COMPLEX POWERS OF ANALYTIC FUNCTIONS AND
MEROMORPHIC RENORMALIZATION IN QFT.

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To the memory of Louis Boutet de Monvel.

Abstract. In this article, we study functional analytic properties of the meromorphic families of distributions \( \prod_{j=1}^{p}(f_j + i0)^{\lambda_j} \) \( (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p \) using Hironaka’s resolution of singularities, then using recent works on the decomposition of meromorphic germs with linear poles, we renormalize products of powers of analytic functions \( \prod_{j=1}^{p}(f_j + i0)^{\lambda_j} \), \( k_j \in \mathbb{Z} \) in the space of distributions. We also study microlocal properties of \( \prod_{j=1}^{p}(f_j + i0)^{\lambda_j} \) \( (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p \) and \( \prod_{j=1}^{p}(f_j + i0)^{\lambda_j} \), \( k_j \in \mathbb{Z} \). In the second part, we argue that the above families of distributions with regular holonomic singularities provide a universal model describing singularities of all Feynman amplitudes and give a new proof of renormalizability of quantum field theory on convex analytic Lorentzian spacetimes as applications of ideas from the first part.

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Introduction.

To renormalize perturbative quantum field theories (QFT) on Minkowski space $\mathbb{R}^{n+1}$, physicists often use a classical method, called dimensional regularization and axiomatized by K. Wilson [15], which can be roughly described as follows: we work in momentum space and replace all integrals $\int_{\mathbb{R}^d} dp f(p)$ of rational functions $f(p)$ on $\mathbb{R}^d$ by integrals $\int_{\mathbb{R}^{d+\epsilon}} dp f(p)$ on the "space" $\mathbb{R}^{d+\epsilon}$ where the dimension is treated as a complex parameter. For example, for a rotation invariant function $f$ on $\mathbb{R}^d$, $\int_{\mathbb{R}^d} dp f(p) = v_d \int_{\mathbb{R}_{\geq 0}} dr d^{d-1}f(r)$ where $v_d = \frac{(2\pi)^{d/2}}{\Gamma(d/2)}$ is the $(d-1)$–volume of the unit sphere which is calculated in such a way that $\int_{\mathbb{R}^d} d^dp e^{-|p|^2} = \pi^{d/2}$. By analytic continuation, these integrals depend meromorphically in $\epsilon$ and renormalization consists in subtracting the poles in Feynman amplitudes following the famous $R$–operation algorithm of Bogoliubov. Despite its efficiency, this procedure is difficult to interpret mathematically, due to the fact that renormalization is performed in momentum space. However, the reason why dimensional regularization works is intuitively quite clear since we integrate rational functions over semialgebraic sets! This suggests that in depth studies of dimensional regularization make use of algebraic geometry [13, 12, 14].

The purpose of the present paper is to understand the meaning of analytic regularization techniques for QFT on an analytic Lorentzian spacetime $M$ in the philosophy of Epstein–Glaser renormalization. In this point of view, we work in position space and interpret renormalization as the operation of extension of distributions on the configuration spaces $(M^n)_{n \in \mathbb{N}}$. At this point, we should refer to several exciting recent works which explore analytic techniques in the Epstein–Glaser framework [34, 35, 19] in the flat case, especially the papers [6, 5] which, as in the present paper, use the resolution of singularities.

In the physics terminology, Feynman amplitudes are formally defined as products of the form

$$\prod_{1 \leq i < j \leq \infty} G(x_i, x_j)^{n_{ij}}, n_{ij} \in \mathbb{N}$$

of Feynman propagators $G(x, y)$ which are distributions on the configuration space $M^2$, where $M$ is our Lorentzian spacetime. The main idea of our work is to exploit the fact that Feynman amplitudes living on configuration spaces $(M^n)_{n \in \mathbb{N}}$ have singularities of regular holonomic type i.e.

**Definition 0.1.** A function $u$ on some open set $U \subset \mathbb{C}^n$, is regular holonomic near a point $z_0$ of some smooth hypersurface defined by some equation $\{ \Gamma = 0 \}, \Gamma(z_0) = 0, d\Gamma(z_0) \neq 0$ if $u$ is near $z_0$ a finite linear combination with coefficients in $\mathcal{O}_{z_0}$ (the algebra of holomorphic germs at $z_0$) of functions of the form $\Gamma^\alpha, \Gamma^\alpha \log \Gamma$.

These generalize meromorphic functions of several complex variables. In modern terms $\Gamma^\alpha$ (resp. $\log \Gamma$) would be defined as the distributions $(\Gamma + i0)^\alpha$ (resp.
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log(Γ + i0)). Our approach, which goes back to Hadamard \cite{28,3} and pre-dates the Schwartz theory of distributions, uses the description of the Feynman propagator as a branched meromorphic function (possibly logarithmically branched) on the complexified spacetime. Indeed, the singularity of \( G \) has the representation:

\[
G(x, y) = \frac{U}{\Gamma + i0} + V \log (\Gamma + i0) + W
\]

where \( \Gamma, U, V, W \) are analytic functions and it follows that \( G \) has regular holonomic singularity along the null cone. Inspired by the work of Borcherds \cite{8}, our idea is to regularize \( G \) by considering the modified propagator:

\[
G_\lambda(x, y) = \left( \frac{U}{\Gamma + i0} + V \log (\Gamma + i0) + W \right) (\Gamma + i0)^\lambda
\]

which is still of holonomic type. Then we consider regularized Feynman amplitudes on configuration space \( M^n \) depending on several complex variables \((\lambda_{ij})_{1 \leq i < j \leq n} \in \mathbb{C}^{\binom{n}{2} - 1}\):

\[
\prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}}, n_{ij} \in \mathbb{N}
\]

so our goal in the present paper is to show that:

- the regularized Feynman amplitude \( \prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}}, n_{ij} \in \mathbb{N} \) depends meromorphically on \((\lambda_{ij})_{1 \leq i < j \leq n} \in \mathbb{C}^{\binom{n}{2} - 1}\) with value distribution.
- Outside the big diagonal \( D_n = \{(x_1, \ldots, x_n) \in M^n \text{ s.t. } \exists (i < j), x_i = x_j\} \), it is holomorphic in \( \lambda \) and

\[
\lim_{\lambda \to 0} \prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}} = \prod_{1 \leq i < j \leq n} G(x_i, x_j)^{n_{ij}}
\]

where the above equality only holds in \( \mathcal{D}(M^n \setminus D_n) \) i.e. on the configuration space of \( n \)-points which are all distinct.

- We can define a collection of renormalization maps \( R_{M^n} \) which are linear maps from the space of Feynman amplitudes to \( \mathcal{D}(M^n) \) such that \( R_{M^n} \left( \prod_{1 \leq i < j \leq n} G(x_i, x_j)^{n_{ij}} \right) \) is a distributional extension of \( \prod_{1 \leq i < j \leq n} G(x_i, x_j)^{n_{ij}} \) which satisfies the consistency axioms \cite{10,11} (also elegantly described in \cite{3}) ensuring that the renormalization satisfies physical requirements such as causality.

0.0.1. Contents of the paper. Our paper is devoted to the realization of the above program and is divided in two parts: the first part is of independent interest and of purely mathematical nature whereas the second part presents applications of the first part to the renormalization of QFT on analytic spacetimes.

Let us start with the first part. In the first two sections, we study the universal model which describes the singularities of all Feynman amplitudes which consists in ill-defined products of powers of real analytic functions of the form \( \prod_{j=1}^{p} (\log(f_j + i0))^{k_j} (f_j + i0)^{\lambda_j} \) where \( p_j \) are nonnegative integers and \( k_j \) negative integers. Then we show how to make sense of the above ill-defined product of distributions by analytic continuation as follows:

1. we consider the family \( \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right) \) where \((\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p\)

and use the resolution of singularities of Hironaka to show in Theorem \cite{13} that the family \( \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right) \) depends meromorphically on \((\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p\) with linear poles with value distribution.

2. Motivated by the problem of renormalization of conical multiple zeta functions at integers, Guo–Paycha–Zhang \cite{27} were able to generalize the Laurent series decomposition to meromorphic germs with linear poles. Then we
use their recent results to decompose the meromorphic family \((\prod_{j=1}^p (f_j + i0)^{\lambda_j})\) in a regular part which is holomorphic in \(\lambda\) and a singular part which contains the polar singularity then we define a renormalization \(\mathcal{R}_{\pi} \left( \prod_{j=1}^p (f_j + i0)^{\lambda_j} \right)\) by letting the complex parameter \((\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p\) go to \((k_1, \ldots, k_p) \in \mathbb{C}^p\) in the regular part.

3. \(\mathcal{R}_{\pi}\) satisfies the following factorization identity of central importance: let \(U, V\) be open sets in \(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}\) respectively and \(f_1, \ldots, f_p\) (resp \(g_1, \ldots, g_p\)) real analytic functions on \(U\) (resp \(V\)) then:

\[
\mathcal{R}_{\pi} \left( f_1^{k_1} \cdots f_p^{k_p} g_1^{j_1} \cdots g_p^{j_p} \right) = \mathcal{R}_{\pi} \left( f_1^{k_1} \cdots f_p^{k_p} \right) \otimes \mathcal{R}_{\pi} \left( g_1^{j_1} \cdots g_p^{j_p} \right).
\]

where the tensor product \(\otimes\) is the exterior tensor product: \(\mathcal{D}'(U) \otimes \mathcal{D}'(V) \hookrightarrow \mathcal{D}'(U \times V)\).

Our philosophy is to hide the complicated combinatorics of renormalization behind two deep results in analytic geometry: the resolution of singularities of Hironaka and the generalized decomposition in Laurent series of [27].

However, for our applications to QFT it is necessary to show that our renormalization satisfies the axioms [10, 11] hence we must study the microlocal properties of the family \((\prod_{j=1}^p (f_j + i0)^{\lambda_j})\) and of the renormalized distribution \(\mathcal{R}_{\pi} \left( \prod_{j=1}^p (f_j + i0)^{k_j} \right)\). We start in section 3 by giving easy results on products of distributions in the setting of Sobolev spaces and we give simple bounds in Theorem 3.2 on the wave front of products. Then in section 4, we apply these tools to study the microlocal properties of the family \(((f + i0)^{\lambda})\). In Theorem 4.1 we bound the wave front set of \(((f + i0)^{\lambda})\) for generic values of \(\lambda\):

\[
WF((f + i0)^{\lambda}) \subset \{(x; \xi) \mid \exists \{(x_k, a_k)\} \in (\mathbb{C}^N \times \mathbb{R}_+) \times \mathbb{R}_+^N, x_k \to x, f(x_k) \to 0, a_k df(x_k) \to \xi\}.
\]

In section 5, based on the recent work [16] we present a functional calculus of meromorphic functions with value \(\mathcal{D}'\), where \(\mathcal{D}'\) is the space of distributions whose wave front set is contained in the conic set \(\Gamma\). Using this functional calculus, we prove two Theorems about functional analytic properties of the families \(((f + i0)^{\lambda})\) and \((\prod_{j=1}^p (f_j + i0)^{\lambda_j})\). In section 6, we show that

**Theorem 0.1.** Let \(f\) be a real valued analytic function s.t. \(\{df = 0\} \subset \{f = 0\}\), \(Z \subset \mathbb{C}\) a discrete subset containing the poles of the meromorphic family \(((f + i0)^{\lambda})\).

Set

\[
\Lambda_f = \{(x; \xi) \mid \exists \{(x_k, a_k)\} \in (\mathbb{C}^N \times \mathbb{R}_+) \times \mathbb{R}_+^N, x_k \to x, f(x_k) \to 0, a_k df(x_k) \to \xi\}.
\]

For all \(z \in Z\), let \(a_k\) to be the coefficients of the Laurent series expansion of \(\lambda \mapsto (f + i0)^{\lambda}\) around \(z\)

\[
(f + i0)^{\lambda} = \sum_{k \in \mathbb{Z}} a_k (\lambda - z)^k.
\]

Then for all \(k \in \mathbb{Z}\), \(WF(a_k) \subset \Lambda_f\) and if \(k < 0\) then \(a_k\) is a distribution supported by the critical locus \(\{df = 0\}\).

In the multiple functions case \((f_1, \ldots, f_p)\), which is the case of interest, we describe in paragraph 6.1.1 geometric constraints on the zero sets of \((f_1, \ldots, f_p)\) and the critical sets \(\{df_1 = 0\}, \ldots, \{df_p = 0\}\) which allow us to give an optimal result in Theorem 6.3.
Theorem 0.2. Under the assumptions of paragraph [6.1.1], the family \( \left( \prod_{j=1}^{p} (f_j + i0)^{k_j} \right)_{\lambda \in \mathbb{C}^p} \) depends meromorphically on \( \lambda \) with linear poles with value \( D^*_\Lambda \) where
\[
\Lambda = \bigcup_j \{(x; \xi)|j \in J, f_j(x) = 0, df_j(x) \neq 0, \xi = \sum_{j \in J} a_j df_j(x), a_j > 0\} \cup N^* \Sigma_j,
\]
\[
\Sigma_j = \cap_{j \in J} \{df_j = 0\}.
\]

The distribution
\[
\mathcal{R}_\pi \left( \prod_{j=1}^{p} (f_j + i0)^{k_j} \right) \in \mathcal{D}'(U)
\]
is a distributional extension of \( \prod_{j=1}^{p} (f_j + i0)^{k_j} \in \mathcal{D}'(U \setminus X) \) and has wave front contained in \( \Lambda \).

The above bound on the wave front set of \( \mathcal{R}_\pi \left( \prod_{j=1}^{p} (f_j + i0)^{k_j} \right) \) is quite natural from the point of view of symplectic geometry. Indeed, motivated by problems in representation theory, Aizenbud and Drinfeld [11] introduced the class of WF-holonomic distribution (which contains Fourier transform of algebraic measures for instance):

Definition 0.2. A distribution \( t \) on a smooth analytic manifold \( M \) is called WF-holonomic if \( WF(t) \) is locally contained in some finite union of conormal bundles of some smooth analytic submanifolds of \( M \), said differently, for all bounded open set \( U \subset M \), there is a finite number of analytic submanifolds \( (N_i)_i \), s.t. \( WF(t) \subset \bigcup_i N^*(N_i) \).

The main Theorem of section 6 shows that both \( \left( \prod_{j=1}^{p} (f_j + i0)^{k_j} \right)_{\lambda \in \mathbb{C}^p} \) and \( \mathcal{R}_\pi \left( \prod_{j=1}^{p} (f_j + i0)^{k_j} \right) \) are WF-holonomic.

Example 0.1. The Feynman propagator on \( \mathbb{R}^{3+1} \) has the form \( G = C(Q + i0)^{-1} \) where \( Q \) is the quadratic form of signature \( (1,3) \) and its wave front set is contained in the union of the conormal \( N^*\{Q = 0\} \setminus \{0\} \) of the cone \( \{Q = 0\} \setminus \{0\} \) (with vertex at the origin removed) and the conormal of the origin \( T^*_0[\mathbb{R}^{3+1}] = N^*(\{0\}) \).

It follows that \( G \) is WF-holonomic.

In the second part of our paper, we apply all results of the first part to prove the existence in Theorem [10.1] of renormalization maps \( (\mathcal{R}_M)_{\mu \in \mathbb{N}} \) compatible with the axioms [10.1] following our philosophy of analytic continuation explained at the beginning of the introduction. Let us explain the central novel feature of our approach: unlike Borchers' work, then we would be forced to subtract divergences in a hierarchical manner using either the Stueckelberg–Bogoliubov renormalization group or the Bogoliubov R-operation since renormalization of Feynman amplitudes must take into account subtle phenomena such as nested subdivergences, overlapping divergences...It is well known that a na"ive subtraction of all poles would not satisfy the axiom of causality in [10.1] However, the effect of introducing many regularization parameters resolves the singularities and using the generalized decomposition in [27], it is sufficient to subtract all singular parts all at once as done in our main Theorem [10.1]. To conclude our paper, we show that unlike the methods of Brunetti–Fredenhagen [10] and of our thesis [17], analytic techniques make no use of partitions of unity which shows that our meromorphic renormalization is functorial when restricted to a category \( \mathbf{M}_{\text{ac}} \) defined in subsection [5.1] whose objects are geodesically convex analytic.
Lorentzian spacetimes $(M, g)$ equipped with a Feynman propagator $G$, this functoriality emphasizes the local character of our renormalization techniques.

0.0.2. Future projects. In the sequel of the present paper [18], we will relate our meromorphic regularization techniques with the renormalization group of Bogoliubov, discuss the specific examples of static spacetimes where our renormalization can be made global using the Wick rotation and finally, more importantly, we plan to discuss important extensions of our results to the case of smooth globally hyperbolic spacetimes following suggestions of C. Guillarmou.

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Part I: Analytic Continuation Techniques.

This part forms the analytical core of our paper since all techniques like “dimensional regularization” in quantum field theory rely more or less on the same idea of analytic continuation: we introduce some parameter $\lambda$ that will smooth out singularities of Feynman propagators then we show that all quantities depend meromorphically in the complex parameter $\lambda$. In mathematics, this is related to Atiyah’s approach [2] to the problem of division of distributions and also the analytic continuation techniques described in [7] based on the existence of Bernstein Sato polynomials.

0.1. Meromorphic functions.

0.1.1. Meromorphic functions in several variables. Before we move on, let us recall basic facts about meromorphic functions in several complex variables. To define meromorphic functions in several variables, we first need to define the notion of thin set. A set $Z \subset \Omega$ is called a thin set if for all $x \in Z$, there is some neighborhood $V_x$ of $x$ such that $(V_x \cap Z) \subset \{g = 0\}$ for some non zero holomorphic function $g$ defined on $V_x$. A function $f$ is meromorphic on $\Omega$ if there exists a thin set $Z \subset \Omega$ such that $f$ is holomorphic on $\Omega \setminus Z$ and near any point $x \in \Omega$, there is some neighborhood $V_x$ of $x$ s.t. $f|_{V_x \setminus Z} = \frac{\psi}{\varphi}$ where $(\varphi, \psi)$ are holomorphic on $V_x$. However in meromorphic regularization in QFT, we encounter more restrictive classes of meromorphic functions.

0.1.2. Meromorphic functions with linear poles. In our paper, we call meromorphic functions of several variables $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p$ have polar singularities along countable union of affine hyperplanes of certain types. They are meromorphic functions with linear poles in the terminology of Guo–Paycha–Zhang [27].

Consider the dual space $(\mathbb{C}^p)^*$ of $\mathbb{C}^p$ where each element $L \in (\mathbb{C}^p)^*$ defines a linear map $L : \lambda \in \mathbb{C}^p \mapsto L(\lambda)$. Consider the lattice of covectors with integer coefficients $\mathbb{N}^p \subset (\mathbb{C}^p)^*$ then to every element $L \in \mathbb{N}^p$, consider the linear map $L : \lambda \in \mathbb{C}^p \mapsto L(\lambda)$.
Definition 0.3. Let \( k = (k_1, \ldots, k_p) \) be some element in \( \mathbb{Z}^p \), then a germ of meromorphic function \( f \) at \( k \) has linear poles if there are \( m \) vectors \( (L_i)_{1 \leq i \leq m} \in (\mathbb{N}^p)^m \) in the lattice \( \mathbb{N}^p \), such that

\[
(\prod_{i=1}^{m} L_i(i, + k)) f
\]
is a holomorphic germ at \( k = (k_1, \ldots, k_p) \in \mathbb{C}^p \). An element \( \frac{1}{\prod_{i=1}^{m} L_i(i, + k)} \) is called a simplicial fraction of order \( m \) at \( k \).

Geometrically such meromorphic germ \( f \) is singular along \( m \) affine hyperplanes of equation \( \{ \lambda \in \mathbb{C}^p \text{ s.t. } L_i(\lambda + k) = 0 \} \) intersecting at point \( k = (k_1, \ldots, k_p) \) with integer coordinates in \( \mathbb{C}^p \).

0.1.3. Distributions depending meromorphically on extra parameters. The core of our analytic regularization method in position space is the concept of distribution depending holomorphically (resp meromorphically) w.r.t. some parameter \( \lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p \) introduced in \( \mathbb{CP} \).

Definition 0.4. Let \( U \) be an open set in a smooth oriented manifold \( M \) and \( \Omega \) an open subset of \( \mathbb{C}^p \). Then a family \( (t_\lambda)_{\lambda \in \Omega} \) is holomorphic (resp meromorphic) with value distribution if for all test function \( \varphi \in \mathcal{D}(U) \), \( \lambda \in \Omega \mapsto t_\lambda(\varphi) \in \mathbb{C} \) is holomorphic (resp meromorphic) in \( \lambda \in \Omega \).

If \( (t_\lambda)_{\lambda} \) depends holomorphically on \( \lambda \in \Omega \subseteq \mathbb{C}^p \) with value \( \mathcal{D}' \), let \( \gamma = \gamma_1 \times \cdots \times \gamma_p \) be a cartesian product where each \( \gamma_i \) is a continuous curve in \( \mathbb{C} \), then we can define weak integrals \( \int_{\gamma \subseteq \mathbb{C}^p} d\lambda t_\lambda \) as limits of Riemann sums which converge to some element in \( \mathcal{D}' \) since for all test function \( \varphi \in \mathcal{D} \), the element \( \int_{\gamma} d\lambda t_\lambda(\varphi) \) exists as a limit of Riemann sums by continuity of \( \lambda \in \gamma \mapsto t_\lambda(\varphi) \).

0.1.4. A gain of regularity: when weak holomorphicity becomes strong holomorphicity. Now we give an easy

Proposition 0.1. Let \( U \) be an open set in \( \mathbb{R}^n \), \( \Omega \subseteq \mathbb{C}^p \), \( (t_\lambda)_{\lambda \in \Omega} \) a holomorphic family of distributions in \( \mathcal{D}'(U) \). Then near every \( z \in \Omega \), \( t_\lambda \) admits a Laurent series expansion \( t_\lambda = \sum_\alpha (\lambda - z)^\alpha t_\alpha \) where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) and each coefficient \( t_\alpha \) is a distribution in \( \mathcal{D}'(U) \) such that for all test function \( \varphi \), \( \sum_\alpha (\lambda - z)^\alpha t_\alpha(\varphi) \) converges as power series near \( z \).

Proof. Without loss of generality assume that \( z = 0 \). It suffices to observe that by weak holomorphicity of \( t \) and the multidimensional Cauchy’s formula \( \text{[20, p. 3]} \) for any polydisk \( D_1 \times \cdots \times D_p \) such that \( \partial D_i \) is a circle surrounding \( z_i \), for all test function \( \varphi \in \mathcal{D}(U) \):

\[
(7) \quad t_\lambda(\varphi) = \frac{1}{(2\pi i)^p} \int_{\partial D_1 \times \cdots \times \partial D_p} \frac{t_\lambda(\varphi)dz_1 \wedge \cdots \wedge dz_p}{(z_1 - \lambda_1) \cdots (z_p - \lambda_p)}.
\]

For all test function, set \( t_\alpha(\varphi) = \frac{\alpha!}{(2\pi i)^p} \int_{\partial D_1 \times \cdots \times \partial D_p} \frac{t_\lambda(\varphi)dz_1 \wedge \cdots \wedge dz_p}{(z_1 - \lambda_1)^{\alpha_1} \cdots (z_p - \lambda_p)^{\alpha_p}} \), then \( t_\alpha \) is linear on \( \mathcal{D}(U) \). Let us prove it defines a genuine distribution. By a simple application of the uniform boundedness principle, for every compact \( K \subset U \) there exist a \( C > 0 \) and some continuous seminorm \( P \) for the Fréchet topology of \( \mathcal{D}_K(U) \) such that:

\[
(8) \quad \forall \varphi \in \mathcal{D}_K(U), \sup_{\lambda \in \partial D_1 \times \cdots \times \partial D_p} |t_\lambda(\varphi)| \leq CP(\varphi).
\]

Assuming that all discs \( \partial D_i \) have radius \( r \), it immediately follows that \( t_\alpha \) satisfies a distributional version of Cauchy’s bound:

\[
(9) \quad \forall \varphi \in \mathcal{D}_K(U), |t_\alpha(\varphi)| \leq \frac{\alpha!}{r^{\alpha}} CP(\varphi).
\]
This immediately implies that \((t_\alpha)_\alpha\) are distributions and also that the power series 
\[\sum_\alpha \lambda^\alpha t_\alpha(\varphi)\] converges near 0 \(\in\Omega\). \(\square\)

0.1.5. Meromorphic functions with linear poles with value distribution. In the present
work, we deal with families of distributions \((t_\lambda)_{\lambda \in \mathbb{C}^p}\) in \(\mathcal{D}'(U)\) depending meromor-
phically on \(\lambda \in \mathbb{C}^p\) with linear poles.

**Definition 0.5.** A family of distributions \((t_\lambda)_{\lambda \in \mathbb{C}^p}\) in \(\mathcal{D}'(U)\) depends meromor-
phically on \(\lambda \in \mathbb{C}^p\) with linear poles if for every \(x \in U\), there is a neighborhood \(U_x\) of \(x\), a collection \((L_i)_{1 \leq i \leq m} \in (\mathbb{N}^p)^m \subset (\mathbb{C}^p)^m\) of linear functions with integer coefficients on \(\mathbb{C}^p\) such that for any element \(z = (z_1, \ldots, z_p) \in \mathbb{Z}^p\), there is a neighborhood \(\Omega \subset \mathbb{C}^p\) of \(z\), such that

\[
\lambda \in \Omega \mapsto \prod_{i=1}^m (L_i(\lambda + z)) t_\lambda
\]

is holomorphic with value distribution.

The above expansion is a useful substitute to the Laurent series expansion in
the one variable case. In particular, \((\prod_{i=1}^m L_i(\lambda + z)) t_\lambda|_{U_x}\) is a holomorphic germ
near \(z\) with value distribution. Locally near any element \(z = (z_1, \ldots, z_p) \in \mathbb{Z}^p\), the
polar set of \(t\) is the union of exactly \(m\) affine hyperplanes.

0.2. The fundamental example of hypergeometric distributions. Next, we
will study the fundamental example of such analytic continuation procedure for
the simplest kind of hypergeometric distributions, we work in \(\mathbb{R}^n\) with coordinates
\((y_1, \ldots, y_n)\):

**Lemma 0.1.** Let \(\Gamma \subset \mathbb{R}^n\) be a quadrant \(\cap_{1 \leq i \leq n} \{y_i \geq 0\}\) for \(\varepsilon \in \{-1,1\}^n\). The
family of distributions \((t_\mu)_\mu\) defined as

\[
t_\mu = 1_\Gamma y_1^{\mu_1} \cdots y_n^{\mu_n} \text{ for } Re(\mu_i) > -1
\]

extends meromorphically in \(\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n\) with polar set \(\cup_{1 \leq i \leq n, k \in \mathbb{N}} \{\mu_i + k = 0\}\).

**Proof.** The proof follows from an easy integration by parts argument, for all test
function \(\varphi \in \mathcal{D}(\mathbb{R}^n)\), for \(-1 < Re(\mu_i) \leq 0\) and for any integers \((k_1, \ldots, k_n) \in (\mathbb{N}^*)^n\):

\[
t_\mu(\varphi) = \int_{\Gamma} dy_1 \cdots dy_n y_1^{\mu_1} \cdots y_n^{\mu_n} \varphi(y_1, \ldots, y_n)
\]

\[
= \left(\prod_{i=1}^n \frac{1}{\mu_i + k_i} \cdots \frac{1}{\mu_i + 1}\right) \int_{\Gamma} dy_1 \cdots dy_n y_1^{\mu_1 + k_1} \cdots y_n^{\mu_n + k_n} \varphi(y_1, \ldots, y_n)
\]

where both sides are holomorphic in the domain \(-1 < Re(\mu_i)\). However for \(-k_i - 1 < Re(\mu_i)\), the right hand side is well defined and meromorphic with poles at
\(\mu_i = -k_i, \ldots, \mu_i = -1\). It is thus an analytic continuation of the distribution \((t_\mu)_\mu\)
on the right hand side which yields the desired result. \(\square\)

Moreover, the distribution \((t_\mu)_\mu\) exhibits an interesting separation of variables
property since it admits a Laurent series expansion around elements of the form
\((k_1, \ldots, k_n) \in (\mathbb{N}^*)^n\) as the product of \(n\) meromorphic functions in each variable
\(\mu_i\):

**Lemma 0.2.** Let us consider again the distribution \(t_\mu\) of Lemma 0.1. Near any
element \((-k_1, \ldots, -k_n) \in \mathbb{C}^n, k_i \in \mathbb{N}^*,\) the polar set of the family \((t_\mu)_\mu\) is a divisor
with normal crossings \(\cup_{1 \leq i \leq n} \{\mu_i = -k_i\}\) i.e. it is the union of \(n\) affine coordinates
\(y_1, \ldots, y_n\).
hyperplanes and \( t_\mu \) admits a Laurent series expansion in \((\mu_i + k_i), 1 \leq i \leq n\) of the form

\[
t_\mu = \sum_\alpha u_\alpha \prod_{i=1}^{n}(\mu_i + k_i)^{\alpha_i - 1}.
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) is a multi-index and \( u_\alpha \in \mathcal{D}'(U) \). In particular, \( t \) is meromorphic with linear poles with value \( \mathcal{D}'(U) \).

**Proof.** Near an element \((-k_1, \ldots, -k_n) \in (-\mathbb{N}^*)^n \subset \mathbb{C}^n\), \( t_\mu \) writes as a product

\[
t_\mu = \prod^n_{i=1} \frac{u_\mu}{\mu_i + k_i}
\]

of a simplicial fraction \( \prod_{i=1}^n \frac{1}{\mu_i + k_i} \) with the distribution \( u_\mu \) defined as:

\[
u_\mu(\varphi) = \left( \prod_{i=1}^n \frac{1}{\mu_i + k_i} \right) \int \prod_{\alpha} dy_1 \ldots dy_n y_1^{\mu_1 + k_1} \ldots y_n^{\mu_n + k_n} \varphi(y_1, \ldots, y_n)
\]

which is a distribution depending holomorphically on \( \mu \) provided that for all \( i \in \{1, \ldots, n\} \), \(-k_i - 1 < \text{Re}(\mu_i) < -k_i + 1\). It means that for every test function \( \varphi \), \( \mu \mapsto u_\mu(\varphi) \) is a holomorphic germ near \((-k_1, \ldots, -k_n)\).

We restrict to a small polydisk near \((-k_1, \ldots, -k_n)\) and by Lemma 0.1, \( u_\mu \) admits a power series expansion \( u_\mu = \sum_\alpha (\mu + k)^\alpha u_\alpha \) near \((-k_1, \ldots, -k_n)\) where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) is a multi-index and \( u_\alpha \) are distributions. Finally, we deduce that

\[
t_\mu = \left( \prod_{i=1}^n \frac{1}{\mu_i + k_i} \right) \sum_\alpha (\mu + k)^\alpha u_\alpha
\]

\[
= \sum_\alpha (\prod_{i=1}^n (\mu_i + k_i)^{\alpha_i - 1}) u_\alpha.
\]

\( \square \)

1. The meromorphic family \( \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)_{\lambda \in \mathbb{C}^p} \).

Let \( U \) be some open set in \( \mathbb{R}^n \) and \( f_1, \ldots, f_p \) be some real valued analytic functions on \( U \). The goal of the first part of our paper is to show that the family of distributions \( \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)_{\lambda \in \mathbb{C}^p} \) depends meromorphically on \( \lambda \), our proof relies on Hironaka’s resolution of singularities. Let us quote the content of the resolution theorem as it is stated in Atiyah’s paper [2] p. 147:

**Theorem 1.1.** Let \( F \neq 0 \) be a real analytic function defined in a neighborhood of \( 0 \in \mathbb{R}^n \). Then there exists an open neighborhood \( U \) of \( 0 \), a real analytic manifold \( \tilde{U} \) and a proper analytic map \( \varphi : \tilde{U} \to U \) such that

1. \( \varphi : \tilde{U} \setminus \{ F \circ \varphi = 0 \} \to U \setminus \{ F = 0 \} \) is an isomorphism,
2. for each \( p \in U \), there are local analytic coordinates \( (y_1, \ldots, y_n) \) centered at \( p \) so that, locally near \( p \), we have

\[ F \circ \varphi = \varepsilon \prod y_i^{k_i} \]

where \( \varepsilon \) is an invertible analytic function and \( k_i \) are non-negative integers.

This Theorem is central for QFT applications since it explains why regularized Feynman amplitudes should depend meromorphically on the regularization parameter \( \lambda \).
Theorem 1.2. Let $U$ be some open set in $\mathbb{R}^n$ and $(f_1, \ldots, f_p)$ be some real valued analytic functions on $U$. For every $(k_1, \ldots, k_p) \in \mathbb{N}^p$, the map $(\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p \mapsto \prod_{j=1}^p \log^{k_j}(f_j + i0)(f_j + i0)^{\lambda_j}$ is meromorphic in $\mathbb{C}^p$ with value distribution.

Proof. We closely follow Atiyah's exposition [2] based on Hironaka's Theorem [11] of resolution of singularities. The proof is essentially local hence we might reduce to a smaller open set $U$ on which Theorem [11] applies.

Step 1 note that $\prod_{j=1}^p \log^{k_j}(f_j + i0)(f_j + i0)^{\lambda_j} = \frac{d}{dx_j^{k_1}} \cdots \frac{d}{dx_j^{k_p}} \prod_{j=1}^p (f_j + i0)^{\lambda_j}$, therefore it suffices to prove the claim for $\prod_{j=1}^p (f_j + i0)^{\lambda_j}$.

Step 2 recognize that for complex $\lambda$, we choose the determination of the log which gives the identity

$$f + i0)^{\lambda} = 1_{(f \geq 0)} f^{\lambda} + 1_{(f \leq 0)} e^{i\pi \lambda} (-f)^{\lambda}.

Step 3 therefore by expanding brutally the product:

$$\prod_{j=1}^p (f_j + i0)^{\lambda_j} = \prod_{j=1}^p \left(1_{(f_j \geq 0)} f_j^{\lambda_j} + 1_{(f_j \leq 0)} e^{i\pi \lambda_j} (-f_j)^{\lambda_j}\right)$$

we may reduce to the problem of meromorphic extension of a product of the form

$$\prod_{j=1}^p g_{j}^{\lambda_j} 1_{\Gamma},$$

where $(g_j)$ are real analytic, $1_{\Gamma}$ is the indicator function of the domain $\Gamma = \bigcap_{1 \leq j \leq p} \{g_j \geq 0\}$ and all functions $g_j \geq 0$ on $\Gamma$.

Step 4. Following Atiyah, we shall apply Hironaka’s Theorem [11] to the function $F = \prod_j g_j$ to resolve simultaneously the collection of real analytic functions $(g_j)$. Assume $\forall j, g_j \neq 0$. Denote by $\Sigma = \bigcup_{j \in \{1, \ldots, p\}} \{g_j = 0\}$ the zero set of all the above functions. Then there is a proper analytic map $\varphi : \hat{U} \to U$, coordinate functions $(y_i)_i$ on $\hat{U}$ such that $\varphi^{-1}(\Sigma) = \{\prod_i y_i = 0\}$, $\varphi$ is a diffeomorphism from $\hat{U} \setminus \{\prod_i y_i = 0\} \to U \setminus \Sigma$ and for all $j$, every pulled–back function $\varphi^* g_j$ has the form $\varepsilon(y) y^{\alpha_j}$ where $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$ is a multi–index, $y^{\alpha_j} = \prod_{i=1}^n y_i^{\alpha_{ji}}$ and $\varepsilon$ does not vanish in some neighborhood of 0.

Step 5 the above pulled–back function $\varphi^* g_j$ reads $\varphi^* g_j = \varepsilon_j y^{\alpha_j}$ hence the pulled–back product $\varphi^* \left( \prod_{j=1}^p g_j^{\lambda_j} 1_{\Gamma} \right)$ can be further be expressed as a finite sum of products of the form:

$$\prod_{j=1}^p (\varepsilon_j y^{\alpha_j})^{\lambda_j} 1_{\Gamma} = \prod_{j=1}^p \varepsilon_j^{\lambda_j} \prod_{j=1}^p y^{\alpha_j^{\lambda_j}} 1_{\Gamma}, \Gamma = \{y^{\alpha_j} \geq 0, \forall j\}.$$

Dropping the factor $\prod_{j=1}^p \varepsilon_j^{\lambda_j}$ which does not vanish near 0 and is analytic for all $\lambda \in \mathbb{C}^p$ we are reduced to study the singular term:

$$\left( y^{\sum_{j=1}^p \alpha_j^{\lambda_j}} 1_{\Gamma} \right) = \prod_{j=1}^p y^{\alpha_j^{\lambda_j}} 1_{\Gamma}, \Gamma = \{y^{\alpha_j} \geq 0, \forall j\}$$

where for every $j \in \{1, \ldots, p\}$, $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$ is a multi–index and $\lambda_j$ a complex number. The above distribution is a typical example of hypergeometric distributions. And it is immediate to prove that the above expression is meromorphic in $\lambda$.
with value $\mathcal{D}'(U)$ by successive integration by parts as in Lemma 0.1 (see also [23]), or by the existence of the functional equation

$$\frac{d^n}{dy^n} \left( y^{\sum_{j=1}^{p} \alpha_j} \lambda \right)_1 \Gamma = \prod_{i=1}^{n} \left( \frac{d}{dy_i} \right)^{\beta_i} \left( \Gamma \left( \sum_{j=1}^{p} \lambda_j \alpha_i^j \right) \right) \left( y^{\sum_{j=1}^{p} \alpha_j \lambda_j - \beta_1} \right),$$

and the poles come from the poles at negative integers of the Euler $\Gamma$ function.

Step 6 We admit that $\Sigma = \{ g_j = 0, h_j = 0 \}$ has null measure as a consequence of Lemma 1.2.

Step 7 Let $u_\lambda$ denote the pulled-back distribution $\varphi^* \prod_{j=1}^{p} g_j^{\lambda_j} \lambda \Gamma$ on $\tilde{U}$. Then for $Re(\lambda_j)$ large enough both distributions $\varphi^* u_\lambda$ and $\prod_{j=1}^{p} g_j^{\lambda_j} \lambda \Gamma$ are holomorphic in $\lambda$ and coincide on $U \setminus \Sigma$. However when $Re(\lambda_j)$ are large enough, both distributions are locally integrable and since $\Sigma$ has null measure, the equality $\varphi^* u_\lambda = \prod_{j=1}^{p} g_j^{\lambda_j} \lambda \Gamma$ holds in $L^1_{loc}(U)$ hence in $\mathcal{D}'(U)$ and both sides are holomorphic in $\lambda$ with value $\mathcal{D}'(U)$. Finally we proved in Step 5 that $\left( y^{\sum_{j=1}^{p} \alpha_j \lambda_j} \lambda \Gamma \right) \in \mathcal{D}'(\tilde{U})$ extends meromorphically in $\lambda \in \mathbb{C}^p$ hence so does $\varphi^* u_\lambda = \prod_{j=1}^{p} g_j^{\lambda_j} \lambda \Gamma$. By uniqueness of the analytic continuation process, this proves the claim. \hfill $\square$

1.0.1. More general examples of hypergeometric distributions. The next result refines Theorem 1.2 and concerns the location of the poles of the meromorphic continued distributions.

**Lemma 1.1.** Let us work in $\mathbb{R}^n$ with coordinates $(y_1, \ldots, y_n)$. Consider the meromorphic family of distributions:

$$\left( y^{\sum_{j=1}^{p} \alpha_j \lambda_j} \lambda \Gamma \right) = 1 \Gamma \prod_{i=1}^{n} y_i^{\sum_{j=1}^{p} \alpha_j^i \lambda_i}, \quad \Gamma = \{ y^{\sum_{j=1}^{p} \alpha_j} \geq 0, \forall y \},$$

where for every $j \in \{1, \ldots, p\}$, $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j) \in \mathbb{N}^n$ is a multi-index and $\lambda_i \in \mathbb{C}$, $1 \leq j \leq p$. Then define the collection $(\mu_1)_{1 \leq i \leq n} \in (\mathbb{N}^p)^n$ of linear functions on $\mathbb{C}^p$:

$$\left( \mu_i : \lambda \in \mathbb{C}^p \mapsto \sum_{j=1}^{p} \alpha_i^j \lambda_j \right)_{1 \leq i \leq n}$$

then:

1. the polar set $Z$ of the family $\left( y^{\sum_{j=1}^{p} \alpha_j \lambda_j} \lambda \Gamma \right)$ is contained in the union of affine hyperplanes

$$Z = \bigcup_{1 \leq i \leq n, k \in \mathbb{N}^p} \{ \lambda \ s.t. \ \mu_i(\lambda) = -k \},$$

2. in some neighborhood of any element $z = (z_1, \ldots, z_p) \in \mathbb{Z}^n$ there is a neighborhood $\Omega \subset \mathbb{C}^p$ of $z$, some distributions $(u_\beta)_{\beta \in \mathbb{N}^p}$ in $\mathcal{D}'(U)$ such that:

$$\forall \lambda \in \Omega, \left( y^{\sum_{j=1}^{p} \alpha_j \lambda_j} \lambda \Gamma \right) = \sum_{\beta \in \mathbb{N}^p} \prod_{i=1}^{n} \left( \mu_i(\lambda + z) \right)^{\beta_i - 1} u_\beta.$$

**Proof.** It is an easy consequence of Lemmas 0.1 and 1.2 for $(\mu_i = \sum_{j=1}^{p} \alpha_i^j \lambda_j)_{1 \leq i \leq n}$. \hfill $\square$
The above result yields that the hypergeometric distributions \( g^n_{\sum_{j=1}^p \alpha_j \lambda_j \Gamma_j} \) depend \textbf{meromorphically of} \( \lambda \) with \textbf{linear poles}. Finally, we can state an extended version of our main Theorem

\textbf{Theorem 1.3.} Let \( U \) be some open set in \( \mathbb{R}^n \) and \( (f_1, \ldots, f_p) \) be some real valued analytic functions on \( U \). Then the family of distributions \( \prod_{j=1}^p (f_j + i0)^{\lambda_j} \) depends \textbf{meromorphically on} \( \lambda \) with linear poles.

\textbf{Proof.} In fact we prove the following stronger result: for all \( x \in U \), there is a neighborhood \( U_x \) of \( x \), \( n2^p \) linear functions with integer coefficients \( (\mu_{i,\epsilon})_{1 \leq i \leq n, \epsilon \in \{-1,1\}^p} \) s.t. for all \( z \in \mathbb{Z}^p \), there is a neighborhood \( \Omega \subset \mathbb{C}^p \) of \( z \) and distributions \( (u_{\beta,\epsilon}), \beta \in \mathbb{N}^n, \epsilon \in \{-1,1\}^p \) such that

\begin{equation}
\prod_{j=1}^p (f_j + i0)^{\lambda_j}|_{U_x} = \sum_{\epsilon \in \{-1,1\}^p, \beta} u_{\beta,\epsilon} \prod_{i=1}^n \mu_{i,\epsilon}(\lambda + \epsilon z)^{\beta_i}.
\end{equation}

The result follows from Step 3 of the proof of Theorem 1.2 where we decomposed \( (f_j + i0)^{\lambda_j} \) as a sum of \( 2^p \) elementary distributions of the form \( \prod_{j=1}^p g^{\lambda_j}_j \Gamma_j \) where every elementary distribution \( \prod_{j=1}^p g^{\lambda_j}_j \Gamma_j \) is the pushforward by the resolution \( \varphi \) of a hypergeometric distribution of the form studied in Lemma 1.1.

The main result of the above Theorem is the existence of a natural Laurent series expansion in \( (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p \) for the family \( \prod_{j=1}^p (f_j + i0)^{\lambda_j} \).

1.0.2. \textit{Appendix to section 1: analytic sets have measure zero.} We give here the key easy Lemma which states that the zero set of a non zero real valued analytic function has measure zero on \( U \).

\textbf{Lemma 1.2.} Let \( F \) be a nonzero real analytic function on \( U \subset \mathbb{R}^n \) then \( \{ F = 0 \} \) has zero Lebesgue measure.

\textbf{Proof.} The proof can be found in Federer [22], but we sketch a simple proof following Atiyah [2] based on Hironaka’s resolution of singularities. It suffices to show that near any point \( x \in \{ F = 0 \} \cap U \) there is some neighborhood \( V_x \) of \( x \) s.t. \( V_x \cap \{ F = 0 \} \) has measure zero. Then it follows by paracompactness of \( U \) that \( \{ F = 0 \} \cap U \) can be covered by a countable number of zero measure sets hence it has measure zero! Locally near any \( x \in U \cap \{ F = 0 \} \), there is a proper analytic map \( \varphi : U \subset \mathbb{R}^n \to U \) such that the set \( \Sigma = \varphi^{-1}(\{ F = 0 \}) \) is contained in the coordinate cross of the form \( D = \{ \prod_{i=1}^n t_i = 0 \} \) and the set \( \Sigma \subset D \) has zero measure since \( D \) has measure zero. Therefore by [23, Proposition 1.3 p. 30], its image by the \( C^1 \) map \( \varphi \) has measure zero in particular it contains \( \{ F = 0 \} \subset \varphi(D) \) which therefore has zero measure.

\section{The main construction.}

The main problem of renormalization in QFT is to define \( \prod_{j=1}^p (f_j + i0)^{-\epsilon_j} \) for values of \( \epsilon_j \) which are \textbf{positive integers} which boils down to evaluate the \textbf{meromorphic} family \( \prod_{j=1}^p (f_j + i0)^{\lambda_j} \) exactly at its poles. Motivated by exciting recent works of Paycha–Guo–Zhang [27], we follow in this section their definition of regularization and construct an abstract framework in which one can regularize meromorphic functions with integral linear poles. This construction will be used in the second part of our paper to renormalize quantum field theories. The philosophy is to introduce as many complex variables in our problem as there are propagators and renormalize with meromorphic functions with integral linear poles of an arbitrary number of variables.
2.1. **Algebras of cylindrical functions.** Our goal is to construct an algebra of functions \( \mathcal{M}_k(\mathbb{C}^N) \) depending on arbitrary number of complex variables \((\lambda_1, \ldots, \lambda_p)\) which contains all meromorphic germs obtained by meromorphic regularization of the first section. More precisely, for all real analytic functions \((f_1, \ldots, f_p)\) on some open set \(U\), for all test function \(\varphi \in \mathcal{D}(U)\), the meromorphic germ \(\lambda \mapsto \prod_{j=1}^p (f_j + i0)^{\lambda_j} \varphi\) at \((k_1, \ldots, k_p)\) whose existence is guaranteed by Theorem 1.3 is contained in the algebra \(\mathcal{M}_k(\mathbb{C}^N)\). We also construct a subalgebra \(\mathcal{O}_k(\mathbb{C}^N)\) of \(\mathcal{M}_k(\mathbb{C}^N)\) which contains all regular elements i.e. holomorphic germs \(f(\lambda_1, \ldots, \lambda_p) \in \mathcal{M}_k(\mathbb{C}^N)\) whose limit exists at \((k_1, \ldots, k_p)\).

Let us consider the space \(\mathbb{C}^N\) of sequences of complex numbers and a fixed sequence of integers \(k \in \mathbb{Z}^N\). We construct an algebra of cylindrical functions on \(\mathbb{C}^N\) as follows. Let \(p\) be a fixed integer. Let \(k \leq p = (k_1, \ldots, k_p)\) be the first \(p\) coefficients of the sequence \(k\) viewed as an element of \(\mathbb{C}^m\) then we define two algebras \(\mathcal{O}_{k \leq p}(\mathbb{C}^p)\) and \(\mathcal{M}_{k \leq p}(\mathbb{C}^p)\) of germs of functions at \(k \leq p = (k_1, \ldots, k_p)\).

**Definition 2.1.** \(\mathcal{O}_{k \leq p}(\mathbb{C}^p)\) is the algebra of holomorphic germs \(f\) at \(k \leq p\). \(\mathcal{M}_{k \leq p}(\mathbb{C}^p)\) is the algebra of meromorphic germs at \(k \leq p\) with linear poles, \(f\) belongs to \(\mathcal{M}_{k \leq p}(\mathbb{C}^p)\) if there are \(m\) integral vectors \((L_i)_{1 \leq i \leq m} \in (\mathbb{N}^p)^m\) such that

\[
\lambda \mapsto f(\lambda) \prod_{i=1}^m L_i(\lambda + k)
\]

is a holomorphic germ at \(k \leq p = (k_1, \ldots, k_p) \in \mathbb{C}^p\).

For all integer \(p\), a germ \(f(\lambda_1, \ldots, \lambda_p)\) can always be viewed as a function of the \(p+1\) variables \((\lambda_1, \ldots, \lambda_p+1)\) which does not depend on the last variable \(\lambda_{p+1}\). It follows that there are obvious inclusions \(\mathcal{O}_{k \leq p}(\mathbb{C}^p) \hookrightarrow \mathcal{O}_{k \leq p+1}(\mathbb{C}^{p+1})\) and \(\mathcal{M}_{k \leq p}(\mathbb{C}^p) \hookrightarrow \mathcal{M}_{k \leq p+1}(\mathbb{C}^{p+1})\) which imply the existence of the inductive limits \(\mathcal{O}_k(\mathbb{C}^N) = \lim_{\rightarrow} \mathcal{O}_{k \leq p}(\mathbb{C}^p)\) and \(\mathcal{M}_k(\mathbb{C}^N) = \lim_{\rightarrow} \mathcal{M}_{k \leq p}(\mathbb{C}^p)\). It is simple to check the following properties.

**Proposition 2.1.** Both \(\mathcal{O}_k(\mathbb{C}^N), \mathcal{M}_k(\mathbb{C}^N)\) are algebras, \(\mathcal{M}_k(\mathbb{C}^N)\) is a \(\mathcal{O}_k(\mathbb{C}^N)\) module and contains \(\mathcal{O}_k(\mathbb{C}^N)\) as a subalgebra.

2.2. **A projector and the factorization property.** By definition of the inductive limit, elements of \(\mathcal{M}_k(\mathbb{C}^N)\) are meromorphic germs with integral linear poles depending on a finite number of variables.

2.2.1. **The notion of independence.** We will say that two elements \((f, g) \in \mathcal{M}_k(\mathbb{C}^N)^2\) are independent if they depend on different sets of variables. It follows that if \((f, g)\) are independent, then they satisfy condition (c) of [27, Theorem 4.4].

2.2.2. **Subtraction of poles and projectors.** Recall that our final goal is to evaluate \(\prod_{j=1}^p (f_j + i0)^{k_j}\) for values of \(k_j\) which are negative integers which requires to subtract the poles of elements from \(\mathcal{M}_k(\mathbb{C}^N)\). An elegant way to reformulate the operation of subtraction of poles is in terms of a projection

\[
\pi : \mathcal{M}_k(\mathbb{C}^N) \mapsto \mathcal{O}_k(\mathbb{C}^N).
\]

2.2.3. **The factorization condition.**

**Definition 2.2.** A projection \(\pi : \mathcal{M}_k(\mathbb{C}^N) \mapsto \mathcal{O}_k(\mathbb{C}^N)\) satisfies the factorization condition if for all \((f, g) \in \mathcal{M}_k(\mathbb{C}^N)^2\), if \(f\) and \(g\) are independent then

\[
\pi(fg) = \pi(f)\pi(g).
\]
2.3. The main existence Theorem. In this subsection, we explain the existence of a projection which satisfies the factorization condition. This is exactly the content of [27 Theorem 4.4]. Let us state their Theorem in our notations:

**Theorem 2.1. Guo–Paycha–Zhang**

Let $Q$ be the quadratic form defined on all the vector spaces $\mathbb{C}^p$ for $p \in \mathbb{N}$ as $Q(z_1, \ldots, z_p) = \sum_{i=1}^{p} |z_i|^2$.

(1) For all $p \in \mathbb{N}$, we have the direct sum decomposition

$$\mathcal{M}_{k \otimes p}(\mathbb{C}^p) = \mathcal{O}_{k \otimes p}(\mathbb{C}^p) \oplus \mathcal{M}_{-k \otimes p}(\mathbb{C}^p)$$

where the space $\mathcal{M}_{-k \otimes p}(\mathbb{C}^p)$ contains all singular functions, in particular any element $f = \frac{h}{L_{k_1} \ldots L_{k_m}} \in \mathcal{M}_{k \otimes p}(\mathbb{C}^p)$ can be written as a sum

$$f = \sum_{i} h_i(\ell_i) + \phi_i(L_{k_1}, \ldots, L_{k_m}, \ell_i)$$

where for each $i$, $(s_1, \ldots, s_{m_i}) \in \mathbb{N}^{m_i}$, the collection of linear forms $(L_{k_1}, \ldots, L_{k_m})$ is a linearly independent subset of $(L_1, \ldots, L_m)$, the collection of linear forms $(\ell_1, \ldots, \ell_{m_i})$ is a basis of the orthogonal complement (for $Q$) of the subspace spanned by the $(L_{k_1}, \ldots, L_{k_m})$, $h_i$ is holomorphic in the independent variables $\ell_i$ so that $\frac{h_i(\ell_i)}{L_{k_1} \ldots L_{k_m}}$ belongs to $\mathcal{M}_{-k \otimes p}(\mathbb{C}^p)$.

(2) The coefficients

$$h_i, \phi_i$$

depend linearly on finite number of partial derivatives of $h$.

(3) Taking a direct limit yields

$$\mathcal{M}_{k}(\mathbb{C}^\mathbb{N}) = \mathcal{O}_k(\mathbb{C}^\mathbb{N}) \oplus \mathcal{M}_{-k}(\mathbb{C}^\mathbb{N})$$

(4) The projection map $\pi: \mathcal{M}_{k}(\mathbb{C}^\mathbb{N}) \rightarrow \mathcal{O}_k(\mathbb{C}^\mathbb{N})$ onto $\mathcal{O}_k(\mathbb{C}^\mathbb{N})$ along the subspace $\mathcal{M}_{-k}(\mathbb{C}^\mathbb{N})$ factorizes on independent functions. If $(f, g) \in \mathcal{M}_{k}(\mathbb{C}^\mathbb{N})^2$ are independent then

$$\pi(f, g) = (f)\pi(g).$$

**Proof.** We refer to [27] for the proof of this beautiful Theorem but will only show the property (2) which explains how to define $\pi$ in an algorithmic fashion closely following the original proof in [27]. Thanks to [27 Lemma 4.1], without loss of generality we can reduce the proof to germs of functions of the type

$$f = \frac{h}{L_{k_1} \ldots L_{k_m}}$$

with $h$ holomorphic, linearly independent linear forms $(L_1, \ldots, L_m)$ and $(s_1, \ldots, s_m)$ positive integers. The system $(L_1, \ldots, L_m, \ell_{m+1}, \ldots, \ell_p)$ is a coordinate system on $\mathbb{C}^p$. Consider a partial Taylor expansion with remainder of $h$ in the first $m$ coordinates $(L_1, \ldots, L_m)$:

$$h = \sum \frac{L_{k_1} \ldots L_{k_p}}{k_1! \ldots k_m!} \partial_{L_{k_1}} \ldots \partial_{L_{k_m}} h(0, \ell_{m+1}, \ldots, \ell_p) + L_{s_1} \ldots L_{s_m} \phi(L_1, \ldots, L_m, \ell_{m+1}, \ldots, \ell_p)$$

where $\phi$ is holomorphic and $k = (k_1, \ldots, k_m) < s = (s_1, \ldots, s_m)$ means that for some $i \in \{1, \ldots, m\}$, $k_i < s_i$ and $k_j \leq s_j, \forall j \neq i$. Then it follows that

$$\frac{h}{L_{s_1} \ldots L_{s_m}} = \sum \frac{1}{k_1! \ldots k_m!} \partial_{L_{k_1}} \ldots \partial_{L_{k_m}} h(0, \ell_{m+1}, \ldots, \ell_p) + \phi(L_1, \ldots, L_m, \ell_{m+1}, \ldots, \ell_p)$$
hence:

\[
\frac{h}{L_1^{s_1} \ldots L_m^{s_m}} = \frac{h}{L_1^{s_1} \ldots L_m^{s_m}} - \sum_{k<s} \frac{1}{k_1! \ldots k_m!} \frac{\partial_{L_{k_1}}^{k_1} \ldots \partial_{L_{k_m}}^{k_m} h(0, \ell_{m+1}, \ldots, \ell_p)}{L_1^{s_1-k_1} \ldots L_m^{s_m-k_m}}.
\]

Theorem 2.2. Let \( U \) be an open set in \( \mathbb{R}^n, \Omega \subset \mathbb{C}^p \) open and \((t_\lambda)_{\lambda \in \Omega}\) a meromorphic family with linear poles at \( k \in \Omega \) with value \( \mathcal{D}'(U) \). Then the family \( \pi(t_\lambda)_\lambda \) defined as

\[
\forall \varphi \in \mathcal{D}(U), \pi(t_\lambda)(\varphi) = \pi(t_\lambda(\varphi))
\]
is holomorphic at \( k \) with value \( \mathcal{D}'(U) \).

Proof. Proposition 1.1 implies that if \( h(\lambda)_\lambda \) is holomorphic in \( \lambda \in \mathbb{C}^p \) with value \( \mathcal{D}'(U) \) then the truncated Laurent series

\[
\sum_{k\geq s} \frac{L_1^{k_1} \ldots L_m^{k_m}}{k_1! \ldots k_m!} \partial_{L_{k_1}}^{k_1} \ldots \partial_{L_{k_m}}^{k_m} h_k(0, \ell_{m+1}, \ldots, \ell_p)
\]
absolutely converge in \( \mathcal{D}'(U) \) by Cauchy’s bound 10. Then dividing the above truncated Laurent series by \( L_1^{s_1} \ldots L_m^{s_m} \) and by definition of the projection \( \pi \) of Theorem 2.1 we find that:

\[
\frac{h}{L_1^{s_1} \ldots L_m^{s_m}} = \frac{h}{L_1^{s_1} \ldots L_m^{s_m}} - \sum_{k<s} \frac{1}{k_1! \ldots k_m!} \frac{\partial_{L_{k_1}}^{k_1} \ldots \partial_{L_{k_m}}^{k_m} h(0, \ell_{m+1}, \ldots, \ell_p)}{L_1^{s_1-k_1} \ldots L_m^{s_m-k_m}}
\]
is also holomorphic in \( \lambda \in \mathbb{C}^p \) with value \( \mathcal{D}'(U) \).

The above Theorem allows us to define a renormalization operator \( \mathcal{R}_\pi \) of the complex powers \( \prod_{j=1}^p (f_j + i0)^{k_j} \) for \( k_j \in -\mathbb{N}^* \) as follows:

**Definition 2.3.** For all test function \( \varphi \in \mathcal{D}(U) \),

\[
\mathcal{R}_\pi(\prod_{j=1}^p (f_j + i0)^{k_j})(\varphi) = \pi(\lambda \mapsto \prod_{j=1}^p (f_j + i0)^{k_j}(\varphi))(k).
\]

2.3.1. The fundamental tensor factorization property. It is immediate by construction that the renormalization operator \( \mathcal{R}_\pi \) satisfies the following factorization identity: let \( U, V \) be open sets in \( \mathbb{R}^{n_1}, \mathbb{R}^{n_2} \) respectively and \( f_1, \ldots, f_p \) (resp \( g_1, \ldots, g_p \)) real analytic functions on \( U \) (resp \( V \)) then

\[
\mathcal{R}_\pi(f_1^{k_1} \ldots f_p^{k_p} g_1^{l_1} \ldots g_p^{l_p}) = \mathcal{R}_\pi(f_1^{k_1} \ldots f_p^{k_p}) \otimes \mathcal{R}_\pi(g_1^{l_1} \ldots g_p^{l_p}).
\]

where the tensor product \( \otimes \) is the exterior tensor product: \( \mathcal{D}'(U) \otimes \mathcal{D}'(V) \mapsto \mathcal{D}'(U \times V) \).

3. \( u = 0 \) Theorem.

3.0.2. Motivation for these Theorems. In QFT, we need to multiply Feynman propagators, which are distributions, in order to define Feynman amplitudes. The control of their wave front sets give sufficient conditions under which one can multiply these distributions. Therefore we are let to study the wave front set of the family \((f + i0)^{k}\). Unfortunately to bound the wave front of the family \((f + i0)^{k}\), we must bound wave front sets of products of distributions which are well defined but fail to satisfy Hörmander’s transversality condition on wave front sets. The \( u = 0 \) Theorem which originates from the work of Iagolnitzer will help us give bounds on
wave front sets of products of distributions $(uv)$ which are well defined but whose wave front set fail to satisfy the transversality condition $WF(u) \cap -WF(v) = \emptyset$ of Hörmander.

3.1. Products in Sobolev spaces. The goal of this part is to recall some well known results on Sobolev spaces. We denote by $H^s(\mathbb{R}^d)$ the usual $L^2$ Sobolev space and $t \in \mathcal{D}'(\mathbb{R}^d)$ belongs to $H^s_{loc}(\mathbb{R}^d)$ if for all test function $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $t\varphi \in H^s(\mathbb{R}^d)$. Recall that the usual multiplication of smooth functions extends naturally to $H^s_{loc}(\mathbb{R}^d) \times H^s_{loc}(\mathbb{R}^d)$ when $s_1 + s_2 \geq 0$. Indeed

**Lemma 3.1.** Let $(u, v) \in H^s_{loc}(\mathbb{R}^d) \times H^s_{loc}(\mathbb{R}^d)$ for $s_1 + s_2 \geq 0$, $s_1 \leq 0 \leq s_2$ then the product $uv$ makes sense in $\mathcal{D}'(\mathbb{R}^d)$ and for all test function $\varphi$, the Fourier transform $\widehat{uv\varphi^2}$ is well defined by an absolutely convergent convolution integral which satisfies the bound:

\[
|\widehat{uv\varphi^2}(\xi)| \leq \int_{\mathbb{R}^d} d^d\eta |\hat{u}(\xi - \eta)\hat{v}\varphi(\eta)| \leq (1 + |\xi|)^{-s_1}\|u\varphi\|_{H^{s_1}}\|v\varphi\|_{H^{s_2}}.
\]

**Proof.** Let $\varphi$ be a test function then from $\widehat{uv\varphi^2} = \hat{u}\hat{v}\varphi$, we deduce the estimates:

\[
|\widehat{uv\varphi^2}(\xi)| \leq \int_{\mathbb{R}^d} d^d\eta |\hat{u}(\xi - \eta)\hat{v}\varphi(\eta)|
\]

\[
\leq \sup_{\eta} (1 + |\xi - \eta|)^{-s_1}(1 + |\eta|)^{-s_2} \int_{\mathbb{R}^d} d^d\eta |(1 + |\xi - \eta|)^{s_1}\hat{u}\varphi(\xi - \eta)(1 + |\eta|)^{s_2}\hat{v}\varphi(\eta)|
\]

\[
\leq \sup_{\eta} \frac{(1 + |\xi - \eta|)^{-s_1}}{(1 + |\eta|)^{s_1}} \|u\varphi\|_{H^{s_1}}\|v\varphi\|_{H^{s_2}}
\]

\[
\leq \frac{(1 + |\xi|)^{-s_1}}{(1 + |\eta|)^{s_1}} \|u\varphi\|_{H^{s_1}}\|v\varphi\|_{H^{s_2}}
\]

The above shows that $\widehat{uv\varphi^2}$ is well defined by an absolutely convergent convolution integral and has polynomial growth in $\xi$. Hence $uv\varphi^2 \equiv \mathcal{F}^{-1}\left(\widehat{uv\varphi^2}\right)$ is a well defined distribution in $\mathcal{E}'(\mathbb{R}^d)$. Now let $(\varphi_j)_j$ be a partition of unity of $\mathbb{R}^d$ such that $\forall j, \varphi_j \in \mathcal{D}(\mathbb{R}^d)$ and $\sum_j \varphi_j^2 = 1$ where the sum is locally finite. Then the identity

\[
uv = \sum_j (uv\varphi_j^2) = \sum_j \mathcal{F}^{-1}\left(\widehat{uv\varphi_j^2}\right)
\]

shows that the product $uv$ makes sense in $\mathcal{D}'(\mathbb{R}^d)$. \hfill \Box

Denote by $H^s_0(\Omega)$ the space of functions in $H^s(\mathbb{R}^d)$ whose support is contained in $\Omega$ endowed with the topology of the Sobolev space $H^s(\mathbb{R}^d)$.

**Proposition 3.1.** Let $s_1 + s_2 \geq 0$, $s_1 \leq 0 \leq s_2$. Then the multiplication $(u, v) \in H^s_0(\Omega) \times H^s_0(\Omega) \mapsto (uv) \in \mathcal{E}'(\mathbb{R}^d)$ is bilinear continuous where $\mathcal{E}'(\mathbb{R}^d)$ is endowed with the strong topology.

**Proof.** Recall that the strong topology of $\mathcal{E}'(\mathbb{R}^d)$ is the topology of uniform convergence on bounded sets of $C^\infty(\mathbb{R}^d)$. Let $B$ be some arbitrary bounded set in $C^\infty(\mathbb{R}^d)$ for its Fréchet space topology. Pick a test function $\chi \in \mathcal{D}(\mathbb{R}^d)$ s.t. $\chi = 1$
on $\Omega$. Then $\forall \varphi \in B$,

$$|\langle uv, \varphi \rangle| = \left| \int_{\mathbb{R}^d} u\hat{v}(\xi)\hat{\varphi}(\xi)d\xi \right|$$

$$\leq \|u\|_{H^{1}}\|v\|_{H^{2}}\int_{\mathbb{R}^d} d^d\xi (1 + |\xi|)^{s-1} |\hat{\varphi}(\xi)|$$

$$\leq \|u\|_{H^{1}}\|v\|_{H^{2}}\int_{\mathbb{R}^d} d^d\xi (1 + |\xi|)^{-d-1}(1 + |\xi|)^{d+1-s} |\hat{\varphi}(\xi)|$$

$$\leq C\|u\|_{H^{1}}\|v\|_{H^{2}}\sup_{x \in \Omega, |\alpha| \leq m} |\varphi(x)|$$

for $m \geq d+1-s_1$ and where $C$ does not depend on $\varphi$. But $\sup_{\varphi \in B} \sup_{x \in \Omega, |\alpha| \leq m} |\varphi(x)| < +\infty$ therefore $\exists C > 0, \sup_{\varphi \in B} |\langle uv, \varphi \rangle| \leq C\|u\|_{H^{1}}\|v\|_{H^{2}}$ which yields the desired result.

3.1.1. The Fourier transform of compactly supported Sobolev distributions. We will need to compare $C^k$ norms and Sobolev norms and we also often use the following local embeddings:

**Proposition 3.2.** Let $\Omega$ be some bounded open set. Denote by $H^s_0(\Omega)$ (resp $C^k_0(\Omega)$) the space of functions in $H^s(\mathbb{R}^d)$ (resp $C^k(\mathbb{R}^d)$) whose support is contained in $\Omega$. If $k + \frac{d}{2} < s$ then the map:

$$u \in H^s_0(\Omega) \mapsto u \in C^k_0(\Omega)$$

is continuous.

Conversely let $k \in \mathbb{N}$ then for all $s$ such that $s + \frac{d}{2} < k$ the map:

$$u \in C^k_0(\Omega) \mapsto u \in H^s_0(\Omega)$$

is continuous.

**Proof.** The embedding results from the elementary estimates:

$$\forall x \in \Omega, |\partial^k u(x)| \leq \int_{\mathbb{R}^d} d^d\xi |\xi|^k |\hat{u}(\xi)| \leq \int_{\mathbb{R}^d} d^d\xi (1 + |\xi|)^k |\hat{u}(\xi)|$$

$$\leq \int_{\mathbb{R}^d} d^d\xi (1 + |\xi|)^{s} |\hat{u}(\xi)|(1 + |\xi|)^{k-s} \leq \|u\|_{H^s} \left( \int_{\mathbb{R}^d} d^d\xi (1 + |\xi|)^{2(k-s)} \right)^{\frac{1}{2}}$$

where the last estimate follows from Cauchy Schwartz inequality and the fact that $(1 + |\xi|)^{k-s} \in L^2(\mathbb{R}^d)$ since $k-s < -\frac{d}{2}$.

Conversely if $k > \frac{d}{2}$ then:

$$u \in C^k_0(\Omega) \Rightarrow |(1 + |\xi|)^{k} \hat{u}(\xi)| \leq C \sup_{x \in \Omega, |\alpha| \leq k} |u(x)|$$

$$\Rightarrow \forall \varepsilon > 0, \exists C' > 0, \|u(1 + |\xi|)^{k-(\frac{d}{2}+\varepsilon)} \hat{u}(\xi)\|_{L^2(\mathbb{R}^d)} \leq C' \sup_{x \in \Omega, |\alpha| \leq k} |u(x)|$$

Finally this means $\forall k \geq 0, C^k_0(\Omega)$ injects continuously in $H^s_0(\Omega), \forall s < k - \frac{d}{2}$.

The embedding will be important for us since it states that a very regular function in $C^k$ for large $k$ will belong to all Sobolev space $H^s$ for $s < k - \frac{d}{2}$ and that the embedding is continuous. The next lemma gives us a way to control weighted norms of Fourier transform of compactly supported distributions of Sobolev regularity $H^s(\mathbb{R}^d)$.

**Lemma 3.2.** Let $u$ be a distribution in $H^s(\mathbb{R}^d)$ and $B$ the ball of radius $R$. There exists $M > 0$ s.t. for all $u$ supported in $B$, $u$ satisfies the estimate

$$\exists M > 0, |\hat{u}(\xi)| \leq M\|u\|_{H^s(\mathbb{R}^d)}(1 + |\xi|)^k$$

for all $k \geq 0$ if $s \geq 0$, $s + k > \frac{d}{2}$ if $s < 0$. 


Proof. First note that $\hat{u}$ is real analytic by Paley–Wiener–Schwartz. If $s \geq 0$ then $u$ is a compactly supported $L^2$ function, hence a distribution of order 0 and thus $k = 0$ which means that $\hat{u}$ is bounded. Moreover, we have the explicit estimate:

$$|\hat{u}(\xi)| = |u(\chi e^{i(\xi,\cdot)})| \leq \|u\|_{L^2(\Omega)} \|\chi e^{i(\xi,\cdot)}\|_{L^2(\mathbb{R}^d)} \leq (2R)^\frac{d}{2} \|u\|_{L^2(\mathbb{R}^d)} \leq (2R)^\frac{d}{2} \|u\|_{H^s(\mathbb{R}^d)}.$$

If $s < 0$, by duality of Sobolev spaces \[21\] Proposition 13.7], we find that for all test function $\varphi$:

$$|\langle u, \varphi \rangle| \leq \|u\|_{H^s} \|\varphi\|_{H^{-s}}.$$

Hence by the embedding \[35\], for all $k$ satisfying $k > -s + \frac{d}{2}$ there exists $C > 0$ s.t.:

$$\|\varphi\|_{H^{-s}(\mathbb{R}^d)} \leq C\|\varphi\|_{C^k(\Omega)}$$

therefore:

$$|\langle u, \varphi \rangle| \leq \|u\|_{H^s} \|\varphi\|_{H^{-s}} \leq C\|u\|_{H^s(\Omega)} \|\varphi\|_{C^k(\Omega)}$$

therefore choosing $\varphi = \chi e^{i(\xi,\cdot)}$ where $\chi \in \mathcal{D}(\mathbb{R}^d), \chi|_B = 1$ yields

$$|\hat{u}(\xi)| = |\langle u, \chi e^{i(\xi,\cdot)} \rangle| \leq C\|u\|_{H^{-s}_c(\Omega)} \|\chi e^{i(\xi,\cdot)}\|_{C^k(\Omega)} \leq C\|\chi\|_{H^{-s}_c(\Omega)} (1 + |\xi|^k)$$

for some constant $C'$ independent of $u$. \[\square\]

3.2. The $\hat{+}$ operation of Iagolnitzer. We first introduce the $\hat{+}$ operation of Iagolnitzer on closed conic sets. Actually, this operation originates from the $u = 0$ Theorems of Iagolnitzer \[32\] which aim to study the analytic wave front set of products of $u,v$ s.t. $WF_A(u)$ and $WF_A(v)$ are not transverse.

3.2.1. Definition. In what follows we define $\hat{+}$ following Iagolnitzer \[32\]. Our definition of $\hat{+}$ is weaker than the $\hat{\circ}$ operation defined by Kashiwara–Schapira \[33\] and gives a larger conic set for the WF of the product. Let $\Gamma_1, \Gamma_2$ be two closed conic sets in $T^*\mathbb{R}^d$, then

$$\Gamma_1 \hat{+} \Gamma_2 = \{(x, \xi) \text{ s.t. } \exists\{(x_1, n; x_1, n), (x_2, n; x_2, n)\} \in (\Gamma_1 \times \Gamma_2)^N, x_{1, n} \rightarrow x, x_{1, n} + x_{2, n} \rightarrow \xi, \xi \neq 0\}$$

Lemma 3.3. If $\Gamma_1 \cap -\Gamma_2 = \emptyset$ then $\Gamma_1 \hat{+} \Gamma_2 = (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$.

Proof. The proof follows from the definition of $\hat{+}$. \[\square\]

3.2.2. A $u = 0$ Theorem. We want to show that

Theorem 3.1. Let $(u,v) \in H^s_{\text{loc}}(\mathbb{R}^d) \times H^s_{\text{loc}}(\mathbb{R}^d)$ for $s_1 + s_2 \geq 0, s_1 \leq 0 \leq s_2$ then the product $uv$ makes sense in $D'(\mathbb{R}^d)$ and

$$WF(uv) \subset WF(u) \hat{+} WF(v).$$

Proof. The existence of the product $uv$ in $D'$ follows from Lemma 5.1. We use the notation of Hörmander and denote by $\Sigma(u\varphi) \subset \mathbb{R}^{n*}$ the closed cone which is the complement of the codirections where $\hat{w}_\varphi$ has fast decrease. We denote by $\pi_2$ the projection $(x, \xi) \in T^*\mathbb{R}^d \rightarrow \xi \in \mathbb{R}^{d*}$. From Hörmander \[31\], the cone $\Sigma(u\varphi)$ can be expressed in terms of the wave front set of $u$:

$$\Sigma(u\varphi) = \pi_2 (WF(u) \cap T^*_{\text{supp } \varphi}\mathbb{R}^d).$$
If \( (x; ξ) \not\in WF(u) \oplus_i WF(v) \), then we claim that there is a closed conic neighborhood \( V \) of \( ξ \) and a small ball \( B_ε(x) \) centered at \( x \) such that for all \( ϕ \in D(B_ε(x)) \),

\[
((Σ(uϕ) ∪ \{0\}) + (Σ(vϕ) ∪ \{0\})) ∩ V = \emptyset.
\]

By contradiction assume the above claim is not true. Then for all closed conic neighborhood \( V \) of \( ξ \) such that \( \{(x) x V \} \cap WF(u) \oplus_i WF(v) |_x = \emptyset \) where \( WF(u) \oplus_i WF(v) |_x \) lives in the fiber \( T^*_x \mathbb{R}^d \), there is some sequence \( ε_n \to 0 \) such that for every \( n \), there are two elements \( (x_{1,n}; ξ_{1,n}) \in WF(u), (x_{2,n}; ξ_{2,n}) \in WF(v), (x_{1,n}, x_{2,n}) \in B_ε(x) \) such that \( ξ_{1,n} + ξ_{2,n} \in V \). Therefore we have a pair of sequences \( (x_{1,n}; ξ_{1,n}) \) \( (x_{2,n}; ξ_{2,n}) \) \( WF(u), (x_{2,n}; ξ_{2,n}) \) \( WF(v) \) such that \( \xi_{1,n} + \xi_{2,n} \in V \), and a small ball \( B_{ε_n}(x) \). Then we have a pair of sequences \( (x_{1,n}; \xi_{1,n}) \) \( (x_{2,n}; \xi_{2,n}) \) \( WF(u), (x_{2,n}; ξ_{2,n}) \) \( WF(v) \) such that \( \xi_{1,n} + ξ_{2,n} \in V \) and \( (x_{1,n}, x_{2,n}) \) \( WF(u) \oplus_i WF(v) \) \( WF(u) \oplus_i WF(v) \). We are reduced to study the localized product \( (uϕ) \oplus_i (vϕ) \) which is supported in a ball \( B_{ε_n}(x) \) around \( x \). We enlarge \( Σ(uϕ), Σ(vϕ) \) and choose functions \( α_1, α_2 \) smooth in \( C^∞(\mathbb{R}^d \setminus \{0\}) \) and homogeneous of degree 0 s.t. \( (supp α_1) + (supp α_2) \cap V = \emptyset \).

Following the method in Eskin [21] (see also [17]), we decompose the convolution product in four parts:

\[
\hat{uvϕ}^2 |ϕ| = I_1(ϕ) + I_2(ϕ) + I_3(ϕ) + I_4(ϕ)
\]

\[
I_1(ϕ) = \int_{\mathbb{R}^d} α_1 \hat{uvϕ}(ξ - η)α_2 \hat{vϕ}(η) dη
\]

\[
I_2(ϕ) = \int_{\mathbb{R}^d} (1 - α_1) \hat{uvϕ}(ξ - η)α_2 \hat{vϕ}(η) dη
\]

\[
I_3(ϕ) = \int_{\mathbb{R}^d} (1 - α_2) \hat{vϕ}(ξ - η)α_1 \hat{uvϕ}(η) dη
\]

\[
I_4(ϕ) = \int_{\mathbb{R}^d} (1 - α_2) \hat{vϕ}(ξ - η)(1 - α_1) \hat{uvϕ}(η) dη
\]

Note that \( (supp α_1) + (supp α_2) \cap V = \emptyset \) \( ∀ξ \in V, I_1(ϕ) = 0 \) hence \( I_1 \) vanishes and we are thus reduced to estimate the remaining terms. Denote by \( δ \) the distance in the unit sphere between \( supp α_1 \cup supp α_2 \cap S^{d-1} \) and \( V \cap S^{d-1} \). Then we have the following estimates:

\[
|I_2(ϕ)| \leq \|u\|_{L^2, supp (1-α_1)ϕ}(1 + \sin δ|ϕ|)^{-N} \int_{\mathbb{R}^d} dη(1 + \sin δ|η|)^{-N} |\hat{vϕ}(η)|
\]

\[
|I_3(ϕ)| \leq \|v\|_{L^2, supp (1-α_2)ϕ}(1 + \sin δ|ϕ|)^{-N} \int_{\mathbb{R}^d} dη(1 + \sin δ|η|)^{-N} |\hat{uvϕ}(η)|
\]

\[
|I_4(ϕ)| \leq (1 + |ϕ|)^{-N} \|v\|_{L^2, supp (1-α_2)ϕ} \|u\|_{L^2, supp (1-α_1)ϕ} \int_{\mathbb{R}^d} \frac{(1 + |η|)^N}{(1 + |ξ - η|)^{2N}}(1 + |η|)^N dη
\]

\((uϕ, vϕ)\) are compactly supported distributions in \( H^{s_1}(\mathbb{R}^d) \times H^{s_2}(\mathbb{R}^d) \) hence by Lemma 3.2 there are integers \( m_1, m_2 \) and constants \( C_1, C_2 \) such that:

\[
\|(1 + |ϕ|)^{-m_1} \hat{vϕ}\|_{L^∞} \leq C_1 \|vϕ\|_{H^{s_1}(\mathbb{R}^d)}
\]

\[
\|(1 + |ϕ|)^{-m_2} \hat{vϕ}\|_{L^∞} \leq C_2 \|vϕ\|_{H^{s_2}(\mathbb{R}^d)}.
\]
Hence, we can recover our estimates in terms of Sobolev norms:

\[
|I_2(\xi)| \leq \|u\|_{2N,\text{supp } (1-\alpha_1),\varphi} \|(1 + |\xi|)^{-m_1} \hat{u}\|_{L^\infty} \int_{\mathbb{R}^d} d\eta (1 + \sin |\eta|)^{-N} (1 + |\eta|)^{m_1}
\]

\[
\leq \|u\|_{2N,\text{supp } (1-\alpha_1),\varphi} C_1 \|\varphi\|_{H^{1}((\mathbb{R}^d))} \int_{\mathbb{R}^d} d\eta (1 + \sin |\eta|)^{-N} (1 + |\eta|)^{m_1}
\]

\[
|I_3(\xi)| \leq \|v\|_{2N,\text{supp } (1-\alpha_2),\varphi} \|(1 + |\xi|)^{-m_2} \hat{v}\|_{L^\infty} \int_{\mathbb{R}^d} d\eta (1 + \sin |\eta|)^{-N} (1 + |\eta|)^{m_2}
\]

\[
|I_4(\xi)| \leq \|v\|_{2N,\text{supp } (1-\alpha_2),\varphi} C_2 \|\varphi\|_{H^{2}((\mathbb{R}^d))} \int_{\mathbb{R}^d} d\eta (1 + \sin |\eta|)^{-N} (1 + |\eta|)^{m_2}
\]

Set $\Gamma_1, \Gamma_2$ to be two closed conic sets. Hence for all $(x, \xi) \notin \Gamma_1 \oplus \Gamma_2$, for all $N > d + m_1 + m_2$, there is a closed cone $V \subset \mathbb{R}^d$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $(x, \xi) \in \text{supp } \varphi \times V$ and $\text{supp } \varphi \times V$ does not meet $\Gamma_1 \oplus \Gamma_2$, and there are some seminorms of $D_{\Gamma_1}, D_{\Gamma_2}$ and some constant $C_N$ which does not depend on $u, v$ such that

\[
\|uv\|_{N,\varphi^2} \leq (\text{const}) \left( \|u\|_{2N,\text{supp } (1-\alpha_1),\varphi} + \|\varphi\|_{H^{1}((\mathbb{R}^d))} \right) \left( \|v\|_{2N,\text{supp } (1-\alpha_2),\varphi} + \|\varphi\|_{H^{2}((\mathbb{R}^d))} \right)
\]

We define functional spaces which are Sobolev spaces of compactly supported distributions whose wave front set is contained in a closed cone $\Gamma \subset T^*\Omega$.

**Definition 3.1.** Let $\Omega$ be a bounded open set in $\mathbb{R}^d$, $\Gamma$ a closed conic set in $T^*\Omega$, then a distribution $t \in \mathcal{E}'(\Omega)$ belongs to $H^{s}_{0,\Gamma}(\Omega)$ if $t \in H^{s}_{0}(\Omega) \cap \mathcal{E}'_{\Gamma}(\Omega)$. We equip $H^{s}_{0,\Gamma}(\Omega)$ with the weakest topology which makes the injections $H^{s}_{0,\Gamma}(\Omega) \hookrightarrow H^{s}(\mathbb{R}^d)$ and $H^{s}_{0,\Gamma}(\Omega) \hookrightarrow \mathcal{D}'_{\Gamma}(\Omega)$ continuous. Equivalently, the topology of $H^{s}_{0,\Gamma}(\Omega)$ is defined by the Sobolev norm of $H^{s}$ and the seminorms $\|t\|_{N,V,\chi} = \sup_{\xi \in V} (1 + |\xi|)^{N} |\chi(\xi)|$ for all $\chi \in \mathcal{D}(\Omega)$ and cone $V$ of $\mathbb{R}^d \setminus \{0\}$ s.t. $\text{supp } \chi \times V \cap \Gamma = \emptyset$.

It follows from Proposition 3.1 and estimate (10) that:

**Theorem 3.2.** Let $\Omega$ be a bounded open set, $(s_1, s_2)$ real numbers s.t. $s_1 + s_2 > 0$ and $(\Gamma_1, \Gamma_2)$ two closed conic sets in $T^*\Omega$. Then the product

\[
(u, v) \in H^{s_1}_{0,\Gamma_1}(\Omega) \times H^{s_2}_{0,\Gamma_2}(\Omega) \hookrightarrow uv \in \mathcal{E}'_{\Gamma}(\mathbb{R}^d)
\]

is continuous where $\Gamma = \Gamma_1 \oplus \Gamma_2$.

4. THE WAVE FRONT SET OF $(f + i0)^\lambda$.

Recall that our goal is to study from the microlocal point of view models for the singularity of Feynman amplitudes of the form $\prod_{i=1}^{\lambda} (f_i + i0)^\lambda$. Since the proof is quite involved, we will start smoothly by investigating the complex power $(f + i0)^\lambda$ for only one analytic function $f$ where all the main ideas can already be found.

Let $U$ be some open set in $\mathbb{R}^n$ and $f$ be some real valued analytic function on $U$. The goal of this section is to provide a relatively simple geometric bound on $WF(f + i0)^\lambda$. Our main result in this section is related to works of Kashiwara, Kashiwara–Kawal on the characteristic variety of the $\mathcal{D}$-module $\mathcal{D}f^\lambda$. Our proof relies on the existence of the Bernstein Sato polynomial [25] and the bounds on the wave front set of products given by Theorem 3.2.

We start with a useful Lemma.
Lemma 4.1. Let $f$ be a real valued analytic function on an open set $U \subset \mathbb{R}^n$, then there is a discrete set $Z \subset \mathbb{C}$ s.t. meromorphic family $((f + i0)^\lambda)_\lambda$ satisfies the identity:

\begin{equation}
\forall \lambda \in \mathbb{C} \setminus Z, \forall k \in \mathbb{N}, WF((f + i0)^\lambda)^{\lambda + k} = WF(f + i0)^\lambda.
\end{equation}

Proof. To determine the wave front set over $U$, it suffices to determine it locally in some neighborhood of any point $x \in U$. Following the lecture notes of Granger [24], we must complexify the whole situation and consider the holomorphic extension of $f$ to some complex neighborhood $V \subset \mathbb{C}^n$ of $U$ and use existence of a local Bernstein Sato polynomial on $\mathbb{C}^n$.

Let us first discuss some issues about complexification. Assume $f$ was extended by holomorphic continuation on $V \subset \mathbb{C}^n$, consider the open set $V = f^{-1}(\mathbb{C} \setminus i\mathbb{R}_{<0})$, this set contains $U$ since $f|_U$ is real valued, then we choose the branch of the log which avoids the negative imaginary axis $i\mathbb{R}_{<0}$ in the complex plane. Therefore for $\varepsilon > 0$, we can define the complex powers $(f + i\varepsilon)^\lambda = e^{\lambda \log(f + i\varepsilon)}$ for $\lambda \in \mathbb{C}$ on $V \setminus \{f = 0\}$. When $\text{Re}(\lambda) > 0$, $(f + i\varepsilon)^\lambda$ has unique extension as a continuous function on $V$ letting $\varepsilon$ goes to zero. Indeed $(f + i0)^\lambda = 0$ on $\{f = 0\}$ and $(f + i0)^\lambda$ equals $f^\lambda$ on $V \setminus \{f = 0\}$ and $(f + i0)^\lambda$ is thus holomorphic on $V \setminus \{f = 0\}$. In the sequel, we denote by $x = (x_1, \ldots, x_n)$ the coordinates in the real open set $U$ and by $z = (z_1, \ldots, z_n)$ the complex coordinates in $V$.

Assuming that $(U, V)$ are chosen small enough, by the local existence of the Bernstein Sato polynomial [23] Theorem 5.4 p. 257, there exists a holomorphic differential operator $P(z, \partial_z)$ with holomorphic coefficients and a polynomial $b(\lambda)$ s.t.

$$P(z, \partial_z) f^{\lambda + 1} = b(\lambda) f^\lambda.$$  

This relation is valid on $V \setminus \{f = 0\}$.

Going back to the real case, we have an equation

$$P(x, \partial_x) f^{\lambda + 1} = b(\lambda) f^\lambda,$$

on $U \setminus \{f = 0\}$ where the real analytic set $\{f = 0\}$ has null measure in $U$ by Lemma 1.2 when $\text{Re}(\lambda)$ is strictly larger than the order of the differential operator $P$, both $P(x, \partial_x) f^{\lambda + 1}$ and $f^\lambda$ have unique continuation as functions of regularity $C^0$ and $C^k$ on $W$ respectively and the above identity holds true in the sense of distributions. Since $(f + i0)^\lambda$ extends meromorphically in $\lambda$ with value distribution by Theorem 1.2 the following equation holds true at the distributional level:

\begin{equation}
P(x, \partial_x) (f + i0)^{\lambda + 1} = b(\lambda) (f + i0)^\lambda
\end{equation}

for all $\lambda$ avoiding the poles of $f^{\lambda + 1}$, $f^\lambda$ and the zeros of $b$. Therefore for such $\lambda$, one has

$$b^{-1}(\lambda) P(x, \partial_x) (f + i0)^{\lambda + 1} = (f + i0)^\lambda$$

$$\implies WF(f + i0)^\lambda = WF(b^{-1}(\lambda) P(x, \partial_x) (f + i0)^{\lambda + 1})$$

$$\implies WF(f + i0)^\lambda \subset WF(f + i0)^{\lambda + 1}.$$

We used the classical bound on the wave front set $WF(Pu) \subset WF(u)$ where $u \in \mathcal{D}'$ and $P$ is a differential operator. On the other hand $(f + i0)^{\lambda + 1} = f (f + i0)^\lambda$ which implies that $WF(f + i0)^{\lambda + 1} \subset WF(f + i0)^\lambda$, finally set $Z$ to be equal to $\{\text{poles of } (f + i0)^\lambda\} \cup \{\text{zeros of } b\} - \mathbb{N}$, this yields

\begin{equation}
\forall \lambda \notin Z, WF((f + i0)^\lambda)^{\lambda + 1} = WF(f + i0)^\lambda.
\end{equation}
Morality: it suffices to bound $WF(f + i0)^\lambda$ for $Re(\lambda)$ then we would bound $WF(f + i0)^\lambda$ for all $\lambda \notin \mathbb{Z}$. Now we state and prove the main Theorem of this section. The proof relies on the $u = 0$ Theorem.

**Theorem 4.1.** Let $f$ be a real valued analytic function s.t. $\{df = 0\} \subset \{f = 0\}$, assume $f$ is proper then for all $\lambda \notin \mathbb{Z}$.

$Wf((f + i0)^\lambda) \subset \{(x; \xi) \ s.t. \ \exists \{(x_k, \alpha_k)k\} \in (\mathbb{R}^n \times \mathbb{R}_{>0})^N, x_k \to x, f(x_k) \to 0, \alpha_k df(x_k) \to \xi\}$.

**Proof.** We use the very simple idea to convert $(f + i0)^\lambda$ into a slightly more complicated integral which is easier to control:

\[
(f + i0)^\lambda = \int_{\mathbb{R}} dt(t + i0)^\lambda \delta_{t-f}.
\]

Let $\pi$ be the projection $\pi: (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto x \in \mathbb{R}^n$. The above integral formula for $(f + i0)^\lambda$ can also be conveniently reformulated as a pushforward $\pi_* ((t + i0)^\lambda \delta_{t-f})$.

Step 1 First, let us show that for $Re(\lambda)$ large enough the product $(t + i0)^\lambda \delta_{t-f}$ makes sense in $D'$. Let $(U_i)_i$ be an open cover of $\mathbb{R} \times U$ by bounded open sets and $(\varphi_i)_i$ a subordinated partition of unity $\sum \varphi_i^2 = 1$. Then it is enough to consider

\[
\sum_i (t + i0)^\lambda \varphi_i (\delta_{t-f} \varphi_i).
\]

The delta function $\delta_{t-f}$ is supported by the hypersurface $\{t-f = 0\}$, by the usual Sobolev trace Theorem [21, Theorem 13.6], any function in $H^s(\mathbb{R}^{n+1})$ for $s > \frac{1}{2}$ can be restricted on $\{t-f = 0\}$ therefore by duality of Sobolev space [21, Proposition 13.7], $\delta_{t-f}$ belongs to $H^s(\mathbb{R}^{n+1})$ for all $s < -\frac{1}{2}$. If $Re(\lambda) \geq m \in \mathbb{N}$ then $((t + i0)^\lambda \varphi_i)$ is a compactly supported function with regularity $C^m$, hence it belongs to the Sobolev space $H^s(\mathbb{R}^{n+1})$ for $s > m + \frac{1}{2}$ by the continuous injection \[50\] of Proposition 3.2. It follows that for every $s$, $((t + i0)^\lambda \varphi_i) \in H^s(\mathbb{R}^{n+1})$ for $Re(\lambda)$ large enough.

Step 2 To study $WF ((t + i0)^\lambda \delta_{t-f})$, we will use the $u = 0$ Theorems to give bounds on the wave front set of the product of $(t + i0)^\lambda$ with $\delta_{t-f}$. Since $\delta_{t-f} \in H^{-\frac{3}{2},-\frac{3}{2}}, \forall \epsilon > 0$, the product $(t + i0)^\lambda \delta_{t-f}$ makes sense for all $\lambda$ s.t. $Re(\lambda) > \frac{1}{2} + 1$, and by the $u = 0$ Theorem \[52\]

\[
WF ((t + i0)^\lambda \delta_{t-f}) \subset WF(t + i0)^\lambda \mathcal{F}_t WF(\delta_{t-f}).
\]

We start from the elementary wave front sets:

$WF(\delta_{t-f}) = \{(t, x; \tau, \xi) \ s.t. \ f(x) = t, \xi = -\tau df, \tau \neq 0\}$

$WF(t + i0)^\lambda = \{(0, x; \tau, 0) \ s.t. \ \tau > 0\}$

and by definition of the $\mathcal{F}_t$ operation of Iagolnitzer, it is obvious that outside $t = 0$,

$WF(t + i0)^\lambda \mathcal{F}_t WF(\delta_{t-f}) \mid_{\{t \neq 0\}} = WF(\delta_{t-f}) \mid_{\{t \neq 0\}}$.

At $t = 0$, set

\[
\Gamma_f = \{(0, x; \tau, \xi) \ s.t. \ \exists (x_n, \tau_n, \xi_n) \in N, x_n \to x, f(x) = 0, \xi_n = -\tau_n df(x_n) \to \xi, \tau_n + \tau'_n \to \tau, \tau'_n > 0\}.
\]

Then we find that

$WF(t + i0)^\lambda \mathcal{F}_t WF(\delta_{t-f}) \mid_{t=0} = \{(0, x; \tau, \xi) \ s.t. \ \exists (x_n, \tau_n, \xi_n) \in N, x_n \to x, f(x) = 0, -\tau_n df(x_n) \to \xi, \tau_n + \tau'_n \to \tau, \tau'_n > 0\}$

$= \Gamma_f$.

The above yields $\Gamma_f = WF((t + i0)^\lambda) \mathcal{F}_t WF(\delta_{t-f}) \mid_{\{t=0\}}$.

Step 3, we evaluate the wave front set of $(f + i0)^\lambda$ viewed as the push–forward $\pi_* ((t + i0)^\lambda \delta_{t-f})$. Outside $\{t = 0\}$, $WF ((t + i0)^\lambda \delta_{t-f}) \cap T^*((\mathbb{R} \setminus \{0\}) \times U) =$
by definition of $\pi$

Define $T\mathrm{WF}(\delta_{t-f})$ and are calculated as follows:

$T\mathrm{WF}(\delta_{t-f}) \cap T^\star((\mathbb{R} \setminus \{0\}) \times U)$ and by the behaviour of the wave front set under push–forward [9, Proposition], $\pi_\ast T\mathrm{WF}(\delta_{t-f}) = \emptyset$. Hence, only the elements of $\Gamma_f$ of the form $(0, x; \tau = 0, \xi) \in T^\star(\mathbb{R} \times U)$ contribute to the wave front set of $\pi_\ast(\Gamma_f)$ and are calculated as follows:

$$\Gamma_f \cap \{(0, x; \tau = 0, \xi)\} = \{(0, x; 0, \xi) \text{ s.t. } x_n \to x, f(x) = 0, \xi_n = -\tau_n df(x_n) \to \xi, \tau_n + \tau_n' \to 0, \tau_n' > 0\}$$

$$= \{(0, x; 0, \xi) \text{ s.t. } x_n \to x, f(x) = 0, \xi_n = -\tau_n df(x_n) \to \xi, \tau_n > 0\}.$$

Define

$$\Lambda_f = \{x; \xi\} \text{ s.t. } \exists \{(x_k, a_k)\} \in (\mathbb{R}^n \times \mathbb{R}_>0)^N, x_k \to x, f(x_k) \to 0, a_k df(x_k) \to \xi\}.$$

by definition of $\pi_\ast$ it is immediate that $\Lambda_f = \pi_\ast (\Gamma_f)$. It follows that:

$$\mathrm{WF}(\pi_\ast ((t + i0)^{\lambda} \delta_{t-f})) \subset \pi_\ast (\mathrm{WF}(t + i0)^{\lambda} \breve{\gamma}_t \mathrm{WF}(\delta_{t-f}))$$

$$= \pi_\ast (\Gamma_f) = \Lambda_f.$$

5. Functional calculus with value $\mathcal{D}_f^\star$.

In the sequel, for any manifold $M$, we will denote by $T^\star M$ the cotangent space $T^\star M$ minus its zero section. In QFT on curved analytic spacetimes, we will show that the meromorphically regularized Feynman amplitudes in position space are distributions depending meromorphically on the regularization parameter. However in order to renormalize, we need to control the WF of the regularized amplitudes therefore we are let to develop a functional calculus for distributions with value in the space $\mathcal{D}_f^\star$ of distributions whose wave front set is contained in some closed conic set $\Gamma$ of the cotangent cone $T^\star \mathbb{R}^d$.

5.0.3. The space $\mathcal{D}_f^\star$ characterized by duality. We work with the space $\mathcal{D}_f^\star$ of distributions whose wave front set is contained in some closed conic set $\Gamma$ of the cotangent space $T^\star \mathbb{R}^d$ endowed with the normal topology constructed by Brouder Dabrowski [9]. For any closed conic set $\Gamma \subset T^\star \mathbb{R}^d$, we denote by $-\Gamma = \{(x; -\xi) \text{ s.t. } (x; \xi) \in \Gamma\}$ the antipode of $\Gamma$ and by $\Gamma^c$ the complement of $\Gamma$ in $T^\star \mathbb{R}^d$. The space of compactly supported distribution whose wave front set is contained in some conic set $\Lambda$ will be denoted by $\mathcal{E}_\Lambda^\star$. The most important property for us is the following characterization of $\mathcal{D}_f^\star$ by duality.

Proposition 5.1. A set $B$ of distributions in $\mathcal{D}_f^\star$ is bounded if and only if, for every $v \in \mathcal{E}_\Lambda^\star$, $\Lambda = (-\Gamma)^c$, there is a constant $C > 0$ such that $|\langle u, v \rangle| \leq C$ for all $u \in B$.

Such a weakly bounded set is also strongly bounded and equicontinuous. Moreover, the closed bounded sets of $\mathcal{D}_f^\star$ are compact, complete and metrizable. The second important property is the following sufficient condition to describe sequential convergence in $\mathcal{D}_f^\star$:

Proposition 5.2. If $u_n$ is a sequence of elements of $\mathcal{D}_f^\star$ such that, for every $v \in \mathcal{E}_\Lambda^\star$, the sequence $(u_n, v)$ converges to $\lambda(v)$, then $u_n$ converges to a distribution $u$ in $\mathcal{D}_f^\star$ and $(u, v) = \lambda(v)$ for all $v \in \mathcal{E}_\Lambda^\star$.

The above plays the same role as the characterization of sequential convergence in $\mathcal{D}(\Omega)$ by duality, it suffices to verify for all test function $\varphi$, $t_n(\varphi)$ converges as a sequence of real numbers.
5.0.4. Continuous, holomorphic functions with value in $\mathcal{D}'_\Gamma$. Motivated by the above characterizations of $\mathcal{D}'_\Gamma$ by duality, we can give definitions of being continuous or holomorphic with value $\mathcal{D}'_\Gamma$. For applications to QFT we need to consider holomorphic (resp meromorphic) functions depending on several complex variables.

**Definition 5.1.** A family of distributions $(t_\lambda)_\lambda$ depends continuously (resp holomorphically) on a complex parameter $\lambda \in \mathbb{C}^p$ with value $\mathcal{D}'_\Gamma$, if for every test distribution $v \in \mathcal{E}'_\Gamma$, $\Lambda = -\Gamma^c$, $t_\lambda(v)$ is a continuous (resp holomorphic) function of $\lambda$. We will also call such family continuous (resp holomorphic) with value $\mathcal{D}'_\Gamma$.

It follows from Proposition 5.2 that

**Proposition 5.3.** Let $\Omega$ be an open set in $\mathbb{C}^p$ and $(t_\lambda)_{\lambda \in \Omega}$ a family of distributions in $\mathcal{D}'_\Gamma$. If $(t_\lambda)_{\lambda}$ depends continuously on $\lambda \in \Omega \subset \mathbb{C}^p$ with value $\mathcal{D}'_\Gamma$, let $\gamma = \gamma_1 \times \cdots \times \gamma_p \subset \mathbb{C}^p$ be a cartesian product where each $\gamma_i$ is a continuous curve in $\mathbb{C}$, then the weak integrals $\int_{\gamma \subset \mathbb{C}^p} d\lambda t_\lambda$ exists in $\mathcal{D}'_\Gamma$.

**Proof.** For every test distribution $v \in \mathcal{E}'_\Gamma$, $\Lambda = -\Gamma^c$, the function $\lambda \in \gamma \mapsto \langle t_\lambda, v \rangle$ is continuous hence Riemann integrable. Therefore $\int_\gamma d\lambda \langle t_\lambda, v \rangle$ exists as a limit of Riemann sums and the integral $\int_{\gamma} d\lambda t_\lambda$ is well defined by the sequential characterization of convergence in $\mathcal{D}'_\Gamma$. \qed

In that case, we will also say that $(t_\lambda)_{\lambda}$ is meromorphic with linear poles in $\lambda$ with value $\mathcal{D}'_\Gamma$.

5.0.5. Meromorphic functions with linear poles with value $\mathcal{D}'_\Gamma$. In the present work, we deal with families of distributions $(t_\lambda)_{\lambda \in \mathbb{C}^p}$ in $\mathcal{D}'_\Gamma(U)$ depending meromorphically on $\lambda \in \mathbb{C}^p$ with linear poles.

**Definition 5.2.** A family of distributions $(t_\lambda)_{\lambda \in \mathbb{C}^p}$ in $\mathcal{D}'_\Gamma(U)$ depends meromorphically on $\lambda \in \mathbb{C}^p$ with linear poles if for every $x \in U$, there is a neighborhood $U_x$ of $x$, a collection $(L_i)_{1 \leq i \leq m} \in (\mathbb{N})^m \subset (\mathbb{C}^p)^m$ of linear functions with integer coefficients on $\mathbb{C}^p$ such that for any element $z = (z_1, \ldots, z_p) \in \mathbb{Z}^p$, there is a neighborhood $\Omega \subset \mathbb{C}^p$ of $z$, such that

$$\lambda \in \Omega \mapsto \prod_{i=1}^m (L_i(\lambda + z)) t_\lambda |_{U_x}$$

is holomorphic with value $\mathcal{D}'_\Gamma(U_x)$.

5.0.6. A gain of regularity: when continuity becomes holomorphicity. Now we give an easy

**Proposition 5.4.** Let $U$ be an open subset of $\mathbb{R}^n$, an open set $\Omega \subset \mathbb{C}^p$, a family $(t_\lambda)_{\lambda \in \Omega}$ holomorphic in $\lambda$ with value $\mathcal{D}'(U)$. If $(t_\lambda)_\lambda$ is continuous with value $\mathcal{D}'_\Gamma(U)$ then $(t_\lambda)_\lambda$ is holomorphic with value $\mathcal{D}'_\Gamma(U)$.

**Proof.** It suffices to observe that by holomorphicity of $t$ and the multidimensional Cauchy's formula [26, p. 3] for any polydisk $D_1 \times \cdots \times D_p$ such that $\partial D_i$ is a circle surrounding $z_i$:

$$t_\lambda = \frac{1}{(2\pi)^p} \int_{\partial D_1 \times \cdots \times \partial D_p} \frac{t_z dz_1 \wedge \cdots \wedge dz_p}{(z_1 - \lambda_1) \cdots (z_p - \lambda_p)}.$$  

Since $t_z$ is continuous along $\partial D_1 \times \cdots \times \partial D_p$, then for any $v \in \mathcal{E}'_\Gamma$, $\Lambda = -\Gamma^c$, the quantity

$$t_\lambda(v) = \frac{1}{(2\pi)^p} \int_{\partial D_1 \times \cdots \times \partial D_p} \frac{t_z(v) dz_1 \wedge \cdots \wedge dz_p}{(z_1 - \lambda_1) \cdots (z_p - \lambda_p)}$$

is well defined by Proposition 5.3 and holomorphic in $\lambda$ by the integral representation which proves the holomorphicity of $(t_\lambda)_\lambda$ with value $\mathcal{D}'_\Gamma$. \qed
By definition of functions meromorphic with linear poles with value $D'_{\Gamma}$, we obtain:

**Corollary 5.1.** Let $\Omega \subset \mathbb{C}^p$, $z_1 \in \mathbb{C}$, $(t_\lambda)_{\lambda \in \Omega}$ a meromorphic family of distributions with linear poles. Denote by $Z$ the polar set of $t$. If $(t_\lambda)_{\lambda}$ is continuous on $\Omega \setminus Z$ with value $D'_{\Gamma}$ then $(t_\lambda)_{\lambda}$ is meromorphic with linear poles with value $D'_{\Gamma}$.

5.0.7. **Consequences of Riemann’s removable singularity Theorem.**

**Lemma 5.1.** Let $\Omega \subset \mathbb{C}^p$, $z_1 \in \mathbb{C}$, $(t_\lambda)_{\lambda \in \Omega}$ a meromorphic family of distributions with linear poles in $Z \subset \Omega$. If $\lambda \in \mathbb{C}^p \mapsto t_\lambda \in D'$ is locally bounded then $(t_\lambda)_{\lambda}$ is a holomorphic family of distributions.

**Proof.** For every test function $\varphi$, $\lambda \in \mathbb{C}^p \mapsto t_\lambda(\varphi)$ is meromorphic i.e. holomorphic on $\mathbb{C}^p \setminus Z$ where $Z$ is a thin set and locally bounded hence by Riemann’s removable singularity Theorem $\Omega$, $\lambda \in \mathbb{C}^p \mapsto t_\lambda(\varphi)$ is holomorphic. We conclude by showing it is a distribution at the points in $Z$ where singularities were removed. Let $\lambda$ be such a point, then the representation of $t_\lambda$ by Cauchy’s formula

$$t_\lambda = \frac{1}{2\pi i} \int \frac{t_\lambda(\varphi_1) \cdots t_\lambda(\varphi_p)}{(\varphi_1 - \lambda_1) \cdots (\varphi_p - \lambda_p)} dz_1 \cdots dz_p$$

along some contour $\partial D_1 \times \cdots \times \partial D_p$ which does not intersect any neighborhood of $\lambda$ shows that $t_\lambda$ is a weak integral with value distribution hence it is a distribution by Proposition 5.3 applied to the conic set $\Gamma = T^* \mathbb{R}^n$.

5.0.8. **Laurent series expansions of meromorphic distributions with linear poles.** We start by examining Laurent series expansions of families $(t_\lambda)_{\lambda}$ of distributions with value $D'_{\Gamma}$ where $\lambda$ is only one complex variable. We show that the coefficients of the Laurent series expansion of $t$ are also distributions in $D'_{\Gamma}$.

**Proposition 5.5.** Let $(t_\lambda)_{\lambda \in \mathbb{C}}$ be a meromorphic family of distributions with value $D'_{\Gamma}$. Then for all $z_0 \in \mathbb{C}$, there exists $\varepsilon > 0$ and a bounded set $B$ in $D'_{\Gamma}$ s.t. the Laurent series expansion of $t_\lambda$ around $z_0$ reads

$$t_\lambda = \sum_k a_k(\lambda - z)^k$$

where for all $k$,

1. $a_k \in D'_{\Gamma}$
2. moreover $\frac{\lambda^k}{k!} a_k \in B$ if $z_0$ is a regular value of $\lambda \mapsto t_\lambda$ and $\frac{\lambda^k(1 + 2^k)}{k!} a_k \in B$ if $z_0$ is a pole of $\lambda \mapsto t_\lambda$.

It follows that the wave front of $a_k$ is contained in $\Gamma$. We call such series expansion absolutely convergent with value in $D'_{\Gamma}$.

**Proof.** Without loss of generality, we can assume that $z_0 = 0$. First case, $0$ is not a pole of $t$. Choose $\varepsilon > 0$ such that the disc of radius $\varepsilon$ contains only $0$ as pole and denote by $\gamma$ the circle $\{ |z| = \varepsilon \} \subset \mathbb{C}$. Let $t_\gamma = \{ t_\lambda \text{ s.t. } \lambda \in \gamma \} \subset D'_{\Gamma}$ be the curve described by $t$ in $D'_{\Gamma}$ when $\lambda$ runs in $\gamma$, this curve is obviously a bounded subset of $D'_{\Gamma}$ by the continuity of $\lambda \mapsto t_\lambda \in D'_{\Gamma}$ and Proposition 5.1 characterizing bounded sets by duality. We want to consider the set $B$ defined as the closure of the disked hull of the curve $t_\gamma$:

$$B = \{ \alpha t_{\lambda_1} + \beta t_{\lambda_2}, \text{ s.t. } |\alpha| + |\beta| \leq 1, (\alpha, \beta) \in \mathbb{C}^2, |\lambda_1| = |\lambda_2| = |z| \leq \varepsilon \}.$$

It is immediate that the disked hull is still bounded in $D'_{\Gamma}$ by the characterization of bounded sets by duality hence its closure $B$ is bounded in $D'_{\Gamma}$. To summarize $B$ is a closed, bounded disk in $D'_{\Gamma}$. Recall $\gamma$ is the circle $\{ |z| = \varepsilon \} \subset \mathbb{C}$, then by Cauchy’s formula, we have

$$\forall k, a_k = \frac{k!}{2\pi i} \int_\gamma t_\lambda d\lambda \frac{1}{\lambda^{k+1}}.$$
By the definition of the weak integral as limit of Riemann sums, we find that:

\[
\frac{k!}{2\pi} \int_{\gamma} t^{k+1} d\lambda = \lim_{n \to \infty} \frac{k!}{2\pi} \sum_{j=1}^{n} \varepsilon^{-(k+1)} \exp(-i2\pi \frac{j(k+1)}{n}) t \varepsilon \exp(i2\pi \frac{j}{n})
\]

hence \( \lim_{n \to \infty} \sum_{j=1}^{n} \varepsilon^{-(k+1)} \exp(-i2\pi \frac{j(k+1)}{n}) t \varepsilon \exp(i2\pi \frac{j}{n}) \) belongs to \( B \) by construction of the closed disk \( B \) and it follows that \( a_k \in \frac{k!B}{2\pi} \).

In case 0 is a pole, we must repeat the above proof for a corona of the form \( \{ z : \varepsilon \leq |z| \leq \varepsilon \} \). So the Cauchy formula gives an integral over two circles of radius \( \frac{\varepsilon}{2} \) and \( \varepsilon \) respectively. And the same argument as above gives that \( \frac{k!}{2\pi} \int_{\gamma} \frac{t \varepsilon \exp(i2\pi \frac{j}{n})}{n} \) belongs to \( \frac{k!}{\varepsilon^2}(B + 2kB) \subset \frac{k!(1+2k)}{\varepsilon}B \) since \( B \) is a disk. \( \square \)

The same result holds true for holomorphic distributions depending on several complex variables by the same type of argument transposed to the multivariable complex case.

**Proposition 5.6.** Let \( (t_\lambda)_{\lambda \in \Omega} \) be a holomorphic family of distributions with value \( D'_k \). Then for all \( z_0 \in \Omega \), there exists \( \varepsilon > 0 \) and a bounded set \( B \) in \( D'_k \) s.t. the power series expansion of \( t_\lambda \) around \( z_0 \) reads

\[
t_\lambda = \sum_{k \in \mathbb{N}^p} a_k (\lambda - z)^k
\]

where for all multi-index \( k \),

1. \( a_k \in D'_k \)
2. and \( \lim_{|k|} a_k \in B \).

The bound \( \lim_{|k|} a_k \in B \) is a functional version of Cauchy’s bound in our functional context.

In the meromorphic case with linear poles, we must use the analogue of Laurent series decomposition for meromorphic functions with linear poles given by Theorem 2.1 and we obtain:

**Theorem 5.1.** Let \( U \) be an open set in \( \mathbb{R}^n \), \( \Omega \subset \mathbb{C}^p \) open and \( (t_\lambda)_{\lambda \in \Omega} \) a meromorphic family with linear poles at \( k \in \Omega \) with value \( D'_k(U) \). Then the element \( t_\lambda = \frac{h}{\ell_{i_1} \cdots \ell_{i_p}} \) can be written as a sum

\[
t = \sum_{i} \frac{h_i (\ell_{i_1}, \ldots, \ell_{i_p})}{L_{i_1} \cdots L_{i_p}} + \phi_i (L_{i_1}, \ldots, L_{i_n}, \ell_{i_1}, \ldots, \ell_{i_p})
\]

where for each \( i \), \( (s_{i_1}, \ldots, s_{i_n}) \in \mathbb{N}^{n_i} \), the collection of linear forms \( (L_{i_1}, \ldots, L_{i_n}) \) is a linearly independent subset of \( (L_1, \ldots, L_n) \), the collection of linear forms \( (\ell_{i_1}, \ldots, \ell_{i_p}) \) is a basis of the orthogonal complement of the subspace spanned by the \( (L_{i_1}, \ldots, L_{i_n}) \) and \( h_i, \phi_i \) are holomorphic distributions with value \( D'_k(U) \).

**Proof.** This is an immediate consequence of the fact that \( h \) is holomorphic in \( \lambda \) with value \( D'_k \) and that \( h_i, \phi_i \) are linear combinations in finite partial derivatives of \( h \) in \( \lambda \) and are therefore holomorphic distributions with value \( D'_k(U) \). \( \square \)
5.0.9. *When Hörmander products of holomorphic distributions are holomorphic.*

The next proposition aims to prove the holomorphicity of a product of holomorphic distributions with specific conditions on their wave front set.

**Proposition 5.7.** Let \((\Omega_1, \Omega_2)\) be two subsets of \((\mathbb{C}^p_1, \mathbb{C}^p_2)\) respectively, \(a(\lambda_1), b(\lambda_2)_{\lambda_1 \in \Omega_1, \lambda_2 \in \Omega_2}\) be two families of distributions which are holomorphic with value \(\mathcal{D}'_{\Gamma_1}, \mathcal{D}'_{\Gamma_2}\). If \(\Gamma_1 \cap -\Gamma_2 = \emptyset\), set \(\Gamma = (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2\) then the product \(a(\lambda_1)b(\lambda_2)\) is holomorphic on \(\Omega_1 \times \Omega_2\) with value \(\mathcal{D}'_{\Gamma}\).

**Proof.** First by transversality of wave front sets, the product \(a(\lambda_1)b(\lambda_2)\) is well defined pointwise for every \((\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2\). Moreover, by Cauchy’s formula and hypocontinuity of the product \([9]\) the integral representation

\[
a(\lambda_1)b(\lambda_2) = \int_{\gamma_1} \frac{dz_1}{z_1 - \lambda_1} a(z_1) \int_{\gamma_2} \frac{dz_2}{z_2 - \lambda_2} b(z_2)
\]

is well defined: use Riemann sum’s argument to express the two integrals

\[(f_{\gamma_1} a(\lambda_1) b_{\lambda_1}, f_{\gamma_2} b(\lambda_2) b_{\lambda_2})\]

as convergent sequences in \(\mathcal{D}'_{\Gamma_1}, \mathcal{D}'_{\Gamma_2}\) respectively then the sequential continuity of the product ensures the convergence of the multiplication

\[(a, b) \in \mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2} \mapsto (ab) \in \mathcal{D}'_{\Gamma}\]

hence the product \(\int_{\gamma_1} \frac{dz_1}{z_1 - \lambda_1} a(z_1) \int_{\gamma_2} \frac{dz_2}{z_2 - \lambda_2} b(z_2)\) is holomorphic in \((\lambda_1, \lambda_2) \in (\Omega_1 \times \Omega_2) \subset \mathbb{C}^{p_1 + p_2} \]

**\(\square\)**

**Proposition 5.8.** Let \((u_{\lambda_1}), (v_{\lambda_2})_{\lambda_2}\) be two families of distributions with value in \((\mathcal{D}'_{\Gamma_1}, \mathcal{D}'_{\Gamma_2})\) respectively with \(\Gamma_1 \cap -\Gamma_2 = \emptyset\) depending meromorphically on \(\lambda_1 \in \mathbb{C}^p_1, \lambda_2 \in \mathbb{C}^p_2\) with linear poles. Set \(\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_1 \cup \Gamma_2\) then the product \(u_{\lambda_1}v_{\lambda_2}\) is meromorphic in \((\lambda_1, \lambda_2) \in \mathbb{C}^{p_1 + p_2}\) with linear poles with value \(\mathcal{D}'_{\Gamma}\).

**Proof.** The proof follows immediately from the decomposition \([55]\) applied to both \(u\) and \(v\) separately and application of Proposition \([5,7]\). **\(\square\)**

5.1. *Functional properties of \(((f + i0)^{\lambda})_{\lambda \in \mathbb{C}}\).* In this subsection, we use the newly defined functional calculus to investigate the functional properties of the family \(((f + i0)^{\lambda})_{\lambda \in \mathbb{C}}\).

**Proposition 5.9.** Let \(f \neq 0\) be a real valued analytic function s.t. \(\{df = 0\} \subset \{f = 0\}\), \(Z\) some discrete set which contains the poles of the meromorphic family \(((f + i0)^{\lambda})_{\lambda}\) and

\[
\Lambda_f = \{(x; \xi) \ s.t. \ \exists \{(x_k, a_k) \in (\mathbb{R}^n \times \mathbb{R}_{>0})^N, x_k \rightarrow x, f(x_k) \rightarrow 0, a_kdf(x_k) \rightarrow \xi\}.
\]

Then \(((f + i0)^{\lambda})_{\lambda \in \mathbb{C}\setminus Z}\) is meromorphic with value \(\mathcal{D}'_{\Lambda_f}\).

**Proof.** By Theorem \([12]\) we already know that the family \(((f + i0)^{\lambda})_{\lambda \in \mathbb{C}}\) is meromorphic with value \(\mathcal{D}'\), then it suffices to show that \(((f + i0)^{\lambda})_{\lambda \in \mathbb{C}\setminus Z}\) is continuous in \(\mathcal{D}'_{\Lambda_f}\) and by Proposition \([5,1]\) we deduce that \(((f + i0)^{\lambda})_{\lambda \in \mathbb{C}}\) is meromorphic with value \(\mathcal{D}'_{\Lambda_f}\).

We want to show that it is sufficient to prove that \(\lambda \in K \mapsto (f + i0)^{\lambda} \in \mathcal{D}'_{\Lambda_f}\) is continuous for arbitrary compact subsets \(K \subset \mathbb{C}\) s.t. \(Re(K)\) is large enough. Choose \(Z\) to be the discrete subset of \(\mathbb{C}\) defined in the proof of Lemma \([11]\). For any differential operator \(P(x, \partial_x)\), note that the linear map \(u \in \mathcal{D}'_{\Lambda_f} \mapsto P(x, \partial_x)u \in \mathcal{D}'_{\Lambda_f}\) is continuous. If we could prove that the map \(\lambda \in \{Re(\lambda) > k\} \mapsto (f + i0)^{\lambda}\) is continuous for some integer \(k \in \mathbb{N}\), then by existence of the functional equation for \(\lambda \notin Z\), we would find some differential operator \(P\) and a polynomial \(b\) such that \(b(\lambda)^{-1}P(x, \partial_x)(f + i0)^{\lambda+1} = (f + i0)^{\lambda}\) then it follows that the map

\[
\lambda \in \{Re(\lambda) > k - 1\} \mapsto (f + i0)^{\lambda+1} = (f + i0)^{\lambda} \in \mathcal{D}'_{\Lambda_f},
\]
is continuous. In summary, if \( \Lambda \in \{ Re(\lambda) > k \} \setminus Z \mapsto (f + i0)^{\lambda} \in D'_{\Lambda_f} \) is continuous then so is \( \lambda \in \{ Re(\lambda) > k - 1 \} \setminus Z \mapsto (f + i0)^{\lambda} \in D'_{\Lambda_f} \) which means by an easy induction that it is sufficient to prove the result for arbitrary compact subsets \( K \subset \mathbb{C} \) s.t. \( Re(K) \) is large enough. Now if we inspect the first step of the proof of Theorem 4.1, the crucial point relies on the product

\[
\sum_i ((t + i0)^{\lambda} \varphi_i(\delta_{t - f} \varphi_i)).
\]

If \( \lambda \) lies in a compact set \( K \subset \mathbb{C} \) s.t. \( Re(K) > m_1 > m_1 \frac{s + 1}{2} + s \), then \( \lambda \in K \mapsto ((t + i0)^{\lambda} \varphi_i) \in C^m \) is continuous hence \( \lambda \in K \mapsto ((t + i0)^{\lambda} \varphi_i) \in H^s(\mathbb{R}^{n+1}) \) is continuous by the continuous injection from Lemma 5.5. Now choose \( s > \frac{1}{2} \) then since \( \varphi_i \delta_{t - f} \) belongs to \( H^{-\frac{1}{2} + \epsilon}(\mathbb{R}^{n+1}) \) for \( \epsilon = \frac{1}{2}(s - \frac{1}{2}) \) by the Sobolev trace theorems [21, Theorem 13.6], and by Lemma 5.1 applied to \( ((t + i0)^{\lambda} \varphi_i, \varphi_i \delta_{t - f}) \in H^{-\frac{1}{2} + 2\epsilon} \times H^{-\frac{1}{2} - \epsilon} \), the map \( \lambda \mapsto ((t + i0)^{\lambda} \varphi_i(\delta_{t - f} \varphi_i)) \) is continuous with value in \( \mathcal{E}'(\mathbb{R}^d) \). By the \( u = 0 \) Theorem 5.2, the map \( \lambda \in K \mapsto ((t + i0)^{\lambda} \varphi_i(\delta_{t - f} \varphi_i)) \) is bounded in \( D'_{\Lambda_f} \) then we can conclude by following the proof of Theorem 4.1 for the family \( (f + i0)^{\lambda} \varphi_i(\delta_{t - f} \varphi_i), \lambda \in K \) that \( ((f + i0)^{\lambda})_{\lambda \in \mathbb{C} \setminus Z} \) is continuous in \( \lambda \in K \) with value \( D'_{\Lambda_f} \).

By Lemma 5.5 we can deduce that:

**Corollary 5.2.** Let \( f \) be a real valued analytic function s.t. \( \{ df = 0 \} \subset \{ f = 0 \} \), \( Z \subset \mathbb{C} \) a discrete subset containing the poles of the meromorphic family \( (f + i0)^{\lambda} \), for all \( z \in Z \), set \( a_k \) to be coefficients of the Laurent series expansion of \( \lambda \mapsto (f + i0)^{\lambda} \) around \( z \)

\[
(f + i0)^{\lambda} = \sum_{k \in \mathbb{Z}} a_k(\lambda - z)^k.
\]

Then \( \forall k \),

\[
WF(a_k) \subset \Lambda_f.
\]

Furthermore, we can localize the distributional support of the coefficients \( a_k(z) \) of the Laurent series expansion of \( (f + i0)^{\lambda} \) around poles for negative values of \( k \).

**Theorem 5.2.** Let \( f \) be a real valued analytic function s.t. \( \{ df = 0 \} \subset \{ f = 0 \} \), \( Z \subset \mathbb{C} \) a discrete subset containing the poles of the meromorphic family \( (f + i0)^{\lambda} \). Set

\[
\Lambda_f = \{ (x; \xi) \ \text{s.t.} \ \exists \{(x_k, a_k)\} \in (\mathbb{R}^n \times \mathbb{R}_{>0})^N, x_k \to x, f(x_k) \to 0, a_k df(x_k) \to \xi \}.
\]

For all \( z \in Z \), let \( a_k \) be the coefficients of the Laurent series expansion of \( \lambda \mapsto (f + i0)^{\lambda} \) around \( z \)

\[
(f + i0)^{\lambda} = \sum_{k \in \mathbb{Z}} a_k(\lambda - z)^k.
\]

Then for all \( k \in Z \), \( WF(a_k) \subset \Lambda_f \) and if \( k < 0 \) then \( a_k \) is a distribution supported by the critical locus \( \{ df = 0 \} \).

**Proof.** Let us prove that the singular terms \( a_k, k < 0 \) in the Laurent series expansion around \( \lambda \in Z \) are distributions supported by the critical locus \( \{ df = 0 \} \). If \( x \) is a nondegenerate point for \( f \) i.e. \( f(x) = 0 \) but \( df(x) \neq 0 \), then \( df \neq 0 \) in some neighborhood \( U_x \) of \( x \) and \( (f + i0)^{\lambda} = f^s(t + i0)^{\lambda} \) is well defined by the pull-back Theorem of Hörmander. It is easy to check that \( \lambda \mapsto (t + i0)^{\lambda} \in D'_{\mathbb{C} \setminus \mathbb{R}} \) is continuous for the normal topology, it follows by continuity of the pull-back of Hörmander for the normal topology [9] that \( (f + i0)^{\lambda} \) depends continuously on
are not supported at $x$.

6. The wave front set of $\left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)_{\lambda \in \mathbb{C}^p}$.

Let $U$ be some open set in $\mathbb{R}^n$ and $(f_1, \ldots, f_p)$ be some real valued analytic functions on $U$ s.t. $\{df_j = 0\} \subset \{f_j = 0\}$. The goal of this section is to provide a relatively simple geometric bound on the wave front set of the family of distributions $\left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)_{\lambda \in \mathbb{C}^p}$ depending meromorphically on $\lambda \in \mathbb{C}^p$. Our proof closely follows the case of one function $f$.

We start by recalling a particular case of some general result of Sabbah [39, Theorem 2.1] on the existence of a multivariate Bernstein Sato polynomial.

**Theorem 6.1.** Let $f_1, \ldots, f_p$ be some analytic functions then there exists functional relations of the type

$$\forall k \in \{1, \ldots, p\}, \text{be } (b_k f_1 + i0)^{\lambda_1} \cdots (f_p + i0)^{\lambda_p} = P_k(x, \partial x, \lambda)f_k(f_1 + i0)^{\lambda_1} \cdots (f_p + i0)^{\lambda_p},$$

where $\lambda = (\lambda_1, \ldots, \lambda_p)$.

The polynomials $(b_k)_{k \in \{1, \ldots, p\}}$ are the Bernstein Sato polynomials. The above Theorem follows from [39, Theorem 2.1] (see also [40, 4]) applied to the holonomic distribution $u = 1$. The existence of the functional equation immediately implies that

**Lemma 6.1.** Let $U$ be some open set in $\mathbb{R}^n$, $f_1, \ldots, f_p$ be some real valued analytic functions on $U$, $Z \subset \mathbb{C}^p$ some thin set which contains the poles of $\prod_{j=1}^{p} (f_j + i0)^{\lambda_j}$ then $WF \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)$ does not depend on $\lambda \in \mathbb{C}^p \setminus Z$.

The proof of the above Lemma is a simple adaptation of the proof of Lemma [43]. In the multivariable case, the zeros of the polynomials $(b_j)_j$ are contained in some thin set $Z$ contained in $\mathbb{C}^p$.

**Theorem 6.2.** Let $U$ be some open set in $\mathbb{R}^n$, $f_1, \ldots, f_p$ be some real valued analytic functions on $U$ s.t. $\{df_j = 0\} \subset \{f_j = 0\}$, $\prod_{j=1}^{p} \log^k (f_j + i0)(f_j + i0)^{\lambda_j}$ some family of distributions depending meromorphically on $\lambda \in \mathbb{C}^p$. Set

$$\Gamma = \bigcup_J Z_J$$

where $J$ ranges over subsets of $\{1, \ldots, p\}$ and

$$Z_J = \{(x, \xi) \in T^*U \text{ s.t. } (x, \alpha_{j,p}^{(j)}(x))_p \in (U \times \mathbb{R}^p)^N, \forall j \in J, f_j(x) = 0, x_p \rightarrow x, \sum_{j \in J} a_j dw_j(x_p) \rightarrow \xi\}.$$

Then there exists a thin set $Z \subset \mathbb{C}^p$ containing the poles of $\prod_{j=1}^{p} \log^k (f_j + i0)(f_j + i0)^{\lambda_j}$ such that for all $\lambda \notin Z$:

$$WF \left( \prod_{j=1}^{p} \log^k (f_j + i0)(f_j + i0)^{\lambda_j} \right) \subset \bigcup_{J \subset \{1, \ldots, p\}} Z_J.$$
Proof. It is enough to establish the Theorem for the family $\prod_{j=1}^{p} (f_j + i0)^{\lambda_j}$ since
\[
\prod_{j=1}^{p} \left( \frac{d}{d\lambda_j} \right)^{k_j} \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} = \left( \prod_{j=1}^{p} \text{log}^{k_j} (f_j + i0)(f_j + i0)^{\lambda_j} \right).
\]
We follow closely the architecture of the proof of Theorem 4.1. First, by Lemma 6.1, we can consider that $Re(\lambda_j)$ is chosen large enough. We write $\prod_{j=1}^{p} (f_j + i0)^{\lambda_j}$ as the integral formula:
\[
\prod_{j=1}^{p} (f_j + i0)^{\lambda_j} = \int_{\mathbb{R}^p} dt_1 \ldots dt_p \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \right).
\]
Let $\pi$ be the projection $\pi : (t_1, \ldots, t_p, x) \in \mathbb{R}^p \times \mathbb{R}^n \mapsto x \in \mathbb{R}^n$, then the above formula writes as the pushforward:
\[
\prod_{j=1}^{p} (f_j + i0)^{\lambda_j} = \pi_\ast \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \right).
\]
Step 1 First, let us show that for $Re(\lambda_j), j \in \{1, \ldots, p\}$ large enough the product $\left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \right)$ makes sense in $\mathcal{D}'$. The seperate distributional products $\prod_{j=1}^{p} \delta_{t_j-f_j}$ and $\prod_{j=1}^{p} (t_j + i0)^{\lambda_j}$ both make sense since they satisfy the Hörmander condition. For $Re(\lambda_j)$ large enough, arguing as in the proof of Theorem 4.1 one can easily prove that the product $\prod_{j=1}^{p} (t_j + i0)^{\lambda_j}$ can be made sufficiently regular in the Sobolev sense so that the distributional product $\left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \right)$ makes sense. Indeed it suffices that $\prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \in H^s(\mathbb{R}^{n+p})$ for $s > \frac{n}{2}$ since $\prod_{j=1}^{p} \delta_{t_j-f_j} \in H^{-\frac{n}{2} - \epsilon}(\mathbb{R}^{n+p}), \forall \epsilon > 0$ by the Sobolev trace theorem.

Step 2 We study $WF \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \right)$.
\[
WF \left( \prod_{j=1}^{p} \delta_{t_j-f_j} \right) = \bigcup_{J \subset \{1, \ldots, p\}} \Gamma_J, \quad \Gamma_J = \{(t, x; \tau, \xi) \text{ s.t. } f_j(x) = t_j, \xi = -\sum_{j \in J} \tau_j df_j, \tau_j \neq 0\}
\]
\[
WF \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \right) = \bigcup_{J \subset \{1, \ldots, p\}} \{(t, x; \tau, 0) \text{ s.t. } t_j = 0, \tau_j > 0 \text{ if } j \in J, \tau_j = 0 \text{ otherwise }\}.
\]
By the $u = 0$ Theorem:
\[
WF \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \right) \subset WF \left( \prod_{j=1}^{p} \delta_{t_j-f_j} \right) \overset{\sim}{=} WF \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \right)
\]
The wave front set of $WF \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \prod_{j=1}^{p} \delta_{t_j-f_j} \right)$ is not interesting outside $\cup_j \{f_j = 0\}$ since it will not contribute after push-forward by $\pi$.
\[
\Gamma_{J,F} = \{(t, x; \tau, \xi) \text{ s.t. } \forall j \in J, t_j = 0, x_n \to x, f_j(x) = 0, \xi_n = -\sum \tau_j^l df_j(x_n) \to \xi, \tau_j^l < \tau^l\}.
\]
In fact, by definition of the $\overset{\sim}{=}$ operation of Igolnizer:
\[
\left( WF \left( \prod_{j=1}^{p} \delta_{t_j-f_j} \right) \overset{\sim}{=} WF \left( \prod_{j=1}^{p} (t_j + i0)^{\lambda_j} \right) \right) \cap T^*_x \{t_j=0\} (\mathbb{R}^p \times U) \subset \bigcup \Gamma_{J,F}.
\]
Step 3, we evaluate the wave front set of the push-forward. The interesting elements of \( \Gamma_{J,f} \) are the \( \tau = 0 \) points and are calculated as follows

\[
\Gamma_{J,f} \cap \{(0, x; \tau, \xi) \text{ s.t. } \forall j \in J, \tau_j = 0\} = \{(0, x; \tau, \xi) \text{ s.t. } \forall j \in J, \tau_j = 0, x_n \to x, f_j(x) = 0, \xi_n = \sum \tau_n \delta f_j(x_n) \to \xi, \tau_n \geq 0\}.
\]

Then it is immediate that \( \pi_\ast (\Gamma_{J,f}) = Z_J \).

We can easily deduce from the above proof and the \( u = 0 \) Theorem 6.3 that when \( \Re(\lambda_j), \forall j \) is large enough, the product \( \left( \prod_{j=1}^p (t_j + i0)^{\lambda_j} \right) \) is continuous in \( \lambda \) with value \( \mathcal{D}'(U_{\lambda}) \), and therefore by continuity of the pushforward \( \mathcal{D}'_{U_{\lambda}} \), the family \( \left( \prod_{j=1}^p (f_j + i0)^{\lambda_j} \right)_{\lambda \in \mathbb{C}^p} \) is continuous in \( \lambda \) with value \( \mathcal{D}'(U_{\lambda}) \). We also know by Theorem 1.2 that the family \( \prod_{j=1}^p (f_j + i0)^{\lambda_j} \) is both continuous in \( \lambda \) with value in \( \mathcal{D}'(U_{\lambda}) \). We further assume that \( \Gamma \) satisfies a strong convexity if for any pair of sequence \( (x_n; \xi_n), (x_n, \eta_n) \) in \( \Gamma \) such that \( (x_n; \xi_n + \eta_n) \to (x; \xi) \), both \( |\xi_n| \) and \( |\eta_n| \) are bounded.

**Theorem 6.3.** Under the assumptions of the above Theorem, the family \( \prod_{j=1}^p (f_j + i0)^{\lambda} \) is meromorphic with value \( \mathcal{D}'(U_{\lambda}) \).

6.1. Geometric assumptions and functional properties. We recall the objects of our study. Let \( U \) be some open set in \( \mathbb{R}^n \), \( (f_1, \ldots, f_p) \) be some real valued analytic functions on \( U \) s.t. \( \{df_j = 0\} \subset \{f_j = 0\} \), then we showed that \( \prod_{j=1}^p (f_j + i0)^{\lambda} \) is a family of distributions depending meromorphically on \( \lambda \in \mathbb{C}^p \) with value \( \mathcal{D}'(U_{\lambda}) \) where:

\[
\Gamma = \bigcup_j Z_J
\]

where \( J \) ranges over subsets of \( \{1, \ldots, p\} \) and

\[
Z_j = \{(x; \xi) \in T^*U \text{ s.t. } \{(x, a^p_j(x)) \in (U \times \mathbb{R}^2) \} \}, \forall j \in J, f_j(x) = 0, x_p \to x, \sum_{j \in J} a^p_j df_j(x_p) \to \xi\}.
\]

In this part, our goal is to add geometric assumptions on the critical loci \( \bigcup \{df_j = 0\} \) in order to give a nicer description of the conic set \( \Gamma \).

6.1.1. Stratification, regularity condition and polarization. We define the following three geometric conditions:

1. **Stratification:** The critical loci \( \{df_j = 0\} \) are smooth analytic submanifolds and for every \( J \subset \{1, \ldots, p\} \) the submanifolds \( \{df_j = 0\} \) for \( j \in J \) intersect cleanly. Define the submanifolds

\[
\Sigma_J = \bigcap_{j \in J} \{df_j = 0\}
\]

2. **Polarization:** let \( \Sigma = \bigcup_j \{df_j = 0\} \), then for all \( x \in \bigcup_j \{f_j = 0\}, x \not\in \Sigma \), for all \( a_j > 0 \), \( \sum a_j df_j(x) \neq 0 \) and there is a closed convex conic subset \( \Gamma \) of \( T^*U \setminus \Sigma \) s.t. \( (x; \sum a_j df_j(x)) \in \Gamma_x \). We further assume that \( \Gamma \) satisfies a strong convexity condition which reads as follows:

**Definition 6.1.** Let \( U \) be an open manifold and \( \Gamma \subset T^*U \) a closed conic set. Then \( \Gamma \) is strongly convex if for any pair of sequence \( (x_n; \xi_n), (x_n, \eta_n) \) in \( \Gamma \) such that \( (x_n; \xi_n + \eta_n) \to (x; \xi) \), both \( |\xi_n| \) and \( |\eta_n| \) are bounded.
(3) **Regularity**: a microlocal regularity condition on the strata which is a particular version of Verdier’s \( w \) condition [13].

\[
\forall (x, y) \in \mathbb{R}^n \setminus \{(f_j = 0) \times \{f_j = 0\} \setminus \{df_j = 0\}\}, \delta(N^*_x(\{df_j = 0\}), \frac{df_j(y)}{|df_j(y)|}) \leq C|x - y|.
\]

where for two vector spaces \( V, W \), \( \delta(V, W) = \sup_{x \in V, |x| = 1} \text{dist} (x, W) \).

**Proposition 6.1.** Let \( U \) be some open set in \( \mathbb{R}^n \), \( (f_1, \ldots, f_p) \) be some real valued analytic functions on \( U \) s.t. \( \{df_j = 0\} \subset \{f_j = 0\} \) Assume the above three conditions are satisfied, then the set \( \Gamma \) defined by equation (63) satisfies the identity:

\[
\bigcup_j \{(x; \xi) | j \in J, f_j(x) = 0, df_j(x) \neq 0, \xi = \sum_{j \in J} a_j df_j(x), a_j > 0 \} \cup N^* \Sigma J
\]

where \( \Sigma_J \) is the submanifold obtained as the clean intersection of the critical submanifolds \( \{df_j = 0\}, \forall j \in J \).

**Proof.** It suffices to evaluate each set \( Z_J \) separately. By polarization, on the analytic set \( \cup_j \{f_j = 0\} \) minus the critical locus \( \cup_j \{df_j = 0\} \), \( Z_J \) is easily calculated and equals

\[
\bigcup_j \{(x; \xi) | j \in J, f_j(x) = 0, df_j(x) \neq 0, \xi = \sum_{j \in J} a_j df_j(x), a_j > 0 \}.
\]

The difficulty resides in the study of \( Z_J \) over the critical locus. First use the assumption that there is some convex conic set \( \Gamma \) s.t. \( \sum_{j \in J} a^2_j df_j(x_p) \in \Gamma_{x_p} \), the strong convexity condition implies that the convergence \( \sum_{j \in J} a^2_j df_j(x_p) \rightarrow \xi \) prevents the sequences \( a^p_j df_j(x_p) \) from blowing up. Up to extraction of a convergent subsequence, assume w.l.o.g. \( a^2_j df_j(x_p) \rightarrow \xi_j \), then the regularity condition implies that \( \xi_j \in N^*_x(\{df_j(x) = 0\}) \) and

\[
\sum_j a^p_j df_j(x_p) \xrightarrow{p \rightarrow +\infty} \sum_j \xi_j \in \sum_{j \in J} N^*_x(\{df_j(x) = 0\})
\]

\[
\Rightarrow \sum_j a^p_j df_j(x_p) \xrightarrow{p \rightarrow +\infty} \xi \in N^*_x(\Sigma_J)
\]

because the submanifolds \( \{df_j = 0\}, j \in J \) cleanly intersect on the submanifold \( \Sigma_J \).

In practical applications for QFT, we will have to check that the above conditions are always satisfied in order to apply the following Theorem:

**Theorem 6.4.** Under the assumptions of paragraph 6.1.1, the family \( \left( \prod_{j=1}^p (f_j + i \lambda)^{k_j} \right)_{\lambda \in \mathbb{C}^p} \)

depends meromorphically on \( \lambda \) with linear poles with value \( D'_\lambda \) where

\[
\Lambda = \bigcup_j \{(x; \xi) | j \in J, f_j(x) = 0, df_j(x) \neq 0, \xi = \sum_{j \in J} a_j df_j(x), a_j > 0 \} \cup N^* \Sigma J.
\]

The distribution

\[
\mathcal{R}_\xi \left( \prod_{j=1}^p (f_j + i \lambda)^{k_j} \right) \in D'(U)
\]

is a distributional extension of \( \prod_{j=1}^p (f_j + i \lambda)^{k_j} \in D'(U \setminus X) \).

**Proof.** We already know by Theorem 6.2 that \( \Lambda = WF \left( \prod_{j=1}^p (f_j + i \lambda)^{k_j} \right) \subset \bigcup_j Z_J \) and \( \Lambda \) is determined from Proposition 6.1.
The meromorhpicity with value $D'_{\lambda} \Lambda$ is a consequence of Proposition Theorem 6.3.

Finally, the fact that the singular part is supported on the critical locus results from the fact that outside $\Sigma = \bigcup_j \{ df_j = 0 \}$, the distributional products $\left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)$ is well defined and is bounded in $\lambda$ by hypocontinuity of the Hörmander product $[9]$ and therefore the family $\left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)_\lambda$ is both meromorphic in $\lambda$ by [13] (by the resolution of singularities of Hironaka) and locally bounded it is thus holomorphic in $\lambda$ by [5] It follows that for all test function $\varphi \in D(U \setminus \Sigma)$, $\pi \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j}(\varphi) \right) = \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j}(\varphi) \right)$ since $\pi$ is a projection on holomorphic functions and it follows that

$$\mathcal{R}_\pi \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j} \right)(\varphi) = \lim_{\lambda \to k} \left( \prod_{j=1}^{p} (f_j + i0)^{\lambda_j}(\varphi) \right)$$

where the limit exists since the wave front set are transverse outside $\Sigma$. $\square$

---

**Part II: Application to meromorphic regularization in QFT.**

### 7. Causal manifolds and Feynman relations.

The goal of this part is to give a definition of Feynman propagators which are needed to calculate vacuum expectation values (VEV) of times ordered products (T-products) in QFT. Our exposition will stress the importance of the causal structure of the Lorentzian manifolds considered.

To define a causal structure on a smooth manifold $M$, we will essentially follow Schapira’s exposition [11] [20] (strongly inspired by Leray’s work) which makes use of no metric since the causal structure is more fundamental than a metric structure and define some cone $\gamma$ in cotangent space $T^*M$ which will induce a partial order on $M$. This presentation is convenient since the same cone will be used to describe wave front sets of Feynman propagators and Feynman amplitudes.

#### 7.0.2. Admissible cones in cotangent space.

For a manifold $M$ we denote by $q_1$ and $q_2$ the first and second projection defined on $M \times M$. We denote by $d_2$ the diagonal of $M \times M$. A cone $\gamma$ in a vector bundle $E \to M$ is a subset of $E$ which is invariant by the action of $\mathbb{R}_+$ on this vector bundle. We denote by $-\gamma$ the opposite cone to $\gamma$, and by $\gamma^\circ$ the polar cone to $\gamma$, a closed convex cone of the dual vector bundle $\gamma^\circ = \{ (x, \xi) \in E^* ; \langle \xi, v \rangle \geq 0, \forall v \in \gamma \}$. In all this section, we assume that $M$ is connected. A closed relation on $M$ is a closed subset of $M \times M$.

**Definition 7.1**. Let $Z$ be a closed subset of $M \times M$ and $A \subset M$ a closed set.

**Definition 7.2**. A cone $\gamma \subset T^*M \setminus \mathcal{Q}$ is admissible if it is closed proper convex, $\gamma \cap -\gamma = \emptyset$ and $\operatorname{Int}(\gamma_x) \neq \emptyset$ i.e. the interior of $\gamma_x \subset T^*_x M$ is non empty for any $x \in M$.

7.0.3. A preorder relation. In the literature, one often encounters time-orientable Lorentzian manifolds to which one can associate a cone in $TM$ or its polar cone in $T^*M$. Here, we only assume that: $M$ is a $C^\infty$ real connected manifold and we are given an admissible cone $\gamma$ in $T^*M$.

**Definition 7.3**. A $\gamma$-path is a continuous piecewise $C^1$-curve $\lambda : [0, 1] \to M$ such that its derivative $\lambda'(t)$ satisfies $\langle \lambda'(t), v \rangle \geq 0$ for all $t \in [0, 1]$ and $v \in \gamma$. Here $\lambda'(t)$ means as well the right or the left derivative, as soon as it exists (both exist on $[0, 1]$ and are almost everywhere the same, and $\lambda'_+(0)$ and $\lambda'_-(1)$ exist).
To $\gamma$ one associates a preorder on $M$ as follows: $x \leq y$ if and only if there exists a $\gamma$-path $\lambda$ such that $\lambda(0) = x$ and $\lambda(1) = y$.

For a subset $A$ of $M$, we set:

$$A_\geq = \{x \in M; \exists y \in A, x \leq y\},$$
$$A_\leq = \{x \in M; \exists y \in A, x \geq y\}.$$ 

Intuitively, $A_\geq$ (resp $A_\leq$) represents the past (resp the future) of the set $A$ for the causal relation.

7.0.4. Topological assumptions. We may assume that the relation $\leq$ is closed and that it is proper:

- $x_n \leq y_n, \forall n$ and $(x_n, y_n) \rightarrow (x, y) \implies x \leq y$,
- for compact sets $A, B, A_\geq \cap B_\leq$ is compact.

**Definition 7.4.** A pair $(M, \gamma)$ where $\gamma \subset T^\ast M$ is an admissible cone whose induced preorder relation $\leq$ is closed and proper is called causal.

An admissible cone $\gamma$ induces a subset $Z_\gamma \subset M \times M$ that we call the graph of the preorder relation $\leq$:

$$Z_\gamma = \{(x, y) \in M \times M \text{ s.t. } x \leq y\} \tag{70}$$

The topological assumptions on $\leq$ imply that $Z_\gamma$ is closed and that for all compact subset $A \times B \subset M \times M$, $q_1^{-1}(A) \cap q_2^{-1}(B) \cap Z_\gamma$ is compact. Lorentzian manifolds are particular cases of causal manifolds. The globally hyperbolic spacetimes defined by Leray are particular cases of causal manifolds where [3 Definition 1.3.8 p. 23]:

- the preorder relation is a **partial order relation** i.e.
  $$(x \leq y, y \leq x) \implies x = y$$
- the relation is **strongly causal**, for all open set $U \subset M$, for all $x \in M$ there is some neighborhood $V$ of $x$ in $U$ such that all causal curves whose endpoints are in $V$ are in fact contained in $V$
- and the space of $\gamma$-paths is **compact** in the natural topology on the space of rectifiable curves induced from any smooth metric on $M$.

7.1. **Feynman relations and propagators.** We assume that $(M, \gamma)$ is a causal manifold. Relations are subsets of the cotangent space $T^\ast (M \times M)$. We denote by $N^\ast (d_2)$ the conormal bundle of the diagonal $d_2 \subset M \times M$. If $(M, g)$ is a Lorentzian manifold, we denote by $(x_1; \xi_1) \sim (x_2; \xi_2)$ if the two elements $(x_1; \xi_1), (x_2; \xi_2)$ are connected by a bicharacteristic curve of $\Box_g$ in cotangent space $T^\ast M$.

**Definition 7.5.** Let $(M, \gamma)$ be a causal manifold. A subset $\Lambda \subset T^\ast (M \times M)$ is a polarized relation if

$$\Lambda \subset \{x_1 < x_2 \text{ and } \xi_2 \in \gamma_{x_2}, \xi_1 \in -\gamma_{x_1}\} \cup \{x_2 < x_1 \text{ and } \xi_1 \in \gamma_{x_2}, \xi_2 \in -\gamma_{x_2}\} \cup N^\ast (d_2).$$

If we assume moreover that $(M, g)$ is Lorentzian then a subset $\Lambda \subset T^\ast (M \times M)$ is a Feynman relation if

$$\Lambda \subset \{(x_1, x_2; \xi_1, \xi_2) \text{ s.t. } (x_1; \xi_1) \sim (x_2; -\xi_2) \text{ and } \xi_2 \in \gamma \text{ if } x_2 > x_1 \text{ and } \xi_2 \in -\gamma \text{ if } x_1 > x_2\} \cup N^\ast (d_2).$$

Feynman relations are particular cases of polarized relations.

**Definition 7.6.** Let $(M, g)$ be a Lorentzian manifold, $\gamma$ the corresponding admissible cone and $\Box_g$ the corresponding wave operator. Then $G \in D^\prime(M \times M)$ is called **Feynman propagator** if $G$ is a fundamental bisolution of $\Box_g + m^2$.

$$\Box_g + m^2)G(x, y) = \delta(x, y) \tag{71}$$
$$\Box_g + m^2)G(x, y) = \delta(x, y) \tag{72}$$
and $WF(G)$ is a Feynman relation in $T^\bullet (M \times M)$.

7.2. Wave front set of Feynman amplitudes outside diagonals. We develop a machinery which allows us to describe wave front sets of Feynman amplitudes which are distributions living on configuration spaces of causal manifolds.

7.2.1. Configuration spaces. For every finite subset $I \subset \mathbb{N}$ and open subset $U \subset M$, we define the configuration space $U^I = \text{Maps} (I \to U) = \{(x_i)_{i \in I} \text{ s.t. } x_i \in U, \forall i \in I\}$ of $|I|$ particles in $U$ labelled by the subset $I \subset \mathbb{N}$. In the sequel, we will distinguish two types of diagonals in $U^I$, the big diagonal $D_I = \{(x_i)_{i \in I} \text{ s.t. } \exists (i \neq j) \in I^2, x_i = x_j\}$ which represents configurations where at least two particles collide, and the small diagonal $d_I = \{(x_i)_{i \in I} \text{ s.t. } \forall (i, j) \in I^2, x_i = x_j\}$ where all particles in $U^I$ collapse over the same element. The configuration space $M^{(1,...,n)}$ and the corresponding big and small diagonals $D_{\{1,...,n\}}, d_{\{1,...,n\}}$ will be denoted by $M^n, D_n, d_n$ for simplicity.

For QFT, we are let to introduce the concept of polarization to describe subsets of the cotangent of configuration spaces $T^*M^n$ for all $n$ where $(M, \gamma)$ is a causal manifold: this generalizes the concept of positivity of energy for the cotangent space of configuration space.

7.2.2. Polarized subsets. In order to generalize this condition to the wave front set of Feynman amplitudes, we define the right concept of positivity of energy which is adapted to conic sets in $T^*M^n$:

**Definition 7.7.** Let $(M, \gamma)$ be a causal manifold. We define a reduced polarized part (resp reduced strictly polarized part) as a conical subset $\Xi \subset T^*M$ such that, if $\pi : T^*M \to M$ is the natural projection, then $\pi(\Xi)$ is a finite subset $A = \{a_1, \ldots, a_r\} \subset M$ and, if $a \in A$ is maximal (in the sense there is no element $\bar{a}$ in $A$ s.t. $\bar{a} > a$), then $(\Xi \cap T^*_a M) \subset (\gamma \cup 0)$ (resp $\Xi \cap T^*_a M \subset \gamma$).

We define the trace operation as a map which associates to each element $p = (x_1, \ldots, x_k; \xi_1, \ldots, \xi_k) \in (T^*M)^k$ some finite part $Tr(p) \subset T^*M$.

**Definition 7.8.** For all elements $p = ((x_1, \xi_1), \ldots, (x_k, \xi_k)) \in T^*M^k$, we define the trace $Tr(p) \subset T^*M$ defined by the set of elements $\eta \in T^*M$ such that $\exists i \in [1,k]$ with the property that $x_i = a, \xi_i \neq 0$ and $\eta = \sum_{i:x_i=a} \xi_i$.

Then finally, we can define polarized subsets $\Gamma \subset T^*M^k$:

**Definition 7.9.** A conical subset $\Gamma \subset T^*M^k$ is polarized (resp strictly polarized) if for all $p \in \Gamma$, its trace $Tr(p)$ is a reduced polarized part (resp reduced strictly polarized part) of $T^*M$.

We enumerate easy to check properties of polarized subsets:

- the union of two polarized (resp strictly polarized) subsets is polarized (resp strictly polarized),
- if a conical subset is contained in a polarized subset it is also polarized,
- the projection $p : M^I \to M^J$ for $J \subset I$ acts by pull–back as $p^* : T^*M^J \to T^*M^I$ and sends polarized (resp strictly polarized) subsets to polarized (resp strictly polarized) subsets.

The role of polarization is to control the wave front set of the Feynman amplitudes of the form $\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j) \in D'(M^n \setminus D_n), n_{ij} \in \mathbb{N}$ where $G$ is a Feynman propagator.

**Proposition 7.1.** Let $(M, \gamma)$ be a causal manifold. If $\Lambda \subset T^\bullet (M \times M)$ is a Feynman relation, then $\Lambda$ is polarized and $\Lambda \cap T^\bullet (M^2 \setminus d_2)$ is strictly polarized.
Theorem 7.1. Let $u,v$ be two distributions in $\mathcal{D}'(\Omega)$, for some subset $\Omega \subset M^n$, s.t. $WF(u) \cap T^*\Omega$ is polarized and $WF(v) \cap T^*\Omega$ is strictly polarized. Then the product $uv$ makes sense in $\mathcal{D}'(\Omega)$ and $WF(uv) \cap T^*\Omega$ is polarized. Moreover, if $WF(u)$ is also strictly polarized then $WF(uv)$ is strictly polarized.

Proof. Step 1: we prove $WF(u) + WF(v) \cap T^*\Omega$ does not meet the zero section. For any element $p = (x_1,\ldots,x_n;\xi_1,\ldots,\xi_n) \in T^*M$ we denote by $-p$ the element $(x_1,\ldots,x_n;\xi_1,\ldots,-\xi_n) \in T^*M$. Let $p_1 = (x_1,\ldots,x_n;\xi_1,\ldots,\xi_n) \in WF(u)$ and $p_2 = (x_1,\ldots,x_n;\eta_1,\ldots,\eta_n) \in WF(v)$, necessarily we must have $(\xi_1,\ldots,\xi_n) \neq 0, (\eta_1,\ldots,\eta_n) \neq 0$. We will show by a contradiction argument that the sum $p_1 + p_2 = (x_1,\ldots,x_n;\xi_1 + \eta_1,\ldots,\xi_n + \eta_n)$ does not meet the zero section. Assume that $\xi_1 + \eta_1 = 0, \ldots, \xi_n + \eta_n = 0$ i.e. $p_1 = -p_2$ then we would have $\xi_i = -\eta_i \neq 0$ for some $i \in \{1,\ldots,n\}$ since $(\xi_1,\ldots,\xi_n) \neq 0, (\eta_1,\ldots,\eta_n) \neq 0$. We assume w.l.o.g. that $\eta_1 \neq 0$, thus $Tr(p_2)$ is non empty! Let $B = \pi(Tr(p_1)), C = \pi(Tr(p_2))$, we first notice $B = C$ since $p_2 = -p_1 \implies Tr(p_1) = -Tr(p_2) \implies \pi \circ Tr(p_1) = \pi \circ Tr(p_2)$. Thus if $a$ is maximal in $B$, $a$ is also maximal in $C$ and we have

$$0 = \sum_{x_i=a} \xi_i + \eta_i = \sum_{x_i=a} \xi_i + \sum_{x_i=a} \eta_i \in (\gamma_a \cup \underline{\gamma} + \gamma_a) = \gamma_a,$$

(since $p_1$ is polarized and $p_2$ is strictly polarized) contradiction!

Step 2, we prove that the set

$$(WF(u) + WF(v)) \cap T^*\Omega$$

is strictly polarized. Recall $B = \pi \circ Tr(p_1), C = \pi \circ Tr(p_2)$ and we denote by $A = \pi \circ Tr(p_1 + p_2)$ hence in particular $A \subset B \cup C$. We denote by $max A$ (resp max $B$, max $C$) the set of maximal elements in $A$ (resp $B$, $C$). The key argument is to prove that $max A = max B \cap max C$. Because if $max A = max B \cap max C$ holds then for any $a \in max A, \sum_{x_i=a} \xi_i + \eta_i = \sum_{x_i=a} \xi_i + \sum_{x_i=a} \eta_i \in \gamma_a$ since $a \in max B \cap max C$ and $Tr(p_1)$ is a reduced polarized part and $Tr(p_2)$ is reduced strictly polarized. Thus $max A = max B \cap max C$ implies that $p_1 + p_2$ is strictly polarized.

We first establish the inclusion $(maxB \cap max C) \subset max A$. Let $a \in max B \cap max C$, then $\sum_{x_i=a} \xi_i \in \gamma_a \cup \{0\}$ and $\sum_{x_i=a} \eta_i \in \gamma_a$. Thus $\sum_{x_i=a} \xi_i + \eta_i \in \gamma_a \implies \sum_{x_i=a} \xi_i + \eta_i \neq 0$ so there must exist some $i$ for which $x_i = a$ and $\xi_i, \eta_i \neq 0$. Hence $a \in A$. Since $A \subset B \cup C$, $a \in max B \cap max C$, we deduce that $a \in max A$ (if there were $\tilde{a}$ in $A$ greater than $a$ then $\tilde{a} \in B$ or $\tilde{a} \in C$ and $a$ would not be maximal in $B$ and $C$).

We show the converse inclusion $max A \subset (max B \cap max C)$ by contraposition. Assume $a \notin max B$, then there exists $x_{j_1} \in max B$ s.t. $x_{j_1} > a$ and $\xi_{j_1} \neq 0$. There are two cases.
• either \( x_{j_1} \in \text{max } C \) as well, then \( \sum_{x_{j_1} = x_i} \xi_i + \eta_i \in \gamma_{x_{j_1}} \implies \sum_{x_{j_1} = x_i} \xi_i + \eta_i \neq 0 \) and there is some \( i \) for which \( x_i = x_{j_1} \) and \( \xi_i + \eta_i \neq 0 \) thus \( x_{j_1} \in A \) and \( x_{j_1} > a \) hence \( a \notin \text{max } A \).

• or \( x_{j_1} \notin \text{max } C \) then there exists \( x_{j_2} \in \text{max } C \) s.t. \( x_{j_2} > x_{j_1} \) and \( \eta_{j_2} \neq 0 \). Since \( x_{j_1} \in \text{max } B \), we must have \( \xi_{j_2} = 0 \) so that \( x_{j_2} \notin B \). But we also have \( \xi_{j_2} + \eta_{j_2} = \eta_{j_2} \neq 0 \) so that \( x_{j_2} \in A \). Thus \( x_{j_2} \in A \) is greater than \( a \) hence \( a \notin \text{max } A \).

We thus proved

\[
a \notin \text{max } B \implies a \notin \text{max } A
\]

and by symmetry of the above arguments in \( B \) and \( C \), we also have

\[
a \notin \text{max } C \implies a \notin \text{max } A.
\]

We established that \((\text{max } B)^c \subset (\text{max } A)^c \) and \((\text{max } C)^c \subset (\text{max } A)^c \), thus \((\text{max } B)^c \cup (\text{max } C)^c \subset (\text{max } A)^c \) therefore \( \text{max } A \subset \text{max } B \cap \text{max } C \), from which we deduce the equality \( \text{max } A = \text{max } B \cap \text{max } C \) which implies that \( \text{WF}(u) + \text{WF}(v) \) is strictly polarized and \( \text{WF}(uv) \) is polarized. \( \square \)

An immediate corollary of the above Theorem is that Feynman amplitudes are well defined outside diagonals

**Corollary 7.1.** Let \( G \in \mathcal{D}'(M^2) \) be a distribution whose wave front set is a Feynman relation. Then for all \( n \in \mathbb{N}^* \), the distributional products

\[
\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j)
\]

are well defined in \( \mathcal{D}'(M^n \setminus D_n) \) and \( \text{WF} \left( \prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j) \right) \) is strictly polarized on \( M^n \setminus D_n \).

**Proof.** This follows from the fact that Feynman relations are strictly polarized outside \( D_n \) hence all wave front sets are transverse by Theorem [7.1] and the wave front of products are strictly polarized. \( \square \)

8. **Meromorphic regularization of the Feynman propagator on Lorentzian manifolds.**

Let \((M, g)\) be a real analytic manifold with real analytic Lorentzian metric. Our construction of meromorphic regularization will not work on every globally hyperbolic manifold but on a category of “convex analytic Lorentzian spacetimes equipped with a Feynman propagator”.

8.1. **A category from convex Lorentzian spacetimes.** The language of category theory is not really necessary but rather convenient for our discussion of the functorial behaviour of our renormalizations. Let us introduce the category \( M_{ca} \) which is contained in the category \( M_a \) of open analytic Lorentzian spacetimes. An **object** \((M, g, G)\) of \( M_{ca} \) is

1. an open real analytic manifold \( M \).
2. \( M \) is endowed with a real analytic Lorentzian metric \( g \) s.t. \((M, g)\) is geodesically convex i.e. for every pair \((x, y)\) \( \in M^2 \), there is a unique geodesic of \( g \) connecting \( x \) and \( y \). For all \( x \in M \), we denote by \( \exp_x \) the exponential map based at \( x \). Since \( M \) is convex, the range of \( \exp_x \) is the whole manifold \( M \).
3. A Feynman propagator \( G \) which is a bisolution of the Klein Gordon operator:

\[
(\Box_x + m^2) G(x, y) = \delta(x, y)
\]

\[
(\Box_y + m^2) G(x, y) = \delta(x, y)
\]
and \( G \) admits a \textit{representation} for \((x, y) \in M^2\) sufficiently close:

\[
G(x, y) = \frac{U}{\Gamma + i0} + V \log (\Gamma + i0) + W
\]

where \( \Gamma(x, y) \) is the Synge world function defined as

\[
\Gamma(x, y) = \langle \exp^{-1}_x(y), \exp^{-1}_x(y) \rangle_{g_x}
\]

and \( \Gamma, U, V, W \) are all analytic functions.

The \textbf{morphisms} of \( M_{\text{ca}} \) are defined to be the analytic embeddings \( \Phi : (M, g, G) \mapsto (M', g', G') \) such that \( \Phi^* g' = g \), in other words \( \Phi \) is an \textit{isometric embedding} and \( \Phi^* G' = G \). Note that geodesics are sent to geodesics under isometries, hence a Lorentzian manifold isometric to a convex Lorentzian manifold is automatically convex.

\subsection{Holonomic singularity of the Feynman propagator along diagonals.}

Once we have defined a suitable category of spacetimes on which we could work, we can discuss the asymptotics of Feynman propagators near the diagonal of configuration space \( M^2 \). A classical result which goes back to Hadamard \cite{28, 3} states that one can construct a Feynman propagator \( G \) which admits a \textit{representation} for \((x, y) \in M^2\) sufficiently close:

\[
G(x, y) = \frac{U}{\Gamma + i0} + V \log (\Gamma + i0) + W
\]

where \( \Gamma(x, y) \) is the Synge world function defined as

\[
\Gamma(x, y) = \langle \exp^{-1}_x(y), \exp^{-1}_x(y) \rangle_{g_x}
\]

and \( \Gamma, U, V, W \) are all analytic functions. As explained in the introduction, the key idea is that this asymptotic expansion is of \textit{regular holonomic type} i.e it is in the \( \mathcal{O} \) module generated by distributions defined as boundary values of holomorphic functions: \( (\Gamma + i0)^{-1}, \log(\Gamma + i0) \). The function \( \Gamma \) should be thought of as the square of the pseudodistance in the pseudoriemannian setting and replaces the quadratic form of signature \((1, 3)\) used in Minkowski space \( \mathbb{R}^{3+1} \). Since \( M \) belongs to the category \( M_{\text{ca}} \), \( M \) is convex therefore the inverse exponential map \( \exp^{-1}_x(y) \) associated to the metric \( g \) is well defined for all \((x, y) \in M^2\) and \( \Gamma \) is globally defined on \( M^2 \). The analytic variety \( \{ \Gamma(x, y) = 0 \} \subset M^2 \) is the null conoid associated to the Lorentzian metric \( g \).

We denote by \( d_2 \) the diagonal \( \{ x = y \} \subset M^2 \) of configuration space \( M^2 \). The next step is to define the regularization \( (G_\lambda)_{\lambda} \) of the propagator \( G \). A simple solution consists in multiplying with some complex powers of the function \( \Gamma \):

\begin{definition}
Let \( (M, g, G) \in M_{\text{ca}} \), we define the meromorphic regularization of \( G \) as the distribution

\[
G_\lambda = G(\Gamma + i0)^\lambda.
\]

If \( M \in M_{\text{ca}} \) is not convex, then we choose a cut–off function \( \chi \) such that \( \chi = 1 \) in some neighborhood of the diagonal and \( \chi = 0 \) outside some neighborhood \( V \) of the diagonal \( d_2 \) such that for any \((x, y) \in V\) there is a unique geodesic connecting \( x \) and \( y \) which implies that \( \Gamma \) is well–defined on \( V \). Then define

\[
G_\lambda = G(\Gamma + i0)^\lambda \chi + G(1 - \chi).
\]

Intuitively, the role of the factor \( (\Gamma + i0)^\lambda \) is to smooth out the singularity of the Feynman propagator \( G \) along the null conoid \( \{ \Gamma(x, y) = 0 \} \) when \( \text{Re}(\lambda) \) is large enough. We assume our Lorentzian manifold to be time oriented and to be foliated by Cauchy hypersurfaces corresponding to some time function \( t \). The Lorentzian
metric $g$ induces the existence of the natural causal partial order relation $\preceq$, and some convex cone $\gamma \subset T^* M$ of covectors of positive energy:

\begin{equation}
\gamma = \{ (x; \xi, \eta) \text{ s.t. } g_{\xi, \eta}(\xi, \eta) \geq 0, dt(\xi) \geq 0 \}. \tag{81}
\end{equation}

We denote by $d_2 \subset M \times M$ the diagonal $\{ x = y \}$ in $M^2$. We describe the conic set which contains the wave front set of the two point functions and we study its main properties.

**Proposition 8.1.** Let $\Gamma \in C^\infty(M^2)$ be the function defined as

\begin{equation}
\Gamma(x,y) = \langle \exp_x^{-1}(y), \exp_y^{-1}(y) \rangle_{g_s}. \tag{82}
\end{equation}

Then:

1. $\{ (x, y; \xi, \eta) \text{ s.t. } \Gamma(x, y) = 0, (x; \xi) \sim (y; -\eta), (x - y)^0 \xi^0 > 0 \} = \{ (x, y; \xi, \eta) \text{ s.t. } \xi = \lambda d_2 \Gamma, \eta = \lambda d_2 \Gamma, \lambda \in \mathbb{R}_{>0} \}.$

2. Set $\Lambda_2 = \{ (x; y; \xi, \eta) \text{ s.t. } \xi = \lambda d_2 \Gamma, \eta = \lambda d_2 \Gamma, \lambda \in \mathbb{R}_{>0} \} \cup N^*(d_2)$ then $\Lambda_2$ is strictly polarized over $M^2 \setminus d_2$.

**Proof.** It is classical and follows from the fact that $\Gamma$ satisfies the first order differential equation

\[ g^{\mu\nu} d_\nu \Gamma d_\mu \Gamma(x, y) = 4 \Gamma(x, y) \]

which dates back to the work of Hadamard [23]. \hfill \Box

Then we show that the families $(\Gamma + i0)^{\lambda-1}, (\Gamma + i0)^{\lambda \log(\Gamma + i0)}$ are meromorphic with value $D'_{\Lambda_2}$.

**Proposition 8.2.** Let $\Gamma$ be the function defined as

\begin{equation}
\Gamma(x,y) = \langle \exp_x^{-1}(y), \exp_y^{-1}(y) \rangle_{g_s}. \tag{83}
\end{equation}

then

- the families $(\Gamma + i0)^{\lambda-1}, (\Gamma + i0)^{\lambda \log(\Gamma + i0)}$ are meromorphic of $\lambda$ with value $D'_{\Lambda_2}$
- all coefficients of its Laurent series expansion around $\lambda = 0$ belong to $D'_{\Lambda_2}$
- its residues are conormal distributions supported by the diagonal $d_2$.

**Proof.** The fact that $\Lambda_2$ is polarized and $\Lambda_2 \setminus N^*(d_2)$ is strictly polarized follows from Proposition [7,1] which is an immediate consequence of the definition of being polarized. The three other claims are consequences of Theorem [6,4] we have to check the three assumptions of Theorem [6,4]

- **Stratification:** the critical manifold $\{ d\Gamma = 0 \}$ is the diagonal $d_2 \subset M^2$ and is a real analytic submanifold of $\{ \Gamma = 0 \}$
- **Polarization:** $\Lambda_2$ is polarized by Proposition [8,1]
- **Regularity:** we perform a local coordinate change as follows,

\[ (x, y) \in M^2 \mapsto (x, h = \exp_x^{-1}(y)) \in M \times \mathbb{R}^{3+1}. \]

In this new set of coordinates $(x, h) \in M \times \mathbb{R}^{3+1}$, the Synge world function $\Gamma$ reads $\Gamma(x, h) = h^\mu h^\nu \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the usual symmetric tensor representing the quadratic form of signature $(1,3)$. It follows that the conormal of $\{ \Gamma = 0 \}$ reads in this new coordinate system:

\begin{equation}
\{(x, h; 0, \xi) \text{ s.t. } \eta_{ij} h^i h^j = 0, \xi = \tau \eta_{ij} h^i, \tau \neq 0 \} \tag{84}
\end{equation}

and the diagonal $\{ x = y \}$ reads $\{ h = 0 \}$ hence the conormal $N^*(d_2)$ reads in this new coordinate system:

\begin{equation}
\{(x, 0; 0, \xi) \text{ s.t. } \xi \neq 0 \}. \tag{85}
\end{equation}
Hence it is immediate that $\delta(dt_{(x_1, h_1)}, N^*(d_2)(x_2, 0)) = 0$ and the regularity condition is thus verified w.r.t. the conormal $N^*(d_2)$.

\[ G_{\lambda} = G(\Gamma + i0)^{\lambda}. \]

Set $\Lambda_2 = \{(x, y; \xi, \eta) \text{ s.t. } \xi = \lambda d_x \Gamma, \eta = \lambda d_y \Gamma, \lambda \in \mathbb{R}_{>0} \} \cup N^*(d_2)$ then the family $(G_{\lambda})_{\lambda \in \mathbb{C}}$ is meromorphic with value $D_{\Lambda_2}$.

8.3. The meromorphic regularization of Feynman amplitudes. Our strategy to regularize a Feynman amplitude $\prod_{1 \leq i < j \leq n} G(x_i - x_j)^{\alpha_{ij}}$ goes as follows. For every pair of points $1 \leq i < j \leq n$, let us consider the regularized product

\[ G_{\lambda_{ij}}(x_i - x_j)^{\alpha_{ij}} = G(x_i, x_j)^{\alpha_{ij}}(\Gamma(x_i, x_j) + i0)^{\lambda_{ij}} \]

depending on the complex parameter $\lambda_{ij} \in \mathbb{C}$. Then the regularization of the whole Feynman amplitude reads:

\[ \prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i - x_j)^{\alpha_{ij}} \]

which is a family of distributions which depends \textit{meromorphically} on the multivariable complex parameter $\lambda = (\lambda_{ij})_{1 \leq i < j \leq n} \in \mathbb{C}^{\frac{n(n-1)}{2}}$ with linear poles. This follows immediately from the existence of the Hadamard expansion and Theorem 12 on the analytic continuation of complex powers of real analytic functions.

9. The regularization theorem.

Our first structure theorem claims that Feynman amplitudes depend meromorphically in the \textit{complex dimensions} $(\lambda_{ij})_{1 \leq i < j \leq n}$ with linear poles. But before we prove our first main Theorem, we need to check that the wave front sets of Feynman amplitudes on $M^n$ denoted by $\Lambda_n$ satisfies the strong convexity condition of definition 6.1.

**Definition 9.1.** We denote by $\Lambda_{ij} = \{(x_i, x_j; \xi_i, \xi_j) \text{ s.t. } \Gamma(x_i, x_j) = 0, \xi_i = \lambda d_x \Gamma, \xi_j = \lambda d_y \Gamma, \lambda \in \mathbb{R}_{>0} \} \cup N^*(d_{ij})$ the wave front set of the family $(\Gamma(x_i, x_j) + i0)^{\lambda}$ in $T^*(M^n \setminus D_n)$. Define $\Lambda_I = \left(\sum_{i<j \in I} (\Lambda_{ij} + 0)\right) \cap T^*M$ \cup $\bigcup_{J \subset I} N^*(d_J)$.

9.0.1. \textit{Strong convexity of the wave front set of Feynman amplitudes outside $D_n$}. We prove a fundamental Lemma about the conic set $\Lambda_n \cap T^*(M^n \setminus D_n)$. Recall that the Lorentzian metric $g$ induces the existence of the natural \textit{causal partial order relation} $\preceq$, and some convex cone $\gamma \subset T^*M$ of covectors of positive energy:

\[ \gamma = \{(x; \xi) \text{ s.t. } g_{\xi}(\xi, \xi) \geq 0, dt(\xi) \geq 0\}. \]

We denote by $\Lambda_{ij} = \{(x_i, x_j; \xi_i, \xi_j) \text{ s.t. } \Gamma(x_i, x_j) = 0, \xi_i = \lambda d_x \Gamma, \xi_j = \lambda d_y \Gamma, \lambda \in \mathbb{R}_{>0} \} \cup N^*(d_{ij})$ the wave front set of the family $(\Gamma(x_i, x_j) + i0)^{\lambda}$ in $T^*(M^n \setminus D_n)$.

**Lemma 9.1.** Let $\Lambda_n = \left(\sum_{1 \leq i < j \leq n} (\Lambda_{ij} + 0)\right) \cap T^*M$ \cup $\bigcup_{J \subset I} N^*(d_J)$. Then the conic set $\Lambda_n \cap T^*(M^n \setminus D_n)$ is \textit{strongly convex} in the sense of definition 6.1.

**Proof.** Let us first reformulate the strong convexity condition in our case. Let us consider the sequences

\[ (x_1(k), \ldots, x_n(k))_k, (a_{ij}(k))_k \in \mathbb{R}^N_{>0} \]
and the sequence of elements of $\Lambda_n$:

\[
(x_1(k), \ldots, x_n(k); \sum_{1 \leq i < j \leq n} a_{ij}(k) d_{x_i, x_j} \Gamma(x_i(k), x_j(k)))_{k \in \mathbb{N}}
\]

such that $(x_1(k), \ldots, x_n(k); \sum_{1 \leq i < j \leq n} a_{ij}(k) d_{x_i, x_j} \Gamma(x_i(k), x_j(k)))$ converges to $(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n) \in T^* M^n$ when $k$ goes to $\infty$. Then for all $1 \leq i < j \leq n$ the sequence of covectors $a_{ij}(k) d_{x_i, x_j} \Gamma(x_i(k), x_j(k))$ remains bounded.

Without loss of generality, we assume that $(x_1(k), \ldots, x_n(k)) \in U^n$ for some open set $U \subset M$, such that the cone $\gamma|_U \subset T^* U$ satisfies the following convexity estimate: there exists $\varepsilon > 0$ such that for all $((x; \xi), (x; \eta)) \in \gamma^2 \subset (T^* M)^2$, $\varepsilon \left(\|\xi\| + \|\eta\|\right) \leq \|\xi + \eta\|$.

We proceed by induction on $n$. Let us assume that the property holds true on all configuration spaces $M^I$ for $|I| < n$. Let us consider the sequences in $\Lambda_n$

\[
(x_1(k), \ldots, x_n(k))_k; (a_{ij}(k))_k \in \mathbb{R}_{>0}^N
\]

such that $\sum_{1 \leq i < j \leq n} a_{ij}(k) d_{x_i, x_j} \Gamma(x_i(k), x_j(k))$ converges to $\xi = (\xi_1, \ldots, \xi_n)$ when $k$ goes to $\infty$. By renumbering and extracting a subsequence, we can assume w.l.o.g that $x_1(k) = \max(x_1(k), \ldots, x_n(k))$ is always maximal for the poset relation on $M$ and that $\sum_{1 \leq i < j \leq n} a_{ij}(k) d_{x_i, x_j} \Gamma(x_1(k), x_j(k))$ does not vanish for all $k$.

\[
\sum_{1 \leq j \leq n} a_{1j}(k) d_{x_1, x_j} \Gamma(x_1(k), x_j(k)) = (\sum_{1 \leq j \leq n} a_{1j}(k) d_{x_1, x_j} \Gamma(x_1(k), x_j(k)), \ldots, \sum_{1 \leq j \leq n} a_{nj}(k) d_{x_j, x_j} \Gamma(x_1(k), x_j(k)))
\]

Since $\sum_{1 \leq j \leq n} a_{1j}(k) d_{x_1, x_j} \Gamma(x_1(k), x_j(k)) \to \xi_1$ and for all $k$, $d_{x_1} \Gamma(x_1(k), x_j(k)) \in \gamma_{x_1(k)}$, each term $a_{ij}(k) d_{x_i, x_j} \Gamma(x_1(k), x_j(k))$ cannot blow up. Moreover,

\[
\forall j, |a_{1j}(k) d_{x_1, x_j} \Gamma(x_1(k), x_j(k))| \leq \varepsilon^{-1} (1 + |\xi_1|)
\]

for $k$ large enough by the convexity estimate on $\gamma$. We combine with the fact that both elements $(x_1, a_{1j}(k) d_{x_1, x_j} \Gamma(x_1(k), x_j(k)))$ and $(x_j, -a_{1j}(k) d_{x_j, x_j} \Gamma(x_1(k), x_j(k)))$ lie on the same bicharacteristic curve which means that

\[
\sum_{1 \leq j \leq n} a_{1j}(k) d\Gamma(x_1(k), x_j(k)) \to_{k \to \infty} (\xi_1, \ldots, \eta_0, \ldots).
\]

It follows that $\sum_{2 \leq i < j \leq n} a_{ij}(k) d_{x_i, x_j} \Gamma(x_i(k), x_j(k))$ converges to $(0, \xi'_2, \ldots, \xi'_n) \in T^* M^n$, $\xi'_j = \xi_j - \delta^{\eta}_j$, and that we can identify with an element $(\xi'_2, \ldots, \xi'_n) \in T^*(M^n)^{n-1}$. Then we can finish the proof using the inductive argument. □

**Theorem 9.1.** Let $\prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i - x_j)^{n_{ij}}$ be a regularized Feynman amplitude then

- the family $\Lambda_n$ is polarized in $T^* M^n$ and strictly in $T^* (M^n \setminus D_n)$
- the family of distributions $\left(\prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i - x_j)^{n_{ij}}\right)_{\lambda \in \mathbb{C}^{n(n-1)/2}}$ is meromorphic with linear poles with value $\mathcal{D}_{\Lambda_n}^\prime (M^n)$
- the family $\left(\prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i - x_j)^{n_{ij}}\right)_{\lambda \in \mathbb{C}^{n(n-1)/2}}$ is holomorphic in $\lambda$ with value $\mathcal{D}_{\Lambda_n} (M^n \setminus D_n)$.

**Proof.** The only thing we need is to check the three assumptions, given in paragraph 6.1.1 of Theorem 6.3 applied to the product:

\[
\left(\prod_{1 \leq i < j \leq n} \log^{k_{ij}}(\Gamma(x_i, x_j) + i0)(\Gamma(x_i, x_j) + i0)^{\lambda_{ij}}\right).
\]
The stratification property is easy to check since the critical locus of \( x \mapsto \Gamma(x_i, x_j) \) is just the diagonal \( d_{ij} \) which is an analytic submanifold of \( M^n \) and any finite intersection of diagonals of the form \( d_{ij} \) is a clean analytic submanifold.

Recall we denoted by \( \Lambda_i \) the wave front set of the family \( (\Gamma(x_i, x_j) + i0)^{\lambda_{ij}} \) in \( T^*M^n \). We already know by Theorem 8.1 that \( \Lambda_i \) is polarized in \( T^*M^n \) and strictly polarized in \( T^*(M^n \setminus D_n) \). It follows from Theorem 9.1 that:

\[
(\prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i, x_j)^{\lambda_{ij}})_{\lambda \in \mathbb{C}^{\binom{n}{2} - 1}}
\]

is holomorphic in \( \lambda \) with value \( \mathcal{D}'(M^n \setminus D_n) \). By corollary 7.1, the conic set \( \Lambda_n \) is strictly polarized on \( M^n \setminus D_n \) and by Lemma 9.1, \( \Lambda_n \) is strongly convex therefore the product \( (\prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i, x_j)^{\lambda_{ij}})_{\lambda \in \mathbb{C}^{\binom{n}{2} - 1}} \) satisfies the second polarization assumption needed for Theorem 9.1.

Finally, we must check the third regularity assumption. The critical locus \( \{d_{ij}, \Gamma(x_i, x_j) = 0\} \) is the diagonal \( d_{ij} = \{x_i = x_j\} \) and we must consider its conormal \( N^*(d_{ij}) \). We must compare it with \( \Lambda_{ij} = \{(y_i, y_j; \lambda_d y_i, \lambda_d y_j, \Gamma) \text{ s.t. } \Gamma(y_i, y_j) = 0, \lambda > 0\} \). But the regularity property was already checked in the proof of Proposition 8.2.

The fact that \( (\prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i - x_j)^{\lambda_{ij}})_{\lambda \in \mathbb{C}^{\binom{n}{2} - 1}} \) is holomorphic in \( \lambda \) with value \( \mathcal{D}'(M^n \setminus D_n) \) implies that it has a nice limit when \( \lambda \to (0, \ldots, 0) \in \mathbb{C}^{\binom{n}{2} - 1} \), the limit being the well defined distribution

\[
(\prod_{1 \leq i < j \leq n} G(x_i, x_j)^{\lambda_{ij}}) \in \mathcal{D}'(M^n \setminus D_n).
\]

It follows from Theorem 9.1 that:

**Corollary 9.1.** Let \( \mathcal{R}_\pi \) be the renormalization operator defined in 2.3 then

\[
\mathcal{R}_\pi \left( (\prod_{1 \leq i < j \leq n} G_0(x_i, x_j))^{\lambda_{ij}} \right) \in \mathcal{D}'(M^n)
\]

at \( \lambda = (0, \ldots, 0) \) is a distributional extension of \( (\prod_{1 \leq i < j \leq n} G(x_i, x_j)^{\lambda_{ij}}) \).

The above corollary gives a geometric meaning to the regularization by analytic continuation.

**10. The renormalization Theorem.**

The goal of this section is to prove that the renormalization operator \( \mathcal{R}_\pi \) defined in the previous section satisfies the axioms 10.1 needed for quantum field theory especially the factorization equation (9.2).

**10.1. Renormalization maps, locality and the factorization property.**

10.1.1. *The vector subspace \( \mathcal{O}(D_I, \Omega) \) generated by Feynman amplitudes.* In QFT, renormalization is not only extension of Feynman amplitudes in configuration space but our extension procedure should satisfy some consistency conditions in order to be compatible with the fundamental requirement of locality.

We introduce the vector space \( \mathcal{O}(D_I, \Omega) \) generated by the Feynman amplitudes

\[
\mathcal{O}(D_I, \Omega) = \left\{ (\prod_{i < j \in I^2} G^{\lambda_{ij}}(x_i, x_j))_{\lambda_{ij}} \right\}_{\mathbb{C}}.
\]

By Corollary 7.1, elements of \( \mathcal{O}(D_I, \Omega) \) are distributions in \( \mathcal{D}'(M^I \setminus D_I) \).
10.1.2. Axioms for renormalization maps: factorization property as a consequence of locality. We define a collection of renormalization maps \( \{ \mathcal{R}_{\Omega \subset M^I} \}_{\Omega, I} \) where \( I \) runs over the finite subsets of \( \mathbb{N} \) and \( \Omega \) runs over the open subsets of \( M^I \) which satisfy the following axioms which are simplified versions of those figuring in \( [37, 2.3 \text{ p. } 12–14; 36, \text{Section 5 p. } 33–35] \):

**Definition 10.1.** For every finite subset \( I \subset \mathbb{N} \), let \( \Lambda_I \) be the conic set in \( T^*M^I \) of definition \( [37] \).

1. For every \( I \subset \mathbb{N}, |I| < +\infty \), \( \Omega \subset M^I \), \( \mathcal{R}_{\Omega \subset M^I} \) is a linear extension operator:

   \[
   \mathcal{R}_{\Omega \subset M^I} : \mathcal{O}(D_I, \Omega) \longrightarrow D'_{\Lambda_I}(\Omega).
   \]

2. For all inclusion of open subsets \( \Omega_1 \subset \Omega_2 \subset M^I \), we require that:

   \[\forall f \in \mathcal{O}(D_I, \Omega_1), \forall \varphi \in \mathcal{D}(\Omega_1) \quad (\mathcal{R}_{\Omega_1 \subset \Omega_2}(f), \varphi) = \langle \mathcal{R}_{\Omega_2}(f), \varphi \rangle.\]

3. The renormalization maps satisfy the factoring property. If \((U, V)\) are disjoint open subsets of \( M \), and \((I, J)\) are disjoint finite subsets of \( \mathbb{N} \), \( \forall (f, g) \in \mathcal{O}(D_I, U^I) \times \mathcal{O}(D_J, V^J) \) and \( \forall \prod_{(i,j) \in I \times J} G^{n_{ij}}(x_i, x_j), n_{ij} \in \mathbb{N} \):

   \[
   \mathcal{R}_{(U^I \times V^J) \subset M^{I \times J}}((f \otimes g) \prod_{(i,j) \in I \times J} G^{n_{ij}}(x_i, x_j)) = \mathcal{R}_{U^I \subset M^I}(f) \otimes \mathcal{R}_{V^J \subset M^J}(g) \prod_{(i,j) \in I \times J} G^{n_{ij}}(x_i, x_j)
   \]

The most important property is the factorization property (3) which is imposed in \( [36, \text{equation (2.2) p. 5}] \).

10.1.3. Remarks on the axioms of the Renormalization maps. The wave front condition

\[
WF(\mathcal{R}_{\Omega \subset \mathcal{M}^I}(\mathcal{O}(D_I, \Omega))) \subset \Lambda_I
\]

is central since it allows the product

\[
\mathcal{R}_{U^I \subset M^I}(f) \otimes \mathcal{R}_{V^J \subset M^J}(g) \prod_{(i,j) \in I \times J} G^{n_{ij}}(x_i, x_j)
\]

\( \in D'_{\Lambda_{I \cup J}}(U^I \times V^J) \)

to make sense over \( U^I \times V^J \) by polarization of \( \Lambda_I, \Lambda_J \) and strict polarization of the wave front set of \( \prod_{(i,j) \in I \times J} G^{n_{ij}}(x_i, x_j) \).

To define \( \mathcal{R} \) on \( M^I \), it suffices to define \( \mathcal{R}_{\Omega \subset \mathcal{M}^I} \) for an open cover \((\Omega_i)_i\) of \( M^I \), by construction they necessarily coincide on the overlaps \( \Omega_i \cap \Omega_j \) and the determinations can be glued together by a partition of unity.

10.1.4. Uniqueness property of renormalization maps. The following Lemma is proved in \( [36, \text{Lemmas 2.2, 2.3 p. 6}] \) and tells us that if a collection of renormalization maps \( \{ \mathcal{R}_{\Omega \subset \mathcal{M}^I} \}_{\Omega, I} \) exists and satisfies the list of axioms \( [10.1.] \) then the restriction of \( \mathcal{R}_{M^I}(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j)) \) on \( M^I \setminus d_{\epsilon} \) would be uniquely determined by the renormalizations \( \mathcal{R}_{\Lambda_I} \) for all \( |I| < n \) because of the factorization axiom.
Lemma 10.2. Let \((\mathcal{R}_{\Pi \subseteq M^1})_{\Omega, I}\) be a collection of renormalization maps satisfying the axioms \([10.1]\). Then for any Feynman amplitude \(\prod_{i<j \in I} G_{\lambda, i}(x_i, x_j)\), the renormalization \(\mathcal{R}_{M^1 \setminus d_n}(\prod_{i<j \in I} G_{\lambda, i}(x_i, x_j))\) is uniquely determined by the renormalizations \(\mathcal{R}_{M^1}(\prod_{i<j \in I} G_{\lambda, i}(x_i, x_j))\) for all \(|I| < n\).

Proof. See [36, p. 6-7] for the detailed proof. \(\square\)

Beware that the above Lemma does not imply the existence of renormalization maps but only that they must satisfy certain consistency conditions if they exist.

10.1.5. Covering lemma. The following Lemma is due to Popineau and Stora [36, Lemma 2.2 p. 6] [32, 38] and states that \(M^n \setminus d_n\) can be partitioned as a union of open sets on which the renormalization map \(\mathcal{R}_n\) can factorize.

Lemma 10.2. Let \(M^n\) be a smooth manifold. For all \(I \subseteq \{1, \ldots, n\}\), let \(C_I = \{(x_1, \ldots, x_n) \mid \forall i \in I, j \notin I, x_i \neq x_j \} \subset M^n\). Then

\[
\bigcup_{I \subseteq \{1, \ldots, n\}} C_I = M^n \setminus d_n.
\]

Proof. The key observation is the following, \((x_1, \ldots, x_n) \in d_n \iff \forall U \subseteq \{1, \ldots, n\}\), let \(C_I = \{(x_1, \ldots, x_n) \mid \forall i \in I, j \notin I, x_i \neq x_j \} \subset M^n\). Then

\[
\bigcup_{I \subseteq \{1, \ldots, n\}} C_I = M^n \setminus d_n.
\]

10.2. Definition of the meromorphic renormalization maps. The Theorem 9.1 motivates us to define Renormalization maps as follows.

Definition 10.2. Let \(\prod_{(i<j) \in I^2} G_{\lambda, i}(x_i, x_j)^{n_{ij}}\) be a Feynman amplitude in \(\mathcal{O}(M^1)\). Then by Theorem 2.7 it is a family of distributions depending meromorphically on \(\lambda \in \mathcal{C}^{|I| (n-1)}/2\) with linear poles, then we define the action of the renormalization map \(\mathcal{R}_{\Pi^1}\) on \(\prod_{(i<j) \in I^2} G_{\lambda, i}(x_i, x_j)^{n_{ij}}\) as follows:

\[
\mathcal{R}_{\Pi} \left( \prod_{(i<j) \in I^2} G_{\lambda, i}(x_i, x_j)^{n_{ij}} \right)
\]

at \(\lambda = (0, \ldots, 0)\) where \(\mathcal{R}_{\Pi}\) is the regularization operator defined in Corollary 9.1.

10.3. The main renormalization Theorem. We next show that the renormalization maps \((\mathcal{R}_{M^1})_{\Pi^1}\) defined in 10.2 satisfies the axioms of 10.1 hence they define a genuine renormalization in QFT in the sense they are compatible with the locality axioms in QFT.

Theorem 10.1. The collection of renormalization maps defined in 10.2 satisfies the collection of axioms \([10.1]\).

Proof. The proof is by induction on \(n\) and relies on Theorem 9.1.

We also need the property established in Theorem 7.1 that the conic set \(A_2\) which contains the wave front sets of all powers of the regularized Feynman propagator is strictly polarized in \(T^* (M^2 \setminus d_2)\) and polarized in \(T^* M^2\).
It suffices to check the factorization identity over each region $C_I \subset M^n \setminus d_n$ of configuration space for some $I \subseteq \{1, \ldots, n\}$ since the collection $(C_I)_I$ forms an open cover of $M^n \setminus d_n$. The key idea is to consider the formal decomposition:

\[
G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}} = \prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}} \prod_{(i<j) \in I^2} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}} \prod_{(i<j) \in I \times I^c} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}}
\]

that we write shortly as:

\[
t_n(\lambda_n) = t_I(\lambda_I) t_{I^c}(\lambda_{I^c})
\]

\[
t_n = \prod_{1 \leq i < j \leq n} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}},
\]

\[
t_I = \prod_{(i<j) \in I^2} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}},
\]

\[
t_{I^c} = \prod_{(i<j) \in I \times I^c} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}}
\]

\[
\lambda_n = (\lambda_{ij})_{1 \leq i < j \leq n}, \lambda_I = (\lambda_{ij})_{(i<j) \in I^2}, \lambda_{I^c} = (\lambda_{ij})_{(i<j) \in I \times I^c}.
\]

Let us explain how to make sense of this decomposition. By Theorem 5.1 the left hand side $t_n(\lambda_n)$ is meromorphic in $\lambda_n$ with value $D_{\Lambda_n}$, and so are each terms $t_I, t_{I^c}, t_{I,I^c}$ w.r.t. the variables $\lambda_I, \lambda_{I^c}, \lambda_{I,I^c}$.

The product on the right hand side makes sense since:

1. By Theorem 9.1 $t_I(\lambda_I)$ is meromorphic with value $D_{\Lambda_I}$, $t_{I^c}(\lambda_{I^c})$ is meromorphic with value $D_{\Lambda_{I^c}}$ and $\Lambda_I, \Lambda_{I^c}$ are polarized

2. The interaction term $\left(\prod_{(i<j) \in I \times I^c} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}}\right)$ is holomorphic with value $D_{\Lambda_I, I^c}$ where $\Lambda_I, I^c = \sum_{(i<j) \in I \times I^c}(\lambda_{ij} + 0) \cap T^* M^n$ is strictly polarized therefore the conic sets $\Lambda_I, \Lambda_{I^c}, \Lambda_{I,I^c}$ are transverse in $T^* C_I$ by Theorem 7.1 which implies that the distributional product $t_I t_{I^c} \left(\prod_{(i<j) \in I \times I^c} G_{\lambda_{ij}}(x_i, x_j)^{n_{ij}}\right)$ makes sense in $D_{\Lambda_n}$ for every $\lambda_n$ avoiding the poles. Moreover by proposition 5.3 the product is meromorphic in $\lambda_n$ with value $D_{\Lambda_n}$, hence equation (94) holds true in the sense of distributions depending meromorphically on $\lambda_n$. In order to conclude, we make two central observations:

- On $C_I$, for every $(i, j) \in I \times I^c$, the Feynman propagator $G_{\lambda_{ij}}(x_i, x_j)$ is holomorphic in $\lambda_{ij}$ with value $D_{\Lambda_I}(C_I)$. Hence by strict polarization of $\Lambda_{ij} \cap T^* C_I$ and Proposition 5.7, $t_{I,I^c}$ is holomorphic in $\lambda_I, I^c$ with value $D_{\Lambda_{I,I^c}}(C_I)$.

- By Theorem 2.1 there exists a projection $\pi$ from meromorphic functions with linear poles on holomorphic functions satisfying the factorization property of definition 2.2 and used to construct the renormalization operator $\mathcal{R}_\pi$, hence:

\[
\pi(t_n(\lambda_n)) = \pi(t_I(\lambda_I) t_{I^c}(\lambda_{I^c}))
\]

\[
= \pi(t_I(\lambda_I)) \pi(t_{I^c}(\lambda_{I^c})) \pi(t_{I,I^c}(\lambda_{I,I^c}))
\]

by factorization property and Proposition 5.8

\[
= \pi(t_I(\lambda_I)) \pi(t_{I^c}(\lambda_{I^c})) t_{I,I^c}(\lambda_{I,I^c})
\]

since $t_{I,I^c}$ holomorphic and $\pi$ acts as the identity map on holomorphic functions thus

\[
\lim_{\lambda_n \to 0} \pi(t_n(\lambda_n)) = \lim_{\lambda_I \to 0} \pi(t_I(\lambda_I)) \lim_{\lambda_{I^c} \to 0} \pi(t_{I^c}(\lambda_{I^c})) \lim_{\lambda_{I,I^c} \to 0} t_{I,I^c}(\lambda_{I,I^c}).
\]
It follows by definition of the renormalization maps that
\[ R_{M^n}(t) \big|_{C_I} = R_{M^I}(t) R_{M^I_c}(t_{I_I}) t_{I_I} \]
which exactly means that \( R \) factorizes on \( M^n \setminus d_n \) since \( (C_I)_I \) forms an open cover of \( M^n \setminus d_n \). □

11. The functorial behaviour of renormalizations.

In this last section, we investigate the functorial behaviour of the renormalization maps previously constructed. We can add a new axiom on renormalization maps which states that renormalizations should behave functorially w.r.t. morphisms of our category \( M_{ca} \).

**Proposition 11.1.** Given \((M, g, G), (M', g', G') \in M_{ca}^2\) and a morphism
\[ \Phi : (M', g', G') \mapsto (M, g, G), \]
then
1. \( \Phi^* \Gamma = \Gamma' \).
2. \( \Phi \) acts by pull–back on \( \mathcal{O}(M^I) \) and sends the Feynman amplitudes in \( \mathcal{O}(M^I) \) to Feynman amplitudes in \( \mathcal{O}((M')^I) \).

**Proof.** The above claims are straightforward consequences from the fact that \( \Gamma \) depends only on the metric \( g \) via the exponential map and from the definition of morphisms which gives \( \Phi^* G = G' \). □

What follows is a definition of **covariant** renormalizations in the spirit of the seminal works \[11\] 29 50

**Definition 11.1.** A family of collection of renormalization maps \((R_M^I)_{I(M,g,G) \in M_{ca}}\) indexed by \((M, g, G) \in M_{ca}\) is **covariant** if for all morphisms \( \Phi : (M', g', G') \mapsto (M, g, G) \) where \((M', g', G'), (M, g, G) \in M_{ca}^2\):
\[ \forall t \in \mathcal{O}(M^I), R_{(M')^I}(\Phi^* t) = \Phi^* (R_M^I t). \]

In section 10, all renormalization maps constructed depend only on the element \((M, g, G) \) in the category \( M_{ca} \) since the only ingredients we used were the Feynman propagator \( G \) and the Synge world function \( \Gamma \) which depends only on the metric \( g \). Therefore, it follows that:

**Theorem 11.1.** The family of collection of renormalization maps \((R_M^I)_{I(M,g,G) \in M_{ca}}\) indexed by \((M, g, G) \in M_{ca}\) constructed in Theorem 10.1 is covariant.

**References**

[1] A. Aizenbud and V. Drinfeld. The wave front set of the Fourier transform of algebraic measures. 2014. [arXiv:1212.3630]
[2] Michael Francis Atiyah. Resolution of singularities and division of distributions. *Communications on pure and applied mathematics*, 23(2):145–150, 1970.
[3] C. Bär, N. Ginoux, and F. Pfäffle. *Wave equations on Lorentzian manifolds and quantization*, EMS, 2007.
[4] Rouchdi Bahloul. Démonstration constructive de l’existence de polynômes de Bernstein-Sato pour plusieurs fonctions analytiques. *Compositio Mathematica*, 141(1):175–191, 2005.
[5] Prakash Belkale and Patrick Brosnan. Periods and Igusa local zeta functions. *International Mathematics Research Notices*, 2003(9):2655–2670, 2003.
[6] Christoph Bergbauer, Romeo Brunetti, and Dirk Kreimer. Renormalization and resolution of singularities. *arXiv preprint arXiv:1008.1935*, 2009.
[7] Joseph Bernstein. The analytic continuation of generalized functions with respect to a parameter. *Functional Analysis and its Applications*, 6(4):26–40, 1972.
[8] R. E. Borcherds. Renormalization and quantum field theory. *Algebra & Number Theory*, 5:627–58, 2011. [arXiv:1008.0129]
[9] Christian Brouder, Nguyen Viet Dang, and Frédéric Hélein. Boundedness and continuity of the fundamental operations on distributions having a specified wave front set. (with a counter example by Semyon Alesker). arXiv preprint arXiv:1409.7662, 2014.

[10] R. Brunetti and K. Fredenhagen. Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds. Commun. Math. Phys., 208:623–61, 2000.

[11] R. Brunetti, K. Fredenhagen, and R. Verch. The generally covariant locality principle – A new paradigm for local quantum field theory. Commun. Math. Phys., 237:31–68, 2003.

[12] Ozgür Ceyhan and Matilde Marcolli. Feynman integrals and motives of configuration spaces. Communications in Mathematical Physics, 313(1):35–70, 2012.

[13] Ozgür Ceyhan and Matilde Marcolli. Feynman integrals and periods in configuration spaces. arXiv preprint arXiv:1207.3844, 2012.

[14] Ozgür Ceyhan and Matilde Marcolli. Algebraic renormalization and Feynman integrals in configuration spaces. arXiv preprint arXiv:1308.5687, 2013.

[15] J.C. Collins. Renormalization: an introduction to renormalization, the renormalization group and the operator-product expansion. Cambridge University Press, 1984.

[16] Y. Dabrowski and Ch. Brouder. Functional properties of Hörmander’s space of distributions having a specified wavefront set. Commun. Math. Phys., 2014.

[17] N. V. Dang. Renormalization of quantum field theory on curved space-times, a causal approach. Ph.D. thesis, Paris Diderot University, 2013. http://arxiv.org/abs/1312.5674.

[18] N.V. Dang. Analytic renormalization of QFT on smooth spacetimes. 2015. In preparation.

[19] Michael Dütsch, Klaus Fredenhagen, and Katarzyna Rejzner. Dimensional regularization in position space and a forest formula for Epstein-Glaser renormalization. Journal of Mathematical Physics, 55(12):122303, 2014.

[20] A. D’Agnolo and P. Schapira. Global propagation on causal manifolds, Asian J. Math., 2(1):641–653, 1998.

[21] G. Eskin. Lectures on Linear Partial Differential Equations, volume 123 of Graduate Studies in Mathematics. Amer. Math. Soc., Providence, 2011.

[22] Herbert Federer. Geometric measure theory. 1969.

[23] I. M. Gel’fand and G. E. Shilov. Generalized Functions: Volume I - Properties and Operations. Academic Press, New York, 1964.

[24] M. Golubitsky and V. Guillemin. Stable mappings and their singularities. Graduate texts in Mathematics. Springer, 1973.

[25] Michel Granger. Bernstein-Sato polynomials and functional equations. Algebraic approach to differential equations, page 225–291, 2010.

[26] Robert Clifford Gunning and Hugo Rossi. Analytic functions of several complex variables, volume 368. American Mathematical Soc., 2009.

[27] Li Guo, Sylvie Paycha, and Bin Zhang. Decompositions and residue of meromorphic functions with linear poles in the light of the geometry of cones. arXiv preprint arXiv:1501.00426, 2015.

[28] J. Hadamard. Lectures on Cauchy’s problem in linear partial differential equations, Dover, 2003.

[29] S. Hollands and R. M. Wald. Local Wick polynomials and time ordered products of quantum fields in curved spacetime. Commun. Math. Phys., 223:289–326, 2001.

[30] S. Hollands and R. M. Wald. Existence of local covariant time ordered products of quantum fields in curved spacetime. Commun. Math. Phys., 231:309–45, 2002.

[31] L. Hörmander. The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis. Springer Verlag, Berlin, second edition, 1990.

[32] D Iagolnitzer. The u= 0 structure theorem. Communications in Mathematical Physics, 63(1):49–96, 1978.

[33] Masaki Kashiwara and Pierre Schapira. Microlocal study of sheaves. Number 51. Société mathématique de France, 1985.

[34] K. J. Keller. Dimensional regularization in position space and forest formula for regularized Epstein-Glaser renormalization. Ph.D. thesis, University of Hamburg, 2010.

[35] K. J. Keller. Euclidean Epstein-Glaser renormalization. J. Math. Phys., 50:103503, 2009.

[36] Ivan Todorov, Nikolay M. Nikolov, Raymond Stora. Renormalization of Massless Feynman Amplitudes in Configuration Space. 2013. http://arxiv.org/abs/1307.6854

[37] Nikolay M. Nikolov. Anomalies in quantum field theory and cohomologies of configuration spaces. 2009. http://arxiv.org/abs/0903.0187

[38] G. Popineau and R. Stora, 1982. A pedagogical remark on the main theorem of perturbative renormalization theory, (unpublished preprint).

[39] C Sabbah. Polynômes de Bernstein-Sato à plusieurs variables. Sém. Equations aux dérivées partielles, pages 1–6, 1986.

[40] C Sabbah. Proximité évanescente. II. Équations fonctionnelles pour plusieurs fonctions analytiques. Compositio Mathematica, 64(2):213–241, 1987.
[41] Pierre Schapira. Hyperbolic systems and propagation on causal manifolds. *Letters in Mathematical Physics*, 103(10):1149–1164, 2013.

[42] R. Stora. Pedagogical experiments in renormalized perturbation theory, contribution to the conference *Theory of Renormalization and Regularization*, Hesselberg, Germany, [http://wwwthep.physik.uni-mainz.de/scheck/hessbg02.html](http://wwwthep.physik.uni-mainz.de/scheck/hessbg02.html) 2002.

[43] Jean-Louis Verdier. Stratifications de Whitney et théorème de Bertini-Sard. *Inventiones mathematicae*, 36(1):295–312, 1976.