COLLAPSING HYPERKÄHLER MANIFOLDS

VALENTINO TOSATTI AND YUGUANG ZHANG

ABSTRACT. Given a projective hyperkähler manifold with a holomorphic Lagrangian fibration, we prove that hyperkähler metrics with volume of the torus fibers shrinking to zero collapse in the Gromov-Hausdorff sense (and smoothly away from the singular fibers) to a compact metric space which is a half-dimensional special Kähler manifold outside a singular set of real Hausdorff codimension 2.

1. Introduction

Let $M^m$ be a compact Calabi-Yau manifold, which for us is a compact Kähler manifold $M^m$ with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Yau’s Theorem [64] shows that given any Kähler class $[\alpha]$ on $M$ we can find a unique representative $\omega$ of $[\alpha]$ which is a Ricci-flat Kähler metric. The basic problem that we study in this paper is to understand the limiting behavior of such Ricci-flat metrics if we degenerate the class $[\alpha]$. More precisely, we fix a class $[\alpha_0]$ on the boundary of the Kähler cone and for $0 < t \leq 1$ we let $\tilde{\omega}_t$ be the unique Ricci-flat Kähler metric in the class $[\alpha_0] + t[\omega_M]$, where $\omega_M$ is a fixed Ricci-flat Kähler metric on $M$, and we wish to understand the behavior of $(M, \tilde{\omega}_t)$ as $t \to 0$. The metrics $\tilde{\omega}_t$ satisfy the equation

\[ \tilde{\omega}_t^m = c_t t^{m-n} \omega_M^m, \]

for some explicit constants $c_t$ which approach a positive constant as $t \to 0$. Up to scaling the whole setup, we may assume without loss of generality that $c_t \to 1$ when $t \to 0$.

This question has been extensively studied in the literature, the most relevant works being [25, 56, 57, 23, 24, 27, 61, 63, 9, 52], see also the surveys [58, 59, 66]. In particular, decisive results in the non-collapsing case when $\int_M \alpha_0^n > 0$ have been obtained in [56, 9, 52]. In this paper we consider the more challenging collapsing case when $\int_M \alpha_0^n = 0$, and we will always assume that $[\alpha_0] = f^*[\omega_N]$ where $(N^n, \omega_N)$ is a compact Kähler manifold with $0 < n < m$ and $f : M \to N$ is a holomorphic surjective map with connected fibers (i.e. a fiber space). This is the same setup as in [57, 23, 24, 27, 61, 63], and as explained there, in this case there are proper analytic subvarieties $S' \subset N$ and $S = f^{-1}(S') \subset M$ such that $f : M \setminus S \to N_0 := N \setminus S'$ is a proper submersion with fibers $M_y = f^{-1}(y), y \in N \setminus S'$, smooth Calabi-Yau $(m-n)$-folds.

In [61], building upon the earlier [57], it is shown that there is a Kähler metric $\omega$ on $N_0$ such that as $t \to 0$ the metrics $\tilde{\omega}_t$ converges to $f^* \omega$ uniformly
on compact subsets of $M\setminus S$, and $\omega$ satisfies
\[ \text{Ric}(\omega) = \omega_{\text{WP}} \geq 0, \]
on $N_0$, where $\omega_{\text{WP}}$ is a Weil-Petersson form which measures the variation of the complex structures of the fibers $M_y$ (see e.g. [57, 53]). This is improved to smooth convergence on compact subsets of $M\setminus S$ in [23, 27, 63] when the fibers $M_y$ are tori (or finite étale quotients of tori). Explicit estimates are also obtained for $\bar{\omega}$ near $S$, but these blow up very fast near $S$.

Our main concern is understanding the possible collapsed Gromov-Hausdorff limits of $(M,\bar{\omega}_t)$ as $t \to 0$, and their singularities. In this regard, we have the following conjecture (see [58, Question 4.4] [59, Question 6]), which is motivated by an analogous conjecture by Gross-Wilson [25], Kontsevich-Soibelman [36, 37] and Todorov [41] for collapsed limits of Ricci-flat Kähler metrics on Calabi-Yau manifolds near a large complex structure limit:

**Conjecture 1.1.** If $(X,d_X)$ denotes the metric completion of $(N_0,\omega)$, and $S_X = X\setminus N_0$, then $(X,d_X)$ is a compact length metric space, $S_X$ has real Hausdorff codimension at least 2, and
\[ (M,\bar{\omega}_t) \xrightarrow{d_{\omega_{\text{WP}}}} (X,d_X), \]
when $t \to 0$.

This conjecture was proved by Gross-Wilson [25] when $f : M \to N$ is an elliptic fibration of $K3$ surfaces with only $I_1$ singular fibers. In our earlier work with Gross [24] we proved Conjecture 1.1 completely in the case when $\dim N = 1$. There it is also proved that $X$ is homeomorphic to $N$. Very recently a new proof was obtained in [39] when $\dim M = 3, \dim N = 1$, the generic fibers $M_y$ are $K3$ surfaces and the singular fibers are nodal $K3$ surfaces, which also gives better estimates near and on the singular fibers.

For bases $N$ of general dimension $n$, the only known partial result towards Conjecture 1.1 is the one proved in [23]: if $(X,d_X)$ is the Gromov-Hausdorff limit of a sequence $(M,\bar{\omega}_{t_i}), t_i \to 0$ (such limits always exist up to passing to subsequences), then there is a homeomorphism $\psi : N_0 \to X_0$ onto a dense open subset $X_0 \subset X$ such that $\psi : (N_0,\omega) \to (X_0,d_X|_{X_0})$ is a local isometry.

Our main result is the following:

**Theorem 1.2.** Conjecture 1.1 holds when $M$ is a projective hyperkähler manifold.

As proved in [24], in this case the limiting metric $\omega$ on $N_0$ is a special Kähler metric in the sense of [13]. In this case, the base $N$ is always $\mathbb{CP}^n$ [31] and the fibers $M_y, y \in N_0$, are holomorphic Lagrangian $n$-tori [44, 45], so that $f$ is an algebraic completely integrable system over $N_0$. A classical result of Donagi-Witten [12] (see also [13, 30]) shows that the base of an algebraic completely integrable system admits a special Kähler metric, and our result shows that this metric arises as the collapsed limit of hyperkähler metrics on the total space.
An application of our result is the revised Strominger-Yau-Zaslow (SYZ) conjecture due to Gross-Wilson [25], Kontsevich-Soibelman [36, 37] and Todorov [41]. As explained in [23], Theorem 1.2 implies a positive solution to such conjecture for collapsed limits of hyperkähler metrics near large complex structure limits which arise via hyperkähler rotation from our setting above:

**Corollary 1.3.** *The conjecture of Gross-Wilson [25, Conjecture 6.2], Kontsevich-Soibelman [36, Conjectures 1 and 2] and Todorov [41, p. 66] holds for those large complex structure families of hyperkähler manifolds which arise from the setup of Theorem 1.2 via hyperkähler rotation as in [23, Theorem 1.3].*

Indeed, this follows exactly as in [23, Theorem 1.3], using Theorem 1.2 together with our earlier results in [24, Theorem 1.2]. The key new information provided by Theorem 1.2, which was not available in [23, 24], is the uniqueness of the Gromov-Hausdorff limit, which is identified with the metric completion of the smooth part, and the fact that it has singularities in real codimension at least 2. This completes the program we started in [24] to extend Gross-Wilson’s theorem on large complex structure limits of $K3$ surfaces [25] to higher-dimensional hyperkähler manifolds. Furthermore, by combining it with [24], Theorem 1.2 gives more precise information about the limit space, which was predicted by [25, 36, 37]. This is explained in detail in section 5 (see Theorem 5.2 there) in a slightly more general setup than [23].

We now give a brief outline of the paper. In section 2 we extend and sharpen a method introduced in our earlier work [24] (when dim $N = 1$) and show that to prove Conjecture 1.1 in full generality it suffices to obtain an upper bound for the limiting metric $\omega$ near $S'$ (which may be assumed to be a simple normal crossings divisor after a modification) in terms of an orbifold Kähler metric up to a logarithmic factor. In section 3 we give some improvements of earlier results of ours, and state the precise estimate that we obtain in the hyperkähler case, which implies the estimate needed in section 2. This estimate is then proved in section 4 by showing that the coefficients of the special Kähler metric $\omega$ are essentially given by periods of the Abelian varieties which are the fibers of $f$, and the blowup rate of these periods can be controlled using degenerations of Hodge structures. Lastly, in section 5 we explain how Theorem 1.2 fits into the SYZ picture of mirror symmetry for hyperkähler manifolds.

**Acknowledgments.** We thank H.-J. Hein, M. Popa and J. Song for useful discussions, and Y.S. Zhang for comments. Part of this work was done during the first-named author’s visit to the Yau Mathematical Sciences Center at Tsinghua University in Beijing and during the second-named author’s visit to the Department of Mathematics at Northwestern University, which we would like to thank for the hospitality. The first-named author was partially supported by NSF grant DMS-1610278.
2. Gromov-Hausdorff Collapsing

In this section we reduce Conjecture 1.1 in general to proving a suitable bound for the limiting metric near the discriminant locus of the map $f$, in terms of an orbifold Kähler metric on some log resolution of the discriminant locus of $f$. This bound will then be proved in section 4 for hyperkähler manifolds.

Let $f : M^m \to N^n$ be as in the Introduction, so $M$ is a compact Calabi-Yau manifold, $N$ is a compact Kähler manifold, $f$ is holomorphic surjective with connected fibers, and $0 < n < m$. The discriminant locus of $f$ (i.e. the locus of critical values of $f$) is denoted by $S' \subset N$, and is a proper analytic subvariety of $N$. As an aside, we will see in Theorem 3.3 below (a small extension of our earlier result in [62]) that $S' = \emptyset$ happens if and only if $f$ is a holomorphic fiber bundle, with base $N$ and fiber also Calabi-Yau manifolds. This is a very special situation, and in fact never happens if $M$ is hyperkähler (since in this case $N \cong \mathbb{CP}^n$ [91]). In any case, we may assume in the following that $S' \neq \emptyset$, since otherwise Conjecture 1.1 follows immediately from the results in [61] which give uniform convergence of $\tilde{\omega}_t$ to $f^*\omega$.

Let $\pi : \tilde{N} \to N$ be a modification such that $E = \pi^{-1}(S')$ is a divisor with simple normal crossings and $\pi : \pi^{-1}(N_0) \to N_0$ is biholomorphic, where $N_0 = N \setminus S'$. Write $E = \bigcup_{j=1}^{\mu} E_j$ for the decomposition of $E$ in irreducible components, so each $E_j$ is a smooth irreducible divisor and the $E_j$'s intersect in normal crossings. There is an integer $0 \leq \ell \leq \mu$ so that the divisors $E_j$ with $1 \leq j \leq \ell$ are $\pi$-exceptional, while $E_j$ with $\ell + 1 \leq j \leq \mu$ are proper transforms of divisors in $S'$. The limit cases $\ell = 0, \mu$ are allowed, where $\ell = 0$ means that $S'$ is already a simple normal crossings divisor and $\pi = \text{Id}$, while $\ell = \mu$ means that $S'$ is of (complex) codimension at least 2 in $N$.

Given natural numbers $m_i \in \mathbb{N}_{>0}$, $1 \leq i \leq \mu$, there is a well-defined notion of orbifold Kähler metric $\omega_{orb}$ on $\tilde{N}$ with singularities along $E$ with orbifold order $m_i$ along each component $E_i$. Any such metric is a smooth Kähler metric on $\tilde{N} \setminus E$ such that in any local chart $U$ (a unit polydisc with coordinates $(w_1, \ldots, w_n)$) centered at a point of $E$ adapted to the normal crossings structure (so $E \cap U$ is given by $w_1 \cdots w_k = 0$ for some $1 \leq k \leq n$, and say that $\{w_i = 0\} = E_j \cap U$ for some $1 \leq j_i \leq \mu$ and all $1 \leq i \leq k$) we have that pulling back $\omega_{orb}$ by the local uniformizing map $q : \tilde{U} \to U$ ($\tilde{U}$ is also the unit polydisc in $\mathbb{C}^n$) given by

$$q(w_1, \ldots, w_n) = (w_1^{m_{j_1}}, \ldots, w_k^{m_{j_k}}, w_{k+1}, \ldots, w_n),$$

the resulting metric on $\tilde{U} \setminus \{w_1 \cdots w_k = 0\}$ extends smoothly to a Kähler metric on $\tilde{U}$. This implies that on $U \setminus \{w_1 \cdots w_k = 0\}$ the metric $\omega_{orb}$ is uniformly equivalent to the model

$$\sum_{i=1}^{k} \frac{\sqrt{-1}dw_i \wedge d\bar{w}_i}{|w_i|^{2(1-1/m_{j_i})}} + \sum_{i=k+1}^{n} \sqrt{-1}dw_i \wedge d\bar{w}_i.$$
We also fix a defining section \( s_i \) of the divisor \( E_i \) and a smooth Hermitian metric \( h_i \) on \( O(E_i) \), for all \( 1 \leq i \leq \mu \). Then a short calculation shows that given any Kähler metric \( \omega_N \) on \( \tilde{N} \) and any \( \varepsilon > 0 \) sufficiently small, the formula

\[
\omega_{\text{orb}} = \omega_N + \varepsilon \sum_{i=1}^{\mu} \sqrt{-1} \partial \bar{\partial} |s_i|_{h_i}^{m_i},
\]

defines an orbifold Kähler metric on \( \tilde{N} \) with orbifold order \( m_i \) along each \( E_i \). This shows that we can always find orbifold Kähler metrics adapted to any given orbifold structure.

The following result can be viewed as a generalization of [24, Section 3] to higher dimensions:

**Theorem 2.1.** Suppose that there is a constant \( C > 0 \) and natural numbers \( d \in \mathbb{N} \) and \( m_i \in \mathbb{N}_{>0}, \ 1 \leq i \leq \mu \), such that on \( \pi^{-1}(N_0) \) we have

\[
(2.1) \quad \pi^* \omega \leq C \left( 1 - \sum_{i=1}^{\mu} \log |s_i|_{h_i} \right)^d \omega_{\text{orb}},
\]

where \( \omega_{\text{orb}} \) is an orbifold metric with orbifold order \( m_i \) along each component \( E_i \). Then Conjecture 1.1 holds.

**Proof.** Let us define \( E'_{n+1} = \emptyset \) and for \( 1 \leq p \leq n \) define recursively

\[
(2.2) \quad E'_p = \bigcup_{|J|=p} (E_{j_1} \cap \cdots \cap E_{j_p}) \setminus E'_{p+1},
\]

where the union is over all multiindices \( J = (j_1, \ldots, j_p) \) with \( 1 \leq j_1, \ldots, j_p \leq \mu \). Therefore each \( E'_p \) is a (possibly empty) disjoint union of smooth connected \((n-p)\)-dimensional relatively compact complex submanifolds of \( \tilde{N} \) (the real \((2n-2p)\)-Hausdorff measure of \( E'_p \) is therefore finite), and we can write \( E = \bigcup_{p=1}^n E'_p \). Note also that if \( U \) is any small open neighborhood of \( E'_{p+1} \), then \( E'_p \setminus U \) is compact. Using this we see that for every small \( \rho, \beta > 0 \) we can find a covering of \( E \) by \( N(\rho) \) open sets \( \{V_i(\rho)\} \subset \tilde{N} \) such that we have

\[
(2.3) \quad \rho^{2n-2+\beta} N(\rho) \to 0,
\]
as \( \rho \to 0 \), and each \( V_i(\rho) \) is contained in a chart with coordinates \((w_1, \ldots, w_n)\) defined in the unit polydisc \( \Delta^n \) where \( E \) is given locally by \( w_1 \cdots w_k = 0 \) for some \( 1 \leq k \leq n \), and in this chart we have

\[
V_i(\rho) = \{ w \in \Delta^n \mid |w_j| < \rho^{m_i}, \text{ for } 1 \leq j \leq k, \text{ and } |w_j| < \rho, \text{ for } k+1 \leq j \leq n \},
\]

where for simplicity of notation we will write \( m_j \) for the orbifold order along \( \{w_j = 0\} \). Define a map \( q: \Delta^n \to \Delta^n \) by

\[
q(w_1, \ldots, w_n) = (w_1^{m_1}, \ldots, w_k^{m_k}, w_{k+1}, \ldots, w_n),
\]
so that \( q^{-1}(V_i(\rho)) \) equals the polydisc of radius \( \rho \), which we will denote by \( \Delta^n(\rho) \). Then our assumption (2.1) implies that on \( \Delta^n \setminus E \) we have that

\[
q^* \pi^* \omega \leq C \left( 1 - \sum_{i=1}^{k} \log |w_i| \right)^d \omega_E,
\]

for some \( C > 0, d \in \mathbb{N} \), where \( \omega_E \) is the Euclidean metric on \( \Delta^n \). Using this, we claim that any two points \( q_1, q_2 \in \Delta^{*k}(\rho) \times \Delta^{n-k}(\rho) \subset \Delta^n(\rho) \) there is a path \( \tilde{\gamma} \) connecting them such that \( \tilde{\gamma} \subset \Delta^{*k}(\rho) \times \Delta^{n-k}(\rho) \) and

\[
\text{length}_{\pi^*\omega}(\tilde{\gamma}) \leq C \rho(-\log \rho)^d.
\]

Indeed, if we denote \( w_i = r_i e^{\sqrt{-1} \theta_i} \), then on \( \Delta^n \setminus E \) the estimate (2.4) translates to

\[
q^* \pi^* \omega \leq C \left( 1 - \sum_{i=1}^{k} \log r_i \right)^d \sum_{j=1}^{n} (d r_{j}^2 + r_{j}^2 d \theta_{j}^2).
\]

Write

\[
q_1 = (r_1 e^{\sqrt{-1} \theta_1}, \ldots, r_n e^{\sqrt{-1} \theta_n}),
\]

where \( 0 < r_j < \rho \) for \( 1 \leq j \leq k \), while \( 0 \leq r_j \leq \rho \) for \( k + 1 \leq j \leq n \), and if \( r_j = 0 \) for any such \( j \) then we set \( \theta_j = 0 \). We then define a path

\[
\gamma_1(s) = ((sp/2 + (1-s)r_1)e^{\sqrt{-1} \theta_1}, \ldots, (sp/2 + (1-s)r_n)e^{\sqrt{-1} \theta_n}),
\]

with \( 0 \leq s \leq 1 \) gives a path in \( \Delta^{*k}(\rho) \times \Delta^{n-k}(\rho) \) whose initial point is \( q_1 \) and whose endpoint lies on the distinguished boundary of \( \Delta^n(\rho/2) \), given by

\[
S(\rho/2) = \{ w \in \Delta^n | |w_j| = \rho/2, 1 \leq j \leq n \}.
\]

The Euclidean norm of \( \gamma_1 \) is at most \( \rho \), and so using (2.6) we obtain

\[
\text{length}_{q^*\pi^*\omega}(\gamma_1) \leq C \rho \int_0^1 \left( 1 - \sum_{i=1}^{k} \log (sp/2 + (1-s)r_i) \right)^{\frac{d}{2}} ds
\]

\[
\leq C \rho \int_0^1 (1 - k \log (sp/2))^{\frac{d}{2}} ds
\]

\[
\leq C \rho (-\log \rho)^d,
\]

where we used the fact that \( \rho \) is small to increase the power of \(-\log\). On the other hand the distinguished boundary \( S(\rho/2) \) is diffeomorphic to the real torus \( T^n \) (in particular it is connected), and using again (2.6) we see that

\[
\text{diam}_{q^*\pi^*\omega}(S(\rho/2)) \leq C \rho(-\log \rho)^{\frac{d}{2}} \leq C \rho(-\log \rho)^d.
\]

Therefore we conclude that \( q_1 \) and \( q_2 \) can indeed be joined by a curve \( \tilde{\gamma} \subset \Delta^{*k}(\rho) \times \Delta^{n-k}(\rho) \) satisfying (2.3). Considering the image \( \gamma = q(\tilde{\gamma}) \), whose \( \pi^*\omega \)-length is equal to the \( q^*\pi^*\omega \)-length of \( \tilde{\gamma} \), we conclude that every two points in \( V_i(\rho) \setminus E \) can be joined by a path \( \gamma \) contained in \( V_i(\rho) \setminus E \) with

\[
\text{length}_{\pi^*\omega}(\gamma) \leq C \rho(-\log \rho)^d.
\]
Since the open sets \( \{ V_i(\rho) \} \) cover \( E \), and since \( \rho(-\log \rho)^d \to 0 \) as \( \rho \to 0 \), it follows in particular that
\[
\sup_{y_1,y_2 \in N_0} d_\omega(y_1,y_2) \leq C,
\]
where \( d_\omega \) is the metric space structure on \( N_0 \) induced by \( \omega \). This implies that the metric completion of \((N_0,d_\omega)\) is compact.

Now pick any sequence \( t_k \to 0 \) such that \((M,\tilde{w}_{t_k})\) converges in the Gromov-Hausdorff topology to a compact length metric space \((X,d_X)\). As we recalled in the Introduction, in [23, Corollary 1.4] (see also [23, Theorem 1.2]) we constructed a local isometric embedding of \((N_0,\omega)\) into \((X,d_X)\) with open dense image \( X_0 \subset X \) via a homeomorphism \( \psi : N_0 \to X_0 \). Call \( S_X = X \setminus X_0 \). The density of \( X_0 \) implies that for every fixed \( \rho > 0 \) the set
\[
\bigcup_{i=1}^{N(\rho)} \psi(\pi(V_i(\rho)) \cap N_0),
\]
covers \( S_X \). Then the fact that \((X,d_X)\) is a length space implies that for every \( i \) we have
\[
diam_{d_X}(\psi(\pi(V_i(\rho)) \cap N_0)) = diam_{d_X}(\psi(\pi(V_i(\rho)) \cap N_0))
= \sup_{\rho,q \in \psi(\pi(V_i(\rho)) \cap N_0)} \inf_{\eta} \text{length}_{d_X}(\eta),
\]
where the infimum is over all curves \( \eta \) in \( X \) joining \( p \) and \( q \). But we have just shown that \( \rho \) and \( q \) can be joined by curves of the form \( \psi(\pi(\gamma)) \), with \( \gamma \subset V_i(\rho) \setminus E \) satisfying (2.7), and since \( \psi \) is a local isometry we have that
\[
\text{length}_{d_X}(\psi(\pi(\gamma))) = \text{length}_\omega(\pi(\gamma)) = \text{length}_{\pi^*\omega}(\gamma),
\]
for any such curve \( \gamma \). We then conclude that
\[
diam_{d_X}(\psi(\pi(V_i(\rho)) \cap N_0)) \leq C\rho(-\log \rho)^d,
\]
for all \( \rho > 0 \) small and for \( C > 0 \) independent of \( \rho \).

Fix now small \( \beta, \varepsilon > 0 \), and given any \( \eta > 0 \), choose \( \rho > 0 \) small so that \( C\rho(-\log \rho)^d < \eta \). We estimate
\[
\mathcal{H}^{2n-2+\beta+\varepsilon}_{d_X,N}(S_X) \leq \sum_{i=1}^{N(\rho)} \omega_{2n} \text{diam}_{d_X}^{2n-2+\beta+\varepsilon}(\psi(\pi(V_i(\rho)) \cap N_0))
\leq C N(\rho) \rho^{2n-2+\beta+\varepsilon}(-\log \rho)^{d(2n-2+\beta+\varepsilon)}
= C \rho^{2+(-\log \rho)^{d(2n-2+\beta+\varepsilon)}(N(\rho)\rho^{2n-2+\beta})}
\to 0,
\]
as \( \rho \to 0 \) thanks to (2.3), where \( \omega_{2n} \) denotes the volume of unit ball in \( \mathbb{R}^{2n} \).

Note that as \( \eta \to 0 \) then \( \rho \to 0 \) as well. Thus
\[
\mathcal{H}^{2n-2+\beta+\varepsilon}_{d_X,N}(S_X) = \lim_{\eta \to 0} \mathcal{H}^{2n-2+\beta+\varepsilon}_{d_X,N}(S_X) = 0,
\]
for any small \( \beta, \varepsilon > 0 \), and so we conclude that \( \dim \mathcal{H} S_X \leq 2n - 2 \).
Note also that for any two points \(x, y \in N_0\) we have
\[
(2.10) \quad d_X(\psi(x), \psi(y)) \leq d_\omega(x, y).
\]
Indeed, for any \(\varepsilon > 0\) there is a path \(\gamma_\varepsilon\) in \(N_0\) joining \(x\) and \(y\) with length\(_\omega(\gamma_\varepsilon) \leq d_\omega(x, y) + \varepsilon\). From (2.8) we see that
\[
\text{length}_\omega(\gamma_\varepsilon) = \text{length}_{d_X}(\psi(\gamma_\varepsilon)) \geq d_X(\psi(x), \psi(y)),
\]
and letting \(\varepsilon \to 0\) proves (2.10).

If we let \(\omega_M\) be any Ricci-flat Kähler metric on \(M\), and let \(\nu\) be the reduced measure constructed in [23, Section 5], then \(\nu(S_X) = 0\) [23, Remark 5.3] and [23, Section 5] shows that there exist constants \(\nu, c > 0\) such that for any \(K \subset N_0\),
\[
\nu(K) = \nu \int f^{-1}(K) \omega^n_M = \nu \int_K f_*(\omega^n_M) = c
\]
because, as explained for example in Section 4 in [57], on \(N_0\) the metric \(\omega\) satisfies
\[
(2.11) \quad \omega^n = cf_*(\omega^n_M),
\]
for some explicit constant \(c > 0\). Therefore
\[
\nu(K) = \lambda\text{Vol}_\omega(K) = \lambda\mathcal{H}^{2n}_{d_X}(K),
\]
for some constant \(\lambda > 0\). Thanks to [5, Theorem 1.10] we have that \(\nu\) is a Radon measure, and then the same argument as in [24, p.105] shows that \(\nu = \lambda\mathcal{H}^{2n}_{d_X}\) as measures on \(X\). If we let \(\nu_{-1}\) be the measure induced by \(\nu\) “in codimension 1”, as defined in [6, Section 2] (see also the discussion in [24, p.106]), then we deduce that
\[
\nu_{-1}(S_X) = \lambda'\mathcal{H}^{2n-1}_{d_X}(S_X) = 0,
\]
using (2.9), for some positive constant \(\lambda'\). We then apply [6, Theorem 3.7] which shows that given any \(x_1 \in X_0\) for \(\mathcal{H}^{2n}_{d_X}\)-almost all \(y \in X_0\) there exists a minimal geodesic from \(x_1\) to \(y\) which lies entirely in \(X_0\). In particular, given any two points \(x_1, x_2 \in X_0\) and \(\delta > 0\), there is a point \(y \in X_0\) with \(d_X(x_1, y) < \delta\) which can be joined to \(x_1\) by a minimal geodesic \(\eta_1\) contained in \(X_0\). Furthermore we can take \(y\) close enough to \(x_2\) so that it can also be joined to \(x_2\) by a curve \(\eta_2\) contained in \(X_0\) with \(d_X\)-length at most \(\delta\). Concatenating \(\eta_1\) and \(\eta_2\) we obtain a curve \(\eta\) in \(X_0\) joining \(x_1\) to \(x_2\) with
\[
\text{length}_{d_X}(\eta) \leq d_X(x_1, y) + \delta \leq d_X(x_1, x_2) + 2\delta.
\]
Since \(\psi : N_0 \to X_0\) is a homeomorphism, we conclude that given any two points \(q_1, q_2 \in N_0\) and \(\delta > 0\), there is a curve \(\gamma\) in \(X_0\) joining \(q_1\) and \(q_2\) with
\[
\text{length}_\omega(\gamma) \leq d_X(\psi(q_1), \psi(q_2)) + 2\delta.
\]
Therefore, thanks to (2.10), we conclude that
\[
d_X(\psi(q_1), \psi(q_2)) \leq d_\omega(q_1, q_2) \leq \text{length}_\omega(\gamma) \leq d_X(\psi(q_1), \psi(q_2)) + 2\delta.
\]
Letting $\delta \to 0$, we conclude that
\[ d_\omega(q_1, q_2) = d_X(\psi(q_1), \psi(q_2)). \]
Hence $\psi : (N_0, \omega) \to (X_0, d_X)$ is a global isometry, and since $X_0$ is dense in $X$ this implies that $(X, d_X)$ is isometric to the metric completion of $(N_0, \omega)$. □

The method that we developed to prove Theorem 2.1 is quite robust, and it applies to other setups as well, see e.g. [65] for a very recent work that uses our result in different settings.

3. Metrics on torus fibrations

In this section we prove some general results about metrics on torus fibrations, extending our earlier work in [23, 62], and in Theorem 3.4 we state the main estimate which holds in the hyperkähler case, which implies estimate (2.1), and which will be proved in section 4.

3.1. Semi-flat forms on torus fibrations. We start with a general discussion. Let $(M^m_0, \omega_M)$ be a possibly noncompact Kähler manifold with a proper holomorphic submersion $f : M^m_0 \to N^n_0$ with connected fibers onto a complex manifold $N_0$, $0 < n < m$. Assume that all the fibers $M_y = f^{-1}(y)$, $y \in N_0$ are complex tori, so that $M_y \cong \mathbb{C}^{m-n}/\Lambda_y$, for some lattice $\Lambda_y \subset \mathbb{C}^{m-n}$, and that $f$ admits a holomorphic section $\sigma : N_0 \to M_0$.

Theorem 3.1. Under these assumptions, there is a unique closed semipositive $(1,1)$ form $\omega_{SF}$ on $M_0$, such that $\omega_{SF}|_{M_y}$ is the unique flat Kähler metric cohomologous to $\omega_M|_{M_y}$, and such that given any coordinate ball $B \subset N_0$, and any trivialization $f^{-1}(B) \cong (B \times \mathbb{C}^{m-n})/\Lambda$ which maps $\sigma$ to the zero section, the form $\omega_{SF}$ is given by the explicit formula of [23, 26, 27].

The last point means the following: the universal cover of $f^{-1}(B)$ is in fact biholomorphic to $p : B \times \mathbb{C}^{m-n} \to (B \times \mathbb{C}^{m-n})/\Lambda \cong f^{-1}(B)$ (see [23]), we may assume that $p$ pulls back $\sigma$ to the zero section, and we can then write $p^*\omega_{SF} = \sqrt{-1} \partial \bar{\partial} \eta$ where
\[ \eta(y, z) = -\frac{1}{4} \sum_{i,j=1}^{m-n} (\text{Im } Z(y))^{-1}(z_i - \bar{z}_i)(z_j - \bar{z}_j), \]
and $Z : B \to \mathcal{H}_{m-n}$ is a holomorphic period map from $B$ to the Siegel upper half space which was constructed in [23, 26, 27].

The form $\omega_{SF}$ is called semi-flat. It was first introduced in [18] in the context of elliptically fibered K3 surfaces.

This result follows easily from the arguments of [23, 26, 27]; in particular, $\omega_{SF}$ is implicitly constructed in [26] but without the explicit formula over small balls, while in [23, 27] we only considered the case when $N_0$ is a small ball. For the reader’s convenience we give the proof.
Proof. We initially follow the construction in [26, Section 3.2]. For that construction to apply, we need the existence of a section (which we assume), and of a “constant polarization” (which exists because \( \omega_M \) is Kähler, as in [27, Proposition 2.1]). If \( g_y \) denotes the unique flat Kähler metric on \( M_y \) in the class \([\omega_M|_{M_y}]\), for any \( y \in N_0 \), then the restriction of \( g_y \) to \( T^{(1,0)}M_y \) defines a Hermitian metric on the holomorphic vector bundle \( E = \sigma^*T_{M_0/N_0} \) over \( N_0 \). As indicated above, we have a biholomorphism \( M_0 \cong E/L \) for a holomorphic lattice bundle \( \Lambda \subset E \). Then \( \Lambda \) induces a flat Gauss-Manin connection on \( E \), with horizontal space \( \mathcal{H} \). Let \( P : T^{(1,0)}M_0 \to E \) be the projection along \( \mathcal{H} \), and for any \( x \in M_0, u, v \in T_{x}^{(1,0)}M_0 \) define \( g_{SF}(u, v) = g_{f(x)}(Pu, Pv) \). Then \( g_{SF} \) defines a semipositive closed real \((1,1)\) form \( \omega_{SF} \) on \( M_0 \), as verified in [26], which restricts to the correct flat Kähler metric on each fiber \( M_y \).

If now \( B \subset N_0 \) is a coordinate ball, and \( p : B \times \mathbb{C}^{m-n} \to f^{-1}(B) \) is the universal covering map, and assume that \( p \) pulls back \( \sigma \) to the zero section, then the construction in [24, 27] gives us a function \( \eta \) on \( B \times \mathbb{C}^{m-n} \) defined by (3.1), such that \( \sqrt{-1} \partial \bar{\partial} \eta \) descends to a closed semipositive \((1,1)\) form \( \omega_{SF}^{(1,0)} \) on \( f^{-1}(B) \), with \( \omega_{SF}^{(1,0)}|_{M_y} = \omega_{SF}|_{M_y} \) for all \( y \in B \). Both \( \omega_{SF} \) and \( \omega_{SF}^{(1,0)} \) are invariant under translation by flat sections of the Gauss-Manin connection, and at every point on the zero section they are equal because they both vanish in the horizontal directions and are equal to the same flat Kähler metric on each fiber. Therefore we conclude that \( \omega_{SF} = \omega_{SF}^{(1,0)} \) on all of \( f^{-1}(B) \), as required. Uniqueness of \( \omega_{SF} \) follows from the fact that, locally on the base, it is given by this explicit formula, and that two different trivializations of \( f^{-1}(B) \) which both map \( \sigma \) to the zero section, must differ by fiberwise translation by a flat section, which leaves \( \omega_{SF} \) unchanged.

As a consequence of the explicit formula (3.1), we see that if over a coordinate ball \( B \subset N_0 \) we define \( \lambda_t : B \times \mathbb{C}^{m-n} \to B \times \mathbb{C}^{m-n} \) by \( \lambda_t(y, z) = (y, t^{-\frac{1}{2}}z) \), then we have that

\[
(3.2) \quad t \lambda_t^* p^* \omega_{SF} = p^* \omega_{SF}.
\]

From now on we specialize to the setting of Theorem 1.2, so that \( M \) is projective hyperkähler, and \( f : M \to N \) is a holomorphic fiber space (and here we also assume the existence of a holomorphic section \( \sigma : N_0 \to M_0 \), so that Theorem 3.1 applies), and \( \tilde{\omega}_t \) is the hyperkähler metric on \( M \) in the class \( f^*[\omega_N] + t[\omega_N], 0 < t \leq 1 \). In this case it follows from [31] that in fact \( N \cong \mathbb{CP}^n \), but we will not need this fact. As before, we let \( S' \subset N \) be the discriminant locus of \( f \) and we set \( N_0 = N \setminus S, M_0 = f^{-1}(N_0) \).

Using (3.2), in [27] it is proves that given any compact \( K \subset B \times \mathbb{C}^{m-n} \) and any \( k \geq 0 \), there is a constant \( C \) independent of \( t \) such that on \( K \) we have

\[
C^{-1} p^*(\omega_N + \omega_{SF}) \leq \lambda_t^* p^* \tilde{\omega}_t \leq C(\omega_N + \omega_{SF}),
\]
and

\begin{equation}
\| \lambda_t^* p^* \tilde{\omega}_t \|_{C^k(K, g_E)} \leq C,
\end{equation}

which is the same result as in [23] Lemma 4.2 and Proposition 4.3] but without the need to use translations by holomorphic sections.

For later use, let us also assume that the fibration $f$ admits a holomorphic Lagrangian section $\sigma : N_0 \to M_0$. As explained for example in [13, p.43, 31 Proposition 3.5] or [32, Proposition 2.4], using this section and the holomorphic symplectic form on $M_0$, we get an isomorphism $M_0 \cong T^{* (1,0)} N_0 / \Lambda$, where $\Lambda \subset T^{* (1,0)} N_0$ is a lattice bundle, and let $p : T^{* (1,0)} N_0 \to M_0$ be the natural projection (over any coordinate ball $B \subset N_0$ where the bundle is trivial, this agrees with the map $p$ as above). The dilations $\lambda_t$ are in fact well-defined as biholomorphisms $\lambda_t : T^{* (1,0)} N_0 \to T^{* (1,0)} N_0$, which in trivializations over $B$ as above are given by the same formula as above.

We can now identify the smooth limit of the metrics $\lambda_t^* p^* \tilde{\omega}_t$ on $T^{* (1,0)} N_0$ as $t \to 0$. The following is an improvement of [23 Lemma 4.7], again without the need to use translations by holomorphic sections:

**Proposition 3.2.** As $t \to 0$ we have

\begin{equation}
\lambda_t^* p^* \tilde{\omega}_t \to p^* (f^* \omega + \omega_{SF}),
\end{equation}

smoothly on compact sets of $T^{* (1,0)} N_0$, where $\omega_{SF}$ is given by Theorem 2.1.

**Proof.** Fix any coordinate ball $B \subset N \setminus S'$. Thanks to [23 Proposition 3.1] we can find a holomorphic section $\tilde{\sigma} : B \to f^{-1}(B)$ of $f$ and a smooth function $\xi$ on $f^{-1}(B)$ so that

\[ \omega_M = T^*_\tilde{\sigma} \omega_{SF} + \sqrt{-1} \partial \bar{\partial} \xi, \]

holds on $f^{-1}(B)$, where $T_\tilde{\sigma} : f^{-1}(B) \to f^{-1}(B)$ is given by fiberwise translation by $\tilde{\sigma}$. We can now follow the argument of [23, Lemma 4.7], with some small modifications. Recall that we have

\[ \tilde{\omega}_t = f^* \omega + t \omega_M + \sqrt{-1} \partial \bar{\partial} \varphi_t, \]

with $|\varphi_t| \leq C$ (see [57]). On $p^{-1}(f^{-1}(B)) \cong B \times \mathbb{C}^{m-n}$ we may then write

\[ \lambda_t^* p^* \tilde{\omega}_t = p^* f^* \omega + t \lambda_t^* p^* T^*_\tilde{\sigma} \omega_{SF} + \sqrt{-1} \partial \bar{\partial} u_t, \]

where we have set $u_t = \varphi_t \circ p \circ \lambda_t - t \xi \circ p \circ \lambda_t$, and thanks to Theorem 3.1 we also have $p^* \omega_{SF} = \sqrt{-1} \partial \bar{\partial} \eta$, where $\eta$ is explicitly given by (3.1). The map $T_\tilde{\sigma}$ is induced by the translation $(y, z) \mapsto (y, z + \tilde{\sigma}(y))$ on $B \times \mathbb{C}^{m-n}$ (which we will also denote by $T_\tilde{\sigma}$), and so this gives

\[ t \lambda_t^* p^* T^*_\tilde{\sigma} \omega_{SF} = t \sqrt{-1} \partial \bar{\partial} (\eta \circ T_\tilde{\sigma} \circ \lambda_t), \]

and by direct inspection we see that this converges smoothly on compact sets to $p^* \omega_{SF}$ as $t \to 0$. Indeed, using (3.1), we see that

\[ t(\eta \circ T_\tilde{\sigma} \circ \lambda_t)(y, z) = t \eta(y, t^{-\frac{1}{2}} z + \tilde{\sigma}(y)) = \eta(y, z + t^{\frac{1}{2}} \tilde{\sigma}(y)), \]
which converges to $\eta(y, z)$ smoothly on compact sets. Thanks to (3.3), we see that $\sqrt{-1} \partial \bar\partial u_t$ also has uniform local $C^\infty$ bounds, and since $|u_t| \leq C$, we conclude that the functions $u_t$ are themselves locally uniformly bounded in $C^\infty$. Arguing then exactly as in [23, Lemma 4.7] we conclude that $u_t \to \varphi \circ f \circ p$ smoothly on compact sets, where $\omega_N + \sqrt{-1} \partial \bar\partial \varphi = \omega$ is the limiting metric on $N_0$. This concludes the proof.

Of course, the same proof shows that even if $f : M_0 \to N_0$ does not have a holomorphic Lagrangian section, we still obtain the convergence in (3.4) but just on the preimage of any coordinate ball $B \subset N_0$ (since on its universal cover $B \times \mathbb{C}^{m-n}$ we still have the stretching maps $\lambda_t$).

3.2. The discriminant locus. As in the Introduction, let $M$ be a projective hyperkähler manifold, and $f : M \to N$ a surjective holomorphic map with connected fibers onto a compact Kähler manifold $N$ with $\dim N < \dim M$. Then we know by Matsushita [44, 45] that $\dim N = \frac{1}{2} \dim M$ and that $f$ is an equidimensional holomorphic Lagrangian torus fibration, which in particular gives an algebraic completely integrable system where it is a submersion (see section 4.2). Also, by Hwang [31] we have that $N$ is biholomorphic to $\mathbb{C}P^n$ where $\dim M = 2n$.

If $S' \subset N$ denotes the discriminant locus of $f$, then it is well-known that $S'$ must be nonempty (see e.g. [31, Proposition 4.1], where it is shown that $S'$ is in fact necessarily a divisor). We note here the following even stronger result, proved by the authors in [62] with an extra hypothesis:

**Theorem 3.3.** Let $f : M \to N$ be a holomorphic submersion with connected fibers between compact Kähler manifolds with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$ (i.e. $M$ is a Calabi-Yau manifold). Then $f$ is a holomorphic fiber bundle with fiber $F$ and base $N$ both Calabi-Yau manifolds.

**Proof.** It is well-known (see e.g. [60]) that $K_M$ is torsion in $\text{Pic}(M)$, so there is a finite étale cover $\pi : \tilde{M} \to M$ with $\tilde{M}$ connected and with $K_{\tilde{M}}$ trivial. The composition $f \circ \pi : \tilde{M} \to M \to N$ is a holomorphic submersion with possibly disconnected fibers, so we consider its Stein factorization $\tilde{M} \xrightarrow{p} \tilde{N} \xrightarrow{q} N$ where $\tilde{N}$ is a (connected) compact Kähler manifold, $p$ is a holomorphic submersion with connected fibers and $q$ is a finite étale cover (see e.g. [14, Lemma 2.4]). Therefore $p : \tilde{M} \to \tilde{N}$ satisfies the hypothesis of [62, Theorem 1.3], and so it is a holomorphic fiber bundle with base $\tilde{N}$ and fiber $\tilde{F}$ both Calabi-Yau manifolds. By [15, Lemma 4.5], it follows that $f : M \to N$ is a holomorphic fiber bundle as well. But we have finite unramified coverings $\tilde{N} \xrightarrow{q} N$ and $\tilde{F} \xrightarrow{p|_{\tilde{F}}} F$, and so $N$ and $F$ are Calabi-Yau manifolds as well. \qed

3.3. Estimates for the special Kähler metric near the discriminant locus. In this subsection we state our main estimate in the hyperkähler setting, which implies the estimate (2.1). This estimate is then proved in section 4.
As in the assumptions of Theorem 1.2, let $M$ be a compact projective hyperkähler $2n$-manifold, with holomorphic symplectic 2-form $\Omega$, and let $[\alpha]$ be an integral Kähler class on $M$. Assume that $M$ admits a surjective holomorphic map $f : M \to N$ with connected fibers onto a compact Kähler $n$-manifold $N$. As before we let $S' \subset N$ be the discriminant locus of $f$, and let $N_0 = N \setminus S'$, $M_0 = f^{-1}(N_0)$. The fibers $M_y = f^{-1}(y), y \in N_0$, are holomorphic Lagrangian $n$-tori \cite{44, 45} so that $f : M_0 \to N_0$ is an algebraic completely integrable system (see section 4.2). The base $N$ is known to be isomorphic to $\mathbb{CP}^n$ by \cite{31}, but we will not need this. The fibers $M_y, y \in N_0$, are Abelian varieties with the polarization $[\alpha_y] = [\alpha|_{M_y}]$, which is of type $(d_1, \cdots, d_n)$ for some $d_i \in \mathbb{N}$. Let $\pi : \tilde{N} \to N$ be a modification, which is an isomorphism over $N_0$, such that $E = \pi^{-1}(S')$ is a divisor with simple normal crossings, so near any point of $E$ there are coordinates $(u_1, \cdots, u_n)$ on an open set $U \subset \tilde{N}$ (which in these coordinates is the unit polydisc in $\mathbb{C}^n$) such that $E \cap U = \{u_1 = u_k = 0\}$, for some $1 \leq k \leq n$, and $\{u_i = 0\} = E_{j_i} \cap U$ for some $1 \leq j_i \leq \mu$ and all $1 \leq i \leq k$.

By Section 3 of \cite{13}, there is a special Kähler metric $\omega$ on $N_0$ induced by the algebraic completely integrable system $(f : M_0 \to N_0, [\alpha], \Omega)$, and in \cite{24} we shows that this metric is equal to the collapsed smooth limit of the Ricci-flat metrics $\omega_t$ as $t \to 0$, obtained in \cite{57, 23}. The goal of this subsection is to study the asymptotic behaviour of $\pi^* \omega$ near $E$. As in section 2 we write $E = \bigcup_{j=1}^{\mu} E_j$ for the decomposition of $E$ into irreducible components.

**Theorem 3.4.** There are positive integers $m_i \in \mathbb{N}, 1 \leq i \leq \mu$, and a constant $C > 0$ such that given any $y \in E$ and any local chart on $U$ as above, if we define the local uniformizing map $q : \tilde{U} \to U$ ($\tilde{U}$ is also the unit polydisc in $\mathbb{C}^n$) by

$$q(t_1, \ldots, t_n) = (t_1^{m_{i_1}}, \ldots, t_k^{m_{i_k}}, t_{k+1}, \ldots, t_n),$$

then on $\tilde{U} \setminus q^{-1}(E)$ we have

$$q^* \pi^* \omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{ij} dt_i \wedge dt_j$$

where

$$|g_{ij}| \leq C(1 - \varepsilon(i) \log |t_i| - \varepsilon(j) \log |t_j|), \quad i, j = 1, \cdots, n,$$

where $\varepsilon(x) = 1$ if $1 \leq x \leq k$, and $\varepsilon(x) = 0$ if $k + 1 \leq x \leq n$. In particular, (2.1) holds.

In particular, Theorem 3.4 implies that the estimate (2.1) holds, and so once we complete the proof of Theorem 3.4, it will follow that Theorem 1.2 holds, thanks to Theorem 2.1. We prove Theorem 3.4 in the next section, which requires a detailed study of special Kähler metrics.
3.4. A further remark about the Gromov-Hausdorff limit. In the setting of Theorem 1.2 it is natural to ask whether the Gromov-Hausdorff limit \((X, d_X)\) is in fact homeomorphic to \(N \cong \mathbb{CP}^n\). This was proved in [24] when \(\dim N = 1\).

Here we make the following observations. Let \((X, d_X)\) be the Gromov-Hausdorff limit of \((M, \tilde{\omega}_t)\), which by Theorem 1.2 is isometric to the metric completion of \((N_0, \omega)\) and let \(\psi : (N_0, \omega) \hookrightarrow (X, d_X)\) be the isometric embedding.

Given any Kähler metric \(\omega_N\) on \(N\), the Schwarz Lemma estimate \(\tilde{\omega}_t \geq C^{-1} f^* \omega_N\) proved in [57] gives a uniform Lipschitz constant bound for the map \(f : (M, \tilde{\omega}_t) \to (N, \omega_N)\), independent of \(t\) and so, up to passing to a sequence \(t_i \to 0\), we obtain in the limit a Lipschitz surjective map \(h : (X, d_X) \to (N, \omega_N)\) with \(h \circ \psi = \text{Id}\).

Let also \(\pi : \tilde{N} \to N\) be the blow-up such that the discriminant locus \(E = \tilde{N} \setminus \pi^{-1}(N_0)\) is a simple normal crossings divisor.

Proposition 3.5. There is a continuous surjective map \(p : \tilde{N} \to X\) such that \(\pi = h \circ p\).

Proof. We have the homeomorphism \(p = \psi \circ \pi : \pi^{-1}(N_0) \to \psi(N_0) \subset X\). We claim that \(p\) extends to the desired map \(p : \tilde{N} \to X\). We use some of the notation as in the proof of Theorem 2.1.

To see this, let \(y \in \tilde{N} \setminus \pi^{-1}(N_0)\), and let \(y_i \in \pi^{-1}(N_0)\) such that \(y_i \to y\) in \(\tilde{N}\). For any small neighborhood \(\Delta^n(\rho)\) of \(y\), we have \(y_i \in \Delta^n(\rho)\) for \(i\) sufficiently large, and

\[
d_\omega(y_i, y_{i+m}) \leq C \rho (\log \rho)^d \to 0.
\]

Thus \(y_i\) is a Cauchy sequence in \((N_0, \omega)\), and \(y_i\) converges to a unique point \(\bar{y}\) in \(X\). We define \(p(y) = \bar{y}\).

If \(y' = \pi(y) \in N\), then \(\pi(y_i) = h(p(y_i)) \to y'\), and thus \(h(\bar{y}) = y'\). This completes the proof. \(\square\)

4. Special Kähler geometry

The goal of this section is to prove Theorem 3.4 and therefore also Theorem 1.2. The proof depends heavily on the geometry of special Kähler metrics. The notion of a special Kähler metric was introduced by physicists (cf. [13, 11, 30, 54]), and an intrinsic definition was given in [13]. Special Kähler metrics exist on the base of algebraic completely integrable systems, and conversely such metrics, at least locally, induce algebraic completely integrable systems. We review some background of special Kähler geometry, following [13] closely, and then we prove Theorem 3.4.

4.1. Special Kähler metrics. Let \((N_0, \omega)\) be a (possibly noncompact) Kähler manifold of dimension \(n\). A special Kähler structure is a real torsion-free flat connection \(\nabla\) on \(T N_0\) such that

\[
\nabla \omega = 0, \quad d \nabla I = 0,
\]
where $I$ is the complex structure of $N_0$.

For a special Kähler manifold $(\{N_0, \omega\})$, it is shown in [13] that $N_0$ admits local flat Darboux coordinates, i.e. for any point $y \in N_0$, there are real coordinates $y_1, \ldots, y_{2n}$ on a neighborhood of $y$ such that $\nabla dy_i = 0$, $i = 1, \ldots, 2n$, and

\[
\omega = \sum_{i=1}^{2n} dy_i \wedge dy_{i+n}.
\]

The transition functions between two such coordinates are of the form $y'_i = \sum_{j=1}^{2n} A_{ij} y_j + b_j$, $b_j \in \mathbb{R}$, and $A = (A_{ji}) \in Sp(2n, \mathbb{R})$. Hence the local flat Darboux coordinates covering gives a real affine manifold structure on $N_0$.

If $y_1, \ldots, y_{2n}$ are local flat Darboux coordinates, there are two holomorphic coordinates systems $\{w_1, \ldots, w_n\}$ and $\{\bar{w}_1^*, \ldots, \bar{w}_n^*\}$ satisfying that

\[
dy_i = \text{Re} \ dw_i, \quad dy_i + \bar{dy}_i = -\text{Re} \ dw_i^* \quad i = 1, \ldots, n.
\]

We call $\{w_i\}$ the special coordinates system and $\{w_i^*\}$ the conjugate coordinates system. We define a complex matrix $Z = [Z_{ij}]$ by

\[
Z_{ij} = \frac{\partial w_i^*}{\partial w_j}.
\]

The Kähler form $\omega$ being a $(1, 1)$-form implies that $Z_{ij} = Z_{ji}$, and there is a holomorphic function $\phi$, called a holomorphic prepotential function, such that $w_i^* = \frac{\partial \phi}{\partial w_i}$, $Z_{ij} = \frac{\partial^2 \phi}{\partial w_i \partial w_j}$. The Kähler potential is given by

\[
\phi = \frac{1}{2} \text{Im} \left( \sum_{i=1}^{n} \bar{w}_i^* \bar{w}_i \right),
\]

and the Kähler metric is

\[
\omega = \sqrt{-1} \partial \bar{\partial} \phi = \frac{\sqrt{-1}}{2} \sum_{ij} \text{Im} Z_{ij} dw_i \wedge d\bar{w}_j.
\]

Thus $Z$ satisfies the Riemann relations

\[
Z^T = Z, \quad \text{Im} Z > 0,
\]

i.e. $Z$ belongs to the Siegel upper half space $\mathcal{H}_n$.

If $g$ denotes the corresponding Riemannian metric of $(\omega, I)$, then $g$ is an affine Kähler metric with respect to the local flat Darboux coordinates $y_1, \ldots, y_{2n}$ (cf. [13, 30]), i.e.

\[
g = \sum_{ij} \frac{\partial^2 \phi}{\partial y_i \partial y_j} dy_i dy_j.
\]
Furthermore, $g$ is a Monge-Ampère metric, i.e. the potential function $\phi$ satisfies the real Monge-Ampère equation
\[
\det \left( \frac{\partial^2 \phi}{\partial y_i \partial y_j} \right) \equiv \text{const},
\]
since
\[
\sqrt{\det(g_{ij})} dy_1 \wedge \cdots \wedge dy_n = \frac{1}{n!} \omega^n.
\]

In [40], it is proved that if $(N_0, \omega)$ is complete, then $\omega$ is a flat metric (see also [10]), which can also be obtained by using Cheng-Yau’s theorem for Monge-Ampère metrics in the case when $N_0$ is compact [7, Corollary 2.3]. Therefore, many interesting examples of special Kähler manifolds are not complete, and it is a natural question to study their completions.

### 4.2. Algebraic completely integrable systems.

An algebraic completely integrable system is a holomorphic Lagrangian fibration $f : M_0 \to N_0$ from a quasiprojective manifold $M_0$ with dim$_C M_0 = 2n$, to a Kähler manifold $N_0$ with dim$_C N_0 = n$, i.e. $M_0$ admits a holomorphic symplectic form $\Omega$ and the class $[\alpha]$ of an ample line bundle, $f : M_0 \to N_0$ is a proper holomorphic submersion with connected fibers, and every fiber $M_y = f^{-1}(y)$, $y \in N_0$, is a complex Lagrangian submanifold with respect to $\Omega$. This forces every fiber $M_y$ to be an Abelian variety (without a specified origin) with the polarization $[\alpha_y] = [\alpha|_{M_y}]$, and the polarization is of type $(d_1, \cdots, d_n)$, for some $d_i \in \mathbb{N}$.

It is shown in [13, Section 3] or [30, Theorem 2] that a special Kähler structure exists on the base of an algebraic completely integrable system, and more generally on the moduli space of holomorphic Lagrangian submanifolds in hyperkähler manifolds (see also [42]). We recall the construction.

The vector bundle $E = f_* (T_{M_0/N_0}^{(1,0)})$ is isomorphic to $T^{(1,0)} N_0$ via the pairing induced by $\Omega$, and the fiberwise action of $E$ on $M_0$ by exponentiation has as kernel a lattice subbundle $\Lambda \subset T^{(1,0)} N_0$, with fiber $\Lambda_y \cong H_1(M_y, \mathbb{Z})$ for any $y \in N_0$, i.e. $\Lambda = \text{Hom}_\mathbb{Z}(R^1 f_* \mathbb{Z}, \mathbb{Z})$. The quotient $T^{(1,0)} N_0 / \Lambda$ is called the Jacobian family of $f$ (see e.g. [8, Section 2.1], [13, Section 3], [43, Proposition 2.1]), and is also a holomorphic Lagrangian fibration from a hyperkähler quasi-projective manifold (with a polarization which is induced by $[\alpha]$ on $M_0$) with fibers isomorphic to those of $f$ (as polarized Abelian varieties) for all $y \in N_0$. The Jacobian fibration comes with a holomorphic Lagrangian section, and over every coordinate ball $B \subset N_0$ the original family $f$ also admits a holomorphic Lagrangian section, and therefore is isomorphic to the Jacobian family over any such $B$ (cf. [13, p.43], [31, Proposition 3.5], [32, Proposition 2.4]), but there may be no such isomorphism globally over $N_0$ precisely because $f$ need not have a holomorphic Lagrangian section on all of $N_0$.

The special Kähler metric induced by $f$ is then defined purely using the Jacobian family as follows. There is a canonical holomorphic symplectic...
form $\Omega_{\text{can}}$ on the holomorphic cotangent bundle $T^{*(1,0)}N_0$, which is characterized by $d\zeta = -\zeta^*\Omega_{\text{can}}$ for any 1-form $\zeta$. Under a local trivialization $T^{*(1,0)}N_0|_U \cong U \times \mathbb{C}^n$ by $\sum z_i dw_i \mapsto (w_1, \ldots, w_n, z_1, \ldots, z_n)$, over some open subset $U \subset N_0$, we have

$$\Omega_{\text{can}} = \sum_{i=1}^n dw_i \wedge dz_i.$$  

Any local section of $\Lambda$ is holomorphic Lagrangian with respect to $\Omega_{\text{can}}$.

Since $T^{*(1,0)}N_0$ is canonically isometric to the tangent bundle $T^*N_0$ as real smooth vector bundles, we have that $\Lambda \otimes \mathbb{Z} \cong T^*N_0$, and the embedding $\Lambda \hookrightarrow T^{*(1,0)}N_0$ is given by $\tilde{\nu} \mapsto \tilde{\nu}^{(1,0)} = \tilde{\nu} - \sqrt{-1}\bar{\nu}$. The dual lattice bundle of $\Lambda$ is $\Lambda = R^1f_*\mathbb{Z} \cong \text{Hom}_Z(\tilde{\Lambda}, \mathbb{Z})$, and is a lattice subbundle of the tangent bundle $TN_0$. The torsion-free flat connection $\nabla$ in the special Kähler structure is the connection such that any local section of $\Lambda$ is parallel.

Let $\omega_{SF}$ be the $(1,1)$-form on $T^{*(1,0)}N_0/\Lambda$ constructed in Theorem 3.1 so that the restriction $\omega_{SF,y} = \omega_{SF}|_{M_y}$ on $T^{*(1,0)}N_0/\Lambda_y \cong M_y$ is the flat Kähler metric representing $[\alpha_y]$. Therefore $\omega_{SF}$ is a real bundle symplectic form on $T^{*(1,0)}N_0$, i.e. $\omega_{SF,y}$ is a linear symplectic form on $T^{*(1,0)}N_0$. Since $\omega_{SF}$ represents an integral cohomology class on the fibers, it defines an integral symplectic form on $\Lambda$. If $\tilde{v}_1, \ldots, \tilde{v}_{2n}$ are local sections of $\Lambda$, which are symplectic basis with respect to $\omega_{SF}$, i.e.

$$\omega_{SF}(\tilde{v}_i, \tilde{v}_j) = 0, \quad \omega_{SF}(\tilde{v}_{i+n}, \tilde{v}_{j+n}) = 0 \quad \omega_{SF}(\tilde{v}_i, \tilde{v}_{j+n}) = d_i \delta_{ij},$$

$1 \leq i, j \leq n$, then local flat Darboux coordinates $y_1, \ldots, y_{2n}$ are defined such that $dy_i = d_i^{-1} \tilde{v}_i$ and $dy_{i+n} = -\tilde{v}_{i+n}$, $i = 1, \ldots, n$. Here we regard $\Lambda$ as a lattice subbundle of $T^*N_0$, and $\tilde{v}_i$, $i = 1, \ldots, 2n$, as real 1-forms.

We have the special coordinates $w_1, \ldots, w_n$ and the conjugate coordinates $w_1^*, \ldots, w_n^*$, and $Z_{ij} = \frac{\partial w_i^*}{\partial w_j}$. The image of $\Lambda_y \hookrightarrow T^{*(1,0)}N_0$ is generated by $dw_i$ and $dw_i^*$, $i = 1, \ldots, n$, and the period matrix of the Abelian variety $M_y$ is $[\Delta_d, Z_{ij}(y)]$, where $\Delta_d = \text{diag}(d_1, \ldots, d_n)$. By (4.1) and (4.4), the special Kähler form is given as

$$\omega = \sum_{i=1}^n dy_i \wedge dy_{i+n} = \frac{\sqrt{-1}}{2} \sum_{ij} \text{Im}Z_{ij} dw_i \wedge d\bar{w}_j.$$  

We end this subsection by showing the following local calculation of the Ricci curvature of the special Kähler metric. A more general result is proved by the first-named author in [57] Proposition 4.1 (see also [53]) in the case when $M \to N$ is a holomorphic fiber space with $M$ Calabi-Yau, and $N$ compact.

**Proposition 4.1.** The Ricci form of the special Kähler metric $\omega$ is the Weil-Petersson form $\omega_{WP}$ of the family of Abelian varieties $f : M_0 \to N_0$, i.e.

$$\text{Ric}(\omega) = \omega_{WP}.$$
Proof. Locally on $N_0$ we have
\[
\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log |\text{det}(\text{Im} Z_{ij})|.
\]

For a sufficiently small open subset $U \subset N_0$, $f^{-1}(U) \cong U \times \mathbb{C}^n / \tilde{\Lambda}_U$, where $\tilde{\Lambda}_U = \text{Span}_\mathbb{Z}(\text{diag}(d_1, \ldots, d_n)|Z_{ij})$, and $f$ is the projection to $U$. We denote $z_1, \ldots, z_n$ the coordinates on $\mathbb{C}^n$, and $\Theta = dz_1 \wedge \cdots \wedge dz_n$. The Weil-Petersson form (see e.g. [57, 53]) is defined by
\[
\omega_{WP} = -\sqrt{-1} \partial \bar{\partial} \log \int_{M_y} (-1)^n \frac{\Theta}{n!} \wedge \bar{\Theta} = -\sqrt{-1} \partial \bar{\partial} \log V_y,
\]
where $V_y$ is the Euclidean volume of $M_y \cong \mathbb{C}^n / \tilde{\Lambda}_{U,y}$. We obtain the conclusion by $V_y = \text{det}(\text{Im} Z_{ij}) \prod_k d_k$. \hfill \Box

4.3. Estimates from degenerations of Hodge structures. Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. Let $\omega$ be the special Kähler metric on $N_0$ induced by the algebraic completely integrable system $(f : M_0 \rightarrow N_0, [\alpha], \Omega)$, as explained in previous subsection. Recall that $f$ comes from a map $f : M \rightarrow N$ as in the Introduction, with $M$ projective hyperkähler, that $N_0 = N \setminus S'$, where $S'$ is the discriminant locus of $f$, and that $\pi : \tilde{N} \rightarrow N$ is a modification with $E = \pi^{-1}(S')$ a simple normal crossings divisor.

Let $\nabla$ be the flat connection of the special Kähler structure, and let $\Lambda$ and $\tilde{\Lambda}$ be the lattice bundles as in last subsection, i.e. $\Lambda \cong \mathbb{R}^1 f_* \mathbb{Z}$ and $\tilde{\Lambda} \cong \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$. We have the canonical identifications $\tilde{\Lambda}_\mathbb{R} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong T^* N_0$, $\Lambda_\mathbb{R} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^1 f_* \mathbb{R} \cong T N_0$ and $H^{1,0}(M_y) \cong T_y^{(1,0)} N_0$ for any $y \in N_0$. There is a weight-one integral polarized variation of Hodge structures on $N_0$ given by the quadruple
\[
(F^1 = T^{(1,0)} N_0 \subset F^0 = T N_0 \otimes_{\mathbb{R}} \mathbb{C}, \Lambda \subset T N_0, \nabla, \omega)
\]
(in fact this data is equivalent to a special Kähler structure, cf. [11, 8.4], [28, Section 3.3] or [2, Section 3]). Since $\pi : \tilde{N} \setminus E \rightarrow N_0$ is a biholomorphism, we can also view this as a variation of Hodge structures on $\tilde{N} \setminus E$. To alleviate notation, we will denote $\pi^* \omega$ simply by $\omega$.

Let $U$ be a local chart near a point of $E$, and $\{t'_{1}, \cdots, t'_{\mu}\}$ be coordinates on $U$ such that $E \cap U$ is given by $\{t'_{1} \cdots t'_{j_i} = 0\}$, and $\{t'_{i} = 0\} = E_{j_i} \cap U$ for some $1 \leq j_i \leq \mu$, and all $1 \leq i \leq k$. For a fixed $y \in U \setminus E$, the lattice bundles $\Lambda$ and $\tilde{\Lambda}$ induce monodromy representations $\tilde{\rho} : \pi_1(U \setminus E) \rightarrow \text{Aut} \Lambda_y$ and $\rho : \pi_1(U \setminus E) \rightarrow \text{Aut} \tilde{\Lambda}_y$ respectively. For $\gamma \in \pi_1(U \setminus E)$, we denote $T_{\gamma} : \Lambda_y \rightarrow \Lambda_y$ and $\tilde{T}_{\gamma} : \tilde{\Lambda}_y \rightarrow \tilde{\Lambda}_y$ the corresponding monodromy operators of $\rho$ and $\tilde{\rho}$. If $\langle \cdot, \cdot \rangle$ denotes the pairing between $\Lambda$ and $\Lambda$, we have
\[
\langle \tilde{T}_{\gamma} \alpha, \beta \rangle = \langle \alpha, T_{\gamma^{-1}} \beta \rangle
\]
for any $\alpha \in \tilde{\Lambda}_y$ and $\beta \in \Lambda_y$ by $\langle \tilde{T}_{\gamma} \alpha, \beta \rangle = \langle \tilde{P}_{\gamma^{-1}(t)} \tilde{T}_{\gamma} \alpha, \tilde{P}_{\gamma^{-1}(t)} \beta \rangle, t \in [0, 1]$, where $P_{\gamma^{-1}(t)}$ and $\tilde{P}_{\gamma^{-1}(t)}$ are parallel transports of $\Lambda$ and $\tilde{\Lambda}$ respectively.
Lemma 4.2. There are positive integers $m_i \in \mathbb{N}$, $1 \leq i \leq \mu$, such that the following holds. Let $q : \tilde{U} \to U$ be the branched covering given by

$$q(t_1, \ldots, t_n) = (t_1^{m_1}, \ldots, t_k^{m_k}, t_{k+1}, \ldots, t_n),$$

where $t_1, \ldots, t_n$ are coordinates on $\tilde{U}$. We denote still by $\Lambda$ and $\tilde{\Lambda}$ the pullbacks of the respective lattice bundles via $q$. Then we have

i) For any $\gamma \in \pi_1(\tilde{U}\setminus q^{-1}(E))$, the monodromy operators of $\Lambda$ and $\tilde{\Lambda}$, still denoted by $T_{\gamma}$ and $\tilde{T}_{\gamma}$, satisfy

$$(T_{\gamma} - I)^2 = 0, \quad (\tilde{T}_{\gamma} - I)^2 = 0.$$  

ii) There are sections $\tilde{v}_1, \ldots, \tilde{v}_{2n}$ of $\tilde{\Lambda} \subset T^*\tilde{U}\setminus q^{-1}(E)$ such that

$$q^* \omega = -\sum_{i=1}^n d_i^{-1} \tilde{v}_i \wedge \tilde{v}_{n+i}.$$  

iii) For any $\gamma \in \pi_1(\tilde{U}\setminus q^{-1}(E))$, $\tilde{T}_{\gamma} \tilde{v}_j = \tilde{v}_j$, $j = 1, \ldots, n$. Hence $\tilde{v}_1, \ldots, \tilde{v}_n$ extend to 1-forms on $\tilde{U}$.

Proof. By shrinking $U$ if necessary, we may assume that $\pi_1(U \setminus E) \cong \mathbb{Z}^k$, generated by loops which wrap once around each component of $E$ that intersects $U$. The Monodromy theorem (cf. Chapter II of [19]) shows that for every $\gamma \in \pi_1(U \setminus E)$ the eigenvalues of $T_{\gamma}$ are roots of unity, and if we take $\gamma$ to be a loop which wraps once around the irreducible component $E_i$ of $E$ then we get one eigenvalue which is an $m_i^{th}$ root of unity (and the others vanish), with $m_i$ being independent of where which particular loop we choose (once we fix its orientation). This way we obtain the positive integers $m_i$, $1 \leq i \leq \mu$, and we can locally define the branched covering $q : \tilde{U} \to U$ as in (4.5), such that for any $\gamma \in \pi_1(\tilde{U}\setminus q^{-1}(E))$, $(T_{\gamma} - I)^2 = 0$. By duality, $(\tilde{T}_{\gamma} - I)^2 = 0$, and thus both $T_{\gamma}$ and $\tilde{T}_{\gamma}$ are unipotent.

If we let $\mathcal{N}_{\gamma} = \log T_{\gamma} = T_{\gamma} - I$, then $\mathcal{N}_{\gamma}$ is an element in the Lie algebra

$$\mathfrak{g}_0 = \{ \mathcal{N} \in \text{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R},y}, \Lambda_{\mathbb{R},y}) | \omega(\mathcal{N}, \cdot) + \omega(\cdot, \mathcal{N}) = 0 \}$$

(cf. Chapter V of [19]). Let $\gamma_i, i = 1, \ldots, k$, be the generators of $\pi_1(\tilde{U}\setminus q^{-1}(E))$, and denote $T_i = T_{\gamma_i}, N_i = \log T_i, i = 1, \ldots, k$. The monodromy cone is the open cone

$$\Sigma = \left\{ \sum_i \nu_i \mathcal{N}_i | \nu_i > 0 \right\} \subset \mathfrak{g}_0.$$  

Note that for any $\mathcal{N} \in \Sigma$, we have $\mathcal{N}^2 = 0$. If $\mathcal{N} = \sum \nu_i \mathcal{N}_i$, then $\mathcal{N} = \log \prod_i T_i^{\nu_i}$. Since $T = \prod_i T_i^{\nu_i}$ is the monodromy operator of the curve $\prod_i \gamma_i^{\nu_i}$, we have $\mathcal{N}^2 = (T - I)^2 = 0$.

For any $\mathcal{N} \in \Sigma$, we have $W_0 = \text{Im}\mathcal{N} = \oplus_{i=1}^k \text{Im}\mathcal{N}_i$ and $W_1 = \ker\mathcal{N} = \bigcap_{i=1}^k \ker\mathcal{N}_i$ by [3] Lemma 2.2, which give the monodromy weight filtration

$$\{0\} \subset W_0 \subset W_1 \subset W_2 = \Lambda_{\mathbb{R},y}.$$
Then $W_0$ is an isotropic subspace and
$$W_1 = W_0^\perp = \{ u \in \Lambda_{\mathbb{R},g} | \omega(u, v) = 0, \forall v \in W_0 \},$$
by $\omega(\mathcal{N}_+, \cdot) + \omega(\cdot, \mathcal{N}_-) = 0$. Thus we have a Lagrangian subspace $W_0 \subset L \subset W_1$, and a symplectic basis $\{v_1, \cdots, v_{2n}\}$ of $\omega$ such that $\omega(v_i, v_j) = 0, \omega(v_{i+n}, v_{j+n}) = 0$ and $\omega(v_i, v_{j+n}) = -d_i^{-1} \delta_{ij}, 1 \leq i, j \leq n$, and $L = \mathbb{R}\{v_n, \cdots, v_{2n}\}$. By varying $y$ in $\tilde{U} \setminus q^{-1}(E)$, we regard $v_n, \cdots, v_{2n}$ as sections of $\Lambda$ on $\tilde{U} \setminus q^{-1}(E)$, and $v_1, \cdots, v_n$ as multi-valued sections. If $\tilde{v}_1, \cdots, \tilde{v}_{2n}$ are dual sections of $\Lambda$, we have $q^* \omega = -\sum_i d_i^{-1} \tilde{v}_i \wedge \tilde{v}_{n+i}$.

Note that $T_y v_i = d_i n, \text{ and } T_y v_i - v_i \in W_0 \subset L, i = 1, \cdots, n$, for any $\gamma \in \pi_1(\tilde{U} \setminus q^{-1}(E))$. We have $\left( T_y \tilde{v}_i, v_{j+n} \right) = \langle \tilde{v}_i, v_{j+n} \rangle = 0$, and $\left( T_y \tilde{v}_i, v_j \right) = \langle \tilde{v}_i, v_{j+n} + (T_y v_i - v_j) \rangle = \delta_{ij}, 1 \leq i, j \leq n$. Thus $T_y \tilde{v}_i = \tilde{v}_i$ for any $\gamma$, and $\tilde{v}_1, \cdots, \tilde{v}_n$ extend across $q^{-1}(E)$ to sections of $T^* \tilde{U}$.

Note that $\Lambda_{\mathbb{R}} \cong T^* \tilde{U} \setminus q^{-1}(E)$, $\Lambda_{\mathbb{R}} \cong T^* \tilde{U} \setminus q^{-1}(E)$, and $\Lambda$ is identified as a lattice subbundle in $T^* \tilde{U} \setminus q^{-1}(E)$, where $\Lambda$ is the complex structure of $\tilde{U}$. Since $\tilde{v}_1^{(1,0)}, \cdots, \tilde{v}_n^{(1,0)}$ are holomorphic Lagrangian sections of $T^* \tilde{U} \setminus q^{-1}(E)$ with respect to the canonical holomorphic symplectic form $\Omega_{\text{can}}$, the extensions $\tilde{v}_1^{(1,0)}, \cdots, \tilde{v}_n^{(1,0)}$ are holomorphic Lagrangian sections of $T^* \tilde{U}$. Then $d\tilde{v}_j^{(1,0)} = (\tilde{v}_j^{(1,0)})^* \Omega_{\text{can}} = 0, j = 1, \cdots, n$, and there are holomorphic functions $w_1, \cdots, w_n$ on $\tilde{U}$ such that $\tilde{v}_j = d_j \text{Red} w_j, j = 1, \cdots, n$, which give the special holomorphic coordinates on $\tilde{U} \setminus q^{-1}(E)$.

Locally, we define the conjugate coordinates $w_1^*, \cdots, w_n^*$ by $\text{Red} w_i^* = \tilde{v}_{j+n}$. We define a holomorphic multivalued matrix function $Z = [Z_{ij}]$ by $Z_{ij} = \partial w_i^* / \partial w_j$, and we have $\tilde{v}_j^{(1,0)} = \sum_{i=1}^n Z_{ji} d_i^{-1} \tilde{v}_i^{(1,0)}, j = 1, \cdots, n$. We have the Riemann relations
$$Z^T = Z, \text{ Im} Z > 0,$$
and
$$q^* \omega = -\sum_{i=1}^n d_i^{-1} \tilde{v}_i \wedge \tilde{v}_{n+i} = \frac{\sqrt{-1}}{2} \sum_{ij} \text{ Im} Z_{ij} dw_i \wedge dw_j.$$

Since the Abelian varieties $M_y \cong T_y^{(1,0)} \tilde{U}/\tilde{\Lambda}_y$, and $\tilde{\Lambda}_y \cong \text{Span}_\mathbb{Z}(\tilde{v}_1^{(1,0)}, \cdots, \tilde{v}_{2n}^{(1,0)})$, the period matrices of these Abelian varieties are $[\Delta_d, Z]$, where $\Delta_d = \text{ diag}(d_1, \cdots, d_n)$.

The following lemma is a standard consequence of the Nilpotent Orbit Theorem [51], and we present the brief proof for the sake of completeness.

**Lemma 4.3.** There are rational matrices $\eta_i, i = 1, \cdots, k$, and a holomorphic matrix valued function $Q(t)$ on $\tilde{U}$ such that
$$Z(t) = Q(t) + \sum_{i=1}^k \frac{\log t_i}{2 \pi \sqrt{-1}} \Delta_d \eta_i.$$
Proof. Note that there are unique local sections \( \theta_1, \cdots, \theta_n \) of \( F^1 = T^{(1,0)} \hat{U}\backslash \gamma^{-1}(E) \) such that \( \langle \theta_i, \bar{v}_j \rangle = d_i \delta_{ij}, \) \( 1 \leq i, j \leq n, \) and equivalently \( \langle \theta_i, \bar{v}^{(1,0)}_j \rangle = 2d_i \delta_{ij}. \) Hence \( \langle \theta_i, \bar{v}^{(1,0)}_{j+n} \rangle = 2Z_{ji}, \) \( \langle \theta_i, \bar{v}_{j+n} \rangle = Z_{ji}, \) and

\[
\theta_i = d_i v_i + \sum_{j=1}^{n} Z_{ij} v_{j+n}, \quad i = 1, \cdots, n.
\]

Let \( \mathcal{D} \) be the classifying space of the polarized variation of Hodge structures \( (T^{(1,0)}N_0, TJ_0) \subset TN_0 \otimes_{\mathbb{R}} \mathbb{C}, \Lambda, \omega), \) and \( \mathcal{P} : \hat{U}\backslash \gamma^{-1}(E) \to \mathcal{D}/\Gamma \) be the the period map, where \( \Gamma \) is a discrete subgroup of \( Sp(2n, \mathbb{R}) \) depending on the polarization \( \omega. \) Note that \( \mathcal{D} \) can be identified as the Siegel upper half space

\[
\mathfrak{D}_n = \{ n \times n \text{ complex matrix } A | A^T = A, \quad \text{Im} A > 0 \},
\]

via \( T^{(1,0)}N_0 = \text{Span}_{\mathbb{C}}(\theta_1, \cdots, \theta_n) \mapsto Z(t). \)

The universal covering \( \hat{U} \to \hat{U}\backslash \gamma^{-1}(E) \) is given by \( t_i = \exp 2\pi \sqrt{-1} q_i, \) \( i = 1, \cdots, k, \) and \( t_j = q_j, \) \( j = k + 1, \cdots, n. \) If \( \hat{P} : \hat{U} \to \mathcal{D} \) is the lifting of \( \mathcal{P}, \) we have

\[
\hat{P}(\cdots, q_i + 1, \cdots) = T_i \hat{P}(\cdots, q_i, \cdots), \quad i = 1, \cdots, k.
\]

Equivalently,

\[
[\Delta_d, Z(q_i + 1)] = [\Delta_d, Z(q_i)] \begin{bmatrix} I & \eta_i \\ 0 & I \end{bmatrix},
\]

where \( \begin{bmatrix} I & 0 \\ \eta_i & I \end{bmatrix} \) is the matrix of \( T_i \) with respect to the basis \( v_1, \cdots, v_{2n}, \) which are rational matrices, and thus

\[
Z(q_i + 1) = Z(q_i) + \Delta_d \eta_i.
\]

If we define

\[
\tilde{Q} = \exp \left( -\sum_{i=1}^{k} q_i N_i \right) \hat{P},
\]

where \( N_i = \log T_i = T_i - I, \) then \( \tilde{Q}(\cdots, q_i + 1, \cdots) = \tilde{Q}(\cdots, q_i, \cdots), \) and \( \tilde{Q} \) descends to a holomorphic map \( \bar{Q} : \hat{U}\backslash \gamma^{-1}(E) \to \mathcal{D}. \)

By the Schmid’s Nilpotent Orbit Theorem \cite{51} (see also Chapter V of \cite{19}), \( \bar{Q} \) extends to a holomorphic map \( Q : \hat{U} \to \mathcal{D}, \) where as in \cite{51} \( \mathcal{D} \) denotes the compact dual of \( \mathcal{D}, \) which implies that there is a symmetric-matrices valued holomorphic function \( Q(t) \) on \( \hat{U} \) such that

\[
[\Delta_d, Q(t)] = [\Delta_d, Z(t)] \prod_{i=1}^{k} \begin{bmatrix} I & -q_i \eta_i \\ 0 & I \end{bmatrix}.
\]

Thus

\[
Z(t) = Q(t) + \sum_{i=1}^{k} \frac{\log t_i}{2\pi \sqrt{-1}} \Delta_d \eta_i.
\]

\( \Box \)
This lemma implies that there are $b^p_{ij} \in \mathbb{Q}$, $p = 1, \cdots, k$, $1 \leq i, j \leq n$, such that
\[ Z_{ij}(t) = Q_{ij}(t) + \sum_{p=1}^{k} \frac{\log t_p}{2\pi \sqrt{-1}} b^p_{ij}, \]
If we denote $G_{ij} = \sum_p Z_{ip} \frac{\partial w_p}{\partial t_j} = \frac{\partial w^*_i}{\partial t_j}$, then $dw^*_i = \sum_j G_{ij} dt_j$, and
\[ G_{ij}(t) = A_{ij}(t) + \sum_{p=1}^{k} \frac{\log t_p}{2\pi \sqrt{-1}} B^p_{ij}(t), \]
where $A_{ij}(t)$ and $B^p_{ij}(t)$ are holomorphic functions on $\tilde{U}$. By $0 = ddw^*_i = d\left( \sum_j G_{ij} dt_j \right)$, we have
\[ \frac{\partial G_{ij}}{\partial t_\ell} = \frac{\partial G_{i\ell}}{\partial t_j}, \]
for any $j \neq \ell$, and
\[ \frac{\partial A_{ij}}{\partial t_\ell} + \frac{\varepsilon(\ell)}{2\pi \sqrt{-1} t_\ell} + \sum_{p=1}^{k} \frac{\log t_p}{2\pi \sqrt{-1}} \frac{\partial B^p_{ij}}{\partial t_\ell} = \frac{\partial A_{i\ell}}{\partial t_j} + \frac{\varepsilon(\ell)}{2\pi \sqrt{-1} t_j} + \sum_{p=1}^{k} \frac{\log t_p}{2\pi \sqrt{-1}} \frac{\partial B^p_{i\ell}}{\partial t_j}, \]
where $\varepsilon(x) = 1$ if $1 < x < k$, and $\varepsilon(x) = 0$ if $k + 1 < x < n$. Therefore $B^\ell_{ij} = t_\ell \tilde{B}^\ell_{ij}$ and $B^j_{i\ell} = t_j \tilde{B}^j_{i\ell}$, for any $j \neq \ell < k$, where $\tilde{B}^\ell_{ij}$ are holomorphic functions. We obtain that
\[ G_{ij}(t) = \frac{\varepsilon(j) \log t_j}{2\pi \sqrt{-1}} B^j_{ij} + o(1). \]
Here we use the fact $|t \log t| < 1$ when $t$ is close to 0. Since
\[ q^* \omega = \frac{1}{4} \sum_{ij} (Z_{ij} - \bar{Z}_{ij}) dw_i \wedge d\bar{w}_j = \frac{1}{4} \sum_i (dw^*_i \wedge d\bar{w}_i - dw_i \wedge d\bar{w}_i^*), \]
we have
\[ g_{ij} = -\frac{1}{4} \sum_p \left( \frac{\partial w^*_i}{\partial t_j} \frac{\partial w_p}{\partial t_j} - \frac{\partial w_p}{\partial t_i} \frac{\partial w^*_i}{\partial t_j} \right) = -\frac{1}{4} \sum_p \left( G^p_{i\ell} \frac{\partial w_p}{\partial t_j} - \frac{\partial w_p}{\partial t_i} G^p_{i\ell} \right). \]
We obtain the conclusion
\[ |g_{ij}| \leq C(1 - \varepsilon(i) \log |t_i| - \varepsilon(j) \log |t_j|), \quad i, j = 1, \cdots, n, \]
for a uniform constant $C > 0$. \hfill \qed

**Remark 4.4.** In the case when $f : M \to N$ is an elliptic surface, then Kodaira’s classification of the possible singular fibers [33] gives explicit formulas for the (multivalued) period function $\tau(y)$, such that $\Lambda_y = \text{Span}_\mathbb{Z}(1, \tau(y))$, from which one can explicitly see that, after a branched covering $y = t^k$, the only possible singularities of $\tau(t)$ are of the form $\log t$ (see e.g. [26, p.377]). Lemma 4.3 is a (well-known) higher-dimensional generalization of
this observation. Very similar (and more precise) estimates were obtained by Hwang-Oguiso \cite{32} under more restrictive assumptions.

5. Applications to SYZ for hyperkähler manifolds

In this section we apply Theorem 1.2 to a refined SYZ conjecture due to Gross-Wilson (\cite[Conjecture 6.2]{25}), Kontsevich-Soibelman (\cite[Conjectures 1 and 2]{36}) and Todorov (\cite[p. 66]{41}) (see also \cite{16}).

5.1. Metric SYZ. Let $X$ be a Calabi-Yau $n$-manifold, and $\mathfrak{M}_X$ be the moduli space of complex deformations of $X$. If $\overline{\mathfrak{M}}_X$ denotes a certain compactification, then a large complex limit point $p \in \overline{\mathfrak{M}}_X$ is a point representing the ‘worst possible degeneration’ of the complex structures, which can be formulated via Hodge theory (cf. \cite{46}). Mirror symmetry predicts that for any large complex limit point $p \in \mathfrak{M}_X$, there is another Calabi-Yau manifold $\tilde{X}$, called the mirror, and an isomorphism between a neighborhood of $p$ in $\overline{\mathfrak{M}}_X$ and a neighborhood of a large radius limit in the complexified Kähler moduli space of $\tilde{X}$, which preserves some additional structures such as Yukawa couplings. Here a large radius limit point means the limit of $\exp 2\pi \sqrt{-1}(B + s\sqrt{-1}\tilde{\omega})$, when $s \to \infty$, in a certain compactification of $H^2(\tilde{X},U(1)) + \sqrt{-1}\mathbb{K}_\tilde{X}$, where $\mathbb{K}_\tilde{X}$ is the Kähler cone, $\tilde{\omega}$ is a Kähler metric, and $B \in H^2(\tilde{X},U(1))$ is called a B-field.

In \cite{55}, Strominger, Yau and Zaslow proposed a conjecture, the so called SYZ conjecture, for constructing mirror Calabi-Yau manifolds via dual special Lagrangian fibrations. More precisely, the SYZ conjecture says that near a large complex structure limit point $p$, the corresponding Calabi-Yau manifolds should admit a special Lagrangian torus fibration such that the mirror $\tilde{X}$ should be obtained as a compactification of the dual torus fibration, after suitable instanton corrections induced from the singular fibers. This has generated an immense amount of work, and we refer the reader to the surveys \cite{1,20,21} and references therein for more information.

Later, a metric version of the SYZ conjecture was proposed by Gross, Wilson, Kontsevich, Soibelman and Todorov \cite{25,36,37,41} by using the collapsing of Ricci-flat Kähler metrics, which is also related to non-Archimedean geometry (cf. \cite{3}). Let $X_t$, $t \in (0,1]$, be a family of $n$-dimensional Calabi-Yau manifolds such that the complex structures of $X_t$ converge to a large complex limit point $p$ in $\overline{\mathfrak{M}}_X$. The metric SYZ conjecture \cite[Conjecture 6.2]{25}, \cite[Conjecture 1]{36}, asserts that there are Ricci-flat Kähler metrics $\tilde{\omega}_t$ on $X_t$, for $t \neq 0$, such that $(X_t, \text{diam}_\mathbb{C}_\omega(X_t)\tilde{\omega}_t)$ converges to a compact metric space $(Y,d_Y)$ in the Gromov-Hausdorff sense, when $t \to 0$. Furthermore, there is an open and dense subset $Y_0 \subset Y$ which is a smooth real $n$-dimensional Riemannian manifold $(Y_0,g)$, and admits a real affine structure. The singular locus $S_Y = Y \setminus Y_0$ is of Hausdorff codimension at least 2. The metric space $(Y,d_Y)$ is the metric completion of $(Y_0,g)$, and $g$ is a Monge-Ampère metric on $Y_0$, i.e. in local affine coordinates $(y_1,\cdots,y_n)$,
there is a potential function $\phi$ such that
\[
g = \sum_{ij} \frac{\partial^2 \phi}{\partial y_i \partial y_j} dy_i dy_j, \quad \text{and} \quad \det \left( \frac{\partial^2 \phi}{\partial y_i \partial y_j} \right) = c,
\]
for some $c \in \mathbb{R}_{>0}$. When $\tilde{\omega}_t$ have holonomy $SU(n)$ (resp. hyperkähler), $Y$ should be homeomorphic to an $n$-sphere (resp. $\mathbb{CP}^n$). It is not hard to see that the conjecture is true when $X_t$ are Abelian varieties (see e.g. [50, Section 3]). This conjecture was verified by Gross and Wilson for elliptically fibered K3 surfaces with only type $I_1$ singular fibers in [23], for large complex structure limits which arise as hyperkähler rotations from our setup in the Introduction. In [24], Gross-Wilson’s result was extended to all elliptically fibered K3 surfaces, and a partial results for higher dimensional hyperkähler manifolds were obtained in [23, 24]. As mentioned in the Introduction, Corollary 1.3 proves this conjecture for all large complex structure limits of projective hyperkähler manifolds which arise from our setup via hyperkähler rotation. An analogue of this conjecture was proved for canonically polarized manifolds in [67].

The next step in the SYZ program is to construct the mirror $\tilde{X}$ as a certain compactification of $TY_0/\Lambda$, for a lattice subbundle $\Lambda$ of $TY_0$. This is the so called the reconstruction problem, and is of great interest in mirror symmetry (see [20, 37, 22]). The reconstruction problem suggests a more explicit behaviour of the Ricci-flat Kähler metrics $\tilde{\omega}_t$ near the collapsing limit [36, Conjecture 2], which asserts that $\tilde{\omega}_t$ is asymptotic to certain semi-flat Ricci-flat Kähler metrics.

5.2. Semi-flat hyperkähler structures. In this subsection, we recall the construction of semi-flat hyperkähler structures on algebraic completely integrable systems [13], and the semi-flat SYZ construction studied in [30, 29]. The next subsection applies Theorem 1.2 to the metric version of SYZ conjecture for compact hyperkähler manifolds, and shows that the hyperkähler structures approach such semi-flat hyperkähler structures near the limit. We use the same notations as in subsection 4.2.

Let $(f : M_0 \to N_0, [\alpha], \Omega)$ be an algebraic completely integrable system, $\omega$ be the induced special Kähler metric on $N_0$, and $g$ be the corresponding Riemannian metric. The fibers $M_y = f^{-1}(y), y \in N_0$, are Abelian varieties of type $(d_1, \ldots, d_n)$. Assume furthermore that there is a holomorphic Lagrangian section $\sigma : N_0 \to M_0$, which as mentioned in subsection 4.2 implies that $M_0 \cong T^{\ast(1,0)} N_0/\hat{\Lambda}$ for a lattice subbundle $\hat{\Lambda}$, and as before let $p : T^{\ast(1,0)} N_0 \to M_0$ be the holomorphic covering map, which satisfies $\Omega_{can} = p^\ast \Omega$, and $\hat{\Lambda} = \ker p$.

For any $t \in (0, 1]$, we define a family of semi-flat Kähler metrics on $M_0$ by
\[
(5.1) \quad \omega_{SF, t} = t^{-\frac{1}{2}} f^\ast \omega + t^{\frac{3}{2}} \omega_{SF},
\]
where \( \omega_{\text{SF}} \) is given by Theorem 5.1, which satisfies \( \omega_{\text{SF}}|_{M_y} = \sqrt{t} \omega_{\text{SF}}|_{M_y} \in \sqrt{t}[\alpha_y] \), for any \( y \in N_0 \), and we denote by \( g_{\text{SF},t} \) the corresponding Riemannian metric. Note that \( t^{\frac{3}{2}} \omega_{\text{SF},t} \) is the semi-flat Kähler metric constructed in Section 2 of [23]. For any fiber \( M_y \), the diameter

\[
\operatorname{diam}(M_y, g_{\text{SF},t}|_{M_y}) \leq C t^{\frac{1}{2}} \to 0, \quad \text{when } t \to 0,
\]

and thus \((M_0, g_{\text{SF},t})\) collapses the torus fibers.

For an open subset \( U \subset N_0 \), let \( y_1, \cdots, y_{2n} \) be the flat Darboux coordinates such that \( dy_i \) satisfy (4.5). For the local trivialization \( T^*N_0|_U \cong U \times \mathbb{R}^{2n} \) by \( \sum_i x_idy_i \to (y_1, \cdots, y_{2n}, x_1, \cdots, x_{2n}) \), \( dx_1, \cdots, dx_{2n} \) are well-defined closed 1-forms on \( f^{-1}(U) \), and we have

\[
(5.2) \quad \omega_{\text{SF}} = -\sum_{i=1}^{n} dx_i \wedge dx_{i+n} = \frac{\sqrt{-1}}{2} \sum_{i,j} \Im Z_{ij}^{-1} \partial_i \wedge \bar{\partial}_j,
\]

where \( \partial_i = dx_i - \sum_{j=1}^{n} Z_{ij} dx_{j+n}, \ i = 1, \cdots, n \), which may not be closed (cf. [26, Lemma 3.3]). Thus

\[
\omega_{\text{SF},t} = t^{-\frac{1}{2}} \sum_{i=1}^{n} dy_i \wedge dy_{i+n} - t^{\frac{3}{2}} \sum_{i=1}^{n} dx_i \wedge dx_{i+n}.
\]

In particular, we see that \( p^* \omega_{\text{SF},t}^2 = \Omega_{\text{can}}^n \wedge \bar{\Omega}_{\text{can}}^n \), which shows that, by changing variables if necessary, \((p^* \omega_{\text{SF},t}, \Omega_{\text{can}} = p^* \Omega)\) is the hyperkähler structure on \( T^*(1,0)N_0 \) constructed in Section 2 of [13] (see also [26, Section 3.2]), and \((\omega_{\text{SF},t}, \Omega, g_{\text{SF},t})\), \( t \in (0,1] \), is a family of hyperkähler structures on \( M_0 \).

By hyperkähler rotation, we define a family of complex structures \( J_t \) with hyperkähler structures

\[
(5.3) \quad \omega_{J_t} = \Re \Omega, \quad \Omega_{J_t} = \Im \Omega + \sqrt{-1} \omega_{\text{SF},t},
\]

and the fibration \( f : M_0 \to N_0 \) is a special Lagrangian fibration with respect to \((\omega_{J_t}, \Omega_{J_t}^2)\).

Note that \( \omega = \sum_i dy_i \wedge dy_{i+n} \) under the flat Darboux coordinates \( y_1, \cdots, y_{2n} \), and \( g \) is a Monge-Ampère metric with the local potential \( \phi \), i.e.

\[
\phi_{ij} = \frac{\partial^2 \phi}{\partial y_i \partial y_j} = g_{ij} \ 	ext{as shown in subsection 4.1.}
\]

The Legendre transform of the local potential function \( \phi \) gives the dual affine structure, which is defined by the local dual affine coordinates \( \phi_1 = \frac{\partial \phi}{\partial y_1}, \cdots, \phi_{2n} = \frac{\partial \phi}{\partial y_{2n}} \).

**Lemma 5.1.** The Kähler form \( \omega_{J_t} \) is induced by the canonical symplectic form on \( T^*N_0 \), i.e.

\[
\omega_{J_t} = \sum_{i=1}^{2n} dy_i \wedge dx_i.
\]
If we let
\[ \chi_{t,i} = \exp 2\pi \sqrt{-1}(x_i + \sqrt{-1}t^{-\frac{1}{2}}\phi_i), \quad i = 1, \ldots, 2n, \]
then \(\chi_{t,1}, \ldots, \chi_{t,2n}\) are holomorphic Darboux coordinates on \(f^{-1}(U)\) with respect to \(J_t\), and
\[ \Omega_{J_t} = \frac{t^{\frac{1}{2}}}{4\pi^2\sqrt{-1}} \sum_{i=1}^{n} \frac{d\chi_{t,i}}{\chi_{t,i}} \wedge \frac{d\chi_{t,i+n}}{\chi_{t,i+n}}. \]

Proof. Denote by \(I\) the complex structure on \(N_0\). By \(g(\cdot, \cdot) = \omega(\cdot, I\cdot)\), if
\[ I \left( \frac{\partial}{\partial y_i} \right) = \sum_j \frac{\partial}{\partial y_j} I_{ji}, \]
then
\[ \begin{bmatrix} 0, \ -\id \end{bmatrix} \begin{bmatrix} \phi_{ij} \end{bmatrix} = \begin{bmatrix} I_{ij} \end{bmatrix}, \]
i.e. \(I_{ij} = -\phi_{i+n,j}, \ I_{i,j+n} = -\phi_{i+n,j+n}, \ I_{i+n,j} = \phi_{i,j}, \ I_{i+n,j+n} = \phi_{i,j+n}\), for \(1 \leq i \leq n\). We have
\[ I(dy_i) = \sum_j I_{ij} dy_j = -d\phi_{i+n}, \quad I(dy_{i+n}) = \sum_j I_{i+n,j} dy_j = d\phi_i, \]
\[ dw_i = d(y_i + \sqrt{-1}\phi_{i+n}), \quad dw_i^* = -d(y_{i+n} - \sqrt{-1}\phi_i), \]
for \(i = 1, \ldots, n\), where the special coordinates \(w_1, \ldots, w_n\) and their conjugates \(w_1^*, \ldots, w_n^*\) are defined as in subsection 4.2. By \(\omega(\cdot, \cdot) = \omega(I\cdot, I\cdot)\),
\[ \omega = \sum_i dy_i \wedge dy_{i+n} = \sum_i I(dy_i) \wedge I(dy_{i+n}) = \sum_i d\phi_i \wedge d\phi_{i+n}, \quad \text{and} \]
\[ \omega_{SF,t} = \sum_i (t^{-\frac{1}{2}}d\phi_i \wedge d\phi_{i+n} - t^\frac{1}{2}dx_i \wedge dx_{i+n}). \]

Under the local trivialization \(T^*(1,0)N_0|_U \cong U \times \mathbb{C}^n\) by \(\sum_i z_idw_i \mapsto (w_1, \ldots, w_n, z_1, \ldots, z_n)\), we have
\[ \Omega_{can} = \sum_i dw_i \wedge dz_i, \quad \text{and} \quad z_i = x_i - \sum_{j=1}^{n} Z_{ij}x_{j+n} \]
by \(\sum_{i=1}^{2n} x_idy_i^{(1,0)} = \sum_{i=1}^{n} \left( x_i - \sum_{j=1}^{n} Z_{ij}x_{j+n} \right) dw_i, \quad 1 \leq i \leq n\). Then
\[ \Omega = -d(\sum_i z_i dw_i) = -d(\sum_i (x_i dw_i - x_{i+n} dw_i^*)) \]
\[ = \sum_{i=1}^{2n} dy_i \wedge dx_i + \sqrt{-1} \sum_{i=1}^{n} (d\phi_{i+n} \wedge dx_i - d\phi_i \wedge dx_{i+n}). \]

Thus
\[ \omega_{J_t} = \sum_{i=1}^{2n} dy_i \wedge dx_i, \quad \text{and} \]
\[ \Omega_{J_t} = \sum_{i=1}^{n} (d\phi_{i+n} \wedge dx_i - d\phi_i \wedge dx_{i+n} + \sqrt{-1}(t^{-\frac{1}{2}}d\phi_i \wedge d\phi_{i+n} - t^{\frac{1}{2}}dx_i \wedge dx_{i+n})). \]

We obtain the conclusion by using
\[ \frac{d\chi_{t,i}}{\chi_{t,i}} = 2\pi \sqrt{-1}(dx_i + t^{-\frac{1}{2}}\sqrt{-1}d\phi_i). \]

We remark that on \( f^{-1}(U) \), \( f \) is the logarithmic map
\[ 2\pi \log |\chi_{t,i}|, \quad i = 1, \ldots, 2n, \]
with respect to the dual affine structure, which converts algebro geometric objects in \( f^{-1}(U) \) into tropical geometric objects on \( U \) when \( t \to 0 \).

Now we recall the semi-flat SYZ construction of \((M_0, \Omega, J_t)\) (cf. [30, 29]). We ignore B-fields in the following discussion. Note that \( J_t \frac{\partial}{\partial x_i} = t^{\frac{1}{2}} \frac{\partial}{\partial \phi_i} \)
and
\[ \omega_{J_t} = \sum_{i=1}^{2n} dy_i \wedge dx_i = \sum_{ij} \frac{\partial y_i}{\partial \phi_j} d\phi_j \wedge dx_i = \sum_{ij} \phi_{ij}^{-1} d\phi_j \wedge dx_i. \]

Thus for any \( y \in N_0 \), we obtain that
\[ g_{SF}\big|_{M_y} = t^{\frac{1}{2}} \sum_{ij} \phi_{ij}^{-1}(y)dx_i dx_j. \]

Following [55], we now construct the semi-flat SYZ mirror of \((M_0, \omega, J_t)\), as follows. Let \( \tilde{M}_0 = TN_0/\Lambda \), and \( f : \tilde{M}_0 \to N_0 \) be the fibration induced by \( TN_0 \to N_0 \). For any \( y \in N_0 \), the fiber \( M_y = T_y N_0/\Lambda_y \) is the dual Abelian variety of \( M_y \), which is of type \((d_n/d_n, \ldots, d_1/d_n)\). On \( TN_0 \), there is a natural complex structure induced by the flat affine structure on \( N_0 \), which gives a complex structure \( \tilde{J} \) on \( \tilde{M}_0 \). Under a local trivialization \( TN_0|_{U} \cong U \times \mathbb{R}^{2n} \) by \( \sum_i \tilde{x}_i \frac{\partial}{\partial y_i} \mapsto (y_1, \ldots, y_{2n}, \tilde{x}_1, \ldots, \tilde{x}_{2n}) \), the complex structure \( \tilde{J} \) is given by the holomorphic coordinates \( \xi_i = \exp 2\pi \sqrt{-1}(\tilde{x}_i + \sqrt{-1}y_i) \), \( i = 1, \ldots, 2n \). Note that if \( y'_i \), \( i = 1, \ldots, 2n \), are another flat Darboux coordinates, and \( \tilde{x}'_i \) are induced coordinates with \( \sum_i \tilde{x}_i \frac{\partial}{\partial y_i} = \sum_i \tilde{x}'_i \frac{\partial}{\partial y'_i} \), then
\[ y_i = \sum a_{ij} y'_j + b_i \]
and \( \tilde{x}_i = \sum_j a_{ij} \tilde{x}'_j \), where \((a_{ij}) \in Sp(2n, \mathbb{R})\) and \( b_i \in \mathbb{R} \). Therefore
\[ \tilde{\Omega} = -\frac{1}{4\pi^2} \sum_{i=1}^{n} \frac{d\xi_i}{\xi_i} \wedge \frac{d\xi_{i+n}}{\xi_{i+n}} \]
is a well-defined holomorphic symplectic form on \( \tilde{M}_0 \).

A natural Kähler metric on \( \tilde{M}_0 \) is
\[ \tilde{\omega} = t^{-\frac{\phi}{2}} \sum_{i=1}^{2n} d\phi_i \wedge d\tilde{x}_i = t^{-\frac{\phi}{2}} \sum_{ij} \frac{\partial^2 \phi}{\partial y_i \partial y_j} dy_i \wedge d\tilde{x}_j = t^{-\frac{1}{2}} \omega, \]
which gives a hyperkähler structure \((\tilde{\omega}_t, t^{-\frac{1}{2}}\tilde{\Omega})\) on \(\tilde{M}_0\) since \(\det(\phi_{ij}) \equiv \text{const.}\)

If \(\tilde{g}_{SF,t}\) denotes the Riemannian metric determined by \(\tilde{\omega}_t\) and \(\tilde{J}\), then

\[\tilde{g}_{SF,t}|_{\tilde{M}_y} = t^{-\frac{1}{2}} \sum_{ij} \phi_{ij}(y) d\tilde{x}_i d\tilde{x}_j,\]

by \(\tilde{J}_{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial y_i}\), for a \(y \in N_0\). The semi-flat SYZ mirror of \((M_0, \omega_{J_t}, J_t)\) is \((\tilde{M}_0, \tilde{\omega}_t, \tilde{J})\) in the sense of T-duality (cf. [55] and Chapter 1.3 in [1]), i.e. \((\tilde{M}_0, \tilde{g}_{SF,t}|_{\tilde{M}_y})\) is the dual torus of \((M_y, g_{SF,t}|_{M_y})\) for any \(y \in N_0\). When \(t \to 0\), we say that the complex structures \(J_t\) tends to a large complex limit, while the symplectic structure \(\omega_{J_t}\) is fixed, in the sense that its semi-flat SYZ mirror has the symplectic structures \(\tilde{\omega}_t = t^{-\frac{1}{2}}\tilde{\omega}\) tending to a large radius limit while keeping the complex structure \(\tilde{J}\) fixed.

5.3. Collapsing hyperkähler metrics are close to semi-flat. Now we show how Theorem 1.2 together with [23, 24], fits into this refined version of SYZ conjecture for hyperkähler manifolds. The setup is now the same as in Theorem 1.2 so \(f : M^{2n} \to N^n \cong \mathbb{C}P^n\) is a holomorphic fiber space with \(M\) projective hyperkähler, with the extra assumption that there is a holomorphic Lagrangian section \(\sigma : N_0 \to M_0\). We denote by \(\Omega\) the holomorphic symplectic form on \(M\), the fibration \(f\) is then an algebraic completely integrable system over \(N_0\), the complement of the discriminant locus of \(f\) in \(N\), and \([\alpha]\) is an integral Kähler class on \(M\). The fibers \(M_y = f^{-1}(y), y \in N_0\), are Abelian varieties, and the polarization \([\alpha_y] = [\alpha|_{M_y}]\) is of type \((d_1, \ldots, d_n)\). As we mentioned in subsection 1.2 the existence of \(\sigma\) implies that \(M_0 \cong T^{*-1(1,0)}N_0/\Lambda\), and let then \(p : T^{*-1(0,1)}N_0 \to M_0\) be the holomorphic covering map, which satisfies \(\Omega_{can} = p^*\Omega\), and \(\Lambda = \ker p\).

Let \([\alpha_0]\) be the ample class on \(N\) such that

\([\alpha]^n \cdot [f^*\alpha_0]^n = \int_M \Omega^n \wedge \hat{\Omega}^n,\]

and \(\tilde{\omega}_t\) be the unique Ricci-flat hyperkähler metric on \(M\) in the class \(f^*\alpha_0 + t[\alpha], 0 < t \leq 1\), which satisfies the complex Monge-Ampère equation

\[\tilde{\omega}_t^{2n} = c_t t^n \Omega^n \wedge \hat{\Omega}^n\]

with \(c_t \to 1\) when \(t \to 0\). Therefore, \((c_t^{-\frac{1}{4}} t^{-\frac{1}{2}} \tilde{\omega}_t, \Omega)\) is a hyperkähler structure, and we denote \(\tilde{g}_t\) the corresponding hyperkähler metric of \(c_t^{-\frac{1}{4}} t^{-\frac{1}{2}} \tilde{\omega}_t\).

By hyperkähler rotation, we have a family of complex structures \(\tilde{J}_t\) with hyperkähler structures

\[\omega_{\tilde{J}_t} = \text{Re}\Omega, \quad \Omega_{\tilde{J}_t} = \text{Im}\Omega + \sqrt{-1} c_t^{-\frac{1}{2}} t^{-\frac{1}{2}} \tilde{\omega}_t.\]

A well-known simple calculation shows that the fibration \(f : M \to N\) becomes a special Lagrangian fibration with respect to \((\omega_{\tilde{J}_t}, \Omega^n_{\tilde{J}_t})\), and \(\sigma\) becomes a special Lagrangian section.
By [23] Theorem 1.2 and \( c_t \to 1 \), we have that by passing to subsequences, \((M,t^{\frac{1}{2}} \bar{g}_t, \bar{\omega}_t)\) converges to a compact metric space \((X,d_X)\) in the Gromov-Hausdorff sense, and there is a locally isometric embedding \((N_0,\omega) \hookrightarrow (X,d_X)\), and [24] Theorem 1.2 asserts that \(\omega\) is a special Kähler metric on \(N_0\). Furthermore, Lemma 4.1 in [24] shows that \(\omega\) is the special Kähler metric induced by the algebraic completely integrable system \((f:M_0 \to N_0,[\alpha],\Omega)\), where \(M_0 = f^{-1}(N_0)\). Now Theorem 1.2 shows that \((X,d_X)\) is the metric completion of \((N_0,\omega)\), the singular set \(X \setminus N_0\) has Hausdorff codimension at least 2, and there is no need to passing to any subsequence in the convergence, i.e.

\[
(M,t^{\frac{1}{2}} \bar{g}_t, \bar{\omega}_t) \to (X,d_X), \text{ when } t \to 0.
\]

Furthermore, as predicted by [35] Conjecture 2], we claim that \(t^{\frac{1}{2}} \bar{g}_t\) approaches some semi-flat metrics in a certain sense that we now explain.

As in (5.1), for any \(t \in (0,1]\) we define a family of semi-flat Kähler metrics on \(M_0 = f^{-1}(N_0)\) by

\[
(5.5) \quad \omega_{SF,t} = t^{-\frac{1}{2}} f^* \omega + t^{\frac{1}{2}} \omega_{SF},
\]

where \(\omega_{SF}\) is given by Theorem 3.1, and we denote by \(g_{SF,t}\) the corresponding Riemannian metric. Following [23], we define the dilation map \(\lambda_t : T^*^{(1,0)}N_0 \to T^*^{(1,0)}N_0\) by \(\lambda_t(y,z) = (y, t^{\frac{1}{2}} z)\) and the covering map \(p : T^*^{(1,0)}N_0 \to M_0\) such that \(\Omega_{can} = p^* \Omega\), as in subsection 1.2. Thanks to Proposition 3.2, we have that

\[
\lambda_t^* p^* \bar{\omega}_t \to p^* \omega_{SF,1},
\]

smoothly on compact sets. Direct calculations show that

\[
\lambda_t^* p^* \sqrt{n} \bar{\omega}_t = \Omega_{can}, \quad \text{and} \quad \lambda_t^* p^* t^{\frac{1}{2}} \omega_{SF,t} = \omega_{SF,1}
\]

(cf. Section 4 in [24]), which implies

\[
\|c_t^{-\frac{1}{2n}} \bar{\omega}_t - t^{\frac{1}{2}} \omega_{SF,t}\|_{C^\infty_0(M_0,t^{\frac{1}{2}} g_{SF,t})} \to 0.
\]

Note that \((\omega_{SF,t},\Omega,g_{SF,t})\), \(t \in (0,1]\), is a family of semi-flat hyperkähler structures on \(M_0\). By hyperkähler rotation, we define a family of complex structures \(J_t\) with hyperkähler structures

\[
\omega_{J_t} = \operatorname{Re}\Omega = \omega_{\bar{J}_t}, \quad \Omega_{J_t} = \operatorname{Im}\Omega + \sqrt{-1} \omega_{SF,t}.
\]

By Lemma 5.1, the Kähler form \(\omega_{J_t}\) is induced by the canonical symplectic form on \(T^*N_0\).

From this discussion together with Theorem 1.2, we obtain the following theorem:

**Theorem 5.2.** In the above setup we have

i) On \(M_0\), when \(t \to 0\), we have \(\omega_{J_t} = \omega_{\bar{J}_t}\),

\[
\|t^{\frac{1}{2}} (\bar{g}_t - g_{SF,t})\|_{C^\infty_0(M_0,t^{\frac{1}{2}} g_{SF,t})} \to 0, \quad \text{and} \quad \|\bar{J}_t - J_t\|_{C^\infty_0(M_0,t^{\frac{1}{2}} g_{SF,t})} \to 0.
\]
There is a special Kähler metric $\omega$ on $N_0$ such that the metric completion $(N_0, g)$ is compact, and

$$(M, t^{\frac{1}{2}} \tilde{g}_t) \to (N_0, g),$$

in the Gromov-Hausdorff sense, where $g$ denotes the corresponding Riemannian metric of $\omega$ on $N_0$.

The singular set $S_{N_0} = (N_0, g) \setminus N_0$ has the Hausdorff codimension at least 2.

g is a real Monge-Ampère metric with respect to the real affine structure determined by the special Kähler metric $\omega$.

Note that the semi-flat symplectic form $\omega_{\tilde{J}_t}$ on $M_0$ can be extended to a symplectic form on $M$, which equals $\omega_{\tilde{J}_t}$. However, the complex structure $J_t$ usually cannot be extended to a complex structure on $M$. In order to extend $J_t$, one must add to it certain additional terms called instanton corrections (these equal $\tilde{J}_t - J_t$ in the present case), which are determined by certain tropical geometric objects on $N_0$ constructed inductively from the initial information of the singularities $S_{N_0}$. See [16] in the analytic setting, [37, 22] in the algebro geometric setting, and [17, 48, 38] for the current case of hyperkähler manifolds.

Let us also remark that, as shown in the last subsection, the complex structures $J_t$ (therefore also $\tilde{J}_t$) tends to a large complex limit, when $t \to 0$, in the sense that its semi-flat SYZ mirror $\tilde{M}_0$ has the symplectic structures $\tilde{\omega}_t = t^{-\frac{1}{2}} \tilde{\omega}$ tending to a large radius limit. Furthermore, we expect that $\tilde{\omega}_t$ extends to a symplectic form on a certain compactification of $\tilde{M}_0$, if the SYZ mirrors of $M$ indeed exist, for example the case of Section 2 in [23].

Lastly, let us mention that the conjecture of Gross-Wilson, Kontsevich-Soibelman and Todorov has directly inspired a purely algebro-geometric conjecture, which is as follows: let $X \to C$ be a projective family of Calabi-Yau manifolds over a quasiprojective curve, smooth over $C \setminus \{o\}$, such that $X_o$ is a large complex structure limit. After applying semistable reduction and a relative MMP, the dual intersection complex of the new central fiber is denoted by $Sk(X)$, the essential skeleton of $X$. It is a connected $n$-dimensional simplicial complex, whose topological type does not depend on the choices we made. The conjecture is then that $Sk(X)$ should topologically be an $n$-sphere when $\tilde{\omega}_t$ (the Ricci-flat Kähler metric on $X_t$ in the polarization class) have holonomy $SU(n)$, and topologically $\mathbb{C}P^n$ when $\tilde{\omega}_t$ are hyperkähler. In particular, it should be homeomorphic to the Gromov-Hausdorff limit $(X, d_X)$ of the collapsing Ricci-flat Kähler metrics (normalized to have unit diameter). See [3, 17, 49] for more details, and [34, 35] for very recent progress on these questions.
References

[1] P. Aspinwall, T. Bridgeland, A. Craw, M. Douglas, M. Gross, A. Kapustin, G. Moore, G. Segal, B. Szendrői, P. Wilson, Dirichlet branes and mirror symmetry. Clay Mathematics Monographs, 4. American Mathematical Society, (2009).

[2] C. Bartocci, I. Mencattini, Some remarks on special Kähler geometry, J. Geom. Phys. 59 (2009), no. 7, 755–763.

[3] S. Boucksom, M. Jonsson, Tropical and non-Archimedean limits of degenerating families of volume forms, J. Éc. polytech. Math. 4 (2017), 87–139.

[4] J. Carlson, E. Cattani, A. Kaplan, Mixed Hodge Structures and compactifications of Siegel’s space, Journées de géométrie algébrique d’Angers, 1979, Sijthoff Noordhoff, 1980, 77–105.

[5] J. Cheeger, T. Colding, On the structure of space with Ricci curvature bounded below I, J. Differential Geom. 46 (1997), 406–480.

[6] J. Cheeger, T. Colding, On the structure of space with Ricci curvature bounded below II, J. Differential Geom. 52 (1999), 13–35.

[7] S.Y. Cheng, S.T. Yau, The real Monge-Ampère equation and affine flat structures, Chern S.S., ed. Proc. 1980 Beijing Sympos. Diff. Geom. Diff. Eq., Vol. 1 (1982), 339–370.

[8] B. Claudon, Smooth families of tori and linear Kähler groups, to appear in Ann. Fac. Sci. Toulouse Math.

[9] T.C. Collins, V. Tosatti, Kähler currents and null loci, Invent. Math. 202 (2015), no.3, 1167–1198.

[10] V. Cortés, Special Kähler manifolds: a survey, Rend. Circ. Mat. Palermo (2) Suppl. 2002, no. 69, 11–18.

[11] P. Dalakov, Lectures on Higgs moduli and Abelisation, preprint, arXiv:1609.00646.

[12] R. Donagi, E. Witten, Supersymmetric Yang-Mills theory and integrable systems, Nuclear Phys. B 460 (1996), no. 2, 299–334.

[13] D. Freed, Special Kähler Manifolds, Comm. Math. Phys. 203 (1999), no. 1, 31–52.

[14] O. Fujino, Y. Gongyo, On images of weak Fano manifolds, Math. Z. 270 (2012), no. 1-2, 531–544.

[15] T. Fujita, On Kähler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), no. 4, 779–794.

[16] K. Fukaya, Multivalued Morse theory, asymptotic analysis and mirror symmetry, in Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., 73, Amer. Math. Soc., (2005), 205–278.

[17] D. Gaiotto, G. Moore, A. Neitzke, Four-dimensional wall-crossing via three-dimensional field theory, Comm. Math. Phys. 299 (2010), no. 1, 163–224.

[18] B. Greene, A. Shapere, C. Vafa, S.-T. Yau, Stringy cosmic strings and noncompact Calabi-Yau manifolds, Nuclear Phys. B 337 (1990), no. 1, 1–36.

[19] P. Griffiths, Topics in transcendental algebraic geometry, Annals of Mathematics Studies, 106. Princeton University Press, Princeton, NJ, 1984.

[20] M. Gross, Mirror symmetry and the Strominger-Yau-Zaslow conjecture, in Current developments in mathematics 2012, 133–191, Int. Press, Somerville, MA, 2013.

[21] M. Gross, D. Huybrechts, D. Joyce, Calabi-Yau manifolds and related geometries, Springer-Verlag 2003.

[22] M. Gross, B. Siebert, From real affine geometry to complex geometry, Ann. of Math. (2) 174 (2011), no. 3, 1301–1428.

[23] M. Gross, V. Tosatti, Y. Zhang, Collapsing of abelian fibred Calabi-Yau manifolds, Duke Math. J. 162 (2013), no. 3, 517–551.

[24] M. Gross, V. Tosatti, Y. Zhang, Gromov-Hausdorff collapsing of Calabi-Yau manifolds, Comm. Anal. Geom. 24 (2016), no. 1, 93–113.
[25] M. Gross, P.M.H. Wilson, Large complex structure limits of $K3$ surfaces, J. Differ. Geom. 55 (2000), 475–546.
[26] H.-J. Hein, Gravitational instantons from rational elliptic surfaces, J. Amer. Math. Soc. 25 (2012), no. 2, 355–393.
[27] H.-J. Hein, V. Tosatti, Remarks on the collapsing of torus fibered Calabi-Yau manifolds, Bull. Lond. Math. Soc. 47 (2015), no. 6, 1021–1027.
[28] C. Hertling, tt* geometry, Frobenius manifolds, their connections, and the construction for singularities, J. Reine Angew. Math. 555 (2003), 77–161.
[29] N. Hitchin, The moduli space of special Lagrangian submanifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 3-4, 503–515.
[30] N. Hitchin, The moduli space of complex Lagrangian submanifolds, Asian J. Math. 3 (1999), no. 1, 77–91.
[31] J.-M. Hwang, Base manifolds for fibrations of projective irreducible symplectic manifolds, Invent. Math. 174 (2008), no. 3, 625–644.
[32] J.-M. Hwang, K. Oguiso, Local structure of principally polarized stable Lagrangian fibrations, preprint, arXiv:1007.2043.
[33] K. Kodaira, On compact analytic surfaces, II, Ann. of Math. (2) 77 (1963), 563–626.
[34] J. Kollár, R. Laza, G. Sacca, C. Voisin, Remarks on degenerations of hyper-Kähler manifolds, preprint, arXiv:1704.02731.
[35] J. Kollár, C. Xu, The dual complex of Calabi-Yau pairs, Invent. Math. 205 (2016), no. 3, 527–557.
[36] M. Kontsevich, Y. Soibelman, Homological mirror symmetry and torus fibrations, in Symplectic geometry and mirror symmetry, 203–263, World Sci. Publishing 2001.
[37] M. Kontsevich, Y. Soibelman, Affine Structures and Non-Archimedean Analytic Spaces, in The Unity of Mathematics, Progress in Mathematics Volume 244, Springer, (2006), 321–385.
[38] M. Kontsevich, Y. Soibelman, Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and mirror symmetry, in Homological mirror symmetry and tropical geometry, 197–208, Lect. Notes Unione Mat. Ital., 15, Springer, Cham, 2014.
[39] Y. Li, On collapsing Calabi-Yau fibrations, preprint, 2017.
[40] Z. Lu, A note on special Kähler manifolds, Math. Ann. 313 (1999), no. 4, 711–713.
[41] Y.I. Manin Moduli, motives, mirrors, in European Congress of Mathematics, Vol. I (Barcelona, 2000), 53–73, Progr. Math., 201, Birkhäuser, Basel, 2001.
[42] E. Markman, Algebraic geometry, integrable systems, and Seiberg-Witten theory, in Integrability: the Seiberg-Witten and Whitham equations, Edinburgh (2000), 23–41.
[43] D. Markushevich, Lagrangian families of Jacobians of genus 2 curves, J. Math. Sci. 82 (1996), no. 1, 3268–3284.
[44] D. Matsushita, On fibre space structures of a projective irreducible symplectic manifold, Topology 38 (1999), no. 1, 79–83; Addendum 40 (2001), no. 2, 431–432.
[45] D. Matsushita, Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds, Math. Res. Lett. 7 (2000), no. 4, 389–391.
[46] D. Morrison, Compactifications of moduli spaces inspired by mirror symmetry, in Journées de géométrie Algébrique d’Orsay, Astérisque 218 (1993), 243–271.
[47] M. Mustaţă, J. Nicaise, Weight functions on non-Archimedean analytic spaces and the Kontsevich-Soibelman skeleton, Algebr. Geom. 2 (2015), no. 3, 365–404.
[48] A. Neitzke, Notes on a new construction of hyperkahler metrics, in Homological mirror symmetry and tropical geometry, 351–375, Lect. Notes Unione Mat. Ital., 15, Springer, Cham, 2014.
[49] J. Nicaise, C. Xu, The essential skeleton of a degeneration of algebraic varieties, Amer. J. Math. 138 (2016), no. 6, 1645–1667.
[50] Y. Odaka, Tropically compactify moduli via Gromov-Hausdorff collapse, preprint, arXiv:1406.7772.
[51] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Invent. Math. 22 (1973), 211–319.

[52] J. Song, Riemannian geometry of Kähler-Einstein currents, preprint, arXiv:1404.0445

[53] J. Song, G. Tian, Canonical measures and Kähler-Ricci flow, J. Amer. Math. Soc. 25 (2012), no. 2, 303–353.

[54] A. Strominger, Special geometry, Comm. Math. Phys. 133 (1990), no. 1, 163–180.

[55] A. Strominger, S.-T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nuclear Phys. B 479 (1996), no. 1-2, 243–259.

[56] V. Tosatti, Limits of Calabi-Yau metrics when the Kähler class degenerates, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 4, 755–776.

[57] V. Tosatti, Adiabatic limits of Ricci-flat Kähler metrics, J. Differential Geom. 84 (2010), no. 2, 427–453.

[58] V. Tosatti, Degenerations of Calabi-Yau metrics, in Geometry and Physics in Cracow, Acta Phys. Polon. B Proc. Suppl. 4 (2011), no.3, 495–505.

[59] V. Tosatti, Calabi-Yau manifolds and their degenerations, Ann. N.Y. Acad. Sci. 1260 (2012), 8–13.

[60] V. Tosatti, Non-Kähler Calabi-Yau manifolds, in Analysis, complex geometry, and mathematical physics: in honor of Duong H. Phong, 261–277, Contemp. Math., 644, Amer. Math. Soc., Providence, RI, 2015.

[61] V. Tosatti, B. Weinkove, X. Yang, The Kähler-Ricci flow, Ricci-flat metrics and collapsing limits, to appear in Amer. J. Math.

[62] V. Tosatti, Y. Zhang, Triviality of fibered Calabi-Yau manifolds without singular fibers, Math. Res. Lett. 21 (2014), no. 4, 905–918.

[63] V. Tosatti, Y. Zhang, Infinite time singularities of the Kähler-Ricci flow, Geom. Topol. 19 (2015), no. 5, 2925–2948.

[64] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31 (1978), 339–411.

[65] Y.S. Zhang, The Kähler-Ricci flow on certain Calabi-Yau fibrations, preprint, arXiv:1705.01434

[66] Y. Zhang, Degeneration of Ricci-flat Calabi-Yau manifolds and its applications, preprint, arXiv:1507.07674

[67] Y. Zhang, Collapsing of negative Kähler-Einstein metrics, Math. Res. Lett. 22 (2015), no. 6, 1843–1869.

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208
E-mail address: tosatti@math.northwestern.edu

Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, P.R.China.
E-mail address: yuguangzhang76@yahoo.com