Birational geometry of symplectic resolutions of nilpotent orbits II

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1 Introduction

Let $G$ be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra. A nilpotent orbit $O$ of $\mathfrak{g}$ is an orbit of a nilpotent element $x \in \mathfrak{g}$ by the adjoint action of $G$ on $\mathfrak{g}$. The closure $\bar{O}$ of $O$ becomes a symplectic singularity via the Kostant-Kirillov form. By Fu [Fu], any symplectic resolution of $\bar{O}$ is obtained as a Springer resolution $T^*(G/P) \to \bar{O}$ for a parabolic subgroup $P \subset G$. In Part I [Na 2], when $\mathfrak{g}$ is classical, we have proved that any two symplectic resolutions of $\bar{O}$ are connected by a sequence of Mukai flops of type $A$ or of type $D$. In this paper (Part II), we shall improve and generalize all arguments in Part I so that the exceptional Lie algebras can be dealt with. We shall replace all arguments of [Na 2] which uses flags, by those which use only Dynkin diagrams. In the classical case, we already know which parabolic subgroups $P$ appear as the polarizations of $O$ and when the Springer map $\mu : Y_P := T^*(G/P) \to \bar{O}$ has degree 1 ([He]); so, in [Na 2], we only had to study the relationship between such polarizations. But, for the exceptional Lie algebras, no complete answer seems to be known. In this paper, we will start with a nilpotent orbit closure $\bar{O}$ which has a Springer resolution $Y_{P_0} := T^*(G/P_0) \to \bar{O}$. Even when $\mathfrak{g}$ is classical, we will not use the classification of polarizations [He]. First we introduce an equivalence relation in the set of parabolic subgroups of $G$ in terms of marked Dynkin diagrams (Definition 1). Our main theorem (Theorem 4.1) then claims that a parabolic subgroup $P \subset G$ always gives a Springer resolution of $\bar{O}$ if $P$ is equivalent to $P_0$. Moreover, any symplectic resolution of $\bar{O}$ actually has this form, which will be proved as a corollary of the fact that the movable cone
\Mov(Y_{P_0}/\mathcal{O}) is the union of nef cones $\Amp(Y_{P}/\mathcal{O})$ with $P \sim P_0$. Here all $Y_{P}$ ($P \sim P_0$) are connected by a sequence of certain Mukai flops (cf. Example 3.5, Theorem 4.1, (v)). When $g$ is of type $E_6$, new Mukai flops (which are called of type $E_{6,I}, E_{6,II}$) appear.

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**Notation.** For a proper birational map $f$ of algebraic varieties, we say that $f$ is **divisorial** if $\text{Exc}(f)$ contains a divisor, and otherwise, we say that $f$ is **small**. Note that the terminology of ”small” is, for example, different from that in [B-M].

## 2 Nilpotent orbits and Springer’s correspondence

Let $G$ be a complex simple Lie group and let $B$ be a Borel subgroup of $G$. Let $g$ (resp. $b$) be the Lie algebra of $G$ (resp. $B$). The set of nilpotent elements $\mathcal{N}$ of $g$ is called the nilpotent variety. It coincides with the closure of the regular nilpotent orbit in $g$. The (original) Springer resolution

$$\pi : T^*(G/B) \to \mathcal{N}$$

is constructed as follows. Let $n(b)$ be the nil-radical of $b$. Then the cotangent bundle $T^*(G/B)$ of $G/B$ is identified with $G \times^B n(b)$, which is, by definition, the quotient space of $G \times n(b)$ by the equivalence relation $\sim$. Here $(g, x) \sim (g', x')$ if $g = gb$ and $x' = Ad_b^{-1}(x)$ for some $b \in B$. Then we define $\pi([(g, x)]) := Ad_g(x)$. According to Borho-MacPherson [B-M], we shall briefly review Springer’s correspondence [Sp]. The nilpotent variety $\mathcal{N}$ is decomposed into the disjoint union of nilpotent orbits $\mathcal{O}_x$, where $x$ is a distinguished base point of the orbit $\mathcal{O}_x$. We put $d_x := \dim \pi^{-1}(x)$. Now $\pi_1(\mathcal{O}_x)$ acts on $H^{2d_x}(\pi^{-1}(x), \mathbb{Q})$ by monodromy. Decompose $H^{2d_x}(\pi^{-1}(x), \mathbb{Q})$ into irreducible representations of $\pi_1(\mathcal{O}_x)$:

$$H^{2d_x}(\pi^{-1}(x), \mathbb{Q}) = \oplus_{\phi} (V_{\phi} \otimes V_{(x,\phi)}),$$

where $\phi : \pi_1(\mathcal{O}_x) \to \text{End}(V_{\phi})$ are irreducible representations and $V_{(x,\phi)} = \text{Hom}_{\pi_1(\mathcal{O}_x)}(V_{\phi}, H^{2d_x}(\pi^{-1}(x), \mathbb{Q}))$. By definition, $\dim V_{(x,\phi)}$ coincides with the multiplicity of $\phi$ in $H^{2d_x}(\pi^{-1}(x), \mathbb{Q})$. We call $(x, \phi)$ is $\pi$-**relevant** if $V_{(x,\phi)} \neq \emptyset$. 
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Fix a maximal torus $T$ in $B$, and let $W$ be the Weyl group relative to $T$. Then there is a natural action of $W$ on $H^2_{\text{d}x}(\pi^{-1}(x), \mathbb{Q})$ commuting with the action of $\pi_1(\mathcal{O}_x)$. Each factor $V_\phi \otimes V_{(x, \phi)}$ becomes a $W$-module, where $W$ acts trivially on $V_\phi$ and $V_{(x, \phi)}$ is an irreducible representation of $W$. These representations were originally constructed by Springer. In [B-M], they are given in terms of the decomposition theorem of intersection cohomology by Beilinson, Bernstein, Deligne and Gabber. The following theorem is called Springer’s correspondence:

**Theorem 2.1.** Any irreducible representation of $W$ is isomorphic to $V_{(x, \phi)}$ for a unique $\pi$-relevant pair $(x, \phi)$.

One can find the tables on Springer’s correspondence in [C, 13.3] for each simple Lie group (see also [A-L], [B-L]).

3 Parabolic subgroups and marked Dynkin diagrams

Let $G$ be a complex reductive Lie group and let $\mathfrak{g}$ be its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the root space decomposition. Let $\Delta \subset \Phi$ be a base of $\Phi$ and denote by $\Phi^+$ (resp. $\Phi^-$) the set of positive roots (resp. negative root). We define a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ as

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

For a subset $\Theta \subset \Delta$, let $\langle \Theta \rangle$ be the sub-root system generated by $\Theta$. We put $\langle \Theta \rangle^+ := \langle \Theta \rangle \cap \Phi^+$ and $\langle \Theta \rangle^- := \langle \Theta \rangle \cap \Phi^-$. We define

$$\mathfrak{p}_\Theta := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \langle \Theta \rangle^+} \mathfrak{g}_\alpha.$$

By definition, $\mathfrak{p}_\Theta$ is a parabolic subalgebra containing $\mathfrak{b}$. Moreover, any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $G$-conjugate to $\mathfrak{p}_\Theta$ for some $\Theta \subset \Delta$. $\mathfrak{p}_\Theta$ and
are $G$-conjugate if and only if $\Theta = \Theta'$. Therefore, there is a one-to-one correspondence between subsets of $\Delta$ and the conjugacy classes of parabolic subalgebras of $\mathfrak{g}$. An element of $\Delta$ is called a simple root, which corresponds to a vertex of the Dynkin diagram attached to $\mathfrak{g}$. A Dynkin diagram with some vertices being marked is called a marked Dynkin diagram. If $\Theta \subset \Delta$ is given, we have a marked Dynkin diagram by marking the vertices which correspond to $\Delta \setminus \Theta$. A marked Dynkin diagram with only one marked vertex is called a single marked Dynkin diagram. A conjugacy class of parabolic subgroups $P \subset G$ with $b_2(G/P) = 1$ corresponds to a single marked Dynkin diagram.

Example 3.1. When $G = SL(n)$, the parabolic subgroup of flag type $(k, n-k)$ corresponds to the marked Dynkin diagram

```
  o-----o-----o
   \    /     /   \\
    \  /     /     \\
     k
```

When $G = SO(2n+1)$, the parabolic subgroup of flag type $(k, 2n-2k+1, k)$ corresponds to the marked Dynkin diagram

```
  o-----o-----o
   \    /     /   \\
    \  /     /     \\
     k
```

When $G = Sp(2n)$, the parabolic subgroup of flag type $(k, 2n-2k, k)$ corresponds to the marked Dynkin diagram

```
  o-----o-----o
   \    /     /   \\
    \  /     /     \\
     k
```

Finally, assume that $G = SO(2n)$. Then the parabolic subgroup corresponding to the marked Dynkin diagram ($k \geq 3$)

```
  o \   / \   / \\
  \ 1/   /  \\
  o
  2
```

has flag type $(n-k+1, 2k-2, n-k+1)$. On the other hand, two marked Dynkin diagrams

```
  o \   / \   / \\
  \ 1/   /  \\
  o
  2
```

both give parabolic subgroups of flag type $(n, 0, n)$ which are not $G$-conjugate.
For a parabolic subgroup $P$ of $G$, let $\mathfrak{p}$ be its Lie algebra and let $n(\mathfrak{p})$ be the nil-radical of $\mathfrak{p}$. There is a unique nilpotent orbit $O \subset \mathfrak{g}$ such that $O \cap n(\mathfrak{p})$ is an open dense subset of $n(\mathfrak{p})$. This nilpotent orbit is called the Richardson orbit for $P$. The cotangent bundle $T^*(G/P)$ of the homogenous space $G/P$ is naturally isomorphic to $G \times n(\mathfrak{p})$, which is the quotient space of $G \times n(\mathfrak{p})$ by the equivalence relation $\sim$. Here $(g, x) \sim (g', x')$ if $g' = gp$ and $x' = Ad_{p^{-1}}(x)$ for some $p \in P$. The Springer map

$$\mu : T^*(G/P) \to \mathfrak{O}$$

is defined as $\mu([g, x]) = Ad_g(x)$. The Springer map $\mu$ is a generically finite surjective proper map. When $\deg \mu = 1$, it is called a Springer resolution.

For a nilpotent orbit $O_x \subset O$, we call $O_x$ is $\mu$-relevant if $\dim \mu^{-1}(x) = \text{codim}(O_x \subset O)/2$.

For the Springer resolution $\pi$ for a Borel subgroup $B$, every nilpotent orbit is $\pi$-relevant. However, this is not the case for a general parabolic subgroup $P$. The $\mu$-relevancy is closely related to Springer’s correspondence. In order to state the result, we shall prepare some terminology. Let $L$ be a Levi subgroup of $P$. Fix a maximal torus $T$ of $L$. Then $T$ is also a maximal torus of $G$. Let $W(L)$ be the Weyl group for $L$ relative to $T$ and let $W$ be the Weyl group for $G$ relative to $T$. Now we have a natural inclusion $W(L) \subset W$. Let $\epsilon_{W(L)}$ be the sign representation of $W(L)$. Denote by $\epsilon^W_{W(L)}$ the induced representation of $\epsilon_{W(L)}$ to $W$. By section 1, every irreducible representation of $W$ has the form $V_{(x, \phi)}$ for a $\pi$-relevant pair $(x, \phi)$. Recall that $\phi$ is an irreducible representation of $\pi_1(O_x)$. Denote by 1 the trivial representation. Then $(x, 1)$ is a $\pi$-relevant pair (cf. [B-M, Lemma 1.2]).

**Proposition 3.2.** A nilpotent orbit $O_x \subset O$ is $\mu$-relevant if and only if $V_{(x, 1)}$ occurs in $\epsilon^W_{W(L)}$.

**Proof.** See [B-M, Collorary 3.5, (b)].

**Proposition 3.3.** Let $G$ be a complex simple Lie group. Assume that $b_2(G/P) = 1$. Then the following are equivalent.

(i) $\deg \mu = 1$ and $\text{Codim}(\text{Exc}(\mu)) \geq 2$,

(ii) The single marked Dynkin diagram associated with $P$ is one of the following:
Remark 3.4. In (ii) there are exactly two different markings for each Dynkin diagram $A_{n-1}$ with $k < n/2$, $D_n$, $E_{6,I}$ or $E_{6,II}$. They are called dual marked Dynkin diagrams. Let $P$ and $P'$ be the corresponding (conjugacy classes of) parabolic subgroups of $G$. Then $p$ and $p'$ have conjugate Levi factors. This implies that $P$ and $P'$ have the same Richardson orbit.

Proof of Proposition 3.3. Assume that the single marked Dynkin diagram is one of first two series in (ii). Then, by [Na 2], Lemmas 3.1 and 3.3, we already know that the Springer map $\mu : T^*(G/P) \to \bar{O}$ becomes a small resolution (cf. Notation). If the single marked diagram is of type $E_{6,I}$, then the Richardson orbit $\mathcal{O}$ of $P$ coincides with orbit $\mathcal{O}_{2A_1}$ in the list of [C-M], p.129, which has dimension 10. The maximal orbit contained in $\mathcal{O}_{2A_1} - \mathcal{O}_{2A_1}$ is $\mathcal{O}_{A_1}$, which has dimension 22. This shows that Sing($\bar{O}$) has codimension $\geq 10$ in $\bar{O}$. On the other hand, since $\pi_1(\mathcal{O}_{2A_1}) = 1$ (cf. [C-M], p.129), $\deg(\mu) = 1$. If $\mu$ is a divisorial birational contraction, then $\text{Codim(Sing(} \bar{O} \subset \bar{O}) = 2$ (cf.}
of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that \( \dim O = 1 \). Consider the nilpotent orbit \( O \) (cf. [Na 2, Theorem 2.7]). Since now we already know that \( \mu \) of the Jordan type \([3, k, q, k] \) (resp. \([3k, 1^{2n-3k+1}] \)) when \( k > 1/3(2n+1) \) (resp. \( k < 1/3(2n+1) \), \( k = 1/3(2n+1) \)). In any case, we have \( O' \subseteq \mathcal{O} \). By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that \( \dim O' = \dim O - 2 \).

Assume that \( g \) is of type \( D_n \). The Levi type of \( P \) is given by

\[
\pi := \begin{cases} 
[3^{2n+1-2k}, 2^{3k-2n-1}] & (k > 1/3(2n+1)) \\
[3k, 1^{2n-3k+1}] & (k \leq 1/3(2n+1)) 
\end{cases}
\]

When \( k > 1/3(2n+1) \), \( k \) must be an odd number. In fact, if \( k \) is even, then \( I(\pi) \neq \emptyset \) and \( \deg(\mu) > 1 \) (cf. [Na 2, Theorem 2.8]). Recall that the Richardson orbit \( O \) of \( P \) has the Jordan type \( S(\pi) \), where \( S \) is the Spaltenstein map (cf. [Na 2, Theorem 2.7]). Since now \( I(\pi) = \emptyset \), \( S(\pi) = \pi \). Let us consider the nilpotent orbit \( O' \) of the Jordan type \([3^{2n+1-2k}, 2^{3k-2n-3}, 1^4] \) (resp. \([3^{k-1}, 2^2, 1^{2n-3k}], [3^{k-1}, 1^4] \)) when \( k > 1/3(2n+1) \) (resp. \( k < 1/3(2n+1) \), \( k = 1/3(2n+1) \)). In any case, we have \( O' \subseteq \mathcal{O} \). By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that \( \dim O' = \dim O - 2 \).

Assume that \( g \) is of type \( C_n \). The Levi type of \( P \) is given by

\[
\pi := \begin{cases} 
[3^{2n-2k}, 2^{3k-2n}] & (k > 2n/3) \\
[3^k, 1^{2n-3k}] & (k \leq 2n/3) 
\end{cases}
\]
When $k \leq 2n/3$, $k$ must be an even number. In fact, if $k$ is odd, then $I(\pi) \neq \emptyset$ and $\deg(\mu) > 1$ (cf. [Na 2, Theorem 2.8]). The Richardson orbit $O$ has the Jordan type $\pi$. Let us consider the nilpotent orbit $O'$ of the Jordan type $[3^{2n-2k}, 2^{3k-2n-1}, 1^2]$ (resp. $[3^{k-2}, 2^4, 1^{2n-3k-2}], [3^{k-2}, 2^3]$) when $k > 2n/3$ (resp. $k < 2n/3$, $k = 2n/3$). In any case, we have $O' \subset O'$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\dim O' = \dim O - 2$.

Assume that $g$ is of type $D_n$. First assume that the Levi type of $P$ is $[2^k]$. The single marked Dynkin diagram is not contained in the list of (ii) exactly when $k$ is even. In this case, $\mu$ is a divisorial birational contraction map by [Na 2, Remark 3.4]. We next assume $k < n$. In this case, the Levi type of $P$ is given by

$$\pi := \begin{cases} [3^{2n-2k}, 2^{3k-2n}] & (n > k > 2n/3) \\ [3^k, 1^{2n-3k}] & (k \leq 2n/3) \end{cases}$$

When $k > 2n/3$, $k$ must be an even number. In fact, if $k$ is odd, then $I(\pi) \neq \emptyset$ and $\deg(\mu) > 1$ (cf. [Na 2, Theorem 2.8]). Recall that the Richardson orbit $O$ of $P$ has the Jordan type $S(\pi)$, where $S$ is the Spaltenstein map (cf. [Na 2, Theorem 2.7]). Since now $I(\pi) = \emptyset$, $S(\pi) = \pi$. Let us consider the nilpotent orbit $O'$ of the Jordan type $[3^{2n-2k}, 2^{3k-2n-2}, 1^4]$ (resp. $[3^{k-1}, 2^2, 1^{2n-3k-1}], [3^{k-1}, 1^3]$) when $k > 2n/3$ (resp. $k < 2n/3$, $k = 2n/3$).

In any case, we have $O' \subset O$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\dim O' = \dim O - 2$.

When $g$ is of type $G_2$, there are exactly two single marked Dynkin diagrams. In the table of $G_2$ nilpotent orbits in [C-M, p.128], $O_{G_2(a_1)}$ is the Richardson orbit of the parabolic subgroups corresponding to these diagrams. The orbit $O_{A_1}$ is contained in $O_{G_2(a_1)}$. Note that $\dim O_{G_2(a_1)} = 10$ and $\dim O_{A_1} = 8$.

When $g$ is of type $F_4$, there are exactly four single marked Dynkin diagrams. Richardson orbits of the parabolic subgroups corresponding to them are $O_{A_2}$, $O_{A_2}$, $O_{F_4(a_3)}$ in the table of [C-M, p.128]. Note that two non-conjugate parabolic subgroups have the same Richardson orbit $O_{F_4(a_3)}$. By looking at the closure ordering of $F_4$ orbits [C, p.440], we see that the closure of each orbit contain a codimension 2 orbit.

When $g$ is of type $E_6$, there are exactly 6 single marked Dynkin diagrams. Four of them are already contained in the list of (ii). The Richardson orbits corresponding to other diagrams are $O_{A_2}$ and $O_{D_4(a_1)}$ in the list of $E_6$. 

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nilpotent orbits in [C-M, p.129]. $\mathcal{O}_{A_2}$ contains a codimension 2 orbit $\mathcal{O}_{3A_1}$. $\mathcal{O}_{D_4(a_1)}$ contains a codimension 2 orbit $\mathcal{O}_{A_3+A_1}$.

When $\mathfrak{g}$ is of type $E_7$, there are exactly 7 single marked Dynkin diagrams. Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{(3A_1)^n}$, $\mathcal{O}_{A_2}$, $\mathcal{O}_{2A_2}$, $\mathcal{O}_{A_2+3A_1}$, $\mathcal{O}_{D_4(a_1)}$, $\mathcal{O}_{A_3+A_2+A_1}$ and $\mathcal{O}_{A_4+A_2}$ in the table of [C-M, p.130-p.131]. By looking at the closure ordering of $E_7$ orbits [C, p.442], we see that the closure of each orbit contains a codimension 2 orbit.

When $\mathfrak{g}$ is of type $E_8$, there are exactly 8 single marked Dynkin diagrams. In the table of [C-M, p.132-p.134], Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{A_2}$, $\mathcal{O}_{2A_2}$, $\mathcal{O}_{D_4(a_1)}$, $\mathcal{O}_{D_4(a_1)+A_2}$, $\mathcal{O}_{A_4+A_2}$, $\mathcal{O}_{A_4+A_2+A_1}$, $\mathcal{O}_{E_6(a_7)}$ and $\mathcal{O}_{A_6+A_1}$. By looking at the closure ordering of $E_8$ orbits, we see that the closure of each orbit contains a codimension 2 orbit.

STEP 2: Assume that $\mathfrak{g}$ is classical. Let $f : \bar{\mathcal{O}} \to \mathcal{O}$ be the normalization map. By STEP 1 we may assume that $\bar{\mathcal{O}}$ contains a codimension 2 orbit $\mathcal{O}'$. In the classical case, by [K-P, 14], we see that $\bar{\mathcal{O}}$ has actually singularities along $f^{-1}(\mathcal{O}')$. The Springer map $\mu$ is factorized as

$$T^*(G/P) \xrightarrow{\mu'} \bar{\mathcal{O}} \overset{\mu}{\rightarrow} \mathcal{O}.$$ 

If $\deg(\mu) = 1$, then $\mu'$ is a birational maps of normal varieties. Then, by Zariski’s main theorem, $\mu'$ must have a positive dimensional fiber over a point of $f^{-1}(\mathcal{O}')$. This implies that $\mu$ is a divisorial birational map.

Assume that $\mathfrak{g}$ is of exceptional type. As explained above, the codimension 2 orbit $\mathcal{O}'$ of $\bar{\mathcal{O}}$ can be specified. It is enough to show that $\mathcal{O}'$ is $\mu$-relevant. By the previous proposition, we have to check that $V_{(x,1)}$ occurs in $\epsilon_{W(L)}^W$ for $x \in \mathcal{O}'$. In [Al], Alvis describes an irreducible decomposition of the induced representation $\text{Ind}_W^L(\rho)$ for any irreducible representation $\rho$ of $W(L)$. Hence, this can be done by using the tables of [Al] (see also the tables in [A-L], [B-L] and [C, 13.3]). Note that Spaltenstein [S] (cf. the footnote of p.68, [B-M]) has already checked that a special orbit is $\mu$-relevant by using these tables. Hence it is enough to check for non-special orbits $\mathcal{O}'$. One can find which orbits are non-special in the tables of [C-M, 8.4].

Example 3.5. (Mukai flops): Let $P$ and $P'$ be two parabolic subgroups of $G$ which correspond to dual marked Dynkin diagrams in the proposition above. Let $\mathcal{O}$ be the Richardson orbit of them. Then we have a diagram

$$T^*(G/P) \xrightarrow{\mu} \bar{\mathcal{O}} \overset{\mu'}{\rightarrow} T^*(G/P').$$
The birational maps \( \mu \) and \( \mu' \) are both small. Moreover, \( T^*(G/P) \to T^*(G/P') \) is not an isomorphism. In fact, \( T^*(G/P), T^*(G/P') \) and \( \mathcal{O} \) all have \( G \) actions, and \( \mu \) and \( \mu' \) are \( G \)-equivariant. If the birational map is an isomorphism, this would become a \( G \)-equivariant isomorphism. This implies that \( G/P \) and \( G/P' \) are isomorphic as \( G \)-varieties. In particular, \( P \) and \( P' \) are \( G \)-conjugate, which is absurd. Since the relative Picard numbers
\[
\rho(T^*(G/P)/\mathcal{O}) = \rho(T^*(G/P')/\mathcal{O}) = 1,
\]
we see that the diagram above is a flop. The diagram is called a Mukai flop of type \( A_{n-1,k} \) (resp. \( D_n, E_{6,1}, E_{6,II} \)) according to the type of the corresponding marked Dynkin diagram.

**Definition 1.** (i) Let \( D \) be a marked Dynkin diagram with exactly \( l \) marked vertices. Choose \( l-1 \) marked vertices from them. Making the remained one vertex unmarked, we have a new marked Dynkin diagram \( \bar{D} \). This procedure is called a contraction of a marked Dynkin diagram. Next remove from \( D \) these \( l-1 \) vertices and edges touching these vertices. We then have a (non-connected) diagram; one of its connected component is a single marked Dynkin diagram. Assume that this single marked Dynkin diagram is one of those listed in Proposition 3.3. Replace this single marked Dynkin diagram by its dual and leave other components untouched. Connecting again removed edges and vertices as before, we obtain a new marked Dynkin diagram \( D' \). Note that \( D' \) (resp. \( \bar{D} \)) has exactly \( l \) (resp. \( l-1 \)) marked vertices. Now we say that \( D' \) is adjacent to \( D \) by means of \( \bar{D} \).

(ii) Two marked Dynkin diagrams \( D \) and \( D' \) are called equivalent and are written as \( D \sim D' \) if there is a finite chain of adjacent diagrams connecting \( D \) and \( D' \).

(iii) Let \( P \) be a parabolic subgroup of \( G \) and let \( D_P \) be the corresponding marked Dynkin diagram. Two parabolic subgroups \( P \) and \( P' \) of \( G \) are called equivalent and are written as \( P \sim P' \) if \( D_P \sim D_{P'} \).

**Example 3.6.** Let us consider the marked Dynkin diagram

\[
D: \quad \circ \longrightarrow \bullet_2 \longrightarrow \bullet_3 \Rightarrow \circ
\]

where vertices 2 and 3 are marked. We choose the vertex 3. Making the remained one vertex (the vertex 2) unmarked, we have a marked Dynkin diagram

\[
\bar{D}: \quad \circ \longrightarrow \bullet_2 \longrightarrow \bullet_3 \Rightarrow \circ
\]

Now the following marked Dynkin diagram \( D' \) is adjacent to \( D \) by \( \bar{D} \).
4 Main Theorem

The following is our main theorem. For the notion of a relative ample cone and a relative movable cone, see [Ka], where some elementary roles of these cones in birational geometry are discussed.

**Theorem 4.1.** Let $\mathcal{O} \subset \mathfrak{g}$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Assume that its closure $\bar{\mathcal{O}}$ has a Springer resolution $\mu_{P_0} : T^*(G/P_0) \to \bar{\mathcal{O}}$. Then the following hold.

(i) For a parabolic subgroup $P$ of $G$ such that $P \sim P_0$, $Y_P := T^*(G/P)$ gives a symplectic resolution of $\bar{\mathcal{O}}$. Conversely, any symplectic resolution is a Springer resolution of this form.

(ii) The closure $\overline{\text{Amp}}(Y_P/\bar{\mathcal{O}})$ of the relative ample cone is a simplicial polyhedral cone.

(iii) $\overline{\text{Mov}}(Y_{P_0}/\mathcal{O}) = \bigcup_{P \sim P_0} \overline{\text{Amp}}(Y_P/\bar{\mathcal{O}})$, where $\overline{\text{Mov}}(Y_{P_0}/\mathcal{O})$ is the closure of the relative movable cone of $Y_{P_0}$ over $\mathcal{O}$.

(iv) A codimension 1 face of $\overline{\text{Amp}}(Y_P/\bar{\mathcal{O}})$ corresponds to a small birational contraction map when it is a face of another ample cone, and corresponds to a divisorial contraction map when it is not a face of any other ample cone.

(v) $\{Y_P\}_{P \sim P_0}$ are connected by Mukai flops of type $A$, $D$, $E_{6,1}$ and $E_{6,11}$.

**Remark 4.2.** For a classical complex Lie algebra, it is already known which nilpotent orbit closure has a Springer resolution (cf. [Na 2, Theorem 2.8]). When $\mathfrak{g}$ is $G_2$, there are exactly 2 nilpotent orbits $\mathcal{O}_{G_2}$ and $\mathcal{O}_{G_2(a_1)}$ whose closures admit Springer resolutions. When $\mathfrak{g}$ is $F_4$, such orbits are $\mathcal{O}_{A_2}$, $\mathcal{O}_{A_2}$, $\mathcal{O}_{A_2}$, $\mathcal{O}_{B_3}$, $\mathcal{O}_{C_3}$, $\mathcal{O}_{F_4(a_2)}$, $\mathcal{O}_{F_4(a_1)}$ and $\mathcal{O}_{F_4}$. When $\mathfrak{g}$ is $E_6$, such orbits are $\mathcal{O}_{A_2}$, $\mathcal{O}_{A_2}$, $\mathcal{O}_{A_2+2A_1}$, $\mathcal{O}_{A_3}$, $\mathcal{O}_{D_4(a_1)}$, $\mathcal{O}_{A_4}$, $\mathcal{O}_{D_4}$, $\mathcal{O}_{A_4+1A_1}$, $\mathcal{O}_{D_5(a_1)}$, $\mathcal{O}_{E_6(a_1)}$, $\mathcal{O}_{D_5}$, $\mathcal{O}_{E_6(a_1)}$, and $\mathcal{O}_{E_6}$.

The statement (ii) of Theorem 4.1 follows from the next Lemma.

**Lemma 4.3.** Let $G$ be a complex simple Lie group and let $P$ be a parabolic subgroup. Let $\hat{\mathcal{O}}$ be the Stein factorization of a Springer map $\mu : Y_P := T^*(G/P) \to \bar{\mathcal{O}}$. Then $\overline{\text{Amp}}(Y_P/\hat{\mathcal{O}})$ is a simplicial polyhedral cone.
Proof. Let $D$ be the marked Dynkin diagram corresponding to $P$. Assume that $D$ has $k$ marked vertices, say, $v_1, ..., v_k$. Then $b_2(G/P) = k$. Choose $l$ vertices $v_{i_1}, ..., v_{i_l}, 1 \leq i_1 < ... < i_l \leq k$ and let $D_{i_1, ..., i_l}$ be the marked Dynkin diagram such that exactly these $l$ vertices are marked and its underlying diagram is the same as $D$. We denote by $X_{i_1, ..., i_l}$ the image of $Y_P \subset G/P \times \bar{O}$ by the projection

$$G/P \times \bar{O} \to G/P_{i_1, ..., i_l} \times \bar{O}.$$ 

Let $\nu_{i_1, ..., i_l} : Y_P \to X_{i_1, ..., i_l}$ be the induced map. Then the Stein factorization of $\nu_{i_1, ..., i_l}$ is a birational contraction map, which corresponds to a codimension $k-l$ face of $\text{Amp}(Y_P/\bar{O})$. We shall denote by $F_{i_1, ..., i_l}$ this face. Then $\text{Amp}(Y_P/\bar{O})$ is a simplicial polyhedral cone generated by $F_1, F_2, ..., F_k$. In fact, any $l$ dimensional face generated by $F_{i_1}, ..., F_{i_l}$ corresponds to the Stein factorization of $\nu_{i_1, ..., i_l}$, which is not an isomorphism.

Next assume that two marked Dynkin diagrams $D$ and $D'$ are adjacent by means of $\bar{D}$. We have three parabolic subgroups $P$, $P'$ and $\bar{P}$ of $G$ corresponding to $D$, $D'$ and $\bar{D}$ respectively. One can assume that these subgroups contain the same Borel subgroup $B$ of $G$ and $\bar{P}$ contains both $P$ and $P'$. Let $\mu : T^*(G/P) \to g$ and $\mu' : T^*(G/P') \to g$ be the Springer maps.

**Proposition 4.4.** (i) The Richardson orbits $O$ of $P$ is the Richardson orbit of $P'$

(ii) Let $\nu$ be the composed map

$$T^*(G/P) \to G/P \times \bar{O} \to G/\bar{P} \times \bar{O}$$

and let $\nu'$ be the composed map

$$T^*(G/P') \to G/P' \times \bar{O} \to G/\bar{P} \times \bar{O}.$$ 

Then $\text{Im}(\nu) = \text{Im}(\nu').$

(iii) If we put $X := \text{Im}(\nu)$, then

$$T^*(G/P) \to X \leftarrow T^*(G/P')$$

is a locally trivial family of Mukai flops of type $A, D, E_{6,1}$ or $E_{6,II}$. In particular, $\nu$ and $\nu'$ are both small birational maps. If $\deg(\mu) = 1$, then $\deg(\mu') = 1$. 

Proof. (i): Take a Levi decomposition

\[ \bar{p} = l(\bar{p}) \oplus n(\bar{p}). \]

In the reductive Lie algebra \( l(\bar{p}) \), \( p \cap l(\bar{p}) \) and \( p' \cap l(\bar{p}) \) are parabolic subalgebras corresponding to dual marked Dynkin diagrams in Proposition 3.3. Hence they have conjugate Levi factors. On the other hand, we have

\[ l(p) = l(p \cap l(\bar{p})), \]

and

\[ l(p') = l(p' \cap l(\bar{p})). \]

Therefore, \( l(p) \) and \( l(p') \) are conjugate. Since \( p \) and \( p' \) have conjugate Levi factors, their Richardson orbits coincide.

(ii): Let \( \mathcal{O} \) be the Richardson orbit of \( p \) and \( p' \). Springer maps \( \mu : T^*(G/P) \to \bar{O} \) and \( \mu' : T^*(G/P') \to \bar{O} \) are both \( G \)-equivariant with respect to natural \( G \)-actions. Then \( U := \mu^{-1}(\mathcal{O}) \) and \( U' := (\mu')^{-1}(\mathcal{O}) \) are open dense orbits of \( T^*(G/P) \) and \( T^*(G/P') \) respectively. Since \( \nu \) and \( \nu' \) are proper maps, \( \text{Im}(\nu) = \overline{\nu(U)} \) and \( \text{Im}(\nu') = \overline{\nu'(U')} \). In the following we shall prove that \( \nu(U) = \nu'(U') \).

(ii-1): We regard \( T^*(G/P) \) (resp. \( T^*(G/P') \)) as a closed subvariety of \( G/P \times \bar{O} \) (resp. \( G/P' \times \bar{O} \)). By replacing \( P' \) by a suitable conjugate in \( \bar{P} \), we may assume that there exists an element \( x \in \mathcal{O} \) such that \( ([P], x) \in U \) and \( ([P'], x) \in U' \). In fact, for a Levi decomposition

\[ \bar{p} = l(\bar{p}) \oplus n(\bar{p}), \]

we have a direct sum decomposition

\[ n(p) = n(p \cap l(\bar{p})) \oplus n(\bar{p}). \]

Let \( p_1 : n(p) \to n(p \cap l(\bar{p})) \) be the 1-st projection. Let \( \mathcal{O}' \subset l(\bar{p}) \) be the Richardson orbit of the parabolic subalgebra \( p \cap l(\bar{p}) \) of \( l(\bar{p}) \). Since \( p_1^{-1}(n(p) \cap \mathcal{O}') \) and \( n(p) \cap \mathcal{O} \) are both Zariski open subsets of \( n(p) \), we can take an element

\[ x \in p_1^{-1}(n(p) \cap \mathcal{O}') \cap (n(p) \cap \mathcal{O}). \]

Since \( x \in n(p) \cap \mathcal{O} \), we have \( ([P], x) \in U \). Decompose \( x = x_1 + x_2 \) according to the direct sum decomposition. Then \( x_1 \in \mathcal{O}' \). The orbit \( \mathcal{O}' \) is also the
Richardson orbit of $p' \cap l(\bar{p})$. Therefore, for some $g \in L(\bar{P})$ (the Levi factor of $\bar{P}$ corresponding to $l(\bar{P})$),

$$x_1 \in n(\text{Ad}_g(p' \cap l(\bar{p}))).$$

The Levi decomposition of $\bar{p}$ induces a direct sum decomposition

$$n(\text{Ad}_g(p')) = n(\text{Ad}_g(p' \cap l(\bar{p}))) \oplus n(\bar{p}).$$

Note that $\text{Ad}_g(p' \cap l(\bar{p})) = \text{Ad}_g(p')$. Hence we see that $x_1 + x_2 \in n(\text{Ad}_g(p'))$. Now, for $\text{Ad}_g(P') \subset \bar{P}$, we have $([\text{Ad}_g(P')], x) \in U'$.

(ii-2): Any element of $U$ can be written as $([gP], \text{Ad}_g(x))$ for some $g \in G$. Then

$$\nu([gP], \text{Ad}_g(x)) = ([g\bar{P}], \text{Ad}_g(x)).$$

For the same $g \in G$, we have $([gP'], \text{Ad}_g(x)) \in U'$ and

$$\nu'([gP'], \text{Ad}_g(x)) = ([gP], \text{Ad}_g(x)).$$

Therefore, $\nu(U) \subset \nu'(U')$. By the same argument, we also have $\nu'(U') \subset \nu(U)$.

(iii): For $g \in G$, $\text{Ad}_g(n(\bar{p}))$ is the nil-radical of $\text{Ad}_g(\bar{p})$. Since $\text{Ad}_g(\bar{p})$ depends only on the class $[g] \in G/\bar{P}$, $\text{Ad}_g(n(\bar{p}))$ also depends on the class $[g] \in G/\bar{P}$. We denote by $\text{Ad}_g(l(\bar{p}))$ the quotient of $\text{Ad}_g(\bar{p})$ by its nil-radical $\text{Ad}_g(n(\bar{p}))$. Let us consider the vector bundle over $G/\bar{P}$

$$\bigcup_{[g] \in G/\bar{P}} \text{Ad}_g(\bar{p}) \rightarrow G/\bar{P}.$$ 

Let $L$ be its quotient bundle whose fiber over $[g] \in G/\bar{P}$ is $\text{Ad}_g(l(\bar{p}))$. We call $L$ the Levi bundle. Let $O'$ be the Richardson orbit of the parabolic subalgebra $p \cap l(\bar{p})$ of $l(\bar{p})$. Note that $O'$ is also the Richardson orbit of $p' \cap l(\bar{p})$. In $L$, we consider the fiber bundle

$$W := \bigcup_{[g] \in G/\bar{P}} \text{Ad}_g(O')$$

whose fiber over $[g] \in G/\bar{P}$ is $\text{Ad}_g(O')$. Put $X := \text{Im}(\nu)$. Define a map

$$f : X \rightarrow W$$

as $f([g], x) := ([g], x_1)$, where $x_1$ is the first factor of $x$ under the direct sum decomposition

$$\text{Ad}_g(\bar{p}) = \text{Ad}_g(l(\bar{p})) \oplus n(\text{Ad}_g(\bar{p})).$$
Note that \( x_1 \in Ad_g(\bar{O}') \). In fact, in the direct sum decomposition, we have
\[
n(Ad_g(p)) = n(Ad_g(p) \cap Ad_g(l(\bar{p}))) \oplus n(Ad_g(\bar{p})).
\]
Therefore
\[
x_1 \in n(Ad_g(p) \cap Ad_g(l(\bar{p}))) \subset Ad_g(\bar{O}').
\]
Since \( W \rightarrow G/\bar{P} \) is an \( \bar{O}' \) bundle, we have a family of Mukai flops parametrized by \( G/\bar{P} \):
\[
Y \rightarrow W \leftarrow Y'.
\]
By pulling back this diagram by \( f : X \rightarrow W \), we have the diagram
\[
T^*(G/P) \rightarrow X \leftarrow T^*(G/P').
\]

Let \( D \) be a marked Dynkin diagram and let \( \bar{D} \) be the diagram obtained from \( D \) by a contraction. Let \( P \) and \( \bar{P} \) be parabolic subgroups of \( G \) corresponding to \( D \) and \( \bar{D} \) respectively. One can assume that \( \bar{P} \) contains \( P \). Let \( \mathcal{O} \) be the Richardson orbit of \( P \) and let \( \nu \) be the composed map
\[
T^*(G/P) \rightarrow G/P \times \mathcal{O} \rightarrow G/\bar{P} \times \mathcal{O}.
\]
We put \( X := \text{Im}(\nu) \). As above, \( \mu : T^*(G/P) \rightarrow \mathcal{O} \) is the Springer map.

**Proposition 4.5.** Let \( \mathfrak{g} \) be a complex simple Lie algebra. Assume that no marked Dynkin diagram is adjacent to \( D \) by means of \( \bar{D} \). If \( \text{deg}(\mu) = 1 \), then \( \nu : T^*(G/P) \rightarrow X \) is a divisorial birational contraction map.

**Proof.** As in the proof of Proposition 4.4, (iii), we construct an \( \bar{O}' \) bundle \( W \) over \( G/\bar{P} \) and define a map \( f : X \rightarrow W \). There is a family of Springer maps
\[
Y \xrightarrow{\sigma} W \rightarrow G/\bar{P}.
\]
By pulling back \( Y \xrightarrow{\sigma} W \) by \( f : X \rightarrow W \), we have the \( \nu : T^*(G/P) \rightarrow X \). Since \( \text{deg}(\mu) = 1 \), \( \nu \) is a birational map. Hence \( \sigma \) should be a birational map. Hence \( \sigma : Y \rightarrow W \) is a family of Springer resolutions. By the assumption, there are no marked Dynkin diagrams adjacent to \( D \) by means of \( \bar{D} \). Now Proposition 3.3 shows that the Springer resolution is divisorial. Therefore, \( \nu \) is also divisorial.
Now let us prove Theorem 4.1. By Proposition 4.4, (iii), \( Y_P := T^*(G/P) \) all give symplectic resolutions of \( \mathcal{O} \) for \( P \sim P_0 \). Hence the first statement of (i) has been proved. Moreover, \( \{Y_P\} \) are connected by Mukai flops, which is nothing but (v). Let us consider \( \cup_{P \sim P_0} \text{Amp}(Y_P/\mathcal{O}) \) in \( N^1(Y_{P_0}/\mathcal{O}) \). Then (iv) follows from Proposition 4.4, (iii) and Proposition 4.5. For an \( \mathcal{O} \)-movable divisor \( D \) on \( Y_{P_0} \), a \( K_{Y_{P_0}} + D \)-extremal contraction is a small birational map. Therefore, the corresponding codimension 1 face of \( \text{Amp}(Y_{P_0}/\mathcal{O}) \) becomes a codimension 1 face of another \( \text{Amp}(Y_P/\mathcal{O}) \). For this small birational map, there exists a flop. Replace \( D \) by its proper transform and continue the same. We shall prove that this procedure ends in finite times. Suppose to the contrary. Since the flops occur between finite number of varieties \( \{Y_P\} \), a variety, say \( Y_{P_1} \), appears at least twice in the sequence of flops:

\[
Y_{P_1} \rightarrow Y_{P_2} \rightarrow ... \rightarrow Y_{P_1}.
\]

For the first flop

\[
Y_{P_1} \xrightarrow{\nu_1} X_1 \leftarrow Y_{P_2},
\]

take a discrete valuation \( v \) of the function field of \( K(Y_{P_1}) \) in such a way that its center is contained in the exceptional locus \( \text{Exc}(\nu_1) \) of \( \nu_1 \). Let \( D_i \subset Y_{P_i} \) be the proper transforms of \( D \). Then we have inequalities for discrepancies (cf. [KMM], Proposition 5-1-11):

\[
a(v, D_1) < a(v, D_2) \leq ... \leq a(v, D_1).
\]

Here the first inequality is a strict one since the center of \( v \) is contained in \( \text{Exc}(\nu_1) \). This is absurd. Hence the procedure ends in finite times, which implies that \( D \in \text{Amp}(Y_P/\mathcal{O}) \) for some \( P \). Therefore, (iii) has been proved. The second statement of (i) immediately follows from (iii).

**Example 4.6. ([Na 2, Example 4.6]):** Assume that \( g = \mathfrak{sl}(6) \). The marked Dynkin diagram \( D \)

\[
\begin{tikzpicture}
    \node at (0,0) [circle, fill=black, inner sep=2pt] {};
    \node at (1,0) [circle, fill=black, inner sep=2pt] {};
    \node at (2,0) [circle, fill=black, inner sep=2pt] {};
    \node at (3,0) [circle, fill=black, inner sep=2pt] {};
    \node at (4,0) [circle, fill=black, inner sep=2pt] {};
    \node at (5,0) [circle, fill=black, inner sep=2pt] {};
    \draw (0,0) -- (1,0);
    \draw (1,0) -- (2,0);
    \draw (2,0) -- (3,0);
    \draw (3,0) -- (4,0);
    \draw (4,0) -- (5,0);
\end{tikzpicture}
\]

gives a parabolic subgroup \( P_{1,2,3} \subset SL(6) \) of flag type \((1,2,3)\). We put \( Y_{1,2,3} := T^*(G/P_{1,2,3}) \). There are 5 other marked Dynkin diagrams which are equivalent to \( D \):
Five parabolic subgroups $P_{1,3,2}$, $P_{3,1,2}$, $P_{3,2,1}$, $P_{2,3,1}$, $P_{2,1,3}$ correspond to the marked Dynkin diagrams above respectively. We put $\mathcal{Y}_{i,j,k} := T^*(SL(6)/P_{i,j,k})$. Let $\mathcal{O}$ be the Richardson orbit of these parabolic subgroups. Then $\overline{\text{Mov}(\mathcal{Y}_{1,2,3}/\mathcal{O})} \cong \mathbb{R}^2$, which is divided into six chambers by the ample cones of $\mathcal{Y}_{i,j,k}$ in the following way:

![Diagram](image)

**Example 4.7.** ([Na 2, Example 4.7]): Assume that $\mathfrak{g} = \mathfrak{so}(10)$. The marked Dynkin diagram

![Diagram](image)

gives a parabolic subgroup $P_{3,2,2,3}^+$ of flag type $(3,2,2,3)$. There are three marked Dynkin diagrams equivalent to this marked diagram:

![Diagram](image)
Three parabolic subgroups $P_{2,3,3,2}^+, P_{2,3,3,2}^-$, $P_{3,2,2,3}^{-1}$ correspond to these marked Dynkin diagrams respectively. Note that there are exactly two conjugacy classes of parabolic subgroups with the same flag type (cf. Example 2.1). We put $Y_{i,j}^+ := T^*(SO(10)/P_{i,j,i,j}^+)$ and put $Y_{i,j}^- := T^*(SO(10)/P_{i,j,i,j}^-)$. Let $O$ be the Richardson orbit of these parabolic subgroups. Then $\text{Mov}(Y_{3,2}^+/ar{O})$ is divided into four chambers by the ample cones of $Y_{3,2}^+, Y_{2,3}^+, Y_{2,3}^-, Y_{3,2}^-$ in the following way:

![Diagram of Dynkin diagrams]

**Example 4.8.** Assume that $g$ is of type $E_6$. Consider the nilpotent orbit $O := O_{A_3}$ (cf. [C-M], p.129). This is the unique orbit with dimension 52. By a dimension count, we see that $O$ is the Richardson orbit of the parabolic subgroup $P_1 \subset G$ associated with the marked Dynkin diagram

![Marked Dynkin diagram with_degrees]

Since $\pi_1(O) = 1$ ([C-M], p.129), the Springer map $\nu_1 : T^*(G/P_1) \to \bar{O}$ has degree 1. The following marked Dynkin diagrams are equivalent to the diagram above:

![Equivalent Dynkin diagrams]
Denote by $P_2$, $P_3$, $P_4$ respectively the parabolic subgroups corresponding to the diagrams above. We put $Y_i := T^*(G/P_i)$ for $i = 1, 2, 3, 4$. Then $\text{Mov}(Y_1/\mathcal{O})$ is divided into four chambers by the ample cones of $Y_i$:

$Y_1$ and $Y_2$ are connected by a Mukai flop of type $D_5$ (cf. Proposition 4.4, (iii)). $Y_2$ and $Y_3$ are connected by a Mukai flop of type $A_{5,1}$ (for the notation, see Example 2.5). $Y_3$ and $Y_4$ are connected by a Mukai flop of type $D_5$.

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