Entropy of Three-Dimensional Black Holes in String Theory

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Abstract

It is observed that the three-dimensional BTZ black hole is a supersymmetric solution of the low-energy field equations of heterotic string theory compactified on an Einstein space. The solution involves a non-zero dilaton and NS-NS H-field. The entropy of the extreme black hole can then be computed using string theory and the asymptotic properties of anti-de Sitter space, without recourse to a D-brane analysis. This provides an explicit example of a black hole whose entropy can be computed using fundamental string theory, as advocated by Susskind.

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1 Introduction

Recently, there has been considerable interest in the microscopic derivation of the Bekenstein-Hawking entropy formula for black holes in string theory [1]-[4]. As pointed out in [3], for example, a key property in trying to compute the entropy of a string theory black hole is the presence of supersymmetry. However, as observed there, the known supersymmetric black holes in string theory typically have zero horizon area at extremality, unless a number of RR fields are excited. With the understanding that D-branes are the carriers of RR charge [5], it has become possible to reliably compute the entropy in certain examples. However, according to the philosophy of [6], it would be appealing to see directly an example of a supersymmetric string theory black hole which has a non-zero horizon area at extremality, without the presence of RR fields, and with the entropy computable directly in terms of fundamental string states. The purpose of the present note is to observe that indeed there does exist a three-dimensional black hole with these properties. The black hole under consideration is the Bañados-Teitelboim-Zanelli (BTZ) black hole [7, 8], which has the local geometry of anti-de Sitter spacetime, adS_3; for a review, see [9]. A further exploration of the nature of black holes within the context of conformal field theory was presented in [10], where the decay rate of certain four- and five-dimensional black holes was understood from the perspective of conformal field theory. This viewpoint was analyzed for the BTZ black hole in [11].

In order to understand the entropy of the BTZ black hole from the point of view of string theory, we must first show that it can be obtained as a compactified solution of string theory. In [12, 13, 14], the BTZ black hole was given an interpretation as a solution of the three-dimensional string action with non-zero cosmological constant. Our aim here is different, however; we seek a compactified solution of the low-energy 10-dimensional string equations of motion, of the form adS_3 × K⁷, where K⁷ is a compact internal space. In order to achieve this, one requires a non-constant dilaton, in contrast to the solution presented in [12, 13, 14]. We should remark here that the solution presented is precisely that given in [15]; see [16] for a review.

The main observation here is to see that by compactifying heterotic string theory on a round S⁷, for example, which has trivial holonomy group, we obtain an N = 8 theory defined on adS_3. The BTZ black hole is then given by the standard identification of adS_3 [3]. The only non-zero fields are the metric, the dilaton, and the NS-NS 3-form H. We thus obtain a concrete example of a black hole which has non-zero horizon length at extremality, and which can be obtained as a compactified supersymmetric solution of string theory without RR fields. We will confine our attention to the case of the extreme black hole, and thus we seek a compactification which preserves supersymmetry, since it has been shown in [17] that the BTZ black hole is supersymmetric if and only if it is extreme. Furthermore, it was observed in [18] that the asymptotic symmetry algebra of three-dimensional anti-de Sitter gravity consists of left-moving and right-moving Virasoro algebras, with the mass of the black hole being related to the Hamiltonian. Thus, based on string theory and supersymmetry, and the special asymptotic properties of three-dimensional anti-de Sitter space, we have a realization of the argument presented in [1].
2 Compactification of String Theory on adS$_3$

In this section, we review the solution presented in [15]. We seek a compactification of string theory to a product of three-dimensional anti-de Sitter spacetime with a compact internal space. We write the 10-dimensional coordinates as $x^M = (x^\mu, y^m)$, with $\mu = 0, 1, 2$, and $m = 3, ..., 9$. We follow the conventions of [16], and use the gamma matrix representation presented in [19]. The action of $N = 1$ $d = 10$ supergravity in the Einstein frame is given by

$$S(\text{Einstein frame}) = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{\hat{g}} \left( \hat{R} - \frac{1}{2}(\hat{\nabla}\phi)^2 - \frac{1}{12}e^{-\phi}\hat{H}^2 \right).$$  \hspace{1cm} (1)$$

Here, $\phi$ is the dilaton field, and the 3-form field $H_{MNP}$ is the NS-NS field which couples to the string. It is convenient in the following to use the so-called 5-brane frame, with metric $\hat{g}_{MN}$ conformally related to the Einstein metric $\hat{g}_{MN}$ by

$$\hat{g}_{MN} = e^{\phi/6}g_{MN}. \hspace{1cm} (2)$$

The action in the 5-brane frame then takes the form

$$S(5 - \text{brane frame}) = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left( \Phi R - \frac{1}{12}\Phi^{-1}H^2 \right),$$  \hspace{1cm} (3)$$

where $\Phi = e^{2\phi/3}$. The equations of motion in the 5-brane frame are given by

$$R_{MN} = \Phi^{-1} \left( \nabla_M \nabla_N \Phi - g_{MN} \nabla^2 \Phi \right) + \frac{1}{4}\Phi^{-2} \left( H_{MPQ}H_{NPQ} - \frac{1}{3}g_{MN}H^2 \right),$$  \hspace{1cm} (4)$$

$$\nabla_M \left( \Phi^{-1}H^{MNP} \right) = 0, \hspace{1cm} (5)$$

$$R = -\frac{1}{12}\Phi^{-2}H^2. \hspace{1cm} (6)$$

We wish to obtain a product metric of the form

$$g_{\mu\nu} = g_{\mu\nu}(x), \quad g_{mn} = g_{mn}(y), \quad g_{\mu m} = 0. \hspace{1cm} (7)$$

To obtain the required solution, we let the dilaton depend only on the spacetime coordinates, i.e., $\Phi = \Phi(x)$, and we make the ansatz

$$H_{\mu\nu} = A\Phi\epsilon_{\mu\nu}, \hspace{1cm} (8)$$

where $A$ is a constant, and all other components are zero. Clearly, (8) is automatically satisfied by this ansatz. Taking the trace of (4), and using (6), we find

$$\nabla^2 \Phi = -\frac{1}{18}\Phi^{-1}H^2.$$  \hspace{1cm} (9)$$

Hence, using (8), we find

$$\left( \nabla^2 - \frac{A^2}{3} \right) \Phi = 0.$$  \hspace{1cm} (10)$$

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To solve the remaining equation, we impose the condition
\[ \nabla_\mu \nabla_\nu \Phi = \frac{1}{3} g_{\mu\nu} \nabla^2 \Phi. \quad (11) \]

Then, (4) yields
\[ R_{\mu\nu} = -\frac{2A^2}{9} g_{\mu\nu}, \quad (12) \]
\[ R_{mn} = \frac{A^2}{6} g_{mn}, \quad (13) \]
with \( R_{\mu m} = 0 \). Choosing \( A = 3/\ell \) yields the normalization of \( \Phi \). Thus, we have obtained a solution of string theory compactified to a product of three-dimensional anti-de Sitter spacetime with a compact seven-dimensional Einstein space.

In order to show that the solution is a supersymmetric solution, we must show that the Killing spinor equations are satisfied. Namely, we must show that the supersymmetry variations of the fermionic fields vanish in the compactified background. In the Einstein frame, the supersymmetry transformations are
\[ \delta \hat{\psi}_M = \hat{\nabla}_M \hat{\epsilon} + \frac{1}{96} e^{-\phi/2} \left( \hat{\Gamma}_M^{NPQ} - 9 \delta_M^N \hat{\Gamma}^{PQ} \right) H_{NPQ} \hat{\epsilon}, \quad (14) \]
\[ \delta \hat{\lambda} = -\frac{1}{2\sqrt{2}} \left( \hat{\Gamma}_M \hat{\nabla}_M \phi \right) \hat{\epsilon} + \frac{1}{24\sqrt{2}} e^{-\phi/2} \hat{\Gamma}^{MNP} H_{MNP} \hat{\epsilon}. \quad (15) \]

To transform these to the 5-brane frame, we note that with \( \hat{g}_{MN} = e^{\phi/6} g_{MN} \), we have
\[ \hat{\nabla}_M = \nabla_M + \frac{1}{24} \Gamma_M^N \nabla_N \phi. \quad (16) \]
We define
\[ \epsilon = e^{-\phi/24} \hat{\epsilon}, \]
\[ \psi_M = e^{-\phi/24} \left( \hat{\psi}_M + \frac{1}{6\sqrt{2}} \hat{\Gamma}_M \hat{\lambda} \right), \]
\[ \lambda = e^{\phi/24} \hat{\lambda}. \quad (17) \]

Then, in the 5-brane frame, the supersymmetry transformations take the form
\[ \delta \psi_M = \nabla_M \epsilon + \frac{1}{96} e^{-2\phi/3} \left( \Gamma_M^{NPQ} - 9 \delta_M^N \Gamma^{PQ} + \frac{1}{3} \Gamma_M \Gamma^{NPQ} \right) H_{NPQ} \epsilon, \quad (18) \]
\[ \delta \lambda = -\frac{1}{2\sqrt{2}} \left( \Gamma_M \nabla_M \phi \right) \epsilon + \frac{1}{24\sqrt{2}} e^{-2\phi/3} \Gamma^{MNP} H_{MNP} \epsilon. \quad (19) \]

A representation of the gamma matrices relevant to the 3 + 7 split is given in [19]. We write the 10-dimensional spinor as \( \epsilon(x, y) = \eta(x) \otimes \chi(y) \). Then, we find
\[ \delta \psi_\mu = \nabla_\mu \eta - \frac{A}{6} \gamma_\mu \eta = 0, \quad (20) \]
\[ \delta \psi_m = \nabla_m \chi \pm i \frac{A}{12} \Sigma_m \chi = 0, \]  
(21)

\[ \delta \lambda = -\frac{1}{2\sqrt{2}} (\gamma^\mu \nabla_{\mu} \phi) \eta + \frac{A}{4\sqrt{2}} J \eta = 0, \]  
(22)

with \( \gamma^4 J \eta = \pm i \eta \), where \( \gamma^4 \) and \( J \) are defined in [19]. We also note that \( \gamma_{\mu} \) and \( \Sigma_m \) are the three- and seven-dimensional gamma matrices, respectively. The integrability conditions implied by (20) and (21) are precisely the conditions (12) and (13), respectively. Also, eqns. (20) and (22) imply that

\[ (\nabla \phi)^2 = \frac{A^2}{4}, \quad \nabla^2 \phi = \frac{A^2}{3}, \]  
(23)

which in turn is precisely the condition (10). Hence, we see that the above solution is indeed a supersymmetric solution of low-energy string theory. Indeed, this is precisely the solution presented in [13, 16], given there in terms of the dual 7-form field defined by \( H = e^{\phi^*} K \).

The internal space must be an Einstein space, and the number of supersymmetries in three dimensions depends on the holonomy of the internal space. For the case of seven-manifolds, this has been analyzed in some detail within the context of compactifications of 11-dimensional supergravity [20]. We note here that by choosing the round metric on \( S^7 \) which has trivial holonomy, we obtain an \( N = 8 \) supersymmetric theory in three dimensions.

### 3 The Entropy

In units with \( 8G = 1 \), the entropy of the BTZ black hole is given by [3]

\[ S = 4\pi r_+, \]  
(24)

where

\[ r_+^2 = \frac{M\ell^2}{2} \left\{ 1 \pm \left[ 1 - \left( \frac{J}{M\ell} \right)^2 \right]^{1/2} \right\}. \]  
(25)

For the extreme (BPS) black hole, we have \( J = \pm M\ell \). Hence, the entropy is given by

\[ S = 2\pi \sqrt{2M\ell^2}. \]  
(26)

Now, according to [13, 17], we have

\[ M\ell = \mathcal{L}_0 + L_0, \quad J = \mathcal{L}_0 - L_0, \]  
(27)

where the left and right Virasoro generators \( L_0 \) and \( \mathcal{L}_0 \) have central charge \( c = 12\ell \). Such a central charge can be realized by an \( N = 8 \) superconformal field theory, and thus we see that the \( S^7 \) compactification is one way to achieve this. Hence, in the extreme case \( J = M\ell \), we have \( L_0 = 0 \), and \( \mathcal{L}_0 = M\ell \). Also, in the extreme case \( J = -M\ell \), we have \( \mathcal{L}_0 = 0 \), and \( L_0 = M\ell \). For very massive black holes, the degeneracy of left-moving and right-moving string states is given by [21]

\[ S = 2\pi \sqrt{\frac{c_{n_L}}{6}} = 2\pi \sqrt{\frac{c_{n_R}}{6}} = 2\pi \sqrt{2M\ell^2}, \]  
(28)

where \( n_L(n_R) \) is the eigenvalue of \( L_0(\mathcal{L}_0) \). We thus have agreement between the string entropy and the Bekenstein-Hawking area formula.
4 Conclusions

The relevance of three-dimensional anti-de Sitter spacetime to the microscopic derivation of the entropy of certain higher-dimensional black holes in string theory has been discussed in [22]. The derivation presented in [22] relies on the entropy calculation of the BTZ black hole given in [23]. In [24], the emergence of the BTZ black hole from D-brane configurations of higher-dimensional black holes was discussed. It has also been shown in [24] that adS$_3$ plays a role in the study of a certain limit of superconformal field theories. Recently, there has been a construction of four- and five-dimensional anti-de Sitter black holes [26]-[29]. In particular, it would be very interesting to understand the entropy of the four-dimensional case from the point of view of M Theory. In this regard, we observe that there is the well-known compactification of 11-dimensional supergravity to adS$_4$ [20]. Also, in the five-dimensional case, it has been shown that the entropy is not proportional to the area [29], and it would be interesting to see this result emerge from string theory.

Note Added
After this work was completed, we noticed Ref. [30]. The present work provides a concrete realization of the observation of Strominger.

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