ON THE INJECTIVITY OF MEAN VALUE MAPPING BETWEEN CONVEX QUADRILATERALS

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Abstract. We prove that Mean Value mapping between convex quadrilaterals is injective, affirmatively proving a conjecture stated in [5].

1. Introduction

Let $n, d$ be integers, with $n \geq d$ and let

$$\Lambda := \{ \lambda \in \mathbb{R}^n : 1^\top \lambda = 1 \}, \quad \Lambda_+ := \{ \lambda \in [0, 1]^n : 1^\top \lambda = 1 \},$$

where $1$ is the vector in $\mathbb{R}^n$ with all components equal to one. It follows that $\Lambda_+ \subseteq \Lambda$.

Given a convex polytope $P = \text{conv}\{v_i\}_{i=1}^n$ in $\mathbb{R}^d$, a set of nonnegative generalized barycentric coordinates for a point $p \in P$ (e.g., see [4]) is an $n$-tuple $\mu \in \Lambda_+$ satisfying the underdetermined, linear system

$$\begin{bmatrix} V \\ 1^\top \end{bmatrix} \mu = \begin{bmatrix} p \\ 1 \end{bmatrix}, \quad V := \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

with the assumption that system (1.1) is full rank.

If for each $i = 1, \ldots, n$ we have $\mu_i \geq 0$ then we call $\mu$ a set of generalized barycentric coordinates for $p \in P$.

Hereafter, $\text{conv}V$ will indicate the convex hull of the $v_i$’s, that is the polytope $P$. Further, let $\nu \in \mathbb{R}^{n \times (n-d-1)}$ be such that

$$\langle \nu \rangle = \ker \begin{bmatrix} V \\ 1^\top \end{bmatrix}.$$

Now, take two different sets of $n$ vertices in $\mathbb{R}^d$, $V$ and $\tilde{V}$, such that $\begin{bmatrix} V \\ 1^\top \end{bmatrix}$, $\begin{bmatrix} \tilde{V} \\ 1^\top \end{bmatrix}$ are full rank, and consider the map

$$f : P \to \tilde{P}, \quad p \mapsto \tilde{p} := \tilde{V} \mu(p)$$

where $\mu(p)$ is a set of nonnegative generalized barycentric coordinates for $p \in P$ and $\tilde{V} := \text{conv}\tilde{V}$. Such a map $f$ is called barycentric mapping between the polytopes $P, \tilde{P}$. The problem addressed in [5] (see also [8] for a recent numerical attempt to tackle it) is whether or not this map is injective.

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In this work, we will restrict to the case of a polygon \( P \) (that is, \( d = 2 \)) and to the barycentric coordinates given by the \textbf{mean-value coordinates}. These were originally proposed by M. Floater in 2005 (see [3, 7]), who defined them, for any \( p \in P \), as

\[
\lambda_i(p) := \frac{w_i(p)}{\sum_{j=1}^{n} w_j(p)}, \quad w_i(p) := \tan\left(\frac{\alpha_i - 1}{2}\right) + \tan\left(\frac{\alpha_i}{2}\right) \frac{\|v_i - p\|}{\|v_i - p\|}, \quad i = 1, \ldots, n,
\]

where the angles \( \alpha_i \)'s are as in Figure 1 and the norm is the Euclidean norm.

As already thoroughly explained in [5], any map of the form (1.3) injectively maps the boundary of \( P \) to the boundary of \( \tilde{P} \). In the same paper [5], Floater and Kosinka showed that if \( n \geq 5 \), then the mean-value coordinates are not injective, unlike other nonnegative barycentric coordinates, such as Wachspress coordinates [10], that are injective for any pair of strictly convex polygons. Yet, Floater and Kosinka left open the important case of quadrilaterals (i.e, \( n = 4 \)), which can be stated as follows.

**Conjecture 1.1.** Let \( v_i, i = 1, 2, 3, 4 \) be such that no three of them are aligned, and let the same hold for \( \tilde{v}_i, i = 1, 2, 3, 4 \). Then the mapping

\[
f : \text{int} P \to \text{int} \tilde{P}, \quad p \mapsto \tilde{p} := \tilde{V} \mu(p).
\]

relative to the mean-value coordinates \( \mu(p) \) between convex quadrilaterals is injective.

**Remark 1.2.** The fact that (1.3) maps \( \text{int} P \) to \( \text{int} \tilde{P} \) is a consequence of the fact that \( \mu(p) \) are the mean value coordinates, in particular all its components are positive, and of the following reasoning. Let \( f(p) \) belong to the boundary of \( \tilde{P} \), say the edge \( \tilde{v}_1\tilde{v}_2 \) without loss of generality, and let \( \tilde{\lambda} = \begin{bmatrix} 1 - \alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^4 \) be its barycentric coordinates for some \( \alpha \in [0, 1] \). Therefore \( \tilde{V} \mu(p) = \tilde{V} \tilde{\lambda} \), and thus there must exist some \( c \neq 0 \) such that

\[
\mu(p) = \tilde{\lambda} + c \tilde{v},
\]

where \( \tilde{v} \) spans \( \text{ker} \begin{bmatrix} \tilde{V} \\ 1 \end{bmatrix} \). According to Theorem 1.3 below it follows that \( \text{sgn} \mu_3(p) \neq \text{sgn} \mu_4(p) \), which is not possible.

Although the authors of [5] reported on extensive numerical simulations leading them to believe the conjecture to be true, a rigorous proof of Conjecture 1.1 is still lacking and our purpose in this note is to prove that Conjecture 1.1 holds true.

Our proof of Conjecture 1.1 is motivated by the following result, that gives an equivalence between the mean-value coordinates on convex quadrilaterals and the solution of
the following regularized linear system

\begin{equation}
\begin{bmatrix}
V \\
1^T \\
d^T
\end{bmatrix} \lambda(p) = 
\begin{bmatrix}
p \\
1 \\
0
\end{bmatrix},
\end{equation}

where

\[d(p) := \begin{bmatrix} d_1(p) & -d_2(p) & d_3(p) & -d_4(p) \end{bmatrix}^T,\]

\[d_i(p) := \|v_i - p\|,\]

\section*{Theorem 1.3.}
For each \(p \in Q\) the system (1.5) is nonsingular, and its unique solution 
\(\lambda_{MV}(p)\) is given by the mean-value coordinates (1.4). In particular, all the components of \(\lambda_{MV}(p)\) are nonnegative for \(p \in Q\), and are strictly positive for \(p \in \text{int}Q\). Moreover, the general solution \(\mu\) to (1.1) can be written as \(\mu = \lambda_{MV} + c\nu\), where \(\nu \in \mathbb{R}^4\) is as in (1.2) and \(\text{sgn}(\nu) = \pm \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T\).

\begin{proof}
What is left to prove is that \(\text{sgn}(\nu) = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T\), as the remaining parts come from [2, Theorem 3.9]. Let \(\tau\) be the barycentric coordinates of \(v_4\) with respect to the triangle \(\text{conv}\{v_1, v_2, v_3\}\), that is the unique solution to

\[\begin{bmatrix} v_1 & v_2 & v_3 \\
1 & 1 & 1
\end{bmatrix} \tau = \begin{bmatrix} v_4 \\
1
\end{bmatrix}.
\]

Using Cramer’s rule we get

\[\tau = \frac{1}{A_{123}} \begin{bmatrix} A_{143} \\
A_{124}
\end{bmatrix},\]

where \(A_{ijk} := \frac{1}{2} \det \begin{bmatrix} v_i & v_j & v_k \\
1 & 1 & 1
\end{bmatrix}\) represents the signed area of the triangle \(v_i, v_j, v_k\). Since \(Q\) is convex, it then follows that \(\text{sgn}(\tau) = [1 \ -1 \ 1 \ -1]^T\) (e.g., see [1]). Let \(\nu' := \begin{bmatrix} \tau \\
-1
\end{bmatrix}\). Then

\[\begin{bmatrix} V \\
1^T
\end{bmatrix} \nu' = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix},\]

which implies that \(\nu' \in \ker \begin{bmatrix} V \\
1^T
\end{bmatrix}\). Thus, there exists \(\alpha \neq 0\) such that \(\nu' = \alpha \nu\), and the claim follows.
\end{proof}

Hereafter, we choose \(\nu\) so that \(\text{sgn}(\nu) = [1 \ -1 \ 1 \ -1]^T\).

For proving Conjecture 1.1 we need an intermediate result.

\begin{lemma}
Let \(P = \text{conv}V\) be a convex quadrilateral. Any set of barycentric coordinates \(\mu : P \to \Lambda\) is bijective.
\end{lemma}

\begin{proof}
Surjectivity follows from the definition of \(P\), see [9]. We show injectivity.
\end{proof}
Let $p, q \in P$, $p \neq q$. If, by contradiction, $\mu(p) = \mu(q)$, then $\lambda(p) + c_p \nu = \lambda(q) + c_q \nu$ for some values of $c_p, c_q \in \mathbb{R}$, where $\lambda$ are the mean-value coordinates. Then
\[
\begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} V \\ I^\top \end{bmatrix} (\lambda(p) + c_p \nu) = \begin{bmatrix} V \\ I^\top \end{bmatrix} (\lambda(q) + c_q \nu) = \begin{bmatrix} q \\ 1 \end{bmatrix},
\]
from which $p = q$, yielding a contradiction. 

Further, we need a general result from projective geometry, for which we refer to classical references.

**Lemma 1.5.** Let $Q \subseteq \mathbb{R}^2$ be a convex polytope. Let $P$ be a nonsingular projective transformation

\[
P : Q \to \mathbb{R}^2
\]

\[
p \mapsto \frac{Ap + b}{k^\top p + \delta}
\]

where $A \in \mathbb{R}^{2\times 2}$ is a linear transformation, $b, k \in \mathbb{R}^2$, $\delta \in \mathbb{R}$. Let also $P$ be permissible for $Q$, i.e. $k^\top p + \delta \neq 0$ for $p \in Q$. Then $P$ is injective.

**Proof.** We are going to prove that the determinant of the Jacobian of $P$ is
\[
\det JP(p) = \frac{1}{(k^\top p + \delta)^3} \det \begin{bmatrix} A & b \\ k^\top & \delta \end{bmatrix}, \quad p \in Q.
\]

In fact, let $m := k^\top p + \delta$, with $p = \begin{bmatrix} x \\ y \end{bmatrix}$. Thus $m = k_1 x + k_2 y + \delta$ and
\[
P(p) = \frac{1}{m} \begin{bmatrix} a_{11} x + a_{12} y + b_1 \\ a_{21} x + a_{22} y + b_2 \end{bmatrix}.
\]

Therefore we have
\[
JP(p) = \frac{1}{m^2} \begin{bmatrix} a_{11} m - k_1 (a_{11} x + a_{12} y + b_1) & a_{12} m - k_2 (a_{11} x + a_{12} y + b_1) \\ a_{21} m - k_1 (a_{21} x + a_{22} y + b_2) & a_{22} m - k_2 (a_{21} x + a_{22} y + b_2) \end{bmatrix},
\]
and hence
\[
\det JP(p) = \frac{1}{m^4} (m^2 \det A - a_{11} m k_2 (a_{21} x + a_{22} y + b_2) - a_{22} m k_1 (a_{11} x + a_{12} y + b_1) \\
+ a_{12} m k_1 (a_{21} x + a_{22} y + b_2) + a_{21} m k_2 (a_{11} x + a_{12} y + b_1))
\]
\[
= \frac{1}{m^4} (m \det A + (a_{11} x + a_{12} y + b_1) (k_2 a_{21} - k_1 a_{22}) \\
+ (a_{21} x + a_{22} y + b_2) (k_1 a_{12} - k_2 a_{11}))
\]
\[
= \frac{1}{m^4} (m \det A - k_1 x \det A - k_2 y \det A \\
+ k_1 (a_{12} b_2 - a_{22} b_1) - k_2 (a_{11} b_2 - a_{21} b_1))
\]
\[
= \frac{1}{m^4} (\delta \det A + k_1 (a_{12} b_2 - a_{22} b_1) - k_2 (a_{11} b_2 - a_{21} b_1))
\]
\[
= \frac{1}{m^4} \det \begin{bmatrix} A & b \\ k^\top & \delta \end{bmatrix}.
\]
Since the last quantity is nonzero for every \( p \in Q \), being \( P \) nonsingular and permissible for \( Q \), the claim is proved.

2. Proof of Conjecture 1.1

We are ready to prove the main result of the paper.

**Theorem 2.1.** Conjecture 1.1 is true.

**Proof.** Let \( V, \tilde{V} \) be such that their respective convex hulls are the two convex quadrilaterals \( Q, \tilde{Q} \). Moreover, let \( p, q \in \text{conv} V \), \( p \neq q \), and let \( \tilde{p} := \tilde{V} \lambda(p), \tilde{q} := \tilde{V} \lambda(q) \), where \( \lambda \) are the mean-value coordinates. Let us assume by contradiction that \( f(p) = \tilde{p} = \tilde{q} = f(q) \).

Let also \( \langle \nu \rangle = \ker \begin{bmatrix} V \\ I^\top \end{bmatrix} \), \( \langle \tilde{\nu} \rangle = \ker \begin{bmatrix} \tilde{V} \\ I^\top \end{bmatrix} \). We claim that \( \tilde{p} \neq \tilde{q} \).

If either \( p, q \) or both belongs to \( \partial Q \), then the claim easily follow; thus, hereafter we assume \( p, q \in \breve{Q} \).

We have to consider two cases.

(i): \( \langle \nu \rangle = \langle \tilde{\nu} \rangle \). Since we are assuming \( \tilde{p} = \tilde{q} \), then \( \tilde{V}(\lambda(p) - \lambda(q)) = 0 \), hence \( \lambda(p) - \lambda(q) = \tilde{c} \nu = c \nu \), for some \( \tilde{c}, c \). Then, it follows that, for some appropriate constants, \( \lambda(p) + c_\nu \nu = \lambda(q) + c_\nu \nu \), and so

\[
\begin{bmatrix} V \\ I^\top \end{bmatrix} \lambda(p) + c_\nu \nu = \begin{bmatrix} V \\ I^\top \end{bmatrix} \lambda(q) + c_\nu \nu,
\]

giving \( p = q \), which proves the claim.

(ii): \( \langle \nu \rangle \neq \langle \tilde{\nu} \rangle \). Without loss of generality, let \( \lambda(p) \neq \lambda(q) \), otherwise Lemma 1.4 gives the result. Since \( \text{sgn}(\nu) = \text{sgn}(\tilde{\nu}) \) then, for each \( i = 1, 2, 3, 4 \), there exist \( \rho_i > 0 \) such that

\[
\tilde{v}_i = \rho_i v_i.
\]

By [6, Sec. 5.4, Th. (vi)], there exist \( k \in \mathbb{R}^2 \) and \( \delta \in \mathbb{R} \) such that \( \rho_i = k^\top v_i + \delta \) for \( i = 1, 2, 3, 4 \), an invertible linear transformation \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) and a vector \( b \in \mathbb{R}^2 \) such that

\[
Pv = \frac{Av + b}{k^\top v + \delta}
\]

is a regular projective transformation satisfying

\[
\tilde{v}_i = Pv_i, \quad i = 1, 2, 3, 4,
\]

and permissible for \( \text{conv} V \), i.e., such that \( k^\top v + \delta \neq 0 \) for each \( v \in \text{conv} V \). It also follows that \( Q, \tilde{Q} \) are projectively equivalent. Now, letting \( \rho := [\rho_1 \ \rho_2 \ \rho_3 \ \rho_4]^\top \) and \( R := \text{diag}(\rho) \), and defining

\[
\bar{\lambda}(p) := R^{-1} \lambda(p), \quad \tilde{p} := V\bar{\lambda}(p), \quad p \in Q,
\]

we prove that

\[
P\tilde{p} = \tilde{V}\lambda(p).
\]
In fact:

\[ P\bar{p} = \frac{A\bar{p} + b}{k^\top \bar{p} + \delta} = \frac{\sum_{i=1}^{4} (Av_i + b)\bar{x}_i(p)}{\sum_{i=1}^{4} (k^\top v_i + \delta)\bar{x}_i(p)} = \frac{\sum_{i=1}^{4} \rho_i \bar{x}_i(p) \bar{v}_i}{\sum_{i=1}^{4} \rho_i \bar{x}_i(p)} = \sum_{i=1}^{4} \lambda_i(p) \bar{v}_i = \bar{V}\lambda(p). \]

Therefore, from \( \bar{V}\lambda(p) = \bar{V}\lambda(q) \), we have \( P\bar{p} = P\bar{q} \), and so \( \bar{p} = \bar{q} \) on the account of Lemma 1.5. Moreover, by definition of \( \bar{p}, \bar{q} \), we deduce that

\[ \begin{bmatrix} V \\ 1^\top \end{bmatrix} \lambda(p) = \begin{bmatrix} V \\ 1^\top \end{bmatrix} \lambda(q), \]

from which, for some \( \bar{c} \neq 0 \),

\[ \bar{\lambda}(p) = \bar{\lambda}(q) + \bar{c} \nu. \]

Now, let us consider the application \( F : \mathbb{R}^4 \to \mathbb{R} \) defined as

\[ F(\lambda) := d(V\lambda)^\top \lambda, \]

where \( d_i(p) := d_i(p)\rho_i \), \( i = 1, 2, 3, 4 \), for \( p \in Q \).

Letting \( j = 1, 2, 3, 4 \) be given, we have that

\[ \frac{\partial F}{\partial \lambda_j}(\lambda) = \sum_{i=1}^{4} (-1)^{i+1} \lambda_i \frac{(V\lambda - v_i)^\top}{d_i(V\lambda)} v_j + (-1)^{j+1} \frac{d_j(V\lambda)}{\rho_j} \]

Hence, computing the first derivative of \( F(\bar{\lambda} + c\nu) \) with respect to \( c \), we obtain

\[ \frac{d}{dc} F(\bar{\lambda} + c\nu) = JF(\bar{\lambda} + c\nu) \cdot \nu \]

\[ = \sum_{j=1}^{4} \sum_{i=1}^{4} (-1)^{i+1} (\bar{\lambda}_i + c\nu_j) \frac{(V\bar{\lambda} - v_i)^\top}{d_i(V\bar{\lambda})} v_j \nu_j + \sum_{j=1}^{4} (-1)^{j+1} \frac{d_j(V\bar{\lambda})}{\rho_j} \nu_j \]

\[ > \sum_{i=1}^{4} (-1)^{i+1} (\bar{\lambda}_i + c\nu_j) \frac{(V\bar{\lambda} - v_i)^\top}{d_i(V\bar{\lambda})} \sum_{j=1}^{4} v_j \nu_j \]

\[ = 0, \]

since, exploiting the sign pattern of \( \nu \),

\[ \sum_{j=1}^{4} (-1)^{j+1} \frac{d_j(V\bar{\lambda})}{\rho_j} \nu_j > 0. \]

We then deduce that the functional \( F \) is injective along the line \( \bar{\lambda}(p) + c\nu \), and since \( F(\bar{\lambda}(p)) = 0 = F(\bar{\lambda}(q)) \), then \( \bar{\lambda}(p) = \bar{\lambda}(q) \). By Lemma 1.4 it follows \( p = q \), which is a contradiction, and this proves the claim. \( \square \)
3. Injectivity of barycentric mapping on quadrilaterals

Here we prove that also bilinear barycentric mappings between convex quadrilaterals are injective.
We are going to need the following.

**Proposition 3.1.** For any \( p \in \text{int}Q \) there exist \( \alpha_p, \beta_p, \varepsilon_p \in (0,1) \) such that

\[
p = (1 - \varepsilon_p) a_p + \varepsilon_p b_p,
\]

with

\[
a_p := (1 - \alpha_p) v_1 + \alpha_p v_2,
\]

\[
b_p := (1 - \beta_p) v_3 + \beta_p v_4
\]

and

\[
d(p)^\top \lambda(p) = 0, \quad \lambda(p) := [(1 - \varepsilon_p)(1 - \alpha_p)(1 - \beta_p) \varepsilon_p(1 - \beta_p)]^\top,
\]

where \( d(p) \) is as in (1.5).

Analogously, there exist \( \gamma_p, \delta_p, \varphi_p \in (0,1) \) such that

\[
p = (1 - \varphi_p) c_p + \varphi_p d_p,
\]

with

\[
c_p := (1 - \gamma_p) v_2 + \gamma_p v_3,
\]

\[
d_p := (1 - \delta_p) v_4 + \delta_p v_1
\]

and

\[
d(p)^\top \lambda(p) = 0, \quad \lambda(p) := [\varphi_p \delta_p (1 - \varphi_p)(1 - \gamma_p)(1 - \varphi_p)\gamma_p \varphi_p(1 - \delta_p)]^\top,
\]

where \( d(p) \) is as in (1.5).

**Proof.** Let \( \lambda(p) \) be the unique solution to (1.5). Therefore, the claim is proved by setting

\[
\alpha_p := \frac{\lambda_2(p)}{\lambda_1(p) + \lambda_2(p)}, \quad \beta_p := \frac{\lambda_4(p)}{\lambda_3(p) + \lambda_4(p)}, \quad \varepsilon_p := \lambda_3(p) + \lambda_4(p),
\]

\[
\gamma_p := \frac{\lambda_3(p)}{\lambda_2(p) + \lambda_3(p)}, \quad \delta_p := \frac{\lambda_1(p)}{\lambda_4(p) + \lambda_1(p)}, \quad \varphi_p := \lambda_4(p) + \lambda_1(p),
\]

which are all well defined since \( \lambda(p) > 0 \) componentwise from Theorem 1.3. \( \square \)

Let \( Q := \text{conv}\{v_i : i = 1, 2, 3, 4\} \) be a strictly convex quadrilateral. It is known that, given \( p \in Q \) there exist unique \( \alpha, \beta \in [0,1] \) such that the bilinear barycentric coordinates of \( p \) are given by

\[
\lambda(p) = \begin{bmatrix}
(1 - \alpha)(1 - \beta) \\
\alpha(1 - \beta) \\
\alpha \beta \\
(1 - \alpha) \beta
\end{bmatrix}.
\]

This implies that, letting

\[
a_p := (1 - \alpha) v_1 + \alpha v_2, \quad b_p := (1 - \alpha) v_4 + \alpha v_3,
\]
we have $p = (1 - \beta)ap + \beta bp$. It is an easy consequences of uniqueness that such $ap \in v_1v_2, bp \in v_4v_3$ are uniquely determined.

Now, let $p, q \in Q, p \neq q$. Then there exist unique $\alpha_p, \beta_p, \alpha_q, \beta_q$ such that, letting $a_p := (1 - \alpha_p)v_1 + \alpha_p v_2, b_p := (1 - \alpha_p)v_4 + \alpha_p v_3, a_q := (1 - \alpha_q)v_1 + \alpha_q v_2, b_q := (1 - \alpha_q)v_4 + \alpha_q v_3$, we have

$$p = (1 - \beta_p)ap + \beta_p bp, \quad q = (1 - \beta_q)aq + \beta_q bq.$$  

Since $p \neq q$, we have, without loss of generality, that $\alpha_p < \alpha_q$ or $\alpha_p = \alpha_q$ and $\beta_p \neq \beta_q$. If $\alpha_p = \alpha_q$ and $\beta_p \neq \beta_q$, then $ap = aq = a$ and $bp = bq = b$ and, since $f$ is linear and injective on the boundary of $Q$, it follows

$$f(p) = (1 - \beta_p)f(a) + \beta_p f(b) \neq (1 - \beta_q)f(a) + \beta_q f(b) = f(q),$$

since the mapping is restricted on the segment $ab$, and the claim is proved.

If $\alpha_p < \alpha_q$ then $ap$ would precede $aq$ on $v_1v_2$ and $bp$ would precede $bq$ on $v_4v_3$ (see Figure 2), and thus the segments $apbp$ and $aqbq$ do not intersect each other in $Q$, implying that neither do $f(ap)f(bp), f(aq)f(bq)$. Therefore, if by contradiction $f(p) = f(q)$, it would follow

$$f(ap)f(bp) \cap f(aq)f(bq) \neq \emptyset,$$

which is not possible, and the claim is proved.

Let us observe that bilinear barycentric coordinates, as well as Wachspress coordinates, are differentiable.

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