Abstract. Let $H_V = -\Delta + V$ be a Schrödinger operator on an arbitrary open set $\Omega$ of $\mathbb{R}^d$, where $d \geq 3$, and $\Delta$ is the Dirichlet Laplacian and the potential $V$ belongs to the Kato class on $\Omega$. The purpose of this paper is to show $L^p$-boundedness of an operator $\varphi(H_V)$ for any rapidly decreasing function $\varphi$ on $\mathbb{R}$. $\varphi(H_V)$ is defined by the spectral theorem. As a by-product, $L^p$-$L^q$-estimates for $\varphi(H_V)$ are also obtained.

1. Introduction and main result

Let $\Omega$ be an open set of $\mathbb{R}^d$, where $d \geq 3$. We consider the Schrödinger operator

$$H_V = H + V(x),$$

where

$$H := -\Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$$

is the Dirichlet Laplacian with domain

$$\mathcal{D}(H) = \{u \in H^1_0(\Omega) \mid \Delta u \in L^2(\Omega)\}$$

and $V(x)$ is a real-valued measurable function on $\Omega$. If we impose an appropriate assumption on $V(x)$, $H_V$ will be a self-adjoint operator on $L^2(\Omega)$. Let $\{E_{H_V}(\lambda)\}_{\lambda \in \mathbb{R}}$ be the spectral resolution of the identity for $H_V$. Then $H_V$ is written as

$$H_V = \int_{-\infty}^{\infty} \lambda dE_{H_V}(\lambda).$$

Hence, for any Borel measurable function $\varphi$ on $\mathbb{R}$, we can define an operator $\varphi(H_V)$ by letting

$$\varphi(H_V) = \int_{-\infty}^{\infty} \varphi(\lambda) dE_{H_V}(\lambda).$$

This operator is initially defined on $L^2(\Omega)$. The present paper is devoted to investigation of functional calculus for Schrödinger operators on $\Omega$. More precisely, our purpose is to prove that $\varphi(H_V)$ is extended uniquely to a bounded linear operator on $L^p(\Omega)$ for $1 \leq p \leq \infty$ and that $L^p$-boundedness of $\varphi(\theta H_V)$ is uniform with respect to a parameter $\theta > 0$. 

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When $\Omega = \mathbb{R}^d$, Simon considered the Kato class $K_d$ of potentials to reveal $L^p$-mapping properties of the Schrödinger operators $H_V$ and $e^{-tH_V}$ for $t > 0$ (see [9, Section A.2]). We now define a Kato class $K_d(\Omega)$ on an open set $\Omega$ as follows: We say that a real-valued measurable function $V$ on $\Omega$ belongs to the class $K_d(\Omega)$ if
\[
\lim_{r \to 0} \sup_{x \in \Omega} \int_{\Omega \cap \{ |x - y| < r \}} \frac{|V(y)|}{|x - y|^{d-2}} dy = 0.
\]
Throughout this paper, defining the “Kato norm”:
\[
\|V\|_{K_d(\Omega)} := \sup_{x \in \Omega} \int_{\Omega} \frac{|V(y)|}{|x - y|^{d-2}} dy,
\]
we impose an assumption on $V$ as follows:

**Assumption A.** Let $d \geq 3$. A real-valued measurable function $V(x)$ on $\Omega$ is decomposed into $V = V_+ - V_-$, $V_\pm \geq 0$, belongs to $K_d(\Omega)$ and satisfies
\[
\|V_-\|_{K_d(\Omega)} < \frac{\pi^{d/2}}{\Gamma(d/2 - 1)},
\]
where $\Gamma(s)$ is the Gamma function for $s > 0$.

If the potential $V$ satisfies assumption A, it is proved in Proposition 2.1 that $H_V$ is non-negative and self-adjoint on $L^2(\Omega)$ (see §2). For a Borel measurable function $\varphi$ on $\mathbb{R}$, we define the operator $\varphi(H_V)$ on $L^2(\Omega)$ by letting
\[
D(\varphi(H_V)) = \left\{ f \in L^2(\Omega) \mid \int_0^\infty |\varphi(\lambda)|^2 d\langle E_{H_V}(\lambda)f, f \rangle_{L^2(\Omega)} < \infty \right\},
\]
\[
\langle \varphi(H_V)f, g \rangle_{L^2(\Omega)} = \int_0^\infty \varphi(\lambda) d\langle E_{H_V}(\lambda)f, g \rangle_{L^2(\Omega)}, \quad \forall f \in D(\varphi(H_V)), \forall g \in L^2(\Omega),
\]
where $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ stands for the inner product in $L^2(\Omega)$. Formally we write
\[
\varphi(H_V) = \int_0^\infty \varphi(\lambda)dE_{H_V}(\lambda).
\]

In this paper we denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from a Banach space $X$ to another one $Y$. When $X = Y$, we denote by $\mathcal{B}(X) = \mathcal{B}(X, X)$. We use the notation $D(T)$ for the domain of an operator $T$.

Denoting by $\mathcal{S}(\mathbb{R})$ the space of rapidly decreasing functions on $\mathbb{R}$, we shall prove here the following:

**Theorem 1.1.** Let $d \geq 3$, $\varphi \in \mathcal{S}(\mathbb{R})$ and $1 \leq p \leq \infty$. Assume that the measurable potential $V$ satisfies assumption A. Then there exists a constant $C = C(d, \varphi) > 0$ such that
\[
\|\varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega))} \leq C
\]
for any $\theta > 0$. 
Let us give a few remarks on Theorem 1.1. We have restricted the result in this theorem to high space dimensions. So, one would expect the result to hold also for low space dimensions, i.e., $d = 1, 2$. However, in the present paper we use the pointwise estimates for kernel of $e^{-t\Delta}$ on $\mathbb{R}^d$ that D’Ancona and Pierfelice proved for $d \geq 3$ (see [2]). Hence low dimensional cases will be a future problem. When $V = 0$, Theorem 1.1 also holds in the cases $d = 1, 2$ by using the pointwise estimates for classical heat kernel of $e^{t\Delta}$.

One can easily see that $\varphi(H_V)$ is bounded on $L^2(\Omega)$ via direct application of the spectral resolution (1.2). From the point of view of harmonic analysis, it would be important to obtain $L^p$-boundedness ($p \neq 2$). For instance, Theorem 1.1 provides a generalization of $L^p$-boundedness for the Fourier multiplier in $\mathbb{R}^d$:

$$\left\| \mathcal{F}^{-1}\left[\hat{\varphi}(\theta \cdot \cdot \cdot)^2 \right]\right\|_{L^p(\mathbb{R}^d)} \leq C_p \| f \|_{L^p(\mathbb{R}^d)}, \quad \forall \theta > 0,$$

where $\varphi \in \mathfrak{S}(\mathbb{R}^d)$, $\hat{\cdot}$ denotes the Fourier transform, and $\mathcal{F}^{-1}$ is the Fourier inverse transform. $L^p$-boundedness of $\varphi(\theta H_V)$ also plays a fundamental role in defining the Besov spaces generated by $H_V$ (see, e.g., [2, 4, 6]). Thus Theorem 1.1 would be a starting point of the study of spectral multiplier and Besov spaces on open sets.

When $\Omega = \mathbb{R}^d$, there are some known results on uniform $L^p$-estimates for $\varphi(\theta H_V)$ with respect to $\theta$. For $0 < \theta \leq 1$, Jensen and Nakamura proved the uniform estimates for $d \geq 1$, under the assumption that the potential $V = V_+ - V_-, V_\pm \geq 0$, satisfies $V_+ \in K^d_\text{loc}^{(\mathbb{R}^d)}$ and $V_- \in K_d(\mathbb{R}^d)$ (see [6, 7]). Here $K^d_\text{loc}^{(\mathbb{R}^d)}$ is the local Kato class, which is the space of all $f \in L^1_\text{loc}(\mathbb{R}^d)$ such that $f$ belongs to the Kato class on any compact set in $\mathbb{R}^d$. For $\theta > 0$, Georgiev and Visciglia proved the uniform estimates under the assumption that the potential $V$ satisfies

$$0 \leq V(x) \leq \frac{C}{|x|^2(|x|^\varepsilon + |x|^{-\varepsilon})} \quad (C > 0, \varepsilon > 0)$$

when $d = 3$ (see [4]). D’Ancona and Pierfelice proved the uniform estimates for $d \geq 3$, under the assumption that the potential $V = V_+ - V_-, V_\pm \geq 0$, satisfies $V_\pm \in K_d(\mathbb{R}^d)$ and

$$\|V_-\|_{K_d(\mathbb{R}^d)} \leq \frac{\pi^{d/2}}{\Gamma(d/2 - 1)}$$

(see [2]). As far as we know, Theorem 1.1 is new in the sense that there would not be no results on $L^p$-estimates for $\varphi(H_V)$ in open sets.

Let us overview the strategy of proof of Theorem 1.1. For the sake of simplicity, we consider the case $V \equiv 0$, since the case $V \neq 0$ is similar. The original idea of proof of $L^p$-boundedness goes back to Jensen and Nakamura [7]. The method for the boundedness of $\varphi(\theta \Delta)$ is to use the amalgam spaces $\ell^p(L^q)$, pointwise estimates for the kernel of $e^{-t\Delta}$ and the commutator estimates for $-\Delta$ and polynomials. As to the uniform boundedness of $\varphi(\theta \Delta)$ with respect to $\theta$, the estimates for operator $\varphi(-\Delta)$ are reduced to those on $\varphi(-\Delta)$ via the following equality

$$\varphi(-\Delta)(\theta)(x) = \left(\varphi(-\Delta)\left(f(\theta^{1/2}x)\right)\right)(\theta^{-1/2}x), \quad x \in \mathbb{R}^d, \quad \theta > 0$$ (1.4)
(see [2, 4, 6]). There, scaling invariance of $\mathbb{R}^d$, i.e., $\mathbb{R}^d = \theta^{1/2}\mathbb{R}^d$, plays an essential role in the argument. On the other hand, when one tries to get (1.4) on open sets $\Omega \subsetneq \mathbb{R}^d$, the scaling invariance breaks down, i.e., $\Omega \neq \theta^{1/2}\Omega$. To avoid this problem, we shall introduce the scaled amalgam spaces $\ell^p(L^q)_\theta(\Omega)$ to estimate the operator norm of $\varphi(-\theta\Delta)$ directly. A scale exponent $1/2$ in $\theta^{1/2}$ of the spaces $\ell^p(L^q)_\theta(\Omega)$ is chosen to fit the scale exponent of the operator $\varphi(-\theta\Delta)$; thus we define the scaled amalgam spaces as follows:

**Definition 1.2** (Scaled amalgam spaces $\ell^p(L^q)_\theta$). Let $1 \leq p, q \leq \infty$ and $\theta > 0$. The space $\ell^p(L^q)_\theta$ is defined by letting

$$\ell^p(L^q)_\theta = \ell^p(L^q)_\theta(\Omega) := \left\{ f \in L^q_{\text{loc}}(\Omega) \mid \sum_{n \in \mathbb{Z}^d} \| f \|_{L^q(C_\theta(n))}^p < \infty \right\}$$

with norm

$$\| f \|_{\ell^p(L^q)_\theta} = \begin{cases} \left( \sum_{n \in \mathbb{Z}^d} \| f \|_{L^q(C_\theta(n))}^p \right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \sup_{n \in \mathbb{Z}^d} \| f \|_{L^q(C_\theta(n))} & \text{for } p = \infty, \end{cases}$$

where $C_\theta(n)$ is the cube centered at $\theta^{1/2}n \in \theta^{1/2}\mathbb{Z}^d$ with side length $\theta^{1/2}$;

$$C_\theta(n) = \left\{ x = (x_1, x_2, \ldots, x_d) \in \Omega \mid \max_{j=1,\ldots,d} |x_j - \theta^{1/2}n_j| \leq \frac{\theta^{1/2}}{2} \right\}.$$ 

Here we adopt the Euclidean norm for $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$;

$$|n| = \sqrt{n_1^2 + n_2^2 + \cdots + n_d^2}.$$ 

It can be checked that $\ell^p(L^q)_\theta$ is a Banach space with norm $\| \cdot \|_{\ell^p(L^q)_\theta}$ having the property that

$$\ell^p(L^q)_\theta \hookrightarrow L^p(\Omega) \cap L^q(\Omega)$$

for $1 \leq p \leq q \leq \infty$.

This paper is organized as follows. In §2 the self-adjointness of Schrödinger operator $H_V$ is shown. In §3 we prepare the pointwise estimate for the kernel of $e^{-tH_V}$. §4 is devoted to the uniform estimates in $\ell^p(L^q)_\theta$ for the resolvent of $H_V$. In §5 we derive the commutator estimates for our problem in the open set $\Omega$. In §6 the proof of Theorem 1.1 is given. As a by-product of Theorem 1.1, $L^p-L^q$-boundedness for $\varphi(H_V)$ is proved in §7.

**2. Self-adjointness of Schrödinger operators**

In this section we show that operator $H_V$ is self-adjoint and non-negative under assumption A.

Our purpose is to prove the following.
**Proposition 2.1.** Let $d \geq 3$. Assume that the measurable potential $V$ is a real-valued function on $\Omega$ and satisfies $V = V_+ - V_-$, $V_\pm \geq 0$ such that $V_\pm \in K_d(\Omega)$ and

\begin{equation}
\|V_-\|_{K_d(\Omega)} < \frac{4\pi^{d/2}}{\Gamma(d/2 - 1)}.
\end{equation}

Let $H_V$ be the operator with domain

\[ D(H_V) = \{ u \in H^1_0(\Omega) \mid H_Vu \in L^2(\Omega) \}, \]

so that

\begin{equation}
\langle H_Vu, v \rangle_{L^2(\Omega)} = \int_{\Omega} \nabla u(x) \cdot \nabla \overline{v(x)} \, dx + \int_{\Omega} V(x) u(x) \overline{v(x)} \, dx
\end{equation}

for any $u \in D(H_V)$ and $v \in H^1_0(\Omega)$. Then $H_V$ is non-negative and self-adjoint on $L^2(\Omega)$.

We need a notion of quadratic forms on Hilbert spaces (see p.276 in Reed and Simon [8]).

**Definition 2.2.** Let $H$ be a Hilbert space with the norm $\| \cdot \|$. A quadratic form is a map $q : Q(q) \times Q(q) \to \mathbb{C}$, where $Q(q)$ is a dense linear subset in $H$ called the form domain, such that $q(\cdot, v)$ is conjugate linear and $q(u, \cdot)$ is linear for $u, v \in Q(q)$. We say that $q$ is symmetric if $q(u, v) = q(v, u)$. A symmetric quadratic form $q$ is non-negative if $q(u, u) \geq 0$ for any $u \in Q(q)$. A non-negative quadratic form $q$ is closed if $Q(q)$ is complete with respect to the norm:

\begin{equation}
\|u\|_{+1} := \sqrt{q(u, u) + \| u \|^2}.
\end{equation}

The proof of Proposition 2.1 is done by using the following two lemmas.

**Lemma 2.3.** Let $H$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and let $q : Q(q) \times Q(q) \to \mathbb{C}$ be a densely defined semi-bounded closed quadratic form. Then there exists a self-adjoint operator $T$ on $H$ uniquely such that

\[ D(T) = \{ u \in Q(q) \mid \exists w_u \in H \text{ such that } q(u, v) = \langle w_u, v \rangle, \forall v \in Q(q) \}, \]

\[ Tu = w_u, \quad u \in D(T). \]

We note that $D(T)$ can be simply written as

\[ D(T) = \{ u \in Q(q) \mid Tu \in H \}. \]

For the proof of Lemma 2.3, see [8, Theorem VIII.15].

The following lemma states that $V_\pm$ are relatively form bounded with respect to the Dirichlet Laplacian $-\Delta$.

**Lemma 2.4.** Let $V_+$ and $V_-$ be as in Proposition 2.1. Then for any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that the following estimates hold:

\begin{equation}
\int_{\Omega} V_+(x)|u(x)|^2 \, dx \leq \varepsilon \| \nabla u \|^2_{L^2(\Omega)} + b_\varepsilon \| u \|^2_{L^2(\Omega)},
\end{equation}

\begin{equation}
\int_{\Omega} V_-(x)|u(x)|^2 \, dx \leq \varepsilon \| \nabla u \|^2_{L^2(\Omega)} + b_\varepsilon \| u \|^2_{L^2(\Omega)}.
\end{equation}
\begin{equation}
\int_{\Omega} V_- (x) |u(x)|^2 \, dx \leq \frac{\|V_-\|_{K_d(\Omega)} \Gamma(d/2 - 1)}{4\pi^{d/2}} \|\nabla u\|^2_{L^2(\Omega)}
\end{equation}

for any \( u \in H^1_0(\Omega) \).

**Proof.** The proof is similar to that of Lemma 3.1 from [2]. Let \( u \in C^\infty_0(\Omega) \), and let \( \tilde{u} \) and \( \tilde{V}_\pm \) be the zero extensions of \( u \) and \( V_\pm \) to \( \mathbb{R}^d \), respectively. First, we prove that for any \( \varepsilon > 0 \), there exists a constant \( b_\varepsilon > 0 \) such that

\begin{equation}
\int_{\mathbb{R}^d} \tilde{V}_+ (x) |\tilde{u}(x)|^2 \, dx \leq \varepsilon \|\nabla \tilde{u}\|^2_{L^2(\mathbb{R}^d)} + b_\varepsilon \|\tilde{u}\|^2_{L^2(\mathbb{R}^d)}.
\end{equation}

We divide the proof of (2.6) into two cases: \( d = 3 \) and \( d > 3 \). When \( d = 3 \), the inequality (2.6) is equivalent to

\begin{equation*}
\int_{\mathbb{R}^3} \tilde{V}_+ (x) |\tilde{u}(x)|^2 \, dx \leq \varepsilon \langle \tilde{u}, -\Delta \tilde{u}\rangle_{L^2(\mathbb{R}^3)} + b_\varepsilon \|\tilde{u}\|^2_{L^2(\mathbb{R}^3)}
\end{equation*}

\begin{equation*}
= \varepsilon \left\| \left( H_0 + \frac{b_\varepsilon}{\varepsilon} \right)^{1/2} \tilde{u} \right\|^2_{L^2(\mathbb{R}^3)},
\end{equation*}

where \( H_0 = -\Delta \) is the self-adjoint operator with domain \( H^2(\mathbb{R}^3) \). Put

\[ v = \left( H_0 + \frac{b_\varepsilon}{\varepsilon} \right)^{1/2} \tilde{u}. \]

Then the estimate (2.6) takes the following form:

\[ \left\| \tilde{V}_+^{1/2} \left( H_0 + \frac{b_\varepsilon}{\varepsilon} \right)^{-1/2} v \right\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon \|v\|^2_{L^2(\mathbb{R}^3)}. \]

This estimate can be obtained if we show that

\begin{equation}
\|TT^*\|_{\mathcal{B}(L^2(\mathbb{R}^3))} \leq \varepsilon,
\end{equation}

where we set

\[ T := \tilde{V}_+^{1/2} \left( H_0 + \frac{b_\varepsilon}{\varepsilon} \right)^{-1/2}. \]

Thus, it suffices to show that for any \( \varepsilon > 0 \), there exists a constant \( b_\varepsilon > 0 \) such that the estimate (2.7) holds. Let \( \varepsilon > 0 \) be fixed and \( b > 0 \). By using the formula:

\[ \left( H_0 + \frac{b}{\varepsilon} \right)^{-1} v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{b}|x-y|}}{|x-y|} v(y) \, dy \]
and Schwarz inequality, we estimate
\[ \|TT^* v\|_{L^2(\mathbb{R}^3)}^2 \]
\[ = \left\| \tilde{V}_+^{1/2} \left( H_0 + \frac{b}{\varepsilon} \right)^{-1} \tilde{V}_+^{1/2} v \right\|_{L^2(\mathbb{R}^3)}^2 \]
\[ = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \tilde{V}_+(x) \left( \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3} |x-y|}}{|x-y|} \tilde{V}_+^{1/2}(y) v(y) \, dy \right)^2 \, dx \]
\[ \leq \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \tilde{V}_+(x) \left( \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3} |x-y|}}{|x-y|} \tilde{V}_+(y) \, dy \right) \left( \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3} |x-y|}}{|x-y|} |v(y)|^2 \, dy \right) \, dx. \]

Now we estimate the first integral on the right. We split the integral as follows:
\[ \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3} |x-y|}}{|x-y|} \tilde{V}_+(y) \, dy = \int_{|x-y|<r} + \int_{|x-y|\geq r} =: I_1 + I_2 \]
for any \( r > 0 \). Let \( \delta > 0 \) be fixed. Then, if we choose \( r > 0 \) small enough, we have \( I_1 \leq \delta \), since \( \tilde{V}_+ \in K_d(\Omega) \). Then, choosing \( b = b_\delta > 0 \) large enough, we have \( I_2 \leq \delta \).
Thus we obtain
\[ (2.8) \quad \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3} |x-y|}}{|x-y|} \tilde{V}_+(y) \, dy \leq 2\delta. \]
Using this estimate, we can estimate
\[ \|TT^* v\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{2\delta}{(4\pi)^2} \int_{\mathbb{R}^3} \tilde{V}_+(x) \left( \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3} |x-y|}}{|x-y|} |v(y)|^2 \, dy \right) \, dx. \]
Moreover, by using Fubini-Tonelli theorem and the inequality (2.8) once more, we estimate
\[ \|TT^* v\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{2\delta}{(4\pi)^2} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3} |x-y|}}{|x-y|} \tilde{V}_+(x) \, dx \right) |v(y)|^2 \, dy \]
\[ \leq \left( \frac{2\delta}{4\pi} \right)^2 \|v\|_{L^2(\mathbb{R}^3)}^2. \]
Thus, by choosing \( \delta = 2\pi \varepsilon \), we get (2.7), which implies (2.6) for \( d = 3 \).

When \( d > 3 \), we can also prove the estimate (2.6) in the same argument as in the case when \( d = 3 \), if we note that the kernel \( K_M(x) \) of \( (-\Delta+M)^{-1} \) for \( M > 0 \) satisfies
\[ |K_M(x)| \leq \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \frac{1}{|x|^{d-2}} \quad \text{and} \quad \lim_{M \to +\infty} \sup_{|x| > r} e^{\frac{b_\delta}{\varepsilon} |x|} K_M(x) = 0 \]
for each \( r > 0 \) (see [9, p.454]). Indeed, we can perform the argument involving \( H_0 + \frac{b_\delta}{\varepsilon} \) by using the previous asymptotics, and as a result, we get also (2.6).
Based on (2.6), we can prove the required estimates (2.4). In fact, we estimate, by using (2.6),
\[
\int_{\Omega} V_+(x)|u(x)|^2 \, dx = \int_{\mathbb{R}^d} \tilde{V}_+(x)|\tilde{u}(x)|^2 \, dx
\]
\[
\leq \varepsilon \|
abla \tilde{u}\|_{L^2(\mathbb{R}^d)}^2 + b_\varepsilon \|	ilde{u}\|_{L^2(\mathbb{R}^d)}^2
\]
\[
= \varepsilon \|
abla u\|_{L^2(\Omega)}^2 + b_\varepsilon \|u\|_{L^2(\Omega)}^2.
\]
As a consequence, by density argument, the inequality (2.4) is proved.

The proof of (2.5) is almost identical to that of (2.4) by regarding $b_\varepsilon$ as 0. The only difference is the estimate (2.8). Instead of (2.8), we can apply the following estimate:
\[
\int_{\mathbb{R}^3} \frac{\tilde{V}_-(y)}{|x - y|} \, dy = \int_{\Omega} \frac{V_-(y)}{|x - y|} \, dy 
\]
\[
\leq \|V_-\|_{K_3(\Omega)},
\]
whence the argument in (2.4) works well in this case, and we get (2.5). The proof of Lemma 2.4 is complete.

We are now in a position to prove Proposition 2.1.

**Proof of Proposition 2.1.** Let $q : H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{C}$ be the quadratic form by letting
\[
q(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + Vu v) \, dx, \quad u, v \in H^1_0(\Omega).
\]
It is clear that $q$ is densely defined and semi-bounded. Hence, as a consequence of Lemma 2.3, it suffices to show that the quadratic form $q$ is closed. Hence all we have to do is to show that the norm $\| \cdot \|_{+1}$ is equivalent to that of $H^1_0(\Omega)$, where $\| \cdot \|_{+1}$ is defined in (2.3), i.e.,
\[
\|u\|_{+1} = \sqrt{q(u, u) + \|u\|_{L^2(\Omega)}^2}.
\]
In fact, by using Lemma 2.4 we have
\[
\|u\|_{+1}^2 \leq \|\nabla u\|_{L^2(\Omega)}^2 + \int_{\Omega} V(x)|u(x)|^2 \, dx + \|u\|_{L^2(\Omega)}^2
\]
\[
\leq C \left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}^2\right),
\]
and by using assumption (2.1) on $V$, we see that
\[
\|u\|_{+1}^2 \geq \|\nabla u\|_{L^2(\Omega)}^2 - \int_{\Omega} V_-(x)|u(x)|^2 \, dx + \|u\|_{L^2(\Omega)}^2
\]
\[
\geq \left(1 - \frac{\|V_\varepsilon\|_{K_3(\Omega)} \Gamma(d/2 - 1)}{4\pi d/2}\right) \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2
\]
for any $u \in H^1_0(\Omega)$, which implies that $\| \cdot \|_{+1}$ is equivalent to $\| \cdot \|_{H^1(\Omega)}$. The proof of Proposition 2.1 is complete. \qed
3. $L^p$-$L^q$-estimates and pointwise estimates for $e^{-tH_V}$

In this section we shall prove $L^p$-$L^q$-estimates for $e^{-tH_V}$ and pointwise estimates for the integral kernel of $e^{-tH_V}$ on $\Omega$. Throughout this section we use the following notation

$$\gamma_d = \frac{\pi^{d/2}}{\Gamma(d/2 - 1)}.$$

We have the following:

**Proposition 3.1.** Assume that the measurable potential $V = V_+ - V_-$ satisfies $V_+ \in K_d(\Omega)$. Let $1 \leq p \leq q \leq \infty$. Suppose that

\begin{equation}
\|V_+\|_{K_d(\Omega)} < 2\gamma_d.
\end{equation}

Then

\begin{equation}
\|e^{-tH_V}f\|_{L^q(\Omega)} \leq \frac{(2\pi t)^{-(d/2)(1/p - 1/q)}}{\left(1 - \|V_+\|_{K_d(\Omega)}/2\gamma_d\right)^2} \|f\|_{L^p(\Omega)}, \quad \forall t > 0
\end{equation}

for any $f \in L^p(\Omega)$. In addition, if we further assume that $V_-$ satisfies assumption (1.1), i.e.,

$$\|V_+\|_{K_d(\Omega)} < \gamma_d,$$

then the kernel $K(t, x, y)$ of $e^{-tH_V}$ enjoys the property that

\begin{equation}
0 \leq K(t, x, y) \leq \frac{(2\pi t)^{-d/2}}{1 - \|V_+\|_{K_d(\Omega)}/\gamma_d} e^{-\frac{|x-y|^2}{8t}}, \quad \forall t > 0
\end{equation}

for any $x, y \in \Omega$.

The following lemma is crucial in the proof of Proposition 3.1.

**Lemma 3.2.** Let $d \geq 3$. Assume that the measurable potential $V = V_+ - V_-$ satisfies $V_+ \in K_d(\Omega)$ and

\begin{equation}
\|V_+\|_{K_d(\Omega)} < 4\gamma_d.
\end{equation}

Let $\tilde{V}$ be the zero extension of $V$ to $\mathbb{R}^d$ and $\tilde{H}_V$ the self-adjoint extension of $H_V$ on $L^2(\mathbb{R}^d)$. Then for any non-negative function $f \in L^2(\Omega)$, the following estimates hold:

\begin{equation}
(e^{-tH_V}f)(x) \geq 0,
\end{equation}

\begin{equation}
(e^{-tH_V}f)(x) \leq (e^{-\tilde{H}_V}\tilde{f})(x)
\end{equation}

for $t > 0$ and almost everywhere $x \in \Omega$, where $\tilde{f}$ is the zero extension of $f$ to $\mathbb{R}^d$.

The proof of Lemma 3.2 is rather long, and will be postponed. Let us prove Proposition 3.1.

**Proof of Proposition 3.1.** Let $f \in C^\infty_0(\Omega)$. Applying (3.5) from Lemma 3.2 to non-negative functions $|f| - f$ and $|f| + f$, we obtain

$$-(e^{-tH_V}|f|)(x) \leq (e^{-tH_V}f)(x) \leq (e^{-tH_V}|f|)(x)$$
for any $t > 0$ and almost everywhere $x \in \Omega$. Hence the above inequality and (3.6) from Lemma 3.2 imply that
\begin{equation}
|\left(e^{-tHv} f\right)(x)| \leq \left(e^{-tHv} |f|\right)(x)
\end{equation}
for any $t > 0$ and almost everywhere $x \in \Omega$. Here we recall the result of $L^p$-$L^q$-estimates for $e^{-tHv}$ on $\mathbb{R}^d$:
\begin{equation}
\|e^{-tHv} |f|\|_{L^q(\mathbb{R}^d)} \leq \frac{(2\pi t)^{(d/2)(1/p - 1/q)}}{1 - \|V_-\|_{K_d(\mathbb{R}^d)/\gamma_d}^2} \|f\|_{L^p(\mathbb{R}^d)}, \quad \forall t > 0,
\end{equation}
provided $1 \leq p \leq q \leq \infty$. (see Proposition 5.1 from [2]). Combining (3.7) and (3.8), we obtain the estimate (3.2) for $f \in C^0_\infty(\Omega)$. Thus, by density argument, we conclude the estimates (3.2) for any $f \in L^p(\Omega)$ if $p < \infty$. The case $p = \infty$ follows from the duality argument.

We now turn to prove (3.3). We adopt a sequence $\{j_\varepsilon(x)\}_{\varepsilon > 0}$ of functions defined by letting
\begin{equation}
j_\varepsilon(x) := \frac{1}{\varepsilon^d} j\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d,
\end{equation}
where
\begin{equation}
j(x) = \begin{cases} C_d e^{-1/(1 - |x|^2)}, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}
\end{equation}
with
\begin{equation}
C_d := \left(\int_{\mathbb{R}^d} e^{-\frac{1}{1 - |x|^2}} \, dx\right)^{-1}.
\end{equation}
As is well-known, the sequence $\{j_\varepsilon(x)\}_\varepsilon$ enjoys the following property:
\begin{equation}
j_\varepsilon(\cdot - y) \rightarrow \delta_y \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \quad (\varepsilon \rightarrow 0),
\end{equation}
where $\delta_y$ is the Dirac delta function at $y \in \Omega$. Let $y \in \Omega$ be fixed, and let $K(t, x, y)$ and $\bar{K}(t, x, y)$ be kernels of $e^{-tHv}$ and $e^{-tHv}$, respectively. Taking $\varepsilon > 0$ sufficiently small so that $\text{supp} \ j_\varepsilon(\cdot - y) \Subset \Omega$, and applying (3.5) and (3.6) from Lemma 3.2 to both $f$ and $\bar{f}$ replaced by $j_\varepsilon(\cdot - y)$, we get
\begin{equation}
0 \leq \int_{\Omega} K(t, x, z)j_\varepsilon(z - y)dz \leq \int_{\mathbb{R}^d} \bar{K}(t, x, y)j_\varepsilon(z - y)dz
\end{equation}
for any $x \in \Omega$. Noting (3.10) and taking the limit of the previous inequality as $\varepsilon \rightarrow 0$, we get
\begin{equation}
0 \leq K(t, x, y) \leq \bar{K}(t, x, y)
\end{equation}
for any $t > 0$ and $x \in \Omega$. Finally, by using the pointwise estimates:
\begin{equation}
\bar{K}(t, x, y) \leq \frac{(2\pi t)^{-d/2}}{1 - \|V_-\|_{K_d(\mathbb{R}^d)/\gamma_d}^2} e^{-\frac{|x - y|^2}{\varepsilon t}} \leq \frac{(2\pi t)^{-d/2}}{1 - \|V_-\|_{K_d(\Omega)/\gamma_d}^2} e^{-\frac{|x - y|^2}{\varepsilon t}}
\end{equation}
(see Proposition 5.1 from [2]), we obtain the estimate (3.3), as desired. The proof of Proposition 3.1 is finished.

In the rest of this section we shall prove Lemma 3.2. To prove Lemma 3.2 we need further the following two lemmas. The first one is concerned with the existence and uniqueness of solutions for evolution equations in abstract setting.
Lemma 3.3. Let $\mathcal{H}$ be a Hilbert space. Assume that $A$ is a non-negative self-adjoint operator on $\mathcal{H}$. Let $\{T(t)\}_{t \geq 0}$ be the semigroup generated by $A$, and let $f \in \mathcal{H}$ and $u(t) = T(t)f$. Then $u$ is the unique solution of the following problem:

$$
\begin{align*}
&u \in C([0, \infty); \mathcal{H}) \cap C^1((0, \infty); \mathcal{H}), \\
u(t) \in D(A), \quad t > 0, \\
u'(t) + Au(t) = 0, \quad t > 0, \\
u(0) = f.
\end{align*}
$$

For the proof of Lemma 3.3, see, e.g., Cazenave and Haraux [1, Theorem 3.2.1].

The second one is about the differentiability properties for composite functions of Lipschitz continuous functions and $W^{1, p}$-functions.

Lemma 3.4. Let $\Omega$ be an open set in $\mathbb{R}^d$, where $d \geq 1$, and let $1 \leq p \leq \infty$. Consider the positive and negative parts of a real-valued function $u \in W^{1, p}(\Omega)$:

$$
u^+ = \chi_{\{u > 0\}}u \quad \text{and} \quad \nu^- = -\chi_{\{u < 0\}}u.
$$

Then $\nu^, \nu^\pm \in W^{1, p}(\Omega)$ and

$$
\partial_{x_j} \nu^+ = \chi_{\{u > 0\}} \partial_{x_j} u, \quad \partial_{x_j} \nu^- = -\chi_{\{u < 0\}} \partial_{x_j} u \quad (1 \leq j \leq d),
$$

where $\partial_{x_j} = \partial / \partial x_j$.

For the proof of Lemma 3.4, see Gilbarg and Trudinger [5, Lemma 7.6].

To prove (3.5), we show that the negative part of $e^{-tH_V}f$ vanishes in $\Omega$, provided $f \geq 0$. For this purpose, we prepare the following lemma.

Lemma 3.5. Let $V$ be as in Lemma 3.2, and $f \in L^2(\Omega)$ a non-negative function. Put

$$
u(t) = e^{-tH_V}f, \quad t > 0.
$$

Then the negative part $\nu^-(t)$ of $\nu(t)$ belongs to $H^1_0(\Omega)$ for each $t > 0$.

Proof. Lemma 3.3 assures that

$$
\begin{align*}
u \in C([0, \infty); L^2(\Omega)) \cap C^1((0, \infty); L^2(\Omega)), \\
u(t) \in H^1_0(\Omega), \quad H_V u(t) \in L^2(\Omega), \quad t > 0, \\
\partial_t \nu(t) + H_V u(t) = 0, \quad t > 0, \\
u(0) = f.
\end{align*}
$$

Since $\nu(t) \in H^1_0(\Omega)$ for each $t > 0$, there exist $\varphi_n(t) \in C^\infty_0(\Omega) \ (n = 1, 2, \ldots)$ such that

$$
\varphi_n(t) \to \nu(t) \quad \text{in} \ H^1(\Omega)
$$

as $n \to \infty$ for each $t > 0$. Here $\{\varphi_n\}_n$ also depends on $t$. For the sake of simplicity, we may omit the time variable $t$ of $\varphi_n$ without any confusion. Let us take a non-negative
function $\psi \in C^\infty(\mathbb{R})$ as

$$\psi(x) = \begin{cases} = -x, & x \leq -1, \\ \leq -x, & -1 < x < 0, \\ = 0, & x \geq 0, \end{cases}$$

and put

$$(3.12) \quad \psi_n(x) := \frac{\psi(nx)}{n}, \quad n = 1, 2, \ldots .$$

Then there exists a constant $C > 0$ such that

$$(3.13) \quad |\psi'_n(x)| \leq C, \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}.$$ 

Let us consider two kinds of composite functions $\psi_n \circ \varphi_n$ and $\psi_n \circ u$. We show that

$$(3.14) \quad \psi_n \circ \varphi_n - \psi_n \circ u \to 0 \quad \text{in } H^1(\Omega),$$

$$(3.15) \quad \psi_n \circ u - u^- \to 0 \quad \text{in } H^1(\Omega)$$

as $n \to \infty$. In fact, by the mean value theorem, we have

$$(3.16) \quad \|\psi_n \circ \varphi_n - \psi_n \circ u\|_{L^2(\Omega)} = \left\| \int_0^1 \psi'_n(\theta \varphi_n + (1 - \theta)u)(\varphi_n - u) \, d\theta \right\|_{L^2(\Omega)} \leq C\|\varphi_n - u\|_{L^2(\Omega)},$$

and the derivative of $\psi_n \circ \varphi_n - \psi_n \circ u$ is written as

$$(3.17) \quad \|\partial_{x_j}(\psi_n \circ \varphi_n - \psi_n \circ u)\|_{L^2(\Omega)}$$

$$= \|\psi'_n(\varphi_n)\partial_{x_j} \varphi_n - \psi'_n(u)\partial_{x_j} u\|_{L^2(\Omega)}$$

$$\leq \|\psi'_n(\varphi_n)(\partial_{x_j} \varphi_n - \partial_{x_j} u)\|_{L^2(\Omega)} + \|\{\psi'_n(\varphi_n) - \psi'_n(u)\}\partial_{x_j} u\|_{L^2(\Omega)}$$

$$\leq C\|\partial_{x_j} \varphi_n - \partial_{x_j} u\|_{L^2(\Omega)} + \|\{\psi'_n(\varphi_n) - \psi'_n(u)\}\partial_{x_j} u\|_{L^2(\Omega)},$$

where we used (3.13) in the last step. Noting the pointwise convergence and uniform boundedness with respect to $n$:

$$\{\psi'_n(\varphi_n)(x) - \psi'_n(u)(x)\}\partial_{x_j} u(x) \to 0 \quad \text{as } n \to \infty \text{ for a.e. } x \in \Omega,$$

$$\{|\psi'_n(\varphi_n)(x) - \psi'_n(u)(x)\| \partial_{x_j} u(x)| \leq 2C|\partial_{x_j} u(x)| \in L^2(\Omega),$$

we can apply Lebesgue’s dominated convergence theorem to obtain

$$(3.18) \quad \|\{\psi'_n(\varphi_n) - \psi'_n(u)\}\partial_{x_j} u\|_{L^2(\Omega)} \to 0$$

as $n \to \infty$. Hence, summarizing (3.11) and (3.16)–(3.18), we obtain (3.14).

As to the latter convergence (3.15), since

$$|(\psi_n \circ u)(x) - u^-(x)| \leq 2|u(x)| \in L^2(\Omega),$$

$$|\partial_{x_j}(\psi_n \circ u)(x) - \partial_{x_j} u^-(x)| \leq (C + 1)|\partial_{x_j} u^-(x)| \in L^2(\Omega),$$

and

$$(\psi_n \circ u)(x) - u^-(x) \to 0,$$

$$\partial_{x_j}(\psi_n \circ u)(x) - \partial_{x_j} u^-(x) = \{\psi'_n(u) + \chi_{\{u < 0\}}\}\partial_{x_j} u(x) \to 0$$
as \( n \to \infty \) for almost everywhere \( x \in \Omega \), Lebesgue’s dominated convergence theorem allows us to obtain (3.15). Thus (3.14) and (3.15) imply that
\[
\psi_n \circ \varphi_n - u^- \to 0 \quad \text{in } H^1(\Omega) \quad (n \to \infty).
\]
Since \( \psi_n \circ \varphi_n \in C_0^\infty(\Omega) \), we conclude that \( u^- \in H^1_0(\Omega) \). The proof of Lemma 3.5 is finished. \( \square \)

We are now in a position to prove Lemma 3.2.

**Proof of Lemma 3.2.** Let \( f \in L^2(\Omega) \) be non-negative almost everywhere on \( \Omega \). We recall that
\[
u(t) = e^{-tH_V} f \quad \text{for } t \geq 0.
\]
If we show that \( \|u^-(t)\|_{L^2(\Omega)}^2 \) is monotonically decreasing with respect to \( t \geq 0 \), then we obtain
\[
u^-(t, x) = 0
\]
for each \( t > 0 \) and almost everywhere \( x \in \Omega \), since \( u^-(0, x) = f^-(x) = 0 \) for almost everywhere \( x \in \Omega \). This means that
\[
u(t, x) \geq 0
\]
for each \( t > 0 \) and almost everywhere \( x \in \Omega \); thus we conclude (3.5). Hence it is sufficient to show that
\[(3.19) \quad \frac{d}{dt} \int_\Omega (u^-)^2 \, dx \leq 0.\]

By the definition of \( u^+ \), we have \( \partial_t u^+(t, x) = 0 \) for \( x \in \{ u < 0 \} \) and each \( t > 0 \). We compute
\[(3.20) \quad \frac{d}{dt} \int_\Omega (u^-)^2 \, dx = 2 \int_\Omega u^- \partial_t u^- \, dx
\]
\[
= 2 \int_{\{u < 0\}} u^- \partial_t (u^+ - u) \, dx
\]
\[
= -2 \int_{\{u < 0\}} u^- \partial_t u \, dx
\]
\[
= 2 \int_\Omega (H_V u) u^- \, dx
\]
where we use the equation \( \partial_t u + H_V u = 0 \) in the last step. Since \( u^- \in H^1_0(\Omega) \) by Lemma 3.5, we have, by going back to (2.2) in the definition of \( H_V \),
\[(3.21) \quad \int_\Omega (H_V u) u^- \, dx = \int_\Omega \nabla u \cdot \nabla u^- \, dx + \int_\Omega Vuu^- \, dx.
\]
Here we see from Lemma 3.4 that
\[
\nabla u^- = -\chi_{\{u < 0\}} \nabla u,
\]
and hence, the first term on the right of (3.21) is written as
\[
\int_\Omega \nabla u \cdot \nabla u^- \, dx = -\int_\Omega |\nabla u^-|^2 \, dx.
\]
As to the second, by the estimate (2.5) from Lemma 2.4, we have
\[
\int_{\Omega} Vu_+- dx = - \int_{\Omega} V|u^-|^2 dx \\
\leq \int_{\Omega} V_-|u^-|^2 dx \\
\leq \frac{\|V\|_{K_d(\Omega)}}{4\gamma_d} \int_{\Omega} |\nabla u^-|^2 dx;
\]
thus we find from assumption (3.4) on $V$ that
\[
\int_{\Omega} (H_V u)_- dx \leq - \left( 1 - \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d} \right) \int_{\Omega} |\nabla u^-|^2 dx \\
\leq 0,
\]
and hence, combining this inequality and (3.20), we conclude (3.19).

Next, we prove (3.6). Let us define two functions $v^{(1)}(t)$ and $v^{(2)}(t)$ as follows:
\[
v^{(1)}(t) := e^{-t\tilde{H}_V} \tilde{f} \quad \text{and} \quad v^{(2)}(t) := e^{-tH_V} f
\]
for $t \geq 0$. Then it follows from Lemma 3.3 that $v^{(1)}$ and $v^{(2)}$ satisfy
\[
\begin{cases}
  v^{(1)} \in C([0, \infty); L^2(\mathbb{R}^d)) \cap C^1((0, \infty); L^2(\mathbb{R}^d)), \\
  v^{(1)}(t) \in H^1(\mathbb{R}^d), \quad \tilde{H}_V v^{(1)}(t) \in L^2(\mathbb{R}^d), \\
  \partial_t v^{(1)}(t) + \tilde{H}_V v^{(1)}(t) = 0, \\
  v^{(1)}(0) = \tilde{f}
\end{cases}
\]
and
\[
\begin{cases}
  v^{(2)} \in C([0, \infty); L^2(\Omega)) \cap C^1((0, \infty); L^2(\Omega)), \\
  v^{(2)}(t) \in H^1_0(\Omega), \quad H_V v^{(2)}(t) \in L^2(\Omega), \\
  \partial_t v^{(2)}(t) + H_V v^{(2)}(t) = 0, \\
  v^{(2)}(0) = f
\end{cases}
\]
for each $t > 0$, respectively. We define a new function $v$ as
\[
v(t) := v^{(1)}(t)|_{\Omega} - v^{(2)}(t)
\]
for $t \geq 0$, where $v^{(1)}(t)|_{\Omega}$ is the restriction of $v^{(1)}(t)$ to $\Omega$. Let us consider the negative part of $v$:
\[
v^- = -\chi_{\{v<0\}} v.
\]
Then, thanks to (3.22) and (3.23), we have
\[
v^- \in C([0, \infty); L^2(\Omega)) \cap C^1((0, \infty); L^2(\Omega)).
\]
Moreover, by using Lemma 3.4, we have $v^- \in H^1(\Omega)$, since $v \in H^1(\Omega)$. Once we prove that
\[
v^- \in H^1_0(\Omega),
\]
we can get, by the previous argument,
\[
\frac{d}{dt} \int_{\Omega} (v^-)^2 dx \leq 0.
\]
In fact, by the definition of $v^-$, we have
\[
\frac{d}{dt} \int_{\Omega} (v^-)^2 \, dx = -2 \int_{\{v<0\}} v^- \partial_t v^{(1)} \, dx + 2 \int_{\{v<0\}} v^- \partial_t v^{(2)} \, dx
\]
\[
= 2 \int_{\mathbb{R}^d} (\tilde{H}_V v^{(1)}) \tilde{v}^- \, dx - 2 \int_{\Omega} (H_V v^{(2)}) v^- \, dx,
\]
where $\tilde{v}^-$ is the zero extension of $v^-$ to $\mathbb{R}^d$, and we use equations $\partial_t v^{(1)} + \tilde{H}_V v^{(1)} = 0$ and $\partial_t v^{(2)} + H_V v^{(2)} = 0$ in the last step. Since $v^- \in H^1_0(\Omega)$ by (3.24), we have, by definitions of $\tilde{H}_V$ and $H_V$,
\[
\int_{\mathbb{R}^d} (\tilde{H}_V v^{(1)}) \tilde{v}^- \, dx - \int_{\Omega} (H_V v^{(2)}) v^- \, dx
\]
\[
= \int_{\mathbb{R}^d} \nabla v^{(1)} \cdot \nabla \tilde{v}^- \, dx + \int_{\mathbb{R}^d} \tilde{V} v^{(1)} \tilde{v}^- \, dx - \int_{\Omega} \nabla v^{(2)} \cdot \nabla v^- \, dx - \int_{\Omega} V v^{(2)} v^- \, dx
\]
\[
= \int_{\Omega} \nabla v \cdot \nabla v^- \, dx + \int_{\Omega} V v v^- \, dx
\]
\[
\leq - \left( 1 - \frac{\|V\|_{K_d(\Omega)}}{4 \gamma_d} \right) \int_{\Omega} |\nabla v^-|^2 \, dx
\]
\[
\leq 0,
\]
where we used assumption A in the last step. Hence we obtain (3.25), which implies the required inequality (3.6).

It remains to prove (3.24). The proof is similar to that of Lemma 3.5. Since $v^{(2)}(t) \in H^1_0(\Omega)$ for each $t > 0$ by (3.23), there exist $\varphi_n = \varphi_n(t) \in C^\infty(\Omega)$ such that
\[
\varphi_n \to v^{(2)} \quad \text{in} \quad H^1(\Omega)
\]
as $n \to \infty$. Put
\[
v_n(t) := v^{(1)}(t)|_\Omega - \varphi_n(t), \quad n = 1, 2, \ldots,
\]
for each $t > 0$. Let $\psi_n$ be as in (3.12). As in the proof of Lemma 3.5, we can show that
\[
\psi_n \circ v_n^- - v^- \to 0 \quad \text{in} \quad H^1(\Omega)
\]
as $n \to \infty$. Since $v_n^-$ have compact supports in $\text{supp} \varphi_n$ by $v^{(1)} \geq 0$ on $\Omega$, the functions $\psi_n \circ v_n^-$ also have compact supports in $\Omega$. Let $\psi_n \circ v_n^-$ be the zero extension of $\psi_n \circ v_n^-$ to $\mathbb{R}^d$, and let $j_\varepsilon$ be Friedrichs' mollifier: For $u \in L^1_{\text{loc}}(\mathbb{R}^d)$,
\[
(j_\varepsilon u)(x) := (j_\varepsilon * u)(x) = \int_{\mathbb{R}^d} j_\varepsilon(x - y) u(y) \, dy, \quad x \in \mathbb{R}^d,
\]
where $\{j_\varepsilon(x)\}_\varepsilon$ are functions defined in (3.9). Taking $\varepsilon = \varepsilon_n$ sufficiently small so that $\varepsilon_n \to 0$ ($n \to \infty$) and $\text{supp} j_\varepsilon \left( \psi_n \circ v_n^- \right)$ is contained in $\Omega$, we have
\[
J_{\varepsilon_n} \left( \psi_n \circ v_n^- \right)|_\Omega \in C^\infty(\Omega).
\]
Since
\[
J_{\varepsilon_n} \left( \psi_n \circ v_n^- \right)|_\Omega - v^- \to 0 \quad \text{in} \quad H^1(\Omega)
\]
as $n \to \infty$, we conclude (3.24). The proof of Lemma 3.2 is complete. \qed
4. $L^p-\ell^q(L^q)_\theta$-boundedness for the resolvent of $\theta H_V$

In this section we shall prove the boundedness of resolvent $(\theta H_V - z)^{-\beta}$ ($\beta > 0$) in scaled amalgam spaces. The result in this section plays an important role in the proof of Theorem 1.1.

More precisely, we have:

**Theorem 4.1.** Assume that the measurable potential $V = V_+ - V_-$ satisfies $V_\pm \in K_d(\Omega)$, and that $V_-$ satisfies assumption (3.1) in Proposition 3.1. Let $1 \leq p \leq q \leq \infty$ and $\beta > (d/2)(1/p - 1/q)$, and let $z \in \mathbb{C}$ with $\text{Re}(z) < 0$. Then there exists a constant $C = C(d, p, q, \beta, z) > 0$ such that

\[
(4.1) \quad \| (\theta H_V - z)^{-\beta} \|_{\mathcal{B}(L^p(\Omega), \ell^q(\Omega))} \leq C \theta^{-(d/2)(1/p - 1/q)}
\]

for any $\theta > 0$. If we further assume that $V_-$ satisfies assumption (1.1), then

\[
(4.2) \quad \| (\theta H_V - z)^{-\beta} \|_{\mathcal{B}(L^p(\Omega), \ell^p(L^q)_\theta)} \leq C \theta^{-(d/2)(1/p - 1/q)}
\]

for any $\theta > 0$.

**Proof.** Let us first prove (4.1). We use the following well-known formula: For $z \in \mathbb{C}$ with $\text{Re}(z) < 0$ and $\beta > 0$,

\[
(4.3) \quad (H_V - z)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{z t} e^{-t H_V} dt.
\]

Since $V$ satisfies assumption (3.1), thanks to the formula (4.3) and $L^p-L^q$-estimates (3.2) for $e^{-t H_V}$ in Proposition 3.1, we can estimate

\[
\| (\theta H_V - z)^{-\beta} f \|_{L^q(\Omega)} \leq \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{\text{Re}(z) t} \| e^{-t \theta H_V} f \|_{L^q(\Omega)} dt
\]

\[
\leq C \theta^{-(d/2)(1/p - 1/q)} \int_0^\infty t^{\beta-1} e^{\text{Re}(z) t} t^{-(d/2)(1/p - 1/q)} dt \cdot \| f \|_{L^p(\Omega)}.
\]

Since $\beta > (d/2)(1/p - 1/q)$ and $\text{Re}(z) < 0$, the integral on the right is absolutely convergent. Hence we obtain

\[
\| (\theta H_V - z)^{-\beta} f \|_{L^q(\Omega)} \leq C \theta^{-(d/2)(1/p - 1/q)} \| f \|_{L^p(\Omega)}.
\]

This proves (4.1).

Let us turn to the proof of (4.2). If we can prove that

\[
(4.4) \quad \| e^{-t \theta H_V} f \|_{\ell^p(L^q)_\theta} \leq C \theta^{-(d/2)(1/p - 1/q)} \left\{ t^{-(d/2)(1/p - 1/q)} + 1 \right\} \| f \|_{L^p(\Omega)}, \quad \forall t > 0
\]

for any $f \in L^p(\Omega)$ provided $1 \leq p \leq q \leq \infty$, then the estimate (4.2) is obtained by combining (4.3) and (4.4). In fact, we estimate

\[
\| (\theta H_V - z)^{-\beta} f \|_{\ell^p(L^q)_\theta}
\]

\[
\leq \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{\text{Re}(z) t} \| e^{-t \theta H_V} f \|_{\ell^p(L^q)_\theta} dt
\]

\[
\leq C \theta^{-(d/2)(1/p - 1/q)} \int_0^\infty t^{\beta-1} e^{\text{Re}(z) t} \left\{ t^{-(d/2)(1/p - 1/q)} + 1 \right\} dt \cdot \| f \|_{L^p(\Omega)}.
\]
Since $\beta > (d/2)(1/p - 1/q)$ and $\text{Re}(z) < 0$, the integral on the right is absolutely convergent. Hence we conclude that

$$\|(\theta H - z)^{-\beta}\|_{L^p(L^q)} \leq C\theta^{-(d/2)(1/p - 1/q)} \|f\|_{L^p(\Omega)}.$$  

This proves (4.2). Therefore, all we have to do is to prove the estimate (4.4). To this end, we prove the following estimate: For $1 \leq q \leq \infty$ and any $\theta > 0$,

$$(4.5) \quad \|K_0(\theta t, \cdot)\|_{L^q(C^\theta(n))} \leq C\theta^{-(d/2)(1/p - 1/q)} \left\{ t^{-(d/2)(1/p - 1/q)} + 1 \right\}, \quad \forall t > 0,$$

where $K_0(t, x)$ is defined by letting

$$K_0(t, x) = \frac{(2\pi t)^{-d/2}}{1 - \|V\|_{K_d(\Omega)/\gamma_d}} e^{-\frac{|x|^2}{8t}} = C_1 t^{-d/2} e^{-\frac{|x|^2}{8t}}$$

for any $t > 0$ and $x \in \mathbb{R}^d$. Here, recalling that

$$\gamma_d = \frac{\pi^{d/2}}{\Gamma(d/2 - 1)},$$

we note from assumption (1.1) on $V$ that

$$C_1 = \frac{(2\pi)^{-d/2}}{1 - \|V\|_{K_d(\Omega)/\gamma_d}} > 0.$$

For the proof of (4.5), we compute $\|K_0(\theta t, \cdot)\|_{L^q(C^\theta(n))}$ for the case $n = 0$ and $n \neq 0$, separately:

**The case $n = 0$:** We estimate

$$\|K_0(\theta t, \cdot)\|_{L^q(C^\theta(0))} \leq C_1(\theta t)^{-d/2} \left( \int_{|x| < \sqrt{\theta t}} e^{-\frac{|x|^2}{8t}} \, dx \right)^{1/q}$$

$$= C_1(\theta t)^{-d/2} \left( \int_{|x| < \sqrt{t}} e^{-\frac{|x|^2}{8\theta}} (\theta t)^{d/2} \, dx \right)^{1/q}$$

$$\leq C(\theta t)^{-(d/2)(1-1/q)} \left( \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{8\theta}} \, dx \right)^{1/q}$$

$$\leq C(\theta t)^{-(d/2)(1-1/q)}.$$  

The case $n \neq 0$: We estimate

$$(4.7) \quad \sum_{n \neq 0} \|K_0(\theta t, \cdot)\|_{L^q(C^\theta(n))} = C_1(\theta t)^{-d/2} \left( \int_{C_\theta(n)} e^{-\frac{|x|^2}{8\theta_t}} \, dx \right)^{1/q}$$

$$\leq C_1(\theta t)^{-d/2} \left( \sup_{x \in C_\theta(n)} e^{-\frac{|x|^2}{8\theta_t}} \right) \left( \int_{C_\theta(n)} \, dx \right)^{1/q}.$$  

Here, observing that

$$\frac{|\theta^{1/2} n|}{2} \leq |x| (\leq 2|\theta^{1/2} n|), \quad x \in C_\theta(n),$$
we can estimate the right member of (4.7) as

\[ C_1(\theta t)^{-d/2} \left( \sum_{n \neq 0} e^{-\frac{|n|^2}{2\theta t}} \right) (\theta^{d/2})^{1/q}, \]

and hence, we get

\[ \sum_{n \neq 0} \|K_0(\theta t, \cdot)\|_{L^q(C_\theta(n))} \leq C_1(\theta t)^{-d/2} \left( \sum_{n \neq 0} e^{-\frac{|n|^2}{2\theta t}} \right) (\theta^{d/2})^{1/q}. \]

Here, by an explicit calculation, we see that

\[ \sum_{n \neq 0} e^{-\frac{|n|^2}{2\theta t}} = \sum_{n \neq 0} e^{-\frac{n_1^2+n_2^2+\cdots+n_3^2}{2\theta t}} = 2^d \left( \sum_{j=1}^{\infty} e^{-\frac{j^2}{2\theta t}} \right)^d \leq 2^d \left( \int_0^\infty e^{-\frac{x^2}{2\theta t}} \, dx \right)^d \]

\[ = (8\sqrt{2})^d n^{d/2} t^{d/2}. \]

Summarizing the estimates obtained now, we conclude that

\[ (4.8) \quad \sum_{n \neq 0} \|K_0(\theta t, \cdot)\|_{L^q(C_\theta(n))} \leq C_1(\theta t)^{-d/2} \cdot (8\sqrt{2})^d n^{d/2} t^{d/2} \cdot (\theta^{d/2})^{1/q} \]

\[ = (8\sqrt{2})^d n^{d/2} t^{d/2} C_1 \theta^{-(d/2)(1/p-1/q)}. \]

Combining the estimates (4.6)–(4.8), we obtain (4.5), as desired.

We are now in a position to prove the key estimate (4.4). Let \( f \in L^p(\Omega) \) and \( \tilde{f} \) be a zero extension of \( f \) to \( \mathbb{R}^d \). Thanks to the estimate (3.3) in Proposition 3.1, we have

\[ \|e^{-i\theta HV} f\|_{L^p(\Omega)} = \left\| \int \Omega K(\theta t, \cdot, y) f(y) \, dy \right\|_{L^p(\Omega)} \leq \left\| \int \Omega K(\theta t, \cdot, y) |f(y)| \, dy \right\|_{L^p(\Omega)} \leq \left\| \int_{\mathbb{R}^d} K_0(\theta t, \cdot - y) |\tilde{f}(y)| \, dy \right\|_{L^p(\Omega)}. \]

Applying the Young inequality (A.1) (see appendix A) to the right member, and using the inequality (4.5), we estimate

\[ \|e^{-i\theta HV} f\|_{L^p(\Omega)} \leq 3^d \|K_0(\theta t, \cdot)\|_{L^1(\mathbb{R}^d)} \|\tilde{f}\|_{L^p(\mathbb{R}^d)} \leq C \theta^{-(d/2)(1-1/r)} \left\{ t^{-(d/2)(1-1/r)} + 1 \right\} \|f\|_{L^p(\mathbb{R}^d)} \]

\[ = C \theta^{-(d/2)(1/p-1/q)} \left\{ t^{-(d/2)(1/p-1/q)} + 1 \right\} \|f\|_{L^p(\Omega)}, \]

provided that \( p, q, r \) satisfy \( 1 \leq p, q, r \leq \infty \) and \( 1/p + 1/r - 1 = 1/q \). This proves (4.4). The proof of Theorem 4.1 is finished. \( \square \)
5. Commutator estimates

In this section we shall prepare commutator estimates. These estimates will be also an important tool in the proof of Theorem 1.1. Among other things, we introduce an operator $A_d$ as follows:

**Definition.** Let $X$ and $Y$ be topological vector spaces, and let $A$ and $B$ be continuous linear operators from $X$ and $Y$ into themselves, respectively. For a continuous linear operator $L$ from $X$ into $Y$, the operator $A_d^k(L)$ from $X$ into $Y$, $k = 0, 1, \ldots$, is successively defined by

$$A_d^0(L) = L, \quad A_d^k(L) = A_d^{k-1}(BL - LA), \quad k \geq 1.$$ 

The result in this section is concerned with $L^2$-boundedness for $A_d^k(e^{-itR_V,\theta})$, where $R_V,\theta$ is the resolvent operator defined by letting

$$R_V,\theta := (\theta H_V + M)^{-1}, \quad \theta > 0$$

for a fixed $M > 0$. Hereafter, operators $A$ and $B$ are taken as

$$(5.1) \quad A = B = x_j - \theta^{1/2}n_j \quad \text{for some } j \in \{1, \ldots, d\}.$$ 

Then we shall prove here the following.

**Proposition 5.1.** Let $d \geq 3$. Assume that the measurable potential $V = V_+ - V_-$ satisfies $V_+ \in K_d(\Omega)$, and that $V_-$ satisfies assumption (3.4) in Lemma 3.2. Let $A$ and $B$ be the operators as in (5.1), and let $L = e^{-itR_V,\theta}$. Then for any non-negative integer $k$, there exists a constant $C = C(d, M, k) > 0$ such that

$$\|A_d^k(e^{-itR_V,\theta})\|_{\mathcal{B}(L^2(\Omega))} \leq C\theta^{k/2}(1 + t)^k$$

for any $t > 0$ and $\theta > 0$.

First, we prepare $L^2$-boundedness for $R_V,\theta$ and $\partial_{x_j}R_V,\theta$ to prove Proposition 5.1.

**Lemma 5.2.** Let $d \geq 3$ and $V$ be as in Proposition 5.1. Then the following estimates hold:

$$\|R_V,\theta\|_{\mathcal{B}(L^2(\Omega))} \leq M^{-1},$$

$$\|
abla R_V,\theta\|_{\mathcal{B}(L^2(\Omega))} \leq M^{-1/2} \left( 1 - \frac{\|V_\perp\|_{K_d(\Omega)}\Gamma(d/2 - 1)}{4\pi^{d/2}} \right)^{-1/2} \theta^{-1/2}$$

for any $\theta > 0$.

**Proof.** Since $H_V$ is the self-adjoint operator with domain

$$\mathcal{D}(H_V) = \{u \in H^1_0(\Omega) \mid H_Vu \in L^2(\Omega)\},$$

we obtain (5.3) and (5.4) by the spectral resolution. In fact, we have

$$\|R_V,\theta f\|_{L^2(\Omega)}^2 = \int_0^\infty \frac{1}{(\theta^2 + M^2)^2} d \|E_{H_V}(\lambda)f\|_{L^2(\Omega)}^2 \leq M^{-2} \int_0^\infty d \|E_{H_V}(\lambda)f\|_{L^2(\Omega)}^2 \leq M^{-2} \|f\|_{L^2(\Omega)}^2.$$
for any \( f \in L^2(\Omega) \). This proves (5.3).

Since \( R_{V,\theta} f \in D(H_V) \) for any \( f \in L^2(\Omega) \), we can write
\[
\| \nabla R_{V,\theta} f \|_{L^2(\Omega)}^2 = \int_{\Omega} \left( \nabla R_{V,\theta} f \cdot \nabla R_{V,\theta} f + V|R_{V,\theta} f|^2 - V|R_{V,\theta} f|^2 \right) dx
\]
\[
= \langle H_V R_{V,\theta} f, R_{V,\theta} f \rangle_{L^2(\Omega)} - \int_{\Omega} V|R_{V,\theta} f|^2 dx
\]
\[
= I + II.
\]

Then we estimate the first term \( I \) as
\[
I \leq \int_{0}^{\infty} \frac{\lambda}{(\theta \lambda + M)^2} d\|E_{H_V}(\lambda) f\|_{L^2(\Omega)}^2
\]
\[
= \int_{0}^{\infty} \theta^{-1} \cdot \frac{\theta \lambda}{\theta \lambda + M} \cdot \frac{1}{\theta \lambda + M} d\|E_{H_V}(\lambda) f\|_{L^2(\Omega)}^2
\]
\[
\leq \theta^{-1} M^{-1} \int_{0}^{\infty} d\|E_{H_V}(\lambda) f\|_{L^2(\Omega)}^2
\]
\[
\leq \theta^{-1} M^{-1} \|f\|_{L^2(\Omega)}^2.
\]

As to \( II \), we have, by Lemma 2.4,
\[
II \leq \int_{\Omega} V_- |R_{V,\theta} f|^2 dx
\]
\[
\leq \frac{\|V_-\|_{K_{d}(\Omega)} \Gamma(d/2 - 1)}{4 \pi d/2} \int_{\Omega} |\nabla R_{V,\theta} f|^2 dx.
\]

Combining the previous estimates, we conclude that
\[
\| \nabla R_{V,\theta} f \|_{L^2(\Omega)}^2 \leq \theta^{-1} \left( 1 - \frac{\|V_-\|_{K_{d}(\Omega)} \Gamma(d/2 - 1)}{4 \pi d/2} \right)^{-1} M^{-1} \|f\|_{L^2(\Omega)}^2
\]
for any \( f \in L^2(\Omega) \). This proves (5.4). The proof of Lemma 5.2 is complete.

We are now in a position to prove Proposition 5.1.

**Proof of Proposition 5.1**. Let us denote by \( \mathcal{D}(\Omega) \) the totality of the test functions on \( \Omega \), and by \( \mathcal{D}'(\Omega) \) its dual space. We regard \( X \) as \( \mathcal{D}(\Omega) \) and \( Y \) as \( \mathcal{D}'(\Omega) \) in the definition of operator \( \text{Ad} \). Then we have, by Lemma B.2 in appendix B,
\[
\text{Ad}^0(R_{V,\theta}) = R_{V,\theta}, \quad \text{Ad}^1(R_{V,\theta}) = -2\theta R_{V,\theta} \partial_x R_{V,\theta},
\]
\[
\text{Ad}^k(R_{V,\theta}) = \theta \left\{ -2k \text{Ad}^{k-1}(R_{V,\theta}) \partial_x R_{V,\theta} + k(k-1) \text{Ad}^{k-2}(R_{V,\theta}) R_{V,\theta} \right\}
\]
for \( k \geq 2 \). Since \( R_{V,\theta} \) and \( \partial_x R_{V,\theta} \) are bounded on \( L^2(\Omega) \) by Lemma 5.2, \( \text{Ad}^k(R_{V,\theta}) \) are also bounded on \( L^2(\Omega) \) for any \( k \geq 0 \). Before going to prove (5.2), we prepare the following estimates for \( \text{Ad}^k(R_{V,\theta}) \): For any non-negative integer \( k \), there exists a constant \( C_k > 0 \) such that
\[
\| \text{Ad}^k(R_{V,\theta}) \|_{\mathcal{A}(L^2(\Omega))} \leq C_k \theta^{k/2}
\]
for any \( \theta > 0 \). We prove (5.7) by induction. For \( k = 0, 1 \), we have, by using (5.5) and Lemma 5.2,
\[
\| \text{Ad}^0(R_{V,\theta}) \|_{\mathcal{A}(L^2(\Omega))} = \| R_{V,\theta} \|_{\mathcal{A}(L^2(\Omega))} \leq C_0,
\]
\[ \|\text{Ad}^1(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))} = 2\theta \|R_{V,\theta}\partial_{x_j}R_{V,\theta}\|_{\mathcal{B}(L^2(\Omega))} \]
\[ \leq 2\theta M^{-1} \cdot M^{-1/2} \left( 1 - \frac{\|V\|_{L^2(\Omega)}\Gamma(d/2 - 1)}{4\pi^{d/2}} \right)^{-1/2} \theta^{-1/2} = C_1 \theta^{1/2}. \]

Let us suppose that (5.7) is true for \( k \in \{0, 1, \ldots, \ell\} \). Combining identities (5.6) and estimates (5.3) and (5.4) from Lemma 5.2, we get (5.7) for \( k = \ell + 1 \):

\[ \|\text{Ad}^{\ell+1}(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))} = \|\theta \{-2(\ell + 1)\text{Ad}^\ell(R_{V,\theta})\partial_{x_j}R_{V,\theta} + \ell(\ell + 1)\text{Ad}^{\ell-1}(R_{V,\theta})R_{V,\theta}\}\|_{\mathcal{B}(L^2(\Omega))} \]
\[ \leq 2\ell(\ell + 1)\theta \{ \|\text{Ad}^\ell(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))}\|\partial_{x_j}R_{V,\theta}\|_{\mathcal{B}(L^2(\Omega))} \]
\[ + \|\text{Ad}^{\ell-1}(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))}\|R_{V,\theta}\|_{\mathcal{B}(L^2(\Omega))} \} \]
\[ \leq C_{\ell+1} \theta \left\{ \theta^{\ell/2} \cdot \theta^{-1/2} + \theta^{(\ell-1)/2} \right\} \]
\[ \leq C_{\ell+1} \theta^{(\ell+1)/2}. \]

Thus (5.7) is true for any \( k \geq 0 \).

We prove (5.2) also by induction. Clearly, (5.2) is true for \( k = 0 \). As to the case \( k = 1 \), by using the estimate (5.7) and the formula (B.7) from Lemma B.3 in appendix B:

\[ \text{Ad}^1(e^{-itR_{V,\theta}}) = -i \int_0^t e^{-isR_{V,\theta}} \text{Ad}^1(R_{V,\theta})e^{-i(t-s)R_{\theta,v}} ds, \]

we have

\[ \|\text{Ad}^1(e^{-itR_{V,\theta}})\|_{\mathcal{B}(L^2(\Omega))} \]
\[ \leq \int_0^t \|e^{-isR_{V,\theta}}\|_{\mathcal{B}(L^2(\Omega))}\|\text{Ad}^1(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))}\|e^{-i(t-s)R_{\theta,v}}\|_{\mathcal{B}(L^2(\Omega))} ds \]
\[ \leq C_1 \int_0^t \theta^{1/2} ds \]
\[ \leq C_1 \theta^{1/2}(1 + t). \]

Hence, (5.2) is true for \( k = 1 \). Let us suppose that (5.2) holds for \( k \in \{0, 1, \ldots, \ell\} \). Then, by using the estimate (5.7) and the formula (B.8) from Lemma B.3:

\[ \text{Ad}^{\ell+1}(e^{-itR_{V,\theta}}) \]
\[ = -i \int_0^t \sum_{\ell_1 + \ell_2 + \ell_3 = \ell} \Gamma(\ell_1, \ell_2, \ell_3) \text{Ad}^{\ell_1}(e^{-isR_{\theta,v}}) \text{Ad}^{\ell_2+1}(R_{V,\theta}) \text{Ad}^{\ell_3}(e^{-i(t-s)R_{\theta,v}}) ds, \]

where constants \( \Gamma(\ell_1, \ell_2, \ell_3) \) are trinomial coefficients:

\[ \Gamma(\ell_1, \ell_2, \ell_3) = \frac{\ell!}{\ell_1!\ell_2!\ell_3!}. \]
we estimate

\[ \| \text{Ad}^{\ell+1}(e^{-itR_V,\theta}) \|_{\mathcal{B}(L^2(\Omega))} \]

\[ \leq C_{\ell+1} \int_0^l \sum_{\ell_1+\ell_2+\ell_3=\ell} \| \text{Ad}^{\ell_1}(e^{-isR_V,\theta}) \|_{\mathcal{B}(L^2(\Omega))} \| \text{Ad}^{\ell_2+1}(R_V,\theta) \|_{\mathcal{B}(L^2(\Omega))} \times \]

\[ \times \| \text{Ad}^{\ell_3}(e^{-i(t-s)R_V,\theta}) \|_{\mathcal{B}(L^2(\Omega))} \] \, ds

\[ \leq C_{\ell+1} \int_0^l \sum_{\ell_1+\ell_2+\ell_3=\ell} \theta^{\ell_1/2}(1+s)^{\ell_1} \cdot \theta^{(\ell_2+1)/2} \cdot \theta^{\ell_3/2}(1+t-s)^{\ell_3} \, ds \]

\[ \leq C_{\ell+1} \theta^{(\ell+1)/2}(1+t)^{\ell+1}. \]

Hence (5.2) is true for \( k = \ell + 1 \). Thus (5.2) holds for any \( k \geq 0 \). The proof of Proposition 5.1 is complete.

\[ \square \]

6. PROOF OF THEOREM 1.1.

In this section we shall prove Theorem 1.1. To begin with, let us introduce a family of operators which is useful to prove the theorem. For any non-negative integer \( N \), we define a family \( \mathcal{A}_N \) of operators as follows: We say that \( A \in \mathcal{A}_N \) if \( A \in \mathcal{B}(L^2(\Omega)) \) and

\[ \| A \|_N := \sup_{n \in \mathbb{Z}^d} \| (\cdot - \theta^{1/2} n)^N A \chi_{C_{\theta}(n)} \|_{\mathcal{B}(L^2(\Omega))} < \infty, \]

where \( \chi_{C_{\theta}(n)} \) are the characteristic functions of cubes \( C_{\theta}(n) \).

First, we prepare two lemmas.

**Lemma 6.1.** For any integer \( N \) with \( N > d/2 \), there exists a constant \( C(d, N) > 0 \) such that

\[ \sum_{m \in \mathbb{Z}^d} \| \chi_{C_{\theta}(m)} A \chi_{C_{\theta}(n)} f \|_{L^2(\Omega)} \]

\[ \leq C(d, N) \left( \| A \|_{\mathcal{B}(L^2(\Omega))} + \theta^{-d/4} \| A \|_N^{d/2N} \| A \|_{\mathcal{B}(L^2(\Omega))}^{1-d/2N} \right) \| \chi_{C_{\theta}(n)} f \|_{L^2(\Omega)} \]

for all \( n \in \mathbb{Z}^d \), \( A \in \mathcal{A}_N \) and \( f \in L^2(\Omega) \).

**Proof.** Let \( n \in \mathbb{Z}^d \) be fixed. For any \( \omega > 0 \), we write

\[ \sum_{m \in \mathbb{Z}^d} \| \chi_{C_{\theta}(m)} A \chi_{C_{\theta}(n)} f \|_{L^2(\Omega)} \]

\[ = \sum_{|m-n| > \omega} \| \theta^{1/2} m - \theta^{1/2} n \|^{-N} \theta^{1/2} m - \theta^{1/2} n \|^{N} \| \chi_{C_{\theta}(m)} A \chi_{C_{\theta}(n)} f \|_{L^2(\Omega)} \]

\[ + \sum_{|m-n| \leq \omega} \| \chi_{C_{\theta}(m)} A \chi_{C_{\theta}(n)} f \|_{L^2(\Omega)} \]

\[ =: I(n) + II(n). \]
By using Schwarz inequality, we estimate $I(n)$ as

\[
I(n) \leq \theta^{-N/2} \left( \sum_{|m-n| > \omega} |m-n|^{-2N} \right)^{1/2} \times \\
\left( \sum_{|m-n| > \omega} \left| \theta^{1/2}m - \theta^{1/2}n \right|^{2N} \left\| \chi_{C_\theta(m)}A\chi_{C_\theta(n)}f \right\|_{L^2(\Omega)}^2 \right)^{1/2}.
\]

The first factor of (6.3) is estimated as

\[
\sum_{|m-n| > \omega} |m-n|^{-2N} = \sum_{|m| > \omega} |m|^{-2N}
\leq C(d, N)\omega^{-2N+d}.
\]

In fact, since $N > d/2$, the right member of (6.4) is estimated as

\[
\sum_{|m| > \omega} |m|^{-2N} \leq \prod_{j=1}^{d} \sum_{|m_j| > \omega^{1/d}} |m_j|^{-2N/d}
\leq C(d, N) \prod_{j=1}^{d} \sum_{|m_j| > \omega^{1/d}} (1 + |m_j|)^{-2N/d}
\leq C(d, N) \prod_{j=1}^{d} \int_{\sigma > \omega^{1/d}} \sigma^{-2N/d} d\sigma
\leq C(d, N) \prod_{j=1}^{d} \omega^{-2N/d+1}
= C(d, N)\omega^{-2N+d},
\]

which implies (6.4). As to the second factor of (6.3), noting that

\[
\frac{|\theta^{1/2}m - \theta^{1/2}n|}{2} \leq |x - \theta^{1/2}n|
\]

for any $x \in C_\theta(m)$, we estimate as

\[
\sum_{|m-n| > \omega} |\theta^{1/2}m - \theta^{1/2}n|^{2N} \left\| \chi_{C_\theta(m)}A\chi_{C_\theta(n)}f \right\|_{L^2(\Omega)}^2
= \sum_{|m-n| > \omega} |\theta^{1/2}m - \theta^{1/2}n|^{2N} \int_{C_\theta(m)} |A\chi_{C_\theta(n)}f|^2 dx
\leq 2^{2N} \sum_{|m-n| > \omega} \int_{C_\theta(m)} \left| |x - \theta^{1/2}n|^N A\chi_{C_\theta(n)}f \right|^2 dx.
\]
Moreover, by the definition (6.1) of $\|A\|_N$, we estimate as

$$
\sum_{|m-n|>\omega} \int_{C_\nu(m)} \left| x - \theta^{1/2} n \right|^N A \chi_{C_\nu(n)} f \, dx \leq \left\| x - \theta^{1/2} n \right|^N A \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}^2 \\
\leq \left| A \right|_{N}^2 \left\| \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}^2.
$$

Hence, summarizing the above two estimates, we deduce that

$$
(6.5) \quad \sum_{|m-n|>\omega} \left| \theta^{1/2} n - \theta^{1/2} m \right|^N \| \chi_{C_\nu(m)} A \chi_{C_\nu(n)} f \|_{L^2(\Omega)}^2 \leq 2^N \left| A \right|_{N}^2 \left\| \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}^2.
$$

Thus we find from (6.3)–(6.5) that

$$
(6.6) \quad I(n) \leq C(d, N) \theta^{-N/2} \omega^{-(N-d/2)} \| A \|_N \left\| \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}.
$$

Let us turn to the estimation of $II(n)$. We estimate

$$
II(n) \leq \left( \sum_{|m-n| \leq \omega} 1 \right)^{1/2} \left( \sum_{|m-n| \leq \omega} \left\| \chi_{C_\nu(m)} A \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}^2 \right)^{1/2}.
$$

Since

$$
\sum_{|m-n| \leq \omega} 1 \leq 1 + \omega^d,
$$

we deduce from the same argument as in $I(n)$

$$
(6.7) \quad II(n) \leq (1 + \omega^{d/2}) \left( \sum_{|m-n| \leq \omega} \left\| \chi_{C_\nu(m)} A \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}^2 \right)^{1/2}
\leq (1 + \omega^{d/2}) \| A \chi_{C_\nu(n)} f \|_{L^2(\Omega)}
\leq (1 + \omega^{d/2}) \| A \|_{\mathcal{B}(L^2(\Omega))} \left\| \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}.
$$

Combining the estimates (6.6) and (6.7), we get

$$
\sum_{m \in \mathbb{Z}^d} \left\| \chi_{C_\nu(m)} A \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)} \leq C(d, N) \left\{ \theta^{-N/2} \omega^{-(N-d/2)} \| A \|_N + (1 + \omega^{d/2}) \| A \|_{\mathcal{B}(L^2(\Omega))} \right\} \left\| \chi_{C_\nu(n)} f \right\|_{L^2(\Omega)}.
$$

Finally, taking $\omega = (\| A \|_N / \| A \|_{\mathcal{B}(L^2(\Omega))})^{1/N} \cdot \theta^{-1/2}$, we obtain the required estimate (6.2). The proof of Lemma 6.1 is complete. \hfill \square

**Lemma 6.2.** Assume that the measurable potential $V = V_+ - V_-$ satisfies $V_+ \in K_d(\Omega)$, and that $V_-$ satisfies assumption (3.4) in Lemma 3.2. Let $N$ be a positive integer, and let $\psi \in \mathcal{S}(\mathbb{R})$. Then $\psi(R_{V, \theta}) \in \mathcal{S}_N$. Furthermore, there exists a constant $C_{\psi} > 0$ such that

$$
(6.8) \quad \| \psi(R_{V, \theta}) \|_{\mathcal{B}(L^2(\Omega))} \leq C_{\psi}, \quad \forall \theta > 0,
$$

and there exists a constant $C_N > 0$ such that

$$
(6.9) \quad \| \psi(R_{V, \theta}) \|_N \leq C_N \theta^{N/2} \int_{-\infty}^{\infty} (1 + |t|)^N |\hat{\psi}(t)| \, dt, \quad \forall \theta > 0.
$$
Proof. The proof is based on the well-known formula:

\[
(6.10) \quad \psi(R_{V,\theta}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-itR_{V,\theta}} \hat{\psi}(t) \, dt,
\]

where \( \hat{\psi} \) is the Fourier transform of \( \psi \) on \( \mathbb{R} \). The estimate (6.8) is an immediate consequence of the unitarity of \( e^{-itR_{V,\theta}} \), the formula (6.10) and \( \psi \in \mathcal{S}(\mathbb{R}) \).

As to the estimate (6.9), applying the formula (6.10), we obtain

\[
\|\psi(R_{V,\theta})\|_N = \sup_{n \in \mathbb{Z}^d} \left\| -\theta^{1/2} n |N| \psi(R_{V,\theta}) \chi_{C_\theta(n)} \right\|_{\mathcal{B}(L^2(\Omega))} \leq (2\pi)^{-1/2} \sup_{n \in \mathbb{Z}^d} \int_{-\infty}^{\infty} \left\| -\theta^{1/2} n |N| e^{-itR_{V,\theta}} \chi_{C_\theta(n)} \right\|_{\mathcal{B}(L^2(\Omega))} \left| \hat{\psi}(t) \right| \, dt.
\]

Resorting to Lemma B.1 for \( A = B = x_j - \theta^{1/2} n_j \) and \( L = e^{-itR_{V,\theta}} \), we find from Proposition 5.1 that

\[
\| -\theta^{1/2} n |N| e^{-itR_{V,\theta}} \chi_{C_\theta(n)} \|_{\mathcal{B}(L^2(\Omega))} \leq \sum_{k=0}^{N} C(N, k) \| \text{Ad}^k(e^{-itR_{V,\theta}}) \|_{\mathcal{B}(L^2(\Omega))} \left\| -\theta^{1/2} n |N-k| \chi_{C_\theta(n)} \right\|_{\mathcal{B}(L^2(\Omega))} \leq \sum_{k=0}^{N} C(N, k) \theta^{k/2} (1 + |t|)^k \theta^{(N-k)/2},
\]

thus we conclude that

\[
\|\psi(R_{V,\theta})\|_N \leq \theta^{N/2} \sum_{k=0}^{N} C(N, k) \int_{-\infty}^{\infty} (1 + |t|)^k |\hat{\psi}(t)| \, dt,
\]

which proves (6.9). The proof of Lemma 6.2 is finished.

We are now in a position to prove the main theorem.

Proof of Theorem 1.1. It suffices to show \( L^1 \)-boundedness of \( \varphi(\theta H_V) \). Let \( \beta > d/4 \) and \( M > 0 \). Let us define \( \psi \in \mathcal{S}(\mathbb{R}) \) as

\[
\psi(\mu) := \mu^{-\beta} \varphi(\mu^{-1} - M), \quad \mu \in (0, 1/M].
\]

Then we can write

\[
(6.11) \quad \psi((\lambda + M)^{-1}) = \varphi(\lambda)(\lambda + M)^{\beta}, \quad \lambda \geq 0.
\]

Now we estimate, by Hölder’s inequality and the definition of amalgam spaces \( \ell^p(L^q)_\theta \),

\[
\|\varphi(\theta H_V)f\|_{L^1(\Omega)} = \sum_{n \in \mathbb{Z}^d} \|\varphi(\theta H_V)f\|_{L^1(C_\theta(n))} \leq \sum_{n \in \mathbb{Z}^d} |C_\theta(n)|^{1/2} \|\varphi(\theta H_V)f\|_{L^2(C_\theta(n))} \leq \theta^{d/4} \|\varphi(\theta H_V)f\|_{\ell^1(L^2)_{\theta}},
\]
where we used $|C_\theta(n)|^{1/2} = \theta^{d/4}$. The right member in the above inequality is estimates as
\[ \|\varphi(\theta H_V)f\|_{\ell^1(L^2)_\theta} = \|\varphi(\theta H_V)(\theta H_V + M)^\beta R_{V,\theta}^\beta f\|_{\ell^1(L^2)_\theta} \]
\[ = \|\psi(R_{V,\theta})R_{V,\theta}^\beta f\|_{\ell^1(L^2)_\theta} \leq \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \|\chi_{C_\theta(m)} \psi(R_{V,\theta}) \chi_{C_\theta(n)} R_{V,\theta}^\beta f\|_{L^2(\Omega)}, \]
where we used (6.11) in the second step. Resorting to Lemma 6.1 for $A$ and $f$ replaced by $\psi(R_{V,\theta})$ and $R_{V,\theta}^\beta f$, respectively, we estimate
\[ \sum_{m \in \mathbb{Z}^d} \|\chi_{C_\theta(m)} \psi(R_{V,\theta}) \chi_{C_\theta(n)} R_{V,\theta}^\beta f\|_{L^2(\Omega)} \leq C \left( \|\psi(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))} + \theta^{-d/4} \|\psi(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))}^{d/2N} \|\psi(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))}^{1-d/2N} \right) \|\chi_{C_\theta(n)} R_{V,\theta}^\beta f\|_{L^2(\Omega)} \]
for any $N > d/2$. Thus we obtain
\[ \|\varphi(\theta H_V)f\|_{L^1(\Omega)} \leq C \theta^{d/4} \left( \|\psi(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))} + \theta^{-d/4} \|\psi(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))}^{d/2N} \|\psi(R_{V,\theta})\|_{\mathcal{B}(L^2(\Omega))}^{1-d/2N} \right) \|R_{V,\theta}^\beta f\|_{\ell^1(L^2)_\theta} \]
for any $N > d/2$. Applying Theorem 4.1 and Lemma 6.2 to the above estimate, we conclude that
\[ \|\varphi(\theta H_V)f\|_{L^1(\Omega)} \leq C \theta^{d/4} \left\{ 1 + \theta^{-d/4} \cdot (\theta^{N/2})^{d/2N} \right\} \theta^{-d/4} \|f\|_{L^1(\Omega)} \leq C \|f\|_{L^1(\Omega)}, \]
where the constant $C$ is independent of $\theta$. The proof of Theorem 1.1 is complete. \qed

7. A FINAL REMARK

As a consequence of Theorems 1.1 and 4.1, we have $L^p$-$L^q$-boundedness of $\varphi(\theta H_V)$. $L^p$-$L^q$-boundedness of $\varphi(\theta H_V)$ is useful to prove the embedding theorem for Besov spaces.

**Proposition 7.1.** Let $\varphi$ and $V$ be as in Theorem 1.1. Then there exists a constant $C = C(d, \varphi) > 0$ such that for $1 \leq p \leq q \leq \infty$,
\[ \|\varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega), L^q(\Omega))} \leq C \theta^{-(d/2)(1/p-1/q)}, \quad \forall \theta > 0. \]

**Proof.** Let us define $\bar{\varphi} \in \mathcal{S}'(\mathbb{R}^d)$ as
\[ \bar{\varphi}(\lambda) = (\lambda + M)^\beta \varphi(\lambda), \quad \lambda \geq 0. \]
By Theorems 1.1 and 4.1, for $1 \leq p \leq q \leq \infty$ and $\beta > (d/2)(1/p-1/q)$, we estimate
\[ \|\varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega), L^q(\Omega))} = \|\varphi(\theta H_V)(\theta H_V + M)^\beta (\theta H_V + M)^{-\beta}\|_{\mathcal{B}(L^p(\Omega), L^q(\Omega))} \]
\[ \leq \|\bar{\varphi}(\theta H_V)\|_{\mathcal{B}(L^q(\Omega))} \|(\theta H_V + M)^{-\beta}\|_{\mathcal{B}(L^p(\Omega), L^q(\Omega))} \leq C \theta^{-(d/2)(1/p-1/q)}. \]
The proof of Proposition 7.1 is complete. \qed
In this appendix we introduce the Young inequality for scaled amalgam spaces.

**Lemma A.1.** Let \( d \geq 1 \), and let \( 1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty \) be such that

\[
\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p} \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} - 1 = \frac{1}{q}.
\]

If \( f \in \ell^{p_1}(L^{q_1})_\theta(\mathbb{R}^d) \) and \( g \in \ell^{p_2}(L^{q_2})_\theta(\mathbb{R}^d) \), then

\[
\int_{\mathbb{R}^d} f(\cdot - y) g(y) \, dy \in \ell^{p(L^q)_\theta}(\mathbb{R}^d)
\]

and

\[
\left\| \int_{\mathbb{R}^d} f(\cdot - y) g(y) \, dy \right\|_{\ell^{p(L^q)_\theta}(\mathbb{R}^d)} \leq 3^d \|f\|_{\ell^{p_1}(L^{q_1})_\theta(\mathbb{R}^d)} \|g\|_{\ell^{p_2}(L^{q_2})_\theta(\mathbb{R}^d)}.
\]

For the proof of Lemma A.1, see Fournier and Stewart [3].

**APPENDIX B. (Recursive formula of operators)**

In this appendix we introduce some formulas on the operator \( \text{Ad} \).

**Lemma B.1** (Lemma 3.1 from [7]). Let \( X \) and \( Y \) be topological vector spaces, and let \( A \) and \( B \) be continuous linear operators from \( X \) and \( Y \) into themselves, respectively. If \( L \) is a continuous linear operator from \( X \) into \( Y \), then there exists a set of constants \( \{C(n, m) \mid n \geq 0, 0 \leq m \leq n\} \) such that

\[
B^n L = \sum_{m=0}^{n} C(n, m) \text{Ad}^m(L) A^{n-m}.
\]

We shall derive two kind of recursive formulas of operator

\[
R_{V,\theta} = (\theta H_V + M)^{-1},
\]

where \( M \) is a certain large constant. Hereafter we put

\[
X = \mathcal{D}(\Omega), \quad Y = \mathcal{D}'(\Omega),
\]

where we denote by \( \mathcal{D}(\Omega) \) the totality of the test functions on \( \Omega \), and by \( \mathcal{D}'(\Omega) \) its dual space, and we take

\[
A = B = x_j - \theta^{1/2} n_j \quad \text{for some} \ j \in \{1, \ldots, d\}.
\]

**Lemma B.2.** Let \( V \) be a measurable function on \( \Omega \) such that \( H_V \) is a self-adjoint operator on \( L^2(\Omega) \) whose domain is given by

\[
\mathcal{D}(H_V) = \{u \in H^1_0(\Omega) \mid H_V u \in L^2(\Omega)\}.
\]

Let \( M \) be an element of resolvent set of \(-\theta H_V\), and let us denote by \( R_{V,\theta} \) the resolvent operator defined by (B.2). Then the sequence \( \{\text{Ad}^k(R_{V,\theta})\}_{k=0}^\infty \) of operators satisfies the following recursive formula:

\[
\text{Ad}^0(R_{V,\theta}) = R_{V,\theta}, \quad \text{Ad}^1(R_{V,\theta}) = -2\theta R_{V,\theta} \partial_{x_j} R_{V,\theta},
\]

and for \( k \geq 2,

\[
\text{Ad}^k(R_{V,\theta}) = \theta \{-2k \text{Ad}^{k-1}(R_{V,\theta}) \partial_{x_j} R_{V,\theta} + k(k-1) \text{Ad}^{k-2}(R_{V,\theta}) R_{V,\theta} \}.
\]
Proof. When \( k = 0 \), the first equation in (B.3) is trivial. Hence it is sufficient to prove the case when \( k > 0 \). For the sake of simplicity, we perform a formal argument without considering the domain of operators. The rigorous argument is given in the final part.

Let us introduce the generalized binomial coefficients \( \Gamma(k, m) \) as follows:

\[
\Gamma(k, m) = \begin{cases} 
  \frac{k!}{(k-m)!m!}, & k \geq m \geq 0, \\
  0, & k < m \text{ or } k < 0.
\end{cases}
\]

Once the following recursive formula is established:

(B.5) \( \text{Ad}^k(R_{V,\theta}) = - \sum_{m=0}^{k-1} \Gamma(k, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{k-m}(\theta H_V) R_{V,\theta}, \quad k = 1, 2, \ldots \),

identities (B.3) and (B.4) are an immediate consequence of (B.5), since

\[
\text{Ad}^1(\theta H_V) = 2 \theta \partial_{x_j}, \quad \text{Ad}^2(\theta H_V) = -2 \theta, \quad \text{Ad}^k(\theta H_V) = 0, \quad k \geq 3.
\]

Hence, all we have to do is to prove (B.5). We proceed the argument by induction. For \( k = 1 \), it can be readily checked that

\[
\text{Ad}^1(R_{V,\theta}) = x_j R_{V,\theta} - R_{V,\theta} x_j = R_{V,\theta}(\theta H_V + M)x_j R_{V,\theta} - R_{V,\theta} x_j R_{V,\theta},
\]

\[
= R_{V,\theta}(\theta H_V x_j - x_j \cdot \theta H_V) R_{V,\theta} = - R_{V,\theta} \text{Ad}^1(\theta H_V) R_{V,\theta} = - \Gamma(1, 0) \text{Ad}^0(R_{V,\theta}) \text{Ad}^1(\theta H_V) R_{V,\theta}.
\]

Hence (B.5) is true for \( k = 1 \). Let us suppose that (B.5) holds for \( k = 1, \ldots, \ell \).

Writing

(B.6) \( \text{Ad}^{\ell+1}(R_{V,\theta}) = x_j \text{Ad}^\ell(R_{V,\theta}) - \text{Ad}^\ell(R_{V,\theta}) x_j, \)

we see that the first term becomes

\[
x_j \text{Ad}^\ell(R_{V,\theta})
= x_j \left\{ - \sum_{m=0}^{\ell-1} \Gamma(\ell, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m}(\theta H_V) \right\} R_{V,\theta}
= - \sum_{m=0}^{\ell-1} \Gamma(\ell, m) \left\{ \text{Ad}^{m+1}(R_{V,\theta}) \text{Ad}^{\ell-m}(\theta H_V) + \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m+1}(\theta H_V) \right\} R_{V,\theta}
= - \sum_{m=0}^{\ell-1} \Gamma(\ell, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m}(\theta H_V) x_j R_{V,\theta}
= : I_1 + I_2.
\]
Here \( I_1 \) is written as
\[
I_1 = - \sum_{m=1}^{\ell} \Gamma(\ell, m-1) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m+1}(\theta H_V) R_{V,\theta}
\]
\[
- \sum_{m=0}^{\ell-1} \Gamma(\ell, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m+1}(\theta H_V) R_{V,\theta}
\]
\[
= - \sum_{m=0}^{\ell} \Gamma(\ell, m-1) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m}(\theta H_V) R_{V,\theta}
\]
\[
- \frac{\ell}{\ell} \sum_{m=0}^{\ell} \Gamma(\ell, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m}(\theta H_V) + \text{Ad}^\ell(R_{V,\theta}) \text{Ad}^1(\theta H_V) R_{V,\theta}
\]
\[
= - \sum_{m=0}^{\ell} \Gamma(\ell + 1, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell+1-m}(\theta H_V) + \text{Ad}^\ell(R_{V,\theta}) \text{Ad}^1(\theta H_V) R_{V,\theta},
\]
where we used
\[
\Gamma(\ell, m-1) + \Gamma(\ell, m) = \Gamma(\ell + 1, m)
\]
in the last step. As to \( I_2 \), we write as
\[
I_2 = - \left\{ \sum_{m=0}^{\ell-1} \Gamma(\ell, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell-m}(\theta H_V) R_{V,\theta} \right\} (\theta H_V + M) x_j R_{V,\theta}
\]
\[
= \text{Ad}^\ell(R_{V,\theta})(\theta H_V + M) x_j R_{V,\theta}.
\]
Hence, summarizing the previous equations, we get
\[
x_j \text{Ad}^\ell(R_{V,\theta}) = - \sum_{m=0}^{\ell} \Gamma(\ell + 1, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell+1-m}(\theta H_V)
\]
\[
+ \text{Ad}^\ell(R_{V,\theta}) \left\{ \text{Ad}^1(\theta H_V) + (\theta H_V + M) x_j \right\} R_{V,\theta}.
\]
Therefore, going back to (B.6), and noting
\[
\text{Ad}^1(\theta H_V) + (\theta H_V + M) x_j = x_j (\theta H_V + M),
\]
we conclude that
\[
\text{Ad}^{\ell+1}(R_{V,\theta}) = - \sum_{m=0}^{\ell} \Gamma(\ell + 1, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell+1-m}(\theta H_V)
\]
\[
+ \text{Ad}^\ell(R_{V,\theta}) \left\{ \text{Ad}^1(\theta H_V) + (\theta H_V + M) x_j \right\} R_{V,\theta} - \text{Ad}^\ell(R_{V,\theta}) x_j
\]
\[
= - \sum_{m=0}^{\ell} \Gamma(\ell + 1, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell+1-m}(\theta H_V)
\]
\[
+ \text{Ad}^\ell(R_{V,\theta}) x_j (\theta H_V + M) R_{V,\theta} - \text{Ad}^\ell(R_{V,\theta}) x_j
\]
\[
= - \sum_{m=0}^{\ell} \Gamma(\ell + 1, m) \text{Ad}^m(R_{V,\theta}) \text{Ad}^{\ell+1-m}(\theta H_V).
\]
Therefore (B.5) is true for \( k = \ell + 1 \).

The above proof is formal in the sense that the domain of operators is not taken into account in the argument. In fact, even for \( f \in C_0^\infty(\Omega) \), each \( x_jR_{V,\theta}f \) does not necessarily belong to the domain of \( H_V \), since we only know the fact that

\[
R_{V,\theta}f \in \mathcal{D}(H_V) = \{ u \in H_0^1(\Omega) \mid H_Vu \in L^2(\Omega) \}.
\]

Therefore, we should perform the argument by using a duality pair \( \varphi(\Omega) \langle \cdot, \cdot \rangle_{\varphi(\Omega)} \) of \( \mathcal{D}'(\Omega) \) and \( \mathcal{D}(\Omega) \) in a rigorous way. We may prove the lemma only for \( k = 1 \). For, as to the case \( k > 1 \), the argument is done in a similar manner. Now we write

\[
\varphi'(\Omega) \langle \text{Ad}^1(R_{V,\theta})f, g \rangle_{\varphi(\Omega)} = \langle R_{V,\theta}f, x_jg \rangle_{L^2(\Omega)} - \langle x_jf, R_{V,\theta}g \rangle_{L^2(\Omega)} =: I - II
\]

for \( f, g \in C_0^\infty(\Omega) \). Since \( R_{V,\theta}f, R_{V,\theta}g \in H_0^1(\Omega) \), there exist two sequences \( \{ f_n \} \), \( \{ g_m \} \) in \( C_0^\infty(\Omega) \) such that

\[
f_n \to R_{V,\theta}f \quad \text{and} \quad g_m \to R_{V,\theta}g \quad \text{in} \quad H^1(\Omega) \quad (n, m \to \infty).
\]

Hence we obtain by \( x_jf_n, x_jg_m \in C_0^\infty(\Omega) \),

\[
I = \lim_{n \to \infty} \langle f_n, x_jg \rangle_{L^2(\Omega)}
= \lim_{n \to \infty} \langle x_jf_n, (\theta H_V + M)R_{V,\theta}g \rangle_{L^2(\Omega)}
= \lim_{n \to \infty} \left\{ \theta \langle \nabla(x_jf_n), \nabla R_{V,\theta}g \rangle_{L^2(\Omega)} + \langle (\theta V + M)x_jf_n, R_{V,\theta}g \rangle_{L^2(\Omega)} \right\}
= \lim_{n,m \to \infty} \left\{ \theta \langle \nabla(x_jf_n), \nabla g_m \rangle_{L^2(\Omega)} + \langle (\theta V + M)x_jf_n, g_m \rangle_{L^2(\Omega)} \right\}
= \lim_{n,m \to \infty} \left\{ \theta \langle f_n, \partial x_j g_m \rangle_{L^2(\Omega)} + \theta \langle x_j \nabla f_n, \nabla g_m \rangle_{L^2(\Omega)} + \langle (\theta V + M)x_jf_n, g_m \rangle_{L^2(\Omega)} \right\}
\]

and

\[
II = \lim_{m \to \infty} \langle x_jf, g_m \rangle_{L^2(\Omega)}
= \lim_{m \to \infty} \langle (\theta H_V + M)R_{V,\theta}f, x_jg_m \rangle_{L^2(\Omega)}
= \lim_{m \to \infty} \left\{ \theta \langle \nabla R_{V,\theta}f, \nabla(x_jg_m) \rangle_{L^2(\Omega)} + \langle (\theta V + M)x_jR_{V,\theta}f, g_m \rangle_{L^2(\Omega)} \right\}
= \lim_{n,m \to \infty} \left\{ \theta \langle \nabla f_n, \nabla(x_jg_m) \rangle_{L^2(\Omega)} + \langle (\theta V + M)x_jf_n, g_m \rangle_{L^2(\Omega)} \right\}
= \lim_{n,m \to \infty} \left\{ \theta \langle \partial x_j f_n, g_m \rangle_{L^2(\Omega)} + \theta \langle x_j \nabla f_n, \nabla g_m \rangle_{L^2(\Omega)} + \langle (\theta V + M)x_jf_n, g_m \rangle_{L^2(\Omega)} \right\}.
\]

Then, combining the above equations, we deduce that

\[
\varphi'(\Omega) \langle \text{Ad}^1(R_{V,\theta})f, g \rangle_{\varphi(\Omega)} = \lim_{n,m \to \infty} \theta \left\{ \langle f_n, \partial x_j g_m \rangle_{L^2(\Omega)} - \langle x_jf_n, g_m \rangle_{L^2(\Omega)} \right\}
= \lim_{n,m \to \infty} \theta \langle -2\partial x_j f_n, g_m \rangle_{L^2(\Omega)}
= \langle -2\theta \partial x_j R_{V,\theta}f, R_{V,\theta}g \rangle_{L^2(\Omega)}
= \langle -2\theta R_{V,\theta} \partial x_j R_{V,\theta}f, g \rangle_{L^2(\Omega)}
\]
for any $f, g \in C_0^\infty(\Omega)$. Thus (B.3) is valid in a distributional sense. In a similar
way, (B.4) can be also shown in a distributional sense. The proof of Lemma B.2 is
finished. □

**Lemma B.3.** Assume that $V$ satisfies the same assumption as in Lemma B.2. Let $A$, $B$ and $L$
be as in Lemma B.2. Then the following formula holds for each $t > 0$:

\[
(B.7) \quad \text{Ad}^1(e^{-itRV,\theta}) = -i \int_0^t e^{-isRV,\theta} \text{Ad}^1(RV,\theta)e^{-i(t-s)RV,\theta} \, ds.
\]

Furthermore, the following formulas hold for $k > 1$:

\[
(B.8) \quad \text{Ad}^{k+1}(e^{-itRV,\theta}) = -i \int_0^t \sum_{k_1+k_2+k_3=k} \Gamma(k_1, k_2, k_3) \text{Ad}^{k_1}(e^{-isRV,\theta}) \text{Ad}^{k_2+1}(RV,\theta) \text{Ad}^{k_3}(e^{-i(t-s)RV,\theta}) \, ds,
\]

where the constants $\Gamma(k_1, k_2, k_3) \quad (k_1, k_2, k_3 \geq 0)$ are trinomial coefficients:

\[
\Gamma(k_1, k_2, k_3) = \frac{k!}{k_1!k_2!k_3!}.
\]

**Proof.** It is sufficient to prove the lemma without taking account of the domain of
operators as in the proof of Lemma B.2. We write

\[
\text{Ad}^1(e^{-itRV,\theta}) = x_j e^{-itRV,\theta} - e^{-itRV,\theta} x_j
\]

\[
= - \int_0^t \frac{d}{ds} (e^{-isRV,\theta} x_j e^{-i(t-s)RV,\theta}) \, ds
\]

\[
= -i \int_0^t e^{-isRV,\theta} (x_j RV,\theta - RV,\theta x_j) e^{-i(t-s)RV,\theta} \, ds
\]

\[
= -i \int_0^t e^{-isRV,\theta} \text{Ad}^1(RV,\theta)e^{-i(t-s)RV,\theta} \, ds.
\]

This proves (B.7). The proof of (B.8) is performed by induction argument. So we
may omit the details. The proof of Lemma B.3 is complete. □

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