary. Given the schedule $X^k$, let $J_1, \ldots, J_l \subseteq [T]$ be the inclusion maximal time intervals such that $|\hat{x}_t^k - x_t^*| \geq 2^k$ holds for all $t \in J_j$ and the sign of $\hat{x}_t^k - x_t^*$ remains the same during $J_j$. The set of all $J_j$ with $j \in [l]$ is denoted by $\mathcal{J}$. If $\mathcal{J}$ is empty, then the condition $|\hat{x}_t^k - x_t^*| < 2^k$ is
fulfilled for all $t \in [T]$. We divide $J$ into the disjunct sets $J^+$ and $J^-$ such that $J^+$ contains the intervals where $\hat{x}_t^k - x_t^*$ is positive and $J^-$ the others.

Given a schedule $X$, the corresponding interval set is denoted by $J(X)$, the set of all time slots by $S(X) := \{ t \in J \mid J \in J(X) \}$, and the number of time slots in $J$ by $L(X) := |S(J(X))| = \sum_{J \in J} |J|$. We will use a recursive transformation $\phi$ that reduces $L(X)$ at least by one for each step, while the cost of $X$ is not increased. Formally, we have to show that $L(\phi(X)) \leq L(X) - 1$ and $C^p(\phi(X)) \leq C^p(X)$ holds. The first inequality ensures that the recursive procedure will terminate. The transformation described below will produce fractional schedules, however for each $t \in [T] \setminus S(X)$ it is ensured that $x_t \in M_k$. Therefore, if $L(X) = 0$, the corresponding schedule fulfills $|x_t - x_t^*| < 2^k$ and $x_t \in M_k$ for all $t \in [T]$. To describe the transformation, we will use the following notation: A given schedule $Y = (y_1, \ldots, y_T)$ with $L(Y) > 0$ is transformed to $Z = \phi(Y) = (z_1, \ldots, z_T)$.

Let $t := (t_t + 1, \ldots, t_{t+1} - 1) \in J(Y)$. We differ between two cases, in case 1 we handle the intervals in $J^+$, i.e. $y_t > x_t^* + 2^k$ holds for all $t \in J$ and in case 2 we handle the intervals in $J^-$, i.e. $y_t < x_t^* - 2^k$. We will handle case 1 first.

Let $[x]_n := n \cdot [x/n]$ with $x \in \mathbb{R}$ and $n \in \mathbb{N}$ be the smallest value that is divisible by $n$ and greater than or equal to $x$. The schedule $Y$ is transformed to $Z$ with

$$z_t := \begin{cases} y_t & \text{if } t \notin J \\ \lambda \cdot y_t + (1 - \lambda) \cdot x_t^* & \text{if } t \in J \end{cases}$$

where $\lambda \in [0, 1]$ is as small as possible such that $z_t \geq [x_t^*]_2$ holds for all $t \in J$, so at least one time slot $t_\infty \in J$ satisfies this condition with equality. This transformation ensures that $L(Z) \leq L(Y) - 1$ holds, because the interval $J$ is split into at least two intervals and one time slot $(t_\infty)$ between them that fulfills $|z_t - x_t^*| < 2^k$.

We still have to show that the total cost is not increased by this operation. The total cost can be written as

$$C^p(X) = C^p_{[0,t_1]}(X) + \beta(x_{t_1+1} - x_t^*) + C^p_{[t_1+1,t_1+1]}(X) + \beta(x_{t_1+1} - x_{t_1+1}) + C^p_{[t_1+1,T]}(X).$$

(5)

We have $C^p_{[0,t_1]}(Y) = C^p_{[0,t_1]}(Z)$ and $C^p_{[t_1+1,T]}(Y) = C^p_{[t_1+1,T]}(Z)$.

Consider the time slot $t_1$. By the definition of the interval $J$, the condition $|y_{t_1+1} - x_{t_1+1}^*| \geq 2^k$ is fulfilled. Thus we can apply Lemma 2 which says that the terms $(y_{t_1+1} - y_{t_1})$ and $(x_{t_1+1}^* - x_{t_1}^*)$ are both either non-negative or non-positive, so in Equation (5) the term $\beta(x_{t_1+1} - x_{t_1})^*$ can be replaced by $\beta(x_{t_1+1} - x_{t_1})$ or zero, respectively. Analogously, for the time slot $t_{t_1+1}$, the condition $|y_{t_{t_1+1}} - x_{t_{t_1+1}}^*| \geq 2^k$ is fulfilled, so by Lemma 2 the term $\beta(x_{t_{t_1+1}} - x_{t_{t_1+1}})$ in Equation (5) can be replaced by $\beta(x_{t_{t_1+1}} - x_{t_{t_1+1}})$ or zero. In the former cases, the cost function is

$$C^p(X) = C^p_{[0,t_1]}(X) + \beta x_{t_1+1} - x_t^* + C^p_{[t_1+1,t_{t_1+1}]}(X) + \beta x_{t_1+1} - \beta x_{t_{t_1+1}} + C^p_{[t_{t_1+1},T]}(X).$$

Given a schedule $X = (x_1, \ldots, x_t)$, we define $X_{[a,b]} := (x_a, \ldots, x_b)$ and $X_J = X_{[t_1+1,t_{t_1+1}]}$. Since there is no summand that contains both $x_t$ and $x_{t_1+1}$, the function

$$D_{X^*}((x_{t_1+1}, \ldots, x_{t_{t_1+1}})) := C^p_{[0,t_1]}(X^*) - \beta x_{t_1}^* + \beta x_{t_{t_1+1}} + C^p_{[t_1+1,t_{t_1+1}]}(X^*) + \beta x_{t_{t_1+1}} - \beta x_{t_{t_1+1}} + C^p_{[t_{t_1+1},T]}(X^*)$$



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with $x^t_{i+1} \geq x^t_{i+1}$ and $x^t_{i+1} \geq x^*_{i+1}$ is convex and has a minimum at $X^\text{min}_j := (x^*_{t+1}, \ldots, x^*_{t+1})$.

Due to convexity, $D_X(Y_j) \geq D_X(Z_j) \geq D_X(X^\text{min}_j)$, because $Z_j = \lambda Y_j + (1 - \lambda) X^\text{min}_j$. Therefore $C^P(Z) \leq C^P(Y)$ holds. If $\beta(x^t_{i+1} - x^t_i)^+ = 0$ or $\beta(x^t_{i+1} - x^t_{i+1})^+ = 0$ we can use the same argument.

We still have to handle the second case, i.e. $y_t < x^*_i$. The proof works almost analogously, the difference is that we choose $\lambda$ as small as possible such that $z_t \leq [x^*_i]_{2^k}$ (where $[x]_n := n \cdot [x/n]$). Then we have a time slot $t_\pi$ with $z_t = [x^*_i]_{2^k}$, so $L(Z) \leq L(Y) - 1$. The proof that shows $C^P(Z) \leq C^P(Y)$ holds for both cases.

We use the transformation $\phi$ until $L(Z) = 0$. Then $J(Z)$ is empty, so all states of $Z$ are multiples of $2^k$, i.e. $z_t \in M_k$ for all $t \in [T]$. Since $\hat{X}^k$ was defined to be optimal, $C^P(\hat{X}^k) = C^P(Z)$ holds. By our assumption, $Z \neq \hat{X}^k$ holds (because otherwise $|\hat{x}_t^k - x^*_i| < 2^k$ would be fulfilled for all $t \in [T]$), so there was a transformation with $\lambda < 1$. Thus we moved towards the optimal schedule, however by $C^P(\hat{X}^k) = C^P(Z)$, the cost does not change. As $D_X(X')$ is a convex function, $C^P(\hat{X}^k) = C^P(Z)$ implies that $C^P(Z) = C^P(X^*)$, because $X^*$ minimizes $C^P$. In this case $\hat{X}^k$ is also optimal for $\mathcal{P}$, so the condition $|\hat{x}_t^k - x^*_i| < 2^k$ is already fulfilled.

In all cases we get a contradiction, so our assumption was wrong and the lemma is proven. □

The next lemma shows how an optimal fractional schedule can be rounded to an integral schedule such that it is still optimal.

**Lemma 4.** Let $X^* \in \Omega(\mathcal{P})$. The schedules $[X^*] := ([x^*_1], \ldots, [x^*_T])$ and $[X^*] := ([x^*_1], \ldots, [x^*_T])$ are optimal too, i.e. $[X^*], [X^*] \in \Omega(\mathcal{P})$.

**Proof.** Let $X^* \in \Omega(\mathcal{P})$ be arbitrary. Let $\mathcal{I}(X^*) = \{I_1, \ldots, I_l\}$ be the set of time intervals such that for each $I_i := \{a_i, a_i+1, \ldots, b_i\}$ with $i \in [l]$ the following conditions are fulfilled

1. All states of $X^*$ have the same value during $I_i$, i.e. $x^*_t = v_i$ for all $t \in I_i$.
2. The value is fractional, i.e. $v_i \notin \mathbb{N}$.
3. Each $I_i$ is inclusion maximal, i.e. $x^*_{a_i-1} \neq v_i$ and $x^*_{b_i+1} \neq v_i$.
4. The intervals are sorted, i.e. $b_i < a_{i+1}$ for all $i \in [l-1]$.

If $\mathcal{I}(X^*) = \emptyset$, then $X^*$ is an integral schedule, so $[X^*] = X^* = [X^*]$. Otherwise let $I_i \in \mathcal{I}(X^*)$ be an arbitrary interval. We will transform $X^*$ to $X'$ by changing the states at $I_i$ such that $|\mathcal{I}(X^*)| < |\mathcal{I}(X^*)|$ and $[x^*_t] \leq x^*_t \leq [x^*_t]$ for all $t \in I_i$. Let $g(x) := \sum_{t=a_i}^{b_i} \delta_t(x)$. Since each $\delta_t(x)$ is linear for $x \in [v_i], [v_i]$, the slope of $g(x)$ is constant for $x \in [v_i], [v_i]$ and denoted by $g'(v_i)$. According to $I_i$, we have to differ between different cases:

1. $x^*_{a_i-1} < v_i < x^*_{b_i+1}$

   Let $\tilde{x}^*_{a_i-1} := \max\{x^*_{a_i-1}, [v_i]\}$ and $\tilde{x}^*_{b_i+1} := \min\{x^*_{b_i+1}, [v_i]\}$. By using any schedule with $x'_{a_i} = x^*_{a_i+1} = \cdots = x'_{b_i} \in [\tilde{x}^*_{a_i-1}, \tilde{x}^*_{b_i+1}]$ (and $x^*_t = x^*_t$ otherwise), the switching cost is unchanged. Since $I_i$ is inclusion maximal and $X^*$ is optimal, we can conclude that $g'(v_i) = 0$, so $C(X') = C(X^*)$. To show that $[X^*]$ is optimal, we set $x'_t = \tilde{x}^*_{b_i+1}$ for all $t \in I_i$. To show that $[X^*]$ is optimal, we set $x'_t = \tilde{x}^*_{a_i-1}$ for all $t \in I_i$.

2. $x^*_{a_i-1} > v_i > x^*_{b_i+1}$

   This case works analogously to the first case.
3. $x_{a_i-1}^* > v_i < x_{b_i+1}^*$

Let $v^+ = \min\{x_{a_i-1}^*, x_{b_i+1}^*, [v_i]\}$. Let $v_i' = \{[v_i], v^+\}$. By using the schedule $x_{a_i}^* = x_{a_i+1}^* = \cdots = x_{b_i} = v_i'$ for all $t \in I_i$, the switching cost is increased by $\beta(v_i - v_i')$, but the operating cost is reduced by $g'(v_i) \cdot (v_i - v_i')$. Since $v_i$ is fractional, $v_i \notin \{x_{a_i-1}^*, x_{b_i+1}^*\}$. As $X^*$ is optimal, we can conclude that $g'(v_i) = \beta$, so the total cost of $X'$ does not change for $v_i' \in \{[v_i], v^+\}$.

To show that $[X^*]$ is optimal, we set $x_t' = [v_i]$ for all $t \in I_i$. To show that $[X^*]$ is optimal, we set $x_t' = v^+$ for all $t \in I_i$.

4. $x_{a_i-1}^* < v_i > x_{b_i+1}^*$

This case works analogously to the third case, but $[x]$ and $[x]$ are swapped as well as min and max. Furthermore, $g'(v_i) = -\beta$ and we replace $(v_i - v_i')$ with $(v_i' - v_i)$.

By using the transformation described above, we can reduce the number $|I|$ of fractional intervals is at least reduced by 1. By applying the transformation several times until $|I| = 0$, we receive $[X^*]$ or $[X^*]$. The total cost is not increased by the operations.

So far, we have shown in Lemma 3 that for each optimal solution of the discrete problem instance $\mathcal{P}_k$ there is an optimal solution of the continuous problem instance $\mathcal{P}$ that is not far away. In the following lemma we expand this statement: Given an optimal solution for $\mathcal{P}_k$, there is not only a fractional solution for $\mathcal{P}$ that is not far away, but also an optimal solution of the discrete problem instance $\mathcal{P}_l$ for the subsequent iterations $l < k$.

Lemma 5. Let $k > l$ with $k, l \in [K]_0$. Let $\hat{X}^k \in \Omega(\mathcal{P}_k)$ be an arbitrary optimal schedule for $\mathcal{P}_k$ with $k \in [K]_0$. There exists an optimal schedule $\hat{X}^l \in \Omega(\mathcal{P}_l)$ for $\mathcal{P}_l$ such that $|\hat{x}_t^k - \hat{x}_t^l| \leq 2^k$ for all $t \in [T]$. Formally, $\forall k \in [K]_0 : \forall l \in [k - 1] : \forall \hat{X}^k \in \Omega(\mathcal{P}_k) : \exists \hat{X}^l \in \Omega(\mathcal{P}_l) : \forall t \in [T] : |\hat{x}_t^k - \hat{x}_t^l| \leq 2^k$.

Proof. Consider the reduced problem instance $\mathcal{Q} := \Psi_l(\mathcal{P}_l)$ as well as the problem instance $\mathcal{Q}_{k-l} := \Phi_{k-l}(\Psi_l(\mathcal{P}_l))$ which is equivalent to $\Psi_l(\mathcal{P}_k)$ due to Lemma 4. Let $\hat{X}^k_{\mathcal{Q}} = (\hat{x}_1^k/2, \ldots, \hat{x}_T^k/2)$ be an optimal schedule for $\mathcal{Q}_{k-l}$. We apply Lemma 3 but we use $\hat{X}^k_{\mathcal{Q}}$ and $\mathcal{Q}$ instead of $\hat{X}^k$ and $\mathcal{P}$. By Lemma 3, there exists an optimal fractional schedule $\hat{X}^k_{\mathcal{Q}} = (x_1^*, \ldots, x_T^*)$ for $\mathcal{Q}$ such that $|\hat{x}_1^k/2 - x_1^*| \leq 2^{k-l}$. By Lemma 4, $|X^k_{\mathcal{Q}}|$ is also an optimal schedule for $\mathcal{Q}$ and therefore it is also optimal for $\mathcal{Q}$. The inequality $|\hat{x}_1^k/2 - x_1^*| \leq 2^{k-l}$ still holds, because the terms $\hat{x}_1^k/2$ and $2^{k-l}$ are integer and therefore adding a value less than 1 to the left side cannot invalidate the inequality. Let $\hat{X}^l := ((x_1^* \cdot 2, \ldots, x_T^* \cdot 2^l)$. As $|X^k_{\mathcal{Q}}|$ is optimal for $\mathcal{Q}$, $\hat{X}^l$ must be optimal for $\mathcal{P}_l$. Furthermore $|x_t^l| = \hat{x}_t^k/2^l$ holds, so we can insert it into the above inequality and get $|\hat{x}_1^k/2^l - \hat{x}_1^l/2^l| \leq 2^{k-l}$ which is equivalent to $|\hat{x}_1^k - \hat{x}_1^l| \leq 2^k$.

Now, we have proven all parts to show the correctness of our polynomial-time algorithm:

Theorem 1. The algorithm described in Section 2.2 is correct.

Proof. We will show the correctness by induction. In the first iteration, the algorithm finds an optimal schedule for $\mathcal{P}_K$, because all states of $M_K$ are considered.

Given an optimal schedule $\hat{X}^k$, in the next iteration the algorithm only considers the states $x_t$ with $|\hat{x}_t^k - x_t| \leq 2^k$. By Lemma 5 there exists an optimal schedule $\hat{X}^l$ with $l = k - 1$ such that $|\hat{x}_t^k - \hat{x}_t^l| \leq 2^k$ holds. Therefore the schedule found in iteration $k - 1$ must be optimal for $\mathcal{P}_{k-1}$ (although some states are ignored by the algorithm). Thus, by induction, the algorithm will find an optimal schedule for $\mathcal{P}_0 = \mathcal{P}$ in the last iteration.
3 Deterministic online algorithm

Lin et. al. \cite{lin17, lin19} developed an algorithm called Lazy Capacity Provisioning (LCP) that achieves a competitive ratio of 3 for the continuous setting (i.e. \( x_t \in \mathbb{R} \)). In this section we adapt LCP to the discrete data-center optimization problem and prove the algorithm is 3-competitive for this problem as well.

The general approach of our proof is similar to the proof of the continuous setting in \cite{lin17}. Some lemmas (e.g., Lemma 6 and 11) were adopted, however, their proofs are completely different. Lin et. al. use the properties of the convex program, especially duality and the complementary slackness conditions. This approach cannot be adapted to the discrete setting.

3.1 Algorithm

First, we will define lower and upper bounds for the optimal offline solution that can be calculated online. For a given time slot \( \tau \) let \( X^L_\tau := (x^L_{\tau,1}, \ldots, x^L_{\tau,\tau}) \) be the vector that minimizes

\[
C^L_\tau(X) = \sum_{t=1}^{\tau} f_t(x_t) + \beta \sum_{t=1}^{\tau} (x_t - x_{t-1})^+
\]

with \( X = (x_1, \ldots, x_\tau) \). This term describes the cost of a workload that ends at \( \tau \leq T \). For \( \tau = T \) this equation is equivalent to (1). Let \( x^U_\tau := x^L_{\tau,\tau} \) be the last state for this truncated workload. If there is more than one vector that minimizes (6), then \( x^U_\tau \) is defined as the smallest possible value.

Similarly, let \( X^U_\tau := (x^U_{\tau,1}, \ldots, x^U_{\tau,\tau}) \) be the vector that minimizes

\[
C^U_\tau(X) = \sum_{t=1}^{\tau} f_t(x_t) + \beta \sum_{t=1}^{\tau} (x_{t-1} - x_t)^+.
\]

The difference to the equation (6) is that we pay the switching cost for powering down. Powering up does not cost anything. The last state is denoted by \( x^*_\tau := x^U_{\tau,\tau} \). If there is more than one vector that minimizes (7), then \( x^U_\tau \) is the largest possible value.

Define \( [x]_a^b := \max\{a, \min\{b, x\}\} \) as the projection of \( x \) into the interval \([a, b]\). The LCP algorithm is defined as follows:

\[
x^{LCP}_\tau := \begin{cases} 
0, & \tau = 0 \\
[x]^{LCP}_{x^L_{\tau-1}} x^U_\tau, & \tau \geq 1
\end{cases}
\]

Before we can prove that this algorithm is 3-competitive, we have to introduce some notation.

3.2 Notation

Let \( X^* = (x^*_1, \ldots, x^*_\tau) \) be an optimal offline solution that minimizes equation (1) (i.e. the whole workload). Note that \( C^L_\tau(X^*) \) indicates the cost of the optimal solution until \( \tau \).

Let \( R_\tau(X) := \sum_{t=1}^{\tau} f_t(x_t) \) with \( X = (x_1, \ldots, x_\tau) \) denote the operating cost until \( \tau \), let \( S^L_\tau(X) := \beta \sum_{t=1}^{\tau} (x_t - x_{t-1})^+ \) denote the switching cost in \( C^L_\tau(X) \) and let \( S^U_\tau(X) := \beta \sum_{t=1}^{\tau} (x_{t-1} - x_t)^+ \) denote the switching cost in \( C^U_\tau(X) \). Note that \( C^L_\tau(X) = R_\tau(X) + S^L_\tau(X) \) and \( C^U_\tau(X) = R_\tau(X) + S^U_\tau(X) \). Furthermore,

\[
S^L_\tau(X) = S^U_\tau(X) + \beta x_\tau
\]

as well as \( C^L_\tau(X) = C^U_\tau(X) + \beta x_\tau \) holds, because in \( C^L_\tau \) we have to pay the missing switching cost to reach the final state \( x_\tau \). Note that \( \beta x_\tau \) equals the cost for powering up in \( C^L_\tau \) minus the cost for powering down in \( C^U_\tau \).
Given an arbitrary function \( f : [m] \to \mathbb{R} \), we define
\[
\Delta f(x) := f(x) - f(x - 1)
\]
as the slope of \( f \) at \( x \). Let
\[
\hat{C}^Y_{\tau}(x) := \min_{x_1, \ldots, x_{\tau-1}} C^Y_{\tau}((x_1, \ldots, x_{\tau-1}, x))
\]
with \( Y \in \{L, U\} \) be the minimal cost achievable with \( x_{\tau} = x \).

### 3.3 Competitive ratio

In this section we prove that the LCP algorithm described by equation (8) achieves a competitive ratio of 3. First, we show that the optimal solution is bounded by the upper and lower bounds defined in the previous section.

**Lemma 6.** For all \( \tau \), \( x^L_{\tau} \leq x^*_\tau \leq x^U_{\tau} \) holds.

**Proof.** We prove both parts of the inequality by contradiction:

**Part 1** \((x^L_{\tau} \leq x^*_\tau)\): Assume that \( x^L_{\tau} > x^*_\tau \). By the definition of the lower bound, \( C^L_{\tau}(X^L_{\tau}) < C^L_{\tau}(X^*) \) holds and we can replace \((x^*_1, \ldots, x^*_\tau)\) by \((x^L_{\tau_1}, \ldots, x^L_{\tau_\tau})\). This reduces the total cost of \( x^* \), because the cost up to \( \tau \) is reduced and for \( \tau + 1 \) there are no additional switching costs because \( x^L_{\tau} > x^*_\tau \) holds. Thus \( x^* \) would not be an optimal solution which is a contradiction, so \( x^L_{\tau} \leq x^*_\tau \) must be fulfilled.

**Part 2** \((x^*_\tau \leq x^U_{\tau})\): Assume that \( x^*_\tau > x^U_{\tau} \). By definition of the upper bound, \( C^U_{\tau}(X^U_{\tau}) < C^U_{\tau}(X^*) \) and, thus,
\[
R_{\tau}(X^U_{\tau}) + S^U_{\tau}(X^U) < R_{\tau}(X^*) + S^U_{\tau}(X^*)
\]
holds. The cost of the optimal solution until \( \tau \) is \( R_{\tau}(X^*) + S^L_{\tau}(X^*) \). If the states \((x^*_1, \ldots, x^*_\tau)\) are replaced by \( X^U_{\tau} \) and afterwards \( x^*_\tau - x^U_{\tau} \) servers are powered up (to ensure that we end in the same state), then the cost is \( R_{\tau}(X^U_{\tau}) + S^L_{\tau}(X^U) + \beta(x^*_\tau - x^U_{\tau}) \). This cost must be greater than or equal to the cost of the optimal solution, so
\[
R_{\tau}(X^U_{\tau}) + S^L_{\tau}(X^U) + \beta(x^*_\tau - x^U_{\tau}) \geq R_{\tau}(X^*) + S^L_{\tau}(X^*)
\]
holds. By using equation (9), we get
\[
R_{\tau}(X^U_{\tau}) + S^U_{\tau}(X^U) + \beta x^U_{\tau} + \beta(x^*_\tau - x^U_{\tau}) \geq R_{\tau}(X^*) + S^U_{\tau}(X^*) + \beta x^*_\tau.
\]
We eliminate identical terms and get
\[
R_{\tau}(X^U_{\tau}) + S^U_{\tau}(X^U) \geq R_{\tau}(X^*) + S^U_{\tau}(X^*)
\]
which is a contradiction to inequality (10). Therefore our assumption was wrong, so \( x^*_\tau \leq x^U_{\tau} \) must be fulfilled.

The following four lemmas show important properties of \( \hat{C}^L_{\tau}(x) \). First, we prove that the relation between \( C^L_{\tau}(X) \) and \( C^U_{\tau}(X) \) described by equation (8) still holds for \( \hat{C}^L_{\tau}(x) \) and \( \hat{C}^U_{\tau}(x) \).

**Lemma 7.** For all \( \tau \), \( \hat{C}^L_{\tau}(x) = \hat{C}^U_{\tau}(x) + \beta x \) holds.
Proof. Let \( X^L \) be a corresponding solution for \( \hat{C}^L_t(x) \) such that \( C^L_t(X^L) = \hat{C}^L_t(x) \) and let \( X^U \) be a corresponding solution for \( \hat{C}^U_t(x) \) such that \( C^U_t(X^U) = \hat{C}^U_t(x) \). Note that the last state of \( X^L \) and \( X^U \) is \( x \). Since \( X^U \) is optimal for \( C^U_t \), the inequality \( C^U_t(X^U) \leq C^U_t(X) \) holds for all \( X = (x_1, \ldots, x_{\tau-1}, x) \). By equation (9), we get

\[
C^L_t(X^U) - \beta x \leq C^L_t(X) - \beta x
\]

which is equivalent to \( C^L_t(X^U) \leq C^L_t(X) \). With \( X := X^L \) we get \( C^L_t(X^U) \leq C^L_t(X^L) \). Since \( X^L \) is optimal for \( C^L_t \), \( X^U \) must be optimal too, so \( C^L_t(X^U) = C^L_t(X^L) \) holds. All in all, we get

\[
\hat{C}^L_t(x) = C^L_t(X^L) = C^L_t(X^U) = C^U_t(X^U) + \beta x = \hat{C}^U_t(x) + \beta x.
\]

\( \square \)

Obviously, the cost functions \( C^L_t(X) \) and \( C^U_t(X) \) are convex, since convexity is closed under addition. The following lemma shows that also \( \hat{C}^L_t(x) \) and \( \hat{C}^U_t(x) \) are convex.

**Lemma 8.** For all \( \tau \) and \( Y \in \{L, U\} \), \( \hat{C}^Y_t(x) \) is a convex function.

We will prove this lemma together with the next lemma:

**Lemma 9.** The slope of \( \hat{C}^L_t(x) \) is at most \( \beta \) for \( x \leq x^U_1 \) and at least \( \beta \) for \( x > x^U_1 \), i.e. \( \Delta \hat{C}^L_t(x^U_1) \leq \beta \) and \( \Delta \hat{C}^L_t(x^U_1 + 1) \geq \beta \)

Proof. First, we will prove the case \( Y = L \) by induction. The function

\[
\hat{C}^L_1(x) = f_1(x) + \beta x
\]

is convex, because all \( f_t \) are convex and \( \beta \) is a linear function which is also convex (note that convexity is closed under addition). For \( \hat{C}^U_1 \) there are no costs for powering up, so \( x^U_1 = \arg \min_x f_1(x) \) and therefore

\[
\Delta \hat{C}^L_1(x^U_1) = \hat{C}^L_1(x^U_1) - \hat{C}^L_1(x^U_1 - 1)
= f_1(x^U_1) - f_1(x^U_1 - 1) + \beta
\leq \beta
\]

and

\[
\Delta \hat{C}^L_1(x^U_1 + 1) = \hat{C}^L_1(x^U_1 + 1) - \hat{C}^L_1(x^U_1)
= f_1(x^U_1 + 1) - f_1(x^U_1) + \beta
\geq \beta
\]

so for \( \tau = 1 \) both lemmas are fulfilled.

Assume that \( \hat{C}^L_{\tau-1} \) is convex, \( \Delta \hat{C}^L_{\tau-1}(x^U_{\tau-1}) \leq \beta \) and \( \Delta \hat{C}^L_{\tau-1}(x^U_{\tau-1} + 1) \geq \beta \). By definition we have

\[
\hat{C}^L_{\tau}(x) = \min_{x'} \left\{ \hat{C}^L_{\tau-1}(x') + \beta(x - x')^+ \right\} + f_{\tau}(x)
\]

Let \( x'_{\min} := \arg \min_{x'} \hat{C}^L_{\tau-1}(x') \). If \( x \leq x'_{\min} \), then

\[
\min_{x'} \left\{ \hat{C}^L_{\tau-1}(x') + \beta(x - x')^+ \right\} = C^L_{\tau-1}(x'_{\min})
\]

holds, so

\[
\hat{C}^L_{\tau}(x) = \hat{C}^L_{\tau-1}(x'_{\min}) + f_{\tau}(x)
\]

(11)
is convex for $x \leq x'_{\min}$.

Now we consider the case $x > x'_{\min}$. It is clear that $x'_\min \leq x' \leq x$, because for $x' > x$ the term $\beta(x-x')^+$ is zero. We differ between two cases:

If $x \leq x'_{\min}$, then $\Delta \hat{C}_{\tau-1}^L(x) \leq \beta$ holds, since $\Delta \hat{C}_{\tau-1}^L(x'_{\min}) \leq \beta$ and $\hat{C}_{\tau-1}^L$ is convex. Therefore $x' = x$ minimizes the term $\hat{C}_{\tau-1}^L(x') + \beta(x-x')^+$ because using a smaller state $\hat{x} < x$ instead of $x' = x$ would increase the switching cost by $\beta(\hat{x} - x')$ while the decrease of $\hat{C}_{\tau-1}^L$ is less than or equal to $\beta(\hat{x} - x')$. Thus

$$\min_{x'} \left\{ \hat{C}_{\tau-1}^L(x') + \beta(x-x')^+ \right\} = \hat{C}_{\tau-1}^L(x)$$

and

$$\hat{C}_{\tau}^L(x) = \hat{C}_{\tau-1}^L(x) + f_{\tau}(x) \tag{12}$$

is convex for $x'_{\min} < x \leq x'_{U_{\tau-1}}$.

If $x > x'_{\min}$, then $x' = x'_{\min}$, because using a greater state $\hat{x} > x'_{\min}$ would increase the value of $C_{\tau-1}^L(x')$ by at least $\beta(\hat{x} - x'_{\min})$ while the switching cost is decreased by $\beta(\hat{x} - x'_{\min})$. Analogously, using a smaller state $\hat{x} < x'_{\min}$ would decrease the value of $C_{\tau-1}^L(x')$ by at most $\beta(x'_{\min} - \hat{x})$ while the switching cost is increased by $\beta(x'_{\min} - \hat{x})$. Thus

$$\min_{x'} \left\{ \hat{C}_{\tau-1}^L(x') + \beta(x-x')^+ \right\} = \hat{C}_{\tau-1}^L(x'_{\min}) + \beta(x - x'_{\min})$$

and

$$\hat{C}_{\tau}^L(x) = \hat{C}_{\tau-1}^L(x'_{\min}) - \beta x'_{\min} + \beta x + f_{\tau}(x) \tag{13}$$

is convex for $x > x'_{\min}$.

To show that $\hat{C}_{\tau}^L(x)$ is convex for all $x$, we have to compare the slopes of the edge cases. Note that equation \(\text{(11)}\) and \(\text{(12)}\) as well as \(\text{(12)}\) and \(\text{(13)}\) have the same values for $x = x'_{\min}$ and $x = x'_{U_{\tau-1}}$, respectively. We have to show that

$$\Delta \hat{C}_{\tau}^L(x'_{\min}) \leq \Delta \hat{C}_{\tau}^L(x'_{\min} + 1) \tag{14}$$

$$\Delta \hat{C}_{\tau}^L(x'_{U_{\tau-1}}) \leq \Delta \hat{C}_{\tau}^L(x'_{U_{\tau-1}} + 1) \tag{15}$$

First, we will prove \(\text{(14)}\):

$$\Delta \hat{C}_{\tau}^L(x'_{\min} + 1) = \hat{C}_{\tau-1}^L(x'_{\min} + 1) - \hat{C}_{\tau-1}^L(x'_{\min}) + f_{\tau}(x'_{\min} + 1) - f_{\tau}(x'_{\min}) \geq f_{\tau}(x'_{\min} + 1) - f_{\tau}(x'_{\min}) \geq f_{\tau}(x'_{\min}) - f_{\tau}(x'_{\min} - 1) = \Delta \hat{C}_{\tau}^L(x'_{\min}).$$

The first equality uses equation \(\text{(12)}\). The first inequality holds because $x'_{\min}$ minimizes $\hat{C}_{\tau-1}^L$. The second inequality uses the convexity of $f_{\tau}$ and the last equality uses equation \(\text{(11)}\).

Inequality \(\text{(15)}\) can be shown as follows:

$$\Delta \hat{C}_{\tau}^L(x'_{U_{\tau-1}}) = \hat{C}_{\tau-1}^L(x'_{U_{\tau-1}}) - \hat{C}_{\tau-1}^L(x'_{U_{\tau-1}} - 1) + f_{\tau}(x'_{U_{\tau-1}}) - f_{\tau}(x'_{U_{\tau-1}} - 1) \leq \beta + f_{\tau}(x'_{U_{\tau-1}}) - f_{\tau}(x'_{U_{\tau-1}} - 1) = \Delta \hat{C}_{\tau}^L(x'_{U_{\tau-1}}).$$
The first equality uses equation \([12]\), the first inequality holds by the induction hypothesis, the second inequality uses the convexity of \(f_x\) and the last equality uses equation \([13]\).

Now, we know that \(\hat{C}_r^L\) is convex. We still have to show the slope property of Lemma \([9]\). We begin showing \(\Delta \hat{C}_r^L(x_r^U) \leq \beta\).

\[
\Delta \hat{C}_r^L(x_r^U) = \hat{C}_r^L(x_r^U) - \hat{C}_r^L(x_r^U - 1)
\]
\[
= C_r^U(x_r^U) + \beta x_r^U - \hat{C}_r^U(x_r^U - 1) - \beta(x_r^U - 1)
\]
\[
\leq \beta.
\]

The second equality uses Lemma \([7]\). The inequality holds, because \(x_r^U\) minimizes \(\hat{C}_r^U\), so \(\hat{C}_r^U(x_r^U) - \hat{C}_r^L(x_r^U - 1) \leq 0\).

The same arguments can be used to show that \(\Delta \hat{C}_r^L(x_r^U + 1) \geq \beta\).

\[
\Delta \hat{C}_r^L(x_r^U + 1) = \hat{C}_r^L(x_r^U + 1) - \hat{C}_r^L(x_r^U)
\]
\[
= \hat{C}_r^U(x_r^U + 1) + \beta(x_r^U + 1) - \hat{C}_r^U(x_r^U) - \beta(x_r^U)
\]
\[
\geq \beta
\]

The inequality holds because \(\hat{C}_r^U(x_r^U + 1) - \hat{C}_r^L(x_r^U) \geq 0\).

Since \(\hat{C}_r^L(x)\) is convex, by Lemma \([4]\), \(\hat{C}_r^U(x) = \hat{C}_r^L(x) + \beta x\) is convex too, because convexity is closed under addition.

\[\square\]

**Lemma 10.** For \(x \leq x_r^U\), the slope of \(\hat{C}_r^L(x)\) is at most \(\beta\), i.e. \(\Delta \hat{C}_r^L(x) \leq \beta\) holds.

**Proof.** By Lemma \([9]\) \(\Delta \hat{C}_r^L(x_r^U) \leq \beta\) holds and by Lemma \([8]\) \(\hat{C}_r^L\) is convex, so \(\Delta \hat{C}_r^L(x) \leq \beta\) for \(x \leq x_r^U\).

The next lemma characterizes the behavior of the optimal solution backwards in time.

**Lemma 11.** A solution vector \((\hat{x}_1, \ldots, \hat{x}_T)\) that fulfills the following recursive equality for all \(t \in \{1, \ldots, T\}\) is optimal:

\[
\hat{x}_t := \begin{cases} 0, & t = T + 1 \\ \frac{x_t}{x_{t+1}}, & t \leq T \end{cases}
\]

**Proof.** We will prove the lemma by induction in reverse time. Powering down does not cost anything, so setting \(\hat{x}_{T+1} = 0\), does not produce any additional costs. Assume that \((\hat{x}_{\tau+1}, \ldots, \hat{x}_T)\) can lead to an optimal solution, i.e. there exists an optimal solution \(X^*\) with \(x_t^* = \hat{x}_t\) for \(t \geq \tau + 1\). We will show that the vector \((\hat{x}_\tau, \ldots, \hat{x}_T)\) can still lead to an optimal solution.

We have to examine three cases:

**Case 1:** If \(\hat{x}_{\tau+1} \leq x_{\tau}^L\), then \(\hat{x}_\tau = x_{\tau}^L\). By Lemma \([6]\) \(x_t^* \geq x_{\tau}^L\) holds. Since \(X_{\tau}^L\) minimizes \(C_{\tau}^L\), we know that \(C_{\tau}^L(X_{\tau}^L) \leq C_{\tau}^L(X)\) for all \(X = (x_1, \ldots, x_T)\). Thus there is no benefit to use a state \(x' \geq x_{\tau}^L\), because afterwards we have to power down some servers to reach \(\hat{x}_{\tau+1}\). Therefore \(\hat{x}_\tau = x_{\tau}^L\) can still lead to an optimal solution.

**Case 2:** If \(\hat{x}_{\tau+1} > x_{\tau}^L\), then \(\hat{x}_\tau = x_{\tau}^U\). By Lemma \([6]\) \(x_t^* \geq x_{\tau}^U\) holds. Since \(X_{\tau}^U\) minimizes \(C_{\tau}^U\), we know that \(C_{\tau}^U(X_{\tau}^U) \leq C_{\tau}^U(X)\) for all \(X\). By using the solution \(X_{\tau}^U\) and then switching to state \(\hat{x}_{\tau+1}\), the resulting cost is

\[
C_{\tau}^L(X_{\tau}^U) + \beta(\hat{x}_{\tau+1} - x_{\tau}^U) + f_{\tau+1}(\hat{x}_{\tau+1})
\]
\[
= C_{\tau}^U(X_{\tau}^U) + \beta \hat{x}_{\tau+1} + f_{\tau+1}(\hat{x}_{\tau+1})
\]
\[
\leq C_{\tau}^U(X) + \beta \hat{x}_{\tau+1} + f_{\tau+1}(\hat{x}_{\tau+1})
\]
\[
= C_{\tau}^L(X) + \beta(\hat{x}_{\tau+1} - x_\tau) + f_{\tau+1}(\hat{x}_{\tau+1}).
\]

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The last line describes the cost until $\tau + 1$ by using the vector $X = (x_1, \ldots, x_\tau)$ with $x_\tau \leq x^U_\tau$ instead of $X^U$. The cost is not reduced by using $X$, so $\hat{x}_\tau = x^U_\tau$ can still lead to an optimal solution.

**Case 3:** If $x^L_\tau \leq \hat{x}_{\tau+1} \leq x^U_\tau$, then $\hat{x}_\tau = \hat{x}_{\tau+1}$. Assume that there is a better state $\hat{x}_\tau^- < \hat{x}_\tau$ with $\hat{C}^L_\tau(\hat{x}_\tau^-) < \hat{C}^L_\tau(\hat{x}_\tau)$. Since $\hat{x}_{\tau+1}$ leads to an optimal solution, after the time slot $\tau$ we have to power up $\hat{x}_\tau - \hat{x}_\tau^-$ servers, so even $\hat{C}^L_\tau(\hat{x}_\tau^-) + \beta(\hat{x}_\tau^--\hat{x}_\tau^-) < \hat{C}^L_\tau(\hat{x}_\tau)$ holds. By Lemma 14 we know that the slope of $\hat{C}^L_\tau(x)$ is at most $\beta$ for $x < x^U_\tau$. This leads to the contradiction $\hat{C}^L_\tau(\hat{x}_\tau^-) \leq \hat{C}^L_\tau(\hat{x}_\tau^-) + \beta(\hat{x}_\tau^- - \hat{x}_\tau^-)$. Therefore there is no $\hat{x}_\tau^-$ with the desired properties.

The other direction is more simple: Assume that there is a better state $\hat{x}_\tau^+ > \hat{x}_\tau$ with $\hat{C}^L_\tau(\hat{x}_\tau^+) < \hat{C}^L_\tau(\hat{x}_\tau)$, then $x^L_\tau$ (which minimizes $\hat{C}^L_\tau$) must be greater than $\hat{x}_\tau$, because by Lemma 8 $\hat{C}^L_\tau$ is a convex function. However, this is a contradiction to $x^L_\tau \leq \hat{x}_{\tau+1} = \hat{x}_\tau$.

In the following $X^* = (x^*_1, \ldots, x^*_\tau)$ denotes an optimal solution that fulfills the recursive equality of Lemma 11. The next lemma describes time slots where $X^*_{LCP}$ and $X^*$ are in same state. Informally, the lemma says that if the LCP curve cuts the optimal solution, then there is one time slot $\tau$ where both solutions are in the same state.

**Lemma 12.** If $x^L_{\tau-1} = x^*_{\tau-1}$ and $x^L_{\tau-1} \geq x^*_\tau$ then $x^L_{\tau-1} = x^*_\tau$. If $x^L_{\tau-1} > x^*_\tau$ then $x^L_{\tau+1} = x^*_\tau$.

**Proof.** We will only show the first statement of the lemma, since the other one works exactly analogously. Assume that $x^L_{\tau-1} < x^*_\tau$ and $x^L_{\tau-1} \geq x^*_\tau$ holds. We differ between two cases.

**Case 1:** If $x^L_{\tau-1} < x^*_\tau$ then $x^L_{\tau-1} = x^L_\tau$ (by the definition of the LCP algorithm). By $x^L_{\tau-1} \geq x^*_\tau$ and Lemma 6 (which says that $x^L_{\tau-1} \leq x^*_\tau$), we get $x^L_{\tau-1} = x^*_\tau$.

**Case 2:** If $x^L_{\tau-1} \geq x^L_\tau$ then $x^L_{\tau-1} = x^L_\tau$. By Lemma 11 $x^L_{\tau-1} = x^*_\tau$ holds which is a contradiction to $x^L_{\tau-1} > x^L_\tau$.

The time slots where $x^L_{\tau-1} = x^*_\tau$ are denoted by $0 = t_0 < t_1 < \cdots < t_\kappa$. Between these time slots it is not possible that $X^*_{LCP}$ powers one or more servers down and $X^*$ powers servers up or vice versa. In the following $[a : b]$ with $a, b \in \mathbb{N}$ denotes the set $\{a, a+1, \ldots, b\}$. Analogously, we define $[a : b[ := \{a, a+1, \ldots, b-1\}$, $]a : b[ := \{a+1, a+2, \ldots, b\}$ and $]a : b] := \{a+1, a+2, \ldots, b-1\}$.

**Lemma 13.** For all time intervals $|t_i : t_{i+1}|$ with $i \geq 0$, either

(i) $x^L_{\tau} > x^*_\tau$ and both $x^L_{\tau}$ and $x^*_\tau$ are non-increasing for all $\tau \in |t_i : t_{i+1}|$, or

(ii) $x^L_{\tau} < x^*_\tau$ and both $x^L_{\tau}$ and $x^*_\tau$ are non-decreasing for all $\tau \in |t_i : t_{i+1}|$.

**Proof.** First, we consider the case (i), i.e., $x^L_{\tau} > x^*_\tau$.

If $x^L_{\tau+1} > x^L_\tau$, then $x^L_{\tau+1} = x^L_{\tau+1}$ by the LCP algorithm and $x^L_{\tau+1} \geq x^L_\tau$ by Lemma 6. By Lemma 11 we get $x^L_\tau = x^*_\tau$ which leads to the contradiction $x^L_\tau = x^*_\tau < x^L_{\tau+1} \leq x^L_\tau$ (the last inequality uses the definition of the LCP algorithm). Thus $x^L_{\tau+1}$ is non-increasing for all $\tau \in |t_i : t_{i+1}|$.

If $x^L_{\tau+1} = x^L_\tau$ by Lemma 11 which is a contradiction to $x^L_\tau \geq x^L_{\tau+1} > x^*_\tau$, so $x^*_\tau$ is also non-decreasing for all $\tau \in |t_i : t_{i+1}|$. By Lemma 12 the inequality $x^L_{\tau+1} > x^*_\tau$ is fulfilled for all $\tau \in |t_i : t_{i+1}|$.

Case (ii) works analogously.

Now we can calculate the switching cost of the LCP algorithm.

**Lemma 14.** $S^L_{\tau}(X^*_{LCP}) \leq S^L_{\tau}(X^*)$
Proof. By Lemma 13 both \( x^*_τ \) and \( x^*_τ \) are either non-increasing or non-decreasing until there is a time slot \( t \) with \( x^*_τ = x^*_τ \). Therefore, the switching cost during the time interval \([t_i : t_{i+1}]\) is \( β(x^*_τ - x^*_τ) \) for both \( X^*_τ \) and \( X^*_τ \). At the end of the workload, \( X^*_τ \) and \( X^*_τ \) are in different states. If \( x^*_τ < x^*_τ \), then \( x^*_τ \) and \( x^*_τ \) are non-decreasing in the corresponding time interval \([t : T]\), so the switching cost of LCP is less than the switching cost of the optimal solution. If the \( x^*_τ > x^*_τ \), both \( x^*_τ \) and \( x^*_τ \) are non-increasing, so there are no switching costs for this time interval. All in all, we get \( S^L_t(X^*_τ) \leq S^L_t(X^*_τ)\).

Lemma 13 divides the intervals \([t_i : t_{i+1}]\) into two sets: Intervals of case (i) are called decreasing intervals, the set of those intervals is denoted by \( T^- \). Intervals of case (ii) are called increasing intervals and the set is denoted by \( T^+ \). The following lemma is needed to estimate the operating cost of the LCP algorithm.

Lemma 15. For all \( τ \in [t_i : t_{i+1}] \subseteq T^+ \),
\[
\hat{C}^L( X^*_τ ) + f_{τ+1} ( x^*_τ ) \leq \hat{C}^L( x^*_τ ) + f_{τ+1} ( x^*_τ ) ,
\]

Analogously, for all \( τ \in [t_i : t_{i+1}] \subseteq T^- \),
\[
\hat{C}^U( X^*_τ ) + f_{τ+1} ( x^*_τ ) \leq \hat{C}^U( x^*_τ ) + f_{τ+1} ( x^*_τ ) .
\]

Proof. First, we will prove equation (16). We differ between \( x^*_τ < x^*_τ \) (case 1) and \( x^*_τ = x^*_τ \) (case 2).

Case 1: If \( x^*_τ < x^*_τ \), then \( x^*_τ = x^*_τ \) by the definition of the LCP algorithm. Furthermore,
\[
C^L_{τ+1}( X^*_τ ) = \hat{C}^L( x^*_τ ) + f_{τ+1} ( x^*_τ ) + β(x^*_τ - x^*_τ)
\]

holds by the definition of the lower bound. If \( x^*_τ \geq x^*_τ \), then \( \hat{C}^L( x^*_τ ) \leq \hat{C}^L( x^*_τ ) \) holds because \( \hat{C}^L( x^*_τ ) \) is convex (Lemma 8) with a minimum at \( x^*_τ \). If \( x^*_τ < x^*_τ \), then
\[
\hat{C}^L( x^*_τ ) \leq \hat{C}^L( x^*_τ ) + β(x^*_τ - x^*_τ)
\]

holds because the slope of \( \hat{C}^L( x^*_τ ) \) is at least \( β \) for \( x \leq x^*_τ \) (Lemma 10). By using this in equation (18), we get
\[
C^L_{τ+1}( X^*_τ ) \geq \hat{C}^L( x^*_τ ) + f_{τ+1} ( x^*_τ )
\]

With \( \hat{C}^L_{τ+1}( X^*_τ ) \geq C^L_{τ+1}( X^*_τ ) \) we get equation (16).

Case 2: If \( x^*_τ = x^*_τ \) is an optimal solution for \( C^L_{τ+1} \) that ends in state \( x^*_τ \). Let \( \hat{X} = ( \ldots , \hat{x}_τ , x^*_τ ) \) be an optimal solution for \( C^L_{τ+1} \) that ends in state \( x^*_τ \). Therefore,
\[
C^L_{τ+1}( \hat{X} ) = C^L_{τ+1}( x^*_τ ) \]

holds. If \( \hat{x}_τ \geq x^*_τ \), then \( \hat{C}^L( \hat{x}_τ ) \geq \hat{C}^L( x^*_τ ) \) holds because \( \hat{C}^L( x^*_τ ) \) is convex (Lemma 8) with a minimum at \( x^*_τ \). If \( \hat{x}_τ < x^*_τ \), then
\[
\hat{C}^L( \hat{x}_τ ) + β(x^*_τ - \hat{x}_τ) \geq \hat{C}^L( x^*_τ )
\]

holds because the slope of \( \hat{C}^L( x^*_τ ) \) is at least \( β \) for \( x \leq x^*_τ \) (Lemma 10). By using this in equation (19), we get
\[
\hat{C}^L_{τ+1}( x^*_τ ) \geq \hat{C}^L( x^*_τ ) + f_{τ+1} ( x^*_τ )
\]

which is exactly equation (16).

The proof of equation (17) works analogously by using the upper bound cost \( \hat{C}^U( x^*_τ ) \) and reversing the inequality signs.
We can use Lemma 15 to estimate the operating cost of the LCP algorithm.

**Lemma 16.** $R_T(X^{LCP}) \leq R_T(X^*) + \beta \sum_{t=1}^{T} |x_t^* - x_{t-1}^*|$

**Proof.** Consider the time interval $[t_i : t_{i+1}] \in T^+$. By adding the inequalities of Lemma 15 for $\tau = [t_i : t_{i+1}]$, we get

$$\sum_{t=t_i}^{t_{i+1}-1} \bar{C}_t^L(x_t^{LCP}) + \sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^{LCP}) \leq \sum_{t=t_i}^{t_{i+1}-1} \hat{C}_t^L(x_t^{LCP})$$

Subtracting the first sum gives

$$\sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^{LCP}) \leq \hat{C}_{t_{i+1}}^L(x_{t_{i+1}}^{LCP}) - \hat{C}_{t_i}^L(x_{t_i}^{LCP})$$

$$= \hat{C}_{t_{i+1}}^L(x_{t_{i+1}}^* - \hat{C}_{t_i}^L(x_{t_i}^*) = \sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^*) + \beta(x_{t_{i+1}}^* - x_{t_i}^*)$$

The first equality holds because $x_t^{LCP} = x_t^*$ and $x_{t+1}^{LCP} = x_{t+1}^*$. Considering the time interval $[t_i : t_{i+1}] \in T^-$ yields to the following inequality:

$$\sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^{LCP}) \leq \hat{C}_{t_{i+1}}^U(x_{t_{i+1}}^{LCP}) - \hat{C}_{t_i}^U(x_{t_i}^{LCP})$$

$$= \hat{C}_{t_{i+1}}^U(x_{t_{i+1}}^* - \hat{C}_{t_i}^U(x_{t_i}^*) = \sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^*) + \beta(x_{t_{i+1}}^* - x_{t_{i+1}}^*)$$

In both (20) and (21) the factor after $\beta$ is positive, so we can write

$$\sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^{LCP}) \leq \sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^*) + \beta| x_{t_{i+1}}^* - x_{t_i}^* |$$

$$= \sum_{t=t_i}^{t_{i+1}-1} f_{t+1}(x_{t+1}^*) + \beta \sum_{t=t_i}^{t_{i+1}-1} | x_{t_{i+1}}^* - x_{t_i}^* |$$

By adding all intervals in $T^+ \cup T^-$ we get

$$\sum_{t=1}^{T} f_t(x_t^{LCP}) \leq \sum_{t=1}^{T} f_t(x_t^*) + \beta \sum_{t=1}^{T} | x_t^* - x_{t-1}^* | \quad \square$$

The term $\beta \sum_{t=1}^{T} | x_t^* - x_{t-1}^* |$ in Lemma 16 is upper bounded by twice the switching cost of the optimal schedule:

**Lemma 17.** $\beta \sum_{t=1}^{T} | x_t^* - x_{t-1}^* | \leq 2 \cdot S_T^L(X^*)$

**Proof.** Since we start at $x_0 = 0$ and end at $x_T \geq 0$, the number of servers that are powered up is at least as great as the number of servers that are powered down, i.e.

$$\sum_{t=1}^{T} (x_t^* - x_{t-1}^*)^+ \geq \sum_{t=1}^{T} (x_{t-1}^* - x_t^*)^+$$

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Thus
\[ T \sum_{t=1}^{T} |x_t^* - x_{t-1}^*| = T \sum_{t=1}^{T} (x_t^* - x_{t-1}^*)^+ + \sum_{t=1}^{T} (x_t^* - x_{t-1}^*)^+ \]
\[ \leq 2 \sum_{t=1}^{T} (x_t^* - x_{t-1}^*)^+ \]
and
\[ 2 \cdot S^L_T(X^*) = 2 \beta \sum_{t=1}^{T} (x_t^* - x_{t-1}^*)^+ \]
\[ \geq \beta \sum_{t=1}^{T} |x_t^* - x_{t-1}^*| \]
\[ \square \]

Now, we are able to show that LCP is 3-competitive.

**Theorem 2.** The LCP algorithm is 3-competitive.

**Proof.** By using Lemma 14 [16] and 17 we get
\[ C^L_T(X^{LCP}) = R_T(X^{LCP}) + S^L_T(X^{LCP}) \]
\[ \leq R_T(X^*) + \beta \sum_{t=1}^{T} |x_t^* - x_{t-1}^*| + S^L_T(X^*) \]
\[ \leq R_T(X^*) + 3 \cdot S^L_T(X^*) \leq C^L_T(X^*) . \]

4 A randomized offline algorithm

In the last section we presented a deterministic online algorithm for the dynamic data-center optimization problem that achieves a competitive ratio of 3. This result can be improved by using randomization. In this section we present a randomized online algorithm that is 2-competitive against an oblivious adversary. The basic idea is to use the algorithm of Bansal et al. [5] to get a 2-competitive schedule for the continuous extension of the given problem instance. Then we round the particular states of the schedule randomly to achieve an integral schedule. The expected total cost of the resulting schedule is at most twice as much as the cost of an oblivious adversary.

4.1 Algorithm

Consider the continuous extension \( \tilde{P} \) of the original problem instance \( P \) as introduced in Section 2.3 (see equation (3)).

For this continuous problem, Bansal et. al. [5] give a 2-competitive (deterministic) online algorithm. Let \( \tilde{X} = (\tilde{x}_1, \ldots , \tilde{x}_T) \) be the schedule calculated by the algorithm of Bansal et al. We will convert this this solution to an integral schedule \( X = (x_1, \ldots , x_T) \).

To describe our algorithm we use the following notation: Let \( [x]_a^b := \max \{a, \min \{b, x\} \} \) be the projection of \( x \) into the interval \( [a, b] \), let \( \text{frac}(x) := x - \lfloor x \rfloor \) denote the fractional part of \( x \) and let \( \tilde{x}_{t-1}' := [\tilde{x}_{t-1}]_{[\tilde{x}_t]} \) be the projection of the previous state into the interval of the current state.

We distinguish between time slots where the number of active servers increases and those where the number of active servers decreases. In the first case, we have \( \tilde{x}_{t-1} \leq \tilde{x}_t \). If \( x_{t-1} \) is already in
the upper state $[\bar{x}_t]$, we keep this state, so $x_t = [\bar{x}_t]$. Otherwise, with probability $p_t^\uparrow := \frac{x_t - x_{t-1}'}{1 - \text{frac}(x_{t-1}')}$, we set $x_t$ to the upper state $[\bar{x}_t]$ and with probability $1 - p_t^\uparrow$ we keep the lower state $[\bar{x}_t]$.

The other case (i.e. $\bar{x}_{t-1} > \bar{x}_t$) is handled symmetrically: If $x_{t-1} = [\bar{x}_t]$, then we keep the state, i.e. $x_t = [\bar{x}_t]$, and otherwise with probability $p_t^\downarrow := \frac{x_{t-1}' - \bar{x}_t}{\text{frac}(x_{t-1}')}$, we set $x_t$ to the lower state $[\bar{x}_t]$ and with probability $1 - p_t^\downarrow$ we keep the upper state $[\bar{x}_t]$.

Obviously, $X$ is an integral schedule. In the following section we show that this schedule is 2-competitive against an oblivious adversary.

### 4.2 Analysis

To show that the algorithm described in the previous section is 2-competitive, we have to prove that the expected cost of our algorithm is at most twice the cost of an optimal offline solution. Let $C_Q(Y)$ denote the total cost of the schedule $Y$ for the problem instance $Q$, so we want to prove that

$$\mathbb{E}[C_P(X)] \leq 2 \cdot C_P(X^*) \tag{22}$$

Let $\bar{X}^*$ be an optimal offline solution for $\bar{P}$. By Lemma 4, we know that this solution can be easily rounded to an integral solution $X^*$ without increasing the cost, i.e.

$$C_P(\bar{X}^*) = C_P(X^*) \tag{23}$$

Furthermore, we know that the algorithm of Bansal et al. is 2-competitive for the continuous setting, so we have

$$C_P(\bar{X}) \leq 2 \cdot C_P(\bar{X}^*). \tag{24}$$

Thus it is sufficient to show that $\mathbb{E}[C_P(X)] = C_P(\bar{X})$.

The following lemma describes the probability that a value $\bar{x}_t$ is rounded up.

**Lemma 18.** The probability that $x_t$ equals the upper state $[\bar{x}_t]$ of the fractional schedule is $\text{frac}(\bar{x}_t)$. Formally, $\Pr[x_t = [\bar{x}_t]] = \text{frac}(\bar{x}_t)$.

**Proof.** We prove the lemma by induction. It is clear that $x_t$ is either $[\bar{x}_t]$ or $[\bar{x}_t]$. For $t = 1$ the probability for $x_1 = [\bar{x}_1]$ is $p_1^\uparrow = \frac{\bar{x}_1 - \bar{x}_0'}{1 - \text{frac}(\bar{x}_0')} = [\bar{x}_1] - [\bar{x}_1] = \text{frac}(\bar{x}_1)$, because $\bar{x}_0 = 0$ and therefore $\bar{x}_0' = [\bar{x}_1]$.

Assume that the claim of Lemma 18 holds for $t - 1$, so $\Pr[x_{t-1} = [\bar{x}_{t-1}]] = \text{frac}(\bar{x}_{t-1})$. We differ between increasing time slots where $\bar{x}_{t-1} \leq \bar{x}_t$ holds (case 1) and decreasing time slots where $\bar{x}_{t-1} > \bar{x}_t$ (case 2). In case 1, the probability $\Pr[x_t = [\bar{x}_t]]$ can be written as

$$\Pr[x_t = [\bar{x}_t]] = \Pr[x_t = [\bar{x}_t] \mid x_{t-1} = [\bar{x}_t]] \cdot \Pr[x_{t-1} = [\bar{x}_t]] + \Pr[x_t = [\bar{x}_t] \mid x_{t-1} \leq [\bar{x}_t]] \cdot \Pr[x_{t-1} \leq [\bar{x}_t]] \tag{25}$$

Note that $x_{t-1}$ is integral and cannot be greater than $[\bar{x}_t]$. If $\bar{x}_{t-1} \leq [\bar{x}_t]$, then $\Pr[x_{t-1} = [\bar{x}_t]] = 0$ and $\Pr[x_{t-1} \leq [\bar{x}_t]] = 1$, so similar to the base case we get

$$\Pr[x_t = [\bar{x}_t]] = p_t^\uparrow = \bar{x}_t - [\bar{x}_t] = \text{frac}(\bar{x}_t)$$

Thus, $\Pr[x_t = [\bar{x}_t]] = \text{frac}(\bar{x}_t)$.
If $\bar{x}_{t-1} > \lfloor \bar{x}_t \rfloor$, then by our induction hypothesis $\Pr[x_{t-1} = \lfloor \bar{x}_t \rfloor] = \frac{\bar{x}_t - \bar{x}_{t-1}}{1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}}$ and $\Pr[x_{t-1} \leq \lfloor \bar{x}_t \rfloor] = \Pr[x_{t-1} = \lfloor \bar{x}_t \rfloor] = 1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}$. By the definition of our algorithm, $\Pr[x_t = \lfloor \bar{x}_t \rfloor \mid x_{t-1} = \lfloor \bar{x}_t \rfloor] = 1$, because we keep the state if we are already in the upper state. Furthermore, we get

$$\Pr[x_t = \lfloor \bar{x}_t \rfloor \mid x_{t-1} \leq \lfloor \bar{x}_t \rfloor] = \Pr[x_t = \lfloor \bar{x}_t \rfloor \mid x_{t-1} = \lfloor \bar{x}_t \rfloor] = p_t^1$$

$$= \frac{\bar{x}_t - \bar{x}_{t-1}}{1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}}$$

By inserting this results in equation (25), we get

$$\Pr[x_t = \lfloor \bar{x}_t \rfloor] = 1 \cdot \frac{\bar{x}_t - \bar{x}_{t-1}}{1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}} \cdot (1 - \frac{\bar{x}_{t-1}}{\bar{x}_t})$$

$$= \bar{x}_{t-1} - \lfloor \bar{x}_{t-1} \rfloor + \bar{x}_t - \bar{x}_{t-1}$$

$$= \bar{x}_t - \lfloor \bar{x}_t \rfloor$$

$$= \frac{\bar{x}_t - \bar{x}_{t-1}}{1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}}$$

The third equation holds, because $\lfloor \bar{x}_t \rfloor < \bar{x}_{t-1} < \bar{x}_t$, so $\lfloor \bar{x}_t \rfloor = \lfloor \bar{x}_{t-1} \rfloor$.

The second case $\bar{x}_{t-1} > \bar{x}_t$ works analogously.

The proof of $\mathbb{E}[C_P(X)] = C_P(\bar{X})$ is divided into two parts. First, in the following lemma, we will show that the expected operating costs of our algorithm are equal to the operating costs of the algorithm of Bansal et al. for the continuous version of the problem instance. Then, in Lemma 20, we will show the same for the switching costs. Let $R_Q(Y)$ and $S_Q(Y)$ denote the operating and switching cost of schedule $Y$ for the problem instance $Q$, respectively.

**Lemma 19.** The expected operating cost of our algorithm is equal to the operating cost of the algorithm of Bansal et al. for the continuous extension of the problem instance, i.e. $\mathbb{E}[R_P(X)] = R_P(\bar{X})$.

**Proof.** The expected operating cost of our algorithm can be written as

$$\mathbb{E}[R_P(X)] = \sum_{t=1}^{T} \left( \Pr[x_t = \lfloor \bar{x}_t \rfloor] \cdot f_t(\lfloor \bar{x}_t \rfloor) + \Pr[x_t = \bar{x}_t] \cdot f_t(\lfloor \bar{x}_t \rfloor) \right).$$

By using Lemma 18 we get

$$\mathbb{E}[R_P(X)] = \sum_{t=1}^{T} \left( \frac{1 - \frac{\bar{x}_t}{\bar{x}_t}}{1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}} \cdot f_t(\lfloor \bar{x}_t \rfloor) + \frac{\bar{x}_t}{\bar{x}_t} \cdot f_t(\lfloor \bar{x}_t \rfloor) \right)$$

$$= \sum_{t=1}^{T} \frac{\bar{x}_t}{\bar{x}_t} \cdot f_t(\bar{x}_t)$$

$$= R_P(X).$$

The second equality follows form the definition of the continuous extension of the operating cost functions.
Now, we will determine the expected switching cost of our algorithm for each time slot.

**Lemma 20.** The expected switching cost of our algorithm is equal to the switching cost of the continuous schedule, i.e. $\mathbb{E}[S_P(X)] = S_P(X)$

**Proof.** We calculate the switching cost for each time slot separately. We distinguish between the cases (1) $\bar{x}_{t-1} < \lfloor \bar{x}_t \rfloor$, (2) $\bar{x}_{t-1} \in [\lfloor \bar{x}_t \rfloor, \bar{x}_t]$ and (3) $\bar{x}_{t-1} > \bar{x}_t$. The last case is trivial, because no servers are powered up, so there are no switching costs.

In case 1, we can separate the expected switching cost into three parts: The expected cost for powering up from $\bar{x}_{t-1}$ to $[\bar{x}_{t-1}]$, the cost from $[\bar{x}_{t-1}]$ to $[\bar{x}_t]$ (can be zero) and the expected cost from $[\bar{x}_t]$ to $\bar{x}_t$. The expected number of servers powered up is

$$\mathbb{E}[(x_t - x_{t-1})^+] = \Pr[x_{t-1} = \lfloor \bar{x}_{t-1} \rfloor] + (\lfloor \bar{x}_t \rfloor - \lfloor \bar{x}_{t-1} \rfloor) + \Pr[x_t = \lfloor \bar{x}_t \rfloor]$$

$$= 1 - \frac{\bar{x}_{t-1}}{\bar{x}_t} + \lfloor \bar{x}_t \rfloor - \lfloor \bar{x}_{t-1} \rfloor + \frac{\bar{x}_t}{1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}}$$

The second equation uses Lemma [18] and the third equation follow from the definition of frac.

For case 2, let $l := \lfloor \bar{x}_t \rfloor$ be the lower and $u := \lfloor \bar{x}_t \rfloor$ the upper state of the fractional state $\bar{x}_t$. Since $\bar{x}_{t-1} \in [\lfloor \bar{x}_t \rfloor, \bar{x}_t]$ holds, we only switch the state, if we are in the lower state during time slot $t-1$ and in the upper state during time slot $t$. Thus, the expected number of servers powered up is

$$\mathbb{E}[(x_t - x_{t-1})^+] = \Pr[x_{t-1} = l] \cdot \Pr[x_t = u \mid x_{t-1} = l]$$

By Lemma [18] we know $\Pr[x_{t-1} = l] = 1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}$. Furthermore, by the definition of our algorithm, we have $\Pr[x_t = u \mid x_{t-1} = l] = p^+_t$, so we get

$$\mathbb{E}[(x_t - x_{t-1})^+] = (1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}) \cdot \frac{\bar{x}_t - \bar{x}_{t-1}}{1 - \frac{\bar{x}_{t-1}}{\bar{x}_t}}$$

$$= (\bar{x}_t - \bar{x}_{t-1})^+$$

So for all cases, $\mathbb{E}[\beta(x_t - x_{t-1})^+] = \beta(\bar{x}_t - \bar{x}_{t-1})^+$ holds. By summing over all time slots, we get $\mathbb{E}(S_P(X)) = S_P(X)$. \qed

**Theorem 3.** The algorithm described in Section 4.1 is 2-competitive against an oblivious adversary

**Proof.** We have to show that $\mathbb{E}[C_P(X)] \leq 2 \cdot C_P(X^*)$. By using Lemma [19] and [20] we get

$$\mathbb{E}[C_P(X)] = \mathbb{E}[R_P(X)] + \mathbb{E}[S_P(X)]$$

$$\stackrel{\text{[19]}}{=} R_P(\bar{X}) + S_P(\bar{X})$$

$$\stackrel{\text{[20]}}{=} C_P(\bar{X})$$

$$\leq 2 \cdot C_P(\bar{X}^*)$$

$$\leq 2 \cdot C_P(\bar{X}^*).$$ \qed
5 Lower bounds

In this section we will show lower bounds for both the discrete and continuous data-center optimization problem. First, in Section 5.1 we prove that there is no deterministic online algorithm that achieves a competitive ratio better than 3 for the discrete problem. This lower bound demonstrates that the LCP algorithm analyzed in the previous section is optimal. Afterwards, we show that this lower bound also holds for the restricted model introduced by Lin et al. [17] where the operating cost functions are more restricted than in the general model investigated in Section 3. A formal definition of the restricted model is given in Section 5.1.2. Moreover, we give a lower bound for the continuous setting and show that this lower bound holds again for the restricted model (see Section 5.2). A lower bound of 2 for the general continuous setting was independently shown by Antoniadis et. al. [3]. Based on our result for the continuous setting, we show in Section 5.3 that there is no randomized algorithm that achieves a competitive ratio better than 2 in the discrete setting. Again, this lower bound still holds for the restricted model. Finally, in Section 5.4 we extend our lower bounds to the scenario that an online algorithm has a finite prediction window.

To simplify the analysis, the switching costs are paid for both powering up and powering down. At the end of the workload all servers have to be powered down. This ensures that the total cost remains the same. We will set $\beta = 2$, so changing a server’s state will cost $\beta/2 = 1$. Thus the cost of a schedule is defined by

$$ C(X) := \sum_{t=1}^{T} f_t(x_t) + \sum_{t=1}^{T+1} |x_t - x_{t-1}| $$

with $x_0 := x_{T+1} := 0$.

5.1 Discrete setting, deterministic algorithms

First, we analyze the discrete setting for deterministic online algorithms. We begin with the general model and afterwards show in Section 5.1.2 how our construction can be adapted to the restricted model.

5.1.1 General model

**Theorem 4.** There is no deterministic online algorithm that achieves a competitive ratio of $c < 3$ for the discrete data-center optimization problem.

**Proof.** Assume that there is a deterministic algorithm $A$ that is $(3 - \delta)$-competitive with $\delta > 0$. The adversary will use the functions $\varphi_0(x) = \epsilon |x|$ and $\varphi_1(x) = \epsilon |x - 1|$ with $\epsilon \to 0$, so we only need the states 0 and 1, there is no benefit to use other states. If $A$ is in state 0 or 1, the adversary will send $\varphi_1$ or $\varphi_0$, respectively.

Let $S$ be the number of time slots where algorithm $A$ changes the state of a server, i.e. $S$ is the switching cost of $A$. Let $T$ be length of the whole workload (we will define $T$ later), so for $T - S$ time slots the operating costs of $A$ are $\epsilon$. Thus, the total cost of $A$ is

$$ C(A) = (T - S)\epsilon + S. $$

The cost of the optimal offline schedule can be bounded by the minimum of the following two strategies. The first strategy is to stay at one state for the whole workload. If $\varphi_0$ is sent more often than $\varphi_1$, then this is state 0, else it is state 1. The operating cost is at most $T\epsilon/2$, the switching cost is at most 2, because if we use state 1, we have to switch the state at the beginning and end of the workload. The second strategy is to always switch the state, such that there are no operating
costs. In this case the switching cost is at most \( S + 2 \), because we switch the state after each time \( \mathcal{A} \) switches its state as well as possibly at the beginning and the end of the workload. Thus, the cost of the optimal offline schedule is

\[
C(X^*) \leq \min(T\epsilon/2 + 2, S + 2).
\] (26)

We want to find a lower bound for the competitive ratio \( \frac{C(A)}{C(X^*)} \). We distinguish between \( S \geq T\epsilon/2 \) (case 1) and \( S < T\epsilon/2 \) (case 2).

In case 1 the competitive ratio of \( A \) is

\[
\frac{C(A)}{C(X^*)} \geq \frac{(T - S)\epsilon + S}{T\epsilon/2 + 2} = 2 + \frac{S(1 - \epsilon) - 4}{T\epsilon/2 + 2}
\]

\[
\geq 2 + \frac{(T\epsilon/2)(1 - \epsilon) - 4}{T\epsilon/2 + 2} = 2 - (1 - \epsilon) \frac{2(1 - \epsilon) + 4}{T\epsilon/2 + 2}
\]

The last inequality uses \( S \geq T\epsilon/2 \) that holds for case 1. By setting \( T \geq \frac{1}{\epsilon^2} \), we get \( \lim_{\epsilon \to 0} T\epsilon = \infty \) and thus \( \lim_{\epsilon \to 0} \frac{C(A)}{C(X^*)} = 3 \).

In case 2, we get

\[
\frac{C(A)}{C(X^*)} \geq \frac{(T - S)\epsilon + S}{S + 2} = (1 - \epsilon) + \frac{T\epsilon - 2(1 - \epsilon)}{S + 2}
\]

\[
\geq (1 - \epsilon) + \frac{T\epsilon - 2(1 - \epsilon)}{T\epsilon/2 + 2} = 3 - \epsilon - \frac{2(1 - \epsilon) + 4}{T\epsilon/2 + 2}
\]

Again, we set \( T \geq \frac{1}{\epsilon^2} \) and get \( \lim_{\epsilon \to 0} \frac{C(A)}{C(X^*)} = 3 \).

Therefore there is no algorithm with a competitive ratio that is less than 3. We can set \( T \) to an arbitrarily large value, so the total cost of \( A \) converges to infinity.

5.1.2 Restricted model

Lin et. al. [17] introduced a more restricted setting as described by equation (2). In this section we show that the lower bound of 3 still holds for this model. The essential differences of the restricted model to the general model are: (1) There is only one convex function for the whole problem instance and (2) there is the additional condition that \( x_t \geq \lambda_t \). The different definition of the switching cost does not influence the total cost as already mentioned in the beginning of Section 5.

**Theorem 5.** There is no deterministic online algorithm for the discrete setting of the restricted model with a competitive ratio of \( c < 3 \).

**Proof.** The general model (examined in the previous sections) is denoted by \( \mathcal{G} \) and the restricted model by Lin et al. is denoted by \( \mathcal{L} \). The states of the model \( X \in \{\mathcal{G}, \mathcal{L}\} \) are indicated by \( x^X_t \). We will use the same idea as in the proof of Theorem 4 but we have to modify it such that it fits for the restricted model.

We use 2 servers, so the states are \( x^X_t \in \{0, 1, 2\} \). Instead of switching between the states 0 and 1 in \( \mathcal{G} \), we will switch between 1 and 2 in \( \mathcal{L} \), so for \( t \in \{1, \ldots, T\} \) we have \( x_t^X = x_t^Y + 1 \). In \( \mathcal{L} \) the state 0 is only used at the beginning (\( t = 0 \)) of the workload. This leads to additional switching costs of 1 for both the optimal offline solution and the online algorithm. However, for a sufficiently long workload the total cost converges to infinity, so the constant extra cost does not influence the competitive ratio.
We will apply the same adversary strategy used in the proof of Theorem 4. Let \( f(z) := \epsilon |1 - 2z| \) with \( \epsilon \to 0 \), let \( \beta = 2 \). If the adversary in \( G \) sends \( \varphi_0(x) = \epsilon |x| \) as function, then we will use \( \lambda_t = l_0 := 0.5 \) which leads to operating cost of

\[
x^\epsilon_t \cdot f\left( l_0/x^\epsilon_t \right) = x^\epsilon_t \cdot \epsilon \left| 1 - \frac{1}{x^\epsilon_t} \right| = \epsilon \left| x^\epsilon_t - 1 \right| = \epsilon \left| x^G_t \right|
\]

If the adversary sends \( \varphi_1(x) = \epsilon |1 - x| \), then we will use \( \lambda_t = l_1 := 1 \) which leads to operating cost of

\[
x^\epsilon_t \cdot f\left( l_1/x^\epsilon_t \right) = x^\epsilon_t \cdot \epsilon \left| 1 - \frac{2}{x^\epsilon_t} \right| = \epsilon \left| x^\epsilon_t - 2 \right| = \epsilon \left| 1 - x^G_t \right|
\]

Thus the difference (1) between both models is solved. For \( t \geq 1 \) it is not allowed to use the state \( x^\epsilon_t = 0 \), because both \( l_0 \) and \( l_1 \) are greater than 0. If \( x^\epsilon_t \in \{1, 2\} \) the inequality \( x_t \geq \lambda_t \) is always fulfilled, so the difference (2) is solved too.

5.2 Continuous setting

In this section we determine a lower bound for the continuous data-center optimization problem. Again, we begin with the general model and analyze the restricted model afterwards in Section 5.2.2.2

5.2.1 General model

**Theorem 6.** There is no deterministic online algorithm for the continuous data-center optimization problem that achieves a competitive ratio that is less than 2.

The proof consists of two parts. First we will construct an algorithm \( B \) whose competitive ratio is greater than \( 2 - \delta \) for an arbitrary small \( \delta \). Then we will show that the competitive ratio of any deterministic algorithm that differs from \( B \) is greater than 2.

For the first part we use an adversary that uses \( \varphi_0(x) = \epsilon |x| \) and \( \varphi_1(x) = \epsilon |1 - x| \) as functions where \( \epsilon \to 0 \). Let \( b_t \) be the state of \( B \) at time \( t \). If the function \( \varphi_0 \) arrives, then the next state \( b_{t+1} \) is \( \max\{b_t - \epsilon/2, 0\} \). If \( \varphi_1 \) arrives, the next state is \( b_{t+1} := \min\{b_t + \epsilon/2, 1\} \), so formally

\[
b_{t+1} := \begin{cases} 
\max\{b_t - \epsilon/2, 0\} & \text{if } f_t = \varphi_0 \\
\min\{b_t + \epsilon/2, 1\} & \text{if } f_t = \varphi_1.
\end{cases}
\]

The algorithm starts at \( b_0 = 0 \), so \( b_t \in [0, 1] \) is fulfilled for all \( t \). Note that algorithm \( B \) is equivalent to the algorithm of Bansal et al. [3] for the special case of \( \varphi_0 \) and \( \varphi_1 \) functions. To simplify the calculations we assume that \( \epsilon^{-1} \) is an integer, so the algorithm \( B \) is able to use \( 2\epsilon + 1 \) different states. Note that \( \epsilon \) can be chosen arbitrarily, so this is not a restriction.

**Lemma 21.** The competitive ratio of \( B \) is at least \( 2 - \delta \) for an arbitrary small \( \delta > 0 \), so \( C(B) \geq (2 - \delta) \cdot C(X^\epsilon) \).

**Proof.** Let \( S_0(t) \) be the number of time slots \( f' \leq t \) where \( f' = \varphi_0 \) and let \( S_1(t) \) be the number of time slots where \( f' = \varphi_1 \). Note that \( S_0(t) + S_1(t) = t \) for all \( t \).

Let \( T > 0 \) denote the first time slot, when \( b_t \) reaches 0 (case 1) or 1 (case 2). Case 3 handles the case that there is no such time slot.

**Case 1:** If \( b_T = 0 \), then \( S_0(T) = S_1(T) \). In each time step the algorithm \( B \) either increases or decreases its state by \( \epsilon/2 \), so the switching cost during the whole workload is \( T\epsilon/2 \). For each
time slot $t$ with $f_t = \varphi_1$ there is exactly one unique corresponding time slot $t'$ with $f_{t'} = \varphi_0$ and $b_{t'} = b_t - \epsilon/2$. The operating costs for both time slots are

$$f_t(b_t) + f_{t'}(b_{t'}) = \epsilon |1 - b_t| + \epsilon |b_{t'}| = \epsilon(1 - \epsilon/2)$$

As $T$ must be even, the operating cost is $T/2 \cdot \epsilon(1 - \epsilon/2)$.

Assume that there is a time slot $t$ where $x_t^i \neq 0$. We know that $b_t$ does not reach $x = 1$, so for each time slot $t \leq T$ the inequality $S_1(t) < S_0(t) + 2/\epsilon$ must be fulfilled. If the optimal algorithm uses state $k$ instead of 0 within any time interval, there are switching costs of $2k$ and reduced operating costs of $k\epsilon$ for each time slot in that time interval. All in all, the reduced costs are at most $k\epsilon(S_1(t) - S_0(t)) - 2k < k\epsilon \cdot 2/\epsilon - 2k = 0$, so there is no benefit to leave the state $x = 0$. Thus the cost of the optimal solution is $C(X^*) = \epsilon S_1(T) = \epsilon T/2$.

The competitive ratio of $B$ is

$$\frac{C(B)}{C(X^*)} = \frac{T\epsilon/2 + T/2 \cdot \epsilon(1 - \epsilon/2)}{\epsilon T/2} = 2 - \epsilon/2$$

**Case 2:** If $b_T = 1$, then $S_1(T) = S_0(T) + 2/\epsilon$. The switching cost during the time interval is again $T\epsilon/2 = S_0(T)\epsilon + 1$. For each time slot $t$ with $f_t = \varphi_1$ there exists either one corresponding time slot $t'$ with $f_{t'} = \varphi_0$ and $b_{t'} = b_t - \epsilon/2$ or $b_{t'} > b_t$ for all $t' > t$. Analogous to case 1 the operating costs of the corresponding pairs are $S_0(T) \cdot \epsilon(1 - \epsilon/2)$. For each level $x \in \{\epsilon/2, 2\epsilon/2, \ldots, 1\}$ there is exactly one time slot where $b_t$ has no corresponding time slot $t'$. This leads to operating costs of

$$\sum_{i=1}^{2/\epsilon} \varphi_1(i \cdot \epsilon/2) = \sum_{i=1}^{2/\epsilon} \epsilon \cdot |1 - i \cdot \epsilon/2|$$

$$= \sum_{i=0}^{2/\epsilon - 1} \epsilon \cdot |i \cdot \epsilon/2|$$

$$= \epsilon^2/2 \sum_{i=0}^{2/\epsilon - 1} i$$

$$= \epsilon^2/2 \cdot (2/\epsilon - 1) \cdot (2/\epsilon)$$

$$= 1 - \epsilon/2.$$

The optimal solution is to switch directly to 1. Analogous to case 1 there is no benefit for switching to a value $k < 1$. Therefore the cost of the optimal solution is $1 + \epsilon S_0(T)$, so the competitive ratio of $B$ is:

$$\frac{C(B)}{C(X^*)} = \frac{(\epsilon S_0(T) + 1) + S_0(T) \cdot \epsilon(1 - \epsilon/2) + 1 - \epsilon/2}{1 + \epsilon S_0(T)} = 2 - \epsilon/2$$

**Case 3:** It is possible that $b_t$ never reaches 0 or 1, for example if the adversary sends $\varphi_0$ and $\varphi_1$ alternately. Let $T$ be an arbitrary time slot. The state of $B$ is $b_T$, so $S_1(T) = S_0(T) + 2b_T/\epsilon$ holds. The switching cost of $B$ is again $T\epsilon/2 = \epsilon S_1(T) - b_T$. Similar to case 2 there are corresponding pairs with operating costs of $S_0(T) \cdot \epsilon(1 - \epsilon/2) = (\epsilon S_1(T) - 2b_T)(1 - \epsilon/2)$. It is not necessary to consider the operating costs of the time slots without a corresponding partner.
Since $B$ does not reach 1 there is no benefit for the optimal strategy to change the state after $t = 1$. The cost of the optimal solution is

$$\min\{1 + \epsilon S_0(T), \epsilon S_1(T)\} \leq \epsilon S_1(T)$$

Thus the competitive ratio is

$$\frac{C(B)}{C(X^*)} \geq \frac{\epsilon S_1(T) - b_T + (\epsilon S_1(T) - 2b_T)(1 - \epsilon/2)}{\epsilon S_1(T)}$$

$$= 2 - \epsilon/2 - \frac{b_T(1 + 2(1 - \epsilon/2))}{\epsilon S_1(T)}$$

$$\geq 2 - \epsilon/2 - \frac{6}{T}.$$  

The last inequality holds because $b_T < 1$ and $S_1(T) > T/2$. We set $T \geq \epsilon/12$ and get $\frac{C(B)}{C(X^*)} = 2 - \epsilon$. As $\epsilon \to 0$, the competitive ratio of $B$ converges to 2.

Instead of ending at the states 0 or 1, we can extend the workload such that the competitive ratio is still at least 2, but the total cost of $B$ converges to infinity. This leads to the following lemma which contains a stronger definition of the competitive ratio:

**Lemma 22.** For all $\delta > 0$ and $\alpha \geq 0$, there exists a workload such that

$$C(B) \geq (2 - \delta) \cdot C(X^*) + \alpha$$

is fulfilled.

**Proof.** We prove the lemma by extending the construction used in the proof of Lemma 21. If $B$ reaches the state 0 (case 1), the situation is the same as at the beginning (i.e. $t = 0$). We can repeat the argumentation of the proof by sending $\varphi_1$ as next function, which leads to a competitive ratio of 2 for the new interval, so the overall competitive ratio is not reduced. If $B$ reaches the state 1 (case 2), then we can use the same construction but the states and functions are switched, i.e. the next function is $\varphi_0$. This is possible, since both the algorithm $B$ and the adversary strategy are symmetrical to $x = 0.5$.

Each workload extension (case 1 and 2) increases the total cost of the optimal solution by at least

$$C(X^*) \geq \min\{\epsilon S_1(T), 1 + \epsilon S_0(T)\} \geq \epsilon$$

The first term of the minimum expression handles case 1, the second term handles case 2 (the values were taken from the proof of Lemma 21). The second inequality holds, because $S_1(T) \geq 1$ due to the adversary strategy. By repeating case 1 or 2, the total cost converges to infinity.

Case 3 already contains an arbitrarily long workload. Algorithm $B$ does not reach 0 or 1 in case 3 by definition, so $S_0(T) < S_1(T) < S_0(T) + 2/\epsilon$ holds and by $S_0(T) + S_1(T) = T$ we get $T < 2S_0(T) + 2/\epsilon$ and $S_0(T) > T/2 - 1/\epsilon$. The cost of the optimal solution is

$$C(X^*) = \min\{1 + \epsilon S_0(T), \epsilon S_1(T)\}$$

$$> \min\{1 + \epsilon S_0(T), \epsilon S_0(T)\}$$

$$= \epsilon S_0(T)$$

$$\geq T/2 - 1/\epsilon.$$  

We choose $\epsilon \geq 4/T$, so $C(X^*) \geq T/4$, so the total cost converges to infinity as $T \to \infty$.

Therefore for all $\alpha \geq 0$ there exists a workload such that $C(B) \geq (2 - \delta) \cdot C(X^*) + \alpha$ holds.  

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So far, we have shown that the competitive ratio of algorithm $B$ is at least $2 - \delta$ for an arbitrary small $\delta > 0$. Now, in the second part of the proof of Theorem 6 we will show that any deterministic online algorithm that differs from $B$ causes more cost than $B$. Thus, 2 is a lower bound for the competitive ratio of the continuous data-center optimization problem.

**Lemma 23.** Any deterministic online algorithm $A$ that differs from the states of $B$ produces more cost than $B$, so $C(A) \geq C(B)$.

*Proof.* Let $A$ be an arbitrary online algorithm. The states of $A$ are denoted by $a_t$. The adversary will use the following strategy: It sends $\varphi_1$ functions as long as $a_t \leq b_t$. If $a_t > b_t$, the adversary will send $\varphi_0$. If $a_t$ reaches 0 or 1, the adversary will send $\varphi_1$ or $\varphi_0$, respectively.

At time slot $t$ the adversary can send $\varphi_0$ (case 1) or $\varphi_1$ (case 2) as function. First, we will consider case 1, so $f_t = \varphi_0$ and therefore $b_t = \max\{b_{t-1} - \epsilon/2, 0\}$. If the function at time slot $t - 1$ was $\varphi_0$, then the condition $a_{t-1} \geq b_{t-1}$ must be fulfilled, because otherwise the adversary had sent $\varphi_1$ at time slot $t$. If the function at time slot $t - 1$ was $\varphi_1$, then $a_{t-1} > b_{t-1}$ holds, because otherwise the adversary had not switched to $\varphi_0$ at time slot $t$. So in both cases $a_{t-1} \geq b_{t-1}$ is fulfilled. During the time slot $t$ the cost of $B$ is

$$C_B^t = f(b_t) + |b_t - b_{t-1}| = eb_t + b_{t-1} - b_t$$

(27)

Let $\delta := b_t - a_t$ denote the difference from $b_t$ to $a_t$. If $a_t < b_t$, then $\delta > 0$ and due to $a_{t-1} \geq b_{t-1}$ we get $a_{t-1} > a_t$. At the time slot $t$ the cost of algorithm $A$ is

$$C_A^t = f(a_t) + |a_t - a_{t-1}|$$

$$= ea_t + a_{t-1} - a_t$$

$$= \epsilon(b_t - \delta) + a_{t-1} - b_t + \delta$$

$$\geq eb_t + b_{t-1} - b_t + \delta(1 - \epsilon)$$

$$> eb_t + b_{t-1} - b_t$$

$$\geq C_B^t.$$

The second equality uses $a_{t-1} > a_t$. The first inequality follows by $a_{t-1} \geq b_{t-1}$. The second inequality holds, because we can choose $\epsilon < 1$. The last inequality uses equation (27).

If $a_t \geq b_t$, then the operating cost of $A$ is at least as large as the operating cost of $B$, because $f(a_t) = ea_t > eb_t = f(b_t)$. The adversary will continue sending $\varphi_0$. If there is a time slot $t' > t$ with $a_{t'} \leq b_{t'}$ then the switching cost of $A$ in the time interval $\{t, t+1, \ldots, t'\}$ is $a_t - a_{t'} \geq b_t - b_{t'}$ and thus greater than or equal to the switching cost of $B$, so $C_A^{[t,t']} \geq C_B^{[t,t']}$, where $C_X^{[i,j]}$ denotes the cost of algorithm $X$ in the time interval $\{i, i+1, \ldots, j\}$. If there is no such time slot, then $a_t > b_t$ for all $t > t$. If there is a constant $c$ such that $a_t > c$, then the operating cost of $A$ goes towards infinity. If there is no such constant, the difference of the switching costs of $A$ and $B$ goes towards zero, while the operating cost of $A$ is greater than the operating cost of $B$. So in all cases we get $C(A) \geq C(B)$ in case of $\varphi_0$ functions.

The case 2 where the adversary sends $\varphi_1$ at time slot $t$ works almost analogously. We have $b_t = \min\{b_{t-1} + \epsilon/2, 1\}$, the condition $a_{t-1} \leq b_{t-1}$ must hold and the cost of $B$ is $C_B^t \leq \epsilon(1 - b_t) + b_t - b_{t-1}$. Let $\delta = a_t - b_t$. If $a_t > b_t$, then $\delta > 0$ and

$$C_A^t = \epsilon(1 - a_t) + a_t - a_{t-1}$$

$$\geq \epsilon(1 - b_t - \delta) + b_t + \delta - b_{t-1}$$

$$\geq C_B^t + \delta(1 - \epsilon)$$

$$\geq C_B^t.$$
If \( a_t \leq b_t \) then the operating cost of \( A \) is at least as large as the operating cost of \( B \). If there is a time slot \( t' > t \) with \( a_{t'} \geq b_{t'} \), then the switching cost of \( A \) in the time interval \([t : t']\) is greater than or equal to the switching cost of \( B \). If there is no such \( t' \), the operating cost of \( A \) is greater than the operating cost of \( B \). So in all cases we get \( C(A) \geq C(B) \) in case of \( \varphi_1 \) functions.

Thus algorithm \( A \) produces at least as many costs as \( B \).

**Proof of Theorem 6** By using Lemma 21 and 23 we get \( C(A) \geq C(B) \geq (2 - \delta) \cdot C(X^*) + \alpha \) for all \( \delta > 0 \) and \( \alpha \geq 0 \).

**5.2.2 Restricted model**

Analogously to the discrete setting, in this section we want to show that the lower bound of 2 for the continuous data-center optimization problem still holds for the restricted model described in Section 5.1.2.

**Theorem 7.** There is no deterministic online algorithm for the continuous setting of the restricted model with a competitive ratio of \( c < 2 \).

**Proof.** The restricted model is denoted by \( L \), the general model is denoted by \( G \). Let \( f(z) := \epsilon \cdot |1-kz| \) with \( \epsilon \to 0 \) and \( k \to \infty \), let \( \beta = 2 \). If the adversary in \( G \) sends \( \varphi_0(x) = \epsilon \cdot |x| \) as function, then we will use \( \lambda_t = l_0 := 0 \) which leads to operating costs of

\[ x_t f(l_0/x_t) = x_t \cdot \epsilon \cdot |1| = \epsilon |x_t| \]

The last equality holds, because \( x \geq l_0 = 0 \). If the adversary sends \( \varphi_1(x) = \epsilon \cdot |1-x| \), then we will use \( \lambda_t = l_1 := 1/k \) which leads to operating costs of

\[ x_t f(l_1/x_t) = x_t \cdot \epsilon \left| 1 - \frac{1}{x_t} \right| = \epsilon |x_t - 1| = \epsilon |1 - x_t| \]

So the difference (1) between both models is solved.

The additional condition that \( x_t \geq \lambda_t \) does not change anything, because both \( l_0 \) and \( l_1 \) are arbitrary close to 0 as \( k \to \infty \), so the difference (2) is solved too.

**5.3 Discrete setting, randomized algorithms**

In this section we determine a lower bound for randomized online algorithms in the discrete setting. We begin with the analysis of the general model and afterwards show how our construction can be adapted to the restricted model (see Section 5.3.2).

**5.3.1 General model**

In this section, we show that there is no randomized online algorithm that achieves a competitive ratio that is smaller than 2 in the discrete setting against an oblivious adversary. The construction is similar to the continuous setting (Section 5.2.1). We have only one single server and the adversary will use the functions \( \varphi_0(x) = \epsilon \cdot |x| \) and \( \varphi_1(x) = \epsilon \cdot |1-x| \) with \( \epsilon > 0 \) and \( \epsilon^{-1} \in \mathbb{N} \).

The lower bound is proven as follows: First, we will construct an algorithm \( B \) that solves the continuous setting with a competitive ratio of \( 2 - \delta \) for an arbitrary small \( \delta > 0 \). Then we consider an arbitrary randomized online algorithm \( A \) for the discrete setting and show how to convert its probabilistic discrete schedule \( X^A \) to a deterministic continuous schedule \( \bar{X}^A \) without increasing
the costs. Finally, we show how the adversary constructs the problem instance in dependence on the current state of $X^A$ and $X^B$.

Consider algorithm $B$ described in Section 5.2.1 By Lemma 22 the competitive ratio of $B$ for the continuous setting is at least $2 - \delta$ for an arbitrary small $\delta > 0$. Formally,

$$C_\mathcal{P}(X^B) \geq (2 - \delta) \cdot C_\mathcal{P}(\hat{X}^*) + \alpha$$  \hspace{1cm} (28)$$

for all $\delta > 0$ and $\alpha \geq 0$.

Let $A$ be an arbitrary randomized online algorithm and let $\mathcal{P}$ be the problem instance created by the adversary (we will define later, how this problem instance is determined). For each time slot $t$ the oblivious adversary knows the probability $\bar{x}_t^A$ that $A$ is in state 1. Note that there is only one server, so the probability that $A$ is in state 0 is given by $1 - \bar{x}_t^A$. Now, consider the fractional schedule $X^A = (\bar{x}_1^A, \ldots, \bar{x}_t^A)$. The following lemma shows that the cost of $X^A$ for the continuous problem instance $\mathcal{P}$ is smaller than or equal to the expected cost of $A$ for the discrete problem instance $\mathcal{P}$.

**Lemma 24.** $E[C_\mathcal{P}(X^A)] \geq C_\mathcal{P}(X^A)$

**Proof.** First, we will analyze the operating costs. The expected operating cost of $X^A$ for time slot $t$ is $(1 - \bar{x}_t^A)f_t(0) + \bar{x}_t^Af_t(1) = f_t(\bar{x}_t^A)$. The last term describes the operating cost of $X^A$ in the continuous setting for time slot $t$. Thus, $E[R_\mathcal{P}(X^A)] = R_\mathcal{P}(X^A)$.

The switching cost of $X^A$ for time slot $t$ is $|\bar{x}_t^A - \bar{x}_{t-1}^A|$. The probability that $X^A$ switches its state from 0 to 1 is at least $(\bar{x}_t^A - \bar{x}_{t-1}^A)^+$. Analogously, the probability for switching the state from 1 to 0 is at least $(\bar{x}_{t-1}^A - \bar{x}_t^A)^+$. The actual probability can be greater, because we do not know the exact behavior of $A$. All in all, the probability that $X^A$ switches its state is at least $|\bar{x}_t^A - \bar{x}_{t-1}^A|$, so over all time slots we get $E[S_\mathcal{P}(X^A)] \geq S_\mathcal{P}(X^A)$ and therefore $E[C_\mathcal{P}(X^A)] \geq C_\mathcal{P}(X^A)$.

Now we have constructed a continuous schedule $\tilde{X}^A$ from the probabilities of $X^A$. The adversary behaves like in Section 5.2.1 that is if $\bar{x}_t^A$ equals 1 or 0, it will send $\varphi_0$ or $\varphi_1$, respectively, and otherwise if $\bar{x}_t^A$ is greater than or smaller than $\bar{x}_t^B$, it will send $\varphi_0$ or $\varphi_1$. If $\bar{x}_t^A = \bar{x}_t^B$, then the adversary can choose an arbitrary state. By Lemma 23

$$C_\mathcal{P}(X^A) \geq C_\mathcal{P}(X^B)$$  \hspace{1cm} (29)$$

holds. Now, we are able to prove that 2 is a lower bound for randomized online algorithms.

**Theorem 8.** There is no randomized online algorithm for the discrete data-center optimization problem that achieves a competitive ratio that is less than 2 against an oblivious adversary.

**Proof.** Let $A$ be an arbitrary randomized online algorithm. By using Lemma 24 as well as equation (28), (29) and (23), we get

$$E[C_\mathcal{P}(X^A)] \geq C_\mathcal{P}(X^A)$$

$$\geq C_\mathcal{P}(X^B)$$

$$\geq (2 - \delta) \cdot C_\mathcal{P}(\hat{X}^*) + \alpha$$

$$= (2 - \delta) \cdot C_\mathcal{P}(X^*) + \alpha$$

where $\delta > 0$ and $\alpha \geq 0$ can be chosen arbitrarily.

The theorem shows that the randomized algorithm given in Section 4.1 is optimal.
5.3.2 Restricted model

In this section we show that the lower bound of 2 presented above still holds for the restricted model. The basic idea is very similar to the proof of Theorem 5 in Section 5.1.2.

**Theorem 9.** There is no randomized online algorithm for the discrete setting of the restricted model with a competitive ratio of \( c < 2 \).

**Proof.** The general model is denoted by \( \mathcal{G} \) and the restricted model is denoted by \( \mathcal{L} \). The states of the model \( X \in \{ \mathcal{G}, \mathcal{L} \} \) are indicated by \( x_t^X \). In the restricted model we use 2 servers, the operating cost function \( f(z) := \epsilon|1-2z| \) with \( \epsilon \to 0 \) and \( \beta = 2 \). Instead of switching between the states 0 and 1 in \( \mathcal{G} \), we will switch between 1 and 2 in \( \mathcal{L} \), so for \( t \in \{1, \ldots, T \} \) we have \( x_t^L = x_t^G + 1 \).

If the adversary in \( \mathcal{G} \) sends \( \phi_0(x) \) as function, then we will use \( \lambda_t = l_0 := 0.5 \), and if he sends \( \phi_1(x) \), then we will use \( \lambda_t = l_1 := 1 \). As already shown in the proof of Theorem 5, the operating cost \( x_t^G f(l_k^G) \) in \( \mathcal{L} \) (with \( k \in \{0, 1\} \)) is equal to the operating cost \( \phi_k(x_t^G) \) in \( \mathcal{G} \).

In the continuous extension of \( \mathcal{L} \) we are allowed to use the states \( x_t^L \geq 0.5 \), if \( \lambda_t = l_0 = 0.5 \). Since \( x_0 = 0 \), the first function the adversary sends in \( \mathcal{G} \) is \( \phi_1 \), so we have \( \lambda_1 = 1 \) and thus even in the continuous extension \( x_t^L \geq 1 \) must be fulfilled. For \( t \geq 2 \), there is no benefit to use states smaller than 1 in \( \mathcal{L} \), since \( x_t^L f(l_0^L) = \epsilon|x_t^L - 1| \) which is minimal for \( x_t^L = 1 \). Moving to states below 1 always increases the operating and switching costs. Therefore, the inequality \( x_t \geq \lambda_t \) is always fulfilled and the proof of Theorem 5 holds for the restricted model. \( \square \)

5.4 Online algorithms with prediction window

So far, we have considered online algorithms that at time \( t \) only know the arriving function \( f_t \) in determining the next state. In contrast, an offline algorithm knows the whole function sequence \( F \). There are models between these edge cases. An online algorithm with a prediction window of length \( w \), at any time \( t \), can not only use the function \( f_t \), but the function set \( \{f_t, \ldots, f_{t+w}\} \) to choose the state \( x_t \). This problem extension was also defined by Lin et al. \[17, 19\]. If \( w \) has a constant size (i.e. \( w \) is independent of \( T \)), then the lower bounds developed in the previous sections still holds as the following theorem shows. We will prove the lower bounds for the restricted model, thus they hold for the general model as well.

**Theorem 10.** Let \( w \in \mathbb{N} \) and \( \delta > 0 \) be arbitrary constants. There is no deterministic online algorithm with a prediction window of length \( w \) that achieves a competitive ratio of \( 3 - \delta \) in the discrete setting or \( 2 - \delta \) in the continuous setting for the restricted model.

**Proof.** Let \( c \) be the lower bound for the competitive ratio without prediction window, i.e. we have \( c = 2 \) for the continuous setting and \( c = 3 \) for the discrete setting. By Theorem 7 and 5, there exists a function sequence \( F \) such that there is no online algorithm that achieves a competitive ratio of \( c - \delta / 2 \) for an arbitrary small \( \delta > 0 \). Let \( \mathcal{A} \) be an optimal online algorithm without prediction window and let \( \mathcal{B}_w \) be an online algorithm with a prediction window of length \( w \geq 1 \). We will construct a function sequence \( F' \) such that the competitive ratio of \( \mathcal{B}_w \) is at least \( c - \delta \).

Let \( m \in \mathbb{N} \). Each function \( f_t \) in \( F \) is replaced by the function sequence \( \{f_{t,1}^l, \ldots, f_{t,mw}^l\} \) with \( f_{t,u}^l(z) := \frac{1}{mw} f_t(z) \) where \( u \in [m \cdot w] \). So we have

\[ F' = \{f_{t,1}^l, \ldots, f_{t,mw}^l, f_{t+1,1}^l, \ldots, f_{t+1,mw}^l, \ldots, f_{T,1,1}^l, \ldots, f_{T,1,mw}^l\} \]

Since the functions in the subsequence \( \{f_{t,1}^l, \ldots, f_{t,mw}^l\} \) are equal and since

\[ \sum_{u=1}^{mw} f_{t,u}^l(x) = f_t(x) \]

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holds for all \( t \in [T] \) and \( x \in \mathbb{R} \), the cost of an optimal online algorithm without prediction window are equal for both function sequences, i.e. \( C_F(A) = C_{F'}(A) \). Furthermore, the inequality \( C(X^*)_F \geq C(X^*)_{F'} \) holds, because in \( F' \) we have more possibilities to choose from.

Only for the last \( w \) functions in the sequence \((f'_t,1,\ldots,f'_t,mw)\) the algorithm \( B_w \) has an extra knowledge in comparison to \( A \). The operating cost of \( B_w \) is at least zero for these functions, so we can bound the cost of \( B_w \) by

\[
C_{F'}(B_w) \geq \frac{(m-1) \cdot w}{mw} \cdot C_{F'}(A) \\
= (1 - 1/m) \cdot C_F(A) \\
> (1 - 1/m) \cdot \left( c - \delta/2 \right) \cdot C(X^*)_F \\
= \left( c - \delta/2 - \frac{c - \delta/2}{m} \right) \cdot C(X^*)_F \\
> \left( c - \delta/2 - c/m \right) \cdot C(X^*)_F \\
\geq \left( c - \delta/2 - c/m \right) \cdot C(X^*)_{F'} 
\]

By using \( m := 2c/\delta \) we get

\[
C_{F'}(B_w) > (c - \delta) \cdot C(X^*)_{F'} 
\]

Thus, there is no online algorithm with a prediction window of length \( w \) that achieves a competitive ratio of \( c - \delta \).

\[\square\]

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