A MIXED FINITE ELEMENT METHOD FOR NEARLY INCOMPRESSIBLE ELASTICITY AND STOKES EQUATIONS USING PRIMAL AND DUAL MESHES WITH QUADRILATERAL AND HEXAHEDRAL GRIDS

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Abstract. We consider a mixed finite element method for approximating the solution of nearly incompressible elasticity and Stokes equations. The finite element method is based on quadrilateral and hexahedral triangulation using primal and dual meshes. We use the standard bilinear and trilinear finite element space enriched with element-wise defined bubble functions with respect to the primal mesh for the displacement or velocity, whereas the pressure space is discretised by using a piecewise constant finite element space with respect to the dual mesh.

Key words. mixed finite elements, nearly incompressible elasticity, primal and dual meshes, Stokes equations, inf-sup condition

AMS subject classifications. 65N30, 65N15, 74B10

1. Introduction. Although there are many mixed finite element methods for nearly incompressible elasticity and Stokes equations leading to an optimal convergence, the search for simple, efficient and optimal finite element schemes is still an active area of research. In this article we present a mixed finite element method for nearly incompressible elasticity and Stokes equations using quadrilateral and hexahedral meshes. The displacement or velocity field is discretised by using the standard bilinear or trilinear finite element space enriched with element-wise defined bubble functions, whereas the pressure space is discretised by the piecewise constant finite element space based on a dual mesh. Such a finite element space for the simplicial mesh is presented in [12], where the inf-sup condition is proved by using the fact that the mini finite element \[ \text{[1]} \] satisfies the inf-sup condition. Note that the mini finite element \[ \text{[1]} \] consists of the linear finite element space enriched with element-wise defined bubble functions for the displacement or velocity and the linear finite element space for the pressure space. The enrichment of the displacement or velocity field increases one vector degree of freedom per element. A main hindrance to extend this approach to the case of quadrilateral and hexahedral meshes is that the displacement or velocity space should be enriched by more than a single bubble function to obtain the inf-sup condition \[ \text{[2]} \]. In this article we show that a similar discretisation scheme can be applied to quadrilateral and hexahedral meshes. We prove that if the pressure space is discretised by using the piecewise constant function space with respect to the dual mesh, it is sufficient to enrich the standard bilinear and trilinear finite element space with a single bubble function per element.

2. The boundary value problem of linear elasticity. We introduce the boundary value problem of linear elasticity in this section. In particular, we present the standard weak formulation and a mixed formulation of a linear elastic problem. We consider a homogeneous isotropic linear elastic material body occupying a bounded domain \( \Omega \subset \mathbb{R}^d \), \( d = \{2, 3\} \), with Lipschitz boundary \( \Gamma \). For a prescribed body force \( f \in [L^2(\Omega)]^d \), the governing equilibrium equation in \( \Omega \) reads

\[- \text{div} \sigma = f, \quad (2.1)\]
where $\sigma$ is the symmetric Cauchy stress tensor. The stress tensor $\sigma$ is defined as a function of the displacement $u$ by the Saint-Venant Kirchhoff constitutive law

$$
\sigma = \frac{1}{2} C (\nabla u + [\nabla u]^t),
$$

(2.2)

where $C$ is the fourth-order elasticity tensor. The action of the elasticity tensor $C$ on a tensor $d$ is defined as

$$
\sigma = C d := \lambda (\text{tr} d) 1 + 2 \mu d.
$$

(2.3)

Here, $1$ is the identity tensor, and $\lambda$ and $\mu$ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and they are assumed to be positive. For simplicity of exposition we assume that the displacement or velocity satisfies homogeneous Dirichlet boundary condition

$$
u = 0 \quad \text{on} \quad \Gamma.
$$

(2.4)

However, the approach works also for mixed boundary conditions.

**Standard weak formulation.**

Let $L^2(\Omega)$ be the set of square-integrable functions defined on $\Omega$, where the inner product and norm on this space is denoted by $(\cdot, \cdot)_0$ and $\|\cdot\|_0$, respectively. The Sobolev space $H^1(\Omega)$ is defined in terms of the space $L^2(\Omega)$ as

$$
H^1(\Omega) = \{ u \in L^2(\Omega), \nabla u \in [L^2(\Omega)]^d \},
$$

and $H^1_0(\Omega) \subset H^1(\Omega)$, where a function in $H^1_0(\Omega)$ vanishes on the boundary in the sense of traces. The space $L^2_0(\Omega)$ is the subset of $L^2(\Omega)$ defined as

$$
L^2_0(\Omega) = \{ p \in L^2(\Omega) : \int_\Omega p \, dx = 0 \}.
$$

To write the weak or variational formulation of the boundary value problem, we introduce the space $V := [H^1_0(\Omega)]^d$ of displacement or velocity with inner product $(\cdot, \cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(u, v)_1 := \sum_{i=1}^d (u_i, v_i)_1$, with the norm being induced by this inner product.

We define the bilinear form $A(\cdot, \cdot)$ and the linear functional $\ell(\cdot)$ by

$$
A : V \times V \rightarrow \mathbb{R}, \quad A(u, v) := \int_\Omega C \varepsilon(u) : \varepsilon(v) \, dx,
$$

$$
\ell : V \rightarrow \mathbb{R}, \quad \ell(v) := \int_\Omega f \cdot v \, dx.
$$

Then the standard weak form of linear elasticity problem is as follows: given $\ell \in V'$, find $u \in V$ that satisfies

$$
A(u, v) = \ell(v), \quad v \in V.
$$

(2.5)

The assumptions on $C$ guarantee that $A(\cdot, \cdot)$ is symmetric, continuous, and $V$-elliptic. Hence by using standard arguments it can be shown that (2.5) has a unique solution $u \in V$. Furthermore, if the the domain $\Omega$ is convex with polygonal or polyhedral boundary, $u \in [H^2(\Omega)]^d \cap V$, and there exists a constant $C$ independent of $\lambda$ such that $\|u\|_2 + \lambda \|\text{div} \, u\|_1 \leq C \|f\|_0$.

(2.6)

**Mixed formulation.** There are many mixed formulation for the linear elasticity problem. The simplest one is given by introducing pressure as an extra variable, which leads to
penalized Stokes equations. Defining \( p := \lambda \text{div} \mathbf{u} \), a mixed variational formulation of linear elastic problem (2.5) is given by: find \((\mathbf{u}, p) \in \mathbf{V} \times L_0^2(\Omega)\) such that

\[
\begin{align*}
    a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \ell(\mathbf{v}), & \mathbf{v} &\in \mathbf{V}, \\
    b(\mathbf{v}, q) - \frac{\lambda}{2} c(p, q) &= 0, & q &\in L_0^2(\Omega),
\end{align*}
\]

where

\[
a(\mathbf{u}, \mathbf{v}) := 2\mu \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx, \quad b(\mathbf{v}, q) := \int_{\Omega} \text{div} \mathbf{v} \, q \, dx, \quad c(p, q) := \int_{\Omega} p \, q \, dx.
\]

Using a standard theory of mixed finite elements [4], the existence and uniqueness of the solution of the problem (2.7) can be shown. Of particular interest for us is the incompressible limit, which corresponds to \( \lambda \to \infty \). We note that the saddle point equations (2.7) reduce to Stokes equations of fluid flow when \( \lambda \to \infty \). In this case, \( \gamma = 2\mu \) denotes the kinematic viscosity, and \( \mathbf{u} \) represents the velocity of the fluid.

3. Finite element discretizations. We consider a quasi-uniform triangulation \( \mathcal{T}_h \) - called the primal mesh - of the polygonal or polyhedral domain \( \Omega \), where \( \mathcal{T}_h \) consists of convex quadrilaterals or hexahedras. The finite element meshes are defined by maps from a reference square \( K = (0,1)^2 \) or reference cube \( K = (0,1)^3 \).

For nonnegative integer \( k \), we let \( P_k(\cdot) \) denote the space of polynomials in two or three variables of total degree less than or equal to \( k \), and \( Q_k(\cdot) \) the space of polynomials in two variables of total degree less than or equal to \( k \) in each variable. A typical element \( K \in \mathcal{T}_h \) is generated by an iso-parametric map \( F_K \) from the reference element \( \hat{K} \), in which \( F_K \) is defined using the basis functions corresponding to \( Q_1 \). It is clear that if \( \mathbf{v} \in Q_1(\hat{K}) \), then \( \mathbf{v} \circ F_K^{-1} \) is in general not a polynomial on the quadrilateral \( K \). However, in the following we assume that the map \( F_K \) is affine for all \( K \in \mathcal{T}_h \).

The finite element space of displacements is taken to be the space of continuous functions whose restrictions to an element \( K \) are obtained by maps of bilinear or trilinear functions from the reference element:

\[
S_h := \left\{ \mathbf{v}_h \in H^1_0(\Omega), \mathbf{v}_h|_K = \hat{\mathbf{v}}_h \circ F_K^{-1}, \hat{\mathbf{v}}_h \in Q_1(\hat{K}), \ K \in \mathcal{T}_h \right\}.
\]

Let \( N \) be the number of vertices in \( \mathcal{T}_h \), and the set of all vertices in \( \mathcal{T}_h \) be denoted by \( N_h := \{ \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \} \).

A dual mesh \( \mathcal{T}_h^* \) is introduced based on the primal mesh \( \mathcal{T}_h \) so that the elements of \( \mathcal{T}_h^* \) are called control volumes. Each control volume element \( V_i \in \mathcal{T}_h^* \) is associated with a vertex \( \mathbf{x}_i \in N_h \). For simplicity we explain the construction of the dual mesh for quadrilateral meshes. The idea can be extended to hexahedral meshes in the standard way as the extension from triangular meshes to tetrahedral meshes in [6] [12]. For a vertex \( \mathbf{x}_i \in N_h \) let \( \mathcal{T}_h^i \) and \( E_h^i \) be the set of elements and edges touching \( \mathbf{x}_i \), respectively. Then the volume element \( V_i \) corresponding to the vertex \( \mathbf{x}_i \in N_h \) is the polygonal domain joining centroids of all elements in \( \mathcal{T}_h^i \) and centroids of all edges in \( E_h^i \). If \( \mathbf{x}_i \in N_h \) is a boundary vertex, all the boundary edges touching \( \mathbf{x}_i \) will also form the boundary of \( V_i \). The set of all volume elements in \( \mathcal{T}_h^* \) will form a non-overlapping decomposition of the polygonal domain \( \Omega \). We refer to [6] [12] for a similar construction in simplicial case. A dual mesh for a quadrilateral grid is shown in Figure 3.1.

In the following we use a generic constant \( C \) which will take different values at different places but will always be positive and independent of the mesh-size \( h \). We call the control volume mesh \( \mathcal{T}_h^* \) regular or quasi-uniform if there exists a positive constant \( C > 0 \) such that

\[
C h^d \leq |V_i| \leq C h^d, \quad V_i \in \mathcal{T}_h^*.
\]
where $h$ is the maximum diameter of all elements $T \in T_h$. It can be shown that, if $T_h$ is locally regular, i.e., there is a constant $C$ such that

$$C h_T^d \leq |T| \leq h_T^d, \ T \in T_h$$

with diam$(T) = h_T$ for all elements $T \in T_h$, then this dual mesh $T_h^*$ is also locally regular. The dual volume element space $S_h^*$ to discretize the pressure is now defined by

$$S_h^* := \{ p \in L^2_0(\Omega) : p|_\Gamma \in P_0(V), \ V \in T_h^* \}.$$

Now any element $p_h \in S_h^*$ and $u_h \in S_h$ can be written as $u_h = \sum_{i=1}^N u_i \phi_i$ and $p_h = \sum_{i=1}^N p_i \chi_i$, where $\phi_i$ are the standard nodal basis functions associated with the vertex $i$, and $\chi_i$ are the characteristic functions of the volume $V_i$. Let $b_T \in Q_2(T)$ with $b_T = 0$ on $\partial T$ and $b_T(x_T) = 1$, where $x_T$ is the centroid of $T$, be a bubble function corresponding to the element $T \in T_h$. Let $\phi_T = \hat{\phi}_T \circ F_T^{-1}$, where $\hat{\phi}_T$ is the standard linear or trilinear basis function corresponding to the reference element $\hat{T} = (0,1)^d$ associated with the origin. Defining the space of bubble functions

$$B_h := \{ b_h \in [C^0(\Omega)]^d : b_h|_T = c_T \nabla \phi_T b_T, \ c_T \in \mathbb{R}, \ T \in T_h \},$$

we introduce our finite element space for displacement or velocity as $V_h = [S_h]^d \oplus B_h$.

Then, the finite element approximation of (2.7) is defined as a solution to the following problem: find $(u_h, p_h) \in V_h \times S_h^*$ such that

$$a(u_h, v_h) + b(v_h, p_h) = \ell(v_h), \quad v_h \in V_h,$$

$$b(u_h, q_h) - \frac{1}{2} c(p_h, q_h) = 0, \quad q_h \in S_h^*.$$  \hspace{1cm} (3.3)

To establish a priori estimates for the discretization errors, we consider the saddle point formulation (3.3) of the elasticity problem and apply the theory of mixed finite elements. The continuity of the bilinear form $a(\cdot, \cdot)$ on $V_h \times V_h$, of $b(\cdot, \cdot)$ on $V_h \times S_h^*$ and of $c(\cdot, \cdot)$ on $S_h^* \times S_h^*$ is straightforward. By using the Korn’s inequality, it is standard that the ellipticity of the bilinear form $a(\cdot, \cdot)$ holds on $V_h \times V_h$. It remains to show that the uniform inf-sup condition holds for the bilinear form $b(\cdot, \cdot)$ on $V_h \times S_h^*$. That means, for any $q_h \in S_h^*$, there exists a constant $\beta > 0$ independent of the mesh-size such that

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1} \geq \beta \|q_h\|_0.$$  \hspace{1cm} (3.4)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\linewidth]{fig3_1}
\caption{Primal and dual meshes with a vertex $x_i$ and four elements of $T_h$ touching the vertex $x_i$.}
\end{figure}
We note that we have proved optimal a priori error estimate for the finite element scheme (3.3) in [12] for simplicial meshes using the stability of the mini finite element [1], where the standard linear finite element space is enriched by an element-wise defined bubble function per element. For quadrilateral and hexahedral meshes the stability is not attained by enriching the standard bilinear or trilinear finite element space by a single element-wise defined bubble function, see [2]. Therefore, we recourse to another method to prove the stability of the scheme here. We prove the inf-sup condition (3.4) using a domain decomposition technique as in [14].

In the following we assume that we have a decomposition of $\Omega$ in $M$ disjoint subdomains $\{S_i\}_{i=1}^M$, where each subdomain consists of four quadrilaterals or eight hexahedra touching the vertex $x_i$, see the right picture of Figure 3.1 for the quadrilateral case. Let $V^i_h \subset H_0^1(S_i)$ be the restriction of the finite element space $V_h$ to the set $S_i$ satisfying the homogeneous Dirichlet boundary condition on the boundary of $S_i$. First we observe that the necessary condition for the patch test is satisfied as the velocity space has 10 degrees of freedom in two dimensions and 27 in three dimensions, whereas the pressure space on $S_i$ has 8 degrees of freedom in two dimensions and 25 in three dimensions after excluding the constant functions on each $S_i$. The proof of the following lemma can be obtained by a direct computation on one $S_i$.

**Lemma 3.1.** The dimension of the space

$$B_i = \{ q_h \in S^*_h : b(v_h, q_h) = 0, \ v_h \in V^i_h \}$$

is one.

**Proof.** We outline the proof in two dimensions. Each bubble function yields two equations leading to eight equations. Due to the symmetry we get linearly independent seven equations. We get additional two equations by using the vertex basis function associated with the vertex $x_i$. However, only one of them is linearly independent to the previous seven equations as constant functions are in $B_i$. Thus we have eight linearly independent equations leading to the fact that $B_i$ contains only constant functions. Note that the factor $\nabla \varphi_T$ in the definition of the space of bubble functions in (3.2) is used to get that there are eight linearly independent equations. If we use the standard bubble function $b_T$ for discretising all components of the displacement, there will be only seven linearly independent equations.

Thus we have the following lemma.

**Lemma 3.2.** There exists a constant $C > 0$ independent of the mesh-size $h$ such that

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1} \geq C \sqrt{\int_{S_i} q_h^2 \, dx}, \quad q_h \in L_0^2(S_i) \cap S^*_h, \quad i = 1, \ldots, M. \quad (3.5)$$

This lemma is used to prove the inf-sup condition.

**Theorem 3.3.** There exists $\beta > 0$ independent of the mesh-size $h$ such that

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1} \geq \beta \|q_h\|_0, \quad q_h \in S^*_h.$$

**Proof.** Let

$$q^i_h = \frac{1}{|S_i|} \int_{S_i} q_h \, dx, \quad \text{and} \quad \tilde{q}_h = \sum_{i=1}^M q^i_h \chi_{S_i},$$
where $\chi_{S_i}$ is the characteristic function of the set $S_i$. Then for $q_h \in S_h^*$
\[
\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1} = \sup_{v_h \in V_h} \frac{b(v_h, q_h - \tilde{q}_h)}{\|v_h\|_1} + \sup_{v_h \in V_h} \frac{b(v_h, \tilde{q}_h)}{\|v_h\|_1}.
\]

Note that
\[
\sup_{v_h \in V_h} \frac{b(v_h, q_h - \tilde{q}_h)}{\|v_h\|_1} \geq \sum_{i=1}^{M} \sup_{v_h \in V_h} \frac{b(v_h, q_h - \tilde{q}_h)}{\|v_h\|_1}.
\]

Since $v_h^i$ is supported only on $S_i$ for $i = 1, \cdots, M$, we get
\[
\sup_{v_h \in V_h} \frac{b(v_h, q_h - \tilde{q}_h)}{\|v_h\|_1} \geq C\|q_h - \tilde{q}_h\|_0
\]
from Lemma 3.5 and since $\tilde{q}_h$ is a piecewise constant function associated with one level coarser mesh, we have
\[
\sup_{v_h \in V_h} \frac{b(v_h, \tilde{q}_h)}{\|v_h\|_1} \geq C\|\tilde{q}_h\|_0.
\]

Note that the final result follows from the fact that
\[
\|q_h - \tilde{q}_h\|_0 + \|\tilde{q}_h\|_0 \geq C\|q_h\|_0.
\]

The immediate consequence of the above discussion is the well-posedness of the discrete problem (3.3). From the theory of saddle point problem, see, e.g., [4], we have the following theorem.

**Theorem 3.4.** The discrete problem (3.3) has exactly one solution $(u_h, p_h) \in V_h \times S_h^*$, which is uniformly stable with respect to the data $f$, and there exists a constant $C$ independent of Lamé parameter $\lambda$ such that
\[
\|u_h\|_1 + \|p_h\|_0 \leq C\|f\|_0.
\]

The convergence theory is provided by an abstract result about the approximation of saddle point problems, see [4].

**Theorem 3.5.** Assume that $(u, p)$ and $(u_h, p_h)$ be the solutions of problems (2.7) and (3.3), respectively. Then, we have the following error estimate uniform with respect to $\lambda$:
\[
\|u - u_h\|_1 + \|p - p_h\|_0 \leq C \left( \inf_{v_h \in V_h} \|u - v_h\|_1 + \inf_{q_h \in S_h} \|p - q_h\|_0 \right).
\]

Since the space $V_h$ contains the space of piece-wise linear polynomials and $S_h^*$ contains the space of piece-wise constant functions with respect to the dual mesh $T_h^*$, Theorem 3.5 yields the linear convergence of the discrete solution with respect to the energy norm for the displacement and with respect to the $L^2$-norm for the pressure.

We note that the displacement field on the primal mesh and the stress field in the dual mesh in the Hellinger-Reissner problem of finding $(u_h, \sigma_h) \in V_h \times S_h^*$ such that
\[
\begin{align*}
\int_{\Omega} C^{-1} \sigma_h : \tau_h \, dx &- \int_{\Omega} \varepsilon(u_h) : \tau_h \, dx = 0, & \tau_h \in S_h^*, \\
\int_{\Omega} \varepsilon(v_h) : \sigma_h \, dx &- \ell(v_h) \, dx = 0, & v_h \in V_h,
\end{align*}
\]
where $S_h^* := (S_h^*)^{d 	imes d}$, we arrive at the node-based uniform strain elements [7, 5]. However, the formulation is not stable [11].

Now we briefly describe how a displacement-based formulation is achieved for a nearly incompressible elasticity problem. From the second equation of (3.3), we can write $p_h = \sum_{i=1}^{N} p_i \chi_i$ with

$$p_i = \frac{\lambda}{|V_i|} \int_{V_i} \nabla \cdot u_h \, dx.$$ 

Hence, after condensing out the pressure from the formulation, we arrive at a problem of finding $u_h \in V_h$ so that

$$a(u_h, v_h) + \sum_{i=1}^{N} \frac{\lambda}{|V_i|} \left( \int_{V_i} \nabla \cdot u_h \, dx \right) \left( \int_{V_i} \nabla \cdot v_h \, dx \right) = \ell(v_h), \quad v_h \in V_h.$$ 

If we look at the algebraic formulation of the finite element scheme for a nearly incompressible elasticity problem, we have

$$\begin{pmatrix} A & B^T \\ B & -\frac{1}{\lambda} C \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix},$$

where $A, B$ and $C$ are matrices associated with the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$, respectively, and $f_h$ is the discrete vector associated with the linear form $\ell(\cdot)$. The important thing to note is that the matrix $C$ is diagonal and the degrees of freedom corresponding to the pressure can easily be condensed out from the system leading to a positive definite formulation.

4. Numerical Results.

Example 1: Cook’s membrane problem. Our first test example is the popular benchmark problem known as Cook’s membrane problem [13, 10, 8]. Let $\Omega$ be the quadrilateral connecting four points

\{(0, 0), (48, 44), (48, 60), (0, 44)\}.

The left boundary of $\Omega$ is clamped, and the right one is subjected to an in-plane shearing load of 100N along the $y$-direction, as shown in the left picture of Figure 4.1. The material properties are taken to be $E = 250$ and $\nu = 0.49999$, so that a nearly incompressible response is obtained. We have presented the vertical tip displacements at the point $T$ computed using the mixed formulation and the standard displacement formulation are presented in the right picture of Figure 4.1 for different levels of uniform refinement, where the computation is started with the initial triangulation shown in the left picture of Figure 4.1. As can be seen from the right picture of Figure 4.1, the standard displacement approach exhibits extreme locking whereas the new mixed formulation shows rapid convergence.

Example 2: Rectangular beam. The second example is concerned with a linear elastic beam of rectangular size subjected to a couple at one end, as shown in Figure 4.2. Along the edge $x = 0$, the horizontal displacement and vertical surface traction are zero. At the point $(0,0)$, the vertical displacement is also zero. The exact solution is given by

$$u(x, y) = \frac{2f(1 - \nu^2)}{E l} x \left( \frac{l}{2} - y \right), \quad \text{and} \quad v(x, y) = \frac{f(1 - \nu^2)}{E l} \left[ x^2 + \frac{\nu}{1-\nu} y(y - l) \right].$$

We set $L = 10$, $l = 2$, $E = 1500$, $\nu = 0.4999$, and $f = 3000$. We have shown the setting of the problem in Figure 4.2 and the discretization errors with respect to the number of
5. Conclusion. We have presented a finite element approach based on primal and dual meshes using quadrilateral and hexahedral meshes to approximate the solution of nearly incompressible elasticity or Stokes equations. Working with the space of bilinear or trilinear
finite elements enriched with bubble functions for the displacement or velocity field we have proved the uniform inf-sup condition. As we have an orthogonal basis for the piecewise constant finite element space on the dual mesh, we can statically condense out the pressure variable from the system leading to a displacement-based formulation. The resulting displacement-based formulation is symmetric and positive-definite.

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