Surgery Obstructions to Seifert Fibered Homology Spheres

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Abstract

We examine surgery on a knot in $S^3$ to determine surgery obstructions to Seifert fibered integral homology spheres. We find such surgery obstructions using Heegaard Floer, Knot Floer homology and the mapping cone formula for computing Heegaard Floer homology of surgery on a knot. Here however, we take a different approach and use the number of singular fibers of a Seifert fibered integral homology sphere, which is the toroidal structure, to find obstructions. This approach allows us to show that genus one knots cannot yield Seifert fibered integral homology spheres with six or more singular fibers. Some other obstructions are also presented for higher genus knots.

1 Introduction

1.1 Background

Consider a knot $k$ in a closed oriented 3-manifold $M$. We can perform Dehn surgery on $k$ in $M$ by cutting $M$ open along the neighborhood of our knot, which is isomorphic to a solid torus, and gluing back in $D^2 \times S^1$. This gluing is entirely determined by sending the meridian of $D^2 \times S^1$ to some simple closed curve, $pm + ql$, given by $p$ times the meridian and $q$ times the longitude of the knot exterior. This is called $p/q$ surgery on the knot $k$. The space resulting from this construction is a closed oriented manifold, generally different from $M$. For this paper we will only be considering surgery on knots in $S^3$. Lickorish and Wallace showed that every closed orientable 3-manifold can be obtained by an integral surgery on a link in $S^3$. Due to this result, Dehn surgery has become a fundamental method of representing 3-manifolds. It is then natural to ask which manifolds can be represented by surgery on a knot in $S^3$, as opposed to a link.

Knots can be partitioned into three categories: torus, satellite and hyperbolic. Many surgery problems are understood for torus and satellite knots. For example, Moser completely classified surgery on torus knots in $S^3$ in [17], and Gabai was able to first prove the Property P conjecture for satellite knots in [6]. (The Property P conjectures has since been completely proved by Kronheimer and Mrowka [13].) Since surgeries on torus and satellite knots are well understood, the most interesting surgeries to consider are those on hyperbolic knots. Surgeries on hyperbolic knots that do not yield hyperbolic manifolds are called exceptional. Exceptional surgeries are reducible, toroidal, or Seifert fibered. Thurston’s hyperbolic Dehn surgery theorem says that there are only finitely many exceptional surgery slopes on a hyperbolic knot [29]. We call Dehn surgery Seifert fibered, toroidal or reducible if it yields a Seifert fibered, toroidal or reducible manifold respectively [10].
Since exceptional surgeries have remained the most elusive, there has been much work put into their study. Dean introduced a condition on knots in $S^3$ that guarantees a hyperbolic surgery \[4\], while Eudave-Muñoz extended this to include surgeries producing Seifert fibered manifolds with a projective plane orbit surface and two exceptional fibers \[3\]. In this same paper Eudave-Muñoz finds a collection of hyperbolic knots that yield toroidal Seifert fibered manifolds \[5\]. It can be checked that these knots will never yield Seifert fibered integral homology 3 spheres. Teragaito showed that any positive integers can arise as the toroidal surgery slope of a hyperbolic knot in \[28\], and Wu classified toroidal surgeries on length 3 Montesinos knots \[33\]. For further general discussions of exceptional surgeries we suggest \[2, 11, 18\].

We are particularly interested in toroidal, Seifert fibered, integral homology 3-sphere surgeries, so let us explore these specifics further. Gordon and Luecke showed that the denominator of toroidal surgery slope is at most 2 for hyperbolic knots in \[8\], and Miyazaki and Motegi built on this result to show that if $K$ is a hyperbolic, periodic knot with period 2, only an integer coefficient can yield a toroidal surgery \[16\]. Ichihara and Jong showed there is no toroidal, Seifert fibered surgery on pretzel knots except for the trefoil in \[10\]. Ozsváth and Szabó showed that the family of Kinoshita-Terasaka knots $KR_{r,n}$ with $|r| \geq 2$ and $n \neq 0$ cannot yield integral Seifert fibered homology 3-spheres \[22\]. Wu found the only three large arborescent knots that yield exceptional toroidal surgeries in \[32\]. Wu also found the Montesinos knots that yield toroidal Seifert fibered surgeries in \[31\].

In this paper our goal is to find obstructions to which knots in $S^3$ yield integral Seifert fibered integral homology spheres. Seifert fibered integral homology spheres are a result of exceptional surgery \[27\]. One of the main tools used is Heegaard Floer homology, and the knot invariant: knot Floer homology. Heegaard Floer Homology is an invariant of closed 3 manifolds, introduced by Ozsváth and Szabó. This invariant is isomorphic to Seiberg-Witten Floer Homology \[23\]. Ozsváth and Szabó and many others have used Heegaard Floer homology to find such obstructions \[22, 31, 32\]. Here, however we use the number of singular fibers of a Seifert fibered integral homology sphere to find obstructions. This is using the toroidal structure of the manifold, as Seifert fibered homology spheres are toroidal if and only if they have 4 or more singular fibers.

The other tools used in this paper are Némethi’s graded root \[20\], and Ozsváth and Szabó’s mapping cone formula \[24\]. The graded root is a combinatorial object that allows us to compute Heegaard Floer homology more easily. This always exists when looking at Heegaard floer homology of Seifert fibered homology spheres, making it a very useful tool for our purposes. The mapping cone formula is used to make the Heegaard Floer homology of a manifold resulting from surgery more computable \[7\].

Our general strategy is to examine the reduced Heegaard Floer homologies of both integral homology spheres, and surgeries on knots. Specifically we observe how each of these behave under the U-action. We use Némethi’s graded root to analyze this for Seifert fibered integral homology spheres and the mapping cone formula to analyze this for surgeries on knots. Then we compare these results in order to establish obstructions to surgeries on knots yielding Seifert Fibered integral homology spheres.

### 1.2 Main Results

Before we give our obstructions, we must introduce some notation. $HF^+(Y)$ is the Heegaard Floer homology of a 3-manifold $Y$, $HF_{red}(Y)$ is the reduced Heegaard Floer homology of $Y$. Our main obstruction is the following.
**Theorem 1.** No surgery on a genus 1 knot, $K$, in $S^3$ can yield a Seifert fibered integral homology sphere with 6 or more singular fibers.

Theorem 1 is proved in two steps. First we use Némethi’s graded root to observe how $HF_{\text{red}}$ behaves under the associated $U^K$ action giving us the following result.

**Theorem 2.** For a Seifert fibered integral homology sphere $Y = \Sigma(p_1, \ldots, p_l)$ we have that
$$U^K \cdot HF_{\text{red}}(Y) \neq 0,$$
when $k < \frac{24l-73}{60}$.

Then we use the mapping cone formula to see how $HF_{\text{red}}$ of surgery on a genus 1 knot behaves under the $U^K$-action, which produces the following.

**Theorem 3.** For a genus 1 knot $K$ and $n \in \mathbb{Z}$ we have that $U^K \cdot HF_{\text{red}}(S^3_{1/n}(K)) = 0$.

Given Theorems 2 and 3, the proof of Theorem 1 is straightforward.

**Proof of Theorem 1.** By Theorem 2 we have that $U^K \cdot HF_{\text{red}}(Y) \neq 0$ where $Y$ is a Seifert fibered integral homology sphere with 6 or more singular fibers. By Theorem 3 we have that $U^K \cdot HF_{\text{red}}(S^3_{1/n}(K)) = 0$ for a genus 1 knot $K$. It follows that surgery on a genus 1 knot cannot yield a Seifert fibered integral homology sphere with 6 or more singular fibers.

We have the following bound for higher genus knots which follows directly from Theorem 2 and Gainullin’s work in [7].

**Theorem 4.** No surgery on a knot $K$ in $S^3$ of genus $g$ can yield a Seifert fibered integral homology sphere with $l$ or more singular fibers if $g \leq \frac{24l-133}{90}$.

**Proof.** Gainullin’s Theorem 3 in [7] tells us that $U^K \cdot HF_{\text{red}} \left( S^3 \left( \frac{g_4(K)}{2} \right) \right) = 0$ where $g(K)$ is the genus of the knot $K$, and $g_4(K)$ is the four ball genus of $K$. Using that $g_4(K) \leq g(K)$ we have the following:
$$g(K) + \left\lfloor \frac{g_4(K)}{2} \right\rfloor \leq g(K) + \frac{g_4(K)}{2} + 1$$
$$\leq g(K) + \frac{g(K)}{2} + 1 \leq g(K) + \frac{3g(K)}{2} = 3g(K) + \frac{1}{2}.$$ 

It follows from Gainullin’s theorem that
$$U^K \cdot HF_{\text{red}} \left( S^3 \left( \frac{3g(K)}{2} \right) \right) = 0.$$

Now by Theorem 2 we have that $U^K \cdot HF_{\text{red}}(Y) \neq 0$ when $k < \frac{24l-73}{60}$. It follows that no surgery on a knot $K$ in $S^3$ of genus $g$ can yield a Seifert fibered integral homology sphere with $l$ or more singular fibers if $g \leq \frac{24l-133}{90}$.

We do have a slightly improved result of Theorem 2 when we look at a large number of singular fibers. We will not be looking at such large homology spheres in this paper, but for the curious reader we refer you to Proposition 3. We also conjecture an improved result of Theorem 1 found at the end of section 2.3.
Organization In section two we introduce Némethi’s graded root and use this to prove Theorem 2. We also discuss some conjectures and the improved bound mentioned above. In section three we use the mapping cone formula for knot Floer homology to prove Theorem 3.

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2 Graded Roots

In this section we introduce Némethi’s graded root [20] and setup the preliminary results necessary for our obstructions. Originally this object was constructed from the plumbing graph for a plumbed 3-manifold. We will predominantly be following Can and Karakurt’s specific construction for Seifert fibered homology spheres in [3].

2.1 Construction

First, a graded root is defined as follows:

**Definition 1.** (Némethi, [20, Definition 3.1.2]) Let $R$ be an infinite tree with vertices $V$ and edges $E$. We denote by $[u, v]$ the edge with end-points $u$ and $v$. We say that $R$ is a **graded root** with grading $\chi: V \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in E$,

(b) $\chi(u) > \min\{\chi(v), \chi(w)\}$ for any $[u, v], [u, w] \in E$, and $v \neq w$,

(c) $\chi$ is bounded below, and $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$ and $|\chi^{-1}(k)| = 1$ for $k$ sufficiently large.

Now we can give the basics for the construction of a graded root for a given Seifert fibered homology sphere, $Y = \Sigma(p_1, \ldots, p_l)$. We define a function $\Delta: \mathbb{N} \to \mathbb{Z}$ by

$$\Delta(n) = 1 + |e_0|n - \sum_{i=1}^{l} \left\lfloor \frac{n p'_i}{p_i} \right\rfloor,$$

where $(e_0, p'_1, \ldots, p'_l)$ is the unique solution to

$$e_0 p_1 \cdots p_l + p'_1 p_2 \cdots p_l + p_1 p'_2 \cdots p_l + \cdots + p_1 p_2 \cdots p'_l = -1,$$

and

$$0 < p'_i \leq p_i - 1 \text{ for } i = 1, 2, \ldots, l.$$

By [3, Theorem 4.1, (2)] we have that $\Delta(n)$ is always positive for $n \geq N_0$ and

$$N_0 = p_1 p_2 \cdots p_l \left( (l - 2) - \sum_{i=1}^{l} \frac{1}{p_i} \right).$$

The general definition follows.
**Definition 2.** (Karakurt-Lidman, [12, Definition 3.1]) A delta sequence is a pair \((X, \delta)\) where \(X\) is a well ordered finite set, and \(\delta: X \rightarrow \mathbb{Z} \setminus \{0\}\) with \(\delta(x_0) > 0\) where \(x_0\) is the minimum of \(X\).

For our purposes, we will have the delta sequence \((X, \Delta)\), where \(\Delta(n)\) is given by (1) and \(X = \{x \in \mathbb{N} | \Delta(x) \neq 0\} \text{ and } x \leq N_0\}. We also have that \(\Delta\) will be symmetric, as \(\Delta(n) = -\Delta(N_0 - n)\) \[3, Theorem 4.1, (2)\]. For the remainder of this paper when we refer to a delta sequence we will be referring to the set \(\{\Delta(x_0), \Delta(x_1), \ldots, \Delta(x_k)\}\), where \(x_k\) is the last integer that satisfies \(\Delta(x_k) < 0\), as we will only be using the function defined in (1).

Once we have our delta sequence, we can define a \(\tau\) function \(\tau_{\Delta}: \{0, 1, \ldots, k\} \rightarrow \mathbb{Z}\) using the recurrence relation \(\tau_{\Delta}(n + 1) - \tau_{\Delta}(n) = \Delta(x_n)\), with the initial condition \(\tau_{\Delta}(0) = 0\). Let \(\{\tau_{\Delta}(0), \ldots, \tau_{\Delta}(k)\}\) be called the \(\tau\) sequence.

Now that we know how to get a \(\tau\) sequence for a Seifert fibered homology sphere, we can use this to construct our graded root, following [20, Example 3.1.3]. For every \(n \in \mathbb{Z}\), let \(R_n\) be the infinite graph with vertex set \(\mathbb{Z} \cap [\tau(n), \infty)\) and edge set \(\{[k, k + 1]: k \in \mathbb{Z} \cap [\tau(n), \infty)\}\). We then identify all common vertices and edges of \(R_n\) and \(R_{n+1}\). This gets us an infinite tree \(\Gamma_{\tau}\), and we assign a function \(\chi(v)\) that gives the unique integer corresponding to the grading of the vertex \(v\).

**Example 1.** The construction of \(\Gamma_{\tau}\) for the \(\tau\) sequence \(\{0, -1, 0, -1, -2, -3, -4, -3\}\), is seen in Figure 1.

![Figure 1: Construction of \(\Gamma_{\tau}\)](image)

We now define some operations on delta sequences which will be used to prove our results.

**Definition 3.** (Karakurt-Lidman, [12, Section 3]) Let \((X, \Delta)\) be a delta sequence. Let \(t\) be a positive integer and \(x \in X\) with \(|\Delta(x)| \geq t\). From this we construct a new delta sequence \((X', \Delta')\) as follows. The set \(X'\) is obtained by removing \(x\) from \(X\) and putting \(t\) consecutive elements \(x_1, \ldots, x_t\) in its place. Now, choose nonzero integers \(n_1, \ldots, n_t\) each with the same sign as \(\Delta(x)\) such that \(n_1 + \cdots + n_t = \Delta(x)\). The new \(\Delta'\) agrees with \(\Delta\) on \(X \setminus \{x\}\) and satisfies \(\Delta'(x_i) = n_i\) for \(i = 1, \ldots, t\). \((X', \Delta')\) is called a refinement of \((X, \Delta)\) at \(x\). For a delta sequence \((X, \Delta)\) we call \((X', \Delta')\) the fully refined delta sequence if \(\Delta'(x) = \pm 1\) for all \(x \in X'\).
2.2 Properties

Now that we can construct our graded root, we will introduce some properties and terminology. To any graded root $\Gamma$, we associate a $\mathbb{Z}$-graded, $\mathbb{Z}[U]$-module. We let $\mathbb{H}^+(\Gamma)$ be the free abelian group generated by the vertex set of $\Gamma$, and the degree of the generator corresponding to the vertex, $v$ has degree $2\chi(v)$. The $U$ action on $\mathbb{H}^+(\Gamma)$ is a degree $-2$ endomorphism that sends each vertex $v$ to the sum of all vertices $w$ connected to $v$ by an edge where $\chi(w) < \chi(x)$. If no vertices $w$ satisfy this, then $U$ sends $v$ to zero. We can now define two finitely generated groups:

$$H_{\text{red}}(\Gamma) = \text{Coker}(U^n), \text{ for large } n,$$

$$\widehat{H}(\Gamma) = \text{Ker}(U) \oplus \text{Coker}(U)[-1].$$

We will also refer to the image of $U^n$ for large $n$ as the tower, denoted $T^+$, which will eventually stabilize and become constant. We can then view $H_{\text{red}}(\Gamma)$, which is the primary focus of this paper, as $\mathbb{H}^+(\Gamma)/T^+$.

These groups encode Heegaard Floer homology as follows:

**Theorem 5.** (Némethi, [14, Section 11.3]) Let $Y$ be a positively oriented integral homology sphere, then we have the following isomorphisms of $\mathbb{Z}[U]$-modules up to an overall degree shift:

1. $HF^+(\gamma) \cong H^+(\Gamma)$,
2. $HF_{\text{red}}(\gamma) \cong H_{\text{red}}(\Gamma)$,
3. $\widehat{HF}(\gamma) \cong \widehat{H}(\Gamma)$.

**Proposition 1.** (Karakurt-Lidman, [12, Proposition 3.6]) Refinements and merges do not change $H^+(\Gamma)$ or $H_{\text{red}}(\Gamma)$.

Finally, we introduce some terminology before giving our results.

**Definition 4.** Let a branch of a graded root be a chain of connected edges and vertices (1 at each grading), such that the vertex $x$ at the lowest grading in the branch satisfies $U \cdot x = 0$. We say a branch has length $k$ if it consists of $k$ edges and $k + 1$ vertices, and we say a branch originates at vertex $a$ if $a$ has the highest grading of the branch. Two vertices are connected if there exists a branch containing both vertices.

**Definition 5.** An induced root of $\Gamma$ is a set, $\Gamma'$ of vertices and edges where the vertex set of $\Gamma'$ is a subset of the vertex set of $\Gamma$ and an edge exists in $\Gamma'$ only if that same edge exists in $\Gamma$.

2.3 Results

**Lemma 1.** In a graded root $\Gamma = (R, \chi)$, an element $h \in \mathbb{H}^+(\Gamma)$ of relative grading $g$ is in $T^+$ if and only if it is equal to the sum of every vertex of grading $g$ in $\Gamma$.

**Remark 1.** We extend $\chi$ from vertices in the graded root, to mean the grading of an element in the module. We will refer to $\chi(h)$ for $h \in \mathbb{H}^+(\Gamma)$ as the height of $h$. 
Proof. Let \( h \in H(\Gamma) \) satisfy \( \chi(h) = g \) and be in \( T^+ \). By definition \( T^+ \) is the image of \( U^n \) for large \( n \). Using this, and that \( |\chi^{-1}(k)| = 1 \) for large enough \( k \), we have that \( h = U^n(v) \) for some single vertex \( v \), and large \( n \). If \( \chi(v) = j \), then all vertices of grading \( j - 1 \) must be connected to \( v \). Thus \( U(v) \) is the sum of all vertices in grading \( j - 1 \). We also have that each vertex at grading \( j - 2 \) must be connected to some vertex at grading \( j - 1 \) by construction. It follows that \( U^m(v) \) is the sum of all vertices in grading \( g \) as desired.

Let \( h \) be equal to the sum of all vertices of grading \( g \). We want to show \( h = U^n(v) \) for some \( v \) and large \( n \). Since \( |\chi^{-1}(k)| = 1 \) for large enough \( k \), we can find some grading \( m > g \) that satisfies \( |\chi^{-1}(m)| = 1 \). Let \( v \) be the vertex that satisfies \( \chi(v) = m \). Then, as above, \( U^j(v) \) is equal to the sum of all vertices at grading \( m - j \). It follows that \( h = U^n(v) \) where \( n = m - g \) and thus \( h \in T^+ \).

Lemma 2. Let \( \Gamma \) be a graded root with a branch of length \( k \) originating at vertex \( a \). Let \( \chi(a) = n \) and \( a \) be connected to both \( b \) and \( c \) with \( \chi(b) = \chi(c) = n - 1 \), and \( b \neq c \). Then \( b \neq 0 \in H^+_{\text{red}}(\Gamma) \) as a \( \mathbb{Z} \)-module and \( U^k(b) = 0 \). See Figure 2.

![Figure 2](image)

**Figure 2:** A branch of length \( k \) originating at vertex \( a \).

**Remark 2.** Vertex \( a \) may be incident to more vertices than just \( b \) and \( c \). We have no additional conditions on \( c \).

**Proof.** First we show that \( U^k(b) = 0 \). We know that for any vertex \( v \) with grading \( n - 1 \), \( U(v) \) is given by the sum of all vertices connected to \( v \) with grading \( n - 2 \), and so \( U^k(b) \) is the sum of all vertices connected to \( b \) at grading \( n - 1 - k \). Since \( b \) is connected to no vertices of grading \( n - 1 - k \), it follows that \( U^k(b) = 0 \).

Now we show that \( b \) is a generator of \( H^+_{\text{red}}(\Gamma) \). To find \( H^+_{\text{red}}(\Gamma) \) we mod out our tower \( T^+ \) from \( H^+(\Gamma) \). Let \( v_1, \ldots, v_k \) be all other vertices in \( \Gamma \) that satisfy \( \chi(v_i) = n - 1 \), then we have \( b + c + v_1 + \cdots + v_k = 0 \), by \( H \) (We may have that \( k = 0 \)). Thus \( b \neq 0 \) in \( H^+_{\text{red}}(\Gamma) \).

Before we proceed, we must introduce another definition.

**Definition 6.** An \( A_k \) structure is two branches of length \( k \), originating from a common vertex. If a graded root \( \Gamma \), contains an \( A_k \) structure as an induced graded root, we say that \( \Gamma \) has an \( A_k \) structure. See Figure 6.

7
Lemma 3. A graded root \( \Gamma \) will satisfy \( U^k \cdot \mathbb{H}_{\text{red}}(\Gamma) \neq 0 \) if and only if it has an \( A_{k+1} \) structure.

Proof. If an \( A_{k+1} \) structure does not appear as an induced graded root, we are in one of two cases. The graded root could have only one vertex at each height, in which case \( \mathbb{H}_{\text{red}}(\Gamma) = 0 \), so our claim is trivially true. The other case is that \( \Gamma \) has only branches of length \( k \) or smaller. Here we have that \( \mathbb{H}_{\text{red}}(\Gamma) \) vanishes under the \( U^k \)-action by Lemma 2.

If we do have an induced \( A_{k+1} \) structure, as in Figure 3, by Lemma 1 we have the sum of all vertices at grading \( n-1 \), \( a + b + w_1 + \cdots + w_n \), is in \( T^+ \). It follows that \( a + b + w_1 + \cdots + w_n = 0 \in \mathbb{H}_{\text{red}}(\Gamma) \). We know that \( a \) is not in the tower by Lemma 1, and \( a \neq 0 \in \mathbb{H}^+(\Gamma) \) because \( \mathbb{H}^+(\Gamma) \) is the free \( \mathbb{Z} \)-module on the vertex set of \( \Gamma \), and \( a \) is a vertex of \( \Gamma \). Therefore \( a \) is a generator of \( \mathbb{H}_{\text{red}}(\Gamma) \) as \( a \neq 0 \) in \( \mathbb{H}_{\text{red}}(\Gamma) \), and \( U^k(a) \neq 0 \) as \( a \) is connected to a vertex at grading \( n-2 = \chi(a) - 1 \). Thus \( \mathbb{H}_{\text{red}}(\Gamma) \) does not vanish under the \( U^k \)-action if we have an \( A_{k+1} \) structure.

Proposition 2. Consider a delta sequence, \( \Delta \), that produces a graded root \( \Gamma \), and let \( \Delta' \) be the full refinement of \( \Delta \). \( \Gamma \) has an \( A_k \) structure if and only if \( \Delta' \) has \( k \) consecutive negative ones preceded by \( k \) consecutive positive ones.

Remark 3. Note that by our definition of a branch an \( A_{k+1} \) structure does not imply an \( A_k \) structure. We do however have that if \( U^{k+1} \cdot \mathbb{H}_{\text{red}}(\Gamma) \neq 0 \), then \( U^k \cdot \mathbb{H}_{\text{red}}(\Gamma) \neq 0 \).

Proof. Let \( \Gamma \) be a graded root with an \( A_k \) structure, and let \( \Delta = \{d_0, d_1, d_2, \ldots, d_r\} \), and \( \tau = \{t_0, t_1, \ldots, t_r\} \) be the associated \( \Delta \) and \( \tau \) sequences respectively. The presence of an \( A_k \) structure, implies that we must have \( t_l = g - k \), \( t_m = g \) and \( t_n = g - k \) for some \( l < m < n \). Additionally, we may assume that \( t_j > g - k \) for \( l < j < m \). This comes from the fact that if some \( t_j \leq g_k \), then we simply let \( t_j = t_l \). Now, using the difference relation between the \( \tau \) and \( \Delta \) sequences we have that

\[
\begin{align*}
    t_l &= \sum_{i=0}^{l-1} d_i, \\
    t_i &= \sum_{i=0}^{m-1} d_i,
\end{align*}
\]

and therefore

\[
    t_m - t_l = \sum_{i=l}^{m-1} d_i = k.
\]

By the assumption that \( t_j > g - k \) for \( l < j < m \), we know that \( d_i \geq 0 \) for \( l \leq i \leq m - 1 \). Then we have that \( d_l, d_{l+1}, \ldots, d_{m-1} \) gives us consecutive non-negative values that sum to \( k \), and thus in the fully refined delta sequence we will have \( k \) consecutive positive ones as desired. Similarly we
can see that there are $k$ consecutive negative ones later in the sequence by looking at the difference between $t_m$ and $t_n$.

Now assume that
\[ \Delta' = \{ \ldots, +1, \ldots, +1, \ldots, -1, \ldots, -1, \ldots \}. \]

Let us say that $d_1$ through $d_{i+k-1}$ represent our consecutive positive ones, and $d_{N_0-i}$ through $d_{N_0-i-k+1}$ represent our consecutive negative ones by symmetry. Using the difference relation we have that
\[ t_{i+k} - t_i = \sum_{j=i}^{i+k-1} d_j = k, \quad t_{N_0-i-k+1} - t_{N_0-i} = \sum_{j=N_0-i}^{N_0-i-k} d_j = -k. \]

Now let $t_i = g$, then $t_{i+k} = g + k$ and by $\Delta(n) = -\Delta(N_0 - n)$, we have that $t_{N_0-i} = g + k$ and therefore $t_{N_0-i-k+1} = g$. Now we are in the same case as above and it can be easily seen that this gives us an $A_k$ structure in our graded root $\Gamma$.

Now we are ready to prove Theorem 2. Our strategy is to find some value $x$ such that $\Delta(x) = k + 1$, and $x$ occurs before $N_0/2$ in our delta sequence. Then we use the symmetry of the delta sequence to show that in the fully refined delta sequence we would have $k + 1$ negative ones, proceeded by $k + 1$ positive ones. Then we apply Proposition 2 to get the desired result.

**Proof of Theorem 2.** Consider $x = kp_1 \cdots p_l$. By (1) it is easily seen that $\Delta(x) = 1 + k$. Thus when we consider the fully refined delta sequence we will have $k + 1$ consecutive positive ones. It remains to show $x < N_0/2$, then since $\Delta(n) = -\Delta(N_0 - n)$ the conditions of Proposition 2 are satisfied and our result will follow. Assume that $k < \frac{24l - 73}{60}$ where $l$ is the number of singular fibers. It follows that
\[ \sum_{i=1}^{l} \frac{1}{p_i} \leq \sum_{i=1}^{l} \frac{1}{q_i} \leq l - 2 - 2k, \]
where $q_i$ is the $i$th prime. Consider the following:
\[ \sum_{i=1}^{l} \frac{1}{p_i} < l - 2 - 2k \]
\[ 2k < l - 2 - \sum_{i=1}^{l} \frac{1}{p_i} \]
\[ 2kp_1p_2 \cdots p_l < p_1 \cdots p_l \left( l - 2 - \sum_{i=1}^{l} \frac{1}{p_i} \right) \]
\[ kp_1p_2 \cdots p_l < \frac{N_0}{2} \]

Thus we have $x = kp_1 \cdots p_l < \frac{N_0}{2}$ when $k \leq \frac{24l - 73}{60}$, and our proof is complete.\qed
We do have an improved bound from [25] for when \( l > e^4 \approx 54.6 \), using Pollack’s main result:

\[
\left| \sum_{p \text{ prime} \leq x} \frac{1}{p} - \ln(\ln(x)) \right| < 6.
\]

**Proposition 3.** For a Seifert fibered homology sphere \( Y = \Sigma(p_1, \ldots, p_l) \), when \( k < \frac{\ln(\ln(l)) - 8}{2} \) we have that

\[
U^k \cdot HF_{\text{red}}(Y) \neq 0.
\]

Using the same strategy as we did to prove Theorem 2, we can get a similar result for a Seifert fibered homology sphere with 5 singular fibers.

**Proposition 4.** For the Seifert fibered homology sphere \( Y = \Sigma(p_1, \ldots, p_l) \) we have that

\[
U \cdot HF_{\text{red}}(Y) \neq 0
\]

when one of the following holds.

1. \( p_1 \geq 4 \),
2. \( p_1 = 3 \) and \( p_5 \geq 17 \),
3. \( p_1 = 2, p_2 = 3, \) and \( p_3 \geq 17 \),
4. \( p_1 = 2, p_2 = 3, p_3 = 7 \) and \( p_4 \geq 83 \),
5. \( p_1 = 2, p_2 = 3, p_3 = 7, p_4 = 43, p_5 \geq 1811 \),
6. \( p_1 = 2, p_2 = 3, p_3 = 11, p_4 \geq 15 \) and \( p_5 \geq 101 \).

**Proof.** This simply involves testing that \( p_1p_2 \cdots p_5 \leq N_0/2 \) in the given cases. This is easily verified and since \( \Delta(p_1 \cdots p_5) = 2 \), the proposition follows easily.

It would seem that the next logical step would be to prove that \( U \cdot HF_{\text{red}}(Y) = 0 \), where \( Y = \Sigma(p_1, p_2, p_3, p_4, p_5) \). We would only need to address the Seifert fibered integral homology spheres of the form \( Y = \Sigma(2, 3, 7, p, q) \) and \( \Sigma(2, 3, 5, p, q) \) to do this, as these are the only infinite cases remaining after the above proposition. The remaining finite cases have already been checked to satisfy \( U \cdot HF_{\text{red}}(Y) = 0 \). Unfortunately the strategy used to prove Theorem 2 does not work in these cases, as \( p_1 \cdots p_5 \) will always occur after \( N_0/2 \). Thus we have the following conjecture:

**Conjecture 1.** For a Seifert fibered homology sphere, \( Y = \Sigma(p_1, \ldots, p_5) \), we have that

\[
U \cdot HF_{\text{red}}(Y) \neq 0.
\]

This has held true in every computation and example tried, and has been proved for 7 out of 8 identified cases, but a complete proof is still in progress.
3 Mapping Cone

In this section we give the proper notation and theorems used to set up the mapping cone for rational surgery on a knot in $S^3$. The mapping cone formula was introduced by Ozsváth and Szabó in [24], but we follow Gainullin’s general structure in [7].

3.1 Background

Given a knot $K$ in $S^3$ we associate a doubly-filtered complex $CFK^\infty$. The generators of this complex are denoted $[x, i, j]$ where the Alexander grading is $j - i$. We say that $[x, i, j]$ corresponds to the generator $U^{-i}x$ and has filtration $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. We have an action on $CFK^\infty$ by $U$ which lowers both the $i$ and $j$ gradings by 1.

We now can define subcomplexes of $CFK^\infty$ used in the mapping cone formula. Let $A^+_k(K) = CFK^\infty \{ i \geq 0 \text{ or } j \geq k \}$, $k \in \mathbb{Z}$, and $B^+ = CFK^\infty \{ i \geq 0 \} \cong CF^+(S^3)$.

We also have two chain maps, from $A^+_k$ to $B^+$. The first is $v_k: A^+_k \to B^+$ via the natural projection. For the second map, first consider $\tilde{B}^+ = CFK^\infty \{ j \geq 0 \}$, which is also a complex for $S^3$. Thus there is a homotopy equivalence between $B^+$ and $\tilde{B}^+$. The second is $h_k: A^+_k \to B^+$ given by the composition of our chain homotopy and the quotient map [15, Section 6.4]. There exists an isomorphism $\phi: A^+_0 \to A^+_0$ that is $U$-equivariant, grading preserving, and satisfies $v_0 \circ \phi = h_0$, this comes from spin$^c$ conjugation invariance, [24, Section 4].

Now we can define chain complexes

$$\bigoplus_{n \in \mathbb{Z}} \left( n, A^+_{\lfloor i + pn \rfloor}(K) \right), \quad \bigoplus_{n \in \mathbb{Z}} (n, B^+),$$

and a chain map $D^+_i$ between them. This chain map is given by the appropriate sums of the various $h_k$ and $v_k$ maps. Explicitly we have

$$D^+_{i, \frac{1}{p}}(\{(k, a_k)\}_{k \in \mathbb{Z}}) = \{(k, b_k)\}_{k \in \mathbb{Z}},$$

where

$$b_k = v^+_i(a_k) + h_{\lfloor \frac{i \pm pt - 1}{q} \rfloor}(a_{k-1}).$$

Let $X^+_i$ denote the mapping cone of $D^+_i$. We give this a relative $\mathbb{Z}$-grading by requiring that $D^+_i$ decrease grading by 1. Here $i$ represents the spin$^c$ structure, based on Ozsváth and Szabó’s identification of $\text{Spin}^c(Y^3_\mathbb{Q}(K))$ with $\mathbb{Z}/p\mathbb{Z}$.

Theorem 6 (Ozsváth-Szabó, [24]). There is a relatively graded isomorphism of $\mathbb{F}[U]$-modules

$$H_*\left(X^+_i\right) \cong HF^+\left(S^3_{\frac{p}{q}}(K), i\right).$$

11
Now let $\mathbb{A}^+_i(K) = H_*(\bigoplus_{n \in \mathbb{Z}} (n, A^+_{i+i\frac{p}{q}}(K)))$ and $B^+ = H_*(\bigoplus_{n \in \mathbb{Z}} (n, B^+))$, and $v_k, h_k$ and $D^+_i \xi$ denote the maps induced by $v_k, h_k$ and $D^+_i \xi$ on homology respectively.

We have that the short exact sequence

$$0 \rightarrow \bigoplus_{n \in \mathbb{Z}} (n, B^+) \xrightarrow{i_*} \mathbb{X}^+_i \xrightarrow{j_*} \bigoplus_{n \in \mathbb{Z}} (n, A^+_{i+i\frac{p}{q}}(K)) \rightarrow 0$$

induces the exact triangle

$$\mathbb{A}^+_i(K) \xrightarrow{D^+_i \xi} B^+ \xrightarrow{j_*} H_*(\mathbb{X}^+_i) \cong HF^+ \left( S^3_{\frac{p}{q}}(K), i \right).$$

All maps above are $U$-equivariant. From the exact triangle we have the following:

**Corollary 1.** (Gainullin, [2], Section 2) If the surgery slope $\frac{p}{q}$ is positive, then the map $D^+_i \xi$ will be surjective, so $HF^+ \left( S^3_{\frac{p}{q}}(K), i \right) \cong \ker \left( D^+_i \xi \right)$.

We need to establish some important decompositions of the maps given earlier. First we have that $\mathbb{A}^+_k(K) = H_*(\mathbb{A}^+_k(K))$ is an $HF^+$ of a rational homology sphere in a certain Spin$^c$-structure, and thus we can decompose it as $\mathbb{A}^+_k(K) \cong \mathbb{A}^+_k(T(K) \oplus \mathbb{A}^+_{red}(K))$, where $\mathbb{A}^+_{red}(K)$ is a finite-dimensional vector space in the kernel of some power of $U$, and $\mathbb{A}^+_k(T(K) \oplus \mathbb{A}^+_{red}(K)) \cong T^+$. Note, we will often drop $K$ from the above notation and refer to $\mathbb{A}^+_k(K)$ as simply $\mathbb{A}^+_k$.

We have that

$$D^+_i \xi = D^+_i T^+ \oplus D^+_{red}$$

where the first map is the restriction of $D^+_i \xi$ to $\mathbb{A}^+_k(T(K) = \bigoplus_{n \in \mathbb{Z}} T^+$ and the second one is the restriction to $\mathbb{A}^+_{red}(K)$. Similarly we have restrictions of $v_k$ and $h_k$ to $T^+$ which will just be some powers of $U$. These powers of $U$ will be denoted by $V_k$ and $H_k$. We will need to following properties of these integers, (see [21, Section 2], [26, Section 7], [9, Lemma 2.5])

$$V_k \geq V_{k+1} \text{ and } H_k \leq H_{k+1}, \quad \forall k \in \mathbb{Z}, \tag{2}$$

$$V_k = H_{-k}, \quad \forall k \in \mathbb{Z}, \tag{3}$$

$$V_k \rightarrow +\infty \text{ as } k \rightarrow -\infty \quad \text{and} \quad H_k \rightarrow +\infty \text{ as } k \rightarrow +\infty \tag{4}$$

$$V_k = 0 \text{ for } k \geq g(K) \quad \text{and} \quad H_k = 0 \text{ for } k \leq -g(K), \tag{5}$$

$$V_{k-1} \leq V_{k+1} \leq V_k, \forall k \in \mathbb{Z}, \tag{6}$$

$$H_k = V_k + k, \forall k \in \mathbb{Z}. \tag{7}$$

**Corollary 2.** (Gainullin, [2], Corollary 30) $U^9 \cdot \mathbb{A}^+_{red} = 0.$
3.2 Results

The goal of this section is to prove Theorem 3, thus we restrict our attention to genus 1 knots in $S^3$. Our strategy is to first limit the possible values for $V_0$ and $H_0$. Using this, and that $U \cdot HF_{red} (S^3_{-1}(K)) = 0$, we are able to restrict the possible maps for $v_0$ and $h_0$ for a genus 1 knot.

Putting all of this together we are able to prove the $U \cdot HF_{red} (S^3_{-1}(K)) = 0$ for genus 1 knots.

Lemma 4. Let $K$ be a knot in $S^3$ with genus $g$. Then $H_i \leq g$ when $i \leq g$ and $V_i \leq g$ when $i \geq 0$.

Proof. By (7) $H_g = V_g + g$ and by (5) $V_g = 0$. Therefore $H_g = g$. By (2) we have that $H_i \leq g$ for $i \leq g$. Since $V_0 = H_0$ by (7), and $H_0 \leq g$, we know that $V_0 \leq g$. Then by (2) we have that $V_i \leq g$ when $i \geq 0$.

Lemma 5. For a genus 1 knot $K$ we have that $U \cdot HF_{red} (S^3_{+1}(K)) = 0$.

Proof. By the mapping cone formula $HF^+ (S^3_1(K)) = H_* (A_0^+)$ and by Corollary 2 we have $U^{g(K)} \cdot H_* (A_0^+) = 0$,

where $g(K)$ is the genus of $K$, thus

$U \cdot HF_{red} (S^3_1(K)) = 0$.

We know that $S^3_1(K) = -S^3_{-1}(\overline{K})$, where $\overline{K}$ denotes the reflection of $K$. [22]. Thus it follows that

$S^3_{+1}(K) = -S^3_{-1}(\overline{K})$.

Since $K$ and $\overline{K}$ are both genus 1 knots it follows that

$U \cdot HF_{red} (S^3_{-1}(K)) = 0$,

by the arguments for $+1$ surgery.

We will refer to levels of the $A^+$ and $B^+$ complexes. This is simply a way to discuss the relative grading of these complexes. Level 0 will contain the bottom of $T^+$, and those elements that have this same grading, call this grading $g$. Then level 1 will consist of those elements in grading $g+2$, level 2 will consist of those elements in grading $g+4$. Thus in general elements in grading $g+2k$, will be in level $k$. Elements in different parity of grading will be in half levels. So an element in grading $g+2k+1$ will be in level $2k+1$. Linear combinations of these elements will always be sent to 0 by the $U$-action by Corollary 2. Thus we will not be considering elements in different parity of grading than the tower, as they can never contribute to $U$ of $HF_{red}$ being nonzero.

Example 2. Consider $A_0^+$ in figure 4. The $a_i$’s are the vertices in $A_0^+$, while the other elements are in $A_0^{red}$. We see that $a_0$ is the bottom of the tower, and thus is in level 0. The only elements in different parity of grading are $w$ and $z$. We can then say the other elements are in the following levels:
Before we proceed with our proof of Theorem 3, we will use Lemma 5, and look at $-1$ surgery in order to restrict the allowable maps for $v_0$ and $h_0$. Note, by Lemma 4 and (7), we only need to check when $V_0 = H_0 = 0$ and when $V_0 = H_0 = 1$. Figure 5 is the truncated mapping cone for $-1$ surgery. This means we have deleted the acyclic subcomplexes or quotient complexes; this process yields a new complex with the same homology [15]. Note in the figure, we have only drawn boxes to represent the reduced Heegaard Floer homologies.

**Remark 4.** Figure 5 gives the labeling we will use to refer to elements in $A^+_0, B^+_0$ and $B^T_{-1}$. The subscript refers to the level that that element is in.

First we consider when $V_0 = H_0 = 0$. Due to the grading restrictions on $v_0$ and $h_0$, and the fact that they are both $U$-equivariant maps, it is clear that each element of $A^+_0$ not at level 0 must be
sent to 0 by both $v_0$ and $h_0$. Elements at level 0 may either be sent to 0 or $\beta_0$ by $h_0$, and to 0 or $\gamma_0$ by $v_0$.

Now consider $V_0 = H_0 = 1$. Due to the grading restrictions on $v_0$ and $h_0$ it is clear that each element of $A_{0}^{red}$ at level 2 or higher must be sent to 0 by both $v_0$ and $h_0$. The same is true for each element of $A_{0}^{red}$ at or below level 0. Elements of $A_{0}^{red}$ at level 1 will be sent to either 0 or $\beta_0$ by $h_0$, and to 0 or $\gamma_0$ by $v_0$. We wish to explore these elements at level 1 further so let $x \in A_{0}^{red}$ be at level 1. Then we have the following cases.

Case 1: Let $h_0(x) = \beta_0$ and $v_0(x) = \gamma_0$. Then $a_1 + x$ is in the kernel of $D_{i}^{+}$, but $U(a_1 + x) \neq 0$. We know $a_1 + x$ is in $A_{0}^{red}$ of $-1$ surgery, so if $U(a_1 + x) \neq 0$ is true, $U \cdot HF_{red}(S_{1}^{3}(K)) \neq 0$. This contradicts Lemma 3 so this is not an allowed map.

Case 2: Let $v_0(x) = \gamma_0$, then $h_0(x) = 0$ for the same reasons as above.

Case 3: Let $v_0(x) = \beta_0$, $h_0(x) = 0$. We know there exists an isomorphism $\phi: A_{0}^{+} \rightarrow A_{0}^{+}$ that is $U$-equivariant, grading preserving, and satisfies $v_0 \circ \phi = h_0$. So $v_0(\phi(x)) = h_0(x) = \beta_0$. It follows that $\phi(x) \neq x$ and must be mapped to $h_0$. Since $V_0 = H_0 = 1$, $\phi(x)$ must be in level 1 of $A_0$ and also must satisfy $U \cdot \phi(x) = 0$ by the $U$-equivariance of $\phi$. Thus $\phi(x)$ must not be in the tower. In this case we have $U(x + \phi(x)) \neq 0$ and so this is also not allowed.

Case 4: The case where $h_0(x) = \beta_0$ and $v_0(x) = 0$ can be argued as case 3.

All of these above restrictions tell us that the only acceptable $v_0$ and $h_0$ maps must send every element of $A_{0}^{red}$ in level 1 to 0.

Given these restrictions, we are now ready to prove Theorem 3. Figure 4 depicts the truncated mapping cone for $\frac{1}{n}$ surgery of genus 1 knots for positive $n$. We consider the element $a^{i}_{(0,i)}$ to be in level $j$ of the $i$th copy of $A_{0}^{+}$, denoted $A_{(0,i)}^{red}$. We have a similar notation for $b^{i}_{(0,k)}$ in level $l$ of $B_{0,k}$.

Now we are ready to prove our main result for genus 1 knots.

**Proof of Theorem 3** First consider $V_0 = H_0 = 0$. By our case study above, the only acceptable maps are those that send every element in $A_{(0,i)}^{red}$, for all $i$, not at level 0 to 0. Thus it is clear that the only elements in $\ker(D_{1/n}^{+})$ are also elements of $A_{(0,i)}^{red}$ and so by Corollary 2 we have $U \cdot HF_{red}(S_{1/n}^{3}(K)) = 0$.

Now consider $V_0 = H_0 = 1$. By our earlier case study of $v_0$ and $h_0$ we know that every element of $A_{(0,i)}^{red}$, for all $i$, is in level 1 must be sent to 0. Thus the elements in $\ker(D_{1/n}^{+})$ consist of all elements of $A_{(0,i)}^{red}$, elements of the form $a_{(0,1)}^{i} + a_{(0,2)}^{i} + \cdots + a_{(0,n)}^{i}$ and sums of these elements. We know that $a_{(0,1)}^{i} + a_{(0,2)}^{i} + \cdots + a_{(0,n)}^{i}$ is in the tower as

$$U^{k} \left( a_{(0,1)}^{i+k} + a_{(0,2)}^{i+k} + \cdots + a_{(0,n)}^{i+k} \right) = a_{(0,1)}^{i} + a_{(0,2)}^{i} + \cdots + a_{(0,n)}^{i}.$$

Thus $HF_{red}(S_{1/n}^{3}(K)) = A_{0}^{red}$ and so $U \cdot HF_{red}(S_{1/n}^{3}(K)) = 0$, for positive $n$.

Now consider negative surgery. We know that $S_{\frac{3}{n}}^{3}(K) = -S_{\frac{3}{n}}^{3}(\overline{K})$ where $\overline{K}$ is the mirror of $K$. Since $K$ and $\overline{K}$ are both genus 1 knots it follows that

$$U \cdot HF_{red}(S_{\frac{3}{n}}^{3}(K)) = 0$$

by the arguments for positive $\frac{1}{n}$ surgery above. \qed
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