UNIQUENESS OF THE COMPACTLY SUPPORTED WEAK SOLUTIONS OF THE RELATIVISTIC VLASOV-DARWIN SYSTEM

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Abstract. We use optimal transportation techniques to show uniqueness of the compactly supported weak solutions of the relativistic Vlasov-Darwin system. Our proof extends the method used by Loeper in [8] to obtain uniqueness results for the Vlasov-Poisson system.

1. Introduction

The relativistic Vlasov-Darwin (RVD) system describes the evolution of a collisionless plasma whose particles interact only through the electromagnetic field they induce. In contrast to the Vlasov-Maxwell system, the particle interaction is assumed to be a low-order relativistic correction (i.e., the Darwin approximation) of the full Maxwell case. In [14], we showed that the model equations for a single-species can be written in terms of a scalar and vector potentials \( \Phi, A \) according to

\[
\partial_t f + v_A \cdot \nabla_x f - \left[ \nabla \Phi - c^{-1} v_A \nabla A \right] \cdot \nabla_p f = 0, \quad v_A = \frac{p - c^{-1} A}{\sqrt{1 + c^{-2} |p - c^{-1} A|^2}}
\]

coupled with

\[
\Phi(t, x) = \int_{\mathbb{R}^3} \rho(t, y) \frac{dy}{|y - x|},
\]

\[
A(t, x) = \frac{1}{2c} \int_{\mathbb{R}^3} \left[ \mathbf{id} + \omega \otimes \omega \right] j_A(t, y) \frac{dy}{|y - x|},
\]

via

\[
\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, p) dp, \quad j_A(t, x) = \int_{\mathbb{R}^3} v_A f(t, x, p) dp.
\]

Here \( f = f(t, x, p) \) denotes the one-particle distribution function at a time \( t \in ]0, \infty[ \), position \( x \in \mathbb{R}^3 \) and (generalized) momentum \( p \in \mathbb{R}^3 \). The scalar and vector potentials \( \Phi \) and \( A \) are induced by \( f \) via the charge and current densities \( \rho \) and \( j_A \), respectively. The relativistic particle velocity is denoted by \( v_A \), and satisfies \( |v_A| \leq c \) where \( c \) is the speed of light. For simplicity, we have set the charge and mass of the particles to one. Also, we have denoted the 3-by-3 identity matrix as \( \mathbf{id} \) and the unit vector \( \omega = (y - x) / |y - x| \). As usual, repeated indexes means summation, which is the case in (1.1), and the symbol \( \otimes \) in (1.3) stands for the tensor product. We define the weak solutions to the RVD system as follows:

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Definition 1. Let \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^6; \mathbb{R}) \) and \( f_0 \geq 0 \). For \( T > 0 \) consider \( f \in C([0,T], L^\infty(\mathbb{R}^6) - w^*; \mathbb{R}) \) such that \( f \geq 0 \). We call \( f \) a weak solution of the relativistic Vlasov-Darwin system with initial datum \( f_0 \), if

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, p) \, dx \, dp = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_0(x, p) \, dx \, dp \quad \forall t \in [0, T],
\]

\( \Phi \) and \( A \) are given by (1.2)-(1.4), and for all \( \varphi \in C_0^\infty([0,T] \times \mathbb{R}^6; \mathbb{R}) \) we have

\[
\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \left\{ \partial_t \varphi + v_A \cdot \nabla_x \varphi - \left[ \nabla \Phi - c^{-1} v_A^i \nabla A^i \right] \cdot \nabla_p \varphi \right\} (t, x, p) \, dt \, dx \, dp = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_0(x, p) \varphi(0, x, p) \, dx \, dp.
\]

(1.5)

The notation \( \varphi \in C_0^\infty([0,T] \times \mathbb{R}^6; \mathbb{R}) \) means that \( \varphi \) is of class \( C^\infty \), \( \varphi(T) = 0 \) and \( \varphi(t) \) has a compact support in \( \mathbb{R}^6 \) for all \( t \in [0, T] \). Also, \( L^\infty(\mathbb{R}^6) - w^* \) means that the space \( L^\infty(\mathbb{R}^6) \) is equipped with the weak-* topology.

The global in time existence of weak solutions to the RVD system without size restriction on the Cauchy datum was established by Pallard in [12]. This result uses the formulation of the RVD system in terms of the electromagnetic field, which is formally equivalent to (1.1)-(1.4); see details in [14]. Here we do not discuss the existence problem nor the equivalence of the two formulations in the context of weak solutions. We are only concerned with the uniqueness of solutions to (1.1)-(1.4) in the sense of Definition 1. We emphasize, however, that for a bounded charge density (and in particular for a compactly supported distribution function), Definition 1 makes perfect sense. We postpone this discussion to the next section (see Lemmas 2 and 3).

If we formally let \( c \to \infty \) in (1.1)-(1.4), the model equations reduce to the Vlasov-Poisson system, which is the zeroth-order relativistic correction of the Vlasov-Maxwell system. A uniqueness result for weak solutions of the Vlasov-Poisson system based on optimal transportation, under the assumption that the charge density remains bounded, was established by Loeper in [8], (see also Robert [13] for the uniqueness of compactly supported weak solutions). Here, we extend the method of [8] to produce a uniqueness of weak solutions to the RVD system under the assumption of a compactly supported distribution function. To the best of our knowledge, this is the first uniqueness result for weak solutions of the RVD system. Our main result is as follows:

Theorem 1. Let \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^6; \mathbb{R}) \), \( f_0 \geq 0 \), have compact support and let \( f \) be a weak solution on \([0,T]\) of the RVD system (1.1)-(1.4) in the sense of Definition 1, such that \( f |_{t=0} = f_0 \). If for all \( t \in [0,T] \) the support of \( f(t) \) is compact in \( \mathbb{R}^6 \), then this solution is unique.

The proof of the theorem will be postponed to Section 5. It uses techniques of optimal transportation theory and extends the proof by Loeper in [8] for the uniqueness of solutions to the Vlasov-Poisson system. These techniques have been useful in the recent paper by Carrillo and Rosado [4] where uniqueness of solutions to several equations containing aggregation terms and aggregation/diffusion competition (e.g. swarming models, chemotaxis) was established. The main new difficulty in our proof is the vector potential \( A \), which for a given \( f \) is defined by a non-linear integral equation. The existence of a solution to this integral equation,
the regularity of this solution, and the corresponding a-priori estimates used in our proof, require some elaborated work.

The rest of the paper is organized as follows. In Section 2 we present some preliminary results on the scalar and vector potentials $\Phi$ and $A$. We also discuss results on the linear Vlasov equation \([11]\) and the associated characteristic system. In Section 3 we recall well-known results of optimal transportation that we use in Section 4 to establish some crucial estimates on the potentials, which are needed for the proof of Theorem 1. Hereafter, $L^\infty(X;Y)$ will denote the set of $L^\infty$- functions $f : X \rightarrow Y$, and $C_0^\infty(X;Y)$ denotes the set of such functions of class $C^\infty$ with compact support in $X$. By $g \in L^\infty([0,T[, C_b(X); Y)$, we mean that $[0,T]\ni t \mapsto g(t) \in C_b(X; Y)$ belongs to $L^\infty([0,T])$. For a function $\mathbb{R}^3 \times \mathbb{R}^3 \ni (x, p) \mapsto g(x, p)$, we denote by $\|g\|_{L^\infty}$ (resp. $\|g\|_{L^\infty}$) the $L^\infty$-norm of $x \mapsto g(x, p)$ (resp. $p \mapsto g(x, p)$), and by $\text{supp} g$ the support of $g$. Also $\partial_2 g$ (resp. $\partial_3 g$) denotes the matrix gradient of $g$ with respect to $x$ (resp. $p$). If $D = [d_{ij}]$ is a real matrix, we define $|D| = \sum_{i,j} |d_{ij}|$.

2. Preliminaries

Henceforth, we set the speed of light to $c = 1$. For simplicity and without loss of generality we shall omit the time dependence throughout this section, unless we specify otherwise. We first recall an estimate that will be used later on. For a proof see [12, Lemma 2.7].

**Lemma 1.** For $1 \leq m < 3$ let $r_0 = 3/(3 - m)$ and $r < r_0 < s$. Set $\lambda = (1 - r/r_0)/(1 - r/s)$. Then there exists a constant $C = C(m, r, s) > 0$ such that for any $\Psi \in L^r \cap L^s(\mathbb{R}^d; \mathbb{R})$

$$\left\| \int_{\mathbb{R}^d} \Psi(y) \left[\frac{dy}{|y-x|^m}\right] \right\|_{L^\infty} \leq C(m, r, s) \|\Psi\|_{L^s}^{1 - \lambda} \|\Psi\|_{L^r}^\lambda.$$  

In particular, $C(m, 1, \infty) = 3 (4\pi/m)^{m/3} / (3 - m)$.

The next lemma shows that the potentials $\Phi$ and $A$ of a weak solution to the RVD system are well-defined and satisfy the appropriate regularity.

**Lemma 2.** Let $f$ be given as in Definition 2 and define $\rho$ according to \([14]\). If $\rho \in L^\infty(\mathbb{R}^3; \mathbb{R})$ has compact support, then there exists bounded potentials $\Phi$ and $A$ that solve \([\ref{12}] - \([\ref{14}]\). Moreover, $(\partial_2 \Phi, \partial_3 A) \in C_b(\mathbb{R}^3; \mathbb{R} \times \mathbb{R}^3)$ and therefore $f \nabla \Phi$ and $f v^A \nabla A^i$ are in $L^1(\mathbb{R}^6, \mathbb{R}^3)$, with $v^A$ defined by \([\ref{17}]\). In addition, $\nabla \cdot A = \text{Trace}(\partial_3 A) = 0$ where

$$\partial_3 A(x) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left\{ \omega \otimes v_A - v_A \otimes \omega + \left[ 3\omega \otimes \omega - \mathbf{i} \mathbf{d} \right] (v_A \cdot \omega) \right\} f(y, p) \frac{dpdy}{|y-x|^2}$$

is the matrix gradient of $A$.

**Proof.** The results corresponding to the scalar potential $\Phi$ are well-known, see for instance [7, Chapter 10]; therefore we center our attention on the vector potential $A$. First, we prove that the integral equation

$$A(t, x) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[ \mathbf{i} \mathbf{d} + \omega \otimes \omega \right] v_A f(t, y, p) \frac{dpdy}{|y-x|}, \quad v_A = \frac{p - A}{\sqrt{1 + |p - A|^2}}.$$
has a solution which satisfies the claimed regularity. The proof is similar to the one in [13, Lemma 7] for the classical solution of the RVD system. We reproduce it here for convenience.

Let $\bar{C} > 0$ be a constant that may depend on $f$, to be fixed later on. Define the set

$$D_{\bar{C}} = \left\{ A \in C_b(\mathbb{R}^3; \mathbb{R}^3) : \|A\|_{L^\infty_x} \leq \bar{C} \right\}.$$ 

We show that there exists a fixed point $A_\infty \in D_{\bar{C}}$ which solves (2.1).

To start with, denote the kernel $K(x, y) = |y - x|^{-1} \left[ \text{id} + \omega \otimes \omega \right]$ and let $A \in D_{\bar{C}}$. Note that $|K(x, y)| \leq 2 |y - x|^{-1}$. Consider the mapping $A \mapsto T[A]$ defined by

$$T[A](x) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x, y) v_A f(y, p) dp dy, \quad v_A = \frac{p - A}{\sqrt{1 + |p - A|^2}}.$$ 

We claim that $T[A] \in D_{\bar{C}}$. Indeed, let $(K)_{ij}(x, y)$ be the $ij$-entry of $K(x, y)$. For some $u_1$, $u_2$ and $u_3$ on the line segment between $x$ and $z$, the mean value theorem implies

$$\left| (K)_{ij}(x, y) - (K)_{ij}(z, y) \right| \leq 2 \left| \frac{1}{|y - x|} - \frac{1}{|y - z|} \right| + \left| \frac{y^i - x^i}{|y - x|^2} - \frac{y^j - z^j}{|y - z|^2} \right| + \left| \frac{y^i - x^i}{|y - x|} - \frac{y^j - z^j}{|y - z|} \right|$$

$$\leq C |x - z| \left( \frac{1}{|y - u_1|^2} + \frac{1}{|y - u_2|^2} + \frac{1}{|y - u_3|^2} \right).$$

Hence, since $|v_A| \leq 1$, Lemma 1 implies

$$|T[A](x) - T[A](z)| \leq \frac{1}{2} \int_{\mathbb{R}^3} |K(x, y) - K(z, y)| \rho(y) dy$$

$$\leq C |x - z| \left\| \int_{\mathbb{R}^3} \rho(y) \frac{dy}{|y - .|^2} \right\|_{L^\infty_x} \leq C(\rho) |x - z|.$$

Thus, $T[A]$ is a continuous vector valued function. Also, it is a simple consequence of Lemma 1 that

$$\|T[A]\|_{L^\infty_x} \leq 3(4\pi)^{1/3} \|\rho\|_{L^1_\rho}^{2/3} \|\rho\|_{L^\infty}^{1/3} \equiv \bar{C}.$$

Therefore, $T[A] \in D_{\bar{C}}$ as claimed.

Now, by virtue of the Schauder fixed point theorem [11, Theorem 3, Section 9.1], $T$ has a fixed point $A_\infty \in D_{\bar{C}}$ if $T$ is a continuous mapping and the closure of the image of $T$ is compact in $D_{\bar{C}}$. To show continuity, suppose that $A_k \to A$ in $D_{\bar{C}}$. Since the mapping $g \mapsto v(g) = g \left( 1 + |g|^2 \right)^{-1/2}$ is $C^1_b$, by Lemma 1 we have

$$|T[A_k](x) - T[A](x)| \leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v_{A_k} - v_A| |f(y, p)| \frac{dp dy}{|y - x|} \leq C(\rho) \|A_k - A\|_{L^\infty_x}.$$ 

To show that $\overline{T D_{\bar{C}}} \subset D_{\bar{C}}$ is compact, we first notice that for $A \in D_{\bar{C}},$

$$|T[A](x)| \leq \|\rho\|_{L^\infty_x} \int_{\text{supp} \rho} \frac{dy}{|x - y|} \leq C(\rho) \frac{1}{1 + |x|^2}.$$
Consider the sequence \( \{B_n\} \subset TD_\varepsilon \), and let \( R > 0 \) be fixed. By (2.2) and (2.3), the restriction
\[
\{B_n\}_{\{x \in \mathbb{R}^3 : |x| \leq R\}
\]
is equicontinuous and bounded. Then, by Arzelà-Ascoli’s theorem and a standard diagonal argument we can find a subsequence \( \{B_{n_k}\} \) and a continuous, bounded limit vector field \( B \) such that \( \{B_{n_k}\} \to B \) uniformly on compact sets, and in particular pointwise. Clearly, \( ||B||_{L^\infty} \leq \overline{C} \), and since \( \{B_{n_k}\} \) satisfies the estimate (2.4), so does \( B \). We only need to show that the convergence \( \{B_{n_k}\} \to B \) is uniform.

Indeed, let \( \varepsilon > 0 \). Choose \( R > 0 \) such that the right-hand side of (2.4) is less than \( \varepsilon /2 \) for \( |x| > R \). Then, for all \( k \) we have \( |B_{n_k}(x) - B(x)| < \varepsilon \) for \( |x| > R \), and we can find a \( k_0 = k_0(R, \varepsilon) \) such that for all \( k > k_0 \)
\[
\sup_{|x| \leq R} |B_{n_k}(x) - B(x)| < \varepsilon.
\]
This proves uniform convergence. Hence, all the hypotheses for the Schauder fixed point theorem are fulfilled, and thus \( T \) has a fixed point \( A_\infty \) in \( D_\varepsilon \).

Next, we show that \( A_\infty \) has the required regularity. To that end, define \( v_{A_\infty} \) and then \( j_{A_\infty} \) according to (1.1) and (1.4), respectively. Since \( |v_{A_\infty}| \leq 1 \), then \( |j_{A_\infty}| \leq \rho \). Thus, \( j_{A_\infty} \in L^\infty(\mathbb{R}^3; \mathbb{R}^3) \) has compact support. On the other hand, \( |K(x, y)| \leq 2 |y - x|^{-1} \), and since the \( imk \)-th entry of \( \partial_x K \) reads
\[
(\partial_x K)_{imk}(x, y) = \partial_{x_k} \left\{ |y - x|^{-1} [\delta_{im} + \omega^i \omega^m] \right\} = |y - x|^{-2} [\delta_{im} \omega^k - \delta_{km} \omega^i - \delta_{ik} \omega^m + 3 \omega^i \omega^k \omega^m],
\]
the kernel \( K(x, y) \) satisfies the derivative estimate \( |\partial_x K(x, y)| \leq 6 |y - x|^{-2} \). Hence, we can use the standard theory for Poisson’s equation to find that \( A_\infty \in C^1(\mathbb{R}^3; \mathbb{R}^3) \) as claimed; see, for instance, [3] Lemma 4.1 or [7] Theorem 10.2 (iii). The remaining assertions in Lemma 2 are easy to check. In particular, since \( |v_{A_\infty}| \leq 1 \) and \( \rho \in L^\infty(\mathbb{R}^3; \mathbb{R}) \), we have by Lemma 1 that \( A_\infty \) and \( \partial_x A_\infty \) are bounded. This completes the proof of the lemma.

The potentials satisfy the following estimates:

**Lemma 3.** Let \( f \) be given as in Definition 4 and define \( \rho \) according to (1.4). Suppose that \( \Phi \) and \( A \) are as in Lemma 2. If \( \rho \in L^\infty(\mathbb{R}^3; \mathbb{R}) \), then
\[
\|A\|_{L^\infty} \leq C \|\rho\|_{L^1}^{2/3} \|\rho\|_{L^\infty}^{1/3}, \quad \|\partial_x A\|_{L^\infty} \leq C \|\rho\|_{L^1}^{1/3} \|\rho\|_{L^\infty}^{2/3}.
\]
These estimates also hold for the scalar potential \( \Phi \). Moreover, there exists a positive constant \( C \) that depends on \( \|\rho\|_{L^1} \) and \( \|\rho\|_{L^\infty} \) such that,
\[
|A(x) - A(z)| + |\partial_x A(x) - \partial_x A(z)| + |\partial_x \Phi(x) - \partial_x \Phi(z)| \leq -C |x - z| \ln |x - z|,
\]
for any \( (x, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \) with \( |x - z| \leq 1/2 \).

**Proof.** These are standard results for the scalar potential \( \Phi \) which were already used in [8] to prove the uniqueness of solutions to the Vlasov-Poisson system. Therefore, we only work here with the vector potential \( A \). Following the notation in the proof of Lemma 2 we have that \( |K(x, y)| \leq 2 |y - x|^{-1} \) and \( |\partial_x K(x, y)| \leq 6 |y - x|^{-2} \). Then, since \( |v_A| \leq 1 \) and thus \( |j_A| \leq \rho \), the estimates in (2.5) readily follow by Lemma 1.
To prove \(2.6\) we rely on a similar result discussed in [9, Lemma 8.1] for the 2D Euler equation. To begin with, let \(h = |x-z| \leq 1/2\) and \(B_r(x)\) be a ball of radius \(r\) centered at \(x\). A lengthy but elementary computation shows that the kernel \(K(x,y)\) and its derivative satisfy

\[
|K(x,y) - K(z,y)| \leq C \left( \left| \frac{y - x}{|y-x|^2} - \frac{y - z}{|y-z|^2} \right| + \left| \frac{1}{|y-x|} - \frac{1}{|y-z|} \right| \right)
\]

and

\[
|\partial_y K(x,y) - \partial_y K(z,y)| \leq C \left( \left| \frac{y - x}{|y-x|^3} - \frac{y - z}{|y-z|^3} \right| + \left| \frac{1}{|y-x|^2} - \frac{1}{|y-z|^2} \right| \right).
\]

As a result,

\[
|A(x) - A(z)| + |\partial_x A(x) - \partial_x A(z)|
\]

\[
\leq \int_{\mathbb{R}^3} \left( |K(x,y) - K(z,y)| + |\partial_y K(x,y) - \partial_y K(z,y)| \right) \rho(y) dy
\]

\[
\leq \int_{\mathbb{R}^3} \left( \left| \frac{y - x}{|y-x|^3} - \frac{y - z}{|y-z|^3} \right| + \left| \frac{1}{|y-x|^2} - \frac{1}{|y-z|^2} \right| \right) \rho(y) dy
\]

\[
= I + II + III + IV.
\]

In the remainder of the proof we shall only estimate \(I\) since it is slightly more involved than the other three integrals and they can all be estimated in the same fashion. To proceed, consider

\[
I = \left[ \int_{B_{2h}(x)} + \int_{B_2(x)/B_{2h}(x)} + \int_{\mathbb{R}^3/B_2(x)} \right] \left| \frac{y - x}{|y-x|^3} - \frac{y - z}{|y-z|^3} \right| \rho(y) dy
\]

\[
= I_1 + I_2 + I_3.
\]

We estimate one integral at a time. \(I_1\)

\[
I_1 \leq \|\rho\|_{L^\infty} \left( \int_{B_{2h}(x)} \frac{dy}{|y-x|^2} + \int_{B_{2h}(x)} \frac{dy}{|y-z|^2} \right)
\]

\[
\leq C \|\rho\|_{L^\infty} \left( \int_0^{2h} dr + \int_0^{3h} dr \right) \leq C \|\rho\|_{L^\infty} |x-z|.
\]

As for \(I_2\), let \(y \in B_2(x)/B_{2h}(x)\). The mean value theorem yields

\[
\left| \frac{y^i - x^i}{|y-x|^3} - \frac{y^i - z^i}{|y-z|^3} \right| \leq C \frac{|x-z|}{|y-u|^3}
\]

for some \(u\) on the line segment between \(x\) and \(z\). Then, since for some constant \(C > 0\) we have \(|y-x| \leq C |y-u|\),

\[
I_2 \leq C \|\rho\|_{L^\infty} |x-z| \int_{B_2(x)/B_{2h}(x)} \frac{dy}{|y-x|}
\]

\[
\leq C \|\rho\|_{L^\infty} |x-z| \int_0^2 \frac{dr}{r} \leq -C \|\rho\|_{L^\infty} |x-z| \ln |x-z|.
\]
To estimate $I_3$, let $y \in \mathbb{R}^3/B_2(x)$. Then $|y - x| \geq 2$ and we use the mean value theorem and a standard estimate [9, Lemma 8.1] to find that

\[
\left| \frac{y - x}{|y - x|^3} - \frac{y - z}{|y - z|^3} \right| 
\leq \frac{1}{|y - x|} \left| \frac{1}{|y - x|} - \frac{1}{|y - z|} \right| + \frac{1}{|y - z|} \left| \frac{y - x}{|y - x|^2} - \frac{y - z}{|y - z|^2} \right| 
\leq \frac{|x - z|}{|y - x||y - z|^2} + \frac{|x - z|}{|y - x||y - z|^2} \leq \frac{1}{2} |x - z| \left( \frac{1}{|y - u|^2} + \frac{1}{|y - z|^2} \right)
\]

for some other $u$ on the line segment between $x$ and $z$. Hence, we have by Lemma 1

\[
I_3 \leq |x - z| \left\| \int_{\mathbb{R}^3} \rho(y) \frac{dy}{|y - z|^2} \right\|_{L^\infty} \leq C \|\rho\|_{L^1}^{\frac{1}{2}} \|\rho\|_{L^2}^{\frac{1}{2}} |x - z|.
\]

We gather these estimates and use the fact that $|x - z| \leq 1/2$ to find that for some constant $C(\rho)$ that depends on $\|\rho\|_{L^1}$ and $\|\rho\|_{L^2}$,

\[
I \leq -C(\rho) |x - z| \ln |x - z|.
\]

Thus, since the same rationale shows that (2.7) also holds for the integrals $II, III$ and $IV$, we conclude that

\[
|\partial_x A(x) - \partial_x A(z)| \leq -C(\rho) |x - z| \ln |x - z|,
\]

and the proof of the lemma is complete. \hfill \Box

The next lemma characterizes the weak solutions of the RVD system via the associated characteristic system. We recall that the speed of light has been set to $c = 1$.

**Lemma 4.** Let $\Phi$ and $A$ be given as in Lemma 2. Then, there exists a unique solution $(X, P)(s, t, x, p)$, $0 \leq s \leq t < T$, to the characteristic system

\[
\begin{align*}
\dot{x} &= v_A(s, x, p), \\
\dot{p} &= \left[ -\nabla \Phi + v_A \nabla A \right](s, x, p)
\end{align*}
\]

with $(X, P)(t, t, x, p) = (x, p)$, associated to the (linear!) Vlasov equation (2.1). Moreover, since the right-hand side of (2.8) - (2.9) is an incompressible vector field, the mapping $(x, p) \mapsto (X, P)$ is measure preserving. Conversely, the weak solution of (1.7) in the sense of (1.7) with $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6; \mathbb{R})$, $f_0 \geq 0$, is uniquely determined by $f(t, x, p) = f_0((X, P)(0, t, x, p))$ on $[0, T]$ for all $(x, y) \in B \subset \mathbb{R}^6$, $B$ a Borel set.

**Proof.** Set $z = (x, p)$. Since the mapping $g \mapsto v(g) = g \left( 1 + |g|^2 \right)^{-1/2}$ is $C_b^1$, Lemma 3 guarantees that the vector field $G = (v_A, -\nabla \Phi + v_A \nabla A)$ on the right-hand side of the equations (2.8) - (2.9) is Lipschitz continuous in the momentum variable and Log-Lipschitz in space. This implies that there exists a unique solution $[0, T] \ni s \mapsto Z = (X, P)(s, \cdot, \cdot, \cdot)$ to the characteristic system (2.8) - (2.9) and the characteristic flow $Z(s, t, z)$ is H"older continuous with respect to $z$ [9, Chapter 8]. Moreover, since by [13, Lemma 1] we have in the classical sense

\[
\nabla_z \cdot G = \nabla_x \cdot v_A + \nabla_p \cdot \left( -\nabla \Phi + v_A \nabla A \right) \equiv 0,
\]

...
the identity (2.12). □

Then, for \( f_0 \) given as in the lemma, we have (as a corollary of [11, Theorem 8.2.1]) that the function \( f(t, z) = f_0(0, t, z) \), \( t \in [0, T[; z = (x, y) \in B \subset \mathbb{R}^6 \), where \( B \) is a Borel set, with \( f|_{t=0} = f_0 \), is the unique solution to the equation (2.11) (and so (1.1)) in the sense of (1.5). □

**Remark 1.** Since by Lemma 4, the weak solution \( f \) of (1.1) is measure-preserving (9, Chapter 8). By (2.10), we can write the Vlasov equation in divergence form, i.e.,

\[
(2.11) \quad \partial_t f(t, z) + \nabla_z \cdot (G(t, z) f(t, z)) = 0.
\]

Then, for \( f_0 \) as given in the lemma, we have (as a corollary of [11, Theorem 8.2.1]) that the function \( f(t, z) = f_0(0, t, z) \), \( t \in [0, T[; z = (x, y) \in B \subset \mathbb{R}^6 \), where \( B \) is a Borel set, with \( f|_{t=0} = f_0 \), is the unique solution to the equation (2.11) (and so (1.1)) in the sense of (1.5). □

**Lemma 5.** Denote \( \omega = (\omega^1, \omega^2, \omega^3) \); \( \nabla_x = (\partial_1, \partial_2, \partial_3) \) and let \( \delta_{ik} \equiv 1 \) if \( i = k \) else \( \delta_{ik} = 0 \). For any \( \varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}) \) and \( y \in \mathbb{R}^3 \), we have

\[
\int_{\mathbb{R}^3} \Delta \varphi(x) \left[ \delta_{ik} - \omega^i \omega^k \right] \frac{dx}{|y - x|} = 2 \int_{\mathbb{R}^3} \partial_k \partial_i \varphi(x) \frac{dx}{|y - x|}.
\]

**Proof.** Clearly, both integrals are well defined. Let the support of \( \varphi \) be contained in a ball centered at \( y \) with radius \( R > 0 \). Let \( 0 < r < R \). It is easy to check that \( \partial_k \omega^i = -|y - x|^{-1} \left[ \delta_{ik} - \omega^i \omega^k \right] \) and \( \Delta \omega^i = -2|y - x|^{-2} \omega^i \) for \( |y - x| > r \). To prove (2.12), we will show that

\[
2 \int_{\mathbb{R}^3} \partial_k \partial_i \varphi(x) \frac{dx}{|y - x|} = -\lim_{r \to 0} \int_{|y - x| \geq r} \Delta \varphi(x) \partial_k \omega^i dx.
\]

Denote the integral on the right-hand side of (2.13) by \( I(r) \). In view of the compact support of \( \varphi \), we can restrict the domain of integration of \( I(r) \) to \( r \leq |y - x| \leq R \). Then, integration by parts in \( \partial_k \) and then twice for \( \Delta \) yield

\[
I(r) = -\int_{r \leq |y - x| \leq R} \partial_k \varphi(x) \Delta \omega^i dx + o(r)
\]

\[
= 2 \int_{r \leq |y - x| \leq R} \partial_k \varphi(x) \frac{\omega^i dx}{|y - x|^2} + o(r),
\]

where \( o(r) \) stands for all boundary terms at \( |x - y| = r \). Note that the boundary terms at \( |y - x| = R \) vanish due to the compact support of \( \varphi \). It is not difficult to check that \( o(r) \to 0 \) as \( r \to 0 \). Then, since we also have \( |y - x|^{-2} \omega^i = \partial_i |y - x|^{-1} \) for \( |y - x| > r \), another integration by parts and a standard limiting process produce the identity (2.12). □
Lemma 6. Let \( f \) be given as in Definition 1. Then, the solution \( A \) to the integral equation (1.3) for the vector potential satisfies

\[
\Delta A(x) = -4\pi j_A(x) - \nabla \left\{ \nabla \cdot \int_{\mathbb{R}^3} j_A(y) \frac{dy}{|y-x|} \right\}
\]

in the sense of distributions, i.e., for any \( \varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}) \), we have

\[
-\int_{\mathbb{R}^3} A(x) \Delta \varphi(x) dx = 4\pi \int_{\mathbb{R}^3} \varphi(x) j_A(x) dx - \int_{\mathbb{R}^3} \nabla \varphi(x) \left\{ \nabla \cdot \int_{\mathbb{R}^3} j_A(y) \frac{dy}{|y-x|} \right\} dx.
\]

(2.14)

\( \varphi \)

Proof. Since \( |v_A| \leq 1 \), we have \( |j_A| \leq \rho \) and thus \( j_A \in L^\infty(\mathbb{R}^3; \mathbb{R}^3) \) has compact support. Then, the integral in the curly brackets on the right-hand side of (2.14) is of class \( C^1(\mathbb{R}^3; \mathbb{R}^3) \). Therefore, the last term in (2.14) is well defined and so are the other terms of this equation. Now, substitution of (1.3) into the left-hand side of (2.14) yields,

\[
-\int_{\mathbb{R}^3} A(x) \Delta \varphi(x) dx = \int_{\mathbb{R}^3} \varphi(x) j_A(x) dx,
\]

(2.15)

where, \( I_1 + I_2 \),

(2.16)

\( \varphi \)

In the last step we have used that the integral in the curly brackets in (2.16) is a solution to the Poisson equation \( \Delta u = -4\pi j_A \) in the sense of distributions. As for the integral \( I_2 \), we work per components:

\[
I_2^i = \frac{1}{2} \int_{\mathbb{R}^3} \Delta \varphi(x) \left\{ \int_{\mathbb{R}^3} [\delta_{ik} - \omega^i \omega^k] j_A^k(y) \frac{dy}{|y-x|} \right\} dx
\]

(2.17)

where in the second and third steps we have used Fubini’s theorem and Lemma 5 respectively, and Fubini’s theorem and integration by parts in the last step. Note that the resulting boundary term vanishes in view of the compact support of \( \varphi \). The relation (2.14) readily follows from (2.15), (2.17).
3. Tools from Optimal Transportation

Denote by $P_2(\mathbb{R}^3 \times \mathbb{R}^3)$ the set of probability densities $f(x, p)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ with finite second moment, $\int_{\mathbb{R}^3 \times \mathbb{R}^3} (|x|^2 + |p|^2) f(x, p) \, dx \, dp < \infty$. The $L^2$-Wasserstein distance between two densities $f_1(x, p)$ and $f_2(x, p)$ in $P_2(\mathbb{R}^3 \times \mathbb{R}^3)$ is defined by

$$W_2^2(f_1, f_2) = \inf \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} |(x, p) - (y, q)|^2 \, d\gamma((x, p), (y, q)) \mid \gamma \in \Gamma(\mu_1, \mu_2) \right\}$$

(3.1) where $d\mu_1 = f_1(x, p) \, dx \, dp$, $d\mu_2 = f_2(x, p) \, dx \, dp$, $\Gamma(\mu_1, \mu_2)$ denotes the set of all probability measures on $\mathbb{R}^6 \times \mathbb{R}^6$ with marginals $\mu_1$ and $\mu_2$, and $T \# f_1 = f_2$ means that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(x, p) f_2(x, p) \, dx \, dp = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(T(x, p)) f_1(x, p) \, dx \, dp$$

for all test functions $\varphi \in C_0(\mathbb{R}^3 \times \mathbb{R}^3)$. In [3], Brenier proved that the minimization problem (3.1), the so-called Monge-Kantorovich problem, has a unique solution $T$, which is characterized $\mu_1$-a.e. by the gradient of a convex function $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$, i.e., $T$ is uniquely determined $\mu_1$-a.e. by $T = \nabla \phi$ with $(\nabla \phi) \# f_1 = f_2$ for some convex function $\phi$. Note that in (3.1), the minimizers $\gamma$ and $T$ are related by $\gamma = (1d_{\mathbb{R}^6}, T) \# f_1$. Moreover, if $\theta \in [1, 2]$ and

$$f_\theta = T_\theta \# f_1, \quad T_\theta = (2 - \theta)1d_{\mathbb{R}^6} + (\theta - 1)T = \nabla \left( (2 - \theta)\frac{|\cdot|^2}{2} + (\theta - 1)\phi \right)$$

(3.3) denotes McCann’s interpolation [10], then the curve $[1, 2] \ni \theta \rightarrow f_\theta \in P_2(\mathbb{R}^3 \times \mathbb{R}^3)$ is the unique length minimizing geodesic joining $f_1$ to $f_2$ in the Wasserstein space $(P_2(\mathbb{R}^3 \times \mathbb{R}^3), W_2)$, in the sense that $W_2^2(f_1, f_2) = W_2^2(f_1, f_\theta) + W_2^2(f_\theta, f_2)$. Furthermore, the interpolant $f_\theta$ satisfies the continuity equation in a weak sense,

$$\partial_\theta f_\theta(x, p) + \nabla_{x\cdot p} \cdot (u_\theta(x, p) f_\theta(x, p)) = 0 \quad \forall \theta \in [1, 2],$$

(3.4) where $u_\theta \in L^2_{\#T_\theta}(\mathbb{R}^3 \times \mathbb{R}^3)$ is the velocity field associated with the trajectory $T_\theta$, i.e.,

$$u_\theta \left( T_\theta(x, p) \right) = \frac{\partial T_\theta(x, p)}{\partial \theta} = \nabla \phi(x, p) - (x, p).$$

(3.5) Indeed, (3.4) can be formally seen as follows. For any test function $\varphi \in C^1_0(\mathbb{R}^3 \times \mathbb{R}^3)$, using (3.2) with $f_\theta = (T_\theta) \# f_1$ and then (3.3), we have:

$$\frac{d}{d\theta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(x, p) f_\theta(x, p) \, dx \, dp$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \varphi \left( T_\theta(x, p) \right) u_\theta \left( T_\theta(x, p) \right) f_1(x, p) \, dx \, dp$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \varphi(x, p) u_\theta(x, p) f_\theta(x, p) \, dx \, dp$$

$$= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(x, p) \nabla_{x\cdot p} \cdot (f_\theta(x, p) u_\theta(x, p)) \, dx \, dp,$$

(3.6) where we use an integration by parts on the right-hand side integral in (3.4).
Combining (3.1) - (3.5), we have that

\[
W^2_2(f_1, f_2) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\nabla \Phi(x, p) - (x, p)|^2 f_1(x, p) \, dx \, dp \\
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u_\theta(T_\theta(x, p))|^2 f_1(x, p) \, dx \, dp \\
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u_\theta(x, p)|^2 f_\theta(x, p) \, dx \, dp.
\]

(3.7)

Formula (3.7) is commonly known as the Benamou-Brenier [2] characterization of the \(L^2\)-Wasserstein distance, namely,

\[
W^2_2(f_1, f_2) = \min \left\{ \int_1^2 \int_{\mathbb{R}^3} f(\theta, x, p)|u(\theta, x, p)|^2 \, dx \, dp \, d\theta : f(\theta) \in \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},
\]

where the minimum is taken over all absolutely continuous curves \( f : [1, 2] \ni \theta \mapsto f(\theta) \in \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \) satisfying the constraints \( f(1) = f_1, \ f(2) = f_2 \) and \( \partial_\theta f + \nabla_{x, p} \cdot (uf) = 0 \). For a development on this topic, we refer to [1].

In the next lemma, we collect some well-known results in optimal transport theory that will be needed later in the paper.

**Lemma 7.** Let \( f_1, f_2 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}) \) be two probability densities in \( \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \) with compact supports. For any \( \theta \in [1, 2] \), define the interpolant \( f_\theta \) as in (3.3). Then

(i). For all \( \theta \in [1, 2] \), \( f_\theta \) has a compact support in \( \mathbb{R}^3 \times \mathbb{R}^3 \), and

\[
\|f_\theta\|_{L^\infty_{x,p}} \leq \max\{\|f_1\|_{L^\infty_{x,p}}, \|f_2\|_{L^\infty_{x,p}}\}.
\]

That is, \( [1, 2] \ni \theta \mapsto f_\theta \) belongs to \( L^\infty([1, 2], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3); \mathbb{R}) \).

(ii). Moreover, \( [1, 2] \ni \theta \mapsto f_\theta \) is differentiable at every point \( \theta \in [1, 2] \), and its derivative \( \partial_\theta f_\theta \), defined in the weak sense by (3.7), satisfies

\[
\|\partial_\theta f_\theta\|_{H^{-1}(\mathbb{R}^3 \times \mathbb{R}^3)} \leq \max\{\|f_1\|_{L^\infty_{x,p}}, \|f_2\|_{L^\infty_{x,p}}\}^{1/2} W_2(f_1, f_2), \ \forall \theta \in [1, 2].
\]

That is, \( [1, 2] \ni \theta \mapsto \partial_\theta f_\theta \) belongs to \( L^\infty([1, 2], H^{-1}(\mathbb{R}^3 \times \mathbb{R}^3); \mathbb{R}) \).

**Proof.** The proofs of (3.8) and (3.9) are done in [3]. Here we only show that \( f_\theta \) has a compact support in \( \mathbb{R}^3 \times \mathbb{R}^3 \). Indeed, assume that the support of \( f_i, \ \text{supp} f_i \), is contained in the ball \( B_{R_i} \) centered at the origin with radius \( R_i \) for \( i = 1, 2 \). By the definition (3.3) of \( f_\theta, \ T_\theta \) is the optimal map in \( W^2_2(f_1, f_\theta) \). Then for \( \mu_1\text{-a.e.} \), \( T_\theta \) is invertible and \( \partial_{x,p} T_\theta(x, p) \) is diagonalizable with positive eigenvalues (see [1] Thm 6.2.4 & Prop 6.2.12). Moreover, the following Monge-Ampère equation holds for \( \mu_1\text{-a.e.} \ (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \),

\[
f_1(x, p) = f_\theta(T_\theta(x, p)) \det \partial_{x,p} T_\theta(x, p).
\]

It follows that for \( \mu_1\text{-a.e.} \ (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \), if \( f_\theta(T_\theta(x, p)) \neq 0 \), then \( f_1(x, p) \neq 0 \) so that

\[
\{(x, p) : f_\theta(x, p) \neq 0 \} \subset T_\theta(\{(x, p) : f_1(x, p) \neq 0 \})
\]

i.e. \( \text{supp} f_\theta \subset T_\theta(B_{R_1}) \). But \( T_\theta(B_{R_1}) \subset (\theta - 2)B_{R_1} \), \( (\theta - 1)B_{R_2} \subset B_R \) where \( R = R_1 + R_2 \). Therefore \( \text{supp} f_\theta \subset \overline{B_R} \), i.e. \( f_\theta \) has a compact support. \[\square\]
4. Final Estimates

We first note that if \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}) \) is such that \( f_0 \geq 0 \), and if \( f \) is a weak solution of the RVD system in the sense of Definition 6 then \( f(t) \geq 0 \) and \( \|f(t)\|_{L^1_p} = \|f_0\|_{L^1_p} \) for all \( t \in [0, T] \), so that \( f(t) \) can be viewed as a probability density on \( \mathbb{R}^3 \times \mathbb{R}^3 \) up to normalizing the \( L^1 \)-norm of \( f_0 \) to 1. Moreover, under the assumption that the support of \( f(t) \) is compact in \( \mathbb{R}^3 \times \mathbb{R}^3 \) for all \( t \in [0, T] \), then \( f(t) \in \mathbf{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \).

For simplicity and without loss of generality, we shall omit the time dependence in \( f \) throughout this section. The next lemma gives estimates on the relative scalar and vector potentials of two solutions to the RVD system.

**Lemma 8.** Let \( f_1, f_2 \in \mathbf{P}_2 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}) \). Define \( \rho_i \) and \( j_{A_i} \) according to (1.4), and let \( \Phi_i \) and \( A_i, i = 1, 2 \), satisfy respectively the equations

\[
\Delta \Phi_i(x) = -4\pi \rho_i(x), \quad \lim_{|x| \to \infty} \Phi_i(x) = 0
\]

and

\[
\Delta A_i(x) = -4\pi j_{A_i}(x) - \nabla \left\{ \nabla \cdot \int_{\mathbb{R}^3} j_{A_i}(y) \frac{dy}{|y-x|} \right\}, \quad \lim_{|x| \to \infty} |A_i(x)| = 0,
\]

in the sense of (1.2)-(1.4) (see Lemma 6). Assume that for some \( R > 0 \), \( \text{supp} f_1 \cup \text{supp} f_2 \subset B_R \times B_R \). Then there exists a constant \( C > 0 \) which depends on \( R \), \( \|f_1\|_{L^\infty_{x,p}} \) and \( \|f_2\|_{L^\infty_{x,p}} \), such that

\[
\|\partial_x \Phi_1 - \partial_x \Phi_2\|_{L^2_p} + \|\partial_x A_1 - \partial_x A_2\|_{L^2_p} \leq C W_2(f_1, f_2),
\]

and, for \( i \in \{1, 2\},
\[
\int_{\mathbb{R}^3} \rho_i(x) |A_1(x) - A_2(x)|^2 \, dx \leq C W_2^2(f_1, f_2).
\]

**Proof.** The estimate on the scalar potential, that is, the first term on the left-hand side of (4.4), is essentially proved in [8] under the weaker assumption of the boundedness of the charge density. Here we only prove the estimates on the vector potential, which are (4.3) and the second term on the left-hand side of (4.4). Indeed, for \( i \in \{1, 2\}, \) set

\[
j_{A_i}(x) = \int_{\mathbb{R}^3} v_{A_i}(x, p) f_i(x, p) \, dp, \quad v_{A_i} = \frac{g_{A_i}}{\sqrt{1 + |g_{A_i}|^2}}, \quad g_{A_i}(x, p) = p - A_i(x).
\]

By the assumptions on \( f_i \), we have that \( j_{A_i} \in L^1 \cap L^\infty(\mathbb{R}^3; \mathbb{R}^3) \) and its support is included in \( B_R \). Then, \( j_{A_i} \in L^2(\mathbb{R}^3; \mathbb{R}^3) \), and by the generalized theory of Poisson’s equation [5] chap. 8, the Newtonian potential

\[
\int_{\mathbb{R}^3} j_{A_i}(y) \frac{dy}{|y-x|} \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3),
\]

which implies that

\[
\nabla \left\{ \nabla \cdot \int_{\mathbb{R}^3} j_{A_i}(y) \frac{dy}{|y-x|} \right\} \in H^{-1}(\mathbb{R}^3; \mathbb{R}^3).
\]

In view of Lemma 8

\[
\Delta A_i(x) = -4\pi j_{A_i}(x) - \nabla \left\{ \nabla \cdot \int_{\mathbb{R}^3} j_{A_i}(y) \frac{dy}{|y-x|} \right\}, \quad i = 1, 2,
\]
in the sense of distributions, and since the right-hand side of (4.3) belongs to $H^{-1}(\mathbb{R}^3; \mathbb{R}^3)$, we deduce by the generalized theory of Poisson’s equation that $A_i \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3)$. Taking the difference of the two equations in (4.5), we obtain

$$-\Delta (A_1 - A_2)(x) = 4\pi (j_{A_1} - j_{A_2})(x) + \nabla \left\{ \nabla \cdot \int_{\mathbb{R}^3} (j_{A_1} - j_{A_2})(y) \frac{dy}{|y-x|} \right\},$$

and since $A_1 - A_2 \in W^{1,2}(\mathbb{R}^3)$, then integration by parts against $A_1 - A_2$ yields

$$\int_{\mathbb{R}^3} |\partial_x A_1 - \partial_x A_2|^2 \, dx = 4\pi \int_{\mathbb{R}^3} (A_1 - A_2) \cdot (j_{A_1} - j_{A_2}) \, dx + \int_{\mathbb{R}^3} \nabla \cdot (A_1 - A_2) \nabla \cdot I \, dx$$

where

$$I(x) = \int_{\mathbb{R}^3} (j_{A_1} - j_{A_2})(y) \frac{dy}{|y-x|}.$$

Notice that the boundary terms vanish. Indeed, standard arguments show that both $\partial_x I(x)$ and $\partial_x I(x)$ have a decay $O(|x|^{-2})$, and $A_i(x)$ has a decay $O(|x|^{-1})$. Then, the products $A_i \partial_x I(x)$ and $A_i \partial_x A_i(x)$ have a decay $O(|x|^{-3})$, which suffice to make the boundary terms equal to zero. By Lemma 2, $\nabla \cdot A_i = 0$. Thus the last integral on the right-hand side vanishes as well.

On the other hand, we write

$$j_{A_1} - j_{A_2} = \int_{\mathbb{R}^3} f_1(v_{A_1} - v_{A_2}) \, dp + \int_{\mathbb{R}^3} v_{A_2}(f_1 - f_2) \, dp$$

$$_{} = \int_{\mathbb{R}^3} f_1(v_{A_1} - v_{A_2}) \, dp - \int_{\mathbb{R}^3} \int_1^2 v_{A_2} \, \partial_\theta f_\theta \, d\theta \, dp,$$

where $f_\theta$ is the interpolant between $f_1$ and $f_2$. Inserting this identity into the above expression, we have

$$\int_{\mathbb{R}^3} |\partial_x A_1 - \partial_x A_2|^2 \, dx - 4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(A_1 - A_2) \cdot (v_{A_1} - v_{A_2}) \, dp \, dx$$

$$= -4\pi \int_1^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (A_1 - A_2) \cdot v_{A_2} \, \partial_\theta f_\theta \, d\theta \, dp \, dx \, d\theta.$$ 

Therefore,

$$\int_{\mathbb{R}^3} |\partial_x A_1 - \partial_x A_2|^2 \, dx - 4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(A_1 - A_2) \cdot (v_{A_1} - v_{A_2}) \, dp \, dx$$

$$= 4\pi \int_1^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\theta u_\theta \cdot \partial_\theta [(A_1 - A_2) \cdot v_{A_2}] \, dp \, dx \, d\theta$$

$$+ 4\pi \int_1^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\theta u_\theta \cdot \partial_p [(A_1 - A_2) \cdot v_{A_2}] \, dp \, dx \, d\theta,$$

where we use the continuity equation (3.4) with the velocity field $u_\theta$ denoted by $u_\theta = (u_\theta^x, u_\theta^p)$. Here $u_\theta^x$ and $u_\theta^p$ are the $x$ and $p$-components of $u_\theta$ in $\mathbb{R}^3$, respectively.

We now estimate each of the integral terms in (4.6). First of all, consider the vector-valued function

$$\mathbb{R}^3 \ni z \mapsto v(z) = \frac{z}{\sqrt{1 + |z|^2}} \in \mathbb{R}^3.$$ 

Clearly, $v \in C^1(\mathbb{R}^3; \mathbb{R}^3)$ and $v_{A_i} = v(g_{A_i})$ where $g_{A_i}(x, p) = p - A_i(x)$ and $(x, p) \in B_R \times B_R$. It is easy to check that the derivative $Dv(z)$ of $v$ at any point $z \in \mathbb{R}^3$ is
given by the 3-by-3 real symmetric matrix

\[ Dv(z) = \frac{1}{\sqrt{1 + |z|^2}} \begin{bmatrix} 1 & \frac{z}{1 + |z|^2} \\ \frac{z}{1 + |z|^2} & 1 \end{bmatrix}. \]

Then \( Dv(z) \) is real orthogonally diagonalizable [6, Theorem 2.5.6]. Moreover, if \( z = (z_1, z_2, z_3) \in \mathbb{R}^3 \), then \( \det Dv(z) = (1 + |z|^2)^{-5/2} > 0 \), the determinant of the 1-by-1 leading principal submatrix of \( Dv(z) \) is \( (1 + z_3^2)(1 + |z|^2)^{-3/2} > (1 + |z|^2)^{-3/2} > (1 + |z|^2)^{-2} > 0 \), and that of its 2-by-2 leading principal submatrix is \( (1 + z_3^2)(1 + |z|^2)^{-2} > 0 \), where the leading principal submatrices of \( Dv(z) \) are the upper-left square submatrices of \( Dv(z) \). Therefore the matrix \( Dv(z) \) is positive definite [6, Theorem 7.2.5], and we have

\[ Dv(z) \xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall (z, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3, \]

where \( \lambda > 0 \) can be chosen as the lowest eigenvalue of \( Dv(z) \). By the mean value theorem, we have at every point \((x, p) \in B_R \times B_R\),

\[
(A_1 - A_2) \cdot (v_{A_1} - v_{A_2}) = \left[ v(g_{A_1}) - v(g_{A_2}) \right] \cdot (A_1 - A_2) = Dv \left( g_{A_1} + \delta (g_{A_1} - g_{A_2}) \right) (g_{A_1} - g_{A_2}) \cdot (A_1 - A_2) = -Dv \left( g_{A_1} + \delta (g_{A_1} - g_{A_2}) \right) (A_1 - A_2) \cdot (A_1 - A_2)
\]

for some \( \delta \in (0, 1) \). But since by Lemma [3] \( |g_{A_1}(x, p)| \leq K_R \) for all \((x, p) \in B_R \times B_R\) and for some \( K_R > 0 \), then \( |(g_{A_1} + \delta (g_{A_1} - g_{A_2})) (x, p)| \leq K_R \) uniformly on \((x, p) \in B_R \times B_R\). Then using (4.7) and the above identity, we have

\[
(A_1 - A_2) \cdot (v_{A_1} - v_{A_2}) \leq -C_R |A_1 - A_2|^2
\]

uniformly on \( B_R \times B_R \) and for some constant \( C_R > 0 \). Then we deduce that the left-hand side of (4.6) is bounded below as

\[
\int_{\mathbb{R}^3} |\partial_x A_1 - \partial_x A_2|^2 \, dx - 4\pi \int_{B_R} \int_{B_R} f_1(A_1 - A_2) \cdot (v_{A_1} - v_{A_2}) \, dp \, dx \geq C_R \left( \left\| \partial_x A_1 - \partial_x A_2 \right\|_{L^2}^2 + \left\| p_1^{1/2}(A_1 - A_2) \right\|_{L^2}^2 \right)
\]

for some other constant \( C_R > 0 \). On the other hand, by inserting the identities

\[
\partial_x [(A_1 - A_2) \cdot v_{A_1}] = v_{A_2} \cdot \partial_x (A_1 - A_2) \cdot \partial_x v_{A_1} \quad \text{and} \quad \partial_x [(A_1 - A_2) \cdot v_{A_2}] = (A_1 - A_2) \cdot \partial_x v_{A_2}
\]

into (4.6), it is easy to see that the expression on the right-hand side of (4.6) is dominated by

\[
I_1 + I_2 + I_3 = 4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\theta |u_\theta| |\partial_x A_1 - \partial_x A_2| \, dx \, dp \, d\theta + 4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\theta |u_\theta| |A_1 - A_2| |\partial_x A_2| \, dx \, dp \, d\theta + 4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\theta |u_\theta| |A_1 - A_2| \, dx \, dp \, d\theta.
\]

Since \( \|\rho\|_{L^\infty} \leq (4/3)\pi R^3 \max\{\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty}\} = K \) (see Lemma [7], by Cauchy-Schwarz inequality and Eq. (3.77), the integral \( I_1 \) can be estimated as

\[
I_1 \leq C_K \|\partial_x A_1 - \partial_x A_2\|_{L^2} \|W_2(f_1, f_2)\|_{L^2}.
\]
Similarly, using Cauchy-Schwarz’ inequality and Lemma\textsuperscript{7} we have that
\begin{align*}
I_3 &\leq 4\pi W_2(f_1, f_2) \int_1^2 d\theta \left( \int_{R^3} \int_{R^3} f_0 |A_1 - A_2|^2 dx \, dp \right)^{1/2} \\
&\leq 4\pi K W_2(f_1, f_2) \left( \int_{R^{3\times R}} |A_1 - A_2|^2 dx \right)^{1/2}
\end{align*}
and we deduce by Poincaré’s inequality that
(4.10) \hspace{1cm} I_3 \leq C_K \|\partial_x A_1 - \partial_x A_2\|_{L^2_x} W_2(f_1, f_2).

As for $I_2$, Cauchy-Schwarz’ inequality yields
\begin{align*}
I_2 &\leq 4\pi W_2(f_1, f_2) \|\partial_x A_2\|_{L^2_x}^{1/2} \left( \int_1^2 d\theta \left( \int_{R^3} \int_{R^3} f_0 |A_1 - A_2|^2 dx \right)^{1/2} \right) \\
\end{align*}
Then we use the second estimate in (2.5) and Poincaré’s inequality to get, as for $I_3$,
(4.11) \hspace{1cm} I_2 \leq C_K \|\partial_x A_1 - \partial_x A_2\|_{L^2_x} W_2(f_1, f_2).

Combining (4.10) - (4.11), we have
\begin{align*}
(4.12) \hspace{1cm} \|\partial_x A_1 - \partial_x A_2\|_{L^2_x}^2 + \|\rho_1^{1/2}(A_1 - A_2)\|_{L^2_x}^2 &\leq C \|\partial_x A_1 - \partial_x A_2\|_{L^2_x} W_2(f_1, f_2),
\end{align*}
for some constant $C > 0$, which implies that
(4.13) \hspace{1cm} \|\partial_x A_1 - \partial_x A_2\|_{L^2_x} \leq C W_2(f_1, f_2).

Finally, inserting (4.13) in the right hand side of (4.12), we obtain
\begin{align*}
\|\rho_1^{1/2}(A_1 - A_2)\|_{L^2_x} &\leq C W_2(f_1, f_2).
\end{align*}
This completes the proof of the lemma. \hfill \Box

5. Proof of Theorem \textbf{1}

For a non-negative function $f_0 \in L^1 \cap L^\infty(R^3 \times R^3; R^3)$ with compact support, let $f_1$ and $f_2$ be two compactly supported weak solutions of the RVD system with the same Cauchy datum $f_0$. Let $(\Phi_i, A_i)$ be the potentials induced by $f_i, i = 1, 2$, respectively. Denote $z = (x, p)$ and let $0 \leq s \leq t < T$. To ease notation, write $Z_i(s, t)$ instead of $Z_i(s, t, z)$ for the solution of the characteristic system (2.8) - (2.9) associated to the Vlasov equation (1.1). Equivalently, $Z_i(s, t)$ with inverse $Z_i^{-1}(s, t) = Z_i(t, s)$ is the characteristic flow associated to the solution $f_i(t) = Z_i(t, s) \# f_i(s)$ of the Vlasov equation; see Lemma\textsuperscript{4} and Remark\textsuperscript{7}. In particular $Z_i(t, 0)$ is the flow associated to $f_i(t) = Z_i(t, 0) \# f_0$, that is, $f_i(t, z) = f_0(Z_i(0, t, z))$ for all $z \in B \subset R^6, B$ a Borel set. We further denote $Z_i(t)$ instead of $Z_i(t, 0)$ and define the function
(5.1) \hspace{1cm} Q(t) = \frac{1}{2} \int_{R^6} f_0(z) |Z_1(t) - Z_2(t)|^2 dz.

We have $W_2^2(f_1(t), f_2(t)) \leq 2Q(t)$ because $\gamma = (Z_1(t), Z_2(t)) \# f_0$ is admissible in (5.1); here the function $(Z_1(t), Z_2(t)) : R^6 \to R^6 \times R^6$ is defined by $(Z_1(t), Z_2(t))(z) = (Z_1(t, z), Z_2(t, z))$. Clearly, $Q(0) = 0$. Our goal is to show that $Q(t) = 0$ for every $t \in [0, T]$. If so, then $W_2(f_1(t), f_2(t)) = 0$ which implies that $f_1 = f_2$ on $[0, T] \times R^6$ and therefore uniqueness.
Take the time derivative on both sides of (5.1). By Lemma 4 we have

\[ \dot{Q}(t) = \int_{\mathbb{R}^6} f_0(z) \left[ Z_1(t) - Z_2(t) \right] \cdot \left[ \dot{Z}_1(t) - \dot{Z}_2(t) \right] \, dz \]
\[ = \int_{\mathbb{R}^6} f_0(z) \left[ X_1(t) - X_2(t) \right] \cdot \left[ v_{A_1}(t, Z_1(t)) - v_{A_2}(t, Z_2(t)) \right] \, dz \]
\[ - \int_{\mathbb{R}^6} f_0(z) \left[ P_1(t) - P_2(t) \right] \cdot \left[ \nabla \Phi_1(t, X_1(t)) - \nabla \Phi_2(t, X_2(t)) \right] \, dz \]
\[ + \int_{\mathbb{R}^6} f_0(z) \left[ P_1(t) - P_2(t) \right] \cdot \left[ v_{A_1}^i \nabla A_1^i(t, Z_1(t)) - v_{A_2}^i \nabla A_2^i(t, Z_2(t)) \right] \, dz \]
\[ =: I_1(t) + I_2(t) + I_3(t). \]

In [8], it is shown that for some constant \( C > 0 \) that depends only on \( \|\rho_i\|_{L^\infty_t} \),

\[ I_2(t) \leq CQ(t) (1 - \ln Q(t)), \tag{5.2} \]

provided \( \|Z_1(t) - Z_2(t)\|_{L^\infty_t} \leq e^{-1} \). This is essentially the result in [8] which yields uniqueness of weak solutions of the Vlasov-Poisson system under the assumption that the charge density stays bounded. As for the RVD system, it remains to estimate \( I_1 \) and \( I_3 \). To estimate \( I_1 \), recall that

\[ v_{A_1}(t, X(t), P(t)) = v(P(t) - A(t, X(t))). \]

where \( v(g) = g (1 + g^2)^{-1/2} \). Then, since \( g \mapsto v(g) \) is \( C^1_b \), we have

\[ |v_{A_1}(t, X_1(t), P_1(t)) - v_{A_2}(t, X_2(t), P_2(t))| \]
\[ \leq C (|P_1(t) - P_2(t)| + |A_1(t, X_1(t)) - A_2(t, X_2(t))|) \]
\[ \leq C \left( |P_1(t) - P_2(t)| + |X_1(t) - X_2(t)| + |A_1(t, X_1(t)) - A_2(t, X_1(t))| \right), \]

where \( C = C(\|\rho_i\|_{L^\infty_t}) \). Note the use of the second estimate in (2.5) for the last step. Thus, Cauchy-Schwarz’ inequality yields

\[ I_1 \leq C \left( Q(t) + Q^{1/2}(t) T^{1/2}(t) \right), \tag{5.3} \]

where, in view of (3.2) and (4.4) in Lemma 8

\[ T(t) = \int_{\mathbb{R}^6} f_0(z) |A_1(t, X_1(t)) - A_2(t, X_1(t))|^2 \, dz \]
\[ = \int_{\mathbb{R}^3} \rho_0(x) |A_1(t, X_1(t)) - A_2(t, X_1(t))|^2 \, dx \]
\[ = \int_{\mathbb{R}^3} \rho_1(t, x) |A_1(t, x) - A_2(t, x)|^2 \, dx \]
\[ \leq CW_2^2(f_1, f_2), \tag{5.4} \]

with \( C = C(R, \|f_i\|_{L^\infty_{t,x,p}}) \). Therefore, since \( W_2^2(f_1(t), f_2(t)) \leq 2Q(t) \), we find that \( I_1 \leq CQ(t) \).
The integral $I_3$ can be estimated just as $I_1$ and $I_2$. Indeed, we have

$$I_3 = \int_{\mathbb{R}^6} f_0(z) \left[ P_1(t) - P_2(t) \right] \cdot \left[ v_{A_1}^i(t, Z_1(t)) - v_{A_2}^i(t, Z_2(t)) \right] \nabla A_1^i(t, X_1(t)) dz$$

$$+ \int_{\mathbb{R}^6} f_0(z) \left[ P_1(t) - P_2(t) \right] \cdot v_{A_2}^i(t, Z_2(t)) \left[ \nabla A_1^i(t, X_1(t)) - \nabla A_2^i(t, X_2(t)) \right] dz.$$ 

In view of (2.5), the first integral on the right-hand side can be estimated exactly as $I_1$. On the other hand, the second integral on the right-hand side is analogous to $I_2$, with the vector potential instead of the scalar potential. Hence, since $|v_{A_2}| \leq 1$, we can use mutatis mutandi the arguments in [8] and Lemmas 3 and 8 to estimate this integral as in (5.2).

Then, we gather all previous estimates to find that for some constant $C = C(R, \|f_1\|_{L^\infty_{t,x,p}}) > 0$

$$\dot{Q}(t) \leq CQ(t) \left( 1 - \ln Q(t) \right),$$

whenever $\|Z_1(t) - Z_2(t)\|_{L^\infty} \leq e^{-1}$. This is a Gronwall’s-type inequality which yields $Q(t) \equiv 0$ on $[0, T]$ and therefore uniqueness. □

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