Simple Lie algebras, Drinfeld–Sokolov hierarchies, and multi-point correlation functions

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Abstract

For a simple Lie algebra $\mathfrak{g}$, we derive a simple algorithm for computing logarithmic derivatives of tau-functions of Drinfeld–Sokolov hierarchy of $\mathfrak{g}$-type in terms of $\mathfrak{g}$-valued resolvents. We show, for the topological solution to the lowest-weight-gauge Drinfeld–Sokolov hierarchy of $\mathfrak{g}$-type, the resolvents evaluated at zero satisfy the topological ODE.

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1 Introduction

1.1 Simple Lie algebra and Drinfeld–Sokolov hierarchy

Let \( g \) be a simple Lie algebra over \( \mathbb{C} \) of rank \( n \), with the Lie bracket denoted by \([\cdot,\cdot]\). Let \( \text{ad} : g \to gl(g) \) be the adjoint representation of \( g \). We denote by \( h, h^\vee \) the Coxeter and dual Coxeter numbers [38] of \( g \), and \( m_1 = 1 < m_2 \leq \cdots \leq m_{n-1} < m_n = h - 1 \) the exponents. Denote \( (\cdot,\cdot) : g \times g \to \mathbb{C} \) the normalized Cartan–Killing [13] form

\[
(x|y) := \frac{1}{2h^\vee} \text{tr}(\text{ad}_x \text{ad}_y), \quad \forall x, y \in g.
\]  

(1.1.1)

Fix a Cartan subalgebra \( h \subset g \), and let \( \Delta \subset h^* \) be the root system. We choose a set of simple roots \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \subset h^* \). Then \( g \) has the root space decomposition

\[
g = h \oplus \bigoplus_{\alpha \in \Delta} g_\alpha.
\]

For any \( \alpha \in \Delta \), denote by \( H_\alpha \) the unique element in \( h \) such that \( (H_\alpha|X) = \alpha(X), \quad \forall X \in h \). The normalized Cartan–Killing form induces naturally a non-degenerate bilinear form on \( h^* \):

\[
(\alpha|\beta) = (H_\alpha|H_\beta), \quad \forall \alpha, \beta \in h^*.
\]

Denote by \( E_i \in g_{\alpha_i}, F_i \in g_{-\alpha_i}, H_i = 2H_{\alpha_i}/(\alpha_i|\alpha_i) \) the Weyl generators of \( g \). They satisfy

\[
[E_i, F_j] = H_i \delta_{ij}, \quad [H_i, E_j] = A_{ij} E_j, \quad [H_i, F_j] = -A_{ij} F_j,
\]

where \( (A_{ij}) \) denotes the Cartan matrix associated to \((g, \Pi)\), and \( \delta_{ij} \) is the Kronecker delta. Here and in what follows, free Latin indices take the integer values from 1 to \( n \) unless otherwise indicated.

Let \( \theta \) be the highest root w.r.t. \( \Pi \); recall that \( (\theta|\theta) = 2 \). We choose \( E_{-\theta} \in g_{-\theta}, E_\theta \in g_\theta \), normalized by the conditions \( (E_\theta|E_{-\theta}) = 1 \) and \( \omega(E_{-\theta}) = -E_\theta \), where \( \omega : g \to g \) is the Chevalley involution. Let

\[
I_+ := \sum_{i=1}^n E_i
\]

be a principal nilpotent element of \( g \). Define

\[
\Lambda = I_+ + \lambda E_{-\theta}.
\]  

(1.1.2)

Denote by \( L(g) = g \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) the loop algebra of \( g \). The Lie bracket \([\cdot,\cdot]\) and the Cartan–Killing form \((\cdot,\cdot)\) extend naturally to \( L(g) \). We have

\[
L(g) = \text{Ker ad}_\Lambda \oplus \text{Im ad}_\Lambda, \quad \text{Ker ad}_\Lambda \perp \text{Im ad}_\Lambda.
\]  

(1.1.4)

Recall that the \textit{principal gradation} on \( L(g) \) is defined by

\[
\deg \lambda = h, \quad \deg E_i = -\deg F_i = 1.
\]

Observe that

\[
\deg \Lambda = 1.
\]

This gradation is of course also defined on \( g = g \otimes 1 \). With the principal gradation, the loop algebra \( L(g) \) and the simple Lie algebra \( g \) decompose into direct sums of homogeneous subspaces \( L(g)^j, g^j, j \in \mathbb{Z} : \)

\[
L(g) = \bigoplus_{j \in \mathbb{Z}} L(g)^j, \quad g = \bigoplus_{j = -(h-1)}^{h-1} g^j.
\]

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We will denote the projection onto the nonnegative subspace by $\langle \bullet \rangle^+ : L(\mathfrak{g}) \rightarrow \sum_{j \geq 0} L(\mathfrak{g})^j$, and onto the negative subspace by $\langle \bullet \rangle^-$. It is known [37] that $\text{Ker ad}_\Lambda \subset L(\mathfrak{g})$ admits the following decomposition
\[
\text{Ker ad}_\Lambda = \bigoplus_{j \in E} \mathcal{C} \Lambda_j, \quad \Lambda_j \in L(\mathfrak{g})^j, \quad j \in E,
\]
\[
[\Lambda_i, \Lambda_j] = 0, \quad \forall i, j \in E.
\]
Here, $E := \bigsqcup_{i=1}^{n} (m_i + h \mathbb{Z})$. The meaning of the symbol $\bigsqcup$ here is that of “disjoint union”: this means that if the exponents are distinct then $\bigsqcup$ denotes the ordinary union, but if an element appears in more than one set, it is actually considered a new element. This is relevant only for the case of the Lie algebra of type $D_n$ with even $n = 2k$: in this case $m_{n/2+1}, m_{n/2+1} + h, \ldots$ should be written as $m'_{n/2+1}, (m_{n/2+1} + h)', \ldots$ because, as integers, $m_{n/2+1} = m_{n/2}$.

We choose normalizations of $\Lambda_j, \; j \in E$ satisfying
\[
\Lambda_{m_a + kh} = \Lambda_{m_a} \lambda^k, \quad k \in \mathbb{Z}, \quad (\Lambda_{m_a} | \Lambda_{m_b}) = h \eta_{ab} \lambda.
\]
Here and below,
\[
\eta_{ab} := \delta_{a+b,n+1}.
\]
Since $\Lambda \in L(\mathfrak{g})^1$, we fix the normalization of $\Lambda_1$ such that $\Lambda_1 = \Lambda$.

It is useful to notice that $\Lambda_{m_a}, \; a = 1, \ldots, n$ have the form [41]
\[
\Lambda_{m_a} = L_{m_a} + \lambda K_{m_a - h}, \quad L_{m_a} \in \mathfrak{g}^{m_a}, \quad K_{m_a - h} \in \mathfrak{g}^{m_a - h}, \quad L_{m_a} \neq 0, \quad K_{m_a - h} \neq 0.
\]

In [17], Drinfeld and Sokolov associate to $\mathfrak{g}$ an integrable hierarchy of Hamiltonian evolutionary PDEs, known as the Drinfeld–Sokolov (DS) hierarchy of $\mathfrak{g}$-type. Let us briefly review their construction in the form suitable for subsequent considerations. Denote by $\mathfrak{b} = \mathfrak{g}^{\leq 0}$ a Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{n} = \mathfrak{g}^{< 0}$ a nilpotent subalgebra. Let
\[
\mathcal{L} = \partial_x + \Lambda + q(x), \quad q(x) \in \mathfrak{b}.
\]

**Definition 1.1.1.** The basic resolvents $R_a, \; a = 1, \ldots, n$ of $\mathcal{L}$ are defined as the unique solutions to
\[
[\mathcal{L}, R_a] = 0, \quad R_a \in \mathcal{A}^{q} \otimes \mathfrak{g}((\lambda^{-1})), \quad R_a(\lambda; q, q_x, \cdots) = \Lambda_{m_a} + \text{ lower order terms w.r.t. deg},
\]
\[
(R_a(\lambda; q, q_x, \cdots) | R_b(\lambda; q, q_x, \cdots)) = h \eta_{ab} \lambda.
\]

(here and below, $\mathcal{A}^{q}$ denotes the ring of differential polynomials in $q$, namely, an element of $\mathcal{A}^{q}$ is a polynomial in the entries of $q, q_x, q_{2x}, \cdots$), together with the requirements that $R_a$ are homogeneous of degree $m_a$ with respect to the extended principal gradation defined by further assigning degrees to entries of $q$ so that $q$ is homogeneous of degree 1.

Existence and uniqueness of the basic resolvents will be shown in Proposition 2.2.3. Note that (1.1.11) can be alternatively replaced by the no-integration constant condition $R_a(\lambda; 0, 0, \cdots) = \Lambda_{m_a}$, which gives rise to a different algorithm of computing $\tilde{R}_a$.

The DS flows for the $\mathfrak{b}$-valued function $q = q(x, \mathbf{T})$, $\mathbf{T} = (T^q_k)_{k \geq 1, \ldots, n}$ are an infinite set of compatible evolutionary PDEs of the form
\[
\frac{\partial \mathcal{L}}{\partial T^q_k} = \left[ \left( \lambda^k R_a \right)_{\text{even}}, \mathcal{L} \right], \quad k \geq 0.
\]
The notation $(\bullet)_+$ stands for the polynomial part of the expression in the variable $\lambda$ (similarly, $(\bullet)_-$ will stand for Laurent tail in the variable $\lambda$). To see that these flows are well defined, we note that the property $[\mathcal{L}, R_a] = 0$ implies that

$$
\left[\left(\lambda^k R_a\right)_+, \mathcal{L}\right] = \partial_x \left(\left(\lambda^k R_a\right)_-\right) + \left[\Lambda, \left(\lambda^k R_a\right)_-\right] + \left[q, \left(\lambda^k R_a\right)_-\right].
$$

(1.1.13)

Then, observing that the RHS contains only non-positive powers in $\lambda$ (here (1.1.3) is used), and that the LHS contains only non-negative powers in $\lambda$, we find that $\left[\left(\lambda^k R_a\right)_+, \mathcal{L}\right]$ takes value in $g \otimes \lambda^0$. Furthermore this contribution can only come from the term $\left[\lambda E_{-\theta}, \left(\lambda^k R_a\right)_-\right]$ (here (1.1.3) is used again): recalling that $E_{-\theta}$ has the principal degree $-(h - 1)$, we conclude that $\left[\lambda E_{-\theta}, \left(\lambda^k R_a\right)_-\right] \in \mathfrak{b} \otimes \lambda^0$. An important property of these flows is that they pairwise commute [17]; they form the pre-DS hierarchy.

Consider transformations of the dependent variable $q(x) \mapsto \tilde{q}(x)$ of the pre-DS hierarchy induced by gauge transformations of the form

$$
\mathcal{L} = \partial_x + \Lambda + q(x) \mapsto \tilde{\mathcal{L}} = e^{ad_{N(x)}} \mathcal{L} = \partial_x + \Lambda + \tilde{q}(x)
$$

(1.1.14)

for arbitrary $n$-valued smooth functions $N(x)$. A crucial point of the Drinfeld–Sokolov construction is the following statement.

**Lemma 1.1.2.** The gauge transformations (1.1.14) are symmetries of the pre-DS flows of (1.1.12). In particular, they map solutions to solutions.

In our approach the proof of this simple but important statement easily follows by observing that the basic resolvents $\tilde{R}_a$ of the gauge-transformed operator $\tilde{\mathcal{L}}$ satisfy

$$
\tilde{R}_a(\lambda; \tilde{q}, \tilde{q}_x, \cdots) = e^{ad_{N(x)}} R_a(\lambda; q, q_x, \cdots), \quad a = 1, \ldots, n.
$$

(1.1.15)

The DS hierarchy is obtained from (1.1.12) by considering suitably chosen gauge invariant functions $q_{\text{can}}$ (see below for more details).

### 1.2 From resolvents to tau-function

We start from defining tau-functions of an arbitrary solution $q(x, T)$ of the pre-DS hierarchy. Then we verify its independence from the choice of the gauge with respect to the transformations of the form (1.1.14).

**Definition 1.2.1.** Define a sequence of functions $\Omega_{a,k;b,\ell} = \Omega_{a,k;b,\ell}(q, q_x, \cdots) \in \mathcal{A}^q$, $k, \ell \geq 0$ by means of the generating function expression below

$$
\sum_{k,\ell \geq 0} \frac{\Omega_{a,k;b,\ell}}{\lambda^{k+1} \mu^{\ell+1}} = \frac{(R_a(\lambda)| R_b(\mu))}{(\lambda - \mu)^2} - n_{ab} \frac{m_a \lambda + m_b \mu}{(\lambda - \mu)^2}.
$$

(1.2.1)

We call $\Omega_{a,k;b,\ell}$ the two-point correlation functions.

**Lemma 1.2.2.** The two-point correlation functions $\Omega_{a,k;b,\ell}$ satisfy the following properties

$$
\partial_{T^a_m} \Omega_{a,k;b,\ell} = \partial_{T^a_k} \Omega_{b,\ell; a, k}, \quad \forall a, b, \forall k, \ell \geq 0,
$$

(1.2.2)

$$
\partial_{T^c_m} \Omega_{a,k;b,\ell} = \partial_{T^c_k} \Omega_{b,\ell; a, k}, \quad \forall a, b, c, \forall k, \ell, m \geq 0.
$$

(1.2.3)
Lemma 1.2.3. For an arbitrary solution $q(x, T)$ to (1.1.12), there exists $\tau = \tau(x, T)$ such that

$$\frac{\partial^2 \log \tau}{\partial T^a_k \partial T^b_\ell} = \Omega_{a, k; b, \ell} (q(x, T), q_x(x, T), \cdots), \quad (1.2.4)$$

$$\frac{\partial \tau}{\partial x} = -\frac{\partial \tau}{\partial T^a_0}. \quad (1.2.5)$$

The proofs are provided later in the paper.

In view of (1.2.5) we will henceforth identify $x$ with $-T^1_0$ for $\tau(x, T)$. So we will use the short notation $\tau = \tau(T)$. Note that the scalar function $\tau(T)$ advocated for in Lemma 1.2.3 is uniquely determined by the solution $q(x, T)$ only up to a factor of the form

$$\exp \left( d_0 + \sum_{a=1}^n \sum_{k \geq 0} d_{a,k} T^a_k \right), \quad d_0, d_{a,k} \text{ arbitrary constants.} \quad (1.2.6)$$

Definition 1.2.4. We call $\tau(T)$ the tau-function of the solution $q(x, T)$ of the pre-DS hierarchy.

For related aspects on tau-functions, see for example [6] [11] [16] [18] [20] [24] [28] [31] [32] [33] [34] [36] [39] [51].

Definition 1.2.5. For an arbitrary solution to the pre-DS hierarchy, let $\tau(T)$ be a tau-function of this solution in the sense of Definition 1.2.4. The $N$-point correlation functions of $\tau(T)$ are defined by

$$\langle \langle \tau_{\alpha_1,k_1} \cdots \tau_{\alpha_N,k_N} \rangle \rangle_{DS} = \frac{\partial^N \log \tau}{\partial T^{\alpha_1}_{k_1} \cdots \partial T^{\alpha_N}_{k_N}}, \quad k_1, \ldots, k_N \geq 0, \quad N \geq 1. \quad (1.2.7)$$

From (1.1.15) it easily follows the following lemma.

Lemma 1.2.6. The tau-function of a solution to the pre-DS hierarchy is invariant, up to a factor of the form (1.2.6), with respect to the gauge transformations (1.1.14).

Thus $\tau(T)$ will also be called tau-function of the solution $q^\text{can}$ of the DS hierarchy corresponding to a gauge-fixed Lax operator. The usual procedure [17] to fix the gauge is by choosing a subspace $\mathcal{V} \subset \mathfrak{b}$ transversal to the adjoint action of the nilpotent subgroup so that $q^\text{can}(x)$ restricts to a $\mathcal{V}$-valued function (see below).

1.3 Main results

For any $a = 1, \ldots, n$ introduce the following differential operator depending on a parameter $\lambda$

$$\nabla_a(\lambda) = \sum_{k \geq 0} \frac{\partial T^a_k}{\lambda^{k+1}}. \quad (1.3.1)$$

For a given $N \geq 1$ and a collection of integers $a_1, \ldots, a_N \in \{1, \ldots, n\}$, we define the following generating series of $N$-point correlations functions by

$$F_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N; T) = \nabla_{a_1}(\lambda_1) \cdots \nabla_{a_N}(\lambda_N) \log \tau(T). \quad (1.3.2)$$

Observe that, for $N \geq 2$ the correlation functions (1.2.7) depend only on the solution $q(x, T)$ of the pre-DS hierarchy. Our goal is to derive an explicit expression for these generating functions for $N \geq 2$ in terms of the basic resolvents defined above.
For any \( N \geq 2 \) define a cyclic-symmetric \( N \)-linear form \( B : \mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow \mathbb{C} \) by
\[
B(x_1, \ldots, x_N) = \text{tr} \left( \text{ad}_{x_1} \cdots \text{ad}_{x_N} \right), \quad \forall x_1, \ldots, x_N \in \mathfrak{g}.
\] (1.3.3)
The normalized Cartan–Killing form (see (1.1.1)) and \( B \) are related by \( B(x, y) = 2h^y(x|y) \).

**Theorem 1.3.1.** For an arbitrary solution \( q^{\text{can}}(T) \) to the DS-hierarchy, let \( \tau(T) \) be a tau-function of this solution. Then \( \forall N \geq 2 \), we have
\[
F_{a_1, \ldots, a_N}(\lambda_1, \ldots, \lambda_N; T) = -\frac{1}{2N} \sum_{s \in S_N} \frac{B \left( R_{a_1}^{\text{can}}(\lambda_1; T), \ldots, R_{a_N}^{\text{can}}(\lambda_N; T) \right)}{\prod_{j=1}^{N} (\lambda_j - \lambda_{j+1})}
\]
\[- \delta_{N2} m_1 \lambda_1 + m_2 \lambda_2, \quad \delta_{N2} = m_1 (\lambda_1 - \lambda_2)^2 ,
\] (1.3.4)
where \( R_{a}^{\text{can}}(\lambda) \), \( a = 1, \ldots, n \) are the basic resolvents of \( \mathcal{L}^{\text{can}} := \partial_x + \Lambda(\lambda) + q^{\text{can}} \), and it is understood that \( s_{N+1} = s_1 \). In particular, \( \forall N \geq 2, \forall a_1, \ldots, a_N \in \{1, \ldots, n\} \), we have \( F_{a_1, \ldots, a_N}(\lambda_1, \ldots, \lambda_N; T) \in \mathcal{A}^{\text{can}}[[\lambda_1^{-1}, \ldots, \lambda_N^{-1}]] \).

**The partition function.** We now consider a particular tau-function that we shall call the partition function: it will be denoted by \( Z(t) \), where the new time variables \( t \) differ from the original \( T \) by a rescaling (see eq. (1.3.6)). This particular tau-function is uniquely specified up to a multiplicative constant by the following string equation:
\[
\sum_{a=1}^{n} \sum_{k \geq 0} t_k^{a} \frac{\partial Z}{\partial t_k^{a}} + \frac{1}{2} \sum_{a, b = 1}^{n} \eta_{ab} t_0^{a} t_0^{b} Z = \frac{\partial Z}{\partial t_0^{a}}
\] (1.3.5)
(see details in Section 4.2 below). Here, the time variables \( t_k^{a} \) and \( T_k^{a} \) are related by
\[
\frac{\partial}{\partial t_k^{a}} = c_{a,k} \frac{\partial}{\partial T_k^{a}}, \quad c_{a,k} = \frac{(-1)^k}{\sqrt{-h}^{m_a + k + 1} (\frac{m_a}{k})_{k+1}} , \quad k \geq 0,
\] (1.3.6)
where \( (\cdot)_k \) denotes the Pochhammer symbol, i.e., \( (y)_k := y(y + 1) \cdots (y + k - 1) \).

**Theorem 1.3.2.** Let the subspace \( \mathcal{V} := \ker \text{ad}_{L} \subset \mathfrak{g} \) be the lowest weight gauge (see eq. (3.1.1) for the definition of \( L_- \)), and \( \mathcal{L}^{\text{can}} \) the associated Lax operator. Let \( R_{a}^{\text{can}}, a = 1, \ldots, n \) be the basic resolvents of \( \mathcal{L}^{\text{can}} \). For the partition function \( Z \), define \( M_a(\lambda) = \lambda^{-\frac{m_a}{h}} R_{a}^{\text{can}}(\lambda; t = 0) \). Then \( \forall a \in \{1, \ldots, n\} \), \( M_a(\lambda) \) satisfies the topological ODE of \( \mathfrak{g} \)-type
\[
M' = \kappa [M, A], \quad \kappa = \left( \sqrt{-h} \right)^{-1} , \quad : = \frac{d}{d\lambda}
\] (1.3.7)
See [7] for the definition and properties of the topological ODE of \( \mathfrak{g} \)-type. Observe that, as \( \lambda \rightarrow \infty \), the solutions \( M_a(\lambda) \) admit the expansions
\[
M_a = \lambda^{-\frac{m_a}{h}} \left[ \Lambda_{m_a} + \text{lower degree terms w.r.t. deg} \right].
\]
Thus, \( M_a \) coincide with the basis of regular solutions to the topological ODE constructed in [7].
1.4 Applications to the FJRW theory

Let \( f : \mathbb{C}^m \to \mathbb{C} \) be a quasi-homogeneous polynomial, i.e., there exist positive integers \( d, n_1, \ldots, n_m \), s.t.

\[
f(z^{n_1}x_1, \ldots, z^{n_m}x_m) = z^d f(x_1, \ldots, x_m), \quad \forall z \in \mathbb{C}.
\]

The weight of \( x_i \) is defined to be \( q_i = \frac{n_i}{d} \), \( i = 1, \ldots, m \). In general the gradient of \( f \) vanishes at the origin and hence the zero level-set \( f^{-1}(0) \) is a singular variety and defines a “singularity” in the sense of singularity theory [3]. The function \( f \) is called non-degenerate if the choice of weights \( q_i \) is unique and \( x = 0 \) is the only singularity of \( f \). Let \( G_f \) (or \( G_{\text{max}} \)) denote the maximal diagonal symmetry group of \( f \), which is the subgroup of \( \text{Aut}(f) \) consisting of diagonal matrices \( \gamma \) such that \( f(\gamma x) = f(x) \). It is easy to see that the matrix

\[
J = \text{diag} \left( e^{2\pi i q_1}, \ldots, e^{2\pi i q_m} \right) \in G_f.
\]

Let \( G \) be a subgroup of \( G_f \) containing \( \langle J \rangle \). Let \( n \) be the dimension of the Fan–Jarvis–Ruan cohomology ring [26] associated to \( (f, G) \). Fan–Jarvis–Ruan associate with the pair \( (f, G) \) a certain generalized Witten class, called the Fan–Jarvis–Ruan–Witten class

\[
\Lambda^{f,G}_{g,N}(a_1, \ldots, a_N) \in H^n(M_{g,N}), \quad a_1, \ldots, a_N \in \{1, \ldots, n\},
\]

such that incorporation of these cohomological classes to \( \overline{M}_{g,N} \) gives rise to a cohomological filed theory [44, 26] (cf. also [18], [19], [48]). The FJRW invariants are defined by

\[
\langle \tau_{a_1k_1} \cdots \tau_{a_Nk_N} \rangle^{f,G} = \int_{\overline{M}_{g,N}} \psi_1^{k_1} \cdots \psi_N^{k_N} \Lambda^{f,G}_{g,N}(a_1, \ldots, a_N),
\]

where \( \psi_i, i = 1, \ldots, N \) are \( \psi \)-classes.

**Definition 1.4.1.** The partition function \( Z^{f,G} \) of FJRW invariants is defined by

\[
Z^{f,G}(t) = \exp \left( \sum_{g,N \geq 0} \frac{1}{N!} \sum_{a_1, \ldots, a_N = 1}^{n} \sum_{k_1, \ldots, k_N \geq 0} \langle \tau_{a_1k_1} \cdots \tau_{a_Nk_N} \rangle^{f,G} t_1^{a_1} \cdots t_N^{a_N} \right).
\]

Now we consider an important subclass of singularities, called simple singularities. They are classified by the ADE Dynkin diagrams [1, 2]. In particular, we consider

\[
A_k : f = x^{k+1}, \quad k \geq 1; \quad D_k : f = x^{k+1} + x y^2, \quad k \geq 4;
\]

\[
E_6 : f = x^3 + y^4; \quad E_7 : f = x^3 + x y^3; \quad E_8 : f = x^3 + y^5.
\]

We are also interested in the mirror singularity of \( D_k \) [26], denoted by \( D_k^{T} \):

\[
D_k^{T} : f = x^{k-1} y + y^2, \quad k \geq 4.
\]

The maximal diagonal symmetry groups \( G_f \) of the above polynomials will be denoted by \( G_{A_k}, G_{D_k}, G_{D_k^{T}} \) and \( G_{E_n}, n = 6, 7, 8 \).

**Theorem-ADE ([26, 27]).** The following statements hold true

A. The partition function \( Z^{A_n,G}(t), n \geq 1 \) with \( G = \langle J \rangle = G_{A_n} \) is a particular tau-function of the Drinfeld–Sokolov hierarchy of \( A_n \)-type satisfying the string equation (1.3.5).
D. The partition function $Z^{D_n,G}(t)$, $n \geq 4$ with $n$ even and $G = \langle J \rangle$ is a particular tau-function of the DS hierarchy of $D_n$-type satisfying (1.3.5).

D’. The partition function $Z^{D_k,G}(t)$, $k \geq 4$ with $G = G_{D_k}$ is a particular tau-function of the DS hierarchy of $A_{2k-3}$-type satisfying (1.3.5).

D”. The partition function $Z^{D^T_k,G}(t)$, $n \geq 4$ with $G = G_{D^T_k}$ is a particular tau-function of the DS hierarchy of $D_n$-type satisfying (1.3.5).

E. The partition function $Z^{E_n,G}(t)$, $n = 6, 7, 8$, with $G = \langle J \rangle = G_{E_n}$ is a particular tau-function of the DS hierarchy of $E_n$-type satisfying (1.3.5).

Summarizing, the partition function $Z^{X_k,Gx_k}(t)$ with $X = A, D, D^T$, or $E$ is a particular tau-function of the DS hierarchy of $X_k$-type satisfying (1.3.5).

In the case that $f = x^r$ with $G = \langle J \rangle = G_f$, the FJRW invariants $\langle \tau_{a_1, \ldots, a_N} \rangle^G$ coincide with Witten’s $r$-spin correlators. The statement A of Theorem-ADE justifies Witten’s $r$-spin conjecture [50], which was first proved by Faber–Shadrin–Zvonkine [25]; see “Theorem $r$-spin” below.

For convenience of the reader let us recall some details in the definition of Witten’s $r$-spin correlators. For a given $N \geq 1$, let $a_1, \ldots, a_N \in \{1, \ldots, r\}$ be integers satisfying the following divisibility condition

$$a_1 + \cdots + a_N - N - (2g - 2) = mr, \quad m \in \mathbb{Z}. \quad (1.4.1)$$

For any smooth algebraic curve $C$ of genus $g$ with $N$ marked points $x_1, \ldots, x_N$ there exists a line bundle $\mathcal{T}$ over $C$ such that

$$\mathcal{T}^\otimes = K_C \otimes \mathcal{O}((1 - a_1)x_1) \otimes \cdots \otimes \mathcal{O}((1 - a_N)x_N). \quad (1.4.2)$$

Here $K_C$ is the canonical class of the curve $C$. Moreover, there are $r^{2g}$ such line bundles. A choice of such an “$r$-th root” of the bundle (1.4.2) defines a point in a covering of the moduli space. After a suitable compactification this covering is denoted by

$$p : \overline{\mathcal{M}}_{g,N}^{1/r}(a_1, \ldots, a_N) \to \overline{\mathcal{M}}_{g,N}. \quad (1.4.3)$$

In genus zero, for a point $(C, x_1, \ldots, x_N, \mathcal{T})$ in the covering space, denote $V = H^1(C, \mathcal{T})$. This defines a vector bundle $V \to \overline{\mathcal{M}}_{0,N}^{1/r}(a_1, \ldots, a_N)$ because the space $V$ has constant dimension thanks to the fact that $H^0(C, \mathcal{T})$ vanishes. Put

$$c_W(a_1, \ldots, a_N) := p_* \left( e(V^\vee) \right) \in H^{2(m-1)}(\overline{\mathcal{M}}_{0,N}),$$

where $e(V^\vee)$ is the Euler class of the dual vector bundle $V^\vee$. The $c_W(a_1, \ldots, a_N)$ is called the Witten class. In higher genus, this is not completely correct because $H^0(C, \mathcal{T})$ is only generically zero and hence the vector bundle is only defined on a generic stratum. The Witten class $c_W(a_1, \ldots, a_N)$ could still be defined as a particular cohomology class in $H^{2(m+g-1)}(\overline{\mathcal{M}}_{g,N})$, but the construction is more involved (see e.g. [50, 25, 35, 47, 46]). The $r$-spin intersection numbers are defined by

$$\langle \tau_{a_1, a_2, \ldots, a_{rN}} \rangle^{r-\text{spin}}_g := \int_{\overline{\mathcal{M}}_{g,N}} c_W(a_1, \ldots, a_N)\psi_1^{p_1} \cdots \psi_N^{p_N}, \quad a_1, \ldots, a_N \in \{1, \ldots, r\}, \quad p_1, \ldots, p_N \geq 0. \quad (1.4.4)$$

The numbers $\langle \tau_{a_1, a_2, \ldots, a_{rN}} \rangle^{r-\text{spin}}_g$ are zero unless

$$\frac{a_1 - 1}{r} + \cdots + \frac{a_N - 1}{r} + \frac{r - 2}{r} (g - 1) + p_1 + \cdots + p_N = 3g - 3 + N. \quad (1.4.5)$$
The so-called Vanishing Axiom conjectured in [35] and proven in [47, 46] tells that the Witten class vanishes if any of \( a_i, i = 1, \ldots, N \) reaches \( r \). Hence, below, we only consider the case of \( a_1, \ldots, a_N \) belonging to \( \{1, \ldots, r-1\} \).

For computing Witten’s \( r \)-spin correlators, we use Theorems 1.3.1–1.3.2 for a particular tau-function along with the following result.

**Theorem** \( r \)-spin ([50, 25]). The partition function of \( r \)-spin intersection numbers

\[
Z^{r\text{--spin}}(t) := \exp \left( \sum_{g,N \geq 0} \frac{1}{N!} \sum_{a_1, \ldots, a_N = 1}^{n} \sum_{k_1, \ldots, k_N \geq 0} \langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_{g}^{r\text{--spin}} t_{a_1}^{k_1} \cdots t_{a_N}^{k_N} \right)
\]

is a particular tau-function of the DS hierarchy of \( A_{n} \)-type, \( n = r-1 \) satisfying (1.3.5).

In [42], Liu–Ruan–Zhang introduced cohomological field theories with finite symmetry, associated with simple singularities and certain symmetry groups, and with a \( \Gamma \)-invariant sector, where \( \Gamma \) is the group of automorphisms of the Dynkin diagram. These theories are proved to be related to the DS integrable hierarchies associated to the non-simply laced simple Lie algebras.

**Theorem-BCFG** ([42]). The partition function of the \( \Gamma \)-invariant sector of \( D_{n+1}^{T}, A_{2n-1}, E_{6} \) FJRW theory with \( G_{\text{max}} \) is a particular tau-function of the Drinfeld–Sokolov hierarchy of \( B_{n}, C_{n}, F_{4} \)-type satisfying (1.3.5); the partition function of the \( \mathbb{Z}/3\mathbb{Z} \)-invariant sector of \( (D_{4}, (J)) \) FJRW theory is a particular tau-function of the Drinfeld–Sokolov hierarchy of \( G_{2} \)-type satisfying (1.3.5).

Note that the common feature of Theorem-ADE and Theorem-BCFG claims that the partition function of FJRW invariants associated to a simple singularity with a symmetry group (possibly also with an invariant sector) is a tau-function of the DS hierarchy of \( \mathfrak{g} \)-type, where \( \mathfrak{g} \) is a simple Lie algebra. We call these numbers the FJRW invariants of \( \mathfrak{g} \)-type, denoted by

\[
\langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_{g}^{FJRW-\mathfrak{g}}, \text{ or simply by } \langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_{g}.
\]

As before, let \( n \) denote the rank of \( \mathfrak{g} \). For a given \( N \geq 1 \) and for a collection of integers \( a_1, \ldots, a_N \in \{1, \ldots, n\} \), we define the following generating functions of \( N \)-point FJRW invariants of \( \mathfrak{g} \)-type

\[
F_{a_1, \ldots, a_N}^{FJRW} (\lambda_1, \ldots, \lambda_N) := (\kappa^{\frac{1}{n+1}} \sqrt{-h})^{N} \sum_{g,k_1, \ldots, k_N \geq 0} \prod_{t=1}^{N} (-1)^{k_t} \left( \frac{a_{r_t}}{h} \right)^{k_t} \left( \frac{\lambda_{r_t}}{\kappa^{\frac{1}{n+1}} + k_t + 1} \right) \langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_{g}^{\mathfrak{g}} \tag{1.4.6}
\]

Here \( \kappa := (\sqrt{-h})^{-h} \).

Combining the results of Theorems 1.3.1 and 1.3.2 with the statements of Theorem-ADE and Theorem-BCFG we arrive at the following formula for the FJRW invariants of \( \mathfrak{g} \)-type.

**Theorem 1.4.2.** Let \( \mathfrak{g} \) be a simple Lie algebra and \( n \) the rank of \( \mathfrak{g} \). Let \( M_{a} = M_{a}(\lambda) \), \( a = 1, \ldots, n \) be the generalized Airy resolvents of \( \mathfrak{g} \)-type, which are the unique solutions to

\[
M' = [M, \Lambda], \tag{1.4.7}
\]

subjected to

\[
M_{a}(\lambda) = \lambda^{-\frac{m_{a}}{n}} [\Lambda_{m_{a}}(\lambda) + \text{lower degree terms w.r.t. } \text{deg}].
\]
Here, $h$ is the Coxeter number and $m_a$ are the exponents of $\mathfrak{g}$. Then the generating functions (1.4.6) for the $N$-point FJRW invariants of $\mathfrak{g}$-type have the following expressions

\[
\frac{dF^{FJRW}_{\mathfrak{g}}}{d\lambda} (\lambda) = -\frac{1}{2h^r} B\left( E_{-\theta}, M_\lambda \right) + \lambda \frac{1}{r} \delta_{n}, \quad N = 1, \quad (1.4.8)
\]

\[
F^{FJRW}_{\alpha_1, \ldots, \alpha_N}(\lambda_1, \ldots, \lambda_N) = -\frac{1}{2Nh^r} \sum_{\alpha \in SN} B\left( M_{\alpha_1}(\lambda_{s_1}), \ldots, M_{\alpha_N}(\lambda_{s_N}) \right) \prod_{j=1}^{N} (\lambda_{s_j} - \lambda_{s_{j+1}}) \lambda_1^{\frac{m_a}{r}} \lambda_2^{\frac{m_a}{r}} (a_1 \lambda_1 + a_2 \lambda_2) \quad N \geq 2. \quad (1.4.9)
\]

Eqs. (1.4.7)–(1.4.9) are equivalent to the proposed formulae in [7] (eq. 4.2.4) of the current paper. For other methods towards computing related invariants, see [4] [8] [9] [10] [12] [24] [23] [30] [43] [52].

In particular, for given integers $r \geq 2$, $N \geq 1$ and a given collection of indices $a_1, \ldots, a_N$ belonging to $\{1, \ldots, r - 1\}$, define

\[
F^{r-spin}_{\alpha_1, \ldots, \alpha_N}(\lambda_1, \ldots, \lambda_N) := \left( \kappa^{\frac{1}{r+1}} \sqrt{-r} \right)^N \sum_{k_1, \ldots, k_N \geq 0} \prod_{t=1}^{N} \frac{(-1)^{k_t} \frac{a_t}{r}}{r_1^{k_t + 1}} \left( \lambda_1^{\frac{1}{r}}, \lambda_2^{\frac{1}{r}} \right) (\tau_{a_1 k_1} \cdots \tau_{a_N k_N})^{r-spin}. \quad (1.4.10)
\]

Here $\kappa = \left( \sqrt{-r} \right)^{-r}$. Note that we have omitted the genus labelling in the notation of correlator, since it can be obtained from the degree-dimension matching (1.4.5).

**Theorem 1.4.3.** Let $n = r - 1$, $\mathfrak{g} = sl_{n+1}(\mathbb{C})$, $\Lambda = \sum_{a=1}^{n} E_{i,i+1} + \lambda E_{n+1,1}$, and let $M_\lambda = M_\lambda(\lambda)$ be the basis of generalized Airy resolvents of $\mathfrak{g}$-type, uniquely determined by the topological ODE

\[
M' = [M, \Lambda],
\]

subjected to

\[
M_\lambda = \lambda^{-\frac{n}{2}} \left[ \Lambda^n + \text{lower degree terms w.r.t. } \text{deg} \right].
\]

Then the $N$-point functions (1.4.10) of $r$-spin intersection numbers have the following expressions

\[
\frac{dF^{r-spin}_{\mathfrak{g}}}{d\lambda} (\lambda) = -(M_\lambda)_{1,n+1}(\lambda) + \lambda^{\frac{r+1}{r}} \delta_{a,n}, \quad N = 1, \quad (1.4.12)
\]

\[
F^{r-spin}_{\alpha_1, \ldots, \alpha_N}(\lambda_1, \ldots, \lambda_N) = -\frac{1}{N} \sum_{\alpha \in SN} \text{Tr}\left( M_{\alpha_1}(\lambda_{s_1}) \cdots M_{\alpha_N}(\lambda_{s_N}) \right) \prod_{j=1}^{N} (\lambda_{s_j} - \lambda_{s_{j+1}}) \lambda_1^{\frac{m_a}{r}} \lambda_2^{\frac{m_a}{r}} (a_1 \lambda_1 + a_2 \lambda_2) \quad N \geq 2. \quad (1.4.13)
\]

**Example 1.4.4** ($r = 2$). Witten’s $2$-spin invariants coincide with intersection numbers of $\psi$-classes over $\overline{M}_{g,N} [49, 40, 25]$. So Theorem 1.4.3 with the choice $r = 2$ recovers the result of [6, 53]:

\[
\sum_{g \geq 0} \sum_{p_1, \ldots, p_N \geq 0} \frac{(2p_1 + 1)!! \cdots (2p_N + 1)!!}{2^{g+2} N^N} \int_{\overline{M}_{g,N}} \psi_{p_1}^{p_1} \cdots \psi_{p_N}^{p_N} \lambda_1^{\frac{2p_1+3}{2}} \cdots \lambda_N^{\frac{2p_N+3}{2}}
\]

\[
= -\frac{1}{N} \sum_{\alpha \in SN} \text{Tr}\left( M(\lambda_{r_1}) \cdots M(\lambda_{r_N}) \right) \prod_{j=1}^{N} (\lambda_{r_j} - \lambda_{r_{j+1}}) \lambda_1^{\frac{m_a}{r}} \lambda_2^{\frac{m_a}{r}} (a_1 \lambda_1 + a_2 \lambda_2) \quad N \geq 2.
\]
where

\[ M = \lambda^{-\frac{1}{2}} \begin{pmatrix} \sum_{g=0}^{\infty} \frac{(6g-5)!!}{96^g g!} \lambda^{-3g} & 2 \sum_{g=0}^{\infty} \frac{(6g-1)!!}{96^g g!} \lambda^{-3g+2} \\ \sum_{g=0}^{\infty} \frac{6g+1}{6g-1} \frac{(6g-1)!!}{96^g g!} \lambda^{-3g+1} & \frac{1}{2} \sum_{g=0}^{\infty} \frac{(6g-5)!!}{96^g g!} \lambda^{-3g+2} \end{pmatrix}. \]

For \( N = 1 \), it follows easily from (1.4.12) the well-known formula

\[ \langle \tau_{3g-2} \rangle_g = \frac{1}{24g!} \text{ for } g \geq 1. \]

**Example 1.4.5** \((r = 3)\). We obtain from Theorem 1.4.3 that the only nontrivial one-point correlators have the following explicit expressions

\[ \int M_{3m-2,1} c_W(1) \psi_1^{8m-7} = \frac{1}{6^{m-4} (m-1)! \left( \frac{1}{3} \right)_m}, \quad m \geq 1, \]

\[ \int M_{3m,1} c_W(2) \psi_1^{8m-2} = \frac{1}{6^m m! \left( \frac{2}{3} \right)_m}, \quad m \geq 1. \]

For \( N \geq 2 \), Witten's 3-spin correlators can be computed from the formulae

\[ F_{i_1, \ldots, i_N}^{3-\text{spin}}(\lambda_1, \ldots, \lambda_N) = -\frac{1}{N} \sum_{s \in SN} \frac{\text{Tr} \left( M_{i_1} (\lambda_{s_1}) \cdots M_{i_N} (\lambda_{s_N}) \right)}{\prod_{j=1}^N (\lambda_{s_j} - \lambda_{s_{j+1}})} - \delta_{N2} \eta_{i_1i_2} \frac{\lambda_1^{i_1} \lambda_2^{i_2} (i_1 \lambda_1 + i_2 \lambda_2)}{(\lambda_1 - \lambda_2)^2} \]

with explicit formulae of \( M_a(\lambda) \) given in Appendix A.

**Organization of the paper.** In Section 2 we introduce the definition of tau-function and prove Theorem 1.3.1. In Section 3 we define the essential series of \( \mathfrak{g} \). In Section 4, we prove Theorem 1.3.2.

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## 2 Tau-function of Drinfeld–Sokolov hierarchy

### 2.1 Fundamental lemma

Let \( \mathfrak{g} \) be a simple Lie algebra of rank \( n \), \( L(\mathfrak{g}) \) its loop algebra. Fix \( \mathfrak{h} \) a Cartan subalgebra of \( \mathfrak{g} \). We denote by \( \rho^\vee \in \mathfrak{h} \) the *Weyl co-vector* of \( \mathfrak{g} \), which is uniquely determined by the following equations

\[ \alpha_i(\rho^\vee) = 1, \quad i = 1, \ldots, n. \]  

(2.1.1)

Here \( \alpha_i \in \mathfrak{h}^* \) are simple roots. We define the *principal* grading operator \( \text{gr} \) on \( L(\mathfrak{g}) \) by

\[ \text{gr} = h\lambda \frac{d}{d\lambda} + \text{ad}_{\rho^\vee}. \]
It follows that \( \deg a = j \in \mathbb{Z} \) iff \( \gr a = j a, \forall a \in \mathcal{L}(g) \). We have the decomposition

\[
\mathcal{L}(g) = \bigoplus_{j \in \mathbb{Z}} \mathcal{L}(g)^j, \quad a \in \mathcal{L}(g)^j \iff \gr a = j a, \quad j \in \mathbb{Z}.
\]

For any \( a \in \mathcal{L}(g) \), we denote its principal decomposition by

\[
a = \sum_{j \in \mathbb{Z}} a[j], \quad a[j] \in \mathcal{L}(g)^j.
\]

The following lemma is elementary but it will be frequently used.

**Lemma 2.1.1.** Let \( x, y \) be any two elements in \( g = g \otimes 1 \) satisfying \( \gr x = k_1 x, \gr y = k_2 y \). If \( k_1 + k_2 \neq 0 \), then we have \( (x|y) = 0 \).

**Proof.** Suppose \( k_1 \neq 0 \). By definition, \( \gr x = k_1 x \) implies \( [\rho^\vee, x] = k_1 x \). So we have

\[
(x|y) = \frac{1}{k_1}([\rho^\vee, x]|y) = -\frac{1}{k_1}(x| [\rho^\vee, y]) = -\frac{k_2}{k_1} \frac{k_1 + k_2}{k_1}(x|y) \Rightarrow \frac{k_1 + k_2}{k_1}(x|y) = 0.
\]

The lemma is proved. \( \square \)

**Lemma 2.1.2** (fundamental lemma, [17]). Let \( q = q(x) \) be a \( b \)-valued smooth function, where \( b := g^{\leq 0} \). Let \( \mathcal{L} = \partial_x + \Lambda + q(x) \). Then there exists a unique pair \((U, H)\) of the form

\[
U = \sum_{k \geq 1} U^{[-k]}(\lambda; q, q_x, \cdots) \in A^q \otimes \text{Im ad}_A, \\
H = \sum_{j \in E_+} H^{[-j]}(\lambda; q, q_x, \cdots) \in A^q \otimes \text{Ker ad}_A,
\]

where \( \text{Im, Ker} \) are taken in \( g((\lambda^{-1})) \), and \( E_+ := \{ j \geq 0 | j \in E \} \), such that

\[
e^{-\text{ad}_U} \mathcal{L} = \partial_x + \Lambda + H.
\]

**Proof.** Eq. (2.1.4) is equivalent to

\[
e^{-U} \circ \partial_x \circ e^U + e^{-\text{ad}_U} (q + \Lambda) = \partial_x + \Lambda + H.
\]

More explicitly this reads

\[
\sum_{j=0}^{\infty} \frac{(-\text{ad}_U)^j}{j!} \left( \frac{U_x}{j+1} + q + \Lambda \right) = \Lambda + H.
\]

Comparing components with principal degree \(-k\) of both sides of (2.1.5) we obtain

\[
H^{[-k]} + \left[ U^{[-k-1]}, \Lambda \right] = G_k \left( \lambda; q; U^{[-1]}, \ldots, U^{[-k]}; \partial_x(U^{[-1]}), \ldots, \partial_x(U^{[-k]}) \right), \quad k \geq 0.
\]

Here, \( G_k \in \mathcal{L}(g) \), \( k \geq 0 \). Moreover, entries of \( G_k \) are polynomials in the entries of

\[
q, U^{[-1]}, \ldots, U^{[-k]}, \partial_x(U^{[-1]}), \ldots, \partial_x(U^{[-k]})
\]

whose coefficients are polynomials in \( \lambda \). The proof proceeds by induction on the principal degree. First, for \( k = 0 \) eq. (2.1.6) reads

\[
H^{[0]} + \left[ U^{[-1]}, \Lambda \right] = q^{[0]}.
\]
Observe that an element \( x \in \mathfrak{g} \) has zero principal degree iff \( x \in \mathfrak{h} \). So \( q^0 \) belongs to \( \mathfrak{h} \). Let us show that \( \mathfrak{h} \subset \text{Im} \text{ad}_\Lambda \). This is equivalent to orthogonality

\[
(x \mid \Lambda_m) = 0 \quad \text{for any } x \in \mathfrak{h}, \quad a = 1, \ldots, n. \tag{2.1.8}
\]

Indeed, by Lemma 2.1.1, any element \( y \in \mathfrak{g} \) of nonzero principal degree is orthogonal to \( \mathfrak{h} \). It remains to recall that any \( \Lambda_m \) has the form \( \Lambda_m = L_m + \lambda K_{m^{-h}} \), where \( L_m \) and \( K_{m^{-h}} \) belong to \( \mathfrak{g} \) and have nonzero principal degree. This proves orthogonality (2.1.8). So we have \( H^0 = 0 \). Noting that the map \( \text{ad}_\Lambda : \text{Im} \text{ad}_\Lambda \rightarrow \text{Im} \text{ad}_\Lambda \) is invertible, and we have

\[
U^{[-1]} = \text{ad}^{-1}_\Lambda (q^0) \in \text{Im} \text{ad}_\Lambda. \tag{2.1.9}
\]

The induction step clearly follows from eq. (2.1.6) and the decomposition

\[
L(\mathfrak{g}) = \text{Ker} \text{ad}_\Lambda \oplus \text{Im} \text{ad}_\Lambda.
\]

The lemma is proved.

Example 2.1.3. Looking at equation (2.1.5) with principal degree \(-1\), we have

\[
H^{[-1]} + [U^{[-2]}, \Lambda] = \frac{1}{2} [U^{[-1]}, [U^{[-1]}, \Lambda]] + \partial_x (U^{[-1]}) - [U^{[-1]}, q^0] + q^{[-1]}.
\]

Since \( U^{[-2]} \) is assumed to be orthogonal to \( \text{Ker} \text{ad}_\Lambda \), this equation uniquely determines \( H^{[-1]} \) and \( U^{[-2]} \) as indicated in the above proof.

2.2 \( \mathfrak{g} \)-valued resolvents

Definition 2.2.1. Let \( q = q(x) \in \mathfrak{b} \). An element \( R \in A^q \otimes \mathfrak{g}((\lambda^{-1})) \) is called a resolvent of \( \mathcal{L} \) if

\[
[\mathcal{L}, R] = 0. \tag{2.2.1}
\]

The set of all resolvents of \( \mathcal{L} \) is denoted by \( \mathcal{M}_\mathcal{L} \), called the resolvent manifold.

For more about resolvents see for example [6] [14] [15] [17] [29].

Lemma 2.2.2 ([17]). We have

\[
\mathcal{M}_\mathcal{L} = e^{\text{ad}_U} (\text{Ker} \text{ad}_\Lambda),
\]

where we note that the kernel\(^1\) is taken in \( \mathfrak{g}((\lambda^{-1})) \).

Proof. Lemma 2.1.2 reduces the problem to considering the resolvent manifold of \( \partial_x + \Lambda + H \). So, let us look at the following equation for \( R_H \in A^q \otimes \mathfrak{g}((\lambda^{-1})) \):

\[
[R_H, \partial_x + \Lambda + H] = 0.
\]

Decompose

\[
R_H = R^\ker_H + R^\text{im}_H, \quad R^\ker_H \in A^q \otimes \text{Ker} \text{ad}_\Lambda, \quad R^\text{im}_H \in A^q \otimes \text{Im} \text{ad}_\Lambda.
\]

It follows that

\[
\frac{\partial R^\ker_H}{\partial x} + \frac{\partial R^\text{im}_H}{\partial x} = [R^\text{im}_H, \Lambda + H].
\]

\(^1\)In the published version of this paper, the kernel is taken in \( L(\mathfrak{g}) \), so the resolvent manifold considered there is smaller and the homogeneity condition for Definition 1.1.1 (cf. Proposition 2.2.3) is not needed. The corrections made here and the addition of the homogeneity condition in Definition 1.1.1 are more consistent with Definition 2.2.1.
The right hand side of the above equation is in the image of \( \text{ad}_\Lambda \), so we have

\[
\frac{\partial R^\ker_H}{\partial x} = 0, \quad (2.2.2)
\]
\[
\frac{\partial R^\im_H}{\partial x} = [R^\im_H, \Lambda + H]. \quad (2.2.3)
\]

Equation (2.2.2) implies that \( R^\ker_H \) can only depend on \( \lambda \). The rest is to show that \( R^\im_H \) must vanish. If it does not vanish, then there exists an integer \( d \) such that

\[
R^\im_H = \sum_{i=-\infty}^{d} R^\im_H, \quad R^\im_H, \neq 0.
\]

Noting that \( \deg H < 0 \), then looking at the highest degree term on both sides of eq. (2.2.3) we obtain

\[
[\Lambda, R^\im_H, \neq 0].
\]

So we have \( R^\im_H, \neq 0 \). This produces a contradiction. The lemma is proved. \( \square \)

**Proposition 2.2.3.** There exist unique series \( R_1, \ldots, R_n \) satisfying the following system of equations

\[
\begin{align*}
[\mathcal{L}, R_a] &= 0, \quad R_a \in \mathcal{A}^q \otimes \mathfrak{g}((\lambda^{-1})), \quad (2.2.4) \\
R_a(\lambda; q, q_x, \ldots) &= \Lambda_{m_a} + \text{lower order terms w.r.t. } \deg, \quad (2.2.5) \\
(R_a(\lambda; q, q_x, \ldots) \mid R_b(\lambda; q, q_x, \ldots)) &= h \eta_{ab} \lambda, \quad (2.2.6)
\end{align*}
\]

together with the requirements that \( R_a \) are homogeneous of the extended principal degrees \( m_a \).

This unique system of solutions \( R_1, \ldots, R_n \) is called in Section 1 the basic resolvents of the operator \( \mathcal{L} \).

**Proof.** The existence follows from the fact that \( e^{\text{ad}_U}(\Lambda_{m_a}) \) is a solution, where (2.2.6) is due to (1.1.6), and (2.2.5) is due to (2.1.2). The uniqueness follows from Lemma 2.2.2. \( \square \)

**Corollary 2.2.4.** Let \( U \) be defined as in Lemma 2.1.2. Then the basic resolvents \( R_a \) satisfy

\[
R_a = e^{\text{ad}_U}(\Lambda_{m_a}), \quad a = 1, \ldots, n.
\]

From this corollary we promptly deduce the following commutativity between the basic resolvents:

\[
[R_a, R_b] = 0. \quad (2.2.7)
\]

**Definition 2.2.5.** Define \( P_{m_a+kh} := \lambda^k R_a = e^{\text{ad}_U}(\Lambda_{m_a+kh}) \), \( k \geq 0 \).

The pre-DS hierarchy can be written as

\[
\frac{\partial \mathcal{L}}{\partial T^a_k} = [(P_{m_a+kh})_+, \mathcal{L}], \quad k \geq 0.
\]

As customary in the literature, we will sometimes write \( T^a_k \) as \( T_{m_a+kh} \), \( a = 1, \ldots, n, k \geq 0 \).

**Lemma 2.2.6.** For every \( i, j \in E_+ \), we have

\[
\begin{align*}
\frac{\partial P_j}{\partial T^+_i} &= [(P_i)_+, P_j], \quad (2.2.8) \\
\frac{\partial (P_i)_+}{\partial T^+_j} - \frac{\partial (P_j)_+}{\partial T^+_i} + [(P_i)_+, (P_j)_+] &= 0. \quad (2.2.9)
\end{align*}
\]
Proof. Using the fundamental lemma 2.1.2 we have
\[
\frac{\partial L}{\partial T_i} = [(P_i)_+, L] \Rightarrow [\partial T_i - (P_i)_+, L] = 0 \Rightarrow [\partial T_i + S_i, \partial x + \Lambda + H] = 0,
\]
where \( S_i := \sum_{k=0}^{\infty} (-1)^k a d_U^k \left( \frac{\partial L}{\partial T_i} \right) - e^{-ad_U} [(P_i)_+] \). Clearly, \( S_i \) takes values in \( A^q \otimes g((\lambda^{-1})) \). Decompose
\[
S_i = S_i^{\text{ker}} + S_i^{\text{im}}, \quad S_i^{\text{ker}} \in A^q \otimes \text{Ker} \text{ad}_A, \quad S_i^{\text{im}} \in A^q \otimes \text{Im} \text{ad}_A.
\]
Then we have
\[
\frac{\partial H}{\partial T_i} - \partial S_i + [S_i, \Lambda + H] = 0 \Rightarrow \begin{cases}
\frac{\partial H}{\partial T_i} - \frac{\partial S_i^{\text{ker}}}{\partial x} = 0, \\
\frac{\partial S_i^{\text{im}}}{\partial x} = [S_i^{\text{im}}, \Lambda + H].
\end{cases}
\]
Using the same argument as in the proof of Lemma 2.2.7 we find from the above equation for \( S_i^{\text{im}} \) that \( S_i^{\text{im}} \) must vanish. So \( S_i \) belongs to \( A^q \otimes \text{Ker} \text{ad}_A \). On another hand,
\[
\frac{\partial P_j}{\partial T_i} = [(P_j)_+, P_j] \iff [\partial T_i - (P_j)_+, P_j] = 0 \iff [\partial T_i + S_i, A_j] = 0.
\]
Hence eq. (2.2.8) is proved. Clearly eq. (2.2.8) implies eq. (2.2.9); this is because
\[
\text{LHS of eq. (2.2.9)} = [(P_j)_+, P_i] + [(P_i)_+, P_j] + [(P_i)_+, (P_j)_+] = 0.
\]
\[ \square \]

Lemma 2.2.7. \( \forall a = 1, \ldots, n \), we have
\[
\nabla_a(\lambda) R_b(\mu) = \frac{[R_a(\lambda), R_b(\mu)]}{\lambda - \mu} - [Q_a, R_b(\mu)], \quad Q_a := \text{Coef}(R_a(\lambda), \lambda^1). \quad (2.2.10)
\]
Proof. We have
\[
\nabla_a(\lambda) R_b(\mu) = \sum_{k \geq 0} \frac{\partial T_i^k R_b(\mu)}{\lambda^{k+1}} = \sum_{k \geq 0} \frac{[\mu^k R_a(\mu)]]_+, R_b(\mu)]}{\lambda^{k+1}} \\
= - \sum_{k \geq 0} \left[ \text{res}_{\rho=\infty} \frac{\rho^k R_a(\rho)}{\rho - \mu} d\rho, R_b(\mu) \right] \\
= \frac{1}{2\pi i} \oint_{|\rho|<|\lambda|} d\rho \frac{[R_a(\rho), R_b(\mu)]}{(\lambda - \rho)(\rho - \mu)} \\
= \frac{[R_a(\lambda), R_b(\mu)]}{\lambda - \mu} - [\text{Coef}(R_a(\lambda), \lambda^1) R_b(\mu)].
\]
\[ \square \]

2.3 Two-point correlation functions

Recall that in Definition 1.2.1, the two-point correlation functions \( \Omega_{a,b} \) was defined by
\[
\sum_{k,\ell \geq 0} \frac{\Omega_{a,k; b,\ell}}{\lambda^{k+1} \mu^{\ell+1}} = \frac{(R_a(\lambda) | R_b(\mu))}{(\lambda - \mu)^2} - \eta_{ab} m_a \lambda + m_b \mu. \quad (2.3.1)
\]
Lemma 2.3.1. Definition 1.2.1, i.e., the above formula (2.3.1) is well-posed.

Proof. Noting that\(^2\)

\[ R_b(\mu) = R_b(\lambda) + R'_b(\lambda)(\mu - \lambda) + (\mu - \lambda)^2 \partial_\lambda \left( \frac{R_b(\lambda) - R_b(\mu)}{\lambda - \mu} \right) \tag{2.3.2} \]

and using eqs. (1.1.6) we have

\[ \frac{(R_a(\lambda) | R'_b(\lambda))}{(\lambda - \mu)^2} = \eta_{ab} \frac{h \lambda}{(\lambda - \mu)^2} - \frac{(R_a(\lambda) | R_b(\lambda))}{\lambda - \mu} + \left( R_a(\lambda) \mid \partial_\lambda \left( \frac{R_b(\lambda) - R_b(\mu)}{\lambda - \mu} \right) \right) \]

In the above formulae, prime, “\(\prime\)”, denotes derivative w.r.t. the spectral parameter. Since \(R_a(\lambda) = O(\lambda^1)\), \(a = 1, \ldots, n\), we know that the third term in the above identity has the form as the left hand side of (1.2.1). Therefore it remains to show

\[ \frac{h \lambda}{(\lambda - \mu)^2} - \frac{(R_a(\lambda) | R_b(\lambda))}{\lambda - \mu} - \eta_{ab} \frac{m_a \lambda + m_b \mu}{(\lambda - \mu)^2} \]

has the form as the left hand side of (1.2.1). We will actually prove that the above expression vanishes. Indeed,

\[ \partial_x (R_a(\lambda) | R'_b(\lambda)) = \left( [R_a(\lambda), \Lambda + q] | R'_b(\lambda) \right) + \left( R_a(\lambda) \right) | \left[ R'_b(\lambda), \Lambda + q \right] + \left[ R_b(\lambda), \Lambda' \right] = 0. \tag{2.3.3} \]

Here we have used the ad-invariance of the Cartan–Killing form and the commutativity (2.2.7) between resolvents. Noting that \(R_a \in \mathcal{A}^\vee \otimes \mathfrak{g}(\lambda^{-1})\), we find that (2.3.3) implies that \((R_a(\lambda) | R'_b(\lambda))\) does not depend on \(q, q_e, q_{2e}, \ldots\), i.e. it is just a function of \(\lambda\). Hence

\[ (R_a(\lambda) | R'_b(\lambda)) = (R_a(\lambda) | R'_b(\lambda))_{q(x) \equiv 0} = (\Lambda_{ma} | \Lambda'_{mb}) . \]

The second equality uses (2.2.6). To compute \((\Lambda_{ma} | \Lambda'_{mb})\), as before, write

\[ \Lambda_{ma} = L_{ma} + \lambda K_{ma-h}, \quad L_{ma} \in \mathfrak{g}^{ma}, \quad K_{ma-h} \in \mathfrak{g}^{ma-h}, \quad a = 1, \ldots, n. \]

Using Lemma 2.1.1 we have

\[ (\Lambda_{ma} | \Lambda'_{mb}) = (L_{ma} | K_{mb-h}) . \]

Note that \((\Lambda_{ma} | \Lambda_{mb}) = \eta_{ab} h \lambda\) implies that

\[ (L_{ma} | K_{mb-h}) + (L_{mb} | K_{ma-h}) = \eta_{ab} h. \tag{2.3.4} \]

The commutativity \([\Lambda_{ma}, \Lambda_{mb}] = 0\) implies that

\[ [K_{ma-h}, L_{ma}] + [L_{ma}, K_{mb-h}] = 0. \]

Applying \((\rho^\vee \mid \cdot)\) to the above equation and using the ad-invariance of \((\cdot | \cdot)\) we have

\[ ([\rho^\vee, K_{ma-h}] | L_{ma}) + ([\rho^\vee, L_{ma}] | K_{mb-h}] = 0 \Rightarrow (m_a - h) (K_{ma-h} | L_{mb}) + m_a (L_{ma} | K_{mb-h}) = 0. \]

Combining eqs. (2.3.4) and the above equation we obtain

\[ (L_{ma} | K_{mb-h}) = \eta_{ab} m_b, \quad \forall a, b. \tag{2.3.5} \]

Hence

\[ \eta_{ab} \frac{h \lambda}{(\lambda - \mu)^2} - \frac{(R_a(\lambda) | R'_b(\lambda))}{\lambda - \mu} - \eta_{ab} \frac{m_a \lambda + m_b \mu}{(\lambda - \mu)^2} = 0. \]

The lemma is proved. \(\blacksquare\)

\(^2\)We would like to thank Anton Mellit for bringing our attention to the useful formula (2.3.2).
Proposition 2.3.2. The following formulae hold true
\[
\sum_{k \geq 0} \frac{\Omega_{a,k:b,0}}{\lambda^{k+1}} = (R_{a}(\lambda) | Q_{b}) - \eta_{ab} m_{b}, \quad \forall a, b.
\] (2.3.6)

In particular, we have
\[
\sum_{k \geq 0} \frac{\Omega_{a,k:1,0}}{\lambda^{k+1}} = (R_{a}(\lambda) | E_{-\theta}) - \eta_{a1}.
\] (2.3.7)

Proof. Taking in (2.3.1) the residue w.r.t. \( \mu = \infty \) we obtain (2.3.6). Noticing that
\[
R_{1}(\mu) = \mu E_{-\theta} + I_{+} + \text{terms with principal degree lower than 1}
\]
we must have \( Q_{1} = \text{Coef} (R_{1}(\mu), \mu^{1}) = E_{-\theta} \). This proves (2.3.7). \( \blacksquare \)

2.4 Tau-function: Proof of Lemmas 1.2.2, 1.2.3

We are ready to introduce our definition of tau-function. We begin with the proof of Lemma 1.2.2.

Proof of Lemma 1.2.2. First of all we have
\[
\sum_{k,\ell \geq 0} \frac{\Omega_{a,k:b,\ell}}{\lambda^{k+1} \mu^{\ell+1}} = \frac{(R_{a}(\lambda) | R_{b}(\mu))}{(\lambda - \mu)^{2}} - \eta_{ab} \frac{m_{a} \lambda + m_{b} \mu}{(\lambda - \mu)^{2}} = \frac{(R_{b}(\mu) | R_{a}(\lambda))}{(\mu - \lambda)^{2}} - \eta_{ba} \frac{m_{b} \mu + m_{a} \lambda}{(\mu - \lambda)^{2}}
\]
\[
= \sum_{k,\ell \geq 0} \frac{\Omega_{b,k:a,\ell}}{\mu^{k+1} \lambda^{\ell+1}} = \sum_{k,\ell \geq 0} \frac{\Omega_{b,\ell:a,k}}{\mu^{\ell+1} \lambda^{k+1}},
\]

where we have used the symmetry property of \( \eta_{ab} \) and \( (\cdot | \cdot) \). It follows \( \Omega_{a,k:b,\ell} = \Omega_{b,\ell:a,k} \).

Secondly, by using Lemma 2.2.7 we have
\[
\sum_{k,\ell,m \geq 0} \frac{\partial \tau_{m}}{\xi^{m+1} \lambda^{k+1} \mu^{\ell+1}} \sum_{k,\ell \geq 0} \frac{\Omega_{a,k:b,\ell}}{\lambda^{k+1} \mu^{\ell+1}} = \nabla_{c}(\xi) \sum_{k,\ell \geq 0} \frac{\Omega_{a,k:b,\ell}}{\lambda^{k+1} \mu^{\ell+1}}
\]
\[
= \frac{(\nabla_{c}(\xi) R_{a}(\lambda) | R_{b}(\mu))}{(\lambda - \mu)^{2}} + \frac{(R_{a}(\lambda) | \nabla_{c}(\xi) R_{b}(\mu))}{(\lambda - \mu)^{2}}
\]
\[
= \frac{([R_{c}(\xi), R_{a}(\lambda)] | R_{b}(\mu))}{(\lambda - \mu)^{2} (\xi - \lambda)} - \frac{([Q_{c}, R_{a}(\lambda)] | R_{b}(\mu))}{(\lambda - \mu)^{2}}
\]
\[
+ \frac{(R_{a}(\lambda) | [R_{c}(\xi), R_{b}(\mu))}{(\lambda - \mu)^{2} (\xi - \mu)} - \frac{(R_{a}(\lambda) | [Q_{c}, R_{b}(\mu))}{(\lambda - \mu)^{2}}.
\]

Clearly the two terms with negative signs give a zero contribution due to the ad-invariance of the Cartan–Killing form. The remaining two terms simplify to
\[
\frac{([R_{c}(\xi), R_{a}(\lambda)] | R_{b}(\mu))}{(\lambda - \mu)^{2}} \left( \frac{1}{\xi - \lambda} - \frac{1}{\xi - \mu} \right) = \frac{([R_{c}(\xi), R_{a}(\lambda)] | R_{b}(\mu))}{(\lambda - \mu)^{2} (\mu - \xi) (\xi - \lambda)}.
\]

So we have
\[
\sum_{k,\ell,m \geq 0} \frac{\partial \tau_{m}}{\xi^{m+1} \lambda^{k+1} \mu^{\ell+1}} = - \frac{([R_{c}(\xi), R_{a}(\lambda)] | R_{b}(\mu))}{(\lambda - \mu)^{2} (\mu - \xi) (\xi - \lambda)}.
\]

This gives also
\[
\sum_{k,\ell,m \geq 0} \frac{\partial \tau_{m}}{\lambda^{k+1} \xi^{m+1} \mu^{\ell+1}} = - \frac{([R_{a}(\lambda), R_{c}(\xi)] | R_{b}(\mu))}{(\xi - \mu) (\lambda - \xi)}.
\]

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Hence
\[ \partial T_m^a(\Omega_{a,k;b,\ell}) = \partial T_m^a(\Omega_{c,m;b,\ell}) \] (2.4.1)
due to skew-symmetry of the Lie bracket. The lemma is proved.

**Proof** of Lemma 1.2.3. It suffices to show the compatibility between (1.2.5) and (1.2.4), namely, to show that
\[ \frac{\partial \Omega_{a,k;b,\ell}}{\partial T_{1,0}} = -\frac{\partial \Omega_{a,k;b,\ell}}{\partial x}. \] (2.4.2)
Taking \( c = 1, m = 0 \) in the already proved identity (2.4.1) we have
\[ \partial T_m^a(\Omega_{1,0;b,\ell}) = \partial T_m^a(\Omega_{a,k;b,\ell}). \]
Hence (2.4.2) is equivalent to
\[ \frac{\partial \Omega_{1,0;b,\ell}}{\partial T_{a,k}} = -\frac{\partial \Omega_{a,k;b,\ell}}{\partial x}. \]
Let us make a generating function. Then the above identity is equivalent to
\[ \sum_{k,\ell} \frac{\partial \Omega_{1,0;b,\ell}}{\partial T_{a,k}} z^{-k-1} w^{-\ell-1} = -\sum_{k,\ell} \frac{\partial \Omega_{a,k;b,\ell}}{\partial x} z^{-k-1} w^{-\ell-1}. \]
We have
\[ -\text{RHS} = \left( \frac{\partial_x R_a(z) | R_b(w)}{(z-w)^2} + \frac{(R_a(z)|\partial_x R_b(w))}{(z-w)^2} \right) \]
\[ = \left( \frac{[R_a(z), \Lambda(z) + q] | R_b(w)}{(z-w)^2} + \frac{(R_a(z)|[R_b(w), \Lambda(w) + q])}{(z-w)^2} \right) \]
\[ = \left( \frac{\Lambda(z) + q[R_b(w), R_a(z)]}{(z-w)^2} - \frac{(\Lambda(w) + q[R_b(w), R_a(z)])}{(z-w)^2} \right) \]
\[ = \left( \frac{\Lambda(z) - (\Lambda(w)|[R_b(w), R_a(z)])}{(z-w)^2} \right). \]
Recall that
\[ \Lambda(z) = I_+ + zE_{-\theta}, \quad \Lambda(w) = I_+ + wE_{-\theta}. \]
So we have
\[ -\text{RHS} = \left( \frac{(z-w)E_{-\theta}|[R_b(w), R_a(z)]}{(z-w)^2} \right) = \left( \frac{(E_{-\theta}|[R_b(w), R_a(z)])}{z-w} \right). \]
On another hand, we have
\[ \text{LHS} = \nabla_a(z) \sum_{\ell} \Omega_{1,0;b,\ell} w^{-\ell-1} \]
\[ = \nabla_a(z) \left[ (E_{-\theta}|R_b(w)) + \text{const} \right] \]
\[ = \left( E_{-\theta}|\nabla_a(z) [R_b(w)] \right) \]
\[ = \left( E_{-\theta}|[R_a(z), R_b(w)] \right) \frac{1}{z-w} + \left( E_{-\theta}|[Q_a, R_b(w)] \right). \]
We note that the second term of the last expression must be zero because
\[ \deg Q_a + h \leq m_a \quad \Rightarrow \quad [E_{-\theta}, Q_a] = 0. \] (2.4.3)
The lemma is proved.

Hence we have arrived at our definition of tau-function, i.e., Definition 1.2.4.
2.5 Gauge invariance

In this subsection, we show that the tau-function in Definition 1.2.4 is, in fact, gauge invariant. Recall that the change of the Lax operator

\[ L = \partial_x + \Lambda + q(x) \quad \mapsto \quad \tilde{L} = e^{\text{ad}_x N(x)} L = \partial_x + \Lambda + \tilde{q}(x), \quad N(x) \in \mathfrak{n} \]  

(2.5.1)

is called a gauge transformation: \( q \mapsto \tilde{q} \). It will also be convenient to deal with the infinitesimal form of (1.1.14), \( \delta L = L + \delta \),

\[ \delta \L := [N(x), \L] = \left[ N(x), q(x) + I_+ \right] - \frac{\partial N(x)}{\partial x}. \]  

(2.5.2)

Let \( \tilde{R}_a, a = 1, \ldots, n \) be the basic resolvents of \( \tilde{\L} \). It is not difficult to verify that \( \tilde{R}_a = e^{\text{ad}_x N(x)} R_a \).

**Lemma 2.5.1.** The gauge transformations (1.1.14) are symmetries of the pre-DS hierarchy.

**Proof.** We have to prove the commutativity

\[ \frac{\partial}{\partial s} \frac{\partial \L}{\partial T} = \frac{\partial}{\partial T} \frac{\partial \L}{\partial s} \]

between the \( j \)-th flow of the pre-DS hierarchy

\[ \frac{\partial \L}{\partial T_j} = [(P_j)_+, \L], \quad j \in \mathcal{E}_+ \]

and the flow given by the infinitesimal gauge transformation

\[ \frac{\partial \L}{\partial s} = [N, \L] \]

for some \( n \)-valued function \( N = N(x) \). Using (1.1.15) we derive

\[ \frac{\partial P_j}{\partial s} = [N, P_j]. \]

So, after simple calculations with the help of the Jacobi identity we compute the difference between the mixed derivatives

\[ \frac{\partial}{\partial s} \frac{\partial \L}{\partial T} - \frac{\partial}{\partial T} \frac{\partial \L}{\partial s} = [[N, P_j]_+ - [N, (P_j)_+], \L] = 0. \]

The two-point correlation functions \( \tilde{\Omega}_{a,k;b,\ell}, k, \ell \geq 0 \) associated to \( \tilde{\L} \) are defined by

\[ \sum_{k,\ell \geq 0} \tilde{\Omega}_{a,k;b,\ell} = \frac{\tilde{\Omega}_{a,k;b,\ell}}{\lambda^{k+\mu+1}} = \left( \frac{\tilde{R}_a(\lambda)}{\lambda-\mu} \right) \left( \frac{\tilde{R}_b(\mu)}{\lambda-\mu} \right) - \eta_{ab} \frac{m_a \lambda + m_b \mu}{(\lambda-\mu)^2}. \]  

(2.5.3)

**Lemma 2.5.2.** \( \forall a, b, \forall k, \ell \geq 0, \) we have \( \tilde{\Omega}_{a,k;b,\ell} = \Omega_{a,k;b,\ell} \).

**Proof.** \( \left( \tilde{R}_a(\lambda) \mid \tilde{R}_b(\mu) \right) = \left( e^{\text{ad}_x N(x)} R_a(\lambda) \mid e^{\text{ad}_x N(x)} R_b(\mu) \right) = (R_a(\lambda) \mid R_b(\mu)). \)

In a similar way one can easily prove that \( \forall N \geq 2 \) the correlation functions \( \langle \tau_{a_1 k_1} \ldots \tau_{a_N k_N} \rangle^{DS} \) are gauge invariant.

Now we are ready to prove Lemma 1.2.6. 

**Proof of Lemma 1.2.6.** The lemma can be proved by using Lemma 2.5.2 and Definition 1.2.4. 

Due to Lemma 1.2.6, \( \forall N \geq 3 \) the correlation functions \( \langle \tau_{a_1 k_1} \ldots \tau_{a_N k_N} \rangle^{DS} \) are gauge invariant.
2.6 Gauge fixing and Drinfeld–Sokolov hierarchy

We consider in this section a particular family of gauges [17, 5, 21].

**Definition 2.6.1.** A linear subspace \( \mathcal{V} \subset \mathfrak{b} \) is called a gauge of DS-type if \( \mathfrak{b} = \mathcal{V} \oplus [I_+, \mathfrak{n}] \).

Let \( \mathcal{V} \) be a gauge of DS-type. The fact that \( \text{ad}_{I_+} : \mathfrak{n} \to \mathfrak{b} \) is injective implies \( \dim_{\mathbb{C}} \mathcal{V} = n \). Write

\[
\mathcal{V} = \bigoplus_{j = -(h-1)}^n \mathcal{V}^j, \quad \mathcal{V}^j \subset \mathfrak{g}^j.
\]

Denote \( \mathfrak{b}^j = \mathfrak{b} \cap \mathfrak{g}^j \). We have \( \mathfrak{b}^j = \mathcal{V}^j \oplus [I_+, \mathfrak{b}^{j-1}] \), \( j = -(h-1), \ldots, 0 \). Clearly, \( \mathcal{V}^{-(h-1)} = \mathbb{C}E_{-\theta} \). Noticing that for \( j = -(h-1), \ldots, 0 \), the dimension \( \dim \mathfrak{b}^j \) can be different from \( \dim \mathfrak{b}^{j-1} \) iff \( -j \) is an exponent of \( \mathfrak{g} \) [45, 17], we find that \( \mathcal{V}^j \) is a null space unless \( (-j) \) is an exponent. Thus

\[
\mathcal{V} = \bigoplus_{a=1}^n V_a, \quad \dim_{\mathbb{C}} V_a = 1,
\]

where non-zero elements in \( V_a \) have principal degree \( -m_a \). We now take a basis \( \{X^1, \ldots, X^n\} \) of \( \mathcal{V} \) satisfying \( \text{deg} X^a = -m_a \). It has been proved in [17] that for any Lax operator \( \mathcal{L} = \partial_x + \Lambda + q(x) \), there exists a unique \( N^\text{can}(x) \in \mathcal{A}_q \otimes \mathfrak{n} \) such that

\[
e^{\text{ad}_{N^\text{can}(x)}} \mathcal{L} = \partial_x + \Lambda + q^\text{can}(x) =: \mathcal{L}^\text{can}, \quad \text{for some } \mathcal{V} \text{-valued function } q^\text{can}.
\]

Write \( q^\text{can} = \sum_{a=1}^n w_a X^a = (w_1, \ldots, w_n) \). The DS-flows of \( q^\text{can} \), or say of \( w_a \), can be written as

\[
\frac{\partial q^\text{can}}{\partial T^a_k} = \left[ (\lambda^k R^\text{can}_a)_{+}, \mathcal{L}^\text{can} \right] + \left[ \frac{\partial e^{N^\text{can}_a}}{\partial T^a_k} e^{-N^\text{can}_a}, \mathcal{L}^\text{can} \right].
\]

A priori, the right hand side of (2.6.2) has a dependence on \( q \), as we can see from the second term that it contains flow of components of \( N^\text{can} \). However, Lemma 2.5.1 says that the gauge transformation is a symmetry of the pre-DS hierarchy. So the right hand side of (2.6.2) depends only on \( q^\text{can} \), i.e., \( w_a, a = 1, \ldots, n \) satisfy equations of the form

\[
\frac{\partial w_a}{\partial T^b_k} = G_{a,b,k} (q^\text{can}, q^\text{can}_x, q^\text{can}_{xx}, \ldots), \quad k \geq 0.
\]

**Definition 2.6.2.** Equations (2.6.3) are called the **DS hierarchy of g-type** associated to \( \mathcal{V} \).

Let \( R^\text{can}_a \) be the basic resolvents of \( \mathcal{L}^\text{can} \), and \( \Omega^\text{can}_{a,k;b,\ell} \) the two-point correlation functions of \( \mathcal{L}^\text{can} \), i.e.,

\[
\sum_{k,\ell \geq 0} \frac{\Omega^\text{can}_{a,k;b,\ell}}{\lambda^{k+1} \mu^{\ell+1}} = \frac{\langle R^\text{can}_a(\lambda) | R^\text{can}_b(\mu) \rangle}{(\lambda - \mu)^2} - \eta_{ab} \frac{m_a \lambda + m_b \mu}{(\lambda - \mu)^2}.
\]

**Corollary 2.6.3.** Let \( \tau(T) \) be a tau-function of the DS hierarchy. The following formulae hold true

\[
\frac{\partial^2 \log \tau}{\partial T^a_k \partial T^b_\ell} = \Omega^\text{can}_{a,k;b,\ell}, \quad \forall \, a, b = 1, \ldots, n, \, k, \ell \geq 0.
\]

**Proof.** By gauge invariance of two-point correlation functions.

We also call \( \tau(T) \) a tau-function of the solution \( q^\text{can}(T) = (w^1(T), \ldots, w^n(T)) \).
2.7 Proof of Theorem 1.3.1

The proof will be almost identical to the proof for the case $\mathfrak{g} = A_1$ case [6].

**Proof of Theorem 1.3.1.** For any permutation $s = [s_1, \ldots, s_p] \in S_p$, $p \geq 2$, define

$$P(s) := - \prod_{j=1}^{p} \frac{1}{\lambda_{s_j} - \lambda_{s_{j+1}}}, \quad \lambda_{s_{p+1}} \equiv \lambda_{s_1}.$$ 

We first prove the generating formula of multi-point correlation functions of a solution of the pre-DS hierarchy, then we use the ad-invariance of $B$ for the gauge-fixed case.

Let $\mathcal{L} = \partial_x + \Lambda + q(x)$, $q(x) \in \mathfrak{g}$ be a linear operator, $R_\alpha$ the basic resolvents of $\mathcal{L}$. For an arbitrary solution $q(x, T)$ to the pre-DS hierarchy (1.1.12), let $\tau(T)$ be the corresponding tau-function, and $F_{\alpha_1, \ldots, \alpha_N}(T)$, $N \geq 1$ the generating series of $N$-point correlations functions of $\tau(T)$.

We now use mathematical induction to prove formula (1.3.4) with $R^\text{can}$ replaced by $R$. For $N = 2$, the formula is true by definition. Suppose it is true for $N = p \ (p \geq 2)$, then for $N = p + 1$, we have

$$F_{\alpha_1, \ldots, \alpha_{p+1}}(\lambda_1, \ldots, \lambda_{p+1}; T) = \nabla_{\alpha_{p+1}}(\lambda_{p+1})F_{\alpha_1, \ldots, \alpha_p}(\lambda_1, \ldots, \lambda_p; T)$$

$$= - \frac{1}{2h^p} \nabla_{\alpha_{p+1}}(\lambda_{p+1}) \sum_{s \in S_p} B\left( R_{\alpha_{s_1}}(\lambda_{s_1}), \ldots, R_{\alpha_{s_p}}(\lambda_{s_p}) \right)$$

$$= - \frac{1}{2h^p} \sum_{s \in S_p} \sum_{q=1}^{p} B\left( R_{\alpha_{s_1}}(\lambda_{s_1}), \ldots, R_{\alpha_{s_{q-1}}}(\lambda_{s_{q-1}}), R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_q}}(\lambda_{s_q}) \right.$$ 

$$\left. \ldots, R_{\alpha_{s_p}}(\lambda_{s_p}) \right) \prod_{j=1}^{p} \left( \lambda_{s_j} - \lambda_{s_{j+1}} \right).$$

Recall that the elements $Q_\alpha \in \mathfrak{g}$ were defined in eq. (2.2.10). Now we observe that the terms containing the commutator with $Q_{\alpha_{p+1}}$ sum up to zero due to the ad–invariance of $B$, namely due to the formula

$$\sum_{q=1}^{p} \left( [A, X_q], X_{q+1}, \ldots, X_p \right) = 0, \ \forall X_1, \ldots, X_p, A \in \mathfrak{g}.$$ 

Thus we are left with

$$= \frac{1}{2h^p} \sum_{s \in S_p} P(s) \sum_{q=1}^{p} \left( B\left( R_{\alpha_{s_1}}(\lambda_{s_1}), \ldots, R_{\alpha_{s_{q-1}}}(\lambda_{s_{q-1}}), R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_q}}(\lambda_{s_q}) \right.$$

$$\left. \ldots, R_{\alpha_{s_p}}(\lambda_{s_p}) \right) \prod_{j=1}^{p} \left( \lambda_{s_j} - \lambda_{s_{j+1}} \right) \right)$$

$$= \frac{1}{2h^p} \sum_{s \in S_p} P(s) \sum_{q=1}^{p} (\lambda_{s_q} - \lambda_{s_{q-1}})$$

$$= \frac{1}{2h^p} \sum_{s \in S_p} P(s) \sum_{q=1}^{p+1} \prod_{j=1}^{q} \left( \lambda_{s_j} - \lambda_{s_{j-1}} \right)$$

$$= \frac{1}{2h^p} \sum_{q=1}^{p+1} \sum_{s \in S_p} P([p+1, s_q, \ldots, s_p, s_1, \ldots, s_{p-1}])$$

$$B\left( R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_q}}(\lambda_{s_q}), \ldots, R_{\alpha_{s_p}}(\lambda_{s_p}), R_{\alpha_{s_1}}(\lambda_{s_1}), \ldots, R_{\alpha_{s_{p-1}}}(\lambda_{s_{p-1}}) \right)$$

$$= \frac{1}{2h^p} \sum_{s \in S_p} P([p+1, s]) B\left( R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_1}}(\lambda_{s_1}), \ldots, R_{\alpha_{s_p}}(\lambda_{s_p}) \right).$$
For any gauge $V$ of DS-type, there exists a unique $N(x) \in A_g \otimes n$ such that
$$e^{ad_{N(x)}} \mathcal{L} = \mathcal{L}^\text{can}.$$ Observing that $\tilde{R}_a = e^{ad_{N(x)}} R_a$ and using the Ad-invariance of $B$ we obtain
$$F_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N; T) = -\sum_{s \in S_N} B \left( \frac{R^\text{can}_{a_1}(\lambda_{s_1}) \cdots R^\text{can}_{a_N}(\lambda_{s_N})}{2N} \prod_j^{N} (\lambda_{s_j} - \lambda_{s_{j+1}}) \right) - \delta_{N2} \eta_{a_1a_2} \frac{m_{a_1} \lambda_1 + m_{a_2} \lambda_2}{(\lambda_1 - \lambda_2)^2}.$$ Finally, $F_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N; T) \in \mathcal{A}^\text{can}_V[[\lambda_1^{-1}, \ldots, \lambda_N^{-1}]]$ due to Lemma 2.2.3 (with $\mathcal{L}$ replaced by $\mathcal{L}^\text{can}$).

Theorem is proved.

**Corollary 2.7.1.** Let $V$ be a gauge of DS-type. For an arbitrary solution $q^\text{can}_V$ to the DS hierarchy of $g$-type associated to $V$, let $\tau$ be the tau-function of this solution. The following formulae hold true
$$\sum_{k \geq 0} \frac{\langle (\tau_{a,k} \tau_{b,0}) \rangle^\text{DS}_{\lambda^{k+1}}}{\lambda^{k+1}} = (R^\text{can}_a(\lambda) \mid Q^\text{can}_{b}) - \eta_{ab} m_b, \quad a, b = 1, \ldots, n. \quad (2.7.1)$$

In particular, we have
$$\sum_{k \geq 0} \frac{\langle (\tau_{a,k} \tau_{1,0}) \rangle^\text{DS}_{\lambda^{k+1}}}{\lambda^{k+1}} = (R^\text{can}_a(\lambda) \mid E_{-\theta}) - \eta_{a1}, \quad a = 1, \ldots, n. \quad (2.7.2)$$

**Proof.** Taking in (1.3.4) with $N = 2$ the residue w.r.t. $\mu$ at $\mu = \infty$ we obtain (2.7.1). To show (2.7.2), we only need to notice that for $b = 1$, $\text{Coeff}(R^\text{can}_1(\mu), \mu^1) = E_{-\theta}$. Indeed,
$$R^\text{can}_1(\mu) = \lambda E_{-\theta} + I_+ + \cdots.$$ Here, the dots denote terms with principal degree lower than 1 which contain no more $\lambda^1$-power.

More explicitly, let $(U^\text{can}, H^\text{can})$ be the unique pair associated to $\mathcal{L}^\text{can}$. Note that
$$R^\text{can}_a = e^{ad_{U^\text{can}}} \Lambda_{m_a}. \quad (2.7.3)$$ Also note that eq. (2.1.2) implies that $U^\text{can}$ must have the following decomposition
$$U^\text{can} = \sum_{k \geq 0} U^\text{can}_{-k} \lambda^{-k}, \quad U^\text{can}_0 \in n, U^\text{can}_{-k} \in g, \quad k \geq 1.$$ Hence we have
$$Q^\text{can}_b = \text{Coeff}(R^\text{can}_b(\mu), \mu^1) = e^{ad_{U^\text{can}}} \Lambda_{m_b - h}, \quad b = 1, \ldots, n. \quad (2.7.4)$$ Before ending this section, we consider taking a faithful irreducible matrix realization $\pi$ of $g$. Let $\chi$ be the unique constant satisfying
$$(a \mid b) = \chi \text{Tr}(\pi(a) \pi(b)), \quad \forall a, b \in g. \quad (2.7.5)$$ For simplicity we will write $\pi(a)$ just as $a$, for $a \in g$. Similarly as Theorem 1.3.1 we have

**Proposition 2.7.2.** Let $V$ be a gauge of DS-type, $\mathcal{L}^\text{can}$ the gauge fixed Lax operator (2.6.1), and $R^\text{can}_a$, $a = 1, \ldots, n$ the basic resolvents of $\mathcal{L}^\text{can}$. For an arbitrary solution $q^\text{can}(T)$ to the DS hierarchy associated to $V$, we have
$$F_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N; T) = -\frac{1}{\chi N} \sum_{s \in S_N} \text{Tr} \frac{R^\text{can}_{a_1}(\lambda_{s_1}) \cdots R^\text{can}_{a_N}(\lambda_{s_N})}{\prod_{j=1}^{N} (\lambda_{s_j} - \lambda_{s_{j+1}})} - \delta_{N2} \eta_{a_1a_2} \frac{m_{a_1} \lambda_1 + m_{a_2} \lambda_2}{(\lambda_1 - \lambda_2)^2}. \quad (2.7.6)$$

**Remark 2.7.3.** The right hand side of (1.3.4) and the right hand side of (2.7.6) coincide. However, this does not mean the summands coincide with each other.
2.8 An algorithm for writing the DS-hierarchy

Let $\mathcal{V}$ be any gauge of DS-type, $\{X^1, \ldots, X^n\}$ a basis of $\mathcal{V}$ s.t. $\deg X^a = -m_a$ and let

$$L^{\text{can}} = \partial_x + \Lambda + q^{\text{can}}(x), \quad q^{\text{can}}(x) = \sum_{a=1}^{n} w_a(x) X^a.$$  

Denote by $R_a^{\text{can}}, a = 1, \ldots, n$ the basic resolvents of $L^{\text{can}}$. The corresponding DS-hierarchy will be defined as in (2.6.2). Although we know that the right hand side of (2.6.2) depends only on $q^{\text{can}}, q_x^{\text{can}}, \ldots$, the second term of the right hand side of (2.6.2) contains evolution of general components in $n$.

So the following question is under consideration:

For any given gauge $\mathcal{V}$, can we write down the DS-hierarchy for $q^{\text{can}}$ using only the information of $R_a^{\text{can}}$?

Let us give a positive answer to this question by using our definition of tau-function.

1. Compute the basic resolvents $R_a^{\text{can}}, a = 1, \ldots, n$.
2. Calculate the Miura transformation $w_a \mapsto r_a$ from eq. (2.7.2). Recall that the normal coordinates are defined by $r_a := \langle\langle \tau_a, 0 \tau_1, 0 \rangle\rangle_{DS}$.
3. Calculate $\langle\langle \tau_b, k \tau_a, 0 \rangle\rangle_{DS}$ from eqs. (2.7.1). Note that the DS-flows for the normal coordinates $r_a$ are

$$\frac{\partial r_a}{\partial T_k} = -\partial_x \langle\langle \tau_b, k \tau_a, 0 \rangle\rangle_{DS}, \quad a, b = 1, \ldots, n, k \geq 0. \quad (2.8.1)$$

The right hand sides of eqs. (2.8.1) are differential polynomials in $w$. Substituting $w_a \mapsto r_a$ in the right hand sides of eqs. (2.8.1) we obtain the DS hierarchy for $r_a$.

4. Substitute the inverse Miura transformation to the DS hierarchy for $r_a$ we obtain the DS hierarchy.

3 Computational aspect of resolvents

3.1 The lowest weight gauge

Recall that there is a particular choice of a gauge of DS-type [5], called the lowest weight gauge. Let us review its construction. Write the Weyl co-vector as $\rho^\vee = \sum_{i=1}^{n} x_i H_i$, $x_i \in \mathbb{C}$ and define

$$I_- = 2 \sum_{i=1}^{n} x_i F_i. \quad (3.1.1)$$

Then $I_+, I_-, \rho^\vee$ generate an $sl_2(\mathbb{C})$ Lie subalgebra of $\mathfrak{g}$:

$$[\rho^\vee, I_+] = I_+, \quad [\rho^\vee, I_-] = -I_-, \quad [I_+, I_-] = 2 \rho^\vee. \quad (3.1.2)$$

According to [41, 5] there exist elements $\gamma^1, \ldots, \gamma^n \in \mathfrak{g}$ such that

$$\ker \text{ad}_{I_-} = \text{Span}_\mathbb{C} \{\gamma^1, \ldots, \gamma^n\}, \quad [\rho^\vee, \gamma^i] = -m_i \gamma^i.$$  

Since $\gamma^n \in \mathbb{C}E_{-\theta}$ we could and will normalize it to be

$$\gamma^n = E_{-\theta}. \quad (3.1.3)$$
The subspace $\text{Ker} \text{ad}_{I_-} \subset \mathfrak{b}$ is a gauge of DS-type, which is called the lowest weight gauge. Denote by
\[ \mathcal{L}^{\text{can}} = \partial_x + \Lambda + q^{\text{can}}(x) \]
the gauge fixed Lax operator associated to $\text{Ker} \text{ad}_{I_-}$, where $q^{\text{can}}(x) := \sum_{a=1}^{n} u_a(x) \gamma^a$.

**Definition 3.1.1.** The functions $u_a$, $a = 1, \ldots, n$ are called the lowest weight coordinates.

### 3.2 Extended principal gradation

**Definition 3.2.1.** Define the extended principal degree by the following degree assignments
\[
\begin{align*}
\deg^e \partial_x &= 1, \\
\deg^e \lambda &= h, \\
\deg^e u_i &= m_i + 1, \\
\deg^e E_i &= 1, \\
\deg^e F_i &= -1, \quad i = 1, \ldots, n.
\end{align*}
\]

It is easy to see that, if $a \in L(\mathfrak{g})^j$ then $\deg^e a = \deg a = j$. Namely, the extended principal degree coincides with the principal degree for any loop algebra element. In particular,
\[
\deg^e \gamma^i = -m_i, \\
\deg^e \text{ad}_{I_+}^i \gamma^i = -m_i + j, \quad j = 0, \ldots, 2m_i.
\]

**Lemma 3.2.2.** For the gauge-fixed Lax operator $\mathcal{L}^{\text{can}}$, we have $\deg^e \mathcal{L}^{\text{can}} = 1$.

Let $(U^{\text{can}}, H^{\text{can}})$ be the unique pair associated to $\mathcal{L}^{\text{can}}$, and $R^{\text{can}}_a$ the basic resolvents.

**Lemma 3.2.3.** The following formulae hold true
\[
\deg^e U^{\text{can}} = 0, \\
\deg^e H^{\text{can}} = 1, \\
\deg^e R^{\text{can}}_a = m_a, \quad a = 1, \ldots, n.
\]

**Proof.** By using the recursion procedure (2.1.6) and by the mathematical induction. \(\square\)

**Corollary 3.2.4.** The $N$-point ($N \geq 2$) generating series of correlation functions $F_{a_1, \ldots, a_N}(\lambda_1, \ldots, \lambda_N; \mathbf{T})$ are homogenous of degree $-Nh + \sum_{i=1}^{N} m_{a_i}$ w.r.t. the extended principal gradation.

### 3.3 Essential series of the Drinfeld–Sokolov hierarchy

Recall that the simple Lie algebra $\mathfrak{g}$ admits the lowest weight decomposition \[ \mathfrak{g} = \bigoplus_{a=1}^{n} \mathfrak{L}^a, \quad \mathfrak{L}^a = \text{Span}_\mathbb{C} \left\{ \gamma^a, \text{ad}_{I_+} \gamma^a, \ldots, \text{ad}_{I_+}^{2m_a} \gamma^a \right\}, \]
where each $\mathfrak{L}^a$ is an $sl_2(\mathbb{C})$-module w.r.t. the $sl_2(\mathbb{C})$ Lie subalgebra generated by $I_+, I_-, 2\rho^\vee$, called a lowest weight module. It is then clear that any $\mathfrak{g}$-valued function $R(\lambda)$ can be uniquely written as
\[
R(\lambda) = \sum_{a=1}^{n} \sum_{j=0}^{2m_a} K_{am}(\lambda) \text{ad}_{I_+}^j \gamma^a.
\]

**Theorem 3.3.1.** Let $\mathcal{L}^{\text{can}} = \partial_x + \Lambda + q^{\text{can}} = \partial_x + \Lambda + \sum_{a=1}^{n} u_a \gamma^a$ be a Lax operator associated to the lowest weight gauge. Let $R^{\text{can}} \in \mathcal{A}^n \otimes \mathfrak{g}(\mathbf{T}^{-1})$ be any resolvent of $\mathcal{L}^{\text{can}}$. Write
\[
R^{\text{can}} = \sum_{i=1}^{n} \mathcal{R}_i \text{ad}_{I_+}^{2m_i} \gamma^i + \sum_{i=1}^{n} \sum_{m=0}^{2m_i-1} K_{im} \text{ad}_{I_+}^m \gamma^i.
\]

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We have 1) \( \forall i \in \{1, \ldots, n\}, m \in \{0,1, \ldots, 2m_i - 1\}, K_{im} \) has the following expression

\[
K_{im} = \sum_{j=1}^{n} \sum_{\ell=0}^{2m_i-m} \left( s_{i,\ell,0}^j + \lambda s_{i,\ell,1}^j \right) \partial_x^\ell (R_j),
\]

where the coefficients \( s_{i,\ell,0}^j, s_{i,\ell,1}^j \) belong to \( \mathcal{A}^n \), and they do not depend on the choice of the resolvent.

2) The ODE \( [\mathcal{L}^\text{can}, R^\text{can}] = 0 \) is equivalent to \( n \) scalar linear ODEs for \( R_1, \ldots, R_n \).

3) The following formulae hold true for the degrees of the coefficients (3.3.1) of the basic resolvents

\[
\deg^e R_{a;i} = m_a - m_i, \quad \deg^e K_{a;im} = m_a + m_i - m, \quad i = 1, \ldots, n; \ m = 0, \ldots, 2m_i - 1. \quad (3.3.2)
\]

**Proof of Theorem 3.3.1.** Write

\[
R^\text{can}(\lambda; u; u_x, \ldots) = \sum_{i=1}^{n} \sum_{m=0}^{2m_i} K_{im}(\lambda; u; u_x, \ldots) \text{ad}^m_{I^0} \gamma^i, \quad K_{i,2m_i} := R_i.
\]

Substituting the above expressions into (2.2.4) we obtain

\[
\sum_{i=1}^{n} \sum_{m=0}^{2m_i} \frac{\partial K_{im}}{\partial x} \text{ad}^m_{I^0} \gamma^i + \sum_{i=1}^{n} \sum_{m=1}^{2m_i} K_{i,m-1} \text{ad}^m_{I^0} \gamma^i + \left[ \lambda \gamma^n + \sum_{\ell=1}^{n} u_\ell \gamma^\ell, \sum_{i=1}^{n} \sum_{m=0}^{2m_i} K_{im} \text{ad}^m_{I^0} \gamma^i \right] = 0. \quad (3.3.3)
\]

Introduce the lowest weight structure constants \( c^m_{i\ell js} \) by

\[
[\gamma^\ell, \text{ad}^m_{I^0} \gamma^j] = \sum_{j=1}^{n} \sum_{s=0}^{2m_j} c^m_{i\ell js} \text{ad}^s_{I^0} \gamma^j, \quad i, \ell = 1, \ldots, n, \ m = 0, \ldots, 2m_i. \quad (3.3.4)
\]

Substituting (3.3.4) into (3.3.3) we obtain

\[
\sum_{i=1}^{n} \sum_{m=0}^{2m_i} \frac{\partial K_{im}}{\partial x} \text{ad}^m_{I^0} \gamma^i + \sum_{i=1}^{n} \sum_{m=1}^{2m_i} K_{i,m-1} \text{ad}^m_{I^0} \gamma^i \quad + \sum_{\ell=1}^{n} \sum_{i=1}^{n} \sum_{m=0}^{2m_i} \sum_{j=1}^{n} \sum_{s=0}^{2m_j} \tilde{u}_\ell K_{im} c^m_{i\ell js} \text{ad}^s_{I^0} \gamma^j = 0, \quad (3.3.5)
\]

where \( \tilde{u}_\ell = u_\ell + \lambda \delta_{\ell,n} \). It follows that

\[
K_{j,s-1} + \frac{\partial K_{js}}{\partial x} + \sum_{\ell=1}^{n} \sum_{i=1}^{n} \sum_{m=0}^{2m_i} \tilde{u}_\ell \lambda K_{im} c^m_{i\ell js} = 0, \quad j = 1, \ldots, n; \ s = 0, \ldots, 2m_j. \quad (3.3.6)
\]

Here \( K_{j,-1} := 0 \). Noting that the structure constant \( c^m_{i\ell js} \) are zero unless

\[
0 \leq m = m_i + m_\ell + s - m_j \leq 2m_i.
\]

Hence we obtain

\[
K_{j,s-1} = -\frac{\partial K_{js}}{\partial x} - \sum_{\ell, i=1}^{n} \tilde{u}_\ell K_{i,m_i+m_\ell+s-m_j} c^m_{i\ell js}^m_{m_i+m_\ell+s-m_j} , \quad j = 1, \ldots, n; \ s = 0, \ldots, 2m_j. \quad (3.3.8)
\]
Define an ordering for pairs of integers \( \{(j, s) | j = 1, \ldots, n, s = 0, \ldots, 2m_j \} \): we say \((j_1, s_1) > (j_2, s_2)\), if \(s_1 > s_2\), or \(s_1 = s_2\) and \(j_1 < j_2\). Noting that \(K_{i,2m_i} := R_i\) we can use (3.3.8) to solve out \(K_{j,s-1}\) in terms of \(R_j\) and their \(x\)-derivatives starting from the largest pair \((j, s - 1) = (n, 2m_n - 1)\) to the smallest pair \((j, s - 1) = (n, 0)\). This proves Part 1) of the theorem.

Taking \(s = 0\) in (3.3.8) we obtain the system of ODEs for \(R_1, \ldots, R_n\), which proves Part 2).

Formulae (3.3.2) follow from Lemma 3.2.3 and eq. (3.3.1), which proves Part 3).

\[\text{Definition 3.3.2.} \text{ We call } R_{a;1}, \ldots, R_{a;n} \text{ the essential series of the DS hierarchy of the } g\text{-type.}\]

Using the same argument as in [7], the essential series \(R_{a,a}\) never vanishes.

\[\text{Definition 3.3.3.} \text{ We call } R_{a;a} \text{ the fundamental series of the DS hierarchy.}\]

4 Proof of Theorem 4.1.2

4.1 Relation between normal coordinates and lowest weight coordinates

The concept of normal coordinates was introduced in [24]; see also [22].

\[\text{Definition 4.1.1.} \text{ We call } r_a := \langle \langle \tau_{a,0}\tau_{1,0} \rangle \rangle^{DS} \text{ the normal coordinates of the DS hierarchy.}\]

Recall that

\[\Lambda_{ma}(\lambda) = L_{ma} + \lambda K_{ma-h}, \quad L_{ma} \in g^{ma}, \quad K_{ma-h} \in g^{ma-h}.\]

Using the commutativity between \(\Lambda_{m1}, \ldots, \Lambda_{mn}\) along with the normalization (1.1.6) we have

\begin{align*}
[L_{ma}, L_{mb}] &= 0, \quad [K_{ma-h}, K_{mb-h}] = 0, \\
[K_{ma-h}, L_{mb}] + [L_{ma}, K_{mb-h}] &= 0
\end{align*}

(4.1.1)

and (2.3.5). Note that \(L_{m1} = I_+\), we have in particular

\[\left[I_+, L_{ma}\right] = 0, \quad \forall a = 1, \ldots, n.\]

(4.1.3)

Therefore, the elements \(L_{ma}\) are the highest weight vectors of the lowest weight module \(L^a\), i.e.,

\[L_{ma} = \text{const } ad^{2ma}_{I_+} \gamma^a, \quad \text{const} \neq 0.\]

\[\text{Lemma 4.1.2.} \text{ The lowest weight vectors } \gamma^a \text{ can be normalized such that}\]

\[\langle \gamma^a | L_{ma} \rangle = 1.\]

(4.1.4)

\[\text{Proof.} \text{ We know that different irreducible representations of } sl_2(\mathbb{C}) \text{ are orthogonal w.r.t. to } (\cdot | \cdot) \text{ and, hence, the nondegeneracy of } (\cdot | \cdot) \text{ implies the nondegeneracy of its restriction to each irreducible representation. Note that}\]

\[\langle \gamma^a | \text{ad}^k_{I_-} L_{ma} \rangle = -\left(I_- | [\gamma^a, \text{ad}^{k-1}_{I_-} L_{ma}] \right) = 0, \quad \forall k \in \{1, \ldots, 2m_a\}.\]

So \((\gamma^a | L_{ma}) \neq 0\) since otherwise we obtain a contradiction with the nondegeneracy of \((\cdot | \cdot)\). Hence for \(a = 1, \ldots, n - 1\), we can normalize \(\gamma^a\) such that \((\gamma^a | L_{ma}) = 1\). Particular consideration must be addressed for \(\gamma^n\), since we have already defined \(\gamma^n = E_{-\theta}\). Taking in (2.3.5) \(a = n, b = 1\) we obtain

\[\langle L_{mn} | K_{m1-h} \rangle = 1 \Rightarrow \langle L_{mn} | E_{-\theta} \rangle = 1,\]

which finishes the proof. \[\square\]
From now on we fix a choice of \( \gamma^1, \cdots, \gamma^n \) satisfying (4.1.4). Then Lemmas 2.1.1, 4.1.2 imply

\[
(\gamma^a \mid L_m) = \delta^a_0. \tag{4.1.5}
\]

Here it should be noted that for the case of \( D_n \) with \( n \) even with appearance of an equal pair of exponents \( m_{n/2} = m_{n/2+1} \), eq. (4.1.5) is valid under a suitable choice of \( \gamma^{n/2}, \gamma^{n/2+1} \).

According to Corollary 3.2.4 and Theorem 1.3.1, \( \langle \tau_{a,k} \tau_{1,0} \rangle \) are differential polynomials in \( u \), homogeneous of degree

\[
m_a + 1 + kh
\]

w.r.t. to \( \deg^e \). In particular, we have

\[
\deg^e r_a = m_a + 1.
\]

We arrive at the following lemma.

**Lemma 4.1.3.** There exists a Miura transformation \( u \to r \) of the form

\[
r_a = c_a u_a + P_a [u_1, \ldots, u_{a-1}] \tag{4.1.6}
\]

for some non-zero constants \( c_a \), where \( P_a \) are differential polynomials in \( u_1, \ldots, u_{a-1} \) satisfying

\[
\deg^e P_a [u_1, \ldots, u_{a-1}] = m_a + 1. \tag{4.1.7}
\]

**Remark 4.1.4.** For \( D_n \) with \( n \) even, Lemma 4.1.3 is valid under a suitable choice of \( \gamma^{n/2}, \gamma^{n/2+1} \).

**Remark 4.1.5.** The inverse Miura transformation has the form

\[
u_a = c_a^{-1} r_a + \tilde{P}_a [r_1, \ldots, r_{a-1}], \tag{4.1.8}
\]

thanks to the triangular nature of the transformation (4.1.6).

**Lemma 4.1.6.** The constants \( c_a \) in Lemma 4.1.3 have the following explicit expressions

\[
c_a = -\frac{m_a}{h}. \tag{4.1.9}
\]

**Proof.** Fix \( a \in \{1, \ldots, n\} \). We are to compute \( r_a|_{u_1=\cdots=u_{a-1}=0} \). Assume \( u_1 \equiv 0, \ldots, u_{a-1} \equiv 0 \). Looking at equation (2.1.5) for the pair \( (U,H) \) we obtain

\[
U[-1] = \cdots = U[-m_a] = 0 = H[-1] = \cdots = H[1-m_a].
\]

The first nontrivial equation arises from the component of principal degree \(-m_a\) in (2.1.5):

\[
H[-m_a] + \left[ U[-m_a-1], \Lambda \right] = u_a \gamma^a \quad \text{(no summation over } a). \tag{4.1.10}
\]

Let us decompose the elements \( H[-m_a], U[-m_a-1] \) as follows

\[
H[-m_a] = \frac{g_a(x)}{\lambda} \Lambda_{h-m_a} = g_a(x) \ K_{-m_a} + \frac{g_a(x)}{\lambda} L_{h-m_a}, \quad a = 1, \ldots, n,
\]

\[
U[-m_a-1] = \frac{1}{\lambda} Y_{h-m_a-1} + W_{-m_a-1}, \quad a = 1, \ldots, n-1,
\]

\[
U[-m_a-1] = \frac{1}{\lambda} Y_0.
\]

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Substituting these expressions in (4.1.10) and comparing the coefficients of powers of \( \lambda \) we obtain

\[
\begin{align*}
\lambda^{-1} : & \quad g_a(x) L_{h-m_a} + [Y_{h-m_a-1}, I_+] = 0, \\
\lambda^0 : & \quad g_a(x) K_{-m_a} + [Y_{h-m_a-1}, E_{-\theta}] + [W_{-m_a-1}, I_+] = u_a \gamma^a, \\
\lambda^1 : & \quad [W_{-m_a-1}, E_{-\theta}] = 0 \quad \text{(automatic!)}.
\end{align*}
\] (4.1.11 - 4.1.13)

Since \( L_{h-m_a} \) is the highest weight vector of the irreducible \( sl_2(\mathbb{C}) \)-module \( \mathcal{L}^{n+1-a} \), the solution to eq. (4.1.11) is

\[
Y_{h-m_a-1} = \frac{g_a(x)}{2(h-m_a)} [I_-, L_{h-m_a}] + f(x) L_{h-m_a-1}
\]

for some function \( f(x) \) which is a differential polynomial in \( u \). Here \( L_{h-m_a-1} \) is defined to be 0 if \( h-m_a-1 \) is not an exponent. We thus have

\[
[Y_{h-m_a-1}, E_{-\theta}] = \frac{g_a(x)}{2(h-m_a)} [I_-, [L_{h-m_a}, E_{-\theta}]] + f(x) [L_{h-m_a-1}, E_{-\theta}]
\] (4.1.12)

Plugging (4.1.14) into (4.1.12) we find

\[
g_a(x) K_{-m_a} + \frac{g_a(x)}{2(h-m_a)} [I_-, [I_+, K_{-m_a}]] + [W_{-m_a-1}, I_+] + f(x) [L_{h-m_a-1}, E_{-\theta}] = u_a \gamma^a.
\]

Employing the Jacobi identity we obtain

\[
g_a(x) \frac{h}{h-m_a} K_{-m_a} + \left[ I_+, \frac{g_a(x)}{2(h-m_a)} [I_-, K_{-m_a}] - W_{-m_a-1} \right] + f(x) [L_{h-m_a-1}, E_{-\theta}] = u_a \gamma^a.
\]

Taking the inner products of both sides of the above equation with \( L_{m_a} \) we have

\[
\left( L_{m_a} \right| \frac{h}{h-m_a} g_a(x) K_{-m_a} + \left[ I_+, \frac{g_a(x)}{2(h-m_a)} [I_-, K_{-m_a}] - W_{-m_a-1} \right] + f(x) [L_{h-m_a-1}, E_{-\theta}] \right) = u_a (L_{m_a} | \gamma^a).
\] (4.1.15)

Noticing that \( L_{m_a} \) is a highest weight vector of the \( sl_2(\mathbb{C}) \)-module \( \mathcal{L}^a \), i.e.,

\[
[L_{m_a}, I_+] = 0, \quad [L_{m_a}, L_{h-m_a-1}] = 0,
\]

and using (2.3.5), (4.1.4) we obtain

\[
g_a(x) = \frac{h}{h(L_{m_a} | K_{-m_a})} (L_{m_a} | \gamma^a) u_a(x) = \frac{1}{h} u_a(x).
\]

Using Definition 4.1.1 and eq. (2.7.2) we have

\[
-r_a = \res_{\lambda=\infty} \left( e^U \Lambda_{m_a} e^{-U} | E_{-\theta} \right) = \res_{\lambda=\infty} \left( \Lambda_{m_a}(\lambda) | E_{-\theta} - [U(\lambda), E_{-\theta}] + \frac{1}{2} [U(\lambda), [U(\lambda), E_{-\theta}]] + \cdots \right).
\]

The only possible contribution to the residue comes from the terms of principal degree \(-h-m_a\) and the first one in the series is easily seen to be residueless

\[
\res_{\lambda=\infty} (\Lambda_{m_a}(\lambda) | E_{-\theta}) d\lambda = 0.
\]
Note that we have already shown that $U$ has the form

$$U = U^{[-m_a-1]} + \sum_{j \leq -m_a-2} U^{[j]}.$$  

Therefore only the very next term $-(\Lambda_{m_a}(\lambda) \mid [U^{[-m_a-1]}(\lambda), E_{-\theta}])$ can contribute to the residue. Thus

$$r_a = \text{res}_{\lambda=\infty} \left( \Lambda_{m_a}(\lambda) \mid [U^{[-m_a-1]}(\lambda), E_{-\theta}] \right) = \text{res}_{\lambda=\infty} \left( \Lambda_{m_a}(\lambda) \mid [U^{[-m_a-1]}(\lambda), E_{-\theta}] \right). \quad (4.1.16)$$

Now substituting

$$\Lambda_{m_a}(\lambda) = \lambda K_{m_a-h} + L_{m_a}, \quad U^{[-m_a-1]} = \frac{1}{\lambda} Y_{h-m_a-1} + W_{-m_a-1} \quad (4.1.17)$$

in (4.1.16) we obtain

$$-r_a(x) = \left( L_{m_a} \mid [Y_{h-m_a-1}, E_{-\theta}] \right) = \left( L_{m_a} \mid \left[ \frac{g_a(x)}{2(h-m_a)} [I_-, L_{h-m_a}] + f(x) L_{h-m_a-1}, E_{-\theta} \right] \right)$$

$$= \frac{g_a(x)}{2(h-m_a)} \left( L_{m_a} \mid [[E_{-\theta}, L_{h-m_a}], I_-] \right)$$

$$= \frac{g_a(x)}{2(h-m_a)} \left( L_{m_a} \mid [[K_{-m_a}, I_+], I_-] \right) = \frac{g_a(x)}{2(h-m_a)} \left( [I_+, [I_-, L_{m_a}], K_{-m_a}] \right)$$

$$= g_a(x) \frac{m_a}{h-m_a} \left( L_{m_a} \mid K_{-m_a} \right) = \frac{m_a}{h} u_a(x).$$

The lemma is proved. \hfill \Box

**Remark 4.1.7.** For the particular $A_n$ case, a similar lemma on relations between normal coordinates and Wronskian-gauge coordinates was obtained for example in [8] (see Lemma 3.1 therein). However, except the $A_1$ case, the Wronskian-gauge coordinates are not the lowest weight coordinates.

### 4.2 Partition function and topological ODE

Recall that the partition function $Z$ of the DS hierarchy of $\mathfrak{g}$-type is a particular tau-function specified (up to a constant factor) by the string equation (1.3.5). The compatibility between the string equation and the DS hierarchy follows from the fact that the flow $\partial_{s_{-1}}$ defined via

$$\partial_{s_{-1}} \tau := \sum_{a=1}^{n} \sum_{k \geq 0} t_k^a \frac{\partial \tau}{\partial t_k^a} + \frac{1}{2} \sum_{a,b=1}^{n} \eta_{ab} t_0^a t_0^b \tau - \frac{\partial \tau}{\partial t_0^a}$$

gives rise to an additional infinitesimal symmetry of the DS hierarchy.

The function $u = u(T) = u(t)$ associated to $Z(t)$ is called the topological solution to the lowest-weight-gauge DS hierarchy, and $\tau = r(t) = r(T)$ the topological solution in normal coordinates.

**Lemma 4.2.1.** The normal coordinates associated to the partition function $Z$ satisfy

$$r_a(t) \big|_{t_k^a = \delta_{k,0} t_0^a} = -\delta_{a,n} \frac{h-1}{h \kappa} t_0^1, \quad \kappa := \sqrt{-h}. \quad (4.2.1)$$
Proof. By applying the $t^a_0$-derivative on both sides of eq. (1.3.5) we have
\[
\frac{\partial^2 \log Z}{\partial t^a_0 \partial t^b_0} \bigg|_{t^a_0 = \delta^a_k \delta^b_0} = \delta_{a,n} t^1_0.
\]
Hence from (1.3.6) we obtain
\[
\frac{\partial^2 \log Z}{\partial T^a_0 \partial T^b_0} \bigg|_{t^a_0 = \delta^a_k \delta^b_0} = -\delta_{a,n} \frac{h - 1}{h} \sqrt{-h} t^1_0.
\]
The lemma is proved.

\[\square\]

**Lemma 4.2.2.** The topological solution to the lowest-weight-gauge DS hierarchy of $g$-type satisfies
\[
u_a(t)|_{t^a_0 = \delta^a_k \delta^b_0} = \frac{\delta_{a,n}}{\kappa} t^1_0.
\] (4.2.2)

**Proof.** By applying Lemma 4.1.3, Lemma 4.1.6 and Lemma 4.2.1.

\[\square\]

**Topological ODE of $g$-type.** Let $u = u(T) = u(t)$ be the topological solution to the lowest-weight-gauge DS hierarchy, and $R^\text{can}_{a}(\lambda; t)$ be the basic resolvents of $\mathcal{L}^\text{can}$ (see Definition 1.1.1). Note that
\[t^1_0 = -T^1_0 = x.
\] Define
\[R^\text{can}_{a}(\lambda, x) = \lambda^{-\frac{m_a}{\kappa}} R^\text{can}_{a}(\lambda; t)|_{t^a_0 = \delta^a_k \delta^b_0}, \quad a = 1, \ldots, n.
\] Clearly, $R^\text{can}_{a}(\lambda, x)$ is the unique solution to (2.2.4)–(2.2.6) with $\mathcal{L}$ replaced by $\mathcal{L}^\text{can} = \partial_x + \Lambda + \frac{\kappa}{\delta} E_{-\theta}$.

**Lemma 4.2.3 (Key Lemma).** The following formulae hold true
\[\partial_x (R^\text{can}_{a}) = \frac{1}{\kappa} \partial_\lambda (R^\text{can}_{a}), \quad a = 1, \ldots, n.
\] (4.2.3)

**Proof.** For each $a \in \{1, \ldots, n\}$, let $M^\ast_{a}(\lambda)$ be the unique solution to the topological ODE (1.3.7) satisfying
\[M^\ast_{a}(\lambda) = \lambda^{-\frac{m_a}{\kappa}} [\Lambda_{m_a}(\lambda) + \text{lower degree terms w.r.t. } \deg] .
\] See in [7] for the proof of existence and uniqueness of $M^\ast_{a}(\lambda)$. Define $R^\text{can,\ast}_{a}(\lambda, x) = \lambda^{\frac{m_a}{\kappa}} M^\ast_{a}(\lambda + \frac{\kappa}{\delta})$. Then $R^\text{can,\ast}_{a}$ satisfies equations (2.2.4)–(2.2.6) with $\mathcal{L} = \partial_x + \Lambda + \frac{\kappa}{\delta} \gamma^\ast$. Hence the uniqueness statement of Proposition 2.2.3 implies that $R^\text{can}_{a}(\lambda, x) \equiv R^\text{can,\ast}_{a}(\lambda, x), \quad a = 1, \ldots, n$. The lemma is proved.

\[\square\]

**Proof of Theorem 1.3.2.** Note that $M_a(\lambda) := R^\text{can}_{a}(\lambda; x = 0)$. So from the above proof of Lemma 4.2.3 we already see that $M_a$ satisfies the topological ODE (1.3.7). The theorem is proved.

**Proof of Theorem 1.4.2.** By using Theorem-ADE, Theorem-BCFG, Theorem 1.3.1, and by using Theorem 1.3.2 we obtain
\[
(\kappa \sqrt{-h})^N \sum_{g, k_1, \ldots, k_N \geq 0} (-1)^{k_1 + \cdots + k_N} \prod_{\ell = 1}^N \frac{(m_{\ell \ell})}{\kappa \lambda_{\ell}} \frac{k_{\ell} + 1}{m_{\ell \ell}} \frac{1}{\kappa \lambda_{\ell}} \langle \tau_{i_1} k_1 \cdots \tau_{i_N} k_N \rangle^g
\]
\[= -\frac{1}{2N h^v} \sum_{s \in S_N} B \langle \tilde{M}_{i_{s1}}(\lambda_{s1}), \ldots, \tilde{M}_{i_{SN}}(\lambda_{sN}) \rangle
\]
\[\prod_{j = 1}^N \frac{1}{\lambda_{s_j} - \lambda_{s_{j+1}}} \frac{m_{i_{s1}}}{\kappa} \frac{\lambda_{s1}}{\lambda_{s_{s2}}} \frac{m_{i_{s2}}}{\kappa} \frac{\lambda_{s2}}{\lambda_{s_{s3}}} \cdots \frac{m_{i_{sN}}}{\kappa} \frac{\lambda_{sN}}{\lambda_{s_{s1}}} - \delta_{N2} \eta_{i_{s1}i_{s2}} \frac{\lambda_{s1}}{\lambda_{s2}} \frac{m_{i_{s1}}}{\kappa} \frac{\lambda_{s1}}{\lambda_{s2}} + \frac{m_{i_{s2}}}{\kappa} \frac{\lambda_{s2}}{\lambda_{s1}} \frac{m_{i_{s1}}}{\kappa} \frac{\lambda_{s1}}{\lambda_{s2}} \lambda_{s2}^2, \quad N \geq 2.
\] (4.2.4)
where $\tilde{M}_a = \tilde{M}_a(\tilde{\lambda})$, $a = 1, \ldots, n$ are the unique solutions to
\[
\frac{d\tilde{M}}{d\tilde{\lambda}} = \kappa \left[ \tilde{M}, \Lambda(\tilde{\lambda}) \right], \quad \kappa = \left( \sqrt{-h} \right)^{-h},
\]
\[
\tilde{M}_a(\tilde{\lambda}) = \tilde{\lambda}^{-\frac{m_a}{h}} \left[ \Lambda_{m_a}(\tilde{\lambda}) + \text{lower degree terms w.r.t. deg} \right].
\]

Now consider the following conjugation of $\tilde{M}_a$ together with a rescaling in $\tilde{\lambda}$:
\[
M_a(\lambda) = \sigma^{\rho^\vee} \tilde{M}_a(\tilde{\lambda}) \sigma^{-\rho^\vee},
\]
\[
\lambda = \sigma^{-h} \tilde{\lambda},
\]
where $\sigma := \kappa^{-1/h}$. It is straightforward to check that
\[
\frac{dM}{d\lambda} = [M, \Lambda(\lambda)],
\]
\[
M_a(\lambda) = \lambda^{-\frac{m_a}{h}} \left[ \Lambda_{m_a}(\lambda) + \text{lower degree terms w.r.t. deg} \right].
\]

Combining with (4.2.4), this proves the validity of the formula (1.4.9). To prove formula (1.4.8), one further needs to observe the following identity obtained from the string equation (1.3.5)
\[
\langle \tau_{a,k+1} \tau_{1,0} \rangle_{\text{FJR}}^{\text{W}-g} = \langle \tau_{a,k} \rangle_{\text{FJR}}^{\text{W}-g}, \quad a = 1, \ldots, n, \ k \geq 0.
\]

The rest of proving (1.4.8) follows from the identity (2.7.2) and the above conjugation of $\tilde{M}_a$ with the rescaling in $\tilde{\lambda}$. $\Box$

**Proof** of Theorem 1.4.3. The theorem is a particular case of Theorem 1.4.2 (cf. Remark 2.7.3) with the particular realization of $A_n$ Lie algebra being consistent with normalization of flows suggested by Witten [50]. $\Box$

**Example 4.2.4** (Rationality of Witten’s $r$-spin intersection numbers). *It is known that Witten’s $r$-spin intersection numbers are rational numbers. Let us verify the rationality through (1.4.12) and (1.4.13). Indeed, our definition of $N$-point $r$-spin correlators reads*
\[
F_{a_1, \ldots, a_N}^{r-\text{spin}}(\lambda_1, \ldots, \lambda_N) = \left( \left( \kappa^{1/r} \right)^N \sum_{k_1, \ldots, k_N \geq 0} \prod_{\ell=1}^N \frac{(-1)^{k_\ell} (\frac{a_\ell}{r})^{k_\ell+1}}{(\kappa^{1/r} \lambda_\ell)^{\frac{a_\ell}{r}+k_\ell+1}} \langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle^{r-\text{spin}}
\]
\[
= \sum_{g \geq 0} (-r)^{g-1+N} \sum_{k_1, \ldots, k_N \geq 0} \prod_{\ell=1}^N \frac{(-1)^{k_\ell} (\frac{a_\ell}{r})^{k_\ell+1}}{\lambda_\ell^{\frac{a_\ell}{r}+k_\ell+1}} \langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_{g}^{r-\text{spin}},
\]

where we have used $\kappa = \left( \sqrt{-r} \right)^{-1/r}$ and the dimension-degree matching (1.4.5). Clearly, all the coefficients are rational. On the other hand, the right hand side of (1.4.12) or of (1.4.13) belongs to $\mathbb{Q}[\lambda_1^{-1/r}, \ldots, \lambda_N^{-1/r}]$ as our regular solutions $M_a(\lambda)$, $a = 1, \ldots, n$ to the topological ODEs of $\mathfrak{sl}_n(\mathbb{C})$-type (1.4.11) are of rational coefficients. The rationality of $r$-spin correlators is verified.*

### A 3-spin

The matrices $M_i(\lambda)$, $i = 1, 2$ for the Witten’s 3-spin invariants have the following explicit expressions. Denote $M_i(\lambda) = (M_i(\lambda)^a_b)_{a,b=1,\ldots,3}$. Then we have
\[
(-M_1)^2 = \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{7}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}} - \frac{1}{72} \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{19}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}}
\]

\[
(-M_2)^2 = \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{11}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}} + \frac{1}{24} \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{19}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}}
\]

\[
(-M_3)^2 = \frac{1}{12} \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{19}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}}
\]

and

\[
(-M_1)^2 = \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{11}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}} - \frac{1}{72} \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{19}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}}
\]

\[
(-M_2)^2 = \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{11}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}} + \frac{1}{24} \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{19}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}}
\]

\[
(-M_3)^2 = \frac{1}{12} \sum_{g \geq 0} \frac{(-1)^g \Gamma(8g + \frac{19}{2})}{12^g g! \Gamma(g + \frac{3}{2})} \lambda^{2g+\frac{1}{2}}
\]

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