Stability of a horizontal shear flow

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Abstract. In this work we study the stability of horizontal shear flows of an ideal fluid in an open channel. Stability conditions are derived in terms of the theory of generalized hyperbolicity of motion equations. We show that flows with monotonic convex profile are always stable, whereas flows with an inflexion point in the velocity profile might become unstable. To illustrate the criteria we give simple examples for stable and unstable flows. Then we derive a multilayered model that is an approximation of the original model and features a continuous piecewise linear velocity profile. We also formulate sufficient hyperbolicity conditions for the multilayered model.

1. Introduction
When modelling flows of fluid in channels and riverbeds it is sometimes necessary to investigate non-linear integro-differential systems of equations. This is caused mainly by a heterogeneity of the flow due to the flow geometry and friction effects. Here we study the horizontal-shear flow model of an ideal fluid in an open channel proposed in [1] that allows to generalize the concept of subcritical and supercritical flow and to study a flow around local constrictions and expansions [2]. Classes of exact solutions of this model are built and investigated in [3]. The current work is focused on the stability analysis of the flow. The stability criteria are derived in terms of generalized hyperbolicity — a theoretical approach developed by V. M. Teshukov [4, 5]. First we derive criteria similar to the Rayleigh and Fjortoft theorem [6]. Then we consider a multilayered approximation of the motion equations, characterized by a piecewise constant potential vorticity, which has proven to be efficient when describing shear flows [7, 8]. We also derive sufficient hyperbolicity conditions for the multilayered model using the same theoretical approach. A similar analysis has been performed in [9] for Benney equations [10] but for a wider class of flows that also features slippage between the layers of the multilayered model.

2. Mathematical model
We consider a system of equations that describes flow of a fluid in an open narrow channel [1, 2]:

\[
\begin{align*}
  u_t + uu_x + vu_y + gh_x &= 0, \quad h_y = 0, \\
  h_t + (uh)_x + (vh)_y &= 0, \quad v|_{y=0} = v|_{y=Y} = 0,
\end{align*}
\]

(1)

here $t, x, y$ are time and Cartesian coordinates; $u(t, x, y), v(t, x, y)$ are horizontal velocity vector components; $h(t, x)$ is the flow depth; $g$ is the gravitational constant; $Y$ is the channel width.
The spatial configuration of the flow is depicted in Figure 1. From equations (1) follows the conservation of potential vorticity $\Omega = u_y/h$ along the trajectories:

$$\Omega_t + u\Omega_x + v\Omega_y = 0.$$  \hfill (2)

From (2) it follows that system (1) allows a class of solutions with piecewise constant vorticity.

![Figure 1. The spatial configuration of the flow.](image)

The stability criteria are derived using the generalization of hyperbolicity theory [4, 5]. Because the hyperbolicity conditions for the system (1) are derived and studied in [1], here we only present them briefly. The characteristic function $\chi(k)$ of the system (1) has the form:

$$\chi(k) = 1 - \frac{gh}{Y} \int_0^Y \frac{dy}{(u-k)^2}.$$  \hfill (3)

And the system of equations (1) is hyperbolic in terms of [4], if

$$\Delta \arg \frac{\chi^+(u)}{\chi^-(u)} = 0, \quad \chi^\pm \neq 0.$$  \hfill (4)

Here $\chi^+(u)$ and $\chi^-(u)$ are the limit values of function $\chi(u)$ from the upper $\chi^+$ and lower $\chi^-$ complex half-plane on segment $[u_0,u_1]$. The functions $\chi^+(u)$ and $\chi^-(u)$ are derived from integrating (3) by parts and applying Sokhotski–Plemelj theorem [11]:

$$\chi^+(u) = 1 + \frac{g}{Y} \left[ \frac{1}{\Omega_1(u_1 - u)} - \frac{1}{\Omega_0(u_0 - u)} - \int_0^Y \left( \frac{1}{\Omega(y')} \right) \frac{dy'}{u(y') - u} \pm \frac{\pi i}{u_y} \left( \frac{1}{\Omega} \right) \right].$$

The indices “0” and “1” correspond to the values on channel walls $y = 0$ and $y = Y$ respectively. So if the conditions (4) hold, then the characteristic equation $\chi(k) = 0$ has real roots only [1, 5].

3. Stability condition

It is convenient to verify hyperbolicity conditions for a flow with a monotonic velocity profile $u = U(y)$, $U'(y) > 0$ using the following function:

$$\Psi^\pm(U) = m(U)\chi^\pm(U), \quad m(U) = (U_1 - U)(U - U_0) > 0.$$
Significant here is that functions

\[
\Psi^\pm(U) = m(U) + \frac{gh}{Y} \left( \frac{U - U_0}{U'_1} + \frac{U_1 - U}{U'_0} + \right.
\]

\[
+ m(U) \int_0^Y \frac{U(\nu)^n d\nu}{(U''(\nu))^2(U(\nu) - U(y))} \pm \pi i m(U) \frac{U''(U')^3}{U''(U')^3}
\]

do not have any singularities at \( U = U_0, U = U_1 \) due to the choice of \( m(U) \) which, in its turn, obviously has no effect on the conditions (4). On the complex plane \((z_1, z_2)\) let us construct a closed contour \( C \), which consists of contours \( C^+ \) and \( C^- \), which are set parametrically:

\[
C^+ : \quad z_1 = \text{Re}[\Psi^+(U)], \quad z_2 = \text{Im}[\Psi^+(U)];
\]

\[
C^- : \quad z_1 = \text{Re}[\Psi^-(U)], \quad z_2 = \text{Im}[\Psi^-(U)];
\]

Note that \( C^+ \) and \( C^- \) are symmetric with respect to the \( z_1 \) axis. If the zero point \((0, 0)\) of the complex plane lies inside the contour \( C \), then the argument of \( \Psi^\pm \) will increase which indicates existence of complex characteristics. Stability conditions for flows that are described by Benney equations are formulated in [9]. Here we formulate similar stability criteria for the model (1). These models are mathematically equivalent, so only short explanations are given.

**Statement 1** Any flow with a monotonic convex velocity profile is stable (Rayleighs inflexion point theorem analogue).

Indeed, let \( U' > 0, U'' \neq 0 \), then \( \Psi^\pm(U_0) > 0, \Psi^\pm(U_1) > 0 \), and \( \text{Im} \Psi^\pm \) does not change its sign on the interval \([U_0, U_1]\). Thus, given the inequality \( \Psi^\pm(U_c) > 0 \), where \( U_c \) is any point from \([U_0, U_1]\), the argument of \( \Psi \) does not grow.

To illustrate this, consider a flow with a monotonic convex velocity profile given by equation:

\[
U(y) = \frac{3y}{2y + 1}, \quad y \in [0, 1].
\]

For the sake of simplicity let us consider \( h = g = Y = 1 \). Then the velocity profile of the flow is shown in Figure 2, and the corresponding contour \( C^- \) in Figure 3.

![Figure 2. Velocity profile given by (5)](image1)

![Figure 3. Contour C^- corresponding to (5)](image2)
**Statement 2** If $U''(U - U_c) \geq 0$, where $U_c = U(y_c)$, and $y_c$ is the inflexion point $U''(y_c) = 0$, then the flow is stable (Fjortoft's theorem analogue).

Similar to the previous statement, $\Psi^\pm(U_0) > 0$, $\Psi^\pm(U_1) > 0$, and the sign of $\text{Im}\Psi^\pm$ is the same as of $U''$. Therefore, verification of stability reduces to determination of the sign of $\Psi^\pm(U_c)$.

$$\Psi^\pm(U_c) = m(U_c) + \frac{gh}{Y} \left( \frac{U_c - U_0}{U_1} + \frac{U_1 - U_c}{U_0} \right) + m(U_c) \int_0^Y \frac{U'(\nu)(U(\nu) - U_c)d\nu}{(U''(\nu))^2(U(\nu) - U(\nu))^2}.$$ 

Considering that $m(U_c) \geq 0$ and $U''(U - U_c) \geq 0$, the following inequality holds $\Psi^\pm(U_c) > 0$. This means that the flow is stable.

To illustrate the second statement, consider a flow with a velocity profile given by equation:

$$U(y) = y^3 - y^2 + y, \quad y \in [0, 1]. \quad (6)$$

For the sake of simplicity let us consider $h = g = Y = 1$. Then the velocity profile of the flow is shown in Figure 4, and the corresponding contour $C^-$ in Figure 5.

![Figure 4. Velocity profile given by (6)](image)

![Figure 5. Contour $C^-$ corresponding to (6)](image)

**Statement 3** If $U''(U - U_c) \leq 0$, where $U_c = U(y_c)$, and $y_c$ is the inflexion point $U''(y_c) = 0$ then the flow might be unstable.

Though in this case all the same conditions as in Statement 2 hold, here we cannot guarantee that the following inequality holds: $\Psi^\pm(U_c) > 0$. Let us construct an example where a non-zero argument increase is possible, and hence the flow might become unstable. This will prove Statement 3. Consider a flow with a velocity profile given by equation:

$$U(y) = \frac{\tanh((y - y_c)a) + \tanh(ay_c)}{\tanh((1 - y_c)a) + \tanh(ay_c)}, \quad y \in [0, 1]. \quad (7)$$

Here we also consider $h = g = Y = 1$. The second derivative $U''(y)$ will be equal zero at the inflection point $y_c = 0.53$. Consider two different cases: $a = 2.4$ and $a = 3.6$. The velocity profiles of the flow are shown in Figure 6 and the corresponding contour $C^-$ in Figure 7.

The dotted lines correspond to $a = 2.4$. In this case the point $(0, 0)$ does not lie inside the contour $C^-$, which means that the argument increase of the function $\chi$ is equal zero and the
Figure 6. Velocity profile given by (7) for $a = 2.4$ (dotted) and $a = 3.6$ (solid)

Figure 7. Contour $C^-$ corresponding to (7) for $a = 2.4$ (dotted) and $a = 3.6$ (solid)

flow is stable. Solid lines represent the velocity profile and the contour $C^-$ when $a = 3.6$. In this case the point $(0, 0)$ lies inside the contour $C^-$, so the flow is unstable.

In this way, stability conditions similar to Rayleigh’s (Statement 1) and Fjortoft’s (Statement 2) theorems [6] are formulated for the model (1).

4. Multilayered model

Now let us consider a class of flows with a piecewise linear velocity profile:

$$u = (y - y_{i-1})\Omega_i h + u_{i-1}, \quad y \in (y_{i-1}, y_i), \quad i = 1, ..., N. \tag{8}$$

Here the vorticities $\Omega_i$ in every single layer $\eta_i = y_i - y_{i-1}$ are constant; the values $u_i, v_i$ are the velocity vector components at $y = y_i$ ($y_0 = 0, y_N = Y$). At the border between layers the following kinematic conditions hold:

$$\frac{\partial y_i}{\partial t} + u_i \frac{\partial y_i}{\partial x} = v_i, \quad i = 1, ..., N + 1. \tag{9}$$

The substitution of the profile (8) into equations (1) result in a system of $N + 1$ equations:

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad \frac{\partial m_i}{\partial t} + \frac{\partial (\bar{u}_i m_i)}{\partial x} = 0, \tag{10}$$

that is used to determine the mass $m_i = \eta_i h$ of the fluid in each layer and velocity $u_0$ at the channel border $y = 0$. The depth of the fluid $h$ and the average velocities in each layer $\bar{u}_i = (u_i + u_{i-1})/2$ are given by formulae:

$$h = \frac{1}{V} \sum_{j=1}^{N} m_j, \quad \bar{u}_i = u_0 - \frac{1}{2} \Omega_i m_i + \sum_{j=1}^{i} \Omega_j m_j.$$

Horizontal velocities at the channel borders are calculated using the formula $u_i = y_i h \Omega_i + u_{i-1}$.
To analyse the stability of a flow in the framework of the model (10), we also use the more general result presented in [1]. Considering the piecewise linear velocity profile (8), the characteristic function (3) is rewritten as follows:

$$\chi(k) = 1 - \frac{g}{Y} \frac{1}{\Omega_1} u_0 - k + \frac{g}{Y} \frac{1}{\Omega_N} u_N - k - \frac{g}{Y} \sum_{i=1}^{N-1} \left( \frac{1}{\Omega_{i+1}} - \frac{1}{\Omega_i} \right) \frac{1}{u_i - k}. \quad (11)$$

Let us rewrite motion equations (10) in the form $U_t + A(U)U_x = 0$, where $U$ is the vector of unknown functions and $A$ is a square matrix of size $N + 1$. Then the equation that determines eigenvalues of $A$ is connected to the characteristic function $\chi(k)$ in the following way:

$$\det(A - kI) = \chi(k) \prod_{i=0}^{N}(u_i - k).$$

**Statement 4.** If all the layer vorticities $\Omega_i$ are ordered

$$\Omega_N < \Omega_{N-1} < \ldots < \Omega_1 \quad \text{or} \quad \Omega_N > \Omega_{N-1} > \ldots > \Omega_1 \quad (12)$$

then equations (10) are hyperbolic.

To prove this statement one has to analyse the behaviour of the function $\chi(k)$ given by equation (11) when the conditions (12) hold. The function $\chi(k)$ has one root inside each interval $k \in (-\infty, u_0)$ and $k \in (u_N, \infty)$, because $\chi(k) \to 1$ when $k \to \pm \infty$ and $\chi(k) \to -\infty$ when $k \to u_0 - 0$, and $\chi(k) \to \infty$ when $k \to u_N + 0$. Furthermore, $\chi(k)' < 0$ on the interval $(-\infty, u_0)$, and $\chi(k)' > 0$ on $(u_N, \infty)$. A similar analysis shows that $\chi(k)$ has $N - 1$ roots on the intervals $(u_i, u_{i+1})$, $i = 0, \ldots, N$. A common form of $\chi(k)$ is shown in Figure 8.

![Figure 8. Common form of $\chi(k)$ when $\Omega_1 > \ldots > \Omega_N > 0.$](image)

As in Section 3, conditions (12) resemble the Rayleigh’s inflexion point theorem. Note that a similar result has recently been obtained for flows within the multilayered approximation of Benney equations [9], where possible slippage between the layers is also taken into account.

As for an example let us consider a case where the conditions (12) do not hold and the flow might become unstable, like in Statement 3. We consider $N = 3$, because in case of $N = 2$ all the velocity profiles are convex and thus the flow is always stable. In case of $N = 3$ the characteristic equation (11) is rewritten as follows:

$$\chi(k) = 1 - \frac{g}{Y} \left( \frac{u_1 - u_0}{\Omega_1(u_1 - k)(u_0 - k)} + \frac{u_2 - u_1}{\Omega_2(u_2 - k)(u_1 - k)} + \frac{u_3 - u_2}{\Omega_3(u_3 - k)(u_2 - k)} \right).$$
For the sake of simplicity here we take $g = Y = 1$, $\eta_1 = \eta_3 = 0.35$, $\eta_2 = 0.3$ and $u_0 = 0$.

Now, for the first example let us consider the following distribution of vorticities $\Omega_1 = 1.2$, $\Omega_2 = 0.2$ $\Omega_3 = 0.6$. This vorticity distribution does not satisfy the condition (12). The velocity profile and the characteristic function of this flow are shown in Figure 9. Here we see that the characteristic function $\chi(k)$ has four real roots and we can conclude that the flow is stable, although the condition (12) does not hold.

For the second example let $\Omega_1 = 0.6$, $\Omega_2 = 1.2$ $\Omega_3 = 0.2$. Here the condition (12) also does not hold. The velocity profile and the characteristic function are shown in Figure 10. Here we see that the characteristic function $\chi(k)$ has only two real roots, and thus this flow is unstable.

**Figure 9.** Velocity profile (left) and $\chi(k)$ (right) when $\Omega_1 = 1.2$, $\Omega_2 = 0.2$ $\Omega_3 = 0.6$.

**Figure 10.** Velocity profile (left) and $\chi(k)$ (right) when $\Omega_1 = 0.6$, $\Omega_2 = 1.2$ $\Omega_3 = 0.2$. 
5. Conclusion
For horizontal shear flows of ideal fluid in an open channel we obtain stability conditions, which are derived in terms of the theory of generalized hyperbolicity. These conditions are similar to Rayleigh and Fjortoft theorems. In particular we show that flows with a monotonic convex profile are always stable, whereas flows with an inflexion point in the velocity profile might become unstable. Illustrative examples for both stable and unstable flows are given.

We also study a class of solutions that is used in a number of works [7, 8] and is described by a piecewise linear velocity profile. For this class we derive a multilayered model that is an approximation of the original model. We also formulate sufficient hyperbolicity conditions for this model, verification of which reduces to analysis of the characteristic function. Illustrative examples that feature both stable and unstable flows are shown.

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