High excursion probabilities for Gaussian fields on smooth manifolds.

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Abstract: Gaussian random fields on finite dimensional smooth manifolds whose variances reach their maximum values at smooth submanifolds are considered. Exact asymptotic behaviors of large excursion probabilities have been evaluated. Vector Gaussian processes, chi-square processes, Bessel process, fractional Bessel process, Bessel bridge are examples of application of this result.

Keywords: Non-stationary random field; Gaussian vector process; Gaussian field; large excursion; Pickands’ method; Double sum method.

1 Introduction.

This work is a continuation of papers [5], [6] and [4], where asymptotic behavior of large excursions probabilities were evaluated for Gaussian nonhomogeneous fields given on Euclidean space. An essential condition in these and many previous works is that the variance of a considered Gaussian field reaches its maximum at unique point in the considered parameter set. In the mentioned three papers, general conditions where given on behavior the variance near its maximum point. It has been shown there that whereas introduced first in 1978, [11], regular behavior of the correlation function near the maximum variance point of the variance is crucial for all known general tools for the asymptotic behavior evaluation, introduced there behavior of the variance near this point can be much more general.

From the other hand, in many situations one have to consider Gaussian fields with a set of maximum variance points. A typical situation is the large excursions probability evaluation for norms of Gaussian vector processes, when one uses the duality and passes to Gaussian fields on a cylinder, and theirs variance can reach theirs maximum at a submanifold of the cylinder. Well known examples are $\chi^2$ process and its generalizations, see [9] for examples and references, and Bessel process and bridge, with generalizations to corresponding fracttion processes, [13]. These examples motivated the present work.

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1.1 General setting

Let $X(t), t \in S \subset \mathbb{R}^n$, be a zero mean a.s. continuous Gaussian random field given on a $d$-dimensional closed differentiable manifold $S$, $d < n$. Denote the covariance function of $X$ by $R(s, t) = \mathbb{E}X(s)X(t)$, and by $\sigma^2(t) = R(t, t)$, the variance function of $X$. Assume the following,

**Condition 1** Maximum points set of $\sigma(t), t \in S$,

$$\mathcal{M} := \text{argmax}_{t \in S} \sigma(t)$$

is a differentiable finitely connected submanifold of $S$ of dimension $r < d$. In particular, $r = 0$ means that $\mathcal{M}$ consists of finite number points on $S$.

**Remark 1** The case $r = 0$ is considered in [2], see also [8], so that we assume further that $r \geq 1$. The case $r = d$, in particular $\sigma(t)$ is a constant on $S$, is called a locally homogeneous (stationary) case, provided the below Condition 3 is fulfilled. We shall consider this case as well.

Without loss of generality assume that $\sigma(t) \equiv 1, t \in \mathcal{M}$.

We study the asymptotic behavior of the probability

$$P(S; u) := P(\max_{t \in S} X(t) > u)$$

as $u \to \infty$. Remark immediately, that we shall use analogous notation, $P(A; u)$, for any $A$, meaning closure of $A$ if it is not close. Remind that $S$ is bounded.

1.2 Examples. $\chi^2$ processes. Bessel process.

Let $X(t) = (X_1(t), ..., X_d(t)), t \in [0, T]$, be a Gaussian vector process, $|| \cdot ||$ be a norm in some $d$-dimensional linear space $\mathbb{L}^d$, generated by a scalar product $\langle \cdot, \cdot \rangle$. Denote by $\mathbb{M}^d$ the corresponding dual space. By duality,

$$P(\max_{t \in [0, T]} ||X(t)|| > u) = P(\max_{(t, \mathbf{v}) \in T \times S_d} \langle \mathbf{v}, X(t) \rangle > u),$$

with $S_d$, the unit sphere in $\mathbb{M}^d$.

**Example.** $\chi^2$ processes. Let $X_i(t), t \in [0, T], i = 1, ..., d$, be independent Gaussian zero mean processes $EX_i^2(t) = \sigma_i^2(t)$, consider generalized $\chi^2$ process,

$$\chi^2_b(t) = \sum_{j=1}^{d} b_j^2 X_j^2(t), t \in [0, T].$$

If all $b_j$ are equal, we have usual $\chi^2$ process, the corresponding field $\langle \mathbf{v}, X(t) \rangle$ is homogeneous with respect to turns and shifts, if not, the variance of the field is equal to

$$\sigma^2(t, b, \mathbf{v}) = \sum_{j=1}^{d} \sigma_i^2(t)b_j^2v_j^2, b = (b_1, ..., b_d),$$

with various structures of the set of maximum points.
For example, in case all $\sigma^2_i(t)$ are equal to $\sigma^2(t)$, with unique maximum point, and $b_1 > b_j, j > 1$, the set $\mathcal{M}$ consists of two points, $\max \sigma^2(t, b, v) = b_1^2$; in case $\sigma^2(t) \equiv 1$ with the same $b_j$s, we have $\mathcal{M} = S_d$, [12].

**Example. Bessel process.** For $X_i(t) = W_i(t), t \in [0,1]$, i.i.d. Brownian motions, $\beta(t) = ||X(t)||$ is Bessel process. We have,

$$P(\max_{[0,1]} \beta(t) > u) = P(\max_{(t,v) \in T \times S_d} \sum_{i=1}^{d} v_i W_i(t) > u).$$

Here $\mathcal{M} = \{1\} \times S_d$. Using behavior of the covariance function of $X(t)$ near this set, it is shown in [13] that

$$P(\max_{[0,1]} \beta(t) > u) = \frac{\pi^{(d-1)/2}}{2^{d/2-1} \Gamma(d/2)} u^{d-2} e^{-u^2/2}(1 + o(1)),$$

as $u \to \infty$. Also Bessel bridge with zero end point, fractional Bessel processes and bridges have been similarly considered in [13], definitions are clear.

### 1.3 Tools. Pickands’ Double Sum Method.

An idea how to consider non-smooth stationary Gaussian processes belong to J. Pickands III, 1969. He first considered the probability on infinitely small intervals and then passed to arbitrary interval, using semi-additivity of the probability with Bonferroni inequalities. It is why it is called Double Sun Method. A generalization to non-stationary processes has been done in [11]. Let $X(t), t \in [0,1]$ be a zero mean Gaussian process with correlation function $r(s,t)$ and variance $\sigma^2(t)$, which has a unique maximum point $t_0 \in (0,1)$. Assume for some positive $a, \beta$ and $\alpha \leq 2$ that

$$\sigma(t) = 1 - a|t - t_0|^\beta(1 + o(1)), \quad t \to t_0;$$

$$r(s,t) = 1 - |t - s|^\alpha(1 + o(1)), \quad s \to t_0, \quad t \to t_0;$$

and for some positive $\gamma, G$ and all $s, t$,

$$E(X(t) - X(s))^2 \leq G|t - s|^{\gamma}.$$

Denote

$$\Psi(u) = \frac{1}{\sqrt{2\pi u}} e^{-u^2/2},$$

the asymptotic of the tail of Gaussian standard distribution. In the above conditions, for some constants $C_1 > 0$ and $C_2 > 1$, see Lemma [2] below, we have as $u \to \infty$ the following.

If $\beta > \alpha$ (stationary like case),

$$P([0,1], u) = C_1 u^{2/\alpha - 2/\beta} \Psi(u)(1 + o(1)).$$

If $\beta = \alpha$ (transition case),

$$P([0,1], u) = C_2 \Psi(u)(1 + o(1)).$$

If $\beta < \alpha$ (Talagrand case),

$$P([0,1], u) = \Psi(u)(1 + o(1)).$$
From the proof of assertions below it will be clear some physical sense of these asymptotic relations.

In the stationary like case, since the correlation is sharper at $t_0$ than variance, trajectories oscillate near $t_0$, do not paying much attention on slow behavior of the variance, therefore we have a degree of $u$ before $\Psi$.

In the Talagrand case, conversely, the variance does not pay any attention on the slow trajectories oscillations, M. Talagrand, [14], showed this relation in the maximal generality.

In transition case, the behaviors of variance and correlation are equal one to another, so the order of asymptotic behavior is the same, but with constant greater than one.

It turned out, that, while power behavior of the correlation is critical for Pickands’ method, behavior of the variance can be much more general. We shall see, that the only important point is whether the variance behavior is slower or faster, or equivalent the behavior of correlation. In case of processes, it was described in [4]. Then in [5], with corrections in [6], this have been generalized on Gaussian fields with single maximum variance point.

The present work is an immediate continuation of these works, several assertions here have almost the same proofs, therefore they are mainly omitted. The only constructions are given in details which are significantly different in case $\dim M > 0$ than the corresponding ones in case $\dim M = 0$.

2 Extracting of an informative set.

Our first aim is to extract an informative small neighbourhood of $M$ in $S$ such that the probability (1) is equivalent to the same probability but with this neighbourhood instead of $S$.

Denote by $M_\varepsilon \subset S$, an $\varepsilon$-neighborhood of $M$ in $S \subset \mathbb{R}^n$, that is,

$$M_\varepsilon := \left\{ s : \min_{t \in M} ||t - s|| < \varepsilon \right\}.$$  

(2)

It is easy to show, using Borell-Tsirelson-Ibragimov-Sudakov (Borell-TIS) inequality, that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$P(S; u) = P(\max_{t \in M_\varepsilon} X(t) > u) (1 + O(e^{-\delta u^2}), \ u \to \infty,$$  

(3)

see, for example, Theorem D.1, [9].

Further, we need in proofs a more narrow neighbourhood of $M$. To this end, assume a slightly stronger condition than a.s. continuity of sample paths.

**Condition 2** There exists $\varepsilon > 0$ such that Dudley’s entropy integral with respect to pseudo-semi-metrics generated by the standardized field $\bar{X}(t) = X(t)/\sigma(t)$, $t \in M_\varepsilon$, is finite. For definitions see [3], [10], [9].

Notice that for homogeneous Gaussian fields this condition is also necessary for existing of a.s. continuous version of the field (X. Fernique [7]).

From this condition it follows V. A. Dmitrovsky’s inequality, [1], [2], [10], which is more accurate than Borell-TIS one. Using this inequality, it is proved in [5] the following Lemma.

**Lemma 1** In the above conditions and notations, there exists $\gamma(u)$ with $\gamma(u) \to 0$ as $u \to \infty$, such that

$$P(S; u) = P(M_u; u) \left( 1 + O\left(e^{\log^2 u}\right) \right).$$  

(4)
as $u \to \infty$, where

$$M_u := M_{\varepsilon(u)} \text{ with } \varepsilon(u) = 2u^{-1}\gamma(u) + 2u^{-2}\log^2 u.$$  \hfill (5)

Thus we are in a position to study the asymptotic behavior of the latter probability.

### 3 Exact asymptotic behavior.

Denote

$$r(s_1, s_2) = \frac{R(s_1, s_2)}{\sigma(s_1)\sigma(s_2)},$$

the correlation function of $X$.

**Condition 3** *(Local homogeneity)*. There exists $\varepsilon > 0$ such that for any $t \in M_\varepsilon$ there exists a correlation function $r_t(s)$, $s \in \mathbb{R}^n$ of a homogeneous random field with $r_t(s) < 1$ for all $s \neq 0$ and such that

$$\lim_{s_1, s_2 \to t} \frac{1 - r(s_1, s_2)}{1 - r_t(s_2 - s_1)} = 1.$$  \hfill (6)

The atlas on $M_\varepsilon$. Remind some definitions related to the smooth manifold $M_\varepsilon$, see for example [15]. Any $t \in M$ has a neighbourhood $U \subset S$ which is homeomorphic to $\mathbb{R}^d$, and the corresponding homeomorphism $\varphi : \mathbb{R}^d \to U$ is differentiable. We call the set $U = \varphi^r(U)$, the map of $U$. Since $M_\varepsilon$ is closed, there exists a finite collection of such neighbourhoods which cover whole $M$ and such that for all sufficiently small $\varepsilon$ the union of them cover whole $M_\varepsilon$. Such the collection together with corresponding maps forms an atlas on $M_\varepsilon$, which we fix for future considerations.

**Agreement on coordinates.** Now consider $U$ from the fixed atlas, with corresponding $\varphi$, and choose the coordinate system in $\mathbb{R}^n$ such that in the coordinates of $s = (s_i, i = 1, \ldots, n) \in \mathbb{R}^n$,

$$U := \varphi^r(U) \subset \mathbb{R}^d := \{s \in \mathbb{R}^n : s_{d+1} = \ldots = s_n = 0\}$$

and

$$M := \varphi^r(U \cap M) \subset \mathbb{R}^r := \{s \in \mathbb{R}^n : s_{r+1} = \ldots = s_n = 0\}.$$  \hfill (6)

From now on we use these coordinates and write $s = (s_1, \ldots, s_d) \in U$. For such $s$ denote

$$s^1 = (s_1, \ldots, s_r, 0, \ldots, 0) \in M,$$  \hfill (7)

the projection of $s$ on $M$, and

$$s^2 = (0, \ldots, 0, s_{r+1}, \ldots, s_d),$$  \hfill (8)

so that $s = s^1 + s^2$, the orthogonal decomposition of $s$. We will write sometimes $s = (s^1, s^2)$.

Thus for any map from the fixed atlas,

$$P(U; u) = P(\max_{s \in U} X(s) > u) = P(U; u),$$  \hfill (9)

where $s \in \mathbb{R}^d$ is scripted in the selected coordinates. As well, with (3, 4) we have in mind the same relation $s$ with $M_\varepsilon$ and $M_u$ instead of $M_\varepsilon$ and $M_u$, correspondingly.
Notice immediately, that the notation $dt$ is used in integrals below for volume elements in $U$, $\mathcal{U}$ and theirs subsets, do not depending of dimensions of the integral sets. Transitions from one integral to the corresponding another one perform in standard way.

We find first the asymptotic behavior of probability of $P(M; u)$, and then, using introduced Conditions and the fact that the atlas is finite, pass to the probability (1). Write for any $h$ with continuous vector function $C$, $t \in \mathbb{R}$ set $\{t \in U \cap M, \text{ such that for any } \alpha > 0, \text{ it follows a regular variation property of } r_t(\cdot) \text{ defined in Condition 3} \}$

\begin{equation}
\lim_{u \to \infty} u^2(1 - r_t(C_t q(u)s)) = h(s)
\end{equation}

uniformly in $s$.

Sometimes we shall write for short $q_t(u) = C_t q(u)$ with corresponding components $q_t(u) = C_t q_i(u), i = 1, \ldots, d$.

Remark that by definition of uniform convergence to a positive for all non-zero $s$ function with $h_t(0) = 0$, from Condition 4 it follows that $h_t(s)$ is continuous, and for any continuous function $c_s$ with $\lim_{s \to 0} c_s = 1$,

\begin{equation}
\lim_{s \to 0} \frac{1 - r_t(c_s s)}{1 - r_t(s)} = 1.
\end{equation}

Remind that we assumed that the basis in $\mathbb{R}^d$ satisfies Condition 4. From Conditions 3 and 4 it follows a regular variation property of $r_t(s)$. The following proposition is proved in [5].

**Proposition 1** Let Conditions 3 and 4 be fulfilled for a covariance function $r(t)$. Then for any vector $f$, function $1 - r_t(s f)$ regularly varies in $s$ at zero with degree $\alpha(f) \in (0, 2]$, and

$h(s f) = A_f |s|^{\alpha(f)}, \ A_f > 0$.

The function $q(u)f$ is regularly varying in $u$ at infinity with degree $-2/\alpha(f)$.

It is easy to see that $\alpha(f)$ is equal to one of the $\alpha(e_i), \{e_i, i = 1, \ldots, d\}$ is the basis of $\mathbb{R}^d$. As it is shown in [5], if $\alpha(f) = 2$, then $\lim_{t \to 0} t^{-2}(1 - r(t f)) > 0$, and for some $q_+ > 0$,

\begin{equation}
q(u)f \geq q_- u^{-1}, \ i = 1, \ldots, d.
\end{equation}

Now assume a behavior of $\sigma(t)$ near the points of $M$. We shall see from the proof of Lemma 2 that the crucial point is the behavior of the ratio

\[
\frac{1 - \sigma(t + C_t q(u)s)}{1 - r_t(C_t q(u)s)}
\]

as $u \to \infty$, where $t \in M$ and $s \in M_e$.

In view of Condition 4 we assume the following.
**Condition 5** For any \( t \in \mathcal{M} \) and all \( s \in \mathcal{M}_\varepsilon \) there exists in the corresponding \((M, M_\varepsilon)\) the limit
\[
h_{1t}(s) := \lim_{u \to \infty} u^2 (1 - \sigma(t + C_t q(u)) s) \in [0, \infty].
\] (13)

As in Subsection 1.3, in case when the limit is equal to zero we speak about the *stationary like case*. If the limit is equal to infinity, we refer to the *Talagrand case*, since M. Talagrand, [14], see comments in [3] and [9]. At last, we say about the *transition case* if \( h_{1t}(s) \) is neither zero nor infinity.

Denote correspondingly for any \( t \in M \),
\[
K_{0t} := \{ s \in M_\varepsilon \setminus M : h_{1t}(s) = 0 \}, \quad K_{ct} := \{ s \in M_\varepsilon \setminus M : h_{1t}(s) \in (0, \infty) \},
\]
\[
K_{\infty t} := \{ s \in M_\varepsilon \setminus M : h_{1t}(s) = \infty \}. \tag{14}
\]

We shall see that properties of union in \( t \) of these sets together with all above conditions follow asymptotic behavior of the probability \( P(S; u) \).

**Pickands’ Lemma.** Introduce a Gaussian a.s. continuous field \( \chi(s) \) with \( \chi(0) = 0 \), and
\[
\text{var}(\chi(s_1) - \chi(s_2)) = 2h(s_1 - s_2), \quad E\chi(s) = -h(s).
\]
The existence of such the field follows from Condition [14] and the proof of Lemma below which, in fact, is a generalization of Lemma 6.1, [9]. For any \( T \subset \mathbb{R}^d \) and any \( t \in M \) introduce the Pickands’ type constant,
\[
P_{q,t}(T) = E \exp(\max_{s \in T \cap U} \chi(s) - h_{1t}(s)). \tag{15}
\]

**Lemma 2** In the above notations and conditions, for any bounded closed set \( T \subset \mathbb{R}^d \) and any point \( t \in U \cap M, \ t \notin \partial U \)
\[
P(\max_{s \in t + q_t(u)T} X(s) > u) = (1 + o(1))P_{q,t}(T)\Psi(u) \tag{16}
\]
as \( u \to \infty \), where \( q_t(u) = C_t q(u) \), as above. If \( t \in \partial U \cap M \) and \( h_{1t}(s) \equiv 0 \) this relation is valid as well. If \( t \in \partial U \cap M \) and \( h_{1t}(s) \neq 0 \) for some \( s \), the set \( T \) on the right will be correspondingly truncated, denote it by \( T_t \).

**Outline of proof.** The proof mainly follows the proof of Lemma 6.1, [9]. Fix \( \varepsilon \) in (2). Observe that since \( M_\varepsilon \) is open, \( t + q_t(u)T \subset M_\varepsilon \) for all sufficiently large \( u \). Further proof can be proceed in \( \mathbb{R}^d \) similarly to the Lemma 6.1 proof with obvious adition of the case \( t \in M \cap \partial U \). We repeat here initial evaluations which are based on the conditions here. We have,
\[
P(\max_{s \in t + q_t(u)T} X(s) > u)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-v^2/2} P\left(\max_{s \in t + q_t(u)T} X(s) > u \mid X(t) = v\right) dv
\]
\[
= \frac{1}{\sqrt{2\pi u}} e^{-u^2/2} \int_{-\infty}^\infty e^{u - w^2/2u^2} P\left(\max_{s \in t + q_t(u)T} X(s) > u \mid X(t) = u - \frac{w}{u}\right) dw, \tag{17}
\]
with change \( v = u - w/u \). Introduce the Gaussian process
\[
\chi_{ut}(s) := u(X(t + q_t(u))s - u) + w. \tag{18}
\]
The latter integral can be rewritten as
\[
\int_{-\infty}^{\infty} e^{w-w^2/2u^2} P\left(\max_{s \in \mathcal{T}} \chi_{ut}(s) > w \bigg| X(t) = u - \frac{w}{u}\right) dw.
\]  \hspace{1cm} (19)

Further, by (18), using formulas for conditional mean and variance,
\[
E\left(\chi_{ut}(s) \big| X(t) = u - \frac{w}{u}\right) = -u^2(1 - R(t, t + q_t(u)s)) + w(1 - R(t, t + q_t(u)s)),
\]  \hspace{1cm} (20)

\[
\text{var}\left(\chi_{ut}(s_1) - \chi_{ut}(s_2) \big| X(t) = u - \frac{w}{u}\right)
= u^2(2 \text{var}[X(t + q_t(u)s_1)] - X(t + q_t(u)s_2))
- [R(t, t + q_t(u)s_1) - R(t, t + q_t(u)s_2)]^2.
\]  \hspace{1cm} (21)

Obviously,
\[
E\left(\chi_{ut}(0) \big| X(t) = u - \frac{w}{u}\right) = E\left(\chi_{ut}^2(0) \big| X(t) = u - \frac{w}{u}\right) = 0.
\]

Since \(\sigma(t) = 1\), by Conditions 4 and 5
\[
u^2(1 - R(t, t + q_t(u)s)) = u^2(1 - r(t, t + q_t(u)s)\sigma(t + q_t(u)s))
= u^2(1 - \sigma(t + q_t(u)s)) + u^2(1 - r(t, t + C_t q_t(u)s))\sigma(t + C_t q_t(u)s)
= (h_{1t}(s) + h(s))(1 + o(1)), \text{ as } u \to \infty.
\]  \hspace{1cm} (22)

Further,
\[
\text{var}[X(t + q_t(u)s_1)] - X(t + q_t(u)s_2)] = \sigma^2(t + q_t(u)s_1) + \sigma^2(t + q_t(u)s_2)
- 2\sigma(t + q_t(u)s_1)\sigma(t + q_t(u)s_2)r(t + q_t(u)s_1), t + q_t(u)s_2)
= ((\sigma(t + q_t(u)s_1) - 1) + (1 - \sigma(t + q_t(u)s_2)))^2
+ 2\sigma(t + q_t(u)s_1)\sigma(t + q_t(u)s_2)(1 - r(t + q_t(u)s_1), t + q_t(u)s_2)).
\]

By Condition 5 the first summand on the right is equal to \(O(u^{-4})\) uniformly in \(s_1, s_2\) as \(u \to \infty\). By conditions 3 and 4 the second summand on the right is equal to \(2u^{-2}h(s_2-s_1)(1+o(1))\) uniformly in \(s_1, s_2\) as \(u \to \infty\).

The further proof is just a repetition of the Lemma 6.1 proof, including weak and majorized convergence as \(u \to \infty\) of the process \(\chi_{ut}(\cdot)\) and the integral in (17).

**Remark 2** In the homogeneous like case, \(h_{1t}(s) \equiv 0\), we have,
\[
P_{q_t}(T) =: H_q(T) = E \exp(\max_{s \in \mathcal{T}} \chi(s) - h(s)).
\]

Moreover, from the proof of standard Pickands theorem for homogeneous Gaussian fields, see for example [3] and a slight its extension in [5], it follows that
\[
\lim_{T \to \infty} T^{-d} H_q([0, T]d) =: H_q \in (0, \infty),
\]  \hspace{1cm} (23)

the Pickands’ constant. In Talagrand case, \(h_{1t}(s) \equiv \infty\), that is, \(K_{\infty} \neq \emptyset\)
\[
P_{q_t}(T) = 1.
\]
Remark 3 In fact, by Lemma 1 we need this lemma and the following splitting and other construction only in the set $M_{\varepsilon}$, but for convenience we assume the informative set to be independent of $u$, that is, simply write $M_{\varepsilon}$.

Splitting. In order to apply local Lemma 2 we construct a splitting of $M_{\varepsilon}$ into small sets. First split $M$, and begin with the first axis of $\mathbb{R}^d$, $\{ t = (t_1, 0, ..., 0), t_1 \in \mathbb{R} \} \subset M$.

Denote $T^1 := (T, 0, ..., 0)$ and take for $k_1 = 0, 1, 2, ...$

$$t_0^1 = 0, t_1^1 = q_{t_0^1}(u)T^1, t_2^1 = t_1^1 + q_{t_1^1}(u)T^1, ..., t_{k_1+1}^1 = t_{k_1}^1 + q_{t_k^1}(u)T^1,$$

till $t_{k_1+1}^1$ such that $t_{k_1}^1 \in M$, that is the last point is out of $M$. Similarly backward for $k_1 = 1, 2, ..., till t_{k_1-1}^1$ such that $t_{k_1}^1 \in M$. Further, at any point $t_{k_1}^1 = (t_{k_1}^1, 0, ..., 0), \text{construct a similar grid on the axis } \{(t_{k_1}^1, t_2, ..., 0), t_2 \in \mathbb{R} \} \subset M$. Namely, denote $T^2 := (0, T, 0, ..., 0)$, and put for $k_2 = 0, 1, ...$

$$t_{k,0}^2 = (t_{k,0}^1, 0, ..., 0), ..., t_{k,k_2+1} = (t_{k,1}^1, t_{k,2}^1), 0, ..., 0) = t_{k_1,k_2}^2 + q_{t_k^2}(u)T^2, ...$$

till $t_{k,k_2+1}^2$ such that $t_{k_2}^k \in M$. Similarly backward for $k_2 = 1, 2, ..., till t_{k_1}^2$ such that $t_{k_2}^k \in M$. So on, till $T^r := (0, 0, ..., 0, T, ..., 0)$, with $T$ on $r$th place. Denote by $N_r := \{ t_k^r, k \in \mathbb{Z}^r \}$, the constructed net on $M$. Observe that for any $t_k^r, 2^r$ points $t_{k+\kappa}^r$ with coordinates of $\kappa$ are 0 or 1 form an $r$-dimensional polygon with parallel faces, trapezoid if $r = 2$, denote it by $\Delta_{\kappa}^r$.

Now supplement the constructed splitting $\{ \Delta_{\kappa}^k, k = (k_1, ..., k_r, 0, ..., 0) \in \mathbb{Z}^d \}$ of $M$ with splitting of $M_{\varepsilon}$. For any polygon $\Delta_{\kappa}^k$, consider the hyperspace $t_k^r + \mathbb{R}^{d-r}$ and take in $\mathbb{R}^{d-r}$ a uniform rectangular net with sides all equal to $q_{t_k^r}(u)T_{d-r}$, where $T_{d-r} = (0, 0, ..., 0, T, ..., T) \in \mathbb{R}^d$, with first $r$ zeros. That is, denote $T^d = [0, T]^d$, $T_{d-r} = T^d \cap \mathbb{R}^{d-r}$, and take in $t_k^r + \mathbb{R}^{d-r}$,

$$\Delta_{kl}^{d-r} = t_k^r + q_{t_k^r}(u)T_{d-r} + q_{t_k^r}(u)T_{d-r}, 1 = (0, ..., 0, l_1+1, ..., l_d) \in \mathbb{Z}^d.$$  \hfill (24)

Finaly denote

$$\Delta_{kl}^d := \Delta_{k}^r \times \Delta_{kl}^{d-r}, \quad \Delta_{kl}^d \cap M_{\varepsilon} \neq \emptyset.$$  \hfill (25)

Denoting $t_k^{d-r} := q_{t_k^r}(u)T_{d-r}$, since $T^d = T_d \otimes T_{d-r}$, we can write this as

$$\Delta_{kl}^d = t_k^r + q_{t_k^r}(u)((k + 1)T + T),$$

so that Lemma 2 is applicable with $t = t_k^r$ and $T = (k + 1)T + T$. In case $t_k^r \notin M$ (some end point of the net $N^r$), take $t \in M$, another vertex of the same polygon and let $q_{t_k^r}(u) = q_{t}(u)$.

Introduce the events

$$A_{k,1} := \{ \sup_{t \in \Delta_{kl}^d} X(t) > u \}.$$ 

We have for any $A \subset M_{\varepsilon}$,

$$\bigcup_{A_{k,1} \subset A} A_{k,1} \subseteq \left\{ \max_{s \in A} X(s) > u \right\} \subseteq \bigcup_{A \subset \emptyset} A_{k,1}.$$  \hfill (26)

3.1 Locally homogeneous Gaussian fields.

Denote by $X_0(t)$, a Gaussian a.s. continuous zeromean field with variance 1 and covariation function $r(s_1, s_2)$. That is,

$$X(t) \overset{d}{=} \sigma(t)X_0(t),$$
in distributions. Assuming that Condition \ref{cond} is fulfilled for all $t \in S_1 \subset S$, we can say that the field $\xi(t)$, $t \in S_1$, is \textit{locally homogeneous}. In particular, in Condition \ref{cond}, the Gaussian field $X(t)$, $t \in M \cap U$, is locally homogeneous. The following theorem is formulated for $X_0(t)$, but in fact it is valid for any locally homogeneous Gaussian fields satisfying the above conditions.

**Theorem 1** Let Condition \ref{cond} be fulfilled for all $t \in U$. Then for any $U_1 \subset U$ which is the closure of an open set,

$$P(\max_{t \in U_1} X_0(t) > u) = H_q \int_{U_1} \prod_{i=1}^r C_{t_i}^{-1}(u)dt \prod_{i=1}^r q_i(u)^{-1}\Psi(u)(1 + o(1))$$

(27)

as $u \to \infty$, with

$$H_q = \lim_{T \to \infty} T^{-d}H_q([0, T]^d) \in (0, \infty).$$

This assertion holds even if corresponding $U_1$ depends of $u$, $U_1 = U_1(u)$, provided there exist boxes $U_i^\pm(u) = \bigotimes_{i=1}^d [-U_i^\pm(u), U_i^\pm(u)]$ such that $U_1^-(u) \subset U_1(u) \subset U_1^+(u)$ with $U_i^-(u)q_i(u) \to \infty$, $u \to \infty$, $i = 1, \ldots, d$.

**Remark 4** Remark that the assertion is valid also in case the parametric set $U_1$ expands unboudedly, with some restrictions on the expanding, it follows from the proof. See corresponding conditions in \cite{5}.

**Outline of proof.** Introduce a positive increasing function $\kappa(u)$, $u > 0$, with

$$\lim_{u \to \infty} \kappa(u) = \infty, \quad \text{but} \quad \lim_{u \to \infty} q_i(u)\kappa(u) = 0, \quad i = 1, \ldots, d.$$  

Cover the set $U_1$ with boxes

$$B_k(u) = \kappa(u)kq(u) + B_0(u), \quad k \in \mathbb{Z}^d, u > 0,$$

where

$$B_0 := \kappa(u) \bigotimes_{i=1}^d [0, q_i(u)].$$

Denote

$$C_k^\pm := (C_{ik}^\pm, i = 1, \ldots, d),$$

with

$$C_{ik}^+ := \max_{t \in B_k(u)} C_{it}, \quad C_{ik}^- := \min_{t \in B_k(u)} C_{it}, \quad i = 1, \ldots, d.$$  

Fix $k$, and consider Gaussian zeromean homogeneous fields $\xi^\pm_k(t)$, $t \in B_k(u)$, with covariance functions $\rho^\pm_k(t)$ satisfying Condition \ref{cond} with $C_k^\pm$ instead of $C_k$, correspondingly. Applying the constructed above splitting of $B_k(u)$ with polygons $\Delta^d_i$ and using Lemma \ref{lem} we get similarly to the Theorem 7.1, \cite{9}, proof, the asymptotic behavior of

$$P(\max_{t \in B_k(u)} \xi^\pm_k(t) > u).$$

Then, using Slepian inequality to bound the probability for $X_0(t)$, $t \in B_k(u)$, and applying again Double Sum Method for the probability in question, we get Theorem. See quite similar proofs in \cite{5}, \cite{4}, as well as in \cite{9}, \cite{10}.
Remark that since (9), it is easier to perform the proof for the corresponding $U_1 = \varphi^{-1}(U_1)$, which leads to the integral over $U_1$ with the functions $C_t$ and $q_t(u)$ constructed for $U = \varphi^{-1}(U_1)$ and the fixed at the agreement coordinate axes. Finally we simply pass to the integral over $M_1$, with another, of course, volume element $dt$ (changing variables $t \to \varphi(t)$), but leaving the same $q(u)$ constructed in Condition $\mathbb{I}$.

We need also the following corollary of Theorem 1.

**Proposition 2** In the assumptions of this section, for any $M_1 \subset U \cap M$ which is the closure of an open set,

$$P(\max_{t \in M_1} X(t) > u) = H_{q,r} \int_{M_1} \prod_{i=1}^{r} C^{-1}_t(u) dt \prod_{i=1}^{r} q_t(u)^{-1} \Psi(u)(1 + o(1))$$

(28)

as $u \to \infty$, with

$$H_{q,r} = \lim_{T \to \infty} T^{-r} H_q([0, T]^r \times [0, 0]^{d-r}) \in (0, \infty).$$

This assertion holds even if corresponding $M_1$ depends of $u$, $M_1 = M_1(u)$, like in the Theorem $\mathbb{I}$.

### 3.2 Homogeneous like Gaussian fields.

Consider the homogeneous like case, that is $h_{1t}(s) = 0$ for all $s \in M_\varepsilon$ and $t \in M$. Take $\varepsilon > 0$ sufficiently small, and denote

$$f(s) = \frac{1}{2}(1 - \sigma^2(s)), \quad s \in M_\varepsilon.$$  

(29)

Introduce the Laplace type integral,

$$L_f(\lambda) := \int_{U} e^{-\lambda f(s)} ds, \quad \lambda > 0.$$  

(30)

Remark that by standard asymptotic Laplace method, its asymptotic behavior as $\lambda \to \infty$ depends only on behavior $f(t)$ at zero, that is, as $\rho(s, M) \to 0$.

**Proposition 3** Let Conditions $\mathbb{I}$ be fulfilled. If further $h_1(t) = 0$ for all $t$, that is, $\cup_{t \in M} K_0t = U$, we have for any map $U$, from the fixed atlas on $M_\varepsilon$,

$$P(U; u) = (1 + o(1)) H_q \int_{M_\varepsilon \cap U} \prod_{i=1}^{d} C^{-1}_t(u) dt L_f(u^2) \prod_{i=1}^{d} q_t^{-1}(u) \Psi(u)$$

(31)

as $u \to \infty$.

Proof of this proposition is based on the same constructions as in Theorem $\mathbb{I}$ proof. The only for the approximation of $P(B_\rho(u); u)$ the probabilities $P(\max_{t \in B_\rho(u)} \sigma^±_k X_0(t) > u)$ are used directly, with $\sigma^±_k$, upper and bound bounds of $\sigma(t)$, $t \in B_k(u)$, with followed Theorem $\mathbb{I}$ application and standard limit passage to integral as $u \to \infty$. Very similar detailed proof in case dim $M = 0$ is given in $\mathbb{I}$, Proposition 3, which combined both above proofs for locally homogeneous and homogeneous like cases. Notice that the proof is performed for the manifold $M_u$, $u \to \infty$ with following application Lemma $\mathbb{I}$. Most technical detailes here are related to double sums estimations in corresponding Bonferroni inequalities.
3.3 Talagrand case.

Proposition 4. In Conditions \([1, 2]\) if \(\cup_{t \in M} K_{\infty} = M\), then for any map \(U\) from the fixed atlas on \(M\),

\[
P(U; u) = H_{q, r} \int_{M \setminus \partial U} \prod_{i=1}^{r} C_{i}^{-1}(u) dt \prod_{i=1}^{r} q_i(u)^{-1} \Psi(u)(1 + o(1)),
\]

(32)
as \(u \to \infty\). The constant \(H_{q, r}\) is defined in \([28]\).

In other words, the asymptotical behavior of the probability in question is coincides with the asymptotic behavior of \(P(M; u)\), compare with \([28]\). Notice that it is proved in \([5]\) that in case \(M\) consists of one point the right part is equal simply to \(\Psi(u)(1 + o(1))\). Therefore it follows from Lemma \([1]\) and trivial application of Bonferroni inequalities that in case \(\dim M = 0\) since \(M\) is finitely connected, the probability is equivalent to \(\text{card}(M) \Psi(u), u \to \infty\). See Remark \([1]\).

Outline of proof. In this case, the informative set \(M_u\) is contained in the two closest to \(M\) layers of the constructed above splitting and moreover of any small thickness. Namely, take the same splitting \(\{\Delta^r_k: k = (k_1, \ldots, k_r, 0, ..., 0) \in \mathbb{Z}^d\}\) of \(M\), but now denote \(\epsilon_{d-r} = (0, ..., 0, \epsilon, ..., \epsilon) \in \mathbb{R}^d, \epsilon > 0\), with first \(r\) zeros. Similarly the above, denote \(E^d = [-\epsilon, \epsilon]^d\), \(E^{d-r} = E^d \cap \mathbb{R}^{d-r}\), and take in \(t^r_k + \mathbb{R}^{d-r}\),

\[
\Delta^{d-r}_{k0} = t^r_k + q_{t^r_k}(u)E^{d-r},
\]

compare with \([24]\). Finally denote again

\[
\Delta^{d}_{k0} := \Delta^r_k \times \Delta^{d-r}_{k0}, \quad \Delta^{d}_{k0} \cap M_u \neq \emptyset,
\]

and

\[
A_{k,0} := \{ \sup_{t \in \Delta^r_k} X(t) > u \}, A^r_k := \{ \sup_{t \in \Delta^r_k} X(t) > u \}.
\]

We have in Talagrand case, that for any \(A \subset U \cap M_u\), any small \(\epsilon > 0\), and all sufficiently large \(u\),

\[
\bigcup_{\Delta^r_k \subset M \cap A} A^r_k \subseteq \left\{ \max_{s \in A} X(s) > u \right\} \subseteq \bigcup_{\Delta^{d}_{k0} \cap A \neq \emptyset} A_{k,0}.
\]

Asymptotic behavior of the left side event probability have been already computed, Proposition \([2]\). The right hand probability is bounded by the sum of probabilities of \(A_{k,0}\) with following Lemma \([2]\) application and then taking \(\epsilon\) arbitrarily small. This part is absolutely similar to the corresponding part of Theorem 8.2, \([9]\) (Lemma 8.4), see also \([10]\), Lecture 10, and \([3]\).

3.4 Transition case.

In this case we have to consider separately two parts of \(M\), the first one belongs to \(\partial U\), and the second one located inside of \(U\). Denote \(\Pi_t := [0, T]^r \times [-S, S]^{d-r} I_{t \in \partial U} + [0, T]^r \times [0, S]^{d-r} I_{t \in \partial U}\),

\[
P_{q, t}(\Pi) = E \exp(\max_{s \in \Pi_t} \chi(s) - h_{1t}(s)).
\]

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Proposition 5 In the above conditions, if $\cup_{t \in M} K_t = M$, then

$$P(U; u) = \int_{\mathcal{M}} P_{q,t} \prod_{i=1}^{r} C_{t_i}^{-1}(u) dt \prod_{i=1}^{r} q_i(u)^{-1} \Psi(u)(1 + o(1))$$

(33)

as $u \to \infty$, with

$$P_{q,t} = \lim_{T,S \to \infty} T^{-r} P_{q,t}(\Pi_t) \in (0, \infty),$$

see (15).

Outline of proof. We again concentrate ourselves on the construction of corresponding splitting. In this case the informative set $M_u$ is much wider than in the previous case. Take again the same splitting $\{\Delta_k^r, k = (k_1, \ldots, k_r, 0, \ldots, 0) \in \mathbb{Z}^d\}$ of $M$, but now denote $S_{d-r} = (0, \ldots, 0, S, \ldots, S) \in \mathbb{R}^d$, $S > 0$, with first $r$ zeros, $S$ will be taken large. Denote $S^d = [-S, S]^d$, $S^{d-r} = S^d \cap \mathbb{R}^{d-r}$, and repeat (24) (25), taking

$$\Delta_{kl}^{d-r} = t_k^r + q_{t_k}^r(u)lS_{d-r} + q_{t_k}^r(u)S_{d-r}, 1 = (0, \ldots, 0, l_{d-r+1}, \ldots, l_d) \in \mathbb{Z}^d$$

and

$$\Delta_{kl}^d := \Delta_k^r \times \Delta_{kl}^{d-r}, \Delta_{kl}^d \cap M_u \neq \emptyset.$$

Denote again

$$A_{k,1} := \{ \sup_{t \in \Delta_{kl}^d} X(t) > u \},$$

We have in this case, that for any $A \subset M$, and all sufficiently large $u$,

$$\bigcup_{A \subset A} A_{k,0}^r \subseteq \left\{ \max_{s \in A} X(s) > u \right\} \subseteq \bigcup_{\Delta_{kl}^d \cap A \neq \emptyset} A_{k,0} \cup \bigcup_{\Delta_{kl}^d \cap A \neq \emptyset, l \neq 0} A_{k,1}.$$

Asymptotic behavior of the probability of left side event can be evaluated by standard Double Sum Method, using Lemma 2 and standard estimation of the corresponding double sum, which is infinitely smaller that the single one as $u \to \infty$ for non-closed $\Delta_{k0}^d$ and the rest one, with closed $\Delta_{k0}^d$, becomes small as $T \to \infty$, even with fixed $S$, the same for the first summand in the right hand part. See [9], [10]. The probability of the second summand in the right hand part can be estimated from above by sum of probabilities of $A_{k,1}$. Then one should estimate theirs asymptotic behavior as $u \to \infty$, and then it turns out that this estimation is negligibly smaller as $S \to \infty$. This estimation is practically identical to the corresponding place in [5]. See again the corresponding parts of [9], [10].

3.5 General case.

Denote

$$K_0(M) = \{ t \in M : K_t \neq \emptyset \}, K_c(M) = \{ t \in M : K_t = \emptyset, K_{ct} \neq \emptyset \};$$

$$K_\infty(M) = \{ t \in M : K_t = \emptyset, K_{ct} = \emptyset, K_\infty \neq \emptyset \},$$

see (14). Furthermore, denote

$$K_0 := \bigcup_{t \in K_0(M)} K_t, K_c := \bigcup_{t \in K_c(M)} K_t, K_\infty := \bigcup_{t \in K_\infty(M)} K_\infty.$$

Assume that all these sets are simply designed in the following sense.
Condition 6 (Structure Condition) Assume that all these sets are finitely connected smooth (two times differentiable) manifolds. Furthermore, assume that $K_0$ is either empty or consists of finite number of smooth disjoint manifolds with positive dimensions, namely,

$$K_0 = \bigcup_{i=1}^{n} K_{0i}, \dim K_{0i} = k_i, \ 0 < k_1 \leq k_2 \leq \ldots \leq k_n \leq d. \quad (34)$$

Assume also that for any $i$, $k_i$-dimensional volume of $K_{0i}$ is finite, $|K_{0i}| < \infty$.

We use here caligraphic $K_i$ because of these manifolds in $U$ can be indeed of general type, not necessary linear subspaces. Fix $i$ with $k_i < d$, and consider in $K_{0i}$ curvilinear coordinates. For $t \in K_{0i}$, using Proposition 11 choose coordinate vectors $e_j(t)$, $j = 1, \ldots, k_i$ of this curvilinear coordinates and complete them to a basis $\{e_j(t), j = 1, \ldots, k_i, \tilde{e}_j(t), j = k_i + 1, \ldots, d\}$ in $\mathbb{R}^d$. Write vector $q_i(t)$ in these coordinates, remind that $q_i(t) = C_i t q_0(u)$, see 4

$$q_i^i(u) = (C_i^i)^{q_0j}(u), \ j = 1, \ldots, k_i, (C_i^i q_0j)(u), j = k_i + 1, \ldots, d, \quad (35)$$

where, in accordance with Proposition 11 $(q_0j, j = 1, \ldots, d)$ is just an permutation of $(q_0j, j = 1, \ldots, d)$, whereas $C_i^j$ are other positive continuous bounded functions. Denote corresponding positive limits by

$$h_i^j(s) := \lim_{u \to \infty} u^2(1 - r(q_i^j(u)s)), \quad (36)$$

where $s = (s_1, \ldots, s_d)$ is written in these coordinates. In fact $h_i^j(s) = h(O_i t s)$, where $O_i t$ is an orthogonal transition matrix to the curvilinear coordinates with the orthogonal complement to a basis in $\mathbb{R}^d$, as above.

We have,

$$\lim_{u \to \infty} \frac{1 - \sigma(q_i^j(u)s)}{1 - r(q_i^j(u)s)} = 0. \quad (37)$$

By analogy with proofs of Theorem 11 and Proposition 3 proof, we build a partition of $K_{0i}$ with $k_i$-dimensional blocks, similar to $B_0$ and $B_k$, with the same $\kappa(u)$ and $q_i^j(u)$ from (35). Denote these blocks $B(t_\nu)$, where $\{t_\nu, \nu = 1, \ldots, N\}$ is the corresponding to this partition grid, that is,

$$K_{0i} = \bigcup_{\nu=1}^{N} B(t_\nu). \quad (38)$$

Similarly to the same proofs, but for $K_{0i} \cap U \cap M_u$ instead of $U \cap M_u$, using Theorem 11 the second part we get asymptotic behavior of the probability $P(B(t_\nu); u)$ for all $\nu$. Then, thinning the grid unboundedly, we get, using Condition 1, the following Lemma.

Lemma 3 For any $K_{0i}$ from the partition (34) with $k_i < d$,

$$P(K_{0i} \cap U; u) = \int_{K_{0i} \cap U} H_{q_i} \prod_{j=1}^{k_i} (q_i^{q_0j}(u))^{-1} e^{-u^2 f(t)} dt \Psi(u)(1 + o(1))$$

as $u \to \infty$. Remind that, by our agreement, here $dt$ is an elementary $k_i$-dimensional volume of $K_{0i}$, the corresponding integration domain, and $f(t)$ is given by (29).

Turning to (34), we have, using again Bonferroni inequalities, with some additional standard techniques for probabilities of large excursions, see [5], [0], we get the following.
Proposition 6 If $\dim K_{0i} < d, i = 1, \ldots, n,$

$$P(K_0 \cap U; u) = (1 + o(1)) \sum_{i=1}^{n} P(K_{0i} \cap U; u),$$

(39)
as $u \to \infty$.

Remark 5 Remark that if for some $i$ in (34), $\dim K_{0i} = d$, Proposition 3 should be used with another Laplace integral (30), namely,

$$L_f(\lambda) := \int_{K_0} e^{-\lambda f(t)} \, dt, \quad \lambda > 0.$$  

(40)

Remark 6 Remark that the summands in (39) can have different orders in $u$ depending on the dimension of the corresponding component $K_{0i}$, on behavior of $d_j^i(t)$'s and on behavior of $\sigma^2(t)$. Hence only summands with slowest order play a role. Remark also that by Remark 5, if for some $i$, $\dim K_{0i} = d$, no summands with $\dim K_{0i} < d$ in (39) contribute to the asymptotic behavior of $P(U; u)$.

Remark 7 Evaluations of asymptotic behaviors of $P(K_\infty, u)$ and $P(K_c, u)$ are obvious, with using Condition 6 and Propositions 4 and 5, respectively.

4 Conclusion. Summary of main results.

Thus, using again Bonferroni, we get for any map $U$ from the atlas $A$ on $M_\varepsilon$ that

$$P(U, u) = P(U, u) =$$

$$= (P(K_0 \cap U; u) + P(K_c \cap U; u) + P(K_\infty \cap U; u))(1 + o(1))$$
as $u \to \infty$, and the asymptotic behavior of the first summand is evaluated in Proposition 6 for the asymptotic behaviors of the rest two summands see Remark 7.

The final steps are

1) to scale the atlas making maps pairwise nonintersected, namely number the maps, $U_i$, $i = 1, \ldots, M, M < \infty$, and take $U_1, U_2 = U_2 \setminus U_1, \ldots, U'_k = U_k \setminus (\cup_{i=1}^{k-1} U'_i), \ldots$ as the new atlas;

2) to write

$$\{\max_{t \in M_\varepsilon} X(t) > u\} = \bigcup_{U' \in A} \{\max_{t \in U'} X(t) > u\};$$

3) to use Bonferroni inequalities with estimating of the double sum of the joint probabilities;

and

4) to use back passage from the informative manifold $M_\varepsilon$ to whole $S$.

Remark that our results here agree with assertions of Theorem 3, [4], where $d = 1$, and Theorem 1 in [5], [6] for $\dim M = 0$. 

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References

[1] Dmitrovsky V. A.: Estimates for the distribution of the maximum of a Gaussian Field. Random Processes and Fields, Moscow State University Ed., Moscow, 22–31 (in Russian) (1979)

[2] Dmitrovsky V. A.: On the integrability of the maximum and the local properties of Gaussian fields. In: Probability Theory and Mathematical Statistics, Vol. I (B. Grigelionis, Yu. V. Prohorov, V. V. Sazonov, and V. Statulevičius, eds.), ”Mokslas”, Vilnius, 271–284 (1990).

[3] Dudley, R.M.: The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. J. Funct. Anal. 1, 290–330 (1967)

[4] Hashorva E., Kobelkov S. G., Piterbarg V. I.: On maximum of Gaussian process with unique maximum point of its variance. Fundamentalnaya i prikladnaya matematika, (in Russian), 2020, 23, 1, 161-174, see also arXiv:1901.09753v1[math.PR]

[5] Kobelkov S. G., Piterbarg V. I. On maximum of Gaussian random field having unique maximum point of its variance. Extremes, 2019, 22, no. 4, 413–432.

[6] Kobelkov S. G., Piterbarg V. I., Rodionov I. V. Correction to: On maximum of Gaussian random fields having unique maximum point of its variance. Extremes, 2021, 24, no. 1, 85–90.

[7] Fernique X.: Régularité des trajectoires des fonctions aléatoires gaussiennes, Lecture Notes in Mathematics, 480, 2-187, (1975)

[8] Peng Liu. Extremes of Gaussian random fields with maximum variance attained over smooth curves. 22 Dec. 2016. arXiv:1612.07780 [math.PR]

[9] Piterbarg V.I.: Asymptotic Methods in Theory of Gaussian Random Processes and Fields, Providence, American Mathematical Society, Ser. Translations of Mathenatical Monographies, 148 (2012)

[10] Piterbarg V. I.: Twenty Lectures About Gaussian Processes. Atlantic Financial Press London, NewYork, (2015)

[11] Piterbarg V. I. and Prisjažnijk V. P.: Asymptotic behavior of the probability of a large excursion for a nonstationary Gaussian process, Theory of Probability and Mathematical Statistics, Kyiv, 18, 121–134, (1978)

[12] Piterbarg V. I. High excursion for non-stationary generalized chi-square processes: Stochastic Processes and their Applications. 1994. 53, no. 2, 307–337.

[13] Piterbarg V. I., Rodionov I. V. High excursions of Bessel and related random processes: Stochastic Processes and their Applications. 2020, 130, 4859–4872.

[14] Talagrand, M.: Small tails for the supremum of a Gaussian process. Annales de l’I.H.P. Probabilités et statistiques, Volume 24 (1988) no. 2, pp. 307-315

[15] Vladimir A. Zorich: Mathematical Analysis II. Springer-Verlag Berlin Heidelberg New York, (2004)
[16] Wanli Qiao: Extremes of locally stationary Gaussian and chi fields on manifolds. May, 2020, https://arxiv.org/abs/2005.07185, pp. 1–27.