Linear Optical Universal Quantum Gates with Higher Success Probabilities

Wen-Qiang Liu and Hai-Rui Wei*

Universal quantum gates lie at the heart of designing quantum computer. Here two compact quantum circuits for implementing post-selected controlled-phase-flip (CPF) gate and Toffoli gate with linear optics are constructed assisted by one and two single photons, respectively. The current existing maximum success probability of 1/4 for linear optical CPF gate is achieved by resorting to an ancillary single photon rather than an entangled photon pair or two single photons. Remarkably, the presented Toffoli gate is accomplished with current maximum success probability of 1/30 and unity fidelity in principle, without using additional entangled photon pairs and the standard decomposition-based approach. Linear optical implementations of the presented two universal gates are feasible and economical under current technology, and provide a potential application in large-scale optical quantum computing.

1. Introduction

Quantum computing [1] has the remarkable potential to dramatically surpass its classical counterpart on solving certain complex tasks in terms of the processing speed or resource overhead. Universal quantum gates are crucial building blocks in quantum computing,[2–6] quantum algorithms,[7–9] quantum simulations,[10] and quantum communication.[11] Photon is generally viewed as one of the promising candidates for flying and solid-state quantum computing owing to its outstanding low decoherence, high-speed transmission, natural information carrier, flexible single-qubit manipulations, and available atom-like qubit interconnector.[12,13] Strong interactions between individual photons are the key resources for nontrivial multi-photon quantum gates, and the prohibited photon–photon interactions can be remedied efficiently by using linear optics[14] or solid-state media.[15–17] Unfortunately, solid-state platforms are challenged by inefficiency and imperfection in experiments. The probabilistic character of universal quantum gates with linear optics is unavoidable. Hence, minimizing the quantum resources required to implement quantum gates with higher success probability is a central problem of linear optical quantum computing, and tremendous efforts have been made on it.[18–45]

Controlled phase flip (CPF) gate or its equivalent controlled-NOT (CNOT) gate is the most quintessential universal quantum gate.[1] CNOT gates together with single-qubit rotations are sufficient to implement any quantum computation.[2] Nowadays, CNOT gate has been experimentally demonstrated in several physical systems.[36–38] The KLM scheme [14] served as a stepping stone for implementing CPF gate with a sheer number of linear optics, large and good quantum memory, and giant interferometer phase stable. Various improved works were later proposed both in theory and experiment.[18–35] So far, it has been demonstrated that CPF gate can be completed with a success probability of 1/9, which is the existing maximum value achievable without ancillary photons,[25–31] and the success probability can be improved to 1/8 via two additional independent single photons.[32,33] The current existing highest success probability of 1/4 for a CNOT gate has been achieved assisted by a necessary entangled photon pair.[19–24] Deterministic generation of entangled photon pairs based on spontaneous parametric down-conversion remains a key technical obstacle in experiments due to multi-photon emissions and probabilistic properties.[39]

Toffoli gate supplemented with Hadamard gate can simulate any multi-qubit quantum computing.[1] Toffoli gate is also served as an essential part in quantum factoring algorithm,[40] quantum search algorithm,[41] quantum half-adder,[42] quantum error correction,[43] quantum fault tolerance,[44] etc. Much attention has been paid to the realization of Toffoli gate.[45–47] It has been confirmed theoretically that the optimal cost of a Toffoli gate is six CNOT gates[2] or five two-qubit entangling gates.[48] Such synthesis might be helpful to design complex quantum gate, but it makes the gate further susceptible to the environmental noise and increases the time scale of the system. Without using the standard decomposition-based approach, early in 2006, Fiurášek[49] first showed a three-photon polarization Toffoli gate with a success probability of 0.75% (≈1/133) using linear optics. Using higher-dimensional Hilbert spaces, Ralph et al.[50] improved the success probability of a linear optical Toffoli gate
to 1/72 in 2007, and this interesting approach was later experimentally demonstrated in linear optics\cite{51} and superconducting systems\cite{46} respectively. In 2022, Liu et al.\cite{32} further enhanced the success probability of the Toffoli gate to 1/64 assisted by auxiliary higher-dimensional spaces and a combination of a CNOT and two partial-swap gates. In the same year, a post-selected Toffoli gate was also reported in recent years.\cite{54–59}

In this paper, we propose two compact quantum circuits to implement post-selected CPF and Toffoli gates in the coincidence basis using solely polarizing beam splitters (PBSs), half-wave plates (HWPbs), beam splitters (BSs), and single-photon detectors. Assisted by one and two independent single photons, our CPF and Toffoli gates are accomplished respectively when exactly one photon appears in each output mode. Our schemes are appealing for higher success probabilities and less quantum resource requirements. The existing highest success probability of a linear optical CPF gate 1/4 is achieved resorting to an auxiliary single photon in our scheme rather than an auxiliary entangled photon pair.\cite{19–24} The presented CPF gate also beats the ones with the success probability of 1/8 assisted by two single photons\cite{32,33} and the ones with 1/9 without auxiliary photons.\cite{25–31} In addition, the average success probability of our Toffoli gate is high to 1/30, which far exceeds all previous results for the same works.\cite{49–54}

2. Post-Selected CPF Gate with Linear Optics

It is well-known that CPF gate introduces a π phase shift when the first qubit and the second qubit are both |1\rangle, and the rest remains unchanged. We encode the gate qubit in two polarization DOFs of a single photon, that is, the horizontally polarized photon |H⟩ = |0⟩ and vertically polarized photon |V⟩ = |1⟩, respectively.

Our scheme described in Figure 1 shows a polarization-based post-selected CPF gate can be completed in the following three steps.

First, the two gate photons and one auxiliary photon in the states

\[ |\Phi_\text{in} = a_1 |H_\text{in} + b_1 |V_\text{in} \]

(1)

\[ |\Phi_\text{in} = a_2 |H_\text{in} + b_2 |V_\text{in} \]

(2)

\[ |\Phi_\text{a} = \frac{1}{\sqrt{2}} (|H_\text{a} + |V_\text{a} \rangle \]

(3)

are injected into the spatial modes c_in, t_in, and a, respectively. Here coefficients \( a_1, b_1, a_2, b_2 \) satisfy the conditions \( |a_1|^2 + |b_1|^2 = 1 \) and \( |a_2|^2 + |b_2|^2 = 1 \). The subscripts denote the spatial modes of photons (also named photon’s paths).

The photons emitted from spatial modes c_in and a are fed into PBS_1, simultaneously. The PBS_1 can transmit the H-polarized photon and reflect the V-polarized photon, which transforms the state of the whole system from \( |\Phi_\text{in} = |\Phi_\text{in} \otimes |\Phi_\text{a} \) to

\[ |\Phi_\text{2} = \frac{1}{\sqrt{2}} (a_1 |H_2⟩ |H_1⟩ + b_1 |V_2⟩ |V_1⟩ + a_2 |H_2⟩ |V_1⟩ + b_2 |V_2⟩ |H_1⟩) \]

\[ + β_i |V_1⟩ |V_2⟩ + β_i |H_1⟩ |H_2⟩ |V_2⟩ \]

(4)

Based on Equation (4), one can see that PBS_1 can complete a parity-check measurement on the polarization photons by choosing the instance in which each of the spatial mode contains exactly one photon in post-selection principle, and then the system would be changed into the state

\[ |\Phi_\text{3} = (a_1 |H_1⟩ |H_2⟩ + β_i |V_1⟩ |V_2⟩) \otimes (a_2 |H_1⟩ |H_2⟩ + β_i |V_1⟩ |V_2⟩) \]

(5)

with a probability of 1/2. While the instance in which each spatial mode involves two photons or none photon indicates the gate operation fails.

Second, as shown in Figure 1, before and after the photons from modes 2 and t_in pass through PBS_2, two polarization Hadamard operations are performed on them by using HWP_22° and HWP_22°, respectively. Here half-wave plate oriented at 22.5° (HWP_22°) completes the transformations

\[ |H⟩ \xrightarrow{\text{HWP}_{22.5}°} \frac{1}{\sqrt{2}} (|H⟩ + |V⟩), |V⟩ \xrightarrow{\text{HWP}_{22.5}°} \frac{1}{\sqrt{2}} (|H⟩ - |V⟩) \]

(6)

Operations HWP_122° → PBS_2 → HWP_22° transform the state \( |\Phi_\text{3} \) into

\[ |\Phi_\text{4} = \frac{1}{2} (a_1 a_2 |H_1⟩ |H_2⟩ + a_1 b_2 |H_1⟩ |V_1⟩ + β_i a_2 |V_1⟩ |H_1⟩ \]

\[ - β_i b_2 |V_1⟩ |V_2⟩) \otimes |H_2⟩ + \frac{1}{2} (a_1 a_2 |H_1⟩ |H_2⟩ \]

\[ - a_1 b_2 |V_1⟩ |V_2⟩ + β_i a_2 |V_1⟩ |H_2⟩ + b_2 |H_1⟩ |H_2⟩ |V_1⟩ |V_2⟩ \otimes |V_2⟩ \]

\[ + \frac{1}{2} \sqrt{2} (a_1 a_2 |H_1⟩ - β_i a_2 |V_1⟩) \otimes (|H_2⟩ + |V_2⟩) \]

(7)
We choose the case where exactly one photon in each of the spatial modes 3 and 4, and then the system would be in a normalization state

\[|\Phi_3\rangle = \frac{1}{\sqrt{2}} (|a_1a_2\rangle |H\rangle_1 |H\rangle_4 + |a_1\beta_2\rangle |H\rangle_1 |V\rangle_4 + \beta_1|a_2\rangle |V\rangle_1 |H\rangle_4 - \beta_1|\beta_2\rangle |V\rangle_1 |H\rangle_4) \]

\[= |\Phi\rangle (|H\rangle_4) \alpha_1 |\beta_2\rangle + \beta_1 |\beta_2\rangle |\alpha_1\rangle (|H\rangle_4) \]

\[= |\Phi\rangle (|H\rangle_4) \alpha_1 |\beta_2\rangle + \beta_1 |\beta_2\rangle |\alpha_1\rangle (|H\rangle_4) \]

(8)

with a probability of \(\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}\).

Finally, the photon emitted from spatial mode 4 will be detected by using PBS3 and photon detectors D2t and D2v. Based on Equation (8), one can see that when D2t is fired, the photons emitted from c2out and t2out kept are in the state

\[|\Phi_6\rangle = \alpha_1 |\alpha_2\rangle |H\rangle_1 |H\rangle_4 + \alpha_1 |\beta_2\rangle |H\rangle_1 |V\rangle_4 + \beta_1 |\alpha_2\rangle |V\rangle_1 |H\rangle_4 - \beta_1 |\beta_2\rangle |V\rangle_1 |H\rangle_4 \]

\[= |\Phi\rangle (|H\rangle_4) \alpha_1 |\beta_2\rangle + \beta_1 |\beta_2\rangle |\alpha_1\rangle (|H\rangle_4) \]

(9)

with a probability of \(\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}\). And then, the performance of CPF gate is completed.

When D2v is fired, the system will collapse into the state

\[|\Phi'_6\rangle = \alpha_2 |\alpha_2\rangle |H\rangle_1 |H\rangle_4 - \alpha_2 |\beta_2\rangle |H\rangle_1 |V\rangle_4 + \beta_1 |\alpha_2\rangle |V\rangle_1 |H\rangle_4 + \beta_1 |\beta_2\rangle |V\rangle_1 |H\rangle_4 \]

\[= |\Phi\rangle (|H\rangle_4) \alpha_1 |\beta_2\rangle + \beta_1 |\beta_2\rangle |\alpha_1\rangle (|H\rangle_4) \]

(10)

with a probability of \(\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}\). It is easily to convert Equations (10) to (9) by applying a feed-forward \(a_2\) operation on the photon emitted from spatial mode 4, which can be achieved by setting an HWP\(^{\circ}\) on spatial mode 4.\(^{166-62}\) In experiment, the feed-forward technique has been experimentally demonstrated by using fiber delays and fast active switches to correct introduced Pauli errors and increase the success probability of linear optical quantum computing.\(^{63,64}\)

Putting all the pieces together, one can see that the quantum circuit shown in Figure 1 completed a CPF operation when exactly one photon in each of the output spatial modes. This means that the gate success is heralded by simultaneous successful detection of exactly one photon for each qubit. The total success probability of the presented gate can reach the current existing best result of \(\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\), and only one additional single photon is required.

### 3. Post-Selected Toffoli Gate with Linear Optics

Toffoli gate flips the state of the target qubit if the two controlled qubits both are in \(|1\rangle\), and has no effect otherwise. Figure 2 depicts a scheme for implementing a Toffoli gate with an average success probability of 1/30 in the linear optical system.

Suppose two controlled photons, one target photon, and two auxiliary photons are initially prepared in the following states

\[|\psi\rangle_{c_{2s}} = |\alpha_1\rangle |H\rangle_{c_{2s}} + |\beta_1\rangle |V\rangle_{c_{2s}} \]

(11)

\[|\psi\rangle_{c_{2s}} = |\alpha_2\rangle |H\rangle_{c_{2s}} + |\beta_2\rangle |V\rangle_{c_{2s}} \]

(12)

\[|\psi\rangle_t = |\alpha_1\rangle |H\rangle_t + |\beta_1\rangle |V\rangle_t \]

(13)

\[|\psi\rangle_{a_1} = \frac{1}{\sqrt{2}} (|H\rangle_{a_1} + |V\rangle_{a_1}) \]

(14)

\[|\psi\rangle_{a_2} = \frac{1}{\sqrt{2}} (|H\rangle_{a_2} + |V\rangle_{a_2}) \]

(15)

where \(|\alpha_1|^2 + |\beta_1|^2 = 1\), \(|\alpha_2|^2 + |\beta_2|^2 = 1\), and \(|\alpha_1|^2 + |\beta_1|^2 = 1\).

In the first step, we employ PBS3 to complete the parity-check measurement on the first controlled photon (emitted from spatial mode 6, 12, 13, and 14) (if the photon has been detected at PBS3) and the second controlled photon (emitted from spatial mode 5, 6, 15, and 16) (if the photon has been detected at PBS3) and the photon detectors D5, D6, D7, and D8. At the end of the parity-check measurement, the photons in the states 

\[|\psi\rangle_{c_{2s}} = |\alpha_1\rangle |H\rangle_{c_{2s}} + |\beta_1\rangle |V\rangle_{c_{2s}} \]

(16)

\[|\psi\rangle_{c_{2s}} = |\alpha_2\rangle |H\rangle_{c_{2s}} + |\beta_2\rangle |V\rangle_{c_{2s}} \]

(17)

\[|\psi\rangle_t = |\alpha_1\rangle |H\rangle_t + |\beta_1\rangle |V\rangle_t \]

(18)

\[|\psi\rangle_{a_1} = \frac{1}{\sqrt{2}} (|H\rangle_{a_1} + |V\rangle_{a_1}) \]

(19)

\[|\psi\rangle_{a_2} = \frac{1}{\sqrt{2}} (|H\rangle_{a_2} + |V\rangle_{a_2}) \]

(20)

are kept and the photons emitted from PBS3 and photon detectors D5, D6, D7, and D8 are kept.
and the first additional photon (emitted from spatial mode $a_t$), and choose the instance in which each outgoing mode contains exactly one photon. And then, PBS$_5$ converts the whole system from the initial state $|\Psi_{in}\rangle = |\psi\rangle_{c_{in}} \otimes |\psi\rangle_{a_1} \otimes |\psi\rangle_{c_{in}} \otimes |\psi\rangle_{a_2} \otimes |\psi\rangle_{c_{in}}$ to

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}} \left( |\gamma\rangle_{c_{in}} \otimes |\gamma\rangle_{a_1} \otimes |\gamma\rangle_{c_{in}} \otimes |\gamma\rangle_{a_2} \otimes |\gamma\rangle_{c_{in}} \right) \otimes (|H\rangle_{c_{in}} + |V\rangle_{c_{in}} \otimes (a_1|H\rangle_{c_{in}} + \beta_1|V\rangle_{c_{in}}),$$

with a probability of 1/2.

In the second step, PBS$_2$ transmits $H_{c_{in}}$-polarized component to PBS$_4$ and reflects $V_{c_{in}}$-polarized component to spatial mode 3 for mixing with the components emitted from spatial mode 2 at PBS$_1$. After the photons emitted from spatial modes 2 and 3 experience the block composed of four HWPs$^{22,5}$ and PBS$_3$, we choose the instance in which each of spatial modes 5 and 6 contains exactly one photon, and then the system will become the normalization state

$$|\Psi_{in}\rangle = \frac{1}{2} \left( |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right.$$  
$$+ |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ \otimes (|H\rangle_{c_{in}} + |V\rangle_{c_{in}} \otimes (a_1|H\rangle_{c_{in}} + \beta_1|V\rangle_{c_{in}})$$

(16)

with a probability of $\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}$. In order to complete the Toffoli gate with unity fidelity in principle, we next reduce the amplitude of the photon emitted from spatial mode 5 to half by using a 50:50 beam splitter (BS). The unitary transformations of the BS can be described as

$$|H\rangle_5 \rightarrow \frac{1}{\sqrt{2}} (|H\rangle_5 + |V\rangle_5), \quad |V\rangle_5 \rightarrow \frac{1}{\sqrt{2}} (|V\rangle_5 + |H\rangle_5)$$

(18)

That is, BS yields the state

$$|\Psi_{in}\rangle = \frac{1}{2} \left( |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|V\rangle_5 \right)$$

$$+ \otimes (|H\rangle_{c_{in}} + |V\rangle_{c_{in}} \otimes (a_1|H\rangle_{c_{in}} + \beta_1|V\rangle_{c_{in}})$$

(19)

If $D_{V_{a_2}}$ is triggered and the photon emitted from spatial mode 5’ is led to PBS$_3$ to mix with the photon emitted from mode $a_2$, Equation (19) will collapse into the state

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}} \left( (\alpha_1|a_2|H\rangle_4|H\rangle_4 + \alpha_2|a_2|H\rangle_4|V\rangle_4 \right) + \beta_1|a_2|V\rangle_4|H\rangle_4 + \beta_2|a_2|V\rangle_4|V\rangle_4 \right)$$

$$\otimes (|H\rangle_{a_2} + |V\rangle_{a_2}) \otimes (a_1|H\rangle_{a_2} + \beta_1|V\rangle_{a_2})$$

(20)

with a probability of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$. If $D_{V_{a_2}}$ is triggered, the photon emitted from spatial mode 5’ will be applied a feedback $\sigma_3$ operation to convert the state

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}} \left( (\alpha_1|a_2|H\rangle_4|H\rangle_4 + \alpha_2|a_2|H\rangle_4|V\rangle_4 \right) + \beta_1|a_2|V\rangle_4|H\rangle_4 + \beta_2|a_2|V\rangle_4|V\rangle_4 \right)$$

$$\otimes (a_1|H\rangle_{a_2} + \beta_1|V\rangle_{a_2})$$

(21)

into Equation (20) with a probability of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$. The $\sigma_3$ operation can be achieved easily by an HWPs$^{22,5}$ setting in mode 5’.

In the third step, after PBS$_4$ completes the parity-check measurement on the photons emitted from spatial modes 5’ and $a_2$, the instance in which the spatial mode 8 involves exactly one photon is chosen, and then the system will be in the following normalization state

$$|\Psi_{in}\rangle = \frac{1}{2} \left( |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|V\rangle_5 \right)$$

$$+ \otimes (|H\rangle_{c_{in}} + |V\rangle_{c_{in}} \otimes (a_1|H\rangle_{c_{in}} + \beta_1|V\rangle_{c_{in}})$$

(22)

with a probability of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$. The even-parity is chosen for the polarized photons in modes 11 and 12 after PBS$_5$. Therefore, before $D_{V_{a_2}}$ or $D_{V_{a_2}}$ is fired, these elements induce the outgoing photons in the state

$$|\Psi_{in}\rangle = \frac{1}{2} \left( |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|H\rangle_4|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|H\rangle_4|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|V\rangle_5 \right)$$

$$+ |\alpha\rangle_{a_2}|H\rangle_1|V\rangle_5 + |\gamma\rangle_{a_2}|V\rangle_1|V\rangle_5 \right)$$

$$+ \otimes (|H\rangle_{c_{in}} + |V\rangle_{c_{in}} \otimes (a_1|H\rangle_{c_{in}} + \beta_1|V\rangle_{c_{in}})$$

(23)
with a probability of \( \frac{1}{2} \times 1^{a} \times \left( \frac{1}{2} + \frac{i}{2} \right) \times \left( \frac{1}{2} + \frac{1}{2} \right) \times \frac{1}{2} \). Here, for simplicity, the coefficients are written as \( \gamma_1 = \alpha_1 \beta_1 \), \( \gamma_2 = \alpha_2 \beta_2 \), \( \gamma_3 = \alpha_3 \beta_3 \), \( \gamma_4 = \alpha_4 \beta_4 \), \( \gamma_5 = \beta_5 \alpha_5 \), \( \gamma_6 = \beta_6 \alpha_6 \), \( \gamma_7 = \beta_7 \alpha_7 \), \( \gamma_8 = \beta_8 \alpha_8 \), and \( \gamma_9 = \beta_9 \alpha_9 \). Half-wave plate HWP induces the transformations
\[
|H\rangle_{HW}^{\text{HWP}} = \frac{1}{\sqrt{2}} (|V\rangle - |H\rangle), \quad |V\rangle_{HW}^{\text{HWP}} = \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle)
\]
(24)

Based on Equation (23), one can see that when \( D_{V_1} \) is fired, the output photons from the spatial modes 1, 9, and 11 keep are in the state
\[
|\Psi_7\rangle = \gamma_1 |H\rangle_1 |H\rangle_9 |H\rangle_{11} + \gamma_2 |H\rangle_1 |H\rangle_9 |V\rangle_{11}
+ \gamma_3 |V\rangle_1 |H\rangle_9 |H\rangle_{11} + \gamma_4 |V\rangle_1 |H\rangle_9 |V\rangle_{11}
+ \gamma_5 |V\rangle_1 |V\rangle_9 |H\rangle_{11} + \gamma_6 |V\rangle_1 |V\rangle_9 |V\rangle_{11}
+ \gamma_7 |V\rangle_1 |V\rangle_9 |H\rangle_{11} + \gamma_8 |V\rangle_1 |V\rangle_9 |V\rangle_{11}
\]
(25)

with a probability of \( \frac{1}{2} \times 1^{a} \times \left( \frac{1}{2} + \frac{i}{2} \right) \times \left( \frac{1}{2} + \frac{1}{2} \right) \times \frac{1}{2} \). That is, the probabilistic Toffoli gate is completed.

When \( D_{V_2} \) is fired, the output photons from the spatial modes 1, 9, and 11 keep are in the state
\[
|\Psi_7\rangle = \gamma_1 |H\rangle_1 |V\rangle_9 |V\rangle_{11} + \gamma_2 |H\rangle_1 |V\rangle_9 |H\rangle_{11}
+ \gamma_3 |V\rangle_1 |V\rangle_9 |V\rangle_{11} + \gamma_4 |V\rangle_1 |V\rangle_9 |H\rangle_{11}
+ \gamma_5 |V\rangle_1 |H\rangle_9 |V\rangle_{11} + \gamma_6 |V\rangle_1 |H\rangle_9 |H\rangle_{11}
+ \gamma_7 |V\rangle_1 |V\rangle_9 |H\rangle_{11} + \gamma_8 |V\rangle_1 |V\rangle_9 |V\rangle_{11}
\]
(26)

with a probability of \( \frac{1}{2} \times 1^{a} \times \left( \frac{1}{2} + \frac{i}{2} \right) \times \left( \frac{1}{2} + \frac{1}{2} \right) \times \frac{1}{2} \). To complete the Toffoli gate, a feed-forward \( \sigma_X \) operation is applied to photon in spatial mode 11. The outcomes of measurement and corresponding feed-forward operations for completing Toffoli gate are summarized in Table 1. When \( D_{V_3} \) is fired, it means that the scheme fails.

We evaluate the performance of the scheme by characterizing success probability and fidelity of Toffoli gate, respectively. The success probability of Toffoli gate is defined as \( P_{\text{Toff}} = n_{\text{out}}/n_{\text{in}} \), where \( n_{\text{out}} \) and \( n_{\text{in}} \) are the number of output photon and input photon, respectively. The fidelity of any two quantum states \( \rho \) and \( \sigma \) is defined as \( F(\rho, \sigma) = \text{tr} \sqrt{\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}}} = \langle \psi | \varphi \rangle \). Here \( \rho = |\psi\rangle \langle \psi| \), \( \sigma = |\varphi\rangle \langle \varphi| \), and \( \varphi \) and \( \varphi \) are the real and ideal normalized output states of the gate, respectively. As shown in Figure 2, there are three alternatives as follows.

(i) If only the photons emitting from \( S' \) are led to P5S (we call it with BS but without recycling), the success probability of Toffoli gate is \( P_{\text{Toff}} = \frac{1}{2} \times 1^{a} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \) with unity fidelity in principle.

(ii) If we both direct the photons emitting from \( S' \) and \( S'' \) to P5S (we call it with BS and with recycling), the same argument as that made for the photons emitted from spatial mode 5, we find that recycling the photons emitting from \( S'' \) can also complete a Toffoli gate. Therefore, in this case, the success probability of gate is enhanced to \( P_{\text{Toff}} = \frac{1}{2} \times 1^{a} \times \frac{2^{2-a}}{2} \times \frac{2^{2-a}}{2} \times \frac{1}{2} = \frac{1}{4} \) with unity fidelity in principle, and the output ports will be double.

(iii) If we remove BS (that is, photons from spatial mode 5 are led to P5S, directly), the success probability of Toffoli gate can also be enhanced to \( P_{\text{Toff}} = \frac{1}{64} \times \frac{1}{64} \times \frac{1}{64} \times \frac{1}{64} \), but the fidelity will be decreased to \( P_{\text{Toff}} = 1 - \frac{a^2}{2} + \frac{\sqrt{a^2 - 1}}{2} \).

Table 1. Measurement outcomes and corresponding feed-forward operations in mode \( S' \) or 11 for realizing a Toffoli gate. \( I_x \) is an identity operation and \( \sigma_x \) is a Pauli X operation, which can be realized by an HWP setting at 45°.

| Measurement | Feed-forward | Achieved |
|-------------|--------------|----------|
| \( D_{V_1} \) | \( D_{V_2} \) | \( I_x \) \( I_x \) | Toffoli gate |
| \( D_{V_2} \) | \( D_{V_1} \) | \( I_x \) \( \sigma_x \) | Toffoli gate |
| \( D_{V_1} \) | \( D_{V_2} \) | \( \sigma_x \) \( I_x \) | Toffoli gate |
| \( D_{V_2} \) | \( D_{V_1} \) | \( \sigma_x \) \( \sigma_x \) | Toffoli gate |

\( a \) is the identity operation and \( \sigma_x \) is a Pauli X operation, which can be realized by an HWP setting at 45°.
The fidelity of the Toffoli gate is a function of parameter $\alpha_2$, which is dependent on the initial state of the second controlled photon.

As shown in Figure 3 and Table 2, the success probability and fidelity of the Toffoli gate are the functions of parameter $\alpha_2$, that is, dependent on the initial state of the second controlled photon.

(i) For the case of with BS but without recycling, the minimum $P'_{\text{Toff}} = \frac{1}{64}$ with $\alpha_2 = 0$ (i.e., $|\psi\rangle_{c2a} = \pm |V\rangle_{c2a}$), the maximum $P''_{\text{Toff}} = \frac{1}{32}$ with $\alpha_2 = 0$ (i.e., $|\psi\rangle_{c2a} = \pm |H\rangle_{c2a}$), and the average fidelity $F_{\text{Toff}} = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{16} \sin^2 \alpha_2 d\alpha_2 = \frac{1}{48}$.

(ii) For the case of with BS and with recycling, the minimum $P'_{\text{Toff}} = \frac{1}{32}$ with $\alpha_2 = 0$ or $\pm 1$, the maximum $P''_{\text{Toff}} = \frac{3}{256}$ with $\alpha_2 = \pm \frac{1}{\sqrt{2}}$, and $F_{\text{Toff}} = \frac{1}{50}$. The success probability of our Toffoli gate is much higher than previous works.\[49-54\]

(iii) For the case of without BS, different from above, the unity fidelity $F''_{\text{Toff}} = 1$ can be achieved only with $\alpha_2 = 0$ or $\pm 1$, and the average fidelity is $F''_{\text{Toff}} = \frac{3 + \sqrt{2}}{12} \approx 95.2\%$.

4. Conclusion

We have proposed two schemes to implement post-selected CPF and Toffoli gates in the coincidence basis by solely using linear optics. The comparisons between our proposed CPF gate and Toffoli gate and previous schemes are presented in Tables 3 and 4, respectively. A maximally entangled photon pair\[19-24\] (or two single photons\[32,33\]) is necessary for implementing a CNOT gate with the current existing maximum success probability of $1/4$ (or $1/8$). Only one auxiliary single photon is introduced to accomplish our CPF gate with the success probability of $1/4$. In addition, our approach to implement Toffoli gate is much more efficient than the synthesis one. Assisted by two independent single photons, our
Tofoli gate is constructed with the current maximum success probability of 1/30.

Our method can be generalized to realize $n$-qubit ($n > 3$) Tofoli gate by using parity check and post-selection technique, and provides a convenient way to achieve optical quantum computing with higher success probability and lower cost by solely using linear optics. We note that the success of the gates can be verified by the final detection of outing photons using the photon-number-resolving detector (PNRD) or by post-selection principle. The PNRD can distinguish the number of photons, which is an important resource for many quantum information processing tasks. It has been realized in experiments. Unfortunately, the destructive measurement technique results in unscalable quantum computing. Our presented two architectures open an alternative insight into probabilistic quantum gates using linear optical elements and suggest that they maybe have various applications in photonic quantum information processing.

Acknowledgements
This work was supported by the National Natural Science Foundation of China under Grant No. 11604012, the Fundamental Research Funds for the Central Universities under Grants FRF-TP-19-011A3.

Conflict of Interest
The authors declare no conflict of interest.

Data Availability Statement
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords
linear optics, quantum circuit, quantum computing, quantum gate

Received: January 11, 2023
Revised: February 21, 2023
Published online: March 23, 2023

[1] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition, Cambridge University Press, Cambridge, 2010.
[2] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, H. Weinfurter, Phys. Rev. A 1995, 52, 3457.
[3] M. Møttönen, J. J. Vartiainen, V. Bergholm, M. M. Salomaa, Phys. Rev. Lett. 2004, 93, 130502.
[4] W. Q. Liu, H. R. Wei, New J. Phys. 2020, 22, 063026.
[5] W. Q. Liu, H. R. Wei, L. C. Kwek, Phys. Rev. Appl. 2020, 14, 054057.
[6] A. S. Nikolaeva, E. O. Kitkenko, A. K. Fedorov, Phys. Rev. A 2022, 105, 032621.
[7] L. K. Grover, Phys. Rev. Lett. 1997, 79, 325.
[8] P. W. Shor, SIAM J. Comput. 1997, 26, 1484.
[9] K. Bhatti, A. Cervera-Lierta, T. H. Kyaw, T. Haug, S. Alperin-Lea, A. Anand, M. Degroote, H. Heimonen, J. S. Kottmann, T. Menke, W. K. Mok, S. Sim, L. C. Kwek, A. Aspuru-Guzik, Rev. Mod. Phys. 2022, 94, 015004.
[10] I. M. Georgescu, S. Ashhab, F. Nori, Rev. Mod. Phys. 2014, 86, 153.
[11] J. W. Pan, Z. B. Chen, C. Y. Lu, H. Weinfurter, A. Zeilinger, M. Žukowski, Rev. Mod. Phys. 2012, 84, 777.
[12] J. L. O’Brien, Science 2007, 318, 1567.
[13] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, G. J. Milburn, Rev. Mod. Phys. 2007, 79, 135.
[14] E. Knill, R. Laamme, C. J. Milburn, Nature 2001, 409, 46. 
[15] X. Li, Y. Wu, D. Steel, D. Gammon, T. H. Stievater, D. S. Katzer, D. Park, C. Piermarocchi, L. J. Sharm, Science 2003, 301, 809.
[16] H. R. Wei, F. G. Deng, Phys. Rev. A 2013, 88, 042323.
[17] I. I. Beterov, I. N. Ashkharin, E. A. Yakshina, D. B. Tretyakov, V. M. Entin, I. I. Ryabtsev, P. Cheinet, P. Pilet, M. Saffman, Phys. Rev. A 2018, 98, 042704.
[18] T. C. Ralph, A. G. White, W. J. Munro, C. J. Milburn, Phys. Rev. A 2001, 65, 012314.
[19] T. B. Pittman, B. C. Jacobs, J. D. Franson, Phys. Rev. A 2001, 64, 062311.
[20] T. B. Pittman, B. C. Jacobs, J. D. Franson, Phys. Rev. Lett. 2002, 88, 257902.
[21] S. Gasparoni, J. W. Pan, P. Walther, T. Rudolph, A. Zeilinger, Phys. Rev. Lett. 2004, 93, 020504.
[22] Z. Zhao, A. N. Zhang, Y. A. Chen, H. Zhang, J. F. Du, T. Yang, J. W. Pan, Phys. Rev. Lett. 2005, 94, 030501.
[23] X. Q. Zhou, T. C. Ralph, P. Kalasuwan, M. Zhang, A. Peruzzo, B. P. Lanyon, J. L. O’Brien, Nat. Commun. 2011, 2, 413.
[24] J. Zeuner, A. N. Sharma, M. Tillmann, R. Heilmann, M. Gräfe, A. Mochanaki, A. Szameit, P. Walther, npj Quantum Inf. 2018, 4, 13.
[25] H. F. Hofmann, S. Takeuchi, Phys. Rev. A 2002, 66, 024308.
[26] N. K. Langford, T. J. Weinhold, R. Prevedel, K. J. Resch, A. Gilchrist, J. L. O’Brien, G. J. Pryde, A. G. White, Phys. Rev. Lett. 2005, 95, 210504.
[27] N. Kiesel, C. Schmid, U. Weber, R. Ursin, H. Weinfurter, Phys. Rev. Lett. 2005, 95, 210505.
[28] K. Lern, A. Černoch, J. Soubusta, K. Kielping, J. Eisert, M. Dušek, Phys. Rev. Lett. 2011, 106, 013602.
[29] J. L. O’Brien, G. J. Pryde, A. G. White, T. C. Ralph, D. Branning, Nature 2003, 426, 264.
[30] R. Okamoto, H. F. Hofmann, S. Takeuchi, K. Sasaki, Phys. Rev. Lett. 2005, 95, 210506.
[31] Y. M. He, Y. He, Y. J. Wei, D. Wu, M. Atature, C. Schneider, S. Höfling, M. Kamp, C. Y. Lu, J. W. Pan, Nat. Nanotechnol. 2013, 8, 213.
[32] X. H. Bao, T. Y. Chen, Q. Zhang, J. Yang, H. Zhang, T. Yang, J. W. Pan, Phys. Rev. Lett. 2007, 98, 170502.
[33] J. P. Li, X. Gu, J. Qin, D. Wu, X. You, H. Wang, C. Schneider, S. Höfling, Y. H. Huo, C. Y. Lu, N. L. Liu, L. J. J. Wu, Pan, Phys. Rev. Lett. 2021, 126, 140501.
[34] R. Okamoto, J. L. O’Brien, H. F. Hofmann, S. Takeuchi, Proc. Natl. Acad. Sci. U. S. A. 2011, 108, 10067.
[35] S. U. Shringarpure, J. D. Franson, Sci. Rep. 2021, 11, 22067.
[36] J. H. Plantenberg, P. C. de Groot, C. J. P. M. Harmsen, J. E. Mooij, Nature 2007, 447, 836.
[37] L. Isenhower, E. Urban, X. L. Zhang, A. T. Gill, T. Henage, T. A. Johnson, T. C. Walker, M. Saffman, Phys. Rev. Lett. 2010, 104, 010503.
[38] Y. Wan, D. Kienzler, S. D. Erickson, K. H. Mayer, T. R. Tan, J. J. Wu, H. M. Vasconcelos, S. Glancy, E. Knill, D. J. Wineland, A. C. Wilson, D. Leibfried, Science 2019, 364, 875.
[39] C. Couteau, Contemp. Phys. 2018, 59, 291.
[40] L. M. K. Vandersypen, M. Steffen, G. Breyta, C. S. Yannoni, M. H. Sherwood, I. L. Chuang, Nature 2001, 414, 833.
[41] K. A. Brickman, P. C. Haljan, P. J. Lee, M. Acton, L. Deslauriers, C. Monroe, Phys. Rev. A 2005, 72, 050306.
[42] G. A. Barbosa, Phys. Rev. A 2006, 73, 052321.
[43] D. C. Cory, M. D. Price, W. Maas, E. Knill, R. Laflamme, W. H. Zurek, T. F. Havel, S. S. Somaroo, Phys. Rev. Lett. 1998, 81, 2152.
[44] A. Paetznick, B. W. Reichardt, Phys. Rev. Lett. 2013, 111, 090505.
[45] T. Monz, K. Kim, W. Hänsel, M. Riebe, A. S. Villar, P. Schindler, M. Chwalla, M. Hennrich, R. Blatt, Phys. Rev. Lett. 2009, 102, 040501.

[46] A. Fedorov, L. Steffen, M. Baur, M. P. da Silva, A. Wallraff, Nature 2012, 481, 170.

[47] H. Levine, A. Keesling, G. Semeghini, A. Omran, T. T. Wang, S. Ebadi, H. Bernien, M. Greiner, V. Vuletić, H. Pichler, M. D. Lukin, Phys. Rev. Lett. 2019, 123, 170503.

[48] N. Yu, R. Duan, M. Ying, Phys. Rev. A 2013, 88, 010304(R).

[49] J. Fiurášek, Phys. Rev. A 2006, 73, 062313.

[50] T. C. Ralph, K. J. Resch, A. Gilchrist, Phys. Rev. A 2007, 75, 022313.

[51] B. P. Lanyon, M. Barbieri, M. P. Almeida, T. Jennewein, T. C. Ralph, K. J. Resch, J. L. O’Brien, A. Gilchrist, A. G. White, Nat. Phys. 2009, 5, 134.

[52] W. Q. Liu, H. R. Wei, L. C. Kwek, Adv. Quantum Technol. 2022, 5, 2100136.

[53] Y. Li, L. Wan, H. Zhang, H. Zhu, Y. Shi, L. K. Chin, X. Zhou, L. C. Kwek, A. Q. Liu, npj Quantum Inf. 2022, 8, 112.

[54] M. Li, C. Li, Y. Chen, L. T. Feng, L. Yan, Q. Zhang, J. Bao, B. H. Liu, X. F. Ren, J. Wang, S. Wang, Y. Gao, X. Hu, Q. Gong, Y. Li, Photonics Res. 2022, 10, 1533.

[55] M. Mičuda, M. Sedláčk, I. Straka, M. Miková, M. Dušek, M. Ježek, J. Fiurášek, Phys. Rev. Lett. 2013, 111, 160407.

[56] M. Mičuda, M. Miková, I. Straka, M. Sedláčk, M. Dušek, M. Ježek, J. Fiurášek, Phys. Rev. A 2015, 92, 032312.

[57] Q. Zeng, T. Li, X. Song, X. Zhang, Opt. Express 2016, 24, 8186.

[58] H. L. Huang, W. S. Bao, T. Li, F. G. Li, X. Q. Fu, S. Zhang, H. L. Zhang, X. Wang, Phys. Lett. A 2017, 381, 2673.

[59] S. Ru, Y. Wang, M. An, F. Wang, P. Zhang, F. Li, Phys. Rev. A 2021, 103, 022606.

[60] R. Simon, N. Mukunda, Phys. Lett. A 1989, 138, 474.

[61] R. Simon, N. Mukunda, Phys. Lett. A 1990, 143, 165.

[62] B. N. Simon, C. M. Chandrashekar, S. Simon, Phys. Rev. A 2012, 85, 022323.

[63] T. B. Pittman, B. C. Jacobs, J. D. Franson, Phys. Rev. A 2002, 66, 052305.

[64] R. Prevedel, P. Walther, F. Tiefenbacher, P. Böhi, R. Kaltenbaek, T. Jennewein, A. Zeilinger, Nature 2007, 445, 65.

[65] A. Divochiy, F. Marsili, D. Bitauld, A. Gaggero, R. Leoni, F. Mattioli, A. Kornee, V. Selezn, N. Kaur, G. Mina, G. Gol’tsman, K. G. Lagoudakis, M. Benkhaoul, F. Lévy, A. Fiore, Nat. Photonics 2008, 2, 302.

[66] R. Nehra, C. H. Chang, Q. Yu, A. Beling, O. Pfister, Opt. Express 2020, 28, 3660.

[67] R. Cheng, Y. Zhou, S. Wang, M. Shen, T. Taher, H. X. Tang, Nat. Photonics 2023, 17, 112.

[68] J. W. Pan, C. Simon, Č. Brukner, A. Zeilinger, Nature 2001, 410, 1067.