The complex side of the TS/ST correspondence

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Abstract

The TS/ST correspondence relates the spectral theory of certain quantum mechanical operators, to topological strings on toric Calabi–Yau threefolds. So far the correspondence has been formulated for real values of Planck’s constant. In this paper we start to explore the validity of the correspondence when \(\hbar\) takes complex values. We give evidence that, for threefolds associated to supersymmetric gauge theories, one can extend the correspondence and obtain exact quantization conditions for the operators. We also explore the correspondence for operators involving periodic potentials. In particular, we study a deformed version of the Mathieu equation, and we solve for its band structure in terms of the quantum mirror map of the underlying threefold.

Keywords: topological string, supersymmetric gauge theory, periodic potentials, spectral theory

(Some figures may appear in colour only in the online journal)

1. Introduction

The topological string/spectral theory (TS/ST) correspondence is a conjectural relationship between topological string theory on toric Calabi–Yau (CY) manifolds and the spectral theory of certain trace class operators on the real line. This correspondence was built on previous work in [1–13] formulated in detail in [14, 15] (see [16] for a review), and further developed in [17–43]. The operator appearing in the correspondence is obtained by quantization of the mirror curve to the toric CY, as originally envisaged in [1]. The TS/ST correspondence provides, among other things, explicit and exact quantization conditions for the spectrum of the
operator, in terms of BPS invariants of the CY manifold, as well as a non-perturbative definition of the topological string partition function.

So far, studies of the TS/ST correspondence have focused on the ‘physical’ case in which \( h \) is real. In addition, the quantization of the mirror curve is made along a real slice. As long as appropriate positivity constraints are imposed on some of the parameters, the resulting operator is self-adjoint and trace class, and it has a discrete, real spectrum. From the point of view of operator theory, this is the simplest situation. However, it is natural to explore other possibilities, in which the quantities specifying the model are allowed to take more general values, or the quantization of the curve is made with a different prescription. For example, in [36] the positivity constraints on some of the parameters were relaxed, and this led to resonant-like states in the spectrum.

In this paper we consider two additional cases. We first study the case in which \( h \) is allowed to take complex values. When this happens, the operators obtained by quantization of the mirror curve are no longer self-adjoint, but we can still make sense of their spectral problem (in particular, their spectrum can be computed numerically). In this paper we give evidence that the exact quantization condition of [14, 21] still captures the exact spectrum of the operator when the underlying toric CY engineers a five-dimensional gauge theory. This requires a partial resummation of the BPS expansion which is natural from the gauge theory point of view (such a resummation was first considered in topological string theory in [44, 45] to reproduce Nekrasov’s results [46] from the topological vertex [47]).

In addition, we consider a mixed quantization of the mirror curve, in which one of the coordinates becomes purely imaginary. For simplicity, we focus on the operator associated to local \( F_0 \), which provides a quantum, or deformed version, of the Mathieu equation. Since this operator involves a periodic potential, there is a band structure for the eigenvalue problem which can be analyzed with standard tools. Our main result is an exact expression for the band energies in terms of a resummation of the quantum mirror map. This generalizes well-known results for the Mathieu equation [2, 48–50].

This paper is organized as follows. In section 2 we review the basics of the TS/ST correspondence. In section 3 we consider the correspondence for complex values of Planck’s constant, and we study in detail the spectral problem associated to local \( F_0 \). In section 4 we analyze the band spectrum of the quantum Mathieu operator. Finally, in section 5 we conclude and list some open problems. The appendix gives some technical details on the grand potential of a local CY threefold.

2. Topological strings and spectral problems

In this paper we are interested in spectral problems arising in the quantization of mirror curves to toric CY manifolds. Let us start by recalling some facts concerning this quantization. We consider a toric CY \( X \) whose complexified Kähler moduli space is parametrized by \( g_{\Sigma} \) ‘true’ moduli

\[
\kappa_i, \quad i = 1, \cdots, g_{\Sigma};
\]

and \( r_{\Sigma} \) mass parameters

\[
\xi_i, \quad i = 1, \cdots, r_{\Sigma}.
\]

We will denote by \( n_{\Sigma} = g_{\Sigma} + r_{\Sigma} \) the total number of Kähler parameters. A more precise definition of these two types of moduli can be found in [51, 52]. The mirror curve to \( X \) has genus \( g_{\Sigma} \), and there are \( g_{\Sigma} \) different ‘canonical’ forms to represent it, namely
\[ O_i(x, p) + \kappa_i = 0, \quad i = 1, \cdots, g_\Sigma, \]  
(2.3)

where \( O_i(x, p) \) is a sum of monomials of the form \( e^{ax+bp} \). The coefficients of such monomials depend on the mass parameters (2.2) and on the moduli \( \kappa_j, j \neq i \). We promote the variables \( x, p \) to operators \( x, p \) such that

\[ [x, p] = i\hbar. \]  
(2.4)

By using Weyl quantization

\[ e^{ax+bp} \rightarrow e^{ax+bp} \]  
(2.5)

we can promote \( O_i(x, p) \) to an operator \( O_i \), \( i = 1, \cdots, g_\Sigma \). For example, the mirror curve to the canonical bundle over \( \mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) has genus one and one mass parameter \( \xi \), and we have

\[ O_1(x, p) + \kappa = e^x + \xi e^{-x} + e^p + e^{-p} + \kappa. \]  
(2.6)

The corresponding operator is

\[ O_1 = e^x + \xi e^{-x} + e^p + e^{-p}. \]  
(2.7)

It was conjectured in [14], and proved in [17, 24] for many examples, that the inverse operators

\[ \rho_i = O_i^{-1} \]  
(2.8)

are self-adjoint, positive and of trace class, provided some positivity constraints are imposed on the mass parameters. Therefore, they have a discrete spectrum which is encoded in the generalized spectral determinant [15]

\[ \Xi_X(\kappa, \hbar, \xi) = \det \left( 1 + \sum_{j=1}^{g_\Sigma} A_{ij} \kappa_j \right) \]  
(2.9)

where the operators \( A_{ij} \) are defined by

\[ O_i + \kappa_i = O_j^{(0)} \left( 1 + \sum_{j=1}^{g_\Sigma} A_{ij} \kappa_j \right), \]  
(2.10)

and \( O_j^{(0)} \) does not depend on the true moduli \( \kappa \) (see [15] for more details). The conjecture of [14, 15] states that

\[ \Xi_X(\kappa, \hbar, \xi) = \sum_{n \in \mathbb{Z}/g_\Sigma} e^{J_X(\mu+2\pi i n \hbar, \xi)}, \quad \kappa_i = e^{\mu_i}, \quad \hbar \in \mathbb{R}_+ \]  
(2.11)

where \( J_X \) is the topological string grand potential [10] and it is completely determined by the (refined) BPS invariants of \( X \). The precise definition of \( J_X \) is given in the appendix. This construction, and in particular (2.11), has led to a new and exact relation between topological strings and the spectral theory of the quantum operators arising in the quantization of mirror curves. We will refer to it as TS/ST correspondence, and it has passed a large number of successful tests.

In this paper we focus only on one particular consequence of this correspondence which concerns the quantization condition for the spectrum of the operators \( \rho_i \). As explained in [14, 15] the eigenvalues of these operators are determined by the vanishing locus of the spectral determinant (2.11). In particular the WKB part of the spectrum is encoded in the NS limit of the refined topological string, as already anticipated in [2–4, 53]. However there are
additional non-perturbative corrections which are encoded in the unrefined (or GV) limit of topological string theory [11, 14, 15]. When the mirror curve has genus one, by using the blowup equations [54], it is possible to express the vanishing locus of $\Xi_X(\kappa, \hbar, \xi)$ in a way which displays S-duality [21, 35, 37]. More precisely, the blowup equations allow one to write the vanishing condition

$$\Xi_X(\kappa, \hbar, \xi) = 0$$

(2.12) as

$$\sum_{i=1}^{g_\Sigma} c_i \left( \frac{\partial}{\partial \tilde{t}_i} F_{NS}(\tilde{t}(h), h) + \frac{\hbar}{2\pi} \frac{\partial}{\partial \tilde{t}_i} F_{NS, inst} \left( \frac{2\pi}{\hbar} \tilde{t}(h), \frac{4\pi^2}{\hbar} \right) \right) = n + \frac{1}{2}, \quad n = 0, 1, \cdots.$$  

(2.13)

In this equation, the coefficients $c_i$ are given by $c_i = C_{ij}$, where the matrix $C_{ij}$ is defined in (A.2), $F_{NS}, F_{NS, inst}$ denote the twisted NS free energy (A.14), (A.13), and $\tilde{t}_i(h)$ is the twisted quantum mirror map (A.15). Equation (2.12), or equivalently (2.13), should be viewed as a quantization condition for the complex modulus $\kappa$, and computing the spectrum of $\rho_1$. Let us denote by

$$\{e^{-E_n}\}_{n \geq 0}$$

(2.14)

the eigenvalues of $\rho_1$. Then,

$$\{-e^{E_n}\}_{n \geq 0} = \{\kappa^{(n)} : \Xi_X(\kappa^{(n)}, h, \xi) = 0\}.$$  

(2.15)

The higher genus situation is more subtle and there are in principle two different spectral problems associated to a given CY. The first one was studied in [15], where one considers the $g_\Sigma$ non-commuting operators $O_i$, $i = 1, \cdots, g_\Sigma$ acting on $L^2(\mathbb{R})$. The spectrum of these operators is encoded in the vanishing locus of the generalized spectral determinant (2.9), which defines a codimension one submanifold in the moduli space. The second one is the spectral problem associated to the quantum cluster integrable system of Goncharov and Kenyon [55]. In this case, one has $g_\Sigma$ commuting Hamiltonians acting on $L^2(\mathbb{R}^{g_\Sigma})$, whose spectrum is determined by the following $g_\Sigma$ exact quantization conditions [25, 26]

$$\sum_{i=1}^{g_\Sigma} C_{ij} \left( \frac{\partial}{\partial \tilde{t}_j} F_{NS}(\tilde{t}(h), h) + \frac{\hbar}{2\pi} \frac{\partial}{\partial \tilde{t}_j} F_{NS, inst} \left( \frac{2\pi}{\hbar} \tilde{t}(h), \frac{4\pi^2}{\hbar} \right) \right) = n_j + \frac{1}{2}, \quad j = 1, \cdots, g_\Sigma.$$  

(2.16)

where $n_j = 0, 1, \cdots$ are non-negative integers. It was pointed out in [25, 26], and verified in [35, 37], that the spectral problem of [15] is more general than the one of [55], in the sense that (2.16) defines a subset of $g_\Sigma$ points lying on the vanishing locus of (2.9). This point of view was confirmed recently by [43] in the study of eigenfunctions.

3. Complexifying Planck’s constant

In the current formulation of the TS/ST correspondence it is always assumed that

$$\hbar \in \mathbb{R}^+.$$  

(3.1)

One of the reasons for this restriction is that, as first pointed out in [10], some of the expansions appearing in the construction of [14, 15] diverge when $\hbar$ is complex. In this section we will see that, by using insights from gauge theory, we can overcome this difficulty and easily extend some aspects of the TS/ST correspondence to complex values of $\hbar$, at least when the underlying CY can be used to engineer gauge theories [56]. We will focus, for concreteness
and simplicity, on the spectral problem associated to local $\mathbb{F}_0$, which corresponds to the pure $\mathcal{N} = 2$, $U(2)$ gauge theory in five dimensions.

3.1. Convergent expansions

Let us first consider the topological string partition function, as computed for example from the topological vertex [47]. This quantity has the following structure:

$$ Z^{\text{GV}}(t, g_s) = \exp[F^{\text{GV}}(t, g_s)] = \sum_{\mathbf{m}} d_{\mathbf{m}}(q)e^{-\mathbf{m} \cdot \mathbf{t}}, \quad q = e^{i\epsilon_1}, \quad t = e^{-i\epsilon_2}, $$ (3.2)

where $F^{\text{GV}}(t, g_s)$ is defined in (A.6), and $\mathbf{m}$ is an $n_\Sigma$-uple of non-negative integers (by convention, $d_{(0, \cdots, 0)} = 1$). The coefficients $d_{\mathbf{m}}$ have poles at $g_s \in \pi\mathbb{Q}$, and as consequence the partition function is ill-defined on the real axis. When $g_s \in \mathbb{C}\setminus\mathbb{R}$ the coefficients $d_{\mathbf{m}}$ do not have poles, however the formal power series (3.2) diverges [10]. A similar analysis can be made for the NS partition function (A.12) as shown in [30]. Nevertheless when the CY $X$ can be used to engineer a five dimensional gauge theory, it is possible to partially resum (3.2), as in [45, 57, 58] by using the insights coming from gauge theory [46].

Let us consider for simplicity five dimensional $U(N)$ gauge theory on $S^1 \times \mathbb{R}^4$, and let $a_I$ be the parameters for the Coulomb branch, $I = 1, \cdots, N$. Let $Y_I$ be the Young tableau describing the contribution of the instanton in the $I$th factor of the gauge group, and let us denote by $|Y_I|$ the total number of boxes in the tableau. We group the different Young tableaux in a vector of partitions

$$ Y = (Y_1, \cdots, Y_N). $$ (3.3)

Then the partition function of the five dimensional theory can be written as a sum over Young tableaux,

$$ Z = \sum_Y \Lambda^{|Y|} z_Y, $$ (3.4)

where $z_Y$ is an appropriate function which we will make explicit below for local $\mathbb{F}_m$ (i.e. for $N = 2$), and

$$ |Y| = \sum_{I=1}^N |Y_I|. $$ (3.5)

Let us first list the ingredients needed in order to write down the refined topological string partition function for local $\mathbb{F}_m$. The first ingredient is

$$ Z_\mu(t, q) = \prod_{(ij) \in \mu} \left(1 - q^{\mu_j - i + 1} t^{\mu_i - j}\right)^{-1}, $$ (3.6)

where $\mu$ is a partition or Young tableau, and the parameters $q, t$ encode the $\epsilon_1, \epsilon_2$ deformations:

$$ q = e^{i\epsilon_1}, \quad t = e^{-i\epsilon_2}. $$ (3.7)

The second ingredient depends on two partitions $\mu, \nu$, and an extra parameter $Q$. It is given by

$$ R_{\mu
u}(Q) = \prod_{i,j=1}^{\infty} \frac{1 - Q^{i+1/2} q^{j+1/2}}{1 - Q^{i+1/2} \mu q^{j+1/2} \nu}. $$ (3.8)
It is easy to see that the product gets truncated and only a finite number of factors get involved. We also introduced, for a given partition $\mu$, the quantities

$$|\mu| = \sum_i \mu_i,$$

$$\|\mu\|^2 = \sum_i \mu_i^2,$$

$$\kappa_\mu = \sum_i \mu_i (\mu_i - 2i + 1),$$

and the refined framing factor

$$f_\mu = (-1)^{|\mu|} \left( \frac{q}{t} \right)^{\|\mu\|^2/2} q^{-\kappa_\mu/2},$$

where $\mu'$ denotes the transposed partition in which one exchange rows and columns of the corresponding Young diagram. The building block of the partition function is

$$Z_{\mu_1, \mu_2} = q^{\sum_{\mu'} \|\mu'||^2/2} t^{\sum_{\mu'} \|\mu'||^2/2} \prod_{i=1}^2 Z_\mu (t, q) Z_{\mu'} (q, t) R_{\mu'} R_\mu \left( \sqrt{\frac{t}{q}} Q_2 \right) R_{\mu'} \left( \sqrt[2]{q} T Q_2 \right).$$

Then, we have

$$z_{(\mu_1, \mu_2)} (Q_2) = \sum_{\mu'} f_{\mu'}^{-m-1} f_{\mu'}^{-m+1} Z_{\mu', \mu} Q_{\mu'} Q_2,$$

so that the partition function of the local $\mathbb{F}_m$ geometry is given by

$$Z_{\mathbb{F}_m} (Q_1, Q_2, q, t) = \sum_{Y_1, Y_2} z_{(Y_1, Y_2), (-Q_1) \frac{1}{\sum_{|Y|}}}.$$

For instance, for local $\mathbb{F}_0$, i.e. $m = 0$, in the NS limit we have

$$- \lim_{\epsilon \to 0} \text{ir}^2 \log \left[ Z_{\mathbb{F}_0} (Q_1, Q_2, q, e^{-\text{ir}\epsilon}) \right] = \sum_{n \geq 1} \frac{q^n + 1}{n^2 (q^n - 1)} Q_2^n + \sum_{m \geq 1} b_m (Q_2, q) Q_2^m$$

where the first coefficient reads

$$b_1 (Q_2, q) = \frac{q (q + 1)}{(1 - q) (q - Q_2) (qQ_2 - 1)}.$$
Another important ingredient in the TS/ST duality is the quantum mirror map. This was introduced in [4] and has the following form

\[-t_i(\hbar) = \log z_i + \Pi_i(z, \hbar), \tag{3.18}\]

where $z_i$ are the Batyrev coordinates defined in (A.2) and $\Pi_i$ is a power series in $z_j$. When $\hbar$ is real this series has a finite radius of convergence as discussed for instance in [10]. When $\hbar$ is complex, this is no longer the case. Nevertheless, as for the free energy, we can overcome this problem by using instanton calculus in toric CY threefolds with a gauge theory realization. More precisely, the quantum mirror map can be computed by using Wilson loop vevs in the gauge theory, as pointed out in [31]. We follow [60, 61] for the Wilson loop vev computation. Let us define

\[W = \sum_{I=1}^{N} e^{a_I}, \]
\[V_Y = \sum_{I=1}^{N} e^{a_I} \sum_{(k,l) \in Y} e^{i(l-1)\epsilon_1 + i(k-1)\epsilon_2}. \tag{3.19}\]

The indices $(k, l)$ label the boxes of the Young tableau. We have changed slightly the formula in [60, 61] to fit our own conventions. We also need the equivariant Chern character,

\[\text{Ch}_Y(\mathcal{E}) = W - (1 - e^{i\epsilon_1})(1 - e^{i\epsilon_2})V_Y. \tag{3.20}\]

The vev of a Wilson loop in the fundamental representation is then given by

\[W = \frac{1}{Z} \sum_{\mathcal{C}} \Lambda^{[\Pi]} \text{Ch}_Y(\mathcal{E})z_Y. \tag{3.21}\]

It turns out that this can be used to compute the quantum mirror map, at least in the simple case of $U(2)$ gauge theories, where there is only one non-trivial mirror map (for theories of higher rank, one probably has to use Wilson loops in higher representations).

Let us look at the explicit form of (3.21) for local $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, the ingredients for the Wilson loop give, when expressed in terms of the natural exponentiated Kähler parameters,

\[W = Q_1^{1/2} + Q_2^{-1/2}, \]
\[\text{Ch}_Y(\mathcal{E}) = Q_1^{1/2} + Q_2^{-1/2} - (1 - q)(1 - r^{-1}) \left( Q_1^{1/2} \sum_{(k,l) \in Y_1} q^{-1}r^{-k+1} + Q_2^{-1/2} \sum_{(k,l) \in Y_2} q^{-1}r^{-k+1} \right). \tag{3.22}\]

We find, in the NS limit, a result of the form

\[Q_2^{1/2}W(Q_1, Q_2, q) = 1 + Q_2 + \frac{(1 + Q_2)Q_1}{(q^{1/2}Q_2 - q^{-1/2})(q^{-1/2}Q_2 - q^{1/2})} + \cdots \tag{3.23}\]

which can be re-expanded in powers of $Q_2, Q_1$:

\[Q_2^{1/2}W(Q_1, Q_2, q) = 1 + Q_2 + Q_1 + (q^{-1} + 1 + q)Q_2Q_1 + (q^2 + q + 1 + q^{-1} + q^{-2})Q_2Q_1(Q_2 + Q_1) + \cdots. \tag{3.24}\]

We note that
\[ Q_2 = \xi Q_1, \quad Q_1 = e^{-t_1(h)}, \quad q = e^{ih}, \]  
\begin{equation}
\tag{3.25}
\end{equation}

where \( \xi \) is a mass parameter and \( t_1(h) \) is the quantum mirror map (A.4). We can write

\[ W^2(Q_1, \xi Q_1, q) = Q_1^{-1} \xi^{-1} \left( 1 + (2\xi + 2)Q_1 + Q_1^2 \left( \xi^2 + 4\xi q + \frac{2\xi}{q} + 1 \right) + \cdots \right). \]  
\begin{equation}
\tag{3.26}
\end{equation}

It follows that \( W \) can be identified with

\[ W^2(Q_1, \xi Q_1, q) = \frac{1}{\xi q}, \]  
\begin{equation}
\tag{3.27}
\end{equation}

where \( z = e^{-2\mu} \) is the bare modulus of the CY, which can be expressed in terms of \( Q_1 \) and \( \xi \) by using the quantum mirror map. This can be checked against explicit calculations.

Let us note that \( W(Q_1, Q_2, q) \) has poles when \( Q_2 = q^{2m} \) for \( m \in \mathbb{N} \). However, these poles do not occur at the values of the Kähler parameters selected by the quantization condition. More precisely, if \( t_1(h, \xi) \) satisfies (2.16) for some fixed value of \( h, \xi \), then

\[ Q_2 = \xi e^{-t_1(h)} \neq e^{+mh}. \]  
\begin{equation}
\tag{3.28}
\end{equation}

As we will see in the next section, in the case of the quantum Mathieu operator, this is no longer the case and these poles occur at the edges of the energy bands.

### 3.2. Exact quantization condition for complex \( h \)

Given the observations above, it is natural to expect that the quantization condition of [14], in the form of [21], still holds when \( h \) takes complex values, provided we consider a partial resummed version of the free energies and of the quantum mirror map. We will test this expectation in the case of local \( \mathbb{P}^1 \times \mathbb{P}^1 \) where the quantization condition (2.13) can be written as

\[ -2 \left\{ \partial_{t_1} F^{\text{NS}}(t(h), h) + \frac{h}{2\pi} \partial_{t_1} F^{\text{NS, inst}}(\frac{2\pi}{h} t(h), \frac{4\pi^2}{h}) \right\} = 2\pi \left( n + \frac{1}{2} \right), \quad n = 0, 1, \ldots. \]  
\begin{equation}
\tag{3.29}
\end{equation}

We need to consider the resummed form of \( F^{\text{NS}} \) given by the NS limit of (3.13). The B–field can be set to zero in this geometry [10, 14]. The very first orders of the expansion of (3.29) in \( Q_1, Q_1^0 = Q_1^{2\pi/h} \) are given by

\[ \frac{t_1(h)^2}{h} - \frac{h}{6} - 2\pi^2 \frac{\log \xi}{3h} + 2 \sum_{W \geq 1} \frac{1}{W} \left( -\frac{Q_2^2}{2h} \cot \left( \frac{2\pi^2}{h} \right) + Q_2^0 \cot \left( \frac{\pi h}{2} \right) \right) \]

\[ \frac{2Q_1^2 ((Q_1^{2\pi/h})^2 - 1) \cot \left( \frac{2\pi^2}{h} \right)}{(Q_1^{2\pi/h})^2 - 2Q_1^2 \cos \left( \frac{2\pi^2}{h} \right) + 1} - \frac{4iQ_1 (Q_2^2 - 1) \cos \left( \frac{2\pi}{h} \right) (-1 + e^{ih})(-Q_2 + e^{ih})^2 (-1 + Q_2 e^{ih})}{(-1 + e^{ih})(-Q_2 + e^{ih})^2 (-1 + Q_2 e^{ih})} + O(Q_1^3, (Q_1^{2\pi/h})^3) \]

\[ = 2\pi \left( n + \frac{1}{2} \right), \quad n = 0, 1, \ldots \]  
\begin{equation}
\tag{3.30}
\end{equation}

where

\[ Q_1^0 = Q_1^{2\pi/h}, \quad Q_2 = \xi Q_1, \quad Q_1 = e^{-t_1(h)} \]  
\begin{equation}
\tag{3.31}
\end{equation}

and \( t_1(h) \) is the quantum mirror map (A.4). Therefore, for fixed \( h \) and \( \xi \), the quantization condition (3.29) is an equation for \( t_1(h) \). Let us denote by

\[ t_1^{(n)}(h, \xi) \]  
\begin{equation}
\tag{3.32}
\end{equation}
the solution to (3.29) for a fixed value of $\hbar, \xi, n$. It follows that the spectrum of (2.7), denoted by $e^{E_n}$, is given by

$$\xi^{-1}e^{2E_n} = \left( W\left( e^{-i\xi(h,\xi)} ; e^{-i\xi(h,\xi)} , e^{ih}\right) \right)^2.$$  

(3.33)

In the following we will assume by simplicity that the mass parameter in (2.7) is set to one $\xi = 1$.

In tables 1 and 2 we list some results for the spectrum of (2.7) with $\xi = 1$, which are obtained by solving (3.29), for different values of $\hbar$. Note that the eigenvalues are now complex, and we order them by their absolute value, i.e. $|E_0| < |E_1| < \cdots$. These eigenvalues can be also computed numerically. To do this, we perform a standard diagonalization of the operator in the harmonic oscillator basis, analytically continued to complex values of $\hbar$. The numerical results are in perfect agreement with the results obtained from the exact quantization condition (3.29).

When $\hbar \in \mathbb{R}_+$, the non-perturbative corrections to the all-orders WKB quantization condition of [2] are crucial to eliminate poles and to even write down a sensible answer, as first pointed out in [11]. However, when $\hbar$ is complex, there are no poles to start with, and the all-orders WKB quantization condition (given by the first term in the lhs of (3.29)) makes sense. However, the non-perturbative corrections are still required to obtain the right spectrum. In table 3 we show the ground state energy obtained by neglecting non-perturbative effects in (3.29), i.e. by neglecting the term

$$\frac{\partial}{\partial t_f} e^{\text{NS,inst}} \left( \frac{2\pi}{\hbar} t(h), \frac{4\pi^2}{\hbar} \right),$$  

(3.34)
and computed for a particular complex value of \( h \). As we increase the number of terms in the \( Q_1 \) series, the putative ground state energy converges, albeit to a wrong value. In view of this, we conclude the following:

1. If \( h \) is real, the non-perturbative terms (3.34) cancel the poles appearing in the all-orders WKB quantization condition, and lead to a convergent expansion for the lhs of (2.16).
2. If \( h \) is not real, the lhs of (2.16), after the resummation implemented by the gauge theory instanton calculation, has good convergence properties, even without the non-perturbative terms (3.34). Nevertheless, if we do not include the non-perturbative corrections, the quantization condition does not lead to the correct energy levels of the operator.

In [62], Kashaev and Sergeev have studied the spectral problem of the operator (2.7) for \( \xi = 1 \) and complex values of \( h \) of the form

\[
h = 2\pi e^{2i\theta}, \quad \theta \in [0, \pi/2).
\]

In particular, they have obtained exact quantization conditions for the spectrum, which they have studied numerically for \( h = 2\pi i \). We have checked that the results obtained with our proposal agree with their results. It turns out that, when \( h = \pm 2\pi i \) and the quantum number \( n \) is of the form

\[
n = 2m(m + 1), \quad m \in \mathbb{N}
\]

the quantization condition (3.29) becomes simply

\[
\begin{align*}
l_0^{(n)}(2\pi i, 1) &= \pm (1 + i)(1 + 2m)\pi, \\
l_0^{(n)}(-2\pi i, 1) &= \pm (1 - i)(1 + 2m)\pi.
\end{align*}
\]

In particular, for these values of the Kähler parameter (3.37), the series in \( Q_1 \) and the one in \( Q^P_1 \) inside (3.29) mutually cancel. When \( m = 0 \), by using (3.37) and (3.33), we have

\[
E_0(\pm 2\pi i) = 1.5248292052302074168 \cdots \pm 1.5707963267948966192i \cdots
\]

which agrees with the numerical computation and with the result of [62].

### 4. The quantum Mathieu operator

So far, topological string theory has been used to solve spectral problems that arise by taking a ‘real’ slice of the mirror curve and applying Weyl quantization to the mirror curve, such as for

| \( n \) | \( E_0^{\text{NS}} \) |
|---|---|
| 3 | 2.123 315 452 1796 + 0.238 128 282 7804 i |
| 5 | 2.123 315 763 7682 + 0.238 128 385 1298 i |
| 6 | 2.123 315 763 6916 + 0.238 128 385 2383 i |
| Num | 2.123 499 196 8312 + 0.238 200 118 3808 i |
instance (2.7). However, as already pointed out in [2] in the case of spectral problems arising from four dimensional gauge theories, one can obtain other, related spectral problems by considering ‘imaginary’ slices of the variables $x, p$. This involves the following transformation of the operators $x, p$ introduced in (2.4):

$$x \rightarrow ix, \quad p \rightarrow ip.$$  

(4.1)

Such transformations change completely the nature of the spectral problem. For instance, the rotation of $x$ leads to the analogue of a periodic potential. In this section we study the operator obtained from (2.7) after such a rotation, i.e. we study the spectrum of

$$O_qM = R(e^{ix} + e^{-ix}) + e^p + e^{-p}, \quad [x, p] = i\hbar,$$  

(4.2)

where $R = \xi^{1/2}$. This is a quantum deformation of the standard Mathieu operator

$$O_M = p^2 + 2R \cos(x),$$  

(4.3)

and we will call it the quantum Mathieu operator. The eigenvalue equation for this operator

$$O_qM\psi(x) = \epsilon\psi(x)$$  

(4.4)

can be also written as

$$\psi(x + i\hbar) + \psi(x - i\hbar) + V(x)\psi(x) = \epsilon\psi(x),$$  

(4.5)

where

$$V(x) = 2R \cos(x)$$  

(4.6)

is the potential appearing in the standard Mathieu equation

$$-\hbar^2 \frac{d^2\psi(x)}{dx^2} + 2R \cos(x)\psi(x) = u\psi(x).$$  

(4.7)

Since (4.5) involves a periodic potential $V(x)$, with period $a = 2\pi$, we will have a band structure as for the standard Mathieu equation (4.7). In this section we first solve the spectral problem for (4.2) by using standard techniques for periodic potentials, and then we will express the result in terms of topological string quantities. As we will see, the spectrum is determined exactly by the quantum mirror map. In particular, no non-perturbative corrections seem to be needed in this case. We note that the Mathieu and modified Mathieu equation, when solved in terms of gauge theory, involve the quantum A and B periods of the Seiberg–Witten curve, respectively [2]. The solution of the quantum Mathieu case is similar, since the quantum mirror map can be regarded as the quantum A-period of the mirror curve.

4.1. The spectrum from the central equation

A standard tool to solve a periodic potential (see for example [64], chapter 8) consists of writing a Fourier expansion for the Bloch wavefunction and solve for its coefficients. This leads to the so-called central equation, which can be used to calculate the spectrum numerically and, sometimes, analytically. Let us then write down the Bloch wavefunction

$$\psi(x) = \sum_d c_d e^{ik_d x},$$  

(4.8)

\[\dagger\] We have shifted $x$ by $\frac{1}{2} \log \xi$ w.r.t. (2.7).

\[\ddagger\] The quantum Mathieu operator has been also studied in [63]. However, the analysis in [63] seems to be restricted to a particular value of the quasimomentum, and is based on an asymptotic WKB expansion.
where $q$ is of the form

$$q = k + \frac{2\pi}{a} \ell, \quad \ell \in \mathbb{Z}, \quad k \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right].$$

(4.9)

$k$ is the quasimomentum, and $a$ is the period of the potential. We will denote the corresponding coefficient by

$$c_q = c^{(k)}_\ell.$$

(4.10)

For a general periodic potential, one uses the Fourier decomposition

$$V(x) = \sum_{r \in \mathbb{Z}} V_r e^{2\pi i r x / a}.$$  

(4.11)

In our case, $a = 2\pi$ and $V_r = 0$ unless $r = \pm 1$. We have in addition a non-standard kinetic term $2 \cosh(p)$. By plugging (4.8) inside (4.5), we find a set of algebraic equations for the coefficients $c_q$:

$$2 \cosh(\hbar q) c_q + R (c_{q-1} + c_{q+1}) = \epsilon(k, \hbar, R)c_q.$$  

(4.12)

This is the analogue of the central equation for the quantum Mathieu operator. By using the explicit expression for $q$, we can also write it as

$$2 \cosh(h(k + \ell)) c^{(k)}_{\ell} + R \left(c_{\ell-1}^{(k)} + c_{\ell+1}^{(k)} \right) = \epsilon(k, \hbar, R)c^{(k)}_{\ell}.$$  

(4.13)

For a fixed $k$, this gives an infinite dimensional matrix equation for $c^{(k)}_{\ell}$. In practice, one truncates the equation for a given $\ell_{\text{max}} \geq |\ell|$, to obtain $2\ell_{\text{max}} + 1$ values of $\epsilon(k, \hbar, R)$. These are the energy bands, as a function of the quasi-momentum in the Brillouin zone

$$k \in [-1/2, 1/2].$$

(4.14)

For values of $h, R \sim 1$, truncating this matrix equation to finite size gives extremely efficient results for the energy bands. Even $\ell_{\text{max}} = 5$ gives more than 100 correct digits for the ground state energy. In figure 1 we show the first three energy bands for $h = \pi/2, R = 1$.

The central equation can also be used to solve the band spectrum analytically, in terms of a power series. In the case of the Mathieu equation, this is nicely explained in section 2 of [65]. This analysis can be easily extended to the quantum Mathieu operator. There are two different cases, depending on whether the quasi-momentum is in the interior or at the edges of the bands. If $k$ is an interior value, we can write the following ansatz for the band energy,

$$\epsilon(k, \hbar, R) = 2 \cosh(k h) + \sum_{\ell \geq 1} \epsilon_{\ell}(k, h) R^{2\ell},$$

(4.15)

where the first term is the dispersion relation for the free theory. We will now assume the normalization condition that $c_k = 1$ for a fixed $k$, and that

$$c_{k \pm n} = R^n \sum_{\ell \geq 0} \epsilon^{(\ell)}_{k \pm n}(k) R^{2\ell}.$$  

(4.16)

The central equation for $k \pm 1$ and at leading order in $R$ gives

$$(2 \cosh(k \pm 1) - 2 \cosh(k h)) R c^{(0)}_{k \pm 1}(k) + R + O(R^2) = 1,$$  

(4.17)
so we find

$$c^{(0)}_{\pm 1}(k) = -\frac{1}{2 \cosh(k \pm 1) h - 2 \cosh(k h)}. \quad (4.18)$$

In addition, by using the central equation for $k$, we find at leading order

$$\epsilon_1(k, h) = c^{(0)}_1(k) + c^{(0)}_{-1}(k) = \frac{(Q_1^{1/2} + Q_2^{-1/2}) Q_1 R^{-2}}{(q^{1/2} Q_2 - q^{-1/2})(q^{-1/2}Q_2 - q^{1/2})}, \quad (4.19)$$

with

$$Q_1 = R^2 e^{2k h}, \quad Q_2 = e^{2k h}, \quad q = e^{-h}. \quad (4.20)$$

In particular, (4.19) reproduces precisely the first term in the expansion of $W(Q_1, Q_2, q)$ as computed by instanton calculus in section 3. It can be easily checked that this is also the case for the next few higher order terms. We conjecture that this is true to all orders in the series expansion (4.15), so that

$$\epsilon(k, h, R) = W(R^2 e^{2k h}, e^{2k h}, e^{-h}) \quad (4.21)$$

We can then use the quantum mirror map to compute the band energies. An example of such a calculation is shown in table 4. We also observe that the solution (4.21) gives the correct band energies for complex values of $h$ as far as $\text{Re}(h) \neq 0^6$.

The equation (4.21) gives an analytic expression for the band energies, in terms of a series expansion for $W(Q_1, Q_2, q)$. This series is convergent, so we do not expect non-perturbative corrections to (4.21). It has however a finite radius of convergence. Let us consider for example the center of the first band $k = 0$, where we expect the maximal difference w.r.t. the free theory. We have the following structure,

$$\epsilon_\ell(0, h) = f_\ell(q) \left( \frac{q}{q - 1} \right)^{2\ell-2}, \quad (4.22)$$

where $f_\ell(q)$ is regular at $q = 1$. In addition, numerical experiments indicate that,

$$f_\ell(q) \sim (-1)^\ell (f(q))^\ell, \quad \ell \gg 1. \quad (4.23)$$

\(^6\text{When } h \text{ is purely imaginary, the spectral problem of the quantum Mathieu operator might be related to the one for the fully periodic operator (5.1).}\)
where \( f(q) > 0 \) whenever \( q \in (0, 1) \). This means that the coefficients \( \epsilon_\ell(0, h) \) grow, for \( \ell \) large, as

\[
\epsilon_\ell(0, h) \sim \left( -\frac{f(q)}{(q-1)^4} \right)^\ell,
\]

so the radius of convergence of this series is approximately given by

\[
R^2 \sim \frac{(q-1)^4}{f(q)}.
\]

This goes to zero as \( h \to 0 \), so we do not expect convergence for small values of \( h \), as we indeed observe numerically by computing the coefficients of (4.15) to high order. Nevertheless, since the series is alternating, it is expected that it can be resummed to a convergent function on the positive real axis of \( R \). In particular, the convergence of (4.15) as \( h \to 0 \) can be remarkably improved by using standard algorithms such as Padé approximants or Shanks transformations (see for example [66]). Let us illustrate this in one example, namely \( k = 0 \) and \( h = \pi/3 \). It is useful to define the truncated series

\[
\epsilon^{(N)}[R] = \sum_{n \geq 0} \epsilon_n(0, \pi/3) R^n
\]

and its Shanks transformation\(^7\) as

\[
S(N, R) = \frac{\epsilon^{(N+1)}[R] \epsilon^{(N-1)}[R] - (\epsilon^{(N)}[R])^2}{\epsilon^{(N+1)}[R] - 2 \epsilon^{(N)}[R] + \epsilon^{(N-1)}[R]}.
\]

Likewise we denote by

\[
S_n(N, R), \quad N > n
\]

the quantity obtained after applying \( n \) times the Shanks transformation. For instance

\[
S_2(N, R) = \frac{S(N + 1, R) S(N - 1, R) - (S(N, R))^2}{S(N + 1, R) - 2S(N, R) + S(N - 1, R)}.
\]

For \( R = 1 \) (4.26) diverges at large \( N \) as expected from (4.25) and shown in the first column of table 5. Nevertheless by applying Shanks transformation repeatedly we can improve the convergence remarkably as shown for instance in table 5.

The above solution for the energies (4.21) is valid in the interior of the band. As in the standard Mathieu equation, there are poles at the edges of the bands, i.e. when \( k = \pm n/2, n \in \mathbb{Z}_{>0} \) (the poles at \( k = \pm 1/2 \) are manifest in (4.19)). This case has then to be treated separately, and

\(^7\) In this example, the Shanks transform turns out to be a bit more efficient than the Padé approximant.

### Table 4. The energy \( \epsilon(0, 2\pi, 1) \) for the quantum Mathieu equation at the center of the band, as computed from (4.21). The order denotes the number of terms retained in the expansion of \( W(Q_1, Q_2, q) \) in powers of \( Q_1 \).

| Order | \( \epsilon_0(0, 2\pi, 1) \)          |
|-------|-------------------------------------|
| 1     | 1.196251 125995 166296 078333       |
| 3     | 1.196251 152313 733297 269664       |
| 5     | 1.196251 152313 733393 3761514      |
| Num   | 1.196251 152313 733393 761514       |

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we can follow again the methods of [65]. To solve the central equation at the edges we have to change the ansatz (4.16) and (4.15). We fix $k = 1/2$ at the edge of the Brillouin zone and we consider the following ansatz

$$
\epsilon(\alpha, 1/2, h, R) = 2 \cosh(h/2) + \sum_{\ell \geq 1} \epsilon(\alpha, 1/2, h)R^{\ell},
$$

(4.30)

with

$$
c_{1/2} = c_0^{(1/2)} = 1, \quad c_{-1}^{(1/2)} = \alpha, \quad \alpha^2 = 1
$$

(4.31)

$$
c_n^{(1/2)} = R^n \sum_{\ell \geq 0} d_n R^{\ell}, \quad n > 0.
$$

(4.32)

By solving (4.13) with this ansatz we find

$$
\begin{align*}
\epsilon(\alpha, 1/2, h, R) &= 2 \cosh(h/2) + \alpha R + \frac{R^2}{2 \cosh(h/2) - 2 \cosh(h/2)} - \frac{\alpha R^3}{4 (\cosh(h/2) - \cosh(h/2))^2} \\
&\quad - \frac{R^4 \csc(h/2) - \csc(h/2)}{128 \cosh(h/2) + 64} + \frac{\alpha R^3 (4 \cosh(h) + 4 \cosh(2h) + 3 \coth(\frac{h}{2}) \csc(h/2))}{64(2 \cosh(h) + 1)^2} + O(R^6).
\end{align*}
$$

(4.33)

By setting $\alpha = -1$ we obtain the energy at the edge of the first band, while $\alpha = 1$ gives the energy at the edge of the second band. Some results are shown in tables 6 and 7 and are in perfect agreement with the numerical computations. Notice that for $k = -\frac{1}{2} < 0$ one has to replace the ansatz (4.32) by

$$
c_{1/2} = c_0^{(1/2)} = 1, \quad c_{-1}^{(1/2)} = \alpha
$$

(4.34)

$$
c_n^{(1/2)} = R^n \sum_{\ell \geq 0} d_n R^{\ell}, \quad n > 0.
$$

(4.35)

However, at the end the result is the same, namely $\epsilon(\alpha, 1/2, h, R) = \epsilon(\alpha, -1/2, h, R)$, as expected. We note that the solution (4.33) also holds for complex values of $h$ as far as $\text{Re}(h) \neq 0$. Other values of $k$, corresponding to other edges of the bands, can be worked out similarly.

The series (4.30) is convergent with a finite radius of convergence, so we do not expect to have non-perturbative corrections. Numerically we observe a behaviour similar to (4.24), with a zero radius of convergence as $h \to 0$. As before, we can improve the convergence by

| $N$ | $\epsilon^{(N)}[1]$ | $S_1(N + 3, 1)$ | $S_6(N + 6, 1)$ | $S_9(N + 9, 1)$ |
|-----|-------------------|-----------------|-----------------|-----------------|
| 1   | 0.334             | 1.0228922       | 1.0201454       | 1.02014601733678 |
| 2   | 2.423             | 1.0187015       | 1.0201458       | 1.02014600931881 |
| 3   | -3.038            | 1.0212857       | 1.0201461       | 1.02014600851393 |
| 4   | 14.864            | 1.0188895       | 1.0201458       | 1.02014600944477 |
| 5   | -50.900           | 1.0219159       | 1.0201462       | 1.02014600942947 |
| Num |                   |                 |                 | 1.02014600942838 |
using standard algorithms. For the series (4.30), the Padé approximant turns out to be well-suited. An example is shown on table 8. We use the same convention as in [67] for the Padé approximant.

Finally, we note that the even powers of $R$ in (4.30) can be obtained from the Wilson loop as

$$
\sum_{n \geq 0} \epsilon_{2n}(a, 1/2, R) R^{2n} = \text{Res}_{k=1/2} W(e^{2k R^2}, e^{2k R}, e^{-k}) (k - 1/2)^{-1}.
$$

(4.36)

It would be interesting to see whether there is a similar interpretation for the odd powers of $R$ in (4.30).

### 4.2. Four dimensional limit

In order to recover the Mathieu equation (4.7) from the quantum Mathieu equation (4.5) one has to implement the standard four dimensional limit, i.e. we replace

$$
h \rightarrow \beta h, \quad R \rightarrow \beta^2 R,
$$

(4.37)

and we take the limit $\beta \rightarrow 0$. Then, away from the edges of the bands, we obtain the well-known result expressing the band energies for the Mathieu equation in terms of the NS free energy of the $SU(2)$, $\mathcal{N} = 2$ theory, namely (see for example [48–50, 68])

$$
u(k, R, h) = k^2 h^2 - \Lambda \partial_\Lambda F^{\text{NS}}(\Lambda, kh, ih)
$$

(4.38)

where we set $\Lambda = R^2$, and

$$
F^{\text{NS}}(\Lambda, a, h) = -\frac{2\Lambda}{4a^2 + h^2} + \frac{\Lambda^2 (7h^2 - 20a^2)}{4 (a^2 + h^2) (4a^2 + h^2)^3} + O(\Lambda^4).
$$

(4.39)

---

**Table 6.** The energy $\epsilon(-1, 1/2, 2\pi, 1)$ for the quantum Mathieu equation at the edge of the first band, as computed from (4.33). The order denotes the number of terms retained in the expansion of $\epsilon(-1, 1/2, 2\pi, R)$ in powers of $R$.

| Order | $\epsilon(-1, 1/2, 2\pi, 1)$ |
|-------|-----------------------------|
| 1     | 22.18390655104304152555035041051 |
| 3     | 22.1838257067966848185142066748 |
| 5     | 22.1838257067966837489560885 |
| 7     | 22.1838257067966837481053206977 |
| Num   | 22.1838257067966837481053206975 |

**Table 7.** The energy $\epsilon(1, 1/2, 2\pi, 1)$ for the quantum Mathieu equation at the edge of the second band, as computed from (4.33). The order denotes the number of terms retained in the expansion of $\epsilon(1, 1/2, 2\pi, R)$ in powers of $R$.

| Order | $\epsilon(1, 1/2, 2\pi, 1)$ |
|-------|-----------------------------|
| 1     | 24.18390655104304152555035041051 |
| 3     | 24.18382569372298652860218359466 |
| 5     | 24.18382569372298562878563284976 |
| 7     | 24.18382569372298562879599690871 |
| Num   | 24.18382569372298562879599690846 |
These results can be easily obtained by solving the central equation for (4.7) instead of taking the four dimensional limit (4.37). More precisely, the central equation for (4.7) reads
\[ h^2 \left( k + \frac{2\pi}{a} \right) c_{\ell}^{(k)} + R \left( c_{\ell-1}^{(k)} + c_{\ell+1}^{(k)} \right) = u(k, R, h)c_{\ell}^{(k)}. \] (4.40)
By solving (4.40) at the edge of the first zone, i.e. for \( k = 1/2 \), one finds [65, 69]
\[ u\left(1/2, R, h\right) = u_o(\alpha, 1/2, R, h) + u_e(1/2, R, h), \] (4.41)
where \( u_{e,o} \) are the even and odd powers of \( R \), respectively. They have the explicit expression,
\[ u_o(\alpha, 1/2, R, h) = R\alpha - \frac{\alpha R^3}{4h^2} + \frac{11R^5}{144h^6} + \frac{55R^7}{2304h^{12}} + \cdots \]
\[ u_e(1/2, R, h) = \frac{1}{4} h^2 - \frac{R^2}{24h^6} - \frac{R^4}{24h^2} + \frac{49R^6}{576h^{10}} + \cdots. \] (4.42)
By choosing \( \alpha = -1 \) one obtains the energy at the edge of the first band, while \( \alpha = 1 \) gives the energy at the edge of the second band. As in the quantum Mathieu case, the even powers are still determined by the NS free energy. Indeed, one has
\[ u_e(1, 2, R, h) = \lim_{k \to 1/2} \left( \frac{u(k, R, h)}{k - 1/2} \right) = \lim_{k \to 1/2} \left( \frac{k^2h^2 - \Lambda \partial_\Lambda F^{NS}(\Lambda, kh, Ih)}{k - 1/2} \right) \bigg|_{\Lambda = R^2}. \] (4.43)
Again, it would be interesting to find an expression for the odd powers \( u_o(\alpha, 1/2, R, h) \) in terms of gauge theory.

As a final observation, we note that, as in the case of quantum Mathieu equation, the series (4.41) and (4.38) have a finite radius of convergence which goes to zero at \( h = 0 \). Nevertheless, one can also use Padé approximants to calculate the band energies.

5. Conclusions

In this paper we have taken the first steps to extend the TS/ST correspondence, as formulated in [14, 15], to complex values of the Planck constant. We have noted that, at least when the toric CY engineers a five dimensional gauge theory, one can resum the series appearing in the exact quantization condition, in the form given in [21]. The results obtained in this way are in agreement with numerical calculations of the spectrum. One interesting outcome of our computations is that the non-perturbative effects detected in [14, 15] are crucial to obtain

| Table 8. The Padé approximant of order \( n \) for the series (4.30) with \( k = 1/2, \ h = \pi/6, \ R = 1 \). We computed the coefficients in the series expansion (4.30) up to \( R^{56} \). |
|-------------|------------------|
| \( n \)    | \( P^*_n(1) \)   |
| 10         | 0.5156117750532916 |
| 20         | 0.5205803657529506  |
| 30         | 0.5205897527191174  |
| 40         | 0.5205897346606964  |
| 50         | 0.5205897353650857  |
| 55         | 0.5205897353651356  |
| Num        | 0.5205897353651301  |
the correct results, even in situations where the all-orders WKB quantization condition of [2]
makes sense.

Clearly, there are many problems that one should address in order to have a full understand-
ing of the complex side of the TS/ST correspondence. First of all, our results cover a small subset of all toric geometries, namely those directly related to gauge theories. For other geometries, like local $\mathbb{CP}^2$, it is not clear how to resum the (refined) topological free energies, the grand potential and the quantum mirror map, in order to obtain convergent expansions. We should also note that, when $\hbar$ is real, the correspondence allows one to calculate the spectral traces and the spectral determinant of the operators. It would be interesting to see how this can be achieved in the complex case, even in examples related to gauge theories.

In this paper we have also considered other slices in the quantization of the mirror cuve, focusing for simplicity in the quantum Mathieu operator associated to local $F_0$. The solution for the resulting band structure only involves the quantum mirror map and is much simpler than the solution of the trace class operators considered in [14]. The underlying reason, as pointed out in [49], is that we can regard the full periodic potential as a perturbation of a free particle on a circle. This leads to the convergent expansion (4.15), which is nothing but the quantum mirror map. Non-perturbative effects are not required to compute the energies of the bands, in contrast to what happens in the quantization scheme of [14], where these effects are crucial. Since the expansions break down at the edges, one can however consider the energy splitting of the bands as some sort of non-perturbative effect, as advocated in [49, 50, 70] for the standard Mathieu equation. It would be interesting to study the splitting for the quantum Mathieu equation from that point of view.

The analysis of the operators along ‘imaginary’ slices should be extended to more general geometries. One could also study operators obtained by going to the imaginary slices for both $x$ and $p$. In the case of local $F_0$, this gives the ‘fully periodic’ operator

$$O = 2 \cos(p) + 2R \cos(x).$$

(5.1)

It is easy to see that this is equivalent to the Harper operator, leading to the famous Hofstadter butterfly. It has been observed that the quantum geometry of local $F_0$ is closely related to some aspects of this problem [33]. It would be interesting to see whether topological string theory sheds further light on the subtle spectral properties of the Harper operator.

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Appendix. The grand potential

In this section we review the definition of the topological string grand potential. For this we first need to introduce various quantities appearing in topological string theory and in quantum geometry. We follow [15, 16], where more details and references can be found.

Let $X$ be a toric CY geometry as in section 2. It is convenient to write the ‘true’ complex moduli (2.1) of $X$ as

$$\kappa_i = e^{\mu_i},$$

(A.1)
and to introduce the Batyrev coordinates $z_i$ such that
\[ - \log z_i = \sum_{j=1}^{g_S} C_{ij} t_j + \sum_{k=1}^{g_S} \alpha_k \log \xi_k, \quad i = 1, \ldots , n_S, \] (A.2)
where $C_{ij}$ is a $n_S \times g_S$ matrix which is determined by the toric data of $X$ as explained in [71].

The Kahler moduli $t_i$ and the Batyrev coordinates are related by the mirror map
\[ - t_i = \log z_i + \Pi_i(z), \] (A.3)
where $\Pi_i$ is a power series in $z_j$. We also denote $Q_i = e^{-t_i}$. As explained in [4], we can promote (A.3) to a quantum mirror map
\[ - t_i(h) = \log z_i + \Pi_i(z, h). \] (A.4)

We introduce the topological string free energy
\[ F^{\text{top}}(t, g_s) = \frac{1}{6 g_s} a_{jk} t_j t_k + b_i t_i + F^{\text{GV}}(t, g_s) \] (A.5)
where
\[ F^{\text{GV}}(t, g_s) = \sum_{g \geq 0} \sum_{d} \sum_{w=1}^{\infty} \frac{1}{w^d} \left( 2 \sin \frac{w g_s}{2} \right)^{2g-2} e^{-w d t} \] (A.6)
is written in terms of the Gopakumar–Vafa (GV) invariants $n^d_g$ of the underlying geometry $X$. Similarly, the Nekrasov–Shatashvili (NS) free energy of $X$ is defined as
\[ F^{\text{NS}}(t, h) = \frac{1}{6 h} a_{jk} t_j t_k + b_i^{\text{NS}} t_i + \sum_{h, a, w} n^d_{h,a} \sin \frac{w}{2} (2 j_k + 1) \sin \frac{w}{2} (2 j_k + 1) \] (A.7)
where $n^d_{h,a}$ are refined BPS invariants. We also need a B-field satisfying
\[ (-1)^{3g_s+2h+1} = (-1)^B, \] (A.8)
for the indices labelling non-vanishing invariants $n^d_{h,a}$.

The topological string grand potential is defined as
\[ J_X(\mu, \xi, h) = J_X^{\text{WKB}}(\mu, \xi, h) + J_X^{\text{WS}}(\mu, \xi, h), \] (A.9)
where
\[ J_X^{\text{WS}}(\mu, \xi, h) = F^{\text{SW}} \left( \frac{2 \pi}{h} t(h) + \pi i \mathbf{B}, \frac{4 \pi^2}{h} \right), \] (A.10)
and
\[ J_X^{\text{WKB}}(\mu, \xi, h) = \frac{t(h)}{2 \pi} \frac{\partial F^{\text{NS}}(t(h), h)}{\partial t_i} + \frac{h^2}{2 \pi} \frac{\partial}{\partial h} \left( \frac{F^{\text{NS}}(t(h), h)}{h} \right) + \frac{2 \pi}{h} b_i t_i(h) + A(\xi, h). \] (A.11)

Here, $A(\xi, h)$ is a moduli-independent contribution which can be related to the contribution of constant maps in topological string theory.

We will also use the following notation
\[ F^{\text{NS, inst}}(t, h) = \sum_{j, a, w} n^d_{j, a} \sin \frac{w}{2} (2 j_k + 1) \sin \frac{w}{2} (2 j_k + 1) \] (A.12)
as well as the twisted free energy

\[ \hat{F}_{\text{NS,inst}}(t, h) = \sum_{j_L, j_R} \sum_{w, d} N_{j_L, j_R}^d \sin \frac{h w}{2} (2j_L + 1) \sin \frac{h w}{2} (2j_R + 1) \frac{1}{2w^2 \sin^3 \frac{h w}{2}} e^{-w (t + i \pi B)}, \]  
(A.13)

\[ \hat{F}_{\text{NS}}(t, h) = \frac{1}{6\hbar} a_{ijkl} t_{ij} t_{kl} + b_{\text{NS}}^{t_i} t_i h + \hat{F}_{\text{NS,inst}}(t, h), \]  
(A.14)

and the twisted quantum mirror map

\[ -\hat{\gamma}_i(h) = \log z_i + \Pi_i(z_{B_i}, h), \]  
(A.15)

where \( z_{B_i} = (z_1(-1)^{\theta_1}, \cdots, z_{g+\Sigma+\Sigma}(-1)^{\theta_{g+\Sigma+\Sigma}}). \)

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