Optimal Hölder regularity for the \( \bar{\partial} \) problem on product domains in \( \mathbb{C}^2 \)

Yuan Zhang

Abstract

The note concerns the \( \bar{\partial} \) problem on product domains in \( \mathbb{C}^2 \). We show that there exists a bounded solution operator from \( C^{k,\alpha} \) into itself, \( k \in \mathbb{Z}^+ \cup \{0\} \), \( 0 < \alpha < 1 \). The regularity result is optimal in view of an example of Stein-Kerzman.

1 Introduction

Let \( \Omega \subset \mathbb{C}^n \) be the product of planar domains whose boundaries consist of finite number of non-intersecting rectifiable Jordan curves. Then \( \Omega \) is weakly pseudoconvex with at most Lipschitz boundary. A natural question is to look for a solution operator to the \( \bar{\partial} \) problem on \( \Omega \) that achieves the optimal regularity.

As indicated by Example 3.2 of Stein-Kerzman [12], the \( \bar{\partial} \) problem on product domains does not gain regularity in general. This phenomenon is in sharp contrast with some well-understood domains bearing with nice geometry (such as strict pseudoconvexity, convexity and/or finite types), on which solutions with gained regularity always exist. See [4, 7, 8, 10, 12, 13] et al and the references therein.

Initiated by the work of Henkin [9] on the bidisc, Bertrams [1], Chen-McNeal [2, 3], Fassina-Pan [5] and Jin-Yuan [11] etc investigated uniform \( C^k \) and Sobolev norms of solutions on product domains. In the Hölder category, the celebrated work of Nijenhuis and Woolf [14] constructed optimal Hölder solutions in some special iterated Hölder spaces for polydiscs. Pan and the author [15] recently proved existence of (the standard) Hölder solutions with an infinitesimal loss of Hölder regularity by analysing the parameter dependence of the Cauchy singular integrals.

2010 Mathematics Subject Classification. Primary 32W05; Secondary 32A26, 32A55.

Keywords. \( \bar{\partial} \) problem, product domains, Hölder spaces, optimal.
In this note, we prove that for product domains in $\mathbb{C}^2$, the solution operator in [15] must attain the same regularity as that of the Hölder data. Thus the operator achieves the optimal regularity in view of Example 3.2. The proof relies on a careful inspection of the Hölder regularity along each direction.

**Theorem 1.1.** Let $\Omega = D_1 \times D_2$, where $D_1$ and $D_2$ are two bounded domains in $\mathbb{C}$ with $C^{k+1,\alpha}$ boundaries, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$. Assume $f \in C^{k,\alpha}(\Omega)$ is a $\bar{\partial}$-closed $(p, q)$ form on $\Omega$ (in the sense of distributions if $k = 0$), $0 \leq p \leq 2, 1 \leq q \leq 2$. Then there exists a solution operator $T$ to $\bar{\partial}u = f$ on $\Omega$ such that $Tf \in C^{k,\alpha}(\Omega)$, $\bar{\partial}Tf = f$, and $\|Tf\|_{C^{k,\alpha}(\Omega)} \leq C\|f\|_{C^{k,\alpha}(\Omega)}$, where the constant $C$ depends only on $\Omega, k$ and $\alpha$.

It is not clear whether the same result extends to general product domains in $\mathbb{C}^n, n \geq 3$, as Example 3.3 demonstrates. As a direct consequence of Theorem 1.1, the following regularity corollary holds for smooth $(0, 1)$ forms up to the boundary.

**Corollary 1.2.** Let $\Omega := D_1 \times D_2$, where $D_1$ and $D_2$ are two bounded domains in $\mathbb{C}$ with smooth boundaries. Assume $f \in C^{\infty}(\overline{\Omega})$ is a $\bar{\partial}$-closed $(p, q)$ form on $\Omega$, $0 \leq p \leq 2, 1 \leq q \leq 2$. Then there exists a solution $u \in C^{\infty}(\overline{\Omega})$ to $\bar{\partial}u = f$ on $\Omega$ such that for each $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$, $\|u\|_{C^{k,\alpha}(\Omega)} \leq C\|f\|_{C^{k,\alpha}(\Omega)}$, where the constant $C$ depends only on $\Omega, k$ and $\alpha$.

**Acknowledgement:** The author thanks Professor Yifei Pan for valuable suggestions. The author dedicates the paper to the memory of her father, Baoguo Zhang, who had consistently supported her in life and work.

### 2 Notations and preliminaries

Let $\Omega$ be an open subset of $\mathbb{C}^n$. For $0 < \alpha < 1$, define the $(\alpha)$-Hölder semi-norm of a function $f$ on $\Omega$ to be

$$H^\alpha[f] := \sup_{z, z' \in \Omega, z \neq z'} \frac{|f(z) - f(z')|}{|z - z'|^\alpha}.$$  

Given any $f \in C^k(\Omega), k \in \mathbb{Z}^+ \cup \{0\}$, its $C^k$ norm is denoted by $\|f\|_{C^k(\Omega)} := \sum_{|\beta|=0}^k \sup_{z \in \Omega} |D^\beta f(z)|$, where $D^\beta$ represents any $|\beta|$-th derivative operator. A function $f \in C^k(\Omega)$ is said to be in $C^{k,\alpha}(\Omega)$ if

$$\|f\|_{C^{k,\alpha}(\Omega)} := \|f\|_{C^k(\Omega)} + \sum_{|\beta|=k} H^\alpha[D^\beta f] < \infty.$$  

We say a $(p, q)$ form to be in $C^{k,\alpha}(\Omega)$ if all its coefficients are in $C^{k,\alpha}(\Omega)$. When $k = 0$, we suppress $k$ in the notations by writing $C^{0,\alpha}(\Omega)$ as $C^{\alpha}(\Omega)$, and $C^0(\Omega)$ as $C(\Omega)$.
Assume that \( \Omega := D_1 \times \ldots \times D_n \) is a product of planar domains \( D_j, 1 \leq j \leq n \). Fixing \((z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in D_1 \times \ldots \times D_{j-1} \times D_{j+1} \times \ldots \times D_n\), denote the Hölder semi-norm of a function \( f \) on \( \Omega \) along the \( z_j \) variable by

\[
H_j^\alpha[f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) := \sup_{\zeta, \zeta' \in D_j, \zeta \neq \zeta'} \left| \frac{f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) - f(z_1, \ldots, z_{j-1}, \zeta', z_{j+1}, \ldots, z_n)}{\zeta' - \zeta} \right|^{\alpha}.
\]

Then one has by triangle inequality that

\[
H^\alpha[f] \leq \sum_{j=1}^n \sup_{z_l \in D_j} H_j^\alpha[f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n). \tag{1}
\]

Suppose in addition that each slice \( D_j \) of \( \Omega \) is bounded with \( C^{k+1, \alpha} \) boundary, \( 1 \leq j \leq n \). We define the solid and boundary Cauchy integral of a function \( f \in C^{k, \alpha}(\Omega) \) along the \( z_j \) variable to be

\[
T_jf(z) := -\frac{1}{2\pi i} \int_{D_j} \frac{f(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z_j} d\zeta_j, \quad z \in \Omega;
\]

\[
S_jf(z) := \frac{1}{2\pi i} \int_{bD_j} \frac{f(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z_j} d\zeta_j, \quad z \in \Omega.
\]

The classical one-dimensional singular integral theory (see [18], or [15, Lemma 4.1]) states that for each \( 1 \leq j \leq n \),

\[
\sup_{z_l \in D_j} H_j^\alpha[D_j^k T_j f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \lesssim \begin{cases} 
\|f\|_{C^k(\Omega)}, & k = 0; \\
\|f\|_{C^{k-1, \alpha}(\Omega)}, & k \geq 1; 
\end{cases} \tag{2}
\]

\[
\sup_{z_l \in D_j} H_j^\alpha[D_j^k S_j f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \lesssim \|f\|_{C^{k, \alpha}(\Omega)}. \tag{3}
\]

Here \( D_j^k \) represents \( k \)-th order derivative operator with respect to the \( z_j \) variable, and the two quantities \( a \) and \( b \) are said to satisfy \( a \lesssim b \) if there exists a constant \( C \) dependent only on \( \Omega, k, \) and \( \alpha \), such that \( a \leq Cb \).

It was further proved in [15, Theorem 1.1] that for each \( 1 \leq j \leq n \), the operator \( T_j \) sends \( C^{k, \alpha}(\Omega) \) into \( C^{k, \alpha}(\Omega) \) with

\[
\|T_j f\|_{C^{k, \alpha}(\Omega)} \lesssim \|f\|_{C^{k, \alpha}(\Omega)} \tag{4}
\]
for any $f \in C^{k,\alpha}(\Omega)$; and for any small $\epsilon$ with $0 < \epsilon < \alpha$, the operator $S_j$ sends $C^{k,\alpha}(\Omega)$ into $C^{k,\alpha-\epsilon}(\Omega)$ with
\[ \|S_j f\|_{C^{k,\alpha-\epsilon}(\Omega)} \lesssim \|f\|_{C^{k,\alpha}(\Omega)} \] (5)
for any $f \in C^{k,\alpha}(\Omega)$. It is worth mentioning that both (4) and (5) are sharp estimates (see Example 4.2-4.3 in [15]), in the sense that the Hölder regularity in neither inequality can be further improved.

Finally, given any $\bar{\partial}$ closed $(0,1)$ form $f = \sum_{j=1}^{n} f_j dz_j \in C^{k,\alpha}(\Omega)$, define as in [14]
\[ T f := T_1 f_1 + T_2 f_2 + \cdots + T_n f_n. \] (6)
It is not hard to verify that $T$ is a solution operator to $\bar{\partial}$ on $\Omega$ (in the sense of distributions if $k = 0$), using the identities $\bar{\partial}_j T_j = S_j + T_j \bar{\partial}_j = id$. As a consequence of (4) and (5), the solution operator $T$ achieves the Hölder regularity with at most an infinitesimal loss from that of the data.

3 The optimal Hölder estimates

Let $\Omega = D_1 \times D_2$, where $D_j \subset \mathbb{C}$ is a bounded domain with $C^{k+1,\alpha}$ boundary, $j = 1, 2$, $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha < 1$. Despite a loss of Hölder regularity of $S_j$ in $C^{k,\alpha}(\Omega)$ as in (5), the following proposition shows that the composition operator $S_j T_l, j \neq l$ preserves exactly the same Hölder regularity. The key observation of the proof is that the loss of Hölder regularity of $S_j$ only occurs along the $z_j$ direction, which is compensated by a gain of Hölder regularity of $T_l$ in this same direction.

**Proposition 3.1.** For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \alpha < 1$, $1 \leq j \neq l \leq 2$, there exists some constant $C$ dependent only on $\Omega, k$ and $\alpha$, such that for any $f \in C^{k,\alpha}(\Omega)$,
\[ \|S_j T_l f\|_{C^{k,\alpha}(\Omega)} \leq C \|f\|_{C^{k,\alpha}(\Omega)}. \]

**Proof.** For simplicity, assume $j = 1$ and $l = 2$. Let $\gamma := (\gamma_1, \gamma_2)$ with $|\gamma| \leq k$. Since $S_1 T_2 f$ is holomorphic with respect to the $z_1$ variable, we only need to estimate $\|D_2^{\gamma_2} \partial_1^{\gamma_1} S_1 T_2 f\|_{C^\alpha(\Omega)}$.

Write $bD_1 = \cup_{m=1}^{N} \Gamma_m$, where each Jordan curve $\Gamma_m$ is connected, positively oriented with respect to $D_1$, and of length $s_m$. Let $\zeta_1|_{s \in [\sum_{j=1}^{m-1} s_j, \sum_{j=1}^{m} s_j]}$ be a $C^{k+1,\alpha}$ parametrization of $\Gamma_m$ with respect to the arclength variable $s$, and $s_0 = \sum_{m=1}^{N} s_m$ is the total length of $bD_1$. In particular, $\zeta_1' = 1/\zeta_1$ on the interval $[\sum_{j=1}^{m-1} s_j, \sum_{j=1}^{m} s_j]$ for each $1 \leq m \leq N$. For any
$$(z_1, z_2) \in \Omega$$, integration by parts on $(\sum_{j=1}^{m-1} s_j, \sum_{j=1}^{m-1} s_j)$ for each $1 \leq m \leq N$ gives

$$\partial_1 S_1 T_2 f(z_1, z_2) = \frac{1}{2\pi i} \int_{bD_1} \partial_{z_1} \left( \frac{1}{\zeta_1(s) - z_1} \right) T_2 f(\zeta_1(s), z_2) \zeta_1'(s) ds$$

$$= -\frac{1}{2\pi i} \sum_{m=1}^{N} \int_{\sum_{j=1}^{m-1} s_j} \frac{\partial_s \left( \frac{1}{\zeta_1(s) - z_1} \right) T_2 f(\zeta_1(s), z_2) ds}{\zeta_1(s) - z_1}$$

$$= \frac{1}{2\pi i} \sum_{m=1}^{N} \int_{\sum_{j=1}^{m-1} s_j} T_2 \left( \partial_1 f(\zeta_1(s), z_2) \zeta_1'(s) + \tilde{\partial}_1 f(\zeta_1(s), z_2) \zeta_1'(s) \right) ds$$

$$= \frac{1}{2\pi i} \sum_{m=1}^{N} \int_{\sum_{j=1}^{m-1} s_j} T_2 \left( \partial_1 f(\zeta_1(s), z_2) + \tilde{\partial}_1 f(\zeta_1(s), z_2) \zeta_1'(s)^2 \right) ds$$

$$= : \frac{1}{2\pi i} \int_{bD_1} \frac{T_2 \tilde{f}(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1 = S_1 T_2 \tilde{f}(z_1, z_2),$$

where the function $\tilde{f} \in C^{k-1,\alpha}(\Omega)$ such that $\tilde{f}(\zeta_1(s), z_2) = \partial_1 f(\zeta_1(s), z_2) + \tilde{\partial}_1 f(\zeta_1(s), z_2) (\zeta_1'(s))^2$ on $[0, s_0] \times D_2$ and $\|\tilde{f}\|_{C^{k-1,\alpha}(\Omega)} \lesssim \|f\|_{C^{k,\alpha}(\Omega)}$ (see [6] Lemma 6.38 on page 137 for the construction of an extension). Repeating the above process, the proposition is reduced to prove for each $\gamma \in \mathbb{Z}^+ \cup \{0\}$, $\gamma \leq k, 0 < \alpha < 1$,

$$\|D_2^\gamma S_1 T_2 f\|_{C^\alpha(\Omega)} \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}$$

for all $f \in C^{\gamma,\alpha}(\Omega)$.

Firstly, choose an $\epsilon$ such that $0 < \epsilon < \alpha$. Applying the estimates (3) and (4) to $S_1 T_2 f$, we get

$$\|D_2^\epsilon S_1 T_2 f\|_{C^0(\Omega)} \leq \|S_1 T_2 f\|_{C^{\gamma,\alpha-\epsilon}(\Omega)} \lesssim \|T_2 f\|_{C^{\gamma,\alpha}(\Omega)} \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}.$$

We next verify that $H\alpha[D_2^\lambda S_1 T_2 f] \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}$. Fixing $z_2 \in D_2$, since $D_2^\lambda S_1 T_2 f = S_1 D_2^\lambda T_2 f$,

$$H\alpha[D_2^\lambda S_1 T_2 f](z_2) = H\alpha[S_1 D_2^\lambda T_2 f](z_2) \lesssim \|D_2^\lambda T_2 f\|_{C^\alpha(\Omega)}.$$

Here the last inequality has used (3) for the estimate of $S_1$ on $D_1$. Consequently, applying (4) to the operator $T_2$ in the last term, we obtain

$$H\alpha[D_2^\lambda S_1 T_2 f](z_2) \lesssim \|T_2 f\|_{C^{\gamma,\alpha}(\Omega)} \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}.$$
We further show for each $z_1 \in D_1$, $H_2^0[D_2^\gamma S_1 T_2 f](z_1) \lesssim \|f\|_{C^\gamma,\alpha}(\Omega)$. If $\gamma \geq 1$, making use of the identity $D_2^\gamma S_1 T_2 f = D_2^\gamma T_2 S_1 f$ by Fubini’s theorem, and the second case of (2) for $T_2$ along the $z_2$ direction, one deduces

$$H_2^0[D_2^\gamma S_1 T_2 f](z_1) = H_2^0[D_2^\gamma T_2 S_1 f](z_1) \lesssim \|S_1 f\|_{C^{\gamma-1,\alpha}(\Omega)}.$$  

Together with (3) for $S_1$ on $\Omega$, we infer

$$H_2^0[D_2^\gamma S_1 T_2 f](z_1) \lesssim \|f\|_{C^\gamma,\alpha}(\Omega).$$  

When $\gamma = 0$, the first case of (2) for $T_2$ and (5) for $S_1$ together give

$$H_2^0[D_2^\gamma S_1 T_2 f](z_1) = H_2^0[T_2 S_1 f](z_1) \lesssim \|S_1 f\|_{C(\Omega)} \lesssim \|f\|_{C^\alpha(\Omega)}.$$  

The proof of the proposition is complete in view of (1).

**Proof of Theorem 1.1 and Corollary 1.2.** We only need to prove the case when $p = 0$. If $q = 2$, for any datum $f = f d\bar{z}_1 \wedge d\bar{z}_2$, it is easy to verify that $T_1 f d\bar{z}_2$ is a solution to $\bar{\partial}$ on $\Omega$. The optimal Hölder estimate follows from that of $T_1$ operator demonstrated in (4). For $q = 1$, the Hölder estimate of the solution given by (6) is a consequence of (4) and Proposition 3.1, from which the theorem and the corollary follow.

Motivated by an $L^\infty$ example of Stein and Kerzman [12], it was shown in [15] that the following $\bar{\partial}$ problem on the bidisc does not gain regularity in Hölder spaces, according to which the Hölder regularity in Theorem 1.1 is optimal.

**Example 3.2.** [12] Let $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the bidisc. For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \alpha < 1$, consider $\bar{\partial} u = f := \bar{\partial}((z_1 - 1)^{k+\alpha} \bar{z}_2)$ on $\Delta^2$, $\frac{1}{2}\pi < \arg(z_1 - 1) < \frac{3}{2}\pi$. Then $f \in C^{k,\alpha}(\Delta^2)$ is $\bar{\partial}$-closed $(0, 1)$. However, there does not exist a solution $u \in C^{k,\alpha'}(\Delta^2)$ to $\bar{\partial} u = f$ for any $\alpha'$ with $1 > \alpha' > \alpha$.

Unfortunately, our method does not obtain optimal Hölder estimates for product domains of dimension larger than 2. For instance, the solution operator of the $\bar{\partial}$ problem on product domains when $n = 3$ is in the form of $T f = T_1 f_1 + T_2 S_1 f_2 + T_3 S_1 S_2 f_3$. Yet not all three operators involved on the right hand side of the formula are bounded in $C^\alpha(\Omega)$ space. In fact, in the following we adopt an example of Tumanov [17] to show that $T_2 S_1$ fails to send $C^\alpha(\Omega)$ into itself, due to the unboundedness of its Hölder semi-norm along the $z_3$ variable. As a result of this, Proposition 3.1 holds only when $n = 2$. 

\[ \]
Example 3.3. For \((e^{i\theta}, \lambda) \in b\Delta \times \Delta\), let

\[
\tilde{h}(e^{i\theta}, \lambda) := \begin{cases} 
|\lambda|^\alpha, & -\pi \leq \theta \leq -\frac{1}{2}\pi; \\
\theta^{2\alpha}, & -\frac{1}{2}\pi \leq \theta \leq 0; \\
\theta^\alpha, & 0 \leq \theta \leq |\lambda|; \\
|\lambda|^\alpha, & |\lambda| \leq \theta \leq \pi,
\end{cases}
\]

and \(h\) be a \(C^\alpha\) extension of \(\tilde{h}\) onto \(\Delta^2\). Define \(f(z_1, z_2, z_3) := h(z_1, z_3)\) for \((z_1, z_2, z_3) \in \Delta^3\). Then \(f \in C^\alpha(\Delta^3)\). However, \(T_2S_1f \notin C^\alpha(\Delta^3)\).

Proof. It is clear to see that \(\tilde{h} \in C^\alpha(b\Delta \times \Delta)\). For each \(z' = (z_1, z_3) \in \Delta^2\), let \(h(z') := \inf_{w \in b\Delta \times \Delta} \{\tilde{h}(w) + M|z' - w|^\alpha\}\), where \(M = \|\tilde{h}\|_{C^\alpha(b\Delta \times \Delta)}\). Then \(h \in C^\alpha(\Delta^2)\) is a \(C^\alpha\) extension of \(\tilde{h}\) onto \(\Delta^2\) and \(f \in C^\alpha(\Delta^3)\).

In [16, Section 3], it was verified that \(H_3^\alpha[S_1h](z_1)\) is unbounded near \(1 \in b\Delta\), and so \(S_1h \notin C^\alpha(\Delta^2)\). On the other hand, making use of the fact that \(T_21(z) = \bar{z}_2, \bar{z} \in \Delta^3\) (see [14, Appendix 6.1b] for instance), we get \(T_2S_1f(z) = T_21(z) \cdot S_1h(z_1, z_3) = \bar{z}_2S_1h(z_1, z_3)\), which does not belong to \(C^\alpha(\Delta^3)\).

References

[1] Bertrams, J.: Randregularität von Lösungen der \(\bar{\partial}\)-Gleichung auf dem Polyzyldner und zweidimensionalen analytischen Polyedern. Bonner Math. Schriften, 176(1986), 1–164.

[2] Chen, L.; McNeal, J.: A solution operator for \(\bar{\partial}\) on the Hartogs triangle and \(L^p\) estimates. Math. Ann. 376 (2020), no. 1-2, 407–430.

[3] Chen, L.; McNeal, J.: Product domains, multi-Cauchy transforms, and the \(\bar{\partial}\) equation. Adv. Math. 360 (2020), 106930, 42 pp.

[4] Diederich, K.; Fischer, B.; Fornæss, J. E.: Hölder estimates on convex domains of finite type. Math. Z. 232 (1999), no. 1, 43–61.

[5] Fassina, M.; Pan, Y.: Supnorm estimates for \(\bar{\partial}\) on product domains in \(\mathbb{C}^n\). Preprint. https://arxiv.org/pdf/1903.10475.pdf.

[6] Gilbarg, D.; Trudinger, N. S.: Elliptic partial differential equations of second order. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp.
[7] Grauert, H.; Lieb, I.: Das Ramirezsche Integral und die Lösung der Gleichung $\bar{\partial}f = \alpha$ im Bereich der beschränkten Formen. (German) Rice Univ. Studies 56 (1970), no. 2, (1971), 29–50.

[8] Henkin, G. M.: Integral representation of functions in strictly pseudoconvex domains and applications to the $\bar{\partial}$-problem. Mat. Sbornik. 124 (1970), no. 2, 300–308.

[9] Henkin, G. M.: A uniform estimate for the solution of the $\bar{\partial}$-problem in a Weil region. (Russian) Uspehi Mat. Nauk 26 (1971), no. 3(159), 211–212.

[10] Henkin, G. M.; Romanov, A. V.: Exact Hölder estimates of the solutions of the $\bar{\delta}$-equation. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1171–1183.

[11] Jin, M.; Yuan, Y.: On the canonical solution of $\bar{\partial}$ on polydisks. C. R. Math. Acad. Sci. Paris 358 (2020), no. 5, 523–528.

[12] Kerzman, N.: Hölder and $L^p$ estimates for solutions of $\bar{\partial}u = f$ in strongly pseudo-convex domains. Comm. Pure Appl. Math. 24 (1971) 301–379.

[13] Lieb, I; Range, R. M.: Lösungsoperatoren für den Cauchy-Riemann-Komplex mit $C^k$-Abschätzungen. (German) Math. Ann. 253 (1980), no. 2, 145–164.

[14] Nijenhuis, A.; Woolf, W.: Some integration problems in almost-complex and complex manifolds. Ann. of Math. (2) 77 (1963), 424–489.

[15] Pan, Y.; Zhang, Y.: Hölder regularity of $\bar{\partial}$ problem on product domains. Internat. J. Math., to appear.

[16] Pan, Y.; Zhang, Y.: Cauchy singular integral operator with parameters in Log-Hölder spaces. J. Anal. Math., to appear.

[17] Tumanov, A.: On the propagation of extendibility of CR functions. Complex analysis and geometry (Trento, 1993), 479–498, Lecture Notes in Pure and Appl. Math., 173, Dekker, New York, 1996.

[18] Vekua, I. N.: Generalized analytic functions, vol. 29, Pergamon Press Oxford, 1962.

Yuan Zhang, zhangyu@pfw.edu, Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA