CYCLIC \((v; r, s; \lambda)\) DIFFERENCE FAMILIES WITH TWO BASE BLOCKS AND \(v \leq 50\)

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Abstract. We construct many new cyclic \((v; r, s; \lambda)\) difference families with \(v \geq 2r \geq 2s \geq 4\) and \(v \leq 50\). In particular we construct the difference families with parameters

\[
\begin{align*}
(45; 18, 10; 9), & \quad (45; 22, 22; 21), & \quad (47; 21, 12; 12), \\
(47; 19, 15; 12), & \quad (47; 22, 14; 14), & \quad (48; 20, 10; 10), \\
(48; 24, 4; 12), & \quad (50; 25, 20; 20)
\end{align*}
\]

for which the existence question was an open problem.

We point out that the \((45; 22, 22; 21)\) difference family gives a balanced incomplete block design (BIBD) with parameters \(v = 45, b = 90, r = 44, k = 22\) and \(\lambda = 21\), and that the one with parameters \((50; 25, 20; 20)\) gives a pair of binary sequences of length 50 with zero periodic autocorrelation function (the periodic analog of a Golay pair). The new SDSs include nine new D-optimal designs.

A normal form for cyclic difference families (with base blocks of arbitrary sizes) is proposed and used effectively in compiling our selective listings in Tables 3-6 of known and new difference families in the above range.

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1. Introduction

We consider difference families in finite abelian groups whose base blocks may be of different sizes, also known as supplementary difference sets (SDS). A few infinite families have been constructed, most of them by using cyclotomy. They have been studied for long time and have been used to construct balanced incomplete block designs (BIBD), Hadamard matrices, skew-Hadamard matrices, and other designs. We recommend [11, 19, 20, 27, 30] for an overview of this topic.

We shall restrict our scope here to the case of SDSs with exactly two base blocks. Their sizes will be denoted by \(r\) and \(s\). The underlying group will be cyclic of order \(v\), identified with \(\mathbb{Z}_v\). As usual, we attach to such SDS its parameters \((v; r, s; \lambda)\) and the order \(n\), where \(\lambda\) is the

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index of the family and \( n = r + s - \lambda \). For precise definition see the next section. The systematic search for these SDSs in the range \( v \leq 50 \), but restricted to odd values of \( v \), was initiated in [8]. In our paper [13] we have continued this work and extended the search to include the even values of \( v \).

There is a simple necessary condition on \((v; r, s; \lambda)\) for the existence of an SDS with this set of parameters (see (4.1) below). We refer to the parameters satisfying that condition as feasible parameters. In a more recent joint paper [21] the question of existence of SDSs with feasible parameter sets has been settled for all \( v \leq 40 \). In the range \( 40 < v \leq 50 \) there remained 23 undecided cases.

Our objective in this paper is twofold. First we feel that there is a need to collect the known results in ready for use tabular form. For that purpose we introduce a normal form for cyclic difference families (with arbitrary number of base blocks of various sizes) in order to be able to compare the results from different sources and avoid the duplication. While doing this we also constructed many new SDSs, not equivalent to the known ones. For instance, each of the Tables 5 and 6 contains 59 representatives of new equivalence classes of SDSs. These two tables contain new D-optimal designs with parameters

\[
(31;15,10;10), \quad (37;16,13;11), \quad (41;16,16;12), \\
(43;18,16;13), \quad (43;21,15;15), \quad (49;22,18;16)
\]

(nine new designs in total).

Our second objective is to make further contribution to the question of existence of SDSs with the given feasible set of parameters. We have constructed SDSs proving the existence in the following 8 of the 23 previously undecided cases with \( v \leq 50 \):

\[
(45;18,10;9), \quad (45;22,22;21), \quad (47;21,12;12), \quad (47;19,15;12), \\
(47;22,14;14), \quad (48;20,10;10), \quad (48;24,4;12), \quad (50;25,20;20).
\]

We pay special attention to three types of these difference families that were investigated separately in the past. The first type are the SDSs with \( r = s \), i.e., the difference families with two base blocks of the same size. They are important because they can be used to construct BIBD’s. For instance, the \((45;22,22;21)\) difference family gives rise to a BIBD with parameters mentioned in the abstract. The second type are the SDSs with \( v = 2n + 1 \). They are used to construct D-optimal designs of order \( 2v \) of circulant i.e., Ehlich type. The third type are the SDSs with \( v = 2n \). They are equivalent to pairs of binary sequences of length \( v \) having zero periodic autocorrelation function (see [5] for
a more general fact). The latter are important in various engineering applications. Apart from the lengths $v$ of known Golay pairs, there are only two known even integers $v$ for which such binary sequences are known: 34 and 50. The ones with $v = 50$ are constructed in this paper.

Our results are presented in Tables 3-6. The most interesting ones are in the last table.

All new SDSs recorded in Tables 3-6 have been constructed by running our genetic type algorithm, sometimes for several days. For a short description of the algorithm we refer the reader to [24], but see also [4].

2. Equivalence of difference families

As we are interested in recording known cyclic difference families having exactly two base blocks (not necessarily distinct or of the same size), we are faced with the problem of testing different families for equivalence. Of course, first we have to define the notion of equivalence for difference families. Then, we have to construct a suitable normal form. We shall be more general and allow any number of base blocks.

Let $v$ be a positive integer and denote by $\mathbb{Z}_v = \{0, 1, \ldots, v - 1\}$ the ring (and also the additive group) of integers modulo $v$. By $\mathbb{Z}_v^*$ we denote the (multiplicative) group of invertible elements of $\mathbb{Z}_v$. To any subset $X$ of $\mathbb{Z}_v$ we associate a function $N_X : \mathbb{Z}_v \rightarrow \mathbb{Z}$ whose value at a point $a \in \mathbb{Z}_v$ is equal to the cardinal of the set $\{(x, y) \in X \times X : y - x = a\}$. Note that $N_X(0) = |X|$, the size of $X$.

**Definition 2.1.** We say that a sequence $\mathcal{X} = (X_1, \ldots, X_m)$, with $X_i \subseteq \mathbb{Z}_v$ for all $i$, is a difference family or that $X_1, \ldots, X_m$ are supplementary difference sets (SDS) if the function

$$N_{\mathcal{X}} = N_{X_1} + \cdots + N_{X_m}$$

takes the same value, say $\lambda$, on all nonzero elements $a \in \mathbb{Z}_v$. In that case, if $k_i = |X_i|$, we say that

$$(v; k_1, \ldots, k_m; \lambda)$$

is the set of parameters of $\mathcal{X}$, the $X_i$’s are its base blocks, $\lambda$ is its index and

$$n = k_1 + \cdots + k_m - \lambda$$

is its order. If also $m = 1$, we say that $X_1$ is a difference set. We denote by $\mathcal{F}_m$ the set of all difference families $(X_1, \ldots, X_m)$ in $\mathbb{Z}_v$.

Let us introduce the elementary transformations $\mathcal{X} \rightarrow \mathcal{Y}$ on the set $\mathcal{F}_m$. There are five types of such transformations. For $X \subseteq \mathbb{Z}_v$, we
denote by $X^\dagger$ the symmetric difference of $X$ and the set of odd integers in $Z_v$.

Let $\mathcal{X} = (X_1, \ldots, X_m) \in F_m$ and let $\mathcal{Y} = (Y_1, \ldots, Y_m)$ be obtained from $\mathcal{X}$ by one of the following elementary transformations:

(i) Replace an $X_i$ with $X_i + t$ for some $t \in Z_v$.
(ii) For some $a \in Z_v^*$, replace each $X_i$ with $a \cdot X_i$.
(iii) Replace an $X_i$ with $(-1) \cdot X_i$.
(iv) Replace an $X_i$ with its complement in $Z_v$.
(v) If $v$ is even and $mv = 4n$, replace each $X_i$ with $X_i^\dagger$.

It is easy to verify that $\mathcal{Y} \in F_m$. If needed, consult e.g. [5] for more details about the case (v).

**Definition 2.2.** We say that $\mathcal{X}, \mathcal{Y} \in F_m$ are equivalent if there exists a finite sequence of elementary transformations which sends $\mathcal{X}$ to $\mathcal{Y}$. This is an equivalence relation on $F_m$.

Note that equivalent SDSs $\mathcal{X}$ and $\mathcal{Y}$ may have different sets of parameters. For instance, if $\mathcal{X}$ has parameters (2.1) and $\mathcal{Y}$ is obtained from $\mathcal{X}$ by the elementary transformation (iv), then the set of parameters of $\mathcal{Y}$ is

$$(v; k_1, \ldots, k_{i-1}, v - k_i, k_{i+1}, \ldots, k_m; \lambda + v - 2k_i).$$

In the case when $m = 2$ and $v$ is odd, this definition of equivalence is different from the one adopted in [22], which additionally allows the $X_i$'s to be permuted.

### 3. Normal Form for Difference Families

Let $X$ be a $k$-subset of $Z_v$ and let $d$ be any nonnegative integer. As we have identified $Z_v$ with a subset of $Z$, it makes sense to raise each $x \in X$ to power $d$ in the ring $Z$ and add all these powers in $Z$. By convention, $x^0 = 1$ for all $x \in Z_v$. We denote this sum by $\sigma_d(X)$. Thus

$$\sigma_d(X) = \sum_{x \in X} x^d \in Z.$$

Now let $X$ and $Y$ be any two subsets of $Z_v$. If $\sigma_d(X) = \sigma_d(Y)$ for all $0 \leq d < v$, then $X = Y$. We shall write $X < Y$ if there exists a nonnegative integer $l$ such that $\sigma_d(X) = \sigma_d(Y)$ for $0 \leq d < l$ and $\sigma_l(X) < \sigma_l(Y)$. If $X < Y$ or $X = Y$, then we shall write $X \leq Y$. Clearly this defines a total order on the set of subsets of $Z_v$.

For $\mathcal{X} = (X_1, \ldots, X_m) \in F_m$ and any nonnegative integer $d$ we set

$$\sigma_d(\mathcal{X}) = \sigma_d(X_1) + \cdots + \sigma_d(X_m).$$

For $\mathcal{X}, \mathcal{Y} \in F_m$, we shall write $\mathcal{X} < \mathcal{Y}$ if there exists a nonnegative integer $l$ such that $\sigma_d(\mathcal{X}) = \sigma_d(\mathcal{Y})$ for $1 \leq d < l$ and $\sigma_l(\mathcal{X}) < \sigma_l(\mathcal{Y})$. If
$X < Y$ or $X = Y$, then we shall write $X \leq Y$. The binary relation $\leq$ on $F_m$ is reflexive and transitive. Moreover, if $X, Y \in F_m$ then at least one of the inequalities $X \leq Y$ and $Y \leq X$ holds.

We can now define our normal form.

**Definition 3.1.** Let $X \in F_m$ and let $E \subseteq F_m$ be its equivalence class. If there exists a $Y \in E$ such that

(i) $Y \leq Z$ for all $Z \in E$,
(ii) $Y$ is unique up to a permutation,

then we say that $Y$ is the *normal form* of $X$. We shall denote this normal form by $\nu(X)$.

Let us point out some peculiar features of this normal form.

First of all, due to the condition (ii), it is not clear whether this normal form always exists. It does in all cases that we have encountered so far. It would be of interest to prove that (ii) is a consequence of (i).

Our normal form exhibits a very strong bias towards the small integers. In particular each nonempty subset in the normal form contains 0. All difference families in this paper will be given in this normal form.

To illustrate the normal form in the simplest case, when $m = 1$, we shall give in Table 1 the normal forms of all cyclic $(v, k, \lambda)$ difference sets with $v \leq 50$.

**Remark 3.2.** One can use cyclic difference sets to construct difference families. For instance, $X = (\{0, 1, 4, 6\}, \{0, 1, 4, 6\})$ is a $(13; 4, 4; 2)$ difference family. By replacing the second base block with the equivalent difference set $\{0, 2, 8, 12\} = 2 \cdot \{0, 1, 4, 6\}$, we obtain the difference family $Y = (\{0, 1, 4, 6\}, \{0, 2, 8, 12\})$ having the same parameters $(13; 4, 4; 2)$. Its normal form is $\nu(Y) = (\{0, 1, 4, 6\}, \{0, 2, 3, 7\})$. It is easy to check that these two difference families are equivalent neither according to the definition [22] (e.g., they have different normal forms) nor the one adopted in [22]. Hence, the claim made there that there is only one equivalence class of $(13; 4, 4; 2)$ difference families is in error. This example shows that both definitions are inadequate when dealing with the difference families having additional symmetry properties. One can refine the notion of equivalence and normal form to handle also such cases but we shall not attempt to do it in this paper.
Table 1: Cyclic \((v,k,\lambda)\) difference sets with \(v \leq 50\)

\[(v, k, \lambda) \quad \text{Difference set in the normal form}\]

\[
egin{align*}
(7,3,1) & \quad \{0,1,3\} \\
(11,5,2) & \quad \{0,1,2,4,7\} \\
(13,4,1) & \quad \{0,1,4,6\} \\
(15,7,3) & \quad \{0,1,2,4,5,8,10\} \\
(19,9,4) & \quad \{0,1,2,3,5,7,12,13,16\} \\
(21,5,1) & \quad \{0,3,4,9,11\} \\
(23,11,5) & \quad \{0,1,2,3,5,7,8,11,12,15,17\} \\
(31,6,1) & \quad \{0,1,3,8,12,18\} \\
(31,15,7) & \quad \{0,1,3,4,7,8,9,10,12,15,17,19,20,21,25\} \\
(35,17,8) & \quad \{0,1,3,4,5,6,8,10,13,14,16,17,19,23,24,25,31\} \\
(37,9,2) & \quad \{0,1,4,6,10,15,17,18,25\} \\
(40,13,4) & \quad \{0,1,2,4,5,8,13,14,17,19,24,26,34\} \\
(43,21,10) & \quad \{0,1,2,4,8,9,10,11,12,14,15,16,19,21,24,27,28,30,32,33,37\} \\
(47,23,11) & \quad \{0,1,2,3,5,6,7,8,11,13,15,16,17,20,23,24,26,27,31,33,35,36,41\}
\end{align*}
\]

By using this normal form, we found that some families or their equivalents have been listed more than once in the same paper or by another author. For instance, in the paper [9] one finds a list of four SDSs \((B_i, D_i)\) with parameters \((31;15,10;10)\). The last three of them are all equivalent:

\[
B_3 = 3B_2, \quad D_3 = 3D_2, \quad B_4 = 7B_3, \quad D_4 = 7D_3. \quad \text{(mod 31)}
\]

The first SDS, \((B_1, D_1)\), has been rediscovered in [18].

There are 10 SDSs in [9] with parameters \((43;21,15;15)\). All of them are indeed non-equivalent. The first one of them again has been rediscovered in [18].

We also found a few errors. For instance, in the paper [8] in the cases \((25;4,12;6)\) and \((27;3,5;1)\) the base blocks \(D\) have wrong size, and in the case \((49;13,21;12)\) both blocks \(C\) and \(D\) have wrong size. We have constructed the SDSs in all three cases (see our tables below).

The often quoted table by Takeuchi [29] also has an error: The item No. 31 is supposed to give an SDS with parameters \((17;8,8;7)\) but the first base block given there is of size 7.
4. Difference families with two base blocks

From now on we shall consider only difference families with exactly two base blocks (not necessarily distinct) and we shall write their parameter sets as \((v; r, s; \lambda)\). These parameters must satisfy the condition

\[
(4.1) \quad r(r - 1) + s(s - 1) = \lambda(v - 1).
\]

We refer to parameter sets satisfying this necessary condition as {	extit{feasible}}. There exist feasible parameter sets for which there are no difference families. The first such example is \((18; 9, 6; 6)\), due to Young [32]. This also follows from a non-existence criterion of Arasu and Xiang [3].

There are a few infinite families of SDSs. One of them is due to Szekeres, see e.g. [26, p. 152], another one to Koukouvinos, Koukias and Seberry [21], and several other families have been constructed by Wilson, Seberry, and Ding [30, 28, 12] using cyclotomy.

Clearly we may assume that \(r \geq s\). Omitting the cases \(s = 0, 1\) as trivial, we shall consider only the parameter sets satisfying the inequalities

\[
(4.2) \quad v \geq 2r \geq 2s \geq 4.
\]

For \(v \leq 50\) there are exactly 227 feasible parameter sets \((v; r, s; \lambda)\) satisfying these inequalities. The difference families in this range with odd \(v\) have been constructed in many cases in the paper [8].

This work has been continued, including the even \(v\)'s, in the papers [13, 24]. The existence question has been resolved for all feasible parameter sets with \(v \leq 40\), but in the range \(40 < v \leq 50\) there remained 23 undecided cases. We shall settle eight of these cases by constructing the required difference families.

In the cases that we consider, the order is given by \(n = r + s - \lambda\). We mention that it plays an important role in the criterion of Arasu and Xiang. We shall now introduce three special types of SDSs which have been studied extensively. Before we do that, let us recall some basic definitions.

Let \(A\) be a binary sequence of length \(v\), i.e., \(A = (a_0, a_1, \ldots, a_{v-1})\) where each \(a_i\) is \(\pm 1\). The \textit{periodic autocorrelation function} (PACF) \(\tilde{\phi}\) of \(A\) is defined by

\[
\tilde{\phi}(i) = \sum_{j=0}^{v-1} a_j a_{i+j}, \quad 0 \leq i < v,
\]
where \( i + j \) should be reduced modulo \( v \). The non-periodic autocorrelation function (NACF) \( \phi \) of \( \mathcal{A} \) is defined by

\[
\phi(i) = \sum_{j=0}^{v-1-i} a_j a_{i+j}, \quad 0 \leq i < v.
\]

The first type is defined by the condition \( r = s \), i.e., the two base blocks are of the same size. Such SDSs give BIBD’s on \( \mathbb{Z}_v \) by an old result of Bose [7].

The second type is defined by the relation \( v = 2n + 1 \). They are known as D-optimal SDSs because they can be used to construct D-optimal designs (of circular type) of size \( 2v \). The D-optimal parameter sets in our range are listed in Table 2. The columns with heading “#” give the number of equivalence classes of SDSs. If this number is not known, the question mark is entered.

The equivalence classes of the D-optimal SDSs have been enumerated in [31] for \( v \leq 19 \), and later this has been extended in [22, 23] to all \( v \leq 27 \) and \( v = 33, 45 \).

**Table 2: D-optimal parameters \((v; r, s; \lambda)\)**

| #  | (9;3,2;1) 1 | (13;4,4;2) 2 | (13;6,3;3) 2 |
|----|-------------|-------------|-------------|
| 3  | (15;6,4;3) 3 | (19;7,6;4) 8 | (21;10,6;6) 31 |
| 17 | (23;10,7;6) 17 | (25;9,9;6) 39 | (27;11,9;7) 48 |
| ?  | (31;15,10;10) ? | (33;13,12;9) 509 | (33;15,11;10) 516 |
| ?  | (37;16,13;11) ? | (41;16,16;12) ? | (43;18,16;13) ? |
| ?  | (43;21,15;15) ? | (45;21,16;15) 1358 | (49;22,18;16) ? |

The third type of SDSs is defined by the relation \( v = 2n \). They are essentially the same objects as the pairs \( \mathcal{A} = (A_1, A_2) \) of binary sequences of length \( v \) for which the sum of the PACF of \( A_1 \) and \( A_2 \) is zero (more precisely, a \( \delta \)-function). We shall refer to this sum as the periodic autocorrelation function of \( \mathcal{A} \). Due to their importance in engineering applications, such binary sequences have been studied for long time (see e.g. [17] and its references).

If one replaces above the word “periodic” by “aperiodic” one obtains the definition of Golay pairs. We refer to the lengths \( v \) of Golay pairs as Golay numbers. The equivalence classes of Golay pairs have been described in our paper [15] and the representatives of these classes given for Golay numbers 2,4,8,10,16,20,26,32 and 40. For further results in this direction see [6]. Each Golay pair \((A_1, A_2)\) of length \( v \) gives a...
cyclic \((v; r, s; \lambda)\) difference family \((X_1, X_2)\) by taking \(X_i\) to be the set of positions of terms \(-1\) in the sequence \(A_i\). Here is an example.

**Example 4.1.** Let us start with the Golay pair of length 10:

\[
A_1 = +, +, -, +, -, +, -, +, +, +, +; \\
A_2 = +, +, +, -, +, +, +, +, +, -, -.
\]

We label the positions by integers 0 to 9 from left to right. The positions of the negative signs (which stand for \(-1\)) are recorded in the two base blocks \(X_1 = \{2, 4, 6, 7\}\) and \(X_2 = \{2, 8, 9\}\). Then \((X_1, X_2)\) is a difference family in \(\mathbb{Z}_{10}\). Its normal form \((\{0, 1, 3, 5\}, \{0, 1, 4\})\) appears in Table 3.

**Remark 4.2.** There is a natural definition of equivalence for Golay pairs, see e.g. \([6, 15]\). While equivalent Golay pairs always give equivalent SDSs, two non-equivalent Golay pairs may also give rise to equivalent SDSs. For instance, there are 5 equivalence classes of Golay pairs of length 8 but they give only 2 equivalence classes of SDSs.

We shall split the table of difference families into four parts, Tables 3-6 covering the ranges \(v \leq 20, 20 < v \leq 30, 30 < v \leq 40\) and \(40 < v \leq 50\), respectively. The parameter sets are arranged in the lexicographic order by the parameters \(v, \lambda\) and \(n\). The asterisk in the last column of the tables means that the family has been constructed by using our genetic type program and that we were not able to locate an equivalent family in the literature.

### 5. The Range \(v \leq 20\)

In Table 3 we list the 30 feasible parameter sets \((v; r, s; \lambda)\) with \(v \leq 20\) satisfying \([1.2]\) and give examples of cyclic difference families for them. In four of these cases the difference families do not exist. This is indicated in Table 3 (and subsequent tables) by the word “None”.

In each case we provide in the last column a reference where the family has been constructed (or just listed) or where it was established that the families do not exist. No effort has been made to assign priorities for these results. The symbol “Sz” indicates that the difference family is a classical Szekeres family, and “DS” means that the two blocks are in fact difference sets. Whenever possible, we have tried to avoid using the “DS” type examples.

The symbol “GP” indicates that the difference family is obtained from a Golay pair. In all four cases \(v = 8, 10, 16, 20\) we have included in the table all equivalence classes of SDSs arising from Golay pairs of length \(v\).
Table 3: $(v; r, s; \lambda)$ difference families with $v \leq 20$

| $(v; r, s; \lambda)$ | $n$ | Base blocks | Ref. |
|----------------------|-----|-------------|------|
| $(5;2;1)$            | 3   | $\{0,1\}$  | $\{0,2\}$ | [28] |
| $(7;3;2)$            | 4   | $\{0,1,3\}$ | $\{0,1,3\}$ | DS  |
| $(8;4;2)$            | 4   | $\{0,1,2,4\}$ | $\{0,3\}$ | GP  |
|                     |     | $\{0,1,3,4\}$ | $\{0,2\}$ | GP  |
| $(9;3;2)$            | 4   | $\{0,1,4\}$ | $\{0,2\}$ | [16] |
| $(9;4;3)$            | 5   | $\{0,1,3,4\}$ | $\{0,1,3,5\}$ | Sz  |
| $(10;4;2)$           | 5   | $\{0,1,3,5\}$ | $\{0,1,4\}$ | GP  |
| $(11;5;4)$           | 6   | $\{0,1,2,4,6\}$ | $\{0,1,2,5,8\}$ | Sz  |
| $(12;5;2)$           | 5   | $\{0,1,2,5,8\}$ | $\{0,2\}$ | [13] |
| $(13;3;3)$           | 5   | $\{0,1,4\}$ | $\{0,2,7\}$ | [29] |
| $(13;4;4)$           | 6   | $\{0,1,4,6\}$ | $\{0,1,4,6\}$ | DS  |
|                     |     | $\{0,1,4,6\}$ | $\{0,2,3,7\}$ | DS  |
| $(13;6;3)$           | 6   | $\{0,1,2,4,7,9\}$ | $\{0,1,4\}$ | [28] |
|                     |     | $\{0,1,3,5,7,8\}$ | $\{0,1,4\}$ | [16] |
| $(13;6;5)$           | 7   | $\{0,1,2,3,6,9\}$ | $\{0,1,2,4,6,9\}$ | [29] |
|                     |     | $\{0,1,2,3,6,10\}$ | $\{0,1,3,5,7,8\}$ | [28] |
|                     |     | $\{0,1,2,4,5,8\}$ | $\{0,1,2,4,7,9\}$ | [28] |
|                     |     | $\{0,1,2,4,5,8\}$ | $\{0,1,3,5,7,8\}$ | *   |
| $(14;5;2)$           | 6   | None        |      | [13] |
| $(15;4;2)$           | 5   | $\{0,1,4,9\}$ | $\{0,2\}$ | [8]  |
| $(15;6;4)$           | 7   | $\{0,1,2,4,6,9\}$ | $\{0,1,4,9\}$ | [16] |
|                     |     | $\{0,1,3,4,8,10\}$ | $\{0,1,4,6\}$ | [31] |
|                     |     | $\{0,1,3,5,7,8\}$ | $\{0,1,4,10\}$ | [31] |
| $(15;7;6)$           | 8   | $\{0,1,2,3,4,8,11\}$ | $\{0,1,3,5,6,9,11\}$ | Sz  |
|                     |     | $\{0,1,2,4,5,7,9\}$ | $\{0,1,2,5,7,8,11\}$ | *   |
|                     |     | $\{0,1,2,4,5,9,11\}$ | $\{0,1,3,4,6,8,9\}$ | *   |
| $(16;6;4)$           | 8   | $\{0,1,2,3,6,10\}$ | $\{0,1,3,6,8,12\}$ | GP  |
|                     |     | $\{0,1,2,4,5,10\}$ | $\{0,1,4,7,9,11\}$ | GP  |
|                     |     | $\{0,1,2,4,6,9\}$ | $\{0,1,2,6,9,12\}$ | GP  |
|                     |     | $\{0,1,2,4,6,9\}$ | $\{0,1,5,7,8,11\}$ | GP  |
|                     |     | $\{0,1,2,5,9,11\}$ | $\{0,1,3,5,6,9\}$ | *   |
|                     |     | $\{0,1,3,4,7,9\}$ | $\{0,1,5,6,8,10\}$ | GP  |
|                     |     | $\{0,1,3,5,6,9\}$ | $\{0,1,3,5,9,10\}$ | GP  |
|                     |     | $\{0,1,3,5,7,8\}$ | $\{0,1,4,6,9,10\}$ | GP  |
| $(17;6;2)$           | 6   | None        |      | [13] |
| $(v; r, s; \lambda)$ | $n$ | Base blocks | Ref. |
|---------------------|------|-------------|------|
| $(17; 5, 4; 2)$     | 7    | $\{0, 1, 4, 6, 10\}$ | $\{0, 1, 3, 8\}$ | * |
|                     |      | $\{0, 1, 4, 7, 9\}$ | $\{0, 1, 5, 7\}$ | S |
| $(17; 7, 3; 3)$     | 7    | None        |      | 13 |
| $(17; 8, 8; 7)$     | 9    | $\{0, 1, 2, 4, 5, 7, 10, 11\}$ | $\{0, 1, 2, 4, 6, 8, 9, 14\}$ | * |
|                     |      | $\{0, 2, 3, 4, 7, 8, 9, 11\}$ | $\{0, 2, 3, 5, 6, 8, 12, 13\}$ | 28 |
| $(18; 8, 4; 4)$     | 8    | $\{0, 1, 2, 3, 5, 8, 9, 13\}$ | $\{0, 2, 6, 9\}$ | 13 |
|                     |      | $\{0, 1, 2, 4, 6, 7, 10, 11\}$ | $\{0, 3, 5, 11\}$ |      |
| $(18; 9, 6; 6)$     | 9    | None        |      | 3 32 |
| $(19; 4, 3; 1)$     | 6    | $\{0, 3, 5, 9\}$ | $\{0, 1, 8\}$ | S |
| $(19; 6, 3; 2)$     | 7    | $\{0, 1, 2, 6, 10, 13\}$ | $\{0, 2, 5\}$ | * |
|                     |      | $\{0, 1, 2, 7, 8, 11\}$ | $\{0, 2, 5\}$ | S |
| $(19; 7, 4; 3)$     | 8    | $\{0, 1, 2, 4, 7, 9, 13\}$ | $\{0, 1, 4, 9\}$ | * |
|                     |      | $\{0, 1, 3, 4, 8, 10, 14\}$ | $\{0, 1, 3, 8\}$ | S |
| $(19; 7, 6; 4)$     | 9    | $\{0, 1, 2, 3, 7, 11, 14\}$ | $\{0, 2, 5, 6, 9, 11\}$ | 31 |
|                     |      | $\{0, 1, 2, 4, 5, 10, 13\}$ | $\{0, 1, 4, 6, 8, 13\}$ | 31 |
|                     |      | $\{0, 1, 2, 5, 7, 11, 12\}$ | $\{0, 2, 4, 5, 8, 11\}$ | 31 |
|                     |      | $\{0, 1, 2, 5, 7, 11, 14\}$ | $\{0, 2, 3, 4, 8, 11\}$ | 16 |
|                     |      | $\{0, 1, 3, 4, 7, 12, 14\}$ | $\{0, 1, 2, 5, 7, 11\}$ | 16 |
|                     |      | $\{0, 1, 3, 4, 8, 10, 14\}$ | $\{0, 1, 2, 4, 7, 12\}$ | 31 |
|                     |      | $\{0, 1, 3, 4, 8, 10, 14\}$ | $\{0, 2, 3, 7, 8, 10\}$ | 31 |
|                     |      | $\{0, 2, 3, 4, 6, 9, 14\}$ | $\{0, 1, 2, 6, 10, 13\}$ | 31 |
| $(19; 9, 9; 8)$     | 10   | $\{0, 1, 2, 3, 4, 8, 10, 13, 15\}$ | $\{0, 1, 2, 4, 5, 6, 9, 12, 15\}$ | 13 |
|                     |      | $\{0, 1, 2, 4, 6, 8, 9, 11, 12\}$ | $\{0, 1, 2, 5, 6, 7, 9, 12, 15\}$ | * |
| $(20; 8, 5; 4)$     | 9    | $\{0, 1, 2, 4, 5, 8, 11, 13\}$ | $\{0, 2, 6, 7, 12\}$ | * |
|                     |      | $\{0, 1, 2, 5, 6, 8, 11, 13\}$ | $\{0, 1, 3, 7, 11\}$ | 13 |
| $(20; 9, 7; 6)$     | 10   | $\{0, 1, 2, 3, 4, 6, 9, 10, 14\}$ | $\{0, 1, 4, 8, 10, 13, 15\}$ | GP |
|                     |      | $\{0, 1, 2, 3, 5, 6, 9, 13, 15\}$ | $\{0, 2, 3, 5, 9, 10, 14\}$ | GP |
|                     |      | $\{0, 1, 2, 3, 5, 7, 9, 12, 13\}$ | $\{0, 1, 3, 6, 7, 10, 15\}$ | GP |
|                     |      | $\{0, 1, 2, 3, 5, 7, 10, 11, 14\}$ | $\{0, 1, 3, 5, 8, 9, 15\}$ | GP |
|                     |      | $\{0, 1, 2, 3, 6, 7, 9, 11, 15\}$ | $\{0, 2, 3, 5, 9, 10, 13\}$ | * |
|                     |      | $\{0, 1, 2, 3, 6, 7, 10, 12, 14\}$ | $\{0, 1, 4, 5, 7, 10, 12\}$ | GP |
|                     |      | $\{0, 1, 2, 3, 6, 7, 11, 13, 15\}$ | $\{0, 2, 3, 5, 8, 9, 12\}$ | GP |
|                     |      | $\{0, 1, 2, 3, 6, 8, 10, 11, 15\}$ | $\{0, 1, 3, 4, 7, 9, 13\}$ | GP |
|                     |      | $\{0, 1, 2, 3, 6, 9, 11, 13, 17\}$ | $\{0, 1, 2, 5, 7, 8, 12\}$ | GP |
|                     |      | $\{0, 1, 2, 4, 6, 7, 9, 10, 14\}$ | $\{0, 1, 2, 5, 9, 11, 14\}$ | GP |
|                     |      | $\{0, 1, 2, 4, 7, 9, 10, 11, 15\}$ | $\{0, 1, 3, 5, 6, 9, 13\}$ | * |
Table 3 (continued)

| $(v; r, s; \lambda)$ | $n$ | Base blocks | Ref. |
|---------------------|-----|-------------|------|
| $(20; 9, 7; 6)$     | 10  | $\{0,1,2,5,6,7,10,12,14\}$ | GP   |
|                     |     | $\{0,1,2,5,6,7,10,12,14\}$ | GP   |
|                     |     | $\{0,1,2,5,7,10,12,16\}$   | GP   |
|                     |     | $\{0,1,2,5,7,10,13\}$      | GP   |
|                     |     | $\{0,1,2,4,6,10,13\}$      | GP   |
|                     |     | $\{0,1,2,3,6,9,13\}$       | GP   |
|                     |     | $\{0,1,2,3,6,9,13\}$       | GP   |
|                     |     | $\{0,1,3,4,5,7,9,13,14\}$  | GP   |
|                     |     | $\{0,1,3,4,6,8,9,13,14\}$  | GP   |
|                     |     | $\{0,1,3,4,6,9,11,13,17\}$ | GP   |
|                     |     | $\{0,1,3,4,7,8,9,12,14\}$  | GP   |
|                     |     | $\{0,1,2,4,6,11,14\}$      | GP   |
|                     |     | $\{0,2,3,6,7,9,10,12,14\}$ | GP   |

The difference families in the five D-optimal cases in this range have been enumerated, up to equivalence, in [31]. As there are not many of them, we have included all of them in the table.

As a curiosity, let us mention that Ehlich gives in his paper [16] three examples of D-optimal designs of circular type and size $2v = 38$. Among the corresponding three SDSs, all with parameters $(19; 7, 6; 4)$, the first two are equivalent. Similarly, his two examples in the case $2v = 18$ are equivalent. (In the case $2v = 26$, his two SDSs are indeed non-equivalent.)

6. The range $20 < v \leq 30$

There are 45 feasible parameter sets in this range. Only four of them are D-optimal (marked by the symbol “DO”). The representatives of the equivalence classes of these D-optimal SDSs have been computed in [22]. The number of classes is 31, 17, 39 and 48 when $v$ is 21, 23, 25 and 27, respectively. We do not list them and refer the reader to that paper.

There is only one Golay number, $v = 26$, in this range and there is up to equivalence a unique Golay pair of that length. The corresponding difference family with parameters $(26; 11, 10; 8)$ is included in the table (and marked with “GP”).
| $(v; r, s; \lambda)$ | $n = 20 \leq v \leq 30$ |
|---------------------|------------------|
| $(21; 5, 5; 2)$, $n = 8$ | None |
| $(21; 6, 6; 3)$, $n = 9$ | 
| \{0,1,3,7,10,15\}, \{0,2,3,4,8,13\}; | 29 |
| \{0,1,3,8,10,14\}, \{0,1,4,6,9,10\}; | * |
| $(21; 10, 6; 6)$, $n = 10$ | DO |
| $(21; 10, 10; 9)$, $n = 11$ | 
| \{0,1,2,3,4,7,8,10,13,14\}, \{0,2,4,5,7,8,10,12,16,17\}; | Sz |
| \{0,1,2,3,6,7,9,11,12,14\}, \{0,1,3,4,5,8,9,11,15,17\}; | * |
| \{0,1,2,4,5,6,7,10,13,15\}, \{0,1,2,4,7,8,11,12,14,16\}; | 11 |
| $(22; 6, 4; 2)$, $n = 8$ | \{0,1,4,6,10,15\}, \{0,2,3,10\}; | 13 |
| $(22; 9, 4; 4)$, $n = 9$ | 
| \{0,1,2,4,6,10,11,14,17\}, \{0,1,3,8\}; | * |
| \{0,1,2,5,7,8,10,12,16\}, \{0,1,4,13\}; | * |
| \{0,1,2,5,7,9,11,12,15\}, \{0,1,6,9\}; | 13 |
| $(22; 7, 7; 4)$, $n = 10$ | 
| \{0,1,2,3,7,9,12\}, \{0,3,4,8,10,14,17\}; | 11 |
| \{0,1,2,3,7,9,13\}, \{0,2,5,6,9,14,17\}; | * |
| \{0,1,2,4,8,10,13\}, \{0,1,5,6,8,11,15\}; | * |
| \{0,1,3,4,8,11,17\}, \{0,1,3,5,7,12,13\}; | 13 |
| $(23; 5, 2; 1)$, $n = 6$ | \{0,1,3,8,14\}, \{0,4\}; | 8 |
| $(23; 7, 2; 2)$, $n = 7$ | None |
| $(23; 10, 5; 5)$, $n = 10$ | 
| \{0,1,2,3,5,7,8,12,13,16\}, \{0,2,6,9,15\}; | * |
| \{0,1,2,3,5,7,8,12,14,18\}, \{0,1,2,8,11\}; | * |
| \{0,1,2,4,6,10,11,12,15,18\}, \{0,2,3,7,10\}; | 13 |
| $(23; 10, 7; 6)$, $n = 11$ | DO |
Table 4 (continued)

(23; 11, 11; 10), \( n = 12 \)
\{0,1,2,3,4,6,9,10,14,15,17\}, \{0,1,3,5,6,7,9,11,12,16,19\}; Sz
\{0,1,2,3,6,9,10,11,13,15,18\}, \{0,1,2,4,6,7,8,11,12,14,20\}; *

(24; 10, 2; 4), \( n = 8 \)
\{0,1,2,3,5,8,9,13,15,18\}, \{0,4\}; *
\{0,1,2,5,7,8,11,12,14,16\}, \{0,8\}; *
\{0,1,4,5,6,7,10,13,15,17\}, \{0,8\}; [13]

(24; 9, 5; 4), \( n = 10 \)
\{0,1,2,3,5,8,10,14,18\}, \{0,1,6,10,13\}; *
\{0,1,2,4,5,7,11,12,16,19\}, \{0,2,8,11,18\}; *
\{0,1,2,4,5,10,12,16,19\}, \{0,2,5,6,13\}; *
\{0,1,2,4,5,11,12,16,19\}, \{0,2,6,8,11\}; *
\{0,1,2,5,6,9,11,16,18\}, \{0,2,3,6,14\}; *
\{0,1,3,4,8,10,12,13,18\}, \{0,2,5,6,13\}; *
\{0,1,3,5,9,10,11,14,17\}, \{0,1,3,7,12\}; [13]

(24; 12, 3; 6), \( n = 9 \)
\{0,1,2,5,6,8,9,11,13,14,15,18\}, \{0,2,10\}; [13]

(25; 4, 4; 1), \( n = 7 \) None [13]
(25; 7, 3; 2), \( n = 8 \) None [13]

(25; 7, 6; 3), \( n = 10 \)
\{0,1,2,5,8,12,14\}, \{0,1,5,8,10,16\}; [8]
\{0,1,3,8,9,12,15\}, \{0,2,3,7,8,12\}; *

(25; 10, 3; 4), \( n = 9 \) None [13]

(25; 10, 6; 5), \( n = 11 \)
\{0,1,3,4,5,8,11,15,17,20\}, \{0,1,2,6,8,15\}; *
\{0,1,3,4,5,9,10,12,16,18\}, \{0,1,4,6,11,14\}; [8]

(25; 12, 4; 6), \( n = 10 \)
\{0,1,2,4,5,6,8,10,13,15,18,19\}, \{0,1,7,10\}; *

(25; 9, 9; 6), \( n = 12 \) DO
Table 4 (continued)

(25; 12, 12; 11), $n = 13$
- \{0, 1, 2, 3, 5, 7, 11, 12, 14, 17, 19, 20\},
- \{0, 1, 2, 4, 5, 6, 8, 10, 11, 14, 15, 18\}; $^\text{[13]}$
- \{0, 1, 2, 3, 7, 8, 11, 12, 13, 15, 17, 20\},
- \{0, 1, 2, 4, 5, 7, 8, 10, 11, 14, 16, 18\};
- $^\ast$

(26; 6, 5; 2), $n = 9$
- \{0, 1, 2, 8, 13, 17\}, \{0, 2, 5, 8, 12\}; $^\text{[13]}$
- \{0, 2, 3, 8, 11, 15\}, \{0, 1, 5, 7, 17\};
- $^\ast$

(26; 11, 10; 8), $n = 13$
- \{0, 1, 2, 3, 4, 7, 9, 12, 14, 16, 20\}, \{0, 1, 4, 5, 9, 10, 11, 13, 16, 19\}; $^\text{[13]}$
- \{0, 1, 2, 4, 5, 8, 10, 14, 16, 19, 21\}, \{0, 1, 2, 4, 5, 8, 11, 12, 13, 18\}; $\text{GP}$
- \{0, 2, 3, 4, 5, 7, 10, 11, 14, 18, 20\}, \{0, 1, 2, 5, 6, 7, 11, 14, 17, 19\};
- $^\ast$

(27; 5, 3; 1), $n = 7$
- \{0, 2, 5, 11, 15\}, \{0, 1, 8\};
- $^\ast$

(27; 9, 3; 3), $n = 9$ None $^\text{[3, 24]}$

(27; 11, 5; 5), $n = 11$
- \{0, 1, 2, 4, 5, 8, 10, 11, 16, 18, 23\}, \{0, 1, 4, 11, 13\};
- \{0, 1, 2, 5, 6, 7, 10, 13, 16, 18, 20\}, \{0, 3, 4, 10, 12\}; $^\text{[8]}$

(27; 11, 9; 7), $n = 13$ DO

(27; 13, 13; 12), $n = 14$
- \{0, 1, 2, 3, 4, 6, 7, 8, 13, 14, 17, 19, 22\},
- \{0, 1, 3, 4, 5, 8, 10, 11, 12, 14, 18, 20, 23\};
- \{0, 1, 2, 3, 5, 6, 8, 9, 12, 16, 17, 19, 21\},
- \{0, 1, 3, 4, 5, 7, 9, 10, 13, 14, 15, 20, 22\}; $^\text{[13]}$

(28; 7, 4; 2), $n = 9$
- \{0, 2, 3, 5, 9, 13, 19\}, \{0, 1, 8, 13\}; $^\text{[13]}$

(28; 13, 3; 6), $n = 10$
- \{0, 1, 2, 3, 6, 7, 8, 10, 12, 15, 17, 18, 21\}, \{0, 4, 12\};
- \{0, 1, 3, 4, 6, 7, 11, 12, 13, 16, 18, 20, 21\}, \{0, 2, 6\}; $^\text{[13]}$
Table 4 (continued)

| (28; 12, 6; 6),  n = 12 | None |
|------------------------|------|
| (28; 10, 9; 6),  n = 13 | {0,1,2,5,6,7,9,14,17,19}, {0,1,5,8,11,12,14,18,20}; |
|                        | {0,2,3,4,5,9,13,15,16,23}, {0,2,4,5,9,10,13,16,22}; |
|                        | * |
| (29; 9, 4; 3),  n = 10 | {0,1,2,4,7,8,12,18,21}, {0,2,7,16}; |
| (29; 7, 7; 3),  n = 11 | {0,1,2,5,9,18,24}, {0,2,4,7,10,18,19}; |
|                        | {0,1,4,6,11,12,20}, {0,1,4,7,12,14,16}; |
|                        | * |
| (29; 11, 2; 4),  n = 9 | None |
| (29; 8, 8; 4),  n = 12 | {0,1,2,4,8,10,15,20}, {0,1,4,7,8,13,16,18}; |
|                        | {0,1,5,6,7,10,14,23}, {0,2,4,7,10,18,19,21}; |
|                        | * |
| (29; 11, 6; 5),  n = 12 | {0,1,2,3,4,8,9,13,17,20,23}, {0,2,4,7,12,18}; |
|                        | {0,1,2,3,6,9,11,13,17,20,25}, {0,1,4,6,13,14}; |
|                        | * |
| (29; 13, 4; 6),  n = 11 | {0,1,2,3,6,7,10,11,13,15,16,18,22}, {0,2,8,19}; |
| (29; 14, 7; 8),  n = 13 | {0,1,2,5,6,7,9,11,13,14,15,18,21,24}, {0,2,3,4,11,14,19}; |
|                        | {0,2,3,4,6,7,9,10,11,13,18,19,23,24}, {0,2,4,7,10,18,19}; |
|                        | * |
| (29; 14, 14; 13),  n = 15 | {0,1,2,3,4,7,9,11,12,15,17,19,20,26}, |
|                        | {0,1,2,4,5,6,8,9,11,14,18,22,23,24}; |
|                        | {0,1,2,3,4,7,9,11,12,15,17,19,20,26}, |
|                        | {0,2,3,4,6,7,9,10,11,13,18,19,23,24}; |
|                        | {0,1,2,3,4,7,9,12,13,16,17,19,21,22}, |
|                        | {0,1,2,4,5,7,8,10,11,12,16,18,23,25}; |
|                        | {0,1,2,3,5,6,10,12,13,15,16,17,21,23}, |
|                        | {0,1,2,4,5,7,9,10,11,13,16,17,21,24}; |
|                        | * |
Table 4 (continued)

(29; 14, 14; 13), $n = 15$
\{0, 1, 2, 4, 5, 6, 9, 11, 13, 14, 15, 17, 21, 24\},
\{0, 1, 3, 4, 6, 8, 9, 11, 12, 16, 17, 18, 19, 23\};
\{0, 2, 3, 4, 5, 8, 10, 13, 14, 15, 16, 18, 22, 25\},
\{0, 2, 3, 4, 6, 7, 9, 10, 11, 13, 18, 19, 23, 24\};

(30; 8, 2; 2), $n = 8$ None

(30; 11, 3; 4), $n = 10$ None

(30; 12, 7; 6), $n = 13$
\{0, 1, 2, 4, 5, 7, 8, 9, 14, 17, 19, 23\}, \{0, 2, 6, 11, 12, 19, 22\};
\{0, 1, 2, 4, 5, 8, 10, 14, 17, 19, 23, 24\}, \{0, 1, 2, 4, 7, 12, 20\};
\{0, 1, 3, 6, 7, 8, 10, 11, 13, 18, 22, 24\}, \{0, 1, 4, 5, 13, 15, 21\};

7. The range $30 < v \leq 40$

In this range we have 63 feasible parameter sets, and only two Golay numbers: 32 and 40. There are 330 equivalence classes of Golay pairs of length 32, and 220 of length 40. The difference families that arise from these pairs have parameters (32; 16, 12; 12) and (40; 18, 16; 14) respectively (up to equivalence). We have included in Table 5 only one example of such difference families for each of these parameter sets.

There is one more set of parameters, namely (34; 16, 13; 12), which belongs to the special type defined by the condition $v = 2n$. This is an important case since the periodic autocorrelation function of the binary sequences $A = (A_1, A_2)$ constructed from the SDSs $X = (X_1, X_2)$ with these parameters is zero. We have constructed two non-equivalent such SDSs in our paper \[14\]. In fact one of them appears in our earlier paper \[13\] but at that time its significance was not recognized. Now we have constructed an additional non-equivalent SDS with the same parameters (see Table 5).

We also have in this range four D-optimal parameter sets. The equivalence classes of SDSs for the two sets with $v = 33$ have been enumerated in \[22\]. We do not give any difference families in these two cases. For $v = 31$ we have included the two non-equivalent D-optimal SDSs from \[9\], and for $v = 37$ the two known D-optimal SDSs \[10, 18\]. We have contributed two new D-optimal designs for each of the cases $v = 31$ and $v = 37$. 
Table 5: \((v; r, s; \lambda)\) difference families with \(30 < v \leq 40\)

| \((31; 6, 6; 2)\), \(n = 10\) | \{0,1,4,11,12,17\}, \{0,2,4,9,12,18\} | \{0,1,4,11,13,17\}, \{0,3,5,10,11,19\}; \[13\] |
| \((31; 10, 6; 4)\), \(n = 12\) | \{0,1,2,6,8,9,11,14,18,22\}, \{0,1,5,7,17,20\} | \{0,3,4,9,11,16,21,22,25\}, \{0,1,2,4,8,16\}; \[8\] |
| \((31; 10, 10; 6)\), \(n = 14\) | \{0,1,2,3,5,7,11,15,16,23\}, \{0,1,4,6,7,11,14,20,23,25\}; \[8\] |
| \((31; 15, 6; 8)\), \(n = 13\) | \{0,1,2,3,4,5,9,10,12,15,19,21,23,26,27\}, \{0,2,3,6,13,18\}; \[13\] |
| \((31; 15, 10; 10)\), \(n = 15\) | \{0,1,2,3,5,6,7,11,13,15,16,18,23,24,27\}, \{0,2,3,5,6,8,12,19,20,27\}; \[9\] |
| \((31; 15, 15; 14)\), \(n = 16\) | \{0,1,2,3,4,5,7,12,13,14,17,20,22,26,28\}, \{0,1,2,3,6,7,9,10,12,14,16,17,20,21,25\}; \[13\] |
| \((32; 8, 3; 2)\), \(n = 9\) | None \[24\] |
| \((32; 7, 5; 2)\), \(n = 10\) | \{0,2,3,5,10,14,18\}, \{0,1,7,13,22\}; \[13\] |
| \((32; 13, 6; 6)\), \(n = 13\) | \{0,1,2,3,4,8,9,11,13,15,18,19,23\}, \{0,3,6,12,19,24\}; \[13\] |
Table 5 (continued)

| Family | n   | Comments |
|--------|-----|----------|
| (32; 16, 12; 12), n = 16 | |  
| \( \{0,1,2,3,6,7,10,11,12,14,16,18,20,23,25,26\} \) | \( \{0,1,2,3,6,8,11,13,14,17,18,21\} \) | GP |
| \( \{0,1,2,4,5,6,7,10,12,13,15,17,18,22,24,26\} \) | \( \{0,1,3,4,5,9,12,13,16,18,19,26\} \) | * |
| \( \{0,1,2,4,6,7,8,10,12,13,15,17,18,21,22,26\} \) | \( \{0,1,2,4,5,9,11,12,18,19,21,24\} \) | [13] |
| (33; 6, 2; 1), n = 7 | None | [13] |
| (33; 5, 4; 1), n = 8 | None | [13] |
| (33; 10, 3; 3), n = 10 | |  
| \( \{0,1,2,5,7,8,12,16,22,25\} \) | \( \{0,2,14\} \) | [8] |
| (33; 9, 8; 4), n = 13 | |  
| \( \{0,1,2,7,10,11,14,17,18,19,22,24\} \) | \( \{0,2,4,6,11,14,19,20\} \) | * |
| \( \{0,2,5,7,8,14,17,18,22\} \) | \( \{0,1,2,6,9,13,15,23\} \) | [24] |
| (33; 14, 7; 7), n = 14 | |  
| \( \{0,1,2,3,5,6,8,9,13,15,18,19,22,27\} \) | \( \{0,1,8,10,12,18,23\} \) | [8] |
| \( \{0,2,3,4,5,8,9,11,13,16,17,19,23,26\} \) | \( \{0,1,2,8,13,18,22\} \) | * |
| (33; 13, 12; 9), n = 16 | DO |  
| (33; 15, 11; 10), n = 16 | DO |  
| (33; 16, 16; 15), n = 17 | |  
| \( \{0,1,2,3,6,7,8,12,13,16,18,20,23,25,28\} \) | \( \{0,1,2,3,4,6,8,9,10,13,14,17,20,21,25,27\} \) | * |
| \( \{0,1,2,3,5,6,8,10,12,14,15,18,22,23,25,29\} \) | \( \{0,1,2,3,4,6,8,9,10,13,14,17,20,21,25,27\} \) | * |
| \( \{0,1,2,3,6,8,10,11,13,15,17,18,19,20,21,24,30\} \) | \( \{0,1,2,3,5,6,8,10,12,14,15,18,19,24,25,27\} \) | Sz |
| \( \{0,1,2,4,5,6,8,11,12,13,16,18,19,21,22,26\} \) | \( \{0,1,2,4,5,6,8,11,12,13,16,18,19,21,22,26\} \) | * |
| (34; 10, 7; 4), n = 13 | |  
| \( \{0,1,3,4,7,8,13,16,18,24\} \) | \( \{0,2,4,9,15,16,24\} \) | * |
| \( \{0,1,3,6,8,12,15,16,17,22\} \) | \( \{0,1,3,9,13,17,24\} \) | * |
| \( \{0,1,3,6,10,13,15,19,26,27\} \) | \( \{0,2,3,4,8,14,19\} \) | [13] |
Table 5 (continued)

\[
\begin{array}{lll}
(34; 13, 7; 6), & n = 14 & \text{None} \\
\hline
(34; 12, 12; 8), & n = 16 & \\
\{0, 1, 2, 3, 6, 8, 12, 15, 16, 17, 25, 29\}, \\
\{0, 1, 3, 6, 7, 9, 11, 14, 18, 19, 21, 25\}; \quad [13] \\
\{0, 1, 2, 3, 7, 8, 13, 16, 18, 21, 25, 28\}, \\
\{0, 1, 2, 4, 5, 8, 9, 11, 13, 19, 23, 25\}; \quad * \\
\{0, 1, 2, 4, 6, 11, 12, 15, 17, 20, 24, 28\}, \\
\{0, 1, 4, 5, 6, 8, 11, 13, 14, 20, 21, 23\}; \quad [11] \\
\hline
(34; 16, 10; 10), & n = 16 & \\
\{0, 1, 2, 3, 4, 5, 7, 8, 10, 14, 15, 19, 20, 25, 27, 28\}, \\
\{0, 2, 4, 5, 8, 12, 16, 18, 21, 27\}; \quad [13] \\
\{0, 1, 2, 3, 4, 7, 8, 10, 12, 13, 16, 19, 20, 21, 24, 26\}, \\
\{0, 1, 3, 6, 8, 10, 14, 15, 21, 25\}; \quad * \\
\{0, 1, 2, 3, 5, 6, 10, 11, 12, 14, 16, 17, 18, 20, 24, 27\}, \\
\{0, 1, 3, 5, 8, 12, 15, 20, 21, 26\}; \quad * \\
\hline
(34; 16, 13; 12), & n = 17 & \\
\{0, 1, 2, 3, 5, 6, 8, 12, 13, 14, 15, 18, 20, 22, 24, 31\}, \\
\{0, 1, 4, 5, 7, 8, 9, 14, 15, 18, 23, 26, 28\}; \quad [13] \\
\{0, 1, 2, 3, 5, 7, 9, 11, 13, 14, 16, 17, 20, 23, 24, 29\}, \\
\{0, 1, 3, 4, 5, 8, 9, 10, 15, 18, 23, 25, 26\}; \quad * \\
\{0, 1, 3, 4, 5, 7, 8, 10, 13, 14, 16, 18, 22, 23, 24, 30\}, \\
\{0, 2, 3, 4, 6, 7, 11, 12, 13, 19, 22, 24, 27\}; \quad [14] \\
\hline
(35; 8, 4; 2), & n = 10 & \text{None} \\
\hline
(35; 10, 4; 3), & n = 11 & \\
\{0, 1, 2, 6, 11, 13, 15, 18, 21, 29\}, \quad \{0, 3, 4, 13\}; \quad [13] \\
\hline
(35; 9, 6; 3), & n = 12 & \\
\{0, 1, 2, 4, 10, 16, 17, 22, 27\}, \quad \{0, 2, 5, 9, 13, 16\}; \quad [13] \\
\{0, 1, 2, 6, 8, 11, 14, 21, 25\}, \quad \{0, 2, 5, 9, 17, 18\}; \quad * \\
\hline
(35; 12, 9; 6), & n = 15 & \\
\{0, 1, 2, 3, 6, 10, 12, 15, 16, 17, 20, 28\}, \quad \{0, 1, 6, 8, 12, 14, 17, 21, 24\}; \quad * \\
\{0, 1, 2, 5, 7, 10, 11, 15, 16, 18, 22, 28\}, \quad \{0, 2, 3, 4, 11, 13, 16, 19, 23\}; \quad [8] \\
\end{array}
\]
Table 5 (continued)

| Description                        | Value   | Ref  |
|------------------------------------|---------|------|
| (35; 14, 8; 7), n = 15             |         | [13] |
| {0,1,2,3,4,5,10,11,13,16,18,22,26,30}, |         |      |
| {0,1,5,7,12,16,19,22};             |         |      |
| {0,1,2,4,5,7,10,12,13,17,21,23,27,28}, |         |      |
| {0,1,4,5,7,14,16,22};              |         |      |
| (35; 14, 10; 8), n = 16            |         | [13] |
| {0,1,2,3,6,8,10,12,13,16,19,20,21,25}, |         |      |
| {0,1,3,4,8,13,15,21,24,29};        |         |      |
| {0,1,2,4,7,9,10,12,13,17,19,23,24,28}, |         |      |
| {0,1,2,3,5,8,14,18,22,28};         |         |      |
| (35; 17, 17; 16), n = 18           |         | Sz   |
| {0,1,2,3,4,5,8,9,10,13,14,17,19,21,24,25,27}, |         |      |
| {0,2,3,6,7,8,9,12,13,15,17,19,20,22,26,27,29}; |         |      |
| {0,1,2,3,4,6,8,9,10,11,12,17,20,23,24,27,30}, |         |      |
| {0,2,3,4,6,7,8,12,13,15,17,20,22,25,26,27,31}; |         |      |
| (36; 11, 6; 4), n = 13             |         | [13] |
| {0,1,2,6,8,11,13,20,21,24,28},    |         |      |
| {0,1,3,4,6,8,12,16,17,22,29},     |         |      |
| {0,1,7,10,19,21};                 |         |      |
| (36; 16, 11; 10), n = 17           |         | [24] |
| {0,1,2,3,5,6,7,10,11,13,14,21,23,24,27,29}, |         |      |
| {0,2,3,7,9,11,14,18,19,24,30};    |         |      |
| {0,1,2,3,6,7,8,11,12,14,17,20,21,24,26,28}, |         |      |
| {0,1,2,6,8,9,11,13,21,24,28};     |         |      |
| (36; 15, 15; 12), n = 18           |         | [3]  |
| (37; 6, 3; 1), n = 8               |         | [8]  |
| {0,3,5,9,17,24},                  |         |      |
| {0,1,11};                         |         |      |
| {0,4,6,13,18,21},                 |         |      |
| {0,1,11};                         |         |      |
| (37; 7, 6; 2), n = 11              |         | [8]  |
| {0,1,2,7,11,19,23},               |         |      |
| {0,2,5,8,15,28};                  |         |      |
| {0,1,4,9,11,17,23},               |         |      |
| {0,2,3,7,18,27};                  |         |      |
Table 5 (continued)

| (37; 12; 4; 4), $n = 12$ | \{0,1,3,5,6,7,11,17,20,22,29,30\}, \{0,4,7,16\}; [13] |
| (37; 9; 9; 4), $n = 14$ | \{0,1,2,3,5,10,16,20,27\}, \{0,1,5,9,11,14,17,23,30\}; [13] |
| (37; 10, 10; 5), $n = 15$ | \{0,1,2,6,10,15,18,20,31\}, \{0,2,3,4,9,12,15,19,26\}; [8] |
| (37; 15, 3; 6), $n = 12$ | \{0,1,2,4,5,7,8,11,13,15,18,23,24,25,33\}, \{0,3,7\}; [8] |
| (37; 16, 4; 7), $n = 13$ | \{0,1,2,3,6,7,9,11,12,14,16,20,21,24,27,28\}, \{0,2,3,4,5,7,10,11,14,15,16,20,21,24,27,28\}; [8] |
| (37; 15, 7; 7), $n = 15$ | \{0,1,2,5,6,8,9,11,14,16,18,20,21,25,29,31\}, \{0,2,3,8,14,15,18,21,24,25,30\}; [8] |
| (37; 13, 12; 8), $n = 17$ | \{0,1,2,3,4,9,10,12,15,16,20,26,30\}, \{0,2,4,6,7,9,12,17,21,24,25,30\}; [8] |
| (37; 18, 10; 11), $n = 17$ | \{0,1,2,4,5,6,7,9,10,11,16,18,19,22,26,29,30,32\}, \{0,1,2,6,8,10,15,20,23,26\}; [8] |
Table 5 (continued)

| (37; 16, 13; 11), n = 18 |  |
|--------------------------|---|
| \{0,1,2,3,4,7,8,11,13,15,16,18,23,24,27,33\}, \{0,1,2,4,8,10,13,14,18,20,21,23,32\}; | 10 |
| \{0,1,2,3,4,7,9,13,14,17,18,19,22,25,30,31\}, \{0,1,3,4,6,7,9,11,16,18,20,26,30\}; | * |
| \{0,1,2,3,4,8,11,12,14,17,18,20,23,25,27,32\}, \{0,1,3,4,5,7,11,12,16,17,20,22,30\}; | * |
| \{0,1,2,4,5,7,10,12,16,18,20,23,24,25,29,32\}, \{0,1,2,4,5,8,11,12,13,15,21,22,27\}; | 18 |

| (37; 18, 18; 17), n = 19 |  |
|--------------------------|---|
| \{0,1,2,3,4,5,7,8,10,14,15,16,19,21,23,28,32\}, \{0,1,2,4,5,7,8,10,11,12,16,17,20,22,24,27,28,30\}; | * |
| \{0,1,2,3,4,5,7,8,11,12,14,17,18,21,28,29,30,33\}, \{0,1,2,3,5,8,10,14,16,18,19,20,22,24,25,27,32,34\}; | * |
| \{0,1,2,3,4,6,7,9,12,14,17,18,19,20,24,27,28,31\}, \{0,1,2,4,5,6,8,9,10,13,15,16,20,22,24,25,31,33\}; | * |
| \{0,1,2,3,5,8,9,11,13,15,16,19,21,22,23,24,28,33\}, \{0,1,2,4,5,6,7,9,10,11,16,18,19,22,26,29,30,32\}; | 28 |

| (38; 9, 2; 2), n = 9 | None |
|---------------------|------|

| (38; 15, 4; 6), n = 13 |  |
|------------------------|---|
| \{0,1,2,3,5,6,9,14,15,18,20,22,25,30,32\}, \{0,1,7,11\}; | * |
| \{0,1,2,4,5,7,10,13,15,17,21,24,25,26,33\}, \{0,1,7,11\}; | 13 |

| (38; 12, 10; 6), n = 16 |  |
|-------------------------|---|
| \{0,1,2,4,5,8,13,14,19,21,24,31\}, \{0,2,4,6,7,13,16,17,22,30\}; | 13 |
| \{0,1,4,5,7,11,13,16,18,20,21,26\}, \{0,1,3,4,10,14,15,22,24,30\}; | * |

| (38; 16, 8; 8), n = 16 |  |
|------------------------|---|
| \{0,1,2,4,5,6,7,10,14,15,17,21,24,26,30,32\}, \{0,1,2,7,11,14,19,22\}; | * |
| \{0,1,2,4,5,7,10,11,13,15,17,21,24,25,26,33\}, \{0,1,2,7,11,14,19,22\}; | 13 |

| (39; 8, 5; 2), n = 11 |  |
|-----------------------|---|
| \{0,2,4,5,10,16,19,26\}, \{0,1,8,12,21\}; | * |
| \{0,3,4,5,12,18,22,28\}, \{0,7,9,12,20\}; | 8 |
Table 5 (continued)

(39; 9; 7; 3), \( n = 13 \)
\[
\begin{align*}
\{0,1,2,5,13,16,22,26,32\}, &\quad \{0,5,6,8,10,17,24\}; \quad \text{[S]} \\
\{0,1,4,5,10,12,19,27,30\}, &\quad \{0,1,3,5,11,17,24\}; \quad \ast
\end{align*}
\]

(39; 12; 5; 4), \( n = 13 \)
\[
\begin{align*}
\{0,1,3,5,9,10,16,17,18,20,23,28\}, &\quad \{0,4,10,13,25\}; \quad \text{[24]} \\
\{0,2,3,4,9,10,13,15,17,22,25,31\}, &\quad \{0,1,4,15,20\}; \quad \ast
\end{align*}
\]

(39; 11; 7; 4), \( n = 14 \)
\[
\begin{align*}
\{0,1,2,4,7,8,14,18,23,28,31\}, &\quad \{0,2,7,8,11,20,22\}; \quad \ast \\
\{0,1,4,6,8,13,15,18,21,31,32\}, &\quad \{0,1,2,5,11,17,21\}; \quad \text{[24]}
\end{align*}
\]

(39; 13; 9; 6), \( n = 16 \)
\[
\begin{align*}
\{0,1,3,6,7,9,14,18,19,21,23,29,33\}, &\quad \{0,1,2,4,10,11,15,18,23\}; \\
\{0,1,4,5,7,10,11,16,18,19,24,26,28\}, &\quad \ast \\
\{0,2,3,4,8,13,16,23,30\}; &\quad \text{[S]}
\end{align*}
\]

(39; 15; 8; 7), \( n = 16 \)
\[
\begin{align*}
\{0,1,2,3,4,7,10,12,15,16,18,22,23,28,32\}, &\quad \{0,2,5,6,10,17,19,26\}; \quad \ast \\
\{0,2,3,6,7,8,10,11,16,18,19,24,26,28\}, &\quad \{0,1,3,9,13,16,22,27\}; \quad \text{[S]}
\end{align*}
\]

(39; 13; 11; 7), \( n = 17 \)
\[
\begin{align*}
\{0,1,3,5,7,9,12,16,17,20,22,23,30\}, &\quad \{0,1,4,5,7,12,13,15,24,25,30\}; \\
\{0,1,4,5,7,10,11,16,18,19,24,26,28\}, &\quad \ast \\
\{0,1,2,3,5,9,13,16,22,27,32\}; &\quad \text{[S]}
\end{align*}
\]

(39; 15; 12; 9), \( n = 18 \)
\[
\begin{align*}
\{0,1,2,3,5,6,10,11,13,16,17,20,24,26,32\}, &\quad \{0,1,4,6,8,12,13,20,22,23,25,34\}; \\
\{0,2,3,4,6,8,9,13,16,18,19,22,23,30,31\}, &\quad \ast \\
\{0,1,2,4,7,8,10,15,19,21,26,31\}; &\quad \text{[S]}
\end{align*}
\]

(39; 19; 19; 18), \( n = 20 \)
\[
\begin{align*}
\{0,1,2,3,4,5,6,9,13,15,17,20,22,23,25,28,30,32,36\}, &\quad \{0,1,2,3,5,6,9,10,11,12,15,16,17,19,23,24,29,30,33\}; \quad \text{Sz}
\end{align*}
\]
Table 5 (continued)

(39; 19, 19; 18), $n = 20$
- \{0, 1, 2, 3, 5, 7, 8, 9, 13, 14, 16, 18, 19, 20, 22, 23, 30, 32, 33\},
- \{0, 1, 2, 4, 5, 6, 7, 9, 12, 13, 16, 17, 19, 22, 24, 25, 27, 31, 35\}; *
- \{0, 1, 2, 3, 6, 7, 8, 9, 11, 15, 18, 19, 21, 22, 23, 25, 30, 31, 36\},
- \{0, 1, 2, 3, 4, 5, 8, 9, 12, 14, 15, 17, 19, 22, 28, 30, 32, 34, 35\}; [3]

(40; 9, 3; 2), $n = 10$
- \{0, 1, 2, 5, 8, 13, 17, 19, 26\}, \{0, 10, 20\}; [13]

(40; 18, 3; 8), $n = 13$ None [21]

(40; 16, 9; 8), $n = 17$
- \{0, 1, 2, 3, 6, 8, 10, 14, 17, 18, 19, 21, 24, 28, 29, 34\},
- \{0, 1, 3, 4, 6, 10, 15, 23, 31\}; *
- \{0, 1, 3, 4, 5, 9, 10, 11, 15, 17, 18, 21, 25, 27, 30, 32\},
- \{0, 2, 3, 4, 9, 13, 16, 21, 24\}; *
- \{0, 2, 3, 4, 5, 6, 8, 9, 12, 16, 17, 21, 23, 28, 30, 34\},
- \{0, 3, 5, 10, 11, 19, 20, 27, 30\}; [13]

(40; 13, 13; 8), $n = 18$
- \{0, 1, 2, 3, 4, 7, 10, 14, 18, 20, 23, 28, 35\},
- \{0, 1, 2, 3, 6, 10, 12, 16, 17, 21, 29, 32, 34\}; *
- \{0, 1, 2, 4, 5, 12, 14, 16, 18, 19, 24, 25, 33\},
- \{0, 3, 4, 6, 8, 10, 11, 15, 16, 21, 24, 30, 33\}; *
- \{0, 1, 2, 5, 6, 8, 10, 16, 17, 18, 21, 27, 28\},
- \{0, 2, 3, 4, 7, 9, 12, 15, 18, 22, 26, 28, 35\}; *
- \{0, 2, 3, 5, 6, 7, 12, 14, 16, 23, 24, 27, 30\},
- \{0, 2, 4, 5, 8, 10, 13, 14, 18, 19, 25, 32, 33\}; [13]

(40; 18, 16; 14), $n = 20$
- \{0, 1, 2, 3, 4, 6, 7, 8, 12, 14, 15, 17, 19, 22, 23, 26, 31, 32\},
- \{0, 1, 2, 3, 6, 8, 10, 13, 14, 18, 21, 23, 24, 27, 34, 36\}; *
- \{0, 1, 3, 4, 6, 8, 9, 10, 13, 16, 18, 21, 22, 23, 29, 32, 33, 34\},
- \{0, 1, 2, 5, 6, 7, 9, 11, 15, 17, 19, 20, 22, 23, 26, 33\}; *
- \{0, 2, 3, 4, 5, 6, 8, 11, 13, 14, 16, 17, 20, 21, 22, 26, 30, 33\},
- \{0, 1, 2, 3, 7, 8, 11, 14, 18, 22, 23, 24, 26, 28, 33, 35\}; GP
- \{0, 2, 3, 4, 5, 7, 11, 13, 16, 17, 19, 21, 22, 24, 28, 29, 35, 36\},
- \{0, 1, 2, 4, 5, 8, 10, 11, 13, 14, 15, 20, 23, 24, 28, 30\}; [13]
8. The range $40 < v \leq 50$

Our most important results are given in this section. The SDSs having the parameter sets mentioned in the Introduction are all included in Table 6. We list all 89 feasible parameter sets in the above range, and in each case list (or give reference to) all known SDSs.

We have inserted in this table the three SDSs constructed by Morales [25] with parameters

$$(45; 11, 11; 5), \quad (46; 10, 10; 4), \quad (49; 9, 9; 3)$$

and one of the SDSs constructed by Abel [1], with parameters $(47; 7, 7; 3)$. Morales found several families in some cases but only one was reported in his paper. Apparently he did not check the families for equivalence. In each of the four cases mentioned above we have constructed a non-equivalent SDS.

Moreover, we found an SDS with parameters $(45; 22, 22; 21)$. By a well known old result of Bose [7], it gives a BIBD with parameters mentioned in the abstract. However, the most interesting SDS that we have constructed is probably the very last one in the table as it gives the first example of a pair of binary sequences of length 50 having zero periodic autocorrelation function. Our genetic program was running for about 7-8 days in order to find this particular SDS. While Golay pairs of length 50 do not exist [2, 6], we see that the periodic analog of them exists. The first example of this phenomenon has been observed earlier for length 34 (see [14]).

There are five D-optimal cases in this range, two with $v = 43$ and one for each of $v = 41, 45, 49$. For $v = 45$ the equivalence classes have been enumerated in [23]. As there are 1358 equivalence classes of SDSs with these parameters, we refer the interested reader to that paper. For $v = 41$ the first D-optimal SDS has been constructed in [8], and we have now constructed a new one. For $v = 43$ we list the 5 D-optimal SDSs with $\lambda = 13$ and the 10 with $\lambda = 15$ constructed in [8]. We have contributed one new SDS in the former and two in the latter case. Finally, the first D-optimal SDS with $v = 49$ has been constructed in [10], and we have found a new one.
Table 6: \((v; r, s; \lambda)\) difference families with \(40 < v \leq 50\)

\[
\begin{align*}
\text{(41; 5, 5; 1), } n &= 9 \\
&\{0,1,4,11,29\}, \quad \{0,2,8,17,22\}; \quad [29]
\end{align*}
\]

\[
\begin{align*}
\text{(41; 10, 6; 3), } n &= 13 \\
&\{0,1,2,6,13,15,17,22,25,35\}, \quad \{0,1,4,10,15,18\}; \quad * \\
&\{0,1,3,6,8,14,15,24,25,29\}, \quad \{0,2,6,10,13,22\}; \quad [24]
\end{align*}
\]

\[
\begin{align*}
\text{(41; 11, 10; 5), } n &= 16 \\
&\{0,1,2,3,5,9,13,18,19,23,34\}, \quad \{0,2,5,8,11,15,17,22,29,30\}; \quad [8] \\
&\{0,1,2,4,5,9,12,18,20,25,31\}, \quad \{0,2,3,6,8,12,19,20,29,34\}; \quad * \\
\end{align*}
\]

\[
\begin{align*}
\text{(41; 15, 6; 6), } n &= 15 \\
&\{0,1,2,3,5,6,7,8,9,13,18,19,22,23,26,32,34\} \\
&\{0,2,5,8,11,15,17,18,22,29,30\}; \quad [8] \\
&\{0,1,3,4,5,7,10,11,14,15,23,26,28,32,34\}, \quad \{0,1,2,4,7,9,16,17,22,28,33\}; \quad * \\
\end{align*}
\]

\[
\begin{align*}
\text{(41; 20, 5; 10), } n &= 15 \\
&\{0,1,2,3,5,6,7,9,11,13,14,16,17,21,25,26,28,31,34,35\}, \quad \{0,1,6,17,19\}; \quad [24] \\
\end{align*}
\]

\[
\begin{align*}
\text{(41; 16, 16; 12), } n &= 20 \\
&\{0,1,2,3,5,7,8,9,13,18,19,22,23,26,32,34\}, \quad \{0,1,3,4,6,8,11,13,15,16,17,23,24,27,30,36\}; \quad [18] \\
&\{0,1,2,4,5,6,8,10,11,14,20,21,27,29,32,34\}, \quad \{0,1,3,5,6,7,12,13,15,17,20,23,24,28,31,32\}; \quad * \\
&\{0,1,3,4,5,6,10,14,15,17,21,23,24,29,31,36\}, \quad \{0,2,3,4,6,7,11,12,13,15,19,20,23,26,29,31\}; \quad [10] \\
\end{align*}
\]

\[
\begin{align*}
\text{(41; 20, 20; 19), } n &= 21 \\
&\{0,1,2,3,4,5,7,9,13,14,16,20,21,22,26,27,30,31,33,36\}, \quad \{0,1,2,3,4,6,7,9,10,11,13,14,17,19,22,25,27,29,33,34\}; \quad * \\
&\{0,1,2,3,4,6,7,9,10,11,14,18,19,20,22,23,27,30,36,37\}, \quad \{0,1,2,3,5,7,9,11,13,14,15,16,21,24,26,27,30,31,33,36\}; \quad Sz \\
&\{0,1,2,3,4,6,8,11,13,15,16,17,18,19,23,24,27,33,36,37\}, \quad \{0,1,2,5,6,8,9,11,12,14,15,18,19,20,26,28,30,31,33,35\}; \quad [28] \\
\end{align*}
\]

Table 6 (continued)

(42; 16, 3; 6), \( n = 13 \)
\{0,1,3,4,5,6,9,10,12,16,20,22,25,27,34,35\}, \{0,14,28\}; \[24\]

(42; 17, 8; 8), \( n = 17 \)
\{0,1,2,4,7,9,10,11,13,16,17,20,21,26,28,30,31\}, \{0,1,5,8,13,19,25,27\}; \[24\]
\{0,1,3,4,5,7,10,12,13,14,16,22,23,27,28,30,35\}, \{0,2,6,10,11,16,24,27\}; \* 
\{0,2,3,4,5,6,8,12,13,14,17,19,22,26,27,29,35\}, \{0,1,4,8,15,20,26,32\}; \* 

(42; 13, 10; 6), \( n = 17 \)
\{0,1,2,3,4,7,8,13,18,21,26,33,35\}, \{0,1,4,7,9,16,20,22,28,32\}; \* 
\{0,1,6,7,8,11,15,17,19,20,26,29,32\}, \{0,2,3,5,7,13,17,21,22,29\}; \[24\]

(42; 20, 6; 10), \( n = 16 \)
\{0,1,2,3,5,6,8,10,12,13,16,18,19,20,22,25,26,31,33,34\}, \{0,4,5,9,20,27\}; \* 
\{0,1,2,3,5,7,9,11,12,16,17,19,20,23,24,25,27,30,32,33\}, \{0,3,4,9,15,32\}; \* 
\{0,1,2,4,5,6,9,12,14,15,17,19,20,21,22,26,28,29,32,38\}, \{0,1,8,12,19,21\}; \* 
\{0,1,2,5,6,7,8,9,11,13,17,18,20,21,23,25,28,31,34,35\}, \{0,1,6,10,19,21\}; \[24\]

(42; 21, 9; 12), \( n = 18 \)
\{0,1,2,4,5,6,9,12,13,15,16,17,18,22,23,24,25,27,30,32,38\}, \{0,2,3,6,11,13,20,26,30\}; \* 
\{0,1,2,4,6,7,9,10,11,13,14,17,20,21,22,25,27,29,30,31,37\}, \{0,2,3,8,12,17,18,20,31\}; \[13\]
\{0,1,3,4,5,8,9,10,11,13,14,16,17,21,22,24,28,31,33,35,37\}, \{0,1,2,4,12,17,18,24,27\}; \* 

(43; 6, 4; 1), \( n = 9 \) None \[24\]
(43; 9, 4; 2), \( n = 11 \) ?
**Table 6 (continued)**

| (43; 7, 7; 2), n = 12 |  |
|------------------------|---|
| \{0,1,3,8,12,18,24\}, \{0,1,5,8,19,21,34\}; | [1] |
| \{0,1,4,10,12,20,25\}, \{0,2,3,7,14,20,29\}; | * |

| (43; 13, 4; 4), n = 13 |  |
|------------------------|---|
| \{0,1,2,5,6,9,12,16,18,21,23,29,31\}, \{0,1,9,19\}; | [24] |

| (43; 16, 4; 6), n = 14 |  |
|------------------------|---|
| \{0,2,3,4,6,7,10,11,13,15,20,23,28,29,34,35\}, \{0,2,12,29\}; | [8] |

| (43; 15, 7; 6), n = 16 |  |
|------------------------|---|
| \{0,1,3,4,6,8,12,14,17,18,21,26,27,28,33\}, \{0,3,5,12,13,20,24\}; | * |
| \{0,1,3,5,6,7,8,11,15,17,20,26,27,30,38\}, \{0,1,8,15,17,26,30\}; | [8] |

| (43; 18, 6; 8), n = 16 |  |
|------------------------|---|
| \{0,1,2,3,5,6,9,10,11,12,17,20,23,24,28,30,33,37\}, \{0,2,4,12,17,31\}; | [8] |
| \{0,1,3,5,6,7,11,12,13,15,17,20,24,25,28,31,33,34\}, \{0,3,4,11,18,20\}; | * |

| (43; 18, 9; 9), n = 18 |  |
|------------------------|---|
| \{0,1,2,4,5,6,8,10,11,14,18,19,21,26,27,33,34,38\}, \{0,2,3,5,14,17,21,23,35\}; | [8] |
| \{0,2,3,4,5,7,9,10,13,15,18,19,22,25,26,33,34,36\}, \{0,1,2,6,8,14,19,24,28\}; | * |

| (43; 15, 15; 10), n = 20 |  |
|------------------------|---|
| \{0,1,2,3,4,6,12,16,18,21,23,26,30,34,37\}, \{0,1,2,4,7,8,9,13,14,17,18,24,26,29,37\}; | [8] |
| \{0,1,2,3,5,7,9,15,18,21,23,24,28,34,35\}, \{0,1,3,5,7,8,10,13,14,18,19,22,26,33,34\}; | * |
| \{0,1,2,3,8,9,12,13,15,17,22,24,26,32,40\}, \{0,1,3,5,6,10,11,13,18,19,22,26,28,29,32\}; | [11] |
| \{0,1,2,4,5,9,10,12,14,16,17,24,27,28,34\}, \{0,1,2,4,6,8,13,14,17,22,23,25,28,31,38\}; | * |
Table 6 (continued)

| (43; 21, 7; 11), n = 17 |   |   |
|-------------------------|--|---|
| \{0,0,1,3,4,5,6,7,10,11,13,15,18,19,21,23,25,26,30,32,34,35\}, |   |   |
| \{0,1,7,10,17,18,23\}; \[9\] |   |   |
| \{0,2,3,4,5,6,7,10,11,12,15,17,18,20,24,26,27,31,32,35,36\}, |   |   |
| \{0,2,4,10,13,20,26\}; * |   |   |

| (43; 18, 13; 11), n = 20 |   |   |
|-------------------------|--|---|
| \{0,1,2,3,5,8,9,13,14,15,18,21,23,24,28,32,35,39\}, |   |   |
| \{0,1,2,3,6,8,9,12,17,19,27,29,31\}; * |   |   |
| \{0,1,2,4,6,8,9,10,11,13,16,19,21,22,28,29,33,35\}, |   |   |
| \{0,1,3,6,7,11,16,17,20,24,28,29,31\}; * |   |   |
| \{0,1,4,5,6,7,10,12,13,14,16,20,24,27,28,29,31,38\}, |   |   |
| \{0,1,3,8,9,11,14,17,19,21,26,30,31\}; \[9\] |   |   |

| (43; 18, 16; 13), n = 21 |   |   |
|-------------------------|--|---|
| \{0,1,2,3,4,7,9,11,12,13,16,19,22,24,25,29,30,36\}, |   |   |
| \{0,1,2,4,5,6,9,14,16,17,20,24,26,31,33,39\}; \[9\] |   |   |
| \{0,1,2,3,4,7,9,11,12,13,16,19,22,24,25,29,30,36\}, |   |   |
| \{0,1,3,5,8,9,14,17,18,19,21,24,25,29,31,33\}; \[9\] |   |   |
| \{0,1,2,3,6,7,9,10,13,16,18,20,21,22,27,28,30,32\}, |   |   |
| \{0,2,3,4,7,8,10,12,15,20,21,23,27,32,36,37\}; \[9\] |   |   |
| \{0,1,2,4,5,6,9,10,11,16,17,18,21,24,26,30,34,37\}, |   |   |
| \{0,1,2,3,4,7,10,12,14,16,21,22,25,28,30,33\}; \[9\] |   |   |
| \{0,1,2,4,5,7,8,10,11,14,16,18,22,23,28,31,33,35\}, |   |   |
| \{0,1,4,5,6,7,11,14,18,19,20,22,27,29,30,38\}; * |   |   |
| \{0,1,2,4,5,7,8,11,12,16,18,20,21,26,27,31,33,35\}, |   |   |
| \{0,1,3,5,6,7,8,13,14,17,21,23,24,26,35,38\}; \[18\] |   |   |
| \{0,2,4,5,6,7,10,11,14,15,19,21,22,24,30,31,35,37\}, |   |   |
| \{0,1,2,3,4,7,11,13,16,17,19,22,24,25,29,36\}; \[9\] |   |   |

| (43; 21, 15; 15), n = 21 |   |   |
|-------------------------|--|---|
| \{0,1,2,3,4,7,9,11,12,13,14,17,20,24,25,28,30,31,34,39\}, |   |   |
| \{0,2,3,4,7,9,12,14,16,22,24,30,31,34,39\}; * |   |   |
| \{0,1,2,3,4,6,7,9,10,11,13,14,17,20,22,23,27,28,29,34,35\}, |   |   |
| \{0,1,2,4,6,9,13,15,17,20,21,25,30,33,35\}; \[9\] |   |   |
| \{0,1,2,3,4,6,7,9,10,11,13,14,17,20,22,23,27,28,29,34,35\}, |   |   |
| \{0,1,3,5,9,11,14,15,16,19,23,28,31,33,35\}; \[9\] |   |   |
Table 6 (continued)

(43; 21, 15; 15), \( n = 21 \)
\{0,1,2,3,4,6,8,9,13,15,16,18,19,21,24,25,28,33,35,38,39\},
\{0,1,2,4,5,12,13,15,16,17,21,23,30,37,39\}; \quad 9
\{0,1,2,3,4,6,8,9,13,15,16,18,19,21,24,25,28,33,35,38,39\},
\{0,1,2,5,9,10,16,17,18,20,21,23,27,29,31\}; \quad 9
\{0,1,2,3,4,7,10,11,12,14,16,19,21,22,24,27,28,30,32,33,37\},
\{0,1,2,4,5,6,8,12,15,19,20,21,27,34,39\}; \quad 9
\{0,1,2,3,4,7,10,11,12,14,16,19,21,22,24,27,28,30,32,33,37\},
\{0,1,5,6,7,9,10,12,13,14,16,20,24,28,29,31\}; \quad 9
\{0,1,2,3,4,7,10,11,13,14,16,19,21,22,23,24,27,28,32,37,39\},
\{0,1,2,4,5,7,8,12,14,16,20,26,29,31,36\}; \quad 9
\{0,1,2,3,5,6,7,9,12,15,16,18,19,20,22,26,27,28,29,34,38\},
\{0,1,2,5,8,10,11,13,15,19,23,24,26,31,39\}; \quad 9
\{0,1,2,3,5,6,7,9,10,11,14,17,18,20,23,25,26,27,28,33,36,39\},
\{0,1,2,3,5,7,8,12,14,16,20,22,31,32,36\}; \quad 9
\{0,1,2,3,5,7,10,11,12,13,14,17,19,22,25,26,27,30,33,35,39\},
\{0,1,3,5,6,7,9,15,16,19,20,22,26,27,34\}; \quad 9
\{0,1,2,3,6,7,8,10,12,14,16,17,21,22,24,25,27,30,33,34,36\},
\{0,1,3,4,5,8,11,12,16,17,18,23,26,28,30\}; \quad 9
\{0,1,2,3,6,7,8,10,12,14,16,17,21,22,24,25,27,30,33,34,36\},
\{0,1,3,4,5,8,11,12,16,17,18,23,26,28,30\}; \quad *
\{0,1,2,4,5,6,9,10,11,14,16,17,18,21,24,26,30,32,34,35,37\},
\{0,2,3,5,6,7,9,12,15,16,20,26,27,28,34\}; \quad 9
\{0,1,2,4,5,7,8,9,10,13,14,17,18,19,22,24,26,29,30,32,37\},
\{0,1,2,3,4,6,11,13,17,20,23,27,29,34,35\}; \quad 9
\{0,1,3,4,5,6,8,9,10,14,15,17,18,21,25,27,28,32,33,35,37\},
\{0,2,3,4,6,8,9,13,15,16,21,26,28,29,37\}; \quad 10

(43; 21, 21; 20), \( n = 22 \)
\{0,1,2,3,4,6,8,9,11,14,15,18,20,23,24,25,28,33,34,36,40\},
\{0,1,2,3,5,7,8,9,11,13,14,15,18,19,22,26,28,29,30,31,38\}; \quad *
\{0,1,2,3,5,6,9,10,11,12,17,20,21,23,24,26,28,30,33,34,37\},
\{0,1,2,4,5,6,7,8,9,13,14,15,18,20,21,23,28,31,33,35,39\}; \quad 13

(44; 8, 6; 2), \( n = 12 \) None \quad \text{[24]}
(44; 19, 2; 8), \( n = 13 \)
Table 6 (continued)

(44; 17, 9; 8)), $n = 18$
\{0,1,2,3,4,7,9,11,15,17,20,22,23,27,28,32,37\}, \{0,1,4,10,12,13,16,23,30\}; $\star$
\{0,1,2,4,6,7,12,13,18,20,22,23,26,27,30,35,37\}, \{0,2,3,6,10,15,16,18,27\}; \[24\]
\{0,1,2,5,6,7,8,9,15,18,19,22,24,26,29,34,38\}, \{0,1,4,6,12,14,17,26,35\}; $\star$

(44; 18, 15; 12), $n = 21$ None \[24\]
(44; 21, 14; 14), $n = 21$ None \[24\]
(45; 7, 2; 1), $n = 8$ None \[24\]

(45; 10, 7; 3), $n = 14$
\{0,1,2,4,9,11,14,19,25,31\}, \{0,4,8,11,17,29,30\}; $\star$
\{0,3,4,5,10,17,19,21,27,30\}, \{0,1,6,9,13,21,35\}; \[24\]

(45; 13, 5; 4), $n = 14$
\{0,1,4,6,8,12,13,14,21,24,30,31,34\}, \{0,3,5,14,19\}; \[24\]

(45; 11, 11; 5), $n = 17$
\{0,1,2,8,10,11,14,19,24,27,31\}, \{0,2,3,4,7,9,16,22,27,33,37\}; $\star$
\{0,1,4,7,8,16,17,22,27,31,33\}, \{0,1,5,7,8,10,12,18,20,29,32\}; \[25\]

(45; 12, 12; 6), $n = 18$
\{0,1,2,3,8,10,14,15,19,24,27,30\}, \{0,1,3,4,6,10,14,19,26,28,34,38\}; \[24\]
\{0,1,2,6,7,9,13,18,21,23,26,36\}, \{0,1,4,7,8,10,16,18,20,27,31,32\}; $\star$

(45; 18, 2; 7), $n = 13$ ?

(45; 18, 10; 9), $n = 19$
\{0,1,2,5,7,9,11,12,13,15,16,20,21,28,30,33,34,36\}, \{0,1,4,6,9,15,16,23,26,33\}; $\star$
Table 6 (continued)

(45; 16, 13; 9), \( n = 20 \)
\{0, 1, 2, 5, 10, 11, 12, 16, 17, 18, 20, 23, 25, 31, 34, 37\};
\{0, 1, 3, 4, 5, 10, 13, 17, 20, 22, 24, 28, 32\};
\{0, 1, 3, 4, 6, 10, 12, 14, 15, 17, 21, 22, 26, 27, 30, 32\};
\{0, 1, 2, 8, 9, 10, 14, 18, 21, 25, 28, 31, 33\};

(45; 21, 5; 10), \( n = 16 \)
\{0, 1, 2, 3, 4, 5, 6, 9, 10, 14, 15, 17, 21, 22, 23, 25, 28, 31, 35, 38, 40\};
\{0, 4, 6, 15, 33\};

(45; 22, 11; 13), \( n = 20 \)
\{0, 1, 2, 3, 4, 7, 9, 11, 14, 15, 18, 19, 20, 21, 23, 26, 27, 28, 31, 34, 36, 42\};
\{0, 1, 2, 3, 7, 10, 12, 16, 25, 31, 35\};
\{0, 2, 3, 4, 5, 6, 8, 9, 11, 13, 16, 17, 18, 19, 23, 26, 29, 30, 34, 35, 38, 39\};
\{0, 1, 3, 7, 9, 11, 14, 21, 23, 28, 29\};

(45; 21, 16; 15), \( n = 22 \) DO

(45; 22, 22; 21), \( n = 23 \)
\{0, 1, 2, 3, 6, 7, 8, 10, 11, 13, 14, 18, 20, 22, 23, 24, 26, 29, 31, 32, 33, 41\};
\{0, 1, 3, 4, 5, 6, 7, 11, 12, 14, 15, 18, 19, 20, 21, 24, 28, 30, 32, 35, 37, 40\};

(46; 10, 10; 4), \( n = 16 \)
\{0, 1, 2, 5, 10, 13, 19, 25, 29, 31\}; \{0, 1, 2, 7, 9, 12, 16, 20, 23, 33\};
\{0, 1, 3, 4, 13, 15, 19, 20, 26, 28\}; \{0, 3, 5, 8, 12, 13, 19, 23, 29, 37\};

(46; 16, 6; 6), \( n = 16 \)
\{0, 1, 2, 5, 7, 8, 9, 12, 15, 20, 21, 23, 25, 27, 36, 37\};
\{0, 3, 9, 17, 22, 26\};

(46; 21, 6; 10), \( n = 17 \) ?
(46; 16, 15; 10), \( n = 21 \) None
(46; 21, 15; 14), \( n = 22 \) None
(47; 10, 2; 2), \( n = 10 \) None
(47; 9, 5; 2), \( n = 12 \) ?
(47; 12, 3; 3), \( n = 12 \) ?
(47; 14, 2; 4), \( n = 12 \) ?
(47; 15, 5; 5), \( n = 15 \) ?
Table 6 (continued)

\[(47; 19, 9; 9), \quad n = 19\]
\[\{0, 1, 2, 3, 4, 6, 7, 10, 14, 16, 17, 19, 23, 24, 26, 28, 34, 36, 42\},\]
\[\{0, 4, 5, 9, 10, 18, 21, 29, 36\}; \quad \ast \]
\[\{0, 1, 2, 3, 6, 7, 9, 11, 13, 14, 17, 21, 22, 23, 29, 31, 32, 34, 36\},\]
\[\{0, 4, 5, 9, 12, 15, 21, 28, 38\}; \quad \text{[24]} \]

\[(47; 22, 10; 12), \quad n = 20\]
\[\{0, 1, 2, 3, 4, 5, 7, 9, 10, 12, 15, 17, 18, 19, 23, 24, 27, 28, 30, 34, 35, 38\},\]
\[\{0, 2, 8, 9, 13, 16, 22, 26, 28, 38\}; \quad \ast \]
\[\{0, 1, 2, 4, 5, 7, 8, 11, 12, 14, 17, 19, 22, 24, 25, 26, 31, 32, 33, 35, 39, 43\},\]
\[\{0, 1, 2, 5, 10, 11, 14, 16, 27, 29\}; \quad \text{[24]} \]

\[(47; 21, 12; 12), \quad n = 21\]
\[\{0, 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 20, 21, 25, 26, 29, 32, 33, 36, 39\},\]
\[\{0, 1, 3, 5, 10, 14, 15, 16, 23, 29, 31, 41\}; \quad \ast \]
\[\{0, 1, 2, 4, 5, 6, 11, 12, 14, 16, 18, 20, 23, 24, 27, 31, 33, 34, 36, 39, 44\},\]
\[\{0, 1, 2, 5, 10, 11, 12, 19, 25, 26, 31\}; \quad \ast \]

\[(47; 19, 15; 12), \quad n = 22\]
\[\{0, 1, 2, 4, 6, 8, 11, 12, 15, 18, 19, 20, 22, 25, 27, 28, 30, 35, 36\},\]
\[\{0, 2, 3, 5, 8, 9, 12, 14, 18, 22, 23, 27, 33, 34, 35\}; \quad \ast \]

\[(47; 22, 14; 14), \quad n = 22\]
\[\{0, 1, 2, 3, 4, 6, 8, 9, 11, 12, 14, 15, 16, 20, 21, 22, 27, 30, 31, 34, 36, 38\},\]
\[\{0, 1, 2, 5, 8, 10, 15, 19, 23, 25, 26, 31, 35, 38\}; \quad \ast \]

\[(47; 23, 23; 22), \quad n = 24\]
\[(48; 14, 3; 4), \quad n = 13\]
\[(48; 12, 8; 4), \quad n = 16\]
\[\{0, 1, 3, 4, 9, 11, 16, 17, 21, 27, 31, 40\}; \quad \{0, 2, 3, 7, 9, 22, 25, 36\}; \quad \text{[24]} \]
\[\{0, 2, 4, 5, 10, 12, 13, 18, 22, 27, 33, 34\}, \quad \{0, 3, 4, 7, 14, 16, 27, 33\}; \quad \ast \]

\[(48; 16, 7; 6), \quad n = 17\]
\[\{0, 1, 2, 3, 7, 9, 10, 14, 19, 20, 22, 25, 28, 32, 34, 36\},\]
\[\{0, 3, 4, 8, 13, 24, 41\}; \quad \text{[24]} \]
\[\{0, 1, 3, 4, 8, 10, 11, 12, 15, 17, 23, 25, 28, 29, 33, 39\},\]
\[\{0, 2, 3, 9, 18, 31, 36\}; \quad \ast \]
Table 6 (continued)

| (48; 15, 9; 6), n = 18 | \{0,1,2,4,5,10,12,14,17,20,23,24,28,31,37\}, \{0,1,2,6,11,18,20,26,33\}; | * |
|------------------------|-------------------------------------------------|-----|
|                         | \{0,1,2,7,9,12,14,16,22,25,30,31,34,35,38\}, \{0,1,3,4,9,11,15,21,32\}; | 24 |
| (48; 20, 10; 10), n = 20 | \{0,1,2,6,8,9,10,11,12,15,16,19,21,24,27,29,32,34,36,38\}, \{0,1,4,5,11,17,18,25,34,37\}; | * |
| (48; 24, 4; 12), n = 16 | \{0,1,2,4,5,6,8,9,11,13,15,16,18,19,21,24,27,28,31,32,33,38,39,40\}, \{0,2,6,20\}; | * |
| (49; 7, 3; 1), n = 9 | \{0,1,7,10,15,27,31\}, \{0,2,13\}; | 8 |
| (49; 12, 4; 3), n = 13 | \{0,1,2,7,10,12,15,18,22,24,31,35\}, \{0,1,19,23\}; | 8 |
| (49; 9, 9; 3), n = 15 | \{0,1,2,4,7,12,20,27,36\}, \{0,1,4,9,16,18,22,28,39\}; | 25 |
|                         | \{0,1,3,6,12,13,20,28,38\}, \{0,2,4,7,8,18,23,27,36\}; | * |
| (49; 15, 6; 5), n = 16 | \{0,1,2,3,4,6,10,13,17,21,22,27,29,35,40\}, \{0,5,12,15,21,29\}; | 8 |
|                         | \{0,1,3,7,9,10,13,15,18,22,23,32,34,38,39\}, \{0,4,5,7,18,26\}; | * |
| (49; 13, 12; 6), n = 19 | \{0,1,2,4,6,10,11,18,22,23,30,33,36\}, \{0,1,3,6,11,12,15,19,21,26,28,36\}; | 8 |
|                         | \{0,1,3,5,6,9,17,19,21,25,26,36,43\}, \{0,1,2,7,10,12,15,21,22,25,29,38\}; | * |
Table 6 (continued)

| (49; 18, 6; 7), \( n = 17 \) | \( \{0, 1, 2, 3, 7, 9, 12, 15, 16, 17, 20, 23, 27, 28, 30, 32, 34, 40\} \), \( \{0, 5, 6, 9, 19, 28\} \); | 24 |
| (49; 19, 7; 8), \( n = 18 \) | \( \{0, 1, 2, 3, 5, 7, 8, 9, 12, 16, 18, 19, 22, 24, 30, 31, 32, 36, 41\} \), \( \{0, 3, 8, 12, 15, 26, 36\} \); \* \( \{0, 1, 2, 4, 5, 8, 10, 13, 15, 17, 19, 23, 24, 29, 30, 31, 34, 37, 41\} \), \( \{0, 3, 5, 6, 14, 15, 31\} \); | 24 |
| (49; 21, 4; 9), \( n = 16 \) | ? |
| (49; 19, 10; 9), \( n = 20 \) | \( \{0, 1, 2, 3, 5, 8, 10, 12, 13, 16, 20, 21, 22, 25, 27, 33, 34, 36, 40\} \), \( \{0, 4, 6, 7, 10, 14, 22, 23, 28, 33\} \); \( \{0, 1, 2, 4, 5, 7, 9, 11, 16, 17, 19, 20, 23, 26, 27, 31, 32, 37, 40\} \), \( \{0, 1, 3, 5, 11, 13, 18, 24, 25, 33\} \); \* |
| (49; 16, 16; 10), \( n = 22 \) | \( \{0, 1, 2, 4, 6, 9, 10, 12, 13, 16, 18, 20, 23, 30, 34, 35, 43\} \), \( \{0, 1, 3, 4, 5, 6, 11, 12, 18, 22, 25, 27, 31, 35, 38, 40\} \); \* \( \{0, 1, 3, 4, 7, 8, 12, 14, 15, 16, 25, 27, 30, 32, 35, 44\} \), \( \{0, 1, 4, 6, 7, 8, 14, 16, 17, 19, 23, 25, 29, 30, 35, 39\} \); | 24 |
| (49; 21, 13; 12), \( n = 22 \) | \( \{0, 1, 2, 3, 5, 7, 8, 9, 13, 14, 16, 17, 18, 23, 26, 28, 29, 32, 33, 36, 45\} \), \( \{0, 1, 3, 6, 11, 13, 15, 17, 23, 30, 31, 38, 41\} \); \* |
| (49; 24, 9; 13), \( n = 20 \) | \( \{0, 1, 2, 3, 5, 6, 8, 9, 11, 12, 17, 18, 19, 21, 23, 25, 28, 31, 32, 33, 36, 37, 40, 44\} \), \( \{0, 1, 3, 10, 11, 16, 21, 23, 25\} \); \* \( \{0, 1, 2, 3, 6, 7, 9, 11, 12, 13, 15, 16, 17, 19, 21, 26, 28, 29, 30, 33, 36, 37, 39, 44\} \), \( \{0, 3, 4, 5, 12, 15, 21, 29, 34\} \); | 8 |
Table 6 (continued)

(49; 22, 15; 14), \( n = 23 \)
\{0,1,2,3,4,5,7,9,10,14,15,17,19,22,24,25,28,31,33,35,39,41\},
\{0,3,4,6,7,11,12,18,19,20,24,29,33,40\};
\{0,2,3,4,5,8,10,12,13,14,16,20,21,22,23,25,27,32,33,36,38,39\}, \{0,1,5,6,8,9,12,15,19,24,27,29,33,34,41\};  
\[24\]

(49; 22, 18; 16), \( n = 24 \)
\{0,1,2,3,4,5,6,9,11,13,14,19,20,21,23,24,26,27,30,35,38,40,42\},
\{0,1,3,4,5,8,9,13,15,19,21,24,26,27,30,37,43,44\}; \[10\]
\{0,1,2,3,4,5,7,10,12,13,15,18,19,22,23,24,29,30,34,36,38,43\},
\{0,1,3,4,6,7,9,10,14,16,17,24,26,28,32,36,37,41\};
\[24\]

(49; 24, 24; 23), \( n = 25 \)
\{0,1,2,3,4,5,6,8,9,11,13,15,17,18,19,20,23,26,27,28,30,33,37,39,44\}, \{0,1,2,3,5,6,7,10,13,14,15,17,21,22,23,26,27,30,32,33,35,36,38,44\};
\[24\]

(50; 8, 7; 2), \( n = 13 \)
\( ? \)

(50; 20, 4; 8), \( n = 16 \)
\( ? \)

(50; 15, 14; 8), \( n = 21 \) None \[24\]

(50; 20, 11; 10), \( n = 21 \) None \[24\]

(50; 20, 18; 14), \( n = 24 \) None \[24\]

(50; 22, 21; 18), \( n = 25 \)
\( ? \)

(50; 25, 20; 20), \( n = 25 \)
\{0,1,2,3,5,6,7,9,10,11,13,15,17,20,22,23,26,27,28,29,31,36,38,39,45\},
\{0,1,2,3,5,6,8,14,15,17,18,21,25,27,32,35,36,40,44,45\};
\[24\]

References

[1] R.J.R. Abel, Forty-three balanced incomplete block designs, J. Combin. Theory Ser. A \textbf{65} (1994), 252–267.

[2] T.H. Andres, Some combinatorial properties of complementary sequences, M.Sc. Thesis, University of Manitoba, Winnipeg, 1977.

[3] K.T. Arasu and Q. Xiang, On the existence of periodic complementary binary sequences, Designs, Codes and Cryptography \textbf{2} (1992), 257–262.
[4] D. Ashlock, Finding designs with genetic algorithms, in W.D. Wallis (Ed.), Computational and Constructive Design Theory, pp. 49–65, Kluwer Academic Publishers, Dordrecht/Boston/London, 1996.

[5] L. Bömer and M. Antweiler, Periodic complementary binary sequences, IEEE Trans. Inform. Theory 36 (1990), 1487–1494.

[6] P.B. Borwein and R.A. Ferguson, A complete description of Golay pairs for lengths up to 100, Math. Comp. 73 (2003), 967–985.

[7] R.C. Bose, On the construction of balanced incomplete block designs, Ann. Eugenics 9 (1939), 353–399.

[8] S. Chadjiconstantinidis, T. Chadjipadelis and K. Sotirakoglou, Two cyclic supplementary difference sets and optimal designs in linear models, J. Comb. Math. Comb. Comput. 18 (1995), 33–56.

[9] T. Chadjipantelis and S. Konias, Supplementary difference sets and D-optimal designs for $n \equiv 2 \mod 4$, Discrete Math. 57 (1985), 211–216.

[10] J.H.E. Cohn, On determinants with elements $\pm 1$, II, Bull. London Math. Soc. 21 (1989), 36–42.

[11] C.J. Colbourn and J.H. Dinitz, Editors, Handbook of Combinatorial Designs, 2nd edition, Chapman & Hall, Boca Raton/London/New York, 2007.

[12] C. Ding, Two constructions of $(v,(v - 1)/2,(v - 3)/2)$ difference families, J. Combin. Designs 16 (2008), 164–171.

[13] D. Ž. Đoković, Survey of cyclic $(v;r,s;\lambda)$ difference families with $v \leq 50$, Facta Universitatis (Niš), Ser. Mathematics and Informatics 12 (1997), 1–13.

[14] D. Ž. Đoković, Note on periodic complementary sets of binary sequences, Designs, Codes and Cryptography 13 (1998), 251–256.

[15] D. Ž. Đoković, Equivalence classes and representatives of Golay sequences, Discrete Math. 189 (1998), 79–93.

[16] H. Ehlich, Determinantenabschätzungen für binäre Matrizen, Math. Zeitschr. 83 (1964), 123–132.

[17] K. Feng, P.J-S. Shiue, and Q. Xiang, On aperiodic and periodic complementary binary sequences, IEEE Trans. Inform. Theory 45 (1999), 296–303.

[18] M. Gysin, New D-optimal designs via cyclotomy and generalised cyclotomy, Australasian Journal of Combinatorics 15 (1997), 247–255.

[19] M. Gysin and J. Seberry, An experimental search and new combinatorial designs via a generalisation of cyclotomy, J. Comb. Math. Comb. Comput. 27 (1998), 143–160.

[20] M. Gysin and J. Seberry, On new families of supplementary difference sets over rings with short orbits, J. Comb. Math. Comb. Comput. 28 (1998), 161–186.

[21] C. Koukouvinos, S. Konias and J. Seberry, Supplementary difference sets and optimal designs, Discrete Math. 88 (1991), 49–58.

[22] S. Konias, C. Koukouvinos, N. Nicolaou and A. Kakos, The non-equivalent circulant D-optimal designs for $n \equiv 2 \mod 4$, $n \leq 66$, J. Combin. Theory Ser. A 65 (1994), 26–38.

[23] S. Konias, C. Koukouvinos, N. Nicolaou and A. Kakos, The non-equivalent circulant D-optimal designs for $n = 90$, J. Statist. Plann. Inference 53 (1996), 253–259.

[24] L. Martínez, D. Ž. Đoković and A. Vera-López, Existence question for difference families and construction of some new families, J. Combin. Designs 12 (2004), 256–270.
[25] L.B. Morales, Constructing difference families through an optimization approach: Six new BIBDs, J. Combin. Designs 8 (2000), 261–273.
[26] A. Pott, Finite Geometry and Character Theory, Lecture Notes Math. 1601, Springer, New York, 1995.
[27] J. Seberry and M. Yamada, Hadamard matrices, sequences and block designs, in Contemporary Design Theory: A Collection of Surveys, Eds. J.H. Dinitz and D.R. Stinson, J. Wiley, New York, 1992, pp. 431–560.
[28] J. Seberry Wallis, Some remarks on supplementary difference sets. In Colloquia Mathematica Societatis Janos Bolyai 10 (1973), 1503–1526.
[29] K. Takeuchi, A table of difference sets generating balanced incomplete block designs, Review Intern. Statistical Inst. 30 (1962), 361–366.
[30] R.M. Wilson, Cyclotomy and difference families in elementary abelian groups, Journal of Number Theory 4 (1972), 17–47.
[31] C.H. Young, On designs of maximal (+1, −1)-matrices of order $n \equiv 2 \pmod{4}$, Math. Comp. 23 (1968), 174–180.
[32] ______, Maximal binary matrices and sum of two squares, Math. Comp. 30 (1976), 361–366.

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