ON THE CAUCHY PROBLEM FOR THE XFEL SCHröDINGER EQUATION

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(Communicated by Shouhong Wang)

Abstract. In this paper, we consider the Cauchy problem for the nonlinear Schrödinger equation with a time-dependent electromagnetic field and a Coulomb potential, which arises as an effective single particle model in X-ray free electron lasers (XFEL). We firstly show the local and global well-posedness for the Cauchy problem under the assumption that the magnetic potential is unbounded and time-dependent, and then obtain the regularity by a fixed point argument.

1. Introduction. In this paper, we investigate a first-principles model for beam-matter interaction in X-ray free electron lasers (XFEL) [7, 13]. The fundamental model for XFEL is the following nonlinear Schrödinger equation with a time-dependent electromagnetic field and a Coulomb potential

\[
\begin{aligned}
i \partial_t u &= (i \nabla - A(t, x))^2 u + \frac{\lambda_1}{|x - \beta(t)|} u + \lambda_2 (\frac{1}{|\cdot|} \ast |u|^2) u + \lambda_3 |u|^{2\sigma} u, \\
u(0, x) &= u_0(x),
\end{aligned}
\]

where \( u = u(t, x) : [0, \infty) \times \mathbb{R}^3 \to \mathbb{C} \) is a complex valued function, \( \ast \) denotes convolution, \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, 0 < \sigma < 2 \), \( \alpha : [0, \infty) \to \mathbb{R} \) and \( \beta : [0, \infty) \to \mathbb{R}^3 \) are two control functions, the vectorfield \( A \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^3, \mathbb{R}^3) \) represents an external electromagnetic potential and satisfies the following assumption.

Assumption 1. There exist some constants \( C_\gamma > 0 \) such that

1. \( \forall \gamma \in \mathbb{N}^3, \sup_{(t,x) \in \mathbb{R}_t \times \mathbb{R}_x^3} |\partial_{x}^\gamma A| \leq C_\gamma \),
2. \( \forall |\gamma| \geq 1, \sup_{(t,x) \in \mathbb{R}_t \times \mathbb{R}_x^3} |\partial_{x}^\gamma A| \leq C_\gamma \),
3. \( \exists \varepsilon > 0, \forall |\gamma| \geq 1, \sup_{(t,x) \in \mathbb{R}_t \times \mathbb{R}_x^3} |\partial_{x}^\gamma B| \leq C_\alpha (1 + |x|)^{-1-\varepsilon}, \) where \( B(t, x) \) is the matrix defined by \( B_{jk} = \partial_x A_k - \partial_x A_j \).

Note that compactly supported perturbations of linear (with respect to \( x \)) magnetic potentials satisfy the above hypothesis.

2010 Mathematics Subject Classification. 35Q51, 35Q55.
Key words and phrases. The Cauchy problem, time-dependent electromagnetic field, time-dependent Coulomb potential, XFEL Schrödinger equation.

This work is supported by NSFC Grants (No. 11601435, No. 11475073), Gansu Provincial Natural Science Foundation (1606RJZA010) and NWNU-LKQN-14-6.

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Under this assumption, the domain \( D((i\nabla - A(t,x))^2) = \{ u \in L^2(\mathbb{R}^3), \ (i\nabla - A(t,x))^2 u \in L^2(\mathbb{R}^3) \} \) does not depend on time \( t \), see [15]. Therefore, for \( s \in \mathbb{N} \), the space
\[
H^s_{A(t)} := \{ u \in \mathcal{S}'(\mathbb{R}^3), \ (1 + (i\nabla - A(t,x))^2)^{s/2} u \in L^2(\mathbb{R}^3) \}
\]
does not depend on time \( t \), where \( \mathcal{S}'(\mathbb{R}^3) \) denotes the space of tempered distribution on \( \mathbb{R}^3 \). Moreover, \( H^s_{A(t)} \)-norm can be defined by
\[
\| u \|_{H^s_{A(t)}} := \| (i\nabla - A(t))^s u \|_{L^2} + \| u \|_{L^2}.
\]
In particular, if \( A(t,x) \) is a bounded function and satisfies Assumption 1, then \( \| u \|_{H^s_{A(t)}} = \| u \|_{H^s} \). Because the space \( H^s_{A(t)} \) is independent of time \( t \), we often abbreviate \( H^s_{A(t)} \) and \( \| \cdot \|_{H^s_{A(t)}} \) by \( H^s \) and \( \| \cdot \|_{H^s} \) respectively.

Equation (1) has important applications in physics. A solution \( u \) of this Schrödinger equation can be considered as the wave function of an electron beam, under the action of magnetic potential \( A(\alpha(t), x) \), interacting with an atomic nucleus, located at the point \( \beta(t) \), interacting self-consistently through the Coulomb (Hartree) force with strength \( \lambda_2 \), the local Fock approximation with strength \( \lambda_3 \) and exponent \( \sigma \), see [13]. Since XFEL is more powerful by several orders of magnitude than more conventional lasers, the systematic investigation of many of the standard assumptions and approximations has attracted increased attention. For physical reasons we shall only consider the three-dimensional case here, that is the spatial variable \( x \) is assumed to be in \( \mathbb{R}^3 \).

When \( A(\alpha(t), x) = 0, \beta(t) = 0 \), equation (1) is simplified to the following non-linear Schrödinger equation
\[
\begin{gathered}
i\partial_t u = -\Delta u + \frac{\lambda_1}{|x|} u + \lambda_2 \frac{1}{|x|} \ast |u|^2 u + \lambda_3 |u|^2|u|^2 \\
\end{gathered}
\]
(2)
It has received a great deal of attention from mathematicians, mainly with respect to the local and global properties; for instance, see [3, 10, 12, 18] and the references therein.

When the potential \( A \) depends only on spatial variable \( x \) and \( \beta(t) = 0 \), equation (1) is a class of non-linear Schrödinger equations with a magnetic potential. Regarding the study of this kind of equations, the property of \( A \) plays an important role. When the magnetic field potential \( A \) is bounded, the spaces \( H^1_A \) and \( H^1 \) coincide and the Cauchy problem (1) can be solved in \( H^1 \) by using standard techniques. If the magnetic field is unbounded, it is impossible to solve the Cauchy problem in \( H^1 \) since the product \( u \mapsto Au \) is unbounded in \( L^2 \). To solve this problem, Bouard [8], Nakamura and Shimomura [16] considered it in the weighted space \( \Sigma := \{ u \in H^1 \text{ and } xu \in L^2 \} \). However, this space does not reflect the property of \( A \). On the other hand, Cazenave and Esteban [4] investigated the special case where the magnetic field \( A \) is linear with respect to \( x \). In one way, this paper is more satisfactory since they only require \( u_0 \) to belong to the energy space. Nevertheless, their results apply only to the linear magnetic field.

When \( \alpha(t) = t \) and \( \lambda_1 = \lambda_2 = 0 \), Michel in [15] studied the local well-posedness of (1) by approximating the magnetic potential \( A(t,x) \) by potentials which are piecewise constant with respect to time. However, the terms \( \frac{1}{|x - \beta(t)|} u \) and \( (| \cdot |^{-1} \ast |u|^2) u \) in (1) do not satisfy the assumption on nonlinearity in [15]. Recently, when the potential \( A \) depends only on time \( t \) and \( \beta(t) = 0 \), by using a simple change of coordinates and a phase shift, Antonelli et al. in [1] investigated the asymptotic behavior of solution of equation (1) with \( 0 < \sigma < \frac{2}{3} \) in the highly oscillating regime.
Antonelli et al. in [2] considered similar problems by numerical simulations. Feng has extended the study of Antonelli et al. in [1] to power nonlinearity with exponent $\frac{2}{\sigma} \leq \sigma < 2$, see [9].

However, to the best of our knowledge, there are no any rigorous mathematical results on equation (1) with general time-dependent electromagnetic field and Coulomb potentials, despite the important significance of the involved applications (see the references given above). In this paper, we are interested in the local and global well-posedness, the $H^2_A$ regularity for the Cauchy problem (1).

Due to the appearance of the time-dependent unbounded magnetic field and the Coulomb potential, there exist some essential difficulties in studying the Cauchy problem (1). Firstly, since the magnetic field potential $A$ is unbounded, we cannot apply a fixed point argument and Strichartz’s estimates to study the local well-posedness. When we use the compactness method, the Coulomb potential brings some essential difficulties, see Remark 3.1 for a detailed analysis.

Secondly, regarding the nonlinearity $|u|^{2\sigma} u$, we use the idea due to T.Kato [14] (see also [3]), based on the general idea for Schrödinger equation, that two space derivative cost the same as one time derivative. Because of the time-dependence of magnetic Laplacian $(i\nabla - A(\alpha(t), x))^2$, the compactness method can not be used to solve this problem, see section 4.

This paper is organized as follows: in Section 2, we will collect some preliminaries such as Strichartz’s estimates, the compactness results, etc. In section 3, we will establish the local and global well-posedness. In section 4, we will establish the $H^2_A$ regularity for (1).

**Notation.** Throughout this paper, we use the following notation. $C > 0$ will stand for a constant that may be different from line to line when it does not cause any confusion. Since we exclusively deal with $\mathbb{R}^3$, we often use the abbreviations $L^r = L^r(\mathbb{R}^3)$, $H^s = H^s(\mathbb{R}^3)$ in what follows. Given any interval $I \subset \mathbb{R}$, the norm of mixed spaces $L^q(I, L^r(\mathbb{R}^3))$ is denoted by $\| \cdot \|_{L^q(I) L^r(\mathbb{R}^3)}$. We recall that a pair of exponents $(q, r)$ is Schrödinger-admissible if $\frac{2}{q} = 3(\frac{1}{4} - \frac{1}{r})$ and $2 \leq r \leq 6$. Then, for any space-time slab $I \times \mathbb{R}^3$, we can define the Strichartz norm

$$\| u \|_{S(I)} = \sup_{(q,r)} \| u \|_{L^q(I) L^r(\mathbb{R}^3)},$$

where the supremum is taken over all admissible pairs of exponents $(q, r)$. For simplicity, we always denote $V_1 = \frac{1}{|x - \beta(t)|} \chi_{B_{\beta(t)}}$ and $V_2 = \frac{1}{|x - \beta(t)|} (1 - \chi_{B_{\beta(t)}})$, where $B_{\beta(t)}$ is the unit ball in $\mathbb{R}^3$ centered at the point $\beta(t)$, $\chi_{B_{\beta(t)}}$ is the characteristic function. Therefore, $V_1 \in L^{\frac{4}{3+\varepsilon}}$, $\nabla V_1 \in L^{\frac{4}{3+\varepsilon}}$ with $\varepsilon > 0$ sufficiently small, $V_2 \in L^{\infty}$ and $\nabla V_2 \in L^{\infty}$.

2. Preliminaries. In this section, we will recall some known facts and give some elementary results which will be used and play important roles later. Firstly, we recall the following compactness lemma, see [3] for a detailed presentation.

**Lemma 2.1.** [3] Let $X \hookrightarrow Y$ be two Banach spaces, $I$ be a bounded, open interval of $\mathbb{R}$, and $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $C(I, Y)$. Assume that $u_n(t) \in X$ for all $(n, t) \in \mathbb{N} \times I$ and that $\sup \{ \| u_n(t) \|_X, (n, t) \in \mathbb{N} \times I \} = K < \infty$. Assume further that $u_n$ is uniformly equicontinuous in $Y$. If $X$ is reflexive, then there exist
a function $u \in C(\bar{I}, Y)$ which is weakly continuous $\bar{I} \rightarrow X$ and some subsequence $(u_{n_k})_{k \in \mathbb{N}}$ such that for every $t \in \bar{I}$, $u_{n_k}(t) \rightarrow u(t)$ in $X$ as $k \rightarrow \infty$.

From this lemma, we can deduce the following compactness lemma.

**Lemma 2.2.** Let $I$ be a bounded interval of $\mathbb{R}$, and $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence of $L^\infty(I, H^1_A) \cap W^{1,\infty}(I, H^{-1}_A)$.

(i) There exist $u \in L^\infty(I, H^1_A) \cap W^{1,\infty}(I, H^{-1}_A)$ and some subsequence $(u_{n_k})_{k \in \mathbb{N}}$ such that for every $t \in \bar{I}$, $u_{n_k}(t) \rightharpoonup u(t)$ in $H^1_A$ as $k \rightarrow \infty$.

(ii) If $\|u_{n_k}(t)\|_{L^2} \rightarrow \|u(t)\|_{L^2}$ as $k \rightarrow \infty$, uniformly on $I$, then $u_{n_k} \rightharpoonup u$ in $C(\bar{I}, L^2)$ as $k \rightarrow \infty$.

Next, we recall some results established in [15, 19]. Assume that $A$ satisfies Assumption 1 and $\alpha \in C(-\infty, \infty)$. It is established in [19] that one can define $U(t, s)$ as $u(t, x) = U(t, s)\varphi(x)$, where

$$i\partial_t u = (i\nabla - A(\alpha(t), x))^2 u; \ u(s, x) = \varphi(x).$$

**Lemma 2.3.** (Strichartz’s estimates). The following properties hold:

1. For any admissible pair $(q, r)$, there exists $C_q$ such that
   $$\|U(\cdot, s)\varphi\|_{L^q_t L^r_x(s, s+T)} \leq C_q \|\varphi\|_{L^2}, \text{ for all } \varphi \in L^2 \text{ and } T > 0.$$

2. Denote
   $$D_s(f)(t, x) = \int_0^t U(t, \tau)f(\tau, x)d\tau.$$

For all admissible pairs $(\gamma, \rho)$ and $(\gamma_1, \rho_1)$, there exists $C = C_{\gamma, \gamma_1}$ independent of $s$ such that

$$\|D_s(f)\|_{L^\gamma_t L^\rho_x(s, s+T)} \leq C\|f\|_{L^{\gamma_1}_t L^{\rho_1}_x(s, s+T)},$$

for all $f \in L^{\gamma_1}_t L^{\rho_1}_x(s, s+T)$ and $T > 0$.

Finally, we recall the following Gronwall-type estimate which will be used to prove the uniqueness of the weak solution of (1), see [11].

**Lemma 2.4.** Let $T > 0$, $1 \leq p_1 < q_1 \leq \infty$, $1 \leq p_2 < q_2 \leq \infty$, and $a, b_1, b_2 \geq 0$. It follows that there exists $\Gamma = \Gamma(b_1, b_2, p, q, T)$ such that if $f_1 \in L^{p_1}(0, T)$ and $f_2 \in L^{q_2}(0, T)$ satisfy

$$\|f_1\|_{L^{p_1}(0, T)} + \|f_2\|_{L^{q_2}(0, T)} \leq a + b_1 \|f_1\|_{L^{p_1}(0, T)} + b_2 \|f_2\|_{L^{p_2}(0, T)},$$

for all $0 < t < T$, then

$$\|f_1\|_{L^{p_1}(0, T)} + \|f_2\|_{L^{q_2}(0, T)} \leq a \Gamma.$$

3. The local existence and global existence. In this section, we establish the local and global well-posedness for (1). Firstly, we consider the local well-posedness of (1).

**Theorem 3.1.** Let $A$ satisfy Assumption 1, $\alpha \in H^1(0, \infty)$, $\beta \in C^1([0, \infty), \mathbb{R}^3)$ and $0 < \sigma < 2$. For every $u_0 \in H^1_A$, there exist $0 < T_{\text{max}} \leq \infty$ and a unique, maximal solution $u \in C([0, T_{\text{max}}], H^1_A) \cap C^1([0, T_{\text{max}}], H^{-1}_A)$ of (1). In addition, the following properties hold:

(i) There is the blow-up alternative, i.e., either $T_{\text{max}} = \infty$ or $\lim_{t \rightarrow T_{\text{max}}} \|u(t)\|_{H^1_A} = \infty$.

(ii) The following equalities follow.

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$

(3)
and

\[
\frac{d}{dt} E(t) = - \alpha'(t) \text{Re}(\partial_\alpha A(\alpha(t), x)u(t), (i\nabla - A(\alpha(t), x))u(t))_{L^2} \\
- \frac{\lambda_1}{2} \int_{\mathbb{R}^3} \frac{(x - \beta(t)) \cdot \beta'(t)}{|x - \beta(t)|^3} |u(t, x)|^2 dx,
\]

for all \( t \in (0, T_{\text{max}}) \), where

\[
E(t) = \frac{1}{2} \int_{\mathbb{R}^3} |(i\nabla - A(\alpha(t), x))u(t, x)|^2 dx + \frac{\lambda_1}{2} \int_{\mathbb{R}^3} \frac{|u(t, x)|^2}{|x - \beta(t)|} dx \\
+ \frac{\lambda_2}{4} \int_{\mathbb{R}^3} \left[ \frac{1}{|\cdot|} * |u(t)|^2(x) |u(t, x)|^2 dx \right] + \frac{\lambda_3}{2\sigma + 2} \int_{\mathbb{R}^3} |u(t, x)|^{2\sigma + 2} dx.
\]

**Remark 3.1.** Because the magnetic field potential \( A(\alpha(t), x) \) is unbounded with respect to \( x \), we cannot apply a fixed point argument and Strichartz’s estimates to prove the local well-posedness of \( (1) \). Indeed, according to

\[
(i\nabla - A(\alpha))|u|^{2\sigma} = |u|^{2\sigma} \{ i\nabla - A(\alpha) \} u + iu \nabla (|u|^{2\sigma}).
\]

When \( A(\alpha, x) \) is unbounded with respect to \( x \), there is no chance to control \( \nabla (|u|^{2\sigma}) \) by \( (i\nabla - A(\alpha))|u|^{2\sigma} \). Motivated by the ideas in [15, 17], we approximate the magnetic potential \( A(\alpha, x) \) and the Coulomb potential \( \frac{1}{|x - \beta(t)|} \) by potentials which are piecewise constant with respect to time. Michel in [15] applied the boundedness of \( \partial^\alpha_\alpha A \) (see Assumption 1) to obtain a priori estimate and pass to limit. However, the Coulomb potential \( \frac{1}{|x - \beta(t)|} \) is unbounded and does not satisfy this property. The methods of a priori estimate and passing to limit in [15] cannot work for our case.

Therefore, it is interesting to introduce some other methods to solve the related problems. In order to establish the local well-posedness for \( (1) \), we firstly solve the Cauchy problem \( (1) \), where \( A(\alpha(t), x) = A(x) \) and \( \beta(t) \equiv \beta_0 \) are time independent.

**Lemma 3.2.** Let \( A \) be time independent and satisfy Assumption 1, \( \beta(t) \equiv \beta_0 \) for some \( \beta_0 \in \mathbb{R}^3 \). Assume that \( M > 0, 0 < \sigma < 2 \). Then, there exist \( 0 < T_{\text{max}} \leq \infty \) depending only on \( M \) such that for all \( u_0 \in H^1_\Lambda^1 \) such that \( \|u_0\|_{H^1_\Lambda^1} \leq M \), there exists a unique, maximal solution \( u \in C([0, T_{\text{max}}), H^1_\Lambda^1) \cap C^1([0, T_{\text{max}}), H^{-1}_\Lambda^1) \) of \( (1) \). Moreover, either \( T_{\text{max}} = \infty \) or \( \lim_{t \to T_{\text{max}}} \|u(t)\|_{H^1_\Lambda^1} = \infty \), and for all \( t \in [0, T_{\text{max}}] \)

\[
\|u(t)\|_{L^2} = \|u_0\|_{L^2},
\]

where

\[
E_0(u(t)) = \frac{1}{2} \int_{\mathbb{R}^3} |(i\nabla - A(x))u(t, x)|^2 dx + \frac{\lambda_1}{2} \int_{\mathbb{R}^3} \frac{|u(t, x)|^2}{|x - \beta_0|} dx \\
+ \frac{\lambda_2}{4} \int_{\mathbb{R}^3} \left[ \frac{1}{|\cdot|} * |u(t)|^2(x) |u(t, x)|^2 dx \right] + \frac{\lambda_3}{2\sigma + 2} \int_{\mathbb{R}^3} |u(t, x)|^{2\sigma + 2} dx.
\]

**Proof.** This lemma can be proved by a similar method as that of Proposition 2.6 in [15]. Let us notice that the nonlinearity \( |u|^{2\sigma} u \) satisfies Assumption 2 in [15], but the nonlinearities \( \frac{1}{|\cdot|} \) and \( (|\cdot|^{-1} * |u|^2) u \) do not satisfy this assumption. Following [6], let

\[
V_1 = |x|^{-1} \chi_{B_{\beta_0}}, \ V_2 = |x|^{-1}(1 - \chi_{B_{\beta_0}}), \ V_{1,m} = \min\{m, V_1\}.
\]
We decompose $|x|^{-1}u = V_1u + V_2u := g_1(u) + g_2(u)$, $(|x|^{-1} |u|^2)u = (V_1 * |u|^2)u + (V_2 * |u|^2)u := g_3(u) + g_4(u)$, $|u|^2u = g_5(u) + g_6(u)$, where

$$g_5(u) = \chi_{\{|u| \leq 1\}}|u|^\sigma u + \chi_{\{|u| > 1\}}u, \quad g_6(u) = \chi_{\{|u| > 1\}}(|u|^\sigma u - u).$$

Next, we define $g_1,m(u) = V_1,mu, g_2,m(u) = g_2(u), g_3,m(u) = V_1,m * |u|^2u, g_4,m(u) = g_4(u), g_5,m(u) = g_5(u)$ and $g_6,m(u) = \chi_{\{|u| \leq m\}}g_6(u) + \chi_{\{|u| > m\}}g_6(u)$. It is easy to check that $g_1,m - g_6,m$ satisfy the Lemmas 2.3 and 2.4 in [15]. With the above results at hand, one can prove this lemma along the lines of Proposition 2.6 in [15], so we omit it.

**Proof.** We will approximate the magnetic potential $A(\alpha(t), x)$ and the Coulomb potential $\frac{1}{|x-\alpha(t)|}$ by potentials which are piecewise constant with respect to time. The proof proceeds in three steps.

**Step 1.** Estimates on the sequence $(u_n)_{n \in \mathbb{N}}$.

According to Lemma 3.2, for all $M > 0$, there exists $T_1 = T_1(M)$ such that for all $u_0 \in H^1_A$ satisfying $\|u_0\|_{H^1_A} \leq M$, there exists a unique solution $u \in C([0, T_1], H^1_A) \cap C^1([0, T_1], H^1_A)$ of (1), where $A$ is time independent and $\beta(t) \equiv \beta_0$ for some $\beta_0 \in \mathbb{R}^3$.

Let $u_0 \in H^1_A$ be such that $\|u_0\|_{H^1_A} \leq \frac{M}{4}$ and $T \in (0, T_1)$. For $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$, we define $t^k_n = \frac{kT}{n}$ and

$$\alpha_n(t) = \sum_{k=0}^{n-1} \chi_{(t^k_n, t_{k+1}^n)} \alpha(t_{k}^n), \quad \beta_n(t) = \sum_{k=0}^{n-1} \chi_{(t^k_n, t_{k+1}^n)} \beta(t_{k}^n), \quad \forall t \in [0, T].$$

Next, we consider the following equation:

$$\begin{cases}
    i \partial_t u_n = (i \nabla - A(\alpha_n(t), x))^2 u_n + \frac{\lambda_1}{|x-\beta_n(t)|} u_n + \lambda_2 \frac{1}{|u_n|^2} u_n + \lambda_3 |u_n|^{2\sigma} u_n, \\
    u_n(0, x) = u_0(x).
\end{cases}$$

We deduce from uniqueness established in Lemma 3.2 that the solution of this equation given by

$$u_n(t, x) = \sum_{k=0}^{n-1} \chi_{(t^k_n, t_{k+1}^n)} v_{k,n}(t, x),$$

where $v_{0,n}$ is the solution of

$$\begin{cases}
    i \partial_t v_{0,n} = (i \nabla - A(\alpha(0), x))^2 v_{0,n} + \frac{\lambda_1}{|x-\beta(0)|} v_{0,n} \\
    + \lambda_2 \frac{1}{|v_{0,n}|^2} v_{0,n} + \lambda_3 |v_{0,n}|^{2\sigma} v_{0,n}, \\
    v_{0,n}(0, x) = u_0(x),
\end{cases}$$

and for $k \geq 1$, $v_{k,n}$ is the solution of the following equation

$$\begin{cases}
    i \partial_t v_{k,n} = (i \nabla - A(\alpha(t_{k}^n), x))^2 v_{k,n} + \frac{\lambda_1}{|x-\beta_{k,n}(t_{k}^n)|} \\
    + \lambda_2 \frac{1}{|v_{k,n}|^2} v_{k,n} + \lambda_3 |v_{k,n}|^{2\sigma} v_{k,n}, \\
    v_{k,n}(t_{k}^n, x) = v_{k-1,n}(t_{k}^n, x).
\end{cases}$$

In the following, we will prove that the functions $v_{k,n}$, $k = 0, \ldots, n - 1$ are well-defined. Since $\|u_0\|_{H^1_A} \leq \frac{M}{4}$, it follows from Lemma 3.2 that the Cauchy problem (12) is local well-posedness. Moreover, for any $k \in \{0, \ldots, n - 1\}$, in order to prove that $v_{k,n}$ is well defined, it suffices to show that $\|v_{k-1,n}(t_{k}^n)\|_{H^1_A} \leq M$. Let $k_1 \in \{1, \ldots, n - 1\}$ be the greatest integer such that $\|v_{k_1-1,n}(t_{k_1}^n)\|_{H^1_A} \leq M$. Then,
the function \( u_n \) given by (11) is well-defined for \( t \in [0, t_n^{k+1}] \) and is continuous with values in \( H^1_\Lambda \). For \( w \in H^1_\Lambda \), we define
\[
E_n(t, w) = \frac{1}{2} \int_{\mathbb{R}^3} |(i\nabla - A(\alpha(t), x))w(x)|^2 \, dx + \frac{\lambda_1}{2} \int_{\mathbb{R}^3} \frac{|w(x)|^2}{|x - \beta_n(t)|} \, dx + \frac{\lambda_2}{4} \int_{\mathbb{R}^3} \left( \left| \frac{\partial w}{\partial t} \right|^2 + |w|^2 \right) \, dx + \frac{\lambda_3}{2\sigma + 2} \int_{\mathbb{R}^3} |w(x)|^{2\sigma + 2} \, dx. \tag{14}
\]

Then, we deduce from Lemma 3.2 that for all \( k \in \{1, \ldots, k_1\} \) and \( t \in [t_n^k, t_n^{k+1}] \)
\[
E_n(t, v_{k,n}(t)) = E_n(t_n^k, v_{k,n}(t_n^k)).
\]
Therefore, from \( v_{k-1,n}(t_n^k) = v_{k,n}(t_n^k) = u_n(t_n^k) \), we have
\[
E_n(t_n^{k-1}, u_n(t_n^{k-1})) = \frac{1}{2} \int_{\mathbb{R}^3} |(i\nabla - A(\alpha(t_n^{k-1}), x))u_n(t_n^{k-1}, x)|^2 \, dx + \frac{\lambda_1}{2} \int_{\mathbb{R}^3} \frac{|u_n(t_n^{k-1}, x)|^2}{|x - \beta(t_n^{k-1})|} \, dx + \frac{\lambda_2}{4} \int_{\mathbb{R}^3} \left( \left| \frac{\partial u_n}{\partial t} \right|^2 + |u_n|^2 \right) \, dx + \frac{\lambda_3}{\sigma + 2} \int_{\mathbb{R}^3} |u_n(t_n^{k-1}, x)|^{2\sigma + 2} \, dx. \tag{15}
\]
This implies
\[
E_n(t_n^{k}, u_n(t_n^{k})) - E_n(t_n^{k-1}, u_n(t_n^{k-1}))
= -\int_{t_n^{k-1}}^{t_n^k} \text{Re} \langle (i\nabla - A(\alpha(s)))u_n(t_n^k), \alpha'(s)\partial_t A(\alpha(s))u_n(t_n^k) \rangle \, ds
- \frac{\lambda_1}{2} \int_{\mathbb{R}^3} \int_{t_n^{k-1}}^{t_n^k} \frac{\langle (x - \beta(s)), \beta'(s) \rangle}{|x - \beta(s)|^3} |u_n(t_n^k, x)|^2 \, ds \, dx. \tag{16}
\]

Taking the sum of (16) for \( k = 1, \ldots, k_1 \), we obtain
\[
E_n(t, u_n(t)) - E_n(0, u_0)
= -\sum_{k=1}^{k_1} \int_{t_n^{k-1}}^{t_n^k} \text{Re} \langle (i\nabla - A(\alpha(s)))u_n(t_n^k), \alpha'(s)\partial_t A(\alpha(s))u_n(t_n^k) \rangle \, ds
- \frac{\lambda_1}{2} \sum_{k=1}^{k_1} \int_{\mathbb{R}^3} \int_{t_n^{k-1}}^{t_n^k} \frac{\langle (x - \beta(s)), \beta'(s) \rangle}{|x - \beta(s)|^3} |u_n(t_n^k, x)|^2 \, ds \, dx. \tag{17}
\]

On the other hand, since \( u_n \) satisfies (10), there exists \( K(M) \) independent of \( n \in \mathbb{N} \) such that
\[
\|\partial_t u_n\|_{H^{-1}_\Lambda} \leq K(M), \text{ for all } t \in [0, t_n^{k_1+1}).
\]
This yield that
\[
\|u_n - u_0\|_{L^2}^2 \leq \|u_n - u_0\|_{H^1_\Lambda} \|u_n - u_0\|_{H^{-1}_\Lambda} \leq 2MK(M)t, \text{ for all } t \in [0, t_n^{k_1+1}). \tag{18}
\]
By some elementary calculations, we have
\[
\left| \int_{\mathbb{R}^3} |u_n(t, x)|^{2\sigma + 2} \, dx - \int_{\mathbb{R}^3} |u_0(x)|^{2\sigma + 2} \, dx \right|
\leq C(M)\|u_n(t) - u_0\|_{L^2}^{\frac{1}{\sigma + 2}} \|u_0(x)\|_{L^2}^{\frac{\sigma + 2}{\sigma + 2}} \leq 2MK(M)t \frac{1}{\sigma + 2}, \text{ for all } t \in [0, t_n^{k_1+1}),
\]
\[ \left| \int_{\mathbb{R}^3} \frac{|u_n(t, x)|^2 - |u_0(x)|^2}{|x - \beta_n(t)|} \, dx \right| = \left| \int_{\mathbb{R}^3} \frac{|u_n(t, x + \beta_n(t))|^2 - |u_0(x + \beta_n(t))|^2}{|x|} \, dx \right| \]

\[ \leq C(M) \|u_n(t) - u_0\|_{L^2_{\mathbb{R}^3}}^{1-\frac{3\alpha}{2}} + C(M) \|u_n(t) - u_0\|_{L^2} \]

\[ \leq 2MK(M) t^{1-\frac{3\alpha}{2}} + 2MK(M) t^{\frac{1}{2}}, \text{ for all } t \in [0, t_{k_n}^{k_1 + 1}), \tag{20} \]

and

\[ \left| \int_{\mathbb{R}^3} \frac{1}{|x|} \cdot |u_n(t)|^2 \right| \frac{|u_n(t, x)|^2}{|x - \beta_n(t)|} \, dx - \left| \int_{\mathbb{R}^3} \frac{1}{|x|} \cdot |u_0|^2 \right| \frac{|u_0(x)|^2}{|x - \beta_n(0)|} \, dx \]

\[ \leq C \|u_n(t) - u_0\|_{L^2} \leq 2MK(M) t^{1-\frac{3\alpha}{2}}, \text{ for all } t \in [0, t_{k_n}^{k_1 + 1}). \tag{21} \]

Moreover, thanks to \( \beta \in C^1([0, T], \mathbb{R}^3) \), we deduce from the definition of \( \beta_n(t) \) that for any \( \delta > 0 \), there exists \( N \in \mathbb{N} \) such that for any \( n > N \)

\[ |\beta_n(t) - \beta(t)| < \delta, \text{ for all } t \in [0, T]. \]

Therefore, note that the diamagnetic inequality

\[ |\nabla |f(x)| \leq |(i \nabla - A) f(x)| \text{ a.e. } x \in \mathbb{R}^3 \]

and the Hardy inequality

\[ \int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|^2} \, dx \leq 4 \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, dx, \]

it follows from (20) and the mean-valued theorem that for all \( t \in [0, t_{k_n}^{k_1 + 1}) \)

\[ \left| \int_{\mathbb{R}^3} \frac{|u_n(t, x)|^2}{|x - \beta_n(t)|} \, dx - \int_{\mathbb{R}^3} \frac{|u_0(x)|^2}{|x - \beta_n(0)|} \, dx \right| \]

\[ \leq \int_{\mathbb{R}^3} \frac{|u_n(t, x)|^2 - |u_0(x)|^2}{|x - \beta_n(t)|} \, dx \left| \frac{|u_0(x)|^2}{|x - \beta_n(0)|} \, dx \right| + \int_{\mathbb{R}^3} \frac{||\beta_n(t) - \beta_n(0)|| \|u_0(x)\|^2}{|x - \beta_n(t)||x - \beta_n(0)|} \, dx \]

\[ \leq C(M) t^{1-\frac{3\alpha}{2}} + C(M) t^{\frac{1}{2}} + ||\beta_n(t) - \beta(t)|| \|u_0\|^2_{H^\alpha} + ||\beta(t) - \beta(0)|| \|u_0\|^2_{H^\alpha} \]

\[ \leq C(M) t^{1-\frac{3\alpha}{2}} + C(M) t^{\frac{1}{2}} + C(M) \delta + C(M)t. \tag{22} \]

Since \( \partial_\alpha A \) is bounded, the first term of the right hand of (17) is bounded by \( C t M^2 \). By the Hardy and diamagnetic inequalities, the second term of the right hand of (17) is bounded by \( C t M^2 \). Combining (14), (17)-(22), we obtain for any \( n > N \) and \( t \in [0, t_{k_n}^{k_1 + 1}) \)

\[ \|u_n(t)\|^2_{H^\alpha} \leq \frac{M^2}{4} + C(M) (t^{1-\frac{3\alpha}{2}} + t^{\frac{1}{2}} + \delta + t^{\frac{1}{2}} - \frac{3\alpha}{\alpha + 2} + t) \]

\[ \leq \frac{M^2}{4} + C(M) (T^{1-\frac{3\alpha}{2}} + T^{\frac{1}{2}} + \delta + T^{\frac{1}{2}} - T^{\frac{3\alpha}{\alpha + 2}} + T). \]

Taking \( T \) and \( \delta \) sufficiently small, we have for any \( n > N \) and \( t \in [0, T] \)

\[ \|u_n(t)\|_{H^\alpha} \leq M. \]

This proves that \( v_{k,n} \) is well defined for all \( k \in \{1, \ldots, n\} \) and

\[ \|u_n\|_{L^\infty((0, T), H^\alpha)} \leq M, \forall n > N. \tag{23} \]

Combining this estimate and the fact that \( u_n \) is the solution of (10), we have

\[ \|u_n\|_{L^\infty((0, T), H^{-\alpha})} \leq C(M), \forall n > N. \tag{24} \]
Step 2. Passage to the limit. By applying (23), (24), and Lemma 2.2, we deduce that there exist \( u \in L^\infty((0, T), H^1_A) \) and a subsequence, still denoted by \((u_n)_{n \in \mathbb{N}}\), such that, for a.e. \( t \in [0, T] \),

\[
  u_n(t) \rightharpoonup u(t) \text{ in } H^1_A \text{ as } n \to \infty.
\]  

(25)

From the embedding \( W^{1,\infty}((0, T), H^{-1}_A) \hookrightarrow C^{0,1}([0, T], H^{-1}_A) \) and the inequality \( \|u\|_{L^2} \leq \|u\|_{H^1_A} \|u\|_{H^{-1}_A} \), we derive for every \( u \in L^\infty((0, T), H^1_A) \cap W^{1,\infty}((0, T), H^{-1}_A) \)

\[
  \|u(t) - u(s)\|_{L^2} \leq C|t - s|^\frac{1}{2}, \text{ for all } t, s \in (0, T).
\]  

(26)

Next, we note that for all \( z_1, z_2 \in \mathbb{C} \), it holds

\[
  \|z_1|^{2\sigma}z_1 - |z_2|^{2\sigma}z_2\| \leq C(|z_1|^{2\sigma} + |z_2|^{2\sigma})|z_1 - z_2|.
\]  

(27)

It follows from (23), (26), (27), the embedding \( H^1_A \hookrightarrow L^r \), Hölder’s inequality that

\[
  \|u_n(t)^{2\sigma}u_n(t) - |u_n(s)|^{2\sigma}u_n(s)\|_{L^r} \\
  \leq C(\|u_n(t)\|^{2r}_L + \|u_n(s)\|^{2r}_L)\|u_n(t) - u_n(s)\|_{L^r} \\
  \leq C\|u_n(t) - u_n(s)\|_L^{2r} \leq C|t - s|^\frac{a}{r},
\]  

(28)

where \( r = 2\sigma + 2 \) and \( a = 1 - 3\left(\frac{1}{2} - \frac{1}{2r + 2}\right) \). This implies \( \|u_n^{2\sigma}u_n\|_{L^r} \) is a bounded sequence in \( C^{0,\frac{a}{r}}([0, T], L^r) \). Therefore, we deduce from Lemma 2.1 that there exist a subsequence, still denoted by \((u_n)_{n \in \mathbb{N}}\), and \( f_2 \in C^{0,\frac{a}{r}}([0, T], L^r) \) such that, for all \( t \in [0, T] \),

\[
  u_n(t)^{2\sigma}u_n(t) \rightharpoonup f_2(t) \text{ in } L^r \text{ as } n \to \infty.
\]  

(29)

Similarly, we can deduce that there exist a subsequence, still denoted by \((u_n)_{n \in \mathbb{N}}\), and \( f_1 \in C^{0,\frac{a}{r}}([0, T], L^2) \) such that, for all \( t \in [0, T] \),

\[
  \left(\frac{1}{|x|} \ast |u_n(t)|^2\right)u_n(t) \rightharpoonup f_1(t) \text{ in } L^2 \text{ as } n \to \infty.
\]  

(30)

Next, in order to pass to the limit for the term \( \frac{u_n}{|x - \beta_n(t)|} \), we write

\[
  \frac{u_n}{|x - \beta_n(t)|} = \left(\frac{1}{|x - \beta_n(t)|} - \frac{1}{|x - \beta(t)|}\right)u_n + \frac{u_n}{|x - \beta(t)|}
\]  

(31)

and show that

\[
  \left(\frac{1}{|x - \beta_n(t)|} - \frac{1}{|x - \beta(t)|}\right)u_n \to 0 \text{ in } \mathcal{D}((0, T) \times \mathbb{R}^3).
\]  

(32)

Indeed, applying the Hardy and diamagnetic inequalities, it follows from the dominated convergence theorem that for every \( \varphi \in C_c^\infty(\mathbb{R}^3) \) and \( \eta \in \mathcal{D}(0, T) \),

\[
  \left| \int_0^T \int_{\mathbb{R}^3} \left(\frac{1}{|x - \beta_n(t)|} - \frac{1}{|x - \beta(t)|}\right)u_n(t,x)\varphi(x)d\eta(t)dt \right| \\
  \leq \int_0^T \int_{\mathbb{R}^3} \left|\eta(t)\beta_n(t) - \beta(t)\right|\|u_n(t,x)|\varphi(x)|dxdt \\
  \leq C\|u_n\|_{L^\infty((0,T), H^1_A)}\|\nabla \varphi\|_{L^2} \int_0^T |\beta_n(t) - \beta(t)|dt \to 0 \text{ as } n \to \infty.
\]  

(33)
For the second term on the right hand in (31), it follows from Hardy’s inequality, (25) and the compact embedding $H^1_\alpha(B_R) \hookrightarrow L^2(B_R)$ that for every $\varphi \in C^\infty_c(\mathbb{R}^3)$ and a.e. $t \in [0, T]$, 
\[ \left| \int_{\mathbb{R}^3} \frac{(u_n(t,x) - u(t,x))\varphi(x)}{|x - \beta(t)|} \, dx \right| = \left| \int_{\text{Supp} \varphi} \frac{(u_n(t,x) - u(t,x))\varphi(x)}{|x - \beta(t)|} \, dx \right| \leq C\|\nabla \varphi\|_{L^2}^2 \|u_n(t) - u(t)\|_{L^2(B_R)} \to 0, \] 
where $\text{Supp} \varphi \subseteq B_R$.

In addition, for every $t \in [0, T]$, there exists $k$ such that $t \in [t_n^k, t_n^{k+1})$, we consequently deduce from the definition of $A(\alpha_n(t), x)$ and the mean-valued theorem that for every $\varphi \in C^\infty_c(\mathbb{R}^3)$
\[ \int_{\mathbb{R}^3} |A(\alpha_n(t), x) - A(\alpha(t), x)|^2 |\varphi(x)|^2 \, dx = \int_{\mathbb{R}^3} |A(\alpha(t_n^k), x) - A(\alpha(t), x)|^2 |\varphi(x)|^2 \, dx \leq \int_{t_n^k}^{t_n^{k+1}} \int_{\mathbb{R}^3} |\alpha'(t)|^2 |\partial_\alpha A(\alpha(t), x)|^2 |\varphi(x)|^2 \, dx \, dt \to 0 \text{ as } n \to \infty. \] 
This implies that for every $t \in [0, T]$, $A(\alpha_n(t)) \varphi \to A(t) \varphi$ in $L^2$. Similarly, we can show $(i\nabla - A(\alpha_n(t), x))^2 \varphi \to (i\nabla - A(\alpha(t), x))^2 \varphi$ in $L^2$ for every $t \in [0, T]$ and $\varphi \in C^\infty_c(\mathbb{R}^3)$.

On the other hand, since $u_n$ satisfies (10), it follows from that for every $\varphi \in C^\infty_c(\mathbb{R}^3)$ and $\eta \in D(0, T)$,
\[ \int_0^T \langle iu_n, \varphi \rangle_{H^{-1}_\alpha, H^1_\alpha} \eta'(t) \, dt + \int_0^T \langle (i\nabla - A(\alpha_n(t), x))^2 u_n, \varphi \rangle_{H^{-1}_\alpha, H^1_\alpha} \eta(t) \, dt + \int_0^T \left( \frac{\lambda_1}{|x - \beta_n(t)|} u_n + \lambda_2 \left( \frac{1}{|x|} \ast |u_n|^2 \right) u_n + \lambda_3 |u_n|^{2\sigma} u_n, \varphi \right)_{H^{-1}_\alpha, H^1_\alpha} \eta(t) \, dt = 0. \] 
Applying (25), (29)-(36), and the dominated convergence theorem, we deduce easily that
\[ \int_0^T \langle iu, \varphi \rangle_{H^{-1}_\alpha, H^1_\alpha} \eta'(t) \, dt + \int_0^T \langle (i\nabla - A(\alpha(t), x))^2 u, \varphi \rangle_{H^{-1}_\alpha, H^1_\alpha} \eta(t) \, dt + \int_0^T \left( \frac{\lambda_1}{|x - \beta(t)|} u + \lambda_2 f_1 + \lambda_3 f_3, \varphi \right)_{H^{-1}_\alpha, H^1_\alpha} \eta(t) \, dt = 0. \] 
This implies that $u$ satisfies
\[ i\partial_t u = (i\nabla - A(\alpha(t), x))^2 u + \frac{\lambda_1}{|x - \beta(t)|} u + \lambda_2 f_1 + \lambda_3 f_2 \text{ for a.e. } t \in [0, T]. \] 
In order to show that $f_1(t) = (\frac{1}{|x|} \ast |u(t)|^2) u(t)$ and $f_2(t) = |u(t)|^{2\sigma} u(t)$, we need the following lemma.

**Lemma 3.3.** For all $t \in [0, T]$, $\text{Im}((\lambda_2 f_1(t) + \lambda_3 f_2(t)) \overline{u(t)}) = 0$ a.e. on $\mathbb{R}^3$.

**Proof.** It suffices to show that for every bounded subset $B$ of $\mathbb{R}^3$, 
\[ \langle (\lambda_2 f_1(t) + \lambda_3 f_2(t)) |_{B}, \overline{iu(t)} |_{B} \rangle_{L^\infty(B), L^\infty(B)} = 0. \]
To this end, we omit the time dependence and write
\[
\langle f_1, iu \rangle_{L^r(B), L^r(B)} = \left\langle f_1 - \left(\frac{1}{|\cdot|} \cdot |u_n|^2\right)u_n, iu \right\rangle_{L^r(B), L^r(B)} + \left\langle \left(\frac{1}{|\cdot|} \cdot |u_n|^2\right)u_n, iu_n \right\rangle_{L^r(B), L^r(B)} + \left\langle \left(\frac{1}{|\cdot|} \cdot |u_n|^2\right)u_n, i(u - u_n) \right\rangle_{L^r(B), L^r(B)} \to a + b + c \text{ as } n \to \infty.
\] (40)

Firstly, (30) implies \(a = 0\). Next, it is obvious that \(b = 0\). Finally, (25) implies that \(u_n \to u\) in \(L^r(B)\). Therefore, \(c = 0\), and \(\langle f_1, iu \rangle_{L^r(B), L^r(B)} = 0\). Similarly, we can prove that \(\langle f_2, iu \rangle_{L^r(B), L^r(B)} = 0\).

Taking the \(H_A^{-1} - H_A^1\) duality product of (38) with \(iu\), we have
\[
\frac{d}{dt} \|u(t)\|_{L^2} = 0 \text{ for all } t \in [0, T].
\]
and so
\[
\|u(t)\|_{L^2} = \|u_0\|_{L^2}.
\] (41)

We deduce from (25) and Lemma 2.2(ii) that
\[
u_n \to u \text{ in } C([0, T], L^2).
\] (42)

This, together with (28), implies
\[
\|u_n(t)|^{2\sigma}u_n(t) - |u(t)|^{2\sigma}u(t)\|_{L^{r'}} \leq C\|u_n(t) - u(t)\|_{L^2}^2 \to 0 \text{ as } n \to \infty.
\]

Similarly, we have
\[
\left\| \left(\frac{1}{|\cdot|} \cdot |u_n|^2\right)u_n - \left(\frac{1}{|\cdot|} \cdot |u|^2\right)u \right\|_{L^2} \leq C\|u_n - u\|_{L^2} \to 0 \text{ as } n \to \infty.
\]

Therefore, we have
\[
f_1(t) = \left(\frac{1}{|\cdot|} \cdot |u(t)|^2\right)u(t) \text{ and } f_2(t) = |u(t)|^{2\sigma}u(t).
\]

In summary, \(u \in L^\infty((0, T), H_A^1) \cap W^{1, \infty}((0, T), H_A^{-1})\) and satisfies (1).

**Step 3.** Conclusion. By using the classical argument based on Strichartz’s estimates, we can obtain the following uniqueness of the weak solution \(u\) of (1).

**Lemma 3.4.** Let \(A\) satisfy Assumption 1, \(\alpha \in H^1(0, \infty), \beta \in C^1([0, \infty), \mathbb{R}^3)\) and \(0 < \sigma < 2\). If \(u_0 \in H_A^1\) and \(u_1, u_2\) are two weak solutions of (1) on \([0, T]\), then \(u_1 = u_2\).

**Proof.** It follows from Duhamel’s formulation of (1) that
\[
(u_1 - u_2)(t) = i \int_0^t U(t, s) \left(\lambda_1 \frac{u_2 - u_1}{|x - \beta|} + \lambda_2 \left(\frac{1}{|\cdot|} \cdot |u_2|^2\right)u_2 - \left(\frac{1}{|\cdot|} \cdot |u_1|^2\right)u_1\right) \right)sds + i \lambda_3 \int_0^t U(t, s) \left((|u_2|^{2\sigma}u_2 - |u_1|^{2\sigma}u_1)\right)(s)ds.
\] (43)
In the following, we set $\rho_1 = 2\sigma + 2$, taking $\gamma_1$ such that $(\gamma_1, \rho_1)$ is an admissible pair. Applying Lemma 2.3 to (43), we deduce from Hölder’s inequality that for every $t \in (0, T]$

$$
\|u_1 - u_2\|_{L^2_t L^2_x(0, t)} + C \|W_1 \ast |u_2|^{2} u_2 - W_1 \ast |u_1|^{2} u_1\|_{L^{\sigma/3}_t L^{2/3}_x(0, t)}
+ C \|V_2(u_2 - u_1)\|_{L^2_t L^2_x(0, t)} + C \|W_2 \ast |u_2|^{2} u_2 - W_2 \ast |u_1|^{2} u_1\|_{L^1_t L^2_x(0, t)}
+ C \|u_2 \ast |u_2|^{2} u_2 - |u_1|^{2} u_1\|_{L^1_t L^2_x(0, t)}
\leq C \|V_1\|_{L^{3/2}_t L^{\sigma/2}_x(0, t)} \|u_2 - u_1\|_{L^2_t L^2_x(0, t)} + C \|V_2\|_{L^{\infty}_t L^{1}_x} \|u_2 - u_1\|_{L^1_t L^2_x(0, t)}
+ C \|u_2 - u_1\|_{L^{\sigma/3}_t L^{2/3}_x(0, t)} + C \|u_2 - u_1\|_{L^1_t L^2_x(0, t)}
+ C \|u_2 - u_1\|_{L^{\sigma/3}_t L^{2/3}_x(0, t)} + C \|u_2 - u_1\|_{L^1_t L^2_x(0, t)},
(44)
$$

where $V_1 = \frac{1}{|x - \beta(t)|} \chi_{B_\beta(t)}$, $V_2 = \frac{1}{|x - \beta(t)|} (1 - \chi_{B_\beta(t)})$, $W_1 = \frac{1}{|x|} \chi_{B_0} \in L^{3/2}$ and $W_2 = \frac{1}{|x|} (1 - \chi_{B_0})$. Therefore, we deduce from Lemma 2.4 that $u_1 = u_2$ on $[0, T]$. \hfill $\square$

On the other hand, by (17), (25), the weak lower semicontinuity of the magnetic Sobolev norm $\| (i\nabla - A(\alpha(s))) \|_{L^2}$, it follows that

$$
E(u(t)) \leq E_n(t, u_n(t))
$$

$$
\rightarrow E(u_0) - \int_0^t Re((i\nabla - A(\alpha(s)))u(s) , \alpha'(s) \partial_0 A(\alpha(s))u(s)) \, ds
$$

$$
- \frac{\lambda_1}{2} \int_0^t \int_{\mathbb{R}^3} \frac{(x - \beta(s)) \cdot \beta'(s)}{|x - \beta(s)|^3} |u_n(s, x)|^2 \, dx \, ds.
(45)
$$

Finally, $t > 0$ being fixed, we deduce from the uniqueness that $v_n(s) = u_n(t - s)$ is the solution of the following equation

$$
i\partial_t v_n = (i\nabla - A(\alpha_n(t - s), x))^2 v_n + \frac{\lambda_1}{|x - \beta_n(t - s)|} v_n + \lambda_2 |v_n|^2 v_n + \lambda_3 |v_n|^{2\sigma} v_n,
$$

with initial data $v_n(0) = u_n(t)$. Then we perform the same computations as above to get the converse inequality to (45) and hence (4) is proved. This completes the proof. \hfill $\square$

Next, we consider the global existence of (1).

**Theorem 3.5.** Let $A$ satisfy Assumption 1, $\alpha \in H^1(0, \infty)$, $\beta \in C^1([0, \infty), \mathbb{R}^3)$, $\lambda_1, \lambda_2 \in \mathbb{R}$. Assume that $0 < \sigma < 2$ if $\lambda_3 > 0$, or $0 < \sigma < \frac{2}{3}$ if $\lambda_3 < 0$. Then the solution $u$ of (1) exists globally.

**Proof.** In order to show this theorem, it suffices to show

$$
\|u(t)\|_{H^1_\lambda} \leq C(T, \|u_0\|_{H^1_\lambda}) \text{ for every } t \in [0, T],
(46)
$$

for every $T > 0$.\hfill $\square$
Using the diamagnetic and Hardy’s inequalities, we have
\[
\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x - \beta(t)|} dx \leq C\|u\|_{L^2} \left( \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} dx \right)^{1/2}
\leq C\|u\|_{L^2} \|\nabla u\|_{L^2}
\leq C\|u\|_{L^2} \|(i\nabla - A)u\|_{L^2},
\tag{47}
\]
and
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|\cdot|} * |u|^2 \right)(x) |u(x)|^2 dx \leq C\|u\|_{L^2}^2 \sup_{x \in \mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x - y|} dy \right)
\leq C\|u\|_{L^2}^3 \|(i\nabla - A)u\|_{L^2}.
\tag{48}
\]
When \(\lambda_3 > 0\), using Young’s inequality with \(\varepsilon\), from (4), (47), (48), we have
\[
\|(i\nabla - A(\alpha(t)))u(t)\|_{L^2}^2
\leq C + \frac{|\lambda_1|}{2} \int_{\mathbb{R}^3} \frac{|u(t)|^2}{|x - \beta(t)|} dx + \frac{|\lambda_2|}{4} \int_{\mathbb{R}^3} \frac{1}{|\cdot|} * |u(t)|^2 dx
+ C \int_0^t |\alpha'(s)|^2 ds + C \int_0^t \|(i\nabla - A(\alpha(s)))u(s)\|_{L^2}^2 ds
+ C \int_0^t |\beta'(s)| \|(i\nabla - A(\alpha(s)))u(s)\|_{L^2}^2 ds
\leq C + \varepsilon \|(i\nabla - A(\alpha(t)))u(t)\|_{L^2}^2 + C \int_0^t \|(i\nabla - A(\alpha(s)))u(s)\|_{L^2}^2 ds.
\tag{49}
\]
Hence, taking \(\varepsilon < 1\), applying Gronwall’s inequality, (46) follows.

When \(\lambda_3 < 0\) and \(0 < \sigma < \frac{2}{3}\), by using the same argument as above and the following Gagliardo-Nirenberg’s inequality
\[
\|u\|_{L^{2\sigma+2}} \leq C\|u\|_{H_\lambda^1}^{3\sigma} \|u\|_{L^2}^{2\sigma - 3\sigma},
\tag{46}
\]
(46) follows. This completes the proof of this theorem. \(\square\)

4. \textbf{The} \(H_\lambda^2\) \textbf{regularity.} In this section, we investigate the \(H_\lambda^2\) regularity for (1). The \(H^s\) regularity for nonlinear Schrödinger equations is well-known if the nonlinearity is sufficiently smooth, see [3]. The smooth condition on the nonlinearity can be improved (removed, if \(s \leq 2\)) by estimating time derivatives of the equation instead of space derivatives, see [3, 14]. There are two possible techniques for studying the \(H_\lambda^2\) regularity of equation (1). One is the compactness method, see Theorem 7.4.1 in [5]. However, a problem arises when we try to mimic proof of Theorem 7.4.1 in [5]. Indeed, the first step is to obtain a generalization of Theorem 7.2.1 in [5] in the time-dependent framework. In the time-dependent framework, the existence of smooth solution is not easy to prove. Indeed, the key point in [5] is that for any \(g \in C([0, T], X)\) being Lipschitz continuous with respect to time, the function \(v(t) = \int_0^t U_1(t, s) g(s) ds\) is also Lipschitz continuous with respect to time, where \(U_1(t) := e^{it\Delta}\) is the free Schrödinger propagator. This result is easily proved in autonomous case by using the identity \(U_1(t + h, s) = U_1(t, s - h)\). This fails to be true in the non-autonomous case. Moreover, we cannot mimic the argument in [15] by approximating the magnetic potential \(A(t, x)\) by potentials which are piecewise constant with respect to time. Indeed, since the nonlinearities we consider are not sufficiently smooth, we obtain some estimates by differentiating the approximating
equations with respect to time variable. However, this magnetic potential is not differentiable with respect to time variable.

The other is Kato’s method, based on a fixed point argument and Strichartz’s estimates. When we apply this method to show the $H^2_A$ regularity of equation (1), a difficulty occurs if $A(\alpha(t), x)$ is unbounded with respect to $x$. Indeed, we infer from (6) that there is no chance to control $\nabla(\lambda^2 u)$ by $(i\nabla - A(\alpha(t)))(\lambda^2 u)$. Therefore, we consider the $H^2_A$ regularity under the assumption that $A(\alpha(t), x)$ is bounded. How to obtain the $H^2_A$ regularity for (1) when $A$ is a time-dependent unbounded potential is an interesting open problem.

**Theorem 4.1.** Assume that $0 < \sigma < 2$ if $\lambda_3 > 0$, or $0 < \sigma < \frac{2}{3}$ if $\lambda_3 < 0$. The magnetic field $A(\alpha(t), x)$ is a bounded function and satisfies Assumption 1. Let $u_0 \in H^2_A$, $\alpha \in H^1(0,T)$ and $\beta \in C^1([0,T],\mathbb{R}^3)$. Then the solution of (1) satisfies $u \in C([0,T], H^2_A) \cap C^1([0,T], U^2)$ Moreover, $u \in W^{1,\sigma}((0,T), L^2)$ for every admissible pair $(q,r)$.

**Proof.** Firstly, we prove the local existence of $H^2_A$ solutions by a fixed point argument.

Given $M, l > 0$ to be chosen later, we set $I = [0,l]$ and we consider

$$E = \{ u, (i\nabla - A(\alpha(t)))u, \partial_t u \in S(I); u(0) = u_0 \}
and ||u||_{S(I)} + \|(i\nabla - A(\alpha(t)))u\|_{S(I)} + \|\partial_t u\|_{S(I)} \leq M \}.

It follows that $(E, d)$ is a complete metric space, where the distance $d$ is defined by

$$d(u, v) = ||u - v||_{S(I)}$$

Duhamel’s formulation for (1) reads

$$u(t) = U(t, 0)u_0 - i \int_0^t U(t, s) \left( \frac{\lambda_1}{|x - \beta|} u + \lambda_2 \left( \frac{1}{|\cdot|} * |u|^2 \right) u + \lambda_3 |u|^{2\sigma} u \right)(s) ds. \quad (50)$$

Denote the right hand side by $\Phi(u)(t)$. By Lemma 2.3 and Hölder’s inequality, we have

$$\|\Phi(u)\|_{S(0,t)} \leq C\|u_0\|_{L^2} + C I^{\frac{2\sigma}{d}} \|V_1\|_{L_{t}^{\infty}L_{x}^{d+\sigma}} \|u\|_{S(0,t)} + C I\|V_2\|_{L_{t}^{\infty}L_{x}^{2}} \|u\|_{S(0,t)}
+ C I^{\frac{2\sigma}{d}} \|u\|_{S(0,t)}^3 + C I^{\frac{2\sigma}{d}} \|u\|_{L_{t,\infty}^{\infty}(0,t), H^2_A} \|u\|_{S(0,t)}.
\quad (51)$$

Next, in order to estimate $(i\nabla - A(\alpha(t)))\Phi(u)$, we deduce from (1) that

$$(i\nabla - A(\alpha(t)))\Phi(u)(t)
= U(t, 0)(i\nabla - A(\alpha(0)))u_0 + i \int_0^t U(t, s)(\alpha' \partial_\alpha A(\alpha)u)(s) ds
-i \int_0^t U(t, s) \left( \frac{\lambda_1}{|x - \beta|} (i\nabla - A(\alpha))u + i\lambda_1 u \nabla \left( \frac{1}{|x - \beta|} \right) \right)(s) ds
-i \int_0^t U(t, s) \left( \lambda_2 \left( \frac{1}{|\cdot|} * |u|^2 \right) (i\nabla - A(\alpha))u + i\lambda_2 u \nabla \left( \frac{1}{|\cdot|} * |u|^2 \right) \right)(s) ds
-i \int_0^t U(t, s) (\lambda_3 |u|^{2\sigma} (i\nabla - A(\alpha))u + i\lambda_3 u \nabla |u|^{2\sigma})(s) ds. \quad (52)$$
By the similar argument as that of (51), the embedding $W^{1,\frac{2}{\alpha\beta}} \hookrightarrow L^{\frac{2\alpha}{\alpha\beta}}$, we obtain

\[
\|(i\nabla - A(\alpha))\Phi(u)\|_{S(0,t)} \\
\leq C\|(i\nabla - A(\alpha(0)))u_0\|_{L^2} + C\|\frac{1}{\alpha\beta}V_1\|_{L^{\infty}_x L^{\frac{2}{\alpha\beta}}_t} \|(i\nabla - A(\alpha))u\|_{S(0,t)} \\
+ C\|V_2\|_{L^{\infty}_x L^{2}_t} \|(i\nabla - A(\alpha))u\|_{S(0,t)} + C\|\frac{1}{\alpha\beta}\|_{L^{\infty}_x L^{\frac{2}{\alpha\beta}}_t} \|\nabla u\|_{S(0,t)}
\]

Similarly, we have

\[
\frac{d}{dt}\Phi(u)(t) = U(t,0)\partial_t u(0) - i\int_0^t U(t,s)(-2i\alpha^*\partial_s A(\alpha)\nabla u - i\alpha^*\text{div}\partial_s A(\alpha)u)\|u\|_{S(0,t)}ds
\]

\[
- i\int_0^t U(t,s)\left(2\alpha^* A(\alpha)\partial_s A(\alpha)u + \frac{\lambda_1}{x - \beta} \partial_s u + \partial_t \frac{\lambda_1}{x - \beta} u\right)(s)ds
\]

Notice that the following equality

\[
\frac{\partial}{\partial t} \frac{1}{|x - \beta(t)|} = \frac{(x - \beta(t)) \cdot \beta'(t)}{|x - \beta(t)|^3} = -\beta'(t) \cdot \nabla \frac{1}{|x - \beta(t)|}
\]

Therefore, we can estimate as above and obtain

\[
\|\frac{d}{dt}\Phi(u)\|_{S(0,t)} \\
\leq C\|\partial_t u(0)\|_{L^2} + C\|\nabla u\|_{S(0,t)} + C\|u\|_{S(0,t)}
\]

\[
+ C\|\frac{1}{\alpha\beta}V_1\|_{L^{\infty}_x L^{\frac{2}{\alpha\beta}}_t} \|\partial_t u\|_{S(0,t)} + C\|V_2\|_{L^{\infty}_x L^{2}_t} \|\partial_t u\|_{S(0,t)}
\]

\[
+ C\|\nabla V_1\|_{L^{\infty}_x L^{\frac{2}{\alpha\beta}}_t} \|\nabla u\|_{S(0,t)} + C\|\nabla V_2\|_{L^{\infty}_x L^{2}_t} \|u\|_{S(0,t)}
\]

\[
+ C\|u\|_{S(0,t)}^2 + C\|\partial_t u\|_{S(0,t)} + C\|\partial_t u\|_{S(0,t)}
\]

Since $A(\alpha(t),x)$ is a bounded function and satisfies Assumption 1, then $\|u\|_{H^1_\alpha(\cdot)} = \|u\|_{H^1}$. It is thus easy to see that $\Phi$ maps $E$ to itself and is a contraction in the $S(0,l)$ norm, provided $l$ is chosen sufficiently small. The contraction mapping theorem then implies the existence of a unique $H^2_\alpha$ solution to (1) on $[0,l]$.

By a similar argument as Section 5.3 in [3], we have $u \in C([0,T],H^2_\alpha) \cap C^1([0,T], L^2)$. This completes the proof. 

**Acknowledgments.** The authors would like to express sincere thanks to the anonymous referee for his/her carefully reading the manuscript and valuable comments and suggestions.
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Received for publication August 2017.

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