Krein Quantization Approach to Hawking Radiation

H. Pejhan\(^1\) and S. Rahbardehghan\(^2\)

\(^1\)Department of Physics, Science and Research Branch, Islamic Azad University, Tehran, Iran
\(^2\)Department of Physics, Islamic Azad University, Central Branch, Tehran, Iran

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In this paper, we prove that in the context of Krein method, by applying the results of the accelerated-mirror to a black hole, one can recover the result obtained by Hawking for black hole radiation even considering the fact that vacuum expectation value of the energy-momentum tensor of the free theory is zero.

I. INTRODUCTION

Krein space as the generalization of the Hilbert space is defined by \( K = \mathcal{H}_+ \oplus \mathcal{H}_- \), where \( K \) is called the total space and \( \mathcal{H}_- (= \mathcal{H}_+^*) \) stands for the 'anti-Hilbert' space. Krein space quantization then is a canonical quantization of Gupta-Bleuler type in which the Fock space is constructed over the total space \([3,4]\). In this quantization method, the set of states is different from the set of physical states. More precisely, the observables are defined on the total space, while the average values of the observables are calculated on the sub-space of physical states. It should be noted that the total space is equipped with an indefinite inner product which results in some (unphysical) states having a negative norm. In this context, it is not only finite, but actually zero. It is also invariant under Bogoliubov transformations \([1]\). Indeed, in-\(\text{stead of having a multiplicity of vacuum, we have several} \)

We consider the massless scalar wave equation in the Minkowski space-time,

\[ \square \varphi(x) = 0. \quad (1) \]

The inner product of a pair of its solutions is defined by:

\[ (\varphi_1, \varphi_2) = -i \int (\varphi_1(x) \partial_\mu \varphi_2^*(x))d^3x. \quad (2) \]

It is reputed that the gravitational field of a black hole creates an effective potential barrier, which for a nonrotating one is localized near \( r = 3M \) \((M \text{ refers to hole's mass; } c = G = 1)\) \([8]\). The black hole is almost unable to absorb electromagnetic waves when their frequencies are less than \( \omega_c \equiv (2/27)^{1/2} M^{-1} \). Indeed, \( \omega_c \) is the cut-off frequency for the absorption of electromagnetic and scalar fields by a Schwarzschild black hole. Therefore, due to the existence of a potential barrier, it is possible that the calculating procedure of Casimir effect in Krein quantization approach, see \([7]\), be applied to a black hole.

II. APPLYING THE METHOD

As proved in Ref. \([8]\), one can substitute the spherical potential barrier by two plane conductors with proper acceleration \( b^{-1} \cong (3\sqrt{3}M)^{-1} \). (This approximation is legitimized only in the vicinity of event horizon of a schwarzschild black hole.) Therefore, the main purpose of this paper - describing Hawking radiation through Krein quantization approach - would be met by applying the results of the accelerated-mirror \((\text{To see more about the relation between moving mirrors and black holes refer to [12]}\): In order to achieve this goal, we need to calculate vacuum expectation value of the energy-momentum tensor for a massless scalar field subjected to the Dirichlet boundary condition in the Minkowski vacuum induced by an infinite plane conductor that is uniformly accelerated normal to itself.

A) We consider the massless scalar wave equation in the Minkowski space-time

\[ \square \varphi(x) = 0. \quad (1) \]

The inner product of a pair of its solutions is defined by:

\[ (\varphi_1, \varphi_2) = -i \int (\varphi_1(x) \partial_\mu \varphi_2^*(x))d^3x. \quad (2) \]

The general formula for proper acceleration of a particle which is at rest in the gravitational field of a non-rotating black hole is as follows \([3,4]\): \( |a| = (1 - 2M/r)^{-1/2} M/r^2 \), while a stationary distant observer will measure \( |a'| = M/r^2 \). So, the potential barrier localized in the vicinity of \( r = 3M \) has a nonzero proper acceleration \((3\sqrt{3}M)^{-1}\).
There exist a complete set of mode solutions \( \{ \phi_d \} \) of Eq. \([1]\) which are orthonormal in the product \([2]\), i.e.

\[
\langle \phi_P(\vec{k}, x), \phi_P(\vec{k}', x) \rangle = \delta(\vec{k} - \vec{k}'),
\]

\[
\langle \phi_N(\vec{k}, x), \phi_N(\vec{k}', x) \rangle = -\delta(\vec{k} - \vec{k}'),
\]

\[
(\phi_P(\vec{k}, x), \phi_N(\vec{k}, x)) = 0,
\]

then \( \{ \phi_P \} \) and \( \{ \phi_N \} \) will be a set of solutions of positive and negative norm states, respectively. The field operator \( \varphi \) in Krein quantization is defined by \( \varphi = \frac{1}{\sqrt{2}}(\varphi_P + \varphi_N) \), in which

\[
\varphi_P(x) = \int d^3k [a(\vec{k})\phi_P(\vec{k}, x) + a^\dagger(\vec{k})\phi^*_P(\vec{k}, x)],
\]

\[
\varphi_N(x) = \int d^3k [b(\vec{k})\phi_N(\vec{k}, x) + b^\dagger(\vec{k})\phi^*_N(\vec{k}, x)].
\]

\([a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') \), \([b(\vec{k}), b^\dagger(\vec{k}')] = -\delta(\vec{k} - \vec{k}') \), and the other commutation relations are zero. The vacuum state \( |0\rangle \) is defined such as \( a(\vec{k})|0\rangle = 0 \), \( b(\vec{k})|0\rangle = 0 \) and the physical and un-physical states are \( a^\dagger(\vec{k})|0\rangle = |1_k\rangle \), \( b^\dagger(\vec{k})|0\rangle = |1_T\rangle \).

Regarding the presence of un-physical states, unitarity of the theory when interactions are taken into account, is assured by applying the following conditions on quantum states and probability amplitude:

- Un-physical states do not interact with the physical ones or real physical world. It means that in the Feynman diagrams such states only appear in the internal legs and disconnected parts of the diagrams.

- Un-physical states, which appear in the disconnected parts of the S-matrix elements, can be eliminated by renormalizing the probability amplitude as \([13]\)

\[
S_{\text{if}} = \frac{|\text{physical states, in}| |\text{physical states, out} \rangle}{\langle 0, \text{in}|0, \text{out} \rangle}.
\]

The presence of un-physical states in the internal lines, plays a key role in the renormalizing procedure. It has been shown that experimental results obtained through the method are in complete agreement with their (Hilbert space) QFT’s counterparts. See \([10]\) and references therein.

So, imposing physical boundary conditions on the field operator only affects physical states. Consequently, the field operator would be as:

\[
\varphi(x) = \sum_d \left[ a(k_d)\phi_P(k_d, x) + a^\dagger(k_d)\phi^*_P(k_d, x) \right]
\]

\[
+ \int d^3k [b(\vec{k})\phi_N(\vec{k}, x) + b^\dagger(\vec{k})\phi^*_N(\vec{k}, x)],
\]

where \( k_d \) are the eigen-frequencies of the system under consideration.

**B)** From now on, in order to facilitate the calculations and make the results more clear, we exploit the accelerated (Rindler) coordinates defined by the coordinate transformation: \( t = \xi \sinh \tau \), \( x = \xi \cosh \tau \), which cover the region \( |x| > |t| \) of Minkowski space. The line element is as \( ds^2 = -\xi^2 d\tau^2 + dx^2 + dy^2 \), with \( x = (y, z) \). The curves \( \xi = \text{constant} \), \( x = \text{constant} \) are worldlines of constant proper acceleration \( \xi^{-1} \) and the surface \( \xi = b \) represents the trajectory of the barrier which has a proper acceleration \( b^{-1} \).

The starting point to calculate vacuum expectation value of the energy-momentum tensor in view of the accelerated observer in the Minkowski space-time is \([14]\)

\[
<0|T_{\mu}^\nu|0> = -i \lim_{x' \rightarrow x} \left( \frac{2}{3} \nabla_\mu \nabla_\nu' - \frac{1}{3} \nabla_\mu \nabla_\nu - \frac{1}{6} \eta_\mu^\nu \nabla_\alpha \nabla_\alpha' \right) G(x, x'),
\]

in which \( G(x, x') \) is the Feynman Green function that with respect to the quantum field of the theory can be decomposed into two parts as follows:

\[
G(x, x') = G_P(x, x') - G_N(x, x').
\]

The first term labeled by subscripts \( P \) refers to the physical part of the theory which is subjected to the Dirichlet boundary condition. The second term labeled by subscripts \( N \) however refers to the un-physical part of the theory which according to its definition can be considered as the free field part of the theory. (The corresponding propagators for the Dirichlet and the free field ones have been calculated in Refs \([14] \) and \([13]\), respectively.)

Hence, contrary to the conventional approach in which the regularized quantity for \( <0|T_{\mu}^\nu|0> \) is acquired by subtracting the value that it would have if evaluated relative to the Minkowski vacuum \([14]\), in the context of Krein method, interestingly due to the presence of un-physical states, as natural renormalizing tools, this term intrinsically exists. Indeed, in both approaches, the regularizing process would have been accomplished equally, but with different interpretations. So, with regard to the calculations of Ref. \([14]\), we obtain that far from the barrier, there exists a negative energy density, corresponding to the absence from the vacuum of black-body radiation with a temperature \( T = (2\pi \xi)^{-1} \) (in the frame of an observer with proper acceleration \( \xi^{-1} \); \( \xi/b \rightarrow \infty \)), as follows

\[
<0|T_{\mu}^\nu|0> \sim \frac{-1}{2\pi^2 \xi^2} \int_0^\infty \frac{\omega^3 d\omega}{e^{\omega} - 1} \text{diag}(-1, 1, 1, 3, 3, 3).\]

\[3\] It is worth to mention that this coordinate transformation affects both physical and un-physical parts of the quantum field.
This asymptotic form is independent of the acceleration of the barrier, and depends only on the local acceleration. It means that respecting the gravitational analogy, at adequately large distances to the barrier, vacuum expectation value of the energy-momentum tensor depends purely on the local gravitational field.

C) Accordingly, an observer who is at rest next to the horizon \((r \approx r_0)\), will discover a negative flux of black-body radiation with a temperature \(T_0 = (1/2\pi)(M/r_0^2)(1 - 2M/r_0)^{-1/2}\). On the other side, in the perspective of an observer far from the black hole - at future infinite - when the particle is infinitesimally close to the event horizon, this negative flow through the horizon \((T = (1/2\pi)(M/r_0^2))\) is as a positive flux of particles at infinity with \(T = k/2\pi\) \((k\) is the surface gravity on the horizon of a schwarzschild black hole; \(k = 1/4M\)). This result is directly analogous to the black hole thermal radiation computed by Hawking [16].

### III. CONCLUSION

In this work, by taking a simple model to simulate a black hole, we showed that utilizing Krein space quantization does not destroy black holes thermodynamics and the method is capable of recovering the very result for Hawking radiation even regarding the fact that \(\langle 0|T^\mu_\nu|0 \rangle\) of the free theory is zero. However, the authors would like to emphasize that, perusing more realistic situations entails a sophisticated modeling to encompass the following subjects: The assumption of ideal conductivity, which is obviously not the case for a spherical potential barrier of a non-rotating black hole, requires to be adjusted as a real, and not as an ideal, conductor. Moreover, respecting that there exist two branches of turning points for a non-rotating black hole potential barrier [10], the approximation of the barrier by a thin shell should be modified. Calculation of Hawking radiation with respect to these issues in such a subtle model, is beyond the scope of this paper, but would have been performed straightforwardly by following the procedure exploited in Ref. [17].

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