Advanced Mechanics. Mathematical Introduction

G. Giachetta, L. Mangiarotti
Department of Mathematica and Informatics, University of Camerino, Camerino, Italy

G. Sardanashvily
Department of Theoretical Physics, Moscow State University, Moscow, Russia

Abstract

Classical non-relativistic mechanics in a general setting of time-dependent transformations and reference frame changes is formulated in the terms of fibre bundles over the time-axis \( \mathbb{R} \). Connections on fibre bundles are the main ingredient in this formulation of mechanics which thus is covariant under reference frame transformations. The basic notions of a non-relativistic reference frame, a relative velocity, a free motion equation, a relative acceleration, an external force are formulated. Newtonian, Lagrangian, Hamiltonian mechanical systems and the relations between them are defined. Lagrangian and Hamiltonian conservation laws are considered.
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Introduction

We address classical non-relativistic mechanics in a general setting of arbitrary time-dependent coordinate and reference frame transformations [15, 27].

The technique of symplectic manifolds is well known to provide the adequate Hamiltonian formulation of autonomous mechanics [1, 26, 46]. Its familiar example is a mechanical system whose configuration space is a manifold $M$ and whose phase space is the cotangent bundle $T^*M$ of $M$ provided with the canonical symplectic form

$$\Omega = dp_i \wedge dq^i,$$

(0.0.1)

written with respect to the holonomic coordinates $(q^i, p_i = \dot{q}_i)$ on $T^*M$. A Hamiltonian $\mathcal{H}$ of this mechanical system is defined as a real function on a phase space $T^*M$. Any autonomous Hamiltonian system locally is of this type.

However, this Hamiltonian formulation of autonomous mechanics is not extended to mechanics under time-dependent transformations because the symplectic form (0.0.1) fails to be invariant under these transformations. As a palliative variant, one develops time-dependent mechanics on a configuration space $Q = \mathbb{R} \times M$ where $\mathbb{R}$ is the time axis [6, 25]. Its phase space $\mathbb{R} \times T^*M$ is provided with the presymplectic form $\mathrm{pr}_2^* \Omega = dp_i \wedge dq^i$ which is the pull-back of the canonical symplectic form $\Omega$ (0.0.1) on $T^*M$. However, this presymplectic form also is broken by time-dependent transformations.

We consider non-relativistic mechanics whose configuration space is a fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$ endowed with the standard Cartesian coordinate $t$ possessing transition functions $t' = t + \text{const}$. A velocity space of non-relativistic mechanics is the first order jet manifold $J^1Q$ of sections of $Q \to \mathbb{R}$, and its phase space is the vertical cotangent bundle $V^*Q$ of $Q \to \mathbb{R}$ endowed with the canonical Poisson structure [27, 15, 40].

A fibre bundle $Q \to \mathbb{R}$ is always trivial. Its trivialization defines both an appropriate coordinate systems and a connection on this fibre bundle which is associated with a certain non-relativistic reference frame. Formulated as theory on fibre bundles over $\mathbb{R}$, non-relativistic mechanics is covariant under gauge (atlas) transformations of these fibre bundles, i.e., time-dependent coordinate and reference frame transformations.

This formulation of mechanics is similar to that of classical field theory on fibre bundles over a smooth manifold $X$, $\dim X > 1$, [14]. A difference between mechanics and field theory, however, lies in the fact that all connections on fibre bundles over $\mathbb{R}$ are flat and,
consequently, they are not dynamic variables. Therefore, formulation of non-relativistic mechanics is covariant, but not invariant under time-dependent transformations.

Equations of motion of non-relativistic mechanics almost always are first and second order dynamic equations. Second order dynamic equations on a fibre bundle $Q \to \mathbb{R}$ are conventionally defined as the holonomic connections on the jet bundle $J^1Q \to \mathbb{R}$. These equations also are represented by connections on the jet bundle $J^1Q \to Q$. The notions of a free motion equation and a relative acceleration are formulated in terms of connections on $J^1Q \to Q$ and $TQ \to Q$.

Generalizing the second Newton law, one introduces the notion of a Newtonian system characterized by a mass tensor. If a mass tensor is non-degenerate, an equation of motion of a Newtonian system is equivalent to a dynamic equation. We also come to the definition of an external force.

Lagrangian non-relativistic mechanics is formulated in the framework of conventional Lagrangian formalism on fibre bundles [8, 14, 15, 27]. Its Lagrangian is defined as a density on the velocity space $J^1Q$, and the corresponding Lagrange equation is a second order differential equations on $Q \to \mathbb{R}$. Besides Lagrange equations, the Cartan and Hamilton–De Donder equations are considered in the framework of Lagrangian formalism. Note that the Cartan equation, but not the Lagrange one is associated to the Hamilton equation. The relations between Lagrangian and Newtonian systems are established. Lagrangian conservation laws are defined in accordance with the first Noether theorem.

Hamiltonian mechanics on a phase space $V^*Q$ is not familiar Poisson Hamiltonian theory on a Poisson manifold $V^*Q$ because all Hamiltonian vector fields on $V^*Q$ are vertical. Hamiltonian mechanics on $V^*Q$ is formulated as particular (polysymplectic) Hamiltonian formalism on fibre bundles [8, 14, 15, 27]. Its Hamiltonian is a section of the fibre bundle $T^*Q \to V^*Q$. The pull-back of the canonical Liouville form on $T^*Q$ with respect to this section is a Hamiltonian one-form on $V^*Q$. The corresponding Hamiltonian connection on $V^*Q \to \mathbb{R}$ defines the first order Hamilton equations on $V^*Q$.

Note that one can associate to any Hamiltonian system on $V^*Q$ an autonomous symplectic Hamiltonian system on the cotangent bundle $T^*Q$ such that the corresponding Hamilton equations on $V^*Q$ and $T^*Q$ are equivalent. Moreover, the Hamilton equations on $V^*Q$ also are equivalent to the Lagrange equations of a certain first order Lagrangian system on a configuration space $V^*Q$. As a consequence, Hamiltonian conservation laws can be formulated as the particular Lagrangian ones.

Lagrangian and Hamiltonian formulations of mechanics fail to be equivalent, unless a Lagrangian is hyperregular. If a Lagrangian $L$ on a velocity space $J^1Q$ is hyperregular, one can associate to $L$ an unique Hamiltonian form on a phase space $V^*Q$ such that Lagrange equations on $Q$ and the Hamilton equations $V^*Q$ are equivalent. In general, different Hamiltonian forms are associated to a non-regular Lagrangian. The comprehensive relations between Lagrangian and Hamiltonian systems can be established in the case of almost regular Lagrangians.
Chapter 1

Dynamic equations

Equations of motion of non-relativistic mechanics are first and second order differential equations on manifolds and fibre bundles over $\mathbb{R}$. Almost always, they are dynamic equations. Their solutions are called a motion.

This Chapter is devoted to theory of second order dynamic equations in mechanics whose configuration space is a fibre bundle $Q \to \mathbb{R}$. They are defined as the holonomic connections on the jet bundle $J^1Q \to \mathbb{R}$ (Section 1.4). These equations are represented by connections on the jet bundle $J^1Q \to Q$. Due to the canonical imbedding $J^1Q \to TQ$ (1.1.6), they are proved equivalent to non-relativistic geodesic equations on the tangent bundle $TQ$ of $Q$ (Theorem 1.5.2).

The notions of a non-relativistic reference frame, a relative velocity, a free motion equation and a relative acceleration are formulated in terms of connections on $Q \to \mathbb{R}$, $J^1Q \to Q$ and $TQ \to Q$.

Generalizing the second Newton law, we introduce the notion of a Newtonian system (Definition 1.9.1) characterized by a mass tensor. If a mass tensor is non-degenerate, an equation of motion of a Newtonian system is equivalent to a dynamic equation. The notion of an external force also is formulated.

1.1 Preliminary. Fibre bundles over $\mathbb{R}$

This section summarizes some peculiarities of fibre bundles over $\mathbb{R}$.

Let

$$\pi : Q \to \mathbb{R}$$

(1.1.1)

be a fibred manifold whose base is treated as a time axis. Throughout the book, the time axis $\mathbb{R}$ is parameterized by the Cartesian coordinate $t$ with the transition functions $t' = t + \text{const}$. Relative to the Cartesian coordinate $t$, the time axis $\mathbb{R}$ is provided with the standard vector field $\partial_t$ and the standard one-form $dt$ which also is the volume element
on $\mathbb{R}$. The symbol $dt$ also stands for any pull-back of the standard one-form $dt$ onto a fibre bundle over $\mathbb{R}$.

In order that the dynamics of a mechanical system can be defined at any instant $t \in \mathbb{R}$, we further assume that a fibred manifold $Q \to \mathbb{R}$ is a fibre bundle with a typical fibre $M$.

**Remark 1.1.1:** In accordance with Remark 4.3.1, a fibred manifold $Q \to \mathbb{R}$ is a fibre bundle iff it admits an Ehresmann connection $\Gamma$, i.e., the horizontal lift $\Gamma \partial_t$ onto $Q$ of the standard vector field $\partial_t$ on $\mathbb{R}$ is complete. By virtue of Theorem 4.1.5, any fibre bundle $Q \to \mathbb{R}$ is trivial. Its different trivializations $\psi: Q = \mathbb{R} \times M$ differ from each other in fibrations $Q \to M$. ♦

Given bundle coordinates $(t, q^i)$ on the fibre bundle $Q \to \mathbb{R}$ (1.1.1), the first order jet manifold $J^1 Q$ of $Q \to \mathbb{R}$ is provided with the adapted coordinates $(t, q^i, \dot{q}^i)$ possessing transition functions (4.2.1) which read

$$q'^i_t = (\partial_t + q^j_t \partial_j)q'^i.$$ (1.1.3)

In mechanics on a configuration space $Q \to \mathbb{R}$, the jet manifold $J^1 Q$ plays a role of the velocity space.

Note that, if $Q = \mathbb{R} \times M$ coordinated by $(t, \overline{q}^i)$, there is the canonical isomorphism

$$J^1(\mathbb{R} \times M) = \mathbb{R} \times TM,$$ (1.1.4)

that one can justify by inspection of the transition functions of the coordinates $\overline{q}^i_t$ and $\overline{q}'^i$ when transition functions of $q^i$ are time-independent. Due to the isomorphism (1.1.4), every trivialization (1.1.2) yields the corresponding trivialization of the jet manifold

$$J^1 Q = \mathbb{R} \times TM.$$ (1.1.5)

The canonical imbedding (4.2.5) of $J^1 Q$ takes the form

$$\lambda_{(1)} : J^1 Q \to TQ, \quad \lambda_{(1)} : (t, q^i, \dot{q}^i) \to (t, q^i, \dot{t} = 1, \dot{q}^i = q'^i_t),$$ (1.1.6)

where by $d_t$ is meant the total derivative. From now on, a jet manifold $J^1 Q$ is identified with its image in $TQ$. Using the morphism (1.1.6), one can define the contraction

$$J^1 Q \times T^* Q \to Q \times \mathbb{R},$$

$$(q^i_t, \dot{t}, \dot{q}^i) \to \lambda_{(1)}[(\dot{t} + \dot{q}^i_t dq^i)] = \dot{t} + \dot{q}^i_t \dot{q}^i,$$ (1.1.8)

where $(t, q^i, \dot{t}, \dot{q}^i)$ are holonomic coordinates on the cotangent bundle $T^* Q$.

A glance at the expression (1.1.6) shows that the affine jet bundle $J^1 Q \to Q$ is modelled over the vertical tangent bundle $V Q$ of a fibre bundle $Q \to \mathbb{R}$. As a consequence, there
is the following canonical splitting (4.1.27) of the vertical tangent bundle $V_Q J^1 Q$ of the affine jet bundle $J^1 Q \to Q$:
\[
\alpha : V_Q J^1 Q = J^1 Q \times Q V, \quad \alpha(\partial_t^i) = \partial_i,
\]  
(1.1.9)

Together with the corresponding splitting of the vertical cotangent bundle $V_Q^* J^1 Q$ of $J^1 Q \to Q$:
\[
\alpha^* : V_Q^* J^1 Q = J^1 Q \times Q^* V_Q, \quad \alpha^*(\overline{dq}_i^j) = \overline{dq}_i^j, 
\]  
(1.1.10)

where $\overline{dq}_i^j$ and $\overline{dq}_i^j$ are the holonomic bases for $V_Q^* J^1 Q$ and $V_Q^*$, respectively. Then the exact sequence (4.3.30) of vertical bundles over the composite fibre bundle
\[
J^1 Q \to Q \to \mathbb{R} 
\]  
reads
\[
\begin{array}{c}
0 \to V_Q J^1 Q \xrightarrow{i} V J^1 Q \xrightarrow{\pi_V} J^1 Q \times Q V \to 0 \\
\end{array}
\]

Hence, we obtain the following linear endomorphism over $J^1 Q$ of the vertical tangent bundle $V J^1 Q$ of the jet bundle $J^1 Q \to \mathbb{R}$:
\[
\hat{v} = i \circ \alpha^{-1} \circ \pi_V : V J^1 Q \to V J^1 Q, 
\]  
(1.1.12)

\[
\hat{v}(\partial_t^i) = \partial_t^i, \quad \hat{v}(\partial_t^i) = 0. 
\]

This endomorphism obeys the nilpotency rule $\hat{v} \circ \hat{v} = 0$.

Combining the canonical horizontal splitting (4.1.27), the corresponding epimorphism
\[
\text{pr}_2 : J^1 Q \times TQ \to J^1 Q \times Q V = V_Q J^1 Q, 
\]
\[
\partial_t^i \to -q_i^j \partial_t^j, \quad \partial_t^i \to \partial_t^i, 
\]
and the monomorphism $V J^1 Q \to T J^1 Q$, one can extend the endomorphism (1.1.12) to the tangent bundle $T J^1 Q$:
\[
\hat{v} : T J^1 Q \to T J^1 Q, 
\]  
(1.1.13)

\[
\hat{v}(\partial_t^i) = -q_i^j \partial_t^j, \quad \hat{v}(\partial_t^i) = \partial_t^i, \quad \hat{v}(\partial_t^i) = 0. 
\]

This is called the vertical endomorphism. It inherits the nilpotency property. The transpose of the vertical endomorphism $\hat{v}$ (1.1.13) is
\[
\hat{v}^* : T^* J^1 Q \to T^* J^1 Q, 
\]  
(1.1.14)

\[
\hat{v}^*(dt) = 0, \quad \hat{v}^*(dq_i^j) = 0, \quad \hat{v}^*(dq_i^j) = \theta_i, 
\]
where $\theta^i = dq^i - q^i dt$ are the contact forms (4.2.6). The nilpotency rule $\hat{v}^* \circ \hat{v}^* = 0$ also is fulfilled. The homomorphisms $\hat{v}$ and $\hat{v}^*$ are associated with the tangent-valued one-form $\hat{v} = \theta^i \otimes \partial^i_t$ in accordance with the relations (4.1.46) – (4.1.47).

In view of the morphism $\lambda_{(1)}$ (1.1.6), any connection

$$\Gamma = dt \otimes (\partial_t + \Gamma^i \partial_i) \quad (1.1.15)$$

on a fibre bundle $Q \to \mathbb{R}$ can be identified with a nowhere vanishing horizontal vector field

$$\Gamma = \partial_t + \Gamma^i \partial_i \quad (1.1.16)$$
on $Q$ which is the horizontal lift $\Gamma \partial_t$ (4.3.3) of the standard vector field $\partial_t$ on $\mathbb{R}$ by means of the connection (1.1.15). Conversely, any vector field $\Gamma$ on $Q$ such that $dt \Gamma = 1$ defines a connection on $Q \to \mathbb{R}$. Therefore, the connections (1.1.15) further are identified with the vector fields (1.1.16). The integral curves of the vector field (1.1.16) coincide with the integral sections for the connection (1.1.15).

Connections on a fibre bundle $Q \to \mathbb{R}$ constitute an affine space modelled over the vector space of vertical vector fields on $Q \to \mathbb{R}$. Accordingly, the covariant differential (4.3.8), associated with a connection $\Gamma$ on $Q \to \mathbb{R}$, takes its values into the vertical tangent bundle $VQ$ of $Q \to \mathbb{R}$:

$$D^\Gamma : J^1Q \to VQ, \quad q^i \circ D^\Gamma = q^i_t - \Gamma^i. \quad (1.1.17)$$

A connection $\Gamma$ on a fibre bundle $Q \to \mathbb{R}$ is obviously flat. It yields a horizontal distribution on $Q$. The integral manifolds of this distribution are integral curves of the vector field (1.1.16) which are transversal to fibres of a fibre bundle $Q \to \mathbb{R}$.

**Theorem 1.1.1:** By virtue of Theorem 4.3.1, every connection $\Gamma$ on a fibre bundle $Q \to \mathbb{R}$ defines an atlas of local constant trivializations of $Q \to \mathbb{R}$ such that the associated bundle coordinates $(t, q^i)$ on $Q$ possess the transition functions $q^i \to q'^i(q^j)$ independent of $t$, and

$$\Gamma = \partial_t \quad (1.1.18)$$

with respect to these coordinates. Conversely, every atlas of local constant trivializations of the fibre bundle $Q \to \mathbb{R}$ determines a connection on $Q \to \mathbb{R}$ which is equal to (1.1.18) relative to this atlas. □

A connection $\Gamma$ on a fibre bundle $Q \to \mathbb{R}$ is said to be complete if the horizontal vector field (1.1.16) is complete. In accordance with Remark 4.3.1, a connection on a fibre bundle $Q \to \mathbb{R}$ is complete iff it is an Ehresmann connection. The following holds [15, 27].

**Theorem 1.1.2:** Every trivialization of a fibre bundle $Q \to \mathbb{R}$ yields a complete connection on this fibre bundle. Conversely, every complete connection $\Gamma$ on $Q \to \mathbb{R}$ defines its trivialization (1.1.2) such that the horizontal vector field (1.1.16) equals $\partial_t$ relative to the bundle coordinates associated with this trivialization. □
1.1. PRELIMINARY. FIBRE BUNDLES OVER \( \mathbb{R} \)

Let \( J^1J^1Q \) be the repeated jet manifold of a fibre bundle \( Q \to \mathbb{R} \) provided with the adapted coordinates \((t, q^i, q^i_t, q^i_{tt})\) possessing transition functions
\[
q^i_t = d_t q^i, \quad q^i_{tt} = d_{tt} q^i, \quad q^i_t = d_t q^i_t,
\]
\[
d_t = \partial_t + q^i_t \partial_j + q^i_{tt} \partial_j, \quad d_{tt} = \partial_t + \hat{q}^i_t \partial_j + q^i_{tt} \partial_j.
\]

There is the canonical isomorphism \( k \) between the affine fibrations \( \pi_{11} \) (4.2.10) and \( J^1\pi^1_0 \) (4.2.11) of \( J^1J^1Q \) over \( J^1Q \), i.e.,
\[
\pi_{11} \circ k = J^1_0 \pi_{01}, \quad k \circ k = \text{Id} J^1J^1Q,
\]
where
\[
q^i_t \circ k = \hat{q}^i_t, \quad \hat{q}^i_t \circ k = q^i_t, \quad q^i_{tt} \circ k = q^i_{tt}.
\]

In particular, the affine bundle \( \pi_{11} \) (4.2.10) is modelled over the vertical tangent bundle \( VJ^1Q \) of \( J^1Q \to \mathbb{R} \) which is canonically isomorphic to the underlying vector bundle \( J^1VQ \to J^1Q \) of the affine bundle \( J^1\pi^1_0 \) (4.2.11).

For a fibre bundle \( Q \to \mathbb{R} \), the sesquiholonomic jet manifold \( J^2Q \) coincides with the second order jet manifold \( J^2Q \) coordinated by \((t, q^i, q^i_t, q^i_{tt})\), possessing transition functions
\[
q^i_t = d_t q^i, \quad q^i_{tt} = d_{tt} q^i.
\]

The affine bundle \( J^2Q \to J^1Q \) is modelled over the vertical tangent bundle
\[
V_QJ^1Q = J^1Q \times V_Q \to J^1Q
\]
of the affine jet bundle \( J^1Q \to Q \). There are the imbeddings
\[
J^2Q \xrightarrow{\lambda(2)} TJ^1Q \xrightarrow{T\lambda} V_Q TQ = T^2Q \subset TTQ,
\]
\[
\lambda(2): (t, q^i, q^i_t, q^i_{tt}) \to (t, q^i, q^i_t, i = 1, \hat{q}^i = q^i_t, \hat{q}^i_t = q^i_{tt}),
\]
\[
T\lambda(1) \circ \lambda(2): (t, q^i, q^i_t, q^i_{tt}) \to
\]
\[
(t, q^i, i = 1, \hat{q}^i = q^i_t, \hat{i} = 0, \hat{q}^i_t = q^i_{tt}),
\]
where: (i) \((t, q^i, i, \hat{q}^i, \hat{i}, \hat{q}^i_t, \hat{q}^i_{tt})\) are the coordinates on the double tangent bundle \( TTQ \), (ii) by \( V_Q TQ \) is meant the vertical tangent bundle of \( TQ \to Q \), and (iii) \( T^2Q \subset TTQ \) is the second order tangent space given by the coordinate relation \( \hat{i} = i \).

Due to the morphism (1.1.21), any connection \( \xi \) on the jet bundle \( J^1Q \to \mathbb{R} \) (defined as a section of the affine bundle \( \pi_{11} \) (4.2.10)) is represented by a horizontal vector field on \( J^1Q \) such that \( \xi dt = 1 \).

A connection \( \Gamma \) (1.1.16) on a fibre bundle \( Q \to \mathbb{R} \) has the jet prolongation to the section \( J^1\Gamma \) of the affine bundle \( J^1\pi^1_0 \). By virtue of the isomorphism \( k \) (1.1.19), every connection \( \Gamma \) on \( Q \to \mathbb{R} \) gives rise to the connection
\[
J\Gamma = k \circ J^1\Gamma: J^1Q \to J^1J^1Q, \quad J\Gamma = \partial_t + \Gamma^i \partial_i + d_t \Gamma^i \partial_i,
\]
on the jet bundle \( J^1Q \to \mathbb{R} \).

A connection on the jet bundle \( J^1Q \to \mathbb{R} \) is said to be holonomic if it is a section

\[
\xi = dt \otimes (\partial_t + q^i \partial_i + \xi^i \partial^i)
\]

of the holonomic subbundle \( J^2Q \to J^1Q \) of \( J^1J^1Q \to J^1Q \). In view of the morphism (1.1.21), a holonomic connection is represented by a horizontal vector field

\[
\xi = \partial_t + q^i \partial_i + \xi^i \partial^i
\]  

(1.1.24)
on \( J^1Q \). Conversely, every vector field \( \xi \) on \( J^1Q \) such that

\[
dt|\xi = 1, \quad \hat{v}(\xi) = 0,
\]

where \( \hat{v} \) is the vertical endomorphism (1.1.13), is a holonomic connection on the jet bundle \( J^1Q \to \mathbb{R} \).

Holonomic connections (1.1.24) make up an affine space modelled over the linear space of vertical vector fields on the affine jet bundle \( J^1Q \to Q \), i.e., which live in \( V_QJ^1Q \).

A holonomic connection \( \xi \) defines the corresponding covariant differential (1.1.17) on the jet manifold \( J^1Q \):

\[
D^\xi : J^1J^1Q \to V_QJ^1Q \subset VJ^1Q,
\]

\[
\dot{q}^i \circ D^\xi = 0, \quad \dot{q}^i \circ D^\xi = \dot{q}^i - \xi^i,
\]

which takes its values into the vertical tangent bundle \( V_QJ^1Q \) of the jet bundle \( J^1Q \to Q \). Then by virtue of Theorem 4.2.1, any integral section \( \overline{c} : () \to J^1Q \) for a holonomic connection \( \xi \) is holonomic, i.e., \( \overline{c} = c \) where \( c \) is a curve in \( Q \).

### 1.2 Autonomous dynamic equations

Let us start with dynamic equations on a manifold. From the physical viewpoint, they are treated as autonomous dynamic equations in autonomous mechanics.

Let \( Z, \dim Z > 1 \), be a smooth manifold coordinated by \( (z^\lambda) \).

**Definition 1.2.1:** Let \( u \) be a vector field \( u \) on \( Z \). A closed subbundle \( u(Z) \) of the tangent bundle \( TZ \) given by the coordinate relations

\[
\dot{z}^\lambda = u^\lambda(z)
\]  

(1.2.1)
is said to be an autonomous first order dynamic equation on a manifold \( Z \). This is a system of first order differential equations on a fibre bundle \( \mathbb{R} \times Z \to \mathbb{R} \) in accordance with Definition 4.2.6. □

By a solution of the autonomous first order dynamic equation (1.2.1) is meant an integral curve of the vector field \( u \).
1.2. AUTONOMOUS DYNAMIC EQUATIONS

Definition 1.2.2: An autonomous second order dynamic equation on a manifold $Z$ is defined as a first order dynamic equation on the tangent bundle $TZ$ which is associated with a holonomic vector field

$$\Xi = \dot{z}^\lambda \partial_\lambda + \Xi^\lambda(z^\mu, \dot{z}^\mu)\dot{\partial}_\lambda$$

(1.2.2)
on $TZ$. This vector field, by definition, obeys the condition $J(\Xi) = u_{TZ}$, where $J$ is the endomorphism (4.1.49) and $u_{TZ}$ is the Liouville vector field (4.1.34) on $TZ$. □

The holonomic vector field (1.2.2) also is called the autonomous second order dynamic equation.

Let the double tangent bundle $TTZ$ be provided with coordinates $(z^\lambda, \dot{z}^\lambda, \ddot{z}^\lambda, \dddot{z}^\lambda)$. With respect to these coordinates, the autonomous second order dynamic equation defined by the holonomic vector field $\Xi$ (1.2.2) reads

$$\dot{z}^\lambda = \dot{z}^\lambda, \quad \ddot{z}^\lambda = \Xi^\lambda(z^\mu, \dot{z}^\mu).$$

(1.2.3)

By a solution of the second order dynamic equation (1.2.3) is meant a curve $c : (, ) \to Z$ in a manifold $Z$ whose tangent prolongation $\dot{c} : (, ) \to TZ$ is an integral curve of the holonomic vector field $\Xi$ or, equivalently, whose second order tangent prolongation $\ddot{c}$ lives in the subbundle (1.2.3). It satisfies an autonomous second order differential equation

$$\ddot{c}^\lambda(t) = \Xi^\lambda(c^\mu(t), \dot{c}^\mu(t)).$$

Second order dynamic equations on a manifold $Z$ are exemplified by geodesic equations on the tangent bundle $TZ$.

Given a connection

$$K = dz^\mu \otimes (\partial_\mu + K^\nu_\mu \dot{\partial}_\nu)$$

(1.2.4)
on the tangent bundle $TZ \to Z$, let

$$\widehat{K} : TZ \times TZ \to TTZ$$

(1.2.5)

be the corresponding linear bundle morphism over $TZ$ which splits the exact sequence

$$0 \to V_ZTZ \to TTZ \to TZ \times TZ \xrightarrow{z} 0.$$  

Note that, in contrast with $K$ (4.3.20), the connection $K$ (1.2.4) need not be linear.

Definition 1.2.3: A geodesic equation on $TZ$ with respect to the connection $K$ (1.2.4) is defined as the range

$$\dot{z}^\lambda = \dot{z}^\lambda, \quad \ddot{z}^\mu = K^\nu_\mu \dot{z}^\nu$$

(1.2.6)
of the morphism (1.2.5) restricted to the diagonal $TZ \subset TZ \times TZ$. □
CHAPTER 1. DYNAMIC EQUATIONS

By a solution of a geodesic equation on $TZ$ is meant a geodesic curve $c$ in $Z$ whose tangent prolongation $\dot{c}$ is an integral section (a geodesic vector field) over $c \subset Z$ for a connection $K$.

It is readily observed that the morphism $\hat{K}|_{TZ}$ is a holonomic vector field on $TZ$. It follows that any geodesic equation (1.2.5) on $TZ$ is a second order equation on $Z$. The converse is not true in general. Nevertheless, there is the following theorem [35].

**Theorem 1.2.4**: Every second order dynamic equation (1.2.3) on a manifold $Z$ defines a connection $K_\Xi$ on the tangent bundle $TZ \to Z$ whose components are

$$K^\mu_\nu = \frac{1}{2} \dot{\partial}_\nu \Xi^\mu. \quad (1.2.7)$$

However, the autonomous second order dynamic equation (1.2.3) fails to be a geodesic equation with respect to the connection (1.2.7) in general. In particular, the geodesic equation (1.2.6) with respect to a connection $K$ determines the connection (1.2.7) on $TZ \to Z$ which does not necessarily coincide with $K$.

**Theorem 1.2.5**: A second order equation $\Xi$ on $Z$ is a geodesic equation for the connection (1.2.7) iff $\Xi$ is a spray, i.e., $[u_{TZ}, \Xi] = \Xi$, where $u_{TZ}$ is the Liouville vector field (4.1.34) on $TZ$, i.e.,

$$\Xi^i = a_{ij}(q^k)\dot{q}^i \dot{q}^j$$

and the connection $K$ (1.2.7) is linear. □

### 1.3 Dynamic equations

Let $Q \to X$ (1.1.1) be a configuration space of non-relativistic mechanics. Referring to Definition 4.2.6 of a differential equation on a fibre bundle, one defines a dynamic equation on $Q \to \mathbb{R}$ as a differential equation which is algebraically solved for the highest order derivatives.

**Definition 1.3.1**: Let $\Gamma$ (1.1.16) be a connection on a fibre bundle $Y \to \mathbb{R}$. The corresponding covariant differential $D^\Gamma$ (1.1.17) is a first order differential operator on $Y$. Its kernel, given by the coordinate equation

$$q^i_t = \Gamma^i(t, q^j), \quad (1.3.1)$$

is a closed subbundle of the jet bundle $J^1Y \to \mathbb{R}$. By virtue of Definition 4.2.6, it is a first order differential equation on a fibre bundle $Y \to \mathbb{R}$ called the first order dynamic equation on $Y \to \mathbb{R}$. □
1.3. DYNAMIC EQUATIONS

Due to the canonical imbedding \( J^1Q \rightarrow TQ \) (1.1.6), the equation (1.3.1) is equivalent to the autonomous first order dynamic equation

\[
\dot{t} = 1, \quad \dot{q}^i = \Gamma^i(t, q^i)
\]  

(1.3.2)
on a manifold \( Y \) (Definition 1.2.2). It is defined by the vector field (1.1.16). Solutions of the first order dynamic equation (1.3.1) are integral sections for a connection \( \Gamma \).

**Definition 1.3.2:** Let us consider the first order dynamic equation (1.3.1) on the jet bundle \( J^1Q \rightarrow \mathbb{R} \), which is associated with a holonomic connection \( \xi \) (1.1.24) on \( J^1Q \rightarrow \mathbb{R} \). This is a closed subbundle of the second order jet bundle \( J^2Q \rightarrow \mathbb{R} \) given by the coordinate relations

\[
q'^{ij}_{tt} = \xi'^i(t, q'^i, q'^j),
\]  

(1.3.3)

Consequently, it is a second order differential equation on a fibre bundle \( Q \rightarrow \mathbb{R} \) in accordance with Definition 4.2.6. This equation is called a second order dynamic equation or, simply, a dynamic equation if there is no danger of confusion. The corresponding horizontal vector field \( \xi \) (1.1.24) also is termed a dynamic equation. \( \Box \)

The second order dynamic equation (1.3.3) possesses the coordinate transformation law

\[
q'^{ii}_{tt} = \xi'^i, \quad \xi'^i = (\xi^j \partial_j + q'^i \partial_i \partial_j + 2q'^i \partial_j \partial_t + \partial_t^2)q'^{ii}(t, q'^i),
\]  

(1.3.4)
derived from the formula (1.1.20).

A solution of the dynamic equation (1.3.3) is a curve \( c \) in \( Q \) whose second order jet prolongation \( \ddot{c} \) lives in (1.3.3). Any integral section \( \sigma \) for the holonomic connection \( \xi \) obviously is the jet prolongation \( \dot{c} \) of a solution \( c \) of the dynamic equation (1.3.3), i.e.,

\[
\dddot{c} = \xi \circ \dot{c},
\]  

(1.3.5)
and *vice versa.*

**Remark 1.3.1:** By very definition, the second order dynamic equation (1.3.3) on a fibre bundle \( Q \rightarrow \mathbb{R} \) is equivalent to the system of first order differential equations

\[
\dddot{q}^i = \dot{q}^i, \quad q'^{ii}_{tt} = \xi'^i(t, q'^i, q'^j),
\]  

(1.3.6)
on the jet bundle \( J^1Q \rightarrow \mathbb{R} \). Any solution \( \sigma \) of these equations takes its values into \( J^2Q \) and, by virtue of Theorem 4.2.1, is holonomic, i.e., \( \sigma = \dot{c} \). The equations (1.3.3) and (1.3.6) are therefore equivalent. \( \Diamond \)

A dynamic equation \( \xi \) on a fibre bundle \( Q \rightarrow \mathbb{R} \) is said to be conservative if there exist a trivialization (1.1.2) of \( Q \) and the corresponding trivialization (1.1.5) of \( J^1Q \) such that the vector field \( \xi \) (1.1.24) on \( J^1Q \) is projectable over \( M \). Then this projection

\[
\Xi_{\xi} = \dot{q}^i \partial_i + \xi^i(q^j, \dot{q}^j)\dot{q}^j
\]  


is an autonomous second order dynamic equation on the typical fibre $M$ of $Q \to \mathbb{R}$ in accordance with Definition 1.2.2. Conversely, every autonomous second order dynamic equation $\Xi$ (1.2.2) on a manifold $M$ can be seen as a conservative dynamic equation

$$\xi \Xi = \partial_t + \dot{q}^i \partial_i + \xi^i \dot{\partial}_i$$

on the fibre bundle $\mathbb{R} \times M \to \mathbb{R}$ in accordance with the isomorphism (1.1.5).

The following theorem holds [15, 27].

**Theorem 1.3.3:** Any dynamic equation $\xi$ (1.3.3) on a fibre bundle $Q \to \mathbb{R}$ is equivalent to an autonomous second order dynamic equation $\Xi$ on a manifold $Q$ which makes the diagram

$$\begin{array}{ccc}
J^2Q & \longrightarrow & T^2Q \\
\xi & \parallel & \Xi \\
J^1Q & \overset{\lambda^{(1)}}{\longrightarrow} & TQ
\end{array}$$

commutative and obeys the relations

$$\xi^i = \Xi^i(t, q^j, \dot{t} = 1, \dot{q}^i = q^i_\dot{t}), \quad \Xi^t = 0.$$ 

Accordingly, the dynamic equation (1.3.3) is written in the form

$$q^i_{tt} = \Xi^i \big|_{i=1, \dot{q}^i = q^i_\dot{t}},$$

which is equivalent to the autonomous second order dynamic equation

$$\ddot{t} = 0, \quad \dot{t} = 1, \quad \ddot{q}^i = \Xi^i,$$  \hspace{1cm} (1.3.8)

on $Q$. □

### 1.4 Dynamic connections

In order to say something more, let us consider the relationship between the holonomic connections on the jet bundle $J^1Q \to \mathbb{R}$ and the connections on the affine jet bundle $J^1Q \to Q$ (see Propositions 1.4.1 and 1.4.2 below).

By $J^1_Q J^1Q$ throughout is meant the first order jet manifold of the affine jet bundle $J^1Q \to Q$. The adapted coordinates on $J^1_Q J^1Q$ are $(q^\lambda, q^i_\dot{t}, q^i_{\dot{\lambda}t})$, where we use the compact notation $\lambda = (0, i)$, $q^0 = t$. Let

$$\gamma : J^1Q \to J^1_Q J^1Q$$

be a connection on the affine jet bundle $J^1Q \to Q$. It takes the coordinate form

$$\gamma = dq^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial_i),$$  \hspace{1cm} (1.4.1)
together with the coordinate transformation law
\[ \gamma^i_{\lambda} = (\partial_j q^\mu \gamma^j_\mu + \partial_\mu q^\lambda_t) \frac{\partial q^\mu}{\partial q^\lambda}. \] (1.4.2)

**Remark 1.4.1:** In view of the canonical splitting (1.1.9), the curvature (4.3.13) of the connection \( \gamma \) (1.4.1) reads
\[ R: J^1 Q \to \wedge^2 T^* J^1 Q \otimes VQ, \]
\[ R = \frac{1}{2} R^i_{\lambda \mu} dq^\lambda \wedge dq^\mu \otimes \partial_i = \left( \frac{1}{2} R^i_{kj} dq^k \wedge dq^j + R^i_{0j} dt \wedge dq^j \right) \otimes \partial_i, \]
\[ R^i_{\lambda \mu} = \partial_\lambda \gamma^i_\mu - \partial_\mu \gamma^i_\lambda + \gamma^i_j \partial_j \gamma^j_\mu - \gamma^j_\mu \partial_j \gamma^i_\lambda. \] (1.4.3)

Using the contraction (1.1.8), we obtain the soldering form
\[ \lambda_{(1)}(1) R = [(R^i_{kj} q^k_t + R^i_{0j}) dq^j - R^i_{0j} q^j_t dt] \otimes \partial_i \]
on the affine jet bundle \( J^1 Q \to Q \). Its image by the canonical projection \( T^* Q \to V^* Q \) (2.2.5) is the tensor field
\[ \overline{R}: J^1 Q \to V^* Q \otimes VQ, \quad \overline{R} = (R^i_{kj} q^k_t + R^i_{0j}) dq^j \otimes \partial_i, \] (1.4.4)
and then we come to the scalar field
\[ \tilde{R}: J^1 Q \to \mathbb{R}, \quad \tilde{R} = R^i_{kj} q^k_t + R^i_{0t}, \] (1.4.5)
on the jet manifold \( J^1 Q \).

**Proposition 1.4.1:** Any connection \( \gamma \) (1.4.1) on the affine jet bundle \( J^1 Q \to Q \) defines the holonomic connection
\[ \xi_\gamma = \rho \circ \gamma: J^1 Q \to J^2 Q, \]
\[ \xi_\gamma = \partial_t + q^i_t \partial_i + (\gamma^i_0 + q^i_t \gamma^j_0) \partial_i, \] (1.4.6)
on the jet bundle \( J^1 Q \to \mathbb{R} \).

**Proof:** Let us consider the composite fibre bundle (1.1.11) and the morphism \( \rho \) (4.3.25) which reads
\[ \rho: J^1 Q \to (q^\lambda, q^i, q^i_0) \mapsto (q^\lambda, q^i_t, \tilde{q}^j_i = q^i_t, q^i_0 = q^i_0 + q^j_i q^j_0) \in J^2 Q. \] (1.4.7)
A connection \( \gamma \) (1.4.1) and the morphism \( \rho \) (1.4.7) combine into the desired holonomic connection \( \xi_\gamma \) (1.4.6) on the jet bundle \( J^1 Q \to \mathbb{R} \). QED

It follows that every connection \( \gamma \) (1.4.1) on the affine jet bundle \( J^1 Q \to Q \) yields the dynamic equation
\[ q^i_{tt} = \gamma^i_0 + q^i_t \gamma^j_j. \] (1.4.8)
on the configuration bundle \( Q \to \mathbb{R} \). This is precisely the restriction to \( J^2 Q \) of the kernel \( \text{Ker} \tilde{D}^\gamma \) of the vertical covariant differential \( \tilde{D}^\gamma \) (4.3.34) defined by the connection \( \gamma \):

\[
\tilde{D}^\gamma : J^1 J^1 Q \to V_Q J^1 Q, \quad \tilde{\dot{q}}_i^d \circ \tilde{D}^\gamma = \dot{q}_i^d - \gamma^i_0 - q_j^d \gamma^i_j \tag{1.4.9}
\]

Therefore, connections on the jet bundle \( J^1 Q \to Q \) are called the dynamic connections. The corresponding equation (1.3.5) can be written in the form

\[
\ddot{c}_i = \rho \circ \gamma \circ \dot{c},
\]

where \( \rho \) is the morphism (1.4.7).

Of course, different dynamic connections can lead to the same dynamic equation (1.4.8).

**Proposition 1.4.2:** Any holonomic connection \( \xi \) (1.1.24) on the jet bundle \( J^1 Q \to \mathbb{R} \) yields the dynamic connection

\[
\gamma_\xi = dt \otimes \left[ \partial_t + (\xi^i - \frac{1}{2} q_i^d \partial_j^i \xi^j) \partial_t^i \right] + dq^j \otimes \left[ \partial_j + \frac{1}{2} \partial_j^i \xi^i \partial_t^i \right] \tag{1.4.10}
\]

on the affine jet bundle \( J^1 Q \to Q \) [15, 27]. \( \square \)

It is readily observed that the dynamic connection \( \gamma_\xi \) (1.4.10), defined by a dynamic equation, possesses the property

\[
\gamma_k^i = \partial_t^i \gamma_0^k + q_i^d \partial_t^i \gamma_j^k, \tag{1.4.11}
\]

which implies the relation \( \partial_j^i \gamma_k^i = \partial_j^i \gamma_k^i \). Therefore, a dynamic connection \( \gamma \), obeying the condition (1.4.11), is said to be symmetric. The torsion of a dynamic connection \( \gamma \) is defined as the tensor field

\[
T : J^1 Q \to V^* Q \otimes V Q,
\]

\[
T = T^k_i dq^i \otimes \partial_k, \quad T^k_i = \gamma^i_k - \partial_t^i \gamma_0^k - q_i^d \partial_t^i \gamma_j^k. \tag{1.4.12}
\]

It follows at once that a dynamic connection is symmetric iff its torsion vanishes.

Let \( \gamma \) be the dynamic connection (1.4.1) and \( \xi_\gamma \) the corresponding dynamic equation (1.4.6). Then the dynamic connection (1.4.10) associated with the dynamic equation \( \xi_\gamma \) takes the form

\[
\gamma_{\xi_\gamma}^i = \frac{1}{2} (\gamma_k^i + \partial_t^i \gamma_0^k + q_i^d \partial_t^i \gamma_j^k), \quad \gamma_{\xi_\gamma}^i = \gamma_0^k + q_i^d \gamma_j^k - q_i^d \gamma_{\xi_\gamma}^k.
\]

It is readily observed that \( \gamma = \gamma_{\xi_\gamma} \) iff the torsion \( T \) (1.4.12) of the dynamic connection \( \gamma \) vanishes.

**Example 1.4.2:** Since a jet bundle \( J^1 Q \to Q \) is affine, it admits an affine connection

\[
\gamma = dq^\lambda \otimes [\partial_\lambda + (\gamma_{\lambda_0}^i (q^\mu) + \gamma_{\lambda_j}^i (q^j) q^i_j) \partial_t^i]. \tag{1.4.13}
\]
1.5 Non-relativistic geodesic equations

This connection is symmetric iff \( \gamma^i_{\lambda\mu} = \gamma^i_{\mu\lambda} \). One can easily justify that an affine dynamic connection generates a quadratic dynamic equation, and *vice versa*. Nevertheless, a non-affine dynamic connection, whose symmetric part is affine, also defines a quadratic dynamic equation. 

Using the notion of a dynamic connection, we can modify Theorem 1.2.4 as follows. Let \( \Xi \) be an autonomous second order dynamic equation on a manifold \( M \), and let \( \xi \) (1.3.7) be the corresponding conservative dynamic equation on the bundle \( \mathbb{R} \times M \to \mathbb{R} \). The latter yields the dynamic connection (1.4.10) on a fibre bundle

\[ \mathbb{R} \times TM \to \mathbb{R} \times M. \]

Its components \( \gamma^i_j \) are exactly those of the connection (1.2.7) on the tangent bundle \( TM \to M \) in Theorem 1.2.4, while \( \gamma^i_0 \) make up a vertical vector field

\[ e = \gamma^i_0 \partial_i = \left( \Xi^i - \frac{1}{2} \dot{q}^j \partial_j \Xi^i \right) \partial_i \quad (1.4.14) \]

on \( TM \to M \). Thus, we have shown the following.

**Proposition 1.4.3:** Every autonomous second order dynamic equation \( \Xi \) (1.2.3) on a manifold \( M \) admits the decomposition

\[ \Xi^i = K^i_j \dot{q}^j + e^i \]

where \( K \) is the connection (1.2.7) on the tangent bundle \( TM \to M \), and \( e \) is the vertical vector field (1.4.14) on \( TM \to M \). \( \square \)

1.5 Non-relativistic geodesic equations

In this Section, we aim to show that every dynamic equation on a configuration bundle \( Q \to \mathbb{R} \) is equivalent to a geodesic equation on the tangent bundle \( TQ \to Q \).

We start with the relation between the dynamic connections \( \gamma \) on the affine jet bundle \( J^1Q \to Q \) and the connections

\[ K = dq^\lambda \otimes (\partial_\lambda + K^\mu_\lambda \dot{q}^\mu) \quad (1.5.1) \]

on the tangent bundle \( TQ \to Q \) of the configuration space \( Q \). Note that they need not be linear. We follow the compact notation (4.1.30).

Let us consider the diagram

\[ J^1_{0}J^1_{Q} \xrightarrow{J^1_{1}(\gamma)} J^1_{Q}TQ \]

\[ \gamma \]

\[ J^1_{Q} \xrightarrow{\lambda_{(1)}} TQ \]

\[ K \]
where $J^1_Q TQ$ is the first order jet manifold of the tangent bundle $TQ \to Q$, coordinated by

$$(t, q^i, \dot{t}, \dot{q}^i, (\dot{t})_\mu, (\dot{q}^i)_\mu).$$

The jet prolongation over $Q$ of the canonical imbedding $\lambda_{(1)}$ (1.1.6) reads

$$J^1 \lambda_{(1)} : (t, q^i, \dot{q}^i, q^i_\mu) \mapsto (t, q^i, \dot{t} = 1, \dot{q}^i = q^i_t, (\dot{t})_\mu = 0, (\dot{q}^i)_\mu = q^i_\mu).$$

Then we have

$$J^1 \lambda_{(1)} \circ \gamma : (t, q^i, \dot{q}^i, q^i_\mu) \mapsto (t, q^i, \dot{t} = 1, \dot{q}^i = q^i_t, (\dot{t})_\mu = 0, (\dot{q}^i)_\mu = \gamma^i_\mu),$$

$$K \circ \lambda_{(1)} : (t, q^i, \dot{q}^i, q^i_\mu) \mapsto (t, q^i, \dot{t} = 1, \dot{q}^i = q^i_t, (\dot{t})_\mu = K^0_\mu, (\dot{q}^i)_\mu = K^i_\mu).$$

It follows that the diagram (1.5.2) can be commutative only if the components $K^0_\mu$ of the connection $K$ (1.5.1) on the tangent bundle $TQ \to Q$ vanish.

Since the transition functions $t \to t'$ are independent of $q^i$, a connection

$$\tilde{K} = dq^\lambda \otimes (\partial_\lambda + K^i_\lambda \dot{q}^i)$$

with $K^0_\mu = 0$ may exist on the tangent bundle $TQ \to Q$ in accordance with the transformation law

$$K'^i_\lambda = (\partial_j q^i K^j_\mu + \partial_\mu q^j K^i_j) \frac{\partial q^\mu}{\partial q'^\lambda}. \quad (1.5.4)$$

Now the diagram (1.5.2) becomes commutative if the connections $\gamma$ and $\tilde{K}$ fulfill the relation

$$\gamma^i_\mu = K^i_\mu \circ \lambda_{(1)} = K^i_\mu(t, q^i, t = 1, \dot{q}^i = q^i_t). \quad (1.5.5)$$

It is easily seen that this relation holds globally because the substitution of $\dot{q}^i = q^i_t$ in (1.5.4) restates the transformation law (1.4.2) of a connection on the affine jet bundle $J^1 Q \to Q$. In accordance with the relation (1.5.5), the desired connection $\tilde{K}$ is an extension of the section $J^1 \lambda \circ \gamma$ of the affine jet bundle $J^1_Q TQ \to TQ$ over the closed submanifold $J^1 Q \subset TQ$ to a global section. Such an extension always exists by virtue of Theorem 4.1.2, but it is not unique. Thus, we have proved the following.

**Proposition 1.5.1:** In accordance with the relation (1.5.5), every dynamic equation on a configuration bundle $Q \to \mathbb{R}$ can be written in the form

$$q^i_{tt} = K^i_0 \circ \lambda_{(1)} + q^i_0 K^j_0 \circ \lambda_{(1)}, \quad (1.5.6)$$

where $\tilde{K}$ is the connection (1.5.3) on the tangent bundle $TQ \to Q$. Conversely, each connection $\tilde{K}$ (1.5.3) on $TQ \to Q$ defines the dynamic connection $\gamma$ (1.5.5) on the affine
jet bundle $J^1Q \to Q$ and the dynamic equation (1.5.6) on a configuration bundle $Q \to \mathbb{R}$.

Then we come to the following theorem.

**Theorem 1.5.2**: Every dynamic equation (1.3.3) on a configuration bundle $Q \to \mathbb{R}$ is equivalent to the geodesic equation

\[
\ddot{q}^0 = 0, \quad \dot{q}^0 = 1, \\
\dddot{q}^i = K^i_{\lambda}(q^\mu, \dot{q}^\mu)\dot{q}^\lambda,
\]

(1.5.7)
on the tangent bundle $TQ$ relative to a connection $\tilde{K}$ with the components $K^0_\lambda = 0$ and $K^i_\lambda$ (1.5.5). Its solution is a geodesic curve in $Q$ which also obeys the dynamic equation (1.5.6), and vice versa.

In accordance with this theorem, the autonomous second order equation (1.3.8) in Theorem 1.3.3 can be chosen as a geodesic equation. It should be emphasized that, written in the bundle coordinates $(t, q^i)$, the geodesic equation (1.5.7) and the connection $\tilde{K}$ (1.5.5) are well defined with respect to any coordinates on $Q$.

From the physical viewpoint, the most relevant dynamic equations are the quadratic ones

\[
\xi^i = a^i_{jk}(q^\mu)\dot{q}^j\dot{q}^k + b^i_{j}(q^\mu)\dot{q}^j + f^i(q^\mu).
\]

(1.5.8)

This property is global due to the transformation law (1.3.4). Then one can use the following two facts.

**Proposition 1.5.3**: There is one-to-one correspondence between the affine connections $\gamma$ on the affine jet bundle $J^1Q \to Q$ and the linear connections $K$ (1.5.3) on the tangent bundle $TQ \to Q$.

**Proof**: This correspondence is given by the relation (1.5.5), written in the form

\[
\gamma^i_{\mu} = K^i_{\mu} = K^i_{\mu 0}(q^\nu)\dot{t} + K^i_{\mu j}(q^\nu)\dot{q}^j |_{i=1, q^\nu = q^i} = K^i_{\mu 0}(q^\nu) + K^i_{\mu j}(q^\nu)\dot{q}^j,
\]

i.e., $\gamma^i_{\mu \lambda} = K^i_{\mu \lambda}$. QED

In particular, if an affine dynamic connection $\gamma$ is symmetric, so is the corresponding linear connection $K$.

**Corollary 1.5.4**: Every quadratic dynamic equation (1.5.8) on a configuration bundle $Q \to \mathbb{R}$ of mechanics gives rise to the geodesic equation

\[
\ddot{q}^0 = 0, \quad \dot{q}^0 = 1, \\
\dddot{q}^i = a^i_{jk}(q^\mu)\dot{q}^j\dot{q}^k + b^i_{j}(q^\mu)\dot{q}^j + f^i(q^\mu)\dot{q}^0\dot{q}^0
\]

(1.5.9)
on the tangent bundle $TQ$ with respect to the symmetric linear connection

$$K_\lambda^0\nu = 0, \quad K^0_i = f^i, \quad K^i_j = \frac{1}{2}b^i_j, \quad K^0_i = a^i_k$$

(1.5.10)
on the tangent bundle $TQ \rightarrow Q$. □

The geodesic equation (1.5.9), however, is not unique for the dynamic equation (1.5.8).

**Proposition 1.5.5:** Any quadratic dynamic equation (1.5.8), being equivalent to the geodesic equation with respect to the symmetric linear connection $\tilde{K}$ (1.5.10), also is equivalent to the geodesic equation with respect to an affine connection $K'$ on $TQ \rightarrow Q$ which differs from $\tilde{K}$ (1.5.10) in a soldering form $\sigma$ on $TQ \rightarrow Q$ with the components

$$\sigma^0_\lambda = 0, \quad \sigma^i_k = h^i_k + (s - 1)h^i_q^0, \quad \sigma^i_0 = -sh^i_k\dot{q}^k - h^i_q^0\dot{q}^0 + h^i_0,$$

where $s$ and $h^i_\lambda$ are local functions on $Q$. □

Proposition 1.5.5 also can be deduced from the following lemma.

**Lemma 1.5.6:** Every affine vertical vector field

$$\sigma = [f^i(q^\mu) + b^j_i(q^\mu)\dot{q}^j]\partial_i$$

(1.5.11)
on the affine jet bundle $J^1Q \rightarrow Q$ is extended to the soldering form

$$\sigma = (f^i dt + b^k_i dq^k) \otimes \dot{\partial}_i$$

(1.5.12)
on the tangent bundle $TQ \rightarrow Q$. □

Now let us extend our inspection of dynamic equations to connections on the tangent bundle $TM \rightarrow M$ of the typical fibre $M$ of a configuration bundle $Q \rightarrow \mathbb{R}$. In this case, the relationship fails to be canonical, but depends on a trivialization (1.1.2) of $Q \rightarrow \mathbb{R}$.

Given such a trivialization, let $(t, \underline{q}^i)$ be the associated coordinates on $Q$, where $\underline{q}^i$ are coordinates on $M$ with transition functions independent of $t$. The corresponding trivialization (1.1.5) of $J^1Q \rightarrow \mathbb{R}$ takes place in the coordinates $(t, \underline{q}^i, \dot{\underline{q}}^i)$, where $\dot{\underline{q}}^i$ are coordinates on $TM$. With respect to these coordinates, the transformation law (1.4.2) of a dynamic connection $\gamma$ on the affine jet bundle $J^1Q \rightarrow Q$ reads

$$\gamma^i_0 = \frac{\partial \underline{q}^i}{\partial \underline{q}^j} \gamma^j_0 \quad \gamma^i_k = \left(\frac{\partial \underline{q}^i}{\partial \underline{q}^n} \gamma^j_n + \frac{\partial \underline{q}^j}{\partial \underline{q}^n} \right) \frac{\partial \underline{q}^n}{\partial \underline{q}^k}.$$

It follows that, given a trivialization of $Q \rightarrow \mathbb{R}$, a connection $\gamma$ on $J^1Q \rightarrow Q$ defines the time-dependent vertical vector field

$$\gamma^i_0(t, \underline{q}^i, \dot{\underline{q}}^i) \frac{\partial}{\partial \underline{q}^i} : \mathbb{R} \times TM \rightarrow VTM$$
and the time-dependent connection

$$
\frac{\partial}{\partial \tilde{q}^i} \otimes \left( \frac{\partial}{\partial \tilde{q}^i} + \gamma^i_k(t, \tilde{q}^j, \tilde{q}^j) \frac{\partial}{\partial \tilde{q}^j} \right) : \mathbb{R} \times TM \to J^1TM \subset TTM
$$

(1.5.13)
on the tangent bundle $TM \to M$.

Conversely, let us consider a connection

$$
\mathcal{K} = \frac{\partial}{\partial \tilde{q}^k} \otimes \left( \frac{\partial}{\partial \tilde{q}^k} + K^i_k(\tilde{q}^j, \tilde{q}^j) \frac{\partial}{\partial \tilde{q}^j} \right)
$$
on the tangent bundle $TM \to M$. Given the above-mentioned trivialization of the configuration bundle $Q \to \mathbb{R}$, the connection $\mathcal{K}$ defines the connection $\bar{K}$ (1.5.3) with the components

$$
K^i_0 = 0, \quad K^i_k = \bar{K}^i_k,
$$
on the tangent bundle $TQ \to Q$. The corresponding dynamic connection $\gamma$ on the affine jet bundle $J^1Q \to Q$ reads

$$
\gamma^i_0 = 0, \quad \gamma^i_k = \bar{K}^i_k.
$$

(1.5.14)

Using the transformation law (1.4.2), one can extend the expression (1.5.14) to arbitrary bundle coordinates $(t, q^i)$ on the configuration space $Q$ as follows:

$$
\gamma^i_k = \left[ \frac{\partial q^i}{\partial \tilde{q}^l} \mathcal{K}^l_n(\tilde{q}^j(\tilde{q}^r, \tilde{q}^r)) + \frac{\partial^2 q^i}{\partial \tilde{q}^l \partial \tilde{q}^m} \tilde{q}^m + \frac{\partial \Gamma^i}{\partial \tilde{q}^l} \right] \partial_k \tilde{q}^l,
$$

$$
\gamma^i_0 = \partial_t \Gamma^i + \partial_j \Gamma^i q^j - \gamma^i_k \Gamma^k,
$$

(1.5.15)

where $\Gamma^i = \partial_t q^i(t, \tilde{q}^j)$ is the connection on $Q \to \mathbb{R}$, corresponding to a given trivialization of $Q$, i.e., $\Gamma^i = 0$ relative to $(t, \tilde{q}^j)$. The dynamic equation on $Q$ defined by the dynamic connection (1.5.15) takes the form

$$
q^i_{tt} = \partial_t \Gamma^i + q^j_t \partial_j \Gamma^i + \gamma^i_k(q^k_t - \Gamma^k).
$$

(1.5.16)

By construction, it is a conservative dynamic equation. Thus, we have proved the following.

**Proposition 1.5.7:** Any connection $\mathcal{K}$ on the typical fibre $M$ of a configuration bundle $Q \to \mathbb{R}$ yields a conservative dynamic equation (1.5.16) on $Q$. □
1.6 Reference frames

From the physical viewpoint, a reference frame in non-relativistic mechanics determines a tangent vector at each point of a configuration space \( Q \), which characterizes the velocity of an observer at this point. This speculation leads to the following mathematical definition of a reference frame in mechanics [15, 27, 33, 40].

**Definition 1.6.1**: A non-relativistic reference frame is a connection \( \Gamma \) on a configuration space \( Q \to \mathbb{R} \).

By virtue of this definition, one can think of the horizontal vector field (1.1.16) associated with a connection \( \Gamma \) on \( Q \to \mathbb{R} \) as being a family of observers, while the corresponding covariant differential (1.1.17):

\[
\dot{q}_t^i = D^\Gamma(q_t^i) = q_t^i - \Gamma^i,
\]
determines the relative velocity with respect to a reference frame \( \Gamma \). Accordingly, \( q_t^i \) are regarded as the absolute velocities.

In particular, given a motion \( c : \mathbb{R} \to Q \), its covariant derivative \( \nabla^\Gamma c \) (4.3.9) with respect to a connection \( \Gamma \) is a velocity of this motion relative to a reference frame \( \Gamma \). For instance, if \( c \) is an integral section for a connection \( \Gamma \), a velocity of the motion \( c \) relative to a reference frame \( \Gamma \) is equal to \( 0 \). Conversely, every motion \( c : \mathbb{R} \to Q \) defines a reference frame \( \Gamma_c \) such that a velocity of \( c \) relative to \( \Gamma_c \) vanishes. This reference frame \( \Gamma_c \) is an extension of a section \( c(\mathbb{R}) \to J^1Q \) of an affine jet bundle \( J^1Q \to Q \) over the closed submanifold \( c(\mathbb{R}) \in Q \) to a global section in accordance with Theorem 4.1.2.

By virtue of Theorem 1.1.1, any reference frame \( \Gamma \) on a configuration bundle \( Q \to \mathbb{R} \) is associated with an atlas of local constant trivializations, and vice versa. A connection \( \Gamma \) takes the form \( \Gamma = \partial_t \) (1.1.18) with respect to the corresponding coordinates \((t, \vec{q})\), whose transition functions \( \vec{q} \to \vec{q}' \) are independent of time. One can think of these coordinates as also being a reference frame, corresponding to the connection (1.1.18). They are called the adapted coordinates to a reference frame \( \Gamma \). Thus, we come to the following definition, equivalent to Definition 1.6.1.

**Definition 1.6.2**: In mechanics, a reference frame is an atlas of local constant trivializations of a configuration bundle \( Q \to \mathbb{R} \).

In particular, with respect to the coordinates \( \vec{q} \) adapted to a reference frame \( \Gamma \), the velocities relative to this reference frame coincide with the absolute ones

\[
D^\Gamma(\vec{q}_t^i) = \dot{\vec{q}}_t^i = \vec{q}_t^i.
\]

**Remark 1.6.1**: By analogy with gauge field theory, we agree to call transformations of bundle atlases of a fibre bundle \( Q \to \mathbb{R} \) the gauge transformations. To be precise, one should call them passive gauge transformations, while by active gauge transformations are meant automorphisms...
1.6. REFERENCE FRAMES

of a fibre bundle. In mechanics, gauge transformations also are reference frame transformations in accordance with Theorem 1.1.1. An object on a fibre bundle is said to be gauge covariant or, simply, covariant if its definition is atlas independent. It is called gauge invariant if its form is maintained under atlas transformations.

A reference frame is said to be complete if the associated connection $\gamma$ is complete. By virtue of Proposition 1.1.2, every complete reference frame defines a trivialization of a bundle $Q \rightarrow \mathbb{R}$, and vice versa.

**Remark 1.6.2:** Given a reference frame $\gamma$, one should solve the equations

$$
\Gamma^i(t, q^j(t, \bar{q}^a)) = \frac{\partial q^i(t, \bar{q}^a)}{\partial t}, \quad (1.6.1)
$$

$$
\frac{\partial \bar{q}^a(t, q^j)}{\partial q^i} \Gamma^i(t, q^j) + \frac{\partial q^a(t, q^j)}{\partial t} = 0 \quad (1.6.2)
$$

in order to find the coordinates $(t, \bar{q}^a)$ adapted to $\gamma$. Let $(t, q^i_1)$ and $(t, q^i_2)$ be the adapted coordinates for reference frames $\gamma_1$ and $\gamma_2$, respectively. In accordance with the equality (1.6.2), the components $\Gamma^i_1$ of the connection $\gamma_1$ with respect to the coordinates $(t, q^i_2)$ and the components $\Gamma^i_2$ of the connection $\gamma_2$ with respect to the coordinates $(t, q^i_1)$ fulfill the relation

$$
\frac{\partial q^i_2}{\partial q^i_1} \Gamma^i_1 + \Gamma^a_2 = 0.
$$

Using the relations (1.6.1) – (1.6.2), one can rewrite the coordinate transformation law (1.3.4) of dynamic equations as follows. Let

$$
\bar{q}^a_{tt} = \xi^a
$$

be a dynamic equation on a configuration space $Q$ written with respect to a reference frame $(t, \bar{q}^a)$. Then, relative to arbitrary bundle coordinates $(t, q^i)$ on $Q \rightarrow \mathbb{R}$, the dynamic equation (1.6.3) takes the form

$$
q^i_{tt} = d_t \Gamma^i + \partial_j \Gamma^i (q^j_t - \Gamma^j) - \frac{\partial q^j}{\partial q^a} \frac{\partial \bar{q}^a}{\partial q^j} (q^j_t - \Gamma^j) (q^k_t - \Gamma^k) + \frac{\partial q^i}{\partial \bar{q}^a} \xi^a, \quad (1.6.4)
$$

where $\gamma$ is a connection corresponding to the reference frame $(t, \bar{q}^a)$. The dynamic equation (1.6.4) can be expressed in the relative velocities $\dot{q}^i_\gamma = q^i_t - \Gamma^i$ with respect to the initial reference frame $(t, \bar{q}^a)$. We have

$$
d_t \dot{q}^i_\gamma = \partial_j \Gamma^i \dot{q}^j_\gamma - \frac{\partial q^j}{\partial q^a} \frac{\partial \bar{q}^a}{\partial q^j} \dot{q}^j_\gamma \dot{q}^k_\gamma + \frac{\partial q^i}{\partial \bar{q}^a} \xi^a (t, q^j, \dot{q}^j_\gamma). \quad (1.6.5)
$$

Accordingly, any dynamic equation (1.3.3) can be expressed in the relative velocities $\dot{q}^i_\gamma = q^i_t - \Gamma^i$ with respect to an arbitrary reference frame $\gamma$ as follows:

$$
d_t \dot{q}^i_\gamma = (\xi - J\gamma)_i = \xi^i - d_t \Gamma, \quad (1.6.6)
$$
where $J\Gamma$ is the prolongation (1.1.23) of a connection $\Gamma$ onto the jet bundle $J^1Q \to \mathbb{R}$.

For instance, let us consider the following particular reference frame $\Gamma$ for a dynamic equation $\xi$. The covariant derivative of a reference frame $\Gamma$ with respect to the corresponding dynamic connection $\gamma_\xi$ (1.4.10) reads

$$\nabla^{\gamma_\Gamma} = Q \to T^*Q \times VQ J^1Q,$$

$$\nabla^{\gamma_\Gamma} = \nabla_\lambda^k dq^\lambda \otimes \partial_k, \quad \nabla_\lambda^k = \partial_\lambda^k - \gamma_k^\lambda \circ \Gamma.$$  

A connection $\Gamma$ is called a geodesic reference frame for the dynamic equation $\xi$ if

$$\Gamma \rceil \nabla^{\gamma_\Gamma} = \Gamma_\lambda^i (\partial_\lambda^k - \gamma_k^\lambda \circ \Gamma) = (d_t \Gamma_i - \xi_i \circ \Gamma) \partial_i = 0.$$  

**Proposition 1.6.3**: Integral sections $c$ for a reference frame $\Gamma$ are solutions of a dynamic equation $\xi$ iff $\Gamma$ is a geodesic reference frame for $\xi$. 

**Remark 1.6.3**: The left- and right-hand sides of the equation (1.6.6) separately are not well-behaved objects. This equation is brought into the covariant form (1.8.6).

Reference frames play a prominent role in many constructions of mechanics. They enable us to write the covariant forms: (1.8.5) – (1.8.6) of dynamic equations and (3.1.16) of Hamiltonians of mechanics.

With a reference frame, we obtain the converse of Theorem 1.5.2.

**Theorem 1.6.4**: Given a reference frame $\Gamma$, any connection $K$ (1.5.1) on the tangent bundle $TQ \to Q$ defines a dynamic equation

$$\xi^i = (K_\lambda^i - \Gamma_\lambda^i K_\lambda^0) \dot{q}^\lambda \big|_{q^0=1, \dot{q}^i=\dot{q}^i_0}.$$  

This theorem is a corollary of Proposition 1.5.1 and the following lemma.

**Lemma 1.6.5**: Given a connection $\Gamma$ on a fibre bundle $Q \to \mathbb{R}$ and a connection $K$ on the tangent bundle $TQ \to Q$, there is the connection $\tilde{K}$ on $TQ \to Q$ with the components

$$\tilde{K}^0_\lambda = 0, \quad \tilde{K}^i_\lambda = K_\lambda^i - \Gamma^i_\lambda K^0_\lambda.$$  

**1.7 Free motion equations**

Let us point out the following interesting class of dynamic equations which we agree to call the free motion equations.

Definition 1.7.1: We say that the dynamic equation (1.3.3) is a free motion equation if there exists a reference frame \((t, \mathbf{q}^i)\) on the configuration space \(Q\) such that this equation reads
\[
\mathbf{q}_{tt}^i = 0. \tag{1.7.1}
\]

With respect to arbitrary bundle coordinates \((t, \mathbf{q}^i)\), a free motion equation takes the form
\[
\mathbf{q}_{tt}^i = d_t \Gamma^i + \partial_j \Gamma^i (\mathbf{q}_{tt}^j - \Gamma^j) - \frac{\partial \mathbf{q}^i}{\partial \mathbf{q}^m} \frac{\partial \mathbf{q}^m}{\partial \mathbf{q}^j} (\mathbf{q}_{tt}^j - \Gamma^j)(\mathbf{q}_{tt}^k - \Gamma^k), \tag{1.7.2}
\]
where \(\Gamma^i = \partial_i \mathbf{q}^i(t, \mathbf{q}^j)\) is the connection associated with the initial frame \((t, \mathbf{q}^i)\) (cf. (1.6.4)).

One can think of the right-hand side of the equation (1.7.2) as being the general coordinate expression for an inertial force in mechanics. The corresponding dynamic connection \(\gamma_\xi\) on the affine jet bundle \(J^1Q \to Q\) reads
\[
\gamma_k^i = \partial_k \Gamma^i - \frac{\partial \mathbf{q}^i}{\partial \mathbf{q}^m} \frac{\partial \mathbf{q}^m}{\partial \mathbf{q}^j} (\mathbf{q}_{tt}^j - \Gamma^j), \tag{1.7.3}
\]
\[
\gamma_0^i = \partial_t \Gamma^i + \partial_j \Gamma^i \mathbf{q}_{tt}^j - \gamma_k^i \Gamma^k.
\]

It is affine. By virtue of Proposition 1.5.3, this dynamic connection defines a linear connection \(K\) on the tangent bundle \(TQ \to Q\), whose curvature necessarily vanishes. Thus, we come to the following criterion of a dynamic equation to be a free motion equation.

**Proposition 1.7.2:** If \(\xi\) is a free motion equation on a configuration space \(Q\), it is quadratic, and the corresponding symmetric linear connection (1.5.10) on the tangent bundle \(TQ \to Q\) is a curvature-free connection. □

This criterion is not a sufficient condition because it may happen that the components of a curvature-free symmetric linear connection on \(TQ \to Q\) vanish with respect to the coordinates on \(Q\) which are not compatible with a fibration \(Q \to \mathbb{R}\).

The similar criterion involves the curvature of a dynamic connection (1.7.3) of a free motion equation.

**Proposition 1.7.3:** If \(\xi\) is a free motion equation, then the curvature \(R\) (1.4.3) of the corresponding dynamic connection \(\gamma_\xi\) is equal to 0, and so are the tensor field \(\overline{R}\) (1.4.4) and the scalar field \(\tilde{R}\) (1.4.5). □

Proposition 1.7.3 also fails to be a sufficient condition. If the curvature \(R\) (1.4.3) of a dynamic connection \(\gamma_\xi\) vanishes, it may happen that components of \(\gamma_\xi\) are equal to 0 with respect to non-holonomic bundle coordinates on an affine jet bundle \(J^1Q \to Q\).
Nevertheless, we can formulate the necessary and sufficient condition of the existence of a free motion equation on a configuration space \( Q \).

**Proposition 1.7.4**: A free motion equation on a fibre bundle \( Q \to \mathbb{R} \) exists iff the typical fibre \( M \) of \( Q \) admits a curvature-free symmetric linear connection.

**Proof**: Let a free motion equation take the form (1.7.1) with respect to some atlas of local constant trivializations of a fibre bundle \( Q \to \mathbb{R} \). By virtue of Proposition 1.4.2, there exists an affine dynamic connection \( \gamma \) on the affine jet bundle \( J^1Q \to Q \) whose components relative to this atlas are equal to 0. Given a trivialization chart of this atlas, the connection \( \gamma \) defines the curvature-free symmetric linear connection (1.5.13) on \( M \). The converse statement follows at once from Proposition 1.5.7. QED

The free motion equation (1.7.2) is simplified if the coordinate transition functions \( \overline{q}^i \to q^i \) are affine in coordinates \( \overline{q}^i \). Then we have

\[
q^{i}_{tt} = \partial_t \Gamma^i - \Gamma^j \partial_j \Gamma^i + 2q^j_t \partial_j \Gamma^i. \tag{1.7.4}
\]

The following lemma shows that the free motion equation (1.7.4) is affine in the coordinates \( q^i \) and \( q^i_t \) [15, 27].

**Lemma 1.7.5**: Let \((t, \overline{q}^i)\) be a reference frame on a configuration bundle \( Q \to \mathbb{R} \) and \( \Gamma \) the corresponding connection. Components \( \Gamma^i \) of this connection with respect to another coordinate system \((t, q^i)\) are affine functions in the coordinates \( q^i \) iff the transition functions between the coordinates \( \overline{q}^i \) and \( q^i \) are affine. □

One can easily find the geodesic reference frames for the free motion equation

\[
q^{i}_{tt} = 0. \tag{1.7.5}
\]

They are \( \Gamma^i = v^i = \text{const} \). By virtue of Lemma 1.7.5, these reference frames define the adapted coordinates

\[
\overline{q}^i = k^i_j q^j - v^i t - a^i, \quad k^i_j = \text{const}., \quad v^i = \text{const}., \quad a^i = \text{const}. \tag{1.7.6}
\]

The equation (1.7.5) obviously keeps its free motion form under the transformations (1.7.6) between the geodesic reference frames. It is readily observed that these transformations are precisely the elements of the Galilei group.

### 1.8 Relative acceleration

In comparison with the notion of a relative velocity, the definition of a relative acceleration is more intricate.

To consider a relative acceleration with respect to a reference frame \( \Gamma \), one should prolong a connection \( \Gamma \) on a configuration space \( Q \to \mathbb{R} \) to a holonomic connection \( \xi_\Gamma \) on the jet bundle \( J^1Q \to \mathbb{R} \). Note that the jet prolongation \( J\Gamma \) (1.1.23) of \( \Gamma \) onto \( J^1Q \to \mathbb{R} \)
1.8. RELATIVE ACCELERATION

is not holonomic. We can construct the desired prolongation by means of a dynamic connection \( \gamma \) on an affine jet bundle \( J^1Q \to Q \).

**Lemma 1.8.1:** Let us consider the composite bundle (1.1.11). Given a reference frame \( \Gamma \) on \( Q \to \mathbb{R} \) and a dynamic connections \( \gamma \) on \( J^1Q \to Q \), there exists a dynamic connection \( \tilde{\gamma} \) on \( J^1Q \to Q \) with the components

\[
\tilde{\gamma}_k^i = \gamma_k^i, \quad \tilde{\gamma}_0^i = d_t \Gamma^i - \gamma_k^i \Gamma^k.
\]  

\( \Box \)

Now, we construct a certain soldering form on an affine jet bundle \( J^1Q \to Q \) and add it to this connection. Let us apply the canonical projection \( T^*Q \to V^*Q \) and then the imbedding \( \Gamma : V^*Q \to T^*Q \) to the covariant derivative (1.6.7) of the reference frame \( \Gamma \) with respect to the dynamic connection \( \gamma \). We obtain the \( V_Q^*J^1Q \)-valued one-form

\[
\sigma = [-\Gamma^i (\partial_i \Gamma^k - \gamma_i^k \circ \Gamma) dt + (\partial_i \Gamma^k - \gamma_i^k \circ \Gamma)dq^i] \otimes \partial_k^i
\]
on \( Q \) whose pull-back onto \( J^1Q \) is a desired soldering form. The sum

\[
\gamma_T = \tilde{\gamma} + \sigma,
\]
called the frame connection, reads

\[
\gamma_T^i_0 = d_t \Gamma^i - \gamma_k^i \Gamma^k - \Gamma^k (\partial_k \Gamma^i - \gamma_k^i \circ \Gamma), \quad \gamma_T^i_k = \gamma_k^i + \partial_k \Gamma^i - \gamma_k^i \circ \Gamma.
\]  

This connection yields the desired holonomic connection

\[
\xi_T^i = d_t \Gamma^i + (\partial_k \Gamma^i + \gamma_k^i - \gamma_k^i \circ \Gamma)(dq^k - \Gamma^k)
\]
on the jet bundle \( J^1Q \to \mathbb{R} \).

Let \( \xi \) be a dynamic equation and \( \gamma = \gamma_\xi \) the connection (1.4.10) associated with \( \xi \). Then one can think of the vertical vector field

\[
a_T = \xi - \xi_T = (\xi^i - \xi_T^i) \partial_i^T
\]
on the affine jet bundle \( J^1Q \to Q \) as being a relative acceleration with respect to the reference frame \( \Gamma \) in comparison with the absolute acceleration \( \xi \).

For instance, let us consider a reference frame \( \Gamma \) which is geodesic for the dynamic equation \( \xi \), i.e., the relation (1.6.8) holds. Then the relative acceleration of a motion \( c \) with respect to a reference frame \( \Gamma \) is

\[
(\xi - \xi_T) \circ \Gamma = 0.
\]
Let $\xi$ now be an arbitrary dynamic equation, written with respect to coordinates $(t, q^i)$ adapted to a reference frame $\Gamma$, i.e., $\Gamma^i = 0$. In these coordinates, the relative acceleration with respect to a reference frame $\Gamma$ is

$$a_i^\Gamma = \xi_i(t, q^j, q_j^i) - \frac{1}{2} q_k^i (\partial_k \xi^i - \partial_k \xi^i |_{q^j=0}).$$  \hfill (1.8.4)

Given another bundle coordinates $(t, q'^i)$ on $Q \to \mathbb{R}$, this dynamic equation takes the form (1.6.5), while the relative acceleration (1.8.4) with respect to a reference frame $\Gamma$ reads

$$a'_i^\Gamma = \partial_j q'^i a_j^\Gamma.$$  

Then we can write the dynamic equation (1.3.3) in the form which is covariant under coordinate transformations:

$$\bar{D}_{\gamma^\Gamma} q^i_t = d_t q^i_t - \xi^i_t = a^\Gamma,$$  \hfill (1.8.5)

where $\bar{D}_{\gamma^\Gamma}$ is the vertical covariant differential (1.4.9) with respect to the frame connection $\gamma^\Gamma$ (1.8.2) on an affine jet bundle $J^1 Q \to Q$.

In particular, if $\xi$ is a free motion equation which takes the form (1.7.1) with respect to a reference frame $\Gamma$, then

$$\bar{D}_{\gamma^\Gamma} q^i_t = 0$$

relative to arbitrary bundle coordinates on the configuration bundle $Q \to \mathbb{R}$.

The left-hand side of the dynamic equation (1.8.5) also can be expressed in the relative velocities such that this dynamic equation takes the form

$$d_t q^i_t - \gamma^{i}_t q^k_{t} = a^\Gamma$$  \hfill (1.8.6)

which is the covariant form of the equation (1.6.6).

The concept of a relative acceleration is understood better when we deal with a quadratic dynamic equation $\xi$, and the corresponding dynamic connection $\gamma$ is affine.

**Lemma 1.8.2:** If a dynamic connection $\gamma$ is affine, i.e.,

$$\gamma^i_t = \gamma^i_{t0} + \gamma^i_{t0} q^k_t,$$

so is a frame connection $\gamma^\Gamma$ for any frame $\Gamma$. $\square$

In particular, we obtain

$$\gamma^{i}_{jk} = \gamma^i_{jk}, \quad \gamma^{i}_{t0} = \gamma^{i}_{k0} = \gamma^{i}_{00} = 0$$

relative to the coordinates adapted to a reference frame $\Gamma$. 


COROLLARY 1.8.3: If a dynamic equation $\xi$ is quadratic, the relative acceleration $a_\Gamma$ (1.8.3) is always affine, and it admits the decomposition

$$a^i_\Gamma = -(\Gamma^\lambda \nabla^j_\lambda \Gamma^i + 2\dot{q}^j_\Gamma \nabla^j_\Gamma \Gamma^i),$$

(1.8.7)

where $\gamma = \gamma_\xi$ is the dynamic connection (1.4.10), and

$$\dot{q}^\lambda_\Gamma = q^\lambda_\Gamma - \Gamma^\lambda, \quad \dot{q}^0_\Gamma = 1, \quad \Gamma^0 = 1,$$

is the relative velocity with respect to the reference frame $\Gamma$. □

Note that the splitting (1.8.7) gives a generalized Coriolis theorem. In particular, the well-known analogy between inertial and electromagnetic forces is restated. Corollary 1.8.3 shows that this analogy can be extended to an arbitrary quadratic dynamic equation.

1.9 Newtonian systems

Equations of motion of non-relativistic mechanics need not be exactly dynamic equations. For instance, the second Newton law of point mechanics contains a mass. The notion of a Newtonian system generalizes the second Newton law as follows.

Let $m$ be a fibre metric (bilinear form) in the vertical tangent bundle $V_Q \to J^1Q$. It reads

$$m : J^1Q \to \mathbb{R}^*, \quad m = \frac{1}{2} m_{ij} dq_i^j \wedge dq_j^i,$$

(1.9.1)

where $dq_i^j$ are the holonomic bases for the vertical cotangent bundle $V_Q^{*} \to J^1Q$. It defines the map

$$\hat{m} : V_Q \to \mathbb{R}^{*} \to J^{1}Q.$$

DEFINITION 1.9.1: Let $Q \to \mathbb{R}$ be a fibre bundle together with:

(i) a fibre metric $m$ (1.9.1) satisfying the symmetry condition

$$\partial_k^i m_{ij} = \partial_j^i m_{ik},$$

(1.9.2)

(ii) and a holonomic connection $\xi$ (1.1.24) on a jet bundle $J^1Q \to \mathbb{R}$ related to the fibre metric $m$ by the compatibility condition

$$\xi \mid dm_{ij} + \frac{1}{2} m_{ik} \partial_j^i \xi^k + m_{jk} \partial_i^j \xi^k = 0.$$

(1.9.3)

A triple $(Q, m, \xi)$ is called the Newtonian system. □

We agree to call a metric $m$ in Definition 1.9.1 the mass tensor of a Newtonian system $(Q, m, \xi)$. The equation of motion of this Newtonian system is defined to be

$$\hat{m}(D^\xi) = 0, \quad m_{ik}(q^k_\Gamma - \xi^k) = 0.$$

(1.9.4)
Due to the conditions (1.9.2) and (1.9.3), it is brought into the form
\[ dt(m_{ik}q^k_t) - m_{ik}\xi^k = 0. \]

Therefore, one can think of this equation as being a generalization of the second Newton law.

If a mass tensor \( m \) (1.9.1) is non-degenerate, the equation of motion (1.9.4) is equivalent to the dynamic equation
\[ D\xi = 0, \quad q_{tt}^k - \xi^k = 0. \]

Because of the canonical vertical splitting (1.1.10), the mass tensor (1.9.1) also is a map
\[ m : J^1Q \to \frac{\mathcal{V}^*Q}{J^1Q}, \quad m = \frac{1}{2}m_{ij}dq^i \vee dq^j, \tag{1.9.5} \]

A Newtonian system \((Q, \tilde{m}, \xi)\) is said to be standard, if its mass tensor \( m \) is the pull-back onto \( V_QJ^1Q \) of a fibre metric
\[ m : Q \to \frac{\mathcal{V}^*Q}{Q} \tag{1.9.6} \]

in the vertical tangent bundle \( VQ \to Q \) in accordance with the isomorphisms (1.1.9) and (1.1.10), i.e., \( m \) is independent of the velocity coordinates \( q^i_t \).

Given a mass tensor, one can introduce the notion of an external force.

**Definition 1.9.2:** An external force is defined as a section of the vertical cotangent bundle \( V^*_QJ^1Q \to J^1Q \). Let us also bear in mind the isomorphism (1.1.10). \( \Box \)

It should be emphasized that there are no canonical isomorphisms between the vertical cotangent bundle \( V^*_QJ^1Q \) and the vertical tangent bundle \( V_QJ^1Q \) of \( J^1Q \). One must therefore distinguish forces and accelerations which are related by means of a mass tensor.

Let \((Q, \tilde{m}, \xi)\) be a Newtonian system and \( f \) an external force. Then
\[ \xi_f^i = \xi^i + (m^{-1})^{ik} f_k \tag{1.9.7} \]

is a dynamic equation, but the triple \((Q, m, \xi_f)\) is not a Newtonian system in general. As it follows from a direct computation, iff an external force possesses the property
\[ \partial^i f_j + \partial^j f_i = 0, \tag{1.9.8} \]
then \( \xi_f \) (1.9.7) fulfills the compatibility condition (1.9.3), and \((Q, \tilde{m}, \xi_f)\) also is a Newtonian system.
1.10 Integrals of motion

Let an equation of motion of a mechanical system on a fibre bundle $Y \rightarrow \mathbb{R}$ be described by an $r$-order differential equation $\mathcal{E}$ given by a closed subbundle of the jet bundle $J^r Y \rightarrow \mathbb{R}$ in accordance with Definition 4.2.6.

**Definition 1.10.1**: An integral of motion of this mechanical system is defined as a $(k < r)$-order differential operator $\Phi$ on $Y$ such that $\mathcal{E}$ belongs to the kernel of an $r$-order jet prolongation of the differential operator $d_t \Phi$, i.e.,

$$J^{r-k-1}(d_t \Phi)|_{\mathcal{E}} = 0. \quad (1.10.1)$$

It follows that an integral of motion $\Phi$ is constant on solutions $s$ of a differential equation $\mathcal{E}$, i.e., there is the differential conservation law

$$(J^k s)^* \Phi = \text{const.}, \quad (J^{k+1} s)^* d_t \Phi = 0. \quad (1.10.2)$$

We agree to write the condition (1.10.1) as the weak equality

$$J^{r-k-1}(d_t \Phi) \approx 0, \quad (1.10.3)$$

which holds on-shell, i.e., on solutions of a differential equation $\mathcal{E}$ by the formula (1.10.2).

In mechanics, we can restrict our consideration to integrals of motion $\Phi$ which are functions on $J^k Y$. As was mentioned above, equations of motion of mechanics mainly are of first or second order. Accordingly, their integrals of motion are functions on $Y$ or $J^k Y$. In this case, the corresponding weak equality (1.10.1) takes the form

$$d_t \Phi \approx 0 \quad (1.10.4)$$

of a weak conservation law or, simply, a conservation law.

Different integrals of motion need not be independent. Let integrals of motion $\Phi_1, \ldots, \Phi_m$ of a mechanical system on $Y$ be functions on $J^k Y$. They are called independent if

$$d\Phi_1 \wedge \cdots \wedge d\Phi_m \neq 0 \quad (1.10.5)$$

everywhere on $J^k Y$. In this case, any motion $J^k s$ of this mechanical system lies in the common level surfaces of functions $\Phi_1, \ldots, \Phi_m$ which bring $J^k Y$ into a fibred manifold.

Integrals of motion can come from symmetries. This is the case of Lagrangian and Hamiltonian mechanics (Sections 2.4 and 3.5).

**Definition 1.10.2**: Let an equation of motion of a mechanical system be an $r$-order differential equation $\mathcal{E} \subset J^r Y$. Its infinitesimal symmetry (or, simply, a symmetry) is defined as a vector field on $J^r Y$ whose restriction to $\mathcal{E}$ is tangent to $\mathcal{E}$. □

For instance, let us consider first order dynamic equations.
CHAPTER 1. DYNAMIC EQUATIONS

PROP 1.10.3: Let $\mathcal{E}$ be the autonomous first order dynamic equation (1.2.1) given by a vector field $u$ on a manifold $Z$. A vector field $\vartheta$ on $Z$ is its symmetry iff $[u, \vartheta] \approx 0$. \hfill $\square$

One can show that a smooth real function $F$ on a manifold $Z$ is an integral of motion of the autonomous first order dynamic equation (1.2.1) (i.e., it is constant on solutions of this equation) iff its Lie derivative along $u$ vanishes:

$$L_u F = u^\lambda \partial_\lambda \Phi = 0.$$ (1.10.6)

PROP 1.10.4: Let $\mathcal{E}$ be the first order dynamic equation (1.3.1) given by a connection $\Gamma$ (1.1.16) on a fibre bundle $Y \to \mathbb{R}$. Then a vector field $\vartheta$ on $Y$ is its symmetry iff $[\Gamma, \vartheta] \approx 0$. \hfill $\square$

A smooth real function $\Phi$ on $Y$ is an integral of motion of the first order dynamic equation (1.3.1) in accordance with the equality (1.10.4) iff

$$L_\Gamma \Phi = (\partial_t + \Gamma^i \partial_i) \Phi = 0.$$ (1.10.7)

Following Definition 1.10.2, let us introduce the notion of a symmetry of differential operators in the following relevant case. Let us consider an $r$-order differential operator on a fibre bundle $Y \to \mathbb{R}$ which is represented by an exterior form $\mathcal{E}$ on $J^rY$ (Definition 4.2.5). Let its kernel $\text{Ker} \mathcal{E}$ be an $r$-order differential equation on $Y \to \mathbb{R}$.

PROP 1.10.5: It is readily justified that a vector field $\vartheta$ on $J^rY$ is a symmetry of the equation $\text{Ker} \mathcal{E}$ in accordance with Definition 1.10.2 iff

$$L_\vartheta \mathcal{E} \approx 0.$$ (1.10.8)

Motivated by Proposition 1.10.5, we come to the following.

DEF 1.10.6: Let $\mathcal{E}$ be the above mentioned differential operator. A vector field $\vartheta$ on $J^rY$ is called a symmetry of a differential operator $\mathcal{E}$ if the Lie derivative $L_\vartheta \mathcal{E}$ vanishes. \hfill $\square$

By virtue of Proposition 1.10.5, a symmetry of a differential operator $\mathcal{E}$ also is a symmetry of the differential equation $\text{Ker} \mathcal{E}$.

Note that there exist integrals of motion which are not associated with symmetries of an equation of motion (Example 2.4.5).
Chapter 2

Lagrangian mechanics

Lagrangian mechanics on a velocity phase space is formulated in the framework of Lagrangian formalism on fibre bundles \([8, 14, 15, 27]\). This formulation is based on the variational bicomplex and the first variational formula, without appealing to the variational principle. Besides Lagrange equations, the Cartan and Hamilton–De Donder equations are considered in the framework of Lagrangian formalism. Note that the Cartan equation, but not the Lagrange one is associated to the Hamilton equation (Section 3.4). The relations between Lagrangian and Newtonian systems are investigated. Lagrangian conservation laws are defined by means of the first Noether theorem.

2.1 Lagrangian formalism on \(Q \rightarrow \mathbb{R}\)

Let \(\pi : Q \rightarrow \mathbb{R}\) be a fibre bundle (1.1.1). The finite order jet manifolds \(J^kQ\) of \(Q \rightarrow \mathbb{R}\) form the inverse sequence

\[
Q \xleftarrow{\pi_1} J^1Q \leftarrow \cdots J^{r-1}Q \xleftarrow{\pi_r} J^rQ \leftarrow \cdots,
\]

(2.1.1)

where \(\pi_{r-1}\) are affine bundles. Its projective limit \(J^\infty Q\) is a paracompact Fréchet manifold. One can think of its elements as being infinite order jets of sections of \(Q \rightarrow \mathbb{R}\) identified by their Taylor series at points of \(\mathbb{R}\). Therefore, \(J^\infty Q\) is called the infinite order jet manifold. A bundle coordinate atlas \((t, q^i)\) of \(Q \rightarrow \mathbb{R}\) provides \(J^\infty Q\) with the manifold coordinate atlas

\[
(t, q^i, q^i_t, q^i_{tt}, \ldots), \quad q^i_{t\Lambda} = d_t q^i_{\Lambda},
\]

(2.1.2)

where \(\Lambda = (t \cdots t)\) denotes a multi-index of length \(|\Lambda|\) and

\[
d_t = \partial_t + q^i_t \partial_t + q^i_{tt} \partial_t^2 + \cdots + q^i_{t\Lambda} \partial_t^\Lambda + \cdots \]

is the total derivative.
CHAPTER 2. LAGRANGIAN MECHANICS

Let \( O_r^* = O^*(J^rQ) \) be a graded differential algebra of exterior forms on a jet manifold \( J^rQ \). The inverse sequence (2.1.1) of jet manifolds yields the direct sequence of differential graded algebras \( O_r^* \):
\[
O^*(Q) \xrightarrow{\pi_0^*} O_1^* \rightarrow \cdots O_{r-1}^* \xrightarrow{\pi_{r-1}^*} O_r^* \rightarrow \cdots
\]
(2.1.3)
where \( \pi_{r-1}^* \) are the pull-back monomorphisms. Its direct limit
\[
O^*_{\infty} Q = \lim_{\rightarrow} O_r^*
\]
(2.1.4)
(or, simply, \( O^*_{\infty} \)) consists of all exterior forms on finite order jet manifolds modulo the pull-back identification. In particular, \( O^0_{\infty} \) is the ring of all smooth functions on finite order jet manifolds. The \( O^*_{\infty} \) (2.1.4) is a differential graded algebra which inherits the operations of the exterior differential \( d \) and exterior product \( \wedge \) of exterior algebras \( O_r^* \).

**Theorem 2.1.1:** The cohomology \( H^*(O^*_{\infty}) \) of the de Rham complex
\[
0 \rightarrow \mathbb{R} \rightarrow O^0_{\infty} \xrightarrow{d} O^1_{\infty} \xrightarrow{d} \cdots
\]
(2.1.5)
of the differential graded algebra \( O^*_{\infty} \) equals the de Rham cohomology \( H^*_{\text{DR}}(Q) \) of a fibre bundle \( Q \) [14].

**Corollary 2.1.2:** Since \( Q \) (1.1.1) is a trivial fibre bundle over \( \mathbb{R} \), the de Rham cohomology \( H^*(O^*_{\infty}) \) of \( Q \) equals the de Rham cohomology of its typical fibre \( M \) in accordance with the well-known K"unneth formula. Therefore, the cohomology \( H^*(O^*_{\infty}) \) of the de Rham complex (2.1.5) equals the de Rham cohomology \( H^*_{\text{DR}}(M) \) of \( M \). \( \square \)

Since elements of the differential graded algebra \( O^*_{\infty} \) (2.1.4) are exterior forms on finite order jet manifolds, this \( O^0_{\infty} \)-algebra is locally generated by the horizontal form \( dt \) and contact one-forms
\[
\theta^i = dq^i - q^i dt.
\]
Moreover, there is the canonical decomposition
\[
O^*_{\infty} = \oplus O^{k,m}_{\infty}, \quad m = 0, 1,
\]
of \( O^*_{\infty} \) into \( O^0_{\infty} \)-modules \( O^{k,m}_{\infty} \) of \( k \)-contact and \((m = 0, 1)\)-horizontal forms together with the corresponding projectors
\[
h_k : O^*_{\infty} \rightarrow O^{k,*}_{\infty}, \quad h^m : O^*_{\infty} \rightarrow O^{*,m}_{\infty}.
\]
Accordingly, the exterior differential on \( O^*_{\infty} \) is decomposed into the sum \( d = d_V + d_H \) of the vertical differential
\[
d_V : O^{k,m}_{\infty} \rightarrow O^{k+1,m}_{\infty}, \quad d_V \circ h^m = h^m \circ d \circ h^m,
\]
\[
d_V(\phi) = \theta^i \wedge \partial^i \phi, \quad \phi \in O^{*}_{\infty},
\]
2.1. LAGRANGIAN FORMALISM ON $Q \to \mathbb{R}$

and the total differential

\[
d_H : \mathcal{O}^{k,m}_\infty \to \mathcal{O}^{k,m+1}_\infty, \quad d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H \circ h_0 = h_0 \circ d,
\]

\[
d_H(\phi) = dt \wedge d_t \phi, \quad \phi \in \mathcal{O}^*_\infty.
\]  

(2.1.6)

These differentials obey the nilpotent conditions

\[
d_H \circ d_H = 0, \quad d_V \circ d_V = 0, \quad d_H \circ d_V + d_V \circ d_H = 0,
\]

and make $\mathcal{O}^{*,*}_\infty$ into a bicomplex.

One introduces the following two additional operators acting on $\mathcal{O}^{*,*}_\infty$.

(i) There exists an $\mathbb{R}$-module endomorphism

\[
\varrho = \sum_{k > 0} \frac{1}{k} \overline{\varrho} \circ h_k \circ h_1 : \mathcal{O}^{>0,1}_\infty \to \mathcal{O}^{>0,1}_\infty,
\]

(2.1.7)

\[
\overline{\varrho}(\phi) = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta_i \wedge [d_\Lambda(\partial^{\Lambda}_i \phi)], \quad \phi \in \mathcal{O}^{>0,1}_\infty,
\]

possessing the following properties.

**Lemma 2.1.3**: For any $\phi \in \mathcal{O}^{>0,1}_\infty$, the form $\phi - \varrho(\phi)$ is locally $d_H$-exact on each coordinate chart (2.1.2). The operator $\varrho$ obeys the relation

\[
(\varrho \circ d_H)(\psi) = 0, \quad \psi \in \mathcal{O}^{>0,0}_\infty.
\]  

(2.1.8)

\[\Box\]

It follows from Lemma 2.1.3 that $\varrho$ (2.1.7) is a projector, i.e., $\varrho \circ \varrho = \varrho$.

(ii) One defines the variational operator

\[
\delta = \varrho \circ d : \mathcal{O}^{>1,1}_\infty \to \mathcal{O}^{>1,1}_\infty.
\]  

(2.1.9)

**Lemma 2.1.4**: The variational operator $\delta$ (2.1.9) is nilpotent, i.e., $\delta \circ \delta = 0$, and it obeys the relation

\[
\delta \circ \varrho = \delta.
\]  

(2.1.10)

\[\Box\]

With operators $\varrho$ (2.1.7) and $\delta$ (2.1.9), the bicomplex $\mathcal{O}^{*,*}_\infty$ is brought into the variational bicomplex. Let us denote $E_k = \varrho(\mathcal{O}^{k,1}_\infty)$. We have

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 \to & \mathcal{O}^{1,0}_\infty \overset{d_V}{\to} & \mathcal{O}^{1,1}_\infty \overset{\varrho}{\to} & E_1 \to 0 \\
0 \to \mathbb{R} \to & \mathcal{O}^{0,0}_\infty \overset{d_H}{\to} & \mathcal{O}^{0,1}_\infty \overset{-\delta}{\to} & \mathcal{O}^{0,1}_\infty \equiv \mathcal{O}^{0,1}_\infty
\end{array}
\]  

(2.1.11)
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This variational bicomplex possesses the following cohomology [14].

**Theorem 2.1.5**: The bottom row and the last column of the variational bicomplex (2.1.11) make up the variational complex

\[
0 \rightarrow \mathbb{R} \rightarrow O_\infty^0 \xrightarrow{d_H} O_\infty^{0,1} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \rightarrow \cdots \tag{2.1.12}
\]

Its cohomology is isomorphic to the de Rham cohomology of a fibre bundle \(Q\) and, consequently, the de Rham cohomology of its typical fibre \(M\) (Corollary 2.1.2). □

**Theorem 2.1.6**: The rows of contact forms of the variational bicomplex (2.1.11) are exact sequences. □

Note that Theorem 2.1.6 gives something more. Due to the relations (2.1.6) and (2.1.10), we have the cochain morphism

\[
\begin{array}{ccccccccc}
O_\infty^0 & d & O_\infty^1 & d & O_\infty^2 & d & O_\infty^3 & \rightarrow & \cdots \\
\downarrow{h_0} & & \downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \\
O_\infty^{0,0} & d_H & O_\infty^{0,1} & \delta & E_1 & \delta & E_2 & \rightarrow & \cdots
\end{array}
\]

of the de Rham complex (2.1.5) of the differential graded algebra \(O_\infty^*\) to its variational complex (2.1.12). By virtue of Theorems 2.1.1 and 2.1.5, the corresponding homomorphism of their cohomology groups is an isomorphism. A consequence of this fact is the following.

**Theorem 2.1.7**: Any \(\delta\)-closed form \(\phi \in O^{k,1}\), \(k = 0, 1\), is split into the sum

\[
\begin{align*}
\phi &= h_0 \sigma + d_H \xi, \quad k = 0, \quad \xi \in O_\infty^{0,0}, \quad \tag{2.1.13} \\
\phi &= \varrho(\sigma) + \delta(\xi), \quad k = 1, \quad \xi \in O_\infty^{0,1}, \quad \tag{2.1.14}
\end{align*}
\]

where \(\sigma\) is a closed \((1 + k)\)-form on \(Q\). □

In Lagrangian formalism on a fibre bundle \(Q \rightarrow \mathbb{R}\), a finite order Lagrangian and its Lagrange operator are defined as elements

\[
\begin{align*}
L &= L dt \in O_\infty^{0,1}, \quad \tag{2.1.15} \\
\mathcal{E}_L &= \delta L = \mathcal{E}_i \theta^i \land dt \in E_1, \quad \tag{2.1.16} \\
\mathcal{E}_i &= \sum_{\theta \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda (\partial^\Lambda L), \quad \tag{2.1.17}
\end{align*}
\]

of the variational complex (2.1.12). Components \(\mathcal{E}_i\) (2.1.17) of the Lagrange operator (2.1.16) are called the variational derivatives. Elements of \(E_1\) are called the Lagrange-type operators.

We agree to call a pair \((O_\infty^*, L)\) the Lagrangian system.
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**Corollary 2.1.8**: A finite order Lagrangian $L$ (2.1.15) is variationally trivial, i.e.,

$$\delta(L) = 0$$

iff

$$L = h_0\sigma + d_H\xi,$$

where $\sigma$ is a closed one-form on $Q$. □

**Corollary 2.1.9**: A finite order Lagrange-type operator $E \in E_1$ satisfies the Helmholtz condition

$$\delta(E) = 0$$

iff

$$E = \delta L + \varrho(\sigma),$$

where $\sigma$ is a closed two-form on $Q$. □

Given a Lagrangian $L$ (2.1.15) and its Lagrange operator $\delta L$ (2.1.16), the kernel $\text{Ker} \delta L \subset J^2_{r\mathbb{R}}Q$ of $\delta L$ is called the Lagrange equation. It is locally given by the equalities

$$\mathcal{E}_i = \sum_{0 \leq |\Lambda| \leq 1} (-1)^{|\Lambda|} d_{\Lambda}(\partial_{i}^{\Lambda} L) = 0.$$  

(2.1.20)

However, it may happen that the Lagrange equation is not a differential equation in accordance with Definition 4.2.3 because $\text{Ker} \delta L$ need not be a closed subbundle of $J^2_{r\mathbb{R}}Q \to \mathbb{R}$.

**Example 2.1.1**: Let $Q = \mathbb{R}^2 \to \mathbb{R}$ be a configuration space, coordinated by $(t, q)$. The corresponding velocity phase space $J^1Q$ is equipped with the adapted coordinates $(t, q, q_t)$. The Lagrangian

$$L = \frac{1}{2}q^2q_t^2dt$$

on $J^1Q$ leads to the Lagrange operator

$$\mathcal{E}_L = [qq_t^2 - d_t(q^2q_t)]dq \wedge dt$$

whose kernel is not a submanifold at the point $q = 0$. □

**Theorem 2.1.10**: Owing to the exactness of the row of one-contact forms of the variational bicomplex (2.1.11) at the term $O^{1,1}_\infty$, there is the decomposition

$$dL = \delta L - d_H L,$$

(2.1.21)

where a one-form $L$ is a Lepage equivalent of a Lagrangian $L$ [14]. □

Let us restrict our consideration to first order Lagrangian theory on a fibre bundle $Q \to \mathbb{R}$. This is the case of Lagrangian non-relativistic mechanics.

A first order Lagrangian is defined as a density

$$L = \mathcal{L}dt, \quad \mathcal{L} : J^1Q \to \mathbb{R},$$

(2.1.22)
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on a velocity space $J^1Q$. The corresponding second-order Lagrange operator (2.1.16) reads

$$\delta L = (\partial_i \mathcal{L} - d_t \partial^i_t \mathcal{L}) \theta^i \wedge dt.$$  \hspace{1cm} (2.1.23)

Let us further use the notation

$$\pi_i = \partial^i_t \mathcal{L}, \quad \pi_{ji} = \partial^j_t \partial^i_t \mathcal{L}.$$  \hspace{1cm} (2.1.24)

The kernel $\text{Ker} \delta L \subset J^2Q$ of the Lagrange operator defines the second order Lagrange equation

$$(\partial_i - d_t \partial^i_t) \mathcal{L} = 0.$$  \hspace{1cm} (2.1.25)

Its solutions are (local) sections $c$ of the fibre bundle $Q \to \mathbb{R}$ whose second order jet prolongations $\dot{c}$ live in (2.1.25). They obey the equations

$$\partial_i \mathcal{L} \circ \dot{c} - \frac{d}{dt} (\pi_i \circ \dot{c}) = 0.$$  \hspace{1cm} (2.1.26)

**Definition 2.1.11**: Given a Lagrangian $L$, a holonomic connection

$$\xi_L = \partial_t + q^i_t \partial_i + \xi^i \partial^i_t$$
on the jet bundle $J^1Q \to \mathbb{R}$ is said to be the Lagrangian connection if it takes its values into the kernel of the Lagrange operator $\delta L$, i.e., if it satisfies the relation

$$\partial_i \mathcal{L} - \partial_t \pi_i - q^i_t \partial_j \pi_i - \xi^j \pi_{ji} = 0.$$  \hspace{1cm} (2.1.27)

\[\square\]

A Lagrangian connection need not be unique.

Let us bring the relation (2.1.27) into the form

$$\partial_i \mathcal{L} - d_t \pi_i + (q^i_{tt} - \xi^i) \pi_{ji} = 0.$$  \hspace{1cm} (2.1.28)

If a Lagrangian connection $\xi_L$ exists, it defines the second order dynamic equation

$$q^i_{tt} = \xi_L$$  \hspace{1cm} (2.1.29)
on $Q \to \mathbb{R}$, whose solutions also are solutions of the Lagrange equation (2.1.25) by virtue of the relation (2.1.28). Conversely, since the jet bundle $J^2Q \to J^1Q$ is affine, every solution $c$ of the Lagrange equation also is an integral section for a holonomic connection $\xi$, which is a global extension of the local section $J^1c(\mathbb{R}) \to J^2c(\mathbb{R})$ of this jet bundle over the closed imbedded submanifold $J^1c(\mathbb{R}) \subset J^1Q$. Hence, every solution of the Lagrange equation also is a solution of some second order dynamic equation, but it is not necessarily a Lagrangian connection.
Every first order Lagrangian \( L \) (2.1.22) yields the bundle morphism

\[ \hat{L} : J^1Q \rightarrow V^*Q, \quad p_i \circ \hat{L} = \pi_i, \] (2.1.30)

where \((t, q^i, p_i)\) are holonomic coordinates on the vertical cotangent bundle \( V^*Q \) of \( Q \rightarrow \mathbb{R} \). This morphism is called the Legendre map, and

\[ \pi^\Pi : V^*Q \rightarrow Q, \] (2.1.31)

is called the Legendre bundle. As was mentioned above, the vertical cotangent bundle \( V^*Q \) plays a role of the phase space of mechanics on a configuration space \( Q \rightarrow \mathbb{R} \). The range \( N_L = \hat{L}(J^1Q) \) of the Legendre map (2.1.30) is called the Lagrangian constraint space.

**Definition 2.1.12:** A Lagrangian \( L \) is said to be:

- **hyperregular** if the Legendre map \( \hat{L} \) is a diffeomorphism;
- **regular** if \( \hat{L} \) is a local diffeomorphism, i.e., \( \det(\pi_{ij}) \neq 0 \);
- **semiregular** if the inverse image \( \hat{L}^{-1}(p) \) of any point \( p \in N_L \) is a connected submanifold of \( J^1Q \);
- **almost regular** if a Lagrangian constraint space \( N_L \) is a closed embedded subbundle \( i_N : N_L \rightarrow V^*Q \) of the Legendre bundle \( V^*Q \rightarrow Q \) and the Legendre map

\[ \hat{L} : J^1Q \rightarrow N_L \] (2.1.32)

is a fibred manifold with connected fibres (i.e., a Lagrangian is semiregular).

**Remark 2.1.2:** A glance at the equation (2.1.27) shows that a regular Lagrangian \( L \) admits a unique Lagrangian connection

\[ \xi_L = (\pi^{-1})^{ij}(\partial_i \mathcal{L} + \partial_i \pi_j + q^k \partial_k \pi_i). \] (2.1.33)

In this case, the Lagrange equation (2.1.25) for \( L \) is equivalent to the second order dynamic equation associated to the Lagrangian connection (2.1.33).

### 2.2 Cartan and Hamilton–De Donder equations

Given a first order Lagrangian \( L \), its Lepage equivalent \( \mathcal{L} \) in the decomposition (2.1.21) is the Poincaré–Cartan form

\[ H_L = \pi_i dq^i - (\pi_i q^i - \mathcal{L}) dt \] (2.2.1)

(see the notation (2.1.24)). This form takes its values into the subbundle \( J^1Q \times T^*Q \) of \( T^*J^1Q \). Hence, we have a morphism

\[ \tilde{H}_L : J^1Q \rightarrow T^*Q, \] (2.2.2)
whose range
\[ Z_L = \ddot{H}_L(J^1Q) \] (2.2.3)
is an imbedded subbundle \( i_L : Z_L \rightarrow T^*Q \) of the cotangent bundle \( T^*Q \). This morphism is called the homogeneous Legendre map. Let \( (t, q^i, p_0, p_i) \) denote the holonomic coordinates of \( T^*Q \) possessing transition functions
\[ p'_i = \frac{\partial q^j}{\partial q^i} p_j, \quad p'_0 = \left(p_0 + \frac{\partial q^j}{\partial t} p_j \right). \] (2.2.4)

With respect to these coordinates, the morphism \( \ddot{H}_L \) (2.2.2) reads
\[ (p_0, p_i) \circ \ddot{H}_L = (\mathcal{L} - q^i \pi_i, \pi_i). \]

A glance at the transition functions (2.2.4) shows that \( T^*Q \) is a one-dimensional affine bundle \( \zeta : T^*Q \rightarrow V^*Q \) (2.2.5) over the vertical cotangent bundle \( V^*Q \) (cf. (4.1.19)). Moreover, the Legendre map \( \dot{L} \) (2.1.30) is exactly the composition of morphisms
\[ \ddot{L} = \zeta \circ H_L : J^1Q \rightarrow V^*Q. \] (2.2.6)

It is readily observed that the Poincaré–Cartan form \( H_L \) (2.2.1) also is the Poincaré–Cartan form \( H_L = H_\ddot{L} \) of the first order Lagrangian
\[ \ddot{L} = \dot{h}_0(H_L) = (\mathcal{L} + (q^i_{(t)} - q^i) \pi_i) dt, \quad \dot{h}_0(dq^i) = q^i_{(t)} dt, \] (2.2.7)
on the repeated jet manifold \( J^1J^1Y \) [8, 14]. The Lagrange operator for \( \ddot{L} \) (called the Lagrange–Cartan operator) reads
\[ \delta \ddot{L} = [(\partial_t \mathcal{L} - \ddot{\partial}_t \pi_i + \partial_i \pi_j(q^i_{(t)} - q^i))(dq^j + \partial_t \pi_j(q^i_{(t)} - q^i)) dq^i] \wedge dt. \] (2.2.8)

Its kernel \( \text{Ker} \delta \ddot{L} \subset J^1J^1Q \) defines the Cartan equation
\[ \begin{align*}
\partial_t \pi_j(q^i_{(t)} - q^i) &= 0, \quad (2.2.9) \\
\partial_t \mathcal{L} - \ddot{\partial}_t \pi_i + \partial_i \pi_j(q^i_{(t)} - q^i) &= 0 \quad \text{(2.2.10)}
\end{align*} \]
on \( J^1Q \). Since \( \delta \ddot{L}|_{J^2Q} = \delta L \), the Cartan equation (2.2.9) – (2.2.10) is equivalent to the Lagrange equation (2.1.25) on integrable sections of \( J^1Q \rightarrow X \). It is readily observed that these equations are equivalent if a Lagrangian \( L \) is regular.
The Cartan equation (2.2.9) – (2.2.10) on sections \( \sigma : \mathbb{R} \to J^1Q \) is equivalent to the relation
\[
\sigma^*(u) dH_L = 0,
\] (2.2.11)
which is assumed to hold for all vertical vector fields \( u \) on \( J^1Q \to \mathbb{R} \).

The cotangent bundle \( T^*Q \) admits the Liouville form
\[
\Xi = p_0 dt + p_i dq^i.
\] (2.2.12)
Accordingly, its imbedded subbundle \( Z_L \) (2.2.3) is provided with the pull-back De Donder form \( \Xi_L = i_L^* \Xi \). There is the equality
\[
H_L = \hat{H}^*_L \Xi_L = \hat{H}^*_L(i_L^* \Xi).
\] (2.2.13)
By analogy with the Cartan equation (2.2.11), the Hamilton–De Donder equation for sections \( \sigma \) of \( Z_L \to \mathbb{R} \) is written as
\[
\sigma^*(u) d\Xi_L = 0,
\] (2.2.14)
where \( u \) is an arbitrary vertical vector field on \( Z_L \to \mathbb{R} \).

**Theorem 2.2.1:** Let the homogeneous Legendre map \( \hat{H}_L \) be a submersion. Then a section \( \sigma \) of \( J^1Q \to \mathbb{R} \) is a solution of the Cartan equation (2.2.11) iff \( \hat{H}_L \circ \sigma \) is a solution of the Hamilton–De Donder equation (2.2.14), i.e., the Cartan and Hamilton–De Donder equations are quasi-equivalent [14, 16]. \( \square \)

### 2.3 Lagrangian and Newtonian systems

Let \( L \) be a Lagrangian on a velocity space \( J^1Q \) and \( \hat{L} \) the Legendre map (2.1.30). Due to the vertical splitting (4.1.27) of \( VV^*Q \), the vertical tangent map \( V\hat{L} \) to \( \hat{L} \) reads
\[
V\hat{L} : V_QJ^1Q \to V^*Q \times V^*Q.
\]
It yields the linear bundle morphism
\[
\hat{m} = (\text{Id}_{J^1Q}, \text{pr}_2 \circ V\hat{L}) : V_QJ^1Q \to V^*_QJ^1Q, \quad \hat{m} : \partial_i \mapsto \pi_i \hat{d}q_i^j,
\] (2.3.1)
and consequently a fibre metric
\[
m : J^1Q \to \frac{2}{J^1Q} V^*_QJ^1Q
\]
in the vertical tangent bundle \( V_QJ^1Q \to J^1Q \). This fibre metric \( m \) obviously satisfies the symmetry condition (1.9.2).
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Let a Lagrangian \( L \) be regular. Then the fibre metric \( m_{ik} \) (2.3.1) is non-degenerate. In accordance with Remark 2.1.2, if a Lagrangian \( L \) is regular, there exists a unique Lagrangian connection \( \xi_L \) for \( L \) which obeys the equality

\[
m_{ik} \xi^k_L + \partial_i \pi_i + \partial_j \pi_i q^j_t - \partial_i \mathcal{L} = 0. \tag{2.3.2}
\]

The derivation of this equality with respect to \( q^j_t \) results in the relation (1.9.3). Thus, any regular Lagrangian \( L \) defines a Newtonian system characterized by the mass tensor \( m_{ij} = \pi_{ij} \).

Now let us investigate the conditions for a Newtonian system to be the Lagrangian one.

The equation (1.9.4) is the kernel of the second order differential Lagrange type operator

\[
\mathcal{E} : \mathcal{F}^2 Q \rightarrow V^* Q, \quad \mathcal{E} = m_{ik} (\xi^k - q^k_{tt}) \theta^i \wedge dt. \tag{2.3.3}
\]

A glance at the variational complex (2.1.12) shows that this operator is a Lagrange operator of some Lagrangian only if it obeys the Helmholtz condition

\[
\delta(\mathcal{E}_i \theta^i \wedge dt) = [(2 \partial_j - \partial^j_i \partial^j_t) \mathcal{E}_i \theta^i \wedge \theta^j + \partial^j_i \mathcal{E}_j - 2 \partial^j_i \partial^j_t \mathcal{E}_i) \theta^i \wedge \theta^j + (\partial^j_i \mathcal{E}_i - \partial^j_i \mathcal{E}_i) \theta^j_t \wedge \theta^i] \wedge dt = 0.
\]

This condition falls into the equalities

\[
\begin{align*}
\partial_j \mathcal{E}_i - \partial_i \mathcal{E}_j + \frac{1}{2} d_t (\partial^j_i \mathcal{E}_j - \partial^j_i \mathcal{E}_i) &= 0, \tag{2.3.4} \\
\partial^j_t \mathcal{E}_i + \partial^j_i \mathcal{E}_j - 2 d_t \partial^j_t \mathcal{E}_i &= 0, \tag{2.3.5} \\
\partial^j_i \mathcal{E}_i - \partial^j_i \mathcal{E}_j &= 0. \tag{2.3.6}
\end{align*}
\]

It is readily observed, that the condition (2.3.6) is satisfied since the mass tensor is symmetric. The condition (2.3.5) holds due to the equality (1.9.3) and the property (1.9.2). Thus, it is necessary to verify the condition (2.3.4) for a Newtonian system to be a Lagrangian one. If this condition holds, the operator \( \mathcal{E} \) (2.3.3) takes the form (2.1.19) in accordance with Corollary 2.1.9. If the second de Rham cohomology of \( Q \) (or, equivalently, \( M \)) vanishes, this operator is a Lagrange operator.

**Example 2.3.1:** Let us consider a one-dimensional motion of a point mass \( m_0 \) subject to friction. It is described by the equation

\[
m_0 q_{tt} = -k q_t, \quad k > 0, \tag{2.3.7}
\]

on the configuration space \( \mathbb{R}^2 \rightarrow \mathbb{R} \) coordinated by \((t, q)\). This mechanical system is characterized by the mass function \( m = m_0 \) and the holonomic connection

\[
\xi = \partial_t + q_t \partial_q - \frac{k}{m} q_t \partial^j_q, \tag{2.3.8}
\]
but it is neither a Newtonian nor a Lagrangian system. The conditions (2.3.4) and (2.3.6) are satisfied for an arbitrary mass function \( m(t, q, \dot{q}) \), whereas the conditions (1.9.3) and (2.3.5) take the form

\[
-kq_i \partial_i m - km + \partial_t m + q_i \partial_q m = 0.
\]

(2.3.9)

The mass function \( m = \text{const.} \) fails to satisfy this relation. Nevertheless, the equation (2.3.9) has a solution

\[
m = m_0 \exp \left[ \frac{k}{m_0} t \right].
\]

(2.3.10)

The mechanical system characterized by the mass function (2.3.10) and the holonomic connection (2.3.8) is both a Newtonian and Lagrangian system with the Havas Lagrangian

\[
\mathcal{L} = \frac{1}{2} m_0 \exp \left[ \frac{k}{m_0} t \right] q_i^2
\]

(2.3.11)

[38]. The corresponding Lagrange equation is equivalent to the equation of motion (2.3.7). ♦

In conclusion, let us mention mechanical systems whose motion equations are Lagrange equations plus additional non-Lagrangian external forces. They read

\[
(\partial_i - d_i \partial_i^j) \mathcal{L} + f_i(t, q^j, \dot{q}^j) = 0.
\]

(2.3.12)

Let a Lagrangian system be the Newtonian one, and let an external force \( f \) satisfy the condition (1.9.8). Then the equation (2.3.12) describe a Newtonian system.

### 2.4 Lagrangian conservation laws

In Lagrangian mechanics, integrals of motion come from variational symmetries of a Lagrangian (Theorem 2.4.8) in accordance with the first Noether theorem (Theorem 2.4.6). However, not all integrals of motion are of this type (Example 2.4.5).

#### 2.4.1 Generalized vector fields

Given a Lagrangian system \((\mathcal{O}_\infty, L)\) on a fibre bundle \( Q \rightarrow \mathbb{R} \), its infinitesimal transformations are defined to be contact derivations of the real ring \( \mathcal{O}_\infty^0 \) [11, 14, 15].

Let us consider the \( \mathcal{O}_\infty^0 \)-module \( \mathfrak{d} \mathcal{O}_\infty^0 \) of derivations of the real ring \( \mathcal{O}_\infty^0 \). This module is isomorphic to the \( \mathcal{O}_\infty^0 \)-dual \( \mathcal{O}_\infty^1 \) of the module of one-forms \( \mathcal{O}_\infty^1 \). Let \( \partial \downarrow \phi, \ \partial \in \mathfrak{d} \mathcal{O}_\infty^0, \phi \in \mathcal{O}_\infty^1 \), be the corresponding interior product. Extended to a differential graded algebra \( \mathcal{O}_\infty^\bullet \), it obeys the rule (4.1.42).

Restricted to the coordinate chart (2.1.2), any derivation of a real ring \( \mathcal{O}_\infty^0 \) takes the coordinate form

\[
\partial = \partial_i \partial_i + \sum_{\theta < |A|} \partial^\Lambda_i \partial_i^A,
\]
where
\[ \partial^\Lambda_i (q^j) = \partial^\Lambda_i dq^j = \delta^j_i \delta^\Lambda_i. \]

Not considering time reparametrization, we can restrict our consideration to derivations
\[ \vartheta = u^t \partial_t + \vartheta^i \partial_i + \sum_{0<|\Lambda|} \vartheta^i_{\Lambda} \partial^\Lambda_i, \quad u^t = 0, 1. \] (2.4.1)

Their coefficients \( \vartheta^i, \vartheta^i_{\Lambda} \) possess the transformation law
\[ \vartheta^i'_{\Lambda} = \partial q^i_{\Lambda} \vartheta^j + \partial q^i_{\Lambda} \partial_t u^t, \quad \vartheta^i_{\Lambda}' = \sum_{|\Sigma| \leq |\Lambda|} \frac{\partial q^i_{\Lambda}}{\partial q^j_{\Sigma}} \vartheta^j_{\Sigma} + \frac{\partial q^i_{\Lambda}}{\partial t} u^t. \]

Any derivation \( \vartheta \) (2.4.1) of a ring \( \mathcal{O}_{0,\infty} \) yields a derivation (a Lie derivative) \( L_\vartheta \) of a differential graded algebra \( \mathcal{O}_{*\infty} \) which obeys the relations (4.1.43) – (4.1.44).

A derivation \( \vartheta \in \mathfrak{d} \mathcal{O}_{0,\infty} \) (2.4.1) is called contact if the Lie derivative \( L_\vartheta \) preserves an ideal of contact forms of a differential graded algebra \( \mathcal{O}_{*\infty} \), i.e., the Lie derivative \( L_\vartheta \) of a contact form is a contact form.

**Lemma 2.4.1**: A derivation \( \vartheta \) (2.4.1) is contact iff it takes the form
\[ \vartheta = u^t \partial_t + \vartheta^i(t, q^j, q^i_{\Lambda}) \partial_i + \sum_{0<|\Lambda|} [d_{\Lambda}(u^t - q^i_{\Lambda} u^t) + q^i_{\Lambda} u^t] \partial^\Lambda_i. \] (2.4.2)

\[ \Box \]

A glance at the expression (2.4.2) enables one to regard a contact derivation \( \vartheta \) as an infinite order jet prolongation \( \vartheta = J^\infty u \) of its restriction
\[ u = u^t \partial_t + \vartheta^i(t, q^j, q^i_{\Lambda}) \partial_i, \quad u^t = 1, 0, \] (2.4.3)
to a ring \( C^\infty(Q) \). Since coefficients \( u^i \) of \( u \) (2.4.3) generally depend on jet coordinates \( q^i_{\Lambda} \), \( 0 < |\Lambda| \leq r \), one calls \( u \) (2.4.3) the generalized vector field. It can be represented as a section of the pull-back bundle
\[ J^r Q \times TQ \rightarrow J^r Q. \]

In particular, let \( u \) (2.4.3) be a vector field
\[ u = u^t \partial_t + \vartheta^i(t, q^j) \partial_i, \quad u^t = 0, 1, \] (2.4.4)
on a configuration space \( Q \rightarrow \mathbb{R} \). One can think of this vector field as being an infinitesimal generator of a local one-parameter group of local automorphisms of a fibre bundle \( Q \rightarrow \mathbb{R} \). If \( u^t = 0 \), the vertical vector field (2.4.4) is an infinitesimal generator of a local one-parameter group of local vertical automorphisms of \( Q \rightarrow \mathbb{R} \). If \( u^t = 1 \), the vector field \( u \)
(2.4.4) is projected onto the standard vector field $\partial_t$ on a base $\mathbb{R}$ which is an infinitesimal generator of a group of translations of $\mathbb{R}$.

Any contact derivation $\vartheta$ (2.4.2) admits the horizontal splitting

$$\vartheta = \vartheta_H + \partial V = u^i d_t + \left[ u^i_t \partial_t + \sum_{0<|\Lambda|} d_{\Lambda} u^i_\Lambda \partial^\Lambda_i \right], \quad (2.4.5)$$

$$u = u_H + u_V = u^i (\partial_t + q^i_t \partial_i) + (u^i - q^i_t u^i) \partial_i. \quad (2.4.6)$$

**Lemma 2.4.2:** Any vertical contact derivation

$$\vartheta = u^i \partial_i + \sum_{0<|\Lambda|} d_{\Lambda} u^i \partial^\Lambda_i \quad (2.4.7)$$

obeys the relations

$$\vartheta | d_H \phi = - d_H (\vartheta | \phi), \quad L_{\vartheta} (d_H \phi) = d_H (L_{\vartheta} \phi), \quad \phi \in \mathcal{O}_\infty^*. \quad (2.4.8)$$

We restrict our consideration to first order Lagrangian mechanics. In this case, contact derivations (2.1.1) can be reduced to the first order jet prolongation

$$\vartheta = J^1 u = u^i \partial_i + u^i \partial_t + d_t u^i \partial^t_i \quad (2.4.9)$$

of the generalized vector fields $u$ (2.4.3).

### 2.4.2 First Noether theorem

Let $L$ be a Lagrangian (2.1.22) on a velocity space $J^1 Q$. Let us consider its Lie derivative $L_{\theta} L$ along the contact derivation $\vartheta$ (2.4.9).

**Theorem 2.4.3:** The Lie derivative $L_{\theta} L$ fulfils the first variational formula

$$L_{J^1 u} L = u^i | \delta L + d_H (u | H_L), \quad (2.4.10)$$

where $\mathcal{L} = H_L$ is the Poincaré–Cartan form (2.2.1). Its coordinate expression reads

$$[u^i \partial_i + u^i \partial_t + d_t u^i \partial^t_i] \mathcal{L} = (u^i - q^i_t u^i) \mathcal{E}_i + d_t [\pi_i (u^i - u^i q^i_t) + u^i \mathcal{L}]. \quad (2.4.11)$$

The generalized vector field $u$ (2.4.3) is said to be the variational symmetry of a Lagrangian $L$ if the Lie derivative $L_{J^1 u} L$ is $d_H$-exact, i.e.,

$$L_{J^1 u} L = d_H \sigma. \quad (2.4.12)$$

Variational symmetries of $L$ constitute a real vector space which we denote $\mathcal{G}_L$. 
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PROPOSITION 2.4.4: A glance at the first variational formula (2.4.11) shows that a generalized vector field \( u \) is a variational symmetry iff the exterior form

\[
\mathcal{F}_t = (u^i - q_t^i u^t) \mathcal{E}_i dt
\]

is \( d_H \)-exact. \( \square \)

PROPOSITION 2.4.5: The generalized vector field \( u \) (2.4.3) is a variational symmetry of a Lagrangian \( L \) iff its vertical part \( u_V \) (2.4.6) also is a variational symmetry. \( \square \)

**Proof:** A direct computation shows that

\[
L_{J^1 u} L = L_{J^1 u_V} L + d_H (u^t \mathcal{L}).
\]  

\( QED \)

A corollary of the first variational formula (2.4.10) is the first Noether theorem.

THEOREM 2.4.6: If a contact derivation \( \vartheta \) (2.4.2) is a variational symmetry (2.4.12) of a Lagrangian \( L \), the first variational formula (2.4.10) restricted to the kernel of the Lagrange operator \( \text{Ker} \delta L \) yields a weak conservation law

\[
0 \approx d_H (u^t H_L - \sigma),
\]

\[
0 \approx d_t (\pi_i (u^i - u^t q_t^i) + u^t \mathcal{L} - \sigma),
\]

of the generalized symmetry current

\[
\mathcal{T}_u = u^t H_L - \sigma = \pi_i (u^i - u^t q_t^i) + u^t \mathcal{L} - \sigma
\]

along a generalized vector field \( u \). The generalized symmetry current (2.4.17) obviously is defined with the accuracy of a constant summand. \( \square \)

The weak conservation law (2.4.15) on the shell \( \delta L = 0 \) is called the Lagrangian conservation law. It leads to the differential conservation law (1.10.2):

\[
0 = \frac{d}{dt} [\mathcal{T}_u \circ J^{r+1} c],
\]

on solutions \( c \) of the Lagrange equation (2.1.26).

PROPOSITION 2.4.7: Let \( u \) be a variational symmetry of a Lagrangian \( L \). By virtue of Proposition 2.4.5, its vertical part \( u_V \) is so. It follows from the equality (2.4.14) that the conserved generalized symmetry current \( \mathcal{T}_u \) (2.4.17) along \( u \) equals that \( \mathcal{T}_{u_V} \) along \( u_V \). \( \square \)

A glance at the conservation law (2.4.16) shows the following.

THEOREM 2.4.8: If a variational symmetry \( u \) is a generalized vector field independent of higher order jets \( q^t_{\Lambda} \), \( |\Lambda| > 1 \), the conserved generalized current \( \mathcal{T}_u \) (2.4.17) along \( u \) plays a role of an integral of motion. \( \square \)
Therefore, we further restrict our consideration to variational symmetries at most of first jet order for the purpose of obtaining integrals of motion. However, it may happen that a Lagrangian system possesses integrals of motion which do not come from variational symmetries (Example 2.4.5).

A variational symmetry \( u \) of a Lagrangian \( L \) is called its exact symmetry if

\[
L_{J^1 u} L = 0. \tag{2.4.18}
\]

In this case, the first variational formula (2.4.10) takes the form

\[
0 = u_V \delta L + d_H(u[H_L]). \tag{2.4.19}
\]

It leads to the weak conservation law (2.4.15):

\[
0 \approx d_t \pi_u, \tag{2.4.20}
\]

of the symmetry current

\[
\pi_u = u[H_L] = \pi_i(u^i - u^i q_i^t) + u^t L \tag{2.4.21}
\]

along a generalized vector field \( u \).

**Remark 2.4.1:** In accordance with the standard terminology, if variational and exact symmetries are generalized vector fields (2.4.3), they are called generalized symmetries [3, 7, 20, 36]. Accordingly, by variational and exact symmetries one means only vector fields \( u \) (2.4.4) on \( Q \). We agree to call them classical symmetries. Classical exact symmetries are symmetries of a Lagrangian, and they are named the Lagrangian symmetries. ♦

**Remark 2.4.2:** Let us describe the relation between symmetries of a Lagrangian and and symmetries of the corresponding Lagrange equation. Let \( u \) be the vector field (2.4.4) and

\[
J^2 u = u^i \partial_i + u^i q_i^t + d_t u^i \partial_t^i + d_{tt} u^i \partial_{tt}^i
\]

its second order jet prolongation. Given a Lagrangian \( L \) on \( J^1 Q \), the relation

\[
L_{J^2 u} \delta L = \delta(L_{J^1 u} L) \tag{2.4.22}
\]

holds [8, 36]. Note that this equality need not be true in the case of a generalized vector field \( u \). A vector field \( u \) is called the local variational symmetry of a Lagrangian \( L \) if the Lie derivative \( L_{J^1 u} L \) of \( L \) along \( u \) is variationally trivial, i.e.,

\[
\delta(L_{J^1 u} L) = 0.
\]

Then it follows from the equality (2.4.22) that a local (classical) variational symmetry of \( L \) also is a symmetry of the Lagrange operator \( \delta L \), i.e.,

\[
L_{J^2 u} \delta L = 0,
\]

and *vice versa*. Consequently, any local classical variational symmetry \( u \) of a Lagrangian \( L \) is a symmetry of the Lagrange equation (2.1.25) in accordance with Proposition 1.10.5. By virtue of Theorem 2.1.5, any local classical variational symmetry is a classical variational symmetry if a typical fibre \( M \) of \( Q \) is simply connected. ♦
2.4.3 Noether conservation laws

It is readily observed that the first variational formula (2.4.11) is linear in a generalized vector field $u$. Therefore, one can consider superposition of the identities (2.4.11) for different generalized vector fields.

For instance, if $u$ and $u'$ are generalized vector fields (2.4.3), projected onto the standard vector field $\partial_t$ on $\mathbb{R}$, the difference of the corresponding identities (2.4.11) results in the first variational formula (2.4.11) for the vertical generalized vector field $u - u'$.

Conversely, every generalized vector field $u$ (2.4.4), projected onto $\partial_t$, can be written as the sum

$$u = \Gamma + v$$

of some reference frame

$$\Gamma = \partial_t + \Gamma^i \partial_i$$

and a vertical generalized vector field $v$ on $Q$.

It follows that the first variational formula (2.4.11) for the generalized vector field $u$ (2.4.4) can be represented as a superposition of those for a reference frame $\Gamma$ (2.4.24) and a vertical generalized vector field $v$.

If $u = v$ is a vertical generalized vector field, the first variational formula (2.4.11) reads

$$(v^i \partial_i + d_t v^i \partial_t^i) L = v^i \mathcal{E}_i + d_t (\pi_i v^i).$$

If $v$ is an exact symmetry of $L$, we obtain from (2.4.20) the weak conservation law

$$0 \approx d_t (\pi_i v^i).$$

By analogy with field theory [14, 15], it is called the Noether conservation law of the Noether current

$$\Sigma_v = \pi_i v^i.$$ (2.4.26)

If a generalized vector field $v$ is independent of higher order jets $q^i_\Lambda$, $|\Lambda| > 1$, the Noether current (2.4.26) is an integral of motion by virtue of Theorem 2.4.8.

**Example 2.4.3:** Let us consider a free motion on a configuration space $Q$. It is described by a Lagrangian

$$L = \left(\frac{1}{2} \overline{m}_{ij} \overline{q}_i^t \overline{q}_j^t \right) dt,$$

written with respect to a reference frame $(t, \overline{q}^i)$ such that the free motion dynamic equation takes the form (1.7.1). This Lagrangian admits $\dim Q - 1$ independent integrals of motion $\pi_i$. ◊
Example 2.4.4: Let us consider a point mass in the presence of a central potential. Its configuration space is

\[ Q = \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \]  
(2.4.28)

depicted with the Cartesian coordinates \((t, q^i)\). A Lagrangian of this mechanical system reads

\[ L = \frac{1}{2} \left( \sum_i (q^i_t)^2 \right) - V(r), \quad r = \left( \sum_i (q^i)^2 \right)^{1/2}. \]  
(2.4.29)

The vector fields

\[ v^a_b = q^a \partial_b - q^b \partial_a \]  
(2.4.30)

are infinitesimal generators of the group \(SO(3)\) acting in \(\mathbb{R}^3\). Their jet prolongation (2.4.9) reads

\[ J^1 v^a_b = q^a \partial_b - q^b \partial_a + q^a_i \partial^b_t - q^b_i \partial^a_t. \]  
(2.4.31)

It is readily observed that vector fields (2.4.30) are symmetries of the Lagrangian (2.4.29). The corresponding conserved Noether currents (2.4.26) are orbital momenta

\[ M^a_b = \Sigma_a^b = (q^a \pi^b_b - q^b \pi^a_a) = q^a_i q^b_t - q^b_i q^a_t. \]  
(2.4.32)

They are integrals of motion, which however fail to be independent. ♦

Example 2.4.5: Let us consider the Lagrangian system in Example 2.4.4 where

\[ V(r) = -\frac{1}{r} \]  
(2.4.33)

is the Kepler potential. This Lagrangian system possesses the integrals of motion

\[ A^a = \sum_b (q^a q^b_t - q^b q^a_t) q^b_t - \frac{q^a}{r}, \]  
(2.4.34)

besides the orbital momenta (2.4.32). They are components of the Rung–Lenz vector. There is no Lagrangian symmetry whose generalized symmetry currents are \(A^a\) (2.4.34). ♦

2.4.4 Energy conservation laws

In the case of a reference frame \(\Gamma\) (2.4.24), where \(u^i = 1\), the first variational formula (2.4.11) reads

\[(\partial_t + \Gamma^i \partial_i + d_t \Gamma^i \partial^i) \mathcal{L} = (\Gamma^i - q^i_t) \mathcal{E}_i - d_t (\pi_i (q^i_t - \Gamma^i) - \mathcal{L}), \]
(2.4.35)

where

\[ E_\Gamma = -\Sigma_\Gamma = \pi_i (q^i_t - \Gamma^i) - \mathcal{L} \]  
(2.4.36)
is the energy function relative to a reference frame $\Gamma$ [5, 15, 27, 40].

With respect to the coordinates adapted to a reference frame $\Gamma$, the first variational formula (2.4.35) takes the form
\[
\partial_t L = (\Gamma^i - q^i_t)E_i - d_t(\pi_i q^i_t - \mathcal{L}),
\]
and $E_\Gamma$ (2.4.36) coincides with the canonical energy function
\[
E_L = \pi_i q^i_t - \mathcal{L}.
\]
A glance at the expression (2.4.37) shows that the vector field $\Gamma$ (2.4.24) is an exact symmetry of a Lagrangian $L$ iff, written with respect to coordinates adapted to $\Gamma$, this Lagrangian is independent on the time $t$. In this case, the energy function $E_\Gamma$ (2.4.37) relative to a reference frame $\Gamma$ is conserved:
\[
0 \approx -d_t E_\Gamma.
\]
It is an integral of motion in accordance with Theorem 2.4.8.

**Example 2.4.6:** Let us consider a free motion on a configuration space $Q$ described by the Lagrangian (2.4.27) written with respect to a reference frame $(t, \bar{q}^i)$ such that the free motion dynamic equation takes the form (1.7.1). Let $\Gamma$ be the associated connection. Then the conserved energy function $E_\Gamma$ (2.4.36) relative to this reference frame $\Gamma$ is precisely the kinetic energy of this free motion. With respect to arbitrary bundle coordinates $(t, q^i)$ on $Q$, it takes the form
\[
E_\Gamma = \pi_i (q^i_t - \Gamma^i) - \mathcal{L} = \frac{1}{2}m_{ij}(t, q^j)(q^i_t - \Gamma^i)(q^j_t - \Gamma^j).
\]

**Example 2.4.7:** Let us consider a one-dimensional motion of a point mass $m_0$ subject to friction on the configuration space $\mathbb{R}^2 \rightarrow \mathbb{R}$, coordinated by $(t, q)$ (Example 2.3.1). It is described by the dynamic equation (2.3.7) which is the Lagrange equation for the Lagrangian $L$ (2.3.11). It is readily observed that the Lie derivative of this Lagrangian along the vector field
\[
\Gamma = \partial_t - \frac{k}{2m_0}q\partial_q
\]
vanishes. Consequently, we have the conserved energy function (2.4.36) with respect to the reference frame $\Gamma$ (2.4.39). This energy function reads
\[
E_\Gamma = \frac{1}{2}m_0 \exp \left[ \frac{k}{m_0}t \right] q_t(q_t + \frac{k}{m_0}q) = \frac{1}{2}(m\dot{q}^2_t - \frac{mk^2}{2m_0^2}q^2),
\]
where $m$ is the mass function (2.3.10).

Since any generalized vector field $u$ (2.4.3) can be represented as the sum (2.4.23) of a reference frame $\Gamma$ (2.4.24) and a vertical generalized vector field $v$, the symmetry current (2.4.21) along the generalized vector field $u$ (2.4.4) is the difference
\[
\mathfrak{T}_u = \mathfrak{T}_v - E_\Gamma
\]
of the Noether current $\mathcal{T}_v$ (2.4.26) along the vertical generalized vector field $v$ and the energy function $E_\Gamma$ (2.4.36) relative to a reference frame $\Gamma$ [5, 15, 40]. Conversely, energy functions relative to different reference frames $\Gamma$ and $\Gamma'$ differ from each other in the Noether current along the vertical vector field $\Gamma' - \Gamma$:

$$E_\Gamma - E_{\Gamma'} = \mathcal{T}_{\Gamma - \Gamma'}.$$ 

One can regard this vector field $\Gamma' - \Gamma$ as the relative velocity of a reference frame $\Gamma'$ with respect to $\Gamma$.

### 2.5 Gauge symmetries

Treating gauge symmetries of Lagrangian field theory, one is traditionally based on an example of the Yang–Mills gauge theory of principal connections on a principal bundle. This notion of gauge symmetries is generalized to Lagrangian theory on an arbitrary fibre bundle [13, 14], including mechanics on a fibre bundle $Q \to \mathbb{R}$.

**Definition 2.5.1:** Let $E \to \mathbb{R}$ be a vector bundle and $E(\mathbb{R})$ the $C^\infty(\mathbb{R})$ module of sections $\chi$ of $E \to \mathbb{R}$. Let $\zeta$ be a linear differential operator on $E(\mathbb{R})$ taking its values into the vector space $\mathcal{G}_L$ of variational symmetries of a Lagrangian $L$. Elements

$$u_\chi = \zeta(\chi) \quad (2.5.1)$$

of $\text{Im} \zeta$ are called the gauge symmetry of a Lagrangian $L$ parameterized by sections $\chi$ of $E \to \mathbb{R}$. These sections are called the gauge parameters. 

**Remark 2.5.1:** The differential operator $\zeta$ in Definition 2.5.1 takes its values into the vector space $\mathcal{G}_L$ as a subspace of the $C^\infty(\mathbb{R})$-module $\mathcal{D}\mathcal{O}_\infty^0$, but it sends the $C^\infty(\mathbb{R})$-module $E(\mathbb{R})$ into the real vector space $\mathcal{G}_L \subset \mathcal{D}\mathcal{O}_\infty^0$. The differential operator $\zeta$ is assumed to be at least of first order (Remark 2.5.2).

Equivalently, the gauge symmetry (2.5.1) is given by a section $\tilde{\zeta}$ of the fibre bundle

$$(J^r Q \times J^m E) \times TQ \to J^r Q \times J^m E$$

(see Definition 4.2.4) such that

$$u_\chi = \zeta(\chi) = \tilde{\zeta} \circ \chi$$

for any section $\chi$ of $E \to \mathbb{R}$. Hence, it is a generalized vector field $u_\chi$ on the product $Q \times E$ represented by a section of the pull-back bundle

$$J^k(Q \times E) \times T(Q \times E) \to J^k(Q \times E), \quad k = \max(r, m),$$
which lives in
\[ TQ \subset T(Q \times E). \]
This generalized vector field yields the contact derivation \( J^\infty u_\zeta \) (2.4.2) of the real ring \( \mathcal{O}^0_\infty [Q \times E] \) which obeys the following condition.

Given a Lagrangian
\[ L \in \mathcal{O}^{0,n}_\infty E \subset \mathcal{O}^{0,n}_\infty [Q \times E], \]
let us consider its Lie derivative
\[ L_{J^\infty u_\zeta} L = J^\infty u_\zeta | dL + d(J^\infty u_\zeta | L) \] (2.5.2)
where \( d \) is the exterior differential of \( \mathcal{O}^*_\infty [Q \times E] \). Then for any section \( \chi \) of \( E \to \mathbb{R} \), the pull-back \( \chi^* L_{J^\infty u_\zeta} L \) is \( d_H \)-exact.

It follows at once from the first variational formula (2.4.10) for the Lie derivative (2.5.2) that the above mentioned condition holds only if \( u_\zeta \) is projected onto a generalized vector field on \( Q \) and, in this case, if the density \( (u_\zeta)_V | \mathcal{E} \) is \( d_H \)-exact (Proposition 2.4.4). Thus, we come to the following equivalent definition of gauge symmetries.

**Definition 2.5.2:** Let \( E \to \mathbb{R} \) be a vector bundle. A gauge symmetry of a Lagrangian \( L \) parameterized by sections \( \chi \) of \( E \to \mathbb{R} \) is defined as a contact derivation \( \vartheta = J^\infty u \) of the real ring \( \mathcal{O}^*_\infty [Q \times E] \) such that:

(i) it vanishes on the subring \( \mathcal{O}^0_\infty E \),

(ii) the generalized vector field \( u \) is linear in coordinates \( \chi^a_{\Lambda} \) on \( J^\infty E \), and it is projected onto a generalized vector field on \( Q \), i.e., it takes the form
\[ u = \partial_t + \left( \sum_{0 \leq |\Lambda| \leq m} u^i_{a\Lambda} (t, q^j_{\Sigma}) \chi^a_{\Lambda} \right) \partial_i, \] (2.5.3)

(iii) the vertical part of \( u \) (2.5.3) obeys the equality
\[ u_V | \delta L = d_H \sigma. \] (2.5.4)

For the sake of convenience, the generalized vector field (2.5.3) also is called the gauge symmetry. In accordance with Proposition 2.4.5, the \( u \) (2.5.3) is a gauge symmetry iff its vertical part is so. Owing to this fact and Proposition 2.4.7, we can restrict our consideration to vertical gauge symmetries
\[ u = \left( \sum_{0 \leq |\Lambda| \leq m} u^i_{a\Lambda} (t, q^j_{\Sigma}) \chi^a_{\Lambda} \right) \partial_i. \] (2.5.5)
2.5. GAUGE SYMMETRIES

Gauge symmetries possess the following particular properties.

(i) Let $E' \to \mathbb{R}$ be another vector bundle and $\zeta'$ a linear $E(\mathbb{R})$-valued differential operator on a $C^\infty(\mathbb{R})$-module $E'(\mathbb{R})$ of sections of $E' \to \mathbb{R}$. Then

$$u_{\zeta'(\chi')} = (\zeta \circ \zeta')(\chi')$$

also is a gauge symmetry of $L$ parameterized by sections $\chi'$ of $E' \to \mathbb{R}$. It factorizes through the gauge symmetry $u_\chi$ (2.5.1).

(ii) Given a gauge symmetry, the corresponding conserved symmetry current $\Sigma_u$ (2.4.17) vanishes on-shell (Theorem 2.5.4 below).

(iii) The second Noether theorem associates to a gauge symmetry of a Lagrangian $L$ the Noether identities of its Lagrange operator $\delta L$.

**Theorem 2.5.3**: Let $u$ (2.5.5) be a gauge symmetry of a Lagrangian $L$, then its Lagrange operator $\delta L$ obeys the Noether identities (2.5.6).

**Proof**: The density (2.5.4) is variationally trivial and, therefore, its variational derivatives with respect to variables $\chi^a$ vanish, i.e.,

$$\mathcal{E}_a = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_{\Lambda}(u^I_{a} \mathcal{E}_i) = 0. \quad (2.5.6)$$

These are the Noether identities for the Lagrange operator $\delta L$ [14]. QED

For instance, if the gauge symmetry $u$ (2.5.3) is of second jet order in gauge parameters, i.e.,

$$u = (u^i_a \chi^a + u^a_t \chi^a + u^{it}_{a} \chi^a) \partial_i, \quad (2.5.7)$$

the corresponding Noether identities (2.5.6) take the form

$$u^i_a \mathcal{E}_i - d_t(u^i_a \mathcal{E}_i) + d_{tt}(u^{it}_{a} \mathcal{E}_i) = 0. \quad (2.5.8)$$

**Remark 2.5.2**: A glance at the expression (2.5.8) shows that, if a gauge symmetry is independent of derivatives of gauge parameters (i.e., the differential operator $\zeta$ in Definition 2.5.1 is of zero order), then all variational derivatives of a Lagrangian equals zero, i.e., this Lagrangian is variationally trivial. Therefore, such gauge symmetries usually are not considered.

If a Lagrangian $L$ admits a gauge symmetry $u$ (2.5.5), i.e., $L_{Jt}u L = \sigma$, the weak conservation law (2.4.16) of the corresponding generalized symmetry current $\Sigma_u$ (2.4.17) holds. We call it the gauge conservation law. Because gauge symmetries depend on derivatives of gauge parameters, all gauge conservation laws in first order Lagrangian mechanics possess the following peculiarity.

**Theorem 2.5.4**: If $u$ (2.5.5) is a gauge symmetry of a first order Lagrangian $L$, the corresponding conserved generalized symmetry current $\Sigma_u$ (2.4.17) vanishes on-shell, i.e., $\Sigma_u \approx 0$ [14, 15]. QED

Note that the statement of Theorem 2.5.4 is a particular case of the fact that symmetry currents of gauge symmetries in field theory are reduced to a superpotential [14, 42].
CHAPTER 2. LAGRANGIAN MECHANICS
Chapter 3

Hamiltonian mechanics

As was mentioned above, a phase space of non-relativistic mechanics is the vertical cotangent bundle \( V^*Q \) of its configuration space \( Q \to \mathbb{R} \). This phase space is provided with the canonical Poisson structure (3.1.7). However, Hamiltonian mechanics on a phase space \( V^*Q \) is not familiar Poisson Hamiltonian theory on a Poisson manifold \( V^*Q \) because all Hamiltonian vector fields on \( V^*Q \) are vertical. Hamiltonian mechanics on \( V^*Q \) is formulated as particular (polysymplectic) Hamiltonian formalism on fibre bundles [8, 14, 15, 27]. Its Hamiltonian is a section of the fibre bundle \( T^*Q \to V^*Q \) (2.2.5). The pull-back of the canonical Liouville form (2.2.12) on \( T^*Q \) with respect to this section is a Hamiltonian one-form on \( V^*Q \). The corresponding Hamiltonian connection (3.1.20) on \( V^*Q \to \mathbb{R} \) defines a first order Hamilton equations on \( V^*Q \).

Note that one can associate to any Hamiltonian system on \( V^*Q \) an autonomous symplectic Hamiltonian system on the cotangent bundle \( T^*Q \) such that the corresponding Hamilton equations on \( V^*Q \) and \( T^*Q \) are equivalent (Section 3.2). Moreover, a Hamilton equations on \( V^*Q \) also ia equivalent to the Lagrange equation of a certain first order Lagrangian on a configuration space \( V^*Q \) (Section 3.3).

Lagrangian and Hamiltonian formulations of mechanics fail to be equivalent, unless a Lagrangian is hyperregular. The comprehensive relations between Lagrangian and Hamiltonian systems can be established in the case of almost regular Lagrangians (Section 3.4).

3.1 Hamiltonian formalism on \( Q \to \mathbb{R} \)

As was mentioned above, a phase space of mechanics on a configuration space \( Q \to \mathbb{R} \) is the vertical cotangent bundle (2.1.31):

\[
V^*Q \xrightarrow{\pi_1} Q \xrightarrow{\pi} \mathbb{R},
\]

of \( Q \to \mathbb{R} \) equipped with the holonomic coordinates \((t, q^i, p_i = \dot{q}_i)\) with respect to the fibre bases \( \{dq^i\} \) for the bundle \( V^*Q \to Q \) [15, 27].
The cotangent bundle $T^*Q$ of the configuration space $Q$ is endowed with the holonomic coordinates $(t, q^i, p_0, p_i)$, possessing the transition functions (2.2.4). It admits the Liouville form $\Xi$ (2.2.12), the symplectic form
\begin{equation}
\Omega_T = d\Xi = dp_0 \wedge dt + dp_i \wedge dq^i,
\end{equation}
and the corresponding Poisson bracket
\begin{equation}
\{f, g\}_T = \partial_0^i f \partial_i g - \partial_0^i g \partial_i f + \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(T^*Q).
\end{equation}
Provided with the structures (3.1.1) – (3.1.2), the cotangent bundle $T^*Q$ of $Q$ plays a role of the homogeneous phase space of Hamiltonian mechanics.

There is the canonical one-dimensional affine bundle (2.2.5):
\begin{equation}
\zeta : T^*Q \to V^*Q.
\end{equation}
A glance at the transformation law (2.2.4) shows that it is a trivial affine bundle. Indeed, given a global section $h$ of $\zeta$, one can equip $T^*Q$ with the global fibre coordinate
\begin{equation}
I_0 = p_0 - h, \quad I_0 \circ h = 0,
\end{equation}
possessing the identity transition functions. With respect to the coordinates
\begin{equation}
(t, q^i, I_0, p_i), \quad i = 1, \ldots, m,
\end{equation}
the fibration (3.1.3) reads
\begin{equation}
\zeta : \mathbb{R} \times V^*Q \ni (t, q^i, I_0, p_i) \to (t, q^i, p_i) \in V^*Q.
\end{equation}
Let us consider the subring of $C^\infty(T^*Q)$ which comprises the pull-back $\zeta^*f$ onto $T^*Q$ of functions $f$ on the vertical cotangent bundle $V^*Q$ by the fibration $\zeta$ (3.1.3). This subring is closed under the Poisson bracket (3.1.2). Then by virtue of the well known theorem, there exists the degenerate Poisson structure
\begin{equation}
\{f, g\}_V = \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(V^*Q),
\end{equation}
on a phase space $V^*Q$ such that
\begin{equation}
\zeta^*\{f, g\}_V = \{\zeta^*f, \zeta^*g\}_T.
\end{equation}
The holonomic coordinates on $V^*Q$ are canonical for the Poisson structure (3.1.7).

With respect to the Poisson bracket (3.1.7), the Hamiltonian vector fields of functions on $V^*Q$ read
\begin{equation}
\vartheta f = \partial^i f \partial_i - \partial_i f \partial^i, \quad f \in C^\infty(V^*Q).
\end{equation}
They are vertical vector fields on $V^*Q \to \mathbb{R}$. Accordingly, the characteristic distribution of the Poisson structure (3.1.7) is the vertical tangent bundle $VV^*Q \subset TV^*Q$ of a fibre
bundle $V^*Q \rightarrow \mathbb{R}$. The corresponding symplectic foliation on the phase space $V^*Q$ coincides with the fibration $V^*Q \rightarrow \mathbb{R}$.

It is readily observed that the ring $C(V^*Q)$ of Casimir functions on a Poisson manifold $V^*Q$ consists of the pull-back onto $V^*Q$ of functions on $\mathbb{R}$. Therefore, the Poisson algebra $C^\infty(V^*Q)$ is a Lie $C^\infty(\mathbb{R})$-algebra.

**Remark 3.1.1:** The Poisson structure (3.1.7) can be introduced in a different way [15, 27]. Given any section $h$ of the fibre bundle (3.1.3), let us consider the pull-back forms

\[ \Theta = h^*(\Xi \wedge dt) = p_i dq^i \wedge dt, \]
\[ \Omega = h^*(d\Xi \wedge dt) = dp_i \wedge dq^i \wedge dt \] (3.1.10)

on $V^*Q$. They are independent of the choice of $h$. With $\Omega$ (3.1.10), the Hamiltonian vector field $\vartheta_f$ (3.1.9) for a function $f$ on $V^*Q$ is given by the relation

\[ \vartheta_f \lrcorner \Omega = -df \wedge dt, \]

while the Poisson bracket (3.1.7) is written as

\[ \{f, g\}_V dt = \vartheta_g \lrcorner \vartheta_f \lrcorner \Omega. \]

Moreover, one can show that a projectable vector field $\vartheta$ on $V^*Q$ such that $\vartheta \lrcorner dt =$const. is a canonical vector field for the Poisson structure (3.1.7) iff

\[ L_\vartheta \Omega = d(\vartheta \lrcorner \Omega) = 0. \] (3.1.11)

\[ \Diamond \]

In contrast with autonomous Hamiltonian mechanics, the Poisson structure (3.1.7) fails to provide any dynamic equation on a fibre bundle $V^*Q \rightarrow \mathbb{R}$ because Hamiltonian vector fields (3.1.9) of functions on $V^*Q$ are vertical vector fields, but not connections on $V^*Q \rightarrow \mathbb{R}$ (see Definition 1.3.1). Hamiltonian dynamics on $V^*Q$ is described as a particular Hamiltonian dynamics on fibre bundles [15, 27, 40].

A Hamiltonian on a phase space $V^*Q \rightarrow \mathbb{R}$ of mechanics is defined as a global section

\[ h : V^*Q \rightarrow T^*Q, \quad p_0 \circ h = \mathcal{H}(t, q^i, p_i), \] (3.1.12)

of the affine bundle $\zeta$ (3.1.3). Given the Liouville form $\Xi$ (2.2.12) on $T^*Q$, this section yields the pull-back Hamiltonian form

\[ H = (-h)^*\Xi = p_k dq^k - \mathcal{H} dt \] (3.1.13)

on $V^*Q$. This is the well-known invariant of Poincaré–Cartan [2].

It should be emphasized that, in contrast with a Hamiltonian in autonomous mechanics, the Hamiltonian $\mathcal{H}$ (3.1.12) is not a function on $V^*Q$, but it obeys the transformation law

\[ \mathcal{H}'(t, q^i, p_i) = \mathcal{H}(t, q^i, p_i) + p_i \partial_t q^i. \] (3.1.14)
**Remark 3.1.2:** Any connection \( \Gamma \) (1.1.16) on a configuration bundle \( Q \to \mathbb{R} \) defines the global section \( h_\Gamma = p_i \Gamma^i \) (3.1.12) of the affine bundle \( \zeta \) (3.1.3) and the corresponding Hamiltonian form

\[
H_\Gamma = p_k dq^k - \mathcal{H}_\Gamma dt = p_k dq^k - p_i \Gamma^i dt. \tag{3.1.15}
\]

Furthermore, given a connection \( \Gamma \), any Hamiltonian form (3.1.13) admits the splitting

\[
H = H_\Gamma - \mathcal{E}_\Gamma dt, \tag{3.1.16}
\]

where

\[
\mathcal{E}_\Gamma = \mathcal{H} - \mathcal{H}_\Gamma = \mathcal{H} - p_i \Gamma^i. \tag{3.1.17}
\]

is a function on \( V^*Q \). One can think of \( \mathcal{E}_\Gamma \) (3.1.17) as being an energy function relative to a reference frame \( \Gamma \) [15, 31]. With respect to the coordinates adapted to a reference frame \( \Gamma \), we have \( \mathcal{E}_\Gamma = \mathcal{H} \). Given different reference frames \( \Gamma \) and \( \Gamma' \), the decomposition (3.1.16) leads at once to the relation

\[
\mathcal{E}_{\Gamma'} = \mathcal{E}_\Gamma + \mathcal{H}_{\Gamma'} - \mathcal{H}_\Gamma = \mathcal{E}_\Gamma + (\Gamma^i - \Gamma'^i)p_i \tag{3.1.18}
\]

between the energy functions with respect to different reference frames. ♦

Given a Hamiltonian form \( H \) (3.1.13), there exists a unique horizontal vector field (1.1.16):

\[
\gamma_H = \partial_t - \gamma^i \partial_i - \gamma_\iota \partial_\iota, \tag{3.1.19}
\]

on \( V^*Q \) (i.e., a connection on \( V^*Q \to \mathbb{R} \)) such that

\[
\gamma_H |_dH = 0. \tag{3.1.19}
\]

This vector field, called the Hamilton vector field, reads

\[
\gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k. \tag{3.1.20}
\]

In a different way (Remark 3.1.1), the Hamilton vector field \( \gamma_H \) is defined by the relation

\[
\gamma_H |_\Omega = dH. \tag{3.1.20}
\]

Consequently, it is canonical for the Poisson structure \( \{,\}_V \) (3.1.7). This vector field yields the first order dynamic Hamilton equation

\[
\dot{q}_t^k = \partial^k \mathcal{H}, \tag{3.1.21}
\]

\[
p_t^k = -\partial_k \mathcal{H} \tag{3.1.22}
\]

on \( V^*Q \to \mathbb{R} \) (Definition 1.3.1), where \((t, q^k, p_k, q_\iota^k, \dot{p}_\iota^k)\) are the adapted coordinates on the first order jet manifold \( J^1V^*Q \) of \( V^*Q \to \mathbb{R} \).
Due to the canonical imbedding $J^1 V^*Q \rightarrow TV^*Q$ (1.1.6), the Hamilton equation (3.1.21) – (3.1.22) is equivalent to the autonomous first order dynamic equation

$$\dot{t} = 1, \quad \dot{q}^i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H}$$

on a manifold $V^*Q$ (Definition 1.2.1).

A solution of the Hamilton equation (3.1.21) – (3.1.22) is an integral section $r$ for the connection $\gamma_H$.

**Remark 3.1.3:** Similarly to the Cartan equation (2.2.11), the Hamilton equation (3.1.21) – (3.1.22) is equivalent to the condition

$$r^*(u \lrcorner d\mathcal{H}) = 0$$

for any vertical vector field $u$ on $V^*Q \rightarrow \mathbb{R}$. ♦

We agree to call $(V^*Q, \mathcal{H})$ the Hamiltonian system of $m = \dim Q - 1$ degrees of freedom.

In order to describe evolution of a Hamiltonian system at any instant, the Hamilton vector field $\gamma_H$ (3.1.20) is assumed to be complete, i.e., it is an Ehresmann connection (Remark 1.1.1). In this case, the Hamilton equation (3.1.21) – (3.1.22) admits a unique global solution through each point of the phase space $V^*Q$. By virtue of Theorem 1.1.2, there exists a trivialization of a fibre bundle $V^*Q \rightarrow \mathbb{R}$ (not necessarily compatible with its fibration $V^*Q \rightarrow Q$) such that

$$\gamma_H = \partial_t, \quad H = p_i d\bar{q}^i$$

with respect to the associated coordinates $(t, \bar{q}^i, \bar{p}_i)$. A direct computation shows that the Hamilton vector field $\gamma_H$ (3.1.20) satisfies the relation (3.1.11) and, consequently, it is an infinitesimal generator of a one-parameter group of automorphisms of the Poisson manifold $(V^*Q, \{,\}_V)$. Then one can show that $(t, \bar{q}^i, \bar{p}_i)$ are canonical coordinates for the Poisson manifold $(V^*Q, \{,\}_V)$ [27], i.e.,

$$w = \frac{\partial}{\partial \bar{p}_i} \wedge \frac{\partial}{\partial \bar{q}^i}.$$

Since $\mathcal{H} = 0$, the Hamilton equation (3.1.21) – (3.1.22) in these coordinates takes the form

$$\bar{q}^i_t = 0, \quad \bar{p}_{ti} = 0,$$

i.e., $(t, \bar{q}^i, \bar{p}_i)$ are the initial data coordinates.
3.2 Homogeneous Hamiltonian formalism

As was mentioned above, one can associate to any Hamiltonian system on a phase space \( V^*Q \) an equivalent autonomous symplectic Hamiltonian system on the cotangent bundle \( T^*Q \) (Theorem 3.2.1).

Given a Hamiltonian system \( (V^*Q, H) \), its Hamiltonian \( H \) (3.1.12) defines the function

\[
H^* = \partial_t [\Xi - \zeta^*(-\rho)^*\Xi)] = p_0 + \rho = p_0 + \mathcal{H}
\]  

on \( T^*Q \). Let us regard \( H^* \) (3.2.1) as a Hamiltonian of an autonomous Hamiltonian system on the symplectic manifold \( (T^*Q, \Omega_T) \). The corresponding autonomous Hamilton equation on \( T^*Q \) takes the form

\[
\dot{i} = 1, \quad \dot{p}_0 = -\partial_i \mathcal{H}, \quad \dot{q}^i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H}.
\]  

Remark 3.2.1: Let us note that the splitting \( H^* = p_0 + \mathcal{H} \) (3.2.1) is ill defined. At the same time, any reference frame \( \Gamma \) yields the decomposition

\[
H^* = (p_0 + \mathcal{H}_\Gamma) + (\mathcal{H} - \mathcal{H}_\Gamma) = H^*_\Gamma + \mathcal{E}_\Gamma,
\]  

where \( \mathcal{H}_\Gamma \) is the Hamiltonian (3.1.15) and \( \mathcal{E}_\Gamma \) (3.1.17) is the energy function relative to a reference frame \( \Gamma \).

The Hamiltonian vector field \( \mathcal{H}^* \) (3.2.1) on \( T^*Q \) is

\[
\mathcal{H}^* = \partial_i - \partial_t \mathcal{H} \partial^0 + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i.
\]  

Written relative to the coordinates (3.1.5), this vector field reads

\[
\mathcal{H}^* = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i.
\]  

It is identically projected onto the Hamilton vector field \( \gamma^* \) (3.1.20) on \( V^*Q \) such that

\[
\zeta^*(L_{\gamma^*} f) = \{\mathcal{H}^*, \zeta^* f\}_T, \quad f \in C^\infty(V^*Q).
\]  

Therefore, the Hamilton equation (3.1.21) - (3.1.22) is equivalent to the autonomous Hamilton equation (3.2.2).

Obviously, the Hamiltonian vector field \( \mathcal{H}^* \) (3.2.5) is complete if the Hamilton vector field \( \gamma^* \) (3.1.20) is complete.

Thus, the following has been proved [4, 15, 29].

Theorem 3.2.1: A Hamiltonian system \( (V^*Q, H) \) of \( m \) degrees of freedom is equivalent to an autonomous Hamiltonian system \((T^*Q, \mathcal{H}^*)\) of \( m+1 \) degrees of freedom on a symplectic manifold \((T^*Q, \Omega)\) whose Hamiltonian is the function \( \mathcal{H}^* \) (3.2.1).

We agree to call \((T^*Q, \mathcal{H}^*)\) the homogeneous Hamiltonian system and \( \mathcal{H}^* \) (3.2.1) the homogeneous Hamiltonian.
3.3 Lagrangian form of Hamiltonian formalism

It is readily observed that the Hamiltonian form \( H \) (3.1.13) is the Poincaré–Cartan form of the Lagrangian

\[
L_H = h_0(H) = (p_i q^i_t - \mathcal{H})dt
\]

on the jet manifold \( J^1 V^* Q \) of \( V^* Q \to \mathbb{R} \) [15, 31].

**Remark 3.3.1:** In fact, the Lagrangian (3.3.1) is the pull-back onto \( J^1 V^* Q \) of the form \( L_H \) on the product \( V^* Q \times Q J^1 Q \).

The Lagrange operator (2.1.16) associated to the Lagrangian \( L_H \) reads

\[
\mathcal{E}_H = \delta L_H = [(q^i_t - \partial^i \mathcal{H}) dp_i - (p_{ti} + \partial_i \mathcal{H}) dq^i] \wedge dt.
\]

(3.3.2)

The corresponding Lagrange equation (2.1.20) is of first order, and it coincides with the Hamilton equation (3.1.21) – (3.1.22) on \( J^1 V^* Q \).

Due to this fact, the Lagrangian \( L_H \) (3.3.1) plays a prominent role in Hamiltonian mechanics.

In particular, let \( u \) (2.4.4) be a vector field on a configuration space \( Q \). Its functorial lift (4.1.32) onto the cotangent bundle \( T^* Q \) is

\[
\tilde{u} = u^t \partial_t + u^i \partial_i - p_j \partial_i u^j \partial^i
\]

(3.3.3)

This vector field is identically projected onto a vector field, also given by the expression (3.3.3), on the phase space \( V^* Q \) as a base of the trivial fibre bundle (3.1.3). Then we have the equality

\[
L_{\tilde{u}} H = L_{J^1 \tilde{u}} L_H = (-u^t \partial_t \mathcal{H} + p_i \partial_t u^i - u^i \partial_t \mathcal{H} + p_i \partial_j u^i \partial^j \mathcal{H}) dt.
\]

(3.3.4)

This equality enables us to study conservation laws in Hamiltonian mechanics similarly to those in Lagrangian mechanics (Section 3.5).

3.4 Associated Lagrangian and Hamiltonian systems

As was mentioned above, Lagrangian and Hamiltonian formulations of mechanics fail to be equivalent. The comprehensive relations between Lagrangian and Hamiltonian systems can be established in the case of almost regular Lagrangians [15, 27, 29, 40]. This is a particular case of the relations between Lagrangian and Hamiltonian theories on fibre bundles [9, 14].

In order to compare Lagrangian and Hamiltonian formalisms, we are based on the facts that:

(i) every first order Lagrangian \( L \) (2.1.15) on a velocity space \( J^1 Q \) induces the Legendre map (2.1.30) of this velocity space to a phase space \( V^* Q \);
(ii) every Hamiltonian form $H$ (3.1.13) on a phase space $V^*Q$ yields the Hamiltonian map

$$\hat{H} : V^*Q \to J^1Q, \quad \hat{H}^i_q = \delta^i \mathcal{H}$$

(3.4.1)

of this phase space to a velocity space $J^1Q$.

**Remark 3.4.1:** A Hamiltonian form $H$ is called regular if the Hamiltonian map $\hat{H}$ (3.4.1) is regular, i.e., a local diffeomorphism. ♦

**Remark 3.4.2:** It is readily observed that a section $r$ of a fibre bundle $V^*Q \to \mathbb{R}$ is a solution of the Hamilton equation (3.1.21) – (3.1.22) for the Hamiltonian form $H$ iff it obeys the equality

$$J^1(\pi_\Pi \circ r) = \hat{H} \circ r,$$

(3.4.2)

where $\pi_\Pi : V^*Q \to Q$. ♦

Given a Lagrangian $L$, the Hamiltonian form $H$ (3.1.13) is said to be associated with $L$ if $H$ satisfies the relations

$$\hat{L} \circ \hat{H} \circ \hat{L} = \hat{L},$$

(3.4.3)

$$\hat{H}^*L_H = \hat{H}^*L,$$

(3.4.4)

where $L_H$ is the Lagrangian (3.3.1).

A glance at the equality (3.4.3) shows that $\hat{L} \circ \hat{H}$ is the projector of $V^*Q$ onto the Lagrangian constraint space $N_L$ which is given by the coordinate conditions

$$p_i = \pi_i(t, q^j, \partial_j \mathcal{H}(t, q^j, p_j)).$$

(3.4.5)

The relation (3.4.4) takes the coordinate form

$$\mathcal{H} = p_i \delta^i \mathcal{H} - \mathcal{L}(t, q^j, \partial^j \mathcal{H}).$$

(3.4.6)

Acting on this equality by the exterior differential, we obtain the relations

$$\partial t \mathcal{H}(p) = -(\partial_t \mathcal{L}) \circ \hat{H}(p), \quad p \in N_L,$$

$$\partial t \mathcal{H}(p) = -(\partial_t \mathcal{L}) \circ \hat{H}(p), \quad p \in N_L,$$

(3.4.7)

$$(p_i - (\partial_i \mathcal{L})(t, q^j, \partial^j \mathcal{H})) \delta^i \partial^a \mathcal{H} = 0.$$

(3.4.8)

The relation (3.4.8) shows that an $L$-associated Hamiltonian form $H$ is not regular outside the Lagrangian constraint space $N_L$.

For instance, let $L$ be a hyperregular Lagrangian, i.e., the Legendre map $\hat{L}$ (2.1.30) is a diffeomorphism. It follows from the relation (3.4.3) that, in this case, $\hat{H} = \hat{L}^{-1}$. Then the relation (3.4.6) takes the form

$$\mathcal{H} = p_i \hat{L}^{-1i} - \mathcal{L}(t, q^j, \hat{L}^{-1j}).$$

(3.4.9)
It defines a unique Hamiltonian form associated with a hyperregular Lagrangian. Let \( s \) be a solution of the Lagrange equation (2.1.25) for a Lagrangian \( L \). A direct computation shows that \( \hat{L} \circ J^1 s \) is a solution of the Hamilton equation (3.1.21) – (3.1.22) for the Hamiltonian form \( H \) (3.4.9). Conversely, if \( r \) is a solution of the Hamilton equation (3.1.21) – (3.1.22) for the Hamiltonian form \( H \) (3.4.9), then \( s = \pi_H \circ r \) is a solution of the Lagrange equation (2.1.25) for \( L \) (see the equality (3.4.2)). It follows that, in the case of hyperregular Lagrangians, Hamiltonian formalism is equivalent to Lagrangian one.

If a Lagrangian is not regular, an associated Hamiltonian form need not exist.

A Hamiltonian form is called weakly associated with a Lagrangian \( L \) if the condition (3.4.4) (namely, the condition (3.4.8) holds on the Lagrangian constraint space \( N_L \).

For instance, any Hamiltonian form is weakly associated with the Lagrangian \( L = 0 \), while the associated Hamiltonian forms are only \( H_\Gamma \) (3.1.15).

A hyperregular Lagrangian \( L \) has a unique weakly associated Hamiltonian form (3.4.9) which also is \( L \)-associated. In the case of a regular Lagrangian \( L \), the Lagrangian constraint space \( N_L \) is an open subbundle of the vector Legendre bundle \( V^*Q \to Q \). If \( N_L \neq V^*Q \), a weakly associated Hamiltonian form fails to be defined everywhere on \( V^*Q \) in general. At the same time, \( N_L \) itself can be provided with the pull-back symplectic structure with respect to the imbedding \( N_L \to V^*Q \), so that one may consider Hamiltonian forms on \( N_L \).

Note that, in contrast with associated Hamiltonian forms, a weakly associated Hamiltonian form may be regular.

In order to say something more, let us restrict our consideration to almost regular Lagrangians \( L \) (Definition 2.1.12) [15, 27, 29].

**Lemma 3.4.1:** The Poincaré–Cartan form \( H_L \) (2.2.1) of an almost regular Lagrangian \( L \) is constant on the inverse image \( \hat{L}^{-1}(z) \) of any point \( z \in N_L \). \( \square \)

A corollary of Lemma 3.4.1 is the following.

**Theorem 3.4.2:** All Hamiltonian forms weakly associated with an almost regular Lagrangian \( L \) coincide with each other on the Lagrangian constraint space \( N_L \), and the Poincaré–Cartan form \( H_L \) (2.2.1) of \( L \) is the pull-back

\[
H_L = \hat{L}^* H, \quad \pi_i q^i_t - \mathcal{L} = \mathcal{H}(t, q^j, \pi_j),
\]

(3.4.10)

of such a Hamiltonian form \( H \). \( \square \)

It follows that, given Hamiltonian forms \( H \) and \( H' \) weakly associated with an almost regular Lagrangian \( L \), their difference is a density

\[
H' - H = (\mathcal{H} - \mathcal{H}') dt
\]

vanishing on the Lagrangian constraint space \( N_L \). However, \( \hat{H}|_{N_L} \neq \hat{H}'|_{N_L} \) in general. Therefore, the Hamilton equations for \( H \) and \( H' \) do not necessarily coincide on the Lagrangian constraint space \( N_L \).
Theorem 3.4.2 enables us to relate the Lagrange equation for an almost regular Lagrangian $L$ with the Hamilton equation for Hamiltonian forms weakly associated to $L$.

**Theorem 3.4.3:** Let a section $r$ of $V^*Q \to \mathbb{R}$ be a solution of the Hamilton equation (3.1.21) – (3.1.22) for a Hamiltonian form $H$ weakly associated with an almost regular Lagrangian $L$. If $r$ lives in the Lagrangian constraint space $N_L$, the section $s = \pi \circ r$ of $\pi : Q \to \mathbb{R}$ satisfies the Lagrange equation (2.1.25), while $\overline{s} = \hat{H} \circ r$ obeys the Cartan equation (2.2.9) – (2.2.10). □

The proof is based on the relation
\[
\hat{L} = (J^1\hat{L})^*L_H,
\]
where $\hat{L}$ is the Lagrangian (2.2.7), while $L_H$ is the Lagrangian (3.3.1). This relation is derived from the equality (3.4.10). The converse assertion is more intricate.

**Theorem 3.4.4:** Given an almost regular Lagrangian $L$, let a section $\overline{s}$ of the jet bundle $J^1Q \to \mathbb{R}$ be a solution of the Cartan equation (2.2.9) – (2.2.10). Let $H$ be a Hamiltonian form weakly associated with $L$, and let $H$ satisfy the relation
\[
\hat{H} \circ \hat{L} \circ \overline{s} = J^1s,
\]
where $s$ is the projection of $\overline{s}$ onto $Q$. Then the section $r = \hat{L} \circ \overline{s}$ of a fibre bundle $V^*Q \to \mathbb{R}$ is a solution of the Hamilton equation (3.1.21) – (3.1.22) for $H$. □

We say that a set of Hamiltonian forms $H$ weakly associated with an almost regular Lagrangian $L$ is complete if, for each solution $s$ of the Lagrange equation, there exists a solution $r$ of the Hamilton equation for a Hamiltonian form $H$ from this set such that $s = \pi_H \circ r$. By virtue of Theorem 3.4.4, a set of weakly associated Hamiltonian forms is complete if, for every solution $s$ of the Lagrange equation for $L$, there exists a Hamiltonian form $H$ from this set which fulfills the relation (3.4.11) where $\overline{s} = J^1s$, i.e.,
\[
\hat{H} \circ \hat{L} \circ J^1s = J^1s.
\]

(3.4.12)

In the case of almost regular Lagrangians, one can formulate the following necessary and sufficient conditions of the existence of weakly associated Hamiltonian forms.

**Theorem 3.4.5:** A Hamiltonian form $H$ weakly associated with an almost regular Lagrangian $L$ exists iff the fibred manifold (2.1.32):
\[
\hat{L} : J^1Q \to N_L,
\]
(3.4.13)
admits a global section. □

In particular, any point of $V^*Q$ possesses an open neighborhood $U$ such that there exists a complete set of local Hamiltonian forms on $U$ which are weakly associated with an almost regular Lagrangian $L$. Moreover, one can construct a complete set of local $L$-associated Hamiltonian forms on $U$ [39].
3.5 Hamiltonian conservation laws

As was mentioned above, integrals of motion in Lagrangian mechanics usually come from variational symmetries of a Lagrangian (Theorem 2.4.8), though not all integrals of motion are of this type (Section 2.4). In Hamiltonian mechanics, all integrals of motion are conserved generalized symmetry currents (Theorem 3.5.12 below).

An integral of motion of a Hamiltonian system \((\mathcal{V}^*Q, \mathcal{H})\) is defined as a smooth real function \(F\) on \(\mathcal{V}^*Q\) which is an integral of motion of the Hamilton equation (3.1.21) – (3.1.22) (Section 1.10). Its Lie derivative

\[ L_{\gamma_H} F = \partial_t F + \{\mathcal{H}, F\}_V \] (3.5.1)

along the Hamilton vector field \(\gamma_H\) (3.1.20) vanishes in accordance with the equation (1.10.7). Given the Hamiltonian vector field \(\partial_F\) of \(F\) with respect to the Poisson bracket (3.1.7), it is easily justified that

\[ [\gamma_H, \partial_F] = \partial_{L_{\gamma_H} F}. \] (3.5.2)

Consequently, the Hamiltonian vector field of an integral of motion is a symmetry of the Hamilton equation (3.1.21) – (3.1.22).

One can think of the formula (3.5.1) as being the evolution equation of Hamiltonian mechanics.

Given a Hamiltonian system \((\mathcal{V}^*Q, H)\), let \((T^*Q, \mathcal{H}^*)\) be an equivalent homogeneous Hamiltonian system. It follows from the equality (3.2.6) that

\[ \zeta^*(L_{\gamma_H} F) = \{\mathcal{H}^*, \zeta^* F\}_T = \zeta^*(\partial_t F + \{\mathcal{H}, F\}_V) \] (3.5.3)

for any function \(F \in C^\infty(\mathcal{V}^*Z)\). This formula is equivalent to the evolution equation (3.5.1). It is called the homogeneous evolution equation.

**Proposition 3.5.1**: A function \(F \in C^\infty(\mathcal{V}^*Q)\) is an integral of motion of a Hamiltonian system \((\mathcal{V}^*Q, H)\) iff its pull-back \(\zeta^* F\) onto \(T^*Q\) is an integral of motion of a homogeneous Hamiltonian system \((T^*Q, \mathcal{H}^*)\).

**Proof**: It follows from the equality (3.5.3) that

\[ \{\mathcal{H}^*, \zeta^* F\}_T = \zeta^*(L_{\gamma_H} F) = 0. \] (3.5.4)

*QED*

**Proposition 3.5.2**: If \(F\) and \(F'\) are integrals of motion of a Hamiltonian system, their Poisson bracket \(\{F, F'\}_V\) also is an integral of motion.

Consequently, integrals of motion of a Hamiltonian system \((\mathcal{V}^*Q, H)\) constitute a real Lie subalgebra of the Poisson algebra \(C^\infty(\mathcal{V}^*Q)\).
Let us turn to Hamiltonian conservation laws. We are based on the fact that the Hamilton equation (3.1.21) – (3.1.22) also is the Lagrange equation of the Lagrangian $L_H$ (3.3.1). Therefore, one can study conservation laws in Hamiltonian mechanics similarly to those in Lagrangian mechanics [15, 31].

Since the Hamilton equation (3.1.21) – (3.1.22) is of first order, we restrict our consideration to classical symmetries, i.e., vector fields on $V^*Q$. In this case, all conserved generalized symmetry currents are integrals of motion.

Let
\[ v = u^i \partial_i + v^i \partial_i + v_i \partial^i, \quad u^i = 0,1, \] (3.5.5)
be a vector field on a phase space $V^*Q$. Its prolongation onto $V^*Q \times Q JQ$ (Remark 3.3.1) reads
\[ J^1v = u^i \partial_i + v^i \partial_i + v_i \partial^i + d_t v^i \partial^i. \]

Then the first variational formula (2.4.11) for the Lagrangian $L_H$ (3.3.1) takes the form
\[ -u^i \partial_i \mathcal{H} - v^i \partial_i \mathcal{H} + v_i(q^j_t - \partial^j \mathcal{H}) + p_i d_t v^i = \]
\[ - (v^i - q^i_t u^t)(p_i + \partial_i \mathcal{H}) + (v_i - p_i u^t) (q^i_t - \partial^i \mathcal{H}) + d_t (p_i v^i - u^i \mathcal{H}). \] (3.5.6)

If $v$ (3.5.5) is a variational symmetry, i.e.,
\[ L_{J^1v}L_H = d_H \sigma, \]
we obtain the weak conservation law, called the Hamiltonian conservation law,
\[ 0 \approx d_t \Xi_v \] (3.5.7)
of the generalized symmetry current (2.4.17) which reads.
\[ \Xi_v = p_i v^i - u^i \mathcal{H} - \sigma. \] (3.5.8)
This current is an integral of motion of a Hamiltonian system.

The converse also is true. Let $F$ be an integral of motion, i.e.,
\[ L_{\gamma_H} F = \partial_t F + \{ \mathcal{H}, F \}_V = 0. \] (3.5.9)
We aim to show that there is a variational symmetry $v$ of $L_H$ such that $F = \Xi_v$ is a conserved generalized symmetry current along $v$.

In accordance with Proposition 2.4.4, the vector field $v$ (3.5.5) is a variational symmetry iff
\[ v^i (p_i + \partial_i \mathcal{H}) - v_i (q^j_t - \partial^j \mathcal{H}) + u^j \partial_j \mathcal{H} = d_t (\Xi_u + u^i \mathcal{H}). \] (3.5.10)
A glance at this equality shows the following.

**PROPOSITION 3.5.3**: The vector field \( \upsilon \) (3.5.5) is a variational symmetry only if

\[
\partial_i \upsilon_i = -\partial_i \upsilon^i. \tag{3.5.11}
\]

\[ \square \]

For instance, if the vector field \( \upsilon \) (3.5.5) is projectable onto \( Q \) (i.e., its components \( \upsilon^i \) are independent of momenta \( p_i \)), we obtain that \( u_i = -p_j \partial_i u^j \). Consequently, \( v \) is the canonical lift \( \bar{\upsilon} \) (3.3.3) onto \( V^*Q \) of the vector field \( u \) (2.4.4) on \( Q \). Moreover, let \( \bar{\upsilon} \) be a variational symmetry of a Lagrangian \( L_H \). It follows at once from the equality (3.5.10) that \( \bar{\upsilon} \) is an exact symmetry of \( L_H \). The corresponding conserved symmetry current reads

\[
\mathcal{F}_{\bar{\upsilon}p} = p_i u^i - u^t H. \tag{3.5.12}
\]

We agree to call the vector field \( u \) (2.4.4) the Hamiltonian symmetry if its canonical lift \( \bar{\upsilon} \) (3.3.3) onto \( V^*Q \) is a variational (consequently, exact) symmetry of the Lagrangian \( L_H \) (3.3.1). If a Hamiltonian symmetry is vertical, the corresponding conserved symmetry current \( \mathcal{F}_\upsilon = p_i u^i \) is called the Noether current.

**PROPOSITION 3.5.4**: The Hamilton vector field \( \gamma_H \) (3.1.20) is a unique variational symmetry of \( L_H \) whose conserved generalized symmetry current equals zero. \[ \square \]

It follows that, given a non-vertical variational symmetry \( \upsilon, u^t = 1 \), of a Lagrangian \( L_H \), there exists a vertical variational symmetry \( \upsilon - \gamma_H \) possessing the same generalized conserved symmetry current \( \mathcal{F}_\upsilon = \mathcal{F}_{\upsilon - \gamma_H} \) as \( \upsilon \).

**THEOREM 3.5.5**: Any integral of motion \( F \) of a Hamiltonian system \( (V^*Q, H) \) is a generalized conserved current \( F = \mathcal{F}_\partial F \) of the Hamiltonian vector field

\[
\partial_F \partial F \partial F - \partial_i F \partial F
\]

of \( F \). \[ \square \]

**Proof**: If \( \upsilon = \partial F \) and \( \mathcal{F}_{\partial F} = F \), the relation (3.5.10) is satisfied owing to the equality (3.5.9).

\[ QED \]

It follows from Theorem 3.5.5 that the Lie algebra of integrals of motion of a Hamiltonian system in Proposition 3.5.2 coincides with the Lie algebra of conserved generalized symmetry currents with respect to the bracket

\[
\{ F, F' \}_V = \{ \mathcal{F}_{\partial F}, \mathcal{F}_{\partial F'} \}_V = \mathcal{F}_{[\partial F, \partial F']}.\]

In accordance with Theorem 3.5.5, any integral of motion of a Hamiltonian system can be treated as a conserved generalized current along a vertical variational symmetry. However, this is not convenient for the study of energy conservation laws.
Let $E_\Gamma$ (3.1.17) be the energy function of a Hamiltonian system relative to a reference frame $\Gamma$. Given bundle coordinates adapted to $\Gamma$, its evolution equation (3.5.1) takes the form

$$L_{\gamma H} E_\Gamma = \partial_t E_\Gamma = \partial_t H.$$  

(3.5.13)

It follows that, an energy of a Hamiltonian system relative to a reference frame $\Gamma$ is an integral of motion iff a Hamiltonian, written with respect to the coordinates adapted to $\Gamma$, is time-independent. By virtue of Theorem 3.5.5, if $E_\Gamma$ is an integral of motion, it is a conserved generalized symmetry current of the variational symmetry

$$\gamma_H + \psi E_\Gamma = -(\partial_t + \Gamma^i \partial_i - p_j \partial_i \Gamma^j \partial^i) = -\tilde{\Gamma}.$$  

This is the canonical lift (3.3.3) onto $V^*Q$ of the vector field $-\Gamma$ (1.1.16) on $Q$. Consequently, $-\Gamma$ is an exact symmetry, and $-\Gamma$ is a Hamiltonian symmetry.

**Example 3.5.1:** Let us consider the Kepler system on the configuration space $Q$ (2.4.28) in Example 2.4.5. Its phase space is

$$V^*Q = \mathbb{R} \times \mathbb{R}^6$$

coordinated by $(t, q^i, p_i)$. The Lagrangian (2.4.29) and (2.4.33) of the Kepler system is hyper-regular. The associated Hamiltonian form reads

$$H = p_i dq^i - \left[ \frac{1}{2} \sum_i (p_i)^2 - \frac{1}{r} \right] dt.$$  

(3.5.14)

The corresponding Lagrangian $L_H$ (3.3.1) is

$$L_H = \left[ p_i q^i_t - \frac{1}{2} \left( \sum_i (p_i)^2 \right) + \frac{1}{r} \right] dt.$$  

(3.5.15)

The Kepler system possesses the following integrals of motion:

- an energy function $\mathcal{E} = \mathcal{H}$;
- orbital momenta
  $$M^a_b = q^a p_b - q^b p_a$$  
  (3.5.16)
- components of the Rung–Lenz vector
  $$A^a = \sum_b (q^a p_b - q^b p_a) p_b - \frac{q^a}{r}.$$  
  (3.5.17)

These integrals of motions are the conserved currents of:

- the exact symmetry $\partial_t$,
- the exact vertical symmetries
  $$v^a_b = q^a \partial_b - q^b \partial_a - p_b \partial^a + p_a \partial^b,$$  
  (3.5.18)
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- the variational vertical symmetries

\[
v^a = \sum_b [p_b \nu^a_b + (q^a p_b - q^b p_a) \partial_b] + \partial_b \left( \frac{q^a}{r} \right) \partial^b, \tag{3.5.19}
\]

respectively. Note that the vector fields \( \nu^a_b \) (3.5.18) are the canonical lift (3.3.3) onto \( V^* Q \) of the vector fields

\[ u^a_b = q^a \partial_b - q^b \partial_a \]

on \( Q \). Thus, these vector fields are vertical Hamiltonian symmetries, and integrals of motion \( M^a_b \) (3.5.16) are the Noether currents. \( \Diamond \)

Let us remind that, in contrast with the Rung–Lenz vector (3.5.19) in Hamiltonian mechanics, the Rung–Lenz vector (2.4.34) in Lagrangian mechanics fails to come from variational symmetries of a Lagrangian. There is the following relation between Lagrangian and Hamiltonian symmetries if they are the same vector fields on a configuration space \( Q \).

**Theorem 3.5.6:** Let a Hamiltonian form \( H \) be associated with an almost regular Lagrangian \( L \). Let \( r \) be a solution of the Hamilton equation (3.1.21) – (3.1.22) for \( H \) which lives in the Lagrangian constraint space \( N_L \). Let \( s = \pi_{\Pi} \circ r \) be the corresponding solution of the Lagrange equation for \( L \) so that the relation (3.4.12) holds. Then, for any vector field \( u \) (2.4.4) on a fibre bundle \( Q \to \mathbb{R} \), we have

\[
\mathcal{L}_u^\Pi(r) = \mathcal{L}_u(\pi_{\Pi} \circ r), \quad \mathcal{L}_u^\Pi(\hat{L} \circ J^1 s) = \mathcal{L}_u(s), \tag{3.5.20}
\]

where \( \mathcal{L}_u \) is the symmetry current (2.4.21) on \( J^1 Y \) and \( \mathcal{L}_u^\Pi \) is the symmetry current (3.5.12) on \( V^* Q \). \( \square \)

By virtue of Theorems 3.4.3 – 3.4.4, it follows that:

- if \( \mathcal{L}_u \) in Theorem 3.5.6 is a conserved symmetry current, then the symmetry current \( \mathcal{L}_u^\Pi \) (3.5.20) is conserved on solutions of the Hamilton equation which live in the Lagrangian constraint space;
- if \( \mathcal{L}_u^\Pi \) in Theorem 3.5.6 is a conserved symmetry current, then the symmetry current \( \mathcal{L}_u \) (3.5.20) is conserved on solutions \( s \) of the Lagrange equation which obey the condition (3.4.12).
Chapter 4
Appendixes

For the sake of convenience of the reader, this Chapter summarizes the relevant material on differential geometry of fibre bundles and modules over commutative rings [14, 18, 30, 44].

4.1 Geometry of fibre bundles

Throughout this Section, all morphisms are smooth (i.e., of class $C^\infty$), and manifolds are smooth real and finite-dimensional. A smooth manifold is customarily assumed to be Hausdorff and second-countable. Consequently, it is locally compact and paracompact. Unless otherwise stated, manifolds are assumed to be connected (and, consequently, arc-wise connected).

Given a smooth manifold $Z$, by $\pi_Z : TZ \to Z$ is denoted its tangent bundle. Given manifold coordinates $(z^\alpha)$ on $Z$, the tangent bundle $TZ$ is equipped with the holonomic coordinates

$$(z^\lambda, \dot{z}^\lambda), \quad \dot{z}^\lambda = \frac{\partial z^\lambda}{\partial z^\mu} \dot{z}^\mu,$$

with respect to the holonomic frames $\{\partial_\lambda\}$ in the tangent spaces to $Z$. Any manifold morphism $f : Z \to Z'$ yields the tangent morphism

$$Tf : TZ \to TZ', \quad \dot{z}^\lambda \circ Tf = \frac{\partial f^\lambda}{\partial z^\mu} \dot{z}^\mu,$$

of their tangent bundles.

4.1.1 Fibred manifolds

Let $M$ and $N$ be smooth manifolds and $f : M \to N$ a manifold morphism. Its rank $\text{rank}_p f$ at a point $p \in M$ is defined as the rank of the tangent map

$$T_p f : T_p M \to T_{f(p)} N, \quad p \in M.$$
Since the function \( p \rightarrow \text{rank}_p f \) is lower semicontinuous, a manifold morphism \( f \) of maximal rank at a point \( p \) also is of maximal rank on some open neighborhood of \( p \). A morphism \( f \) is said to be an immersion if \( T_p f, p \in M \), is injective and a submersion if \( T_p f, p \in M \), is surjective. Note that a submersion is an open map (i.e., an image of any open set is open).

If \( f : M \to N \) is an injective immersion, its range is called a submanifold of \( N \). A submanifold is said to be imbedded if it also is a topological subspace. In this case, \( f \) is called an imbedding. For the sake of simplicity, we usually identify \( (M, f) \) with \( f(M) \). If \( M \subset N \), its natural injection is denoted by \( i_M : M \to N \).

There are the following criteria for a submanifold to be imbedded.

**Theorem 4.1.1**: Let \( (M, f) \) be a submanifold of \( N \).

(i) A map \( f \) is an imbedding iff, for each point \( p \in M \), there exists a (cubic) coordinate chart \( (V, \psi) \) of \( N \) centered at \( f(p) \) so that \( f(M) \cap V \) consists of all points of \( V \) with coordinates \((x^1, \ldots, x^m, 0, \ldots, 0)\).

(ii) Suppose that \( f : M \to N \) is a proper map, i.e., the inverse images of compact sets are compact. Then \( (M, f) \) is a closed imbedded submanifold of \( N \). In particular, this occurs if \( M \) is a compact manifold.

(iii) If \( \dim M = \dim N \), then \( (M, f) \) is an open imbedded submanifold of \( N \). \( \square \)

If a manifold morphism

\[
\pi : Y \to X, \quad \dim X = n > 0,
\]

is a surjective submersion, one says that: (i) its domain \( Y \) is a fibred manifold, (ii) \( X \) is its base, (iii) \( \pi \) is a fibration, and (iv) \( Y_x = \pi^{-1}(x) \) is a fibre over \( x \in X \).

By virtue of the inverse function theorem [47], the surjection (4.1.1) is a fibred manifold iff a manifold \( Y \) admits an atlas of fibred coordinate charts \((U_Y; x^\lambda, y^j)\) such that \((x^\lambda)\) are coordinates on \( \pi(U_Y) \subset X \) and coordinate transition functions read

\[
x^\lambda = f^\lambda(x^\mu), \quad y^i = f^i(x^\mu, y^j).
\]

The surjection \( \pi \) (4.1.1) is a fibred manifold iff, for each point \( y \in Y \), there exists a local section \( s \) of \( Y \to X \) passing through \( y \). Recall that by a local section of the surjection (4.1.1) is meant an injection \( s : U \to Y \) of an open subset \( U \subset X \) such that \( \pi \circ s = \text{Id} U \), i.e., a section sends any point \( x \in X \) into the fibre \( Y_x \) over this point. A local section also is defined over any subset \( N \subset X \) as the restriction to \( N \) of a local section over an open set containing \( N \). If \( U = X \), one calls \( s \) the global section. A range \( s(U) \) of a local section \( s : U \to Y \) of a fibred manifold \( Y \to X \) is an imbedded submanifold of \( Y \). A local section is a closed map, sending closed subsets of \( U \) onto closed subsets of \( Y \). If \( s \) is a global section, then \( s(X) \) is a closed imbedded submanifold of \( Y \). Global sections of a fibred manifold need not exist.
4.1. **GEOMETRY OF FIBRE BUNDLES**

**Theorem 4.1.2**: Let $Y \to X$ be a fibred manifold whose fibres are diffeomorphic to $\mathbb{R}^m$. Any its section over a closed imbedded submanifold (e.g., a point) of $X$ is extended to a global section [44]. In particular, such a fibred manifold always has a global section. □

Given fibred coordinates $(U_Y; x^\lambda, y^i)$, a section $s$ of a fibred manifold $Y \to X$ is represented by collections of local functions $\{s^i = y^i \circ s\}$ on $\pi(U_Y)$.

Morphisms of fibred manifolds, by definition, are fibrewise morphisms, sending a fibre to a fibre. Namely, a fibred morphism of a fibred manifold $\pi : Y \to X$ to a fibred manifold $\pi' : Y' \to X'$ is defined as a pair $(\Phi, f)$ of manifold morphisms which form a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi} & Y' \\
\pi \downarrow & & \downarrow \pi' \\
X & \xrightarrow{f} & X'
\end{array}
\]

Fibred injections and surjections are called monomorphisms and epimorphisms, respectively. A fibred diffeomorphism is called an isomorphism or an automorphism if it is an isomorphism to itself. For the sake of brevity, a fibred morphism over $f = \text{Id} X$ usually is said to be a fibred morphism over $X$, and is denoted by $Y \to X$. In particular, a fibred automorphism over $X$ is called a vertical automorphism.

### 4.1.2 Fibre bundles

A fibred manifold $Y \to X$ is said to be trivial if $Y$ is isomorphic to the product $X \times V$. Different trivializations of $Y \to X$ differ from each other in surjections $Y \to V$.

A fibred manifold $Y \to X$ is called a fibre bundle if it is locally trivial, i.e., if it admits a fibred coordinate atlas $\{((\pi^{-1}(U_\xi); x^\lambda, y^i))\}$ over a cover $\{\pi^{-1}(U_\xi)\}$ of $Y$ which is the inverse image of a cover $\mathcal{U} = \{U_\xi\}$ of $X$. In this case, there exists a manifold $V$, called a typical fibre, such that $Y$ is locally diffeomorphic to the splittings

\[
\psi_\xi : \pi^{-1}(U_\xi) \to U_\xi \times V,
\]

glued together by means of transition functions

\[
\varrho_{\xi\zeta} = \psi_\xi \circ \psi_\zeta^{-1} : U_\xi \cap U_\zeta \times V \to U_\zeta \cap U_\xi \times V
\]

on overlaps $U_\xi \cap U_\zeta$. Transition functions $\varrho_{\xi\zeta}$ fulfil the cocycle condition

\[
\varrho_{\xi\zeta} \circ \varrho_{\zeta\iota} = \varrho_{\xi\iota}
\]

on all overlaps $U_\xi \cap U_\zeta \cap U_\iota$. Restricted to a point $x \in X$, trivialization morphisms $\psi_\xi$ (4.1.2) and transition functions $\varrho_{\xi\zeta}$ (4.1.3) define diffeomorphisms of fibres

\[
\psi_\xi(x) : Y_x \to V, \quad x \in U_\xi,
\]

\[
\varrho_{\xi\zeta}(x) : V \to V, \quad x \in U_\xi \cap U_\zeta.
\]
Trivialization charts \((U_\xi, \psi_\xi)\) together with transition functions \(g_{\xi\zeta}\) (4.1.3) constitute a bundle atlas
\[
\Psi = \{(U_\xi, \psi_\xi), g_{\xi\zeta}\}
\] (4.1.7)
of a fibre bundle \(Y \to X\). Two bundle atlases are said to be equivalent if their union also is a bundle atlas, i.e., there exist transition functions between trivialization charts of different atlases. All atlases of a fibre bundle are equivalent.

Given a bundle atlas \(\Psi\) (4.1.7), a fibre bundle \(Y\) is provided with the fibred coordinates
\[
x^\lambda(y) = (x^\lambda \circ \pi)(y), \quad y^i(y) = (y^i \circ \psi_\xi)(y), \quad y \in \pi^{-1}(U_\xi),
\]
called the bundle coordinates, where \(y^i\) are coordinates on a typical fibre \(V\).

A fibre bundle \(Y \to X\) is uniquely defined by a bundle atlas. Given an atlas \(\Psi\) (4.1.7), there exists a unique manifold structure on \(Y\) for which \(\pi : Y \to X\) is a fibre bundle with a typical fibre \(V\) and a bundle atlas \(\Psi\).

There are the following useful criteria for a fibred manifold to be a fibre bundle.

**Theorem 4.1.3**: If a fibration \(\pi : Y \to X\) is a proper map, then \(Y \to X\) is a fibre bundle. In particular, a compact fibred manifold is a fibre bundle. \(\square\)

**Theorem 4.1.4**: A fibred manifold whose fibres are diffeomorphic either to a compact manifold or \(\mathbb{R}^r\) is a fibre bundle [34]. \(\square\)

A comprehensive relation between fibred manifolds and fibre bundles is given in Remark 4.3.1. It involves the notion of an Ehresmann connection.

Forthcoming Theorems 4.1.5 – 4.1.7 describe the particular covers which one can choose for a bundle atlas [18].

**Theorem 4.1.5**: Any fibre bundle over a contractible base is trivial. \(\square\)

Note that a fibred manifold over a contractible base need not be trivial. It follows from Theorem 4.1.5 that any cover of a base \(X\) by domains (i.e., contractible open subsets) is a bundle cover.

**Theorem 4.1.6**: Every fibre bundle \(Y \to X\) admits a bundle atlas over a countable cover \(\Omega\) of \(X\) where each member \(U_\xi\) of \(\Omega\) is a domain whose closure \(\overline{U_\xi}\) is compact. \(\square\)

If a base \(X\) is compact, there is a bundle atlas of \(Y\) over a finite cover of \(X\) which obeys the condition of Theorem 4.1.6.

**Theorem 4.1.7**: Every fibre bundle \(Y \to X\) admits a bundle atlas over a finite cover \(\Omega\) of \(X\), but its members need not be contractible and connected. \(\square\)

A fibred morphism of fibre bundles is called a bundle morphism. A bundle monomorphism \(\Phi : Y \to Y''\) over \(X\) onto a submanifold \(\Phi(Y)\) of \(Y''\) is called a subbundle of a fibre bundle \(Y'' \to X\). There is the following useful criterion for an image and an inverse image of a bundle morphism to be subbundles.
4.1. GEOMETRY OF FIBRE BUNDLES

Theorem 4.1.8: Let $\Phi : Y \to Y'$ be a bundle morphism over $X$. Given a global section $s$ of the fibre bundle $Y' \to X$ such that $s(X) \subset \Phi(Y)$, by the kernel of a bundle morphism $\Phi$ with respect to a section $s$ is meant the inverse image

$$\text{Ker}_s \Phi = \Phi^{-1}(s(X))$$

of $s(X)$ by $\Phi$. If $\Phi : Y \to Y'$ is a bundle morphism of constant rank over $X$, then $\Phi(Y)$ and Ker$_s \Phi$ are subbundles of $Y'$ and $Y$, respectively. □

The following are the standard constructions of new fibre bundles from old ones.

- Given a fibre bundle $\pi : Y \to X$ and a manifold morphism $f : X' \to X$, the pull-back of $Y$ by $f$ is called the manifold

$$f^*Y = \{ (x', y) \in X' \times Y : \pi(y) = f(x') \}$$

(4.1.8)

together with the natural projection $(x', y) \to x'$. It is a fibre bundle over $X'$ such that the fibre of $f^*Y$ over a point $x' \in X'$ is that of $Y$ over the point $f(x') \in X$. There is the canonical bundle morphism

$$f_Y : f^*Y \ni (x', y)_{|\pi(y)=f(x')} \to y \in Y.$$  

(4.1.9)

Any section $s$ of a fibre bundle $Y \to X$ yields the pull-back section

$$f^*s(x') = (x', s(f(x'))$$

of $f^*Y \to X'$.

- If $X' \subset X$ is a submanifold of $X$ and $i_{X'}$ is the corresponding natural injection, then the pull-back bundle

$$i_{X'}^*Y = Y|_{X'}$$

is called the restriction of a fibre bundle $Y$ to the submanifold $X' \subset X$. If $X'$ is an imbedded submanifold, any section of the pull-back bundle

$$Y|_{X'} \to X'$$

is the restriction to $X'$ of some section of $Y \to X$.

- Let $\pi : Y \to X$ and $\pi' : Y' \to X$ be fibre bundles over the same base $X$. Their bundle product $Y \times_X Y'$ over $X$ is defined as the pull-back

$$Y \times Y' = \pi^*Y' \quad \text{or} \quad Y \times Y' = \pi'^*Y'$$

(4.1.10)

together with its natural surjection onto $X$. Fibres of the bundle product $Y \times Y'$ are the Cartesian products $Y_x \times Y'_x$ of fibres of fibre bundles $Y$ and $Y'$.

- Let us consider the composite fibre bundle

$$Y \to \Sigma \to X.$$
4.1.3 Vector and affine bundles

A fibre bundle \( \pi : Y \to X \) is called a vector bundle if both its typical fibre and fibres are finite-dimensional real vector spaces, and if it admits a bundle atlas whose trivialization morphisms and transition functions are linear isomorphisms. Then the corresponding bundle coordinates on \( Y \) are linear bundle coordinates \( (y^i) \) possessing linear transition functions

\[
y' = A^i_j(x)y^j, \quad \pi(y) \in U_\xi, \quad (4.1.11)
\]

where \( \{e_i\} \) is a fixed basis for a typical fibre \( V \) of \( Y \) and \( \{e_i(x)\} \) are the fibre bases (or the frames) for the fibres \( Y_x \) of \( Y \) associated to a bundle atlas \( \Psi \).

By virtue of Theorem 4.1.2, any vector bundle has a global section, e.g., the canonical global zero-valued section \( \hat{0}(x) = 0 \).

**Theorem 4.1.9**: Let a vector bundle \( Y \to X \) admit \( m = \dim V \) nowhere vanishing global sections \( s_i \) which are linearly independent, i.e., \( m \wedge s_i \neq 0 \). Then \( Y \) is trivial. \( \square \)

Global sections of a vector bundle \( Y \to X \) constitute a projective \( C^\infty(X) \)-module \( Y(X) \) of finite rank. It is called the structure module of a vector bundle. The well-known Serre–Swan theorem [12] states the categorial equivalence between the vector bundles over a smooth manifold \( X \) and projective \( C^\infty(X) \)-modules of finite rank.

There are the following particular constructions of new vector bundles from the old ones.

- Let \( Y \to X \) be a vector bundle with a typical fibre \( V \). By \( Y^* \to X \) is denoted the dual vector bundle with the typical fibre \( V^* \), dual of \( V \). The interior product of \( Y \) and \( Y^* \) is defined as a fibred morphism

\[
\mathcal{I} : Y \otimes X Y^* \to X \times \mathbb{R}.
\]

- Let \( Y \to X \) and \( Y' \to X \) be vector bundles with typical fibres \( V \) and \( V' \), respectively. Their Whitney sum \( Y \oplus_X Y' \) is a vector bundle over \( X \) with the typical fibre \( V \oplus V' \).
- Let \( Y \to X \) and \( Y' \to X \) be vector bundles with typical fibres \( V \) and \( V' \), respectively. Their tensor product \( Y \otimes_X Y' \) is a vector bundle over \( X \) with the typical fibre \( V \otimes V' \).

Similarly, the exterior product of vector bundles \( Y \wedge_X Y' \) is defined. The exterior product

\[
\wedge Y = X \times \mathbb{R} \oplus_X Y \oplus_X Y \wedge_X \cdots \oplus_X Y, \quad k = \dim Y - \dim X, \quad (4.1.12)
\]

is called the exterior bundle.

- If \( Y' \) is a subbundle of a vector bundle \( Y \to X \), the factor bundle \( Y/Y' \) over \( X \) is defined as a vector bundle whose fibres are the quotients \( Y_x/Y'_x, \ x \in X \).

By a morphism of vector bundles is meant a linear bundle morphism, which is a linear fibrewise map whose restriction to each fibre is a linear map.
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Given a linear bundle morphism \( \Phi : Y' \to Y \) of vector bundles over \( X \), its kernel \( \text{Ker } \Phi \) is defined as the inverse image \( \Phi^{-1}(\hat{0}(X)) \) of the canonical zero-valued section \( \hat{0}(X) \) of \( Y \). By virtue of Theorem 4.1.8, if \( \Phi \) is of constant rank, its kernel and its range are vector subbundles of the vector bundles \( Y' \) and \( Y \), respectively. For instance, monomorphisms and epimorphisms of vector bundles fulfill this condition.

**Remark 4.1.1:** Given vector bundles \( Y \) and \( Y' \) over the same base \( X \), every linear bundle morphism \( \Phi : Y \ni \{ e_i(x) \} \to \{ \Phi_i^k(x) e'_k(x) \} \in Y' \)
over \( X \) defines a global section

\[
\Phi : x \to \Phi_i^k(x) e^i(x) \otimes e'_k(x)
\]
of the tensor product \( Y \otimes Y^* \), and vice versa. \( \triangleleft \)

A sequence \( Y' \xrightarrow{i} Y \xrightarrow{j} Y'' \) of vector bundles over the same base \( X \) is called exact at \( Y \) if \( \text{Ker } j = \text{Im } i \). A sequence of vector bundles

\[
0 \to Y' \xrightarrow{i} Y \xrightarrow{j} Y'' \to 0
\]
over \( X \) is said to be a short exact sequence if it is exact at all terms \( Y', Y \), and \( Y'' \). This means that \( i \) is a bundle monomorphism, \( j \) is a bundle epimorphism, and \( \text{Ker } j = \text{Im } i \). Then \( Y'' \) is isomorphic to a factor bundle \( Y/Y' \). Given an exact sequence of vector bundles (4.1.13), there is the exact sequence of their duals

\[
0 \to Y''^* \xrightarrow{j^*} Y^* \xrightarrow{i^*} Y'^* \to 0.
\]

One says that the exact sequence (4.1.13) is split if there exists a bundle monomorphism \( s : Y'' \to Y \) such that \( j \circ s = \text{Id } Y'' \) or, equivalently,

\[
Y = i(Y') \oplus s(Y'') = Y' \oplus Y''.
\]

**Theorem 4.1.10:** Every exact sequence of vector bundles (4.1.13) is split [19]. \( \square \)

The tangent bundle \( TZ \) and the cotangent bundle \( T^*Z \) of a manifold \( Z \) exemplify vector bundles. Given an atlas \( \Psi_Z = \{(U_i, \phi_i)\} \) of a manifold \( Z \), the tangent bundle is provided with the holonomic bundle atlas

\[
\Psi_T = \{(U_i, \psi_i = T\phi_i)\}.
\]

(4.1.14)
The associated linear bundle coordinates are holonomic coordinates \((\dot{z}^\lambda)\).

The cotangent bundle of a manifold \( Z \) is the dual \( T^*Z \to Z \) of the tangent bundle \( TZ \to Z \). It is equipped with the holonomic coordinates

\[
(z^\lambda, \dot{z}_\lambda).
\]

\[
\dot{z}_\lambda = \frac{\partial z^\mu}{\partial z^\lambda} \dot{z}_\mu,
\]
with respect to the coframes \( \{dz^\lambda\} \) for \( T^*Z \) which are the duals of \( \{\partial_\lambda\} \).

The tensor product of tangent and cotangent bundles

\[
T = (\otimes T^*Z) \otimes (\otimes T^*Z), \quad m, k \in \mathbb{N},
\]

(4.1.15)
is called a tensor bundle, provided with holonomic bundle coordinates \( \dot{z}^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_k} \) possessing transition functions

\[
\dot{z}^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_k} = \frac{\partial z^{\alpha_1}}{\partial \dot{x}^{\mu_1}} \cdots \frac{\partial z^{\alpha_m}}{\partial \dot{x}^{\mu_1}} \frac{\partial \dot{z}^{\nu_1}}{\partial \dot{x}^{\nu_1}} \cdots \frac{\partial \dot{z}^{\nu_k}}{\partial \dot{x}^{\nu_k}} \dot{z}^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_k}.
\]

Let \( \pi_Y : TY \to Y \) be the tangent bundle of a fibred manifold \( \pi : Y \to X \). Given fibred coordinates \( (x^\lambda, y^i) \) on \( Y \), it is equipped with the holonomic coordinates \( (x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i) \). The tangent bundle \( TY \to Y \) has the subbundle \( VY = \text{Ker}(T\pi) \), which consists of the vectors tangent to fibres of \( Y \). It is called the vertical tangent bundle of \( Y \), and it is provided with the holonomic coordinates \( (x^\lambda, y^i, \dot{y}^i) \) with respect to the vertical frames \( \{\dot{\partial}_i\} \). Every fibred morphism \( \Phi : Y \to Y' \) yields the linear bundle morphism over \( \Phi \) of the vertical tangent bundles

\[
V\Phi : VY \to VY', \quad \dot{y}^i \circ V\Phi = \frac{\partial \Phi^i}{\partial y^j} \dot{y}^j. 
\]

(4.1.16)

It is called the vertical tangent morphism.

In many important cases, the vertical tangent bundle \( VY \to Y \) of a fibre bundle \( Y \to X \) is trivial, and it is isomorphic to the bundle product

\[
VY = Y \times_X Y',
\]

(4.1.17)

where \( Y' \to X \) is some vector bundle. One calls (4.1.17) the vertical splitting. For instance, every vector bundle \( Y \to X \) admits the canonical vertical splitting

\[
VY = Y \oplus_X Y.
\]

(4.1.18)

The vertical cotangent bundle \( V^*Y \to Y \) of a fibred manifold \( Y \to X \) is defined as the dual of the vertical tangent bundle \( VY \to Y \). It is not a subbundle of the cotangent bundle \( T^*Y \), but there is the canonical surjection

\[
\zeta : T^*Y \ni \dot{x}^\lambda dx^\lambda + \dot{y}_i dy^i \to \dot{y}_i dy^i \in V^*Y,
\]

(4.1.19)

where the bases \( \{\bar{dy}^i\} \), possessing transition functions

\[
\bar{dy}^i = \frac{\partial y^i}{\partial y^j} \bar{dy}^j,
\]

are the duals of the vertical frames \( \{\partial_i\} \) of the vertical tangent bundle \( VY \).
For any fibred manifold $Y$, there exist the exact sequences of vector bundles
\[
0 \to VY \longrightarrow TY \xrightarrow{\pi_T} Y \times TX \to 0, \quad (4.1.20)
\]
\[
0 \to Y \times T^*X \to T^*Y \to V^*Y \to 0. \quad (4.1.21)
\]
Their splitting, by definition, is a connection on $Y \to X$ (Section 4.3.1).

Let us consider the tangent bundle $TT^*X$ of $T^*X$ and the cotangent bundle $T^*TX$ of $TX$. Relative to coordinates $(x^\lambda, p_\lambda = \dot{x}_\lambda)$ on $T^*X$ and $(x^\lambda, \dot{x}^\lambda, \ddot{x}_\lambda, \dot{p}_\lambda)$ and $(x^\lambda, \dot{x}_\lambda, \ddot{x}_\lambda, \dddot{x}_\lambda, \dddot{p}_\lambda)$, respectively. By inspection of the coordinate transformation laws, one can show that there is an isomorphism
\[
\alpha : TT^*X = T^*TX, \quad p_\lambda \leftrightarrow \dddot{x}_\lambda, \quad \dot{p}_\lambda \leftrightarrow \dot{x}_\lambda \quad (4.1.22)
\]
of these bundles over $TX$. Given a fibred manifold $Y \to X$, there is the similar isomorphism
\[
\alpha_V : VV^*Y = V^*VY, \quad p_i \leftrightarrow \dddot{y}_i, \quad \dot{p}_i \leftrightarrow \dot{y}_i \quad (4.1.23)
\]
over $VV^*Y$, where $(x^\lambda, y^i, p_i, \dot{y}_i, \dot{p}_i)$ and $(x^\lambda, \dot{y}_i, \dddot{y}_i, \dddot{y}_i, \dddot{p}_i)$ are coordinates on $VV^*Y$ and $V^*VY$, respectively.

Let $\pi : \overline{Y} \to X$ be a vector bundle with a typical fibre $\overline{V}$. An affine bundle modelled over the vector bundle $Y \to X$ is a fibre bundle $\pi : Y \to X$ whose typical fibre $V$ is an affine space modelled over $\overline{V}$, all the fibres $Y_x$ of $Y$ are affine spaces modelled over the corresponding fibres $\overline{Y}_x$ of the vector bundle $\overline{Y}$, and there is an affine bundle atlas
\[
\Psi = \{(U_\alpha, \psi_\lambda), \varphi_{\chi\zeta}\}
\]
of $Y \to X$ whose local trivializations morphisms $\psi_\lambda$ (4.1.5) and transition functions $\varphi_{\chi\zeta}$ (4.1.6) are affine isomorphisms.

Dealing with affine bundles, we use only affine bundle coordinates $(y^i)$ associated to an affine bundle atlas $\Psi$. There are the bundle morphisms
\[
Y \times \overline{Y} \xrightarrow{\pi_Y} Y, \quad (y^i, \overline{y}^j) \to y^i + \overline{y}^j,
\]
\[
Y \times Y \xrightarrow{\pi_Y} \overline{Y}, \quad (y^i, \overline{y}^j) \to y^i - \overline{y}^j,
\]
where $(\overline{y}^j)$ are linear coordinates on a vector bundle $\overline{Y}$.

By virtue of Theorem 4.1.2, affine bundles have global sections, but in contrast with vector bundles, there is no canonical global section of an affine bundle. Let $\pi : Y \to X$ be an affine bundle. Every global section $s$ of an affine bundle $Y \to X$ modelled over a vector bundle $\overline{Y} \to X$ yields the bundle morphisms
\[
Y \ni y \to y - s(\pi(y)) \in \overline{Y}, \quad (4.1.24)
\]
\[
\overline{Y} \ni \overline{y} \to s(\pi(y)) + \overline{y} \in Y. \quad (4.1.25)
\]
In particular, every vector bundle $Y$ has a natural structure of an affine bundle due to the morphisms (4.1.25) where $s = \hat{0}$ is the canonical zero-valued section of $Y$.

**Theorem 4.1.11**: Any affine bundle $Y \to X$ admits bundle coordinates $(x^\lambda, \tilde{y}^i)$ possessing linear transition functions $\tilde{y}^i = A^i_j(x)\tilde{y}^j$ [14]. □

By a morphism of affine bundles is meant a bundle morphism $\Phi : Y \to Y'$ whose restriction to each fibre of $Y$ is an affine map. It is called an affine bundle morphism. Every affine bundle morphism $\Phi : Y \to Y'$ of an affine bundle $Y$ modelled over a vector bundle $\overline{Y}$ to an affine bundle $Y'$ modelled over a vector bundle $\overline{Y}$ yields an unique linear bundle morphism

$$\overline{\Phi} : \overline{Y} \to \overline{Y'}, \quad \overline{\tilde{y}}^i \circ \overline{\Phi} = \frac{\partial \Phi^i}{\partial \tilde{y}^j} \overline{\tilde{y}}^j, \quad (4.1.26)$$

called the linear derivative of $\Phi$.

Every affine bundle $Y \to X$ modelled over a vector bundle $\overline{Y} \to X$ admits the canonical vertical splitting

$$VY = Y \times \overline{Y_X}. \quad (4.1.27)$$

### 4.1.4 Vector and multivector fields

Vector fields on a manifold $Z$ are global sections of the tangent bundle $TZ \to Z$.

The set $T_1(Z)$ of vector fields on $Z$ is both a $C^\infty(Z)$-module and a real Lie algebra with respect to the Lie bracket

$$u = u^\lambda \partial_\lambda, \quad v = v^\lambda \partial_\lambda, \quad [v, u] = (v^\lambda \partial_\lambda u^\mu - u^\lambda \partial_\lambda v^\mu)\partial_\mu.$$

Given a vector field $u$ on $X$, a curve

$$c : \mathbb{R} \supset (, ) \to Z$$

in $Z$ is said to be an integral curve of $u$ if $Tc = u(c)$. Every vector field $u$ on a manifold $Z$ can be seen as an infinitesimal generator of a local one-parameter group of local diffeomorphisms (a flow), and *vice versa* [22]. One-dimensional orbits of this group are integral curves of $u$.

A vector field is called complete if its flow is a one-parameter group of diffeomorphisms of $Z$.

**Theorem 4.1.12**: Any vector field on a compact manifold is complete. □

A vector field $u$ on a fibred manifold $Y \to X$ is called projectable if it is projected onto a vector field on $X$, i.e., there exists a vector field $\tau$ on $X$ such that

$$\tau \circ \pi = T\pi \circ u.$$
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A projectable vector field takes the coordinate form
\[ u = u^\lambda (x^\mu) \partial_\lambda + u^i (x^\mu, y^j) \partial_i, \quad \tau = u^\lambda \partial_\lambda. \]  
\hspace{1cm} (4.1.28)

A projectable vector field is called vertical if its projection onto \( X \) vanishes, i.e., if it lives in the vertical tangent bundle \( VY \).

A vector field \( \tau = \tau^\lambda \partial_\lambda \) on a base \( X \) of a fibred manifold \( Y \to X \) gives rise to a vector field on \( Y \) by means of a connection on this fibre bundle (see the formula (4.3.3)). Nevertheless, every tensor bundle (4.1.15) admits the functorial lift of vector fields
\[ \tilde{\tau} = \tau^\mu \partial_\mu + [\partial_\nu \tau^{\alpha_1} \dot{x}_{\beta_1\cdots\beta_k} + \ldots - \partial_{\beta_1} \tau^\nu \dot{x}_{\alpha_1\cdots\alpha_m} - \ldots] \dot{\delta}_{\alpha_1\cdots\alpha_m}, \]  
\hspace{1cm} (4.1.29)

where we employ the compact notation
\[ \dot{\delta}_\lambda = \frac{\partial}{\partial \dot{x}^\lambda}. \]  
\hspace{1cm} (4.1.30)

This lift is an \( \mathbb{R} \)-linear monomorphism of the Lie algebra \( T_1(X) \) of vector fields on \( X \) to the Lie algebra \( T_1(Y) \) of vector fields on \( Y \). In particular, we have the functorial lift
\[ \tilde{\tau} = \tau^\mu \partial_\mu + \partial_\nu \tau^{\alpha_1} \frac{\partial}{\partial \dot{x}^\alpha} \]  
\hspace{1cm} (4.1.31)

of vector fields on \( X \) onto the tangent bundle \( TX \) and their functorial lift
\[ \tilde{\tau} = \tau^\mu \partial_\mu - \partial_\beta \tau^\nu \dot{x}_\nu \frac{\partial}{\partial \dot{x}_\beta} \]  
\hspace{1cm} (4.1.32)

onto the cotangent bundle \( T^*X \).

Let \( Y \to X \) be a vector bundle. Using the canonical vertical splitting (4.1.18), we obtain the canonical vertical vector field
\[ u_Y = y^i \partial_i \]  
\hspace{1cm} (4.1.33)

on \( Y \), called the Liouville vector field. For instance, the Liouville vector field on the tangent bundle \( TX \) reads
\[ u_{TX} = \dot{x}^\lambda \dot{\delta}_\lambda. \]  
\hspace{1cm} (4.1.34)

Accordingly, any vector field \( \tau = \tau^\lambda \partial_\lambda \) on a manifold \( X \) has the canonical vertical lift
\[ \tau_V = \tau^\lambda \dot{\delta}_\lambda \]  
\hspace{1cm} (4.1.35)

onto the tangent bundle \( TX \).

A multivector field \( \vartheta \) of degree \(|\vartheta| = r \) (or, simply, an \( r \)-vector field) on a manifold \( Z \) is a section
\[ \vartheta = \frac{1}{r!} \vartheta^{\lambda_1\cdots\lambda_r} \partial_{\lambda_1} \wedge \cdots \wedge \partial_{\lambda_r} \]  
\hspace{1cm} (4.1.36)
of the exterior product \( \wedge T^*Z \rightarrow Z \). Let \( \mathcal{T}_r(Z) \) denote the \( C^\infty(Z) \)-module space of \( r \)-vector fields on \( Z \). All multivector fields on a manifold \( Z \) make up the graded commutative algebra \( \mathcal{T}_*(Z) \) of global sections of the exterior bundle \( \wedge T^*Z \) (4.1.12) with respect to the exterior product \( \wedge \).

Given an \( r \)-vector field \( \vartheta \) (4.1.36) on a manifold \( Z \), its tangent lift \( \tilde{\vartheta} \) onto the tangent bundle \( T^*Z \) of \( Z \) is defined by the relation

\[
\tilde{\vartheta}(\tilde{\sigma}^r, \ldots, \tilde{\sigma}^1) = \vartheta(\sigma^r, \ldots, \sigma^1)
\]  

(4.1.37)

where [17]:

- \( \sigma^k = \sigma^k_\lambda dz^\lambda \) are arbitrary one-forms on a manifold \( Z \),
- by

\[
\tilde{\sigma}^k = \dot{z}^\mu \partial_\mu \sigma^k_\lambda dz^\lambda + \sigma^k_\lambda \dot{z}^\lambda
\]

are meant their tangent lifts (4.1.41) onto the tangent bundle \( T^*Z \) of \( Z \),

- the right-hand side of the equality (4.1.37) is the tangent lift (4.1.39) onto \( T^*Z \) of the function \( \vartheta(\sigma^r, \ldots, \sigma^1) \) on \( Z \).

The tangent lift (4.1.37) takes the coordinate form

\[
\tilde{\vartheta} = \frac{1}{r!} [\dot{z}^\mu \partial_\mu \vartheta^{\lambda_1 \ldots \lambda_r} \partial_{\lambda_1} \wedge \cdots \wedge \partial_{\lambda_r} + \\
\vartheta^{\lambda_1 \ldots \lambda_r} \sum_{i=1}^r \partial_{\lambda_i} \wedge \cdots \wedge \partial_{\lambda_i} \wedge \cdots \wedge \partial_{\lambda_r}].
\]

(4.1.38)

In particular, if \( \tau \) is a vector field on a manifold \( Z \), its tangent lift (4.1.38) coincides with the functorial lift (4.1.31).

### 4.1.5 Differential forms

An exterior \( r \)-form on a manifold \( Z \) is a section

\[
\phi = \frac{1}{r!} \phi_{\lambda_1 \ldots \lambda_r} dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r}
\]

of the exterior product \( \wedge T^*Z \rightarrow Z \), where

\[
dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r} = \frac{1}{r!} \epsilon^{\lambda_1 \ldots \lambda_r}_{\mu_1 \ldots \mu_r} dz^{\mu_1} \otimes \cdots \otimes dz^{\mu_r},
\]

\[
\epsilon^{\ldots \lambda_i \ldots \lambda_j \ldots \ldots \mu_p \ldots \mu_k \ldots} = -\epsilon^{\ldots \lambda_j \ldots \lambda_i \ldots \ldots \mu_p \ldots \mu_k \ldots} = -\epsilon^{\ldots \lambda_i \ldots \lambda_j \ldots \ldots \mu_p \ldots \mu_k \ldots},
\]

\[
\epsilon^{\lambda_1 \ldots \lambda_r}_{\lambda_1 \ldots \lambda_r} = 1.
\]

Sometimes, it is convenient to write

\[
\phi = \phi'_{\lambda_1 \ldots \lambda_r} dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r}
\]
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without the coefficient $1/r!$.

Let $\mathcal{O}^r(Z)$ denote the $C^\infty(Z)$-module of exterior $r$-forms on a manifold $Z$. By definition, $\mathcal{O}^0(Z) = C^\infty(Z)$ is the ring of smooth real functions on $Z$. All exterior forms on $Z$ constitute the graded algebra $\mathcal{O}^*(Z)$ of global sections of the exterior bundle $\bigwedge T^*Z$ (4.1.12) endowed with the exterior product

$$
\phi = \frac{1}{r!} \phi_{\lambda_1 \ldots \lambda_r} dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r}, \quad \sigma = \frac{1}{s!} \sigma_{\mu_1 \ldots \mu_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_s},
$$

$$
\phi \wedge \sigma = \frac{1}{r! s!} \phi_{\nu_1 \ldots \nu_r} \sigma_{\nu_{r+1} \ldots \nu_{r+s}} dz^{\nu_1} \wedge \cdots \wedge dz^{\nu_{r+s}} = \frac{1}{r! s!(r+s)!} \epsilon^{\nu_1 \ldots \nu_{r+s}} \alpha_1 \ldots \alpha_{r+s} \phi_{\nu_1 \ldots \nu_r} \sigma_{\nu_{r+1} \ldots \nu_{r+s}} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{r+s}},
$$

such that

$$
\phi \wedge \sigma = (-1)^{|\phi||\sigma|} \sigma \wedge \phi,
$$

where the symbol $|\phi|$ stands for the form degree. The algebra $\mathcal{O}^*(Z)$ also is provided with the exterior differential

$$
d\phi = dz^\mu \wedge \partial_\mu \phi = \frac{1}{r!} \partial_\mu \phi_{\lambda_1 \ldots \lambda_r} dz^\mu \wedge dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r},
$$

which obeys the relations

$$
d \circ d = 0, \quad d(\phi \wedge \sigma) = d(\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge d(\sigma).
$$

The exterior differential $d$ makes $\mathcal{O}^*(Z)$ into a differential graded algebra, called the exterior algebra.

Given a manifold morphism $f : Z \to Z'$, any exterior $k$-form $\phi$ on $Z'$ yields the pull-back exterior form $f^* \phi$ on $Z$ given by the condition

$$
f^* \phi(v^1, \ldots, v^k)(z) = \phi(T_f(v^1), \ldots, T_f(v^k))(f(z))
$$

for an arbitrary collection of tangent vectors $v^1, \ldots, v^k \in T_z Z$. We have the relations

$$
f^*(\phi \wedge \sigma) = f^* \phi \wedge f^* \sigma, \quad df^* \phi = f^*(d \phi).
$$

In particular, given a fibred manifold $\pi : Y \to X$, the pull-back onto $Y$ of exterior forms on $X$ by $\pi$ provides the monomorphism of graded commutative algebras $\mathcal{O}^*(X) \to \mathcal{O}^*(Y)$. Elements of its range $\pi^*\mathcal{O}^*(X)$ are called basic forms. Exterior forms

$$
\phi : Y \to \bigwedge T^*X, \quad \phi = \frac{1}{r!} \phi_{\lambda_1 \ldots \lambda_r} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r},
$$

on $Y$ such that $u^\diamond \phi = 0$ for an arbitrary vertical vector field $u$ on $Y$ are said to be horizontal forms. Horizontal forms of degree $n = \text{dim } X$ are called densities.
In the case of the tangent bundle $TX \to X$, there is a different way to lift exterior forms on $X$ onto $TX$ [17, 25]. Let $f$ be a function on $X$. Its tangent lift onto $TX$ is defined as the function

$$\tilde{f} = \dot{x}^\lambda \partial_\lambda f.$$  \hfill (4.1.39)

Let $\sigma$ be an $r$-form on $X$. Its tangent lift onto $TX$ is said to be the $r$-form $\tilde{\sigma}$ given by the relation

$$\tilde{\sigma}(\tilde{\tau}_1, \ldots, \tilde{\tau}_r) = \sigma(\tau_1, \ldots, \tau_r),$$  \hfill (4.1.40)

where $\tau_i$ are arbitrary vector fields on $X$ and $\tilde{\tau}_i$ are their functorial lifts (4.1.31) onto $TX$.

We have the coordinate expression

$$\sigma = \frac{1}{r!} \sigma_{\lambda_1 \ldots \lambda_r} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r},$$

$$\tilde{\sigma} = \frac{1}{r!} [\dot{x}^\mu \partial_\mu \sigma_{\lambda_1 \ldots \lambda_r} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} + \sum_{i=1}^{r} \sigma_{\lambda_1 \ldots \lambda_r} dx^{\lambda_1} \wedge \cdots \wedge d\dot{x}^{\lambda_i} \wedge \cdots \wedge dx^{\lambda_r}].$$  \hfill (4.1.41)

The following equality holds:

$$d\tilde{\sigma} = \tilde{d}\sigma.$$

The interior product (or contraction) of a vector field $u$ and an exterior $r$-form $\phi$ on a manifold $Z$ is given by the coordinate expression

$$u \lrcorner \phi = \sum_{k=1}^{r} \frac{(-1)^{k-1}}{r!} u^\lambda \phi_{\lambda_1 \ldots \lambda_k \ldots \lambda_r} dz^{\lambda_1} \wedge \cdots \wedge d\dot{z}^{\lambda_k} \wedge \cdots \wedge dz^{\lambda_r} = \frac{1}{(r-1)!} u^\mu \phi_{\mu \alpha_2 \ldots \alpha_r} dz^{\alpha_2} \wedge \cdots \wedge dz^{\alpha_r},$$

where the caret $\hat{}$ denotes omission. It obeys the relations

$$\phi(u_1, \ldots, u_r) = u_r \lrcorner \cdots \lrcorner u_1 \lrcorner \phi,$$

$$u \lrcorner (\phi \wedge \sigma) = u \lrcorner \phi \wedge \sigma + (-1)^{|\phi|} \phi \wedge u \lrcorner \sigma.$$  \hfill (4.1.42)

A generalization of the interior product to multivector fields is the left interior product

$$\vartheta \lrcorner \phi = \phi(\vartheta), \quad |\vartheta| \leq |\phi|, \quad \phi \in \mathcal{O}^*(Z), \quad \vartheta \in \mathcal{T}_r(Z),$$

of multivector fields and exterior forms. It is defined by the equalities

$$\phi(u_1 \wedge \cdots \wedge u_r) = \phi(u_1, \ldots, u_r), \quad \phi \in \mathcal{O}^*(Z), \quad u_i \in \mathcal{T}_1(Z),$$
and obeys the relation
\[ \vartheta \wedge \varphi \wedge = (\varphi \wedge \vartheta) \wedge (\vartheta \wedge \varphi) = (-1)^{|\vartheta||\varphi|} \varphi, \vartheta, \varphi \in \mathcal{O}^*(Z), \vartheta, \varphi \in \mathcal{T}_*(Z). \]

The Lie derivative of an exterior form \( \phi \) along a vector field \( u \) is
\[ L_u \phi = u \dagger d \phi + d(u \dagger \phi), \]
\[ L_u (\phi \wedge \sigma) = L_u \phi \wedge \sigma + \phi \wedge L_u \sigma. \]

In particular, if \( f \) is a function, then
\[ L_u f = u \dagger (d f). \]

An exterior form \( \phi \) is invariant under a local one-parameter group of diffeomorphisms \( G_t \) of \( Z \) (i.e., \( G_t^* \phi = \phi \)) iff its Lie derivative along the infinitesimal generator \( u \) of this group vanishes, i.e.,
\[ L_u \phi = 0. \]

Following physical terminology (Definition 1.10.6), we say that a vector field \( u \) is a symmetry of an exterior form \( \phi \).

A tangent-valued \( r \)-form on a manifold \( Z \) is a section
\[ \phi = \frac{1}{r!} \phi^\mu_{\lambda_1...\lambda_r} dz^{\lambda_1} \wedge ... \wedge dz^{\lambda_r} \otimes \partial_\mu \]

of the tensor bundle
\[ \wedge^r T^*Z \otimes TZ \rightarrow Z. \]

**Remark 4.1.2:** There is one-to-one correspondence between the tangent-valued one-forms \( \phi \) on a manifold \( Z \) and the linear bundle endomorphisms
\[ \hat{\phi} : TZ \rightarrow TZ, \quad \hat{\phi} : T_z Z \ni v \rightarrow \phi(z) \in T_z Z, \]
\[ \hat{\phi}^* : T^*Z \rightarrow T^*Z, \quad \hat{\phi}^* : T^*_z Z \ni v^* \rightarrow \phi(z) | v^* \in T^*_z Z, \]

over \( Z \) (Remark 4.1.1). For instance, the canonical tangent-valued one-form
\[ \theta_Z = dz^\lambda \otimes \partial_\lambda \]
on \( Z \) corresponds to the identity morphisms (4.1.46) and (4.1.47).

**Remark 4.1.3:** Let \( Z = TX \), and let \( TTX \) be the tangent bundle of \( TX \). It is called the double tangent bundle. There is the bundle endomorphism
\[ J(\partial_\lambda) = \hat{\partial}_\lambda, \quad J(\hat{\partial}_\lambda) = 0 \]
of \( TTX \) over \( X \). It corresponds to the canonical tangent-valued form
\[
\theta_J = dx^\lambda \otimes \partial_\lambda
\]  
(4.1.50)
on the tangent bundle \( TX \). It is readily observed that \( J \circ J = 0 \). ◯

The space \( \mathcal{O}^s(Z) \otimes T_1(Z) \) of tangent-valued forms is provided with the Frölicher–Nijenhuis bracket
\[
[\cdot, \cdot]_{FN} : \mathcal{O}^s(Z) \otimes T_1(Z) \times \mathcal{O}^s(Z) \otimes T_1(Z) \to \mathcal{O}^{s+1}(Z) \otimes T_1(Z),
\]
\[
[\alpha \otimes u, \beta \otimes v]_{FN} = (\alpha \wedge \beta) \otimes [u, v] + (\alpha \otimes L_u \beta) \otimes v - (L_v \alpha \wedge \beta) \otimes u + (-1)^r(d\alpha \wedge u) \beta \otimes v + (-1)^r(v \wedge d\beta) \otimes u,
\]
\[
\alpha \in \mathcal{O}^s(Z), \quad \beta \in \mathcal{O}^s(Z), \quad u, v \in T_1(Z).
\]
Its coordinate expression is
\[
[\phi, \sigma]_{FN} = \frac{1}{r!s!}(\phi_{\lambda_1, \ldots, \lambda_r} \sigma_\mu^{\lambda_1, \ldots, \lambda_{r+s}} - \sigma_\mu^{\lambda_1, \ldots, \lambda_{r+s}} \phi_{\lambda_1, \ldots, \lambda_r} - r \phi_{\lambda_1, \ldots, \lambda_{r-1}, \lambda_r} \sigma_\mu^{\lambda_1, \ldots, \lambda_{r+s}} \partial_\mu \phi_{\lambda_1, \ldots, \lambda_r} - s \sigma_{\nu \lambda_1, \ldots, \lambda_{r-1}, \lambda_r} \phi^{\nu \lambda_1, \ldots, \lambda_{r+s}} \partial_\nu \sigma_{\lambda_1, \ldots, \lambda_r}, d\lambda^\lambda \wedge \cdots \wedge d\lambda^{\lambda+s} \otimes \partial_\mu,
\]
\[
\phi \in \mathcal{O}^s(Z) \otimes T_1(Z), \quad \sigma \in \mathcal{O}^s(Z) \otimes T_1(Z).
\]
There are the relations
\[
[\phi, \sigma]_{FN} = (-1)^{|\phi||\sigma|+1}[\sigma, \phi]_{FN},
\]
\[
[\phi, [\sigma, \theta]_{FN}]_{FN} = [[\phi, \sigma]_{FN}, \theta]_{FN} + (-1)^{|\phi||\sigma|}[\sigma, [\phi, \theta]_{FN}]_{FN},
\]
\[
\phi, \sigma, \theta \in \mathcal{O}^s(Z) \otimes T_1(Z).
\]

Given a tangent-valued form \( \theta \), the Nijenhuis differential on \( \mathcal{O}^s(Z) \otimes T_1(Z) \) is defined as the morphism
\[
d\theta : \psi \to d\theta \psi = [\theta, \psi]_{FN}, \quad \psi \in \mathcal{O}^s(Z) \otimes T_1(Z).
\]
By virtue of (4.1.53), it has the property
\[
d\phi[\psi, \theta]_{FN} = [d\phi \psi, \theta]_{FN} + (-1)^{|\phi||\psi|}[\psi, d\phi \theta]_{FN}.
\]
In particular, if \( \theta = u \) is a vector field, the Nijenhuis differential is the Lie derivative of tangent-valued forms
\[
L_u \sigma = d_u \sigma = [u, \sigma]_{FN} = \frac{1}{s!}(u^\nu \partial_\nu \sigma_\mu^{\lambda_1, \ldots, \lambda_s} - \sigma_\mu^{\lambda_1, \ldots, \lambda_s} \partial_\nu u^\mu + s \sigma_{\nu \lambda_2, \ldots, \lambda_s} \partial_\nu \sigma_{\lambda_1, \ldots, \lambda_s} dx^\lambda \wedge \cdots \wedge dx^\lambda \otimes \partial_\mu), \quad \sigma \in \mathcal{O}^s(Z) \otimes T_1(Z).
\]

Let \( Y \to X \) be a fibred manifold. We consider the following subspaces of the space \( \mathcal{O}^s(Y) \otimes T_1(Y) \) of tangent-valued forms on \( Y \):
• horizontal tangent-valued forms
\[ \phi : Y \to \wedge T^* X \otimes T Y, \]
\[ \phi = dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \frac{1}{r!} \left[ \phi^\mu_{\lambda_1...\lambda_r} (y) \partial_\mu + \phi^i_{\lambda_1...\lambda_r} (y) \partial_i \right], \]

• projectable horizontal tangent-valued forms
\[ \phi = dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \frac{1}{r!} \left[ \phi^\mu_{\lambda_1...\lambda_r} (x) \partial_\mu + \phi^i_{\lambda_1...\lambda_r} (y) \partial_i \right], \]

• vertical-valued form
\[ \phi : Y \to \wedge T^* X \otimes V Y, \]
\[ \phi = \frac{1}{r!} \phi^i_{\lambda_1...\lambda_r} (y) dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \partial_i, \]

• vertical-valued one-forms, called soldering forms,
\[ \sigma = \sigma^i_{\lambda} (y) dx^{\lambda} \otimes \partial_i, \quad (4.1.54) \]

• basic soldering forms
\[ \sigma = \sigma^i_{\lambda} (x) dx^{\lambda} \otimes \partial_i. \]

**Remark 4.1.4:** The tangent bundle \( TX \) is provided with the canonical soldering form \( \theta_I \) (4.1.50). Due to the canonical vertical splitting
\[ VTX = TX \times TX, \quad (4.1.55) \]
the canonical soldering form (4.1.50) on \( TX \) defines the canonical tangent-valued form \( \theta_X \) (4.1.48) on \( X \). By this reason, tangent-valued one-forms on a manifold \( X \) also are called soldering forms. ◊

We also mention the \( TX \)-valued forms
\[ \phi : Y \to \wedge T^* X \otimes TX, \quad (4.1.56) \]
\[ \phi = \frac{1}{r!} \phi^i_{\lambda_1...\lambda_r} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \partial_i, \]

and \( V^* Y \)-valued forms
\[ \phi : Y \to \wedge T^* X \otimes V^* Y, \quad (4.1.57) \]
\[ \phi = \frac{1}{r!} \phi^i_{\lambda_1...\lambda_r} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes dy^i. \]

It should be emphasized that (4.1.56) are not tangent-valued forms, while (4.1.57) are not exterior forms. They exemplify vector-valued forms. Given a vector bundle \( E \to X \), by a \( E \)-valued \( k \)-form on \( X \), is meant a section of the fibre bundle
\[ (\wedge T^* X) \otimes E^* \to X. \]
4.1.6 Distributions and foliations

A subbundle $T$ of the tangent bundle $TZ$ of a manifold $Z$ is called a regular distribution (or, simply, a distribution). A vector field $u$ on $Z$ is said to be subordinate to a distribution $T$ if it lives in $T$. A distribution $T$ is called involutive if the Lie bracket of $T$-subordinate vector fields also is subordinate to $T$.

A subbundle of the cotangent bundle $T^*Z$ of $Z$ is called a codistribution $T^*$ on a manifold $Z$. For instance, the annihilator $\text{Ann}T$ of a distribution $T$ is a codistribution whose fibre over $z \in Z$ consists of covectors $w \in T^*_z$ such that $v \rfloor w = 0$ for all $v \in T_z$.

There is the following criterion of an involutive distribution [47].

**Theorem 4.1.13**: Let $T$ be a distribution and $\text{Ann}T$ its annihilator. Let $\land \text{Ann}T(Z)$ be the ideal of the exterior algebra $O^*(Z)$ which is generated by sections of $\text{Ann}T \to Z$. A distribution $T$ is involutive iff the ideal $\land \text{Ann}T(Z)$ is a differential ideal, i.e.,

$$d(\land \text{Ann}T(Z)) \subset \land \text{Ann}T(Z).$$

The following local coordinates can be associated to an involutive distribution [47].

**Theorem 4.1.14**: Let $T$ be an involutive $r$-dimensional distribution on a manifold $Z$, $\dim Z = k$. Every point $z \in Z$ has an open neighborhood $U$ which is a domain of an adapted coordinate chart $(z^1, \ldots, z^k)$ such that, restricted to $U$, the distribution $T$ and its annihilator $\text{Ann}T$ are spanned by the local vector fields $\partial/\partial z^1, \ldots, \partial/\partial z^r$ and the local one-forms $dz^{r+1}, \ldots, dz^k$, respectively.

A connected submanifold $N$ of a manifold $Z$ is called an integral manifold of a distribution $T$ on $Z$ if $TN \subset T$. Unless otherwise stated, by an integral manifold is meant an integral manifold of dimension of $T$. An integral manifold is called maximal if no other integral manifold contains it. The following is the classical theorem of Frobenius [22, 47].

**Theorem 4.1.15**: Let $T$ be an involutive distribution on a manifold $Z$. For any $z \in Z$, there exists a unique maximal integral manifold of $T$ through $z$, and any integral manifold through $z$ is its open subset.

Maximal integral manifolds of an involutive distribution on a manifold $Z$ are assembled into a regular foliation $F$ of $Z$.

A regular $r$-dimensional foliation (or, simply, a foliation) $F$ of a $k$-dimensional manifold $Z$ is defined as a partition of $Z$ into connected $r$-dimensional submanifolds (the leaves of a foliation) $F_i, i \in I$, which possesses the following properties [37, 45].

A manifold $Z$ admits an adapted coordinate atlas

$$\{(U_\xi; z^\lambda, z^i)\}, \quad \lambda = 1, \ldots, k - r, \quad i = 1, \ldots, r, \quad (4.1.58)$$

such that transition functions of coordinates $z^\lambda$ are independent of the remaining coordinates $z^i$. For each leaf $F$ of a foliation $F$, the connected components of $F \cap U_\xi$ are
given by the equations $z^\lambda = \text{const}$. These connected components and coordinates $(z^i)$ on them make up a coordinate atlas of a leaf $F$. It follows that tangent spaces to leaves of a foliation $\mathcal{F}$ constitute an involutive distribution $TF$ on $Z$, called the tangent bundle to the foliation $\mathcal{F}$. The factor bundle

$$VF = TZ/TF;$$

called the normal bundle to $\mathcal{F}$, has transition functions independent of coordinates $z^i$. Let $TF^* \to Z$ denote the dual of $TF \to Z$. There are the exact sequences

$$0 \to TF \xrightarrow{i_F} TX \longrightarrow VF \to 0, \quad (4.1.59)$$

$$0 \to \text{Ann} TF \longrightarrow T^*X \xrightarrow{i^*_F} TF^* \to 0 \quad (4.1.60)$$

of vector bundles over $Z$.

A pair $(Z, \mathcal{F})$, where $\mathcal{F}$ is a foliation of $Z$, is called a foliated manifold. It should be emphasized that leaves of a foliation need not be closed or imbedded submanifolds. Every leaf has an open saturated neighborhood $U$, i.e., if $z \in U$, then a leaf through $z$ also belongs to $U$.

Any submersion $\zeta : Z \to M$ yields a foliation

$$\mathcal{F} = \{F_p = \zeta^{-1}(p)\}_{p \in \zeta(Z)}$$

of $Z$ indexed by elements of $\zeta(Z)$, which is an open submanifold of $M$, i.e., $Z \to \zeta(Z)$ is a fibred manifold. Leaves of this foliation are closed imbedded submanifolds. Such a foliation is called simple. Any (regular) foliation is locally simple.

### 4.2 Jet manifolds

This Section collects the relevant material on jet manifolds of sections of fibre bundles [14, 23, 30, 43].

#### 4.2.1 First order jet manifolds

Given a fibre bundle $Y \to X$ with bundle coordinates $(x^\lambda, y^i)$, let us consider the equivalence classes $j^1_xs$ of its sections $s$, which are identified by their values $s^i(x)$ and the values of their partial derivatives $\partial_\mu s^i(x)$ at a point $x \in X$. They are called the first order jets of sections at $x$. One can justify that the definition of jets is coordinate-independent. A key point is that the set $J^1Y$ of first order jets $j^1_xs$, $x \in X$, is a smooth manifold with respect to the adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$ such that

$$y^i_\lambda(j^1_xs) = \partial_\lambda s^i(x), \quad y^i_\lambda = \frac{\partial x^\mu}{\partial x^\mu_\lambda}(\partial_\mu + y^j_\mu \partial_j)y^i. \quad (4.2.1)$$
It is called the first order jet manifold of a fibre bundle $Y \to X$. We call $(y^i_\lambda)$ the jet coordinate.

A jet manifold $J^1 Y$ admits the natural fibrations

$$\pi^1 : J^1 Y \ni j^1_x s \to x \in X, \quad (4.2.2)$$
$$\pi^0 : J^1 Y \ni j^1_x s \to s(x) \in Y. \quad (4.2.3)$$

A glance at the transformation law (4.2.1) shows that $\pi^0_1$ is an affine bundle modelled over the vector bundle

$$T^* X \otimes V_Y \to Y. \quad (4.2.4)$$

It is convenient to call $\pi^1$ (4.2.2) the jet bundle, while $\pi^0_1$ (4.2.3) is said to be the affine jet bundle.

Let us note that, if $Y \to X$ is a vector or an affine bundle, the jet bundle $\pi^1_1$ (4.2.2) is so.

Jets can be expressed in terms of familiar tangent-valued forms as follows. There are the canonical imbeddings

$$\lambda^{(1)} : J^1 Y \to T^* X \otimes T_Y Y,$$
$$\lambda^{(1)} = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) = dx^\lambda \otimes d\lambda, \quad (4.2.5)$$
$$\theta^{(1)} : J^1 Y \to T^* Y \otimes V_Y,$$
$$\theta^{(1)} = (dy^i - y^i_\lambda dx^\lambda) \otimes \partial_i = \theta^i \otimes \partial_i, \quad (4.2.6)$$

where $d\lambda$ are said to be total derivatives, and $\theta^i$ are called contact forms.

We further identify the jet manifold $J^1 Y$ with its images under the canonical morphisms (4.2.5) and (4.2.6), and represent the jets $j^1_x s = (x^\lambda, y^i, y^i_\mu)$ by the tangent-valued forms $\lambda^{(1)}$ (4.2.5) and $\theta^{(1)}$ (4.2.6).

Sections and morphisms of fibre bundles admit prolongations to jet manifolds as follows.

Any section $s$ of a fibre bundle $Y \to X$ has the jet prolongation to the section

$$(J^1 s)(x) = j^1_x s, \quad y^i_\lambda \circ J^1 s = \partial_\lambda s^i(x),$$

of the jet bundle $J^1 Y \to X$. A section of the jet bundle $J^1 Y \to X$ is called integrable if it is the jet prolongation of some section of a fibre bundle $Y \to X$.

Any bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism $f$ admits a jet prolongation to a bundle morphism of affine jet bundles

$$J^1 \Phi : J^1 Y \xrightarrow{\Phi} J^1 Y', \quad y^i_\lambda \circ J^1 \Phi = \frac{\partial(f^{-1})^\mu}{\partial x^\lambda} d\mu \Phi^i. \quad (4.2.7)$$
4.2. JET MANIFOLDS

Any projectable vector field $u$ (4.1.28) on a fibre bundle $Y \to X$ has a jet prolongation to the projectable vector field

$$J^1u = r_1 \circ J^1u : J^1Y \to J^1TY \to TJ^1Y,$$

$$J^1u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y^j_\mu \partial_\lambda u^\mu) \partial^j_\lambda,$$  

(4.2.8)
on the jet manifold $J^1Y$. In order to obtain (4.2.8), the canonical bundle morphism

$$r_1 : J^1TY \to TJ^1Y,$$

$$\dot{y}^i_\lambda \circ r_1 = (\dot{y}^i)_\lambda - y^j_\mu \dot{x}^j_\lambda$$
is used. In particular, there is the canonical isomorphism

$$VJ^1Y = J^1VY,$$

$$\dot{y}^i_\lambda = (\dot{y}^i)_\lambda.$$  

(4.2.9)

4.2.2 Second order jet manifolds

Taking the first order jet manifold of the jet bundle $J^1Y \to X$, we obtain the repeated jet manifold $J^1J^1Y$ provided with the adapted coordinates

$$(x^\lambda, y^i, \dot{y}^i_\lambda, \ddot{y}^i_\lambda, y^j_\mu, \dot{y}^j_\mu, \ddot{y}^j_\mu)$$
possessing transition functions

$$y^i_\lambda = \frac{\partial x^\alpha}{\partial x^\alpha} d_\alpha y^i, \quad \dot{y}^i_\lambda = \frac{\partial x^\alpha}{\partial x^\alpha} \dot{d}_\alpha y^i, \quad y^j_\mu = \frac{\partial x^\alpha}{\partial x^\alpha} \ddot{d}_\alpha y^j_\lambda,$$

$$d_\alpha = \partial_\alpha + y^i_\alpha \partial_i + y^j_\nu \partial_j^\nu, \quad \dot{d}_\alpha = \partial_\alpha + \ddot{y}^i_\alpha \partial_i + \ddot{y}^j_\nu \partial_j^\nu.$$There exist two different affine fibrations of $J^1J^1Y$ over $J^1Y$:

- the familiar affine jet bundle (4.2.3):

$$\pi_{11} : J^1J^1Y \to J^1Y, \quad y^i_\lambda \circ \pi_{11} = y^i_\lambda,$$

(4.2.10)

- the affine bundle

$$J^1\pi^1_0 : J^1J^1Y \to J^1Y, \quad y^i_\lambda \circ J^1\pi^1_0 = \ddot{y}^i_\lambda.$$  

(4.2.11)

In general, there is no canonical identification of these fibrations. The points $q \in J^1J^1Y$, where

$$\pi_{11}(q) = J^1\pi^1_0(q),$$
form an affine subbundle $J^2Y \to J^1Y$ of $J^1J^1Y$ called the sesquiholonomic jet manifold. It is given by the coordinate conditions $\ddot{y}^i_\lambda = y^i_\lambda$, and is coordinated by $(x^\lambda, y^i, \dot{y}^i_\lambda, y^j_\mu, \dot{y}^j_\mu).$
The second order (or holonomic) jet manifold $J^2Y$ of a fibre bundle $Y \to X$ can be defined as the affine subbundle of the fibre bundle $\hat{J}^2Y \to J^1Y$ given by the coordinate conditions $y^i_{\lambda\mu} = y^i_{\mu\lambda}$. It is modelled over the vector bundle $\bigvee^2 T^*X \otimes VY \to J^1Y$, and is endowed with adapted coordinates $(x^\lambda, y^i, y^i_{\lambda}, y^i_{\lambda\mu})$, possessing transition functions

$$y^i_{\lambda}(j^2_x s) = \partial_{\lambda} s^i(x), \quad y^i_{\lambda\mu}(j^2_x s) = \partial_{\lambda\mu} s^i(x).$$

Theorem 4.2.1: Let $\overline{s}$ be a section of the jet bundle $J^1Y \to X$, and let $J^1\overline{s}$ be its jet prolongation to a section of the repeated jet bundle $J^1J^1Y \to X$. The following three facts are equivalent:

- $\overline{s} = J^1s$ where $s$ is a section of a fibre bundle $Y \to X$,
- $J^1\overline{s}$ takes its values into $\hat{J}^2Y$,
- $J^1\overline{s}$ takes its values into $J^2Y$.

$\square$

4.2.3 Higher order jet manifolds

The notion of first and second order jet manifolds is naturally extended to higher order jet manifolds.

The $k$-order jet manifold $J^kY$ of a fibre bundle $Y \to X$ comprises the equivalence classes $j^k_x s$, $x \in X$, of sections $s$ of $Y$ identified by the $k + 1$ terms of their Taylor series at points $x \in X$. The jet manifold $J^kY$ is provided with the adapted coordinates

$$(x^\lambda, y^i, y^i_{\lambda}, \ldots, y^i_{\lambda_2\ldots\lambda_1}),$$

$$y^i_{\lambda_1\ldots\lambda_l}(j^k_x s) = \partial_{\lambda_1} \cdots \partial_{\lambda_l} s^i(x), \quad 0 \leq l \leq k.$$
Every section $s$ of a fibre bundle $Y \to X$ gives rise to the section $J^k s$ of a fibre bundle $J^k Y \to X$ such that

$$y^i_{\lambda_1 \cdots \lambda_l} \circ J^k s = \partial_{\lambda_1} \cdots \partial_{\lambda_l} s^i, \quad 0 \leq l \leq k.$$ 

The following operators on exterior forms on jet manifolds are utilized:

- the total derivative operator

$$d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_\lambda \partial^\mu_i + \cdots, \quad (4.2.13)$$ 

obeying the relations

$$d_\lambda (\phi \wedge \sigma) = d_\lambda (\phi) \wedge \sigma + \phi \wedge d_\lambda (\sigma),$$

$$d_\lambda (d\phi) = d(d_\lambda (\phi)),$$

in particular,

$$d_\lambda (f) = \partial_\lambda f + y^i_\lambda \partial_i f + y^i_\lambda \partial^\mu_i f + \cdots, \quad f \in C^\infty(J^k Y),$$

$$d_\lambda (dx^\mu) = 0, \quad d_\lambda (dy^i_{\lambda_1 \cdots \lambda_l}) = dy^i_{\lambda_1 \cdots \lambda_l};$$

- the horizontal projection $h_0$ given by the relations

$$h_0(dx^\lambda) = dx^\lambda, \quad h_0(dy^i_{\lambda_1 \cdots \lambda_l}) = y^i_{\mu \lambda_1 \cdots \lambda_l} dx^\mu, \quad (4.2.14)$$

in particular,

$$h_0(dy^i) = y^i_j dx^\mu, \quad h_0(dy^i_\lambda) = y^i_\mu dx^\mu;$$

- the total differential

$$d_H(\phi) = dx^\lambda \wedge d_\lambda (\phi), \quad (4.2.15)$$

possessing the properties

$$d_H \circ d_H = 0, \quad h_0 \circ d = d_H \circ h_0.$$ 

**4.2.4 Differential operators and differential equations**

Jet manifolds provide the standard language for the theory of differential equations and differential operators [3, 8, 24].

**Definition 4.2.2:** Let $Z$ be an $(m+n)$-dimensional manifold. A system of $k$-order partial differential equations (or, simply, a differential equation) in $n$ variables on $Z$ is defined to be a closed smooth submanifold $E$ of the $k$-order jet bundle $J^k_n Z$ of $n$-dimensional submanifolds of $Z$. □
By its solution is meant an \( n \)-dimensional submanifold \( S \) of \( Z \) whose \( k \)-order jets \( [S]_z^k \), \( z \in S \), belong to \( \mathcal{E} \).

**DEFINITION 4.2.3:** A \( k \)-order differential equation in \( n \) variables on a manifold \( Z \) is called a dynamic equation if it can be algebraically solved for the highest order derivatives, i.e., it is a section of the fibration \( J^k_n Z \rightarrow J^{k-1}_n Z \). \( \square \)

In particular, a first order dynamic equation in \( n \) variables on a manifold \( Z \) is a section of the jet bundle \( J^1_n Z \rightarrow Z \). Its image in the tangent bundle \( TZ \rightarrow Z \) is an \( n \)-dimensional vector subbundle of \( TZ \). If \( n = 1 \), a dynamic equation is given by a vector field

\[
\dot{z}^\lambda(t) = u^\lambda(z(t)) \tag{4.2.16}
\]

on a manifold \( Z \). Its solutions are integral curves \( c(t) \) of the vector field \( u \).

Let \( Y \rightarrow X \) be a fibre bundle. There are several equivalent definitions of (non-linear) differential operators. We start with the following.

**DEFINITION 4.2.4:** Let \( E \rightarrow X \) be a vector bundle. A \( k \)-order \( E \)-valued differential operator on a fibre bundle \( Y \rightarrow X \) is defined as a section \( \mathcal{E} \) of the pull-back bundle

\[
\text{pr}_1 : E^k_Y \times_X E \rightarrow J^k_Y. \tag{4.2.17}
\]

\( \square \)

Given bundle coordinates \( (x^\lambda, y^i) \) on \( Y \) and \( (x^\lambda, \chi^a) \) on \( E \), the pull-back (4.2.17) is provided with coordinates \( (x^\lambda, y^i_\Sigma, \chi^a) \), \( 0 \leq |\Sigma| \leq k \). With respect to these coordinates, a differential operator \( \mathcal{E} \) seen as a closed imbedded submanifold \( \mathcal{E} \subset E^k_Y \) is given by the equalities

\[
\chi^a = \mathcal{E}^a(x^\lambda, y^i_\Sigma). \tag{4.2.18}
\]

There is obvious one-to-one correspondence between the sections \( \mathcal{E} \) (4.2.18) of the fibre bundle (4.2.17) and the bundle morphisms

\[
\Phi : J^k_Y \xrightarrow{\text{pr}_2} E, \quad \Phi = \text{pr}_2 \circ \mathcal{E} \iff \mathcal{E} \equiv (\text{Id} \cdot J^k_Y, \Phi). \tag{4.2.19}
\]

Therefore, we come to the following equivalent definition of differential operators on \( Y \rightarrow X \).

**DEFINITION 4.2.5:** A fibred morphism

\[
\mathcal{E} : J^k_Y \xrightarrow{\text{pr}_2} E \tag{4.2.20}
\]

is called a \( k \)-order differential operator on the fibre bundle \( Y \rightarrow X \). It sends each section \( s(x) \) of \( Y \rightarrow X \) onto the section \( (\mathcal{E} \circ J^k s)(x) \) of the vector bundle \( E \rightarrow X \) [3, 24]. \( \square \)
The kernel of a differential operator is the subset
\[ \text{Ker } \mathcal{E} = \mathcal{E}^{-1}(\hat{0}(X)) \subset J^kY, \] (4.2.21)
where \( \hat{0} \) is the zero section of the vector bundle \( E \to X \), and we assume that \( \hat{0}(X) \subset \mathcal{E}(J^kY) \).

**Definition 4.2.6:** A system of \( k \)-order partial differential equations (or, simply, a differential equation) on a fibre bundle \( Y \to X \) is defined as a closed subbundle \( \mathcal{E} \) of the jet bundle \( J^kY \to X \). \( \square \)

Its solution is a (local) section \( s \) of the fibre bundle \( Y \to X \) such that its \( k \)-order jet prolongation \( J^k s \) lives in \( \mathcal{E} \).

For instance, if the kernel (4.2.21) of a differential operator \( \mathcal{E} \) is a closed subbundle of the fibre bundle \( J^kY \to X \), it defines a differential equation
\[ \mathcal{E} \circ J^k s = 0. \]

The following condition is sufficient for a kernel of a differential operator to be a differential equation.

**Theorem 4.2.7:** Let the morphism (4.2.20) be of constant rank. Its kernel (4.2.21) is a closed subbundle of the fibre bundle \( J^kY \to X \) and, consequently, is a \( k \)-order differential equation. \( \square \)

### 4.3 Connections on fibre bundles

There are different equivalent definitions of a connection on a fibre bundle \( Y \to X \). We define it both as a splitting of the exact sequence (4.1.20) and a global section of the affine jet bundle \( J^1Y \to Y \) [14, 30, 43].

#### 4.3.1 Connections

A connection on a fibred manifold \( Y \to X \) is defined as a splitting (called the horizontal splitting)
\[ \Gamma : Y \times TX \to TY, \quad \Gamma : \dot{x}^\lambda \partial_\lambda \mapsto \dot{x}^\lambda (\partial_\lambda + \Gamma_i^\lambda(y) \partial_i), \] (4.3.1)
\[ \dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i = \dot{x}^\lambda (\partial_\lambda + \Gamma_i^\lambda(y) \partial_i) + (\dot{y}^i - \dot{x}^\lambda \Gamma_i^\lambda(y)) \partial_i. \]

of the exact sequence (4.1.20). Its range is a subbundle of \( TY \to Y \) called the horizontal distribution. By virtue of Theorem 4.1.10, a connection on a fibred manifold always exists. A connection \( \Gamma \) (4.3.1) is represented by the horizontal tangent-valued one-form
\[ \Gamma = dx_\lambda \otimes (\partial_\lambda + \Gamma_i^\lambda(y) \partial_i) \] (4.3.2)
on $Y$ which is projected onto the canonical tangent-valued form $\theta_X$ (4.1.48) on $X$.

Given a connection $\Gamma$ on a fibred manifold $Y \to X$, any vector field $\tau$ on a base $X$ gives rise to the projectable vector field

$$\Gamma \tau = \tau |_Y = \tau^\lambda (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

on $Y$ which lives in the horizontal distribution determined by $\Gamma$. It is called the horizontal lift of $\tau$ by means of a connection $\Gamma$.

The splitting (4.3.1) also is given by the vertical-valued form

$$\Gamma = (dy^i - \Gamma^i_\lambda dx^\lambda) \otimes \partial_i,$$

which yields an epimorphism $TY \to VY$. It provides the corresponding splitting

$$\Gamma : V^*Y \ni dy^i \mapsto dy^i - \Gamma^i_\lambda dx^\lambda \in T^*Y,$$

of the dual exact sequence (4.1.21).

In an equivalent way, connections on a fibred manifold $Y \to X$ are introduced as global sections of the affine jet bundle $J^1Y \to Y$. Indeed, any global section $\Gamma$ of $J^1Y \to Y$ defines the tangent-valued form $\lambda^1 \circ \Gamma$ (4.3.2). It follows from this definition that connections on a fibred manifold $Y \to X$ constitute an affine space modelled over the vector space of soldering forms $\sigma$ (4.1.54). One also deduces from (4.2.1) the coordinate transformation law of connections

$$\Gamma'_i^\lambda = \partial_x^\mu \partial_x'^\lambda (\partial_\mu + \Gamma^j_\mu \partial_j) y'^n.$$

**Remark 4.3.1:** Any connection $\Gamma$ on a fibred manifold $Y \to X$ yields a horizontal lift of a vector field on $X$ onto $Y$, but need not defines the similar lift of a path in $X$ into $Y$. Let

$$\mathbb{R} \ni [\cdot] \ni t \to x(t) \in X, \quad \mathbb{R} \ni t \to y(t) \in Y,$$

be smooth paths in $X$ and $Y$, respectively. Then $t \to y(t)$ is called a horizontal lift of $x(t)$ if

$$\pi(y(t)) = x(t), \quad \dot{y}(t) \in H_{y(t)}Y, \quad t \in \mathbb{R},$$

where $HY \subset TY$ is the horizontal subbundle associated to the connection $\Gamma$. If, for each path $x(t)$ ($t_0 \leq t \leq t_1$) and for any $y_0 \in \pi^{-1}(x(t_0))$, there exists a horizontal lift $y(t)$ ($t_0 \leq t \leq t_1$) such that $y(t_0) = y_0$, then $\Gamma$ is called the Ehresmann connection. A fibred manifold is a fibre bundle iff it admits an Ehresmann connection [18]. ◦

Hereafter, we restrict our consideration to connections on fibre bundles. The following are two standard constructions of new connections from old ones.
4.3. CONNECTIONS ON FIBRE BUNDLES

\- Let \( Y \) and \( Y' \) be fibre bundles over the same base \( X \). Given connections \( \Gamma \) on \( Y \) and \( \Gamma' \) on \( Y' \), the bundle product \( Y \times Y' \) is provided with the product connection

\[
\Gamma \times \Gamma' = dx^\lambda \otimes \left( \partial_\lambda + \Gamma^i_\lambda \frac{\partial}{\partial y^i} + \Gamma'^j_\lambda \frac{\partial}{\partial y'^j} \right). \tag{4.3.6}
\]

\- Given a fibre bundle \( Y \rightarrow X \), let \( f: X' \rightarrow X \) be a manifold morphism and \( f^*Y \) the pull-back of \( Y \) over \( X' \). Any connection \( \Gamma \) (4.3.4) on \( Y \rightarrow X \) yields the pull-back connection

\[
f^*\Gamma = \left( dy^i - \Gamma^i_\lambda(f^\mu(x^\nu), y^j) \frac{\partial f^\lambda}{\partial x^\mu} dx^\mu \right) \otimes \partial_i \tag{4.3.7}
\]
on the pull-back bundle \( f^*Y \rightarrow X' \).

Every connection \( \Gamma \) on a fibre bundle \( Y \rightarrow X \) defines the first order differential operator

\[
D^\Gamma : J^1Y \rightarrow T^*X \otimes VY, \tag{4.3.8}
\]

\[
D^\Gamma = \lambda_1 - \Gamma \circ \pi^i_0 = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i,
\]
on \( Y \) called the covariant differential. If \( s: X \rightarrow Y \) is a section, its covariant differential

\[
\nabla^\Gamma s = D^\Gamma \circ J^1s = (\partial_\lambda s^i - \Gamma^i_\lambda \circ s) dx^\lambda \otimes \partial_i \tag{4.3.9}
\]
and its covariant derivative \( \nabla^\Gamma \tau s = \tau \rfloor \nabla^\Gamma s \) along a vector field \( \tau \) on \( X \) are introduced. In particular, a (local) section \( s \) of \( Y \rightarrow X \) is called an integral section for a connection \( \Gamma \) (or parallel with respect to \( \Gamma \)) if \( s \) obeys the equivalent conditions

\[
\nabla^\Gamma s = 0 \quad \text{or} \quad J^1s = \Gamma \circ s. \tag{4.3.10}
\]

Let \( \Gamma \) be a connection on a fibre bundle \( Y \rightarrow X \). Given vector fields \( \tau, \tau' \) on \( X \) and their horizontal lifts \( \Gamma \tau \) and \( \Gamma \tau' \) (4.3.3) on \( Y \), let us consider the vertical vector field

\[
R(\tau, \tau') = [\Gamma \tau, \Gamma \tau'] = \tau^\lambda \tau'^\mu R^i_{\lambda \mu} \partial_i, \tag{4.3.11}
\]

\[
R^i_{\lambda \mu} = \partial_\lambda \Gamma^i_\mu - \partial_\mu \Gamma^i_\lambda + \Gamma^j_\lambda \partial_j \Gamma^i_\mu - \Gamma^j_\mu \partial_j \Gamma^i_\lambda. \tag{4.3.12}
\]
It can be seen as the contraction of vector fields \( \tau \) and \( \tau' \) with the vertical-valued horizontal two-form

\[
R = \frac{1}{2}[\Gamma, \Gamma]_{FN} = \frac{1}{2} R^i_{\lambda \mu} dx^\lambda \wedge dx^\mu \otimes \partial_i \tag{4.3.13}
\]
on \( Y \) called the curvature form of a connection \( \Gamma \).

Given a connection \( \Gamma \) and a soldering form \( \sigma \), the torsion of \( \Gamma \) with respect to \( \sigma \) is defined as the vertical-valued horizontal two-form

\[
T = [\Gamma, \sigma]_{FN} = (\partial_\lambda \sigma^i_\mu + \Gamma^j_\lambda \partial_j \sigma^i_\mu - \partial_j \Gamma^i_\lambda \sigma^j_\mu) dx^\lambda \wedge dx^\mu \otimes \partial_i. \tag{4.3.14}
\]
4.3.2 Flat connections

A flat (or curvature-free) connection is a connection $\Gamma$ on a fibre bundle $Y \to X$ which satisfies the following equivalent conditions:

- its curvature vanishes everywhere on $Y$;
- its horizontal distribution is involutive;
- there exists a local integral section for the connection $\Gamma$ through any point $y \in Y$.

By virtue of Theorem 4.1.15, a flat connection $\Gamma$ yields a foliation of $Y$ which is transversal to the fibration $Y \to X$. It called a horizontal foliation. Its leaf through a point $y \in Y$ is locally defined by an integral section $s_y$ for the connection $\Gamma$ through $y$. Conversely, let a fibre bundle $Y \to X$ admit a horizontal foliation such that, for each point $y \in Y$, the leaf of this foliation through $y$ is locally defined by a section $s_y$ of $Y \to X$ through $y$. Then the map

$$\Gamma : Y \ni y \mapsto J^1_{\pi(y)} s_y \in J^1 Y$$

sets a flat connection on $Y \to X$. Hence, there is one-to-one correspondence between the flat connections and the horizontal foliations of a fibre bundle $Y \to X$.

Given a horizontal foliation of a fibre bundle $Y \to X$, there exists the associated atlas of bundle coordinates $(x^\lambda, y^i)$ on $Y$ such that every leaf of this foliation is locally given by the equations $y^i = \text{const}.$, and the transition functions $y^i \to y'^i(y'^j)$ are independent of the base coordinates $x^\lambda$ [14]. It is called the atlas of constant local trivializations. Two such atlases are said to be equivalent if their union also is an atlas of the same type. They are associated to the same horizontal foliation. Thus, the following is proved.

**Theorem 4.3.1**: There is one-to-one correspondence between the flat connections $\Gamma$ on a fibre bundle $Y \to X$ and the equivalence classes of atlases of constant local trivializations of $Y$ such that $\Gamma = dx^\lambda \otimes \partial_\lambda$ relative to the corresponding atlas. □

**Example 4.3.2**: Any trivial bundle has flat connections corresponding to its trivializations. Fibre bundles over a one-dimensional base have only flat connections. ◇

4.3.3 Linear connections

Let $Y \to X$ be a vector bundle equipped with linear bundle coordinates $(x^\lambda, y^i)$. It admits a linear connection

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda (x) y^j \partial_j). \quad (4.3.15)$$

There are the following standard constructions of new linear connections from old ones.

- Any linear connection $\Gamma$ (4.3.15) on a vector bundle $Y \to X$ defines the dual linear connection

$$\Gamma^* = dx^\lambda \otimes (\partial_\lambda - \Gamma_j^i (x) y_j \partial^i) \quad (4.3.16)$$
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on the dual bundle \( Y^* \to X \).

- Let \( \Gamma \) and \( \Gamma' \) be linear connections on vector bundles \( Y \to X \) and \( Y' \to X \), respectively. The direct sum connection \( \Gamma \oplus \Gamma' \) on the Whitney sum \( Y \oplus Y' \) of these vector bundles is defined as the product connection (4.3.6).

- Similarly, the tensor product \( Y \otimes Y' \) of vector bundles possesses the tensor product connection

\[
\Gamma \otimes \Gamma' = dx^\lambda \otimes \left[ \partial_\lambda + (\Gamma^i_j y^j + \Gamma_{ab}^i y^j y^b) \frac{\partial}{\partial y^a} \right].
\] (4.3.17)

The curvature of a linear connection \( \Gamma \) (4.3.15) on a vector bundle \( Y \to X \) is usually written as a \( Y \)-valued two-form

\[
R = \frac{1}{2} R_{\lambda\mu}^{\alpha\beta} d x^\lambda \wedge d x^\mu \otimes \partial_\alpha,
\] (4.3.18)

\[
R_{\lambda\mu}^{\alpha\beta} = \partial_{\lambda} \Gamma_{\mu}^{\alpha\beta} - \partial_{\mu} \Gamma_{\lambda}^{\alpha\beta} + \Gamma_{\lambda}^{\gamma\beta} \Gamma_{\mu}^{\alpha\gamma} - \Gamma_{\mu}^{\gamma\beta} \Gamma_{\lambda}^{\alpha\gamma}.
\]

due to the canonical vertical splitting \( VY \cong Y \times Y \), where \( \{ \partial_\lambda \} = \{ e_i \} \). For any two vector fields \( \tau \) and \( \tau' \) on \( X \), this curvature yields the zero order differential operator

\[
R(\tau, \tau') s = (\nabla^\Gamma_\tau, \nabla^\Gamma_{\tau'}) s
\] (4.3.19)
on section \( s \) of a vector bundle \( Y \to X \).

An important example of linear connections is a connection \( K \) (4.3.20) on the tangent bundle \( TX \) of a manifold \( X \). It is called a world connection or, simply, a connection on a manifold \( X \). The dual connection (4.3.16) on the cotangent bundle \( T^*X \) is

\[
K^* = dx^\lambda \otimes (\partial_\lambda - K_{\lambda\mu}^{\rho\nu} \hat{x}_\mu \hat{\partial}_\nu).
\] (4.3.21)

The curvature of a world connection \( K \) (4.3.20) reads

\[
R = \frac{1}{2} R_{\lambda\mu}^{\alpha\beta} \hat{x}^\lambda \wedge d x^\mu \otimes \partial_\alpha,
\] (4.3.22)

\[
R_{\lambda\mu}^{\alpha\beta} = \partial_{\lambda} K_{\mu}^{\alpha\beta} - \partial_{\mu} K_{\lambda}^{\alpha\beta} + K_{\lambda}^{\gamma\beta} K_{\mu}^{\alpha\gamma} - K_{\mu}^{\gamma\beta} K_{\lambda}^{\alpha\gamma}.
\]

Its Ricci tensor \( R_{\lambda\beta} = R_{\lambda\mu}^{\mu\beta} \) is introduced.

A torsion of a world connection is defined as the torsion (4.3.14) of the connection \( K \) (4.3.20) on the tangent bundle \( TX \) with respect to the canonical vertical-valued form \( dx^\lambda \otimes \hat{\partial}_\lambda \). Due to the vertical splitting of \( VTX \), it also is written as a tangent-valued two-form

\[
T = \frac{1}{2} T_{\mu\lambda}^{\nu} dx^\lambda \wedge d x^\mu \otimes \partial_\nu,
\] (4.3.23)

\[
T_{\mu\lambda}^{\nu} = K_{\mu}^{\nu\lambda} - K_{\lambda}^{\nu\mu}.
\]
on $X$. A world connection (4.3.20) is called symmetric if its torsion (4.3.23) vanishes. For instance, let a manifold $X$ be provided with a non-degenerate fibre metric 

$$g \in \bigotimes V O^1(X), \quad g = g_{\mu \nu} dx^\lambda \otimes dx^\mu,$$

in the tangent bundle $TX$, and with the dual metric 

$$g \in \bigotimes V T^1(X), \quad g = g^{\mu \nu} \partial_\lambda \otimes \partial_\mu,$$

in the cotangent bundle $T^*X$. Then there exists a world connection $K$ such that $g$ is its integral section, i.e.,

$$\nabla_\lambda g^{\alpha \beta} = \partial_\lambda g^{\alpha \beta} - g^{\alpha \gamma} K_\lambda^\beta_{\, \gamma} - g^{\beta \gamma} K_\lambda^\alpha_{\, \gamma} = 0.$$ 

It is called the metric connection. There exists a unique symmetric metric connection

$$K_\lambda^\nu_{\mu} = \{\lambda^\nu_{\mu}\} = -\frac{1}{2} g^{\nu \rho} (\partial_\lambda g_{\rho \mu} + \partial_\mu g_{\rho \lambda} - \partial_{\rho} g_{\lambda \mu}). \tag{4.3.24}$$

This is the Levi–Civita connection, whose components (4.3.24) are called Christoffel symbols.

### 4.3.4 Composite connections

Let us consider the composite bundle $Y \rightarrow \Sigma \rightarrow X$ (4.1.10), coordinated by $(x^\lambda, \sigma^m, y^i)$. Let us consider the jet manifolds $J^1 \Sigma, J^1_2 Y$, and $J^1 Y$ of the fibre bundles $\Sigma \rightarrow X, Y \rightarrow \Sigma$ and $Y \rightarrow X$, respectively. They are parameterized respectively by the coordinates

$$(x^\lambda, \sigma^m, \sigma^m_\lambda), \quad (x^\lambda, \sigma^m, y^i, \bar{y}^i_\lambda, y^i_m), \quad (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, \bar{y}^i_\lambda).$$

There is the canonical map

$$\varrho : J^1 \Sigma \times J^1_2 Y_\Sigma \rightarrow J^1 Y, \quad y^i_\lambda \circ \varrho = y^i_m \sigma^m_\lambda + \bar{y}^i_\lambda. \tag{4.3.25}$$

Using the canonical map (4.3.25), we can consider the relations between connections on fibre bundles $Y \rightarrow X, Y \rightarrow \Sigma$ and $\Sigma \rightarrow X$ [30, 43].

Connections on fibre bundles $Y \rightarrow X, Y \rightarrow \Sigma$ and $\Sigma \rightarrow X$ read

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^m_\lambda \partial_m + \gamma^i_\lambda \partial_i), \tag{4.3.26}$$

$$A_Y = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i), \tag{4.3.27}$$

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m). \tag{4.3.28}$$
4.3. CONNECTIONS ON FIBRE BUNDLES

The canonical map \( \varrho \) (4.3.25) enables us to obtain a connection \( \gamma \) on \( Y \to X \) in accordance with the diagram

\[
\begin{array}{c}
J^1\Sigma \times J^1\Sigma Y \xrightarrow{\varrho} J^1Y \\
\downarrow \quad \downarrow \gamma \\
\Sigma \times X \leftarrow Y
\end{array}
\]

This connection, called the composite connection, reads

\[
\gamma = dx^\lambda \otimes [\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_\lambda + A^i_m \Gamma^m_\lambda) \partial_i].
\]

(4.3.29)

It is a unique connection such that the horizontal lift \( \gamma_\tau \) on \( Y \) of a vector field \( \tau \) on \( X \) by means of the connection \( \gamma \) (4.3.29) coincides with the composition \( A^\Sigma (\Gamma \tau) \) of horizontal lifts of \( \tau \) onto \( \Sigma \) by means of the connection \( \Gamma \) and then onto \( Y \) by means of the connection \( A^\Sigma \). For the sake of brevity, let us write \( \gamma = A^\Sigma \circ \Gamma \).

Given the composite bundle \( Y \) (4.1.10), there is the exact sequence

\[
0 \to V^\Sigma Y \to VY \to Y \times V\Sigma \to 0,
\]

(4.3.30)

where \( V^\Sigma Y \) denotes the vertical tangent bundle of a fibre bundle \( Y \to \Sigma \) coordinated by \((x^\lambda, \sigma^m, \dot{y}^i, \ddot{y}^i)\). Let us consider the splitting

\[
B : VY \ni v = \dot{y}^i \partial_i + \dot{\sigma}^m \partial_m \mapsto v \bigr| B = (\dot{y}^i - \dot{\sigma}^m B^i_m) \partial_i \in V^\Sigma Y,
\]

(4.3.31)

\[
B = (dy^i - B^i_m \partial_i) \otimes \partial_i \in V^*Y \otimes V^\Sigma Y,
\]

of the exact sequence (4.3.30). Then the connection \( \gamma \) (4.3.26) on \( Y \to X \) and the splitting \( B \) (4.3.31) define a connection

\[
A^\Sigma = B \circ \gamma : TY \to VY \to V^\Sigma Y,
\]

(4.3.32)

\[
A^\Sigma = dx^\lambda \otimes (\partial_\lambda + (\gamma^i_\lambda - B^i_m \sigma^m) \partial_i) + d\sigma^m \otimes (\partial_m + B^i_m \partial_i),
\]

on the fibre bundle \( Y \to \Sigma \).

Conversely, every connection \( A^\Sigma \) (4.3.27) on a fibre bundle \( Y \to \Sigma \) provides the splitting

\[
VY = V^\Sigma Y \oplus A^\Sigma (Y \times V\Sigma),
\]

(4.3.33)

\[
\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A^i_m \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A^i_m \partial_i),
\]

of the exact sequence (4.3.30). Using this splitting, one can construct the first order differential operator

\[
\overline{D} : J^1Y \to T^*X \otimes V^\Sigma Y, \quad \overline{D} = dx^\lambda \otimes (\dot{y}^i - A^i_\lambda - A^i_m \sigma^m_\lambda) \partial_i,
\]

(4.3.34)
called the vertical covariant differential, on the composite fibre bundle $Y \to X$.

The vertical covariant differential (4.3.34) possesses the following important property. Let $h$ be a section of a fibre bundle $\Sigma \to X$, and let $Y_h \to X$ be the restriction of a fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$. This is a subbundle $i_h : Y_h \to Y$ of a fibre bundle $Y \to X$. Every connection $A_{\Sigma}$ (4.3.27) induces the pull-back connection $A_h = i_h^* A_{\Sigma} = dx^\lambda \otimes [\partial_\lambda + ((A^i_m \circ h) \partial h^m + (A \circ h)_\lambda^i) \partial_i]$ (4.3.35)
on $Y_h \to X$. Then the restriction of the vertical covariant differential $\tilde{D}$ (4.3.34) to $J^1 i_h(J^1 Y_h) \subset J^1 Y$ coincides with the familiar covariant differential $D^{A_h}$ (4.3.8) on $Y_h$ relative to the pull-back connection $A_h$ (4.3.35).
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