The Hilbert-Space Structure of Non-Hermitian Theories with Real Spectra‡

Ralph Kretschmer†§ and Lech Szymanowski∥¶
† Fachbereich Physik, Universität Siegen, Germany
∥ Soltan Institute for Nuclear Studies, Warsaw, Poland
E-mail: § kretschm@hepth2.physik.uni-siegen.de,
¶ lech.szymanowski@fuw.edu.pl

Abstract. We investigate the quantum-mechanical interpretation of models with non-Hermitian Hamiltonians and real spectra. After describing a general framework to reformulate such models in terms of Hermitian Hamiltonians defined on the Hilbert space $L_2(-\infty, \infty)$, we discuss the significance of the algebra of physical observables.

PACS numbers: 03.65.-w, 03.65.Ca

1. Introduction

In the past few years, models with non-Hermitian Hamiltonians (the term non-Hermiticity meant here in the sense of the space $L_2(-\infty, \infty)$ of square-integrable functions) have attracted a lot of interest, because many examples are known in which such models have real spectra [1, 2, 3]. Therefore, they may describe realistic physical systems.

Despite this, the physical interpretation of these models remains unclear: The eigenstates $\psi_n$ of a non-Hermitian Hamiltonian $H$ are not mutually orthogonal (their scalar product may even be undefined) and the time evolution generated by $H$ is non-unitary, so that the usual probabilistic interpretation of wave functions cannot be applied. In addition, in some models [1, 2] it is necessary to extend the definition of position-space wave functions to complex values of the coordinate. This means that the wave functions are not elements of the Hilbert space $L_2(-\infty, \infty)$, so that non-Hermiticity in the sense of the space $L_2$ has no obvious meaning here.

Recently, it seems that a consensus has been reached that—provided the spectrum of the Hamiltonian is real—one can always define a positive-definite scalar product under which the eigenstates are orthogonal. Among the approaches investigated so far is

‡ Talk given by R. Kretschmer at the *1st International Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics*, Prague, Czech Republic, June 16-17, 2003.
The notion of pseudo-Hermiticity, advocated in the work of Mostafazadeh [4] (see also [5] for an early discussion of this concept). Here a metric operator $\eta$ is used to define a modified scalar product,

\[(\psi, \varphi) := (\psi, \eta \varphi)_{L_2}\]  

((.,.)_{L_2} is the scalar product in $L_2$),

- the introduction of a $CPT$ transformation [6]

\[(\psi, \varphi) := \int_C d x [CPT \psi(x)] \varphi(x) ,\]  

and

- the direct construction of the Hilbert space $\mathcal{H} = \text{span}\{\psi_1, \psi_2, \ldots\}$ (that is the closure of the space of finite superpositions of the eigenfunctions) with a scalar product defined by [7]

\[(\psi_n, \psi_m)_{\mathcal{H}} := \delta_{nm} \quad \text{for all} \ n, m \] .

In this contribution we want to explain the last method in some detail. We will only treat the case of a discrete, infinite spectrum of $H$.

### 2. The canonical formulation

Definition [3] leads to a separable Hilbert space $\mathcal{H}$ in which the Hamiltonian $H$ is Hermitian (provided its spectrum is real), because for two vectors $\varphi = \sum_n a_n \psi_n, \psi = \sum_n b_n \psi_n \in \mathcal{H}$ that are in the domain of definition of $H$ one finds

\[(\varphi, H \psi)_{\mathcal{H}} = \sum_{n,m} a_n^* b_m (\psi_n, H \psi_m)_{\mathcal{H}} = \sum_{n,m} a_n^* b_m E_n \delta_{nm} = (H \varphi, \psi)_{\mathcal{H}} .\]  

This construction is very general, and one can easily find trivial, in general physically insignificant transformations $T$. An example is the linear map $T : \mathcal{H} \to L_2$ that fulfills $T \psi_n = \psi_n^{(ho)}$ for all $n$, where the $\psi_n^{(ho)}$ are the eigenstates of the harmonic oscillator. Then

\[\hat{H} \psi_n^{(ho)} = THT^{-1} \psi_n = E_n T \psi_n = E_n \psi_n^{(ho)} .\]
We want to stress that in order to find physically acceptable transformations $T$, one has to take into account the fact that the Hamiltonian has to be a function of physical observables.

Initially, one usually starts with a Hamiltonian that is of the form

$$H(x, p) = p^2 + V(x)$$

in which $p$ and $x$ are the usual position and momentum operators that are Hermitian with respect to $L_2$, $V(x)$ is non-Hermitian. In the space $\mathcal{H}$, $H$ is Hermitian, but $x$ and $p$ will in general be non-Hermitian. (Consider, for example, the model investigated by Bender et al. \[1\], $H = p^2 + x^2(ix)^\nu$, $\nu \geq 0$. Here the operators $H$, $x$ and $p$ cannot all be simultaneously Hermitian.) This means that within the formulation in $\mathcal{H}$, the operators $x$ and $p$ can no longer be observables. Thus, the physical meaning of the Hamiltonian $H(x, p)$ in the space $\mathcal{H}$ is quite unclear.

In order to understand the physical content of $H(x, p)$, one has to express it as a function of two operators $x^c$, $p^c$ that correspond to the observables of position and momentum,

$$H = \tilde{H}(x^c, p^c).$$

Necessary conditions for the operators $x^c$, $p^c$ are that

- $x^c$, $p^c$ are Hermitian in $\mathcal{H}$,
- they fulfill canonical commutation relations, $[x^c, p^c] = i$.

If such operators are found, one may call the representation (7) the canonical formulation of the model. As we will show, this formulation leads to some insight into the structure of the transformation $T$.

Let us illustrate this with a very simple example: The Hamiltonian

$$H(x, p) = p^2 + 2i x p - 2 \frac{x^2}{x^2} + \omega^2 x^2$$

is non-Hermitian in $L_2$, but (as will become evident below) has a real spectrum and square-integrable eigenfunctions. We claim that in the space $\mathcal{H}$, it is physically equivalent to the harmonic oscillator.

The reason is that (8) can be written as

$$H = x(p^2 + \omega^2 x^2)x^{-1} =: x\hat{H}x^{-1},$$

i.e. the transformation $T$ between $\mathcal{H}$ and $L_2$ can be chosen to be $T = x^{-1}$.

In $\mathcal{H}$, which is here defined as the image of $L_2$ under $T^{-1}$, the scalar product is given by

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = (T\varphi, T\psi)_{L_2} = \int_{-\infty}^{\infty} \frac{dx}{x^2} \varphi^*(x)\psi(x).$$

One easily finds that with respect to this scalar product, the relations $x^\dagger = x$ and $p^\dagger = p + 2i/x$ hold, so that $(p + i/x)^\dagger = p + i/x$. In fact, (8) can be expressed as

$$H(x, p) = \left(p + \frac{i}{x}\right)^2 + \omega^2 x^2.$$
Table 1. Properties of the operators involved in the canonical formulation.

| operator | in $L_2$ | in $\mathcal{H}$ |
|----------|----------|------------------|
| $H$      | non-Hermitian | Hermitian |
| $x, p$   | Hermitian   | non-Herm. (in general) |
| $\hat{H} = THT^{-1}$ | Hermitian | non-Hermitian |
| $x^c = T^{-1}xT$ | non-Herm. (in general) | Hermitian |
| $p^c = T^{-1}pT$ | non-Herm. (in general) | Hermitian |

Now define $x^c := x$, $p^c := p + i/x$. Note that

$$x^c = T^{-1}xT, \quad p^c = T^{-1}pT = xpx^{-1}, \quad (11)$$

so that $[x^c, p^c] = i$. Thus $x^c$ and $p^c$ fulfill the two necessary conditions given above.

The canonical formulation of the Hamiltonian (8) is therefore

$$H = (p^c)^2 + \omega^2(x^c)^2 \equiv \tilde{H}(x^c, p^c), \quad (12)$$

which makes it evident that we are describing nothing but a harmonic oscillator. A summary of the properties of the various operators is given in Table 1.

This simple example demonstrates some general features: Given a Hermitian Hamiltonian $\hat{H}(x, p)$ that acts in $L_2$, one can make a similarity transformation $H = T^{-1}\hat{H}T$ with an operator $T$ that is non-unitary if considered as an endomorphism $L_2 \rightarrow L_2$. Then the spectrum of $H$ remains real, but the Hermiticity of $H$ is destroyed. If $\hat{H}(x, p)$ is an analytic function of $x$ and $p$, then

$$H = T^{-1}\hat{H}(x, p)T = \hat{H}(T^{-1}xT, T^{-1}pT) = \hat{H}(x^c, p^c) \equiv \tilde{H}(x^c, p^c), \quad (13)$$

i.e. the canonical formulation can be found by substituting $x \rightarrow x^c$ and $p \rightarrow p^c$ in the $L_2$-Hermitian Hamiltonian $\hat{H}(x, p)$.

This emphasizes the importance of the canonical formulation: Turning the argument around, one starts with a non-Hermitian Hamiltonian $\hat{H}(x, p)$. Once a set of canonical operators $x^c$ and $p^c$ is found, and $H$ has been expressed as a function of these operators, $H(x, p) = \hat{H}(x^c, p^c)$, the model can be formulated in the space $L_2$ as an ordinary quantum-mechanical problem with the Hermitian Hamiltonian $\hat{H}(x, p)$ that is a function of the usual Hermitian position and momentum operators $x$ and $p$. Thus the physical meaning of the model is clear. The transformation $T$ has to be chosen such that in addition to (5) it also fulfills (11).

If, on the other hand, a canonical formulation cannot be found, the physical interpretation of $H(x, p)$ is unclear.

3. Summary and outlook

The canonical formulation of the Hamiltonian is meaningful as a relation between observables. The non-Hermiticity completely disappears from the model.
The crucial ingredient of this formulation is the correct choice of the transformation $T : \mathcal{H} \to L_2$ (or equivalently the correct choice of the Hilbert space $\mathcal{H}$). Transformations that render a given non-Hermitian Hamiltonian $H$ with real spectrum Hermitian can be easily found, but we require in addition that $T$ maps between $x$, $x^c$ and $p$, $p^c$, resp., via a similarity transformation.

One immediate question concerns the uniqueness of $T$. The Stone-von Neumann uniqueness theorem [8] states that all irreducible representations of two self-adjoint operators $x^c$ and $p^c$ that are defined in a separable Hilbert space and fulfill canonical commutation relations are unitarily equivalent. Finding such a set of operators removes the arbitrariness in $T$ completely (up to unitary equivalence). But here the self-adjointness (as opposed to Hermiticity) of $x^c$ and $p^c$ is crucial.

In [7] we have given explicit expressions for $T$ for a number of models that have been recently discussed. For some of these examples, the canonical formulation shows that they are physically equivalent to well-known quantum-mechanical problems. In other models with non-Hermitian Hamiltonians, interesting effects like a transition from real to complex eigenvalues occur [1], or the wave functions are analytically continued along complicated paths in the complex domain [2]. We believe that in such cases the construction of a canonical formulation may lead to interesting new insights.

Acknowledgments

R. K. wishes to thank M. Znojil for organizing this very stimulating workshop. L. S. is grateful for the warm hospitality extended to him at the Ruhr-Universität Bochum. The work of L. S. is supported in part by the German-Polish scientific and technological agreement (WTZ Deutschland-Polen).

References

[1] C. M. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243; C. M. Bender, S. Boettcher, P. Meisinger, J. Math. Phys. 40 (1999) 2201.
[2] F. Cannata, G. Junker, J. Trost, Phys. Lett. A 246 (1998) 219.
[3] M. Znojil, Phys. Lett. A 259 (1999) 220.
[4] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205; ibid. 43 (2002) 2814; ibid. 43 (2002) 3944.
[5] F. G. Scholtz, H. B. Geyer, F. J. W. Hahne, Ann. Phys., NY 213 (1992) 74.
[6] C. M. Bender, D. C. Brody, H. F. Jones, Phys. Rev. Lett. 89 (2002) 270401.
[7] R. Kretschmer, L. Szymanowski, Preprint quant-ph/0105054 (2001); Preprint quant-ph/0305123 (2003).
[8] E. Prugovečki, Quantum Mechanics in Hilbert Space, Academic Press, New York, 1981.