On stochastic Euler-Poincaré equations driven by pseudo-differential/multiplicative noise

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Abstract

In this paper we focus on the stochastic Euler-Poincaré equations with pseudo-differential/multiplicative noise. We first establish two new cancellation properties on pseudo-differential operators, which play a key role in energy estimate. Then, we obtain results on local solution, blow-up criterion and global existence. The interplay between stability on exiting times and continuous dependence of solution on initial data is also studied for the multiplicative noise case.

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Contents

1 Introduction and main results
   1.1 Pseudo-differential/multiplicative noise .................................................. 2
   1.2 Dependence on the initial data ................................................................. 3
   1.3 Plan and remark on the paper ................................................................. 4

2 Notations and definitions ................................................................. 5
   2.1 Notations .................................................................................................... 5
   2.2 Definitions ................................................................................................ 7

3 Cancellation of singularities ............................................................ 8

4 Local and global results ................................................................. 15
   4.1 Approximation scheme and estimates ....................................................... 16
   4.2 Solving the cut-off problem ...................................................................... 19
   4.3 Finish the proof for Theorem 4.1 ............................................................ 23

5 Noise effect on the dependence on initial data .................................. 24
   5.1 Estimates on the errors ............................................................................ 26
   5.2 Construction of actual solutions ............................................................. 28
   5.3 Estimates on the error ............................................................................. 28
   5.4 Proof for Theorem 5.1 ............................................................................ 30

A Auxiliary results ................................................................. 32
1 Introduction and main results

Let $I$ be the identity operator. Consider the following Euler-Poincaré (EP) equations:

\begin{equation}
\partial_t m + (u \cdot \nabla) m + (\nabla u)^T m + (\text{div} u) m = 0, \quad m = (I - \alpha \Delta)u,
\end{equation}

where $u = (u_j)_{1 \leq j \leq d}$ is the velocity, $m = (m_j)_{1 \leq j \leq d}$, with $m_j = (I - \alpha \Delta)u_j(t, x)$ represents the momentum, $(\nabla u)^T$ denotes the transpose of $\nabla u$ and $\alpha$ corresponds to the square of the length scale. The EP equations (1.1) were first studied by Holm et al. [25, 26] as a higher dimensional Camassa-Holm (CH) system for modeling and analyzing the nonlinear shallow water waves, see also [27]. When $d \geq 2$ and $m > d/2 + 3$, we refer to [9] for the local existence and uniqueness of a strong solution belonging to $H^m$. Blow-up phenomenon for the case $\alpha = 0$ was also obtained in [9]. For the case $\alpha > 0$, the blow-up and global existence of the solutions to (1.1) were studied in [34]. For convenience, in this paper we assume $\alpha = 1$ in (1.1).

Now we rewrite (1.1) into the following form (see [9, 58, 60] for the computation):

\begin{equation}
\partial_t u + (u \cdot \nabla) u + F(u) = 0,
\end{equation}

where

\begin{equation}
\begin{cases}
F(u) = (I - \Delta)^{-1} \text{div} F_1(u) + (I - \Delta)^{-1} F_2(u), \\
F_1(u) = \nabla u(\nabla u + \nabla u^T) - \nabla u^T \nabla u - \nabla u(\text{div} u) + \frac{1}{2} I_{d \times d} |\nabla u|^2, \\
F_2(u) = u(\text{div} u) + u \cdot \nabla u^T.
\end{cases}
\end{equation}

In (1.3), $I_{d \times d}$ is the $d \times d$ identity matrix and $f = (I - \Delta)^{-1} g$ means $f = G * g$ with the Green function $G$ for the Helmholtz operator $I - \Delta$.

1.1 Pseudo-differential/multiplicative noise

The introduction of stochasticity into fluid PDEs has received special attention over the past two decades. The additional stochastic noise can be a way of representing model uncertainty and turbulence. For instance, phenomena in weather forecast including cloud formation are to this day poorly understood and the inclusion of stochastic noise has become an essential tool for gaining better understanding about it. In this paper, we are interested in the following stochastic version of (1.2):

\begin{equation}
\partial_u + [(u \cdot \nabla) u + F(u)] dt = \sum_{k=1}^{\infty} \left( Q_k u \circ d\tilde{W}_k + h_k(t, u) dW_k \right),
\end{equation}

where $\{\{\tilde{W}_k, W_k\}\}_{k \geq 1}$ is a family of independent 1-D Brownian motions, $\{Q_k\}_{k \geq 1}$ is a sequence of pseudo-differential operators (see Section 2.1 for details), $\{h_k(\cdot, \cdot)\}_{k \geq 1}$ is a sequence of nonlinear functions, $\circ d\tilde{W}_k$ is the Stratonovich stochastic differential and $dW_k$ is the Itô stochastic differential.

We note the form (1.4) extends many recent results. If

\[ Q_k \equiv 0, \quad k \geq 1, \quad \sum_{k=1}^{\infty} \|h_k\|^2_{H^s} < \infty \text{ for all } s > 0, \]

where $H^s$ is the Sobolev space with regularity index $s$ (see Section 2.1 for precise definition), then (1.4) in 1-D becomes the stochastic CH equation investigated in [40, 41, 45]:

\[ \partial_t u + \left[ uu_x + (I - \partial_x^2)^{-1} \partial_x \left( u^2 + \frac{1}{2} (\partial_x u)^2 \right) \right] dt = \sum_{k=1}^{\infty} h_k(t, u) dW_k. \]

For the closely related stochastic models, we refer to [10, 11, 23, 40, 50, 59, 59]. If

\[ h_k \equiv 0, \quad Q_k = (I - \partial_x^2)^{-1} \left\{ c_k (I - \partial_x^2) + d_k \partial_x (I - \partial_x^2) \right\}, \quad k \geq 1 \]

H. Tang  
Stochastic Euler-Poincaré equations
for some nice functions $c_k(x), d_k(x)$, then (1.4) in 1-D reduces to the stochastic Camassa-Holm (CH) equation with transport noise:

$$dm + (um_x + 2u_x m) \, dt = \sum_{k=1}^{\infty} (c_k m + d_k \partial_x m) \circ d\tilde{W}_k, \quad m = u - u_{xx},$$

which has been studied recently in [2]. We also refer to [6, 23, 24] for similar models.

Here we remark that, as a sequence of pseudo-differential operators, $Q_k$ considerably extends the well-known transport noise. For almost all the known results, the transport noise coefficient can be formulated as:

$$Q_k = c_k I + \sum_{j=1}^{d} d_{k,j} \partial_{x_j}$$

for some smooth functions $c_k = c_k(x), d_{k,j} = d_{k,j}(x) : \mathbb{R}^d \to \mathbb{R}$ ($1 \leq j \leq d, k \geq 1$). We refer to [2, 4–6, 12, 13, 15–17, 23, 33] and the references therein for different examples with transport noise.

In the following, we consider the case that $\{Q_k\}_{k \geq 1}$ is a sequence of pseudo-differential operators (see Section 2.1 for precise definition) generalizing (1.5). In this case, even the problem how to close the a priori estimate in $H^s$ is non-trivial. To see this, rewriting (1.4) into Itô’s form (see (2.8)), and then applying the Itô formula to $\|u\|_{H^s}^2$, we will be confronted with the following two terms:

$$\sum_{k=1}^{\infty} (Q_k u, u)_{H^s} \, d\tilde{W}_k$$

$$+ \sum_{k=1}^{\infty} \sum_{j=1}^{d} \langle Q_k u, \partial_{x_j} u \rangle_{H^s} \, dt,$$

which are a priori singular in terms of $H^s$ since derivatives of order more than $s$ is involved. However, to close the a priori estimate for $\|u\|_{H^s}^2$, one has to control the above two terms by $H^s$-norm of $u$. Such estimates are called cancellation of singularities. And the first main result in this work is

**Main Result (A)** (see Theorems 3.1 and 3.2 for the precise statement). *Cancellation of singularities for two classes of pseudo-differential operators* $\{Q_k\}_{k \geq 1}$, i.e.,

$$\sum_{k=1}^{\infty} \langle P_n Q_k f, P_n f \rangle_{L^2}^2 \lesssim \|f\|_{H^s}^4, \quad \sum_{k=1}^{\infty} \|P_n Q_k f \|_{L^2} + \|P_n Q_k f \|_{L^2} \lesssim \|f\|_{H^s}^2,$$

where $\{P_n\}_{n \geq 1} \subset \text{OPS}^s$ is bounded and $f$ is sufficiently regular.

Even though there are already many papers on the existence and uniqueness of solutions to different kinds of SPDEs, as far as we know, there is almost no result on SPDEs with coefficients involving pseudo-differential operators except the recent one [49]. The above result in this work further extends the cancellation properties in [49]. With the above cancellation properties, we can obtain the second result in the paper. Roughly speaking, we obtain

**Main Result (B)** (see Theorem 4.1 for the detailed statement). *Existence, uniqueness, blow-up criterion and global existence of solutions to the following Cauchy problem of the stochastic EP equations* (1.4),

$$du + [(u \cdot \nabla)u + F(u)] \, dt = \sum_{k=1}^{\infty} \left( Q_k u \circ d\tilde{W}_k + h_k(t, u) \, dW_k \right), \quad u|_{t=0} = u_0, \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d,$$

when $Q_k$ and $h_k$ satisfy certain conditions.

### 1.2 Dependence on the initial data

For SPDEs, noise effect is one of the probabilistically important questions worthwhile to study and many regularization effects have been observed. We refer to [15–17, 31, 32, 41, 50] and the references therein. In this paper, we will consider this noise effect on (1.7) associated with the dependence on the initial data.
According to Hadamard, the classical notion of well-posedness of an abstract Cauchy problem requires the existence, uniqueness and the continuous dependence of such solution on initial data. For nonlinear stochastic evolution equations, the property of dependence on initial conditions turns out to be a problem much more complicated than linear growth case or deterministic case. This is because the solution may only exist up to some random time interval \([0, \tau]\) and in general we do not have estimates on such \(\tau\), cf. \([18, 40, 50]\). Indeed, as we will see in Definition 2.2, in stochastic case the solution is a pair \((u, \tau)\). Therefore the dependence on initial conditions is more complicated since we will be confronted with \(u_0 \mapsto (u, \tau)\) rather than \(u_0 \mapsto u\) on some \([0, \tau]\).

In contrast to most of the previous works where the effects of noise are considered in terms of the regularity or uniqueness of solutions, in this work we consider the dependence on initial conditions. The question whether (and how) noise can affect initial-data dependence becomes interesting by comparing noise and Laplacian: On one hand, “regularization by noise” may formally be related to the regularization produced by an additional Laplacian; On the other hand, if one can indeed add a Laplacian to the governing equations in some cases, then the dependence on initial data may be improved. For example, for the deterministic Euler equations, the dependence on initial data cannot be better than continuous \([22]\), but for the deterministic Navier-Stokes equations, it is at least Lipschitz, see pp. 79–81 in \([19]\). So far we have not been able to identify the effect from \(Q_k u \circ d\tilde{W}_k\), hence we consider the following case of (1.7):

\[(1.8) \quad du + [(u \cdot \nabla) u + F(u)] dt = \sum_{k=1}^{\infty} h_k(t, u) dW_k, \quad u\big|_{t=0} = u_0.\]

The second result in this paper can be roughly stated as follows:

**Main Result (C)** (The detailed statement is in Theorem 5.1). The solution map \(u_0 \mapsto (u, \tau)\) defined by (1.8) on \(\mathbb{T}^d\) is weakly unstable in the sense that:

- Either the exiting time of solution \(u \equiv 0\) is not strongly stable (see Definition 2.3);
- Or the dependence on initial data is not uniformly continuous.

### 1.3 Plan and remark on the paper

- In Section 2, we introduce notations and precise the definition of solutions.
- In Section 3, we establish the cancellation properties (1.6) for two board classes of \(Q_k\) not far away from anti-symmetric. Even from the aspect of pure analysis in pseudo-differential operators, the results are new, as far as we know. In Theorem 3.1, the operators depend on \(x\) with order \(\beta \in [0, 1]\). In Theorem 3.2, the operators are independent of \(x\) with order \(\alpha \geq 0\). Section 3 is ended by Remark 3.1 and Lemma 3.1, where we provide some explanations and examples regarding the hypotheses on the operators.
- We prove existence, uniqueness, blow-up criterion and global existence of solutions to (1.7) (cf. Theorem 4.1) in Section 4. It is worthwhile mentioning that the conditional expectation \(E[\cdot|\mathcal{F}_0]\) will be used to replace the expectation \(E\) in the construction of solutions and hence we do not assume any moment condition on initial data. It seems that conditional expectation has been rarely used in the literature of SPDEs. Besides, we remark that the proof for Theorem 4.1 does not require any compactness on Sobolev embeddings, which is needed in well-known martingale approach (cf. Prokhorov’s Theorem and Skorokhod’s Theorem). Hence Theorem 4.1 holds true not only on torus \(\mathbb{T}^d\) but also on the whole space \(\mathbb{R}^d\).
- We study the noise effect on the solution map in Section 5. Our main result is stated in Theorem 5.1, which tells us that the multiplicative noise (in Itô’s sense) cannot improve both the stability of the exiting time and the continuity of the dependence on initial data simultaneously. Results of this type seem to experience less attention in the literature of SPDEs.
- Some necessary estimates employed in the proofs are formulated and proved in Appendix A.
2 Notations and definitions

2.1 Notations

To begin with, we list some notations used subsequently. Let \( K = \mathbb{R} \) or \( T := \mathbb{R}/\mathbb{Z} \) and \( d, m, n \in \mathbb{N} \). For \( 1 \leq p < \infty \), we denote by \( L^p(K; \mathbb{R}^m) \) the standard Lebesgue space of measurable \( p \)-integrable \( \mathbb{R}^m \)-valued functions with domain \( K \), and we let \( L^\infty(K; \mathbb{R}^d) \) be the space of essentially bounded functions. Particularly, \( L^2(K; \mathbb{R}^m) \) has an inner product
\[
\langle f, g \rangle_{L^2} := \sum_{i=1}^m \int_{K^d} f_i \cdot \overline{g}_i \, dx,
\]
where \( \overline{g} \) denotes the complex conjugate of \( g \). If there is no ambiguity, in the following we denote by \( \langle f, g \rangle_{L^2} \) the inner product for both \( f, g \in L^2(K; \mathbb{R}^m) \) and \( f, g \in L^2(K; \mathbb{R}) \) with the customary abuse of notation.

Let \( i = \sqrt{-1} \) be the imaginary unit. The Fourier transform and inverse Fourier transform are defined by
\[
(\mathcal{F} f)(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} \, dx, \quad (\mathcal{F}^{-1} f)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi) e^{i(x, \xi)} \, d\xi, \quad x, \xi \in \mathbb{R}^d.
\]

On torus, i.e., \( x \in \mathbb{T}^d \), the Fourier and inverse Fourier transforms are defined as
\[
(\mathcal{F} f)(k) := (2\pi)^d \int_{\mathbb{T}^d} f(x) e^{-2\pi i (x, k)} \, dx, \quad (\mathcal{F}^{-1} f)(x) = \sum_{k \in \mathbb{Z}^d} f(k) e^{2\pi i (x, k)}, \quad x \in \mathbb{T}^d, \ k \in \mathbb{Z}^d.
\]

We remark that the factor \( 2\pi \) appears here to guarantee the periodicity of \( f(x) \), which can be dropped if we take \( T = \mathbb{R}/(2\pi \mathbb{Z}) \) instead of \( \mathbb{R}/\mathbb{Z} \).

Recall that \( \mathbf{I} \) stands for the identity mapping. For any \( s \in \mathbb{R} \), the operator \( \mathcal{D}^s = (\mathbf{I} - \Delta)^{s/2} \) is defined by
\[
\mathcal{D}^s := \mathcal{F}^{-1} \left( (1 + |\cdot|^2)^{s/2} \mathbf{I} \right) \mathcal{F}.
\]

For \( s \geq 0 \), \( d, m \geq 1 \), the Sobolev spaces \( H^s \) on \( K^d \) with values in \( \mathbb{R}^m \) are defined as
\[
\begin{align*}
H^s(K^d; \mathbb{R}^m) := C_0^\infty(K^d; \mathbb{R}^m)^{\| \|_s}, \quad H^s(T^d; \mathbb{R}^m) := C_0^\infty(T^d; \mathbb{R}^m)^{\| \|_s},
\end{align*}
\]
where
\[
\langle f, g \rangle_{H^s} := \sum_{i=1}^m \langle \mathcal{D}^s f_i, \mathcal{D}^s g_i \rangle_{L^2}.
\]

If \( d, m \in \mathbb{N} \) are fixed in the context, for \( s \in \mathbb{R}, \ p \in [1, \infty] \) we will simply write
\[
H^s = H^s(K^d; \mathbb{R}^m), \quad W^{1, \infty} = W^{1, \infty}(K^d; \mathbb{R}^m), \quad L^p = L^p(K^d; \mathbb{R}^m), \quad s \in \mathbb{R}, \quad p \in [1, \infty],
\]
where \( W^{1, \infty}(K^d; \mathbb{R}^m) \) is the set of weakly differential functions \( f : K^d \to \mathbb{R}^m \) such that
\[
\| f \|_{W^{1, \infty}} := \sum_{j=1}^m \sum_{|\alpha| = 0, 1} \| \partial_x^\alpha f_j \|_{L^\infty} < \infty.
\]

Let \( N_0^d := (\mathbb{N} \cup \{0\})^d \). For two multi-indexes \( \alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in N_0^d \) with \( \beta \leq \alpha \) (which means \( \beta_i \leq \alpha_i \) with \( 1 \leq i \leq d \)), we define
\[
|\alpha| := \sum_{k=1}^d \alpha_k, \quad \partial_x^\alpha := \prod_{k=1}^d \partial_{x_k}^{\alpha_k}, \quad \partial_x^\beta := \prod_{k=1}^d \partial_{x_k}^{\beta_k}, \quad \left( \alpha \atop \beta \right) := \prod_{i=1}^d \frac{\alpha_i}{\beta_i} \prod_{i=1}^d \beta_i! (\alpha_i - \beta_i)!.
\]

Then for any \( s \in \mathbb{R} \), we define the symbol class \( S^s(K^d \times K^d; \mathbb{C}^{m \times m}) \) as
\[
(2.1) \quad S^s(K^d \times K^d; \mathbb{C}^{m \times m}) := \left\{ p : \forall \alpha, \beta \in N_0^d, \ \exists C(\alpha, \beta) > 0 \text{ such that } \left| \partial_{x}^\beta \partial_{\xi}^\alpha p(x, \xi) \right|_{m \times m} < C(\alpha, \beta) \right\}.
\]
Here and in the sequel, $| \cdot |_{m \times m}$ and $| \cdot |$ are usual norms in $\mathbb{C}^{m \times m}$ and $\mathbb{R}^d$, respectively. It is well-known that $\mathcal{S}^*(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m \times m})$ is a Fréchet space equipped with the topology generated by seminorms $\{ | \cdot |_{\alpha,\beta, s} \}_{\alpha,\beta \in \mathbb{N}_0^d}$, where

$$|p|_{\alpha,\beta, s} := \sup_{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d} \frac{|\partial_\xi^\alpha \partial_\eta^\beta p(x,\xi)|}{(1 + |\xi|)^{s - |\alpha|}}.$$ 

For any $\alpha \in \mathbb{N}_0^d$, the partial difference operator $\Delta^\alpha$ for a function $g : \mathbb{Z}^d \to \mathbb{C}$ is given by

$$(\Delta^\alpha g)(k) = \Delta^\alpha_k g(k) := \sum_{\gamma \in \mathbb{N}_0^d, \gamma \leq \alpha} (-1)^{\alpha - \gamma} \gamma(\alpha \gamma) g(k + \gamma), \quad k \in \mathbb{Z}^d.$$ 

Then the (toroidal) symbol class of order $s$ for $s \in \mathbb{R}$ is defined as (cf. [43]):

$$\mathcal{S}^*(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C}^{m \times m}) := \left\{ p : \forall \alpha, \beta \in \mathbb{N}_0^d, \exists C(\alpha, \beta) > 0 \text{ such that } \frac{|\partial_\xi^\alpha \partial_\eta^\beta p(x,\xi)|}{(1 + |\xi|)^{s - |\alpha|}} < \infty \right\}$$

Again, this is a Fréchet space under the topology generated by seminorms $\{ | \cdot |_{\alpha,\beta, s} \}_{\alpha,\beta \in \mathbb{N}_0^d}$ with

$$|p|_{\alpha,\beta, s} := \sup_{(x,\xi) \in \mathbb{T}^d \times \mathbb{Z}^d} \frac{|\partial_\xi^\alpha \partial_\eta^\beta p(x,\xi)|}{(1 + |\xi|)^{s - |\alpha|}}.$$ 

Then the pseudo-differential operator with symbol $p$ is defined by

$$\text{OP}(p) := \mathcal{P}, \quad [\mathcal{P} f](x) := \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} p(x,\xi) (\mathcal{F} f)(\xi) e^{ix \cdot \xi} \, d\xi, \quad \text{if } p(x,\xi) \in \mathcal{S}^*(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m \times m}),$$

$$= \sum_{k \in \mathbb{Z}^d} p(x,k) (\mathcal{F} f)(k) e^{ix \cdot k}, \quad \text{if } p(x,k) \in \mathcal{S}^*(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C}^{m \times m}).$$

Throughout this paper, all pseudo-differential operators are assumed to be real-valued, i.e., when $f$ is real, $[\text{OP}(p) f]$ is also real. Equivalently, it is required that

$$p(x,-\xi) = \overline{p(x,\xi)} \quad \text{if } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \quad \text{and} \quad p(x,-k) = \overline{p(x,k)} \quad \text{if } (x,k) \in \mathbb{T}^d \times \mathbb{Z}^d.$$

When $m = 1$, we remark that, $p(x,k) \in \mathcal{S}^*(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C})$ if and only if there exists $\overline{p} \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C})$ such that $\overline{p}(x,\xi)$ is periodic in $x$ with period 1 for all $\xi \in \mathbb{R}^d$ (hence $x \in \mathbb{T}^d$), $\overline{p} |_{\mathbb{T}^d \times \mathbb{Z}^d} = p(x,k)$ (and see for example [43, Theorem 4.5.3 and Corollary 4.5.7] and [44, Theorem 7.2.1]).

$$|p|_{\alpha,\beta, s} \simeq |p|_{\alpha,\beta, s}.$$ 

Therefore the bounded subset in $\mathcal{S}^*(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C})$ coincides with the restriction to $\mathbb{T}^d$ of bounded subset in $\mathcal{S}^*(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C})$. If $m \geq 1$, this also holds true by considering each element in the matrix. Therefore, we simplify notations if there is no ambiguity in the context and we will simply write

$$(\alpha) \mathcal{S}^* := \left\{ \mathcal{S}^*(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m \times m}) : p(x,-\xi) = \overline{p(x,\xi)} \right\} \quad \text{or} \quad \left\{ \mathcal{S}^*(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C}^{m \times m}) : p(x,-k) = \overline{p(x,k)} \right\}.$$ 

In the following we will also consider symbols only depending on the frequency variable $\xi$ (if $x \in \mathbb{R}^d$) or $k$ (if $x \in \mathbb{T}^d$). To highlight the differences, we let

$$(\alpha) \mathcal{S}_0 := \left\{ \mathcal{S}^*(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}) : p(x,\xi) = \overline{p(x,\xi)}, \quad \text{if } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \right\}$$

To emphasize the scalar symbols (i.e., $m = 1$ in (2.1)), as in (2.4), we simply write

$$(\alpha) \mathcal{S}^* := \left\{ \mathcal{S}^*(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}) : p(x,-\xi) = \overline{p(x,\xi)} \right\} \quad \text{or} \quad \left\{ \mathcal{S}^*(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C}) : p(x,-k) = \overline{p(x,k)} \right\}.$$
Then for $s \in \mathbb{R}$, we recall (2.3) and define

\[(2.6)\quad \text{OP}^s := \left\{ \text{OP}(p) : p \in S^s \right\}, \quad \text{OP}^s_0 := \left\{ \text{OP}(p) : p \in S^s_0 \right\}.\]

In the same way, $\text{OP}^s$ and $\text{OP}^s_0$ can be defined as pseudo-differential operators with symbols in $S^s$ and $S^s_0$, respectively.

For linear operators $\mathcal{A}$ and $\mathcal{B}$, $[\mathcal{A}, \mathcal{B}] := A\mathcal{B} - \mathcal{B}\mathcal{A}$. $\mathcal{A}^\ast$ denotes the $L^2$-adjoint operator of the linear operator $\mathcal{A}$. Let $\lesssim$ and $\gtrsim$ denote estimates that hold up to some universal deterministic constant which may change from line to line. Let $X$ and $Y$ be two Banach spaces. We denote by $\mathcal{L}(X; Y)$ the class of bounded linear operators from $X$ to $Y$. To conclude this part, we recall, cf. [53, Page 53] and [1, Theorem 3.41],

\[(2.7)\quad \text{OP} : S^s \rightarrow \mathcal{L}(H^{r+s}; H^r) \quad \text{is continuous,} \quad r, s \in \mathbb{R}.\]

### 2.2 Definitions

Although (2.7) means that $\text{OP}^s$ can be measured by $\mathcal{L}(H^r; H^{r-s})$, it is also convenient to consider boundedness in the following sense:

**Definition 2.1.** Let $s \in \mathbb{R}$. \{$p_n \} \in S^s$ is said to be bounded if $p_n = \text{OP}(p_n)$ and $\{p_n\} \in S^s$ is bounded in the sense of boundedness in Fréchet space (cf. [42]).

To avoid any confusion, for two separable Banach spaces $X$ and $Y$, $\| \cdot \|_{\mathcal{L}(X; Y)}$ will always be mentioned if boundedness of $\mathcal{L}(X; Y)$ is considered.

Next, we give the precise definition of the solutions. To this end, we first rewrite (1.7). By using the following formula for a semi-martingale $\xi(t)$:

\[\xi \circ d\hat{W}_k = \xi d\hat{W}_k + \frac{1}{2} \langle \xi, \hat{W}_k \rangle,\]

where $\langle \cdot, \cdot \rangle$ is the quadratic variation, (1.7) can be reformulated as

\[(2.8)\quad du = \left[-(u \cdot \nabla)u - F(u) + \frac{1}{2} \sum_{k=1}^{\infty} Q_k^2 u \right] dt + \sum_{k=1}^{\infty} \left( Q_k u d\hat{W}_k + h_k(t, u) dW_k \right), \quad u|_{t=0} = u_0, \quad x \in \mathbb{R}^d.\]

Then we will try to find solutions to (2.8) in the following sense:

**Definition 2.2.** Let $d \geq 2$ and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$. Let $u_0$ be an $H^s(\mathbb{K}^d; \mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable with $s > \frac{d}{2} + 1$. A local pathwise solution to (2.8) is a pair $(u, \tau)$, where

1. $\tau$ is a stopping time satisfying $\mathbb{P}(\tau > 0) = 1$ and $(u(t))_{t \in [0, \tau)}$ is an $\mathcal{F}_t$-progressively measurable such that

\[\sup_{t' \in [0, t]} \|u(t')\|_{H^s} < \infty, \quad t \in [0, \tau) \quad \mathbb{P}\text{-a.s.},\]

and the following equation holds for all $(t, x) \in [0, \infty) \times \mathbb{K}^d$:

\[u(t) - u_0 + \int_0^t \left[ (u \cdot \nabla)u + F(u) - \frac{1}{2} \sum_{k=1}^{\infty} Q_k^2 u \right] (t') dt' = \int_0^{t \wedge \tau} \sum_{k=1}^{\infty} Q_k u(t') d\hat{W}_k(t') + \int_0^t \sum_{k=1}^{\infty} h_k(t', u) dW_k(t'), \quad t \in [0, \tau) \quad \mathbb{P}\text{-a.s.}\]

2. Additionally, a local solution $(u, \tau^\ast)$ is called maximal, if $\tau^\ast > 0$ almost surely and

\[\limsup_{t \to \tau^\ast} \|u(t)\|_{H^s} = \infty \text{ a.s. on } \{\tau^* < \infty\}.\]

If $\tau^\ast = \infty$ almost surely, then such a solution is called global.
We also introduce the following notions on the stability of exiting time.

**Definition 2.3 (Stability of exiting time).** Let \( d \geq 2 \) and let \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{T} \). Let \( u_0 \) be an \( H^s(\mathbb{K}^d, \mathbb{R}^d) \)-valued \( \mathcal{F}_0 \)-measurable random variable with \( s > \frac{d}{2} + 1 \). Assume that \( \{u_{0,n}\} \) is an arbitrary sequence of \( H^s(\mathbb{K}^d, \mathbb{R}^d) \)-valued \( \mathcal{F}_0 \)-measurable random variables. For each \( n \), let \( u \) and \( u_n \) be the unique solutions to (1.8) with initial value \( u_0 \) and \( u_{0,n} \), respectively. For any \( R > 0 \) and \( n \in \mathbb{N} \), define the \( R \)-exiting time as

\[
\tau_n^R := \inf \{ t \geq 0 : \|u_n(t)\|_{H^s} > R \}, \quad \tau^R := \inf \{ t \geq 0 : \|u(t)\|_{H^s} > R \},
\]

where \( \inf \emptyset = \infty \).

1. Let \( R > 0 \). If \( u_{0,n} \to u_0 \) in \( H^s \) almost surely implies

\[
\lim_{n \to \infty} \tau_n^R = \tau^R \text{ P-a.s.},
\]

then the \( R \)-exiting time is said to be stable at \( u \).

2. Let \( R > 0 \). If \( u_{0,n} \to u_0 \) in \( H^{s'} \) for all \( s' < s \) almost surely also implies (2.9), then the \( R \)-exiting time is said to be strongly stable at \( u \).

## 3 Cancellation of singularities

In this section, we will develop two abstract cancellation properties to achieve (1.6) (see Theorems 3.1 and 3.2). For the well-known transport noise case, we refer to [2, 4-6, 12] and the references therein.

Recall that \( \{P_n\}_{n \geq 1} \subset \text{OPS}^s \) is said to be bounded if \( P_n = \text{OP}(p_n) \) and \( p_n \in \text{S}^s \) is bounded. Remember that \( \text{P}^s \) is the \( L^2 \)-adjoint operator of the linear operator \( \mathcal{P} \), and \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{T} \). Then we make the following assumptions:

**Hypothesis H1.** Let \( d, m \geq 1 \) and \( \beta \in [0, 1] \). Let \( b_k = \text{diag}(b_{k,1}, \cdots, b_{k,m}) \) with \( b_{k,i} \in H^s(\mathbb{K}^d, \mathbb{R}) \) \( (1 \leq i \leq m) \) satisfying \( \sum_{k=1}^{\infty} \sum_{i=1}^{m} \|b_{k,i}\|_{H^s} < \infty \) for all \( s \geq 0 \). Assume that

\[
B_k := \text{diag}(B_{k,1}, \cdots, B_{k,m}), \quad k \geq 1,
\]

such that \( \{B_{k,i}\}_{k \geq 1} \subset \text{OPS} \beta \) \( (1 \leq i \leq m) \) are bounded. Besides, we suppose that the following conditions hold true:

**H1(1)** There is a sequence of operators \( \{\mathcal{H}_k\}_{k \geq 1} \) such that \( \mathcal{H}_k = \text{diag}(\mathcal{H}_{k,1}, \cdots, \mathcal{H}_{k,m}) \), \( \{\mathcal{H}_{k,i}\}_{k \geq 1} \subset \text{OPS}^0 \) \( (1 \leq i \leq m) \) is bounded, and

\[
B_{k,i}^* = \mathcal{H}_k - B_k, \quad k \geq 1.
\]

**H1(2)** There are constants \( \sigma_0, c \geq 0 \) such that for any \( h \in C_0(\mathbb{R}^d, \mathbb{R}) \) if \( \mathbb{K} = \mathbb{R} \) or \( h \in C(\mathbb{T}^d, \mathbb{R}) \) if \( \mathbb{K} = \mathbb{T} \),

\[
\left\| \left[ (hI)B_{k,i}, B_{k,i}^*(hI) \right] \right\|_{L_c(L^2)} \leq c\|h\|_{H^s}^2, \quad 1 \leq i \leq m, \quad k \geq 1, \quad \sigma > \sigma_0.
\]

As extensions of (1.5), where \( \{\partial_{x_j}\}_{1 \leq j \leq d} \) are skew-adjoint operators, we assume that \( \{B_k\}_{k \geq 1} \) are not far away from skew-adjoint operators. In [49], \( b_k \) is assumed to be constant. Since \( B_k \) already depends on \( x \), at first glance it might not be necessary to assume that \( b_k \) also depends on \( x \). However, we prefer to do so not only because it can be easily compared to the well-known case (1.5) but also because the extension is non-trivial and there are subtle adjustments necessary to be clarified in the new situation. We refer to Remark 3.1 for more details explaining the conditions in H1.

For clarity, in the following we denote by \( b_k B_k := (b_kI)B_k \) and \( \|b_k\|_{H^s} := \sum_{i=1}^{m} \|b_{k,i}\|_{H^s}^2, \ s \geq 0 \). The cancellation properties (1.6) for \( b_k B_k \) is stated in the following

**Theorem 3.1.** Let \( s \geq 0 \) and \( \{P_n\}_{n \geq 1} \subset \text{OPS}^s \) be bounded. We let

\[
s_0 := 1 + \frac{d}{2} + \varepsilon_0 \quad (\forall \varepsilon_0 > 0) \text{ if } s \leq \frac{d}{2} + 1, \text{ and } s_0 := s, \text{ if } s > \frac{d}{2} + 1.
\]
Then we have the following assertions:

1. If Hypothesis $H_1$ without $H_1(2)$ holds true, then there is a constant $C > 0$ independent of $n$ such that

$$
\sum_{k=1}^{\infty} \langle \mathcal{P}_n(b_k B_k) f, \mathcal{P}_n f \rangle_{L^2} \leq C \sum_{k=1}^{\infty} \|b_k\|_{H^\sigma}^2 \|f\|_{H^r}^2, \quad f \in H^{s+\beta}(\mathbb{R}^d; \mathbb{R}^m).
$$

2. Let Hypothesis $H_1$ hold and $s \geq 1 - \beta$. There is a constant $C > 0$, independent of $k$ and $n$, such that

$$
\sum_{k=1}^{\infty} \left| \langle \mathcal{P}_n(b_k B_k)^2 f, \mathcal{P}_n f \rangle_{L^2} + \langle \mathcal{P}_n(b_k B_k) f, \mathcal{P}_n(b_k B_k) f \rangle_{L^2} \right| \leq C \sum_{k=1}^{\infty} B_k \|f\|_{H^r}^2, \quad f \in H^{s+2\beta}(\mathbb{R}^d; \mathbb{R}^m),
$$

where $\sigma_0$ is given in $H_1(2)$ and $B_k := \|b_k\|_{H^\sigma} + \|b_k\|_{H^\sigma \cap (\sigma_0+1)}^2$.

**Proof.** 1. One first infers from (2.7) and $H_1(1)$ that

$$
\sup_{n \geq 1} \|\mathcal{P}_n\|_{L(H^{s+\sigma}; H^r)}, \sup_{k \geq 1} \|B_k\|_{L(H^{s+\sigma}; H^r)}, \sup_{k \geq 1} \|H_k\|_{L(H^r; H^r)} < \infty, \quad r \in \mathbb{R}.
$$

Since $b_k B_k f = (b_{k,i} B_{k,i} f_i)_{1 \leq i \leq m}$, we find

$$
\langle \mathcal{P}_n(b_k B_k) f, \mathcal{P}_n f \rangle_{L^2} = I_{1,k} + I_{2,k},
$$

$$
I_{1,k} := \sum_{i=1}^{m} \langle [\mathcal{P}_n, b_{k,i}] B_{k,i} f_i, \mathcal{P}_n f_i \rangle_{L^2},
$$

$$
I_{2,k} := \sum_{i=1}^{m} \langle b_{k,i} B_{k,i} f_i, \mathcal{P}_n f_i \rangle_{L^2}.
$$

Since $s_0 - s \geq 0$ and $\beta \leq 1$, we can infer from Lemma A.9 (with $q = 0$, $\sigma = s_0$ and $r = s$) and (3.4) that

$$
I_{1,k} \lesssim \|b_k\|_{H^\sigma} \|B_k f\|_{H^r} \|f\|_{H^r} \lesssim \|b_k\|_{H^\sigma} \|f\|_{H^r}^2.
$$

Now we estimate $I_{2,k}$. Via $H_1(1)$, one can observe that

$$
I_{2,k} = \sum_{i=1}^{m} \left( \langle [\mathcal{P}_n, B_{k,i}] f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2} + \langle B_{k,i} \mathcal{P}_n f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \right)
$$

$$
= \sum_{i=1}^{m} \left( \langle [\mathcal{P}_n, B_{k,i}] f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, -B_{k,i}(b_{k,i} \mathcal{P}_n f_i) \rangle_{L^2} + \langle \mathcal{P}_n f_i, H_{k,i}(b_{k,i} \mathcal{P}_n f_i) \rangle_{L^2} \right)
$$

$$
= \sum_{i=1}^{m} \left( \langle [\mathcal{P}_n, B_{k,i}] f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2} - \langle \mathcal{P}_n f_i, [B_{k,i}, b_{k,i}] \mathcal{P}_n f_i \rangle_{L^2}
$$

$$
- \langle \mathcal{P}_n f_i, b_{k,i} B_{k,i} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, H_{k,i}(b_{k,i} \mathcal{P}_n f_i) \rangle_{L^2} \right).
$$

Since all the functions and operators are real, $\langle \mathcal{P}_n f_i, b_{k,i} B_{k,i} \mathcal{P}_n f_i \rangle_{L^2} = \langle B_{k,i} \mathcal{P}_n f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2}$, which brings us

$$
I_{2,k} = \sum_{i=1}^{m} \left( \langle [\mathcal{P}_n, B_{k,i}] f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2} - \langle \mathcal{P}_n f_i, [B_{k,i}, b_{k,i}] \mathcal{P}_n f_i \rangle_{L^2}
$$

$$
- \langle B_{k,i} \mathcal{P}_n f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, H_{k,i}(b_{k,i} \mathcal{P}_n f_i) \rangle_{L^2} \right)
$$

$$
= \sum_{i=1}^{m} \left( 2 \langle [\mathcal{P}_n, B_{k,i}] f_i, b_{k,i} \mathcal{P}_n f_i \rangle_{L^2} - \langle \mathcal{P}_n f_i, [B_{k,i}, b_{k,i}] \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, H_{k,i}(b_{k,i} \mathcal{P}_n f_i) \rangle_{L^2} \right) - I_{2,k}.
$$
Hence
\[ I_{2,k} = \sum_{i=1}^{m} \left( \langle [P_n, B_{k,i}] f_i, b_{k,i} P_n f_i \rangle_{L^2} - \frac{1}{2} \langle P_n f_i, [B_{k,i}, b_{k,i}] P_n f_i \rangle_{L^2} + \frac{1}{2} \langle P_n f_i, H_{k,i} (b_{k,i} P_n f_i) \rangle_{L^2} \right). \]

On account of (3.4), $H^{s_0} \hookrightarrow W^{1,\infty}$, Lemmas A.8 and A.9 (with $q = 0$, $\sigma = s_0$ and $r = \beta$), we have
\[ \sum_{i=1}^{m} \langle [P_n, B_{k,i}] f_i, b_{k,i} P_n f_i \rangle_{L^2} \lesssim \| f \|_{H^{s+\beta-1}} \| b_k \|_{L^\infty} \| f \|_{H^\beta} \lesssim \| b_k \|_{H^{s_0}} \| f \|_{H^\beta}^2, \]
\[ \sum_{i=1}^{m} \langle P_n f_i, [B_{k,i}, b_{k,i}] P_n f_i \rangle_{L^2} \lesssim \| f \|_{H^\beta} \| b_k \|_{H^{s_0}} \| P_n f \|_{H^{s_0}} \lesssim \| b_k \|_{H^{s_0}} \| f \|_{H^\beta}^2, \]
and
\[ \sum_{i=1}^{m} \langle P_n f_i, H_{k,i} (b_{k,i} P_n f_i) \rangle_{L^2} \lesssim \| b_k \|_{H^{s_0}} \| f \|_{H^\beta}^2. \]

In conclusion we derive
\[ |I_{2,k}| \lesssim \| b_k \|_{H^{s_0}} \| f \|_{H^\beta}^2. \]

Combining the estimates for $I_{i,k}$ with $i = 1, 2$ and then taking summation $\sum_{k \geq 1}$, we obtain (3.2).

(2) The proof for (3.3) includes the following steps.

**Step (1).** For $k, n \in \mathbb{N}$ and $1 \leq i \leq m$, we let
\[ \begin{align*}
& \mathcal{Z}_{k,i} = [b_{k,i}I, B_{k,i}] + H_{k,i} (b_{k,i}I), \\
& \mathcal{R}_{1,k}^{(i)} = [b_{k,i}B_{k,i}, \mathcal{Z}_{k,i}], \\
& \mathcal{R}_{2,k}^{(i)} = [P_n, b_{k,i}B_{k,i}], \\
& \mathcal{R}_{3,k}^{(i)} = [b_{k,i}B_{k,i}], \\
& \mathcal{R}_{4,k}^{(i)} = [b_{k,i}B_{k,i}], \\
& \mathcal{R}_{5,k}^{(i)} = [b_{k,i}B_{k,i}],
\end{align*} \]
where $H_k$ is given in $H_1(1)$. We claim that
\[ \langle P_n (b_k B_k)^2 f, P_n f \rangle_{L^2} + \langle P_n (b_k B_k) f, P_n (b_k B_k) f \rangle_{L^2} = \sum_{j=1}^{6} \sum_{i=1}^{m} N_{j,i}, \]
where
\[ \begin{align*}
N_{1,i} := & \langle \mathcal{R}_{3,k,n}^{(i)} f_i, P_n f_i \rangle_{L^2}, \\
N_{2,i} := & \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2}, \\
N_{3,i} := & \langle P_n f_i, \mathcal{Z}_{k,i} \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2}, \\
N_{4,i} := & -\langle P_n f_i, \mathcal{R}_{1,k}^{(i)} P_n f_i \rangle_{L^2}, \\
N_{5,i} := & \frac{1}{2} \langle P_n f_i, \mathcal{Z}_{k,i} P_n f_i \rangle_{L^2}, \\
N_{6,i} := & \langle \mathcal{R}_{3,k,n}^{(i)} f_i, \mathcal{Z}_{k,i} P_n f_i \rangle_{L^2},
\end{align*} \]
To simplify notation, we let
\[ \mathcal{T}_k := \text{diag} (\mathcal{T}_{k,1}, \cdots, \mathcal{T}_{k,m}), \quad \mathcal{T}_{k,i} := b_{k,i} B_{k,i} = (b_{k,i}I) B_{k,i}, \quad 1 \leq i \leq m. \]
By $H_1(1)$, one can immediately find that $\mathcal{T}_{k,i}^* = -\mathcal{T}_{k,i} + \mathcal{Z}_{k,i}$. Therefore we arrive at
\[ \langle P_n \mathcal{T}_{k,i}^2 f_i, P_n f_i \rangle_{L^2} = \langle (\mathcal{T}_{k,i} P_n + \mathcal{R}_{2,k,n}^{(i)}) \mathcal{T}_{k,i} f_i, P_n f_i \rangle_{L^2} \]
\[ = \langle P_n \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{T}_{k,i} P_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} P_n f_i, P_n f_i \rangle_{L^2} \]
\[ = -\langle P_n \mathcal{T}_{k,i} f_i, \mathcal{T}_{k,i} P_n f_i \rangle_{L^2} + \langle P_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} P_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i, P_n f_i \rangle_{L^2} \]
\[ = -\langle P_n \mathcal{T}_{k,i} f_i, P_n \mathcal{T}_{k,i} f_i \rangle_{L^2} + \langle P_n \mathcal{T}_{k,i} f_i, \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle P_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} P_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i, P_n f_i \rangle_{L^2} \]
\[ + \langle P_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} P_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i, P_n f_i \rangle_{L^2}. \]
which means
\[ \langle \mathcal{P}_n \mathcal{T}_{k,i}^2, f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{P}_n \mathcal{T}_{k,i} f_i \rangle_{L^2} \]
\[ = \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i, \mathcal{P}_n f_i \rangle_{L^2}. \]

Note that \( \mathcal{P}_n \) is of order \( s \). Then \( \mathcal{P}_n \mathcal{T}_{k,i} \) is of order \( s + \beta \geq s \). Similarly, the order of \( \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} \) may be bigger than \( s \). Therefore, by commuting \( \mathcal{P}_n \) and \( \mathcal{T}_{k,i} \) and using \( \mathcal{T}_{k,i}^* \) again, we have
\[ \langle \mathcal{P}_n \mathcal{T}_{k,i}^2 f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{P}_n \mathcal{T}_{k,i} f_i \rangle_{L^2} \]
\[ = \langle \mathcal{T}_{k,i} \mathcal{P}_n f_i, \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]
\[ = - \langle \mathcal{P}_n f_i, \mathcal{T}_{k,i} \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i} \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]
\[ = \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i} \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2}. \]

(3.6) where in the last step the fact that all the functions are real-valued is used to obtain
\[ -\langle \mathcal{P}_n f_i, \mathcal{T}_{k,i} \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i, \mathcal{P}_n f_i \rangle_{L^2} = \langle \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i, \mathcal{P}_n f_i \rangle_{L^2}. \]

Note that \( \mathcal{Z}_{k,i} \) is self-adjoint. Then, once again, we find
\[ \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]
\[ = \langle \mathcal{T}_{k,i} \mathcal{P}_n f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]
\[ = - \langle \mathcal{P}_n f_i, \mathcal{T}_{k,i} \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i}^2 \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]
\[ = - \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i} \mathcal{T}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} - \langle \mathcal{P}_n f_i, \mathcal{R}_{1,k}^{(i)} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i}^2 \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]
\[ = - \langle \mathcal{Z}_{k,i} \mathcal{P}_n f_i, \mathcal{T}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} - \langle \mathcal{P}_n f_i, \mathcal{R}_{1,k}^{(i)} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i}^2 \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2}. \]

Therefore, adding \( \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \) to both sides of the above equation and then using \( \mathcal{P}_n \mathcal{T}_{k,i} - \mathcal{T}_{k,i} \mathcal{P}_n = \mathcal{R}_{2,k,n}^{(i)} \) give rise to
\[ 2\langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} = - \langle \mathcal{P}_n f_i, \mathcal{R}_{1,k}^{(i)} \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i}^2 \mathcal{P}_n f_i \rangle_{L^2} + 2\langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]

Combining (3.6) and (3.7) gives
\[ \langle \mathcal{P}_n \mathcal{T}_{k,i}^2 f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n \mathcal{T}_{k,i} f_i, \mathcal{P}_n \mathcal{T}_{k,i} f_i \rangle_{L^2} \]
\[ = \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i} \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{R}_{2,k,n}^{(i)} f_i \rangle_{L^2} \]
\[ - \frac{1}{2} \langle \mathcal{P}_n f_i, \mathcal{R}_{1,k}^{(i)} \mathcal{P}_n f_i \rangle_{L^2} + \frac{1}{2} \langle \mathcal{P}_n f_i, \mathcal{Z}_{k,i}^2 \mathcal{P}_n f_i \rangle_{L^2} + \langle \mathcal{R}_{2,k,n}^{(i)} f_i, \mathcal{Z}_{k,i} \mathcal{P}_n f_i \rangle_{L^2} \]
\[ = \sum_{j=1}^{N_{j,i}} N_{j,i}. \]

Taking summation \( \sum_{i=1}^{m} \) to above identity gives (3.5).

**Step (2).** Now we claim that there is a constant \( C > 0 \) independent of \( n \) and \( k \) such that for \( 1 \leq i \leq m \),
\[ \| \mathcal{Z}_{k,i} \|_{L^2} \leq C \| \mathcal{b}_{k,i} \|_{H^s}, \quad k \in \mathbb{N}, \]
(3.8)
\[ \sup_{n \geq 1} \| \mathcal{R}_{2,k,n}^{(i)} \|_{L(H^{s+\beta-1})} \leq C \| \mathcal{b}_{k,i} \|_{H^s}, \quad k \geq 1, \]
(3.9)
where \( s_0 \) is given in (3.1), and
\[ \sup_{k,n \geq 1} \| \mathcal{R}_{2,k,n}^{(i)} \|_{L(H^{s+2\beta-2})} < \infty, \quad 1 \leq i \leq m. \]
(3.10)
For $\mathcal{R}_{2,k,n}^{(i)}$, it holds that
\[ \| \mathcal{R}_{2,k,n}^{(i)} g \|_{L^2} = \| [c_n, b_{k,i}] g \|_{L^2} \leq \| [c_n, b_{k,i}] g \|_{L^2} + \| b_{k,i} [c_n, b_{k,i}] g \|_{L^2}. \]
Since the symbols of $c_n$ is bounded in $\mathcal{S}^s$, by $\beta \leq 1$, Lemma A.9 (with $\sigma = 0$ and $q = 0$) and (3.4), we obtain that for sufficiently regular function $g$,
\[ \| [c_n, b_{k,i}] g \|_{L^2} \leq C \| b_{k,i} \|_{H^0} \| g \|_{H^{s+\beta-1}}. \]
Similarly, we infer from $H_1(1)$ and Lemma A.8 that $[c_n, b_{k,i}] \in \text{OPS}^{s+\beta-1}$ and its operator norm is bounded. Therefore, on account of Lemmas A.6 and A.8, for sufficiently regular function $g$, we have
\[ \| b_{k,i} [c_n, b_{k,i}] g \|_{L^2} \leq \| b_{k,i} \|_{H^0} \| g \|_{H^{s+\beta-1}}. \]
Collecting the above estimates, we obtain (3.9).

The estimate (3.8) may be obtained in much the same way as (3.9). Using Lemma A.9 to $[b_{k,i}, b_{k,i}] g$, we arrive at
\[ \| [b_{k,i}, b_{k,i}] g \|_{L^2} \leq C \| b_{k,i} \|_{H^0} \| g \|_{H^{s+\beta-1}} \leq C \| b_{k,i} \|_{H^0} \| g \|_{L^2}. \]
From this and (3.4), it is easy to see that (3.8) holds.

Since the symbol of $\mathcal{R}_{2,k,n}^{(i)} = [c_n, b_{k,i}]$ can be explicitly written down (see for example [54, (0.3.6)] or [38, Theorem 1.2.16]) and they form a bounded sequence in $\mathcal{S}^{s+\beta-1}$. Then, by Lemma A.8 once again, we obtain (3.10).

**Step (3).** In this step we finish the proof.

We recall that $s_0 > \frac{4}{2} + 1$ is given in (3.1). We first note that
\[
\sum_{i=1}^{m} |N_{1,i}| \leq \sum_{i=1}^{m} \left[ \left\| \mathcal{R}_{2,k,n}^{(i)} \mathcal{T}_{k,i} f_i \right\|_{L^2} \left\| f_i \right\|_{H^s} \right] 
\leq \sum_{i=1}^{m} \left[ \left\| \mathcal{R}_{2,k,n}^{(i)} b_{k,i} f_i \right\|_{L^2} \left\| b_{k,i} f_i \right\|_{H^s} + \left\| b_{k,i} \mathcal{R}_{2,k,n}^{(i)} b_{k,i} f_i \right\|_{L^2} \left\| f_i \right\|_{H^s} \right].
\]
As in **Step (2)**, the sequence of symbols corresponding to $\{\mathcal{R}_{2,k,n}^{(i)}\}$ is bounded in $\mathcal{S}^{s+\beta-1}$. Applying Lemma A.9 to $\mathcal{R}_{2,k,n}^{(i)} b_{k,i} f_i$ (with $q = 0$, $\sigma = 0$ and $r = s + \beta - 1 \geq 0$) yields
\[ \left\| \mathcal{R}_{2,k,n}^{(i)} b_{k,i} f_i \right\|_{L^2} \leq \| b_{k,i} \|_{H^0} \| b_{k,i} f_i \|_{H^{s+\beta-2}} \leq \| b_{k,i} \|_{H^0} \| f_i \|_{H^s}. \]
Then we can infer from (3.10) and $\beta \leq 1$ that
\[ \left\| b_{k,i} \mathcal{R}_{2,k,n}^{(i)} b_{k,i} f_i \right\|_{L^2} \leq \| b_{k,i} \|_{H^0} \left\| \mathcal{R}_{2,k,n}^{(i)} b_{k,i} f_i \right\|_{L^2} \leq \| b_{k,i} \|_{H^0} \| f_i \|_{H^s}. \]
To sum up,
\[ \sum_{i=1}^{m} |N_{1,i}| \leq C \| b_{k,i} \|_{H^0} \| f \|_{H^s}^2. \]

For $\sum_{i=1}^{m} N_{2,i}$, we simply use (3.9) to find that
\[ \sum_{i=1}^{m} |N_{2,i}| = \sum_{i=1}^{m} \left\| \mathcal{R}_{2,k,n}^{(i)} f_i \right\|_{L^2} \leq \| b_{k,i} \|_{H^0} \| f \|_{H^s}^2. \]
Then we use (3.4), (3.8) and (3.9) that
\[ \sum_{i=1}^{m} |N_{3,i}| \leq C \sum_{i=1}^{m} \left\| \mathcal{P}_n f_i \right\|_{L^2} \| b_{k,i} \|_{H^0} \left\| \mathcal{R}_{2,k,n}^{(i)} f_i \right\|_{L^2} \leq \sum_{i=1}^{m} \| b_{k,i} \|_{H^0} \| f_i \|_{H^s}^2 \leq \| b_{k,i} \|_{H^0} \| f \|_{H^s}^2. \]
For $N_{k,i}$, we observe that
\begin{equation}
(3.11) \quad [T_{k,i}, Z_{k,i}] = [(b_{k,i})_{E_{k,i}}, (b_{k,i})_{E_{k,i}} - E_{k,i}(b_{k,i}) + \mathcal{H}_{k,i}(b_{k,i})] = [(b_{k,i})_{E_{k,i}}, E_{k,i}^* (b_{k,i})].
\end{equation}
Therefore $H_1(2)$ and (3.4) give rise to
\begin{equation}
(3.12) \quad \sum_{i=1}^{m} |N_{k,i}| \lesssim \sum_{i=1}^{m} \|b_{k,i}\|_{H^{\sigma_0}} \|P_n f_i\|_{L^2} \lesssim \|b_k\|_{H^{\sigma_0}} \|f\|_{H^2},
\end{equation}
where $\sigma_0$ is given in $H_1(2)$. Once again, (3.4), (3.8) and (3.9) enable us to derive
\begin{equation}
\sum_{i=1}^{m} (|N_{5,i}| + |N_{6,i}|) \leq C \|b_k\|_{H^{\sigma_0}} \|f\|_{H^2}.
\end{equation}
Collecting all these estimates for (3.5), we see that there is a constant $C > 0$ independent of $n$ and $k$ such that
\begin{equation}
\langle P_n(b_k)_{E_{k,i}}^2 f, P_n f \rangle_{L^2} + \langle P_n(b_k)_{E_{k,i}} f, P_n(b_k)_{E_{k,i}} f \rangle_{L^2} \leq C \left( \|b_k\|_{H^{\sigma_0}} + \|b_k\|_{H^{\sigma_0}} + \|b_k\|_{H^{\sigma_0}} \right) \|f\|_{H^2}.
\end{equation}
Hence we obtain (3.3). \qed

From the above proof, we see that, if $P_n \in \text{OPS}^0_0$, the cancellation properties hold true for another class of operators described by

**Hypothesis $H_2$.** Let $d, m \geq 1$, $\alpha \geq 0$ and $a_k := \text{diag}(a_{k,1}, \cdots, a_{k,m})$, with $k \geq 1$ such that $\{a_{k,i}\}_{k \geq 1} \subset \text{OPS}^0_0$ ($1 \leq i \leq m$) are bounded. Besides, suppose that
\begin{equation}
A_{k} = M_k - A_k, \quad k \geq 1,
\end{equation}
for some $M_k := \text{diag}(M_{k,1}, \cdots, M_{k,m})$ such that $\{M_{k,i}\}_{k \geq 1} \subset \text{OPS}^0_0$ ($1 \leq i \leq m$) are bounded.

Indeed, if $P_n \in \text{OPS}^0_0$, repeating the proof for Theorem 3.1 with noting that in this case $A_k \in \text{OPS}^0_0$, we have $|P_n, A_k| = |P_n, a_k| = |A_k, a_k| = 0$ and hence
\begin{equation}
\langle P_n(a_k A_k) f, P_n f \rangle_{L^2} = \frac{a_k^2}{2} \langle P_n f, M_k P_n f \rangle_{L^2},
\end{equation}
and
\begin{equation}
\left| \langle P_n(a_k A_k) f, P_n f \rangle_{L^2} + \langle P_n(a_k A_k) f, P_n(a_k A_k) f \rangle_{L^2} \right| \leq \frac{a_k^2}{2} \left| \langle P_n f, M_k^2 P_n f \rangle_{L^2} \right|.
\end{equation}
Therefore we have actually established the following cancellation properties for $x$-independence operators:

**Theorem 3.2.** Let $s \geq 0$ and $\{P_n\}_{n \geq 1} \subset \text{OPS}^0_0$ be bounded. If Hypothesis $H_2$ holds true, then there is a constant $C > 0$ independent of $n$ such that
\begin{equation}
\sum_{k=1}^{\infty} \langle P_n(a_k A_k) f, P_n f \rangle_{L^2}^4 \leq C \|a_k\|_{L^2}^4 \|f\|_{H^s}^4, \quad f \in H^{s+\alpha}(\mathbb{R}^d; \mathbb{R}^m),
\end{equation}
and
\begin{equation}
\sum_{k=1}^{\infty} \left| \langle P_n(a_k A_k)^2 f, P_n f \rangle_{L^2} + \langle P_n(a_k A_k) f, P_n(a_k A_k) f \rangle_{L^2} \right| \leq C \|a_k\|_{L^2}^2 \|f\|_{H^s}^2, \quad f \in H^{s+2\alpha}(\mathbb{R}^d; \mathbb{R}^m).
\end{equation}

**Remark 3.1.** A few comments are in order regarding Hypothesis $H_1$ and Theorem 3.1.

1. When we construct approximation scheme on (2.8) (see (4.11) below), mollifiers $J_n$ can not commute with $B_k$. Therefore, to obtain uniform estimate in $H^s$, we have to deal with $P_n = D^s J_n$ (see Lemma 4.3 below). Therefore we state the uniform (in $n$) estimate for a sequence $\{P_n\}_{n \geq 1}$ rather than just one $P$. 

13
Let \( b(x) \in H^{\infty}(\mathbb{R}^d; \mathbb{R}) \). Now we consider the following question: Is there a \( q \geq 0 \) such that for all \( \mathcal{P} \in \text{OPS}^s \) and \( Q \in \text{OPS}^l \),

\[
(3.12) \quad \left\| \left[ [\mathcal{P}, b(x)Q], b(x)Q \right] \right\|_{L(H^s; L^2)} \lesssim \|b\|^2_{H^q}?
\]

So far we have only been able to show (see the estimate for \( L_1 \) in Step (3) in the proof for (3.3)) that

\[
\left\| \left[ [\mathcal{P}, b(x)Q], b(x)Q \right] \right\|_{L(H^s; L^2)} \lesssim \|b\|^2_{H^0},
\]

which is weaker than (3.13) in the case \( \|b\|_{H^q} < 1 \). And this is why we have to assume \( \sum_{k=1}^{\infty} \|b_k\|_{H^s} < \infty \), stronger than \( \sum_{k=1}^{\infty} \|b_k\|^2_{H^s} < \infty \). In [12], \( P_n = \mathcal{P} = \Delta \) and \( B_k \) is of the form (1.5). Then Leibniz rule holds true. Without Leibniz rule, so far it is not clear how to verify (3.13).

(3) Now we explain \( \text{H}_1(2) \). On account of \( \text{H}_1(1) \) and Lemma A.6, we see that for each \( k \geq 1 \)

\[
[(hI)B_{k,i}, B_{k,i}^*(hI)] = [(hI)B_k, hIB_k + B_k^*(hI)] = [(hI)B_k, hIB_k - B_k(hI) + \mathcal{H}_k(hI)] = [(hI)B_k, [hI, B_k]] + [(hI)B_{k,i}, \mathcal{H}_{k,i}, (hI)]
\]

is actually an operator of order 0 (since \( \beta \leq 1 \)). Even though its operator norm can be independent of \( k \) (via Hypothesis \( \text{H}_1(1) \) and Lemma A.8), it depends on \( h \). Namely, for \( k \geq 1 \), there is a constant \( C = C(h) > 0 \) such that

\[
\left\| [(hI)B_{k,i}, B_{k,i}^*(hI)] g \right\|_{L^2} \leq C(h) \|g\|_{L^2}.
\]

Then \( \text{H}_1(2) \) actually means that constant \( C(h) \) can be replaced by \( c\|h\|^{3/2} \) with \( \sigma \geq \sigma_0 \) and \( c > 0 \). This enables us to take summation \( \sum_{k=1}^{\infty} \) (see (3.11) and (3.12) in the proof). Without \( \text{H}_1(2) \), even if \( \sum_{k=1}^{\infty} \|b_k\|_{H^s} < \infty \), one can only take summation of finitely many \( k \) in (3.3) since (3.12) becomes \( \sum_{i=1}^{\infty} |\|N_{1,i}\|_{L^2}^2 | \leq C(b_k) \|f\|_{H^s} \), for some constant \( C(b_k) > 0 \) and we do not a priori know whether or not \( \sum_{k=1}^{\infty} C(b_k) < \infty \). For other qualitative research on second-order commutator in stochastic setting, we refer to [24].

A broad class of operators satisfying \( \text{H}_1(2) \) are given by

**Lemma 3.1.** If \( B_k \) satisfies \( \text{H}_1(1) \) with \( 0 \leq \beta \leq \frac{1}{2} \) \( \Rightarrow \) \( B_k \) satisfies \( \text{H}_1(2) \) with \( \sigma_0 = \frac{d}{2} + 1 \).

**Proof.** Keep in mind that \( 1 \leq i \leq m \) in the following. We note that

\[
[hIB_{k,i}, B_{k,i}^*(hI)] = -[hIB_{k,i}, B_{k,i}(hI)] + [hIB_{k,i}, \mathcal{H}_{k,i}, (hI)] := \Theta_1 + \Theta_2.
\]

Direct computation shows that

\[
\Theta_1 = -h[B_{k,i}^2, hI] + [B_{k,i}, hI][hIB_{k,i}] + h[B_{k,i}, hI]B_{k,i} := \sum_{j=1}^{3} \Theta_{1,j},
\]

\[
\Theta_2 = h[B_{k,i}, \mathcal{H}_{k,i}, hI] + h^2[B_{k,i}, \mathcal{H}_{k,i}] + [h^2I, \mathcal{H}_{k,i}]B_{k,i} := \sum_{i=1}^{3} \Theta_{2,j}.
\]

Again, (2.7) and \( \text{H}_1(1) \) give us

\[
\sup_{k \geq 1} \|B_{k,i}\|_{L(H^{s+r}, H^r)} \cdot \sup_{k \geq 1} \|\mathcal{H}_{k,i}\|_{L(H^r, H^s)} < \infty, \quad r \in \mathbb{R}.
\]

Remember the above estimate and let \( \eta > \frac{d}{2} + 1 \). By Lemmas A.6 and A.9 (with \( \sigma = \eta > r = 2\beta \) and \( q = 0 \)) and \( H^\eta \hookrightarrow W^{1,\infty} \), we see that

\[
\|\Theta_{1,1}g\|_{L^2} \lesssim \|h\|_{L^\infty} \|B_{k,i}^2, hI\|_{L^2} \lesssim \|h\|_{H^\eta}^2 \|g\|_{L^2}.
\]
Similarly, using $H^n \hookrightarrow W^{1,\infty}$ and Lemma A.9 (with $\sigma = \eta > r = \beta$ and $q = 0$) and Lemma A.3 (with $s_1 = \beta - 1$ and $s_2 = \eta$), we arrive at

\[ \| \Theta_{1,2} g \|_{L^2} \lesssim \| h \|_{H^n} \| h B_{k,i} g \|_{H^{\beta - 1}} \lesssim \| h \|_{H^n} \| h B_{k,i} g \|_{H^{\beta - 1}} \lesssim \| h \|_{H^n} \| g \|_{L^2}, \]

and

\[ \| \Theta_{1,3} g \|_{L^2} \lesssim \| h \|_{L^\infty} \| h B_{k,i} g \|_{H^{\beta - 1}} \lesssim \| h \|_{H^n} \| g \|_{L^2}. \]

Due to $H_1(1)$, as in the proof for Lemma A.8, the symbols of the product operator $B_{k,i} H_{k,i}$ is also bounded in $S^0$. Hence we apply Lemma A.9 (with $\sigma = \eta > r = \beta$ and $q = 0$) to find

\[ \| \Theta_{2,1} g \|_{L^2} \lesssim \| h \|_{L^\infty} \| (B_{k,i} H_{k,i}, h I) g \|_{L^2} \lesssim \| h \|_{H^n} \| g \|_{H^{\beta - 1}} \lesssim \| h \|_{H^n} \| g \|_{L^2}. \]

Using Lemma A.8 to $\Theta_{2,2}$ yields

\[ \| \Theta_{2,2} g \|_{L^2} \lesssim \| h \|_{L^\infty} \| (B_{k,i} H_{k,i}, h I) g \|_{L^2} \lesssim \| h \|_{H^n} \| g \|_{H^{\beta - 1}} \lesssim \| h \|_{H^n} \| g \|_{L^2}. \]

Finally, Lemma A.9 (with $\sigma = \eta > s = 0$ and $q = 0$) gives rise to

\[ \| \Theta_{2,3} g \|_{L^2} \lesssim \| h \|_{H^n} \| B_{k,i} g \|_{H^{\beta - 1}} \lesssim \| h \|_{H^n} \| g \|_{H^{\beta - 1}} \lesssim \| h \|_{H^n} \| g \|_{L^2}. \]

Hence $H_1(2)$ holds true with $\sigma_0 = \frac{d}{2} + 1$.

\[ \square \]

4 Local and global results

In this section we focus on (1.7) and we will use Theorems 3.1 and 3.2 with $m = d$. As before we simply write

\[ H^s = H^s(\mathbb{K}^d, \mathbb{R}^d), \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{T}. \]

We recall the following estimates for $F(\cdot)$:

**Lemma 4.1** ([58, 60]). Let $s > d/2$ with $d \geq 2$. The non-local term $F(\cdot)$ defined in (1.3) satisfies

\[
\| F(v) \|_{H^s} \lesssim \| v \|_{W^{1,\infty}} \| v \|_{H^s}, \quad s > d/2 + 1, \quad v \in H^s,
\]

\[
\| F(v_1) - F(v_2) \|_{H^s} \lesssim \| v_1 \|_{H^{s+1}} + \| v_2 \|_{H^{s+1}} \| v_1 - v_2 \|_{H^s}, \quad s > d/2 + 1, \quad v_1, v_2 \in H^s
\]

\[
\| F(v_1) - F(v_2) \|_{H^s} \lesssim \| v_1 \|_{H^{s+1}} + \| v_2 \|_{H^{s+1}} \| v_1 - v_2 \|_{H^s}, \quad s > d/2 + 1, \quad v_1, v_2 \in H^{s+1}.
\]

Let $J_n$ be the Friedrichs mollifier defined in Appendix A (cf. (A.1)). Then we have

**Lemma 4.2.** For all $\sigma > \frac{d}{2} + 1$, there is a constant $\Lambda = \Lambda(\sigma, d') > 0$ such that

\[
\| (u \cdot \nabla)u + F(u) \|_{H^s} \leq \Lambda \| u \|_{H^s}^2 \| u \|_{W^{1,\infty}}, \quad u \in H^{s+1},
\]

\[
\| J_n [(u \cdot \nabla)u] + J_n F(u), J_n u \|_{H^s} \leq \Lambda \| u \|_{H^s}^2 \| u \|_{W^{1,\infty}}, \quad u \in H^s.
\]

**Proof.** We only prove (4.2) since (4.1) can be proved in the same way. Using Lemmas A.1, A.2 and A.4, integration by parts and $H^s \hookrightarrow W^{1,\infty}$, we obtain that for some $\Lambda = \Lambda(\sigma, d') > 0$,

\[
(D^\sigma J_n [(u \cdot \nabla)u], D^\sigma J_n u)_{L^2} = \left( \langle [D^\sigma, (u \cdot \nabla)]u, D^\sigma J_n u \rangle_{L^2} + \langle [J_n, (u \cdot \nabla)]D^\sigma u, D^\sigma J_n u \rangle_{L^2} + \langle (u \cdot \nabla)D^\sigma J_n u, D^\sigma J_n u \rangle_{L^2} \right)
\]

\[
\leq \Lambda \left( \| u \|_{H^s} \| \nabla u \|_{L^\infty} \| J_n u \|_{H^s} + \| u \|_{H^s} \| \nabla u \|_{L^\infty} \| J_n u \|_{H^s} + \| J_n u \|_{H^s}^2 \| \nabla u \|_{L^\infty} \right)
\]

\[
\leq \Lambda \| u \|_{H^s}^2 \| u \|_{W^{1,\infty}}.
\]

Similarly, Lemma 4.1 implies

\[
(D^\sigma J_n F(u), D^\sigma J_n u)_{L^2} \leq \Lambda \| u \|_{H^s}^2 \| u \|_{W^{1,\infty}}.
\]

Combining the above estimates gives (4.2).
To obtain a solution, we need the following

**Hypothesis H₃.** Let \( d \geq 1 \). We assume

\[
Q_k = a_kA_k + b_kB_k, \quad a_kb_k = 0, \quad k \geq 1,
\]

and \((b_k, B_k)\) and \((a_k, A_k)\) satisfy Hypotheses H₁ and H₂ with \( m = d \), respectively.

**Hypothesis H₄.** For all \( k, h_k : [0, \infty) \times H^s \ni (t, u) \mapsto h_k(t, u) \in H^s \) is continuous for \( s > \frac{d}{2} + 1 \). Moreover, there is a function \( K : [0, \infty) \times [0, \infty) \rightarrow (0, \infty) \) increasing in both variables such that

\[
\sum_{k=1}^{\infty} \| h_k(t, u) \|_{H^s}^2 \leq K(t, \| u \|_{W^{1, \infty}})(1 + \| u \|_{H^s}^2), \quad t \geq 0, \quad u \in H^s,
\]

\[
\sum_{k=1}^{\infty} \| h_k(t, u) - h_k(t, v) \|_{H^s}^2 \leq K(t, \| u \|_{H^s} + \| v \|_{H^s})\| u - v \|_{H^s}^2, \quad t \geq 0, \quad u, v \in H^s.
\]

Recall \((\beta, b_k)\) and \((\alpha, a_k)\) in Hypotheses H₁ and H₂, respectively. Let

\[
p_0 := \max \left\{ \alpha \mathbf{1}_{\{\| a_k \|_2 > 0\}}, \beta \mathbf{1}_{\{\sum_{k=1}^{\infty} \| b_k \|_{H^{s0}} > 0\}} \right\}.
\]

The main results for (1.7) (or (2.8)) is the following

**Theorem 4.1.** Let Hypotheses H₃ and H₄ be verified. Let \( s > \frac{d}{2} + 1 + \max\{2p_0, 1\} \) with \( d \geq 2 \). For any \( H^s \)-valued \( F_0 \)-measurable random variable \( u_0 \),

(I) (1.7) admits a unique maximal solution \((u, \tau^*)\) in the sense of Definitions 2.2. Besides, \((u, \tau^*)\) satisfies \( \mathbb{P}(u \in C([0, \tau^*]; H^s)) = 1 \) and

\[
1\{\limsup_{t \to \tau^*} \| u(t) \|_{H^s} = \infty\} = 1\{\limsup_{t \to \tau^*} \| u(t) \|_{W^{1, \infty}} = \infty\} \quad \mathbb{P}\text{-a.s.}
\]

(II) \( u \) exists globally, i.e., \( \mathbb{P}(\tau^* = \infty) = 1 \), if

\[
\limsup_{\| f \|_{H^s} \to \infty} \frac{\Xi(T, f, \eta)}{2a\| f \|_{W^{1, \infty}}\| f \|_{H^s}} < -1, \quad T \in (0, \infty), \quad \eta \in \left(\frac{d}{2} + 1, s - \max\{2p_0, 1\}\right),
\]

where \( \Lambda \) is given in Lemma 4.2 and

\[
\Xi(T, v, \sigma) := \sup_{t \in [0, T]} \sum_{k=1}^{\infty} \left( \| h_k(t, v) \|_{H^s}^2 - \frac{2\langle h_k(t, v), v \rangle_{H^s}}{e + \| v \|_{H^s}^2} \right), \quad \sigma > \frac{d}{2} + 1.
\]

The proof for Theorem 4.1 can be carried out in a way similar to [49]. However, since the pseudo-differential operators in this paper are extended, here we also provide the details and the proof is divided into three subsections.

### 4.1 Approximation scheme and estimates

For convenience, we recall that (1.7) is equivalent to (2.8). Because \( a_kb_k = 0 \), we have \( Q_k^2 = (a_kA_k)^2 + (b_kB_k)^2 \) and then we further rewrite (2.8) as

\[
\begin{align*}
\left\{ 
\begin{array}{c}
du = - (u \cdot \nabla)u - F(u) + \frac{1}{2} \sum_{k=1}^{\infty} \left[ (a_kA_k)^2 u + (b_kB_k)^2 u \right] dt \\
+ \sum_{k=1}^{\infty} \left( a_kA_k u d\overline{W}_k + b_kB_k u d\overline{W}_k + h_k(t, X(t)) dW_k(t) \right), \quad u|_{t=0} = u_0, \quad t \geq 0,
\end{array}
\right.
\end{align*}
\]

where \( \overline{W}_k = W_k - E[W] \) are independent Brownian motions.
where
\[ \overline{W}_k(t) = \overline{W}_{2k-1}(t), \quad \hat{W}_k(t) = \hat{W}_{2k}(t), \quad k \geq 1. \]

Let \( U \) be a separable Hilbert space with a complete orthonormal basis \( \{e_k\}_{k \geq 1} \). Let
\[
\begin{cases}
G(u) = -(u \cdot \nabla)u + \frac{1}{2} \sum_{k=1}^{\infty} [(a_k A_k)^2 u + (b_k B_k)^2 u], & k \geq 1, \\
H(t, u)e_{3k-2} := a_k A_k u, & \quad H(t, u)e_{3k-1} := b_k B_k X, & \quad h(t, u)e_{3k} := h_k(t, u), & k \geq 1, \\
W(t) := \sum_{k=1}^{\infty} (\overline{W}_k(t)e_{3k-2} + \hat{W}_k(t)e_{3k-1} + \hat{W}_k(t)e_{3k}).
\end{cases}
\]

With the above notations, (4.8) reduces to
\[
du = [G(u) - F(u)] \, dt + H(t, u) \, dW, \quad u|_{t=0} = u_0, \quad t > 0.
\]

Let \( d \geq 2 \) and recall \( p_0 \) in (4.4). Let \( s > \frac{d}{2} + 1 + \max\{2p_0, 1\} \). According to Hypotheses \( \textbf{H}_3 \) and \( \textbf{H}_4 \), if \( u \in H^s \), then \( G(u) \in H^{(s-1) \vee (s-2p_0)} \) and \( H(t, u) \in L_2(U; H^{s-p_m}) \), while by Lemma 4.1, \( F(u) \in H^s \). To apply the theory for SDEs in Hilbert space, we need to mollify \( G(u) \) and \( H(t, u) \). To this end, we will use the mollifier \( J_n \) defined in (A.1) and construct the following regularization
\[
\begin{align*}
G_n(u) &:= -J_n[(J_n u \cdot \nabla)J_n u] + \frac{1}{2} \sum_{k=1}^{\infty} J_n^3(a_k A_k)^2 J_n u + \frac{1}{2} \sum_{k=1}^{\infty} J_n^3(b_k B_k)^2 J_n u, \\
H_n(t, u)e_{3k-2} &:= J_n(a_k A_k)J_n u, \quad H_n(t, u)e_{3k-1} := J_n(b_k B_k)J_n u, \quad H_n(t, u)e_{3k} := h_n(t, u), \quad k \geq 1.
\end{align*}
\]

We also need a cut-off function to split the expectation. Hence for any \( R > 1 \), we take a cut-off function \( \chi_R \in C^\infty([0, \infty); [0, 1]) \) such that
\[
\chi_R(y) = 1 \text{ for } |y| \leq R, \quad \text{and } \chi_R(y) = 0 \text{ for } y > 2R,
\]
and then consider
\[
du = \chi_R^2(u(t) - u_0)_{W^{1, \infty}}[G_n(u) - F(u)] \, dt + \chi_R^2(u(t) - u_0)_{W^{1, \infty}}H_n(t, u) \, dW, \quad u|_{t=0} = u_0.
\]

Keep in mind that \( p_0 \) is in (4.4) and the following

**Lemma 4.3.** Let \( d \geq 2 \) and \( s > \frac{d}{2} + 1 + \max\{2p_0, 1\} \). Let Hypotheses \( \textbf{H}_3 \) and \( \textbf{H}_4 \) be verified. For any \( R > 1 \), \( n \geq 1 \) and \( \mathcal{F}_0 \)-measurable \( H^s \)-valued random variable \( u_0 \), (4.13) has a unique global solution \( u_n = u_n^{(R)}(t, x) \in C([0, \infty); H^s) \) such that for some function \( V : [0, \infty) \times [0, \infty) \to (0, \infty) \) increasing in both variables,
\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n\|^2_{H^s} \mathcal{F}_0 \right] \leq V(T, 2R + \|u_0\|_{W^{1, \infty}})(1 + \|u_0\|^2_{H^s}), \quad T, R > 0.
\]

**Proof.** By Lemma A.1, it is easy to see that \( G_n : H^s \to H^s \) and \( H_n : [0, \infty) \times H^s \to L_2(U; H^s) \) is locally Lipschitz. Hence for any deterministic initial data (4.13) admits a unique solution, and the solution is continuous in \( H^s \) (See for instance [39, 57]). Combining this and the fact that \( \mathcal{F}_0 \) is independent of the equation, we see that for any \( \mathcal{F}_0 \)-measurable \( H^s \)-valued random variable \( u_0 \), (4.13) also admits a unique solution \( u_n = u_n(t) \), which is continuous in \( H^s \).

Now we verify (4.14). To begin with, we can infer from (4.11), Hypothesis \( \textbf{H}_4 \), Lemma A.1, Theorem 3.1 and 3.2 (with \( p_n \equiv D^* \), \( f = J_n u \)) that
\[
\sum_{k=1}^{\infty} \langle H_n(t, u_n)e_k, u_n \rangle_{H^s}^2
= \sum_{k=1}^{\infty} \left( \langle J_n(a_k A_k)J_n u_n, u_n \rangle_{H^s} + \langle J_n(b_k B_k)J_n u_n, u_n \rangle_{H^s} + \langle h_k(t, u_n), u_n \rangle_{H^s}^2 \right)
\leq (1 + K^2(t, \|u_n\|_{W^{1, \infty}}))(1 + \|u_n\|^2_{H^s}).
\]
Besides, it follows from (4.11) that
\[
2 \langle G_n(u_n), u_n \rangle_{H^s} + \|H_n(t, u_n)\|_{L^2(U_H)}^2
\]
\[
= -2 \langle J_n[(J_n u_n \cdot \nabla) J_n u_n], u_n \rangle_{H^s} + \sum_{k=1}^\infty \langle J_n^2(a_k A_k)^2 J_n u_n, u_n \rangle_{H^s} + \sum_{k=1}^\infty \langle J_n^3(b_k B_k)^2 J_n u_n, u_n \rangle_{H^s}
\]
\[
+ \sum_{k=1}^\infty \|J_n(a_k A_k) J_n u_n\|_{H^s}^2 + \sum_{k=1}^\infty \|J_n(b_k B_k) J_n u_n\|_{H^s}^2 + \sum_{k=1}^\infty \|h_k(t, u_n)\|_{H^s}^2
\]
\[
:= 6 I_i.
\]
On account of Hypothesis H4, Lemmas A.1 and A.4, it holds that
\[
|I_1| \leq 2 \langle |(J_n u_n \cdot \nabla) J_n u_n|, |J_n u_n| \rangle_{H^s} \lesssim \|u_n\|_{W^{1,\infty}} \|u_n\|_{H^s}^2, \quad |I_6| \leq K(t, \|u_n\|_{W^{1,\infty}}) (1 + \|u_n\|_{H^s}^2).
\]
It follows from Theorem 3.1 and 3.2 (with \(\mathcal{P}_n = D^2 J_n\) and \(f = J_n u_n\)) that
\[
|I_2 + I_4| + |I_3 + I_5| \lesssim \|J_n u\|_{H^s}^2 \leq \|u_n\|_{H^s}^2.
\]
By the above estimates and Itô’s formula, we obtain that for some function \(\tilde{V}: [0, \infty) \times [0, \infty) \to (0, \infty)\),
\[
d\|u_n(t)\|_{H^s}^2 - dM_n(t) = \chi_{\mathbb{R}}^2 \left(\|u_n(t) - u_0\|_{W^{1,\infty}}\right) \left\{ \sum_{i=1}^6 I_i \right\} dt \leq \tilde{V}(t, 2R + \|u_0\|_{W^{1,\infty}}) (1 + \|u_n(t)\|_{H^s}^2) dt,
\]
where
\[
dM_n(t) := 2 \chi_{\mathbb{R}}^2 \left(\|u_n(t) - u_0\|_{W^{1,\infty}}\right) \langle u_n(t), H(t, u_n) dW(t) \rangle_{H^s},
\]
satisfies
\[
d\langle M_n(t) \rangle \leq \tilde{V}(t, 2R + \|u_0\|_{W^{1,\infty}}) (1 + \|u_n(t)\|_{H^s}) dt.
\]
Define
\[
\tau_n := \lim_{N \to \infty} \tau_{n,N}, \quad \tau_{n,N} := \inf \left\{ t \geq 0 : \|u_n(t)\|_{H^s} \geq N \right\}, \quad n, N \geq 1.
\]
For any \(T > 0\), we use BDG’s inequality to find constants \(c_1, c_2 > 0\) such that for any \(t \in [0, T]\) and \(N \geq 1\),
\[
E \left[ \sup_{t' \in [0, T \wedge \tau_{n,N}]} \|u_n(t')\|_{H^s}^2 \bigg| F_0 \right] - \|u_0\|_{H^s}^2
\]
\[
\leq c_1 E \left[ \left( \int_0^{T \wedge \tau_{n,N}} \tilde{V}(t', 2R + \|u_0\|_{W^{1,\infty}}) \left( 1 + \|u_n(t')\|_{H^s}^4 \right) dt' \right)^{\frac{1}{2}} \bigg| F_0 \right]
\]
\[
+ c_2 E \left[ \int_0^{T \wedge \tau_{n,N}} \tilde{V}(t', 2R + \|u_0\|_{W^{1,\infty}}) \left( 1 + \|u_n(t')\|_{H^s}^4 \right) dt' \bigg| F_0 \right]
\]
\[
\leq \frac{1}{2} \left[ \sup_{t' \in [0, T \wedge \tau_{n,N}]} \|u_n(t')\|_{H^s}^2 \bigg| F_0 \right] + c_2 + c_2 \int_0^T \tilde{V}(t', 2R + \|u_0\|_{W^{1,\infty}}) E \left[ \sup_{t \in [0, t' \wedge \tau_{n,N}]} \|u_n(r)\|_{H^s} \bigg| F_0 \right] dt'.
\]
By Grönwall’s inequality, there exists a function \(V : [0, \infty) \times [0, \infty) \to (0, \infty)\) increasing in both variables such that
\[
E \left[ \sup_{t \in [0, T \wedge \tau_{n,N}]} \|u_n(t)\|_{H^s}^2 \bigg| F_0 \right] \leq V(T, 2R + \|u_0\|_{W^{1,\infty}}) (1 + \|u_0\|_{H^s}^2), \quad n, N \geq 1.
\]
This implies that for all \(n, N \geq 1\),
\[
\mathbb{P} \left( \tau_{n,N} < T \bigg| F_0 \right) \leq \mathbb{P} \left( \sup_{t \in [0, T \wedge \tau_{n,N}]} \|u_n(t)\|_{H^s} \geq N \bigg| F_0 \right) \leq \frac{V(T, 2R + \|u_0\|_{W^{1,\infty}}) (1 + \|u_0\|_{H^s}^2)}{N^2},
\]
18
so that \( \tau_n = \lim_{N \to \infty} \tau_{n,N} \) satisfies

\[
P(\tau_n < T | F_0) \leq \lim_{N \to \infty} P(\tau_{n,N} < T | F_0) = 0.
\]

Hence, \( P(\tau_n \geq T) = \mathbb{E}[P(\tau_n \geq T | F_0)] = 1 \) for all \( T > 0 \), which means \( P(\tau_n = \infty) = 1 \). Letting \( N \to \infty \) in (4.15) yields (4.14).

### 4.2 Solving the cut-off problem

In this section we will take limit in (4.13) to find a solution to the following cut-off problem

\[
du = \chi_R^2(\|u(t) - u_0\|_{W_s^1}) [G(u) - F(u)] \, dt + \chi_R^2(\|u(t) - u_0\|_{W_s^1}) H(t, u) \, dW, \quad u|_{t=0} = u_0, \quad t > 0,
\]

where \( \chi_R, F \) and \( (G, H) \) are given in (4.12), (1.3) and (4.9), respectively.

**Lemma 4.4.** Let \( u_n \) be the approximate solution as in Lemma 4.3. For any \( n, l \geq 1, \delta_0 \in \left( \frac{d}{2} + 1, s - \max \{2p_0, 1 \} \right) \) and \( T, N > 0 \), let

\[
\tau_{n,l,T}^N := T \wedge \inf \{ t \geq 0 : \|u_n(t)\|_{H^s} \vee \|u_l(t)\|_{H^s} \geq N \}.
\]

Then P-a.s.,

\[
\lim_{n \to \infty} \sup_{t \in [0, \tau_{n,l,T}^N]} \mathbb{E} \left[ \sup_{t \in [0, \tau_{n,l,T}^N]} \|u_n(t) - u_l(t)\|_{H^s}^2 | F_0 \right] = 0, \quad T, N > 0.
\]

**Proof.** Let \( v_{n,l} = u_n - u_l \) for \( n, l \geq 1 \). We have that

\[
dv_{n,l}(t) = \sum_{i=1}^4 A_{i}^{n,l}(t) \, dt + \sum_{i=1}^2 B_{i}^{n,l}(t) \, dW(t), \quad v_{n,l}(0) = 0,
\]

where

\[
A_1^{n,l}(t) := -\left[ \chi_R^2(\|u_n(t) - u_0\|_{W^1_s}) - \chi_R^2(\|u_l(t) - u_0\|_{W^1_s}) \right] F(t, u_n(t)),
\]

\[
A_2^{n,l}(t) := -\chi_R^2(\|u_l(t) - u_0\|_{W^1_s}) [F(t, u_n(t)) - F(t, u_l(t))],
\]

\[
A_3^{n,l}(t) := \left[ \chi_R^2(\|u_n(t) - u_0\|_{W^1_s}) - \chi_R^2(\|u_l(t) - u_0\|_{W^1_s}) \right] G_n(t, u_n(t)),
\]

\[
A_4^{n,l}(t) := \chi_R^2(\|u_l(t) - u_0\|_{W^1_s}) [G_n(t, u_n(t)) - G_l(t, u_l(t))],
\]

and

\[
B_1^{n,l}(t) := \left[ \chi_R(\|u_l(t) - u_0\|_{W^1_s}) - \chi_R(\|u_l(t) - u_0\|_{W^1_s}) \right] H_n(t, u_n(t)),
\]

\[
B_2^{n,l}(t) := \chi_R(\|u_l(t) - u_0\|_{W^1_s}) [H_n(t, u_n(t)) - H_l(t, u_l(t))].
\]

By the Itô formula, we obtain

\[
d\|v_{n,l}(t)\|_{H^s}^2 = 2 \sum_{i=1}^2 \langle v_{n,l}(t), B_{i}^{n,l}(t) dW(t) \rangle_{H^s} + \left\{ \sum_{i=1}^2 \left\| B_{i}^{n,l}(t) \right\|_{L^2(\Omega; H^s)}^2 + 2 \sum_{i=1}^4 \langle A_{i}^{n,l}(t), v_{n,l}(t) \rangle_{H^s} \right\} dt.
\]

**Claim:** There is a function \( Q : [0, \infty) \times [0, \infty) \to [0, \infty) \) increasing in both variables and a function \( \lambda : \mathbb{N} \times \mathbb{N} \to [0, \infty) \) with \( \lim_{n,l \to \infty} \lambda_{n,l} = 0 \) such that for all \( n, l \geq 1, \)

\[
\sum_{i=1}^2 \sum_{k=1}^2 \langle v_{n,l}(t), B_{i}^{n,l}(t) e_k \rangle_{H^s} \leq Q(t, N) \|v_{n,l}(t)\|_{H^s}^2 \left\{ \lambda_{n,l} + \|v_{n,l}(t)\|_{H^s}^2 \right\}, \quad t \in [0, \tau_{n,l,T}^N],
\]

\[
\sum_{i=1}^2 \left\| B_{i}^{n,l}(t) \right\|_{L^2(\Omega; H^s)}^2 + 2 \sum_{i=1}^4 \langle A_{i}^{n,l}(t), v_{n,l}(t) \rangle_{H^s} \leq Q(t, N) \left\{ \lambda_{n,l} + \|v_{n,l}(t)\|_{H^s}^2 \right\}, \quad t \in [0, \tau_{n,l,T}^N].
\]
If (4.19) and (4.20) hold true, then we use BDG’s inequality to (4.18) to find constants $a_1, a_2 > 0$ depending on $N$ and $T$ such that for all $n, l \geq 1$ and $t \in [0, N],$

$$E \left[ \sup_{t' \in [0, t \wedge \tau_{n,l}^{N,T}] \cap \mathcal{F}_t} \left\| v_{n,l}(t') \right\|_{H^{\delta_0}}^2 \right] \leq a_1 E \left[ \int_0^{t \wedge \tau_{n,l}^{N,T}} Q(t', N) \left\{ \lambda_{n,l} + \left\| v_{n,l}(t') \right\|_{H^{\delta_0}}^2 \right\} dt' \bigg| \mathcal{F}_0 \right]$$

$$\leq a_1 E \left[ \left( \int_0^{t \wedge \tau_{n,l}^{N,T}} Q(t', N) \left\| v_{n,l}(t') \right\|_{H^{\delta_0}} \left\{ \lambda_{n,l} + \left\| v_{n,l}(t') \right\|_{H^{\delta_0}}^2 \right\} dt' \right) \frac{1}{2} \bigg| \mathcal{F}_0 \right] + a_2 \lambda_{n,l}$$

$$\leq \frac{1}{2} E \left[ \sup_{t' \in [0, t \wedge \tau_{n,l}^{N,T}] \cap \mathcal{F}_t} \left\| v_{n,l}(t') \right\|_{H^{\delta_0}}^2 \bigg| \mathcal{F}_0 \right] + a_2 \lambda_{n,l}$$

(4.21)

$$+ a_2 \int_0^t Q(t', N) E \left[ \sup_{r \in [0, t \wedge \tau_{n,l}^{N,T}] \cap \mathcal{F}_r} \left\| v_{n,l}^{(R)}(r) \right\|_{H^{\delta_0}}^2 \bigg| \mathcal{F}_0 \right] dt'. $$

By Grönwall’s inequality and noting $\lambda_{n,l} \to 0$ as $n, l \to \infty$, we prove (4.17). Therefore now it suffices to prove (4.19) and (4.20).

We only prove (4.20) since (4.19) can be verified similarly. We note that $\chi_R(\cdot)$ is bounded and Lipschitz, $F(\cdot)$ is locally Lipschitz (cf. Lemma 4.1) and $H_{\delta_0} \to W^{1,\infty}$. Then we use Hypothesis $H_3$, (4.11) and Lemma A.1 to obtain that for all $n, l \geq 1,$

$$\left\| B_n(t, t) \right\|_{L^2(\mathbb{R}; H^{\delta_0})}^2 + 2 \sum_{i=1}^3 \left\langle B_n(t, t) \right\|_{H^{\delta_0}} \leq Q(t, N) \left\| v_{n,l}(t) \right\|_{H^{\delta_0}}^2, \quad t \in [0, \tau_{n,l}^{N,T}]$$

for some increasing function $Q : [0, \infty) \times [0, \infty) \to (0, \infty)$ increasing in both variables. Once again, since $\chi_{R}(\cdot) \leq 1$, we only need to prove that for all $n, l \geq 1$ and $t \in [0, \tau_{n,l}^{N,T}],$

(4.22)

$$2 \left\langle G_n(u_n) - G_t(u_l), v_{n,l} \right\|_{H^{\delta_0}} + \left\| H_n(t, u_n) - H_t(t, u_l) \right\|_{L^2(\mathbb{R}; H^{\delta_0})}^2 \leq Q(t, N) \left\{ \lambda_{n,l} + \left\| v_{n,l}(t) \right\|_{H^{\delta_0}}^2 \right\}. $$

To this end, we find

$$2 \left\langle G_n(u_n) - G_t(u_l), v_{n,l} \right\|_{H^{\delta_0}} + \left\| H_n(t, u_n) - H_t(t, u_l) \right\|_{L^2(\mathbb{R}; H^{\delta_0})}^2 \leq \Psi_1 + \sum_{k=1}^{\infty} \sum_{i=2}^{\infty} \Psi_{i,k},$$

where

$$\Psi_1 = \Psi_{n,l}^1 \cdot 2 \left\{ \left\langle J_n([J_n u_n \cdot \nabla]) J_n u_n - J_l([J_l u_l \cdot \nabla]) J_l u_l, u_n - u_l \right\|_{H^{\delta_0}} \right.$$ \n
$$\Psi_{2,k} = \Psi_{n,l}^2 \cdot \left\{ \left\langle J_n^3 (a_k A_k) J_n^2 u_n - J_l^3 (a_k A_k) J_l^2 u_l, u_n - u_l \right\|_{H^{\delta_0}} \right.$$ \n
$$\Psi_{3,k} = \Psi_{n,l}^3 \cdot \left\{ \left\langle J_n^3 (b_k B_k) J_n^2 u_n - J_l^3 (b_k B_k) J_l^2 u_l, u_n - u_l \right\|_{H^{\delta_0}} \right.$$ \n
$$\Psi_{4,k} = \Psi_{n,l}^4 \cdot \left\{ \left\langle [h_n(t, u_n)]_{e_k} - [h_l(t, u_l)]_{e_k}, u_n - u_l \right\|_{H^{\delta_0}} \right.$$ \n
$$\Psi_{5,k} = \Psi_{n,l}^5 \cdot \left\{ \left\langle [h_n(t, u_n)]_{e_k} - [h_l(t, u_l)]_{e_k}, u_n - u_l \right\|_{H^{\delta_0}} \right.$$ \n
$$\Psi_{6,k} = \Psi_{n,l}^6 \cdot \left\{ \left[ h_n(t, u_n) - h_l(t, u_l) \right] \left\|_{H^{\delta_0}} \right.$$ \n
For $\Psi_1$, one can show that for $\epsilon \in (0, s - \delta_0),$

$$\left| \Psi_1 \right| \leq \left( \left\| u_n \right\|_{H^r} + \left\| u_l \right\|_{H^r} \right) \left( \left\| v_{n,l} \right\|_{H^{\delta_0}} + (l \wedge n)^{-2(s - 1 - \delta_0 - \epsilon)} \right).$$

The proof for this estimate is similar to [50, Lemma 3.1], and here we omit the details to save space. It suffices to estimate the other two terms. To control $\sum_{k=1}^{\infty} \left\{ \Psi_{3,k} + \Psi_{5,k} \right\},$ we find

$$\Psi_{3,k} = \sum_{j=1}^{3} \Psi_{3,k,j}, \quad \Psi_{5,k} = \sum_{i,j=1}^{3} \left\langle \Psi_{5,k,i} \right\|_{H^{\delta_0}}.$$
where
\[
\begin{align*}
\Psi_{3,1} &= \langle (J_n^2 - J_l^2)(b_k B_k)^2 J_n u_l, v_{n,l} \rangle_{H^{\delta_0}}, \\
\Psi_{3,2} &= \langle J_n^2 (b_k B_k)^2 (J_n - J_l)u_l, v_{n,l} \rangle_{H^{\delta_0}}, \\
\Psi_{3,3} &= \langle J_l^2 (b_k B_k)^2 J_l v_{n,l}, v_{n,l} \rangle_{H^{\delta_0}},
\end{align*}
\]
\[
\Psi_{5,1} := (J_n - J_l)(b_k B_k) J_n u_n, \quad \Psi_{5,2} := J_l (b_k B_k)(J_n - J_l) u_n, \quad \Psi_{5,3} := (b_k B_k) J_l v_{n,l}.
\]

By Hypothesis \(H_3\) and Lemma A.1, we have for any \(\epsilon \in (0, s - 2p_0 - \delta_0)\),
\[
\sum_{k=1}^{\infty} \sum_{i=1,2} \langle \Psi_{5,k,i}, \Psi_{5,k,3} \rangle_{H^{\delta_0}} \lesssim (l \wedge n)^{-2(s_0 - 2p_0 - \delta_0 - \epsilon)} \| u_n \|_{H^s} \| v_{n,l} \|_{H^{\delta_0}},
\]
\[
\sum_{k=1}^{\infty} \sum_{i,j=1,2} \langle \Psi_{5,k,i}, \Psi_{5,k,j} \rangle_{H^{\delta_0}} \lesssim (l \wedge n)^{-2(s_0 - 2p_0 - \delta_0 - \epsilon)} \| u_n \|_{H^s}^2.
\]

Then we apply Proposition 3.1 (with \(s = \delta_0\), \(P_n = D^{\delta_0} J_l \) and \( f = J_l v_{n,l} \)) to find
\[
\sum_{k=1}^{\infty} \{ \Psi_{3,k,3} + \langle \Psi_{5,k,3}, \Psi_{5,k,3} \rangle_{H^{\delta_0}} \} \lesssim \| v_{n,l} \|_{H^{\delta_0}}^2.
\]

Hence we find an increasing function \( Q : [0, \infty) \times [0, \infty) \to (0, \infty) \) increasing in both variables such that
\[
\sum_{k=1}^{\infty} \{ \Psi_{3,k} + \Psi_{5,k} \} \lesssim Q(t, N) \{(l \wedge n)^{-2(s_0 - 2p_0 - \delta_0 - \epsilon)} + \| u_n \|_{H^{\delta_0}}^2 \}, \quad n, l \geq 1, \ t \in [0, T].
\]

Similarly, the same estimate holds for \( \sum_{k=1}^{\infty} \{ \Psi_{4,k} + \Psi_{6,k} \} \). Obviously, the desired upper bound of \( \Psi_{6,k} \) follows from Hypothesis \(H_4\). In conclusion, (4.22) holds true.

**Lemma 4.5.** Let \( u_n \) be the approximate solution as in Lemma 4.3. There exists a \( \mathcal{F}_t \)-progressive measurable \( H^s \)-valued process \( u(t) = (u(R))(t) \) such that, up to a subsequence, \( P \) a.s.,
\[
(4.23) \quad \lim_{n \to \infty} u_n \to u \text{ in } C([0, \infty); H^{\delta_0}).
\]

**Proof.** For any \( T > 0, N \geq 1 \) and \( \epsilon > 0 \), by using (4.14) in Lemma 4.3 and Chebyshev’s inequality, we have
\[
P(\tau_n^{n,l,T} < T | \mathcal{F}_0)
\leq P \left( \sup_{t \in [0, T]} \| u_n(t) \|_{H^s} \geq N \right) + P \left( \sup_{t \in [0, T]} \| u_l(t) \|_{H^s} \geq N \right)
\leq \frac{2V(T, 2R + \| u_0 \|_{W^{1,\infty}})(1 + \| u_0 \|_{H^s}^2)}{N^2}.
\]

Since \( \tau_n^{n,l,T} \leq T \) \( P \) a.s., for any \( T > 0, N \geq 1, n, l \geq 1 \), we have
\[
P \left( \sup_{t \in [0, T]} \| u_n(t) - u_l(t) \|_{H^{\delta_0}} > \epsilon \right)
\leq P \left( \tau_n^{n,l,T} < T \right) + P \left( \sup_{t \in [0, \tau_n^{n,l,T}]} \| u_n(t) - u_l(t) \|_{H^{\delta_0}} > \epsilon \right)
\leq \frac{2V(T, 2R + \| u_0 \|_{W^{1,\infty}})(1 + \| u_0 \|_{H^s}^2)}{N^2} + P \left( \sup_{t \in [0, \tau_n^{n,l,T}]} \| u_n(t) - u_l(t) \|_{H^{\delta_0}} > \epsilon \right).
\]

On account of Lemma 4.4, we first let \( n, l \to \infty \) and then \( N \to \infty \) to find
\[
\lim_{n,l \to \infty} P \left( \sup_{t \in [0, T]} \| u_n(t) - u_l(t) \|_{H^{\delta_0}} > \epsilon \right) = 0, \quad \epsilon, T > 0.
\]
According to the reverse Fatou lemma, this gives rise to

\[
\limsup_{n,l \to \infty} \mathbb{P} \left( \sup_{t \in [0,T]} \| u_n(t) - u_l(t) \|_{H^{s_0}} > \varepsilon \right) = \limsup_{n,l \to \infty} \mathbb{E} \left( \mathbb{P} \left( \sup_{t \in [0,T]} \| u_n(t) - u_l(t) \|_{H^{s_0}} > \varepsilon \bigg| \mathcal{F}_0 \right) \right) \leq \mathbb{E} \left[ \limsup_{n,l \to \infty} \mathbb{P} \left( \sup_{t \in [0,T]} \| u_n(t) - u_l(t) \|_{H^{s_0}} > \varepsilon \bigg| \mathcal{F}_0 \right) \right] = 0, \ \varepsilon, T > 0.
\]

Therefore, up to a subsequence, (4.23) holds for certain progressively measurable process \( u \) on \( H^s \).

**Lemma 4.6.** Let \( d \geq 2 \) and \( s > \frac{d}{4} + 1 + \max \{ 2p_0, 1 \} \). Let Hypotheses \( H_3 \) and \( H_4 \) hold. For any \( R > 1 \), \( n \geq 1 \) and \( \mathcal{F}_0 \)-measurable \( H^s \)-valued random variable, (4.16) has a unique global solution \( u = u^{(n)} \) such that for any \( T > 0 \),

\[
P \left( u \in C([0,T]; H^s) \right) = 1,
\]

and

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| u(t) \|_{H^s}^2 \right] \leq V(T, 2R + \| u_0 \|_{W^{1,\infty}})(1 + \| u_0 \|_{H^s}^2),
\]

where \( V \) is given in Lemma 4.3.

**Proof.** For any \( R > 1 \), by Lemma 4.5, we can take limit to see that the limit process \( u \) obtained in Lemma 4.5 is a solution to (4.16). Uniqueness of \( u \) can be obtained in the same way as we estimate (4.21). Besides, (4.25) comes from (4.23) and (4.14).

Now we prove (4.24). By (4.23), we know that \( u \in C([0,T]; H^{b_0}) \), which, together with the fact that \( H^s \hookrightarrow H^{b_0} \) is dense, means that \( u \) is weakly continuous in \( H^s \). In order to prove (4.24), we only need to prove that \( [0,T] \ni t \mapsto \| u(t) \|_{H^s} \) is continuous almost surely. Let

\[
\tau_N := N \land \inf \{ t \geq 0 : \| u(t) \|_{H^s} \geq N \}, \quad N \geq 1.
\]

Note that (4.25) implies \( \lim_{N \to \infty} \tau_N = \infty \) \( \mathbb{P} \)-a.s. It suffices to prove

\[
\| u(\cdot) \|_{H^s} \in C([0, \tau_N \land T]; \mathbb{R}), \quad N \geq 1.
\]

However, since we only know \( u \in H^s \) (by (4.25)), one cannot use Itô’s formula to \( \| u \|_{H^s}^2 \) since \( (H(t,u)e_k, u)_{H^s} \) and \( (G(u) - F(u), u)_{H^s} + \| H(t,u) \|^2_{L^2(U; H^s)} \) are not well-defined. Then we apply Lemma 4.2, (4.9), Hypothesis \( H_4 \), Lemma A.1, Theorem 3.1 and 3.2 (with \( \mathcal{P}_n = \mathcal{D}^s J_n, f = u \)) to find

\[
\sum_{k=1}^{\infty} \langle J_n H(t,u) e_k, J_n u \rangle_{H^s}^2 = \sum_{k=1}^{\infty} \left( \langle J_n(a_k A_k)u, J_n u \rangle_{H^s} + \langle J_n(b_k B_k)u, J_n u \rangle_{H^s} + \langle J_n h_k(t,u), J_n u \rangle_{H^s}^2 \right) \lesssim (1 + K^2(t, \| u \|_{W^{1,\infty}}))(1 + \| u \|_{H^s}^4),
\]

and

\[
\left| 2\langle J_n [G(u) - F(u)], J_n u \rangle_{H^s} + \| J_n H(t,u) \|^2_{L^2(U; H^s)} \right| \leq 2A \| u \|_{H^s}^2 \| u \|_{W^{1,\infty}} + \sum_{k=1}^{\infty} \langle J_n(a_k A_k)^2 u, J_n u \rangle_{H^s} + \sum_{k=1}^{\infty} \langle J_n(a_k A_k) u \rangle_{H^s}^2 + \sum_{k=1}^{\infty} \| J_n(b_k B_k) u \|_{H^s}^2 + \sum_{k=1}^{\infty} \| J_n h_k(t,u) \|_{H^s}^2 \lesssim (1 + \| u \|_{W^{1,\infty}} + K^2(t, \| u \|_{W^{1,\infty}}))(1 + \| u \|_{H^s}^4).
\]
Therefore, by Itô’s formula to \( \|J_n u\|_{H^s}^2 \) for (4.16), for any \( n, N \geq 1 \) we find a martingale \( M^{(n)}_t \) such that for some constant \( Q_N > 0 \) such that
\[
(4.28) \quad -Q_N dt \leq d\|J_n u(t)\|_{H^s}^2 + dM^{(n)}(t) \leq Q_N dt, \quad d\langle M^{(n)} \rangle(t) \leq Q_N dt, \quad t \in [0, \tau_N], \quad n \geq 1.
\]
That is to say, for some constant \( C_N > 0, \)
\[
\mathbb{E}\left[ \|J_n u(t) \wedge \tau_N\|_{H^s}^2 - \|J_n u(t') \wedge \tau_N\|_{H^s}^2 \right] \leq C_N|t - t'|^2, \quad t, t' \geq 0, \quad n \geq 1.
\]
By Lemma A.1 and Fatou’s lemma with \( n \to \infty, \) we derive
\[
\mathbb{E}\left[ \|u(t \wedge \tau_N)\|_{H^s}^2 - \|u(t') \wedge \tau_N\|_{H^s}^2 \right] \leq C_N|t - t'|^2, \quad t, s \geq 0.
\]
From this and Kolmogorov’s continuity theorem, we obtain (4.27).

### 4.3 Finish the proof for Theorem 4.1

Now we are in the position to prove Theorem 4.1.

**Proof for Theorem 4.1.** (I) Let \( u = u^{(R)} \) be the solution to (4.16) as in Lemma 4.6. Now we remove the cut-off. To this end, we let
\[
\tau^{(R)} := \inf \left\{ t \geq 0 : \|u^{(R)}(t) - u_0\|_{W^{1, \infty}} \geq R \right\}.
\]
By the continuity of \( u^{(R)}(t) \) in \( H^b \) and \( H^b \hookrightarrow W^{1, \infty}, \) we have \( \mathbb{P}(\tau^{(R)} > 0) = 1 \) for any \( R > 0. \) Since \( \chi_R^2(\|u^{(R)}(t) - u_0\|_{W^{1, \infty}}) = 1 \) for \( t \leq \tau, \) \( u^{(R)}, \tau^{(R)} \) is a local solution to (2.8) (or equivalently, (4.8)). The uniqueness of \( u^{(R)} \) implies
\[
u^{(R)}(t) = u^{(R+1)}(t), \quad t \leq \tau^{(R)}, \quad R \geq 1 \quad \mathbb{P}\text{-a.s.}
\]
Let \( \tau^* := \lim_{R \to \infty} \tau^{(R)}, \) \( \tau^{(0)} = 0 \) and we define
\[
u(t) := \sum_{R=1}^{\infty} 1_{[\tau^{(R-1)}, \tau^{(R)}]}(t) u^{(R)}(t), \quad t \in [0, \tau^*).
\]
Then one can conclude that \( (u, \tau^*) \) is a local solution to (2.8). Again, by the uniqueness of \( u^{(R)} \) and (4.24), \( \mathbb{P}(u \in C([0, \tau^*); H^s)) = 1 \). Moreover, the construction of \( \tau^* \) and (4.25) immediately tell us
\[
limit_{t \to \tau^*} \|u(t)\|_{W^{1, \infty}} = \limsup_{t \to \tau^*} \|u(t)\|_{H^s} = \infty \quad \text{on} \quad \{\tau^* < \infty\} \quad \mathbb{P}\text{-a.s.},
\]
which gives (4.5).

(II). Recalling (4.9), and then using Itô’s formula to \( \log(e + \|u(t)\|_{H^s}^2) \) with noting (4.1) in Lemma 4.2 and (4.9), we arrive at
\[
d\log(e + \|u(t)\|_{H^s}^2) = \frac{1}{(e + \|u(t)\|_{H^s}^2)^2} \left\{ \left( 2\langle G(u(t)) + F(u(t)), u(t) \rangle_{H^s} + \|H(t, u(t))\|_{L^2(U; H^s)}^2 \right) dt \\
- \frac{2}{(e + \|u(t)\|_{H^s}^2)^2} \sum_{k=1}^{\infty} \langle H(t, u(t))e_k, u(t) \rangle_{H^s}^2 dt + dM_t \\
\leq \frac{1}{(e + \|u(t)\|_{H^s}^2)^2} \left\{ \left( 2\lambda \|u(t)\|_{H^s}^2 \|u(t)\|_{W^{1, \infty}} + \sum_{k=1}^{\infty} \left| \langle a_k A_k^2 u(t), u(t) \rangle_{H^s} + \|a_k A_k u(t)\|_{H^s}^2 \right| \\
+ \sum_{k=1}^{\infty} \left| \langle b_k B_k^2 u(t), u(t) \rangle_{H^s} + \|b_k B_k u(t)\|_{H^s}^2 \right| + \sum_{k=1}^{\infty} \|h_k(t, u(t))\|_{H^s}^2 \right) dt \\
- \frac{2}{(e + \|u(t)\|_{H^s}^2)^2} \sum_{k=1}^{\infty} \langle h_k(t, u(t)), u(t) \rangle_{H^s}^2 dt + dM_t, \quad t \in [0, \tau^*),
\]
(4.31)
where $M_t$ is a martingale up to $\tau_N$ defined in (4.26) for any $N \geq 1$. According to (4.6) and (4.7), one can find a bounded function $Q : [0; \infty) \to (0, \infty)$ such that

$$
\frac{1}{e + \|u(t)\|_{H^N}} \left\{ 2 A \|u\|_{H^N}^2 \|u\|_{W^{1, \infty}} + C \|u(t)\|_{H^N}^2 + \sum_{k=1}^{\infty} \|h_k(t, u(t))\|_{H^N}^2 \frac{2}{(e + \|u(t)\|_{H^N})^2} \sum_{k=1}^{\infty} \langle h_k(t, u(t)), u(t) \rangle_{H^N} \right\} \leq Q(t), \quad t \in [0, \tau^*).
$$

Using this, Theorems 3.1 and 3.2 in (4.31) yields

$$
d \log(e + \|u(t)\|_{H^N}) \leq \frac{1}{e + \|u(t)\|_{H^N}} \left\{ 2 A \|u\|_{H^N}^2 \|u\|_{W^{1, \infty}} + C \|u(t)\|_{H^N}^2 + \sum_{k=1}^{\infty} \|h_k(t, u(t))\|_{H^N}^2 \right\} dt - \frac{2}{(e + \|u(t)\|_{H^N})^2} \sum_{k=1}^{\infty} \langle h_k(t, u(t)), u(t) \rangle_{H^N}^2 dt + dM_t
$$

$$
\leq Q(t) dt + dM_t, \quad t \in [0, \tau^*),
$$

which means that for some function $V : [0, \infty) \times [0, \infty) \to (0, \infty)$ increasing in both variables,

$$
E \left[ \log(e + \|u(t \land \tau_N)\|_{H^N}^2) \big| F_0 \right] \leq V(t, \|u_0\|_{H^s}), \quad t \geq 0, \quad N \geq 1.
$$

Consequently, by the continuity of $u$ in $H^s$ (hence also in $H^N$), we derive

$$
P(\tau^* < t | F_0) \leq P(\tau_N < t | F_0) \leq \frac{E \left[ \log(e + \|u(t \land \tau_N)\|_{H^N}^2) \big| F_0 \right]}{\log(e + N^2)} \leq \frac{V(t, \|u_0\|_{H^s})}{\log(e + N^2)}, \quad N \geq 1, \quad t > 0.
$$

Letting $N \to \infty$ and then $t \to \infty$ we see that $P(\tau^* < \infty | F_0) = 0$ and hence, $P(\tau^* < \infty) = 0$. \hfill \Box

5 Noise effect on the dependence on initial data

In this section, we consider the problem (1.8) on $\mathbb{T}^d$. For simplicity, we fix a separable Hilbert space $U$ with the complete orthonormal basis $\{e_k\}_{k \geq 1}$. Then we reformulate (1.8) on $\mathbb{T}^d$ as

$$
\begin{cases}
\mathrm{d} u + [(u \cdot \nabla) u + F(u)] \, \mathrm{d} t = B(t, u) \, \mathrm{d} W(t), \quad t > 0, \quad u|_{t=0} = u_0, \quad x \in \mathbb{T}^d, \\
\mathcal{W}(t) := \sum_{k=1}^{\infty} W_k e_k, \\
B(t, u) e_k := h_k(t, u).
\end{cases}
$$

We assume that $h_k(t, \cdot)$ is controlled by $F$ in the following sense:

**Hypothesis $H_5$.** For all $k$, $h_k : [0, \infty) \times H^s \ni (t, u) \mapsto h_k(t, u) \in H^s$ is continuous for $s > \frac{d}{2}$, and

$$
\sum_{k=1}^{\infty} \|h_k(t, u)\|_{H^s}^2 \leq \|F(u)\|_{H^s}^2, \quad \sum_{k=1}^{\infty} \|h_k(t, u) - h_k(t, v)\|_{H^s}^2 \leq \|F(u) - F(v)\|_{H^s}^2,
$$

where $F$ is defined in (1.3).

With the above notations at hand, Hypothesis $H_5$ is equivalent to

**Hypothesis $H_5$.** $B : (t, u) \mapsto B(t, u) \in \mathcal{L}_2(U; H^s)$ is continuous for $s > \frac{d}{2}$ and

$$
\|B(t, u)\|_{\mathcal{L}_2(U; H^s)} \leq \|F(u)\|_{H^s}, \quad \|B(t, u) - B(t, v)\|_{\mathcal{L}_2(U; H^s)} \leq \|F(u) - F(v)\|_{H^s}, \quad s > \frac{d}{2}.
$$
For (5.1), we have the following

**Proposition 5.1.** Let \( s > \frac{d}{2} + 1 \). Let Hypothesis \( H_5' \) (equivalently, Hypothesis \( H_5 \)) hold. If \( u_0 \) is an \( H^s \)-valued \( \mathcal{F}_0 \)-measurable random variable with \( \mathbb{E}[\|u_0\|_{H^s}^2] < \infty \), then there is a unique maximal solution \((u, \tau^*)\) to (5.1) in the sense of Definitions 2.2, and \((u, \tau^*)\) satisfies (4.5).

**Proof.** Since Hypothesis \( H_5 \) implies Hypothesis \( H_4 \), existence, uniqueness and the blow-up criterion (4.5) in \( H^s \) with \( s > \frac{d}{2} + 2 \) come from Theorem 4.1. The extension from \( s > \frac{d}{2} + 3 \) to \( s > \frac{d}{2} + 1 \) can be done, as in [18, 37], by mollifying initial data and then passing to the limit, as in the same way. Here we omit the details to avoid redundancy. \( \square \)

For the noise effect on the solution map \( u_0 \mapsto (u, \tau) \), we consider (1.8) and we have

**Theorem 5.1.** Let \( s > d/2 + 1 \) with \( d \geq 2 \). Let Hypothesis \( H_5 \) be satisfied. Then there is at least one of the following properties holding true for the problem (5.1):

(i) For any \( R \gg 1 \), the \( R \)-exiting time is not strongly stable at the zero solution in the sense of Definition 2.3.

(ii) The solution map \( u_0 \mapsto u \) defined by (1.8) is not uniformly continuous, as a map from \( L^p(\Omega; H^s) \) \( (p \in [1, \infty]) \) into \( L^1(\Omega; C([0, T]; H^s)) \) for any \( T > 0 \). More precisely, there exist two sequences of solutions \( u_{1,n}(t) \) and \( u_{2,n}(t) \), and two sequences of stopping times \( \tau_{1,n} \) and \( \tau_{2,n} \), such that

\[
\begin{align*}
&\text{(a)} \quad \mathbb{P}\{\tau_{i,n} > 0\} = 1 \text{ for each } n > 1 \text{ and } i = 1, 2. \quad \text{Besides,} \\
&\quad \lim_{n \to \infty} \tau_{1,n} = \lim_{n \to \infty} \tau_{2,n} = \infty \quad \mathbb{P}\text{-a.s.} \\
&\text{(b)} \quad \text{For } i = 1, 2, \; u_{i,n} \in C([0, \tau_{i,n}]; H^s) \quad \mathbb{P}\text{-a.s., and} \\
&\quad \left\| \sup_{t \in [0, \tau_{1,n}]} \|u_{1,n}(t)\|_{H^s} \right\|_{L^p(\Omega)} + \left\| \sup_{t \in [0, \tau_{2,n}]} \|u_{2,n}(t)\|_{H^s} \right\|_{L^p(\Omega)} \lesssim 1, \quad p \in [1, \infty]. \\
&\text{(c)} \quad \text{At } t = 0, \\
&\quad \lim_{n \to \infty} \|u_{1,n}(0) - u_{2,n}(0)\|_{L^p(\Omega; H^s)} = 0, \quad p \in [1, \infty]. \\
&\text{(d)} \quad \text{For any } T > 0, \quad \text{we have} \\
&\quad \liminf_{n \to \infty} \mathbb{E} \sup_{t \in [0, T \wedge \tau_{1,n} \wedge \tau_{2,n}]} \|u_{1,n}(t) - u_{2,n}(t)\|_{H^s} \gtrsim \sup_{t \in [0, T]} |\sin 2\pi t|.
\end{align*}
\]

**Remark 5.1.** We give the following remarks concerning Theorem 5.1.

(1) It is worthwhile noting that in deterministic cases, the issue of the optimal dependence of solutions (for example, the solution map is continuous but not uniformly continuous) to various nonlinear dispersive and integrable equations has been the subject of many papers. One of the first results of this type dates back at least as far as to Kato [28], where Kato proved that the solution map \( H^s(\mathbb{T}) \ni u_0 \mapsto u \) \((s > 3/2)\) given by the inviscid Burgers equation is not Hölder continuous regardless of the Hölder exponent. Since then different techniques have been successfully applied to various problems. Particularly, for the incompressible Euler equation, we refer to [22, 46], and for CH type equations, we refer to [20, 21, 47, 48, 51] and the references therein.

(2) To prove Theorem 5.1, we assume that for some \( R_0 \gg 1 \), the \( R_0 \)-exiting time of the zero solution is strongly stable. Then we will construct an example to show that the solution map \( u_0 \mapsto u \) defined by (1.8) is not uniformly continuous. This example involves the construction (for each \( s > d/2 + 1 \)) of two sequences of solutions which are converging at time zero but remain far apart at any later time. Actually, we will first construct two sequences of approximation solutions \( u^{i,n}(l \in \{-1, 1\}) \) such that the actual solutions \( u_{i,n}(l \in \{-1, 1\}) \) starting from \( u_{i,n}(0) = u^{i,n}(0) \) satisfy that as \( n \to \infty \),

\[
\lim_{n \to \infty} \mathbb{E} \sup_{[0, \tau_{i,n}]} \|u_{i,n} - u^{i,n}\|_{H^s}^2 = 0,
\]

\[
\text{(5.4)}
\]
where \( u_{l,n} \) exists at least on \([0, \tau_{l,n}]\). Due to the lack of life span estimate in stochastic setting, in order to obtain (5.4), we first connect the property \( \inf_{n} \tau_{l,n} > 0 \) with the stability property of the exiting time of the zero solution. In deterministic case, we have uniform lower bounds for the existence times of a sequence of solutions (see (4.7)–(4.8) in [48] and (3.8)–(3.9) in [51] for example). If (5.4) holds true, then we can estimate the approximation solutions instead of the actual solutions and obtain \( (d) \) by showing that the error in \( H^{2s-\sigma} \) behaves like \( n^{s-\sigma} \), but the error in \( H^s \) is \( O(1/n^{r_s}) \), where \( d/2 < \sigma < s - 1 \) and \( -r_s + s - \sigma < 0 \). These two estimates and interpolation give (5.4). Theorem 5.1 is proved for \( d \geq 2 \). However, the proof holds true also for \( d = 1 \), namely the stochastic CH equation case (see Remark 5.2).

(3) Theorem 5.1 implies that for the issue of the dependence on initial data, we cannot expect the multiplicative noise (in Itô sense) to improve the continuity of the exiting time of the zero solution, and simultaneously improve the continuity of the dependence on initial data. Formally speaking, the “regularization by (Itô sense) noise” actually preserves the hyperbolic structure of the equations. As for the noise in the sense of Stratonovich, whether it can improve the dependence on initial data is our future work.

Now we proceed to prove Theorem 5.1. We assume that for some \( R_0 \gg 1 \), the \( R_0 \)-exiting time is strongly stable at the zero solution. Then we will show that the solution map \( u_0 \mapsto u \) defined by (1.8) is not uniformly continuous. We will firstly assume that the dimension \( d \geq 2 \) is even.

5.1 Estimates on the errors

Let \( l \in \{-1, 1\} \). Define divergence-free vector field as

\[
(5.5) \quad u^{l,n} = (ln^{-1} + n^{-s} \cos \theta_1, ln^{-1} + n^{-s} \cos \theta_2, \ldots, ln^{-1} + n^{-s} \cos \theta_d),
\]

where \( \theta_i = 2\pi(nx_{d+1-i} - lt) \) with \( 1 \leq i \leq d \) and \( n \geq 1 \). Substituting \( u^{l,n} \) into (1.8), we see that the error \( \mathcal{E}^{l,n}(t) \) can be defined as

\[
(5.6) \quad \mathcal{E}^{l,n}(t) = u^{l,n}(t) - u^{l,n}(0) + \int_0^t [(u^{l,n} \cdot \nabla)u^{l,n} + F(u^{l,n})] \, dt' - \int_0^t B(t', u^{l,n}) \, dW.
\]

Now we analyze the error as follows.

**Lemma 5.1.** Let \( d \geq 2 \) be even and \( s > 1 + \frac{d}{2} \geq 2 \). For \( \sigma \in \left( \frac{d}{2}, \min \{s - 1, \frac{d}{2} + 1\} \right) \), we have that for any \( T > 0 \) and \( n \gg 1 \),

\[
(5.7) \quad \mathbb{E} \sup_{t \in [0,T]} \|\mathcal{E}^{l,n}(t)\|_{H^{\sigma}} \leq Cn^{-2r_s}, \quad C = C(T),
\]

where

\[
r_s = \begin{cases} 
2s - \sigma - 1 & \text{if } 1 + \frac{d}{2} < s \leq 3, \\
\sigma - 2 & \text{if } s > 3.
\end{cases}
\]

**Proof.** Direct computation shows that

\[
(u^{l,n} \cdot \nabla)u^{l,n} = \left( -2\pi ln^{-s} \sin \theta_i - 2\pi n^{-s+1} \sin \theta_i \cos \theta_{d+1-i} \right)_{1 \leq i \leq d},
\]

which means that

\[
u^{l,n}(t) - u^{l,n}(0) + \int_0^t (u^{l,n} \cdot \nabla)u^{l,n} \, dt' = 2\pi \int_0^t \left( -n^{-s+1} \sin \theta_i \cos \theta_{d+1-i} \right)_{1 \leq i \leq d} \, dt'.
\]

Then we have

\[
(5.8) \quad \mathcal{E}^{l,n}(t) + \int_0^t \left[ (2\pi n^{-s+1} \sin \theta_i \cos \theta_{d+1-i})_{1 \leq i \leq d} - F(u^{l,n}) \right] \, dt' + \int_0^t B(t, u^{l,n}) \, dW = 0.
\]

We note that by Lemma A.5,

\[
(5.9) \quad \left\| ( -2\pi n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i} )_{1 \leq i \leq d} \right\|_{H^{\sigma}} \leq C \sum_{i=1}^d \left\| n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i} \right\|_{H^s} \lesssim n^{-2s+1+\sigma} \lesssim n^{-r_s}.
\]
For \( F(\cdot) = (I - \Delta)^{-1} \text{div} F_1(u) + (I - \Delta)^{-1} F_2(u) \) given by (1.3), some calculations reveal that \( F_1(u^{l,n}) \) is a diagonal matrix such that

\[
F_1(u^{l,n}) = 4\pi^2 n^{-2s+2} \times \text{diag}(\kappa_1, \ldots, \kappa_d),
\]

\[
\kappa_i := \sin \theta_i (\sin \theta_i + \sin \theta_{d+1-i}) - \sin^2 \theta_{d+1-i} + \frac{1}{2} (\sin^2 \theta_1 + \cdots + \sin^2 \theta_d), \quad 1 \leq i \leq d.
\]

Therefore

\[
\text{div} F_1(u^{l,n}) = 8\pi^3 n^{-2s+3} (\sin \theta_i \cos \theta_{d+1-i} - \sin \theta_{d+1-i})_{1 \leq i \leq d}.
\]

Similarly, since \( \text{div} u^{l,n} = 0 \), we have

\[
F_2(u^{l,n}) = (-2\pi l n^{-s} \sin \theta_{d+1-i} - 2\pi n^{-2s+1} \sin \theta_{d+1-i})_{1 \leq i \leq d}.
\]

Therefore

\[
F(u^{l,n}) = (I - \Delta)^{-1} \Gamma_i)_{1 \leq i \leq d},
\]

where

\[
\Gamma_i = (8\pi^3 n^{-2s+3} \sin \theta_i \cos \theta_{d+1-i} - (\pi n^{-2s+1} + 4\pi^3 n^{-2s+3}) \sin 2\theta_{d+1-i} - 2\pi l n^{-s} \sin \theta_{d+1-i}).
\]

Since \( (I - \Delta)^{-1} \) is bounded from \( H^\sigma \) to \( H^{\sigma+2} \), we can use Lemma A.5 to derive that

\[
\| F(u^{l,n}) \|_{H^\sigma} \leq C \sum_{i=1}^{d} \left( \| n^{-2s+3} \sin \theta_i \cos \theta_{d+1-i} \|_{H^{\sigma-2}} + \| n^{-2s+3} \sin 2\theta_{d+1-i} \|_{H^{\sigma-2}} \right)
\]

\[
+ C \sum_{i=1}^{d} \left( \| n^{-2s+1} \sin 2\theta_{d+1-i} \|_{H^{\sigma-2}} + \| n^{-s} \sin \theta_{d+1-i} \|_{H^{\sigma-2}} \right)
\]

\[
\lesssim n^{-2s+3+\sigma-2} + n^{-2s+1+\sigma-2} + n^{-s+\sigma-2} \lesssim n^{-r_2}.
\]

Then we can use the Itô formula to (5.8) to find that for any \( T > 0 \) and \( t \in [0,T] \),

\[
\mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}^{l,n}(t) \|_{H^\sigma}^2 \leq \mathbb{E} \sup_{t \in [0,T]} \left| -2 \int_0^t \langle B(t', u^{l,n}) dW, \mathcal{E}^{l,n}(t') \rangle_{H^\sigma} \right| + \sum_{i=2}^{4} \int_0^T \mathbb{E} |P_i| dt,
\]

where

\[
P_2 = -2 \langle D^\sigma (n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i})_{1 \leq i \leq d}, D^\sigma \mathcal{E}^{l,n} \rangle_{L^2},
\]

\[
P_3 = 2 \langle D^\sigma F(u^{l,n}), D^\sigma \mathcal{E}^{l,n} \rangle_{L^2},
\]

\[
P_4 = \| B(t, u^{l,n}) \|_{L_2}^2.
\]

Using (5.2) and the BDG inequality, we find that

\[
\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \langle -2B(t', u^{l,n}) dW, \mathcal{E}^{l,n} \rangle_{H^\sigma} \right| \leq \frac{1}{2} \mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}^{l,n}(t) \|_{H^\sigma}^2 + CT n^{-2r_2}.
\]

We use (5.9) and (5.10) to find that,

\[
\int_0^T \mathbb{E} |P_2| dt \leq C \int_0^T \mathbb{E} \| (n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i})_{1 \leq i \leq d} \|_{H^\sigma}^2 dt + C \int_0^T \mathbb{E} \| \mathcal{E}^{l,n}(t) \|_{H^\sigma}^2 dt
\]

\[
\leq CT n^{-2r_2} + C \int_0^T \mathbb{E} \| \mathcal{E}^{l,n}(t) \|_{H^\sigma}^2 dt,
\]

27
\[
\int_0^T \mathbb{E}|P_3|dt \leq C \int_0^T \mathbb{E} \left( \|F(u^{l,n})\|_{H^{r}} \|\xi^{l,n}(t)\|_{H^{r}} \right) dt \leq CTn^{-2r} + C \int_0^T \mathbb{E}\|\xi^{l,n}(t)\|_{H^{r}}^2 dt,
\]
and
\[
\int_0^T \mathbb{E}|P_4|dt \leq C \int_0^T \mathbb{E}\|F(u^{l,n})\|_{H^{r}}^2 dt \leq CTn^{-2r}.
\]

Collecting the above estimates into (5.11), we arrive at
\[
\mathbb{E} \sup_{t \in [0,T]} \|\xi^{l,n}(t)\|_{H^{r}}^2 \leq CTn^{-2r} + C \int_0^T \mathbb{E} \sup_{t \in [0,t]} \|\xi^{l,n}(t')\|_{H^{r}}^2 dt.
\]
Then it follows from the Grönwall inequality that
\[
\mathbb{E} \sup_{t \in [0,T]} \|\xi^{l,n}(t)\|_{H^{r}}^2 \leq Cn^{-2r}, \quad C = C(T),
\]
which is the desired result.

5.2 Construction of actual solutions

Now we consider the problem (1.8) with deterministic initial data \(u^{l,n}(0, x)\), i.e.,
\[
\begin{align*}
\begin{cases}
u + ([u \cdot \nabla] u + F(u)] dt = B(t, u) dW, & t > 0, \ x \in \mathbb{T}^d, \\
u(0, x) = u^{l,n}(0, x), & x \in \mathbb{T}^d,
\end{cases}
\end{align*}
\]
(5.12)
where
\[
u^{l,n}(0, x) = (ln^{-1} + n^{-s} \cos n(2\pi x_{d+1-i})), 1 \leq i \leq d.
\]
Then Proposition 5.1 means that for each \(n\), (5.12) has a unique maximal solution \((u_l, \tau_{l,n}^t)\).

5.3 Estimates on the error

**Lemma 5.2.** Let \(d \geq 2\) be even, \(s > 1 + \frac{d}{4}\), \(\sigma \in \left(\frac{2}{3}, \min \{s - 1, \frac{d}{2} + 1\}\right)\) and \(r_s > 0\) be given in Lemma 5.1. For \(R \gg 1\), we define
\[
\tau_{l,n}^R := \inf \{t \geq 0 : \|u_l\|_{H^{r}} > R\} , \quad l \in \{-1, 1\}.
\]
Then for any \(T > 0\) and \(n \gg 1\), we have that for \(l \in \{-1, 1\},
\[
\mathbb{E} \sup_{t \in [0,T]} \|u^{l,n} - u_{l,n}\|_{H^{r}}^2 \leq Cn^{-2r}, \quad C = C(R, T),
\]
and
\[
\mathbb{E} \sup_{t \in [0,T]} \|u^{l,n} - u_{l,n}\|_{H^{2s-\sigma}}^2 \leq Cn^{2s-2\sigma}, \quad C = C(R, T).
\]

**Proof.** We first note that by Lemma A.5, for \(l \in \{1, -1\},
\[
\|u^{l,n}(t)\|_{H^{r}} \lesssim 1, \quad t \geq 0, \ n \geq 1,
\]
which means \(P\{\tau_{l,n}^R > 0\} = 1\) for any \(n \geq 1\) and \(l \in \{-1, 1\}\). Let \(v = u^{l,n} - u_{l,n}\). In view of (5.6), (5.8) and (5.12), we see that \(v\) satisfies
\[
v(t) + \int_0^t \left[ (u^{l,n} \cdot \nabla) v + (v \cdot \nabla) u_{l,n} + (-F(u_{l,n})) \right] dt' \\
= \int_0^t [-B(t', u_{l,n})] dW - 2\pi \int_0^t \left[ (n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i}) \right] dt'.
\]
For any $T > 0$, we use the Itô formula on $[0, T \wedge \tau_{t,n}^R]$, take a supremum over $t \in [0, T \wedge \tau_{t,n}^R]$ and use the BDG inequality to find

$$
\mathbb{E} \sup_{t \in [0, T \wedge \tau_{t,n}^R]} \|v\|_{H^s}^2 \leq 2\mathbb{E} \sup_{t \in [0, T \wedge \tau_{t,n}^R]} \left| \int_0^t \langle -B(t', u_{t,n})dW, v \rangle_{H^s} + \sum_{i=2}^{6} \mathbb{E} \int_0^{T \wedge \tau_{t,n}^R} |N_i|dt, \right|
$$

where

$$
N_2 = 2 \left\langle D^\sigma \left( -2\pi n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i} \right) \right\rangle_{L^2},
$$

$$
N_3 = -2 \left\langle D^\sigma [(v \cdot \nabla)u_{t,n}], D^\sigma v \right\rangle_{L^2}, \quad N_4 = -2 \left\langle D^\sigma [(u_{t,n} \cdot \nabla)v], D^\sigma v \right\rangle_{L^2},
$$

$$
N_5 = 2 \left\langle D^\sigma F(u_{t,n}), D^\sigma v \right\rangle_{L^2}, \quad N_6 = \|B(t, u_{t,n})\|_{L^2(\Omega; H^s)}^2.
$$

We can first infer from Lemma 4.1 that

$$
\|F(u_{t,n})\|_{H^s}^2 \lesssim (\|F(u_{t,n}) - F(u_{t,n})\|_{H^s} + \|F(u_{t,n})\|_{H^s})^2 \lesssim (\|u_{t,n}\|_{H^s} + \|u_{t,n}\|_{H^s})^2 \|v\|_{H^s}^2 + \|F(u_{t,n})\|_{H^s}^2.
$$

From the above estimate, (5.2), the BDG inequality, (5.10), (5.13) and (5.16), we have

$$
\mathbb{E} \sup_{t \in [0, T \wedge \tau_{t,n}^R]} \left| \int_0^t \langle -2B(t', u_{t,n})dW, v \rangle_{H^s} \right|
$$

$$
\leq 2\mathbb{E} \left( \int_0^{T \wedge \tau_{t,n}^R} \|v\|_{H^s}^2 \|F(u_{t,n})\|_{H^s}^2 dt \right)^{\frac{1}{2}}
$$

$$
\leq C\mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_{t,n}^R]} \|v\|_{H^s}^2 \int_0^{T \wedge \tau_{t,n}^R} \left( \|u_{t,n}\|_{H^s} + \|u_{t,n}\|_{H^s} \right)^2 \|v\|_{H^s}^2 dt \right)^{\frac{1}{2}}
$$

$$
+ C\mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_{t,n}^R]} \|v\|_{H^s}^2 \right)^{\frac{1}{2}} \int_0^{T \wedge \tau_{t,n}^R} \|F(u_{t,n})\|_{H^s}^2 dt
$$

$$
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, T \wedge \tau_{t,n}^R]} \|v\|_{H^s}^2 + C\mathbb{E} \int_0^{T \wedge \tau_{t,n}^R} \sup_{t' \in [0, t \wedge \tau_{t,n}^R]} \|v(t')\|_{H^s}^2 dt + CTn^{-2r_s}.
$$

Applying Lemma 4.1, $H^s \hookrightarrow L^\infty$, integration by parts and (5.9), we have

$$
|N_2| \lesssim \|(n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i}) \|_{H^s}^2 + \|v\|_{H^s}^2 \lesssim n^{-2r_s} + \|v\|_{H^s}^2,
$$

$$
|N_3| \lesssim \|(v \cdot \nabla)u_{t,n}\|_{H^s} \|v\|_{H^s} \lesssim \|v\|_{H^s}^2 \|u_{t,n}\|_{H^s},
$$

$$
|N_5| \lesssim (\|u_{t,n}\|_{H^s} + \|u_{t,n}\|_{H^s}) \|v\|_{H^s}^2 + \|F(u_{t,n})\|_{H^s} + \|v\|_{H^s}^2,
$$

and

$$
|N_6| \lesssim (\|u_{t,n}\|_{H^s} + \|u_{t,n}\|_{H^s}^2)^2 \|v\|_{H^s}^2 + \|F(u_{t,n})\|_{H^s}^2.
$$

With Lemma A.4 at hand, we consider the following two cases:

$$
|N_4| \lesssim \|u_{t,n}\|_{W^s_{d,2}} \|\nabla v\|_{L^d} \|v\|_{H^s} + \|\nabla u_{t,n}\|_{L^\infty} \|v\|_{H^s} \lesssim \|u_{t,n}\|_{H^s} \|v\|_{H^s}^2 \quad \text{for even } d \geq 4,
$$

and

$$
|N_4| \lesssim \|u_{t,n}\|_{W^s_{d,2}} \|\nabla v\|_{L^d} \|v\|_{H^s} + \|\nabla u_{t,n}\|_{L^\infty} \|v\|_{H^s}^2 \quad \text{for } d = 2,
$$

29
where in the case $d = 2$, $p$ will be chosen such that $\sigma - \frac{d}{2} = \sigma - 1 > 1 - \frac{2}{p} > 0$ and $q$ is determined by $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$. We use $H^s \hookrightarrow H^s+1 \hookrightarrow W^{\sigma,2q}$, $H^s \hookrightarrow W^{1,d}$ for the case $d \geq 4$ and use $H^s \hookrightarrow W^{\sigma+\frac{2q}{d},q} \hookrightarrow W^{\sigma,q}$ and $H^s \hookrightarrow W^{1,p}$ for the case $d = 2$ to obtain
\[
|N_4| \lesssim \|u^{l,n}\|_{H^s} \|v\|^2_{H^s}.
\]
Therefore we can infer from Lemma 4.1, (5.10), (5.13) and (5.16) that
\[
\mathbb{E} \int_0^{T \wedge \tau^n_{l,n}} (|N_2| + |N_3| + |N_4|) \, dt \leq C T n^{-2r_2} + C R \int_0^T \mathbb{E} \sup_{t' \in [0,t \wedge \tau^n_{l,n}]} \|v(t')\|^2_{H^s} \, dt,
\]
and
\[
\mathbb{E} \int_0^{T \wedge \tau^n_{l,n}} (|N_2| + |N_4|) \, dt \leq C R \int_0^T \mathbb{E} \sup_{t' \in [0,t \wedge \tau^n_{l,n}]} \|v(t')\|^2_{H^s} \, dt.
\]
Over all, we arrive at
\[
\mathbb{E} \sup_{t \in [0,T \wedge \tau^n_{l,n}]} \|v(t)\|^2_{H^s} \leq C n^{-2r_2} + C R \int_0^T \mathbb{E} \sup_{t' \in [0,t \wedge \tau^n_{l,n}]} \|v(t')\|^2_{H^s} \, dt.
\]
Via the Grönwall inequality, we have
\[
\mathbb{E} \sup_{t \in [0,T \wedge \tau^n_{l,n}]} \|v(t)\|^2_{H^s} \leq C n^{-2r_2}, \quad C = C(R,T),
\]
which is (5.14). For (5.15), we first note that $u_{l,n}$ is the unique solution to (5.12) and $2s - \sigma > d / 2 + 1$. For each fixed $n \geq 1$, similarly we use (5.13) to find
\[
\mathbb{E} \sup_{t \in [0,T \wedge \tau^n_{l,n}]} \|u_{l,n}(t)\|^2_{H^{2s-\sigma}} \leq 2 \mathbb{E} \|u^{l,n}(0)\|^2_{H^{2s-\sigma}} + C R T \int_0^T \left( \mathbb{E} \sup_{t' \in [0,t \wedge \tau^n_{l,n}]} \|u(t')\|^2_{H^{2s-\sigma}} \right) \, dt.
\]
From the above estimate, we can use the Grönwall inequality and Lemma A.5 to infer
\[
\mathbb{E} \sup_{t \in [0,T \wedge \tau^n_{l,n}]} \|u_{l,n}(t)\|^2_{H^{2s-\sigma}} \leq C \mathbb{E} \|u^{l,n}(0)\|^2_{H^{2s-\sigma}} \leq C n^{2s-2\sigma}, \quad C = C(R,T).
\]
Then it follows from Lemma A.5 that for some $C = C(R,T)$ and $l \in \{-1,1\}$,
\[
\mathbb{E} \sup_{t \in [0,T \wedge \tau^n_{l,n}]} \|v\|^2_{H^{2s-\sigma}} \leq C \mathbb{E} \sup_{t \in [0,T \wedge \tau^n_{l,n}]} \|u_{l,n}\|^2_{H^{2s-\sigma}} + C \mathbb{E} \sup_{t \in [0,T \wedge \tau^n_{l,n}]} \|u^{l,n}\|^2_{H^{2s-\sigma}} \leq C n^{2s-2\sigma},
\]
which is (5.15).

\section{5.4 Proof for Theorem 5.1}

\textbf{Lemma 5.3.} Let $d \geq 2$ be even and $B(t,u)$ satisfy Hypothesis H$_5$. If for some $R_0 > 1$, the $R_0$-exiting time is strongly stable at the zero solution to (1.8), then for $l \in \{1,-1\}$, we have
\[
(5.17) \quad \lim_{n \to \infty} \tau^{R_0}_{l,n} = \infty \quad \mathbb{P}\text{-a.s.},
\]
where $\tau^{R_0}_{l,n}$ is given in (5.13).

\textbf{Proof.} Since $F(0) = 0$, it is clear that zero is the unique solution to (1.8) with zero initial data under Hypothesis H$_5$. Due to (5.12), it follows that
\[
\lim_{n \to \infty} \|u_{l,n}(0) - 0\|_{H^{s'}} = \lim_{n \to \infty} \|u^{l,n}(0)\|_{H^{s'}} = 0 \quad \forall \, s' < s,
\]

\[\text{note that the } R_0\text{-exiting time at the zero solution is } \infty. \text{ Therefore we see that if the } R_0\text{-exiting time is strongly stable at the zero solution to (1.8), then (5.17) holds true.} \]
With the above result at our disposal, now we can prove Theorem 5.1.

**Proof for Theorem 5.1.** Let us first consider the case $d \geq 2$ is even. We will show that, if the $R_0$-exiting time is strongly stable at the zero solution for some $R_0 \gg 1$, then $(u_{-1,n}, \tau_{-1,n})$ and $(u_{1,n}, \tau_{1,n})$ satisfy (a)–(d) in Theorem 5.1.

(a) For each $n > 1$, for $l \in \{-1, 1\}$ and for the fixed $R_0 \gg 1$, Lemma A.5 and (5.13) give us $\mathbb{P}\{\tau_{R_0} \gg 1\} = 1$ and Lemma 5.3 implies (a).

(b) Besides, Theorem 4.1 and (5.13) show that $u_{l,n} \in C([0, \tau_{l,n}]; H^s)$ $\mathbb{P}$-a.s.

\[ \sup_{t \in [0, \tau_{l,n}]} \|u_{l,n}\|_{H^s} \leq R_0 \quad \mathbb{P}-a.s., \]

which gives (b).

(c) Since $u_{-1,n}(0)$ and $u_{1,n}(0)$ are deterministic and

\[ \|u_{-1,n}(0) - u_{l,n}(0)\|_{H^s} = \|u_{-1,n}(0) - u_{1,n}(0)\|_{H^s} \lesssim n^{-1}, \]

we obtain (c) holds.

(d) For any $T > 0$, using the interpolation inequality and Lemma 5.2, we see that for $l \in \{-1, 1\}$ and $v = v^{l,n} = u^{l,n} - u_{l,n}$,

\[ \left( \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|v\|_{H^s} \right)^2 \leq \left( \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|v\|_{H^s}^2 \right)^{\frac{2}{s}} \left( \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|v\|_{H^2 - \sigma}^2 \right)^{\frac{1}{s}} \lesssim n^{-r_+ + (s - \sigma)}. \]

It follows from

\[ 0 > -r_+ + s - \sigma = \begin{cases} 1 - s & \text{if } 1 + \frac{d}{2} < s \leq 3, \\ -2 & \text{if } s > 3, \end{cases} \]

that for $l \in \{-1, 1\}$,

\[ \lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|u^{l,n} - u_{l,n}\|_{H^s} = 0. \]

For any given $T > 0$, on account of (5.18), Lemmas A.5 and 5.3, we have

\[ \lim_{n \to \infty} \inf \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s} \geq \lim_{n \to \infty} \inf \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s} \]

\[ \geq \lim_{n \to \infty} \inf \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s} \]

\[ \geq \lim_{n \to \infty} \inf \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|n^{-s} \cos (2\pi n x_{d+1-i} + 2\pi t) - n^{-s} \cos (2\pi n x_{d+1-i} - 2\pi t)\|_{H^s} \]

\[ \geq \lim_{n \to \infty} \inf \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \left( n^{-s} \|\sin 2\pi n x_{d+1-i}\|_{H^s} \|\sin 2\pi t| - \|2n^{-1}\|_{H^s} \right). \]

Using the Fatou’s lemma, we arrive at

\[ \lim_{n \to \infty} \inf \mathbb{E} \sup_{t \in [0, T \wedge \tau_{R_0}]} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s} \geq \sup_{t \in [0, T]} \|\sin 2\pi t|, \]

which implies (d).

Now we consider the case that $d \geq 3$ is odd. Instead of (5.5), we define the following divergence-free vector field as

\[ u^{l,n} = (ln^{-1} + n^{-s} \cos \theta_1, ln^{-1} + n^{-s} \cos \theta_2, \cdots, ln^{-1} + n^{-s} \cos \theta_{d-1}, 0), \]
where \( \theta_i = 2\pi(nx_{d-i} - lt) \) with \( 1 \leq i \leq d - 1 \), \( n \geq 1 \), \( t \in \{-1, 1\} \). In this case, \( d - 1 \) is even and we can repeat the proof for Lemma 5.1 to find that the error \( E^{l,n}(t) \) also enjoys (5.7). Moreover, for the pathwise solutions \( u_{l,n} \) to (5.12) with

\[
\begin{align*}
\pi_n(0) &= u_{l,n}(0) = (ln^{-1} + n^{-s} \cos 2\pi nx_{d-i}, 0)_{1 \leq i \leq d-1},
\end{align*}
\]

we can basically repeat the previous procedure to show that Lemmas 5.2 and 5.3 also hold true. Therefore one can establish (a)–(d) for \( u_{l,n} \) similarly.

In conclusion, we see that if for some \( R_0 \gg 1 \), the \( R_0 \)-exiting time is strongly stable at the zero solution, then the solution map defined by (1.8) is not uniformly continuous when \( B(t, \cdot) \) satisfies Hypothesis \( H_5 \).

**Remark 5.2.** From the above proof for Theorem 5.1, it is clear that if \( d = 1 \), one can use

\[
\begin{align*}
u^{l,n} = ln^{-1} + n^{-s} \cos 2\pi(nx - lt), \quad n \geq 1
\end{align*}
\]

as a sequence of approximation solutions and repeat the other part of the proof correspondingly to obtain the similar statements in \( d = 1 \). Therefore Theorem 5.1 also holds true for \( d = 1 \), namely the stochastic CH equation case.

**Declaration**

The author declare that data sharing is not applicable to this article since no datasets were generated or analyzed during the current study.

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**A Auxiliary results**

In this appendix we recall and establish some auxiliary results from analysis employed in the proofs above. We begin with introducing mollifiers. For \( n \geq 1 \), we define the Friedrichs mollifier \( J_n \) as

\[
J_n := \text{OP}(j(\cdot/n)), \quad n \geq 1,
\]

where \( j \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}) \) (the Schwarz space of rapidly decreasing \( C^\infty \) functions on \( \mathbb{R}^d \)) satisfies \( 0 \leq j(y) \leq 1 \) for all \( y \in \mathbb{R}^d \) and \( j(y) = 1 \) for any \( |y| \leq 1 \).

**Lemma A.1 ([35, 45, 54]).** The following properties for \( J_n \) hold true:

\[
\begin{align*}
\|I - J_n\|_{L(H^r; H^s)} &\lesssim \frac{1}{n^{s-r}}, \quad r < s, \\
\|J_n\|_{L(H^r; H^s)} &\sim O(n^{r-s}), \quad r > s,
\end{align*}
\]

and

\[
[D^s, J_n] = 0, \quad \langle J_n f, g \rangle_{L^2} = \langle f, J_n g \rangle_{L^2}, \quad \|J_n\|_{L(\ell_\infty; \ell_\infty)} \lesssim 1, \quad \|J_n\|_{L(H_r; H^s)} \leq 1, \quad n \geq 1, \quad s \geq 0.
\]

** Lemma A.2 (Page 3 in [56]). Let \( d \geq 1 \) and \( f, g : \mathbb{K}^d \to \mathbb{R}^d \) such that \( g \in W^{1,\infty} \) and \( f \in L^2 \). Then for some \( C = C(d) > 0 \),

\[
\|\langle J_n, (g \cdot \nabla) f \rangle_{L^2} \leq C\|g\|_{W^{1,\infty}}\|f\|_{L^2}, \quad n \geq 1.
\]

Then we recall some estimates in Sobolev spaces \( H^s \).
Lemma A.3 ([7, 14]). For any $s > 0$, and $s_1, s_2 \in \mathbb{R}$ with $s_1 + s_2 > 0$ and $s_1 < \frac{d}{2} < s_2$,

$$\|fg\|_{H^{s_1}} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}, \quad f \in H^{s_1}, g \in H^{s_2}.$$  

Lemma A.4 ([29, 30]). If $f, g \in H^s \cap W^{1,\infty}$ with $s > 0$, then for $p, p_1 \in (1, \infty)$ with $i = 2, 3$ and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2} \), we have

$$\|D^s f, D^s g\|_{L^p} \leq C_s(\|f\|_{L^{p_1}} \|D^{s-1} g\|_{L^{p_2}} + \|D^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}),$$

and

$$\|D^s(fg)\|_{L^p} \leq C_s(\|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}} + \|D^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}).$$

We also recall the following

Lemma A.5 ([47, 60]). Let $\sigma, \alpha \in \mathbb{R}$. If $n \gg 1$, then

$$\|\sin(n(2\pi x) - \alpha)\|_{H^r(\mathbb{T}; \mathbb{R})}, \quad \|\cos(n(2\pi x) - \alpha)\|_{H^r(\mathbb{T}; \mathbb{R})}, \quad \|\cos(n(2\pi y) - \alpha)\|_{H^r(\mathbb{T}^2; \mathbb{R})} \approx n^r.$$  

The following lemmas with single $\mathcal{P}$ and a pair $(\mathcal{P}_1, \mathcal{P}_2)$ on $\mathbb{R}^d$ are well known in the literature, and they can be easily extended to the case on $\mathbb{T}^d$ (cf. see for example [43, Theorem 4.5.3, Corollaries 4.5.7 and 4.6.13] and [36]).

Lemma A.6 ([8, 52, 54]). Let $\mathcal{P} \in \text{OPS}^s$, $s \in \mathbb{R}$. Then $\mathcal{P}^* \in \text{OPS}^s$, and $\mathcal{P} \in \mathcal{L}(H^q; H^{q-s})$ for any $q \in \mathbb{R}$. For any $\mathcal{P}_i \in \text{OPS}^{r_i}$, $r_i \in \mathbb{R}$, $i = 1, 2$,

$$\mathcal{P}_1 \mathcal{P}_2 \in \text{OPS}^{r_1 + r_2}.$$  

Moreover, if their symbols are commuting matrices, then

$$[\mathcal{P}_1, \mathcal{P}_2] \in \text{OPS}^{r_1 + r_2 - 1}.$$  

Lemma A.7 (Proposition 4.2 in [55]). Let $\mathcal{P} \in \text{OPS}^r$ with $r \geq 0$. For any $\sigma > 1 + \frac{d}{2}$ and $q \in [0, \sigma - r]$,

$$\|\mathcal{P}, gI\|_{L^q} \lesssim \|g\|_{H^q}, \quad g \in H^q, \quad u \in H^{q+r-1}.$$  

Lemma A.8. Let $r_1, r_2 \in \mathbb{R}$. Assume that $\{(p_n, q_n)\}_{n \geq 1} \subset S^{r_1} \times S^{r_2}$ is bounded. If for all $n, k \geq 1$, $p_n$ and $q_k$ are commuting matrices, then

$$\sup_{n,k} \|\text{OP}(p_n), \text{OP}(q_k)\|_{\mathcal{L}(H^{r_1+r_2-1}; L^2)} < \infty.$$  

Proof. Since $S^{r_1} \times S^{r_2}$ is a Fréchet space, it suffices to show that the mapping $(p, q) \to \text{OP}(p), \text{OP}(q)$ is bilinear and continuous from $S^{r_1} \times S^{r_2}$ to $\mathcal{L}(H^{r_1+r_2-1}; L^2)$. Bilinearity is obvious, and now we prove the continuity. To this end, we denote by $p_1 \# p_2$ the symbol of the operator product $\text{OP}(p_1)\text{OP}(p_2)$, i.e., $\text{OP}(p_1)\text{OP}(p_2) = \text{OP}(p_1 \# p_2)$.

On one hand, for two symbols $p_1$ and $p_2$, it is well-known that the mapping $S^{r_1} \times S^{r_2} \ni (p_1, p_2) \mapsto p_1 \# p_2 \in S^{r_1+r_2}$ is continuous (cf. [38, Theorem 1.2.16], [1, Page 72]).

On the other hand, when $p_1$ and $p_2$ are commuting matrices, some direct computations (cf. [3, Corollary 4.1] or [8, Theorem C.3]) yield $p_1 \# p_2 - p_2 \# p_1 \in S^{r_1+r_2-1}$.

Therefore, the mapping

$$\mathbf{T} : S^{r_1} \times S^{r_2} \ni (p_1, p_2) \mapsto \mathbf{T}(p_1, p_2) := p_1 \# p_2 - p_2 \# p_1 \in S^{r_1+r_2-1}$$

is continuous, which together with (2.7) implies that $(p, q) \to [\text{OP}(p), \text{OP}(q)] = \text{OP}(\mathbf{T}(p, q))$ is continuous from $S^{r_1} \times S^{r_2}$ to $\mathcal{L}(H^{r_1+r_2-1}; L^2)$. \hfill \square

Lemma A.9. Let $r \geq 0, \sigma > d/2 + 1$ and $q \in [0, \sigma - r]$. If $\{p_n\}_{n \geq 1} \subset S^{r}$ is bounded and $g \in H^{\sigma}$, then there is a constant $C > 0$ independent of $n$ such that

$$\|\text{OP}(p_n), gI\|_{\mathcal{L}(H^{\sigma})} \leq C \|g\|_{H^{\sigma}} \|u\|_{H^{q+r-1}}, \quad u \in H^{q+r-1}.$$
Proof. To begin with, we define a sequence of linear operator $\mathfrak{T}: \mathcal{OPS}^* \ni \mathcal{P} \mapsto [\mathcal{P}, g_\mathcal{I}]$. Then Lemma A.7 shows

$$\|\mathfrak{T}(\mathcal{P})u\|_{H^q} \leq C\|g\|_{H^r}\|u\|_{H^{q+r-1}}, \quad C = C(\mathcal{P}),$$

which means

$$\|\mathfrak{T}(\mathcal{P})\|_{\mathcal{L}(H^{q+r-1}; H^q)} = \sup_{\|u\|_{H^{q+r-1}} \neq 0} \frac{\|\mathfrak{T}(\mathcal{P})u\|_{H^q}}{\|u\|_{H^{q+r-1}}} \leq C\|g\|_{H^r}, \quad C = C(\mathcal{P}).$$

The above estimate and (2.7) imply that under the above assumptions for $r, \sigma$ and $q$, $\mathfrak{T}\mathcal{OPS}: \mathcal{S} \to \mathcal{L}(H^{q+r-1}; H^q)$ is continuous,

which implies the desired estimate. \qed

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