A Survey on the Complemented Subspace Problem

Mohammad Sal Moslehian
Dept. of Math., Ferdowsi Univ., P.O.Box 1159, Mashhad 91775, Iran
E-mail: msalm@math.um.ac.ir
http://www.um.ac.ir/~moslehian/

Abstract

The complemented subspace problem asks, in general, which closed subspaces $M$ of a Banach space $X$ are complemented; i.e. there exists a closed subspace $N$ of $X$ such that $X = M \oplus N$? This problem is in the heart of the theory of Banach spaces and plays a key role in the development of the Banach space theory. Our aim is to investigate some new results on complemented subspaces, to present a history of the subject, and to introduce some open problems.

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1 Introduction.

The problem related to complemented subspaces are in the heart of the theory of Banach spaces. These are more than fifty years old and play a key role in the development of the Banach space theory. Our aim is to review of results on complemented subspaces, to present a history of the subject, and to introduce some open problems.

We start with simple observations concerning definition and properties of complemented subspaces. Some useful sources are [8], [17], [28].

Let $X$ be a normed space, $M, N$ be algebraically complemented subspaces of $X$ (i.e. $M + N = X$ and $M \cap N = \{0\}$), $\pi : X \rightarrow \frac{X}{M}$ be the quotient map, $\phi : M \times N \rightarrow X$ be the natural isomorphism $(x, y) \mapsto x + y$ and $P : X \rightarrow M, P(x + y) = x, x \in M, y \in N$ be the projection of $X$ on $M$ along $N$. Then the following statements are equivalent:

(i) $\phi$ is a homeomorphism.

(ii) $M$ and $N$ are closed in $X$ and $\pi|_N$ is a homeomorphism.

(iii) $M$ and $N$ are closed and $P : X \rightarrow M$ is a bounded projection.

The Subspaces $M$ and $N$ are called topologically complemented or simply complemented if each of the above equivalent statements holds. If $N_1, N_2$ are complemented subspaces of a closed subspace $M$, then $N_1$ and $N_2$ are isomorphic Banach spaces.

It is known that every finite dimensional subspace is complemented and every algebraic complement of a finite codimension subspace is topologically complemented.

In a Banach space $X$, applying the closed graph theorem we can establish that two closed subspace are algebraically complemented if and only if they are complemented. Moreover, if $M$ is a closed subspace of $X$, then $M$ is complemented if and only if the following equivalent assertions hold:

(I) The quotient map $i : M \hookrightarrow X$ has a left inverse as a continuous
operator.

(II) The natural projection $\pi : M \to \frac{X}{M}$ has a right inverse as a continuous operator.

$l^\infty$ is complementary in every normed space $X$ containing it isomorphically as a closed subspace [28]. Also, if $c_0$ is subspace of a separable Banach space $X$, then there is a bounded projection $P$ of $X$ onto $c_0$ of norm $\leq 2$, cf. [44].

Suppose now that $F$ is a retract of a Banach space $X$, i.e. $F$ is a Banach subspace of $X$ and there is a continuous linear map $\phi : X \to F$ such that for all $x \in F, \phi(x) = x$. Then $C_0(X - F) = \{f \in C(X) : f(x) = 0$ for all $x \in F\}$ is complemented in $C(X)$. In fact, by defining $P : C(X) \to C(X)$ by $P(g) = g \circ \phi$, we have $P^2 = P, \|P(g)\| = \sup_{x \in X} |g(\phi(x))| \leq \|g\|$ and $\text{Ker}P = \{g \in C(X)|g(\phi(x)) = 0$ for all $x \in X\} = C_0(X - F)$.

Hence we may say that "complemented ideal" is the Gelfand dual of "retract closed subspace" (see [31]).

There are non-complemented closed subspaces. For example, let $X$ be the disk algebra, i.e. the space of all analytic functions on $\{z \in \mathbb{C}; |z| < 1\}$ which are continuous on the closure of $D$. Then the subspace of $C(T)$ consisting of the restrictions of functions of $X$ to $T = \{z \in \mathbb{C}; |z| = 1\}$ is not complemented in $X$ (see [18]).

Throughout the paper $c_0, c, l^\infty, l_p$ denote the space of all complex sequences $\{x_n\}$ such that $\lim_{n \to \infty} x_n = 0$, $\{x_n\}$ is convergent, $\{x_n\}$ is bounded, and $\sum_{n=1}^{\infty} |x_n|^p < \infty$, respectively. In addition, $L_p$ denotes the $L_p$-space over the Lebesgue interval $[0, 1]$. The reader is referred to [20] and [26] for undefined terms and notation.
2 Complementary subspace problem and related results.

This problem asks, in general, which closed subspaces of a Banach space are complemented?

In 1937, Murray [32] proved, for the first time, that $l_p, p \neq 2, p > 1$ has non-complemented subspace.

Phillips [38] proved that $c_0$ is non-complemented in $l^\infty$. This significant fact has been refined, reproved or generalized by many mathematicians, cf. [37], [16], [12] and [34].

Banach and Mazur showed that all subspaces in $C[0,1]$ which are isometrically isomorphic to $l_1$ or $L^1[0,1]$ are non-complemented, cf. [43] and [1].

In 1960, Pelczynski [36] showed that complemented subspaces of $l_1$ are isomorphic to $l_1$. Köthe [22] generalized this result to the non-separable case.

In 1967, Lindenstrauss [25] proved that every infinite dimensional complemented subspace of $l^\infty$ is isomorphic to $l^\infty$. This also holds if $l^\infty$ is replaced by $l_p, 1 \leq p < \infty, c_0$ or $c$.

It is shown by Lindenstrauss [24] that if the Banach space $X$ and its closed subspace $Y$ are generated by weakly compact sets (in particular, if $X$ is reflexive), then $Y$ is complemented in $X$.

In 1971, Lindenstrauss and Tzafriri [26] proved that every infinite dimensional Banach space which is not isomorphic to a Hilbert space contains a closed non-complemented subspace.

Johnson and Lindenstrauss [13] proved the existence of a continuum of non-isomorphic separable $L^1$-spaces. (An $L^1$-space is a space $X$ for which $X^{**}$ is a complemented subspace of an $L^1$-space)

Classically known complemented subspaces of $L_p, 1 < p < \infty, p \neq 2$ are
$l_p, l_2, l_p \oplus l_2$ and $L_p$ itself. In 1981, Bourgain, Rosenthal and Schechtman proved that up to isomorphism, there exist uncountably many complemented subspaces of $L_p$.

It is shown that a complemented subspace $M$ of $l^*_\infty$ is isomorphic to $l^*_\infty$ provided $M$ is either $w^*$-closed or isomorphic to a bidual space, cf. [29].

Pisier established that any complemented reflexive subspace of a $C^*$-algebra is necessarily linearly isomorphic to a Hilbert space.

In 1993, Gowers and Maurey showed that there exists a Banach space $X$ without non-trivial complemented subspaces.

If $E$ is one of the spaces $l_p$, $(1 \leq p \leq \infty)$ or $c_0$, and $X$ is a vector space complemented in $E$ which contains a vector subspace $Y$ complemented in $X$ and isomorphic to $E$, then $X$ is isomorphic to $E$. Moreover, each infinite dimensional vector subspace complemented in $E$ is isomorphic to $E$. Conversely, if $Y$ is a vector subspace of $E = l^2$ or $c_0$ which is isomorphic to $E$, then $Y$ is complemented in $E$.

If $X$ is an infinite dimensional vector subspace complemented in some space $C(S)$, then $X$ contains a vector subspace isomorphic to $c_0$.

Randrianantoanina showed that if $X$ and $Y$ are isometric subspaces of $L_p$ $(p \neq 4, 6, ...)$, and $X$ is complemented in $L_p$ then so is $Y$. Moreover, the projection constant does not change. This number is defined to be $\inf\{\|T\| : T : L_p \to X$ is a bounded linear projection of $L_p$ onto $X\}$.

The above theorem fails in the case $p \geq 4$ is an even integer, i.e. there exist pairs of isomorphic subspaces $X$ and $Y$ of $L_p$ to itself so that $X$ is complemented and $Y$ is not.

### 3 Schroeder-Bernstein Problem.

If two spaces are isomorphic to complemented subspaces of each other, are then they isomorphic?
There are negative solutions to this problem. (see [15] and [?])

4 Basis and complemented subspaces.

A Schauder basis for a Banach space $X$ is a sequence $\{x_n\}$ in $X$ with the property that every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} \alpha_n x_n; \alpha_n \in \mathbb{C}$ in which the sum is convergent in the norm topology, cf. [20]. For example, the trigonometrical system is a basis in each space $L^p[0, 1], 1 < p < \infty$.

Pelczynski [36] showed that any Banach space with a basis is a complemented subspace of an isomorphically unique space.

In 1987, Szarek [45] showed that there is a complemented subspace without basis of a space with a basis and answered therefore to a problem of fifty years old.

5 Approximation property and complemented subspaces.

A Banach space $X$ has the approximation property (AP) if for every $\epsilon > 0$ and each compact subset $K$ of $X$ there is a finite rank operator $T$ in $X$ such that for each $x \in K$, $\|Tx - x\| < \epsilon$. If there is a constant $C > 0$ such that for each such $T$, $\|T\| \leq C$, then $X$ is said to have bounded approximation property (BAP), cf. [20]. For example, every Banach space with a basis has BAP.

Pelczynski [36] proved that every Banach space with the BAP can be complementably embedded in a Banach space with a basis.
6 Complemented minimal subspaces.

A Banach space $X$ is called minimal if every infinite dimensional subspace $Y$ of $X$ contains a subspace $Z$ isomorphic to $X$. For example $c_0$ is minimal. If $Z$ is also complemented then $X$ is said to be complementary minimal. Casazza and Odell [5] showed that Tsirelson’s space $T$ (see [10] and [12]) have no minimal subspaces.

Casazza, Johnson and Tzafriri [4] showed that the dual $T^*$ of $T$ is minimal but not complementary minimal.

7 quasi-complemented subspaces.

A closed subspace $Y$ of a Banach space $X$ is called quasi-complemented if there exists a closed subspace $Z$ of $X$ such that $Y \cap Z = \{0\}$ and $Y + Z$ is dense in $X$.

Then such a subspace $Z$ is said to be a quasi-complement of $Y$. Those notions are first introduced by Murray [33].

Every closed subspace of $l_\infty$ is quasi-complemented, cf. [12]. Also Mackey [27] proved that in a separable Banach space every subspace is quasi-complemented.

Rosenthal [41] showed that if $X$ is a Banach space, $Y$ is a closed subspace of $X$, $Y^*$ is $W^*$-separable and the annihilator $Y^\perp$ of $Y$ in $X^*$ has an infinite dimensional reflexive subspace, then $Y$ is quasi-complement in $X$.

8 Weakly complemented subspaces.

A closed subspace of a Banach space $X$ is called weakly complemented if the dual $i^*$ of the natural embedding $i : M \rightarrow X$ has a right inverse as a bounded operator.

For example, $c_0$ is weakly complemented in $l_\infty$, not complemented in $l_\infty$ (see [17]).
If $M$ is complemented in $X$ with the corresponding projection $P$, then the adjoint of $id_X - P$ is a projection in $B(X)$ with the range $M^o = \{f \in X^*; f|_M = 0\}$. Hence $M$ is weakly complemented in $X$.

9 contractively complemented subspaces.

As mentioned before, a closed subspace $Y$ of a Banach space $X$ is said to be complemented if it is the range of a bounded linear projection $P : X \to X$. If $\|P\| = 1$, $Y$ is called a contractively complemented or 1-complemented subspace of $X$.

Let $X$ be a Banach space with $\dim X \geq 3$. Then $X$ is isometrically isomorphic to a Hilbert space iff every subspace of $X$ is the range of a projection of norm 1 (see [21] and [2]).

In 1969, Zippin [18] proved that every separable infinite dimensional $L_1$-predual space (i.e a Banach space whose dual is isometric to $L_1(\mu)$ for some measure space $(\Omega, \Sigma, \mu)$) contains a contractively complemented subspace isomorphic to $c_0$.

Lindenstrauss and Lazar [23] proved that $X$ contains a contractively complemented subspace isometric to some space $C(S)$ when $X^*$ is non-separable.

Question. Let $X$ be a Banach space and $T : X \to X$ be an isometry. Is the range of $T$ contractively complemented in $X$?

In Hilbert and $L^p, (1 \leq p < \infty)$ spaces, we have an affirmative answer. In case $C[0, 1]$, however, it may happen that the range of an isometry is not complemented, cf. [1].

Pisier [39] proved that if $M$ is a Von Neumann subalgebra of $B(H)$ which is complemented in $B(H)$ and isomorphic to $M \otimes M$, then $M$ is contractively complemented.
10 Prime Banach spaces and complemented subspaces.

A Banach space $X$ is called prime if each infinite dimensional complemented subspace of $X$ is isomorphic to $X$, cf. [26].

Pelczynski [36] proved that $c_0$ and $l_p$ ($1 \leq p < \infty$) are prime. Lindenstrauss [25] proved that $l^\infty$ is also prime. Gowers and Maurey [13] constructed some new prime spaces.

11 Complemented subspaces of topological products and sums of Banach spaces.

Metafune and Moscatelli [30] proved that when $X$ is one of the Banach spaces $l_p$ ($1 \leq p \leq \infty$) or $c_0$, then each infinite dimensional complemented subspace of $X^N$ is isomorphic to one of the spaces $\omega, \omega \times X^N$ or $X^N$, where $\omega = K^N$ (K is the scalar field) and $X^N$ is the product of countably many copies of $X$.

In [11], the authors obtained a complete description of the complemented subspace of the topological product $l^m_\infty$ where $m$ is an arbitrary cardinal number.

Every complemented subspace of a product $V = \prod_{i \in I} X_i$ of Hilbert spaces is isomorphic to a product of Hilbert spaces ($I$ is a set of arbitrary cardinal), cf. [10].

Ostraskii [35] showed that not all complemented subspaces of countable topological products of Banach spaces are isomorphic to topological products of Banach spaces.

Chigogidze [6] proved that complemented subspaces of a locally convex direct sum of arbitrary collection of Banach spaces are isomorphic to locally
convex direct sum of complemented subspaces of countable subsums.

Chigogidze \[7\] proved that a complemented subspace of an uncountable topological product of Banach spaces is isomorphic to a topological product of complemented subspaces of countable subproducts and hence isomorphic to a topological product of Frechet spaces.

12 Some interesting problems.

The following problems in this area arise:

1) Given a Banach space $X$, characterize the isomorphic types of its complemented subspaces.

2) Given a Banach space $X$, characterize the isomorphic types of such Banach space $Z$ that every vector subspace of $Z$ isomorphic to $X$ is complemented in $Z$.

3) Is every complemented vector subspace of $C(S)$ isomorphic to some $C(S_1)$?

4) If a Banach space $X$ is complemented in every Banach space containing it, is $X$ isomorphism to some $C(S)$ over a Stone space $S$? (A space is Stonian if the closure of every open set is open)

5) Does every complemented subspace of a space with an unconditional basis have an unconditional basis? Recall that an unconditional basis for a Banach space is a basis $\{x_n\}$ such that every permutation of $\{x_n\}$ is also a basis or equivalently, the convergence of $\sum \alpha_n x_n$ implies the convergence of every rearrangement of the series, cf. \[20\].

6) If a von Neumann algebra is a complemented subspace of $B(H)$, is it then injective?

7) Are $l_p, 1 \leq p \leq \infty$ and $c_0$ the only prime Banach spaces with an unconditional basis? is still open.
Remark. Some pieces of information are taken from Internet-based resources without mentioning the URL’s.

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