DIMENSION OF THE MODULI SPACE OF A GERM OF CURVE IN $\mathbb{C}^2$.

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Abstract. In this article, we prove a formula that computes the generic dimension of the moduli space of a germ of irreducible curve in the complex plane. It is obtained from the study of the Saito module associated to the curve, which is the module of germs of holomorphic 1-forms letting the curve invariant.

1. Introduction

In 1973, in its lecture [26], Zariski started the systematic study of the analytic classification of the branches of the complex plane, which are germs of irreducible curves at the origin of $\mathbb{C}^2$. The general purpose was to describe as accurately as possible the moduli space of $S$ that is the quotient of the topological class of $S$ by the action of the group $\text{Diff} (\mathbb{C}^2, 0)$,

$$M (S) = \{ S' | S' \text{ topologically equivalent to } S \} / \text{Diff} (\mathbb{C}^2, 0)$$

The Puiseux parametrization of a branch $S = \{ \gamma (t) | t \in (\mathbb{C}, 0) \}$ written

$$\gamma : \begin{cases} x = t^p \\ y = t^q + \sum_{k > q} a_k t^k \end{cases} \quad , \quad p < q, \ p \nmid q, \ t \in (\mathbb{C}, 0)$$

(1.1)

highligths two basic topological invariants, namely the integers $p$ and $q$. In the whole article, we will denote them by $p (S)$ and $q (S)$, or simply, $p$ and $q$ when no confusion is possible. The integer $p (S)$ corresponds to the algebraic multiplicity of the branch $S$. This is also the algebraic multiplicity at $(0, 0)$ of any irreducible function $f \in \mathbb{C} \{ x, y \}$ that vanishes along $S$. Actually, Zariski proved that the whole topological classification depends on a sub-semigroup $\Gamma_S$ of $\mathbb{N}$ defined by

$$\Gamma_S = \{ \nu (f \circ \gamma) | f \in \mathbb{C} \{ x, y \} , \ f (0) = 0 \}$$

where $\nu$ is the standard valuation of $\mathbb{C} \{ t \}$.

Beyond the topological classification, Zariski proposed in [26] various approaches to achieve the analytical classification, introducing in particular the set $\Lambda_S$ of valuations of Khler differential forms for $S$

$$\Lambda_S = \{ \nu (\gamma^* \omega) + 1 | \omega \in \Omega^1 (\mathbb{C}^2, 0) \} \supset \Gamma_S$$

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Fixing the topological - and thus the semigroup $\Gamma_S$ above -, Zariski gave a precise
description of the associated moduli space for, for instance,
$$\Gamma_S = \langle 2, 3 \rangle, \langle 4, 5 \rangle, \langle 4, 6, \beta_2 \rangle$$
or more generally $\langle n, n + 1 \rangle$ and $\langle n, hn + 1 \rangle$. According to him, is of special interest,
the generic component of the moduli space: a finite determinacy property ensures
that $\gamma$ is analytically equivalent to a parametrization whose Taylor expansion is
truncated at an order depending on the sole topological class. Having so a finite
dimension family of branches, the theory of geometric invariant provides an open
set of orbits of same dimension under the action of $\text{Diff}(\mathbb{C}^2, 0)$. The image of this
open set in the moduli space is the generic component studied by Zariski. In some
sense, its dimension is the minimal number of parameters on which a universal
family for the deformation of $S$ depends. In the particular cases mentioned above,
Zariski found an explicit formula of this dimension.

In fact, as far as we know, the first example of computation of the dimension
of the generic component of the moduli space of a branch goes back to Ebey [5]
who, anticipating in 1965 some ideas of Zariski, described not only the generic
component, but the whole moduli space of the branch whose semigroup is $\langle 5, 9 \rangle$.
In 1978, Delorme [4] studied extensively the case of one Puiseux pair - $\Gamma_S = \langle m, n \rangle$
with $m \wedge n = 1$ - and established some formulas to compute the generic dimension.
In 1979, Granger [11] and later, in 1988, Brianon, Granger and Maisonobe [2]
produced an algorithm to compute the generic dimension of the moduli space of a
non irreducible quasi-homogeneous curve defined by $x^m + y^n = 0$ first, for $m$ and $n$
relatively prime, and then in the general case. The common denominator of the two
previous works is the algorithmic approach based upon arithmetic properties of the
continuous fraction expansion associated to the pair $\langle m, n \rangle$. In 1988, Laudal, Martin
and Pfister in [13], improved the work of Delorme and gave an explicit description
of a universal family for $S$ with $\Gamma_S = \langle m, n \rangle$, $m \wedge n = 1$ and a stratification of the
moduli space. Finally, in 1998, Peraire exhibited an algorithm in [23] to compute the
Tijuna number for a curve in its generic component when $\Gamma_S = \langle m, n \rangle$, $m \wedge n = 1$,
which is linked to the dimension of the generic component.

From 2009, in a series of papers [14, 15, 16]. Hefez and Hernandes achieved a break-
through in the problem of Zariski. They completed the analytical classification of
irreducible germs of curves thanks to the set of valuations of Kähler differential
forms. Moreover, they built an algorithm that describes very precisely the stratifi-
cation of the moduli space in terms of the possible $\Lambda_S$ for a given topological class,
computes the dimension of each stratum and produces some normal forms corre-
sponding to each stratum. One could consider that these works gave a definitive
answer to the initial problem addressed by Zariski. Nevertheless, the disadvantage
of the algorithmic approach is twofold: first, the high complexity of the algorithm -
based upon Groebner basis routine - prevents its actual effectiveness as soon as the
degree of the curve is big. Second, it is difficult - not to say impossible - to extract
general geometric informations or formulas from it.

In 2010 and 2011, in [8, 9], Paul and the author described the moduli space of
a topologically quasi-homogeneous curve $S$ as the spaces of leaves of an algebraic
foliation defined on the moduli of a foliation whose analytic invariant curve is
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precisely \( S \). These works initiated an approach based upon the theory of foliations, which is at stake here.

In this article, we propose a construction relying basically, on one hand, on the desingularization of the curve \( S \), on the other hand, on technics from the framework of the theory of holomorphic foliations. We intend to obtain an explicit formula for the generic dimension of the moduli space - the dimension of the generic stratum -, that can be performed by hand.

Let \( S \) be a germ of irreducible curve in the complex plane.

**Theorem 1.** Let \( E = E_1 \circ \cdots \circ E_N \) be the minimal desingularization of \( S \). Let \( c_i \) be the center of \( E_i \). Then

\[
\dim_{\text{gen}} \mathcal{M}(S) = \sum_{i=1}^{N} \sigma \left( \nu_{c_i} \left( (E_1 \circ \cdots \circ E_{i-1})^{-1}(S) \right) \right)
\]

where \( \nu_{\star} \) is the algebraic multiplicity at \( \star \) and \( \sigma(k) = \begin{cases} \frac{(k-3)^2}{4} & \text{if } k \text{ is odd} \\ \frac{(k-2)(k-4)}{4} & \text{else} \end{cases} \).

Notice that this formula depends only on some topological invariants of the curve \( S \): in particular, it is not necessary to exhibit a curve in the generic component of the moduli space of \( S \) - that is in general difficult - to perform the computation above. One can take any curve in the topological class of \( S \) to compute the multiplicities involved in Theorem 1.

**Example.** In [26], Zariski showed that the dimension of the generic component of the moduli space of \( S = \{y^n - x^{n+1} = 0\} \) is \( \sigma(n) \). After one blowing-up \( E_1 \), the strict transform of \( S \) by \( E_1 \) is a smooth curve tangent to the exceptional divisor, thus for any \( i \geq 2 \), the multiplicity satisfy

\[
\nu_{c_i} \left( (E_1 \circ \cdots \circ E_{i-1})^{-1}(S) \right) \leq 3.
\]

**Example.** More generally, for the semi-group \( \Gamma_S = \langle n, nh + 1 \rangle \) with \( h \geq 1 \), the desingularization of \( S \) consists first in \( h \) successive blowing-ups, after which the curve is smooth. The algebraic multiplicity of the curve \( S \) is \( n \). After \( k \leq h \) blowing-ups, the strict transform of \( S \) is a curve whose topological class is given by the semi-group \( \langle n, n(h-k) + 1 \rangle \) that is transverse to the exceptional divisor. Thus, according to Theorem 1 one has

\[
\dim_{\text{gen}} \mathcal{M}(S_{(n, nh+1)}) = \sigma(n) + \frac{\sigma(n+1) + \cdots + \sigma(n+1)}{h-1} + \sigma(3) + \cdots
\]

\[
= \sigma(n) + (h-1) \sigma(n+1).
\]

This formula coincides with the one in [26].

**Example.** Let us consider the following Puiseux parametrization

\[
S : \begin{cases} x = t^8 \\ y = t^{20} + t^{30} + t^{35} \end{cases}.
\]
Its semigroup is \( \langle 8, 20, 50, 105 \rangle \) and its Puiseux pairs are \( (2,5) \), \( (2,15) \) and \( (2,35) \). Thus, \( S \) is not topologically quasi-homogeneous. The successive multiplicities 
\[
\nu_i \left( (E_1 \circ \cdots \circ E_{i-1})^{-1} (S) \right)
\]
are 
\[
8, 9, 5, 6, 5, 5, 3, \ldots
\]
Thus the generic dimension of the moduli space is 
\[
\sigma (8) + \sigma (9) + \sigma (5) + \sigma (6) + \sigma (5) + \sigma (5) = 20
\]
which is confirmed by the algorithm of Hefez and Hernandes.

The inductive form of the formula in Theorem \( \text{(ii)} \) comes naturally from the inductive structure of the desingularization. At each step, the theory of foliations is involved through the theory of logarithmic vector fields or forms introduced by Saito in 1980 in [24]. Let us consider the set 
\[
\Omega^1 (S)
\]
that invariant \( S, \gamma \omega = 0 \). Saito proved that \( \Omega^1 (S) \) is a free \( \mathcal{O}_2 \)-module of rank 2. If \( f \) is a reduced equation of \( S \), the family \( \{ \omega_1, \omega_2 \} \) is a basis of \( \Omega^1 (S) \) if and only if there exists a germ of unity \( u \in \mathcal{O}, u(0) \neq 0 \) such that the exterior product of \( \omega_1 \) and \( \omega_2 \) is written 
\[
\omega_1 \wedge \omega_2 = u f dx \wedge dy.
\]
In other words, the tangency locus between \( \omega_1 \) and \( \omega_2 \) is reduced to the sole curve \( S \). Beyond this characterization, very few is known about these two generators. At first glance, we can say the following: among all the possible basis \( \{ \omega_1, \omega_2 \} \), there is one for which the sum of the algebraic multiplicities 
\[
\nu (\omega_1) + \nu (\omega_2) \leq \nu (S)
\]
is maximal. It can be seen that

**Proposition.** The couple of multiplicities \( (\nu (\omega_1), \nu (\omega_2)) \), up to order, that maximizes its sum is an analytic invariant of \( S \).

However, these two integers are not topologically invariant and in the topological class of a curve, they may vary widely.

**Example.** Let \( S \) be the curve \( y^6 - x^7 = 0 \). Then the family 
\[
\{ 7xdy - 6ydx, d (y^6 - x^7) \}
\]
is a basis of the Saito module since 
\[
(6xdy - 7ydx) \wedge d (y^6 - x^7) = -42 (y^6 - x^7) dx \wedge dy.
\]
In that case, the couple of valuation is \( (1, 5) \) whose sum is exactly 6. However, perturbing a bit \( S \) leads to different values of the multiplicities. For instance, if \( S \) if the curve \( y^6 - x^7 + x^4y^4 = 0 \) which is topologically but not analytically equivalent to \( y^6 = x^7 \), one can show that the couple 
\[
\omega_1 = \frac{5}{3} x^4 dx - \frac{20}{21} x^2 y^3 dy + \frac{8}{21} xy^3 + y (6xdy - 7ydx)
\]
\[
\omega_2 = \frac{20}{21} x^3 y^3 dx + \left( \frac{10}{7} y^4 - \frac{80}{147} xy^3 \right) dy + \left( x^2 + \frac{32}{147} y^6 \right) (6xdy - 7ydx)
\]
is a basis for \( \Omega^1 (S) \). The multiplicities are respectively 2 and 3 whose sum is strictly smaller than the multiplicity of \( S \). Finally, if \( S \) is given by \( y^6 - x^7 + y^2x^5 = 0 \) then \( S \) admits a basis \( \{ \omega_1, \omega_2 \} \) with \( \nu (\omega_1) = \nu (\omega_2) = 3 \).
Definition. A curve $S$, reducible or not, is said to admit a balanced basis if there exists a basis $\{\omega_1, \omega_2\}$ of $\Omega^1(S)$ with

- $\nu(\omega_1) = \nu(\omega_2) = \frac{\nu(S)}{2}$ if $\nu(S)$ is even,
- $\nu(\omega_1) = \nu(\omega_2) - 1 = \frac{\nu(S) - 1}{2}$ else.

A direction $d$ for $S$ is either an empty set, a smooth germ of curve or the union of two transverse smooth curves. The interest of $d$ will be highlighted in the course of the article. We will denote by $S_d$ the union $S \cup d$.

Theorem 2. For a generic irreducible curve $S$ and any direction $d$, one has

$$\min_{\omega \in \Omega^1(S_d)} \nu(\omega) = \left\lfloor \frac{\nu(S_d)}{2} \right\rfloor$$

where $\lfloor \cdot \rfloor$ stands for the integer part function. Moreover, if $S$ is generic then the curve $S_d$ admits a balanced basis.

The first section of this article is devoted to the proof of Theorem 2. The second focuses on the proof of Theorem 1 as a consequence of Theorem 2.

2. Balanced basis for a generic irreducible curve.

For any basis $\{\omega_1, \omega_2\}$ of $\Omega^1(S_d)$, the criterion of Saito ensures that

$$\nu(\omega_1) + \nu(\omega_2) \leq \nu(S_d).$$

Thus at least one of these multiplicities is smaller or equal to $\left\lfloor \frac{\nu(S_d)}{2} \right\rfloor$, which proves one part of the equality in Theorem 2. However, to obtain the whole equality we will need some more informations about these generators. In this section, we are going to construct quite explicitly an element of $\Omega^1(S_d)$ with multiplicity $\left\lfloor \frac{\nu(S_d)}{2} \right\rfloor$.

One of the purpose of this section is to obtain the following result

Proposition 3. If $\nu(S_d)$ is even or if $d$ is irreducible, then there exists a 1-form of multiplicity $\left\lfloor \frac{\nu(S_d)}{2} \right\rfloor$ in $\Omega^1(S_d)$ whose induced foliation is not dicritical along the exceptional divisor of the standard blowing-up of its singularity.

We recall that a foliation $\mathcal{F}$ is said to be dicritical along a divisor $\Sigma$ if and only if $\mathcal{F}$ is generically transverse to $\Sigma$.

Let us give a sketch of the proof of Proposition 3: first, we construct an auxiliary foliation $\mathcal{F} [S_d]$ tangent to some curve $S_d$ topologically equivalent to $S_d$ with the desired algebraic multiplicity. Then, we study the deformations of $\mathcal{F} [S_d]$ by means of cohomological tools. In particular, considering a deformation linking $S_d$ to $S_d$, we prove that it can be followed by a deformation of $\mathcal{F} [S_d]$ that preserves the algebraic multiplicity. The resulting foliation is tangent to $S_d$ with $\left\lfloor \frac{\nu(S_d)}{2} \right\rfloor$ as algebraic multiplicity. Among other properties, we obtain Proposition 3.
2.1. The auxiliary foliation $\mathcal{F}[S_d]$. In this section, we are going to construct a foliation associated to $S_d$, denoted by $\mathcal{F}[S_d]$, thanks to a result of Alcides Lins-Neto \cite{19, 20} that is a kind of recipe to construct germs of singular foliations in the complex plane.

Let $E$ be the minimal desingularization of $S$. We denote it by

$$E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0).$$

The map $E$ is a finite sequence of elementary blowing-ups of points

$$E = E_1 \circ E_2 \circ \cdots \circ E_N.$$

**Notation.** If $\Sigma$ is a germ of curve at $(\mathbb{C}^2, 0)$ or a divisor, $\Sigma^E$ will stand for the strict transform of $\Sigma$ by $E$, i.e., the closure in $\mathcal{M}$ of $E^{-1}(\Sigma \setminus \{0\})$.

The exceptional divisor of $E$, $D = E^{-1}(0)$, is an union of a finite number of exceptional smooth rational curves intersecting transversely

$$D = \bigcup_{i=1}^N D_i, \quad D_i \simeq \mathbb{P}^1(\mathbb{C}).$$

The components are numbered such that $D_i$ appears exactly after $i$ blowing-ups.

We can encode the map $E$ in a square matrix $E$ of size $N$ called by Wall the *proximity matrix* \cite[p. 52]{25}. The first two columns of $E$ are

$$
\begin{pmatrix}
1 & -1 \\
0 & 1 & \ldots \\
& 0 & 0 \\
& 0 & 1 \\
& \ddots & \ddots \\
& & 0 & 0
\end{pmatrix}_N
$$

The $i$th column $C_i$ is defined by $(C_i)_i = 1$ and $(C_i)_{i-1} = -1$; if $E_i$ is the blowing-up of the point $D_{i-1} \cap D_j$ then $(C_i)_j = -1$; for any other index $j$, $(C_i)_j = 0$.

Notice that, since the curve $S$ is irreducible, the proximity matrix has the following property: if $i < j$ and $C_{ij} = 0$ then $C_{ik} = 0$ for $k \geq j$.

We will denote by $E^k$ the truncated process $E_k \circ \cdots \circ E_N$ and $D^k = \bigcup_{i=k}^N D_i$ the exceptional divisor of $E^k$. Let $S_i$ be the strict transform of $S$ by $E_1 \circ \cdots \circ E_{i-1}$ for $i \geq 2$ and $S_1 = S$. The map $E^k$ is the minimal desingularization of the total transform of $S_1$ by $E_1 \circ \cdots \circ E_{i-1}$. The following lemma is in \cite[p. 53]{25}

**Lemma 4.** The inverse of the proximity matrix $E^{-1}$ has the following form

$$
\begin{pmatrix}
1 & -1 & e_{kl} \\
0 & \ddots & \ddots \\
& \ddots & 1 \\
& 0 & 0 & 1
\end{pmatrix}
$$

where for $k < l$, $e_{kl}$ is the multiplicity of $D_l$ in the blowing-ups process $E^k$. Furthermore, the matrix $E^tE$ is the intersection matrix of $D$. 
The exception divisors of the sequence of processes of blowing-ups associated to the desingularization of $y^5 - x^{13} = 0$.

**Example 5.** Let us consider $S = \{y^5 = x^{13}\}$. Then the proximity matrix $E$ is written

$$E = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
$$

The inverse matrix is written

$$E^{-1} = \begin{pmatrix}
1 & 1 & 1 & 2 & 3 & 5 \\
0 & 1 & 1 & 2 & 3 & 5 \\
0 & 0 & 1 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
$$

The exceptional divisors of the associated sequence of processes of blowing-ups $\{E^k\}_{k=1.5}$ are presented in Figure (2.1).

The next proposition is the one upon which the construction of the auxiliary foliation $F[S_d]$ is based.

**Proposition 6.** Let $\delta_i \in \{0, 1, 2\}$ be the number of components of the direction $d$. In the same way, consider the number $\delta_i$ of branches of $(E_1 \circ \cdots \circ E_{i-1})^{-1}(d)$ meeting $S_i$ for $2 \leq i \leq N$. For $i \geq 2$, $\delta_i \in \{1, 2\}$. Let us denote $n_i - 1$ the number of $-1$ on the $i$-th row.

Let us consider the vector of integers defined by

$$\begin{pmatrix}
p_1 \\
p_2 \\
\vdots \\
p_N
\end{pmatrix} = \begin{pmatrix}
\nu(S_1) - \delta_1 \\
\nu(S_2) - \delta_2 \\
\vdots \\
\nu(S_N) - \delta_N
\end{pmatrix} + 1.
$$

Then

1. any integer $p_i$ is bigger or equal to $-1$. The case $p_i = -1$ occurs if only if
   (a) either, $n_i = 2$, $\delta_i = 2$, $\delta_{i+1} = 1$ and $\nu(S_i) = p(S_i)$ is odd.
   (b) or, $n_i = 3$, $\delta_i = 2$, $\delta_{i+1} = 1$, $\nu(S_i)$ is odd and $q(S_i)$ is even.
(2) If $D_i \cap D_j \neq \emptyset$ then one cannot have both $p_i = -1$ and $p_j = -1$.

(3) Let us consider $D$ the exceptional divisor $D$ deprived of $D_N$ and of the components $D_i$ for which $p_i = -1$. Then in each connected component of $D$, there exists at least one component $D_j$ for which, either $p_j > 0$ or, that meets a component of $dE$.

Proof. The proof is an induction on the length of the desingularization of $S_d$. Let us consider that $E$ is written

$$E = \begin{pmatrix}
1 & -1 & -1 & \cdots & -1 & 0 \\
1 & -1 & \cdots & \cdots \\
1 & \cdots & -1 & 1 & -1 \\
\vdots & \ddots & \cdots & \ddots & \ddots
\end{pmatrix}$$

Expanding the expression of $p_1$, we find

$$p_1 = \left\lfloor \frac{\nu(S_1) - \delta_1}{2} \right\rfloor + 1 - \sum_{j=2}^{n} \left( \left\lfloor \frac{\nu(S_j) - \delta_j}{2} \right\rfloor + 1 \right)$$

where for the sake of simplicity $n = n_1$. Consider a Puiseux parametrization of $S_1 = S$,

$$S_1 : \begin{cases}
x = t^p \\
y = t^q + \cdots
\end{cases}$$

with $p = p(S_1) < q = q(S_1)$. Following to the desingularization of $S_1$, encoded in the proximity matrix, the multiplicities and the $\delta_i$’s satisfy

$$\begin{align*}
\nu(S_1) &= p \\
\nu(S_j) &= q - p & \text{for } 2 \leq j \leq n - 1 \\
\nu(S_n) &= (n - 1) p - (n - 2) q \\
\delta_1 &\in \{0, 1, 2\} \\
\delta_2 &\in \{1, 2\} \\
\delta_j &= 2 & \text{for } 2 \leq j \leq n.
\end{align*}$$

Thus, the integer $p_1$ is written

$$p_1 = \left\lfloor \frac{p - \delta_1}{2} \right\rfloor - \sum_{j=2}^{n-1} \left\lfloor \frac{q - p - \delta_j}{2} \right\rfloor - \left\lfloor \frac{(n - 1) p - (n - 2) q - \delta_n}{2} \right\rfloor - n + 2.$$

The following lemma is straightforward

\footnote{Actually, $n_1$ is equal to $\left\lfloor \frac{q}{q-p} \right\rfloor$, but we will not need this expression.}
Table 1. Values of $p_1$.

| $\delta_1 = 0$, $\delta_2 = 1$ | $\delta_1 = 1$, $\delta_2 = 1$ | $\delta_1 = 1$, $\delta_2 = 2$ | $\delta_1 = 2$, $\delta_2 = 1$ | $\delta_1 = 2$, $\delta_2 = 2$ |
|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| $p$ and $q$ both odd          | $p$ and $q$ both even         | $p$ even, $q$ odd             | $p$ odd, $q$ even             | $p$ odd, $q$ even             |
| $1$                           | $1$                           | $\frac{n-2}{2}$, $\frac{n-1}{2}$ | $\frac{n-2}{2}$, $\frac{n-3}{2}$ | $\frac{n-2}{2}$, $\frac{n-3}{2}$ |

**Lemma 7.** If $n = 2$, then $p_1$ is equal to

$$p_1 = \left\lfloor \frac{p - \delta_1}{2} \right\rfloor - \left\lfloor \frac{p - \delta_2}{2} \right\rfloor = \begin{cases} \delta_1 = 0, \delta_2 = 1 & \begin{cases} 0 & \text{if } p \text{ is odd} \\ 1 & \text{else} \end{cases} \\ \delta_2 = 1 & 0 \\ \delta_2 = 2 & \begin{cases} 1 & \text{if } p \text{ is odd} \\ 0 & \text{else} \end{cases} \\ \delta_1 = 2 & \begin{cases} 0 & \text{if } p \text{ is odd} \\ \delta_2 = 1 & -1 \\ \delta_2 = 2 & 0 \end{cases} \end{cases}.$$ 

If $n \geq 3$ then the values of $p_1$ are given in Table 1. When the value depends on $n$, it is precised the value of $p_1$ if $n$ is even or odd. In particular, $p_1 = -1$ if and only if one of the following case occurs,

- $n = 2$, $\delta_1 = 2$, $\delta_2 = 1$ and $p$ is odd.
- $n = 3$, $\delta_1 = 2$, $\delta_2 = 1$ and $p$ is odd and $q$ is even.

Now, we are able to study the general behavior of $p_1$ and to prove Proposition 6.

The property (1) can be seen by reading inductively Lemma 7.

The property (2) is proved as follows. Suppose that $p_1 = -1$. According to property (1), two cases may occur

- if $n = 2$, $\delta_1 = 2$ and $\delta_1 = 1$, then $D_1$ meets $D_2$ in $D$. Since $\delta_2 = 1$, $p_2$ cannot be equal to $-1$. Proposition 6 applied inductively to $S_2$ yields the proposition for $S_2$.  

• if \( n = 3, \delta_1 = 2, \delta_2 = 1, p \) is odd and \( q \) is even, then \( D_1 \) meets \( D_3 \) and \( \delta_3 = 2 \). Suppose that \( \delta_4 = 1 \) then \( S_3 \) is neither tangent to \( D_1 \) nor to \( D_2 \). 
Looking at the Puiseux parametrization of \( S_3 \) yields 
\[
q - p = 2p - q
\]
which is impossible since \( p \) is odd. Thus \( \delta_4 = 2 \), and \( p_3 \) cannot be equal to \(-1\). We conclude by induction.

Let us now focus on property (3).

• Suppose first that \( \delta_1 = 2 \). If \( p_1 > 0 \), then the connected component of \( D_1 \) in \( \mathcal{D} \) contains \( D_1 \) as component with \( p_1 > 0 \). Applying inductively Proposition \( \text{[6]} \) to \( S_2 \) with the sequence of \( \delta \)'s equal to 
\( \delta_2, \delta_3, \ldots \)
yields the proposition for \( S_1 \) with the sequence of \( \delta \)'s equal to \( \delta_1, \delta_2, \ldots \).
If \( p_1 = 0 \) then at least one of the component of \( d^E \) is attached to \( D_1 \). Thus the same argument as before ensures the proposition. If \( p_1 = -1 \), then two cases may occur:
- if \( n = 2 \) then \( \nu(S_2) = \nu(S_1) \) is odd and \( \delta_2 = 1 \). Applying inductively Proposition \( \text{[6]} \) to \( S_2 \) with the sequence of \( \delta \)'s equal to 
\( 0, \delta_3, \delta_4, \ldots \)
yields the result: indeed, one has
\[
\left\lfloor \frac{\nu(S_2) - 0}{2} \right\rfloor = \left\lfloor \frac{\nu(S_2) - 1}{2} \right\rfloor = \left\lfloor \frac{\nu(S_2) - \delta_2}{2} \right\rfloor
\]
and for \( j \geq 3 \), since \( n = 2 \), one has \( \delta_j = \delta_j \) where the \( \delta_j \) would be the sequence obtained following the desingularization of \( S_2 \) with \( \delta_j = 0 \).
- if \( n = 3 \), then \( \delta_2 = 1, \nu(S_1) = p \) is odd and \( q \) is even. Moreover, since \( n = 3 \), one has \( \delta_3 = 2 \). Following the desingularization of \( S_1 \), one has \( \nu(S_2) = q - p \) that is odd and \( \nu(S_3) = 2p - q \) that is even. Applying inductively Proposition \( \text{[6]} \) to \( S_2 \) with the sequence of \( \delta \)'s equal to 
\( 0, 1, \delta_4, \ldots \)
yields the result: indeed, one has
\[
\left\lfloor \frac{\nu(S_2) - 0}{2} \right\rfloor = \left\lfloor \frac{\nu(S_2) - 1}{2} \right\rfloor = \left\lfloor \frac{\nu(S_2) - \delta_2}{2} \right\rfloor, \\
\left\lfloor \frac{\nu(S_3) - 1}{2} \right\rfloor = \left\lfloor \frac{\nu(S_3) - 2}{2} \right\rfloor = \left\lfloor \frac{\nu(S_2) - \delta_3}{2} \right\rfloor,
\]
and for \( j \geq 4 \), since \( n = 3 \), one has \( \delta_j = \delta_j \) where the \( \delta_j \) would be the sequence obtained following the desingularization of \( S_2 \) with \( \delta_j = 0 \) and \( \delta_j = 1 \).

• Suppose now that \( \delta_1 = 1 \). Then according to property (2), \( p_1 \geq 0 \). If \( \delta_2 = 1 \) then the component of \( d^E \) meets \( D_1 \). So applying inductively Proposition \( \text{[6]} \) to \( S_2 \) with the sequence \( \delta_2, \delta_3, \ldots \) yields the proposition. Let us suppose that \( \delta_2 = 2 \). If \( p_1 > 0 \), then inductively the proposition is proved. If \( p_1 = 0 \) then according to Lemma \( \text{[4]} \) two cases may occur.
- if \( n = 2 \) and \( \nu(S_2) = \nu(S_1) \) is even, then \( D_2 \) meets \( D_1 \) in \( D \) and \( p_2 \) cannot be equal to \(-1\). Applying inductively Proposition to \( S_2 \) with the sequence 
\[
1, \delta_3, \ldots
\]
yields the result. The arguments are the same as before.
- if \( n \geq 3 \), then \( p \) and \( q \) are even and the curve \( S \) cannot be topologically quasi-homogeneous. While \( \delta_i \neq 1 \), no component \( D_j \) with \( p_j = -1 \) can appear. If at some point, one has \( \delta_j = 1 \) then the multiplicity of \( \nu(S_j) \) is written \( \alpha p + \beta q \) for some \( \alpha, \beta \) in \( \mathbb{Z} \). Thus it is even and \( p_j \) cannot be equal to \(-1\). Therefore, \( D_2 \) and \( D_1 \) belongs to the same connected component \( \overline{D} \), which inductively proved the proposition since \( d^E \) is attached to \( D_2 \).

\[\begin{align*}
\text{Suppose finally that } \delta_1 &= 0. \text{ One has } \delta_2 = 1. \text{ If } p_1 > 0 \text{ then the proposition is proved inductively. If not, two cases may occur:} \\
- &\text{if } n = 2 \text{ then } \nu(S_2) = \nu(S_1) \text{ is odd. The proposition is proved applying it inductively to } S_2 \text{ with the sequence} \\
&0, \delta_3, \ldots
\end{align*}\]

The arguments are the same as above noticing that
\[
\left[ \frac{\nu(S_2)}{2} \right] = \left[ \frac{\nu(S_2) - \delta_2}{2} \right].
\]
- if \( n \geq 3 \) and \( p_1 = 0 \) then \( n = 3, p \) is odd and \( q \) is even. The proposition is proved applying it inductively to \( S_2 \) with the sequence 
\[
0, 1, \delta_4, \ldots
\]
Again, the arguments are the same as before.

\[\Box\]

Now, we introduce a foliation associated to \( S_d \) prescribing some topological data.

**Definition 8.** The numbered dual tree \( \mathbb{A}[F] \) of a foliation \( F \) is a numbered graph constructed as follows. Let \( E \) be the minimal desingularization of \( F \). The vertices of \( \mathbb{A}[F] \) are in one-to-one correspondence with the irreducible components of the exceptional divisor of \( E \). There is an edge between \( D_i \) and \( D_j \) if and only if \( D_i \cap D_j \neq \emptyset \). Each vertex is numbered following the next rule:

- if \( E^*F \) is dicritical along \( D_i \), then \( D_i \) is numbered \(+\infty\)
- else it is numbered by the number of irreducible invariant curves of \( E^*F \) intersecting \( D_i \) transversely.

**Proposition 9.** Let \( \mathbb{A} \) the dual tree of \( S_d \) numbered the following way:

- if \( p_i = -1 \) then \( D_i \) is numbered \( \infty \).
- if not, \( D_i \) is numbered \( p_i + ( \text{ the number of component of } d^E \text{ meeting } D_i) \)

There exists a foliation \( F[S_d] \) whose singularities are linearizable and such that 
\[
\mathbb{A}[F[S_d]] = \mathbb{A}.
\]
Proof. Using a result of Lins-Neto [19, 20], we consider a foliation \( \mathcal{F}[S_d] \) whose desingularization has the same topology as the desingularization of \( S_d \). For the sake of simplicity, we keep denoting by \( D = \bigcup_{i=1}^{N} D_i \) the exceptional divisor of its desingularization. We require that

- \( \mathcal{F}[S_d] \) is dicritical and regular along \( D_N \).
- If \( p_i = -1 \), then \( \mathcal{F}[S_d] \) is dicritical and smooth along \( D_i \). If not, \( D_i \) is invariant.
- At each corner point of \( D \) that does not meet a dicritical component, \( \mathcal{F}[S_d] \) admits a linear singularity written in some local coordinates \((x, y)\)

\[
\lambda x dy + y dx, \quad \lambda \notin \mathbb{Q}^-
\]

where \( xy = 0 \) is a local equation of \( D \).

- For each \( D_i \) with \( p_i \geq 0 \), \( \mathcal{F}[S_d] \) admits \( p_i \) more linear singularities along \( D_i \) and written in some local coordinates \((x, y)\)

\[
\lambda x dy + y dx, \quad \lambda \notin \mathbb{Q}^-
\]

where \( x = 0 \) is a local equation of \( D_i \).

The local analytic class of the singularities added above depends on the value of \( \lambda \) which is called the Camacho-Sad index [3] of the singularity \( s \) along \( D \). It is denoted by

\[
\lambda = CS_s (\mathcal{F}[S_d], D)
\]

where \( s \) is the singularity.

- Finally, for each component of \( d^E \) attached to \( D_j \) with \( p_j \geq 0 \), \( \mathcal{F}[S_d] \) admits one more linear singularity along \( D_j \).

The above data must satisfy some compatibility conditions stated in the theorem of Lins-Neto: first, two dicritical components cannot meet which is ensured by the second property of Proposition 6. Second, the Camacho-Sad indexes of the singularities along a given component \( D_j \) have to satisfy a relation known as the Camacho-Sad relation

\[
\sum_{s \in D_j} CS_s (\mathcal{F}[S_d], D_j) = -D_j \cdot D_j.
\]

The third property in Proposition 6 allows us to choose the Camacho-Sad indices of the linear singularities added at (2.2) and at (2.3) in order to ensure the Camacho-Sad relation for any component \( D_j \). By construction, one has

\[
\mathbb{A}[\mathcal{F}[S_d]] = \mathbb{A}.
\]

\[\square\]

A lot of foliations can be constructed as above. Indeed, we do not prescribe the way these local data are glued together. Hence, there is a big number of non equivalent choices. However, all the foliations build the way above share some properties. In any case, \( \mathcal{F}[S_d] \) is dicritical along \( D_N \). Its singularities are all linearizable and thus \( \mathcal{F}[S_d] \) is of second kind as defined in [21, 6]. Its desingularization has the same topological type as the desingularization of \( S_d \). Moreover, the foliation \( \mathcal{F}[S_d] \) is tangent to some curve topologically equivalent to \( S_d \). Finally, the algebraic multiplicity is the desired one. Indeed, one has the following result:
**Lemma 10.** Regardless the foliation $\mathcal{F}[S_d]$ constructed as above, one has

$$\nu(\mathcal{F}[S_d]) = \left\lfloor \frac{\nu(S_d)}{2} \right\rfloor.$$ 

**Proof.** Following a formula in [17, 9] gives us

$$\nu(\mathcal{F}[S_d]) = \sum_{i=1}^{N-1} p_i e_{1i} + \delta_1 - 1.$$ 

Since $\nu(S_N) = 1$ and $\delta_N = 2$, one has $p_N = 0$. Writing the first line of the relation [2.1] in Proposition 8 yields

$$\sum_{i=1}^{N-1} p_i e_{1i} + \delta_1 - 1 = \left\lfloor \frac{\nu(S_1) - \delta_1}{2} \right\rfloor + \delta_1 = \left\lfloor \frac{\nu(S_d)}{2} \right\rfloor.$$ 

$\square$

2.2. **Deformations of $\mathcal{F}[S_d]$.** In this section, we are interested in the deformations of foliations with a cohomological approach.

2.2.1. **Basic vector fields and deformations.** Let $\omega$ be a germ of 1–form and $X$ a vector field. The vector field $X$ is said to be basic for $\omega$ if and only if

$$(L_X \omega) \wedge \omega = d(\omega(X)) \wedge \omega - \omega(X) d\omega = 0.$$ 

The property of being basic for the 1–form $\omega$ depends only on the foliation induced by $\omega$, since for any function $f$, one has

$$L_X (f \omega) \wedge \omega = f^2 (L_X \omega) \wedge \omega.$$ 

The following lemma is classical.
Lemma 11. Let $X$ be a germ of vector field. It is basic for $\omega$ if and only if for any $t \in (\mathbb{C}, 0)$, the flow at time $t$ of $X$, denoted by $e^{[t]X}$, is an automorphism of $\omega$, i.e.,

\[
\left( (e^{[t]X})^* \omega \right) \land \omega = 0.
\]

More generally, a germ of automorphism of $\omega$ is a germ of automorphism $\phi$ such that $(\phi^* \omega) \land \omega = 0$. If $\phi$ is tangent to $\text{Id}$, then there exists a formal basic vector field $X$ such that $e^{[t]X} = \phi$. In what follows, we will simply denote the flow at time 1 of $X$ by $e^X$. If $X$ is singular at $p$, then the flow $e^X$ is convergent in a neighborhood of $p$.

Thanks to basic automorphisms, we can describe a surgery construction that produces many non-equivalent germs of foliations from a given one. Consider the desingularization $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$ of some singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$.

For any covering $\{U_i\}_{i \in J}$ of a neighborhood of $D$ in $\mathcal{M}$ and for any 2-intersection $U_{ij} = U_i \cap U_j$, we consider $\phi_{ij}$ a basic automorphism of $E^* \mathcal{F}$ which is the identity map along $U_{ij} \cap D$. We suppose that the family $\{\phi_{ij}\}_{i,j}$ satisfies the cocycle relation: on any 3-intersection $U_{ijk}$, one has

\[
\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \text{Id}.
\]

We construct a manifold with the following gluing

\[
\mathcal{M} [\phi_{ij}] = \coprod_i U_i / x \sim \phi_{ij}(x)
\]

which is a neighborhood of some divisor isomorphic to $D$. This manifold is foliated by a foliation $\mathcal{F}'$ obtained by gluing with the same collection of maps the family of restricted foliations $\{ E^* \mathcal{F}|_{U_i} \}_{i}$.

Lemma 12. There exists a germ of singular foliation at the origin of $(\mathbb{C}^2, 0)$ denoted by $\mathcal{F}[\phi_{ij}]$ and a process of blowing-ups $E'$ such that $(E')^* \mathcal{F}[\phi_{ij}]$ is analytically equivalent to $\mathcal{F}'$.

Proof. The manifold $\mathcal{M} [\phi_{ij}]$ is an open neighborhood of a divisor whose intersection matrix is the same as the one of $D$. In particular, the intersection matrix is definite negative. Following the Grauert’s contraction result [12], there exists a process of blowing-ups $E' : (\mathcal{M}', D') \to (\mathbb{C}^2, 0)$ such that $\mathcal{M}'$ is analytically equivalent to $\mathcal{M} [\phi_{ij}]$. Being analytically equivalent to $\mathcal{M} [\phi_{ij}]$, the manifold $\mathcal{M}'$ is foliated. Since $E'$ is an isomorphism between $\mathcal{M}' \setminus D'$ and $(\mathbb{C}^2, 0) \setminus \{0\}$, there exists a foliation in $(\mathbb{C}^2, 0) \setminus \{0\}$ whose pull-back by $E'$ coincides with the foliation of $\mathcal{M}'$ on $\mathcal{M}' \setminus D'$. The Hartogs’s extension result allows us to extend this foliation in $(\mathbb{C}^2, 0)$. The obtained foliation is $\mathcal{F} [\phi_{ij}]$. \qed

A foliation built the way above is said to be a basic surgery of $\mathcal{F}$. Our goal is to study the basic surgeries of $\mathcal{F}[S_d]$ and in particular to prove the following

Proposition 13. For any curve $C$ topologically equivalent to $S_d$, there is a 1-form $\omega \in \Omega^1(C)$ defining a foliation obtained from a basic surgery of $\mathcal{F}[S_d]$.

The proof is based upon the study of deformations of $\mathcal{F}[S_d]$ with a cohomological point of view, that is developed below.
2.2.2. The sheaf $TS_d$. In the desingularization $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$, let us consider the sheaf $TS_d$, with $D$ as basis, of vector fields tangent to $D$ and to $S_N = S^E$ that vanish along the strict transform $d^E$.

For any divisor $\Sigma = \sum n_i \Sigma_i$ in $\mathcal{M}$, we denote by $\Omega^2(\Sigma)$ the sheaf with $D$ as basis, of 2–forms $\omega$ such that the multiplicity of $\omega$ along $\Sigma_i$ satisfies

$$\nu_{\Sigma_i}(\omega) \geq -n_i.$$  

Let $F$ be a balanced equation of $\mathcal{F}[S_d]$ as defined in [6]. First, we prove the following proposition.

**Proposition 14.** In Čech cohomology, one has

$$H^1(D, \Omega^2(2(F)^E - S^E_d + D)) = 0$$

where the divisor $(F)^E$ is $(F = 0)^E - (F = \infty)^E$.

The proof is an induction on the length of the desingularization $E$. The first step is the following lemma.

**Lemma 15.** Let us consider a germ of divisor $\Sigma$ at the origin of $(\mathbb{C}^2, 0)$. Let $E_1 : (\mathcal{M}_1, D_1) \to (\mathbb{C}^2, 0)$ be the standard blowing-up of the origin. Then, for any $n \geq 0$, the following are equivalent

- The multiplicity of $\Sigma$ at the origin satisfies $\nu(\Sigma) \geq n$.
- The first cohomology group of $\Omega^2(\Sigma_{E_1} + nD_1)$ on $D_1$ vanishes

$$H^1(D_1, \Omega^2(\Sigma_{E_1} + nD_1)) = 0.$$  

**Proof.** Let $l$ be an equation of $\Sigma$. Consider the standard coordinates of the blowing-up together with its standard covering.

$$U_1 : \begin{cases} y = y_1x_1 \\ x = x_1 \end{cases} \quad U_2 : \begin{cases} y = y_2 \\ x = y_2x_2 \end{cases}.$$  

The global sections of $\Omega^2(\Sigma_{E_1} + nD_1)$ on each associated open sets are written

$$\Omega^2(\Sigma_{E_1} + nD_1)(U_1) = \left\{ f(x_1, y_1) \frac{1}{l_1x_1^2} dx_1 \wedge dy_1 \mid f \in \mathcal{O}(U_1) \right\}$$

$$\Omega^2(\Sigma_{E_1} + nD_1)(U_2) = \left\{ g(x_2, y_2) \frac{1}{l_2y_2^2} dx_2 \wedge dy_2 \mid g \in \mathcal{O}(U_2) \right\}$$

$$\Omega^2(\Sigma_{E_1} + nD_1)(U_1 \cap U_2) = \left\{ h(x_1, y_1) \frac{1}{l_1x_1^2} dx_1 \wedge dy_1 \mid h \in \mathcal{O}(U_1 \cap U_2) \right\}$$

where $l_1 = \frac{lo_{E_1}}{x_1^{(E_1)}}, l_2 = \frac{lo_{E_1}}{y_2^{(E_1)}}$. Since the covering $\{U_1, U_2\}$ is acyclic, one has the following isomorphism

$$H^1(D_1, \Omega^2(\Sigma_{E_1} + nD_1)) \simeq \frac{\Omega^2(\Sigma_{E_1} + nD_1)(U_1 \cap U_2)}{\Omega^2(\Sigma_{E_1} + nD_1)(U_1) \oplus \Omega^2(\Sigma_{E_1} + nD_1)(U_2)}.$$
Therefore, the dimension of (2.4) is the number of obstructions to the following cohomological equation
\[
   h(x_1, y_1) \frac{1}{l_1} dx_1 \wedge dy_1 = g(x_2, y_2) \frac{1}{l_2} dx_2 \wedge dy_2
   - f(x_1, y_1) \frac{1}{l_1} dx_1 \wedge dy_1
\]
which is equivalent to
\[
   (2.5) \quad h(x_1, y_1) = - f(x_1, y_1) - \frac{1}{y_1^{-\nu(\Sigma) + n + 1}} g \left( \frac{1}{y_1}, y_1 x_1 \right).
\]
Let \( h = x_1^i y_1^j \). Then \( h \) is an obstruction to (2.5) if and only if \( j_0 < 0 \) and the following system cannot be solved in \( \mathbb{N} \)
\[
   \begin{cases}
   i_0 = j \\
   j_0 = j - i - \nu (\Sigma) - n - 1
   \end{cases}
\implies
   \begin{cases}
   j = i_0 \\
   i = i_0 - j_0 + \nu (\Sigma) - n - 1
   \end{cases}
\]
Thus, \( \nu (\Sigma) \geq n \) if and only if there is no obstruction. \( \square \)

Now let us prove Proposition 14.

\textbf{Proof}. The proof of the proposition is an induction on the length of the desingularization of \( S_d \). Let us write \( E = E_1 \circ E^2 \).

Let \( U_1 \) be \( D_1 \setminus \text{Sing} \, (S_2) \) and \( U_2 \) a very small neighborhood of \( \text{Sing} \, (S_2) \). We defined the following open sets
\[
   (2.6) \quad U_1 = (E^2)^{-1} (U_1) \quad U_2 = (E^2)^{-1} (U_2)
\]
The system \( \{U_1, U_2\} \) is an open covering of \( D \). The associated Mayer-Vietoris sequence for the sheaf \( \Omega^2 \left( 2(F)^E - S_d^E + \overline{D} \right) \) is written
\[
   (2.7) \quad H^0 \left( U_1, \Omega^2 \left( 2(F)^E - S_d^E + \overline{D} \right) \right) \bigoplus H^0 \left( U_2, \Omega^2 (\cdots) \right)
   \to H^0 \left( U_1 \cap U_2, \Omega^2 (\cdots) \right) \to 0
\]
and
\[
   (2.8) \quad 0 \to \mathcal{N} \to H^1 (D, \Omega^2 (\cdots)) \to H^1 (U_1, \Omega^2 (\cdots)) \bigoplus H^1 (U_2, \Omega^2 (\cdots)) \to 0.
\]
We are going to identify each term of the above exact sequences.

The manifold \( D_1 \setminus \text{Sing} \, (S_2) \) is isomorphic to \( \mathbb{C} \). Thus, it is a Stein. Since, the sheaf \( \Omega^2 (\cdots) \) is coherent, its cohomology vanishes on \( U_1 \) and, in (2.8), the following relation holds,
\[
   H^1 \left( U_1, \Omega^2 \left( 2(F)^E - S_d^E + \overline{D} \right) \right) = 0.
\]
Let \( F_2 \) be defined by the germ of foliation \( E_1^* F \mid S_d \) at \( \text{Sing} \, (S_2) \). By construction, the foliation \( F_2 \) let invariant \( S_2 \). Let \( F_2 \) be a balanced equation of \( F_2 \). Let \( h \) be a local equation of \( D_1 \) at \( \text{Sing} \, (S_2) \). Two cases have to be considered.
Let us prove that (2.9)

Thus, if the direction $d_2$ of $S_2$ is chosen to be the local trace at $\text{Sing} (S_2)$ of the union of $d^{E_1}$ and $D_1$, then the next equalities hold

$$
\left( 2 (F)^E - S^E_{d_2} + D \right) |_{U_2} = 2 \left( (F)^E_2 - (h)^E \right) - S^E_{d_2} |_{U_2} + D |_{U_2} \\
= 2 (F)^E_2 - 2 (h)^E - S^E_{d_2} + D + (h)^E \\
= 2 (F)^E_2 - S^E_{2,d_2} + D^2
$$

- If $D_1$ is not invariant for $\mathcal{F} [S_d]$ then $F_2$ can be chosen so that

$$(F)^E_2 = (F)^E$$

Thus setting for the direction $d_2$ of $S_2$ the local trace at $\text{Sing} (S_2)$ of the sole $d^{E_1}$ still yields

$$
\left( 2 (F)^E - S^E_{d_2} + D \right) |_{U_2} = 2 (F)^E_2 - S^E_{2,d_2} + D^2
$$

since here $D |_{U_2} = D^2$.

In any case, applying inductively Proposition 14 to $S_2$ and to the associated divisor $2 (F)^E_2 - S^E_{2,d_2} + D^2$ ensures that, in (2.8), one has

$$
H^1 \left( U_2, \Omega^2 \left( 2 (F)^E - S^E_{d_2} + D \right) \right) = H^1 \left( U_2, \Omega^2 \left( 2 (F)^E_2 - S^E_{2,d_2} + D^2 \right) \right) = 0.
$$

The map $E^2$ induces isomorphisms in cohomology

$$
H^0 \left( U_1, \Omega^2 \left( 2 (F)^E - S^E_{d_2} + D \right) \right) \simeq H^0 \left( U_1, \Omega^2 \left( 2 (F)^E_1 - S^E_{d_1} + D_1 \right) \right) \\
H^0 \left( U_1 \cap U_2, \Omega^2 (\cdots) \right) \simeq H^0 \left( U_1 \cap U_2, \Omega^2 (\cdots) \right).
$$

Let us prove that $E^2$ induces also an isomorphism on the set of global sections along $U_2$ and $U_2$. If $\eta$ is a global section of $\Omega^2 \left( 2 (F)^E - S^E_{d_2} + D \right)$ on $U_2$ then the push-forward of $\eta$ by $E^2$ can be extended analytically at $\text{Sing} (S_2)$ by Hartogs’s extension result. It induces naturally a section of $\Omega^2 \left( 2 (F)^E_1 - S^E_{d_1} + D_1 \right)$ on $U_2$.

Thus, $E^2$ induces a injective map

$$
H^0 \left( U_2, \Omega^2 \left( 2 (F)^E - S^E_{d_2} + D \right) \right) \longrightarrow H^0 \left( U_2, \Omega^2 \left( 2 (F)^E_1 - S^E_{d_1} + D_1 \right) \right).
$$

By induction, it is enough to prove that (2.10) is onto when $E^2$ is the simple blowing-up of $\text{Sing} (S_2)$ and $D$ reduced to $D_1 \cup D_2$.

Let $\eta$ be a section of $\Omega^2 \left( 2 (F)^E_1 - S^E_{d_1} + D_1 \right)$ on $U_2$.

- If $D_1$ is not dicritical for $\mathcal{F} [S_d]$ then $\eta$ is written in coordinates

$$
\eta = h f \frac{dx \wedge dy}{x},
$$

where $x$ is a local equation of $D_1$, $f$ is any meromorphic function whose local divisor is $2 (F)^E_1 - S^E_{d_1}$ and $h$ is any holomorphic function. If $\delta = 2$ then
the possible component of $d$ meets $D_1$ at a different point from $\text{Sing}(S_2)$. Thus the valuation of $f$ is equal to

$$\nu(f) = e_{2n} - 2 \sum_{i=2}^{n-1} p_i e_{2i} = e_{2n} - 2 \left[ \frac{e_{2n} - 1}{2} \right] - 1$$

Now, after the blowing-up $E^2$ which is written in adapted coordinates $E^2(x, t) = (x, tx)$, the pull back of $\eta$ is written

$$E^{2*}\eta = h^* f^* dx \wedge dt.$$

Thus, the valuation of $E^{2*}\eta$ along $D_2$ is at least $-1$. The exceptional divisor of $E^2$ cannot be dicritical for $\mathcal{F}[S_d]$ since $\delta_2 = 1$. Therefore, $E^{2*}\eta$ is a section of $\Omega^2 \left( 2 (F)^{E^2} - S_d^{E^2} + D_1 \cup D_2 \right)$ along $D_1 \cup D_2$. Now, if $\delta_2 = 2$ then one of the components of $d^{E_1}$, say $d_1^{E_1}$, meets $S_2$. Whether or not the component $d_1^{E_1}$ meets a dicritical component, the valuation of $f$ is at least

$$\nu(f) \geq e_{2n} - 2 \sum_{i=2}^{n-1} p_i e_{2i} - 1 = e_{2n} - 2 \left[ \frac{e_{2n}}{2} \right] - 1$$

If the exceptional divisor of $E^2$ is dicritical then $e_{2n}$ is odd and $\nu(f) \geq 0$. If not, $\nu(f) \geq -1$. Thus, whether the exceptional divisor of $E^2$ is dicritical or not, $E^{2*}\omega$ is a section of $\Omega^2 \left( 2 (F)^{E^2} - S_d^{E^2} + D_1 \cup D_2 \right)$ along $D_1 \cup D_2$.

- if $D_1$ is dicritical then $\delta_2 = 1$. Moreover, $\eta$ is written

$$\eta = hf dx \wedge dy,$$

where

$$\nu(f) + 1 = e_{2n} - \sum_{i=2}^{n-1} p_i e_{2i} + 1 = e_{2n} - 1 - 2 \left[ \frac{e_{2n} - 1}{2} \right] \geq 0.$$ 

Hence, $E^{2*}\omega$ is still a section of $\Omega^2 \left( 2 (F)^{E^2} - S_d^{E^2} + D_1 \cup D_2 \right)$ along $D_1 \cup D_2$.

By induction on the length of $E^2$, the isomorphism (2.10) is proved. Thus, the isomorphisms (2.4) and the exact sequence (2.7) identify $\mathcal{N}$ with the cohomology group

$$H^1 \left( D_1, \Omega^2 \left( 2 (F)^{E_1} - S_d^{E_1} + D_1 \right) \right).$$

Let us prove that the latter vanishes. If $p_1 = -1$, then $D_1$ is dicritical and $\delta_1 = 2$ and $\delta_2 = 1$. Therefore,

$$\nu \left( 2 (F)^{E_1} - S_d^{E_1} \right) = e_{1n} - 2 \sum_{i=2}^{n-1} p_i e_{1i} = e_{1n} - 2 \left( \left[ \frac{e_{1n} - 2}{2} \right] + 2 \right) = -1$$

since $e_{1n}$ is odd. If $p_1 \neq -1$, then

$$\nu \left( 2 (F)^{E_1} - S_d^{E_1} \right) = e_{1n} - 2 \sum_{i=1}^{n-1} p_i e_{1i} - \delta_1 = e_{1n} - \delta_1 - 2 \left[ \frac{e_{1n} - \delta_1}{2} \right] \leq -1.$$ 

Therefore, according to Lemma (15) $\mathcal{N}$ vanishes, which completes the proof of Proposition (14).
To compare the deformations of $\mathcal{F}[S_d]$ and of the underlying curve $S_d$, we introduce the following operator.

**Definition 16.** The operator of basic vector fields for $\mathcal{F}[S_d]$ is a morphism of sheaves defined by

$$B : X \in TS_d \mapsto L_X E^* \omega_F \wedge E^* \omega_F \in \Omega^2$$

where $\omega$ is any 1–form with an isolated singularity defining $\mathcal{F}[S_d]$ and $F$ any balanced equation of $\mathcal{F}[S_d]$.

**Proposition 17.** Let $\mathcal{B}_n(\mathcal{F}[S_d])$ be the sheaf defined by the kernel

$$\mathcal{B}_n(\mathcal{F}[S_d]) = \ker (\mathcal{B}|_{\mathcal{M}_n \cdot TS_d})$$

where $\mathcal{M}_n$ is the $n^{th}$ power of the sheaf of $\mathcal{O}$–module generated by the functions $E^* f$ with $f(0) = 0$. There is an exact sequence of sheaves

$$0 \rightarrow \mathcal{B}_n(\mathcal{F}[S_d]) \rightarrow \mathcal{M}_n \cdot TS_d \rightarrow \mathcal{M}_n \cdot \Omega^2(2(F)^E - S_d^E + D) \rightarrow 0$$

where $\mathcal{I}_n$ is a sheaf whose support is contained in the singular locus of $\mathcal{F}[S_d]$. Moreover, one has

$$H^1(D, \mathcal{M}_n \cdot \Omega^2(2(F)^E - S_d^E + D)/\mathcal{I}_n) = 0.$$ 

In particular, extracted from the long exact in cohomology associated to (2.12), there is an exact sequence

$$H^1(D, \mathcal{B}_n(\mathcal{F}[S_d])) \rightarrow H^1(D, \mathcal{M}_n \cdot TS_d) \rightarrow 0$$

**Proof.** The first part of the proposition is a computation in local coordinates. We describe the image of $\mathcal{M}_n \cdot TS_d$ by the operator $\mathcal{B}$. Since, $\mathcal{F}[S_d]$ is of second kind [21], the multiplicities of $\mathcal{F}[S_d]$ and of the balanced equation $F$ along any irreducible component $D_i$ of the exceptional divisor satisfy [3]

- $\nu_{D_i}(\mathcal{F}[S_d]) = \nu_{D_i}(E^* F)$ if $D_i$ is dicritical
- $\nu_{D_i}(\mathcal{F}[S_d]) = \nu_{D_i}(E^* F) + 1$ else.

**Case 1.** Let $p$ be a regular point of $D$ where $\mathcal{F}[S_d]$ is regular and tangent to exceptional divisor. In some local coordinates $(x, y)$ around $p$, the pullback $E^* \omega_F$ is written

$$E^* \omega_F = u \frac{dx}{x}$$

where $x$ is a local equation of $D$. Now, a local section $X$ of $\mathcal{M}_n \cdot TS_d$ is written

$$X = x^m \left( ax \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right), \ a, b \in \mathbb{C} \{x, y\}.$$
Therefore, applying the basic operator leads to
\[ \mathcal{B}(X) = x^m y^n \frac{\partial^2}{\partial y} \left( \frac{dx \wedge dy}{x} \right) \]
which is a local section of \( \mathcal{M}^\cdot \Omega^2 \left( 2(F)^E - S^E_d + \overline{D} \right) \). Since the equation \( \frac{\partial^2}{\partial y} = h \) can be solved for any \( h \), the operator \( \mathcal{B} \) is onto locally around \( p \). This property is true for any type of regular points for \( \mathcal{F}[S_d] \).

**Case 2.** Now suppose that \( p \) is a singular point of \( \mathcal{F}[S_d] \). By construction, \( \mathcal{F}[S_d] \) is locally linearizable. Let us fix some coordinates \( (x, y) \) such that
\[ E \omega = u \left( \frac{dx}{x} + \frac{dy}{y} \right) \]
and \( xy \) is a local equation of \( D \). A local section of \( \mathcal{M}^\cdot \cdot \mathcal{T}S_d \) is written
\[ X = x^k y^l \left( ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} \right), \quad a \in \mathbb{C} \{x, y\} \]
where \( b \in \mathbb{C} \{x, y\} \) if \( p \) is a corner point and \( b \in \frac{1}{y} \mathbb{C} \{x, y\} \) if not. Let us write \( E \omega \mathcal{F}(X) = u x^k y^l \cdot \mathcal{G}(a \lambda + b) = u \mathcal{G} \). Then
\[ \mathcal{B}(X) = u^2 \left( x \frac{\partial \mathcal{G}}{\partial x} - \lambda y \frac{\partial \mathcal{G}}{\partial y} \right) \frac{dx}{x} \wedge \frac{dy}{y} \]
Writing \( \mathcal{G} = \sum_{i,j} g_{ij} x^{i+k} y^{j+l} \) yields
\[ x \frac{\partial \mathcal{G}}{\partial x} - \lambda y \frac{\partial \mathcal{G}}{\partial y} = \sum_{i,j} g_{ij} ((i + k) - \lambda (j + l)) x^{i+k} y^{j+l} \]
Therefore, \( \mathcal{B}(X) \) lies in \( \mathcal{M}^\cdot \cdot \mathcal{O} \left( 2(F)^E - S^E_d + \overline{D} \right) \).

In case (2), \( \mathcal{B} \) is onto \( \mathcal{M}^\cdot \cdot \mathcal{O} \left( 2(F)^E - S^E_d + \overline{D} \right) \). However, we can identify a supplementary subspace of its image. Indeed, fixing some coordinates \( (x_p, y_p) \) as above for each singular point, we introduce the sheaf \( \mathcal{I}_n \) defined by the following properties

- if \( U \) does not meet any singular point of \( \mathcal{F}[S_d] \) then \( \mathcal{I}_n(U) = 0 \).
- if \( U \) meets the singular points \( p_1, \ldots, p_j \) then \( \mathcal{I}_n(U) \) is the set of 2-forms \( \eta \) defined in a neighborhood of \( U \) such that in the coordinates \( (x, y) \), it is locally written
  \[ \eta = u^2 \left( \sum_{(i+k) - \lambda (j+l) = 0} g_{ij} x^{i+k} y^{j+l} \right) \frac{dx_p}{x_p} \wedge \frac{dy_p}{y_p} \]
for some \( g_{ij} \). For instance, if \( \lambda_p \notin \mathbb{Q} \), then for \( n = 0 \), the stack \( (\mathcal{I}_0)_p \) is simply the finite vector space \( \mathbb{C} \cdot u^2 \frac{dx}{x} \wedge \frac{dy}{y} \).

By construction, for any \( p \), the stack \( (\mathcal{I}_n)_p \) is a supplementary subspace of the image \( \mathcal{B} \left( (\mathcal{M}^\cdot \cdot \mathcal{T}S_d)_p \right) \) in \( \left( \mathcal{M}^\cdot \cdot \mathcal{O} \left( 2(F)^E - S^E_d + \overline{D} \right) \right)_p \). Thus, the sequence (2.12) is exact.
The sheaf $\mathcal{M}^n$ is generated by its global sections. Therefore, Proposition 14 ensures that

$$H^1 \left( D, \mathcal{M}^n \cdot \Omega^2 \left( 2 \left( F \right)^E - S_d^E + \mathcal{D} \right) \right) = 0.$$ 

The short exact sequence associated to the quotient of $\mathcal{M}^n \cdot \Omega^2 \left( 2 \left( F \right)^E - S_d^E + \mathcal{D} \right)$ by $\mathcal{I}_n$ induces a long exact sequence in cohomology that is written

$$\cdots \to 0 \to H^1 \left( D, \frac{\mathcal{M}^n \cdot \Omega^2 \left( 2 \left( F \right)^E - S_d^E + \mathcal{D} \right)}{\mathcal{I}_n} \right) \to H^2 (D, \mathcal{I}_n) \to \cdots$$

Since the support of $\mathcal{I}_n$ contains only isolated points, its cohomology vanishes in rank 2 [10]. Therefore, the first term in the sequence above vanishes also. Finally, the long exact sequence in cohomology associated to (2.12) proves the end of Proposition 17. □

2.3. Deformations of $\mathcal{F}[S_d]$. Proposition 17 can be expressed as follows: any infinitesimal deformation of $S_d$ tangent to $D$ at order $n$ can be followed by an infinitesimal deformation of the foliation $\mathcal{F}[S_d]$ at the same level of tangency. Roughly speaking, the proof of Proposition 13 consists in a non-commutative analog. Actually, let us consider the following sheaves of non-abelian groups

**Definition 18.** For any involutive sub-sheaf $\mathcal{J}$ of the sheaf of tangent vector fields to $S_d^E$ that vanish along $d$ and $D$, we consider

$$\mathfrak{G}(\mathcal{J})$$

the sheaf of non-abelian groups generated by the flows of vector fields in $\mathcal{J}$.

According to the Campbell-Hausdorff formula,

$$e^{X_{ij}} = e^{X_{ij}} e^{Y_{ij}} = e^{X_{ij} + Y_{ij} + \frac{1}{2} [X_{ij}, Y_{ij}] + \frac{1}{12} ([X_{ij}, [X_{ij}, Y_{ij}]] - [Y_{ij}, [X_{ij}, Y_{ij}]]) + \cdots}$$

any element of $\mathfrak{G}(\mathcal{J})$ is a flow of an element of $\mathcal{J}$.

The first step of the proof is the following:

**Proposition 19.** Extracted from the long exact sequence in cohomology induced by the embedding $\mathfrak{G}(\mathcal{B}_1 (\mathcal{F}[S_d])) \rightarrow \mathfrak{G}(\mathcal{M} \cdot T S_d)$, the following sequence

$$H^1 (D, \mathfrak{G}(\mathcal{B}_1 (\mathcal{F}[S_d]))) \rightarrow H^1 (D, \mathfrak{G}(\mathcal{M} \cdot T S_d))$$

is exact.

**Proof.** Let us consider a 1–cocycle $\{\phi_{ij}\}_{ij} \in Z^1 (D, \mathfrak{G}(\mathcal{M} \cdot T S_d))$. By definition, this is a flow

$$\phi_{ij} = e^{X_{ij}}$$

where $\{X_{ij}\}_{ij} \in Z^1 (D, \mathcal{M} \cdot T S_d)$. By induction on $n$, we are going to prove that there exist $\{B^n_{ij}\}_{ij} \in Z^1 (D, \mathcal{B}_1 (\mathcal{F}[S_d]))$, $\{X^n_i\}_i \in Z^0 (D, \mathcal{M} \cdot T S_d)$ and $\{X^n_{ij}\}_{ij} \in Z^1 (D, \mathcal{M}^n \cdot T S_d)$ such that

$$e^{-X^n_i} \phi_{ij} e^{X^n_i} = e^{B^n_{ij} e^{X^n_{ij}},}$$
For $n = 1$, this is the relation (2.13). Now, suppose this is true for $n$. According to Proposition 17 there exist $\{\tilde{B}^n_{ij}\} \in Z^1(D, B_1(\mathcal{F}[S_d]))$ and $\{Y^n_i\} \in Z^0(D, \mathcal{M} \cdot TS_d)$ such that

$$X^n_{ij} = Y^n_i + \tilde{B}^n_{ij} - Y^n_j.$$ **Proposition 20.** Let $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$ be the desingularization of $\mathcal{F}[S_d]$. Consider the sheaf $\mathcal{M} \cdot TS_d$, where $\mathcal{M}$ is the ideal of functions vanishing along $D$ and $B_0(\mathcal{F}[S_d]) = \ker (B|_{\mathcal{M} \cdot TS_d})$. Then for every $\{\phi_{ij}\} \in Z^1(D, \mathcal{M} \cdot TS_d)$ there exists a family $\{\psi_{ij}^k\} \in Z^1(D, \mathcal{M} (B_0(\mathcal{F}[S_d])))$ such that

$$\mathcal{M} [\phi_{ij}] \cong \mathcal{M} [\psi_{ij}^0] \cdots [\psi_{ij}^k].$$

In particular, $\mathcal{M} [\phi_{ij}]$ is the support of a foliation obtained by successive basic surgeries of $\mathcal{F}[S_d]$.

**Proof.** The proof is an induction on the length of the resolution of $S_d$. Let us consider a 1-cocyle $\{\phi_{ij}\}$ in $Z^1(D, \mathcal{M} \cdot TS_d)$, $\{\phi_{ij}\}$ the restriction of the cocyle $\{\phi_{ij}\}$ to $D^2$. We are going to apply inductively the property to $S_{d_2}$ and of 1-cocycles in $\mathcal{M} \cdot TS_{d_2}$ such that

$$\bar{\phi}_{ij} = \phi_1^1 \phi_2^1 \phi_3^2 \cdots \phi_i^M (\phi_j^M)^{-1} (\phi_j^{M-1})^{-1} \cdots (\phi_j^1)^{-1},$$

a relation that is equivalent to (2.17) for $\{\phi_{ij}\}$. Now, consider the following 1-cocyle

$$\bar{\phi}_{ij} = \begin{cases} \phi_{12} \phi_{1j}^1 \phi_{2j}^2 \cdots \phi_{ij}^M & \text{for } i = 1 \text{ and } j = 2 \\ \text{Id} & \text{else.} \end{cases}$$
It belongs to $Z^1 (\mathfrak{S} (\mathfrak{T} \cdot TS_d))$. Since $\mathfrak{M}$ and $\mathfrak{T}$ coincide along $D_1$, it belongs also to $Z^1 (\mathfrak{S} (\mathfrak{T} \cdot TS_d))$. Therefore, Proposition (13) yields a 0-cocycle and 1-cocycle respectively in $\mathfrak{S} (\mathfrak{T} \cdot TS_d)$ and $\mathfrak{S} (\mathfrak{B}_1 (\mathcal{F} [S_d]))$ such that

$$\tilde{\phi}_{ij} = \phi_i \psi_{ij} \phi_j^{-1}.$$  

In particular, if $(i, j) \neq 2$, then $\phi_j^{-1} \psi_i = \psi_{ij}$. Therefore, for any $(i, j) \neq (1, 2)$, one can write

$$\phi_{ij} = \phi_1 \psi_{ij} \phi_2 \psi_{23} \cdots \psi_{ij} \phi_i \psi_{ij} \phi_j^{-1} (\phi_j^M)^{-1} (\phi_j^{M-1})^{-1} \cdots (\phi_j^{1})^{-1}$$

and

$$\phi_{12} = \phi_1 \psi_{12} \phi_2^{-1} (\phi_2^M)^{-1} (\phi_2^{M-1})^{-1} \cdots (\phi_2^{1})^{-1}$$

which is equivalent to (2.17) for $\{\phi_{ij}\}_{ij}$. The proposition is proved. \hfill $\square$

Finally, we can prove Theorem 13. Let $E' : (\mathcal{M}', D') \to (\mathbb{C}^2, 0)$ be the desingularization of $C$. The curves $C$ and $S_d$ are topologically equivalent. Since $S$ is irreducible, the exceptional divisors $D$ and $D'$ are analytically equivalent. Following section 3.2, there exists a 1-cocycle $\{\phi_{ij}\}_{ij}$ in $\mathfrak{S} (\mathfrak{T} \cdot TS_d)$ such that

$$\mathcal{M}' \simeq \mathcal{M} [\phi_{ij}].$$

According to Proposition (20), $\mathcal{M}'$ is the support of a foliation obtained from a basic surgery of $\mathcal{F} [S_d]$ that lets invariant the curve $C$, which completes the proof of Proposition 13.

As a corollary, we obtain Proposition 3 since under the hypothesis mentionned, $p_1$ cannot be equal to $-1$ and $\mathcal{F} [S_d]$ is not dicritical along the exceptional divisor of the first blowing-up.

2.4. Proof of Theorem 2. The proof consists in an argument by contradiction.

Let $S$ be an irreducible germ of curve in the generic component of its moduli space and let $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$ be its minimal desingularization. Let $d$ be any direction for $S$. Suppose that for a generic curve $S_d$ in the topological class of $S_d$, there exists a germ of 1-form in $\Omega^1 (S_d)$ of multiplicity $\nu < \left[ d(S_d) \right]$ \frac{d(S_d)}{2}]. We suppose $\nu$ as small as possible with that property. We can choose $S_d$ in the generic stratum of the moduli space $\mathcal{M} (S_d)$ so that, there exists an open neighborhood $U$ of $S_d$ in $\mathcal{M} (S_d)$, such that for any $C \in U$, there exists a germ of 1-form in $\Omega^1 (C)$ of multiplicity $\nu$. Taking a local parametrization of $\mathcal{M} (S_d)$ around $S_d$,

$$\epsilon \in (\mathbb{C}^P, 0) \to S_d (\epsilon) \in \mathcal{M} (S_d)$$

with $S_d (0) = S_d$, we obtain a miniversal equisingular deformation of $S_d$. Moreover, for any $\epsilon \in (\mathbb{C}^P, 0)$, there exists $\omega (\epsilon) \in \Omega^1 (S_d (\epsilon))$ such that $\nu (\omega (\epsilon)) = \nu$.

**Lemma 21.** We can suppose the family $\omega (\epsilon) : \epsilon \in (\mathbb{C}^P, 0) \to \Omega^1 (S_d (\epsilon))$ being analytic in $\epsilon$. 

Proof. Up to some change of coordinates \((x, y) \in (\mathbb{C}^2, 0)\), we can suppose that the direction \(d\) is a fixed curve equal to \(\emptyset\), \(\{x = 0\}\) or \(\{xy = 0\}\) that does not depend on \(\epsilon\). In these three respective cases, any element in \(\Omega^1 (S_d (\epsilon))\) can be written in coordinates

\[
\omega (\epsilon) = \begin{cases}
A_d dx + B_d dy, & d = \emptyset \\
A_d dx + xB_d dy, & d = \{x = 0\} \\
yA_d dx + xB_d dy, & d = \{xy = 0\}
\end{cases}
or

Let \(\gamma_\epsilon\) be a Puiseux parametrization of \(S(\epsilon)\) depending analytically on \(\epsilon\). The hypothesis ensures that for any \(M \in \mathbb{N}\) and for any \(\epsilon\), the following system has a solution \(\omega\)

\[
(S_\epsilon) : \begin{cases}
\text{Jet}_{t=0}^M (\gamma_\epsilon^* \omega) = 0 \\
\text{Jet}^{\nu-1}_{(x,y)} \omega = 0 \\
\text{Jet}^{\nu}_{(x,y)} \omega \neq 0
\end{cases}
\]

The family \((S_\epsilon)_{\epsilon \in (\mathbb{C}^P, 0)}\) is an analytical family of linear systems with a finite number of unknown variables, say \(M\), which are some coefficients of the Taylor expansion of \(A_d\) and \(B_d\) - (1) and (2) - and an open condition (3). The solutions can be viewed as a semi-analytic set \(Z\) of \((\mathbb{C}^{M+P}, 0)\) that projects onto \((\mathbb{C}^P, 0)\) through the projection \(\mathbb{C}^{M+P} \to \mathbb{C}^P\). Hence, there exists an analytical section \(\sigma : (\mathbb{C}^P, 0) \to (\mathbb{C}^{M+P}, 0)\) such that for all \(\epsilon \in (\mathbb{C}^P, 0)\), one has \(\sigma (\epsilon) \in Z\). This provides two functions \(A_d\) and \(B_d\) in \(\mathbb{C} \{\epsilon\}[x, y]\) such that \(\omega_\epsilon\) is a solution of \((S_\epsilon)\). Since the family \(\gamma_\epsilon\) is topologically trivial, taking a bigger integer \(M\) if necessary, we can find a family of functions \(f_k \in \mathbb{C} \{x, y\}\) with \(\nu (df_k) > \nu, \nu (df_k) \xrightarrow{k \to \infty} +\infty\) such that for any \(k \geq M\) and any \(\epsilon\), one has

\[
\nu (\gamma_\epsilon^* df_k) = k.
\]

Considering a form written

\[
(2.18) \quad \Omega = \omega_\epsilon + \sum_{k \geq M} \alpha_k (\epsilon) df_k,
\]

we can choose inductively \(\alpha_k (\epsilon)\) such that \((2.18)\) becomes a formal solution \(\Omega \in \mathbb{C} \{\epsilon\}[x, y]\) of the system

\[
\begin{cases}
\gamma_\epsilon^* \Omega = 0 \\
\text{Jet}^{\nu-1}_{(x,y)} \Omega = 0 \\
\text{Jet}^\nu_{(x,y)} \Omega \neq 0
\end{cases}
\]

According to the Artin’s approximation theorem [1], we can take \(\Omega\) analytic as a whole, \(\Omega \in \mathbb{C} \{\epsilon, x, y\}\).

For \(\epsilon\) generic, we can also suppose that \(\omega(\epsilon)\) is unirreducible [22]. Let

\[
E (\epsilon) : (\mathcal{M} (\epsilon), D (\epsilon)) \to (\mathbb{C}^2, 0)
\]

be the equisingular family of minimal desingularizations of the foliations \(\mathcal{F} (\epsilon)\) defined by \(\omega (\epsilon)\). In particular, \(E (\epsilon)\) is also an equisingular family of desingularizations of \(S_d (\epsilon)\). For the sake of simplicity, we still denote by \(\mathcal{M}\), \(E\) and \(S_d\) respectively the manifold \(\mathcal{M} (0)\), the desingularization \(E (0)\) and the curve \(S_d (0)\). \(\square\)
Let \( \{ T_{ij} \} \) be a \( 1 \)-cocycle in \( Z^1(\mathcal{M}, TS_d) \). Let us consider the deformation obtained by the gluing

\[
\mathcal{M} \left[ e^{(t)T_{ij}} \right].
\]

Since the flow \( e^{(t)T_{ij}} \) lets globally invariant \( S_d \), the manifold \( \mathcal{M} \left[ e^{(t)T_{ij}} \right] \) admits an invariant curve topologically equivalent to \( S_d^E \). By versality, the so defined topologically trivial deformation is equivalent to a deformation \( S_d^E(\epsilon(t)) \) for some analytic factorization \( \epsilon(t) : (\mathbb{C}, 0) \to (\mathbb{C}^P, 0) \). The deformation \( S_d^E(\epsilon(t)) \) is followed by the deformation of foliations \( F(\epsilon(t)) \). Therefore on the open set \( \mathcal{M}(\epsilon)^* \) which is \( \mathcal{M}(\epsilon) \) deprived of the singular locus of \( E(\epsilon)^* F(\epsilon) \), the cocycle \( \{ e^{(t)T_{ij}} \} \) is equivalent to a cocycle of basic automorphisms. Thus, there exist a \( 0 \)-cocycle of \( \{ \phi_i(t) \} \) letting globally invariant \( S_d^E(\epsilon(t)) \) and \( D(\epsilon(t)) \) and a \( 1 \)-cocycle of basic automorphisms \( \{ B_{ij}(t) \} \) for \( F \), such that on \( \mathcal{M}(\epsilon(t))^* \), one has

\[
e^{(t)T_{ij}} = \phi_i(t) B_{ij}(t) \phi^{-1}_j(t).
\]

(2.19)

Taking the derivative at \( t = 0 \) of the above expression yields to a cohomological relation on \( \mathcal{M}(0) = \mathcal{M} \).

\[
T_{ij} = T_i + b_{ij} - T_j
\]

where \( \{ T_i \} \) is a \( 0 \)-cocycle in \( TS_d \) and \( \{ b_{ij} \} \) is a \( 1 \)-cocycle with values in the sub-sheaf of basic vector fields for \( F \) tangent to \( S_d \), denoted simply by \( B(F) \).

Let us denote by \( \Omega \) the image sheaf of \( TS_d \) by the basic operator \( 2.11 \) for \( F \) with a given balanced equation \( F \).

The following diagram

\[
\begin{array}{ccc}
H^1(\mathcal{M}^*, B(F)) & \to & H^1(\mathcal{M}^*, TS_d) \\
\downarrow & & \downarrow \\
H^1(\mathcal{M}, TS_d) & \xrightarrow{\alpha} & H^1(\mathcal{M}^*, TS_d) \\
\downarrow & & \downarrow \\
H^1(\mathcal{M}, \Omega) & \xrightarrow{\gamma} & H^1(\mathcal{M}^*, \Omega) \\
\downarrow & & \downarrow \\
H^2(\mathcal{M}, B(F)) & \xrightarrow{\delta} & H^2(\mathcal{M}, B(F))
\end{array}
\]

(2.20)

is commutative. Since for any \( 1 \)-cocycle \( \{ T_{ij} \} \in Z^1(\mathcal{M}, TS_d) \), a relation such as \( 2.19 \) exists, one has

\[
\text{Im} \alpha \subset \text{Im} \gamma.
\]

Thus, the composed map \( B \circ \alpha \) is the zero map.

The sheaf \( \Omega \) on \( \mathcal{M}^* \) can be described as follows

\[
\Omega = \Omega^2 \left( 2(F)^E - S_d^E + \sum n_i D_i \right)
\]

where \( D = \sum D_i \) and the \( n_i \)'s are some integers depending on \( F \). This sheaf can be extended analytically on \( \mathcal{M} \). The Mayer-Vietoris sequence applied to the covering
\{M^*, U\} of M where U is an union of some small open balls around each singularity is written
\[
\cdots \to H^0 (M^*, \Omega) \bigoplus H^0 (U, \Omega) \xrightarrow{\Delta} H^0 (M^* \cap U, \Omega) \\
\to H^1 (M, \Omega) \to H^1 (M^*, \Omega) \bigoplus H^1 (U, \Omega) \to \cdots
\]
The Hartogs’s extension result ensures that \(\Delta\) is onto. Moreover, since U can be supposed to be Stein and \(\Omega\) is coherent, we deduce that in the diagram (2.20) the map \(\gamma\) is injective.

**Lemma 22.** We have
\[H^2 (M, B(F)) = 0.\]

**Proof.** Taking small flow-boxes on the regular part of \(E^*F\), we can find a finite Stein covering \(\{U_\alpha\}_{\alpha \in I} \cup \{U_s\}_{s \in \text{Sing}(E^*F)}\) of M such that

- for any \(s \in \text{Sing}(E^*F)\), \(U_s\) is a very small neighborhood of \(s\).
- on any open set \(U_\alpha\) with \(\alpha \in I\), there exists a biholomorphism on its image \(\psi_\alpha : U_\alpha \to \mathbb{C}^2\) such that \((\psi_\alpha^{-1})^* E^*F|_{U_\alpha}\) is the trivial regular foliation given by \(dx = 0\).

Applying the Mayer-Vietoris sequence to the covering leads to the long exact sequence in cohomology from which is extracted
\[(2.21) \bigoplus_{\alpha, \beta} H^1 (U_\alpha \cap U_\beta, B(F)) \to H^2 (M, B(F)) \to \bigoplus_{\alpha \in I} H^2 (U_\alpha, B(F)) \bigoplus_{s \in \text{Sing}(E^*F)} H^2 (U_s, B(F)) \to \bigoplus_{\alpha, \beta} H^2 (U_\alpha \cap U_\beta, B(F))\]

The foliation is analytically equivalent to the trivial foliation \(dx = 0\) on any 2-intersection \(U_\alpha \cap U_\beta\) and on \(U_\alpha\) with \(\alpha \in I\). Therefore, the cohomology of \(B(F)\) vanishes in rank 1 and 2 on these open sets. Finally, (2.21) is written
\[H^2 (M, B(F)) \simeq \bigoplus_{s \in \text{Sing}(E^*F)} H^2 (U_s, B(F)).\]

The open sets \(U_s\) can be taken as small as needed. Thus, the inductive limit on the family of open sets containing the singular locus of \(E^*F\) is written
\[0 = \bigoplus_{s \in \text{Sing}(E^*F)} H^2 (\{s\}, B(F)) \simeq \bigoplus_{s \in \text{Sing}(E^*F)} \lim_{U_s \to s} H^2 (U_s, B(F)) \simeq H^2 (M, B(F)).\]
from which the lemma follows. \(\square\)

The previous lemma and the properties of the diagram (2.20) ensure that
\[H^1 (M, \Omega) = 0.\]
Now, let us consider $E_1 : (\mathcal{M}_1, D_1) \to (\mathbb{C}^2, 0)$ the first blowing-up in the resolution $E$. The Mayer-Vietoris sequence of some adapted covering shows that

$$H^1 \left( \mathcal{M}_1, \Omega^2 \left( 2 (F)^{E_1} - S_d^{E_1} + n_1 D_1 \right) \right) \hookrightarrow H^1 \left( \mathcal{M}, \Omega^2 \left( 2 (F)^E - S_d^E + \sum n_i D_i \right) \right)$$

and therefore,

$$(2.22) \quad H^1 \left( \mathcal{M}_1, \Omega^2 \left( 2 (F)^{E_1} - S_d^{E_1} + n_1 D_1 \right) \right) = 0.$$

We are going to prove that the latter equality leads to a contradiction with $\nu (\mathcal{F}) < \left\lfloor \frac{\nu (S_d)}{2} \right\rfloor$.

We recall that $F$ being a balanced equation of $\mathcal{F}$, the next relation holds

$$\nu (\mathcal{F}) = \nu (F) - 1 + \tau (\mathcal{F})$$

where $\tau (\mathcal{F})$ is a positive integer called the tangency excess of $\mathcal{F}$.

- Suppose that $\mathcal{F}$ is not dicritical along the exceptional divisor of the blowing-up of its singularity. A computation in coordinates ensures that $n_1 = 1 - 2\tau (\mathcal{F})$. However, if $(2.22)$ is true, Lemma 15 shows that

$$2\nu (F) - \nu (S_d) \geq n_1 \iff 2\nu (F) - \nu (S_d) \geq -1.$$

But $\nu (\mathcal{F}) \leq \left\lfloor \frac{\nu (S_d)}{2} \right\rfloor - 1$ gives us

$$2\nu (\mathcal{F}) - \nu (S_d) \leq 2 \left\lfloor \frac{\nu (S_d)}{2} \right\rfloor - \nu (S_d) - 2 < -1$$

which is a contradiction.

- Suppose now that $\mathcal{F}$ is dicritical along the exceptional divisor of the blowing-up of its singularity. Then $n_1 = -2\tau (\mathcal{F})$. Again, Lemma 15 ensures that

$$2\nu (\mathcal{F}) - \nu (S_d) \geq -2.$$

If $\nu (\mathcal{F}) \leq \left\lfloor \frac{\nu (S_d)}{2} \right\rfloor - 2$ then we are led to a contradiction. Suppose that

$$\nu (\mathcal{F}) = \left\lfloor \frac{\nu (S_d)}{2} \right\rfloor - 1.$$ If $\nu (S_d)$ is odd then

$$2\nu (\mathcal{F}) - \nu (S_d) = 2 \left( \frac{\nu (S_d) - 1}{2} - 1 \right) - \nu (S_d) = -3,$$

which is still a contradiction. Suppose that $\nu (S_d)$ is even. Then, $\nu (\mathcal{F}) = \frac{\nu (S_d)}{2} - 1$. The multiplicity $\nu (\mathcal{F})$ being as small as possible in $\Omega^1 (S_d)$, a basis of $\Omega^1 (S_d)$ can be written $\{\omega_1, \omega_2\}$ with

$$\nu (S_d) - 1 = \nu (\omega_1) \leq \nu (\omega_2)$$

and $\nu (\omega_1) + \nu (\omega_2) \leq \nu (S_d)$.

Thus there are only three possibilities for $\nu (\omega_2)$. 

According to (2.23) there are some

- if \( \nu(\omega_2) = \frac{\nu(S_d)}{2} + 1 \), then any 1-form \( \omega \) of multiplicity \( \frac{\nu(S_d)}{2} \) in \( \Omega^1(S_d) \) is written

\[
\omega = a_1 \omega_1 + b_2 \omega_2,
\]

where \( a_1 \) is a function of multiplicity 1 and \( b_2 \) is any function. In particular, its jet of smallest order is written

\[
(a_1)_i \cdot (\omega_1)_{\nu(S_d)_2},
\]

where \( (\star)_i \) stands for the jet of order \( i \). Thus, as \( \omega_1 \) is, \( \omega \) is dicritical. This would imply that any element of multiplicity \( \frac{\nu(S_d)}{2} \) in the Saito module is dicritical along the exceptional divisor of the blowing-up of its singularity. This is a contradiction with Proposition 3.

- if \( \nu(\omega_2) = \frac{\nu(S_d)}{2} \) or \( \nu(\omega_2) = \frac{\nu(S_d)}{2} - 1 \) then using the criterion of Saito we have

\[
(\omega_1)_{\nu(S_d)} \cap (\omega_2)_{\nu(S_d)} = 0.
\]

This completes the proof of the first part of Theorem 2.

Let us prove now the existence of balanced basis for \( \Omega^1(S_d) \).

Let us suppose first that \( \nu(S_d) \) is even. Consider a basis \( \{\omega_1, \omega_2\} \) of \( \Omega^1(S_d) \). According to (2.23) there are some 1-forms with multiplicity \( \frac{\nu(S_d)}{2} \) in \( \Omega^1(S_d) \). Hence, at least one of the forms in the basis, say \( \omega_1 \), has a multiplicity equal to \( \frac{\nu(S_d)}{2} \). The multiplicity of \( \omega_2 \) is greater or equal to \( \frac{\nu(S_d)}{2} \). If it is equal, then the basis is balanced. If not, \( \{\omega_1, \omega_1 + \omega_2\} \) is still a basis and is balanced.
Suppose now that $\nu(S_d)$ is odd. If the direction of $S_d$ is empty or contains one component, let us consider $\tilde{S} = S_d \cup L$ where $L$ is a smooth curve transverse to the direction of $S_d$. Since the multiplicity of $\tilde{S}$ is even, according to the previous case, the module $\Omega^1(\tilde{S})$ admits a balanced basis. Therefore there exists a couple a 1-forms $\{\omega_1, \omega_2\}$ of multiplicity $\frac{\nu(S_d)+1}{2}$ such that

$$\omega_1 \wedge \omega_2 = uf dx \wedge dy,$$

where $l$ is an irreducible equation of $L$ and $f$ a reduced equation of $S_d$. Now, according to Proposition 1, there exists $\varpi$ tangent to $S_d$ such that $\nu(\varpi) = \frac{\nu(S_d)-1}{2}$. The 1-form $\varpi$ is tangent to $\tilde{S}$. Hence, there exist two germs of functions $a_1$ and $a_2$ such that

$$\varpi = a_1 \omega_1 + a_2 \omega_2.$$

The functions $a_1$ and $a_2$ cannot both vanish. Suppose by symmetry that $a_1$ does not vanish, then $\{\varpi, \omega_2\}$ is a basis of $\Omega^1(\tilde{S})$. Thus

$$\varpi \wedge \omega_2 = vf dx \wedge dy, \quad v(0) \neq 0.$$

Dividing by $l$ the above expression leads to the criterion of Saito for the balanced basis $\{\varpi, \omega_2\}$ of $\Omega^1(S_d)$.

If the direction of $S_d$ contains two components $L_1$ and $L_2$, then let us consider $\tilde{S} = S \cup L_1$. The module $\Omega^1(\tilde{S})$ admits a balanced basis $\{\omega_1, \omega_2\}$ with $\nu(\omega_1) = \nu(\omega_2) = \frac{\nu(S_d)}{2} = \frac{\nu(S_d)+1}{2}$. Now, there exist $\varpi$ in $\Omega^1(S_d)$ with $\nu(\varpi) = \frac{\nu(S_d)-1}{2}$. Since $\varpi$ is also tangent to $S \cup L_1$, there exist two functions $a_1$ and $a_2$ such that

$$\varpi = a_1 \omega_1 + a_2 \omega_2.$$

The functions $a_1$ and $a_2$ cannot both vanish so we can suppose that $a_1(0) \neq 0$. The family $\{\varpi, \omega_2\}$ is still a basis of $\Omega^1(\tilde{S})$ that satisfies

$$\varpi \wedge \omega_2 =wf dx \wedge dy, \quad w(0) \neq 0.$$

Thus, multiplying by $l_2$ leads to

$$\varpi \wedge l_2 \omega_2 =wf l_2 dx \wedge dy, \quad w(0) \neq 0$$

and $\{\varpi, l_2 \omega_2\}$ is a balanced basis of $\Omega^1(S_d)$.

This ends the proof of Theorem 2.

3. Generic dimension of the moduli space of an irreducible curve.

The proof of Theorem 1 relies on the following

**Proposition 23.** Let $S$ be a curve - irreducible or not - such that $\Omega^1(S)$ admits a basis $\{\omega_1, \omega_2\}$ with

$$\nu(\omega_1) + \nu(\omega_2) = \nu(S).$$

Then

$$\dim_{\mathbb{C}} H^1(D_1, TS) = \frac{(\nu_1 - 1)(\nu_1 - 2)}{2} + \frac{(\nu_2 - 1)(\nu_2 - 2)}{2}.$$
with \( \nu_i = \nu (\omega_i) \).

Proof. Since \( \{\omega_1, \omega_2\} \) is a basis of \( \Omega^1 \) \((S)\), the criterion of Saito ensures that

\[
(3.1) \quad \omega_1 \wedge \omega_2 = u f dx \wedge dy.
\]

for some unity \( u \) and some reduced equation \( f \) of \( S \). Let us consider the standard covering of \( D_1 \) by two open sets \( U_1 \) and \( U_2 \) and two charts \((x_1, y_1)\) and \((x_2, y_2)\) with

\[
y_2 = y_1 x_1, \quad x_2 = \frac{1}{y_1} \quad E_1 (x_1, y_1) = (x_1, y_1 x_1).
\]

The pull-back of \((3.1)\) by \( E_1 \) is written

\[
x_1^{\nu_1 + \nu_2} E_1^* \omega_1 \wedge E_1^* \omega_2 = x_1^{\nu} E_1^* u E_1^* f x_1 dx_1 \wedge dy_1
\]

Simplifying by \( x^{\nu} = x_1^{\nu_1 + \nu_2} \) yields the relation

\[
\tilde{\omega}_1^1 \wedge \tilde{\omega}_2^1 = E_1^* u \tilde{f} x_1 dx_1 \wedge dy_1.
\]

The two forms \( \tilde{\omega}_1^1 \) and \( \tilde{\omega}_2^1 \) are tangent to the exceptional divisor regardless of their dicriticalness. Obviously, they are also tangent to \( \tilde{f} = 0 \). According to the Saito criterion, at any point \( c \) of the exceptional divisor, the germ of \( \{\tilde{\omega}_1^1, \tilde{\omega}_2^1\} \) at \( c \) is a basis of the module \( \Omega^1 \left( (\tilde{S} \cup D_1)_c \right) \). Since, the covering \( \{U_1, U_2\} \) is acyclic for the coherent sheaf \( TS \), one has

\[
H^1 (D_1, TS) = H^1 (\{U_1, U_2\}, TS) = \frac{H^0 \left( U_1 \cap U_2, TS \right)}{H^0 \left( U_1, TS \right) + H^0 \left( U_2, TS \right)}.
\]

Now, the spaces of global sections on \( U_1, U_2 \) and the intersection can be described as follows

\[
H^0 \left( U_1 \cap U_2, TS \right) = \{ \phi_{12} \tilde{\omega}_1^1 + \psi_{12} \tilde{\omega}_2^1 : \phi_{12}, \psi_{12} \in \mathcal{O} (U_1 \cap U_2) \}
\]

\[
H^0 (U_1, TS) = \{ \phi_1 \tilde{\omega}_1^1 + \psi_1 \tilde{\omega}_1^2 : \phi_1, \psi_1 \in \mathcal{O} (U_1) \}
\]

\[
H^0 (U_2, TS) = \{ \phi_2 \tilde{\omega}_2^1 + \psi_2 \tilde{\omega}_2^2 : \phi_2, \psi_2 \in \mathcal{O} (U_2) \}.
\]

Thus, the cohomological equation is written

\[
\phi_{12} \tilde{\omega}_1^1 + \psi_{12} \tilde{\omega}_2^1 = \phi_1 \tilde{\omega}_1^1 + \psi_1 \tilde{\omega}_1^2 - \phi_2 \tilde{\omega}_2^1 + \psi_2 \tilde{\omega}_2^2
\]

\[
= \phi_1 \tilde{\omega}_1^1 + \psi_1 \tilde{\omega}_2^1 - \phi_2 y_1^{-\nu_1} \tilde{\omega}_1^1 + \psi_2 y_1^{-\nu_2} \tilde{\omega}_2^1.
\]

Since, \( \{\tilde{\omega}_1^1, \tilde{\omega}_2^1\} \) is a basis of \( \mathcal{O} \)-module, the above leads to the system

\[
\begin{cases}
\phi_{12} = \phi_1 - \phi_2 y_1^{-\nu_1} \\
\psi_{12} = \psi_1 - \psi_2 y_1^{-\nu_2}
\end{cases}
\]

Writing these equations using Taylor expansions leads to the checked number of obstructions. \(\square\)
Finally, the proof of the Theorem relies on the remark that the generic dimension of $M(S)$ is equal to

$$\dim_C H^1(D, TS_{gen})$$

where $S_{gen}$ is a generic point in $M(S)$. The Mayer-Vietoris sequence associated to the covering $(2,0)$ and applied to the sheaf $TS_{gen}$ decomposes $H^1(D, TS_{gen})$ along the desingularization of $S_{gen}$

$$H^1(D, TS_{gen}) \simeq H^1(D_1, TS_{gen}) \bigoplus H^1(D_2, TS_{gen})$$

The curve $S_{gen}$ admits a balanced basis according to Theorem 2. Hence, Theorem 1 is an inductive application of Proposition noticing that in that case

$$\dim H^1(D_1, TS_{gen}) = \sigma (\nu(S_{gen})).$$

As a corollary, Theorem 1 give a new proof of the following result contained in [14].

**Corollary.** A germ of irreducible curve $S$ is generically rigid if and only if

- $\nu(S) \in \{1, 2, 3\}$ or
- $\nu(S) = 4$ and its Puiseux pairs are $(4, 5), (4, 7)$ or $(2, 3), (2, 2k + 1)$ with $k \geq 3$.

Indeed, one can check that the cases above are the only one for which the formula in Theorem 1 yields 0.

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