A $\mathbb{Z}_2$ Structure in the Configuration Space of Yang-Mills Theories

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Abstract

We argue for the presence of a $\mathbb{Z}_2$ topological structure in the space of static gauge-Higgs field configurations of $SU(2n)$ and $SO(2n)$ Yang-Mills theories. We rigorously prove the existence of a $\mathbb{Z}_2$ homotopy group of mappings from the 2-dim. projective sphere $\mathbb{R}P^2$ into $SU(2n)/\mathbb{Z}_2$ and $SO(2n)/\mathbb{Z}_2$ Lie groups respectively. Consequently the symmetric phase of these theories admits infinite surfaces of odd-parity static and unstable gauge field configurations which divide into two disconnected sectors with integer Chern-Simons numbers $n$ and $n + 1/2$ respectively. Such a $\mathbb{Z}_2$ structure persists in the Higgs phase of the above theories and accounts for the existence of $CS = 1/2$ odd-parity saddle point solutions to the field equations which correspond to spontaneous symmetry breaking mass scales.
1 Introduction

The electroweak interactions of quarks and leptons are well described by a gauge theory of an $SU(2) \times U_Y(1)$ of isospin ($I$) and hypercharge ($Y$) spontaneously broken to $U_{em}(1)$ of electromagnetism. It is a theory of the Yang-Mills $SU(N)$ type. As a consequence the vacuum has a nontrivial periodic structure \[1\]. The distinct ground states are labeled by integer values of the Chern-Simons number (CS) of the three dimensional (spatial component) $SU(N)$ gauge field $W_i$, with the index $i = 1, 2, 3$ corresponding to spatial directions in the four dimensional Minkowski space. The Chern-Simons number is given by the following functional

$$CS[W] = \frac{1}{16\pi^2} \int_{D^3} \text{Tr}(WdW - \frac{2i}{3}W^3)$$

where $D^3$ is a 3-dimensional disk. This structure follows from the topological property that $\pi_3(SU(N)) = \mathbb{Z}$ \[1\].

The height of the potential barrier between the adjacent vacua is arbitrary for an unbroken $SU(N)$ gauge theory. Quantum tunneling is induced by instantons, finite action solutions to the 4-d Euclidean equations of motion. Large (small) size instantons transverse low (high) barriers respectively. In the case of a spontaneously broken $SU(N)$ theory the height of the barrier is fixed and is roughly proportional to the mass scale of the theory $M_w/\alpha$ ($\alpha =$ gauge coupling constant, and $M_w$ is a mass of the vector bosons). Quantum tunneling through instantons is exponentially suppressed by their finite action \[4\]. More specifically for a chiral theory such as the electroweak Weinberg-Salam model Baryon number is violated through the chiral anomaly. A change in Chern-Simons number through transitions between different vacua results in a change of the net Baryon number.

While at zero temperature B-violating transitions are exponentially suppressed at sufficiently high temperature $T < M_w/\alpha$ it has been argued \[3\] that they become quite rapid as they are dominated by a static, finite energy unstable solution to the electroweak equations of motion, the sphaleron \[4\]. This is because at precisely this temperature range $M_w < T < M_w/\alpha$ the sphaleron configuration is a saddle point on the energy surface, located at the highest point of a continuous set of static configurations that interpolates between the topologically distinct vacua with $CS = n, n + 1 (n \in \mathbb{Z})$ and has $CS = n + 1/2$. In thermal equilibrium and at temperatures $T \leq E_{sp} = M_w/\alpha$ the probability of forming a coherent sphaleron in the hot plasma is given by the Boltzmann weight of the classical sphaleron energy $P \sim \exp(-E_{sp}/T)$ \[5\].

Though the sphaleron configuration in the standard electroweak theory is homotopically trivial because of $\pi_2(SU(2)) = 0$, the existence of saddle point solutions in spontaneously broken gauge theories is a consequence of a nontrivial topological structure in their configuration space. (Configuration space is hereby taken to be the space of all static, finite energy 3-dim. configurations of Yang-Mills-Higgs fields $W_i(\vec{x}), \phi(\vec{x})$.) This, in turn, is a reflection of their nontrivial periodic vacuum structure. It was in fact rigorously argued that saddle point solutions must
be a consequence of the existence of noncontractible loops in the bosonic sector of $SU(2)$ Yang-Mills theories such as in the classical Weinberg-Salam model. While the existence of such a noncontractible loop along with a sphaleron saddle point is a symptom of the nontrivial topology of the $SU(2)$ configuration space yet by themselves they tell us nothing more about what is its actual character. A crucial condition for the existence of saddle point solutions with $CS = 1/2$ is the presence of a mass scale, or equivalently of a mass gap in the subspace of configurations with $CS = n + 1/2$ which is introduced through the Higgs mechanism. This very condition is somewhat restrictive and not a necessary ingredient of the nontrivial structure we are after as we will argue shortly.

Indeed in a nonabelian pure $SU(2)$ gauge theory with no such mass scale ($M_w \rightarrow 0$), and in the absence of a saddle point sphaleron solution it is unclear which finite energy static configurations with $CS = 1/2$ characterize the nontrivial topological structure of its configuration space. The latter is of course expected to be present and reflect the existence of a periodic vacuum. In the context of the electroweak theory it is unknown which are the sphaleron-like configurations which lie deep in its high temperature symmetric phase and mediate unsuppressed baryon violating thermal transitions. Moreover the very origin of the B-violating single normalizable fermionic zero mode in the presence of the $CS = 1/2$ electroweak sphaleron has also been obscure. We have recently addressed both of these issues in the context of the $SU(2) \times U(1)$ Weinberg-Salam model. There we observed that the electroweak sphaleron written in an appropriate gauge is odd under parity reflection symmetry with an odd pure gauge behavior at infinity. This was sufficient for it to possess a $CS = 1/2$ and a single fermionic zero mode. Moreover we showed that the number of fermionic zero modes modulo 2 in the background of gauge and Higgs static configurations odd under a properly defined parity is a topological invariant which is determined by the parity properties of the pure gauge behavior at spatial infinity. Configurations with even pure gauge behavior possess a $CS = n$ ($n$ is an integer) and an even number of fermionic zero modes. The ones with odd pure gauge behavior at infinity have a $CS = n + 1/2$ and an odd number of fermion zero modes mediating as a consequence fermionic level crossing. We pointed out that the above classification reflects the existence of nontrivial discrete groups of maps $[\mathbb{R}P^2, SU(2)/\mathbb{Z}_2] = \mathbb{Z}_2$ as well as $[\mathbb{R}P^3, SU(2)/\mathbb{Z}_2] = \hat{\mathbb{Z}}$ ($\hat{\mathbb{Z}}$ is a double cover of $\mathbb{Z}$) in the configuration space of gauge fields (for a strict mathematical treatment of these statements see next section).

In our present study we will provide rigorous proofs for the existence of such nontrivial groups of mappings. We will do it for the most general case of $SU(N)$ and $SO(N)$ Lie groups. For the case of $N = even$, ($N = 2n$) we will establish the existence of a nontrivial two element ($\mathbb{Z}_2$) group of maps from the 2-dimensional projective sphere $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$ at spatial infinity into the $SU(2n)/\mathbb{Z}_2$ and $SO(2n)/\mathbb{Z}_2$ groups respectively. More schematically we will demonstrate that $[\mathbb{R}P^2, SU(2n)/\mathbb{Z}_2] = [\mathbb{R}P^3, SO(2n)/\mathbb{Z}_2] = \mathbb{Z}_2$. Moreover we will argue that the presence of a $U(1)$ factor in a product group of the type $SU(2n) \times U(1)$ preserves such a $\mathbb{Z}_2$ structure. As our arguments are based on abstract topological properties
of Lie groups they are completely general. More specifically for the case of the electroweak theory they demonstrate rigorously the existence of a \( \mathbf{Z}_2 \) structure in its configuration space of static, finite energy gauge fields alone which are odd under a properly defined parity. Such a structure provides us with a novel top-bottom argument for the existence of sphalerons and their nontrivial deformations such as the W and Z vortex loops. As such it does not depend on the existence of any mass scale in the theory it characterizes its complete configuration space. Our arguments identify \( SU(2n) \), \( SO(2n) \) and \( E_7 \) as the gauge groups which admit an similar \( \mathbf{Z}_2 \) homotopy group of mappings.

The paper is organized as follows: in section 2 we summarize our physical arguments for the existence of a \( \mathbf{Z}_2 \) structure in a pure \( SU(2) \) Yang-Mills theory. They mean to physically motivate our subsequent rigorous mathematical treatment. In section 3 we prove that for \( SU(2n) \) groups \([\mathbb{R}P^2, SU(2n)/\mathbf{Z}_2] = \mathbf{Z}_2 \) and that \([\mathbb{R}P^3, SU(2n)/\mathbf{Z}_2] = \hat{\mathbf{Z}} \), where \( \hat{\mathbf{Z}} \) in the r.h.s. of the latter acts as a double covering of \( \mathbf{Z} \) on the space of odd-parity gauge field configurations. Such a properly understood double covered \( \mathbf{Z} \) group can be interpreted from a physical point of view as \( \mathbf{Z} \times \mathbf{Z}_2 \) which agrees with the statement of ref.\[7\]. In section 4 we show that the presence of a \( U(1) \) factor group does not affect the \( \mathbf{Z}_2 \) structure. We also show that \([\mathbb{R}P^2, SO(2n)/\mathbf{Z}_2] = \mathbf{Z}_2 \). We close with some final comments with regard to the physical implications of the presence of a \( \mathbf{Z}_2 \) topological structure in the configuration space of \( SU(2n) \) and \( SO(2n) \) gauge theories. In Appendix we illustrate a difference between \( SU(even) \) from \( SU(odd) \) gauge groups by proving the existence of a 3-element homotopy group \( \mathbf{Z}_3 \) of maps from \( S^3/\mathbf{Z}_3 \) to \( SU(3)/\mathbf{Z}_3 \) (more precisely \([S^3/\mathbf{Z}_3, SU(3)/\mathbf{Z}_3] = \mathbf{Z} \oplus \mathbf{Z}_3 \)). In contrast to the \( SU(2n) \) gauge theories it implies a \( \mathbf{Z}_3 \) "parity" classification on the space of 4-dimensional gauge field configurations of an \( SU(3) \) gauge theory.

## 2 \( \mathbf{Z}_2 \) structure of an \( SU(2) \) Configuration Space

We start with the observation that our familiar static sphaleron configuration has an odd-parity gauge field everywhere in space. By imposing the same property on all its possible deformations (they may not be solutions) we find two topologically distinct sectors of configurations that depend on the (even-odd) parity properties of their pure gauge behaviour at spatial infinity.

We hereby give a summary of our arguments. The Chern-Simons number (CS) for the \( SU(2) \) weak sphaleron is defined as the functional \( CS(W') \) which is given by

\[
CS(W') = CS(W) + S_{WZW}(U'),
\]

where \( W' \) is given by

\[
W'_k = U'W_k(U')^{-1} + i\partial_k U' (U')^{-1}, \quad k = 1, 2, 3.
\]
and $S_{WZW}$ is the Wess-Zumino-Witten functional

$$S_{WZW}(U') = \frac{1}{24\pi^2} \int_{D^3} \text{Tr}(dU' U'^{-1})^3. \quad (2.3)$$

Here $W$ is the sphaleron gauge field

$$W_k = f(r) \frac{\epsilon_{ijk} x^i \tau^j}{r^2} = -i f(r) \partial_k U_{\text{sph}} U_{\text{sph}}^{-1}, \quad (2.4)$$

where $k = 1, 2, 3$, $r = |x|$, and

$$U_{\text{sph}} = i \frac{\vec{\tau}, \vec{x}}{r}. \quad (2.5)$$

We take $U'$ to be an $SU(2)$ group element which is smooth everywhere and coincides with $U_{\text{sph}}$ at spatial infinity. It is easy to see that $CS(W) = 0$ due to the oddness of $W$ under parity reflections. Therefore the behavior of $U'$ at the boundary $S^2$ of a $D^3$ disk determines the value of $CS(W')$ and makes it equal to $1/2 + n$ ($n$ is an integer). For the particular choice of

$$U' = \exp \left( \frac{i\pi}{2} \frac{\vec{\tau}, \vec{x}}{\sqrt{x^2 + \rho^2}} \right) \quad (2.6)$$

it gives $S_{WZW}(U') = 1/2$. There are two ingredients necessary for the result $CS(W') = 1/2$. Firstly from eq.(2.1) it is necessary that $CS(W) = 0$ which the case as the field $W$ is odd under parity, i.e. $W(-x) = -W(x)$. Secondly it is the oddity of $U'$ under parity ($U'(-x) = -U'(x)$) that renders $S_{WZW}(U') = 1/2$ in eq.(2.3). Indeed for the choice of an even $U'$ under parity it is that $S_{WZW}(U') = 0$.

In fact we have argued [7] that for any $SU(2)$ gauge field with pure gauge behavior on a $S^2$ sphere at infinity

$$A_i = -i(\partial_i U) U^{-1}, \quad (2.7)$$

there exists a nontrivial $\mathbb{Z}_2$ homotopic classification in the space of 3-dim. odd-parity gauge fields. It should be noted that the statement of oddity of a gauge field configuration under parity reflections is certainly not a gauge invariant one. For our arguments to be meaningful though it suffices that there exists a gauge in which this is true. As such among the spatially odd gauge fields the ones with odd $U$ fields have $CS = 1/2$ and the ones with even $U$ fields possess $CS = 0$. It was pointed out that this classification is a consequence of the existence of a nontrivial homotopy group of maps from the projective sphere $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is a group of parity reflections with respect to some point in 3-dim. space) to the group $SO(3) = SU(2)/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is the center of $SU(2)$). In short $[\mathbb{R}P^2, SO(3)] = \mathbb{Z}_2$. Consequently there exists odd parity $SU(2)$ gauge fields which split into two topologically disconnected classes. It is not possible to get from one to the other continuously through odd-parity gauge field configurations.

In fact configurations with even $U$ fields are continuously connected to the vacua $A^n_i = i(\partial_i U_n) U_n^{-1}$ where the group element $U_n$ is given by the even-parity (at infinity) group elements [8]

$$U_n = \exp(\left\{i\pi \frac{\vec{\tau}, \vec{x}}{\sqrt{x^2 + \rho^2}}\right\} \quad (2.8)$$
with \( \rho \) being a constant parameter and \( n \) an integer. Vacuum configurations are associated with the different Chern-Simons numbers given by \( n \). Such a classification is a manifestation of the non-triviality of the homotopic group \( \pi_3(SU(2)) = \mathbb{Z} \). The group element \( U_n \) is a constant matrix at infinity and hence it corresponds to a compactification of \( D_3 \) into \( S^3 \). In turn by taking into consideration the group elements which are odd under parity at infinity we compactify \( D^3 \) into \( \mathbb{R}P^3 \). Thus the relevant homotopy group is in this case \( [\mathbb{R}P^3, SO(3)] = \mathbb{Z} \times \mathbb{Z}_2 \). It is worth noticing that this statement seems to be different from that which appears in the next section where this homotopy group is shown to be \( \mathbb{Z} \). However this \( \mathbb{Z} \) group is shown there to be double covered and can be understood as given above on the physical grounds.

The appearance of an above \( \mathbb{Z} \) factor in the homotopy group can be better understood in the following way. The Chern-Simons number of a gauge field configuration is defined as the gauge-independent difference in the value of the Chern-Simons functional of a given gauge field relative to the one of the vacuum. In order to make this comparison, we must transform to a gauge in which the gauge fields decrease more rapidly than \( 1/|x| \) at infinity (similar to the case of sphaleron). Actually such a gauge transformation is defined modulo a large gauge transformation which changes the Chern-Simons number by an integer. The above \( \mathbb{Z} \) factor is actually a manifestation of such an ambiguity. It is worth emphasizing that the fractional part of the Chern-Simons functional is gauge invariant.

A common feature of all odd-parity, even-\( U \) configurations is that they have an integer valued Chern-Simons functional. Indeed similarly with the case of the sphaleron we can make a nonsingular gauge rotation so that we remove the gauge field at infinity. The Wess-Zumino-Witten functional would give us a Chern-Simons number for the gauge field configuration. It is easy to see that the value of \( S_{WZW}(U) \) is invariant under even-parity smooth deformations of the \( U \) field at infinity. Indeed a variation of the Wess-Zumino-Witten functional reads

\[
\delta S_{WZW} = \frac{1}{8\pi^2} \int_{D^3} dTr((U^{-1}\delta U)(U^{-1}dU)^2) .
\] (2.9)

Since the variation of the group element on the surface \( S^2 \) is odd under parity and its value depends only on the values of the fields at the boundary we immediately conclude that the present variation of the Wess-Zumino-Witten functional equals zero. On the other hand let us consider a product of even-parity (at infinity) group elements \( U_1 \) and \( U_2 \) that correspond to any two such gauge fields. We have

\[
S_{WZW}(U_1U_2) = S_{WZW}(U_1) + S_{WZW}(U_2) + \frac{1}{8\pi^2} \int_{D^3} Trd((U_1^{-1}dU_1)(dU_2U_2^{-1})) .
\] (2.10)

The third term in the left hand side of this equation equals zero due to the odd parity of the integrand at infinity. Hence we see that the Wess-Zumino-Witten functional acts as a homomorphism from the group of maps \( U \) to a discrete subgroup of the group of real numbers which is obviously isomorphic to \( \mathbb{Z} \). As we argued before the even-\( U \) (at infinity) group element is contractible to the vacuum. In turn as
it is well known that the vacuum can have any integer value of the Chern-Simons number we conclude that even-parity $U$-fields are indeed classified by $\mathbb{Z}$. Thus all odd-parity even-$U$ gauge fields split into an infinite set of disconnected equivalence classes which are labeled by integer values of their Chern-Simons numbers.

Let us now consider odd-parity odd-$U$ gauge fields. A similar argument shows that the value of the Chern-Simons functional is a topological invariant while the Wess-Zumino-Witten functional maps the odd-parity $U$ fields to a discrete subgroup of the group of real numbers according to eq.(2.10). On the other hand a product of two odd-parity group elements $U_1$ and $U_2$ is even under parity. By taking also into account that the sphaleron has $S_{WZW}(U) = 1/2$ we conclude that the odd-parity odd-$U$ gauge fields have half-integer values of the Wess-Zumino-Wess functional and hence are classified by $n+1/2$ ($n \in \mathbb{Z}$) while these equivalence classes are themselves topologically disconnected one from the other for different values of $n$.

Thus we see that the Chern-Simons functional plays the role of a topological charge: it takes values in $\mathbb{Z}$ for even-$U$ and in $\mathbb{Z} + 1/2$ for odd-$U$ fields respectively.

An immediate implication of such a topological index for the fermionic spectrum of a 3-dimensional Dirac operator is the following. Let us consider a Dirac operator $D = \gamma_i(\partial_i - iA_i)$ in an external odd-parity gauge field $A_i$. Its non-zero eigenvalues are paired up ($\lambda, -\lambda$). Hence when the external field varies continuously the number of zero modes of the Dirac operator is invariant modulo 2. For the sphaleron background this topological invariant is equal to one while for the vacuum its value is zero. This means that it is not possible to get to the vacuum from the sphaleron configuration continuously through odd-parity gauge field configurations.

From the above considerations we conclude that in the presence of an odd-parity external gauge field the number of fermionic zero modes is 0 mod 2 for even-$U$ and 1 mod 2 for odd-$U$ configurations.

In ref.[10] we extended such a classification to the gauge field configurations which are odd under a generalized parity reflection symmetry. This property means an oddity under the parity reflections up to a gauge transformation. We have shown that the $\mathbb{Z}_2$ structure holds in this case too and corresponds to integer and half-integer values of the Chern-Simons functional.

### 3 Compact 1-connected Lie groups

We study the topological classification of the maps from $\mathbb{R}P^2$ and $\mathbb{R}P^3$ into $G/\mathbb{Z}_2$, where $G$ is a compact 1-connected Lie group. These maps correspond to purely gauge behavior of the gauge connections at infinity.

Let $H$ be any compact connected semisimple Lie group, i.e.

$$H = G/K,$$  \hspace{1cm} (3.1)

where $K$ is a subgroup of the center $Z(G)$ of $G$. Let us consider the associated
fibration sequence

\[ K \to G \to H \to BK \to BG \]  \hspace{1cm} (3.2)

BK stands for the classifying space of K. For any pointed space S let \([S, X]\) be the set of pointed homotopy classes of maps of S into X. If X is a topological, or Lie group, this set is a group.

**Proposition 1.** Let S be any closed 2-dimensional manifold. Then

\[ \pi_1 : [S, H] \to \text{Hom}(\pi_1(S), \pi_1(H)) \]  \hspace{1cm} (3.3)

is a group isomorphism.

**Proof.** Note that \(K = \pi_1(H)\). The map \(H \to BK\) is a 3-connected \(H\)-map. Therefore it induces a bijection

\[ [S, H] \to [S, BK] \]  \hspace{1cm} (3.4)

of the groups. Here,

\[ [S, BK] = H^1(S; K) = \text{Hom}(\pi_1(S), K) = \text{Hom}(\pi_1(S), \pi_1(H)). \square \]  \hspace{1cm} (3.5)

**Corollary 2.** Let G be any compact simply connected Lie group which has a center with \(Z_2\) as a subgroup. Then

\[ [RP^2, G/Z_2] \cong Z_2. \]  \hspace{1cm} (3.6)

**Proof.** By proposition 1,

\[ [RP^2, G/Z_2] \cong \text{Hom}(Z_2, Z_2) = Z_2. \]  \hspace{1cm} (3.7)

Thus we see that the space of the gauge field configurations odd under the parity reflections splits into two homotopically inequivalent classes which correspond to the \(Z_2\) group.

We now want to extend such a classification to the maps from \(RP^3\) into H.

Suppose that S is a 3-dimensional closed manifold. The functor \([S, *]\) applied to (3.2) yields an exact sequence

\[ 0 \to [S, G] \to [S, H] \to [S, BK] \to 0 \]  \hspace{1cm} (3.8)

of groups. Here we have used that \([S, K] = 0\) and \([S, BG] = 0\) since BG is 3-connected. Note that

\[ [S, BK] = H^1(S, K) \]

as before and that

\[ [S, G] = H^3(S; H_3(G)) \]

since the Postnikov fibration \(BG \to K(H_3(G), 4)\) which is a 5-connected \(H\)-map induces a 4-connected \(H\)-map \(G \to K(H_3(G), 3)\). Thus eq. (3.8) is equivalent to the short exact sequence

\[ 0 \to H^3(S; H_3(G)) \to [S, H] \to H^1(S; K) \to 0 \]  \hspace{1cm} (3.9)
of groups.

Example.
(a). With \( G = \text{Spin}(2n) \), \( n \) even, and \( K = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \),
\[
H = G/K = \text{PSpin}(2n) = \text{SO}(2n)/\mathbb{Z}_2
\]
and we obtain the exact sequence
\[
0 \to \mathbb{Z} \to [\mathbb{R}P^3, \text{SO}(2n)/\mathbb{Z}_2] \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0
\]
of groups.
(b). With \( G = \text{Spin}(2n) \), \( n \) odd, and \( K = \mathbb{Z}_4 \),
\[
H = G/K = \text{PSpin}(2n) = \text{SO}(2n)/\mathbb{Z}_2
\]
and we obtain the exact sequence
\[
0 \to \mathbb{Z} \to [\mathbb{R}P^3, \text{SO}(2n)/\mathbb{Z}_2] \to \mathbb{Z}_2 \to 0
\]
of groups.

We now specialize to the case where \( S = \mathbb{R}P^3 \) is the real projective sphere and \( K = \mathbb{Z}_2 \) is of order 2.

Proposition 3. The map
\[
H_3 : [\mathbb{R}P^3, H] \to \text{Hom}(H_3(\mathbb{R}P^3), H_3(H)) \tag{3.10}
\]
is a group isomorphism.

Proof. The Serre spectral sequence for the fibration
\[
G \to H \to \mathbb{R}P^\infty = B\mathbb{Z}_2, \tag{3.11}
\]
which is part of eq. (3.2) contains the short exact sequence
\[
0 \to H_3(G) \to H_3(H) \to H_3(\mathbb{Z}_2) \to 0 \tag{3.12}
\]
of abelian groups. Here \( H_3(\mathbb{Z}_2) = \mathbb{Z}_2 \). Moreover the functor \( H_3 \) induces a commutative diagram
\[
\begin{array}{cccccc}
0 & \to & H^3(\mathbb{R}P^3, H_3(G)) & \to & [\mathbb{R}P^3, H] & \to & H^1(\mathbb{R}P^3, \mathbb{Z}_2) & \to & 0 \\
\cong \downarrow & & \downarrow & & \cong \downarrow & & \downarrow & & \\
0 & \to & \mathcal{H}_G & \to & \mathcal{H}_H & \to & \mathcal{H}_\mathbb{Z}_2 & \to & 0
\end{array}
\]
where
\[
\mathcal{H}_G = \text{Hom}(H_3(\mathbb{R}P^3), H_3(G)),
\]
\[ \mathcal{H}_H = \text{Hom}(H_3(\mathbb{R}P^3), H_3(H)), \]
\[ \mathcal{H}_{Z_2} = \text{Hom}(H_3(\mathbb{R}P^\infty), Z_2). \]
The upper line is eq.(3.9) with \( S = \mathbb{R}P^3 \) and the bottom line is the result of applying the exact functor \( \text{Hom}(H_3(\mathbb{R}P^3), \ast) \) to eq. (3.12). The two vertical maps are group isomorphisms; use the Universal Coefficient Theorem for the left one and simply inspect the right one. Inspection also shows that the middle vertical map is a group homomorphism. Now the 5-lemma shows that the middle map is a group isomorphism.

Assume now that \( H \) is simple, i.e. that \( G \) is simple and simply connected.

**Lemma 4.**
\[ H_3(H) \cong \mathbb{Z}. \]  
(3.13)

**Proof.** The Serre spectral sequence for the orientable fibration
\[ H \to BZ_2 \to BG \]  
(3.14)
implies that
\[ H_3(H; Z_2) \cong H_3(BZ_2; Z_2) \cong Z_2. \]  
(3.15)
The extension (3.12) shows that \( H_3(H) \) is isomorphic to \( \mathbb{Z} \) or to \( \mathbb{Z} \oplus Z_2 \). The latter is impossible since
\[ H_3(H) \otimes Z_2 \subseteq H_3(H; Z_2) \]
by the Universal Coefficient Theorem. Hence \( H_3(H) \cong \mathbb{Z} \).

Since also \( H_3(\mathbb{R}P^3) = \mathbb{Z} \) (which is a special case of Lemma 4.), we conclude that double covering homomorphism \( G \to H \) induces a commutative diagram of abelian groups
\[
\begin{array}{ccccccc}
0 & \to & [\mathbb{R}P^3, G] & \to & [\mathbb{R}P^3, H] & \to & Z_2 & \to & 0 \\
& \cong & \downarrow & & \cong & \downarrow & || \\
0 & \to & \mathbb{Z} & \times^Z_2 & \mathbb{Z} & \to & Z_2 & \to & 0
\end{array}
\]
with exact rows. Here we could take \( G \to H \) to be \( SU(2) \to SO(3) \) or \( SU(2n) \to SU(2n)/Z_2 \), or \( E_7 \to E_7/Z_2 \).

This double covering corresponds exactly to the sets of integer and half integer values of the Chern-Simons functional as it was described in the introduction.

The double covering homomorphism \( G \to G/Z_2 \) induces an isomorphism \( \pi_i(G) = \pi_i(G/Z_2) \) for \( i \geq 2 \)! In the case of \( SU(2n) \) group \( \pi_3(SU(2n)) = \pi_3(SU(2n)/Z_2) = \mathbb{Z} \).

For the case \( i = 1 \) however there exists an exact sequence
\[ 1 \to \pi_1(G) \to \pi_1(G/Z_2) \to Z_2 \to 1 \]
which implies that \( \pi_1(G) \) is smaller than \( \pi_1(G/Z_2) \).
4 \quad SU(2n) \times U(1) and SO(2n) Lie groups

Let us now consider the case of $G = SU(N) \times U(1)$ groups. Assume that $\mathbb{Z}_2 \subseteq G$ is a central group of order 2. Form $H = G/\mathbb{Z}_2$. We are interested in $[\mathbb{R}P^2, H]$. There are two cases: $A$ and $B$.

A. Suppose that the composite $\mathbb{Z}_2 \hookrightarrow G \twoheadrightarrow U(1)$ is trivial. Then $\mathbb{Z}_2 \subseteq SU(N)$, (this is possible of course only for even $N = 2n$) so

$$H = SU(N)/\mathbb{Z}_2 \times U(1) \quad (4.1)$$

and hence

$$[\mathbb{R}P^2, H] = [\mathbb{R}P^2, SU(N)/\mathbb{Z}_2] \times [\mathbb{R}P^2, U(1)], \quad (4.2)$$

where the first factor is $\cong \mathbb{Z}_2$ by Corollary 2, and the second factor $[\mathbb{R}P^2, U(1)] = H^1(\mathbb{R}P^2, \mathbb{Z}) = 0$.

B. Suppose that the composite $\mathbb{Z}_2 \hookrightarrow G \twoheadrightarrow U(1)$ is nontrivial (i.e. that $\mathbb{Z}_2 \not\subseteq SU(N)$). In the diagram

$$\begin{array}{cccccc}
\mathbb{Z}_2 & \to & G & \to & H & \to & B\mathbb{Z}_2 & \to & BG \\
\downarrow & & \downarrow Bpr & & & & & & \\
\mathbb{B}U(1) & & & & & & & & \\
\end{array}$$

the upper row is a fibration sequence and the slanted arrow is essential (i.e. not null-homotopic). Apply the functor $[\mathbb{R}P^2, \ast]$ to this diagram and obtain the commutative diagram

$$\begin{array}{cccccc}
[\mathbb{R}P^2, G] & \to & [\mathbb{R}P^2, H] & \to & [\mathbb{R}P^2, B\mathbb{Z}_2] & \to & [\mathbb{R}P^2, BG] \\
\downarrow & & \downarrow & & \downarrow \cong & & \\
[\mathbb{R}P^2, \mathbb{B}U(1)] & & & & & & \\
\end{array}$$

where the upper row is an exact sequence of sets and the vertical arrow is a bijection for dimensional reasons. Here,

$$[\mathbb{R}P^2, G] = [\mathbb{R}P^2, SU(N)] \times [\mathbb{R}P^2, U(1)] = 0 \quad (4.3)$$

since $[\mathbb{R}P^2, SU(N)] = 0$ by Proposition 1 and also $[\mathbb{R}P^2, U(1)] = 0$ (as explained under $A$), $[\mathbb{R}P^2, B\mathbb{Z}_2] \cong H^1(\mathbb{R}P^2, \mathbb{Z}) \cong \mathbb{Z}_2$, and

$$[\mathbb{R}P^2, BG] \cong [\mathbb{R}P^2, BSU(N)] \times [\mathbb{R}P^2, \mathbb{B}U(1)] \cong \quad (4.4)$$

$$\cong 0 \times H^2(\mathbb{R}P^2, \mathbb{Z}) \cong \mathbb{Z}_2.$$
Moreover, the slanted arrow is a bijection (this follows, for instance, by applying $[\mathbb{R}P^2, \ast]$ to the fibration sequence $U(1) \to B\mathbb{Z}_2 \to BU(1)$). Then also

$$[\mathbb{R}P^2, B\mathbb{Z}_2] \to [\mathbb{R}P^2, BG]$$

(4.5)
is a bijection. Hence $[\mathbb{R}P^2, H] = 0$ by exactness.

Thus we see that the presence of $U(1)$ group does not affect the homotopy.

We now proceed to examine the case of $SO(N)$ groups whose center is $\mathbb{Z}_2$, i.e. $N = 2n$. The quotient group $H = SO(2n)/\mathbb{Z}_2$ is the semi-spinor group $PSO(2n)$. Its fundamental group is given by

$$\pi_1(PSO(2n)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2, & n \text{ even} \\ \mathbb{Z}_4, & n \text{ odd} \end{cases}$$

(4.6)

By Proposition 1,

$$[\mathbb{R}P^2, PSO(2n)] = \text{Hom}(\pi_1(\mathbb{R}P^2), \pi_1(PSO(2n))) = \text{Hom}(\mathbb{Z}_2, \pi_1(PSO(2n))) \cong$$

$$\begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2, & n \text{ even} \\ \mathbb{Z}_2, & n \text{ odd} \end{cases}.$$  

(4.7)

Let us discuss some physical aspects of the above classification. The nontriviality of the above homotopy groups implies the existence of infinite surfaces of gauge-higgs fields as well as of gauge field configurations alone which are homotopically equivalent to the sphaleron type configurations for the gauge theories with $SU(2n)$, $SO(2n)$ and $E_7$ gauge symmetries. Moreover in the presence of such a topologically non-trivial gauge field configuration the three dimensional Dirac operator has an odd number of normalizable zero modes. This conclusion obviously applies to the phases both with broken and unbroken gauge symmetry. Strictly speaking this fact itself is not sufficient to prove an existence of saddle points of the energy functional in the broken (Higgs) phase. But with a plausible assumption that the height of the potential barrier between vacua is non-zero but not infinitely high we may expect the presence of saddle point $CS = 1/2$ sphalerons along with their equivalent deformations at each symmetry breaking mass scale above the $SU(2) \times U(1)$ Weinberg-Salam one. In other words in their Higgs phase these theories possess a hierarchy of odd-parity saddle point solutions to the field equations corresponding to a sequence of spontaneous symmetry breaking mass scales.

## 5 Conclusions

We rigorously proved the existence of a topological classification for odd-parity gauge field configurations in pure $SU(2n) \times U(1)$ and $SO(2n)$ Yang-Mills theories. This is an automatic consequence of the existence of cyclic homotopy groups of maps from the 2-dim. and 3-dim. projective spheres into the respective Lie groups appropriately modded out by $\mathbb{Z}_2$. This is a sufficient condition for the existence of a nontrivial
$\mathbb{Z}_2$ topological structure in the configuration space of a theory with spontaneously broken $SU(2) \times U(1)$ gauge symmetry. Such a structure is characterized by the already found $CS = 1/2$ electroweak sphaleron. More importantly it survives in its absence too for the case of an unbroken gauge symmetry as it reflects the homotopic properties of the space of gauge field configurations alone. In this sense it points to the existence of infinite surfaces of static, finite energy and unstable 3-dim. configurations in the symmetric high temperature phase of the standard electroweak theory which are homotopically equivalent to the saddle point sphaleron configuration and have $CS = 1/2$.

More generally as the $\mathbb{Z}_2$ classification applies to all $SU(2n)$ and $SO(2n)$ groups as well as for the $E_7$ exceptional gauge group it indicates the existence of a hierarchy of saddle point sphaleron configurations associated with higher gauge groups and symmetry breaking scales beyond the one of the standard model such as in Grand Unified Gauge theories \cite{12}. It is worth noticing however that such “sphalerons” are not necessarily responsible for baryon number violation in realistic models. The point is that $B + L$ can be gauged (for example in the $SO(10)$ model). While a hierarchy of saddle point $B$-violating sphaleron configurations could have amusing implications for existing scenarios of Baryogenesis the physical consequences of $B$ preserving ones are probably of minimal significance.

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A Appendix

Classification by maps of lens spaces

Here we illustrate the difference between $SU(even)$ and $SU(odd)$ gauge groups by considering maps from the lens space $L = S^3/\mathbb{Z}_3$ into the group $P = PSU(3)$ = $SU(3)/\mathbb{Z}_3$ (the adjoint form of $SU(3)$), where $\mathbb{Z}_3$ is a center of $SU(3)$. In particular we prove that $[S^3/\mathbb{Z}_3, SU(3)/\mathbb{Z}_3] = \mathbb{Z} \oplus \mathbb{Z}_3$.

Before to proceed we notice that one can not consider $\tilde{L} = S^2/\mathbb{Z}_3$ because it is not a smooth manifold. Indeed for a 3-fold covering map $S^2 \to \tilde{L}$ the equation $3\chi(\tilde{L}) = \chi(S^2) = 2$ ($\chi(Y)$ is an Euler characteristic of $Y$) has no integer solution. Hence $\tilde{L}$ can not exist. Therefore we consider $L = S^3/\mathbb{Z}_3$ which is a smooth manifold. We now can proceed to prove our claim.

**Theorem 1.** $[L, P] \cong \mathbb{Z} \oplus \mathbb{Z}_3$. 

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Before the proof, we recall some facts about the homology of these spaces. The cellular chain complex of $L$ shows that
\[ H_1(L) \cong \mathbb{Z}_3, \quad H_2(L) = 0, \quad H_3(L) \cong \mathbb{Z}. \] (A.1)
The Serre spectral sequence
\[ E^2 = H_*(\mathbb{Z}_3, H_*(SU(3))) \longrightarrow H_*(P) \] (A.2)
for the fibration
\[ SU(3) \to P \to B\mathbb{Z}_3 \] (A.3)
has a simple structure partly because
\[ H_i(B\mathbb{Z}_3) = \begin{cases} \mathbb{Z}_3, & i \text{ odd} \\ 0, & i \text{ even} \end{cases} \] (A.4)
The spectral sequence shows that
\[ H_2(P) = 0 \] (A.5)
and that the sequence
\[ 0 \to H_3(SU(3)) \to H_3(P) \to H_3(B\mathbb{Z}_3) \to 0 \] (A.6)
is exact. Here $H_3(SU(3)) \cong \mathbb{Z}$ and $H_3(B\mathbb{Z}_3) \cong \mathbb{Z}_3$ so $H_3(P)$ is either isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_3$. In fact

**Lemma 2.**
\[ H_3(P) \cong \mathbb{Z} \oplus \mathbb{Z}_3. \] (A.7)

**Proof.** According to R.M.Kane [13]
\[ H^*(P, \mathbb{F}_3) \cong E(V_3) \otimes E(V_1) \otimes \mathbb{F}_3[V_2]/(U_2^3) \] (A.8)
where the subscript indicates degree. In particular
\[ H^3(P, \mathbb{F}_3) \cong \mathbb{F}_3 \oplus \mathbb{F}_3. \] (A.9)
Since $H_2(P) = 0$, $H_3(P) \otimes \mathbb{F}_3 \cong H_3(P, \mathbb{F}_3) \cong H^3(P, \mathbb{F}_3) \cong H^3(P, \mathbb{F}_3)$ by the Universal Coefficient Formula. Therefore we must have $H_3(P) \cong \mathbb{Z} \oplus \mathbb{Z}_3$.

**Proof of Theorem 1.** Applying the functor $[L, \ast]$ to fibration eq. (A.3) and the functor $\text{Hom}(H_3(L), \ast)$ to the exact sequence eq. (A.6) yields two exact sequences
\[ 0 \to [L, SU(3)] \to [L, P] \to [L, B\mathbb{Z}_3] \to 0 \]
\[ H_3 \downarrow \quad H_3 \downarrow \quad H_3 \downarrow \]
\[ 0 \to \mathcal{F}_{SU(3)} \to \mathcal{F}_P \to \mathcal{F}_{B\mathbb{Z}_3} \to 0 \]
where
\[ \mathcal{F}_{SU(3)} = \text{Hom}(H_3(L), H_3(SU(3))), \]
\[ \mathcal{F}_P = \text{Hom}(H_3(L), H_3(P)), \]
\[ \mathcal{F}_{BZ_3} = \text{Hom}(H_3(L), H_3(BZ_3)), \]

Connected by a homomorphism determined by the functor \( H_3 \). The upper sequence is exact since \( P \to BZ_3 \) is 3-connected so that \([L, P] \to [L, BZ_3]\) is surjective. The lower sequence is exact since \( H_3(L) \cong \mathbb{Z} \). The left vertical homomorphism is an isomorphism since \( L \) is a compact orientable 3-manifold. An inspection shows that the right vertical homomorphism is an isomorphism. Hence also the middle homomorphism
\[ H_3 : [L, P] \to \text{Hom}(H_3(L), H_3(P)) \] (A.11)
is an isomorphism. We have
\[ \text{Hom}(H_3(L), H_3(P)) \cong H_3(P) \cong \mathbb{Z} \oplus \mathbb{Z}_3 \] (A.12)
by Lemma 2 and since \( H_3(L) \cong \mathbb{Z} \).

Let us discuss the physical interpretation of the above conclusion. When we consider the lens space \( L \) we restrict ourselves to the four dimensional gauge field configurations which are invariant under \( Z_3 \) transformations up to gauge transformations. The above computed homotopy group implies that this subset of the gauge field configurations has a fine structure governed by the \( Z_3 \) group. That means that this subset is splitted into three disconnected classes. It is important to note that the classification applies to four dimensional gauge field configurations in contrast to the \( SU(2) \) case where the existence of a \( Z_2 \) structure was a property of the space of \( 3 - \text{dim} \) static configurations. Notice that this structure does not literally have anything to do with sphalerons because it corresponds to a classification of the boundary conditions for 4-dimensional gauge fields. Instead we have to assign this structure to the vacuum of the 4-dimensional Yang-Mills theory. It would be interesting to have an interpretation of this structure in terms of the spectrum of the 4-dimensional Dirac operator.

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