Precise Determination of Quantum Critical Points by the Violation of the Entropic Area Law

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Finite-size scaling analysis turns out to be a powerful tool to calculate the phase diagram as well as the critical properties of two dimensional classical statistical mechanics models and quantum Hamiltonians in one dimension. The most used method to locate quantum critical points is the so called crossing method, where the estimates are obtained by comparing the mass gaps of two distinct lattice sizes. The success of this method is due to its simplicity and the ability to provide accurate results even considering relatively small lattice sizes. In this paper, we introduce an estimator that locates quantum critical points by exploring the known distinct behavior of the entanglement entropy in critical and non critical systems. As a benchmark test, we use this new estimator to locate the critical point of the quantum Ising chain and the critical line of the spin-1 Blume-Capel quantum chain. The tricritical point of this last model is also obtained. Comparison with the standard crossing method is also presented. The method we propose is simple to implement in practice, particularly in density matrix renormalization group calculations, and provides us, like the crossing method, amazingly accurate results for quite small lattice sizes. Our applications show that the proposed method has several advantages, as compared with the standard crossing method, and we believe it will become popular in future numerical studies.

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I. INTRODUCTION

In recent years, the connection between the quantum correlations and the entanglement properties of quantum critical systems motivated the research on the possible characterization of critical phenomena via quantum information concepts. In particular, several of these studies have been devoted to the identification of possible measures of entanglement that give a precise localization of the quantum critical point in quantum chains. These studies basically considered local measures of entanglement like the concurrence and the negativity, or other measures of quantum correlations that are not always simple to calculate. The concurrence and the negativity are easy to measure, however can not be considered universal measures, since depending on the distance of the sites they are measured, and the model, they may detect or not the critical points.

On the other hand, the standard theory of finite-size scaling (FSS) gives us a quite simple way to locate quantum phase transitions. The procedure, usually named as the crossing method (CM), is based on the crossing of the functions $L G(L) = L' G(L')$ at the critical points, where $G(L)$ and $G(L')$ are the mass gaps of the quantum chains with lattice sizes $L$ and $L'$, respectively. This is the most used method to locate a phase transition in a quantum chain. The success of this method is due to the fact that it provides precise estimates of critical points even considering small system sizes. This point is very important because, with the exception of the exactly soluble models, we do not have access of the ground state properties of large systems. Normally, for a given lattice size, the CM method gives the most accurate localization of critical points.

A natural question concerns the existence of an universal estimator, based on the changing of entanglement at the quantum critical points, that gives the same order of precision, for a given numerical effort, as the standard CM. In this paper, we are going to present such estimator for the quantum chains. The estimator will be obtained by exploiting the well known violation of the area law of the entanglement entropy of a subsystem with the rest of the quantum chain that happens in a critical quantum point.

The entropy itself could, in principle, be used to locate the quantum critical points. However, as we are going to show in two examples, we need to use much larger lattices, to achieve the same precision, as compared with the CM. The estimator we propose is given by the difference of the entanglement entropy when we split the quantum chain in subsystems of different sizes. As a benchmark test, we are going to use this estimator to determine the localization of critical points of two well known models: the quantum Ising chain and the spin-1 Blume-Capel model (BCM). These are good examples since in the Ising model we have a well known critical point, and in the BCM we possess a line of second order phase transition in the Ising universality, that ends in a tricritical point in the universality class of the tricritical Ising model. Surprisingly, the estimates for the critical and tricritical points obtained by using the proposed estimator are as precise as the standard CM, for the same
lattice sizes. Moreover, the new method in comparison to the CM has some advantages. We need only to calculate the ground state eigenfunction, instead of the mass gaps as in the CM. Furthermore at the same time as the quantum critical point is located the conformal anomaly of the underlying conformal field theory governing the critical point is also evaluated. Our applications also shows that in the case of first order transitions (gapped) the proposed estimator, differently from the CM, shows a distinct finite-size behavior already for quite small lattices, that in principle can be used to detect if the phase transition is discontinuous (first order) or not. The paper is organized as follows. Since the proposed method to locate quantum phase transitions is going to be compared with the standard CM, we present both methods in the next section. In the sections III and IV, we present our results for the quantum Ising chain and the BCM, respectively. Finally, in section V we summarize our results and present our conclusions.

II. ESTIMATORS OF QUANTUM CRITICAL POINTS

Let us consider a quantum chain whose Hamiltonian $H(\lambda)$ has a quantum critical point at $\lambda = \lambda_c$. The FSS theory tell us that at the quantum critical point, the finite chain with $L$ sites will have a mass gap $G(L,\lambda) \sim 1/L$, as $L \to \infty$. This gives us the most popular method used to locate phase transitions. The finite-size estimates of the critical point ($\lambda_L,L'$) are obtained from the mass gaps of two distinct lattice sizes. They are obtained from the crossing of the function $F_L(\lambda) = LG(L,\lambda)$ for two lattice sizes $L$, i.e.,

$$LG_L(\lambda)|_{\lambda_L,L'} = L'G_{L'}(\lambda)|_{\lambda_L,L'}.$$  \hspace{1cm} (1)

For this reason, this method is also known as the CM. Its application is quite simple because it is necessary to evaluate only the two lowest eigenvalues of $H(\lambda)$. A difficulty of this method happens in the case where the quantum chain undergoes a first-order phase transition with a small (but finite) mass gap $G_\infty$. In this case, we also find crossing points $\lambda_{L,L'}$, since for $L < 1/G_\infty$ we should expect $G_L(\lambda) \sim 1/L$, as in a true critical point.

The purpose in this paper is to introduce and test an alternative method to locate critical points in quantum chains. The method we propose exploits the changing of the behavior of the entanglement entropy of a segment of the quantum chain, as we cross a quantum critical point.

In order to present our method, we first define the well known entanglement entropy. Consider a quantum chain with $L$ sites, described by a pure state whose density operator is $\rho$. Let us consider that the system is composed by the subsystems $A$ with $\ell$ sites ($\ell = 1, \ldots, L$) and $B$ with $L-\ell$ sites. The entanglement entropy is defined as the von Neumann entropy $S_L(\ell) = -\text{Tr}_{\rho_A} \ln \rho_A$, associated to the reduced density matrix $\rho_A = \text{Tr}_B \rho$.

Most of the critical chains besides being scale invariant are also conformal invariant, its long-distance physics is governed by a conformal field theory (CFT) with central charge $c$. In the scaling regime $1 << \ell << L$ the entanglement entropy $S_L(\ell,\lambda)$ of the ground state behaves differently if the system is critical ($\lambda = \lambda_c$) or not. It increases logarithmically with the correlation length $\xi(\lambda)$, or equivalently, with the decrease of the mass gap $G_{L}(\lambda) = 1/\xi_L(\lambda)$, namely, \[ S_L(\ell,\lambda) = \begin{cases} \gamma \ln \left[ \frac{\ell}{\xi_L(\lambda)} \right] + \beta, & \lambda = \lambda_c, \\ -\gamma \ln \left[ G_{L}(\lambda) \right] + \beta', & \lambda \neq \lambda_c \end{cases} \] \hspace{1cm} (2)

where $\gamma$ is an universal constant related with the central charge $c$, namely $\gamma = c/3$ for periodic chains and $\gamma = c/6$ for chains with open boundaries. The constants $\beta$ and $\beta'$ are non universal and model dependent. Here, we consider only PBC in order to avoid boundary effects.

The estimator we propose to locate a quantum critical point $\lambda_c$ is given by the coupling $\lambda_L^\ell$ that gives the maximum value of the entanglement entropy difference (MVEED) given by:

$$\Delta S_L(\lambda) = S_L(\ell,\lambda) - S_L(\ell',\lambda),$$ \hspace{1cm} (3)

with $\ell = L/2$ and $\ell' = L/4$. According to Eq. (2), as $L \to \infty$, we should have

$$\Delta S_L(\lambda) = \begin{cases} \gamma \ln(2), & \lambda = \lambda_c, \\ 0, & \lambda \neq \lambda_c. \end{cases}$$ \hspace{1cm} (4)

This means that as $L$ increases the estimator goes to zero, except at the critical point, where its value is proportional to the central charge $c$. Although in the definition of $\beta$ it is assumed that the lattice sizes are multiples of 4, for other lattice sizes a simple variation of Eq. (4) can also be defined. In principle, we could choose two arbitrary subsystem sizes $\ell$ and $\ell'$. However, the signature of the violation of the entropic area law is observed more clear for $\ell, \ell' \gg 1$. Then, we have to choose $\ell, \ell' \sim L$. On the other hand, we also want to distinguish $\Delta S_L(\ell,\ell')$ from zero. This means that we have to choose small values of $\ell'$, when compared with $\ell \sim L$, and satisfying $\ell' \gg 1$. So, it seems to be natural to choose $\ell = L/2$ and $\ell' = L/4$.

In the following sections, we are going to use the MVEED estimator to locate the critical points of two different models, and compare them with those obtained through the standard CM. Note that in the same line, we could, in principle, also use as estimators the difference of the $\alpha$-Rényi entropies. However, for $\alpha > 1$ unusual corrections appears and the finite-size effects became more relevant (see also Refs. 24,25).

The key ingredient in the MVEED method is the violation of the entropic area law at the critical point. This violation have been observed in several systems such as
the conformal invariant critical chains. We could naively expect that the entanglement entropy of a $d$-dimensional system would behave as $S \sim L^d$, since the entropy is usually an extensive quantity. However this is not correct, the entanglement entropy of the subsystems $A$ and $B$, with arbitrary volumes, are identical, i.e., $S_A = -\text{Tr} \rho_A \ln \rho_A = S_B = -\text{Tr} \rho_B \ln \rho_B$. Since both subsystems share the same area, we should expect the area-law dependence of the entropy, i.e., $S_A = S_B \sim L^{d-1}$.[26–28]

This means that the information is shared only among the degrees of freedom localized around the surface separating both systems. However, at the critical point the correlation length diverges and both subsystems share much more information among themselves. We thus expect, in this case, a violation of the entropic area law. As we already mention, the one dimension conformal invariant critical systems [see Eq. (2)] are celebrated examples where this law is violated. There are also examples, in dimension $d > 1$, where the entropic area law is violated, such as some gapless fermionic systems with a finite Fermi surface.[29–35] It is interesting also to point out that in some disordered one-dimensional systems the entropic law can be violated in a different way than the one presented in Eq. (2).[26–28]

We should mention that the difference of the entanglement entropy of chains with distinct sizes were used to extract the central charge $c$ in one dimension quantum systems with good accuracy.[37–40] Läuchli and Kollath in Ref. [41] also used a similar procedure to detect the continuum line of critical points of the bosonic Hubbard chain. In their procedure they considered the following estimator: $\Delta S_{LK}(L, L' = L/2, \lambda) = S_L(L/2, \lambda) - S_{L/2}(L/4, \lambda)$.

For the above model, and also for the $XXZ$ quantum chain, where a continuum line of critical points appears by tuning a single coupling, the difference of entropies tends toward a constant, appearing a plateau along the critical line. In the non critical region they go to zero, as the lattice size increases. In this case, we should not use the maximum of $\Delta S_L(\lambda)$ [or the maximum of $\Delta S_{LK}(\lambda)$] but its overall size dependence to locate the endpoint of the critical line. We also have calculated $\Delta S_L(\lambda)$ for the $XXZ$ chain, for $L < 100$, and we confirmed that the end point of the critical line is better estimated by the overall size dependence of $\Delta S_L(\lambda)$, as compared with the MVEED.

For the sake of comparison, we also test the estimator proposed by Läuchli and Kollath to detect the critical point of the quantum Ising chain. As our estimator, $\Delta S_{LK}(L, L', \lambda)$ also has a maximum close to the critical point [see next section and Fig. 1(e)]. So, it is also possible to estimate the critical point by considering the coupling that gives the maximum values of $\Delta S_{LK}(L, L')$. However, note that to obtain one estimate of the critical point, with this estimator, we need to consider two distinct lattice sizes, one being twice the size of the other. This demand larger numerical effort. Besides that we verify, for both models with $L = 16$, that the errors of our estimates are approximately two orders of magnitude smaller than the ones obtained by considering the estimator $\Delta S_{LK}(L, L' = L/2, \lambda)$.[26]

### III. QUANTUM ISING CHAIN - CRITICAL POINT DETERMINATION

The Ising quantum chain describes the dynamics of spin-$1/2$ localized spins whose Hamiltonian is given by

$$H_{\text{Ising}} = -\sum_j (\sigma_j^x \sigma_{j+1}^x + \lambda \sigma_j^z)$$

where $\sigma^x, \sigma^z$ are spin-$1/2$ Pauli matrices. It depends on the parameter $\lambda$, and for simplicity, hereafter we are going to consider only periodic chains. This model has a critical point $\lambda_c = 1$ that can be obtained from its exact solution, or even more simply from its self dual property.[44]

Let us show first the difficulty of extracting the localization of the critical point by directly using the Eq. (2). In Fig. 1(a), we present for the lattice size $L = 64$ the entanglement entropy $S_L(\ell, \lambda)$ of the Ising model for $\lambda = 1.0, 1.01$ and $\lambda = 1.1$. These results can be obtained from free fermion technique.[44,45] We also use these results.
to test the precision of our density matrix renormalization group (DMRG) calculation. For the critical coupling \( \lambda = \lambda_c = 1 \) the entanglement entropy has a quite good fit with the expected behavior given in Eq. (2) with the values \( c = 0.5003 \) and \( \beta = 0.4781 \), which are close to the exact ones.\(^{45}\) On the other hand, for a point clearly away from the critical point, like \( \lambda = 1.1 \), the entropy tends towards a constant as \( \ell \) increases, as expected from Eq. (2). Although we see a clear distinct behavior of the entropy for such anisotropies, for anisotropies closer to the critical point it is quite difficult to distinguish such behavior even for relatively large lattices. For example, as we can see in Fig. 1(a), it is difficult to discern if we are at the critical point or not at \( \lambda = 1.01 \) for a chain with lattice size \( L = 64 \). In fact, for this coupling, we also obtain a nice fit with the predicted critical behavior given in Eq. (2). The fit to our numerical data gives us \( c = 0.489 \) and \( \beta = 0.472 \), which are close to the values at the critical point. The reason, for not seeing a saturation, as we increase \( \ell \), for \( \lambda = 1.01 \), is that at this coupling and for such lattice size the correlation length \( \xi_L(\lambda) = 1/\xi_L(\lambda) > L \). Actually, it is possible to distinguish the critical and the non critical behavior only when one consider lattice sizes \( L >> \xi \). This means that for relative small lattices \( (L \sim 20) \), differently from the CM, it is not possible to obtain reasonable estimates for the critical coupling constants only from the behavior of \( \xi_L(\ell, \lambda) \). Indeed, as shown in Fig. 1(b), for the lattice size \( L = 12 \), it is even difficult to distinguish the behavior at the anisotropies \( \lambda = 1 \) and \( \lambda = 1.1 \). Certainly, the distinct behavior of \( S_L(\ell, \lambda) \) as a function of \( \ell \), should be more evident at \( \ell = L/2 \). So, let us check if the signature of the critical point is more evident for \( \ell = L/2 \). In Fig. 1(c), we present \( S_L(L/2, \lambda) \) as a function of \( \lambda \) for lattice sizes \( L = 8, 12, 16, 24 \) and 36. In this figure only for \( L \geq 24 \) we start to see a clear change of behavior around the expected critical point. We observed that the maximum occurs at \( \lambda \approx 0.97 \). This means that the simple use of \( S_L(L/2, \lambda) \) will give an acceptable estimate only for quite larger lattices, as compared with those necessary to obtain reasonable estimates through the use of the CM. The fact that we need to consider large systems to observe in \( S_L(L/2, \lambda) \) the signature of the critical point is not particular of the Ising model. This behavior have been observed also in other models\(^{39,47,48}\) [see also Fig. 2(a)]. We should also mention that we did not observe a clear signature of the quantum critical point, for small lattice sizes, in the derivative of the entanglement entropy \( \frac{d S_L(L/2, \lambda)}{d \lambda} \) [see Fig. 1(d)].

Now, we are going to show that \( \Delta S_L(\lambda) \), defined in Eq. (3), already presents the signature of the phase transition for small system sizes. This fact is highly desirable, because it will not need a huge numerical effort to detect the localization of the phase transition. In Fig. 1(e), we depict the difference \( \Delta S_L(\lambda) \) as a function of \( \lambda \), for the same lattice sizes presented in Fig. 1(c). We notice clearly two interesting features: (i) the maximum of \( \Delta S_L(\lambda) \) appears close to the true critical point \( \lambda_c = 1 \), and more important (ii) the signature of the phase transition is observed already for small lattices, as happens in the CM (see Table I).

In Table I, we present the finite-size estimates of \( \lambda^c_L \) and \( c^L \) for the quantum Ising model acquired from the MVEED method and CM. The results in parentheses are from the CM. The exact values of these quantities are also shown.

| \( L \) | \( \lambda^c_L \) | \( c^L \) |
|---|---|---|
| 8 | 0.99307 | 0.50374 |
| (1.00197) |
| 12 | 0.99813 | 0.50137 |
| (1.00049) |
| 16 | 0.99939 | 0.50073 |
| (1.00014) |
| 24 | 0.99984 | 0.50031 |
| (1.00005) |
| 36 | 0.99996 | 0.50014 |
| (1.00001) |
| 48 | 0.99997 | 0.50008 |
| (1.00001) |

Table I: The finite-size estimates of \( \lambda^c_L \) and \( c^L \) for the quantum Ising model acquired from the MVEED method and CM. The results in parentheses are from the CM. The exact values of these quantities are also shown.

IV. BLUME-CAPEL QUANTUM CHAIN - CRITICAL POINT DETERMINATION

The Blume-Capel quantum chain is obtained by the time-continuum limit of the Blume-Capel model in two dimensions. It describes the dynamics of spin-1 localized particles, with Hamiltonian given by

\[
H_{BC} = -\sum_j (s_j^x s_{j+1}^x - \delta(s_j^z)^2 - \gamma s_j^y),
\]
where $s^x$ and $s^z$ are the spin-1 operators. This Hamiltonian, differently from the quantum Ising chain, has two coupling constants ($\delta$ and $\gamma$) and is not exactly integrable. The phase diagram in the plane $\delta - \gamma$ is known from earlier numerical studies based in the CM (see Fig. 1 of Ref. [43]). For values $\gamma > \gamma_{tr}$ the Hamiltonian has a quantum critical line $\delta_c(\gamma)$ governed by a CFT in the same universality class of the quantum Ising chain ($c = 1/2$). At $\gamma_{tr}$ the model has a quantum tricritical point at $\delta_{tr}$ in the universality class of the tricritical Ising model, having central charge $c \approx 7/10$. For $\gamma < \gamma_{tr}$ there is a line $\delta_c(\gamma)$ of first-order phase transitions (gapped). The numerical estimate of the tricritical point ($\gamma_{tr}, \delta_{tr}$) was acquired by a generalization of the CM. It was obtained either by the crossing of the two lowest mass gaps of two lattice sizes, or by the crossing of the first mass gap of three lattice sizes. These previous results were obtained for small lattice sizes ($L \leq 9$) and the mass gaps were calculated by using the Lanczos method. In this work, we also used Lanczos method together with the CM to extend the previous finite-size estimates of ($\gamma_{tr}, \delta_{tr}$) up to lattice size $L = 15$. Some of these estimates are presented in Table II.

Now, let us calculate the critical line of the model by using the MVEED [Eq. (1)]. Although it is also possible to calculate the entanglement entropy with the Lanczos method, here, we used the DMRG technique to do this task. In the DMRG the entanglement entropy is easily calculated in all sweeps of the DMRG. In our DMRG, we kept up to $m = 600$ states per block in the final sweep. We have done $\sim 4 - 6$ sweeps, and the discarded weight was typically $10^{-12} - 10^{-15}$ at that final sweep.

Like in the case of the quantum Ising chain, let us consider first the behavior of the entanglement entropy for the subsystem with size $\ell = L/2$, i.e., $S_L(L/2, \gamma, \delta)$. We consider initially a value of $\gamma$ where a second-order phase transition in the Ising universality is expected. This is the case for $\gamma = 1.1$, where a critical point happens for $\delta_c \approx 0.3135$ (see Table II).

In Fig. 2(a), we show the entropy $S_L(L/2, \gamma = 1.1, \delta)$ as a function of $\delta$. Similarly as in the application for the quantum Ising chain, this entropy does not present any trace of a quantum phase transition for small system sizes. We start to observe the signature of the phase transition only for relative large lattice sizes ($L \gg 12$). Similar results are also seen in the derivative $\frac{d^2 S_L(\gamma, \delta)}{d\delta^2}$, as depicted in Fig. 2(b) for $\gamma = 1.1$. On the other hand, if we now use our estimator $\Delta S_L(\gamma, \delta)$, as shown in Fig. 2(c), we already see a clear appearance of a peak for the small lattice size $L = 8$. The finite-size estimates $\delta_{tr}$, for three values of $\gamma$, obtained from the MVEED method are presented in Table II. It is also given in this table, for comparison, the values of $\delta_{tr} - 2L$ obtained through the standard CM [Eq. (1)]. We clearly see in the table that the results acquired from the MVEED method converged already to almost five digits for small lattice sizes like $L = 24$.

Note that the results of Table II tell us that the estimated results obtained from the MVEED method are as good, if not better, than those coming from the CM. This table also includes the extrapolated results acquired by using the VBS method.

As we already mentioned, at the same time we get the finite-size estimates of $\delta_{tr}$, we also obtain the finite-size estimates of the central charge $c^L$. In Fig. 2(d), we depict these estimates for the lattice sizes $L = 16, 24$ and $36$ for several values of $\gamma(\delta)$. We clearly see, in this figure, that the estimates of $c^L$ tend to $c = 1/2$ for $\gamma > 0.6$ and, for $0.4 < \gamma < 0.6$, exhibit a quite large change with a peak in a value close to $c^L = 7/10$, which is the expected value of the central charge for the tricritical point of the BCM.

It is interesting to note that like the CM, the method we propose in this paper also gives us estimates of critical quantum points, in regions where they do not exist (gapped regions), like the ones presented in Fig. 2(d) for $\gamma \lesssim 0.4$. This happens in both methods due to the fact that correlation length is large than the system size considered, i.e., $\zeta = 1/G_L > L$ in this region. However, differently from the CM the MVEED method, at least in this application, allows us to distinguish these fake results. In the case of a true critical point the maximum of $\Delta S_L$, as $L$ increases, should saturate on a finite value, while in the non critical case it should decrease to zero. In Fig. 2(e), we show $\Delta S_L$ as a function of $\delta$ for $\gamma = 0.39$. 

![Figure 2](image-url)
Table II: The finite-size estimates of $\delta^c_L$ for the Blume-Capel model for three values of $\gamma$ obtained from the MVEED method and the CM. The results in parentheses are from the CM.

| $\gamma$ | $\delta^c_L$     | $c_L$    | $\Delta S_L$ |
|---------|------------------|----------|--------------|
| 1.1     | 0.31072 (0.30997) | 0.91260 (0.91317) | 0.92121 (0.92240) |
| 0.41    | 0.31279 (0.31254) | 0.91271 (0.91292) | 0.92132 (0.92190) |
| 0.39    | 0.31321 (0.31315) | 0.91273 (0.91286) | 0.92133 (0.92171) |

Table III: The finite-size estimates of $\gamma^{\text{tric}}$ and $\delta^{\text{tric}}$, and $c^L$ for the tricritical point of the Blume-Capel model obtained with the MVEED. The finite-size estimate of the tricritical couplings taken from the Ref. [49] for sizes $L = 8, 9$ is also presented.

| $L$ | $\gamma^{\text{tric}}$ | $\delta^{\text{tric}}$ | $c^L$ |
|-----|-------------------------|-------------------------|------|
| 8   | 0.41047 (0.41405)       | 0.91240 (0.91091)       | 0.7065 |
| 12  | 0.41496 (0.41551)       | 0.91052 (0.91028)       | 0.7014 |
| 24  | 0.41551 (0.41560)       | 0.91024 (0.91024)       | 0.7008 |
| 36  | 0.41560 (0.41563)       | 0.91024 (0.91024)       | 0.7003 |

For this coupling we expect a first order phase transition, with a small gap. It is clear that for quite small lattices like $L = 8$ and $L = 12$ we observe the peak of $\Delta S_L$ tending towards zero.

Our estimate for the tricritical point $(\gamma^{\text{tric}}, \delta^{\text{tric}})$ comes from the maximum value of $\Delta S_L(\gamma, \delta)$ (in the $\gamma$-$\delta$ plane). In Table III, we present the estimated values of the tricritical couplings obtained by this procedure. We also show in this table the finite-size estimates of the central charge $c^L$ obtained with the MVEED method. Notice the quite good convergence towards the predicted value $c = 7/10$. One finite-size estimate of the tricritical point obtained from the CM is also presented.

The evaluation of those tricritical points, by using the CM, is much more difficult since in this case we should find the crossing of two gaps in a pair of lattice sizes or use the crossing of the first gap in three lattice sizes. This means that to obtain a sequence with a given number of estimates we need larger lattice sizes in the CM. The available finite-size estimates derived from the CM (up to $L = 8$) are of the same order of precision as those obtained by the MVEED method by using the same lattice sizes. However, in the MVEED method we get a larger sequence to extrapolated, since in this case there is one estimate for each lattice size. For this reason, in principle, we should obtain more precise extrapolated values.

V. CONCLUSIONS

We have introduced a practical and simple method, called maximum value of the entanglement entropy difference (MVEED), to estimate the quantum critical points of quantum chains. The MVEED method explores the distinct behaviors of the entanglement entropy at critical (gapless) and non critical (gapped) couplings. We made two applications of this method. We calculated the critical points for the quantum Ising chain and for the quantum Hamiltonian of the Blume-Capel model. We compared, in both models, the obtained results with those coming from the standard crossing method (CM). Our results show clearly that the MVEED method gives us accurate estimates for small lattice sizes, similar as those obtained with the standard CM. However, the MVEED method has some advantages as compared with the CM:

(i) Simplicity of calculation. In order to obtain a finite-size estimate for the critical coupling constants with the MVEED method we only need the eigenfunction of the ground state, in contrast with the CM where it is necessary to calculate the mass gaps (two different energies) for two distinct lattice sizes. For this reason the CM demands more computational effort, as compared with the MVEED method.

(ii) Identification of the universality class. Differently from the CM, at the same time the critical point is obtained the central charge $c$, that identify the universality class of the critical behavior, is also calculated. In the usual finite-size scaling where we use the CM, the central charge can also be estimated by using the consequences of conformal invariance in the eigenspectra of the finite quantum chain. We need, in this case, to evaluate additional mass gaps, in order to estimate the sound velocity.

(iii) Effectiveness for the localization of tricritical points. Due to the property (ii) the method is quite effective to locate tricritical points since the universality class of critical behavior changes at the tricritical points. This was illustrated in the example of the BCM (section 4). In the usual CM the numerical effort is higher, because the finite-size estimates are obtained by using the two lowest mass gaps of two lattice sizes or the lowest mass gaps of three distinct lattice sizes.

(iv) Sensibility to detect the first-order phase transitions. Gapped quantum chains with small gaps $(\zeta_L = \ldots$
$1/G_L > L$), produce crossing points in the usual CM, but also a maximum in the proposed estimator $\Delta S_L$. In the case of the CM this can be decided only by further analysis of the finite-size dependence of the mass gaps, and usually it is necessary large lattices. Surprisingly in the application we did for the BCM, our proposed method shows, already for quite small lattice sizes, a distinct behavior for the gapped and non-gapped systems. Even for non-critical points that are close to the tricritical point (small mass gaps) the estimator, instead of saturating in a non-zero value, goes to zero, as we should expect in a gapped system. This is a surprise because, similarly as the CM, the proposed method is also based on the variations of the correlation length, which are quite small. However, distinct form the CM, our method shows a large sensitivity already in quite small lattice sizes. Future applications of the proposed method will confirm if this is a general feature of the method or an accident of the present application.

It is interesting to stress that differently from several existing estimators, based on the entanglement properties, the MVEED method was tested in small lattice sizes producing surprisingly good results. For the above reasons, we believe the use of this method may become popular in future applications.

In order to conclude, we mention that a generalization of the presented method, based on the distinct behavior of the entanglement entropy, to locate quantum critical points, can also be introduced for higher dimensions at least for fermionic systems with a finite Fermi surface. In this case, a natural estimator would be $\Delta S_L/L^{d-1}$.

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1. A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature 416, 608 (2002).
2. T. Osborne and M. A. Nielsen, Phys. Rev. A 66, 032110 (2002).
3. G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
4. A. Amico, R. Fazio, Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
5. V. E. Korepin, Phys. Rev. Lett. 92, 096402 (2004).
6. P. Calabrese and J. Cardy, J. Stat. Mech., P06002(2004).
7. F. C. Alcaraz and M. S. Sarandy, Phys. Rev. A 78, 032319 (2008).
8. H. Wichterich, J. Molina-Vilaplana, and S. Bose, Phys. Rev. A 80, 010304 (2009).
9. J. Vidal, G. Palacios, and R. Mosseri, Phys. Rev. A 69, 022107 (2004).
10. J. Vidal, R. Mosseri, and J. Dukelsky, Phys. Rev. A 69, 054101 (2004).
11. H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).
12. F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. 92, 027901 (2004).
13. T. R. de Oliveira, G. Rigolin, M. C. de Oliveira, and E. Miranda, Phys. Rev. Lett. 97, 170401 (2006).
14. R. Dillenschneider, Phys. Rev. B 78, 224413 (2008).
15. M. S. Sarandy, Phys. Rev. A 80, 022108 (2009).
16. T. Werlang, C. Trippe, G. A. P. Ribeiro, and G. Rigolin, Phys. Rev. Lett. 105, 095702 (2010).
17. C. C. Rulli and M. S. Sarandy, Phys. Rev. A 81, 032334 (2010).
18. M. Barber, Phase transition and Critical Phenomena, Vol. 8 (New York;Academic, 1993).
19. Actually, for general quantum critical points we should have the crossing of $L^z G(L)$, where $z$ is the dynamical critical exponent. However, most of the quantum critical chains are conformally invariant, and in this case $z = 1$.
20. C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B424, 443 (1994).
21. P. Calabrese and J. Cardy, J. Phys. A: Math. Theor. 42, 504005 (2009).
22. I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. 67, 161 (1991).
23. P. Calabrese, M. Campostrini, F. Essler, and B. Nienhuis, Phys. Rev. Lett. 104, 095701 (2010).
24. J. C. Xavier and F. C. Alcaraz, Phys. Rev. B 83, 214425 (2011).
25. M. Dalmonte, E. Ercolessi, and L. Taddia, arXiv:1105.3101(2011).
26. M. Srednicki, Phys. Rev. Lett. 71, 666 (1993).
27. M. Cramer, J. Eisert, M. B. Plenio, and J. Dreißig, Phys. Rev. A 73, 012309 (2006).
28. M. B. Plenio, J. Eisert, J. Dreißig, and M. Cramer, Phys. Rev. Lett. 94, 060503 (2005).
29. M. M. Wolf, Phys. Rev. Lett. 96, 010404 (2006).
30. D. Gioev and I. Klich, Phys. Rev. Lett. 96, 100503 (2006).
31. L. Ding, N. Bray-Ali, R. Yu, and S. Haas, Phys. Rev. Lett. 100, 215701 (2008).
32. T. Barthel, M.-C. Chung, and U. Schollwöck, Phys. Rev. A 74, 022329 (2006).
33. W. Li, L. Ding, R. Yu, T. Roscilde, and S. Haas, Phys. Rev. B 74, 073103 (2006).
34. B. Swingle, Phys. Rev. Lett. 105, 050502 (2010).
35. S. Farkas and Z. Zimborás, J. Math. Phys. 48, 102110 (2007).
36. G. Vitagliano, A. Riera, and J. I. Latorre, New Journal of Physics 12, 113049 (2010).
37. J. Zhao, I. Peschel, and X. Wang, Phys. Rev. B 73, 024417 (2006).
38. J. C. Xavier, Phys. Rev. B 81, 224404 (2010).
39. C. J. Pearson, W. Barford, and R. J. Bursill, Phys. Rev. B 83, 195105 (2011).
40. J. Ren, S. Zhu, and X. Hao, J. Phys. B: At. Mol. Opt. Phys. 42, 015504 (2009).
41. A. M. Läuchli and C. Kollath, J. Stat. Mech., P05018(2008).
42. The estimate obtained by considering $\Delta S_{L,K}$ instead of $\Delta S_L$ with
$L = 16$ for the Ising quantum chain is $\lambda_c = 0.98279$ ($\lambda_c = 0.99939$) and for the Blume Capel Model with $\gamma = 1.1$ is $\delta_c = 0.29809$ ($\delta_c = 0.31321$).

43 J. Kogut, Reviews of Modern Physics 51, 659 (1979).
44 M.-C. Chung and I. Peschel, Phys. Rev. B 64, 064412 (2001).
45 I. Peschel, Journal of Physics A: Mathematical and General 36, L205 (2003).
46 F. Iglói and Y. -C. Lin, J. Stat. Mech., P06004(2008).
47 G.-H. Liu, H.-L. Wang, and G.-S. Tian, Phys. Rev. B 77, 214418 (2008).
48 M.-H. Chung and D. P. Landau, Phys. Rev. B 83, 113104 (2011).
49 F. C. Alcaraz, J. R. Drugowich de Felício, R. Köberle, and J. F. Stilck, Phys. Rev. B 32, 7469 (1985).
50 E. Dagotto, Rev. Mod. Phys, 66, 763 (1994).
51 S. R. White, Phys. Rev. Lett. 69, 2863 (1992).
52 J. M. van den Broeck and L. W. Swartz, SIAM J. Math. Anal. 10, 658 (1979).
53 J. C. Xavier, F. C. Alcaraz, D. P. Lara, and J. A. Plascak, Phys. Rev. B 57, 11575 (1998).
54 H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. 56, 742 (1986).
55 I. Affleck, Phys. Rev. Lett. 56, 746 (1986).