Uniqueness of radial centers of parallel bodies

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September 26, 2011

Abstract

We show the uniqueness of the radial centers of any order \( \alpha \) of a parallel body of a convex body \( \Omega \) in \( \mathbb{R}^m \) at distance \( \delta \) if \( \delta \) is greater than the diameter of \( \Omega \) multiplied by a constant which depends only on the dimension \( m \).

Key words and phrases. Riesz potential, renormalization, centroid.

2010 Mathematics Subject Classification: 53C65, 53A99, 31C12, 52A40, 51M16, 51P05.

1 Introduction

Let \( \Omega \) be a body in \( \mathbb{R}^m \) (\( m \geq 2 \)), i.e. a compact set which is a closure of its interior, with a piecewise \( C^1 \) boundary. Consider a potential of the form

\[
V^{(\alpha)}_\Omega(x) = \int_\Omega |x - y|^{\alpha - m} \, d\mu(y) \quad (\alpha > 0),
\]

where \( \mu \) is the standard Lebesgue measure of \( \mathbb{R}^m \). It is a singular integral when \( \alpha < m \) and \( x \in \Omega \). When \( 0 < \alpha < m \) it is the Riesz potential of \( \Omega \).

In particular, when \( \Omega \) is convex and \( x \in \Omega \), \( V^{(\alpha)}_\Omega(x) \) can be expressed as

\[
V^{(\alpha)}_\Omega(x) = \frac{1}{\alpha} \int_{S^{m-1}} (\rho_{\Omega - x}^{\alpha - m}(v))^{\alpha} \, d\sigma(v)
\]

where \( \sigma \) is the standard Lebesgue measure of \( S^{m-1} \) and \( \rho_{\Omega - x} : S^{m-1} \to \mathbb{R}^{\geq 0} \) is a radial function of \( \Omega - x = \{ y - x \mid y \in \Omega \} \) given by \( \rho_{\Omega - x}^{\alpha - m}(v) = \sup \{ a \geq 0 \mid x + av \in \Omega \} \). Thus \( V^{(\alpha)}_\Omega(x) \) coincides with the dual mixed volume as introduced by Lutwak \([L1, L2]\) up to multiplication by a constant.

In \([O3]\) we defined an \( r^{\alpha-m} \)-center of \( \Omega \). It is a point where the extreme value of \( V^{(\alpha)}_\Omega \) (minimum or maximum according to the value of \( \alpha \)) is attained when \( \alpha \neq m \). (The case when \( \alpha = m \) will be addressed later.) For example, the center of mass is an \( r^2 \)-center. When \( \Omega \) is convex, an \( r^{\alpha-m} \)-center (\( \alpha \neq m \)) coincides with the radial center of order \( \alpha \), which was introduced in \([M]\) for \( 0 < \alpha \leq 1 \) and in \([H]\) in general (\( \alpha \neq 0 \)). An \( r^{\alpha-m} \)-center of a body \( \Omega \) exists for any \( \alpha \) and is unique if \( \alpha \geq m + 1 \) or if \( \alpha \leq 1 \) and \( \Omega \) is convex \((O3)\). It was conjectured that a convex subset \( \Omega \) has a unique \( r^{\alpha-m} \)-center for any \( \alpha \).

In this paper we show the uniqueness of an \( r^{\alpha-m} \)-center for any \( \alpha \) when \( \Omega \) is close to a ball in some sense. To be precise, we show that there is a positive function \( \varphi(m) \) such that for any convex body \( \Omega \) with a piecewise \( C^1 \) boundary, if \( \delta \geq \varphi(m) \cdot \text{diam}(\Omega) \) then a \( \delta \)-parallel body of \( \Omega \) has a unique \( r^{\alpha-m} \)-center for any \( \alpha \). Here, a \( \delta \)-parallel body of \( \Omega \) is the closure of a \( \delta \)-tubular neighborhood of \( \Omega \), and is denoted by \( \Omega + \delta B^m \). The proof has two steps. First we show that a center can appear only in \( \Omega \) by the so-called moving plane method in analysis \((GNN)\). Then we show that \( V^{(\alpha)}_{\Omega + \delta B^m} \) is convex (or concave according to the value of \( \alpha \)) on \( \Omega \) using the boundary integral expression of the second derivatives of \( V^{(\alpha)}_{\Omega} \).

Throughout the paper, \( X, \mathring{X}, X^c, \) and \( \text{conv}(X) \) denote the interior, the closure, the complement, and the convex hull of \( X \) respectively. We denote the standard Lebesgue measure of \( \mathbb{R}^m \) by \( \mu \), and that of \( \partial \Omega \) and other \((m-1)\)-dimensional spaces like \( S^{m-1} \) by \( \sigma \).
2 Preliminaries from \[O3\]

In this section we introduce some of the results of \[O3\] which are necessary for our study.

First remark that if we define

\[ X - Y = (X \setminus (X \cap Y)) \cup -(Y \setminus (X \cap Y)) \quad (X, Y \subset \mathbb{R}^m), \]

where the second term is equipped with the reverse orientation, then

\[ V_{\Omega_1 - \Omega_2}^{(\alpha)} (x) = V_{\Omega_1}^{(\alpha)} (x) - V_{\Omega_2}^{(\alpha)} (x) \quad (x \in \partial \Omega_1 \cap \partial \Omega_2) \]

for any \( \alpha \).

2.1 Boundary integral expression of the derivatives

The first derivatives of \( V_{\Omega_1}^{(\alpha)} \) can be expressed by the boundary integral as

\[
\frac{\partial V_{\Omega_1}^{(\alpha)}}{\partial x_j} (x) = - \int_{\partial \Omega} |x - y|^{\alpha - m} e_j \cdot n \, d\sigma(y)
\]

for any \( j \) (1 \( \leq j \leq m \)) if \( x = (x_1, \ldots, x_m) \notin \partial \Omega \), where \( n \) is a unit outer normal vector to \( \partial \Omega \) at \( y \), \( e_j \) is the \( j \)-th unit vector of \( \mathbb{R}^m \), and \( \sigma \) denotes the standard Lebesgue measure of \( \partial \Omega \). This is because

\[
\frac{\partial \rho^{\alpha - m}}{\partial x_j} = - \frac{\partial \rho^{\alpha - m}}{\partial y_j} = - \text{div}_x (\rho^{\alpha - m} e_j).
\]

It follows that the second derivatives satisfy

\[
\frac{\partial^2 V_{\Omega_1}^{(\alpha)}}{\partial x_j^2} (x) = - (\alpha - m) \int_{\partial \Omega} |x - y|^{\alpha - m - 2} (x_j - y_j) e_j \cdot n \, d\sigma(y)
\]

for any \( \alpha \) if \( x \notin \partial \Omega \) (or for any \( x \) if \( \alpha > 2 \)). Furthermore, if \( x \in \Omega^c \) then for any \( \alpha \)

\[
\frac{\partial^2 V_{\Omega_1}^{(\alpha)}}{\partial x_j^2} (x) = (\alpha - m) \int_{\Omega} |x - y|^{\alpha - m - 4} \left( (\alpha - m - 2)(x_j - y_j)^2 + |x - y|^2 \right) d\mu(y)
\]

\[
= (\alpha - m) \int_{\Omega} |x - y|^{\alpha - m - 4} \left( (\alpha - m - 1)(x_j - y_j)^2 + \sum_{i \neq j} (x_i - y_i)^2 \right) d\mu(y).
\]

2.2 Definition of the \( r^{\alpha - m} \)-centers

When \( \alpha \neq m \) we call a point \( r^{\alpha - m} \)-center of \( \Omega \) if it gives the minimum value of \( V_{\Omega}^{(\alpha)} \) when \( \alpha > m \) and the maximum value of \( V_{\Omega}^{(\alpha)} \) when \( 0 < \alpha < m \). When \( \alpha = m \) it is meaningless to use \( V_{\Omega}^{(m)} \) as it is constantly equal to \( \text{Vol} (\Omega) \). We call a point an \( r^0 \)-center if it gives the maximum value of the log potential

\[ V_{\Omega}^{\text{log}} (x) = \int_\Omega \log \frac{1}{|x - y|} \, d\mu(y) = - \int_\Omega \log |x - y| \, d\mu(y). \]

As we noticed in the introduction, the center of mass is an \( r^2 \)-center, and if \( \Omega \) is convex and \( \alpha \neq m \), an \( r^{\alpha - m} \)-center coincides with the radial center of order \( \alpha \), which was introduced in \[M\] for \( 0 < \alpha \leq 1 \) and in \[H\] for \( \alpha \neq 0 \).

We remark that the statements of \( r^{\alpha - m} \)-centers in the case when \( \alpha = m \) in this paper can be obtained exactly in the same way as in the case when \( 0 < \alpha < m \). This is because we only use the estimate on the second derivative in our study, and that of the log potential

\[
\frac{\partial^2 V_{\Omega}^{\text{log}}}{\partial x_j^2} (x) = \int_{\partial \Omega} |x - y|^{-2} (x_j - y_j) e_j \cdot n \, d\sigma(y)
\]

can be considered as the limit of \( 1/(m - \alpha) \) times the second derivative of \( V_{\Omega}^{(\alpha)} \) as \( \alpha \) approaches \( m \) (see \[2.2\]).
2.3 Minimal unfolded regions

Let $v$ be a unit vector in $S^{m-1}$ and $b$ be a real number. Put

$$H_{v,b} = \{ x \in \mathbb{R}^m \mid x \cdot v = b \}, \quad H_{v,b}^+ = \{ x \in \mathbb{R}^m \mid x \cdot v > b \}, \quad H_{v,b}^- = \{ x \in \mathbb{R}^m \mid x \cdot v < b \}. $$

Let $\text{Refl}_{v,b}$ be a reflection of $\mathbb{R}^m$ in $H_{v,b}$. Let $\Omega$ be a compact set in $\mathbb{R}^m$. Put $M_v = M_v(\Omega) = \max_{x \in \Omega} x \cdot v$ and

$$u_v = u_v(\Omega) = \inf \left\{ a \mid a \leq M_v, \text{Refl}_{v,b}(\Omega \cap H_{v,b}^+) \subset \Omega \right\}.$$

The minimal unfolded region of $\Omega$ is given by

$$Uf(\Omega) = \bigcap_{v \in S^{m-1}} H_{v,u_v}^-.$$

It is a non-empty compact convex set and is contained in the convex hull of $\Omega$. It is not necessarily contained in $\Omega$.

![Figure 1: Folding a convex set like origami](image1)

![Figure 2: A minimal unfolded region of a non-obtuse triangle. Bold lines are angle bisectors, dotted lines are perpendicular bisectors.](image2)

2.4 Existence and uniqueness of $r^{\alpha-m}$-centers

**Theorem 2.1** ([GNN]) Let $\Omega$ be a body in $\mathbb{R}^m$ with a piecewise $C^1$ boundary $\partial \Omega$.

1. There exists an $r^{\alpha-m}$-center of $\Omega$ for any $\alpha$.
2. An $r^{\alpha-m}$-center is contained in the minimal unfolded region of $\Omega$ for any $\alpha$.
3. An $r^{\alpha-m}$-center of $\Omega$ is unique if $\alpha \geq m + 1$.
4. An $r^{\alpha-m}$-center of $\Omega$ is unique if $\alpha \leq 1$ and $\Omega$ is convex.

The second statement is essentially based on the so-called moving plane method in analysis ([GNN]) as the integrands appearing in the formulae of $V_\Omega^{(\alpha)}$ and its derivatives are symmetric. The uniqueness of an $r^{\alpha-m}$-center follows from $\frac{\partial^2 V_\Omega^{(\alpha)}}{\partial x^2} > 0$ when $\alpha \geq m + 1$ and the strong concavity $V_\Omega^{(\alpha)}$ on $\Omega$ when $\alpha \leq 1$ and $\Omega$ is convex.

3 Uniqueness of the centers of $\Omega + \delta B^m$

We conjectured that if $\Omega$ is convex then it has only one $r^{\alpha-m}$-center for any $\alpha$, although it was proved that $V_\Omega^{(\alpha)}$ is not necessarily convex nor concave. In this section we show that the conjecture holds for a $\delta$-parallel bodies $\tilde{\Omega} + \delta B^m \{ x + u \mid x \in \tilde{\Omega}, u \in B^m \}$ provided that $\delta$ is large enough compared with the diameter of $\tilde{\Omega}$. To be precise, we prove the following theorem:
Theorem 3.1 For any natural number \( m \geq 2 \) there is a positive constant \( \varphi(m) \) such that for any compact convex set \( \tilde{\Omega} \) in \( \mathbb{R}^m \) with piecewise \( C^1 \) boundary, if \( \delta \geq \varphi(m) \cdot \text{diam}(\tilde{\Omega}) \) then \( \tilde{\Omega} + \delta B^m \) has a unique \( r^{\alpha-m} \)-center for any \( \alpha \).

By Theorem 2.1 it is enough to show
\[
\frac{\partial^2 V(\alpha)}{\partial x_j^2} < 0 \quad (1 < \alpha < m), \quad \frac{\partial^2 V^{\log}(\alpha)}{\partial x_j^2} < 0, \quad \frac{\partial^2 V(\alpha)}{\partial x_j^2} > 0 \quad (m < \alpha < m + 1) \quad (3.1)
\]
for any \( j \) on the minimal unfolded region of \( \tilde{\Omega} + \delta B^m \).

Lemma 3.2 Let \( \tilde{\Omega} \) be any compact subset of \( \mathbb{R}^m \). The minimal unfolded region of \( \tilde{\Omega} + \delta B^m \) is contained in the convex hull of \( \tilde{\Omega} \) for any \( \delta > 0 \).

Proof. Let us use the notation in Subsection 2.3. Let \( v \in S^{m-1} \) be any vector and \( b \) any real number that satisfies
\[
M_v(\tilde{\Omega}) < b \leq M_v(\tilde{\Omega} + \delta B^m) = M_v(\tilde{\Omega}) + \delta.
\]
Then for any point \( Q \) in \( \tilde{\Omega} \) we have \( \text{Refl}_{v,b}(B_\delta(Q) \cap H^+_{v,b}) \subset B_\delta(Q) \cap H^+_{v,b} \) as the center \( Q \) is in \( H^+_{v,b} \). As \( (\tilde{\Omega} + \delta B^m) \cap H^+_{v,b} = \bigcup_{Q \in \tilde{\Omega}} (B_\delta(Q) \cap H^+_{v,b}) \) we have
\[
\text{Refl}_{v,b}((\tilde{\Omega} + \delta B^m) \cap H^+_{v,b}) = \bigcup_{Q \in \tilde{\Omega}} \text{Refl}_{v,b}(B_\delta(Q) \cap H^+_{v,b}) \subset \bigcup_{Q \in \tilde{\Omega}} (B_\delta(Q) \cap H^+_{v,b}) \subset (\tilde{\Omega} + \delta B^m) \cap H^+_{v,b}.
\]
Consequently we have \( u_v(\tilde{\Omega} + \delta B^m) \leq M_v(\tilde{\Omega}) \). It follows that
\[
Uf(\tilde{\Omega} + \delta B^m) = \bigcap_{v \in S^{m-1}} H^+_{v,u_v(\tilde{\Omega} + \delta B^m)} \subset \bigcap_{v \in S^{m-1}} H^+_{v,M_v(\tilde{\Omega})} = \text{conv}(\tilde{\Omega}).
\]

Next we proceed to the proof of 3.1 on \( \tilde{\Omega} \).

Definition 3.3 Let \( m \) be a natural number with \( m \geq 2 \) and let \( a > 0 \). Let \( L_a \) denote an oriented line segment in \( \mathbb{R}^2 \) which starts from \( (a,0) \) and ends at \( (0,1) \). For real numbers \( \alpha \) and \( \xi \) with \( 0 \leq \xi < a \) define
\[
F(m,\alpha,\xi,\zeta) = \int_{L_a} |x - y|^{\alpha-m-2}(\xi - y_1) y_2^{m-2} dy_2,
\]
where \( x = (\xi,0) \).

Lemma 3.4 Suppose \( m = 2 \) and \( 1 < \alpha < 3 \). For any \( a > 0 \), if \( 0 \leq \xi \leq \frac{a}{4} \) then \( F(2,\alpha,a,\xi) < 0 \).

Proof. Divide \( L_a \) into three parts;
\[
L_1 = L_a \cap \{ \xi \leq x_1 \leq a \}, \quad L_2 = L_a \cap \{ \xi \leq x_1 \leq 2\xi \}, \quad L_3 = L_a \cap \{ 0 \leq x_1 \leq \xi \}.
\]
If we put
\[
p(t) = \left( \xi + t, 1 - \frac{\xi}{a} - \frac{t}{a} \right) \in L_2, \quad q(t) = \left( \xi - t, 1 - \frac{\xi}{a} + \frac{t}{a} \right) \in L_3 \quad (0 \leq t \leq \xi)
\]
then \( |x - p(t)| < |x - q(t)| \) (Figure 3). Hence, as \( y_1 = a(1 - y_2) \) on \( L_a \),
\[
\int_{L_2 \cup L_3} |x - y|^{\alpha-4}(\xi - y_1) dy_2 = \int_{0}^{\xi} \left( -|x - p(t)|^{\alpha-4} + |x - q(t)|^{\alpha-4} \right) \cdot t \cdot \frac{dt}{a} < 0.
\]
Since \( \int_{L_1} |x - y|^{\alpha-4}(\xi - y_1) dy_2 < 0 \), it completes the proof. \( \square \)
Lemma 3.5 Suppose \( m \geq 3 \) and \( 1 < \alpha < m + 1 \). There is a map \( \psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for any \( c > 0 \) if \( a \geq \psi_\alpha(c)c \) then

\[
\int_{-a}^{a} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt < 0. \tag{3.3}
\]

Proof. Let \( c > 0 \). Assume \( a > 2c \). Note that the integrand of (3.3) is positive (or negative) if \( t > \frac{1}{a} \) (or respectively \( t < \frac{-1}{a} \)). Therefore we have only to consider the case when \( \frac{1}{a} < c \). Observe that

\[
\int_{-a}^{a} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt = - \int_{-2c,-\epsilon}\int_{[\epsilon,-\epsilon]} \frac{|t - \frac{1}{a}|(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt + \int_{\frac{1}{a}}^{a} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt, \tag{3.4}
\]

where the right hand side can be estimated by

\[
\int_{-c}^{-\frac{1}{a}} \frac{|t - \frac{1}{a}|(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt \geq \left( \frac{a - c}{a + c} \right)^{m-2} \int_{\frac{1}{a}}^{c} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt, \]

\[
\int_{-2c}^{-c} \frac{|t - \frac{1}{a}|(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt \geq \left( \frac{a - 2c}{a + c} \right)^{m-2} \cdot \frac{1}{(4c^2 + 1)^{m+2-\alpha}} \int_{\frac{1}{a}}^{c} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-\alpha}} \, dt.
\]

As

\[
\left( \frac{a - c}{a + c} \right)^{m-2} + \left( \frac{a - 2c}{a + c} \right)^{m-2} \frac{1}{(4c^2 + 1)^{m+2-\alpha}} \geq \left( \frac{a - 2c}{a + c} \right)^{m-2} \left( 1 + (4c^2 + 1)^{-\frac{m+2-\alpha}{2}} \right), \tag{3.5}
\]

if we put

\[
\psi_\alpha(c) = 2\left(1 + (4c^2 + 1)^{-\frac{m+2-\alpha}{2}}\right)^{\frac{1}{m-2}} + \frac{3}{\left(1 + (4c^2 + 1)^{-\frac{m+2-\alpha}{2}}\right)^{\frac{m}{m-2}} - 1} \tag{3.6}
\]

then \( a \geq \psi_\alpha(c)c \) is equivalent to

\[
a - 2c \geq \left(1 + (4c^2 + 1)^{-\frac{m+2-\alpha}{2}}\right)^{-\frac{1}{m-2}},
\]

which implies that the right hand side of (3.4) is negative; thus the proof is completed. Remark that as \( \psi_\alpha(c) > 2 \), if \( a \geq \psi_\alpha(c)c \) then \( a \) satisfies the assumption \( a > 2c \) which appeared at the beginning of the proof. \( \square \)
Corollary 3.6 Suppose $m \geq 3$ and $1 < \alpha < m + 1$. For any $\xi_0 > 0$ there is $a_0 > 0$ such that if $0 \leq \xi \leq \xi_0$ and $a \geq a_0$ then $F(m, \alpha, a, \xi) < 0$, where $F(m, \alpha, a, \xi)$ is given by (3.2). In fact, we can take
\[
a_0 = \psi_\alpha \left( \frac{2\xi_0^2 + 1}{\xi_0} \right) \frac{2\xi_0^2 + 1}{\xi_0},
\]
where $\psi_\alpha$ is given by (3.6).

Proof. As $y_1 = a(1 - y_2)$ on $L_\alpha$,
\[
F(m, \alpha, a, \xi) = \int_0^1 \left( (a(1 - y_2) - \xi)^2 + y_2^2 \right)^{\frac{m-1}{2m}} (\xi - a(1 - y_2)) y_2^{m-2} dy_2. \tag{3.7}
\]
Put $y_2 - \frac{a(\xi - \xi_0)}{1 + a^2} = \frac{a - \xi}{1 + a^2} t$. Then, as
\[
(a(1 - y_2) - \xi)^2 + y_2^2 = \left( \frac{a - \xi}{1 + a^2} t^2 + 1 \right), \quad \xi - a(1 - y_2) = \frac{a - \xi}{1 + a^2} (at - 1), \quad y_2 = \frac{a - \xi}{1 + a^2} (t + a),
\]
$F(m, \alpha, a, \xi) < 0$ is equivalent to
\[
\int_{-a}^{\frac{a - \xi}{1 + a^2}} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-a}} dt < 0.
\]

First remark that if $0 < \xi < \xi_0$ then $F(m, \alpha, a, \xi) < F(m, \alpha, a, \xi_0)$ since $\frac{a - \xi}{1 + a^2}$ is an increasing function of $\xi$ ($0 < \xi < a$) with limit $\xi \to 0 \frac{a - \xi}{1 + a^2} = \frac{1}{a}$ and the integrand is positive when $t > \frac{1}{a}$.

On the other hand, if we put $c(a) = \frac{\xi_0^2 + 1}{\xi_0}$, it is a decreasing function of $a$ as $c(a) = \xi_0 + \frac{1 + \xi_0^2}{a - \xi_0}$. Put
\[
c_0 = c(2\xi_0) = \frac{2\xi_0^2 + 1}{\xi_0}, \quad a_0 = \psi_\alpha (c_0) c_0 = \psi_\alpha \left( \frac{2\xi_0^2 + 1}{\xi_0} \right) \frac{2\xi_0^2 + 1}{\xi_0},
\]
where $\psi_\alpha$ is given by (3.6). If $a \geq a_0$ then $c(a) < c(2\xi_0) = c_0$ as $a > 2\xi_0$. Since $c(a) > \frac{1}{a}$ we have
\[
\int_{-a}^{c(a)} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-a}} dt < \int_{-a}^{c_0} \frac{(t - \frac{1}{a})(t + a)^{m-2}}{(\sqrt{t^2 + 1})^{m+2-a}} dt < 0,
\]
where the second inequality follows from Lemma 3.5 as $a \geq \psi_0 (c_0) c_0$. It implies $F(m, \alpha, a, \xi_0) < 0$, which completes the proof. \hfill \Box

Lemma 3.7 Suppose $m \geq 2$. There is a function $f : (1, m+1) \to \mathbb{R}_+$ with the following property.

Suppose $\Omega$ is a convex set of $\mathbb{R}^2_{\geq 0} \cap \mathbb{R}^2_{\geq 0}$ with a non-empty intersection $\overline{\Omega}$ with the $x_1$-axis. Put $\Omega_\delta = (\overline{\Omega} + \delta D^2) \cap \mathbb{R}^2_{\geq 0}$ and let $\Gamma_\delta$ be the closure of the intersection of $\partial \Omega_\delta$ and the upper half plane (Figure 4). Then if $1 < \alpha < m + 1$ and if $\delta \geq f(\alpha) \cdot \text{diam}(\overline{\Omega})$ then for any point $x = (\xi, 0) \in \Gamma_\delta$ we have
\[
\int_{\Gamma_\delta} |x - y|^{\alpha - m - 2} (\xi - y_1) y_2^{m-2} dy_2 < 0.
\]

Proof. Let us write $\Omega = \Omega_\delta$ and $\Gamma = \Gamma_\delta$ in what follows.

Suppose $x \in \Gamma_\delta$. If $\Omega''$ is a compact domain that does not contain $x$ then
\[
\int_{\partial \Omega''} |x - y|^{\alpha - m - 2} (\xi - y_1) y_2^{m-2} dy_2
\]
\[
= \int_{\Omega''} |x - y|^{\alpha - m - 4} \left( (m + 1 - \alpha) (y_1 - \xi)^2 - y_2^2 \right) y_2^{m-2} dy_1 dy_2
\]
Figure 4: $\delta$-parallel body $\Omega = \Omega_{\delta}$. $\Gamma_{\delta}$ is an envelope of circles with radius $\delta$ whose centers lie on $\partial \Omega \cap \mathbb{R}^2_+$.  

Note that the integrand above is positive if $|y_2| < \sqrt{m + 1 - \alpha} |y_1 - \xi|$ and negative if $|y_2| > \sqrt{m + 1 - \alpha} |y_1 - \xi|$.

Two lines $y_2 = \pm \sqrt{m + 1 - \alpha} (y_1 - \xi)$ intersect $\Gamma$ in a point each as $\Omega$ is convex. Let $L$ be a line through the two intersection points. Remark that the line $L$ does not have any other intersection points with $\partial \Omega$ as $\Omega$ is convex. We construct a new domain $\Omega'$, which is a rectangle or a triangle, according to whether $L$ is parallel to the $y_1$-axis or not, bounded by a line segment of $L$ (denoted by $\Gamma_L$), a line segment of the $y_1$-axis, and some vertical line segments as is indicated in figures 5 and 6. When $\Omega'$ is a triangle, we take the vertical line segment as close to the point $x$ as possible.

Put $\Omega'' = \Omega' \setminus (\Omega' \cap \Omega)$ and $\Omega''' = \Omega \setminus (\Omega' \cap \Omega)$ (domains in blue (light gray) and red (dark gray) respectively in figures 5 and 6), and $\Omega'' = \Omega'' \cap \Omega'$. Note that $\Omega'' = (-\Omega'' \cup \Omega''').$ Then,

$$
\int_{\Gamma} |x-y|^{\alpha-m-2}(\xi-y_1)y_2^{-m-2} dy_2 = \int_{\partial \Omega} |x-y|^{\alpha-m-2}(\xi-y_1)y_2^{-m-2} dy_2 = \int_{\partial \Omega'} |x-y|^{\alpha-m-2}(\xi-y_1)y_2^{-m-2} dy_2 + \int_{\partial \Omega''} |x-y|^{\alpha-m-2}(\xi-y_1)y_2^{-m-2} dy_2.
$$

(3.8)

As

$$
\Omega'' \subset \{(y_1, y_2) | |y_2| < \sqrt{m + 1 - \alpha} |y_1|\}, \quad \Omega''' \subset \{(y_1, y_2) | |y_2| > \sqrt{m + 1 - \alpha} |y_1|\},
$$
the first term of the right hand side of (3.8) satisfies
\[
\int_{\partial \Omega'} |x - y|^{\alpha - m - 2} (\xi - y_1) y_2^{m-2} \, dy_2
= -\int_{\partial \Omega''} |x - y|^{\alpha - m - 4} \left\{ (m + 1 - \alpha)(y_1 - \xi)^2 - y_2^2 \right\} y_2^{m-2} \, dy_1 dy_2
+ \int_{\partial \Omega''} |x - y|^{\alpha - m - 4} \left\{ (m + 1 - \alpha)(y_1 - \xi)^2 - y_2^2 \right\} y_2^{m-2} \, dy_1 dy_2
< 0.
\]

The second term of the right hand side of (3.8) can be estimated as follows. Notice that \(\partial \Omega'\) consists of a line segment of \(L\), which we denote by \(\Gamma_L\), vertical edges, which we denote by \(\Gamma_v\), and a horizontal edge on the \(y_1\)-axis, where the integral vanishes. As the orientation of \(\Gamma_v\) is upward on the right edge and downward on the left edge, we have
\[
\int_{\Gamma_v} |x - y|^{\alpha - m - 2} (\xi - y_1) y_2^{m-2} \, dy_2 < 0.
\]

Therefore, it remains to show
\[
\int_{\Gamma_L} |x - y|^{\alpha - m - 2} (\xi - y_1) y_2^{m-2} \, dy_2 < 0
\]
when \(\Gamma_L\) is not parallel to the \(y_1\)-axis. It is equivalent to show that \(F(m, \alpha, a, \xi) < 0\) if \(a\) and \(\xi\) satisfy some conditions which are derived from the condition for \(\delta\).

We may assume without loss of generality that the slope of \(\Gamma_L\) is negative. Put \(d = \text{diam}(\Omega)\). Let \(z\) (or \(w\)) be the intersection point of the \(y_1\)-axis and \(\Gamma_v\) (or \(\Gamma_L\) respectively). Let \(p\) and \(q\) be the intersection points of \(\Gamma_L\) and the lines through \(x\) with slopes \(\pm \sqrt{m + 1 - \alpha}\) (Figure 7). Then

|\(x - p|, |x - q| \leq \delta + d.
\]

Therefore, the slope of \(\Gamma_L\) is not greater than \(\frac{d}{2\delta + d} \sqrt{m + 1 - \alpha}\), and hence
\[
|x - w| \geq \frac{\delta}{\sqrt{m + 2 - \alpha}} + \frac{\sqrt{m + 1 - \alpha}}{\sqrt{m + 2 - \alpha}} \delta \frac{2\delta + d}{d\sqrt{m + 1 - \alpha}} = \frac{2\delta(\delta + d)}{d\sqrt{m + 2 - \alpha}}. \tag{3.9}
\]

On the other hand, as we take \(\Gamma_v\) as close to \(x\) as possible, we have
\[
|x - z| \leq \delta + d. \tag{3.10}
\]

(i) Suppose \(m = 2\). Put \(f(\alpha) = \frac{1}{2} \sqrt{4 - \alpha}\). Then, if \(\delta \geq f(\alpha) d\) then
\[
|x - w| \geq \frac{2\delta(\delta + d)}{d\sqrt{4 - \alpha}} \geq \delta + d \geq |x - z|,
\]

Figure 7:
Lemma 3.4 implies $f_{\Omega_2} |x - y|^{\alpha - 4}(\xi - y_1) \, dy_2 < 0$.

(ii) Suppose $m \geq 3$. Let $u$ be the intersection point of $\Gamma_L$ and $\Gamma_v$. Then $|u - z| \geq \delta \sqrt{\frac{m+\alpha}{m+2-\alpha}}$. Put

$$\xi_0 = 2 \sqrt{\frac{m+2-\alpha}{m+1-\alpha}}.$$ 

If we assume $\frac{\delta}{\alpha} \geq 1$ then (3.9) and (3.10) imply

$$\frac{|x - z|}{|u - z|} \leq \frac{\delta + d}{\delta} \sqrt{\frac{m+2-\alpha}{m+1-\alpha}} \leq \xi_0.$$  

(3.11)

On the other hand, as the the slope of $\Gamma_L$ is not greater than $\frac{d}{2\delta + d} \sqrt{m+1-\alpha}$ we have

$$\frac{|w - z|}{|u - z|} \geq \frac{2(\delta + d)}{d \sqrt{m+1-\alpha}} = 2 \frac{1 + \frac{\delta}{d}}{\sqrt{m+1-\alpha}}.$$  

(3.12)

Put

$$f(\alpha) = \frac{\sqrt{m+1-\alpha}}{2} \psi_0 \left( \frac{2\xi_0^2 + 1}{\xi_0} \right) \frac{2\xi_0^2 + 1}{\xi_0} - 1$$

$$= \frac{\sqrt{m+1-\alpha}}{2} \left( 2 + \frac{3}{\left[ 1 + \left( 4\sqrt{\frac{m+2-\alpha}{m+1-\alpha}} + \frac{1}{2} \sqrt{\frac{m+1-\alpha}{m+2-\alpha}} \right)^2 + 1 \right]^{\frac{m+2-\alpha}{2}} - 1 \right)$$

$$\times \left( 4 \sqrt{\frac{m+2-\alpha}{m+1-\alpha}} + \frac{1}{2} \sqrt{\frac{m+1-\alpha}{m+2-\alpha}} \right) - 1.$$  

(3.13)

Remark that $f(\alpha) \geq 3$ and hence if $\frac{\delta}{\alpha} \geq f(\alpha)$ then the assumption $\frac{\delta}{\alpha} \geq 1$ above is satisfied.

If $\frac{\delta}{\alpha} \geq f(\alpha)$ then (3.12) implies

$$\frac{|w - z|}{|u - z|} \geq \psi_0 \left( \frac{2\xi_0^2 + 1}{\xi_0} \right) \frac{2\xi_0^2 + 1}{\xi_0}.$$  

(3.14)

Then by (3.11) and (3.14), Corollary 3.6 implies

$$\int_{\Gamma_L} |x - y|^{\alpha - m - 2}(\xi - y_1) \, y_2^{m-2} \, dy_2 < 0.$$

\[\square\]

Theorem 3.8 Suppose $m \geq 2$ and $1 < \alpha < m + 1$. Let $\tilde{\Omega}$ be a compact convex set in $\mathbb{R}^m$ with a piecewise $C^1$ boundary. If $\delta \geq f(\alpha) \cdot \text{diam}(\tilde{\Omega})$, where $f(\alpha)$ is given in Lemma 3.7, then (3.11) holds on $\tilde{\Omega}$ for any $j$ ($1 \leq j \leq m$).

Proof. Put $\Omega = \tilde{\Omega} + \delta B^m$ in what follows. Suppose $x \in \tilde{\Omega}$. By the symmetry, we may assume that $j = 1$ and that $x$ is on the $x_1$-axis. We omit the proof for the case when $\alpha = m$ as it is same as that for the case when $1 < \alpha < m$.

(i) The case when $m = 2$. Recall (2.2):

$$\frac{\partial^2 V^{(\alpha)}_\Omega}{\partial x_1^2}(x) = (2 - \alpha) \int_{\partial \Omega} |x - y|^{\alpha - 4}(x_1 - y_1) \, dy_2.$$
Divide $\partial \Omega$ into two parts by the $x_1$-axis, and lemma 3.7 implies the conclusion.

(ii) The case when $m \geq 3$. We use the orthogonal decomposition

$$\mathbb{R}^m = \mathbb{R} \oplus \mathbb{R}^{m-1} = \langle x_1 \rangle \oplus \langle x_2, \ldots, x_m \rangle.$$ 

Suppose the intersection of $\Omega$ and the $x_1$-axis is given by $[x_1^{\min}, x_1^{\max}]$.

Let $S_{m-2}$ be the unit sphere in $\mathbb{R}^{m-1}$. Suppose $\theta_2, \ldots, \theta_{m-1}$ are local coordinates of $S_{m-2}$.

Put $\theta = (\theta_2, \ldots, \theta_{m-1})$, and let $\gamma(\theta)$ be the corresponding point on $S_{m-2}$. Let $\Pi_\gamma(\theta)$ be a half 2-plane in $\mathbb{R}^m$ with the axis being the $x_1$-axis that contains the point $\gamma(\theta)$.

Assume that $\partial \Omega$ can locally be parametrized by

$$\Phi(t, \theta) = (f(t, \theta), g(t, \theta)\gamma(\theta)) \in \mathbb{R} \oplus \mathbb{R}^{m-1} \quad (t_0(\theta) \leq t \leq t_1(\theta))$$ 

so that the following conditions are satisfied.

- $f$ and $g$ are piecewise $C^1$-functions with $(f_t)^2 + (g_t)^2 > 0$,
- $f(t_0(\theta), \theta) = x_1^{\max}$, $f(t_1(\theta), \theta) = x_1^{\min}$,
- $g(t, \theta) \geq 0$, namely $\Phi(t, \theta) \in \Pi_\gamma(\theta)$, and $g(t_0(\theta), \theta) = g(t_1(\theta), \theta) = 0$.

Then, if we put $\Gamma_\gamma(\theta) = \partial \Omega \cap \Pi_\gamma(\theta)$ then $\Gamma_\gamma(\theta)$ can be expressed with respect to the $x_1$-axis and an orthogonal axis in $\Pi_\gamma(\theta)$ by (Figure 8)

$$\tilde{y}(t, \theta) = (\tilde{y}_1, \tilde{y}_2) = (f(t, \theta), g(t, \theta)) \quad (t_0(\theta) \leq t \leq t_1(\theta)).$$

Put

$$\nu = \frac{\partial \Phi}{\partial t} \times \frac{\partial \Phi}{\partial \theta_2} \times \cdots \times \frac{\partial \Phi}{\partial \theta_{m-1}},$$

which is a normal vector to $\partial \Omega$. Then $\nu$ is an outer normal vector if and only if

$$(\Phi(t, \theta) - p_0) \cdot \nu = \left| \Phi(t, \theta) - p_0 \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial \theta_2} \cdots \frac{\partial \Phi}{\partial \theta_{m-1}} \right| > 0 \quad (3.15)$$

for any point $p_0$ in $\Omega$ as $\Omega$ is convex. When $f_t \neq 0$ we can take $p_0$ in $\Pi_\gamma(\theta)$ so that $p_0$ has the same $x_1$-coordinate as $\Phi(t, \theta)$. Then $\Phi(t, \theta) - p_0$ is a positive multiple of $(0, -(\text{sgn } f_t)\gamma(\theta))$. Therefore, if $f_t \neq 0$ then (3.15) is equivalent to

$$0 < \left| \begin{array}{cccccccc} 0 & f_t & g_{\theta_2} & \cdots & g_{\theta_{m-1}} \\ -(\text{sgn } f_t)\gamma & g_{\theta_2} & g_{\theta_3} & \cdots & g_{\theta_{m-1}} \end{array} \right| = g^{m-2}|f_t| \left| \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & \gamma & \gamma_{\theta_2} & \cdots & \gamma_{\theta_{m-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \right|.$$
Assume \( \theta_2, \ldots, \theta_{m-1} \) are positive local coordinates of \( S^{m-2} \), i.e. \( |\gamma \, \gamma_{\theta_2} \cdots \gamma_{\theta_{m-1}}| > 0 \). Then \( \nu \) is an outer normal vector to \( \partial \Omega \). This holds even when \( f_1 = 0 \) because in this case \( \nu \) is outer normal if and only if \( (\text{sgn} \, g_1) e_1 \cdot \nu > 0 \), which follows from \( 3.10 \) below.

On the other hand,

\[
\begin{align*}
e_1 \cdot \nu &= \begin{vmatrix}
1 & f_t & g_{\theta_1} & \cdots & g_{\theta_{m-1}} \\
0 & g_{\nu} & g_{\theta_2} + g_{\gamma_{\theta_2}} & \cdots & g_{\theta_{m-1}} + g_{\gamma_{\theta_{m-1}}} \\
\end{vmatrix} \\
&= g^{m-2} g_t |\gamma \, \gamma_{\theta_2} \cdots \gamma_{\theta_{m-1}}|.
\end{align*}
\]

Since

\[
dS^{m-2} = |\gamma \, \gamma_{\theta_2} \cdots \gamma_{\theta_{m-1}}| \, d\theta_2 \cdots d\theta_{m-1},
\]

\[
d\sigma = |\nu| \, dt \, d\theta_2 \cdots d\theta_{m-1},
\]

we have

\[
e_1 \cdot n \, d\sigma = g^{m-2} g_t |\gamma \, \gamma_{\theta_2} \cdots \gamma_{\theta_{m-1}}| \, dt \, d\theta_2 \cdots d\theta_{m-1}
\]

\[
= g^{m-2} g_t \, dt \, dS^{m-2}.
\]

Therefore, \( 2.2 \) implies that

\[
\frac{\partial^2 V^{(\alpha)}_{\Omega}(x)}{\partial x_1^2} = (m - \alpha) \int_{\partial \Omega} |x - y|^{\alpha - m - 2} (x_1 - y_1) \, e_1 \cdot \nu(y) \, d\sigma(y)
\]

\[
= (m - \alpha) \int_{S^{m-2}} \left( \int_{\Gamma(y)} |x - \bar{y}|^{\alpha - m - 2} (x_1 - \bar{y}_1) g^{m-2} g_t \, dt \right) dS^{m-2}.
\]

By lemma \( 3.7 \)

\[
\int_{\Gamma(y)} |x - \bar{y}|^{\alpha - m - 2} (x_1 - \bar{y}_1) g^{m-2} g_t \, dt = \int_{\Gamma(y)} |x - \bar{y}|^{\alpha - m - 2} (x_1 - \bar{y}_1) \, \bar{g}_2^{m-2} \, d\bar{g}_2 < 0
\]

for each point \( \gamma(\theta) \) in \( S^{m-2} \), which completes the proof. \( \square \)

**Corollary 3.9** Suppose \( m \geq 2 \) and \( 1 < \alpha < m + 1 \). For any compact convex set \( \tilde{\Omega} \) in \( \mathbb{R}^m \) with a piecewise \( C^1 \) boundary, if \( \delta \geq f(\alpha) \cdot \text{diam}(\tilde{\Omega}) \), where \( f(\alpha) \) is given in Lemma \( 3.7 \) then \( \tilde{\Omega} + \delta B^m \) has a unique \( r^{\alpha - m} \)-center.

When \( m = 2 \) we have \( \sup_{1 < \alpha < 3} f(\alpha) = \sqrt{3} \), so if we put \( \varphi(2) = \sqrt{3} \) we completes the proof of Theorem \( 3.1 \) for the case when \( m = 2 \).

When \( m \geq 3 \), unfortunately we have \( \sup_{1 < \alpha < m+1} f(\alpha) = +\infty \) as \( \lim_{\alpha \downarrow m+1} f(\alpha) = +\infty \).

**Lemma 3.10** Suppose \( m \geq 3 \). For any \( b > 0 \) there is \( \alpha_0 = \alpha_0(b) \) with \( m < \alpha_0 < m + 1 \) such that for any compact convex set \( \tilde{\Omega} \) in \( \mathbb{R}^m \) with a piecewise \( C^1 \) boundary, if \( \delta \geq b \cdot \text{diam}(\tilde{\Omega}) \) then \( \tilde{\Omega} + \delta B^m \) has a unique \( r^{\alpha - m} \)-center if \( \alpha_0 \leq \alpha < m + 1 \).

**Proof.** Suppose \( \tilde{\Omega} \) has diameter \( d \) and \( x \in \tilde{\Omega} \). Let \( C_j(\alpha) \) be the cone with vertex \( x \) given by

\[
C_j(\alpha) = \left\{ y \mid -(m + 1 - \alpha)(x_j - y_j)^2 + \sum_{i \neq j} (x_i - y_i)^2 \leq 0 \right\}.
\]
The radial function of $\tilde{\Omega} + \delta B^m$ with respect to $x$ defined by $\rho(v) = \sup\{t \geq 0 \mid x + tv \in \tilde{\Omega} + \delta B^m\}$ ($v \in S^{m-1}$) satisfies $\delta \leq \rho(v) \leq \delta + d$ for any $v$. Therefore

$$\frac{1}{\alpha - m} \cdot \frac{\partial^2 V^r_{\tilde{\Omega} + \delta B^m}}{\partial x_j^2} (x) = \int_{\tilde{\Omega} + \delta B^m} |x - y|^{\alpha - m - 4} \left( -(m + 1 - \alpha)(x_j - y_j)^2 + \sum_{i \neq j} (x_i - y_i)^2 \right) d\mu(y)$$

$$\geq \int_{B^{\alpha}_{R + \delta}(x) \cap C_j(\alpha)} |x - y|^{\alpha - m - 4} \left( -(m + 1 - \alpha)(x_j - y_j)^2 + \sum_{i \neq j} (x_i - y_i)^2 \right) d\mu(y)$$

$$+ \int_{B^m(x) \cap C_j(\alpha)^c} |x - y|^{\alpha - m - 4} \left( -(m + 1 - \alpha)(x_j - y_j)^2 + \sum_{i \neq j} (x_i - y_i)^2 \right) d\mu(y).$$

Define $g: (m, m + 1) \times \mathbb{R}_+ \to \mathbb{R}$ by

$$g(\alpha, \beta) = \int_{X_{\alpha, \beta}} \frac{-(m + 1 - \alpha)y_1^2 + \sum_{i \geq 1} (x_i - y_i)^2}{|y|^{m+4-\alpha}} d\mu(y),$$

where

$$X_{\alpha, \beta} = B^m \cup \left( B^m_{1 + \frac{\beta}{\alpha}} \cap C_1(\alpha) \right).$$

Remark that $g(\alpha, \beta)$ is an increasing function of $\beta$. Fix $b > 0$. As $g(\alpha, b)$ is continuous with respect to $\alpha$ and $g(m + 1, b) > 0$, there is $\alpha_0 \in (m, m + 1)$ such that if $\alpha_0 \leq \alpha < m + 1$ then $g(\alpha, b) > 0$, which completes the proof as the right hand side of (3.17) is proportional to $g(\alpha, b)$.

Suppose $m \geq 3$. Put

$$\varphi(m) = \max \left\{ 10, \sup_{1 < \alpha \leq \alpha_0(10)} f(\alpha) \right\} = \max \left\{ 10, \max_{1 \leq \alpha \leq \alpha_0(10)} f(\alpha) \right\},$$

where $f(\alpha)$ is given by (3.13) and $\alpha_0$ is given in Lemma 5.10. Then, Theorem 5.8 and Lemma 5.10 implies Theorem 5.7 for the case when $m \geq 3$.

Remark 3.11 In [O3], using the same renormalization process of defining energy functionals of knots ([O1], [O2]), we renormalized $V^r_{\tilde{\Omega}}(x)$ so that it is well-defined for $\alpha \leq 0$ and $x \in \tilde{\Omega}$. The $r^{\alpha-m}$-center of $\tilde{\Omega}$ for $\alpha \leq 0$ can be defined in a similar way: it is a point in $\tilde{\Omega}$ where $V^r_{\tilde{\Omega}}$ attains the maximum value. We can show $\frac{\partial^2 V^r_{\tilde{\Omega}}}{\partial x_j^2} < 0$ on $\tilde{\Omega}$ for any $j$ if $\alpha \leq 1$ and $\tilde{\Omega}$ is convex by a similar way as in Lemma 3.7 and Theorem 3.8. This gives an alternative proof of the uniqueness of the $r^{\alpha-m}$-center of $\tilde{\Omega}$ when $\alpha \leq 1$ and $\tilde{\Omega}$ is convex.

References

[GNN] B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979), 209–243.

[H] I. Herbur, On the Uniqueness of Gravitational Centre, Mathematical Physics, Analysis and Geometry 10 (2007), 251–259.

[L1] E. Lutwak, Dual mixed volumes, Pacific J. Math. 58 (1975), 531–538.

[L2] E. Lutwak, Intersection bodies and dual mixed volumes, Advances in Mathematics 71 (1988), 232–261.

[M] M. Moszyńska, Looking for selectors of star bodies, Geom. Dedicata 81 (2000), 131–147.
[O1] J. O’Hara, *Energy of a knot*, Topology 30 (1991), 241–247.

[O2] J. O’Hara, *Family of energy functionals of knots*, Topology Appl. 48 (1992), 147–161.

[O3] J. O’Hara, *Renormalization of potentials and generalized centers*, to appear in Adv. Appl. Math., available at arXiv:1008.2731.

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