PERFORMANCE BOUNDS OF THE INTENSITY-BASED ESTIMATORS FOR NOISY PHASE RETRIEVAL

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ABSTRACT. The aim of noisy phase retrieval is to estimate a signal $x_0 \in \mathbb{C}^d$ from $m$ noisy intensity measurements $b_j = |\langle a_j, x_0 \rangle|^2 + \eta_j$, $j = 1, \ldots, m$, where $a_j \in \mathbb{C}^d$ are known measurement vectors and $\eta = (\eta_1, \ldots, \eta_m)^\top \in \mathbb{R}^m$ is a noise vector. A commonly used estimator for $x_0$ is to minimize the intensity-based loss function, i.e., $\hat{x} := \arg\min_{x \in \mathbb{C}^d} \sum_{j=1}^m \left( |\langle a_j, x \rangle|^2 - b_j \right)^2$. Although one has developed many algorithms for solving the intensity-based estimator, there are very few results about its estimation performance. In this paper, we focus on the performance of the intensity-based estimator and prove that the error bound satisfies

$$\min_{\theta \in \mathbb{R}} \| \hat{x} - e^{i\theta} x_0 \|_2 \lesssim \min \left\{ \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}} \right\}$$

under the assumption of $m \gtrsim d$ and $a_j \in \mathbb{C}^d$, $j = 1, \ldots, m$, being complex Gaussian random vectors. We also show that the error bound is rate optimal when $m \gtrsim d \log m$. For the case where $x_0$ is an $s$-sparse signal, we present a similar result under the assumption of $m \gtrsim s \log(\alpha d/s)$. To the best of our knowledge, our results are the first theoretical guarantees for the intensity-based estimator and its sparse version. Our proofs employ Mendelson’s small-ball method which can deliver an effective lower bound on a nonnegative empirical process.

1. Introduction

1.1. Phase retrieval. Assume that

$$b_j := |\langle a_j, x_0 \rangle|^2 + \eta_j, \quad j = 1, \ldots, m$$

where $a_j \in \mathbb{C}^d$ are known measurement vectors and $\eta := (\eta_1, \ldots, \eta_m)^\top \in \mathbb{R}^m$ is a noise vector. Throughout this paper, we assume that the noise $\eta$ is a fixed or random vector independent of measurement vectors $a_j$, $j = 1, \ldots, m$.

To estimate $x_0 \in \mathbb{C}^d$ from $b := (b_1, \ldots, b_m)^\top \in \mathbb{R}^m$ is referred to as phase retrieval. Due to the physical limitations, optical sensors can record only the modulus of Fraunhofer diffraction pattern while losing the phase information, and hence phase retrieval has many applications in fields of physical sciences and engineering, which includes X-ray crystallography [19, 28], microscopy [27], astronomy [10], coherent diffractive imaging [17, 33] and optics [39] etc. Despite its simple
mathematical form, it has been shown that to reconstruct a finite-dimensional discrete signal from its Fourier transform magnitudes is generally $NP$-complete [32].

Based on the least squares criterion, one can employ the following intensity-based empirical loss to estimate $x_0$:

$$\min_{x \in \mathbb{C}^d} \sum_{j=1}^{m} \left( |\langle a_j, x \rangle|^2 - b_j \right)^2.$$  

For the case where $x_0$ is sparse, the following Lasso-type program can be employed to estimate $x_0$:

$$\min_{x \in \mathbb{C}^d} \sum_{j=1}^{m} \left( |\langle a_j, x \rangle|^2 - b_j \right)^2 \text{ s.t. } \|x\|_1 \leq R,$$

where $R$ is a parameter which specifies a desired sparsity level of the solution. An advantage of the intensity-based estimator (1) is that the objective function is differentiable based on the Wirtinger derivatives. One can therefore try to find its minimum with high-order algorithms such as trust-region and Gauss-Newton methods. Though the objective functions are non-convex, the strategy of spectral initialization plus local gradient descent can be adopted to solve (1) and (2) efficiently under Gaussian random measurements. For instance, it has been proved that when $m \gtrsim d$ and $\|\eta\|_\infty \lesssim \|x_0\|_2^2$, with high probability the truncated spectral method given in [8] can return an initial guess $z_0$ which is close to the target signal in the real case, namely, $\|z_0 - x_0\|_2 \leq \delta \|x_0\|_2$ for any fixed relative error tolerance $\delta$. With this in place, the update rules such as Wirtinger Flow [5], Trust-Region [34] and Gauss-Newton [16] methods could find a global solution to (1) at least in the noiseless case.

In the noiseless case, i.e., $b_j = |\langle a_j, x_0 \rangle|^2$, $j = 1, \ldots, m$, the solution to (1) is exactly $x_0$ (up to a unimodular constant) if $m \geq 4d - 4$ and $a_j$, $j = 1, \ldots, m$, are generic vectors in $\mathbb{C}^d$ [7, 43]. However, one still does not know the distance between the solution to (1) and the true signal $x_0$ in the noisy case. The aim of this paper is to study the performance of (1) and (2) from the theoretical viewpoint.

1.2. Algorithms for phase retrieval. For the last two decades, many algorithms have been designed for phase retrieval, especially in the noiseless case, which falls into two categories: convex methods and non-convex ones.
The convex methods rely on the “matrix-lifting” technique which lifts the quadratic system to a linear rank-one positive semi-definite program. More specifically, a rank one matrix \( X = xx^* \) is introduced to linearize the quadratic constrains and then a nuclear norm minimization is adopted as a convex surrogate of the rank constraint. Such methods include PhaseLift [4, 6], PhaseCut [38] etc. Although the convex methods have good theoretical guarantees, they require to solve a semi-definite program in the “lifted” space \( \mathbb{C}^{d^2} \) rather than \( \mathbb{C}^d \), where \( d \) is the dimension of signals. Thus the memory requirements and computational complexity become quite high, which makes it prohibitive for large-scale problems in practical applications.

The non-convex methods operate directly on the original space, which achieves significantly improved computational performance. The oldest non-convex algorithms for phase retrieval are based on alternating projection including Gerchberg-Saxton [17] and Fineup [13], but lack of theoretical guarantees. The first non-convex algorithm with theoretical guarantees was given by Netrapalli et al who showed that the AltMinPhase [29] algorithm converges linearly to the true solution up to a global phase with \( O(d \log^3 d) \) resampling Gaussian random measurements. In [5], Candès, Li and Soltanolkotabi developed the Wirtinger Flow (WF) to solve (1) and proved WF algorithm can achieve the linear convergence with \( O(d \log d) \) Gaussian random measurements. Lately, Chen and Candès improved the result to \( O(d) \) Gaussian random measurements by Truncated Wirtinger Flow (TWF) [8]. In [16], Gao and Xu proposed a Gauss-Newton algorithm to solve (1) and proved the Gauss-Newton method can achieve quadratic convergence for the real-valued signals with \( O(d \log d) \) Gaussian random measurements. In [34], Sun, Qu, and Wright proved that, for \( O(d \log^3 d) \) Gaussian random measurements, the objective function of (1) has a benign geometric landscape: (1) all local minimizers are global; and (2) the objective function has a negative curvature around each saddle point\(^1\). They also developed the Trust-Region method to find a global solution.

Another alternative approach for phase retrieval is to solve the following amplitude-based empirical loss:

\[
\min_{x \in \mathbb{C}^d} \sum_{j=1}^{m} \left( |\langle a_j, x \rangle| - \psi_j \right)^2,
\]

where \( \psi_j := \sqrt{b_j}, \ j = 1, \ldots, m \). For Gaussian random measurements, through an appropriate initialization, many algorithms can be used to solve (3) successfully such as Truncated Amplitude

\(^1\)We do not differentiate between saddle points and local maximizers.
Flow (TAF) [40], Reshaped Wirtinger Flow (RWF) [45] and Perturbed Amplitude Flow (PAF) [15]. It has been proved that TAF, RWF and PAF algorithms converge linearly to the true solution up to a global phase under $O(d)$ Gaussian random measurements.

For sparse phase retrieval, a standard $\ell_1$ relaxation technique leads to the corresponding sparse intensity-based estimator (2). It has been shown that when the noises are independent centered sub-exponential random variables with maximum sub-exponential norm $\sigma$ and $m \gtrsim \alpha_\sigma^2 s^2 \log(md)$, in the real case, the initialization procedure given in [2] can return an initial guess $z_0$ which satisfies $\text{supp}(z_0) \subset \text{supp}(x_0)$ and is very close to the target signal with high probability, where $s$ is the sparsity level and $\alpha_\sigma$ is a constant related to $\sigma$. Next, the projection gradient method [12] could be used to find a global minimizer to (2). Such two-step procedure has also been used in various signal processing and machine learning problems, such as Blind Deconvolution [26], matrix completion [35] and sparse recovery [31].

We refer the reader to survey papers [18,41,44] for accounts of recent developments in the algorithms of sparse phase retrieval. The theoretical results concerning the injectivity of sparse phase retrieval can be found in [22,42].

1.3. Related work.

1.3.1. PhaseLift. We first introduce the estimation performance of PhaseLift for noisy phase retrieval. In [4], Candès and Li suggest using the following empirical loss to estimate $x_0$:

$$\min_{X \in \mathbb{C}^{d \times d}} \sum_{j=1}^{m} |a_j^* X a_j - b_j| \quad \text{s.t.} \quad X \succeq 0.$$  \hspace{1cm} (4)

They prove that the solution $\hat{X}$ to (4) obeys

$$\|\hat{X} - x_0^* x_0\|_F \lesssim \frac{\|\eta\|_1}{m}$$

with high probability provided $m \gtrsim d$ and $a_j \in \mathbb{C}^d$, $j = 1, \ldots, m$, are complex Gaussian random vectors. Though (4) is a convex optimization problem, one needs to solve it in a “lifted” space $\mathbb{C}^{d^2}$. The computational cost typically far exceeds the order of $d^3$, which is not suitable for large-dimensional data. However, for the intensity-based estimator, one just needs to operate on the original space $\mathbb{C}^d$ rather than lifting the problem into higher dimensions.
1.3.2. **The amplitude-based estimator.** As shown before, the amplitude-based empirical loss (3) is an alternative estimator for phase retrieval. In [20], Huang and Xu studied the estimation performance of the amplitude-based estimator (3) for real-valued signals. They prove that the solution $\hat{x}$ to (3) satisfies

$$\min\{\|\hat{x} + x_0\|_2, \|\hat{x} - x_0\|_2\} \lesssim \frac{\|\eta\|_2}{\sqrt{m}}$$

with high probability provided $m \gtrsim d$ and $a_j \in \mathbb{R}^d, j = 1, \ldots, m$, are Gaussian random vectors. They also prove that the reconstruction error $\|\eta\|_2/\sqrt{m}$ is sharp. Furthermore, in [20], Huang and Xu consider the following constrained nonlinear Lasso to estimate $s$-sparse signals $x_0$:

(5) $$\min_{x \in \mathbb{R}^d} \sum_{j=1}^{m} (|\langle a_j, x \rangle| - \psi_j)^2 \quad \text{s.t.} \quad \|x\|_1 \leq R.$$ 

They show that any global solution $\hat{x}$ to (5) with $R := \|x_0\|_1$ obeys

$$\min\{\|\hat{x} + x_0\|_2, \|\hat{x} - x_0\|_2\} \lesssim \frac{\|\eta\|_2}{\sqrt{m}}$$

with high probability provided $m \gtrsim s \log(ed/s)$ and $a_j \in \mathbb{R}^d, j = 1, \ldots, m$, are Gaussian random vectors.

The results from [20] hold only for real-valued signals and it seems highly nontrivial to extend them to the complex case. However, the results in this paper hold for both real-valued and complex-valued signals. Furthermore, the intensity-based estimator considered in this paper is differentiable comparing with amplitude-based estimator, which admits high-order algorithms.

1.3.3. **Poisson log-likelihood estimator.** In [8], Chen and Candès consider the Poisson log-likelihood function

$$\ell_j(x, b_j) = b_j \log(|a_j^\top x|^2) - |a_j^\top x|^2$$

and use

(6) $$\min_{x \in \mathbb{R}^d} - \sum_{j=1}^{m} \ell_j(x, b_j)$$

to estimate real-valued signals $x_0$. They establish stability estimates using Truncated Wirtinger Flow approach and show that if $\|\eta\|_\infty \lesssim \|x_0\|_2^2$ then the solution $\hat{x}$ to (6) satisfies

$$\text{dist}(\hat{x}, x_0) \lesssim \frac{\|\eta\|_2}{\|x_0\|\sqrt{m}}$$
with high probability provided \( m \gtrsim d \) and \( \mathbf{a}_j \in \mathbb{R}^d, j = 1, \ldots, m \), are Gaussian random vectors. Furthermore, a lower bound on the minimax estimation error is also derived under the real-valued Poisson noise model, namely, with high probability

\[
\inf_{\hat{x}} \sup_{x_0 \in \gamma(K)} \mathbb{E} \text{dist} (\hat{x}, x_0) \gtrsim \sqrt{\frac{d}{m}},
\]

where \( \gamma(K) := \{ x_0 \in \mathbb{R}^d : \| x_0 \|_2 \in (1 \pm 0.1)K \} \) for any \( K \geq \log^{1.5} m \).

The result (7) presents a lower bound on the expectation of the minimax estimation error under Poisson noise structure, whereas the result in Theorem 1.2 gives a tail bound with a reasonably large measurements and holds for a wide range of noises. Moreover, all the results in [8] are for real-valued signals, while ours hold for complex-value ones.

1.4. Our contributions. As stated earlier, the two-step strategy of spectral initialization plus local gradient descent can be use to solve the intensity-based estimator (1). To our knowledge, there is no result concerning the reconstruction error of (1) for noisy phase retrieval from the theoretical viewpoint. The goal of this paper is to study the estimation performance of the intensity-based estimator (1) and its sparse version (2).

Our first result shows that the estimation error is quite small and bounded by the average noise per measurement, as stated below. We emphasize that this theorem does not assume any particular structure on the noise \( \eta \).

**Theorem 1.1.** Suppose that the measurements \( \mathbf{a}_j \sim 1/\sqrt{2} \cdot \mathcal{N}(0, I_d) + i/\sqrt{2} \cdot \mathcal{N}(0, I_d) \) are i.i.d. complex Gaussian random vectors and the measurement number \( m \gtrsim d \). Then the following holds with probability at least \( 1 - \exp(-cm) \): For all \( x_0 \in \mathbb{C}^d \), the solution \( \hat{x} \in \mathbb{C}^d \) to (1) with \( b_j = |\langle \mathbf{a}_j, x_0 \rangle|^2 + \eta_j, j = 1, \ldots, m \), satisfies

\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - e^{i\theta} x_0 \|_2 \leq C \min \left\{ \frac{\sqrt{\| \eta \|_2^2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \cdot \sqrt{m}} \right\}.
\]

Here, \( C \) and \( c \) are positive absolute constants.

According to Theorem 1.1, the following holds with high probability:

\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - e^{i\theta} x_0 \|_2 \lesssim \min \left\{ \frac{\sqrt{\| \eta \|_2^2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \cdot \sqrt{m}} \right\} = \begin{cases} \frac{\sqrt{\| \eta \|_2}}{m^{1/4}} & \text{if } \| x_0 \|_2 < \frac{\sqrt{\| \eta \|_2}}{m^{1/4}} \\ \frac{\| \eta \|_2}{\| x_0 \|_2 \cdot \sqrt{m}} & \text{if } \| x_0 \|_2 \geq \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}. \end{cases}
\]
In practical applications, the signals of interest \( x_0 \) usually fall in the second regime where \( \|x_0\|_2 \geq \sqrt{|\log m|/m} \).

The next theorem presents a lower bound for the estimation error for any fixed \( x_0 \), under the assumption of \( m \gtrsim d \log m \) and noise with the structures \( \|\eta\|_2 \asymp \sqrt{m}, \|\eta\|_\infty \lesssim \log m, \quad |\sum_{j=1}^m \eta_j| \lesssim m \) and \( |\sum_{j=1}^m \eta_j| \gtrsim \sqrt{m}\|\eta\|_2 \gtrsim d\|\eta\|_\infty \). The result shows that the estimator (1) is rate optimal for some \( \eta \) provided \( m \gtrsim d \log d \).

**Theorem 1.2.** Suppose that the measurements \( a_j \sim 1/\sqrt{2} \cdot N(0, I_d) + i/\sqrt{2} \cdot N(0, I_d) \) are i.i.d. complex Gaussian random vectors and the measurement number \( m \gtrsim d \log m \). Assume that \( \eta \in \mathbb{R}^m \) is a noise vector satisfying \( \|\eta\|_2 \asymp \sqrt{m}, \|\eta\|_\infty \lesssim \log m, \quad |\sum_{j=1}^m \eta_j| \lesssim m \) and \( d\|\eta\|_\infty \lesssim \sqrt{m}\|\eta\|_2 \lesssim |\sum_{j=1}^m \eta_j| \). For any fixed \( x_0 \in \mathbb{C}^d \) satisfying \( \|x_0\|_2 \gtrsim 2C\|\eta\|_2/\sqrt{m} \) the following holds with probability at least \( 1 - c' \exp(-c''d) - c'''m^{-1} \): any solution \( \hat{x} \in \mathbb{C}^d \) to (1) with \( b_j = |\langle a_j, x_0 \rangle|^2 + \eta_j, \quad j = 1, \ldots, m \), satisfies

\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - e^{i\theta} x_0 \|_2 \gtrsim \min \left\{ \frac{\sqrt{\|\eta\|_2}}{m^{1/4}}, \frac{\|\eta\|_2}{\|x_0\|_2 \cdot \sqrt{m}} \right\}.
\]

Here, \( c', c'', c''' \) are universal positive constants and \( C \) is the constant in Theorem 1.1.

**Remark 1.3.** In Theorem 1.2, we require \( \eta \) satisfies the conditions \( \|\eta\|_2 \asymp \sqrt{m}, \|\eta\|_\infty \lesssim \log m, \quad |\sum_{j=1}^m \eta_j| \lesssim m \) and \( d\|\eta\|_\infty \lesssim \sqrt{m}\|\eta\|_2 \lesssim |\sum_{j=1}^m \eta_j| \). In fact, there exist many noises satisfying them. For instance, if \( \eta_j \) are generated independently according to the biased Gaussian distribution, i.e., \( \eta_j \sim N(\mu, \sigma^2) \) for some non-zero constant \( \mu \), then the noise vector \( \eta \) satisfies those conditions with high probability.

We next turn to the phase retrieval for sparse signals. This is motivated by the signal \( x_0 \in \mathbb{C}^d \) admitting a sparse representation under some linear transformation in many applications. Without loss of generality, we assume that \( x_0 \in \mathbb{C}^d \) is an \( s \)-sparse vector and wish to estimate \( x_0 \) from \( b = (b_1, \ldots, b_m) \) by solving

\[
(8) \quad \min_{x \in \mathbb{C}^d} \sum_{j=1}^m \left( |\langle a_j, x \rangle|^2 - b_j \right)^2 \quad \text{s.t.} \quad \|x\|_1 \leq R.
\]

The estimation performance of (8) is stated as follows.
Theorem 1.4. Suppose that the measurements $a_j \sim 1/\sqrt{2} \cdot \mathcal{N}(0, I_d) + i/\sqrt{2} \cdot \mathcal{N}(0, I_d)$ are i.i.d. complex Gaussian random vectors and the measurement number $m \geq s \log(\frac{ed}{s})$. Then the following holds with probability at least $1 - 5 \exp(-cm)$ where $c$ is a constant: for any $s$-sparse vector $x_0 \in \mathbb{C}^d$,

$$
\min_{\theta \in [0, 2\pi]} \|\hat{x} - e^{i\theta} x_0\|_2 \lesssim \min \left\{ \frac{\sqrt{\|\eta\|_2}}{m^{1/4}}, \frac{\|\eta\|_2}{\|x_0\|_2 \cdot \sqrt{m}} \right\},
$$

where $\hat{x} \in \mathbb{C}^d$ is a solution to (8) with parameter $R := \|x_0\|_1$.

Remark 1.5. In [2], the authors establish the estimation error using the Thresholded Wirtinger Flow approach under the centered sub-exponential noise for the real-valued signals. In short, they show that, with probability at least $1 - 47/m - 10/e^s$, the estimator $\hat{x}$ given by the Thresholded Wirtinger Flow algorithm obeys

$$
\min \{ \hat{x} - x_0, \hat{x} + x_0 \} \lesssim \sigma \|x_0\|_2 \sqrt{\frac{s \log d}{m}} \quad \text{provided} \quad m \geq O(s^2 \log(md)),
$$

where $\sigma := \max_{1 \leq j \leq m} \|\eta_j\|_{\psi_1}$. Although the estimation error is slightly better than the upper bound given in Theorem 1.4, however, our result holds for any noise structure and complex-valued signals. Moreover, the probability of failure in Theorem 1.4 is exponentially small in the number of measurements.

1.5. Numerical Experiments. In this subsection, we report some numerical experiments to verify that the global solutions to (1) and (2) can be obtained efficiently and the results given in Subsection 1.4 are rate optimal. In our experiments, the target signal $x_0$ and the measurement vectors $a_1, \ldots, a_m$ are independent standard complex Gaussian random vectors, whereas the noise vector $\eta \in \mathbb{R}^m$ is a real Gaussian random vector with entries $\eta_j \sim N(1, 1)$.

Example 1.6. In this example, we verify the estimation error presented in Theorem 1.1 is rate optimal. We consider the case where $d = 500$ and vary $m$ within the range $[4d, 50d]$. To solve the estimator (1), we use the truncated spectral method proposed in [8] to obtain a good initial guess and then refine it by Wirtinger Flow [5]. Figure 1 depicts the ratio $\rho_m$ against the number of measurements $m$, when averaged over 100 times independent trials. Here, the ratio $\rho_m$ is defined as

$$
\rho_m := \frac{\text{dist}(\hat{x}, x_0)}{\|\eta\|_2 / (\|x_0\|_2 \cdot \sqrt{m})}.
$$

Numerical results show that $\rho_m$ tends to be a constant around 0.37, which verifies the estimation error $\frac{\|\eta\|_2}{\|x_0\|_2 \cdot \sqrt{m}}$ presented in Theorem 1.1 is rate optimal.
The ratio $\rho_m$ versus the number of measurements $m$ under Gaussian noises with $d = 500$.

**Example 1.7.** The purpose of this numerical experiment is to verify the estimation bound given in Theorem 1.4 is rate optimal when $m = O(s \log(ed/s))$. We choose $d = 1000$ and take the sparsity level $s = 100$. The support of $x_0$ is uniformly distributed at random. The non-zero entries of $x_0$ are chosen randomly according to a standard normal distribution. We vary $m$ between $[6s \log(ed/s)]$ and $[20s \log(ed/s)]$. For each fixed $m$, we run 100 times trials and calculate the average ratio $\rho_m$ defined in (9). The constrained optimization problem (8) is solved by combining the initialization method introduced in [2] and the projection gradient descent onto the $\ell_1$-ball [12]. The result is plotted in Figure 2. We can see that $\rho_m$ tends to be a constant around 0.72, which verifies the estimation error $\frac{\|\eta\|_2}{\|x_0\|_2 \sqrt{m}}$ presented in Theorem 1.4 is rate optimal for sparse signals.

**Figure 2.** The ratio $\rho_m$ versus the number of measurements $m$ for $s$-sparse signals with $d = 1000$, $s = 100$. 
1.6. Notations. Throughout this paper, we assume the measurements \( \mathbf{a}_j \in \mathbb{C}^d, j = 1, \ldots, m \) are i.i.d. complex Gaussian random vectors. Here we say \( \mathbf{a} \in \mathbb{C}^d \) is a complex Gaussian random vector if \( \mathbf{a} \sim \frac{1}{\sqrt{2}} \cdot \mathcal{N}(0, I_d) + i \cdot \frac{1}{\sqrt{2}} \cdot \mathcal{N}(0, I_d) \). We write \( \mathbf{z} \in \mathbb{S}_{\mathbb{C}}^{d-1} \) if \( \mathbf{z} \in \mathbb{C}^d \) and \( \|\mathbf{z}\|_2 = 1 \). We use the notations \( \|\cdot\|_2 \) and \( \|\cdot\|_* \) to denote the operator norm and nuclear norm of a matrix, respectively. For any \( A, B \in \mathbb{R} \), we use \( A \preceq B \) to denote \( A \leq C_0 B \) where \( C_0 \in \mathbb{R}^+ \) is an absolute constant. The notion \( \succeq \) can be defined similarly. Moreover, \( A \asymp B \) means that there exist constants \( C_1, C_2 > 0 \) such that \( C_1 A \leq B \leq C_2 A \). In this paper, we use \( C, c \) and the subscript (superscript) form of them to denote universal constants whose values vary with the context.

1.7. Organization. The paper is organized as follows. In Section 2, after introducing some definitions, we study the recovery of low-rank matrices from rank-one measurements, which plays a key role in the proofs of main results. We also believe that the results in Section 2 are of independent interest. Combining the Mendelson’s small-ball method and the results in Section 2, we present the proofs of Theorem 1.1 and Theorem 1.2 in Section 3. The proof of Theorem 1.4 is given in Section 4. A brief discussion is presented in Section 5. Appendix collects the technical lemmas needed in the proof.

2. The recovery of low-rank matrices from rank-one measurements

For convenience, we let \( \mathcal{A} : \mathcal{H}^{d \times d}(\mathbb{C}) \rightarrow \mathbb{R}^m \) be a linear map which is defined as
\[
\mathcal{A}(X) := (\mathbf{a}_1^* X \mathbf{a}_1, \mathbf{a}_2^* X \mathbf{a}_2, \ldots, \mathbf{a}_m^* X \mathbf{a}_m)^\top,
\]
where \( \mathcal{H}^{d \times d}(\mathbb{C}) := \{ X \in \mathbb{C}^{d \times d} : X^* = X \} \). Its dual operator \( \mathcal{A}^* : \mathbb{R}^m \rightarrow \mathcal{H}^{d \times d}(\mathbb{C}) \) is given by
\[
\mathcal{A}^*(z) = \sum_{j=1}^m z_j \mathbf{a}_j \mathbf{a}_j^*.
\]
In this section, we focus on the following minimization problem:
\[
\min_{X \in \mathcal{H}^{d \times d}(\mathbb{C})} \| \mathcal{A}(X) - \mathbf{b} \|_2^2 \quad \text{s.t.} \quad \text{rank}(X) \leq r.
\]
Set \( \mathcal{H}_r^{d \times d}(\mathbb{C}) := \{ X \in \mathbb{C}^{d \times d} : X^* = X, \text{rank}(X) \leq r \} \). A simple observation is that \( \hat{\mathbf{x}} \) is a solution to (1) if and only if \( \hat{X} := \hat{\mathbf{x}} \hat{\mathbf{x}}^* \) is a solution to (12) with \( r = 1 \). Hence, (12) can be regarded as a lifted version of (1). To prove the main results of this paper, we first characterize the estimation performance of (12).
2.1. **The performance of (12).** The main result of this section is Corollary 2.4 which presents the estimation performance of (12). We believe some results in this section are also of independent interest in the area of low-rank matrix recovery from rank-one measurements [3, 9, 21, 25].

We first introduce the definition of Lower Restricted Isometry Property (see, e.g., [1, 24]).

**Definition 2.1.** [Lower Restricted Isometry Property] A linear map \( A : \mathcal{H}_d \rightarrow \mathbb{R}^m \) is said to have the Lower Restricted Isometry Property (LRIP) condition of order \( r \) and constant \( c_0 \) if the following holds

\[
\|A(X)\|_2 / \sqrt{m} \geq c_0 \|X\|_F
\]

for all non-zero matrices \( X \in \mathcal{H}_r \).

With the LRIP condition in place, we can demonstrate that the optimization (12) is stable, as stated in the following theorem.

**Theorem 2.2.** Suppose \( A \) satisfies the LRIP condition with order \( 2r \) and constant \( c_0 > 0 \), then the solution \( \hat{X} \) to (12) satisfies

\[
\|\hat{X} - X_0\|_F \lesssim \|\eta\|_2 / \sqrt{m}
\]

for all matrices \( X_0 \in \mathcal{H}_r \) and \( b = A(X_0) + \eta \) with the noise vector \( \eta \in \mathbb{R}^m \).

**Proof.** Since \( \hat{X} \) is the global solution to (12) and \( X_0 \) is a feasible point, we have

\[
\|A(\hat{X}) - b\|_2 \leq \|A(X_0) - b\|_2.
\]

Noting \( b = A(X_0) + \eta \), we obtain that

\[
\|A(H) - \eta\|_2 \leq \|\eta\|_2,
\]

where \( H = \hat{X} - X_0 \in \mathcal{H}_r \). Since \( A \) satisfies the LRIP condition, we have

\[
c_0 \sqrt{m} \|H\|_F \leq \|A(H)\|_2 \leq \|A(H) - \eta\|_2 + \|\eta\|_2 \leq 2\|\eta\|_2.
\]

Consequently,

\[
\|H\|_F \lesssim \|\eta\|_2 / \sqrt{m}.
\]

We arrive at the conclusion. \( \square \)
The next result shows that $A$ satisfies the LRIP condition with high probability provided $a_j, j = 1, \ldots, m$, are i.i.d. complex Gaussian random vectors. We postpone its proof to the end of this section.

**Theorem 2.3.** Suppose that $a_j \in \mathbb{C}^d, j = 1, \ldots, m$, are i.i.d. complex Gaussian random vectors and $t, r \in \mathbb{Z}_{\geq 1}$ satisfy $t \cdot r < d$. If $m \gtrsim tdr$ then with probability at least $1 - \exp(-cm)$, the linear map $A$ defined in (10) satisfies LRIP condition of order $t \cdot r$ and constant $c_0$, where $c, c_0 > 0$ are constants independent of $d, r$ and $t$.

As a direct consequence of Theorem 2.2 and Theorem 2.3, the estimation performance of optimization (12) is given below.

**Corollary 2.4.** Suppose that $a_j \in \mathbb{C}^d, j = 1, \ldots, m$, are i.i.d. complex Gaussian random vectors. If $m \gtrsim dr$, then the following holds with probability at least $1 - \exp(-cm)$: for any $X_0 \in \mathcal{H}_{t \cdot r}(\mathbb{C})$, the solution $\hat{X}$ to (12) with noisy measurements $b_j = a_j^* X_0 a_j + \eta_j, j = 1, \ldots, m$, satisfies

$$
\| \hat{X} - X_0 \|_F \lesssim \frac{\| \eta \|_2}{\sqrt{m}},
$$

where $\eta := (\eta_1, \ldots, \eta_m)^T \in \mathbb{R}^m$ is a noise vector and $c$ is a positive constant.

**Proof.** Taking $t = 2$ in Theorem 2.3, it then follows that with probability at least $1 - \exp(-cm)$, the linear map $A$ defined in (10) satisfies LRIP condition of order $2r$ and constant $c_0$ for some constants $c, c_0 > 0$. Combining with Theorem 2.2, we complete the proof.

□

2.2. **Proof of Theorem 2.3.** In this subsection, we will establish the LRIP condition of $A$. Before proceeding, we gather some lemmas which are useful in our arguments.

2.2.1. **Lemmas.** The Mendelson’s small-ball method (see [36]) plays a key role in our proof, which is a strategy to establish a lower bound for $\inf_{x \in E} \sum_{j=1}^m | \langle x, \phi_j \rangle |^2$ where $\phi_j \in \mathbb{R}^d$ are independent random vectors and $E$ is a subset of $\mathbb{R}^d$.

**Lemma 2.5.** [36, Proposition 5.1] Fix $E \subset \mathbb{R}^d$ and let $\phi_1, \ldots, \phi_m$ be independent copies of a random vector $\phi$ in $\mathbb{R}^d$. For any $\xi \geq 0$, set

$$
Q_\xi(E, \phi) := \inf_{u \in E} \mathbb{P} \{ | \langle \phi, u \rangle | \geq \xi \}
$$

with $\phi, u \in \mathbb{R}^d$. If $\phi$ is a random vector in $\mathbb{R}^d$ and $E \subset \mathbb{R}^d$ is a subset, then $Q_\xi(E, \phi) \geq \mathbb{P} \{ | \langle \phi, u \rangle | \geq \xi \}$ for any $u \in E$. Moreover, if $E$ is a subset of $\mathbb{R}^d$, then $Q_\xi(E, \phi) \geq \mathbb{P} \{ | \langle \phi, u \rangle | \geq \xi \}$ for any $u \in E$. Finally, if $E$ is a subset of $\mathbb{R}^d$, then $Q_\xi(E, \phi) \geq \mathbb{P} \{ | \langle \phi, u \rangle | \geq \xi \}$ for any $u \in E$.
and
\[ W_m(E, \phi) := \mathbb{E} \sup_{u \in E} \langle h, u \rangle \quad \text{where} \quad h := \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \epsilon_j \phi_j \]
where \( \epsilon_1, \ldots, \epsilon_m \) are independent Rademacher random variables. Then for any \( \xi > 0 \) and \( t > 0 \) the following holds with probability at least \( 1 - \exp(-t^2/2) \):
\[
\inf_{u \in E} \left( \sum_{j=1}^{m} |\langle \phi_j, u \rangle|^2 \right)^{1/2} \geq \xi \sqrt{mQ_2(E, \phi)} - 2W_m(E, \phi) - \xi t.
\]

The following lemma is a consequence of the classical Paley-Zygmund inequality (e.g., [11, 23]).

**Lemma 2.6.** [14, Lemma 7.16] If a nonnegative random variable \( Z \) has finite second moment, then
\[
P(Z > t) \geq \frac{(\mathbb{E}Z - t)^2}{\mathbb{E}Z^2}, \quad 0 \leq t \leq \mathbb{E}Z.
\]

In addition, we also need the following lemma which presents an upper bound for the spectral norm of \( A^*(\epsilon) \) for a fixed independent Rademacher random vector \( \epsilon \in \mathbb{R}^m \).

**Lemma 2.7.** Suppose that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors and \( \epsilon_1, \ldots, \epsilon_m \) are independent Rademacher random variables. If \( m \gtrsim d \), then \( \mathbb{E}_{\epsilon,A} \|A^*(\epsilon)\|_2 \lesssim \sqrt{md} \), namely,
\[
\mathbb{E} \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \epsilon_j a_j^* a_j^\top \lesssim \sqrt{d}.
\]

**Proof.** We assume that \( \mathcal{N} \) is a 1/4-net of the complex unit sphere \( \mathbb{S}^{d-1} \subset \mathbb{C}^d \). It then follows from [37, Lemma 4.4.3] that
\[
\| \sum_{j=1}^{m} \epsilon_j a_j^* a_j^\top \|_2 \leq 2 \max_{x \in \mathcal{N}} \left| \sum_{j=1}^{m} \epsilon_j |a_j^\top x|^2 \right|.
\]
For any fixed \( x \in \mathbb{S}^{d-1} \), the terms \( \epsilon_j |a_j^\top x|^2, j = 1, \ldots, m \) are independent centered sub-exponential random variables with the sub-exponential norm being a constant. Using the Bernstein’s inequality [37, Theorem 2.8.1], we obtain that, for any \( t \geq 0 \), it holds
\[
P \left\{ \left| \sum_{j=1}^{m} \epsilon_j |a_j^\top x|^2 \right| \geq \frac{C \sqrt{md} + t \sqrt{m}}{2} \right\} \leq 2 \exp \left( -c \min \left( \frac{\lambda^2}{m}, \lambda \right) \right).
\]
where $\lambda := C\sqrt{md} + t\sqrt{m}$ and $C \geq 1$ is a constant to be chosen later. Noting that

$$\lambda^2/m = C^2d + 2Ct\sqrt{d} + t^2 \geq Cd + t$$

and $\lambda \geq Cd + t$ for any $m \gtrsim d$, we have

$$\mathbb{P}\left\{ \left| \sum_{j=1}^{m} \epsilon_j a_j^* x \right| \geq \frac{C\sqrt{md} + t\sqrt{m}}{2} \right\} \leq 2\exp\left(-c(Cd + t)\right).$$

Recall that $|N| \leq 9d^2$. Taking the constant $C$ such that $C \cdot c \geq 2\ln 9$, we obtain

(14) $$\mathbb{P}\left\{ \max_{x \in N} \left| \sum_{j=1}^{m} \epsilon_j a_j^* x \right| \geq \frac{C\sqrt{md} + t\sqrt{m}}{2} \right\} \leq 2\exp\left(-ct\right).$$

Combining (13) and (14), we obtain that if $m \gtrsim d$ then with probability at least $1 - 2\exp(-ct)$ it holds

$$\frac{1}{\sqrt{m}} \left\| \sum_{j=1}^{m} \epsilon_j a_j a_j^* \right\|_2 \leq C\sqrt{d} + t$$

for all $t \geq 0$. According to the definition of expectation, we have

$$E \frac{1}{\sqrt{m}} \left\| \sum_{j=1}^{m} \epsilon_j a_j a_j^* \right\|_2 = \int_{0}^{\infty} \mathbb{P}\left\{ \frac{1}{\sqrt{m}} \left\| \sum_{j=1}^{m} \epsilon_j a_j a_j^* \right\|_2 \geq t \right\} dt$$

$$= \int_{0}^{C\sqrt{d}} \mathbb{P}\left\{ \frac{1}{\sqrt{m}} \left\| \sum_{j=1}^{m} \epsilon_j a_j a_j^* \right\|_2 \geq t \right\} dt + \int_{C\sqrt{d}}^{\infty} \mathbb{P}\left\{ \frac{1}{\sqrt{m}} \left\| \sum_{j=1}^{m} \epsilon_j a_j a_j^* \right\|_2 \geq C\sqrt{d} + t \right\} dt$$

$$\leq C\sqrt{d} + 2\int_{0}^{\infty} e^{-ct} dt \leq \sqrt{d}.\qed$$

2.2.2. Proof of Theorem 2.3. We next present a proof of Theorem 2.3. We would like to mention that one can prove Theorem 2.3 based on RUB condition and the results in [3] (see Section 2.2.3 for details). For completeness, we provide a proof which employs Mendelson’s small-ball method.

Proof of Theorem 2.3 According to Definition 2.1, it is sufficient to prove that

(15) $$\|A(H)\|_2 \gtrsim \sqrt{m}\|H\|_F$$

holds with high probability. Due to homogeneity, without loss of generality, we can assume $\|H\|_F = 1$. We employ Mendelson’s small-ball method to prove the conclusion (see Lemma 2.5 and Lemma
2.6). To see this, we identify \( \mathcal{H}_{d \times d}(\mathbb{C}) \subseteq \mathbb{C}^{d \times d} \) with \( \mathbb{R}^{d^2} \) and let

\[
\mathcal{S} \mathcal{H}_{d \times d} := \left\{ H \in \mathcal{H}_{d \times d}(\mathbb{C}) : \|H\|_F = 1 \right\}.
\]

For any \( \xi \geq 0 \) define

\[
Q_\xi := \inf_{H \in \mathcal{S} \mathcal{H}_{d \times d}} \mathbb{P}\left\{ |a_j^* H a_j| \geq \xi \right\}
\]

\[
W_m := \mathbb{E} \sup_{H \in \mathcal{S} \mathcal{H}_{d \times d}} \langle H, A \rangle \quad \text{where} \quad A := \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \epsilon_j a_j a_j^*.
\]

Here, the \( \epsilon_1, \ldots, \epsilon_m \) are independent Rademacher random variables. Then Lemma 2.5 implies that, with probability at least \( 1 - \exp(-\gamma^2/2) \), it holds

\[
\inf_{H \in \mathcal{S} \mathcal{H}_{d \times d}} \|A(H)\|_2 = \inf_{H \in \mathcal{S} \mathcal{H}_{d \times d}} \left( \sum_{j=1}^{m} (a_j^* H a_j)^2 \right)^{1/2} \geq \xi \sqrt{m} Q_2 \xi - 2W_m - \xi \gamma
\]

for any \( \xi > 0 \) and \( \gamma > 0 \). We take \( \xi = \sqrt{2}/4, \gamma = c\sqrt{m} \) for a sufficiently small positive constant \( c \) in (16) and claim that

\[
Q_{1/\sqrt{2}} \geq 1/52, \quad W_m \lesssim \sqrt{tdr}.
\]

Combining (16), (17) and \( m \gtrsim tdr \), we arrive at (15).

It remains to prove (17). For the term \( Q_{1/\sqrt{2}} \), according to the Payley-Zygmund inequality (Lemma 2.6), we have

\[
P\left\{ |a_j^* H a_j|^2 \geq \frac{1}{2} \mathbb{E} |a_j^* H a_j|^2 \right\} \geq \frac{1}{4} \cdot \frac{(\mathbb{E}|a_j^* H a_j|^2)^2}{\mathbb{E}|a_j^* H a_j|^4}.
\]

By spectral decomposition, we can write \( H := \sum_{j=1}^{t_r} \lambda_j v_j v_j^* \) where \( \lambda_1, \ldots, \lambda_{t_r} \in \mathbb{R} \) are eigenvalues and \( v_1, \ldots, v_{t_r} \in \mathbb{C}^d \) are the corresponding orthonormal eigenvectors. For a standard complex Gaussian random variable \( Z \sim 1/\sqrt{2} : \mathcal{N}(0, 1) + i/\sqrt{2} : \mathcal{N}(0, 1) \), we have \( \mathbb{E}|Z|^{2k} = k! \), \( k \in \mathbb{Z}_+ \).

By the unitary invariance of complex Gaussian random vectors, we have

\[
\mathbb{E}|a_j^* H a_j|^2 = \mathbb{E} \left( \sum_{k=1}^{t_r} \lambda_k |a_{j,k}|^2 \right)^2 = 2 \sum_{k=1}^{t_r} \lambda_k^2 + 2 \sum_{1 \leq k < l \leq t_r} \lambda_k \lambda_l
\]

\[
= \sum_{k=1}^{t_r} \lambda_k^2 + \left( \sum_{k=1}^{t_r} \lambda_k \right)^2 \geq \|H\|_F^2.
\]
Noting that \( \|H\|_F^2 = \sum_{k=1}^{t \cdot r} \lambda_k^2 = 1 \), we have \( |\lambda_k| \leq 1 \). Then \( \sum_{k=1}^{t \cdot r} \lambda_k^2 \leq \sum_{k=1}^{t \cdot r} \lambda_k^2 = 1 \) and \( \sum_{k=1}^{t \cdot r} \lambda_k^4 \leq \sum_{k=1}^{t \cdot r} \lambda_k^2 = 1 \). Thus we have

\[
\begin{align*}
\mathbb{E} |a_j^* H a_j|^4 &= \sum_{j \neq k \neq l \neq s} \lambda_j \lambda_k \lambda_l \lambda_s + 12 \sum_{k \neq l} \lambda_k^2 \lambda_l \lambda_s + 12 \sum_{k \neq l} \lambda_k^2 \lambda_l^2 + 24 \sum_{k \neq l} \lambda_k^3 \lambda_l + 24 \sum_{k \neq l} \lambda_k^4 \\
&= \sum_{j,k,l,s} \lambda_j \lambda_k \lambda_l \lambda_s + 6 \sum_{k \neq l} \lambda_k^2 \lambda_l \lambda_s + 9 \sum_{k \neq l} \lambda_k^2 \lambda_l^2 + 20 \sum_{k \neq l} \lambda_k^3 \lambda_l + 23 \sum_{k \neq l} \lambda_k^4 \\
&= \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^4 + 6 \|H\|_F^2 \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^2 + 3 \|H\|_F^4 + 8 \left( \sum_{k=1}^{t \cdot r} \lambda_k^3 \right) \left( \sum_{k=1}^{t \cdot r} \lambda_k \right) + 6 \sum_{k=1}^{t \cdot r} \lambda_k^4 \\
&\leq \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^4 + 6 \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^2 + 4 \left( 1 + \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^2 \right) + 9 \\
&\leq 13 \left( \mathbb{E} |a_j^* H a_j|^2 \right)^2.
\end{align*}
\]

Here, we use the fact that \( 2 \left| \sum_{k=1}^{t \cdot r} \lambda_k \right| \leq 1 + \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^2 \) in the first inequality and \( \langle a_j^* H a_j \rangle^2 = \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^4 + 2 \left( \sum_{k=1}^{t \cdot r} \lambda_k \right)^2 + 1 \) in the last inequality. Putting (19) into (18), we obtain

\[
\mathbb{P} \left\{ |a_j^* H a_j|^2 \leq \frac{1}{2} \|H\|_F^2 \right\} \geq \frac{1}{52},
\]

which implies

\[
Q_{1/\sqrt{2}} \geq \frac{1}{52}.
\]

We next show \( W_m = \mathbb{E} \sup_{H \in \mathcal{SH}_{t \cdot r}^{d \times d}} \langle H, A \rangle \leq \sqrt{t \cdot r} \). For any \( H \in \mathcal{SH}_{t \cdot r}^{d \times d} \), by spectral decomposition, we can write \( H = \sum_{j=1}^{t \cdot r} \lambda_j v_j v_j^* \). Then

\[
\langle H, A \rangle = \sum_{j=1}^{t \cdot r} \lambda_j v_j^* A v_j \leq \|A\|_2 \cdot \sum_{j=1}^{t \cdot r} |\lambda_j| = \|A\|_2 \|H\|_\ast \leq \sqrt{t \cdot r} \|A\|_2.
\]

Here, we use the fact that the nuclear norm \( \|H\|_\ast \leq \sqrt{t \cdot r} \|H\|_F \) due to \( \text{rank}(H) \leq t \cdot r \). It gives

\[
W_m = \mathbb{E} \sup_{H \in \mathcal{SH}_{t \cdot r}^{d \times d}} \langle H, A \rangle \leq \sqrt{t \cdot r} \cdot \mathbb{E} \|A\|_2.
\]

Recall that \( A = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \epsilon_j a_j a_j^* \) where \( \epsilon_1, \ldots, \epsilon_m \) are independent Rademacher random variables, independent from everything else. From Lemma 2.7, we have

\[
\mathbb{E} \left\| \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \epsilon_j a_j a_j^* \right\|_2 \leq \sqrt{d}.
\]
which implies
\[ W_m \lesssim \sqrt{tdr}. \]
This completes the claim. \(\square\)

2.2.3. The connection between LRIP and RUB. In [3], Cai and Zhang introduce the definition of Restricted Uniform Boundedness (RUB) and prove that the Gaussian rank-one projection satisfies such condition with high probability in the real case. One can prove Theorem 2.3 based on RUB condition and the results in [3], as shown below.

A linear map \( \mathcal{A} : \mathbb{R}^{p_1 \times p_2} \to \mathbb{R}^m \) has RUB condition of order \( r \) if there exist uniform constants \( C_1 \) and \( C_2 \) such that
\[ C_1 \| M \|_F \leq \frac{1}{m} \| \mathcal{A}(M) \|_1 \leq C_2 \| M \|_F \]
holds for all non-zero rank-\( r \) matrices \( M \in \mathbb{R}^{p_1 \times p_2} \). Using the notation RUB and the results under “Rank-One Projection” model in [3], we can present an alternative proof for Theorem 2.3. To see this, recognize that the linear map defined in Theorem 2.3 is \( \mathcal{A} : \mathcal{H}^{d \times d}(\mathbb{C}) \to \mathbb{R}^m \) with
\[ [\mathcal{A}(H)]_j = a_j^* H a_j, \quad j = 1, \ldots, m. \]
If we let \( a_j = a_j^R + i a_j^3 \), \( H = H^R + i H^3 \) with \( H^R = (H^R)^\top \) and \( H^3 = -(H^3)^\top \), then we could rewrite the linear map \( \mathcal{A} \) as
\[ [\mathcal{A}(H)]_j = a_j^* H a_j = \tilde{a}_j^\top \tilde{H} \tilde{a}_j, \quad j = 1, \ldots, m, \]
where \( \tilde{H} = \begin{pmatrix} H^R & -H^3 \\ H^3 & H^R \end{pmatrix} \in \mathbb{R}^{2d \times 2d} \) and \( \tilde{a}_j = \begin{pmatrix} a_j^R \\ a_j^3 \end{pmatrix} \in \mathbb{R}^{2d} \). Let \( \tilde{\mathcal{A}} : \mathbb{R}^{2d \times 2d} \to \mathbb{R}^{|m/2|} \) be an operator which is defined by
\[ [\tilde{\mathcal{A}}(M)]_j := \beta_j^\top M \gamma_j, \quad j = 1, \ldots, |m/2| \]
with \( \beta_j := \tilde{a}_{2j-1} + \tilde{a}_{2j} \) and \( \gamma_j := \tilde{a}_{2j-1} - \tilde{a}_{2j} \). Since \( a_j^R, a_j^3 \sim N(0, I_d/2) \), it leads to \( \beta_j, \gamma_j \sim N(0, I_{2d}) \) with \( \beta_j \) and \( \gamma_j \) independent because they are Gaussian random vectors and \( \mathbb{E}\langle \beta_j, \gamma_j \rangle = 0 \).
Thus, the linear map \( \tilde{\mathcal{A}} \) is exactly a Gaussian rank-one projection model as defined in [3]. Theorem 2.2 in [3] shows that with probability at least \( 1 - \exp(-cm) \) for some constant \( c > 0 \), \( \tilde{\mathcal{A}} \) satisfies RUB condition of order \( t \cdot r \) provided \( m \gtrsim tdr \), namely, there exist constants \( C_1, C_2 > 0 \) such that
\[ C_1 \| M \|_F \leq \frac{1}{|m/2|} \| \tilde{\mathcal{A}}(M) \|_1 \leq C_2 \| M \|_F \]
holds for all rank-\((t \cdot r)\) matrices \(M \in \mathbb{R}^{2d \times 2d}\). It follows from (20) that the connection between \(\tilde{A}\) and \(A\) is

\[
[A(H)]_{2j-1} - [A(H)]_{2j} = \tilde{A}(\tilde{H}), \quad j = 1, \ldots, \lfloor m/2 \rfloor.
\]

This means that, for all \(H \in \mathcal{H}_{d \times d}^d(\mathbb{C})\), directly associated with the Hermitian \(\tilde{H} \in \mathbb{R}^{2d \times 2d}\), we have

\[
\frac{1}{\sqrt{m}}\|A(H)\|_2 \geq \frac{1}{m}\|A(H)\|_1 \geq \frac{1}{m}\sum_{j=1}^{\lfloor m/2 \rfloor} \left| [A(H)]_{2j-1} - [A(H)]_{2j} \right| = \frac{1}{m}\|\tilde{A}(\tilde{H})\|_1 \geq \frac{C_1}{2}\|\tilde{H}\|_F = \frac{C_1}{\sqrt{2}}\|H\|_F,
\]

which implies the result in Theorem 2.3.

### 3. Proofs of Theorem 1.1 and Theorem 1.2

Motivated by the observation

\[
b_j = |\langle a_j, x_0 \rangle|^2 + \eta_j = a_j^*X_0a_j + \eta_j, \quad j = 1, \ldots, m,
\]

where \(X_0 := x_0x_0^*\), we can employ Corollary 2.4 to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \(X_0 := x_0x_0^*\). Since \(\tilde{x}\) is the global solution to (1), \(\tilde{X} := \tilde{x}\tilde{x}^*\) is the global solution to (12) with \(r = 1\). From Corollary 2.4, we obtain that with probability at least \(1 - \exp(-cm)\), it holds

\[
\|\tilde{x}\tilde{x}^* - x_0x_0^*\|_F \lesssim \frac{\|\eta\|_2}{\sqrt{m}}
\]

provided \(m \gtrsim d\). We claim that for any \(u, v \in \mathbb{C}^d\), we have

\[
\min_{\theta \in [0, 2\pi]} \|u - e^{i\theta}v\|_2 \leq \frac{2\|uu^* - vv^*\|_F}{\|v\|_2}.
\]

Indeed, choosing \(\theta := \text{Phase}(v^*u)\) and setting \(\bar{v} := e^{i\theta}v\), then \(\langle u, \bar{v} \rangle \geq 0\). Let \(h := u - \bar{v}\). Then we have

\[
\|uu^* - vv^*\|_F^2 = \|uu^* - \bar{v}v^*\|_F^2 = \|hh^* + hv^* + vh^*\|_F^2 = \|h\|^2 + 4\|h\|^2\langle h, \bar{v} \rangle + 2\|h\|^2\|\bar{v}\|^2 \\
\geq (4 - 2\sqrt{2})\|h\|^2\langle h, \bar{v} \rangle + 2\|h\|^2\|\bar{v}\|^2 \\
= (4 - 2\sqrt{2})\|h\|^2\langle u, \bar{v} \rangle + (2\sqrt{2} - 2)\|h\|^2\|\bar{v}\|^2 \\
\geq \frac{1}{2}\|h\|^2\|\bar{v}\|^2.
\]
where the last line follows from the fact $\langle u, v \rangle \geq 0$. Thus we obtain (23). Combining (22) and (23), we arrive at

\begin{equation}
\min_{\theta \in [0, 2\pi)} \| \hat{x} - e^{i \theta} x_0 \|_2 \leq \frac{2}{\| x_0 \|_2} \| \hat{x} \bar{x}^* - x_0 x_0^* \|_F \lesssim \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}}.
\end{equation}

We next show

\[ \min_{\theta \in [0, 2\pi)} \| \hat{x} - e^{i \theta} x_0 \|_2 \lesssim \| x_0 \|_2 + \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}. \]

Indeed, according to (22), we have

\[ \| \hat{x} \|^2 = \| \hat{x} \bar{x}^* \|_F \leq \| x_0 x_0^* \|_F + \| \hat{x} \bar{x}^* - x_0 x_0^* \|_F \lesssim \| x_0 \|^2 + \frac{\| \eta \|^2}{\sqrt{m}}, \]

which implies

\[ \| \hat{x} \|_2 \lesssim \| x_0 \|_2 + \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}. \]

Hence, we have

\begin{equation}
\min_{\theta \in [0, 2\pi)} \| \hat{x} - e^{i \theta} x_0 \|_2 \leq \| \hat{x} \|_2 + \| x_0 \|_2 \lesssim \| x_0 \|_2 + \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}.
\end{equation}

Combining (24) and (25), we obtain

\[ \min_{\theta \in [0, 2\pi)} \| \hat{x} - e^{i \theta} x_0 \|_2 \lesssim \min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}, \frac{\| \eta \|^2}{\| x_0 \|_2 \cdot \sqrt{m}} \right\}. \]

A simple calculation shows that

\[ \min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}, \frac{\| \eta \|^2}{\| x_0 \|_2 \cdot \sqrt{m}} \right\} = \frac{\| \eta \|^2}{\| x_0 \|_2 \cdot \sqrt{m}} \]

holds provided $\| x_0 \|^2 \geq \frac{\sqrt{5} - 1}{2} \cdot \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}$. For the case where $\| x_0 \|^2 < \frac{\sqrt{5} - 1}{2} \cdot \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}$, we have

\[ \min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}, \frac{\| \eta \|^2}{\| x_0 \|_2 \cdot \sqrt{m}} \right\} = \| x_0 \|_2 + \frac{\sqrt{\| \eta \|^2}}{m^{1/4}} \leq \frac{\sqrt{5} + 1}{2} \cdot \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}. \]

Hence, we obtain

\[ \min_{\theta \in [0, 2\pi)} \| \hat{x} - e^{i \theta} x_0 \|_2 \lesssim \min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}, \frac{\| \eta \|^2}{\| x_0 \|_2 \cdot \sqrt{m}} \right\} \lesssim \min \left\{ \frac{\sqrt{\| \eta \|^2}}{m^{1/4}}, \frac{\| \eta \|^2}{\| x_0 \|_2 \sqrt{m}} \right\}. \]

\[ \square \]
3.1. **Proof of Theorem 1.2.** We first introduce some lemmas which play a key role in our proof. The following lemma is a nonuniform result about the upper bound of the fourth power of complex Gaussian variables.

**Lemma 3.1.** [34, Lemma 21] Let $a_j \in \mathbb{C}^d, j = 1, \ldots, m$, be i.i.d complex Gaussian random vectors. Suppose that $v \in \mathbb{C}^d$ is a fixed vector. For any $\delta \in (0, 1)$ the following holds with probability at least $1 - c_v \delta^{-2} m^{-1} - c_v \exp(-c_v \delta^2 m / \log m)$

$$\left\| \frac{1}{m} \sum_{j=1}^{m} |a_j^* v|^2 a_j a_j^* - (v v^* + \|v\|_2^2 I) \right\| \leq \delta \|v\|^2$$

provided $m \geq C(\delta) d \log d$. Here $C(\delta)$ is a constant depending on $\delta$ and $c_v$. $c_v$ and $c_c$ are positive absolute constants.

**Lemma 3.2.** [34, Lemma 22] Let $a_j \in \mathbb{C}^d, j = 1, \ldots, m$, be i.i.d complex Gaussian random vectors. For any $\delta \in (0, 1)$ the following holds with probability at least $1 - c_v \delta^{-2} m^{-1} - c_v \exp(-c_v \delta^2 m / \log m)$

$$\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 |a_j^* w|^2 \geq (1 - \delta) \left( \|w\|^2\|z\|^2 + |w^* z|^2 \right) \quad \text{for all } z, w \in \mathbb{C}^d$$

provided $m \geq C(\delta) d \log d$. Here $C(\delta)$ and $c(\delta)$ are constants depending on $\delta$ and $c_v$, $c_v'$ are positive absolute constants.

**Lemma 3.3.** Suppose that $a_j \in \mathbb{C}^d, j = 1, \ldots, m$, are i.i.d. complex Gaussian random vectors and $\eta_j \in \mathbb{R}, j = 1, \ldots, m$. For any fixed $\delta \in (0, 1)$, there exists a constant $\rho > 0$ depending only on $\delta$ such that the following holds with probability at least $1 - 2 \exp(-c(\delta) d)$:

$$\|\sum_{j=1}^{m} \eta_j (a_j a_j^* - I)\|_2 \leq \rho \cdot \delta \cdot (\sqrt{d} \|\eta\|_2 + d \|\eta\|_{\infty})$$

Here, $c(\delta) > 0$ is a constant depending only on $\delta$.

**Proof.** We assume that $\mathcal{N}$ is a 1/4-net of the complex unit sphere $S^{d-1} \subset \mathbb{C}^d$. It then follows from [37, Lemma 4.4.3] that

$$\|\sum_{j=1}^{m} \eta_j (a_j a_j^* - I)\|_2 \leq 2 \max_{x \in \mathcal{N}} \left| \sum_{j=1}^{m} \eta_j (|a_j^* x|^2 - 1) \right|,$$

where the cardinality $|\mathcal{N}| \leq 9^{2d}$. For any fixed $x \in S^{d-1}$, the terms $|a_j^* x|^2 - 1, j = 1, \ldots, m$ are independent centered sub-exponential random variables with the sub-exponential norm being a
constant. Using the Bernstein’s inequality [37, Theorem 2.8.1], we obtain that
\[
\mathbb{P}\left\{ \left| \sum_{j=1}^{m} \eta_j (|a_j^* x|^2 - 1) \right| \geq t \right\} \leq 2 \exp\left( -c' \min\left( \frac{t^2}{\|\eta\|_2^2}, \frac{t}{\|\eta\|_\infty} \right) \right)
\]
for some positive constant \( c' \). We assume that \( \rho > 1 \) is a constant which will be specified later. Taking \( t := \rho(\sqrt{d}\|\eta\|_2 + d\|\eta\|_\infty)\delta \), we obtain that
\[
\left| \sum_{j=1}^{m} \eta_j (|a_j^* x|^2 - 1) \right| \leq \rho(\sqrt{d}\|\eta\|_2 + d\|\eta\|_\infty)\delta
\]
holds with probability at least \( 1 - 2 \exp(-c' \cdot \rho \cdot \delta^2 d) \). Choosing the constant \( \rho \) such that \( c' \cdot \rho \cdot \delta^2 \geq 2 \ln 9 \) and taking the union bound, we can obtain that, with probability at least \( 1 - 2 \exp(-c(\delta)d) \), it holds that
\[
\left\| \sum_{j=1}^{m} \eta_j (a_j a_j^* - I) \right\|_2 \leq \rho(\sqrt{d}\|\eta\|_2 + d\|\eta\|_\infty)\delta
\]
where \( c(\delta) := c' \rho \delta^2 - 2 \ln 9 > 0 \) is a constant depending only on \( \delta \). \( \square \)

The following lemma states that if \( \hat{x} \) is the solution to (1), then the upper-tail of \( \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \) is well-behaved, although it involves the fourth power of the Gaussian variables. We present its proof in Appendix A.

**Lemma 3.4.** Suppose that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors. Assume that the noise vector \( \eta \) satisfies \( \|\eta\|_2 \leq \sqrt{m}, \|\eta\|_\infty \leq \log m \) and \( \sum_{j=1}^{m} \eta_j \lesssim m \). Suppose that \( x_0 \in \mathbb{C}^d \) is a fixed vector satisfying \( \|x_0\|_2 \geq 2C\|\eta\|_2 / \sqrt{m} \). For any \( \gamma > 0 \), if \( m \geq c'(\gamma)d \log m \) then with probability at least \( 1 - c''(\gamma)m^{-1} - c''\exp(-c'''(\gamma)d) \) the following holds: any solution \( \hat{x} \in \mathbb{C}^d \) to (1) with \( b_j = |\langle a_j, x_0 \rangle|^2 + \eta_j, j = 1, \ldots, m \), satisfies
\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \leq (2 + \gamma)\|\hat{x}\|^4.
\]
Here \( c'(\gamma), c''(\gamma), c'''(\gamma) \) are constants depending on \( \gamma \), and \( c'' \) is an absolute constant, and \( C \) is the constant in Theorem 1.1.

We next present the proof of Theorem 1.2.
Proof of Theorem 1.2. Without loss of generality, we assume \( \|x_0\|_2 = 1 \) (the general case can be obtained via a simple rescaling). Let

\[
f(x) = \sum_{j=1}^{m} \left( |\langle a_j, x \rangle|^2 - b_j \right)^2.
\]

Then the Wirtinger gradient (see, eg, \([5]\)) of \( f \) is

\[
\nabla f(x) = 2 \sum_{j=1}^{m} \left( |\langle a_j, x \rangle|^2 - b_j \right) a_j^* a_j x.
\]

Since \( \hat{x} \) is the solution to (1), we have

\[
\nabla f(\hat{x}) = 2 \sum_{j=1}^{m} \left( |\langle a_j, \hat{x} \rangle|^2 - b_j \right) a_j^* a_j \hat{x} = 0.
\]

Noting that \( b_j = |\langle a_j, x_0 \rangle|^2 + \eta_j \), \( j = 1, \ldots, m \), then \( \langle \nabla f(\hat{x}), \hat{x} \rangle = 0 \) gives

\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 - \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^2 |a_j^* x_0|^2 = \frac{1}{m} \sum_{j=1}^{m} \eta_j |a_j^* \hat{x}|^2.
\]

We claim that, when \( m \gtrsim d \log m \), with probability at least \( 1 - c' m^{-1} - c'' \exp(-c'''d) \) it holds: any solution \( \hat{x} \) to (1) obeys

\[
\left| \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 - \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^2 |a_j^* x_0|^2 \right| \leq 3 \| \hat{x} \|^4 - \| \hat{x} \|^2 \leq 3 \| \hat{x} \|^2 \| H \|_F
\]

and

\[
\frac{1}{m} \left| \sum_{j=1}^{m} \eta_j |a_j^* \hat{x}|^2 \right| \gtrsim \| \eta \|_2 \| \hat{x} \|^2 / \sqrt{m},
\]

where \( H = \hat{x} \hat{x}^* - x_0 x_0^* \) and \( c', c'', c''' \) are universal positive constants. Combining (26), (27) and (28), we obtain that

\[
\| H \|_F = \| \hat{x} \hat{x}^* - x_0 x_0^* \|_F \gtrsim \| \eta \|_2 \sqrt{m}
\]

holds with probability at least \( 1 - c' m^{-1} - c'' \exp(-c'''d) \) provided \( m \gtrsim d \log m \). We next use (29) to derive the conclusion. For any \( \theta \in [0, 2\pi) \), we have

\[
\| \hat{x} \hat{x}^* - x_0 x_0^* \|_F = \| \hat{x} \hat{x}^* - \hat{x} \hat{x}^* e^{-i\theta} + \hat{x} x_0^* e^{-i\theta} - x_0 x_0^* \|_F
\]

\[
\leq \| \hat{x} \|_2 \| \hat{x} \hat{x}^* - x_0 e^{i\theta} \|_2 + \| x_0 \|_2 \| \hat{x} - x_0 e^{i\theta} \|_2.
\]

If \( \| x_0 \|_2 < \sqrt{\| \eta \|_2 / m^{1/4}} \), then Theorem 1.1 gives

\[
\| \hat{x} \|_2 \lesssim \| x_0 \|_2 + \sqrt{\| \eta \|_2 / m^{1/4}} < 2 \sqrt{\| \eta \|_2 / m^{1/4}}.
\]
Combining (29), (30) and (31), we obtain that
\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - e^{i\theta} x_0 \|_2 \geq \frac{m^{1/4}}{\sqrt{\|\eta\|_2}} \cdot \| \hat{x}^* - x_0^* \|_F \geq \frac{\|\eta\|_2}{m^{1/4}}.
\]
On the other hand, if \( \| x_0 \|_2 \geq \frac{\sqrt{\|\eta\|_2}}{m^{1/4}} \), then Theorem 1.1 gives
\[
\| \hat{x} \|_2 \lesssim \| x_0 \|_2 + \frac{\|\eta\|_2}{\| x_0 \|_2 \cdot \sqrt{m}} \leq 2 \| x_0 \|_2.
\]
According to (29) and (30), we obtain
\[
\min_{\theta \in [0, 2\pi]} \| \hat{x} - e^{i\theta} x_0 \|_2 \geq \frac{1}{\| x_0 \|_2} \| \hat{x}^* - x_0^* \|_F \geq \frac{\|\eta\|_2}{\| x_0 \|_2 \sqrt{m}}.
\]
We arrive at the conclusion.

It remains to prove (27) and (28). We first consider (27). According to Lemma 3.4, if \( m \gtrsim d \log m \) then the following holds
\[
(32) \quad \frac{1}{m} \sum_{j=1}^{m} |a_j \hat{x}|^4 \leq 3 \| \hat{x} \|^4
\]
with probability at least \( 1 - c_1 m^{-1} - c_2 \exp(-c_3 d) \) for some positive constants \( c_1, c_2 \) and \( c_3 \). On the other hand, Lemma 3.1 implies that if \( m \gtrsim d \log d \) then with probability at least \( 1 - c_1 m^{-1} - c_2 \exp(-c_3 m/ \log m) \) it holds
\[
(33) \quad \frac{1}{m} \sum_{j=1}^{m} |a_j \hat{x}|^2 |a_j x_0|^2 \geq \| \hat{x} \|^2.
\]
Combining (32) and (33), we obtain that, with probability at least \( 1 - 2c_1 m^{-1} - 2c_2 \exp(-c_3 d) \),
\[
(34) \quad \frac{1}{m} \sum_{j=1}^{m} |a_j \hat{x}|^4 - \frac{1}{m} \sum_{j=1}^{m} |a_j \hat{x}|^2 |a_j x_0|^2 \leq 3 \| \hat{x} \|^4 - \| \hat{x} \|^2 \leq 3 \| \hat{x} \|^2 \| H \|_F
\]
holds provided \( m \gtrsim d \log m \). Similarly, we can use Lemma 3.1 and Lemma 3.2 to obtain that
\[
(35) \quad \frac{1}{m} \sum_{j=1}^{m} |a_j \hat{x}|^2 |a_j x_0|^2 - \frac{1}{m} \sum_{j=1}^{m} |a_j \hat{x}|^4 \leq 3 \| \hat{x} \|^2 - \| \hat{x} \|^4 \leq 3 \| \hat{x} \|^2 \| H \|_F
\]
holds with probability at least \( 1 - 2c_1 m^{-1} - 2c_2 \exp(-c_3 d) \) provided \( m \gtrsim d \log m \). Combining (34) and (35), we arrive at (27).

We next consider (28). To prove (28), it is enough to show that the following holds with high probability:
\[
(36) \quad \left| \sum_{j=1}^{m} \eta_j |a_j x|^2 \right| \gtrsim \sqrt{m} \| \eta \|_2 \quad \text{for all } x \in \mathbb{C}^d \text{ with } \| x \| = 1.
\]
A simple observation is that for any fixed $x \in \mathbb{C}^d$ with $\|x\| = 1$ the terms $|a_j^* x|^2$ are independent sub-exponential random variables with the sub-exponential norm $C_0$ where $C_0$ is a constant. According to Bernstein’s inequality, we have

$$\mathbb{P}\left\{ \left| \sum_{j=1}^{m} \eta_j \left( |a_j^* x|^2 - 1 \right) \right| \geq t \right\} \leq 2 \exp \left( -c_4 \min \left( \frac{t^2}{C_0^2 \|\eta\|^2}, \frac{t}{C_0 \|\eta\|_\infty} \right) \right).$$

(37)

Assume that $c_5 > 1$ is a constant which will be specified later. Taking $t := \sqrt{c_5 d \|\eta\|_2 + c_5 d \|\eta\|_\infty}$, we have

$$\frac{t^2}{C_0^2 \|\eta\|^2} \geq \frac{c_5 d}{C_0^2} \quad \text{and} \quad \frac{t}{C_0 \|\eta\|_\infty} \geq \frac{c_5 d}{C_0}.$$  

Then (37) implies that, with probability at least

$$1 - 2 \exp \left( -c_4 \min \left( \frac{t^2}{C_0^2 \|\eta\|^2}, \frac{t}{C_0 \|\eta\|_\infty} \right) \right) \geq 1 - 2 \exp \left( -c_4 c_5 d \cdot \min \left( \frac{1}{C_0^2}, \frac{1}{C_0} \right) \right),$$

it holds that

$$\left| \sum_{j=1}^{m} \eta_j |a_j^* x|^2 \right| \geq \left| \sum_{j=1}^{m} \eta_j \right| - \sqrt{c_5 d \|\eta\|_2 - c_5 d \|\eta\|_\infty}.$$ 

Hence, for any fixed $c_5 > 0$ there exists a constant $C' > 0$ such that if $m \geq C' d \log m$ then

$$\left| \sum_{j=1}^{m} \eta_j |a_j^* x|^2 \right| \geq \sqrt{m} \|\eta\|_2$$

(38)

holds with probability at least $1 - 2 \exp(-cd)$, where $c := c_4 c_5 \min \left( 1/C_0^2, 1/C_0 \right)$. Here, we use $\left| \sum_{j=1}^{m} \eta_j \right| \geq \sqrt{m} \|\eta\|_2 \geq \sqrt{C' d \|\eta\|_\infty}, \|\eta\|_2 \geq \sqrt{m}$ and $\|\eta\|_\infty \leq \log m$.

Next, we give a uniform bound for (38). We assume that $\mathcal{N}$ is an $\epsilon$-net of the unit complex sphere in $\mathbb{C}^d$ and hence the covering number $\# \mathcal{N} \leq (1 + \frac{2}{\epsilon})^{2d}$. For any $x' \in \mathbb{C}^d$ with $\|x'\|_2 = 1$, there exists a $x \in \mathcal{N}$ such that $\|x' - x\|_2 \leq \epsilon$. Taking $\delta = 1/2$ in Lemma 3.3, we can obtain that if $m \geq d \log m$ then the following holds with probability at least $1 - 2 \exp(-cd)$

$$\left| \sum_{j=1}^{m} \eta_j |a_j^* x'|^2 \right| - \left| \sum_{j=1}^{m} \eta_j |a_j^* x|^2 \right| \leq \left| \sum_{j=1}^{m} \eta_j a_j^* (x' x' - x x^*) a_j \right|$$

$$\lesssim (\sqrt{d} \|\eta\|_2 + d \|\eta\|_\infty) \|x' x' - x x^*\|_F$$

$$\lesssim (\sqrt{m} \|\eta\|_2 + d \|\eta\|_\infty) \|x' x' - x x^*\|_F$$

$$\lesssim (\sqrt{m} \|\eta\|_2 + d \|\eta\|_\infty) \epsilon$$

$$\lesssim \epsilon \sqrt{m} \|\eta\|_2,$$
where we take \( \epsilon \) to be some positive constant in the third inequality and use the fact that \( \sqrt{m}\|\eta\|_2 \gtrsim d\|\eta\|_\infty \) in the last inequality. Here, \( c_6 \) is a positive constant. We can choose the constant \( c_5 \) such that
\[
c = c_4c_5 \min \left( \frac{1}{C_0^2}, \frac{1}{C_0} \right) > 2 \log(1 + 2/\epsilon).
\]
Combining (38) and (39), we obtain that (36) holds with probability at least
\[
1 - 2 \exp(-cm) \cdot (1 + \frac{2}{\epsilon})^{2d} - 2 \exp(-c_6 d) \geq 1 - 4 \exp(-c_7 d)
\]
for some positive constant \( c_7 \), provided \( m \gtrsim d \log m \).

\[ \Box \]

4. Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4. Before proceeding, we introduce several notations and technical lemmas which are useful in the proof. For convenience, we set
\[
S_{d,s} := \left\{ x \in \mathbb{C}^d : \|x\|_2 \leq 1, \|x\|_0 \leq s \right\},
\]
and
\[
K_{d,s} := \left\{ x \in \mathbb{C}^d : \|x\|_2 \leq 1, \|x\|_1 \leq \sqrt{s} \right\}.
\]
The relationship of the two sets are characterized by the following lemma:

**Lemma 4.1.** [30, Lemma 3.1] It holds that \( \text{conv}(S_{d,s}) \subset K_{d,s} \subset 2\text{conv}(S_{d,s}) \), where \( \text{conv}(S_{d,s}) \) denotes the convex hull of \( S_{d,s} \).

The following lemma presents an upper bound for the spectral norm of \( \mathcal{A}^*(\epsilon) \) which is defined in (11).

**Lemma 4.2.** Suppose that \( a_j \in \mathbb{C}^s, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors and \( \epsilon_1, \ldots, \epsilon_m \) are independent Rademacher random variables. If \( m \gtrsim s \), then with probability at least \( 1 - 2 \exp(-cm) \) we have \( \|\mathcal{A}^*(\epsilon)\|_2 \lesssim m \), that is,
\[
\| \sum_{j=1}^{m} \epsilon_j a_j a_j^* \|_2 \lesssim m,
\]
where \( \epsilon := (\epsilon_1, \ldots, \epsilon_m)^\top \).

**Proof.** Assume that \( \mathcal{N} \) is a 1/4-net of the complex unit sphere \( S^{s-1} \subset \mathbb{C}^s \). Then we have
\[
\| \sum_{j=1}^{m} \epsilon_j a_j a_j^* \|_2 \leq 2 \max_{x \in \mathcal{N}} \left| \sum_{j=1}^{m} \epsilon_j |a_j^* x|^2 \right|.
\]
For any fixed \( x \in \mathcal{N} \), the terms \( \epsilon_j |a_j^* x|^2, j = 1, \ldots, m \) are independent centered sub-exponential random variables with maximum sub-exponential norm \( C_1 \), where \( C_1 \) is a constant. We use Bernstein’s inequality [37, Theorem 2.8.1] to obtain

\[
P \left\{ \left| \sum_{j=1}^m \epsilon_j |a_j^* x|^2 \right| \geq Cm \right\} \leq 2 \exp \left( -c' \min \left( \frac{C_2^2 m^2}{C_1 C_2}, \frac{C_2 m}{C_1} \right) \right) \leq 2 \exp (-cm)
\]

for some positive constants \( C, c, c' \). Recognize that \( |\mathcal{N}| \leq 9^{2s} \). Taking the union bound over \( \mathcal{N} \), we can obtain that for \( m \gtrsim s \) with probability at least \( 1 - 2 \exp (-cm) \) it holds

\[
\| \sum_{j=1}^m \epsilon_j a_j a_j^* \|_2 \lesssim m,
\]

which completes the proof.

Following the spirit of LRIP condition, we need to present a lower bound \( \sum_{j=1}^m (a_j^* H a_j)^2 \) for simultaneously sparse and low-rank matrix \( H \) under the optimal sampling complexity, as demonstrated in the theorem below.

**Lemma 4.3.** Suppose that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors and \( \epsilon_1, \ldots, \epsilon_m \) are independent Rademacher random variables. Assume that \( \eta \in \mathbb{R}^m \) is a vector. If \( m \gtrsim s \log(ed/s) \), then the followings hold with probability at least \( 1 - 4 \exp(-cm) \):

(i) There exists a sufficiently small constant \( c_0 > 0 \) such that

\[
\left| \sum_{j=1}^m \epsilon_j a_j^* u v^* a_j \right| \leq c_0 m \quad \text{for all } u, v \in K_{d,s}.
\]

(ii)

\[
\inf_{H \in M_{d,s}} \| A(H) \|_2 \gtrsim \sqrt{m},
\]

where

\[
M_{d,s} := \left\{ \frac{h h^* + h x^* + x h^*}{\|h h^* + h x^* + x h^*\|_F} \in \mathcal{H}_{d \times d}^d(\mathbb{C}) : h/\|h\|_2 \in K_{d,s}, x \in K_{d,s} \text{ and } \|h\|_2 \leq 2 \|h h^* + h x^* + x h^*\|_F \right\}.
\]

**Proof.** We first prove (i). According to Lemma 4.1 we have

\[
\sup_{u,v \in K_{d,s}} \left| \sum_{j=1}^m \epsilon_j a_j^* u v^* a_j \right| \lesssim \sup_{u,v \in S_{d,s}} \left| \sum_{j=1}^m \epsilon_j a_j^* u v^* a_j \right|.
\]
It suffices to present an upper bound for \( \sup_{\mathbf{u}, \mathbf{v} \in S_{d,s}} | \sum_{j=1}^{m} \epsilon_j a_j^* \mathbf{u} \mathbf{v}^* \mathbf{a}_j | \). For any fixed \( \mathbf{u}_0, \mathbf{v}_0 \in S_{d,s} \), the terms \( \epsilon_j a_j^* \mathbf{u}_0 \mathbf{v}_0 \mathbf{a}_j \) are independent centered sub-exponential random variables with maximum sub-exponential norm \( C \), where \( C \) is a constant. The Bernstein’s inequality gives

\[
(40) \quad \mathbb{P} \left\{ \left| \sum_{j=1}^{m} \epsilon_j a_j^* \mathbf{u}_0 \mathbf{v}_0 \mathbf{a}_j \right| \geq c_1 m \right\} \leq 2 \exp \left( -c' \min \left( \frac{c_1^2 m^2}{C^2 m^2}, \frac{c_1 m}{C} \right) \right) \leq 2 \exp(-cm)
\]

for some sufficiently small constants \( c, c_1, c' > 0 \). Suppose that \( \mathcal{N} \) is an \( \epsilon \)-net of \( S_{d,s} \times S_{d,s} \). Hence, for any \( \mathbf{u}, \mathbf{v} \in S_{d,s} \), there exist \( \mathbf{u}_0, \mathbf{v}_0 \in \mathcal{N} \) satisfying \( \| \mathbf{u} - \mathbf{u}_0 \|_2 \leq \epsilon \) and \( \| \mathbf{v} - \mathbf{v}_0 \|_2 \leq \epsilon \). Note that the matrix \( \mathbf{u} \mathbf{v}^* - \mathbf{u}_0 \mathbf{v}_0^* \) has at most \( 2s \) nonzero columns and \( 2s \) nonzero rows because of \( \mathbf{u}, \mathbf{v}, \mathbf{u}_0, \mathbf{v}_0 \in S_{d,s} \). Using Lemma 4.2, we obtain that if \( m \gtrsim 2s \) then, with probability at least \( 1 - 2 \exp(-cm) \), it holds

\[
\left| \sum_{j=1}^{m} \epsilon_j a_j^* \mathbf{u} \mathbf{v}^* \mathbf{a}_j \right| - \left| \sum_{j=1}^{m} \epsilon_j a_j^* \mathbf{u}_0 \mathbf{v}_0 \mathbf{a}_j \right| \leq \left| \sum_{j=1}^{m} \epsilon_j a_j^* (\mathbf{u} \mathbf{v}^* - \mathbf{u}_0 \mathbf{v}_0^*) \mathbf{a}_j \right|
\]

\[
\leq \| \mathbf{u} \mathbf{v}^* - \mathbf{u}_0 \mathbf{v}_0^* \|_F \left( \sum_{j=1}^{m} \epsilon_j a_j^* a_j \right) \leq \sqrt{2} \left( \sum_{j=1}^{m} \epsilon_j a_j^* a_j \right) \| \mathbf{u} \mathbf{v}^* - \mathbf{u}_0 \mathbf{v}_0^* \|_F \lesssim m \| \mathbf{u} \mathbf{v}^* - \mathbf{u}_0 \mathbf{v}_0^* \|_F \lesssim m \epsilon,
\]

where we use \( \langle A, B \rangle \leq \sqrt{r} \| B \|_2 \| A \|_F \) for any Hermitian matrices \( A, B \) with \( \text{rank}(A) \leq r \). Note that the covering number \( \# \mathcal{N} \leq \exp(C s \log(ed/s)/\epsilon^2) \). Choosing a sufficiently small constant \( \epsilon > 0 \) and taking the union bound over \( \mathcal{N} \), we obtain that if \( m \gtrsim s \log(ed/s) \) then with probability at least \( 1 - 4 \exp(-cm) \), it holds

\[
\sup_{\mathbf{u}, \mathbf{v} \in S_{d,s}} \left| \sum_{j=1}^{m} \epsilon_j a_j^* \mathbf{u} \mathbf{v}^* \mathbf{a}_j \right| \leq c_0 m
\]

for some sufficiently small positive constant \( c_0 \). Here, we use (40) and (41). This completes the proof of (i).

We next turn to prove (ii). Let

\[
Q_\xi := \inf_{H \in M_{d,s}} \mathbb{P} \left\{ \left| a_j^* H \mathbf{a}_j \right| \geq \xi \right\}
\]

\[
W_m := \mathbb{E} \sup_{H \in M_{d,s}} \langle H, A \rangle \quad \text{where} \quad A := \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \epsilon_j a_j^* a_j,
\]
Here, the $\epsilon_1, \ldots, \epsilon_m$ are independent Rademacher random variables. Then Lemma 2.5 shows that, with probability at least $1 - \exp(-t^2/2)$, it holds
\begin{equation}
\inf_{H \in M_{d,s}} \left( \sum_{j=1}^{m} (a_j^* H a_j)^2 \right)^{1/2} \geq \xi \sqrt{m Q_2} - 2W_m - \xi t
\end{equation}
for any $\xi > 0$ and $t > 0$. Taking $\xi = \sqrt{2}/4$, we can employ the method in the proof of Theorem 2.3 to obtain
\begin{equation}
Q_{1/\sqrt{2}} \geq \frac{1}{52}.
\end{equation}
We next proceed to obtain an upper bound for $W_m$. According to the definition of $M_{d,s}$, the matrix $H \in M_{d,s}$ is in the form of
\[H = \frac{hh^* + hx^* + xh^*}{\|hh^* + hx^* + xh^*\|_F}\]
with $h/\|h\|_2 \in K_{d,s}$ and $x \in K_{d,s}$. Recall that $A = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \epsilon_j a_j a_j^*$. This immediately leads to
\[
\langle H, A \rangle = \frac{1}{\|hh^* + hx^* + xh^*\|_F} \cdot \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \left( \epsilon_j a_j^* hh^* a_j + \epsilon_j a_j^* hx^* a_j + \epsilon_j a_j^* xh^* a_j \right).
\]
According to the result (i), there exists a sufficiently small constant $c_0 > 0$ such that the following holds with probability at least $1 - 4 \exp(-cm)$
\begin{equation}
\langle H, A \rangle \leq c_0 \sqrt{m} \cdot \frac{\|h\|_2^2 + 2\|h\|_2}{\|hh^* + hx^* + xh^*\|_F}.
\end{equation}
provided $m \gtrsim s \log(ed/s)$. On the other hand, $H \in M_{d,s}$ implies
\begin{equation}
\|h\|_2 \leq 2\|hh^* + hx^* + xh^*\|_F.
\end{equation}
We next show $\|h\|_2^2 \leq 5\|hh^* + hx^* + xh^*\|_F$. Indeed, by triangle inequality, we have
\begin{equation}
\|hh^* + hx^* + xh^*\|_F \geq \|hh^*\|_F - \|hx^* + xh^*\|_F \geq \|h\|_2^2 - 2\|h\|_2.
\end{equation}
Combining (45) and (46), we have
\begin{equation}
\|h\|_2^2 \leq \|hh^* + hx^* + xh^*\|_F + 2\|h\|_2 \leq 5\|hh^* + hx^* + xh^*\|_F.
\end{equation}
Putting (45) and (47) into (44), we obtain that
\begin{equation}
W_m = \mathbb{E} \sup_{H \in M_{d,s}} \langle H, A \rangle \leq 9c_0 \sqrt{m}.
\end{equation}
Choosing $t = c\sqrt{2m}$ for a sufficiently small positive constant $c$ and putting (43) and (48) into (42), we arrive at the conclusion. \hfill \Box
Based on the above lemmas, we are now ready to present the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Without loss of generality, we assume $\|x_0\|_2 = 1$ (the general case can be obtained via a simple rescaling) and $\langle \hat{x}, x_0 \rangle \geq 0$ (Otherwise, we can choose $e^{i\theta} x_0$ for an appropriate $\theta$). Set $h := \hat{x} - x_0$. We first show that $\|h\|_1 \leq 2\sqrt{s}\|h\|_2$. Indeed, let $S := \text{supp}(x_0)$. Then we have

$$\|\hat{x}\|_1 = \|x_0 + h\|_1 = \|x_0 + h_S\|_1 + \|h_{S^c}\|_1 \geq \|x_0\|_1 - \|h_S\|_1 + \|h_{S^c}\|_1.$$  

Here $h_S$ denotes the restriction of the vector $h$ onto the set of coordinates $S$. Then the constraint condition $\|\hat{x}\|_1 \leq R := \|x_0\|_1$ implies that $\|h_{S^c}\|_1 \leq \|h_S\|_1$. Using Hölder inequality, we have

$$\|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \leq 2\|h_S\|_1 \leq 2\sqrt{s}\|h\|_2.$$  

Set $H = \hat{x} x^* - x_0 x_0^*$. It is straightforward to check that

$$H = hh^* + hx_0^* + x_0 h^*.$$  

From the claim (23), we know $\|h\|_2 \leq 2\|H\|_F$. Recall that $\|h\|_1 \leq 2\sqrt{s}\|h\|_2$ and $x_0 \in K_{d,s}$. It implies that $H/\|H\|_F \in M_{d,4s}$, where the set $M_{d,s}$ is defined in Lemma 4.3.

Since $\hat{x}$ is the global solution to (8) and $x_0$ is a feasible point, we have

$$\|A(\hat{x} x^*) - b\|_2 \leq \|A(x_0 x_0^*) - b\|_2$$  

which implies

$$\|A(H) - \eta\|_2 \leq \|\eta\|_2.$$  

Noting that $H/\|H\|_F \in M_{d,4s}$, by Lemma 4.3, we obtain

$$\sqrt{\lambda} \|H\|_F \lesssim \|A(H)\|_2 \leq \|A(H) - \eta\|_2 + \|\eta\|_2 \leq 2\|\eta\|_2$$  

with probability at least $1 - 4\exp(-cm)$, provided $m \gtrsim s\log(ed/s)$. Thus, (50) gives

$$\|H\|_F = \|\hat{x} x^* - x_0 x_0^*\|_F \lesssim \frac{\|\eta\|_2}{\sqrt{m}},$$  

which implies

$$\min_{\theta \in [0,2\pi]} \|\hat{x} - e^{i\theta} x_0\|_2 \leq \frac{2}{\|x_0\|_2} \|\hat{x} x^* - x_0 x_0^*\|_F \lesssim \frac{\|\eta\|_2}{\|x_0\|_2 \sqrt{m}}.$$  

Here, we use (23). Based on (51), similar to the proof of Theorem 1.1, we have

$$\min_{\theta \in [0,2\pi]} \|\hat{x} - e^{i\theta} x_0\|_2 \lesssim \|x_0\|_2 + \frac{\sqrt{\|\eta\|_2}}{m^{1/4}}.$$
It means that
\[
\min_{\theta \in [0, 2\pi)} \| \hat{x} - e^{i\theta} x_0 \|_2 \lesssim \min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}} \right\}.
\]

Finally, note that if \( \| x_0 \|_2 \geq \frac{\sqrt{5} - 1}{2} \cdot \sqrt{\| \eta \|_2 m^{1/4}} \) then
\[
\min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}} \right\} = \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}}
\]
and if \( \| x_0 \|_2 < \frac{\sqrt{5} - 1}{2} \cdot \sqrt{\| \eta \|_2 m^{1/4}} \) then
\[
\min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}} \right\} = \| x_0 \|_2 + \frac{\sqrt{\| \eta \|_2}}{m^{1/4}} \leq \frac{\sqrt{5} + 1}{2} \cdot \sqrt{\| \eta \|_2 m^{1/4}}.
\]
We obtain the conclusion that
\[
\min_{\theta \in [0, 2\pi)} \| \hat{x} - e^{i\theta} x_0 \|_2 \lesssim \min \left\{ \| x_0 \|_2 + \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}} \right\} \lesssim \min \left\{ \frac{\sqrt{\| \eta \|_2}}{m^{1/4}}, \frac{\| \eta \|_2}{\| x_0 \|_2 \sqrt{m}} \right\}.
\]

5. Discussions

This paper considers the performance of the intensity-based estimators for phase retrieval and its sparse version. The upper and lower bounds are obtained under complex Gaussian random measurements.

There are some interesting problems for future research. First, in the presence of noises, many numerical experiments show that gradient descent algorithms can solve estimators (1) and (2), however, it is of practical interest to provide some theoretical guarantees for it. Second, a more practical scenario is the case where the measurements are Fourier vectors. Since there is much less or even no randomness to be exploited in this scenario, we conjecture the estimation error would be no less than the lower bound given in this paper, namely, \( O(\| \eta \|_2 / \sqrt{m}) \). To establish the precise upper and lower bounds for Fourier measurements is the future work.
Appendix A. Proof of Lemma 3.4

The goal of this section is to prove Lemma 3.4. Before continuing, we introduce some lemmas. The following result is a complex version of Lemma 5.8 in [8] and the proof is the same as that of Lemma 5.8 in [8].

**Lemma A.1.** Let \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), be i.i.d complex Gaussian random vectors. For any \( \epsilon > 0 \), there exist some universal constants \( c_0, c_1, C > 0 \) such that

\[
\frac{1}{m} \sum_{j=1}^{m} \mathbb{1}\{|a_j^* z| \geq \gamma \|z\|\} \leq \frac{1}{0.49\gamma} \exp(-0.485\gamma^2) + \frac{\epsilon}{\gamma^2} \quad \text{for all } z \in \mathbb{C}^d \setminus \{0\}, \gamma \geq 2
\]

holds with probability at least \( 1 - C \exp(-c_0\epsilon^2 m) \), provided \( m \geq c_1 \epsilon^{-2} \log \epsilon^{-1} d \).

**Lemma A.2.** Suppose that \( a_j \in \mathbb{C}^d, j = 1, \ldots, m \), are i.i.d. complex Gaussian random vectors. For any \( \epsilon \in (0, 1) \), if \( m \geq c(\epsilon)d \log d \) then the following holds with probability at least \( 1 - c'_a m^{-1} - c'_b \exp(-c'_c(\epsilon)m/\log m) \):

\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \leq 2 \Re(x_0^* z) + \epsilon + 2 \epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^4 \right)^{\frac{3}{4}}
\]

for all \( z \in S_{\mathbb{C}}^{d-1} \), where \( S_{\mathbb{C}}^{d-1} := \{ x \in \mathbb{C}^d : \|x\| = 1 \} \), \( c'_a, c'_b \) are positive absolute constants and \( c(\epsilon), c'_c(\epsilon) \) are positive constants depending on \( \epsilon \).

**Proof.** Suppose that \( \phi \in C^\infty_c(\mathbb{R}) \) is a Lipschitz continuous function satisfying \( 0 \leq \phi(x) \leq 1 \) for all \( x \in \mathbb{R} \). We furthermore require \( \phi(x) = 1 \) for \( |x| \leq 1 \) and \( \phi(x) = 0 \) for \( |x| \geq 2 \). For any \( \beta > 0 \), we have

\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) = T + r
\]

where

\[
T := \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right),
\]

\[
r := \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \left( 1 - \phi \left( \frac{|a_j^* z|}{\beta} \right) \right) \leq \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^3 |a_j^* x_0| \mathbb{1}\{|a_j^* z| \geq \beta\}.
\]
We claim that for any $0 < \epsilon < 1$ there exists a sufficiently large $\beta > 1$ such that if $m \geq c(\epsilon)d \log d$ then the following holds with probability at least $1 - c'_a m^{-1} - c'_b \exp(-c'_c(\epsilon)m/\log m)$:

$$T \leq 2 \Re(x_0^* z) + \epsilon, \quad r \leq 2\epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^4 \right)^{\frac{3}{4}} \quad \text{for all } z \in \mathbb{S}^{d-1}_{C}. \quad (53)$$

Here $c(\epsilon), c'_a(\epsilon), c'_b, c'_c$ are constants depending on $\epsilon$ and $c'_a, c'_b, c'_c$ are positive absolute constants. Substituting (53) into (52), we obtain the conclusion that with probability at least $1 - c'_a m^{-1} - c'_b \exp(-c'_c(\epsilon)m/\log m)$, it holds

$$\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \leq 2 \Re(x_0^* z) + \epsilon + 2\epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^4 \right)^{\frac{3}{4}} \quad \text{for all } z \in \mathbb{S}^{d-1}_{C}$$

provided $m \geq c(\epsilon)d \log d$.

It remains to prove (53). We first show $T \leq 2 \Re(x_0^* z) + \epsilon$. Due to the cut-off $\phi \left( \frac{|a_j^* z|}{\beta} \right)$, the terms $|a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right)$ are independent sub-gaussian random variables with the sub-gaussian norm $O(\beta^2)$. According to Hoeffding’s inequality, we obtain that the following holds with probability at least $1 - 2 \exp(-c(\beta)\epsilon^2 m)$

$$\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right) \leq 2 \Re(x_0^* z) + \epsilon, \quad (54)$$

where $c(\beta) > 0$ is a constant depending on $\beta$. Here we use the fact

$$\mathbb{E} \left( |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right) \right) \leq \mathbb{E} \left( |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \right) + \frac{\epsilon}{6} = 2 \Re(x_0^* z) + \frac{\epsilon}{6}$$

for some sufficiently large $\beta$ depending only on $\epsilon$. We next show that (54) holds for all unit vectors $z \in \mathbb{C}^d$, for which we adopt a basic version of a $\delta$-net argument. We assume that $\mathcal{N}$ is a $\delta$-net of the unit complex sphere in $\mathbb{C}^d$ and hence the covering number $\#\mathcal{N} \leq (1 + \frac{\beta}{3})^{2d}$. For any $z' \in \mathbb{S}^{d-1}_{C}$, there exists a $z \in \mathcal{N}$ such that $\|z' - z\|_2 \leq \delta$. Noting $f(\tau) := \tau^2 \phi(\tau/\beta)$ is a bounded function with Lipschitz constant $O(\beta)$, we obtain that when $m \geq d \log d$, with probability at least
$1 - c_a m^{-1} - c_b \exp(-c_c m / \log m)$, it holds

$$\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right) - \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right) \leq 1 \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right) | x_0^* a_j a_j^* z' - x_0^* a_j a_j^* z |$$

$$+ \frac{1}{m} \sum_{j=1}^{m} |a_j^* z| |a_j^* x_0| |a_j^* z'|^2 \phi \left( \frac{|a_j^* z'|}{\beta} \right) - |a_j^* z|^2 \phi \left( \frac{|a_j^* z|}{\beta} \right) | |a_j^* z'| - |a_j^* z|$$

$$\leq \frac{\beta^2}{m} \sum_{j=1}^{m} |a_j^* x_0| |a_j^* z'| - |a_j^* z| + \frac{\beta}{m} \sum_{j=1}^{m} |a_j^* z| |a_j^* x_0| |a_j^* z'| - |a_j^* z|$$

$$\leq \frac{\beta^2}{m} \|A x_0\| \|A(z' - z)\| + \beta \sqrt{\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 |a_j^* x_0|^2} \cdot \frac{1}{\sqrt{m}} A(z' - z)$$

$$\leq 2\beta^2 \|z' - z\| + 3\beta \|z' - z\| \leq 5\beta^2 \delta,$$

where the fourth inequality follows from Lemma 3.1 and the fact that $\frac{1}{\sqrt{m}} \|A\| \leq \sqrt{2}$ with probability at least $1 - 2 \exp(-cm)$ provided $m \gtrsim d$. Here, the matrix $A := [a_1, \ldots, a_m]^*$ and $c_a, c_b, c_c$ are absolute constants. Taking $\delta = \epsilon/(15\beta^2)$, we use (54) and (55) to obtain that if $m \geq c(\epsilon, \beta) d \log d$ then with probability at least $1 - c_a m^{-1} - c_b \exp(-c'(\beta) \epsilon^2 m / \log m)$ it holds

$$\frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^2 \Re(x_0^* a_j a_j^* z) \phi \left( \frac{|a_j^* z|}{\beta} \right) \leq 2\Re(x_0^* z) + \epsilon$$

for all $z \in \mathbb{S}_C^{d-1}$, where $c(\epsilon, \beta)$ is a positive constant depending on $\epsilon, \beta$ and $c'(\beta)$ is a positive constant depending on $\beta$.

We next show that $r \leq 2\epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^4 \right)^{\frac{1}{4}}$. By Lemma A.1, for any $\epsilon > 0$ there exists a sufficiently large $\beta$ such that if $m \geq c_1(\epsilon) d$ then with probability at least $1 - C \exp(-c_0(\epsilon) m) -$
\(c_2m^{-1}\) it holds
\[
\begin{align*}
 r & \leq \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^3 |a_j^* x_0| \mathbb{I}_{\{|a_j^* z| \geq \beta\}} \\
 & \leq \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^4 \right)^{\frac{3}{4}} \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* x_0|^8 \right)^{\frac{1}{8}} \left( \frac{1}{m} \sum_{j=1}^{m} \mathbb{I}_{\{|a_j^* z| \geq \beta\}} \right)^{\frac{1}{8}} \\
 & \leq 2\epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* z|^4 \right)^{\frac{1}{4}},
\end{align*}
\]
where we use the Chebyshev’s inequality in the last line to deduce that with probability at least 
\(1 - c_2m^{-1}\),
\[
\frac{1}{m} \sum_{j=1}^{m} |a_j(1)|^8 \leq 25.
\]
Here, \(c_0(\epsilon)\) and \(c_1(\epsilon)\) are constants depending on \(\epsilon\), and \(C, c_2\) are absolute constants.

We are now ready to prove Lemma 3.4.

**Proof of Lemma 3.4.** Without loss of generality, we assume \(\|x_0\|_2 = 1\) (the general case can be obtained via a simple rescaling) and \(\langle \hat{x}, x_0 \rangle \geq 0\) (otherwise, we can choose \(e^{i\theta}x_0\) for an appropriate \(\theta\)). Recall that the loss function is
\[
f(z) = \sum_{j=1}^{m} \left( |\langle a_j, z \rangle|^2 - b_j \right)^2.
\]
Since \(\hat{x}\) is a global minimizer of \(f(z)\), we have
\[
\nabla f(\hat{x}) = 2 \sum_{j=1}^{m} \left( |\langle a_j, \hat{x} \rangle|^2 - b_j \right) a_j a_j^* \hat{x} = 0.
\]
Let \(\hat{z} := R \hat{z}\), where \(R \geq 0\) and \(\|\hat{z}\| = 1\). Recall that \(b_j = |a_j^* x_0|^2 + \eta_j, j = 1, \ldots, m\). Then \(\langle \nabla f(\hat{x}), \hat{x} \rangle = 0\) implies
\[
R^2 \sum_{j=1}^{m} |a_j^* \hat{z}|^4 = \sum_{j=1}^{m} |a_j^* \hat{z}|^2 |a_j^* x_0|^2 + \sum_{j=1}^{m} \eta_j |a_j^* \hat{z}|^2.
\]
Similarly, according to \(\langle \nabla f(\hat{x}), x_0 \rangle = 0\) we have
\[
R^2 \sum_{j=1}^{m} |a_j^* \hat{z}|^2 x_0^* a_j a_j^* \hat{z} = \sum_{j=1}^{m} |a_j^* x_0|^2 x_0^* a_j a_j^* \hat{z} + \sum_{j=1}^{m} \eta_j x_0^* a_j a_j^* \hat{z}.
\]
Combining (55) and (56), we obtain
\[ U := \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \cdot \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* x_0|^2 \Re(x_0^* a_j a_j^* \hat{x}) + \frac{1}{m} \sum_{j=1}^{m} \eta_j \Re(x_0^* a_j a_j^* \hat{x}) \right) \]
(57)
\[ = \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^2 \Re(x_0^* a_j a_j^* \hat{x}) \cdot \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^2 |a_j^* x_0|^2 + \frac{1}{m} \sum_{j=1}^{m} \eta_j |a_j^* \hat{x}|^2 \right) =: V. \]

Since \( \langle \hat{x}, x_0 \rangle \geq 0 \), without loss of generality, we may assume \( \hat{x} = s x_0 + s_1 x_0^\perp \), where \( x_0^\perp \in S_{-1} \) satisfies \( \langle x_0^\perp, x_0 \rangle = 0 \) and \( s, s_1 \) are positive real numbers obeying \( s^2 + s_1^2 = 1 \). A simple observation is that \( s := \langle \hat{x}, x_0 \rangle \in [0, 1] \). We claim that for any \( 0 < \epsilon < 1 \), when \( m \geq c(\epsilon) d \log m \), with probability at least \( 1 - c_a \epsilon^2 m^{-1} - c_b \exp(-c_c(\epsilon)d) \), the followings hold:
\[ U \geq \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \cdot \left( 2s + s \cdot \frac{1}{m} \sum_{j=1}^{m} \eta_j - \lambda \epsilon \right) \]
(58)
and
\[ V \leq \left( 2s + \epsilon + 2 \epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \right)^{\frac{3}{4}} \right) \cdot \left( 1 + s^2 + \frac{1}{m} \sum_{j=1}^{m} \eta_j + \lambda \epsilon \right), \]
(59)
where \( \lambda \) is a universal positive constant. Here, \( c_a, c_b \) are absolute constants and \( c(\epsilon), c_c(\epsilon) \) are constants depending on \( \epsilon \). Combining (57), (58) and (59), we obtain
\[ \left( 2s + \epsilon + 2 \epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \right)^{\frac{3}{4}} \right) \cdot \left( 1 + s^2 + \frac{1}{m} \sum_{j=1}^{m} \eta_j + \lambda \epsilon \right) \]
\[ \geq \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \cdot \left( 2s + s \cdot \frac{1}{m} \sum_{j=1}^{m} \eta_j - \lambda \epsilon \right). \]
(60)
According to Lemma 3.2, when \( m \geq c(\epsilon) d \log d \), with probability at least \( 1 - c_1 \exp(-c_2(\epsilon)m) - c_3 m^{-d} \), it holds
\[ \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{x}|^4 \geq 2 - \epsilon > 1, \]
(61)
where \( c_2(\epsilon) \) is a constant depending on \( \epsilon \) and \( c_1, c_3 \) are absolute constants. Since \( \frac{1}{m} \sum_{j=1}^{m} \eta_j \) is bounded, there exists a constant \( C_0 \) so that
\[ 1 + s^2 + \frac{1}{m} \sum_{j=1}^{m} \eta_j + \lambda \epsilon \leq C_0 \]
(62)
for some positive constant $C_0$. We can use (61) and (62) to obtain

$$2\epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 \right)^{\frac{2}{3}} \cdot \left( 1 + s^2 + \frac{1}{m} \sum_{j=1}^{m} \eta_j + \lambda \epsilon \right) \leq 2C_0 \epsilon \cdot \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 ,$$

(63)

$$\epsilon \cdot \left( 1 + s^2 + \frac{1}{m} \sum_{j=1}^{m} \eta_j + \lambda \epsilon \right) \leq C_0 \epsilon \cdot \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 ,$$

$$2s \lambda \epsilon \leq 2 \lambda \epsilon \cdot \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 .$$

Substituting (63) into (60), we have

$$2s \left( 1 + s^2 + \frac{1}{m} \sum_{j=1}^{m} \eta_j \right) \geq \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 \cdot \left( 2s + s \cdot \frac{1}{m} \sum_{j=1}^{m} \eta_j - C_1 \epsilon \right) ,$$

(64)

where $C_1 := 3 \lambda + 3C_0$ is bounded. Assume that $c_0$ is a constant satisfying $\frac{1}{m} \sum_{j=1}^{m} \eta_j \leq c_0$. Using (61) again, we have

$$\frac{2}{m} \sum_{j=1}^{m} \eta_j \leq \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 \cdot \left( \frac{1}{m} \sum_{j=1}^{m} \eta_j + c_0 \epsilon \right) .$$

(65)

Combining (64) and (65), we have

$$2s(1 + s^2) \geq \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 (2s - C_2 \epsilon) \geq (2 - \epsilon) (2s - C_2 \epsilon) ,$$

(66)

where $C_2 := 3 \lambda + 3C_0 + c_0$ is bounded. We claim that $s > \frac{\sqrt{\epsilon}}{2}$. Recall that $s \leq 1$. By taking $\epsilon > 0$ sufficiently small, it then follows from (66) that $s$ must be sufficient close to 1. Then (66) implies that, for any $\gamma > 0$, the following holds with probability at least $1 - c' \gamma m^{-1} - c'' \exp(-c''' \gamma d)$

$$\frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 \leq 1 + s^2 + \frac{C_3 \epsilon}{s} \leq 2 + \gamma ,$$

provided $m \geq c' \gamma d \log m$, where $c'(\gamma), c''(\gamma), c'''(\gamma)$ are constants depending $\gamma$, $C_3$ and $c''$ are sufficiently large constant.

It remains to prove (58), (59) and $s \geq \frac{\sqrt{\epsilon}}{2}$.

We first show that (58) holds. Lemma 3.1 implies that for any $0 < \epsilon < 1$, when $m \geq c(\epsilon) d \log d$, with probability at least $1 - c_1(\epsilon)^{-2} m^{-1} - c_1' \exp(-c_1(\epsilon)^2 m / \log m)$,

$$\frac{1}{m} \sum_{j=1}^{m} |a_j^* x_0|^2 \Re(x_0^* a_j^* \hat{z}) \geq 2s - \epsilon .$$

(67)
Here, $c(\epsilon)$ is a constant depending on $\epsilon$ and $c_a, c'_b, c_c$ are absolute constants. On the other hand, note that $|\eta| \lesssim \sqrt{m}$ and $|\eta|_\infty \lesssim \log m$. Taking $\delta = \epsilon$ in Lemma 3.3, we obtain that the following holds with probability at least $1 - 2 \exp(-c_c \epsilon^2 d)$:

$$
\frac{1}{m} \sum_{j=1}^{m} \eta_j \Re(x_0^* a_j \hat{a}_j^* \hat{z}) \geq s \cdot \frac{1}{m} \sum_{j=1}^{m} \eta_j - \rho \epsilon \cdot \left( \frac{\sqrt{d}}{m} \|\eta\| + \frac{d}{m} \|\eta\|_\infty \right)
$$

(68)

for some universal positive constant $C'$, provided $m \geq C'' \rho d \log m$. Here, $C''$ is a universal constant and $\rho$ is a constant depending on $\epsilon$. Combining (67) and (68), we arrive at (58).

We next turn to (59). By Lemma 3.1, for any $0 < \epsilon < 1$, when $m \geq c(\epsilon) d \log d$, with probability at least $1 - c_a \epsilon^{-2} m^{-1} - c'_b \exp(-c_c \epsilon^2 m / \log m)$,

$$
\frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^2 |a_j^* x_0|^2 \leq 1 + s^2 + \epsilon.
$$

Lemma 3.3 implies that, with probability at least $1 - 2 \exp(-c_c \epsilon^2 d)$, we have

$$
\frac{1}{m} \sum_{j=1}^{m} \eta_j |a_j^* \hat{z}|^2 \leq \frac{1}{m} \sum_{j=1}^{m} \eta_j + \rho \epsilon \cdot \left( \frac{\sqrt{d}}{m} \|\eta\| + \frac{d}{m} \|\eta\|_\infty \right)
$$

(70)

provided $m \geq C'' \rho^{-1} d \log m$. According to Lemma A.2, we obtain that the following holds with probability at least $1 - c_a m^{-1} - c'_b \exp(-c'(\epsilon)m / \log m)$,

$$
\frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^2 \Re(x_0^* a_j \hat{a}_j^* \hat{z}) \leq 2 s + \epsilon + 2 \epsilon \left( \frac{1}{m} \sum_{j=1}^{m} |a_j^* \hat{z}|^4 \right)^{1/4}
$$

(71)

provided $m \geq c(\epsilon) d \log d$. Here $c'(\epsilon)$ is a positive constant depending on $\epsilon$. Combining (69), (70) and (71), we obtain (59).

We still need to show that $s \geq \frac{\sqrt{\epsilon}}{d}$. From Theorem 1.1, we know that for $m \gtrsim d$, with probability at least $1 - \exp(-cm)$,

$$
\|\hat{x} - x_0\| \leq C \frac{\|\eta\|}{\sqrt{m}}.
$$

It immediately gives

$$
\langle \hat{x}, x_0 \rangle \geq \frac{1}{2} \|\hat{x}\| + \frac{1 - C^2 \|\eta\|^2 / m}{2 \|\hat{x}\|}.
$$

(72)
We claim that \( s := \langle \hat{z}, x_0 \rangle \geq \sqrt{5}/5 \) where \( \hat{z} := \frac{\hat{x}}{\|\hat{x}\|} \). Indeed, if \( C\|\eta\|/\sqrt{m} \geq 2/\sqrt{5} \) then \((72)\) gives

\[
s := \langle \hat{z}, x_0 \rangle \geq \frac{1}{2}\|\hat{x}\| \geq \frac{C\|\eta\|}{2\sqrt{m}} \geq \sqrt{5}/5,
\]

where we use the fact that \( 2C\|\eta\|/\sqrt{m} \leq \|x_0\| = 1 \) and \( \|\hat{x}\| \geq 1 - C\|\eta\|/\sqrt{m} \). On the other hand, if \( C\|\eta\|/\sqrt{m} < 2/\sqrt{5} \) then \((72)\) implies

\[
s := \langle \hat{z}, x_0 \rangle \geq \sqrt{1 - C^2\|\eta\|^2/m} > \sqrt{5}/5,
\]

where we use the inequality \( a + b \geq 2\sqrt{ab} \) for any positive real numbers \( a, b \). In summary, we obtain \( s \geq \sqrt{5}/5 \).

\[\square\]

**Bibliography**

[1] Bourrier, A.; Davies, M. E.; Peleg, T.; Pérez, P.; Gibonval, R. Fundamental performance limits for ideal decoders in high-dimensional linear inverse problems. *IEEE Trans. Inf. Theory* 60 (2014), no. 12, 7928–7946.

[2] Cai, T. T.; Li, X.; Ma, Z. Optimal rates of convergence for noisy sparse phase retrieval via thresholded Wirtinger flow. *Ann. Statist.* 44 (2016), no. 5, 2221–2251.

[3] Cai, T. T.; Zhang, A. ROP: Matrix recovery via rank-one projections. *Ann. Statist.* 43 (2015), no. 1, 102–138.

[4] Candès, E. J.; Li, X. Solving quadratic equations via PhaseLift when there are about as many equations as unknowns. *Found. Comput. Math.* 14 (2014), no. 5, 1017–1026.

[5] Candès, E. J.; Li, X.; Soltanolkotabi, M. Phase retrieval via Wirtinger flow: Theory and algorithms. *IEEE Trans. Inf. Theory* 61 (2015), no. 5, 1985–2007.

[6] Candès, E. J.; Strohmer, T.; Voroninski, V. Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming. *Commun. Pure Appl. Math.* 66 (2013), no. 8, 1241–1274.

[7] Conca, A.; Edidin, D.; Hering, M.; Vinzant, C. An algebraic characterization of injectivity in phase retrieval. *Appl. Comput. Harmon. Anal.* 38 (2015), no. 2, 346–356.

[8] Chen, Y.; Candès, E. J. Solving random quadratic systems of equations is nearly as easy as solving linear systems. *Commun. Pure Appl. Math.* 70 (2017), no. 5, 822–883.

[9] Chen, Y.; Chi, Y.; Goldsmith, A. J. Exact and stable covariance estimation from quadratic sampling via convex programming. *IEEE Trans. Inf. Theory* 61 (2015), no. 7, 4034–4059.

[10] Dainty, J. C.; Fienup, J. R. Phase retrieval and image reconstruction for astronomy. *Image Recovery: Theory and Application* 231 (1987), 275.

[11] De la Pena, V.; Giné, E. Decoupling: from dependence to independence. Springer Science and Business Media, 2012.

[12] Duchi, J.; Shalev-Shwartz, S.; Singer, Y.; Chandra, T. Efficient projections onto the \( \ell_1 \)-ball for learning in high dimensions. In *Proceedings of the 25th international conference on Machine learning* (2008), 272–279.

[13] Fienup, J. R. Phase retrieval algorithms: a comparison. *Appl. Opt.* 21 (1982), no. 15, 2758–2769.

[14] Foucart, S.; Rauhut, H. A mathematical introduction to compressive sensing. *Bull. Am. Math.* 54 (2017), 151–165.

[15] Gao, B.; Sun, X.; Wang, Y.; Xu, Z. Perturbed Amplitude Flow for Phase Retrieval. *IEEE Trans. Signal Process.* 68 (2020), 5427–5440.

[16] Gao, B.; Xu, Z. Phaseless recovery using the Gauss–Newton method. *IEEE Trans. Signal Process.* 65 (2017), no. 22, 5885–5896.

[17] Gerchberg, R. W.; Saxton, W. O. A practical algorithm for the determination of the phase from image and diffraction plane pictures. *Optik* 35 (1972), 237–246.

[18] Hand, P.; Voroninski, V. Compressed sensing from phaseless gaussian measurements via linear programming in the natural parameter space. arXiv preprint arXiv:1611.05985 (2016).

[19] Harrison, R. W. Phase problem in crystallography. *JOSA A* 10 (1993), no. 5, 1046–1055.
[20] Huang, M.; Xu, Z. The estimation performance of nonlinear least squares for phase retrieval. *IEEE Trans. Inf. Theory* **66**(2020), no. 12, 7967-7977.

[21] Huang, M.; Xu, Z. Solving Systems of Quadratic Equations via Exponential-type Gradient Descent Algorithm. *J. Comp. Math.* **38**(2020), no. 4, 638–660.

[22] Iwen, M.; Viswanathan, A.; Wang, Y. Robust sparse phase retrieval made easy. *Appl. Comput. Harmon. Anal.* **42**(2015), no. 1, 135–142.

[23] Kahane, C.; Kahane, J. P. *Some random series of functions.* Cambridge University Press, 1993.

[24] Keriven, N; Gribonval, R. Instance Optimal Decoding and the Restricted Isometry Property. *Journal of Physics: Conference Series*, **1131**(2018), no. 1, 012002.

[25] Kueng, R.; Rauhut, H.; Terstiege, U. Low rank matrix recovery from rank one measurements. *Appl. Comput. Harmon. Anal.* **42**(2017), no. 1, 88-116.

[26] Ling, S; Strohmer, T. Regularized gradient descent: a non-convex recipe for fast joint blind deconvolution and demixing. *Information and Inference: A Journal of the IMA*, **8**(2019), no. 1, 1-49.

[27] Miao, J.; Ishikawa, T.; Shen, Q.; Earnest, T. Extending X-ray crystallography to allow the imaging of noncrystalline materials, cells, and single protein complexes. *Annu. Rev. Phys. Chem.* **59**(2008), no. 3, 394-411.

[28] Millane, R. P. Phase retrieval in crystallography and optics. *J. Optical Soc. America A* **7**(1990), no. 3, 394-411.

[29] Netrapalli, P.; Jain, P.; Sanghavi, S. Phase retrieval using alternating minimization. *IEEE Trans. Signal Process.* **63**(2015), no. 18, 4814–4826.

[30] Plan, Y.; Vershynin, R. One-bit compressed sensing by linear programming. *Commun. Pure Appl. Math.* **66**(2011), no. 8, 1275–1297.

[31] Qu, Q.; Wright, J. Finding a sparse vector in a subspace: Linear sparsity using alternating directions. *IEEE Trans. Inf. Theory* **62**(2016), no. 10, 5855–5880.

[32] Sahinoglou, H.; Cabrera, S. D. On phase retrieval of finite-length sequences using the initial time sample. *IEEE Trans. Circuits and Syst.* **38**(1991), no. 8, 954–958.

[33] Shechtman, Y.; Eldar, Y. C.; Cohen, O.; Chapman, H. N.; Miao, J.; Segev, M. Phase retrieval with application to optical imaging: a contemporary overview. *IEEE Signal Process. Mag.* **32**(2015), no. 3, 87–109.

[34] Sun, J.; Qu, Q.; Wright, J. A geometric analysis of phase retrieval. *Found. Comut. Math.* **18**(2018), no. 5, 1131–1198.

[35] Sun, R.; Luo, Z. Q. Guaranteed matrix completion via non-convex factorization. *IEEE Trans. Inf. Theory* **62**(2016), no. 11, 6535–6579.

[36] Tropp, J. A. Convex recovery of a structured signal from independent random linear measurements. *Sampling Theory, a Renaissance* (2015), 67–101.

[37] Vershynin, R. *High-dimensional probability: An introduction with applications in data science.* U.K.:Cambridge Univ. Press, 2018.

[38] Waldspurger, I.; d’Aspremont, A.; Mallat, S. Phase recovery, maxcut and complex semidefinite programming. *Math. Prog.* **149**(2015), no. 1-2, 47–81.

[39] Walther, A. The question of phase retrieval in optics. *J. Mod. Opt.* **10**(1963), no. 1, 41–49.

[40] Wang, G.; Giannakis, G. B.; Eldar, Y. C. Solving systems of random quadratic equations via truncated amplitude flow. *IEEE Trans. Inf. Theory* **64**(2018), no. 2, 773–794.

[41] Wang, G.; Zhang, L.; Giannakis,G. B.; Akcakaya, M.; Chen, J. “Sparse phase retrieval via truncated amplitude flow,” *IEEE Trans. Signal Process.*, vol. 66, no. 2, pp. 479–491, 2018.

[42] Wang, Y.; Xu, Z. Phase Retrieval for Sparse Signals. *Appl. Comput. Harmon. Anal.* **37**(2014), no. 3, 531–544.

[43] Wang, Y.; Xu, Z. Generalized phase retrieval : measurement number, matrix recovery and beyond. *Appl. Comput. Harmon. Anal.* **47**(2019), no. 2, 423–446.

[44] Wu, F.; Rebeshcini, P. Hadamard wirtinger flow for sparse phase retrieval.. arXiv preprint arXiv:2006.01065 (2020).

[45] Zhang, H.; Zhou, Y.; Liang, Y.; Chi, Y. A nonconvex approach for phase retrieval: Reshaped wirtinger flow and incremental algorithms. *The Journal of Machine Learning Research* **18**(2017), no. 1, 5164–5189.
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