To the Linearization Problem for Single-Input Control Affine Systems

D. Fetisov
Bauman Moscow State Technical University, Russia
E-mail: dfetisov@yandex.ru

Abstract. State feedback linearization is well-known as an effective tool to solve various problems for nonlinear control systems. An affine system is called to be state feedback linearizable if there exist smooth invertible changes of the state and the inputs which transform the system into a linear controllable system. If an affine system is not state feedback linearizable one can try to use orbital feedback linearization. An affine system is called to be orbital feedback linearizable if there exists time scaling which transforms the system into a state feedback linearizable system. As usual for affine systems, time scaling is considered to be depending only on the state. Recently, it has been shown that if time scaling depends both on the state and on the inputs, then it becomes possible to linearize affine systems which are not orbital feedback linearizable. In this paper, while considering such transformations, we suggest the new sufficient condition for linearizability of single-input control affine systems. We derive a system of partial differential equations which has to be solved in order to find an appropriate time scaling. We provide an example how the proposed approach can be applied to solve a terminal problem.

1. Introduction

Various problems for nonlinear control systems can be efficiently solved if the system in question can be transformed to a linear controllable system. An affine system is said to be state feedback linearizable if it can be transformed to a linear controllable system by means of smooth nonsingular changes of the state and of the inputs. The necessary and sufficient condition for state feedback linearizability of affine systems is known from [1]. If a system is not state feedback linearizable, it was proposed to use changes of the independent variable [2]. An affine system is called to be orbital feedback linearizable if there exists a change of the independent variable (depending on the state) which transforms the system to a state feedback linearizable system. The conditions for orbital feedback linearization of single-input control affine systems were obtained in [3–5]. Changes of the independent variable in single-input control affine systems were applied to solve a stabilization problem for the Acrobot [6], a path following problem for unmanned aerial vehicles [7] and wheeled robots [8]. The conditions for orbital feedback linearization of multi-input control affine systems one can find in [9, 10]. It has been shown in [11, 12] that using changes of the independent variable depending both on the state and on the inputs allows to linearize systems which are not orbital feedback linearizable. However, the linearizability conditions derived in [11, 12] deal only with some specific cases. In the present paper, while considering changes of the independent variable which depend both on the state
and on the input, we generalize the class of transformations suggested in [12] and develop the sufficient condition for linearizability of single-input control affine systems.

2. Problem formulation
Consider a single-input control affine system

\[ \dot{x} = f(x) + g(x)u, \]
\[ x = (x_1, \ldots, x_n)^T \in \Omega \subset \mathbb{R}^n, \quad u \in \mathbb{R}, \quad \dot{\cdot} = d(\cdot)/dt, \quad n \geq 3, \]
\[ f(x) = (f_1(x), \ldots, f_n(x))^T, \quad g(x) = (g_1(x), \ldots, g_n(x))^T, \quad f_i(x), g_i(x) \in C^\infty(\Omega), \quad i = 1, \ldots, n, \]

where \( \Omega \) is an open subset of the state space \( \mathbb{R}^n \). Hereafter smooth means \( C^\infty \)-smooth.

The system (1) is called to be state feedback linearizable on \( \Omega \) if there exist diffeomorphism \( \Phi : \Omega \rightarrow P \) and functions \( A, B \in C^\infty(\Omega) \) \( (B(x) \neq 0 \text{ for all } x \in \Omega) \) such that the change of the state \( z = \Phi(x) \) and the change of the input \( v = A(x) + B(x)u \) transform the system (1) on \( \Omega \) into the linear controllable system

\[ \dot{z}_1 = z_2, \ldots, \dot{z}_{n-1} = z_n, \quad \dot{z}_n = v. \]

We shall use the notation \( \text{ad}^0 g = g, \text{ad}_f g = [f, \text{ad}_f^{k-1} g], k = 1, 2, \ldots, \) where \([f, g] \) is the Lie bracket of the vector fields \( f \) and \( g \). Consider the sequence of the distributions

\[ S_i = \text{span}\{g, \text{ad}_f g, \ldots, \text{ad}_f^{i-1} g\}, \quad i = 1, 2, \ldots. \]

The necessary and sufficient condition for state feedback linearizability is known from [1].

Theorem 1. The system (1) is state feedback linearizable in a neighbourhood of the point \( x_0 \) if and only if:

i) the distribution \( S_{n-1} \) is involutive in a neighbourhood of \( x_0 \);
ii) \( \text{dim} S_n(x_0) = n. \)

A change of the independent variable given by the relation \( \dot{\tau} = h(x, u) \) transforms the system (1) to the nonlinear system

\[ x' = \frac{f(x) + g(x)u}{h(x, u)}, \]

which is not an affine one. However, it is easy to verify that the following statement holds.

Lemma 1. For any two functions \( \lambda, \mu \in C^\infty(\Omega) \) such that \( \mu(x) \neq 0 \) for all \( x \in \Omega \) the change of the independent variable

\[ \dot{\tau} = \lambda(x) + \mu(x)u \]

and the change of the input

\[ v = \frac{1}{\lambda(x) + \mu(x)u} \]

transform the system (1) on the set \( Q = \{(x, u) : x \in \Omega, \lambda(x) + \mu(x)u \neq 0\} \) to the affine system

\[ x' = \frac{1}{\mu(x)} g(x) + \left( f(x) - \frac{\lambda(x)}{\mu(x)} g(x) \right) v, \]

defined on the set \( \tilde{Q} = \{(x, v) : x \in \Omega, v \neq 0\} \).

In the case of \( \mu = 1 \), the necessary and sufficient condition for state feedback linearizability of the system (5) has been derived in [12]. In the present paper, we consider the general case and prove the sufficient condition for state feedback linearizability of the system (5) in terms of the vector fields \( f \) and \( g \).
3. Linearizability condition
The following lemma is known from [2].

**Lemma 2.** Let \( \text{dim } S_n = n \) on an open subset \( V \) of the state space \( R^n \). Then the following conditions are equivalent:

i) there exist functions \( \theta_{ij} \in C^\infty(V) \), \( j = 1, \ldots, n-2 \), \( i = 0, \ldots, j \), such that

\[
[\text{ad}_{\tilde{g}}^j f, \text{ad}_{\tilde{g}}^j f] = \sum_{i=0}^j \theta_{ij} \text{ad}_{\tilde{g}}^i f, \quad j = 1, \ldots, n-2
\]

for all \( x \in V \);

ii) the distribution \( S_{n-1} \) is involutive on \( V \).

Let us introduce the notations

\( k = -\lambda/\mu \), \( l = 1/\mu \).

Then the vector fields \( \tilde{f} \) and \( \tilde{g} \) associated with the system (5) are given by the expressions

\[
\tilde{f} = lq, \quad \tilde{g} = f + kg.
\]

The sequence of the distributions \( P_i(k, l) = \text{span}\{\tilde{g}, \text{ad}_{\tilde{g}}^j \tilde{g}, \ldots, \text{ad}_{\tilde{g}}^{i-1} \tilde{g}\}, i = 1, 2, \ldots \) corresponds to each pair of functions \( k \) and \( l \). According to Lemma 2, the state feedback linearizability condition for the system (5) acquires the following form.

**Lemma 3.** A necessary and sufficient condition for the existence of smooth functions \( \lambda \) and \( \mu \) such that the system (5) is state feedback linearizable in a neighbourhood of \( x_0 \in \Omega \) is the existence of smooth functions \( k \) and \( l \), \( l(x_0) \neq 0 \), such that:

i) there exist smooth functions \( \theta_{ij} \), \( j = 1, \ldots, n-2 \), \( i = 0, \ldots, j \), satisfying the equality

\[
[\text{ad}_{\tilde{g}}^j \tilde{g}, \text{ad}_{\tilde{g}}^j \tilde{g}] = \sum_{i=0}^j \theta_{ij} \text{ad}_{\tilde{g}}^i \tilde{g}, \quad j = 1, \ldots, n-2
\]

in a neighbourhood of \( x_0 \).

ii) \( \text{dim } P_n(k, l)(x_0) = n \).

**Remark 1.** If there exist smooth functions \( k \) and \( l \), \( l(x_0) \neq 0 \), satisfying the conditions i) and ii) in Lemma 3, then the system (5) with \( \lambda = -k/l \), \( \mu = 1/l \) is state feedback linearizable in a neighbourhood of \( x_0 \).

Let us prove some auxiliary properties of the vector fields \( \text{ad}_{\tilde{g}}^j \tilde{g} \). For a smooth function \( \varphi \), we shall denote the derivative of \( \varphi \) along the vector field \( g \) as \( L_g \varphi \).

**Lemma 4.** For any functions \( k \), \( l \in C^\infty(\Omega) \) the following equalities hold on the set \( \Omega \):

\[
\text{ad}_{\tilde{g}}^j \tilde{g} = \sum_{i=0}^j \alpha_{ij} \text{ad}_{\tilde{g}}^i f + \alpha_{gj} g, \quad j = 0, 1, 2, \ldots
\]

where \( \alpha_{jj} = l^j \), \( \alpha_{0,j+1} = 0 \), \( \alpha_{i,j+1} = lL_g \alpha_{ij} + l\alpha_{i-1,j} \), \( j = 0, 1, 2, \ldots \), \( i = 1, \ldots, j \), \( \alpha_{g0} = k \), \( \alpha_{g,j+1} = lL_g \alpha_{gj} - \alpha_{gj} L_g l - \sum_{i=0}^j \alpha_{ij} L_{\text{ad}_{\tilde{g}}^i f} l^i \), \( j = 0, 1, \ldots \)

**Proof.** We shall prove the equality (7) by induction with respect to \( j \). If \( j = 0 \), the relation (7) coincides with the definition of the vector field \( \tilde{g} \). Now, assume that for some arbitrary \( r \) the equalities

\[
\text{ad}_{\tilde{g}}^r \tilde{g} = \sum_{i=0}^r \alpha_{ir} \text{ad}_{\tilde{g}}^i f + \alpha_{gr} g, \quad \alpha_{0r} = \text{const}, \quad \alpha_{rr} = l^r
\]
hold on $\Omega$. Then,
\[
ad_{\tilde{f}}^{r+1} \tilde{g} = [\tilde{f}, \ad_{\tilde{g}}]^{r+1} = \left[ l g, \sum_{i=0}^{r} \alpha_{ir} \ad_{g} f + \alpha_{gr} g \right] = l L g \alpha_{0r} f + \sum_{i=1}^{r} \left( l g \alpha_{ir} + l \alpha_{i-1, r} \right) \ad_{g} f + l \alpha_{r} \ad_{g}^{r+1} f + \left( l g \alpha_{gr} - \alpha_{gr} L g l - \sum_{i=0}^{r} \alpha_{ir} L \ad_{g} f \right) g.
\]

Since $\alpha_{0r} = \text{const}$, then $L g \alpha_{0r} = 0$. Taking into account the equality $\alpha_{rr} = l^{r}$ we obtain
\[
ad_{\tilde{f}}^{r+1} \tilde{g} = \sum_{i=1}^{r} \left( l g \alpha_{ir} + l \alpha_{i-1, r} \right) \ad_{g} f + l^{r+1} \ad_{g}^{r+1} f + \left( l g \alpha_{gr} - \alpha_{gr} L g l - \sum_{i=0}^{r} \alpha_{ir} L \ad_{g} f \right) g.
\]

The proof is complete.

**Lemma 5.** For any functions $k, l \in C^\infty(\Omega)$ the following relations hold on the set $\Omega$:
\[
[\ad_{\tilde{f}}^{j-1} \tilde{g}, \ad_{\tilde{g}}^{j}] = \sum_{i=0}^{j-1} \sum_{m=0}^{j} \alpha_{i, j-1} \alpha_{mj} [\ad_{g} f, \ad_{g} m f] + \sum_{s=1}^{j+1} \xi_{sj} \ad_{g} f + \xi_{gj} g, \quad j = 1, \ldots, n - 2, \quad (8)
\]

where
\[
\xi_{gj} = \alpha_{g, j-1} L g \alpha_{gj} - \alpha_{gj} L g \alpha_{g, j-1} - \sum_{m=0}^{j} \alpha_{mj} L \ad_{g}^{m} f \alpha_{g, j-1} + \sum_{m=0}^{j-1} \alpha_{m, j-1} L \ad_{g}^{m} f \alpha_{gj}, \quad j = 1, \ldots, n - 2;
\]
\[
\xi_{jj} = \sum_{r=0}^{j-1} \alpha_{r, j-1} L \ad_{g}^{r} f \alpha_{jj} - \alpha_{j-1, j-1} \alpha_{gj} + \alpha_{j-1, j} \alpha_{g, j-1} + \alpha_{g, j-1} L g \alpha_{jj}, \quad j = 1, \ldots, n - 2;
\]
\[
\xi_{j+1,j} = \alpha_{g, j-1} \alpha_{jj}, \quad j = 1, \ldots, n - 2;
\]
\[
\xi_{sj} = \sum_{r=0}^{j-1} \alpha_{r, j-1} L \ad_{g}^{r} f \alpha_{sj} - \alpha_{g, j-1} \alpha_{sj} - \sum_{r=0}^{j-1} \alpha_{r, j-1} L \ad_{g}^{r} f \alpha_{s, j-1} - \alpha_{s-1, j-1} \alpha_{gj} + \alpha_{s-1, j} \alpha_{g, j-1} + \alpha_{g, j-1} L g \alpha_{sj}, \quad s = 1, \ldots, j - 1;
\]

**Proof.** Taking into account the relations (7) we have
\[
[\ad_{\tilde{f}}^{j-1} \tilde{g}, \ad_{\tilde{g}}^{j}] = \left[ \sum_{i=0}^{j-1} \alpha_{i, j-1} \ad_{g} f + \alpha_{g, j-1} g, \sum_{m=0}^{j} \alpha_{mj} \ad_{g} m f + \alpha_{gj} g \right]
\]
\[
= \sum_{i=0}^{j-1} \sum_{m=0}^{j} \left( \alpha_{i, j-1} \alpha_{mj} \ad_{g} f, \ad_{g} m f \right) + \alpha_{i, j-1} L \ad_{g} f \alpha_{mj} \ad_{g} m f - \alpha_{mj} L \ad_{g} f \alpha_{i, j-1} \ad_{g} f \right) \\
+ \sum_{i=0}^{j-1} \left( \alpha_{i, j-1} \alpha_{gj} \ad_{g} f, \ad_{g} g \right) + \alpha_{i, j-1} L \ad_{g} f \alpha_{gj} g - \alpha_{gj} L g \alpha_{i, j-1} \ad_{g} f \right) + \sum_{m=0}^{j} \left( \alpha_{g, j-1} \alpha_{mj} \ad_{g} m f - \alpha_{mj} L \ad_{g} f \alpha_{g, j-1} g \right) + \alpha_{g, j-1} L g \alpha_{gj} g - \alpha_{gj} L g \alpha_{g, j-1} g.
\]

Since $[g, \ad_{g} f] = \ad_{g}^{j+1} f$, we obtain
\[
[\ad_{\tilde{f}}^{j-1} \tilde{g}, \ad_{\tilde{g}}^{j}] = \sum_{i=0}^{j-1} \sum_{m=0}^{j} \alpha_{i, j-1} \alpha_{mj} \ad_{g} f, \ad_{g} m f \right) + \sum_{m=0}^{j} \left( \sum_{i=0}^{j-1} \alpha_{i, j-1} L \ad_{g} f \alpha_{mj} \right) \ad_{g} m f
\]
\[- \sum_{i=0}^{j-1} \left( \sum_{m=0}^{j} \alpha_{mj} L_{ad^{m}_{y} f} \alpha_{i,j-1} \right) \text{ad}^{j}_{y} f - \sum_{i=0}^{j-1} \alpha_{i,j-1}(x) \alpha_{gj} \text{ad}^{i+1}_{y} f + \sum_{i=0}^{j-1} \alpha_{i,j-1} L_{ad^{i}_{y} f} \alpha_{gj} g \]

\[- \sum_{i=0}^{j-1} \alpha_{gj} L_{y} \alpha_{i,j-1} \text{ad}^{i}_{y} f + \sum_{i=0}^{j} \alpha_{g,j-1} \alpha_{ij} \text{ad}^{i+1}_{y} f + \sum_{i=0}^{j} \alpha_{g,j-1} L_{y} \alpha_{ij} \text{ad}^{i}_{y} f - \sum_{i=0}^{j} \alpha_{ij} L_{ad^{i}_{y} f} \alpha_{g,j-1} g \]

\[+ (\alpha_{g,j-1} L_{g} \alpha_{gj} - \alpha_{gj} L_{g} \alpha_{g,j-1}) g = \sum_{i=0}^{j} \sum_{m=0}^{j} \alpha_{i,j-1} \alpha_{mj} [\text{ad}^{i}_{y} f, \text{ad}^{m}_{y} f] + \sum_{s=1}^{j} \xi_{s} \text{ad}^{s}_{y} f + \xi_{gj} g. \]

The proof is complete.

**Corollary 1.** Assume that for any number \( j = 1, \ldots, n - 2 \) there exist smooth functions \( \gamma^{ir}_{s}, \gamma^{ir}_{g}, s = 0, \ldots, j + 1, i, r = 0, \ldots, j \), such that the relations

\[ [\text{ad}_{y}^{i} f, \text{ad}_{y}^{r} f] = \sum_{s=0}^{j+1} \gamma^{ir}_{s} \text{ad}_{y}^{s} f + \gamma^{ir}_{g} g, \quad i, r = 0, \ldots, j \quad (9) \]

hold on the set \( \Omega \subset R^{n} \). Then for any functions \( k, l \in C^{\infty}(\Omega) \) the following equalities hold on \( \Omega \):

\[ [\text{ad}_{y}^{i} g, \text{ad}_{y}^{j} g] = \sum_{s=0}^{j+1} \delta_{sj} \text{ad}_{y}^{s} f + \delta_{gj} g, \quad j = 1, \ldots, n - 2, \quad (10) \]

where

\[ \delta_{gj} = \xi_{gj} + \sum_{r=0}^{j} \sum_{m=0}^{j} \alpha_{r,j-1} \alpha_{mj} \gamma^{rm}_{g}, \quad \delta_{hj} = \sum_{r=0}^{j} \sum_{m=0}^{j} \alpha_{r,j-1} \alpha_{mj} \gamma^{rm}_{g}, \quad j = 1, \ldots, n - 2, \]

\[ \delta_{sj} = \xi_{sj} + \sum_{r=0}^{j} \sum_{m=0}^{j} \alpha_{r,j-1} \alpha_{mj} \gamma^{rm}_{s}, \quad j = 1, \ldots, n - 2, \quad s = 1, \ldots, j + 1. \]

**Proof.** Replacing the Lie brackets \([\text{ad}_{y}^{i} f, \text{ad}_{y}^{m} f] \) with the expressions (9) in the relations (8) we obtain the equalities (10). The proof is complete.

**Theorem 2.** Assume that for any number \( j = 1, \ldots, n - 2 \) there exist smooth functions \( \gamma^{ir}_{s}, \gamma^{ir}_{g}, s = 0, \ldots, j + 1, i, r = 0, \ldots, j \), such that:

i) the equalities (9) hold in a neighbourhood of \( x_{0} \);

ii) there exist smooth functions \( k, l, \beta_{ij}, j = 1, \ldots, n - 2, i = 0, \ldots, j, l(x_{0}) \neq 0, \) satisfying the system of partial differential equations

\[ \delta_{0j} = \beta_{0j}, \quad \delta_{sj} = \sum_{i=s}^{j} \beta_{ij} \alpha_{si}, \quad s = 1, \ldots, j, \quad \delta_{j+1,j} = 0, \quad \delta_{gj} = \beta_{0j} k + \sum_{i=1}^{j} \beta_{ij} \alpha_{gi}, \quad \delta_{j+1,j} = \sum_{i=1}^{j} \beta_{ij} \alpha_{gi}, \quad j = 1, \ldots, n - 2 \quad (11) \]

in a neighbourhood of \( x_{0} \) and such that \( \dim \mathcal{P}(k, l)(x_{0}) = n. \)

Then the system (5) with \( \lambda = -k/l, \mu = 1/l \) is state feedback linearizable in a neighbourhood of \( x_{0}. \)

**Proof.** By using (10) and (11), we obtain

\[ [\text{ad}_{y}^{j-1} g, \text{ad}_{y}^{j} g] = \sum_{s=0}^{j+1} \delta_{sj} \text{ad}_{y}^{s} f + \delta_{gj} g = \beta_{0j} f + \sum_{s=1}^{j} \sum_{i=s}^{j} \beta_{ij} \alpha_{si} \text{ad}_{y}^{s} f + \left( \beta_{0j} k + \sum_{i=1}^{j} \beta_{ij} \alpha_{gi} \right) g \]
It is easily seen that any first integral
\[\lambda = \frac{k}{l}\]

According to ii), there exists a first integral
\[l\]

Since \(\mu = \frac{1}{l}\), the system (5) with \(\lambda = -\frac{k}{l}\) is state feedback linearizable in a neighborhood of \(x_0\). The proof is complete.

**Corollary 2.** Assume that \(n = 3\) and there exist smooth functions \(\gamma_0^1, \gamma_1^0, \gamma_2^0, \gamma_9^1\), such that:

i) the equalities
\[f, ad_g f] = \gamma_0^1 f + \gamma_1^0 ad_g f + \gamma_2^0 ad_g^2 f + \gamma_9^1 g\]

and
\[\gamma_9^1 + L_{ad_g} \gamma_2^0 - L_f \gamma_0^1 g = \gamma_0^1 \gamma_2^0 + \gamma_0^1 \gamma_1^0 L_g \gamma_2^0 = 0\]

hold in a neighborhood of \(x_0\);

ii) there exists a first integral \(l\) of the vector field \(f - \gamma_2^0 g\), such that \(l(x_0) \neq 0\) and \(\dim \mathcal{P}_3(\gamma_2^0, l)(x_0) = 3\).

Then the system (5) with \(\lambda = \gamma_2^0/l\), \(\mu = 1/l\) is state feedback linearizable in a neighborhood of \(x_0\).

**Proof.** In the case of \(n = 3\) the system (11) acquires the form
\[\delta_{01} = \beta_{01}, \quad \delta_{11} = \beta_{11} l, \quad \delta_{21} = 0, \quad \delta_{g1} = \beta_{01} k + \beta_{11} \alpha_{g1} = 0\]

Since
\[\alpha_{g1} = L_g k - L_f + k l, \quad \delta_{01} = \gamma_0^1, \quad \delta_{11} = \gamma_1^0 + 2L_f + k l - L_g k, \quad \delta_{21} = \gamma_2^0 + k l, \quad \delta_{g1} = -\alpha_{g1} L_g k - L_{ad_g} f k + L_f + k g \alpha_{g1} + L_{g1}^0\]

we can rewrite the system (14) as
\[\gamma_0^1 k = \beta_{01}, \quad \gamma_1^0 + 2L_f + k l - L_g k = \beta_{11} l, \quad \gamma_2^0 + k l = 0, \quad -\alpha_{g1} L_g k - L_{ad_g} f k + L_f + k g \alpha_{g1} + L_{g1}^0 = \beta_{01} k + \beta_{11} \alpha_{g1}\]

It follows from the third equation of the system (15) that \(k = -\gamma_2^0\). By eliminating \(\beta_{01}\) and \(\beta_{11}\) from the first and second equations of the system (15), we obtain the equation
\[\gamma_0^1 L_g - L_f - \gamma_0^1 l + L_{ad_g} f \gamma_2^0 + \gamma_1^0 l L_g \gamma_2^0 = 0\]

Since the equality (13) holds, the equation (16) acquires the form
\[\gamma_0^1 L_f - \gamma_0^1 l + L_f \gamma_2^0 + \gamma_1^0 l L_g \gamma_2^0 = 0\]

It is easily seen that any first integral \(l\) of the vector field \(f - \gamma_0^1 g\) is a solution of the equation (17). According to ii), there exists a first integral \(l\) of the vector field \(f - \gamma_0^1 g\), such that \(l(x_0) \neq 0\) and \(\dim \mathcal{P}_3(\gamma_2^0, l)(x_0) = 3\). It follows from Theorem 2 that the system (5) with \(\lambda = \gamma_2^0/l\) and \(\mu = 1/l\) is state feedback linearizable in a neighborhood of \(x_0\). The proof is complete.
4. Example
Consider the system
\[ \dot{x}_1 = x_2 + e^{x_1}x_2u, \quad \dot{x}_2 = x_3 + e^{x_1}x_3u, \quad \dot{x}_3 = 1 - x_3 - e^{x_1}x_3u \] (18)
and the terminal problem in the following setting: given the boundary conditions
\[ x_1(0) = 5, \quad x_2(0) = 12, \quad x_3(0) = 0, \quad x_1(t_s) = 24, \quad x_2(t_s) = 36, \quad x_3(t_s) = 60, \] (19)
find a time \( t_s \) and an input \( u(t) \in C[0, t_s] \), such that the corresponding trajectory of the system (18) satisfies the conditions (19).

Associated with the system (18) are the vector fields
\[ \begin{align*}
    f &= x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + (1 - x_3) \frac{\partial}{\partial x_3}, \\
    g &= e^{x_1}x_2 \frac{\partial}{\partial x_1} + e^{x_1}x_3 \frac{\partial}{\partial x_2} - e^{x_1}x_3 \frac{\partial}{\partial x_3}.
\end{align*} \]

Let us compute the vector fields \( \text{ad}_f g \) and \( \text{ad}_f^2 g \):
\[ \begin{align*}
    \text{ad}_f g &= e^{x_1} x_2^2 \frac{\partial}{\partial x_1} + e^{x_1}(1 + x_2 x_3) \frac{\partial}{\partial x_2} - e^{x_1}(1 + x_2 x_3) \frac{\partial}{\partial x_3}, \\
    \text{ad}_f^2 g &= e^{x_1} (x_2^2 + 1 - x_2 x_3) \frac{\partial}{\partial x_1} + e^{x_1} (x_2^2 x_3 - x_3 - 1) \frac{\partial}{\partial x_2} - e^{x_1} (x_2^2 x_3 - x_3 - 1) \frac{\partial}{\partial x_3}.
\end{align*} \]

It is easily seen that \( \text{dim span} \{ g, \text{ad}_f g, \text{ad}_f^2 g \} < 3 \) on \( R^3 \). Hence the system (18) is not state feedback linearizable on any open subset of \( R^3 \).

At the same time,
\[ \begin{align*}
    \text{ad}_g f &= -\text{ad}_f g = -e^{x_1} x_2^2 \frac{\partial}{\partial x_1} - e^{x_1}(1 + x_2 x_3) \frac{\partial}{\partial x_2} + e^{x_1}(1 + x_2 x_3) \frac{\partial}{\partial x_3}, \\
    \text{ad}_g^2 f &= e^{2x_1} (x_2 x_3 - 1) \frac{\partial}{\partial x_1} + e^{2x_1}(1 + x_2 + x_3^2) \frac{\partial}{\partial x_2} - e^{2x_1}(1 + x_2 + x_3^2) \frac{\partial}{\partial x_3}, \\
    [f, \text{ad}_g f] &= e^{x_1} (-x_2^3 - 1 + x_2 x_3) \frac{\partial}{\partial x_1} - e^{x_1} (x_2^3 x_3 - x_3 - 1) \frac{\partial}{\partial x_2} + e^{x_1} (x_2^3 x_3 - x_3 - 1) \frac{\partial}{\partial x_3} \\
    &= x_2 \text{ad}_g f + e^{-x_1} \text{ad}_g^2 f.
\end{align*} \]

Thus, the equalities (12) hold, where the functions \( \gamma^0_0, \gamma^0_1, \gamma^0_2 \) and \( \gamma^0_g \) are given by the formulas \( \gamma^0_0(x) = 0, \gamma^0_1(x) = x_2, \gamma^0_2(x) = e^{-x_1}, \gamma^0_g(x) = 0 \). The vector field \( f - \gamma^0_2 g \) acquires the form
\[ f - \gamma^0_2 g = \frac{\partial}{\partial x_3}. \]

Since \( L_{\text{ad}_f} \gamma^0_2(x) = x_2^2, L_g \gamma^0_2(x) = -x_2, L_{f-\gamma^0_2 g} L_g \gamma^0_2(x) = 0 \), the equality (13) holds.

We take the function \( l(x) = e^{-x_1} \) as the first integral of the vector field \( f - \gamma^0_2 g \). In this case, the functions \( \lambda \) and \( \mu \) are given by the expressions \( \lambda(x) = 1, \mu(x) = e^{-x_1} \).

The change of the independent variable \( \tau = 1 + e^{x_1}u \) and the change of the input \( v = 1/(1 + e^{x_1}u) \) transform the system (18) on the set \( Q = \{(x, u) : 1 + e^{x_1}u \neq 0\} \) to the system
\[ \begin{align*}
    x'_1 &= x_2, \\
    x'_2 &= x_3, \\
    x'_3 &= -x_3 + v.
\end{align*} \] (20)
defined on the set \( \tilde{Q} = \{(x, v) : v \neq 0\} \). We note that the inverse change of the input is given by the relation
\[
u = e^{-x_1(1 - v)/v}. \tag{21}\]
The change of the input \( w = -x_3 + v \) transforms the system (20) to the linear controllable system
\[
x_1' = x_2, \quad x_2' = x_3, \quad x_3' = w,
\]
defined on the set \( \tilde{Q} = \{(x, w) : w \neq -x_3\} \).

Let us consider the terminal problem for the system (20) with the boundary conditions
\[
x_1(0) = 5, \quad x_2(0) = 12, \quad x_3(0) = 0, \quad x_1(1) = 24, \quad x_2(1) = 36, \quad x_3(1) = 60, \tag{22}\]
and the restriction \( v \neq 0 \). It easily seen that the function \( p(\tau) = 3\tau^4 + 4\tau^3 + 12\tau + 5 \) satisfies the conditions \( p(0) = 5, p'(0) = 12, p''(0) = 0, p(1) = 24, p'(1) = 36, p''(1) = 60 \). Hence the relations
\[
x_1 = p(\tau) = 3\tau^4 + 4\tau^3 + 12\tau + 5, \quad x_2 = p'(\tau) = 12\tau^3 + 12\tau^2 + 12, \quad x_3 = p''(\tau) = 36\tau^2 + 24\tau
\]
are \( \tau \)-parametric equations of the trajectory of the system (20) satisfying the conditions (22). From the last equation of the system (20), we obtain
\[
v = p''(\tau) + p''(\tau) = 36\tau^2 + 96\tau + 24. \tag{23}\]

It is obvious that the function (23) satisfies the condition \( v \neq 0 \) for all \( \tau \in [0, 1] \). Now, we have to substitute the function (23) to the relation \( \tilde{\tau} = 1/v \). As a result, we obtain the equation \( \tilde{\tau} = 1/(36\tau^2 + 96\tau + 24) \). Let \( \tau(t) \) be the solution of this equation satisfying the initial condition \( \tau(0) = 0 \). From the equality \( \tau(t_*) = 1 \), we have \( t_* = \int_0^1 (36\tau^2 + 96\tau + 24)d\tau = 84 \). According to (21), the solution of the original problem for the system (18) is given by the formula
\[
u = e^{-p(\tau)} \left. \frac{1 - p''(\tau) - p''(\tau)}{p''(\tau) + p''(\tau)} \right|_{\tau=\tau(t)}.
\]
The relations \( x_1 = p(\tau(t)), x_2 = p'(\tau(t)), x_3 = p''(\tau(t)) \) are \( t \)-parametric equations of the corresponding trajectory of the system (18).

5. Acknowledgments
The study was supported by the Russian Foundation for Basic Research (Project 17-07-00653).

References
[1] Jakubczyk B and Respondek W 1980 Bull. Acad. Polon. Sci. Ser. Math. 28 517
[2] Sampei M and Furuta K 1986 IEEE Trans. Automat. Control 31 459
[3] Respondek W 1998 Proc. IFAC NOLCOS'98 499
[4] Guay M 1999 Systems Control Lett. 38(4-5) 271
[5] Krishchenko A P 2014 Differ. Equations 50(11) 1508
[6] Saito A, Sekiguchi K and Sampei M 2010 IEEE Conf. on Decision and Control 5408
[7] Tkachev S B and Lin W 2015 IFAC-PapersOnline 48(11) 10
[8] Pesterev A V, Rapoport L B and Tkachev S B 2015 Journal of Computer and Systems Sciences Int. 54(4) 656
[9] Guay M 2001 Proc. of the American Control Conference 3630
[10] Li S J and Respondek W 2015 Int. Journal of Robust and Nonlinear Control 25(9) 1352
[11] Fetisov D A 2017 IFAC-PapersOnLine 50(1) 2677
[12] Fetisov D A 2017 Differ. Equations 53(11) 1483