Comment on “Filters in ordered Γ-semigroups”

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ABSTRACT. This is about the paper in the title by Kostaq Hila in Rocky Mt. J. Math. 41, no. 1 (2011), 189–203 [1] for which corrections should be done.

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Throughout the paper in [1], $M$ stands for an ordered Γ-semigroup (po-Γ-semigroup) [1, page 191].

If $A$ is a left (or right) ideal of a po-Γ-semigroup $M$, then $A$ is a subsemigroup of $M$. So the ideals of $M$ are subsemigroups of $M$ as well. In the proof of Theorem 2.5 in [1], it has been proved that $(a)_N$ is a subsemigroup and at the same time an ideal of $N(a)$. Since the ideals are subsemigroups, the proof that $(a)_N$ is a subsemigroup of $N(a)$ should be deleted from the proof of the theorem and keep only the fact that $(a)_N$ is a nonempty subset of $N(a)$. It might be emphasized in the theorem that the $N(a)$ is a po-Γ-semigroup.

This is the Theorem 2.6 in [1]:

**Theorem 2.6.** Let $a, b \in M$. Then $(a)_N \leq (b)_N$ if and only if $N(b) \subseteq N(a)$.

In the proof of the “$\Rightarrow$” part of the theorem “it is clear that $(a)_N \leq (b)_N$ implies $(a)_N \gamma (b)_N \subseteq (a)_N \forall \gamma \in \Gamma$” is written. The correct is that $(a)_N \leq (b)_N$ implies $(a)_N \gamma (b)_N = (a)_N \forall \gamma \in \Gamma$. For the class $(a)_N$ instead of using the (undefined concept) “(a)$_N$ is a semilattice congruence class” the phrase “$N$ is a semilattice congruence on $M$” should be used.

In the proof of the “$\Leftarrow$” part of the theorem “we only need to prove that $(a)_N \gamma (b)_N \subseteq (a)_N$, $(b)_N \gamma (a)_N \subseteq (a)_N \forall \gamma \in \Gamma$” is written. As $N$ is a semilattice congruence on $M$, $(a)_N \gamma (b)_N = (b)_N \gamma (a)_N$. So the author has only proved that $(a)_N \gamma (b)_N \subseteq (a)_N \forall \gamma \in \Gamma$ and to prove the theorem, he had to prove that for all $\gamma \in \Gamma$, the inclusion $(a)_N \subseteq (a)_N \gamma (b)_N$ also holds.

The Theorem 2.6 in [1] can be read as follows:
Proposition. Let $M$ be a po-$\Gamma$-semigroup and $a, b \in M$. Then we have the following:

1. If $(a)_N \preceq (b)_N$, then $N(b) \subseteq N(a)$.
2. If $N(b) = N(a)$, then $(a)_N \preceq (b)_N$.

Proof. The proof of (2): Let $N(b) = N(a)$ and $\gamma \in \Gamma$. Since $(a, b) \in N$, we have $(a\gamma b, b\gamma b) \in N$. Since $(b\gamma b, b) \in N$, we get $(a\gamma b, b) \in N$, then $(a\gamma b)_N = (b)_N = (a)_N$, and $(a)_N \preceq (b)_N$.

This is Theorem 2.7 in [1]:

Theorem 2.7. Let $a \in M$. Then the following sets are equal:

1. $K(a) = \{b \in M \mid (b)_N \succ (a)_N\}$.
2. $A = \{b \in N(a) \mid (b)_N \succ (a)_N\}$.
3. $B = \bigcup \{(b)_N \mid (b)_N \succ (a)_N\}$.
4. $C = N(a) \setminus (a)_N$.

To prove that $K(a) \subseteq A$ or $B \subseteq C$ is not necessary to say that $N(b) \not\subseteq N(a)$ to have $b \in N(a)$. In the proof of $C \subseteq A$ the “$\Leftarrow$” part of Theorem 2.6 has been used. So we do not know if the Theorem 2.7 in [1] is true. The Corollary 2.8 of the paper is based on Theorem 2.7, so we cannot say that that corollary is true as well. The Corollary 2.8 has been also used in Corollary 2.10 and in the Example 2.9.

Before the Theorem 2.11, the author wrote: “To prove the following theorems we use some important notions and results proved in [7, 1.3.2] for ordered semigroups, the modification of which can be easily done for the ordered $\Gamma$-semigroups”. The [7, 1.3.2] mentioned above is the Reference [2] of the present note but there are no such results in the book by J. Howie. In the Proposition 1.3.2 Howie showed the well known that every lower semilattice is an idempotent and commutative semigroup and conversely, that every commutative and idempotent semigroup is a lower semilattice (which is true for upper semilattices as well).

And this is Theorem 2.11 in [1]:

Theorem 2.11. The following statements are equivalent:

1. $M$ is a semilattice such that $a \leq a\gamma a$ for every $a \in M$ and every $\gamma \in \Gamma$.
2. For every $a \in M$, $N(a) = [a]$.
3. $N$ is the equality relation on $M$.

First of all a $\Gamma$-semigroup $M$ is a semilattice means that the $\Gamma$-semigroup has the properties $a\gamma a = a$ and $a\gamma b = b\gamma a$ for every $a, b \in M$ and every $\gamma \in \Gamma$. So property (1) is not true as there is no idempotent (and so idempotent and commutative) $\Gamma$-semigroup such that $a < a\gamma a$ for every $a \in M$ and every $\gamma \in \Gamma$. The same is mentioned
in the introduction of the paper as well, where the author wrote: “Also, we will consider a structure of principal filter on ordered \( \Gamma \)-semigroups and by using the relation \( N \), we will observe that \( N \) on any ordered \( \Gamma \)-semigroup \( M \) is the equality relation on \( M \) if and only if \( M \) is a semilattice having the property \( a \leq a\gamma a \) for all \( a \in M \), \( \gamma \in \Gamma \).” However, in the proof of \( (3) \Rightarrow (1) \) of the same theorem he shows that \( a\gamma a = a \) and \( a\gamma b = b\gamma a \) for every \( a, b \in M \) and says “this shows that \( M \) is a semilattice as required”.

In the proof of the implication \( (1) \Rightarrow (2) \), to prove that \( [a] \) is a filter containing \( a \), the author considers \( b, c \in M \) such that \( b\gamma c \in [a] \) “for all \( \gamma \in \Gamma \)” (and uses the “for all \( \gamma \in \Gamma \)” in the rest of the proof as well). The correct is: Suppose \( b, c \in M \) and \( \gamma \in \Gamma \) such that \( b\gamma c \in [a] \). From the fact that \( M \) is a semilattice, he concludes that there exist \( \gamma_1, \gamma_2 \in \Gamma \) such that \( b = b\gamma_1 b \) and \( c = c\gamma_2 c \) which is wrong. Then he wrote: “Since \( b\gamma c \geq a \) for all \( \gamma \in \Gamma \) (for which we already said is not true), we have \( b\gamma_1 c \geq a \) and there exists \( \gamma_2 \in \Gamma \) such that \( a = a\gamma_2 b\gamma_1 c \)” (which is also wrong as the order on \( M \) does not have this property). This being wrong, the rest of the proof that \( a\gamma_1 b = b\gamma_1 a = a\gamma_2 b\gamma_1 c = a\gamma_3 b\gamma_2 c = a \) cannot be true. Besides, from \( a\gamma_1 b = a \) and from \( a\gamma_2 c = a \) we cannot conclude that \( b \geq a \) and \( c \geq a \) to have \( b, c \in [a] \).

In the proof of the implication \( (3) \Rightarrow (1) \), the phrase “since \( (a)_N \) and \( (b)_N \) are both semilattice congruence classes on \( M \)” should be replaced by “since \( N \) is a semilattice congruence on \( M \)”.

We denote by \( [a] \) the subset of \( M \) defined by \( [a] := \{ t \in M \mid t \geq a \} \).

A \( \text{po}-\Gamma \)-groupoid \( M \) is said to be a \textit{band} if \( a\gamma a = a \) for every \( a \in M \) and all \( \gamma \in \Gamma \).

**Proposition.** Let \( M \) be a \( \text{po}-\Gamma \)-groupoid in which the order “\( \leq \)” has the following property:

\[ a \leq b \implies a\gamma b = a \quad \forall \gamma \in \Gamma. \]

Then \( M \) is a band.
Proof. Let \( a \in M \). Take an element \( \gamma \in \Gamma \) (\( \Gamma \neq \emptyset \)). Since \( a \leq a \), by hypothesis, we have \( a\gamma a = a \).

This is the corrected form of the Theorem 2.11 in [1]:

**Theorem 2.11.** Let \( M \) be a po-\( \Gamma \)-semigroup. Then we have the following:

1. If \( M \) is a band then, for any \( a \in M \), the set \([a]\) is a subsemigroup of \( M \).
2. If \( N(a) = [a] \) for every \( a \in M \), then the relation \( N \) is the equality relation on \( M \).
3. If \( N \) is the equality relation on \( M \), then \( M \) is a semilattice.
4. In particular, if \( M \) is a commutative band (i.e. a semilattice) and the order “\( \leq \)” on \( M \) satisfies the relation

\[
a \leq b \iff a\gamma b = a \ \forall \gamma \in \Gamma
\]

then, for every \( a \in M \), we have \( N(a) = [a] \).

Proof. (1) First of all, \([a]\) is a nonempty subset of \( M \) as \( a \in [a] \). Let \( x, y \in [a] \) and \( \gamma \in \Gamma \). Since \( x \geq a, y \geq a, \gamma \in \Gamma \) and \( \Gamma \) is a band, we have \( x\gamma y \geq a\gamma a = a \), so \( x\gamma y \in [a] \). Thus \([a]\) is a subsemigroup of \( M \).

(2) Let \((a, b) \in N \). Since \( a \in N(a) = N(b) = [b] \), we have \( a \leq b \). Since \( b \in N(b) = N(a) = [a] \), we have \( b \leq a \). Thus we have \( a = b \), and \( N \) is the equality relation on \( M \).

(3) Let \( a, b \in M \) and \( \gamma \in \Gamma \). Since the relation \( N \) is a semilattice congruence on \( M \), we have \((a\gamma a, a) \in N \) and \((a\gamma b, b\gamma a) \in N \). Since \( N \) is the equality relation on \( M \), we have \( a\gamma a = a \) and \( a\gamma b = b\gamma a \), thus \( M \) is a semilattice.

(4) Let \( a \in M \). As we have already seen in (1), the set \([a]\) is a subsemigroup of \( M \). Let \( x, y \in [a] \) and \( \gamma \in \Gamma \) such that \( x\gamma y \in [a] \). Then \( x \in [a] \) and \( y \in [a] \). Indeed: Let \( \mu \in \Gamma \) (\( \Gamma \neq \emptyset \)). Since \( x\gamma y \geq a \), by hypothesis, we have \( a\mu(x\gamma y) = a \). Since \( M \) is a band, we have

\[
a\mu x = (a\mu(x\gamma y))\mu x = a\mu((x\gamma y)\mu x) = a\mu(x\mu(x\gamma y)) = (a\mu x)\mu(x\gamma y) = a\mu(x\gamma y) = a,
\]

and

\[
a\mu y = (a\mu x\gamma y)\mu y = a\mu x\gamma(y\mu y) = a\mu x\gamma y = a,
\]

so \( x \geq a \) and \( y \geq a \), that is \( x, y \in [a] \). If \( x \in [a] \) and \( M \ni y \geq x \), then \( y \in [a] \). Indeed: Since \( x \in [a] \), we have \( x \geq a \). Then \( y \geq x \geq a \), so \( y \in [a] \). Let \( T \) be a filter of \( M \) such that \( a \in T \). Then \([a] \subseteq T \). Indeed: If \( x \in [a] \), then \( x \in M \) and \( x \geq a \in T \). Since \( T \)
is a filter of \( M \), we have \( x \in T \). As \( \langle a \rangle \) is the smallest (with respect to the inclusion relation) filter of \( M \) containing \( a \), we have \( N(a) = \langle a \rangle \).

Taking into account the Proposition above we notice that if \( M \) is a commutative \( \text{po-} \Gamma \)-semigroup satisfying the relation

\[
a \leq b \iff a\gamma b = a \text{ for all } \gamma \in \Gamma,
\]

then \( M \) is a commutative band (and so a semilattice) and, for every \( a \in M \), we have \( N(a) = \langle a \rangle \).

This is the Theorem 2.13 in [1]:

**Theorem 2.13.** Let \( \sigma \) be a complete semilattice congruence on an ordered \( \Gamma \)-semigroup \( M \) and \( Y \) the semilattice \( M/\sigma \). Then for any \( x \in Y \), we have

1. \( M_x \) is the union of some \( N \)-classes.
2. The set \( T = \bigcup \{ M_y \mid y \succeq x, y \in Y \} \) is a filter.
3. For any \( a \in M_x \), \( N(a) = T \) if and only if \( \sigma \) is the smallest complete semilattice congruence on \( M \).

The relation “\( y \succeq x \)” for the elements of \( M/\sigma \) has not defined in the paper. On p. 194 in [1] the definition is only for \( \sigma = N \).

We define \( \langle a \rangle_x \leq \langle b \rangle_x \iff \langle a \rangle_x = \langle a \rangle_x \gamma \langle b \rangle_x := \langle a \gamma b \rangle_x \) for all \( \gamma \in \Gamma \).

The \( M_x \) has not defined in the paper, apparently it is the \( \langle x \rangle_x \). As far as the property (2) is concerned, it should be clarified in (2) if the filter mentioned in it was in \( M \) or in \( M/\sigma \). It seems that it is in \( M/\sigma \) while according to the proof of (2) it is in \( M \). In fact, the author tried to prove in (2) that “\( T \) is a subsemigroup of \( M \)” and that if \( \langle a, b \rangle \in M \) and \( \gamma \in \Gamma \) such that \( a\gamma a \in T \), then \( a \in T \) and \( b \in T \”), which means that he considers the filter in \( M \). The proof contains serious mistakes in it. For example, in several parts of the proof \( M_\Gamma M_t \subseteq M_\Gamma M_t \) is written.

The Theorem 2.13 in [1] could be replaced by the following:

**Theorem 2.13.** Let \( M \) be a \( \text{po-} \Gamma \)-semigroup and \( \sigma \) a semilattice congruence on \( M \). Then the following property is satisfied:

If \( x \in M \), then the set \( T := \{ (y)_x \mid y \in M, (y)_x \succeq (x)_x \} \) is a filter in \( M/\sigma \).

**Proof.** Take an element \( \gamma \in \Gamma \) (\( \Gamma \neq \emptyset \)). Since \( \sigma \) is a semilattice congruence on \( M \), we have \( (x, x\gamma x) \in \sigma \), then \( (x)_\sigma = (x\gamma x)_\sigma \), so \( (x)_\sigma \succeq (x)_\sigma \), and \( (x)_\sigma \in T \). Thus \( T \) is a nonempty subset of \( M/\sigma \). Let now \( (y)_\sigma, (z)_\sigma \in T \) and \( \gamma \in \Gamma \). Then \( (y)_\sigma \gamma (z)_\sigma \in T \),
that is \((y\gamma z)_\sigma \in T\). Indeed: Since \((y)_\sigma \succeq (x)_\sigma\), \((z)_\sigma \succeq (x)_\sigma\) and \(\sigma\) is a semilattice congruence on \(M\), we have
\[
(y\gamma z)_\sigma = (y)_\sigma \gamma (z)_\sigma \succeq (x)_\sigma \gamma (x)_\sigma = (x)_\sigma,
\]
so \((y\gamma z)_\sigma \in T\). Let \(a, b \in M\) and \(\gamma \in \Gamma\) such that \((a)_\sigma \gamma (b)_\sigma \in T\). We have to prove that \((a)_\sigma \in T\) and \((b)_\sigma \in T\), that is \((a)_\sigma \succeq (x)_\sigma\) and \((b)_\sigma \succeq (x)_\sigma\), which means that \((x)_\sigma = (x\gamma a)_\sigma\) and \((x)_\sigma = (x\gamma b)_\sigma\) for every \(\mu \in \Gamma\).

Let now \(\mu \in \Gamma\). By hypothesis, we have \((a\gamma b)_\sigma \in T\), then \((a\gamma b)_\sigma \succeq (x)_\sigma\), and so \((x)_\sigma = (x\gamma a\gamma b)_\sigma\) for every \(\xi \in \Gamma\). Thus we have \((x)_\sigma = (x\mu a\gamma b)_\sigma\). Then we get
\[
(x\mu a)_\sigma = (x)_\sigma \mu (a)_\sigma = (x\mu a\gamma b)_\sigma \mu (a)_\sigma = (x\mu a\gamma b\mu a)_\sigma.
\]
Since \(\sigma\) is a semilattice congruence on \(M\), we have \((a\gamma b\mu a, a\mu (a\gamma b)) \in \sigma\) and \((a\mu a, a) \in \sigma\). Then \((a\mu a\gamma b, a\gamma b) \in \sigma\), and \((a\gamma b\mu a, a\gamma b) \in \sigma\). Since \((a\gamma b\mu a, a\gamma b) \in \sigma\) and \(\sigma\) is a congruence on \(M\), we have \((x\mu a\gamma b\mu a, x\mu a\gamma b) \in \sigma\). Hence we obtain
\[
(x\mu a)_\sigma = (x\mu a\gamma b)_\sigma = (x)_\sigma.
\]
Similarly we prove that \((x\mu b)_\sigma = (x)_\sigma\) for all \(\mu \in \Gamma\). Finally, let \((y)_\sigma \in T\) and \(M/\sigma \ni (z)_\sigma \ni (y)_\sigma\). Then \((z)_\sigma \in T\). Indeed: Take an element \(\gamma \in \Gamma\) \((\Gamma \neq \emptyset)\). Since \((y)_\sigma \ni (x)_\sigma\) and \((z)_\sigma \ni (y)_\sigma\), we have \((x)_\sigma = (x)_\sigma \gamma (y)_\sigma\) and \((y)_\sigma = (y)_\sigma \gamma (z)_\sigma\). Hence we obtain
\[
(x)_\sigma = (x)_\sigma \gamma (y)_\sigma = (x)_\sigma \gamma ((y)_\sigma \gamma (z)_\sigma) = ((x)_\sigma \gamma (y)_\sigma) \gamma (z)_\sigma = (x)_\sigma \gamma (z)_\sigma,
\]
so \((z)_\sigma \ni (x)_\sigma\), and \((z)_\sigma \in T\).

The Corollaries 2.14 and 2.15 of the paper are based on Theorem 2.13. The Example 2.16 is based on Theorem 2.11. In Examples 1.3 and 1.6 the author defines an order on “\(\Gamma\)” while for a \(\Gamma\)-semigroup, the set \(\Gamma\) is just a nonempty set and not an ordered set.

Finally it should be noted that except of the case in which we search for a counterexample in which case an example for an ordered semigroup is enough, examples of ordered \(\Gamma\)-semigroups in which the set \(\Gamma\) consists only by one element are actually examples of ordered semigroups. A sufficient example of an ordered \(\Gamma\)-semigroup should be an example in which the set \(\Gamma\) has at least two elements. The examples of the paper in [1] are examples of ordered semigroups. A counterexample is given in the Example 1.11, but this also being an example of an ordered semigroup, just looking at the table and the figure of it one can immediately concludes that this is an example of a commutative ordered semigroup (and so of a commutative ordered \(\Gamma\)-semigroup as well) while the author gets the assumption that it is not a \(\Gamma\)-semigroup to prove that it is.

Many of the results of the paper in [1] hold in po-\(\Gamma\)-groupoids in general.
References

[1] K. Hila, Filters in ordered Γ-semigroups, Rocky Mt. J. Math. 41, no. 1 (2011), 189–203.

[2] J. M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, Clarendon Press · Oxford 1995.