A generalisation of the Gilbert-Varshamov bound and its asymptotic evaluation

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Abstract. The Gilbert-Varshamov (GV) lower bound on the maximum cardinality of a $q$-ary code of length $n$ with minimum Hamming distance at least $d$ can be obtained by application of Turán’s theorem to the graph with vertex set $\{0, 1, \ldots, q-1\}^n$ in which two vertices are joined if and only if their Hamming distance is at least $d$. We generalize the GV bound by applying Turán’s theorem to the graph with vertex set $C^n$, where $C$ is a $q$-ary code of length $m$ and two vertices are joined if and only if their Hamming distance at least $d$. We asymptotically evaluate the resulting bound for $n \to \infty$ and $d \sim \delta mn$ for fixed $\delta > 0$, and derive conditions on the distance distribution of $C$ that are necessary and sufficient for the asymptotic generalized bound to beat the asymptotic GV bound. By invoking the Delsarte inequalities, we conclude that no improvement on the asymptotic GV bound is obtained.

By using a sharpening of Turán’s theorem due to Caro and Wei, we improve on our bound. It is undecided if there exists a code $C$ for which the improved bound can beat the asymptotic GV bound.

Keywords: Error-correcting codes, minimum Hamming distance, Gilbert-Varshamov bound, Turán’s theorem, Delsarte inequalities

1 Introduction

Let $A_q(n, d)$ denote the maximum cardinality of code of length $n$ and minimum Hamming distance at least $d$ over an alphabet $Q$ with $q$ letters. Moreover, for $0 \leq \delta \leq 1$, let $\alpha_q(\delta)$ denote the limes superior of the maximum rate of $q$-ary codes of relative distance $\delta$, that is,

$$\alpha_q(\delta) = \limsup_{n \to \infty} \frac{1}{n} \log_q A_q(n, \delta n).$$

According to the asymptotic Plotkin bound [1, Thm. 5.2.5] $\alpha_q(\delta) = 0$ for $\delta \geq 1 - \frac{1}{q}$; for $0 < \delta < 1 - \frac{1}{q}$, the value of $\alpha_q(\delta)$ is unknown.

The Gilbert-Varshamov (GV) bound [1, Thm. 5.1.7] states that

$$A_q(n, d) \geq q^n \left( \sum_{i=0}^{d-1} \binom{n}{i} (q-1)^i \right).$$
The asymptotic version of the GV bound [1, Thm. 5.1.9] reads as follows:

\[
\text{for } 0 \leq \delta \leq 1 - \frac{1}{q}, \text{ we have that } \alpha_q(\delta) \geq 1 - h_q(\delta),
\]

(1)

where \( h_q \) is the \( q \)-ary entropy function, defined as

\[
h_q(x) = -x \log_q(x) - (1 - x) \log_q(1 - x) + x \log_q(q - 1).
\]

To the best of the author’s knowledge, no lower bound on \( \alpha_q(\delta) \) improving on (1) is known for \( q < 46 \). For an extensive survey on literature on the Gilbert-Varshamov bound and improvements on it, we refer to [2].

In [3], it was observed that the GV bound can be obtained by application of Turán’s theorem [4, Thm. 3.2.1] to the graph with vertex set \( Q^n \), in which two vertices are joined by an edge if and only if their Hamming distance is at least \( d \). By using this graph-theoretical approach and applying a refined version of Turán’s theorem for locally sparse graphs, Jiang and Vardy [2] obtained an improvement of the GV bound for binary codes with a multiplicative factor \( n \). This result was generalized to \( q \)-ary codes by Vu and Wu who proved the following [5, Thm. 1.2].

**Theorem 1.** Let \( q \) be a fixed positive integer and let \( \beta, \beta' \) be constants satisfying \( 0 < \beta' < \beta < \frac{q - 1}{q} \). There is a positive constant \( c \) depending on \( q \) and \( \beta \) such that for any \( \beta' n < d < \beta n \),

\[
A_q(n, d + 1) \geq c \frac{q^n}{\sum_{i=0}^{d-1} \binom{n}{i}(q-1)^i} n.
\]

In this paper, we use the graph-theoretical approach to obtain a generalization of the GV bound. Again, we consider a graph in which two vertices are joined by an edge if and only if their Hamming distance is at least \( d \). The vertex set does not equal \( Q^n \), but it equals \( C \), where \( C \subseteq Q^m \) is a fixed \( q \)-ary code of length \( m \). We use Turán’s theorem to obtain a lower bound on the size of the largest clique in this graph, and employ a bounding technique from [6] to obtain a manageable asymptotic expression. We analyze the generalized asymptotic GV bound, and by employing the Delsarte inequalities [1, Sec. 5.3], we infer that it cannot improve the asymptotic GV bound. We end with an improvement of our bound based on a sharpening of Turán’s theorem due to Caro and Wei, and derive a necessary and sufficient condition on \( C \) for this improved bound to beat the asymptotic GV bound. We have not been able to decide if there exist codes \( C \) satisfying this condition.

## 2 Prerequisites

### 2.1 Turán’s theorem

Let \( G \) be a simple graph without loops. A clique in \( G \) is a set of vertices of which each pair is joined by an edge. It is intuitively clear that a graph with many edges should contain a large clique. This is quantified by Turán’s theorem, of which we use the following version (for a proof, see e.g. [4 Thm. 3.2.1]).
Theorem 2. A simple graph without loops with $v$ vertices and $e$ edges contains a clique of size at least $\frac{v^2}{(v^2 - 2e)}$.

2.2 Distance enumerator of a code

Let $C \subseteq Q^m$. For $0 \leq j \leq m$, we define $B_j$ as

$$B_j = \frac{1}{|C|} |\{(x, y) \in C^2 \mid d(x, y) = j\}|.$$

We define the distance enumerator polynomial $B(x)$ of $C$ as $B(x) = \sum_{j=0}^{m} B_j x^j$. It is easy to check that for each $n \geq 1$, the code $C^n$ has distance enumerator polynomial $(B(x))^n$.

2.3 A bounding lemma

The following lemma is similar to a bounding technique that can be found in [6].

Lemma 3. Let $f(x) = \sum_{j=0}^{n} f_j x^j$ be a polynomial with non-negative coefficients, i.e., $f_j \geq 0$ for all $j$. For each $k$, $0 \leq k \leq n$ and each $x \in (0, 1]$, we have

$$\sum_{j=0}^{k} f_j \leq \frac{f(x)}{x^k}.$$ 

Proof. Let $0 \leq x \leq 1$. Whenever $0 \leq j \leq k$, we have that $x^j \geq x^k$, and so $f(x) = \sum_{j=0}^{n} f_j x^j \geq \sum_{j=0}^{k} f_j x^j \geq \sum_{j=0}^{k} f_j x^k$. $\square$

3 Main result and its proof

Theorem 4. Let $C \subseteq Q^m$ have distance enumerator polynomial $B(x)$. For each $x \in (0, 1]$, we have

$$\alpha_q(\delta) \geq \frac{1}{m} \log_q \left( \frac{|C|}{B(x)} \right) + \delta \log_q x.$$

Proof. For each integer $n$, we consider the graph $G$ with vertex set $C^n$ in which two vertices are joined by an edge if and only if they have Hamming distance at least $d = \lceil \delta mn \rceil$. The number of edges $e$ thus is given by

$$e = |C|^n \frac{1}{2} \sum_{j=0}^{m} B_j^{(n)},$$

where $B_j^{(n)}$ is the $j$-th coefficient of the distance enumerator of $C^n$. Application of Turán’s bound to $G$ yields the existence of a subcode $D$ of $C^n$ with minimum distance at least $d$ and cardinality at least

$$\frac{|C|^n}{\sum_{j=0}^{d-1} B_j^{(n)}}.$$
4 Ludo Tolhuizen

We now invoke Lemma 3 and find that for each $x \in (0, 1]$, 
\[ \sum_{j=0}^{d-1} B_j^{(n)} \leq \frac{B_C(x)}{x^{d-1}} = \left( \frac{B(x)}{x^{d-1}} \right)^n. \]

Combining the above inequalities, we find that the code $D$ of length $mn$ satisfies 
\[ \frac{1}{mn} \log_q |D| \geq \frac{1}{m} \log_q \left( \frac{|C|}{B(x)} \right) + \delta \log_q x. \]

It can be readily verified that Theorem 4 with $m = 1$, $C = Q$ and $x = \delta/(q - 1)(1 - \delta)$ yields the asymptotic GV bound (1).

Theorem 4 contains a parameter $x$ that can be optimized over. By straightforward differentiation, one finds that the optimizing value $x$ satisfies the equation 
\[ xB'\left(\delta \frac{1}{q}\right) - \delta mB(x) = 0. \]

For a given code $C$, it seems in general a hopeless task to obtain a closed expression for the largest bound on $\alpha_q(\delta)$ that can be obtained from Theorem 4. We set ourselves instead a different goal, viz. to determine if Theorem 4 can improve on the asymptotic GV bound. To this end, for each $\delta \in (0, 1)$ and $x \in (0, 1)$, we define
\[ F(x, \delta) = \frac{1}{m} \log_q \left( \frac{|C|}{B(x)} \right) + \delta \log_q x - (1 - h_q(\delta)). \]

For a pair $(x, \delta)$ optimizing $F(x, \delta)$, we have that 
\[ 0 = \partial F(x, \delta)/\partial \delta = \log_q x + h_q'(\delta), \]

implying that $x = x_\delta := \frac{\delta}{(q - 1)(1 - \delta)}$.

Note that $x_\delta \leq 1$ whenever $\delta \leq 1 - \frac{1}{q}$. As $\delta \log_q x_\delta + h_q(\delta) = -\log_q (1 - \delta)$, we have that 
\[ F(x_\delta, \delta) = \frac{1}{m} \log_q \left( \frac{|C|}{B(x_\delta)} \right) - 1 - \log_q (1 - \delta) = -\frac{1}{m} \log_q \left( B(x_\delta)(1 - \delta)^m \frac{q^m}{|C|} \right). \]

As a result, we have proved the following lemma.

**Lemma 5.** There exists a $\delta$ for which Theorem 4 yields an improvement on the asymptotic GV bound if and only if for some $\delta \in (0, 1 - \frac{1}{q})$
\[ \frac{q^m}{|C|}(1 - \delta)^m B\left(\frac{\delta}{(q - 1)(1 - \delta)}\right) < 1. \]

**Theorem 6.** The largest lower bound on $\alpha_q(\delta)$ that can be obtained from Theorem 4 is the asymptotic GV bound $1 - h_q(\delta)$. 

\[ \Box \]
Proof. By substituting $z = 1 - \frac{q}{q-1} \delta$ in the condition of Lemma 5, we obtain the equivalent condition that for some $z \in (0, 1)$,

$$\frac{1}{|C|} (1 + (q - 1)z) m B \left( \frac{1 - z}{1 + (q - 1)z} \right) < 1.$$  \hspace{1cm} (2)

We write the left hand side of (2) as the polynomial $A(z) = \sum_{i=0}^{m} A_i z^i$. By choosing $z = 0$, we find that $A_0 = A(0) = \frac{1}{|C|} B(1) = 1$. According to the Delsarte inequalities, that form the basis of the linear programming bound [1, Sec. 5.3], $A_i \geq 0$ for all $i$. So for $z \geq 0$, we have that $A(z) \geq A_0 = 1$. \hfill \Box

4 Extension of the main result

We extend our result by using the sharpening of Turán’s theorem from Theorem 7 below; an elegant proof, attributed to Caro and Wei, can be found in [4, p. 95].

**Theorem 7.** Let $G = (V, E)$ be a simple graph without loops. For each $v \in V$, let $d_v$ be the number of neighbours of $v$. Then $G$ contains a clique of size at least

$$\sum_{v \in V} \frac{1}{|V| - d_v}.$$

By using a convexity argument and the fact that $\sum_{v \in V} d_v = 2|E|$, it can be shown that Theorem 7 implies Theorem 2 and that they give the same result for regular graphs, i.e., if all vertices have equally many neighbours. If we thus apply our construction with a code $C$ for which the number of codewords at a given distance from a word $c \in C$ actually depends on the choice of $c$, we may improve our main result. For describing this improvement, we introduce the following notion: for a given code $C$ and $c \in C$, the local distance enumerator $B_c(x)$ is defined as

$$B_c(x) = \sum_{j=0}^{m} |\{y \in C \mid d(c, y) = j\}| x^j.$$

**Theorem 8.** Let $C \subseteq Q^m$. For each $\delta \in (0, 1)$ and $x \in (0, 1)$, we have

$$\alpha_q(\delta) \geq \frac{1}{m} \log \left( \sum_{c \in C} \frac{1}{B_c(x)} \right) + \delta \log_q x.$$

Note that Theorem 8 reduces to Theorem 4 if all local distance distributions are equal.

Proof. We apply Theorem 7 to the graph with vertex set $C^n$, in which two vertices are joined by an edge if and only if they have Hamming distance at
least \( d \). In this way, we infer the existence of a code of length \( mn \) and minimum Hamming distance at least \( d \) of size at least

\[
\sum_{c \in C^n} \frac{1}{|\{y \in C^n \mid d(c, y) \leq d\}|}
\]

For each \( c \in C^n \) and \( x \in (0, 1] \), we have that

\[
|\{y \in C^n \mid d(c, y) \leq d\}| \leq \frac{B_c^n(x)}{x^d},
\]

where \( B_c^n(x) = \sum_j |\{y \in C^n \mid d(c, y) = j\}|x^j \). It is easy to see that for \( c = (c_1, c_2, \ldots, c_n) \in C^n \), we have that

\[
B_c^n(x) = \prod_{i=1}^n B_{c_i}(x),
\]

and so there exists a code of length \( mn \) and size at least

\[
\sum_{c=e_1, \ldots, e_n \in C^n} x^d \prod_{c_{i \in 1}} \frac{1}{B_{c_i}(x)} = x^d \left( \sum_{c \in C} B_c(x) \right)^n,
\]

where the final equality can be proved by induction on \( n \). \( \square \)

**Lemma 9.** Theorem 8 improves on the asymptotic GV bound for some \( \delta \in (0, 1 - \frac{1}{q}) \) if and only if for some \( z \in (0, 1) \),

\[
\sum_{c \in C} \frac{1}{(1 + (q - 1)z)^m B_c \left( \frac{1-z}{1+(q-1)z} \right)} > 1.
\]

**Proof.** Similar to the proof of Lemma 5 and Theorem 6. \( \square \)

We have not been able to decide if there exist codes for which the inequality in Lemma 9 is met. We note that for certain codes \( C \), e.g. for \( C = \{0, 1\}^3 \setminus \{(0, 0, 0)\} \), the left-hand side of the condition in Lemma 9 is monotonically decreasing in \( z \), although not all individual terms have this property.

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