Experimental Study of the Thermodynamics of Entangled Photons under Kolmogorov Turbulence

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Optical modes possessing orbital angular momentum constitute a very useful platform for experimental studies on the quantum limits of thermodynamics. Here, we present experimental results for entangled photon pairs subjected to thin turbulence as a thermodynamical process. By measuring the orbital angular momentum correlations, we investigate Jarzynski’s equality for single and double-sided turbulence channels. Since the process is not unital, the usual Jarzynski equality does not hold and we employ a generalized version of it in order to describe the experimental results. We also use Klyshko’s advanced wave picture to interpret the experimental scheme as two-way processes in a fully quantum picture.

I. INTRODUCTION

Light beams possessing orbital angular momentum (OAM) have become an important subject in optical sciences. They are considered promising candidates for several applications, notably classical and quantum communication. Regarding quantum thermodynamics, it has been shown that OAM beams can be naturally employed to experimentally investigate Jarzynski’s fluctuation relation. The link between photonic OAM states and thermodynamics is made through the analogy between the paraxial wave equation and the time-dependent Schrödinger equation in two dimensions.

The interest in quantum thermodynamics has hugely increased in the last decade, specially due to experimental developments in controlling quantum systems, quantum computation and quantum communication. From the experimental point of view, investigations of thermodynamics of quantum systems were reported in nuclear magnetic resonance, superconducting devices and ion traps and other platforms are being considered. Recently, the OAM degree of freedom of light was employed in order to simulate the thermodynamics behaviour of a two-dimensional quantum mechanical oscillator undergoing a given unitary process, thus establishing a new experimental platform. Among other features, this system lives in a high dimensional Hilbert space, and thus can be employed to increase the rate of information transmission through a quantum channel.

In this article we go far beyond Ref. by considering entangled photons (in the OAM degree of freedom) undergoing a turbulent process that mimics the atmospheric turbulence. As we show, this process is non-unital (process that do not preserve the identity) and the generalized form of Jarzynski equality must be employed. The results reported here show that the OAM degree of freedom of light can be employed to the experimental investigation of thermodynamic systems in the deep quantum limit.

II. QUANTUM THERMODYNAMICS WITH LIGHT BEAMS

We start by defining the class of protocols we are interested in. Let a system, described by an initial Hamiltonian \( H_I \), be prepared in a thermal equilibrium state (Gibbs ensemble). The protocol starts with an energy measurement (a projective measurement performed on the energy eigenbasis of the initial Hamiltonian). After this, a given process is applied to the system by some external agent (for instance, by changing some parameter of the Hamiltonian). The process can change the state, the Hamiltonian, or both. At the end of the protocol, a second energy measurement is performed, as a projection onto the eigenbasis of the final Hamiltonian \( H_F \). This the so called two-measurement protocol. Notice that even though one starts with a Gibbs state, the first energy measurement projects it onto an energy eigenstate, so that the process is applied to pure states.

In our experiment, we consider the orbital angular momentum of light as the system, which undergoes a turbulence process. The system is described by the two-dimensional quantum harmonic oscillator, whose Hamiltonian is given by

\[
H = (N_r + N_l + 1)\hbar\omega, \tag{1}
\]

where \( N_r(l) \) is the number operator for right (left) circular quanta. The eigenfunctions of this Hamiltonian are the Laguerre-Gaussian modes, which are characterized by an azimuthal quantum number \( \ell \) (the topological charge).
and by the radial quantum number $p$. For $p = 0$, such modes have a ring-like transversal intensity profile, whose radii increase with $ℓ$ \[^{[12]}\], and a phase singularity at the center (along the propagation axis). Since the eigenenergies of our system are given by $\epsilon_ℓ = (|ℓ| + 1)\hbarω$, projections onto the OAM basis are equivalent to projections in the energy eigenbasis. In the two-measurement protocol, in each run of the experiment, the system is initially prepared in a Gibbs state and the first measurement projects it onto some OAM value $ℓ$. The process is applied to this pure OAM state and the second measurement projects it onto a pure OAM state $ℓ'$ again. The work done per run of the experiment when the system is driven by the process from an initial $ℓ$ to a final $ℓ'$, can be defined as $W_{ℓ,ℓ'} = |ℓ' − ℓ| \[4\]$ (for simplicity, we write from now on all energies in units of $\hbarω$). Since the outcome of each measurement is described by a random variable, the associated work probability distribution is then defined as

$$P(W) = \sum_ℓ p_{ℓℓ′} δ(W − W_{ℓℓ′}),$$

where $p_{ℓℓ′} = p_ℓ p_{ℓ′}\mid ℓ$ is the joint probability of observing $ℓ$ in the first measurement and $ℓ'$ in the second one. $p_ℓ$ is the probability of measuring $ℓ$ before the process while $p_{ℓℓ′}$ is the conditional probability of obtaining $ℓ'$ after the process given that we observed $ℓ$ before it. In the context of thermodynamics, $p_ℓ$ is usually given by a thermal distribution associated with the state $ρ_β = e^{-βH}/Z$, with the partition function being given by $Z = e^{-β\coth(β/2)} − 1$ (note that the whole spectrum, except for $ℓ = 0$, is doubly degenerated), while $β$ stands for the inverse temperature. For the optical system in our set-up, the physical realization of the thermal state corresponds to a single photon mixed state, where the populations of the density matrix represented in the energy eigenbasis are given by $e^{-β|ℓ|}/Z$. The transition probabilities $p_{ℓℓ′}$ contain the information about how the process acts on each energy eigenmode. This is actually the quantity we measure in the experiment. While the coefficients $p_ℓ$ do have a physical meaning in terms of the single photon thermal state as we just described, in the context of the two-measurement protocol they are just calculated and in this way the inverse temperature $β$ is a parameter that can be varied.

In this regime, where fluctuations are important, thermodynamic quantities like work, heat and entropy becomes stochastic variables, thus implying that the usual laws of thermodynamics do not hold. However, when fluctuations are properly taken into account, stronger laws emerge, in the form of fluctuation relations such as the Jarzynski equality \[^{[5]}\]

$$\langle e^{-βW} \rangle = e^{−βΔF},$$

where $ΔF$ is the free energy difference between the final and initial states (after and before the process is applied). The ensemble average is taken over the probability distribution \[^{[2]}\]. The usual form of the second law of thermodynamics, $⟨W⟩ − ΔF ≥ 0$, follows from the convexity of the exponential function. This relation clearly states the statistical nature of the second law, that must hold only on average, and not on each realization of the process. It is important to observe here that this equation is valid as long as the dynamics is unital, i.e., the dynamical map describing the process preserves the identity.

This result was later extended for processes described by any completely positive and trace-preserving map, resulting in \[^{[13]}\]

$$\langle e^{βW} \rangle = e^{−βΔF(1 + δ)},$$

where $δ = \text{Tr}[ρ_β G_φ]$, with $G_φ = Φ(ρ∗) − ρ∗$ (ρ∗ is the maximally mixed state), is a measure of how much the dynamical map $Φ$ deviates from an unital one. For unital maps, $Φ(ρ∗) = ρ∗$, thus implying $δ = 0$. Again, from the convexity of the exponential function we get the modified second law of thermodynamics

$$⟨W⟩ − ΔF ≥ −β^{-1} \ln(1 + δ).$$

Our goal in this article is to investigate the thermodynamic properties of a turbulent process through these fluctuation relations.

### III. EXPERIMENT

Figure 1 displays the experimental setup that was conceived to study the effect of atmospheric turbulence on quantum correlations between signal and idler photons, exploiting the OAM degree of freedom \[^{[10]}\]. A 355-nm wavelength laser (mode-locked, average power of 350 mW) pumped a 3-mm-thick type-I β-barium borate (BBO) crystal, producing degenerate entangled photon pairs via spontaneous parametric down-conversion (SPDC). Each photon was sent through a 4f telescope (lenses $L_1$ and $L_2$ with 200-mm and 400-mm focal distances, respectively), arranged to image the crystal plane onto the surfaces of two spatial light modulators (SLMs). The SLMs were used for generating and measuring the Laguerre-Gaussian modes, and for applying a turbulence process to one or both photons. Each SLM plane was imaged downstream using a second 4f telescope (lenses $L_3$ and $L_4$ with 500-mm and 2-mm focal distances, respectively) onto an end of a single-mode optical fiber, into which only the fundamental Gaussian optical fiber, into which only the fundamental Gaussian modes were coupled. The fibers terminate onto avalanche photodiodes (APDs), which were then connected to a photon coincidence counter. The coincidence counts registering the photon pairs were accumulated for 10 s with a gating time of 12 ns. Fluctuations on the pump beam caused an uncertainty in the measured counts of about 5%.

For simulating the atmospheric turbulence as a one-sided or two-sided channel, random phase fluctuations were added to the phase mask of one or both SLMs. The fluctuations were distributed according to the Kolmogorov theory of turbulence \[^{[10]}\]. Briefly, the random phase function is characterized by the scintillation
strength $w_0/r_0$, where $w_0$ is the beam waist radius and $r_0$ is the Fried parameter [17]. Here, we used $w_0/r_0$ ranging from 0 to 4 with incremental steps of 0.2. The measurements for every scintillation strength were repeated 30 times, and the density matrix was reconstructed in each step via full quantum state tomography. After removing experimental imperfections leading to negative eigenvalues [18], a mean density matrix for each scintillation strength was computed as the average of the reconstructed matrices. The effect of the turbulence on the orbital-angular-momentum entanglement was analysed elsewhere [16]. Here, we focus on investigating the thermodynamics of the turbulence process.

IV. STATE PREPARATION AND KLYSHKO ADVANCED WAVE PICTURE

The quantum state of the entangled photons produced in the SPDC process can be written in a simplified form, in terms of an entangled state of the OAM degrees of freedom, as $|\psi\rangle = \sum_{\ell_p,\ell_a,\ell_b} C_{\ell_p,\ell_a,\ell_b} |\ell_a,\ell_b\rangle$, where $\ell_p,\ell_a$ and $\ell_b$ are the pump, signal and idler OAM azimuthal indices, respectively, while $C_{\ell_p,\ell_a,\ell_b}$ stands for the coefficients that take into account the phase matching function and the OAM of the pump beam [19]. We will restrict ourselves to states where the radial index is $p = 0$.

Considering the case where $\ell_p = 0$, the down-conversion state can be well approximated by [19]

$$|\psi\rangle = \sum_{\ell=-\infty}^{\infty} C_{\ell} |+\ell\rangle_a |-\ell\rangle_b , \quad (6)$$

where $C_{-\ell} = C_{\ell} \equiv C_{0}^{\ell,-\ell}$. This is a high-dimensional entangled state where signal and idler modes have OAM with the same absolute value $|\ell|$, but with opposite signs. Note that this quantum correlation can be used for the preparation of the idler photon state (in an OAM state $|+\ell\rangle_b$) by projecting the signal photon onto the appropriate OAM state $|-\ell\rangle_a$.

Remote state preparation — The idea here is to avoid the first measurement of the work protocol described above by employing the entanglement present in the state shown in Eq. (3). Instead of performing the projective measurement directly on the idler photon that will undergo the process, we can measure the signal photon. Due to entanglement, this will result in the remote state preparation of the idler photon in the (unnormalized) state

$$C_{\ell} |+\ell\rangle_b = (\ell') \sum_{\ell=-\infty}^{\infty} C_{\ell} |+\ell\rangle_a |-\ell\rangle_b . \quad (7)$$

Figure 2 illustrates the procedure: A laser pumps a nonlinear crystal and a signal-idler photon pair is generated. The signal photon is reflected by a spatial light modulator (SLM) which applies masks designed to implement the angular momentum operators $\mathcal{L}_{\ell_a}$ and $\mathcal{L}_{\ell_b}$ that increase and decrease, respectively, the azimuthal number by $\ell$ units. The beam is coupled to an optical fiber and then detected with a single photon counting module (SPCM). When the SPCM clicks and the SLM mask is set to $\mathcal{L}_{\ell}$, it announces the idler single photon state with azimuthal number $-\ell$, since $\ell_a + \ell_b = \ell_p$, and $\ell_p = 0$. The state of the idler is remotely prepared by the measurement of the signal.

Klyshko’s Advanced-Wave Picture — The remote state preparation scenario can be understood in terms of Klyshko’s Advanced-Wave Picture (AWP) [20, 21]. The AWP is a method that makes use of the quantum correlations between the transverse spatial degrees of freedom of signal and idler beams to provide a classical analog of the experimental picture [22]. Figure 2 illustrates the AWP for the remote state preparation scheme shown in Fig. 2b. The SPCM detector acts like a light source emitting an advanced photonic wave packet that propagates backwards in the optical fiber and that is outcoupled and collimated by the objective. At this point, it is a zero-order Gaussian mode that propagates back towards the crystal and then forward to the idler detector. In its way back to the crystal it finds the SLM, which applies an operation $\mathcal{L}^\dagger_{\ell}$ preparing the advanced beam with OAM = $-\hbar \ell$ per photon. The crystal acts like a SLM (transmission in this picture) that is controlled by the angular spectrum of the pump beam (not shown in this picture). In the present scheme, it acts as a transparent plate coupling the signal advanced wave with the idler retarded wave, which is thus prepared with OAM = $-\hbar \ell$ per photon.

Figure 2 is an extension of Fig. 2b, with the addition of an OAM detection scheme on the idler side. It indicates that the application of the operation $\mathcal{L}^\dagger_{\ell}$ should bring the idler OAM to zero, then coupling to the fiber with high success rate. It is worth mentioning that the AWP works only for fields generating coincidence counting events, that is to say, for twin photon pairs. This implies that the AWP actually does not impose any particular direction for creating the picture, meaning that
changing the roles of signal and idler beams as advanced and retarded waves (see Fig. 2) should give rise to the same photon counting rate. Therefore, the scheme shown in Fig. 2 is equivalent to that of Fig. 2. However, in this case it is the idler SPCM detector that is interpreted as the light source, while the signal SPCM is viewed as the actual detector.

The work protocol — Figure 3 illustrates the AWP of the two-measurement protocol for a single-sided turbulence channel acting as a process on the idler photon. The advanced photon is emitted by the signal SPCM and comes out of the fiber with OAM = 0. It propagates towards the signal SLM, which applies \( \ell \), thus preparing a OAM state with index \(-\ell\). Notice that the OAM for the signal retarded wave is \(+\ell\), while the OAM for the advanced wave has the opposite sign, because they propagate in opposite senses and the OAM sign is defined depending on the sense of propagation using a right-hand rule. The crystal acts like a transparent window and the idler retarded wave is prepared with OAM = \(-h\ell\) per photon. This corresponds to the first measurement of the protocol, which actually prepares a pure energy eigenstate, represented here by a mode with given OAM. The photon propagates through the turbulence and couples to other OAM modes. The idler SLM is scanned, that is to say, measurements are made using a sequence of masks applying rising and lowering operators ranging from \(\ell_m\) to \(\ell_n\), where \(\pm m\) are the maximal/minimal

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**FIG. 2.** Remote state preparation and Klyshko’s advanced-wave picture (AWP). a) Remote state preparation. b) AWP of a). c) Direct AWP of OAM detection scheme. d) Reverse AWP of OAM detection scheme. See main text for details.

**FIG. 3.** Klyshko’s advanced wave picture (AWP) of two-measurement protocol of a turbulent process. a) Direct and b) reverse AWP with single-photon turbulence. c) Direct and d) reverse AWP with two-photon turbulence. See main text for details.
azimuthal indices. Coupling to the fiber and knowledge about the applied mask provides the measurement result. This corresponds to the second measurement of the protocol. From the outcome of these measurement we obtain the transition probabilities that are necessary to compute the relevant quantities in Jarzynski’s relation, as it will be explained in detail later.

Figure 3b illustrates the backwards version of the process in Fig. 3a. Due to the symmetry in the AWP, it is also possible to interpret the two-measurement protocol supposing that the source of advanced waves is the idler SPCM and that the signal SPCM is the actual detector. Moreover, there are lenses (not shown in Figs. 2 and 3 for simplicity) that image input planes onto output planes (e.g. crystal plane is imaged onto both SLM planes), cancelling all free propagation effects. Therefore, AWP symmetry between signal-to-idler and idler-to-signal senses is practically perfect.

Figure 3c shows a scheme similar to Fig. 3b, but using a two-sided turbulence channel. The interpretation in terms of AWP is identical to the case of single-sided channel. The main difference is that, now, both signal and idler photons are affected by turbulence, increasing the mode scattering. In other words, this process tends to couple the input OAM mode to a larger number of output OAM modes. Figure 3d shows the backwards version of Fig. 3c, again based on the AWP symmetry.

FIG. 4. Transition probability matrices of turbulence process given by the coincidence counts for simultaneous measurements of modes with azimuthal index $\ell_A$ in the signal beam and $\ell_B$ in the idler beam when only one of the two photons propagates through a) no turbulence, b), c) and d) with increasingly stronger scintillation strengths. The transitions to higher OAM modes increases with the increase of the scintillation strength.

V. RESULTS AND DISCUSSION

The measurement results are shown in Fig. 4 for single-sided turbulence channels and in Fig. 5 for double-sided channels. They provide the conditional transition probabilities $p(\ell|\ell)$. They are used to compute the work probability distribution according to Eq. 2 and $P(W)$ is used to compute the quantity $\langle e^{-\beta W} \rangle$, with the average taken over $P(W)$. From the measurements shown in the absence of a turbulence process (Fig. 4a or Fig. 5a), we can say that the free energy does not change, since the initial and final Hamiltonian eigenstates perfectly overlap. Therefore, with $\Delta F = 0$, one expects the quantity $\langle e^{-\beta W} \rangle$ to be one in accordance to Jarzynski equality (Eq. 3).

The results for $\langle e^{-\beta W} \rangle$ are shown in Fig. 4b for the forward process described in Fig. 3a, and in Fig. 5b, for the backward process illustrated in Fig. 3b. The results shown in Fig. 4c correspond to the forward process described in Fig. 3c, and in Fig. 5c, to the backward process illustrated in Fig. 3d. $\beta$ was varied from small values up to 10. We recall that we are dealing with an infinite dimensional system, which is truncated in $|r| \leq 10$. For $\beta$ in the region below 2, the truncation effects are more pronounced, while for $\beta$ in the region above 2, it is negligible. For temperatures low enough (i.e., for $\beta$ high enough), the coefficients of the thermal distribution are very small for $|r| > 10$, making the contribution of high order modes negligible for thermodynamics purposes. The shaded area represents the measurement uncertainties within 95% confidence level, calculated by propagating the coincidence counting rate uncertainty using Monte Carlo processing.
We can see that in both cases, for single and double-sided turbulence channels, the value of $\langle e^{-\beta W} \rangle$ is different for the forward and backward processes. However, the sense of the process is just a matter of interpretation. In fact, the measurements are the same for the two cases because they are provided by photon coincidence events. The different results for forward and backward processes arise when the measured frequencies given by the coincidence counting rates are converted into probabilities. For instance, consider one of the measurement results expressed in terms of a correlation matrix (Figs. 4 and 5). Each cell contains a coincidence rate between OAM $l_A$ and $l_B$. If we consider the forward process $l_A \rightarrow l_B$, the lines should be normalized, while the backward process $l_B \rightarrow l_A$ requires normalization of the columns.

Another conclusion from the results is that in many cases, the turbulence processes do not respect Jarzynski’s relation for unitary and unital processes. Therefore, the more general relation in Eq. 4 must be used. The insets in Figs. 6 and 7 show the difference $\langle e^{-\beta W} \rangle - (1 - \delta)$, which is practically zero, meaning that the non-unitarity of the process is well characterized by the quantity $\delta$.

We can conclude that the turbulent processes considered here is not unital. It is somewhat surprising that a turbulence process emulated by a phase mask does not result in a unitary process. Specially because we are considering only the events that result in a successful coincidence count, meaning that there are no photon losses.

We have also computed the average work $\langle W \rangle$, which respects the generalized form of the second law of Thermodynamics given in Eq. 5. In Fig. 8 we can see $\langle W \rangle$ and $-\beta^{-1} \ln(1 + \delta)$ as a function of $\beta$. The error bars were also calculated using Monte Carlo simulation as before and are represented by the thickness of the curves. In all cases (backward processes not shown) $\langle W \rangle$ is much bigger than $-\beta^{-1} \ln(1 + \delta)$. Therefore, we can conclude that despite the non-unitality, which would allow $\langle W \rangle < 0$ meaning that the light beam would realize work on the turbulence, the average work is always positive.

Let us now discuss the possible physical reasons why some turbulence processes are non-unitary/non-unital: i) the Hamiltonian would be actually changing. This is probably not the case. When the Hamiltonian changes, the energy eigenstates change and this results in overlap...
between eigenmodes with different energies; in our case, values of OAM ($\ell$). The calibration procedure shown in Figs. 4 and 5 shows that without turbulence the overlap between modes with different OAM ($\ell$) is very small. It is indeed the detection system that determines the family of modes (eigenfunctions) that are being measured and the good calibration results show that they are very approximately the same for signal and idler photons; ii) effect of experimental imperfections. It is expected that the experimental imperfections produce errors in the measurements. These errors can be random or systematic. Random errors would not only increase the value of $\langle e^{-\beta W} \rangle$ above 1, but also increase the entropy. Figures 9 and 10 (shaded area) show the effect of random errors coming from coincidence counting statistics. Systematic errors can in fact make $\langle e^{-\beta W} \rangle$ greater or smaller than 1. However, the calibration procedure displaying nearly perfect correlation between input and output modes without turbulence shows that systematic errors are very small; iii) measurement and feedback. In the experiment there was no mechanism designed to realize measurements and act back on the system according to the results of these measurements; iv) sampling of turbulence masks. The procedure of simulating atmospheric turbulence with phase masks include the application of different masks with the same scintillation strength and averaging over the measurement results. Each phase mask realizes a unitary operation, but averaging over different unitary processes may lead to another type of operation described by a quantum map. This is the most probable reason for the observed non-unitarity behaviour of our turbulent channel.

VI. CONCLUSION

We analyse an experiment that measures the loss of correlations between signal and idler photons from parametric down-conversion due to the action of Kolmogorov turbulence masks from the point of view of quantum thermodynamics via fluctuation relations. Specifically, we investigate the generalized Jarzynski’s fluctuation relation. We interpret the scheme in terms of remote state preparation and Klyshko’s advanced-wave picture (AWP). We observe that, even though the turbulence masks were expected to act as unitary operations, it was necessary to use the generalized Jarzynski’s equality for quantum stochastic maps to fit the experimental data. For stronger scintillation masks and for double-sided channels, the deviation from unital processes was more significant. We have also analyzed the second law of Thermodynamics in terms of the average value of the work $\langle W \rangle$ and observed that, while the fluctuation relation allows $\langle W \rangle < 0$ for the non unital process, $\langle W \rangle$ is always greater than zero for our measurements. In the context of AWP interpretation, the information can flow either from signal to idler or vice-versa due to the coincidence measurement scheme and it is expected to be invariant under reversion. However, our results show considerable deviation from the expected symmetry. In conclusion, we have presented a new experimental scheme that allows the investigation of quantum thermodynamics and demonstrates a few subtle aspects resulting from nonclassical correlations in the system.

ACKNOWLEDGMENTS

The authors would like to thank the Brazilian Agencies CNPq, FAPESC, and the Brazilian National Institute of Science and Technology of Quantum Information (INCT/IQ). This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. LCC would like to also acknowledge support from Spanish MCIU/AEI/FEDER (PGC2018-095113-B-I00), Basque Government IT986-16, the projects QMiCS (820505) and OpenSuperQ (820363) of the EU Flagship on Quantum Technologies and the EU FET Open Grant Quromorphic and the U.S. Department of Energy, Office of Science, Of-

FIG. 8. $\langle W \rangle$ and $-\beta^{-1}\ln(1 + \delta)$ (circles) plotted as a function of $\beta$ for a) forward one-photon process and b) forward two-photon process.
office of Advanced Scientific Computing Research (ASCR) quantum algorithm teams program, under field work proposal number ERKJ333.

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