1. Introduction.

This note extends the work of J. Mingo and A. Nica [5] on graded independence of random variables and the graded \( R \)-transform. We generalize the results of Mingo and Nica from the context of \( \mathbb{Z}_2 \)-graded noncommutative probability spaces to that of \( \mathbb{Z}_n \)-graded noncommutative probability spaces.

A very general setting for noncommutative probability is an algebraic noncommutative probability space consisting of a not necessarily commutative unital complex algebra \( A \) and a linear functional \( \varphi : A \to \mathbb{C} \) satisfying \( \varphi(1) = 1 \). A noncommutative random variable is simply an element \( x \) of the algebra \( A \), and its expectation is \( \varphi(x) \).

The general question addressed in [5] is the existence of notions of independence of random variables other than (and possibly interpolating between) the classical concept of independence and the concept of free independence of Voiculescu [15]. Several authors have provided axiomatizations and classification theorems for independence on algebraic noncommutative probability spaces ([13], [8], [9], [2], [3]). The common conclusion of these works (under various axiomatizations) is that the only possibilities are classical independence and free independence. (There are additional possibilities, namely, monotone and boolean independence, for nonunital algebras.)

Nevertheless, for more restricted categories of algebras, with additional structure, there can be notions of independence specific to these categories, which take into account the additional structure. Here we consider \( \mathbb{Z}_n \)-graded independence of \( \mathbb{Z}_n \)-graded noncommutative probability spaces.

1.1. \( \mathbb{Z}_n \)-graded independence.

Definition 1.1. An \( \mathbb{Z}_n \)-graded algebraic noncommutative probability space \((A, \varphi, \gamma)\) consists of

1. a unital complex algebra \( A \),
2. a linear functional \( \varphi \) on \( A \) satisfying \( \varphi(1) = 1 \), and
3. a unital algebra automorphism \( \gamma \) of \( A \) of order \( n \) satisfying \( \varphi \circ \gamma = \varphi \).

A morphism \( \Phi : (A_1, \varphi_1, \gamma_1) \to (A_2, \varphi_2, \gamma_2) \) of \( \mathbb{Z}_n \)-graded algebraic noncommutative probability spaces is a unital algebra morphism satisfying \( \gamma_2 \circ \Phi = \Phi \circ \gamma_1 \) and \( \varphi_2 \circ \Phi = \varphi_1 \).
If \((A, \varphi, \gamma)\) is a \(\mathbb{Z}_n\)-graded algebraic noncommutative probability space, then \(A\) is the direct sum of eigenspaces for \(\gamma\),

\[
A = \bigoplus_{q \in \mathbb{Z}_n^*} A_q,
\]

where \(\mathbb{Z}_n^*\) denotes the group of \(n^{th}\) roots of unity in \(\mathbb{C}\), and for \(q \in \mathbb{Z}_n^*\),

\[
A_q = \{ a \in A : \gamma(a) = qa \}.
\]

The elements of any eigenspace are said to be homogeneous. The eigenspaces satisfy

\[
A_q A_r \subseteq A_{qr} \quad \text{for } q, r \in \mathbb{Z}_n^*.
\]

The eigenspace \(A_1\) is the subalgebra of fixed points for \(\gamma\). Moreover, for \(r \neq 1\), the restriction of \(\varphi\) to \(A_r\) is identically zero, since for \(x \in A_r\), we have \(\varphi(x) = \varphi(\gamma(x)) = r \varphi(x)\). Equivalently, \(\varphi = \varphi \circ E\), where \(E\) is the projection of \(A\) onto \(A_1\) defined by

\[
E(x) = (1/n) \sum_{0 \leq i \leq n-1} \gamma^i(x).
\]

Let \(q\) be any primitive \(n^{th}\) root of unity, so \(\mathbb{Z}_n^*\) is generated by \(q\) as a multiplicative group. If \(a \in A_q\), we say the degree of \(a\) with respect to \(q\) is \(r\), and we write \(\delta_q(a) = r\). For homogeneous elements \(a\) and \(b\), we have \(\delta_q(ab) \equiv \delta_q(a) + \delta_q(b) \mod n\).

**Definition 1.2.** Let \(A\) and \(B\) be two unital subalgebras of a \(\mathbb{Z}_n\)-graded noncommutative probability space \((C, \varphi, \gamma)\), with \(A\) and \(B\) invariant under the grading automorphism \(\gamma\). The algebras \(A\) and \(B\) are said to be \(\mathbb{Z}_n\)-graded independent if

1. \(A\) and \(B\) gradedly commute; that is, there exists a primitive \(n^{th}\) root of unity \(q\) such that if for all \(a \in A\) and \(b \in B\),

\[
ba = q^{\delta_q(a)\delta_q(b)}ab.
\]

2. \(\varphi(ab) = \varphi(a)\varphi(b)\) for all \(a \in A\) and \(b \in B\).

Two examples of algebras with \(\mathbb{Z}_n\)-graded independent subalgebras are the \(\mathbb{Z}_n\)-graded tensor product of two \(\mathbb{Z}_n\)-graded graded algebras and the rotation algebras with parameter an \(n^{th}\) root of unity. See Section 3 for a discussion of these examples.

1.2. **Linearization.** In both classical and free probability, there exist transforms which linearize the addition of independent random variables. Let \((A, \varphi)\) be a noncommutative probability space. For \(a \in A\), the **moment sequence** of \(a\) is the sequence of numbers \((\mu_k = \varphi(a^k))_{k \geq 0}\). The moment sequence determines a linear functional \(\mu_a : \mathbb{C}[X] \to \mathbb{C}\) by \(\mu_a(X^k) = \mu_k\) for \(k \geq 0\). A **linearizing transform** is a bijective correspondence \((\mu_k)_{k \geq 0} \mapsto (\alpha(a))_{k \geq 1}\) such that for independent random variables \(a, b\), we have \(\alpha(a + \delta) = \alpha(a) + \alpha(b)\). The sequence \(\alpha\) is called the **cumulant sequence**. To obtain explicit formulas and to facilitate computation, it is useful to use introduce an (exponential or ordinary) generating function \(R(z) = R[\mu_a](z)\) for the cumulant sequence \(\alpha(a)\). The linearizing property of the transform is then expressed as

\[
R[\mu_{a+b}](z) = R[\mu_a](z) + R[\mu_b](z),
\]
for independent random variables $a$ and $b$.

1.2.1. **Classical cumulants and classical independence.** Let $\mu : \mathbb{C}[x] \rightarrow \mathbb{C}$ be a unital linear functional, with moments $\mu_k = \mu(X^k)$, $k \geq 0$. The classical cumulant sequence corresponding to $\mu$ is determined recursively by the relations

$$
\mu(X^k) = \sum_{P \in \mathcal{P}[k]} \prod_{B \text{ a block of } P} \alpha_{\text{card}(B)} \quad (k \geq 1),
$$

(1.1)

where $\mathcal{P}[k]$ denotes the family of set partitions of the interval $[k] = 1, 2, \ldots, k$. For example,

$$
\begin{align*}
\mu_1 &= \alpha_1 \\
\mu_2 &= \alpha_2^2 + \alpha_2 \\
\mu_3 &= \alpha_3^2 + 3\alpha_1\alpha_2 + \alpha_3 \\
\mu_4 &= \alpha_4^4 + 4\alpha_1\alpha_3 + 3\alpha_2^2 + 6\alpha_2\alpha_1^2 + \alpha_4.
\end{align*}
$$

The exponential generating function for the classical cumulants

$$
R_1[\mu](z) = \sum_{k \geq 1} \frac{\alpha_k}{k!} z^k
$$

satisfies

$$
R_1[\mu](z) = \log \left( \sum_{k \geq 0} \frac{\mu_k(X^k)}{k!} z^k \right).
$$

(1.2)

See, for example, Section II.12.8 in [11]. For classically independent random variables $x$ and $y$, we have

$$
R_1[\mu_{x+y}](z) = R_1[\mu_x](z) + R_1[\mu_y](z)
$$

(1.3)

Formally, we have $R_1[\mu_x](z) = \log \varphi \exp(zx)$, and

$$
R_1[\mu_{x+y}](z) = \log[\varphi \exp(z(x+y))] = \log[\varphi(\exp(zx)\exp(zy))]
$$

$$
= \log[\varphi(\exp(zx))\varphi(\exp(zy))]
$$

$$
= \log[\varphi(\exp(zx))] + \log[\varphi(\exp(zy))]
$$

$$
= R_1[\mu_x](z) + R_1[\mu_y](z).
$$

These computations can be justified by formal power series manipulations.

1.2.2. **Free cumulants and free independence.** The free cumulants of a moment sequence $\mu_k = \mu(X^k)$ are determined recursively by the requirements

$$
\mu(X^k) = \sum_{P \in \mathcal{NC}[k]} \prod_{B \text{ a block of } P} \alpha_{\text{card}(B)} \quad (k \geq 1),
$$

(1.4)
where now the sum is over noncrossing partitions of \([k]\). Free cumulants were introduced by Speicher \[12\]. For example,

\[
\begin{align*}
\mu_1 &= \alpha_1 \\
\mu_2 &= \alpha_1^2 + \alpha_2 \\
\mu_3 &= \alpha_1^3 + 3\alpha_1\alpha_2 + \alpha_3 \\
\mu_4 &= \alpha_1^4 + 4\alpha_1\alpha_3 + 2\alpha_2 + 6\alpha_2\alpha_1 + \alpha_4.
\end{align*}
\]

The ordinary generating function for the free cumulants

\[
R_0[\mu](z) = \sum_{k \geq 1} \alpha_k z^k
\]

is the \(R\)-transform of Voiculescu \[14\], in the formulation due to Speicher \[12\]. The \(R\)-transform satisfies

\[
\sum_{k \geq 0} \frac{\mu(X^k)}{\zeta^k} = \left[ \frac{1 + R_0[\mu](z)}{z} \right]<-1> (\zeta),
\]

where \(<-1>\) indicates inversion with respect to composition of power series. \(R_0\) linearizes addition of free–independent random variables,

\[
R_0[\mu_{a+b}](z) = R_0[\mu_a](z) + R_0[\mu_b](z),
\]

when \(a\) and \(b\) are free–independent random variables. See \[14\] and \[12\] for details and proofs.

1.2.3. \(q\)-cumulants and graded independence. Nica and Mingo (\[10\] and \[5\]) observed that there exists a simultaneous \(q\)-deformation of the formulas (Equations (1.1) and (1.4)) defining the classical and free cumulants. For a parameter \(q\), they define the \(q\)-cumulants as follows.

**Definition 1.3.** The \(q\)-cumulants \((\alpha_k(q))_{k \geq 1}\) of a moment sequence \(\mu_k = \mu(X^k)\) are determined recursively by the requirements

\[
\mu(X^k) = \sum_{P \in \Pi[k]} q^{c_0(P)} \prod_{B \text{ a block of } P} \alpha_{\text{card}(B)}^{(q)} (k \geq 1),
\]

where \(c_0(P)\) is the reduced crossing number of the partition \(P\), defined in Definition \[3\].

For example,

\[
\begin{align*}
\mu_1 &= \alpha_1^{(q)} \\
\mu_2 &= (\alpha_1^{(q)})^2 + \alpha_2^{(q)} \\
\mu_3 &= (\alpha_1^{(q)})^3 + 3\alpha_1^{(q)}\alpha_2^{(q)} + \alpha_3^{(q)} \\
\mu_4 &= (\alpha_1^{(q)})^4 + 4\alpha_1^{(q)}\alpha_3^{(q)} + (2 + q)(\alpha_2^{(q)})^2 + 6\alpha_2^{(q)}(\alpha_1^{(q)})^2 + \alpha_4^{(q)}.
\end{align*}
\]

For \(q = 1\), one recovers the classical cumulants, while for \(q = 0\), one obtains the free cumulants, since \(c_0(P) = 0\) precisely when \(P\) is noncrossing.
Mingo and Nica asked whether $\mu \mapsto \alpha(q)$ is a linearizing transform for some species of independent random variables. They showed that, in fact, for $q = -1$, $\mu \mapsto \alpha(-1)$ linearizes addition of $\mathbb{Z}_2$–graded independent random variables. The main result of the present paper is that for $q$ a primitive $n$th root of unity, $\mu \mapsto \alpha(q)$ linearizes addition of $\mathbb{Z}_n$–graded independent random variables.

For $q$ a proper, but not necessarily primitive, $n$th root of unity, we define the $R_q^{(n)}$–transform of a moment sequence $\mu_k = \mu(X^k)$ by

$$R_q^{(n)}[\mu](z) = \sum_{k \geq 1} \frac{\alpha(q)^k}{k!} z^{nk}.$$ 

Moreover, for $a$ a homogeneous element of a $\mathbb{Z}_n$–graded algebraic noncommutative probability space, and for $q$ a primitive $n$th root of unity we define the graded $R_q^{(n)}$ transform of $a$ to be

$$R_q^{(n)}[a](z) = R_q^{(n')}[\mu_a](z),$$

where $n' = n/\gcd(\delta_q(a), n)$ and $\delta(a)$ is the $q$–degree of $a$.

We show that

$$R_q^{(n)}[a](z) = \log \left[ \sum_{k \geq 0} \frac{\mu_a(X^{kn'})}{k!} z^{kn'} \right],$$

and

$$R_q^{(n)}[a_1 + a_2] = R_q^{(n)}[a_1] + R_q^{(n)}[a_2]$$

for homogeneous $\mathbb{Z}_n$–graded independent random variables of the same degree (Theorems 5.3 and 5.4).

Our approach closely follows that of [5], but several technical innovations are necessary for the proofs.

2. Examples of $\mathbb{Z}_n$–graded independence

2.1. Graded tensor product. The $\mathbb{Z}_n$–graded tensor product of two $\mathbb{Z}_n$–graded algebraic noncommutative probability spaces depends on the choice of a primitive $n$th root of unity $q$. Let $(\mathcal{A}, \varphi, \alpha)$ and $(\mathcal{B}, \psi, \gamma)$ be two $\mathbb{Z}_n$–graded algebraic noncommutative probability spaces. Define $\mathcal{A} \otimes_q \mathcal{B}$ to be $\mathcal{A} \otimes \mathcal{B}$ as a vector space, with the algebra structure given by

$$(a \otimes b)(a' \otimes b') = q^\delta_q(b)\delta_q(a') ab' \otimes bb'.$$

Then $(\mathcal{A} \otimes_q \mathcal{B}, \varphi \otimes \psi, \alpha \otimes \gamma)$ is a $\mathbb{Z}_n$–graded algebraic noncommutative probability space, and the injections $\mathcal{A} \to \mathcal{A} \otimes_q \mathcal{B}$ and $\mathcal{B} \to \mathcal{A} \otimes_q \mathcal{B}$ given by $a \mapsto a \otimes_q 1$, $b \mapsto 1 \otimes b$ are morphisms of $\mathbb{Z}_n$–graded algebraic noncommutative probability spaces.

The tensor product $\otimes_q$ is associative, but not commutative.

A more general construction of graded tensor products has been discussed, for example, in [4]. The $\mathbb{Z}_n$ graded tensor product has the following universal property:

**Proposition 2.1.** Suppose

1. $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}' \subseteq C$. 


2. \( \varphi : A \to A' \) and \( \psi : B \to B' \) are morphisms of \( \mathbb{Z}_n \)-graded algebras.

3. \( q \) is a primitive \( n \)th root of unity.

4. \( A' \) and \( B' \) \( q \)-commute in \( C \); that is, for homogeneous elements \( a \in A' \) and \( b \in B' \),
   \[ ba = q^{\delta_q(a)\delta_q(b)}ab. \]

Then there is a unique morphism of \( \mathbb{Z}_n \)-graded algebras \( \varphi \otimes q \psi : A \otimes_q B \to C \) such that \((\varphi \otimes q \psi)(a \otimes 1) = \varphi(a)\) for all \( a \in A \) and \((\varphi \otimes q \psi)(1 \otimes b) = \psi(b)\) for all \( b \in B \).

**Proof.** Left to the reader. \( \square \)

**Proposition 2.2.** Let \((C,\varphi,\gamma)\) be a \( \mathbb{Z}_n \)-graded algebraic noncommutative probability space, and let \( A,B \) be subalgebras of \( C \) invariant under \( \gamma \). Suppose \( q \) is a primitive \( n \)th root of unity such that \( A' \) and \( B' \) \( q \)-commute. The following are equivalent:

1. \( A \) and \( B \) are \( \mathbb{Z}_n \)-graded independent.

2. There is a morphism of \( \mathbb{Z}_n \)-graded algebraic noncommutative probability spaces
   \[ h : (A \otimes_q B,\varphi \otimes \varphi) \to (C,\varphi) \]
   such that for all \( a \in A \) and \( b \in B \), \( h(a \otimes b) = ab \).

**Proof.** Straightforward. Compare [3], Proposition 3.7. \( \square \)

### 2.2. Rational rotation algebras.

A rotation algebra \( \mathcal{R} \) is a unital algebra generated by two invertible elements \( u,v \) satisfying
\[
vu = quv
\]
for some complex number \( q \) of modulus 1. (The \( C^* \)-algebra version, which has been much studied, is a unital \( C^* \)-algebra generated by two unitaries \( u,v \) satisfying the relation (2.1); see [1].) A rotation algebra \( \mathcal{R} \) has a basis consisting of (certain) monomials \( u^m v^n \).

We endow \( \mathcal{R} \) with the unital linear functional \( \varphi \) determined by
\[
\varphi(u^m v^n) = \begin{cases} 0 & u^m v^n \neq 1 \\
1 & u^m v^n = 1. \end{cases}
\]

Consider the algebra \( \mathcal{L} = \mathbb{C}[[x,x^{-1}]] \) of formal Laurent series, with the linear functional \( \tau : \sum a_i x^i \mapsto a_0 \). Give the linear space \( \mathcal{L} \otimes \mathcal{L} \cong \mathbb{C}[[x,x^{-1},y,y^{-1}]] \) the algebra structure determined by
\[
x^m y^n x^{m'} y^{n'} = q^{m'n} x^{m+m'} y^{n+n'}.
\]

Denote this algebra by \( \mathcal{L} \otimes_q \mathcal{L} \). The linear functional \( \tau \otimes \tau \) on \( \mathcal{L} \otimes_q \mathcal{L} \) satisfies
\[
\tau \otimes \tau(x^m y^n) = \begin{cases} 0 & (m,n) \neq (0,0) \\
1 & (m,n) = (0,0). \end{cases}
\]

For any rotation algebra \( \mathcal{R} \), with parameter \( q \), there is a unique unital algebra homomorphism \( \psi : \mathcal{L} \otimes_q \mathcal{L} \to \mathcal{R} \) such that \( \psi(x) = u \) and \( \psi(y) = v \). Moreover, \( \varphi \circ \psi = \tau \otimes \tau \); that is, \( \psi \) is a morphism of noncommutative probability spaces.
The algebra $L$ has a unital algebra automorphism $\gamma$ determined by $\gamma : x^m \mapsto q^m x^m$ and satisfying $\tau \circ \gamma = \tau$. The tensor square of $\gamma$ is an algebra automorphism of $L \otimes_q L$; in fact, $\gamma \otimes \gamma$ is the inner automorphism $\gamma \otimes \gamma = \text{Ad}(x^{-1}y)$, as is easily checked.

Now suppose that $q$ is a primitive $n$th root of unity; in this case, the automorphism $\gamma$ of $L$ has order $n$, and thus $(L, \tau, \gamma)$ is a $\mathbb{Z}_n$-graded algebraic noncommutative probability space. Furthermore, $(L \otimes_q L, \tau \otimes \tau, \gamma \otimes \gamma)$ is the $\mathbb{Z}_n$-graded tensor square of $(L, \tau, \gamma)$.

Consider any rotation algebra $R$ generated by invertible elements $u, v$ satisfying $(2.1)$, with $q$ a primitive $n$th root of unity. Such an algebra is called a rational rotation algebra. $R$ has the inner automorphism $\alpha = \text{Ad}(u^{-1}v)$, which is of order $n$ and which satisfies $\varphi \circ \alpha = \alpha$. Thus, $(R, \varphi, \alpha)$ is a $\mathbb{Z}_n$-graded algebraic noncommutative probability space.

**Proposition 2.3.** There is a unique homomorphism of $\mathbb{Z}_n$-graded noncommutative probability spaces $\psi : (L \otimes_q L, \tau \otimes \tau, \gamma \otimes \gamma) \to (R, \varphi, \alpha)$ satisfying $\psi(x) = u$ and $\psi(y) = v$. Furthermore, the subalgebras $A$ generated by $u$ and $B$ generated by $v$ are $\mathbb{Z}_n$-graded independent in $R$.

**Proof.** Straightforward. \hfill $\square$

**Remark 2.4.** Unfortunately, the computation of the moments of the Harper operator $u + u^{-1} + v + v^{-1}$, carried out in $[8]$ for the universal rotation algebra with parameter $q = \pi$, cannot be repeated here for $q$ an arbitrary root of unity, using the techniques of this paper. The reason is that for $n > 2$, the elements $u + u^{-1}$ and $v + v^{-1}$ are no longer homogeneous.

**2.3. Generalized Clifford algebras.** Let $q$ be a primitive $n$th root of unity. The generalized Clifford algebra $C_m^{(n)}$ (see [3], [7]) is the universal unital algebra with generators $e_i$, $(1 \leq i \leq m)$ subject to the relations

$$e_i^n = 1, \quad e_je_i = qe_ie_j \quad \text{for} \ i < j.$$ 

For $m = 2$, this is just the rotation algebra

$$C_2^{(n)} \cong \mathbb{C}[\mathbb{Z}_n^*] \otimes_q \mathbb{C}[\mathbb{Z}_n^*].$$

For $m > 2$, it is the $m$-fold graded tensor power of $\mathbb{C}[\mathbb{Z}_n^*]$:

$$C_m^{(n)} \cong \mathbb{C}[\mathbb{Z}_n^*] \otimes_q \mathbb{C}[\mathbb{Z}_n^*] \otimes \cdots \otimes_q \mathbb{C}[\mathbb{Z}_n^*].$$

Let $\varphi$ denote the unital linear functional $\varphi$ on $C_m^{(n)}$ whose value on nontrivial monomials in the $e_i$ is zero. For any $k$, the two subalgebras generated by $\{e_i : i \leq k\}$ and $\{e_i : i > k\}$ are $\mathbb{Z}_n$-graded independent with respect to $\varphi$.

3. Crossings and ordered partitions.

This section contains the combinatorial results on crossings of set partitions which underlie the main results on additivity of the graded $R$ transform.

We recall the concepts of a crossing and of the restricted crossing number $[8]$ of a set partition of an interval $[N] = \{1, 2, \ldots, N\}$. The restricted crossing number plays
a key role in the combinatorics of set partitions related to ideas of independence in noncommutative probability.

**Definition 3.1.** Let \( P \) be a partition of \([N]\). A **crossing** of \( P \) is a quadruple \( a_1 < b_1 < a_2 < b_2 \), where \( a_i \in A, b_i \in B \), and \( A, B \) are distinct blocks of the partition. The number of crossings of \( P \) is the **crossing number** \( c(P) \).

Note that if one draws the points of \([N]\) in cyclic order on a circle and connects each pair of points belonging to the same block by a straight line, then crossings correspond to pairs of crossing lines. The crossing number \( c(P) \) is therefore invariant under cyclic permutations of the points of \([N]\). A partition \( P \) is called noncrossing if it has no crossings, \( c(P) = 0 \).

**Definition 3.2.** A crossing \( a_1 < b_1 < a_2 < b_2 \) is **left reduced** if \( a_1 \) and \( b_1 \) are minimum in the blocks which contain them. The number of left reduced crossings is the **restricted crossing number** of \( P \), denoted \( c_0(P) \).

Note that a partition is noncrossing if, and only if, its restricted crossing number is zero. The restricted crossing number does not share the symmetries of the crossing number; it is not invariant under cyclic permutations of \([N]\). For this reason, it seems a less natural statistic on set partitions than the full crossing number. Nevertheless, our main technical results involve the restricted crossing number rather than the full crossing number; these results concern the evaluation of quantities of the form \( \sum_{P \in \mathcal{X}} q^{c_0(P)} \), where \( \mathcal{X} \) is a collection of set partitions of an interval, and \( q \) is a parameter. It seems that the results cannot be revised to remain valid for the full crossing number in place of the restricted crossing number.

We will evaluate \( c_0(P) \) by a sum, as follows. For disjoint subsets \( A, B \) of \([N]\), set
\[
c_0(A, B) = \begin{cases} 
0 & \text{if } \min B < \min A \\
\text{card}(\{(a, b) \in A \times B : \min B < a < b\}) & \text{if } \min A < \min B.
\end{cases}
\]

Then for a partition \( P = \{A_1, \ldots, A_s\} \) of \([N]\), we have
\[
c_0(P) = \sum_{i \neq j} c_0(A_i, A_j).
\]

It is convenient to have the auxiliary notions of an **ordered set partition** and the **sorting number** of an ordered set partition.

**Definition 3.3.** An **ordered set partition** \( P = (A_1, \ldots, A_s) \) of a set \( S \) is a sequence of subsets of \( S \) such that \( \{A_1, \ldots, A_s\} \) is a partition of \( S \).

An ordered set partition \( P \) with \( s \) parts determines a surjective map \( f_P : S \to [s] \); namely, \( f(x) = j \) if, and only if, \( x \in A_j \). Conversely, a surjective map \( f : S \to [s] \) determines the ordered set partition \((f^{-1}(1), \ldots, f^{-1}(s))\).
Definition 3.4. Given an ordered set partition $P = (A_1, \ldots, A_s)$ of an interval $[n]$, define the sorting number of $P$ to be
\[ x(P) = \sum_{i<j} \text{card}(\{(x, y) \in A_i \times A_j : x < y\}). \]

Note that if $f = f_P$, then
\[ x(P) = \text{card}(\{(x, y) \in [n] \times [n] : x < y \text{ and } f(x) < f(y)\}). \]

Lemma 3.5. Let $P = (A, B)$ be an ordered partition of $[n]$ with two parts. Let $q$ be a proper $n^{\text{th}}$ root of unity. Let $\sigma$ denote the cyclic permutation of $[n]$, $\sigma = (1, 2, \ldots, n)$, in cycle notation. Then
\[
\sum_{0 \leq k \leq n-1} q^{x(\sigma^k(P))} = 0.
\]

Proof. Let $r = \text{card}(A)$. If $n \in B$, then $x(\sigma(P)) = x(P) - r$. If $n \in A$, then $x(\sigma(P)) = x(P) + (n - r) \equiv x(P) - r \mod n$. It follows that for all $k$, $x(\sigma^k(P)) \equiv x(P) - kr \mod n$. Therefore,
\[
\sum_{0 \leq k \leq n-1} q^{x(\sigma^k(P))} = q^{x(P)} \sum_{0 \leq k \leq n-1} q^{-kr} = 0.
\]

Definition 3.6. We define an action of $\mathbb{Z}_n$ on ordered set partitions of $[n]$, as follows. Let $P = (A_1, \ldots, A_s)$. Put $B = [n] \setminus A_1$. Put $\sigma(j) = j + 1 \mod n$. Put $B' = \sigma(B)$, and let $\theta : B \rightarrow B'$ be the unique order preserving bijection. Define $\sigma(P) = (A_1', \ldots, A_s')$, where $A_1' = \sigma(A_1)$, and $A_j' = \theta(A_j)$, for $j > 1$.

Remark 3.7. It is easy to see that $\sigma$ has order $n$ and that $P$ is fixed if, and only if, $P$ has only one part.

Example 3.8. $n = 7$, $P = (A_1, A_2, A_3)$, where $A_1 = \{1, 5, 6\}$, $A_2 = \{2, 7\}$, and $A_3 = \{3, 4\}$. Then $\sigma(P) = (A_1', A_2', A_3')$, where $A_1' = \{2, 6, 7\}$, $A_2' = \{1, 5\}$, $A_3' = \{3, 4\}$. The map $f : [n] \rightarrow [3]$ corresponding to $P$ is \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 3 & 2 & 1 & 1 
\end{pmatrix}.
\]
The map corresponding to $\sigma(P)$ is \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 3 & 2 & 1 & 1 
\end{pmatrix}.
\]

Lemma 3.9. Let $P = (A_1, \ldots, A_s)$ be an ordered partition of $[n]$ with $s \geq 2$ parts. Let $q$ be a proper $n^{\text{th}}$ root of unity. Let $\sigma$ denote the generator of the $\mathbb{Z}_n$ action on ordered partitions of $[n]$. Then
\[
\sum_{0 \leq k \leq n-1} q^{x(\sigma^k(P))} = 0.
\]
Proof. For two subsets $A, B$ of $[n]$, let $x(A, B) = \text{card}((x, y) \in A \times B : x < y)$. Write $\sigma(P) = (A_1', \ldots, A_s')$. Set $B = [n] \setminus A_1$, and $B' = [n] \setminus A_1' = \sigma(B)$. We have
\[
x(P) = \sum_{i<j} x(A_i, A_j) = x(A_1, B) + \sum_{2 \leq i<j} x(A_i, A_j).
\]
Similarly,
\[
x(\sigma(P)) = \sum_{i<j} x(A_i', A_j') = x(A_1', B') + \sum_{2 \leq i<j} x(A_i', A_j').
\]
But $x(A_i', A_j') = x(A_i, A_j)$ if $2 \leq i < j$. Note that $(A_1, B)$ is an ordered partition with 2 parts and $\sigma(A_1, B) = \sigma(A_1, B) = (A_1', B')$. By the argument of Lemma \[\text{Lemma} 3.10\], we are going to define an action of $\mathbb{Z}_n$ on $P_n([mn])$ such that each part has cardinality divisible by $n$. Denote the collection of such set partitions by $\mathcal{P}_n([mn])$. For $0 \leq k \leq m-1$, let $J_k$ denote the interval $\{kn+1, \ldots, (k+1)n\}$. Let $\mathcal{P}_n^0([mn])$ denote those set partitions $P \in \mathcal{P}_n([mn])$ such that for all $k$ and for all $A \in P$, either $J_k \subseteq A$, or $J_k \cap A = \emptyset$. Equivalently, the requirement on $P$ is that for each $k$, the set partition $P|_{J_k}$ induced on $J_k$ by $P$ is the trivial set partition $\{J_k\}$.

We are going to define an action of $\mathbb{Z}_n$ on $P_n([mn])$ whose fixed point set is precisely $\mathcal{P}_n^0([mn])$. If $P \in \mathcal{P}_n^0([mn])$, set $\sigma(P) = P$.

For $P \in (\mathcal{P}_n([mn]) \setminus \mathcal{P}_n^0([mn]))$, there exists $k$ ($1 \leq k \leq m$) such that the induced partition $P|_{J_k}$ is not the trivial partition of $J_k$. Let $k_0$ be the maximum of such $k$.

Write $P = (A_1, \ldots, A_s)$, with the order given by $i < j$ if, and only if, $A_i < \min A_j$. Consider the induced partition on $J_{k_0}$, with the induced order; namely, $P|_{J_{k_0}} = (B_1, \ldots, B_s)$, with $B_j = A_i \cap J_{k_0}$, and $i_1 < i_2 < \cdots < i_s$. By definition of $k_0$, we have $s \geq 2$.

We let $\mathbb{Z}_n$ act on $P$ via its action on $P|_{J_{k_0}}$. More precisely, let the generator $\sigma$ act on $P|_{J_{k_0}}$ as in Definition \[\text{Definition} 3.9\], and put $\sigma(P|_{J_{k_0}}) = (B_1', \ldots, B_s')$. Define $\sigma(P) = (A_1', \ldots, A_s')$

where
\[
A_i' = \begin{cases} A_i & \text{if } A_i \cap J_{k_0} = \emptyset, \\
B_j' \cup \bigcup_{k \neq k_0} (A_i \cap J_k) & \text{if } A_i \cap J_{k_0} = B_j.
\end{cases}
\]

Evidently, $\sigma$ has order $n$, and $P$ is fixed by $\sigma$ if, and only if, $P \in \mathcal{P}_n^0([mn])$.

Proposition 3.10. Let $P \in (\mathcal{P}_n([mn]) \setminus \mathcal{P}_n^0([mn]))$, and let $q$ be a nontrivial $n^{th}$ root of unity. Then
\[
\sum_{0 \leq k \leq n-1} q^{\text{ord}(\sigma^k(P))} = 0.
\]
Proof. Write \( P = (A_1, \ldots, A_t) \), with \( i < j \) if, and only if, \( \min A_i < \min A_j \). Write \( \sigma(P) = (A'_1, \ldots, A'_t) \). Let \( k_0 \) be the maximum of those \( k \) such that the restricted partition \( P|_{J_k} \) is nontrivial.

Set \( c_0(A_i, A_j)_{k, \ell} = \text{card}(\{(a, b) : a \in A_i \cap J_k, b \in A_j \cap J_\ell, \text{ and } \min A_i < \min A_j < a < b\}) \).

Then

\[
c_0(A_i, A_j) = \sum_{k \leq \ell} c_0(A_i, A_j)_{k, \ell}.
\]

Observe that \( c_0(A_i, A_j)_{k, \ell} = c_0(A'_i, A'_j)_{k, \ell} \), unless \( k = \ell = k_0 \) and both \( A_i \cap J_{k_0} \) and \( A_j \cap J_{k_0} \) are nonempty.

It follows that

\[
c_0(\sigma(P)) = c_0(P) = \sum_{i < j} c_0(A'_i, A'_j)_{k_0, k_0} - \sum_{i < j} c_0(A_i, A_j)_{k_0, k_0}.
\]

Note that if \( A \in P \) and \( A \cap J_{k_0} \neq \emptyset \), then \( \min A < k_0n + 1 \). This is because \( \text{card}(A) \equiv 0 \mod n \) by definition of \( P_n \), \( \text{card}(A \cap J_k) \equiv 0 \mod n \) if \( k > k_0 \), by definition of \( k_0 \), and \( \text{card}(A \cap J_{k_0}) \neq 0 \mod n \) by assumption. This implies there exists \( k < k_0 \) such that \( A \cap J_k \neq \emptyset \). It follows from this observation that if \( A_i \cap J_{k_0} \) and \( A_j \cap J_{k_0} \) are nonempty, then

\[
c_0(A_i, A_j)_{k_0, k_0} = x(A_i \cap J_{k_0}, A_i \cap J_{k_0}),
\]

and, therefore,

\[
\sum_{i < j} c_0(A_i, A_j)_{k_0, k_0} = \sum_{i < j} x(A_i \cap J_{k_0}, A_i \cap J_{k_0}) = x(P|_{J_{k_0}}),
\]

and similarly for \( \sigma(P) \) replacing \( P \). It follows that

\[
c_0(\sigma(P)) - c_0(P) = x(\sigma(P)|_{J_{k_0}}) - x(P|_{J_{k_0}}),
\]

and the conclusion of the proposition follows from this and Lemma 3.9.

\[ \square \]

Corollary 3.11. Fix natural numbers \( m \) and \( n \). For \( q \) a proper \( n^{th} \) root of unity,

\[
\sum_{P \in P_n[mn]} q^{c_0(P)} = \text{card}(P_n^0[mn]).
\]

Proof. It is easy to see that for \( P \in P_n^0[mn] \), \( c_0(P) \equiv 0 \mod n \), so \( q^{c_0(P)} = 1 \). On the other hand, if \( P \in (P_n[mn] \setminus P_n^0[mn]) \), then \( P \) belongs to a nontrivial orbit \( \mathcal{O} \) for the \( \mathbb{Z}_n \) action, and, by Proposition 3.10, the sum over the orbit is zero, \( \sum_{Q \in \mathcal{O}} q^{c_0(Q)} = 0 \).

\[ \square \]

Definition 3.12. Let \( \lambda \) be a partition of the natural number \( N \); that is, \( \lambda \) is a sequence of natural numbers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \) such that \( \sum_i \lambda_i = N \). The \( \lambda_i \) are called the parts of \( \lambda \). A set partition \( P \) of \([N]\) is said to be of type \( \lambda \) if the cardinalities of the parts of \( P \), listed in decreasing order, are the parts of \( \lambda \). If \( \lambda \) is a partition of \( N \) and \( m \) is a natural number, the partition \((m\lambda_1, m\lambda_2, \ldots, m\lambda_r)\) of \( mN \) is denoted by \( m\lambda \).
Corollary 3.13. Fix natural numbers $m$ and $n$. Let $\lambda$ be a partition of $m$. For $q$ a proper $n^{\text{th}}$ root of unity,

$$\sum_{P \in \mathcal{P}_n[mn] \atop \text{type}(P) = n\lambda} q^{c_0(P)} = \text{card} \{ P \in \mathcal{P}[m] : \text{type}(P) = \lambda \}.$$ 

Proof. The set of $P \in \mathcal{P}_n[mn]$ that are of type $n\lambda$ is invariant under the $\mathbb{Z}_n$ action on $\mathcal{P}_n[mn]$. The proof of the previous corollary shows that

$$\sum_{P \in \mathcal{P}_n[mn] \atop \text{type}(P) = n\lambda} q^{c_0(P)} = \text{card} \{ P \in \mathcal{P}_n[0mn] : \text{type}(P) = n\lambda \}.$$ 

But $\{ P \in \mathcal{P}_n[0mn] : \text{type}(P) = n\lambda \}$ is in one to one correspondence with $\{ P \in \mathcal{P}[m] : \text{type}(P) = \lambda \}$. \hfill \qed

Remark 3.14. For $n = 2$ and $P \in \mathcal{P}_n[nm]$, Mingo and Nica show that $c_0(P) \equiv c(P) \mod n$; therefore for $n = 2$, Corollaries 3.11 and 3.13 are also valid with the restricted crossing number replaced by the full crossing number. However, for $n \geq 3$, the congruence $c_0(P) \equiv c(P) \mod n$ does not hold, and the analogues of Corollaries 3.11 and 3.13 with $c_0(P)$ replaced by $c(P)$ are not valid, as shown by computations.

4. $\mathbb{Z}_n$-Graded Noncommutative Probability Spaces.

Let $(\mathcal{A}, \varphi, \gamma)$ be a $\mathbb{Z}_n$-graded algebraic noncommutative probability space. If $a$ is homogeneous, say $a \in \mathcal{A}_q$, then the only moments $\varphi(a^k)$ which are (possibly) nonzero are those for which $a^k \in \mathcal{A}_1$, or, equivalently, $q^k = 1$. This is true if, and only if, $k$ divides the order of $q$ in $\mathbb{Z}_n^*$, which is also the order of $\gamma|_{\mathcal{A}_q}$.

Lemma 4.1. Let $n$ be a natural number. Suppose $\mu$ satisfies $\mu(X^k) = 0$ unless $n$ divides $k$. For any $q$, the sequence $(\alpha_k^{(q)})$ of $q$-cumulants of $\mu$ also satisfies $\alpha_k^{(q)} = 0$ unless $n$ divides $k$.

Proof. This is easily seen by induction on $k$. \hfill \qed

Definition 4.2. Let $n$ be a natural number and let $q$ be a proper $n^{\text{th}}$ root of unity. Let $\mu : \mathbb{C}[X] \to \mathbb{C}$ be a linear functional with $\mu(1) = 1$ and $\mu(X^k) = 0$ unless $n$ divides $k$. Let $(\alpha_k^{(q)})_{k \geq 1}$ be the sequence of $q$-cumulants of $\mu$. The $R_q^{(n)}$-transform of $\mu$ is the power series

$$R_q^{(n)}[\mu](z) = \sum_{k \geq 1} \frac{\alpha_k^{(q)}}{k!} z^k.$$
For an arbitrary linear functional \( \mu : \mathbb{C}[X] \to \mathbb{C} \) with \( \mu(1) = 1 \), let \( (\alpha^{(1)}_k)_{k \geq 1} \) be the sequence of 1–cumulants of \( \mu \). The \( R_1 \)-transform of \( \mu \) is the power series

\[
R_1[\mu](z) = \sum_{k \geq 1} \frac{\alpha^{(1)}_k}{k!} z^k.
\]

**Remark 4.3.** It is necessary to specify \( n \) as well as \( q \) in \( R_q^{(n)} \), as it is not assumed that \( q \) is a primitive \( n^{th} \) root of unity.

**Theorem 4.4.** Let \( n \) be a natural number and let \( q \) be a proper \( n^{th} \) root of unity. Let \( (\mathcal{A}, \varphi) \) be an algebraic noncommutative probability space. Suppose \( a \in \mathcal{A} \) satisfies \( \varphi(a^k) = 0 \) unless \( n \) divides \( k \). Let \( (\alpha^{(q)}_k)_{k \geq 1} \) be the sequence of \( q \)-cumulants of \( \mu_a \). (Recall that \( \alpha^{(q)}_k = 0 \) unless \( n \) divides \( k \).) Let \( (\beta_k)_{k \geq 1} \) be the sequence of 1–cumulants of \( \mu_{a^n} \). Then for all \( k \geq 1 \),

\[
\alpha^{(q)}_{nk} = \beta_k.
\]

**Proof.** The proof is the same, *mutatis mutandis*, as that of Corollary 1.8 in [5]. We give the proof here for the sake of completeness. The proof goes by induction on \( k \). For \( k = 1 \), we have

\[
\alpha^{(q)}_n = \mu_a(X^n) = \varphi(a^n) = \mu_{a^n}(X) = \beta_1.
\]

Fix \( k > 1 \) and suppose the equality \( \alpha^{(q)}_{nk'} = \beta_{k'} \) holds for all \( k' < k \). We have

\[
\varphi(a^{kn}) = \mu_a(X^{kn}) = \sum_{P \in \mathcal{P}[kn]} q^{c_0(P)} \prod_{B \text{ a block of } P} \alpha^{(q)}_{\text{card}(B)}.
\]

Since \( \alpha^{(q)}_r = 0 \) unless \( n \) divides \( r \), the latter sum reduces to

\[
\sum_{P \in \mathcal{P}[kn]} q^{c_0(P)} \prod_{B \text{ a block of } P} \alpha^{(q)}_{\text{card}(B)} = \sum_{\lambda \text{ partition of } k} \sum_{P \in \mathcal{P}_n[kn]} q^{c_0(P)} \prod_{i} \alpha^{(q)}_{n\lambda_i}.
\]

Applying Corollary 3.13 to the inner sum, we obtain

(4.1)

\[
\varphi(a^{kn}) = \sum_{\lambda \text{ partition of } k} \left( \sum_{P \in \mathcal{P}[k]} 1 \right) \prod_{i} \alpha^{(q)}_{n\lambda_i}.
\]

On the other hand, we have

\[
\varphi(a^{kn}) = \mu_{a^n}(X^k) = \sum_{P \in \mathcal{P}[k]} \prod_{B \text{ a block of } P} \beta_{\text{card}(B)}
\]

(4.2)

\[
= \sum_{\lambda \text{ partition of } k} \left( \sum_{P \in \mathcal{P}[k]} 1 \right) \prod_{i} \beta_{\lambda_i}.
\]
Comparing Equations (4.1) and (4.2) gives

\[(4.3) \quad \sum_{\lambda \text{ partition of } k} \left( \sum_{P \in \mathcal{P}[k]} 1 \right) \left[ \prod_{i} \beta_{\lambda_i} - \prod_{i} \alpha_{n\lambda_i}^{(q)} \right] = 0.\]

All of the expressions \( \left[ \prod_{i} \beta_{\lambda_i} - \prod_{i} \alpha_{n\lambda_i}^{(q)} \right] \) are zero, by the induction assumption, with the exception of that corresponding to \( \lambda = (k) \), which is \( \beta_k - \alpha_{nk}^{(q)} \). It follows that \( \beta_k - \alpha_{nk}^{(q)} = 0 \) as well.

**Corollary 4.5.** Let \( n \) be a natural number and let \( q \) be a proper \( n^{th} \) root of unity. Let \((\mathcal{A}, \varphi)\) be an algebraic noncommutative probability space. Suppose \( a \in \mathcal{A} \) satisfies \( \varphi(a^k) = 0 \) unless \( n \) divides \( k \). Then

\[ R_{q^n}^{(n)}[\mu_a](z) = R_{1}[\mu_{a^n}](z^n). \]

**Proof.** This is immediate from the theorem and the definition of the \( R \)-transforms. \( \square \)

### 5. \( \mathbb{Z}_n \)-Graded Independence and the \( \mathbb{Z}_n \)-Graded \( R \)-Transform.

Let \( n \) be a natural number and let \((\mathcal{C}, \varphi, \gamma)\) be a \( \mathbb{Z}_n \)-graded algebraic noncommutative probability space.

**Lemma 5.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \mathbb{Z}_n \)-graded independent subalgebras of \( \mathcal{C} \). Let \( q \) be a primitive \( n^{th} \) root of unity such that \( \mathcal{A} \) and \( \mathcal{B} \) \( q \)-commute. Let \( a \in \mathcal{A} \) and \( b \in \mathcal{B} \) be homogenous elements of the same \( q \)-degree \( r \). Write \( n' = n/\gcd(r, n) \). Then \( (a + b)^{n'} = a^{n'} + b^{n'} \).

**Proof.** The elements \( a \) and \( b \) satisfy \( ba = q^{r^2}ab \). Write \( q' \) for \( q^{r^2} \), and note that \( q' \) is a primitive \( n^{th} \) root of unity. Let \( 0 < k < n' \), and consider the sum \( S \) of those terms in the expansion of \( (a + b)^{n'} \) that are of degree \( k \) in \( a \) and degree \( n' - k \) in \( b \). It follows from the relation \( ba = q'ab \) that

\[ S = a^k b^{n'-k} \sum_{(A,B)} (q')^{x((B,A))}, \]

where the sum is over all ordered set partitions \((B, A)\) of \([n']\) with two parts of cardinalities \( \text{card}(B) = n' - k, \text{card}(A) = k, \) and \( x((B,A)) \) is as in Definition 3.4. Consider the \( \mathbb{Z}_{n'} \) action on such ordered partitions, induced by the action on \([n']\) by powers of the cyclic permutation \((1, 2, \ldots, n')\). According to Lemma 3.5, for any orbit \( \mathcal{O} \), the sum over the orbit \( \sum_{P \in \mathcal{O}} (q')^{x(P)} \) is equal to zero, since \( q' \) is a proper \( n^{th} \) root of unity. But then

\[ \sum_{(A,B)} (q')^{x((B,A))} = \sum_{\mathcal{O}} \sum_{P \in \mathcal{O}} (q')^{x(P)} = 0. \]

\( \square \)
Definition 5.2. Let $q$ be a fixed primitive $n^{th}$ root of unity. Let $a \in \mathcal{A}$ be homogeneous. If $a \not\in \mathcal{A}_1$, put $n' = n / \gcd(\delta_q(a), n)$. The graded $R^{(n)}_q$-transform of $a$ is defined by

$$R^{(n)}_q[a](z) = \begin{cases} R_1[\mu_a](z) & \text{if } a \in \mathcal{A}_1 \\ R^{(n')}_q[\mu_a](z) & \text{if } a \in \mathcal{A}_{q^{r'}} \neq \mathcal{A}_1. \end{cases}$$

The proofs of the following two results follow those of the corresponding results in [5].

Theorem 5.3. Let $a$ be a homogeneous element of $\mathcal{A}$, and let $q$ be a primitive $n^{th}$ root of unity. Set

$$n' = \begin{cases} 1 & a \in \mathcal{A}_1 \\ n / \gcd(\delta_q(a), n) & \text{otherwise}. \end{cases}$$

The graded $R^{(n)}_q$-transform of $a$ satisfies

$$R^{(n)}_q[a](z) = \log \left[ \sum_{k \geq 0} \frac{\mu_a(X^{kn'})}{k!} z^{kn'} \right].$$

Proof. If $a \in \mathcal{A}_1$, this is immediate from Equation (1.2). Otherwise, put $r = \delta_q(a)$. We have

$$R^{(n)}_q[a](z) = R^{(n')}_{q^r}[\mu_a](z) = R_1[\mu_{a^{r'}}](z^{r'})$$

$$= \log \left[ \sum_{k \geq 0} \frac{\mu_{a^{r'}}(X^k)}{k!} (z^{r'})^k \right]$$

$$= \log \left[ \sum_{k \geq 0} \frac{\mu_a(X^{kn'})}{k!} z^{kn'} \right],$$

where the second equality follows from Corollary 4.5 and the third from Equation (1.2).

Theorem 5.4. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{Z}_n$-graded independent subalgebras of $\mathcal{C}$. Let $a \in \mathcal{A}$ and $b \in \mathcal{B}$ be homogeneous elements of the same degree. Then

$$R^{(n)}_q[a + b] = R^{(n)}_q[a] + R^{(n)}_q[b].$$

Proof. Note that the fixed point algebras $\mathcal{A}_1$ and $\mathcal{B}_1$ are classically independent; that is, they mutually commute, and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x \in \mathcal{A}_1$, $y \in \mathcal{B}_1$. Furthermore, for classically independent random variables $x$ and $y$, one has

$$R_1[\mu_{x+y}] = R_1[\mu_x] + R_1[\mu_y].$$  \hspace{1cm} (5.1)

Therefore, for $a, b \in \mathcal{A}_1$, the result follows from Definition 5.2 and Equation (5.1).

If $a, b \not\in \mathcal{A}_1$, let $r$ denote their $q$-degree and put $n' = n / \gcd(r, n)$. We have

$$R^{(n)}_q[a + b](z) = R^{(n')}_{q^r}[\mu_{a+b}](z) = R_1[\mu_{(a+b)^{r'}}](z^{n'})$$

where the first equality is by Definition 5.2 and the second by Corollary 4.5.
According to Lemma 5.1, we have \((a + b)^{n'} = a^{n'} + b^{n'}\), so
\[
R_1[\mu_{(a+b)^{n'}}](z^{n'}) = R_1[\mu_{a^{n'}+b^{n'}}](z^{n'}).
\]
Observe that \(a^{n'}\) and \(b^{n'}\) are elements of the classically independent fixed point algebras \(A_1, B_1\), so
\[
R_1[\mu_{a^{n'}+b^{n'}}](z^{n'}) = R_1[\mu_{a^{n'}}](z^{n'}) + R_1[\mu_{b^{n'}}](z^{n'})
= R_q^{(n')}[\mu_a](z) + R_q^{(n')}[\mu_b](z)
= R_q^{(n')}[a](z) + R_q^{(n')}[b](z),
\]
using Equation (5.1), Corollary 4.5 and Definition 5.2.

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Department of Mathematics, University of Iowa, Iowa City, Iowa

E-mail address:  goodman@math.uiowa.edu