Integrability in the Hamiltonian Chern-Simons theory

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Dedicated to L.D. Faddeev
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Abstract

We consider the moduli space of flat connections on the Riemann surface with marked points. The new efficient parametrization is suggested and used to construct an integrable model on the moduli space. A family of commuting Hamiltonians is extracted from the trace of the transfer matrix built from the Wilson line observables of the Chern-Simons theory. Our model appears to be gauge equivalent to XXZ magnetic chain with finite number of sites.

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1 Introduction

Quantization of the Chern-Simons theory in 3 dimensions became the object of intensive investigations when the relation between the Chern-Simons functional integral and the theory of knots had been discovered [1]. Let us start with a brief description of the model.

1.1 Chern-Simons theory

The Chern-Simons model corresponding to the Lie algebra $\mathcal{G}$ is defined as a theory of the gauge connection $A$ with the action

\[ CS(A) = \frac{k}{4\pi} Tr \int_M (AdA + \frac{2}{3} A^3), \]

where the integration region $M$ is a 3-dimensional manifold. The theory (1.1) appears to be topological because the action is written in terms of differential forms and hence there is no metric dependence from the very beginning. In principle, metric may influence the theory through regularization procedure but we don’t touch these subtleties here.

The model (1.1) enjoys two symmetries: the gauge symmetry (for integer values of $k$) and reparametrization symmetry. So, we should construct observables in such a way that they have the same symmetries as the action. The simplest example is provided by the Wilson line observables

\[ W_I(\Gamma) = tr_I Pexp \int_{\Gamma} A^I. \]

Here $\Gamma$ is a closed curve in the manifold $M$ and the connection $A^I = A^i_T dx^i$ is evaluated in the representation $I$ of the algebra $\mathcal{G}$. So in general we deal with evaluation of the Wilson lines correlator of the following type:

\[ Z_k(I_1, \ldots, I_n) = \int DA exp^{iCS(A)} W_{I_1} \cdots W_{I_n}. \]

Let us simplify the problem and assume that at least locally the manifold $M$ looks like a cylinder $\Sigma \times R$, where $\Sigma$ is a Riemann surface. Then the problem may be treated in the framework of Hamiltonian mechanics. We shall refer to the Chern-Simons model on the cylinder as to Hamiltonian Chern-Simons model.
It is well-known that in the Hamiltonian CS theory the gauge field $A$ is constrained by the flatness condition on the equal-time surface $\Sigma$:

$$F = dA + A^2 = 0.$$  \hfill (1.4)

The reduced phase space may be obtained as a quotient of the space of flat connections over the gauge group action

$$A^g = gAg^{-1} - dgg^{-1}.$$  \hfill (1.5)

In this way the moduli space of flat connection appears. If we have some Wilson lines intersecting $\Sigma$, the curvature may develop $\delta$-function singularities at the intersection points. In this case we deal with the moduli space of flat connections with marked points. Marked points are exactly those where the Wilson lines intersect $\Sigma$. Each marked point is equipped with the representation of $G$ carried by the corresponding Wilson line.

In this paper I consider quantization of the Hamiltonian Chern-Simons theory. Let me briefly describe the plan of the paper. Section 2 is devoted to the algebra of observables which may be interpreted as quantized algebra of functions on the moduli space of flat connections (moduli algebra). In this part I extensively use the material of [2],[3]. In Section 3 I introduce a new integrable model on the moduli space of flat connections. Being topological the Chern-Simons theory itself has the Hamiltonian equal to zero. However, one may look for the complete set of commuting observables. A certain number of them may be constructed starting from Wilson line observables. At this point the technique of Inverse Scattering method [4] appears to be useful. I postpone to the next paper more serious discussion of completeness of the commutative family. The equivalence of the integrable model of Section 3 and and XXZ magnetic chain is established in Section 4. In Section 5 I return to the moduli algebra and construct its irreducible representations for generic value of the deformation parameter $q$ (noninteger coupling constant $k$ in the CS theory).

### 1.2 Background

Some basic references are collected in this subsection.

The main technical tool of this paper is the lattice simulation of the Hamiltonian Chern-Simons theory. The approach I am following here origi-
nates from the construction of the lattice current algebra [5] where the lattice simulation of the Wess-Zumino-Novikov-Witten (WZNW) model has been suggested. The current algebra may be simulated on the finite lattice by the quadratic $R$-matrix algebra. In this approach the relation to the Quantum groups is especially transparent. Later the spectral dependent $L$-operator has been introduced and the family of commuting integrals of motion in the lattice WZNW model has been constructed [6]. Here we fulfil the same program but for the different algebra.

The next important step towards the correct lattice approximation of the CS theory has been made in [2] where the proper discretization of the moduli space of flat connections has been suggested. The idea is to draw a graph on the Riemann surface and consider the set of parallel transports along the graph links (link variables) instead of the two-dimensional gauge field. It is possible to introduce the quadratic $R$-matrix algebra for the link variables consistent with the standard quantization of the Chern-Simons model. In the simplest case when the graph is just a polygon we get the same $R$-matrix algebra as we had for the lattice version of the WZNW model. Because of the topological nature of the CS theory its lattice simulation has an advantage in comparison with the lattice WZNW model. The lattice CS model is expected to give exactly the same results as the continuous theory. In this connection it is worth mentioning that the similar lattice simulation has been successfully used in the construction of invariants of 3-dimensional manifolds [7].

The $R$-matrix algebra of link variables in the lattice CS theory belongs to the class of nonultralocal quadratic algebras. It means that the variables assigned to different links do not necessarily commute. The general theory of nonultralocal quadratic algebras has been developed in [8],[9].

Let me finish with the remark that the attempt to represent the CS theory as a lattice gauge model has been described in [10]. The link algebra in this model is ultralocal. As a consequence the model does not enjoy gauge invariance. However, the paper [10] includes many useful observations (e.g. the canonical integral on the link algebra).

### 1.3 Notations

Here I introduce some useful notations. First of all we refer to the finite dimensional algebra $G$ and its set of irreducible representations $\mathcal{I}$. Particular irreducible representations will be usually denoted by $I_1, \ldots, I_n \in \mathcal{I}$.
We shall often deal with parallel transports defined by the flat connection $A$ on the Riemann surface. For a given closed curve $\Gamma$ we introduce the monodromy

$$M_\Gamma = P \exp \int_\Gamma A.$$  \hfill (1.6)

It is convenient to have the matrix $M$ in any representation $I \in \Im$:

$$M^I_\Gamma = P \exp \int_\Gamma A^I.$$  \hfill (1.7)

The next important object which will be used to describe the moduli algebra is the quantum $R$-matrix. One can treat it as an element of the tensor square of the quantized universal enveloping algebra $U_q(\mathfrak{g}) \otimes^2$. If we evaluate the first multiplier of $R$ in the representation $I$ and the second multiplier in the representation $J$, we get the numerical matrix $R^{IJ}$. The set of numerical $R$-matrices satisfies the quantum Yang-Baxter equation:

$$R^{IJ}_{12} R^{IK}_{13} R^{JK}_{23} = R^{JK}_{23} R^{IK}_{13} R^{IJ}_{12}.$$  \hfill (1.8)

The $R$-matrix corresponding to $U_q(\mathfrak{g})$ depends on the deformation parameter $q$. For example, let us write the $4 \times 4$ $R$-matrix corresponding to the algebra $sl(2)$:

$$R = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & (q - q^{-1}) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q \\
\end{pmatrix}.$$  \hfill (1.9)

If we have a solution of (1.8) it is always possible to get 3 more solutions in the following simple way. Let us redenote the matrix (1.9) by $R_+$ and consider $R_+^{-1}$, $R_-$ and $R_-^{-1}$, where

$$R_- = P R_+^{-1} P.$$  \hfill (1.10)

All of them are solutions of (1.8). Here $P$ is the permutation matrix which exchanges two representation spaces in the tensor square. The $R$-matrices $R_\pm$ and $R_\pm^{-1}$ play the role of structure constants in the $R$-matrix algebras.

It is worth mentioning that when the deformation parameter $q$ is closed to unity, all $R$-matrices approach the unit matrix.
\[ R_\pm = I + (q - 1)r_\pm + \ldots \]  

(1.11)

Now we can turn to the description of the moduli algebra.

2 Moduli algebra

One way to deal with the moduli algebra is to choose a particular graph on the Riemann surface. There is no canonical choice and we do it trying to get the most economic description. Let us choose a graph to be a bunch of circles \( B \) intersecting at the only point \( P \). In this bunch we have two circles for each handle (corresponding to \( a_- \) and \( a_+ \)-cycles) and one circle for each marked point. We shall denote the circles corresponding to the \( i \)'s handle by \( a_i \) and \( b_i \) (\( i = 1, \ldots, g \)) and we shall use symbols \( m_i \) (\( i = 1, \ldots, n \)) for the circles surrounding marked points. We shall assume that the circles on \( \Sigma \) are chosen in such a way that the only defining relation in \( \pi_1(\Sigma_{g,n}) \) looks as

\[
  m_1 \ldots m_n(a_1b_1^{-1}a_1^{-1}b_1) \ldots (a_gb_g^{-1}a_g^{-1}b_g) = id.
\]  

(2.1)

In particular, equation (2.1) fixes the cyclic order of links incident to the graph vertex \( P \).

Now we can define the algebra \( \mathcal{F} \) of link variables corresponding to the graph \( B \). To each circle we assign the corresponding monodromy matrix. Let us denote these matrices by \( A_i, B_i \) and \( M_i \) for \( a_- \), \( b_- \) and \( m_- \)-circles. The set of monodromy matrices provides the representation of the fundamental group \( \pi_1(\Sigma_{g,n}) \). It implies the relation

\[
  M_1 \ldots M_n(A_1B_1^{-1}A_1^{-1}B_1) \ldots (A_gB_g^{-1}A_g^{-1}B_g) = I
\]  

(2.2)

imposed on the values of \( A_i, B_i \) and \( M_i \). In principle, we may introduce a set of matrices for each link so that any monodromy (for example \( A_1 \)) appears in any representation \( I \) of the finite dimensional algebra \( \mathcal{G} \). We shall denote corresponding matrices by \( A^I_i, B^I_i \) and \( M^I_i \). Apparently, relation (2.2) holds true for any representation \( I \).

We define the algebra \( \mathcal{F} \) by the set of quadratic \( R \)-matrix relations for the matrix elements of monodromies. It is useful to introduce notation \( X_i \) referring to arbitrary monodromy matrix. So the symbol \( X_i \) may denote \( A_i, B_i \) or \( M_i \). It is convenient to introduce the partial order in the set of
circles. If the letter $x_i$ appears from the left hand side of the letter $x_j$ in the
word (2.1), we can express it as $i < j$. This definition should not be applied
to $a$- and $b$- cycles winding around the same handle as it gets ambiguous.

**Definition 1** The algebra $F_{g,n}$ is generated by the matrix elements of the
monodromy matrices $A^l_i, B^l_i$ and $M^l_i$ subject to the following set of quadratic
relations:

\[ R_+X_i^1R_-^1X_i^2 = X_i^2R_+X_i^1R_-^1 \]  

(2.3)

for any monodromy matrix $X_i$;

\[ R_+X_i^1R_-^1X_j^2 = X_j^2R_+X_i^1R_-^1 \]  

(2.4)

for the matrix elements of two different monodromies $X_i$ and $X_j$ if $i < j$;

\[ R_+A_i^1R_-^1B_i^2 = B_i^2R_+A_i^1R_-^1 \]  

(2.5)

for monodromies $A_i$ and $B_i$ corresponding to $a$- and $b$-cycles of the same
handle.

Let us make several comments concerning this definition.

**Remark 1.** The classical analogue of the algebra $F$ may be easily defined
if we consider the limit when the deformation parameter $q$ which enters all
$R$-matrices tends to 1. Exchange relations (2.3–2.5) will be replaced by
quadratic $r$-matrix Poisson brackets:

\[ \{X_i^1, X_j^2\} = -r_-X_i^1X_j^2 - X_i^1X_j^2r_+ + X_i^1r_-X_j^2 + X_i^2r_+X_i^1, \]  

(2.6)

\[ \{X_i^1, X_j^2\} = -r_+X_i^1X_j^2 - X_i^1X_j^2r_+ + X_i^1r_+X_j^2 + X_j^2r_+X_i^1, \]  

(2.7)

for $i < j$ and

\[ \{A_i^1, B_i^2\} = -r_+A_i^1B_i^2 - A_i^1B_i^2r_+ + A_i^1r_-B_i^2 + B_i^2r_+A_i^1. \]  

(2.8)

The Poisson brackets (2.6–2.8) admit the restriction to the set of functions
invariant with respect to simultaneous conjugations of monodromies:
\[ X_i^g = gX_ig^{-1}. \] (2.9)

Being restricted they coincide with the canonical Poisson brackets on the moduli space of flat connections \([2]\).

Remark 2. The transformations (2.9) may be implemented in the quantum algebra \(\mathcal{F}\) as well. To this end one should impose the quantum group exchange relation on the transformation parameter \(g\):

\[ R_+g_1^2g_2 = g^2g_1^1R_+. \] (2.10)

Definition 2 The algebra \(I_{g,n}\) is an invariant subalgebra of \(\mathcal{F}_{g,n}\) with respect to the quantum group action (2.9).

Definition 3 The moduli algebra \(M_{g,n}\) is a factor algebra of \(I_{g,n}\) over the ideal generated by the relation (2.2).

The algebra \(M\) provides a natural quantization of the moduli space of flat connections with marked points. For more extended description one can look the reference \([3]\).

Remark 3. One can assign to each link a subalgebra of \(\mathcal{F}_{g,n}\) generated by the matrix elements of \(X_i\). It is worth mentioning that all these subalgebras are isomorphic to \(U_q(G)\) \([11]\). For the case of \(G = sl(2)\) the isomorphism may be easily written:

\[
X = \begin{pmatrix}
K^2 + q^{-1}(q - q^{-1})^2FE & (q - q^{-1})FK^{-1} \\
(q - q^{-1})K^{-1}E & K^{-2}
\end{pmatrix}
\]

Here \(K, E, F\) are standard generators of \(sl_q(2)\):

\[
KE = qEK, \\
KF = q^{-1}FK, \\
EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}.
\]

It is convenient to introduce Gauss decomposition for the quantum matrix \(X\):

\[
7
\]
\[ X = X_+X_-^{-1}. \quad (2.12) \]

The diagonal parts of \( X_+ \) and \( X_- \) are inverse to each other. The matrices \( X_+ \) and \( X_- \) satisfy nice exchange relations

\[
\begin{align*}
R_\pm X_\pm^1 X_\pm^2 &= X_\pm^2 X_\pm^1 R_\pm, \\
R_+ X_+^1 X_+^2 &= X_+^2 X_+^1 R_+.
\end{align*} \quad (2.13)
\]

Matrix elements of \( X_\pm \) generate upper- and lower-triangular Borel subalgebras in \( U_q(\mathcal{G}) \). We shall assume that for any matrix which satisfies the exchange relation (2.3) one can introduce the Gauss components \( X_\pm \).

To the construction of the integrable model which will appear in the next section we introduce the following notations:

\[
M_{n+2i-1} = A_i, \quad M_{n+2i} = B_i^{-1} A_i^{-1} B_i. \quad (2.14)
\]

The definition (2.14) may be motivated by the observation that now the exchange relations for matrices \( M_i \) look uniformly:

\[
\begin{align*}
R_- M_i^1 R_i^{-1} M_i^2 &= M_i^2 R_i^{-1} M_i^1 R_i^{-1}, \\
R_+ M_i^1 R_i^{-1} M_j^2 &= M_j^2 R_i^{-1} M_i^1 R_i^{-1}.
\end{align*} \quad (2.15)
\]

The last relation as usually holds for \( i < j \). It is worth mentioning that the matrix elements of the monodromies \( M_i \) generate a subalgebras in \( \mathcal{F} \). In principle, one can choose \( M_i \) for \( i > n \) in a different way:

\[
M'_{n+2i-1} = A_i B_i^{-1} A_i^{-1}, \quad M'_{n+2i} = B_i^{-1}. \quad (2.16)
\]

The relations (2.15) are still valid but the subalgebra generated by \( M'_i \) is different from the subalgebra generated by \( M_i \).

### 3 Transfer matrix

The purpose of this section is to define classical and quantum completely integrable systems on the moduli space of flat connections. To this end one
should find a set of functionally independent commuting Hamiltonians $H_k$ ($k = 1, \ldots, N/2$) so that its number is equal to a half of the dimension $N$ of the phase space. In the framework of the Inverse Scattering method one prefers to start with a generating function which depends on the complex parameter $\lambda$ (spectral parameter) and produces the Hamiltonians $H_k$ in the expansion around some point $\lambda_0$ (usually $\lambda_0$ may be chosen to be equal to infinity). So our nearest goal is to introduce the dependence on the spectral parameter into the definition of monodromies. For simplicity we shall restrict ourselves to the case of $SL(N)$ and even more specifically to its $N$-dimensional representation and briefly discuss other situations in the next section.

Let us introduce a new object

$$M_i(\lambda) = M_i + \lambda I,$$  \hspace{1cm} \text{(3.1)}

where $I$ denotes the $n$ by $n$ unit matrix. Exchange properties of the matrices $M_i(\lambda)$ may be described as follows.

**Theorem 1** The matrix elements of spectral dependent monodromies $M_i(\lambda)$ satisfy the following set of quadratic relations:

$$R(\lambda, \mu) M_i^1(\lambda) R^{-1}_- M_i^2(\mu) = M_i^2(\mu) R_+ M_i^1(\lambda) \tilde{R}(\lambda, \mu),$$

$$R_+ M_i^1(\lambda) R^{-1}_+ M_i^2(\mu) = M_i^2(\mu) R_+ M_i^1(\lambda) R^{-1}_+.$$ \hspace{1cm} \text{(3.2)}

As usual, the last equation is valid for $i < j$. It does not get changed comparativley to (2.15). As about the first relation, the new structure constants $R(\lambda, \mu)$ and $\tilde{R}(\lambda, \mu)$ appear. These are matrices in the tensor square of the fundamental representation:

$$R(\lambda, \mu) = \lambda R_+ - \mu R_-,$$

$$\tilde{R}(\lambda, \mu) = \lambda R^{-1}_- - \mu R^{-1}_+.$$ \hspace{1cm} \text{(3.3)}

Both $R(\lambda, \mu)$ and $\tilde{R}(\lambda, \mu)$ are solutions of the quantum Yang-Baxter equation with the spectral parameter:

$$R_{12}(\lambda, \mu) R_{13}(\lambda, \nu) R_{23}(\mu, \nu) = R_{23}(\mu, \nu) R_{13}(\lambda, \nu) R_{12}(\lambda, \mu).$$ \hspace{1cm} \text{(3.4)}
It is convenient that $R(\lambda, \mu)$ and $\tilde{R}(\lambda, \mu)$ may be efficiently inverted:

\begin{align*}
(\lambda R_+ - \mu R_-)(\lambda R_+^{-1} - \mu R_-^{-1}) &= f(\lambda, \mu)I, \\
(\lambda R_-^{-1} - \mu R_+^{-1})(\lambda R_- - \mu R_+) &= f(\lambda, \mu)I,
\end{align*}

(3.5)

where the scalar function $f(\lambda, \mu)$ is equal to:

\[ f(\lambda, \mu) = \lambda^2 + \mu^2 - \lambda \mu (q^2 + q^{-2}). \]

(3.6)

Formulae (3.5) are based on the important property of $SL(N)$ $R$-matrices in the $N$-dimensional representation:

\[ R_+ R_-^{-1} + R_- R_+^{-1} = (q^2 + q^{-2})I. \]

(3.7)

We postpone the proof of the Theorem 1 until the next section where the proper technique will be developed and turn to the definition of the transfer matrix.

**Definition 4** The transfer matrix is defined as an ordered product of spectral dependent monomodromies $M_i(\lambda)$:

\[ T(\lambda) = \prod_{i=1}^{n+2g} M_i(\lambda) = M_1(\lambda) \ldots M_{n+2g}(\lambda). \]

(3.8)

The matrix elements of the transfer matrix satisfy the exchange relation which is similar to (3.2):

\[ R(\lambda, \mu) T^1(\lambda) R_+^{-1} T^2(\mu) = T^2(\mu) R_+ T^1(\lambda) \tilde{R}(\lambda, \mu). \]

(3.9)

Proof of the formula (3.9) is straightforward. In the course of commuting $T(\lambda)$ and $T(\mu)$ one have to use the equality:

\[ R_+^{-1} R(\lambda, \mu) = \tilde{R}(\lambda, \mu) R_. \]

(3.10)

The very important consequence of formula (3.9) reads as follows:

**Theorem 2** The $q$-trace of the transfer matrix $T(\lambda)$

\[ F(\lambda) = Tr_q T(\lambda) \]

(3.11)
provides a family of commuting variables on the moduli space:

\[ F(\lambda)F(\mu) = F(\mu)F(\lambda). \]  

(3.12)

Proof. Let us multiply the identity (3.9) by \( R_+^{-1} \) from the left and by \( (\lambda R_+ - \mu R_+) \) from the right. Then we apply the \( q \)-trace in the first and in the second space. The result looks as follows:

\[ Tr_q^{1,2} R_+^{-1} (\lambda R_+ - \mu R_+) T^1(\lambda) R_+^{-1} T^2(\mu) (\lambda R_+ - \mu R_+) = f(\lambda, \mu) Tr_q^{1,2} R_+^{-1} T^2(\mu) R_+ T^1(\lambda). \]  

(3.13)

It is easy to check that (3.13) may be rewritten as

\[ f(\lambda, \mu) Tr_q^{1,2} R_+ T^1(\lambda) R_+^{-1} T^2(\mu) = f(\lambda, \mu) Tr_q^{1,2} R_+^{-1} T^2(\mu) R_+ T^1(\lambda). \]  

(3.14)

Here we have used the fact that \( R \)-matrices commute with the kernel of \( Tr_q^{1,2} \) and have applied equations (3.5, 3.10). The \( q \)-trace has a remarkable property which enables us to cancel all \( R \)-matrices both in the l.h.s and in the r.h.s. of (3.14). If we take into account this fact, formula (3.14) implies commutativity of \( F(\lambda) \). Obviously \( F(\lambda) \) is invariant with respect to quantum group action (2.9) and defines a family of commuting Hamiltonians on the moduli space of flat connections. This remark completes the proof of Theorem 2 and we turn to discussion of possible generalizations and applications.

Remark 1. It is possible to generalize the described procedure for any simple Lie group \( \mathcal{G} \) and for any representation \( I \). The form of spectral dependent monodromies \( M_i(\lambda) \) will be more sophisticated. It will consist of \( d \) terms, where \( d \) is the number of irreducible representations which appear in the tensor product \( I \otimes I^* \). It is easy to check that for \( N \)-dimensional representation of the group \( SL(N) \) \( d = 2 \). The matrix \( M_i(\lambda) \) is always polynomial in \( \lambda \) but the degree will increase with the number of terms. We don’t give more details in this paper but we shall use the fact that one can construct the quantity \( F^I(\lambda) \) for any representation \( I \):

\[ F^I(\lambda) = Tr_q M_1^I(\lambda) \cdots M_{n+2g}^I(\lambda). \]  

(3.15)

so that all of them commute with each other.
Remark 2. Actually, formula (3.11) defines a family of commuting families parametrized by \( n + 2g - 1 \) parameter. In order to construct them, let us modify the definition of the transfer matrix:

\[
T_{z_1, \ldots, z_{n+2g}}(\lambda) = \prod_{i=1}^{n+2g} M_i(z_i \lambda). \quad (3.16)
\]

The transfer matrix (3.16) provides a family of commuting variables which depends on the point in \( \mathbb{C}P^{n+2g-1} \) defined by the parameters \( z_1, \ldots, z_{n+2g} \). The simultaneous dilatation of the parameters \( z_i \) is equivalent to renormalization of \( \lambda \) and does not change the commuting family. Let us consider the simplest example when the shift parameters \( z_1, \ldots, z_{n+2g} \) are ordered as follows \( |z_1| \ll |z_2| \ll \ldots \ll |z_{n+2g}| \). Then the family of commuting variables corresponding to this choice of shift parameters is generated by \( q \)-traces of matrices \( T_i \)

\[
T_i = M_1 M_2 \ldots M_i. \quad (3.17)
\]

This construction resembles the polarization on the moduli space of \( SU(2) \) connections obtained in [12].

Remark 3. In the classical case we obtain a system of Hamiltonians in involution. It means that their pairwise Poisson brackets vanish. In general, the number of independent functions in our family is less than half of the dimension of the moduli space. In the case of the group \( SU(2) \) the number of variables is sufficient and the family of polarizations similar to considered in [12] may be extracted from the transfer matrix (3.16).

4 Equivalence to XXZ magnetic chain

Here we shall establish the equivalence between the integrable model which appeared in the last section and XXZ magnetic chain. The change of variables on the moduli space which provides this equivalence is quite important itself because it furnishes a nice representation for the symplectic structure [13].

Let us introduce new matrix generators \( L_i (i = 1, \ldots, n+2g) \). Each of them may be decomposed into the product upper- and lower-triangular multipliers:
\[ L_i = L_+ (i) L_- (i)^{-1}. \]  

(4.1)

We introduce a new algebra.

**Definition 5** The algebra \( \mathcal{L}_{n+2g} \) is generated by the matrix elements of \( L_i \) subject to quadratic relations:

\[
R_- L_i^1 R_i^{-1} L_i^2 = L_i^2 R_+ L_i^1 R_i^1, \\
L_i^1 L_j^2 = L_j^2 L_i^1.
\]

(4.2)

As we see, the algebra \( \mathcal{L}_{n+2g} \) is isomorphic to \( U_q(\mathcal{G})^{\otimes(n+2g)} \). However, it appears to be very useful. The following theorem explains the importance of \( \mathcal{L}_{n+2g} \) for the moduli algebra.

**Theorem 3** The algebra \( \mathcal{L}_{n+2g} \) is isomorphic to the subalgebra in \( \mathcal{F}_{0,n+2g} \) generated by matrix elements of monodromies \( M_i \).

**Proof.** Let us introduce the new set of matrices \( K_i \) as an ordered product of lower-triangular components \( L_- (i) \):

\[ K_i = L_- (1) \ldots L_- (i-1), K_1 = I. \]

(4.3)

The isomorphism between \( \mathcal{L}_{n+2g} \) and the corresponding subalgebra in \( \mathcal{F}_{n+2g} \) is defined by the explicit formulae:

\[ M_i = K_i L_i K_i^{-1}. \]

(4.4)

One can easily check that the defining relations (2.13) follow from (4.2)(4.4). The fact that one can always start with a direct product of quantum algebras instead of the complicated moduli algebra is in intimate relation with the representation of the states in the CS theory as invariants in the tensor product of representations of the quantum group.

The important question is what happens to the relation (2.2) after the change of variables. Separating upper- and lower-triangular components one finds two relations:
\[
\begin{align*}
L_+(1) \ldots L_+(n+2g) &= I, \\
L_-(1) \ldots L_-(n+2g) &= I. 
\end{align*}
\]  

(4.5)

Now we can clarify the construction of spectral dependent monodromy:

\[
M_i(\lambda) = K_i^{-1}(L_+(i) + \lambda L_-(i))K_i^{-1}. 
\]  

(4.6)

It appears to be gauge equivalent to the standard \(L\)-operator for the XXZ magnetic chain:

\[
L_i(\lambda) = L_+(i) + \lambda L_-(i). 
\]  

(4.7)

Formulae (4.5) assure that the gauge parameter \(K_i\) is periodic on our finite lattice \(K_{n+2g+1} = I\). Thus, the trace of the monodromy matrix does not change when we substitute \(M_i(\lambda)\) instead of \(L_i(\lambda)\). So one may say that the Theorem 2 follows from the commutativity theorem for the trace of monodromy for XXZ magnet. However, we prefer the formulation of the previous section because the change of variables from \(M_i\) to \(L_i\) is obviously not canonical.

Theorem 3 enables us to derive exchange relations for \(M_i(\lambda)\) in a simple way.

\textit{Proof of Theorem 1.} It is well known that the \(L\)-operator of XXZ model (4.7) satisfies the basic relation:

\[
R(\lambda, \mu)L_1^1(\lambda)L_2^2(\mu) = L_2^2(\mu)L_1^1(\lambda)R(\lambda, \mu). 
\]  

(4.8)

If we take into account formula (4.8) the proof of relations (3.2) becomes straightforward. In the course of commuting \(M_i(\lambda)\) the following identities are especially useful:

\[
\begin{align*}
R_+L_1^1(\lambda)L_2^2(i) &= L_2^2(i)L_1^1(\lambda)R_+, \\
R_\pm K_i^1K_i^2 &= K_i^2K_i^1R_\pm, \\
K_i^1L_i^2(\lambda) &= L_i^2(\lambda)K_i^1. 
\end{align*}
\]  

(4.9)
5 Factorization of the moduli algebra

The idea of this section is to extend the Theorem 3 to the algebra $F_{g,n}$ corresponding to any genus $g$. This algebra can not be represented as a direct product of several copies of $U_q(G)$. So we have to introduce one more basic object [14].

**Definition 6** The quantized algebra of functions of the cotangent bundle of the group $G$ $Fun(T^*G_q)$ is generated by the matrix elements of matrices $g$ and $L$ subject to quadratic relations:

\[
R_- L^1 R_-^1 L^2 = L^2 R_+ L^1 R_+^{-1},
\]

\[
R_\pm g^1 g^2 = g^2 g^1 R_\pm,
\]

\[
R_\pm L_\pm^1 g^2 = g^2 L_\pm^1 R_\pm.
\]

(5.1)

The algebra $Fun(T^*G_q)$ may be considered as a deformation of the algebra of differential operators of finite order on the group $G$. One may interpret this algebra as an algebra of observables for the quantum system which consists of the point particle which moves in the quantum group $G_q$. From this point of view it is natural to treat $g$ as coordinates and $L_\pm$ as components of the left momentum. The algebra $Fun(T^*G_q)$ has a unique representation which may be realized for example by left multiplication on the algebra itself. It is convenient to introduce a matrix $\tilde{L}$:

\[
\tilde{L} = g^{-1} L^{-1} g.
\]

(5.2)

Formula (5.2) introduces the right momentum into the theory:

\[
R_- \tilde{L}^1 R_-^1 \tilde{L}^2 = \tilde{L}^2 R_+ \tilde{L}^1 R_+^{-1},
\]

\[
L^1 \tilde{L}^2 = \tilde{L}^2 L^1.
\]

(5.3)

As usual, left and right momenta commute with each other and realize two independent copies of $U_q(G)$. If we regard the only representation of $Fun(T^*G_q)$ as a representation of the subalgebra generated by the matrix elements of left and right momenta $L$ and $\tilde{L}$, we discover the regular representation of $U_q(G)$:
\[ \mathcal{R}_G = \oplus_{\mathcal{I} \in \mathcal{I}} I \otimes I^*, \quad (5.4) \]

where \( \mathcal{I} \) is the set of irreducible representations of \( \mathcal{G} \).

The algebra \( \mathcal{F}_{g,n} \) may be handled in a way similar to how we handle the algebra \( \mathcal{F}_{0,n+2g} \).

**Theorem 4** The algebra \( \mathcal{F}_{g,n} \) is isomorphic to the algebra \( \mathcal{L}_{g,n} = U_q(\mathcal{G})^\otimes n \otimes \text{Fun}(T^*G_q)^\otimes g \).

**Proof.** We describe the isomorphism explicitly. The algebra \( \mathcal{L}_{g,n} \) is generated by the matrix elements of \( L_1, \ldots, L_n \) corresponding to \( n \) copies of \( U_q(\mathcal{G}) \) and \( (L_{n+1}, g_1), (L_{n+3}, g_2), \ldots, (L_{n+2g-1}, g_g) \) corresponding to \( g \) copies of the algebra \( \text{Fun}(T^*G_q) \). We have enumerated left momenta by the indices \( n + 2i - 1 \) and reserved the indices \( n + 2i \) for right momenta:

\[ L_{n+2i} = g_i^{-1} L_{n+2i-1} g^i. \quad (5.5) \]

As in the previous section we define the variables \( K_i \):

\[ K_i = L_{-1} L_{-2} \ldots L_{-(i-1)}, \quad K_1 = I. \quad (5.6) \]

The isomorphism which we need looks as:

\[
\begin{align*}
M_i &= K_i L_i K_i^{-1}, \\
A_i &= K_{n+2i-1} L_{n+2i-2} K_{n+2i-1}^{-1}, \\
B_i &= K_{n+2i-1} g_i K_{n+2i-1}^{-1}.
\end{align*} \quad (5.7)
\]

As usual, the checking of relations (5.7) is straightforward.

Let us state an important property of the algebra \( \mathcal{M}_{g,n} \). The elements

\[ c_i^I = \text{Tr}_q M_i^I \quad (5.8) \]

for \( i = 1, \ldots, n \) belong to the centre of the algebra \( \mathcal{M}_{g,n} \). Indeed, due to the first formula (5.4) we have

\[ c_i^I = \text{Tr}_q L_i^I. \quad (5.9) \]

It is known that the elements of the type \( c_i^I \) for given \( i \) generate the centre of \( U_q(\mathcal{G}) \). As the algebra \( \mathcal{L}_{g,n} \) includes a direct product of several copies of
$U_q(G)$, we conclude that the elements lie in the centre of $\mathcal{M}_{g,n}$. Let us remark that the algebra $\text{Fun}(T^*G_q)$ has no centre.

Now we are equipped to construct the representation theory of $\mathcal{M}_{g,n}$. It is convenient to start with some irreducible representation of $\mathcal{F}_{g,n}$, or equivalently $\mathcal{L}_{g,n}$. The latter is a direct product and its representations are enumerated by tuples $(I_1, \ldots, I_n)$ of irreducible representations of $G$ and may be realized in the space

$$V_{I_1, \ldots, I_n} = I_1 \otimes \ldots \otimes I_n \otimes \mathbb{R}^{2g},$$

where $n$ copies of $U_q(G)$ act in the multipliers $I_1, \ldots, I_n$ and each copy of $\text{Fun}(T^*G_q)$ is realized in $\mathbb{R}$.

The algebra $U_q(G)$ may be naturally embedded into $\mathcal{L}_{g,n}$ in the following way:

$$L_\pm = \prod_{i=1}^{n+2g} L_\pm(i).$$

The algebra of invariants $\mathcal{I}_{g,n}$ may be defined as a commutant of the image of $U_q(G)$ embedded by the formula (5.11). Obviously, the representation $V_{I_1, \ldots, I_n}$ gets reducible when we restrict it to the algebra $\mathcal{I}_{g,n}$. Moreover, the representation $V_{I_1, \ldots, I_n}$ may be decomposed into the direct sum of irreducible representations of $U_q(G)$ and $\mathcal{I}_{g,n}$:

$$V_{I_1, \ldots, I_n} = \oplus_{I \in \mathcal{I}} I \otimes W_{I_1, \ldots, I_n}^{I}.$$  

The algebra $\mathcal{M}_{g,n}$ is defined as a factorialgebra of $\mathcal{I}_{g,n}$ over the ideal defined by relations

$$L_\pm = \prod_{i=1}^{n+2g} L_\pm(i) = I.$$  

So only those representations of the algebra $\mathcal{I}_{g,n}$ may be reinterpreted as representations of $\mathcal{M}_{g,n}$ where the ideal generated by

$$c^I = Tr_q L_+^I (L_-^I)^{-1}$$  

acts by the trivial representation. There is only one representation which satisfies this condition in each sum (5.12). Namely, one should pick up a partner of the trivial representation $W_{e I_1, \ldots, I_n}^e$ which is actually isomorphic to
the space of $q$-invariants in the space $V_{I_1,...,I_n}$ regarded as a representation of $U_q(G)$ realized by the formula (5.11). Thus we conclude.

**Theorem 5** The irreducible representations of the algebra $M_{g,n}$ are enumerated by tuples $(I_1,\ldots,I_n)$ of representations of $G$. Each representation of this type may be realized in the space

$$W_{I_1,...,I_n}^c = \text{Inv}_q(I_1 \otimes \ldots \otimes I_n \otimes \mathcal{R} \otimes g)$$

(5.15)

by the standard formulae which define the action of $U_q(G)$ in its representations $I_1,\ldots,I_n$ and $\text{Fun}(T^*G_q)$ in the representation $\mathcal{R}$.

Let us emphasize that we have proved this Theorem only for generic values of the deformation parameter $q$. The most interesting case of $q$ being a root of unity needs some further consideration (see also [3]).

### 6 Discussion

This Section is devoted to open problems related to quantization of the Hamiltonian Chern-Simons theory and to the integrable model on the moduli space of flat connections introduced in Section 3.

The first problem that I would like to mention is the construction of the representation theory of the moduli algebra for $q$ being a root of unity. The difference in comparison with generic $q$ is that we have to work with quasi-associative algebras instead of associative if we want to have nondegenerate scalar product in the representation space. The other possibility is to deal with the same moduli algebra as in Section 2 but now it has a nontrivial ideal which should be factorized out. The first step in this direction is presented in [3].

Another problem is related to the question of completeness of the family of commuting Hamiltonians. For my knowledge, it is possible to complete this simplest family corresponding to infinite values of spectral shifts (see Section 3) in order to get polarization. However, it would be interesting to fulfil the same program for arbitrary spectral shifts. Such construction would provide a new family of polarizations on the moduli space of flat connections. If we compare this hypothetical family with polarizations induced by complex structures on the Riemann surface, we discover that in these two families...
the number of parameters corresponding to a marked point coincide (it is equal to 1), whereas the number of parameters corresponding to a handle is different (3 in the case of complex structures and only 2 in our case). This is a hint that it may be possible to construct $L$-operator of more general form than the one which we used in Section 3. Actually, there we treat a handle as a couple of marked points. On the language of complex structures it would mean that we consider only hyperelliptic surfaces. So, it is quite possible that some extra spectral shifts may be introduced into the definition of the transfer matrix.

If we assume that there exists a family of polarizations parametrized by spectral shifts (at least for $SL(2)$ it is indeed the case) we face the problem how to deal with different quantizations provided by these polarizations simultaneously. The experience obtained in the course of investigation of another family of polarizations on the same moduli space [15] tells that one of possible ways is to construct the flat connection on the space of parameters which is designed to identify canonically the Hilbert spaces obtained upon quantization starting from different polarizations. It is the Knizhnik-Zamolodchikov (KZ) equation which provides such a connection in the case of the family of polarizations related to complex structures. Now the exiting question arises what kind of equation one can get instead of the KZ for the family of polarizations parametrized by spectral shifts? I hope to return to this question in some other paper.

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