Three-body system in leading order Effective Field Theory without three-body forces

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Abstract

The use of leading order effective field theory (EFT) to describe neutron-deuteron scattering leads to integral equations that have unusual behaviour: when only two-body interactions are included, the scattering amplitude does not approach a limit when the cutoff used to solve the equations is removed. It has recently been shown that this cutoff dependence can be eliminated by the careful inclusion of a three-body force. In this paper we show that the cutoff dependence is just a reflection of the fact that the aforementioned integral equations admit an infinite number of solutions amongst which only one corresponds to the physical scattering amplitude. We show how to numerically extract the physical scattering amplitude from the general solution and in this way explicitly demonstrate that the amplitude for a particle scattering off a two-body bound state, in leading order EFT, is in fact determined entirely by two-body forces.

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I. INTRODUCTION

Effective field theory (EFT) approaches to problems in nuclear physics have been under intensive investigation during the last few years. A review of recent developments (and references to the relevant papers) can be found in [1]. In the two-body sector, most of the theoretical problems encountered initially have now been resolved with recent EFT calculations of nucleon-nucleon scattering giving a particularly successful description [2–6]. On the other hand, application of the EFT program to the three-body sector remains problematic. Here we would like to address an especially intriguing problem encountered in the lowest order EFT calculation of neutron-deuteron scattering in the $J = 1/2$ channel [7].

For very low energy EFT calculations of nuclear systems, one integrates out all particles other than the nucleon, and at leading order one is left with just a constant contact interaction between the nucleons. A problem arises immediately when one tries to use this contact interaction in a standard calculation of three-body scattering. In this regard we note that the spinor nature of the nucleons plays no essential role here and that exactly the same problem occurs in the case where the three particles are all scalars. Thus for the sake of simplicity and without losing generality, we shall present our discussion for the case of three bosons, described by EFT, where at very low energies all other particles have been integrated out and the leading order interaction is a constant contact term.

The problem in question can be described as follows. To find the leading order amplitude for a boson scattering off a two-boson bound state we need to sum up an infinite number of diagrams involving the two-body constant contact interaction. While each leading order three-body diagram with re-summed two-body interactions is individually finite, when one tries to sum up these diagrams by using an integral equation

$$a(p, k) = M(p, k) + \int_0^\infty dq \, Z(p, q)a(q, k), \quad (1)$$

where $a$ is the summed amplitude, $M$ is the Born term, and $Z$ is the kernel, one finds that the operator $(1 - Z)^{-1}$ does not exist. Thus one cannot express the amplitude as $a = (1 - Z)^{-1}M$ with the consequence that the numerical solution of Eq. (1) is particularly difficult to construct.

If one tries to handle this problem by introducing an ultraviolet cutoff for the integral in Eq. (1), one then finds that although the modified equation is easily solved (e.g. by matrix inversion), its solution is sensitive to the chosen cutoff and the limit as the cutoff is removed does not exist. To resolve this problem, Bedaque et al. [7] introduced a one-parameter three-body force counter-term into their leading order EFT calculations. Indeed, they argued that the introduction of this three-body force is a necessary and sufficient condition to eliminate the cut-off dependence.

In the present paper we solve the problem of sensitivity to the ultraviolet cut-off without introducing three-body forces into the leading order Lagrangian. We do this by recognising that Eq. (1) has an infinite number of solutions, only one of which corresponds to the physical amplitude for particle-bound state scattering. Furthermore, we show that the introduction of a particular ultraviolet cutoff into Eq. (1), results in an equation whose solution is an approximation to just one of the infinite number of solutions of the original Eq. (1). Moreover, by varying the ultraviolet cutoff, we end up obtaining approximations...
that jump between *different* solutions of Eq. (1). It is this jumping between solutions which is responsible for the sensitivity to the ultraviolet cut-off observed in Ref. [7]. By performing a straightforward numerical analysis of the solutions with different cutoff parameters, we are able to construct the actual physical solution of Eq. (1) where the input consists of two-body interactions only. In this way the present paper extends the ideas presented in Ref. [8] and also complements these through explicit numerical calculations.

Although we have considered the case of three boson scattering for simplicity, exactly the same considerations hold for the problem of $J = 1/2$ channel neutron-deuteron scattering in the framework of EFT. We will address the specific case of neutron-deuteron scattering in a separate paper.

II. THREE-BODY SYSTEM IN LEADING ORDER EFT

A. Integral equations for boson-dimeron scattering

The leading order Lagrangian for the considered EFT of non-relativistic self-interacting bosons is given by [9]

$$\mathcal{L} = \phi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m}\right)\phi - \frac{C_0}{2}(\phi^\dagger\phi)^2$$

(2)

where $\phi$ is the boson field, $m$ is its mass, and $C_0$ is a coupling constant. For the sake of convenience [10] one can rewrite this theory by introducing a dummy field $\Phi$ with the quantum numbers of two bosons, referred to as a “dimeron” [9]:

$$\mathcal{L} = \phi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m}\right)\phi + \Delta\Phi^\dagger\Phi - \frac{g}{\sqrt{2}}(\Phi^\dagger\phi\phi + \text{h.c.})$$

(3)

The scale parameter $\Delta$ is included to give the field $\Phi$ the usual mass dimension of a heavy field. Observables depend on the parameters of Eq. (3) only through the combination $C_0 \equiv g^2/\Delta$.

The (bare) dimeron propagator is a constant $i/\Delta$ and the boson propagator is given by the usual non-relativistic expression $i/(p^0 - p^2/2m + i\epsilon)$ where $p = |\mathbf{p}|$ (similar three-vector notation is used below for other momentum variables). The dressing of the dimeron propagator is illustrated in Fig. [1(a)]. Summing loop-diagrams, subtracting the divergent integral at $P^0 = P = 0$ (where $P^0$ and $P = |\mathbf{P}|$ refer to the momentum of the dimeron) and removing the cut-off, one gets the following dressed dimeron propagator [9]:

$$iS(P^0, P) = \frac{-i}{-\Delta^R + \frac{mg^2}{4\pi}\sqrt{-mP^0 + P^2/4 - i\epsilon + i\epsilon}}$$

(4)

where $\Delta^R$ is the renormalised parameter ($\Delta$ has absorbed the linear divergence). Attaching four boson lines to this dressed dimeron propagator one gets the two-particle scattering amplitude at leading order. This amplitude has the form of an effective range expansion truncated at leading order with $g^2/\Delta^R = 4\pi a_2/m$ where $a_2$ is the two-body scattering length.
For the scattering of a particle off a two-body bound state, standard power counting shows that the leading order contribution to the amplitude $T$ is given by the diagrams illustrated by the first equality of Fig. 1(b). The sum of all these diagrams satisfies the equation represented by the second equality in Fig. 1(b). For $s$-wave scattering it is convenient to define the function $a(p, k)$ in terms of the $s$-wave amplitude $T_0(p, k)$ by

$$a(p, k) = \frac{p^2 - k^2}{-1/a_2 + \sqrt{3p^2/4 - mE}} \frac{T_0(p, k)}{mg^2},$$

which can be shown to satisfy the equation

$$a(p, k) = M(p, k) + 2\lambda \pi \int_0^\infty dq \, M(p, q) \frac{q^2}{q^2 - k^2 - i\epsilon} a(q, k),$$

where $k (p)$ is the incoming (outgoing) momentum magnitude, $E = 3k^2/4m - 1/ma^2$ is the total energy, and

$$M(p, q) = \frac{4}{3} \left( \frac{1}{a_2} + \sqrt{\frac{3}{4}p^2 - mE} \right) \frac{1}{pq} \ln \left( \frac{q^2 + pq + p^2 - mE}{q^2 - qp + p^2 - mE} \right).$$

Eq. (5) was first derived by Skorniakov and Ter-Martirosian (S-TM equation) and has $\lambda = 1$ for the three-boson case. Three nucleons in the spin $J = 1/2$ channel obey a pair of integral equations with similar properties to this bosonic equation, and the spin $J = 3/2$ channel corresponds to $\lambda = -1/2$.

It was shown by Danilov that for $\lambda = 1$ the homogeneous equation corresponding to Eq. (6),

$$a_h(p, k) = \frac{2\lambda}{\pi} \int_0^\infty dq \, M(p, q) \frac{q^2}{q^2 - k^2 - i\epsilon} a_h(q, k),$$

has a solution for arbitrary $E$. In particular, there exists a solution of the homogeneous equation for every energy corresponding to the scattering of a projectile off a two-body bound state. Although all such scattering energy solutions are unphysical, it must be emphasised that they are purely an artifact of having two-body effective potentials that are zero-range ($\delta$-function potentials in coordinate space). By contrast, the solution of the corresponding homogeneous equation for non-zero range potentials (namely the bound state Faddeev
equation) has no solutions for energies corresponding to scattering. In turn, it should be remembered that zero range effective potentials are themselves an artifact of restricting the EFT model to the lowest order terms - they are not a property of the full EFT approach [4].

Although unphysical, the existence of a scattering energy solution \( a_h \) of Eq. (8) has practical consequences for finding the physical scattering amplitude \( a_{ph} \) satisfying Eq. (6). The problem is that the existence of \( a_h \) implies that Eq. (8) has an infinite number of solutions given by \( a = a_{ph} + Ca_h \) where \( C \) is an arbitrary parameter. As we shall see, Eq. (8) has actually more than one solution for any given \( E \). Writing these solutions as \( a_i^h \) where \( i = 1, 2, 3, \ldots \), the most general solution of Eq. (6) can therefore be written as

\[
a = a_{ph} + \sum_i C_i a_i^h .
\]  

Thus the sum of the diagrams in Fig. 1(b), which defines the physical amplitude \( a_{ph} \), is only one of an infinite number of solutions to Eq. (6).

Setting \( \lambda = 1 \) and writing Eq. (6) and Eq. (8) in operator form as

\[
a = M + MG_0a
\]  

and

\[
a_h = MG_0a_h
\]  

respectively, it is clear from the existence of a non-zero solution \( a_h \) of Eq. (11) that the inverse operator \( (1 - MG_0)^{-1} \) does not exist. This in turn means that one cannot write the solution of Eq. (10) as \( a = (1 - MG_0)^{-1}M \) with the practical consequence that Eq. (6) cannot be solved by matrix inversion. Indeed we find that many other numerical methods, for example Padé approximants, are likewise unstable for this case.

The task of finding the physical amplitude \( a_{ph} \) appears to be formidable. Even if Eq. (6) could be solved, one would still need to determine the appropriate values of the parameters \( C_i \) in order to extract \( a_{ph} \). Attempts to find \( a_{ph} \) by writing it directly as the sum

\[
a_{ph} = M + MG_0M + MG_0MG_0M + \ldots
\]  

also do not help, as this sum cannot be evaluated numerically due to extreme sensitivity to roundoff errors. Again, this numerical instability appears to be linked to the non-existence of \( (1 - MG_0)^{-1} \). Fortunately, all these numerical difficulties can be overcome. The rest of this paper is devoted to accomplishing this task.

B. Extracting the physical boson-dimeron amplitude

1. Asymptotic behaviour

In order to distinguish the physical boson-dimeron amplitude \( a_{ph} \) from the infinite number of non-physical solutions given by Eq. (4), it is useful to examine the asymptotic behaviour of the general solution \( a(p, k) \) to the S-TM equation for large \( p \). It has the form [13]

\[
a(p, k) = \sum_i A_i(k) p^{k_i} + O(1/p)
\]  

5
where \( s_i \) are roots of the equation

\[
1 - \frac{8\lambda}{\sqrt{3}} \frac{\sin\frac{\pi s}{6}}{s \cos\frac{\pi s}{2}} = 0.
\]  

(14)

The summation in Eq. (13) goes over all solutions of Eq. (14) for which \(|\text{Re}\ s| < 1\). For \( \lambda = 1 \) Eq. (14) has two roots for which \(|\text{Re}\ s| < 1\): \( s = \pm is_0 \), where \( s_0 \approx 1.00624 \), so that Eq. (13) gives the asymptotic behaviour of the boson-dimeron amplitude as

\[
a(p,k) \sim A_1(k) p^{is_0} + A_2(k) p^{-is_0}.
\]  

(15)

It can be shown [15] that for \( \lambda = 1 \) the asymptotic behaviour of the solution \( a_h(p,k) \) of the homogeneous equation, Eq. (8), is also given by the right hand side of Eq. (15).

What distinguishes the physical amplitude \( a_{ph} \) from any other amplitude \( a \) satisfying the S-TM equation is that it is the sum of the diagrams in Fig. 1(b); that is, it is equal to the sum of the series obtained by iterating Eq. (6). In this way \( a_{ph} \) can be expressed as a power series in the parameter \( \lambda \). By contrast, it can easily be seen that for the physical amplitude \( a_{ph} \), the right hand side of Eq. (13) must be identically zero, i.e., it cannot be expressed as a non-trivial power series in \( \lambda \). Thus

\[
a_{ph}(p,k) \to 0 \quad \text{as} \quad p \to \infty.
\]  

(16)

2. Unitarity

Another essential property of the physical amplitude \( a_{ph} \) is that it satisfies unitarity; that is, the on-shell physical amplitude \( a = a_{ph} \) satisfies the relation

\[
a(k,k) - a^*(k,k) = 2i\lambda k |a(k,k)|^2.
\]  

(17)

In this respect it is important to note that if an amplitude \( a \) satisfies the S-TM equation, it does not necessarily mean that \( a \) satisfies unitarity. Indeed, if we write the S-TM equation in operator form as \( a = M + MG_0a \) and then try to prove unitarity in the usual way, we would firstly want to write \( M^{-1} = a^{-1} + G_0 \), then subtract the Hermitian conjugate of this equation, and lastly, rearrange to obtain the unitarity relation \( a - a^\dagger = a(G_0 - G_0^\dagger)a^\dagger \) which reduces to Eq. (17) on-shell. However, if the inverse \( (1-MG_0)^{-1} \) doesn’t exist, then it is easy to show that either operator \( a \) has no right-inverse or \( M \) has not left-inverse. In either case, the usual proof of unitarity breaks down at the first step. On the other hand, because \( a_{ph} \) can be expressed as an iteration of the S-TM equation, \( a_{ph} = M + MG_0M + MG_0MG_0M + \ldots \), the unitarity relation for \( a_{ph} \) can easily be proved directly without the need to take the inverse of any operator.

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\(^1\)For if \( A_1 \) and \( A_2 \) were non-vanishing and we could write \( s_0(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \ldots \) , then substituting this into Eq. (13) we would obtain a power series in \( \lambda \) whose coefficients do not have the same asymptotic \( p \) behaviour as the corresponding diagrams coming from the iteration of Eq. (6).
As the physical amplitude satisfies unitarity, we would like to consider only those solutions $a$ of the S-TM equations that likewise satisfy unitarity. For Eq. (9), this means a strong restriction on the allowed values for the parameters $C_i$. This restriction on the solutions of Eq. (6) can be achieved through the introduction of the $K$ matrix equation

$$K(p, k) = M(p, k) + \frac{2\lambda}{\pi} P \int_0^\infty dq \ M(p, q) \frac{q^2}{q^2 - k^2} K(q, k),$$

where $P$ stands for “principal value” and where only the real solutions for the $K$ matrix $K(p, k)$ are considered. It is then easy to show that the amplitude

$$a(p, k) \equiv \frac{K(p, k)}{1 - i\lambda k K(k, k)}$$

satisfies both the S-TM equation and the unitarity relation of Eq. (17). Conversely, any amplitude $a(p, k)$ that satisfies the S-TM equation and is unitary can be shown to be of the form given by Eq. (19) where $K$ is real and satisfies Eq. (18).

One might hope that the unitarity condition would single out the physical solution from an infinite number of solutions of Eq. (6). This would require Eq. (18) to have a unique solution corresponding to the physical amplitude. Unfortunately the homogeneous equation corresponding to Eq. (18) admits non-trivial solutions, and this means that Eq. (18) has in fact an infinite number of solutions. Nevertheless, we know from the above discussion of the asymptotic behaviour of $a_{ph}$ and the relation between the $a$ amplitude and the $K$ matrix, Eq. (19), that amongst all the solutions to Eq. (18), the one that corresponds to the physical amplitude is the solution which vanishes for large $p$.

### III. NUMERICAL SOLUTION

#### A. Solution to the inhomogeneous equation

As we are specifically interested in the description of three bosons within leading order EFT, we shall implicitly assume that $\lambda = 1$ in all the equations below. Writing Eq. (18) symbolically for this case as $K = M + MG_0^P K$, the fact that the homogeneous equation corresponding to this equation has a solution for an arbitrary value of the energy means that the inverse operator $(1 - MG_0^P)^{-1}$ does not exist. Thus trying to solve Eq. (18) directly presents exactly the same type of numerical difficulties as discussed above for the case of Eq. (6). In order to solve Eq. (18) numerically we introduce regularisation such that, in contrast to Eq. (18), the regularised equation has a unique solution. Although this solution depends strongly on the cutoff parameter(s) of the regularisation, we will see that this is not an indication of any difficulty with leading order EFT, as suggested in Ref. 7, but rather it is simply a consequence of the fact that different regularisations correspond to different solutions of the unregularised equation.

We consider the regularised equation

$$K^\Lambda(p, k) = M(p, k) + \frac{2\Lambda}{\pi} P \int_0^\infty dq \ M^\Lambda(p, q) \frac{q^2}{q^2 - k^2} K^\Lambda(q, k)$$

(20)
FIG. 2. The dot-dashed, dashed, and dotted lines are solutions $K^{\Lambda_i}(p, k)$ of the inhomogeneous $K$ matrix equation, Eq. (20), corresponding to three different cutoffs $\Lambda_1 = 1.6 \times 10^7 a_2^{-1}$, $\Lambda_2 = 3.2 \times 10^7 a_2^{-1}$ and $\Lambda_3 = 6.4 \times 10^7 a_2^{-1}$, respectively. The solid line is the function $K_h(p, k) = K^{\Lambda_1}(p, k) - K^{\Lambda_2}(p, k)$ which forms a solution of the corresponding homogeneous equation. In all cases $k = 0.82 a_2^{-1}$.

where

$$M^\Lambda(p, q) = M(p, q) + \frac{1}{\Lambda + q^2}. \quad (21)$$

For this regularisation the inverse operator $(1 - M^\Lambda G_0^p)^{-1}$ exists and Eq. (20) is easily solved. We find that the solution of this equation, $K \equiv K^\Lambda$, satisfies Eq. (18) with very good accuracy for any $\Lambda$ that is sufficiently large; moreover, by increasing $\Lambda$ one can obtain solutions $K^{\Lambda_i}$ that will satisfy Eq. (18) to any given accuracy. As expected from the asymptotic behaviour specified by Eq. (15), we find that $K^{\Lambda_i}(p, k)$ has oscillating behaviour for large $p$ - see Fig. 2. We also find that $K^{\Lambda}(p, k)$ has an oscillating behaviour with respect to $\Lambda$, in agreement with what was observed in Ref. [7].

More information on the functional form of $K^{\Lambda_i}(p, k)$ can be obtained by constructing solutions $K^{\Lambda_i}(p, k)$ ($i = 1, \ldots, 4$) corresponding to four different values of $\Lambda$. As each of the $K^{\Lambda_i}(p, k)$, for $\Lambda$ large enough, is a solution to Eq. (18), the differences $K^{\Lambda_1}(p, k) - K^{\Lambda_2}(p, k)$ are clearly solutions to the homogeneous equation corresponding to Eq. (18). However, what is particularly interesting about such differences is that their ratio

$$\frac{K^{\Lambda_1}(p, k) - K^{\Lambda_2}(p, k)}{K^{\Lambda_3}(p, k) - K^{\Lambda_4}(p, k)}$$

is found to be totally independent of the momentum variable $p$. From this observation we conclude that the structure of $K^\Lambda$ must be of the form

$$K^\Lambda(p, k) = K_0(p, k) + C(\Lambda)K_h(p, k) \quad (22)$$

where $K_0(p, k)$ is a solution to Eq. (18), $K_h(p, k)$ is a solution to the corresponding homogeneous equation, and $C(\Lambda)$ is purely a function of $\Lambda$ (we consider the initial momentum $k$ to
be fixed). As the normalisation of the function \( K_h(p, k) \) is not determined by the homogeneous equation, for definiteness we take \( K_h(p, k) = K^{\Lambda_1}(p, q) - K^{\Lambda_2}(p, q) \) which is drawn in Fig. 2 together with three of the particular solutions to the inhomogeneous equation. Note that \( K_0(p, k) \) and \( K_h(p, k) \) do not depend on \( \Lambda \). Also, because each \( K^{\Lambda}(p, k) \) oscillates as a function of \( p \) with a phase that depends on \( \Lambda \), \( K_0(p, k) \) and \( K_h(p, k) \) must similarly oscillate but with differing phases. Being a solution to the inhomogeneous \( K \) matrix equation, \( K_0(p, k) \) can be written generally as \( K_0(p, k) \equiv K^{\Lambda}(p, k) \) where \( K_0(p, k) \) is the physical \( K \) matrix which has the vanishing asymptotical behaviour (i.e. the \( K \) matrix corresponding to the physical amplitude \( a_{ph} \)), and \( K_h(p, k) \) is another solution to the homogeneous equation. We may thus write the solution to Eq. (18) for large \( \Lambda \) as

\[
K^{\Lambda}(p, k) = K_{ph}(p, k) + K^*_h(p, k) + C(\Lambda)K_h(p, k).
\] (23)

For large \( p \) the structure of Eq. (23) agrees with the asymptotical behaviour of the solution to the regularised equation obtained in [9]:

\[
K^{\Lambda}(p \gg 1/a_2, k) = -\gamma \cos \left( s_0 \ln \left( p/a_2 \right) + \epsilon \right) - \gamma \tan \left( s_0 \ln \left( \Lambda a_2 \right) - \delta + \epsilon \right) \sin \left( s_0 \ln \left( p a_2 \right) + \epsilon \right)
\] (24)

where \( p_* = \exp (-\delta/s_0) \Lambda \) and \( \delta, \gamma \) and \( \epsilon \) are cutoff-independent constants. Indeed, by writing Eq. (24) as

\[
K^{\Lambda}(p \gg 1/a_2, k) = -\gamma \cos \left( s_0 \ln \left( p a_2 \right) + \epsilon \right) - \gamma \tan \left( s_0 \ln \left( \Lambda a_2 \right) - \delta + \epsilon \right) \sin \left( s_0 \ln \left( p a_2 \right) + \epsilon \right)
\] (25)

and noting that the physical \( K \) matrix \( K_{ph} \) vanishes for large \( p \), we can deduce that \( K^*_h(p, k) \sim \cos \left( s_0 \ln \left( p a_2 \right) + \epsilon \right) \), \( K_h(p, k) \sim \sin \left( s_0 \ln \left( p a_2 \right) + \epsilon \right) \) and \( C(\Lambda) \sim \tan \left( s_0 \ln \left( \Lambda a_2 \right) - \delta + \epsilon \right) \). The deduced asymptotic behaviour of \( K_h \) and \( K^*_h \) is consistent with Eq. (23) and is further borne out by our numerical results.

The existence of (at least) two linearly independent solutions to the homogeneous equation may seem a little surprising in light of the fact that Eq. (22) contains just one solution, \( K_h \), with a \( \Lambda \)-dependent coefficient. To further check that the structure given by Eq. (22) is consistent with there being more than one linearly independent solution to the homogeneous equation, we consider the equation

\[
\ddot{K}(p, k) = M(p, k) + \frac{2\lambda}{\pi} \int_0^\infty dq \ M(p, q) \ddot{K}(q, k),
\] (26)

where

\[
\ddot{M}(p, k) = \frac{2}{\sqrt{3} q} \ln \left( \frac{q^2 + p q + p^2}{q^2 - p q + p^2} \right).
\] (27)

Note that the cutoff used in Ref. [9] is different from ours; however, the functional form of the asymptotical behaviour does not depend on the particular choice of the cutoff.
To obtain $\tilde{M}$ we substituted $a_2 = \infty$ and $mE = 0$ in the expression for $M(p,q)$, and also removed $q^2/(q^2 - k^2)$ from the integrand. The homogeneous equation corresponding to Eq. (24) has solutions and hence Eq. (24) cannot be solved directly. We again introduce regularisation which enables a straightforward numerical solution. Thus we solve the equation

$$\tilde{K}^\Lambda(p,k) = M(p,k) + \frac{2\lambda}{\pi} \int_0^\infty dq \tilde{M}^\Lambda(p,q) \tilde{K}^\Lambda(q,k),$$

(28)

where

$$\tilde{M}^\Lambda(p,q) = \tilde{M}(p,q) + \frac{1}{\Lambda + q^2}. \quad (29)$$

For a sufficiently large cutoff $\Lambda$, the solution $\tilde{K}(p,k) \equiv \tilde{K}^\Lambda(p,k)$ satisfies Eq. (28) with very good accuracy and has oscillating behaviour for large $p$. $\tilde{K}^\Lambda(p,k)$ is also oscillating with respect to $\Lambda$. Just like the $K^\Lambda$ of the original problem, the solutions $\tilde{K}^\Lambda$ have the property that the ratio

$$\frac{\tilde{K}^{\Lambda_1}(p,k) - \tilde{K}^{\Lambda_2}(p,k)}{\tilde{K}^{\Lambda_3}(p,k) - \tilde{K}^{\Lambda_4}(p,k)}$$

does not depend on the momentum $p$. The amplitude structure is given accordingly by

$$\tilde{K}^\Lambda(p,k) = \tilde{K}_0(p,k) + A(\Lambda)\tilde{K}_h(p,k) \quad (30)$$

where $\tilde{K}_0(p,k)$ is a solution to Eq. (24) and $\tilde{K}_h(p,k) = \tilde{K}^{\Lambda_1}(p,k) - \tilde{K}^{\Lambda_2}(p,k)$ is a solution to the corresponding homogeneous equation. Everything is the same as before except that now one can actually solve the homogeneous equation analytically. We find that there are two linearly independent solutions, $\sin [s_0 \ln(a_2 p)]$ and $\cos [s_0 \ln(a_2 p)]$, even though only one solution enters with a $\Lambda$-dependent coefficient in the general numerical form given by Eq. (30). It is easy to see that the other linearly independent solution contributes into $\tilde{K}_0(p,k)$ and is responsible for its oscillating behaviour.

**B. Extracting the physical amplitude**

In the previous subsection we have shown that the inhomogeneous $K$ matrix equation, Eq. (18), can be solved numerically by introducing a sufficiently large cutoff $\Lambda$, and that the resulting solution is of the form given by Eq. (23). Writing Eq. (23) as

$$K^\Lambda(p,k) = K_{ph}(p,k) + K_h(p,k),$$

(31)

we are left with the task of extracting the physical $K$ matrix $K_{ph}(p,k)$ from the numerical values for $K^\Lambda(p,k)$. Because $K_{ph}(p,k)$ vanishes for large $p$, the function $K_h^\Lambda(p,k)$ in Eq. (31) is that solution to the homogeneous equation corresponding to Eq. (18) which has the same asymptotic behaviour as $K^\Lambda(p,k)$; that is, $K_h^\Lambda(p,k)$ satisfies the two equations
\[
K_h^\Lambda(p, k) = \frac{2\lambda}{\pi} P \int_0^\infty dq \ M(p, q) \frac{q^2}{q^2 - k^2} K_h^\Lambda(q, k),
\]
(32)

\[
K_h^\Lambda(p, k) \sim_{p \to \infty} K^\Lambda(p, k).
\]
(33)

Note that \(K_h^\Lambda(p, k)\) cannot be obtained by solving the homogeneous equation corresponding to Eq. (20); this homogeneous equation has no non-trivial solutions since the inverse operator \((1 - M^\Lambda G_0^P)^{-1}\) exists. On the other hand, since Eq. (32) has solutions for every energy, its numerical solution presents a significant technical problem. For example, if one simply discretises this homogeneous equation and then tries to solve the resulting simultaneous equations, one encounters severe numerical difficulties coming from the fact that the eigenvalues of the operator \((1 - MG_0^P)^{-1}\) include not only zero, but also values infinitely close to zero.

To solve Eq. (32), and moreover, to obtain the solution with the asymptotic behaviour required by Eq. (33), we solve a sequence of inhomogeneous equations, like Eq. (20), but with progressively smaller inhomogeneous terms. That is, we solve the equations

\[
K_i^\Lambda(p, k) = M_i(p, k) + \frac{2\lambda}{\pi} P \int_0^\infty dq \ M^\Lambda(p, q) \frac{q^2}{q^2 - k^2} K_i^\Lambda(q, k),
\]
(34)

where \(i = 1, 2, 3, \ldots\) and \(M_i(p, k)\) is taken to be a decreasing sequence of functions. In particular, we take

\[
M_i(p, k) = \frac{a_2}{10^i (pa_2)^6 + 1}.
\]
(35)

Since \(M_i(p, k)\) becomes a vanishingly small function for increasing values of \(i\), we expect \(K_i^\Lambda(p, k)\), with \(\Lambda\) large enough, to approximate the solution of Eq. (32) for sufficiently large values of \(i\). At the same time, since \(M_i(p, k)\) vanishes as \(p \to \infty\), the solution \(K_i^\Lambda(p, k)\) might be expected to have, up to some normalisation, the same asymptotic behaviour (as a function of \(p\)) as \(K^\Lambda(p, k)\). That is in fact what we find empirically, so that

\[
K_i^\Lambda(p, k) \sim_{i \to \infty} C_i K_h^\Lambda(p, k)
\]
(36)

where \(C_i\) is a constant.

It is useful to illustrate the above procedure on the case of the homogeneous equation corresponding to Eq. (26) for which an analytic solution is available. Thus we would first like to consider the equation

\[
\tilde{K}_i^\Lambda(p, k) = M_i(p, k) + \frac{2\lambda}{\pi} P \int_0^\infty dq \ M^\Lambda(p, q) \tilde{K}_i^\Lambda(q, k),
\]
(37)

where \(M_i(p, k)\) is given by Eq. (33) and \(\Lambda\) has a large fixed value. By the above discussion, we expect the solution to Eq. (37) to have the form

\[
\tilde{K}_i^\Lambda(p, k) = A_i \tilde{K}_h^\Lambda(p, k) + R_i^\Lambda(p, k)
\]
(38)
where $\tilde{K}_h^\Lambda(p,k)$ is that solution of the homogeneous equation corresponding to Eq. (26) which has the same asymptotic behaviour as the solution $\tilde{K}_h^\Lambda(p,k)$ of Eq. (28), and $R_\Lambda^i(p,k)$ is a remainder term that becomes progressively smaller with increasing $i$ and that decreases rapidly as a function of $p$ beyond a close neighbourhood of the origin. The constant $A_i$ is expected to depend only on the choice of functions $M_i(p,k)$. To see how these expectations are borne out in practice, we have numerically solved Eq. (37) for the sequence of inhomogeneous functions given by Eq. (35). We find that for different $i$ and up to a normalisation factor, the functions $\tilde{K}_h^\Lambda(p,k)$ all have identical oscillating behaviour for large $p$. We scale these functions so that their oscillating tails have the same amplitude, i.e., we rescale Eq. (38) as

$$\tilde{K}_i^\Lambda(p,k) = \tilde{K}_h^\Lambda(p,k) + R_\Lambda^i(p,k)$$

with $\tilde{K}_i^\Lambda \equiv \tilde{K}_h^\Lambda/A_i$ and $R_\Lambda^i \equiv R_\Lambda^i/A_i$, and then plot the results in Fig. 3. We see that in agreement with our expectations for increasing values of $i$, the remainder term $R_\Lambda^i$ becomes progressively smaller in such a way that two sequential functions $\tilde{K}_i^\Lambda(p,k)$ and $\tilde{K}_{i+1}^\Lambda(p,k)$ coincide beyond some value of $p$ that is progressively approaching zero. According to Eq. (39) this tail, which is common for all functions $\tilde{K}_i^\Lambda$, is just the tail of the solution to the homogeneous equation, $\tilde{K}_h^\Lambda$. Choosing a different sequence of inhomogeneous terms, for example

$$M_i(p,k) = a_2 \exp[-i(a_2p)^4],$$

we obtain a sequence of functions $\tilde{K}_i^{\Lambda'}$ that approaches exactly the same solution $\tilde{K}_h^\Lambda$ to the homogeneous equation. For this non-physical case we are able to solve the homogeneous equation analytically. As the general solution is a linear combination of the independent solutions $\sin[s_0 \ln(a_2p)]$ and $\cos[s_0 \ln(a_2p)]$, we are able to write our sequence of solutions as

$$\tilde{K}_i^\Lambda(p,k) = A_i \sin[s_0 \ln(a_2p) + \alpha] + R_\Lambda^i(p,k).$$

FIG. 3. Solutions to the homogeneous equation corresponding to Eq. (28) that have the same asymptotic behaviour as the solution to Eq. (28) with $\Lambda = 1.6 \times 10^7 a_2^{-1}$. The solid line is the exact solution $\tilde{K}_h^\Lambda$ to the homogeneous equation, the dot-dashed, dashed, and dotted lines correspond to the sequential approximations $\tilde{K}_i^\Lambda$ with $i = 4, 8, \text{ and } 10$, respectively.
FIG. 4. Numerical construction of the function $K_\Lambda(p, k)$ defined by Eq. (32) and Eq. (33). The solid line is the solution $K_\Lambda(p, k)$ to the inhomogeneous $K$ matrix equation with cutoff $\Lambda = 1.6 \times 10^7 a_2^{-1}$ and on-shell momentum $k = 0.82 a_2^{-1}$. The dot-dashed, dashed, and dotted curves correspond to sequential approximations to $K_\Lambda(p, k)$ with $i = 4, 8, \text{and} 10$, respectively.

It is interesting to note that although the constant $\alpha$ depends on the cutoff $\Lambda$, our numerical findings indicate that it does not depend on the value of $i$. The function $\sin[s_0 \ln(a_2 p) + \alpha]$ (scaled appropriately) is also plotted in Fig. 3. As we can see, by using the sequence of functions approach, we can reproduce the particular exact solution to the homogeneous equation with better and better accuracy. Finally, by solving the inhomogeneous equation with cutoff, Eq. (28), we find that the asymptotic behaviour of the solution $\tilde{K}_\Lambda(p, k)$ is identical (up to a normalisation factor) with the asymptotic behaviour of all the solutions $\tilde{K}_i^\Lambda(p, k)$ of Eq. (37). It is thus $\tilde{K}_\Lambda(p, k)$ to which the sequence of functions $\tilde{K}_i^\Lambda(p, k)$ converges.

We now apply the above procedure to our physical case of boson-dimeron scattering described by Eq. (18). Thus we first solve Eq. (20) with a sufficiently large value of $\Lambda$ in order to obtain a particular solution $K_\Lambda(p, k)$. Then we solve Eq. (34) with the same value of $\Lambda$ using the sequence of inhomogeneous terms specified by Eq. (35). As in the simpler case above, the asymptotic tails of $K_\Lambda(p, k)$ and all the solutions $\tilde{K}_i^\Lambda(p, k)$ are oscillatory and, up to a normalisation factor, identical. This can be seen explicitly in Fig. 4 where the $K_i^\Lambda$ for $i = 4, 8, \text{and} 10$ are plotted after having been scaled to have the same asymptotic tail as $K_\Lambda$. From this it is seen how the sequence of functions $K_i^\Lambda$ provides successively better approximations to the function $K_\Lambda$ appearing in Eq. (34). With $i$ sufficiently large, we can subtract $K_i^\Lambda$ from $K_\Lambda$, in this way deducing the desired physical $K$ matrix $K_{ph}$ with vanishing asymptotics.

As evident from Fig. 4, larger and larger values of $i$ are needed in order to approximate the value of $K_i^\Lambda(p, k)$ for progressively smaller values of the momentum $p$. For our purposes it is sufficient to have $i$ only large enough to give an accurate approximation to $K_i^\Lambda(p, k)$ at the on-mass-shell value of $p$, i.e., at $p = k$. By repeating the above calculations for a range of values of the on-mass-shell momentum, we obtain the physical $K$ matrix $K_{ph}(k, k)$ as a smooth function of $k$. The result is drawn in Fig. 5.

The above numerical procedure cannot be used directly to obtain the boson-dimeron scattering length because our sequence of functions method to determine $K_i^\Lambda(p, k)$ does not extend to the value of $p = 0$. However, taking into account the smoothness of the scattering
FIG. 5. The on-mass-shell $K$ matrix $K_{ph}(k, k)$ corresponding to the physical amplitude for boson-dimeron scattering. This numerical result is based on leading order EFT with two-body input only.

amplitude, we can determine the scattering length by extrapolating the amplitude to zero momentum. There is not much sense in making that kind of extrapolation for the present scalar particle EFT model; however, we will be able to compare the predicted value for the scattering length with the experimental value when we complete calculations for the case of neutron-deuteron scattering in the doublet channel.

IV. SUMMARY

Applying leading order EFT to describe the scattering of a boson off a two-boson bound state results in an unusual inhomogeneous integral equation, Eq. (6), originally derived in the 1950's by Skorniakov and Ter-Martirosian [11], whose kernel $MG_0$ is such that the operator $1 - MG_0$ has no inverse. The same equation arises in the application of leading order EFT to neutron-deuteron scattering where its unusual properties have caused difficulties with its numerical solution [7]. The problem encountered in Ref. [7] is that the scattering amplitude has no limit as the cutoff, used to solve the equation, is removed; in particular, it was found that as the cutoff is taken to infinity, the amplitude oscillates with a constant amplitude. It was argued that this undesirable behaviour of the amplitude should be eliminated by the introduction of a carefully chosen three-body force into the leading order Lagrangian [7].

In the present paper we have solved the problem of the ultraviolet cutoff-dependence of the scattering amplitude without the need to introduce a three-body force. We did this by taking into account the fact that, for any given energy, Eq. (6) has an infinite number of solutions amongst which only one corresponds to the physical amplitude for particle-bound state scattering. Rather than solving Eq. (6) directly, we chose to work with the corresponding $K$ matrix equation, Eq. (18). Although Eq. (18) still has an infinite number of solutions, it has the advantage of singling out only those amplitudes which are unitary. By introducing a cutoff $\Lambda$ into Eq. (18) and numerically solving the resulting equation, Eq. (20), we have shown that every $\Lambda$ that is sufficiently large, results in an amplitude that satisfies the original equation, Eq. (6). Thus, the sensitivity of the scattering amplitude to
the ultraviolet cutoff \( \Lambda \) is simply a reflection of the fact that one is in this way obtaining \textit{different} solutions of Eq. (1) for every \( \Lambda \). A simple numerical analysis of our solutions to Eq. (20) has allowed us to deduce the functional form of the \( \Lambda \) dependence of the solution - see Eq. (22).

Amongst the infinite number of solutions \( K^\Lambda(p,k) \) described by Eq. (22), the \( K \) matrix corresponding to the physical amplitude, \( K_{ph} \), is distinguished by the fact that it vanishes in the limit of infinite off-shell momentum \( p \). This asymptotic behaviour can be contrasted with the oscillatory asymptotic behaviour of all other solutions. By introducing a sequence of equations like Eq. (20) but with progressively smaller inhomogeneous terms, we have managed to extract \( K_{ph} \) numerically from our numerical solutions of Eq. (20) - see Fig. 5. In this way we have shown explicitly that EFT in leading order describes the scattering of a particle off a two-body bound state in terms of two-body forces only.

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