METHODS OF SIGNAL PROCESSING

On Detection of Gaussian Stochastic Sequences

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Abstract—We consider the minimax detection problem for a Gaussian random signal vector in white Gaussian additive noise. It is assumed that an unknown vector \( \sigma \) of signal vector intensities belongs to a given set \( E \). We investigate when it is possible to replace the set \( E \) with a smaller set \( E_0 \) without loss of quality (and, in particular, replace it with a single point \( \sigma_0 \)).

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1. INTRODUCTION

1. Simple hypotheses. Two simple hypotheses \( H_0 \) ("noise") and \( H_1 \) ("noise + stochastic signal") on observations \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) are considered:

\[
H_0: \quad y = \xi,
H_1: \quad y = s + \xi,
\]

where \( \xi = (\xi_1, \ldots, \xi_n) \) are independent \( \mathcal{N}(0,1) \)-Gaussian random variables and \( s = (s_1, \ldots, s_n) \) are, independent \( \mathcal{N}(0, \sigma_i^2) \)-Gaussian random variables (i.e., \( E(s_i^2) = \sigma_i^2 \)) independent of \( \xi \), \( i = 1, \ldots, n \).

We denote \( \sigma = (\sigma_1, \ldots, \sigma_n) \), where all \( \sigma_i \geq 0 \), and introduce the functions (this notation will be used throughout what follows)

\[
D(\sigma) = \sum_{i=1}^{n} \ln(1 + \sigma_i^2), \quad T(\sigma) = \sum_{i=1}^{n} \frac{\sigma_i^2}{1 + \sigma_i^2}, \quad B(\sigma) = 2 \sum_{j=1}^{n} \frac{\sigma_j^4}{(1 + \sigma_j^2)^2}.
\] (2)

Then for conditional probability densities we have

\[
p(y|H_0) = (2\pi)^{-n/2} e^{-\frac{1}{2} \sum_{i=1}^{n} y_i^2}, \quad p(y|\sigma) = (2\pi)^{-n/2} e^{-\frac{1}{2} \sum_{i=1}^{n} y_i^2/(1 + \sigma_i^2) - \frac{1}{2} D(\sigma)}.
\] (3)

Denote also

\[
r(y, \sigma) = \ln \frac{p(y|\sigma)}{p(y|H_0)} = \frac{1}{2} \sum_{i=1}^{n} \frac{\sigma_i^2 y_i^2}{1 + \sigma_i^2} - \frac{1}{2} D(\sigma).
\] (4)

An optimal solution of this problem of testing a simple hypothesis \( H_0 \) against a simple alternative \( H_1 \) (the Neyman–Pearson criterion) \([1,2]\) has the form

\[
y \in A(A, \sigma) \Rightarrow H_0, \quad y \notin A(A, \sigma) \Rightarrow H_1,
\] (5)

1 The research was carried out at the Institute for Information Transmission Problems of the Russian Academy of Sciences at the expense of the Russian Science Foundation, project no. 14-50-00150.
where $A(A, \sigma)$ is the set (ellipsoid)

$$
A(A, \sigma) = \left\{ y : \sum_{i=1}^{n} \frac{\sigma_i^2 y_i^2}{1 + \sigma_i^2} \leq D(\sigma) + A \right\}, \quad \sigma = (\sigma_1, \ldots, \sigma_n).
$$

(6)

The threshold level $A$ of this test is determined by a given 1st-kind error probability (false alarm probability) $\alpha = \alpha(A, \sigma)$:

$$
\alpha(A, \sigma) = P(y \notin A \mid H_0) = P \left( \sum_{i=1}^{n} \frac{\sigma_i^2 \xi_i^2}{1 + \sigma_i^2} > D(\sigma) + A \right).
$$

(7)

If hypothesis $H_1$ is true, then $y_i = \xi_i + \sigma_i \eta_i \sim \sqrt{1 + \sigma_i^2} \eta_i$, where $(\eta_1, \ldots, \eta_n)$ are independent $\mathcal{N}(0, 1)$-Gaussian random variables. Therefore, the 2nd-kind error probability (miss probability) $\beta(A, \sigma)$ is given by

$$
\beta(A, \sigma) = P(y \in A \mid H_1) = P \left( \sum_{i=1}^{n} \sigma_i^2 \xi_i^2 < D(\sigma) + A \right).
$$

(8)

For a given value $\alpha$, denote by $\beta(\alpha, \sigma)$ the minimum possible value of $\beta(A, \sigma)$ under the optimal choice of the threshold $A$ (according to equations (7) and (8)).

Since $E \xi_i^2 = 1, i = 1, \ldots, n$, due to the law of large numbers and equations (7) and (8) we find that for sufficiently small $\alpha$ and $\beta$ the quantity $A$ should satisfy the conditions

$$
\sum_{i=1}^{n} \frac{\sigma_i^2}{1 + \sigma_i^2} < D(\sigma) + A < \sum_{i=1}^{n} \sigma_i^2.
$$

(9)

Below we assume both conditions (9) to be satisfied. Note that if the threshold $A$ decreases, then the error probability $\beta(A, \sigma)$ also decreases, but the error probability $\alpha(A, \sigma)$ increases. In particular, the case where $D(\sigma) + A$ is relatively close to the left-hand side of condition (9) will be of interest for us.

2. **Simple hypothesis against a composite alternative.** Let a set $E$ of nonnegative vectors $\sigma = (\sigma_1, \ldots, \sigma_n)$ be given. Assume that for the vector $\sigma$ describing the hypothesis $H_1$ in (1) it is only known that $\sigma \in E$, but the vector $\sigma$ itself is not known (i.e., $H_1$ is a composite hypothesis).

Similarly to (5), for testing the hypotheses $H_0$ and $H_1$ we choose a decision region $A \in \mathbb{R}^n$ such that

$$
y \in A \Rightarrow H_0, \quad y \notin A \Rightarrow H_1.
$$

The 1st- and 2nd-kind error probabilities are defined, respectively, by

$$
\alpha(A) = P(y \notin A \mid H_0),
\beta(A, E) = P(y \in A \mid H_1) = \sup_{\sigma \in E} P(y \in A \mid \sigma).
$$

In other words, the minimax problem of testing the hypotheses $H_0$ and $H_1$ is considered.

For a given 1st-kind error probability $\alpha, 0 < \alpha < 1$, we are interested in the minimal possible 2nd-kind error probability

$$
\beta(\alpha, E) = \inf_{A : \alpha(A) \leq \alpha} \beta(A, E)
$$

(10)

and the corresponding decision region $A(\alpha)$. PROBLEMS OF INFORMATION TRANSMISSION Vol. 53 No. 4 2017
Without loss of generality, we assume the set $E$ to be closed and Lebesgue measurable on $\mathbb{R}^n$. Formally speaking, the optimal solution of the minimax testing problem (10) for hypotheses $H_0$ and $H_1$ is described in Wald’s general theory of statistical decisions [1]. For that solution we need to find the “worst” (least favorable) prior distribution $\pi_{lf}(dE)$ on $E$, replace the composite hypothesis $H_1$ by the simple hypothesis $H_1(\pi)$, and then investigate characteristics of the corresponding Neyman–Pearson criterion for testing the simple hypotheses $H_0$ and $H_1(\pi)$. Unfortunately, all this can be done in some very special cases only. Therefore, it is natural to distinguish cases where that “least favorable” prior distribution on $E$ has the simplest form (for example, is concentrated in a single point from $E$).

Clearly, for $\beta(\alpha, E)$ there is a lower bound

$$\beta(\alpha, E) \geq \sup_{\sigma \in E} \beta(\alpha, \sigma).$$

(11)

The function $\beta(\alpha, \sigma)$, $\alpha \in [0, 1]$, $\sigma \in \mathbb{R}^n_+$, is continuous in both arguments. Since the set $E \in \mathbb{R}^n_+$ is assumed to be closed, there exists $\sigma_0 = \sigma_0(E, \alpha) \in E$ such that

$$\beta(\alpha, \sigma_0) = \sup_{\sigma \in E} \beta(\alpha, \sigma).$$

First of all, we want to know for what $E$ the “least favorable” prior distribution is concentrated in the point $\sigma_0$, in which case we have

$$\beta(\alpha, E) = \beta(\alpha, \sigma_0).$$

(12)

If for the set $E$ equality (12) holds, then without any loss of detection quality we may replace the composite hypothesis $H_1 = \{E\}$ by the simple hypothesis $H_1 = \sigma_0$, and the optimal solution (5)–(6) for the simple hypothesis $H_1 = \sigma_0$ remains optimal (in minimax sense) for the composite hypothesis $H_1 = \{E\}$ as well (see a similar question for shifts of measures in [3]). Some sufficient conditions for equality (12) to hold are given in Section 3 below. Of course, those conditions set rather strong limitations on the set $E$.

It is shown in Section 2 that sometimes it is possible, without any loss of detection quality, to replace $E$ by a smaller set $E_0$ (i.e., make a reduction of $E$).

Usually in the considered problem the probability $\beta(\alpha, E)$ should be very small. Therefore, instead of the strong condition (12), its simpler asymptotic analog is often investigated: exponents of error probabilities (see, for example, [4]) are compared and a weaker condition

$$\ln \beta(\alpha, E) = \ln \beta(\alpha, \sigma) + o(\ln \beta(\alpha, \sigma)), \quad |\ln \beta(\alpha, \sigma)| \rightarrow \infty,$$

(13)

is considered. Below it will be shown that condition (13) holds under weaker restrictions on $E$ than in the case of condition (12).

Note that if asymptotic equality (13) holds for $E$, this does not mean that the optimal solution (5)–(6) for the simple hypothesis $H_1 = \sigma_0$ remains optimal for the composite hypothesis $H_1 = \{E\}$ too. Possibly, another test should be used for that. Some sufficient conditions for equality (13) to hold and the corresponding test are described in Section 4.

Since in the considered problem the probability $\beta(\alpha, E)$ should usually be very small, we also investigate large deviations for $\beta(\alpha, E)$ (i.e., its logarithmic asymptotic as $n \rightarrow \infty$). For this asymptotic be obtain upper bounds in Section 4 lower bounds in the Appendix (whence exact logarithmic asymptotic of $\beta(\alpha, E)$ as $n \rightarrow \infty$ follows). In Section 5, similar upper bounds for $\alpha(A, \sigma)$ are derived. If $\alpha(A, \sigma)$ is not too small, then, to give a complete picture, we investigate it in Section 5.1 using the central limit theorem and Berry–Esseen inequality, which gives tighter
estimates. In Section 6, a special example is considered. Some useful estimates for large deviations of the \( \chi^2 \) distribution, used in the paper, are given in the Appendix.

All formulas in the paper are, in essence, nonasymptotic. All remainder terms can always be estimated.

Below, as usual, \( \sigma \leq \lambda \) means \( \sigma_i \leq \lambda_i, \ i = 1, \ldots, n. \)

2. REDUCTION OF THE SET \( \mathcal{E} \)

We show that sometimes, without any loss of detection quality, it is possible to replace \( \mathcal{E} \) with a smaller set \( \mathcal{E}_0 \). Define such a set \( \mathcal{E}_0 = \mathcal{E}_0(\mathcal{E}) \) as any set having the following property:

\[
\text{for any } \sigma \in \mathcal{E} \text{ there exists } \sigma_0 \in \mathcal{E}_0 \text{ with } \sigma_0 \leq \sigma.
\]

(14)

If \( \mathcal{E} \) is a closed set (this is assumed throughout the paper), then \( \mathcal{E}_0 \subseteq \mathcal{E} \). Generally, the set \( \mathcal{E}_0 \) can be chosen in several ways.

Below we show that for any Bayesian criterion of testing a simple hypothesis \( \mathcal{H}_0 \) against a composite alternative \( \mathcal{H}_1 = \{ E \} \) the set \( \mathcal{E} \) can be replaced with the set \( \mathcal{E}_0 \) without any loss of quality. This remains valid for the likelihood ratio criterion too. In the one-dimensional case, these properties are similar to the case of distributions with monotone likelihood ratio [2, ch. 3.9].

The purpose of introducing the set \( \mathcal{E}_0 \) is to decrease (if possible) the set \( \mathcal{E} \) and thus simplify the test that we use.

1. Bayesian criterion. Consider a Bayesian criterion with a prior distribution \( \pi(dE) \) on \( \mathcal{E} \) and a corresponding decision set \( \mathcal{A} \in \mathbb{R}^n \ (y \in \mathcal{A} \Rightarrow \mathcal{H}_0, \ y \notin \mathcal{A} \Rightarrow \mathcal{H}_1) \) of the form

\[
\mathcal{A} = \mathcal{A}(A) = \{ y : \pi(y, E, \pi) \leq A \},
\]

(15)

where (see (3) and (4))

\[
p(y | \mathcal{H}_1, \pi) = \int_{\sigma \in \mathcal{E}} p(y | \sigma) \pi(dE) = (2\pi)^{-n/2} \int_{\sigma \in \mathcal{E}} e^{-1/2 \sum_{i=1}^{n} y_i^2/(1+\sigma_i^2)-1/2 D(\sigma)} \pi(dE)
\]

and

\[
r(y, E, \pi) = \ln \frac{p(y | \mathcal{H}_1, \pi)}{p(y | \mathcal{H}_0)} = \ln \int_{\sigma \in \mathcal{E}} e^{1/2 \sum_{i=1}^{n} y_i^2/(1+\sigma_i^2)-1/2 D(\sigma)} \pi(dE).
\]

Then \( \mathcal{A} \) is a convex set in \( \mathbb{R}^n \), and if \( y = (y_1, \ldots, y_n) \in \mathcal{A} \), then all \( (\pm y_1, \ldots, \pm y_n) \) belong to \( \mathcal{A} \), i.e., the set \( \mathcal{A} \) is symmetric about any coordinate axis or plane. In particular, such an \( \mathcal{A} \) is also centrally symmetric (i.e., if \( y \in \mathcal{A} \), then \( -y \in \mathcal{A} \)).

Assume that for \( y = s + \xi \) and \( \sigma \in \mathcal{E} \) in (1) a Bayesian criterion with a prior distribution \( \pi(dE) \) on \( \mathcal{E}_0 \) and the corresponding decision region \( \mathcal{A} \in \mathbb{R}^n \) of the form (15) are used. Assume also that for the 2nd-kind error probability and some \( \beta \geq 0 \) we have

\[
\beta(\mathcal{A}, \sigma_0) = \mathbb{P}(y \in \mathcal{A} | \sigma_0) = \mathbb{P}\{ r(y, \mathcal{E}_0, \pi) \leq A | \sigma_0 \} \leq \beta, \quad \sigma_0 \in \mathcal{E}_0.
\]

(16)

Let us show that inequality (16) remains valid for any \( \sigma \in \mathcal{E} \), i.e.,

\[
\beta(\mathcal{A}, \sigma) = \mathbb{P}(y \in \mathcal{A} | \sigma) = \mathbb{P}\{ r(y, \mathcal{E}_0, \pi) \leq A | \sigma \} \leq \beta, \quad \sigma \in \mathcal{E}.
\]

(17)

In other words, for any Bayesian criterion the extension of \( \mathcal{E}_0 \) to the set \( \mathcal{E} \) does not increase the 2nd-kind error probability (the 1st-kind error probability \( \alpha(\mathcal{A}) \) is unchanged). In particular, since \( \mathcal{E}_0 \subseteq \mathcal{E} \), we obtain

\[
\beta(\mathcal{A}, \mathcal{E}_0) = \beta(\mathcal{A}, \mathcal{E}), \quad 0 \leq \alpha \leq 1.
\]

(18)
Let us prove (17). Let \( \sigma \in \mathcal{E} \) but \( \sigma \notin \mathcal{E}_0 \). Then there exists \( \sigma_0 \in \mathcal{E}_0 \) with \( \sigma_0 < \sigma \). Let \( s_0 \) be a Gaussian “signal” in (1) in the case of \( \sigma_0 \). Then in the case of \( \sigma \) such a “signal” \( s \) has the form \( s = s_0 + \eta \), where \( \eta \) is a Gaussian random vector independent of \( s_0 \). Inequality (17) follows from the following auxiliary result (the set \( \mathcal{A} \) satisfies its conditions).

**Lemma 1.** Let \( \mathcal{B} \in \mathbb{R}^n \) be a convex set such that if \( y = (y_1, \ldots, y_n) \in \mathcal{B} \), then all points of the form \((\pm y_1, \ldots, \pm y_n)\) belong to \( \mathcal{B} \). Let also \( \xi \) and \( \eta \) be independent zero-mean Gaussian vectors consisting of independent components (possibly, with different distributions). Then

\[
P(\xi + \eta \in \mathcal{B}) \leq P(\xi \in \mathcal{B}).
\]  

**Proof.** If \( n = 1 \), then \( \mathcal{B} = [-a, a] \), \( a > 0 \), and inequality (19) clearly holds. Let \( n = 2 \) and let vectors \((\xi_1, \xi_2)\) and \((\xi_1 + \eta_1, \xi_2 + \eta_2)\) be compared. Compare first the vectors \((\xi_1, \xi_2)\) and \((\xi_1 + \eta_1, \xi_2)\). Denote

\[
\mathcal{B}_x = \{ y \in \mathcal{B} : y_2 = x \} \in \mathbb{R}^1.
\]

Due to the assumptions of the lemma, for any \( x \) we have \( \mathcal{B}_x = [-a(x), a(x)] \), \( a(x) > 0 \). Therefore, for a fixed \( \xi_2 \) the problem reduces to the case \( n = 1 \) and

\[
P\{\xi_1 + \eta_1 \in \mathcal{B}_x\} \leq P\{\xi_1 \in \mathcal{B}_x\},
\]

and then

\[
P\{(\xi_1 + \eta_1, \xi_2) \in \mathcal{B}\} \leq P\{(\xi_1, \xi_2) \in \mathcal{B}\}.
\]

Compare now the vectors \((\xi_1 + \eta_1, \xi_2)\) and \((\xi_1 + \eta_1, \xi_2 + \eta_2)\). Similarly to (20) and (21), we obtain

\[
P\{\xi_2 + \eta_2 \in \mathcal{B}_{\xi_1 + \eta_1}\} \leq P\{\xi_2 \in \mathcal{B}_{\xi_1 + \eta_1}\}
\]

and

\[
P\{(\xi_1 + \eta_1, \xi_2 + \eta_2) \in \mathcal{B}\} \leq P\{(\xi_1 + \eta_1, \xi_2) \in \mathcal{B}\}.
\]

Then by (21) and (22) inequality (19) follows for \( n = 2 \). Similarly, the case of \( n = 3 \) reduces to the case of \( n = 2 \), and so on. This proves inequality (19) for any \( n \). \( \triangle \)

**2. Likelihood ratio criterion.** For any function \( A(\sigma) \), its critical region \( \mathcal{A}_{ML}(A, \mathcal{E}) \) is defined as follows:

\[
\mathcal{A}_{ML}(A, \mathcal{E}) = \left\{ y : \sup_{\sigma \in \mathcal{E}} [2r(y, \sigma) - A(\sigma)] \leq 0 \right\},
\]

and then \( y \in \mathcal{A}_{ML}(A, \mathcal{E}) \Rightarrow \mathcal{H}_0 \), \( y \notin \mathcal{A}_{ML}(A, \mathcal{E}) \Rightarrow \mathcal{H}_1 \).

Let us show that without any loss of quality we can replace the set \( \mathcal{E} \) in (23) with a smaller set \( \mathcal{E}_0 \) (see (14)), i.e., use the criterion

\[
\mathcal{A}_{MLR}(A, \mathcal{E}) = \left\{ y : \sup_{\sigma \in \mathcal{E}_0} [2r(y, \sigma) - A(\sigma)] \leq 0 \right\},
\]

keeping the same decision-making method. In other words, for the likelihood ratio criterion, the expansion of \( \mathcal{E}_0 \) to a set \( \mathcal{E} \) does not increase the 2nd-kind error probability (the 1st-kind error probability \( \alpha(A) \) is unchanged).
Indeed, if $\sigma \in E$ but $\sigma \notin E_0$, then there exists $\sigma_0 \in E_0$ with $\sigma_0 < \sigma$. Using the definition (24) and equations (26) and (27) below, we have

$$
\beta(A, \sigma) = P \left\{ \sup_{\lambda \in E_0} \left[ 2r(y, \lambda) - A(\lambda) \right] \leq 0 \bigg| \sigma \right\} \\
= P \left\{ \sup_{\lambda \in E_0} \left[ \sum_{i=1}^{n} \frac{\lambda_i^2 (1 + \sigma_i^2) \eta_i^2}{1 + \lambda_i^2} - D(\lambda) - A(\lambda) \right] \leq 0 \right\} \\
\leq P \left\{ \sup_{\lambda \in E_0} \left[ \sum_{i=1}^{n} \frac{\lambda_i^2 (1 + \sigma_0^2) \eta_i^2}{1 + \lambda_i^2} - D(\lambda) - A(\lambda) \right] \leq 0 \right\} \\
= P \left\{ \sup_{\lambda \in E_0} \left[ 2r(y, \lambda) - A(\lambda) \right] \leq 0 \bigg| \sigma_0 \right\} = \beta(A, \sigma_0).
$$

The obtained results (17) and (25) can be formulated as follows.

**Proposition 1.** Consider the minimax problem of testing a simple hypothesis $H_0$ against a composite alternative $H_1 = \{E_n\}$, and let $E_0 \subseteq E$. If condition (14) is satisfied for the set $E$ and if $E_0$ is replaced with $E$, then for any Bayesian criterion and the likelihood ratio criterion the 1st- and 2nd-kind error probabilities are not changed. In particular, equality (18) holds.

**Remark 1.** It seems that in Proposition 1 it would be more natural to start with a set $E$ and replace it with $E_0 \subseteq E$. But in that case it would be necessary to describe a “projection” of a Bayesian criterion from $E$ onto $E_0$.

**Remark 2.** “Reduced” sets red$_1 S$ and red$_2 S$ similar to $E_0$ were earlier introduced in [3], where Gaussian measures differed from each other by shifts only. From the analytical viewpoint, various convexity properties with respect to shifts of Gaussian measures were highly useful in [3]; for example, due to such properties the set red$_1 S$ had a very simple and natural form. Unfortunately, the author does not know similar convexity properties for variances of Gaussian measures, and that is why we only use certain monotonicity properties (which is less productive).

### 3. Exact Equality (12)

1. Equation (12) has also another equivalent interpretation. Assume that for the hypothesis $H_1$ we know in advance that the “signal” has a certain $\sigma$ and therefore we use the optimal solution (5)–(6) for this $\sigma$. Assume also that the “signal” in the hypothesis $H_1$ may also take other values $\lambda$ from a set $E$. For what $E$ the solution (5)–(6) (oriented only on $\sigma$) remains optimal for the set $E$ too?

If $\sigma$ is replaced with $\lambda$ and the decision (5)–(6) is used, then the 1st-kind error probability $\alpha$ is not changed. Therefore, we have only to check how the 2nd-kind error probability $\beta_\sigma(A, \lambda)$ may change:

$$
\beta_\sigma(A, \lambda) = P(y \in A \bigg| \lambda) \\
= P \left( \sum_{i=1}^{n} \frac{\sigma_i^2 (\xi_i + s_i)^2}{1 + \sigma_i^2} - D(\sigma) < A \bigg| \lambda \right) = P \left( \sum_{i=1}^{n} \nu_i^2 \xi_i^2 - D(\sigma) < A \right),
$$

since $(\xi_i + s_i)^2 = (1 + \lambda_i^2) \eta_i^2$, $i = 1, \ldots, n$, where

$$
\nu_i^2 = \frac{\sigma_i^2 (1 + \lambda_i^2)}{1 + \sigma_i^2} = \frac{\sigma_i^2 (\lambda_i^2 - \sigma_i^2)}{1 + \sigma_i^2}, \quad i = 1, \ldots, n,
$$

and $\{\eta_i\}$ are independent $N(0,1)$-Gaussian random variables.
If for any \( \lambda \in \mathcal{E} \) and \( A \) the inequality \( (\nu = (\nu_1, \ldots, \nu_n) \) is defined in (27))

\[
\beta_\sigma(A, \lambda) = \mathbb{P} \left( \sum_{i=1}^{n} \nu_i^2 \xi_i^2 - D(\sigma) < A \right) \leq \mathbb{P} \left( \sum_{i=1}^{n} \sigma_i^2 \xi_i^2 - D(\sigma) < A \right) = \beta(A, \sigma)
\]

holds, then

\[
\beta(A, \mathcal{E}) \leq \sup_{\lambda \in \mathcal{E}} \beta_\sigma(A, \lambda) \leq \beta(A, \sigma),
\]

and therefore formula (12) is valid.

Some results showing the validity of inequality (28) for certain \( \sigma, \nu, \) and \( A \) can be found, for example, in [5–7].

For \( \beta_\sigma(A, \lambda) \leq \beta(A, \sigma) \) to hold for any \( A \) (see (28)), it is in any case necessary (comparing the expectations) that

\[
\sum_{i=1}^{n} \nu_i^2 - \sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \frac{\sigma_i^2 (\lambda_i^2 - \sigma_i^2)}{1 + \sigma_i^2} \geq 0.
\]

By comparing (8), (26), and (27), we obtain the following simple result.

**Proposition 2.** 1. If \( \sigma \geq \lambda \), then \( \beta_\sigma(A, \lambda) \leq \beta(A, \sigma) \) and \( \beta_\sigma(A, \lambda) \leq \beta(A, \sigma) \) for any \( A \).

2. If \( \sigma \leq \lambda \) for any \( \lambda \in \mathcal{E} \), then \( \beta(\alpha, \mathcal{E}) = \beta(\alpha, \sigma) \) for any \( \alpha \).

2. As an example, consider the following result, which is a part of [7, Lemma 1].

**Lemma 2.** Assume that the set of indices \( I = \{1, 2, \ldots, n\} \) of vectors \( \sigma \) and \( \lambda \) can be partitioned in \( k \geq 1 \) groups \( I_1, \ldots, I_k \) such that \( I = \bigcup_{j=1}^{k} I_j, I_i \cap I_j = \emptyset, i \neq j \), and the conditions

\[
\sigma_i \leq \lambda_{0,j}, \quad i \in I_j, \quad j = 1, \ldots, k,
\]

are fulfilled, where

\[
\lambda_{0,j} = \left( \prod_{i \in I_j} \lambda_i \right)^{1/|I_j|}.
\]

Then \( \beta(A, \lambda) \leq \beta(A, \sigma) \) for any \( A \).

Example 1. For a given \( D > 0 \), let

\[
\mathcal{E} = \left\{ \lambda \geq 0 : \prod_{i=1}^{n} \left( 1 + \lambda_i^2 \right) \geq \left( 1 + D^2 \right)^n \right\}.
\]

Then from equation (27) and Lemma 2 with \( k = 1 \) it follows that the set \( \mathcal{E} \) can be replaced (without loss of quality) by the single point \( \sigma_0 = (D, \ldots, D) \in \mathcal{E} \) (in the sense of exact equality (12)).

4. **ASYMPTOTIC EQUALITY (13).** LARGE DEVIATIONS FOR \( \beta(A, \sigma) \) AND \( \beta_\sigma(A, \lambda) \).

Now we consider conditions for equality (13) to hold. For that purpose we investigate the logarithmic asymptotics of the probabilities \( \beta(A, \sigma) \) and \( \beta_\sigma(A, \lambda) \) as \( n \to \infty \).

1. **Large deviations. Upper bounds.** Since for \( \xi \sim \mathcal{N}(0, 1) \) we have

\[
\mathbb{E} e^{a(\xi+b)^2} = \frac{1}{\sqrt{1 - 2a}} \exp \left\{ \frac{2ab^2}{1 - 2a} \right\}, \quad a < 1/2, \quad b \in \mathbb{R}^1,
\]

\[
(29)
\]
by the exponential Chebychev inequality we have, for any \( u \geq 0 \),

\[
\beta(A, \sigma) = P \left( \sum_{i=1}^{n} \frac{\sigma_i^2 \xi_i^2}{1 + u \sigma_i^2} < D(\sigma) + A \right) \leq e^{u[D(\sigma)+A]/2} e^{-u \sum_{i=1}^{n} \sigma_i^2 \xi_i^2/2} = e^{-g_\sigma(u)}, \tag{30}
\]

where

\[
2g_\sigma(u) = \sum_{i=1}^{n} \ln(1 + u \sigma_i^2) - u[D(\sigma) + A],
\]

\[
2g'_\sigma(u) = \sum_{i=1}^{n} \frac{\sigma_i^2}{1 + u \sigma_i^2} - D(\sigma) - A, \quad g''_\sigma(u) < 0.
\tag{31}
\]

Since both conditions (9) are assumed to be fulfilled, we have \( g'_\sigma(0) > 0 \) and \( g'_\sigma(1) < 0 \). Therefore, \( \max_{u \geq 0} g_\sigma(u) \) is attained for \( 0 < u_0 < 1 \) determined by the equation \( g'_\sigma(u_0) = 0 \), i.e.,

\[
\sum_{i=1}^{n} \frac{\sigma_i^2}{1 + u_0 \sigma_i^2} = D(\sigma) + A.
\tag{32}
\]

Then from (30) and (31) we obtain

\[
\beta(A, \sigma) \leq e^{-g_\sigma(u_0)}, \tag{33}
\]

where

\[
g_\sigma(u_0) = \max_{u \geq 0} g_\sigma(u). \tag{34}
\]

Under certain conditions (see Paragraph 3 of the Appendix) this is the exact logarithmic asymptotic of \( \beta(A, \sigma) \) as \( n \to \infty \).

Similarly, from (26) and (27) we have, for any \( v \geq 0 \),

\[
\beta_\sigma(A, \lambda) = P \left( \sum_{i=1}^{n} \nu_i^2 \xi_i^2 < D(\sigma) + A \right) \leq e^{v[D(\sigma)+A]/2} \prod_{i=1}^{n} \frac{1}{\sqrt{1 + v \nu_i^2}} = e^{-g_\sigma(v, \lambda)},
\]

where the \( \{\nu_i^2\} \) are defined in (27) and

\[
2g_\sigma(v, \lambda) = \sum_{i=1}^{n} \ln(1 + v \nu_i^2) - v[D(\sigma) + A],
\]

\[
2g'_\sigma(v, \lambda) = \sum_{i=1}^{n} \frac{\nu_i^2}{1 + v \nu_i^2} - D(\sigma) - A, \quad g''_\sigma(v, \lambda) < 0.
\]

Then

\[
\beta_\sigma(A, \lambda) \leq e^{-g_\sigma(v_0, \lambda)}, \tag{35}
\]

where

\[
g_\sigma(v_0, \lambda) = \max_{v \geq 0} g_\sigma(v, \lambda). \tag{36}
\]

In is reasonable to only consider \( \lambda \) such that \( g'_\sigma(0, \lambda) = \sum_{i=1}^{n} \nu_i^2 - D(\sigma) - A > 0 \) (otherwise, \( v_0 = 0 \)). Then \( \max_{v \geq 0} g_\sigma(v, \lambda) \) is attained for \( v_0 > 0 \) determined by the equation

\[
\sum_{i=1}^{n} \frac{\nu_i^2}{1 + v_0 \nu_i^2} = \sum_{i=1}^{n} \frac{\sigma_i^2(1 + \lambda_i^2)}{1 + \sigma_i^2 + v_0 \sigma_i^2(1 + \lambda_i^2)} = D(\sigma) + A.
\]
If estimates (33)–(34) and (35)–(36) give the correct logarithmic asymptotics (as \( n \to \infty \)) of 
\[ \beta(A, \sigma) \text{ and } \beta_\sigma(A, \lambda), \]
we have \( g_\sigma(u_0) - g_\sigma(v_0, \lambda) \geq 0 \). Hence, condition (13) is equivalent to the following question: If \( \sigma \) is given, then for what \( \lambda \) the condition
\[ g_\sigma(u_0) - g_\sigma(v_0, \lambda) = o(g_\sigma(u_0)) \text{ as } g_\sigma(u_0) \to \infty \tag{37} \]
holds?

If condition (37) is fulfilled and we replace \( \sigma \) with \( \lambda \) using the decision (5)–(6) (oriented on \( \sigma \)), then the 1st-kind error probability \( \alpha \) is not changed, and the 2nd-kind error probability \( \beta_\sigma(A, \lambda) \) changes slightly.

Generally, condition (37) is rather difficult to check (since we must find \( v_0 \) for each \( \lambda \)). A condition sufficient for (37) to hold is a simpler condition
\[ g_\sigma(u_0) - \max\{g_\sigma(u_0, \lambda), g_\sigma(1, \lambda)\} = o(g_\sigma(u_0)), \quad g_\sigma(u_0) \to \infty, \tag{38} \]
or, in particular,
\[ g_\sigma(u_0, \lambda) - g_\sigma(u_0) = \sum_{i=1}^{n} \ln \left[ 1 + \frac{u_0 \sigma_i^2 (\lambda_i^2 - \sigma_i^2)}{(1 + \sigma_i^2)(1 + u_0 \sigma_i^2)} \right] = o(g_\sigma(u_0)). \tag{39} \]

Note that condition (38) (or (39)) is only sufficient but not necessary. It may give satisfactory results if \( \lambda \) is not too different from \( \sigma \). If \( \lambda \) is very far from \( \sigma \), then an essential loss in accuracy is possible (see Example 3 below, where condition (38) is not fulfilled but condition (37) is satisfied). Let us give another similar example (omitting some details).

**Example 2.** Choose \( \sigma \) and \( \lambda \) such that \( u_0 \neq v_0 \) and equality in (37) holds, i.e.,
\[ g_\sigma(u_0) = \max_{u \geq 0} g_\sigma(u) = g_\lambda(v_0) = \max_{v \geq 0} g_\lambda(v). \]

Now, if condition (38) is satisfied, then a similar condition
\[ g_\lambda(v_0) \leq g_\sigma(v_0) \tag{40} \]
cannot be fulfilled. This means that if we exchange \( \sigma \) and \( \lambda \), then condition (40) is no more necessary.

2. **Case of \( u_0 \approx 1 \).** Consider an important particular case where the set \( \mathcal{E} \) can be replaced with a single point \( \sigma \in \mathcal{E} \) and the sufficient condition (39) takes a simple form. Let \( \alpha \) be too small, so that we need only that \( \alpha(A, \mathcal{E}) \) satisfies \( \alpha(A, \mathcal{E}) \leq 6B_n^{-1/2} \), where \( B_n = \inf_{\sigma \in \mathcal{E}} B(\sigma) \) and \( B(\sigma) \) is defined in (2). For that purpose, keeping in mind some \( \sigma \in \mathcal{E} \), we put (see (2) and (43))
\[ A = T(\sigma) - D(\sigma) + \varepsilon, \]

where
\[ \varepsilon = \sqrt{B(\sigma) \ln B(\sigma)}. \]
Denoting \( u_0 = 1 - \delta, \delta \geq 0 \), we show that \( \delta \) is small for large \( B(\sigma) \). Indeed, equation (32) takes the form
\[ \sum_{i=1}^{n} \frac{\sigma_i^2}{1 + u_0 \sigma_i^2} = D(\sigma) + A = T(\sigma) + \varepsilon = \sum_{i=1}^{n} \frac{\sigma_i^2}{1 + \sigma_i^2} + \varepsilon, \]

which implies
\[ \sum_{i=1}^{n} \frac{\delta \sigma_i^4}{(1 + \sigma_i^2 - \delta \sigma_i^2)(1 + \sigma_i^2)} = \varepsilon \geq \sum_{i=1}^{n} \frac{\delta \sigma_i^4}{(1 + \sigma_i^2)^2} = \frac{\delta B}{2}. \]
Therefore,
\[ 0 \leq 1 - u_0 = \delta \leq \frac{2\varepsilon}{B} = 2\sqrt{\frac{\ln B}{B}}. \]

Since \( g_\sigma(1) = -A/2, g_\sigma'(1) = -\varepsilon/2, \) and \( g_\sigma''(u) < 0, \) we have
\[ -A \leq 2g_\sigma(u_0) = 2g_\sigma(1 - \delta) \leq 2g_\sigma(1) - 2\delta g_\sigma'(1) = -A + \delta \varepsilon \leq -A + 2 \ln B. \]

Therefore, in the sufficient condition (39) we can put \( u_0 = 1; \) then it takes the form
\[ g_\sigma(1, \lambda) - g_\sigma(1) = \sum_{i=1}^{n} \ln \left[ 1 + \frac{\sigma_i^2(\lambda_i^2 - \sigma_i^2)}{(1 + \sigma_i^2)^2} \right] = o\left( g_\sigma(1) \right), \quad g_\sigma(1) \to \infty. \quad (41) \]

The obtained results can be formulated as follows.

**Proposition 3.** 1. If there exists \( \sigma \in \mathcal{E} \) such that condition (39) is satisfied for any \( \lambda \in \mathcal{E}, \) then property (13) holds and the set \( \mathcal{E} \) can be replaced with the point \( \sigma \in \mathcal{E} \) without any loss of detection quality.

2. If only \( \alpha(A, \mathcal{E}) \leq 6B_n^{-1/2} \) is required and there exists \( \sigma \in \mathcal{E} \) such that condition (41) is satisfied for any \( \lambda \in \mathcal{E}, \) then property (13) holds and the set \( \mathcal{E} \) can be replaced with the point \( \sigma \in \mathcal{E} \) without any loss of detection quality.

In the case of stationary sequences, a condition similar to (41) appeared in [4, Theorem 1, equation (6)] from different arguments. The authors of [4] called that analog of condition (41) “surprising,” since, in particular, it does not require the set \( \mathcal{E} \) to be convex. But, as was already mentioned (see Remark 2), in the problems with unknown correlations such convexity is not that important. Condition (41) itself is a consequence of a quite natural sufficient condition (38).

It is shown in the Appendix that, under certain assumptions, the upper bounds on \( \beta(A, \sigma) \) and \( \beta_\sigma(A, \lambda) \) used above give exact logarithmic asymptotics for them as \( n \to \infty \) (and then it is sufficient to compare the functions \( g_\sigma(u) \) and \( g_\sigma(v, \lambda) \)).

5. RELATION BETWEEN \( A \) AND \( \alpha(A, \sigma) \)

1. **Central limit theorem.** If a given \( \alpha(A, \sigma) \) is not too small, then it is possible to evaluate it rather accurately using the central limit theorem and Berry–Esseen inequality. Let \( X_1, \ldots, X_n \) be independent random variables, \( E X_j = 0, \) \( E |X_j|^3 < \infty, \) \( j = 1, \ldots, n. \) Denote
\[ b_j^2 = E X_j^2, \quad B_n = \sum_{j=1}^{n} b_j^2, \quad F_n(x) = P \left( \frac{1}{n} \sum_{j=1}^{n} X_j < x \right), \quad L_n = B_n^{-3/2} \sum_{j=1}^{n} E |X_j|^3. \]

Then by the Berry–Esseen inequality [8, ch. V, Section 2, Theorem 3] we have
\[ \sup_x |F_n(x) - \Phi(x)| \leq L_n. \]

In our case,
\[ X_j = \frac{\sigma_j^2(\xi_j^2 - 1)}{1 + \sigma_j^2}, \quad b_j^2 = \frac{2\sigma_j^4}{(1 + \sigma_j^2)^2}, \quad B_n = \sum_{j=1}^{n} b_j^2, \]
\[ E |X_j|^3 \leq \frac{10\sigma_j^6}{(1 + \sigma_j^2)^3} \leq \frac{10\sigma_j^4}{(1 + \sigma_j^2)^2} = 5b_j^2, \quad L_n \leq 5B_n^{-1/2}. \]

Therefore,
\[ \alpha(A, \sigma) = P \left( \sum_{i=1}^{n} \frac{x_i^2}{1 + \sigma_i^2} > D(\sigma) + A \right) = P \left( \sum_{i=1}^{n} X_i > D(\sigma) + A - T(\sigma) \right), \]

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and then we get

$$|α(A, σ) − Q(x)| \leq 5B_n^{−1/2}, \quad x = B_n^{−1/2}[D(σ) + A − T(σ)],$$

where

$$Q(x) = \frac{1}{\sqrt{2π}} \int_{x}^{∞} e^{-u^2/2} du \leq \min \left\{ \frac{1}{2}, \frac{1}{x\sqrt{2π}} \right\} e^{-x^2/2}, \quad x > 0.$$ 

In particular,

$$α(A, σ) \leq \frac{5}{\sqrt{B}} + \min \left\{ \frac{1}{2}, \frac{1}{z\sqrt{2π}} \right\} e^{-z^2/(2B)}, \quad z = D(σ) + A − T(σ) > 0, \quad (42)$$

where $B = B(σ)$ is defined in (2). Equation (42) implies the following.

**Proposition 4.** If $A \geq T(σ) − D(σ) + \sqrt{B(\ln B − \ln \ln B)}$, then

$$α(A, σ) \leq \frac{6}{\sqrt{B(σ)}}, \quad B = B(σ). \quad (43)$$

Estimate (43) shows quite accurately the dependence of $α(A, σ)$ on $B(σ)$ (for large $B$) if the given value satisfies $α > 5B^{−1/2}$. Usually, $B(σ) \sim n$.

2. Large deviations. Upper bound. Since

$$α(A, σ) = P \left( \sum_{i=1}^{n} r_i^2 ξ_i^2 > D(σ) + A \right), \quad r_i^2 = \frac{σ_i^2}{1 + σ_i^2}, \quad (44)$$

for any $t \geq 0$, similarly to (30) and (31), we have

$$α(A, σ) \leq e^{-t[D(σ) + A]/2} \mathbb{E} e^{t \sum_{i=1}^{n} r_i^2 ξ_i^2 / 2} = e^{-f_α(t)},$$

where

$$2f_α(t) = t[D(σ) + A] + \sum_{i=1}^{n} \ln(1 − tr_i^2),$$

$$2f_α'(t) = D(σ) + A − \sum_{i=1}^{n} \frac{r_i^2}{1 − tr_i^2}, \quad f_α''(t) < 0. \quad (45)$$

Since both conditions (9) are assumed to be satisfied, we have $f_α'(0) > 0$ and $f_α'(1) < 0$. Therefore, $\max_{t \geq 0} f_α(t)$ is attained for $0 < t_0 < 1$ determined by the equation

$$\sum_{i=1}^{n} \frac{r_i^2}{1 − t_0 r_i^2} = \sum_{i=1}^{n} \frac{σ_i^2}{1 + (1 − t_0)σ_i^2} = D(σ) + A.$$ 

Then

$$α(A, σ) \leq e^{-f_α(t_0)}. \quad (46)$$

For $t = 1$ we have $f_α(1) = A/2$, which implies the estimate

$$α(A, σ) \leq e^{-f_α(1)} = e^{-A/2}. \quad (47)$$

The simple estimate (47) is sufficiently tight if $t_0$ is close to 1 (i.e., if all $σ_i^2$ are small).
6. ONE MORE EXAMPLE

Now we consider a more complicated example.

Example 3. For a given $R > 0$, let

$$\mathcal{E} = \left\{ \sigma \geq 0 : \sum_{i=1}^{n} \sigma_i^2 \geq nR^2 \right\}.$$ 

Then

$$\mathcal{E}_0 = \left\{ \sigma \geq 0 : \sum_{i=1}^{n} \sigma_i^2 = nR^2 \right\}.$$ 

Denote $\sigma_0 = (R, \ldots, R)$. Then

$$D(\sigma_0) = n \ln(1 + R^2), \quad \min_{\sigma \in \mathcal{E}} D(\sigma) = \ln(1 + nR^2),$$

and the minimum is attained for $\sigma$ having only one nonzero (equal to $R\sqrt{n}$) coordinate. Denote all those vectors by $\sigma_i$, $i = 1, \ldots, n$. For example, $\sigma_1 = (R\sqrt{n}, 0, \ldots, 0)$. Denote also

$$\mathcal{E}_1 = \{ \sigma_i, i = 1, \ldots, n \}.$$ 

We show that without any loss of quality (in the sense of asymptotic equality (13)), we can replace the whole set $\mathcal{E}$ with $\mathcal{E}_1$ and thus obtain the same results as for a single point $\sigma_1$. Note that this does not follow from the sufficient condition (38).

To show the possibility of such reduction of the set $\mathcal{E}$, we use the likelihood ratio criterion with the set $\mathcal{E}_1$ (see (24))

$$A(A, \mathcal{E}, \mathcal{E}_1) = \left\{ y : 2 \sup_{\lambda \in \mathcal{E}_1} r(y, \lambda) \leq A \right\}.$$ 

Now, if $D(\sigma_1) + A = \ln(1 + nR^2) + A \geq 0$, then

$$\beta(A, \sigma_1) = \mathbb{P} \left\{ 2 \max_{\lambda \in \mathcal{E}_1} r(y, \lambda) \leq A | \sigma_1 \right\} = \mathbb{P} \left\{ 2r(y, \sigma_1) \leq A | \sigma_1 \right\} \prod_{i=2}^{n} \mathbb{P} \left\{ 2r(y, \sigma_i) \leq A | \sigma_1 \right\}$$

$$= \mathbb{P} \left\{ \xi_1^2 \leq \frac{D(\sigma_1) + A}{nR^2} \right\} \prod_{i=2}^{n} \mathbb{P} \left\{ \frac{nR^2\xi_i^2}{1 + nR^2} \leq D(\sigma_1) + A \right\} \leq \frac{\sqrt{D(\sigma_1) + A}}{R\sqrt{n}}.$$ 

Estimate (48) gives the correct asymptotic in $n$, since for $n \rightarrow \infty$ and small $\alpha(A)$ we have

$$\prod_{i=2}^{n} \mathbb{P} \left\{ \frac{nR^2\xi_i^2}{1 + nR^2} \leq D(\sigma_1) + A \right\} \sim \mathbb{P}^{n} \left\{ |\xi_1| \leq \sqrt{D(\sigma_1) + A} \right\} \approx \left[ 1 - \frac{\alpha(A)}{n} \right]^{n} \approx e^{-\alpha(A)}.$$ 

We also have (see estimates (62))

$$\alpha(A) = \mathbb{P} \left\{ 2 \max_{\lambda \in \mathcal{E}_1} r(y, \lambda) \geq A | \mathcal{H}_0 \right\} \leq \mathbb{P} \left\{ \max_{i=1, \ldots, n} \xi_i^2 \geq D(\sigma_1) + A \right\} \leq n \mathbb{P} \left\{ \xi_1^2 \geq D(\sigma_1) + A \right\} \leq \frac{n}{\sqrt{D(\sigma_1) + A}} \exp \left\{ -\frac{D(\sigma_1) + A}{2} \right\}.$$ 

To simplify formulas, we choose $A$ as follows:

$$A = 2 \ln n - D(\sigma_1) = 2 \ln n - \ln(1 + nR^2).$$
Then
\[ \alpha(\sigma_1) \leq \frac{1}{\sqrt{2 \ln n}} \quad \text{and} \quad \beta(\sigma_1) \leq \frac{\sqrt{2 \ln n}}{R \sqrt{n}}. \] (49)

Consider \( \beta(\lambda) \) for \( \lambda \in \mathcal{E}_0 \). Denote
\[ z^2 = \frac{2(1 + nR^2) \ln n}{nR^2}. \]
Without loss of generality, assume that \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) and introduce an auxiliary threshold \( \lambda_0 \), \( 0 \leq \lambda_0 \leq z \) (the value of \( \lambda_0 \) to be defined below). Then we have
\[ \ln \beta(A, \mathcal{E}_1, \lambda) = \ln P \left\{ \max_{i=1, \ldots, n} \left[ (1 + \lambda_i^2) \xi_i^2 \right] \leq z^2 \right\} = B_1 + B_2 + B_3, \] (50)
where
\[ B_1 = \ln P \left\{ \max_{\lambda_i \leq \lambda_0} \left[ (1 + \lambda_i^2) \xi_i^2 \right] \leq z^2 \right\} < 0, \]
\[ B_2 = \ln P \left\{ \max_{\lambda_0^2 < \lambda_i^2 \leq z^2 - 1} \left[ (1 + \lambda_i^2) \xi_i^2 \right] \leq z^2 \right\} < 0, \] (51)
\[ B_3 = \ln P \left\{ \max_{\lambda_i^2 > z^2 - 1} \left[ (1 + \lambda_i^2) \xi_i^2 \right] \leq z^2 \right\} < 0. \]

We sequentially estimate \( B_1, B_2, \) and \( B_3 \) in (51). For that purpose, denote
\[ n_1 = \#\{\lambda_i : \lambda_i \leq \lambda_0\}, \quad n_2 = \#\{\lambda_i : \lambda_0^2 < \lambda_i^2 \leq z^2 - 1\}, \quad n_3 = \#\{\lambda_i : \lambda_i^2 > z^2 - 1\}, \]
\[ s_1 n_1 = \sum_{\lambda_i \leq \lambda_0} \lambda_i^2, \quad s_2 n_2 = \sum_{\lambda_0^2 < \lambda_i^2 \leq z^2 - 1} \lambda_i^2, \quad s_3 n_3 = \sum_{\lambda_i^2 > z^2 - 1} \lambda_i^2. \] (52)

Using notation (52) and the inequality \( \ln(1 + z) \leq z \), for \( B_1 \) we have
\[ B_1 = \sum_{i=1}^{n_1} \ln \left[ 1 - P \left\{ (1 + \lambda_i^2) \xi_i^2 \geq z^2 \right\} \right] \leq -2 \sum_{i=1}^{n_1} P \left\{ (1 + \lambda_i^2) \xi_i^2 \geq z^2 \right\} = \frac{2n_1}{n} P \left( \sqrt{1 + \lambda_i^2} \xi \geq z \right). \] (53)

Consider first the variational problem of minimizing the sum of any two terms from the right-hand side of (53)
\[ P \left( \sqrt{1 + \lambda_i^2} \xi \geq z \right) + P \left( \sqrt{1 + \lambda_j^2} \xi \geq z \right) \]
over the variables \( \lambda_i \) and \( \lambda_j \) with a given sum \( \lambda_i^2 + \lambda_j^2 = r^2 - 2 \geq 0 \). Denoting \( u^2 = 1 + \lambda_i^2 \) and \( v^2 = 1 + \lambda_j^2 \), denote also by \( f(u, z, r) \) the sum under consideration. Then we have
\[ f(u, z, r) = P \left\{ \xi \geq \frac{z}{u} \right\} + P \left\{ \xi \geq \frac{z}{v} \right\} = 2 \left[ 1 - \Phi \left( \frac{z}{u} \right) - \Phi \left( \frac{z}{v} \right) \right], \quad v = \sqrt{r^2 - u^2} \geq 1. \]

We are interested in \( \min_{1 \leq u \leq \sqrt{r^2 - 1}} f(u, z, r) \). We have
\[ f'(u) = f'_u + f'_v v_u, \quad f'_u = \frac{2 ze^{-z^2/(2u^2)}}{\sqrt{2\pi u^2}} - \frac{ue^{-z^2/(2u^2)}}{u^3}, \quad v_u = -\frac{u}{v}, \]
\[ f'(u) = \frac{2z}{\sqrt{2\pi}} \left\{ \frac{e^{-z^2/(2u^2)}}{u^2} - \frac{ue^{-z^2/(2u^2)}}{u^3} \right\} = \frac{2u}{\sqrt{2\pi z^2}} \left[ \exp \left\{ t \left( \frac{z}{u} \right) \right\} - \exp \left\{ t \left( \frac{z}{v} \right) \right\} \right], \]
where we denoted
\[ t(x) = 3 \ln x - \frac{x^2}{2}, \quad t'(x) = \frac{3}{x} - x, \quad x > 0. \]

The function \( t(x) \) monotonically decreases for \( x > \sqrt{3} \) and monotonically increases for \( 0 < x < \sqrt{3} \). Without loss of generality, we may assume that \( z/v \leq z/u \), i.e., \( 2u^2 \leq r^2 \). Therefore, if \( z/v = z/\sqrt{r^2 - u^2} \geq z/\sqrt{r^2 - 1} \geq \sqrt{3} \), then \( f'(u) \leq 0 \), \( u \leq r/\sqrt{2} \), and hence the minimum of \( f(u, z, r) \) in \( u \) is attained at \( u = v \) (i.e., when \( \lambda_i = \lambda_j \)). To fulfill these conditions, it is sufficient to set \( r^2 = z^2/3 + 1 \). As a result, we get that if among the \( \{\lambda_i\} \) there is a pair \( \lambda_i, \lambda_j \) such that \( \lambda_i \neq \lambda_j \) and \( \lambda_i^2 + \lambda_j^2 \leq r^2 - 2 = z^2/3 - 1 \), then \( f(u, z, r) \) decreases if we replace each \( \lambda_i^2, \lambda_j^2 \) with their half-sum \( (\lambda_i^2 + \lambda_j^2)/2 \). Therefore, we define the threshold \( \lambda_0 \) as follows:
\[ \lambda_0^2 = z^2/6 - 1/2. \] (54)

Continuing this process of maximizing the right-hand side of (53), we get that its maximum is attained at
\[ \lambda_1^2 = \ldots = \lambda_{n_1}^2 \leq \lambda_0^2. \]

For the remaining \( n - n_1 \) components \( \{\lambda_i\} \) we have
\[ \lambda_i > \lambda_0, \quad i = n_1 + 1, \ldots, n. \]

Therefore, for \( B_1 \) in (53), for large \( n \) and some \( C > 0 \) (see estimates (62)) we obtain
\[ B_1 \leq -2n_1 P \left( \sqrt{1 + \lambda_i^2} \geq z \right) \leq -C n_1 \frac{\lambda_0^2}{z} e^{-\frac{\pi}{2(1+\lambda_0^2)}} \leq -C n \frac{\lambda_0^2}{\sqrt{\ln n}}, \] (55)

since \( (n - n_1)\lambda_0^2 \leq R^2 n \), i.e., \( n_1 \geq n \left( 1 - R^2/\lambda_0^2 \right) \).

For \( B_2 \) in (51), we get
\[ B_2 \leq n_2 \ln P \left\{ (1 + \lambda_0^2)^{\xi^2} \leq z \right\} \leq n_2 \ln P \left\{ |\xi| \leq \sqrt{6} \right\} \leq -\frac{n_2}{100}. \] (56)

Now we estimate \( B_3 \) in (51). We have
\[ B_3 = \sum_{\lambda_i^2 > z^2 - 1} \ln \left( 2 P \left\{ 0 \leq \xi_i \leq \frac{z}{\sqrt{1 + \lambda_i^2}} \right\} \right) \leq -\frac{n_3}{2} \ln \frac{\pi}{2} - \frac{1}{2} I_3, \] (57)

where
\[ I_3 = \sum_{\{\lambda_i^2 \geq z^2 - 1\}} \ln \frac{1 + \lambda_i^2}{z^2}. \]

Consider the value of \( I_3 \) for given \( s_3 \) and \( n_3 \). Since the function \( I_3 \) is \( \cap \)-concave in \( \{\lambda_i^2\} \), its minimum is attained at an extreme point, i.e., where one of the coordinates \( \lambda_j^2 \) equals \( s_3 n_3 - (n_3 - 1)(z^2 - 1) \) and all the remaining coordinates \( \lambda_i^2 \) equal \( z^2 - 1 \). Hence,
\[ I_3 \geq \ln \frac{1 + s_3 n_3 - (n_3 - 1)(z^2 - 1)}{z^2} = \ln \frac{z^2 + n_3(s_3 - z^2 + 1)}{z^2}. \] (58)

Therefore, from (50) and (55)–(58) we obtain, for large \( n \),
\[ \ln \beta(A, \mathcal{E}, \lambda) \leq -\frac{C n_1}{\sqrt{\ln n}} - \frac{n_2}{100} - \frac{n_3}{5} - \frac{1}{2} \ln \frac{z^2 + n_3(s_3 - z^2 + 1)}{z^2}. \] (59)
It remains to show that the right-hand side of (59) satisfies the inequality

$$\min_{n_2,n_3,\lambda_1} \left\{ \frac{\lambda_1^2}{\sqrt{\ln n}} + \frac{n_2}{100} + \frac{n_3}{5} + \frac{1}{2} \ln \frac{z^2 + n_3(s_3 - z^2 + 1)}{z^2} \right\} \geq \frac{1}{2} \ln(R^2n) + o(\ln(R^2n)), \quad (60)$$

where the minimum is taken under the condition $n_2\lambda_0^2 + n_3s_3 \geq R^2n + o(R^2n)$.

We may assume (see (49) and (55)) that $n_2 < 50\ln(R^2n), \quad n_3 < 3\ln(R^2n)$ and $\lambda_1^2 < \frac{2\ln\ln n}{\ln n} + \frac{2\ln R}{\ln^2 n}$ (otherwise, inequality (60) holds). In other words, almost all the power $R^2n$ is distributed over the last $n_3$ components. Hence,

$$n_3s_3 = R^2n - n_1\lambda_1^2 - n_2z^2 = R^2n + o(n).$$

Then inequality (60) holds, and therefore for any $\lambda \in \mathcal{E}$ we obtain, as $n \to \infty$,

$$\ln \beta(A, \mathcal{E}_1, \lambda) \leq -\frac{1}{2} \ln(R^2n) + o(\ln(R^2n)) = (1 + o(1)) \ln \beta(\sigma_1). \quad (61)$$

Equation (61) means that the likelihood ratio criterion with the set $\mathcal{E}_1$ allows to get the same results for the whole set $\mathcal{E}$ as for the single point $\sigma_1$.

**APPENDIX**

1. **Tails of $\mathcal{N}(0, 1)$**. Let $\xi \sim \mathcal{N}(0, 1)$. Then the following estimates are known:

$$\frac{ze^{-z^2/2}}{(z^2 + 1)\sqrt{2\pi}} \leq P \{ \xi \geq z \} \leq \frac{e^{-z^2/2}}{z\sqrt{2\pi}}, \quad z > 0, \quad (62)$$

where the lower bound is derived via integration by parts.

2. **$\chi^2$ distribution. Large deviations.** Consider the quantity

$$\beta(A, n) = P \left( \sum_{i=1}^{n} \xi_i^2 < A \right). \quad (63)$$

**Lemma 3.** For $A \leq n$ and $n \geq 1$ we have the following estimates:

$$-\frac{1}{2} \ln(\pi n) - \frac{1}{3n} \leq \ln \beta(A, n) + \frac{1}{2} \left( n \ln \frac{n}{eA} + A \right) \leq 0. \quad (64)$$

**Proof.** The right-hand inequality in (64) follows from the exponential Chebychev inequality (see (29) and (30)). To prove the left-hand one, denote

$$\mathcal{B}_n(r) = \left\{ y : \sum_{i=1}^{n} y_i^2 \leq r^2 \right\}. \quad (65)$$

Then

$$|\mathcal{B}_n(r)| = \frac{\pi^{n/2}r^n}{\Gamma(n/2 + 1)},$$

$$\ln \Gamma(z) = z \ln \frac{z}{e} - \frac{1}{2} \ln z + \frac{1}{2} \ln(2\pi) + \frac{\theta}{6z}, \quad z > 0, \quad 0 \leq \theta \leq 1.$$
Therefore,
\[
\beta(A, n) = \frac{1}{(2\pi)^{n/2}} \int_0^{\sqrt{A}} e^{-r^2/2} \, d|B_n(r)| = \frac{1}{\Gamma(n/2)} \int_0^{A/2} v^{n/2-1} e^{-v} \, dv.
\]

Integrating by parts, we have \((a = n/2 - 1, B = A/2, 0 < \theta < 1)\)
\[
\int_0^B v^a e^{-v} \, dv = \frac{B^{a+1} e^{-B}}{a + 1} + \frac{1}{(a + 1)} \int_0^B v^{a+1} e^{-v} \, dv = \frac{B^{a+1} e^{-B}}{a + 1} + \frac{\theta B}{(a + 1)} \int_0^B v^a e^{-v} \, dv.
\]
Therefore,
\[
\int_0^B v^a e^{-v} \, dv = \frac{B^{a+1} e^{-B}}{a + 1 - \theta B}, \quad 0 < \theta < 1.
\]

Then
\[
\beta(A, n) = \frac{1}{\Gamma(n/2)} \int_0^{A/2} v^{n/2-1} e^{-v} \, dv = \frac{2 \Gamma(n/2)}{(n - \theta A) \Gamma(n/2)} \left( \frac{A}{2} \right)^{n/2} e^{-A/2}.
\]

Hence \((0 \leq \theta, \theta < 1)\),
\[
\ln \beta(A, n) = -\frac{n}{2} \ln \frac{n}{eA} - \frac{A}{2} + \frac{1}{2} \ln \frac{n}{4\pi} - \frac{\theta_1}{3n} + \ln \frac{2}{n - \theta A},
\]
whence the left-hand inequality in (64) follows. \(\triangle\)

Consider the quantity
\[
\alpha(A, n) = P \left( \sum_{i=1}^{n} \xi_i^2 > A \right).
\]

**Lemma 4.** For \(A \geq n\) and \(n \geq 2\) we have the following estimates:
\[
-\frac{1}{3n} - \frac{1}{2} \ln \frac{\pi A^2}{n} \leq \ln \alpha(A, n) + \frac{1}{2} \left( n \ln \frac{n}{eA} + A \right) \leq 0.
\]  

**Proof.** The right-hand inequality in (67) follows from the exponential Chebychev inequality (see (44)–(46)). For the left-hand one we have, using notation (65),
\[
\alpha(A, n) = \frac{1}{(2\pi)^{n/2}} \int_{\sqrt{A}}^{\infty} e^{-r^2/2} \, d|B_n(r)| = \frac{1}{\Gamma(n/2)} \int_0^{A/2} v^{n/2-1} e^{-v} \, dv.
\]

Integrating by parts, for the last integral \((a = n/2 - 1, B = A/2)\) we obtain
\[
\int_B^\infty v^a e^{-v} \, dv = B^a e^{-B} + \frac{\theta a}{B} \int_B^\infty v^a e^{-v} \, dv, \quad 0 < \theta < 1.
\]
Therefore,
\[
\int_B^\infty v^a e^{-v} \, dv = \frac{B^a e^{-B}}{1 - \theta a/B}, \quad 0 < \theta < 1,
\]
and then
\[
\alpha(A, n) = \frac{2}{A \Gamma(n/2)} \left( \frac{A}{2} \right)^{n/2} e^{-A/2} \frac{1}{1 - \theta(n - 2)/A}, \quad 0 < \theta < 1.
\]
As a result, for \( A \geq n \) and \( n \geq 2 \) we obtain \( 0 < \theta, \theta_1 < 1 \)

\[
\ln \alpha(A, n) = -\frac{n}{2} \ln \frac{n}{eA} - \frac{A}{2} - \frac{1}{2} \ln \frac{\pi}{3n} - \theta_1 - \ln[A - \theta(n - 2)],
\]

whence the left-hand inequality in (67) follows.

3. Large deviations for \( \beta(A, \sigma) \). Lower bound. To estimate \( \beta(A, \sigma) \) from below, we use an approach similar to [9, proof of Theorem 1]. Let \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \). We divide the segment \([1, n]\) into \( K \) equal parts of length \( \Delta = (n - 1)/K \) by points \( n_k = 1 + \Delta k, \ 1 \leq k \leq K \), and represent \( A \) as a sum \( A = A_1 + \ldots + A_K \). Then

\[
\beta(A, \sigma) = \mathbf{P} \left( \sum_{i=1}^{n} \sigma_i^2 \xi_i^2 < A \right) \geq \max_{K} \prod_{k=1}^{K} \mathbf{P} \left( \sigma_{n_k-1+1}^2 \sum_{i=n_k-1+1}^{n_k} \xi_i^2 < A_k \right),
\]

where the maximum is over all \( K \) and \( \{A_k\} \). To evaluate the probabilities on the right-hand side of (68), we use estimate (64). Denoting

\[
b_k = \sigma_{n_k-1+1}^2, \quad k = 1, \ldots, K,
\]

and assuming \( A_k \leq b_k \Delta, \ k = 1, \ldots, K \) (see (63)), from (64) we have

\[
2 \ln \beta(A, \sigma) \geq 2 \max_{K} \sum_{k=1}^{K} \ln \mathbf{P} \left( \sum_{i=n_k-1+1}^{n_k} \xi_i^2 < A_k/b_k \right)
\]

\[
\geq - \min_{\{A_k\}} \sum_{k=1}^{K} \left( \Delta \ln \frac{b_k}{A_k} + \frac{A_k}{b_k} \right) - (n - 1) \ln \frac{\Delta}{e} - K \ln(\pi \Delta).
\]

The minimum on the right-hand side of (69) under the condition \( A = A_1 + \ldots + A_K \) is attained for

\[
A_k = \frac{\Delta b_k}{1 + u_1 b_k}, \quad k = 1, \ldots, K,
\]

where \( u_1 \) is determined by the equation

\[
\sum_{k=1}^{K} \frac{\Delta b_k}{1 + u_1 b_k} = A,
\]

similar to (32). Since \( \Delta b_k \geq A_k \), we have \( u_1 \geq 0 \). Moreover,

\[
\sum_{k=1}^{K} \left( \Delta \ln \frac{b_k}{A_k} + \frac{A_k}{b_k} \right) = \Delta \sum_{k=1}^{K} \ln(1 + u_1 b_k) - (n - 1) \ln \frac{\Delta}{e} - u_1 A.
\]

Since for any \( u \geq 0 \) we have

\[
\Delta \sum_{k=1}^{K} \ln(1 + u b_k) = \sum_{i=1}^{n} \ln(1 + u \sigma_i^2) \leq \Delta \sum_{k=1}^{K} \ln(1 + u b_k),
\]

using (31), we obtain

\[
2 \ln \beta(A, \sigma) + K \ln \frac{\pi n}{K} \geq -\Delta \sum_{k=1}^{K} \ln(1 + u_1 b_k) + u_1 A
\]

\[
= -2g_\sigma(u_0) - \Delta \sum_{k=1}^{K} \ln(1 + u_1 b_k) + \sum_{i=1}^{n} \ln(1 + u_0 \sigma_i^2) + (u_1 - u_0) A
\]

\[
\geq -2g_\sigma(u_0) - \Delta \sum_{k=1}^{K} \ln \frac{1 + u_1 b_k}{1 + u_1 b_{k+1}} + (u_1 - u_0) A
\]

\[
\geq -2g_\sigma(u_0) - \Delta \ln \frac{1 + u_1 b_1}{1 + u_1 b_n} = -2g_\sigma(u_0) - \Delta \ln \frac{b_1}{b_n},
\]

\[\]
where the inequality \( u_1 \geq u_0 \) was used. Indeed, from equation (70) we have
\[
(u_1)'_k = \frac{1}{b_k} \geq 0, \quad k = 1, \ldots, K,
\]
and since \( \sigma_1 \geq \ldots \geq \sigma_n \), we conclude that \( u_1 \geq u_0 \). Therefore, denoting
\[
\delta_{\sigma} = \ln \frac{\max \sigma_i^2}{\min \sigma_i^2} \geq 0,
\]
from (71) we obtain
\[
\ln \beta(A, \sigma) + g_{\sigma}(u_0) \geq \frac{1}{2} \min_{K \geq 1} \left\{ \frac{n \delta_{\sigma}}{K} + K \ln(\pi n) \right\} = - \left[ n \delta_{\sigma} \ln(\pi n) \right]^{1/2},
\]
provided that the maximizer \( K = K_0 \geq 1 \), where
\[
K_0^2 = \frac{n \delta_{\sigma}}{\ln(\pi n)}.
\]
If \( K_0 < 1 \) (i.e., \( \sigma_1^2/\sigma_n^2 \) is close to 1), then, setting \( K = 1 \), we obtain
\[
\ln \beta(A, \sigma) + g_{\sigma}(u_0) \geq - \ln(\pi n).
\]
Both cases \( K_0 \geq 1 \) and \( K_0 < 1 \) and equation (33) can be combined as follows:
\[
-\sqrt{\delta_{\sigma} n \ln(\pi n)} - \ln(\pi n) \leq \ln \beta(A, \sigma) + g_{\sigma}(u_0) \leq 0. 
\]

(72)

Note that usually \( g_{\sigma}(u_0) \sim n \). Then (72) gives the correct logarithmic asymptotic for \( \beta(A, \sigma) \) if \( \delta_{\sigma} = o(n/\ln n), n \to \infty \).

As a result, we obtain the following.

**Proposition 5.**
1. For \( \ln \beta(A, \sigma) \), the upper and lower bounds (72) hold.
2. If \( g_{\sigma}(u_0) \leq g_{\lambda}(u_0) \) (for example, the sufficient condition (39) is fulfilled), then
\[
\ln \beta(A, \sigma) \leq \ln \beta(A, \sigma) + \sqrt{\delta_{\sigma} n \ln(\pi n)} + \ln(\pi n).
\]

(73)

Equation (73) follows from (72):
\[
\ln \beta(A, \lambda) \leq -g_{\lambda}(u_0) \leq -g_{\sigma}(u_0) \leq \ln \beta(A, \sigma) + \sqrt{\delta_{\sigma} n \ln(\pi n)} + \ln(\pi n).
\]

The lower bound on \( \alpha(A, \sigma) \) can be derived similarly.

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