The Basic Elliptic Equations in an Equilateral Triangle

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Abstract

In his deep and prolific investigations of heat diffusion, Lamé was led to the investigation of the eigenvalues and eigenfunctions of the Laplace operator in an equilateral triangle. In particular he derived explicit results for the Dirichlet and Neumann cases using an ingenious change of variables. The relevant eigenfunctions are complicated infinite series in terms of his variables.

Here we first show that boundary value problems with simple boundary conditions, such as the Dirichlet and the Neumann problems, can be solved in an elementary manner. In particular for these problems, the unknown Neumann and Dirichlet boundary values respectively, can be expressed in terms of a Fourier series. Our analysis is based on the so called global relation, which is an algebraic equation coupling the Dirichlet and the Neumann spectral values on the perimeter of the triangle.

As Lamé correctly pointed out, infinite series are inadequate for expressing the solution of more complicated problems, such as mixed boundary value problems. Here we show, utilizing further the global relation, that such problems can be solved in terms of generalized Fourier integrals.
1 Introduction

Solutions of certain linear elliptic boundary value problems can be expanded in complete sets of eigenfunctions. Unfortunately, the actual form of these eigenfunctions is known for only simple geometries. In fact, only geometries that allow separation of variables yield well known expressions for the associated eigenfunctions. But what happens when separation of variables does not apply? Is it possible to construct the spectral characteristics of a fundamental domain that does not fit any separable coordinate system? Some examples where this construction is possible are presented in the present work. The approach used here has its roots in the unified transform method for analysing both lineal and integrable nonlinear PDEs introduced in [4].

A crucial role in this analysis is played by a certain equation coupling all boundary values, which was called the global relation in [4]. The concrete form of this equation for the equilateral triangle was given in the important work of [14], where it was called a functional equation.

A general overview of the problems solved in this paper is presented in the sequel where notations and some elementary formulae are included in order to facilitate the understanding of the new results. We study boundary value problems for the Laplace, the Helmholtz and the modified Helmholtz equations in the interior of an equilateral triangle. These equations are three of the basic equations of classical mathematical physics. In particular, they arise as the reduction of several fundamental parabolic and hyperbolic linear equations. Furthermore, the specific boundary conditions discussed here cover most cases of physical significance.

We first introduce some notations.

1.1 Notations and Useful Identities

(i) $z$ will denote the usual complex variable and $\alpha$ will denote one of the complex roots of unity,

$$z = x + iy, \quad \alpha = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$  

Bar will denote complex conjugation, in particular

$$\bar{z} = x - iy, \quad \bar{\alpha} = e^{-\frac{2\pi i}{3}}.$$ 

$F(\bar{k})$ will denote the Schwarz conjugate of the function $F(k)$.

(ii) The complex numbers

$$z_1 = \frac{l}{\sqrt{3}} e^{\frac{i\pi}{3}}, \quad z_2 = \bar{z}_1, \quad z_3 = -\frac{l}{\sqrt{3}},$$  

will denote the vertices of the equilateral triangle, and $D \subset \mathbb{C}$ will denote the interior of the triangle. The length of each side is $l$.

The sides $(z_2, z_1)$, $(z_3, z_2)$, $(z_1, z_3)$ will be referred to as sides (1), (2), (3) respectively.
(iii) On each side we identify the positive direction $\hat{T}$ and the outward normal $\hat{N}$ as in Figure 1.1. The functions
\[
q^{(j)}(s), \quad q_N^{(j)}(s), \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right], \quad j = 1, 2, 3,
\]
will denote the function $q(x, y)$, as well as the derivative of $q(x, y)$ along the outward normal $\hat{N}$ respectively, for the side $(j)$.

(iv) $E(k)$ and $e(k)$ will denote the following exponential functions
\[
E(k) = \exp \left\{ \left( k + \frac{\lambda}{k} \right) \frac{l}{2\sqrt{3}} \right\}, \quad e(k) = \exp \left\{ \left( k + \frac{\lambda}{k} \right) \frac{l}{2} \right\}.
\]

(v) Using the fact that the numbers $\alpha$ and $\bar{\alpha}$ satisfy the obvious relations
\[
\alpha^2 = \bar{\alpha} = \alpha^{-1}, \quad 1 + \alpha + \bar{\alpha} = 0, \quad i\bar{\alpha} - i\alpha = \sqrt{3}, \quad i\alpha - i = \sqrt{3}\bar{\alpha},
\]
it is straightforward to obtain analogous relations for $E(k)$ and $e(k)$. For example, the last three equations in (1.5) imply
\[
E(k)E(\alpha k)E(\bar{\alpha}k) = 1, \quad E(i\bar{\alpha}k)E(-i\alpha k) = e(k), \quad E(i\alpha k)E(-ik) = e(\bar{k}).
\]

1.2 Formulation of the Problem

We will investigate the basic elliptic equations in the interior of the equilateral triangle $D$, namely we will study the equation
\[
q_{xx} + q_{yy} - 4\lambda q = 0, \quad (x, y) \in D,
\]
where $q(x, y)$ is a real valued function and $\lambda$ is a real constant. The case of $\lambda = 0$, of $\lambda$ negative, and of $\lambda$ positive, correspond to the Laplace, the Helmholtz, and the modified Helmholtz equations, respectively. We will analyze the following problems:
(i) The Dirichlet problem

\[ q^{(j)}(s) = f_j(s), \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right], \quad j = 1, 2, 3. \]  

(ii) The oblique Robin problem,

\[ \sin \beta q^{(j)}(s) + \cos \beta \frac{d}{ds} q^{(j)}(s) + \gamma q^{(j)}(s) = f_j(s), \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right], \quad j = 1, 2, 3, \]  

where \( \beta \) and \( \gamma \) are real constants and \( \sin \beta \neq 0 \). The sum of the first two terms of the lhs of this equation equals the derivative of \( q^{(j)}(s) \) in the direction making an angle \( \beta \) with the positive direction of the side \((j)\). The Neumann and the Robin problems correspond to the following particular choices of \( \beta \) and \( \gamma \),

\[ \text{Neumann: } \beta = \frac{\pi}{2}, \quad \gamma = 0; \quad \text{Robin: } \beta = \frac{\pi}{2}, \quad \gamma \neq 0. \]  

(iii) The Poincaré type problem

\[ \sin \beta q^{(j)}(s) + \cos \beta \frac{d}{ds} q^{(j)}(s) + \gamma q^{(j)}(s) = f_j(s), \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right], \quad j = 1, 2, 3, \]  

where \( \beta_1 \) is a real constant such that \( \sin \beta_1 \neq 0 \), \( \beta_2 \) and \( \beta_3 \) satisfy \( \sin \beta_2 \neq 0 \), \( \sin \beta_3 \neq 0 \) and are given in terms of \( \beta_1 \) by the expressions

\[ \beta_2 = \beta_1 + \frac{n\pi}{3}, \quad \beta_3 = \beta_1 + \frac{m\pi}{3}, \quad m, n \in \mathbb{Z}, \]  

and the real constants \( \{\gamma_j\}_1^3 \) satisfy the relations

\[ \sin 3\beta_1 \left[ \gamma_2(3\lambda - \gamma_2^2) - e^{im\pi} \gamma_1(3\lambda - \gamma_1^2) \right] = 0, \]  

\[ \sin 3\beta_1 \left[ \gamma_3(3\lambda - \gamma_3^2) - e^{im\pi} \gamma_1(3\lambda - \gamma_1^2) \right] = 0. \]

A particular case of such a Poincaré type problem, which is solved in detail, is the modified Helmholtz equation with Neumann values on sides (2) and (3) and with Robin values on side (1), where the constant \( \gamma \) is given by \( \sqrt{3\lambda} \).

We assume that the functions \( f_j \) have sufficient smoothness and that they are compatible at the corners of the triangle. The case of boundary conditions which are discontinuous at the corners will be considered elsewhere.

### 1.3 The Global Relation

As it was mentioned earlier the approach used here is based on the analysis of the global relation, which is the fundamental algebraic relation that couples the Dirichlet and the Neumann values around the perimeter of the triangle. This equation, first derived for the case of equilateral triangle in [14] (see also [5]) is

\[ E(-ik)\Psi_1(k) + E(-i\bar{a}k)\Psi_2(\bar{a}k) + E(-i\alpha k)\Psi_3(\alpha k) \]
where the exponential function $E(k)$ is defined in equation (1.4a), and $\Psi_j$ and $\Phi_j$ are the following transforms of the Neumann and Dirichlet boundary values:

$$\Psi_j(k) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \exp \left\{ (k + \frac{\lambda}{k})s \right\} q_N^{(j)}(s)ds,$$

$$\Phi_j(k) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \exp \left\{ (k + \frac{\lambda}{k})s \right\} \left[ \frac{1}{2} \frac{d}{ds} q^{(j)}(s) + \frac{\lambda}{k} q^{(j)}(s) \right] ds,$$

for each $j = 1, 2, 3$, and every complex $k \neq 0$.

The general methodology introduced in [4], [5] implies that the global relation must be supplemented by its Schwarz conjugate, as well as by the four equations obtained from these two equations by replacing $k$ with $\alpha k$ and with $\bar{\alpha} k$. We will refer to these six equations as the basic algebraic relations. In this paper we present two different techniques for solving these equations.

### 1.3.1 Solutions via Infinite Series

For simple problems it is possible to compute the unknown boundary values by evaluating the basic algebraic relations at particular discrete values of $k$. This yields the unknown boundary values in terms of infinite series. The Dirichlet and the Neumann problems are examples of problems which can be solved using this technique.

We use the Dirichlet problem to illustrate this approach. In this case the functions $\Phi_j$ appearing in the rhs of the global relation (1.13) can be immediately computed in terms of the given boundary conditions $f_j$, thus the global relation becomes a single equation for the three unknown functions $\{\Psi_j\}_{1}^{3}$. Multiplying this equation by $E(i\alpha k)$, and multiplying the Schwarz conjugate of the global relation by $E(-i\alpha k)$, we find the following two equations (where we have used the last two of the identities (1.6))

$$e(\alpha k)\Psi_1(k) + e(-k)\Psi_2(\alpha k) + \Psi_3(\alpha k) = 2iA(k),$$

$$e(-\alpha k)\Psi_1(k) + \Psi_2(\alpha k) + e(k)\Psi_3(\alpha k) = -2iB(k).$$

In these equations $A(k)$ and $B(k)$ are known functions and $k \in \mathbb{C} - \{0\}$.

For the general Dirichlet problem, we will supplement these two equations with the four equations obtained from these equations by replacing $k$ with $\alpha k$ and with $\bar{\alpha} k$. However, there exists a particular case for which it is sufficient to analyze only the above two equations. This is the symmetric Dirichlet problem, namely the problem where the functions $f_j$ are all the same, $f_j = f$, $j = 1, 2, 3$. Then the Neumann values $q_N^{(j)}(s)$ are also the same, $q_N^{(j)} = q_N$ and hence $\Psi_j(k) = \Psi(k)$, $j = 1, 2, 3$. Thus equations (1.15) and (1.16) become two equations for the three unknown functions $\Psi(k)$, $\Psi(\alpha k)$, $\Psi(\bar{\alpha} k)$. Hence, any two of them can be expressed in terms of the remaining one, for example $\Psi(\alpha k)$ and $\Psi(\bar{\alpha} k)$ can be expressed in terms of $\Psi(k)$. In particular, subtracting equations (1.15), (1.16), we find

$$(e(k) - e(-k)) \Psi(\bar{\alpha} k) = (e(\alpha k) - e(-\alpha k)) \Psi(k) - 2iG(k),$$

where $G(k) = A(k) + B(k)$ is a known function. Equation (1.17) is a single equation for the two unknown functions $\Psi(\alpha k)$ and $\Psi(k)$. However, by evaluating this equation at those
values of \( k \) for which the coefficient of \( \Psi(\alpha k) \) vanishes, i.e. at \( e^2(k) = 1 \), or \( k = s_n \),
\[
s_n + \frac{\lambda}{s_n} = \frac{2i\pi}{l}, \quad n \in \mathbb{Z}, \tag{1.18}
\]
it follows that \( \Psi(s_n) \) can be determined. Recalling the definition of \( \Psi(k) \) and evaluating equation (1.17) at \( k = s_n \), we find
\[
\sinh \left[ \left( \frac{\alpha s_n + \lambda}{\alpha s_n} \right) \frac{l}{2} \right] \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{i\pi n + k} q_N(s) ds = iG(s_n), \quad n \in \mathbb{Z}. \tag{1.19}
\]
Thus \( q_N(s) \) can be expressed as a Fourier series.

For the general Dirichlet problem (1.8), the six basic algebraic relations couple the nine unknown functions \( \{ \Psi_j(k), \Psi_j(\alpha k), \Psi_j(\alpha^2 k) \}_1^3 \). Thus any six of them can be expressed in terms of the remaining three. In particular, it is shown in section 3 that \( \Psi(\alpha k) \) can be expressed in terms of \( \{ \Psi_j(k) \}_1^3 \) by the equation
\[
(e^3(-k) - e^3(k)) \Psi_2(\alpha k) = [e(-\alpha k) - e^2(-k)e(\alpha k)] (\Psi_1(k) + e^2(k)\Psi_3(k))
\]
\[+ e^2(-k) [e(-\alpha k) - e^4(k)e(\alpha k)] \Psi_2(k) + 2iX(k), \tag{1.20}
\]
where \( X(k) \) is known. In spite of the fact that this equation is a single equation for four unknown functions, it yields all the three Neumann values \( q_N^{(j)}, \ j = 1, 2, 3 \). Indeed, by evaluating equation (1.20) at those values of \( k \) for which the coefficient of \( \Psi_2(\alpha k) \) vanishes, i.e. at \( e^6(k) = 1 \), or \( k = k_m \), where
\[
k_m + \frac{\lambda}{k_m} = \frac{2i\pi m}{3l}, \quad m \in \mathbb{Z}, \tag{1.21}
\]
equation (1.20) yields
\[
\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{2i\pi m s} \left[ q_N^{(1)}(s) + e^{-2i\pi m} q_N^{(2)}(s) + e^{2i\pi m} q_N^{(3)}(s) \right] ds = M(k_m), \quad m \in \mathbb{Z}, \tag{1.22}
\]
where \( M(k_m) \) is known. This equation in contrast to equation (1.19) involves three unknown functions. However, equation (1.21) gives three times as many values for \( m \) as equation (1.18). Replacing in equation (1.22) \( m \) by \( 3n, 3n - 1, 3n - 2 \), and inverting the left hand sides of the resulting equations, we find
\[
q_N^{(1)}(s) + q_N^{(2)}(s) + q_N^{(3)}(s) = \frac{1}{l} \sum_{-\infty}^{\infty} M(3n)e^{-2i\pi ns},
\]
\[
e^{-2i\pi s} \left[ q_N^{(1)}(s) + \alpha q_N^{(2)}(s) + \alpha^2 q_N^{(3)}(s) \right] = \frac{1}{l} \sum_{-\infty}^{\infty} M(3n - 1)e^{-2i\pi ns},
\]
\[
e^{-2i\pi s} \left[ q_N^{(1)}(s) + \alpha q_N^{(2)}(s) + \alpha^2 q_N^{(3)}(s) \right] = \frac{1}{l} \sum_{-\infty}^{\infty} M(3n - 2)e^{-2i\pi ns}. \tag{1.23}
\]

Thus by solving this system of three algebraic equations it follows that each one of the Neumann boundary values can be represented in terms of a Fourier series (see Proposition 3.2).

The analysis of the oblique Robin problem (equation (1.9)) is similar. However, the values of $s_n$ and of $k_m$ in general cannot be found explicitly. The values $k_m$ satisfy the transcendental equation

$$e^{(k_m + \frac{\lambda}{k_m})} \left( \alpha k_m e^{i\beta} + \frac{\lambda}{\alpha k_m e^{i\beta}} - \gamma \right) \left( \bar{\alpha} k_m e^{-i\beta} + \frac{\lambda}{\bar{\alpha} k_m e^{-i\beta}} - \gamma \right) = e^{\frac{2i\pi m}{3}}, \quad m \in \mathbb{Z}. \quad (1.24)$$

Thus in equation (1.22) instead of $\exp\left\{\frac{2i\pi m}{3}\right\}$ we now have $\exp\left\{k_m + \frac{\lambda}{k_m}\right\}$, where $k_m$ satisfies (1.24). In the particular case of the Neumann problem, $k_m$ satisfies equation (1.21).

1.3.2 Solutions Via Generalized Fourier Integrals

For more complicated problems, such as the problem (1.11), the basic algebraic relations can be solved in terms of a generalized Fourier integral. This technique in generic, in the sense that it can also be used for the solution of simple problems.

We use such a simple problem, namely the symmetric Dirichlet problem, to illustrate this approach: It is shown in Section 4 that the integral defining $\Psi(\bar{\alpha} k)$ can be solved for $q_N(s)$. For $\lambda \geq 0$, $q_N(s)$ is given by

$$q_N(s) = \frac{i\bar{\alpha}}{4\pi} \int_{\infty e^{\frac{4\pi i}{3}}}^{\infty e^{\frac{2\pi i}{3}}} \exp\left\{ - \left( \bar{\alpha} k + \frac{\lambda}{\bar{\alpha} k} \right) s \right\} \left( 1 - \frac{\lambda}{(\bar{\alpha} k)^2} \right) \Psi(\bar{\alpha} k) dk, \quad \lambda \geq 0. \quad (1.25)$$

Replacing in this equation $\Psi(\bar{\alpha} k)$ with the expression obtained by solving equation (1.17) for $\Psi(\bar{\alpha} k)$, it follows that $q_N(s)$ involves a known integral, as well as an integral containing the unknown function $\Psi(k)$. However, using the analyticity properties of the integrant of the latter integral, it can be shown that this integral can be computed in terms of residues. Furthermore these residues can be explicitly calculated in terms of the known function $G(k)$.

The situation for more complicated problems is similar: The unknown boundary values can be expressed in terms of known integrals, as well as integrals containing the three unknown functions $\{\Psi_j(k)\}_{j=1}^3$. Exploiting the analyticity properties of the integrants of the latter integrals, it can be shown that these integrals can be computed explicitly.

1.4 Integral Representations for $q(x, y)$

When both the Dirichlet and the Neumann boundary values are known, the solution $q(x, y)$ can be determined either using the classical integral representation in terms of Green’s functions [3], or using the novel integral representations constructed in [5] and [8]. For completeness, both representations are presented in Section 5.
1.5 Organization of the Paper

In Section 2 we derive the global relation (1.13). In Section 3 we solve the symmetric Dirichlet problem (Proposition 3.1), the general Dirichlet problem (Proposition 3.2), the general Neumann problem (Proposition 3.3), and we also discuss the oblique Robin problem. In Section 4 we discuss the basic algebraic relations associated with the Poincare boundary condition (1.11) and derive the relations (1.12b) and (1.12c). In Section 5 we obtain an alternative representation for the symmetric Dirichlet problem and then analyze the problem defined by equations (1.11) and (1.12). A particular case of this problem, which is solved in detail in Proposition 5.1, is a mixed boundary value problem for the modified Helmholtz equation. In Section 6 we discuss the associated integral representations for \( q(x, y) \). Further discussion of these results is presented in Section 7.

2 The Global Relation

Writing the basic elliptic equation (1.7) in the complex variables \((z, \bar{z})\) we find

\[
q_{zz} - \lambda q = 0.
\]  

It is straightforward to verify that this equation can be rewritten in the form [5]

\[
\left( \exp \left\{ -ikz - \frac{\lambda}{ik} \bar{z} \right\} q_z \right)_z + \frac{\lambda}{ik} \left( \exp \left\{ -ikz - \frac{\lambda}{ik} \bar{z} \right\} q \right)_z = 0, \tag{2.1}
\]

where for the rest of this section \( k \in \mathbb{C} - \{0\} \). Suppose that equation (2.1) is valid in a simply connected bounded domain \( \Omega \subset \mathbb{C} \) with a piecewise smooth boundary \( \partial \Omega \). Then equation (2.2) and the complex form of Green’s theorem imply

\[
\int_{\partial \Omega} \exp \left\{ -ikz - \frac{\lambda}{ik} \bar{z} \right\} \left( q_z dz - \frac{\lambda}{ik} q d\bar{z} \right) = 0. \tag{2.2}
\]

In the particular case that \( \Omega \) is the triangular domain \( D \), equation (2.3) becomes

\[
\sum_{j=1}^{3} \tilde{\rho}_j(k) = 0, \tag{2.3}
\]

where the function \( \tilde{\rho}_j(k) \) is given by the following line integral along the side \((j)\) of the equilateral triangle

\[
\tilde{\rho}_j(k) = \int_{z_{j+1}}^{z_j} \exp \left\{ -ikz - \frac{\lambda}{ik} \bar{z} \right\} \left( q_z dz - \frac{\lambda}{ik} q d\bar{z} \right), \quad j = 1, 2, 3. \tag{2.4}
\]

In what follows we will show that

\[
\tilde{\rho}_1(k) = \rho_1(k), \quad \tilde{\rho}_2(k) = \rho_2(\bar{\alpha}k), \quad \tilde{\rho}_3(k) = \rho_3(\alpha k), \tag{2.5}
\]
where the functions $\rho_j(k)$ are defined in terms of the functions $\Phi_j(k)$ and $\Psi_j(k)$ by the equation

$$
\rho_j(k) = E(-ik) \left[ \frac{i}{2} \Psi_j(k) + \Phi_j(k) \right], \quad j = 1, 2, 3.
$$

(2.7)

For this purpose we will use the following local parameterizations:

**Side 1:** On the side (1) the variable $z$ can be parameterized as

$$
z(s) = \frac{l}{2\sqrt{3}} + is, \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right].
$$

(2.8a)

Then $z(-l/2) = z_2$ and $z(l/2) = z_1$. Since the normal and the tangential derivatives are parallel to the $x$ and to $y$ axes respectively, it follows that

$$
\partial_z = \frac{1}{2} (\partial_x - i\partial_y) = \frac{1}{2} (\partial_N - i\partial_T).
$$

(2.8b)

**Side 2:** If $z$ varies along the side (2) and $\zeta$ varies along the side (1), then $z = \zeta \exp \left\{ -\frac{2\pi}{3} \right\}$. Thus

$$
z(s) = \left( \frac{l}{2\sqrt{3}} + is \right) e^{-i\frac{2\pi}{3}}, \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right].
$$

(2.9a)

Note again that $z(-l/2) = z_3$ and $z(l/2) = z_2$. The equation $\partial_z = \exp \left\{ i\frac{2\pi}{3} \right\} \partial_\zeta$ implies that

$$
\partial_z = \frac{\alpha}{2} (\partial_N - i\partial_T).
$$

(2.9b)

**Side 3:** In analogy with equations (2.9), if $z$ varies along side (3) we find the equations

$$
z(s) = \left( \frac{l}{2\sqrt{3}} + is \right) e^{i\frac{2\pi}{3}}, \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right],
$$

(2.10a)

and

$$
\partial_z = \frac{\bar{\alpha}}{2} (\partial_N - i\partial_T).
$$

(2.10b)

Finally, $z(-l/2) = z_1$ and $z(l/2) = z_3$. Using equations (2.8)-(2.10) in the expressions (2.5) we find equations (2.6), (2.7).

### 3 The Analysis of the Global Relation for Simple Boundary Value Problems

#### 3.1 The Symmetric Dirichlet Problem

We first give the details for the symmetric problem. In this case

$$
q^{(j)}_N(s) = q_N(s), \quad \Phi_j(k) = F(k), \quad \Psi_j(k) = \Psi(k), \quad j = 1, 2, 3,
$$

(3.1)
where the function $\Psi(k)$ is defined in terms of the unknown function $q_N(s)$ by equation (1.14a) (without the superscript ($j$)), and the function $F(k)$ is defined in terms of the given boundary condition $f(s)$ by equation (1.14b), i.e., by the equation

$$F(k) = \int_{-l/2}^{l/2} \exp \left\{ \left( k + \frac{\lambda}{k} \right) s \right\} \left[ \frac{1}{2} f'(s) + \frac{\lambda}{k} f(s) \right] ds, \quad k \in \mathbb{C} - \{0\}. \quad (3.2)$$

Using equations (3.1), the global relation (1.13) and its Schwarz conjugate yield (1.15) and (1.16), with

$$A(k) = e(\bar{\alpha}k)F(k) + e(-k)F(\bar{\alpha}k) + F(\alpha k),$$

$$B(k) = e(-\bar{\alpha}k)F(k) + F(\alpha k) + e(k)F(\bar{\alpha}k).$$

Hence, since $G = A + B$,

$$G(s_n) = 2 \cosh \left[ \left( \bar{\alpha} s_n + \frac{\lambda}{\bar{\alpha} s_n} \right) l/2 \right] F(s_n) + 2e^{in\pi} F(\bar{\alpha} s_n) + 2F(\alpha s_n). \quad (3.3)$$

In summary, we have derived the following result:

**Proposition 3.1** Let the real valued function $q(x, y)$ satisfy equation (1.7) in the triangular domain $D$, with the Dirichlet conditions (1.8), where

$$f_j(s) = f(s), \quad j = 1, 2, 3, \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right], \quad (3.4)$$

and the function $f(s)$ is sufficiently smooth and satisfies the continuity condition $f(-l/2) = f(l/2)$. Then the Neumann boundary values are the same, $q_N^{(j)}(s) = q_N(s)$, $j = 1, 2, 3$, and are given by the Fourier series

$$q_N(s) = \frac{i}{l} \sum_{-\infty}^{\infty} e^{-2\pi j s} \frac{G(s_n)}{\sinh \left[ \left( \bar{\alpha} s_n + \frac{\lambda}{\bar{\alpha} s_n} \right) l/2 \right]}, \quad (3.5)$$

where $s_n$ is defined by equation (1.18) and $G(s_n)$ is given in terms of $f(s)$ by equations (3.2) and (3.3).

### 3.2 The General Dirichlet Problem

The global relation and its Schwarz conjugate yield equations (1.15) and (1.16), where the known functions $A(k)$ and $B(k)$ are now given by the equations

$$A(k) = e(\bar{\alpha}k)F_1(k) + e(-k)F_2(\bar{\alpha}k) + F_3(\alpha k),$$

$$B(k) = e(-\bar{\alpha}k)F_1(k) + F_2(\alpha k) + e(k)F_3(\bar{\alpha}k), \quad (3.6)$$

where

$$F_j(k) = \int_{-l/2}^{l/2} \exp \left\{ \left( k + \frac{\lambda}{k} \right) s \right\} \left[ \frac{1}{2} f_j'(s) + \frac{\lambda}{k} f_j(s) \right] ds, \quad j = 1, 2, 3, \quad k \in \mathbb{C} - \{0\}. \quad (3.7)$$
Replacing in equations (1.15) and (1.16) \( k \) by \( \tilde{\alpha} k \) and then eliminating \( \Psi_1(\tilde{\alpha} k) \) from the resulting two equations we find
\[
e(-\alpha k)\Psi_3(k) + e(k)\Psi_2(\alpha k) - 2ie(-\alpha k)A(\alpha k)
= e(\alpha k)\Psi_2(k) + e(-k)\Psi_3(\alpha k) + 2ie(\alpha k)B(\alpha k).
\] (3.8)

Taking the Schwarz conjugate of this equation (or equivalently eliminating \( \Psi_1(\alpha k) \) from the equations obtained from equations (1.15) and (1.16) by replacing \( k \) with \( \alpha k \) we find an equation involving \( \Psi_2(\alpha k) \) from equation (1.15) and from equation (1.15) we find equation (1.20) with \( \bar{X} \) can be expressed in terms of the given Dirichlet data by the Fourier series
\[
e(-\tilde{\alpha} k)\Psi_3(k) + e(k)\Psi_2(\tilde{\alpha} k) + 2ie(-\tilde{\alpha} k)A(\tilde{\alpha} k)
= e(\tilde{\alpha} k)\Psi_2(k) + e(-k)\Psi_3(\tilde{\alpha} k) - 2ie(\tilde{\alpha} k)B(\tilde{\alpha} k).
\] (3.9)

Substituting \( \Psi_2(\alpha k) \) from equation (3.8) and \( \Psi_3(\tilde{\alpha} k) \) from equation (3.9) into equation (1.16), we find an equation involving \( \Psi_3(\alpha k), \Psi_2(\tilde{\alpha} k), \) and \( \{\Psi_j(k)\}_1^3 \). Eliminating \( \Psi_3(\alpha k) \) from this equation and from equation (1.15) we find equation (1.20) with \( X(k) \) given by the following equation,
\[
X(k) = [e^2(-k)e(\tilde{\alpha} k) + e(-\tilde{\alpha} k)] [F_1(k) + e^2(k)F_3(k)]
+ e^2(-k) [e^2(-k)e(\tilde{\alpha} k)e^6(k) + e(-\tilde{\alpha} k)] F_2(k)
+ 2e^2(k)F_1(\alpha k) + 2F_2(\alpha k) + 2e^2(-k)F_3(\alpha k)
+ e^3(-k) [2e^2(k)F_1(\tilde{\alpha} k) + (e^6(k) + 1)F_2(\tilde{\alpha} k) + 2e^2(-k)e^6(k)F_3(\tilde{\alpha} k)]
\] (3.10a)

Letting \( k = k_n \) we find
\[
X(k_n) = [e^2(-k_n)e(\tilde{\alpha} k_n) + e(-\tilde{\alpha} k_n)] \times
[F_1(k_n) + e^2(-k_n)F_2(k_n) + e^2(k_n)F_3(k_n)]
+ 2e^2(k_n) [F_1(\alpha k_n) + e^2(-k_n)F_2(\alpha k_n) + e^2(k_n)F_3(\alpha k_n)]
+ 2e(-k_n) [F_1(\tilde{\alpha} k_n) + e^2(-k_n)F_2(\tilde{\alpha} k_n) + e^2(k_n)F_3(\tilde{\alpha} k_n)]
\] (3.10b)

Solving the algebraic equations (1.23) we find the following result:

**Proposition 3.2** Let the real valued function \( q(x, y) \) satisfy equation (1.7) in the triangular domain \( D \), with the boundary conditions (1.8), where the given functions \( f_j(s) \) have sufficient smoothness and are continuous at the vertices. Then the Neumann data \( q_N^{(j)}(s), j = 1, 2, 3 \) can be expressed in terms of the given Dirichlet data by the Fourier series
\[
q_N^{(j)}(s) = \frac{1}{3l} \sum_{n=-\infty}^{\infty} \left[ M(k_{3n}) + c_1^{(j)} e^{\frac{2i\pi s}{a}} M(k_{3n-1}) + c_2^{(j)} e^{\frac{4i\pi s}{a}} M(k_{3n-2}) \right] e^{-\frac{2i\pi sn}{a}},
\] (3.11)
where \( k_m \) is defined by equation (1.21),
\[
\begin{align*}
  c_1^{(1)} &= c_2^{(1)} = 1, & c_1^{(2)} &= c_2^{(3)} = \bar{\alpha}, & c_1^{(3)} &= c_2^{(2)} = \alpha,
\end{align*}
\]
(3.12)
\[
M(k_m) = \frac{2iX(k_m)}{\bar{\alpha}e(\bar{\alpha}k_m) - e(-\bar{\alpha}k_m)}
\]
(3.13)
and \( X(k_m) \) is defined in terms of \( f_j(s) \) by equations (3.7) and (3.10).

### 3.3 The Oblique Robin Problem

Suppose that \( q(x, y) \) satisfies the Poincaré boundary condition (1.11), i.e.
\[
q^{(j)}_N(s) = \frac{1}{\sin \beta_j} \left( f^{(j)} - \cos \beta_j \frac{dq^{(j)}}{ds} - \gamma_j q^{(j)} \right).
\]
(3.14)
Substituting this expression in the definition of \( \rho_j(k) \), i.e. in the equation (2.7), we obtain
\[
\rho_j(k) = E(-ik) \int_{\frac{l}{2}}^{\frac{l}{2}} \exp \left\{ \left( k + \frac{\lambda}{k} \right) s \right\} \left[ \frac{i}{2} q^{(j)}_N(s) + \frac{1}{2} \frac{d}{ds} q^{(j)}(s) + \frac{\lambda}{k} q^{(j)}(s) \right] ds.
\]
Integrating by parts we find the following expression for \( \rho_j(k) \):
\[
\rho_j(k) = iE(-ik) \left[ H_j(k) Y_j(k) + F_j(k) + C_j(k) \right], \quad j = 1, 2, 3,
\]
(3.15)
where the function \( H_j(k) \) is defined by
\[
H_j(k) = ke^{i\beta_j} + \frac{\lambda}{ke^{i\beta_j}} - \gamma_j,
\]
(3.16)
the function \( F_j(k) \) is defined in terms of the given boundary conditions \( f_j(s) \) by the equation
\[
F_j(k) = \frac{1}{2\sin \beta_j} \int_{\frac{l}{2}}^{\frac{l}{2}} \exp \left\{ \left( k + \frac{\lambda}{k} \right) s \right\} f_j(s) ds,
\]
(3.17)
the function \( C_j(k) \) involves the values of \( q(x, y) \) at the vertices,
\[
C_j(k) = \frac{e^{i\beta_j}}{2\sin \beta_j} \left[ e(-k)q^{(j)} \left( -\frac{l}{2} \right) - e(k)q^{(j)} \left( \frac{l}{2} \right) \right],
\]
(3.18)
and the function \( Y_j(k) \) involves the unknown Dirichlet boundary values,
\[
Y_j(k) = \frac{1}{2\sin \beta_j} \int_{\frac{l}{2}}^{\frac{l}{2}} \exp \left\{ \left( k + \frac{\lambda}{k} \right) s \right\} q^{(j)}(s) ds.
\]
(3.19)
In equations (3.15)-(3.19), \( k \) is complex and \( k \neq 0 \).
In the particular case of the oblique Robin problem (1.9), $\beta_j = \beta$ and $\gamma_j = \gamma$, $j = 1, 2, 3$, thus $H_j(k) = H(k)$, where $H(k)$ is defined by equation (3.16) without the subscript ($j$). Substituting the expression for $\rho_j(k)$ (with $H_j = H$) in the global relation (2.4) we find

$$E(-ik) [H(k)Y_1(k) + F_1(k) + C_1(k)] + E(-i\bar{\alpha}k) [H(\bar{\alpha}k)Y_2(\bar{\alpha}k) + F_2(\bar{\alpha}k) + C_2(\bar{\alpha}k)]
$$

$$+ E(-iak) [H(\alpha k)Y_3(\alpha k) + F_3(\alpha k) + C_3(\alpha k)] = 0. \quad (3.20)$$

The contribution from the corner terms $C_j$ cancels. Indeed, this contribution is proportional to the following expression

$$E(-ik) \left[ e(-k)q^{(1)} \left( -\frac{l}{2} \right) - e(k)q^{(1)} \left( \frac{l}{2} \right) \right] + E(-i\bar{\alpha}k) \left[ e(-\bar{\alpha}k)q^{(2)} \left( -\frac{l}{2} \right) - e(\bar{\alpha}k)q^{(2)} \left( \frac{l}{2} \right) \right]
$$

$$+ E(-iak) \left[ e(-\alpha k)q^{(3)} \left( -\frac{l}{2} \right) - e(\alpha k)q^{(3)} \left( \frac{l}{2} \right) \right]. \quad (3.21)$$

But, the assumption of continuity at the vertices implies

$$q^{(1)} \left( -\frac{l}{2} \right) = q^{(2)} \left( \frac{l}{2} \right), \quad q^{(2)} \left( -\frac{l}{2} \right) = q^{(3)} \left( \frac{l}{2} \right), \quad q^{(3)} \left( -\frac{l}{2} \right) = q^{(1)} \left( \frac{l}{2} \right). \quad (3.22)$$

Hence the terms $q^{(1)}(-l/2)$ and $q^{(2)}(l/2)$ in the expression (3.21) cancel iff

$$E(-ik)e(-k) = E(-i\bar{\alpha}k)e(\bar{\alpha}k). \quad (3.23)$$

This equation is indeed valid, and it is the consequence of the identity

$$\frac{1}{\sqrt{3}} \left( -ik + \frac{\lambda}{-ik} \right) + \left( -k + \frac{\lambda}{-k} \right) = \frac{1}{\sqrt{3}} \left( -i\bar{\alpha}k + \frac{\lambda}{-i\bar{\alpha}k} \right) + \left( \bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k} \right).$$

Using the fact that the corners term cancel, the global relation and its Schwarz conjugate yield (compare with equations (1.15) and (1.16)) the following equations:

$$e(\bar{\alpha}k)H(k)Y_1(k) + e(-k)H(\bar{\alpha}k)Y_2(\bar{\alpha}k) + H(\alpha k)Y_3(\alpha k) = -A(k),$$

$$e(-\bar{\alpha}k)\overline{H(k)}Y_1(k) + \overline{H(\bar{\alpha}k)}Y_2(\bar{\alpha}k) + e(\alpha k)\overline{H(\alpha k)}Y_3(\alpha k) = -B(k), \quad (3.24)$$

where $A(k)$ and $B(k)$ are defined by equations (3.6) in terms of $F_j$.

Let

$$P(k) = \frac{H(k)}{\overline{H(k)}}. \quad (3.25)$$

Following precisely the same steps used for the general Dirichlet problem we find the following expression for $Y_2(\bar{\alpha}k)$ in terms of $\{Y_j(k)\}_1^3$:

$$\left[ e^3(-k) \frac{P^2(\bar{\alpha}k)}{P^2(\alpha k)} - e^3(k) \frac{P(\alpha k)}{P(\bar{\alpha}k)} \right] \frac{H(\alpha k)}{H(k)} Y_2(\bar{\alpha}k) = T(k)
$$

$$+ \left[ e(-\bar{\alpha}k) - e^2(-k) \frac{P(\bar{\alpha}k)P(k)}{P^2(\alpha k)} e(\bar{\alpha}k) \right] Y_1(k) + e^2(k) \frac{P(\alpha k)}{P(\bar{\alpha}k)} Y_3(k)$$

13
Proposition 3.3

Let the real valued function \( q(x, y) \) satisfy (1.7) in the triangular domain \( D \), with the Neumann boundary conditions

\[
q_N^{(j)}(s) = f_j(s), \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right], \quad j = 1, 2, 3
\]  

(3.30)
where the functions $f_j(s)$ have sufficient smoothness and are continuous at the vertices of the triangle. Then the Dirichlet data $q^{(j)}(s)$, $j = 1, 2, 3$ can be expressed in terms of the given Neumann data by the Fourier series

$$q^{(j)}(s) = \frac{1}{3l} \sum_{n=-\infty}^{\infty} \left[ N(k_{3n}) + c_1^{(j)} e^{\frac{2i\pi s}{3l}} N(k_{3n-1}) + c_2^{(j)} e^{\frac{4i\pi s}{3l}} N(k_{3n-2}) \right] e^{-\frac{2i\pi ns}{l}}$$  \hspace{1cm} (3.31)

where $c_i^{(j)}$ are given by (3.12) and

$$N(k_m) = \frac{2T_N(k_m)}{\tilde{\alpha}^m e(\tilde{\alpha} k_m) - e(-\tilde{\alpha} k_m)}. \hspace{1cm} (3.32)$$

The known function $T_N(k_m)$ is defined by the equation

$$-\left(\frac{ik + \lambda}{ik}\right) T_N(k) = e(-k) \left[ E^3(-i\alpha k) + E^3(i\alpha k) \right] F_1(k) + \left[ E^3(-i\tilde{\alpha} k) + E^3(i\tilde{\alpha} k) \right] F_2(k) + e(k) \left[ E^3(-i\alpha k) + E^3(i\alpha k) \right] F_3(k) + 2e^2(k) F_1(\alpha k) + 2F_2(\alpha k) + 2e^2(-k) F_3(\alpha k) + 2e(-k) F_1(\tilde{\alpha} k) + (e^3(k) + e^3(-k)) F_2(\tilde{\alpha} k) + 2e(k) F_3(\tilde{\alpha} k)$$

where $F_j(k)$ is given by (3.17) with $\sin \beta_j = 1$.

### 4 Poincaré Type Boundary Value Problems

Suppose that $q(x, y)$ satisfies the Poincaré type boundary condition (1.1). Then substituting the expression $\rho_j(k)$ from equation (3.15) into the global relation (2.4), we find an equation similar with equation (3.20), where $H(k)$, $H(\tilde{\alpha} k)$ and $H(\alpha k)$ are replaced by $H_1(k)$, $H_2(\tilde{\alpha} k)$ and $H_3(\alpha k)$, and $F_j$, $C_j$, $Y_j$ are defined by equations (3.17)-(3.19). Proceeding as in Section 3.3, in analogy with equation (3.26), we now find

$$D(k) H_2(\tilde{\alpha} k) Y_2(\tilde{\alpha} k) = \sum_{j=1}^{3} \Gamma_j(k) H_j(k) Y_j(k) + T(k) + C(k), \hspace{1cm} (4.1)$$

where $T(k)$ is defined in terms of the known functions $f_j(s)$, $C(k)$ involves the values of $q$ at the corners, and $D(k)$, $\{\Gamma_j(k)\}^3_1$ are defined by the following equations:

$$D(k) = \frac{P_1(\tilde{\alpha} k)}{P_2(\alpha k) P_3(\alpha k)} \left[ e^3(-k) - e^3(\alpha k) \frac{P_1(\alpha k) P_2(\alpha k) P_3(\alpha k)}{P_1(\tilde{\alpha} k) P_2(\tilde{\alpha} k) P_3(\tilde{\alpha} k)} \right], \hspace{1cm} (4.2)$$

$$\Gamma_1(k) = \frac{1}{P_1(k)} \left[ e(-\tilde{\alpha} k) - e^2(-k) e(\tilde{\alpha} k) \frac{P_1(k) P_1(\tilde{\alpha} k)}{P_2(\alpha k) P_3(\alpha k)} \right], \hspace{1cm} (4.3a)$$

$$\Gamma_2(k) = e^2(-k) \frac{P_1(\tilde{\alpha} k)}{P_2(k) P_2(\alpha k)} \left[ e(-\tilde{\alpha} k) - e^4(k) e(\tilde{\alpha} k) \frac{P_2(k) P_2(\alpha k)}{P_1(\tilde{\alpha} k) P_3(\alpha k)} \right], \hspace{1cm} (4.3b)$$

$$\Gamma_3(k) = e^2(k) \frac{P_1(\alpha k)}{P_3(k) P_3(\alpha k)} \left[ e(-\tilde{\alpha} k) - e^2(-k) e(\tilde{\alpha} k) \frac{P_3(k) P_3(\alpha k)}{P_1(\alpha k) P_2(\alpha k)} \right], \hspace{1cm} (4.3c)$$

15
with

\[ P_j(k) = \frac{H_j(k)}{H_j(k)}. \] (4.4)

In order to be able to solve this problem using a generalized Fourier integral we require that when \( D(k) \) vanishes, then \( \Gamma_2(k) \) and \( \Gamma_3(k) \) are proportional to \( \Gamma_1(k) \). Actually, \( \Gamma_3(k) \) is proportional to \( \Gamma_1(k) \) for all complex \( k \) provided that

\[ P_1(k)P_1(\alpha k)P_1(\tilde{\beta}k) = P_3(k)P_3(\alpha k)P_3(\tilde{\beta}k). \] (4.5a)

Equating the brackets appearing in the definitions of \( \Gamma_1(k) \) and \( \Gamma_2(k) \), and replacing in the resulting expression \( e^3(k) \) by

\[ \frac{P_1(\tilde{\beta}k)P_2(\alpha k)P_3(\tilde{\beta}k)}{P_1(\alpha k)P_2(\alpha k)P_3(\alpha k)}, \]

it follows that \( \Gamma_2(k) \) is proportional to \( \Gamma_1(k) \) provided that

\[ P_1(k)P_1(\alpha k)P_1(\tilde{\beta}k) = P_2(k)P_2(\alpha k)P_2(\tilde{\beta}k). \] (4.5b)

Equation (4.5b) is valid if the following two equations are valid:

\[ \sin 3(\beta_1 - \beta_2) = 0, \quad \gamma_2(3\lambda - \gamma_2^3) \sin 3\beta_1 - \gamma_1(3\lambda - \gamma_1^3) \sin 3\beta_2 = 0. \] (4.6)

Indeed, in order to simplify equation (4.5b) we first compute the product

\[ H_1(k)H_1(\alpha k)H_1(\tilde{\beta}k) = k^3e^{3i\beta_1} + \frac{\lambda^3}{k^3e^{3i\beta_1}} + 3\lambda\gamma_1 - \gamma_1^3. \] (4.7)

The function \( H_1(k) \) can be obtained from \( H_1(k) \) by replacing \( \beta_1 \) with \( -\beta_1 \), thus \( H_1(k)H_1(\tilde{\beta}k) \)

\[ H_1(\alpha k) \]

is given by an expression similar to (4.7) with \( \beta_1 \) replacing by \( -\beta_1 \). Hence equation (4.5b) yields

\[ \frac{k^3e^{3i\beta_1} + \frac{\lambda^3}{k^3e^{3i\beta_1}} + 3\lambda\gamma_1 - \gamma_1^3}{k^3e^{-3i\beta_1} + \frac{\lambda^3}{k^3e^{3i\beta_1}} + 3\lambda\gamma_1 - \gamma_1^3} = \frac{k^3e^{3i\beta_2} + \frac{\lambda^3}{k^3e^{3i\beta_2}} + 3\lambda\gamma_2 - \gamma_2^3}{k^3e^{-3i\beta_2} + \frac{\lambda^3}{k^3e^{3i\beta_2}} + 3\lambda\gamma_2 - \gamma_2^3}. \]

This equation simplifies to the equation

\[ \left( k^6 - \frac{\lambda^6}{k^6} \right) \sin 3(\beta_1 - \beta_2) + \left( k^6 + \frac{\lambda^6}{k^6} \right) \left[ (3\lambda\gamma_2 - \gamma_2^3) \sin 3\beta_1 - (3\lambda\gamma_1 - \gamma_1^3) \sin 3\beta_2 \right] = 0, \]

which is valid for all \( k \) iff equations (4.6) are valid.

Equation (4.6a) implies \( \beta_2 = \beta_1 + n\pi/3 \), then \( \sin 3\beta_2 = \sin 3\beta_1 \exp\{in\pi\} \) and we find equation (1.12b). Similarly equation (4.5a) yields equation (1.12c).

**The Case that the Corner Terms Cancel**

The definition of the corner terms \( C_j(k) \), i.e. equation (3.18), shows that \( C_j(k) \) involves \( \exp\{i\beta_j\}/\sin \beta_j \). Thus the contribution of the corner terms in the global relation (3.20) vanishes iff

\[ e^{2i\beta_1} = e^{2i\beta_2} = e^{2i\beta_3}. \] (4.8)
Example 1.

\[ \beta_1 = \beta_2 = \beta_3 = \frac{2\pi}{3}, \quad \gamma_j \text{ arbitrary.} \] \hspace{1cm} (4.9)

In this case

\[ H_j(k) = k\alpha + \frac{\lambda}{k\alpha} - \gamma_j, \quad P_j(k) = \frac{k\alpha + \frac{\lambda}{k\alpha} - \gamma_j}{k\alpha + \frac{\lambda}{k\alpha} - \gamma_j}. \] \hspace{1cm} (4.10)

Example 2.

\[ \beta_1 = \beta_2 = \beta_3 = \beta, \quad \beta \text{ arbitrary,} \quad \gamma_2 = \gamma_3 = 0, \quad \gamma_1 = (3\lambda)^{1/2}, \quad \lambda > 0. \] \hspace{1cm} (4.11)

In this case

\[ H_1(k) = ke^{i\beta} + \frac{\lambda}{ke^{i\beta}} - \gamma_1, \quad H_2(k) = H_3(k) = ke^{i\beta} + \frac{\lambda}{ke^{i\beta}}. \] \hspace{1cm} (4.12)

In particular if \( \beta = \pi/2 \), then

\[ H_1(k) = i \left( k - \frac{\lambda}{k} \right) - \gamma_1, \quad H_2(k) = H_3(k) = i \left( k - \frac{\lambda}{k} \right). \] \hspace{1cm} (4.13)

Thus

\[ P_1(k) = \frac{i \left( k - \frac{\lambda}{k} \right) - \gamma_1}{-i \left( k - \frac{\lambda}{k} \right) - \gamma_1}, \quad P_2(k) = P_3(k) = -1. \] \hspace{1cm} (4.14)

Hence

\[ D(k) = P_1(\bar{\alpha}k) \left[ e^3(-k) - e^3(k) \frac{P_1(\alpha k)}{P_1(\bar{\alpha} k)} \right], \] \hspace{1cm} (4.15a)

\[ \Gamma_1(k) = \frac{1}{P_1(k)} \left[ e(-\bar{\alpha}k) - e\bar{2}(-k) e(\bar{\alpha}k) P_1(k) P_1(\bar{\alpha} k) \right], \] \hspace{1cm} (4.15b)

\[ \Gamma_2(k) = e\bar{2}(-k) P_1(\bar{\alpha} k) \left[ e(-\bar{\alpha}k) + \frac{e^4(k) e(\bar{\alpha} k)}{P_1(\bar{\alpha} k)} \right] \] \hspace{1cm} (4.15c)

\[ \Gamma_3(k) = e\bar{2}(k) P_1(\alpha k) \left[ e\bar{2}(-k) e(\bar{\alpha} k) \right] / P_1(\alpha k). \] \hspace{1cm} (4.15d)

If \( P_1(k) \) is defined by (4.14) with \( \gamma_j^2 = 3\lambda \), it can be verified that

\[ P_1(k) P_1(\bar{\alpha} k) = -\frac{1}{P_1(\alpha k)}. \]

Hence

\[ \Gamma_3(k) = -\frac{e\bar{2}(k) P_1(\bar{\alpha} k)}{P_1(\bar{\alpha} k)} \Gamma_1(k), \quad \Gamma_2(k)|_{D(k)=0} = -\frac{e\bar{2}(-k) \Gamma_1(k)}{P_1(\alpha k)}. \] \hspace{1cm} (4.16)

The Laplace Equation

In the particular case of the Laplace equation with \( \gamma_j = 0 \), it follows that \( P_j = e^{2i\bar{\beta}} \), i.e. \( P_j \) is independent of \( k \).
5 The Analysis of the Global Relation Via Fourier Integrals

In this section we restrict $\lambda$ to be non-negative. Slightly more complicated formulae can be derived for $\lambda < 0$.

We first derive equation (1.25). Letting $k = |k|e^{it/6}$, the definition of $\Psi(\tilde{\alpha}k)$ yields

$$\Psi(-i|k|) = \frac{1}{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \exp \left\{ \left( -i|k| + \frac{\lambda}{-i|k|} \right) s \right\} q_N(s) ds. \quad (5.1)$$

Suppose that $\lambda > 0$. Letting $t(\lvert k \rvert) = \lvert k \rvert - \lambda/\lvert k \rvert$, it follows that if $\lvert k \rvert \in (0, +\infty)$, then $t \in (-\infty, \infty)$. Thus inverting equation (5.1) we find

$$q_N(s) = \frac{1}{2\pi} \int_0^\infty e^{its} \Psi(-i|k|) dt, \quad s \in \left[ -\frac{l}{2}, \frac{l}{2} \right], \quad q_N(s) = 0 \text{ elsewhere},$$

where $|k|$ (in the argument of $\Psi$) is a function of $t$. Rewriting $t$ in terms of $|k|$ we find

$$q_N(s) = \frac{1}{2\pi} \int_0^\infty \exp \left\{ -\left( \tilde{\alpha}k + \frac{\lambda}{\tilde{\alpha}k} \right) s \right\} \Psi(\tilde{\alpha}k) \left( ke^{-it/6} + \frac{\lambda}{ke^{-it/6}} \right) \frac{dk}{k}. \quad (5.2)$$

The rhs of this equation equals

$$\frac{1}{2\pi} \int_0^\infty e^{-it\xi} \exp \left\{ -\left( \tilde{\alpha}k + \frac{\lambda}{\tilde{\alpha}k} \right) s \right\} \Psi(\tilde{\alpha}k) \left( \xi e^{-it/6} + \frac{\lambda}{\xi e^{-it/6}} \right) \frac{d\xi}{\xi}. \quad (5.3)$$

Indeed, for the derivation of (5.3) we first observe that the function $\Psi(\tilde{\alpha}k)$ remains invariant under the transformation $k \rightarrow \tilde{\alpha}\lambda/k$. Thus making the change of variables $k = \tilde{\alpha}\lambda/\xi$ in the rhs of equation (5.2), and using

$$ke^{-it/6} + \frac{\lambda}{ke^{-it/6}} \rightarrow -\left( \xi e^{-it/6} + \frac{\lambda}{\xi e^{-it/6}} \right), \quad \frac{dk}{k} = -\frac{d\xi}{\xi},$$

we find the expression (5.3). Combining (5.2) and (5.3) we obtain

$$q_N(s) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-is\xi} \exp \left\{ -\left( \tilde{\alpha}k + \frac{\lambda}{\tilde{\alpha}k} \right) s \right\} \Psi(\tilde{\alpha}k) \left( ke^{-it/6} + \frac{\lambda}{ke^{-it/6}} \right) \frac{dk}{k}. \quad (5.4)$$

Using $e^{-it/6} = i\tilde{\alpha}$, this equations becomes equation (1.25).

If $\lambda = 0$, we set $k = t \exp \{i\tilde{\alpha}x/6\}$, $t \in \mathbb{R}$, and rewrite (1.14a) as

$$\Psi(it) = \int_{-l/2}^{l/2} e^{its} q_N(s) ds \quad (5.5)$$
which is inverted to
\[ q_N(s) = \frac{l}{2\pi} \int_{-\infty}^{+\infty} e^{-its}\Psi(it)dt. \] (5.6)

Replacing in (5.6) \( \{it\} \) with \( \{\bar{\alpha}k\} \), we arrive at
\[ q_N(s) = \frac{i\bar{\alpha}}{2\pi} \int_{\infty}^{\infty} e^{-\bar{\alpha}ks}\Psi(\bar{\alpha}k)dk. \]

This equation, in comparison to (1.25) misses a factor of 1/2; this is due to the linearity of the relevant transformation in this case.

### 5.1 The Symmetric Dirichlet Problem

Solving equation (1.17) for \( \Psi(\bar{\alpha}k) \) and substituting the resulting expression in equation (1.25) we find
\[ q_N(s) = \frac{i\bar{\alpha}}{4\pi} \int_{\infty}^{\infty} \exp \left\{ -\left( \bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k} \right) s \right\} \left\{ (e(\bar{\alpha}k) - e(-\bar{\alpha}k))\Psi(k) - 2iG(k) \right\} \left( 1 - \frac{\lambda}{(\bar{\alpha}k)^2} \right) dk, \] (5.7)

where \( \Delta(k) \) denotes the coefficient of \( \Psi(\bar{\alpha}k) \) in equation (1.17), i.e.
\[ \Delta(k) = e(k) - e(-k). \]

The line \( (\infty e^{\frac{2\pi i}{6}}, \infty e^{\frac{7\pi i}{6}}) \) splits the complex \( k \)-plane into the two half planes,
\[ \mathcal{D}^+ = \left\{ k \in \mathbb{C}, \quad \frac{\pi}{6} < \arg k < \frac{7\pi}{6} \right\}, \]
\[ \mathcal{D}^- = \left\{ k \in \mathbb{C}, \quad \frac{7\pi}{6} < \arg k < \frac{13\pi}{6} \right\}. \]

We observe that
\[ \exp \left\{ -\left( \bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k} \right) s \right\} e(\bar{\alpha}k) \text{ is bounded for } k \in \mathcal{D}^-, \]
\[ \exp \left\{ -\left( \bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k} \right) s \right\} e(-\bar{\alpha}k) \text{ is bounded for } k \in \mathcal{D}^+. \] (5.8)

Indeed the exponential of (5.8a) involves \( \bar{\alpha}k \left( \frac{l}{2} - s \right) \), and since \( l/2 - s \geq 0 \), the exponential in (5.8a) is bounded for \( \text{Re}(\bar{\alpha}k) \leq 0 \), i.e. in \( \mathcal{D}^- \). Similarly, the exponential of (5.8b) involves \( -\bar{\alpha}k \left( \frac{l}{2} + s \right) \), which since \( l/2 + s \geq 0 \), is bounded for \( \text{Re}(\bar{\alpha}k) \geq 0 \), i.e. in \( \mathcal{D}^+ \).

We also note that \( \Psi(k)/\Delta(k) \) is bounded for all \( k \in \mathbb{C}, k \neq s_n \). Indeed, for \( \text{Re } k > 0 \), \( \Delta(k) \) is dominated by \( e(k) \), while for \( \text{Re } k < 0 \), \( \Delta(k) \) is dominated by \( e(-k) \), hence
\[ \frac{\Psi(k)}{\Delta(k)} \sim \begin{cases} \Psi(k)e(-k), & \text{Re } k > 0 \\ -\Psi(k)e(k), & \text{Re } k < 0. \end{cases} \] (5.9)
Furthermore $\Psi(k)e(-k)$ involves $k(s-1/2)$ which is bounded for $\Re k \geq 0$, while $\Psi(k)e(k)$ involves $k(s+1/2)$ which is bounded for $\Re k \leq 0$ (recall that $-\frac{1}{2} \leq s \leq \frac{1}{2}$).

The above considerations imply that the parts of the integral (5.7) containing $e(\bar{\alpha}k)\Psi(k)$ and $e(-\bar{\alpha}k)\Psi(k)$ can be computed by using Cauchy’s theorem in $D^+$ and $D^-$ respectively. The associated residues can be computed as follows: Let $s^+_n$ and $s^-_n$ denote the subsets of $s_n$ in $D^+$ and $D^-$, respectively. Evaluating equation (1.17) at $k = s^+_n$ we find

$$e(\bar{\alpha}s^-_n)\Psi(s^-_n) = \frac{2iG(s^-_n)}{-e^2(-\bar{\alpha}s^-_n) + 1}, \quad -e(-\bar{\alpha}s^+_n)\Psi(s^+_n) = \frac{2iG(s^+_n)}{1 - e^2(\bar{\alpha}s^+_n)}.$$

Thus

$$q_N(s) = -\frac{\bar{\alpha}}{2\pi} \int_{\infty e^{\pm i\pi/2}}^{-\alpha} \exp \left\{ - \left( \bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k} \right) s \right\} \frac{G(k)}{\Delta(k)} \left[ 1 - \frac{\lambda}{(\bar{\alpha}k)^2} \right] dk$$

$$-i\bar{\alpha} \sum_{s^+_n} \exp \left\{ - \left( \bar{\alpha}s^+_n + \frac{\lambda}{\bar{\alpha}s^+_n} \right) s \right\} \frac{G(s^+_n)}{\Delta'(s^+_n) [1 - e^2(\bar{\alpha}s^+_n)]} \left[ 1 - \frac{\lambda}{(\bar{\alpha}s^+_n)^2} \right]$$

$$+ i\bar{\alpha} \sum_{s^-_n} \exp \left\{ - \left( \bar{\alpha}s^-_n + \frac{\lambda}{\bar{\alpha}s^-_n} \right) s \right\} \frac{G(s^-_n)}{\Delta'(s^-_n) [1 - e^2(-\bar{\alpha}s^-_n)]} \left[ 1 - \frac{\lambda}{(\bar{\alpha}s^-_n)^2} \right]. \quad (5.10)$$

### 5.2 The Poincaré Problem

Evaluating equation (4.1) at $k = k_m$, where $k_m$ is a zero of $D(k)$, it follows that the unknown terms $Y_j(k)$ appear in the form

$$\Gamma_1(k_m)H_1(k_m) \left\{ Y_1(k_m) + e^2(-k_m)\frac{P_1(\alpha k_m)P_1(k_m)}{P_2(k_m)P_2(\bar{\alpha}k_m)} H_2(k_m) \right\} Y_2(k_m)$$

$$+ e^2(k_m)\frac{P_1(\alpha k_m)P_1(k_m)}{P_3(k_m)P_3(\bar{\alpha}k_m)} H_3(k_m) \right\} Y_3(k_m)$$

The crucial difference of this general case, as compared with the oblique Robin case (1.9), is the following: Using the definition of $Y_j(k_m)$ we find that the coefficients of $q^{(2)}(s)$ and of $q^{(3)}(s)$ involve in general $k_m$-dependent expressions, thus it is not clear how the associated integral can be inverted. In contrast, equation (4.1) can be solved using the approach of section (5.1). The definition of $Y_2(\bar{\alpha}k)$, i.e. equation (3.19), and equation(1.25), imply

$$q^{(2)}(s) = \frac{i\alpha \sin \beta}{2\pi} \int_{\infty e^{\pm i\pi/2}} \exp \left\{ - \left( \bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k} \right) s \right\} \left[ 1 - \frac{\lambda}{(\bar{\alpha}k)^2} \right] Y_2(\bar{\alpha}k) dk. \quad (5.11)$$

Solving equation (4.1) for $Y_2(\bar{\alpha}k)$ and substituting the resulting expression in equation (5.11) we find an integral involving the three unknown functions $\{Y_j(k)\}_j$. The unknown part of this integral involves the factors (5.8) analyzed already, as well as factors of the type

$$e^2(-k)\frac{Y_j(k)}{D(k)}, \quad e^2(k)\frac{Y_j(k)}{D(k)}.$$
These terms are bounded for all \( k \neq k_m \). Indeed, ignoring the terms involving \( P_j(k) \) we find

\[
e^2(-k) \frac{Y_j(k)}{D(k)} \sim \begin{cases} 
Y_j(k)e(-k) \cdot e^4(-k), & \text{Re } k > 0 \\
Y_j(k)e(k), & \text{Re } k < 0,
\end{cases}
\]

which are identical with the expressions (5.9) except for the occurrence of the factors \( e^4(-k) \) and \( e^4(k) \) for \( \text{Re } k > 0 \) and \( \text{Re } k < 0 \), which are bounded.

The above discussion implies that the integral involving

\[
ee(-\bar{\alpha}k) \frac{H_1(k)}{P_1(k)} Y_1(k) + e^2(-k) \frac{P_1(\bar{\alpha}k)H_2(k)}{P_2(k)P_2(\alpha k)} Y_2(k) + e^2(k) \frac{P_1(\alpha k)H_3(k)}{P_3(k)P_3(\bar{\alpha} k)} Y_3(k)
\]

(5.12)
can be computed by using Cauchy’s theorem in \( D^+ \). Evaluating equation (4.1) at \( k_m^+ \) it follows that the associated residue equals

\[
\frac{\frac{1}{1} - \frac{P_1(k_{m^+})P_1(\bar{\alpha} k_{m^+})}{P_2(\alpha k_{m^+})} e^2(-k_{m^+}) e^2(\bar{\alpha} k_{m^+})} \frac{[T(k_{m^+}) + C(k_{m^+})]}{[T(k_{m^+}) + C(k_{m^+})]}
\]

(5.13)
Similarly, the integral involving

\[
-\frac{P_1(k)P_1(\bar{\alpha} k) e(\bar{\alpha} k)}{P_2(\alpha k)P_3(\alpha k) D(k)} \left[ e^2(-k) \frac{H_1(k)}{P_1(k)} Y_1(k) + e^2(k) \frac{P_2(\alpha k)H_2(k)}{P_1(k)P_1(\bar{\alpha} k)} Y_2(k)
\]

(5.14)
can be computed by using Cauchy’s theorem in \( D^- \). Evaluating equation (4.1) at \( k_m^- \), it follows that the associated residue equals

\[
\frac{[T(k_{m^-}) + C(k_{m^-})]}{[T(k_{m^-}) + C(k_{m^-})]}
\]

(5.15)

In what follows we give the details for a mixed Neumann-Robin problem.

We will consider Example 2, as it is described by (4.11) with \( \beta = \frac{\pi}{2} \). On side (1) we assume the Robin condition

\[
q^{(1)}_N(s) + \sqrt{3} \lambda q^{(1)}_N(s) = f_1(s),
\]

(5.16)
and on sides (2) and (3) we assume the Neumann conditions

\[
q^{(2)}_N(s) = f_2(s)
\]

(5.17)
and

\[
q^{(3)}_N(s) = f_3(s).
\]

(5.18)
Then $H_j(k)$, $P_j(k)$, $D(k)$ and $\Gamma_j(k)$, are given by (4.13), (4.14), (4.15a) and (4.15b,c,d) respectively. Furthermore, $\Gamma_3(k)$ is proportional to $\Gamma_1(k)$ (see equation (4.16a)), while $\Gamma_2(k)$ becomes proportional to $\Gamma_1(k)$ only on those $k_m$’s for which $D(k)$ vanishes. These are roots of the transcendental equation

$$
\exp\left\{ 3 \left( k + \frac{\lambda}{k} \right) \right\} = \frac{(k + \frac{1}{k} - \sqrt{\lambda})}{(k + \frac{3}{k} + \sqrt{\lambda})} \left( k + \frac{1}{k} - 2\sqrt{\lambda} \right) \frac{(k + \frac{3}{k} + \sqrt{\lambda})}{(k + \frac{1}{k} + 2\sqrt{\lambda})}.
$$

(5.19)

For $k = k_m$, equation (4.1), in view of (4.16), implies

$$
\Gamma_1(k_m) \left[ H_1(k_m) Y_1(k_m) - \frac{e^2(-k_m)}{P_1(\alpha k_m)} H_2(k_m) Y_2(k_m) - \frac{e^2(k_m)}{P_1(\alpha k_m) H_3(k_m) Y_3(k_m)} \right] = -T(k_m).
$$

(5.20)

By virtue of (4.15b) and the identity

$$
P_1(k_m) P_1(\alpha k_m) = -1
$$

(5.21)

equation (5.20) is written as

$$
\left[ e(-\alpha k_m) + e(\alpha k_m) \frac{e^2(-k_m)}{P_1(\alpha k_m)} \left[ \frac{H_1(k_m) Y_1(k_m)}{P_1(k_m)} \right] \right] + e^2(-k_m) P_1(\alpha k_m) H_2(k_m) Y_2(k_m) + e^2(k_m) P_1(\alpha k_m) H_3(k_m) Y_3(k_m) = -T(k_m).
$$

(5.22)

Since the corner terms $C(k)$ vanish, the representation (5.11) and equation (4.1) yield

$$
q^{(2)}(s) = \frac{i \alpha}{2\pi} \int_{\alpha e^{\frac{\pi}{2}}}^{\alpha e^{\frac{-\pi}{2}}} \exp\left\{ -\left( \alpha k + \frac{\lambda}{\alpha k} \right) s \right\} \left( 1 - \frac{\lambda}{(\alpha k)^2} \right) \frac{1}{H_2(\alpha k) D(k)} \sum_{j=1}^{3} \Gamma_j(k) H_j(k) Y_j(k) + T(k) \right\} dk.
$$

(5.23)

Utilizing the expression (5.22) we arrive at the following result.

**Proposition 5.1.** Let the real valued function $q(x, y)$ satisfy equation (1.7) with $\lambda > 0$ in the triangular domain $D$, with the Robin boundary condition (5.16) on side (1) and the Neumann boundary conditions (5.17) and (5.18) on sides (2) and (3), where the given functions $f_j(s)$ have sufficient smoothness and are continuous at the vertices. Then the Dirichlet value on side (2) is given by

$$
q^{(2)}(s) = \frac{i \alpha}{2\pi} \int_{\alpha e^{\frac{\pi}{2}}}^{\alpha e^{\frac{-\pi}{2}}} \exp\left\{ -\left( \alpha k^+ + \frac{\lambda}{\alpha k^+} \right) s \right\} \left( 1 - \frac{\lambda}{(\alpha k^+)^2} \right) \frac{T(k)}{H_2(\alpha k^+ D(k)} \right\} dk
$$

$$
+ \alpha \sum_{k_m^+} \frac{\exp\left\{ -\left( \alpha k^+_m + \frac{\lambda}{\alpha k^+_m} \right) s \right\} \left( 1 - \frac{\lambda}{(\alpha k^+_m)^2} \right) \frac{T(k^+_m)}{H_2(\alpha k^+_m) D^+(k^+_m)}}{1 + E^\theta(\alpha k^+_m) P_1(\alpha k^+_m)}
$$

22
\[-\bar{\alpha} \sum_{k_m^-} \exp \left\{ -\left( \bar{\alpha} k_m^- + \frac{\lambda}{\bar{\alpha} k_m^-} \right) s \right\} \left( 1 - \frac{\lambda}{(\bar{\alpha} k_m^-)^2} \right) \frac{T(k_m^-)}{1 + P_1(\alpha k_m^-) E^6(-i\alpha k_m^-)}, \tag{5.24} \]

where $D'(k_n^\pm)$ denotes the derivative of $D(k)$ evaluated at $k = k_n^\pm$. The summations are taken over all $k^+_m \in D^+$ and $k^-_m \in D^-$ respectively, and $T$, $H_2$, $P_1$ are defined by equations (3.27), (3.16), (4.4) respectively.

There exist similar formulas for the Dirichlet values on sides (1) and (3).

### 6 The Integral Representations

If $\lambda \geq 0$ the classical Green’s representation is given by [3]

\[ q(r) = \frac{1}{2\pi} \int_{\partial D} \left[ K(2\sqrt{\lambda}|r - r'|) \partial_{n'} q(r') - q(r') \partial_{n'} K(2\sqrt{\lambda}|r - r'|) \right] dl(r') \tag{6.1} \]

where the integration is over the boundary $\partial D$ of the triangle in the positive direction, $\partial_{n'}$ denotes the outward normal derivative on $\partial D$, $dl(r')$ is the line element along $\partial D$, and $K(x)$ is the modified Bessel function of the zeroth order and of the second kind for the modified Helmholtz equation. For the case of the Helmholtz equation $K(x)$ is proportional to the Hankel function of the zeroth order and of the first kind, while for the Laplace’s equation $K(x)$ is proportional to the logarithm of $x$.

For the Laplace equation, the integral representation constructed in [5] is defined as follows:

\[ \frac{\partial q}{\partial z} = \frac{1}{2\pi} \sum_{j=1}^{3} \int_{l_j} e^{i k z} \tilde{\rho}_j(k) dk, \quad z \in D, \tag{6.2} \]

where the contours $l_j$ are the rays from 0 to $\infty$ specified by the arguments $-\pi/2, \pi/6, 5\pi/6$ respectively, and the functions $\tilde{\rho}_j(k)$ are defined by equations (2.6) in terms of $\rho_j(k)$, where the latter functions are defined by equations (2.7), (1.14) with $\lambda = 0$.

![Figure 6.1: The rays $l_j$ in the complex $k$-plane](image)

For the modified Helmholtz equation, the analogue of equation (6.2) is [5]

\[ q(z, \bar{z}) = \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{l_j} e^{ik z + i\frac{\pi}{6} \bar{z} \tilde{\rho}_j(k)} \frac{dk}{k}, \quad z \in D, \tag{6.3} \]

23
where the rays $l_j$ are the same as in (6.2) and $\tilde{\rho}_j(k)$ are defined by equations (2.6) and (2.7).

There exists a similar representation for the Helmholtz equation, which however, in addition to rays, it also involves circular arcs [5].

### 6.1 The Symmetric Dirichlet Problem

Using the integral representation (6.3) it is possible to compute directly $q(z, \bar{z})$, bypassing the computation of the unknown boundary values. For brevity of presentation we will only give details for the symmetric Dirichlet problem. The analysis of the more general boundary value problems (1.8)-(1.11) is similar.

Recalling the definitions of $\tilde{\rho}_j$, i.e. equations (2.6) and (2.7), it follows that the representation $q(z, \bar{z})$ given by equation (6.3) involves the known function $F(k)$ defined by equation (3.2), as well as the unknown function $\Psi(k)$ which on the rays $l_j$ appears as:

$$l_1 : E(-ik)\Psi(k), \quad l_2 : E(-i\tilde{\alpha}k)\Psi(\tilde{\alpha}k), \quad l_3 : E(-i\alpha k)\Psi(\alpha k). \quad (6.4)$$

Solving equation (1.17) for $\Psi(\alpha k)$ in terms of $\Psi(k)$, and then using the Schwarz conjugation of the resulting equations in order to express $\Psi(\alpha k)$ in terms of $\Psi(k)$, it follows that the expressions in (6.4) involve the unknown function $\Psi(k)/\Delta(k)$, $\Delta(k) = e(k) - e(-k)$, times the following expressions:

$$l_1 : E(-ik)[e(k) - e(-k)], \quad l_2 : E(-i\tilde{\alpha}k)[e(\tilde{\alpha}k) - e(-\tilde{\alpha}k)], \quad l_3 : E(-i\alpha k)[e(\alpha k) - e(-\alpha k)]. \quad (6.5)$$

The third of the relations in (1.5) implies $e(k) = E(i\tilde{\alpha}k)E(-i\alpha k)$, thus

$$E(-ik)e(k) = E^2(i\tilde{\alpha}k), \quad E(-ik)e(-k) = E^2(i\alpha k).$$

Replacing $k$ by $\tilde{\alpha}k$ and by $\alpha k$ in these identities, we find

$$E(-i\tilde{\alpha}k)e(\tilde{\alpha}k) = E^2(i\alpha k), \quad E(-i\tilde{\alpha}k)E(-\tilde{\alpha}k) = E^2(ik),$$

$$E(-i\alpha k)e(\alpha k) = E^2(ik), \quad E(-i\alpha k)e(-\alpha k) = E^2(i\tilde{\alpha}k).$$

Thus the expressions in (6.5) involve

$$l_1 : E^2(i\tilde{\alpha}k) - E^2(i\alpha k), \quad l_2 : E^2(i\alpha k) - E^2(ik), \quad l_3 : E^2(ik) - E^2(i\tilde{\alpha}k).$$

Hence, the unknown part of $q(z, \bar{z})$ involves the following integral

$$J(z, \bar{z}) = \sum_{j=1}^{3} J_j(z, \bar{z}), \quad (6.6)$$

$$J_3(z, \bar{z}) = \frac{1}{4\pi} \int_{\{l_1\} \cup \{l_2\}} \exp \left\{ ikz + \frac{\lambda}{ik} \bar{z} \right\} \frac{E^2(i\alpha k)\Psi(k)dk}{k\Delta(k)}, \quad (6.7a)$$

$$J_1(z, \bar{z}) = \frac{1}{4\pi} \int_{\{l_2\} \cup \{l_3\}} \exp \left\{ ikz + \frac{\lambda}{ik} \bar{z} \right\} \frac{E^2(ik)\Psi(k)dk}{k\Delta(k)}, \quad (6.7b)$$
\[ J_2(z, \bar{z}) = \frac{1}{4\pi} \int_{\{-i\} \cup \{i\}} \exp \left\{ ikz + \frac{\lambda}{ik} \bar{z} \right\} E^2(ik) \Psi(k) \frac{dk}{k\Delta(k)}. \] (6.7c)

Each of the above integrals can be computed in terms of residues. Indeed, it was shown in Section 5 that \( \Psi(k)/k\Delta(k) \) is bounded as \( k \to 0 \) and as \( k \to \infty \). Furthermore, it will be verified below that the exponentials,

\[ \exp \left\{ ikz + \frac{\lambda}{ik} \bar{z} \right\} E^2(ik), \quad \exp \left\{ ikz + \frac{\lambda}{ik} \bar{z} \right\} E^2(i\alpha k), \] (6.8)

are bounded as \( k \to 0 \) and \( k \to \infty \), for \( \arg k \) in

\[ \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right], \quad \left[ \frac{\pi}{6}, \frac{5\pi}{6} \right], \quad \left[ \frac{5\pi}{6}, \frac{3\pi}{2} \right], \]

respectively, provided that \((z, \bar{z}) \in D\). We first consider the first exponential in (6.8); since \( z_2 = -\alpha \frac{1}{\sqrt{3}} \), this exponential can be written as

\[ \exp \{ ik(z - z_2) + \lambda(\bar{z} - \bar{z}_2)/ik \}. \]

If \( z \) is in the triangular domain then

\[ \frac{\pi}{2} \leq \arg(z - z_2) \leq \frac{5\pi}{6}. \]

Thus if \(-\frac{\pi}{2} \leq \arg k \leq \frac{\pi}{6}\), we find

\[ 0 \leq \arg[k(z - z_2)] \leq \pi. \]

Hence \( \exp\{ik(z - z_2)\} \) is bounded as \(|k| \to \infty\) and \( \exp\{\lambda(\bar{z} - \bar{z}_2)k/i|k|^2\} \) is bounded as \(|k| \to 0\). Hence there is no contribution from zero and from infinity.

The results for the second and for the third integrals in (6.8) follows from the above result by using appropriate rotations. The roots of \( \Delta(k) = 0 \) lie on the imaginary axis. Denote by \( s_n^+ \) those with positive imaginary part and by \( s_n^- \) those with negative imaginary part. Obviously, the residue from each \( s_n^+ \) has a full contribution to \( J_1(z, \bar{z}) \), while the residue contribution from each \( s_n^- \) is split in two halves, one half is contributed to \( J_2(z, \bar{z}) \) and one half to \( J_3(z, \bar{z}) \).

Tedious but straightforward calculations lead to the expression

\[
J(z, \bar{z}) = \sum_{s_n^+} \exp \left\{ is_n^+ z + \frac{\lambda}{is_n^+} \bar{z} \right\} \frac{E^2(is_n^+ G(s_n^+) s_n^+ \Delta'(s_n^+) \Delta(\bar{s}_n^+) \Delta(\bar{s}_n^+) )}{s_n^+ \Delta'(s_n^+) \Delta(\bar{s}_n^+)}
\]

\[ + \frac{1}{2} \sum_{s_n^-} \exp \left\{ is_n^- z + \frac{\lambda}{is_n^-} \bar{z} \right\} \frac{E^2(i\alpha s_n^-) + E^2(i\bar{s}_n^-) G(s_n^-) s_n^- \Delta'(s_n^-) \Delta(\bar{s}_n^-) \Delta(\bar{s}_n^-) }{s_n^- \Delta'(s_n^-) \Delta(\bar{s}_n^-) \Delta(\bar{s}_n^-) \Delta(\bar{s}_n^-) } . \] (6.9)

In the above calculations, the value of \( \Psi_1(s_n^+) \) is obtained from (1.18).

We observe that for each \( n \in \mathbb{Z}, \)

\[ \exp \left\{ is_n z + \frac{\lambda}{is_n} \bar{z} \right\} = \exp \left\{ \pm 2 \sqrt{\left( \frac{n\pi}{l} \right)^2 + \lambda x} \right\} \cdot \exp \left\{ -2i \frac{n\pi}{l} y \right\} \] (6.10)

thus this expression shows that the equilateral triangle admits separable solutions. It is clear that each eigensolution in (6.10), solves equation (1.7).
Conclusion

Eigenvalues and eigenfunctions for equation (1.7) with homogeneous Dirichlet, Neumann, and Robin boundary conditions were constructed in the classical works of Lamé [10]-[12]. Some of these results have been rederived by several authors, in particular the Dirichlet problem is discussed in the recent review [13]. The Robin problem is analysed in [14]. It is remarkable that Lamé argued, using physical considerations, that it is impossible to solve certain problems using infinite series as opposed to integrals. Indeed Lamé writes [12 p.191]: “The series should therefore express the fact that the temperature remains zero on strips of constant width separated by other strips of double width, in which the temperature may vary. The analytic interpretation of this sort of discontinuity demands the introduction of terms where the variables appear inside integrals\(^1\). These terms, of a nature that we will not consider here, cannot disappear from the total series unless the discontinuity disappears”.

In this paper we have solved several boundary value problems by introducing a novel analysis of the global relation, i.e. of equation (1.13). Although this equation was first derived in the important work [14], where it was also used to solve the Robin problem, our treatment of equation (1.13) is different than that of [14]. As a consequence of our novel analysis of equation (1.13) we are able to first present a straightforward treatment of simple boundary value problems. This treatment, which is based on the evaluation of the basic algebraic relations (see the Introduction) at particular values of \(k\), expresses the unknown boundary values in terms of infinite series. The Dirichlet, Neumann and Robin problems can be solved using this approach. We then show that, in agreement with the above remarks of Lamé, more complicated boundary value problems apparently require the use of generalized Fourier integrals as opposed to infinite series. Proposition 5.1 presents the solution of such a problem.

In this paper, as opposed to the works of [1], [2], [5]-[7], we have introduced a method for determining the generalised Dirichlet to Neumann map, i.e. determining the unknown boundary values as opposed to determining \(q(x, y)\) itself. In this respect we note that: (a) In some applications one requires precisely these unknown boundary values. (b) When both the Dirichlet and the Neumann boundary values are known, it is straightforward to compute \(q(x, y)\).

We emphasize however, that the approach of Section 5 can be used to construct directly \(q(x, y)\). Indeed, if one uses the novel integral representations for \(q(x, y)\) obtained in [5], instead for the representation (1.25) for \(q_N\), and if one follows the approach of Section 5, one can again compute explicitly the contribution of the unknown functions \(Y_j(k)\). This latter approach is illustrated in Section 6.1 for the symmetric Dirichlet problem. More complicated problems using this approach are solved in [2], [6], [7].

In order to compute \(q(x, y)\) from the knowledge of both the Dirichlet and the Neumann boundary values one can use either the classical Green’s formulae or the representations of [5]. Regarding the latter representations we note that they provide a tailor-made transform for the particular problem at hand. In fact the exponential \(\exp\{ikz + (\lambda/ik)\bar{z}\}\) reflects the structure of the PDE, the contours \(l_j\) in the complex \(k\)-plane reflect the geometry of the domain, and the functions \(\rho_j(k)\) describe the boundary conditions.

\(^1\)These words were not italic in the original text
Both the Dirichlet and the Neumann problems involve elementary trigonometric functions. It is interesting that the analysis of the global condition yields these separable solutions without the direct use of separation of variables.

For arbitrary values of the constants $\beta_j$ and $\gamma_j$, the Poincaré problem (1.11) gives rise to a matrix Riemann-Hilbert problem. For the particular case that equations (1.12) are valid, it is possible to avoid this Riemann-Hilbert problem and to solve the problem in closed form. Although equations (1.12) impose severe restrictions on $\beta_j$ and $\gamma_j$, some of the resulting cases appear interesting. These cases include the following:

1. $\beta_1 = \beta_2 = \beta_3 = \frac{2\pi}{3}, \gamma_j$ arbitrary.
   In this case the angles are specified, but $\gamma_j$ are arbitrary.

2. The mixed Neumann-Robin problem analyzed in Section 5.

3. $\beta_2 = \beta + \frac{4\pi}{3}, \beta_3 = \beta + \frac{2\pi}{3}, \gamma_1 = \gamma_2 = \gamma_3 = \gamma$.
   In this case all derivatives are computed along a direction making an angle $\beta$ with the positive vertical axis.

The results presented here can be made rigorous, following a formalism similar to the one used in [9].

Several problems remain open which include the following:

1. The investigation of singularities associated with discontinuous boundary conditions.

2. If the $\beta_j$’s differ, then the global relation (3.20) contains a contribution from $q$ at the three corners. Several approaches for determining these terms are presented in [1] and [6], however, the optimal treatment of these terms remains open.

The approach introduced in [4] and [5] constructs the solutions of a given boundary value problem without using eigenfunction expansions. Similar considerations apply to the approach introduced here for constructing the generalised Dirichlet to Neumann map. However, it turns out that the above approaches can also be used to investigate the existence of eigenfunction expansions and to construct these expansions when they exist. This will be presented elsewhere.

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