AN ALGEBRAIC CHARACTERIZATION
OF THE AFFINE CANONICAL BASIS

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Abstract. The canonical basis for finite type quantized universal enveloping algebras was introduced in [L3]. The principal technique is the explicit construction (via the braid group action) of a lattice $\mathcal{L}$ over $\mathbb{Z}[q^{-1}]$. This allows the algebraic characterization of the canonical basis as a certain bar-invariant basis of $\mathcal{L}$. Here we present a similar algebraic characterization of the affine canonical basis. Our construction is complicated by the need to introduce basis elements to span the “imaginary” subalgebra which is fixed by the affine braid group. Once the basis is found we construct a PBW-type basis whose $\mathbb{Z}[q^{-1}]$-span reduces to a “crystal” basis at $q = \infty$, with the imaginary component given by the Schur functions.

0. Introduction.

The canonical basis of the quantized universal enveloping algebra associated to a simple finite-dimensional Lie algebra was introduced by Lusztig in [L3] via an elementary algebraic definition. The definition was characterized by three main components: 1) the basis was integral, 2) it was bar-invariant, and 3) it spanned a certain $\mathbb{Z}[q^{-1}]$-lattice $\mathcal{L}$ with a specific image in the quotient $\mathcal{L}/q^{-1}\mathcal{L}$. This algebraic definition does not work for quantized universal enveloping algebras of arbitrary Kac–Moody algebras. The difficulty in constructing a basis for $\mathcal{L}$ arises from the need to define suitable analogues of imaginary root vectors. The definition of the canonical basis for arbitrary type was subsequently made using topological methods [L6]. In his paper [L3], Kashiwara gave a suitable algebraic definition of the lattices $\mathcal{L}$ and $\mathcal{L}/q^{-1}\mathcal{L}$, making use of a remarkable symmetric bilinear form on the algebra, (introduced by Drinfeld) which led to an inductive construction of the global crystal basis. It was later shown in [GL] that the two concepts—the global crystal basis (algebraic) and the canonical basis (topological)—coincide.

In this paper we synthesize the two aforementioned techniques and construct a crystal basis for the quantized universal enveloping algebra of (untwisted) affine type. Then we give an elementary algebraic characterization of the canonical basis analogous to the characterization given in the finite type case [L3]. A remarkable feature of our construction is that the part of the crystal basis corresponding to the imaginary root spaces is given by Schur functions in the Heisenberg generators. We were motivated to consider the Schur functions for the following reasons. The imaginary root vectors were constructed in [CP], where it was shown that they could be defined by a certain functional equation in terms of the Heisenberg generators. After a suitable renormalization, this equation is the same as the equation that expresses the complete symmetric functions in terms of the power sums. In Section 4 of this paper we show that the imaginary root vectors generate a polynomial
algebra over \(\mathbb{Z}[q, q^{-1}]\), are group-like with respect to the comultiplication and are quasi-orthonormal with respect to the Drinfeld form on the algebra. It is well-known that the complete symmetric functions are also group-like and orthonormal with respect to the standard Hopf algebra structure and inner product on the ring of symmetric functions, and that the Schur functions form an orthonormal \(\mathbb{Z}\)-basis for the ring. Thus, we are able to identify the imaginary subalgebra with the ring of symmetric functions to construct the crystal basis of the imaginary part. The appearance of Schur functions was anticipated in [L7] in a series of conjectures on level 0 representations of quantum affine algebras. In future work we shall make explicit the connection between our work and these conjectures.

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1. The algebras \(U\) and \(U^+\).

In this section, we recall certain facts about \(\hat{g}\) and \(g\) and their associated quantum groups that will be needed later.

Throughout this paper \(g\) will denote a simply-laced, finite-dimensional complex simple Lie algebra and \((a_{ij})_{i,j\in I}, I = \{1, \ldots, n\}\), will denote its Cartan matrix. Let \((a_{ij})_{i,j\in I}, \hat{I} = I \cup \{0\}\), be the extended Cartan matrix of \(g\) and let \(\hat{g}\) be the corresponding affine Lie algebra. Let \(R\) (resp. \(R^+\)) denote a set of roots (resp. positive roots) of \(g\) and let \(\alpha_i (i \in I)\) be a set of simple roots. Let \(Q\) be the root lattice of \(g\), let \(P\) be the weight lattice, and let \(\omega_i \in P (i \in I)\) be the fundamental weights of \(g\). For \(\omega \in P\), \(\eta \in Q\), define an integer \(|\omega| \cdot |\eta|\) by extending bilinearly the assignment \(|\omega_i| \cdot |\alpha_j| = \delta_{ij}\). Notice that \(|\alpha_i| \cdot |\alpha_j| = a_{ij}\). The corresponding objects for \(\hat{g}\) are defined similarly and we denote them by \(\hat{R}, \hat{Q}\) and so on. Let \(\theta\) be the highest root of \(g\). Then, it is well-known that the element \(\alpha_0 + \theta = \delta \in \hat{R}\) and that \(|\delta| \cdot |\alpha_i| = 0\) for all \(i \in \hat{I}\). Further, the set of roots \(\hat{R}\) of \(\hat{g}\) is given by \(\hat{R} = \hat{R}^+ \cup -\hat{R}^+\), where

\[
\hat{R}^+ = \{\alpha + k\delta \mid k \geq 0, \alpha \in R^+\} \cup \{k\delta \mid k > 0\} \cup \{-\alpha + k\delta \mid k > 0, \alpha \in R^+\}.
\]

Set

\[
\hat{R}_+ = \{k\delta + \alpha \mid k \geq 0, \alpha \in R^+\},
\]

\[
\hat{R}_- = \{k\delta - \alpha \mid k > 0, \alpha \in R^+\},
\]

\[
\hat{R}_0 = \{k\delta \mid k > 0\} \times I,
\]

\[
\hat{R} = \hat{R}_+ \cup \hat{R}_0 \cup \hat{R}_-.
\]

We call \(R\) the set of positive roots (with multiplicity) of \(\hat{g}\). Given an element \(\eta \in \hat{Q}\), let \(re(\eta) \in Q\) be such that \(\eta = k\delta + re(\eta)\) for some \(k \in \mathbb{Z}\). We call \(re(\eta)\) the real part of \(\eta\).

Let \(W\) and \(\hat{W}\) be the Weyl groups of \(g\) and \(\hat{g}\), respectively. It is well-known that they are Coxeter groups generated by simple reflections \(s_i\) for \(i \in I\) and \(s_i\) for \(i \in \hat{I}\), respectively. The Weyl group \(W\) acts on the root lattice \(Q\) by extending \(s_i(\alpha_j) = \alpha_j - \alpha_j \alpha_i\). Then, \(\hat{W}\) is isomorphic to the semi-direct product \(W \rtimes Q\) under the map \(s_i \to (s_i, 0)\) for \(i \in I\), \(s_0 \to (s_0, \theta)\), where \(s_0(\alpha_j) = \alpha_j - (\theta \cdot |\alpha_j|)\theta\). The extended Weyl group \(\hat{W}\) is defined to be the semi-direct product \(\hat{W} \rtimes \hat{Q}\). For any
$w \in W$ we write $(w, 0)$ for the corresponding element in $\tilde{W}$, and for $\omega \in P$ we write $t_\omega$ for the element $(1, \omega)$. The affine Weyl group $\tilde{W}$ is a normal subgroup of $W$, and the quotient $\tilde{T} = W/\tilde{W}$ is a finite group isomorphic to a subgroup of the group of diagram automorphisms of $\tilde{a}$, i.e. the bijections $\tau : \tilde{I} \to \tilde{I}$ such that $a_{\tau(i)\tau(j)} = a_{ij}$ for all $i, j \in \tilde{I}$. Moreover, there is an isomorphism of groups $\tilde{W} \cong \tilde{T} \times \tilde{W}$, where the semi-direct product is defined using the action of $\tilde{T}$ in $\tilde{W}$ given by $\tau.s_i = s_{\tau(i)}\tau$ (see [B]). If $w \in \tilde{W}$, a reduced expression for $w$ is an expression $w = \tau s_{i_1}s_{i_2}\ldots s_{i_m}$ with $\tau \in \tilde{T}$, $i_1, i_2, \ldots, i_m \in \tilde{I}$ and $m$ minimal; we define the length $l(w)$ of $w$ to be $m$. The element $t_\omega$ acts on $\mathcal{Q}$ by extending $t_\omega(\alpha_i) = \alpha_i - (\langle \omega | \alpha_i \rangle)\delta$.

For $i \in I$, let $\tau_i \in \tilde{T}$ be such that $w_i = \tau_i^{-1}t_\omega w_i \in \tilde{W}$. Then, by [B, Lemma 3.1], we know that $l(w'_i s_i) = l(w'_i) - 1$ and that there exists $j$ such that $l(s_j w'_j) = l(w'_j) - 1$ and $\tau_i(s_j) = s_0\tau_i$. Define elements $w_1, w_2, \ldots, w_{2n}$ by

\[
\begin{align*}
  w_i &= \tau_i \tau_{i-1} \ldots \tau_1 w'_i (\tau_1 \tau_{i-1} \ldots \tau_1)^{-1}, \\
  w_{n+i} &= \tau_i \tau_{i-1} \ldots \tau_1 (\tau_i \tau_{i-1} \ldots \tau_1)^{-1} (\tau_1 \tau_{i-1} \ldots \tau_1)^{-2},
\end{align*}
\]

if $1 \leq i \leq n$. Let $2\rho$ denote the sum of all the roots in $R^+$. It is well-known that $2\rho = 2\sum_{i=1}^{n} \omega_i$. The following lemma is easily established by using standard results on Coxeter groups (see [B], for instance).

**Lemma 1.1.**

(i) *The element $t_{2\rho} \in \tilde{W}$, and has length $N = \sum_{i} l(t_{2\omega_i})$. (*

(ii) *$t_{2\rho} = w_1 w_2 \ldots w_{2n}$. (*

(iii) *There exists a reduced expression $s_{i_1}s_{i_2}\ldots s_{i_N}$ for $t_{2\rho}$ such that the expressions $s_{i_1}\ldots s_{i_{l(t_{2\omega_i})}},\ s_{i_{l(t_{2\omega_i})}+1}\ldots s_{i_{l(t_{2\omega_i})}+l(t_{2\omega_i})},\ \ldots$ etc., are reduced expressions for $w_1, w_2, \ldots$ etc.*

(iv) *Define a doubly infinite sequence $h = (\ldots, i_{-1}, i_0, i_1, \ldots)$ by setting $i_k = i_{k(\text{mod} N)}$ for $k \in \mathbb{Z}$. Then, for any integers $m < p$, the product $s_{i_m}s_{i_{m+1}}\ldots s_{i_p}$ is reduced. (*

(v) *We have

\[
\begin{align*}
  \mathcal{R}_{>0} &= \{ \alpha_{i_0}, s_{i_0}(\alpha_{i_1}), s_{i_0}s_{i_1}(\alpha_{i_2}), \ldots \}, \\
  \mathcal{R}_{<0} &= \{ \alpha_{i_1}, s_{i_1}(\alpha_{i_2}), s_{i_1}s_{i_2}(\alpha_{i_3}), \ldots \}. \quad \square
\end{align*}
\]

From now on we fix a reduced expression for $t_{2\rho}$ as in (iii) above. For $k \in \mathbb{Z}$, write $k = k_0 + rN$, with $|k_0| < N$ and $k, k_0$ and $r$ either zero or of the same sign, and set

\[
\begin{align*}
  \beta_k &= s_{i_0}s_{i_1}\ldots s_{i_{k+1}}(\alpha_{i_k}) = t_{2R}^{-r} s_{i_0}s_{i_1}\ldots s_{i_{k_0+1}}(\alpha_{i_{k_0}}) & \text{if } k \leq 0, \\
  \beta_k &= s_{i_1}s_{i_2}\ldots s_{i_{k-1}}(\alpha_{i_k}) = t_{2R}^{r} s_{i_1}s_{i_2}\ldots s_{i_{k_0-1}}(\alpha_{i_{k_0}}) & \text{if } k > 0.
\end{align*}
\]

Define a total order on $\mathcal{R}$ by setting

\[
\begin{align*}
  \beta_0 < \beta_1 < \beta_2 \cdots < \beta^{(1)} < \cdots < \beta^{(n)} < 2\delta^{(1)} < \cdots < \beta_3 < \beta_2 < \beta_1,
\end{align*}
\]

where $k\delta^{(i)}$ denotes $(k\delta, i) \in \mathcal{R}_{0}$.

Let $q$ be an indeterminate, let $\mathbb{Q}(q)$ be the field of rational functions in $q$ with rational coefficients, and let $\mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials with integer coefficients. For $r, m \in \mathbb{N}$, $m \geq r$, define

\[
\begin{align*}
  [m] &= \frac{q^m - q^{-m}}{q - q^{-1}}, & [m]! &= [m][m-1] \ldots [2][1], & \left[ \frac{m}{r} \right] &= \frac{[m]!}{[r]![m-r]!}.
\end{align*}
\]
Then \[ \binom{m}{r} \in \mathbb{Z}[q, q^{-1}] \] for all \( m \geq r \geq 0 \).

**Proposition 1.1.** There is a Hopf algebra \( U \) over \( \mathbb{Q}(q) \) which is generated as an algebra by elements \( E_{\alpha_i}, F_{\alpha_i}, K_i^{\pm 1} \) \((i \in \hat{I})\), with the following defining relations:

- \( K_i K_i^{-1} = K_i^{-1} K_i = 1 \), \( K_i K_j = K_j K_i \),
- \( K_i E_{\alpha_i} K_i^{-1} = q^{a_{ij}} E_{\alpha_j} \),
- \( K_i F_{\alpha_i} K_i^{-1} = q^{-a_{ij}} F_{\alpha_j} \),
- \( [E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} K_i - K_i^{-1} \frac{1}{q - q^{-1}} \),
- \[ \sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r} (E_{\alpha_i})^r E_{\alpha_j} (E_{\alpha_i})^{1-a_{ij}-r} = 0 \] if \( i \neq j \),
- \[ \sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r} (F_{\alpha_i})^r F_{\alpha_j} (F_{\alpha_i})^{1-a_{ij}-r} = 0 \] if \( i \neq j \).

The comultiplication of \( U \) is given on generators by

- \( \Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_i \otimes E_{\alpha_i} \),
- \( \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes K_i^{-1} + 1 \otimes F_{\alpha_i} \),
- \( \Delta(K_i) = K_i \otimes K_i \),

for \( i \in \hat{I} \).

Let \( U^+ \) (resp. \( U^- \), \( U^0 \)) be the \( \mathbb{Q}(q) \)-subalgebras of \( U \) generated by the \( E_{\alpha_i} \) (resp. \( F_{\alpha_i}, K_i^{\pm 1} \)) for \( i \in \hat{I} \). The following result is well-known, see [L4] for instance.

**Lemma 1.2.** \( U \cong U^- \otimes U^0 \otimes U^+ \) as \( \mathbb{Q}(q) \)-vector spaces.

**Definition 1.1.**

(i) Let \( \sigma : U^+ \to U^+ \) denote the \( \mathbb{Q}(q) \)-algebra anti-automorphism obtained by extending \( \sigma(E_{\alpha_i}) = E_{\alpha_i} \).

(ii) Let \( \omega : U \to U \) be the \( \mathbb{Q}(q) \)-algebra automorphism defined by extending

\[
\omega(E_{\alpha_i}) = F_{\alpha_i}, \quad \omega(F_{\alpha_i}) = E_{\alpha_i}, \quad \omega(K_i) = K_i^{-1}.
\]

It is convenient to use the following notation:

\[
E_{\alpha_i}^{(r)} = \frac{E_{\alpha_i}^r}{[r]!}. \]

The elements \( F_{\alpha_i}^{(r)} \) are defined similarly.

Corresponding to each element \( w \in \check{W} \) one can define an automorphism \( T_w : U \to U \) as follows. Let \( T_i \) \((i \in \hat{I})\) be the \( \mathbb{Q}(q) \)-algebra automorphisms of \( U \) defined
as follows (see [L4]):

\[(1.4)\]
\[T_i(F^{(m)}_{\alpha_i}) = (-1)^m q^{-m(m-1)} E^{(m)}_{\alpha_i} K^m_i, \quad T_i(F^{(m)}_{\alpha_i}) = (-1)^m q^{-m(m-1)} K^m_i E^{(m)}_{\alpha_i},\]

\[(1.5)\]
\[T_i(F^{(m)}_{\alpha_i}) = \sum_{r=0}^{-ma_ij} (-1)^r q^{-r} E^{(m)}_{\alpha_i} F^{(-ma_ij-r)}_{\alpha_i} F^{(r)}_{\alpha_i} \text{ if } i \neq j,\]

\[(1.6)\]
\[T_i(F^{(m)}_{\alpha_i}) = \sum_{r=0}^{-ma_ij} (-1)^r q^{-r} F^{(r)}_{\alpha_i} E^{(m)}_{\alpha_i} F^{(-ma_ij-r)}_{\alpha_i} \text{ if } i \neq j.\]

The finite group \(T\) acts as \(\mathbb{Q}(q)\)-Hopf algebra automorphisms of \(U\) by

\[
\tau.E_{\alpha_i} = E_{\tau(\alpha_i)}, \quad \tau.F_{\alpha_i} = F_{\tau(\alpha_i)}, \quad \tau.K_i = K_{\tau(i)}, \quad \text{for all } i \in I.
\]

If \(w \in W\) has a reduced expression \(w = \tau s_{i_1} \ldots s_{i_m}\), let \(T_w\) be the \(\mathbb{Q}(q)\)-algebra automorphism of \(U\) given by \(T_w = \tau T_{s_{i_1}} \ldots T_{s_{i_m}}\). Then, \(T_w\) depends only on \(w\), and not on the choice of its reduced expression. Further, it is shown in [B1] that \(T_\omega(E_{\alpha_j}) = E_{\alpha_j}, T_\omega(F_{\alpha_j}) = F_{\alpha_j}, T_\omega(K_j) = K_j\) if \(i \neq j, T_\omega(K_i) = K_i C^{-1}\).

This leads us to another realization of \(U\), due to [D1], [B1], [F].

**Theorem 1.** There is an isomorphism of \(\mathbb{Q}(q)\)-Hopf algebras from \(U\) to the algebra with generators \(x_{i,r}^\pm (i \in I, r \in \mathbb{Z}), K_i^{\pm 1} (i \in I), h_{i,r} (i \in I, r \in \mathbb{Z} \setminus \{0\})\) and \(C^{\pm 1/2}\), and the following defining relations:

\(C^{\pm 1/2}\) are central,
\[K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad C^{1/2} C^{-1/2} = C^{-1/2} C^{1/2} = 1,\]
\[K_i K_j = K_j K_i, \quad K_i h_{j,r} = h_{j,r} K_i,\]
\[K_i x_{j,r}^\pm K_i = q^{a_{ij}} x_{j,r}^\pm,\]
\[[h_{i,r}, h_{j,s}] = \delta_{r-s} \left[\frac{r a_{ij}}{q - q^{-1}}\right] C^r - C^{-r},\]
\[[h_{i,r}, x_{j,s}^\pm] = \pm \frac{1}{r} [r a_{ij}] C^{r/2} x_{j,r+s}^\pm,\]
\[x_{i,r+1,j,s}^+ x_{j,r}^\pm = q^{a_{ij}} x_{i,r+1,j,s}^+ x_{j,r}^\pm + x_{i,r,j,s}^+ x_{j,r}^\pm x_{i,r+1,j,s}^+,\]
\[x_{i,r,j,s}^- = \delta_{i,j} \frac{C^{(r-s)/2} \psi_{i,r+s}^+ - C^{-(r-s)/2} \psi_{i,r+s}^-}{q - q^{-1}},\]
\[\sum_{\pi \in \Sigma_m} \sum_{k=0}^m (-1)^k \left[\frac{m}{k}\right] x_{i,r_{\pi(1)}}^{\pm} \ldots x_{i,r_{\pi(k)}}^{\pm} x_{j,r_{\pi(k+1)}}^{\pm} \ldots x_{j,r_{\pi(m)}}^{\pm} = 0, \quad \text{if } i \neq j,\]

for all sequences of integers \(r_1, \ldots, r_m\), where \(m = 1 - a_{ij}, \Sigma_m\) is the symmetric group on \(m\) letters, and the \(\psi_{i,r}^\pm\) are determined by equating powers of \(u\) in the formal power series

\[\sum_{r=0}^{\infty} \psi_{i,r}^\pm u^{\pm r} = K_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^{\pm s} \right).\]
The isomorphism is given by [B1]

\[ x_{i,r}^+ = o(i)^r T_{\omega_i}^{-r}(E_{\alpha_i}), \quad x_{i,r}^- = o(i)^r T_{\omega_i}^{-r}(F_{\alpha_i}), \]

where \( o : I \to \{ \pm 1 \} \) is a map such that \( o(i) = -o(j) \) whenever \( a_{ij} < 0 \) (it is clear that there are exactly two possible choices for \( o \)).

The following lemma is easily checked.

**Lemma 1.3.** There exists a \( \mathbb{Q} \)-algebra anti-automorphism \( \Omega \) of \( U \) such that \( \Omega(q) = q^{-1} \) and, for all \( i \in I \), \( r \in \mathbb{Z} \),

\[ \Omega(x_{i,r}^\pm) = x_{i,r}^\pm, \quad \Omega(\psi_{i,r}^\pm) = \psi_{i,r}^\pm, \quad \Omega(h_{i,r}) = -h_{i,r}, \quad \Omega(C^{1/2}) = C^{1/2}. \]

We now define a set of root vectors for each element of \( \mathcal{K} \cup \mathcal{L} \).

\[ E_{\beta_k} = \begin{cases} T_{i_0}^{-1}T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(E_{\alpha_{i_k}}) & \text{if } k \leq 0, \\ T_{i_1}T_{i_2} \cdots T_{i_{k-1}}(E_{\alpha_{i_k}}) & \text{if } k > 0. \end{cases} \]

It follows from [L4, Proposition 40.1.3] that the elements \( E_{\beta_k} \in U^+ \). The elements \( F_{\beta_k} \in U^- \) are defined similarly by replacing \( E_{\alpha_i} \) by \( F_{\alpha_i} \). The set

\[ \{ E_{\beta_k}, F_{\beta_k} \mid \beta_k \in \mathcal{K} \cup \mathcal{L} \} \]

is called the set of real root vectors. The next lemma is obvious from the definition of the root vectors (recall that \( l(T_{2\rho}) = N \)).

**Lemma 1.4.** \( T_{t_2\rho}(E_{\beta_k}) \in U^+ \) if and only if \( \beta_k < \beta_N \). If \( \beta_N \leq \beta_k \leq \beta_1 \), then \( T_{t_2\rho}^{-1}(E_{\beta_k}) \in U^- \). \( \square \)

For \( k > 0 \), \( i \in I \), set

\[ \tilde{\psi}_{k,i} = E_{k\delta} - \alpha_i E_{\alpha_i} - q^{-2} E_{\alpha_i} E_{k\delta} - \alpha_i, \]

and define elements \( E_{k\delta,i} \in U^+ \) by the functional equation

\[ \exp \left( (q - q^{-1}) \sum_{k=1}^\infty E_{k\delta,i} u^k \right) = 1 + \sum_{k=1}^\infty (q - q^{-1}) \tilde{\psi}_{k,i} u^k. \]

The next result relates the generators in Theorem 1 to the root vectors defined above. It is an easy consequence of the choice of the reduced expression for \( t_{2\rho} \) (see [B2]).

**Lemma 1.5.**

\[ x_{i,k}^+ = \begin{cases} o(i)^k E_{k\delta + \alpha_i} & \text{if } k \geq 0, \\ -o(i)^k F_{-k\delta - \alpha_i} K_i^{-1} C^k & \text{if } k < 0, \end{cases} \]

\[ x_{i,k}^- = \begin{cases} -o(i)^k C^{-k} K_i E_{k\delta - \alpha_i} & \text{if } k > 0, \\ o(i)^k F_{-k\delta + \alpha_i} & \text{if } k \leq 0. \end{cases} \]

Further, for \( k > 0 \), \( i \in I \), we have

\[ h_{i,k} = o(i)^k C^{-k/2} E_{k\delta,i} \quad \text{if } k > 0, \]

\[ \tilde{\psi}_{i,k} = o(i)^k (q - q^{-1}) C^{-k/2} K_i \tilde{\psi}_{i,k} \quad \text{if } k > 0. \]

The following is now an immediate consequence of Theorem 1 and Lemma 1.3.
Proposition 1.2. For $i,j \in I$, $k,l \geq 0$, we have
\[
[E_{k,i}, E_{l,j}] = 0, \quad \text{if } k,l > 0,
\]
\[
[E_{k,i}, E_{l,i}] = \pm o(i)^k o(j)^l E_{k+i,l}, \quad \text{if } k > 0, l \delta \pm \alpha_i \in \mathbb{R},
\]
\[
E_{k+i,l} E_{k,i} - E_{l+i,k} E_{k,i} = q^{\pm o(i) - o(j)} E_{k,i} E_{l,i} - E_{k+i,l} E_{k,i} - E_{l+i,k} E_{k,i} = q^{\pm o(i) - o(j)} E_{k,i} E_{l,i} - E_{k+i,l} E_{k,i}.
\]

Corollary 1.1. For $k > 0, i, j \in I$, we have
\[
T_{\omega_i}(E_{k+i}) = E_{k+i},
\]
\[
T_{\omega_j}(E_{k+i}) = E_{k+j}, \quad \text{if } i \neq j,
\]
\[
T_{\omega_i}(E_{k+j}) = E_{k+i+j}, \quad T_{\omega_j}(E_{k+i+j}) = E_{k+j+i}, \quad \text{if } k > 0.
\]

Let $U^+(\langle \rangle)$ (resp. $U^+(\langle \rangle)$, $U^+(0)$) be the $\mathbb{Q}(q)$-subalgebra of $U^+$ generated by the $E_{\beta_k}$ for $k \leq 0$ (resp. $E_{\beta_k}$ for $k > 0$, $E_{k,i}$ for $k > 0$). We now define a basis for these algebras.

Following [CP, Section 3], we introduce elements $P_{k,i}$ ($i \in I$, $k \in \mathbb{Z}$, $k \geq 0$) in $U^+(0)$ by $P_{0,i} = 1$ and
\[
\begin{align*}
P_{k,i} &= -\frac{1}{|k|} \sum_{r=1}^{k} q^{k-r} \tilde{\psi}_{r,i} P_{k-r,i}, \\
\tilde{\psi}_{k,i} &= \Omega(P_{k,i}),
\end{align*}
\]

Remark. The $P_{k,i}$ defined here are actually $q^{-k} o(i)^k C^{k/2}$ times the $P_{i,k}$ in [CP].

Setting $\tilde{P}_{k,i} = \Omega(P_{k,i})$, we find that
\[
\begin{align*}
\tilde{P}_{k,i} &= \frac{1}{|k|} \sum_{r=1}^{k} q^{-k+r} \tilde{\psi}_{r,i} \tilde{P}_{k-r,i}.
\end{align*}
\]

It is proved in [CP, Section 3] that $P_{k,i}$ and $\tilde{P}_{k,i}$ satisfy the functional equations
\[
\begin{align*}
\sum_{k \geq 0} P_{k,i} u^k &= \exp \left( k \sum_{k=1}^{\infty} \frac{-E_{k,i} u^k}{|k|} \right), \\
\sum_{k \geq 0} \tilde{P}_{k,i} u^k &= \exp \left( k \sum_{k=1}^{\infty} \frac{E_{k,i} u^k}{|k|} \right).
\end{align*}
\]

It is clear from the above that $U^+(0)$ is generated as a $\mathbb{Q}(q)$-algebra by the $\tilde{P}_{k,i}$ for $k > 0, i \in I$.

Definition 1.2. (1) Let $N^R$ (resp. $N^{R^+}$, $N^{R^R}$, $N^{R_0}$) be the set of finitely supported maps from $\mathbb{R}$ (resp. $\mathbb{R}^+$, $\mathbb{R}^R$, $\mathbb{R}_0$) to $\mathbb{N}$.

(2) For $c, c' \in N^R$, we say that $c < c'$ if there exists $\beta \in \mathbb{R}$ such that $c(\beta) < c'(\beta)$ and $c(\beta') = c'(\beta')$ if $\beta' > \beta$.

From now on, we shall identify $\alpha$ with the indicator function $1_\alpha \in N^R$.

Definition 1.3. Given $c \in N^R$, define $E_{c}$ (resp. $E_{c}'$) to be the monomial formed by multiplying the $(E_{\beta_k}(c(\beta_k)))$, for real roots $\beta_k$ ($k \in \mathbb{Z}$), and the $\tilde{P}_{k,i}$ ($k > 0$, $i \in I$), (resp. $\psi_{k,i}$, $k > 0$, $i \in I$) in the order $\mathbb{L}_{3}$. 

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Definition 1.4. Set

\[ B = \{ E_c \mid c \in \mathbb{N}^R \}, \]
\[ B_+ = \{ E_c \mid c \in \mathbb{N}^{R^+} \}, \]
\[ B_0 = \{ E_c \mid c \in \mathbb{N}^{R_0} \}, \]
\[ B'_0 = \{ E'_c \mid c \in \mathbb{N}^{R_0} \}, \]
\[ B_\times = \{ E_c \mid c \in \mathbb{N}^{R_\times} \}. \]

Proposition 1.3. (i) \( U^+(0)\) and \( U^+(>)\) are \( \mathbb{Q}(q)\)-subalgebras of \( U^+ \).

(ii) As \( \mathbb{Q}(q)\)-modules, we have \( U^+ \cong U^+(>) \otimes U^+(0) \otimes U^+(<) \).

(iii) The set \( B_+ \) (resp. \( B_\times, B'_0 \)) is a \( \mathbb{Q}(q)\)-basis of \( U^+(>) \) (resp. \( U^+(<), U^+(0) \)).

(iv) The set \( B_0 \) is a \( \mathbb{Q}(q)\)-basis of \( U^+(0) \).

(v) The set \( B \) is a \( \mathbb{Q}(q)\)-basis of \( U^+ \).

Proof. Parts (i) through (iii) were proved in \([B_1], [B_2]\). Part (iv) can be now deduced easily by using the definition of the \( \hat{P}_{k,i} \). Part (v) follows from parts (ii) through (iv).

An element \( x \in U^+ \) is said to have homogeneity \( \sum_i d_i \alpha_i \in \mathbb{Q}^+ \) if \( x \) is a \( \mathbb{Q}(q)\)-linear combination of products of the \( E_{\alpha_i} \) (\( i \in \hat{I} \)) in which \( E_{\alpha_i} \) occurs \( d_i \) times for all \( i \). In that case, \( x \) is said to be homogeneous, we let \( |x| = \sum_{i \in I} d_i \alpha_i \) denote its homogeneity, and we write \( ht(x) = \sum_{i \in I} d_i \). Any \( x \in U^+ \) is a finite sum of homogeneous elements.

We end this section by recalling some results from \([L_3]\) that will be used crucially in later sections.

For \( i \in \hat{I} \), there exists a unique \( \mathbb{Q}(q)\)-linear map \( r_i : U^+ \rightarrow U^+ \) given by \( r_i(1) = 0, \quad r_i(E_{\alpha_j}) = \delta_{i,j} \) for \( j \in \hat{I} \), and satisfying \( r_i(xy) = q^{|y|\alpha_i} \delta_{i,j} r_i(x)y + x r_i(y) \) for all homogeneous \( x, y \in U^+ \) (\([L_3], 1.2.13\)). Similarly, there exists a unique \( \mathbb{Q}(q)\)-linear map \( l : U^+ \rightarrow U^+ \) given by \( l(1) = 0, \quad l(E_{\alpha_j}) = \delta_{i,j} \), and satisfying \( l(xy) = l(x)y + q^{|x|\alpha_i} x l(y) \) for all homogeneous \( x, y \in U^+ \).

The following is proved in \([L_3]\), Lemma 38.1.2, Lemma 38.1.5, Proposition 38.1.6).

Proposition 1.4. (i) \( \{ x \in U^+ \mid T_i(x) \in U^+ \} = \{ x \in U^+ \mid r_i(x) = 0 \} \).

(ii) \( \{ x \in U^+ \mid T_i^{-1}(x) \in U^+ \} = \{ x \in U^+ \mid r_i(x) = 0 \} \).

(iii) Defining \( U^+[i] \) and \( \sigma U^+[i] \) to be the sets in parts (i) and (ii), respectively, we have

\[ U^+ = \bigoplus_{r \geq 0} E_{\alpha_i}^{(r)} U^+[i], \quad U^+ = \bigoplus_{r \geq 0} \sigma U^+[i] E_{\alpha_i}^{(r)}. \]

2. The integral form and the integral PBW basis

Let \( A = \mathbb{Z}[q, q^{-1}] \). We define the divided powers subalgebra \( A U \) to be the \( A \)-subalgebra of \( U \) generated by the \( K_i^{\pm 1}, E_{\alpha_i}^{(r)}, F_{\alpha_i}^{(r)} \), for \( i \in \hat{I} \) and \( r \geq 0 \). The
\(A\)-subalgebras \(A U^\pm\) of \(A U\) are defined in the obvious way. The main result in this section is:

**Theorem 2.** The set \(B\) is an \(A\)-basis of \(A U^+\).

The rest of this section is devoted to proving Theorem 2. We proceed as follows. First we show that \(B\) is contained in \(A U^+\). This allows us to define \(A\)-subalgebras \(A U^+(>)\), \(A U^(<)\) and \(A U^+(0)\). Then we prove that \(B>\), \(B<\) and \(B_0\) are \(A\)-bases of these subalgebras. Finally we prove a triangular decomposition \(A U^+ = A U^+(>) \otimes A U^+(<)\). By Proposition [3], this means that, as \(A\)-modules, \(A U^+ \cong A U^+(>) \otimes A U^+(0) \otimes A U^+(<)\). The theorem follows.

For \(i \in \mathcal{I}, r \geq 1, m \in \mathbb{Z},\) define elements

\[
\left[ K_i, m \right]_r = \prod_{s=1}^{r} \frac{K_i q^{m-s+1} - K_i^{-1} q^{-m+s-1}}{q^s - q^{-s}}.
\]

The following lemma is well-known (see [4, Corollary 3.1.9]).

**Lemma 2.1.** For \(r, s \in \mathbb{Z}, r, s \geq 0,\) we have

\[
E_{a_i}^{(r)} F_{a_i}^{(s)} = \sum_{t=0}^{\min(r,s)} E_{a_i}^{(r-t)} \left[ K_i, 2t - r - s \right]_t F_{a_i}^{(s-t)}.
\]

Let \(A U^0\) be the \(A\)-subalgebra of \(A U\) generated by the \(K_i^{\pm 1}, [K_i, m]_r\) for all \(i \in \mathcal{I}, r \geq 1, m \in \mathbb{Z}\). The following is well-known and can be deduced from the preceding lemma and Lemma [1,2].

**Lemma 2.2.** As \(A\)-modules, we have \(A U \cong A U^- \otimes A U^0 \otimes A U^+\). In particular, we have \(A U^+ = U^+ \cap A U\).

The formulas given in Section 1 show that the restriction of \(T_i\) to \(A U\) defines an \(A\)-algebra automorphism of \(A U\). This proves, in light of the preceding lemma, that \(E_{\beta_k}^{(r)} \in A U^+\) for all \(k \in \mathbb{Z}, r \geq 0\). Let \(A U^+\) (resp. \(A U^+(<)\)) be the \(A\)-subalgebras generated by the \(E_{\beta_k}^{(r)}\) for \(k \leq 0\) (resp. \(k > 0\), \(r \geq 0\)).

We next show that \(\hat{P}_{k,i} \in A U^+\). For this we need several results from [5,6] which we now recall.

Let \(\xi\) be an indeterminate and form the polynomial algebra \(Q(q)[\xi]\) over \(Q(q)\). For \(r \in \mathbb{Z}, r > 0,\) set

\[\xi^{(r)} = \frac{\xi^r}{[r]!}.\]

For \(i \in \mathcal{I},\) define the following elements of the algebra \(U[[u]]\), where \(u\) is another indeterminate:

\[\mathcal{X}_i^+(u) = \sum_{k=0}^{\infty} E_{k}\delta_\alpha u^k, \quad \mathcal{X}_i^-(u) = \sum_{k=0}^{\infty} E_{(k+1)\delta_\alpha} u^k.\]

Let \(D_i^\pm : Q(q)[\xi] \to U[[u]]\) be the \(Q(q)\)-algebra homomorphisms that take \(\xi\) to \(\mathcal{X}_i^\pm(u)\). Writing

\[D_i^\pm = \sum_{k=0}^{\infty} D_{k,i}^\pm u^k,\]
the fact that $D^\pm_{k,i}$ are homomorphisms is equivalent to $D^\pm_{k,i}(1) = \delta_{k,0}$ and

$$D^\pm_{k,i}(fg) = \sum_{m=0}^k D^\pm_{m,i}(f)D^\pm_{k-m,i}(g)$$

for all $f,g \in \mathbb{Q}(q)[\xi]$. The $\mathbb{Q}(q)$-linear maps $D^\pm_{k,i} : \mathbb{Q}(q)[\xi] \to \mathbb{U}$ are uniquely determined by this relation together with

$$D^+_{k,i}(\xi) = E_{k\delta+\alpha_i}, \quad D^-_{k,i}(\xi) = E_{(k+1)\delta-\alpha_i}, \quad D^\pm_{k,i}(1) = \delta_{k,0}.$$  

It is clear from the definition that, for all $r \geq 1$,

$$|D^+_{k,i}(\xi^{(r)})| = k\delta + r\alpha_i, \quad |D^-_{k,i}(\xi^{(r)})| = (k+1)\delta - r\alpha_i.$$  

**Proposition 2.1.** We have

(i) $D^+_{k,i}(\xi^{(r)}) = \sum_{t=1}^{r-1} (-1)^{t+1} q^{t(r-1)} E_{t \xi} D^+_{k,i}(\xi^{(r-t)}) + q^{r(r-1)} T_{\omega_i}^{-1} D^+_{k-r,i}(\xi^{(r)}).$

(ii) $D^-_{k,i}(\xi^{(r)}) = \sum_{t=1}^{r-1} (-1)^{t+1} q^{t(r-1)} D^-_{k,i}(\xi^{(r-t)}) E_{\delta-\alpha_i} + (-1)^r q^{r(r-1)} T_{\omega_i} D^-_{k-r,i}(\xi^{(r)}).$

*The second term on the right-hand side of these equations is omitted if $k < r$.

**Proof.** For this proof only, we write $\mathbb{U}$ as $U_q(\hat{\mathfrak{g}})$ to make explicit its dependence on the underlying finite-dimensional complex simple Lie algebra $\mathfrak{g}$. Using Theorem 1, it is easy to see that there is a unique homomorphism of $\mathbb{Q}(q)$-algebras $\mu_i : U_q(\hat{sl}_2) \to U_q(\hat{\mathfrak{g}})$ such that

$$\mu_i(x^\pm_{\alpha,k}) = o(1)^k x^\pm_{\alpha,k}, \quad \mu_i(K_1) = K_i, \quad \mu_i(C^{1/2}) = C^{1/2}$$

for all $k \in \mathbb{Z}$ (we take $o(1) = 1$ for $sl_2$). By checking on the generators in Theorem 1, it is easy to see that

$$T_{\omega_i} \circ \mu_i = \mu_i \circ T_{\omega_1}.$$  

In our present notation, Proposition 4.2 in [CP] states that

$$D^+_{k,1}(\xi^{(r)}) = \sum_{t=1}^{r-1} (-1)^{t+1} q^{t(r-1)} E_{\alpha_1}^t D^+_{k,1}(\xi^{(r-t)}) + q^{r(r-1)} T_{\omega_1}^{-1} D^+_{k-r,1}(\xi^{(r)}).$$

(The automorphism $T$ used in Proposition 4.2 in [CP] is $T_{\omega_1}^{-1}$; the definition of $T$ that precedes the statement of Proposition 4.2 in [CP] is incorrect.) Applying $\mu_i$ to both sides of this equation gives part (i) of the proposition. Part (ii) is proved similarly.

**Corollary 2.1.** Let $r,k \geq 1$. Then,

$$D^+_{k,i}(\xi^{(r)}) = \sum \mu(s_0,s_1,\ldots)(E_{\alpha_1})^{(s_0)}(E_{\delta-\alpha_i})^{(s_1)} \ldots,$$

where the sum is over those non-negative integers $s_0,s_1,\ldots$ such that $\sum_i s_i = r$ and $\sum_i \gamma_i s_i = k$, and the coefficients $\mu(s_0,s_1,\ldots) \in \mathbb{A}$. In particular, the coefficient
of $E_{k,i}^{(r)}$ in $D_{k,i}^+(\xi^{(r)})$ is $q^{kr(r-1)}$. An analogous statement holds for $D_{k,i}^-(\xi^{(r)})$ and the coefficient of $E_{k,i}^{(r)}$ in $D_{k-1,i}^-(\xi^{(r)})$ is $(-1)^{(k-1)(r-1)}q^{(k-1)r(r-1)}$.

**Proof.** The first part is immediate from the proposition. The second part follows by induction on $r$, noting that the term involving $E_{k,i}^{(r)}$ can arise only from the second term on the right-hand side of the formula for $D_{k,i}^-(\xi^{(r)})$ (and similarly for the other case).

Next, if $x \in U$, let $L_x : U \to U$ (resp. $R_x : U \to U$) be left (resp. right) multiplication by $x$, and define
\[
D_{k,i} = L_{q^k p_{k,i}} D_{0,i}^+ + L_{q^{k-1} p_{k-1,i}} D_{1,i}^+ + \cdots + L_{p_{0,i}} D_{k,i}^+,
\]
(resp. $\tilde{D}_{k,i} = R_{q^{-k} \tilde{p}_{k,i}} D_{0,i}^+ + R_{q^{-1} \tilde{p}_{k-1,i}} D_{1,i}^+ + \cdots + R_{\tilde{p}_{0,i}} D_{k,i}^+.$)

**Proposition 2.2.** Let $r, s \in \mathbb{N}$, $i \in I$. Then,
\[
\begin{align*}
(i) & \quad E_{\alpha_i}^{(r)} E_{\alpha_i}^{(s)} = \sum_{t=0}^{\min(r,s)} \sum_{m+k=t} q^{2rs-tr-ts+t} D_{m,i}^- \xi^{(s-t)} D_{k,i}^-(\xi^{(r-t)}), \\
(ii) & \quad E_{\alpha_i}^{(s)} E_{\alpha_i}^{(r)} = \sum_{t=0}^{\min(r,s)} \sum_{m+k=t} q^{rt+st-2sr+t} \tilde{D}_{k,i} \xi^{(r-t)} D_{m,i}^-(\xi^{(s-t)}).
\end{align*}
\]

**Proof.** Part (i) is a restatement in the current notation of Lemma 5.1 in [CT]. And part (ii) follows from part (i) by applying $\Omega$.

**Corollary 2.2.** $\tilde{p}_{k,i} \in \mathcal{A}U^+$ for all $k > 0$, $i \in I$.

**Proof.** The proof proceeds by induction on $k$. If $k = 0$ the result is obvious. Assume that $\tilde{p}_{m,i} \in \mathcal{A}U^+$ for all $m < k$. Then, taking $r = s = k$ in the identity above, we see using Corollary 2.1 and the induction hypothesis that
\[
E_{\alpha_i}^{(k)} E_{\alpha_i}^{(k)} = \tilde{p}_{k,i} + \text{terms in } \mathcal{A}U^+.
\]

Since the left-hand side is in $\mathcal{A}U^+$ the result follows.

Let $\mathcal{A}U^+(>)$, $\mathcal{A}U^+(<)$ and $\mathcal{A}U^+(0)$ be the $\mathcal{A}$-subalgebras of $\mathcal{A}U^+$ generated by the sets $\{E_{\alpha_i}^{(r)} | k \leq 0, r \geq 1\}$, $\{E_{\alpha_i}^{(r)} | k > 0, r \geq 1\}$ and $\{\tilde{p}_{k,i} | k > 0, i \in I\}$, respectively. The following lemma is now obvious from Proposition 1.3 (iv) and the preceding corollary.

**Lemma 2.3.** The set $B_0$ is an $\mathcal{A}$-basis for $\mathcal{A}U^+(0)$.

The next lemma can be found in [4] and is an obvious consequence of Proposition 1.4 (iii) and the fact that the maps $r$ and $r_i$ preserve $\mathcal{A}U^+$.

**Lemma 2.4.** For all $i \in I$, we have
\[
\mathcal{A}U^+ = \bigoplus_r E_{\alpha_i}^{(r)} \mathcal{A}U^+[i], \quad \mathcal{A}U^+ = \bigoplus_r \mathcal{A}U^+[i] E_{\alpha_i}^{(r)},
\]
where $\mathcal{A}U^+[i] = \{x \in \mathcal{A}U^+ | r(x) = 0\}$ and $\mathcal{A}U^+[i] = \{x \in \mathcal{A}U^+ | r_i(x) = 0\}$.
Proposition 2.3. Let \( s_{j_1}s_{j_2}\ldots s_{j_k} \) be an arbitrary reduced expression in \( \widetilde{W} \). Define an \( \mathcal{A} \)-subalgebra

\[
\mathcal{A}U_{j_1,j_2,\ldots,j_k}^+ = \{ x \in \mathcal{A}U^+ \mid T_{j_k}T_{j_{k-1}}\ldots T_{j_1}(x) \in \mathcal{A}U^-\mathcal{A}U^0 \}.
\]

Then, the elements \( E_{\alpha(j_1)}^{(r_1)}(T_{j_1}^{-1}E_{\alpha(j_2)}^{(r_2)})\ldots(T_{j_1}^{-1}T_{j_2}^{-1}\ldots T_{j_{k-1}}^{-1}E_{\alpha(j_k)}^{(r_k)}) \), for \( r_1,\ldots,r_k \geq 0 \), form an \( \mathcal{A} \)-basis of \( \mathcal{A}U_{j_1,j_2,\ldots,j_k}^+ \).

Proof. For \( 1 \leq l \leq k \), we have

\[
T_{j_k}T_{j_{k-1}}\ldots T_{j_l}(T_{j_l}^{-1}T_{j_{l+1}}^{-1}\ldots T_{j_1}^{-1}E_{\alpha(j_{l+1})}^{(r_{l+1})}) = T_{j_k}T_{j_{k-1}}\ldots T_{j_1}E_{\alpha(j_{l+1})}^{(r_{l+1})}
\]

\[
= (-1)^{r_{l+1}}q^{-r_{l+1}(r_{l+1}-1)/2}T_{j_k}T_{j_{k-1}}\ldots T_{j_{l+1}}E_{\alpha(j_{l+1})}^{(r_{l+1})}T_{j_l}^{-1}\ldots T_{j_1}^{-1}E_{\alpha(j_{l+1})}^{(r_{l+1})}.
\]

By \([4, \text{Proposition 40.2.1}]\), the expression \( s_{j_k}s_{j_{k-1}}\ldots s_{j_{l+2}} \) is reduced and so by \([4, \text{Lemma 40.1.2}]\) (working with \( F \)'s rather than \( E \)'s) we see that the right-hand side of the above equation is in \( \mathcal{A}U^-\mathcal{A}U^0 \) for all \( r_1,\ldots,r_k \geq 0 \). Since \( \mathcal{A}U^-\mathcal{A}U^0 \) is an \( \mathcal{A} \)-subalgebra of \( \mathcal{A}U \), it follows that \( E_{\alpha(j_1)}^{(r_1)}(T_{j_1}^{-1}E_{\alpha(j_2)}^{(r_2)})\ldots(T_{j_1}^{-1}T_{j_2}^{-1}\ldots T_{j_{k-1}}^{-1}E_{\alpha(j_k)}^{(r_k)}) \) is an \( \mathcal{A} \)-subalgebra of \( \mathcal{A}U_{j_1,j_2,\ldots,j_k}^+ \). Further, it is proved in \([4, \text{Proposition 40.2.1}]\) that these elements are linearly independent. So, it remains to prove that they span \( \mathcal{A}U_{j_1,j_2,\ldots,j_k}^+ \).

Let \( x \in \mathcal{A}U_{j_1,j_2,\ldots,j_k}^+ \) be homogeneous, and write it as a sum

\[
x = \sum E_{\alpha(j_1)}^{(r_1)}x_{r_1}
\]

according to the decomposition in Lemma \( [4, \text{Proposition 1.4}] \). Since \( T_{j_1}x_{r_1} \in \mathcal{A}U^+ \) by Proposition \( [4, \text{Proposition 1.4}] \), write

\[
T_{j_1}x_{r_1} = \sum E_{\alpha(j_2)}^{(r_2)}x_{r_1,r_2},
\]

again according to the decomposition in Lemma \( [4, \text{Proposition 1.4}] \). Repeating this process, we define for \( 1 \leq l \leq k \) a set of elements \( x_{r_1,r_2,\ldots,r_l} \in \mathcal{A}U^+[l] \) satisfying

\[
T_{j_1}x_{r_1,r_2,\ldots,r_l} = \sum E_{\alpha(j_{l+1})}^{(r_{l+1})}x_{r_1,r_2,\ldots,r_{l+1}}
\]

for \( 1 \leq l < k \). This gives

\[
T_{j_k}T_{j_{k-1}}\ldots T_{j_1}x = \sum (T_{j_k}T_{j_{k-1}}\ldots T_{j_1}E_{\alpha(j_1)}^{(r_1)})(T_{j_k}T_{j_{k-1}}\ldots T_{j_1}x_{r_1})
\]

\[
= \sum (T_{j_k}T_{j_{k-1}}\ldots T_{j_1}E_{\alpha(j_1)}^{(r_1)})(T_{j_k}T_{j_{k-1}}\ldots T_{j_2}E_{\alpha(j_2)}^{(r_2)})(T_{j_k}T_{j_{k-1}}\ldots T_{j_2}x_{r_1,r_2})
\]

\[
= \sum (T_{j_k}T_{j_{k-1}}\ldots T_{j_1}E_{\alpha(j_1)}^{(r_1)})(T_{j_k}T_{j_{k-1}}\ldots T_{j_2}E_{\alpha(j_2)}^{(r_2)})\ldots
\]

\[
\ldots (T_{j_k}T_{j_{k-1}}E_{\alpha(j_{k-1})}^{(r_{k-1})})(T_{j_k}E_{\alpha(j_k)}^{(r_k)})x_{r_1,r_2,\ldots,r_k}.
\]

The left-hand side of the above equation is in \( \mathcal{A}U^-\mathcal{A}U^0 \), and since \( T_{j_k}T_{j_{k-1}}\ldots T_{j_1}E_{\alpha(j_1)}^{(r_1)} \) is in \( \mathcal{A}U^-\mathcal{A}U^0 \), it follows from Lemma \( [4, \text{Proposition 1.2}] \) that \( x_{r_1,r_2,\ldots,r_k} \in \mathcal{A} \). Applying \( (T_{j_k}T_{j_{k-1}}\ldots T_{j_1})^{-1} \), we now get the statement of the proposition.

Proposition 2.4. \( B > \) is an \( \mathcal{A} \)-basis of \( \mathcal{A}U^+(>) \).
Proof. Let \( x \in \mathcal{A}U^+(>) \). Then, using the definition of the root vectors, we see that there exists an integer \( l \leq 0 \) and \( i_0, i_{-1}, \ldots, i_l \in I \) such that \( x \in \mathcal{A}U^+_{i_0, i_{-1}, \ldots, i_l} \). The result is now immediate from the preceding proposition.

One can prove similarly (working with \( r_i \) and replacing \( T_i \) by \( T_i^{-1} \)):

**Proposition 2.5.** \( B_\prec \) is an \( \mathcal{A} \)-basis of \( \mathcal{A}U^+(<) \).

We omit the details.

Summarizing, we have proved that \( B \) is an \( \mathcal{A} \)-basis of \( \mathcal{A}U^+(>) \mathcal{A}U^+(0) \mathcal{A}U^+(<) \).

It remains then to prove the triangular decomposition. We shall need a number of subalgebras and subspaces of \( \mathcal{A}U \). We collect them in the following definition.

**Definition 2.1.**

(i) For \( i \in I \), let \( \mathcal{A}U^+(\succ) \) (resp. \( \mathcal{A}U^+(\prec) \)) be the \( \mathcal{A} \)-subalgebra of \( \mathcal{A}U^+ \) generated by the elements \( E_{k \delta + \alpha_i}^{(r)} \) (resp. \( E_{(k+1)\delta - \alpha_i}^{(r)} \)) for \( k \geq 0, r \geq 0 \).

(ii) Let \( \mathcal{A}U^+(\succeq) \) (resp. \( \mathcal{A}U^+(\preceq) \)) be the \( \mathcal{A} \)-subalgebras of \( \mathcal{A}U^+ \) (resp. \( \mathcal{A}U^+ \)) generated by \( \mathcal{A}U_i^+(\succ) \) (resp. \( \mathcal{A}U_i^+(\prec) \)) for all \( i \in I \).

(iii) The \( \mathcal{A} \)-subalgebra of \( \mathcal{A}U \) generated by \( \mathcal{A}U_i^+(\prec) \) and the \( F_{\alpha_i}^{(r)} \) for all \( i \in I, r \geq 0 \) will be denoted by \( \mathcal{A}\tilde{U}^{(\prec)} \). Let \( \mathcal{A}\tilde{U}^+ \) be the \( \mathcal{A} \)-subalgebra of \( \mathcal{A}U \) generated by \( \mathcal{A}U^{(\succ)} \) and \( \mathcal{A}\tilde{U}^{(\preceq)} \). Set \( \mathcal{A}\tilde{U}(\Delta) = \mathcal{A}U^{(\succ)} \mathcal{A}U^+(0) \mathcal{A}\tilde{U}^{(\prec)} \).

The following is an obvious consequence of Corollary 2.1.

**Lemma 2.5.** For all \( k, r \geq 0, i \in I \), we have

\[
D_{k,i}^+(\xi^{(r)}) \in \mathcal{A}U_i^+(\succ), \quad D_{k,i}^-(\xi^{(r)}) \in \mathcal{A}U_i^+(\prec).
\]

**Proposition 2.6.** For all \( r \geq 0 \), the element \( E_{\alpha_i}^{(r)} \) is in the \( \mathcal{A} \)-subalgebra generated by the \( (x_i^{\pm})^{(s)} \) for \( i \in I, k, s \geq 0 \). Hence, \( \mathcal{A}U^+ \) is an \( \mathcal{A} \)-subalgebra of \( \mathcal{A}\tilde{U}^+ \).

**Proof.** Assume first that \( \theta = \sqrt{2} \). Then, we can choose

(i) \( i_0 \in I \) such that \( |\omega_{i_0}| \cdot |\theta| = 1 \);

(ii) an element \( w \) in the subgroup of \( W \) generated by \( \{ s_j \mid j \in I, j \neq i_0 \} \) satisfying \( w(\alpha_{i_0}) = \theta \).

Hence, \( t_{\omega_{i_0}} ws_{i_0} \alpha_{i_0} = \alpha_0 \). Setting \( w' = t_{\omega_{i_0}} ws_{i_0} \), it follows that \( l(w') = l(t_{\omega_{i_0}} w) - 1 \), and also from [2] that \( T_{w'} E_{\alpha_{i_0}^{(r)}} = E_{\alpha_0^{(r)}}^{(r)} \). Further, it is easy to check that \( l(T_{\omega_{i_0}} w) = l(t_{\omega_{i_0}} w) + l(w) \), and so we get

\[
T_w E_{\alpha_{i_0}^{(r)}} = T_{\omega_{i_0} w} T_i^{-1} E_{\alpha_0^{(r)}} = T_{\omega_{i_0} w} T_i^{-1} E_{\alpha_{i_0}^{(r)}}.
\]

Now, the element \( T_w T_i^{-1} E_{\alpha_0}^{(r)} \) is in the \( \mathcal{A} \)-subalgebra generated by the \( E_{\alpha_i}^{(l)}, F_{\alpha_i}^{(m)} \) for \( i \in I, l, m \geq 0 \) (since \( ws_{i} \in W \)). The result now follows by using Theorem 3 and the fact that \( T_{\omega_i}(E_{\alpha_j}) = E_{\alpha_j}, T_{\omega_i}(F_{\alpha_j}) = F_{\alpha_j} \), if \( i \neq j \).

The case of \( E_8 \) is somewhat more complicated, since we can only choose an element \( i_0 \) such that \( |\omega_{i_0}| \cdot |\theta| = 2 \). However, the argument above can be modified by using [2] Lemma 2.7. We omit the details.

The second statement in the lemma now follows from Lemma 1.5 and the fact that \( \mathcal{A}U^+ \) contains the generators of \( \mathcal{A}U^+ \). 

**Proposition 2.7.** We have:
We prove this proposition in the remainder of this section.

We need the following commutation relations.

Proposition 2.8. Let $i, j \in I$, $k, r \geq 0$. We have

(i) $\tilde{P}_{k,j} E_{r\delta+\alpha_i} = \sum_{s=0}^{k} o(i)^s o(j)^s \frac{[a_{ji}] [a_{ji} + 1] \cdots [a_{ji} + s - 1]}{[s]!} E_{(r+s)\delta+\alpha_i} \tilde{P}_{k-s,j}$;

(ii) $E_{(r+1)\delta-\alpha_i} \tilde{P}_{k,j} = \sum_{s=0}^{k} o(i)^s o(j)^s \frac{[a_{ji}] [a_{ji} + 1] \cdots [a_{ji} + s - 1]}{[s]!} \tilde{P}_{k-s,j} E_{(r+s+1)\delta-\alpha_i}$.

Proof. This is the same as for Lemmas 3.3 and 3.4 in [CP].

Corollary 2.3. (i) If $a_{ij} = 0$,

\[ \tilde{P}_{k,j} E_r^{(s)} = E_r^{(s)} \tilde{P}_{k,j} \]

(ii) If $a_{ji} = -1$,

\[ \tilde{P}_{k,j} E_r^{(s)} = \sum_{m=0}^{k} q^{m(s-m)} E_r^{(s-m)} E_{(r+1)\delta+\alpha_i} \tilde{P}_{k-m,j} \]

(iii) Fix $i \in I$, let $x \in \mathcal{A} U^+ (>\alpha_i)$ be homogeneous, and assume that $\tilde{P}_{k,i} x \in \mathcal{A} U^+ (>\alpha_i) U^+ (0)$. Then, there exist elements $x_s \in \mathcal{A} U^+ (>\alpha_i)$ of homogeneity $|x| + s\delta$ such that

\[ \tilde{P}_{k,i} x = \sum_s x_s \tilde{P}_{k-s,i} \]

Analogous results hold involving the $E^{(s)}_{(r+1)\delta-\alpha_i}$.

Proof. Parts (i) and (ii) follow from Proposition 2.8 by a direct computation using the relations in Proposition 2.2. If $i = j$, a repeated application of part (i) of Proposition 2.8 implies that we can write

\[ \tilde{P}_{k,i} x = \sum_s y_s \tilde{P}_{k-s,i} \]

for some $y_s$ of homogeneity $|y_s| = |x| + s\delta$ in the $\mathcal{Q}(q)$-subalgebra of $U^+$ generated by $\{ E_{k+i} \mid k \geq 0 \}$. On the other hand, since $\tilde{P}_{k,i} x \in \mathcal{A} U^+ (>\alpha_i) U^+ (0)$, by Lemma 2.3 we can write

\[ \tilde{P}_{k,i} x = \sum_{c \in \mathbb{N}^0} x_c E_c \]

for some elements $x_c \in \mathcal{A} U^+ (>\alpha_i)$. Since $B_0$ is a basis of $U^+ (0)$, we can now equate coefficients to get the result.

Lemma 2.6. Let $i, j \in I, k, r, s \geq 0$. 


Lemma 2.5 and (2.1).

By a repeated application of (i)

(i) For $0 \leq t \leq k$, there exist $x_t \in \mathcal{A}U_i^+(\bowtie)$ (resp. $x_t \in \mathcal{A}U_i^+(\bowtie)$) of homogeneity $|x_t| = (s + k - t)\delta + r\alpha_j$ (resp. $|x_t| = (s + k + 1 - t)\delta - r\alpha_j$) such that

\[ \tilde{P}_{k,i}E_{\alpha_{\delta + \alpha_j}}^{(r)} = \sum_t x_t \tilde{P}_{t,i} \quad \text{(resp. } E_{(s + 1)\delta - \alpha_j}^{(r)} \tilde{P}_{k,i} = \sum_t \tilde{P}_{t,i}x_t). \]

(ii) The elements $D_{k,j}(\xi^{(s)})E_{\alpha_i}^{(r)}$ can be written as $\mathcal{A}$-linear combinations of products $xyz$, where $x \in \mathcal{A}U_i^+(\bowtie)$, $y \in \mathcal{A}U_i^+(0)$, $z \in \mathcal{A}U_i^+(\bowtie)$ are homogeneous and

\[ |x| = r'\alpha_i + l\delta, \quad |y| = m\delta, \quad |z| = p\delta - s'\alpha_i, \]

where $r' \leq r$, $s' \leq s$ and $l + m + p = k + 1$.

(iii) There exist elements $x_t$ of homogeneity $|x_t| = (k - t)\delta - r\alpha_j$ in the $\mathcal{A}$-subalgebra generated by $\mathcal{A}U_i^+(\bowtie)$ and the $F_{\alpha_j}^{(s)}$ for $s \geq 0$, such that

\[ \tilde{P}_{k,i}E_{\alpha_{j}}^{(r)} \tilde{P}_{t,i} = \sum_t \tilde{P}_{t,i}x_t. \]

Proof. If $i \neq j$ part (i) was proved in Corollary 2.3(i), (ii), and part (ii) is obvious from the defining relations. Let $i = j$. From Corollary 1.1 we see that $T_{\alpha_i} \tilde{P}_{k,i} = \tilde{P}_{k,i}$ for all $k \geq 0, i \in I$. Now, observe that (i) is equivalent (by applying $T_{\alpha_i}$) to:

(i') There exist homogeneous elements $x_t \in \mathcal{A}U_i^+(\bowtie)$, with $|x_t| = (k - t)\delta + r\alpha_i$, such that $\tilde{P}_{k,i}E_{\alpha_{j}}^{(r)} = \sum_t x_t \tilde{P}_{t,i}$.

We prove (i') and (ii) simultaneously by induction on $k$.

If $k = 0$, then (i') is obvious for all $r$ and (ii) is clear from Proposition 2.2, Lemma 2.5 and (2.1).

Assume now that (i') and (ii) hold for all smaller values of $k$ and for all $r, s \geq 0$. By a repeated application of (i'), we can write, for $m < k$ and any homogeneous $x \in \mathcal{A}U_i^+(\bowtie)$,

\[ \tilde{P}_{m,i}x = \sum_t x_t \tilde{P}_{m-t,i}, \]

where $x_t \in \mathcal{A}U_i^+(\bowtie)$ and $|x_t| = |x_t| + t\delta$. Next, note that $E_{(k)}^{(s - \alpha_i)}, E_{\alpha_i}^{(r)}$ belongs to $\mathcal{A}U_i^+(\bowtie), \mathcal{A}U_i^+(0), \mathcal{A}U_i^+(\bowtie)$ because it equals

\[ \left[ \begin{array}{c} k + r \end{array} \right] E_{\delta - \alpha_i}^{(k)} E_{\alpha_i}^{(r+k)}, \]

which is in $\mathcal{A}U_i^+(\bowtie), \mathcal{A}U_i^+(0), \mathcal{A}U_i^+(\bowtie)$ by Proposition 2.2. On the other hand, the same proposition implies that

\[ E_{(s - \alpha_i)}^{(k)} E_{\alpha_i}^{(r)} = \tilde{P}_{k,i}E_{\alpha_i}^{(r)} + wE_{\alpha_i}^{(r)}, \]

where $w$ is a linear combination of terms of type $D_{m,i}(\xi^{(l)})D_{l,j}(\xi^{(p)})$, with $m, t < k$. By equation (2.2) and the induction hypothesis, we see that $wE_{\alpha_i}^{(r)}$ belongs to $\mathcal{A}U_i^+(\bowtie), \mathcal{A}U_i^+(0), \mathcal{A}U_i^+(\bowtie)$. Hence, $\tilde{P}_{k,i}E_{\alpha_i}^{(r)} \in \mathcal{A}U_i^+(\bowtie), \mathcal{A}U_i^+(0), \mathcal{A}U_i^+(\bowtie)$. Since $\tilde{P}_{k,i}E_{\alpha_i}^{(r)} \in U^+(\bowtie)U^+(0)$, we conclude by Proposition 1.3 that $\tilde{P}_{k,i}E_{\alpha_i}^{(r)} \in \mathcal{A}U_i^+(\bowtie), \mathcal{A}U_i^+(0)$. But now applying Corollary 2.3(iii), we get (i') for $k$. 


To prove (ii), consider
\begin{equation}
E_{\delta+\alpha_i}^{(k+s)} E_{\alpha_i}^{(k)} E_{\alpha_i}^{(r)} = E_{\delta+\alpha_i}^{(k+s)} E_{\alpha_i}^{(k+r)}.
\end{equation}

By Proposition 2.2 and Lemma 2.3, the right-hand side of (2.3) belongs to \( \mathcal{A} U_+^{(\gg)} A U^+_i(0) \) and the left-hand side equals
\[ q^{-k(s-1)} D_{k,i}(\xi^{(s)}) E_{\alpha_i}^{(r)} + w E_{\alpha_i}^{(r)}, \]
where \( w \) is a linear combination of terms \( D_{m,i}(\xi^{(i)}) D_{l,i}(\xi^{(l)}) \) with \( m \leq k \) and \( t < k \). The induction hypothesis and the fact that (i) holds for \( k \) now implies that \( D_{k,i}(\xi^{(s)}) E_{\alpha_i}^{(r)} \in \mathcal{A} U_+^{(\gg)} A U^+_i(0) \) and \( D_{k,i}(\xi^{(s)}) E_{\alpha_i}^{(r)} \) is homogeneous, then \( D_{k,i}(\xi^{(s)}) E_{\alpha_i}^{(r)} = \sum c E_c \),
where \( c \in \mathcal{A} \). This is possible since we have already proved that \( B \) is an \( \mathcal{A} \)-basis of \( \mathcal{A} U_+^{(\gg)} A U^+_i(0) \). Now, it is easy to see, using Proposition 1.2, that \( c \neq 0 \) only if \( c \) is supported on the set
\[ \{ l \delta + r \alpha_i \mid r' \leq r \} \cup \{ m \delta^{(i)} \mid m \leq s \} \cup \{ n \delta - s' \alpha_i \mid s' \leq s \}. \]

The result follows. The case of \( E_{\alpha_i}^{(r)} \) is similar.

Part (iii) can be deduced from part (i) by taking \( s = 0 \) and applying \( T_{\omega_i}^{-1} \) to part (i) and using Lemma 1.7.

Since \( \tilde{\mathcal{A}} \tilde{U}(\Delta) \) contains the generators of \( \mathcal{A} \tilde{U}_i^{(\Delta)} \), to prove part (i) of Proposition 2.7, it suffices to show that \( \mathcal{A} \tilde{U}(\Delta) \) is an \( \mathcal{A} \)-subalgebra of \( \mathcal{A} \tilde{U}_i^{(\Delta)} \). We prove that, if \( x \in \mathcal{A} U_+^{(\gg)} \), \( y \in \mathcal{A} U^+_i(0) \) and \( z \in \mathcal{A} U^+_i(\ll) \) are homogeneous, then
\[ yx \in \mathcal{A} U_+^{(\gg)} A U^+_i(0), \quad zy \in \mathcal{A} U^+_i(0) \mathcal{A} U_i(\ll), \quad zx \in \mathcal{A} \tilde{U}(\Delta). \]

This is enough, in view of the following scheme (in which \( x \)'s (resp. \( y \)'s, \( z \)'s) denote elements of \( \mathcal{A} U_+^{(\gg)} \) (resp. \( \mathcal{A} U^+_i(0), \mathcal{A} U^+_i(\ll) \))):
\[ (x_1 y_1 z_1)(x_2 y_2 z_2) = (x_1 x_2)(y_1 y_2)(z_1 z_2), \quad \text{if } (x_1 x_2) = \sum x_3 y_3 z_3 \]
\[ = \sum (x_1 x_4)(y_4 y_5)(z_5 z_2), \quad \text{if } (y_4 y_5) = \sum x_4 y_4, \quad z_5 z_2 = \sum z_5 z_5. \]

To see that \( yx \in \mathcal{A} U_+^{(\gg)} A U^+_i(0) \), we proceed by induction on \( |y| = m \delta \). If \( m = 0 \) there is nothing to prove. If \( y = P_{m,j} \) for some \( i \), the assertion is proved by using Lemma 2.6(i) repeatedly. If \( y \) is an \( \mathcal{A} \)-linear combination of terms of type \( y_t P_{l,j} \), for some \( y_t \in \mathcal{A} U^+_i(0), t \geq 0 \), the result follows again by Lemma 2.6(i) and the induction hypothesis. The case of \( zy \) is similar.

To prove that \( zx \in \mathcal{A} \tilde{U}(\Delta) \), we proceed by induction on the height of \( \text{ht}(\text{re}(-|z|)) \). Consider first the case \( \text{ht}(\text{re}(-|z|)) = 1 \), so that for some \( i \in I \) we have
\[ z = E_{l\delta+\alpha_i}, \quad \text{for some } l > 0, \quad \text{or } z = F_{a_i}. \]

We can assume that \( x \equiv E_{m,j+\alpha_i}^{(r)} \) for some \( m, r \geq 0, j \in I \), for if \( x \) is a product \( E_{m,j+\alpha_i}^{(r)} E_{m,j+\alpha_i}^{(r)} \), the proof can then be completed by an obvious induction along with the fact that we have proved that \( \mathcal{A} U^+_i(\gg) \mathcal{A} U^+_i(0) \) is an \( \mathcal{A} \)-subalgebra of \( \mathcal{A} \tilde{U}_i^{(\Delta)} \). Moreover, we can assume that \( i = j \), since otherwise \( z \) and \( x \) commute.
Consider first the case in which \( z = E_{i\delta - \alpha_i} \), with \( l > 0 \). Taking \( s = 1, k = l - 1 \) in Lemma 2.6(ii), we see that \( zx \in \mathcal{A}\tilde{U}(\Delta) \), and this proves the case \( m = 0 \). To deal with the case \( m > 0 \), we start by using the fact that \( B \) is an \( \mathcal{A} \)-basis of \( \mathcal{A}U^+(\rangle)\mathcal{A}U^+(0)\mathcal{A}U^+(\langle) \) to write \( E_{i\delta - \alpha_i}E^{(r)}_{\alpha_i} \), \( l > 0 \), as a sum

\[
\sum_{a} a_{c}E_{c}
\]

where \( a_{c} \in \mathcal{A} \). It is easy to see, by using Lemma 1.2, and an obvious induction on \( r \), that \( a_{c} \neq 0 \) only if the restriction of \( c \) to \( \mathbb{N}^{K_{c}} \) is either zero or supported on the root \( l\delta - \alpha_i \). The result for arbitrary \( m > 0 \) now follows by applying \( T_{\omega_{i}}^{-m} \) to \( E_{(l+m)\delta - \alpha_i}E^{(r)}_{\alpha_i} \) and making use of Corollary 1.1.

If now \( z = F_{\alpha_i} \), the case \( m = 0 \) is contained in [A, Corollary 3.1.9]. The case \( m > 0 \) is deduced from this by applying \( T_{\omega_{i}}^{-m} \) to \( E_{m\delta - \alpha_i}E^{(r)}_{\alpha_i} \).

This completes the proof that \( zx \in \mathcal{A}\tilde{U}(\Delta) \) when \( ht(re(\langle z \rangle)) = 1 \). In fact, more precisely we have proved that \( zE^{(r)}_{m\delta + \alpha_i} \) can be written as an \( \mathcal{A} \)-linear combination of products \( x'z' \), where \( x' \in \mathcal{A}U^+(\rangle)\mathcal{A}U^+(0) \) and \( z' \in \mathcal{A}\tilde{U}(\langle) \) are homogeneous and \( |z'| = 0 \) or \( |z'| = l'\delta - \alpha_i \) for some \( l' \geq 0 \).

Assume now that, for all homogeneous elements \( z \) such that \( ht(re(\langle z \rangle)) < s \) and all \( x \in \mathcal{A}U^+(\rangle) \), \( zx \) can be written as an \( \mathcal{A} \)-linear combination of terms \( x'z' \) where \( x' \in \mathcal{A}U^+(\rangle)\mathcal{A}U^+(0) \) and \( z' \in \mathcal{A}\tilde{U}(\langle) \) has \( ht(re(\langle z' \rangle)) < ht(re(\langle z \rangle)) \). We prove that the result holds for \( ht(re(\langle z \rangle)) = s \). If \( z \) can be written as a product \( z_1z_2 \) such that \( ht(re(\langle z_1 \rangle)) < ht(re(\langle z \rangle)) \), for \( t = 1, 2 \), then we are done by induction. Otherwise, \( z = E^{(p)}_{i\delta - \alpha_i} \), for some \( i \in I, p \geq 0 \). Arguing as in the case when \( ht(re(\langle z \rangle)) = 1 \), we see that it suffices to prove the result when \( z = E^{(r)}_{\alpha_i} \) and \( l > 0 \). Recall from Corollary 2.1 that there exists an integer \( M \) such that

\[
D_{(l-1)s_{1},l'(s_{1})}(\xi^{(s_{1})}) = q^{M}E_{l\delta - \alpha_i}^{(s_{1})} + z'',
\]

where \( z'' \) is a linear combination of products \( E_{l_{1}\delta - \alpha_i}^{(s_{1})}E_{l_{2}\delta - \alpha_i}^{(s_{2})} \ldots \) with \( l_{1}, l_{2}, \ldots \) and \( s_{1}, s_{2}, \ldots \) being positive integers such that \( s_{1}, s_{2}, \ldots < s \) and \( l_{1}s_{1} + l_{2}s_{2} + \cdots = ls \). Hence,

\[
E_{l_{i}\delta - \alpha_i}^{(s_{i})}E^{(r)}_{\alpha_i} = q^{-M}\left( \pm D_{(l-1)s_{i},l'(s_{i})}(\xi^{(s_{i})})E^{(r)}_{\alpha_i} + zE^{(r)}_{\alpha_i} \right).
\]

The first term has the correct form by Lemma 2.4(ii), and the second term can be written (by a repeated application of the induction hypothesis, since \( s_{i} + r < s \)) as an \( \mathcal{A} \)-linear combination of products \( x'z' \), where \( x' \in \mathcal{A}U^+(\rangle)\mathcal{A}U^+(0) \) and \( z' \in \mathcal{A}\tilde{U}(\langle) \) has \( |z'| = l'\delta - s'\alpha_i \) for some \( l' \geq 0, s' \leq s \). This proves part (i) of Proposition 2.7.

To prove part (ii), observe (from the definition of the root vectors) that \( T_{l_{2}r} \) maps \( \mathcal{A}U^+(\langle) \) into \( \mathcal{A}U^+(\langle) \). Hence, for \( x \in \mathcal{A}U^+(\langle) \), we have

\[
T_{l_{2}r}(x) = \sum_{c \in \mathbb{N}^{R_{\langle}}} a_{c}E_{c}
\]

for some \( a_{c} \in \mathcal{A} \). Now, from Lemma 1.4 we see that \( T_{l_{2}r}^{-1}(E_{c}) \in \mathcal{A}U^+(\langle)\mathcal{A}U^{-}\mathcal{A}U^{0} \).

Part (ii) follows. Part (iii) is now obvious.

The proof of Proposition 2.7 is complete. □
3. A Computation of Inner Products on $A\mathbb{U}(0)$.

The canonical basis is characterized by its behavior with respect to a symmetric bilinear form introduced in [K], following Drinfeld. Since we have introduced the imaginary root vectors, a prerequisite to the construction of a crystal basis is an understanding of the behavior of the form on $A\mathbb{U}(0)$. We begin with certain preliminary definitions.

Define an algebra structure on $A\mathbb{U}^+ \otimes A\mathbb{U}^+$ by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{|x_2| - |y_1|} x_1 y_1 \otimes x_2 y_2,$$

where $x_1, y_1$ are homogeneous. Let $r : U^+ \to U^+ \otimes U^+$ be the $\mathbb{Q}(q)$-algebra homomorphism defined by extending $r(E_{\alpha}) = E_{\alpha} \otimes 1 + 1 \otimes E_{\alpha}, \quad (i \in I)$.

The algebra $U^+$ has a unique symmetric bilinear form $(\ , \ ) : U^+ \times U^+ \to \mathbb{Q}(q)$ \cite[1.2.5]{L4} satisfying (1,1) = 1 and

$$(E_{\alpha}, E_{\alpha}) = \delta_{i,j} (1 - q^{-2})^{-1}, \quad (x, yy') = (r(x), y \otimes y'), \quad (xx', y) = (x \otimes x', r(y)),$$

where the form on $U^+ \otimes U^+$ is defined by $(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2)(y_1, y_2)$. The form satisfies

$$(E_{\alpha}, y, x) = (1 - q^{-2})^{-1}(y, r(x)) \quad (yE_{\alpha}, x) = (1 - q^{-2})^{-1}(y, r(x)).$$

Let $A = \mathbb{Q}(q) \cap \mathbb{Q}[q^{-1}]$. The main result of this section is:

**Proposition 3.1.** For $i, j \in I, k, k' > 0$, we have

$$(P_{k,i}', P_{k',j}') = \delta_{k,k'} \delta_{i,j} \quad \text{mod} \quad (q^{-1}A).$$

For $i \in I$, let $A\Lambda_i$ denote the $A$-subalgebra of $U^+$ generated by the $P_{k,i}$ for $k \geq 0$. Since $A\mathbb{U}^+(0)$ is commutative, it follows from Lemma 2.3 that $A\Lambda_i$ is the polynomial algebra generated by the $P_{k,i}$ and that

$$A\mathbb{U}^+(0) \cong A\Lambda_1 \otimes A\Lambda_2 \otimes \cdots \otimes A\Lambda_n.$$

Any $x \in A\mathbb{U}^+(0)$ can thus be written as a finite sum of products $x(1) \ldots x(n)$, where $x(i) \in A\Lambda_i$, and we denote this by

$$x = \sum x(1) \ldots x(n).$$

Let $\Delta_i : A\Lambda_i \to A\Lambda_i \otimes A\Lambda_i$ be the $A$-algebra homomorphism obtained by extending

$$\Delta_i(P_{k,i}) = \sum_{s=0}^{k} P_{s,i} \otimes P_{k-s,i}.$$

Here, the algebra structure on $A\Lambda_i \otimes A\Lambda_i$ is the usual one, namely

$$(x \otimes y)(x' \otimes y') = xx' \otimes yy' \quad \text{for} \quad x, x', y, y' \in A\Lambda_i.$$

**Corollary 3.1.** Let $i, j \in I$. Then:

(i) if $x \in A\Lambda_i$ and $y = y(1) \ldots y(n) \in A\mathbb{U}^+(0)$ is such that $y(j) \neq 1$ for some $j \neq i$, then

$$(x, y) = 0 \quad \text{mod} \quad (q^{-1}A);$$

(ii) if $y \in A\mathbb{U}^+(0)$ is such that $y(i) \neq 1$ for some $i$, then

$$(x, y) = 0 \quad \text{mod} \quad (q^{-1}A);$$

(iii) if $x \in A\Lambda_i$ is such that $x(i) \neq 1$ for some $i$, then

$$(x, y) = 0 \quad \text{mod} \quad (q^{-1}A);$$

(iv) if $x \in A\mathbb{U}^+(0)$ is such that $x(i) \neq 1$ for some $i$, then

$$(x, y) = 0 \quad \text{mod} \quad (q^{-1}A);$$

(v) if $x \in A\mathbb{U}^+(0)$ is such that $x(i) \neq 1$ for some $i$, then

$$(x, y) = 0 \quad \text{mod} \quad (q^{-1}A).$$
(ii) for \( x, y \in \mathcal{A}U^+(0) \), we have
\[
(x, y) = \sum (x(1), y(1))(x(2), y(2)) \ldots (x(n), y(n)) \mod (q^{-1} \mathcal{A}).
\]

The proof of Proposition 3.1 and Corollary 3.1 occupies the rest of this section. We start with some preliminary results.

**Proposition 3.2.** (i) For \( m > 0 \) and \( i \in I \), we have
\[
r_i(E_{\delta-\alpha_i}) = 0, \quad r_i(E_{\delta,i}) = (1 - q^{-4})E_{\delta-\alpha_i},
\]
\[
r_i(E_{2m\delta-\alpha_i}) = q(1 - q^{-2})(1 - q^{-4})
\times \left( \sum_{s=0}^{m-2} q^{2s}E_{(2m-s)\delta-\alpha_i}E_{(s+1)\alpha-\alpha_i} + q^{2m-3}E_{m\delta-\alpha_i} \right),
\]
\[
r_i(E_{(2m+1)\delta-\alpha_i}) = q(1 - q^{-2})(1 - q^{-4}) \sum_{s=0}^{m-1} q^{2s}E_{(2m-s)\delta-\alpha_i}E_{(s+1)\delta-\alpha_i}.
\]

(ii) For \( k > 0, i, j \in I \) and \( a_{ij} = -1 \), we have
\[
i r(E_{k\delta-\alpha_j}) = 0, \quad i r(\tilde{\psi}_{k,j}) = 0,
\]
\[
i r(E_{k\delta,j}) = 0, \quad r_i(k)E_{k\delta,j} = -q^{-1}(1 - q^{-2})E_{k\delta-\alpha_i}.
\]

(iii) For \( k > 0, i, j \in I \) and \( a_{ij} = 0 \), we have
\[
r_i(E_{k\delta-\alpha_i}) = 0, \quad r_i(E_{k\delta,j}) = 0.
\]

**Proof.** The first equality in (i) follows from [31, Lemma 3.4]. The second is now easily deduced from the definition of \( r_i \) and the relation
\[
E_{\delta-\alpha_i}E_{\alpha_i} - q^{-2}E_{\alpha_i}E_{\delta-\alpha_i} = E_{\delta,i}.
\]
The third and fourth equalities are proved by induction using the definition of \( r_i \), the relation
\[
[E_{\delta,i}, E_{k\delta-\alpha_j}] = -[2]E_{(k+1)\delta-\alpha_j},
\]
and the following consequence of the relation in Proposition 1.2 between the \( E_{k\delta-\alpha_i} \) for \( k > 0 \):
\[
E_{\delta-\alpha_i}E_{2m\delta-\alpha_i} - q^{2}E_{2m\delta-\alpha_i}E_{\delta-\alpha_i} = (q^4 - 1) \sum_{s=1}^{m-1} q^{2s}E_{(2m-s)\delta-\alpha_i}E_{(s+1)\delta-\alpha_i},
\]
\[
E_{\delta-\alpha_i}E_{(2m+1)\delta-\alpha_i} - q^{2}E_{(2m+1)\delta-\alpha_i}E_{\delta-\alpha_i}
\]
\[
= (q^4 - 1) \sum_{s=0}^{m-1} q^{2s}E_{(2m+1-s)\delta-\alpha_i}E_{(s+1)\delta-\alpha_i} + q^{2m-3}E_{(m+1)\delta-\alpha_i}.
\]

We omit the details.

The proof of the first two equalities in (ii) is similar, but instead using that \( i r(E_{\delta-\alpha_j}) = 0 \) if \( i \neq j \) ([31, Lemma 3.5]). To prove the third equality we proceed as follows. Assume by induction that \( i r(E_{s\delta,j}) = 0 \) for all \( s < k \). It is easy to deduce from Lemma 1.5 and the functional equation in Theorem 1 relating the \( \psi_{m,j} \) to the \( h_{m,j} \) that \( E_{k\delta,j} \) is in the \( \mathcal{A} \)-subalgebra generated by \( \tilde{\psi}_{k,j} \) and the \( E_{s\delta,j} \) for \( s < k \).
The result is now clear. To prove the fourth equality, recall that for \( x \in \mathbf{U}^+ \) we have, by Proposition 3.6,

\[
[x, F_{\alpha_i}] = \frac{r_i(x)K_i - K_i^{-1}r(x)}{q - q^{-1}}.
\]

The result now follows by taking \( x = \frac{k}{[k]} E_{k\delta,j} \) and using the defining relation

\[
\frac{k}{[k]} E_{k\delta,j}, F_{\alpha_i} = -K_i E_{k\delta-\alpha_i}
\]

in Theorem 1 and Proposition 1.2. The proof of (iii) is similar to that of (i), but instead using that \( r_i(E_{k\delta-\alpha_j}) = 0 \) if \( a_{ij} = 0 \) (B1 Lemma 3.5).

The following result is proved in Proposition 40.2.4.

**Proposition 3.3.** For \( c, c' \in N^\mathbb{R} \), we have

\[
(E_{c}, E_{c'}) = (E_{c_0}, E_{c'_0}) \prod_{s \in \mathbb{Z}} (E_{\alpha_{i_s}}^{(c_s)} E_{\alpha_{i_s}}^{(c'_s)}),
\]

where the \( i_s \) are as in Lemma 1.1(iii) and \( c_0 \) (resp. \( c'_0 \)) denotes the restriction of \( c \) (resp. \( c' \)) to \( \mathcal{R}_0 \).

**Lemma 3.1.** Let \( k > 0, i, j \in I \).

(i) We have

\[
\tilde{\psi}_{k,i}, \tilde{\psi}_{k,i} = \frac{q^{2k-2}(1 - q^{-4})}{(1 - q^{-2})^2}.
\]

(ii) If \( a_{ij} = -1 \), then

\[
\tilde{\psi}_{k,i}, \frac{k}{[k]} E_{k\delta,j} = \frac{q^{-1}}{(1 - q^{-2})^2}.
\]

(iii) If \( a_{ij} = 0 \), then \( \tilde{\psi}_{k,i}, \tilde{\psi}_{k,j} = 0 \).

**Proof.** Using Proposition 1.1, we see that

\[
(\tilde{\psi}_{k,i}, \tilde{\psi}_{k,i}) = (E_{k\delta-\alpha_i} E_{k\delta-\alpha_i}, E_{k\delta-\alpha_i} - q^{-2} E_{\alpha_i} E_{k\delta-\alpha_i})
\]

\[
= (1 - q^{-2})^{-1}(E_{k\delta-\alpha_i}, r_i(E_{k\delta-\alpha_i}))
\]

\[
- 2q^{-2}(1 - q^{-2})^{-1}(E_{k\delta-\alpha_i}, r_i(E_{k\delta-\alpha_i}))
\]

\[
+ q^{-4}(1 - q^{-2})^{-1}(E_{k\delta-\alpha_i}, E_{k\delta-\alpha_i})
\]

\[
= \frac{1 - q^{-4}}{(1 - q^{-2})^2} + \frac{q^2}{(1 - q^{-2})^2} r_i(E_{k\delta-\alpha_i}, r_i(E_{k\delta-\alpha_i})).
\]

The second and third equalities in this computation follow from Proposition 3.3, keeping in mind that \( E_{\alpha_i} E_{m\delta-\alpha_i} \in B \) for all \( m > 0 \), and using the explicit formula for \( r_i(E_{k\delta-\alpha_i}) \) given in Proposition 3.2. The computation can now be completed by using Proposition 3.4, Proposition 3.3, and the following identity in [4, Lemma 1.4.4]

\[
(E_{\alpha_i}^{(p)}, E_{\alpha_i}^{(p)}) = \prod_{s=1}^p (1 - q^{-2s})^{-1}.
\]
To prove (ii), notice that
\[
(\tilde{\psi}_{k,i}, \frac{k}{[k]} E^k_{\delta,j}) = (E^k_{\delta-\alpha_i}E^\alpha_i - q^{-2}E^\alpha_iE^k_{\delta-\alpha_i}, \frac{k}{[k]} E^k_{\delta,j})
\]
\[
= (E^k_{\delta-\alpha_i}E^\alpha_i, \frac{k}{[k]} E^k_{\delta,j}), \text{ since } r(E^k_{\delta,j}) = 0
\]
\[
= (1 - q^{-2})^{-1} (E^k_{\delta-\alpha_i}, q^{-1}(1 - q^{-2})E^k_{\delta-\alpha_i})
\]
\[
= \frac{q^{-1}}{(1 - q^{-2})},
\]
where the penultimate equality follows from Proposition 3.2(ii). The proof of (iii) is similar, using Proposition 3.2(iii).

We need some additional results about the behaviour of \(\Delta\) and \(r\) on the Heisenberg generators.

Consider the polynomial ring \(\mathbb{Q}[x_k, k > 0]\). This has a natural Hopf algebra structure with comultiplication obtained by extending \(x_k \rightarrow x_k\otimes 1 + 1 \otimes x_k\). Defining elements \(\lambda_k\) for \(k \geq 0\) by \(\lambda_0 = 1\) and \(\lambda_k = \frac{1}{k} \sum_{s=1}^{k} x_s \lambda_{k-s}\) for \(k > 0\), and setting \(\Lambda = \sum_{k=0}^{\infty} \lambda_k x_k\), it is a result of [G, appendix] that the comultiplication takes \(\Lambda\) to \(\Lambda \otimes \Lambda\).

For any algebra \(A\), let \(A_+\) denote the augmentation algebra.

**Proposition 3.4.** Let \(k > 0, i \in I\). We have:

(i) \(r(E^k_{\delta,i}) = E^k_{\delta,i} \otimes 1 + 1 \otimes E^k_{\delta,i} + \text{terms in } U^+(<)U^+(0) \otimes U^+(0)U^+(>)_+;\)

(ii) \(r(\tilde{\psi}_{k,i}) = \tilde{\psi}_{k,i} \otimes 1 + 1 \otimes \tilde{\psi}_{k,i} + (q - q^{-1}) \sum_{s=1}^{k-1} \tilde{\psi}_{s,i} \otimes \tilde{\psi}_{k-s,i} + \text{terms in } U^+(<)U^+(0) \otimes U^+(0)U^+(>)_+;\)

(iii) \(r(\tilde{P}_{k,i}) = \sum_{s=0}^{k} \tilde{P}_{s,i} \otimes \tilde{P}_{k-s,i} + \text{terms in } U^+(<)U^+(0) \otimes U^+(0)U^+(>)_+.
\)

**Proof.** The following formula for the coproduct of the \(E^k_{\delta,i}\) is proved in [Da, Proposition 7.1] and can be derived from the explicit coproduct formulas for the loop-like generators ([B1], Proposition 5.3):

\[
\Delta(E^k_{\delta,i}) = E^k_{\delta,i} \otimes 1 + K_{k\delta} \otimes E^k_{\delta,i} + \text{terms in } U^0U^+(<)U^+(0) \otimes U^+(>)_+.
\]

Similar formulas can also be found in [V]. Part (i) now follows from [L4, 3.1.5]. Part (ii) can be deduced from (i) (again a proof can be found in [V]).

To prove (iii), recall from Section 1 that
\[
\tilde{P}_{k,i} = \frac{1}{k} \sum_{s=1}^{k} \frac{s}{[s]} E^s_{\delta,i} \tilde{P}_{k-s,i}.
\]

The proof of (iii) is now easily deduced from Lemma 2.3 and (1.8) by replacing \(x_k\) by \(k/[k] E^k_{\delta,i}\) and using the fact that \(U^+(>)_+U^+(0) \subset U^+(0)U^+(>)_+\).
Corollary 3.2. (i) If \( x \in \Lambda_i \), then
\[
 r(x) = \Delta_i(x) + \text{terms in } U^+ (<) U^+ (0) \otimes U^+ (0) U^+ (>)_+ \\
= x \otimes 1 + 1 \otimes x + \text{terms in } (\Lambda_i)_+ \otimes (\Lambda_i)_+ \\
+ \text{terms in } U^+ (<) U^+ (0) \otimes U^+ (0) U^+ (>)_+.
\]

(ii) If \( x \in \Lambda_i, y, z \in \Lambda U^+ (0) \), then
\[
 (x, yz) = (\Delta_i(x), y \otimes z).
\]

Proof. It suffices to prove (i) when \( x \) is a product of the \( \tilde{P}_{k,i} \) for \( k > 0, i \in I \). If \( x \) has length one this is Proposition 3.4(iii). The proof can be completed easily by an induction on the length of \( x \), keeping in mind that for \( x, y \in \Lambda U^+ (0) \) we have \( |x| \cdot |y| = 0 \). For part (ii), we have, by Proposition 3.3, that \( (U^+ (0) U^+ (>)_+, U^+ (0)) = 0 \).

The result now follows from part (i), since \( (x, yz) = (r(x), y \otimes z) \) for all \( x, y, z \in \Lambda U^+ \).

Lemma 3.2. For \( k > 0 \), we have

(i)
\[
 (\tilde{\psi}_{k,i}, \tilde{P}_{k,i}) = q^{k-1} \frac{(1 - q^{-2k-2})}{(1 - q^{-2})^2};
\]

(ii) if \( i \neq j \),
\[
 (\tilde{\psi}_{k,i}, \tilde{P}_{k,j}) = 0 \mod (q^{-1} A).
\]

Proof. The proof of (i) is by induction on \( k \). The case \( k = 1 \) is contained in Lemma 3.1. Assume the result for all smaller values of \( k \). Using the definition of \( \tilde{P}_{k,i} \) we get

\[
 (\tilde{\psi}_{k,i}, \tilde{P}_{k,i}) = (r(\tilde{\psi}_{k,i}), \frac{q - q^{-1}}{|k|} \sum_{s=1}^{k-1} q^{s-k} \tilde{\psi}_{s,i} \otimes \tilde{P}_{k-s,i}) + (\tilde{\psi}_{k,i}, \tilde{\psi}_{k,i})
\]

(3.2)
\[
 = \frac{q - q^{-1}}{|k|} \sum_{s=1}^{k-1} (q^{s-k} \tilde{\psi}_{s,i}, \tilde{\psi}_{k-s,i}) \tilde{P}_{k-s,i} + (\tilde{\psi}_{k,i}, \tilde{\psi}_{k,i}),
\]

(3.3)

where we use Proposition 3.3 to get the last equality. A direct computation using Lemma 3.1 and the induction hypothesis gives (i).

The proof of (ii) is identical except that we use the relation
\[
 \tilde{P}_{k,i} = \frac{1}{k} \sum_{s=1}^{k} \delta_{s,i} \tilde{P}_{k-s,i}.
\]

We omit the details.

We can now complete the proof of Proposition 3.1 as follows. If \( k \neq k' \) the result is obvious since the elements have different homogeneity. If \( k = k' = 1 \) the result follows from Lemma 3.1. Assume the result for all smaller values of \( k = k' \). By
using the definition of $\tilde{P}_{k,i}$, the properties of the inner product and Proposition 3.4, we get

\[
\langle \tilde{P}_{k,i}, \tilde{P}_{k,j} \rangle = \frac{1}{|k|} \left( \sum_{s=1}^{k} q^{s-k} \psi_{s,i} \otimes \tilde{P}_{k-s,i} + \sum_{s=0}^{k} \tilde{P}_{s,j} \otimes \tilde{P}_{k-s,j} \right)
\]

\[
= \frac{1}{q^{k-1}(1 + q^{-2} + \ldots + q^{-2k+2})} \sum_{s=1}^{k} q^{s-k}(\psi_{s,i}, \tilde{P}_{s,j})(\tilde{P}_{k-s,i}, \tilde{P}_{k-s,j})
\]

\[
= \frac{1}{(1 + q^{-2} + \ldots + q^{-2k+2})} \sum_{r=1}^{k} q^{s-2k+1}(\psi_{s,i}, \tilde{P}_{s,j})(\tilde{P}_{k-s,i}, \tilde{P}_{k-s,j}).
\]

Since $s - 2k + 1 < 0$, it is obvious by Lemma 3.2(ii) that the right-hand side is in $q^{-1}A$ if $i \neq j$. If $i = j$, then by Lemma 3.2(i) we see that the right-hand side is

\[
\frac{1 - q^{-2k-2}}{(1 - q^{-2})^2(1 + q^{-2} + \ldots + q^{-2k+2})} \sum_{s=1}^{k} q^{s-k}(\tilde{P}_{k-s,i}, \tilde{P}_{k-s,i}) = 1 \mod (q^{-1}A),
\]

and the proof is complete. 

We turn now to the proof of Corollary 3.1. To prove (i), we set $y = y(j)y'$ and write $\Delta_i(x) = \sum_s z_s \otimes z'_s$, $z_s, z'_s \in \mathcal{A}_i$. Using Corollary 4.2, we get

\[
(x, y) = (\Delta_i(x), y(j) \otimes y') = \sum_s (z_s, y(j))(z'_s, y').
\]

We are thus reduced to proving that

\[
(x, y) = 0 \mod (q^{-1}A) \text{ if } x \in \mathcal{A}_i, \ y \in \mathcal{A}_j, \ i \neq j.
\]

It is obviously enough to prove this when $x$ and $y$ are products of the $\tilde{P}_{k,i}$ and $\tilde{P}_{k,j}$, respectively. Assume that $x = \tilde{P}_{k,i}$. If $y$ also has length one, this is just the statement of Proposition 3.1. Assume the result for all monomials $y$ of length less than $s$. Write $y = \tilde{P}_{m,j}y_1$, where $y_1 \in \mathcal{A}_j$ has length less than $s$. We have, by Proposition 3.4, that

\[
\langle \tilde{P}_{k,i}, \tilde{P}_{m,j}y_1 \rangle = (r(\tilde{P}_{k,i}), \tilde{P}_{m,j} \otimes y_1) = \left( \sum_{l=0}^{k} \tilde{P}_{k-l,i} \otimes \tilde{P}_{l,j} \right) \otimes y_1
\]

The right-hand side is zero mod $(q^{-1}A)$ by the induction hypothesis. Assume now that we know the result for all monomials $y$ if $x$ has length less than $s$. Write $x = x_1 \tilde{P}_{k,i}$, where $x_1 \in \mathcal{A}_i$ has length $s-1$. Proceeding as before and using Corollary 3.2, we see that

\[
(x, y) = (x_1 \otimes \tilde{P}_{k,i}, r(y)) = (x_1 \otimes \tilde{P}_{k,i}, \Delta_j(y)).
\]

The right-hand side is again zero mod $(q^{-1}A)$ by induction and the proof of (i) is complete.

To prove (ii), assume without loss of generality that $x(1) \neq 1$ and write $x = x(1)x'$. Choose $j$ such that $y(j) \neq 1$ and $y(m) = 1$ if $m < j$. If $j > 1$ then, by Corollary 3.2, we see that

\[
r(y) \in \prod_{m \neq 1} \mathcal{A}_m \otimes \mathcal{A}_m + \text{ terms in } U^+\langle \rangle \otimes U^+(0) \otimes U^+(0)U^+\langle \rangle.
\]
Hence, we get
\[(x, y) = (x(1) \otimes x', r(y)) = 0 \mod (q^{-1}A)\]
by part (i) of Corollary 3.1. If \(j = 1\), write \(y = y(1)y'\). If \(y' = 1\), then we are again done by part (i) of Corollary 3.1. So assume that \(y'(m) \neq 1\) for some \(m \neq 1\). Then, using Corollary 3.2, we can write
\[r(y) = y(1) \otimes y' + 1 \otimes y(1)y' + \sum z_s \otimes z'_s + \text{terms in } U^+(-)U^+(0) \otimes U^+(0)U^+(>),\]
where \(z_s(m) \neq 1\). Since we have already proved that \((x(1), z_s(m)) = 0 \mod (q^{-1}A)\) if \(m \neq 1\), it follows that
\[(x, y) = (x(1), y(1))(x', y') \mod (q^{-1}A)\]
The proof can now be completed by repeating the argument for \(x'\) and \(y'\).

4. Characterization of the canonical basis.

The results of Section 3 allow us to use the theory symmetric functions \(M\) to modify the imaginary root vectors so that we have an orthonormal basis mod \((q^{-1}A)\) for \(_A U^+(0)\). Using Theorem 2 and Proposition 3.3, we then have an orthonormal basis mod \((q^{-1}A)\) for \(_A U^+\). We use this basis to construct our crystal basis and the canonical basis.

Following \(M\) page 41, we define, for \(i \in I\) and a given partition \(\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)\), the corresponding Schur function \(s_{\lambda,i} \in _A A_i\) by
\[s_{\lambda,i} = \det(\tilde{P}_{\lambda_k - k,m,i})_{1 \leq k,m \leq \ell},\]
where \(t \geq l(\lambda)\). Next, given a function \(c_0 \in \mathbb{N}^{R_0}\), consider the \(n\)-tuple of partitions \((\lambda^{(1)}, \ldots, \lambda^{(n)})\) whose \(i\)-th component is the partition with \(c_0(k\delta, i)\) parts equal to \(k\), and define \(S_{c_0} = s_{\lambda^{(1)},1}s_{\lambda^{(2)},2} \cdots s_{\lambda^{(n)},n}\).

**Definition 4.1.** Let \(c \in \mathbb{N}^R\). Denote by \(c_\succ\) (resp. \(c_\prec\)) the restriction of \(c\) to \(R_\succ\) (resp. \(R_\prec\)). Define
\[(4.1)\quad B_c = (E_{c_\succ}) \cdot S_{c_0} \cdot (E_{c_\prec}).\]

**Proposition 4.1.** (i) The set \(\{B_c \mid c \in \mathbb{N}^R\}\) is an \(_A\)-basis of \(_A U^+\).
(ii) Let \(c, c' \in \mathbb{N}^R\). Then
\[(4.2)\quad (B_c, B_{c'}) = \delta_{c,c'} \mod (q^{-1}A).\]

**Proof.** Let \(A_i\) be the \(Z\)-subring of \(_A A_i\) generated by the \(\tilde{P}_{k,i}\). The first statement is an immediate consequence of Theorem 2 and the fact \(M\) Section 3.3] that the \(s_{\lambda,i}\) form a \(Z\)-basis of \(A_i\). For the second, it suffices by Proposition 3.3 to prove that
\[(4.3)\quad (S_{c_0}, S_{c_0'}) = \delta_{c_0, c_0'} \mod (q^{-1}A).\]
By Corollary 3.1, we see that
\[(S_{c_0}, S_{c_0'}) = (s_{\lambda^{(1)},1}, s_{\mu^{(1)},1})(s_{\lambda^{(2)},2}, s_{\mu^{(2)},2}) \cdots (s_{\lambda^{(n)},n}, s_{\mu^{(n)},n}) \mod (q^{-1}A).\]
Now we use \(M\) Page 92, Exercise 25(c)] and \(M\) Chapter 1, Equation 4.8] to conclude that
\[(4.4)\quad (s_{\lambda^{(i)},1}, s_{\mu^{(i)},1}) = \delta_{\lambda^{(i)},\mu^{(i)}} \mod (q^{-1}A).\]
Part (ii) follows.
Let $\mathbf{B}$ be the canonical basis of $\mathbf{U}^+$, and let $\mathcal{L}$ be the $\mathbb{Z}[q^{-1}]$-lattice spanned by $\mathbf{B}$. By Proposition 5.1.3 in [K], we have the following alternative characterization of $\mathcal{L}$:

\[ \mathcal{L} = \{ x \in \mathbf{A} \mathbf{U}^+ | (x,x) \in \mathbf{A} \}. \] (4.5)

**Proposition 4.2.** The set

\[ \{ \mathbf{B}^c | c \in \mathbb{N}^R \} \] (4.6)

is a $\mathbb{Z}[q^{-1}]$-basis of $\mathcal{L}$.

**Proof.** This follows directly from Propositions 3.3 and 4.1, and [L4, Lemma 16.2.5(a)]. \qed

**Theorem 3.** For every $c \in \mathbb{N}^R$, there exists $b \in \mathbf{B}$ such that

\[ b = \mathbf{B}^c \mod (q^{-1}\mathcal{L}), \] (4.7)

and hence $\{ \mathbf{B}^c | c \in \mathbb{N}^R \}$ is a crystal basis of $\mathbf{A} \mathbf{U}^+$.

We begin by observing that this theorem is certainly true up to sign:

**Proposition 4.3.** For every $c \in \mathbb{N}^R$, there exists $b \in \mathbf{B}$ such that

\[ b = \pm \mathbf{B}^c \mod (q^{-1}\mathcal{L}). \] (4.8)

**Proof.** This follows from [L4, Lemma 16.2.5(f)] and Proposition 4.1. \qed

Let $\pi^0$ be the projection of $\mathbf{A} \mathbf{U}^+$ onto $\mathbf{A} \mathbf{U}^+(0)$ corresponding to the decomposition

\[ \mathbf{A} \mathbf{U}^+ = \mathbf{A} \mathbf{U}^+(0) \oplus (\mathbf{A} \mathbf{U}^+)(\mathbf{A} \mathbf{U}^+(0) \mathbf{A} \mathbf{U}^+(<) + \mathbf{A} \mathbf{U}^+(>) + \mathbf{A} \mathbf{U}^+(0) \mathbf{A} \mathbf{U}^+(>) ), \]

which follows from Theorem 2. Note that this theorem also implies that the restriction of $\pi^0$ to the $\mathbf{A}$-subalgebra of $\mathbf{A} \mathbf{U}^+$ consisting of elements of homogeneity in $\mathbb{N} \delta$ is a homomorphism. The following additional properties of $\pi^0$ follow from Proposition 4.3:

**Corollary 4.1.** We have:

(i) $\mathcal{L}$ is the $\mathbb{Z}[q^{-1}]$-lattice spanned by $\{ \mathbf{B}^c \}_{c \in \mathbb{N}^R}$;

(ii) $\pi^0(\mathcal{L}) \subseteq \mathcal{L}$;

(iii) the non-zero elements of $\pi^0(\mathbf{B}) \subset \mathcal{L}$ are linearly independent (mod $q^{-1}\mathcal{L}$);

(iv) $\mathbf{A} \mathbf{U}^+(0) \cap \mathcal{L}$ is closed under products.

**Proof.** Part (i) is immediate from Propositions 4.2 and 4.3. Part (ii) follows from part (i) and the fact that $\pi^0$ takes each $\mathbf{B}^c$ either to itself or zero. Part (iii) follows from Proposition 4.3 and the argument in part (ii). Finally, by Proposition 4.2 and the definition of the $s_{\lambda,i}$, it follows that $\mathcal{L}$ is the $\mathbb{Z}[q^{-1}]$-lattice spanned by $\{ E_c \}_{c \in \mathbb{N}^R}$ (see Section 3). This implies that $\mathbf{A} \mathbf{U}^+(0) \cap \mathcal{L}$ is spanned over $\mathbb{Z}[q^{-1}]$ by the monomials in the $\tilde{P}_{k,i}$. The statement in (iv) is now clear. \qed

For the proof of Theorem 3, we note that by [L3, Proposition 8.3], it suffices to prove the theorem for $c \in \mathbb{N} \delta$. This is done in the following lemmas.
Lemma 4.1. Let $i \in I$. Then, for every $k > 0$, 
\[ \tilde{P}_{k,i} = \beta_{k,i} + b_{k,i}, \]
where $b_{k,i} \in q^{-1}\mathcal{L}$ and $\beta_{k,i} \in \mathcal{B}$.

Proof. First we check that
\begin{equation}
E_{\delta-\alpha_i}^{(k)} F_{\alpha_i}^{(k)} = \tilde{P}_{k,i} \mod (q^{-1}\mathcal{L}).
\end{equation}
We know by Proposition 2.2 that
\begin{equation}
E_{\delta-\alpha_i}^{(k)} F_{\alpha_i}^{(k)} = \tilde{P}_{k,i} + x,
\end{equation}
where $x \in \mathcal{A}U^+(>)\mathcal{A}U^+(0)\mathcal{A}U^<(>)$. By Proposition 4.1, this means that
\[ (E_{\delta-\alpha_i}^{(k)} F_{\alpha_i}^{(k)}, E_{\delta-\alpha_i}^{(k)} F_{\alpha_i}^{(k)}) = 1 \mod (q^{-1}\mathcal{A}), \]
and considering (4.10), this implies that
\[ (\tilde{P}_{k,i}, \tilde{P}_{k,i}) + (x, x) + 2(x, \tilde{P}_{k,i}) = 1 \mod (q^{-1}\mathcal{A}). \]
By Proposition 4.1, this means that
\[ (x, x) + 2(x, \tilde{P}_{k,i}) = 0 \mod (q^{-1}\mathcal{A}). \]
Proposition 4.3 implies that $(x, \tilde{P}_{k,i}) = 0$, and so finally $(x, x) = 0 \mod (q^{-1}\mathcal{A})$. By Lemma 16.2.5(f), it follows that
\begin{equation}
x \in q^{-1}\mathcal{L}.
\end{equation}
This proves (4.9). Now, by Proposition 8.2, we have $E_{\delta-\alpha_i}^{(k)} = b \mod (q^{-1}\mathcal{L})$. Consider the Kashiwara operators $\phi_i : \mathcal{A}U^+ \to \mathcal{A}U^+$ introduced in [K] defined by
\[ \phi_i(E_{\alpha_i}^{(r)} x) = E_{\alpha_i}^{(r+1)} x, \]
for all $r \geq 0$, $x \in \mathcal{A}U^+[i]$ (see Proposition [L]). Since $\sigma$ and $\phi_i$ map $\mathcal{L}$ into itself, and $\sigma(E_{\delta-\alpha_i}) = 0$ by Lemma 3.4], we have
\begin{equation}
E_{\delta-\alpha_i}^{(k)} F_{\alpha_i}^{(k)} = \sigma \tilde{\phi}_i \sigma(E_{\delta-\alpha_i}^{(k)}) = \sigma \tilde{\phi}_i \sigma(b) \mod (q^{-1}\mathcal{L}),
\end{equation}
where $\sigma \tilde{\phi}_i \sigma(b) = b' \mod (q^{-1}\mathcal{L})$ and $b' \in \mathcal{B}$. The lemma now follows from (4.9). \hfill \Box

Lemma 4.2. For $i \in I$, we have
\begin{equation}
s_{\lambda,i} = \beta_{\lambda,i} + b_{\lambda,i},
\end{equation}
where $b_{\lambda,i} \in q^{-1}\mathcal{L}$ and $\beta_{\lambda,i} \in \mathcal{B}$.

Proof. We prove the lemma by induction on the length $\ell(\lambda)$ of $\lambda$. The case $\ell(\lambda) = 1$ is contained in Lemma 4.1. Assume that the statement holds for all $\lambda$ such that $\ell(\lambda) < L$. By the Pieri formulas [M, Chapter 1], we have
\begin{equation}
s_{(k),i} s_{\mu,i} = \sum_{\lambda \supset \mu} s_{\lambda,i},
\end{equation}
where the summation is over those $\lambda$ such that $\lambda - \mu$ is a horizontal $k$-strip. By the inductive assumption, we have
\begin{equation}
s_{\mu,i} = \beta_{\mu,i} + b_{\mu,i}, \quad s_{(k),i} = \beta_{(k),i} + b_{(k),i},
\end{equation}

where $\beta_{\mu,i}, \beta_{(k),i} \in B$ and $b_{\mu,i}, b_{(k),i} \in q^{-1}L$. Multiplying these expressions together and using Corollary 4.1, we obtain

$$s_{(k),i}s_{\mu,i} = \beta_{(k),i}\beta_{\mu,i} + y,$$

where $\pi^0(y) \in q^{-1}L$. By the positivity result for the canonical basis [2, Theorem 14.4.13], there exist $n_r \in \mathbb{N}[q, q^{-1}], b_r \in B, r = 1, \ldots, d$, such that

$$\beta_{(k),i}\beta_{\mu,i} = \sum_{r=1}^{d} n_r b_r.$$

It follows that

$$\sum_{\lambda \supset \mu} s_{\lambda,i} = \sum_{r=1}^{d} n_r b_r + y,$$

where $n_r \in \mathbb{N}[q, q^{-1}], b_r \in B$. Hence,

$$\sum_{\lambda \supset \mu} s_{\lambda,i} = \sum_{r=1}^{d} n_r \pi^0(b_r) + \pi^0(y).$$

Since (4.7) holds up to sign, we have $s_{\lambda,i} = \pm \beta_{\lambda,i} + b_{\lambda,i}$ with $b_{\lambda,i} \in q^{-1}L$, hence

$$\sum_{\lambda \supset \mu} s_{\lambda,i} = \sum_{\lambda \supset \mu} \pi^0(\pm \beta_{\lambda,i}) \mod (q^{-1}L)$$

and so

$$\sum_{\lambda \supset \mu} \pm \pi^0(\beta_{\lambda,i}) = \sum_{r} n_r \pi^0(b_r) \mod (q^{-1}L).$$

On the right-hand side, we sum only over those $r$ for which $\pi^0(b_r) \neq 0$. We claim that, for such $r, n_r \in \mathbb{N}[q^{-1}]$. For otherwise, we may assume that $q^N$ is the highest power of $q$ appearing in $n_{r_1}, \ldots, n_{r_s}$, where $N, s > 0$, and that the highest power of $q$ appearing in the other $n_r$ in (4.20) is $< N$. Multiplying (4.19) on both sides by $q^{-N}$ gives

$$\sum_{i=1}^{s} (q^{-N}n_{r_i})_{\infty} \pi^0(b_{r_i}) = 0 \mod (q^{-1}L),$$

where $(\zeta)_{\infty}$ denotes the constant coefficient of an element $\zeta \in \mathbb{N}[q^{-1}]$. This contradicts the linear independence of the $\pi^0(b_{r_i})$. Thus,

$$\sum_{\lambda \supset \mu} \pm \pi^0(\beta_{\lambda,i}) = \sum_{r} (n_r)_{\infty} \pi^0(b_r).$$

It follows from linear independence again that, for each $r$ such that $\pi^0(b_r) \neq 0$, there exists a $\lambda \supset \mu$ such that $\beta_{\lambda,i} = b_r$ and $(n_r)_{\infty} = \pm 1$. Hence, all the signs must be $\pm$.

Since every $\lambda$ of length $L$ appears in an equation of the form (4.14), this implies the lemma.

**Lemma 4.3.** Given an “imaginary” PBW basis monomial $S_{e_0}$, we have

$$S_{e_0} = \beta_{e_0} + b_{e_0},$$

where $b_{e_0} \in q^{-1}L$ and $\beta_{e_0} \in B$. 

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Proof. Write
\[ S_{c_0} = \prod_i s_{\lambda(i),i} = \pm \beta_{c_0} + y, \]
where \( \beta_{c_0} \in B \) and \( y \in q^{-1}L \). Also write each
\[ s_{\lambda(i),i} = \beta_{\lambda(i)} + b_{\lambda(i)}, \]
where by Lemma \([4.2]\), \( \beta_{\lambda(i)} \in B \) and \( b_{\lambda(i)} \in q^{-1}L \). By the positivity result for the canonical basis \([4.4.13]\) and (i)–(iv) above, we have
\[ \prod_i s_{\lambda(i),i} = \sum_{r=1}^k n_r \beta_r + y, \]
where \( n_r \in \mathbb{N}[g,q^{-1}] \) and \( \pi^0(y) \in q^{-1}L \). Arguing exactly as in Lemma \([4.2]\), the statement follows.

To obtain the canonical basis, let \( \pi : L \to L/q^{-1}L \) be the natural projection. Then, \( \pi \) takes the basis \( \{ B_c \mid c \in \mathbb{N}^R \} \) to a \( \mathbb{Z} \)-basis of \( L/q^{-1}L \) and it is known \([3]\) that \( \pi \) restricts to an isomorphism of \( \mathbb{Z} \)-modules \( \pi' : L \cap \overline{L} \cong L/q^{-1}L \). We have the following immediate corollary of Theorem 3.

**Theorem 4.**
\[ B = \{ \pi'^{-1}\pi(B_c) \mid c \in \mathbb{N}^R \}. \]

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