Lower Bounds for Semialgebraic Range Searching and Stabbing Problems*

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In the semialgebraic range searching problem, we are given a set of \( n \) points in \( \mathbb{R}^d \) and we want to preprocess the points such that for any query range belonging to a family of constant complexity semialgebraic sets (Tarski cells), all the points intersecting the range can be reported or counted efficiently. When the ranges are composed of simplices, then the problem is well-understood: it can be solved using \( S(n) \) space and with \( Q(n) \) query time with \( S(n)Q(n)^d = \tilde{O}(n^d) \) where the \( \tilde{O}(\cdot) \) notation hides polylogarithmic factors and this trade-off is tight (up to \( n^{o(1)} \) factors). Consequently, there exists "low space" structures that use \( O(n) \) space with \( O(n^{1-1/d}) \) query time and "fast query" structures that use \( O(n^d) \) space with \( O(\log^{d+1} n) \) query time. However, for the general semialgebraic ranges, only "low space" solutions are known, but the best solutions match the same trade-off curve as the simplex queries, with \( O(n) \) space and \( \tilde{O}(n^{1-1/d}) \) query time. It has been conjectured that the same could be done for the "fast query" case but this open problem has stayed unresolved.

Here, we disprove this conjecture. We give the first nontrivial lower bounds for semialgebraic range searching and other related problems. More precisely, we show that any data structure for reporting the points between two concentric circles, a problem that we call 2D annulus reporting problem, with \( Q(n) \) query time must use \( S(n) = \tilde{O}(n^{1/2}/Q(n)) \) space where the \( \tilde{O}(\cdot) \) notation hides \( n^{o(1)} \) factors, meaning, for \( Q(n) = \log^{O(1)} n \), \( \tilde{O}(n^{1/2}) \) space must be used. In addition, we study the problem of reporting the subset of input points between two polynomials of the form \( Y = \sum a_i X^i \) where values \( a_0, \ldots, a_\Delta \) are given at the query time, a problem that we call polynomial slab reporting. For this, we show a space lower bound of \( \Omega(n^{\Delta+1}/Q(n)^{(\Delta+3)/2}) \), which shows for \( Q(n) = \log^{O(1)} n \), we must use \( \tilde{O}(n^{\Delta+1}) \) space. We also consider the dual problems of semialgebraic range searching, semialgebraic stabbing problems, and present lower bounds for them. In particular, we show that in linear space, any data structure that solves 2D annulus stabbing problems must use \( \Omega(n^{2/3}) \) query time. Note that this almost matches the upper bound obtained by lifting 2D annuli to 3D. Like semialgebraic range searching, we also present lower bounds for general semialgebraic slab stabbing problems. Again, our lower bounds are almost tight for linear size data structures in this case.

CCS Concepts: • Theory of computation → Computational geometry.

Additional Key Words and Phrases: Semialgebraic Range Searching, Lower Bound, Computational Geometry

1 INTRODUCTION

We address one of the biggest open problems of the recent years in the range searching area. Our main results are lower bounds in the pointer machine model of computation that essentially show that the so-called "fast query" version of the semialgebraic range reporting problem is "imperious" to the algebraic techniques. Our main result reveals that to obtain polylogarithmic query time, the data structure requires \( \tilde{O}(n^{\Delta+1}) \) space\(^1\), where the constant depends on \( \Delta \), \( n \) is the input size, and \( \Delta + 1 \) is the number of parameters of each "polynomial inequality" (these will be defined more clearly later). Thus, we refute a relatively popular recent conjecture that data structures with \( \tilde{O}(n^d) \) space and polylogarithmic query time could exist, where \( d \) is the dimension of the input points. Surprisingly, the proofs behind these lower bounds are simple, and these lower bounds could have been discovered years ago as the tools we use already existed decades ago.

Range searching is a broad area of research in which we are given a set \( P \) of \( n \) points in \( \mathbb{R}^d \) and the goal is to preprocess \( P \) such that given a query range \( R \), we can count or report the subset of \( P \) that lies in \( R \). Often \( R \) is restricted to a fixed family of ranges, e.g., in simplex range counting problem, \( R \) is a simplex in \( \mathbb{R}^d \) and the goal is to report \( |P \cap R| \), or in halfspace range reporting problem, \( R \) is a halfspace and the goal is to report \( P \cap R \). Range searching problems have been studied extensively and they have numerous variants. For an overview of this topic, we refer the readers to an excellent survey by Agarwal [17].

Another highly related problem which can be viewed as the "dual" of this problem is range stabbing: we are given a set \( R \) of ranges as input and the goal is to preprocess \( R \) such that given a query point \( p \), we can count or report the ranges of \( R \) containing \( p \) efficiently. Here, we focus on the reporting version of range stabbing problems.

1.1 Range Searching: A Very Brief Survey

1.1.1 Simplex Range Searching. Simplices are one of the most fundamental family of queries. In fact, if the query is decomposable (such as range counting or range reporting queries), then simplices can be used as "building blocks" to answer more complicated queries: for a query

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\(^1\) \( \tilde{O}(\cdot), \tilde{\Omega}(\cdot), \tilde{\Theta}(\cdot) \) notations hide \( n^{o(1)} \) factors and \( \Omega(\cdot), \tilde{O}(\cdot), \tilde{\Theta}(\cdot) \) notations hide \( \log^{O(1)} n \) factors.

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which is a polyhedral region of $O(1)$ complexity, we can decompose it into $O(1)$ disjoint simplices (with a constant that depends on $d$) and thus answering $\mathcal{A}$ can be reduced to answering $O(1)$ simplicial queries.

Simplicial queries were hotly investigated in 1980s and this led to development of two important tools in computational geometry: cuttings and partition theorem and both of them have found applications in areas not related to range searching.

Cuttings and Fast Data Structures. "Fast query" data structures can answer simplex range counting or reporting queries in polylogarithmic query time but by using $O(n^d)$ space and they can be built using cuttings. In a nut-shell, given a set $H$ of $n$ hyperplanes in $\mathbb{R}^d$, a $\frac{1}{3}$-cutting, is a decomposition of $\mathbb{R}^d$ into $O(n^d)$ simplices such that each simplex is intersected by $O(n/r)$ hyperplanes of $H$. These were developed by some of the pioneers in the range searching area, such as Clarkson [15], Haussler and Welzl [20], Chazelle and Friedman [12], Matoušek [21], finally culminating in a result of Chazelle [10] who optimized various aspects of cuttings. Using cuttings, one can answer simplex range counting, or reporting queries with $O(n^d)$ space and $O((\log n)^{d+1} + k)$ query time (where $k$ is the output size) [22]. The query time can be lowered to $O(\log n)$ by increasing the space slightly to $O(n^{d+\epsilon})$ for any constant $\epsilon > 0$ [14]. An interested reader can refer to a survey on cuttings by Chazelle [11].

The Partition Theorem and Space-efficient Data Structures. At the opposite end of the spectrum, simplex range counting or reporting queries can be answered using linear space but with higher query time of $O(n^{1-1/d})$, using partition trees and the related techniques. This branch of techniques has a very interesting history. In 1982, Willard [25] cleverly used ham sandwich theorem to obtain a linear-sized data structure with query time of $O(n^\gamma)$ for some constant $\gamma < 1$ for simplicial queries in 2D. After a number of attempts that either improved the exponent or generalized the technique to higher dimensions, Welzl [24] in 1982 provided the first optimal exponent for the partition trees, then Chazelle et al. [14] provided the first near-linear size data structure with query time of roughly $O(n^{1-1/d})$. Finally, a data structure with $O(n)$ space and $O(n^{1-1/d})$ query time was given by Matoušek [22]. This was also simplified recently by Chan [7].

Space/Query Time Trade-off. It is possible to combine fast query data structures and linear-sized data structures to solve simplex queries with $S(n)$ space and $Q(n)$ query time such that $S(n)Q(n)^d = \widetilde{O}(n^d)$. This trade-off between space and query time is optimal, at least in the pointer machine model and in the semigroup model [1, 8, 13].

Multi-level Structures, Stabbing and Other Related Queries. By using multi-level data structures, one can solve more complicated problems where both the input and the query shapes can be simplicial objects of constant complexity. The best multi-level data structures use one extra $\log n$ factor in space and query time per level [7] and there exist lower bounds that show space/query time trade-off should blow up by at least $\log n$ factor per level [2]. This means that problems such as simplex stabbing (where the input is a set of simplices and we want to output the simplices containing a given query point) or simplex-simplex containment problem (where the input is a set of simplices, and we want to output simplices fully contained in a query simplex) all have the same trade-off curve of $S(n)Q(n)^d = \tilde{O}(n^d)$ between space $S(n)$ and query time $Q(n)$.

Thus, one can see that the simplex range searching as well as its generalization to problems where both the input and the query ranges are “flat” objects is very well understood. However, there are many natural query ranges that cannot be represented using simplices, e.g., when query ranges are spheres in $\mathbb{R}^d$. This takes us to semialgebraic range searching.

1.1.2 Semialgebraic Range Searching. A semialgebraic set is defined as a subset of $\mathbb{R}^d$ that can be described as the union or intersection of $O(1)$ ranges, where each range is defined by $d$-variate polynomial inequality of degree at most $\Delta$, defined by at most $B$ values given at the query time; we call $B$ the parametric dimension. For instance, with $B = 3$, $\Delta = 2$, and given three values $a, b$ and $c$ at the query time, a circular query can be represented as $\{(X, Y) \in \mathbb{R}^2 : (X^2 + (Y - b)^2 \leq c^2\}$. In semialgebraic range searching, the queries are semialgebraic sets.

Before the recent "polynomial method revolution", the tools available to deal with semialgebraic range searching were limited, at least compared to the simplex queries. One way to deal with semialgebraic range searching is through linearization [26]. This idea maps the input points to $\mathbb{R}^L$ for some potentially large parameter $L$, such that each polynomial inequality can be represented as a halfspace. Consequently, semialgebraic range searching can be solved with the space/query time trade off of $S(n)Q(n)^L = \tilde{O}(n^L)$. The exponent of $Q(n)$ in the trade-off can be improved (increased) a bit by exploiting that in $\mathbb{R}^L$, the input set actually lies in a $d$-dimensional surface [5]. It is also possible to build "fast query" data structures but using $O(n^{d+\epsilon})$ space by a recent result of Agarwal et al. [4].

In 2009, Zeev Dvir [16] proved the discrete Kakeya problem with a very elegant and simple proof, using a polynomial method. Within a few years, this led to revolution in discrete and computational geometry, one that was ushered in by Katz and Guth’s almost tight bound on Erdős distinct distances problem [19]. For a while, the polynomial method did not have much algorithmic consequences but this changed
with the work of Agarwal, Matoušek, and Sharir [6] where they showed that at least as long as linear-space data structures are considered, semialgebraic range queries can essentially be solved within the same time as simplex queries (ignoring some lower order terms). Later developments (and simplifications) of their approach by Matoušek and Patáková [23] lead to the current best results: a data structure with linear size and with query time of $\tilde{O}(n^{1-1/d})$.

Fast Queries for Semialgebraic Range Searching: an Open Problem. Nonetheless, despite the breakthrough results brought on by the algebraic techniques, the fast query case still remained unsolved, even in the plane: e.g., the best known data structures for answering circular queries with polylogarithmic query time still use $\tilde{O}(n^3)$ space, by using linearization to $\mathbb{R}^3$. The fast query case of semialgebraic range searching has been explicitly mentioned as a major open problem in multiple recent publications\(^2\). In light of the breakthrough result of Agarwal et al. [6], it is quite reasonable to conjecture that semialgebraic range searching should have the same trade-off curve of $S(n)Q(n)^d = \tilde{O}(n^d)$.

Nonetheless, the algebraic techniques have failed to make sufficient advances to settle this open problem. As mentioned before, the best known “fast query” result by Agarwal et al. [4] uses $O(n^{B+\epsilon})$ space. In general, $B$ can be much larger than $d$ and thus it leaves a big gap between current best upper bound and the conjectured one. Given that it took a revolution caused by the polynomial method to advance our knowledge of the “low space” case of semialgebraic range searching, it is not too outrageous to imagine that perhaps equally revolutionary techniques are needed to settle the “fast query” case of semialgebraic range searching.

1.1.3 Semialgebraic Range Stabbing. Another important problem is semialgebraic stabbing, where the input is a set of $n$ semialgebraic sets, i.e., “ranges”, and queries are points. The goal is to output the input ranges that contain a query point. Here, “fast query” data structures are possible, for example by observing that an arrangement of $n$ disks in the plane has $O(n^2)$ complexity, counting or reporting the disks stabbed by a query point can be done with $O(n^2)$ space and $O(\log n)$ query time using slab and persistent. However, this simple idea does not work for general semialgebraic sets in higher dimensions. The main bottleneck before the recent “polynomial method revolution” is that for dimensions higher than 4 no technique was able to decompose the arrangement formed by a collection of algebraic surfaces into an arrangement with the property that each cell is a constant-complexity semialgebraic set and the complexity of the decomposed arrangement is close to that of the original arrangement. This was also one of the roadblocks for the “fast query” version of semialgebraic range searching. Recently, Guth [18] showed the existence of polynomials whose connected components give such decompositions for arbitrary dimensions. An efficient algorithm for computing such partitioning polynomials was given very recently by Agarwal et al. [4], and as one of the results, they showed it is possible to build “fast query” data structures for semialgebraic stabbing problems using $O(n^{d+\epsilon})$ space for $n$ semialgebraic sets in $\mathbb{R}^d$. However, comparing to the reporting problems, in this stabbing scenario, it seems difficult to make advancements in the “low space” side of things; e.g., for the planar disk stabbing problem, the only known data structure with $O(n)$ space is one that uses linearization to $3D$ that results in $\tilde{O}(n^{2/3})$ query time.

1.2 Our Results

Our main results are lower bounds in the pointer machine model of computation for four central problems defined below. In the 2D polynomial slab reporting problem, given a set $\mathcal{P}$ of $n$ points in $\mathbb{R}^2$, the task is to preprocess $\mathcal{P}$ such that given a query 2D polynomial slab $\mathcal{B}$, the points contained in the polynomial slab, i.e., $\mathcal{B} \cap \mathcal{P}$, can be reported efficiently. Informally, a 2D polynomial slab is the set of points $(x, y)$ such that $P(x) \leq y \leq P(x) + w$, for some univariate polynomial $P(x)$ of degree $\Delta$ and value $w$ given at the query time. In the 2D polynomial slab stabbing problem, the input is a set of $n$ polynomial slabs and the query is a point $q$ and the goal is to report all the slabs that contain $q$. Similarly, in the 2D annulus reporting problem, the input is a set $P$ of $n$ points in $\mathbb{R}^2$ and the query is an “annulus”, the region between two concentric circles. Finally, in 2D annulus stabbing problem, the input is a set of $n$ annuli, the query is a point $q$ and the goal is to report all the annuli that contain $q$.

For polynomial slab queries, we show that if a data structure answers queries in $Q(n) + O(k)$ time, where $k$ is the output size, using $S(n)$ space, then $S(n) = \tilde{O}(n^{\Delta+1}/Q(n)^{\Delta+1}A/2)$, the hidden constants depend on $\Delta$. So for “fast queries”, i.e., $Q(n) = \tilde{O}(1)$, we have $\tilde{O}(n^{\Delta+1})$ space must be used. This is almost tight as the exponent matches the upper bounds obtained by linearization as well as the recent upper bound of Agarwal et al. [4]. Also, we prove that any structure that answers polynomial slab stabbing queries in $Q(n) + O(k)$ time must use $\Omega(n^{1+2/((\Delta+1)/2)})$ space. In the “low space” setting, when $S(n) = O(n)$, this gives $Q(n) = \Omega(n^{1-1/(\Delta+1)})$. This is once again almost tight as it matches the upper bounds obtained by linearization for when $S(n) = \tilde{O}(n)$.

\(^2\)To quote Agarwal et al. [6], “...a very interesting and challenging problem is, in our opinion, the fast-query case of range searching with constant-complexity semialgebraic sets, where the goal is to answer a query in $O(\log n)$ time using roughly $O(n^d)$ space.” The same conjecture is repeated in a different survey [3] and it is also emphasized that the question is even open for disks in the plane, ”... whether a disk range-counting query in $\mathbb{R}^2$ be answered in $O(\log n)$ time using $O(n^2)$ space?”.
For the annulus reporting problem, we get the same \( S(n) = \tilde{\Omega}(n^{3/4}/Q(n)^{5/4}) \) lower bound as polynomial slab reporting when \( \Delta = 2 \). For the annulus stabbing problem, we show \( S(n) = \Omega(n^{3/2}/Q(n)^{3/4}) \), e.g., in “low space” setting when \( S(n) = O(n) \), we must have \( Q(n) = \Omega(n^{2/3}) \); compare this with simplex stabbing queries can be solved with \( O(n) \) space and \( \tilde{O}(\sqrt{n}) \) query time. As before, this is almost tight, as it matches the upper bounds obtained by linearization to 3D for when \( S(n) = \tilde{O}(n) \).

Somewhat disappointingly, no revolutionary new technique is required to obtain these results. We use novel ideas in the construction of “hard input instances” but otherwise we use the two widely used pointer machine lower bound frameworks by Chazelle [9], Chazelle and Rosenberg [13], and Afshani [1]. Our results are summarized in Table 1.

Table 1. Our Results, * indicates this paper. In the table, \( \Omega(\cdot) \) and \( \tilde{O}(\cdot) \) notations hide \( n^{o(1)} \) factors, and \( \tilde{O}(\cdot) \) notation hides \( \log^{O(1)} n \) factors.

| Problem                                | Lower Bound                  | Upper Bound                  |
|----------------------------------------|------------------------------|------------------------------|
| 2D Polynomial Slab Reporting           | \( S(n) = \tilde{\Omega}\left( \frac{n^{2\Delta+1}}{Q(n)^{3(\Delta+1)/2}} \right) \) * | \( S(n) = \tilde{O}\left( \frac{n^{2\Delta+1}}{Q(n)^{2\Delta/3}} \right) \) [4–6, 22] |
| When \( Q(n) = \tilde{O}(1) \)        | \( S(n) = \tilde{\Omega}(n^{\Delta+1}) \) | \( S(n) = \tilde{O}(n^{\Delta+1}) \) [4–6, 22] |
| 2D Annulus Reporting                   | \( S(n) = \tilde{\Omega}\left( \frac{n^{2\Delta+1}}{Q(n)^{3(\Delta+1)/2}} \right) \) * | \( S(n) = \tilde{O}\left( \frac{n^{2\Delta+1}}{Q(n)^{2\Delta/3}} \right) \) [4] |
| When \( Q(n) = \tilde{O}(1) \)        | \( S(n) = \tilde{\Omega}(n^{\Delta+1}) \) | \( S(n) = \tilde{O}(n^{\Delta+1}) \) [4] |
| 2D Polynomial Slab Stabbing            | \( S(n) = \Omega\left( \frac{n^{2\Delta+1}}{Q(n)^{3(\Delta+1)/2}} \right) \) * | \( S(n) = \tilde{O}\left( \frac{n^{2\Delta+1}}{Q(n)^{2\Delta/3}} \right) \) [4] |
| When \( S(n) = \tilde{O}(n) \)        | \( Q(n) = \tilde{\Omega}\left( \frac{n^{2\Delta+1}}{Q(n)^{3(\Delta+1)/2}} \right) \) * | \( Q(n) = \tilde{O}(n^{2\Delta+1}) \) [4] |

2 PRELIMINARIES

We first review the related geometric reporting data structure lower bound frameworks. The model of computation we consider is (an augmented version of) the pointer machine model.

In this model, the data structure is a directed graph \( M \). Let \( S \) be the set of input elements. Each cell of \( M \) stores an element of \( S \) and two pointers to other cells. Assume a query \( q \) requires a subset \( S_q \subset S \) to be output. For the query, we only charge for the pointer navigations. Let \( M_q \) be the smallest connected subgraph, s.t., every element of \( S_q \) is stored in at least one element of \( M_q \). Clearly, \( |M| \) is a lower bound for space and \( |M_q| \) is a lower bound for query time. Note that this grants the algorithm unlimited computational power as well as full information about the structure of \( M \).

In this model, there are two main lower bound frameworks, one for range reporting [9, 13], and the other for its dual, range stabbing [1]. We describe them in detail here.

2.1 A Lower Bound Framework for Range Reporting Problems

The following result by Chazelle [9] and later Chazelle and Rosenberg [13] provides a general lower bound framework for range reporting problems. In the problem, we are given a set \( S \) of \( n \) points in \( \mathbb{R}^d \) and the queries are from a set \( \mathcal{R} \) of ranges. The task is to build a data structure such that given any query range \( \mathcal{R} \in \mathcal{R} \), we can report the points intersecting the range, i.e., \( \mathcal{R} \cap S \), efficiently.

**Theorem 2.1 (Chazelle [9] and Chazelle and Rosenberg [13]).** Suppose there is a data structure for range reporting problems that uses at most \( S(n) \) space and can answer any query in \( O(n) + O(k) \) time where \( n \) is the input size and \( k \) is the output size. Assume we can show that there exists an input set \( S \) of \( n \) points satisfying the following: There exist \( m \) subsets \( q_1, q_2, \ldots, q_m \subset S \), where \( q_i, i = 1, \ldots, m \), is the output of some query and they satisfy the following two conditions: (i) for all \( i = 1, \ldots, m \), \( |q_i| \geq Q(n) \); and (ii) the size of the intersection of every distinct subsets \( q_{i_1}, q_{i_2}, \ldots, q_{i_d} \) is bounded by some value \( c \geq 2 \), i.e., \( |q_{i_1} \cap q_{i_2} \cap \cdots \cap q_{i_d}| \leq c \). Then \( S(n) = \Omega\left( \frac{mQ(n)^{\frac{d-1}{d}}} {\alpha^{2d-1}c^d} \right) = \Omega\left( \frac{mQ(n)^{\frac{d-1}{d}}} {\alpha^{2d-1}e} \right) \).

To use this framework, we need to exploit the property of the considered problem and come up with a construction that satisfies the two conditions above. Often, the construction is randomized and thus one challenge is to satisfy condition (ii) in the worst-case. This can be done by showing that the probability that (ii) is violated is very small and then using a union bound to prove that with positive probability the construction satisfies (ii) in the worst-case.
2.2 A Lower Bound Framework for Range Stabbing Problems

Range stabbing problems can be viewed as the dual of range reporting problems. In this problem, we are given a set $\mathcal{R}$ of $n$ ranges, and the queries are from a set $\mathcal{Q}$ of $n$ points. The task is to build a data structure such that given any query point $q \in \mathcal{Q}$, we can report the ranges “stabbed” by this query point, i.e., $\{R \in \mathcal{R} : R \cap q \neq \emptyset\}$, efficiently. A recent framework by Afshani [1] provides a simple way to get the lower bound of such problems.

**Theorem 2.2 (Afshani [1]).** Suppose there is a data structure for range stabbing problems that uses at most $S(n)$ space and can answer any query in $O(n + q)$ time where $n$ is the input size and $k$ is the output size. Assume we can show that there exists an input set $\mathcal{R} \subseteq \mathcal{R}$ of $n$ ranges that satisfy the following: (i) every query point of the unit square $U$ is contained in at least $t \geq Q(n)$ ranges; and (ii) the area of the intersection of every $a < t$ ranges is at most $v$. Then $S(n) = \Omega(\frac{t^{\log_2 n}}{v}) = \Omega(\frac{Q(n)}{v^{\log_2 n}})$.

This is very similar to framework of Theorem 2.1 but often it requires no derandomization.

2.3 Derandomization Lemmas

As mentioned before, one challenge of using Chazelle’s framework is to show the existence of a hard instance in the worst case while the construction is randomized. To do that, we present two derandomization lemmas.

**Lemma 2.1.** Let $\mathcal{P}$ be a set of $n$ points chosen uniformly at random in a square $S$ of side length $n$ in $\mathbb{R}^2$. Let $\mathcal{R}$ be a set of ranges in $S$ such that (i) the intersection area of any $t \geq 2$ ranges $R_1, R_2, \ldots, R_t \in \mathcal{R}$ is bounded by $O\left(\frac{n \log n}{2^k}\right)$; (ii) the total number of intersections is bounded by $O\left(n^{2k}\right)$ for $k \geq 1$. Then with probability $\geq \frac{1}{2}$, for all distinct ranges $R_1, R_2, \ldots, R_t \in \mathcal{R}$, $|R_1 \cap R_2 \cap \cdots \cap R_t \cap \mathcal{P}| < 3k\sqrt{\log n}$.

**Proof.** Consider any intersection region $\rho \in \mathcal{S}$ of $t$ ranges with area $A$. Let $X_i$ be an indicator random variable with $X_i = \begin{cases} 1, & \text{the } i\text{-th point is inside } \rho, \\ 0, & \text{otherwise.} \end{cases}$

Let $X = \sum_{i=1}^{n} X_i$. Clearly, $\mathbb{E}[X] = \frac{A}{n}$. By Chernoff’s bound,

$$\Pr\left[ X \geq (1 + \delta)\frac{A}{n} \right] < \left( \frac{e^{\delta}}{(1 + \delta)^{1+\delta}} \right)^{\frac{A}{n}}.$$

for any $\delta > 0$. Let $t = (1 + \delta)^{\frac{A}{n}}$, then

$$\Pr[X \geq t] < \frac{e^{\delta}}{(1 + \delta)^{1+\delta}} < \frac{e^t}{(1 + \delta)^{1+\delta}} = \left( \frac{eA}{nt} \right)^t.$$

Now we pick $t = 3k\sqrt{\log n}$, since $A \leq cn/2^{\log n}$ for some constant $c$, we have

$$\Pr\left[ X \geq 3k\sqrt{\log n} \right] < \left( \frac{ce}{2^{\log n} 3k\sqrt{\log n}} \right)^{3k\sqrt{\log n}} < \frac{(ce)^{3k\sqrt{\log n}}}{n^{3k}}.$$

Since the total number of intersections is bounded by $O(n^{2k})$, the number of cells in the arrangement is also bounded by $O(n^{2k})$ and thus by the union bound, for sufficiently large $n$, with probability $> \frac{1}{2}$, the number of points in every intersection region is less than $3k\sqrt{\log n}$. □

**Lemma 2.2.** Let $\mathcal{P}$ be a set of $n$ points chosen uniformly at random in a square $S$ of side length $n$ in $\mathbb{R}^2$. Let $\mathcal{R}$ be a set of ranges in $S$ such that (i) the intersection area of any range $R \in \mathcal{R}$ and $S$ is at least $ct$ for some constant $c \geq 4k$ and a parameter $t \geq \log n$, where $k \geq 2$; (ii) the total number of ranges is bounded by $O\left(n^{k+1}\right)$. Then with probability $> \frac{1}{2}$, for every range $R \in \mathcal{R}$, $|R \cap \mathcal{P}| \geq t$.

**Proof.** The proof of this lemma is similar to the one for Lemma 2.1. We pick $n$ points in $S$ uniformly at random. Let $X_{ij}$ be the indicator random variable with $X_{ij} = \begin{cases} 1, & \text{point } i \text{ in range } j, \\ 0, & \text{otherwise.} \end{cases}$
We know that the area of each range is at least $cnt$. Then the expected number of points in each range is $ct$. Consider an arbitrary range, let $X_j = \sum_{i=1}^{n} X_{ij}$, then by Chernoff’s bound
\[
\Pr \left[ X_j < t \left(1 - \frac{c - 1}{c}\right) ct \right] < e^{-\frac{(c-1)^2}{2}} t \\
\Rightarrow \Pr \left[ X_j < t \right] < e^{-\frac{(c-1)^2}{2}} t < \frac{1}{n} < \frac{1}{n^{\frac{1}{4}} - \frac{1}{2k+1}/(8k)}.
\]

The second last inequality follows from $t \geq \log n$ and the last inequality follows from $c \geq 4k$. Since the total number of ranges is bounded by $O(n^{k+1})$, by a standard union bound argument, the lemma holds.

\[\Box\]

### 3 2D POLYNOMIAL SLAB REPORTING AND STABBING

We first consider the case when query ranges are 2D polynomial slabs. The formal definition of 2D polynomial slabs is as follows.

**Definition 3.1.** Let $P(x) = \sum_{i=0}^{\Delta} a_i x^i$, where $a_\Delta \neq 0$, be a degree $\Delta$ univariate polynomial. A 2D polynomial slab is a pair $(P(x), w)$, where $P(x)$ is called the base polynomial and $w > 0$ the width of the polynomial slab. The polynomial slab is then defined as $\{(x, y) \in \mathbb{R}^2 : P(x) \leq y \leq P(x) + w\}$.

### 3.1 2D Polynomial Slab Reporting

We consider the 2D polynomial slab reporting problem in this section, where the input is a set $P$ of $n$ points in $\mathbb{R}^2$, and the query is a polynomial slab. This is an instance of semialgebraic range searching where we have two polynomial inequalities where each inequality has degree $\Delta$ and it is defined by $\Delta + 1$ parameters given at the query time (thus, $B = \Delta + 1$). Note that $\Delta + 1$ is also the dimension of linearization for this problem, meaning, the 2D polynomial slab reporting problem can be lifted to the simplex range reporting problem in $\mathbb{R}^{\Delta+1}$. Our main result shows that for fast queries (i.e., when the query time is polylogarithmic), this is tight, by showing an $\tilde{\Omega}(n^{\Delta+1})$ space lower bound, in the pointer machine model of computation.

Before we present the lower bound, we first introduce a simple property of polynomials, which we will use to upper bound the intersection coefficient and the maximum range within which its value is bounded.

**Lemma 3.1.** Let $P(x) = \sum_{i=0}^{\Delta} a_i x^i$ be a degree $\Delta$ univariate polynomial where $\Delta > 0$ and $|a_\Delta| \geq d$ for some positive $d$. Let $w$ be any positive value and $x_1$ be a parameter. If $|P(x)| \leq w$ for all $x \in [x_1, x_1 + t]$, then $t \leq (\Delta + 1)^3 \left(\frac{w}{d^\frac{1}{3}}\right)^{\frac{1}{2}}$.

**Proof.** First note that w.l.o.g., we can assume $x_1 = 0$, because otherwise we can consider a new polynomial $P'(x) = P(x + x_1)$. Since $P'(x)$ is still a degree $\Delta$ univariate polynomial with $|a_\Delta| \geq d$, and for all $x \in [0, t]$, $P'(x) = P(x + x_1)$, to bound $t$, we only need to consider $P'(x)$ on interval $[0, t]$.

Assume for the sake of contradiction that $t > (\Delta + 1)^3 \left(\frac{w}{d^\frac{1}{3}}\right)^{\frac{1}{2}}$. We show that this will lead to $|a_\Delta| < d$.

We pick $\Delta + 1$ different points $(x_i, y_i)$, where $x_i \in [0, t]$ and $y_i = P(x_i)$, on the polynomial. Then $P(x)$ can be expressed as
\[
P(x) = \sum_{i=0}^{\Delta} y_i \prod_{j=0, j \neq i}^{\Delta} \frac{x - x_j}{x_i - x_j}.
\]

The coefficient of the degree $\Delta$ term is therefore
\[
a_\Delta = \sum_{i=0}^{\Delta} y_i \prod_{j=0, j \neq i}^{\Delta} \frac{1}{x_i - x_j}.
\]

We pick $x_i = \frac{d^\frac{1}{3}}{\Delta + 1}$ for $i = 0, 1, \cdots, \Delta$ and we therefore obtain
\[
a_\Delta = \sum_{i=0}^{\Delta} y_i \prod_{j=0, j \neq i}^{\Delta} \frac{1}{\frac{d^\frac{1}{3}}{\Delta + 1} - \frac{d^\frac{1}{3}}{\Delta + 1}}.
\]
We now upper bound
\[
\frac{1}{\mu^2} \leq \frac{1}{\mu^2} \leq \frac{(i+1)(j+1)}{t(j-i)} < \frac{(\Lambda + 1)^2}{t},
\]
where the last inequality follows from \( i, j = 0, 1, \ldots, \Lambda \) and \( j - i \geq 1 \). Also by assumption, \( |y_i| \leq w \), we therefore have
\[
|\alpha_i| < \frac{w(\Lambda + 1)(\Lambda + 1)^2}{t} < d,
\]
where the last inequality follows from \( t > (\Lambda + 1)^3 \left( \frac{w}{d} \right)^2 \). However, in \( P(x) \), \( |\alpha_i| \geq d \), a contradiction. Therefore, \( t \leq (\Lambda + 1)^3 \left( \frac{w}{d} \right)^2 \). \( \square \)

With Lemma 3.1 at hand, we now show a lower bound for polynomial slab reporting.

**Theorem 3.1.** Let \( P \) be a set of \( n \) points in \( \mathbb{R}^2 \). Let \( R \) be the set of all 2D polynomial slabs \( \{(P(x), w) : \deg(P) = \Lambda \geq 2, w > 0\} \). Then any data structure for \( P \) that solves polynomial slab reporting for queries from \( R \) with query time \( Q(n) + O(k) \), where \( k \) is the output size, uses \( S(n) = \Omega\left(n^{\Lambda+1}/Q(n)^{(\Lambda+3)\Lambda/2}\right) \) space.

**Proof.** We use Chazelle’s framework to prove this theorem. To this end, we will need to show the existence of a hard input instance. We do this as follows. In a square \( S \), we construct a set of special polynomial slabs with the following properties: (i) The intersection area of any two slabs is small; and (ii) The area of each slab inside \( S \) is relatively large. Intuitively and consequently, if we sample \( n \) points uniformly at random in \( S \), in expectation, few points will be in the intersection of two slabs, and many points will be in each slab. Intuitively, this satisfies the two conditions of Theorem 2.1. By picking parameters carefully and a derandomization process, we get our theorem. Next, we describe the details.

Consider a square \( S = [0, n] \times [0, n] \). Let \( d_i \) for \( i = 1, 2, \ldots, \Lambda \) and \( w \) be some parameters to be specified later. We generate a set of \( \Theta\left(\frac{n^\Lambda}{2^\Lambda 1! \Lambda^2 d_i \cdot \frac{w}{d}}\right) \) polynomial slabs \( \{(P(x), w)\} \) with

\[
P(x) = \left( \sum_{i=1}^{\Lambda} \frac{j_i d_i x^i}{n^i} \right) + kw
\]

where \( j_i = \lfloor \frac{n}{2d_i} \rfloor, \lfloor \frac{d_i}{w} \rfloor + 1, \ldots, \lfloor \frac{n}{d_i} \rfloor \) for \( 1 \leq i \leq \Lambda \) and \( k = \lfloor \frac{n}{2w} \rfloor + 1, \ldots, \lfloor \frac{n}{w} \rfloor \). Note that we normalize the coefficients such that for any polynomial slab in range \( x \in [0, n] \), a quarter of this slab is contained in \( S \) if \( w < n/6 \). To show this, it is sufficient to show that every polynomial is inside \( S \), for every \( x \in [0, n/4] \). As all the coefficients of the polynomials are positive, it is sufficient to upper bound \( P(n/4) \), among all the polynomials \( P(x) \) that we have generated. Similarly, this maximum is attained when all the coefficients are set to their maximum value, i.e., when \( j_i = n/d_i \) and \( k = n/(2w) \), resulting in the polynomial \( P_w(x) = \left( \sum_{i=1}^{\Lambda} \frac{x^i}{n^i} \right) + \frac{n}{2} \). Now it easily follows that \( P_w(n/4) < n/6 \). Then, the claim follows from the following simple observation.

**Observation 3.1.** The area of a polynomial slab \( \{(P(x), w)\} \) for when \( a \leq x \leq b \) is \((b - a)w\).

**Proof.** The claimed area is \( \left( \int_a^b (P(x) + w)dx \right) - \left( \int_a^b P(x)dx \right) = \int_a^b wdx = (b - a)w. \) \( \square \)

Next, we bound the area of the intersection of two polynomial slabs. Consider two distinct slabs \( \mathcal{R}_p = \{(P(x), w)\} \) and \( \mathcal{R}_q = \{(Q(x), w)\} \). Observe that by our construction, if \( P(x) \) and \( Q(x) \) only differ in their constant terms, their intersection is empty. So we only consider the case that there exists some \( 0 < i \leq \Lambda \), such that the coefficients for \( x^i \) are different in \( P(x) \) and \( Q(x) \). As each slab is created using two polynomials of degree \( \Delta \), \( \mathcal{R}_q \cap \mathcal{R}_p \) can have at most \( O(\Lambda) \) connected regions. Consider one connected region \( \mathcal{R} \) and let the interval \( \eta = [x_1, x_2] \subset [0, n] \), be the projection of \( \mathcal{R} \) onto the X-axis. Define the polynomial \( R(x) = P(x) - Q(x) \) and observe that we must have \( |R(x)| \leq w \) for all \( x \in [x_1, x_2] \). We now consider the coefficient of the highest degree term of \( R(x) \). Let \( j_i d_i/n^i \) (resp. \( j_i d_i/n^i \)) be the coefficient of the degree \( i \) term in \( P(x) \) (resp. \( Q(x) \)). Clearly, if \( j_i = j_i^* \), then the coefficient of \( x^i \) in \( R(x) \) will be zero. Thus, to find the highest degree term in \( R(x) \), we need to consider the largest index \( i \) such that \( j_i \neq j_i^* \); in this case, \( R(x) \) will have degree \( i \) and coefficient of \( x_i \) will have absolute value \( |(j_i - j_i^*)d_i/n^i| \). By Lemma 3.1, \( x_2 - x_1 \leq O(\Lambda^3) \left( \frac{\sqrt{\Lambda}}{d_i} \right)^{1/i} \). Next, by Observation 3.1, the area of the intersection of \( \mathcal{R}_q \) and \( \mathcal{R}_p \) is \( O(\Lambda^3)nw^{1/2}\log n \) and \( w = 16\Lambda Q(n) \), for a large enough constant \( c \). Then, the intersection area of any two polynomial slabs is bounded by \( n/2\sqrt{\log n} \). Since in total we have generated \( O(n^{3\Lambda+1}) \) slabs, the total number of intersections they can form is bounded
by $O(n^{Ω(Δ^4)})$. By Lemma 2.1, with probability $> \frac{1}{2}$, the number of points of $\mathcal{P}$ in any intersection of two polynomial slabs is at most $3(Δ + 1)\sqrt{\log n}$. Also, as we have shown that the intersection area of every slab with $S$ is at least $nw/4 = 4ΔnQ(n)$, by Lemma 2.2, with probability more than $\frac{1}{2}$, each polynomial slab has at least $Q(n)$ points of $\mathcal{P}$.

It thus follows that with positive probability, both conditions of Theorem 2.1 are satisfied, and consequently, we obtain the lower bound of

$$S(n) = \Omega \left( \frac{Q(n)^2}{2^{Ω(Δ)} \log n} \right) = \Omega \left( \frac{n^{Δ+1}}{Q(n)^{(Δ+3)/2}} \right),$$

where the last equality follows from $d_i = cΔ^{2i}w^{i+1/2}\sqrt{\log n}$, $w = 16ΔQ(n)$, and

$$\prod_{i=1}^{Δ} d_i = cΔ^{2(1+Δ)Δ/2}w^{(2+Δ+1)Δ/2}2\sqrt{\log n} = Q(n)^{(Δ+3)Δ/2}n^{Ω(Δ)}. \tag*{□}$$

So for the “fast query” case data structure, by picking $Q(n) = log^{O(1)} n$, we obtain a space lower bound of $S(n) = Ω(n^{Δ+1})$.

### 3.2 2D Polynomial Slab Stabbing

By small modifications, our construction can also be applied to obtain a lower bound for (the reporting version of) polynomial slab stabbing problems using Theorem 2.2.

One modification is that we need to generate the slabs in such a way that they cover the entire square $S$. The framework provided through Theorem 2.2 is more stream-lined and derandomization is not needed and we can directly apply the “volume upper bound” obtained through Lemma 3.1. There is also no $n^{Ω(1)}$ factor loss (our lower bound actually uses $Ω(Δ)$ notation). The major change is that we need to use different parameters since we need to create $n$ polynomial slabs, as now they are the input.

**Theorem 3.2.** Given a set $\mathcal{R}$ of $n$ 2D polynomial slabs $\{(P(x), w) : \deg(P) = Δ ≥ 2, w > 0\}$, any data structure for $\mathcal{R}$ solving the 2D polynomial slab stabbing problem with query time $O(q(n) + O(k))$ uses $S(n) = Ω(\frac{1}{2^{Ω(Δ)} Q(n)^{Δ/2}})$ space, where $k$ is the output size.

**Proof.** We use Afshani’s lower bound framework as described in Theorem 2.2. First we generate $n$ polynomial slabs in a unit square $S = [0, 1] × [0, 1]$ as follows. Consider $Θ \left( \frac{1}{2^{Ω(Δ)} \prod_{i=1}^{Δ} d_i} \right)$ polynomial slabs with their base polynomials being:

$$P(x) = \sum_{i=1}^{Δ} j_i d_i x^i + kw,$$

where $j_i = \lfloor \frac{i}{Δ} \rfloor, \lfloor \frac{i}{Δ} \rfloor + 1, \ldots, \lfloor \frac{i}{Δ} \rfloor$ for $1 ≤ i ≤ Δ$ and $k = -\lfloor \frac{Δ}{w} \rfloor, -\lfloor \frac{Δ}{w} \rfloor + 1, \ldots, \lfloor \frac{Δ}{w} \rfloor$ for parameters $d_i, i = 1, 2, \ldots, Δ$ and $w$ to be specified later. Note that by our construction, any point in the unit square is covered by $t = Θ(1/(2^{Ω(Δ)} \prod_{i=1}^{Δ} d_i))$ polynomial slabs. To see this, consider each polynomial of form

$$P'(x) = \sum_{i=1}^{Δ} j_i d_i x^i,$$

where $j_i = \lfloor \frac{i}{Δ} \rfloor, \lfloor \frac{i}{Δ} \rfloor + 1, \ldots, \lfloor \frac{i}{Δ} \rfloor$ for $1 ≤ i ≤ Δ$. By shifting $P'(x)$ vertically with distance $kw$ for $k = -\lfloor \frac{Δ}{w} \rfloor, -\lfloor \frac{Δ}{w} \rfloor + 1, \ldots, \lfloor \frac{Δ}{w} \rfloor$, we generate a series of adjacent 2D polynomial slabs. For $x ∈ [0, 1]$, $0 ≤ P'(x) ≤ Δ$ for each $P'(x)$. The maximum value is achieved by picking $j_i = \lfloor 1/d_i \rfloor$ for each $i = 1, 2, \ldots, Δ$. Observe that $S$ is completely contained between polynomials $y = P'(x) - \lfloor \frac{Δ}{w} \rfloor w$ and $y = P'(x) + \lfloor \frac{Δ}{w} \rfloor w$. As a result, for each $P'(x)$ (i.e., each choice of indices $j_i$’s), the square $S$ is covered exactly once. This implies $S$ is covered by $t = Π_{i=1}^{Δ} (\lfloor \frac{Δ}{w} \rfloor - \lfloor \frac{i}{Δ} \rfloor + 1) = Ω(1/(2^{Ω(Δ)} \prod_{i=1}^{Δ} d_i))$ times, which is the number of choices we have for the indices $j_1, \ldots, j_Δ$. Therefore, each point in $S$ is covered by $t$ polynomial slabs.

We set $d_i = c_1 \left( \frac{1}{Q(n)^{2/(Δ+Δ+1)}} \right)^{1/t}$ for some sufficiently small constant $c_1$ such that $S$ is covered by $t ≥ Q(n)$ polynomial slabs. We then set $w = c_2 \frac{Q(n)}{n^2}$ for some suitable constant $c_2$ according to $c_1$ such that the total number of polynomial slabs we have generated is exactly $n$.

By Lemma 3.1 and a similar argument as in the proof of Theorem 3.1, the intersection area of any two polynomial slabs in our construction is bounded by

$$u ≤ O(Δ)w(Δ+1)^{1/t} \left( \frac{w}{d_i} \right)^{1/t} = f(Δ) \frac{Q(n)^{1+Δ/2}}{n^{1+Δ/2}}.$$
where \( f(\Delta) = O(1) \) is some value depending on \( \Delta \) only and \( i = 1, 2, \ldots, \Delta \) is the largest degree where the two polynomials differ in their coefficients. Since \( t \geq Q(n) \), then according to Theorem 2.2,

\[
S(n) = \Omega \left( \frac{t}{q} \right) = \Omega \left( \frac{n^{1+\frac{2}{\Delta+2}}}{Q(n)^{\frac{1}{2}}} \right).
\]

So for any data structure that solves the 2D polynomial slab stabbing problem using \( S(n) = O(n) \) space, Theorem 3.2 implies that its query time must be \( Q(n) = \Omega(n^{1-/(\Delta+1)}) \).

4 2D ANNULUS REPORTING AND STABBING

4.1 2D Annulus Reporting

In this subsection, we show that any data structure that solves 2D annulus reporting with \( 2D \) query time must be \( O_1 \) space. The proof sketch.

**Lemma 4.1.** Consider two annuli of width \( w \) with inner radii of \( r_1, r_2 \), where \( r_1 + w \leq r_2, w < r_1 \), and \( r_1, r_2 = \Theta(n) \). Let \( d \) be the distance between the centers of two annuli. When \( w \leq d < r_2 \), the intersection area of two annuli is bounded by \( O(\frac{w^2}{\sqrt{(g+wd)}}) \), where \( g = \max\{r_1 - r_2 + d, 0\} \).

**The proof sketch.** For the complete proof see Appendix A. When \( w \leq d \leq r_2 - r_1 + 2w \), the intersection region consists of two triangle-like regions. We only bound the triangle-like region \( \Delta PQR \) in the upper half annuli as shown in Figure 1. We can show that its area is asymptotically upper bounded by the product of its base length \( |QR| = O(w) \) and its height \( h \). We bound \( h \) by observing that \( \frac{bd}{2} \) is the area of triangle \( \Delta PO_1O_2 \) but we can also obtain its area of using Heron’s formula, given its three side lengths. This gives \( h = O(n\sqrt{w/d}) \). Since in this case \( g \leq 2w \), the intersection area is upper bounded by \( O\left(wn\sqrt{\frac{w^2}{(g+wd)}}\right) \) as claimed.

![Fig. 1. Intersections When d is Small](image)

When \( r_2 - r_1 + 2w \leq d \leq r_2 \), the intersection region consists of two quadrilateral-like regions. Again we only consider \( \tilde{\Delta}ABCD \) in the upper half of the annuli, which is contained in a partial annulus, \( \tilde{\mathcal{R}}_{EFHD} \), as shown in Figure 2a.
We show the area of $\tilde{\mathcal{R}}_{EFHD}$ is asymptotically bounded by $|BH| \cdot w$, where $|BH|$ is the distance between the two endpoints of the inner arc. We upper bound $|BH|$ by $|BD|$. We use the algebraic representation of the two annuli, to bound the length of the projection of $BD$ on the $X$-axis by $\Theta\left(\frac{wT}{T}\right)$; See Figure 2b. We use Heron’s formula to bound the length of the projection of $BD$ on the $Y$-axis by $O\left(n\sqrt{w^2/dg}\right)$. The maximum of the length of the two projections yields the claimed bound.

We use Chazelle’s framework to obtain a lower bound for 2D annulus reporting. Let $S_1$ and $S_2$ be two squares of side length $n$ that are placed $10n$ distance apart and $S_2$ is directly to the left of $S_1$. We generate the annuli as follows. We divide $S_1$ into a $\frac{T}{n} \times \frac{T}{n}$ grid where each cell is a square of side length $T$. For each grid point, we construct a series of circles as follows. Let $O$ be a grid point. The first circle generated for $O$ must pass through a corner of $S_2$ and not intersect the right side of $S_2$, as shown in Figure 3. Then we create a series of circles centered at $O$ by increasing the radius by increments of $w$, for some $w < T$, as long as it does not intersect the left side of $S_2$. Every consecutive two circles defines an annulus centered on $O$. We repeat this for every grid cell in $S_1$ and this makes up our set of queries. The input points are placed uniformly randomly inside $S_2$.

We now show that for the annuli we constructed, the intersection of $\ell$ annuli is not too large, for some $\ell$ we specify later. More precisely we prove the following.

**Lemma 4.2.** There exists a large enough constant $c$ such that in any subset of $\ell = \frac{cw^2}{\sqrt{T}}$ annuli, we can find two annuli such that their intersection has area $O\left(nw\sqrt{\frac{T}{T}}\right)$.

**The proof sketch.** For the complete proof see Appendix B. Let $S$ be a set of $\ell = \frac{cw^2}{\sqrt{T}}$ annuli. Suppose for the sake of contradiction that we cannot find two annuli in $S$ whose intersection area is $O\left(nw\sqrt{\frac{T}{T}}\right)$. Since by Lemma 4.1, the intersection area of any two annuli in our construction with distance $\Omega(wT)$ is $O\left(nw\sqrt{\frac{T}{T}}\right)$. The maximum distance between any two annuli in $S$ must be $o(wT)$. 

---

**Fig. 2.** Cover a Quadrilateral-like Region by a Partial Annulus

**Fig. 3.** Generate a Family of Annuli at Point $O$
Let $P$ be a point in the intersection of annuli in $S$. Consider an arbitrary annulus $A_1 \in S$ centered at $O_1$ and another annulus $A_2 \in S$ centered at $O_2$ for some $O_2 \notin PO_1$. For $A_1, A_2$ to contain $P$, we must have $|PO_1| = r_1 + a, |PO_2| = r_2 + b$ for $0 \leq a, b \leq w$. See Figure 4 for an example. Also $|O_1O_2| = d$, by exploiting the shape of $\triangle PO_1O_2$ and applying Lemma 4.1, we can compute an upper bound for the distance between $O_2$ and $PO_1$, namely, $h = d \sin \alpha = o(w \sqrt{T})$, where $\alpha$ is the angle between $O_1O_2$ and $PO_1$. This implies that $S$ must fit in a rectangle of size $o(w^2) \times o(w \sqrt{T})$. Since the grid cell size is $T \times T$, only $o(w^2 / \sqrt{T})$ annuli are contained in such a rectangle, a contradiction. 

We are now ready to plug in some parameters in our construction. We set $T = w^2 2^{\sqrt{\log n}}$. First, we claim that from each grid cell $O$, we can draw $\Theta(n/w)$ circles; let $C_1, C_2, C_3$, and $C_4$ be the corners of $S_2$ sorted increasingly according to their distance to $O$. As $S_1$ and $S_2$ are placed $10w$ distance apart, an elementary geometric calculation reveals that $C_1$ and $C_2$ are vertices of the right edge of $S_2$, meaning, the smallest circle that we draw from $O$ passes through $C_2$ and we keep drawing circles, by incrementing their radii by $w$ until we are about to draw a circle that is about to contain $C_3$. We can see that $|OC_3| - |OC_2| = \Theta(n)$ and thus we draw $\Theta(n/w)$ circles from $O$. As we have $\Theta((n/T)^2)$ grid cells, it thus follows that we have $\Theta(n^3 / (T^2w))$ annuli in our construction.

Also by our construction, the area of each annulus within $S_2$ is $\Theta(wn)$. To see this, let $P$ be an arbitrary point in $S_1$, let $A, B$ be the intersections of some circle centered at $P$ as in Figure 5.

We connect $AB$ and let $C$ be the center of $AB$. Let $\alpha = \angle APC$. In the triangle $\triangle ABP$, all the sides are within constant factors of each other and thus $\alpha = \Theta(1)$ and so the area of the annulus inside $S_2$ is at least a constant fraction of the area of the entire annulus.

Suppose we have a data structure that answers 2D annulus reporting queries in $O(n) + O(k)$ time. We set $w = c'Q(n)$ for a large enough constant $c'$ such that the area of each annulus within $S_2$ is at least $\Theta(wn) > 8nQ(n)$. By Lemma 2.2, if we sample $n$ points uniformly at random in $S_2$, then with probability more than $1/2$, each annulus contains at least $Q(n)$ points.

Also by our construction, the total number of intersections of two annuli is bounded by $O(n^9)$ and by our choice of $T$, $O\left(nw^{\sqrt{1/T}} \right) = O\left(\frac{n}{2^{\sqrt{\log n}}} \right)$. Then by Lemma 2.1 and Lemma 4.2, with probability $> \frac{1}{2}$, a point set of size $n$ picked uniformly at random in $S_2$ satisfies that the number of points in any of the intersection of $cw^2 / \sqrt{T}$ annuli is no more than $9\sqrt{\log n}$.
Now by union bound, there exist \( \Theta \left( \frac{n^3}{wT^2} \right) \) point sets such that each set is the output of some 2D annulus query and each set contains at least \( Q(n) \) points. Furthermore, the intersection of any \( cw^2/\sqrt{T} \) sets is bounded by \( 9\sqrt{\log n} \). Then by Theorem 2.1, we obtain a lower bound of

\[
S(n) = \Omega \left( \frac{Q(n)n^3\sqrt{T}}{wT^2w^22^\Theta(\sqrt{\log n})} \right) = \Omega \left( \frac{n^3}{Q(n)^3} \right).
\]

This proves the following theorem about 2D annulus reporting.

**Theorem 4.1.** Any data structure that solves 2D annulus reporting on point set of size \( n \) with query time \( Q(n) + O(k) \), where \( k \) is the output size, must use \( \Omega \left( \frac{n^3}{Q(n)^3} \right) \) space.

So for any data structure that solves 2D annulus reporting in time \( Q(n) = \log^{O(1)} n \), Theorem 4.1 implies that \( \Omega \left( \frac{n^3}{Q(n)^3} \right) \) space must be used.

### 4.2 2D Annulus Stabbing

Modifications similar to those done in Subsection 3.2 can be used to obtain the following lower bound.

**Theorem 4.2.** Any data structure that solves the 2D annulus stabbing problem with query time \( Q(n) + O(k) \), where \( k \) is the output size, must use \( S(n) = \Omega(n^{3/2}/Q(n)^{3/4}) \) space.

**Proof.** We use Afshani’s lower bound framework as in Theorem 2.2. We construct annuli similar to the way as we did in the proof of Theorem 4.1, but with some differences. Let \( S_1 \) be a unit square, we decompose \( S_1 \) into a grid where each grid cell is a square of size \( T \times T \), for some parameter \( T \) to be determined later. Let \( S_2 \) be another unit square to the left of \( S_1 \) with distance 10. For each grid point in \( S_1 \), we generate a family of circles where the first circle is the first one tangent to the right side of \( S_2 \). See Figure 6 for an example. Let \( r_0 \) be the radius of this circle. We generate other circles by increasing the radius by \( w \) each time. The last circle is the one with radius at least \( r_0 + (\sqrt{122} - 9) \). The choice of constant \( \sqrt{122} - 9 \) is due to the following. Consider a point \( P \) in the upper half of \( S_1 \), let \( a \) (resp. \( b \)) be the distance from \( P \) to the upper (resp. left) side of \( S_1 \). The case for the lower half of \( S_1 \) is symmetric. Consider the last circle intersecting \( S_2 \) only at the corners. See Figure 7 for an example. The difference of radius lengths between the last circle and the first circle is

\[
f = \sqrt{(1-a)^2 + (11+b)^2} - 10 - b.
\]

![Fig. 6. Generate a Family of Annuli at Point O](image)

![Fig. 7. The Last Valid Circle](image)
Simple analysis shows that $f \leq \sqrt{122} - 10$. This shows as long as the last circle has radius at least $\sqrt{122} - 9 + r_0$, it is completely outside of $S_2$. Like we did previously, we consider the region between two consecutive annuli to be an annulus. Note that by our construction, for any point in $S_2$, it is contained in one of the annuli we generated for a given grid point. The total number of annuli we have generated is therefore

$$\left( \frac{1}{T} + 1 \right)^2 \cdot \frac{\sqrt{122} - 9}{w}.$$ 

We set $T = 1/(2\sqrt{Q(n)} - 1)$ and $w = 4(\sqrt{122} - 9)Q(n)/n$. Then the total number of annuli we have generated is $n$. Furthermore, each query point is contained in $t = (1/T + 1)^2 \geq Q(n)$ annuli.

To use Lemma 4.1, we require $w \leq T$, which implies $Q(n) = O(n^{2/3})$, the intersection area of two annuli in our construction is upper bounded by $v = O(w/\sqrt{T}) = O(Q(n)^{7/4}/n^{3/2})$. Then by Theorem 2.2,

$$S(n) = \Omega \left( \frac{f}{\delta} \right) = \Omega \left( \frac{n^{3/2}}{Q(n)^{3/4}} \right).$$

So for any data structure that solves the 2D annulus stabbing problem using $O(n)$ space, Theorem 4.2 implies that its query time must be $Q(n) = \Omega(n^{2/3})$.

5 CONCLUSION AND OPEN PROBLEMS

We investigated lower bounds for range searching with polynomial slabs and annuli in $\mathbb{R}^2$. We showed space-time tradeoff bounds of $S(n) = \tilde{O}(n^{\Delta+1}/Q(n)^{(\Delta+3)/2})$ and $S(n) = \tilde{O}(n^{3/2}/Q(n)^{3/4})$ for them respectively. Both of these bounds are almost tight in the “fast query” case, i.e., when $Q(n) = \log^{O(1)} n$ (up to a $n^{O(1)}$ factor). This refutes the conjecture of the existence of data structure that can solve semialgebraic range searching in $\mathbb{R}^d$ using $\tilde{O}(n^d)$ space and $\log^{O(1)} n$ query time. We also studied the “dual” polynomial slab stabbing and annulus stabbing problems. For these two problems, we obtained lower bounds $S(n) = \tilde{O}(n^{3+2(\Delta+1)}/Q(n)^{2/3})$ and $S(n) = \tilde{O}(n^{3/2}/Q(n)^{3/4})$ respectively. These bounds are tight when $S(n) = O(n)$. Our work, however, brings out some very interesting open problems.

To get the lower bounds for the polynomial slabs, we only considered univariate polynomials of degree $\Delta$. In this setting, the number of coefficients is at most $\Delta + 1$, and we have also assumed they are all independent. It would be interesting to see if similar lower bounds can be obtained under more general settings. In particular, as the maximum number of coefficients of a bivariate polynomial of degree $\Delta$ is $\binom{\Delta + 2}{2}$, it would interesting to see if a $\tilde{O}(n^{(\Delta+2)/2})$ space lower bound can be obtained for the “fast query” case.

It would also be interesting to consider space-time trade-offs. For instance, by combining the known “fast query” and “low space” solutions for 2D annulus reporting, one can obtain data structures with trade-off curve $S(n) = \tilde{O}(n^3/Q(n)^4)$, however, our lower bound is $S(n) = \tilde{O}(n^{3/2}/Q(n)^{3/4})$ and it is not clear which of these bounds is closer to truth. For the annulus searching problem in $\mathbb{R}^2$, in our lower bound proof, we considered a random input point set, since in most cases a random point set is the hardest input instance and our analysis seems to be tight, we therefore conjecture that our lower bound could be tight, at least when $Q(n)$ is small enough. We believe that it should be possible to obtain the trade-off curve of $S(n) = \tilde{O}(n^3/Q(n)^5)$ when the input points are uniformly random in the unit square and $Q(n)$ is not too big.

Finally, another interesting direction is to study the lower bound for the counting variant of semialgebraic range searching.

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A PROOF OF LEMMA 4.1

Lemma 4.1. Consider two annuli of width w with inner radii of r1, r2, where r1 + w ≤ r2, w < r1, and r1, r2 = O(n). Let d be the distance between the centers of two annuli. When w ≤ d < r2, the intersection area of two annuli is bounded by O \left( \frac{w}{\sqrt{g (1 + w/d)}} \right), where g = \max \{r_1 - r_2 + d, 0\}.

Proof. First note that for two annuli to intersect, d ≥ r_2 - r_1 - w. When w ≤ d ≤ r_2 - r_1 + 2w, observe that g ≤ 2w and the intersections are all bounded by a two triangle-like regions. We consider the triangle like region \triangle PQR in the upper half of the annuli as shown in Figure 8. The area of the other triangle like region can be bounded symmetrically.

To upper bound its area, we first compute its "height" h. Consider \triangle PO_1O_2, by Heron's Formula, its area is

\[ A_{\triangle PO_1O_2} = \sqrt{\frac{r_1 + r_2 + w + d}{2} \cdot \frac{r_2 + d - r_1 - w}{2} \cdot \frac{r_1 + d + w - r_2}{2} \cdot \frac{r_1 + r_2 + w - d}{2}} \]

\[ = O(\sqrt{r_2 \cdot d \cdot w \cdot r_1}) \]

\[ = O\left( n \sqrt{dw} \right), \]

Fig. 8. Intersections When d is Small
where the second equality follows from
\[d \leq r_2 - r_1 + 2w \implies r_1 + r_2 + w + d \leq 2r_2 + 3w \leq 5r_2,\]
\[r_2 - r_1 - w \leq d \implies r_2 + d - r_1 - w \leq 2d,\]
\[d \leq r_2 - r_1 + 2w \implies r_1 + d - r_2 + w \leq 3w,\]
\[r_2 - r_1 - w \leq d \implies r_1 + r_2 + w - d \leq 2r_1 + 2w \leq 4r_1.\]

Then we bound \(h\) by
\[h = \frac{2A_{\Delta PQR}}{d} = O\left(n\sqrt{\frac{w}{d}}\right).
\]
To bound the area of \(\Delta PQR\), we move the outer circle of the annulus centered at \(O_1\) along \(O_1O_2\) such that it passes \(R\) as in Figure 9. Let \(S\) be a point in this new circle such that \(PS \parallel QR\). The area of region \(PQRS\) formed by the outer circle of the annulus centered at \(O_1\) and the new circle as well as \(PS\) and \(QR\), i.e., the shaded region in Figure 9, is \(O(wh)\) by a simple integral argument. Since the curvature of the inner circle of the annulus centered at \(O_2\) is no larger than that of the outer circle of \(O_1\) since \(r_2 \geq r_1 + w\), \(\Delta PQR\) is contained in this region. So \(A_{\Delta PQR} = O(wh) = O(nw\sqrt{w/d}) = O(nw\sqrt{w^2/(g + w)d}).\)

\[\text{Fig. 9. Bound the Intersection Area for Small } d\]

In the case \(r_2 - r_1 + 2w < d < r_2\), the intersections consist of two quadrilateral-like regions, one at the upper half and the other at the lower half. In this case, \(g > 2w\). We show how to bound the area of the quadrilateral-like region at the upper half, the other one can be bounded symmetrically.

First note that each quadrilateral-like region \(\Delta ABCD\) is contained in a partial annulus as in Figure 10. We generate this partial annulus by shooting a ray from the center \(O_1\) and connect a point \(P\) in \(AB\), such that \(\Delta ABCD\) is completely to the right of this ray, to create the left boundary \(EF\), where \(E\) is a point in the outer circle of the annulus \(\mathcal{A}_1\) centered at \(O_1\) and \(F\) is a point in the inner circle of annulus \(\mathcal{A}_1\). And similarly, shoot another ray and connect a point \(Q\) in \(CD\) to create the right boundary \(GH\). Again, \(G\) is a point in the outer circle of annulus \(\mathcal{A}_1\) and \(H\) is a point in the inner circle of annulus \(\mathcal{A}_1\). Note that since \(d < r_2\), by elementary geometry, \(P\) can only be \(B\) and \(Q\) can only be \(D\) as in Figure 10a.

First, we observe that the area of the partial annulus is bounded by the product of the width of the annulus, \(w\), and the length \(FH\), within some constant factor.

To see this, consider the possible partial annulus generated by any two annuli, see Figure 10b for an example. One important observation is that for any intersection, the angle \(\alpha\) of the partial annulus containing it is no more than \(\pi\). The area of the partial annulus is clearly \(a(\alpha w + w^2/2) = \Theta(rw\alpha)\). Note that for \(0 \leq \alpha \leq \pi\), let \(\beta = \alpha/2\), it is a simple fact that
\[\frac{1}{2}\beta \leq \sin \beta \leq \beta\]
for \(0 \leq \beta \leq \pi/2\). So for \(\beta\) in this range, \(\sin \beta = \Theta(\beta)\). Thus we can compute \(|FH| = 2r \sin \beta = \Theta(r\alpha)|\) which implies that indeed \(w \cdot |FH|\) gives us the area of the partial annulus, within some constant factor.

Now we proceed to bound the length of \(|FH|\). Note that \(|FH| \leq |BD|\) because \(\angle BHD = \frac{\pi}{2}\). By Lemma A.1, we know \(|BD| = O\left(n\sqrt{\frac{w^2}{dg}}\right)\). Therefore, the area of intersection is bounded by \(O\left(wn\sqrt{\frac{w^2}{(gw)dg}}\right)\) as claimed. \(\square\)
Lemma A.1. In \( \mathbb{R}^2 \), given two annuli centered at \( O_1, O_2 \) of width \( w \) with inner radius lengths being \( r_1, r_2 \) respectively, where \( r_1 + w \leq r_2 \), \( w < r_1 \), and \( r_1, r_2 = \Theta(n) \). Let \( d \) be the distance between two centers of the annuli and \( g \) be the distance between two inner circles of two annuli along the direction of \( O_1O_2 \). Consider the quadrilateral-like region \( \tilde{\square}ABCD \) formed by four arcs when \( r_2 - r_1 + 2w < d < r_2 \), we have \( |BD| = O\left( \frac{\sqrt{w}}{g^2} \right) \).

Proof. First note that \( g \) is the distance between the two inner circles along \( O_1O_2 \), as shown in Figure 11.

We first compute \( |BT| = x_{BD} = x_D - x_B \). For point \( B = (x_B, y_B) \), it satisfies

\[
\begin{cases}
  x^2 + y^2 = r_1^2 \\
  (x - d)^2 + y^2 = (r_2 + w)^2.
\end{cases}
\]

So \( x_B \) satisfies

\[
2dx - d^2 = r_1^2 - (r_2 + w)^2,
\]

which implies

\[
x_B = \frac{r_1^2 - (r_2 + w)^2 + d^2}{2d}. \tag{1}
\]

Similarly, we obtain

\[
x_D = \frac{(r_1 + w)^2 - r_2^2 + d^2}{2d}. \tag{2}
\]
Therefore, by Equation 3 and Equation 4, we have

\[ |BT| = x_{BD} = x_D - x_B = \frac{w(r_1 + r_2 + w)}{d} = \Theta\left(\frac{wn}{d}\right). \]

Now we compute |DT| = y_{BD} = h_1 - h_2. Let \( \mu_1 \) be the area of \( \Delta DO_1O_2 \) and \( \mu_2 \) be the area of \( \Delta BO_1O_2 \).

By Heron’s formula,

\[
\begin{cases}
\mu_1 = \sqrt{\frac{r_1 + w + r_2 + d}{2} \cdot \frac{r_2 + d - r_1 - w}{2} \cdot \frac{r_1 + w + r_2 - d}{2}} \\
\mu_2 = \sqrt{\frac{r_1 + r_2 + w + d}{2} \cdot \frac{r_2 + d - r_1 - w}{2} \cdot \frac{r_1 + d - r_2 - w}{2}}
\end{cases}
\]

So

\[
\frac{\mu_1}{\mu_2} = \sqrt{\frac{(r_2 + d - r_1 - w)(r_1 + w + d - r_2)}{(r_2 + w + d - r_1)(r_1 + d - r_2 - w)}}.
\]

Observe that \( g = r_1 + d - r_2 \) and so

\[
\frac{\mu_1}{\mu_2} = \sqrt{\frac{(2d - g - w)(g + w)}{(2d - g + w)(g - w)}} = \sqrt{\frac{(X + 2w)Y}{X(Y + 2w)}} = \sqrt{\frac{1 + 2w(Y - X)}{XY + 2wX}}.
\]

where \( X = g - w \) and \( Y = 2d - g - w \). Since in this case, \( d \geq r_2 - r_1 + 2w \geq 3w \), and so \( g = r_1 + d - r_2 \geq 2w \) this implies \( X = g - w = \Theta(g) \).

On the other hand, \( Y - X = 2(d - g) = 2(r_2 - r_1) > 0 \) and also we have \( 3w \leq d \leq Y \). Therefore,

\[
\frac{\mu_1}{\mu_2} = \sqrt{1 + O\left(\frac{w}{g}\right)}.
\]

Note that

\[
\begin{cases}
\mu_1 = \frac{h_1 d}{2} \\
\mu_2 = \frac{h_2 d}{2}
\end{cases}
\]

Since \( 2w \leq g \), we therefore have

\[
\frac{h_1}{h_2} = \frac{\mu_1}{\mu_2} = \sqrt{1 + O\left(\frac{w}{g}\right)} = 1 + O\left(\frac{w}{g}\right) \quad \Rightarrow \quad \frac{h_2}{h_1} = 1 - O\left(\frac{w}{g}\right).
\]

By Heron’s formula, we obtain

\[
h_1 = O\left(n\sqrt{\frac{g}{d}}\right).
\]

So

\[
|DT| = y_{BD} = h_1 - h_2 = h_1 \left(1 - \frac{h_2}{h_1}\right) = h_1 \cdot O\left(\frac{w}{g}\right) = O\left(n\sqrt{\frac{w^2}{dg}}\right).
\]

Therefore, by Equation 3 and Equation 4, we have

\[
|BD| \leq |BT| + |DT| = O\left(n\sqrt{\frac{w^2}{dg}}\right). \tag{4}
\]

\[\Box\]

**B \ PROOF OF LEMMA 4.2**

**Lemma 4.2.** There exists a large enough constant \( c \) such that in any subset of \( \ell = cw^2/\sqrt{T} \) annuli, we can find two annuli such that their intersection has area \( O\left(nw\sqrt{\frac{T}{T}}\right) \).

Proof. We consider for the sake of contradiction that in our construction for every set \( S \) of \( \ell = cw^2/\sqrt{T} \) annuli for any positive constant \( c \), we cannot find two annuli in \( S \) such that their intersection area is \( O(nw\sqrt{\frac{T}{T}}) \).

First observe that by our construction, any two annuli from the same family have zero intersection. Also the distance between the centers of any two annuli is less than the radius of any annulus. Furthermore, the width of an annulus is always less than the distance between the centers of any two annuli. So by Lemma 4.1, the intersection area of any two annuli in our construction with distance \( \Omega(wT) \) is \( O(nw\sqrt{\frac{T}{T}}) \).

So for our assumption to hold, the distance between the centers of any two annuli in \( S \) is \( o(wT) \).
Let $P$ be a point in the intersection of $S$, then $P$ is contained in every annulus in $S$. Now consider an arbitrary annulus $R_1 \in S$ centered at $O_1$ and another annulus $R_2 \in S$ centered at $O_2$ for some $O_2$ not in line $PO_1$. Connect $PO_1$ and $PO_2$, for $R_1, R_2$ to contain $P$, we must have $|PO_1| = r_1 + a$ and $|PO_2| = r_2 + b$ for $0 \leq a, b \leq w$ as shown in Figure 12.

We first consider the case when the distance between the centers of $R_1$ and $R_2$ is no more than $r_2 - r_1 + 2w$. In this case, their intersection area is upper bounded by $O(nw \sqrt{w/d})$ according to Lemma 4.1.

By Heron’s formula,

\[
\text{Area}_{\triangle PO_1 O_2} = \sqrt{\frac{r_1 + a + r_2 + b + d}{2} \cdot \frac{r_2 - r_1 + b - a + d}{2} \cdot \frac{r_1 + a - r_2 - b + d}{2} \cdot \frac{r_1 + a + r_2 + b - d}{2}}
\]

\[= O(r_1 \sqrt{dw})\]

\[= \Theta(r_1 h),\]

where the second equality follows from $a, b \leq w \leq d$ and $r_2 - r_1 - w \leq d \leq r_2 - r_1 + 2w$, and $h$ is the distance between $O_2$ and line $PO_1$. This implies $h = O(\sqrt{d} w)$. Since $d = o(wT)$, we obtain that

\[h = o(w \sqrt{T}).\] (5)

Now we consider the case when the distance between the centers of $R_1$ and $R_2$ is more than $r_2 - r_1 + 2w$. See Figure 13 for an example.

Let $g = r_1 + d - r_2$ be the distance between the inner circle of $R_1$ and the inner circle of $R_2$. Let $\alpha$ be the angle between $O_1 O_2$ and $PO_1$. Note that $\alpha > 0$ since $O_2$ is not in $PO_1$. W.l.o.g., we assume $0 < \alpha \leq \pi/2$. The situations for other values of $\alpha$ are symmetric. Then

\[\cos(\pi - \alpha) = \frac{(r_1 + a)^2 + d^2 - (r_2 + b)^2}{2d(r_1 + a)}.\] (6)
Since $a, b < g = r_1 + d - r_2 < d < r_2$, we obtain from (6) that

$$g = \Theta(d(1 - \cos \alpha)).$$

(7)

So according to Lemma 4.1 the intersection area of $R_1, R_2$ is upper bounded by

$$A = O\left(wn \sqrt{\frac{w^2}{dg}}\right).$$

Let $A = \omega\left(nw \sqrt{\frac{T}{T}}\right)$, by equation (7), we get

$$d^2(1 - \cos \alpha) = o\left(w^2T\right),$$

(8)

Since $d = \frac{h}{\sin \alpha} = \frac{h}{\sqrt{1 - \cos^2 \alpha}}$, and $0 < \alpha \leq \pi/2$, we plug $d$ in inequality (8) and obtain

$$\frac{h^2}{1 - \cos^2 \alpha} (1 - \cos \alpha) = \frac{h^2}{1 + \cos \alpha} = o\left(w^2T\right).$$

This implies

$$h = o\left(\sqrt{w^2T(1 + \cos \alpha)}\right) = o\left(w\sqrt{T}\right).$$

(9)

So in order to have no two annuli having intersection area $O(nw \sqrt{1/T})$, the distance between the centers of annuli in $S$ and $PO_1$ must be $o(w\sqrt{T})$. We have already shown that the distance between any two centers is $o(wT)$. Together they imply that the centers of $S$ fit in a rectangle of shape $o(wT) \times o(w\sqrt{T})$. However, in our construction, each grid cell is of size $T \times T$, this implies we only have $o(w^2/\sqrt{T})$ centers in the rectangle. But we should have $cw^2/\sqrt{T}$ centers, a contradiction.

□