DECOMPOSITIONS OF MODULES LACKING ZERO SUMS

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Abstract. A direct sum decomposition theory is developed for direct summands (and complements) of modules over a semiring \( R \), having the property that \( v + w = 0 \) implies \( v = 0 \) and \( w = 0 \). Although this never occurs when \( R \) is a ring, it always does hold for free modules over the max-plus semiring and related semirings. In such situations, the direct complement is unique, and the decomposition is unique up to refinement. Thus, every finitely generated projective module is a finite direct sum of summands of \( R \) (assuming the mild assumption that 1 is a finite sum of orthogonal primitive idempotents of \( R \)). Some of the results are presented more generally for weak complements and semidirect complements. We conclude by examining the obstruction to the “upper bound” property in this context.

1. Introduction

The motivation of this research is to understand direct sum decompositions of submodules of free modules over the max-plus algebra and related structures in tropical algebra (supertropical algebra \([4, 9, 11]\) and symmetrized algebra \([11]\)), as well as in some other settings in algebra. It turns out that direct sum decompositions are unique (not just up to isomorphism), and thus one can develop a theory of direct sum decompositions analogous to the theory of the socle in customary abstract algebra, using the axiom of “lacking zero sums”:

\[ v + w = 0 \quad \Rightarrow \quad v = w = 0 \]

(termed “zerosumfree” in \([8]\).) This axiom may seem rather peculiar at first glance, but is easily seen to hold in tropical mathematics and also over other semirings of interest, as noted in Examples \([1, 6]\) especially in real algebra, such as the positive cone of an ordered field \([3, \text{p. 18}]\) or a partially ordered commutative ring \([2, \text{p. 32}]\). More instances are given in Examples \([1, 6]\).

After writing the first draft of this paper, we became aware of \([13]\), in which Macpherson already has proved the uniqueness of direct sum decompositions of projective modules in the tropical setting in \([14, \text{Theorem 3.9 and Corollary 4.13}]\), working over the Boolean semifield B. (Then he goes on to prove other interesting results about projective modules). However, our hypotheses are different, based solely on this axiom of “lacking zero sums,” which is in the language of elementary logic is a quasi-identity (with all its formal implications) and our main tool (Theorem \([2, 3]\)) is somewhat stronger than the decomposition property for projective modules (to be compared with \([14, \text{Aside 3.7}]\)). Also, our main results hold for “weak complements,” which are more general than complements in direct sums.

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Definition 1.1. A submodule $T \subset V$ is a **weak complement** (of a submodule $W \subset V$) if $T + W = V$ and $(w + T) \cap T = \emptyset$, $\forall \ w \in W \setminus \{0_V\}$.

Our main theorems for modules lacking zero sums:

**Theorem 2.3** Suppose $V$ has a submodule $T$ of $V$ which is a weak complement. Then any decomposition of $V$ descends to a decomposition of $T$, in the sense that if $V = Y + Z$, then $T = (T \cap Y) + (T \cap Z)$.

**Theorem 2.7** Suppose $V = T \oplus W = Y \oplus Z$. Then

$$V = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y) \oplus (W \cap Z).$$

**Theorem 2.9** Given two decompositions $V = T \oplus W = Y \oplus Z$ of $V$, then $T + Y = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y)$.

(1.1)

**Theorem 3.2** Any indecomposable projective $R$-module $P$ is isomorphic to a direct summand of $R$, and thus has the form $Re$ for some primitive idempotent $e$ of $R$.

For $R$ lacking zero sums, given a module, we start to decompose it, and either we can continue ad infinitum or the process terminates at an indecomposable summand. The direct sum of all the indecomposable summands is called the **decomposition socle**, denoted $\text{dsoc}(V)$, and contains every indecomposable summand of $V$, in analogy to the socle (the sum of the simple submodules) in classical module theory. In fact, this situation is even tighter than with the classical socle, since $\text{dsoc}(V)$ now is written uniquely as a direct sum of indecomposables. Furthermore, under certain conditions, e.g., $R = \mathbb{N}_0$, the set-theoretic complement of $\text{dsoc}(V)$, with $\{0_V\}$ adjoined, also is a submodule of $V$ whose intersection with $\text{dsoc}(V)$ obviously is $\{0_V\}$. (But this is not the direct complement!) Furthermore we can understand $\text{dsoc}(R)$ in terms of the idempotents of $R$. This approach yields a result analogous to the relation of semisimplicity and the socle in classical ring theory:

**Theorem 3.3** $\text{dsoc}(R) = R$ iff $R$ has a finite set of primitive orthogonal idempotents whose sum is $1_R$, iff $R$ is a finite direct sum of indecomposable projective modules.

We can improve these results by strengthening the lacking zero sum hypothesis.

**Definition 1.2.** A subset $W$ of a monoid $(V, +, 0_V)$ is **summand absorbing** (abbreviated **SA** in $V$, if

$$\forall x, y \in V : \ x + y \in W \ \Rightarrow \ x \in W, y \in W.$$  

An analogous argument yields:

**Theorem 4.5** Assume that $V = W + T$, where $T$ is SA in $V$, and $W \cap T = \{0_V\}$.

(i) Then $T$ is the unique weak complement of $W$ in $V$.

(ii) If in addition $U$ is a submodule of $V$ with $W + U = V$ and also $U$ is SA in $V$, then $T \subseteq U$, and $T$ is the unique weak complement of $W \cap U$ in $U$.

Further along section §4 we extend some of these results to more general decompositions, arising from **weak complements** and **semidirect complements** (Definition 4.6). These would all be the same for modules over rings, but have subtle distinctions in this setting.
Proposition 4.8 If \( U := W + S = W \ltimes S \) and \( V = U \ltimes T \), then
\[
S + T = S \ltimes T,
\]
\[
W + T = W \ltimes T,
\]
\[
V = W \ltimes (S \ltimes T) = (W \ltimes S) \ltimes T.
\]
(Here the sign \( \ltimes \) denotes a semidirect decomposition.)

Proposition 4.9 Let \( W, S, T \) be submodules of an \( R \)-module \( V \), and assume that \( S \) is a weak complement of \( W \) in \( U := W + S \), while \( T \) is a semidirect complement of \( U \) in \( V \). Then \( S + T \) is a weak complement of \( W \) in \( V \).

Finally, one could recall that the tropical situation often involves the stronger condition (than lacking zero sums), called upper bound (ub) that \( a + b + c = a \) implies \( a + b = a \). This leads us in \( \S 5 \) to utilize Green’s partial preorder on a semigroup \( (V, +) \) by saying that \( x \preceq y \) if \( x + z = y \) for some \( z \) in \( V \). This yields a congruence, the obstruction for a module to be ub, which is studied in terms of a convexity condition, and given in the context of the earlier results of this paper.

1.1. Background.

We recall that a semiring, denoted in this paper as a \( (R, +, \cdot, 0_R, 1_R) \), is a set \( R \) equipped with two binary operations \( + \) and \( \cdot \), called addition and multiplication, such that:

(i) \( (R, +, 0_R) \) is an abelian monoid with identity element \( 0_R \);
(ii) \( (R, \cdot, 1_R) \) is a monoid with identity element \( 1_R \);
(iii) multiplication distributes over addition.

Modules over semirings (often called “semimodules” in the literature, cf. [9]) are defined just as modules over rings, except that now the additive structure is that of a semigroup instead of a group. (Note that subtraction does not enter into the other axioms of a module over a ring.) To wit:

Definition 1.3. Suppose \( R \) is a semiring. A (left) \( R \)-module \( V \) is a monoid \( (V, +, 0_V) \) together with scalar multiplication \( R \times V \rightarrow V \) satisfying the following properties for all \( r_i \in R \) and \( v, w \in V \):

(i) \( r(v + w) = rv + rw \);
(ii) \( (r_1 + r_2)v = r_1v + r_2v \);
(iii) \( (r_1r_2)v = r_1(r_2v) \);
(iv) \( 1_Rv = v \);
(v) \( 0_Rv = r0_V = 0_V \).

We are concerned with the following property.

Definition 1.4. An additive monoid \( (V, +, 0_V) \) lacks zero sums if \( v_1 + v_2 = 0_V \) implies \( v_1 = v_2 = 0_V \), for any \( v_1, v_2 \in V \).

Although this condition never holds when \( V \) is a group, since we could take \( v_2 = -v_1 \), it always holds in \( R^n \) when \( R \) is one of the semirings mentioned in Examples 1.6 below. In such situations the condition of lacking zero sums actually is rather ubiquitous, being a “quasi-identity” in the language of elementary logic.

Examples 1.5.
a) If \((V_i \mid i \in I)\) is a family of \(R\)-modules which lack zero sums, then \(V := \prod_{i \in I} V_i\) lacks zero sums.

b) If an \(R\)-module \(V\) lacks zero sums, then the same holds for every submodule of \(V\). In particular, over any semiring \(R\) lacking zero sums, every submodule of \(R^n\) lacks zero sums.

c) If \(V\) lacks zero sums, then for any set \(S\) the module \(\text{Fun}(S, V)\) of functions from \(S\) to \(V\), lacks zero sums.

Thus, any semiring lacking zero sums supports a wide range of modules lacking zero sums.

If \(V = R\) then lacking zero sums is precisely the condition of being an “antiring” in the sense of Tan [15] and Dolžna-Oblak [5], and we have the following basic examples:

**Examples 1.6.**

a) Obviously if \(R \setminus \{0_R\}\) is closed under addition then \(R\) lacks zero sums. This happens for the max-plus algebra, the supertropical algebra mentioned above, and the more general layered version [10] when the “sorting set” is non-negative. Other instances of this phenomenon worth explicit mention:

1) The “boolean semifield” \(B = \{\infty, 0\}\) (and thus subalgebras of algebras that are free modules over \(B\)). This shows that our results pertain to “\(\mathcal{F}_1\)-geometry,” treated in [14].

2) Rewriting the boolean semifield instead as \(B = \{0, 1\}\) where \(1 + 1 = 1\), one can generalize it to \(\{0, 1, \ldots, q\}\) \(L = [1, q] := \{1, 2, \ldots, q\}\) the “truncated semiring” of [10] Example 2.14, where \(a + b\) is defined to be the minimum of their numerical sum and \(q\).

3) Function semirings, polynomial semirings, and Laurent polynomial semirings over these semirings.

4) If \(F\) is a formally real field, i.e. \(-1\) is not a sum of squares in \(F\), then the subsemiring \(R = \Sigma F^2\), consisting of all sums of squares in \(F\), lacks zero sums. In fact \(R\) is a semifield; the inverse of a sum of squares 
\[
a = x_1^2 + \cdots + x_r^2 \quad \text{is} \quad a^{-1} = \left(\frac{x_1}{a}\right) + \cdots + \left(\frac{x_r}{a}\right)^2.
\]

5) Let \(\mathbb{Z}[t] = \mathbb{Z}[t_1, \ldots, t_n]\) denote the polynomial ring in \(n\) variables over \(\mathbb{Z}\). We choose a non-constant polynomial \(f \in \mathbb{Z}[t]\). Then the smallest subsemiring of \(\mathbb{Z}[t]\) containing \(f\), namely
\[
\mathbb{N}_0[f] = \mathbb{N}_0 + \mathbb{N}_0 f + \mathbb{N}_0 f^2 + \ldots
\]
lacks zero sums, since \(\mathbb{N}_0[f]\) is a free \(\mathbb{N}_0\)-module.

6) More generally, the set of positive elements of any partially ordered semiring is a sub-semiring lacking zero sums.

7) The set of finite dimensional characters over a field of characteristic 0 of any group is a semiring lacking zero sums.

b) Any abelian monoid \((V, +, 0_V)\) can be viewed as a module over the semiring \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\), which lacks zero sums.

**Definition 1.7.** Suppose that \(T\) is a submodule of a module \(V\). We write \(T'\) for \(V \setminus T\), the **set-theoretic complement** of \(T\). On the other hand, we define the **direct sum** \(T \oplus W\) in the usual way (as the Cartesian product, with componentwise operations).

A submodule \(W \subset V\) is a **direct complement** of \(T\) if \(T \oplus W = V\).
2. Direct sum decompositions of modules lacking zero sums

We assume through the end of §3 that the $R$-module $V$ lacks zero sums.

Suppose that $W$ and $T$ are submodules of $V$ with $W + T = V$ and $W \cap T = \{0_V\}$. In order for $T$ to be a direct complement of $W$ we need the stronger condition that $w_1 + t_1 = w_2 + t_2$ implies $w_1 = w_2$ and $t_1 = t_2$.

The notion of weak complement (Definition 1.1) goes half way.

**Remark 2.1.** $T \subset V$ is a weak complement of $W$, iff $w_1 + t_1 = t_2$ implies $w_1 = 0_V$. In particular $W \cap T = \{0_V\}$, seen by taking $t_1 = 0_V$.

We turn to the main computation of this paper.

**Lemma 2.2.** If $W$ is a submodule of $V$ with weak complement $T$, and $a = a' + (w_1 + w_2)$ for $a, a' \in T$ and $w_1 \in W$, then $w_1 = w_2 = 0_V$ and $a = a'$.

**Proof.** By hypothesis $w_1 + w_2 = 0_V$, implying $w_1 = w_2 = 0_V$ since $W$ lacks zero sums. □

**Theorem 2.3.** Suppose $V$ has a submodule $T$ of $V$ which is a weak complement. Then any decomposition of $V$ descends to a decomposition of $T$, in the sense that if $V = Y + Z$, then $T = (T \cap Y) + (T \cap Z)$.

**Proof.** Namely, take a submodule $W$ of $V$ having weak complement $T$, and for any $a \in T$, write $a = y + z$ for $y \in Y$ and $z \in Z$. In turn $y = t_1 + w_1$ and $z = t_2 + w_2$ for $t_i \in T$ and $w_i \in W$. Hence $a = (t_1 + t_2) + (w_1 + w_2) \in T + (w_1 + w_2)$. By Lemma 2.2, $w_1 = w_2 = 0_V$. Thus $y = t_1 \in T \cap Y$ and $z = t_2 \in T \cap Z$. □

**Corollary 2.4.** Assume that $W$ is a submodule of $V$. Assume furthermore that $T$ is a weak complement of $W$ in $V$ and $U$ is a submodule of $V$ with $W + U = V$. Then $T \subset U$.

**Proof.** Taking $Y = W$ and $Z = U$ in Theorem 2.3 implies $T = T \cap U$ since $T \cap W = \{0_V\}$. □

**Corollary 2.5.** Any submodules $W$ of $V$ lacking zero sums has at most one weak complement in $V$.

**Proof.** If $T$ and $U$ are weak complements of $W$ in $V$, then $T \subset U$ by the theorem. Also $U \subset T$ by symmetry, whence $T = U$. □

We leave further results about weak complements to §4 and turn more specifically to direct complements.

Since direct complements are also weak complements, we may state in consequence of Corollary 2.3 the following.

**Corollary 2.6.** Given submodules $T, W, Z$ of $V$, with $W = W \oplus T = W \oplus Z$, then $T = Z$.

From Theorem 2.3 we draw the following conclusion.

**Theorem 2.7.** Suppose $V = T \oplus W = Y \oplus Z$. Then

$$V = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y) \oplus (W \cap Z).$$

**Proof.** By Theorem 2.3 $T = (T \cap Y) \oplus (T \cap Z)$, and, symmetrically, $W = (W \cap Y) \oplus (W \cap Z)$. We get the assertion by putting these together. □

We note in passing that Theorem 2.3 also leads to a second proof of Corollary 2.6 as follows:
Second proof of Corollary 2.6. Applying Theorem 2.3 to $W$ instead of $T$, we have
\[ W = (W \cap T) \oplus (W \cap Z) = W \cap Z \]
since $W \cap T = 0_V$. Hence $W \subset Z$, and, by symmetry, $Z \subset W$, yielding $Z = W$. \hfill \Box

Thus any direct summand $T$ of $V$ has a unique direct complement, which we denote as $T^c$. Note this is properly contained in $T'$ whenever $T \neq \{0_V\}$, $T \neq V$, since taking $a \notin 0_V$ in $T$ and $b \notin 0_V$ in $T'$ we have $a + b \in T' \setminus T^c$.

**Corollary 2.8.** If $T \subset Y$ then $Y^c \subset T^c$.

**Proof.** Easily seen by refining the decompositions. \hfill \Box

**Theorem 2.9.** Given two decompositions $V = T \oplus W = Y \oplus Z$ of $V$, then
\[ T + Y = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y). \] (2.1)

**Proof.** Write $V = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y) \oplus (W \cap Z)$, and let $\pi$ be the projection of $V$ onto $W \cap Z$ sending $(T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y)$ to $0_V$. Clearly $\pi^{-1}(0_V) \subset T + Y$. On the other hand, Theorem 2.3 applied to the decomposition of $T$ tells us that $T \subset \pi^{-1}(0_V)$. By symmetry also $Y \subset \pi^{-1}(0_V)$, and thus $T + Y \subset \pi^{-1}(0_V)$. This proves that
\[ T + Y = \pi^{-1}(0_V) = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y). \] \hfill \Box

**Corollary 2.10.** If $T$ and $Y$ are direct summands of an $R$-module $V$ lacking zero sums, then both $T \cap Y$ and $T + Y$ are direct summands of $V$ and
\[ (T \cap Y)^c = (T \cap Y^c) \oplus (T \cap Y) \oplus (T^c \cap Y), \] (2.2)
\[ (T + Y)^c = T^c \cap Y^c. \] (2.3)

**Proposition 2.11.** Assume that $(U_i \mid i \in I)$ is a finite family of direct summands of an $R$-module $V$ lacking zero sums, with decompositions $V = U_1 \oplus U_2^c$. For any $J \subset I$ define $U_J := \bigcap_{j \in J} U_j$ and $U_J^c := \bigcap_{j \in J} U_j^c$. Then
\[ V = \bigoplus_{J \subset I} (U_J \cap U_J^c). \]

**Proof.** An easy induction on $|I|$ starting from Theorem 2.9 where we peel off one $U_i$ at a time. \hfill \Box

**Definition 2.12.** As usual, we call an $R$-module $V$ **indecomposable**, if $V \neq \{0_V\}$ and $V$ has no decomposition $V = W_1 \oplus W_2$ with $W_1 \neq \{0_V\}$, $W_2 \neq \{0_V\}$.

Let us turn to the indecomposable direct summands of $V$.

**Lemma 2.13.** If $T$ and $Y$ are indecomposable direct summands of $V$, then either $T = Y$ or $T + Y \cong T \oplus Y$.

**Proof.** We obtain from $V = Y \oplus Y^c$ by Theorem 2.3 that $T = (T \cap Y) \oplus (T \cap Y^c)$, and then, since $T$ is indecomposable, that $T \cap Y = T$ or $T \cap Y = \{0_V\}$, i.e., $T \subset Y$ or $T \cap Y = \{0_V\}$.

If $T \subset Y$ we conclude from $V = T \cap Y \oplus T^c$ in the same way that $Y = T \oplus (T^c \cap Y)$, and then that $Y = T$, since $Y$ is indecomposable. If $T \cap Y = \{0_V\}$ we have from the above that $T = T \cap Y^c$, i.e., $T \subset Y^c$, and now infer from $V = Y \oplus Y^c$ that $Y + T = Y \oplus T$. \hfill \Box
Proposition 2.14. The indecomposable direct summands of $V$ are independent, in the sense that if $T$ and \( \{ T_i : i \in I \} \) are distinct indecomposable direct summands of $V$, then
\[
T \cap (\sum_i T_i) = \{0_V\}.
\]

Proof. Taking direct limits, we may assume that $I$ is finite, and then we are done by Theorem 2.9 and induction. □

Definition 2.15. The decomposition socle $\text{dsoc}(V)$ is the sum of the indecomposable direct summands of $V$.

Now let $\{T_i : i \in I \}$ denote the set of all indecomposable direct summands of $V$.

Proposition 2.16. When $I$ is finite,
\[
\text{dsoc}(V) = \sum_{i \in I} T_i = \bigoplus_{i \in I} T_i,
\]
and is a direct summand of $V$, with direct complement $\bigcap_{i \in I} T_i^c$.

Proof. We may assume that $I = \{1, \ldots, n\}$. Let $V_r = \sum_{i=1}^r T_i$, for $r \leq n$. By an easy induction, we obtain from Corollary 2.10 that every $V_r$ is a direct summand of $V$, written
\[
V = V_r \oplus W_r.
\]
Furthermore, from Proposition 2.14
\[
V_r \cap T_{r+1} = \{0_V\}.
\]
By Theorem 2.3 we conclude that
\[
T_{r+1} = (V_r \cap T_{r+1}) + (W_r \cap T_{r+1}) = W_r \cap T_{r+1},
\]
i.e.,
\[
T_{r+1} \subset W_r.
\]
Given elements $u, u' \in V_r$, $t, t' \in W_r$ with $u + t = u' + t'$ it follows from (2.4) and (2.6) that $u = u'$ and $t = t'$. Thus
\[
V_{r+1} = V_r \oplus T_{r+1}
\]
for every $r < n$. The proposition now follows, up to the last assertion, which can be obtained from (2.3) in Corollary 2.10 by another easy induction. □

For $I$ infinite, it is seen in the same way that $\text{dsoc}(V)$ is the direct sum of the $T_i$, but now it need not be a direct summand of $V$. Furthermore, we must cope with the possibility that
\[
\text{dsoc}(V)_0 := \text{dsoc}(V) \cup \{0_V\}
\]
is not an $R$-submodule of $V$, but just a submonoid.

To rectify the situation, we view $V$ as an $N_0$-module, and let
\[
\text{Indc}(V) := \{W_i : i \in I' \}
\]
denote the set of all indecomposable $N_0$-direct summands of $V$, and
\[
W := \sum_{i \in I'} W_i = \bigoplus_{i \in I'} W_i.
\]
Then
\[
W'_0 := (V \setminus W) \cup \{0_V\}
\]
does not contain any $N_0$-indecomposable direct summands of $V$, and in particular none of the $T_j$.

Let $R^x$ denote the group of units of $R$. Every $\lambda \in R^x$ yields an automorphism $v \mapsto \lambda v$ of $(V, +, 0_V)$, and so the group $R^x$ operates on $\text{Indc}(V)$. When the semiring $R$ is additively generated by $R^x$, the $T_i$ are precisely the sums $\sum_{i \in J} W_i = \bigoplus_{i \in J} W_i$ where $\{W_i : i \in J\}$ is an orbit of $R^x$ on $\text{Indc}(V)$. The following result follows immediately from these observations.

**Theorem 2.17.** Assume that the semiring $R$ is additively generated by $R^x$. Then $\text{dsoc}(V)$ is the direct sum of all indecomposable direct summands of $V$, and the additive monoid

$$\text{dsoc}(V)_0 := (V \setminus \text{dsoc}(V)) \cup \{0_V\}$$

is an $R$-submodule of $V$. (But if there are infinitely many such indecomposable direct summands, $\text{dsoc}(V)$ need not be a direct summand of $V$.)

Examples where the theorem applies are:

- $R = \mathbb{N}_0$;
- $R$ is a semifield;
- $R$ is a so-called supersemifield, i.e., a supertropical semiring (cf. [11], [12]) where both $R \setminus (eR)$ and $(eR) \setminus \{0_R\}$ are groups, with $e = 1_R + 1_R$;
- $R$ is replaced by the semiring $R[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ of Laurent polynomials in $n$ variables over any of the previous semirings $R$.

3. Projective $R$-modules

We are ready to apply these results to the case that $V = R^n$. Assume throughout this section that $R$ lacks zero sums.

**Definition 3.1.** A module $P$ is **projective** if it is a direct summand of a free $R$-module, and is **finitely generated projective** if it is a summand of $R^n$.

An element $e \in R$ is **idempotent** if $e^2 = e$. Two idempotents $e, f$ are **orthogonal** if $ef = fe = 0_R$. An idempotent is **primitive** if it cannot be written as the sum of two nonzero orthogonal idempotents.

Projective modules are treated much more generally in [7]. The same argument as in [13] yields the analogous conclusion.

**Theorem 3.2.** Any indecomposable projective $R$-module $P$ is isomorphic to a direct summand of $R$, and thus has the form $Re$ for some primitive idempotent $e$ of $R$.

**Proof.** Write the free module $F = P \oplus P^c = \bigoplus_{i \in I} R\varepsilon_i$ with base $\{\varepsilon_i : i \in I\}$. For each $i \in I$,

(i) $R\varepsilon_i = ((R\varepsilon_i) \cap P) \oplus ((R\varepsilon_i) \cap P^c)$;
(ii) $P\varepsilon_i = (R\varepsilon_i \cap P) \oplus \sum_{j \neq i}((R\varepsilon_j) \cap P)$.

If $(R\varepsilon_i) \cap P = \{0_F\}$, then (i) yields $(R\varepsilon_i) \cap P^c = R\varepsilon_i$, so $R\varepsilon_i \subset P^c$; if this holds for all $i$ then $P^c = F$ implying $P = 0$.

Thus we may assume that there is $i \in I$ with $(R\varepsilon_i) \cap P \neq \{0_F\}$. Now (ii) implies $(R\varepsilon_i) \cap P = P$, since $P$ is indecomposable, whence $P$ is a direct summand of $R\varepsilon_i$, and we may assume that $F = R$.

Now consider the projection $\pi : R \rightarrow P$ onto $P$. Letting $e = \pi(1_R)$ we have $e^2 = e\pi(1_R) = \pi(e) = e$, so $e$ is idempotent. If $e$ were not primitive then writing $e = e_1 + e_2$ for orthogonal
idempotents $e_1, e_2$ would yield $Re = Re_1 \oplus Re_2$. (The standard proof for modules over rings also holds here.) This would contradict the indecomposability of $P$.

**Theorem 3.3.** $dsoc(R) = R$ iff $R$ has a finite set of orthogonal primitive idempotents whose sum is $1_R$, iff $R$ is a finite direct sum of indecomposable projective modules.

**Proof.** In view of Theorem 3.2, the only thing remaining to check here is the finiteness. But this is another standard argument taken from ring theory. If $R = \bigoplus_i P_i$, then the unit element $1_R$ is in this sum, and thus is some finite sum of elements $\sum_i r_i e_i$, implying $R$ is the sum of these $P_i$. □

**Corollary 3.4.** If $R$ has a finite set of orthogonal primitive idempotents $\{e_1, \ldots, e_m\}$ whose sum is $1_R$, then every finitely generated projective $R$-module $P$ is a finite direct sum of indecomposable projective modules, i.e., $P \cong \bigoplus_{i=1}^m (Re_i)^{n_i}$ for suitable $n_i$, and this direct sum decomposition is unique. □

4. Submodules satisfying the summand absorbing property

Throughout, $R$ is a semiring and $V$ an $R$-module. Perhaps surprisingly at first glance, uniqueness of decompositions can be proved in settings where we drop the requirement that $V$ lacks zero sums (but strengthen the requirement on $W$). So we drop this hypothesis and focus instead on its submodule $W$, first in conjunction with SA (Definition 1.2), and then in terms of “semidirect complements.”

**Proposition 4.1.** Assume that $W$ is a submodule of $V$ and that $T$ is a weak complement of $W$ in $V$. Assume also that $V \setminus T$ is closed under addition. Then $W$ lacks zero sums (and so $T$ is the unique weak complement of $W$ in $V$).

**Proof.** Let $w_1, w_2 \in W \setminus \{0_V\}$. Then $w_i \notin T$ for $i = 1, 2$, implying $w_1 + w_2 \notin T$. Thus certainly $w_1 + w_2 \neq 0_V$. □

**Lemma 4.2.** The following conditions are equivalent for $W \subset V$:

(i) $W$ is SA in $V$;

(ii) The set-theoretic complement $W'$ of $W$ is an additive (monoid) ideal, in the sense that $w + v \in W'$ for all $w \in W'$, $v \in V$;

(iii) If $\sum_{i=1}^m a_i \in W$, then each $a_i \in W$.

**Proof.** (i) $\Leftrightarrow$ (ii) is clear, and (i) $\Rightarrow$ (iii) by induction on $m$. (iii) $\Rightarrow$ (i) is immediate. □

We pass to the case of $R$-modules (which is not much of a transition, since an additive monoid is an $\mathbb{N}_0$-module).

**Lemma 4.3.** Assume that $\varphi : V_1 \to V_2$ is a homomorphism of $R$-modules over an arbitrary semiring $R$. If $T$ is a SA-submodule of $V_2$, then $\varphi^{-1}(T)$ is a SA-submodule of $V_1$.

**Proof.** Let $x, y \in V$, and assume $x + y \in \varphi^{-1}(T)$. Then $\varphi(x) + \varphi(y) = \varphi(x + y) \in T$, and so $\varphi(x), \varphi(y) \in T$, whence $x, y \in \varphi^{-1}(T)$. □

**Remark 4.4.**

a) If $(U_i \mid i \in I)$ is a family of SA submodules of an $R$-module $V$, then the intersection $\bigcap_{i \in I} U_i$ clearly also is SA in $V$. 

b) If \((U_i \mid i \in I)\) is an upward directed family of SA submodules of \(V\), i.e. for any \(i, j \in I\) there exists \(k \in I\) with \(U_i \subseteq U_k\), \(U_j \subseteq U_k\), then the union \(\bigcup_{i \in I} U_i\) is a SA submodule of \(V\).

Note in Definition 4.7 that for \(R = \mathbb{N}_0\), Lemma 4.2(ii) implies that any SA submodule \(T\) is a weak complement of \(T' \cup \{0_V\}\).

**Theorem 4.5.** Assume that \(V = W + T\), where \(T\) is SA in \(V\), and \(W \cap T = \{0_V\}\).

(i) Then \(T\) is the unique weak complement of \(W\) in \(V\).

(ii) If in addition \(U\) is a submodule of \(V\) with \(W + U = V\) and also \(U\) is SA in \(V\), then \(T \subseteq U\), and \(T\) is the unique weak complement of \(W \cap U\) in \(U\).

**Proof.** (i): Let \(w \in W \setminus \{0_V\}\) and \(t \in T\). Suppose that \(w + t \in T\). Then \(w \in T\), contradicting \(W \cap T = \{0_V\}\). Thus \((w + T) \cap T = \emptyset\).

(ii): \(V \setminus T\) is closed under addition. By Proposition 4.1 we know that \(W\) lacks zero sums and thus \(T \subseteq U\). Since \(U\) is SA in \(V\) we conclude from \(V = W + T\) that \(U = (W \cap U) + T\). By part (i), \(T\) is the unique weak complement of \(W \cap U\) in \(U\), since \(T\) is SA in \(U\). \(\square\)

Here is one nice kind of weak complement.

**Definition 4.6.** Let \(W\) and \(T\) be \(R\)-submodules of \(V\). \(T\) is a **semidirect complement** of \(W\) in \(V\) if \(W + T = V\) and

\[
\forall w_1, w_2 \in W: \quad w_1 \neq w_2 \implies (w_1 + T) \cap (w_2 + T) = \emptyset.
\]

In this case, we also say that \(V\) is the **semidirect sum** of \(W\) and \(T\) and write \(V = W \ltimes T\).

Condition (4.1) can be recast as follows: For any \(w_1, w_2 \in W\), \(t_1, t_2 \in T\),

\[
w_1 + t_1 = w_2 + t_2 \implies w_1 = w_2.
\]

This means that there exists an \(R\)-linear projection \(p: V \to V\) given by \(p(w + t) = w\), with image \(p(V) = W\) and kernel \(p^{-1}(0_V) = T\). We sometimes write

\[
p = \pi_{W,T}.
\]

In summary, we have the following hierarchy of conditions on modules \(T, W\) satisfying \(W + T = V\), each implying the next, which are all equivalent in classical module theory over a ring:

(i) \(T\) is a direct complement of \(W\);

(ii) \(T\) is a semidirect complement of \(W\);

(iii) \(T\) is a weak complement of \(W\);

(iv) \(W \cap T = \{0_V\}\).

Here the reverse implications may fail. We now address “transitivity” of these various complements.

**Question 4.7.** Assume that \(W, S, T\) are submodules of an \(R\)-module \(V\) such that \(S\) is a complement of \(W\) in \(U := W + S\) of a certain type (direct, semidirect, weak) and \(T\) is a complement of \(U\) in \(V\) of the respective type. Then is \(S + T\) a complement of \(W\) in \(V\), of this respective type?

This is obviously true for direct complements. It also holds for semidirect complements. More explicitly, we have the following facts.
Proposition 4.8. If \( U := W + S = W \ltimes S \) and \( V = U \ltimes T \), then
\[
S + T = S \ltimes T, \quad (4.3)
\]
\[
W + T = W \ltimes T \quad (4.4)
\]
\[
V = W \ltimes (S \ltimes T) = (W \ltimes S) \ltimes T. \quad (4.5)
\]

We give two proofs of these facts, having different flavors.

First proof. Here we use the definition of semidirect complements given in (1.1). Let \( w_1, w_2 \in W \) and \( w_1 \neq w_2 \). Then \( (w_1 + S) \cap (w_2 + S) = \emptyset \) and so \( w_1 + s_1 \neq w_2 + s_2 \) for any \( s_1, s_2 \in S \). Since \( T \) is a semidirect complement of \( W + S \) in \( V \) we have in turn
\[
(w_1 + s_1 + T) \cap (w_2 + s_2 + T) = \emptyset.
\]
This proves that
\[
(w_1 + S + T) \cap (w_2 + S + T) = \emptyset, \quad (4.6)
\]
and it follows that
\[
(w_1 + T) \cap (w_2 + T) = \emptyset.
\]

If \( s_1 \neq s_2 \) in \( S \) then, since \( s_1 \) and \( s_2 \) are different elements of \( W + S \), we also conclude from (1.6) that \( (s_1 + T) \cap (s_2 + T) = \emptyset \).

Second proof. We employ the projections associated to semidirect decompositions, cf. (1.2), identifying any projection \( p : X \to X \) onto an \( R \)-module \( X \) with the induced surjection \( X \to p(X) \). We have projections \( p := \pi_{U,T} : V \to U \) and \( q := \pi_{W,S} : U \to W \) with respective kernels \( T \) and \( S \). Then \( r := q \circ p : V \to W \) is a projection with kernel \( S + T \), yielding \( V := W \ltimes (S + T) \). The projection \( r : V \to W \) restricts to maps \( r|(S + T) \to S \) and \( r|(W + T) \to T \), which both are projections with kernel \( T \). Thus \( S + T = S \ltimes T \) and \( W + T = W \ltimes T \).

For weak complements we cannot expect a transitivity statement such as (4.5) above. But a “mixed transitivity” holds for weak and semidirect complements.

Proposition 4.9. Let \( W, S, T \) be submodules of an \( R \)-module \( V \), and assume that \( S \) is a weak complement of \( W \) in \( U := W + S \), while \( T \) is a semidirect complement of \( U \) in \( V \). Then \( S + T \) is a weak complement of \( W \) in \( V \).

Proof. Let \( w \in W \setminus \{0_V\} \) and \( s_1, s_2 \in S \). Then \( (w + S) \cap S = \emptyset \). Thus \( w + s_1 \) and \( s_2 \) are different elements of \( W + S = U \), which implies that \( (w + s_1 + T) \cap (s_2 + T) = \emptyset \). This proves that \( (w + S + T) \cap (S + T) = \emptyset \), as desired.

We finally mention a result of independent interest, which can be obtained by a slight amplification of the proof of Proposition 4.8.

Proposition 4.10. Assume that \( W, T, U \) are submodules of an \( R \)-module \( V \) with \( W + T \subset W + U \) and \( W \cap T \subset W \cap U \). Assume furthermore that \( W \) lacks zero sums, and \( T \) is SA in \( V \). Then \( T \subset U \).

Proof. Let \( t \in T \) be given. We write \( t = w + u \) with \( w \in W, u \in U \). Since \( T \) is SA in \( V \), this implies that \( w \in T \), whence \( w \in W \cap T \subset W \cap U \). We conclude that \( t = w + u \in U \). \( \square \)
5. The obstruction to the “upper bound” condition

Recall from [12] that an additive monoid \((V,+,0_V)\) is upper bound if \(x+y+z=x\) implies \(x+y=x\). This property instantly implies “lacking zero sums.” The object of this section is to study the obstruction to this condition.

**Definition 5.1.** Define Green’s partial preorder on a monoid \((V,+,0_V)\) by saying
\[
x \preceq y \text{ if } x+z=y \text{ for some } z \in V.
\]
We write \(x \equiv y\) if \(x \preceq y\) and \(y \preceq x\).

Clearly \(\preceq\) is reflexive and transitive, implying that \(\equiv\) is an equivalence relation; in fact, \(\equiv\) is a congruence, since if \(x \preceq y\) then \(x+a \preceq y+a\) for any \(a \in V\). (Indeed, if \(x+z=x\), then \(x+a+z=y+a\).) Accordingly, \(\overline{V}:=V/\equiv\) also is a monoid, with the induced operation \(\overline{x}+\overline{y}=\overline{x+y}\), where \(\overline{x}\) denotes the equivalence class of \(x\). \(\preceq\) induces a partial order \(\leq\) on \(\overline{V}\), given by
\[
\overline{x} \leq \overline{y} \text{ if } x \preceq y.
\]

**Lemma 5.2.** The monoid \(\overline{V}\) is upper bound.

**Proof.** Suppose \(\overline{x}+\overline{y}+\overline{z}=\overline{x}\). Then \(x+y+z+z'=x\) for some \(z'\), implying \(x+y \leq x\). But clearly \(x \leq x+y\), so \(x+y \equiv x\), and \(\overline{x}+\overline{y}=\overline{x}\).

This construction respects other topological notions.

**Definition 5.3.** A subset \(S \subset V\) is convex if, for any \(s_i \in S\) and \(v \in V\), \(s_1 \preceq v \preceq s_2\) implies \(v \in S\).

**Lemma 5.4.** The convex hull of a point \(s \in S\) is its equivalence class in \(\overline{V}\).

**Proof.** (\(\subseteq\)) If \(s+y+z=s\), then \(s \preceq s+y \preceq s\), implying \(s \equiv s+y\).

(\(\supseteq\)) If \(s \equiv s+y\), then \(s+y+z=s\) for some \(z\), implying \(s \preceq s+y \preceq s\).

**Proposition 5.5.** \(S\) is convex in \(V\) iff \(\overline{S}\) is convex in \(\overline{V}\) and \(S\) is a union of equivalence classes.

**Proof.** Take the convex hull and apply Lemma 5.3.

This also ties in with the SA property.

**Lemma 5.6.** A subset \(S\) containing \(0_V\) is convex in \(V\), iff \(S\) is SA.

**Proof.** (\(\Rightarrow\)) If \(a+b \in S\) then \(0_V \preceq a \preceq a+b\) implies \(a \in S\), and likewise \(b \in S\).

(\(\Leftarrow\)) If \(s_1 \preceq a \preceq s_2\), then writing \(s_2=a+z_2\), we have \(a \in S\).

**Proposition 5.7.** A submodule \(S \subset V\) is SA in \(V\), iff \(S\) is a union of equivalence classes and \(\overline{S}\) is SA in \(\overline{V}\).

**Proof.** This follows from Proposition 5.5 and Lemma 5.6 applied to \(S\) and \(\overline{S}\).

Some concluding observations:

**Remark 5.8.**

(i) If \(R\) is a semiring, then (taking \(V=R\)) the equivalence \(\equiv\) also respects multiplication, so \(R/\equiv\) is an ab semiring.

(ii) If \(V\) is an \(R\)-module, then \(\overline{V}\) is an \(\overline{R}\)-module, where scalar multiplication is given by \(\overline{a}v=\overline{av}\).

(iii) Any decomposition \(V=W_1 \oplus W_2\) induces a decomposition \(\overline{V}=\overline{W_1} \oplus \overline{W_2}\).
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