A remark about orthogonal matching pursuit algorithm

Zhiqiang Xu*

Abstract

In this note, we investigate the theoretical properties of Orthogonal Matching Pursuit (OMP), a class of decoder to recover sparse signal in compressed sensing. In particular, we show that the OMP decoder can give \((p, q)\) instance optimality for a large class of encoders with \(1 \leq p \leq q \leq 2\) and \((p, q) \neq (2, 2)\). We also show that, if the encoding matrix is drawn from an appropriate distribution, then the OMP decoder is \((2, 2)\) instance optimal in probability.

1 Introduction

We consider a signal \(x \in \mathbb{R}^N\) where \(N\) is large and denote by \(\Sigma_k\) the set of \(k\)-sparse vectors, i.e.,

\[ \Sigma_k := \{ x \in \mathbb{R}^N : \# \text{supp}(x) \leq k \}, \]

where \(\text{supp}(x)\) is the set of \(i\) for which \(x_i \neq 0\) and \#\(A\) is the number of elements in the set \(A\). Given a norm \(\| \cdot \|_X\) on \(\mathbb{R}^N\), we set

\[ \beta_k(x) := \arg\min_{z \in \Sigma_k} \| x - z \|_X, \]

and

\[ \sigma_k(x)_X := \| x - \beta_k(x) \|_X, \]

and call \(\beta_k(x)\) and \(\sigma_k(x)_X\) as the the best \(k\)-term approximation and the best \(k\)-term approximation error, respectively.

In Compressed Sensing theory, the information we gather about \(x\) can be described by

\[ y = \Phi x \]

where \(\Phi\) is an \(n \times N\) matrix. To recover \(x\) from \(y\), we use a decoder \(\Delta : \mathbb{R}^n \rightarrow \mathbb{R}^N\) and denote \(x^* := \Delta(y) = \Delta(\Phi x)\). The \(x^*\) can be considered as an approximation

*Supported by the National Natural Science Foundation of China (10871196).
of $x$. Following Cohen, Dahmen and DeVore, we say that a pair $(\Phi, \Delta)$ is instance optimality of order $k$ with constant $C$ for the norm $X$ if it satisfies

$$\| x - \Delta(\Phi x) \|_X \leq C \sigma_k(x)_X, \quad \text{for all } x \in \mathbb{R}^N. \quad (1)$$

In general, one chooses $X$ as $\ell_p$ (quasi-)norm where $p > 0$. To state conveniently, throughout of this paper, we use the subscript $p$ to denote the $\ell_p$ norm. Using the notation of [12], we say that a pair $(\Phi, \Delta)$ is $(q, p)$ instance optimality of order $k$ with constant $C$ if

$$\| x - \Delta(\Phi x) \|_q \leq C \frac{\sigma_k(x)_p}{\sqrt{1/p - 1/q}}, \quad \text{for all } x \in \mathbb{R}^N. \quad (2)$$

The theoretical analysis of instance optimality is presented in [4].

We next introduce a class of encoding matrix. Following Candès and Tao, we say that the matrix $\Phi$ satisfies the Restricted Isometry Property (RIP) of order $k$ and constant $\delta_k \in (0, 1)$ if

$$(1 - \delta_k) \| x \|_2^2 \leq \| \Phi x \|_2^2 \leq (1 + \delta_k) \| x \|_2^2$$

holds for all $x \in \Sigma_k$. Throughout the rest of the paper, using the notation of [12], we say that the matrix $\Phi$ satisfies RIP($k, \delta$) if $\delta_k < \delta$.

In the past, one investigated the instance optimality of $\ell_1$ minimization, which is given by

$$\Delta_1^\epsilon(y) := \arg\min_x \| x \|_1, \quad \text{subject to } \| \Phi x - y \|_2 \leq \epsilon.$$ 

In [2], Candès improved on the work of Candès, Romberg and Tao [3] and showed that

$$\| \Delta_1^\epsilon(\Phi x) - x \|_2 \leq C_0 \sigma_k(x)_1 / \sqrt{s} + C_1 \epsilon$$

provided $\Phi$ satisfies RIP($2k, \sqrt{2} - 1$). The result implies that, in the noise-free case, i.e., $\epsilon = 0$, $(\Phi, \Delta_1^0)$ is $(2, 1)$ instance optimality of order $k$ if $\Phi$ satisfies RIP($2k, \sqrt{2} - 1$). In [9], Saab and Yılmaz extended the results to $\ell_p$ decoder, where $0 < p < 1$, which is defined by

$$\Delta_p^\epsilon(y) := \arg\min_x \| x \|_p, \quad \text{subject to } \| \Phi x - y \|_2 \leq \epsilon,$$

and showed that $(\Phi, \Delta_p^0)$ is $(2, p)$ instance optimality of order $k$ if $\Phi$ satisfies some RIP condition (see [9]). Independently, Foucart and Lai [6] also proved that $(\Phi, \Delta_p^0)$ is $(2, p)$ instance optimality under other sufficient conditions.

In compressed sensing, an alternative decoder is Orthogonal Matching Pursuit (OMP). The major advantages of OMP are its ease of implementation and potentially faster than $\Delta_1^\epsilon$ (see [7][10][11]). However, so far, very few theoretical results about the instance optimality of OMP are known. The aim of the note is the investigation of the instance optimality property of OMP decoder. To state conveniently, we use OMP$_M$ to denote the OMP decoder with $M$ iterations (see Algorithm 1). Then combining the methods developed by Cohen, Dahmen and DeVore [4] and the result obtained by Zhang [13], we can prove the following result, which is the main result of this note:
Theorem 1. Suppose that \( 1 \leq p \leq q \leq 2 \) and \( 0 < \delta \leq 1 \). We furthermore suppose that \( p \neq 2 \). Let \( \Phi \) be any matrix which satisfies RIP condition \( \text{RIP}(L, \delta) \) and \( \delta_k + (1 + \delta)\delta_{2k} \leq \delta \), where \( \alpha := \lceil 16 + 15\delta \rceil \) and \( L := k(N/k)^{2-2/p} \). Then for any signal \( x \) and any permutation \( e \) with \( \|e\|_2 \leq \epsilon \), the solution \( x^* := \text{OMP}_{2(\alpha-1)k}(\Phi x + e) \) obeys

\[
\|x^* - x\|_q \leq C_0 \frac{\sigma_k(x)_p}{k^{1/p-1/q}} + C_1 k^{1/q-1/2} \epsilon,
\]

where \( C_0 = 1 + C_1 + (2\alpha)^{1/q-1/2} \) and \( C_1 = (2\alpha)^{1/q-1/2}(2(1 + \delta)(\sqrt{11} + 20\delta + 1) + 1) \).

Algorithm 1 \( \text{OMP}_M(y) \)

**Input:** encoding matrix \( \Phi \), the vector \( y \), maximum allowed sparsity \( M \)

**Output:** the \( x^* \).

**Initialize:** \( r^0 = y, c^0 = 0, \Lambda^0 = \emptyset, \ell = 0 \).

**while** \( \ell < M \) **do**

**match:** \( h^\ell = \Phi^T r^\ell \)

**identity:** \( \Lambda^{\ell+1} = \Lambda^\ell \cup \{ \arg\max_j |h^\ell(j)| \} \)

**update:** \( c^{\ell+1} = \arg\min_{z: \supp(z) \subseteq \Lambda^{\ell+1}} \|y - \Phi z\|_2 \)

\[
r^{\ell+1} = y - \Phi c^{\ell+1}
\]

\( \ell = \ell + 1 \)

**end while**

\( x^* = c^M \)

By setting \((q, p) = (2, 1)\) and \( \epsilon = 0 \) in Theorem 1, we obtain the following Corollary:

Corollary 1. Suppose that \( \Phi \) satisfies the RIP condition \( \delta_{2k} + (1 + \delta)\delta_{2ak} \leq \delta \). Then

\[
\|\text{OMP}_{2(\alpha-1)k}(\Phi x) - x\|_2 \leq C_2 \sigma_k(x)_1/\sqrt{k},
\]

where \( \alpha = \lceil 16 + 15\delta \rceil \) and \( C_2 = 2(1 + \delta)(\sqrt{11} + 20\delta + 1) + 3 \).

Remark 1. Theorem 1 implies that \((\Phi, \text{OMP})\) is \((q, p)\) instance optimality of order \( k \) provided \( \Phi \) satisfies RIP condition of order \( k(N/k)^{2-2/p} \). Note that \( n \times N \) matrix \( \Phi \) can have RIP of order \( k \) if \( k = O(n/\log(N/n)) \). Then, for OMP decoder, \((q, p)\) instance optimality can be achieved at the price of \( O(k(N/k)^{2-2/p} \log(N/k)) \) measurements. As shown in [4] (Theorem 7.3), the number of measurements is optimal up to a constant.

Remark 2. No recover method can improve the term \( k^{1/q-1/2} \epsilon \) on the right side of (3) for arbitrary perturbations \( e \). To see why this true, suppose that we take \( x \in \Sigma_k \) and we know in advance the support of \( x \), i.e., \( T_0 = \supp(x) \).
Using this additional information, as shown in [3], Least-Square has the best performance to recover $x$. Set $y = \Phi x + e$. Then

$$x^* = \begin{cases} (\Phi_{T_0}^T \Phi_{T_0})^{-1} \Phi_{T_0} y, & \text{on } T_0, \\ 0, & \text{elsewhere}. \end{cases}$$

A simple observation is that $(x^* - x)_{T_0^c} = 0$ and

$$(x^* - x)_{T_0} = (\Phi_{T_0}^T \Phi_{T_0})^{-1} \Phi_{T_0} e.$$  

Here, we use $T_0^c$ to denote the complement of $T$ and $x_{T_0}$ to denote the vector which agrees with $x$ on $T_0$ and has all components equal to zero on $T_0^c$. Then Hölder inequality implies that

$$\|x^* - x\|_q = \|(\Phi_{T_0}^T \Phi_{T_0})^{-1} \Phi_{T_0} e\|_q \leq k^{1/q - 1/2}\|(\Phi_{T_0}^T \Phi_{T_0})^{-1} \Phi_{T_0} e\|_2,$$

where the equality holds for some non-zero permutation $e$. Since $\Phi$ satisfies RIP condition of order $L$,

$$\|(\Phi_{T_0}^T \Phi_{T_0})^{-1} \Phi_{T_0} e\|_2 \approx \|\Phi_{T_0} e\|_2 \approx \epsilon,$$

which implies that $\|x^* - x\|_q \approx k^{1/q - 1/2} \epsilon$ for some non-zero permutation $e$ provided $x \in \Sigma_k$.

We next consider the $(2, 2)$ instance optimality. As pointed out in [4], to obtain $(2, 2)$ instance optimality with order $k = 1$, one has to require the number of measurements is $O(N)$, which is not what we hope. Hence, instance optimality in probability is the proper formulation in $\ell_2$. We let $\Omega$ be a probability space with probability measure $P$ and let $\Phi = \Phi(\omega), \omega \in \Omega$, be an $n \times N$ random matrix and suppose $\Delta(\omega)$ is a corresponding family of decoders. We say that $(\Phi(\omega), \Delta(\omega))$ is $(2, 2)$ instance optimality in probability of order $k$ with constant $C$ if

$$\|x - \Delta(\Phi x)\|_2 \leq C\sigma_k(x)$$

holds with high probability for this particular $x$. To state our results, we first introduce two properties that the random matrix $\Phi$ should satisfy (see [4]).

**Definition 1.** We say that the $n \times N$ random matrix $\Phi$ satisfies RIP of order $k$ with constants $\delta_k$ and probability $1 - \epsilon$ if there is a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) \geq 1 - \epsilon$ such that for all $\omega \in \Omega_0$ the matrix $\Phi(\omega)$ satisfies

$$(1 - \delta_k)\|x\|_2^2 \leq \|\Phi(\omega)x\|_2^2 \leq (1 + \delta_k)\|x\|_2^2,$$

for all $x \in \Sigma_k$.

**Definition 2.** We say that the random matrix $\Phi$ has the boundedness property with constant $C$ and probability $1 - \epsilon$ if for each $x \in \mathbb{R}^N$ there is a set $\Omega_0(x) \subset \Omega$ with $P(\Omega_0(x)) \geq 1 - \epsilon$ such that for all $\omega \in \Omega_0(x)$,

$$\|\Phi(\omega)x\|_2^2 \leq C\|x\|_2^2.$$ (4)
The key thing in Definition 2 is that the set of \( \omega \)'s where (4) holds depends on \( x \). The both properties above have been shown for random matrices of the Gaussian or Bernoulli type (see [1]). Now we can state the result of instance optimality with probability for \( \ell_2 \) norm.

**Theorem 2.** Suppose \( 0 < \delta \leq 1 \) and \( \alpha := \lceil 16 + 15\delta \rceil \). Assume that \( \Phi \) is a random matrix which satisfies the RIP condition \( \delta_k + (1 + \delta)\delta_{\alpha k} \leq \delta \) with probability \( 1 - \epsilon \), and also satisfies the boundedness property with constant \( C \) and probability \( 1 - \epsilon \). Then for each \( x \in \mathbb{R}^N \), there exists a set \( \Omega(x) \subset \Omega \) with \( \mathbb{P}(\Omega(x)) \geq 1 - 2\epsilon \) such that for all \( \omega \in \Omega(x) \) and \( \Phi = \Phi(\omega) \),

\[
\|x^* - x\|_2 \leq C_3 \sigma_k(x)
\]

where \( x^* := \text{OMP}_{(\alpha - 1)k}(\Phi x) \) and \( C_3 = 1 + \sqrt{C(1 + \delta)(1 + 11 + 20\delta)} \).

**Remark 3.** According to Theorem 2, OMP can achieve (2,2) instance optimality in probability of order \( k \) after at least 15\( k \) steps. Suppose that \( \Phi \) satisfy the conditions in Theorem 2. Then, the following question is interesting: what is the minimal value of \( c_0 \) for which OMP can achieve (2,2) instance optimality in probability of order \( k \) after \( c_0 k \) iterations?

## 2 The proofs of Theorem 1 and Theorem 2

We first recall a result obtained by T. Zhang, which plays an important role in our analysis.

**Theorem 3.** ( [13]) Let \( \bar{x} \in \mathbb{R}^N \) and \( 0 < \delta \leq 1 \). If the RIP condition \( \delta \|\bar{x}\|_0 + (1 + \delta)\delta_{\alpha \|\bar{x}\|_0} \leq \delta \) holds, then when \( s = (\alpha - 1)\|\bar{x}\|_0 \), we have

\[
\|\Phi x^* - y\|_2^2 \leq (11 + 20\delta)\|\Phi \bar{x} - y\|_2^2,
\]

where \( x^* = \text{OMP}_s(y) \) and \( \alpha = \lceil 16 + 15\delta \rceil \).

We also need the following lemmas:

**Lemma 1.** Let \( \Phi \) be any matrix which satisfies RIP(\( L, \delta \)) with \( \delta < 1 \) and

\[ L := k \left( N \frac{2}{k} \right)^{2 - \frac{2}{p}}. \]

Then for any \( z \in \mathbb{R}^N \) and \( 1 \leq p < 2 \) we have

\[
\|\Phi z\|_2 \leq \sqrt{1 + \delta}(\|z\|_2 + \|z\|_{p/k^{1/p - 1/2}}).
\]

**Proof.** Let \( T_1 \) denote the set of indices of the \( L \) largest entries of \( z \), \( T_2 \) the next \( L \) largest, and so on. The last set \( T_h \) defined this way may have less than \( L \) elements. Set \( T_0 := \bigcup_{j=2}^{h} T_j \). Then

\[
\|\Phi z\|_2 \leq \|\Phi z_{T_1}\|_2 + \|\Phi z_{T_2}\|_2 + \cdots + \|\Phi z_{T_h}\|_2 \\
\leq \sqrt{1 + \delta}(\|z_{T_1}\|_2 + \|z_{T_2}\|_2 + \cdots + \|z_{T_h}\|_2).
\]
Since for any \( i \in T_{j+1} \) and \( i' \in T_j \), we have \( |z_i| \leq |z_{i'}| \), which implies that
\[
|z_i|^p \leq L^{-1} \|z_{T_j}\|_p^p.
\]

Then we obtain that
\[
\|z_{T_{j+1}}\|_2 \leq L^{1/2-1/p} \|z_{T_j}\|_p, \quad j = 1, \ldots, h - 1.
\]

Noting \( h - 1 \leq N/L \), we obtain that
\[
\sum_{j=2}^{h} \|z_{T_j}\|_2 \leq L^{1/2-1/p} \sum_{j=1}^{h-1} \|z_{T_j}\|_p
\leq L^{1/2-1/p} (h - 1)^{1-1/p} \|z\|_p \leq \|z\|_p/k^{1/p-1/2}.
\] (6)

Then (5) and (6) give
\[
\|\Phi z\|_2 \leq \sqrt{1 + \delta}(\|z\|_2 + \|z\|_p/k^{1/p-1/2}).
\]

Lemma 2. (\[7\]) Set \( r := 1/p - 1/q \) and suppose \( r \geq 0 \). Then
\[
\sigma_k(z)_q \leq \|z\|_p/k^r.
\]

We now have all ingredients to prove our conclusion:

Proof of Theorem 1. We first consider the \((2,p)\) case with \( 1 \leq p < 2 \). Taking \( x := \beta_{2k}(x) \) in Theorem 3, we obtain that
\[
\|\Phi x^* - y\|_2 \leq \sqrt{1 + 20\delta}\|\Phi \beta_{2k}(x) - \Phi x - e\|_2
\leq \sqrt{1 + 20\delta}(\|\Phi \beta_{2k}(x) - \Phi x\|_2 + \|e\|_2),
\]
where \( x^* := \text{OMP}_{2(\alpha-1)k}(\Phi x + e) \). Noting that \#supp\((x^* - \beta_{2k}(x)) \leq 2\alpha k \) and \( 1/\sqrt{1 - \delta_{2\alpha k}} \leq \sqrt{1 + \delta} \), we have
\[
\|x^* - \beta_{2k}(x)\|_2
\leq \sqrt{1 + \delta}\|\Phi x^* - \Phi \beta_{2k}(x)\|_2
\leq \sqrt{1 + \delta}(\|\Phi x^* - y\|_2 + \|\Phi \beta_{2k}(x) - y\|_2)
\leq \sqrt{1 + \delta}(\sqrt{1 + 20\delta} + 1)(\|\Phi \beta_{2k}(x) - \Phi x\|_2 + \|e\|_2)
\leq (1 + \delta)(\sqrt{1 + 20\delta} + 1)(\sigma_{2k}(x)_2 + \sigma_{2k}(x)_p/k^{1/p-1/2} + \|e\|_2),
\] (7)

where the last inequality uses Lemma 4. We now consider the term \( \sigma_{2k}(x)_2 \) which appears on the right side of (7). Lemma 2 gives
\[
\sigma_{2k}(x)_2 = \sigma_k(x - \beta_k(x))_2 \leq \|x - \beta_k(x)\|_p/k^{1/p-1/2} = \sigma_k(x)_p/k^{1/p-1/2},
\] (8)

Combining (7) and (8), we obtain that
\[
\|x^* - \beta_{2k}(x)\|_2 \leq 2(1 + \delta)(\sqrt{1 + 20\delta} + 1)(\sigma_k(x)_p/k^{1/p-1/2} + \|e\|_2).
\] (9)
Equations (8) and (9) imply that
\[
\|x^* - x\|_2 \leq \sigma_{2k}(x)_2 + \|x^* - \beta_{2k}(x)\|_2
\]
\[
\leq \sigma_k(x)_p/k^{1/p-1/2} + \|x^* - \beta_{2k}(x)\|_2
\]
\[
\leq \left(2(1 + \delta)(\sqrt{11} + 20\delta + 1) + 1\right) (\sigma_k(x)_p/k^{1/p-1/2} + \|e\|_2). \quad (10)
\]

We next consider the general case. We first recall that Hölder inequality, which says
\[
(|a_1|^u + \cdots + |a_k|^u)^{1/u}(|b_1|^v + \cdots + |b_k|^v)^{1/v} \geq |a_1b_1| + \cdots + |a_kb_k|
\]
provided $1/u + 1/v = 1$ and $u, v > 0$ where $a, b \in \mathbb{R}^k$. Then, by Hölder inequality, when $1 \leq q < 2$, we have
\[
\left(\left(|b_1|^q\right)^{2/q} + \cdots + \left(|b_k|^q\right)^{2/q}\right)^{q/2} \cdot k^{1-q/2} \geq |b_1|^q + \cdots + |b_k|^q,
\]
which implies that
\[
\|b\|_2 \geq \frac{\|b\|_q}{k^{1/q - 1/2}}. \quad (11)
\]
It follows from expressions (8) and (10) that the condition
\[
\|x^* - \beta_{2k}(x)\|_2 \leq \|x^* - x\|_2 + \sigma_{2k}(x)_2
\]
\[
\leq C'\left(\sigma_k(x)_p/k^{1/p-1/2} + \|e\|_2\right) + \|x^* - \beta_{2k}(x)\|_2
\]
\[
\leq (C' + 1)\sigma_k(x)_p/k^{1/p-1/2} + C'\|e\|_2,
\]
where $C' = 2(1 + \delta)(\sqrt{11} + 20\delta + 1) + 1$. Note that $\#\text{supp}(x^* - \beta_{2k}(x)) \leq 2\alpha k$. The inequality (11) provides the bound
\[
\|x^* - \beta_{2k}(x)\|_2 \geq \frac{\|x^* - \beta_{2k}(x)\|_q}{(2\alpha k)^{1/q - 1/2}}.
\]
Then, combining inequalities above, we arrive at
\[
\|x^* - \beta_{2k}(x)\|_q \leq (2\alpha)^{1/q - 1/2}(C' + 1)k^{1/q - 1/2}\sigma_k(x)_p + C'(2\alpha k)^{1/q - 1/2}\|e\|_2. \quad (12)
\]
From Lemma 2 and expression (12), we obtain the relation
\[
\|x^* - x\|_q \leq \|x - \beta_{2k}(x)\|_q + \|x^* - \beta_{2k}(x)\|_q = \sigma_{2k}(x)_q + \|x^* - \beta_{2k}(x)\|_q
\]
\[
\leq \sigma_k(x)_p/k^{1/p-1/4} + (2\alpha)^{1/q - 1/2}k^{1/q - 1/2}\left(C' + 1\right)\sigma_k(x)_p + C'(2\alpha k)^{1/q - 1/2}\|e\|_2
\]
\[
\leq (1 + (2\alpha)^{1/q - 1/2}(C' + 1))\frac{\sigma_k(x)_p}{k^{1/p-1/4}} + C'(2\alpha k)^{1/q - 1/2}\|e\|_2,
\]
where $C' = 2(1 + \delta)(\sqrt{11} + 20\delta + 1) + 1$. The conclusion follows.
Proof of Theorem 2. We build the proof on the ideas of Cohen, Dahmen and DeVore [4]. We give the full proof for completeness.

Using the triangle inequality, we have
\[ \| x - x^* \|_2 \leq \| x - \beta_k(x) \|_2 + \| \beta_k(x) - x^* \|_2 = \sigma_k(x)_2 + \| \beta_k(x) - x^* \|_2. \] (13)

Let \( \Omega_0 \) and \( \Omega(x - \beta_k(x)) \) be, respectively, the set in the definition of RIP in probability and the set in the definition of boundedness in probability for the vector \( x - \beta_k(x) \). We set \( \Omega' := \Omega_0 \cap \Omega(x - \beta_k(x)) \). A simple observation is
\[ P(\Omega') \geq 1 - 2\epsilon. \]

Also, Theorem 3 implies that
\[ \| y - \Phi x^* \|_2 \leq \sqrt{1 + 20\delta}\| y - \Phi \beta_k(x) \|_2. \]

Noting that \( 1/\sqrt{1 - \delta_{2\alpha k}} \leq \sqrt{1 + \delta} \), for any \( \omega \in \Omega' \), we obtain that
\[ \| \beta_k(x) - x^* \|_2 \leq \| \Phi(\beta_k(x) - x^*)\|_2/\sqrt{1 - \delta_{\alpha k}} \leq \sqrt{1 + \delta}\| \Phi(\beta_k(x) - x^*)\|_2 \]
\[ \leq \sqrt{1 + \delta}(\| y - \Phi x_T \|_2 + \| y - \Phi x^* \|_2) \]
\[ \leq \sqrt{C}\sqrt{1 + \delta}(1 + \sqrt{1 + 20\delta})\| x - \beta_k(x) \|_2 \]
\[ = \sqrt{C}\sqrt{1 + \delta}(1 + \sqrt{1 + 20\delta})\sigma_k(x)_2. \]

Here, the first inequality uses the RIP and \( \#\text{supp}(\beta_k(x) - x^*) \leq \alpha k \) and the last inequality uses the boundedness property in probability for \( x - \beta_k(x) \). Combining (13) and the equation above, we have
\[ \| x - x^* \|_2 \leq \left( 1 + \sqrt{C(1 + \delta)(1 + \sqrt{1 + 20\delta})} \right) \sigma_k(x)_2. \]

\[ \square \]

References
[1] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, A simple proof of the restricted isometry property for random matrices, Constr. Approx. 28(3)(2008) 253-263.

[2] E. J. Candès, The restricted isometry property and its implications for compressed sensing, C. R. Math. Acad. Sci. Paris, Series I, 346(2008)589-592.

[3] E. J. Candès, J. Romberg, and T. Tao, Stable signal recovery from incomplete and inaccurate measurements, Comm. Pure Appl. Math., 59(8)(2006)1207-1223.

[4] A. Cohen, W. Dahmen and R. DeVore, Compressed sensing and best k-term approximation, J. Amer. Math. Soc., 22 (2009)211-231.
[5] D. L. Donoho, For most large underdetermined systems of linear equations the minimal $\ell^1$ solution is also the sparsest solution, Comm. Pure Appl. Anal., 59(6)(2006)797-829.

[6] S. Foucart, M. Lai, Sparsest solutions of underdetermined linear systems via $\ell^q$-minimization for $0 < q \leq 1$, Appl. Comput. Harmon. Anal. 26(3)(2009)395-407.

[7] S. Kunis, H. Rauhut, Random sampling of sparse trigonometric polynomials, II. Orthogonal Matching Pursuit versus basis pursuit, Found. Comput. Math. 8(2008)737-763.

[8] H. Rauhut, On the impossibility of uniform sparse reconstruction using greedy methods, Samp. Theory Signal Image Process., 7(2) (2008)197-215.

[9] R. Saab, Ö. Yılmaz, Sparse recovery by non-convex optimization-instance optimality, Appl. Comput. Harmon. Anal., 29(2010)30-48.

[10] J. A. Tropp, Greed is Good: Algorithmic Results for Sparse Approximation, IEEE Trans. Inform. Theory, 50(11)(2004)2231-2242.

[11] J. A. Tropp, A. C. Gilbert, Signal recovery from random measurements via orthogonal matching pursuit, IEEE Trans. Inform. Theory, 12(2007)4655-4666.

[12] P. Wojtaszczyk, Stability and instance optimality for gaussian measurements in compressed sensing, Found. Comput. Math. 1(10)(2010)1-13.

[13] Tong Zhang, Sparse recovery with orthogonal matching pursuit under RIP, arXiv:1005.2249v1.

Authors’ addresses:
Zhiqiang Xu, LSEC, Inst. Comp. Math., Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100091, China. Email: xuzq@lsec.cc.ac.cn