The Magnetic Effect on Dynamical Tide in Rapidly Rotating Astronomical Objects

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Abstract

By numerically solving the equations of rotating magnetohydrodynamics, we study the magnetic effect on dynamical tide. We find that a magnetic field has a significant impact not only on the flow structure, i.e., the internal shear layers in a rotating flow can be destroyed in the presence of a moderate or stronger magnetic field (in the sense that the Alfvén velocity is at least of the order of 0.1 of the surface rotational velocity), but also on the dispersion relation of waves excited by tidal force such that the range of tidal resonance is broadened by a magnetic field. A major result is that the total tidal dissipation scales as a square of the field strength, which can be used to estimate the strength of the internal magnetic field in the astronomical object of a binary system. Moreover, with a moderate or stronger field, the ratio of magnetic dissipation to viscous dissipation is almost inversely proportional to the magnetic Prandtl number (i.e., the ratio of viscosity to magnetic diffusivity); thus, in the astrophysical situation at a small magnetic Prandtl number magnetic dissipation dominates over viscous dissipation with a moderate or stronger field.

Key words: binaries: general – magnetic fields

1. Introduction

In a binary system, e.g., binary stars, a star and a planet, a planet and a satellite, etc., the angular momentum transfers between the orbital motion and the rotational motion of astronomical objects such that the binary system evolves to the equilibrium state of minimum energy, the so-called synchronization and circulation processes. Tidal dissipation plays an important role in these processes. Tide consists of two parts: equilibrium tide, which is a response to the external tidal potential and a slow (compared to the dynamical timescale) quasi-hydrostatic displacement at the surface of astronomical objects (Ogilvie 2014), and dynamical tide, which is the internal fluid waves resonantly excited by the external tidal force. Equilibrium tide $\xi$ obtained by the hydrostatic balance is equal to the sum of the tidal potential $\Psi$ and the Eulerian perturbation of the self-gravitational potential $\Phi'$ divided by the local gravitational acceleration $g$, i.e., $\xi = -(\Psi + \Phi')/g$ (Souchay et al. 2013; Ogilvie 2014). The self-gravitational potential perturbation $\Phi'$ is an order-of-unit multiple of the tidal potential $\Psi$ and the major component of tidal potential is on the $(\ell = 2, m = 2)$ mode, where $\ell$ and $m$ are respectively degree and order of spherical harmonics. Therefore, equilibrium tide is on a large length scale, but the fluid waves of dynamical tide can have small length scales. Energy can dissipate either through the small-scale turbulent convective eddies on the dynamical state of minimum energy, the so-called large-scale equilibrium tide can have small length scales. Energy can dissipate either through the small-scale turbulent convective eddies or through the small-scale dynamical tide. We obtained by the hydrostatic balance is equal to the sum of the tidal potential $\Psi$ and the Eulerian part

Internal gravity waves have been extensively studied. Zahn (1975; 1977), Goldreich & Nicholson (1989) gave a physical interpretation and applied it to early-type stars (1989), Savonije & Papaloizou (1983) studied the non-adiabatic effect, Goodman & Dickson (1998) applied it to solar-type stars, Fuller & Lai (2012) applied it to white dwarfs with a sharp composition jump, etc. Inertial waves have also been extensively studied in Ogilvie (2007), Goodman & Lackner (2009), Ogilvie & Lin (2004), Ogilvie (2009), Ogilvie (2013), Wu (2005a), Wu (2005b), etc. Nonlinear effects on dynamical tide were considered in Kumar & Goodman (1996), Ogilvie (2007), Barker & Ogilvie (2010), Weinberg et al. (2012), Favier et al. (2014), Essick & Weinberg (2016), Wei (2016b), etc. A summary and details of these studies about dynamical tide can be found in the recent review paper Ogilvie (2014).

However, the magnetic effect on dynamical tide is not very clear. Buffett (2010) estimated the magnetic (or Ohmic) tidal dissipation in the Earth’s fluid core to extrapolate the strength of Earth’s magnetic field in the fluid core. He performed numerical calculations for rotating magnetohydrodynamics (MHD) in a spherical shell and assumed that magnetic dissipation concentrates in the internal shear layers built by the propagation of inertial waves. In his paper, how a magnetic field influences the internal shear layers was not considered,
i.e., the Lorentz force was neglected in the equation of fluid motion, and the dynamics was completely controlled by rotation. In Wei (2016a) I performed a local WKB analysis in rotating MHD studying the magnetic effect on dynamical tide, and found two results. The first is that magnetic dissipation wins out over viscous dissipation even with a weak magnetic field because the presence of a magnetic field modifies the dispersion relation of the waves of dynamical tide, i.e., inertial waves with only rotation become magneto-inertial waves with both rotation and a magnetic field. The second is that the frequency-averaged dissipation is constant regardless of rotation and magnetic field. But in the local analysis the global spherical geometry was not considered. Most recently, in an online published paper (Lin & Ogilvie 2018), the magnetic effect on dynamical tide was thoroughly studied in the presence of either a uniform vertical or a dipolar field in a spherical shell with the vortical forcing method developed by Ogilvie (2005). Lin & Ogilvie (2018) found similar results to Wei (2016a), namely a magnetic field changes the dissipations because wave dispersion is influenced by magnetic field such that magnetic dissipation can win out over viscous dissipation, and the frequency-averaged dissipation is independent of the dissipation mechanism. Moreover, they found that a sufficiently strong magnetic field changes the flow pattern of internal shear layers, and obtained the scaling law for the competition between rotation and the magnetic field, which is very useful for the estimation in the astrophysical situation.

In this paper I perform numerical calculations for rotating MHD in a spherical shell to study the magnetic effect on dynamical tide, and the impact of a magnetic field on flow structure and waves will also be considered. Although the problem and the setup in this study are similar to those in Lin & Ogilvie (2018), there are still some differences. The first is that I use the boundary flow method instead of the vortical forcing method and the time-stepping method instead of a super-large matrix solver (the matrix is very large, especially in cases with a strong field). The second is that I study the regime of a stronger field. The third is that I try to find the scaling laws for the tidal dissipation versus the magnetic field strength. In Section 2 the model of this numerical study is built, in Section 3 the numerical results are shown and discussed, and in Section 4 a brief summary is given.

2. Model

We use a spherical shell with the radius ratio $R_l/R_o = 1/2$ as the model for a spherical astronomical object. The shell rotates rapidly at a constant angular velocity $\Omega = \Omega \hat{z}$, where $\hat{z}$ is the unit vector in the rotational axis, and a conducting fluid resides within the shell to mimic the convective zone of a star or giant planet or the fluid core of a terrestrial planet or satellite. The inner sphere rotates uniformly to mimic the radiative zone of a star or giant planet or the solid inner core of a terrestrial planet or satellite. A uniform vertical magnetic field $B = B\hat{z}$ is imposed throughout the shell and the motion of the conducting fluid in the shell will induce a new field $\mathbf{b}$, which is weak compared to $B$. We numerically solve the equations of rotating MHD in a frame rotating at the angular frequency $\Omega$. Because sound waves in a compressible fluid are too fast to be resonantly excited by tidal force, we consider an incompressible fluid to filter out sound waves. This will bring us a big numerical advantage to avoid the very small time-step for capturing sound waves. We study the linear regime and neglect the nonlinear quadratic terms. We solve the dimensionless rather than dimensional equations, because dimensionless equations can give us more, e.g., scaling laws, to extrapolate to the real parameter regime, which is not accessible with current computational power.

The dimensionless equation of fluid motion in the rotating frame reads

$$\frac{\partial \mathbf{u}}{\partial t} = - \nabla p' + E \nabla^2 \mathbf{u} + 2\mathbf{u} \times \hat{z} + S^2(\nabla \times \mathbf{b}) \times \hat{z}. \quad (1)$$

In Equation (1), the term on the left side is the acceleration of fluid motion, the terms on the right side are successively the Eulerian perturbation of the pressure gradient incorporating the centrifugal potential perturbation, the viscous force, the Coriolis force arising from rotation, and the Eulerian perturbation of Lorentz force arising from magnetic field, and the nonlinear quadratic terms $\mathbf{u} \cdot \nabla \mathbf{u}$ and $(\nabla \times \mathbf{b}) \times \mathbf{b}$ are neglected. The tidal force will be implemented by the outer boundary condition. In Equation (1), length is normalized with the outer radius $R_o$, time is normalized with the inverse of rotational speed $\Omega^{-1}$, velocity is normalized with $\Omega R_o$, and the magnetic field is normalized with the strength of the imposed field $B$. The two dimensionless numbers are the Ekman number,

$$E = \frac{\nu}{\Omega R_o^2}, \quad (2)$$

which is the ratio of the rotational timescale $\Omega^{-1}$ to the viscous timescale $R_o^2/\nu$ ($\nu$ being viscosity). It measures the strength of rotation $\Omega$ relative to viscosity $\nu$, and the $S$ number

$$S = \frac{B}{\sqrt{\mu \nu \Omega R_o}}, \quad (3)$$

which is the ratio of the Alfven speed $B/\sqrt{\mu \nu}$ ($\mu$ being magnetic permeability) to the angular speed $\Omega R_o$. It measures the strength of the imposed field $B$ relative to rotation $\Omega$. The $S$ number is called the Lennert number (Lennert 1954).

On the inner boundary, the horizontal velocity is taken to be stress-free to minimize the effect of the boundary layer and the radial velocity vanishes, i.e., the fluid cannot penetrate across the inner boundary. On the outer boundary, the horizontal velocity is also taken to be stress-free but the radial velocity follows an oscillatory rising-falling equilibrium tide, i.e.,

$$u_r = AP_i^{\text{eq}}(\cos \theta)\cos m\phi - \omega t. \quad (4)$$

This boundary condition enforces the equilibrium tide at the outer surface to excite the dynamical tide in the interior. The interior flow driven by the boundary radial flow consists of both tidal flow (equilibrium tide) and waves (dynamical tide). In Equation (4), $A$ is the amplitude and in the linear problem it is taken to be 1, $P_i^{\text{eq}}$ is the associate Legendre polynomial normalized by $\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}$, and $l$ and $m$ are taken to be the major component of tidal potential, namely $(l = 2, m = 2)$, $\theta$ and $\phi$ are respectively colatitude and longitude in the spherical polar coordinates, and $\omega$ is the tidal frequency in the rotating frame, i.e., the Doppler-shifted frequency.
\[ \omega = \omega_i - m\Omega \] (\( \omega_i \) being the tidal frequency in the inertial frame). The method of using equilibrium tide to excite dynamical tide was proposed and validated by Ogilvie (2009) and was numerically implemented in Favier et al. (2014).

Campbell & Papaloizou (1986) studied the magnetic effect on stellar oscillations and found that a strong magnetic field induces a boundary layer to modify the radial flow at the surface. For stellar oscillations (the free-oscillation problem), the magnetic field is significant for boundary conditions, whereas for the tide (the forced-oscillation problem), it is not so significant because the tidal frequency is too slow compared to the dynamical frequency at the surface, i.e., the low-frequency limit. We now illustrate this point. The normal stress at the disturbed free surface vanishes, i.e.,

\[ -(p' - \rho g \xi_r) - \frac{2B \cdot b}{2\mu} + 2\rho \mu \frac{\partial u_r}{\partial r} + \frac{2B \cdot b}{\mu} = 0. \] (5)

The terms are successively the Eulerian perturbation of fluid pressure, the excess pressure due to the disturbed surface, the Eulerian perturbation of magnetic pressure, the viscous normal stress, and the Eulerian perturbation of magnetic normal stress.

The round bracket consisting of the first two terms is the Lagrangian perturbation of pressure. In the Eulerian perturbations of the magnetic pressure and normal stress, the quadratic terms \( b^2/2\mu \) and \( b^2/\mu \) are neglected in the linear regime. In Ogilvie (2009) it was verified that in the low-frequency limit \( \omega \ll \omega_d \sim (g/R)^{1/2} \), the Eulerian perturbation of fluid pressure and the viscous normal stress are negligible such that the radial displacement is the equilibrium tide with a correction due to self-gravitation. This result is in agreement with Goldreich & Nicholson (1989). Thus, the radial flow at the free surface is proportional to the tidal potential. We now show that the Eulerian perturbation of magnetic pressure is negligible in the low-frequency limit. The ratio of the Eulerian perturbation of magnetic pressure to the excess pressure is

\[ \frac{2B \cdot b}{2\mu} \left| \frac{1}{\rho g \xi_r} \right| \sim \frac{B^2}{\rho \mu (\Omega R)^2} \frac{b}{B (R \omega_d^2) \xi_r} \approx S^2 \frac{B \Omega}{\omega_d} \left( \frac{R}{\xi_r} \right)^2. \] (6)

The radial flow \( u_r \) at surface is \( u_r = \partial \xi_r / \partial t \approx \omega \xi_r \), and on the other hand \( u_r \sim A \Omega R \), where \( A \) is the dimensionless factor for the boundary radial flow (4). We are then led to

\[ \frac{2B \cdot b}{2\mu} \left| \frac{1}{\rho g \xi_r} \right| \sim \frac{S^2 B \Omega}{B \omega_d \omega_d A} \approx \frac{S^2 \Omega}{\omega_d \omega_d A} \approx \frac{A}{\omega_d \omega_d}. \] (7)

In the usual astrophysical situations, the magnetic field is not so strong that \( S \) is less than unity (in our calculations \( S^2 \) is up to 0.2), rotation frequency \( \Omega \) is much less than dynamical frequency \( \omega_d \), and tidal frequency \( \omega \) is also much less than dynamical frequency \( \omega_d \), i.e., the low-frequency limit. Although the factor \( A \) that measures the strength of the equilibrium tide is very small, it is still much larger than \( S^2 (\Omega / \omega_d) (\omega / \omega_d) \). Therefore, this ratio is very small and the magnetic effect at the surface is negligible. Similarly, the Eulerian perturbation of magnetic normal stress is also much less than the excess pressure. Consequently, in the tidal problem the magnetic field is negligible at the surface because of the low-frequency limit. The radial-flow boundary condition is valid for studying the magnetic effect on tide. However, it should be noted that in some situations this ratio is not so small. For example, a star or planet in a binary system rotates so fast that \( \Omega / \omega_d \sim 0.1 \), the orbital motion is so fast that \( \omega / \omega_d \sim 0.1 \) (breaking the low-frequency limit), and the surface displacement is so small that \( A \sim 0.01 \). If the magnetic field at the surface is so strong or density at the surface is so low that \( S^2 \) is of order unity, then this ratio will be of order unity at surface. Therefore, in a situation with fast rotation, fast orbital motion, a strong surface field, and a low surface density, a magnetic field will have a significant effect at the surface and it even deforms the spherical geometry. In our calculations we assume this ratio is much less than unity in the low-frequency limit and neglect the magnetic effect at the surface.

The dimensionless equation of the magnetic field reads

\[ \frac{\partial b}{\partial t} = \nabla \times (u \times b) + \frac{E}{Pm} \nabla^2 b, \] (8)

where the nonlinear term \( \nabla \times (u \times b) \) is neglected and the magnetic Prandtl number

\[ Pm = \frac{\nu}{\eta} \] (9)

is the ratio of viscosity \( \nu \) to magnetic diffusivity \( \eta \). For the magnetic boundary condition, the magnetic field in the insulating exteriors of both the \( r > R_o \) and \( r < R_l \) regions is required to match a potential field.

The numerical method is the standard spectral method with spherical harmonics used on the spherical surface and Chebyshev polynomials used in the radial direction, and the toroidal-poloidal decomposition method is employed for the divergence-free condition of velocity and magnetic field (Hollerbach 2000). The time-stepping calculation is used to find the final “steady” state (here “steady” is in terms of the volume integrals of energy and dissipation). An alternative method is to separate \( \omega \) (namely \( \partial / \partial t = \omega \)) and then to solve a super-large matrix problem; we do not use it because it is numerically difficult in the MHD case. In the longitude direction only one mode \( m = 2 \) is necessary for the linear problem, but in the latitude direction the modes much higher than \( l = 2 \) should be involved because the Coriolis force, the Lorentz force and the induction term will couple more modes in this direction. In our numerical calculations the sufficient resolution for convergence is guaranteed that both the radial and latitude spectra of both kinetic and magnetic energies span more than 10 magnitudes.

In our calculations we focus on the magnetic effect on dynamical tide so that we investigate the two parameters related to the magnetic field, \( S \) and \( Pm \), as well as the tidal frequency \( \omega \), but keep all the other parameters constants. The radius ratio \( R_l / R_o \) is taken to be 1/2, which was the value chosen in Favier et al. (2014). A larger radius ratio leads to a stronger excitation of inertial waves. We choose this value for comparison with the results in Favier et al. (2014) to validate our numerical calculations. The Ekman number is taken to be \( E = 10^{-4} \), which is already sufficiently low for the study of rapid rotation (an Ekman number that is too low is numerically demanding). \( l \) and \( m \) are taken to be 2. The frequency of inertial waves is lower than \( 2\Omega \) such that a tidal frequency higher than \( 2\Omega \) cannot excite inertial waves in a rotating flow. However, in
the presence of a magnetic field the frequency of magneto-inertial waves can exceed 2Ω (Wei 2016a), so a regime of tidal frequency that is higher than 2Ω will be studied. In our calculations the dimensionless tidal frequency ω ranges from 0.1 to 3, with a spacing 0.1, i.e., 30 values for ω in our calculations. We will calculate six values for S, i.e., S^2 = 0, 0.01, 0.02, 0.05, 0.1, and 0.2. S = 0 is a case of a rotating flow without a magnetic field. At S = √0.2 ≈ 0.45 the magnetic field is already sufficiently strong to be comparable to rotation (see Equation (3)). We will calculate four values for Pm, i.e., 1, 0.5, 0.2, and 0.1, i.e., spanning one magnitude. In the astrophysical situation Pm is very low, e.g., Pm is of the order of 10^-6 in the Earth’s liquid-iron core, or in Jupiter’s conducting region, or in the Sun’s convection zone. A Pm that is too low is numerically demanding, so we study a range of one magnitude to try to find scaling laws. Altogether we have 30 × (6 + 4) = 300 calculations that will be presented in the next section.

3. Results

By numerically solving Equations (1) and (8) with boundary condition (4), we obtain velocity u and induced field b. Kinetic energy indicates the tidal response and the kinetic and magnetic dissipations are responsible for the orbital evolution of a linear system. Then we calculate the volume integrals of kinetic energy and the two dissipations over the spherical shell. In the dimensionless expressions, kinetic energy is normalized with ρR^3_eΩ^2 and both dissipations are normalized with ρR^3_eΩ^2, and thus

\[
\text{kinetic energy} = \frac{1}{2} \int \left| u^2 \right| dV, \\
\text{viscous dissipation} = 2 \int S_{ij} S_{ij} dV, \\
\text{magnetic dissipation} = \frac{S^2}{Pm} \int | \nabla \times b |^2 dV, \tag{10}
\]

where S_{ij} is the strain tensor. In the next part of this section we will use the dimensionless expressions (10) as output.

We also calculate the total angular momentum and it conserves in our linear model. Note that in Favier et al. (2014) the angular momentum conserves in the linear model of rotating flow but does not conserve in the nonlinear model, and in our linear model of rotating MHD it conserves as well. It may be inferred that angular momentum does not conserve in the nonlinear model of rotating MHD, which we do not study in this paper. We need to remind readers that this method of using the equilibrium tide at the outer surface to excite the dynamical tide in the interior is applicable in the linear regime but not in the nonlinear regime.

3.1. Investigation of S

First, we investigate the strength of the imposed magnetic field, namely the dimensionless number S. Figure 1 shows the kinetic energy as the tidal response versus frequency at various S^2, with Pm fixed to 1. The black curve at S = 0 denotes the kinetic energy in a rotating flow. The peak values at the tidal frequencies ω = 0.2, 0.8, 1.1, 1.4 are the ones near the resonance of inertial waves and tidal force. It should be noted that the tidal frequencies at these peaks are near, but not exactly equal to, the eigenfrequencies of inertial waves. To obtain accurate eigenfrequencies, as illustrated in Section 2, a superlarge matrix problem should be solved, which is numerically difficult in the MHD case. It is well-known that the frequency of inertial waves is lower than 2Ω, therefore on the black curve at frequencies higher than 2 there does not exist any tidal response of inertial waves. When a magnetic field is present and its strength (S) increases, some new peaks appear, e.g., ω = 1.3 at S^2 = 0.02, ω = 0.3, and 1.8 at S^2 = 0.05, ω = 0.4, and 1.9 at S^2 = 0.1, etc. This is because the magnetic field modifies the dispersion relation of waves, i.e., inertial waves in a rotating flow become magneto-inertial waves in rotating MHD, such that the eigenfrequencies are changed and the tidal resonances appear at different frequencies. Moreover, with a strong magnetic field, e.g., at S^2 = 0.1 and 0.2, there exist tidal responses at tidal frequencies higher than 2. These results indicate that a magnetic field broadens the range of tidal resonance in the global spherical geometry.

Figure 2(a) shows the viscous dissipation. For a tidal frequency higher than 2 (ω > 2), the viscous dissipation is very small in a rotating flow (S = 0) because no inertial wave can be excited in this range of tidal frequency, but it is moderate and increases as the field strength increases in this range. In the presence of a strong field, e.g., S^2 = 0.2, the viscous dissipation for ω > 2 can be even higher than that for ω < 2. Figure 2(b) shows the magnetic dissipation. Its dependence on the tidal frequency is similar to that of viscous dissipation. Figure 2(c) shows the total dissipation, namely viscous dissipation + magnetic dissipation. Not surprisingly, it is very small for ω > 2 in a rotating flow but does not vary significantly with tidal frequency in the presence of a strong field. Figure 2(d) shows the ratio of magnetic dissipation to viscous dissipation. The dashed line indicates the equipartition of the two dissipations. When the tidal frequency increases the two dissipations tend to be close to each other and in the high-frequency range (approximately ω > 2) the two dissipations are almost equal to each other. But in the low-frequency range (approximately ω < 2), magnetic dissipation wins out over viscous dissipation with a strong field and viscous dissipation wins out over magnetic dissipation with a weak field.

The result of Figures 1 and 2 can be applied to the astrophysical situation. A magnetic field broadens the range of tidal resonance. That is, the tidal frequency higher than 2Ω
which cannot lead to significant viscous dissipation without a magnetic field can now lead to significant viscous and magnetic dissipations with a moderate or stronger magnetic field. For example, in a binary system, if the orbital frequency in the rotating frame is faster than twice the rotational frequency, then inertial waves cannot be excited by the tidal force. But in the presence of a strong magnetic field, the tidal force can excite magneto-inertial waves and energy can dissipate very quickly through both viscous and magnetic dissipations.

To better understand the mechanism of tidal dissipations with magnetic field, we plot the contours of kinetic and magnetic energies in a certain meridional plane at $\phi = 90^\circ$. Here, magnetic energy is shown in terms of the induced field, but does not include the contribution of the imposed field. Figure 3 shows the energy contours at the tidal frequency $\omega = 1.0$. Figure 3(a) is for a rotating flow and Figures 3(b)–(f) are for rotating MHD with the field strength gradually increasing. In Figure 3 both tidal flow (equilibrium tide) and waves (dynamical tide) are included. Inertial waves in a rotating flow propagate at an angle $\arcsin(\omega/2\Omega) = 30^\circ$ inclined to the rotational axis and Figure 3(a) exhibits the $30^\circ$ oblique internal shear layers built by the propagation of inertial waves. However, in the presence of a magnetic field, as shown in Figures 3(b)–(f), these oblique internal shear layers disappear. With the field strength gradually increasing, the contours of kinetic energy become more vertical and those of magnetic energy concentrate on the cylinder tangential to the inner sphere. This is because the magneto-inertial waves become more Alfvén-like than inertial-like when the field strength increases. Meanwhile, the kinetic energy spreads outside the tangent cylinder, whereas the magnetic energy concentrates more in a thin internal shear layer on the tangent cylinder. This difference between the flow and field structures explains why magnetic dissipation wins out over viscous

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**Figure 2.** (a) Viscous dissipation, (b) magnetic dissipation, (c) total dissipation, and (d) the ratio of magnetic to viscous dissipations vs. tidal frequency at various $S^2$. $P_m = 1.0$. 

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dissipation in the low-frequency range with a strong field: because the smaller length scale of the magnetic field leads to higher magnetic dissipation. Note that with a very weak magnetic field, e.g., $S = 10^{-4}$, the magnetic effect is negligible and the internal shear layers cannot be destroyed by the field (Lin & Ogilvie 2018). So the magnetic field wins out over rotation at a moderate or stronger field in the sense that the Alfvén velocity is at least of the order of 0.1 of the surface rotational velocity. A more accurate scaling law for this competition was obtained by Lin & Ogilvie (2018).

For comparison with the low tidal frequency, we plot the energy contours at a high tidal frequency $\omega = 3.0$, as shown in Figure 4. Figure 4(a) shows that the contours of kinetic energy of a rotating flow consisting of both equilibrium tide and dynamical tide at the high tidal frequency are almost vertical but do not exhibit the structure of oblique internal shear layers.

Figure 3. (a) Shows the contours of kinetic energy at $S = 0$, and (b)-(f) show the contours of kinetic (left panel) and magnetic (right panel) energies at, respectively, $S^2 = 0.01, 0.02, 0.05, 0.1,$ and $0.2$. The tidal frequency is $\omega = 1.0$. 
at low tidal frequency because a tidal frequency higher than $2\Omega$ cannot excite inertial waves to build the oblique internal shear layers. When a magnetic field is present, as shown in Figures 4(b)–(f), both kinetic and magnetic energies exhibit similar distributions and tend to concentrate inside the tangent cylinder when the field strength increases, which explains why viscous and magnetic dissipations are comparable at high tidal frequency.

The dependence of dissipations on tidal frequency is irregular because the resonance in spherical geometry is complicated, but the dependence of dissipations on magnetic field is not so irregular. Figure 5 shows the log–log relation of kinetic energy and dissipations versus $S^2$ at different tidal frequencies that are almost a geometric sequence. Figure 5(a) shows the kinetic energy versus $S^2$. It is irregular because the resonance occurs irregularly at certain frequencies with certain field strengths. For

Figure 4. Same as Figure 3 but for the tidal frequency $\omega = 3.0$. 

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the dissipations shown in Figures 5(b) and (c), we cannot find accurate scaling laws, i.e., straight lines in the log–log diagram, because certain tidal frequencies are near resonance at certain $S$ values, as shown in Figure 5(a). However, we see that both viscous and magnetic dissipations increase as the field strength increases. This is not surprising because a stronger field provides more magnetic energy to damp. We fit the slopes of this “big trend” for all the curves by appropriately removing a few points far away from this “big trend” and taking the average of these slopes. Interestingly, both viscous and magnetic dissipations obey an identical scaling law:

$$
\text{viscous and magnetic dissipations } \propto S^2. \quad (11)
$$

The dashed lines in Figures 5(b) and (c) denote this scaling law. This scaling law for the relation of the two dissipations and magnetic fields reveals that there exists self-similarity for the magnetic effect on dynamical tide.

3.2. Investigation of $Pm$

In this subsection we investigate $Pm$. Similar to Figure 2, Figure 6 shows viscous dissipation, magnetic dissipation, total dissipation, and the ratio of magnetic to viscous dissipations versus tidal frequency at various $Pm$ with $S^2$ fixed to be 0.1, namely in a strong field regime. The first three subfigures show that for all the tidal frequencies, a larger $Pm$ leads to higher viscous dissipation and lower magnetic dissipation but total dissipation is almost independent of $Pm$. This is reasonable because $Pm$ is the ratio of viscosity to magnetic diffusivity (see Equation (9)) but it cannot influence the total diffusivity. More interestingly, Figure 6(d) shows that magnetic dissipation wins
out over viscous dissipation for $Pm < 1$ and this ratio is approximately equal to $Pm^{-1}$. In the astrophysical situation $Pm$ is very small, and it can be inferred that magnetic dissipation dominates over viscous dissipation with a moderate or stronger magnetic field. Note again that with a very weak field, e.g., $S \lesssim 10^{-4}$, viscous dissipation wins out over magnetic dissipation even at a small $Pm$, e.g., $Pm = 10^{-4}$ (Lin & Ogilvie 2018).

To find the scaling laws for $Pm$, we plot the two dissipations versus $Pm$ as shown in Figure 7. The dependence on $Pm$ looks more regular than the dependence on $S$ in Figure 5 because the field strength influences the resonant frequencies, whereas diffusivities cannot. By taking the average of the slopes for different tidal frequencies, we obtain the scaling laws

$$\text{viscous dissipation} \propto Pm^{0.6},$$
$$\text{magnetic dissipation} \propto Pm^{-0.3}. \tag{12}$$

The dashed lines in Figures 7(a) and (b) denote these two scaling laws. The ratio of magnetic to viscous dissipations is indeed close to $Pm^{-1}$, as shown in Figure 6(d). Note that our parameter regime is not the real regime, e.g., $E$ and $Pm$ are not too small, so these scaling laws might not be realistic. Lacking knowledge of the physical properties in stars and planets, we do not know the real parameters, and even estimating the order of magnitude is not convincing. These scaling laws obtained in the moderate parameter regime at least give us some qualitative evidence that a moderate or stronger magnetic field improves...
both dissipations, and the magnetic dissipation wins out over viscous dissipation at a low $Pm$.

3.3. A Major Result

Now let us come back to the equations governing the system of magnetic tide. In Equations (1), (4), and (8) there are four parameters, $E$, $S$, $Pm$, and $A$. In the linear regime that we studied, the tidal dissipations were proportional to $A^2$. Since we focus on the magnetic effect but not the rotational effect, the Ekman number is not investigated but is fixed to a small value. Through numerical studies we know the dependences on the two magnetic parameters $S$ and $Pm$, i.e., viscous dissipation is proportional to $S^2Pm^{0.6}$, magnetic dissipation is proportional to $S^2Pm^{-0.3}$, and total dissipation is proportional to $S^2$ but independent of $Pm$. Translating to the dimensional expression, we obtain the scaling law for the total dissipation for the strength of magnetic field,

$$\text{total dissipation } \propto B^2.$$  \hfill (13)

Equation (13) is the main result of this paper and has important astrophysical applications. For example, it can be used to estimate the strength of the internal magnetic field of an astronomical object of a binary system by observing the orbital evolution of the binary system. Note that this scaling law is for the magnetic field, and in addition to the magnetic field, some other factors such as diffusivities, radius ratio, and so on can also influence the tidal dissipation. Lin & Ogilvie (2018) discussed the other factors in detail, which we will briefly discuss in the next section.

4. Summary

In this paper we have numerically investigated the magnetic effect on the dynamical tide of a binary system. We tuned the tidal frequency, the field strength, and the two diffusivities to calculate both viscous and magnetic dissipations, which are important for the orbital evolution of a binary system. We found that a moderate or stronger magnetic field (in the sense that the Alfvén velocity is at least of the order of 0.1 of the surface rotational velocity) destroys the internal shear layers built by the propagation of inertial waves. A magnetic field modifies not only the flow structure but also the dispersion relation of the waves excited by the tidal force such that the tidal resonance in a rotating MHD is quite different from that in a rotating fluid, namely the resonance range is broadened by a magnetic field to be out of $2\Omega$. Magnetic dissipation wins out over viscous dissipation at low tidal frequencies with a strong magnetic field but the two dissipations are comparable at high tidal frequencies. The ratio of magnetic to viscous dissipations is almost inversely proportional to $Pm$ such that in the astrophysical situation of very low $Pm$, magnetic dissipation dominates over viscous dissipation with a moderate or stronger field, but the total dissipation does not depend on $Pm$. We obtained three scaling laws: viscous dissipation $\propto S^2Pm^{0.6}$, magnetic dissipation $\propto S^2Pm^{-0.3}$, and total dissipation $\propto S^2$. The last scaling law, in its dimensional expression, shows that total dissipation $\propto B^2$ can be applied to estimating the strength of the internal magnetic field in an astronomical object of a binary system.

Due to computational power limitations, we do not compute the frequency-averaged dissipation, which requires scanning a huge range of tidal frequency, especially in the case of a strong magnetic field. Wei (2016a) confirmed in a periodic box that the frequency-averaged dissipation is constant and interpreted this result with a simple damped harmonic oscillator. Ogilvie (2013) confirmed this result in a spherical shell for a rotating fluid and Lin & Ogilvie (2018) confirmed this for a rotating MHD.

To end this paper, we discuss some topics that we did not consider in this work but can be studied in the future. First, we focused on the two magnetic parameters $S$ and $Pm$ but did not investigate the other parameters, i.e., the Ekman number $E$ measuring rotation and the radius ratio $R_i/R_o$ measuring the
stellar or planetary structure. The dependence on $E$ is not very clear so far (Ogilvie 2014). The dependence of the viscous dissipation of a rotating flow on the core size was discussed in Goodman & Lackner (2009). Their WKB analysis was based on the critical latitude of inertial waves but this dependence is not as clear in rotating MHD because a magnetic field removes the critical latitude. These two parameters, $E$ and $R_\text{m}/R_*$, need to be further studied. Second, the imposed magnetic field in our model is uniform and vertical, which is a simplified geometry, so more complex field geometries should be considered, e.g., dipolar and quadrupolar fields. In Wei & Hollerbach, (2010) various field geometries were studied in a spherical Couette flow, namely the flow between two differentially rotating spheres, and it was found that (1) the flow tends to be along field lines due to Alfvén’s frozen-in theorem, and (2) it can be inferred that the structure of a tidal flow with various magnetic field geometries will be similar, namely the flow along the field lines. Third, our model is linear, neglecting all the quadratic terms, i.e., $u \cdot \nabla u$, $(\nabla \times b) \times b$ and $\nabla \times (u \times b)$. In Wei (2016b) it was found that nonlinear inertial force suppresses the tidal response and dissipation near resonance. In Favier et al. (2014) it was found that the dynamical tide in the nonlinear regime is quite different from that in the linear regime. So the nonlinear regime should also be further studied for comparison with the linear regime.

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