Topological properties of QCD with two dynamical fermions

B. Allés\textsuperscript{a}, M. D’Elia\textsuperscript{b}, A. Di Giacomo\textsuperscript{b}

\textsuperscript{a}Dipartimento di Fisica, Università di Milano–Bicocca and INFN Sezione di Milano, Milano, Italy
\textsuperscript{b}Dipartimento di Fisica, Università di Pisa, Via Buonarroti 2, Ed. B, 56127 Pisa, Italy

We investigate the topological susceptibility of the QCD vacuum with two flavours of dynamical staggered fermions on the lattice both at zero and finite temperature. At zero temperature we study the dependence of the signal on the fermion mass and at finite temperature we analyze the behaviour across the phase transition.

1. INTRODUCTION

The topological susceptibility is an important parameter of the QCD vacuum. In the continuum it is defined as

$$\chi \equiv \int d^4x \partial_\mu \langle 0 | T \{ K_\mu(x) Q(0) \} | 0 \rangle ,$$

where \( K_\mu(x) \) is the Chern current

$$K_\mu(x) = \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \times A^a_\nu \left( \partial_\mu A^a_\sigma - \frac{1}{3} g f^{abc} A^b_\rho A^c_\sigma \right) ,$$

and \( Q(x) = \partial_\mu K_\mu(x) \) is the density of topological charge,

$$Q(x) = \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) .$$

Eq. (1) uniquely defines the prescription for the singularity of the time ordered product when \( x \rightarrow 0 \).

We present results on the topological susceptibility in QCD with \( N_f = 2 \) degenerate flavours.

1.1. The Simulation

We have simulated the theory on a \( 32^3 \times 8 \) lattice with two flavours of staggered fermions with bare mass \( am = 0.0125 \). The R–type HMC algorithm has been utilized for the updating. Each trajectory consisted in 60 steps of \( \Delta \tau = 0.005 \) units of molecular dynamics time. We have performed the simulation at the following values of the lattice bare coupling \( \beta \equiv 6/g^2 = 5.4, 5.5, 5.6 \) and 5.7. To be sure that the topology is well thermalized we have checked the topological charge of our sample of configurations by cooling \( \frac{1}{3} \).

In Fig. 1 we show the distribution of topological charge from our configurations at \( \beta = 5.5 \). The histogram displays a good sampling. To achieve the same ergodicity at larger values of \( \beta \) we had to increase the statistics.

We have measured the Polyakov loop at zero and finite temperatures. With our bare fermion mass and lattice size the deconfining transition seems to occur at \( \beta_c = 5.54(2) \).

![Figure 1. Distribution of topological charge after 30 cooling steps at \( \beta = 5.5 \).](attachment:/9909018v1/9909018v1.pdf)
Table 1
Topological susceptibility, $T/T_c$ and $a^2\sigma$ vs. $\beta$.

| $\beta$ | $a^2\sigma$ | $T/T_c$ | $10^{-8}\chi$/MeV$^4$ |
|---------|-------------|---------|------------------------|
| 5.4     | 0.138(8)    | 0.707   | 5.16(1.66)             |
| 5.5     | 0.082(9)    | 0.917   | 6.21(1.74)             |
| 5.6     | 0.053(3)    | 1.143   | 2.27(0.72)             |
| 5.7     | –           | 1.430   | 1.72(1.00)             |

1.2. The Operators and their Renormalizations

We have measured the topological charge density on the lattice by using the operator $Q_L(x)$

$$Q_L(x) = -\frac{1}{2^{2/3}} \bar{\psi} \gamma^\mu \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma \psi \Pi^\mu^\nu(x) \Pi^\rho^\sigma(x),$$

and applying two smearing steps on it. The lattice topological susceptibility is defined by

$$\chi_L \equiv \frac{\langle Q_L^2 \rangle}{V},$$

where $Q_L$ is the total topological charge and $V$ the space–time volume. To match this definition to the one given in Eq.(1) we must subtract the singularity in the product of the two operators in Eq.(5). We make the subtraction by using the field–theoretical method

$$\chi_L = Z a^4 \chi + M,$$

and evaluating the multiplicative and additive renormalizations $Z$ and $M$ by use of the heating method. $M$ contains mixings with gluonic and fermionic condensates. The lattice topological charge $Q_L$ mixes with fermionic operators during renormalization but it has been shown that the off–diagonal mixing is negligible.

The heating method to evaluate $Z$ and $M$ yields a non–perturbative determination of these renormalization constants. To calculate $Z$ a local updating algorithm is applied on a configuration containing a charge +1 classical instanton. These updatings, being local, thermalize the short distance fluctuations, responsible for the renormalization effects, and due to the slowing down leave large structures, like instantons, unchanged. The measurement of $Q_L$ on such updated configurations yields $Z \cdot Q$. As $Q$ is known, one can extract $Z$. Notice that this procedure is equivalent to imposing the continuum value for the 1–instanton charge (in the $\overline{MS}$ scheme it is +1) and extracting the finite multiplicative renormalization $Z$ by evaluating a matrix element of $Q_L$.

The additive renormalization $M$ is obtained in a similar way. We apply a few heating steps with a local updating algorithm on a zero–field configuration. Then the topological susceptibility is calculated. This provides the value of $M$, if no instantons have been created during the few updating steps. This method agrees with the treatment of the singularity of Eq.(5).

As explained in the paper, cooling tests must be done to check that the background topological charge has not been changed during the local heating.

1.3. Determination of the scale

The lattice spacing was extracted by measuring the string tension at $\beta = 5.4$, 5.5 and 5.6 on a $16^4$ lattice and assuming $\sqrt{\sigma} = 420$ MeV. For $\beta = 5.7$ it was determined by an extrapolation using the 2–loop beta function. Wilson loops were evaluated by using smeared spatial links. The determinations of $a^2\sigma$ are shown in Table 1. From the value at $\beta_c$ ($a = 0.123(6)$ fm) we infer the critical temperature to be $T_c = 200 \pm 10 \pm 15$ MeV where the first error is our statistical one and the second comes from the indetermination in the result for $\beta_c$.

2. RESULTS

The topological susceptibility at zero temperature is shown in Fig. 2. In the lower (upper) axis we show the values of $\beta$ (bare fermion mass). It displays a trend consistent with the theoretical expectation $\chi \propto m(\bar{\psi}\psi)$.

The values for the topological susceptibility at finite temperature are given in Table 1. In Fig. 3...
we show the normalized topological susceptibility \( \chi(T)/\chi(T=0) \) as a function of the temperature. The value at zero temperature for \( \beta = 5.7 \) has been obtained by extrapolating the results shown in Fig. 2. In Fig. 3 the results for the quenched case \([6] \) are shown for comparison. The signal for \( \chi \) drops when crossing the transition temperature. Within the large errors, the drop seems to be as sharp as it was for the quenched case. Our results for \( N_f = 2 \) are in qualitative agreement with the ones shown in \([4] \). For \( N_f = 4 \) the drop in the signal for \( \chi \) turns out to be much steeper \([2] \).

**Figure 2.** Topological susceptibility as a function of the bare fermion mass.

**Figure 3.** Behaviour of the topological susceptibility as a function of the normalized temperature \( T/T_c \). Black dots correspond to \( N_f = 2 \) and white triangles to \( N_f = 0 \) \([9] \). Lines are drawn to guide the eye.

**REFERENCES**

1. E. Witten, Nucl. Phys. B156 (1979) 269.
2. S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, Phys. Rev. D35 (1987) 2531.
3. B. Allés, G. Boyd, M. D’Elia, A. Di Giacomo and E. Vicari, Phys. Lett. B389 (1996) 107.
4. B. Allés, G. Bali, M. D’Elia, A. Di Giacomo, N. Eicker, K. Schilling, A. Spitz, S. Gäsken, H. Hoeber, Th. Lippert, T. Struckmann, P. Ueberholz and J. Viehoff, Phys. Rev. D58 (1998), 071503.

5. S. Gottlieb, A. Krasnitz, U. M. Heller, A. D. Kennedy, J. B. Kogut, R. L. Renken, D. K. Sinclair, R. L. Sugar, D. Toussaint and K. C. Wang, Phys. Rev. D47 (1993) 3619.
6. C. Christou, A. Di Giacomo, H. Panagopoulos and E. Vicari, Phys. Rev. D53 (1996) 2619.
7. M. Campostrini, A. Di Giacomo, H. Panagopoulos and E. Vicari, Nucl. Phys. B329 (1990) 683.
8. A. Di Giacomo and E. Vicari, Phys. Lett. B275 (1992) 429.
9. B. Allés, M. D’Elia and A. Di Giacomo, Nucl. Phys. B494 (1997) 281.
10. B. Allés, A. Di Giacomo, H. Panagopoulos and E. Vicari, Phys. Lett. B350 (1995) 70.
11. P. de Forcrand, M. García–Pérez, J. E. Hetrick and I.–O. Stamatescu, Nucl. Phys. B (Proc. Suppl.) 63A–C (1998) 549.
12. B. Allés, M. D’Elia, A. Di Giacomo and P. Stephenson, Nucl. Phys. B (Proc. Suppl.) 73 (1999) 518.