Improving the Single Scalar Consistency Relation

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ABSTRACT

We propose a test of single-scalar inflation based on using the well-measured scalar power spectrum to reconstruct the tensor power spectrum, up to a single integration constant. Our test is a sort of integrated version of the single-scalar consistency relation. This sort of test can be used effectively, even when the tensor power spectrum is measured too poorly to resolve the tensor spectral index. We give an example using simulated data based on a hypothetical detection with tensor-to-scalar ratio $r = 0.01$. Our test can also be employed for correlating scalar and tensor features in the far future when the data is good.

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1 Introduction

The theory of primordial inflation [1, 2, 3, 4, 5, 6, 7, 8] has had a profound effect on cosmology and fundamental theory. Particularly striking is the prediction that primordial tensor [9] and scalar [10] perturbations derive from quantum gravitational fluctuations which fossilized near the end of inflation. This not only affords us access to quantum gravity at an intoxicating energy scale [11, 12, 13], it also provides information about the mechanism that powered inflation. This information can be accessed by comparing observations of the two power spectra, $\Delta^2_R(k)$ and $\Delta^2_h(k)$, to predictions from the many models [14, 15, 16]. For example, the simplest models of inflation are driven by the potential of a single, minimally coupled scalar. These models all obey the single-scalar consistency relation [17, 18, 19],

$$r \approx -8n_t,$$

where $r$ is the tensor-to-scalar ratio and $n_t$ is the tensor spectral index,

$$r(k) \equiv \frac{\Delta^2_h(k)}{\Delta^2_R(k)} , \quad n_t(k) \equiv \frac{\partial \ln(\Delta^2_h(k))}{\partial \ln(k)} .$$

A statistically significant violation of (1) would falsify the entire class of single-scalar models, as well as all models which are related to them by conformal transformation, such as $f(R)$ inflation [20].

Although the single-scalar consistency relation was a brilliant theoretical insight, the progress of observation has rendered it somewhat inconvenient. The scalar power spectrum was first resolved in 1992 [21], and is now quite well measured [22, 23, 24, 25]. The tensor power spectrum has not yet been resolved [26, 27], but polarization measurements are now providing the strongest limits on it [28]. It is not known if the current generation of polarization experiments [29, 30, 31, 32, 33] can resolve the tensor power spectrum at all, and it is very unlikely that they will measure it well enough to constrain the tensor spectral index with much accuracy.

In view of the observational situation, it makes sense to develop a test of single-scalar inflation that is based primarily on the abundant data for $\Delta^2_R(k)$, and does not require taking derivatives of the sparse data for $\Delta^2_h(k)$ likely to result from the first positive detections. There is no reason not to do this because the close relation between the tensor and scalar mode functions of single-scalar inflation implies that either power spectrum determines the
other, up to some integration constants. That is the purpose of this paper. In the next section we fix notation, recall the relation between the two power spectra, and infer the tensor power spectrum from the scalar one. Section 3 gives a comparison between the single scalar consistency relation and the scatter test we propose, using simulated data based on a hypothetical detection of $r = 0.01$ with the same number of data points and the same fractional error as was in fact reported by the recent spurious BICEP2 detection [34]. The final section mentions applications.

2 Constructing $\Delta^2_R(k)$ from $\Delta^2_R(k)$

We work in spatially flat, co-moving coordinates with scale factor $a(t)$, Hubble parameter $H(t)$ and first slow roll parameter $\epsilon(t)$,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x} \implies H(t) \equiv \frac{\dot{a}}{a}, \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2}. \quad (3)$$

We assume $a(t)$ is known, with the scalar background and potential determined to enforce the background Einstein equations [35, 36, 37, 38, 39],

$$V(\varphi) = \frac{[3 - \epsilon(t)]H^2(t)}{8\pi G} \bigg|_{t=t(\varphi)} \quad (4)$$

We fix the gauge so that the full scalar agrees with its background value and the graviton field $h_{ij}$ is transverse, with $g_{00}$ and $g_{0i}$ regarded as constraints. The two dynamical fields are $h_{ij}$ and $\zeta$, which reside in the 3-metric $g_{ij} = a^2 e^{2\kappa[h]_{ij}}$. At quadratic order their Lagrangian is [40],

$$\mathcal{L}_2 = \frac{a^3}{64\pi G} \left[ h_{ij}\dot{h}_{ij} - \frac{h_{ij,k}h_{ij,k}}{a^2} \right] + \frac{ea^3}{8\pi G} \left[ \dot{\zeta}^2 - \frac{\zeta_k\zeta_k}{a^2} \right]. \quad (6)$$

The spatial plane wave mode functions of the graviton are $u(t, k)$, with exactly the same polarization tensors as in flat space. From (6) we see that the evolution equation, Wronskian and asymptotically early form of the tensor mode functions $u(t, k)$ are,

$$\ddot{u} + 3H\dot{u} + \frac{k^2}{a^2}u = 0, \quad u\dot{u}^* - \dot{u}\ddot{u}^* = \frac{i}{a^3}, \quad u(t, k) \longrightarrow \frac{\exp[-ik\int_{t_i}^{t} \frac{dt'}{a(t')}]}{\sqrt{2ka^2(t)}}. \quad (7)$$
The scalar perturbation $\zeta$ has spatial plane wave mode functions $v(t, k)$. From (6) we see that their evolution equation, Wronskian and asymptotically early form are,

$$\ddot{v} + \left(3H + \frac{\dot{\epsilon}}{\epsilon}\right)\dot{v} + \frac{k^2}{a^2}v = 0,$$

$$v\dot{v} - \dot{v}v^* = \frac{i}{\epsilon a^3}, \quad v(t, k) \rightarrow \frac{\exp[-ik\int_{t_0}^{t} \frac{dt'}{a(t')}}}{\sqrt{2k\epsilon(t)a^2(t)}}.$$  \hspace{1cm} (8)

The two power spectra are determined (at tree order) by evolving their respective mode functions from their early forms through the time $t_k$ of first horizon crossing ($k \equiv H(t_k)a(t_k)$), after which they approach constants,

$$\Delta^2_R(k) = \frac{k^3}{2\pi^2} \times 4\pi G \times |v(t, k)|^2 \bigg|_{t \gg t_k} \approx \frac{GH^2(t_k)}{\pi \epsilon(t_k)},$$  \hspace{1cm} (9)

$$\Delta^2_h(k) = \frac{k^3}{2\pi^2} \times 32\pi G \times 2 \times |u(t, k)|^2 \bigg|_{t \gg t_k} \approx \frac{16GH^2(t_k)}{\pi}.$$  \hspace{1cm} (10)

The relations (7) which define $u(t, k)$ are carried into the relations (8) which define $v(t, k)$ by making simultaneous changes in the scale factor and the co-moving time \[31\], \[32\],

$$a(t) \rightarrow \sqrt{\epsilon(t)} a(t), \quad \frac{\partial}{\partial t} \rightarrow \frac{1}{\sqrt{\epsilon(t)}} \frac{\partial}{\partial t}.$$  \hspace{1cm} (11)

To understand what this means for the power spectra we must consider them as nonlocal functionals of the expansion history $a(t)$, which will involve integrals and derivatives with respect to time. We denote this functional dependence with square brackets, so relation (11) implies,

$$\Delta^2_R[a, dt](k) = \frac{1}{16} \Delta^2_h[\sqrt{\epsilon}a, \sqrt{\epsilon}dt](k).$$  \hspace{1cm} (12)

Relation (12) is easy to check at leading slow roll order by comparing the slow roll approximation for $\Delta^2_R(k)$ on the right hand side of (9) with the effect of making transformation (11) on the Hubble parameter in the right hand side of expression (10),

$$H(t) \equiv \frac{\partial}{\partial t} \ln[a(t)] \rightarrow \frac{1}{\sqrt{\epsilon}} \frac{\partial}{\partial t} \ln[\sqrt{\epsilon}a] = \frac{H + \dot{\epsilon}}{\sqrt{\epsilon} \epsilon}.$$  \hspace{1cm} (13)

However, we stress that relation (12) is exact, not just valid at leading slow roll order, provided one employs the exact expressions for $\Delta^2_h(k)$ and $\Delta^2_R(k)$. 3
Figure 1: The left hand figure shows the first slow roll parameter for a model which was proposed [46, 47] to explain the observed features in the scalar power spectrum at $\ell \approx 22$ and $\ell \approx 40$ which are visible in the data reported from both WMAP [48, 49] and PLANCK [50, 51]. The right hand figure shows the resulting scalar power spectrum (in blue), with the result of our analytic approximation (14) (in yellow). The slow roll approximation (9) does not give a very accurate fit even to the main feature in the range $54.5 < N < 53$ e-foldings before the end of inflation, and it completely misses the secondary oscillations visible in the range $53.5 < N < 51.5$. The nonlocal contributions (17) are essential for correctly reproducing these features.

We should also point out that very accurate functional expressions are now available for the power spectra of single scalar inflation, valid to all orders in the slow roll parameter $\epsilon(t_k)$, and even including nonlocal effects from times before and after first horizon crossing [43, 44]. These expressions take the form [45],

$$\Delta^2_R(k) \approx \frac{G H^2(t_k)}{\pi \epsilon(t_k)} \times C(\epsilon(t_k)) \times \exp[\sigma[\epsilon](k)] , \quad (14)$$

$$\Delta^2_h(k) \approx \frac{16 G H^2(t_k)}{\pi} \times C(\epsilon(t_k)) \times \exp[\tau[\epsilon](k)] , \quad (15)$$

where the local slow roll correction factor is,

$$C(\epsilon) \equiv \frac{1}{\pi} \Gamma^2 \left( \frac{1}{2} + \frac{1}{1-\epsilon} \right) \left[ 2(1-\epsilon) \right]^{\frac{2}{1-\epsilon}} \approx 1 - \epsilon . \quad (16)$$

For the nonlocal corrections $\sigma[\epsilon](k)$ and $\tau[\epsilon](k)$ it is best to abuse the notation by writing the first slow parameter $\epsilon(n) \equiv \epsilon(t(n))$ as a function of $n \equiv$
\[ \ln[a(t)/a_i], \text{ the number of e-foldings since the start of inflation,} \]

\[
\sigma[\epsilon](k) = \int_0^{n_k} dn \left[ \frac{1}{2} \left( \frac{\partial \ln[\epsilon(n)]}{\partial n} \right)^2 + 3 \frac{\partial \ln[\epsilon(n)]}{\partial n} \right] G(e^{\Delta n}) - \frac{\partial_n \ln[\epsilon(n)]}{G(1)} + \int_{n_k}^{\infty} dn \left[ \frac{2G(e^{\Delta n})}{1 + e^{2\Delta n}} - \frac{\partial_n \ln[\epsilon(n)]}{G(1)} \right], \quad (17) \]

\[
\tau[\epsilon](k) = \int_0^{n_k} dn \left[ \mathcal{E}_1(e^{\Delta n}) \epsilon''(n) + \mathcal{E}_2(e^{\Delta n}) \left( \epsilon'(n) \right)^2 + \mathcal{E}_3(e^{\Delta n}) \epsilon'(n) \right] G(e^{\Delta n}) - \epsilon'(n_k) \mathcal{E}_1(1) G(1) - \int_{n_k}^{\infty} \frac{dn}{\Delta n} \left\{ \Delta(n) + \left( \frac{4 + 2e^{2\Delta n}}{1 + e^{2\Delta n}} \right) \int_{n_k}^{m} \Delta(n) \right\} \frac{2G(e^{\Delta n})}{1 + e^{2\Delta n}}. \quad (18) \]

Here \( \Delta_n \equiv n - n_k, \Delta(n) \equiv \epsilon(n) - \epsilon_k \), and the functions of \( x \equiv e^{\Delta n} \) are,

\[
G(x) = \frac{1}{2} \left( x + x^3 \right) \sin \left[ \frac{2}{x} - 2 \arctan \left( \frac{1}{x} \right) \right], \quad (19) \]

\[
\mathcal{E}_1(x) \approx \frac{\frac{1}{2}x^2 - 1.8x^4 + 1.5x^6 - 0.63x^8}{1 + x^2}, \quad (20) \]

\[
\mathcal{E}_2(x) \approx \frac{2.8x^4 - 7x^6 + 3x^8 + 1.8x^{10} - 2.3x^{12} + 0.95x^{14} - 0.20x^{16}}{(1 + x^2)^2}, \quad (21) \]

\[
\mathcal{E}_3(x) \approx \frac{\frac{9}{2}x^2 - 11.9x^4 + 7.1x^6 - 1.3x^8 - 1.9x^{10}}{(1 + x^2)^2}. \quad (22) \]

The 95\% confidence bound on the tensor-to-scalar ratio of \( r < 0.12 \) \cite{27, 28} implies \( \epsilon < 0.0075 \), so \( \tau[\epsilon](k) \) is about a hundred times smaller than \( \sigma[\epsilon](k) \). Models with smooth potentials typically have \( \epsilon' \sim \epsilon^2 \) and \( \epsilon'' \sim \epsilon^3 \), so the leading contributions in \( \sigma[\epsilon](k) \) come from the 3rd and 5th terms of expression (17). In particular the 5th (final) term is needed to correct for a systematic under-prediction of the local slow roll approximation \cite{45}. For models with features the leading contributions to \( \sigma[\epsilon](k) \) come from the 1st, 3rd and 4th terms of expression (17) \cite{45}. These corrections can be very important for realistic models such as the one depicted in Figure 1.

To keep the analysis simple, we illustrate the procedure for predicting \( \Delta_n^2(k) \) from \( \Delta_S^2(k) \) using only the leading slow roll terms in expressions (14-15), without either of the nonlocal corrections or even the slow roll factor \( C(\epsilon) \). The conversion from wave number to time is,

\[
k = H(t_k)a(t_k) \implies \frac{dk}{k} = (1-\epsilon)Hdt \approx Hdt. \quad (23) \]
The leading slow roll approximation \( (14) \) for the scalar power spectrum can be recognized as a differential equation for the Hubble parameter,

\[
\Delta^2_R(k) \simeq \frac{G H^2(t)}{\pi \epsilon(t)} = -\frac{G H^4(t)}{\pi H(t)} .
\] (24)

We can integrate this equation from some arbitrary time \( t_* \) to \( t_k \),

\[
d\left(\frac{1}{H^2}\right) \simeq \frac{2G d \ln(k)}{\pi \Delta^2_R(k)} \implies 
\frac{1}{H^2(t_k)} - \frac{1}{H^2(t_*)} \simeq \frac{2G}{\pi} \int_{0}^{\ln(k/k_*)} d \ln(k') \Delta^2_R(k') .
\] (25)

Substituting the reconstructed Hubble parameter \( (25) \) into the leading slow roll approximation \( (10) \) for the tensor power spectrum gives,

\[
\Delta^2_h(k) \simeq \frac{16GH^2(t_k)}{\pi} \approx \Delta^2_h(k_*) \left[ 1 + \frac{r(k_*)}{8} \int_{0}^{\ln(k/k_*)} \frac{\Delta^2_R(k_*)}{\Delta^2_R(k')} d \ln(k') \right]^{-1} .
\] (26)

Equation \( (26) \) is in some sense an integrated form of the single-scalar consistency relation \( (1) \) which can be applied more reliably. Both relations are valid to leading slow roll order, but whereas \( (1) \) compares a single value of the high quality data in \( \Delta^2_R(k) \) with a derivative of the poor data on \( \Delta^2_h(k) \), our relation \( (26) \) combines a single measurement of the tensor power spectrum at \( k = k_* \) with the high quality scalar data to predict what \( \Delta^2_h(k) \) should be for other wave numbers. This seems to be a better way of exploiting the sparse data on \( \Delta^2_h(k) \) which is likely to persist for some years after a first positive detection.

## 3 Comparison Using Simulated Data

It is illuminating to compare the single scalar consistency relation with the method we propose using simulated data. Let us suppose that the actual tensor power spectrum corresponds to single scalar inflation with \( r = \frac{1}{100} \), and which implies \( n_t = -\frac{1}{800} \). We further suppose the simplest possible \( k \) dependence,

\[
\Delta^2_h(k) = r A_s \left( \frac{k}{k_0} \right)^{n_t} \implies \ln[\Delta^2_h(k)] = \ln[r A_s] + n_t \ln[k/k_0] ,
\] (27)

where the scalar amplitude (at the tensor pivot \( k_0 \)) is \( A_s = 2.5 \times 10^{-9} \). Let us assume that the first positive detection of this tensor power spectrum consists
of results for five binned wave numbers, the same as was in fact reported for the spurious BICEP2 detection [34]. To simplify matters we assume a linear relation for logarithms of the observed wave numbers, \( \ln\left[\frac{k_{i+1}}{k_i}\right] = \frac{1}{3} \), and that each measurement of \( \ln[\Delta_n^2] \) has the same 1-sigma uncertainty of \( \sigma = \frac{1}{4} \). These numbers are roughly consistent with what BICEP2 actually reported [34]. Hence the detection consists of five observations \( y_i \) obeying the relation,

\[
y_i = \ln\left[2.5 \times 10^{-11}\right] - \frac{i}{2400} + e_i, \quad i \in \{1, 2, 3, 4, 5\},
\]

(28)

where the \( e_i \) are independent Gaussian random variables with mean zero and standard deviation \( \sigma = \frac{1}{4} \). Table 1 simulates the five data points using a random number generator to find the \( e_i \).

Because the relation (27) is linear we can use least squares to determine the parameters. The least squares fit for \( N \) data points obeying the relation \( y_i = \alpha + \beta x_i \) (with \( x_i = i/3 \)) is,

\[
\alpha = \frac{\sum_{i=1}^{N} x_i^2 \sum_{j=1}^{N} y_j - \sum_{i=1}^{N} x_i \sum_{j=1}^{N} x_i y_j}{N \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} = \frac{\sum_{i=1}^{N} x_i (x_i - x_j) y_j}{\sum_{i=2}^{N} \sum_{j=1}^{i-1} (x_i - x_j)^2},
\]

(29)

\[
\beta = \frac{N \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{j=1}^{N} y_j}{N \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} = \frac{\sum_{i=2}^{N} \sum_{j=1}^{i-1} (x_i - x_j)(y_i - y_j)}{\sum_{i=2}^{N} \sum_{j=1}^{i-1} (x_i - x_j)^2}.
\]

(30)

Even in this general form it is obvious that expression (29) for \( \alpha \) represents a sort of average whereas expression (30) is a kind of numerical derivative.

| \( i \) | \( \ln(2.5 \times 10^{-11}) \) | \( -\frac{i}{2400} \) | \( e_i \) | \( y_i \) |
|---|---|---|---|---|
| 1 | -24.412145 | -0.000417 | +0.226742 | -24.185820 |
| 2 | -24.412145 | -0.000833 | -0.176041 | -24.589020 |
| 3 | -24.412145 | -0.001250 | -0.091555 | -24.504950 |
| 4 | -24.412145 | -0.001667 | -0.164330 | -24.578142 |
| 5 | -24.412145 | -0.002083 | +0.331640 | -24.082589 |

Table 1: Simulated data from relation (28), representing a hypothetical first detection of the tensor power spectrum with \( r = \frac{1}{100} \) and \( n_t = -\frac{1}{800} \). The random errors \( e_i \) follow a normal distribution with mean zero and standard deviation \( \sigma = \frac{1}{4} \).
Table 2: Predicted results according to relation (36), with the parameters $\alpha$ and $r$ taken from expressions (31) and (33), respectively.

| $i$ | $\alpha$ | $\frac{r}{24} \times i$ | $z_i$ | $y_i - z_i$ |
|-----|---------|-----------------|-----|----------|
| 1   | -24.453306 | -0.000400       | -24.453706 | +0.267886 |
| 2   | -24.453306 | -0.000800       | -24.454106 | -0.134914 |
| 3   | -24.453306 | -0.001200       | -24.454506 | -0.050445 |
| 4   | -24.453306 | -0.001599       | -24.454906 | -0.123236 |
| 5   | -24.453306 | -0.001999       | -24.455306 | +0.372717 |

So we expect the fractional error on $\beta$ to be larger than that on $\alpha$. That becomes even more apparent when specializing to $N = 5$ and $x_i = i/3$,

$$
\alpha \rightarrow \frac{(8y_1+5y_2+2y_3-y_4-4y_5)}{10} \simeq -24.453306 \pm 0.262202 , \quad (31)
$$

$$
\beta \rightarrow \frac{(-6y_1-3y_2+3y_4+6y_5)}{10} \simeq +0.065202 \pm 0.237171 . \quad (32)
$$

Hence the simulated data of Table 1 implies a reasonably accurate reconstruction of the tensor-to-scalar ratio,

$$
r = \exp[\alpha - \ln(2.5 \times 10^{-9})] = 0.0096 \pm 0.0027 , \quad (33)
$$

but a miserably inaccurate value for the tensor spectral index,

$$
n_t = \beta = 0.065 \pm 0.237 . \quad (34)
$$

The resulting check of the single scalar consistency relation is not very sensitive,

$$
0.010 \pm 0.003 = -0.522 \pm 1.897 . \quad (35)
$$

Because of the large (but statistically allowed) positive fluctuation $e_5$ the measured tensor spectral index (34) does not even have the right sign!

We propose to instead use the much better measured scalar spectral index to predict the tensor spectral index, up to an integration constant, and then to compare the fluctuation of the observed data around this prediction. For the model in question this might amount to assuming predictions of the form,

$$
z_i = \alpha - \frac{r}{24} \times i , \quad (36)
$$
where $\alpha$ is (31) and $r$ is (33). Table 2 reports these predictions, along with the difference between each simulated observation $y_i$ and the associated prediction $z_i$. Of course the parameter $r$ comes from the parameter $\alpha$ through relation (33), so the final column of Table 2 represents four statistically independent measurements. The resulting estimate for the scatter between measurement and prediction is,

$$\sqrt{\frac{1}{4} \sum_{i=1}^{5} (y_i - z_i)^2} \simeq 0.246614.$$  (37)

This is quite consistent with our assumed 1-sigma fluctuation of $\sigma = \frac{1}{4}$ for each observation.

4  Discussion

Resolving the tensor power spectrum $\Delta^2_{h}(k)$ is crucial for the progress of inflation because it constrains the causative mechanism. This is already evident from the angst [52, 53, 54, 55] elicited by the increasingly tight bounds on the tensor-to-scalar ratio $r$ [56]. A positive detection at several different wave lengths has the potential to falsify entire classes of models. For example, any model in which inflation is driven by the potential of a minimally coupled scalar must obey relation (1) between $r$ and the tensor spectral index $n_t$ [17, 18, 19]. Unfortunately, relation (1) requires taking a derivative of $\Delta^2_{h}(k)$, and the first generation of detections will probably be too sparse to provide a good bound because numerical differentiation makes bad data worse.

It makes more sense to integrate the high quality data we already possess for $\Delta^2_{R}(k)$. If the leading slow roll expressions (9-10) are assumed then the prediction (26) from $\Delta^2_{R}(k)$ requires only a single integration constant from $\Delta^2_{h}(k)$. (The same thing would be true even if the more accurate approximations (14-15) were employed [15].) Fixing this constant uses up one combination of whatever data we have for $\Delta^2_{h}(k)$, leaving the scatter of the remaining data about the prediction as a legitimate test of single scalar inflation. Hence relation (26) is a sort of integrated form of the single-scalar consistency relation (1) which can be applied more reliably. Section 3 compares this sort of scatter test with checking $r = -8n_t$ for simulated data based on a hypothetical detection of $r = 0.01$ at five wave lengths with fractional errors similar to those reported in the spurious BICEP2 detection [34].
Of course no massaging of poorly resolved data is going to extract a precision bound, but the scatter test seems clearly better.

Note that it is simple to adapt the scatter test to data fits. For example, the usual parameterization of the scalar data \[22, 23, 24, 25\] implies,

\[
\Delta^2_R(k) \sim A_s \left( \frac{k}{k_0} \right)^{n_s - 1} \implies \Delta^2_h(k) \simeq \Delta^2_h(k_\ast) \left[ 1 + \frac{r(k_\ast)}{8(1-n_s)} \left( \frac{k}{k_\ast} \right)^{1-n_s} - 1 \right]^{-1}.
\]

(38)

Here \(A_s\) is the scalar amplitude, \(n_s\) is the scalar spectral index, and \(k_0\) is a fiducial wave number.

Finally, we can look forward to the day, in the far future, when the tensor power spectrum is well resolved. Then the sort of scatter test we propose could be employed to search for correlations between features in the two power spectra. For example, Figure 1 depicts the bump in the first slow roll parameter from a model \[46, 47\] introduced to explain the scalar power spectrum’s dip at \(\ell \approx 22\) and peak at \(\ell \approx 40\) \[48, 49, 50, 51\]. These features are caused by the way the scalar nonlocal corrections \(17\) depend upon derivatives of \(\epsilon(n)\). The tensor nonlocal corrections \(18\) involve the same derivatives — although lacking the large factors of \(1/\epsilon\) — so it is obvious there will be corresponding features \[45\]. Resolving this sort of correlation probes the functional relation between the two power spectra far more deeply than the single scalar consistency relation.

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