Spaces of Dyadic Distributions

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ABSTRACT. This paper studies spaces of distributions on a dyadic half-line, which is the positive half-line equipped with bitwise binary addition and Lebesgue measure. We prove the nonexistence of a space of dyadic distributions which satisfies a number of natural requirements (for instance, the property of being invariant with respect to the Walsh–Fourier transform) and, in addition, is invariant with respect to multiplication by linear functions. This, in particular, is evidence that the space of dyadic distributions suggested by S. Volosivets in 2009 is optimal. We also show applications of dyadic distributions to the theory of refinement equations and wavelets on the dyadic half-line.

KEY WORDS: dyadic half-line, distributions, Walsh functions, Walsh–Fourier transform, refinement equations, wavelets.

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1. Introduction

The existence of a more or less reasonable definition of a space of dyadic distributions has been discussed in the literature and at conferences since the early 2000s. The dyadic half-line is the positive half-line equipped with bitwise binary addition and Lebesgue measure. We prove the nonexistence of a space of dyadic distributions which satisfies a number of natural requirements (for instance, the property of being invariant with respect to the Walsh–Fourier transform) and, in addition, is invariant with respect to multiplication by linear functions. This, in particular, is evidence that the space of dyadic distributions suggested by S. Volosivets in 2009 is optimal. We also show applications of dyadic distributions to the theory of refinement equations and wavelets on the dyadic half-line.

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The main problem in constructing the spaces $D_d$ and $S_d$ is that they are not invariant with respect to the Walsh–Fourier transform. Thus, the Walsh–Fourier transform cannot be well-defined on the corresponding distribution spaces $D'_d$ and $S'_d$. This problem, however, was solved by S. S. Volosivets in 2009. Note that similar constructions in the space $L_2$ over other groups appeared in [4]–[6]. In [1] Volosivets suggested another, narrower than $D_d$, space $H_d$ of test functions, which is the space of the so-called dyadic-analytic functions $f$, that is, finite linear combinations of indicator functions $\chi_{\Delta_{j,k}}$, where $\Delta_{j,k} = [2^{-j}k, 2^{-j}(k+1))$ is the dyadic interval of rank $j \in \mathbb{Z}$. The term “dyadic-analytic” is used because, for all such functions $f$, $f(\cdot + h) - f(\cdot) \equiv 0$ for an arbitrary $h \in (0, 2^{-n})$, where $n$ is the highest rank of the intervals in the linear combination.

The topology is defined by convergence to zero: $f_k \to 0$ as $k \to \infty$ if the ranks of the intervals in the supports of the functions in the sequence $\{f_k\}_{k \in \mathbb{N}}$ are bounded above and contained in a fixed interval and the sequence itself converges to zero pointwise. It is easy to show that $H_d$ is a complete linear space invariant with respect to the Walsh–Fourier transform. Linearity and completeness can be checked directly, and invariance follows from the fact that if $\text{supp } f \subset [0, 2^m]$ and $n$ is the highest rank of the intervals in the support of $f$, then $\text{supp } \hat{f} \subset [0, 2^m]$ and $m$ is the highest rank of the intervals in the support of $\hat{f}$. Thus, the Walsh–Fourier transform maps the space of test functions into itself. The Walsh–Fourier transform on the corresponding space $H_d$ of distributions is defined as usual: for each $f \in H'_d$, we have $(\hat{f}, \varphi) = (f, \hat{\varphi})$, $\varphi \in H_d$, where $\hat{\varphi}$ denotes the inverse Walsh–Fourier transform.

2. On the Possibility of Multiplication by Smooth Functions. The Main Result

The space of distributions suggested by Volosivets is very convenient due to its simplicity and wide range of applications. While the space of test functions is very narrow (it only contains functions generated by binary dilations and shifts of the function $\chi_{[0,1]}$), the space of distributions is rather wide. For instance, every function $f$ locally integrable on $\mathbb{R}_+$ belongs to $H'_d$, and therefore so does its Walsh–Fourier transform. No slow-growth conditions on $f$ are needed. Thus, say, the exponent also belongs to $H'_d$, which is not true for the classical Schwartz space.

The main disadvantage of $H'_d$ is that it is not invariant with respect to multiplication by smooth functions, in particular, by polynomials. For example, it is not generally true that $xf \in H'_d$ for $f \in H'_d$. The natural question is whether there is a space of distributions on the dyadic half-line that is invariant with respect to both the Walsh–Fourier transform and multiplication by smooth functions, e.g., by polynomials. Theorem 1 gives a negative answer to this question; it essentially establishes the nonimprovability of the space $H'_d$. There is no extension of $H'_d$ that allows multiplication even by linear functions.

First we need to introduce some notation. For arbitrary $x, y \in \mathbb{R}_+$, we set $(y, x) = \sum_{k \in \mathbb{Z}} y_k x_{1-k}$, where the $x_i$ and $y_i$ are digits in the binary expansions of $x$ and $y$, respectively. This sum always contains only finitely many nonzero terms. For an integer $k \geq 0$, the Walsh function is defined as $w_k(x) = (-1)^{(k,x)}$. Let $\psi(x, y) = w_{\lceil y \rceil}(x) \cdot w_{\lfloor y \rfloor}(y)$, where $\lceil y \rceil$ is the integer part of $y$. The Walsh–Fourier transform of a function $f \in L_1(\mathbb{R}_+)$ is $\hat{f}(y) = \int_{\mathbb{R}_+} \psi(x, y) f(x) dx$; it is extended to $L_2(\mathbb{R}_+)$ in a standard way. The Walsh–Fourier transform is an invertible orthogonal transform of $L_2(\mathbb{R}_+)$ [3], [14].

What can be the space of test functions for a distribution space on the dyadic half-line? It is natural to require that this space contain the indicator function $\chi_{[0,1]}$ and be invariant with respect to the integer shift $f(x) \mapsto f(x + 1)$, as well as with respect to the binary contraction $f(x) \mapsto f(2x)$ and the binary expansion $f(x) \mapsto f(x/2)$. In this case, it already contains $H_d$. Thus, $H_d$ is the smallest (by inclusion) functional space satisfying these requirements. The property of being invariant with respect to the Walsh–Fourier transform is fulfilled automatically. Indeed, $\hat{\chi}_{[0,1]} = \chi_{[0,1]}$, so the Walsh–Fourier transform maps $H_d$ into itself. The question arises whether it
is possible to extend \( H_d \) so as to make it also invariant with respect to multiplication by algebraic polynomials. If so, then the corresponding space of distributions would also allow multiplication by polynomials: if \( f \in H'_d \), then \( xf \) is defined, as in the classical case, by \( (xf, \varphi) = (f, x\varphi) \), where \((f, \varphi)\) denotes the action of the distribution \( f \) on a test function \( \varphi \in H_d \).

Theorem 1 provides a negative answer to the above question under yet another natural condition: the space of distributions must contain the functional space \( L_2(\mathbb{R}_+) \), which means that each \( g \in L_2(\mathbb{R}_+) \) acts naturally on \( H_d \), i.e., the integral \( \int_{\mathbb{R}_+} fg \, dx \) is defined for each function \( f \) in the new space.

**Theorem 1.** There is no vector space of measurable functions on the dyadic half-line containing the indicator function \( \chi_{[0,1)} \) and invariant with respect to both the Walsh–Fourier transform and multiplication by linear functions such that on this space every element of the space \( L_2(\mathbb{R}) \) acts by the formula of inner product.

**Proof.** Suppose that such a space exists; we denote it by \( \tilde{H}_d \). Computing the Walsh–Fourier transform of the function \( f(x) = x\chi_{[0,1)}(x) \), we obtain

\[
\hat{f}(y) = \int_0^1 x \cdot \psi(x, y) \, dx = \int_0^1 x \cdot w_{[y]}(y) \cdot w_{[y]}(x) \, dx = \int_0^1 x \cdot w_{[y]}(x) \, dx,
\]

since \( w_{[x]}(y) = w_0(y) = 1 \). This integral depends only on the integer part of \( y \). We calculate it for \( y \in [2^n, 2^n + 1) \) such that \( [y] = 2^n \), \( n \in \mathbb{N} \), i.e., \( [y] = 10 \ldots 0 \) \((n\) zeroes); then \( w_{[y]}(x) = (-1)^{(2^n, x)} = (-1)^{2 - n - 1} \). Thus, for \( y \in [2^n, 2^n + 1) \), we obtain

\[
\int_0^1 x \cdot w_{[y]}(x) \, dx = \int_0^1 x \cdot x_{2^{n+1}} \, dx = \sum_{k=0}^{2^{n+1}} (-1)^k \int_{2^{n+1}k}^{2^{n+1}(k+1)} x \, dx
\]

\[
= \sum_{k=0}^{2^{n+1}} (-1)^k \frac{(k+1)^2 - k^2}{2^{n+3}} = -2^{-(n+1)}.
\]

Finally,

\[
\hat{f}(y) = -2^{-(n+1)}, \quad y \in [2^n, 2^n + 1), \quad n \in \mathbb{N}.
\]

By assumption, \( \hat{f}(y) \in \tilde{H}_d \); therefore, \( y\hat{f}(y) \in \tilde{H}_d \). On the other hand, each element of \( L_2(\mathbb{R}_+) \) acts naturally on \( \tilde{H}_d \). We choose the following function \( g \in L_2(\mathbb{R}_+) \):

\[
g(y) = \begin{cases} 
-1/(n+1) & \text{if } y \in [2^n, 2^n + 1), \quad n \in \mathbb{N}, \\
0 & \text{otherwise}.
\end{cases}
\]

Then

\[
(g, y\hat{f}(y)) = -\sum_{n=1}^{\infty} \frac{1}{n+1} \int_{2^n}^{2^{n+1}} \frac{y}{2^{n+1}} \, dy = -\sum_{n=1}^{\infty} \frac{1}{n+1} \frac{y^2}{2^{n+2}} \big|_y^{2^{n+1}}
\]

\[
= -\sum_{n=1}^{\infty} \frac{1}{n+1} \frac{2^{2n} - (2^n + 1)^2}{2^{n+2}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{2^{2n} - (2^n + 1)^2}{2^{n+2}} = \infty.
\]

Consequently, the action \((g, y\hat{f}(y))\) is not defined, which leads to a contradiction. \( \square \)

**Remark.** One may also define the space of test functions to be the space \( Q_d \) of smooth rapidly decreasing dyadic functions. By the dyadic smoothness of a function \( f \) we mean that

\[
\|f(x \oplus t) - f(x)\|_2 \leq C(\alpha) \cdot t^\alpha, \quad t > 0,
\]

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for each $\alpha > 0$; as usual, we say that $f$ is rapidly decreasing if $|f(x)| \leq C(x+1)^{-\alpha}$ for any $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$. The space $Q_d$ is invariant with respect to the Walsh–Fourier transform, and it is also invariant with respect to multiplication by smooth dyadic functions of bounded (at most polynomial) growth. However, dyadic smoothness is not the same as smoothness in the classical sense, and $Q_d$ is not invariant with respect to multiplication by linear functions.

3. Applications to Wavelet Theory

The first examples of systems of wavelets on the dyadic half-line and its various generalizations can be found in the paper [9] of Lang; more general constructions were presented in [15], [12], and [13]. Wavelets on more general Abelian groups have also been studied [10]. To construct a system of wavelets, one must solve the refinement equation, which is a functional difference equation for a function $\varphi$ with binary expansion of the argument $x$:

$$
\varphi(x) = \sum_{k=0}^{2^n} c_k \varphi(2x \ominus k), \quad x \in \mathbb{R}_+.
$$

Such equations are also used when studying dyadic approximation algorithms [8]. The theory of refinement equations on the classical real line $\mathbb{R}$ has largely been completing by the late 1980s ([7], [11]). However, many questions have remained unsolved on the dyadic half-line. Does the refinement equation always have a solution, and if so, in what class of functions? Is a solution unique up to multiplication by a constant? If a solution is compactly supported, what is the length of its support? Using the space $H_d$ of dyadic distributions, we can fully answer these questions (see Theorem 2). First, let us introduce some further notation.

A solution of the refinement equation is called a refinable function; this is a fixed point of

$$
T \psi = \sum_{k=0}^{2^n} c_k \psi(k),
$$

defined by $T \psi = \sum_{k=0}^{2^n} c_k \psi(2x \ominus k)$. We set $m(y) = \frac{1}{2} \sum_{k=0}^{2^n} c_k \mathbf{w}_k(y)$. This Walsh polynomial is called the mask of the refinement equation. It is known [12] that studying general refinement equations can be reduced to the case $\sum_{k} c_k = 2$, which is equivalent to $m(0) = 1$.

**Theorem 2.** For each sequence $\{c_k\}_{k=0}^{2^n}$ of complex coefficients summing up to 2, the refinement equation has a solution $\varphi \in H_d'$ unique up to multiplication by a constant in the class of distributions. The support of this solution $\varphi$ lies in the segment $[0, 2^n]$, and the Walsh–Fourier transform of $\varphi$ is given by the formula

$$
\hat{\varphi}(y) = \prod_{j=1}^{\infty} m(2^{-j}y).
$$

Moreover, for each compactly supported integrable function $f \in L_2(\mathbb{R}_+)$, the sequence $T^k f$ converges in $H_d'$ to the solution of the refinement equation $c \varphi$, where $c = \int_{\mathbb{R}_+} f(x) \, dx$.

**Proof.** Using properties of the Walsh–Fourier transform, we obtain $\widehat{T^k f}(y) = m(y/2) \hat{f}(y/2)$. Consequently, for each $k$, we have

$$
\widehat{T^k f}(y) = \hat{f}(2^{-k}y) \prod_{j=1}^{\infty} m(2^{-j}y) \tag{1}
$$

We show that, for each compactly supported function $f \in L_1(\mathbb{R}_+)$, the product (1) converges uniformly on each segment $[0, 2^N]$. Note that the function $m(y)$ is a Walsh polynomial of degree $2^n$, so it is constant on the dyadic intervals of rank $n$. Since $\sum_i c_i = 2$, it follows that $m(0) = 1$. Thus, $m(z) = 1$ for each $z \in [0, 2^{-n})$. Therefore, given $k > n + N$, we have $2^{-k}y \in [0, 2^{-n})$ for each $y \in [0, 2^N]$, and hence $m(2^{-k}y) = 1$. So, on the segment $[0, 2^N]$ each term in (1) with number $k > n + N$ is identically equal to 1 on $[0, 2^N]$, and the product converges on this segment. Thus, the
product (1) converges uniformly on each compact set in $\mathbb{R}_+$; consequently, it converges in the sense of distributions in $H'_d$. Therefore, the $T^k f$ converge in $H'_d$ to a distribution $\psi$. We have $T\psi = \psi$, i.e., $\psi$ is a solution of the refinement equation.

Since $f$ is compactly supported, we can assume that it is supported on $[0, 2^\ell)$. Then on the segment $[0, 2^\ell)$ the function $f$ is identically equal to $f(0) = \int_{\mathbb{R}_+} f \, dt = c$. Given $k > \ell + N$, we obtain $2^{-k}y \in [0, 2^{-\ell})$ for each $y \in [0, 2^N]$; hence $\hat{f}(2^{-k}y) = c$. Thus, the product (1) converges as $k \to \infty$ to $c \prod_{j=1}^{\infty} m(2^{-j}y)$. We see that the inverse Walsh–Fourier transform of this product is nothing but a solution of the refinement equation. Therefore, the inverse Walsh–Fourier transform of $\prod_{j=1}^{\infty} m(2^{-j}y)$ is also a refinable function, which we denote by $\varphi$. Finally, for each compactly supported function $f$, the sequence $T^k f$ converges to $c \varphi$.

Now let $\varphi_0$ be an arbitrary compactly supported solution of the refinement equation. Since $T \varphi_0 = \varphi_0$, it follows that $T^k \varphi_0 = \varphi_0$ for each $k$. Hence the sequence $T^k \varphi_0$ converges to $\varphi_0$, and the function $\varphi_0$ is proportional to $\varphi$ (namely, $\varphi_0 = \varphi \int_{\mathbb{R}} \varphi_0 \, dx$). This implies the uniqueness of the solution. Finally, to prove that the support of the solution lies in $[0, 2^N]$, it is enough to consider an arbitrary function $f$ supported on this segment and apply the operator $T$. The function $Tf$ is also supported on the same segment, and so do all the functions $T^k f$; therefore, the limit (which is the solution of the refinement equation) of the sequence $T^k f$ is also supported on $[0, 2^N]$.

Theorem 2 provides a tool for studying properties of many refinements equations. For instance, the following fact is useful in probability theory, approximation theory, and the theory of subdivision schemes.

**Corollary 1.** If all the coefficients $c_k$ of a refinement equation are nonnegative, then its solution $\varphi$ normalized by the condition $\int_{\mathbb{R}_+} \varphi \, dx = 1$ is a nonnegative distribution.

**Proof.** Take $f = \chi_{[0,1)}$ and consider the sequence $T^k f$. Since the operator $T$ preserves the nonnegativity of functions, it follows that all the elements of that sequence are also nonnegative. Hence its limit, which is the solution, is nonnegative as well.

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