Abstract

Under the presence of ultra high intensity lasers or other intense electromagnetic fields the motion of particles in the ultrarelativistic regime can be severely affected by radiation reaction. The standard particle-in-cell (PIC) algorithms do not include radiation reaction effects. Even though this is a well known mechanism, there is not yet a definite algorithm nor a standard technique to include radiation reaction in PIC codes. We have compared several models for the calculation of the radiation reaction force, with the goal of implementing an algorithm for classical radiation reaction in the Osiris framework, a state-of-the-art PIC code. The results of the different models are compared with standard analytical results, and the relevance/advantages of each model are discussed. Numerical issues relevant to PIC codes such as resolution requirements, application of radiation reaction to macro particles and computational cost are also addressed. The Landau and Lifshitz reduced model is chosen for implementation.

Keywords: radiation reaction, particle-in-cell, laser-matter interactions, relativistic electron motion

1. Introduction

The next generation of high-power lasers is going to reach intensities that will open new doors for exploring a wide range of physical problems with an even wider range of applications. The ELI project [1] is expected to reach laser intensities several orders of magnitude higher than those available today. At intensities $I \sim 10^{23} - 10^{24}$ W/cm$^2$ one can expect electron-positron pair production [2, 3, 4, 5]. In astrophysics these intensities are relevant for the study of pulsars, blazars, and gamma-ray bursts [6]. High intensity laser-mater interactions can also produce proton and heavy ion beams [7, 8, 9] that are of great significance for many applications, the most important being cancer treatment.

At high intensities particle acceleration can be severely limited by the radiation reaction associated with the energy loss via radiation emission [11]. This is important whenever the radiated energy is comparable to the total particle energy. The threshold electromagnetic field intensities to drive these effects vary in the available literature, but they are usually in the range $10^{22} - 10^{25}$ W/cm$^2$. Many authors have discussed the classical radiation reaction in pursuit of a proper analytical description [12, 13, 14, 15, 16, 17, 18, 19, 20, 21] or experimental signatures [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. There is also a rising interest in the effect of the radiation reaction on particle dynamics in the astrophysical phenomena [33, 34, 35, 36]. In order to perform reliable particle-in-cell (PIC) simulations in the classical radiation-dominated regime, radiation reaction (RR) must be included in the equations of motion for the particles. In this paper we compare several different approaches in order to accomplish this. We test each of them with well studied examples of particle motion in electromagnetic fields where the trajectory can be analytically expressed and estimates for the radiated power/energy can be obtained. We also identify the domains of validity for all the given models and estimate the additional computational cost that the implementation of each model introduces, describing also specific questions associated with the interpretation of classical RR in PIC simulations.

This paper is structured as follows. In section 2 we introduce the radiation reaction models and deal with macro particle interpretation. In section 3, the behaviour of all the models is investigated in the standard cases of synchrotron radiation and bremsstrahlung. Section 4 underlines the difference between the results with and without radiation reaction for an electron in a laser pulse field and gives an estimate of the threshold. The issue of the optimal temporal resolution is addressed in sec-
tion 5, and section 6 contains estimates for the computational overhead for each model. Finally, in section 7 we state the conclusions.

2. Radiation reaction models

The charged particle motion with radiation reaction is expressed by the Lorentz-Abraham-Dirac (LAD) equation \[ \frac{dp_\mu}{d\tau} = F^\text{EXT}_\mu + F^\text{RR}_\mu \] where

\[ F^\text{RR}_\mu = \frac{2e^2}{3mc^3} \left( \frac{d^2 p_\mu}{d\tau^2} + \frac{p_\mu}{m^2 c^2} \left( \frac{dp_\nu}{d\tau} \frac{dp^\nu}{d\tau} \right) \right). \]

Here, \( F^\mu_\mu \) denotes the electromagnetic force four-vector, \( p_\mu \) is the particle momentum four-vector, \( e, m \) are the elementary charge and particle mass respectively, and \( \epsilon \) is the speed of light. Equation (1) is derived for a point charge. Unphysical solutions appear, for example, when \( F^\text{ext}_\mu = 0 \), where in addition to the solution with a constant velocity, eq. (1) has a solution where the particle accelerates infinitely (the so-called "runaway solution"). The principle of causality is also violated here with pre-acceleration solutions - these solutions anticipate the change of the force, so the particle accelerates before the force has been applied. The detailed explanation of these problems and suggestions for possible improvements are given in [38, 39].

However, even if these problems would be solved, the LAD equation is inconvenient for numerical integration. It is possible to integrate it backwards in time \[ \frac{dp}{dt} = F_L + F_{RR} \] where \( F_L \) in CGS units, is given by:

\[ F_L = e \left( \frac{E}{\gamma mc^2} \times B \right). \]

To facilitate the analysis for PIC implementation, we will use the 3-vector form of the equations throughout. The total change of momentum in time depends on the Lorentz force \( (F_L) \) and the radiation reaction force \( (F_{RR}) \):

\[ \frac{dp}{dt} = F_L + F_{RR} \] (2)

where \( F_L \), in CGS units, is presented in Table 1. Here the radiation back reaction is explicitly given as an additional force acting on the particle, expressed as a function of the electromagnetic fields \( E, B \), the particle momentum \( p \), charge \( e \), mass \( m \) and relativistic factor \( \gamma \), the speed of light \( c \) and time \( t \).

Particle-in-cell codes usually employ normalised units to bring all the quantities to similar orders of magnitude and express the physics as a function of fundamental plasma parameters. The normalisation in OSIRIS \[48, 49\] as follows: \( t \rightarrow t \omega_p \), \( x \rightarrow x \omega_p/c \), \( p \rightarrow p/mc = \gamma v/c \), \( E \rightarrow eE/mc\omega_p \), \( B \rightarrow eB/mc\omega_p \). Here, \( x \) represents a vector in coordinate space, while \( \omega_p \) is a chosen reference value for a frequency, which, for instance, can be equal to the electron plasma frequency of the plasma, or the frequency of the laser. In normalized units, the equations of motion of the particles are free of physical constants, except for a dimensionless coefficient

\[ k = \frac{2\omega_p e^2}{3mc^3} \] (10)

which appears in the radiation reaction force term. Therefore, when including the radiation reaction force, the particle motion is no longer dependent only on the charge-to-mass ratio as in the Lorentz force \( F_L \). This poses an additional challenge for the PIC implementation of classical radiation reaction.

In standard PIC codes, the system is represented using macro-particle. Every macro-particle has the same charge-to-mass ratio as a single particle, and therefore the dynamics of the macro-particles is the same as the dynamics of the original particle species. However, after including the radiation reaction, this no longer holds. If we examine equations (2), (3) and (9), we can see that the ratio between the radiation reaction force and the Lorentz force is proportional to the cube of the charge, and to the reciprocal value of the squared mass

\[ \frac{F_{RR}}{F_L} \propto \frac{e^3}{m^2}. \] (11)
A particular particle species. This approach yields the is therefore essential to use the real charge and mass different number of particles per cell or different cell sizes. PIC simulation would be qualitatively different for different number of particles per cell or different cell sizes. To obtain the correct dynamics of a macro-particle, it is therefore essential to use the real charge and mass to calculate the correct radiation reaction coefficient for a particular particle species. This approach yields the same result regardless of the macro-particle weight.

Table 1: Radiation reaction contribution to the equations of motion: $F_{RL} \propto \text{Lorentz force}$; $p$, $e$, $m$ - particle momentum, charge and mass, $\gamma$ - relativistic factor; $E$, $B$ - electromagnetic fields, $c$ - speed of light, $t$ - time.

Let us consider a macro particle that represents $\alpha$ electrons. The charge of the macro particle is $e_m = \alpha e$, and the mass is $m_m = \alpha m_e$. For a single particle with the same mass and charge as the macro particle, the radiation reaction would be $\alpha$ times stronger than in the case of a single electron:

$$
\frac{F_{RL}}{F_L} \propto \frac{(\alpha e)^2}{(\alpha m_e)^2} = \frac{\alpha e^3}{m_e^3}
$$

(12)

and the trajectory of such particle would be different than the trajectory of a single electron (Fig. [1]). This result would be equivalent to assuming that $\alpha$ electrons are radiating coherently. As a consequence, the results of a PIC simulation would be qualitatively different for different number of particles per cell or different cell sizes. To obtain the correct dynamics of a macro-particle, it is therefore essential to use the real charge and mass to calculate the correct radiation reaction coefficient for a particular particle species. This approach yields the same result regardless of the macro-particle weight.

3. Comparison of the models with standard radiation mechanisms

To examine the physics captured by the different models, we compare the dynamics of a single particle with well known examples for which the particle trajectory and radiated power are known. We first consider synchrotron radiation, where a particle moves in a constant external magnetic field. Taking only the Lorentz force into account, we expect the trajectory to be a perfect circumference due to the $v \times B$ term. When the particle has a very high initial momentum, and is moving in an intense magnetic field, it radiates. The radiative energy loss over time is given by [42]:

$$
\frac{d\xi}{dt} = \frac{2e^3 B^2}{3m^4c^5}(\xi^2 - m^2c^4),
$$

(13)

where $\xi = \gamma mc^2$ is the total particle energy and $B$ is the magnetic field. If the particle is not relativistic, then $\xi \approx mc^2$ and the right-hand side of (13) is close to zero, and therefore the energy loss is negligible. This is also
the case when the $B$ field is small. However, an energetic particle in a high intensity magnetic field will lose a significant amount of energy over time. As the particle loses energy, its velocity decreases, and so does the curvature radius of its trajectory. When it loses a sufficient amount of energy, so that the right-hand side of (13) approaches zero i.e. $\xi^2 \approx m^2 c^4$, the particle trajectory then converges to a circumference.

We have performed simulations for the same initial conditions using all the considered models (eqs. (4)-(9)). A typical example for a very strong damping is shown in Fig. 2 where we can see that all the energy loss predictions closely resemble each other and match the analytical expression (13). The strongest difference to eq. (13) can be seen for the model B08 [2], which is still smaller than 0.3%. Therefore, we can confidently state that this effect is well resolved by all the models. This is not surprising because in this configuration, where $B$ is constant and $B \perp p$, equations (4)-(9) reduce to:

![Figure 1](image1.png)

Figure 1: a) Trajectory of a macro-particle without radiation reaction b)-c) Trajectories of macro particles with weights $\alpha = 1$ and $\alpha = 5$ respectively, counter propagating with a laser pulse where $a_0 = 100$. Particle initial momentum is $p_0 = -100$. d) Energy of the same macro particles from a)-c) as a function of time.

![Figure 2](image2.png)

Figure 2: Synchrotron radiation: electron trajectories and energy evolution. $T_g$ - electron gyro period. The normalizing frequency $\omega_p$ is the laser frequency corresponding to $\lambda = 1 \mu m$. The initial conditions are: a) $p_0 = 100, B_0 = 100$, b) $p_0 = 100, B_0 = 1000$, c) $p_0 = 10, B_0 = 5975$. Dotted circle denotes the limit of the spiral motion.
B08 and F93  \[
\left( \frac{dp}{dt} \right)_{RR} = -kB^2 \frac{p^2}{\gamma}.
\]
LL, S09, LLR, H08  \[
\left( \frac{dp}{dt} \right)_{RR} = -kB^2 \frac{p^2 + 1}{\gamma} - p.
\] (14)

For a highly relativistic particle, \( p \gg 1 \) and \( p^2 + 1 \approx p^2 \), so these two equations (14) are expected to yield similar results.

Another illustration of the role of radiation reaction is Bremsstrahlung radiation. We let a charged particle propagate (\( \gamma_0 = 10 \)) along the x-axis starting from the origin in a static external electric field parallel to the particle motion given by (in normalised units):
\[
E(x) = E_0 \left( e^{s/a} - e^{-s/a} \right) \left( e^{a(L-s)/a} - e^{-(a(L-s)/a)} \right) \left( e^{s/a} + e^{-s/a} + e^{a(L-s)/a} + e^{-(a(L-s)/a)} \right)
\] (15)
where \( E_0 = 1 \), \( a = 4 \) and \( L = 30 \) (see Fig. 3). By changing the normalising frequency \( \omega_p \) (and therefore changing \( k \)), we select a stronger or a weaker damping regime. This is equivalent to changing the electric field amplitude. In this case, the dominant term in eqs. (14)-(17) that leads to essentially the same result for the synchrotron radiation (14), is identically equal to zero in this specific configuration. This allows us to explore the conceptual differences between some of the models. In Fig. 3 we observe that the LL model and the F93 model depend on the gradient of the applied electric field. There is no radiation reaction observed in the B09 or in the LLR models. The S09 and H08 models show some energy loss, but the rate at which the particle loses energy is constant where the electric field is constant and is not affected by the field gradient.

This is better understood if we reduce the equations in Table 1 to the case where \( E \parallel p \):

B08 and LLR  \( \left( \frac{dp}{dt} \right)_{RR} = 0 \)
LL and F93  \( \left( \frac{dp}{dt} \right)_{RR} = k\gamma (\frac{\partial E}{\partial t} + \frac{p E}{\gamma}) \) (16)
S09  \( \left( \frac{dp}{dt} \right)_{RR} = -k\gamma \frac{E_p}{\gamma} \)
H08  \( \left( \frac{dp}{dt} \right)_{RR} = -k\gamma \frac{E_p}{\gamma} \)

We see that the LL and F93 models reduce to the same expression. Their radiation reaction force depends only on the field derivatives, not on the absolute value of the field. There is no effect associated with the radiation reaction force in the B09 or in the LLR models. The S09 and H08 models lead to expressions that do not depend on the field derivatives, but only on the absolute value of the field, as can be observed in Fig. 3.

Figure 3: a) Electric field as felt by the particle over time; b) total particle energy vs. time; c) the difference in energy compared to the case without radiation reaction for \( k=0.005 \); d) the difference in energy compared to the case without radiation reaction for \( k=0.001 \).
This analysis shows that if we choose one of the models that are insensitive to the rate of the field change, we can apply it exclusively in a regime where this change is small enough not to significantly influence the motion. The condition that must be satisfied is then:
\[ \frac{p}{mc^2e} |F_\perp|^2 \gg \frac{dF}{dt} \]  
(17)
or equivalently
\[ \frac{\gamma}{mc^2e} \gg \frac{\Delta E_f}{\Delta t} \]  
(18)
where \( \Delta t \) is the acceleration length, \( p \) is the particle momentum, \( F_\perp \) is the perpendicular component of the Lorentz force with respect to \( p \) and \( E_\parallel \) is the parallel component of the electric field.

To show that the difference in the contribution of the longitudinal fields to the radiation reaction in different models will be small in the vast majority of scenarios, we have chosen to illustrate two cases with very strong fields \( E_0 = 0.2 E_C \) (Fig. 3), the corresponding normalization coefficient defined in eq. (10) is \( k = 0.001 \) and \( E_0 = E_C \) (Fig. 4), normalization \( k = 0.005 \). Here \( E_C \) stands for the Schwinger limit (\( E_C = m^2c^3/\epsilon_0 h \)), which marks the transition from the classical to the quantum regime. The applicability of classical radiation reaction models cannot go beyond \( E_C \), and its validity is questionable when close to this value. Figure 3 shows that even for \( E_0 = E_C \), the effect of radiation reaction in this configuration would be smaller than half a percent compared with the total electron energy (Fig. 3b). Our setup is similar as in \([\text{45}]\), but for our parameters there is more significant radiation reaction.

The condition \( \text{18} \) is satisfied in most physical scenarios of interest. In particular, for laser pulses, the inequality \( \text{18} \) is satisfied if \( \alpha_0 \gamma \gg 1 \), which is true in all the scenarios where radiation reaction is significant. In \([\text{46}]\) this was confirmed by comparing the contribution of the particle spin and the contribution of radiation reaction force arising from \( dB/dt \) and \( dE/dt \) in the plane wave scenario. In this comparison, the spin gives a bigger contribution. In the classical regime that we are addressing here, the spin contribution is negligible, and so are the contributions of \( dB/dt \) and \( dE/dt \).

In summary, any of the proposed models can be used to describe the classical radiation reaction dominated regime.

4. Testing the role of classical radiation reaction with the dynamics of electrons in intense laser pulses

In order to examine the role of classical radiation reaction in scenarios with intense laser pulses and to determine the conditions where such models should be used, we consider the dynamics of a single electron interacting with a laser pulse. This is one of the main scenarios where radiation reaction can be explored \([\text{50, 51}]\) and of very high relevance for future laser facilities \([\text{42, 43, 46, 52, 53, 54, 55}]\). The laser pulse normalised vector potential is written as \([\text{50}]\):

\[ A(x,t) = a_0 f(t) \cos \phi \mathbf{e}_x \]  
(19)
where the temporal envelope \( f(t) \) is a slowly varying function relative to the laser cycle (\( df/dt \ll \omega_0 f \)), and the phase of the wave is given by \( \phi = \omega_0 t - \omega_0 x/c \). We choose for \( f(t) \) a polynomial function \( 10^3 t^2 - 15 \tau^2 + 6 \tau^3 \) where \( \tau = t/\tau_{\text{FWHM}} \) takes values in the domain \([0,1]\) to define the envelope rise. Here, the pulse duration \( \tau_{\text{FWHM}} \) is defined as full-width-at-half-maximum in the laser fields. The relativistically invariant normalised vector potential \( a_0 \) can be related to the linearly polarised laser intensity through \( a_0 = 0.8 \sqrt{f[10^{18} \text{W/cm}^2]l[\mu\text{m}]} \).

In the field of a linearly polarized laser pulse, a charged particle undergoes quiver motion. Without the radiation reaction force, the total particle energy remains unchanged after the interaction with the laser pulse. With radiation reaction, the total energy can decrease substantially. If we consider a circularly polarized laser pulse, the situation is similar. In Fig. 4 this is illustrated for a linearly and a circularly polarized laser with identical temporal envelopes and the same intensities \( a_0^{LP} = 100, a_0^{CP} = 100/\sqrt{2}, \lambda_0 = 1 \mu\text{m} \). The initial normalised momentum of the particle is \( p_0 = 100 \), opposite to the laser propagation direction. The total energy that the particle loses while interacting with the laser is about 30% and is the same for the linearly and the circularly polarized case. All the models give similar results (the biggest difference in the final energy in Fig. 5 is 0.03%).

It is important to estimate the threshold when radiation reaction starts to significantly affect the motion. In Fig. 4 we see that for counter-propagating particle and a laser pulse there is already strong radiation reaction for \( a_0 = 100 \) if \( p_0 = 100 \). However, if we repeat the simulation with the same laser parameters for a less energetic particle, the radiation reaction has a weaker effect. For a particle starting at rest, there is no radiation reaction observed for \( a_0 = 100 \). This indicates that in order to estimate whether the radiation reaction is important for a set of particular conditions, it is also required to have the information about the relative motion of the particle and the laser. We can assume that the radiation reaction force is significant when the follow-
ing condition is satisfied:

\[
\frac{|F_{\text{rad}}|}{|F_L|} > 10^{-3}. \quad (20)
\]

In eq. (5), the dominant term due to the radiation for high \( \gamma \) (ultra-relativistic particle) written in units where all quantities are normalized to the laser frequency is:

\[
- \frac{2}{3} \frac{e^2 \omega_0}{mc^2} \gamma^2 p \left( \mathbf{E} + \frac{\mathbf{p} \times \mathbf{B}}{\gamma} \right)^2 - \frac{1}{\gamma^2} (\mathbf{E} \cdot \mathbf{p})^2. \quad (21)
\]

In these units \( E, B \approx \omega_0 \) and \( \gamma \approx p_c \), and thus the left hand side of (20) is on the order of \((2e^2 \omega_0/(3mc^2))\gamma^2 \omega_0 a_0 \) for a particle counter-propagating with the wave. For a laser with a wavelength \( \lambda_0 \approx 1 \mu m \) eq. (20) then becomes:

\[
\left( \frac{1 \mu m}{\lambda_0} \right) \left( \frac{\omega_0}{10^3} \right) \left( \frac{\gamma_0}{\gamma_0^2} \right) \gtrsim 1. \quad (22)
\]

The limits of validity of the classical approach for radiation reaction have been discussed in [53, 57]. When the electric field of the laser approaches the Schwinger field in the rest frame of the particle, quantum effects are expected to dominate. For a head-on collision, where the radiation reaction has the strongest effect on the motion, the classical equations can then be used if

\[
\left( \frac{1 \mu m}{\lambda_0} \right) \left( \frac{\omega_0}{10^3} \right) \left( \frac{\gamma_0}{\gamma_0^2} \right) \lesssim 1. \quad (23)
\]

Therefore, from eq. (22) and eq. (23) we can identify the range of applicability of the classical radiation reaction models described here for the case of an electron colliding head-on with a laser as:

\[
\frac{10^4}{\gamma_0} \lesssim \left( \frac{1 \mu m}{\lambda_0} \right) \left( \frac{\omega_0}{10^3} \right) \lesssim \frac{10^4}{\gamma_0}. \quad (24)
\]

For instance, a 300 J laser pulse with \( \lambda = 1 \mu m \) and 30 fs duration, yields the peak power of 10 PW and when focused to a 10 \( \mu m \) focal spot the peak intensity of \( I = 10^{22} \text{ W/cm}^2 \). The corresponding vector potential is \( a_0 \approx 100 \). Future laser facilities will be able to provide normalised vector potentials of this magnitude, and for \( \gamma_0 \gtrsim 100 \) the inequalities (24) are verified.

5. The role of the timestep for simulating particle motion in an intense electromagnetic wave

The dynamics of electrons in intense electromagnetic waves can be strongly relativistic. A laser can interact with particles initially at rest or with previously accelerated particles. At ultra high intensities the relative dynamics of the particles in the laser field is highly non-linear and the numerical integration of the equations of motion must consider this when identifying the relevant numerical parameters. The higher resolution requirements at \( a_0 \gg 1 \) were reported previously in [58], where they studied the particles in intense laser fields within plasma channels and found the approximate condition \( c\Delta t/\lambda \ll 1/a_0 \) (here \( \lambda \) and \( a_0 \) are the laser wavelength and the normalised vector potential, \( c \) is the speed of light and \( \Delta t \) the simulation time step). This is even more relevant in scenarios where radiation reaction plays an important role because in these scenarios ultra-high intensities are always present. In fact, as discussed below, we will have different resolution requirements depending on the particle energy and the maximum intensity of the laser field. For our case, the resolution requirements could be less or more strict than that of ref. [58] depending on the energy of the interacting particles.

An initially small integration error can, in principle, grow during a long simulation. To assess this effect, we perform temporal resolution tests for realistic simulation timescales. Let us consider a typical LWFA acceleration scenario for a 0.5 GeV electron acceleration stage. For a laser wavelength \( \lambda_L = 1 \mu m \), it is required to have a 1.6 mm plasma length, laser \( a_0 \approx 1.6 \) and the ratio between the laser and plasma frequency \( \omega_L/\omega_p \approx 10 \). This amounts to a total simulation time of \( T_{\text{total}} \approx 10^4 \omega_p^{-1} \), where the laser frequency is given by \( \omega_L = 1.88 \times 10^{15} \text{ s}^{-1} \). Therefore, a typical simulation time for a LWFA is on the order of \( T_{\text{total}} \).

We consider the motion of an electron in a plane electromagnetic wave with a longitudinal envelope that propagates along the x-direction. The electromagnetic wave is expressed analytically according to the eq. (19) and has 30fs duration at FWHM in intensity. The electron is considered as a test-particle that gives no feedback to the fields. The electron initial momentum is

![Figure 4: Particle energy over time in the field of linearly (solid line) and circularly (dashed line) polarized laser pulse.](image-url)
chosen to be always parallel to the wave propagation direction and either co- ($p_0 > 0$) or counter-propagating ($p_0 < 0$) with the laser. We have studied the convergence of the particle trajectories for different initial momenta $p_0$, and different vector potential of the wave $a_0$ varying the timestep, while keeping the simulation duration in the units normalised to the laser frequency $\omega_L$ equal to $T_{\text{total}}$.

In order to establish a quantitative convergence criteria that would allow an automatic evaluation, we have first performed the simulations with a very small time step $dt$, well beyond the convergence limit for each individual case. Then, we chose 1000 reference points equally spaced in time (for $T=10, 20, ...$) along the trajectory which would serve as a benchmark to compare with the runs with longer $dt$. We can then define the relative error as:

$$R = \frac{\sum |\Delta \gamma|}{\sum \gamma}.$$  \hspace{1cm} (25)

Here, $\gamma$ is the Lorentz factor of the converged result, $|\Delta \gamma|$ stands for the absolute difference between this value and the one from the current test-particle run and the sums are over all reference points. We take the error to be acceptable if $R < 1\%$ for which we consider that the result has converged.

Comprehensive tests were done for a wide range of values of $a_0$ and initial momenta. A subset of these is plotted in Fig. 5 where each pixel represents a single simulation (300x100 pixels in each panel). It can be seen that below $a_0 = 5$, the condition of having ~ 30 points per laser period is sufficient (here equivalent to $dt=0.2 \ [1/\omega_L]$). If $a_0$ is higher, the most demanding case is when the particle has very low energy before the interaction with the laser. In that case, the particle can get a strong kick, on the order of its total energy, the interaction with the laser. In that case, the particle can get a strong kick, on the order of its total energy, within a single time step. This can modify its direction of motion and put the particle on a different trajectory, allowing the error to grow as the particle gains energy in the laser field. If a particle is already ultra-relativistic, the same error in the accelerating force does not change the motion significantly. Figure 5a shows that for studying the interaction of relativistic electron beams ($\gamma > 20$) with intense lasers a resolution of 30 timesteps per laser period can still be kept. For smaller values of initial $\gamma$ this no longer holds and special care should be applied when simulating intense lasers interacting with cold plasmas. These are general conclusions that apply whether or not the radiation reaction is accounted for in the code. We have repeated the same procedure with and without radiation reaction, and for lasers where $a_0 < 80$ and we have not observed any differences in the convergence map. A difference between

![Figure 5: a) Convergence map for different timesteps in low intensities. Below $a_0 = 5$, the standard condition ( $dt=0.2 \sim 30$ points per laser period ) is enough. For higher values, the most limiting case is laser in a cold plasma - relativistic particles can still be modelled with $dt=0.2$. b), c) Convergence map for $dt=0.05$, with and without accounting for radiation reaction. Color legend for both b) and c) is presented in panel c). Below $a_0 = 100$, the convergence does not depend on whether radiation reaction is included or not. The difference in the regions with higher values of $a_0$ can be attributed to the particle energy loss that lowers the effective $p$ of the particle.](image-url)
the convergence map with and without RR (Fig. 5b) arises only for very high $a_0$. This is not surprising since in these cases the particles lose significant energy due to the strong radiation reaction, and thus move to the regions in the parameter space of Fig. 5 where smaller $dt$ is required. Therefore, by knowing the approximate energy loss of electrons in the simulation when running with radiation reaction, we can still apply the same resolution as without radiation reaction for corresponding lower particle energy.

### 6. Computational overhead of radiation reaction modelling

When examining the equations (4)-(9) one can expect a significant overhead of the particle pusher as compared to the case where we only had the Lorentz force. To estimate the additional computational cost, we have determined the particle pusher overhead for every model used here. The overhead is defined as the ratio between the number of floating point operations (flop) required for the radiation reaction force and for the Lorentz force.

In state-of-the-art supercomputers, memory access is often more time-consuming than additional flop. The balance depends on a particular computer architecture, but in all cases it is advisable to minimize the amount of saved quantities. The computation of radiation reaction force requires sometimes additional memory. For example, in eq. (5) the Lorentz force from a previous time step needs to be saved. This means that three space components of the force are to be added for each particle in the simulation. The additional computational cost is summarized in Table 2. The models that do not require any additional memory (B08, S09 and LLR) also have low overhead in terms of flop. Therefore, any of these three models can be introduced without affecting significantly the performance of a PIC code.

### 7. Conclusions

We have compared different approaches to include classical radiation reaction in a PIC code. Out of the numerous models available in the literature, we chose a subset compatible with the standard PIC algorithm and compared their performance in modelling the electron motion in physical scenarios of interest.

The first test was on the electron motion in a constant B-field (i.e. synchrotron), where all the models gave the same results, with a minimal departure of B08 at extremely high fields ($B \approx 0.1 B_C$). Interaction of electrons with a counter-propagating laser pulse (both for circular and linear polarisation) also did not show any appreciable distinction between the models. This is not surprising, because for relativistic electrons the leading order term in the radiation reaction force associated with the transverse acceleration is the same for all the compared models. However, differences arise in the presence of a longitudinal electric field with a spatio-temporal gradient. For such fields, the model F93 predicts the same recoil as the LL model, that depends on the electric field gradient. The models S09 and H08 depart from the LL results, but do predict an energy loss that depends solely on the field magnitude (not the gradient). The models B08 and LLR do not account for any energy loss in this configuration. Even though this offers a qualitative distinction between the models, the electric field needs to be on the order of the Schwinger critical field to show it, and even then the differences make for less than a percent of the total electron energy. They are, therefore, unlikely to make any real impact to the simulation results for realistic configurations. For example, these differences are negligible compared to the leading order term of the radiation reaction force for particles in the field of a laser pulse if $a_0 \gamma \gg 1$, which is necessarily satisfied in case of significant radiation reaction.

One additional question that needed to be addressed for PIC implementation is the definition of radiation reaction for a macro-particle. We note that the radiation reaction depends nonlinearly on charge and mass. To avoid overestimating the output radiation by artificially assuming coherent emission, the real charge and mass of a single particle should be used when calculating the radiation reaction force for a macro-particle.

The models are all fit to describe the physics, but the computational cost is minimal (in terms of flop and additional memory required) for the models B08, S09 and LLR. They can be included in a massively parallel PIC code just by re-writing the particle pusher, without any additional quantities to store. We have opted to imple-
ment the LLR model in OSIRIS.

Apart from choosing a model that accounts for radiation reaction, to correctly model the particle dynamics at extreme intensities it is essential to use the appropriate temporal resolution in the simulations. For the same laser frequency, a higher $a_0$ requires higher resolution to account correctly for the electron motion (especially if the electron is not relativistic at the start of the interaction). Detailed analysis supported by test-particle simulations was performed to facilitate the choice of temporal resolution for simulations at high intensities.

Taking all the aforementioned into account, it is possible to simulate the classical radiation reaction dominated regime with PIC codes by keeping the same parallel structure and introducing modifications only at particle push-time. Such modifications should not affect significantly the computational performance of massively parallel PIC simulations, with several models (e.g. LLR or B08) capturing the appropriate dynamics with minimal computational overhead.

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References

[1] [The ELI Project](http://www.extreme-light-infrastructure.eu/)
[2] A. B. Bell, J. G. Kirk, Possibility of prolific pair production with high-power lasers, Phys. Rev. Lett. 101 (2008) 200403. doi:10.1103/PhysRevLett.101.200403.
[3] S. S. Bulanov, T. Z. Esirkepov, A. G. R. Thomas, J. K. Koga, S. V. Bulanov, Schwinger limit attainability with extreme power lasers, Phys. Rev. Lett. 105 (2010) 220407. doi:10.1103/PhysRevLett.105.220407.
[4] C. P. Balcers, C. S. Brady, R. Duclos, J. G. Kirk, K. Bennett, T. D. Arber, A. P. L. Robinson, A. R. Bell, Dense electromagnetic plasma and ultraintense γ rays from laser-irradiated solids, Phys. Rev. Lett. 108 (2012) 165006. doi:10.1103/PhysRevLett.108.165006.
[5] N. V. Elkin, A. M. Fedotov, I. Y. Kostyukov, M. V. Legkov, N. B. Narozhny, E. N. Neush, H. Ruhl, Qed cascades induced by circularly polarized laser fields, Phys. Rev. ST Accel. Beams 14 (2011) 054401. doi:10.1103/PhysRevSTAB.14.054401
[6] K. Noguchi, E. Liang, Radiative effects on particle acceleration in electromagnetic dominated outflows, arXiv: astro-ph/0412310v3 (2008) 0412310v3.
[7] A. Zhidkov, J. Koga, A. Sasaki, M. Usaka, Radiation damping effects on the interaction of ultraintense laser pulses with an overdense plasma, Phys. Rev. Lett. 88 (18) (2002) 185002. doi:10.1103/PhysRevLett.88.185002.
[8] E. L. Clark, K. Krushelnick, H. Ruhl, Radiation reaction effects in the interaction of an intense laser pulse with a relativistic electron beam, Phys. Rev. ST Accel. Beams 17 (2014) 054401. doi:10.1103/PhysRevSTAB.17.054401.
[9] A. Zhidkov, A. Sasaki, T. Tajima, Emission of mev multiple-charged ions from metallic foils irradiated with an ultrashort laser pulse, Phys. Rev. E 61 (3) (2000) R2224–R2227. doi:10.1103/PhysRevE.61.R2224.
[10] M. Chen, A. Pukhov, T.-P. Yu, Z.-M. Sheng, Radiation reaction effects on ion acceleration in laser foil interaction, Plasma Phys. Contr. F. 53 (1) (2011) 014004.
[11] D. Iwanenko, I. Pomeranchuk, On the maximal energy attainable in a betatron, Phys. Rev. 65 (1944) 343–343. doi:10.1103/PhysRev.65.343.
[12] F. Rohrlich, Dynamics of a charged particle, Phys. Rev. E 77 (2008) 046609. doi:10.1103/PhysRevE.77.046609.
[13] A. Di Piazza, Exact solution of the landau-lifshitz equation in a plane wave, Lett. Math. Phys. 83 (3) (2008) 305–313. doi:10.1007/s11005-008-0228-9.
[14] J. Koga, Integration of the lorentz-dirac equation: Interaction of an intense laser pulse with high-energy electrons, Phys. Rev. E 70 (4) (2004) 046502. doi:10.1103/PhysRevE.70.046502.
[15] S. E. Gralla, A. I. Harte, R. M. Wald, Rigorous derivation of electromagnetic self-force, Phys. Rev. D 80 (2009) 024031. doi:10.1103/PhysRevD.80.024031.
[16] S. V. Bulanov, T. Z. Esirkepov, M. Kando, J. K. Koga, S. S. Bulanov, Lorentz-abraham-dirac versus landau-lifshitz radiation friction force in the ultrarelativistic electron interaction with electromagnetic wave (exact solutions), Phys. Rev. E 84 (2011) 056605. doi:10.1103/PhysRevE.84.056605.
[17] K. Seto, H. Nagatomo, J. Koga, K. Mima, Equation of motion with radiation reaction in ultrarelativistic laser-electron interactions, Phys. Plasmas 18 (12) (2011) 123101. doi:http://dx.doi.org/10.1063/1.3663843.
[18] H. Spohn, The critical manifold of the lorentz-dirac equation, Europhys. Lett. 50 (3) (2000) 287. doi:10.1209/epl/i2000-00268-x.
[19] A. M. Fedotov, N. V. Elkin, E. G. Gelfer, N. B. Narozhny, H. Ruhl, Radiation friction versus ponderomotive effect, Phys. Rev. A 90 (2014) 053847. doi:10.1103/PhysRevA.90.053847.
[20] Y. Kravets, D. Noble, Aand Jaroszynski, Radiation reaction effects on the interaction of an electron with an intense laser pulse, Phys. Rev. E 88 (2013) 011201. doi:10.1103/PhysRevE.88.011201.
[21] A. Noble, D. A. Burton, J. Gratus, D. A. Jaroszynski, A kinetic model of radiating electrons, Journal of Mathematical Physics 54 (4) (2013) –. doi:http://dx.doi.org/10.1063/1.4798790.
[22] A. Zhidkov, S. Masuda, S. S. Bulanov, J. Koga, T. Hosokai, R. Kodama, Radiation reaction effects in cascade scattering of intense, tightly focused laser pulses by relativistic electrons: Classical approach, Phys. Rev. ST Accel. Beams 17 (2014) 054001. doi:10.1103/PhysRevSTAB.17.054001.
[23] F. Rohrlich, The correct equation of motion of a classical point charge, Phys. Lett. A 283 (2001) 276 – 278. doi:http://dx.doi.org/10.1016/S0375-9601(01)00264-4.
[24] M. Tamburini, T. V. Liseykina, F. Pegoraro, A. Macchi, Radiation-pressure-dominated acceleration: Polarization and ra-
laser fields, Phys. Rev. E 81 (3) (2010) 036412. doi:10.1103/PhysRevE.81.036412.

[58] A. V. Arefiev, G. E. Cochran, D. W. Schumacher, A. P. L. Robinson, G. Chen, Temporal resolution criterion for correctly simulating relativistic electron motion in a high-intensity laser field, Phys. Plasmas 22 (1) (2015) 013103. doi:http://dx.doi.org/10.1063/1.4905523.

[59] W. Lu, M. Tzoufras, C. Joshi, F. S. Tsung, W. B. Mori, J. Vieira, R. A. Fonseca, L. O. Silva, Generating multi-gev electron bunches using single stage laser wakefield acceleration in a 3d nonlinear regime, Phys. Rev. ST Accel. Beams 10 (2007) 061301. doi:10.1103/PhysRevSTAB.10.061301.