Origin and meaning of quantum nonlocality

L. de la Peña,1 A. M. Cetto,1
A. Valdés-Hernández1, and H. M. França2
1Instituto de Física,
Universidad Nacional Autónoma de México,
A.P. 20-364, México D.F., Mexico
e-mail: luis, ana, andreavh@fisica.unam.mx
2Instituto de Física, Universidade de São Paulo,
CP 66318, São Paulo, Brazil
e-mail: hfranca@if.usp.br

Abstract

Quantum nonlocality is revisited from a novel point of view by studying the problem of an originally classical particle immersed in the stochastic zero-point radiation field (zpf). The entire system is left to evolve until it reaches a state in which the radiative terms cancel each other in the mean in a first approximation. The ensuing approximate statistical description reduced to the particle’s configuration space contains a non-classical term due to the dispersion of the momentum, which depends on the density of particles $\rho(x)$ and thus is nonlocal. This description is shown to be equivalent to Schrödinger’s equation and its complex conjugate. The nonlocal term is recognized as the so-called quantum potential, thus solving the long standing problem of the origin and meaning of this term. Further, the relationship between the Wigner function and a true Kolmogorovian probability density in phase space is discussed from the perspective provided by this theory.

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1 Introduction

The aim of the present paper is to shed light on the physical origin and meaning of quantum nonlocality, one of the major conceptual quandaries of quantum mechanics. We accomplish this within a theoretical framework that identifies the presence of the stochastic zero-point radiation field (zpf) in interaction with matter as the entity that is ultimately responsible for the fluctuating properties of the particle’s momentum, which from this perspective are at the root of both the nonlocal and the indeterministic nature of the quantum description.
The present paper is based on previous work, but supersedes it in a significant way. It starts by considering the full (infinite dimensional), nonrelativistic and stochastic phase-space statistical description of the field and particle in interaction. After averaging over the realizations of the zpf the system is allowed to evolve towards a state of mean energy balance between the particle and the zpf, which takes us in the radiationless limit to Schrödinger’s theory, once the description is reduced to the configuration space of the particle. Of particular importance is the observation that this reduction of the description from the complete phase space to the configuration space of the particle gives rise to a term representing the fluctuations of the momentum impressed by the zpf on the particle. This term is a function of the probability density $\rho(x)$ and hence possesses a nonlocal nature, manifested even in one-particle quantum systems. The origin of the quantum nonlocality appearing in the reduced description is thus traced to the action of the fluctuating background field. Planck’s constant $\hbar$ enters into the quantum-mechanical description precisely through the nonlocal term related to the momentum dispersion, as a measure of the fluctuations driven by the zpf.

It is further shown that under the stated approximations the phase-space probability density — which is explicitly written — is transformed into the Wigner function. In the course of the approximations and partial averaging the original true (that is, Kolmogorovian) probability density loses some of its properties, which explains why the phase space probability functions employed in the quantum description (and particularly the Wigner function) are normally not true probability densities.

The results presented imply that the quantum phenomenon is not innate — it is not intrinsic to matter or to the radiation field alone — but is an emergent phenomenon generated by the permanent matter-field interaction. This observation reinforces the results obtained in Refs. and . In the first two of these references it is shown that the mere introduction of the zero-point energy of the radiation field into an otherwise classical description suffices to explain the Planck distribution and the discrete properties of the field. A detailed study of the dynamics of the mechanical system immersed in the zpf is shown to lead to the Heisenberg formalism of nonrelativistic quantum mechanics, and to entanglement of particles. Altogether, these results show that a careful study of (originally classical) material systems in interaction with a stochastic radiation field endowed with a zero-point contribution allows to achieve a better understanding of the quantum description. A short version of the present paper is given in Ref. (11).

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1The term nonlocality is used here in its standard sense, to refer to a direct, instantaneous (nonmediated) influence between two distant objects or points in space. The zpf, as any Maxwellian field, obeys of course the principle of locality.
2 Reducing the Liouville Equation

Our starting point is the Liouville equation for the density \( R \) of points \((x_s, p_s)\) in the phase space of the complete system composed of a particle (of mass \( m \) and charge \( e \)) and the zero-point field (here \( x_s \) and \( p_s \) stand for the whole set of degrees of freedom \( x_s = \{ x_{\text{part}} \equiv x, x_{\text{zpf}} \}, p_s = \{ p_{\text{part}} \equiv p, p_{\text{zpf}} \} \)):

\[
\frac{\partial}{\partial t} R(x_s, p_s, t) + \frac{\partial}{\partial x_s} (x_s R(x_s, p_s, t)) + \frac{\partial}{\partial p_s} (p_s R(x_s, p_s, t)) = 0. \tag{1}
\]

The motion of the mechanical (one-dimensional) subsystem is governed, in the nonrelativistic limit, by

\[
\dot{x} = p/m, \quad \dot{p} = f(x) + m\tau \dot{x} + eE(t), \tag{2}
\]

where \( f(x) \) is the external force, \( E(t) = \mathbf{E}(t) \cdot \mathbf{x} \) with \( \mathbf{E}(t) \) the electric component of the zpf in the long-wavelength approximation and \( m\tau \dot{x} \) is the radiation reaction force, with \( \tau = 2e^2/3mc^3 \approx 10^{-23} \) s for the electron. We will further approximate this term in the form (although this will be of little importance in what follows) \( m\tau \dot{x} \approx \tau \dot{x} f'(x) \), where the prime denotes derivative with respect to \( x \).

We are not interested in the motions for a certain realization of the random field \( E(t) \), but in the average motion of an ensemble of similar systems. We therefore average \( R(x_s, p_s, t) \) over the realizations of this field and call \( Q(x, p, t) \) the averaged density in the phase space of the particle. The procedure can be performed using the projector technique (see e.g. Refs (4) and (12)). After a somewhat lengthy calculation Eqs. (1) and (2) lead to the Fokker-Planck-type equation

\[
\frac{\partial}{\partial t} Q + \frac{1}{m} \frac{\partial}{\partial x} pQ + \frac{\partial}{\partial p} \left[ \left( f(x) + \frac{\tau}{m} f'(x)p \right) Q \right] = e^2 \frac{\partial}{\partial p} \hat{D}(t)Q, \tag{3}
\]

where \( \hat{D}(t) \) is a complicated integro-differential operator. More precisely the ‘diffusion’ term can be written in the form

\[
e^2 \frac{\partial}{\partial p} \hat{D}(t)Q = e \frac{\partial}{\partial p} \hat{PA} \sum_{k=0}^{\infty} \left[ cG \frac{\partial}{\partial p} (1 - \hat{P}) E \right]^{2k+1} Q. \tag{4}
\]

\( \hat{P} \) stands for the smoothing operator that averages over the realizations of the field (\( \hat{PA} = \mathbf{A}^E \)) and \( \hat{G} \) represents the operator

\[
\hat{G}A(x, p, t) = \left( \frac{\partial}{\partial t} + \hat{L} \right)^{-1} A(x, p, t) = \int_0^t e^{-\hat{L}(t-t')} A(x, p, t') dt', \tag{5}
\]

(\( \hat{L} = \frac{1}{m} \frac{\partial}{\partial x} p + \frac{\partial}{\partial p} (f + \tau f' x/m) \)). The explicit form of \( \hat{D}(t) \) will not be required for the present work. However, it is important to note that the diffusion term \( e^2 \hat{PA} \) comprises at least a correlation of the form \( E(t)E(t') \) averaged over the
realizations of the field, and hence contains a factor proportional to the spectral (energy) density of the zpf, given by
\[ \rho_{zpf}(\omega) = \frac{\hbar \omega}{2\pi^2 c^3} \]
as follows from the fact that the energy per normal mode of frequency \( \omega \) is \( \hbar \omega/2 \). (Incidentally, we recall that this expression is derived without quantum assumptions from the demand of Lorentz invariance; see Ref. [13] and references therein).

3 Transition to configuration space

Our interest here lies in the connection of Eq. (3) with quantum mechanics, which is usually described in configuration space. The transition to configuration space can be performed systematically by means of the characteristic (moment generating) function \( \tilde{Q} \) associated with the density \( Q \),
\[ \tilde{Q}(x, z, t) = \int Q(x, p, t) e^{ipz} dp. \]
From this expression it follows that the probability density \( \rho(x, t) \), or rather the marginal probability in configuration space, is
\[ \rho(x, t) = \int Q(x, p, t) dp = \tilde{Q}(x, 0, t), \]
and the (partially averaged) local moments of \( p \) are given by
\[ \langle p^n \rangle(x) \equiv \langle p^n \rangle_x = \frac{1}{\rho(x)} \int p^n Q dp = (-i)^n \left( \frac{1}{\tilde{Q}} \frac{\partial^n \tilde{Q}}{\partial z^n} \right) \bigg|_{z=0}. \]
Now we introduce the change of variables
\[ z_+ = x + \beta z, \quad z_- = x - \beta z, \]
with an as yet undetermined parameter \( \beta \) that has the dimensions of an action (see Eq. (7)), and rewrite \( \tilde{Q} \) in the general form
\[ \tilde{Q}(x, z, t) = q_+(z_+, t)q_-(z_-, t)\chi(z_+, z_-, t), \]
Note from Eq. (7) that \( \tilde{Q}^*(x, z, t) = \tilde{Q}(x, -z, t) \), so that Eq. (10) leads to
\[ q_+(z_-, t)q_-(z_+, t)\chi(z_-, z_+, t) = q_+^*(z_+, t)q_-^*(z_-, t)\chi^*(z_+, z_-, t). \]
whence
\[ q_+(z_-, t) = q_+^*(z_-, t), \quad q_-(z_+, t) = q_+^*(z_+, t), \quad \chi^*(z_+, z_-, t) = \chi(z_-, z_+, t). \]  
(12)

Consequently \( \tilde{Q}(x, z, t) \) can be rewritten as
\[ \tilde{Q}(x, z, t) = q_+(z_+, t)q_+(z_-, t)\chi(z_+, z_-, t) = q(z_+, t)q_+(z_-, t)\chi(z_+, z_-, t), \]  
(13)

with
\[ q(z_\pm, t) \equiv q_+(z_\pm, t) = q_+^*(z_\pm, t). \]  
(14)

Further, from Eqs. (13) and (8a) we may write
\[ \rho(x, t) = \tilde{Q}(x, 0, t) = q^*(x, t)q(x, t)\chi_0(x, t), \]  
(15)

with \( \chi_0(x, t) = \chi|_{z=0} \) a real function that can be taken as a constant without loss of generality, absorbing its possible time and space dependence into the functions \( q(x, t), q^*(x, t) \). Thus
\[ \chi_0(x, t) = 1, \quad \rho(x, t) = q^*(x, t)q(x, t). \]  
(16)

We now return to Eq. (3). Its Fourier transform is
\[ \frac{\partial \tilde{Q}}{\partial t} - i \frac{1}{m} \frac{\partial^2 \tilde{Q}}{\partial x \partial z} - i z f(x) \tilde{Q} - \frac{\tau}{m} f' \frac{\partial \tilde{Q}}{\partial z} = -ie^2 z (\tilde{D}Q). \]  
(17)

To translate this into a description in configuration space that preserves the information coming from momentum space, we expand Eq. (17) into a power series around \( z = 0 \) and separate the coefficients of \( z^k \) \((k = 0, 1, 2, \ldots)\). In this form we obtain an infinite hierarchy of equations containing moments of \( p \) of increasing order. For the first two equations (coefficients of \( z^0 \) and \( z \)) we get
\[ \frac{\partial \rho}{\partial t} + \frac{1}{m} \frac{\partial}{\partial x} (p_x \rho) = 0, \]  
(18a)
\[ \frac{\partial}{\partial t} (p_z \rho) + \frac{1}{m} \frac{\partial}{\partial x} (p^2_z \rho) - f \rho = \frac{\tau}{m} f' \bigg|_{z=0} + e^2 \bigg|_{z=0}. \]  
(18b)

The subsequent equations (corresponding to higher powers of \( z \)) are linked to the above couple by the same elements \( \rho, \langle p_x \rangle, \langle p^2_x \rangle \) and higher-order moments \( \langle p^n_x \rangle \), in addition of course to contributions deriving from the term \( z (\tilde{D}Q) \). Equation (18a) is the continuity equation (for the transfer of matter) in configuration space, with the local (partially averaged) flow velocity given by \( v(x) = \langle \dot{x} \rangle_x = \frac{1}{m} \langle p_x \rangle_x \). Equation (18b) describes the transfer of momentum density and contains the second moment \( \langle p^2_x \rangle \), whose value reappears in the third equation, the one that describes the transfer of kinetic energy density, and so on. This coupling between successive equations creates a highly difficult

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5In what follows we will assume that all surface terms vanish at infinity.
mathematical problem. However, in the case of interest here, a decoupling of the first two equations from the rest takes place in a certain approximation, as shown in the next section. Therefore only the first two local moments of \( p \) intervene in the resulting (approximate) description, which means that we may concentrate on the behavior of \( \tilde{Q} \) for small values of \( z \) (see Eqs. 8a and 8b).

Resorting to Eqs. 8b and 10 we obtain

\[
\langle p \rangle = -i \left( \frac{1}{Q} \partial_z Q \right) \bigg|_{z=0} = -i \beta [\partial_x \ln q(x,t) - \partial_x \ln q^*(x,t)] - i \partial_z \ln \chi \bigg|_{z=0}
\]

and for the fluctuations of \( p \), given by \( \sigma_p^2(x) = \langle p^2 \rangle_x - \langle p \rangle_x^2 \), we have

\[
\sigma_p^2(x) = -i \beta [\partial_x \ln q(x,t) - \partial_x \ln q^*(x,t)] - i \chi_1.
\]

An expansion of \( \chi(x,z,t) \) as a power series of \( z \)

\[
\chi(x,z,t) = \sum_{s=0}^{\infty} z^s \chi_s(x,t)
\]

allows us to rewrite Eqs. 19 and 20 as

\[
\langle p \rangle = -i \beta [\partial_x \ln q(x,t) - \partial_x \ln q^*(x,t)] - i \chi_1,
\]

\[
\sigma_p^2(x) = -\beta^2 \partial_x^2 \ln \rho(x,t) + \left( \beta^2 \partial_x^2 - \partial_z^2 \right) \ln \chi \bigg|_{z=0}.
\]

4 Radiationless approximation in the time-asymptotic limit

Equation 17, or for that matter, the complete hierarchy of equations in configuration space, provides in principle an exact statistical description of the evolution of the ensemble of particles. It is clear that initially, when particle and field start to interact, there is an irreversible process of energy and momentum exchange during which the field has a randomizing and dissipative effect on the particle dynamics, due to the diffusion and radiation reaction terms (proportional to \( e^2 \) and \( \tau \sim e^2 \), respectively) in Eq. 17. However, we are here specifically interested in the time-asymptotic limit, when the combined action of these two terms has led to a stationary regime in which this exchange is no more irreversible. In this situation a balance should be attained between the mean power radiated and that absorbed by the particle from the field; this balance condition, discussed in detail in section 5.3, means that the equality

\[
\frac{e^2}{m} \langle f' \langle p \rangle_x \rangle = e^2 \left( \langle \tilde{D}Q \rangle \bigg|_{z=0} \right)
\]

should hold (see Eq. 15b below). One may then consider that the remaining effect of the two terms, both proportional to \( e^2 \) and thus of order \( \alpha = e^2/\hbar c \), is reduced to a small (stochastic) radiative
correction. By neglecting the terms on the right-hand side of Eq. (18b) one obtains

\[
\frac{\partial \rho}{\partial t} + \frac{1}{m} \frac{\partial}{\partial x} \langle p \rangle_x \rho = 0, \quad (24a)
\]

\[
\frac{\partial}{\partial t} \langle p \rangle_x \rho + \frac{1}{m} \frac{\partial}{\partial x} \langle p^2 \rangle_x \rho - f \rho = 0. \quad (24b)
\]

As follows from Eqs. (22) and (23), the first of these equations depends on \( \chi_1 \), the second one on \( \chi_1 \) and \( \chi_2 \), the third one on \( \chi_1 \), \( \chi_2 \) and \( \chi_3 \), and so on, so the functions \( \chi_s \) (\( s = 1, 2, \ldots \)) — the non-factorizable part of \( \tilde{Q} \) (see Eq. (10)) — successively couple the equations of the hierarchy. Before proceeding to the decoupling announced above, it is convenient to distinguish for a moment between the canonical and the mechanical momentum of the particle. Note that \( p \) in Eqs. (22) and (23) refers to the mechanical momentum, \( p_m = p = P - \frac{e}{mc} A \), where \( A \) stands for the \( x \)-component of the electromagnetic potential \( A \) (including radiation reaction), whence

\[
\langle p \rangle_x = \langle P \rangle_x - \frac{e}{mc} \langle A \rangle_x, \quad (25)
\]

\[
\sigma^2_P(x) = \sigma^2_c(x) - \frac{e^2}{m^2 c^2} \Sigma(A, p)
\]

where \( \Sigma(A, p) = \sigma^2_c(x) + (2mc/e) \Gamma_{Ap} \), and \( \Gamma_{AB} = \langle AB \rangle_x - \langle A \rangle_x \langle B \rangle_x \). Now, resorting to Eqs. (22) and (23) and using (15), one is led to

\[
\langle P \rangle_x - \frac{e}{mc} \langle A \rangle_x = -i \beta \partial_x \ln \frac{q(x, t)}{q^*(x, t)} - i \chi_1, \quad (26a)
\]

\[
\sigma^2_P(x) - \frac{e^2}{m^2 c^2} \Sigma(A, p) = -\beta^2 \partial^2_x \ln q^*(x, t) q(x, t) - 2 \chi_2 + \chi_1^2. \quad (26b)
\]

At this point we assume (this is our Ansatz) that the (local) moments of \( P \) are determined by the functions \( q(x, t), q^*(x, t) \), so that

\[
\langle P \rangle_x = -i \beta [\partial_x \ln q(x, t) - \partial_x \ln q^*(x, t)], \quad (27)
\]

\[
\sigma^2_P(x) = -\beta^2 \partial^2_x \ln q^*(x, t) q(x, t). \quad (28)
\]

This means, in particular, that the information regarding the phase of \( q(x, t) \) is transferred to the local average of the canonical momentum and the information regarding the magnitude of \( q(x, t) \) is transferred to the dispersion of the canonical momentum. The above assumption implies that the terms containing \( \chi_1 \) and \( \chi_2 \) in Eqs. (26a) and (26b) are the ones that bear information regarding the electromagnetic potential \( A \), so that once energy balance has been reached the following equalities should hold:

\[
- i (\partial_x \ln \chi)_{x=0} = -i \chi_1 = -\frac{e}{mc} \langle A \rangle_x, \quad 2 \chi_2 = \frac{2e}{mc} \Gamma_{Ap} - \frac{e^2}{m^2 c^2} \langle A^2 \rangle_x. \quad (29)
\]

Similarly, the remaining terms \( \chi_s(x, t) \) with \( s \geq 3 \) in Eq. (21) should represent \( A \)-dependent contributions to the higher moments of \( p \). This is equivalent to assuming that the entire function \( \chi \) contributes only radiative corrections to the
resulting equations, hence in the radiationless approximation we should take \( \chi \simeq \chi_0 = 1 \). This is well verified a posteriori, since \( \chi \) leads to the Lamb shift of the energy levels, a very small correction indeed. Thus in what follows we take the radiationless approximation \( P \to p \) and \( \chi = 1 \), whence

\[
\langle p \rangle_x = mv(x) = -i\beta \left[ \partial_x \ln q(x,t) - \partial_x \ln q^*(x,t) \right],
\]

and, using Eq. (16),

\[
\sigma^2_p(x) = \langle p^2 \rangle_x - \langle p \rangle_x^2 = -\beta^2 \partial_x^2 \ln q^*(x,t)q(x,t) = -\beta^2 \partial_x^2 \ln \rho,
\]

or \( \langle p^2 \rangle_x = \langle p \rangle_x^2 - \beta^2 \partial_x^2 \ln \rho \). This means that \( \langle p^2 \rangle_x \) has become a function of \( \langle p \rangle_x \) and \( \rho \) only, so that the first two equations of the hierarchy suffice to determine the two unknown functions \( \rho(x,t) \) and \( \langle p \rangle_x \). Equations (24a), (24b) have decoupled from the rest and become an independent system of equations for the evolution of the ensemble.

In the following section we tackle the problem of solving this system of equations. This is a complicated nonlinear system of equations, so the search for a simpler procedure is an important task. It turns out to be important indeed, since the end result is a couple of linear equations, as shown in the next section.

5 The quantum description

In order to determine \( q(x,t) \) we resort to Eq. (17), which in the current (radiationless) approximation reduces to

\[
\frac{1}{\tilde{Q}} \partial_t \tilde{Q} - \frac{i}{m\bar{Q}} \partial_x \partial_z \tilde{Q} = if(x).
\]

In this equation we should take \( \chi = 1 \) according to our previous observations; however we will write still \( \tilde{Q} = q_+q_-\chi \) (according to Eq. (10)), leaving explicit for a moment the \( \chi \) function, because it is instructive to find the role it plays in a crucial place. We thus obtain

\[
\frac{1}{\chi} \partial_t \chi + \frac{1}{q_+q_-} \partial_t (q_+q_-) - \frac{i\beta}{m} \left[ \frac{1}{q_+\chi} \partial^2_{q_+} (q_+\chi) - \frac{1}{q_-\chi} \partial^2_{q_-} (q_-\chi) \right] = \\
= \frac{i}{2\beta} (z_+ - z_-) f \left[ \frac{1}{2} (z_+ + z_-) \right].
\]

For sufficiently small values of \( z = (z_+ - z_-)/2\beta \) (corresponding to the first moments of \( p \)), the mean-value theorem can be applied to the term on the second line of Eq. (33)

\[
\int_{z_-}^{z_+} f(u)du = (z_+ - z_-) f \left[ \frac{1}{2} (z_+ + z_-) + \frac{1}{2} \epsilon (z_+ - z_-) \right],
\]

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where $\epsilon$ is a number between $-1$ and $+1$. Assuming $f(z)$ to be well behaved in the small interval $[z_-, z_+]$, we have to lowest order in $z$

$$(z_+ - z_-) f\left(\frac{1}{2}(z_+ + z_-)\right) = \int_{z_-}^{z_+} f(u) du = -[V(z_+) - V(z_-)], \quad (35)$$

where $V(z)$ is the potential associated with the force $f(z)$. This result holds for any (well-behaved) potential and not just for quadratic functions, for which it is immediate. Equation (35) now takes on the form

$$\frac{1}{\chi} \partial_t \chi + \frac{1}{q_+ q_-} \partial_t (q_+ q_-) - \frac{i \beta}{m} \left[ \frac{1}{q_+ \chi} \partial_x^2 (q_+ \chi) - \frac{1}{q_- \chi} \partial_x^2 (q_- \chi) \right] = -\frac{i}{2 \beta} [V(z_+) - V(z_-)]. \quad (36)$$

In terms of the couple of functions

$$\Psi_+(z_+, z_-, t) = q_+(z_+, t) \chi(z_+, z_-, t), \quad \Psi_-(z_+, z_-, t) = q_-(z_-, t) \chi(z_+, z_-, t), \quad (37)$$

equation (10) reads

$$\bar{Q}(x, z, t) = \frac{\Psi_+ \Psi_-}{\chi}, \quad (38)$$

and Eq. (36) becomes

$$\frac{1}{\Psi_+} \left[ - \frac{i \beta}{m} \partial_x^2 \Psi_+ + \frac{i}{2 \beta} V(z_+) \Psi_+ + \frac{\partial \Psi_+}{\partial t} \right] +$$

$$+ \frac{1}{\Psi_-} \left[ \frac{i \beta}{m} \partial_x^2 \Psi_- - \frac{i}{2 \beta} V(z_-) \Psi_- + \frac{\partial \Psi_-}{\partial t} \right] = \frac{1}{\chi} \frac{\partial \chi}{\partial t} \quad (39)$$

In this equation, the functions $\Psi_+$ and $\Psi_-$ are coupled through $\chi(z_+, z_-, t)$ and will remain so as long as the latter plays any significant role. In the radiationless approximation the right-hand side reduces to zero and Eq. (39) becomes separable. This is the crucial place referred to above: it is only in the radiationless approximation that the functions $\Psi_+$ and $\Psi_-$ obey each one a separate equation. Knowing this we take the limit $z \to 0$ (so that both $z_+$ and $z_-$ reduce to $x$), $\chi = 1$ and using Eq. (12) put

$$\Psi_+|_{z=0} = q_+(x, t) = e^{iCt} \psi(x, t), \quad \Psi_-|_{z=0} = q_+^*(x, t) = e^{-iCt} \psi^*(x, t), \quad (40)$$

with $C$ the constant of separation. Eq. (39) separates and we finally get

$$-\frac{\beta^2}{m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi = 2i\beta \frac{\partial \psi}{\partial t} \quad (41)$$

and its complex conjugate. The particle density $\rho$ is now given by

$$\rho(x, t) = \psi^*(x, t) \psi(x, t), \quad (42)$$
as follows from Eqs. (16) and (40). Except for the (constant) factor $\beta$ to be determined (see the discussion in section 5.3), we recognize in (41) the Schrödinger equation for the (wave) function $\psi(x,t)$.

According to the above discussion, Eqs. (41) and (42) provide a valid description in the particle’s configuration space only once the combined irreversible effect of the radiative terms has taken place, energy balance has been reached and the radiationless approximation can be taken. Under these circumstances all (reversible) time evolution is confined to the time dependence of the wave functions $\psi$ and $\psi^*$. In a more exact description (the QED description, in particular) it will be necessary to make recourse to the remaining equations of the hierarchy to extract the additional information that has been left out. In particular and as was already stated, it can be demonstrated by a perturbative calculation that the terms that depend on $\chi_1$ and $\chi_2$ give rise to the Lamb shift of the atomic levels plus a nonrelativistic mass correction.

One can consider the above results as a determination of the assumptions, conditions and approximations required to arrive at quantum mechanics from an initially complete phase-space description. From such a perspective this work serves to disclose some of the hidden features behind the quantum phenomenon. Whether in all cases the system under study reaches a regime in which the above conditions hold, remains to be studied. Whenever they do reach it, however, these systems are of utmost importance: they are just the quantum systems.

5.1 The quantum potential

Resorting to Eq. (31) we can rewrite the couple of equations (24a) and (24b), with $v = \frac{1}{m} \langle p \rangle_x$ as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (v \rho) = 0,$$

$$(43a)$$

$$m \frac{\partial}{\partial t} (v \rho) + m \frac{\partial}{\partial x} (v^2 \rho) - \frac{\beta^2}{m} \frac{\partial}{\partial x} \left( \rho \frac{\partial^2}{\partial x^2} \ln \rho \right) - f \rho = 0.$$

$$(43b)$$

According to the above discussions, when the radiationless approximation is taken and the original (statistical) description is reduced to configuration space, it is not a Fokker-Planck-type equation what controls the evolution of the system, but the couple of equations (43) instead, or, equivalently, the Schrödinger equation. In these equations the only (explicit) remaining connection to momentum space is the term containing $\ln \rho$. As Eq. (31) shows, this contribution comes from the local value of the fluctuations of the momentum: a term that is unusual in classical mechanics, but essential in quantum physics. If we write the momentum $p$ of one specific particle of the ensemble at a given point $x$ in the form

$$p|_x = mv(x) + \delta(x),$$

$$(44a)$$

An important remark by Wallstrom (1994) about this equivalence is briefly discussed in section 5.3.
where $\delta(x)$ stands for the deviation, then

$$\sigma_p^2(x) = \langle \delta^2(x) \rangle_x.$$  \hspace{1cm} (44b)

The momentum fluctuation averaged over the whole phase space with probability density $Q(x,p,t)$ is given by

$$\langle \sigma_p^2(x) \rangle = \int \langle \delta^2(x) \rangle_x \rho(x) dx$$

$$= -\beta^2 \int \rho \frac{\partial^2}{\partial x^2} \ln \rho dx = \beta^2 \int \rho \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right)^2 dx,$$  \hspace{1cm} (45)

or

$$\langle \sigma_p^2(x) \rangle = \beta^2 \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right)^2.$$  \hspace{1cm} (46)

This formula (when multiplied by $1/2m$) represents a central contribution to the average kinetic energy of the particle. It is important to recognize that the centered momentum $\sigma_p^2(x)$ at a given point $x$ bears statistical information about the entire (available) space, due to its dependence on the probability density $\rho(x)$, and thus it introduces a nonlocal ingredient into the description through Eq. (43b).

To establish the connection between this term and the quantum potential we write the solution of Eq. (41) in the familiar form

$$\psi(x,t) = \sqrt{\rho} e^{iS(x,t)},$$  \hspace{1cm} (47)

whence

$$v(x) = \frac{1}{m} \langle p \rangle_x = \frac{2\beta}{m} \frac{\partial S}{\partial x},$$  \hspace{1cm} (48)

as follows from Eq. (30). Equation (43b) rewritten with the aid of Eq. (48) gives after an integration (the integration constant is absorbed in $S$)

$$2\beta \frac{\partial S}{\partial t} + \frac{2\beta^2}{m} \left( \frac{\partial S}{\partial x} \right)^2 - \frac{2\beta^2}{m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} + V = 0,$$  \hspace{1cm} (49)

where $V$ is the potential associated with the external force $f(x)$. Except for the third term (originating in the term that contains $\ln \rho$ in Eq. (43b)) this is the Hamilton-Jacobi equation of classical mechanics, with $S$ playing the role of the

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\textsuperscript{5}This contribution appears in the literature under several guises. It is discussed in Refs. \textsuperscript{13}, \textsuperscript{14}, where attention is paid to its nonlocal nature. In stochastic quantum mechanics it is identified as produced by the stochastic velocity $u = (\hbar/2m) (\partial \rho/\rho)$ (see, e.g., Refs. \textsuperscript{17}, \textsuperscript{20}, \textsuperscript{13} and references therein). This same contribution to the kinetic energy is interpreted by Olavo as coming from a local entropy due to spontaneous local fluctuations in positions (see Ref. \textsuperscript{21}). In Bohmian quantum mechanics, the causal version of this theory, extensive use is made of it, giving rise to the quantum potential (e.g. Ref. \textsuperscript{22}); see Eq. (49) below.

Note that the momentum associated with the velocity $u$ is given by $\beta \partial_x \rho/\rho$, hence it follows from Eq. (49) that it is directly related to $\langle \sigma_p^2(x) \rangle^{1/2}$. 

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action function. Hence the statistical (and nonlocal) nature of the description of the ensemble is encapsulated in this term alone, just the one that contains the information about the fluctuations impressed by the zpf and marks the deviation of the system from its classical behaviour. Note also that this equation (together with Eq. (48), known as the “guiding” formula) is precisely the starting point in the usual presentations of Bohmian quantum mechanics. The ‘extra’ term is known in such context as the quantum potential (e.g. Ref. 22, Ch. 3), although it is of kinetic origin. As already stated, this is the main source of quantum nonlocality (due to its dependence on the spatial distribution), even for a single particle, arising at the level of the reduced configuration-space description. As is well known, it is also what makes Bohm’s theory nonlocal.

Notice that for any solution of the Schrödinger equation (with the single exception of the free particle described by a plane wave) the quantum potential is different from zero. If the system is composed of two or more particles described by a nonfactorizable probability density, there are additional (interparticle) nonlocal contributions to the quantum potential, also having no classical analog. For instance, for a two-particle density \( \rho(x_1, x_2) = \rho_1(x_1) \rho_2(x_2) \rho_{12}(x_1, x_2) \) the quantum potential for particle 1 has the form

\[
Q_1(1, 2) = -\frac{2\beta^2}{m} \left[ \frac{1}{\sqrt{\rho_1}} \frac{\partial^2}{\partial x_1^2} \sqrt{\rho_1} + \frac{1}{\sqrt{\rho_{12}}} \frac{\partial^2}{\partial x_1^2} \sqrt{\rho_{12}} + \frac{1}{2} \left( \frac{\partial}{\partial x_1} \ln \rho_1 \right) \left( \frac{\partial}{\partial x_1} \ln \rho_{12} \right) \right].
\]

The first term corresponds to the usual single-particle quantum potential; the remaining two terms, however, are entangled contributions which produce the nonlocal effects characteristic of Bell’s correlations.

### 5.2 Some comments on quantum nonlocality

We have reached a fundamental conclusion, namely that any quantum system manifests nonlocality, the only possible exception being the free particle described by a plane wave. The existence of quantum nonlocalities is a well-known fact. A treatment of them is to be found in the literature related to the causal version of quantum mechanics proposed by Bohm, a theory in which the quantum potential plays a fundamental role. However, the most extensive line of research on quantum nonlocalities during the last decades is related to the Bell inequalities, a fact that unfortunately has led in some circles to the misleading idea that nonlocality is a property exclusive of multipartite quantum systems. It should be stressed that, independently of interpretation, the Schrödinger equation contains the quantum potential — even if not in an explicit form — and hence the associated quantum nonlocalities. What we have demonstrated here is that the so-called potential — really a kinetic contribution — results from the reduction of the statistical description in the phase space of the particle to its

\[^{6}\text{As follows from Eq. (44a), Eq. (45) refers to the local mean velocity, not to the individual local velocity of a given particle. Hence it seems necessary to add a stochastic component } \delta \dot{v}(x) \text{ to } \dot{v}(x) \text{ in Bohmian mechanics to get the true individual (stochastic) velocities. See e.g. Refs. (22), (23).}\]
configuration space. In this space the nonlocality is present, whereas the original description in the full phase space is as local as any statistical description can be.

As for multiparticle systems, as has just been seen, extra nonlocalities arise due to the correlations among variables; these are related to the Bell nonlocalities. The mechanism that gives rise to the entanglement between the components of a bipartite system within the present theory is discussed in detail in Refs. (24), (10).

The cause or origin of both the quantum fluctuations and quantum nonlocality has been a long standing problem. What is fluctuating and why remains undefined in quantum mechanics. Proposals on the origin of the quantum potential — the recognized source of nonlocality — can be counted by the dozen (see e.g. Refs. (17)-(22) and (25)), the mechanism put forward being frequently alien to quantum theory. Here we discover a simple and unifying answer: what fluctuates is the momentum of the particle due to the direct action of the fluctuating electromagnetic vacuum, and these fluctuations give rise to the quantum potential when transferred to configuration space.

It could still be argued that local theories cannot lead to nonlocal results. Of course in the realm of pure dynamics this is true. But the statistical account of an originally local description can have nonlocal features. Take as an example the Brownian motion of particles immersed in a fluid: the (partially averaged) local velocities and accelerations required for the statistical description in configuration space are determined by the full phase-space density. Thus, local laws can lead to nonlocalities associated with the (reduced) statistical description. This is the case in the present theory, that deals with particles immersed in a local field, solution of Maxwell’s equations.

From this perspective, quantum nonlocality does not refer to an ontological property, i.e., an ingredient of Nature, but to a property of the (quantum) description. It is a real nonlocality in the restricted — but operational — sense that it can be “observed”, if the description of the observed phenomenon is made within quantum mechanics. Then the predictions of the theory correspond to what we observe. The point is how we describe (and interpret) what we observe. In a full phase-space theory we would employ a local language; in the reduced, quantum mechanical description, we require a nonlocal one.

Of course ours is not the first and only theory from which it follows that the violation of Bell’s inequalities does not rule out local realism. For example, since long ago it has been argued by a number of authors that the violation of a Bell inequality is due to the failure not of locality, but of the assumption of joint measurability; see in particular Refs. (26) and (27). More recent excellent examples are advanced by Khrennikov (see e.g. Refs. (28)-(30)) and T. M. Nieuwenhuizen, as well as by ’t Hooft in a somewhat different direction.

5.3 Energy balance and the value of $\beta$

Let us now look into the condition required to attain a state of constant mean energy. From Eq. it follows, assuming again that all surface terms vanish
at infinity, that
\[ \frac{1}{2m} \frac{d}{dt} \langle p^2 \rangle = \frac{1}{2m} \frac{d}{dt} \int p^2 Q dx dp = \frac{1}{m} \left\{ fp + \frac{\tau}{m} f' p^2 - \frac{e^2}{2} p \hat{D} \right\}. \]  

(51a)

Since \( \langle fp \rangle / m = -d \langle V \rangle / dt \), the average energy gained or lost by the quantum-mechanical system through radiation exchange is given by
\[ \frac{d}{dt} \langle H \rangle = \frac{d}{dt} \left\{ \frac{1}{2m} p^2 + V \right\} = \frac{\tau}{m^2} \langle f' p^2 \rangle - \frac{e^2}{2m} \langle p \hat{D} \rangle, \]  

(52)

where \( H \) represents the mechanical Hamiltonian of the particle. The first term on the right-hand side of Eq. (52) gives the average power dissipated by the particle along its orbit due to Larmor radiation. The last term, on the other hand, represents the mean power absorbed by the fluctuating particle from the field. In particular, when these terms cancel one another, namely for those averaged motions such that
\[ \frac{\tau}{m^2} \langle f' p^2 \rangle = \frac{e^2}{2m} \langle p \hat{D} \rangle, \]  

(53)

the value of \( \langle H \rangle \) becomes a constant and an energy balance is reached. Note, incidentally, that this equation plays the role of a fluctuation-dissipation relationship for the stochastic (quantum) system.

In particular, when both the background field and the mechanical system are in their ground state, there is no net exchange of energy and Eq. (53) is satisfied. This equation can then be used to determine the value of the parameter \( \beta \), as follows. The calculation of the left-hand side to lowest order in \( e^2 \) (or \( \tau \)) can be done by resorting to the (zero-order) solutions of Eq. (41), which leads to a result that contains the linear factor \( \beta \). The right-hand side, on the other hand, is a complicated expression containing second and higher moments of the electric component of the background field. However, to lowest order in \( e^2 \) the term on the right-hand side becomes proportional to the spectral energy density of the zpf, which as Eq. (6) shows, is linear in Planck’s constant \( \hbar \). The detailed calculation, to be presented in a paper in preparation, then leads directly to the result
\[ \beta = \frac{1}{2} \hbar. \]  

(54)

This completes the identification of Eq. (11) with the Schrödinger equation. Further to indicating the point of entry of Planck’s constant \( \hbar \) into the quantum description, the derivation leading to Eq. (54) exhibits the unique role that the zpf, with the specific spectrum given by Eq. (6), plays in achieving energy balance in the quantum case.

Some time ago Wallstrom criticized theories in which the Schrödinger equation is derived from the pair of equations (43), arguing that the theory described by this couple of equations allows in general for more solutions than
those afforded by the corresponding Schrödinger equation. However this critique does not apply to the present case because in addition to Eqs. (43), the system must comply with Eq. (53), the energy balance condition that can be satisfied in general only by a discrete set of solutions. Quantization has therefore a deep physical root within the present approach. Although the fundamental equations in various phenomenological accounts of quantum mechanics—such as Bohmian mechanics, Nelson’s theory or stochastic quantum mechanics—match with the present ones, it is the crucial condition provided by Eq. (53) what makes the difference. For an additional answer to Wallstrom’s criticism see Ref. (19).

### 5.4 The extremum value of the energy

It is appropriate here to recollect a suggestive derivation related to Eq. (31). Let us consider a stationary state with \( v = 0 \); equation (31) gives then

\[
\langle p^2 \rangle_x = -\frac{\hbar^2}{4} \frac{\partial^2}{\partial x^2} \ln \rho.
\]

The mean energy of the mechanical system is given by the mean value of its Hamiltonian, thus

\[
\langle H \rangle = \int \rho(x) \left[ -\frac{\hbar^2}{8m} \frac{\partial^2}{\partial x^2} \ln \rho + V(x) \right] dx.
\]

We demand now this energy to be an extremum under conservation of probability, \( \int \rho(x) dx = 1 \). With the probability density written in the form \( \rho = \psi^2 \) (which is consistent with the assumption \( v = 0 \)), this variational problem has as solution the Euler-Lagrange equation

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = \langle H \rangle \psi,
\]

where \( \langle H \rangle \) is the Lagrange multiplier. This result emphasizes the remarkable role played by the local dispersion of the momentum, Eq. (55): it guarantees that the stationary (quantized) distribution of particles corresponds to a local extremum (normally a minimum) of the mean energy of the system. We thus discover that the stochastic velocity \( u \) referred to above (see footnote 15 and well known in the context of stochastic quantum mechanics) is instrumental in leading to the quantum description through its divergence, as follows from the relation

\[
-\frac{1}{2} \beta \frac{\partial u}{\partial x} = \frac{\beta^2}{2m} \frac{\partial^2 \ln \rho}{\partial x^2} = \frac{\langle p^2 \rangle}{2m}.
\]
Equation (58) shows that the gradient of the probability density is intimately related to the local description of the stochastic component $\mu_\rho$ of the momentum density. Hence the description of the velocity in terms of the function $\langle p \rangle_x = mv(x)$ alone does not settle the question about the quantum motions. Actually, quantum mechanics deals very smoothly and elegantly with these matters, although not paying too much attention to the ultimate meaning of the results. In fact, from Eqs. (30) and (58) it follows that
\[ -i\hbar \frac{\partial \psi}{\partial x} = m(v - iu) \psi = \hat{p} \psi, \]  
where the last equality has been taken from the usual quantum formalism, $\hat{p} = -i\hbar \partial_x$. The momentum operator takes thus both components of the velocity into account, although the velocity $u$ (and even the velocity $v$) remains concealed in the standard formalism.

6 Why the Wigner distribution is not a true probability density

Since our starting point, Eq. (1), involved a full description in the entire particle-field phase space, it is interesting to explore whether a description in the particle’s phase space can still be recovered from the reduced description of quantum mechanics. For this purpose we invert Eq. (7) and combine it with Eqs. (9) and (10), obtaining
\[ Q(x, p, t) = \frac{1}{2\pi} \int \tilde{Q}(x, z, t)e^{-ipz}dz = \frac{1}{2\pi} \int q_+(x + \beta z, t)q_-(x - \beta z, t)\chi(x + \beta z, x - \beta z, t)e^{-ipz}dz. \]  
Equation (61) furnishes indeed a true (Kolmogorovian) probability density in phase space. On the other hand, the probability density proper of quantum mechanics in the radiationless approximation is not this $Q(x, p, t)$, as we have seen, but its approximate form $W(x, p, t)$ that ensues from putting $\tilde{Q}(x, z, t) = q(z_+, t)q^*(z_-, t) = \psi(z_+, t)\psi^*(z_-, t)$ as follows from Eq. (13) with $\chi = 1$ and Eq. (40). We thus obtain instead
\[ W(x, p, t) = \frac{1}{\pi\hbar} \int \psi^*(x + y, t)\psi(x - y, t)e^{-i2py/\hbar}dy, \]  
with $y = \beta z = hz/2$. This is the well-known Wigner phase-space function (see e.g. Refs. (36), (37)). However, it must be noted that whilst in Eq. (61) the integral runs from $-\infty$ to $+\infty$, the functions $\psi^*$, $\psi$ as given by the Schrödinger equation can be taken as approximate representatives of $q_+, q_-$, respectively, only for small values of $y$ and a constant $\chi$, since only then can Eq. (39) be separated into the equations for $\Psi_+ = q_+\chi(z_+, z_-)$ and $\Psi_- = q_-\chi(z_+, z_-)$. 

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One cannot take $W$ to be a true Kolmogorovian probability in general. And indeed, despite its recognized value, it is not, since as is well known it can take on negative values in some regions of phase space for almost all states and systems. The right solution to this long-standing problem is of course to recognize the intrinsic limitation of $W$ that ensues from its approximate nature and to try revert to the original $Q(x,p,t)$. This, however, being the solution of the integro-differential equation $Q$, is a much more complicated function, which evolves with time as the system approaches the state characterized by quantum mechanics. There remains here an interesting problem to be explored.

7 Concluding remarks and discussion

A first conclusion is that the quantum mechanical description provided by the Schrödinger equation emerges naturally from a Fokker-Planck-type equation (of infinite order) in phase space, namely Eq. (3), under several simplifying assumptions that can be made only when the entire system has reached energy equilibrium. The original equation remains outside the limits of quantum mechanics, the latter emerging as a partially averaged, asymptotic radiationless theory. As a consequence, even though the numerical correctness of quantum mechanics is out of question for present-day use, its physical transparency falls far from the mark. The ultimate reason for this singular situation resides in the fact that the cause of the quantum behavior — identified here with the stochastic zero-point radiation field in interaction with matter — remains hidden in the background once the complete description in phase space is reduced to the particle’s configuration space and the radiationless approximation is taken. This leads to the well-known abstruse narrative of the quantum world, whose statistical origin remains hidden and in which apparently noncausal and nonlocal effects take place. According to the above discussion, the so-called quantum nonlocality arises as soon as the effects of the fluctuations in momentum space (due to the zpf) are transferred to configuration space. In the full phase-space description everything is local; the nonlocality appears only in the reduced description. Indeed, the Schrödinger equation is shown to be intrinsically nonlocal due to the term containing $\sqrt{\rho}$ in Eq. (19), which is but a manifestation (on the mechanical subsystem) of the stochastic nature of the field. These observations — which serve to disclose also the physical cause behind the dispersive nature of the quantum systems — alert us not to interpret nonlocality as inherent to the physical system, but rather as a distinctive feature of the quantum description.

The present work serves to confirm that the stochastic process underlying quantum mechanics is of a different nature (and requires befitting methods to handle it) from that of classical stochastic motions. In particular, it is remarkable that even though there exists a true (integro-differential) Fokker-Planck-type equation in phase space, the appropriate (radiationless) description of quantum mechanics can be made in terms of just a couple of equations, one corresponding to the much simpler continuity equation, the other being close to the classical Hamilton-Jacobi equation, but with a statistical term originat-
ing in the fluctuations in momentum space, over which an average has been performed.

The transition from Eqs. (24a), (24b) to Eqs. (43) is clearly an irreversible procedure: relevant information is lost by disconnecting the former from the rest of the hierarchy. Once this has been done one cannot transit from Eqs. (43) back to Eq. (4) on purely logical steps. In particular, one cannot reconstruct the true probability density in phase space from the (approximate) Wigner function. This explains the origin of the large number of existing phase-space versions of quantum mechanics, each one carrying their correlative correspondence rules, which become an addition to usual quantum mechanics (see e.g. Ref. (12)).

It is appropriate to bear in mind that although the zpf could appear as a sort of collection of hidden variables, introduced with the aim of completing the quantum description, this is not the case here, since the zpf is not an ingredient simply added on top of the quantum-mechanical formalism to make it deterministic. Quite the contrary: nothing is here added to quantum mechanics, but quantum mechanics emerges from a more general theory that embodies the zero-point field. The emerging description is then naturally indeterministic, since in every case the specific realization of the field is unknown, the description having a statistical nature right from the outset. It is important to note that the particle (the electron, say) remains always a particle, the existing wave (the zpf) remaining literally in the background. The trajectories belong to subensembles characterized by the local mean velocity $v(x)$.

The above theory is in principle applicable to an arbitrary number of particles. Yet it is important to realize that the quantum nonlocality discussed here emerges already in the one-particle case. For a problem of several particles, instead of Eqs. (2), we would have a system of (originally local, stochastic) equations coupled through the respective radiation reaction terms. This leads frequently to a nonseparable phase-space distribution function, and eventually to a solution $\psi$ that is nonfactorizable. When this occurs, the probability density (in configuration space) contains several terms, including those describing interferences among single-particle states. The corresponding quantum potential contains, in addition to the various single-particle terms, extra multiparticle contributions produced by correlations among the dynamic variables, that give rise to additional nonlocalities, as Eq. (50) illustrates. A most striking manifestation of such ‘extra’ nonlocalities due to correlations among the particles is related to Bell’s theorem. An explanation of the origin of such additional nonlocalities in the composite problem, or rather of the mechanism that entangles the particles, is particularly important. In Refs. (9), (24) and (10) it

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8 By its form, this equation suggests a kind of fluid. This is true of course only for the one particle case (a 3-D fluid). However in any case a considerable stretch of imagination is required to interpret it in terms of a quantum fluid. Still, the formal analogy was stressed by Madelung (39) from the very inception of quantum mechanics and continues to be used on occasions. See e.g Refs. (40) and (41).

9 The term indeterministic is used consistently in this paper (as in our work in general) to refer to a description that ignores the specific realizations in an ensemble – without this, of course, having any implications on the causality or locality of the system (at the ontological level).
is shown how the zpf can give rise to the entanglement of two noninteracting particles embedded in the common background field. Specifically, it is shown that even when there is no external interaction potential between the particles, their interaction through the background field can correlate and entangle them in such a way that apparent nonlocalities arise at the level of the quantum description.

Our results serve to explain the success of several works within stochastic electrodynamics. From among those of more direct interest we recall the numerical calculations of Cole et al leading to a correct prediction of the ground state orbit for the H-atom, the description of spontaneous transitions and more generally radiative corrections, and other treatments that complete the perspective offered by the present one.

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