Symplectic Geometric Methods for Matrix Differential Equations Arising from Inertial Navigation Problems

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Abstract
This article explores some geometric and algebraic properties of the dynamical system which is represented by matrix differential equations arising from inertial navigation problems, such as the symplecticity and the orthogonality. Furthermore, it extends the applicable fields of symplectic geometric algorithms from the even dimensional Hamiltonian system to the odd dimensional dynamical system. Finally, some numerical experiments are presented and illustrate the theoretical results of this paper.

Keywords: symplectic geometric methods, matrix differential equation, inertial navigation, simultaneous localization and mapping, robotic vision

AMS subject classifications. 65H17, 65J15, 65K05, 65L05

1. Introduction
Simultaneous localization and mapping (SLAM), which is involved in the inertial navigation and robotic vision, is a hot research topic in some engineering fields for which there are many applications such as unmanned aerial vehicles, autonomous vehicles, remote medical operations and so on (see \([3, 12, 16, 21]\)). In the front end of SLAM, in order to track the pose of camera which is fixed to the vehicle, it needs to solve a matrix differential equation arising from the inertial navigation problem. In engineering fields, they usually adopt the explicit Runge-Kutta method for this differential equation owing to its simplicity of implementation (see \([3, 16]\)).

It is well-known that the explicit Runge-Kutta method is not suitable for a Hamiltonian dynamical system, since the non-symplectic Runge-Kutta method can not preserve its geometric and algebraic properties such as the symplecticity and the orthogonality \([5, 7, 8, 9, 11, 12, 13, 18, 19, 20, 24, 25]\). Hong, Liu and Sun \([11]\)
consider the symplecticity of a Hamiltonian system which is represented by a PDEs with the skew-symmetric matrix coefficient.

For the inertial navigation problems, the variables are a $3 \times 3$ matrix. We known that the dimension of variables is odd. Thus, it can not directly apply the classical results of the symplectic geometric algorithm to this problem, since the classical symplectic geometric algorithm is applicable to the even dimensional Hamiltonian system. On the other hand, it is important to preserve the geometric and algebraic properties of the differential equation for the numerical method. Therefore, in this article, we investigate some geometric and algebraic properties of the matrix differential equation such as the law of generalized energy, the pseudo-symplecticity and the orthogonal invariant. Consequently, we discuss the geometric and algebraic properties of the symplectic Runge-Kutta method for the linear matrix differential equation with the skew-symmetric matrix coefficient in Section 3. In Section 4, we compare the symplectic Runge-Kutta method with the non-symplectic Runge-Kutta for the differential equation with the skew-symmetric matrix coefficient. The simulation results illustrate the theoretical results of this article. Finally, in Section 5, we give some conclusions and discuss the future work.

2. Geometric Structure of Linear Matrix Differential Equations

We choose a moving coordinate system connected to the aerial vehicle and consider motions of the aerial vehicle where the origin is fixed. By one of Euler’s famous theorems, any such motion is infinitesimally a rotation around an axis. We represent the rotational axis of the aerial vehicle by the direction of a vector $\omega$ and the speed of rotation by the length of $\omega$. Thus, the velocity of a mass point $x$ of the aerial vehicle is given by the exterior product

$$ v = \omega \times x = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad (1) $$

which is orthogonal to $\omega$, orthogonal to $x$, and of length $\|\omega\|\|x\|\sin \gamma$, $\cos \gamma = \frac{x^T \omega}{\|\omega\|\|x\|}$.

We also regard the motion of the aerial vehicle from a coordinate system stationary in the space. The transformation of a vector $x \in \mathbb{R}^3$ in the aerial vehicle frame, to the corresponding $y \in \mathbb{R}^3$ in the stationary frame, is denoted by

$$ y = R(t)x. \quad (2) $$

Matrix $R(t)$ is orthogonal and describes the rotation of the aerial vehicle. For $x = e_i$ in the aerial vehicle frame, we find that the columns of matrix $R(t)$ are the coordinates of the aerial vehicle’s principal axes in the stationary frame.
From equation (2), we know that these rotate with the velocity
\[
\omega \times e_1, \omega \times e_2, \omega \times e_3 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = W. \tag{3}
\]

Thus, we obtain \( \dot{R} \), which is the rotational velocity expressed in the stationary frame, by the back transformation (2):
\[
R^T \dot{R} = W,
\]
which gives
\[
\dot{R} = RW, \quad R(t_0)^T R(t_0) = I, \tag{4}
\]
where \( S \) is a skew-symmetric matrix, namely \( W^T = -W \), and \( R \) represents the rotational transform matrix (see p. 48 in \[3\], p. 122 in \[15\], or p. 278 in \[8\]). If we denote \( Q(t) = R(t)^T \) and \( S = W^T \), from equation (4), we have
\[
\dot{Q} = SQ, \quad Q(t_0)^T Q(t_0) = I, \tag{5}
\]
where \( S \) is a skew-symmetric matrix. We denote the solution of linear differential equation (5) as
\[
\Phi_t(Q(t_0)) = Q(t; Q(t_0)), \tag{6}
\]
and term the map \( \Phi_t : \mathbb{R}^{M \times M} \to \mathbb{R}^{M \times M} \) as the flow map of the given system (5).

We introduce some concepts before we investigate geometric structures of equation (5). Assume that \( f : \mathbb{R}^{M \times M} \to \mathbb{R} \) is a smooth function. Its directional derivative along a matrix \( X \in \mathbb{R}^{N \times N} \) is denoted here by
\[
df(X) = \lim_{\Delta t \to 0} \frac{f(Z + \Delta tX) - f(Z)}{\Delta t} = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial f}{\partial z_{ij}} x_{ij}, \tag{7}
\]
where the partial derivatives of \( f \) are computed at a fixed location \( Z \in \mathbb{R}^{M \times M} \), and \( z_{ij}, \ x_{ij} \) are the elements of matrix \( Z \) and matrix \( X \), respectively. The linear function \( df(\cdot) \) is called the differential of \( f \) at \( Z \) and is an example of a differential one-form.

Using this denotation, we define the wedge product \( df \wedge dg \) of two differentials \( df \) and \( dg \) as follows (see pp. 61-64 in [11]):
\[
(df \wedge dg)(X, Y) = df(X)dg(Y) - df(Y)dg(X), \tag{8}
\]
where \( X, Y \in \mathbb{R}^{M \times M} \). Thus, we give the definition of the wedge product of two matrix functions \( P \in \mathbb{R}^{M \times M} \) and \( Q \in \mathbb{R}^{M \times M} \) as follows (see p. 30, [23]):
\[
dP \wedge dQ = \sum_{i=1}^{M} \sum_{j=1}^{M} (dp_{ij} \wedge dq_{ij}), \tag{9}
\]
where \( p_{ij} \) and \( q_{ij} \) are the elements of matrix \( P \) and \( Q \), respectively. For the wedge product \([9]\) of two matrix differentials \( dQ \) and \( dP \), it also has some basic properties similar to the wedge product of two vector differentials \( dp \) and \( dq \). We state them as the following Property 2.1.

**Property 2.1.** Let \( dP, dQ, dR \) be \( M \times M \)-matrices of differential one-forms on \( \mathbb{R}^{M \times M} \), then the following properties hold.

1. **Skew-symmetry**
   \[
   dP \wedge dQ = -dQ \wedge dP. \quad (10)
   \]

2. **Bilinearity:** for any \( \alpha, \beta \in \mathbb{R} \),
   \[
   dP \wedge (\alpha dQ + \beta dR) = \alpha dP \wedge dQ + \beta dP \wedge dR. \quad (11)
   \]

3. **Rule of matrix multiplication**
   \[
   dP \wedge (AdQ) = (A^T dP) \wedge dQ, \quad (12)
   \]
   for any \( M \times M \) matrix \( A \).

Now we give the pseudo-symplectic property of equation (5).

**Property 2.2.** Assume that \( Q(t) \) is the solution of equation (5) with an initial orthogonal matrix \( Q(t_0) \), then it satisfies the following geometric property

\[
\dot{dQ} \wedge SdQ = \text{const}, \quad (13)
\]

where the wedge product \( dP \wedge dQ \) is defined by equation (9) and \( \text{const} \) is a constant number.

**Proof.** Actually, if \( Q(t) \) is the solution of equation (5), from properties (10)-(12), we have

\[
\frac{d}{dt}(dQ \wedge SdQ) = \dot{dQ} \wedge SdQ + dQ \wedge \dot{SdQ} = SdQ \wedge SdQ + S^T dQ \wedge \dot{dQ} = -SdQ \wedge SdQ = 0,
\]

which proves the result of equation (13). \(\square\)

**Remark 2.1.** If the skew-symmetric matrix \( S \) equals \( J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \) and \( Q \in \mathbb{R}^{2d} \) is the vector function, equation (13) is the condition of symplecticity (see p. 65 in [11]).

**Theorem 2.1.** (Kang Feng & Zai-Jiu Shang [6]). Assume that \( Q(t) \) is the solution of the matrix differential equation (5) and matrix \( S \) is skew-symmetric. Then, the determinant \( \det(Q(t)) \) is an invariant.
Proof. We denote \( g(t) = \det(Q(t)) \), then it only needs to prove \( \frac{d}{dt} g(t) = 0 \) along the solution curve of the matrix differential equation (5). Actually, from equation (5), we have

\[
g(t + \Delta t) = \det(Q(t + \Delta t)) = \det \left( (Q(t) + \dot{Q}(t) \Delta t + O((\Delta t)^2)) \right)
\]

\[
= \det \left( (Q(t) + SQ(t) \Delta t + O((\Delta t)^2)) \right) = \det \left( I + S \Delta t + O((\Delta t)^2) \right) \det(Q(t))
\]

\[
= \prod_{i=1}^{M} \left( (1 + \lambda_i(S) \Delta t + O((\Delta t)^2)) \det(Q(t)) \right)
\]

\[
= (1 + \text{trace}(S) \Delta t + O((\Delta t)^2)) g(t), \tag{14}
\]

where \( \lambda_i(S) \) represents the eigenvalue of matrix \( S \). Here, we use the property

\[
\text{trace}(S) = \sum_{i=1}^{M} \lambda_i(S).
\]

Since matrix \( S \) is skew-symmetric, we have \( \text{trace}(S) = 0 \). Replacing this result into equation (14), we have

\[
\frac{d}{dt} g(t) = \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t} = 0,
\]

which gives the proof of the result. \( \square \)

For a linear Hamiltonian dynamical system, the particle satisfies the law of conservation of energy. Similarly, the generalized energy of the dynamical system (5) conserves constant if we define its generalized energy as

\[
E(t) = \text{trace} \left( Q(t)^T Q(t) \right) = \sum_{i=1}^{M} \sum_{j=1}^{M} q_{ij}^2(t), \tag{15}
\]

where \( q_{ij}(t) \) \((i, j = 1 : M)\) are entries of matrix \( Q(t) \).

Property 2.3. The generalized energy of the dynamical system (5) conserves constant.

Proof. From the definition of the generalized energy (15), we have

\[
\frac{dE(t)}{dt} = \frac{d}{dt} \text{trace} \left( Q(t)^T Q(t) \right) = 2 \text{trace} \left( Q(t)^T \dot{Q}(t) \right). \tag{16}
\]

Since \( Q(t) \) satisfies the dynamical equation (5), replacing \( \dot{Q}(t) \) with \( SQ(t) \) in equation (16), we obtain

\[
\frac{dE(t)}{dt} = 2 \text{trace} \left( Q(t)^T SQ(t) \right). \tag{17}
\]
Noticing the trace property \( \text{trace}(A) = \text{trace}(A^T) \) and from equation (17), we have

\[
\frac{dE(t)}{dt} = 2 \text{trace} \left( Q(t)^T SQ(t) \right) = 2 \text{trace} \left( Q(t)^T S^T Q(t) \right)
\]

\[
= -2 \text{trace} \left( Q(t)^T SQ(t) \right) = 0,
\]

where we use the assumption \( S^T = -S \) in the above third equality of equation (18). Thus, we prove that the generalized energy of the dynamical system (5) conserves constant. □

Another interesting property is about the inverse proposition of property 2.3. We state this property as the following Property 2.4.

**Property 2.4.** Assume that the generalized energy \( \text{trace} \left( Q(t)^T Q(t) \right) \) of the dynamical system (5) conserves constant, then matrix \( S \) is skew-symmetric.

**Proof.** Since the generalized energy \( \text{trace} \left( Q(t)^T Q(t) \right) \) along the solution of equation (5) conserves constant, from equations (16)-(17) and equation (5), we have

\[
\frac{dE(t)}{dt} = 2 \text{trace} \left( Q(t)^T \dot{Q}(t) \right) = 2 \text{trace} \left( Q(t)^T SQ(t) \right)
\]

\[
= 2 \sum_{i=1}^{n} (q(t)_i^T S q(t)_i) = 0,
\]

where \( Q(t) = [q(t)_1, \ldots, q(t)_n] \). Let vectors \( q(t)_i = 0 \) (\( i = 2, \ldots, n \)) and vector \( q(t)_1 = e_i + e_j \) (\( i, j = 1, \ldots, n \)) in equation (19), we obtain

\[ s_{ij} + s_{ji} = 0. \]

Namely matrix \( S \) is skew-symmetric. □

**Remark 2.2.** For the linear matrix differential equation (5), there is a stronger property than Property 2.3. Namely \( Q(t)^T Q(t) \) is an invariant (see Theorem 1.6, pp. 99 in [8]).

**Proof.** We denote \( F(t) = Q(t)^T Q(t) \), then from equation (5), we have

\[
\frac{dF(t)}{dt} = \dot{Q}(t)^T Q(t) + Q(t)^T \dot{Q}(t) = Q(t)^T S^T Q(t) + Q(t)^T SQ(t)
\]

\[
= -Q(t)^T SQ(t)^T + Q(t)^T SQ(t) = 0,
\]

which gives the proof of the result. □
3. Pseudo-Symplectic Runge-Kutta Methods

Since it does not exist a general linear multiple method to satisfy the symplectic property for a Hamiltonian dynamical system (see [22]), we consider Runge-Kutta methods with the symplecticity for linear matrix differential equation (5). An s-stage Runge-Kutta method for equation (5) has the following form (see [1]):

\[ Y_i = Q_k + \Delta t \sum_{j=1}^{s} a_{ij} SY_j, \quad 1 \leq i \leq s, \quad (20) \]

\[ Q_{k+1} = Q_k + \Delta t \sum_{i=1}^{s} b_i SY_i, \quad (21) \]

where \( \Delta t \) is a time-stepping length and \( b_i, a_{ij} \) \((i, j = 1, 2, \ldots, s)\) are constants. For a Runge-Kutta method, we can write a condensed representation, which is so-called Butcher-array as Table 1 (see [1]).

| \( c \) | \( A \) | \( b^T \) |
|---|---|---|
| \( c_i = \sum_{j=1}^{s} a_{ij} \), \( i = 1, 2, \ldots, s \). |

The symplectic condition of a Runge-Kutta method for the even dimensional Hamiltonian system is

\[ M = BA + A^T B - bb^T = 0, \quad (22) \]

where \( B = \text{diag}(b) \) is a diagonal matrix and \( b, A \) are the coefficients of the Runge-Kutta method (20)-(21) (see [17], or Theorem 4.3, p. 192 in [8], or Theorem 1.4, p. 267 in [5], or equation (6.14), p. 152 in [11]). For the linear matrix differential equation (5) with the odd dimensional variables, we have the same symplectic condition (22) of the Runge-Kutta method. We state it as the following Theorem 3.1.

**Theorem 3.1.** Assume that \( Q_{k+1} \) is the solution of equations (20)-(21), when the coefficients of a Runge-Kutta method satisfy the symplectic condition (22), then we have

\[ dQ_{k+1} \wedge SdQ_{k+1} = dQ_k \wedge SdQ_k. \quad (23) \]
Proof. From equation (21), we have
\[
dQ_{k+1} \wedge SdQ_{k+1} = \left( dQ_k + \Delta t \sum_{i=1}^{s} b_i SdY_i \right) \wedge S \left( dQ_k + \Delta t \sum_{i=1}^{s} b_i SdY_i \right) \\
= dQ_k \wedge SdQ_k + \Delta t \sum_{i=1}^{s} b_i \left( dQ_k \wedge S^2dY_i + SdY_i \wedge SdQ_k \right) \\
+ (\Delta t)^2 \sum_{i=1}^{s} \sum_{j=1}^{s} \left( b_i b_j SdY_i \wedge S^2dY_j \right). \tag{24}
\]

On the other hand, from equation (20), we have
\[
SdY_i \wedge SdQ_k = SdY_i \wedge Sd \left( Y_i - \Delta t \sum_{j=1}^{s} a_{ij} SY_j \right) \\
= SdY_i \wedge SdY_i - \Delta t \sum_{j=1}^{s} a_{ij} \left( SdY_i \wedge S^2dY_j \right) \\
= -\Delta t \sum_{j=1}^{s} a_{ij} \left( SdY_i \wedge S^2dY_j \right), \tag{25}
\]
and
\[
dQ_k \wedge S^2dY_i = \left( dY_i - \Delta t \sum_{j=1}^{s} a_{ij} SdY_j \right) \wedge S^2dY_i \\
= S^T dY_i \wedge SdY_i - \Delta t \sum_{j=1}^{s} a_{ij} \left( SdY_j \wedge S^2dY_i \right) \\
= -SdY_i \wedge SdY_i - \Delta t \sum_{j=1}^{s} a_{ij} \left( SdY_j \wedge S^2dY_i \right) \\
= -\Delta t \sum_{j=1}^{s} a_{ij} \left( SdY_j \wedge S^2dY_i \right). \tag{26}
\]

Replacing the results of equations (25)-(26) into equation (24), we obtain
\[
dQ_{k+1} \wedge SdQ_{k+1} = dQ_k \wedge SdQ_k \\
+ (\Delta t)^2 \sum_{i=1}^{s} \sum_{j=1}^{s} \left( b_i b_j - b_i a_{ij} - b_j a_{ji} \right) \left( SdY_i \wedge S^2dY_j \right). \tag{27}
\]

Thus, from equation (27), we know that the result of equation (23) is true if the coefficients of a Runge-Kutta method satisfy equation (22). □
Theorem 3.2. Assume that the coefficients of a Runge-Kutta method satisfy the symplectic condition (22) and apply this Runge-Kutta method for the linear matrix differential equation (5) to obtain its numerical solution \( Q_k \), then we have

\[
Q_{k+1}^T Q_{k+1} = Q_k^T Q_k = Q(t_0)^T Q(t_0) = I,
\]

which also gives the conservation of the discrete generalized energy trace \( (Q_k^T Q_k) \).

Proof. From the Runge-Kutta method (21), we have

\[
Q_{k+1}^T Q_{k+1} = Q_k^T Q_k + \Delta t \sum_{i=1}^{s} b_i (Q_k^T S Y_i + Y_i^T S^T Q_k) + (\Delta t)^2 \sum_{i=1}^{s} \sum_{j=1}^{s} b_i b_j Y_i^T S^T S Y_j.
\]

According to equation (20), we obtain

\[
Q_k^T S Y_i = Y_i^T S Y_i - \Delta t \sum_{j=1}^{s} a_{ij} Y_j^T S^T S Y_i,
\]

and

\[
Y_i^T S^T Q_k = Y_i^T S^T Y_i - \Delta t \sum_{j=1}^{s} a_{ij} Y_j^T S^T S Y_j.
\]

Inserting the results of equations (30)-(30) into equation (29), and using the symplectic condition (22), we have

\[
Q_{k+1}^T Q_{k+1} = Q_k^T Q_k + (\Delta t)^2 \sum_{i=1}^{s} \sum_{j=1}^{s} (b_i a_{ij} + b_j a_{ji} - b_i b_j) Y_i^T S^2 Y_j = Q_k^T Q_k,
\]

which gives the proof of the result of equation (28) and also gives \( \text{trace} \ (Q_{k+1}^T Q_{k+1}) = \text{trace} \ (Q_k^T Q_k) \). \( \Box \)

When \( s = 1 \) of a Runge-Kutta method (20)-(21) and its coefficients are listed by Table 2, the method is also called the implicit midpoint method with order 2. It is not difficult to verify that the implicit midpoint method satisfies the symplectic condition (22). Therefore, it is a symplectic geometric method.

If we apply the implicit midpoint method to the linear matrix differential equation (5), we have

\[
Q_{k+1} = \left( I - \frac{1}{2} \Delta t S \right)^{-1} \left( I + \frac{1}{2} \Delta t S \right) Q_k.
\]
Here, the Cayley transform

$$\Omega_{\Delta t}(S) = \left( I - \frac{1}{2} \Delta t S \right)^{-1} \left( I + \frac{1}{2} \Delta t S \right)$$

(33)

is commutative. Namely $\Omega_{\Delta t}$ equals $\Omega_{-\Delta t}$ and the adjoint operator $\Omega_{\Delta t}^*$ is defined by

$$\Omega_{\Delta t}^*(S) = \Omega_{-\Delta t}^{-1}(S) = \left( I + \frac{1}{2} \Delta t S \right) \left( I - \frac{1}{2} \Delta t S \right)^{-1}.$$  

(34)

When matrix $S$ is skew-symmetric, from equations (33)-(34), we have

$$\Omega_{\Delta t}(S)^T \Omega_{\Delta t}(S) = I.$$

Therefore, from equation (32), we obtain

$$Q_{k+1}^T Q_{k+1} = Q_k^T \Omega_{\Delta t}(S)^T \Omega_{\Delta t}(S) Q_k = I.$$

Namely the numerical solutions of the implicit midpoint method preserve the orthogonal invariant.

The Cayley transform $\Omega_{\Delta t}$ is also looked as a composition of the explicit Euler transform

$$\Phi_{\Delta t}(S) = (I + \Delta t S)$$

and the implicit Euler transform

$$\Phi_{\Delta t}^*(S) = (I - \Delta t S)^{-1}.$$  

That is to say

$$\Omega_{\Delta t}(S) = \Phi_{\frac{1}{2} \Delta t}^* \Phi_{\frac{1}{2} \Delta t}(S).$$

**Definition 3.1.** The adjoint operator $\Phi_{\Delta t}^*$ of $\Phi_{\Delta t}$ is defined by $\Phi_{-\Delta t}^{-1}$. If the adjoint operator $\Phi_{\Delta t}^*$ equals $\Phi_{\Delta t}$, it is called symmetric.

According to the definition of the symmetric operator, it is not difficult to see that the Cayley transform (33) is symmetric.
Table 3: Coefficients of the explicit second order RK Method.

|   |   |   |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 1 |

4. Numerical Experiments

In order to illustrate the structure-preserving property of the symplectic method for the differential equation (5), we compare the numerical results of the symplectic implicit midpoint method listed by Table 2 with the numerical results of the non-symplectic explicit second order Runge-Kutta method listed by Table 3. When we apply the explicit second order Runge-Kutta method to the linear matrix differential equation (5), we obtain the following iteration formula

\[ Q_{k+1} = \left( I + \Delta t S + \frac{1}{2} \Delta t^2 S^2 \right) Q_k. \]  

(35)

It is not difficult to verify that the coefficients of the explicit Runge-Kutta method can not satisfy the symplectic condition (22). This means that its numerical solutions of the explicit Runge-Kutta method can not preserve geometric structure (23) and can not comply with the conservation of energy of the dynamical system (5).

The test problem is given as

\[ S = \begin{bmatrix} 0 & 2 & -0.1 \\ -2 & 0 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}. \]  

(36)

The integrated interval is [0, 2000] and the fixed time-stepping length is 0.1.

The numerical results of the test problem are presented by Figure 1. The horizontal axis is on time and the vertical axis represents the error of the discrete energy. From Figure 1, we find that the generalized energy of the symplectic implicit midpoint method (32) fluctuates tiny, and the generalized energy of the non-symplectic explicit Runge-Kutta method grows with time. It means that the numerical results conform to the theoretical results of the previous sections.

5. Conclusion

We mainly extend the applicable fields of symplectic geometric algorithms from the even dimensional Hamiltonian system to the odd dimensional dynamical system, and discuss the geometric and algebraic properties of symplectic Runge-Kutta methods for the linear matrix differential equation, such as the
symplecticity and the orthogonality. It is worth noting that the implicit mid-
point rule preserves the Lie group structure of orthogonal matrices (see for ex-
ample p. 118 in [8]) and this is the interested research topic. Another interesting
issue is how to preserve the invariant of $Q_k^TQ_k$ when we use the approximate
technique in [13] if the initial matrix $Q_0$ is not orthogonal. We would like to
consider those issues in our future work.

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