Consumption-investment optimization problem in a Lévy financial model with transaction costs and lâdlâg strategies

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Abstract
We consider the consumption-investment optimization problem for the financial market model with constant proportional transaction rates and Lévy price process dynamics. Contrarily to the recent work of De Vallière (Financ Stoch 20:705–740, 2016), portfolio process trajectories are only left and right limited. This allows us to identify an optimal lâdlâg strategy, e.g. in the two dimensional case, as it is possible to suitably rebalance the portfolio processes when they jump out of the no-trade region in the solvency cone.

Keywords
Consumption-investment problem · Transaction costs · Hamilton–Jacobi–Bellman equation · Viscosity solution · Lévy process

Mathematics Subject Classification 60G44

JEL Classification G11 · G13

1 Introduction

We study a consumption-investment problem with infinite horizon for financial market models including proportional transaction costs and price’s dynamics driven by a Lévy process. This problem originates from the seminal paper of [3]. Davis and Norman [5] rigorously solved the problem and provided the optimal consumption plan in a diffusion model with transaction costs. Although the value function $W$ is in general not smooth, Soner and Shreve [12] show that $W$ is solution to the HJB equation in a weak sense, i.e. is a viscosity solution.
When the risky asset prices follow a one dimensional exponential Lévy processes, Framstad et al. [8] have obtained the same results as those of [12] under some mild conditions. An extension to multidimensional Levy processes is proposed by Kabanov et al. [6] where a general market model with conic constraints is considered. Nevertheless, this paper does not provide an optimal solution.

Traditionally, for general discontinuous price processes such as Levy processes, both the price and the trading strategies are supposed to be càdlàg (right-continuous and left-limited), see the classical book written by Rama Cont and Peter Tankov. However, it is now accepted that trading strategies have to be modeled by làdlàg trading strategies. This allows to establish the characterization of super-hedging prices and the needed duality for portfolio optimization under transaction costs, see [2,4]. Moreover, it is shown in [4] that, if the price process admits jumps at predictable stopping times, then the optimal strategy has left and right jumps. Moreover, contrarily to the case of predictable jumps, a price process modeled by a Levy process only jumps at totally inaccessible times. Right after a jump time, a re-adjustment is then necessary as a response to the new information given by the jump. Therefore, the traditional way of modeling trading strategies by càdlàg processes is not appropriate. The strategies should be làdlàg processes (left and right limited) or, at least, làdcàg (left-continuous and right-limited) processes.

As claimed in [8], the optimal policy for the jump diffusion case has the same form as in the pure diffusion case. In particular, there is a no transaction cone $D$ for the wealth position such that it is optimal to make no transactions inside and to rebalance the position in the trade region as soon as the wealth position gets out of $D$, in particular after an unpredictable jump of the price process.

Why do we write a new paper on optimal consumption with transaction costs and a Levy price process? The aim of this paper is two folds: First, it extends [6,8] to the case where the controls (the portfolio strategies and consumption plans) are only làdlàg. This generalization is necessary to identify the optimal control policy of the problem. Actually, Framstad et al. [8] provide such làdlàg optimal solution [8, Theorem 4.3] under additional assumptions but, unfortunately, it does not belong to the set of controls they consider, i.e. right-continuous processes [8, Definition 1.2]. In our paper, we consider the right set of controls, which is coherent with the conjectured solution of [8]. Secondly, we provide a full characterization of the optimal strategy, which completes the analysis of [8]. This issue will be carefully addressed in the two assets case in the last sections of the paper. By carefully analyzing the arguments in [6,8,11], we study the regularity of the Bellman function and give a rigorous construction of the optimal strategy. Comparatively to the previous works, we need to adapt the stochastic calculus to làdlàg processes. Moreover, we need to consider a new concept of viscosity solution, precisely in a weaker sense than the usual one, see Definition 9.10, in order to prove the regularity of the wealth process. One of the most difficult part is to show that the Bellman function is the viscosity solution to the HJB equation with a non-local operator. In order to obtain the optimal strategy, we also extend the Skorohod problem into the case of a Levy process in the context of a market with two assets and transaction costs.

The paper is organized as follows:

- Section 2: description of the consumption optimization problem.
- Section 3: elementary properties of the value function (Bellman function) $W$.
- Section 4: we show that $W$ is a viscosity solution of a Hamilton–Jacobi–Bellman (HJB) equation.
- Section 5: we show that the HJB equation admits a unique solution under some mild conditions and, as soon as there exists a Lyapunov function.
- Section 6: we propose a condition under which \( W \) is finite. This is in particular the case when there exists a non negative classical supersolution to the HJB equation.
- Section 7: we show that \( W \) is continuous when finite.
- Sections 8 and 9: for the power utility function, we construct a Lyapunov function and a non negative classical supersolution for Sects. 5 and 6.
- Section 10: we apply the general results to a two-dimensional model. Under some conditions, we prove that the value function is continuously twice differentiable and we construct an optimal control. To do so, we solve a Skorokhod problem. Note that we introduce in this part a new definition of viscosity solution in a weaker sense. This allows us to change the global operator by a local one and then deduce that the Bellman function is \( C^2 \).
- Appendix: resolution of the Skorokhod problem for Sect. 10.

Notations

In \( \mathbb{R}^d \), we use standard notations like \( |x| \), the Euclidean norm of \( x \in \mathbb{R}^d \), we define \( d(x, y) = |x - y| \), etc. The Euclidean scalar product between two vectors \( x, y \in \mathbb{R}^d \) is denoted by \( xy \in \mathbb{R} \).

We shall use the notations \( A^+ \) and \( A^- \) to designate the left (resp. right) limit of a process \( A \) and we also denote by \( A_{t-} \) and \( A_{t+} \) its left and right limits at time \( t \). If \( A \) is a càdlàg predictable process of finite variations, the left and right jump processes are denoted by

\[
\Delta A := A^- - A^+, \quad \Delta^+ A := A^+ - A,
\]

and we introduce the associated càdlàg processes:

\[
A^+_t := \sum_{s \leq t} \Delta A_s, \quad A^+_t := \sum_{s < t} \Delta^+ A_s, \quad t \geq 0.
\]

The continuous part of \( A \) is defined as

\[
A^c := A - A^d - A^d^-.
\]

We denote \( \dot{A}^c \) the optional version of the Radon–Nikodym derivative \( dA^c / d\|A^c\| \) where \( \|A^c\| \) is the total variation of \( A^c \).

2 Optimal consumption investment problem

We consider the financial market model with jumps adopted in [6]. The price return process is modeled by a \( d \)-dimensional Lévy process \((Y_t)_{t \geq 0}\) defined on a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) satisfying the usual conditions. We denote by \( p(dz, dt) \) its jump measure and \( q(dz, dt) = \Pi(dz)dt \) its compensator. We suppose that \( \Pi(dz) \) is a non negative measure concentrated on \((-1, \infty)^d\) and

\[
\int_{\mathbb{R}^d} (|z|^2 \wedge |z|) \Pi(dz) < \infty. \tag{2.1}
\]

The dynamics of \( Y \) is given by

\[
dY_t = \mu t + \Xi dW_t + \int_{\mathbb{R}^d} z (p(dz, dt) - q(dz, dt)), \tag{2.2}
\]

where \( \mu \in \mathbb{R}^d \), \( W \) is a \( m \)-dimensional standard Brownian motion and \( \Xi \) is a matrix of dimension \( d \times m \). In the identification of an optimal strategy, we shall only consider a pure jump Lévy process with finite activity, i.e. \( \int_{\mathbb{R}} |z| \pi(dz) < \infty \). The general case remains open.
Two constant cones $K$ and $C$ are given. They are supposed to be closed and proper, i.e. $K \cap (-K) = \{0\}$ and $C \cap (-C) = \{0\}$. We assume that $C \subseteq \text{int} K \neq \emptyset$. In finance $K$ and $C$ stand respectively for the set of transaction constraints (solvency cone of financial positions with non negative liquidation values, see [9]) and consumption constraints, respectively. The dynamics of a portfolio process is defined by:

$$dV^i_t = V^i_{t-}dY^i_t + dB^i_t - dC^i_t, \quad i = 1, \ldots, d, \quad V_0 = V_{0-} = x,$$

where the controls $\pi = (B, C)$ are càdlàg predictable processes of finite variations. The dynamics (2.3) means that the portfolio process $V$ is self-financing. The variations of $V$ are only due to the increments of $Y$. The transaction costs described by $B$ are withdrawn from the portfolio value. At last, $C$ represents the cumulated sum of consumed wealth. If $x \in \mathbb{R}^d$ is an initial endowment, we assume that $\pi = (B, C)$ belongs to the class denoted by $\mathcal{A}_x$ of all admissible controls satisfying the following properties:

1. $\dot{B}^c \in -K, d\mathbb{P}, d\|B^c\| \text{ a.e.}, \dot{C}^c \in C, d\mathbb{P}, d\|C^c\| \text{ a.e.},$
2. $\Delta^+ B \in -K, \Delta^+ C \in C, \text{a.s. for all stopping times } \tau,$
3. $\Delta B \in -K, \Delta C \in C, \text{a.s. for all predictable stopping times } \tau,$
4. If $V = V^\pi$, $\pi = (B, C) \in \mathcal{A}_x$, is such that $V_t \in \text{int} K$ for some stopping time $\tau$, then $V^i_t = V_t + \Delta^+ B^i_t - \Delta^+ C^i_t \in \text{int} K,$
5. If $V = V^\pi$, $\pi = (B, C) \in \mathcal{A}_x$, is such that $V_{\tau-} \in \text{int} K$ (resp. $K$) for some stopping time $\tau$, then $V_{\tau-} + \Delta B_t - \Delta C_t \in \text{int} K$ (resp. $K$),
6. $x + \Delta^+ B_0 \in \text{int} K.$

The three last conditions mean that the portfolio manager does not deliberately get his position out of the solvency cone. It is also assumed that $\Delta B_0^+ = \Delta C_0^+ = \Delta^+ C_0 = 0$ and $dC^c$ is absolutely continuous with respect to the Lebesgue measure and we write $dC^c_t = c_t dt$.

Using the monotonicity of the controls $B$ and $C$ with respect to the partial order induced by $K$ (i.e. $\forall x, y \in \mathbb{R}^d, x \leq y \iff y - x \in K$), we may deduce that $B$ and $C$ are of finite variations. Indeed, since int $K \neq \emptyset$, by an appropriate change of coordinates we may assume w.l.o.g. that all coordinates of $B, C$ are monotonic, hence are of finite variations.

Without loss of generality, we assume that $C = C^c$ is continuous as the jumps of $C$ are ignored in the optimization problem we consider. Precisely, we assume that $dC_t = c_t dt$ almost everywhere w.r.t. the Lebesgue measure $dt$ on $\mathbb{R}$.

For every control $\pi \in \mathcal{A}_x$, let us introduce the stopping time

$$\theta^\pi = \inf \left\{ t : V^\pi_t \notin \text{int} K \right\},$$

where $V^\pi$ is the portfolio process starting from $x \in \mathbb{R}^d$ and satisfying (2.3). We suppose that the strategy $\pi = (B, C)$ is frozen after the exit time, i.e. $\Delta^+ B_\theta = 0$ and $dB_t = c_t = 0$ for $t > \theta$. Throughout the paper, we fix a discount coefficient $\beta > 0$.

For every control $\pi = (B, C) \in \mathcal{A}_x$, and $x \in \text{int} K$, we define the utility process

$$J^\pi_t(x) := \int_0^{t \wedge \theta^\pi} e^{-\beta s} U(c_s)ds,$$

where $U$ is a non-negative utility function defined on $C$. We assume that $U$ is concave, $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. The optimal consumption problem consists in optimizing the utility process $J^\pi(x)$ over the set of all admissible strategies. To do so, we define the Bellman function as

$$W(x) := \sup_{\pi \in \mathcal{A}_x} \mathbb{E}J^\pi_\infty(x), \quad x \in \text{int} K.$$
Showing that the Bellman function is finite is not a trivial task. Indeed, this is based on the existence of classical supersolutions to the associated HJB equation, see Lemma 5.2. Moreover, the continuity of $W$ is proven in Theorem 6.4, Sect. 6.

3 Some elementary properties of the Bellman function

We denote by $\succeq$ the partial order defined by $K$, i.e. if $x, y \in \mathbb{R}^d$, $x \succeq y$ $\iff$ $x - y \in K$.

**Proposition 3.1** The function $W$ is increasing with respect to the partial order $\succeq$.

**Proof** It suffices to adapt the proof of [6] to ladlag strategies with (2.4). $\square$

In the following, we obtain lower bounds for the Bellman function. To do so, let us define the liquidation function associated to the solvency cone $K$, i.e. for $x \in \mathbb{R}^d$,

$$l(x) := \sup\{z \in \mathbb{R}_+ : x - ze_1 \in K\}.$$ 

We have $x - l(x)e_1 \in K$. Then, consider the strategy $\Delta B_0 := l(x)e_1 - x$ and $B_t = B_0$ for $t \geq 0$. For a given consumption plan $c$, the corresponding portfolio process is:

$$V_t = (X_t, 0)$$

where

$$X_t := l(x)S^1_t - S^1_t \int_0^t (S^1_u)^{-1}c_u du.$$ 

If the consumption plan is $c_s = \kappa X_s$, then $Y_t := X_t(S^1_t)^{-1}$ satisfies

$$Y_t := l(x) - \kappa \int_0^t Y_u du,$$

i.e. $Y_t = l(x)e^{-\kappa t}$. It follows that $X_t = S^1_t l(x)e^{-\kappa t}$. We deduce that the process $X$ does not hit zero, i.e. the process $V$ stays in the interior of the solvency cone. With the strategy $\pi := (B, C)$ above,

$$J_\pi^\infty = \int_0^\infty e^{-\beta t} U(\kappa S^1_t l(x)e^{-\kappa t}) dt,$$

and then,

$$W(x) \geq \sup_{\kappa > 0} E \int_0^\infty e^{-\beta t} U(\kappa S^1_t l(x)e^{-\kappa t}) dt.$$ 

In the particular case where $Y^1_t = 0$, i.e. $S^1_t = 1$, we have the following:

**Lemma 3.2** With $S^1 \equiv 1$ and the power utility function $U(x) = u_\gamma (xe_0)$, where $u_\gamma (t) = t^\gamma /\gamma$, $\gamma \in (0, 1)$, we have

$$W(x) \geq \frac{1}{\gamma} \kappa^\gamma - 1 l^\gamma (x) = \frac{1}{\gamma} \left( \frac{\beta}{1 - \gamma} \right)^{\gamma - 1} l^\gamma (x),$$

(3.6)

where $\kappa_* := \beta/(1 - \gamma)$. 

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4 The HJB equation

In the following, we denote by $C_p(K)$ the set of all continuous functions $f$ on $K$ such that $\sup_{x \in K} |f(x)| (1 + |x|)^{-p} < \infty$. The set of all functions $f$ which are $C^2$, i.e. twice continuously differentiable on int $K$ is denoted by $C^2(K)$. We use the notation $C^2(x), x \in \mathbb{R}^2$, for the functions having only these properties on a neighbourhood of $x$. For each $\pi = (B, C) \in \mathcal{A}_x$, $x \in \text{int } K$, and every function $f \in C_1(K) \cap C^2(x)$ which are increasing with respect to the order $\leq_K$, we consider the integro-differential operator

$$H(f, x) := \int_{\mathbb{R}^d} \left[ f(x + \text{diag}(x)z)I(x, z) - f(x) - f'(x)\text{diag}(x)z \right] \Pi(dz), \quad (4.7)$$

where $I(v, z) = I_{v+\text{diag}(v)z \in \text{int } K}$. Applying a Taylor expansion to $f \in C_1(K)$, we claim that

$$|f(x + \text{diag}(x)z)I(x, z) - f(x) - f'(x)\text{diag}(x)z| \leq C_x(|z| \land |z|^2),$$

where $C_x$ is a constant depending on $x$. Therefore, the operator $H$ is well defined. Let us now define

$$G := (-K) \cap \partial \mathcal{C}_1(0),$$
$$\Sigma_G(p) := \sup_{x \in G} px,$$
$$U^*(p) := \sup_{x \in \mathcal{C}} (U(x) - px),$$
$$Av := (\text{diag } v) \cdot \mathcal{E} (\text{diag } v) \cdot \mathcal{E}, \quad v \in \mathbb{R}^d,$$

and the operators:

$$F_0(X, p, H, W, x) := \frac{1}{2} \text{Tr } A(x)X + \mu^T (\text{diag } x) p + H - \beta W, \quad (4.8)$$
$$\mathcal{L}_0 \phi := F_0(\phi''(x), \phi'(x), H(\phi, x), \phi(x), x), \quad (4.9)$$
$$F(X, p, H, W, x) := \max \left\{ F_0(X, p, H, W, x) + U^*(p), \Sigma_G(p) \right\}, \quad (4.10)$$
$$\mathcal{L} \phi := F(\phi''(x), \phi'(x), H(\phi, x), \phi(x), x). \quad (4.11)$$

Let us introduce the Dirichlet problem associated to the HJB equation:

$$F(W''(x), W'(x), H(W, x), W(x), x) = 0, \quad x \in \text{int } K, \quad (4.12)$$
$$W(x) = 0, \quad \text{on } \partial K. \quad (4.13)$$

We consider a possible solution to the HJB equation in the viscosity sense. Observe that the integro-differential operator $\mathcal{L}$ is not locally defined because of the operator $H$. Therefore, we define viscosity solutions in the global sense as follows:

**Definition 4.1** A function $v \in C(K)$ is called a **viscosity supersolution** of (4.12) with operator $\mathcal{L}$ on a subset $\bar{K} \subseteq K$ if for every $x \in \text{int } \bar{K}$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \geq f$ on $K$, the inequality $\mathcal{L}f(x) \leq 0$ holds.

**Definition 4.2** A function $v \in C(K)$ is called a **viscosity subsolution** of (4.12) with operator $\mathcal{L}$ on a subset $\bar{K} \subseteq K$ if for every $x \in \text{int } \bar{K}$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \leq f$ on $K$, the inequality $\mathcal{L}f(x) \geq 0$ holds.

**Definition 4.3** A function $v \in C(K)$ is called a **viscosity solution** of (4.12) on a subset $\bar{K} \subseteq K$ if $v$ is simultaneously a viscosity super- and subsolution on the subset $\bar{K} \subseteq K$.
When the subset \( \tilde{K} = K \), we just say that \( v \) is a viscosity solution (resp. super- or subsolution). At last, a function \( v \in C_1(K) \cap C^2(\text{int} \ K) \) is called classical supersolution of (4.12) with operator \( \mathcal{L} \) on a subset \( \tilde{K} \subseteq K \) if \( \mathcal{L}v \leq 0 \) on \( \text{int} \ K \). We add the adjective strict when \( \mathcal{L}v < 0 \) on the set \( \text{int} \ K \).

**Lemma 4.4** A function \( v \in C(K) \) is a viscosity supersolution of (4.12) with operator \( \mathcal{L} \) on a subset \( \tilde{K} \subseteq K \) if and only if, for every point \( x \in \text{int} \tilde{K} \), the inequality \( \mathcal{L}f(x) \leq 0 \) holds for any function \( \phi \in C^2(x) \), such that the difference \( v - \phi \) attains its global minimum on \( K \) at \( x \).

**Proof** The proof is an adaptation of the proof of [11, Lemma 4.2.4] where we replace the notion of local minimum by the global one. \( \square \)

**Remark 4.5** In the classical theory developed for differential equations, the notion of viscosity solution admits an equivalent formulation in terms of super- and sub-jets \( J^+, J^- \) (see definition in [6]). But this is not the case in our formulation due to the non-local property of the integro-differential operator. Although, there is a link between the notion of viscosity solutions and super- and sub-jets as stated in [6, Section 7].

**Theorem 4.6** Suppose that \( W \) takes finite values. Then,

(i) The Bellman function \( W \) is a viscosity supersolution to (4.12).

(ii) The Bellman function \( W \) is a viscosity subsolution to (4.12).

Theorem 4.6 is proven in Appendix, Sect. 10.1. It implies that the Bellman function is a viscosity solution to (4.12). We shall see conditions under which \( W \) is finite.

**Definition 4.7** We say that a positive function \( \ell \in C_1(K) \cap C^2(\text{int} \ K) \) is a Lyapunov function if the following properties are satisfied:

1. \( \ell'(x) \in \text{int} \ K^* \) and \( \mathcal{L}_0\ell(x) \leq 0 \) for all \( x \in \text{int} K \),
2. \( \ell(x) \rightarrow \infty \) as \( |x| \rightarrow \infty \).

In other words, \( \ell \) is a classical supersolution of the truncated equation (excluding the term \( U^* \)), continuous up to the boundary, and growing to infinity at infinity. Let us introduce the following condition on \( \Pi \) which guaranties the uniqueness of the HJB equation we consider, under the condition that there exists a Lyapunov function as stated in the next theorem.

**Condition \( \Pi^0 \):** \( \forall x \in \text{int} K, \Pi(z : x + \text{diag} xz \in \partial K) = 0 \).

**Remark 4.8** This condition holds in the two dimensional case if the first component of the underlying asset is a bond \( B = 1 \) so that \( \Pi = \delta_0 \otimes \pi \) where we suppose that \( \pi \) does not charge the singletons.

**Theorem 4.9** Suppose that there exists a Lyapunov function \( \ell \) and condition \( \Pi^0 \) holds for the Lévy measure \( \Pi \). Then, the Dirichlet problem (4.12) has at most one viscosity solution in the class of continuous functions satisfying the growth condition

\[
W(x)/\ell(x) \rightarrow 0, \quad |x| \rightarrow \infty.
\]

Moreover, \( W \) is concave.

The proof of this result is given in Appendix, Sect. 10.2.
5 Finiteness of the value function

We denote by $\Phi$ the set of all continuous functions $f : K \mapsto \mathbb{R}_+$ increasing with respect to the partial ordering $\succeq_K$ and such that for every $x \in \text{int } K$ and $\pi = (B, C)$ the positive process $X^f_t = e^{-\beta t} f(V^\pi_t) + J^\pi_t$, $t \geq 0$, is a supermartingale. The following lemma is a consequence of Lemma 10.2. It may be proven as in [6].

**Lemma 5.1** Let $f \in C^2(\mathbb{R}^d)$ be a non negative classical supersolution of (4.12) which vanishes out of $\text{int } K$, then $f \in \Phi$.

The following proposition formulates finiteness and continuity up to boundary $\partial K$ of the Bellman function in term of $\Phi$. This corresponds to [6, Lemma 8.1].

**Proposition 5.2** (a) If $f \in \Phi$, then $W \leq f$ on $K$. Hence, if $\Phi \neq \emptyset$, then $W$ is finite.

(b) If $x_0 \in \partial K$ is such that, for every $\epsilon > 0$, there exists $f_\epsilon \in \Phi$ with $f_\epsilon(x_0) \leq \epsilon$, then $W$ is continuous at $x_0$ and $W(x_0) := 0$.

**Corollary 5.3** Let $f \in C^2(\mathbb{R}^d)$ be a non negative classical supersolution of (4.12) which vanishes out of $\text{int } K$, then $W$ is finite.

**Proof** By Lemma 5.1, $\Phi \neq \emptyset$. We conclude by Proposition 5.2.

6 Continuity of the value function

The following results may be proved as in [6].

**Lemma 6.1** Let us consider $x_0 \in \text{int } K$. Then, $\limsup_{\lambda \to 1} W(\lambda x_0) \leq W(x_0)$.

**Lemma 6.2** We have $V_{t^-} \in \text{int } K$ if $t \in [0, \theta]$.

**Proof** Suppose that $V_{t^-} \in \partial K$ for some $t < \theta$. Then, by Lemma 10.1 and Assumption 5 of the model, we have $\Delta Y_t = \Delta B_t = \Delta C_t = 0$. Therefore, $V_t = V_{t^-} \in \partial K$ hence a contradiction.

Note that we also deduce from Lemma 10.1 that the portfolio process exits $K$ either in a continuous manner (in the case $V_{\theta^-} \in \partial K$) or after a jump (in the case $V_{\theta^-} \in \text{int } K$).

**Lemma 6.3** Let us consider a sequence $x_n \to x_0 \in \text{int } K$ and $T \in (0, \infty)$. Then, for any $\pi \in \mathcal{A}_{x_0}$, the sequence of portfolios $V^{(n)} = V^{\pi, x_n}$ with initial values $V_0^{(n)} = x_n$ is such that

$$T \wedge \theta \leq \liminf_n \theta_n \wedge T$$

where $\theta, \theta_n$ are the stopping times defined by $V := V^{\pi, x}$, $V^{(n)}$ respectively in (2.4).

**Theorem 6.4** Assume that $W(x_0) < \infty$ where $x_0 \in \text{int } K$. Then, $W$ is continuous at $x_0$.

7 Existence of Lyapunov functions

In this subsection, we study the existence of Lyapunov functions. We only focus on the case where the matrix $A = (a_{ij})$ is diagonal with $a_{ii} = \sigma^i$, such that $\sigma^0 = 0, \mu^0 = 0$ and

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\[ \sigma^i \neq 0, i = 1 \ldots d, \text{i.e. the first asset is a numéraire and the others are risky assets. We also suppose that the utility function is } U(x) = u_\gamma(x e_0), \text{ where } u_\gamma(t) = t^{\gamma} / \gamma, \gamma \in (0, 1), \text{ and } C = R_+ e_0. \text{ At last, we suppose the following condition} \]

\[ \int_{R^d} |z| |\Pi(dz)| < \infty. \]

For \( p \in \text{int } K^* \), we construct a Lyapunov function of the form \( v(x) := u_\eta(px) \), where \( u_\eta(x) = \frac{x^{\eta}}{\eta} \) and \( \gamma < \eta < 1 \). Note that the Bellman function is homogeneous of degree \( \gamma \). Therefore, we choose \( \eta \geq \gamma \) so that the Lyapunov function grows faster than the Bellman function. If such a Lyapunov exists, the HJB equation admits a unique concave solution in \( C^1 \) by Theorem 4.9.

We have \( u_\eta'(x) = (px)^{\eta-1} p \in \text{int } K^* \) as required for \( v \) to be a Lyapunov function. Moreover,

\[ \mathcal{L}_0 v(x) = \frac{1}{2} \langle A(x)p , p \rangle u_\eta''(px) + \langle \mu(x), p \rangle u_\eta'(px) - \beta u_\eta(px) \]

\[ + \int_{R^d} \left[ u_\eta(px + t p \text{diag}(x))I(x, z) - u_\eta(px) - u_\eta'(px) t p \text{diag}(x) z \right] \Pi(dz). \]

Let us denote the integral expression above by \( H_\eta(x) \). Our goal is to choose \( u \) such that \( \mathcal{L}_0 v(x) \leq 0 \) on \( K \), or equivalently

\[ \beta \geq \sup_{x \in K} \left[ p \text{diag}(x) \mu_\eta u_\eta'(px) + \frac{1}{2} \langle A(x)p , p \rangle u_\eta''(px) \right. \]

\[ \left. + \left( \frac{1}{2} \langle A(x)p , p \rangle + \frac{H_\eta(x)}{u_\eta(px)} \right) \right]. \]

Let us introduce

\[ L_\eta(p, x) = p \text{diag}(x) \mu_\eta u_\eta'(px) + \frac{1}{2} \langle A(x)p , p \rangle u_\eta''(px). \]

We have

\[ L_\eta(p, x) = \frac{1}{2} \frac{\eta}{1-\eta} \sum_{i=1}^{d} \frac{\mu_i^2}{\sigma_i^2} - \frac{1}{2} \eta (1 - \eta) \sum_{i=1}^{d} \left( \frac{\sigma_i p_i x_i}{px} - \frac{\mu_i}{\sigma_i (1 - \eta)} \right)^2, \]

\[ \leq \frac{1}{2} \frac{\eta}{1-\eta} \sum_{i=1}^{d} \frac{\mu_i^2}{\sigma_i^2}. \]

We then choose \( \beta \) such that

\[ \beta \geq \frac{1}{2} \frac{\eta}{1-\eta} \sum_{i=1}^{d} \frac{\mu_i^2}{\sigma_i^2} + \sup_{x \in K} \frac{H_\eta(x)}{u_\eta(px)}. \]

Let us define \( k(p, x) = \frac{1}{px} p \text{diag}(x) \). We have

\[ \frac{H_\eta(x)}{u_\eta(px)} \leq \int_{R^d} \left[ (1 + k(p, x) z)^{\eta} 1_{\{1 + k(p, x) z > 0\}} - 1 - \eta k(p, x) z \right] \Pi(dz). \]

Note that \( k(p, tx) = k(p, x) \) for \( t > 0 \). Therefore, instead of considering the r.h.s of the equality above on \( K \), we may simply consider it on \( B_1 := K \cap \mathcal{O}_1 \), where \( \mathcal{O}_1 \) is the unit ball \( \{x : |x| = 1\} \). It is easy to prove that this expression is bounded on \( B_1 \). We then define

\[ h(\eta) := \inf_{p \in \text{int } K^*} \sup_{x \in B_1} \int_{R^d} \left[ (1 + k(p, x) z)^{\eta} 1_{\{1 + k(p, x) z > 0\}} - 1 - \eta k(p, x) z \right] \Pi(dz), \]

\[ \square \]
and we chose \( \beta \) such that
\[
\beta > \frac{1}{2} \frac{\eta}{1-\eta} \sum_{i=1}^{d} \frac{\mu_i^2}{\sigma_i^2} + h(\eta). \tag{7.15}
\]
We may deduce the following:

**Proposition 7.1**

(i) *If the condition (7.15) holds, then \( \nu(x) = \frac{(px)^\eta}{\eta}, 1 > \eta > \gamma, \) is a Lyapunov function with respect to \( \mathcal{L}_0 \) for some \( p \in K^* \) with \( p^1 = 1 \).

(ii) *Set \( \bar{h}(\gamma) := \lim \inf \limits_{\eta \searrow \gamma} h(\eta) \) and suppose that \( \beta > \frac{1}{2} \frac{\gamma}{1-\gamma} \sum_{i=1}^{d} \frac{\mu_i^2}{\sigma_i^2} + \bar{h}(\gamma). \tag{7.16} \)

Then, the HJB equation admits a unique solution under the conditions of Theorem 4.9.

### 8 Classical supersolution

The hypothesis of this section are those of Sect. 7 and the notations are the same. By Lemma 5.1, if we construct a non negative classical supersolution \( f \in C^2(\mathbb{R}^d) \) of (4.12) which vanishes out of \( \text{int } K \), then \( f \in \Phi \). Therefore, by Proposition 5.2, \( W \) is finite.

Let us define \( \nu(x) := ku_\gamma(p x) \). By definition,
\[
U^*(\nu'(x)) = \sup_{y \in \mathcal{C}} \left( U(y) - \nu'(x)y \right) = \sup_{y \in \mathcal{C}} \left( U(y) - ku_\gamma(p x)py \right),
\]
\[
= \sup_{y_1 \geq 0} \left( \frac{(y_1)^\gamma}{\gamma} - k(p x)^{\gamma-1} p_1 y_1 \right).
\]

Then, with \( p_1 = 1 \), \( U^*(\nu'(x)) = k \frac{\nu'}{\nu} \ (p x)^\gamma (1/\gamma - 1) \). Since \( u' \geq 0 \), \( G \subseteq -K \), and \( p \in K^* \setminus \{0\} \), we have
\[
\Sigma_G(\nu'(x)) = \sup_{x \in G} u_\gamma'(p x) p x \leq 0.
\]

Our goal is to choose \( p \) and \( k \) so that, on \( \text{int } K \), we have the following inequality
\[
k \frac{\nu'}{\nu} \ (p x)^\gamma (1/\gamma - 1) + k \frac{1}{2} (A(x)p, p) u''_\gamma(p x) + k (\mu(x), p) u'_\gamma(p x) - k \beta u_\gamma(p x)
\]
\[
+ \int_{\mathbb{R}^d} \left[ u_\gamma(p x + t \text{diag}(x)z) I(x, z) - u_\gamma(p x) - u'_\gamma(p x) t \text{diag}(x)z \right] \Pi(dz) \leq 0.
\]

Adapting the reasoning of the previous subsection, we choose \( \beta \) such that
\[
\beta > \frac{1}{2} \frac{\gamma}{1-\gamma} \sum_{i=1}^{d} \frac{\mu_i^2}{\sigma_i^2} + h(\gamma) + k \frac{1}{\gamma} (1 - \gamma). \tag{8.17}
\]

**Proposition 8.1**

(i) *Suppose that condition (8.17) holds. Then, the function \( \nu(x) = \frac{k}{\gamma} (p x)^\gamma \) is a classical supersolution to the HJB equation for some \( p \in K^* \) with \( p^1 = 1 \).
(ii) In the two-dimensional model with the power utility function, assume that the Merton parameter satisfies

\[ \kappa_M := \frac{1}{1 - \gamma} \left( \beta - \frac{\gamma \mu^2}{2\sigma^2(1 - \gamma)} - h(\gamma) \right) > 0. \]

Then, there exists \( p \in K^* \) with \( p^1 = 1 \) such that the function \( f(x) = m(px)^\gamma \) is a classical supersolution of the HJB equation and \( m > (1/\gamma)\kappa_M^{\gamma-1} \).

Combining Propositions 7.1 and 8.1, we obtain the following result

**Corollary 8.2** Let \( h^* \) be defined as \( h^*(\gamma) := \max(h(\gamma), \tilde{h}(\gamma)) \). If

\[ \beta > \frac{1}{2} \frac{\gamma}{1 - \gamma} \sum_{i=1}^{d} \frac{\mu_i^2}{\sigma_i^2} + h^*(\gamma), \] (8.18)

there exists a classical supersolution to the HJB equation and a Lyapunov function with higher growth rate than the Bellman function.

### 9 Application to the two-asset model

We investigate the properties satisfied by the Bellman function and we construct an optimal policy in the case where \( d = 2 \). We use the notation \( z = (x, y)' = (z_1, z_2)' \) to designate a generic element \( z \) of \( \mathbb{R}^2 \) where \( ' \) is the transpose operator. The canonical basis of \( \mathbb{R}^2 \) is \((e_1, e_2)\) where \( e_1 = (1, 0)' \) and \( e_2 = (0, 1)' \). The risk-free asset (a bond) is supposed to be constant, and the risky asset follows a geometric Lévy process:

\[
d S_1^t = 0, \quad S_0^1 = 1, \quad d S_2^t = S_{2-}^t \left( \mu t + \sigma d W_t + \int_{\mathbb{R}} x(p(dy, dt) - q(dy, dt)) \right), \quad S_0^2 = 1, \]

where \( p \) is the jump measure of \( S^2 \) and \( q(dy, dt) = \pi(dy)dt \) is its compensator. We suppose that \( \pi(dy) \) is a positive measure concentrated in \((-1, \infty)\) which does not charge the singletons and satisfies the following condition:

**Condition I**

\[
\int_{-1}^{\infty} \max(1, |t|) \pi(dt) < \infty. \tag{9.19}
\]

The inequality (9.19) ensures that \( \pi \) is a finite measure such that the associated Lévy process has a finite activity. This implies that

\[
Y^1 = 0, \quad Y^2 = \sigma W_t + \mu t + \sum_{i=1}^{N_t} \chi_i
\]

where \( N_t = \sum_{i=1}^{\infty} 1_{T_n \leq t} \) is a Poisson process of intensity \( \lambda > 0 \) and \((\chi_i)_{i \geq 1}\) is a sequence of i.i.d. \( \pi \)-distributed random variables independent of \( N \). A portfolio process satisfies by definition the following dynamics:

\[
d V_1^t = d L_{12}^t - (1 + \lambda_{12}^2) d L_{12}^t - c_t dt, \quad d V_2^t = V_{2-}^t \left( \mu dt + \sigma d W_t + \int_{\mathbb{R}} x(p(dy, dt) - q(dy, dt)) \right) + d L_{12}^t - (1 + \lambda_{21}^2) d L_{21}^t, \]

\( \text{Springer} \)
where $L^{ij}, i, j = 1, 2,$ are the transfer processes supposed to be càdlàg and $(\lambda^{ij})_{i,j=1,2}$ are the transaction cost coefficients. We rewrite the dynamics of a portfolio process under the vector form:

$$dV_t = \text{diag} \{ \mu dt + \sigma dW_t + \int z(\tilde{\mu}(dz, dt) - \tilde{\nu}(dz)dt) \} + dB_t - dC_t.$$ 

In the following, we use the notations $\tilde{\mu} = (0, \mu)', \tilde{\sigma} = (0, \sigma)'.$ Therefore, $z = (x, y)'.$

$$\tilde{\mu}(dz, dt) = \delta_0(dx)dt \otimes \delta_{\lambda^2}((dy)dN_t, \quad \tilde{\nu}(dz) = \lambda \delta_0(dx) \otimes \pi(dy).$$

This means that $A = \text{diag}(0, \sigma^2)$ is the diagonal matrix with diagonal elements 0 and $\sigma^2$. Moreover, we introduce $dC_t = (c_t dt, 0)'$ and

$$dB_t = (dL_t^{11} - (1 + \lambda^{12})dL_t^{12}, dL_t^{12} - (1 + \lambda^{21})dL_t^{21}).$$

The optimization problem reads as

$$\mathbb{E} \int_0^\theta e^{-\beta s}u(c_s)ds \rightarrow \text{max},$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a concave utility function. In the sequel, we consider the case of power utility functions, i.e. $u(r) = r^\gamma, \gamma \in (0, 1)$. Therefore, $W$ is homogeneous of degree $\gamma$:

$$W(\lambda x) = \lambda^\gamma W(x), \quad \forall x \in K, \quad \gamma \geq 0.$$  \hspace{1cm} (9.20)

Note that for other utility functions, the following analysis needs to be reconsidered. For instance, for the logarithmic utility function, a simple adaptation of our work does not seem to be enough.

In our example, the solvency cone $K$ is simply a sector generated by the vectors $g_1 = (1 + \lambda^{12})e_1 - e_2, g_2 = (1 + \lambda^{21})e_2 - e_1.$ The dual cone of $K$ is $K^* = \text{cone} \{ p_1, p_2 \}$ with $p_1 = (1 + \lambda^{12})e_2 + e_1, p_2 = (1 + \lambda^{21})e_1 + e_2.$ For the sake of simplicity, we suppose that $\lambda^{12} = \lambda^{21} = \lambda.$ The consumption region is $C = \mathbb{R}_+e_1.$ Therefore, the HJB equation is given by:

$$F(W''(z), W'(z), H(W, z), W(z), z) = 0, \quad (9.21)$$

$$W(z) = 0 \quad \text{on} \quad \partial K,$$ \hspace{1cm} (9.22)

where

$$F_0(X, p, H, W, z) := \frac{1}{2} \sigma^2 yX_{22} + \mu y p_2 + H - \beta W,$$ \hspace{1cm} (9.23)

$$F(X, p, H, W, z) := \max \left\{ F_0(X, p, H, W, z) + U^*(p), \Sigma_G(p) \right\}. \hspace{1cm} (9.24)$$

By simple computation, we get that $U^*(p) = \frac{1}{\gamma} (p_1)^{\gamma / (\gamma - 1)}$, if $p = (p_1, p_2)'$. Moreover

$$\Sigma_G(p) \leq 0 \Leftrightarrow \max \{-g_1 p, -g_2 p\} \leq 0,$$ 

and

$$H(u, z) = \int_K \left[ u(x, y(1 + t))I(z, t) - u(x, y) - u_y'(x, y)yt \right] \pi(dt),$$

where $I(z, t) := 1_{\{(x, y + yt) \in \text{im}K\}}, z = (x, y)'$. We now formulate properties satisfied by the operator $H(u, z)$.
Lemma 9.1  (i) If \( u \in C_1(K) \cap C^1(K \setminus \mathbb{R}e_1) \) is a non negative concave function such that \( \Sigma_G(u^i) \leq 0 \) on \( K \) (i.e. \( u \) is increasing with respect to the natural order on \( K \)), then the operator \( H(u, z) \) is non positive.

(ii) If \( u \in C^1(K \setminus \mathbb{R}e_1) \) is homogeneous of degree \( \gamma < 1 \) and \( \pi(\mathbb{R}) < \infty \) then \( H(u, .) \) is continuous on \( K \cap \{ (x, y) : y > 0 \} \). Moreover, if \( u_y(x, 0^+) \) exists and is finite, then so is \( H(u, (x, 0^+)) \).

(iii) If \( u \in C^2(K \setminus \mathbb{R}e_1) \) is homogeneous of degree \( \gamma < 1 \) and \( \pi(\mathbb{R}) < \infty \), then \( H(u, .) \) is \( C^1 \) on \( K \cap \{ (x, y) : y > 0 \} \). Moreover, if \( u_y(x, 0^+) \) and \( u_{yy}(x, 0^+) \) exist and are finite, then so is \( H(u, (x, 0^+)) \).

Proof  (i) We aim to prove that \( P(z, t) \leq 0 \) where

\[
P(z, t) := u(x, y(1 + t))I(z, t) - u(x, y) - u'_y(x, y)yt.
\]

(9.25)

It is clear that \( P(z, t) = 0 \) in the case where \( y = 0 \). Otherwise, since \( u \geq 0 \), we deduce that \( P(z, t) \leq u(x, y(1 + t)) - u(x, y) - u'_y(x, y)yt \). So,

\[
P(z, t) \leq u(x, y(1 + t)) - u(x, y) - u'_y(x, y)yt,
\]

\[
\leq \left( u'_x(x, \theta x) - u'_y(x, y) \right) yt,
\]

where \( \theta x = \{ y, y(1 + t) \} \). Since the function \( y \mapsto u(x, y) \) is concave, \( y \mapsto u'_y(x, y) \) is non increasing on \( C^1(K \setminus \mathbb{R}e_1) \) hence \( P(z, t) \leq 0 \).

(ii) Assume that \((x, y) \in K \) and \( y > 0 \). Let \( z_n := (x_n, y_n) \in K, y_n > 0 \) be a sequence convergent to \( z := (x, y) \). Since \( u \) is locally Lipschitz on \( K \setminus \mathbb{R}e_1 \), we have for \( n \) large enough \( P((x_n, y_n), t) - P((x, y), t) = o(1) \) uniformly on \( t \in (-1, 0) \). We now consider \( t \geq 0 \). Using the homogeneity of \( u \), we obtain

\[
u(x_n, y_n + y_n t) - u(x, y + yt) = (1 + t)^\gamma (u(x_n 1 + t, y_n) - u(x 1 + t, y)) = (1 + t)^\gamma o(1),
\]

uniformly on \( t \geq 0 \). Finally,

\[
u(x_n, y_n) - u(x, y) = o(1),
\]

and, similarly,

\[
u'_y(x_n, y_n) y_n t - u'_y(x, y) yt = o(t).
\]

We deduce that

\[
P((x_n, y_n), t) - P((x, y), t) = o(1) \max (1, |t|).
\]

This implies that \( H(u, z_n) \rightarrow H(u, z) \) when \( n \rightarrow \infty \). The case \( y = 0 \) is proved similarly, using the boundedness of \( (u_y(z_n))_{n \geq 1} \) when \( y_n \rightarrow 0 \).

(iii) The proof is similar than (ii).

Let us also suppose the following:

**Condition II**

\[
\beta > \frac{\gamma \mu^2}{2\sigma^2(1 - \gamma)} + h^*(\gamma),
\]

where the function \( h^* \) is given in Corollary 8.2. By Corollary 8.2, conditions I and II imply the existence of a Lyapunov function with higher growth rate than the Bellman function and a classical supersolution to the HJB equation. Therefore, using Proposition 5.2, we deduce that
the Bellman function is finite on \( K \) and continuous up to the boundary \( \partial K \), see Theorem 6.4. Then, we deduce by Theorem 4.9 that the HJB equation admits a unique concave solution in the class \( C^1(K) \). Actually, we prove that \( W \in C^2(K) \) in Sect. 9.1. To do so, we first formulate some well known results from the literature on the Bellman function. Recall that the cone \( K \) can be splitted into three non-empty open disjoint cones \( K_i, i = 0, 1, 2 \), such that \( \bar{K} = \bar{K}_0 \cup \bar{K}_1 \cup \bar{K}_2 \) with \( K_0 = \text{cone} \{ \bar{g}_1, \bar{g}_2 \}, K_1 = \text{cone} \{ g_1, \bar{g}_1 \} \) and \( K_2 = \text{cone} \{ \bar{g}_2, g_2 \} \) for some vectors \( \bar{g}_1, \bar{g}_2 \). Moreover, \( K_1 \) contains \( \text{cone} \{ g_1, e_1 \} \) and \( K_2 \) contains \( \text{cone} \{ 2\mu \sigma^{-2}(1 - \gamma)^{-1}(1 + \lambda)^{-1}g_2 + e_2, g_2 \} \). On \( K_1 \cup K_2, W \) is \( C^\infty \) and is given by

\[
\begin{align*}
W(z) &= a_1 u(p_1 z), \quad \text{on } K_1, \\
W(z) &= a_2 u(p_2 z), \quad \text{on } K_2,
\end{align*}
\]

where \( a_1, a_2 \) are some constants such that

\[
a_1 \geq \left( \frac{\beta}{1 - \gamma} \right)^{\gamma - 1}, \quad a_2 \geq \left( \frac{\beta}{1 - \gamma} \right)^{\gamma - 1} \frac{1}{(1 + \lambda)^\gamma}.
\]

**Proof** We adapt the proof of [11, Proposition 4.8.2]. With \( \psi(x) := a_1 u(p_1 x) \), we need to show that

\[
\mathcal{L}_0 \psi(x) + u^*(\psi, x) + H(\psi, x) \leq 0,
\]

for all \( x \in \text{cone} \{ g_1, e_1 \} \). By Lemma 9.1 (i), we have \( H(\psi, x) \leq 0 \). Moreover, as in the proof of [11, Proposition 4.8.2], \( \mathcal{L}_0 \psi(x) + u^*(\psi, x) \leq 0 \) due to the lower bound given by Lemma 3.2. This implies that \( \psi \) is a classical super solution of our HJB equation on the subset \( \bar{K} = \text{cone} \{ g_1, e_1 \} \). Moreover, by construction, \( W = \psi \) on \( \bar{K} \). By Theorem 10.7, we deduce that \( W \leq \psi \) on \( \bar{K} \). Similarly, we follow the proof of [11, Proposition 4.8.2, statement b)] using Theorem 10.7. We then define \( K_1, K_2 \) as the largest sectors on which (9.26) and (9.27) hold.

**Lemma 9.3** If

\[
W(e_1) = \frac{1}{\gamma} \left( \frac{\beta}{1 - \gamma} \right)^{\gamma - 1},
\]

then the axis of the abscises is not the common boundary of \( K_1 \) and \( K_2 \).

**Proof** We follow the proof of [11, Lemma 4.8.4]. Suppose the opposite. Then, the function \( \psi(z) = a_2(p_2 z)^\gamma \) coincides with \( W(z) \) on the sector cone \( \{ g_2, e_1 \} \). We may deduce the value of \( a_2 \) using the assumption of the lemma. Hence, we deduce with \( z = (x, y) \) that

\[
\mathcal{L}_0 \psi(z) + u^*(\psi, z) = a_2(p_2 z)^\gamma - 1 y \left( \frac{1}{2} \sigma^2 (\gamma - 1) \frac{y}{p_2 z} + \mu \right) + H(\psi, z).
\]

With \( z = (x, y) \) where \( x, y > 0 \) and \( x \) is fixed, the first term in the r.h.s. above admits a lower bound \( c_x y \) provided that \( y \) is sufficiently close to 0 and \( c_x > 0 \) only depends on \( x, a_2, 

---

1 Here \( \bar{K}_i \) denotes the closure of \( K_i, i = 0, 1, 2 \).
\( \gamma, \mu \) and \( \lambda \). On the other hand, we may write

\[
H(\psi, z) = \frac{\gamma^2}{2} \int_{-1}^{M} \psi''(z + y\theta t)t^2 \pi(dt) + \int_{M}^{\infty} P(z, t)\pi(dt), \tag{9.28}
\]

where \(|\theta t| \leq |t| \) for all \( t \geq -1 \), and \( P \) is defined by (9.25). We choose \( M \) large enough such that \( \pi([M, \infty)) \) is close to 0. Precisely, as \( \psi_2(z) = a_2((1 + \lambda)x + y)^{\gamma-1} \) is bounded by a constant depending on \( x \) when \( y \) is small enough, we deduce that \( \psi(x, \cdot) \) is Lipschitz and finally

\[
\left| \int_{M}^{\infty} P(z, t)\pi(dt) \right| \leq \epsilon_x y
\]

where \( \epsilon_x \) is arbitrarily small provided that we choose \( M \) large enough. We fix \( M \) such that \( \epsilon_x \leq c_x / 3 \). If \( t \in (-1, M) \), the factor \( \psi''(z + y\theta t)|t| \) appearing in the first term of the r.h.s. of (9.28) is also bounded by a constant depending on \( M \) but we may choose \( y \) small enough such that \( \psi''(z + y\theta t)|t|y \) is as small as we want. We finally deduce that \( |H(\psi, z)| \leq (2c_x / 3)y \) provided that \( y \) is small enough. This implies that \( \mathcal{L}_0 \psi(z) + u^*(\psi_x(z)) > 0 \) for some \( z \) small enough. This yields a contradiction. \( \square \)

### 9.1 Reduction to one variable and regularity of the value function

Using the homogeneity property of the Bellman function, we reduce our problem to the case of one variable by considering the restriction of the Bellman function on the intersection of the line \( \{z = (x, y) : x + y = 1\} \) with the interior of \( K \). Indeed, if we define \( \psi(t) := W(1 - t, t), t \in \Delta := [-\frac{1}{2}, 1 + \frac{1}{2}] \), we may reconstruct \( W \) from \( \psi \) by the formula

\[
W(x, y) = (x + y)^{\gamma} \psi(\frac{y}{x + y}), \quad (x, y) \in \text{int } K.
\]

As in [11,12], we may show that \( \psi \) is the viscosity solution to the new HJB equation obtained by the change of variables above:

\[
\max_{i=0,1,2} l_i(f) = 0, \tag{9.29}
\]

with the two first-order operators

\[
l_1(f) := -\lambda \gamma f + (1 + \lambda z)f', \quad l_2(f) := -\lambda \gamma f - (1 + \lambda - \lambda z)f',
\]

and the second-order operator

\[
l_0(f) := f_2 f'' + f_1 f' + f_0 f + \frac{1 - \gamma}{\gamma} \left[ f f' - z f' \right]^{\gamma - 1} + H(z, f, f(z), f'(z)),
\]

where

\[
f_2(z) := \frac{1}{2} \sigma^2 z^2 (1 - z)^2,
\]

\[
f_1(z) := -\sigma^2 (1 - \gamma)z(1 - z)(z - \theta), \quad \theta := (1 - \gamma)^{-1} \mu \sigma^{-2},
\]

\[
f_0(z) := \frac{1}{2} \sigma^2 \gamma (\gamma - 1) z^2 + \gamma \mu z - \beta,
\]

\[
H(z, f, v, \nu') := \int_{-1}^{\infty} \left[ (1 + z t)^{\gamma} f \left( \frac{z + z t}{1 + z t} \right) \mathbf{1}_{(1 - z, z(1 + t))} \in K - v - z t (\gamma v + (1 - z) \nu') \right] d\pi(t).
\]
Recall that \( W \) is concave by Theorem 10.8. Then, the function \( \psi \) defined above is also concave on \( \Delta \) and its derivatives \( \psi', \psi'' \) exist almost everywhere. Therefore, (9.29) holds in the classical sense as stated in [6, Lemma 6.1]. Moreover, \( \psi \) being concave, it admits left and right derivatives which are respectively left and right continuous and such that the inequality \( D^+ \psi \leq D^- \psi \) holds and is strict only on a countable set. Moreover, by Theorem 9.2, \( \psi(z) > 0 \) when \( z \) is sufficiently close to \( \lambda^{-1} \) or \( 1 + \lambda^{-1} \). Since \( \psi \) is concave, we deduce that \( \psi > 0 \) on \( \text{int} \, \Delta \).

We now adapt some results from the literature on the regularity of the Bellman function.

**Proposition 9.4** The function \( \psi \) is continuously differentiable on the interval \( \text{int} \, \Delta \) except, maybe at zero. If \( \psi' \) has a discontinuity at 0, then

\[
\psi(0) = \frac{1}{\gamma} \left( \frac{1 - \gamma}{\beta} \right)^{1-\gamma} = \frac{1}{\gamma} k^*_{\gamma}.
\]  

**(Proof)** We adapt the proof of [11, Lemma 4.8.6]. The only difference is due to the operator \( H \). Clearly, this does not change the proof when \( z \notin \{0, 1\} \) as \( H \) takes finite values. When \( z = 1 \), we need to show the extra property that \( H(z_n, \psi, \psi') \to H(1, \psi, D^\pm \psi) \) as \( z_n \to 1 \) with \( z_n < 1 \) and \( z_n > 1 \) respectively. To see it, observe that \( 1 \in (1-z_n, 0) \in K = 1, \pi(dt) \text{ a.s.}, \) then if \( z \in [0, 1] \). If the sequence \( z_n > 1 \) is such that \( z_n \to 1 \), then \( 1 \in (0, 1) \to 1 \) if \( t > -1 \). Moreover, if \( z \) bounded in a neighbourhood of 1, the quantity \( z(1+t)/(1+zt) \) is uniformly bounded in \( t \geq 1 \) and in \( z \). As \( D^\pm \psi(1) \) exists, we then deduce that the integrand defining \( H(z_n, \psi, \psi') \) is uniformly bounded by \( c \min(1, |t|) \) for some constant \( c > 0 \) independent of \( n \). This implies that the dominated convergence theorem applies. Finally, when \( z_n \uparrow 1 \), we get that \( H(z_n, \psi, \psi') \to H(1, \psi, D^- \psi) \). As in [11, Lemma 4.8.6], we finally get that

\[
f_0(1)\psi(1) + u^*(\gamma \psi(1) - D^- \psi(1)) + H(1, \psi, D^- \psi) \leq 0,
\]

(9.31)

\[
f_0(1)\psi(1) + u^*(\gamma \psi(1) - D^+ \psi(1)) + H(1, \psi, D^+ \psi) \leq 0,
\]

(9.32)

where \( u^*(x) := x^{(\gamma-1)^{-1}} \). Let \( \alpha \in (0, 1) \) be such that \( p = \alpha D^- \psi + (1 - \alpha) D^+ \psi \). By (9.31) and (9.32), we deduce that

\[
f_0(1)\psi(1) + \alpha u^*(\gamma \psi(1) - D^- \psi(1)) + (1 - \alpha) u^*(\gamma \psi(1) - D^+ \psi(1)) + H(1, \psi, p) \leq 0.
\]

Since \( u^* \) is strictly convex, we get that \( f_0(1)\psi(1) + u^*(\gamma \psi(1) - p) + H(1, \psi, D^+ \psi) < 0 \), in contradiction with [11, Lemma 4.8.6, Inequality (4.8.20)]. The conclusion follows in the case where \( z = 1 \).

In the case where \( z = 0 \), we also need to verify the convergence of \( H(z_n, \psi, \psi') \) to \( H(0, \psi, \psi') \) as \( z_n \to 0 \) with \( z_n \in (0, 1) \). In this case, \( 1 \in (1-z_n, 0) \in K = 1 \) for all \( n \) if \( t > -1 \). We verify that the quantity \( z(1+t)/(1+zt) \) is uniformly bounded in \( t \geq 1 \) and \( z > 0 \). With \( z > 0 \), we have \( 1 + tz > z + tz = z(1+t) > 0 \) hence \( 0 \leq z(1+t)/(1+zt) \leq 1 \). Therefore, the dominated convergence theorem also applies and we get that \( f_0(0)\psi(0) + u^*(\gamma \psi(0)) + H(0, \psi, 0) \leq 0 \) and finally \( f_0(0)\psi(0) + u^*(\gamma \psi(0)) + H(0, \psi, 0) = 0 \) if \( \psi' \) admits a discontinuity at zero. Since \( H(0, \psi, 0) = 0 \), the conclusion follows.

A direct consequence of the proposition above is that the Bellman function is \( C^1 \) on \( \text{int} \, K \setminus R^+_1 e_1 \). More precisely we have:

**Corollary 9.5** The value function is \( C^1 \) on \( \text{int} \, K \setminus R^+_1 e_1 \). If \( \psi \) is not \( C^1 \) on \( R^+_1 e_1 \), then (9.30) holds. Furthermore, even if \( \psi \) is not \( C^1 \) on \( R^+_1 e_1 \), the partial derivative \( W_x \) is defined and...
continuous, and the one-sided derivatives \( W_y(x, 0 \pm) \) are also defined and satisfy the one-sided continuity conditions
\[
W_y(x, 0 \pm) = \lim_{(\xi, \eta) \to (x, 0 \pm)} W_y(\xi, \eta) = x^{y-1} (y \psi(0) + D^\pm \psi(0)), \quad x > 0.
\]

**Proof** The first claim is an immediate consequence of Proposition 9.4. When \( y = 0 \), \( W(x, y) = x^{y} \psi(0) \) so that \( W_x(x, y) \) exists and is given by \( W_x(x, y) = y x^{y-1} \psi(0) \). Otherwise, when \( y \neq 0 \) or equivalently \( z = z(x, y) = y/(x + y) \neq 0 \), then \( \psi \) is differentiable at the point \( z \) hence \( W(x, y) = (x + y)^{y} \psi(z(x, y)) \) exists and is given by
\[
W_x(x, y) = y(x + y)^{y-1} \psi(z(x, y)) + (x + y)^{y} z_x(x, y) \psi'(z(x, y)). \tag{9.33}
\]
Since \( D^\pm \psi(0) \) exists, it is natural to take the convention \( 0 \times \psi'(0) = 0 \) even if \( \psi'(0) \) does not exist. It follows that (9.33) holds everywhere. In particular, \( W_x \) is continuous.

Similarly, when \( y \neq 0 \), i.e. \( z(x, y) \neq 0 \), \( W_y(x, y) \) exists and is given by
\[
W_y(x, y) = y(x + y)^{y-1} \psi(z(x, y)) + (x + y)^{y} z_y(x, y) \psi'(z(x, y)). \tag{9.34}
\]
The claim follows. \(\square\)

**Lemma 9.6** The interior of \( K_0 \) is nonempty.

**Proof** It suffices to mimic the proof of [11, Lemma 4.8.7] by virtue of Proposition 9.4 and Lemma 9.3. \(\square\)

Since \( K_0 \neq \emptyset \) and \( K_1 \) contains cone \((g_1, e_1)\), there exists two real numbers \( z_1, z_2 \in \Delta \) satisfying
\[
0 \leq z_1 < z_2 < 1 + \frac{1}{\lambda},
\]
such that
\[
K_0 = \left\{ (x, y) \in \text{int } K : z_1 < \frac{y}{x + y} < z_2 \right\}.
\]

Moreover, we have
\[
\psi(z) = \kappa_1 (1 + \lambda z)^y, \quad z \in [-\lambda^{-1}, z_1],
\]
\[
\psi(z) = \kappa_2 (1 + \lambda - \lambda z)^y, \quad z \in [z_2, 1 + \lambda^{-1}].
\]
for some constants \( \kappa_i, i = 1, 2 \).

**Proposition 9.7** The point \( e_1 \) belongs to \( K_1 \).

**Proof** We adapt the proof of [11, Proposition 4.8.8]. In the case where \( y W(e1) > \kappa_1^y - 1 \), the proof is based on [11, Lemma 4.2.5.] that we need to verify for global viscosity solutions, i.e. when replacing local minimum by global minimum. As a real-valued continuous function admits a global minimum on a compact subset, we may easily adapt the proof provided that [11, Lemma 4.2.4] also holds. To see it, we adapt the proof of the latter replacing local by global domination. At last, if \( y W(e1) > \kappa_1^y - 1 \), we observe that \( H(z_n, \psi, \psi') \to 0 \) as \( z_n \to 0 \). This is indeed proved in the last part of the proof of Proposition 9.4. We then conclude. \(\square\)

**Corollary 9.8** The function \( \psi \) is \( C^1 \).
Proposition 9.9 We have $l_1 \psi(z) < 0$ and $l_2 \psi(z) < 0$ for all $z \in (z_1, z_2)$.

Proof We follow the reasoning given in [12, Section 6]. Using the same notations, as $\psi$ is $C^1$, we deduce by [12, Proposition 6.2.] that the sub differential is a singleton, i.e. $\partial W(x, y) = \{W'(x, y)\}$. Therefore, the functions $\theta^+$ and $\theta^-$ defined in [12, Section 6] coincide with the function $\theta^2 := (1 + \lambda)^{-1} g_2 W'^2$. We parametrize the half-line $D_2$ originating at $(0, g_1)$ and parallel to $(0, g_2)$ as in [12], i.e.

$$D_2 := \left\{(x(\rho), y(\rho)) := \left(1 - \frac{\rho}{1 + \lambda}, -\frac{1}{1 + \lambda} + \rho\right), \quad \rho \geq 0\right\}.$$ 

Since $W$ is concave, [12, Property (6.6)] holds hence the function $\rho \mapsto \theta^2(x(\rho), y(\rho))$ is non increasing as stated by [12, Lemma 6.3, (6.7)]. Let us define the coefficient $^3$

$$\rho_2 := \inf\{\rho : \theta^2(x(\rho), y(\rho)) = 0\}.$$

Observe that $\rho_2 \in (0, \infty)$ by Theorem 9.2. By [12, Lemma 6.3, (6.8)], and by continuity, we get the property $\theta^2(x(\rho), y(\rho)) = 0$ for all $\rho \geq \rho_2$ since $\theta^2 \geq 0$ on $K$ and $\rho \mapsto \theta^2(x(\rho), y(\rho))$ is non increasing. Moreover, it is trivial that $\theta^2(x(\rho), y(\rho)) > 0$ for all $\rho < \rho_2$. This implies that the conic sector generated by the points of $D_2$ parametrized by $\rho \geq \rho_2$ coincides with $\overline{K}_2$ of Theorem 9.2. Similarly, We parametrize the half-line $D_1$ originating at $(0, g_2)$ and parallel to $(0, g_1)$:

$$D_1 := \left\{(x(\rho), y(\rho)) := \left(-\frac{1}{1 + \lambda} + \rho, 1 - \frac{\rho}{1 + \lambda}\right), \quad \rho \geq 0\right\}.$$ 

We also define $\theta^1 := (1 + \lambda)^{-1} g_1 W'$ and

$$\rho_1 := \inf\{\rho : \theta^1(x(\rho), y(\rho)) = 0\}.$$ 

Similarly, we obtain that the conic sector generated by the points of $D_1$ parametrized by $\rho \geq \rho_1$ coincides with $\overline{K}_1$ of Theorem 9.2. Moreover, by definition of $\rho_1$, we get that $\theta^1(x(\rho), y(\rho)) > 0$ for all $\rho < \rho_1$. This means that $l_1(\psi) < 0$ and $l_2(\psi) < 0$ on $(z_1, z_2)$. □

In order to apply the Itô formula and construct an optimal control, we need for the value function $W$ to be $C^2$ across the boundary of the cone $K_0$, except at $0$. To do so, we introduce the following local operator deduced from the global operator $l_0$ by freezing the value function $\psi$.

$$\tilde{l}_0(f) := f_2 f'' + f_1 f' + f_0 f + \frac{1 - \gamma}{\gamma} [y f - zf']^\gamma - \tilde{H}(z, f(z), f'(z)), \quad (9.35)$$ 

where $\tilde{H}(z, v, v') = H(z, \psi, v, v')$.

Although, by doing so, $\psi$ is no more a viscosity solution to the (new) local operator but it is only a viscosity solution in a weak sense we precisce as follows. Precisely, we define the notion of weak viscosity solution for an operator $L$. The corresponding weak solution for the one dimensional equation is easily deduced.

Definition 9.10 A function $v \in C(K)$ is a weak viscosity supersolution (resp. subsolution) of (4.12) with operator $L$ on a subset $\tilde{K} \subseteq K$ if and only if, for every point $x \in \text{int } \tilde{K}$,

\[2\text{ Note that the coefficient } (1 + \lambda) \text{ in our paper corresponds to the coefficient } (1 - \lambda)^{-1} = (1 - \mu)^{-1} \text{ of [12].}

\[3\text{ Denoted by } \rho_0 \text{ in [12].} \]
the inequality $\mathcal{L}\phi(x) \leq 0$ (resp. $\mathcal{L}\phi(x) \geq 0$) holds for any function $\phi \in C^2(x)$ satisfying $(v - \phi)(x) < 0$ (resp. $(v - \phi)(x) > 0$) such that the difference $v - \phi$ attains its global minimum (resp. maximum) on $K$ at $x$.

As usual, a weak viscosity solution is both a weak viscosity supersolution and weak viscosity subsolution. Adapting the proof of [11, Lemma 4.2.5], we get a similar result for weak viscosity solutions.

**Lemma 9.11** Let $\psi \in C^1(a, b)$ be a weak viscosity solution of the (local) equation

$$\psi''(z) = G(\psi'(z), \psi(z), z)$$

on a nonempty subinterval $(a', b') \subseteq (a, b)$. Suppose that $z \mapsto G(\psi'(z), \psi(z), z)$ is continuous on $[a', b']$. Then, $\psi \in C^2(a', b')$ and the equation holds in the classical sense.

**Proof** We adapt the proof of [11, Lemma 4.2.5]. Indeed, with the same notations, the case of interest is when the minimum of $\psi - \psi_\epsilon$ is negative. In that case, we use the weak supersolution property. Otherwise, $\psi - \psi_\epsilon \geq 0$, i.e. $\psi \geq \psi_\epsilon$ on $[z_1, z_2]$, which is the desired conclusion. Symmetrically, when the maximum of $\psi - \psi_\epsilon$ is positive, we use the weak subsolution property. On the contrary, we have $\psi \leq \psi_\epsilon$ and the conclusion follows. \hfill $\Box$

In the following proposition, we also need the notion of weak viscosity solution contrarily to [11].

**Proposition 9.12** Suppose that $l_1 \psi(z_0) < 0$ and $l_2 \psi(z_0) < 0$ at some point $z_0 \in (0, 1 + \lambda^{-1}) \backslash \{1\}$. Then, if $\pi(\mathbb{R}) < \infty$, $\psi \in C^2(z_0)$.

**Proof** By continuity, we may assume that $l_1 \psi(z) < 0$ and $l_2 \psi(z) < 0$ for all $z$ in some interval $[a, b]$ where $a < z_0 < b$. As $z_0 \neq 1$, we may suppose that $1 \notin [a, b]$. We show that $\psi$ is a weak viscosity solution to the equation $l_0 \psi = 0$ on $[a, b]$.

First, as $\psi$ is a viscosity supersolution to the equation $\max_{i=0,1,2} l_i \phi = 0$, this is also a viscosity supersolution to $l_0 \phi = 0$. Indeed, $\max_{i=0,1,2} l_i \phi \leq 0$ implies $l_0 \phi \leq 0$ for any test function $\phi$.

Let us now consider any function $f \in C^2(z_0)$ where $z_0 \in (a, b)$ and suppose that that the difference $\psi - \tilde{f}$ attains its global minimum on $K$ at $z_0$ such that $(\psi - f)(z_0) < 0$. Let us consider $\tilde{f}^\epsilon := \tilde{f} + \psi(1 - \xi^\epsilon)$ where, by the one dimensional version of Lemma 10.9, $\xi^\epsilon \in [0, 1]$ is infinitely differentiable, vanishes out of the ball of center $z_0$ and radius $r \to 0$ and $\xi^\epsilon = 1$ on a smaller ball around $z_0$. Note that $\psi - \tilde{f}^\epsilon = (\psi - f)\xi^\epsilon$. Since $\xi^\epsilon \geq 0$, $(\psi - \tilde{f}^\epsilon)(z) \geq (\psi - f)(z_0)\xi^\epsilon(z)$. As $(\psi - f)(z_0) < 0$ and $\xi^\epsilon \leq 1$, $(\psi - f)(z_0)\xi^\epsilon(z) \geq (\psi - f)(z_0)$. This implies that $(\psi - \tilde{f}^\epsilon)(z) \geq (\psi - \tilde{f})(z_0)$, i.e. $\psi - \tilde{f}^\epsilon$ admits a global minimum at $z_0$. Therefore, $l_0 \tilde{f}^\epsilon(z_0) \leq 0$. Notice that by assumption $f(z_0) > \psi(z_0)$ so that, by continuity of $f$, we may choose $r \to 0$ and $\epsilon > 0$ small enough such that $\tilde{f}^\epsilon(z) \geq \psi(z) - \epsilon h(z)$ on $z \in D_h$ where $h$ is a continuous function we may choose arbitrarily. Actually, we choose $h$ on a set $D_h \ni z_0$ such that $h(z_0) = 1$. Precisely, if $z_0 > 1$, $h$ is defined on $D_h = [1, \infty)$ with $h = 0$ on $[1, (z_0 + 1)/2]$. If $z_0 < 1$, $h$ is defined on $D_h = [0, 1]$ and $h = 0$ on $[(z_0 + 1)/2, 1]$. Notice that the range of the mapping $\delta : t \mapsto (z_0 + z_0 t)/(1 + z_0 t)$ is $D_h$ when $1 + z_0 t > 0$ and $\lim_{t \to \infty} \delta(t) = 1$. Therefore, the integral

$$\int_{-1}^{\infty} (1 + z_0 t)^\gamma h \left(\frac{z_0 + z_0 t}{1 + z_0 t}\right) 1_{\{1 - z_0, z_0(1 + t)\}} \in K d\pi(t)$$
is well defined and finite as $h((z_0 + z_0 t)/(1 + z_0 t))$ vanishes when $(z_0 + z_0 t)/(1 + z_0 t)$ is close to 1, i.e. when $t$ is close to $\infty$. From the inequalities $l_0\tilde{f}^* (z_0) \leq 0$ and $\tilde{f} \geq \psi - \epsilon h$ on $z \in D_h$, we then deduce that $\tilde{l}_0 f(z_0) \leq 0$ as $\epsilon \to 0$, i.e. we have replaced $\tilde{f}$ by $\psi$ in the global operator.

Since $\psi$ is a viscosity subsolution of the equation $\max_{i=0,1,2} l_i \psi = 0$, for any test function dominating $\psi$, we have $\max_{i=0,1,2} l_i \phi \geq 0$. Moreover, $\max_{i=0,1,2} l_1 \psi < 0$ since $l_1$ and $l_2$ are local operators only depending on the values of $\psi$ and its first derivative at the considered point $z \in (a, b)$. It follows that $l_0 \phi = \max_{i=0,1,2} l_i \phi \geq 0$. As above, we deduce that $\psi$ is a weak viscosity subsolution of the equation $\tilde{l}_0 \psi = 0$ on $[a, b]$ hence $\psi$ is finally a weak viscosity solution to the equation $\tilde{l}_0 \psi = 0$ on $[a, b]$.

Let us define the local operator $\zeta(f, z) := \tilde{l}_0 f(z) - f_2(z) f''(z)$. Since $\pi(R) < \infty$, we deduce by Corollary 9.8 and Lemma 9.1 (ii) that $z \mapsto \zeta(\psi, z)$ is continuous on $(0, 1 + \lambda^{-1}]$. We deduce that $z \mapsto G(\psi'(z), \psi(z), z) := -\zeta(f, z)/f_2(z)$ is continuous. Recall that the operator which appears in the definition of $G$ is local as we have frozen the dependence in the test function $\psi$ before. Therefore, by the weak solution property and Lemma 9.11, $\psi \in C^2[a, b]$. \hfill $\Box$

The following result corresponds to [12, Proposition 8.5]. Its proof may be easily adapted to the case with jumps. To do so, we use the continuity of the function $z \rightarrow H(z, \psi, \psi')$ as stated by Lemma 9.1 (i).

**Proposition 9.13** Suppose that $\pi(R) < \infty$. The function $\psi$ is $C^2$ on the set $(z_1, z_2)\setminus\{1\}$ and satisfies the HJB equation $\psi = 0$ on this set in the classical sense.

The proposition above implies that the value function satisfies the HJB equation $\mathcal{L}_0(W) + U^*(W_x) = 0$ on $K_0 \setminus \mathcal{R}e_2$ in the classical sense and it is $C^2$ on this set. It remains to study $W$ on the set $\mathcal{R}e_2$. To do so, we follow the proof of in [12, Theorem 9.1].

**Proposition 9.14** The second derivative $W_{yy}$ is well defined and is continuous across $\mathcal{R} + e_2$. Moreover, $W$ satisfies the equation $\mathcal{L}_0(W) + U^*(W_x) = 0$ on $K_0$ in the classical sense.

Finally, we deduce the following:

**Corollary 9.15** Suppose that $\pi(R) < \infty$. Then, the value function $W$ is $C^2$.

**9.2 Optimal control**

The following important result provides an optimal policy for the optimization problem. The proof is based on the resolution of a Skorohod problem described in Appendix, see (10.61).

**Theorem 9.16** Suppose that the boundaries of $K_0$ are different from the x- and y-axes. Let $(x, y) \in \overline{K}_0$, then the Skorokhod problem (10.61) with $\sigma(V_t) := (-W_x(V_t)^{\frac{1}{\gamma-1}}, V_t)$ admits a unique solution. Moreover, the portfolio process $V$ participating in the solution of this problem is an optimal portfolio. An optimal strategy is given by the formula

\[
B_t = \int_0^t g(V_s) dk_s, \tag{9.36}
\]

\[
c_t = W_x^{\frac{1}{\gamma-1}}, \tag{9.37}
\]

where $W$ is the Bellman function.
Proof  Existence of a solution to the Skorohod problem holds by Theorem 10.12. Note that \( \theta^\pi = \infty \) since \( V^+_t \in K_0, \forall t, \) and \( W_x \) is positive [hence (9.37) makes sense]. We shall only consider the case where \( K_0 \) is included in the first quadrant. Otherwise, we refer to Remark 9.18. By the propositions 9.9 and 9.13, the function \( \psi \) coincides on the interval \((z_1, z_2)\) with a \( C^2\)-function \( \tilde{\psi} \) defined on \((-\lambda^{-1}, 1 + \lambda^{-1})\). Indeed, it suffices to replace \( \psi \) by suitable parabolic functions outside \((z_1, z_2)\). In particular, since \( \tilde{\psi}' \) is Lipschitz on \([z_1, z_2]\) and \( W(z) = (x + y)^\gamma \tilde{\psi}(z) \) where \( z = y/(x + y) \) on \( K_0 \), we also deduce that \( W' \) is also Lipschitz on \( K_0 \). Moreover, by [11, Lemma 4.7.1, Corollary 4.7.6], \( W \neq 0 \) implies that \( W'(z) \in K^* \setminus \{0\} \subset (0, \infty)^2 \) hence \( W^*_x(z) > 0 \) for all \( z \). It follows that \( \sigma \) is locally Lipschitz on the set \( K_0 \). We then deduce that the Skorokhod problem admits a unique solution. We check the second assertion. Applying Lemma 10.2, since the term \( R \) of the expansion is negative, we have

\[
e^{-\beta t} W(V^+_t) + J^\pi_t \leq W(x, y) + N_t + \int_0^t e^{-\beta u} \left[ \mathcal{L}_0 W(V_{u-}) - W'(V_{u-})c_u + U(c_u) \right] du.
\]

Moreover, when \( c \) is defined by (9.37), we have \( W_x(V_{u-})c_u + U(c_u) = U^*(c_u) \). Therefore, the last integral term in the equality above is zero by virtue of Proposition 9.14. It remains to prove that \( N \) is a martingale and

\[
\lim_{n \to \infty} \mathbb{E}e^{-\beta t_n} W(V^+_t) \to 0, \tag{9.38}
\]

for a sequence of real numbers \( t_n \to \infty \). To prove (9.38), we observe that \( |W(z)| \leq \kappa |z|^\gamma \) and \( |W_x(z)| \leq \kappa |z|^\gamma - 1 \) where \( \kappa \) is an upper bound of \( W \) and \( W_x \) on the intersection \( \Delta_0 \) of the set \( K_0 \) with the line \( x + y = 1 \). This is deduced from the continuity of \( W \) on \( \Delta_0 \) and the fact that \( \psi'(0+) < \infty \). We deduce the existence of a constant \( \kappa \) such that

\[
\mathbb{E} \int_0^\infty e^{-\beta t} W(V^+_t) dt \leq \kappa \mathbb{E} \int_0^\infty e^{-\beta t} |V^+_t|^\gamma dt \leq \kappa \mathbb{E} \int_0^\infty e^{-\beta t} u(c_t) dt \leq \kappa W(x) \tag{9.39}
\]

Since \( W \) is finite, this implies the existence of a sequence \( t_n \uparrow \infty \) such that (9.38) holds. Details of this assertion are given in Lemma 9.17. We now prove that \( N_t \) is a true martingale. Indeed, by a similar argument, we have

\[
|z W_y(z)| \leq \kappa |z|^\gamma \leq (1 + |z|), \quad z \in K_0.
\]

Hence, we infer that the stochastic process \( \int_0^\infty e^{-\beta u} W_y(V^2_{u-})V^2_{u-} \sigma dW_u \) is a martingale. The second process defining \( N \) is the integral with respect to the Lévy measure. Observe that, for each fixed \( s \), we have \( f(V_{s-}, z) = 1 \) (because \( V^+_s \in K_0, \forall s \)). Moreover, using the finite Taylor expansion, we get

\[
|W(\tilde{V}_{s-} + \text{diag}(\tilde{V}_{s-})z) - W(\tilde{V}_{s-})| \leq |W'(\eta)||\text{diag}(\tilde{V}_{s-})z|,
\]

where \( \eta \in [\tilde{V}_{s-}, \tilde{V}_{s-} + \text{diag}(\tilde{V}_{s-})z] \) satisfies \( |\eta| \leq |\tilde{V}_{s-}|(1 + |z|) \). It follows that

\[
|W(\tilde{V}_{s-} + \text{diag}(\tilde{V}_{s-})z) - W(\tilde{V}_{s-})| \leq \kappa |\eta|^\gamma - 1|\text{diag}(\tilde{V}_{s-})z| \leq \kappa |\tilde{V}_{s-}|^\gamma (1 + |z|)^{\gamma - 1}|z|,
\]

where the last inequality is deduced from the inequality \( |W'(\eta)| \leq \kappa |\eta|^\gamma - 1 \). We then obtain that

\[
|W(\tilde{V}_{s-} + \text{diag}(\tilde{V}_{s-})z) - W(\tilde{V}_{s-})| \leq \kappa |\tilde{V}_{s-}|^\gamma |z|.
\]
Therefore, as the Lévy process is of finite activity and (9.39) holds,
\[ \mathbb{E} \int_0^t \int_{\mathbb{R}} e^{-\mu u} |W(\tilde{V}_{s-} + \text{diag}(\tilde{V}_{s-})z)I(\tilde{V}_{s-}, z) - W(\tilde{V}_{s-})| \pi(dz)ds \leq \kappa W(x) < \infty. \]

By [10, Theorem I. 1.33 b., p. 73], we deduce that the purely discontinuous local martingale \( N \) satisfies \( \mathbb{E} \text{var}(N)_\infty < \infty \) hence is a martingale.

To complete the proof of the theorem, we need the following lemma

**Lemma 9.17** Suppose that \( \int_0^\infty X_u du < \infty \) where \( X_u = \mathbb{E} e^{-\beta u} W(V_u^+ ) \geq 0 \). Then \( \lim_{t \to \infty} X_t \to 0 \).

**Proof** Observe that \( e^{-\beta t} W(V_t^+) + J_1^\pi = W(x) + N_t \). With \( Y_u = e^{-\beta u} W(V_u^+ ) \), we get that
\[ \text{for } u \geq s, \quad Y_u - Y_s = N_u - N_s - (J_u^\pi - J_s^\pi) \]
where \( J_u^\pi - J_s^\pi \geq 0 \). Since \( N \) is a martingale, we deduce that \( X_u - X_s \leq 0 \), i.e. \( X \) is decreasing. Therefore, the integrability of \( \int_0^\infty X_u du \) ensures that \( \lim_{t \to \infty} X_t \to 0 \). Indeed, if \( \lim_{t \to \infty} X_t \to c > 0 \), then \( \int_0^\infty X_u du \geq \int_0^\infty cdu = \infty \), hence a contradiction.

**Remark 9.18** The situations where \( x \in K_i, i = 1, 2 \), are easily reduced to the one treated in the theorem above. Indeed, recall that the function \( W \) restricted on the set \( K_i \) is constant along the direction \( g_i, i = 1, 2 \). Instead of considering the initial position \( x \in K_i, i = 1, 2 \), we consider the point \( \tilde{x} \) lying on the boundary of \( K_0 \) by projecting \( x \) onto \( K_0 \) parallel to \( g_i \). This translation does not change the value of the Bellman function, meaning that \( W(x) = W(\tilde{x}) \). Therefore, the optimal strategy for \( x \) is constructed simply by adding the initial jump \( \Delta B_0 := \tilde{x} - x \) to the optimal strategy given by the Skorokhod problem with the initial point \( \tilde{x} \).

### 10 Appendix

#### 10.1 Proof of Theorem 4.6

The proof is based on preliminary results we formulate before. It is an adaptation of [6] to lâdlâg strategies by considering right limits of the wealth process. The proof of the following lemma is given in [6].

**Lemma 10.1** For every portfolio process \( V = V^\pi, \pi \in A_x \), the process \( I_{V_{-} \in \partial K} |\Delta Y| I_{[0, \theta^\pi]} \) is indistinguishable from zero.

Let us define for each \( n \) the compact set
\[
K_n := \{ x \in K : |x| \leq n \} \cap \left\{ x \in K : d(x, \partial K) \geq \frac{1}{n} \right\}.
\]

Note that \( (K_n)_{n \geq 1} \) is an increasing sequence whose union is \( \text{int} \ K \). For every control \( \pi = (B, C) \in A_x \) and \( x \in \text{int} \ K \), we define \( V_{\theta^\pi}^\pi \) as the stopped portfolio process.
\[
V_{\theta^\pi}^\pi = V_{\pi, x, \theta^\pi} = V_{\pi, x} I_{[0, \theta^\pi]} + V_{\pi, x}^\pi I_{[\theta^\pi, \infty]},
\]
where \( \theta^\pi \) is the first instant when the portfolio exits \( K_n \). We also define \( B_{\theta^\pi}^\pi \) similarly. Note that the value of \( V_{\theta^\pi}^\pi \) may get out of \( \text{int} \ K \) due to a possible jump of the Lévy process at
\(\theta^n\) but, in this case, \(V^{\pi,x}_{\theta_n}\) \(\notin \partial K\) by virtue of Lemma 10.1. From the dynamics of \(V^{\pi,x}\), we deduce that
\[
\begin{align*}
V^{\theta_n}_{t+} &= x + \int_0^t I_{[0,\theta^n]}(s) \text{diag} \left( V^{\theta_n}_{s-} \right) \left( \mu_s ds + \Xi dW_s \right) + B^\theta_{t+} - C^\theta_t, \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \text{diag} \left( V^{\theta_n}_{s-} \right) z I_{[0,\theta^n]}(s) \left( p(dz, ds) - q(dz, ds) \right).
\end{align*}
\]
We propose to study the quantity
\[
\tilde{V}^f_n := e^{-\beta t} f(V^\theta_{t+}) I(V^\theta_{t-}, \Delta Y_{t+\theta^n}) + J^\pi_t,
\]
where, we recall that \(I(v, z) = I_{v+\text{diag}(v)z} \in \text{int} K\). We set \(f(V^\theta_{t+}) I(V^\theta_{t-}, \Delta Y_{t+\theta^n}) = 0\) by convention when \(I(V^\theta_{t-}, \Delta Y_{t+\theta^n}) = 0\) so that \(\tilde{V}^f_n\) is well defined even if \(f\) is only defined on \(K\). If \(K_n\) is replaced by \(K\) we write \(\tilde{V}^f_t\) instead of \(\tilde{V}^f_n\). We also introduce
\[
\tilde{V}^n_{t+} := V_{t+} I_{[0,\theta^n]}(t) + V^\theta_{t+} I_{[\theta^n, \infty]}(t).
\]
We have the following key result:

**Lemma 10.2** (Itô expansion) Let \(f \in C_1(K) \cap C^2(\text{int} K)\) be an increasing function with respect to the order \(\preceq_K\). Then, we have
\[
\tilde{V}^f_n = f(x) + N_{t+\theta^n} + R_{t+\theta^n}
\]
\[
+ \int_0^t e^{-\beta u} \left[ \mathcal{L}_0 f(\tilde{V}^n_{u-}) - f'(\tilde{V}^n_{u-})c_u + U(c_u) \right] du,
\]
where \(N\) is a local martingale and \(R\) is a decreasing process such that \(R_0 = 0\).

**Proof** Note that we do not assume any regularity of \(f\) on \(\partial K\). Therefore, we can not directly apply the Itô formula to \(\tilde{V}^f_n\). To overcome this difficulty, instead of considering \(V^\theta_{t+}\), we study the process \(V^\theta_t\) defined by
\[
V^\theta_t := V_{t+} I_{[0,\theta^n]} + V^\theta_{t+} I_{[\theta^n, \infty]}.
\]
We also have a representation for \(V^\theta_{t+}\):
\[
V^\theta_{t+} = x + \int_0^t I_{[0,\theta^n]}(s) \text{diag} \left( V^\theta_{s-} \right) \left( \mu_s ds + \Xi dW_s \right) + B^\theta_{t+} - C^\theta_t
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \text{diag} \left( V^\theta_{s-} \right) z I_{[0,\theta^n]}(s) \left( p(dz, ds) - q(dz, ds) \right).
\]
For a sake of simplicity, we write \(\tilde{V}_t := V^\theta_{t+} = V^\theta_t + \Delta^+ B^\theta_{t+}\). The Itô formula applied to the process \(e^{-\beta t} f(\tilde{V}_t) := e^{-\beta t} f(V^\theta_{t+})\) yields
\[
e^{-\beta t} f(\tilde{V}_t) = f(x) + \int_0^t e^{-\beta u} f'(\tilde{V}_{u-}) d\tilde{V}_u - \beta \int_0^t e^{-\beta u} f(\tilde{V}_{u-}) I_{[0,\theta^n]}(s) du
\]
\[
+ \frac{1}{2} \int_0^t e^{-\beta s} \text{Tr} A
\frac{\partial}{\partial z} f(\tilde{V}_{s-}) I_{[0,\theta^n]}(s) ds
\]
\[
+ \sum_{s \leq t} e^{-\beta s} \left[ f(\tilde{V}_s) - f(\tilde{V}_{s-}) - f'(\tilde{V}_{s-})(\tilde{V}_s - \tilde{V}_{s-}) \right].
\]
Using (10.42) and the fact that $B_t^\theta + \Delta^+ B_t^\theta = B_{t+1}^1(\theta^n) + B_{t-\theta-1}^\theta$, we deduce that

\[
\int_0^t e^{-\beta u} f'(\tilde{V}_{u-})d\tilde{V}_u = \int_0^t e^{-\beta u} f'(\tilde{V}_{u-}) I_{(0,\theta^n)}(u) \text{diag}(\tilde{V}_{u-})(\mu u du + \mathcal{Z} dW_u) + \int_0^t e^{-\beta u} f'(\tilde{V}_{u-}) B_u^c + \int_0^t e^{-\beta u} f'(\tilde{V}_{u-}) c_u du
\]

\[
+ \sum_{s \leq t} e^{-\beta s} I_{(0,\theta^n)}(s) f'(\tilde{V}_s) (\Delta^+ B_s + \Delta B_s)
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} e^{-\beta u} f'(\tilde{V}_{u-}) \text{diag}(\tilde{V}_{u-}) z I_{(0,\theta^n)}(s) (p(dz, ds) - q(dz, ds)).
\]

Note that $\tilde{V}_s = \tilde{V}_{s_-} + \text{diag} \tilde{V}_{s_-} \Delta Y_s + \Delta^+ B_s + \Delta B_s$. We then rewrite

\[
\sum_{s \leq t} e^{-\beta s} \left[ f(\tilde{V}_s) - f(\tilde{V}_{s_-}) - f'(\tilde{V}_{s_-}) (\tilde{V}_s - \tilde{V}_{s_-}) \right] = \sum_{s \leq t} e^{-\beta s} \left[ f(\tilde{V}_s) - f(\tilde{V}_{s_-} + \text{diag} \tilde{V}_{s_-} \Delta Y_s) \right] I_{(0,\theta^n)}(s)
\]

\[
+ \sum_{s \leq t} e^{-\beta s} \left[ f(\tilde{V}_{s_-} + \text{diag}(\tilde{V}_{s_-}) \Delta Y_s) - f(\tilde{V}_{s_-}) - f'(\tilde{V}_{s_-}) \text{diag}(\tilde{V}_{s_-}) \Delta Y_s \right] I_{(0,\theta^n)}(s)
\]

\[
- \sum_{s \leq t} e^{-\beta s} f'(\tilde{V}_{s_-}) (\Delta^+ B_s + \Delta B_s) I_{(0,\theta^n)}(s).
\]

Moreover,

\[
\sum_{s \leq t} e^{-\beta s} \left[ f(\tilde{V}_{s_-} + \text{diag}(\tilde{V}_{s_-}) \Delta Y_s) - f(\tilde{V}_{s_-}) - f'(\tilde{V}_{s_-}) \text{diag}(\tilde{V}_{s_-}) \Delta Y_s \right] I_{(0,\theta^n)}(s)
\]

\[
= \int_0^t \int_{\mathbb{R}^d} e^{-\beta s} \left[ f(\tilde{V}_{s_-} + \text{diag}(\tilde{V}_{s_-}) z) - f(\tilde{V}_{s_-}) - f'(\tilde{V}_{s_-}) \text{diag}(\tilde{V}_{s_-}) z \right] I_{(0,\theta^n)}(s) p(dz, ds).
\]

Since $I(\tilde{V}_{s_-}, \Delta Y_s) = 1$ for $s < \theta^n$, we may omit the indicator $I$ within the operator $H$ for $s < \theta^n$. We deduce that

\[
e^{-\beta t} f(\tilde{V}_t) = f(x) + \tilde{N}_{t \wedge \theta^n} + \tilde{R}_{t \wedge \theta^n}
\]

\[
+ \int_0^{t \wedge \theta^n} e^{-\beta u} \left[ \mathcal{L}_0 f(\tilde{V}_{u-}) - f'(\tilde{V}_{u-}) c_u \right] du,
\]

where

\[
\tilde{N}_t = \int_0^t e^{-\beta u} f'(\tilde{V}_{u-}) \text{diag}(\tilde{V}_{u-}) \mathcal{Z} I_{(0,\theta^n)}(u) dW_u
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} e^{-\beta u} \left[ f(\tilde{V}_{u-} + \text{diag}(\tilde{V}_{u-}) z) - f(\tilde{V}_{u-}) \right] I_{(0,\theta^n)}(u) (p(dz, du) - q(dz, du)).
\]

The residual term is

\[
\tilde{R}_t := \sum_{s \leq t} e^{-\beta s} I_{(0,\theta^n)}(s) \left[ f(\tilde{V}_{s_-} + \text{diag}(\tilde{V}_{s_-}) \Delta Y_s + \Delta B_s^+ + \Delta B_s) - f(\tilde{V}_{s_-} + \text{diag}(\tilde{V}_{s_-}) \Delta Y_s) \right] + \int_0^t e^{-\beta u} f'(\tilde{V}_{u-}) I_{(0,\theta^n)}(u) dB_u^c.
\]
The process \( \tilde{R} \) is decreasing due to the monotonicity of \( f \) with respect to \( K \). Finally, observe that

\[
\tilde{X}^{f,n}_t - e^{-\beta t} f(\tilde{V}_t) = J^{\pi}_t + e^{-\beta^n} \left[ f(V_{0^n}^+) I(V_{0^n}^-, \Delta Y_{0^n}) - f(V_{0^n}^-) \right] 1_{t=0^n},
\]

\[
= J^{\pi}_t + e^{-\beta^n} \left[ f(V_{0^n}^+) - f(V_{0^n}^+ + \text{diag}(V_{0^n}) \Delta Y_{0^n}) \right] I(V_{0^n}^-, \Delta Y_{0^n}) 1_{t=0^n} + e^{-\beta^n} \left[ f(V_{0^n}^+ + \text{diag}(V_{0^n}) \Delta Y_{0^n}) I(V_{0^n}^-, \Delta Y_{0^n}) - f(V_{0^n}^-) \right] 1_{t=0^n}.
\]

(10.47)

Plugging the r.h.s of (10.47) into (10.43), as the Lesbesgue measure \( dt \) does not charge any theorem, we conclude the theorem with

\[
\tilde{N}_t = \int_0^t e^{-\beta u} f'(V_{u-}) \text{diag}(V_{u-}) \mathbb{E} I_{[0,\theta^n]}(u) dW_u \\
+ \int_0^t \int_{\mathbb{R}^d} e^{-\beta u} \left[ f(V_{u-} + \text{diag}(V_{u-}) z) I(V_{u-}, \Delta Y_{u}) - f(V_{u-}) \right] I_{[0,\theta^n]}(u) (p(dz, d\nu) - q(dz, d\nu)),
\]

and

\[
\tilde{R}_t := \sum_{s \leq t} e^{-\beta s} I_{[0,\theta^n]}(s) \left[ f(V_{s-} + \text{diag}(V_{s-}) \Delta Y_{s}) + \Delta^+ B_s + \Delta B_s \right] - f(V_{s-} + \text{diag}(V_{s-}) \Delta Y_{s}) I(V_{s-}, \Delta Y_s) + \int_0^t e^{-\beta u} f'(V_{u-}) I_{[0,\theta^n]}(s) dB^c_s.
\]

(10.49)

\[\square\]

**Remark 10.3** If the function \( f \in C^2(\mathbb{R}^d) \), we may directly apply the Ito formula to \( X^{f}_t := e^{-\beta f}(V_{i+}) + J^{\pi}_t \) and we obtain the following result:

\[
X^{f}_t = f(x) + N_t + R_t + \int_0^{t \wedge \theta} e^{-\beta u} \left[ \mathcal{L}_0 f(V_{u-}) - f'(V_{u-}) c_u + U(c_u) \right] du,
\]

where \( N \) is a local martingale and \( R \) is a decreasing process with \( R_0 = 0 \). Moreover, \( X^{f}_t = \tilde{X}^{f}_t \) if \( f \) vanishes out of int \( K \).

We now formulate a strict supersolution property which is the key point to deduce that \( W \) is a global viscosity subsolution of the HJB equation by using the dynamic programming principle. We fix a closed ball such that \( \overline{O}_r(x) \subseteq O_{2r}(x) \subseteq \text{int} K \) and we define the stopping time \( \tau^{\pi} := \tau^{\pi}_r \) as the exit time of \( V^{\pi,x} \) from \( O_r(x) \), i.e.

\[
\tau^{\pi} := \inf \{ t \geq 0 : |V_t^{\pi,x} - x| \geq r \}.
\]

(10.50)

**Lemma 10.4** Let \( f \in C^2(K) \cap C_1(K) \) be such that \( \mathcal{L} f \leq -\varepsilon \leq 0 \) on \( \overline{O}_r(x) \). There exists a constant \( \eta := \eta(\varepsilon, r) \) and an interval \( (0, t_0) \) such that

\[
\sup_{\tau \in A_x} E \tilde{X}^{f,n}_{t \wedge \tau} \leq f(x) - \eta t, \quad t \in (0, t_0),
\]

(10.51)

where \( t \mapsto \tilde{X}^{f,n}_{t \wedge \tau} \) is defined by (10.41).
Proof We fix a strategy \( \pi \) and omit its symbol in the notations below. Observe that only the behaviour of the processes we consider does matter on \([0, \tau] \). For \( n \) large enough, we have \( O_{2^n}(x) \subseteq \text{int} \, K_n \). Therefore, \( \tau^n \leq \theta^n \) hence we may apply Lemma 10.2 so that

\[
\tilde{X}_{t \wedge \tau}^{f,n} = f(x) + N_{t \wedge \tau} + \tilde{R}_{t \wedge \tau} + \int_{0}^{t \wedge \tau} e^{-\beta u} \left[ \mathcal{L}_0 f(V_{u-}) + U^*(V_{u-}) \right] du
\]

where \( \tilde{R} \) is a decreasing process such that \( \tilde{R}_0 = 0 \) and \( N \) is a local martingale. As it is shown in [6, Lemma 8.3], the stopped process \( N_{\tau} \) is a martingale hence \( \mathbb{E} N_{t \wedge \tau} = 0 \).

By assumption, \( \mathcal{L} f(y) \leq -\varepsilon \) for all \( y \in \overline{G} \) and so \( \Sigma_G (f(y)) \leq -\varepsilon \) on \([0, \tau] \). It follows that \( f'(y) k \leq -\varepsilon |k| \) whatever \( k \in -K \) so that \( f'(\overline{G})(x) \subseteq \text{int} \, K^* \) on \([0, \tau] \). In particular, for \( s \in [0, \tau] \), \( f'(V_{s-}) \tilde{B}_s^c \leq -\varepsilon |\tilde{B}_s^c| \). We deduce that the following term above [appearing in the expression (10.49) of \( R \)] is bounded as follows:

\[
\int_{0}^{t \wedge \tau} e^{-\beta u} f'(V_{u-}) I_{u \leq \theta} \tilde{B}_u^c d\|B^c\|_u \leq -\varepsilon \int_{0}^{t \wedge \tau} e^{-\beta u} I_{u \leq \theta} \tilde{B}_u^c |d\|B^c\|_u.
\]

On the other hand, the other terms defining \( R \) in (10.49) can be estimated as follows:

\[
\begin{align*}
&f \left( V_{s-} + \text{diag}(V_{s-}) \Delta Y_s + \Delta B^+_s + \Delta B_s \right) - f(V_{s-} + \text{diag}(V_{s-}) \Delta Y_s) \\
= &\left( f(V_{s+}) - f(V_{s-}) \right) 1_{\Delta B_s = 0} + \left( f(V_{s+}) - f(V_{s-}) \right) 1_{\Delta B_s \neq 0} \\
= &f'(\gamma_s) (\Delta B_s + \Delta^+ B_s),
\end{align*}
\]

where \( \gamma_s \in [V_{s-}, V_{s+}] \) on the set \( \{ \Delta B_s = 0 \} \) and \( \gamma_s \in [V_{s-}, V_{s+}] \) on the set \( \{ \Delta B_s \neq 0 \} \). Observe that \( \gamma_s \in \overline{G}(x) \) if \( s < \tau \). If \( s = \tau \) we may assume without loss of generality that the controls \( \Delta B^+_s = \Delta B_s = 0 \) hence \( V_s = V_{s+} \) as we consider the supremum given in (10.51).

Therefore, we have

\[
\begin{align*}
&\left[ f \left( V_{s-} + \text{diag}(V_{s-}) \Delta Y_s + \Delta B^+_s + \Delta B_s \right) - f(V_{s-} + \text{diag}(V_{s-}) \Delta Y_s) \right] I_{s \leq \tau} \\
\leq &-\varepsilon \left( |\Delta B^+_s| + |\Delta B_s| \right) I_{s \leq \tau}.
\end{align*}
\]

Therefore, by equality (10.52), we deduce that

\[
\mathbb{E} \tilde{X}_{t \wedge \tau}^{f,n} \leq f(x) - \varepsilon e^{-\beta t} (t \wedge \tau) - e^{-\beta t} \mathbb{E} Z_t,
\]

\( \diamondsuit \) Springer
where
\[
Z_t := \int_0^{t \wedge \tau} r(c_s, f'(V_{s-}))ds + \epsilon \int_0^{t \wedge \tau} |\dot{B}^c_s|d\|B^c\|_3 + \epsilon \sum_{s \leq t \wedge \tau} (|\Delta B^c_s| + |\Delta B_s|),
\]
r(c, p) := U^*(p) + pc - U(c).

Recall that \(U^*(p) = \sup_{x \in C}(U(x) - px) \geq 0\) since \(U(0) = 0\). Moreover, by assumption,
\[
\inf_{p \in f'(\overline{C}(x)), c \in C, |c| = 1} pc \geq \epsilon.
\]
Since \(U(c)/|c| \to 0\) as \(|c| \to \infty\), we finally deduce that there exists a constant \(\kappa > 1\) such that
\[
\inf_{p \in f'(\overline{C}(x))} r(c, p) \geq \kappa^{-1}|c|, \quad \forall c \in C, \quad |c| \leq \kappa.
\]
Therefore,
\[
\int_0^{t \wedge \tau} r(c_s, f'(V_{s-}))ds \geq \kappa^{-1} \int_0^{t \wedge \tau} I_{|c_s| \geq \kappa}|c_s|ds.
\]
Moreover, the second integral defining \(Z\) dominates \(\kappa_1 \|B^c\|_{1 \wedge \tau}\) for some \(\kappa_1 > 0\). Indeed, recall that all norms in \(\mathbb{R}^d\) are equivalent, in particular \(c^{-1}|.| \leq |.|_1 \leq c|.|\) for some \(c > 0\) where \(|x|_1 := \sum_{i=1}^d |x_i|\) and \(|.|\) is the Euclidean norm. It follows that
\[
c^{-1}\|B^c\| \leq \operatorname{Var} B^c \leq c\|B^c\|,
\]
where \(\operatorname{Var} B^c\) is the total variation of \(B^c\) with respect to \(|.|_1\). At last, we have:
\[
|\dot{B}^c|_1 = \sum_{i=1}^d |\dot{B}^c_i| = \sum_{i=1}^d \left|\frac{dB^c_i}{d\|B^c\|}\right| = \sum_{i=1}^d \left|\frac{dB^c_i}{d\operatorname{Var} B^c}\right| = \frac{d\operatorname{Var} B^c}{d\|B^c\|}.
\]
The claimed property follows. We deduce some constant \(\gamma > 0\) such that
\[
\mathbb{E}X_{t \wedge \tau}^f \leq f(x) - e^{-\beta t} \gamma^{-1}\mathbb{E}\tilde{Z}_t,
\]
where
\[
\tilde{Z}_t := t \wedge \tau + \int_0^{t \wedge \tau} I_{|c_s| \geq \kappa}|c_s|ds + \|B\|_{1 \wedge \tau+}.
\]
Observe that
\[
\tilde{Z}_t \geq t \wedge \tau + \|B\|_{1 \wedge \tau+} - \int_0^{t \wedge \tau} I_{|c_s| \leq \kappa}|c_s|ds \geq (1 - \kappa)t \wedge \tau + \|B\|_{1 \wedge \tau+}.
\]
Using the stochastic representation of \(V\), we immediately get the existence of a number \(t_0 > 0\) and a measurable set \(\Gamma\) with \(\mathbb{P}(\Gamma) > 0\) on which
\[
|V_t^{\pi, x} - x| \leq r/2 + \delta \|B\|_{t+}, \quad t \in [0, t_0],
\]
whatever the control \(\pi = (B, C)\). Diminishing \(t_0\), we may assume without loss of generality that \(\kappa t_0 \leq r/(4\delta)\). For any \(t \leq t_0\), the inequality \(\|B\|_{t+} \geq r/(2\delta)\) holds on the set \(\Gamma \cap \{\tau \leq t\}\). Therefore, if \(t \leq t_0\),
\[
\tilde{Z}_t \geq (1 - \kappa)t + 2\kappa t_0 \geq \kappa t_0 \geq t.
\]
On the set $\Gamma \cap \{ \tau > t \}$, the inequality $\tilde{Z}_t \geq t$ obviously holds. Thus, $E\tilde{Z}_t \geq t^p(\Gamma)$ if $t \in [0, t_0]$ and the result is proven. \hfill \Box

Observe that, if $n$ is large enough, $\tau^\pi \leq \theta^n$ hence $X_{t \wedge \tau^\pi}$ does not depend on $n$.

**Lemma 10.5** Suppose that $W$ is continuous on int $K$. Let $T_f$ be the set of finite stopping times. Then,

$$W(x) \leq \sup_{\pi \in A_x} \inf_{\tau \in T_f} E \left( J_{\tau}^\pi + e^{-\beta \tau} W(V_{\tau +}^{x, \pi}) I_{\tau < \theta} \right).$$

(10.53)

**Proof** This lemma is similar to [6, Lemma 9.1]. The only difference is that we use the right-continuous version of the portfolio process $V_+$ instead of $V$. Moreover, the proof does not depend on the structure imposed on the controls $B$. \hfill \Box

**Lemma 10.6** Suppose that $W$ is continuous on int $K$. Then, for any stopping time $\tau \in T_f$, we have:

$$W(x) \geq \sup_{\pi \in A_x} E \left( J_{\tau}^\pi (x) + e^{-\beta \tau} W(V_{\tau +}^{x, \pi}) I_{\tau < \theta} \right).$$

**Proof** The proof is an adaptation of the proof of [6, Lemma 9.2] but we replace $\rho$ by $\rho := \inf\{j \geq 1 : V_{t}^{x, \pi} \in O_j\}$. \hfill \Box

**Proof of Theorem 4.6** (i) We adapt the proof of [6, Lemma 10.2] since the arguments of the proof are based on a strategy $\pi = (B, C)$ such that $B = 0$ and, in our case, we need Lemma 10.6 to replace [6, Lemma 9.2]). In that case, $V_+ = V_t$ for all $t \geq 0$ and the Itô formula is valid as observed in Remark 10.3 but also in [6].

(ii) Let $x \in \text{int} K$ and $\phi \in C_1(K) \cap C^2(K)$ be a function with $\phi(x) = W(x)$ and $W \leq \phi$ on $K$. Suppose that $\phi$ is not a subsolution, i.e. there exists $x \in \text{int} K$ such that the required inequality fails. Precisely, by continuity, suppose that $\mathcal{L}\phi \leq -\epsilon$, $\epsilon > 0$, on a neighborhood $\overline{O}_r(x) \subseteq \text{int} K$ of $x$. By virtue of Lemma 10.4, there exists a constant $\eta := \eta(\epsilon)$ and an interval $(0, t_0)$ such that

$$\sup_{\pi \in A_x} E \left( J_{\tau^\pi}^\pi + e^{-\beta (t \wedge \tau^\pi)} \phi(V_{t \wedge \tau^\pi}^{x, \pi}, \Delta Y_{t \wedge \tau^\pi}) \right) \leq \phi(x) - \eta t, \quad t \in (0, t_0],$$

(10.54)

where $\tau^\pi := \tau^\pi_r$ is given by (10.50). We may assume w.l.o.g. that $r = r'$. Fix an arbitrary $t \in (0, t_0]$. Applying Lemma 10.5, we deduce that there exists $\pi \in A_x$ such that

$$W(x) \leq \inf_{\tau \in T_f} E \left( J_{\tau}^\pi + e^{-\beta (t \wedge \tau)} W(V_{\tau +}^{x, \pi}) I_{\tau < \theta} \right) + \frac{1}{2} \eta t.$$ (10.55)

As $W \leq \phi$ and, since $I(V_{\tau +}^{x, \pi}, \Delta Y_{\tau +}^\pi) = 0$, we get that $I(V_{\tau +}^{x, \pi} \wedge \theta) = 0$ and we obtain from above that $W(x) \leq \phi(x) - \frac{1}{2} \eta t$, in contradiction since $W(x) = \phi(x)$. \hfill \Box

### 10.2 Proof of Theorem 4.9

Uniqueness is an immediate consequence of Theorem 10.7 below. Moreover, Theorem 10.8 ensures that the unique solution $W$ is concave. Notice that Proposition 7.16 ensures the existence of a Lyapunov function under mild assumptions.
Theorem 10.7 (Maximum principle) Suppose that there exists a Lyapunov function $\ell$ and $\Pi$ satisfies Condition $\Pi'$\textsuperscript{0}. Let $\tilde{K}$ be a nonempty subset of $K$ with nonempty interior. Let $W$ be a continuous viscosity subsolution of the Dirichlet problem (4.12) on $\tilde{K}$ satisfying the growth condition

$$ |W(x)|/\ell(x) \to 0, \quad |x| \to \infty. \quad (10.56) $$

Let $\tilde{W}$ be a continuous viscosity supersolution of the Dirichlet problem (4.12) on $\tilde{K}$ such that $\tilde{W} \geq W$ on $\partial \tilde{K}$. Suppose that either $\tilde{W} \geq 0$ or $\tilde{W}$ satisfies (10.56). Then, $\tilde{W} \geq W$ on $\tilde{K}$.

Proof The proof is an adaptation of the proof of [6, Theorem 11.2] but we consider the supremum of $\Delta_n$ on the set $\tilde{K}$. \hfill $\Box$

Theorem 10.8 If the HJB equation (4.12) admits a unique global viscosity solution in $C_0(K)$, then the Bellman function is concave.

Proof Consider the (modified) Bellman function $\hat{W}$ only defined by the (non empty) class of admissible strategies generating the portfolio processes evolving in $(\text{int } K) \cup \{0\}$ on $R_+$, i.e.$^4$

$$ \hat{\mathcal{A}}_x := \{ \pi \in \mathcal{A} : V^\pi_t \in (\text{int } K) \cup \{0\}, \quad \forall t \geq 0 \} \neq \emptyset, $$

and

$$ \hat{W}(x) := \sup_{\pi \in \hat{\mathcal{A}}_x} E J_{\infty}^\pi(x), \quad x \in \text{int } K. $$

We also consider the corresponding stopping times

$$ \theta = \theta^{x,\pi} := \inf \{ t : V^{x,\pi}_t \notin \text{int } K \} $$

for $\pi \in \hat{\mathcal{A}}_x$ and we have $V^\pi_t = 0$ for $t \geq \theta$. Therefore, the consumption strategy $c$ is zero after $\theta$. We deduce that the Bellman function $\hat{W}$ is a global viscosity solution to the same HJB equation as $W$. Indeed, we use Lemma 10.5 or equivalently [6, Lemma 9.1] and we adapt [6, Lemma 9.2], where we replace the random variable $\rho$ in the proof of [6, Lemma 9.2] by $\rho := \inf \{ j \geq 1 : V^{x,\pi}_{\tau+j} \in \mathcal{O}_j \}$ and the strategy $\tilde{\pi}$ is replaced by

$$ \tilde{\pi} := \pi I_{(0,\tau_k]} + \sum_{n=1}^{\infty} \{(y_n - V^{x,\pi}_{\tau_k}, 0) + \tilde{\pi}^{n,k}\} I_{[\tau_k,\infty)} I_{(\rho=n)} I_{\{V^{x,\pi}_{\tau_k}-y_n \in K\}} I_{[\tau_k,\theta)}. $$

For $t < \tau_k$, it is clear that $V^{x,\pi}_{\tau_k} \in (\text{int } K) \cup \{0\}$ as $\pi \in \hat{\mathcal{A}}_x$. Moreover, on $\{ \rho = n \}$, $V^{x,\pi}_{\tau_k} = y_n \in \text{int } K$ provided that $V^{x,\pi}_{\tau_k} - y_n \in K$ and $\tau_k < \theta$. Otherwise, $V^{x,\pi}_{\tau_k} = V^{x,\pi}_{\tau_k} = 0$ by Lemma 10.1. It follows that $\tilde{\pi} \in \hat{\mathcal{A}}_x$ and we may conclude as in [6, Lemma 9.2] since $u \geq 0$.

By assumption, the global viscosity solution of this HJB equation is unique hence $\hat{W} = W$. It is well known that the function $\hat{W}$ is concave. This is proven by Akian et al. [1] as mentioned by Framstad et al. [8]. Therefore, $W$ is also concave. \hfill $\Box$

10.3 Skorokhod problem for continuous diffusion processes

The construction of the optimal control for the two-dimensional optimal consumption problem we consider in Sect. 9 is based on the resolution of the so-called Skorokhod problem.

$^4$ Observe that $\hat{\mathcal{A}}_x \neq \emptyset$. Indeed, rebalance the portfolio starting from $x \in \text{int } K$ so that $V^+_x = 0$. 
This problem is about existence and uniqueness of the solution to a S.D.E. with reflection. We first recall some known results for the continuous diffusion case. We provide the proof for the sake of completeness. In the next section, we extend these results to finite activity pure-jumps Lévy processes.

Let \( \gamma : \partial K_0 \mapsto \mathbb{R}^2 \) be a vector-valued function with \( g(x) = -g_i \) on \((\partial K_0 \cap \partial K_i)\setminus\{0\}\) and \( \gamma(0) = 0 \). Let \( Y \) be the process \( Y_t = (Y^1_0, Y^2_0) + (t, W_t), \ t \geq 0 \), where \( W \) is a standard Brownian motion. Let \( \sigma = \mathbb{R}^2 \mapsto \mathbb{R}^2 \times \mathbb{R}^2 \) be a matrix-valued function which is Lipschitz-continuous.

On the closed cone \( \bar{K}_0 \) of Sect. 9, we consider the Skorokhod problem formulated as follows: find a pair of adapted continuous processes \( V \in \mathbb{R}^2 \) and \( k \in \mathbb{R} \), starting respectively from \( x \in \bar{K}_0 \) and zero, such that \( k \) is non decreasing and

\[
\begin{align*}
    dV_t &= \sigma(V_t)dY_t + \gamma(V_t)dk_t, \\
    dk_t &= I_{V_t \in \partial K_0}dk_t, \\
    V_t &= K_0, \quad \forall t \geq 0.
\end{align*}
\] (10.57)

For any compact subset \( C \) of \( \mathbb{R}^2 \) and for all \( \epsilon > 0 \), let us define

\[ C_\epsilon := \bigcup_{c \in C} B(c, \epsilon), \] (10.60)

where \( B(c, \epsilon) \) is the closed ball of radius \( \epsilon \) and center \( c \). Recall the well known result:

**Lemma 10.9** For any compact subset \( C \) of \( \mathbb{R}^2 \) and for all \( \epsilon > 0 \), there exists \( \chi^\epsilon \in C^\infty(\mathbb{R}^2, [0, 1]) \) such that \( \chi^\epsilon = 1 \) on \( C_\epsilon \) and \( \chi^\epsilon \) vanishes outside \( C_{2\epsilon} \).

The aim of this section is to show that the R.S.D.E (10.57) admits a solution on the set \( K_0 \) which is trapped at zero. To do so, we shall prove several intermediate lemmas. The main proof is based on the existence of a solution to a R.S.D.E. on a bounded domain \( G \) if the direction of the reflection is given by a function \( \gamma \in C^2 \) satisfying the following condition (see [7]):

\( C_1 \): \( \gamma \in C^2(\mathbb{R}^2, \mathbb{R}^2) \) and there is \( b \in (0, 1) \) such that

\[
\bigcup_{0 \leq t \leq b} B(x - t\gamma(x), t) \subseteq G^c := \mathbb{R}^2 \setminus G, \quad \text{for all } x \in \partial G.
\]

**Theorem 10.10** The Skorokhod problem (10.57) admits a solution which is trapped at zero.

**Proof** Let us introduce the polygons

\[
K^n_0 := K_0 \cap \{(x, y) : \epsilon_{n-1} \leq x + dy \leq \epsilon_n\},
\]

where \( \epsilon_n \to \infty \). Let \( D \) be the bisector of the cone \( K^n_0 \) and \( d > 0 \) such that \( D := \{(x, y) : -dx + y = 0\} \). Let \( x \in \partial K^n_0 \) be a starting point. The case \( x = 0 \) being trivial, we assume that \( x \neq 0 \) hence \( x \in K^n_0 \) if \( n \) is large enough.

**Step 1** There exists closed regions \( \tilde{K}^n_0 \) such that \( K^n_0 \subseteq \tilde{K}^n_0 \subseteq K^{n+1}_0 \) which satisfy condition \( C_1 \) for some reflection function \( \gamma_n \) such that \( \gamma_n(x) \to \gamma(x) \) for all \( x \in \partial K_0 \). Indeed, we denote by \( a_n \) and \( b_n \) the two points of \( \partial K_0 \cap \{(x, y) : \epsilon_n = x + dy\} \) such that \( y_{a_n} > y_{b_n} \). Observe that \( b_n \) is the symmetric of \( a_n \) with respect to the bisector \( D \). Similarly, \( c_n \) and \( d_n \) are the two symmetric points of \( \partial K_0 \cap \{(x, y) : \epsilon_{n-1} = x + dy\} \). We then define \( \tilde{K}^n_0 \) as the polygon

\[
\tilde{K}^n_0 := K_0 \cap \{(x, y) : (\epsilon_{n-1} + \epsilon_{n-1}^{-1})/2 \leq x + dy \leq (\epsilon_n + \epsilon_{n+1})/2\},
\]

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and denote by $\tilde{a}_n$ and $\tilde{b}_n$ the two points of $\partial K_0 \cap \{(x, y) : (\epsilon_n + \epsilon_n+1)/2 = x + dy \}$ such that $\epsilon_n > \epsilon_n+1$. Similarly, $\bar{c}_n$ and $\bar{a}_n$ are the two points of $\partial K_0 \cap \{(x, y) : (\epsilon_n-1+\epsilon_n+1)/2 = x + dy \}$ such that $\epsilon_n > \epsilon_n-1$.

Let $\eta_1$ be the outward normal to $\partial K_0 \cap K_1$ and $\eta_2$ be the outward normal to $\partial K_0 \cap K_2$. We consider a unit vector $g_3$ such that $g_3 \eta_1 > 0$ and $g_3(1, d) > 0$. Similarly, we define $g_4$ as a unit vector such that $g_4 \eta_2 > 0$ and $g_4(1, d) > 0$, $g_5$ is a unit vector such that $g_5 \eta_1 > 0$ and $g_5 \eta_2 > 0$. Let us introduce the smooth function $\gamma_n(x) = (\gamma_n^1(x), \gamma_n^2(x))$ with

$$
\gamma_n^1(x) := -(g_1 \chi^1(x) + g_2 \chi^2(x) + g_3(1 - \chi^1(x))(1 - \chi^4(x))\chi^3(x)),
\gamma_n^2(x) := -(g_4(1 - \chi^2(x))(1 - \chi^3(x))\chi^4(x) + g_5(1 - \chi^1(x))(1 - \chi^2(x))\chi^5(x)).
$$

By Lemma 10.9, $\chi_i \in C^\infty(\mathbb{R}^2, [0, 1])$ for all $i = 1, \ldots, 5$. Moreover, with the notation given by (10.60) above, if $\gamma_n$ is sufficiently closed to 0, these functions satisfy:

- $\chi^1(x) = 1$ on $[d_n, b_n]_{\gamma_n}$ and $\chi^1(x) = 0$ on $\mathbb{R}^2 \setminus [d_n, b_n]_{\gamma_n}$,
- $\chi^2(x) = 1$ on $[c_n, a_n]_{\gamma_n}$ and $\chi^2(x) = 0$ on $\mathbb{R}^2 \setminus [c_n, a_n]_{\gamma_n}$,
- $\chi^3(x) = 1$ on $C^n_0 \gamma_n := ([\bar{c}_n, \tilde{b}_n] \cup [\tilde{b}_n, b_n])_{\gamma_n}$ and $\chi^3(x) = 0$ on $\mathbb{R}^2 \setminus C^n_0 \gamma_n$,
- $\chi^4(x) = 1$ on $C^n_0 \gamma_n := ([\bar{c}_n, \tilde{a}_n] \cup [\tilde{a}_n, a_n])_{\gamma_n}$ and $\chi^4(x) = 0$ on $\mathbb{R}^2 \setminus C^n_0 \gamma_n$,
- $\chi^5(x) = 1$ on $C^n_0 \gamma_n := ([c_n, \bar{c}_n] \cup [\bar{c}_n, \tilde{d}_n] \cup [\tilde{d}_n, d_n]_{\gamma_n})$ and $\chi^5(x) = 0$ on $\mathbb{R}^2 \setminus C^n_0 \gamma_n$.

Let us denote by $\eta(x)$ the outward normal at each point of $\partial K^n_0$. The mapping $\eta : \partial K^n_0 \mapsto \mathbb{R}^2$ is continuous except at the points $\tilde{a}_n$, $\tilde{b}_n$, $\bar{c}_n$, $\bar{a}_n$ where it admits left and righ limits we denote by $\eta(x \pm)$. Moreover, by construction we have $\gamma_n(x) \eta(x-1) > 0$ and $\gamma_n(x) \eta(x+1) > 0$ for all $x \in \partial K^n_0$.

Observe that

$$
d(x - t \gamma_n(x), \partial K^n_0) \geq \min_{x \in \partial K^n_0} d(x - t \gamma_n(x), \partial K^n_0) := m^n, \quad \forall x \in \partial K^n_0.
$$

By a compactness argument, $m^n = d(x_\infty - t \gamma_n(x_\infty), \partial K^n_0)$ for some $x_\infty \in K^n_0$. Since

$$
d(x_\infty - t \gamma_n(x_\infty), \partial K^n_0) \geq -t \gamma_n(x_\infty) \eta(x_\infty \pm) = 2bt,
$$

where $2b := -\gamma_n(x_\infty) \eta(x_\infty \pm) > 0$, we finally deduce that Condition C1 holds.

**Step 2** By virtue of [7, Corollary 5.2], since $\sigma$ is globally Lipschitz on the bounded domain $\tilde{K}^n_0$, there exists a unique strong solution $(V^n, k^n)$, starting from $x$, to the reflected S.D.E. (10.57) on the domain $\tilde{K}^n_0$. Let us introduce

$$
\tau^n := \inf \{t : V^n_1(1, d) = \epsilon_n\},
\rho^n := \inf \{t : V^n_1(1, d) = \epsilon_n\},
$$

and $\mu^n := \tau^n \land \rho^n$. On the intervall $[0, \mu^n]$, the process $(V^n, k^n)$ is solution to (10.57) on the domain $\tilde{K}^{n+1}_0$ with respect to $\gamma^{n+1}$. Indeed, on the intervall $[0, \mu^n]$, the reflection only occurs on the boundary $\partial K_0$ on which $\gamma^{n+1}$ and $\gamma^n$ coincides with $\gamma$. By the uniqueness property given by [7, Corollary 5.2], we deduce that $(V^n, k^n) = (V^{n+1}, k^{n+1})$ on $[0, \mu^n]$. It follows that $\mu^n \leq \mu^{n+1}$. The rest of the proof is done as in [11, p. 229].

### 10.4 Skorokhod problem for pure-jumps Lévy processes

The setting of this subsection is given in Sect. 10.3. Let $\gamma : \partial K_0 \mapsto \mathbb{R}^2$ be a vector-valued function with $g(x) = -g_i$ on $(\partial K_0 \cap \partial K_1) \setminus \{0\}$ and $\gamma(0) = 0$. Recall that, as the Lévy
process we consider is of finite activity, it can be represented as the sum of a compound
poison process and a Wiener process. So, consider a process \( Y \) defined by
\[
Y_t = (Y^1_t, Y^2_t, 0) + (t, W_t, N_t), \quad t \geq 0,
\]
where \( W \) is a standard Brownian motion and \( N \) is a pure jump process of finite activity. This
means that
\[
N_t = \sum_{k=1}^{\tilde N_t} \Delta N_{T_k},
\]
where \( \Delta N_{T_k} \) are i.i.d. random variables and \( \tilde N_t = \sum_{k=1}^t 1_{T_k \leq t} \) is a Poisson process with jump
stopping times \( (T_k)_{k \geq 1} \). Let \( \sigma = \mathbb{R}^2 \mapsto \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \) be a matrix-valued function which
is Lipschitz-continuous.

We consider the Skorokhod problem on \( K_0 \) formulated as follows: find a pair of adapted
làdlàg (resp. càglàd) processes \( V, \) starting from \( x \in K_0 \) and real-valued process \( k \), starting at
zero and increasing, such that
\[
dV_t = \sigma(V_t) dY_t + \gamma(V_t) dk_t,
\]
\[
dk_t = 1_{V_t \in K \setminus \text{int } K_0} dk_t.
\]
\[
V_t^+ \in \tilde K_0, \quad \forall t \geq 0. \tag{10.61}
\]

The goal is to show that this R.S.D.E has a solution on the set \( \tilde K_0 \). To do so, we shall use the following:

**Lemma 10.11** (Projection onto \( K_0 \) parallel to \( -K \)) Assume that \( K \subseteq \mathbb{R}^2 \) is a constant cone
satisfying the hypothesis of the introduction and \( K_0 \subseteq K \) is a closed cone with \( \partial K_0 \subseteq \text{int } K \)
and \( \text{int } K_0 \neq \emptyset \). For every \( x \in K \), there exists a unique \( y := \mathbb{P}_{K_0}^K(x) \in K_0 \) such that
\[
\| x - y \| = \min_{k \in K} \{ \| x - k \| : x - k \in K \}.
\]

We omit the proof which is standard. It is easily observable that the direction of \( x - \mathbb{P}_{K_0}^K(x) \)
is given by \( g_2 \) if \( x \in K_2 \) and \( g_1 \) if \( x \in K_1 \).

**Theorem 10.12** There exists a unique solution to the Skorokhod problem (10.61).

**Proof** Let \( (T_k)_{k \geq 1} \) be the jump stopping times of the process \( Y \). Assume that we have already
constructed a solution \( (V, k) \) to (10.61) on the interval \([0, T_k)\). Define
\[
V_{T_k} := V_{T_k^-} + \sigma(V_{T_k^-}) \Delta Y_{T_k}.
\]
Let us introduce
\[
V_{T_k^+} := \mathbb{P}_{K_0}^K(V_{T_k}) \in L^0(\mathbb{R}^2, \mathcal{F}_{T_k}) \in \partial K_0,
\]
where the projection operator \( \mathbb{P}_{K_0}^K \) is defined in Lemma 10.11. We define \( \Delta^+k_{T_k} \) by the equality
\[
\Delta^+k_{T_k} := \gamma(V_{T_k}) \Delta^+k_{T_k}.
\]
Applying Theorem 10.10 and the strong markov property, there exists a solution \( (\tilde V, \tilde k) \)
to (10.61) from the starting point \( \tilde V_0 := V_{T_k^+} \) with respect to \( \tilde N_t := N_{T_k+t} - N_{T_k} \) and
\( \tilde W_t := W_{T_k+t} - W_{T_k} \) defined on the interval \([0, T^{k+1} - T^k] \). We then define
\[
V_t := \tilde V_{t-T_k}, \quad \text{and} \quad k_t := k_{T_k} + \tilde k_{t-T_k}, \quad t \in (T^k, T^{k+1}).
\]
Uniqueness follows from uniqueness on each interval $[T_k, T_{k+1})$. 

References

1. Akian, M., Taksar, M., Sulem, A.: Ergodic multidimensional diffusion dynamic portfolio selection model. Research Report INRIA, 3626 (Feb 1999)
2. Campi, L., Owen, M.: Multivariate utility maximization with proportional transaction costs. Financ. Stoch. 15(3), 461–499 (2011)
3. Constantinides, G.M.: Capital market equilibrium with transaction costs. J. Political Econ. 94, 842–862 (1996)
4. Czichowsky, C., Schachermayer, W.: Duality theory for portfolio optimisation under transaction costs. Ann. Appl. Probab. 26(3), 1888–1941 (2016)
5. Davis, M., Norman, A.: Portfolio selection with transaction costs. Math. Oper. Res. 15, 676–713 (1990)
6. De Vallière, D., Kabanov, Y., Lépinette, E.: Consumption-investment optimization problem in a Lévy financial model with transaction costs. Financ. Stoch. 20(3), 705–740 (2016)
7. Dupuis, P., Ishii, H.: SDEs with oblique reflection on nonsmooth domains. Ann. Probab. 21(1), 554–580 (1993)
8. Framstad, N., Oksendal, B., Sulem, A.: Optimal consumption and portfolio in a jump diffusion market. J. Math. Econ. 35, 233–257 (2001)
9. Guasoni, P., Lépinette, E., Rásony, M.: The fundamental theorem of asset pricing under transaction costs. Financ. Stoch. 16(4), 741–777 (2012)
10. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. Springer, Berlin (2002)
11. Kabanov, Y., Safarian, M.: Markets with Transaction Costs: Mathematical Theory. Monograph. Springer, Berlin (2010)
12. Shreve, S.E., Soner, H.M.: Optimal investment and consumption with transaction costs. Ann. Appl. Probab. 4(3), 609–692 (1994)

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