Zero sums in restricted sequences

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Abstract

Suppose \( A \subset [1, n-1] \). A sequence \( x = (x_1, \ldots, x_m) \) of elements of \( \mathbb{Z}_n \) is called an \textit{A-weighted Davenport Z-sequence} if there exists \( a := (a_1, \ldots, a_m) \in (A \cup \{0\})^m \setminus \{0_m\} \) such that \( \sum a_i x_i = 0 \). Here \( 0_m = (0, \ldots, 0) \in \mathbb{Z}_n^m \). Similarly, the sequence \( x \) is called an \textit{A-weighted Erdős Z-sequence} if there exists \( a := (a_1, \ldots, a_m) \in (A \cup \{0\})^m \setminus \{0_m\} \) with \( |\text{Supp}(a)| = n \), such that \( \sum a_i x_i = 0 \), where \( \text{Supp}(a) := \{i : a_i \neq 0\} \). A \( \mathbb{Z}_n \)-sequence \( x \) is called \( k \)-restricted if no element of \( \mathbb{Z}_n \) appears more than \( k \) times in \( x \). In this paper, we study the problem of determining the least value of \( m \) for which a \( k \)-restricted \( \mathbb{Z}_n \)-sequence of length \( m \) is an \( A \)-weighted Davenport Z-sequence (resp. an \( A \)-weighted Erdős Z-sequence). We also consider the same problem for random \( \mathbb{Z}_n \) sequences, for certain very natural choices for the set \( A \).

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1 Introduction

By \([n]\) we shall mean the set \( \{1, \ldots, n\} \), and for integers \( a < b \), \([a, b]\) shall denote the set \( \{a, a+1, \ldots, b\} \). By \( \mathbb{Z}_n \) we shall denote the cyclic group of order \( n \). If \( x_i \in \mathbb{Z}_n \) and \( r_i \in \mathbb{N} \) \( (x_i^{r_1}, \ldots, x_i^{r_k}) \) shall denote the sequence consisting of \( r_i \) copies of \( x_i \) for each \( i \). Throughout this paper, we shall use the Landau asymptotic notation: For functions \( f, g \), we write \( f(n) = O(g(n)) \) if there exists an absolute constant \( C > 0 \) and an integer \( n_0 \) such that for all \( n \geq n_0, |f(n)| \leq C|g(n)| \). We write \( f = \Omega(g) \) if \( g = O(f) \), and we write \( f = \Theta(g) \) if \( f = O(g) \) and \( f = \Omega(g) \).

Suppose \( A \subset [1, n-1] \). By a \( \mathbb{Z}_n \) sequence of length \( m \), we mean a sequence \( x := (x_1, \ldots, x_m) \) with \( x_i \in \mathbb{Z}_n \). For a sequence \( x = (x_1, \ldots, x_m) \), and for a subset \( I \subset [m] \) of the set of indices, we

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shall denote by $x_I$ the sum $\sum_{i \in I} x_i$, and for sequences $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m)$ of the same length, we shall denote by $\langle x, y \rangle_I$ the sum $\sum_{i \in I} x_i y_i$ where as before, $I \subset [m]$. In case $I = [m]$ then we shall drop the subscript and simply write $\langle x, y \rangle$ to denote $\langle x, y \rangle_{[m]}$. If $a = (a_1, \ldots, a_m)$ and $x = (x_1, \ldots, x_m)$ then $a \cdot x$ shall denote $(a_1 x_1, \ldots, a_m x_m)$.

A sequence $x = (x_1, \ldots, x_m)$ of elements of $\mathbb{Z}_n$ is called an $A$-weighted Davenport Z-sequence if there exists $a := (a_1, \ldots, a_m) \in (A \cup \{0\})^m \setminus \{0\}_m$ such that $\sum_i a_i x_i = 0$. Here $0_m = (0, \ldots, 0) \in (\mathbb{Z}_n)^m$. Similarly, the sequence $x$ is called an $A$-weighted Erdős Z-sequence if there exists $a := (a_1, \ldots, a_m) \in (A \cup \{0\})^m \setminus \{0\}_m$ with $|\text{Supp}(a)| = n$, such that $\sum_i a_i x_i = 0$, where $\text{Supp}(a) := \{i : a_i \neq 0\}$. When $A = \{a\}$ for some $a$ coprime to $n$, we shall refer to such a sequence simply as a Davenport Z-sequence (resp. an Erdős Z-sequence). When the set $A$ is clear from the context, we shall simply refer to a sequence as a weighted Davenport Z-sequence (resp. weighted Erdős Z-sequence) and drop any mention of the set $A$.

The notion of a weighted Davenport Z-sequence (resp. a weighted Erdős Z-sequence) draws its motivation from two well-studied invariants, the Davenport constant and the Erdős constant of a finite abelian group $G$. For instance, it is an easy exercise to show that every $\mathbb{Z}_n$-sequence $x := (x_1, \ldots, x_n)$ of length $n$ admits a zero-sum subsequence, i.e. there exists a non-empty subset $I \subset [1, m]$ such that $\langle x, x \rangle_I = 0$, so every sequence of length $n$ is a Davenport Z-sequence, while the sequence $(1^{n-1})$ is not a Davenport Z-sequence. A generalization of this somewhat simple exercise goes as follows. For a given $A \subset [1, n-1]$, denote by $D_A(\mathbb{Z}_n)$ the least integer $m$ such that for every $\mathbb{Z}_n$-sequence $x = (x_1, \ldots, x_m)$ of length $m$, there exists $a \in (A \cup \{0\})^m \setminus \{0\}_m$ satisfying $(a, x) = 0$. Then

- (See [2]) For the set $A = \{1, -1\}$ (where $-1 = (n - 1) \pmod{n}$), any $\mathbb{Z}_n$-sequence of length $\lfloor \log_2 n \rfloor + 1$ is a weighted Davenport Z-sequence, and this result is again, best possible: The sequence $(1, 2, \ldots, 2^{k-1})$, for $k = \lfloor \log_2 n \rfloor$, is not a weighted Davenport Z-sequence.

- (See [6], [10]) For $A = \mathbb{Z}_n$, where $n = q_1 \cdots q_a$ is product of $a$ primes (not necessarily distinct) any sequence of length $a + 1$ is a weighted Davenport Z-sequence, and this result is best possible: The sequence $(1, q_1, q_1 q_2, \ldots, q_1 \cdots q_{a-1})$ is not a weighted Davenport Z-sequence.

The notion of an Erdős Z-sequence draws its motivation from the following non-trivial theorem of Erdős-Ginzburg-Ziv [5]: Every sequence of length $2n - 1$ of elements of $\mathbb{Z}_n$ contains a subsequence of size $n$ whose sum equals zero, so that every $\mathbb{Z}_n$-sequence of length $2n - 1$ is an Erdős Z-sequence, and again, this is best possible since the sequence consisting of $(n - 1)$ zeroes and $(n - 1)$ ones is not an Erdős Z-sequence. For an arbitrary set $A \subset [1, n-1]$, one can analogously define the parameter $E_A(\mathbb{Z}_n)$ as the least integer $m$ such that for every $\mathbb{Z}_n$-sequence $x = (x_1, \ldots, x_m)$, there exists $a := (a_1, \ldots, a_m) \in (A \cup \{0\})^m \setminus \{0\}_m$ with $|\text{Supp}(a)| = n$ and $\sum_i a_i x_i = 0$. Then, a result of [17] shows that $E_A(\mathbb{Z}_n) = D_A(\mathbb{Z}_n) + n - 1$. In particular, for instance, it follows that for $A = \{1, -1\}$, every sequence of length $n + \lfloor \log_2 n \rfloor$ is a weighted Erdős Z-sequence.

One distinct feature of all the aforementioned results is that the tightness of $D_A(\mathbb{Z}_n)$ (and also for other natural choices for $A$) is witnessed by sequences $x$ that are ‘constant’ sequences. For instance, for $A = \{a\}$ (for any $a$ co-prime to $n$), the maximal sequences $x$ that are not Davenport
Z-sequences are necessarily of the form $x = (x^{n-1})$ for some $x$ co-prime to $n$. In particular, if we restrict our attention to $\mathbb{Z}_n$-sequences $x$ with a bound on the number of incidences of any particular element of $\mathbb{Z}_n \setminus \{0\}$, then it is conceivable that among this restricted class of sequences, the minimum value of $m$ for which every restricted $\mathbb{Z}_n$ sequence of length $m$ is an $A$ weighted Davenport (resp. Erdős) $Z$-sequence, might be considerably smaller. And this is the focal point of this paper.

In this paper, we consider the following problems: Let $A \subset [1, n - 1]$.

1. Suppose $k \in \mathbb{N}$. Let

$$\mathcal{X}_k(m) := \{(x_1, \ldots, x_m) : \text{no element of } \mathbb{Z}_n \text{ appears more than } k \text{ times}\}.$$

Determine the least integer $m$ such that every $x \in \mathcal{X}_k(m)$ is a weighted Davenport (Erdős) $Z$-sequence.

2. Suppose $\mathcal{X} = (X_1, \ldots, X_m)$ is a random $\mathbb{Z}_n$-sequence, i.e., let each $X_i$ be picked independently and uniformly at random from $\mathbb{Z}_n$. Determine the least $m$ such that a random $\mathbb{Z}_n$ of length $m$ sequence is a weighted Davenport (Erdős) $Z$-sequence with high probability.

The second problem needs some further elucidation. Given a probability space we say that a sequence of events $\mathcal{E}_n$ occurs with high probability (abbreviated as whp) if $\lim_{n \to \infty} \mathbb{P}(\mathcal{E}_n) = 1$. In our results, the parameter $n$ will be explicitly described as part of the statement of the relevant results.

As for the first problem, this is part of a larger umbrella of problems that usually go by the name of Inverse Zero-Sum problems and some related results appear in [7]. In fact, some of our results also appear in [7] though they do not frame their results in the language of our formulation.

Before we state our results precisely, we set up some further notation. For $k \in \mathbb{N}$, by $s^{(k)}(\mathbb{Z}_n)$, we shall mean the least integer $m$ such that every $x \in \mathcal{X}_k(m)$ is an Erdős $Z$-sequence. The case $k = 1$ is usually awarded greater status and is referred to as the Harborth constant of $\mathbb{Z}_n$ (see [11], for instance, for more results on the Harborth constant).

Our first result describes a solution to the first problem posed above:

**Theorem 1.** Suppose $k \geq 2$.

1. $s^{(2)}(\mathbb{Z}_n) = n + 2$.

2. For any prime $p$, $s^{(k)}(\mathbb{Z}_p) \leq p + k$. Furthermore, for each $k$, there exists an integer $p_0(k)$ such that $s^{(k)}(\mathbb{Z}_p) \geq p + k$ for all primes $p \geq p_0(k)$.

3. There exist constants $c, C > 0$ such that every sequence $x \in \mathcal{X}_k(C\sqrt{n}k)$ is a Davenport $Z$-sequence. Also, there exist sequences $y \in \mathcal{X}_k(c\sqrt{n}k)$ that are not Davenport $Z$-sequences. In other words, the maximum size a $\mathbb{Z}_n$-sequence needs to be in order that it is a $k$-restricted Davenport $Z$-sequence is of the order $\Theta(\sqrt{n}k)$.

The next theorem considers the second problem for weighted Erdős $Z$-sequences.
Theorem 2. Let $X = (X_1, X_2, \ldots, X_m)$ be a random $\mathbb{Z}_n$-sequence. Then whp

1. If $m = n + 2$, then $X$ is an Erdős $\mathbb{Z}$-sequence.

2. Let $A = \{a, b\}$ where $a, a + b, a - b \in \mathbb{Z}_n^*$. Then for $m = n$, $X$ is a weighted Erdős $\mathbb{Z}$-sequence $X$.

3. Let $A = \{1, -1\}$. Then for $m = n + 1$, $X$ is a weighted Erdős $\mathbb{Z}$-sequence.

The next theorem describes our results vis-à-vis the problem of random $\mathbb{Z}_n$-sequences that are whp (weighted) Davenport $\mathbb{Z}$-sequences.

Theorem 3. Let $X = (X_1, X_2, \ldots, X_m)$ be a random $\mathbb{Z}_n$-sequence. Suppose $\omega(n)$ is a function that satisfies $\omega(n) \to \infty$ as $n \to \infty$. Then whp

1. $X$ is a Davenport $\mathbb{Z}$-sequence if $m \geq \log_2 n + \omega(n),$
   $X$ is not a Davenport $\mathbb{Z}$-sequence if $m \leq \log_2 n - \omega(n)$.

2. Suppose $A = \{-1, 1\}$. Then whp
   $X$ is a weighted Davenport $\mathbb{Z}$-sequence if $m \geq \frac{1}{2} \log_2 n + \omega(n),$
   $X$ is not a weighted Davenport $\mathbb{Z}$-sequence if $m \leq \frac{1}{2} \log_2 n - \omega(n)$.

3. Suppose $A = \mathbb{Z}_n^*$, and suppose $n = p_1 \cdots p_r$ is a squarefree integer. Then
   (a) There exists a constant $C > 0$ such that if $m \geq C \log r$ then $X$ is a weighted Davenport $\mathbb{Z}$-sequence whp.
   (b) Suppose $n$ is the product of all the odd primes less than or equal to $x$, then there is an absolute constant $c > 0$ such that if $m < c \log \log x$ then the probability that $X$ is not a weighted Davenport $\mathbb{Z}$-sequence is bounded away from zero.

In theorems 2 and 3 the asymptotics of the corresponding whp statements must be clear from the context. For instance, in theorem 2 the parameter that goes to infinity (that underlines the phrase whp) is $n$, whereas in the last part of theorem 3 the parameter that goes to infinity is $r$, the number of distinct prime factors in $n$.

We make a couple of other remarks about the last part of theorem 3 where the corresponding set $A = \mathbb{Z}_n^*$. First, since the $r^{th}$ prime is of the order $r \log r$, the result stated there translates loosely as stating that with probability $\Omega(1)$, a random $\mathbb{Z}_n$-sequence of length $\Omega(\log r)$ is not a Davenport $\mathbb{Z}$-sequence. Note that unlike all the other statements, we do not have a sharp threshold at $\log r$; for $m = c \log r$ (where $c$ is the relevant constant in the statement of the theorem) the probability that a random $\mathbb{Z}_n$-sequence of length $m$ is not a weighted Davenport $\mathbb{Z}$-sequence is in fact bounded away from 1. Moreover, this last statement does not hold for all square-free $n$ with $r$
distinct prime factors. Although our result is stated in a more concrete form for \( n \), our method of proof will indicate that a corresponding statement holds for several other \( n \) as well. At the moment we can only conjecture what exactly dictates the form of \( n \) for which we have this corresponding weak threshold statement. But on a more definitive note, we do show that the upper bound can be very far from sharp for several forms of \( n \), so in that sense our results are somewhat (upto constant factors) tight.

An interesting aspect of our results is in the nature of the results that exhibit a contrast between what we may call the deterministic case versus the random case. For instance the Davenport constant of \( \mathbb{Z}_n \) equals \( n \), whereas for a random sequence, one only requires (with high probability) a sequence of size about \( \log_2 n \) for it to be a Davenport \( \mathbb{Z} \)-sequence. The same contrast works for the weighted Davenport sequence with weight set \( A \) of \( n \) different elements. Although our result is stated in a more concrete form for \( n \), we necessarily have \( |A| + |B| = n \), and consider the sets \( A = \{a_1, a_2, \ldots, a_k, a_{2k+1}, \ldots, a_{2n+3}\} \) and \( B = \{a_1, a_2, \ldots, a_k, a_{2k+(l-1)/2}, \ldots, a_{2k+l}\} \). Clearly, \( |A'\) = \( n + 1 \) and \( |B'\) = \( n + 2 \). Let \( A' \) denote the set of all possible sums of \( n \) different elements of \( A \); similarly, let \( B' \) denote the set of all possible sums of \( n + 1 \) distinct elements of \( B \). Clearly, \( |A'| = n + 1 \) and \( |B'| = n + 2 \), so \( |A'| + |B'| = 2n + 3 \). Therefore by the observation at the beginning of this section, it follows that \( S \) admits a non-trivial zero-sum subsequence, and consequently, \( s^{(2)}(\mathbb{Z}_n) \leq 2n + 3 \).

## 2 Proof of Theorem 1

We start with a simple observation. For any finite abelian group \( G \), and \( A, B \subseteq G \) satisfying \( |A| + |B| > |G| \) we necessarily have \( A + B = G \). This follows since for any \( x \in G \) we have \( |A| + |x - B| > |G| \), so \( A \cap (x - B) \neq \emptyset \), and that implies that \( x \in A + B \). Since \( x \) was arbitrary, the observation follows.

**Proof.** 1. We begin with the proof of the first part, and we shall deal with the case where \( n \) is even, or \( n \) is odd separately. We start with the odd case. Let \( x = (a_1, a_1, \ldots, a_k, a_k, a_{2k+1}, \ldots, a_{2n+3}) \) be a \( \mathbb{Z}_{2n+1} \)-sequence of length \( 2n + 3 \); here each \( a_i \) appears twice (for \( 1 \leq i \leq k \) and the elements \( a_{2k+1}, \ldots, a_{2n+3} \) are distinct and the elements \( a_i \) are pairwise disjoint. Set \( l = 2(n - k) + 3 \), and consider the sets \( A = \{a_1, a_2, \ldots, a_k, a_{2k+1}, \ldots, a_{2k+(l-1)/2}\} \) and \( B = \{a_1, a_2, \ldots, a_k, a_{2k+(l-1)/2}, \ldots, a_{2k+l}\} \). Clearly, \( |A| = n + 1 \) and \( |B| = n + 2 \). Let \( A' \) denote the set of all possible sums of \( n \) different elements of \( A \); similarly, let \( B' \) denote the set of all possible sums of \( n + 1 \) distinct elements of \( B \). Clearly, \( |A'| = n + 1 \) and \( |B'| = n + 2 \), so \( |A'| + |B'| = 2n + 3 \). Therefore by the observation at the beginning of this section, it follows that \( S \) admits a non-trivial zero-sum subsequence, and consequently, \( s^{(2)}(\mathbb{Z}_n) \leq 2n + 3 \).
To show that \( s(2)(\mathbb{Z}_n) \geq 2n + 3 \), consider the sequence \( x = (1, 2, \ldots, n-1, n, n, n+1, n+2, \ldots, 2n) \). \( x \) is a sequence of length \( 2n + 2 \) that has the property that the sum of all elements of the sequence is 0 but no element is equal to 0. Moreover, no element appears in \( x \) more than twice, so this establishes that \( s(2)(\mathbb{Z}_n) \geq 2n + 3 \). Consequently, \( s(2)(\mathbb{Z}_{2n+1}) = 2n + 3 \).

For the even case, as before, let \( x = (a_1, a_1, \ldots, a_k, a_k, a_{2k+1}, \ldots, a_{2n+2}) \) be a \( \mathbb{Z}_{2n} \)-sequence of length \( 2n + 2 \). Let \( l = 2(n - k) + 2 \), and as before, let \( A = \{a_1, a_2, \ldots, a_k, a_{2k+1}, \ldots, a_{2k+l/2}\} \) and \( B = \{a_1, a_2, \ldots, a_k, a_{2k+l/2+1}, \ldots, a_{2k+l}\} \). Note that these are well defined since \( l \) is even. As before define \( A' \) and \( B' \) as the sets of sums of \( n \) distinct elements of \( A \) and \( B \) respectively. Since \( |A| = n + 1 \) and \( |B| = n + 1 \), we have \( |A'| = n + 1 \) and \( |B'| = n + 1 \), so that \( |A'| + |B'| = 2n + 2 > 2n \), and by the observation, it follows that \( s(2)(\mathbb{Z}_n) \leq 2n + 2 \).

To complete the proof of the theorem, consider the sequence \( x = (1, 2, 3, 1, 3, 5, 6, \ldots, n - 1, n, n, n + 1, n + 2, \ldots, 2n - 1) \) of length \( 2n + 1 \); the sum of all elements of the sequence equals 0 but since 0 is itself not in \( x \), it follows that \( s(2)(\mathbb{Z}_n) \geq 2n + 2 \). This completes the proof.

2. We now turn to prove the second part of the theorem. We start with the proof of the upper bound which we shall prove by induction on \( k \).

The case of \( k = 2 \) is just a special case of the first part of theorem \([1]\) that was proved above. Suppose now that \( k \geq 3 \), and suppose that the statement holds for values less than \( k \).

Let \( a \) be a sequence of size \( p + k \), where each element appears at most \( k \) times. If no element in \( a \) has multiplicity \( k \), then since \( p + k > p + k - 1 \), we are through by induction, so we may assume that there is at least one element in \( a \) that appears \( k \) times.

Without loss of generality let us write

\[
a = \left( a_1^k, a_2^k, \ldots, a_l^k, a_{l+1}^{k-1}, \ldots, a_{l+t}^{k-1}, \ldots, a_{l+t}^{k-1} \right),
\]

where the \( a_i \) are distinct for 1 \( \leq i \leq l \) and \( k l + 1 \leq k + k - 1 \).

Consider the sets \( A_i = \{a_1, a_2, \ldots, a_{l+i-k}\} \), for 1 \( \leq i \leq k \). By the Cauchy-Davenport theorem (see \([13]\) for instance) we have,

\[
|\sum_{i=1}^k A_i| \geq \min(p, \sum_{i=1}^k |A_i| - k + 1) = \min(p, p + k - k + 1) = p.
\]

In particular we have \( \sum_{i=1}^k A_i = \mathbb{Z}_p \) which implies that the sum of the elements of \( a \) is also in \( \sum_{i=1}^k A_i \). Consequently, \( s(k)(\mathbb{Z}_p) \leq p + k \), and the induction is complete.

To complete the proof of the second part, we shall construct a sequence \( x \) of length \( p + k - 1 \) such that

i. No element of \( \mathbb{Z}_n \) appearing with multiplicity greater than \( k \) in \( x \)

ii. No subsequence of \( x \) of size \( p \) sums to zero.

Towards that end, we shall in particular, consider \( x \) to be of the form \( x = (-1)^k, a_1, a_2, \ldots, a_{p-1} \).

To establish what we seek of this sequence, we shall impose a couple of further restrictions on the \( a_i \). We shall choose the sequence \( a := (a_1, \ldots, a_{p-1}) \) (with no element of \( \mathbb{Z}_p \) appearing more than \( k \) times in \( a \)) such that

(a) \( a_{[1, p-1]} = 0 \); equivalently, the sum of the elements of \( x \) equals \(-k\).
(b) For each $1 \leq r \leq k$, and any $J \subset \{1, p - 1\}$ with $|J| = p - r$, we must have $a_J \neq r$. In other words, no subsequence of the $a_i$ of size $p - r$ has sum $r$.

It is a straightforward check to see that if the $a_i$ satisfy constraints (2a) and (2b), then indeed, no subsequence of $S$ of size $p$ has zero sum.

Recasting constraint (2b) gives us the equivalent formulation:

$$\text{For any } J \subset \{1, p - 1\} \text{ with } |J| = s, \text{ we have } a_J \neq -(s + 1) \text{ for } 1 \leq s \leq k - 1. \quad (1)$$

Suppose $p$ is large enough (we will make this more precise soon). Write $p = (k - 1)l + r$ for some $l$, and $0 < r < k - 1$, so that $l = \frac{p - r}{k - 1}$. By setting $s = 1$ in (1) it follows that $a_i \neq -2$ for each $i$; setting $s = 2$ in (1) gives us that, from each of the pairs $(0, p - 1), (1, p - 2), \cdots, (\frac{p - 5}{2}, \frac{p - 1}{2}),$ at most one element is picked.

Let us pick among the sequence $0^{(k - 1)}1^{(k - 1)} \cdots (l - 2)^{(k - 1)}$ a subsequence of the $a_i$; that leaves a further choice of $p - 1 - (k - 1)(l - 1) = r + k - 2$ elements to make the sequence $a_i$. Let us denote the remaining elements (the ones that need to be picked) by $x_1, \ldots, x_{r + k - 2}$. We shall pick $x_i \leq l - 2$; once that is the case, then for any $s \leq k - 1$ of the $a_i$’s, their sum is at most $(k - 1)(l - 2) < p - k$, so that constraint (2b) is satisfied.

By virtue of constraint (2a) it follows that we need

$$\sum_{i=1}^{r+k-2} x_i = -(k - 1)(1 + 2 + \cdots + (l - 2)) = \frac{-(p - r - (k - 1))(p - r - 2(k - 1))}{2(k - 1)}$$

in $\mathbb{Z}_p$. Now consider the possible two cases: $p \equiv r \pmod{2(k - 1)}$ or $p \equiv r + k - 1 \pmod{2(k - 1)}$.

In the first case,

$$-(p - r - (k - 1))(p - r - 2(k - 1)) = -(r + k - 1)(t - 1)$$

where $t = \frac{p - r}{2(k - 1)}$, so we need to pick $x_i$ ($1 \leq i \leq r + k - 2$) distinct with $0 \leq x_i \leq l - 2$ and

$$\sum_{i=1}^{r+k-2} x_i = (r + k - 1)(t - 1).$$

Since $l = 2t$, we make the choice as follows. Set $x_{p - 1} = 2t - (k - 1), x_{p - 2} = 2t - (k - 2), x_{p - 3} = k - 2 - r$, and set

$$x_{2i - 1} = t - i, x_{2i} = t + i \text{ for } 1 \leq i \leq \frac{r + k - 5}{2}, \quad \text{if } r + k \text{ is odd}$$

$$x_{2i - 1} = t - i, x_{2i} = t + i \text{ for } 1 \leq i \leq \frac{r + k - 6}{2}, x_{p - 4} = t, \quad \text{if } r + k \text{ is even}.$$

It is a straightforward check to see that these $x_i$ satisfy our constraints. Moreover, if $p$ is large enough, then $t + \frac{r + k - 5}{2} < 2t - (k - 2)$ and $k - 2 - r < t - \frac{r + k - 5}{2}$, so all these choices for $x_i$ are also pairwise distinct.

Now suppose $p \equiv r + k - 1 \pmod{2(k - 1)}$; in this case, we write

$$\frac{-(p - r - (k - 1))(p - r - 2(k - 1))}{2(k - 1)} = -t(r + 2(k - 1))$$

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3. Before we prove the third part of theorem 1 we make a couple of remarks. A conjecture of Erdős and Eggleston:

\[ p \]  

\[ (x_1, \ldots, x_r) \]  

\[ \sum_{i=1}^{r+k-2} x_i = t(r + 2(k - 1)) = tr + p - (r + k - 1) = (t - 1)r - k + 1. \]

Again, this is quite easy to achieve. For instance, set \( a = \left\lfloor \frac{(t-2)r}{t+k-3} \right\rfloor \), and consider \( x_i = a - i \) for \( 1 \leq i \leq r + k - 3 \). This achieves

\[ \sum_{i=1}^{r+k-3} x_i = (t-2)a - \frac{(r+k-3)(r+k-2)}{2} - s(r,k) \]

for some \( 0 \leq s(r,k) < r + k - 3 \). Now pick \( x_{r+k-2} \) so that the summation criterion above is fulfilled. Note that this expression for \( x_{r+k-2} \) depends only on \( r, k \) so that if \( p \) is sufficiently large, \( x_{r+k-2} < a - (r + k - 3) \). In consequence, the \( x_i \) are all distinct, and satisfy all our requirements; this completes the proof.

**Remark:** A quick glance at this proof suggests that our construction of the sequence \( a \) has nothing remotely canonical about it, and in fact there are several such examples. Also, since we make no attempt to optimize for \( p_0(k) \) (in the statement of part 2 of theorem 1), it should be possible to find ‘better’ examples than ours. For instance, our results (upon a little careful scrutiny) reveal that we need \( p_0(k) = \Theta(k^2) \), and it should be possible to improve upon this drastically.

3. Before we prove the third part of theorem 1 we make a couple of remarks. A conjecture of Erdős-Heilbronn, which was settled by Szemerédi \([14]\) states that there exists an absolute constant \( C > 0 \) such that for any abelian group \( G \) of order \( n \), and any subset \( A \subseteq G \) with \( |A| \geq C\sqrt{n} \), there exists some non-trivial subset of \( A \) the sum of whose elements equals zero. Szemerédi remarks \([14]\) that the same methods actually can be extended to also settle another conjecture of Erdős and Eggleston:

**Theorem 4.** There exists an absolute constant \( \varepsilon_0 > 0 \) such that if \( A \subseteq G \) and \( \mathcal{S}(A) \) denotes the set of all sums of elements over all non-trivial subsets of \( A \), then either \( 0 \in \mathcal{S}(A) \) or \( |\mathcal{S}(A)| \geq \varepsilon_0 |A|^2 \).

We shall include a sketch of the proof of this theorem in the appendix.

We will also need another result due to Scherk, settling a problem proposed by L. Moser:

**Theorem 5.** \([13, 16]\) Suppose \( A, B \subseteq \mathbb{Z}_n \) such that \( 0 \in A \cap B \), and suppose the equation \( a + b = 0 \) with \( a \in A, b \in B \) has the unique solution \( a = b = 0 \), then \( |A + B| \geq \min\{|A| + |B| - 1, n\} \).

We start with the lower bound. We shall drop ceiling and floors to make the presentation clear. Consider the sequence \( x = (1^k, 2^k, \ldots, t^k) \), where \( \frac{kt(t+1)}{2} < \frac{n}{4} \). Set \( N = kt \); note that this gives \( N = \Omega(\sqrt{nk}) \). Then note that for any non-trivial \( I \subset [N], 0 < x_I < n/2 \) by choice, so \( x \) does not admit a zero-sum subsequence, and that establishes the lower bound.

Let \( C \) be the constant from Szemerédi’s theorem and let \( \varepsilon_0 \) be the constant from theorem 4. Let \( C^* = 2C/\varepsilon_0 \). We claim that if \( x \) is a \( k \)-restricted \( \mathbb{Z}_n \)-sequence of length \( m = C^*\sqrt{nk} \), then \( x \) is a Davenport \( \mathbb{Z} \)-sequence, and we shall prove this by induction on \( k \). The case of
Proof. Let \( x = (x_1^1, x_2^1, \ldots, x_r^r) \) with \( k \geq \ell_1 \geq \cdots \geq \ell_r \). If \( k > \ell_1 \), then by induction, since \( m \geq C^* \sqrt{n(k-1)} \) so \( x \) is a Davenport Z-sequence. So, we may assume that \( \ell_1 = k \).

Let \( A \) denote the \( r \times k \) array with the \( i \)-th row of \( A \) consisting of the \( \ell_i \) copies of \( x_i \) (with the last \( k - \ell_i \) entries of \( A \) being empty), and let \( R_i \) (\( 1 \leq i \leq k \)) denote the sets comprising of the columns of \( A \). By the preceding discussions, we have \( |R_1| \geq \cdots \geq |R_k| \geq 1 \). Write \( |R_i| = r_i \).

If \(|R_1| \geq C \sqrt{n}\) then by Szemerédi’s theorem \( 0 \in S(R_1) \), and so we are done. So we may assume that \(|R_1| < C \sqrt{n}\). If \( r_k \leq \frac{C^* \sqrt{n}}{\sqrt{k+\sqrt{k}-1}} \) then \( x \) contains a \((k-1)\)-restricted subsequence of length at least \( C^* \sqrt{nk} - r_k \geq C^* \sqrt{n(k-1)} \), so by induction, \( x \) is a Davenport Z-sequence. So, again, we may assume that \( r_k > \frac{C^* \sqrt{n}}{\sqrt{k+\sqrt{k}-1}} \).

But if \( r_k > \frac{C^* \sqrt{n}}{\sqrt{k+\sqrt{k}-1}} \geq (C/\varepsilon_0) \sqrt{n/k} = t \), say, then in particular means that \( \ell_1 = \cdots = \ell_t = k \).

Let \( B_i = S(A_i) \cup \{0\} \) for \( i = 1, \ldots, k \). By theorem 4 either \( 0 \in S(A_i) \) and we are through, or \( |B_i| \geq \varepsilon_0|A_i|^2 \) for each \( i \). Suppose the \( 0 = b_1 + \cdots + b_k \) holds for \( b_i \in B_i \) with at least one of the \( b_i \neq 0 \), then we are done. So we may assume that \( 0 = b_1 + \cdots + b_k \) implies that \( b_i = 0 \) for each \( i \). Then by the result of Scherk (theorem 5),

\[
|\sum_{i=1}^{k} B_i| \geq \min \left( n, \sum_{i=1}^{k} \varepsilon_0|A_i|^2 - k + 1 \right) > n
\]

and that is a contradiction. This completes the proof of the induction, and the theorem as well.

\[\square\]

Remark: The third part of theorem 1 considers the problem of how large a \( Z_n \)-sequence needs to so that it is a Davenport Z-sequence. The analogous problem for weighted Davenport Z-sequences for the set \( A = \{-1, 1\} \) does not offer anything substantially new. Indeed, by the result in [2], any sequence of size \( \lceil \log_2 n \rceil + 1 \) admits an \( A \)-weighted Davenport Z-sequence. Furthermore, the sequence \( x = (1, 2, \ldots, 2^r) \) for \( r = \lceil \log_2 n \rceil - 1 \) has the property that no two distinct subsequences of \( x \) have the same sum, so in particular, \( x \) is not a weighted \( Z \)-sequence.

3 Proof of Theorem 2

Proof. 1. Let \( X = (X_1, \ldots, X_{n+2}) \) be a random \( Z_n \)-sequence. Let

\[
\mathcal{H} := \{ I \subset [n+2] : |I| = n \},
\]

\[
N := \sum_{I \in \mathcal{H}} \mathbb{I}(X_I).
\]

Here \( \mathbb{I}(X_I) = 1 \) if \( X_I = 0 \) and zero otherwise. Then

\[
\mathbb{E}(N) = \sum_{I \in \mathcal{H}} \mathbb{P}(X_I = 0) = \frac{1}{n} \binom{n+2}{n} = \frac{(n+2)(n+1)}{2n}.
\]
and,
\[ Var(N) = \sum_{I \in \mathcal{H}} Var(\mathbb{I}(X_I)) + \sum_{I \neq J, I,J \in \mathcal{H}} \text{Cov}(\mathbb{I}(X_I), \mathbb{I}(X_J)). \]

The main observation is that since \(X_i\)'s are i.i.d, it follows that the \(X_I\) are pairwise independent. Indeed pick \(i \in I \setminus J\) and \(j \in J \setminus I\) and condition on \(\{X_I\}_{I \neq i,j}\); this determines \(X_i, X_j\) uniquely, so the conditional (and hence also the unconditional probability) of \(X_I = X_J = 0\) is \(1/n^2 = \mathbb{P}(X_I = 0) \cdot \mathbb{P}(X_J = 0)\). Consequently, \(\text{Cov}(\mathbb{I}(X_I), \mathbb{I}(X_J)) = 0\) for \(I \neq J \in \mathcal{H}\). Also, \(Var(\mathbb{I}(X_I)) = \frac{1}{n}(1 - \frac{1}{n})\), so
\[ Var(N) = \sum_{I \in \mathcal{H}} Var(\mathbb{I}(X_I)) = \frac{1}{n} \left( 1 - \frac{1}{n} \right) \frac{(n+2)(n+1)}{2}. \]

Therefore, by Chebyshev’s inequality we have,
\[ \mathbb{P}(\lvert N - \mathbb{E}(N) \rvert > \mathbb{E}(N)) \leq \frac{Var(N)}{(\mathbb{E}(N))^2} = \frac{\frac{1}{n}(1 - \frac{1}{n})}{\frac{4}{n}(1 + \frac{4}{n})} = O \left( \frac{1}{n} \right). \]

This implies \(\mathbb{P}(N > 0) \to 1\). This completes the proof.

2. Let \(X = (X_1, \ldots, X_n)\) be a random \(\mathbb{Z}_n\)-sequence and \(A = \{a, b\}\) with \(\{a, a+b, a-b\} \subset \mathbb{Z}_n^*\).

Again
\[ \mathcal{H} = \{ I \subset [n] \} \]
\[ N = \sum_{I \in \mathcal{H}} \mathbb{I}(X_I = -(b/a)X(I)) \]

with \(\mathbb{I}(X_I = -(b/a)X(I))\) being the corresponding indicator function, and \(\bar{T} := [n] \setminus I\).

Again,
\[ \mathbb{E}(N) = \sum_{I \in \mathcal{H}} \mathbb{P}(X(I) = -(b/a)X(I)) = \frac{2^n}{n}, \]

and,
\[ Var(N) = \sum_{I \in \mathcal{H}} Var(\mathbb{I}(X(I) = -(b/a)X(I))) + \sum_{I \neq J, I,J \in \mathcal{H}} \text{Cov}(\mathbb{I}(X(I) = -(b/a)X(I)), \mathbb{I}(X(J) = -(b/a)X(J))). \]

We claim that in this case too \(\text{Cov}(\mathbb{I}(X(I) = -(b/a)X(I)), \mathbb{I}(X(J) = -(b/a)X(J))) = 0\) when \(I \neq J\). Note that
\[ Var(\mathbb{I}(X(I) = -(b/a)X(I))) = \frac{1}{n}(1 - \frac{1}{n}) \]

so
\[ Var(N) = \sum_{I \in \mathcal{H}} Var(\mathbb{I}(X(I) = -(b/a)X(I))) = \frac{1}{n}(1 - \frac{1}{n})(n+1)2^n \]

so again by Chebyshev’s inequality,
\[ \mathbb{P}(\lvert N - \mathbb{E}(N) \rvert > \mathbb{E}(N)) \leq \frac{2^n(1 + \frac{1}{n})(1 - \frac{1}{n})}{2^{2n}(1 + \frac{4}{n})^2} = O(2^{-n}) \]
which gives us what we seek.

So to complete the proof, we need to establish the aforementioned claim.

\[
\text{Cov}(\mathbb{I}_{\mathcal{X}(I) = -(b/a)\mathcal{X}(J)}, \mathbb{I}_{\mathcal{X}(J) = -(b/a)\mathcal{X}(J)}) = \mathbb{P}\left(\mathbb{I}(I) = -(b/a)\mathcal{X}(J) \land \mathbb{I}(J) = -(b/a)\mathcal{X}(I)\right) - \mathbb{P}\left(\mathbb{I}(I) = -(b/a)\mathcal{X}(J)\right) \cdot \mathbb{P}\left(\mathbb{I}(J) = -(b/a)\mathcal{X}(I)\right)
\]

We consider the following cases:

i) \(I \cap J \neq \emptyset\) and \(\overline{I} \cap J \neq \emptyset\): Suppose \(i \in I \cap J\) and \(j \in J \cap \overline{I}\). Then conditioning on \(X_I\) for all \(\ell \neq i, j\) gives us two linear equations of the form \(aX_i + bX_j = \xi_1, aX_i + aX_j = \xi_2\) and since by assumption \(a^2 - b^2 \in \mathbb{Z}_n^\times\), there is a unique choice for \((X_i, X_j)\) that satisfies these conditions, so the conditional probability equals \(1/n^2\), so it follows that the probability in the first term equals \(1/n^2\), so that completes the proof.

ii) Suppose \(I \subset J\), say. Pick \(i \in I, j \in J \setminus I\). Then as before, conditioning on \(X_\ell\) for all \(\ell \neq i, j\) gives us two linear equations of the form \(aX_i + bX_j = \xi_1, aX_i + aX_j = \xi_2\). Again, since \(a - b \in \mathbb{Z}_n^\times\), this gives a unique choice for \((X_i, X_j)\) satisfying these equations. The proof then proceeds as before. If \(J \subset I\), then the proof is similar.

3. Let \(\mathcal{X} = (X_1, \ldots, X_{n+1})\) be a random \(\mathbb{Z}_n\)-sequence, and define

\[
\mathcal{H} := \{I, J \subset [n+1], I, J \neq \emptyset, I \cap J = \emptyset, |I \cup J| = n\},
\]

\[
N := \sum_{(I, J) \in \mathcal{H}} \mathbb{I}(\mathcal{X}_{I, J}),
\]

where as before \(\mathbb{I}(\mathcal{X}_{I, J})\) is the indicator function that equals one if \(\mathcal{X}_I = \mathcal{X}_J\) and equals zero otherwise. Note that \(|\mathcal{H}| = (n+1)(2^n - 2)\).

Again,

\[
\mathbb{E}(N) = \sum_{(I, J) \in \mathcal{H}} \mathbb{P}(\mathcal{X}_I = \mathcal{X}_J) = \frac{1}{n} \left(\begin{array}{c} n+1 \\ n \end{array}\right) (2^n - 2) = \frac{(n+1)(2^n - 2)}{n}.
\]

Again,

\[
\text{Var}(N) = \sum_{(I, J) \in \mathcal{H}} \text{Var}(\mathbb{I}(\mathcal{X}_{I, J})) + \sum_{(I, J) \neq (I', J')} \text{Cov}(\mathbb{I}(\mathcal{X}_{I, J}), \mathbb{I}(\mathcal{X}_{I', J'})).
\]

Unlike the previous cases, we do not always have pairwise independence of the random variables \(\mathcal{X}_{I, J}\) in this case. But there still are many pairs \((I, J), (I', J')\) that are pairwise independent and that is sufficient for our purpose here.

Suppose \((I, J) \neq (I', J')\) are pairs in \(\mathcal{H}\) such that \(I \cup J = [n+1] \setminus \{a\}\) and \(I' \cup J' = [n+1] \setminus \{b\}\) with \(a \neq b\). We claim that the random variables \(\mathbb{I}(\mathcal{X}_{I, J})\) and \(\mathbb{I}(\mathcal{X}_{I', J'})\) are independent. Without loss of generality, suppose \(a \in I'\) and \(b \in I\). Then conditioning on \(X_i\) for \(i \neq a, b\) gives us two equations of the form \(X_a = \mathcal{X}_{J', J} - \mathcal{X}_{I', \{a\}}\) and \(X_b = \mathcal{X}_{J'} - \mathcal{X}_{I, \{b\}}\), which admits a unique solution for the pair \((X_a, X_b)\); consequently, the probability that \(\mathbb{I}(\mathcal{X}_{I, J})\) and \(\mathbb{I}(\mathcal{X}_{I', J'})\) hold conditioned on the values of \(\{X_i\}_{i \neq a, b}\) equals \(1/n^2\), from which it follows that the unconditional probability also equals \(1/n^2\), and that proves that \(\mathcal{X}_{I, J}, \mathcal{X}_{I', J'}\) are independent. In fact, if \(n\) is
odd, then $X_{I,J}$ are all pairwise independent. Indeed, suppose $I \cup J = I' \cup J' = [n]$ (without loss of generality). Pick $i \in I \cap I'$ and $j \in I \cap J'$; again, since at least one of $i, J$ meets both $I'$ and $J'$ nontrivially, so we may assume that it is $I$. Then as before, conditioning on all $X_\ell$ for $\ell \neq i, j$ gives us two equations of the form $X_i + X_j = \xi_1, X_i - X_j = \xi_2$, and if $n$ is odd, this admits a unique solution for $(X_i, X_j)$ and that proves (as before) that $I(X_{I,J}), I(X_{I',J'})$ are independent.

If $n$ is even, then this might not admit any solution at all, in which case the pair $I(X_{I,J}), I(X_{I',J'})$ are negatively correlated. If $\xi_1 + \xi_2$ is even, then the pair of linear equations above admit two possible solutions for $X_i$ (say), and for each of these, a unique value for $X_j$. Consequently, in these cases $\text{Cov}(I(X_{I,J}), I(X_{I',J'})) = 2/n^2 - 1/n^2 = 1/n^2$.

Hence, plugging into the expression for the variance, we have
\[
\text{Var}(N) = \sum_{(I,J) \in \mathcal{H}} \text{Var}(I(X_{I,J})) + \sum_{(I,J) \neq (I',J')} \text{Cov}(I(X_{I,J}), I(X_{I',J'}))
\]
\[
= \frac{1}{n} \left(1 - \frac{1}{n}\right)(n+1)(2^n - 2) + \sum_{(I,J) \neq (I',J')} \text{Cov}(I(X_{I,J}), I(X_{I',J'}))
\]
\[
\leq \frac{2(n+1)(2^n - 2)(2^n - 3)}{n^2}.
\]
Therefore, by Chebyshev’s inequality,
\[
\mathbb{P}(|N - \mathbb{E}(N)| > \mathbb{E}(N)) \leq \frac{\text{Var}(N)}{(\mathbb{E}(N))^2} = O\left(\frac{1}{n}\right),
\]
so again, we have $\mathbb{P}(N > 0) \rightarrow 1$. This completes the proof.

This completes the proof of the claim, and that of the theorem as well.

\[\square\]

**Remark:** Since one needs a sequence of size at least $n$ in order that it is an Erdős Z-sequence, the results of the previous theorem assert that for random $\mathbb{Z}_n$-sequences one does not need much more than the minimum required size for it to be an Erdős Z-sequence whp. One of the immediate consequences of Theorem 2 is that for ‘most’ sets $A$, a random $\mathbb{Z}_n$-sequence of length $n$ is whp an Erdős Z-sequence. This contrasts rather sharply with the fact that $E_A(\mathbb{Z}_n) = n + \lceil \log_2 n \rceil$ for $A = \{-1,1\}$.

### 4 Proof of Theorem 3

Let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be a random $\mathbb{Z}_n$-sequence. The proof methods of this theorem are similar to the ones of the preceding section. In contrast to the results of the previous theorem the results here witness a sharp threshold.

Let $\omega(n)$ be an arbitrary function satisfying $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$. 
Proof. 1. We start with the lower bound. Let \( m = \log_2 n - \omega(n) \). We wish to show that
\[
\Pr(\mathcal{X} \text{ is not a Davenport Z-sequence}) \rightarrow 0.
\]
Let
\[
\mathcal{H} = \{ I : I \subset [m] \}, \\
N = \sum_{I \in \mathcal{H}} \mathbb{I}(\mathcal{X}_I),
\]
where \( \mathbb{I}(\mathcal{X}_I) \) is the indicator function which equals 1 if \( \mathcal{X}_I = 0 \) and zero otherwise.
Then
\[
\mathbb{E}(N) = \sum_{I \in \mathcal{H}} \Pr(\mathcal{X}_I = 0) = \frac{2^m}{n} = \frac{2^{\log_2 n - \omega(n)}}{n} = \frac{1}{2^{\omega(n)}}
\]
and consequently it follows that \( N = 0 \) whp. This establishes what we seek.

For the upper bound, let \( m = \log_2 n + \omega(n) \); then
\[
\text{Var}(N) = \sum_{I \in \mathcal{H}} \text{Var}(\mathbb{I}(\mathcal{X}_I)) + \sum_{I \neq J \in \mathcal{H}} \text{Cov}(\mathbb{I}(\mathcal{X}_I), \mathbb{I}(\mathcal{X}_J)).
\]
Since the \( \mathcal{X}_I \)'s, by a very similar argument as in the proof of part 1 of theorem \([2]\) it follows that \( \mathcal{X}_I \)'s are pairwise independent, so that \( \text{Cov}(\mathbb{I}(\mathcal{X}_I), \mathbb{I}(\mathcal{X}_J)) = 0 \) for \( I \neq J \in \mathcal{H} \). Since
\[
\text{Var}(\mathbb{I}(\mathcal{X}_I)) = \frac{1}{n} \left( 1 - \frac{1}{n} \right) 2^m = \frac{n-1}{n^2} 2^{\log_2 n + \omega(n)},
\]
so by Chebyshev’s inequality
\[
\Pr(|N - \mathbb{E}(N)| > \mathbb{E}(N)) \leq \frac{1}{2^{\omega(n)}} \rightarrow 0
\]
as \( n \rightarrow \infty \). Hence \( \Pr(N > 0) \rightarrow 1 \) and this completes the proof of the upper bound, and the proof of the first part.

2. Again, first let \( m = \frac{1}{2} \log_2 n - \omega(n) \). We shall show that \( \text{whp} \ \mathcal{X} \) is not a Davenport Z-sequence. Let
\[
\mathcal{H} = \{(I, J) : I, J \subset [m], I \neq J \} \\
N = \sum_{(I, J) \in \mathcal{H}} \mathbb{I}(\mathcal{X}_{I,J}),
\]
where \( \mathbb{I}(\mathcal{X}_{I,J}) \) is the same indicator function as considered in the proof of part 3 of theorem \([2]\) Again,
\[
\mathbb{E}(N) = \sum_{(I, J) \in \mathcal{H}} \Pr(\mathcal{X}_I = \mathcal{X}_J) = \frac{2^m(2^m - 1)}{n} \leq \frac{2^{2m}}{n} \leq \frac{1}{4^\omega(n)}
\]
which implies \( \Pr(N > 0) \rightarrow 0 \), or equivalently, \( N = 0 \) whp. Now suppose \( m = \frac{1}{2} \log_2 n + \omega(n) \). Again,
\[
\text{Var}(N) = \sum_{(I, J) \in \mathcal{H}} \text{Var}(\mathbb{I}(\mathcal{X}_{I,J})) + \sum_{(I, J), (I', J') \in \mathcal{H}} \text{Cov}(\mathbb{I}(\mathcal{X}_{I,J}), \mathbb{I}(\mathcal{X}_{I',J'})).
\]
As before, we claim that $\mathbb{I}(X_{I,J}), \mathbb{I}(X_{I',J'})$ are independent unless one of the sets of one pair (say $J$) equals the union of the other two sets (i.e., in this case, $J = I' \cup J'$), and in this case (as in the proof of part 3 of theorem 2 again, the covariance is non-zero only when $n$ is even) we have $\text{Cov}(\mathbb{I}(X_{I,J}), \mathbb{I}(X_{I',J'})) \leq 1/n^2$, so the total contribution from the covariances equals

$$
\sum_{I,I',J' \subset [m]} \text{Cov}(\mathbb{I}(X_{I,J}), \mathbb{I}(X_{I',J'})) = O\left(\frac{2^m}{n^2}\right).
$$

Hence if the claim is established, by Chebyshev’s inequality,

$$
\mathbb{P}(|N - \mathbb{E}(N)| > \mathbb{E}(N)) \leq \sum_{(I,J), (I',J') \in \mathcal{H}} \text{Cov}(\mathbb{I}(X_{I,J}), \mathbb{I}(X_{I',J'})) + \sum_{I,I',J' \neq \emptyset} \text{Cov}(\mathbb{I}(X_{I,J}), \mathbb{I}(X_{I',J'}))
\leq \frac{1}{4^ω(n)} + O(2^{-m}) \leq O(4^{-ω(n)}),
$$

so again $\mathbb{P}(N > 0) \to \infty$, and that completes the proof.

To prove the claim, suppose first that one of $(I, J)$ (say $I$) and one of $(I', J')$ (say $I'$) satisfy that there exists $i \in I \setminus I'$ and $j \in I' \setminus I$, (both these sets are non-empty) then conditioning on $X_\ell$ for all $\ell \neq i,j$ yields a unique solution for $X_i$ and $X_j$ when we set $X_I = X_J$ and $X_{I'} = X_{J'}$, and that establishes the independence. If this condition does not hold then without loss of generality, one of the following holds: $I \subseteq I' \subseteq J$ or $I \subseteq I' \cap J' \subseteq I' \cup J' \subseteq J$. In the first case, suppose first that both $J \subseteq J'$ and $I' \subseteq I$; in this case we pick $i \in I \setminus I'$ and $j \in J \setminus J'$, and condition on all $X_\ell$ for $\ell \neq i,j$ and argue as before. If say $J = J'$ then since $I' \neq J'$ we must have $J \setminus I' \neq \emptyset$, and $I' \setminus I \neq \emptyset$. In this case, we pick $i \in I' \setminus I$, $j \in J \setminus I'$ and again condition on $X_\ell$ for all $\ell \neq i,j$. The two conditions are in this case equivalent to $X_{I' \setminus I} = 0$ and $X_{J \setminus J'} = 0$, so that there is a unique solution to $(X_i, X_j)$.

In the second scenario, if $I' \cup J' \subseteq J$ pick $i \in I \setminus (I' \cup J')$ and $j \in I' \setminus J'$ (say) and argue as before. This exhausts all the possibilities, and the proof is complete.

3. Suppose $n = \prod_{i=1}^r p_i$, where $p_1 < \cdots < p_r$ are primes and let $m = C \log r$, where $C$ is a sufficiently large constant that shall be specified later. As before, let $\mathcal{X} = (X_1, \ldots, X_m)$ be a random $\mathbb{Z}_n$-sequence. We shall show that whp $\mathcal{X}$ is a weighted Davenport $\mathbb{Z}_r$-sequence w.r.t. $A = \mathbb{Z}_n^*$. 

First note that every $x \in \mathbb{Z}_n \setminus \{0\}$ can be written as $x = a \cdot p_A$ for some $A \subset [r]$. Here, $p_A$ denotes $\prod_{i \in A} p_i$. If $A = \emptyset$, then $x \in \mathbb{Z}_n^*$. We start with a lemma.

**Lemma 6.** Suppose $n = \prod_{i=1}^r p_i$, where $p_1 < \cdots < p_r$ are odd primes. Let $\mathbf{x} = (x_1, \ldots, x_m)$ be a $\mathbb{Z}_n$-sequence of pairwise distinct elements of $\mathbb{Z}_n$ such that for each $i$, $x_i = u_i p_{A_i}$ for some sets $A_i \subset \{1, 2, \ldots, r\}$, and $u_i \in \mathbb{Z}_n^*$. If for each $i$ we have $\cap_{j \neq i} A_j = \emptyset$, there exists $\mathbf{a} \in (\mathbb{Z}_n^*)^m$ such that $\langle \mathbf{a}, \mathbf{x} \rangle = 0$ in $\mathbb{Z}_n$.

**Proof.** (Proof of the lemma) First we may assume that $u_i = 1$ for all $i$; if the lemma is proved in this case, i.e., if there exist $a_i \in \mathbb{Z}_n^*$ satisfying the lemma in this case, then in the general case, $b_i = a_i u_i^{-1}$ will satisfy the same. For $m = 2$ the hypothesis implies that $A_1 = A_2 = \emptyset$ so $x_1, x_2 \in \mathbb{Z}_n^*$. Hence taking $a_1 = -x_2, a_2 = x_1$ proves the statement.

Suppose $m > 2$. By the Chinese Remainder theorem, we have $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ as rings, and also $\mathbb{Z}_n^* \cong \mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_r}^*$. Write $a_i = (a_{i1}, \ldots, a_{ir})$ with $a_{ij} \in \mathbb{Z}_{p_i}^*$ for each $i$, and similarly,
let \( x_i = (x_{i1}, \ldots, x_{ir}) \). Since \( x_i = p_{A_i} \) for some sets \( A_i \), we have \( x_{ij} = 0 \) if \( j \notin A_i \) and \( x_{ij} \in \mathbb{Z}^*_p \) otherwise. Consider the equations \( \sum_{i=1}^m a_{ij}x_{ij} = 0 \) in \( \mathbb{Z}_{p_j} \), for \( 1 \leq j \leq r \). By the hypothesis, each such equation admits at least two non-zero \( x_{ij} \). Then by the Cauchy-Davenport theorem we have
\[
\sum_{i \in X_{ij} \neq 0} x_{ij} \cdot \mathbb{Z}^*_p = \mathbb{Z}^*_p,
\]
which implies that there exist \( a_{ij} \in \mathbb{Z}^*_p \) such that \( a \cdot x = 0 \) in \( \mathbb{Z}_n \). This completes the proof. \( \square \)

For a random \( \mathbb{Z}_n \)-sequence \( X \), we shall show that there exists \( I \subset [m] \) of size at least \( m/2 \) such that

i. For each \( 1 \leq i < j \leq m \), \( X_i \neq X_j \),

ii. For all \( i \in I \), \( X_i \)'s are square free, i.e., \( p_j^2 \) does not divide \( X_i \) for all \( 1 \leq j \leq r \),

iii. For each \( i \in I \), if we write \( X_i = u_i p_{A_i} \) for \( A_i \subset \{1, 2, \ldots, r\} \) then \( \cap_{j \neq i} A_i = \emptyset \)

holds with high probability.

First
\[
\mathbb{P}(X_i = X_j \text{ for some } 1 \leq i < j \leq m) \leq \frac{m^2}{2n} \leq \frac{C^2 \log^2 r}{2^r}
\]
so the first condition holds whp. Also, for a fixed \( 1 \leq i \leq m \),
\[
\mathbb{P}(X_i \text{ is not square-free}) \leq \left( \sum_{p \text{ prime}} \frac{1}{p^2} \right) \leq 0.4522.
\]

Let \( I = \{1 \leq i \leq r : X_i \text{ is square free}\} \). Then \( I \sim \text{Bin}(m, \theta) \) where \( \theta \geq 0.5478 \), so by the Chernoff bounds \([8]\) we have
\[
\mathbb{P} \left( |I| < \frac{9m}{20} \right) \leq 2e^{-m/600}.
\]
Fix a prime \( p \), condition on the set \( I \), and let \( N_p := |\{i \in I : p|X_i\}| \). Then \( N_p \sim \text{Bin}(|I|, \frac{1}{p} - \frac{1}{p^2}) \), and \( \mathbb{E}(N_p) \leq |I|/4 \). Furthermore, by the Chernoff bounds \([8]\)
\[
\mathbb{P} \left( N_p > \frac{3\mathbb{E}(N_p)}{2} \mid I \right) \leq 2e^{-\left(\frac{3\mathbb{E}(N_p)}{2}\right)}
\]
so that
\[
\mathbb{P} \left( N_p > \frac{3|I|}{8} \mid I \right) \leq re^{-C_1|I|} \leq re^{-C \log r}.
\]
So if \( p \geq C \log r \) for a suitably large \( C \) then the unconditional probabilit of the aforementioned event also is at most \( O(r^{-c}) \) for some constant \( c > 0 \) and so that establishes that whp all of [i], [ii], [iii] hold for some \( I \subset [m] \) with \( |I| \geq m/2 \). But then by the preceding lemma, it follows that \( X \) is a weighted Davenport \( \mathbb{Z}_n \) sequence (for \( A = \mathbb{Z}^*_n \)) whp.

Now, for the final part of the proof, let \( A_i \subset [r] \) be the set of primes associated with \( X_i \). Let \( x \) be a sufficiently large integer, and let \( n \) being the product of all the odd primes less than or
equal to $x$. We shall show that for some constant $c > 0$, and an absolute constant $\varepsilon > 0$, for $m = c \log \log x$, there is a rearrangement $\pi$ of the $X_i$ such that

$$\mathbb{P}(A_{\pi(1)} \subseteq \cdots \subseteq A_{\pi(m)}) \geq \varepsilon.$$ 

Note that by well known number theoretic estimates, $p_r \sim r \log r$, so this translates as $m = \Omega(\log r)$. For simplicity, let us write $n = \prod_{i=1}^{r} p_i$. If $X$ is picked uniformly from $\mathbb{Z}_n^*$ then

$$\mathbb{P}(X = up \mid u \in \mathbb{Z}_n^*) = \frac{\phi(n/p)}{n}.$$ 

This follows easily from the simple observation that the map $\Psi : \mathbb{Z}_n^* \rightarrow p\mathbb{Z}_n^*$ given by $\Psi(x) = px$ describes a bijective map.

Since the $X_i$ are picked uniformly and independently, it is a straightforward consequence to see that the distribution that induces on the subsets of $[r]$ can be described as follows:

$$\mathbb{P}(A) := \prod_{i \in A} \frac{1}{p_i} \prod_{i \notin A} (1 - \frac{1}{p_i}).$$ 

In other words, each element $i \in [r]$ is picked independently with probability $\frac{1}{p_i}$.

Given the random $\mathbb{Z}_n$-sequence $X$, define the $r \times m$ array $A = (x_{ij})$ where $x_i = (x_{i1}, \cdots, x_{im})$ where $x_{ij} = 1$ if $p_i \in A_j$ and 0 otherwise. Then $(A_1, \ldots, A_m)$ is a chain, i.e., $A_i \subseteq A_j$ for $i < j$ if and only if each $x_i$ is an increasing $0-1$ sequence.

For a fixed $i$,

$$\mathbb{P}(x_i \text{ is increasing}) = \left(1 - \frac{1}{p_i}\right)^m + \frac{1}{p_i} \left(1 - \frac{1}{p_i}\right)^{m-1} + \cdots + \frac{m}{p_i} = \frac{(p_i - 1)^{m+1} - 1}{(p_i - 2)p_i^{m+1}},$$

for $p_i \geq 3$. Hence

$$\mathbb{P}(x_i \text{ is increasing for all } i) = \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)^{m+1} \geq \prod_{i=1}^{r} \left(1 - \frac{1}{(p_i - 1)^2}\right) = \prod_{i=1}^{r} \left(1 - \frac{1}{n^2}\right) = 1/2.$$ 

The last inequality follows since $(p_i/(p_i - 2)) \geq 1/2$ and

$$\prod_{i=1}^{r} \left(1 - \frac{1}{p_i - 1}\right)^{m+1} \geq \prod_{i=1}^{\infty} \left(1 - \frac{1}{n^2}\right) = 1/2.$$ 

For a permutation $\sigma \in S_m$ let $\mathcal{E}_\sigma$ denote the event $A_{\sigma(1)} \subseteq \cdots \subseteq A_{\sigma(m)}$. Then

$$\mathbb{P}(\bigvee_{\sigma \in S_m} \mathcal{E}_\sigma) = m! \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)^{m+1} \left(1 - \frac{1}{p_i^{m+1}}\right) \left(\frac{p_i}{p_i - 2}\right).$$

Since

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + O(1)$$

(19) for example, or for a proof with a concrete bound of 6 for the $O(1)$ term, see [12] we have so

$$(m + 1) \sum_{i=1}^{r} \log \left(1 - \frac{1}{p_i}\right) \geq m \left(\sum_{i=1}^{r} \frac{1}{p_i} + O(1)\right) = \Theta(m(- \log \log r + O(1))).$$

Using Stirling’s approximation, we have

$$\text{RHS of (2)} \geq \exp(m \log m - m \log \log x + O(1)) = \Omega(1)$$

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for $m = c \log \log x$ for some suitably small $c > 0$. By the preceding discussions, this completes the proof.

\[ \square \]

**Remark:** As mentioned in the introduction, the bound $m = \Theta(\log r)$ in the aforementioned theorem does not always hold. Indeed, suppose $\varepsilon > 0$. We claim that for any $r$, there are square-free $n$ for which the random sequence $X = (X_1, X_2)$ is a weighted (w.r.t. $A = \mathbb{Z}_n^*$) Davenport $Z$-sequence in $\mathbb{Z}_n$ holds with probability at least $1 - \varepsilon$. Let $n = \prod_{i=1}^{r} p_i$, where $p_1 < \cdots < p_r$ satisfy where $p_i > p_1^{i}$, and $p_1$ is large enough so that $(1 - 1/p_1)^r > 1 - \varepsilon/2$. Then

$$
P(X_1, X_2 \in \mathbb{Z}_n^*) = \left( \frac{\phi(n)}{n} \right)^2 > 1 - \varepsilon,
$$

and consequently, with probability at least $1 - \varepsilon$, the sequence $X$ is a weighted Davenport $Z$-sequence as claimed.

### 5 Concluding remarks

- As we remarked, our proof of theorem 1, part 2, works only $p$ prime, since we invoke the Cauchy-Davenport theorem there. A more general result due to Kneser (see [15], chapter 2) provides a lower bound for all general $n$, but it is not clear how to improve our proof on to the general case, or even to the case $n = p^m$ for $m \geq 2$, and $p$ prime. But we believe the following conjecture holds:

**Conjecture 7.** For any integer $n$ and any integer $k$,

$$s^{(k)}(C_n) = n + k.
$$

Indeed, another aspect of the $k$-restricted sequence we constructed as an example in the proof of part 2 of theorem 1 was the somewhat ad-hoc nature of the example. It seems in fact that there exist several different examples of $k$-restricted sequences of length $p + k - 1$ that are not Erdős $Z$-sequences. Our examples use the fact that $p$ is prime; however we believe that it should be possible to modify it a bit so that it works for all $n$.

- Our proofs of theorems 2 and 3 also give a bound on the error probability of their corresponding statements. It is sometimes more interesting to fix a desired level of probability, viz., one might insist that the error probability decay is (say) quasi-polynomial, or exponential. It would be a matter of some interest to see how much such a stipulation would change the nature of the results.

- Part 3 of theorem 3 only considers $n$ square-free, and this is again crucial since we invoke the Cauchy-Davenport theorem in the proof of Lemma 6. The general case may be

- In part 3 of theorem 3, we remarked that for any $\varepsilon > 0$ there exist (for each fixed $r$) square-free $n$ for which any random $\mathbb{Z}_n$-sequence of length 2 is already a weighted Davenport $Z$-sequence (w.r.t the weight set $\mathbb{Z}_n^*$) with probability $1 - \varepsilon$. A more careful scrutiny of our proof suggests that the correct magnitude of $m$ such that a random $\mathbb{Z}_n$-sequence of size $m$ is whp a Davenport $Z$-sequence is possibly of the order $\Theta(\sum_i \frac{1}{p_i})$. It would be very interesting if such is indeed the case.
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6 Appendix: Szemerédi’s proof of the Erdős-Eggleston Conjecture

In this section, we present a proof of the Erdős-Eggleston conjecture that was settled by Szemerédi. In fact, Szemerédi’s proof works for any abelian group. Our presentation of the proof alone is ours, as these ideas are all there in Szemerédi’s paper. We make no claim regarding the optimality of the constant that appears here, nor do we make any attempts to optimize. We shall also drop ceilings and floors to make the presentation simpler. Recall that for a set $A \subset G$, by $S(A)$ we mean the set of all sum $\sum_{x \in X} x$ as $X$ varies over all non-empty subsets of $A$.

**Theorem 8. (Szemerédi)** Let $G$ be a finite abelian group and suppose $A \subset G$ such that $0 \notin S(A)$. Then $|S(A)| \geq \frac{|A|^2}{10000}$.

**Proof.** Suppose the statement of the theorem does not hold; in particular we may assume that $|A|$ is sufficiently large. Let us write $|A| = \ell$. For each $\ell/4 \leq k \leq 3\ell/4$ we define the bipartite graph $G_k$ with vertex sets $(A, \binom{A}{k})$ (For a set $A$, $\binom{A}{r}$ denotes the set of all $r$-subsets of $A$) and for $X \in (A)\, Y \in \binom{A}{k+1}$ we have $X \leftrightarrow Y$ in $G_k$ if and only if $X \subset Y$ and $|S(Y) \setminus S(X)| \leq \ell/100$.

The upshot of this definition for the graphs $G_k$ is this: If we consider the union of all the graphs $\mathcal{G} = \bigcup_k G_k$, and consider any chain $X_1 \subset \cdots \subset X_{\ell/2}$ of sets with $|X_i| = \ell/2 + i - 1$, then there are at most $\ell/100$ ‘missing edges’ along the chain in the union graph $\mathcal{G}$. This follows since if there are more than $\ell/100$ missing edges along some chain $(X_1, \ldots, X_{\ell/2})$ then $|\cup_i S(X_{i+1} \setminus S(X_i))| > (\ell/100)(\ell/100)$ and that contradicts the assumption that the statement is false.

Fix $k$ and consider $D \in \binom{A}{k}$. Suppose $D$ has degree at least $t$ in both $G_k$ and $G_{k-1}$; let $D$ have neighbors $B_i$ (resp. $A_i$) in $G_k$ (resp. $G_{k-1}$). Write $A_i = D \setminus \{a_i\}$, and $B_i = D \cup \{b_i\}$. Since $S := \{ \sum_{x \in D} x - a_i + b_j : i, j \in [\ell] \} \subset S(B_j)$ and $|S(B_j) \setminus S(D)| \leq \ell/100$, there are at least $t - \ell/100$ elements in $S \cap S(D)$. Since this holds for each $j$, by averaging, it follows that there exists $i$ such that $t - \ell/100$ elements of the set $\{ \sum_{x \in D} x - a_i + b_j : 1 \leq j \leq t \}$ belong to $S(D)$, and since $|S(D) \setminus S(A_i)| \leq \ell/100$, it follows again that at least $t - \ell/50$ elements of this set lie in $S(A_i)$. If $t > \ell/50$, then picking some $j_0$ in this set gives us

$$\sum_{x \in D} x - a_i + b_{j_0} = \sum_{y \in D_i \subset D \setminus \{a_i\}} y,$$

which implies that $b_{j_0} + \sum_{x \in D \setminus D_i} x = 0$ contradicting the hypothesis that $0 \notin S(A)$.

So, to complete the proof, we need to show that there exists $k$, and $D \in \binom{A}{k}$ such that $D$ has degree at least $\ell/50$ in both $G_k$ and $G_{k-1}$. Towards that end, let us denote by $d(G_k)$ the potential density of $G_k$, i.e., $d(G_k) = \frac{e(G_k)}{m_k}$ where $e(G_k)$ denotes the number of edges in $G_k$ and $m_k = \# \{(X, Y) : X \in \binom{A}{k}, Y \in \binom{A}{k+1}, X \subset Y\}$.

We make the observation, that if we can show that $d(G_k), \ d(G_{k-1}) \geq 2/3$, then we are through. To see why, suppose $BAD_k := \{ D \in \binom{A}{k} : deg_k(D) \leq \ell/48 \}$ where $deg_k(D)$ denotes the degree of $D$ in $G_k$. Then

$$\frac{2}{3} \binom{\ell}{k}(\ell - k) = \frac{2m_k}{3} \leq e(G_k) \leq |BAD_k|(\ell/48) + \left( \binom{\ell}{k} - |BAD_k| \right)(\ell - k)$$
which gives $|BAD_k| \leq \frac{18}{44} \binom{\ell}{k} < \frac{1}{2} \binom{\ell}{k}$. Similarly, we get $|BAD_{k-1}| < \frac{1}{2} \binom{\ell}{k}$, so there exists $D \in \binom{A}{k} \setminus (BAD_k \cup BAD_{k-1})$ and for this $D$, we have $deg_k(D) \geq \ell/48, deg_{k-1}(D) \geq \ell/48$, and that achieves our goal.

So finally, to establish that for some $k$ we have $d(G_k), d(G_{k-1}) \geq 2/3$, we revert to our original observation that every chain $C = (X_1 \subset \cdots \subset X_{\ell/2})$ misses at most $\ell/100$ edges, or equivalently, for every chain, if we uniformly pick a random ‘link’ $1 \leq i \leq \ell/2$, then the probability that the link $(X_i, X_{i+1}) \in C$ is not an edge is at least 0.99.

Call a level $(k, k+1)$ a Bad level, if $d(G_k) < 2/3$. If what we seek does not hold, then every alternate level is Bad. In particular, if $C = (X_1, \ldots, X_{\ell/2})$ is a uniformly random chain and $Miss(C)$ denotes the number of missing edges in $C$, then

$$\frac{\ell}{100} \geq \mathbb{E}(Miss(C)) \geq \sum_{k \text{ Bad}} \mathbb{P}((X_i, X_{i+1}) \text{ is not an edge}) \geq \frac{1}{2} \cdot \frac{\ell}{2}$$

(the first inequality follows by assumption on $Miss(C)$ for all chains $C$, and the last inequality follows since there are at most $\ell/2$ levels, and at least half of those are Bad by assumption) and that is a contradiction. \qed