LOWER ORDER TENSORS IN NON-KÄHLER GEOMETRY AND NON-KÄHLER GEOMETRIC FLOW

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Abstract. In recent years, Streets and Tian introduced a series of curvature flows to study non-Kähler geometry. In this paper, we study how to construct second order curvature flows in a uniform way, under some natural assumptions which holds in Streets and Tian’s works. As a result, by classifying the lower order tensors, we classify the second order curvature flows in almost Hermitian, almost Kähler and Hermitian geometries in certain sense.

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1. Introduction

In 1982, Hamilton [11] introduced Ricci Flow \( \frac{\partial}{\partial t} g = -2Ric \) on Riemannian manifolds and showed that compact 3-manifold with positive Ricci curvature is spherical space form. After that, many people studied Ricci Flow intensively. In 2002, Perelman [16][17][18] did breakthrough that by using Ricci Flow he proved Thurston’s Geometric Conjecture and as a corollary, Poincaré Conjecture. Since the method of curvature flow is so powerful, people tried to use the similar idea to study other geometric objects. In 1985, Cao [3] initialized the study of Ricci Flow on Kähler manifolds, which is Kähler Ricci Flow. He showed that if we fixed the complex structure \( J \), Ricci Flow preserves Kähler structure.

To generalize Kähler Ricci Flow to non-Kähler case, Streets and Tian introduced a series of second order curvature flows on this subject, including Hermitian Curvature Flow [25], Symplectic Curvature Flow [26], Almost Hermitian Curvature Flow [26], Pluriclosed Flow [23], Pluriclosed Flow on generalized Kähler manifolds [27]. Along this direction, other people constructed new flows. Vezzoni generalized Hermitian Curvature Flow to almost Hermitian setting [30]. The author unified Symplectic Curvature Flow and Pluriclosed Flow in almost Hermitian setting [4]. For relevant studies, one may refer [1][3][6][7][8][12][13][19][20][21][22][23][28]. Besides these series of works, there are other ways to define second order curvature flows on non-Kähler geometry. Gill, Tosatti and Weinkove studied Chern Ricci Flow on Hermitian manifolds [10][29]. Wang and Lê studied Anti-Complexified Ricci Flow on almost Kähler manifolds [14].

In this paper, we focus on three kinds of non-Kähler geometries, almost Hermitian geometry, almost Kähler geometry and Hermitian geometry. We discuss how to construct second order curvature flows in certain “canonical” sense. Basically, we show that the geometric flows defined by Streets
and Tian in above three geometries have some “canonical” uniqueness in certain sense.

Let \((g, J, \omega)\) be an almost Hermitian structure. Let \(T = T(g, J, \omega)\) be a tensor defined from \((g, J, \omega)\). Notice that \((g, -J, \omega)\) is also an almost Hermitian structure. We call \(T\) is of even type if \(T(g, J, \omega) = T(g, -J, \omega)\), say \(\text{Ric}(X, Y)\). We call \(T\) is of odd type if \(T(g, J, \omega) = -T(g, -J, \omega)\), say \(\text{Ric}(JX, Y)\). We call \(T\) scales as \(r^0\) if \(T(r^0g, J, r^0\omega) = T(g, J, \omega)\).

Let \(T\) be a 2-tensor. Consider \(J\) action on \(T\), i.e., \(J^\star T(X, Y) = T(JX, JY)\). We denote \(T(1, 1)(X, Y) := \frac{1}{2}(T(X, Y) + T(JX, JY))\), \(T(0, 2) + (2, 0)(X, Y) := \frac{1}{2}(T(X, Y) - T(JX, JY))\).

Consider transposition action on \(T\), i.e., \(t T(X, Y) = T(Y, X)\). We denote \(T_{\text{sym}}(X, Y) := \frac{1}{2}(T(X, Y) + T(Y, X))\), \(T_{\text{skew}}(X, Y) := \frac{1}{2}(T(X, Y) - T(Y, X))\).

Let \((g_t, J_t, \omega_t)\) be a family of almost Hermitian structures. Omit time subscript \(t\), let \(\frac{\partial}{\partial t} g = h, \frac{\partial}{\partial t} J = K, \frac{\partial}{\partial t} \omega = \eta\).

Notice that every two of \((h, K, \eta)\) decide the third one. We may focus on the pair \((h, K)\). We construct geometric flows under the following natural assumption.

**Assumption (A)**

1. \((h, K)\) is tensorial defined from \((g, J, \omega)\) and is differential operator with respect to \((g, J, \omega)\).
2. \((h, K)\) is a second order system with respect to \((g, J)\).
3. \(h\) is of even type, and \(K\) is of odd type.
4. \(h\) scales as \(r^0\) and \(K\) scales as \(r^{-2}\).
5. Modulo gauge transformation, the symbol of the system \((h, K)\) is identity, where the gauge transformation is generated by a vector field defined from \((g, J, \omega, D)\), where \(D\) is a fixed linear connection.

In almost Hermitian setting, we have the following result.

**Theorem 1.1.** In almost Hermitian geometry, under Assumption (A), the geometric flow is defined of the following form.

\[
\begin{align*}
\frac{\partial}{\partial t} g &= -2\text{Ric} + aL_{\partial t}g + Q_1 \\
\frac{\partial}{\partial t} J &= \triangle J + \mathcal{N} + \mathcal{R} + aL_{\partial t}J + Q_2,
\end{align*}
\]

where \(a \in \mathbb{R}, Q_1, Q_2\) are of the form \(DJ \ast DJ\), which satisfies the algebraic necessary condition \(Q_1\) is symmetric, \(Q_2\) is \((0, 2) + (2, 0)\), \(Q_1^{(0,2)}(2,0) = Q_2^{\text{sym}}J\).

\((-2\text{Ric}, \triangle J + \mathcal{N} + \mathcal{R})\) is second order term which gives “good” symbol of this system, where \(\text{Ric}\) is the Ricci curvature with respect to Levi-Civita connection \(D\), \(\triangle\) is the rough Laplacian with respect to Levi-Civita connection \(D\),

\[
\begin{align*}
g(\mathcal{N}(X), Y) &= g^{ab}g(D_aJ(JX), D_bJ(Y)), \\
g(\mathcal{R}(X), Y) &= \text{Ric}(JX, Y) + \text{Ric}(X, JY), \\
\theta(X) &= g^{ij}DJ(e_i, Je_j, X).
\end{align*}
\]
And \((aL_g, aL_{\bar{g}}, J)\) is from gauge transformation. 

We notice that Almost Hermitian Curvature Flow in \([26]\) is in the family above.

In almost Kähler setting, we need to require \(d\eta = 0\). One natural option is \(\eta = P\), where \(P\) is Chern form up to a factor. And from Appendix, we see in almost Kähler setting, Chern connection is the unique Hermitian connection. Then we have the following result.

**Theorem 1.2.** In almost Kähler geometry, under Assumption (A), furthermore we suppose \(\frac{\partial}{\partial t}\omega = P\), then the only geometric flow is defined as Symplectic Curvature Flow in \([26]\).

\[
\frac{\partial}{\partial t}\omega = P \\
\frac{\partial}{\partial t}g = \Delta J + N + R.
\]

Notice that in \([26]\), the authors mentioned that one may modify Symplectic Curvature Flow by adding some first order terms, but here we rule out this possibility.

In Hermitian setting, we should require the integrability of \(J\) preserves. We would like to fix the complex structure \(J\). In this case, the system is not gauge invariant, so we assume \((5')\) the symbol of \(h\) is identity with respect to \(g\) (not modulo gauge transformation).

**Theorem 1.3.** In Hermitian geometry, if we assume \(\frac{\partial}{\partial t}J = 0\), under (1)(2)(3)(4) in Assumption (A) and \((5')\) above, the geometric flow is defined of the following form

\[
\frac{\partial}{\partial t}\omega = S + Q \\
\frac{\partial}{\partial t}J = 0,
\]

where \(Q = a_1B^1J + a_2B^2J + a_3(B^5)^{(1,1)}J + a_4(B^6)^{sym}J\).

Here \(S(X, Y) = g^{ij}g(\Omega(e_i, Je_j, X), Y), \Omega\) is the curvature with respect to Chern connection. And the definition of \(B^i\) is given in section [4].

To establish the above results, first we show algebraic necessary condition and analytic sufficient condition of deforming almost Hermitian structures. Then to find suitable tensors satisfying these conditions, we classify lower order tensors in the corresponding geometry, more precisely, first order tensors, second order tensors and gauge terms. Finally, by calculating symbols, we find out the desired tensors.

We organize the paper as follows. In Section [2] we recall some preliminaries in almost Hermitian geometry and fix some notations. In Section [3] we derive algebraic necessary condition and analytic sufficient condition to deform almost Hermitian structures, which forms the main assumption in constructing the curvature flows. In Section [4] we classify the lower order tensors in almost Hermitian setting, almost Kähler setting and Hermitian setting. In Section [5] we calculate the symbols of second order tensors and then classify the second order curvature flow in almost Hermitian, almost Kähler and Hermitian geometries. In Appendix, we recall the classification of Hermitian connections.

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## 2. Preliminaries

Let \(M\) be a manifold, \(J\) be a section of \(\text{End}(TM)\). We call \(J\) an almost complex structure if \(J^2 = -1\). An almost complex structure \(J\) is called integrable if \(J\) is induced by holomorphic
coordinates. By the theorem of Newlander-Nirenberg [15], \( J \) is integrable if and only if
\[
N_J(X,Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]
\]
is Nijenhuis tensor.

We call \((g, J, \omega)\) is an almost Hermitian structure if the following conditions hold.
1. \( g \) is a Riemannian metric.
2. \( J \) is an almost complex structure.
3. \((g, J)\) is compatible, i.e., \( g(JX, JY) = g(X, Y) \).
4. We define \( \omega(X, Y) = g(JX, Y) \).

Moreover \((g, J, \omega)\) is Hermitian structure if \( J \) is integrable, and \((g, J)\) is almost Kähler structure if
\[ d\omega = 0. \]
Furthermore \((g, J)\) is Kähler structure if \( J \) is integrable and \( d\omega = 0 \).

We fix some notations first.

Notations:
Let \((g, J, \omega)\) be an almost Hermitian structure.
1. Let \( D \) denote Levi-Civita connection and \( D \) is extended to tensor fields. For instance,
\[
DJ(X, Y) = (DX)JY = DJX(Y) - J(DX)Y.
\]
2. We implicitly identify \( TM \) and \( T^*M \) by using \( g \), i.e., for instance,
\[
DJ(X, Y, Z) = g(DJ(X, Y), Z).
\]
3. Usually, we use \( i \) instead of \( e_i = \frac{\partial}{\partial x_i} \) for short. We use orthonormal basis at one point, and often we assume it is normal. The same index means to take (real) trace with respect to \( g \). For complex trace, we mean that \( \omega_{ij} = T_{i, Ji} \), and it equals to \( T(i, Ji) \), where \( i \) goes over all the orthonormal basis.

Remark 2.1. Since \( g \) is parallel with respect to our connection, the notation (2) is safe if we do tensor calculation in a fixed geometric structure. But we should take care if we calculate evolution equations.

Remark 2.2. We call \( T \) is \((1,1)\) \(((0,2) + (2,0))\), if \( T^{(0,2)} + (2,0) = 0 \ (T^{(1,1)} = 0) \). Then by our identification, in almost Hermitian setting,
\[
T \text{ is } (1,1) \iff TJ = JT,
\]
\[
T \text{ is } (0,2) + (2,0) \iff TJ = -JT.
\]

Proof. We only prove the second identity. By definition,
\[
\langle (TJ + JT)X, Y \rangle = T(JX, Y) - T(X, JY) = 2T^{(1,1)}(JX, Y).
\]

For contraction, we have the following lemma.

Lemma 2.3. In the following cases, the tensors will vanish.
(a) Taking trace of a skew 2-tensor.
(b) Taking complex trace of a symmetric 2-tensor.
(c) Either taking trace or complex trace of a \((0,2) + (2,0)\) 2-tensor.

Proof. We only prove (c). Since \( \{e_i\} \) is a orthonormal basis and \( J \) is isometry, \( \{Je_i\} \) is also a orthogonal basis. So to take trace, we may replace \( i \) by \( Ji \). By definition, we have
\[
T(i, i) = T(Ji, Ji) = T(JJi, i) = -T(i, i),
\]
\[
T(i, Ji) = T(Ji, i) = T(JJi, Ji) = -T(i, Ji).
\]
We come back to preliminaries. Let $(g, J, \omega)$ be an almost Hermitian structure. Let $\nabla$ be the corresponding Chern connection, i.e.
$$\nabla g = 0, \quad \nabla J = 0, \quad \text{Tor} \nabla \equiv (0, 2) + (2, 0)$$
for the first two variables,
where $\text{Tor} \nabla (X, Y)$ is $\nabla_X Y - \nabla_Y X - [X, Y]$ is the torsion of \nabla. From [9] or Appendix, we see
$$g(\nabla_X Y, Z) = g(D_X Y, Z) + \frac{1}{2} DJ(X, JY, Z) + \frac{1}{4} (DJ(JY, Z, X) + DJ(JZ, X, Y) - DJ(Y, Z, JX) - DJ(Z, X, JY)).$$
And in almost Kähler setting,
$$g(\nabla_X Y, Z) = g(D_X Y, Z) + \frac{1}{2} DJ(X, JY, Z).$$
Let $\Omega$ denote the curvature of $\nabla$, i.e.
$$\Omega(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W).$$
And we also define Riemmanian curvature $Rm$ in the same manner. Define
$$P(X, Y) = \Omega(X, Y, i, Ji), \quad S(X, Y) = \Omega(i, Ji, X, Y), \quad Ric(X, Y) = Rm(i, X, Y, i).$$
Notice that $-\frac{1}{4\pi} P$ is Chern form, i.e., $[-\frac{1}{4\pi} P] = c_1$, where $c_1$ is the first Chern class of $T^{1,0} M$. And $S$ is a $(1, 1)$ form.

**Remark 2.4.** Here $P$ differs a minus from $P$ in [26] since the definition of curvature differs a minus.

Denote $\rho'(X, Y) = Rm(JX, Y, i, Ji), \quad s' = Rm(i, Ji, j, Jj).$ Let $\theta$ denote the Lee form, i.e., $\theta = DJ(i, Ji, \cdot)$.

Now we recall some basic identities in Riemmannian geometry.
(1) Symmetries of $Rm$.
(2) 1st and 2nd Bianchi identity:
$$1st: \quad Rm(X, Y, Z, W) + Rm(Y, Z, X, W) + Rm(Z, X, Y, W) = 0.$$
$$2nd: \quad DRm(X, Y, Z, W, V) + DRm(Y, Z, X, W, V) + DRm(Z, X, Y, W, V) = 0.$$ (3) Ricci identity: For example, let $T$ be a 2-tensor, then
$$D^2 T(X, Y, Z, W) - D^2 T(Y, X, Z, W) = (Rm(X, Y) T)(Z, W)$$
$$= -T(Rm(X, Y, Z), W) - T(Z, Rm(X, Y, W)).$$

Then we consider some basic identities in almost Hermitian geometry. For the following material, one may refer [9].

Let $(g, J, \omega)$ be an almost Hermitian structure.

**Lemma 2.5.** $DJ$ is skew and $(0, 2) + (2, 0)$ with respect to the last two components.
$$DJ(X, Y, Z) = -DJ(X, Z, Y) \quad DJ(X, JY, JZ) = -DJ(X, Y, Z)$$

**Proof.** By our notation, $DJ = D\omega$, so the first identity holds. For the second one, fix a point $p$, suppose $X, Y, Z$ are in a normal frame at $p$, then
$$0 = D_X g(Y, Z) = D_X (g(JY, JZ)) = g(D_X J(Y), JZ) + g(JY, D_X J(Z))$$
$$= DJ(X, Y, JZ) + DJ(X, Z, JY) = DJ(X, Y, JZ) - DJ(X, JY, Z).$$
Then we finish the proof. \qed

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Lemma 2.6. *(Hermitian condition)* Let \((g, J, \omega)\) be an almost Hermitian structure. Then
\[
N_J = 0 \iff DJ(JX, Y, Z) - DJ(JY, X, Z) + DJ(X, JY, Z) - DJ(Y, JX, Z)
\]
\[
\iff DJ(JX, JY, Z) = DJ(X, Y, Z)
\]
Proof. The first identity is from the definition of \(N_J\) and 
\(D_XY - D_YX = [X, Y]\). For the second identity, adding the following identities together,
\[
DJ(JX, Y, Z) - DJ(JY, X, Z) + DJ(X, JY, Z) - DJ(Y, JX, Z) = 0
\]
\[
DJ(JZ, X, Y) - DJ(JX, Z, Y) + DJ(Z, JX, Y) - DJ(X, JZ, Y) = 0
\]
\[
-DJ(JY, Z, X) + DJ(JZ, Y, X) - DJ(Y, JZ, X) + DJ(Z, JY, X) = 0
\]
we obtain one direction and the other direction is obvious. □

Lemma 2.7. *(almost Kähler condition)* Let \((g, J, \omega)\) be an almost Hermitian structure. Then
\[
d\omega = 0 \iff DJ(X, Y, Z) + DJ(Y, Z, X) + DJ(Z, X, Y) = 0
\]
\[
\iff DJ(JX, JY, Z) = -DJ(X, Y, Z)
\]
Proof. The first identity is directly obtained from definition since 
\(D_XY - D_YX = [X, Y]\) and 
\(DJ = D\omega\). For the second identity, adding the following identities together,
\[
DJ(JX, JY, JZ) + DJ(JY, JZ, JX) + DJ(JZ, JX, JY) = 0
\]
\[
-DJ(JX, Y, Z) - DJ(Y, Z, X) - DJ(Z, X, Y) = 0
\]
\[
DJ(X, JY, Z) + DJ(JY, Z, X) + DJ(Z, X, JY) = 0
\]
\[
DJ(X, Y, JZ) + DJ(Y, JZ, X) + DJ(JZ, X, Y) = 0
\]
and by Lemma 2.5, we obtain the desired result. □

In almost Hermitian setting, modulo first order term, \(\Delta J\) is \((0, 2) + (2, 0)\).

Lemma 2.8. Let \((g, J, \omega)\) be an almost Hermitian structure. Then
\[
(\Delta J)^{(1,1)} = -\mathcal{N}, \text{ where } \mathcal{N}(X, Y) = DJ(i, j, JX)DJ(i, j, Y).
\]
So \(\Delta J + \mathcal{N}\) is \((0, 2) + (2, 0)\).

Proof.
\[
\Delta J(JX, JY) = D_i(DJ(i, JX, JY)) - DJ(i, j, JY)DJ(i, j, X) - DJ(i, JX, j)DJ(i, j, Y)
\]
\[
= -D_i(DJ(i, X, Y)) + 2DJ(i, j, JX)DJ(i, j, Y)
\]
\[
= -\Delta(X, Y) + 2\mathcal{N}(X, Y).
\]
So we finish the proof. □

3. DEFORMATION CONDITIONS IN ALMOST HERMITIAN SETTING

First, we derive some algebraic conditions for the deformation of almost Hermitian structures, which is necessary.

Lemma 3.1. Let \((g, J)\) be a family of almost Hermitian structures, \(\frac{\partial}{\partial t} g = h, \frac{\partial}{\partial t} J = K\). Then \((g, K)\) satisfies the following algebraic conditions.

(a) \(h\) is symmetric,
(b) \(K\) is \((0, 2) + (2, 0)\),
(c) \(K^{sym} J = h^{(0,2) + (2,0)}\).
Proof. Condition (a) is obvious. For (b),
\[ 0 = \frac{\partial}{\partial t} J^2 = KJ + JK. \]
For (c),
\[
0 = \frac{\partial}{\partial t} (g(JX, JY) - g(X, Y)) \\
= h(JX, JY) - h(X, Y) + g(KX, JY) + g(JX, KY) \\
= -2h^{(0,2)+(2,0)}(X, Y) + K(JX, Y) + K(Y, JX) \\
= -2h^{(0,2)+(2,0)}(X, Y) + 2(K^{sym} J)(X, Y).
\]
\[\square\]

**Remark 3.2.** Similarly, we have
\[
\eta^{(0,2)+(2,0)} = K^{skew}, \\
\eta^{(1,1)} = h^{(1,1)} J.
\]

**Lemma 3.3.** Let \((g, J)\) be an almost Hermitian structure. Then \((L_X g, L_X J)\) satisfies the necessary condition of a deformation of \((g, J)\).

*Proof.* Let \(\phi_t\) be the 1-parameter transformation groups generated by \(X\), \(g_t = \phi_t^* g\), \(J_t = \phi_t^* J\), then
\[
\left. \frac{\partial}{\partial t} \right|_{t=0} g_t = \left. L_X g \right|_{t=0}, \quad \left. \frac{\partial}{\partial t} \right|_{t=0} J_t = \left. L_X J \right|_{t=0}.
\]
Then the result follows from Lemma 3.1. \[\square\]

Next, we consider the analytic condition to deform almost Hermitian structures. We only consider second order flow. And to ensure the short-time existence (on compact manifolds), we assume \((h, K)\) satisfies the following conditions,
1. \((h, K)\) is of second order with respect to \((g, J)\),
2. Modulo gauge transformation, the symbol of the linearization of \((h, K)\) is \(|\xi|^2 Id\), i.e., there exists a vector field \(\overline{\sigma}\), s.t,
\[
h + L_{\overline{\sigma}} g = g^{ij} \partial_i \partial_j g + \mathcal{O}(\partial g, \partial J), \\
K + L_{\overline{\sigma}} J = g^{ij} \partial_i \partial_j J + \mathcal{O}(\partial g, \partial J).
\]

**Remark 3.4.** In Ricci flow, \(\overline{\sigma} = g^{ab}(\Gamma^k_{ab} - \Gamma^k_{ab}) \frac{\partial}{\partial x^k}\), where \(\Gamma\) is the Christoffel symbol of a fixed background linear connection.

**Remark 3.5.** If \((h, K)\) satisfies both the algebraic condition and the analytic condition, then we may apply the same method in [26] (see also [4]) to show that there exist a family of almost Hermitian structures on a compact manifold for a short while, with given initial data.

**Remark 3.6.** There are other second order “canonical” curvature flows being constructed, but not satisfying our analytic condition, see [10] [14].

Naturally, we also require that our flow should coincide with Kähler Ricci flow if the initial data is Kähler. (But in fact, in Kähler setting, Kähler Ricci flow is the unique flow satisfies the conditions above. So this requirement is vacant.)

From the discuss above, we see to classify the second order curvature flows, we just need to do the following two steps.
Step 1: Classify the tensors up to order 2, as well as gauge terms. More precisely, we need to classify the tensors of the following type,

first order terms: $\partial J \ast \partial J$, second order terms: $\partial^2 J$, $\partial^2 g$, gauge terms: $X$.

Step 2: Calculate the symbols of the second order terms and gauge terms, and then find the suitable tensors to satisfy the analytic condition.

Remark 3.7. We only consider the quadratic terms of $\partial J$, for the consideration of scaling property.

4. Classification of Lower Order Tensors

We only consider Levi-Civita connection since the difference between two connections also gives a tensor.

First, we consider first order tensors. We require $h$ is of even type and $K$ is of odd type. So we only consider tensors of even or odd type. For instance, we don’t consider tensors like $g + \omega$. So the zero order tensors are only $g$ and $\omega$. And to contraction, we can only use $g$ and $\omega$.

We notice that in almost Hermitian setting, last two variables of $DJ$ is $(0, 2) + (2, 0)$. So it will vanish if we either take trace or complex trace in last two positions.

Lemma 4.1. Consider 2-tensors of even type of form $DJ \ast DJ$. We take twice trace or twice complex trace of $DJ(\cdot, \cdot)DJ(\cdot, \cdot)$, then there are two positions remaining for the variables. Modulo transposition and $J$ action, the tensors described above can be classified as follows,

\[
B^1(X, Y) = DJ(X, i, j)DJ(Y, i, j),
B^2(X, Y) = DJ(i, X, j)DJ(i, Y, j),
B^3(X, Y) = DJ(i, X, j)DJ(j, Y, i),
B^4(X, Y) = DJ(X, i, j)DJ(i, Y, j),
B^5(X, Y) = DJ(i, X, i)DJ(j, Y, j),
B^6(X, Y) = DJ(X, Y, i)DJ(j, i, j),
B^7(X, Y) = DJ(i, X, Y)DJ(j, i, j),
B^8(X, Y) = DJ(JX, i, j)DJ(Y, Ji, j),
B^9(X, Y) = DJ(i, JX, j)DJ(Ji, Y, j),
B^{10}(X, Y) = DJ(i, JX, Y)DJ(Jj, Ji, j).
\]

Furthermore $B^1, B^3, B^5$ are symmetric, $B^2$ is symmetric and $(1, 1)$, $B^7, B^{10}$ are skew and $(0, 2) + (2, 0)$, and $^tB^8 = J^*B^8$, $^tB^9 = J^*B^9$.

Proof. We identify $(X, Y)$, $(Y, X)$, $(JY, JX)$ and $(JY, JX)$. And we frequently use Lemma 2.5 implicitly. First, we consider the case that the variables are $X, Y$, then we need to take twice trace or twice complex trace in remaining four positions. Suppose $X$ is in the first position, i.e.

$DJ(X, \cdot, \cdot)DJ(\cdot, \cdot, \cdot)$.

There are three ways to pose $Y$,

$DJ(X, \cdot, \cdot)DJ(Y, \cdot, \cdot), \quad DJ(X, \cdot, \cdot)DJ(\cdot, Y, \cdot), \quad DJ(X, Y, \cdot)DJ(\cdot, \cdot, \cdot)$.

We obtain $B^1, B^4, B^6$ respectively. If $X$ (and $Y$) is not in the first position, we may assume $X$ is in the second position. There are two positions for $Y$,

$DJ(\cdot, X, \cdot)DJ(\cdot, Y, \cdot), \quad DJ(\cdot, X, Y)DJ(\cdot, \cdot, \cdot)$.

If we take twice trace of $DJ(\cdot, X, \cdot)DJ(\cdot, Y, \cdot)$, it gives $B^2$, $B^3$, and $B^5$. And if we take twice complex trace of $DJ(\cdot, X, \cdot)DJ(\cdot, Y, \cdot)$, it gives $B^9$. We obtain $B^7$ from $DJ(\cdot, X, Y)DJ(\cdot, \cdot, \cdot)$. Next, we
consider the case that the variables are $JX$ and $Y$. Then we need to take once trace and once complex trace in the remaining four positions. If $JX$ is in the first position, we only need to consider

$$DJ(JX, \cdot, \cdot)DJ(Y, \cdot, \cdot),$$

since other cases are reduced to the above situation. Then we obtain $B^8$. If $JX$ is in the second position, we only need to consider

$$DJ(\cdot, JX, Y)DJ(\cdot, \cdot, \cdot),$$

for the same reason. Then we obtain $B^{10}$. From the definition, we can easily obtain the identities of the transposition and $J$ action of $B^1$ to $B^{10}$.

**Lemma 4.2.** The functions of form $DJ \ast DJ$ can be classified as follows,

\[
\begin{align*}
E^1 &= DJ(i, j, k)DJ(i, j, k) = |DJ|^2, \\
E^2 &= DJ(i, j, k)DJ(j, i, k), \\
E^3 &= DJ(i, i, j)DJ(k, k, j) = |\theta|^2, \\
E^4 &= DJ(i, j, k)DJ(Ji, Ji, k).
\end{align*}
\]

**Proof.** We just take trace or complex trace of $B^1$ to $B^{10}$ to obtain the desired functions. Notice that for a symmetric 2-tensor, it will vanish if we take complex trace. And for $B^7$ and $B^{10}$, they vanish either we take trace or complex trace. For $B^4$,

$$DJ(k, i, j)DJ(i, Jk, j) = DJ(k, i, j)DJ(i, k, Ji) = -DJ(i, k, Ji)DJ(k, i, j).$$

So the complex trace of $B^4$ vanishes. The similar reason hold for $B^6, B^8, B^9$. Then we obtain $E^1$ (from $B^1, B^2$), $E^2$ (from $B^3, B^4$), $E^4$ (from $B^5, B^6$), $E^3$ (from $B^8, B^9$).

To summarize

**Lemma 4.3.** Modulo transposition and $J$ action, the 2-tensors of form $DJ \ast DJ$ can be classified as follows,

Even type: $B^i, 1 \leq i \leq 10$, $E^i g, 1 \leq i \leq 4$,

Odd type: $B^i J, 1 \leq i \leq 10$, $E^i \omega, 1 \leq i \leq 4$.

Functions of form $DJ \ast DJ$ can be classified as $E^i, 1 \leq i \leq 4$.

Now we consider second order tensors. To begin with, we consider the tensors in terms of $\partial^2 g$, but without $\partial^2 J$. They are tensors in terms of Riemannian curvature.

**Lemma 4.4.** The 2-tensors in terms of Riemannian curvature can be classified as follows, modulo transposition and $J$ action,

Even type: $\text{Ric}(X, Y), \rho'(JX, Y), Rg, s' g$.

Odd type: $\text{Ric}(JX, Y), \rho'(X, Y), R\omega, s'\omega$.

The functions in terms of Riemannian curvature are given by $R$ and $s'$.

**Proof.** We only consider the case that the tensor is of even type, and $Rm$ is not totally contracted. From the symmetries of Riemannian curvature, we see there are three possibilities, modulo transposition and $J$ action,

$$\text{Ric}(X, Y) = Rm(X, i, i, Y), \quad \text{Rm}(JX, i, Ji, Y), \quad \rho'(X, Y) = Rm(JX, Y, i, Ji).$$

But from Bianchi identity,

$$Rm(JX, i, Ji, Y) = -Rm(JX, Ji, Y, i) - Rm(JX, Y, i, Ji)$$

$$= -Rm(JX, i, Ji, Y) - Rm(JX, Y, i, Ji).$$
So

\[ Rm(JX, i, Ji, Y) = -\frac{1}{2} \rho'(X, Y). \]

Then we can obtain the result. \(\Box\)

Now we consider the tensors in terms of \(\partial^2 J\). Notice that if we take trace or complex trace in two positions of \(D^2 J\), it will reduce to first order terms. For instance,

\[ D^2 J(X, Y, i, Ji) = D_X (DJ(Y, i, Ji)) - DJ(Y, i, D_X (Ji)) = -DJ(Y, i, j) DJ(X, i, j). \]

**Lemma 4.5.** Modulo transposition action and \(J\) action, modulo lower order terms and Riemannian curvature terms, the 2-tensors in terms of \(\partial^2 J\) can be classified as follows

Odd type: \(\triangle J(X, Y), \ D^2 J(X, i, Y, i)\)

Even type: \(\triangle J(JX, Y), \ D^2 J(JX, i, Y, i)\).

There is no functions in terms of \(\partial^2 J\).

**Proof.** Notice that modulo Riemannian curvature, the first two components of \(D^2 J(\cdot, \cdot, \cdot, \cdot)\) is symmetric. And modulo lower order terms in \(DJ\), the last two components of \(D^2 J(\cdot, \cdot, \cdot, \cdot)\) is skew and \((0, 2) + (2, 0)\). So suppose the tensors are of odd type, and \(D^2 J\) is not totally contracted, then the tensors are classified as

\[ \triangle J(X, Y), \ D^2 J(X, i, Y, i). \]

Now we consider the contraction of above tensors. From Lemma 2.8, we see modulo lower order tensors, \(\triangle J\) is \((0, 2) + (2, 0)\). And notice that \(\triangle J = \triangle \omega\) is skew. So either taking trace or complex trace of \(\triangle J\) will vanish. For \(D^2 J(j, i, j, i)\) and \(D^2 J(j, i, Jj, i)\), notice that, modulo Riemannian curvature and modulo lower order terms, the \(i, j\) in first two positions commutes and \(i, j\) in last two positions commutes. So they also vanish. Hence there is no functions in terms of \(\partial^2 J\). \(\Box\)

We finish the classification of lower order tensors in almost Hermitian setting. Now we consider two special cases. First, we consider almost Kähler setting. We notice that in almost Kähler setting, every two variables of \(DJ\) is \((0, 2) + (2, 0)\). So it will vanish if we either take trace or complex trace in any two positions. And if we take trace or complex trace in any two of last three positions of \(D^2 J\), it will reduce to first order terms.

**Lemma 4.6.** In almost Kähler setting, the 2-tensors of form \(DJ \ast DJ\) can be classified as

\[ B^1, \ B^2, \ |DJ|^2 g, \ B^1 J, \ B^2 J, \ |DJ|^2 \omega. \]

The functions of form \(DJ \ast DJ\) can be classified as \(|DJ|^2\).

**Proof.** We just need to show the list in Lemma 4.1 can be reduced to \(B^1, B^2\). From Lemma 2.7, we see \(B^5, B^6, B^7, B^{10}\) vanish, and \(B^8, B^9\) are reduced to \(B^1, B^2\). So we only need express \(B^3, B^4\) in terms of \(B^1, B^2\). In fact,

\[ DJ(X, i, j)DJ(i, Y, j) = -DJ(X, i, j)(DJ(Y, i, i) + DJ(j, i, Y)) = DJ(X, i, j) DJ(Y, i, j) - DJ(X, j, i) DJ(j, Y, i). \]

So \(B^4 = \frac{1}{2} B^1\). And

\[ DJ(i, X, j)DJ(j, Y) = -DJ(i, X, j)(DJ(Y, i, j) + DJ(i, j, Y)) = -DJ(Y, i, j) DJ(i, X, j) + DJ(i, X, j) DJ(i, Y, j). \]

So \(B^3 = -\frac{1}{2} B^1 + B^2\). Notice that in the almost Kähler setting, both \(B^1\) and \(B^2\) are symmetric and \((1, 1)\). So we finish the proof. \(\Box\)
Lemma 4.7. In almost Kähler setting, 2-tensors in terms of \( \partial^2 J \) can be classified as follows
\[
\triangle J(X,Y), \quad \triangle J(JX,Y).
\]

Proof. Notice that \( D^2 J(X,i,Y,i) \) is reduced to first order terms. (In fact, it vanishes.) \( \square \)

Lemma 4.8. In almost Kähler setting, the functions are classified as \( R, |DJ|^2 \).

The following lemma gives the proof.

Lemma 4.9. In almost Kähler setting, We have the following identity.
\[
s' + 2R + |DJ|^2 = 0.
\]

Proof. First we have
\[
\text{Rm}(i,j,k,l) + \text{Rm}(i,j,k,Jl) = D^2 J(i,j,k,l) - D^2 J(j,i,k,l).
\]
To see this, suppose \( i,j,k,l \) are in a local normal coordinate chart. Then
\[
D_i D_j (Jk) = D_i (DJ(j,k) + JD_j k) = D^2 J(i,j,k) + JD_i D_j k
\]
So
\[
\text{Rm}(i,j,k,l) = \langle D_i D_j (Jk) - D_j D_i (Jk), l \rangle
\]
\[
= D^2 J(i,j,k,l) - D^2 J(j,i,k,l) - \langle D_j D_i k, Jl \rangle + \langle D_i D_j k, Jl \rangle
\]
So we obtain the identity. (One may also show it directly from Ricci identity.)

Now let \( k = j, l = Ji \), we see
\[
\text{Rm}(i,j,Jj,Ji) - R = D^2 J(i,j,j,Ji) - D^2 J(j,i,j,Ji).
\]
From \( d\omega = 0 \),
\[
-D^2 J(j,i,j,Ji) = -D_j (DJ(i,j,Ji)) + DJ(i,j,k) DJ(j,i,k)
\]
\[
= D_j (DJ(j,Ji,i) + DJ(Ji,i,j)) + tr B
\]
\[
= D_j (-DJ(i,j,Ji) - DJ(i,j,Jj)) + \frac{1}{2} |DJ|^2
\]
\[
= -D^2 J(j,i,j,Ji) - |DJ|^2 - D^2 J(i,j,i,j) - |\theta|^2 + \frac{1}{2} |DJ|^2.
\]
Since \( DJ(i,i,X) = 0 \), we have
\[
\text{Rm}(i,j,Jj,Ji) - R = -\triangle J(i,Ji) - \frac{1}{2} |DJ|^2.
\]
From Lemma 2.8 we see
\[
\triangle J(i,Ji) = -|DJ|^2.
\]
And from the identity in (4.4), we see
\[
\text{Rm}(i,j,Jj,Ji) = \frac{1}{2} s'
\]
So we finish the proof. \( \square \)

In the case of dimension 4, we can reduce the tensors further in almost Kähler setting. The author learned the well-known result below from R. Bryant in mathoverflow [2].
Lemma 4.10. Let \((M, g, J, \omega)\) be an almost Kähler manifold with \(\dim_{\mathbb{R}} M = 4\). Let \(p \in M\). Then in any local coordinate chart of \(p\), there exists a local unitary frame, i.e.,

\[
g = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},
\]

such that at \(p\),

\[
B^1 = \begin{pmatrix} 4a^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 2a^2 & 0 \\ 0 & 2a^2 \end{pmatrix},
\]

where \(a^2 = \frac{1}{8} |DJ|^2\). In particular, \(B^2 = \frac{1}{4} |DJ|^2 g\).

Proof. We can choose a local coframe \(\{\eta_i\}, i = 1, 2, 3, 4\), s.t.

\[
g = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2, \quad \omega = \eta_1 \wedge \eta_2 + \eta_3 \wedge \eta_4.
\]

Notice that \(\omega\) is self-dual. Let

\[
\kappa = \eta_1 \wedge \eta_3 + \eta_4 \wedge \eta_2, \quad \lambda = \eta_1 \wedge \eta_4 + \eta_2 \wedge \eta_3.
\]

Then \((\omega, \kappa, \lambda)\) forms a pointwise basis for the self-dual 2-forms. First we claim \(D_X \omega\) is self-dual. In fact, in general, we have

\[
D_X (\ast \alpha) = \ast D_X \alpha.
\]

This is from

\[
D_X (\ast \alpha) \wedge \beta = D_X (\ast (\alpha \wedge \beta)) - \ast (\alpha \wedge D_X \beta) = D_X (\langle \alpha, \beta \rangle dV) - \langle \alpha, D_X \beta \rangle dV
= \langle D_X \alpha, \beta \rangle dV = \ast D_X \alpha \wedge \beta.
\]

So set

\[
D\omega = \alpha \otimes \kappa + \beta \otimes \lambda + \gamma \otimes \omega,
\]

where \(\alpha, \beta, \gamma\) are 1-forms. But notice that

\[
D_X \omega \wedge \omega = D_X (dV) = 0,
\]

so \(\gamma = 0\). Hence we have

\[
0 = d\omega = \alpha \wedge \kappa + \beta \wedge \lambda.
\]

Let

\[
\alpha = a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 + a_4 \eta_4.
\]

Then

\[
\beta = a_2 \eta_1 - a_1 \eta_2 + a_4 \eta_3 - a_3 \eta_4.
\]

If \(\alpha(p) = 0\), then there is nothing to prove. If \(\alpha(p) \neq 0\), then we may choose

\[
a_1(p) = a > 0, \quad a_2(p) = a_3(p) = a_4(p) = 0.
\]

Hence at \(p\),

\[
DJ = a \eta_1 \otimes (\eta_1 \wedge \eta_3 + \eta_4 \wedge \eta_2) - a \eta_2 \otimes (\eta_1 \wedge \eta_4 + \eta_2 \wedge \eta_3).
\]

So at \(p\),

\[
D_1 J = a \kappa, \quad D_2 J = -a \lambda, \quad D_3 J = D_4 J = 0.
\]
Therefore at $p$,

$$B^1 = \begin{pmatrix} 4a^2 & 4a^2 \\ 4a^2 & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}. $$

Taking trace, we see $a^2 = \frac{1}{8} |DJ|^2$.

**Remark 4.11.** If $DJ \neq 0$ at $p$, we can choose a local unitary frame such that the above result holds in a neighborhood of $p$, not only at $p$.

**Remark 4.12.** From the proposition above, we see $\frac{1}{2} B^1 - B^2 \leq 0$. So Symplectic Curvature Flow is “slower” than Ricci Flow in $g$.

Now we consider Hermitian setting.

**Lemma 4.13.** In Hermitian setting, the 2-tensors of form $DJ \ast DJ$ can be classified as

$B^i, 1 \leq i \leq 7, E^i g, 1 \leq i \leq 3, B^i J, 1 \leq i \leq 7, E^i \omega, 1 \leq i \leq 3$.

Furthermore $B^1, B^2$ are symmetric and $(1, 1)$, $B^3$ is symmetric and $(0, 2)+(2, 0)$, $B^4$ is $(0, 2)+(2, 0)$, $B^5$ is symmetric, $B^6$ is $(1, 1)$ and $B^7$ is skew and $(0, 2)+(2, 0)$. The functions of type $DJ \ast DJ$ can be classified as $E^i, 1 \leq i \leq 3$.

**Proof.** By taking advantage of Lemma 2.6, the tensors in the list of Lemma 4.1 are reduced to the result above.

Next, we consider the gauge terms. More precisely, we consider the vector fields defined from $(g, J, \omega, D)$, where $\overline{D}$ is a fixed background linear connection. Naturally, we require the vector field is of first order and of even type. First, we have “canonical” gauge.

**Lemma 4.14.** Let $X$ be a first order vector field defined from $(g, J, \omega)$, then $X$ can be classified as

$$\theta^2 = DJ(i, Ji), \quad J\theta^2.$$

**Proof.** We just need take once trace or once complex trace of $DJ$. It will vanish if we contract the last two components, so we have the result above.

Next, we consider the vector field in terms of $\overline{D}$, i.e., $\overline{D} g$ and $\overline{D} J$. Notice that

$$\overline{D} J - DJ = (\overline{D} - D) \ast J.$$

So we only need to classify the vector field in terms of $\overline{D} g$. Notice that the last two components of $\overline{D} g$ is symmetric, so we have the following result.

**Lemma 4.15.** The gauge terms in $(\partial g, J)$ can be classified as

$$X_1 = g^{ij} \overline{D} g(i, j), \quad X_2 = g^{ij} \overline{D} g(;i, j), \quad X_3 = g^{ij} J^k \overline{D} g(i, k, J).$$

**Remark 4.16.** In Ricci Flow, the gauge term is $X = X_1 - \frac{1}{2} X_2$.

We can calculate $L_{\theta^2} J$ as follows.

**Lemma 4.17.** Let $(g, J)$ be an almost Hermitian structure, then

$$L_{\theta^2} J(X, Y) = DJ(Z, X, Y) - DZ(JX, Y) - DJ(Z, X, JY).$$

$$L_{\theta^2} J(X, Y) = -D^2 J(JX, i, Ji, Y) - D^2 J(X, i, Ji, JY) - DJ(JX, i, j) DJ(i, j, Y) - DJ(X, i, j) DJ(i, j, JY) + DJ(j, X, Y) DJ(i, Ji, j).$$
Proof. For the first one,
\[ L_Z J(X) = [Z, JX] - J[Z, X] = D_Z(JX) - D_JXZ - JD_ZX + JD_XZ \]
So we obtain the first identity. For the second one,
\[ D\delta^o(X,Y) = g(D_X(DJ(i, Ji)), Y) = D^2 J(X, i, Ji, Y) + DJ(i, j, Y) DJ(X, i, j), \]
then we finish the proof.

5. Classification of Second Order Curvature Flows

To classify the second order curvature flows, we just need to find the suitable tensor satisfying the conditions in Section 3. We have discussed the algebraic condition in Section 4. Now we consider the parity of the type of Lemma 4.4 and Lemma 4.5, and considering the linearity of the type of Lemma 4.17 we see our candidates of second order terms are in fact
\[ \triangle J(X, Y), \quad D^2 J(X, i, Y, i), \quad D^2 J(JX, i, JY, i), \quad D^2 J(JY, i, JX, i), \]
\[ \text{Ric}(JX, Y), \quad \text{Ric}(JY, X), \quad \rho(JX, Y), \quad \rho(X, JY), \quad RJ, \quad L^J. \]
Here \( \triangle J \) and \( \rho(J, \cdot, \cdot) \) are skew and \( \mathcal{R} \) is symmetric, so we don’t need to consider their transposition. Considering algebraic condition, the tensors of evolution terms should be \((0,2) + (2,0)\) and from Lemma 4.17 we see \( D^2 J(X, i, Y, i) + D^2 J(JX, i, JY, i) \) is just \( L\delta^i J \) modulo lower order terms. So our candidates of second order terms are in fact
\[ \triangle J, \quad t^i(L\delta^i J), \quad \mathcal{R}, \quad (\rho(J, \cdot, \cdot))^{(0,2) + (2,0)}. \]
Now we calculate their symbols.

Lemma 5.1. In local coordinate chart,
\[ Rm_{ijkl} = \frac{1}{2}(\partial_i \partial_k g_{lj} - \partial_i \partial_l g_{kj} - \partial_j \partial_k g_{li} + \partial_j \partial_l g_{ki}) + O(\partial g). \]
\[ (D^2 J)^i_{ijk} = \partial_i \partial_j \Gamma^l_{jk} + \frac{1}{2} J^p_k g^{pq}(\partial_i \partial_j g_{qp} + \partial_i \partial_p g_{jq} - \partial_i \partial_q g_{jp}) \]
\[ -\frac{1}{2} J^i_k g^{pq}(\partial_i \partial_q g_{kp} + \partial_i \partial_k g_{pq} - \partial_i \partial_p g_{kq}) + O(\partial g, \partial J). \]

Proof. In any local coordinate chart,
\[ Rm_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + O(\partial g) \]
\[ = \frac{1}{2}(\partial_i \partial_j g_{lk} + \partial_i \partial_k g_{lj} - \partial_i \partial_l g_{jk}) - \frac{1}{2}(\partial_j \partial_i g_{lk} + \partial_j \partial_k g_{lj} - \partial_j \partial_l g_{ij}) + O(\partial g) \]
\[ = \frac{1}{2}(\partial_i \partial_k g_{lj} - \partial_i \partial_l g_{kj} - \partial_j \partial_k g_{li} + \partial_j \partial_l g_{ki}) + O(\partial g). \]
\[ (D^2 J)^i_{ijk} = (D_i(D_J J(k)))^i + O(\partial g, \partial J) \]
\[ = \partial_i \partial_j \Gamma^l_{jk} + \frac{1}{2} J^p_k g^{pq}(\partial_i \partial_j g_{qp} + \partial_i \partial_p g_{jq} - \partial_i \partial_q g_{jp}) \]
\[ -\frac{1}{2} J^i_k g^{pq}(\partial_i \partial_q g_{kp} + \partial_i \partial_k g_{pq} - \partial_i \partial_p g_{kq}) + O(\partial g, \partial J). \]
To calculate the symbols of our candidates, we notice that symbol is also tensorial, so we just need to calculate the symbols of $Rm$ and $D^2 J$, and then use the corresponding manner to take trace to get the desired symbols. To calculate the symbols of the linearization operators of $Rm$ and $D^2 J$, what we need to do is to replace the second derivative terms by their deformation terms, and replace “$\partial_i$” by $\xi_i$, where $\xi$ is a 1-form. Finally, we simplified the tensors we obtained. We denote $\sigma$ to be the symbol of the linearization operator of a tensor. Then for instance

$$\sigma: g^{ij} \partial_i \partial_j g_{ab} \mapsto g^{ij} \xi_i \xi_j h_{ab} = |\xi|^2 h_{ab}.$$ 

Since symbol is also tensorial, we also use Riemannian metric to identify $TM$ and $T^* M$. We may use orthonormal frame to reduce the calculation, and notice that $J_i^j = - J_j^i$. Then we obtain the following results.

**Lemma 5.2.**

$$\sigma(Rm)_{ijkl} = \frac{1}{2} (\xi_i \xi_k h_{lj} - \xi_i \xi_l h_{kj} - \xi_j \xi_k h_{li} + \xi_j \xi_l h_{ik}),$$

$$\sigma(D^2 J)_{ijkl} = \xi_i \xi_j K_{kl} + \frac{1}{2} (\xi_i \xi_j \partial^p h_{kp} + \xi_i \xi_p \partial^p h_{kj} - \xi_j \xi_p \partial^p h_{kp}) + \xi_i \xi_j \partial^p h_{kp} + \xi_i \xi_k \partial^p h_{pj} - \xi_i \xi_p \partial^p h_{jk}).$$

**Lemma 5.3.**

$$\sigma(\Delta J)_{ab} = |\xi|^2 K(a, b) + \frac{1}{2} (|\xi|^2 h(Ja, b) + h(\xi, b)\xi(Ja) - h(\xi, Ja)\xi(b)),$$

$$+ |\xi|^2 h(Jb, a) + h(\xi, Jb)\xi(a) - h(\xi, a)\xi(Jb)),$$

$$\sigma((L_{g*}) J)_{ab} = K(a, \xi) \xi(b) + \frac{1}{2} \xi(b) \xi(Ja)h + h(J, \xi) \xi(Jb)$$

$$- K(Ja, \xi) \xi(Jb) + \frac{1}{2} \xi(Jb) \xi(a)h - h(J, \xi, ja) \xi(Jb),$$

$$\sigma((\rho^1(J, \cdot))^{(0,2)} + (2,0))_{ab} = h(a, J\xi) \xi(b) - h(b, J\xi) \xi(a) - h(Ja, J\xi) \xi(Jb) + h(Jb, J\xi) \xi(Ja).$$

For gauge terms, the candidates are $L_{g*} J, L_{l*} J, L_{\xi*} J, L_{\xi_0^*} J$.

We have calculated $\sigma((L_{g*} J))$. For other symbols, by definition, we have

**Lemma 5.4.**

$$X_1^k = g^{kl} g^{ij} \partial_i g_{jl} + O(g),$$

$$X_2^k = g^{kl} g^{ij} \partial_i g_{jl} + O(g),$$

$$X_0^k = g^{jk} J^l_j g^{ab} J^i_b \partial_a g_{il} + O(g, J).$$

From the Lemma 5.1 we have

**Lemma 5.5.**

$$(L_{X_1^k})^b_a = - J^a_b g^{pq} \partial_k g_{pq} + J^b_b g^{pq} \partial_a g_{pq} + O(\partial g, \partial J),$$

$$(L_{X_2^k})^b_a = - J^a_b g^{pq} \partial_k g_{pq} + J^b_b g^{pq} \partial_a g_{pq} + O(\partial g, \partial J),$$

$$(L_{X_0^k})^b_a = - J^a_b g^{pq} \partial_k g_{pq} + g^{pq} J^l_j \partial_a g_{jl} + O(\partial g, \partial J).$$
Then the symbols of above tensors are

**Lemma 5.6.**

\[
\begin{align*}
\sigma(L_{X_1})_{ab} &= -\xi(Ja)h(\xi, b) - \xi(a)h(\xi, Jb), \\
\sigma(L_{X_2})_{ab} &= -\xi(Ja)\xi(b)trh - \xi(a)\xi(Jb)trh, \\
\sigma(L_{X_0})_{ab} &= -\xi(Ja)h(J\xi, Jb) + \xi(a)h(J\xi, b).
\end{align*}
\]

We consider the symbol of $K$ with respect to $J$ first. We assume there is no $L_{\partial t}J$ in $K$. Then we only need to consider $\triangle J$ and $\iota L_{\partial t}J$. Notice that the symbol with respect to $J$ of $\triangle J$ is already good and we cannot compensate the symbol with respect to $J$ in $\iota L_{\partial t}J$ by using $L_{\partial t}J$ (other gauge terms only involving $\partial g$), so we can only choose $\triangle J$ to be the symbol term of $K$. As for the symbol of $K$ with respect to $h$, we see that to compensate terms of $[\xi]hJ$ in the symbol of $\triangle J$, we must have $\triangle J + R$, and notice that $\triangle J + R + L_{\overline{X}}J$ gives us desirable symbol. As for $(\rho'(J, \cdot))(0, 2)+(2, 0)$, we can’t compensate $h(Ja, J\xi)\xi(Jb)$ by using gauge terms. To sum up, modulo “canonical” gauge, the second order terms of $K$ can be only chosen as $\triangle J + R$, and the gauge term is also unique, that is $\overline{X} = \overline{X}_1 - \frac{1}{2}\overline{X}_2$.

Next, we consider deformation of $g$. We notice that as in Ricci Flow $-2Ric + L_{\overline{X}}g$ gives good symbol. And from the discussion above, the gauge terms are already chosen, so what we can do is just to find some “canonical” second order tensors whose symbols compensate each other both in $g$ and $J$. But if it happens, it just gives first order tensors. So modulo canonical gauge, $h$ is also unique, that is $-2Ric$. So we finish the proof of Theorem 1.3. \hfill \Box

**Proof of Theorem 1.3.** In almost Kähler setting, we require $\eta = P$. From Remark 3.2 we have $K^{\text{skew}} = P(0, 2)+(2, 0), h^{(1, 1)} = -P(1, 1)J$. And from the calculations in [26], we obtain

\[
\begin{align*}
h^{(1, 1)} &= -2Ric^{(1, 1)} + \frac{1}{2}B^1 - B^2 \\
K^{\text{skew}} &= \triangle J + R.
\end{align*}
\]

So we have the freedom to choose the symmetric part of $K$. For the first order terms, from Lemma 4.6 we see there is no such $(0, 2) + (2, 0)$ and symmetric tensors in almost Kähler setting. For the second order terms, first we notice that the canonical gauge vanishes, and from Lemma 4.3 notice that $(\rho' J)^{\text{sym}}$ is $(1, 1)$, we see the only candidate is $R$. So we just need to investigate it from the consideration of symbol. Comparing to almost Hermitian condition, we have a extra condition $d\omega = 0$. So we need to check what new symbol identities we can obtain from this condition. Notice that

\[
d\omega = 0 \iff DJ(i, j, k) + DJ(j, k, i) + DJ(k, i, j) = 0.
\]

Considering symbol, we obtain

\[
\sigma(DJ)(i, j, k) + \sigma(DJ)(j, k, i) + \sigma(DJ)(k, i, j) = 0.
\]

By direct calculation, we have

\[
\sigma(DJ)(i, j, k) = \xi(i)K(j, k) + \frac{1}{2}(h(Jj, k)\xi(i) + h(Jk, j)\xi(i)) + h(i, k)\xi(Jj) + h(i, Jk)\xi(j) - h(i, Jj)\xi(k) - h(i, j)\xi(Jk)).
\]
So

\[ 0 = \xi(i)K(j, k) + \frac{1}{2}(h(Jj, k)\xi(i) + h(Jk, j)\xi(i)) + h(i, k)\xi(Jj) + h(i, Jk)\xi(j) - h(i, Jj)\xi(k) - h(i, j)\xi(Jk) + \xi(j)K(k, i) + \frac{1}{2}(h(Jk, i)\xi(j) + h(Ji, k)\xi(j)) + h(j, i)\xi(Jk) + h(j, Ji)\xi(k) - h(j, Jk)\xi(i) - h(j, k)\xi(Ji)) + \xi(k)K(i, j) + \frac{1}{2}(h(Ji, j)\xi(k) + h(Jj, i)\xi(k)) + h(k, j)\xi(Ji) + h(k, Ji)\xi(j) - h(k, Ji)\xi(j) - h(k, i)\xi(Jj)). \]

By simplified the identity above, we have

\[ \xi(i)K(j, k) + \xi(j)K(k, i) + \xi(k)K(i, j) + h(k, Jj)\xi(i) + h(i, Jk)\xi(j) + h(j, Ji)\xi(k) = 0 \]

To consider the symbol of the second order 2-tensors, we just need to take tensor product with \( \xi \) and take trace or complex trace. Since we consider the symbol of \( K \), we require it is of odd type. We have

\[ \xi(l)\xi(i)K(j, k) + \xi(l)\xi(j)K(k, i) + \xi(l)\xi(k)K(i, j) + h(k, Jj)\xi(i) + h(i, Jk)\xi(j) + h(j, Ji)\xi(k)\xi(l) = 0. \]

Considering the symmetries, we have the following cases to take trace or complex trace.

Let \( l = i, j = a, k = b \), we obtain

\[ |\xi|^2K(a, b) + \xi(a)K(b, \xi) + \xi(b)K(\xi, a) + |\xi|^2h(Ja, b) + \xi(a)h(\xi, Jb) + \xi(b)h(a, J\xi) = 0. \]

Let \( l = Ji, j = Ja, k = b \), we obtain

\[ -\xi(Ja)K(b, J\xi) + \xi(b)K(\xi, a) + h(\xi, b)\xi(Ja) + h(Ja, \xi)\xi(b) = 0. \]

Let \( i = j, k = a, l = b \), we obtain

\[ \xi(b)K(\xi, a) + \xi(\xi)K(a, \xi) + h(a, J\xi)\xi(b) + h(\xi, Ja)\xi(b) = 0. \]

Let \( i = Jj, k = Ja, l = b \), we obtain

\[ K(\xi, a)\xi(b) - K(a, \xi)\xi(b) + h(Ja, \xi)\xi(b) - h(J\xi, a)\xi(b) - trh(\xi(Ja))\xi(b). \]

Recall Lemma 5.3, we see that, in \( \sigma(\triangle J) \) and \( \sigma(\mathcal{R}) \), the terms involving \( |\xi|^2 \) are in fact, \( |\xi|^2(hJ)^{(0,2)} + (2,0) \).

To cancel this term, the only possibly useful identity is the first one, but when consider the \((0, 2) + (2, 0) \) part, it gives nothing. So for the consideration of symbol, the choice is unique. So we finish the proof of Theorem 1.2. \( \square \)

**Proof of Theorem 1.3:** In Hermitian case, we require \( \delta J = K = 0 \). Consider the second order terms in \( h \), it should satisfy \( h^{(0, 2) + (2, 0)} = 0 \) and \( \sigma(h) \) itself is \( Id \), not modulo gauge. From our classification results, the candidates are

\[ \text{Ric}(X, Y) + \text{Ric}(JX, JX), \quad \text{Rm}(JX, Y, i, Ji) - \text{Rm}(X, JY, i, Ji), \]

\[ D^2J(X, i, Y, i) + D^2J(JY, i, X, i) - D^2J(X, i, JY, i) - D^2J(Y, i, JX, i). \]
Remark 5.7. So we see that the “good” second order term is
\[ \partial \] the symbol term above can be given from Lemma 6.1.
Let (\( J, \xi \)) be a linear connection. Let (\( \nabla \)) be a linear connection. Let \( \nabla = D + A \), i.e., \( g(\nabla_X Y, Z) = g(D_X Y, Z) + A(X, Y, Z) \), where \( A \) is a 3-tensor.

Lemma 6.1.
\[ \nabla g = 0 \iff A(X, Y, Z) + A(X, Z, Y) = 0. \]
\[ \nabla J = 0 \iff A(X, JY, Z) + A(X, Y, JZ) + DJ(X, Y, Z) = 0. \]
Lemma 6.2. Let $\nabla$ be a Hermitian connection.
In almost Hermitian setting,

$$A = \frac{1}{2} DJ(X, Y, Z) + \frac{t}{4} (DJ(JY, Z, X) + DJ(JZ, X, Y) - DJ(Y, Z, JX) - DJ(Z, X, JY)),$$

In Hermitian setting,

$$A = \frac{1}{2} DJ(X, JY, Z) - \frac{t}{2} (DJ(Y, Z, JX) + DJ(Z, X, JY)).$$

In almost Kähler setting,

$$A = \frac{1}{2} DJ(X, JY, Z).$$

Proof. Suppose $(g, J, \omega)$ is an almost Hermitian structure. From our assumption and Lemma 2.5, we have

$$A(X, Y, Z) = a_1 DJ(X, JY, Z) + a_2 DJ(Y, JX, Z) + a_3 DJ(Z, X, JY) + a_4 DJ(JX, Y, Z) + a_5 DJ(JY, Z, X) + a_6 DJ(JZ, X, JY).$$

From Lemma 6.1 $\nabla$ is Hermitian if and only if

$$0 = a_1 DJ(X, JY, Z) + a_2 DJ(Y, JX, Z) + a_3 DJ(Z, X, JY) + a_4 DJ(JX, Y, Z) + a_5 DJ(JY, Z, X) + a_6 DJ(JZ, X, JY),$$

To simplify the above equations, we have

$$0 = (a_2 - a_3)(DJ(Y, Z, JX) - DJ(Z, X, JY)) + (a_5 - a_6)(DJ(JY, Z, X) - DJ(JZ, X, JY)).$$
Therefore, in almost Hermitian setting, \( a_1 = \frac{1}{2}, a_4 = 0, a_2 = a_3 = -a_5 = -a_6 = -\frac{1}{4}. \)

In Hermitian setting, from Lemma 2.6

\[
A(X, Y, Z) = (a_1 - a_4) DJ(X, Y, JZ) + (a_2 - a_5) DJ(Y, Z, JX) + (a_3 - a_6) DJ(Z, X, JY),
\]

and the equations are

\[
0 = (a_2 - a_5 - a_3 + a_6)(DJ(Y, Z, JX) - DJ(Z, X, JY))
\]
\[
0 = (1 - 2a_1 + 2a_4) DJ(X, Y, Z).
\]

Therefore, \( a_1 - a_4 = \frac{1}{2}, a_2 - a_5 = a_3 - a_6 = -\frac{1}{2}. \)

In almost Kähler setting, from Lemma 2.7

\[
A(X, Y, Z) = (a_1 + a_4) DJ(X, Y, JZ) + (a_2 + a_5) DJ(Y, Z, JX) + (a_3 + a_6) DJ(Z, X, JY),
\]

and the equations are

\[
0 = (a_2 + a_5 - a_3 - a_6)(DJ(Y, Z, JX) - DJ(Z, X, JY))
\]
\[
0 = (1 - 2a_1 - 2a_4) DJ(X, Y, Z) - (2a_2 + 2a_5) DJ(Y, Z, X) - (2a_3 + 2a_6) DJ(Z, X, Y).
\]

Therefore, \( a_2 + a_5 = a_3 + a_6, a_1 + a_4 = a_2 + a_5 + \frac{1}{2}. \) Then

\[
A(X, Y, Z) = \frac{1}{2} DJ(X, Y, JZ).
\]

\[\square\]

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