Invariant Manifolds for Competitive Systems in the Plane

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May 12, 2009

Abstract

Let $T$ be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$, and assume $T$ is $C^1$ in a neighborhood of a fixed point $\bar{x} \in \mathcal{R}$. The main results of this paper give conditions on $T$ that guarantee the existence of an invariant curve emanating from $\bar{x}$ when both eigenvalues of the Jacobian of $T$ at $\bar{x}$ are nonzero and at least one of them has absolute value less than one, and establish that $C$ is an increasing curve that separates $\mathcal{R}$ into invariant regions. The results apply to many hyperbolic and nonhyperbolic cases, and can be effectively used to determine basins of attraction of fixed points of competitive maps, or equivalently, of equilibria of competitive systems of difference equations. Several applications to planar systems of difference equations with non-hyperbolic equilibria are given.
1 Introduction and main results

The following system of difference equations is analyzed in [6] for $a > 1$:

\[
\begin{aligned}
x_{n+1} &= \frac{x_n}{a + y_n}, \quad n = 0, 1, 2, \ldots, \quad x_0, y_0 \geq 0. \\
y_{n+1} &= \frac{y_n}{1 + x_n}
\end{aligned}
\]

(1)

It is shown there that every point $(0, \bar{y})$ on the positive section of the $y$-axis is a non-hyperbolic equilibrium point with eigenvalues of the Jacobian of the associated map at the point, or characteristic values, given by $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{a+\bar{y}}$. It is also shown in [6] that for each $\bar{y} > 0$, equation (1) possesses solutions which converge to $(0, \bar{y})$. Parts of the basins of attraction of the equilibrium points $(0, \bar{y})$ were found in [6], and the global behavior of solutions of system (1) was characterized completely, with the exception of determining the basin of attraction of each equilibrium point. Extensive simulations conducted by the authors of [6] suggest that the basin of attraction of each equilibrium point on the $y$-axis is the graph of a continuous increasing function on $[0, \infty)$, but they were not able to prove this fact. A similar phenomenon has been observed in [5] in several special cases of competitive systems of the form

\[
\begin{aligned}
x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \quad n = 0, 1, \ldots, \\
y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}
\end{aligned}
\]

(2)

with nonnegative parameters $\alpha_1, \beta_1, \gamma_1, A_1, B_1, C_1, \alpha_2, \beta_2, \gamma_2, A_2, B_2, C_2$ and with arbitrary nonnegative initial conditions $x_0, y_0$ such that the denominators are always positive. See Open Problems 1-3 in [5].

A first order system of difference equations

\[
\begin{aligned}
x_{n+1} &= f(x_n, y_n) \\
y_{n+1} &= g(x_n, y_n)
\end{aligned}
\]

(3)

where $\mathcal{R} \subset \mathbb{R}^2$, $(f, g) : \mathcal{R} \to \mathcal{R}$, $f$, $g$ are continuous functions is competitive if $f(x, y)$ is non-decreasing in $x$ and non-increasing in $y$, and $g(x, y)$ is non-increasing in $x$ and non-decreasing in $y$. If both $f$ and $g$ are nondecreasing in $x$ and $y$, the system (3) is cooperative. Competitive and cooperative maps are defined similarly. Strongly competitive systems of difference equations or maps are those for which the functions $f$ and $g$ are coordinate-wise strictly monotone. Competitive and cooperative systems of difference equations of the form (3) have been studied by many authors [3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 22, 23, 24, 25, 28, 29, 30, 31] and others.

A classical result of Poincaré, Hadamard, and Sternberg (see Lemma 5.1 in page 234 of [16] and the Notes section in page 271 therein) gives conditions for the existence of a local smooth curve through the fixed point of a smooth map on the plane when the characteristic values are real and distinct and such that one of them is smaller than 1. The well known Stable Manifold Theorem has local character and applies to hyperbolic cases of very general maps in the plane, for example see [1]. The local stable manifold of a diffeomorphism is always a “nice” curve, but in general the corresponding global stable manifold may be a very complex set, for example it is a strange attractor in the case of Henon’s system, see [1] and references therein. See also [22]. H. L. Smith [29] obtained results for fixed points of smooth maps on Banach space and showed that under
certain conditions, if the Frechet derivative of the map at the fixed point is an eigenvalue larger
than one, then there is an invariant curve emanating from the fixed point, which Smith termed the
“most unstable manifold”. Smith also showed that if the map has certain monotonicity conditions,
then the curve is monotone. See also [27, 7].

The first result of this article gives conditions for the existence of a global invariant curve through
a fixed point (hyperbolic or not) of a competitive map that is differentiable in a neighborhood of
the fixed point, when at least one of two nonzero eigenvalues of the Jacobian of the map at the
fixed point has absolute value less than one. Proofs for all theorems in this section will be given in
Section 4. A region \( R \subset \mathbb{R}^2 \) is rectangular if it is the cartesian product of two intervals in \( \mathbb{R} \). By
\( \text{int} A \) we denote the interior of a set \( A \).

**Theorem 1** Let \( T \) be a competitive map on a rectangular region \( R \subset \mathbb{R}^2 \). Let \( \bar{x} \in R \) be a fixed
point of \( T \) such that \( \Delta := R \cap \text{int} (Q_1(\bar{x}) \cup Q_3(\bar{x})) \) is nonempty (i.e., \( \bar{x} \) is not the NW or SE vertex
of \( R \)), and \( T \) is strongly competitive on \( \Delta \). Suppose that the following statements are true.

a. The map \( T \) has a \( C^1 \) extension to a neighborhood of \( \bar{x} \).

b. The Jacobian \( J_T(\bar{x}) \) of \( T \) at \( \bar{x} \) has real eigenvalues \( \lambda, \mu \) such that \( 0 < |\lambda| < \mu \), where \( |\lambda| < 1 \),
and the eigenspace \( E^\lambda \) associated with \( \lambda \) is not a coordinate axis.

Then there exists a curve \( C \subset R \) through \( \bar{x} \) that is invariant and a subset of the basin of attraction
of \( \bar{x} \), such that \( C \) is tangential to the eigenspace \( E^\lambda \) at \( \bar{x} \), and \( C \) is the graph of a strictly increasing
continuous function of the first coordinate on an interval. Any endpoints of \( C \) in the interior of \( R \)
are either fixed points or minimal period-two points. In the latter case, the set of endpoints of \( C \) is
a minimal period-two orbit of \( T \).

We shall see in Theorem 4 and in the Section 3 examples that the situation where the endpoints of
\( C \) are boundary points of \( R \) is of interest. The following result gives a sufficient condition for this
case.

**Theorem 2** For the curve \( C \) of Theorem 1 to have endpoints in \( \partial R \), it is sufficient that at least
one of the following conditions is satisfied.

i. The map \( T \) has no fixed points nor periodic points of minimal period-two in \( \Delta \).

ii. The map \( T \) has no fixed points in \( \Delta \), \( \det J_T(\bar{x}) > 0 \), and \( T(x) = \bar{x} \) has no solutions \( x \in \Delta \).

iii. The map \( T \) has no points of minimal period-two in \( \Delta \), \( \det J_T(\bar{x}) < 0 \), and \( T(x) = \bar{x} \) has no
solutions \( x \in \Delta \).

In many cases one can expect the curve \( C \) to be smooth.

**Theorem 3** Under the hypotheses of Theorem 1 suppose there exists a neighborhood \( U \) of \( \bar{x} \) in \( \mathbb{R}^2 \)
such that \( T \) is of class \( C^k \) on \( U \cup \Delta \) for some \( k \geq 1 \), and that the Jacobian of \( T \) at each \( x \in \Delta \) is
invertible. Then the curve \( C \) in the conclusion of Theorem 1 is of class \( C^k \).

In applications it is common to have rectangular domains \( R \) for competitive maps. If a competitive
map exist has several fixed points, often the domain of the map may be split into rectangular
invariant subsets such that Theorem 1 could be applied to the restriction of the map to one or more
subsets. For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 1 reduces just to \( |\lambda| < 1 \). This follows from a change of variables \( \bar{x} \) that allows the Perron-
Frobenius Theorem to be applied to give that at any point, the Jacobian of a strongly competitive

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map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

Lemma 5.1 in page 234 of [16] is a local version of Theorem 1 for (not necessarily competitive) $C^1$ planar maps, and it gives the existence of a $C^1$ local curve $C$ as in Theorem 1. In the case when the map is a diffeomorphism, H. L. Smith’s results give the conclusions of Theorem 1; see Remark 5 in [29]. The proof of a result analogous to Theorem 1 was given in Theorem 5 of [22] in the hyperbolic case when the equilibrium is a saddle point, but a key feature of Theorem 1 is that the equilibrium may be non-hyperbolic. Theorem 1 refines and extends Theorem 5 from [22] in that it only requires smoothness of the map in a neighborhood of the fixed point, it relaxes the hypothesis that the fixed point is a saddle, and it removes other hypotheses.

The next result is useful for determining basins of attraction of fixed points of competitive maps. If $x \in \mathbb{R}^2$, we denote with $Q_i(x)$, $i \in \{1, 2, 3, 4\}$, the four quadrants in $\mathbb{R}^2$ relative to $x$, i.e., $Q_1(x) = \{ (u, v) \in \mathbb{R}^2 : u \geq x, v \geq y \}$, $Q_2(x) = \{ (u, v) \in \mathbb{R}^2 : x \geq u, v \geq y \}$, $Q_3(x) = \{ (u, v) \in \mathbb{R}^2 : u \geq x, v < y \}$, and $Q_4(x) = \{ (u, v) \in \mathbb{R}^2 : x < u, v \geq y \}$, and so on. Define the South-East partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \leq t$. For $A \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, define the distance from $x$ to $A$ as $\text{dist}(x, A) := \inf \{ ||x - y|| : y \in A \}$.

**Theorem 4** (A) Assume the hypotheses of Theorem 1, and let $C$ be the curve whose existence is guaranteed by Theorem 1. If the endpoints of $C$ belong to $\partial \mathcal{R}$, then $C$ separates $\mathcal{R}$ into two connected components, namely

$$W_- := \{ x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } x \preceq_{se} y \} \quad \text{and} \quad W_+ := \{ x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } y \preceq_{se} x \},$$

such that the following statements are true.

(i) $W_-$ is invariant, and $\text{dist}(T^n(x), Q_2(x)) \to 0$ as $n \to \infty$ for every $x \in W_-$.  

(ii) $W_+$ is invariant, and $\text{dist}(T^n(x), Q_4(x)) \to 0$ as $n \to \infty$ for every $x \in W_+$.  

(B) If, in addition to the hypotheses of part (A), $\bar{x}$ is an interior point of $\mathcal{R}$ and $T$ is $C^2$ and strongly competitive in a neighborhood of $\bar{x}$, then $T$ has no periodic points in the boundary of $Q_1(\bar{x}) \cup Q_3(\bar{x})$ except for $\bar{x}$, and the following statements are true.

(iii) For every $x \in W_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int} Q_2(\bar{x})$ for $n \geq n_0$.  

(iv) For every $x \in W_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int} Q_4(\bar{x})$ for $n \geq n_0$.  

Basins of attraction of period-two solutions or period-two orbits of certain systems or maps can be effectively treated with Theorems 1 and 4. See [22] [23] [24] for the hyperbolic case, for the non-hyperbolic case see Example 3 in Section 3 and reference [1].

If $T$ is a map on a set $\mathcal{R}$ and if $\bar{x}$ is a fixed point of $T$, the stable set $W^s(\bar{x})$ of $\bar{x}$ is the set \{ $x \in \mathcal{R} : T^n(x) \to \bar{x}$ \} and unstable set $W^u(\bar{x})$ of $\bar{x}$ is the set

$$\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^0 \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1}, x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \bar{x} \}$$

When $T$ is non-invertible, the set $W^s(\bar{x})$ may not be connected and made up of infinitely many curves, or $W^u(\bar{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $W^s(\bar{x})$ and $W^u(\bar{x})$ are the stable and unstable manifolds of $\bar{x}$.  

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Theorem 5 In addition to the hypotheses of part (B) of Theorem \[4\], suppose that \( \mu > 1 \) and that the eigenspace \( E^\mu \) associated with \( \mu \) is not a coordinate axis. If the curve \( C \) of Theorem \[7\] has endpoints in \( \partial \mathcal{R} \), then \( C \) is the stable set \( \mathcal{W}^s(\mathfrak{x}) \) of \( \mathfrak{x} \), and the unstable set \( \mathcal{W}^u(\mathfrak{x}) \) of \( \mathfrak{x} \) is a curve in \( \mathcal{R} \) that is tangential to \( E^u \) at \( \mathfrak{x} \) and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of \( \mathcal{W}^u(\mathfrak{x}) \) in \( \mathcal{R} \) are fixed points of \( T \).

The following result gives information on local dynamics near a fixed point of a map when there exists a characteristic vector whose coordinates have negative product and such that the associated eigenvalue is hyperbolic. A point \((x, y)\) is a subsolution if \( T(x, y) \leq_{se} (x, y) \), and \((x, y)\) is a supersolution if \( (x, y) \geq_{se} T(x, y) \). An order interval \([a, b)\) is the cartesian product of the two compact intervals \([a, c]\) and \([b, d]\).

Theorem 6 Let \( T \) be a competitive map on a rectangular set \( \mathcal{R} \subset \mathbb{R}^2 \) with an isolated fixed point \( \mathfrak{x} \in \mathcal{R} \) such that \( \mathcal{R} \cap \text{int}(\mathcal{Q}_2(\mathfrak{x}) \cup \mathcal{Q}_4(\mathfrak{x})) \neq \emptyset \). Suppose \( T \) has a \( C^1 \) extension to a neighborhood of \( \mathfrak{x} \). Let \( v = (v^{(1)}, v^{(2)}) \in \mathbb{R}^2 \) be an eigenvector of the Jacobian of \( T \) at \( \mathfrak{x} \), with associated eigenvalue \( \mu \in \mathbb{R} \). If \( v^{(1)} v^{(2)} < 0 \), then there exists an order interval \( I \) which is also a relative neighborhood of \( \mathfrak{x} \) such that for every relative neighborhood \( U \subset I \) of \( \mathfrak{x} \) the following statements are true.

i. If \( \mu > 1 \), then \( U \cap \text{int} \mathcal{Q}_2(\mathfrak{x}) \) contains a subsolution and \( U \cap \text{int} \mathcal{Q}_4(\mathfrak{x}) \) contains a supersolution. In this case for every \( x \in I \cap (\mathcal{Q}_2(\mathfrak{x}) \cup \mathcal{Q}_4(\mathfrak{x})) \) there exists \( N \) such that \( T^n(x) \notin I \) for \( n \geq N \).

ii. If \( \mu < 1 \), then \( U \cap \text{int} \mathcal{Q}_2(\mathfrak{x}) \) contains a subsolution and \( U \cap \text{int} \mathcal{Q}_4(\mathfrak{x}) \) contains a supersolution. In this case \( T^n(x) \rightarrow \mathfrak{x} \) for every \( x \in I \).

In the non-hyperbolic case, we have the following result.

Theorem 7 Assume that the hypotheses of Theorem \[5\] hold, that \( T \) is real analytic at \( \mathfrak{x} \), and that \( \mu = 1 \). Let \( c_j, d_j, j = 2, 3, \ldots \) be defined by the Taylor series

\[
T(\mathfrak{x} + t v) = \mathfrak{x} + v t + (c_2, d_2) t^2 + \cdots + (c_n, d_n) t^n + \cdots. \tag{5}
\]

Suppose that there exists an index \( \ell \geq 2 \) such that \( (c_\ell, d_\ell) \neq (0, 0) \) and \( (c_j, d_j) = (0, 0) \) for \( j < \ell \). If either

(a) \( c_\ell d_\ell < 0 \), or (b) \( c_\ell \neq 0 \) and \( T(\mathfrak{x}+tv)^{(2)} \) is affine in \( t \), or (c) \( d_\ell \neq 0 \) and \( T(\mathfrak{x}+tv)^{(1)} \) is affine in \( t \),

then there exists an order interval \( I \) which is also a relative neighborhood of \( \mathfrak{x} \) such that for every relative neighborhood \( U \subset I \) of \( \mathfrak{x} \) the following statements are true.

i. If \( \ell \) is odd and \( (c_\ell, d_\ell) \leq_{se} (0, 0) \), then \( U \cap \text{int} \mathcal{Q}_4(\mathfrak{x}) \) contains a supersolution and \( U \cap \text{int} \mathcal{Q}_2(\mathfrak{x}) \) contains a subsolution. In this case, for every \( x \in I \cap (\mathcal{Q}_2(\mathfrak{x}) \cup \mathcal{Q}_4(\mathfrak{x})) \) there exists \( N \) such that \( T^n(x) \notin I \) for \( n \geq N \).

ii. If \( \ell \) is odd and \( (0, 0) \leq_{se} (c_\ell, d_\ell) \), then \( U \cap \text{int} \mathcal{Q}_4(\mathfrak{x}) \) contains a subsolution and \( U \cap \text{int} \mathcal{Q}_2(\mathfrak{x}) \) contains a supersolution. In this case, \( T^n(x) \rightarrow \mathfrak{x} \) for every \( x \in I \).

iii. If \( \ell \) is even and \( (c_\ell, d_\ell) \leq_{se} (0, 0) \), then \( U \cap \text{int} \mathcal{Q}_4(\mathfrak{x}) \) contains a subsolution and \( U \cap \text{int} \mathcal{Q}_2(\mathfrak{x}) \) contains a supersolution. In this case, \( T^n(x) \rightarrow \mathfrak{x} \) for every \( x \in I \cap \mathcal{Q}_4(\mathfrak{x}) \), and for every \( x \in I \cap \text{int}(\mathcal{Q}_2(\mathfrak{x})) \) there exists \( N \) such that \( T^n(x) \notin I \) for \( n \geq N \).

iv. If \( \ell \) is even and \( (0, 0) \leq_{se} (c_\ell, d_\ell) \), then \( U \cap \text{int} \mathcal{Q}_2(\mathfrak{x}) \) contains a supersolution and \( U \cap \text{int} \mathcal{Q}_4(\mathfrak{x}) \) contains a subsolution. In this case, \( T^n(x) \rightarrow \mathfrak{x} \) for every \( x \in I \cap \mathcal{Q}_2(\mathfrak{x}) \), and for every \( x \in I \cap \text{int}(\mathcal{Q}_4(\mathfrak{x})) \) there exists \( N \) such that \( T^n(x) \notin I \) for \( n \geq N \).
The rest of this paper is organized as follows. In Section\textsuperscript{2} some definitions and background on competitive maps and systems of difference equations is given. In Section\textsuperscript{3} we present applications of the main results to several classes of difference equations that depend on parameters. In Example 1 we study system (1), which has a continuum of non-hyperbolic equilibria along a vertical line. Theorem\textsuperscript{1} is used to establish that the stable set of each equilibrium point is an increasing curve, and that the limiting equilibrium of each solution is a continuous function of the initial point. Example 2 completes the analysis of a system of difference equation that was studied in [9] for the hyperbolic equilibria case. Here we consider the case of non-hyperbolic equilibria, for which there exists a line segment of such equilibria, and we show that each of them has a global invariant set given by an increasing curve. In Example 3, we apply our results to a difference equation and obtain stable sets for each of the period-two points, which are non-hyperbolic and consist of all points on a hyperbola. Example 4 exhibits a system of difference equations with a unique equilibrium which is of non-hyperbolic type and semi-stable. The equilibrium is of oscillatory type. Theorems\textsuperscript{1, 4 and 7} are used to establish global behavior of solutions. For this we also used the competitive character of the system, as well as information on eigenvectors associated to the characteristic values at the equilibrium. Example 5 is about a system with a semi-stable non-hyperbolic interior equilibrium. Only qualitative information is assumed about this system (two equilibria exist), yet this is all is needed to characterize the basins of attraction of the two equilibria. Our results here expand and complete the analysis given in [3]. In Section\textsuperscript{4} the proofs of Theorems\textsuperscript{1–7} are presented.

2 Competitive and cooperative systems and maps

We shall restrict our discussion to competitive systems, since if system (3) is cooperative, a simple change of variables yields a competitive system, see [31]. Also, applications require the region $\mathcal{R}$ to be the cartesian product of intervals in $\mathbb{R}$, which we shall assume in our main result.

The most natural way to study properties of competitive and cooperative systems (3) is to consider the corresponding maps $T : \mathcal{R} \to \mathcal{R}$ where $T(x, y) = (f(x, y), g(x, y))$, $(x, y) \in \mathcal{R}$, since such maps are order preserving or monotone, i.e., $T(x^{(1)}, y^{(1)}) \leq T(x^{(2)}, y^{(2)})$ whenever $(x^{(1)}, y^{(1)}) \preceq (x^{(2)}, y^{(2)})$, where $\preceq$ is a suitable partial order in $\mathbb{R}^2$. Consider the “North-East” and “South-East” partial orders in $\mathbb{R}^2$ given by

$$(x^{(1)}, y^{(1)}) \preceq_{ne} (x^{(2)}, y^{(2)}) \text{ if and only if } x^{(1)} \leq x^{(2)} \text{ and } y^{(1)} \leq y^{(2)},$$

and

$$(x^{(1)}, y^{(1)}) \preceq_{se} (x^{(2)}, y^{(2)}) \text{ if and only if } x^{(1)} \leq x^{(2)} \text{ and } y^{(1)} \geq y^{(2)}.$$

We shall use the notation $\overline{0}$ to represent the origin $(0, 0)$ in $\mathbb{R}^2$. The first quadrant $\mathcal{Q}_1(\overline{0}) = \{(x, y) : x \geq 0, y \geq 0\}$ is the nonnegative cone associated to $\preceq_{ne}$, and the fourth quadrant $\mathcal{Q}_4(\overline{0}) = \{(x, y) : x \geq 0, y \leq 0\}$ is the nonnegative cone associated to $\preceq_{se}$.

From the definition of competitive and cooperative systems one can see that maps of cooperative (respectively, competitive) systems are monotone with respect to $\preceq_{ne}$ (resp. $\preceq_{se}$). Note that strongly competitive maps $T$ satisfy the relation $(x, y) \preceq_{se} (w, z) \implies T(w, z) - T(x, y) \in \text{int } \mathcal{Q}_4(\overline{0})$.

Consider $\mathbb{R}^2$ equipped with a partial order $\preceq$ equal to either $\preceq_{ne}$ or $\preceq_{se}$, that is, the nonnegative cone is $P = \mathcal{Q}_1(\overline{0})$ or $P = \mathcal{Q}_4(\overline{0})$. We say that $x, y \in \mathbb{R}^2$ are comparable in the order $\preceq$ if either $x \preceq y$ or $y \preceq x$. For $x, y \in \mathbb{R}^2$ such that $x < y$, the order interval $[x, y]$ is the set of all $z$ such that $x \preceq z \preceq y$. A set $\mathcal{A}$ is said to be linearly ordered if $\preceq$ is a total order on $\mathcal{A}$.
A map $T$ on a set $B \subset \mathbb{R}^2$ is a continuous function $T : B \to B$. A set $A \subset B$ is invariant for the map $T$ if $T(A) \subset A$. The omega-limit set of a point $z \in A$ is the set $\omega(z) = \{ w \in \mathbb{R} : \exists n_k \to \infty$ such that $T^{n_k}(z) \to w \}$. A point $x \in B$ is a fixed point of $T$ if $T(x) = x$, and a minimal period-two orbit if $T^2(x) = x$ and $T(x) \neq x$. The orbit of $x \in B$ is the set $\{ T^k(x) \}_{k=0}^{\infty}$. A minimal period-two orbit is an orbit if the interior of $B$ set that the Jacobian matrix of $T$ of a fixed point $x$ is the set of all $y$ such that $T^n(y) \to x$. A fixed point $x$ is a basin of attraction if a set $A$ if $A$ is a subset of the basin of attraction of $x$.

The map is smooth on $B$ if the interior of $B$ is nonempty and if $T$ is continuously differentiable on the interior of $B$. If $T$ is differentiable, a sufficient condition for $T$ to be strongly competitive is that the Jacobian matrix of $T$ at any $x \in B$ has the sign configuration

$$
\begin{pmatrix}
+ & - \\
- & +
\end{pmatrix}.
$$

System (3) has an associated map $T = (f, g)$ defined on the set $\mathcal{R}$. For additional definitions and results (e.g., repellor, hyperbolic fixed points, stability, asymptotic stability, stable and unstable sets and manifolds) see [26] for maps, [18] and [31] for competitive maps, and [21, 22] for difference equations.

The next two theorems gives sufficient conditions for a competitive system to have solutions that are component-wise eventually monotonic.

Following Smith [31], we introduce

**Definition 1** A competitive map $T : \mathcal{R} \to \mathcal{R}$, $\mathcal{R} \subset \mathbb{R}^2$, is said to satisfy condition $(O+)$ if for every $x, y \in \mathcal{R}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$. The map $T$ is said to satisfy condition $(O-)$ if for every $x, y \in \mathcal{R}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.

The following theorem was proved by DeMottoni-Schiaffino [12] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [28, 29].

**Theorem 8** If $T : \mathcal{R} \to \mathcal{R}$, $\mathcal{R} \subset \mathbb{R}^2$, is a competitive map for which $(O+)$ holds then for all $x \in \mathcal{R}$, $\{ T^n(x) \}$ is eventually componentwise monotone. If the orbit of $x$ has compact closure, then it converges to a fixed point of $T$. If instead $(O-)$ holds, then for all $x \in \mathcal{R}$, $\{ T^{2n}(x) \}$ is eventually componentwise monotone. If the orbit of $x$ has compact closure in $\mathcal{R}$, then its omega limit set is either a period-two orbit or a fixed point.

The following result is Lemma 4.3 from [31] specialized to smooth maps on planar rectangular regions. If $T$ is a map which is differentiable at a point $x$, by $J_T(x)$ we denote the Jacobian matrix of $T$ at $x$.

**Theorem 9** Let $T$ be a $C^1$ competitive map on a rectangular region $\mathcal{R}$. If $T$ is injective and $\det J_T(x) > 0$ for all $x \in \mathcal{R}$ then $T$ satisfies $(O+)$. If $T$ is injective and $\det J_T(x) < 0$ for all $x \in \mathcal{R}$ then $T$ satisfies $(O-)$. 

H. L. Smith performed a systematic study of competitive and cooperative maps and in particular introduced invariant manifolds techniques in his analysis [28, 29, 30] with some results valid for maps on $n$-dimensional space. Smith restricted attention mostly to competitive maps $T$ that satisfy additional constraints. In particular, $T$ is required to be a diffeomorphism of a neighborhood of $\mathbb{R}^n_+$ that satisfies either $(O+)$ or $(O-)$, (this is the case if $T$ is orientation-preserving or orientation-reversing), and that the coordinate semiaxes are invariant under $T$. The latter requirement is a
common feature of many population dynamics applications, where a point on a positive semiaxis is interpreted as one of the populations having no individuals, and thus the corresponding orbit terms having the same characteristic. For such class of maps (as well as for cooperative maps satisfying similar hypotheses) Smith obtained results on invariant manifolds passing through fixed points and a fairly complete description of the phase-portrait when \( n = 2 \), especially for those cases having a unique fixed point on each of the open positive semiaxes.

3 Applications

In this section we present several applications of our main results. The examples are of non-hyperbolic type. The hyperbolic case is well known and has been treated in [7, 8, 22, 23, 24].

Example 1 A system with a continuum of non-hyperbolic equilibria along a vertical line. Consider system (1) with \( a > 1 \). The map of the system is

\[
T(x, y) = \left( \frac{x}{a + y}, \frac{y}{1 + x} \right), \quad (x, y) \in [0, \infty)^2.
\]

The fixed points of \( T \) have the form \((0, \bar{y})\), with \( \bar{y} \geq 0 \). The map is smooth and strongly competitive on \([0, \infty)^2\). One eigenvalue of the Jacobian of the map \( T \) at \((0, \bar{y})\) is 1. The hyperbolic eigenvalue is \( \frac{1}{a + \bar{y}} \), with corresponding eigenvector \((a - 1 + \bar{y}, (a + \bar{y})\)). Thus the hypotheses of Theorem 1 are satisfied. Notice that the conditions of Theorem 9 are satisfied and thus by Theorem 8 all solutions of (1) are eventually componentwise monotone. Indeed, the Jacobian matrix \( J_T(x, y) \) satisfies \( \det J_T(x, y) = \frac{1}{a + y} > 0 \). In addition, a direct verification shows that \( T \) is injective. A consequence of Theorem 8 is that there are no periodic points of minimal period-two. Also, with an argument similar to the one used in [4], one has that the equilibrium depends continuously on the initial condition. That is, if \( T^*(x, y) := \lim T^n(x, y) \), then \( T^* \) is continuous. These considerations lead to the following result.

**Theorem 10** For system (1) with \( a > 1 \),

i. Every solution converges to an equilibrium \((0, \bar{y})\) for some \( \bar{y} \geq 0 \).

ii. At each equilibrium \((0, \bar{y})\) with \( \bar{y} > 0 \), the stable set \( W^s((0, \bar{y})) \) is an unbounded increasing curve \( C \) that starts at \((0, \bar{y})\).

iii. The limiting equilibrium varies continuously with the initial condition.

Statement ii excludes the equilibrium \((0, 0)\) since the hypotheses of Theorem 1 are not satisfied at \((0, 0)\). Theorem 10 has been proved in [6] by using differential equations associated to the map \( T \) and asymptotic estimates of infinite products.

Example 2 A system with a continuum of non-hyperbolic equilibria along a line. Consider the Leslie-Gower competition model with nonhyperbolic equilibrium points

\[
x_{n+1} = \frac{b_1 x_n}{1 + x_n + c_1 y_n}, \quad n = 0, 1, \ldots
\]

\[
y_{n+1} = \frac{b_2 y_n}{1 + y_n + c_2 x_n}
\]
where all parameters are positive and the initial conditions \( x_0, y_0 \) are non-negative. This system was considered in [9, 22] and its global dynamics has been settled with the exception of the nonhyperbolic case which will be considered here. It is shown in [9] that when \( c_1 (b_2 - 1) \neq b_1 - 1 \) or \( c_2 (b_1 - 1) \neq b_2 - 1 \), the map associated to (6),

\[
T(x, y) = \left( \frac{b_1 x}{1 + x + c_1 y}, \frac{b_2 y}{1 + y + c_2 x} \right)
\]

(7) has between one and four fixed points, and that they are of hyperbolic type. The case when

\[
c_1 (b_2 - 1) = b_1 - 1 \quad \text{and} \quad c_2 (b_1 - 1) = b_2 - 1
\]

(8) was not considered in [9]. When (8) holds, a direct calculation gives that the equilibrium points of \( T \) are \( E_0(0,0) \) and the family of points \( \mathcal{E} := \{ E_t : 0 \leq t \leq 1 \} \), where

\[
E_t := ((b_1 - 1)(1 - t), (b_2 - 1)t), \quad 0 \leq t \leq 1.
\]

The eigenvalues of the Jacobian of \( T \) at \( E_t \) are easily calculated to be

\[
1 \quad \text{and} \quad (1 - t) \frac{1}{b_1} + t \frac{1}{b_2}, \quad 0 \leq t \leq 1,
\]

and corresponding eigenvectors are

\[
\left( -\frac{1 - b_1}{1 - b_2}, 1 \right) \quad \text{and} \quad \left( b_2 (1 - b_1)^2 (1 - t), b_1 (1 - b_2)^2 t \right), \quad 0 \leq t \leq 1.
\]

It is shown in [5] that, for system (6), the hypotheses of Theorem 8 are satisfied and that all solutions fall inside an invariant rectangular region. Therefore every solution of (6) converges to an equilibrium point. A direct calculation shows that the origin is a repeller. We conclude that every nonzero solution converges to a point \((x, y) \in E \). Also, with an argument similar to the one used in [4], one has that the equilibrium depends continuously on the initial condition. That is, if \( T^*(x, y) := \lim T^n(x, y) \), then \( T^* \) is continuous. These observations, together with an application of Theorem 1 lead to the following result.

**Theorem 11** Assume inequalities (8) hold. Then,

i. Every nonzero solution to system (6) converges to an equilibrium \((x, y) \in \mathcal{E} \).

ii. For every \((x, y) \in \mathcal{E} \) with \( x \neq 0 \) and \( y \neq 0 \), the stable set \( W_{(x, y)}^s \) is an unbounded increasing curve \( C \) with endpoint \((0,0)\).

iii. The limiting equilibrium varies continuously with the initial condition.

Statement ii excludes equilibria of the form \((0, y) \) and \((x, 0)\) since the hypotheses of Theorem 1 are not satisfied at these points.

**Example 3** A difference equation with a continuum of period-two points along a branch of a hyperbola. Consider the second order difference equation

\[
x_{n+1} = 1 + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots ,
\]

(9)
where the initial conditions \(x_{-1}, x_0\) are positive. This equation was considered in \([2, 4, 20]\) and its global dynamics has been settled completely in \([2]\). Here we give more precise description of the dynamics of \((9)\). The map
\[
T(x, y) = \left( y, 1 + \frac{x}{y} \right)
\]
associated to \((9)\) has a unique fixed point \((2, 2)\). The second iterate of \(T\),
\[
T^2(x, y) = \left( 1 + \frac{x}{y}, 1 + \frac{y^2}{x+y} \right),
\]
is strongly competitive in the interior of first quadrant and has an infinite number of fixed points \((\bar{x}, \bar{y})\). The collection of fixed points of \(T^2\) (period-two points of \(T\)) is the set
\[
\mathcal{H} = \{ (x, y) \in (0, \infty)^2 : x + y = xy \}.
\]
The eigenvalues of \(T^2\) at \((\bar{x}, \bar{y})\) \(\in \mathcal{H}\) are 1 and \(\frac{1}{\bar{y}}\) < 1, and the eigenvector of \(T^2\) at \((\bar{x}, \bar{y})\) associated with \(\lambda = \frac{1}{\bar{y}}\) is \((\bar{x}, 1)\). The Jacobian matrix of \(T^2\) is
\[
J_{T^2}(u, v) = \begin{pmatrix}
\frac{1}{v} & -\frac{u}{v^2} \\
-\frac{u^2}{(u+v)^2} & \frac{v(2u+v)}{(u+v)^2}
\end{pmatrix},
\]
thus \(\det J_{T^2}(u, v) = \frac{1}{u+v} > 0\) for \((u, v) \in (0, \infty)^2\). In addition, direct verification shows that \(T^2\) is injective. Thus all hypotheses of Theorem \([4]\) are satisfied by \(T^2\), so for every fixed point \((\bar{x}, \bar{y})\) of \(T^2\) (consequently, for every period-two point of \(T\)), there exists an unbounded increasing invariant curve \(C_{(\bar{x}, \bar{y})}\) which is a subset of the basin of the attraction of \((\bar{x}, \bar{y})\). Furthermore, it can be shown that all conditions of deMottoni-Schiaffino theorem are satisfied and so every solution of \((9)\) converges to a period-two solution. In addition, with an argument similar to the one used in \([4]\), applied to \(T^2\), we may conclude that, given any solution to Eq.\((9)\), the limiting period-two solution \((\bar{x}, \bar{y})\) depends continuously on the initial condition \((x_0, y_0)\). That is, if \(T^* (x, y) := \lim T^n (x, y)\), then \(T^*\) is continuous. Thus we have the following result.

**Theorem 12** The following statements are true for equation \((9)\).

i. Every period-two solution converges to a period-two solution \((\bar{x}, \bar{y}) \in \mathcal{H}\).

ii. For every period-two solution \((\bar{x}, \bar{y}) \in \mathcal{H}\), the stable set \(W^s_{(\bar{x}, \bar{y})}\) is an unbounded increasing curve \(C_{(\bar{x}, \bar{y})}\).

iii. The limiting period-two solution \((\bar{x}, \bar{y})\) depends continuously on the initial condition \((x_0, y_0)\).

**Example 4** A system with an isolated non-hyperbolic interior equilibrium which is of oscillatory type. The system
\[
x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n}, \quad n = 1, 2, \ldots
\]
where all the parameters are positive and the initial conditions \(x_0, y_0\) are non-negative and such that \(x_0 + y_0 > 0\) was mentioned in \([5]\) as a special case of system \((2)\), and was investigated in detail in \([11]\). When the condition
\[
\beta_1 - B_1 \gamma_2 = 2 \sqrt{B_1 \alpha_2}
\]
is satisfied, system (12) has a unique equilibrium point \( E = \left( \frac{B_1 \gamma_2 + \beta_1}{2B_1}, \frac{\beta_1 - B_1 \gamma_2}{2} \right) \) which is non-hyperbolic. The eigenvalues of the linearized system at \( E \) are

\[
\lambda_1 = 1, \quad \lambda_2 = -\frac{(\beta_1 - B_1 \gamma_2)^2}{2 \beta_1 B_1 (\beta_1 + B_1 \gamma_2)},
\]

with corresponding eigenvectors \( e_1 = (-1, B_1) \), and \( e_2 = (2 \beta_1 B_1 (\beta_1 - B_1 \gamma_2), (\beta_1 + B_1 \gamma_2)^2) \). From (13) we have \( \lambda_2 \in (-1, 0) \), and \( E \) is of oscillatory type. Thus the hypotheses of Theorems 1 and 4 are satisfied at the equilibrium point, and the conclusions of Theorems 1 and 4 follow. Let \( C, W_- \) and \( W_+ \) be the sets given in the conclusion of Theorems 1 and 4. We have the following result.

**Theorem 13** The unique equilibrium \( E \) of system (12) with conditions (13) is non-hyperbolic and semi-stable. The basin of attraction of \( E \) is \( C \cup W_+ \), and the orbit of every point in \( W_- \) is asymptotic to \((0,\infty)\).

**Proof.** Let \( S := \{(x,y) : 0 \leq x \leq \frac{\beta_1}{B_1}, 0 \leq y \} \). Since \( \frac{\beta_1}{B_1} x + y \leq \frac{\beta_1}{B_1} \) for \( x \geq 0, y \geq 0, x + y > 0 \), the map \( T \) of system (12) satisfies \( T([0,\infty)^2 \setminus (0,0)) \subset S \). Thus \( T(C \cup W_+) \subset (C \cup W_+) \cap S \), which implies that \( T(C \cup W_+) \) is bounded. Since every orbit is eventually coordinate-wise monotone \([5]\), it follows that every orbit with initial point in the invariant set \( C \cup W_+ \) must converge to an equilibrium. The only equilibrium in \( C \cup W_+ \) is \( E \), so we have \( C \cup W_+ \) is a subset of the basin of attraction of \( E \). If \((x,y)\) is in \( W_- \), by Theorem 1 the orbit of \((x,y)\) eventually enters \( Q_2(E) \). Assume (without loss of generality) that \((x,y)\) is in \( Q_2(E) \). A calculation gives

\[
T(E + t e_1)^{(1)} - (E + t e_1)^{(1)} = 0 \quad \text{for all } t
\]

and

\[
\frac{1}{2} \frac{d}{dt^2} \left. T(E + t e_1) \right|_{t=0} = \left( 0, \frac{1}{\beta_1 + B_1 \gamma_2} \right).
\]

Since in expansion \([5]\) we have \((c_2,d_2) = (0, \frac{1}{\beta_1 + B_1 \gamma_2}) \) and \( T(E + t e_1)^{(1)} \) is affine in \( t \), by Theorem 7 in any relative neighborhood of \( E \) there exists a subsolution \((w,z) \in Q_2(E) \), i.e., \( T(w,z) \preceq_{se} (w,z) \). Choose one such \((w,z)\) so that \((x,y) \preceq_{se} (w,z) \). Since \( T \) is competitive, \( T^{n+1}(w,z) \preceq T^n(w,z) \) for \( n = 0,1,2, \ldots \). The monotonically decreasing sequence \( \{T^n(w,z)\} \in Q_2(E) \) is unbounded below, since if it weren’t it would converge to the unique fixed point in \( Q_2(E) \), namely \( E \), which is not possible. Let \((w_n,z_n) := T^n(w,z), n = 0,1, \ldots \). Then \((w_n,z_n) \in S \) for \( n = 1,2, \ldots \), hence \( \{w_n\} \) is bounded. It follows that \( \{z_n\} \) is monotone and unbounded. From (12) it follows that \( w_n \to 0 \). Since \( T^n(x,y) \preceq_{se} (w_n,z_n) \), it follows that \( T^n(x,y) \to (0,\infty) \).

**Example 5** A system with a semi-stable non-hyperbolic interior equilibrium. We consider an example where explicit computation of eigenvalues of the Jacobian of the map at equilibrium points is not practical. Nevertheless, it is possible to establish a result on basins of attraction of non-hyperbolic equilibria. The strongly competitive system of difference equations

\[
x_{n+1} = \frac{b_1 x_n}{1 + x_n + c_1 y_n} + h_1, \quad n = 0,1, \ldots, \quad (x_0,y_0) \in [0,\infty) \times [0,\infty), \quad (14)
\]

\[
y_{n+1} = \frac{b_2 y_n}{1 + y_n + c_2 x_n} + h_2
\]
with positive parameters was studied in \cite{I}, where it was shown that the system has between one and three equilibria depending on the choice of parameters, and that the number of equilibria determines global behavior as follows: if there is only one equilibrium, then it is globally asymptotically stable. If there are two equilibria, then one is a locally asymptotically stable (L.A.S.) point and the other is nonhyperbolic. If there are three equilibria, then they are linearly ordered in the south-east ordering of the plane, and consist of a L.A.S. point, a saddle point, and another L.A.S. point. We have the following result for the non-hyperbolic case.

**Theorem 14** In the case where system \( (14) \) has exactly two distinct equilibria in \([0, \infty)^2\) given by a locally asymptotically stable point \((\overline{x}_1, \overline{y}_1)\) and a non-hyperbolic fixed point \((\overline{x}_2, \overline{y}_2)\), there exists a curve \( C \) through \((\overline{x}_2, \overline{y}_2)\) as in Theorem 7 and sets \( \mathcal{W}_- \) and \( \mathcal{W}_+ \) as in Theorem 2 such that one of \( \mathcal{W}_- \) or \( \mathcal{W}_+ \) is the basin of attraction of \((\overline{x}_1, \overline{y}_1)\), and the complement of such set is the basin of attraction of \((\overline{x}_2, \overline{y}_2)\).

**Proof.** It is shown in \cite{I} that under the hypothesis of the theorem, the intersection of \((h_1, \infty) \times (h_2, \infty)\) with each of the critical curves

\[
C_1 = \{(x, y) \in [0, \infty)^2 : x^2 + c_1 xy + (1 - b_1 - h_1) x - c_1 h_1 y - h_1 = 0\}
\]

\[
C_2 = \{(x, y) \in [0, \infty)^2 : y^2 + c_2 xy + (1 - b_2 - h_2) y - c_2 h_2 x - h_2 = 0\}
\]

are the graphs of smooth decreasing functions \(y_1(x)\) and \(y_2(x)\) of \(x\) with common points given by the two equilibrium points of system \((14)\). Moreover, \(C_1\) and \(C_2\) intersect tangentially at the non-hyperbolic fixed point \((\overline{x}_1, \overline{y}_1)\), and transversally at the local attractor \((\overline{x}_2, \overline{y}_2)\). By Lemma 4.2 of \cite{I} and Theorem 2.1 of \cite{I}, the eigenvalues \(\lambda, \mu\) of the Jacobian of the map at \((\overline{x}_1, \overline{y}_1)\) satisfy \(0 < |\lambda| < 1\) and \(\mu = 1\). Note that there are no periodic points of minimal period-two. Thus the hypotheses of Theorem 14 and Theorem 3 are satisfied, hence there exist sets \( \mathcal{C}, \mathcal{W}_- \) and \( \mathcal{W}_+ \) with the properties specified in the conclusions of such theorems. In particular,

\[
(0, \infty)^2 = \mathcal{C} \cup \mathcal{W}_- \cup \mathcal{W}_+ ,
\]

where any two of the sets \( \mathcal{C}, \mathcal{W}_- \), and \( \mathcal{W}_+ \) have no common points. Since the set of equilibrium points is linearly ordered by \(\preceq_{se}\), it follows that either \((\overline{x}_2, \overline{y}_2) \in \mathcal{W}_-\) or \((\overline{x}_2, \overline{y}_2) \in \mathcal{W}_+\). In the rest of the proof we shall assume that \((\overline{x}_2, \overline{y}_2) \in \mathcal{W}_-\); the proof in the case \((\overline{x}_2, \overline{y}_2) \in \mathcal{W}_+\) is similar and will be omitted. Then the only fixed point in the closed invariant set \( \mathcal{C} \cup \mathcal{W}_+\) is \((\overline{x}_1, \overline{y}_1)\), and since every orbit converges to a fixed point \cite{I}, we have that \( \mathcal{C} \cup \mathcal{W}_+\) is a subset of the basin of attraction of \((\overline{x}_1, \overline{y}_1)\). To complete the proof of the theorem it is enough to show that \( \mathcal{W}_-\) is a subset of the basin of \((\overline{x}_2, \overline{y}_2)\). There exists an open interval \(I\) centered at \(x_1\) such that \(y_1(x) < y_2(x)\) for \(x \in I \setminus \{x_1\}\). Set \(y_3(x) = \frac{1}{2}(y_1(x) + y_2(x))\), and let \(C_3\) be the graph of \(y_3(x)\). For small enough \(\delta\), the map \(T\) satisfies \(T(x, y) \preceq_{se} (x, y)\) for every \((x, y) \in C_3 \cap B((\overline{x}_1, \overline{y}_1), \delta)\), where \(B((\overline{x}_1, \overline{y}_1), \delta)\) is the open disk in \(\mathbb{R}^2\) with center \((\overline{x}_1, \overline{y}_1)\) and radius \(\delta\). By Theorem 3 the orbit of every \((x, y) \in \mathcal{W}_-\) enters the invariant set \(Q_2(x_2, y_2) \cap (0, \infty)^2\). Choose \((s, t) \in C_3 \setminus \{(x_1, y_1)\}\) such that \((x, y) \preceq_{se} (s, t) \preceq_{se} (x_1, y_1)\). Since \(T^n(s, t) \preceq_{se} T^n(s, t) \preceq_{se} (s, t)\) for \(n = 0, 1, 2, \ldots\), we have that \(T^n(x, y)\) does not converge to \((x_1, y_1)\). Hence \(T^n(x, y)\) converges to \((x_2, y_2)\). \(\square\)

### 4 Proof of the main results

**Proof of Theorem 14** We assume first that \(\overline{x} \in \partial \mathcal{R}\). Suppose (without loss of generality) that \(Q_1(\overline{x}) \cap \text{int}(\mathcal{R}) \neq \emptyset\). See Figure 1.

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By Lemma 5.1 in page 234 of [10], there exists a small neighborhood $V$ of $\bar{x}$ and a locally invariant $C^1$ manifold $\mathcal{C} \subset V$ that is tangential to $E^\lambda$ at $\bar{x}$ and such that $T^n(x) \to \bar{x}$ for all $x \in \mathcal{C}$. Since $T$ is competitive, a unit eigenvector $\nu^\lambda$ associated with $\lambda$ may be chosen so that $\nu^\lambda$ has non-negative entries. By hypothesis (b), the vector $\nu^\lambda$ has positive entries. If necessary, the diameter of $V$ may be taken to be small enough to guarantee that no two points on $\hat{\mathcal{C}}$ are comparable in the ordering $\preceq_{se}$. This can be done since the tangential vector $\nu^\lambda$ has positive entries. Let $\mathcal{C}$ be the connected component of $\{x \in \mathcal{R} \cap Q_1(\bar{x}) : (\exists n) T^n(x) \in \hat{\mathcal{C}}\}$ that contains $\hat{\mathcal{C}}$. The set $\mathcal{C}$ consists of non-comparable points in the order $\preceq_{se}$. Indeed, if $v$ and $w$ are two distinct points in $\mathcal{C}$, such that $v \preceq_{se} w$, then $T^n(v) \preceq_{se} T^n(w)$ and $T^n(v) \not= T^n(w)$ for $n \geq 0$ since the map $T$ is strongly competitive. But for $n$ large enough, both $T^n(v)$ and $T^n(w)$ belong to $\hat{\mathcal{C}}$, which consists of non-comparable points. Hence $\mathcal{C}$ consists of non-comparable points. The projection of $\mathcal{C}$ onto the first coordinate is a connected set, thus it is an interval $J \subset \mathbb{R}$. Since points on $\mathcal{C}$ are non-comparable, $\mathcal{C}$ is the graph of a strictly increasing function $f(t)$ of $t \in J$. If there is a jump discontinuity at $t_0 \in \mathcal{C}$, let $y_-$ and $y_+$ respectively be the left and right (distinct) limits of $f(t)$ as $x$ approaches $t_0$, respectively. The points $(t_0, y_-)$ and $(t_0, y_+)$ are comparable in the order $\preceq_{se}$, and since $T$ is strongly competitive in $\Delta$, $T(t_0, y_-) \preceq_{se} T(t_0, y_+)$ and $T(t_0, y_-) \not= T(t_0, y_+) \in \text{int} \ Q_4(\bar{x})$. Since both $T(t_0, y_-)$ and $T(t_0, y_+)$ are accumulation points of $\mathcal{C}$, we obtain that $\mathcal{C}$ must have comparable points, a contradiction. Thus $f(t)$ is a continuous function.

If $\mathcal{C}$ is not bounded, then both of its endpoints are in $\partial \mathcal{R}$, as the conclusion of the theorem asserts. If $\mathcal{C}$ is bounded, it has two endpoints, $\bar{x}$ and $x_0$ (say). To show that either $x_0 \in \partial \mathcal{R}$, or $x_0$ is a fixed point of $T$, assume this is not the case. That is, assume $x_0 \in \text{int} \mathcal{R}$ and $T(x_0) \not= x_0$. We shall show that in this case, the curve $\mathcal{C}$ can be extended, contradicting the definition of $\mathcal{C}$ as a connected component.

We first show that $T(x) \not= \bar{x}$ for $x \in \Delta$. To see this, consider points $y$ and $z$ in $\Delta$ so that the points $y, x_0$ and $z$ lie on a vertical line $x = c$ with $y \preceq_{se} x_0 \preceq_{se} z$, and such that both $y$ and $z$ distinct from $x_0$. Then $T(y) \preceq_{se} T(x_0) \preceq_{se} T(z)$. Furthermore, since $T$ is strongly competitive, if $T(x_0) = \bar{x}$, we have $x_0 - T(x) \in \text{int} \ Q_4(\bar{x})$ and $T(z) - x \in \text{int} \ Q_4(\bar{x})$. Since $\bar{x} \in \partial \mathcal{R}$, we conclude that one of the points $T(y)$ and $T(z)$ does not belong to $\mathcal{R}$, a contradiction.

Since $T(x_0) \not= \bar{x}$, $T(x_0) \not= x_0$ and $\mathcal{C}$ is a (forward) invariant connected set, we must have $T(x_0) \in \mathcal{C} \cap \Delta$. Let $\mathcal{R}_1$ be the rectangular region determined by the endpoints of $\mathcal{C}$, and let $\varepsilon > 0$ be such that $B(T(x_0), \varepsilon) \subset \mathcal{R}_1$. Note that $\mathcal{C}$ is a separatrix for $\mathcal{R}_1$. For $y \in \mathbb{R}^2$ and $\eta > 0$, denote with $B(y, \eta)$ the open disk in $\mathbb{R}^2$ with center $y$ and radius $\eta$. By continuity of $T$, there exists $\delta > 0$ such that $T(B(x_0, \delta)) \subset B(T(x_0), \varepsilon)$. Consider the line segment $L_0$ with endpoints $(x_0^{(1)}, x_0^{(2)} + \delta/2)$. Since $L_0$ is linearly ordered by $\preceq_{se}$, so is $T(L_0)$, and the points $T(x_0^{(1)}, x_0^{(2)} + \delta/2)$ are on different
components of $\mathcal{R}_1 \setminus \mathcal{C}$. Find $\varepsilon_2 > 0$ such that $B(T(x_0^{(1)} + \delta/2, \varepsilon_2)$ is a subset of a component of $\mathcal{R}_1 \setminus \mathcal{C}$ and of $B(T(x_0), \varepsilon)$, and choose $\eta > 0$ such that

\[ T/B((x_0^{(1)}, x_0^{(2)} + \delta/2), \eta) \subset B(T[(x_0^{(1)}, x_0^{(2)} + \delta/2)], \varepsilon_2) \]

Now for each $t \in (0, \eta)$ consider the line segment $L_t$ with endpoints $p_{\pm}(t) := (x_0^{(1)} + t, x_0^{(2)} + \delta/2)$. Note that $L_t \subset B(x_1, \delta)$, and that $T(p_{+}(t)), T(p_{-}(t))$ belong to different components of $\mathcal{R}_1 \setminus \mathcal{C}$. For each $t \in (0, \eta)$, the line segment $L_t$ is linearly ordered by $\leq_{as}$, hence so is $T(L_t)$. Thus for each $t \in (0, \eta)$ there exists $x_t \in L_t$ such that $T(x_t) \in \mathcal{C}$. That is, the function $f(x)$ may be extended to a function $\hat{f}(x)$ defined on an interval $\hat{J}$ that includes $J$ as a proper subset. The reasoning used to show continuity and monotonicity of $\hat{f}(x)$ gives continuity and monotonicity of $\hat{f}(x)$. This contradicts the choice of $\mathcal{C}$ as a component, and we conclude that either $T(x_0) = x_0$ or $x_0 \in \partial \mathcal{R}$. See Figure 2.

We have proved the theorem for $\overline{x} \in \partial \mathcal{R}$. If now $\overline{x} \in \text{int} \mathcal{R}$, the argument in the case $\overline{x} \in \partial \mathcal{R}$ may be used to demonstrate the existence of a curve $\mathcal{C}$ with endpoints $w \in \text{int} Q_3(\overline{x})$ and $z \in \text{int} Q_1(\overline{x})$. By the argument used before, if both of $w, z$ are in the interior of $\mathcal{R}$, then the set $\{w, z\}$ is invariant and therefore consists of fixed points or of minimal period-two points. If only one of $w, z$ is in $\text{int} \mathcal{R}$, then such point must be a fixed point of $T$. \hfill \Box

![Figure 2: Sets that appear in the proof of Theorem 1](image)

**Proof of Theorem 2** i. The conclusion follows from Theorem 1. ii. We claim that $\mathcal{C} \cap Q_1(\overline{x})$ and $\mathcal{C} \cap Q_3(\overline{x})$ are invariant. Note first that both of these sets are connected, hence so are their images under $T$. Since $T(x) = \overline{x}$ for $x \in \mathcal{C}$ is not possible by hypothesis, it follows that for $\ell = 1, 2$, $\overline{x}$ is necessarily an endpoint of $T(\mathcal{C} \cap Q_\ell(\overline{x}))$. Hence either $T(\mathcal{C} \cap Q_\ell(\overline{x})) \subset Q_1(\overline{x})$, or, $T(\mathcal{C} \cap Q_\ell(\overline{x})) \subset Q_3(\overline{x})$, $\ell = 1, 2$. We now show $T(\mathcal{C} \cap Q_1(\overline{x})) \subset Q_1(\overline{x})$. Since $T$ is competitive, the largest eigenvalue $\mu$ of $J_T(\overline{x})$ is positive. This fact and the hypothesis det $J_T(\overline{x}) > 0$ implies that both eigenvalues $\lambda, \mu$, of $J_T(\overline{x})$ are positive. Let $v^\lambda$ be an eigenvector associated with $\lambda$. Since the product of the entries of $v^\lambda$ is positive, we may assume without loss of generality the entries are
positive, and in this case for all \( r > 0 \) small enough, \( x = \overline{x} + r v^\lambda \in \Delta \cap Q_1(\overline{x}) \). Now \( x \in C \) satisfies \( T(x) = \overline{x} + r v^\lambda + o(r) \), hence for \( r > 0 \) small, \( T(x) \in \Delta \cap Q_1(\overline{x}) \). This proves \( C \cap Q_1(\overline{x}) \neq \emptyset \), and therefore \( C \cap Q_3(\overline{x}) \subset Q_1(\overline{x}) \). That \( C \cap Q_3(\overline{x}) \subset Q_3(\overline{x}) \) is proved in similar manner. The statement that \( C \) has no endpoints in the interior of \( \mathcal{R} \) follows from the fact that \( C \cap Q_1(\overline{x}) \) and \( C \cap Q_3(\overline{x}) \) are invariant. Indeed, each one of these sets has a set of endpoints which is invariant, and contains \( \overline{x} \). Since \( T(x) = \overline{x} \) only for \( x = \overline{x} \) by hypothesis, the endpoint that is not equal to \( \overline{x} \) must be a fixed point, a contradiction. The proof of iii. is similar to the proof of ii. and we skip it.

**Proof of Theorem 3** By Lemma 5.1 and Exercise 5.1 (a) (ii) in pages 234 and 238 of [16], there exists a neighborhood \( \mathcal{D} \) of \( \overline{x} \) such that \( \mathcal{D} \cap C \) is a class \( C^k \) manifold. For arbitrary \( x \in C \), let \( n \in \mathbb{N} \) be such that \( T^n(x) \in \mathcal{D} \cap C \). By the hypotheses on \( T \), the map \( T^n \) is of class \( C^k \), with \( C^k \) inverse defined on a neighborhood \( \mathcal{E} \) of \( T^n(x) \). Thus \( (T^n)^{-1}(\mathcal{E} \cap \mathcal{C}) \) is a \( C^k \) manifold.

**Proof of Theorem 4** For convenience, in this proof we assume \( \overline{x} = \overline{y} \). It is clear that the sets \( \mathcal{W}_- \) and \( \mathcal{W}_+ \) are connected, disjoint, and satisfy \( \mathcal{C} \cup \mathcal{W}_- \cup \mathcal{W}_+ \subset \mathcal{R} \). To prove the reverse inclusion, let \( w \) and \( z \) be the two endpoints of \( C \), where \( w \leq_{ne} z \). If \( x \in \mathcal{R} \setminus (\mathcal{C} \cup \mathcal{W}_- \cup \mathcal{W}_+) \), then either \( w \in \mathbb{R}^2 \) and \( z \leq_{ne} w \), or \( z \in \mathbb{R}^2 \) and \( z \leq_{ne} x \). Suppose \( z \in \mathbb{R}^2 \) and \( z \leq_{ne} x \). Since \( z \in \partial \mathcal{R} \), \( x \in \partial \mathcal{R} \) and necessarily \( x \) is comparable to \( z \). Since \( z \in C \) contradicts the assumption on \( x \), necessarily \( z \notin C \). But then \( z \notin \mathcal{R} \), which implies that \( x \notin \mathcal{R} \). This contradiction proves the inclusion.

We now prove (i). If \( x \in \mathcal{W}_- \), let \( y \in C \) be such that \( x \leq_{se} y \). Then \( T^n(x) \leq_{se} T^n(y) \) for \( n > 0 \). Hence \( T^n(x)^{(1)} \leq T^n(y)^{(1)} \), and \( T^n(x)^{(2)} \geq T^n(y)^{(2)} \), where for \( v \in \mathbb{R}^2 \) we write \( v = (v^{(1)}, v^{(2)}) \). Since \( T^n(y) \to \overline{y} \) as \( n \to \infty \), it follows that \( \limsup T^n(x)^{(1)} \leq \limsup T^n(y)^{(1)} = 0 \) and \( \liminf T^n(x)^{(2)} \geq \liminf T^n(y)^{(2)} = 0 \), that is, \( \text{dist}(T^n(x), Q_2(\overline{y})) \to 0 \). This proves (i). The proof of (ii) is similar.

To prove (iii), assume first \( \lambda > 0 \) and \( \text{int } Q_1(\overline{y}) \cap \mathcal{R} \neq \emptyset \). We proceed by contradiction and assume there exists \( (x_n, y_n) \in \mathcal{W}_- \) such that

\[
T^n(x_n, y_n) \in \text{int } Q_1(\overline{y}) \quad \text{for } n \in \mathbb{N}.
\]

**Claim 1** \( T^n(x_n, y_n) \to (0, 0) \).

To prove the claim, let \( \epsilon > 0 \) be such that \( T \) is strongly monotonic on \( \mathcal{B}(\overline{y}, \delta) \), and let \( \epsilon \) be an arbitrary positive number in \( (0, \eta) \). Since \( (0, \epsilon) \leq_{se} (0, 0) \), strong monotonicity of \( T \) implies \( T(0, \epsilon) \in \text{int } Q_2((0, 0)) \). By continuity of \( T \), there exists \( \delta(\epsilon) > 0 \) such that \( (s, t) \in \text{clos } \mathcal{B}((0, \epsilon_0), \delta(\epsilon)) \) implies \( T(s, t) \in \text{int } Q_2((0, 0)) \). In particular, \( T(\delta(\epsilon), \epsilon) \in \text{int } Q_2((0, 0)) \). If \( (s, t) \in S(\delta(\epsilon), \epsilon) := \{(w, z) : 0 \leq w < \delta(\epsilon), \epsilon < z\} \), then \( (s, t) \leq_{se} (\delta(\epsilon), \epsilon) \) and \( T(s, t) \leq_{se} T(\delta(\epsilon), \epsilon) \) since \( T \) is competitive. Thus \( T(S(\delta(\epsilon), \epsilon)) \subset \text{int } Q_2((0, 0)) \). Since \( (x_n, y_n) := T^n(x_0, y_0) \) satisfies \( y_n \to 0 \) by part (i) of the Theorem, and since \( (x_n, y_n) \notin S(\delta(\epsilon), \epsilon) \) for all \( \epsilon \in (0, \eta) \), we conclude \( y_n \to 0 \), thus completing the proof of the claim.

By Lemma 5.1 and Exercise 5.1 (a) (ii) in pages 234 and 238 of [16], there exists a \( C^2 \) change of coordinates \( \Theta \) such that the map \( \hat{T} := \Theta T \Theta^{-1} \) is defined in a neighborhood \( \mathcal{B}(\overline{y}, \delta) \) of \( \overline{y} \) where it is of class \( C^2 \), \( \hat{T} \) has eigenvectors \( \mu \) and \( \lambda \) with associated eigenvectors \( (0, 1) \) and \( (1, 0) \), and such that the invariant curve \( \mathcal{C} \) is mapped to the \( x \)-axis. The points \( (\hat{x}_n, \hat{y}_n) := \Theta(x_n, y_n) = \hat{T}^n(\hat{x}_0, \hat{y}_0) \), satisfy

\[
(\hat{x}_n, \hat{y}_n) \in Q_1(\overline{y}), \ n = 0, 1, 2, \ldots, \ \text{and } (\hat{x}_n, \hat{y}_n) \to \overline{y}.
\]

For \( m > 0 \), let \( \Omega(m) \) be the open wedge in the first quadrant limited by the \( x \)-semiaxis and the line \( y = mx \), that is,

\[
\Omega(m) = \{(x, y) \in (0, \infty) : 0 < y < mx \}.
\]
If \( v \in Q_1(\bar{0}) \), by \( \theta(v) \) we denote the measure of the polar angle of \( v \) with respect to the positive horizontal semiaxis. Thus \( \tan \theta(v) = \frac{v^{(2)}}{v^{(1)}} \) whenever \( v^{(1)} \neq 0 \).

**Claim 2** For every \( m \in (0, \infty) \) and every \( q \in (1, \frac{\mu}{\lambda}) \) there exists \( \delta > 0 \) such that

\[
\| \hat{T}(x,y) - \hat{T}(s,t) - A(x-s,y-t) \| < \varepsilon \| (x-s,y-t) \| \quad \text{for} \; (x,y),(s,t) \in B(0,\delta) .
\]

**Proof.** Let \( A \) be the Jacobian operator of \( \hat{T} \) at \( \bar{0} \). Thus the matrix representation of \( A \) in the standard basis of \( \mathbb{R}^2 \) is a diagonal matrix with \( \lambda \) and \( \mu \) on the diagonal. Since \( \hat{T} \) is \( C^2 \) on a neighborhood of the origin, Lemma 10.11 from [1] guarantees that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\| \hat{T}(x,y) - \hat{T}(0,0) - \mu y (0,1) \| < \varepsilon \| y \| \quad \text{for} \; (x,y) \in B(0,\delta_1) .
\]

By continuity of the function \( g(s,t) := \frac{\mu s + \lambda t}{\lambda t + \mu} \) at \( (0,0) \) and since \( g(0,0) = \frac{\mu}{\lambda} > q > 1 \), there exists \( \eta \in (0,\lambda) \) such that

\[
|r(s,t)| > q \quad \text{for} \; |s| < \eta, \; |t| < \eta .
\]

Set \( \varepsilon := \min\{ \frac{\eta}{2m}, \eta, \mu \} \), and let \( \delta_1 > 0 \) be chosen so that the inequality in (19) holds for \( (x,y) \in B(0,\delta_1) \). Note that from (19) we have

\[
|\hat{T}(x,y) - \hat{T}(x,0) - \mu y (0,1)| < \varepsilon \| y \| \quad \text{for} \; (x,y) \in B(\bar{0},\delta_1) .
\]

In this case there exist functions \( \phi(x,y) \) and \( \psi(x,y) \) of \( (x,y) \in B(0,\delta_1) \) such that

\[
\begin{align*}
\phi(x,y) &< \varepsilon , \quad \psi(x,y) < \varepsilon , \\
\hat{T}(x,y)^{(2)} & = \mu y + \phi(x,y) y \\
\hat{T}(x,y)^{(1)} & = \hat{T}(x,0)^{(1)} + \psi(x,y) y
\end{align*}
\]

From the Taylor expansion of \( \hat{T}(x,y) \) about \( (0,0) \) one can see that there exist \( c > 0 \) and \( \delta_2 > 0 \) such that

\[
\| \hat{T}(x,0) - \lambda x (0,1) \| < c x^2 \quad \text{for} \; |x| < \delta_2 .
\]

Since \( (\lambda x (0,1))^{(1)} = \lambda x \), we have from (23) that

\[
| \hat{T}(x,0)^{(1)} - \lambda x | < c x^2 \quad \text{for} \; |x| < \delta_2 .
\]

In this case there exists a function \( \xi(x) \) of \( x \in (-\delta_2,\delta_2) \) such that \( |\xi(x)| < c \) for \( |x| < \delta_2 \), and

\[
\hat{T}(x,0)^{(1)} = \lambda x + \xi(x) x^2 \quad \text{for} \; |x| < \delta_2 .
\]

Set \( \delta_3 := \min(\delta_1,\delta_2) \). Then,

\[
\hat{T}(x,y) = (\lambda x + \xi(x) x^2 + \psi(x,y) y, \mu y + \phi(x,y) y) , \quad (x,y) \in B(\bar{0},\delta_3) ,
\]

(26)
and consequently,
\[
\tan \theta(\hat{T}(x, y)) = \frac{\mu y + \phi(x, y) y}{\lambda x + \xi(x)x^2 + \psi(x, y) y} = \frac{y}{x} \left(\frac{\mu + \phi(x, y)}{\lambda + \xi(x)x + \psi(x, y)x^2}\right), \quad (x, y) \in \mathcal{B}(\bar{0}, \delta_3).
\]  
Set \(\delta := \min\{\frac{m}{2c}, \delta_3\}\). Then
\[
|\xi(x)x| < c \cdot \frac{\eta}{2c} = \frac{\eta}{2} \quad \text{for} \quad |x| < \delta.
\]  
By relation (22) we have
\[
\left|\psi(x, y) \frac{y}{x}\right| \leq \frac{\eta}{2m} \cdot m = \frac{\eta}{2} \quad \text{for} \quad (x, y) \in \mathcal{B}(\bar{0}, \delta) \cap \Omega(m),
\]  
and
\[
|\phi(x, y)| < \eta \quad \text{for} \quad (x, y) \in \mathcal{B}(\bar{0}, \delta).
\]  
By setting \(s = \phi(x, y), t = \xi(x)x + \psi(x, y) \frac{y}{x}\) for \((x, y) \in \mathcal{B}(\bar{0}, \delta),\) relations (20) and (27) through (30) yield
\[
\tan \theta(\hat{T}(x, y)) = \frac{y}{x} g(s, t) = \tan(\theta(x, y))g(s, t) > q \tan(\theta(x, y)) \quad \text{for} \quad (x, y) \in \mathcal{B}(\bar{0}, \delta) \cap \Omega(m).
\]  
We now verify \(\hat{T}(x, y) \in Q_1(\bar{0})\). Since \(\varepsilon < \mu\) by our choice of \(\varepsilon\), then from (21) we have \(\hat{T}(x, y)^{(2)} > 0\). Also, since \(|\xi(x)x + \psi(x, y) \frac{y}{x}| < \eta < \lambda\), we have
\[
\hat{T}(x, y)^{(1)} = x(\lambda + \xi(x)x + \psi(x, y) \frac{y}{x}) > x(\lambda - \eta) > 0 \quad \text{for} \quad (x, y) \in \mathcal{B}(\bar{0}, \delta) \cap \Omega(m).
\]  
Relations (31) and (32) give (18), thus completing the proof of the claim.

\[\square\]

**Proof of Theorem 5** The map \(T\) is a diffeomorphism on a neighborhood \(\mathcal{U}\) of \(\bar{x}\) onto its image. The Unstable Manifold Theorem (page 282 in [19]) guarantees the existence of the local unstable set \(\mathcal{W}^u_{\text{loc}}(\bar{x})\), which is an invariant curve that is tangential to \(v^u\) at \(\bar{x}\) and such that
\[
x \in \mathcal{W}^u_{\text{loc}}(\bar{x}) \implies T^{-n}(x) \in \mathcal{W}^u_{\text{loc}}(\bar{x}) \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad T^{-n}(x) \to \bar{x}
\]  
Since \(\mu > 1\) and the eigenspace \(E^u\) is not a coordinate axis, the entries of eigenvectors \(v^u\) are nonzero. Further, since \(T\) is competitive, the entries of \(v^u\) have different sign. Thus points in \(\mathcal{W}^u_{\text{loc}}(\bar{x})\) that are close enough to \(\bar{x}\) are comparable. It follows from this and from (33) that \(\mathcal{W}^u_{\text{loc}}(\bar{x})\) is linearly ordered by \(\preceq_{\text{se}}\). We claim that points \(x \in Q_2(\bar{x}) \cap \mathcal{W}^u_{\text{loc}}(\bar{x})\) are subsolutions. Indeed, the set \(Q_2(\bar{x}) \cap \mathcal{W}^u_{\text{loc}}(\bar{x})\) is invariant. If \(x \in Q_2(\bar{x}) \cap \mathcal{W}^u_{\text{loc}}(\bar{x})\) had \(x \preceq_{\text{se}} T(x) \preceq \bar{x}\), then \(T^{-n}(x) \preceq_{\text{se}} x \preceq \bar{x}\) for \(n \in \mathbb{N}\), which contradicts (33). Similarly, points \(x \in Q_4(\bar{x}) \cap \mathcal{W}^u_{\text{loc}}(\bar{x})\) are supersolutions. We
have, $\mathcal{W}^u(\mathfrak{x}) = \bigcup_{n=0}^{\infty} T^n(\mathcal{W}^u_{loc}(\mathfrak{x}))$, hence $\mathcal{W}^u(\mathfrak{x})$ is a nested union of connected and linearly ordered sets, hence itself is connected and linearly ordered. Thus $\mathcal{W}^u(\mathfrak{x})$ is the graph of a decreasing function of the first coordinate. Let $y$ and $z$ be the endpoints of $\mathcal{W}^u(\mathfrak{x})$, with $y \leq z$. Suppose $y' \in \text{int} \mathcal{R}$. Since $y$ is an accumulation point of subsolutions in $\mathcal{W}^u(\mathfrak{x})$, $y$ is also a subsolution by continuity of $T$, and it follows that $y$ is a fixed point. Similar reasoning applies to $z$.

To see that $\mathcal{W}^u(\mathfrak{x}) = \mathcal{C}$, note that by part (B) of Theorem 4, iterates of points $x \in \mathcal{R} \setminus \mathcal{C}$ eventually enter $\text{int}(Q_2(\mathfrak{x}) \cup Q_4(\mathfrak{x})) \cap \mathcal{R}$. If for some $n$, $T^n(x) \in \text{int} Q_2(\mathfrak{x})$ (say), then there exists $y \in \mathcal{W}^u(\mathfrak{x})$ such that $T^n(x) \not\leq_se y$, and consequently, $T^{n+k}(x) \not\leq_se T^k(y)$ for $k \in \mathbb{N}$. Since $y$ is a subsolution, it follows that $T^l(x) \not\leq_se \mathfrak{x}$. Similarly, $T^m(x) \in \text{int} Q_4(\mathfrak{x})$ implies $T^m(x) \not\leq_se \mathfrak{x}$. Thus $T^q(x) \not\leq_se x$ if and only if $x \in \mathcal{C}$.

Proof of Theorem 6 Since $\mathfrak{x}$ is a fixed point of $T$ and $v$ is an eigenvector with associated eigenvalue $\mu$, we have

$$T(\mathfrak{x} + tv) = \mathfrak{x} + t\mu v + o(t).$$

(34)

Since $v^{(1)}v^{(2)} < 0$, we may assume without loss of generality that $v \not\leq_se 0$. To prove (i) assume $\mu > 1$, so in particular $(\mu - 1)v \not\leq_se 0$. If $\text{int} Q_2(\mathfrak{x}) \cap \mathcal{R} = \emptyset$, set $t_0 = 0$, and if $\text{int} Q_2(\mathfrak{x}) \cap \mathcal{R} \neq \emptyset$, choose $t_0 > 0$ such that $[\mathfrak{x} + t_0 v, \mathfrak{x}]$ is a subset of $\mathcal{R}$ and has no fixed points other than $\mathfrak{x}$, and

$$
\frac{1}{t} \left( T(\mathfrak{x} + tv) - (\mathfrak{x} + tv) \right) = (\mu - 1) v + \left( \frac{a_1(t)}{t}, \frac{a_2(t)}{t} \right) \not\leq_se 0, \quad t \in (0, t_0].
$$

(35)

Hence $T(\mathfrak{x} + tv) \not\leq_se \mathfrak{x} + tv$ for $t \in (0, t_0)$. Similarly, if $\mathcal{R} \cap \text{int} Q_4(\mathfrak{x}) = \emptyset$ set $t_1 = 0$, and if $\mathcal{R} \cap \text{int} Q_4(\mathfrak{x}) \neq \emptyset$ choose $t_1 < 0$ such that $\mathfrak{x} + tv \not\leq_se T(\mathfrak{x} + tv)$ for $t \in (t_1, 0)$. Note that $t_1$ and $t_2$ are not both 0. Define $I := [\mathfrak{x} + t_0 v, \mathfrak{x} + t_1 v]$. The set $I$ is a relative neighborhood of $\mathfrak{x}$ in $\mathcal{R}$, and $I \cap \text{int} (Q_2(\mathfrak{x}) \cup Q_4(\mathfrak{x}))$ has no fixed points. Then for every relative neighborhood $U \subset I$ of $\mathfrak{x}$, $U \cap \text{int} Q_2(\mathfrak{x})$ contains a subsolution and $U \cap \text{int} Q_4(\mathfrak{x})$ contains a supersolution. For $x \in I \cap (Q_2(\mathfrak{x}) \cup Q_4(\mathfrak{x}))$, there exists a $y \in I$ which is not a fixed point of $T$ such that such that $y$ is a subsolution and $x \not\leq_se y \not\leq_se \mathfrak{x}$ or $y$ is a supersolution and $\mathfrak{x} \not\leq_se y \not\leq_se x$. Then either $T^n(x) \not\leq_se T^n(y)$ or $T^n(y) \not\leq_se x \not\leq_se y \leq_se \mathfrak{x}$. Since there exists $n_0$ such that $T^n(y) \not\in I$ for $n \geq n_0$, we have $T^n(x) \not\in I$ for $n \geq n_0$. The proof of (ii) is similar and we skip it.

Proof of Theorem 7 We prove statement (i) only, as the proof of statements (ii)-(iv) is similar. Assume $\ell$ is odd and $(c_{\ell}, d_{\ell}) \not\leq_se 0$. If $c_{\ell} d_{\ell} < 0$, then there exists $t_\ast > 0$ such that

$$
\frac{1}{\ell} \left( T(\mathfrak{x} + tv) - (\mathfrak{x} + tv) \right) = (c_{\ell} d_{\ell}) + (O_1(t), O_2(t)) \not\leq_se 0, \quad t \in (-t_\ast, t_\ast),
$$

(36)

which implies $T(\mathfrak{x} + tv) \not\leq_se \mathfrak{x} + tv$ for $t \in (0, t_\ast)$ and $\mathfrak{x} + tv \not\leq_se T(\mathfrak{x} + tx)$ for $t \in (-t_\ast, 0)$. The rest of the proof now proceeds as in the proof of Theorem 6. If $c_{\ell} \neq 0$ and $T(\mathfrak{x} + tx)^{(2)}$ is affine, then $d_{\ell} = 0$ and $O_2(t) = 0$ in (36), and the proof proceeds as before. The same reasoning applies to the case where $d_{\ell} \neq 0$ and $T(\mathfrak{x} + tv)^{(1)}$ is affine.

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