THE HAAGERUP PROPERTY FOR DRINFELD DOUBLES

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Abstract. We show that Drinfeld’s double group construction for locally compact quantum groups preserves the Haagerup property. This shows that the Drinfeld doubles of the quantum groups, $C_0(F_2)$, $SU_q(2)$, $SU_q(1,1)_{ext}$, quantum $ax+b$, quantum $az+b$, and $E_q(2)$ have the Haagerup property.

1. Introduction

The Haagerup approximation property for groups is weaker than the notion of amenability. One of the equivalent formulations of the Haagerup property for groups is the existence of a mixing representation that weakly contains the trivial representation. Following this characterisation, Daws, Fima, Skalski and White introduced the Haagerup property for locally compact quantum groups in [7]. Moreover, [7] Proposition 5.2 shows that the Haagerup property for a quantum group $G$ follows from the coamenability of the dual quantum group $\hat{G}$. As an application, they show that quantum $ax+b$ in [20], quantum $az+b$ in [24], and $E_q(2)$ in [14], and their duals have the Haagerup property (see [7, Example 5.4]). All of these examples are amenable and coamenable. Also, like compact groups, all compact quantum groups have the Haagerup property. The quantum group $C_0(F_2)$ associated to the free group $F_2$ is an example of a discrete non-amenable quantum group with the Haagerup property. The recent work of Caspers [6] shows that the extended $SU_q(1,1)$ groups in [9], are examples of non-classical, non-discrete and non-amenable quantum groups that enjoy the Haagerup property. Moreover, their duals also have the Haagerup property.

The quantum double construction of Drinfeld or the Drinfeld double for Hopf algebras is one of the fundamental results of the pioneering work of Drinfeld [8]. Roughly, the Drinfeld double of a finite dimensional Hopf algebra $H$ over a field $k$ is a Hopf algebra $D(H)$ such that the factors $H$ and $H^* := \text{Hom}_k(H,k)$ inside $D(H)$ do not commute. The quantum double construction for analytic quantum groups was developed in many different frameworks, along with the development of a general theory of compact and locally compact quantum groups. In [13], Podleś and Woronowicz introduced the double group construction, as the dual of the Drinfeld double, for compact quantum groups [13]. In [2], Baaj and Vaes obtained Drinfeld double for regular (in the sense of Baaj and Skandalis [1]) $C^*$-algebraic locally compact quantum groups (see [10,11]) as a special case of the dual of the generalised double crossed product construction.

In general, for $C^*$-algebraic locally compact quantum groups (see [10,11]), this is generalised by Masuda, Nakagami and Woronowicz [11, Section 8], under the name quantum codouble. We shall follow this terminology. (What we call quantum codouble is called Drinfeld double in [12, Section 3]).

The main result of this article is the following theorem:

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Result 1.1. Drinfeld’s double group construction preserves the Haagerup property. That is, the Drinfeld double of $G$ has the Haagerup property whenever the quantum group $G$ and its dual, $\hat{G}$ both have the Haagerup property.

After introducing basic definitions in Section 2, we briefly extract the C*-bialgebra structure from their construction in Section 3. In the last Section 4, we prove our main result 1.1. Then, we show that the Drinfeld doubles of the classical group $\mathbb{F}_2$, quantum ax + b, quantum $ax + b$, quantum E(2), quantum SU(2) and extended quantum SU(1,1) have the Haagerup property.

1.1. Basic notation. All Hilbert spaces and C*-algebras are assumed to be separable. For two norm-closed subsets $X$ and $Y$ of a C*-algebra, let

$$X \cdot Y := \{ xy : x \in X, y \in Y \}^{\text{CLS}},$$

where CLS stands for the closed linear span.

For a C*-algebra $A$, let $\mathcal{M}(A)$ be its multiplier algebra and let $\mathcal{U}(A)$ be the group of unitary multipliers of $A$. Let $C^\ast\text{-alg}$ be the category of C*-algebras with nondegenerate *-homomorphisms $\varphi : A \to \mathcal{M}(B)$ as morphisms $A \to B$; let $\text{Mor}(A, B)$ denote this set of morphisms.

For a Hilbert space $\mathcal{H}$, $K(\mathcal{H})$ and $B(\mathcal{H})$ denote the C*-algebras of compact and bounded operators acting on $\mathcal{H}$, respectively. A representation of a C*-algebra $A$ on a Hilbert space is an element of $\text{Mor}(A, K(\mathcal{H}))$. The group of unitary operators on a Hilbert space $\mathcal{H}$ is denoted by $U(\mathcal{H})$.

We write $\Sigma$ for the tensor flip $H \otimes K \to K \otimes H$, $x \otimes y \mapsto y \otimes x$, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We write $\sigma$ for the tensor flip isomorphism $A \otimes B \to B \otimes A$ for two C*-algebras $A$ and $B$, where $\otimes$ denotes the minimal tensor product of C*-algebras.

2. Locally compact quantum groups

For a general theory of C*-algebraic locally compact quantum groups see [10][11].

Definition 2.1 ([11 Définition 0.1]). A C*-bialgebra $(A, \Delta_A)$ is a C*-algebra $A$ and a comultiplication $\Delta_A \in \text{Mor}(A, A \otimes A)$ that is coassociative: $(\text{id}_A \otimes \Delta_A) \circ \Delta_A = (\Delta_A \otimes \text{id}_A) \circ \Delta_A$. Moreover, if $\Delta_A$ satisfies the cancellation property,

$$\Delta_A(A) \cdot (1_A \otimes A) = \Delta_A(A) \cdot (A \otimes 1_A) = A \otimes A,$$

$(A, \Delta_A)$ is a bisimplifiable C*-bialgebra.

Let $\varphi$ be a faithful (approximate) KMS weight (see [11 Section 1]) on $A$. The set of all positive $\varphi$-integrable elements is defined by $\mathcal{M}_\varphi^+ := \{ a \in A^+ : \varphi(a) < \infty \}$. Moreover, $\varphi$ is called

1. left invariant if $\omega((\text{id}_A \otimes \varphi)\Delta_A(a)) = \omega(1)\varphi(a)$ for all $\omega \in A^+_\varphi$, $a \in \mathcal{M}_\varphi^+$;
2. right invariant if $\omega((\varphi \otimes \text{id}_A)\Delta_A(a)) = \omega(1)\varphi(a)$ for all $\omega \in A^+_\varphi$, $a \in \mathcal{M}_\varphi^+$.

Definition 2.2 ([10] Definition 4.1]). A locally compact quantum group (quantum groups from now onwards) is a bisimplifiable C*-bialgebra $G = (A, \Delta_A)$ with left and right invariant approximate KMS weights $\varphi$ and $\psi$, respectively.

By Theorem 7.14 and 7.15 in [10], the invariant weights $\varphi$ and $\psi$ are unique up to a positive scalar factor; hence they are called the left and right Haar weights for $G$. Moreover, there is a unique (up to isomorphism) Pontrjagin dual $\hat{G} = (\hat{A}, \hat{\Delta}_A)$ of $G$, which is again a quantum group.

Next we consider the GNS triple $(L^2(G), \pi, A)$ for $\varphi$. There is a right multiplicative unitary $\mathbb{W} \in U(L^2(G) \otimes L^2(G))$. Equivalently, $\mathbb{W}$ satisfies the pentagon equation:

$$W_{23}W_{12} = W_{12}W_{13}W_{23} \quad \text{in } U(L^2(G) \otimes L^2(G)) \quad (2.3).$$
The right Haagerup weight version of the result [10] Proposition 6.10] ensures the manageability (see [15] Definition 2.1) of \( \mathbb{W} \). The theory of manageable multiplicative unitaries [11] gives:

1. the dual multiplicative unitary \( \hat{W} := \Sigma^* \Sigma \in \mathcal{U}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})) \) is also manageable.
2. the slices of \( \mathbb{W} \) defined by

\[
A := \{ (\omega \otimes \text{id}_{L^2(\mathbb{G})}) \mathbb{W} : \omega \in \mathbb{B}(L^2(\mathbb{G})), \}^{\text{CLS}},
\]

\[
\hat{A} := \{ (\text{id}_{L^2(\mathbb{G})} \otimes \omega) \mathbb{W} : \omega \in \mathbb{B}(L^2(\mathbb{G})), \}^{\text{CLS}},
\]

are nondegenerate C*-subalgebras of \( \mathbb{B}(L^2(\mathbb{G})) \).
3. \( \mathbb{W} \in \mathcal{U}(\hat{A} \otimes A) \subseteq \mathcal{U}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})) \). We write \( \mathbb{W} \) for \( \mathbb{W} \) viewed as a unitary multiplier of \( \hat{A} \otimes A \);
4. the comultiplication maps \( \Delta_A \) and \( \hat{\Delta}_A \) are characterised by the following conditions:

\[
(id_A \otimes \Delta) \mathbb{W} = W_{12} W_{13} \quad \text{in } \mathcal{U}(A \otimes \hat{A} \otimes \hat{A})
\]

\[
(\hat{\Delta} \otimes id_A) \mathbb{W} = W_{23} W_{13} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes A).
\]

5. there exist antiunitary involutive operators \( J \) and \( \hat{J} \) on \( L^2(\mathbb{G}) \). They implement the unitary antipodes \( R \) and \( \hat{R} \) on \( G \) and \( \hat{G} \) as follows:

\[
R(a) := \hat{J} a^* \hat{J} \quad \text{for } a \in A \quad \text{and} \quad \hat{R}(\hat{a}) := J a^* J \quad \text{for } \hat{a} \in \hat{A}.
\]

The unitary \( \mathbb{W} \in \mathcal{M}(\hat{A} \otimes A) \) is called the reduced bicharacter of \( G \).

**Definition 2.6** ([3] Definition 3.1). A quantum group \( G = (A, \Delta_A) \) is coamenable if it has a bounded counit. Equivalently, there is a unique \(*\)-homomorphism \( e^A \in \text{Mor}(A, \mathbb{C}) \) such that

\[
(e^A \otimes \text{id}_A) \Delta_A = \text{id}_A.
\]

3. **Duality between quantum codoubles and Drinfeld doubles**

Let \( G = (A, \Delta_A) \) be a quantum group, let \( \hat{G} = (\hat{A}, \hat{\Delta}_A) \) be its dual, and let \( \mathbb{W} \in \mathcal{U}(\hat{A} \otimes A) \) be the reduced bicharacter.

Define

\[
\hat{D} := \hat{A} \otimes A \quad \text{and} \quad \hat{\Delta}_D(\hat{a} \otimes a) := (\text{id}_A \otimes \sigma^W \otimes \text{id}_A)(\hat{\Delta}_A \otimes \Delta_A).
\]

Here \( \sigma^W \in \text{Mor}(\hat{A} \otimes A, A \otimes \hat{A}) \) denotes the flip twisted by \( \mathbb{W} \) defined by \( \sigma^W(\cdot) := \sigma(W(\cdot)W) \). Define \( U := \mathbb{W}(\hat{J} \otimes J) \mathbb{W}(J \otimes J) \in \mathcal{U}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})) \). Theorem 8.7 in [11] shows that \( (\hat{D}, \Delta_D) \) is a quantum group and that

\[
\hat{W}^D := U_{12} \hat{W}_{13} U_{12}^* W_{24} \in \mathcal{U}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))
\]

is a manageable multiplicative unitary for it. The quantum codouble of \( G \), denoted by \( \mathcal{D}(G)^- \), is the C*-bialgebra \( (\hat{D}, \Delta_D) \).

**Definition 3.2.** A pair of representations \( \rho : A \to \mathbb{B}(\mathcal{H}) \) and \( \theta : \hat{A} \to \mathbb{B}(\mathcal{K}) \) is called a \( G \)-Drinfeld pair if it satisfies the \( G \)-Drinfeld commutation relation:

\[
W_{1\rho} W_{13} W_{\theta_3} = W_{\theta_3} W_{13} W_{1\rho} \quad \text{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes A).
\]

Here \( W_{1\rho} := ((\text{id}_A \otimes \rho) \mathbb{W})_{12} \) and \( W_{\theta_3} := ((\theta \otimes \text{id}_A) \mathbb{W})_{23} \).

Define \( \rho : A \to \mathbb{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})) \) and \( \theta : \hat{A} \to \mathbb{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})) \) by

\[
\rho(a) := \mathbb{U}(a \otimes 1_{L^2(\mathbb{G})}) \mathbb{U}^* \quad \text{and} \quad \theta(\hat{a}) := 1_{L^2(\mathbb{G})} \otimes \hat{a}.
\]

Here we drop the GNS representations of \( A \) and \( \hat{A} \) on \( L^2(\mathbb{G}) \).
Proposition 3.4. The pair \((\rho, \theta)\) is a \(G\)-Drinfeld pair. Define \(D := \rho(A) \cdot \theta(\hat{A}) \subseteq B(L^2(G) \otimes L^2(G))\) and a map \(\Delta_D : D \to B(L^2(G) \otimes L^2(G) \otimes L^2(G) \otimes L^2(G))\) by
\[
\Delta_D(\rho(a) \cdot \theta(\hat{a})) := (\rho \otimes \rho)(\Delta_A(a)(\theta \otimes \theta)(\hat{A})(\hat{a})) \quad \text{for} \quad a \in A, \hat{a} \in \hat{A}.
\]
Then \((D, \Delta_D)\) is the dual quantum group of the quantum codouble \(\mathcal{D}(G)^-\).

Proof. The \(G\)-Drinfeld commutation relation for the pair \((\rho, \theta)\) is equivalent to the following relation
\[
U_{23}W_{12}U_{23}^* W_{14}W_{34} = W_{34}W_{14}U_{23}W_{12}U_{23}^*;
\]
then one is the intermediate steps in the proof of Proposition 8.6 in [11].

The multiplicative unitary in (3.1) is \(\tilde{W}^D = \tilde{W}_{\rho_2}W_{\theta_3}\). The manageability of \(W\) and \(\tilde{W}\) imply:
\[
\{(\iota_{L^2(G)} \otimes \omega \otimes \omega') \tilde{W}_{\rho_2}W_{\theta_3} : \omega, \omega' \in B(L^2(G))\}_\text{CLS} = \theta(\hat{A}) \cdot \rho(A) = D.
\]
Since \(\tilde{W}^D\) is also manageable, \(D\) is a nondegenerate \(C^*\)-subalgebra of \(B(L^2(G) \otimes L^2(G))\).

Furthermore, we get \(\tilde{W}_{\rho_2}W_{\theta_3} \in U(D \otimes \tilde{D})\).

The definition of \(\Delta_D\) gives \(\Delta_D \in \text{Mor}(D, D \otimes D)\). Hence it is sufficient to check that \(\Delta_D\) satisfies (2.5) for \(W = \tilde{W}_{\rho_2}W_{\theta_3}\). We compute
\[
(\Delta_D \otimes \text{id}_{A \otimes A}) \tilde{W}_{\rho_2}W_{\theta_3} = (((\rho \otimes \rho) \hat{\Delta} \otimes \text{id}_{A}) \tilde{W})_{123}(((\theta \otimes \theta) \Delta \otimes \text{id}_A)W)_{124}
\]
\[
= \tilde{W}_{\rho_{i,3}}W_{\rho_{i,4}}W_{\theta_{1,4}}W_{\theta_{2,4}} = \tilde{W}_{\rho_{i,3}}W_{\rho_{i,4}}W_{\theta_{2,4}}.
\]
The first equality follows from (3.3); the second equality uses (2.4) and (2.5), and \(\rho_i\) or \(\theta_i\) means \(\rho\) or \(\theta\) acting on the \(i\)-th leg for \(i = 1, 2\); and the third equality uses the trivial commutation between \(\tilde{W}_{\rho_{i,3}}\) and \(W_{\theta_{i,4}}\). \(\square\)

Definition 3.6. The quantum group \(\mathcal{D}(G) := (D, \Delta_D)\) is the Drinfeld double of \(G\).

Example 3.7. Let \(A = C_0(G)\) and \(\hat{A} = C^*_r(G)\) for a locally compact group \(G\). The underlying \(C^*\)-algebra of the Drinfeld double of \(G\) is \(C(\hat{G}) \rtimes_{\text{conj}} G\).

4. The Haagerup Property for the Drinfeld Double

A (unitary) representation of \(\mathcal{G} = (A, \Delta_A)\) on a Hilbert space \(\mathcal{H}\) is a unitary \(V \in \mathcal{U}(K(\mathcal{H}) \otimes A)\) satisfying the following condition:
\[
(4.1) \quad (\text{id}_H \otimes \Delta_A)V = V_{12}V_{13} \quad \text{in} \quad \mathcal{U}(K(\mathcal{H}) \otimes A \otimes A).
\]

Definition 4.2 ([7, Definition 5.1]). A locally compact quantum group \(\mathcal{G} = (A, \Delta_A)\) has the Haagerup property if there is a representation \(V \in \mathcal{U}(K(\mathcal{H}) \otimes A)\) with the following properties:

1. there is a net \(\{x_i\}\) of almost invariant unit vectors in \(\mathcal{H}\):
\[
(4.3) \quad \|V(x_i \otimes \eta) - x_i \otimes \eta\| \to 0 \quad \text{for all} \quad \eta \in L^2(G).
\]
2. \(V\) is a mixing representation: \((\omega_{x,y} \otimes \text{id}_A)U \in A\) for all \(x, y \in \mathcal{H}\), where \(\omega_{x,y}(T) := \langle Tx, y \rangle\) is the vector functional.

Next we prove our main result:

Theorem 4.4. Let \(\mathcal{G}\) and \(\hat{\mathcal{G}}\) have the Haagerup property. Then the Drinfeld double of \(\mathcal{G}\) also has the Haagerup property.
Proof. Let \( X \in \mathcal{U}(\mathbb{K}(\mathcal{H}_1) \otimes A) \) and \( Y \in \mathcal{U}(\mathbb{K}(\mathcal{H}_2) \otimes \hat{A}) \) be representations of \( G \) and \( \hat{G} \), respectively.

Define \( V := X_{1p}Y_{20} \in \mathcal{U}(\mathbb{K}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{D}_V) \). Equations (3.3) and (1.1) for \( X \) and \( Y \) give:

\[
(id_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes \Delta_{\mathcal{D}_V})X_{1p}Y_{20} = X_{1p}X_{1p}Y_{20} = X_{1p}Y_{20}X_{1p}Y_{20}.
\]

Hence \( V \) is a representation of the Drinfeld double \( \mathcal{D}(G) \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

Let \( X \) and \( Y \) satisfy the mixing condition (2) in Definition 4.2. Then we get the mixing condition for \( V \):

\[
(\omega_{x_1 \otimes y_1, x_2 \otimes y_2} \otimes id_{\mathcal{D}_V})(X_{1p}Y_{20}) = \rho(\omega_{x_1, y_1} \otimes id_A)X \theta(\omega_{y_1, y_2} \otimes id_A)Y \in \mathcal{D}_V
\]

for all \( x_1, x_2 \in \mathcal{H}_1, y_1, y_2 \in \mathcal{H}_2 \).

Let \( X \in \mathcal{L}(\mathcal{H}_1 \otimes A) \) be the image of \( X \in \mathcal{U}(\mathbb{K}(\mathcal{H}_1) \otimes A) \) under the canonical isomorphism between \( \mathcal{M}(\mathbb{K}(\mathcal{H}_1) \otimes A) \) and \( \mathcal{L}(\mathcal{H}_1 \otimes A) \) (the space of adjointable operators on the Hilbert \( A \)-module \( \mathcal{H}_1 \otimes A \)). By [7, Proposition 2.7], condition (4.3) becomes equivalent to the following condition:

(4.5) \[ \|X(x_i \otimes a) - x_i \otimes a\| \to 0 \text{ for all } a \in A. \]

Let \( \{x_i\} \) be the net of almost invariant unit vectors in \( \mathcal{H}_1 \) for \( X \). Let \( d \in \mathcal{D}_V \) and let \( \{e_{x_i}^A\} \) be a bounded approximate identity in \( A \). The construction of \( D \) gives \( \rho \in \text{Mor}(A, \mathcal{D}) \). Given any \( \epsilon > 0 \) there exists \( \lambda' \) such that \( \|\rho(e_{x_i}^A) d - d\| < \epsilon/3 \); hence

\[ \|X_{1p}(x_i \otimes d) - X_{1p}(x_i \otimes \rho(e_{x_i}^A) d)\| < \epsilon/3, \quad \|x_i \otimes \rho(e_{x_i}^A) d - x_i \otimes d\| < \epsilon/3 \]

for each \( x_i \). Furthermore, by (4.3), there exists \( i' \) such that for \( i \geq i' \)

\[ \|X_{1p}(x_i \otimes \rho(e_{x_i}^A) d) - x_i \otimes \rho(e_{x_i}^A) d\| \leq \|d\| \cdot \|X_{1p}(x_i \otimes \rho(e_{x_i}^A)) - x_i \otimes \rho(e_{x_i}^A)\| < \epsilon/3. \]

The last two estimates together give

\[ \|X_{1p}(x_i \otimes d) - (x_i \otimes d)\| \leq \|X_{1p}(x_i \otimes d) - X_{1p}(x_i \otimes \rho(e_{x_i}^A) d)\| + \|X_{1p}(x_i \otimes \rho(e_{x_i}^A) d) - x_i \otimes \rho(e_{x_i}^A) d\| + \|x_i \otimes \rho(e_{x_i}^A) d - x_i \otimes d\| < \epsilon \quad \text{for } i \geq i'. \]

Since \( \epsilon > 0 \) is arbitrary, we get \( \lim_{i \to \infty} \|X_{1p}(x_i \otimes d) - (x_i \otimes d)\| = 0 \) for \( d \in \mathcal{D}_V \).

Similarly, for a net of almost invariant unit vectors \( \{y_j\} \) in \( \mathcal{H}_2 \) for \( Y \), we have \( \lim_{j \to \infty} \|Y_{1p}(y_j \otimes d) - y_j \otimes d\| = 0 \) for \( d \in \mathcal{D}_V \).

Finally, we show that \( \{x_i \otimes y_j\} \) is an almost invariant unit vector for \( X_{1p}Y_{20}. \)

We compute

\[
\|X_{1p}Y_{20}(x_i \otimes y_j \otimes d) - x_i \otimes y_j \otimes d\|
\leq \|X_{1p}Y_{20}(x_i \otimes y_j \otimes d) - X_{1p}(x_i \otimes y_j \otimes d)\| + \|X_{1p}(x_i \otimes y_j \otimes d) - x_i \otimes y_j \otimes d\|.
\]

Therefore, \( \lim_{i,j} \|X_{1p}Y_{20}(x_i \otimes y_j \otimes d) - x_i \otimes y_j \otimes d\| = 0 \) for \( d \in \mathcal{D}_V \).

\[ \square \]

Corollary 4.6. Assume \( G \) and \( \hat{G} \) are coamenable. Then the Drinfeld double \( \mathcal{D}(G) \) has the Haagerup property.

Proof. By [7, Proposition 5.2], both \( G \) and \( \hat{G} \) have the Haagerup property; hence \( \mathcal{D}(G) \) has the Haagerup property. \[ \square \]

Remark 4.7. The proof of Theorem 4.4 also works in more general framework of locally compact quantum groups constructed from manageable multiplicative unitaries [17]. Furthermore, one can generalise the statement of Theorem 4.4 by replacing \( \hat{G} \) by another locally compact quantum group \( \hat{H} \) with the Haagerup property, and the Drinfeld of \( G \) by generalised Drinfeld double of \( G \) and \( \hat{H} \) with respect to a given bicharacter (see [14]) between them.
Proposition 4.8. The Drinfeld doubles of locally compact groups $G$ with the Haagerup property, coamenable compact quantum groups $G$, extended quantum SU(1, 1), quantum $ax+b$, quantum $az+b$ and quantum $E(2)$ groups have the Haagerup property, respectively. In particular, the Drinfeld doubles of $\mathbb{F}_n$ (free group with $n(\geq 2)$ generators) and the duals of quantum Lorentz groups have the Haagerup property.

Proof. Let $G$ be a locally compact group; hence $C_0(G)$ is coamenable. By [2, Proposition 5.2], $C^*_r(G)$ (as a quantum group) has the Haagerup property. Furthermore, if $G$ has the Haagerup property if and only if $C_0(G)$ has the Haagerup property. Consequently, by Theorem 4.4, the Drinfeld double of $G$, namely, $C_0(G) \rtimes \text{conj}G$ has the Haagerup property if and only if $G$ has the Haagerup property. In particular, the Drinfeld doubles of $\mathbb{F}_n$ (free group with $n(\geq 2)$ generators) have the Haagerup property.

It is shown in [6] that the extended quantum SU(1, 1) groups have the Haagerup property. By [6, Theorem 8.3], their duals also have the Haagerup property. By Theorem 4.4, Drinfeld doubles of the extended SU$_q(1, 1)$ has the Haagerup property.

Let $G$ be a compact quantum group. By, [3, Proposition 4.1], the discrete quantum group $\hat{G}$ is always coamenable. Additionally, if $G$ is coamenable, by Corollary 4.6, $D(G)$ is amenable and has the Haagerup property. In particular, SU$_q(2)$ is a compact quantum group and coamenable (see [5]). Hence, the Drinfeld double of SU$_q(2)$, dual of the quantum Lorentz group in [13], has the Haagerup property.

Finally, [7, Example 5.4] shows that quantum $ax+b$, quantum $az+b$ and quantum $E(2)$ groups have the Haagerup property. Therefore, by Corollary 4.6 we conclude that their respective Drinfeld doubles have the Haagerup property. □

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