Topological logics with connectedness over Euclidean spaces

Roman Kontchakov, Birkbeck, University of London, UK
Yavor Nenov, University of Oxford, UK
Ian Pratt-Hartmann, University of Manchester, UK
Michael Zakharyaschev, Birkbeck, University of London, UK

We consider the quantifier-free languages, $Bc$ and $Bc^\circ$, obtained by augmenting the signature of Boolean algebras with a unary predicate representing, respectively, the property of being connected, and the property of having a connected interior. These languages are interpreted over the regular closed sets of $\mathbb{R}^n$ ($n \geq 2$) and, additionally, over the regular closed semilinear sets of $\mathbb{R}^n$. The resulting logics are examples of formalisms that have recently been proposed in the Artificial Intelligence literature under the rubric Qualitative Spatial Reasoning. We prove that the satisfiability problem for $Bc$ is undecidable over the regular closed semilinear sets in all dimensions greater than 1, and that the satisfiability problem for $Bc$ and $Bc^\circ$ is undecidable over both the regular closed sets and the regular closed semilinear sets in the Euclidean plane. However, we also prove that the satisfiability problem for $Bc^\circ$ is NP-complete over the regular closed sets in all dimensions greater than 2, while the corresponding problem for the regular closed semilinear sets is ExpTime-complete. Our results show, in particular, that spatial reasoning is much harder over Euclidean spaces than over arbitrary topological spaces.

Categories and Subject Descriptors: I.2.4 [Knowledge Representation Formalisms and Methods]: Representation languages; F.4.1 [Mathematical Logic]: Computability theory

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1. INTRODUCTION

Let $Bc$ be the quantifier-free fragment of first-order logic in the signature $(+, \cdot, -, 0, 1, c)$, where $c$ is a unary predicate; and let $RCP(\mathbb{R}^n)$ be the collection of polyhedra in $n$-dimensional Euclidean space, where a polyhedron is understood as any finite union of finite intersections of closed half-spaces. (Thus, our polyhedra need not be connected, need not be bounded, and need not have connected complements.) The collection $RCP(\mathbb{R}^n)$ forms a Boolean algebra under the subset ordering; we interpret $Bc$ over $RCP(\mathbb{R}^n)$ by taking $+, \cdot, -, 0, 1$ to have their Boolean algebra meanings, and by taking $c$ to denote the property of being connected. Intuitively, we think of elements of $RCP(\mathbb{R}^n)$ as regions of space, and of formulas of $Bc$ as descriptions of arrangements of these regions. Our primary concern is the satisfiability problem: given a $Bc$-formula, is there an assignment of elements of $RCP(\mathbb{R}^n)$ to its variables making it true?

The motivation for studying this problem comes from the field of Qualitative Spatial Reasoning in Artificial Intelligence, the aim of which is to develop formal languages for...
representing and processing qualitative spatial information. In this context, $\mathcal{B}c$ constitutes a parsimonious language: it has no quantifiers, and its non-logical primitives express only the basic region-combining operations and the property of connectedness. At the same time, the structures $\mathcal{RCP}(\mathbb{R}^n)$—particularly in the cases $n = 2$ and $n = 3$—constitute its most natural domains of interpretation, given current practice in the fields of Qualitative Spatial Reasoning, Geographic Information Systems and Spatial Databases.

For reasons discussed below, we broaden the subject of enquiry slightly. Let $\mathcal{B}c^o$ denote the quantifier-free fragment of first-order logic in the signature $\{+,-,0,1,c^o\}$, where $+,-,0,1$ are as before, and $c^o$ is a unary predicate interpreted as the property of *having a connected interior*. Further, let $\mathcal{RC}(\mathbb{R}^n)$ denote the collection of regular closed sets in $n$-dimensional Euclidean space, where a *regular closed* set is defined as the closure of any open set. Again, $\mathcal{RC}(\mathbb{R}^n)$ forms a Boolean algebra under the subset ordering, and has $\mathcal{RCP}(\mathbb{R}^n)$ as a sub-algebra. Intuitively, we think of $\mathcal{RC}(\mathbb{R}^n)$ as a more liberal model of spatial regions than $\mathcal{RCP}(\mathbb{R}^n)$. In the sequel, we consider the satisfiability problem for $\mathcal{B}c$ and $\mathcal{B}c^o$ over the structures $\mathcal{RC}(\mathbb{R}^n)$ and $\mathcal{RCP}(\mathbb{R}^n)$. The results of this paper are as follows: (i) the satisfiability problem for $\mathcal{B}c$ over $\mathcal{RCP}(\mathbb{R}^n)$ is undecidable for all $n \geq 2$; (ii) the satisfiability problem for $\mathcal{B}c$ over $\mathcal{RC}(\mathbb{R}^2)$ is undecidable, as are the satisfiability problems for $\mathcal{B}c^o$ over both $\mathcal{RC}(\mathbb{R}^2)$ and $\mathcal{RCP}(\mathbb{R}^2)$; (iii) the satisfiability problem for $\mathcal{B}c^o$ over $\mathcal{RC}(\mathbb{R}^n)$ is NP-complete for all $n \geq 3$, while over $\mathcal{RCP}(\mathbb{R}^n)$ the corresponding problem is ExpTime-complete. (It may be of interest to note that, over $\mathcal{RC}(\mathbb{R})$ and $\mathcal{RCP}(\mathbb{R})$, the satisfiability problem for $\mathcal{B}c$ and $\mathcal{B}c^o$ is NP-complete.) The decidability of the satisfiability problems for $\mathcal{B}c$ over $\mathcal{RC}(\mathbb{R}^n)$, for $n \geq 3$, is left open. Results (ii) and (iii) were announced, without proofs, in Kontchakov et al. 2010b; 2011.

Mathematically, it is also meaningful to consider the satisfiability of $\mathcal{B}c$- and $\mathcal{B}c^o$-formulas over the regular closed subsets of *any* topological space. If $T$ is a topological space, we denote the collection of regular closed subsets of $T$ by $\mathcal{RC}(T)$; again, this collection always forms a Boolean algebra under the subset ordering. The satisfiability problem for $\mathcal{B}c$ over the class of structures of the form $\mathcal{RC}(T)$ is known to be ExpTime-complete, while for $\mathcal{B}c^o$, the corresponding problem is NP-complete [Kontchakov et al. 2010a; 2010b]. However, satisfiability over *arbitrary* topological spaces is of at most marginal relevance to Qualitative Spatial Reasoning. Indeed, the results reported here show that, for languages able to express the property of connectedness, reasoning over *Euclidean* spaces is a different kettle of fish altogether. In the remainder of this section, we discuss the significance of these results in the context of recent developments in spatial, algebraic and modal logics.

### 1.1. Spatial logic

A spatial logic is a formal language interpreted over some class of geometrical structures. Spatial logics, thus understood, have a long history, tracing their origins back both to the axiomatic tradition in geometry [Hilbert 1909; Tarski 1959] and also the region-based theory of space [Whitehead 1929; de Laguna 1922], subsequently developed in [Clarke 1981; 1985; Biacino and Gerla 1991]. They were proposed as a formalism for Qualitative Spatial Reasoning in the seminal paper [Randell et al. 1992]. The basic idea is as follows: numerical coordinate-based descriptions of the objects that surround us are hard to acquire, inherently error-prone, and probably unnecessary for everyday spatial reasoning tasks; therefore—so goes the argument—we should employ a representation language whose variables range over spatial regions (rather than points), and whose non-logical primitives are interpreted as qualitative (rather than quantitative) relations and operations. On this view, formulas are to be understood as expressing descriptions of (putative) configurations of objects in space, with the satisfiability of a formula over the space in question equating to the geometrical realizability of the described arrangement. If we imagine an intelligent agent employing such a language to represent spatial arrangements of objects, then the problem of recognizing...
satisfiable formulas amounts to that of eliciting the geometrical knowledge latent in that agent’s operating environment and cognitive design.

The best-known, and most intensively studied, qualitative spatial representation language is \textsc{RCC8} [Egenhofer and Franzosa 1991; Randell et al. 1992; Smith and Park 1992]. This language features predicates for the six topological relations DC (disconnection), EC (external connection), PO (partial overlap), EQ (equality), TPP (tangential proper part) and NTPP (non-tangential proper part) illustrated, for the case of closed discs, in Fig. 1. (The name \textsc{RCC8} becomes less puzzling when we observe that the relations TPP and NTPP are asymmetric.) Note that \textsc{RCC8} has no individual constants or function symbols, and no quantifiers. Traditionally, \textsc{RCC8} is interpreted over the regular closed sets in some topological space.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{RCC8-relations-over-discs.pdf}
\caption{\textsc{RCC8}-relations over discs in \(\mathbb{R}^2\).}
\end{figure}

The satisfiability problem for \textsc{RCC8} over the class of structures of the form RC\((T)\) (for \(T\) a topological space) is NP-complete, though tractable fragments have been explored [Renz and Nebel 2001]. In particular, satisfiability of conjunctions of atomic \textsc{RCC8}-formulas is known to be NLOGSPACE-complete [Bennett 1994; Nebel 1995; Renz 1998; Griffiths 2008]. Further, satisfiability of an \textsc{RCC8}-formula over any structure in this class implies satisfiability over RCP\((\mathbb{R}^n)\), for all \(n \geq 1\) [Renz and Nebel 1999]. Thus, the satisfiability problems for \textsc{RC}(\mathbb{R}^n) and RCP(\mathbb{R}^n), for all \(n \geq 1\), coincide and are NP-complete—a fact which testifies to the restricted expressive power of \textsc{RCC8}.

A word of caution is in order at this point. Satisfiability of an \textsc{RCC8}-formula over \textsc{RC}(\mathbb{R}^2) does not necessarily imply satisfiability by natural or familiar regions—for example, closed disc-homeomorphs. The \textsc{RCC8}-satisfiability problem for such interpretations requires specialized, and highly non-trivial, techniques. A landmark result [Schaefer et al. 2003] in the area shows, however, that the satisfiability problem for \textsc{RCC8} interpreted over the closed disc-homeomorphs in \(\mathbb{R}^2\) is still in NP. The contribution of present paper, with its emphasis on Euclidean spaces and the property of connectedness, imposes severe limits on what further results of this kind we can hope for.

We mentioned above that, if \(T\) is a topological space, the collection RC\((T)\) always forms a Boolean algebra under the subset ordering. This enables us to extend \textsc{RCC8} with the function symbols \(+\), \(\cdot\), \(-\) and constants 0, 1, interpreting these in the natural way over any structure RC\((T)\). Such an extended language was originally introduced in [Wolter and Zakharyaschev 2000] under the name \textsc{BRCC8} (Boolean \textsc{RCC8}). Intuitively, if \(a_1\) and \(a_2\) are regular closed sets, we may think of \(a_1 + a_2\) as the agglomeration of \(a_1\) and \(a_2\), \(a_1 \cdot a_2\) as the common part of \(a_1\) and \(a_2\), \(-a_1\) as the complement of \(a_1\), 0 as the empty region and 1 as the whole space. The satisfiability problem for \textsc{BRCC8} over the class of structures of the form RC\((T)\) is still NP-complete; however, restricting attention to connected spaces \(T\) yields a PSPACE-complete satisfiability problem. Thus, \textsc{BRCC8}, unlike \textsc{RCC8}, has sufficient expressive power to distinguish between satisfiability over arbitrary spaces and satisfiability over connected spaces. But that is about as far as this extra expressive power takes us: satisfiability of a \textsc{BRCC8}-formula over any structure RC\((T)\), for \(T\) connected, implies satisfiability over RCP\((\mathbb{R}^n)\) for all \(n \geq 1\). Hence, the satisfiability problems for \textsc{RC}(\mathbb{R}^n) and RCP(\mathbb{R}^n), for all \(n \geq 1\), coincide, and are PSPACE-complete. Note in particular that \textsc{BRCC8} does not enable us to say that a given region of space is connected.
We end this discussion of RCC8 and BRCC8 with a remark on the absence of quantification from these languages. This restriction is motivated by computability considerations: all reasonable region-based spatial logics with full first-order syntax interpreted over $\mathbb{R}^n$ ($n \geq 2$) have undecidable satisfiability problems, and so are considered unsuitable for Qualitative Spatial Reasoning [Grzegorczyk 1951; Dornheim 1998; Davis 2006; Lutz and Wolter 2006]. To be sure, first-order spatial logics are nevertheless of considerable model-theoretic interest; see [Pratt-Hartmann 2007] for a survey. We note in particular that, if we can quantify over regions, then the RCC8-primitives easily enable us to define, over most interesting classes of interpretations, all of the primitives $+, \cdot, -$, 0, 1, $c$ and $c^\circ$. However, as computability considerations are to the fore in this paper, we too confine ourselves to quantifier-free formalisms in the sequel.

1.2. Algebraic and modal logic

The standard view of topology takes a topological space to consist of a set of points on which a collection of open subsets is defined. However, a dual view is possible, in which one begins with a Boolean algebra, and then adds algebraic structure defining distinctively topological relations between its objects. There are two main approaches to developing this second view. On the first, we think of the underlying Boolean algebra as a field of sets, and we augment this Boolean algebra with a pair of unary operators, conceived of as representing the operations of closure and interior, and assumed to obey the standard Kuratowski axioms [McKinsey and Tarski 1944]. The striking similarity between these axioms and the axioms for the propositional modal logic $S4$ [Orlov 1928; Gödel 1933] led to the development of topological semantics for modal logics. Under this semantics, the (propositional) variables are taken to range over any collection of subsets of a topological space (not just regular closed sets), and the logical connectives are interpreted by the operations of union, intersection, complement and topological interior (for necessity) and closure (for possibility). The extension of this language with the universal modality, denoted $S4_u$ [Goranko and Passy 1992], is known to be a super-logic for RCC8 and BRCC8 [Bennett 1994; Renz and Nebel 1997; Cohn et al. 1997; Nutt 1999; Wolter and Zakharyaschev 2000]. The satisfiability problem for $S4_u$ is the same over every connected, separable, dense-in-itself metric space, and this problem is PSPACE-complete [McKinsey and Tarski 1944; Shehtman 1999; Areces et al. 2000]. We remark that, as for RCC8 and BRCC8, $S4_u$ is unable to express the condition that a region is connected. For a survey of the relationship between spatial and modal logics see [van Benthem and Bezhanishvili 2007; Gabelaia et al. 2005; Kontchakov et al. 2008b] and references therein.

On the second approach, we instead think of the underlying Boolean algebra as an algebra of regular closed sets, and we augment this algebra with a binary predicate $C$, conceived of as representing the relation of contact. (Two sets are said to be in contact if they have a non-empty intersection). This binary predicate is assumed to satisfy the axioms of contact algebras, a category which is known to be dual to the category of dense sub-algebras of regular closed algebras of topological spaces [Düntsch and Winter 2005; Dimov and Vakarelov 2006a; 2006b; Balbiani et al. 2007; Tinchev and Vakarelov 2010; Vakarelov 2007]. The contact relation as a basis for topology actually has a venerable career, having originally been introduced in [Whitehead 1929] under the name ‘extensive connection.’ More relevantly for the present paper, it is straightforward to show that all the RCC8 relations can be expressed, in purely propositional terms, using this signature [Balbiani et al. 2007; Kontchakov et al. 2008b]. (Thus, for example, $EC(\tau_1, \tau_2)$ is equivalent to $C(\tau_1, \tau_2) \land (\tau_1 \cdot \tau_2 = 0)$.) For this reason, we regard the propositional language over the signature $(+, \cdot, -0, 1, C)$, here denoted $C$, as equivalent to the language BRCC8 mentioned above. The purely Boolean fragment of $C$ (without the contact predicate $C$) is denoted by $B$. This language is in fact equivalent to the extension of the spatial logic RCC5 [Bennett 1994] with $+$, $\cdot$ and $-$. 

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1.3. Spatial logics with connectedness

Many spatial regions of interest—plots of land in a cadastre, the space occupied by physical objects, paths swept out by moving objects—are either connected or at least contain few connected components [Cohn and Renz 2008]. It seems, therefore, that to be genuinely useful, logics for Qualitative Spatial Reasoning should possess some means of expressing this notion. The simplest way of proceeding is to consider languages featuring a unary predicate denoting this property. Various such languages have been investigated before [Pratt-Hartmann 2002; Kontchakov et al. 2008a; 2010a; Vakarelov 2007; Tinchev and Vakarelov 2010]; the language $\mathcal{B}c$ is chosen for study here because it is so parsimonious.

It is worth bearing in mind, however, that ‘connectedness,’ in the topologists’ sense may not be exactly what we want. For example, a region consisting of two closed discs externally touching is, in this sense, connected, yet, in certain contexts, may be functionally equivalent to a disconnected region. (Imagine having a garden that shape.) In such contexts, it may be more useful to employ the notion of a region’s having a connected interior, a property we refer to as interior-connectedness. Note that every regular closed, interior-connected set is connected; also, in the space $\mathbb{R}$, connectedness and interior-connectedness coincide. So as not to prejudice the issue here, we employ predicates for both notions: $c$ to denote the standard property of connectedness, $c^\circ$ to denote the property of interior-connectedness. Hence, in addition to the ‘minimal’ language $\mathcal{B}c$, we have its counterpart $\mathcal{B}c^\circ$.

Strikingly, the languages $\mathcal{RCC}8$ and $\mathcal{BRCC}8$, which cannot represent connectedness (or, for that mater, interior-connectedness), are far less sensitive to the underlying geometrical interpretation than the languages $\mathcal{B}c$ and $\mathcal{B}c^\circ$, which can. For example, an $\mathcal{RCC}8$-formula that is satisfiable over the regular closed algebra of any topological space is satisfiable over $\text{RCP}(\mathbb{R}^n)$, for all $n \geq 1$ [Renz 1998]. Or again, a $\mathcal{BRCC}8$-formula that is satisfiable over the regular closed algebra of any connected topological space is satisfiable over $\text{RCP}(\mathbb{R}^n)$, for all $n \geq 1$ [Wolter and Zakharyaschev 2000]. Thus, $\mathcal{RCC}8$ and $\mathcal{BRCC}8$ care neither about the dimension of the (Euclidean) space we are reasoning about, nor about the distinction between arbitrary regular closed sets and polyhedra. Not so with the languages $\mathcal{B}c$ or $\mathcal{B}c^\circ$, which are sensitive both to the dimension of space and to the restriction to polyhedra. This sensitivity is easy to demonstrate for $\mathcal{B}c^\circ$, and we briefly do so here, by way of illustration.

Consider first sensitivity to dimension. The $\mathcal{B}c^\circ$-formula

$$\bigwedge_{1 \leq i \leq 3} (c^\circ(r_i) \land (r_i \neq 0)) \land \bigwedge_{1 \leq i < j \leq 3} (c^\circ(r_i + r_j) \land (r_i \cdot r_j = 0))$$

(1)

says that $r_1$, $r_2$ and $r_3$ are non-empty regions with connected interiors, such that each forms an interior-connected sum with the other two, without overlapping them. It is obvious that this formula is not satisfiable over $\text{RC}(\mathbb{R})$. For the non-empty, (interior-) connected regular closed sets on the real line are precisely the non-punctual, closed intervals, and it is impossible for three such intervals to touch each other without overlapping. On the other hand, (1) is easily seen to be satisfiable over $\text{RC}(\mathbb{R}^n)$ for all $n \geq 2$. Likewise, the $\mathcal{B}c^\circ$-formula

$$\bigwedge_{1 \leq i \leq 5} (c^\circ(r_i) \land (r_i \neq 0)) \land \bigwedge_{1 \leq i < j \leq 5} (c^\circ(r_i + r_j) \land (r_i \cdot r_j = 0)),$$

(2)

which makes the analogous claim for regions $r_1, \ldots, r_5$, is not satisfiable over $\text{RC}(\mathbb{R}^2)$, since any satisfying assignment would permit a plane drawing of the graph $K_5$. On the other hand, (2) is easily seen to be satisfiable over $\text{RC}(\mathbb{R}^n)$ for all $n \geq 3$. Thus, the satisfiability problems for $\mathcal{B}c^\circ$ over $\text{RC}(\mathbb{R})$, $\text{RC}(\mathbb{R}^2)$ and $\text{RC}(\mathbb{R}^3)$ are all different. (We shall see in Sec. 6, however, that the satisfiability problem for $\mathcal{B}c^\circ$ over $\text{RC}(\mathbb{R}^n)$ is the same for all $n \geq 3$.)
Consider next sensitivity to the restriction to polyhedra. The $Bc^o$-formula

$$\bigwedge_{1 \leq i \leq 3} c^o(r_i) \land c^o(r_1 + r_2 + r_3) \land \bigwedge_{i=2,3} -c^o(r_1 + r_i)$$

(3)

is satisfiable over $RC(\mathbb{R}^2)$, as we see from the regular closed sets in Fig. 2, where $r_2$ and $r_3$ lie, respectively, above and below the graph of the function $\sin \frac{1}{x}$ on the interval $(0, 1]$. By contrast, formula (3) is unsatisfiable over $RCP(\mathbb{R}^n)$ for all $n \geq 1$ [Pratt and Lemon 1997, p. 231]. Actually, the result can be sharpened: (3) is unsatisfiable over any Boolean sub-algebra of $RC(\mathbb{R}^n)$ whose regions all satisfy a form of the curve selection lemma from real algebraic geometry [Bochnak et al. 1998]. As we might say, in dimensions 2 and above, $Bc^o$ is sensitive to the presence of ‘non-tame’ regions. And since—at least conceivably—non-tame regions may be thought implausible models of the space occupied by any physical objects—it is natural to consider satisfiability of $Bc^o$-formulas over $RCP(\mathbb{R}^n)$ rather than over $RC(\mathbb{R}^n)$.

The language $Bc$ is similarly sensitive to the dimension of the Euclidean space over which it is interpreted, and also to the restriction to polyhedra. For dimensionality, this sensitivity can be demonstrated by examples similar to (1) and (2); see [Kontchakov et al. 2008b]. For the restriction to polyhedra, this result follows from Sec. 3, where we show that there exists a $Bc$-formula satisfiable in $RC(\mathbb{R}^n)$ for all $n \geq 2$, but only by tuples of regions having infinitely many connected components!

1.4. Plan of the paper and summary of results

The remainder of this paper is organized as follows. Sec. 2 defines the syntax and semantics of $Bc$ and $Bc^o$. To simplify proofs, we also employ the more expressive languages $Cc$ and $Cc^o$, obtained by adding the predicates $c$ and $c^o$, respectively, to $C (= BRCC8)$. In Sec. 3, we prove that there exist $Cc$, $Cc^o$- and $Bc$-formulas satisfiable over $RC(\mathbb{R}^n)$, for all $n \geq 2$, but only by tuples of regions some of which have infinitely many connected components, and hence which cannot belong to $RCP(\mathbb{R}^n)$. By further developing the ideas encountered in this proof, we show in Sec. 4 that $Cc$, $Cc^o$ and $Bc$ (but not $Bc^o$) are r.e.-hard over $RCP(\mathbb{R}^n)$, for all $n \geq 2$. Using a different approach, we show in Sec. 5 that all four of our logics—$Bc$, $Bc^o$, $Cc$ and $Cc^o$—are r.e.-hard over both $RCP(\mathbb{R}^2)$ and $RC(\mathbb{R}^2)$. Finally, we show in Sec. 6 that $Bc^o$ is NP-complete over $RC(\mathbb{R}^n)$, and EXPTime-complete over $RCP(\mathbb{R}^n)$, for all $n \geq 3$. The decidability of satisfiability for $Cc$, $Cc^o$ and $Bc$ over $RC(\mathbb{R}^n)$, for all $n \geq 3$, is left open.

2. PRELIMINARIES

We begin by formally defining the syntax and semantics of the topological logics considered in this paper. This section also contains the basic technical definitions and results we need in what follows.
2.1. Basic topological notions

A topological space is a pair \((T, \mathcal{O})\), where \(T\) is a set and \(\mathcal{O}\) a collection of subsets of \(T\) containing \(\emptyset\) and \(T\), and closed under arbitrary unions and finite intersections. The elements of \(\mathcal{O}\) are referred to as open sets; their complements are closed sets. If \(\mathcal{O}\) is clear from context, we refer to the topological space \((T, \mathcal{O})\) simply as \(T\). If \(X \subseteq T\), the closure of \(X\), denoted \(\overline{X}\), is the smallest closed set including \(X\), and the interior of \(X\), denoted \(X^o\), is the largest open set included in \(X\). These sets always exist. The boundary of \(X\), denoted \(\delta X\), is the set \(\overline{X} \setminus X^o\). The Euclidean space \(\mathbb{R}^n\) is assumed always to have the usual metric topology. We may treat any subset \(X \subseteq T\) as a topological space in its own right by defining the subspace topology on \(X\) to be the collection of sets \(O_X = \{O \cap X \mid O \in \mathcal{O}\}\).

We call \(X\) regular closed if it is the closure of an open set—equivalently, if \(X = (X^o)^c\). We denote by \(RC(T)\) the set of regular closed subsets of \(T\). It is a standard result that \(RC(T)\) forms a complete Boolean algebra, with operations \(\sum A = (\bigcup A)^c\), \(\prod A = (\bigcap A)^c\), and \(-X = (T \setminus X)^c\) (see, e.g., [Koppelberg 1989]). The partial order induced by this Boolean algebra is simply \((T, \subseteq)\); we often write \(X \leq Y\) in preference to \(X \subseteq Y\) where \(X\) and \(Y\) are regular closed. Note that, if \(A = \{X_1, X_2\}\), then \(\sum A = X_1 + X_2 = X_1 \cup X_2\).

A topological space \(T\) is said to be connected if it cannot be decomposed into two disjoint, non-empty closed sets; likewise, \(X\) is connected if it is a connected space under the subspace topology. We call \(X\) interior-connected if \(X^o\) is connected. A maximal connected subset of \(X\) will be called a component of \(X\) (some authorities prefer the term connected component). The following facts are easily verified: every non-empty connected subset of \(X\) is included in a unique component of \(X\); every component of a closed set is closed.

The space \(T\) is said to be locally connected if every neighbourhood of any point of \(T\) includes a connected neighbourhood of that point (a neighbourhood of a point \(p\) is a set \(X\) that includes an open set containing \(p\)). In a locally connected space, every component of an open set is open; note however that components of regular closed sets are closed but, in general, not regular closed, even in locally connected spaces. The space \(T\) is said to be unicoherent if, for any closed, connected subsets \(X_1, X_2\) such that \(T = X_1 \cup X_2\), the set \(X_1 \cap X_2\) is connected. For all \(n \geq 1\), the Euclidean space \(\mathbb{R}^n\) is (obviously) locally connected and (much less obviously) unicoherent [Kuratowski 1928]. A simple example of a non-locally connected space is the rational numbers \(\mathbb{Q}\) under the usual metric topology. Simple examples of non-unicoherent spaces are the Jordan curve and the torus.

The most important properties of local connectedness and unicoherence, from our point of view, are given by the following lemmas.

**Lemma 2.1.** Let \(X\) be a regular closed subset of a topological space \(T\) and \(S\) a component of \(-X\). If \(-X\) has finitely many components, then \(\delta S \subseteq X\). Alternatively, if \(T\) is locally connected, then \(\delta S \subseteq X\).

**Proof.** For the first statement, let \(Z\) be the union of all components of \(-X\) other than \(S\). By definition, \(T = X^o \cup S \cup Z\). Further, both \(S \cap X^o\) and \(S \cap Z\) are empty, whence \(T \setminus S = X^o \cup Z\). Since \(X\) is regular closed and \(Z\) is closed (as the union of finitely many closed sets), \((T \setminus S)^c = X \cup Z\). Finally, since \(S\) is closed, and \(S \cap Z = \emptyset\), \(\delta S = S \cap (T \setminus S)^c \subseteq X\). For the second statement, suppose, to the contrary, that \(\delta S\) contains a point \(p\) lying in \((-X)^o\). Since \(S\) is closed, \(p \in S\). By local connectedness, let \(Y\) be a connected open set such that \(p \in Y \subseteq (-X)^o\). Since \(p \in S\) and \(Y\) is a connected subset of \(-X\), we have \(p \in Y \subseteq S\). But this contradicts the assumption that \(p \in \delta S\).

**Lemma 2.2.** Let \(T\) be a unicoherent space and \(X \in RC(T)\) be connected. Then every component of \(-X\) has a connected boundary.

**Proof.** Let \(S\) be a connected component of \(-X\), and let \(Z\) be the union of all components of \(-X\) other than \(S\). Thus, \(T \setminus S = Z \cup X^o\). We write \(S^* = (T \setminus S)^c\). Since \(X\) is...
regular closed, $S^* = Z^* \cup X$. By connectedness of $T$, $X$ intersects every component of $-X$. 
It follows that $Z \cup X$, and hence $Z^* \cup X = S^*$ are connected. By definition, $S$ is connected, 
whence, by unicoherence of $T$, $\delta S = S \cap S^*$ is connected. \hfill \Box

2.2. Frames

A frame is a pair $(T, S)$, where $T$ is a topological space, and $S$ is a Boolean sub-algebra of 
$\text{RC}(T)$. Where $T$ is clear from context, we refer to $(T, S)$, simply, as $S$. Furthermore, where 
$S$ is clear from context, we refer to elements of $S$ as regions. We denote the class of frames of the form $(T, \text{RC}(T))$ by $\text{RC}$. Note that not all frames are of this form: in particular, when 
working in $n$-dimensional Euclidean spaces, we shall be principally interested in the following proper sub-algebra of $\text{RC}(\mathbb{R}^n)$. Any $(n-1)$-dimensional hyperplane bounds two elements of 
$\text{RC}(\mathbb{R}^n)$ called half-spaces; a polyhedron in $\mathbb{R}^n$ is an element of the Boolean sub-algebra of 
$\text{RC}(\mathbb{R}^n)$ generated by the half-spaces, or, equivalently, a finite union of finite intersections of 
half-spaces. We denote this collection of sets by $\text{RCP}(\mathbb{R}^n)$, and, in the case $n = 2$, we speak 
of polygons, rather than polyhedra. Note that the polyhedra in $\mathbb{R}^n$ may be alternatively 
characterized as the regular closed semilinear sets. Polyhedra will be regarded as ‘well-behaved’ or, in topologists’ parlance, ‘tame.’ We call $(T, S)$ unicoherent if $T$ is unicoherent, 
and finitely decomposable if, for all $s \in S$, there exist connected elements $s_1, \ldots, s_k$ of $S$, 
such that $s = s_1 + \cdots + s_k$. Evidently, $(\mathbb{R}^n, \text{RCP}(\mathbb{R}^n))$ is finitely decomposable, since any 
product of half-spaces is connected. Equally obviously:

**Lemma 2.3.** Suppose the frame $(T, S)$ is finitely decomposable, and $s \in S$. Then every 
component of $s$ is in $S$, and $s$ is equal to the sum of those components.

The following basic concepts will be used repeatedly in the sequel. Let $(T, S)$ be a frame. 
a tuple $(s_0, \ldots, s_{k-1})$, where $k \geq 1$, will be called a partition, provided $s_0 + \cdots + s_{k-1} = 1$ 
and $s_i \cdot s_j = 0$ for $0 \leq i < j < k$. We do not insist that the $s_i$ are non-empty. We call a 
partition $(s_0, \ldots, s_{k-1})$ sub-cyclic if the $s_i$ are non-empty and $s_i \cap s_j = \emptyset$, for $0 \leq i, j < k$ 
such that $1 < j - i < k - 1$. The term ‘sub-cyclic’ refers to an imagined graph with nodes 
$(s_0, \ldots, s_{k-1})$ and edges $\{(s_i, s_j) \mid i \neq j$ and $s_i \cap s_j \neq \emptyset\}$; this graph is required to be a 
(not necessarily proper) subgraph of the cyclic graph on $(s_0, \ldots, s_{k-1})$.

Suppose $s$ is a non-empty element of a frame $(T, S)$, and $\vec{s} = (s_0, \ldots, s_{k-1})$ a partition in 
that frame. We say that $\vec{s}$ is a colouring of the components of $s$ if every component of $s$ is 
included in exactly one of the regions of $\vec{s}$. Colourings will be used repeatedly in the sequel, 
particularly in situations where we may regard the components of $s$ as positions in a finite 
sequence; by regarding the set of elements of $\vec{s}$ as an alphabet, colourings define words over 
that alphabet in the obvious way.

2.3. Topological logics

In this paper, the focus of attention is not on frames themselves, but rather, on frames as 
they are described in some language. The languages considered here all employ a countably 
infinite collection of variables $r_1, r_2, \ldots$. The language $C$ is defined by the following syntax:

$$
\begin{align*}
\tau &::= r \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2 \mid \neg \tau_1 \mid 0 \mid 1; \\
\varphi &::= \tau_1 = \tau_2 \mid C(\tau_1, \tau_2) \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \land \varphi_2 \mid \neg \varphi_1.
\end{align*}
$$

The language $B$ is defined analogously, but without the predicate $C$. Thus, $B$ is the 
quantifier-free language of the variety of Boolean algebras.

An interpretation over a frame $(T, S)$ is a function mapping variables $r$ to elements 
$r^3$ of $S$. We extend $\cdot$ to terms $\tau$ by setting $(\tau_1 + \tau_2)^3 = \tau_1^3 + \tau_2^3$, 
$(\tau_1 \cdot \tau_2)^3 = \tau_1^3 \cdot \tau_2^3$, 
$(-\tau_1)^3 = -(\tau_1^3)$, $0^3 = 0$ and $1^3 = T$. We write $\mathcal{I} \models \tau_1 = \tau_2$ if and only if $\tau_1^3 = \tau_2^3$, and 
$\mathcal{I} \models C(\tau_1, \tau_2)$ if and only if $\tau_1^3 \cap \tau_2^3 \neq \emptyset$, extending this relation to non-atomic formulas in 
the standard way. We read $C(\tau_1, \tau_2)$ as ‘$\tau_1$ contacts $\tau_2$.’ If $\varphi$ is a formula whose variables,
taken in some order, are \( r = (r_1, \ldots, r_n) \), and \( \mathcal{J} \models \varphi \), then the tuple \( \vec{a} = (r_1^2, \ldots, r_n^3) \) is said to satisfy \( \varphi (\vec{r}) \); in such a case, we will often say ‘\( \vec{a} \) satisfies \( \varphi (\vec{r}) \).

The property that a \( k \)-tuple forms a partition is expressible using the \( B \)-formula

\[
\mathrm{part}(r_0, \ldots, r_{k-1}) = \left( \sum_{0 \leq i < k} r_i = 1 \right) \land \bigwedge_{0 \leq i < j < k} (r_i \cdot r_j = 0).
\]

The property that a \( k \)-tuple forms a sub-cyclic partition is expressible using the \( C \)-formula

\[
\mathrm{sc-part}(r_0, \ldots, r_{k-1}) = \mathrm{part}(r_0, \ldots, r_{k-1}) \land \bigwedge_{0 \leq i < k} (r_i \neq 0) \land \bigwedge_{1 < j - i < k - 1} \neg C(r_i, r_j).
\]

And, assuming that \( \mathrm{part}(r_0, \ldots, r_{k-1}) \) is satisfied, the \( C \)-formula

\[
\mathrm{colour}(r; r_0, \ldots, r_{k-1}) = \bigwedge_{0 \leq i < j < k} \neg C((r \cdot r_i), (r \cdot r_j))
\]

ensures that the partition \( r_0, \ldots, r_{k-1} \) colours the components of \( r \). Conversely, over finitely decomposable frames, any colouring of \( r \) by a partition \( r_0, \ldots, r_{k-1} \) must satisfy \( \mathrm{colour}(r; r_0, \ldots, r_{k-1}) \).

Turning to connectedness predicates, we define the languages \( \mathcal{B}c \) and \( \mathcal{C}c \) to be extensions of \( \mathcal{B} \) and \( \mathcal{C} \), respectively, with the unary predicate \( c \). We set \( \mathcal{J} \models c(\tau) \) if and only if \( \tau^3 \) is connected in the topological space under consideration. Similarly, we define \( \mathcal{B}c^o \) and \( \mathcal{C}c^o \) to be extensions of \( \mathcal{B} \) and \( \mathcal{C} \) with the predicate \( c^o \), setting \( \mathcal{J} \models c^o(\tau) \) if and only if \( (\tau^3)^o \) is connected. If \( \mathcal{K} \) is a class of frames, and \( \mathcal{L} \) is one of \( \mathcal{B}c, \mathcal{C}c, \mathcal{B}c^o \) or \( \mathcal{C}c^o \), then \( \text{Sat}(\mathcal{L}, \mathcal{K}) \) is the set of \( \mathcal{L} \)-formulas satisfiable over \( \mathcal{K} \).

In the case \( \mathcal{K} = \mathcal{RC} \), the complexity of \( \text{Sat}(\mathcal{L}, \mathcal{K}) \) is known for all of the languages \( \mathcal{L} \) considered above [Kontchakov et al. 2010a; 2010b]. If \( \varphi \) is a formula of any of the languages \( \mathcal{B}c, \mathcal{C}c \) or \( \mathcal{C}c^o \), and \( \varphi \) is satisfiable over some frame \( \mathcal{RC}(T) \), where \( |T| \), the cardinality of \( T \), is bounded by an exponential function of \( |\varphi| \); and the problems \( \text{Sat}(\mathcal{L}, \mathcal{RC}) \), for \( \mathcal{L} \in \{ \mathcal{B}c, \mathcal{C}c, \mathcal{C}c^o \} \), are all \text{ExpTime}-complete. On the other hand, if \( \psi \) is a \( \mathcal{B}c^o \)-formula satisfiable over \( \mathcal{RC} \), then \( \psi \) is satisfiable over some frame \( \mathcal{RC}(T) \), where \( |T| \) is polynomial in \( |\psi| \); and the problem \( \text{Sat}(\mathcal{B}c^o, \mathcal{RC}) \) is \text{NP}-complete. Thus, we observe a difference between \( \mathcal{B}c, \mathcal{C}c \) and \( \mathcal{C}c^o \) on the one hand, and \( \mathcal{B}c^o \) on the other.

However, satisfiability over \( \mathcal{RC} \) is of little interest from the point of view of spatial logic in \( \mathcal{AI} \), where the majority of applications concern the frames over Euclidean space of dimensions 2 or 3. Accordingly, we shall be concerned with \( \text{Sat}(\mathcal{L}, \mathcal{K}) \), where \( \mathcal{L} \) is any of \( \mathcal{B}c, \mathcal{B}c^o, \mathcal{C}c \) or \( \mathcal{C}c^o \), and \( \mathcal{K} \) is \( \{ \mathcal{RC}(\mathbb{R}^n) \} \) or \( \{ \mathcal{RC}(\mathbb{R}^n) \} \) for \( n \geq 2 \). For ease of reading, we write \( \text{Sat}(\mathcal{L}, \mathcal{RC}(\mathbb{R}^n)) \) and \( \text{Sat}(\mathcal{L}, \mathcal{RC}(\mathbb{R}^n)) \) rather than \( \text{Sat}(\mathcal{L}, \{ \mathcal{RC}(\mathbb{R}^n) \}) \) and \( \text{Sat}(\mathcal{L}, \{ \mathcal{RC}(\mathbb{R}^n) \}) \).

### 2.4. Graphs

Unless explicitly indicated to the contrary, all graphs in this paper are taken to be finite, and to have no multiple edges and no self-loops: i.e., if \( G = (V, E) \) is a graph, \( (v, v') \in E \) if and only if \( (v', v) \in E \). A path in \( G \) is a sequence of distinct vertices \( v_0, \ldots, v_{n-1} \) such that \( (v_i, v_{i+1}) \) is an edge, for all \( 0 \leq i < n - 1 \); further, a cycle in \( G \) is a path \( v_0, \ldots, v_{n-1} \), with \( n \geq 3 \), such that, in addition, \( (v_{n-1}, v_0) \) is an edge. Informally, in this case, we speak of the sequence \( v_0, \ldots, v_{n-1}, v_0 \) as a cycle. A graph is connected if any two nodes are joined by some path; a graph which contains no cycles is acyclic; and a connected, acyclic graph is a tree. If \( G \) is a tree, then any pair of nodes in \( G \) is joined by a unique path. Further, if \( v_0, \ldots, v_{n-1} \) is a sequence of nodes in a tree such that \( (v_i, v_{i+1}) \) is an edge for \( 0 \leq i < n - 1 \), and \( v_i \neq v_{i+2} \) for \( 0 \leq i < n - 2 \), then this sequence contains no duplicates, and thus is a path.

Let \( S \) be a finitely decomposable frame over some topological space, and \( \vec{s} \) a connected partition in \( S \). We can associate a graph with \( \vec{s} \), denoted \( H(\vec{s}) \), as follows: the vertices of
$H(\vec{s})$ are the components of the elements of $\vec{s}$; the edges of $H(\vec{s})$ are the pairs $(X,Y)$ such that $X \neq Y$ and $X \cap Y \neq \emptyset$. We refer to $H(\vec{s})$ as the component graph of $\vec{s}$. Note that the number of vertices of $H(\vec{s})$ is in general larger than the number of elements in $\vec{s}$; however, since $\mathcal{S}$ is finitely decomposable, this number is still finite.

We prove a simple but powerful lemma connecting some of the notions encountered above.

**Lemma 2.4.** Let $T$ be a unicoherent topological space, $\mathcal{S}$ a finitely decomposable frame on $T$, and $\vec{s}$ a sub-cyclic partition in $\mathcal{S}$. Then the component graph, $H(\vec{s})$, is a tree.

**Proof.** Write $\vec{s} = (s_0, \ldots, s_{n-1})$. Since $\mathcal{S}$ is finitely decomposable, and $T$ is connected, $H(\vec{s})$ is obviously finite and connected. We need only show that it contains no cycles. If $n = 1$, then $|H(\vec{s})| = 1$, and this is trivial. We assume, for ease of formulation, that $n \geq 4$, since a similar (and in fact simpler) argument applies if $n = 2$ or $n = 3$.

Suppose $(X_0, X_1)$ is an edge of $H(\vec{s})$. We may assume, without loss of generality, that $X_0$ is a component of $s_0$, and $X_1$ a component of $s_1$. The sub-cyclicity condition ensures that $s_0 \cap s_i = \emptyset$ for $2 \leq i < n-1$, and $s_1 \cap s_i = \emptyset$ for $3 \leq i < n$. Now let $S$ be the component of $-X_1$ containing $X_0$: we claim that $\delta S \subseteq s_0$. By the first statement of Lemma 2.1, $\delta S \subseteq X_1 \subseteq s_1$, whence $\delta S$ contains no point of $s_3 + \cdots + s_{n-1}$. On the other hand, $\delta S$ is obviously included in $-s_1 = s_0 + s_2 + s_3 + \cdots + s_{n-1}$, and hence in $s_0 + s_2$. Since $s_0 \cap s_2 = \emptyset$, and, by Lemma 2.2, $\delta S$ is connected, we have either $\delta S \subseteq s_0$ or $\delta S \subseteq s_2$. Now, since $(X_0, X_1)$ is an edge of $H(\vec{s})$, and any point of $X_0 \cap X_1$ must lie in both $S$ and $-S$, we have $\delta S \cap X_0 \neq \emptyset$, and, therefore, $\delta S \subseteq s_0$, as claimed.

Now suppose $(X_1, X_2)$ is also an edge of $H(\vec{s})$, with $X_0$ and $X_2$ distinct. We claim that $X_0$ and $X_2$ lie in different components of $-X_1$, (i.e., $X_2 \not\subseteq S$). For suppose otherwise. Again, since any point of $X_1 \cap X_2$ lies in both $S$ and $-S$, $\delta S \cap X_2 \neq \emptyset$. Furthermore, since $s_1 \cap s_i = \emptyset$ for $3 \leq i < n$, $X_2$ must be a component of either $s_0$ or $s_2$. But if $X_2 \subseteq s_0$, then the connected set $\delta S \subseteq s_0$ has points in common with the components $X_0, X_2$ of $s_0$, contradicting the assumption that $X_0$ and $X_2$ are distinct. On the other hand, if $X_2 \subseteq s_2$ then $\delta S \subseteq s_2$, which is again impossible.

Finally, suppose that $X_0, X_1, X_2, \ldots, X_m$ is a cycle in $H(\vec{s})$, where $m \geq 3$ and $X_m = X_0$. Then the connected set $X_2 + \cdots + X_{m-1} + X_0$ lies entirely in $-X_1$, contradicting the fact that $X_0$ and $X_2$ lie in different components of $-X_1$. \qed

2.5. Post correspondence problem

In the sequel, we make use of the well-known Post correspondence problem (PCP). Fix finite alphabets $T$ and $U$, where $|T| \geq 7$ and $|U| \geq 2$. A morphism from $T$ to $U$ is a function $w: T \to U^*$ mapping each element of $T$ to a word over $U$. We extend $w$ to a mapping $w^*: T^* \to U^*$ by defining, for any word $\tau = t_1 \cdots t_k \in T^*$, $w(\tau) = w(t_1) \cdots w(t_k)$. An instance of the PCP is a pair of morphisms $W = (w^1, w^2)$ from $T$ to $U$. The instance $W$ is positive if there exists a non-empty word $\tau \in T^*$ such that $w^1(\tau) = w^2(\tau)$. Intuitively, we are invited to think of each element of $T$ as a ‘tile’ inscribed with an ‘upper’ word over $U$, given by $w^1(t)$, and a ‘lower’ word over $U$, given by $w^2(t)$; we are asked to determine, for the given collection of tiles, whether there exists a non-empty, finite sequence of these tiles (repeats allowed) such that the concatenation of their upper words equals the concatenation of their lower words. The set of positive PCP instances is known to be r.e.-complete [Post 1946], and remains so even under the restriction that $w^k(t)$ is non-empty for every $t \in T$. In fact, nothing hinges on the exact choice of $T$ and $U$, subject to the restrictions mentioned above. In particular, we may assume $T$ and $U$ are disjoint.

3. FORCING INFINITELY MANY COMPONENTS IN LOCALLY CONNECTED UNICOHERENT SPACES

In this section, we construct $\varphi^c_\mathcal{C}$, $\varphi^{c^2}_\mathcal{C}$, and $\mathcal{B}_\varphi$-formulas $\varphi$ with the following properties: (i) $\varphi$ is satisfiable over $\mathcal{R}^n$ for all $n \geq 2$; (ii) if $T$ is a locally connected, unicoherent space
and \( \bar{r} \) is a tuple from \( \text{RC}(T) \) satisfying \( \varphi \), then \( \bar{r} \) includes members with \textit{infinitely many} connected components. Since \( \text{RCP}(\mathbb{R}^n) \) is finitely decomposable, these properties entail that \( \text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^n)) \neq \text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^n)) \) for \( \mathcal{L} \) any of \( Cc, Cc^\circ \) or \( Bc \), and all \( n \geq 2 \). Furthermore, the techniques developed in this section will be used in Sec. 4 to prove that satisfiability of \( Cc, Cc^\circ \) and \( Bc \)-formulas over \( \text{RCP}(\mathbb{R}^n) \), for \( n \geq 2 \), is undecidable.

We now construct a \( Cc \)-formula, \( \varphi_\infty \) with properties (i) and (ii). The first conjunct of \( \varphi_\infty \) states that \( r_0, r_1, r_2, r_3 \) form a sub-cyclic partition:

\[
\text{sc-part}(r_0, r_1, r_2, r_3). \tag{4}
\]

We also require non-empty sub-regions \( r'_i \) of \( r_i \) and a non-empty region \( t \):

\[
\bigwedge_{0 \leq i < 4} ( (r'_i \neq 0) \land (r'_i \leq r_i) ) \land (t \neq 0). \tag{5}
\]

As an aid to intuition, take \( T \) to be the Euclidean plane \( \mathbb{R}^2 \), and consider the configuration depicted in Fig. 3, where components of the \( r_i \) are arranged like the layers of an onion (\( r'_i \) is included in \( r_i \)). The ‘innermost’ component of \( r_0 \) is surrounded by a component of \( r_1 \), which in turn is surrounded by a component of \( r_2 \), and so on. The region \( t \) passes through every layer, but avoids the \( r'_i \). To enforce a configuration of this sort, we need the following formulas. Throughout this section, we write \( [i] \) to denote the value of \( i \) modulo 4.

\[
\bigwedge_{0 \leq i < 4} c(r'_i + r_{[i+1]} + t), \tag{6}
\]

\[
\bigwedge_{0 \leq i < 4} -C(r'_i, t), \tag{7}
\]

\[
\bigwedge_{0 \leq i < 4} -C(r'_i, r_{[i+1]} \cdot (r'_{[i+1]})). \tag{8}
\]

Observe that (6)–(8) ensure each component of \( r'_i \) is in contact with \( r'_{[i+1]} \). Denote by \( \varphi_\infty \) the conjunction of (4)–(8).

**Theorem 3.1.** The \( Cc \)-formula \( \varphi_\infty \) is satisfiable over \( \text{RC}(\mathbb{R}^n) \), \( n \geq 2 \). On the other hand, if \( T \) is a locally connected, unicoherent space, then any tuple from \( \text{RC}(T) \) satisfying \( \varphi_\infty \) features sets that have infinitely many components.

**Proof.** Fig. 3 shows how \( \varphi_\infty \) can be satisfied over \( \text{RC}(\mathbb{R}^2) \). By cylindrification, it is also satisfiable over any \( \text{RC}(\mathbb{R}^n) \), for \( n > 2 \). This establishes the first statement of the lemma. For the second statement, we suppose that \( \varphi_\infty \) is satisfied by a tuple \( r_0, \ldots, r_3, r'_0, \ldots, r'_3, t \) over \( \text{RC}(T) \), where \( T \) is a locally connected, unicoherent space. Our strategy is to prove that this tuple looks approximately like the arrangement in Fig. 3. More precisely, we construct a sequence \( X_0, X_1, \ldots \) of elements of \( \text{RC}(T) \), such that: (i) \( X_i \) is a component of \( r'_{[i]} \); and (ii) each \( X_i \) separates any \( X_j \) \( (j < i) \) from any \( X_k \) \( (k > i) \). It follows that each \( r_i \) has infinitely many

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many components. Note however that these regions may in general be badly-behaved. In particular, $r_0, \ldots, r_3$ may have other components in addition to the $X_i$.

We construct the sequence of components $X_i$ of $r_{ij}$ together with open sets $Z_i$ connecting $X_i$ to $X_{i+1}$; see Fig. 4. By the first conjunct of (5), let $X_0$ be a component of $r_0$ containing points in $r_0'$. Suppose $X_i$ has been constructed. By (6)–(8), $X_i$ is in contact with $r'_{i+1}$. By (4) and local connectedness of $T$, one can find a component $X_{i+1}$ of $r_{i+1}$ which has points in $r'_{i+1}$, and a connected open set $Z_i$ such that $Z_i \cap X_i$ and $Z_i \cap X_{i+1}$ are non-empty, but $Z_i \cap r'_{i+2}$ is empty.

To see that the $X_i$ are distinct, let $S_{i+1}$ and $R_{i+1}$ be the components of $-X_{i+1}$ containing $X_i$ and $X_{i+2}$, respectively. It suffices to show that we have $S_{i+1} \subseteq S_{i+2}^{c}$. Note that the connected set $Z_i$ must intersect $\delta S_{i+1}$. By the second statement of Lemma 2.1, $\delta S_{i+1} \subseteq X_{i+1} \subseteq r'_{i+1}$. Also, $\delta S_{i+1} \subseteq -X_{i+1}$; hence, by (4), $\delta S_{i+1} \subseteq r_{i+1} \cup r'_{i+2}$. By Lemma 2.2, $\delta S_{i+1}$ is connected, and therefore, by (4), $\delta S_{i+1}$ is entirely contained either in $r_{i+1}$ or in $r'_{i+2}$. Since $Z_i \cap \delta S_{i+1} \neq \emptyset$ and $Z_i \cap r'_{i+2} = \emptyset$, we have $\delta S_{i+1} \subseteq r_{i+1}$, so $\delta S_{i+1} \subseteq r_{i+1}$. Similarly, $\delta R_{i+1} \subseteq r'_{i+2}$. By (4), then, $\delta S_{i+1} \cap \delta R_{i+1} = \emptyset$, and since $S_{i+1}$ and $R_{i+1}$ are components of the same set, and have non-empty boundaries, they are disjoint. Hence, we obtain $S_{i+1} \subseteq (-R_{i+1})^{c}$, and since $X_{i+2} \subseteq R_{i+1}$, also $S_{i+1} \subseteq (-X_{i+2})^{c}$. So, using local connectedness again, $S_{i+1}$ lies in the interior of a component of $-X_{i+2}$, and since $\delta S_{i+1} \subseteq X_{i+1} \subseteq S_{i+2}$, that component must be $S_{i+2}$. \[\Box\]

Now we show how the $Cc$-formula $\varphi_{\infty}$ can be transformed to $Cc^{\infty}$- and $Bc$-formulas with similar properties. Note first that all occurrences of $c$ in $\varphi_{\infty}$ have positive polarity. Let $\varphi_{\infty}^{c}$ be the result of replacing them with the predicate $c^{\infty}$. In Fig. 3, the connected regions mentioned in (6) are in fact interior-connected; hence $\varphi_{\infty}^{c}$ is satisfiable over $RC(\mathbb{R}^n)$, $n \geq 2$. Since interior-connectedness implies connectedness, $\varphi_{\infty}^{c}$ entails $\varphi_{\infty}$, and we obtain:

**COROLLARY 3.2.** The $Cc^{\infty}$-formula $\varphi_{\infty}^{c}$ is satisfiable over $RC(\mathbb{R}^n)$, $n \geq 2$. On the other hand, if $T$ is a locally connected, unicoherent space, then any tuple from $RC(T)$ satisfying $\varphi_{\infty}^{c}$ features sets that have infinitely many components.

We next consider the language $Bc$. Observe that all occurrences of $C$ in $\varphi_{\infty}$ are negative. We eliminate these using the predicate $c$: we use the fact that, if the sum of two connected regions is not connected, then they are not in contact. If $\tau_1$ and $\tau_2$ are any terms, we employ the abbreviation

$$notC(\tau_1, \tau_2) = c(\tau_1) \land c(\tau_2) \land \neg c(\tau_1 + \tau_2).$$

Observe that $notC(\tau_1, \tau_2)$ is a $Bc$-formula. Furthermore, $notC(\tau_1, \tau_2)$ implies $\neg C(\tau'_1, \tau'_2)$ for any $\tau'_1 \leq \tau_1$ and $\tau'_2 \leq \tau_2$. Now we replace (7) by

$$notC(r'_0 + r'_1 + r'_2 + r'_3, t).$$

(7c)
The resulting formula thus implies the original; on the other hand, it is satisfied by the configuration of Fig. 3. Next, we replace each conjunct \( \neg C(r'_i, r'_{i+1} \cdot (-r'_{i+1})) \) in (8) by

\[
\text{notC}(r'_i + s, r_{i+1} \cdot (-r'_{i+1}) + t),
\]

where \( s \) is a fresh variable. Again, the resulting formula implies the original, and, furthermore, is evidently satisfied by the configuration of Fig. 5, where \( s \) lies inside \( \sum_{j=0}^{3} r'_j \), symmetrically to \( t \) lying inside \( \sum_{j=0}^{3} (r_j \cdot (-r'_j)) \). The only remaining occurrences of the contact predicate \( C \) are in (4). We deal with them by partitioning the regions: instead of each \( \neg C(r_i, r_{i+2}) \) we consider the equivalent conjunction of 4 formulas:

\[
\neg C(r'_i, r'_{i+2}) \land \neg C(r_i \cdot (-r'_i), r_{i+2} \cdot (-r'_{i+2})) \land \neg C(r_i \cdot (-r'_i), r'_{i+2}) \land \neg C(r'_i, r_{i+2} \cdot (-r'_{i+2})),
\]

which are then replaced by

\[
\text{notC}(r'_i + s_i, r'_{i+2} + s_{i+2}) \land \text{notC}(r_i \cdot (-r'_i) + t_i, r_{i+2} \cdot (-r'_{i+2}) + t_{i+2}) \land \\
\text{notC}(r_i \cdot (-r'_i) + t, r'_{i+2} + s) \land \text{notC}(r'_i + s, r_{i+2} \cdot (-r'_{i+2}) + t).
\]

Again, the resulting formula implies the original. The conjuncts of the second row are satisfied by the configuration of Fig. 5. To see that the conjuncts in the first row are still satisfiable, we select regions \( s_0, \ldots, s_3 \), with \( s_i \) and \( s_{i+2} \) disjoint \((i = 0, 1) \), such that each \( s_i \) \((0 \leq i < 4) \) connects together the components of \( r'_i \) as shown in Fig. 6. In a symmetric way, select regions \( t_0, \ldots, t_3 \), with \( t_i \) and \( t_{i+2} \) disjoint \((i = 0, 1) \), such that each \( t_i \) \((0 \leq i < 4) \) connects together the components of \( r_i \cdot (-r'_i) \). Transforming \( \varphi_\infty \) in the way just described, we obtain a \( Bc \)-formula \( \varphi_\infty \) with the required properties.

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Theorem 3.3. The Bc-formula $\varphi_s^\infty$ is satisfiable over $\text{RC}(\mathbb{R}^n)$, $n \geq 2$. On the other hand, if $T$ is a locally connected, unicoherent space, then any tuple from $\text{RC}(T)$ satisfying $\varphi_s^\infty$ features sets that have infinitely many components.

The results of this section make no reference to the language Bc$^c$. In fact, an analogue of Theorem 3.3 for Bc$^c$ will be proved in the special case $n = 2$, in Sec. 5.1, using a planarity argument. For $n \geq 3$, however, this result fails, as we show in Sec. 6. As we observed above, Theorem 3.3 shows that, for all $n \geq 2$, $\text{Sat}(\text{Bc, RC}(\mathbb{R}^n)) \neq \text{Sat}(\text{Bc, RCP}(\mathbb{R}^n))$. The reader will recall from Sec. 1.3 that the corresponding inequations for the language Bc$^c$ hold anyway, by (3). Finally, we remark on the case of the real line, $\mathbb{R}$, which was considered in [Kontchakov et al. 2010b]. The analogue of Theorem 3.1 for the case $n = 1$ holds (though we need to use a different formula to force an infinitude of components); however, the analogue of Theorem 3.3 for $n = 1$ fails: indeed, we have $\text{Sat}(\text{Bc, RC}(\mathbb{R})) = \text{Sat}(\text{Bc, RCP}(\mathbb{R}))$.

4. UNDECIDABILITY: THE POLYHEDRAL CASE

We use the techniques of the previous section to prove that the satisfiability problem for any of the languages Bc, Cc or Cc$^c$ over the frame $\text{RCP}(\mathbb{R}^n)$, $n \geq 2$, is undecidable. Recall that a frame $(T, S)$ is unicoherent if $T$ is unicoherent; and $(T, S)$ is finitely decomposible if, for all $s \in S$, there exist connected elements $s_1, \ldots, s_k$ of $S$, such that $s = s_1 + \cdots + s_k$.

**Theorem 4.1.** Let $\mathcal{K}$ be any class of unicoherent, finitely decomposible frames, such that $\mathcal{K}$ contains some frame of the form $(\mathbb{R}^n, S)$, $n \geq 2$, where $\text{RCP}(\mathbb{R}^n) \subseteq S$. Then the problem $\text{Sat}(\text{Cc, } \mathcal{K})$ is r.e.-hard.

**Proof.** We proceed via a reduction of the Post correspondence problem (PCP), constructing, for any instance $W$, a formula $\varphi_W$ with the property that (i) if $W$ is positive then $\varphi_W$ is satisfiable over $\text{RCP}(\mathbb{R}^n)$, $n \geq 2$, and (ii) if $\varphi_W$ is satisfiable over a unicoherent, finitely decomposable frame then $W$ is positive.

We begin with a sketch of the proof strategy. Let $T$ and $U$ be the alphabets of $W$. Our formula $\varphi_W$ enforces the existence of two finite sequences of regions, $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$, with each sequence forming an ‘onion-like’ configuration, much as in the proof of Theorem 3.1. Each region $A_i$ is ‘coloured’ with some letter of $U$, and each region $B_j$ is likewise coloured with some letter of $T$, thus yielding words $v \in U^*$ and $\tau \in T^*$. In addition, $\varphi_W$ establishes a pair of regions, $e^1$ and $e^2$, ensuring that every element $B_j \cdot e^k$ is the union of one or more of the elements $A_i \cdot e^k$, $k = 1, 2$. Thus, the word $v$ is segmented in two different ways—once by $e^1$ and once by $e^2$—with each segment corresponding to a position in the word $\tau$. Finally, $\varphi_W$ ensures that, if $B_j$ is coloured with some tile $t \in T$, then the regions $A_i$ such that $A_i \cdot e^1 \subseteq B_j \cdot e^1$ are coloured with the letters of $U$ so as to spell out the upper string on tile $t$; similarly, for $e^2$, but spelling out the lower string of $t$. In this way we establish that any satisfying assignment for $\varphi_W$ in $\text{RCP}(\mathbb{R}^n)$ yields a solution to $W$.

For example, let $U = \{0, 1\}$ and $T$ contain the tiles $t_1: \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $t_2: \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $t_3: \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Thus, $W$ is a positive PCP-instance since the tile string $t_1t_2t_3$ yields, on both top and bottom rows, 101011. Notice how $t_1t_2t_3$ segments 101011 as two sequences of substrings with different lengths: 3, 2, 1 on the top, and 1, 1, 4 on the bottom. Fig. 7 shows an arrangement satisfying our formula $\varphi_W$, where $A_1, \ldots, A_6$ are indicated by thin outlines, and $B_1, B_2, B_3$ by thick outlines. (The region $A_7 = B_4$ functions as a full stop, and may be ignored here.) Notice how $e^1$ and $e^2$ segment $A_1, \ldots, A_6$ in different ways, with lengths 3, 2, 1 in $e^1$, and 1, 1, 4 in $e^2$, just like our solution of $W$. By colouring the regions $A_1, \ldots, A_6$ with the letters 1, 0, 1, 0, 1, 1, respectively, and the regions $B_1, B_2, B_3$ with the tiles $t_1, t_2$ and $t_3$, respectively, we satisfy the remaining constraints of $\varphi_W$.

We now proceed with the detailed proof (in which $|i|$ means $i$ modulo 4). Let the PCP-instance $W = (w^1, w^2)$ over alphabets $T$ and $U$ be given, and let $r = (r_1, \ldots, r_3)$ and $s = (s_0, \ldots, s_3)$ be quadruples of variables. The first conjuncts of $\varphi_W$ ensure that $r$ and $s$
are sub-cyclic partitions:

\[ \text{sc-part}(r_0, r_1, r_2, r_3), \]
\[ \text{sc-part}(s_0, s_1, s_2, s_3). \]

By Lemma 2.4, the component graphs \( H(\vec{r}) \) and \( H(\vec{s}) \) are trees. Thus, any two vertices of \( H(\vec{r}) \) are joined by a unique path, and likewise for \( H(\vec{s}) \). The vertices of \( H(\vec{r}) \) will be used to represent letters in some word \( v \in U^* \), and those of \( H(\vec{s}) \), letters in some word \( \tau \in T^* \).

Let \( e^1 \) and \( e^2 \) be fresh variables. We shall use these to represent the morphisms \( w^1 \) and \( w^2 \), respectively. The next conjuncts of \( \varphi_W \) ensure that, for all \( 0 \leq i < 4 \), the components of both \( r_i \cdot e^1 \) and \( r_i \cdot e^2 \) are coloured by the elements \( \vec{s} \), as defined in Sec. 2:

\[ \bigwedge_{k=1,2} \bigwedge_{0 \leq i < 4} \text{colour}(r_i \cdot e^k; s_0, s_1, s_2, s_3). \]

Fig. 7 shows a configuration conforming to conditions (9)--(11). In this arrangement, \( H(\vec{r}) \) has vertices \( \{A_1, \ldots, A_7\} \), where \( A_i \) is a component of \( r_{i,j} \), and \( H(\vec{s}) \) has vertices \( \{B_1, \ldots, B_4\} \) (indicated by thick boundaries), where \( B_j \) is a component of \( s_{i,j} \). Observe that, for \( 0 \leq i < 4 \) and \( 1 \leq k \leq 2 \), each component of \( r_i \cdot e^k \) is included in exactly one of \( s_0, \ldots, s_3 \), and hence in a single vertex \( B_j \) of \( H(\vec{s}) \); however, outside \( e^1 + e^2 \), elements of \( H(\vec{s}) \) may intersect elements of \( H(\vec{r}) \) without including them. A word of warning: in the configuration of Fig. 7, the various sets \( A_i \cdot e^k \) and \( B_j \cdot e^k \) are all connected; however, the formula \( \varphi_W \) does not enforce this. That is, there is nothing to prevent the sets \( e^k \) from chopping elements of \( H(\vec{r}) \) and \( H(\vec{s}) \) into several pieces.

Let \( w^* \) be a fresh variable. The next conjuncts of \( \varphi_W \) ensure that the graphs \( H(\vec{r}) \) and \( H(\vec{s}) \) contain a common vertex, \( w^* \):

\[ e(w^*) \land (w^* \neq 0), \]
\[ \text{colour}(w^*; r_0, r_1, r_2, r_3) \land \bigwedge_{0 \leq i < 4} \text{colour}(r_i; w^*, -w^*), \]
\[ \text{colour}(w^*; s_0, s_1, s_2, s_3) \land \bigwedge_{0 \leq i < 4} \text{colour}(s_i; w^*, -w^*). \]

To see why, note that the first conjunct of (13) ensures that \( w^* \) is included in one of the sets \( r_i \), and hence—since it is connected, by (12)—in one of the vertices of \( H(\vec{r}) \); on the other hand, the remaining conjuncts of (13) ensure that every vertex of \( H(\vec{r}) \) is included in either \( w^* \) or \( -w^* \). Since \( w^* \) is non-empty, it must be identical to a single vertex of \( H(\vec{r}) \). The same conclusion holds for \( H(\vec{s}) \) using (14). In the arrangement of Fig. 7, we have \( w^* = A_7 = B_4 \).
We need to impose a little more structure on the graphs \(H(\vec{r})\) and \(H(\vec{s})\). Let \(r'_0, \ldots, r'_4\) and \(w_1\) be fresh variables, and let \(\varphi_w\) contain the conjuncts:

\[
c(w_1) \land (w_1 \leq r_1) \land (w_1 \leq s_1) \land (w_1 \cdot w^* = 0),
\]

\[
\bigwedge_{0 \leq i < 4} (r'_i \leq r_i),
\]

\[
\bigwedge_{k=1,2} (e^k \cdot r'_1 \cdot w_1 \neq 0).
\]

Since \(w_1\) is a non-empty, connected subset of both \(r_1\) and \(s_1\), let \(A_1\) be the component of \(r_1\) including \(w_1\), and let \(B_1\) be the component of \(s_1\) including \(w_1\). It follows that \(A_1 \leq B_1\); the final conjunct of (15) ensures that \(A_1\) and \(B_1\) are both distinct from \(w^*\). In the sequel, we shall construct a path in the graph \(H(\vec{r})\) from \(A_1\) to \(w^*\), and a path in the graph \(H(\vec{s})\) from \(B_1\) to \(w^*\). The proof will hinge on analysing the properties of these paths.

Let \(t\) be a fresh variable, and let \(\varphi_w\) contain the conjuncts:

\[
\bigwedge_{k=1,2} (e^k \cdot t \neq 0),
\]

\[
\bigwedge_{k=1,2} \bigwedge_{0 \leq i < 4} c(e^k \cdot ((r'_i \cdot (-w^*)) + r_{[i+1]} + t)),
\]

\[
\bigwedge_{0 \leq i < 4} \neg C(r'_i, t),
\]

\[
\bigwedge_{0 \leq i < 4} \neg C(r'_i, r_{[i+1]} \cdot (-r'_{[i+1]})).
\]

Fixing the value of \(k\) for the moment (\(1 \leq k \leq 2\)), from (17), select a point \(q^k_i\) in the interior of \(e^k \cdot r'_i \cdot w_1\). By (15), \(q^k_i \neq w^*\). Let \(X^k_1\) be the component of \(e^k \cdot r_i\) containing \(q^k_1\), and \(A^k_1\) the component of \(r_1\) including \(X^k_1\); note that \(A^k_1\) is a vertex of the graph \(H(\vec{r})\). Evidently, \(A^k_1 \neq w^*\), and so \(X^k_1 \leq (e^k \cdot r_1 \cdot (-w^*))\). Now suppose \(X^k_1\) has been defined and contains some point \(q^k_{i+1} \in (e^k \cdot r'_{[i+1]} \cdot (-w^*))\). From (18)–(20), \(X^k_1\) contains a point \(q^k_{i+1} \in (e^k \cdot r'_{[i+1]} \cdot (-w^*))\), which, by (21), is in fact in \(e^k \cdot r'_{[i+1]}\). Let \(X^k_{i+1}\) be the component of \(e^k \cdot r_{[i+1]}\) containing \(q^k_{i+1}\), and \(A^k_{i+1}\) the component of \(r_{[i+1]}\) including \(X^k_{i+1}\); again, \(A^k_{i+1}\) is a vertex of \(H(\vec{r})\).

Note that either \(q^k_{i+1} \in w^*\) or \(q^k_{i+1} \notin w^*\), and in the latter case, \(q^k_{i+1} \in (e^k \cdot r'_{[i+1]} \cdot (-w^*))\). This process either continues forever, or, at some point, \(q^k_{i+1} \in w^*\). But now consider any sequence \(A^k_1, A^k_2, \ldots, A^k_{i+1}\) obtained in this way. Evidently, \((A_i, A_{i+1})\) is an edge of \(H(\vec{r})\) for all \(1 \leq i < \ell\); moreover, since \(A_i \leq r_{[i]}\), we see from (9) that \(A_i \neq A_{i+2}\) for all \(1 \leq i < \ell - 1\), whence, since \(H(\vec{r})\) is a tree, \(A^k_1, A^k_2, \ldots, A^k_{\ell}\) is a path (i.e., has no repeated nodes). It follows that, for some value of \(i\), denoted by \(n^k\), the condition \(q^k_{i+n^k} \in w^*\) must hold, for otherwise, \(H(\vec{r})\) would contain an infinite path, contradicting the assumption that the frame in question is finitely decomposable. Since \(q^k_{i+n^k} \in r_{n^k+1}\), we have \(A^k_{n^k+1} = w^*\), and hence there is a path \(A^k_1, A^k_2, \ldots, A^k_{n^k+1}\) in \(H(\vec{r})\) from \(A^k_1 = A_1\) to \(A^k_{n^k+1} = w^*\). Indeed, this must be the same path for both \(k = 1, 2\), so that we may drop the \(k\)-superscripts, and write: \(A_1, A_2, \ldots, A_n, A_{n+1}\), where \(A_{n+1} = w^*\). (Note that the letter \(n\) here is simply a convenient label for the length of this path: it has nothing to do with the dimension of the space.) It is important to remember that the sets \(X^1_1\) and \(X^2_1\), for a fixed value of \(i\), will in general be distinct (Fig. 8).

Let us now turn our attention to the graph \(H(\vec{s})\). Fixing the value of \(k\) again, consider the sequence \(X^k_1, \ldots, X^k_{n+1}\). Since \(X^k_1\) is a connected subset of \(e^k \cdot r_{[i]}\), it follows from (11) that
Each $X^k_i$ is included in some vertex of $H(\vec{s})$, say $\hat{B}^k_i$. Thus, we have a sequence $\hat{B}^1_i, \ldots, \hat{B}^k_{i+1}$, where $A_1 \leq \hat{B}^k_i$ and $\hat{B}^k_{i+1} = w^*$. Of course, this sequence may contain adjacent duplicates, since there is nothing to stop $X^k_i$ and $X^k_{i+1}$ being included in the same vertex of $H(\vec{s})$. Furthermore, the two sequences (for $k = 1, 2$) may be distinct, since, for fixed $i$, there is nothing to stop $X^k_1$ and $X^k_2$ lying in different vertices of $H(\vec{s})$; see Figs. 7 and 8. But now suppose we remove adjacent duplicates, obtaining sequences: $\hat{B}^k_1, \ldots, \hat{B}^k_{m+1}$, where $A_1 \leq \hat{B}^k_1$ and $\hat{B}^k_{m+1} = w^*$ with $m^k \leq n$. Thus, every $\hat{B}^k_i$ is the result of coalescing a contiguous block of identical vertices $\hat{B}^k_{i+1}$ of $\hat{B}^k_i$. Evidently, $(\hat{B}^k_i, \hat{B}^k_{i+1})$ must be an edge of $H(\vec{s})$, for $1 \leq j \leq m^k$.

Now let $\varphi_W$ contain the conjuncts:

$$\bigwedge_{k=1,2} \bigwedge_{0 \leq i < 4} \bigwedge_{0 \leq j < 4} -C(e^k \cdot r_i \cdot s_j, e^k \cdot r_{[i+1]} \cdot s_{[j-1]}).$$

We claim that $B^k_j \leq s_{[j]}$ for all $1 \leq j \leq m^k$. The proof is by induction on $j$. By (15), $X^k_i \leq s_1$, and so $B^k_j \leq s_1$. Suppose, then, $B^k_j \leq s_{[j]}$, for some $1 \leq j < m^k$. Let $\hat{B}^k_i$, be the last vertex in the block coalescing to $B^k_j$, so that $\hat{B}^k_{i+1}$ is the first element of the block coalescing to $B^k_{i+1}$. Thus, $X^k_i \leq r_{[j]}$ and $X^k_{i+1} \leq r_{[j+1]}$. But (10) and (22) then ensure that either $B^k_{j+1} \leq s_{[j]}$ or $B^k_{j+1} \leq s_{[j+1]}$; and the former is impossible, since then $\hat{B}^k_i$ and $\hat{B}^k_{i+1}$ would have coalesced to the same block. This proves the claim. By (10) (and the fact that $B^k_{m^k+1} \neq w^*$), we then have $B^k_j \neq B^k_{j+2}$ for $1 \leq j < m^k$. And since $H(\vec{s})$ is a tree, it follows that $B^k_1, \ldots, B^k_{m^k+1}$ is a path through $H(\vec{s})$ with $B^k_1 = B^k_2$ and $B^k_{m^k+1} = B^k_{m^k+2} = w^*$. Indeed, this is the same path through $H(\vec{s})$ for both values of $k$, so that we can again drop the superscripts, and just write: $B_1, \ldots, B_m, B_{m+1}$, where $B_{m+1} = w^*$.

Taking stock, we see that, for each $k = 1, 2$, the path $A_1, \ldots, A_n$ may be grouped into $m$ contiguous blocks $E^k_1, \ldots, E^k_m$ by taking the vertex $A_i$ to be in the $j$th block $E^k_j$ just in case $e^k \cdot X^k_i \leq B_j$. We may depict this grouping as follows:

$$A_1, \ldots, A_n = A^k_{1,1}, \ldots, A^k_{1,h^k_1}, \ldots, A^k_{j,1}, \ldots, A^k_{j,h^k_j}, \ldots, A^k_{m,1}, \ldots, A^k_{m,h^k_m}.$$ 

It is important to realize that, although there is only one path $A_1, \ldots, A_{n+1}$ and one path $B_1, \ldots, B_{n+1}$, the two values $k = 1$ and $k = 2$ will in general give rise to different groupings of the vertices of the former into blocks corresponding to the vertices of the latter, as in the example of Fig. 8 (hence, the two sequences of indices $h^k_1, \ldots, h^k_m$).

Fig. 8. Regions $A_1, \ldots, A_7$ and $e^1$, $e^2$ from Fig. 7; the components of $r_{[j]}$ lying in each $A_i$ are indicated by shading.
Recall the PCP-instance $W = (w^1, w^2)$ over the alphabets $T$ and $U$, which we wish to encode. We regard the elements of these alphabets as fresh variables, and order them in some way to form tuples $\vec{t}$ and $\vec{u}$. We use these variables to colour the vertices of $H(\vec{t})$ and $H(\vec{r})$, respectively, by taking $\varphi_W$ to contain the conjuncts:

$$\text{part}(\vec{t}) \land \bigwedge_{0 \leq j < 4} \text{colour}(s_j; \vec{t}) \land \text{part}(\vec{u}) \land \bigwedge_{0 \leq i < 4} \text{colour}(r_i; \vec{u}).$$

(23)

In this way, the path $A_1, \ldots, A_n$ defines a word $v \in U^*$, and the path $B_1, \ldots, B_m$ defines a word $\tau \in T^*$. Using the groupings of the sequence $A_1, \ldots, A_n$ obtained above, we shall write conjuncts of $\varphi_W$ ensuring that $w^k(\tau) = v$ for $k = 1, 2$. This will mean that, if $\varphi_W$ has a satisfying assignment over some frame $S \in K$, then the PCP-instance $W$ is positive.

For $k = 1, 2$, let $\{p^k_{h, \ell} \mid 1 \leq h \leq |T|, 1 \leq \ell \leq |w^k(t_h)|\}$ be a collection of fresh variables, enumerated in some way as $\vec{p}^k$, which we shall use to colour the vertices of $H(\vec{r})$. That is, we add to $\varphi_W$ the conjuncts:

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell \leq |w^k(t_h)|} \left( t_h \leq \sum_{1 \leq \ell \leq |w^k(t_h)|} p^k_{h, \ell} \right).$$

(25)

We refer to these variables as position colours, because we are to think of $p^k_{h, \ell}$ as denoting the $\ell$th position in the word $w^k(t_h)$. In particular, any position colour $p^k_{h, \ell}$ is naturally associated to the letter $t_h$ of $T$. Fixing $k$ for the moment, consider the vertices $A^k_j, 1 \leq j \leq A^k_i$ grouped into the $j$th block, $E^k_j$. What ensures that these vertices belong to one block is the existence of a single $B_j$ such that, if $A_i$ is one of these vertices, then the corresponding set $X^k_j \leq A_i$ is included in $B_j$. By (23), $X^k_j \leq t_h$, for some $1 \leq h \leq |T|$. It follows that the conjuncts

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell \leq |w^k(t_h)|} \left( (w_1 \cdot p^k_{h, \ell} = 0) \land \bigwedge_{0 \leq i < 4} \neg C(s_i, s_{i+1}, p^k_{h, \ell}) \right).$$

(26)

These ensure that the first vertex of each block is assigned one of the colours $p^k_{h,1}$, for $1 \leq h \leq |T|$. The rules for colouring successive vertices can now be simply stated. Consider the following binary relation on the variables in $\vec{p}^k$:

$$\Psi^k = \left\{(p_{h, \ell}, p^k_{h, \ell+i}) \mid 1 \leq h \leq |T|, 1 \leq \ell < |w^k(t_h)|\right\} \cup \left\{(p^k_{h, \ell}, p^k_{h', 1}) \mid 1 \leq h, h' \leq |T|\right\}.$$

This relation captures the rules of possible succession for colouring by the variables $p^k_{h, \ell}$: if $A_i$ is coloured $p^k_{h, \ell}$, where $\ell$ indicates a non-final position in $w^k(t_h)$, then $A_{i+1}$ must be coloured $p^k_{h, \ell+1}$; and, if $A_i$ is coloured $p^k_{h, \ell+1}$, indicating the final position in $w^k(t_h)$, then $A_{i+1}$ must be coloured $p^k_{h', 1}$ for some $1 \leq h' \leq |T|$. We therefore add to $\varphi_W$ the conjuncts

$$\bigwedge_{k=1,2} \bigwedge_{0 \leq i < 4} \left( (p^k_{h, \ell}, p^k_{h', \ell+i}) \notin \Psi^k \right) \land \neg C(p^k_{h, \ell}, r_i, p^k_{h', \ell+i}, r_{i+1}).$$

(27)

We also ensure that each block spells out only one word. That is, we ensure that no vertex of the sequence $A_1, \ldots, A_n$ can be coloured with the starting position in a word if the previous
vertex belongs to the same block:

$$\bigwedge_{k=1,2} \bigwedge_{0 \leq i < n} \bigwedge_{0 \leq j < 4} \bigwedge_{1 \leq h \leq |T|} \neg C(r_i \cdot s_j, r_{[i+1]} \cdot s_j \cdot p_{h,1}).$$ (28)

Lastly, we ensure that the final vertex of the final block corresponds to the final position in a word. In other words, we ensure that the vertex $A_n$ (which contacts $A_{n+1} = w^*$) is coloured $p_{h,[w^k(t_h)]}$, for some $1 \leq h \leq |T|:

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell < w^k(t_h)} \neg C(p_{h,\ell}, w^*).$$ (29)

At this stage, we have ensured that, for $k = 1$ and $k = 2$, vertices of each block $A_{k,1}, \ldots, A_{k,n_k}$, $1 \leq j \leq m$, are coloured $p_{h,1}, \ldots, p_{h,[w^k(t_h)]}$, where $t_h$ is the $j$th letter of the word $\tau$. This easily enables us to enforce the sought-after conditions $w^k(\tau) = v$ for $k = 1, 2$. Denoting by $u_{h,\ell}$ the variable in $\bar{u}$ (i.e., that letter of the alphabet $U$) that is the $\ell$th letter in the word $w^k(t_h)$, we add to $\varphi_W$ the conjuncts:

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell < w^k(t_h)} (u_{h,\ell}^k \leq u_{h,\ell}).$$ (30)

That $w^k(\tau) = v$ for $k = 1, 2$ then follows from the fact that each vertex $A_i$ is assigned a unique colour from $\bar{u}$. Thus, if $\varphi_W$ is satisfiable over $\mathcal{K}$ then $W$ is positive.

Conversely, if $W$ is positive, it is obvious that $\varphi_W$ may be satisfied over $\text{RCP}(\mathbb{R}^n)$, $n \geq 2$, by suitably extending a configuration similar to that shown in Fig. 7. $\square$

**Corollary 4.2.** Let $\mathcal{K}$ be any class of unicoherent, finitely decomposable frames, such that $\mathcal{K}$ contains some frame of the form $(\mathbb{R}^n, S)$, $n \geq 2$, where $\text{RCP}(\mathbb{R}^n) \subseteq S$. Then the problem $\text{Sat}(\mathcal{K}, \mathcal{K})$ is r.e.-hard.

**Proof.** We start with the formula $\varphi_W$ of Theorem 4.1, and replace all occurrences of $c$ by $c^\circ$. Denote the resulting $\mathcal{K}c^\circ$-formula by $\varphi_{W}^\circ$. Since all atoms of the form $c(\tau)$ in $\varphi_W$ occur with positive polarity, $\varphi_W^\circ$ entails $\varphi_W$. On the other hand, by inspection of Fig. 7, we see that if $W$ is positive, then $\varphi_W^\circ$ will be satisfiable in $\text{RCP}(\mathbb{R}^2)$, and hence in $\text{RCP}(\mathbb{R}^n)$ for all $n \geq 2$. This proves the corollary. $\square$

**Corollary 4.3.** Let $\mathcal{K}$ be any class of unicoherent, finitely decomposable frames, such that $\mathcal{K}$ contains some frame of the form $(\mathbb{R}^n, S)$, $n \geq 2$, where $\text{RCP}(\mathbb{R}^n) \subseteq S$. Then the problem $\text{Sat}(\mathcal{K}, \mathcal{K})$ is r.e.-hard.

**Proof.** Consider again the formula $\varphi_W$ of Theorem 4.1. Since all occurrences of the predicate $C$ in $\varphi_W$ have negative polarity, we can replace them with $Bc$-formulas as we did in the proof of Theorem 3.3. The resulting formula $\varphi_W^c$ implies $\varphi_W$, and is satisfiable in $\text{RCP}(\mathbb{R}^n)$, for all $n \geq 2$, whenever $W$ is positive. $\square$

**5. Undecidability: The Plane Case**

In Sec. 3, we established that, if $\mathcal{L}$ is any of the languages $Bc$, $Cc$ or $Cc^\circ$, there exists an $\mathcal{L}$-formula that is satisfiable over $\text{RC}(\mathbb{R}^n)$, $n \geq 2$, but only by regions having infinitely many components. Nothing was mentioned in this regard about the language $Bc^\circ$. In Sec. 4, we established the undecidability of $\text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^n))$, $n \geq 2$, where $\mathcal{L}$ is any of the languages $Bc$, $Cc$, $Cc^\circ$. Nothing was mentioned in this regard about the problems $\text{Sat}(Bc^\circ, \text{RCP}(\mathbb{R}^n))$, $n \geq 2$, or indeed about the problems $\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^n))$ where $\mathcal{L}$ is any of $Bc$, $Bc^\circ$, $Cc$, $Cc^\circ$.

In this section, we complete the picture in the case $n = 2$. Specifically, we establish the existence of a $Bc^\circ$-formula satisfiable over $\text{RC}(\mathbb{R}^2)$, but only by regions having infinitely
many components; and we establish the undecidability of the problems \( \text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^2)) \) and \( \text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^2)) \), where \( \mathcal{L} \) is any of \( \text{Bc}, \text{Bc}^o, \text{Cc} \) or \( \text{Cc}^o \).

We employ the standard terminology of Jordan arcs and curves: a non-degenerate Jordan arc is a continuous, 1–1 function \( \alpha \) from the unit interval to \( \mathbb{R}^2 \); a degenerate Jordan arc is a constant function from the unit interval to \( \mathbb{R}^2 \); a Jordan arc is a degenerate Jordan arc or a non-degenerate Jordan arc. A Jordan curve is a continuous, 1–1 function from the unit circle to \( \mathbb{R}^2 \). Where no confusion results, we identify Jordan arcs and curves with their loci (ranges). If \( \alpha_1 \) and \( \alpha_2 \) are Jordan arcs which intersect in the unique point \( \alpha_1(1) = \alpha_2(0) \), then we write \( \alpha_1 \circ \alpha_2 \) to denote, ambiguously, any Jordan arc with locus \( \alpha_1 \cup \alpha_2 \) such that \( \alpha(0) = \alpha_1(0) \) and \( \alpha(1) = \alpha_2(1) \). We employ the following notation: if \( \alpha \) is a Jordan arc, \( \alpha^{-1} \) denotes a Jordan arc with the same locus but opposite direction, e.g., \( \alpha^{-1}(t) = \alpha(1-t) \), for all \( 0 \leq t \leq 1 \). If, in addition, \( p_1 = \alpha(t_1), p_2 = \alpha(t_2) \) are points on \( \alpha \) with \( t_2 \geq t_1 \), \( \alpha[p_1, p_2] \) denotes a Jordan arc whose locus is the segment of \( \alpha \) between \( p_1 \) and \( p_2 \), and which has the same direction as \( \alpha \): \( \alpha[p_1, p_2](t) = \alpha(t_1 + t(t_2 - t_1)) \). An end-cut to \( p \) in a set \( X \) is a Jordan arc \( \alpha \subseteq X^o \cup \{p\} \) such that \( \alpha(1) = p \). A cross-cut in \( X \) is a Jordan arc \( \alpha \) in \( X \) intersecting the boundary \( \delta X \) of \( X \) only at its endpoints \( \alpha(0) \) and \( \alpha(1) \).

5.1. Forcing infinitely many components with \( \text{Bc}^o \)

We begin by showing that there exists a \( \text{Bc}^o \)-formula that is satisfiable over \( \text{RC}(\mathbb{R}^2) \), but only by regions having infinitely many components. Many of the techniques we employ will prove useful in Sec. 5.2. Our basic tools are two formulas that enable us to construct Jordan arcs and curves containing points in specified regions. But before presenting these formulas, we need to establish the following property of regular closed sets:

**Lemma 5.1.** Let \( T \) be any topological space, and \( a, b_1 \) and \( b_2 \) elements of \( \text{RC}(T) \) such that \( b_1 \cdot b_2 = 0 \). Then \( (a + b_1)^o \cap (a + b_2)^o = a^o \).

**Proof.** Note that, for any \( s \in \text{RC}(T), -s = T \setminus s^o \). Since \( b_1 \cdot b_2 = 0 \), we have \( -a = \sum_{i=1,2}(-a + b_i) \), which is then equal to \( \bigcup_{i=1,2}(T \setminus (a + b_i)^o) = T \setminus \bigcap_{i=1,2}(a + b_i)^o \). \( \Box \)

Consider now the following \( \text{Bc}^o \)-formula:

\[
\text{frame}^o(r_0, \ldots, r_{n-1}) = \bigwedge_{0 \leq i < n} \left( (r_i \neq 0) \land c^o(r_i + r_{i+1}) \right) \land \bigwedge_{0 \leq i < j < n} (r_i \cdot r_j = 0),
\]

where \([k] \) denotes the value of \( k \) modulo \( n \). This formula allows us to construct Jordan curves that contain points of all regions \( r_0, \ldots, r_{n-1} \):

**Lemma 5.2.** Fix \( n \geq 3 \), and let \( (a_0, \ldots, a_{n-1}) \) be a tuple of elements of \( \text{RC}(\mathbb{R}^2) \) satisfying \( \text{frame}^o(r_0, \ldots, r_{n-1}) \). Then there exist Jordan arcs \( a_0, \ldots, a_{n-1} \) and points \( p_0, \ldots, p_{n-1} \) such that, for all \( 0 \leq i < n \), \( a_i \) is a Jordan arc from \( p_i \) to \( p_{i+1} \), with \( a_i \subseteq (a_i + a_{i+1})^o \); \( a_0 \cdots a_{n-1} \) is a Jordan curve lying in \( (a_0 + \cdots + a_{n-1})^o \); and \( p_i \in a_i^o \), for all \( 0 \leq i < n \).

**Proof.** For every \( 0 \leq i < n \), select points \( p_i' \) in the interior of \( a_i \), and connect each \( p_i' \) to \( p_{i+1}' \) with an arc \( a_i'' \subseteq (a_i + a_{i+1})^o \). Let \( p_1 \) be the first point on \( a_0'' \) that is on \( a_1'' \), let \( a_0' \) be the initial segment of \( a_0'' \) ending at \( p_1 \), and let \( a_1' \) be the final segment of \( a_1'' \) starting at \( p_1 \). Note that \( a_0' \cap a_1' = \{p_1\} \). For \( 2 \leq i < n - 1 \), let \( p_i \) be the first point on \( a_i'-1 \) that is on \( a_i'' \), let \( a_{i-1} \) be the initial segment of \( a_{i-1}' \) ending at \( p_i \), and let \( a_i' \) be the final segment of \( a_i'' \) starting at \( p_i \). Note that \( a_{i-1} \cap a_i' = \{p_i\} \). Finally, let \( p_0 \) be the first point on \( a_{n-2}'' \) that is on \( a_0' \), let \( a_{n-1} \) be the initial segment of \( a_{n-1}' \) ending at \( p_0 \), and let \( a_0 \) be the final segment of \( a_0'' \) starting at \( p_0 \). Note that \( a_{n-1} \cap a_0 = \{p_0\} \). By construction, for every \( 0 \leq i < n \), \( a_i \) connects points \( p_i \) and \( p_{i+1} \), and \( a_i' \cap a_{i-1} = \{p_i\} \), whence, by Lemma 5.1, \( p_i \in a_i^o \). \( \Box \)
Consider now the $C_{e^o}$-formula, for $n > 1$,
\[
    \text{stack}^o(r_1, \ldots, r_n) = \bigwedge_{1 \leq i \leq n} c^o(r_i + \cdots + r_n) \land \bigwedge_{1 \leq i < j \leq n} (r_i \cdot r_j = 0) \land \bigwedge_{j > i > 1} \neg C(r_i, r_j),
\]
which will allow us to construct arcs containing points of all the regions $r_1, \ldots, r_n$.

**Lemma 5.3.** Let $(a_1, \ldots, a_n)$ be a tuple of elements of $\text{RC}(\mathbb{R}^2)$ satisfying $\text{stack}^o(r_1, \ldots, r_n)$. Then every point $p_1 \in a_1^o$ can be connected to every point $p_n \in a_n^o$ by a Jordan arc $\alpha = \alpha_1 \cdots \alpha_{n-1}$ such that, for all $1 \leq i < n$, $\alpha_i$ is a non-degenerate Jordan arc in $(a_i + a_{i+1})^o$, starting at a point $p_i \in a_i^o$.

**Proof.** Since $a_1 + \cdots + a_n$ is interior-connected, let $\alpha_i' \subseteq (a_1 + \cdots + a_n)^o$ be a Jordan arc connecting $p_1$ to $p_i$. Since $\neg C(a_1, (a_3 + \cdots + a_n))$, $\alpha_i'$ must contain a point $p_i' \in a_i^o$ such that $p_i'[p_i, p_i'] \subseteq (a_1 + a_i + a_{i+1})^o$. For convenience, let $p_0 = p_1$, let $\alpha_0$ be the degenerate Jordan arc located at $p_1$, and let $a_0$ be the empty region.

![Fig. 9. The constraint stack^o(a_1, \ldots, a_n) ensures the existence of a Jordan arc \alpha = \alpha_1 \cdots \alpha_{n-1}.](image)

We inductively define, for all $1 \leq i < n$, arcs $\alpha_{i-1}, \alpha_i'$ and points $p_i, p_i'$ with the following properties: $\alpha_{i-1} \subseteq (a_i - a_{i-1})^o$ and runs from $p_i$ to $p_i'$. $\alpha_i' \subseteq (a_i + \cdots + a_n)^o$; $\alpha_0 \cdots \alpha_{n-1}$ is a Jordan arc from $p_0$ to $p_n$; and $p_i' \in a_i' \cap a_{i+1}$ with $\alpha_i'[p_i, p_i'] \subseteq (a_{i+1} + a_{i+2})^o$. Suppose that, for some $1 \leq i \leq n - 2$, the requisite entities have already been defined (notice that this is already the case for $i = 1$). Since $a_{i+1} + \cdots + a_n$ is interior-connected, let $\alpha_{i+1}' \subseteq (a_{i+1} + \cdots + a_n)^o$ be a Jordan arc connecting $p_i'$ to $p_{i+1}$. Since we certainly have $(a_1 + \cdots + a_i)$, $(a_{i+1} + \cdots + a_n) = 0$, $\alpha_i'$ can intersect $\alpha_0 \cdots \alpha_{i-1} \alpha_i'$ only in its final segment $\alpha_i'^{n-i}$. Let $p_{i+1}$ be the first point of $\alpha_i'$ lying on $\alpha_i'^{n-i}$; let $\alpha_i$ be the initial segment of $\alpha_i'$ ending at $p_{i+1}$; and let $\alpha_i'^{n-i}$ be the final segment of $\alpha_i'$ starting at $p_{i+1}$. By construction, then, $\alpha_0 \cdots \alpha_i \alpha_i'^{n-i}$ is a Jordan arc from $p_0$ to $p_{i+1}$, and $\alpha_{i+1} \subseteq (a_{i+1} + a_{i+2})^o$. Moreover, since $\neg C(a_{i+1}, (a_{i+3} + \cdots + a_n))$, $\alpha_{i+1}'$ must contain a point $p_{i+1}' \in a_{i+2}^o$ such that $\alpha_i'[p_i, p_i'] \subseteq (a_{i+1} + a_{i+2})^o$. Continuing up to the value $i = n - 1$, we have defined $\alpha_1, \ldots, \alpha_{n-1}$, $\alpha_{n-1}'$, and $p_1, \ldots, p_{n-1}$. It remains only to define $\alpha_n$; for this we simply set $\alpha_n = \alpha_n'$. For all $1 < i < n$, we have $p_i \in a_{i-1} \cap a_i$, whence, by Lemma 5.1, $p_i \in a_i^o$. It also follows that the $\alpha_i$ are non-degenerate. $\square$

It should be noted that stack^o(r_1, \ldots, r_n) is not a $B_{e^o}$-formula, as it contains (negative) occurrences of the contact predicate $C$. It turns out, however, that we can eliminate them. To this end, consider the $B_{e^o}$-formula
\[
    \text{K5m}(r_1, \ldots, r_5) = \bigwedge_{1 \leq i \leq 5} (c^o(r_i) \land (r_i \neq 0)) \land \bigwedge_{1 \leq i < j \leq 5} (r_i \cdot r_j = 0) \land \bigwedge_{3 \leq j \leq 5} c^o(r_1 + r_j) \land \bigwedge_{2 \leq i < j \leq 5} c^o(r_i + r_j).
\]
This formula is similar to formula (2) encoding the non-planar graph $K_5$ (hence the name); however, there is no requirement that $r_1 + r_2$ is interior-connected.

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Lemma 5.4. (i) For each tuple \((a_1, \ldots, a_5)\) from of \(RC(\mathbb{R}^2)\) satisfying \(K5m(r_1, \ldots, r_5)\), we have \(-C(a_1, a_2)\). (ii) If regions \(b_1\) and \(b_2\) can be separated by a Jordan curve then there exist polygons \((a_1, \ldots, a_5)\) satisfying \(K5m(r_1, \ldots, r_5)\) such that \(b_1 \leq a_1\) and \(b_2 \leq a_2\).

Proof. (i) For all \(i\) \((1 \leq i \leq 5)\), pick a point \(p_i \in a_i^c\). Then, for all \(j\) \((3 \leq j \leq 5)\) let \(\gamma_{1,j}\) be an arc from \(p_1\) to \(p_j\) lying in \((a_1 + a_j)^c\), and, for all \(i\), \(j\) \((2 \leq i < j \leq 5)\), let \(\gamma_{i,j}\) be an arc from \(p_i\) to \(p_j\) lying in \((a_i + a_j)^c\). It is routine to show that the various \(\gamma_{i,j}\) can be chosen so that they intersect only at their endpoints. Thus, \(\Gamma = \gamma_{3,4}, \gamma_{4,5}, \gamma_{3,5}^{-1}\) forms a Jordan curve in \((a_3 + a_4 + a_5)^c\), and \(\gamma_{2,3}, \gamma_{2,4}\) and \(\gamma_{2,5}\) join \(\Gamma\) to the point \(p_2\) lying in one of its residual domains. (Fig. 10 illustrates the situation where \(p_2\) lies in the bounded residual domain.) Since \(a_3\) and \(a_2\) are (interior-) connected and cannot intersect \(\Gamma\), they each lie in one of its residual domains. It suffices to show that \(a_1\) and \(a_2\) lie in different residual domains. To see this, observe that \(\gamma_{2,3}, \gamma_{2,4}\) and \(\gamma_{2,5}\) divide the residual domain of \(\Gamma\) containing \(p_2\) into three regions, bounded by arcs lying in \((a_2 + a_3 + a_4)^c\), \((a_2 + a_4 + a_5)^c\) and \((a_2 + a_3 + a_5)^c\), respectively. But if \(a_1\) and \(a_2\) lie on the same side of \(\Gamma\), then \(p_1\) lies in one of these regions, contradicting the existence of arcs \(\gamma_{1,j} \subseteq (a_1 + a_j)^c\) connecting \(p_1\) to \(p_j\), for \(j = 3, 4, 5\).

(ii) Let \(\Gamma\) be a Jordan curve separating \(b_1\) and \(b_2\). We may assume that \(\Gamma\) is piecewise-linear. Now thicken \(\Gamma\) to form an annular element of \(RCP(\mathbb{R}^2)\), still disjoint from \(b_1\) and \(b_2\), and divide it into the three interior-connected and non-overlapping polygons \(a_3, a_4, a_5\). Choose \(a_1\) and \(a_2\) to be the components of the complement of \(a_3 + a_4 + a_5\) containing \(b_1\) and \(b_2\), respectively. ∎

We remark that Lemma 5.4 (ii) guarantees that \(a_1, \ldots, a_5\) are polygons. This fact will be important in Sec. 5.2, where we prove the undecidability of \(Sat(\mathcal{B}c^\omega, RCP(\mathbb{R}^2))\); for the main result of this section, however, we require only that \(a_1, \ldots, a_5\) are regular closed sets in \(\mathbb{R}^2\):

Theorem 5.5. There is a \(\mathcal{B}c^\omega\)-formula satisfiable over \(RC(\mathbb{R}^2)\), but only by tuples featuring sets with infinitely many components.

Proof. We first write a \(\mathcal{C}c^\omega\)-formula, \(\varphi_\infty^*\), with the required properties, and then show that all occurrences of \(C\) in it can be eliminated. Note that \(\varphi_\infty^*\) is not the same as \(\varphi_\infty^*\) constructed for the proof of Corollary 3.2. If \(k\) is an integer, \([k]\) indicates \(k\) modulo 2.

We begin with a sketch of the proof strategy. Our formula \(\varphi_\infty^*\) will be satisfied by the arrangement shown in Fig. 11, with regions \(s_0, \ldots, s_3\), \(a\) and \(b\) forming a rectangular frame, and the \(a_{ij}, b_{ij}\) \((0 \leq i < 2, 1 \leq j \leq 3)\) having infinitely many components arranged in a repeating pattern, as shown. Our task is to show that any satisfying tuple in \(RC(\mathbb{R}^2)\) looks approximately like this. More precisely, we construct a Jordan curve lying in \((s_0 + \cdots + s_3 + a + b)^c\), together with infinite sequences of Jordan arcs \(\{a_k\}_{k \geq 1}, \{b_k\}_{k \geq 1}\).
this argument depends heavily on the assumption that we are working in \( \mathbb{R}^2 \). In Fig. 12, this ensures that the forcing all regions involved to have pairwise disjoint interiors. Given the existence of the arcs arranged as in Fig. 12. We show that \( \varphi \).

Note that the regions \( a \) as well as formulas are evidently satisfied by the arrangement of Fig. 11. Let \( \sigma \) be the conjunction of (31)–(34). Finally, we write constraints

\[
\varphi \sigma = \alpha_k \cup \beta_k \subseteq (b_{[k],2} + a_{[k],1} + a_{[k],2} + a_{[k],3} + a)^\circ, \quad \text{for all } k \geq 1,
\]

\[
\varphi \sigma = \beta_k \subseteq (a_{[k+1],2} + b_{[k],1} + b_{[k],2} + b_{[k],3} + b)^\circ, \quad \text{for all } k \geq 2.
\]

(The arc \( \beta_1 \) is exceptional, because it contains points in \( s_0^\circ \)).

Forcing all regions involved to have pairwise disjoint interiors. Given the existence of the arcs in Fig. 12, this ensures that the \( a_{i,j} \) and \( b_{i,j} \) have infinitely many components. Of course, this argument depends heavily on the assumption that we are working in \( \mathbb{R}^2 \).

We now proceed to the details, beginning with the conjuncts of \( \varphi \sigma \). The constraints

\[
\text{frame}^\circ(s_0, s_1, b, s_2, a, s_3), \quad (31)
\]

\[
\text{stack}^\circ(s_0, b_{1,1}, b_{1,2}, b_{1,3}, b), \quad (32)
\]

\[
\bigwedge_{i=0,1} \text{stack}^\circ(b_{i,2}, a_{i,1}, a_{i,2}, a_{i,3}, a), \quad (33)
\]

\[
\bigwedge_{i=0,1} \text{stack}^\circ(a_{[i-1],2}, b_{i,1}, b_{i,2}, b_{i,3}, b) \quad (34)
\]

are evidently satisfied by the arrangement of Fig. 11. Let \( \varphi \sigma \) be the conjunction of (31)–(34) as well as formulas

\[
(r \cdot r' = 0), \quad \text{for distinct variables } r \text{ and } r'. \quad (35)
\]

Note that the regions \( a_{i,j} \) and \( b_{i,j} \) have infinitely many components. We will show that this is true for every satisfying tuple of \( \varphi \).

By (31) and Lemma 5.2, there is a Jordan curve \( \sigma \lambda_0 \mu_0^{-1} \) whose segments \( \sigma \), \( \lambda_0 \) and \( \mu_0^{-1} \) are Jordan arcs lying in the respective sets \( (s_3 + s_0 + s_1)^\circ \), \( (s_2 + a + s_3)^\circ \) and \( (s_1 + b + s_2)^\circ \); see Fig. 13. Note that all points in \( s_0 \) that are on \( \sigma \lambda_0 \mu_0^{-1} \) are on \( \sigma \). Let \( \sigma' \) be the common point of \( \mu_0 \) and \( \lambda_0 \) and \( q_{1,1} \in \sigma \cap s_0^\circ \).

A word is required concerning the generality of this and other diagrams in this section. The reader is to imagine the figure drawn on a \textit{spherical} canvas, of which the sheet of paper or computer screen in front of him is simply a small part. This sphere represents the plane with a ‘point’ at infinity, under the usual stereographic projection. We do not say where this point at infinity is, other than that it never lies on a drawn arc. In this way, a diagram in

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By (33) and Lemma 5.3, we can connect $\tilde{\lambda}$ to $\beta$ since the boundary of $\alpha$ and $\beta$ of $\tilde{\alpha}$ is disjoint from $b$.

Moreover, all points of $\alpha$ in the final segment of $\beta$, starting at $q_1$, by Lemma 5.1, $q_1 \in S_0$. Similarly, let $q_1$ be the first point on $\beta$ that is on $\mu_0$ and let $\beta_1$ be the initial segment of $\beta$ ending at $q_1$; by Lemma 5.1, $q_1 \in S_0$. Hence, the arc $\beta_1 = \beta_1 \beta_2 \beta_3$ lies in exactly one of the regions bounded by $\sigma \lambda_0 \mu_0$. For reasons that will emerge in the course of the proof, we denote that region by $R'_0$. Now, $\beta_1$ divides $R'_0$ into two sub-regions: we denote the sub-region whose boundary is disjoint from $a$ by $S_1$, and the other sub-region by $S'_1$. Let $\mu_1 = \beta_1 \mu_0 q_1 \subseteq (b_1, \lambda_1)$.

The arc $\beta_1$ contains a point $p_1 \in S_0$; moreover, all points of $\beta_1$ in $S_0$ lie on $\beta_1$.

We will now construct a cross-cut $\alpha_1 \alpha_2 \alpha_3$ in $S'_1$. Let $p_1$ be a point in $\lambda_0 \cap a^\circ$. By (33) and Lemma 5.3, we can connect $p_1$ to $p_3$ by a Jordan arc $\alpha_1 \alpha_2 \alpha_3$ whose segments lie in the respective sets $(b_1, a_1, \alpha_1, a_2, \alpha_3)$; see Fig. 15. Let $p_1$ be the last point on $\alpha_1$ that is on $\beta_2$ and let $\alpha_1$ be the final segment of $\alpha_1$, starting at $p_1$; by Lemma 5.1, $p_1 \subseteq S_0$. Similarly, let $p_3$ be the first point on $\alpha_3$ that is on $\lambda_0$ and let $\alpha_3$ be the initial segment of $\alpha_3$ ending at $p_3$; by Lemma 5.1, $p_3 \subseteq a^\circ$. Since $\alpha_3 = \alpha_1 \alpha_2 \alpha_3$ does not intersect the boundaries of $S_1$ and $S'_1$ except at its endpoints, it is a cross-cut in one of these regions. Moreover, that region has to be $S'_1$ since the boundary of $S_1$ is disjoint from $a$. So, $\alpha_1$ divides $S'_1$ into two sub-regions: we denote the sub-region whose boundary is disjoint from $b$ by $R_1$, and the other sub-region by $R'_3$. Let $\lambda_1 = \alpha_1 \lambda_0 (p_1, a') \subseteq (a_1, a_3 + a + a + s_2)^\circ$. The arc $\alpha_1$ contains a point $\lambda_2 \in a^\circ$; moreover, all points of $\alpha_1$ in $a^\circ$ lie on $\alpha_1$.
We can now forget about the region $S_1$, and start constructing a cross-cut $\beta_{1,2} \beta_{2,3}$ in $R'_1$. As before, let $\tilde{\varphi}_{2,3} \in \mu_1 \cap b^\circ$. Then there is a Jordan arc $\tilde{\beta}_{2,1} \beta_{2,2} \beta_{2,3}$ connecting $\tilde{\varphi}_{2,3}$ to $\tilde{\varphi}_{2,3}$ such that its segments are contained in the respective sets $(a_{1,2} + b_{0,1})^\circ$, $(b_{0,1} + b_{0,2} + b_{0,3})^\circ$ and $(b_{0,3} + b)^\circ$. As before, we choose $\beta_{2,1} \subseteq \tilde{\beta}_{2,1}$ and $\beta_{2,3} \subseteq \tilde{\beta}_{2,3}$ so that the Jordan arc $\beta_{2,1} \beta_{2,2} \beta_{2,3}$ (apart from its endpoints) is disjoint from the boundaries of $R_1$ and $R'_1$. Hence $\beta_{2,1} \beta_{2,2} \beta_{2,3}$ has to be a cross-cut in $R_1$ or $R'_1$, and since the boundary of $R_1$ is disjoint from $b$ it has to be a cross-cut in $R'_1$. So, $\beta_{2,1} \beta_{2,2} \beta_{2,3}$ separates $R'_1$ into two regions $S_2$ and $S'_2$ so that the boundary of $S_2$ is disjoint from $a$. Let $\mu_2 = \beta_{2,3} \mu_1 [\varphi_{2,3}, \varphi'] \subseteq (b_{0,3} + b_{1,3} + s_1 + b + s_2)^\circ$. Now, we can ignore the region $R_1$, and reasoning as before we can construct a cross-cut $\alpha_2 = \alpha_{2,1} \alpha_{2,2} \alpha_{2,3}$ in $S'_2$, dividing it into two sub-regions $R_2$ and $R'_2$.

Evidently, this process continues forever: $R_{i-1}$ is divided into $S_i$ and $S'_i$ and $S'_2$ is divided into $R_i$ and $R'_i$. Now, the boundary of $S_i$ contains the arc $\beta_{i,1}$, whence the interior of $S_i$ contains points of $b_{i,2}$. On the other hand, $S_i$ certainly lies outside $S'_{i+1}$; moreover, $\delta S'_{i+1}$ is a subset of $\alpha_{i,2} \cup \beta_{i+1,1} \cup \beta_{i+1,2} \cup \mu_{i+1} \cup \lambda_i$, whence

$$\delta S'_{i+1} \subseteq (a_{i,1} + a_{i,2} + a_{i,3})^\circ \cup (a_{i,1} + b_{i+1,1}, 1)^\circ \cup (b_{i+1,1} + b_{i+1,2} + b_{i+1,3})^\circ \cup (b_{0,3} + b_{1,3} + b + s_1 + s_2)^\circ \cup (a_{0,3} + a_{1,3} + a + s_2 + s_3)^\circ.$$

Hence $\delta S'_{i+1}$ contains no points of $b_{i,2}$. Yet $S'_{i+1}$ includes all the regions $S_{i+2k}$ for $k \geq 1$, each of which contains points of $b_{i,2}$. It follows that $b_{i,2}$ has infinitely many components.

So far we know that the $C c^\circ$-formula $\varphi^\circ_{\infty}$ forces infinitely many components. Now we replace every conjunct in $\varphi^\circ_{\infty}$ of the form $\neg C(s_1, s_2)$ by $K s m(\tilde{r}) \land (s_1 \leq r_1) \land (s_2 \leq r_2)$, where $\tilde{r}$ is a vector of fresh variables. By Lemma 5.4 $(i)$, the resulting formula entails $\varphi^\circ_{\infty}$. Conversely, to show that the formula is satisfiable, we apply Lemma 5.4 $(ii)$: it suffices to separate every pair of disjoint regions in Fig. 11 by a Jordan curve. Such a Jordan curve is shown in Fig. 16 for $b_{1,2}$ and $a_{1,2}$. Other pairs of disjoint regions are treated analogously. □

5.2. Undecidability in the plane

We now return to the question of decidability. We know from Sec. 4 that $Sat(\mathcal{L}, RCP(\mathbb{R}))$ is undecidable, where $\mathcal{L}$ is any of the languages $Bc$, $Cc$ or $C c^\circ$. We proceed to establish the undecidability of the problems $Sat(\mathcal{L}, RCP(\mathbb{R}))$, where $\mathcal{L}$ is any of the languages $Bc$, $Cc$ or $C c^\circ$, and also of the problem $Sat(Bc^\circ, RCP(\mathbb{R}))$. Most of the techniques required
have been rehearsed in the proof of Theorem 5.5. However, we face a new difficulty. In the language \( \mathcal{B}c^c \), we can say that the interior of a region (rather than merely the region itself) is connected. Since, for open sets, connectedness implies arc-connectedness, we were able, in the proof of Theorem 5.5, to write formulas enforcing various arrangements of Jordan arcs in the plane. When dealing with \( \mathcal{B}c \) and \( \mathcal{C}c \), however, we can speak merely of the connectedness of a region (rather than of its interior), which, for elements of \( \text{RC}(\mathbb{R}^2) \) does not imply arc-connectedness; this complicates the business of enforcing the requisite arrangements of Jordan arcs.

To overcome this difficulty, we employ the technique of ‘wrapping’ a region inside two bigger ones. If \( a \) and \( b \) are regions such that \( -C(a, -b) \), we write \( a \ll b \) (pronounced: \( a \) is right inside \( b \)). Let us say that a 3-region is a triple \( a = (a, \bar{a}, \hat{a}) \) of elements of \( \text{RC}(\mathbb{R}^2) \) such that \( 0 \neq \bar{a} \ll \hat{a} \ll a \). It helps to think of \( a = (a, \bar{a}, \hat{a}) \) as consisting of a kernel, \( \hat{a} \), encased in two protective layers: an inner shell, \( \bar{a} \) and an outer shell, \( a \). As a simple example, consider the sequence of 3-regions \( a_1, a_2, a_3 \) depicted in Fig. 17, where the kernels form a sequence of externally touching polygons. When describing arrangements of 3-regions, we use the variable \( r \) for the triple of variables \( (r, \hat{r}, \bar{r}) \), taking the following conjuncts to be implicit:

\[
(r \neq 0) \land (\hat{r} \ll \bar{r}) \land (\hat{r} \ll r).
\]

In the sequel, when depicting arrangements of 3-regions, we standardly draw only the kernels of these 3-regions, leaving the reader to imagine the encasing layers of shell. (This is simply to reduce diagrammatic clutter.)

For \( n \geq 2 \), define the \( \mathcal{C}c \)-formula

\[
\text{stack}(r_1, \ldots, r_n) = \bigwedge_{1 \leq i < n} c(\hat{r}_i + \bar{r}_{i+1} + \cdots + \bar{r}_n) \land c(\hat{r}_n) \land \bigwedge_{1 \leq i < j \leq n} -C(r_i, r_j).
\]

(Observe that the term \( c(\hat{r}_i + \bar{r}_{i+1} + \cdots + \bar{r}_n) \) features the inner shell of \( r_1 \), and the kernels of \( r_2, \ldots, r_n \).) Thus, the triple of 3-regions \( (a_1, a_2, a_3) \) in Fig. 17 satisfies \( \text{stack}(r_1, r_2, r_3) \). This formula allows us to construct sequences of arcs with useful properties.

**Lemma 5.6.** Fix \( n \geq 2 \), and let \( a_1, \ldots, a_n \) be a tuple of 3-regions satisfying \( \text{stack}(r_1, \ldots, r_n) \). Then, for every point \( p_0 \in a_1 \) and every point \( p_n \in a_n \), there exist points \( p_1, \ldots, p_{n-1} \) and Jordan arcs \( \alpha_1, \ldots, \alpha_n \) such that: (i) \( \alpha = \alpha_1 \cdots \alpha_n \) is a Jordan arc from \( p_0 \) to \( p_n \); (ii) \( p_i \in \alpha_{i+1} \cap \alpha_i \), for all \( 1 \leq i < n \); and (iii) \( \alpha_i \subseteq a_i \), for all \( 1 \leq i \leq n \).

**Proof.** Let \( v_0 = p_0 \). Since \( v_0 \in a_1^1 \), \( p_n \in a_n^2 \) and \( a_1 + \bar{a}_2 + \cdots + \bar{a}_n \) is connected, we see that \( v_0 \) and \( p_n \) lie in the same component of \( (a_1 + \bar{a}_2 + \cdots + \bar{a}_n)^c \). So let \( \beta_1 \) be a Jordan arc connecting \( v_0 \) to \( p_n \) in \( (a_1 + \bar{a}_2 + \cdots + \bar{a}_n)^c \). Since \( a_1 \) is disjoint from all the \( a_i \) except \( a_2 \), let \( p_1 \) be the first point of \( \beta_1 \) lying in \( \bar{a}_2 \), so \( \beta_1[v_0, p_1] \subseteq a_1^1 \cup \{p_1\} \), i.e., the arc \( \beta_1[v_0, p_1] \) is either included in \( a_1^1 \), or is an end-cut of \( a_1^1 \). (We do not rule out \( v_0 = p_1 \).) Similarly, let \( \beta_2 \) be a Jordan arc connecting \( p_1 \) to \( p_n \) in \( (a_2 + \bar{a}_3 + \cdots + \bar{a}_n)^c \), and let \( q_1 \) be the last point of \( \beta_2 \) lying on \( \beta_1[v_0, p_1] \). If \( q_1 = p_1 \), then set \( v_1 = p_1 \), \( \alpha_1 = \beta_1[v_0, p_1] \), and \( \beta_2 = \beta_2 \), so that the endpoints of \( \beta_2 \) are \( v_1 \) and \( p_n \). Otherwise, we have \( q_1 \in a_1^2 \). We can now construct an
arc $\gamma_1 \subseteq a_0^\circ \cup \{p_1\}$ from $p_1$ to a point $v_1$ on $\beta_2[q_1, p_n]$, such that $\gamma_1$ intersects $\beta_1[v_0, p_1]$ and $\beta_2[q_1, p_n]$ only at its endpoints, $p_1$ and $v_1$; see Fig. 18.

\begin{figure*}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig18}
\caption{Proof of Lemma 5.6.}
\end{figure*}

Let $\alpha_1 = \beta_1[v_0, p_1]\gamma_1$, and let $\beta_2 = \beta_2[q_1, p_n]$. Since $\beta_2$ contains a point $p_2 \in \dot{a}_3$, we may iterate this procedure, obtaining $\alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \beta_n$. We remark that $\alpha_i$ and $\alpha_{i+1}$ have a single point of contact by construction, while $\alpha_1$ and $\alpha_j$ ($i < j - 1$) are disjoint by the constraint $\neg C(a_i, a_j)$. Finally, we let $\alpha_n = \beta_n$; see Fig. 18. \hfill \square

In fact, we can add a ‘switch’ to the formula \text{stack}(t_1, \ldots, t_n), in the following sense. Recall from Sec. 2 that if $a, a_0, \ldots, a_{n-1}$ are regions satisfying \text{colour}(r; r_0, \ldots, r_{n-1}), then every connected subset of $a$—and in particular, any component of $a$—is included in exactly one of the $a_0, \ldots, a_{n-1}$. Let $z$ be a variable, and consider what happens when we replace $\dot{r}_1$ in \text{stack}(t_1, \ldots, t_n) by $(-z) \cdot \dot{r}_1$, and add the conjunct \text{colour}(\dot{r}_1; z, -z)$. The result is

\[
\text{stack}_z(t_1, \ldots, t_n) = \text{colour}(\dot{r}_1; z, -z) \land \text{colour}((-z) \cdot \dot{r}_1 + \dot{r}_2 + \cdots + \dot{r}_n) \land \bigwedge_{1 < i < n} c(\dot{r}_i + \dot{r}_{i+1} + \cdots + \dot{r}_n) \land c(\dot{r}_n) \land \bigwedge_{1 \leq i, j \leq n, j > i} \neg C(r_i, r_j).
\]

Now let $a_1, \ldots, a_n$ be 3-regions and $d$ a region satisfying \text{stack}_z(t_1, \ldots, t_n).$ The first conjunct of the formula ensures that any component of $\dot{a}_1$ is either included in $d$ or included in $-d$. The remaining conjuncts then have the same effect as \text{stack}(t_1, \ldots, t_n)—but only for those components of $\dot{a}_1$ included in $-d$. That is, if $p \in (-d) \cdot \dot{a}_1$, we can find an arc $\alpha_1 \cdots \alpha_n$ starting at $p$, with the properties of Lemma 5.6. However, if $p \in d \cdot \dot{a}_1$, no such arc need exist. Thus, the variable $z$ functions so as to ‘de-activate’ \text{stack}_z(t_1, \ldots, t_n) when we are dealing with a component of $\dot{r}_1$ included in $z$.

As a further application of Lemma 5.6, consider the formula

\[
\text{frame}(t_0, \ldots, t_n) = \text{stack}(t_0, \ldots, t_{n-1}) \land \neg C(r_n, r_1 + \cdots + r_{n-2}) \land c(\dot{r}_n) \land (\dot{r}_0 \cdot \dot{r}_n \neq 0) \land (\dot{r}_{n-1} \cdot \dot{r}_n \neq 0).
\]

This formula allows us to construct Jordan curves in the plane, in the following sense:

\begin{lemma}
Fix $n \geq 3$, and let $a_0, \ldots, a_n$ be a tuple of 3-regions satisfying \text{frame}(t_0, \ldots, t_n).$ Then there exist Jordan arcs $\gamma_0, \ldots, \gamma_n$ such that $\gamma_0 \cdots \gamma_n$ is a Jordan curve and $\gamma_i \subseteq a_i$, for all $0 \leq i \leq n$.
\end{lemma}

\begin{proof}
By Lemma 5.6, let $a_0, \gamma_1, \ldots, \gamma_{n-2}, a_{n-1}$ be Jordan arcs in the respective regions $a_0, \ldots, a_{n-1}$ such that $a_0 \cdots a_{n-1}$ is a Jordan arc connecting a point $\bar{p} \in a_0 \cdot a_n$ to a point $\bar{q} \in a_{n-1} \cdot a_n$; see Fig. 19. Because $\dot{a}_n$ is a connected subset of the interior of $a_n$, let $\alpha_n \subseteq a_n^\circ$ be an arc connecting $\bar{p}$ and $\bar{q}$. Note that $\alpha_n$ does not intersect $\alpha_i$, for $1 \leq i \leq n - 2$. Let $p$ be the last point on $\alpha_0$ that is on $\alpha_n$ (possibly $\bar{p}$); and $q$ be the first point on $\alpha_{n-1}$ that is on $\alpha_n$ (possibly $\bar{q}$). Let $\gamma_0$ be the final segment of $\alpha_0$ starting at $p$ and let $\gamma_{n-1}$ be the
whether an initial segment of \(a_{n-1}\) ending at \(q\). Finally, let \(\gamma_n = \alpha_n[p, q]\) or \(\gamma_n = \alpha_n[q, p]\), depending on whether \(p\) or \(q\) is encountered first on \(\alpha_n\). Then the arcs \(\gamma_i, 0 \leq i \leq n\), are as required.

We are now ready to prove the main result of this section. Again, recall that if \(a, a_0, \ldots, a_{n-1}\) are regions of the alphabet \(\alpha\) —and in particular, any Jordan arc \(a \subseteq a)—is included in exactly one of the \(a_i\), for \(0 \leq i < n\). In this case, it is sometimes helpful to think of \(a\) as being ‘labelled’ by a letter of the alphabet \(a_0, \ldots, a_{n-1}\).

**Theorem 5.8.** The problems \(\text{Sat}(Cc, RC(\mathbb{R}^2))\) and \(\text{Sat}(Cc, RCP(\mathbb{R}^2))\) are r.e.-hard.

**Proof.** Again, we proceed via a reduction of the Post correspondence problem (PCP), constructing, for any instance \(W\), a formula \(\psi_W\) with the property that the following are equivalent: (i) \(W\) is positive; (ii) \(\psi_W\) is satisfiable over \(RC(\mathbb{R}^2)\); (iii) \(\psi_W\) is satisfiable over \(RC(\mathbb{R}^2)\). This establishes the theorem. In this proof, \([k]\) indicates the value of \(k\) modulo 3.

The proof proceeds in six stages. In Stage 1, we introduce constraints satisfied by the arrangement in Fig. 20. We show that any tuple of 3-regions in \(RC(\mathbb{R}^2)\) satisfying these constraints contains the system of arcs shown in Fig. 21 (with each arc included in some specified region). These arcs form a large rectangle divided into an ‘upper window’ and a ‘lower window’. In Stage 2, we introduce constraints satisfied by the finitely repeating arrangement shown in Fig. 22. We then show, using an argument similar to that of Theorem 5.5, that any tuple of 3-regions in \(RC(\mathbb{R}^2)\) satisfying these constraints contains the finite sequence of triples of arcs \((\zeta_i, \eta_i, \kappa_i)\) shown in Fig. 27 (again, with each arc included in some specified region) such that the \(\eta_i\) all lie in the lower window. Stage 3 simply duplicates Stage 2 to obtain the finite sequence of triples of arcs \((\zeta_i', \eta_i', \kappa_i')\), with the \(\eta_i'\) all lying in the upper window. In Stage 4, we introduce constraints that force these two sequences to have the same length, yielding the arrangement of Fig. 28. At this point, the main geometrical work has been accomplished. In Stage 5, we introduce constraints relating specifically to our PCP-instance \(W\), over the alphabets \(T\) and \(U\). These constraints allow us to ‘colour’ the various arcs \(\eta_i\) with the elements of \(U\), thus spelling out a word of \(U^*\), and similarly for the \(\eta_i'\); indeed we show that these two words must be identical. In addition, our constraints allow us to group the arcs \(\eta_i\) into contiguous blocks, with each block coloured with some element of \(T\), thus obtaining a word \(\tau\) of \(T^*\), and similarly for the \(\eta_i'\), obtaining a word \(\tau'\) of \(T^*\).

Using additional constraints, we can ensure that, within any block of the arcs \(\eta_i\) coloured with \(t \in T\), the letters from \(U\) spell out the lower string on the tile \(t\); similarly, any block of the \(\eta_i'\) coloured with \(t \in T\) spells out the upper string on the tile \(t\). Finally, in Stage 6, we line up the \(\eta_i\)-blocks and the \(\eta_i'\)-blocks, using the same technique as employed in Stage 4. We show that the words \(\tau\) and \(\tau'\) formed by the lower and upper block-sequences must be identical. It follows that, if any tuple of regions in \(RC(\mathbb{R}^2)\) satisfies \(\psi_W\), then \(W\) must have a solution. Conversely, if \(W\) has a solution, we can construct a tuple from \(RCP(\mathbb{R}^2)\) satisfying \(\psi_W\), in the style of Fig. 22. This completes the reduction.

**Stage 1.** In the first stage, we define an assemblage of arcs that will serve as scaffolding for the ensuing construction. Consider the arrangement of polygonal 3-regions depicted in Fig. 20, assigned to the 3-region variables \(\varphi_0, \ldots, \varphi_9, \varphi_8, \ldots, \varphi_1, \vartheta_0, \ldots, \vartheta_6\) as indicated. (Note that we have here followed the convention of depicting only the kernels of 3-regions.)

![Fig. 19. Establishing a Jordan curve.](image-url)
It is easy to verify that this arrangement can be made to satisfy the following formulas:

\[
\text{frame}(s_0, s_1, \ldots, s_8, s_9, s_0', \ldots, s_1'), \tag{36}
\]
\[
(s_0 \leq d_0) \land (s_9 \leq d_9), \tag{37}
\]
\[
\text{stack}(d_0, \ldots, d_6). \tag{38}
\]

And obviously, the arrangement can be made to satisfy any formula

\[
-\text{C}(r, r'), \tag{39}
\]

for which the corresponding 3-regions \( r \) and \( r' \) are drawn as not being in contact. (Remember, \( r \) is the outer shell of the 3-region \( r \), and similarly for \( r' \); so we must take these shells to hug the kernels depicted in Fig. 20 quite closely.) Thus, for example, (39) includes \(-\text{C}(s_0, d_1)\), but not \(-\text{C}(s_0, d_0)\) or \(-\text{C}(d_0, d_1)\).

Now suppose \( s_0, \ldots, s_9, s_0', \ldots, s_1' \) is any collection of 3-regions (not necessarily polygonal) satisfying (36)–(39). By Lemma 5.7 and (36), let \( \gamma_0, \ldots, \gamma_9, \gamma_8', \ldots, \gamma_1' \) be Jordan arcs included in the respective regions \( s_0, \ldots, s_9, s_0', \ldots, s_1' \), such that \( \Gamma = \gamma_0 \cdot \gamma_0', \gamma_0, \gamma_8' \cdot \gamma_1' \) is a Jordan curve (note that \( \gamma_0' \) and \( \gamma_0 \) have opposite directions). We select points \( \hat{o} \) on \( \gamma_0 \) and \( \hat{o}' \) on \( \gamma_9 \); see Fig. 21. By (37), \( \hat{o} \in d_0 \) and \( \hat{o}' \in d_9 \). By Lemma 5.6 and (38), let \( \hat{x}_1, \hat{x}_2, \hat{x}_3 \) be Jordan arcs in the respective regions \( (d_0 + d_3), (d_2 + d_3 + d_4), (d_5 + d_6) \) such that \( \hat{x}_1 \hat{x}_2 \hat{x}_3 \) is a Jordan arc from \( \hat{o} \) to \( \hat{o}' \). Let \( o \) be the last point of \( \hat{x}_1 \) lying on \( \Gamma \), and let \( \chi_3 \) be the final segment of \( \hat{x}_3 \), starting at \( o \). Let \( o' \) be the first point of \( \hat{x}_3 \) lying on \( \Gamma \), and let \( \chi_3 \) be the initial segment of \( \hat{x}_3 \), ending at \( o' \). By (39), we see that the arc \( \chi_1 \chi_2 \chi_3 \) intersects \( \Gamma \) only in its endpoints, and is thus a chord of \( \Gamma \), as shown in Fig. 21.

As before, we treat these diagrams as being drawn on a spherical canvas. For ease of reference, we refer to the two rectangles in Fig. 21 as the ‘upper window’ and ‘lower window,’ it being understood that these are simply handy labels: in particular, either (but not both) of these ‘windows’ may be unbounded.
Stage 2. In this stage, we construct a sequence of triples \((\zeta_i, \eta_i, \kappa_i)\) of arcs of indeterminate length \(n \geq 1\), such that the members of \(\zeta_i\) all lie in the lower window. (Recall that \([k]\) denotes \(k\) modulo 3). Let \(a, b, \delta, a_i, j\) and \(b_{i,j}\) \((0 \leq i < 3, 1 \leq j \leq 6)\) be 3-region variables, and consider the formulas

\[
(s_6 \leq \bar{a}) \land (s_6' \leq \bar{b}) \land (s_3 \leq \bar{a}_{0,3}), \tag{40}
\]

\[
\bigwedge_{i=0,1,2}\text{stack}(a_{i-1,3}, b_{i,1}, b_{i,6}, b), \tag{41}
\]

\[
\bigwedge_{i=0,1,2}\text{stack}(b_{i,3}, a_{i,1}, \ldots, a_{i,6}, a). \tag{42}
\]

(Observe that the \text{stack}-formula in (41) has a switch, the inner shell \(\hat{z}\) of \(\delta\).) The arrangement of polygonal 3-regions depicted in Fig. 22 (with \(\delta\) assigned appropriately) is one such satisfying assignment. We stipulate that (39) applies now to all regions depicted in either

Fig. 22. A tuple of 3-regions satisfying (40)--(42) (showing kernels only).

Fig. 20 or Fig. 22, and we further stipulate

\[
z \cdot (s_0 + \cdots + s_9 + s_1' + \cdots + s_6' + d_1 + \cdots + d_4) = 0. \tag{43}
\]

Note that \(d_5\) does not appear in this constraint; thus, \(z^\circ\) may intersect the arc \(\chi_3\). Again, these additional constraints are evidently satisfiable.

Now suppose we are given any collection of regions (not necessarily polygonal) satisfying (36)--(43). And let the arcs \(\gamma_0, \ldots, \gamma_9, \gamma_9', \ldots, \gamma'_1\) and \(\chi_1, \chi_2, \chi_3\) be as defined above. It will be convenient in this stage to rename \(\gamma_0\) and \(\gamma'_0\) as \(\lambda_0\) and \(\mu_0\), respectively. Thus, \(\lambda_0\) forms the bottom edge of the lower window, and \(\mu_0\) the top edge of the upper window. Likewise, we rename \(\gamma_3\) as \(\alpha_0\), forming part of the left-hand side of the lower window. Let \(\tilde{q}_{1,1}\) be any point of \(\alpha_0\), \(p^*\) any point of \(\lambda_0\), and \(q^*\) any point of \(\mu_0\); see Fig. 23. By (40), then, \(\tilde{q}_{1,1} \in \alpha_{0,3}\), \(p^* \in \bar{a}\), and \(q^* \in \bar{b}\). Certainly, the constraint (43) ensures that \(\tilde{q}_{1,1} \in (-\hat{z})\).

By Lemma 5.6 and (41), we may draw an arc \(\tilde{\beta}_1\) from \(\tilde{q}_{1,1}\) to \(q^*\), with successive segments \(\tilde{\beta}_{1,1}, \tilde{\beta}_{1,2}, \ldots, \tilde{\beta}_{1,5}, \tilde{\beta}_{1,6}\) lying in the respective regions \(a_{0,3} + b_{1,1}, b_{1,2}, \ldots, b_{1,5}, b_{1,6} + b\); further, we can guarantee that \(\tilde{\beta}_{1,2}\) contains a point \(\bar{p}_{1,1} \in \bar{b}_{1,3}\). Denote the last point of \(\tilde{\beta}_{1,5}\) by \(q_{1,2}\). Also, let \(q_{1,1}\) be the last point of \(\tilde{\beta}_1\) lying on \(\alpha_0\), and \(q_{1,3}\) the first point of \(\tilde{\beta}_1\) lying on \(\mu_0\). Finally, let \(\beta_1\) be the segment of \(\tilde{\beta}_1\) between \(q_{1,1}\) and \(q_{1,2}\); and let \(\mu_1\) be the segment of \(\tilde{\beta}_1\) from \(q_{1,2}\) to \(q_{1,3}\) followed by the final segment of \(\mu_0\) from \(q_{1,3}\); see Fig. 23. By repeatedly using the constraints in (39), it is easy to see that \(\beta_1\) and the initial segment of \(\mu_1\) up to \(q_{1,3}\) together form a chord of \(\Gamma\). Adding the constraint

\[
c(b_{1,5} + d_3), \tag{44}
\]
and taking into account (39) ensures that this chord divides the residual domain of $\Gamma$ containing $\chi_2$ into the regular closed sets $S_1$ and $S'_1$, as shown in Fig. 23. The wiggly lines indicate that we do not care about the exact positions of $\dot{q}_{1,1}$ or $q^*$; otherwise, Fig. 23 is again completely general. Note that $\mu_1$ lies entirely in $b_{1,6} + b$, and hence certainly in

$$b^* = b_{0,6} + b_{1,6} + b_{2,6} + b.$$  \hfill (45)

![Fig. 23. The arc $\beta_1$.](image)

Recall that $\tilde{p}_{1,1} \in \tilde{b}_{1,3}$ and $p^* \in \tilde{a}$. By Lemma 5.6 and (42), we may draw an arc $\tilde{\alpha}_1$ from $\tilde{p}_{1,1}$ to $p^*$, with successive segments $\tilde{\alpha}_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,5}, \alpha_{1,6}$ lying in the respective regions $b_{1,3} + a_{1,1}, a_{1,2}, \ldots, a_{1,5}, a_{1,6} + a$; further, we can guarantee that the segment $\alpha_{1,2}$ contains a point $\tilde{q}_{2,1} \in \tilde{a}_{1,3}$. (Thus: $\alpha_{1,2}$ lies in $a_{1,2}$, but nevertheless contains at least one point lying in $\tilde{a}_{1,3}$.) Denote the last point of $\tilde{\alpha}_1$ by $p_{1,1}$. Also, let $p_{1,1} \in b_{1,3}$ be the last point of $\tilde{\alpha}_1$ lying on $\lambda_1$, and $p_{1,1}$ the first point of $\tilde{\alpha}_1$ lying on $\lambda_0$. By (39), these points must be arranged as shown in Fig. 24. In particular, the segment of $\tilde{\alpha}_1$ between $p_{1,1}$ and $p_{1,3}$ is a chord in $S'_1$ and divides it into regions $R_1$ and $R'_1$. Let $\alpha_1$ be the segment of $\tilde{\alpha}_1$ between $p_{1,1}$ and $p_{1,2}$. Noting that (39) entails

$$-C(a_{1,1} + \cdots + a_{1,6}, s_0 + s_0 + d_0 + \cdots + d_5),$$

we can be sure that $\alpha_1$ lies entirely in the ‘lower’ window, whence $\beta_1$ crosses the central chord, $\chi_2$ at least once. Let $\alpha_1$ be the first such point (measured along $\chi_2$ from left to right). Finally, let $\lambda_1$ be the segment of $\tilde{\alpha}_1$ between $p_{1,2}$ and $p_{1,3}$, followed by the final segment of $\lambda_0$ from $p_{1,3}$. Note that $\lambda_1$ lies entirely in $a_{1,6} + a$, and hence certainly in the region

$$\alpha^* = a_{0,6} + a_{1,6} + a_{2,6} + a.$$  \hfill (46)

The region $S_1$ may now be forgotten.

By construction, the point $\tilde{q}_{2,1}$ lies in some component of $\tilde{a}_{1,3}$, and, from the presence of the ‘switching’ variable $\tilde{z}$ in (42), that component is either included in $\tilde{z}$ or included in $-\tilde{z}$. Suppose the latter. Then we can repeat the above construction to obtain an arc $\tilde{\beta}_2$ from $\tilde{q}_{2,1}$ to $q^*$, with successive segments $\tilde{\beta}_{2,1}, \beta_{2,2}, \ldots, \beta_{2,5}, \beta_{2,6}$ lying in the respective regions $a_{1,3} + b_{2,1}, b_{2,2}, \ldots, b_{2,5}, b_{2,6} + b$; further, we can guarantee that $\beta_{2,2}$ contains a point $\tilde{p}_{2,1} \in b_{2,3}$. Denote the last point of $\beta_{2,5}$ by $q_{2,2}$. Also, let $q_{2,1} \in b_{2,3}$ be the last point of $\beta_{2}$ lying on $\alpha_1$, and $q_{2,2}$ the first point of $\beta_{2}$ lying on $\mu_1$. Again, we let $\beta_2$ be the segment of $\tilde{\beta}_2$ between $\tilde{q}_{2,1}$ and $q_{2,2}$; and we let $\mu_2$ be the segment of $\tilde{\beta}_2$ from $q_{2,1}$ to $q_{2,2}$, followed by the final segment of $\mu_1$ from $q_{2,2}$. Note that $\mu_2$ lies in the set $b^*$. It is easy to see that the segment of $\beta_2$ from $q_{2,1}$ to $q_{2,3}$ is a cross-cut in $R'_1$ dividing it into regions $S_2$ and $S'_2$, as shown in Fig. 25. Indeed, $\beta_2 = \beta_2[q_{2,1}, q_{2,2}]$ cannot enter the interior of the region $R_1$, for, by construction, it can have only one point of contact with $\alpha_1$, and the constraints (39)
ensure that it cannot intersect any other part of \( \delta R_1 \). Since \( q^* \notin \tilde{a} \) is guaranteed to lie outside \( R_1 \), we evidently have that \( \beta_2 \subseteq -R_1 \). By the constraints (39), \( \beta_2 \) lies in the interior of \( R_1 \) except for its first point, which lies on the boundary of \( R_1 \); hence the reversal of \( \beta_2 \) is an end-cut in \( R_1 \). Similarly, \( \tilde{\beta}_2[q_2,2, q_2,3] \) is an end-cut in \( R_1 \) as well, and thus \( \tilde{\beta}_2[q_2,1, q_2,3] \) is a cross-cut in \( R_1 \). This observation having been made, \( R_1 \) may now be forgotten.

Symmetrically, we construct the arc \( \tilde{\alpha}_2 \) in \( b_2 + a_2 + \cdots + a_6 + a \), and points \( p_{2,1}, p_{2,2}, p_{2,3} \), together with the arcs \( \alpha_2 \) and \( \lambda_2 \). Again, we know from (39) that \( \alpha_2 \) lies entirely in the 'lower' window, whence \( \beta_2 \) must cross the central chord, \( \chi_2 \), at least once. Let \( \alpha_2 \) be the first such point (measured along \( \chi_2 \) from left to right); see Fig. 25.

This process continues, generating arcs \( \beta_1, \beta_2, \ldots, \beta_n \leq a_{i-1,3} + a_{i,1} + \cdots + a_{i,5} \) and \( \alpha_i \leq b_{i,3} + a_{i,3} + \cdots + a_{i,5} \), as long as \( \alpha_i \) contains a point \( q_{i+1,1} \in (-\hat{z}) \). That we eventually reach a value \( i = n \) for which no such point exists follows from (39). For the conjuncts \( -C(b_{i,j}, d_k) \), for \( j \neq 5 \), together entail \( \alpha_i \in b_{i,j,5} \), for every \( i \) such that \( \beta_i \) is defined; and
the constraints (39) guarantee that ζ_i \subseteq \eta_i \subseteq b_{i,j} and \kappa_i \subseteq a_{i,j}, for 1 \leq i \leq n. Observe that the arcs ζ_i are located entirely in the ‘lower window,’ and that each arc \eta_i connects \ζ_i to some point \eta_i, which in turn is connected to \eta_i by an arc in \beta_i.

**Stage 3.** We now repeat Stage 2 symmetrically, with the ‘upper’ and ‘lower’ windows exchanged. Let \alpha_{i,j}, b'_{i,j} be 3-region variables (with indices in the same ranges as for \alpha_{i,j}, b_{i,j}). Let \alpha' = b, b' = a. The formulas

\[
\begin{align*}
(s' &\leq s_{0,3}), \\
\bigwedge_{i=0,1,2} \text{stack}_2(a'_{i-1,3}, b'_{i,1}, \ldots, b'_{i,6}, b'), \\
\bigwedge_{i=0,1,2} \text{stack}(b'_{i,1}, a'_{i,1}, \ldots, b'_{i,6}, a'), \\
c(b'_{i,5} + d_3) 
\end{align*}
\]

then establish sequences of n' triples of arcs (ζ'_i, η'_i, κ'_i) satisfying ζ'_i \subseteq r'_{i,j}, η'_i \subseteq b'_{i,j} and κ'_i \subseteq a'_{i,j}, for 1 \leq i \leq n', where the r'_{i,j}, b'_{i,j} and a'_{i,j} are defined as in (47)–(49) but with the primed variables. The arcs ζ'_i are located entirely in the ‘upper window’, and each arc \eta'_i connects ζ'_i to a point \eta'_i, which in turn is connected to a point q' by an arc in the region b' = b'_{0,6} + b'_{1,6} + b'_{2,6} + b'.

**Stage 4.** Our next task is to write constraints to ensure that n = n’, and that, furthermore, each \eta_i (as well each \eta'_i) connects \ζ_i to ζ'_i, for 1 \leq i \leq n. From (43), the only arc depicted in Fig. 21 that \zeta may intersect is \chi_3. Recalling that \zeta_n and \zeta_{n'} contain points q_{n+1,1} and q'_{n'+1,1}, respectively, both lying in \zeta' \subseteq \zeta, the constraint

\[
c(\hat{z})
\]

ensures that q_{n+1,1} and q'_{n'+1,1} may be joined by an arc, say ζ*, lying in \zeta', and also lying entirely in the upper and lower windows, crossing the chord \chi_1 \chi_2 \chi_3 only in \chi_3. Without
loss of generality, we may assume that \( \zeta^* \) contacts each of \( \zeta_n \) and \( \zeta'_n \) in just one point. Bearing in mind that the constraints (39) force \( \eta_n \subseteq b_0 + b_1 + b_2 \) and \( \eta'_n \subseteq b'_0 + b'_1 + b'_2 \) to cross the chord \( \chi_1 \chi_2 \chi_3 \) in its central section, \( \chi_2 \), and from (43), we see that the following constraint ensures that \( \zeta^* \) is as shown in Fig. 28:

\[
 z \cdot (b^* + b_0 + b_1 + b_2 + b'^* + b'_0 + b'_1 + b'_2) = 0.
\]  

Fig. 28. The arc \( \zeta^* \).

Now consider the arc \( \eta_1 \). Recalling that \( \eta_1 \) crosses \( \chi_2 \) and connects \( \zeta_1 \) to some point \( q_{1,2} \), which in turn is connected to the point \( q^* \) by an arc in \( b^* \), we see by inspection of Fig. 28 that (51) together with

\[
\bigwedge_{i=0,1,2} \neg C(r'_i, b^*)
\]

forces \( \eta_1 \) to cross one of the arcs \( \zeta'_j \subseteq r'_{j-1} \), for \( 1 \leq j' \leq n' \); and the constraints

\[
\bigwedge_{i=0,1,2} \neg C(r'_i, b_{i-1} + b_{i+1})
\]

ensure that \( j' \equiv 1 \) modulo 3. Now suppose \( j' \geq 4 \). We write the constraints

\[
\bigwedge_{i=0,1,2} \neg C(b'_i, r_{i-1} + r_{i+1}),
\]

\[
\bigwedge_{i=0,1,2} \neg C(b'_i, b_{i-1} + b_{i+1}).
\]

The arc \( \eta'_2 \) must connect \( \zeta'_2 \) to the point \( q'_{2,2} \), which in turn is connected to the point \( p^* \) on the bottom edge of the lower window by an arc in \( b'^* \), which is now impossible without \( \eta'_2 \subseteq b'_2 \) crossing either \( \zeta \subseteq r_1 \) or \( \eta \subseteq b_1 \)—both forbidden by (53)–(54). Thus, \( \eta_1 \) intersects \( \zeta'_2 \) if and only if \( j = 1 \). Symmetrically, \( \eta'_2 \) intersects \( \zeta_2 \) if and only if \( j = 1 \). And the reasoning can now be repeated for \( \eta_2, \eta'_2, \eta_3, \eta'_3, \ldots \), leading to the 1–1 correspondence depicted in Fig. 29. In particular, we are guaranteed that \( n = n' \).

**Stage 5.** Recall the given PCP-instance, \( W = (w, w') \) over alphabets \( T \) and \( U \). In the sequel, we use the standard imagery of ‘tiles’, where each tile \( t \in T \) has an ‘upper string’, \( w'(t) \in U^* \) and a ‘lower string’, \( w(t) \in U^* \). Thus, the problem is to determine whether there is some non-empty sequence of tiles such that the concatenated upper and lower strings both spell out the same word of \( U^* \). We shall label the arcs \( \zeta_1, \ldots, \zeta_n \) so as to define a word \( \tau \in T^* \) (with \( |\tau| = m \leq n \)); likewise we shall label the arcs \( \zeta'_1, \ldots, \zeta'_n \) so as to define another word \( \tau' \in T^* \) (with \( |\tau'| = m' \leq n \)). Then the arcs \( \eta_1, \ldots, \eta_n \) will be labelled with the regions in
$\vec{u}$, so to define a string $v \in U^*$, with $|v| = n$. We shall then add conjuncts to $\psi_\mathcal{W}$ ensuring $w(\tau) = W'(\tau') = v$ and $\tau = \tau'$, which will guarantee that $\mathcal{W}$ is positive.

For $1 \leq h \leq |T|$, $1 \leq \ell \leq |w(t_h)|$ and $0 \leq i < 3$, let $p_{h,\ell}$ be a fresh variable, and let these variables be ordered in some way as the tuple $\vec{p}$. As in the proof of Theorem 4.1, we think of $p_{h,\ell}$ as standing for the $\ell$th position in the string $w(t_h)$, where $t_h \in T$. We use $\vec{p}$ to label the components of $r_i$, $0 \leq i < 3$, but since the $r_i$ are not pairwise disjoint, we require a copy of $\vec{p}$ for each $i$. Hence, for $1 \leq h \leq |T|$, $1 \leq \ell \leq |w(t_h)|$ and $0 \leq i < 3$, let $p_{h,\ell}$ be a fresh variable, and let $\vec{p}_i$ be an ordering of the variables with superscript $i$. Consider the constraints

$$\bigwedge_{i=0,1,2} \left( (\vec{r}_i = \sum_{h=1}^{|T|} \sum_{\ell=1}^{|w(t_h)|} p_{h,\ell}^i) \land \text{colour}(r_i; \vec{p}_i) \right) \land \bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell \leq |w^k(t_h)|} (p_{h,\ell} = \sum_{i=0,1,2} p_{h,\ell}^i). \tag{55}$$

The first conjunct ensures that each arc $\beta_{i,3} \subseteq b_{i,j,3}$ ($1 \leq i \leq n$) is included in exactly one of the regions $\vec{p}_{i}^j$ and is disjoint from the rest of the regions in $\vec{p}_{i}^{j-1}$ and all the regions in $\vec{p}_{i}^{j+1}$; the second conjunct then ensures that $\zeta_i$ is contained in exactly one of the regions $\vec{p}$, and that $\beta_{i,3}$ is disjoint from the rest of the regions in $\vec{p}$. Note that the $\vec{p}$ do not actually form a partition, because they cannot be made disjoint; nevertheless, we can think of the $\vec{p}$ as ‘labels’ for arcs $\zeta_i$. The regions in $\vec{p}_0$, $\vec{p}_1$ and $\vec{p}_2$ can now be forgotten.

Next, we organize the arcs $\zeta_i$ into (contiguous) blocks, $E_1, \ldots, E_m$ such that, in the $j$th block, $E_j$, the sequence of labels reads $p_{h,1}, \ldots, p_{h,|w(t_h)|}$, for some fixed $1 \leq h \leq |T|$. This amounts to insisting that: (i) the very first arc, $\zeta_1$, must be labelled with $p_{h,1}$ for some $h$; (ii) if $\zeta_i$ ($1 < n$) is labelled with $p_{h,\ell}$, where $\ell < |w(t_h)|$, then the next arc, namely $\zeta_{i+1}$, must be labelled with the next position in $w(t_h)$, namely $p_{h,\ell+1}$; (iii) if $\zeta_i$ ($1 < n$) is labelled with the final position of $w(t_h)$, then the next arc must be labelled with the initial position of some possibly different word $w(t_h')$; and (iv) $\zeta_n$ must be labelled with the final position of some word $w(t_h)$. To do this we simply write:

$$\bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell \leq |w^k(t_h)|} \neg C(p_{h,\ell}, s_3), \tag{56}$$

$$\bigwedge_{i=0,1,2} \bigwedge_{h=1}^{\bigl|w(t_h)\bigr|} \bigwedge_{\ell=1}^{\bigl|w(t_h)\bigr|-1} \neg C(r_i \cdot p_{h,\ell}; t_{i+1} \cdot (-r_i) \cdot \left( \sum_{h' \neq h} p_{h',\ell} + \sum_{\ell' = 1}^{\bigl|w(t_{h'})\bigr|} p_{h',\ell'} \right)), \tag{57}$$

$$\bigwedge_{i=0,1,2} \bigwedge_{1 \leq h, h' \leq |T|} \bigwedge_{1 \leq \ell \leq |w^k(t_h)|} \neg C(r_i \cdot p_{h,|w(t_h)|}; t_{i+1} \cdot (-r_i) \cdot p_{h',\ell}), \tag{58}$$

$$\bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell < |w^k(t_h)|} \neg C(p_{h,\ell}, z). \tag{59}$$
Supposing the arcs of jth block $E_j$ to have labels reading $p_{h,1} \ldots p_{h,|w(t_h)|}$ (for some fixed $h$), then, we write $h_j$ to denote the common subscript $h$. The sequence of indices $h_1 \ldots h_m$ corresponding to the successive blocks thus defines a word $\tau = t_{h_1} \cdots t_{h_m} \in T^*$. Using corresponding formulas, we label the arcs $\zeta_i^t$ (1 $\leq i \leq n$) with the tuple $p^t$ of variables $p_{h,t}^i$, for $1 \leq h \leq |T|$ and $1 \leq t \leq |w(t_h)|$, so that, in any satisfying assignment over RC($\mathbb{R}^2$), every arc $\zeta_i^t$ is labeled with exactly one of the regions $p^t$ and $\beta_{i,j}^h \subseteq b_i$ is disjoint from the rest of the regions in $p^t$. Further, we can ensure that these labels are organized into (say) $m'$ contiguous blocks, $E'_1 \ldots E'_{m'}$ such that in the jth block, $E'_j$, the sequence of labels reads $p_{h,1}^i \ldots p_{h,|w(t_h)|}^i$, for some fixed $h$. Again, writing $h'_j$ for the common value of $h$, the sequence of indices $h'_1 \ldots h'_{m'}$, corresponding to the successive blocks defines a word $\tau' = t_{h'_1} \cdots t_{h'_{m'}} \in T^*$.

Now, the constraints

$$\text{part}(\bar{u}) \land \bigwedge_{i=0,1,2} \text{colour}(h_i; \bar{u})$$ (60)

ensure that, in any satisfying assignment over RC($\mathbb{R}^2$), every arc $\eta_i \subseteq b_{i|j|}$, for $1 \leq i \leq n$, is included in (‘labelled with’) exactly one of the regions in $\bar{u}$, so that the sequence of arcs $\eta_1 \ldots \eta_n$ defines a word $v \in U^*$, with $|v| = n$.

Securing $w(\tau) = w'(\tau') = v$ is easy. The constraints

$$\bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq t \leq |w(t_h)|} \bigwedge_{u_\ell}$ is not the $\ell$th letter of $w(t_h)$

ensure that, since $\eta_i$ intersects $\zeta_i$, for all $1 \leq i \leq n$, the word $v \in U^*$ defined by the arcs $\eta_i$ must be identical to the word $w(t_{h_1}) \cdots w(t_{h_m})$. But this is just to say that $v = w(\tau)$. The equation $v = w'(\tau')$ is obtained similarly.

Stage 6. In the foregoing stages, we assembled conjuncts of $\psi_W$ in such a way that, given any satisfying assignment for $\psi_W$, we can construct sequences of labelled arcs defining words $v \in U^*$ and $\tau, \tau' \in T^*$ with $w(\tau) = w'(\tau') = v$, as described above. In this stage, we add more conjuncts to $\psi_W$ to enforce the equation $\tau = \tau'$. This shows that, if $\psi_W$ is satisfiable over RC($\mathbb{R}^2$), then $\mathcal{W}$ is positive.

In particular, it remains to show that $m = m'$ and that $h_j = h'_j$, for all $1 \leq j \leq m$. To do so, we re-use the techniques encountered in Stage 4. We first introduce a new pair of variables, $f_0, f_1$, which we refer to as ‘block colours,’ and with which we label the arcs $\zeta_i$. Again, since the regions $r_i$ overlap, we additionally require regions $f_0^i$ and $f_1^i$, for $0 \leq i < 3$. Consider the constraints:

$$\bigwedge_{i=0,1,2} (\{r_i = f_0^i + f_1^i\} \land \text{colour}(r_i; f_0^i, f_1^i)) \land \bigwedge_{i=0,1,2} (f_k = \sum_{i=0,1,2} f_k^i)$$ (62)

It is readily checked that each $\zeta_i \subseteq r_{i|j|}$ is included in exactly one of the regions $f_0$ or $f_1$, and that $\beta_{i,j}^h$ is disjoint from the other. (Again, however, $f_0, f_1$ do not form a partition, because they must overlap.) We force all arcs in each block $E_j$ to have a uniform block colour, and we force the block colours to alternate by writing:

$$\bigwedge_{k=0,1} \bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 \leq \ell < |w(t_h)|} -C(f_k \cdot p_{h,\ell}; f_1 - k \cdot p_{h,\ell + 1})$$ (63)

$$\bigwedge_{i=0,1,2} \bigwedge_{k=0,1} \bigwedge_{1 \leq h, h' \leq |T|} -C(f_k \cdot p_{h,|w(t_h)|} \cdot r_i; f_k \cdot p_{h',1} \cdot r_{i+1} \cdot (-r_i))$$ (64)

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Thus, we may speak unambiguously of the colour \( (f_0 \text{ or } f_1) \) of a block: if \( E_1 \) is coloured \( f_0 \), then \( E_2 \) will be coloured \( f_1 \), \( E_3 \) coloured \( f_0 \), and so on. Using variables \( f_0' \) and \( f_1' \), we similarly establish a block structure \( E_1', \ldots , E_m' \) on the arcs \( \zeta_i' \).

![Diagram](image.png)

Fig. 30. Arc \( \theta_j \) intersecting \( \zeta_1' \cdots \zeta_n' \).

Now we match up the blocks in a 1–1 fashion, just as we matched up the individual arcs in Stage 4. Let \( g_0, g_1, g_0' \) and \( g_1' \) be new 3-region variables. Recall that every arc \( \zeta_i \) contains some point of \( b_{i,3} \) (for instance: \( p_{h,1} \)) and every such point is unambiguously labeled by a region in \( \theta \) and a region in \( (f_0, f_1) \). We wish to connect any such arc that starts a block \( E_j \) (i.e., any \( \zeta_i \) labelled by \( p_{h,1} \) for some \( h \)) to the top edge of the upper window, with the connecting arc depending on the block colour. We can do this using the constraints:

\[
\bigwedge_{k=0,1} ((f_k \cdot (b_{0,3} + b_{1,3} + b_2, 3) \leq g_k) \land \text{stack}(g_k, b)).
\]  

(65)

Specifically, the first (actually: every) arc \( \zeta_i \) in each block \( E_j \), for \( 1 \leq j \leq m \), is connected by an arc \( \theta_j \theta_j \) to some point on the upper edge of the upper window, where \( \theta_j \subseteq g_k \) and \( \theta_j \subseteq b \). Using corresponding formulas, we ensure that the first arc in each block \( E_j' \), for \( 1 \leq j \leq m' \), is connected by an arc \( \theta_j' \theta_j' \) to some point on the bottom edge of the lower window, where \( \theta_j' \subseteq g_k' \) and \( \theta_j' \subseteq b' \).

Recall from Stage 3 that \( q_{n+1,1} \) is connected by an arc \( \kappa_n \subseteq a_0 + a_1 + a_2 \) to \( p_{n,2} \), which is in turn connected to the lower edge of the lower window by an arc in lying in \( a' \). And recall from Stage 4 that \( q_{n+1,1} \) is connected by \( \zeta_i' \subseteq z' \) to \( \eta_{n+1,1} \). Thus, we see from Fig. 30 that the non-contact constraints

\[ -C(g_0 + g_1, s_2' + \cdots + s_i' + s_5 + s_5 + a' + a_0 + a_1 + a_2 + z) \]  

(66)

ensure that each \( \theta_j \theta_j \) (1 \( \leq j \leq m \)) intersects one of the \( \zeta_i' \) (1 \( \leq i \leq n \)). Indeed, since \( \theta_j \cup \theta_j \) cannot intersect any \( \zeta_i' \), we know that all such points of intersection lie on \( \theta_j \). Using a corresponding formula, we ensure that each \( \theta_j' \) (1 \( \leq j \leq m' \)) intersects one of the \( \zeta_i \) (1 \( \leq i \leq n \)).

We now write the constraints

\[
\bigwedge_{k=0,1} (-C(g_k, f_1'-k) \land -C(g_{k}', f_1'-k)).
\]  

(67)

Thus, any \( \theta_j \) included in \( g_k \) must join some arc \( \zeta_i \) in a block with colour \( f_k \) to some arc \( \zeta_i' \) in a block with colour \( f_1' \); and symmetrically for the \( \theta_j' \). Adding

\[ -C(g_0 + g_1, g_1 + g_1') \]  

(68)

then ensures, via reasoning similar to that employed in Stage 4, that \( \theta_1 \) connects the block \( E_1 \) to the block \( E_1', \theta_2 \) connects \( E_2 \) to \( E_2' \), and so on; and similarly for the \( \theta_j' \) (as shown,
schematically, in Fig. 31). Thus, we have a 1–1 correspondence between the two sets of blocks, whence \( m = m' \).

![Diagram](image)

**Fig. 31.** The 1–1 correspondence between the \( E_j \) and the \( E'_j \) established by the \( \theta_j \) and the \( \theta'_j \).

Finally, we regard elements of the alphabet \( T \) as fresh variables and order them to form the tuple \( \vec{t} \). These variables are used for labelling the components of \( g_0 \) and of \( g_1 \), and hence the arcs \( \theta_1, \ldots, \theta_m \):

\[
\text{part}(\vec{t}) \quad \land \quad \text{colour}(g_0; \vec{t}) \quad \land \quad \text{colour}(g_1; \vec{t}). \tag{69}
\]

(Note that this time we can take the regions \( \vec{t} \) to form a partition.) Adding the constraints

\[
\bigwedge_{k=0,1} \bigwedge_{1 \leq h \leq |T|} \neg \mathcal{C}(\sum_{h' \neq h} (g_k \cdot t_{h'}), \sum_{1 \leq \ell \leq |w(t_h)|} p_{h,\ell} + \sum_{1 \leq \ell \leq |w(t_{h'})|} p'_{h,\ell}) \tag{70}
\]

instantly ensures that the sequences of tile indices \( h_1, \ldots, h_m \) and \( h'_1, \ldots, h'_m \) are identical. In other words, \( \tau = \tau' \).

This completes the argument that, if \( \psi_W \) has a satisfying assignment over \( \text{RC}(\mathbb{R}^2) \), then \( W \) is a positive instance of the PCP. By extending the arrangement of Fig. 22 in the obvious way, we see that, if \( W \) is a positive instance of the PCP, then \( \psi_W \) has a satisfying assignment over \( \text{RC}(\mathbb{R}^2) \), and hence (trivially) a satisfying assignment over \( \text{RC}(\mathbb{R}^2) \).

The case \( C^0 \) is dealt with as for Corollary 3.2: we replace all occurrences of \( c \) in \( \psi_W \) with \( c^0 \). Denoting the resulting \( C^0 \)-formula by \( \psi'_W \), we see that the following are equivalent: (i) \( W \) is positive; (ii) \( \psi'_W \) is satisfiable over \( \text{RC}(\mathbb{R}^2) \); (iii) \( \psi'_W \) is satisfiable over \( \text{RC}(\mathbb{R}^2) \). Thus,

**Corollary 5.9.** The problems \( \text{Sat}(C^0, \text{RC}(\mathbb{R}^2)) \) and \( \text{Sat}(C^0, \text{RCP}(\mathbb{R}^2)) \) are r.e.-hard.

Employing the techniques of the proof of Theorem 3.3, we show that

**Theorem 5.10.** The problems \( \text{Sat}(Bc, \text{RC}(\mathbb{R}^2)) \) and \( \text{Sat}(Bc, \text{RCP}(\mathbb{R}^2)) \) are r.e.-hard.

**Proof.** Again, observe that all conjuncts of \( \psi_W \) featuring the predicate \( C \) are negative (remember that there are additional such literals implicit in the use of 3-region variables, e.g., \( \vec{r} \ll \vec{r} \); but let us ignore these for the moment.) Recall the formula \( \neg \mathcal{C}(r, s) \) from the proof of Theorem 3.3 and consider the effect of replacing any literal \( \neg C(r, s) \) in \( \psi_W \) by the corresponding instance of \( \neg \mathcal{C}(r + r', s + s') \), where \( r' \) and \( s' \) are fresh variables; denote the resulting formula by \( \psi' \). It is easy to see that \( \psi' \) entails \( \psi_W \); hence if \( \psi' \) is satisfiable, then \( W \) is a positive instance of the PCP.

We next show that, if \( W \) is a positive instance of the PCP, then \( \psi \) is satisfiable over \( \text{RCP}(\mathbb{R}^2) \). For consider a tuple from \( \text{RCP}(\mathbb{R}^2) \) satisfying \( \psi_W \), and based on the arrangement of Fig. 22. Note that if \( r \) and \( s \) are 3-regions whose outer shells, \( r \) and \( s \) are not in contact (e.g., \( a_{0,1} \) and \( a_{0,3} \), then \( r \) and \( s \) have (i) finitely many components, and (ii) connected complements. Hence, it is easy to find polygons \( r' \) and \( s' \) satisfying \( \neg \mathcal{C}(r + r', s + s') \). Fig. 32 represents the situation schematically. We may therefore assume that all such literals \( \neg C(r, s) \) have been eliminated from \( \psi_W \).
We are not quite done, however. We must show that we can replace the implicit non-contact constraints \((\vec{r} \ll \vec{r})\) and \((\vec{r} \ll r)\) that come with the use of each 3-region variable \(r\) by suitable \(\mathsf{BC}\)-formulas. Since the two conjuncts are identical in form, we only show how to deal with \((\vec{r} \ll r)\), which, we recall, is an alternative notation for \(\neg C(\vec{r}, r)\). Since the complement of \(\neg r\) is in general not connected, a direct use of \(\neg \neg C(\vec{r} + r', (\neg r) + s')\) will result in an unsatisfiable formula. Instead, we represent \(\neg r\) as the sum of two regions \(s_1\) and \(s_2\) with connected complements, and then proceed as before. In particular, we replace \((\vec{r} \ll r)\) by \(((\neg r) = (s_1 + s_2)) \land \neg \neg C(\vec{r} + r_1, s_1) \land \neg \neg C(\vec{r} + r_2, s_2)\). For \(i = 1, 2\), \(\vec{r} + r_i\) is a connected region that is disjoint from \(s_i\). So, \(\vec{r}\) is disjoint from \(s_1 \land s_2\), and hence disjoint from their sum, \((\neg r)\). Fig. 33 shows regions \(r_1\), \(s_1\) satisfying the above formula; the other pair, \(r_2\), \(s_2\) is the mirror image.

\[
\text{Fig. 33. Disjoint connected regions } \vec{r} + r_1 \text{ and } s_1 \leq (\neg r) \text{ for } \vec{r} \text{ right inside } r.
\]

Let \(\psi_W^C\) be the result of replacing in \(\psi_W\) all the (explicit or implicit) conjuncts containing the predicate \(C\), as just described. We have thus shown that if \(\psi_W^C\) is satisfiable over \(\mathsf{RC}(\mathbb{R}^2)\) then \(W\) is positive, and, conversely, if \(W\) is positive then \(\psi_W^C\) is satisfiable over \(\mathsf{RCP}(\mathbb{R}^2)\). \(\square\)

**Theorem 5.11.** The problems \(\mathsf{Sat}(\mathsf{BC}^\circ, \mathsf{RC}(\mathbb{R}^2))\) and \(\mathsf{Sat}(\mathsf{BC}^\circ, \mathsf{RCP}(\mathbb{R}^2))\) are r.e.-hard.

**Proof.** We begin with the \(\mathsf{CC}^\circ\)-formula \(\psi_W^C\) constructed in the proof of r.e.-hardness result in Corollary 5.9. We proceed by eliminating occurrences of \(C\). However, we cannot directly use the same Lemma 5.4 as in the proof of Theorem 5.5 because the regions in question may not necessarily be bounded. For instance, consider the formula \((s_0 \ll s_0)\), which is an alternative notation for \(\neg C(s_0, s_0)\): although the region \(s_0\) in Fig. 20 is evidently bounded, \(-s_0\) is not. We proceed as follows. Say that a region \(r\) is quasi-bounded if either \(r\) itself or its complement, \(\neg r\), is bounded. Since all the polygons in the tuple satisfying \(\psi_W^C\) are quasi-bounded, we can eliminate all occurrences of \(C\) from \(\psi_W^C\) using the following fact [Newman 1964, p. 137]:

**Lemma 5.12.** Let \(F, G\) be disjoint, closed subsets of \(\mathbb{R}^2\) such that both \(\mathbb{R}^2 \setminus F\) and \(\mathbb{R}^2 \setminus G\) are connected. Then \(\mathbb{R}^2 \setminus (F \cup G)\) is connected.

So, suppose we have a conjunct \(\neg C(r, s)\) in \(\psi_W^C\). We consider the following formula:

\[
\chi(r, s, \vec{v}) = (r = r_1 + r_2) \land (s = s_1 + s_2) \land \bigwedge_{1 \leq i, j \leq 2} (K_{5m}(\vec{v}_{ij}) \land (r_i \leq v_{ij}^1) \land (s_j \leq v_{ij}^2)),
\]

where \(\vec{v}\) is a vector of variables containing \(r_1, r_2, s_1, s_2\) and the \(v_{ij}^1, v_{ij}^2\), for \(1 \leq i, j \leq 2\), and \(K_{5m}(v_1, \ldots, v_5)\) is the formula defined before Lemma 5.4. By Lemma 5.4 (i), \(\chi(r, s, \vec{v})\) entails \(\neg C(r, s)\) over \(\mathsf{RC}(\mathbb{R}^2)\). We also show that, conversely, if \(a\) and \(b\) are disjoint quasi-bounded polygons then there exists a tuple of polygons \(\vec{c}\) such that \((a, b, \vec{c})\) satisfies \(\chi(r, s, \vec{v})\). Indeed, it is routine to show that, for each quasi-bounded region \(a\), there exist a pair of polygons \(a_1\) and \(a_2\) such that \(a = a_1 + a_2\) and both \(\mathbb{R}^2 \setminus a_1\) and \(\mathbb{R}^2 \setminus a_2\) are connected. Let
\[ b_1 \text{ and } b_2 \text{ be chosen analogously for } b. \text{ Then, for all } 1 \leq i, j \leq 2, \text{ we have } a_i \cap b_j = \emptyset \text{ and, by Lemma 5.12, } \mathbb{R}^2 \setminus (a_i + b_j) \text{ is connected. Thus, there exists a piecewise-linear Jordan curve in } \mathbb{R}^2 \setminus (a_i + b_j) \text{ separating } a_i \text{ and } b_j. \text{ By Lemma 5.4 (ii), let } e_{ij} \text{ be a tuple of polygons satisfying } K^\text{Saw}(u_{ij}) \text{ and such that } a_i \leq e_{ij}^1 \text{ and } b_j \leq e_{ij}^2. \text{ It should be clear that the tuple of } a_1, a_2, b_1, b_2 \text{ and the } e_{ij}, \text{ for } 1 \leq i, j \leq 2, \text{ is as required.}

By replacing all occurrences of } C \text{ in } \psi_W \text{ as described above, we obtain a } BC^c\text{-formula, say } \psi_W^*, \text{ such that, if } \psi_W \text{ is satisfiable over } RC(\mathbb{R}^2), \text{ then } W \text{ is a positive instance of PCP; and, conversely, if } W \text{ is a positive instance of PCP, then } \psi_W^* \text{ is satisfiable over } RCP(\mathbb{R}^2). \quad \square

6. THE LANGUAGE } BC^c \text{ IN DIMENSIONS GREATER THAN 2

In this section, we consider the complexity of satisfying } BC^c\text{-formulas by polyhedra and regular closed sets in three-dimensional Euclidean space. We proceed by analysing the connections between geometrical and graph-theoretic interpretations of } BC^c.

A topological space } T \text{ in which the intersection of any family of open sets is open is called an Aleksandrov space. Every quasi-order } (W, R), \text{ that is, a transitive and reflexive relation } R \text{ on } W, \text{ can be regarded as an Aleksandrov space by taking } X \subseteq W \text{ to be open just in case } x \in X \text{ and } xRy \text{ imply } y \in X. \text{ (Hence, } X \text{ is closed just in case } x \in X \text{ and } yRx \text{ implies } y \in X.) \text{ It can be shown } [\text{Bourbaki 1966}] \text{ that every Aleksandrov space is the homeomorphic image of one constructed in this way. In the sequel, we shall silently treat any quasi-order } (W, R) \text{ as a topological space. As we shall see, every } BC^- \text{ or } BC^c^-\text{-formula that is satisfiable in } RC \text{ can also be satisfied in an Aleksandrov space of rather primitive structure.}

By a quasi-saw we mean a partial-order } (W, R) \text{ of depth 1, in which every point of depth 0 has at least one } R\text{-predecessor of depth 1; see Fig. 34. We denote the set of points of depth } i \text{ by } W_i, i = 0, 1. \text{ Every regular closed set } X \text{ in such a quasi-saw is uniquely defined by its points of depth 0: a point } z \in W_1 \text{ is in } X \text{ if and only if there is } x \in W_0 \cap X \text{ such that } zRx. \text{ Note also that a set } X \text{ in (the Aleksandrov space of) a quasi-saw } (W, R) \text{ is connected if and only if } X \text{ is connected in the undirected graph with the vertices } W \text{ and the edges given by } R. \text{ For example, the set } \{x_3, x_4, z_2, z_3, z_4\} \text{ in Fig. 34 (shaded) is regular closed and connected, while its interior } \{x_3, x_4, z_3\} \text{ is not connected.}

![Fig. 34. A quasi-saw.](image)

A quasi-saw model is a model based on a quasi-saw (with variables interpreted by regular closed sets). The proof of the following lemma follows from [Kontchakov et al. 2010a, Lemmas 4.1 and 4.2] (see also [Wolter and Zakharyaschev 2000]). But the critical observation can, in essence, be found already in [McKinsey and Tarski 1944] and [Kripke 1963]: for every formula } \varphi \text{ and every topological model } \mathcal{I}, \text{ there exist a finite Aleksandrov model } \mathfrak{A}\text{ and a continuous function } f: \mathcal{I} \rightarrow \mathfrak{A} \text{ such that } \tau^\mathfrak{A} = f(\tau^\mathcal{I}) \text{ for every term } \tau \text{ in } \varphi.

**Lemma 6.1.** Let } \varphi \text{ be a } BC^- \text{ or } BC^c^-\text{-formula. If } \varphi \text{ is satisfiable over } RC \text{ then it can be satisfied in a finite quasi-saw model.}

We begin by briefly discussing the results of [Kontchakov et al. 2010b] for the polyhedral case. Denote by } ConRC \text{ the class of all frames over connected topological spaces with regular closed regions. For a } BC^c\text{-formula } \varphi, \text{ let } \varphi^* \text{ be the result of replacing every occurrence of } c^\circ \text{ in } \varphi \text{ with } c. \text{ Evidently, the mapping } \varphi \mapsto \varphi^* \text{ is a bijection from } BC^c \text{ to } BC.
**Theorem 6.2.** For all $n \geq 3$, the mapping $\varphi \mapsto \varphi^*$ constitutes a reduction of $\text{Sat}(\text{Bc}^c, \text{RCP}(\mathbb{R}^n))$ to $\text{Sat}(\text{Bc}, \text{ConRC})$. Hence, the problems $\text{Sat}(\text{Bc}^c, \text{RCP}(\mathbb{R}^n))$ coincide, and are all $\text{ExpTime}$-complete.

**Proof.** A connected partition in $\text{RCP}(\mathbb{R}^n)$ is a tuple $X_1, \ldots, X_k$ of non-empty polyhedra having connected and pairwise disjoint interiors, which sum to the entire space $\mathbb{R}^n$. The *neighbourhood graph* $(V, E)$ of this partition has vertices $V = \{X_1, \ldots, X_k\}$ and edges

$$E = \{(X_i, X_j) \mid i \neq j \text{ and } (X_i + X_j)^c \text{ is connected};$$

see Fig. 35. Clearly, every connected partition in $\text{RCP}(\mathbb{R}^n)$ has a connected neighbourhood graph; and conversely, one can show that every connected graph is the neighbourhood graph of some connected partition in $\text{RCP}(\mathbb{R}^n)$. Furthermore, every neighbourhood graph $(V, E)$ gives rise to a quasi-saw $(W_0 \cup W_1, R)$, where $W_0 = V$, $W_1 = \{z_{x,y} \mid (x, y) \in E\}$, and $R$ is the reflexive closure of $\{(z_{x,y}, x), (z_{x,y}, y) \mid (x, y) \in E\}$. Note that in this quasi-saw every point of depth 1 has precisely two $R$-successors. Such quasi-saws are called 2-quasi-saws. Conversely, every connected 2-quasi-saw $(W_0 \cup W_1, R)$ can be represented as the neighbourhood graph $(W_0, E)$ of some connected partition, where

$$E = \{(x, y) \mid x \neq y \text{ and there is } z \in W_1 \text{ with } zRx \text{ and } zRy\}.$$  

From this, we see that a $\text{Bc}^c$-formula $\varphi$ is satisfiable over $\text{RCP}(\mathbb{R}^n)$ if and only if $\varphi$ is satisfiable over a connected 2-quasi-saw. But, over 2-quasi-saws, connectedness coincides with interior-connectedness. Thus, $\varphi$ is satisfiable over $\text{RCP}(\mathbb{R}^n)$ if and only if $\varphi^*$ is satisfiable over a connected 2-quasi-saw. The problem $\text{Sat}(\text{Bc}, \text{ConRC})$ is known to be $\text{ExpTime}$-complete [Kontchakov et al. 2010a]. \qed

Similarly, one can show that a $\text{Bc}^c$-formula $\varphi$ is satisfiable over $\text{RCP}(\mathbb{R}^2)$ if and only if there is a connected partition of $\mathbb{R}^2$ such that its neighbourhood graph is planar and $\varphi$ is satisfiable over the corresponding 2-quasi-saw [Kontchakov et al. 2010b, Lemma 4]. However, as we show in Theorem 5.5, a ‘planar’ 2-quasi-saw model of a $\text{Bc}^c$-formula may be infinite, and the satisfiability problem is in fact undecidable (cf. Theorem 5.11).

Having shown that the problem $\text{Sat}(\text{Bc}^c, \text{RCP}(\mathbb{R}^3))$ is $\text{ExpTime}$-complete, we now turn our attention to the satisfiability of $\text{Bc}^c$-formulas over the complete Boolean algebra $\text{RC}(\mathbb{R}^3)$, where the picture changes drastically: for instance, the $\text{Bc}^c$-formula (3) is not satisfiable over 2-quasi-saws, but has a quasi-saw model as in Fig. 36. In fact, it is shown in [Kontchakov et al. 2010b] that every $\text{Bc}^c$-formula $\varphi$ satisfiable over $\text{ConRC}$ can be satisfied in a connected quasi-saw model of size bounded by a polynomial function of $|\varphi|$, and thus the problem $\text{Sat}(\text{Bc}^c, \text{ConRC})$ is $\text{NP}$-complete. The following theorem says, in essence, that such polynomial models also give rise to ‘small’ models over regular closed subsets of $\mathbb{R}^n$, for $n \geq 3$: 

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THEOREM 6.3. The problems $\text{Sat(Bc}^3, \text{RC}(\mathbb{R}^n))$, for all $n \geq 3$, coincide with $\text{Sat(Bc}^3, \text{ConRC})$ and are all NP-complete.

PROOF. We show that, if a Bc$^3$-formula is satisfiable over ConRC, then it can be satisfied over RC($\mathbb{R}^n$), for any $n \geq 3$. (The converse implication is trivial.) Indeed, we need only establish the special case $n = 3$, since the others follow by cylindrification. The second statement of the theorem follows from the complexity of $\text{Sat(Bc}^3, \text{ConRC})$.

So, suppose $\phi$ is satisfied in a model $\mathfrak{A}$ over a connected quasi-saw ($W, R$) of size bounded by a polynomial function of $|\phi|$. Let $W_i$ be the set of points of depth $i = 0, 1$. Without loss of generality we may assume that there is a point $z_0 \in W_1$ with $z_0Rz$ for all $z \in W_0$, since adding such a point cannot change the truth-values of subformulas of $\phi$ of the form $(\tau_1 = \tau_2)$ or $c^i(\tau)$. (The only danger here is that adding $z$ to $W_1$ will change a formula of the form $c^i(\tau)$ from false to true. Now, since $z$ is related by $R$ to every point of $W_0$, the only region that it can cause to become connected is the entire space; but the quasi-saw ($W, R$) is connected by hypothesis.) We show now how $\mathfrak{A}$ can be embedded into a model over RC($\mathbb{R}^3$). In the remainder of this proof, we repeatedly rely on the fact that, if $r$ and $s$ are interior-connected, regular closed subsets of some topological space, with $r \cdot s \neq 0$, then $r + s$ is also interior-connected.

By an open ball we mean a subset of the form $\{x \in \mathbb{R}^3 \mid ||x - y|| < a\}$, where $y \in \mathbb{R}^3$ and $a$ is a positive real. We select open balls $D_z$ for $z \in W_1 \setminus \{z_0\}$ such that their closures are pairwise non-intersecting, and define $D_{z_0} = \mathbb{R}^3 \setminus \bigcup_{z \in W_1 \setminus \{z_0\}} D_z$. Thus, each $D_z$ is connected and open, and the open set $D = \bigcup_{z \in W_1} D_z$ is dense. Then we take pairwise disjoint sets $B^k_x$ for $x \in W_0$, each homeomorphic to a closed ball, and arranged so that, for all $x \in W_0$ and $z \in W_1$, $D_z \not\subset B^k_x$ and $B^k_x \cap D_z \neq \emptyset$ if and only if $zRx$.

We describe a construction in which the regular closed sets $B^k_x$ are expanded to sets $B_x$ so as to exhaust the entire space, $\mathbb{R}^3$. First, let $q_1, q_2, \ldots$ be an enumeration of all the points in $D$ with rational coordinates. Consider any piecewise-linear Jordan arc $\alpha$ such that the endpoints of each linear segment of $\alpha$ have rational coordinates; call such an $\alpha$ rational piecewise-linear; and let $\alpha_1, \alpha_2, \ldots$ be an enumeration of all the rational piecewise-linear arcs with both endpoints in the open set $D$. We define, for all $k \geq 1$, a collection $\{B^k_x \mid x \in W_0\}$ of interior-connected, pairwise disjoint, regular closed sets in $\mathbb{R}^3$. The case $k = 1$ has already been dealt with. Suppose, then, for $k \geq 1$, the sets $B^k_x$ have been defined; we construct the sets $B^{k+1}_x$ in two steps:

1. If $q_k \in B^k_x$ for some $x \in W_0$, let $\tilde{B}^k_{x'} = B^k_{x'}$, for every $x' \in W_0$. Otherwise, $q_k \in D_z$ for some $z \in W_1$. Pick some $x \in W_0$ with $zRx$ and let $C \subseteq D_z$ be a regular closed interior-connected set containing $q_k$ and a point in $(B^k_x)^o \cap D_z$ in its interior (e.g., a closed ball centred on $q_k$ and a regular closed ‘rod’ connecting it to $B^k_x$, as depicted in Fig. 37). Let $\tilde{B}^k_x = \tilde{B}^k_{x'} + C$, and let $\tilde{B}^k_{x'} = B^k_{x'}$ for all other $x' \in W_0$. The sets $\tilde{B}^k_{x'}$, for $x' \in W_0$, are interior-connected, and, since we are working in $\mathbb{R}^3$, $C$ can obviously be chosen so that the $\tilde{B}^k_{x'}$ are pairwise disjoint. (Note that this would not in general be true in $\mathbb{R}$ or $\mathbb{R}^2$.)

2. Let $\tilde{B}^k = \bigcup_{z \in W_0} \tilde{B}^k_z$. For each $z \in W_1$ such that $\alpha_k \cap D_z$ is not contained in $\tilde{B}^k$ and for each $x \in W_0$ such that $zRx$, choose a distinct point $p_{z,x} \in \alpha_k \cap D_z$, not lying in $\tilde{B}^k$. If $p_{z,x}$ is defined, let $C_{z,x} \subseteq D_z$ be a regular closed interior-connected set containing $p_{z,x}$ and a point in $(B^k_x)^o \cap D_z$ in its interior, see Fig. 37; otherwise, let $C_{z,x} = \emptyset$. Set...
$B^{k+1}_z = \hat{B}^k_z + \sum_{x \in W_0} C_{z,x}$, for all $x \in W_0$. The sets $B^{k+1}_z$ are interior-connected; moreover, since we are working in $\mathbb{R}^3$, the $C_{z,x}$ can obviously be chosen such that the $B^{k+1}_z$ are also pairwise disjoint. (Again, this would not in general be true in $\mathbb{R}$ or $\mathbb{R}^2$.)

Since $RC(\mathbb{R}^3)$ is a complete Boolean algebra, define $B_x = \sum_{k=1}^{\infty} B^k_x$, for each $x \in W_0$. We show that the $B_x$ are interior-connected and form a partition (i.e., their pairwise products are empty, and they sum to $R$). Indeed, for distinct $x, y \in W_0$, we certainly have, for all $k, \ell \geq 1$, $B^k_x \cdot B^\ell_y = 0$, whence, by the distributivity law, $B_x \cdot B_y = 0$. And since, for all $k \geq 1$, $B^k_x$ is interior-connected and includes the non-empty, interior-connected set $B^1_x$, the set $\bigcup_{k=1}^{\infty} (B^k_x)^\circ$ is connected. But $B^\circ_x$ lies in between $\bigcup_{k=1}^{\infty} (B^k_x)^\circ$ and its closure, and hence is also connected. Finally, by Step 1 of the above construction, every rational point of the set $D$ lies in some $B_x$, so that $\sum_{x \in W_0} B_x \supseteq D$, whence $\sum_{x \in W_0} B_x = 1$. This completes the definition of the partition $\{B_x \mid x \in W_0\}$.

Now define a function $f: RC(W, R) \to RC(\mathbb{R}^3)$ by taking $f(X) = \sum_{x \in X \cap W_0} B_x$. Let $X \in RC(W, R)$, and let $z$ be a point of $W_1$. We claim that $z \in X^\circ$ implies $D_z \subseteq f(X)$; further, if $zRx$, then $D_z \cup B_z$ is interior-connected. Indeed, if $z \in X^\circ$ with $zRx$, then $z \in X$. And since, by Step 1, every rational point of $D_z$ lies in $B_z$ for some such $x$, it follows that $D_z \subseteq \sum \{B_x \mid x \in X \cap W_0\} = f(X)$. The second statement follows easily from the choice of the sets $B^1_x$ and the fact that the sets $B_x$ are interior-connected.

We can now show that if $f$ is a Boolean algebra homomorphism, and that $X \in RC(W, R)$ is interior-connected if and only if $f(X)$ is interior-connected. Trivially, $f(X + Y) = f(X) + f(Y)$; and since the $B_x$ form a partition, $f(-X) = \sum_{x \in W_0 \setminus X} B_x = -f(X)$. Now suppose $X \in RC(W, R)$ is interior-connected, and let $p, q$ be points in $f(X)^\circ$. Then there exist points $p', q'$ in the same components of $f(X)^\circ$ as $p, q$, respectively, such that, for some $k \geq 1$ and $x, y \in X \cap W_0$, we have $p' \in B^k_x$ and $q' \in B^k_y$. Since $X$ is interior-connected, we can find sequences of points $x_0, \ldots, x_m$ in $X \cap W_0$, and $z_1, \ldots, z_m \in X^\circ \cap W_1$ such that $x = x_0$, $y = x_m$, and $z_i Rx_{i-1}$ and $z_i Rx_i$, for all $1 \leq i \leq m$. But we have shown above that the sets $D_{x_i}$ are subsets of $f(X)$, and that the sets $D_{z_i} + B_{x_i}$ and $D_{z_i} + B_{z_{i-1}}$ are interior-connected. Hence, $p'$ and $q'$ lie in the same component of $f(X)^\circ$, whence $p$ and $q$ do as well. That is: $f(X)$ is interior-connected, as required. Finally, suppose $X \in RC(W, R)$ is not interior-connected, so that we may find elements $x, y \in W_0$ lying in different components of $X^\circ$. We show that $f(X)$ is not interior-connected. For suppose otherwise. Then there exists a rational piecewise-linear arc $\alpha$ with endpoints in the sets $D \cap (B^1_x)^\circ$ and $D \cap (B^1_y)^\circ$, and lying entirely in $f(X)^\circ$. But $\alpha$ occurs as some $\alpha_k$ in our enumeration. It follows that there will be a first point $q'$ of $\alpha_k$ lying in a set $\hat{B}^k_{\alpha}$ such that $x$ and $y'$ lie in different components of $X^\circ$; and there will be a last point $p'$ of $\alpha_k$, occurring strictly before $q'$ and lying in a set.
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