On the admissibility of retarded delay systems

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Abstract

We investigate a Hilbert space dynamical system of the form $\dot{z}(t) = Az(t) + A_1z(t-\tau) + Bu(t)$, where $A$ generates a semigroup of contractions and $A_1$ is a bounded operator, in order to determine whether the operator $B$ is admissible. Our approach is based on the Miyadera–Voigt perturbation theorem and the Weiss conjecture on admissibility of control operators for contraction semigroups. We demonstrate that the retarded delay system can be represented as a well-posed abstract Cauchy problem with a solution formed by an initially log-concave bounded semigroup.

Keywords: admissibility, state delay, retarded dynamical systems, contraction semigroups

1. Introduction

In this article we analyse dynamical systems with delay in state variable from the perspective of admissibility of a control operator. The object of our interest is an abstract retarded system

$$\begin{cases}
\dot{z}(t) = Az(t) + A_1z(t-\tau) + Bu(t) \\
z(0) = z_0,
\end{cases}$$

(1)

where, initially, the closed, densely defined operator $A : D(A) \to X$, $D(A) \subset X$, is a generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ such that $T(t) \in \mathcal{L}(X)$ for every $t \geq 0$, where $X$ is a Hilbert state space, $B$ is the control operator acting on values of control functions $u \in L^2(J, U)$ with $J$ being a time interval and $U$ a Hilbert space.

A system of the form (1) without control input $u$ is frequently used as an example of a positive system describing population dynamics and, either in an abstract or PDE setting, is well analysed - see [6, Chapter VI.6] and references therein.

For a thorough presentation of admissibility results for state-undelayed systems we refer the reader to [10] and a rich list of references therein. In particular,
the results in [8] and [5] form a basis for considerations in [1] in terms of developing a correct setting, adapted also in this article, in which we conduct the admissibility analysis. The Weiss conjecture, based on which a necessary and sufficient condition for admissibility of hypercontractive semigroups was established in [9], was stated in [15].

In this paper we do not relate [11] to any particular positive system, but rather treat it as a general starting point for analysis of linear dynamical systems with delay influencing the state vector. We do not put any other assumptions on the undelayed semigroup $(T(t))_{t \geq 0}$ apart from being a contraction. This allows us to perform analysis in a relatively general case where the necessary and sufficient conditions for admissibility are known, and yet obtain concrete outcomes.

The results presented here form a basis for analysis of a specific case of delay system, namely state delay diagonal systems (full details concerning diagonal systems will be presented elsewhere [13]).

Section 2 contains the necessary background results, leading to the main results in Section 3. An example is given in Section 4, and some conclusions are given in Section 5.

2. Preliminaries

Apart from definitions introduced in the previous section, throughout this paper the notation $(X, \| \cdot \|_X)$ and $(U, \| \cdot \|_U)$ denotes Hilbert spaces with norms coming from appropriate inner products (this is also the case whenever the subscript is omitted). We use the following Sobolev spaces (see [11] for vector valued functions or [7, Chapter 5] for functionals):

$$H^1(\mathbb{J}, X) = W^{1,2}(\mathbb{J}, X) := \{ f \in L^2(\mathbb{J}, X) : \frac{d}{d\tau} f(t) \in L^2(\mathbb{J}, X) \},$$

$$H^1_c(\mathbb{J}, X) = W^{1,2}_c(\mathbb{J}, X) := \{ f \in H^1(\mathbb{J}, X) : f|_{\mathbb{J}\setminus S} = 0 \text{ for every compact } S \subset \mathbb{J} \}$$

and

$$H^1_0(\mathbb{J}, X) = W^{1,2}_0(\mathbb{J}, X) := \{ f \in H^1(\mathbb{J}, X) : f(\partial \mathbb{J}) = 0 \}.$$

2.1. The state delay equation setting

To describe a correct setting in which we will consider system [11], we follow [6, Chapter VI.6] and [1, Chapter 3.1]. Consider a function $z : [-\tau, \infty) \to X$. For each $t \geq 0$ we call the function $z_t : [-\tau, 0) \to X, z_t(\sigma) := z(t + \sigma)$, a history segment with respect to $t \geq 0$. With history segments we consider a function called the history function of $z$, that is $h_z : [0, \infty) \to L^2([-\tau, 0], X), h_z(t) := z_t$.

For the whole of the remaining part of this paper we normalize the delay $\tau$ to $\tau = 1$. In [1, Lemma 3.4] we find the following

**Proposition 2.1.** Let $z : [-1, \infty) \to X$ be a function which belongs to $H^1_{loc}([-1, \infty], X)$. Then the history function $h_z : t \to z_t$ of $z$ is continuously differentiable from $\mathbb{R}_+$ into $L^2([-1, 0], X)$ with derivative

$$\frac{d}{dt} h_z(t) = \frac{dz}{d\sigma} z_t.$$  

(2)
Define the Cartesian product \( \mathcal{X} := X \times L^2([-1, 0], X) \) with an inner product

\[
\left\langle \begin{pmatrix} x \\ f \end{pmatrix}, \begin{pmatrix} y \\ g \end{pmatrix} \right\rangle_{\mathcal{X}} := \langle x, y \rangle_X + \langle f, g \rangle_{L^2}.
\] (3)

Then \( \mathcal{X} \) becomes a Hilbert space \( (\mathcal{X}, \| \cdot \|_{\mathcal{X}}) \) with the norm \( \| \begin{pmatrix} x \\ f \end{pmatrix} \|_{\mathcal{X}}^2 = \| x \|_X^2 + \| f \|_{L^2}^2 \). Consider a linear, autonomous delay differential equation of the form

\[
\begin{aligned}
\dot{z}(t) &= A z(t) + \Psi z_t \\
z(0) &= x, \\
z_0 &= f,
\end{aligned}
\] (4)

where \( \Psi \in \mathcal{L}(H^1([-1, 0], X), X) \) is a delay operator, the pair \( x \in D(A) \) and \( f \in L^2([-1, 0], X) \) forms an initial condition. By Proposition 2.1 equation (4) may be written as an abstract Cauchy problem

\[
\begin{aligned}
\dot{v}(t) &= A v(t) \\
v(0) &= \begin{pmatrix} x \\ f \end{pmatrix},
\end{aligned}
\] (5)

where \( v : t \to \begin{pmatrix} z(t) \\ z_t \end{pmatrix} \in \mathcal{X} \) and \( A \) is an operator on \( \mathcal{X} \) defined as

\[
A := \begin{pmatrix} A & \Psi \\ 0 & \frac{d}{d\sigma} \end{pmatrix},
\] (6)

with domain

\[
D(A) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times H^1([-1, 0], X) : f(0) = x \right\}.
\] (7)

The operator \( (A, D(A)) \) is closed and densely defined on \( \mathcal{X} \) [1, Lemma 3.6]. Let \( A = A_0 + A_{\Psi} \), where

\[
A_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \quad D(A_0) = D(A),
\] (8)

and

\[
A_{\Psi} := \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X \times H^1([-1, 0], X), \mathcal{X}).
\] (9)

We recall the following Proposition from [1, Theorem 3.25], as we will later need the form of a semigroup generated by \( (A_0, D(A)) \).

**Proposition 2.2.** The following are equivalent:

(i) The operator \( (A, D(A)) \) generates a strongly continuous semigroup \( (T(t))_{t \geq 0} \) on \( X \).

(ii) The operator \( (A_0, D(A_0)) \) generates a strongly continuous semigroup \( (T_0(t))_{t \geq 0} \) on \( X \times L^p([-1, 0], X) \) for all \( 1 \leq p < \infty \).
The semigroup \((T_0(t))_{t \geq 0}\) is given by
\[
T_0(t) := \begin{pmatrix} T(t) & 0 \\ S_t & S_0(t) \end{pmatrix} \quad \forall t \geq 0,
\]
where \((S_0(t))_{t \geq 0}\) is the nilpotent left shift semigroup on \(L^p([-1,0], X)\),
\[
S_0(t)f(\tau) := \begin{cases} f(\tau + t) & \text{if } \tau + t \in [-1,0], \\ 0 & \text{otherwise} \end{cases}
\]
and \(S_t : X \to L^p([-1,0], X)\),
\[
(S_t x)(\tau) := \begin{cases} T(\tau + t)x & \text{if } -t < \tau \leq 0, \\ 0 & \text{if } -1 \leq \tau \leq -t. \end{cases}
\]

In order to make use of the Miyadera–Voigt Perturbation Theorem, we need the following

**Definition 2.3.** Let \(\beta \in \rho(A)\) and denote \((X_1, \|\cdot\|_1) := (D(A), \|\cdot\|_1)\) with \(\|\cdot\|_1 := \|\beta I - A\|\) \((x \in D(A))\).

Similarly, we set \(\|x\|_{-1} := \|\beta I - A\|^{-1}x\) \((x \in X)\). Then the space \((X_{-1}, \|\cdot\|_{-1})\) denotes the completion of \(X\) under the norm \(\|\cdot\|_{-1}\). For \(t \geq 0\) we define \(T_{-1}(t)\) as the continuous extension of \(T(t)\) to the space \((X_{-1}, \|\cdot\|_{-1})\).

In the sequel, much of our reasoning is justified by the following proposition, to which we do not refer directly but include here for the reader’s convenience.

**Proposition 2.4.** With notation of Definition 2.3 we have the following

(i) The spaces \((X_1, \|\cdot\|_1)\) and \((X_{-1}, \|\cdot\|_{-1})\) are independent of the choice of \(\beta \in \rho(A)\).

(ii) \((T_1(t))_{t \geq 0}\) is a strongly continuous semigroup on the Banach space \((X_1, \|\cdot\|_1)\) and we have \(\|T_1(t)\|_1 = \|T(t)\|\) for all \(t \geq 0\).

(iii) \((T_{-1}(t))_{t \geq 0}\) is a strongly continuous semigroup on the Banach space \((X_{-1}, \|\cdot\|_{-1})\) and we have \(\|T_{-1}(t)\|_{-1} = \|T(t)\|\) for all \(t \geq 0\).

See [6, Chapter II.5] or [14, Chapter 2.10] for more details on these elements. A sufficient condition for \(P \in \mathcal{L}(X_1, X)\) to be a perturbation of Miyadera-Voigt class, hence implying that \(A + P\) is a generator on \(X\), takes the form of [6, Corollaries III.3.15 and 3.16]

**Proposition 2.5.** Let \((A, D(A))\) be the generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\) and let \(P \in \mathcal{L}(X_1, X)\) be a perturbation which satisfies
\[
\int_0^t \|PT(r)x\|dr \leq q\|x\| \quad \forall x \in D(A)
\]
for some \(0 \leq q < 1\). Then the sum \(A + P\) with domain \(D(A + P) := D(A)\) generates a strongly continuous semigroup \((S(t))_{t \geq 0}\) on \(X\). Moreover, for all \(t \geq 0\) the semigroup \((S(t))_{t \geq 0}\) satisfies
\[
S(t)x = T(t)x + \int_0^t S(s)PT(t-s)xds \quad \forall x \in D(A).
\]
2.2. The admissibility problem

The basic object in the formulation of the admissibility problem is a linear system and its mild solution

\[ \frac{d}{dt} x(t) = Ax(t) + Bu(t); \quad x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \]  

where \( x : [0, \infty) \to X, \ u \in V \) where \( V \) is a space of measurable functions from \([0, \infty)\) to \( U \) and \( B \) is a control operator; \( x_0 \in X \) is an initial state.

In many practical examples the control operator \( B \) is unbounded. In such cases \((15)\) is viewed on an extrapolation space \( X^{-1} \supset X \), where \( B \in L(U, X^{-1}) \).

To ensure that the state \( x(t) \) lies in \( X \) it is sufficient that \( \int_0^t T(t-s)Bu(s)ds \in X \) for all inputs \( u \in V \). Put differently, we have the following

**Definition 2.6.** The control operator \( B \in L(U, X^{-1}) \) is said to be finite-time admissible for a semigroup \((T(t))_{t \geq 0}\) on a Hilbert space \( X \) if for each \( \tau > 0 \) there is a constant \( c(\tau) \) such that the condition

\[ \left\| \int_0^\tau T(\tau-s)Bu(s)ds \right\|_X \leq c(\tau)\|u\|_V \]  

holds for all inputs \( u \), and an infinite-time admissible if the condition \((16)\) holds for all \( \tau > 0 \) with \( c(\tau) \) uniformly bounded.

For contraction semigroups the following proposition was shown in \( [9] \):

**Proposition 2.7.** Let \((T(t))_{t \geq 0}\) be a \( C_0 \)-semigroup of contractions on a separable Hilbert space \( X \) with infinitesimal generator \( A \) and let \( B \in L(U, X_{-1}) \), where \( \dim U < \infty \). Then \( B \) is infinite-time admissible if and only if there exists a constant \( C > 0 \) such that the following resolvent condition holds

\[ \| (\lambda I - A)^{-1}B \| \leq \frac{C}{\sqrt{\text{Re} \lambda}} \quad \forall \lambda \in \mathbb{C}_+. \]  

**Remark 2.8.** Condition \((17)\), which is usually easier to check than admissibility itself has as a consequence the following observation that if the semigroup satisfies \( \| T(t) \| \leq e^{\omega t} \), so that \( A - \omega I \) generates a contraction semigroup, then finite-time admissibility for the pair \((A, B)\) follows from the resolvent condition

\[ \| (\lambda I - A)^{-1}B \| \leq \frac{C}{\sqrt{\text{Re} \lambda} - \omega} \quad \forall \text{Re} \lambda > \omega. \]  

The next result is a useful tool \([11]\) in many norm estimations:

**Theorem 2.9 (Sobolev Embedding Theorem).** Let \( X \) be a Banach space and \( 1 \leq p \leq \infty \), then there exists a constant \( C \) such that

\[ \| f \|_{L^\infty(J, X)} \leq C\| f \|_{W^{1, p}(J, X)} \]

for all \( f \in W^{1, p}(J, X) \), i.e. the embedding \( W^{1, p}(J, X) \hookrightarrow L^\infty(J, X) \) is continuous. Further, the inclusion \( W^{1, p}(J, X) \subset C_b(J, X) \) holds, where \( C_b(J, X) \) is the space of all continuous and bounded functions from \( J \) to \( X \) with the supremum norm.
3. Retarded non-autonomous dynamical systems

We begin with an analysis of retarded non-autonomous dynamical systems of the form

$$\begin{align*}
\dot{z}(t) &= Az(t) + \Psi z_{t} + Bu(t) \\
z(0) &= x, \\
z_{0} &= f,
\end{align*}$$

(19)

where all the elements are as in (4), $u \in L^{2}(0, \infty; U)$, $B$ is a control operator and the delay operator $\Psi \in \mathcal{L}(H^{1}([-1,0], X), X)$,

$$\Psi(f) := A_{1}f(-1),$$

(20)

with $A_{1} \in \mathcal{L}(X)$. Note that a generalization to the case $\Psi(f) := \sum_{k=1}^{n} A_{k}f(-h_{k})$ with $f \in H^{1}([-1,0], X)$, $A_{k} \in \mathcal{L}(X)$ and $h_{k} \in [0,1]$ for each $k = 1, \ldots, n$ is straightforward and will be omitted.

Following the procedure described in the Preliminaries section, for the system (19) we define a non-autonomous abstract Cauchy problem

$$\begin{align*}
\dot{v}(t) &= Av(t) + Bu(t) \\
v(0) &= (x, f),
\end{align*}$$

(21)

which we consider firstly on the space $X$ with $B = \left(\frac{\partial}{\partial \sigma}\right)$, and then on its completion $X_{-1}$ where the control operator $B \in \mathcal{L}(U, X_{-1})$.

The delay operator $\Psi$ defined in (20) is an example of a much wider class of delay operators, with which condition (13) is satisfied and $(A, D(A))$ remains a generator of a strongly continuous semigroup (see [1, Chapter 3.3.3]). Hence (21) is well-posed and we can formally write its $X_{-1}$-valued mild solution as

$$v(t) = T(t)v(0) + \int_{0}^{t} T(t-s)Bu(s)ds,$$

(22)

where $T(t) \in \mathcal{L}(X_{-1})$ is the extension of the semigroup generated by $(A, D(A))$, where the latter semigroup is given by the implicit formula (14). The remaining part is to find the space $X_{-1}$. We begin with determination of the adjoint $A^{*}$, with a reasoning similar to [2, Chapter A.3.64].

**Proposition 3.1.** Let $(A, D(A))$ be as defined by (9) and (7). Then its adjoint operator $A^{*}$ is given by

$$A^{*} := \begin{pmatrix} A^{*} & 0 \\ \Psi^{*} & -\frac{d}{d\sigma} \end{pmatrix},$$

(23)

with domain

$$D(A^{*}) := D(A^{*}) \times H_{0}^{1}([-1,0], X).$$

(24)

**Proof.** From $(A, D(A))$ being closed we have $D(A^{*}) \neq \emptyset$. Let now $v = (v^{*}) \in D(A)$ and $w = (w^{0}) \in D(A^{*})$ and denote $A_{0} := -\frac{d}{d\sigma}$. Then, by the definition of
adjoint operator
\[
\langle Av, w \rangle_X = \langle Ax + \Psi f, y \rangle_X + \langle A_0 f, g \rangle_{L^2} \\
= \langle x, A^* y \rangle_X + \langle f, \Psi^* y + A_0^* g \rangle_{L^2} = \langle v, A^* w \rangle_X. \tag{25}
\]
The calculation in (25) is correct provided that \(D(A^*)\) is defined in an appropriate way. Namely, assuming that \(D(A^*)\) is properly defined, we have to examine only the term
\[
\langle f, \Psi^* y + A_0^* g \rangle_{L^2} = \langle f, \Psi^* y - \frac{d}{dt} g \rangle_{L^2} = \int_{-1}^{0} \left\langle f(t), \Psi^* y(t) - \frac{d}{dt} g(t) \right\rangle_X dt
\]
\[
= \int_{-1}^{0} \left\langle f(t), \Psi^* y(t) \right\rangle_X dt - \int_{-1}^{0} \left\langle f(t), g(t) \right\rangle_X \left| \right. dt + \int_{-1}^{0} \left| \left. \frac{d}{dt} f(t), g(t) \right\rangle_X \right. dt. \tag{26}
\]
Since \(\Psi \in \mathcal{L}(H^1([-1,0], X), X)\) and \(X\) is a Hilbert space, the domain of \(\Psi^*\) is \(D(\Psi^*) = X\), where we identify \(X\) with its dual \(X'\). This results in
\[
\int_{-1}^{0} \left\langle f(t), \Psi^* y(t) \right\rangle_X dt = \langle \Psi f, y \rangle_X \int_{-1}^{0} dt = \langle \Psi f, y \rangle_X.
\]
The remaining term of (26) is
\[
- \left\langle f(t), g(t) \right\rangle_X \left| \right._{-1}^{0} + \int_{-1}^{0} \left| \left. \frac{d}{dt} f(t), g(t) \right\rangle_X \right. dt
\]
\[
= \langle f(-1), g(-1) \rangle_X - \langle f(0), g(0) \rangle_X + \int_{-1}^{0} \left| \left. \frac{d}{dt} f(t), g(t) \right\rangle_X \right. dt \tag{27}
\]
if and only if
\[
\langle f(-1), g(-1) \rangle_X - \langle x, g(0) \rangle_X = 0 \quad \forall v \in D(A). \tag{28}
\]
As \(x\) and \(f\) need only to be in \(D(A)\), for \(g\) to satisfy (28) for every \((\frac{df}{dt})\) \(D(A)\) it has to be \(g \in H^1([-1,0], X)\). As \(D(A^*) \subset X\) densely and \(H^1([-1,0], X) \subset L^2([-1,0], X)\) densely \(7\) we obtain \(D(A^*) := D(A^*) \times H^1([-1,0], X)\). \(\square\)

Due to the fact that \(X_{-1}\) is the dual to \(D(A^*)\) with respect to the pivot space \(X\), we may explicitly write
\[
X_{-1} = (X_{-1} \times H^{-1}([-1,0], X)) \tag{29}
\]
where \(H^{-1}([-1,0], X)\) is the dual to \(H^1([-1,0], X)\) with respect to the pivot space \(L^2([-1,0], X)\) - see \([14]\) Chapter 2.10 and Definition 13.4.7.
3.1. Contraction semigroups

As our main tool for admissibility analysis is expressed in Proposition 2.7 it is important to see if the delay semigroup \((T(t))_{t \geq 0}\) is hypercontractive, i.e., \(\|T(t)\| \leq e^{\omega t}\) for every \(t \geq 0\) and some \(\omega \in \mathbb{R}\). In the case when the operator \((A, D(A))\) in the retarded system (19) generates a contraction semigroup we start the analysis with the following

**Proposition 3.2.** Let \((T(t))_{t \geq 0}\) be a semigroup of contractions generated by \((A, D(A))\). Then the semigroup \((T_0(t))_{t \geq 0}\) generated by \((A_0, D(A_0))\) is hypercontractive and

\[
\|T_0(t)\| \leq e^{\frac{\omega}{2} t} \quad \forall t \geq 0.
\]

**Proof.** Fix \(t > 0\) and \(v = (\tau^T) \in \mathcal{X} \). We can calculate

\[
\|T_0(t)v\|_{X_2}^2 = \|T(t)x\|_{X_2}^2 + \|S_t x\|_{L_2}^2 + 2 \text{Re}\langle S_t x, S_t(t)f\rangle_{L_2} + \|S(t)f\|_{L_2}^2.
\]

The second term of (30) expands to

\[
\|S_t x\|_{L_2}^2 = \int_{-t}^{0} \langle T(\tau + t) x, T(\tau + t) x \rangle_{X} d\tau = \int_{0}^{t} \langle T(\tau) x, T(\tau) x \rangle_{X} d\tau,
\]

while the fourth one expands to

\[
\|S(t)f\|_{L_2}^2 = \int_{-1}^{0} \langle (S_0(t)f)(\tau), (S_0(t)f)(\tau) \rangle_{X} d\tau = \int_{-1+t}^{0} \langle f(\tau), f(\tau) \rangle_{X} d\tau
\]

if \(t \in [0, 1]\) and \(\|S_0(t)f\|_{L_2} = 0\) if \(t > 1\). As for the third term note that according to the definition \((S_t x)(\tau) = 0\) for \(\tau \in [-1, -t]\) and \((S_0(t)f)(\tau) = 0\) for \(\tau \in (-t, \infty)\). Hence,

\[
2 \text{Re}\langle S_t x, S_0(t)f\rangle_{L_2} = 2 \text{Re} \int_{-1}^{0} \langle (S_t x)(\tau), (S_0(t)f)(\tau) \rangle_{X} d\tau = 0
\]

for all \(t \geq 0\). The contraction assumption now gives the following estimation

\[
\|T_0(t)v\|_{X_2}^2 \leq \|T(t)x\|_{X_2}^2 + \int_{0}^{t} \|T(\tau)x\|_{X_2}^2 d\tau + \int_{-1+t}^{0} \langle f(\tau), f(\tau) \rangle_{X} d\tau
\]

\[
\leq \|x\|_{X_2}^2 + t\|x\|_{X_2}^2 + \|f\|_{L_2}^2 \leq (1 + t)(\|x\|_{X_2}^2 + \|f\|_{L_2}^2) < e^{\frac{\omega}{2} t} \|v\|_{X_2}^2.
\]

\[
\square
\]

Proposition 3.2 opens up a wide field of applications of perturbation and approximation of semigroups results. We will continue to follow the Miyadera–Voigt approach given in Proposition 2.7.

**Proposition 3.3.** Let \((T(t))_{t \geq 0}\) be the semigroup of contractions generated by \((A, D(A))\), \((T_0(t))_{t \geq 0}\) be the semigroup generated by \((A_0, D(A_0))\) and suppose...
that \((A_\Psi, D(A_\Psi))\) is the perturbing operator. Then for the semigroup \((T(t))_{t\geq0}\) generated by \((A_0 + A_\Psi, D(A_0))\) the inequality
\[
\|T(t)\| \leq e^{\frac{1}{2}t}(1 + \|A_1\|Mt^{\frac{1}{2}}) \quad \forall t \in [0,1]
\] (31)
holds, where \(A_1\) comes from \((20)\) and \(M \leq \sqrt{2e^{2\|A_1\|^2}}\).

**Proof.** 1. From Proposition (2.5) the semigroup \((T(t))_{t\geq0}\) is given by
\[
T(t)v = T_0(v) + \int_0^t T(s)A_\Psi T_0(t-s)vds \quad \forall t \geq 0, \forall v \in D(A_0).
\] (32)
Due to Proposition (3.2) the operator \((A_0 - \frac{1}{2}I, D(A_0))\) generates a contraction semigroup \((T_1(t))_{t\geq0}\) on \(X\), where \(T_1(t) = e^{-\frac{1}{2}t}T_0(t)\) for all \(t \geq 0\). In consequence, the operator \((A_0 + A_\Psi - \frac{1}{2}I, D(A_0))\) generates a semigroup \((T_r(t))_{t\geq0}\) on \(X\) where \(T_r(t) = e^{-\frac{1}{2}t}T(t)\) for all \(t \geq 0\). Equation (32) for the rescaled semigroup \(T_r(t)\) and \(v = (r) \in D(A_0)\) reads
\[
T_r(t)v = T_1(v) + \int_0^t T_r(s)A_\Psi e^{-\frac{1}{2}(t-s)}T_0(t-s)vds \\
= T_1(v) + \int_0^t e^{-\frac{1}{2}(t-s)}T_r(s)\left(\Psi(S_{t-s}x) + \Psi(S_0(t-s)f)\right)ds
\] (33)
2. Before estimating the norm of \(T_r(t)\) consider the following
\[
\|\Psi(S_{t-s}x) + \Psi(S_0(t-s)f)\|_X \\
= \|A_1(S_{t-s})x(-1) + A_1(S_0(t-s)f)(-1)\|_X \\
\leq \|A_1T(-1 + t - s)x\|_X + \|A_1f(-1 + t - s)\|_X \\
\leq \|A_1\||x|_X + \|A_1\||f(-1 + t - s)\|_X
\] (34)
where we denote by the same symbol a continuous bounded representative of \(f \in H^1([-1,0], X)\). Because of the Sobolev Embedding Theorem (2.3) we know that such representative exists. Due to the Hölder inequality and again the Sobolev Embedding Theorem we have also
\[
\int_0^t \|f(-1 + t - s)\|_X ds = \int_{-1}^{-1+t} \|f(s)\|_X ds \\
\leq t^{\frac{1}{2}} \left(\int_{-1}^{-1+t} \|f(s)\|_X^2 ds\right)^{\frac{1}{2}} \leq t^{\frac{1}{2}} \|f\|_{L^2}.
\] (35)
3. Fix \( v = (\frac{1}{t}) \in D(A_0) \) and let \( t \in [0, 1] \). Using above results we have

\[
\| T_r(t)v \| \leq \| T_I(t)v \| + \int_0^t \| T_r(s)A_0 e^{-\frac{t-s}{t}} T_0(t-s)v \| ds
\]

\[
\leq \| T_I(t)v \| + \int_0^t \| T_r(s) \| \left( \| A_1 \| \| x \|_X + \| A_1 \| \| f(-1 + t - s) \|_X \right) ds
\]

\[
\leq \| v \| + \| A_1 \| \| x \|_X \int_0^t \| T_r(s) \| ds + \| A_1 \| \int_0^t \| T_r(s) \| \| f(-1 + t - s) \|_X ds
\]

\[
\leq \| v \| + \| A_1 \| \| x \|_X M t + \| A_1 \| M \int_0^t \| f(-1 + t - s) \|_X ds
\]

\[
\leq \| v \| + \| A_1 \| M (t \| x \|_X + t^2 \| f \|_{L^2}) \leq (1 + \| A_1 \| M t^2) \| v \|
\]

where \( M := \max \{ \| T_r(s) \| : s \in [0, 1] \} \).

4. Consider a square of norm estimation resulting from (33), namely

\[
\| T_r(t)v \|^2 = \left( \| T_I(t)v \| + \int_0^t \| T_r(s)A_0 e^{-\frac{t-s}{t}} T_0(t-s)v \| ds \right)^2
\]

\[
\leq 2 \| T_I(t)v \|^2 + 2 \left( \int_0^t \| T_r(s) \| \left( \| A_1 \| \| x \|_X + \| A_1 \| \| f(-1 + t - s) \|_X \right) ds \right)^2
\]

\[
\leq 2 \| v \|^2 + 4 \| A_1 \|^2 \left( \int_0^t \| T_r(s) \| ds \right)^2 +
\]

\[
+ 4 \| A_1 \|^2 \left( \int_0^t \| T_r(s) \| \| f(-1 + t - s) \|_X ds \right)^2
\]

\[
\leq 2 \| v \|^2 + 4 \| A_1 \|^2 \left( t \| x \|_X^2 + \| f \|_{L^2}^2 \right) \int_0^t \| T_r(s) \|^2 ds,
\]

where we used the Hölder inequality twice. As \( t \in [0, 1] \) we have \( t \| x \|_X^2 + \| f \|_{L^2}^2 \leq \| x \|_X^2 + \| f \|_{L^2}^2 = \| v \|^2 \), and the above estimation gives

\[
\| T_r(t) \|^2 \leq 2 + 4 \| A_1 \|^2 \int_0^t \| T_r(s) \|^2 ds.
\]

(36)

The Grönwall–Bellman lemma (see, for example, [12] or [4] for an exposition of such inequalities) now gives

\[
\| T_r(t) \|^2 \leq 2 e^{4 \| A_1 \|^2 t} \quad \forall t \in [0, 1].
\]

Hence,

\[
\| T_r(t) \| \leq \sqrt{2} e^{2 \| A_1 \|^2 t} \quad \forall t \in [0, 1],
\]

(37)

and we obtain that \( M \leq \sqrt{2} e^{2 \| A_1 \|^2} \).

5. Getting back to the original delay semigroup \( T(t) \) we finish the proof.
**Corollary 3.4.** Under assumptions of Proposition 3.3 the rescaled semigroup $T_r(t)$ is initially log-concave bounded, that is there exists $v : [0, 1] \to [0, \infty)$ such that $v(t) := \log(N(t)) \geq \log(\|T_r(t)\|)$ for some function $N : [0, 1] \to \mathbb{R}_+$. □

With Proposition 2.7 we may state a necessary and sufficient condition for finite time admissibility of the retarded system given by (19), namely

**Theorem 3.5.** Using the previously defined notation for the retarded non-autonomous dynamical system (19) let the control operator $B := (B_0) \in L(U, X_{-1})$, where $\dim U < \infty$, and there exist $\eta > 0$ and $\omega < \infty$ such that the inequality

$$\|T(t)\| \leq e^{\omega t} \quad \forall t \in (0, \eta)$$

(38)

holds. Then the control operator $B$ is finite-time admissible if and only if there exists a constant $C > 0$ such that the following resolvent condition holds

$$\|(\lambda I - A)^{-1} B\| \leq \frac{C}{\sqrt{\text{Re} \lambda - \omega}} \quad \forall \text{Re} \lambda > \omega.$$

**Proof.** The proof of this theorem follows from Proposition 2.7, Remark 2.8, Proposition 3.3 and semigroup property. □

Note that (38) in Theorem 3.5 does not follow from Proposition 3.3. The necessary and sufficient condition for (38) to hold is

$$\text{Re} \langle Av, v \rangle_X \leq \omega \quad \forall v \in D(A).$$

(39)

Under the relatively weak assumptions made by us (in fact in Theorem 3.5 as in this whole subsection, we assume only the contraction property of the undelayed semigroup $T(t)$ and a simple form of the delay operator $\Psi$) condition (39) takes the form

$$\text{Re} \langle A_1 f(-1), f(0) \rangle_X \leq \omega \quad \forall \left( \begin{array}{c} x \\ f \end{array} \right) \in D(A),$$

(40)

and whether one can draw conclusions on hypercontractivity under such weak assumptions remains an open problem.

A natural way of strengthening the result of Proposition 3.3 and thus Theorem 3.5 would be to add a condition on the differentiability of $T : [0, \eta) \to (L(X), \|\cdot\|_{L(X)})$ in the form

$$\limsup_{t \to 0^+} \frac{d}{dt} \|T(t)\|_{L(X)} < \infty.$$  (41)

However, the question of what properties the undelayed semigroup $T(t)$ must have so that the conclusion (41) can be drawn, remains open in the setting of this paper.

A noticeable fact is that Corollary 3.4 says that the set of log-concave bounds for the delayed semigroup $T(t)$ is not empty. Hence, another way one may look at the hypercontractivity problem is given in [3] by means of the upper log-concave envelope of $\|T(t)\|$.
4. Example

As an example of a retarded dynamical system consider a Lotka–Scharpe or the McKendrick–von Foerster equation as in [1, Example 3.16]. In general, it may be seen as describing a population aging with delay, where the delay can be a result of measuring time or cell development.

\[
\begin{align*}
\frac{\partial}{\partial t} z(t, s) + \frac{\partial}{\partial s} z(t, s) &= -\mu(s) z(t, s) + \nu(s) z(t - 1, s), \quad t \geq 0, \ s \in \mathbb{R}_+ \\
z(t, 0) &= \int_0^\infty \beta(r) z(t, r) dr, \quad t \geq 0, \\
z(t, s) &= f(t, s), \quad (t, s) \in [-1, 0] \times \mathbb{R}_+,
\end{align*}
\]  

where \( \mu, \nu, \beta \in L^\infty(\mathbb{R}_+) \); \( \mu, \beta \) are positive and \( f \) is in \( H^1([-1, 0] \times \mathbb{R}_+) \). In the abstract setting we may specify:

- the Banach space \( X := L^2(\mathbb{R}_+) \),
- the operator \( (Ag)(s) := -g'(s) - \mu(s)g(s), \ s \in \mathbb{R}_+ \) with the domain \( D(A) := \{ g \in H^1(\mathbb{R}_+) : g(0) = \int_0^\infty \beta(r)g(r) dr \} \);

we see that \( A \) generates a contraction semigroup by applying the perturbation result in [6, Theorem III.2.7],

- the delay operator \( \Psi : H^1([-1, 0], X) \to X \) defined as \( \Psi(f) := \nu f(-1) \).

With the above definitions we obtain an autonomous abstract system, to which we can apply a suitable control signal and obtain a well-posed abstract Cauchy problem (42) representing a system of the form (19).

5. Conclusions

The admissibility analysis of the retarded delay system with bounded \( A_1 \) operator presented in this paper is a good starting point in the admissibility analysis of other state-delayed systems. In our future work particular attention among such systems will be paid to systems which have a well-known structure giving additional insight, such as diagonal systems.

The admissibility results obtained here are also a natural starting point for the analysis of controllability or observability of state-delayed systems.

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