STABILITY OF HASIMOTO SOLITONS IN ENERGY SPACE FOR A FOURTH ORDER NONLINEAR SCHRÖDINGER TYPE EQUATION

ZHONG WANG*

School of Mathematics and Big Data, Foshan University
Foshan, Guangdong 528000, China

(Communicated by Adrian Constantin)

Abstract. In this article we investigate the nonlinear stability of Hasimoto solitons, in energy space, for a fourth order Schrödinger equation (4NLS) which arises in the context of the vortex filament. The proof relies on a suitable Lyapunov functional, at the $H^2$ level, which allows us to describe the dynamics of small perturbations. This stability result is also extended to Sobolev spaces $H^m$ for all $m \in \mathbb{Z}_+$ by employing the infinite conservation laws of 4NLS.

1. Introduction. In this paper we consider the following fourth order nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + \nu \partial_x^4 u + \mathcal{N}(u, \bar{u}, \partial_x u, \partial_x \bar{u}, \partial_x^2 u, \partial_x^2 \bar{u}) = 0,$$

(4NLS)

where $u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is an unknown function, $\nu \neq 0$ is a real constant. The nonlinear term $\mathcal{N}$ is given by

$$\mathcal{N}(u, \bar{u}, \partial_x u, \partial_x \bar{u}, \partial_x^2 u, \partial_x^2 \bar{u})$$

$$= \frac{1}{2} |u|^2 u + \nu \left( \frac{3}{8} |u|^4 u + \frac{3}{2} (\partial_x u)^2 \bar{u} + |\partial_x u|^2 u + \frac{1}{2} u^2 \partial_x^2 \bar{u} + 2 |u|^2 \partial_x^2 u \right).$$

(4NLS) arises in the context of the three-dimensional motion of an isolated vortex filament.

Let us denote the centerline of the above isolated vortex filament by $X = X(t, x)$, which is a function of time $t$ and arclength $x$. Let $(\kappa, \tau)$ be the curvature and torsion, and $(t, n, b)$ be the Frenet-Serret frame of centerline of the vortex filament, respectively. By using the method of localized induction approximation, in 1906, Da Rios [7] proposed the following equation

$$\partial_t X = \kappa b,$$

(1.1)

which approximates the three-dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. We refer the reader to [22] for a survey on Da Rios’s work.

2010 Mathematics Subject Classification. Primary: 35Q35, 76W05; Secondary: 35B65.

Key words and phrases. Orbital stability, fourth order, Schrödinger equation, vortex filament.

This work is supported by the China National Natural Science Foundation under grant number 11571381.

* Corresponding author: Zhong Wang.
In 1972, Hasimoto [12] found that the Da Rios equation (1.1) is reduced to the cubic Schrödinger equation
\[ i\partial_t u + \partial_x^2 u + \frac{1}{2}|u|^2 u = 0 \] (1.2)
via the Hasimoto transform:
\[ u(t, x) = \kappa(t, x) \exp(-i \int_0^t A(s) ds + i \int_0^x \tau(t, y) dy), \]
where
\[ A(t) = -i \partial_t \kappa + \partial_x^2 \kappa + 2i\tau \partial_x \kappa + i\kappa \partial_x \tau - \kappa \tau^2 + \frac{1}{2} \kappa^3 \bigg|_{x=0}. \]
To describe the motion of an actual vortex filament precisely, several authors take into account of the effect from higher-order approximate of the equation. In particular, Fukumoto and Moffatt [8] pointed out that the local self-induced flow around the core comprises not only a uniform flow but also a straining field which deforms the core into an ellipse. Taking this phenomenon into account, Fukumoto and Moffatt proposed the following higher order approximate equation
\[ \partial_t X = \kappa b - \nu \left( \kappa^2 \tau t + (2\tau \partial_x \kappa + \kappa \partial_x \tau) \mathbf{n} + (\kappa \tau^2 - \partial_x^2 \kappa - \frac{1}{2} \kappa^3) b \right). \] (1.3)
By employing the Hasimoto transform again, we see that (1.3) is transformed to (4NLS). For more physical background of (4NLS), see Fukumoto and Moffatt [8] and Segata [24].

It is well known that (1.2) is a completely integrable equation and can be solved by the inverse scattering transform (IST), see Zakharov and Shabat [27]. (1.2) has a one parameter family of solitary waves of the form
\[ u_\omega(t, x) = e^{i\omega t} Q_\omega(x), \]
where \( \omega > 0, \)
\[ Q_\omega(x) = 2\sqrt{\omega} \text{sech}(\sqrt{\omega} x). \] (1.4)
Using the fact that (1.2) is invariant under the Galilean transform, namely, if \( u(t, x) \) is a solution of (1.2), then \( v(t, x) = e^{i(\frac{1}{4}c^2 - \frac{1}{4}c^2 t)} u(t, x - ct) \) is also a solution of (1.2).

Combining the above two facts, we obtain a two parameter family of solitary waves \( u_{\omega, c}(t, x) = e^{i(\omega t - \frac{1}{4}c^2 t + \frac{1}{4}c x)} Q_\omega(x - ct). \)

Let us return to (4NLS). It is known that (4NLS) is also a completely integrable equation with an infinite family of conservation laws, see Langer and Perline [16]. Noticing that (4NLS) is a perturbation of (1.2) with high order dispersion and nonlinearities. Usually, the complete integrability will be lost under perturbations and it is a difficult problem to find suitable perturbations which retain the integrability, in this context, (4NLS) is a good example. However, we remark that any modification of the parameters in (4NLS) will break the integrability. The following quantities are conserved formally along the flow of (4NLS):
\[ M(u) := \frac{1}{2} \int_R |u|^2 dx, \] (1.5)
\[ P(u) := \frac{1}{2} \text{Im} \int_R u \partial_x \bar{u} dx, \] (1.6)
\[ E(u) := \frac{1}{2} \int_R |\partial_x u|^2 - \frac{1}{4} |u|^4 dx, \] (1.7)
\[ F(u) := \frac{1}{2} \int_R \left| \frac{\partial_x^2 u}{2} \right|^2 dx + \frac{1}{16} \int_R |u|^6 dx. \]
In general, for $m \geq 3$ the conservation quantities could be expressed as follows:

$$G_m(u) := \frac{1}{4} \int_R |u|^2 \bar{u} \partial_x^2 u \ dx - \frac{1}{2} \int_R |u|^2 |\partial_x u|^2 \ dx.$$  (1.8)

In general, for $m \geq 3$ the conservation quantities could be expressed as follows:

$$G_m(u) := \frac{1}{2} \int_R |\partial_x^m u|^2 \ dx + \int_R q_m(u_R, u_I, ..., \partial_x^{m-1} u_R, \partial_x^{m-1} u_I) \ dx,$$  (1.9)

where $u = u_R + i u_I, u_R, u_I$ are real value functions and $q_m$ is a polynomial.

Before we state our main results, let us recall first the earlier results on (4NLS). Well-posedness of the initial value problems of (4NLS) have been considered by different groups of authors in e.g. [14, 15, 23] and references cited therein, their results indicate that (4NLS) is locally well-posed in $H^s$ for $s \geq \frac{1}{2}$ and whereas ill-posed in $H^s$ for $s < \frac{1}{2}$. By using the conservation laws proposed above, one can easily deduce that (4NLS) is globally well-posed in $H^m$ for $m \in \mathbb{Z}_+$. Compare to (1.2), (4NLS) does not have the Galilean transform, which is one of the crucial differences between (1.2) and (4NLS). In spite of the lack of the Galilean invariance for (4NLS), Hoseini and Marchant [13] found a two parameter family of solitary waves (called Hasimoto soliton) of the form

$$R(t, x) = e^{i\beta t} Q_{\omega, \alpha} (x - ct) := e^{i\beta t} e^{i\alpha x} Q_{\omega} (x - ct),$$  (1.10)

where $\omega > 0, \alpha \in \mathbb{R}$, $Q_{\omega}$ is defined in (1.4) and the above parameters satisfy the following relation:

$$\beta = \nu \alpha^4 + \nu \omega^2 - 6\nu \omega \alpha^2 - \alpha^2 + \omega, \quad c = -4\nu \alpha^3 + 4\nu \omega \alpha + 2\alpha.$$  (1.11)

Concerning the stability of above Hasimoto soliton, Cazenave and Lions [6] proved that Hasimoto soliton in the case $\nu = 0$ is orbitally stable in $H^1$. For the case $\nu \neq 0$, when $\alpha = 0$, Masaya and Segata [17] showed that the corresponding one-parameter solitary wave is orbitally stable in $H^m$ for $m \in \mathbb{Z}_+$. Later on, Segata [21] prove that the two-parameter solitary wave (1.10) is orbitally stable in $H^1$ by employing the variational method due to Cazenave and Lions [6]. However, the stability of two-parameter solitary wave in its energy space $H^2(\mathbb{R})$ is still unknown. In this article, we investigate the case $\nu < 0, \alpha \neq 0$ and show that the above Hasimoto soliton is orbitally stable in the energy space $H^2(\mathbb{R})$. Our main result of this manuscript is the following.

**Theorem 1.1.** Suppose that $\nu < 0$ and $|\alpha|$ large, $\omega > 0$ and let $\beta$ and $c$ be given by (1.11). The Hasimoto soliton $e^{i\beta t} Q_{\omega, \alpha} (x - ct)$ defined by (1.10) is orbitally stable in $H^2(\mathbb{R})$ in the following sense: There exist parameters $\epsilon_0, A_0$ such that the following holds. Consider $u_0 \in H^2(\mathbb{R})$, assume that there exists $\epsilon \in (0, \epsilon_0)$ such that

$$\|u_0 - Q_{\omega, \alpha}\|_{H^2(\mathbb{R})} < \epsilon,$$  (1.12)

then there exists $\theta(t)$ and $y(t)$ such that the solution $u(t) \in C([0, +\infty), H^2(\mathbb{R}))$ of (4NLS), with the initial data $u(0) = u_0$, satisfies

$$\|u(t) - e^{i(\theta(t) + i\alpha x)} Q_{\omega} (x - y(t))\|_{H^2(\mathbb{R})} < A_0 \epsilon,$$  (1.13)

with

$$\sup_{t \in (0, +\infty)} |\theta'(t) - \beta| + |y'(t) - c| \leq CA_0 \epsilon.$$  (1.14)

**Corollary 1.** Suppose that $\nu < 0$ and $|\alpha|$ large, $\omega > 0$ and let $\beta$ and $c$ be given by (1.11). The Hasimoto soliton $e^{i\beta t} Q_{\omega, \alpha} (x - ct)$ is orbitally stable in $H^m(\mathbb{R})$ for $m \in \mathbb{Z}_+$ in the sense of Theorem 1.1.
Remark 1. Our method is valid in the case $|\alpha| \geq \alpha_0$, for some $\alpha_0 > 0$ (for the value of $\alpha_0$, see Remark 3 below), since in this case, we can character the spectral information of the corresponding operator around the Hasimoto solitons and show that the linearized operator has a unique negative eigenvalue. However, for $|\alpha| < \alpha_0$, it is not an easy task. We remark that once we confirm the unique negative eigenvalue property of this operator, the nonlinear stability of the Hasimoto solitons in $H^2(\mathbb{R})$ will be valid for all $\alpha \in \mathbb{R}$.

Roughly speaking, there are two main approaches to prove the stability of standing waves: the method based on the study of a linearized Hamiltonian around the standing waves, see [4, 5, 10, 11, 25, 26, 3, 2, 1, 19] and references cited therein, and the variational method developed in [6]. To prove Theorem 1.1, we use the formal approach because of the lack of a maximum principle of $\partial^4_x$, the ground states are not positive, in addition, the variational approach provides the orbital stability of the set of minimizers and may not the stability of the standing waves itself (uniqueness modula symmetries is unknown).

Recently, Natali and Pastor [19] successfully proved the stability of standing waves of a cubic fourth order Schrödinger equation with mixed dispersion. Their proof relies on the total positivity theory developed in [1, 2] to obtain the spectral properties of linearized operator around the standing wave and a suitable Lyapunov functional to study the dynamics of small perturbations. Our proof is essentially similar to [19], however, the linearized operators we investigated contain derivatives, it seems that we can not apply the total positivity theory in this situation. To overcome this difficulty, we are trying to make full use of the result of the reduced second order operators and follow the approach of Neves and Lopes [20] to have the spectral properties of linearized operator around the Hasimoto soliton.

The manuscript is organized as follows. In section 2, we collect some preliminaries on (4NLS) and review the well-posed theory of (4NLS). We will study the spectral properties of the corresponding linearized operator around the Hasimoto soliton in section 3. Section 4 is devoted to the proof of Theorem 1.1 and Corollary 1.

2. Linearized operator around Hasimoto soliton. In this section we collect some preliminaries on (4NLS), we first calculate the linearized operator around Hasimoto soliton, then we give a brief review the well-posedness issues associated with Cauchy problem of (4NLS).

Let us first introduce some notation, given $s \in \mathbb{R}$, by $H^s := H^s(\mathbb{R})$ we denote the usual Sobolev space. In particular $H^0(\mathbb{R}) \simeq L^2(\mathbb{R})$. The scalar product in $H^s$ will be denoted by $(\cdot, \cdot)_{H^s}$. We also use the subscript odd to denote space of odd functions and the subscript ev to denote space of even functions.

Since the Hasimoto soliton $R(t, x) = e^{i\beta t} Q_{\omega, \alpha}(x - ct)$ in (1.10) is a solution of (4NLS), for fixed $t$, we see that $R$ satisfies the following nonlinear stationary equation

\[
-\partial^2_x R + (\beta + \alpha c) R - \frac{1}{2} |R|^2 R + ic \partial_x R - \nu (\partial^4_x R) + \frac{3}{8} |R|^4 R + \frac{3}{2} (\partial_x R)^2 R + |\partial_x R|^2 R + \frac{1}{2} R^2 \partial^2_x R + 2 |R|^2 \partial^2_x R = 0.
\]  

(2.1)

In view of (2.1), we can check that the function $R$ is a critical point of the following functional

\[
S(u) := E(u) + (\beta + \alpha c) M(u) + c P(u) - \nu F(u).
\]  

(2.2)
Let us introduce a functional $H$ in $\mathbb{R} \times H^2(\mathbb{R})$ as follows: For $\varphi \in H^2(\mathbb{R})$,

$$H(\varphi) := \langle S''(R)\varphi, \varphi \rangle.$$  

(2.3)

It is a lengthy but straightforward computation to show that

$$S''(R)\varphi = -\partial_x^2\varphi + (\beta + \alpha c)\varphi + ic\partial_x \varphi - (|R|^2\varphi + \frac{1}{2} R^2\varphi) \quad \text{and}$$

$$H(\varphi) = \int_{\mathbb{R}} |\partial_x \varphi|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} (|R|^2|\varphi|^2 + 2\text{Re}(R\bar{\varphi})^2) \, dx + (\beta + \alpha c) \int_{\mathbb{R}} |\varphi|^2 \, dx$$

$$+ c\text{Im} \int_{\mathbb{R}} \varphi \partial_x \bar{\varphi} \, dx - \nu \int_{\mathbb{R}} |\partial_x \varphi|^2 + \frac{9}{8} |R|^4 |\varphi|^2 + \frac{3}{4} |R|^2 \text{Re}(R\bar{\varphi})^2 + \frac{3}{2} \text{Re}(\partial_x R \bar{\varphi})^2$$

$$+ 3\text{Re}(\partial_x R \bar{\varphi} R) + 2\text{Re}(\partial_x R \bar{\varphi}(\bar{R} R) + [\partial_x R]^2 |\varphi|^2 + \frac{1}{2} \text{Re}(R^2 \bar{\varphi} \partial_x^2 \bar{\varphi})$$

$$+ 2\text{Re}(\bar{R} R |\varphi|^2 + 4\text{Re}(R\bar{\varphi}) \text{Re}(\partial_x^2 R \bar{\varphi}) + 2 |R|^2 \text{Re}(\partial_x^2 \varphi) \bar{\varphi}) \, dx.$$  

(2.4)

Let us review the case $\nu = 0$. In this case, (2.2) reduces to the Weinstein functional adapted to (1.2), that is

$$S(\varphi) = E(\varphi) + (\omega + \frac{c^2}{4})M(\varphi) + cP(\varphi),$$

$$S''(R)\varphi = -\partial_x^2\varphi + (\omega + \frac{c^2}{4})\varphi + ic\partial_x \varphi - (|R|^2\varphi + \frac{1}{2} R^2\varphi)$$

Since (1.2) has the Galilean invariance, so if we set

$$\varphi(x) = e^{i(\omega t - \frac{1}{4}c^2 t + \frac{1}{4}cx)} \psi(x - ct).$$

Then we have

$$S(\varphi) = E(\psi) + \omega M(\psi),$$

$$S''(R)\varphi = e^{i(\omega t - \frac{1}{4}c^2 t + \frac{1}{4}cx)} ( - \partial_x^2 \psi + \omega \psi - Q^2 \psi - \frac{1}{2} Q^2 \bar{\psi}),$$

where henceforth we replace $Q_{\omega}$ by $Q$ for simplicity. The spectrum information of $S''(R)$ in this case is well understood [25, 26], we summarise it as follows:

**Lemma 2.1.** Suppose $\omega > 0$. The operator $S''(R)$ defined on $L^2(\mathbb{R})$ with domain $H^4(\mathbb{R})$ has a unique negative eigenvalue, which is simple. The eigenvalue zero is double with associated eigenfunctions $R'$ and $iR$. Moreover, the essential spectrum is the interval $[\omega, \infty)$.

However, when $\nu \neq 0$, does not have the Galilean invariance any more, the spectral analysis of $S''(R)$ in this case is unknown. Similar as before, we set

$$\varphi(x) = e^{i(\beta t + \alpha x)} \psi(x - ct).$$

Then a tedious computation yields

$$L_\alpha \psi := e^{-i(\beta t + \alpha x)} S''(R) \varphi$$

$$= -\nu \partial_x^2 \psi - (-6\nu \alpha^2 + 1) \partial_x^2 \psi + (-\nu \alpha^4 + \beta + \alpha^2) \psi + 4\nu \alpha (\omega \partial_x \psi - \partial_x^3 \psi)$$

$$+ 2(\partial_x^2 \varphi + R \partial_x^2 \bar{\varphi} + \bar{R} \partial_x^2 \varphi)$$

and

$$H(\varphi) = \int_{\mathbb{R}} |\partial_x \varphi|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} (|R|^2|\varphi|^2 + 2\text{Re}(R\bar{\varphi})^2) \, dx + (\beta + \alpha c) \int_{\mathbb{R}} |\varphi|^2 \, dx$$

$$+ c\text{Im} \int_{\mathbb{R}} \varphi \partial_x \bar{\varphi} \, dx - \nu \int_{\mathbb{R}} |\partial_x \varphi|^2 + \frac{9}{8} |R|^4 |\varphi|^2 + \frac{3}{4} |R|^2 \text{Re}(R\bar{\varphi})^2 + \frac{3}{2} \text{Re}(\partial_x R \bar{\varphi})^2$$

$$+ 3\text{Re}(\partial_x R \bar{\varphi} R) + 2\text{Re}(\partial_x R \bar{\varphi}(\bar{R} R) + [\partial_x R]^2 |\varphi|^2 + \frac{1}{2} \text{Re}(R^2 \bar{\varphi} \partial_x^2 \bar{\varphi})$$

$$+ 2|\partial_x \varphi|^2 + 4\text{Re}(R\bar{\varphi}) \text{Re}(\partial_x^2 R \bar{\varphi}) + 2 |R|^2 \text{Re}(\partial_x^2 \varphi) \bar{\varphi}) \, dx.$$  

(2.5)
In view of (2.6), we observe that \( \partial_x \psi \) will not be cancelled unless \( \alpha = 0 \), in addition, there exists an additional \( \partial_x^2 \psi \) term. More precisely, by writing \( \psi = f + ig \) and separating real and imaginary parts, we obtain

\[
L_\alpha = \begin{pmatrix} L_{\alpha,+} & P \\ P^* & L_{\alpha,-} \end{pmatrix},
\]

where

\[
L_{\alpha,+} = -(-6\nu \alpha^2 + 1) \partial_x^2 + (-\nu \alpha^4 + \beta + \alpha^2) - \nu \partial_x^4 - \frac{3}{2} - 9\nu \alpha^2)Q^2
\]

\[
L_{\alpha,-} = -(-6\nu \alpha^2 + 1) \partial_x^2 + (-\nu \alpha^4 + \beta + \alpha^2) - \nu \partial_x^4 - \frac{1}{2} - 3\nu \alpha^2)Q^2
\]

\[
P = -2\nu \alpha (2\omega \partial_x - 2\partial_x^3 - 3Q^2 \partial_x),
\]

\[
P^* = 2\nu \alpha (2\omega \partial_x - 2\partial_x^3 - 3Q^2 \partial_x - 6QQ').
\]

The analysis of the spectrum information of the operator \( S''(R) \) is significant in our later proof. In particular, we want to show that \( S''(R) \) has only one negative eigenvalue, which will be done in section 3.

2.1. A simple case \( \alpha = 0 \). We first consider the case \( \alpha = 0 \), in this case \( R = e^{i \beta_0 t} Q(x) \), where \( \beta_0 = \nu \omega^2 + \omega \) by (1.11). Then by (2.7), we have

\[
L_0 = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix},
\]

where the self-adjoint operators \( L_+, L_- \) are defined as follows:

\[
L_+ = -\partial_x^2 + \beta_0 - \frac{3}{2} Q^2
\]

\[
-\nu(\partial_x^4 + \frac{15}{8} Q^4 + \frac{5}{2} Q_x^2 + 5QQ_x \partial_x + 5QQ_{xx} + \frac{5}{2} Q^2 \partial_x^2),
\]

\[
L_- = -\partial_x^2 + \beta_0 - \frac{1}{2} Q^2
\]

\[
-\nu(\partial_x^4 + \frac{3}{8} Q^4 - \frac{1}{2} Q_x^2 + 3QQ_x \partial_x + QQ_{xx} + \frac{3}{2} Q^2 \partial_x^2).
\]

The analysis of its spectrum is a main task in our later section.

2.2. Well-posedness theory of \( \text{(4NLS)} \). For the rest of this section, let us review the well-posedness of the Cauchy problem of \( \text{(4NLS)} \). \( \text{(4NLS)} \) is locally well-posed in \( H^s \) for \( s \geq \frac{1}{2} \) and whereas ill-posed in \( H^s \) for \( s < \frac{1}{2} \), we refer to [14, 15, 23, 17] and references cited therein for details. By using the conservation laws proposed
in (1.7) and (1.8), we deduce that (1NLS) is global well-posed in $H^m$ for $m \in \mathbb{Z}_+$.

More precisely, we have

**Proposition 1.** For $m \in \mathbb{Z}_+$ and $u_0 \in H^m$, (1NLS) has a unique solution $u(t) \in C([0, +\infty); H^m)$. For any $r > 0$, the map $u_0 \mapsto u(t)$ is continuous from the ball

\[ \{ u \in H^m(\mathbb{R}) : \| u \|_{H^m(\mathbb{R})} < r \} \]

to $C([0, +\infty); H^m)$. Furthermore, $u(t)$ satisfies the following conservation laws:

\[ M(u(t)) = M(u_0), \quad P(u(t)) = P(u_0), \quad E(u(t)) = E(u_0), \]

\[ F(u(t)) = F(u_0), \quad \ldots, \quad G_m(u(t)) = G_m(u_0) \]

for all $t > 0$.

3. **Spectral analysis.** In this section we describe the spectrum the spectral properties of linearized operator $L_\alpha$. We want to show that $L_\alpha$ possesses, for all time, only one negative eigenvalue.

Let us introduce first the Greenberg and Maddocks-Sachs method in [9, 18], which is a tool enables us to count the number of negative eigenvalues of a class of self-adjoint, 2n-th order ordinary differential operators considered on the whole real line. We briefly introduce the above test for 2n-th order operator as follows.

Consider a 2n-th order operator of the form

\[ Lv = \sum_{j=1}^{n} (-1)^j \frac{\partial^2}{\partial x^2} (p_j(x)\partial^2_x v) = (-1)^n \frac{\partial^2}{\partial x^2} (p_n(x)\partial^2_x v) + \cdots + p_0(x), \quad (3.1) \]

where $p_n(x) = 1$ and $v \in H^n$. For $j = 0, \ldots, n - 1$, the coefficient functions $p_j(x)$ are smooth and satisfy

\[ \lim_{x \to \pm\infty} p_j(x) = \tilde{p}_j, \]

in addition, $p_j(x) - \tilde{p}_j$ decay exponentially as $x \to \pm\infty$. The corresponding constant coefficient operator is as follows:

\[ \tilde{L} = \sum_{j=1}^{n} (-1)^j \frac{\partial^2}{\partial x^2} (\tilde{p}_j(x)\partial^2_x v) = (-\partial^2_x + a_1)(-\partial^2_x + a_2) \cdots (-\partial^2_x + a_n), \quad (3.2) \]

The constant coefficient operator $\tilde{L}$ is easy to analyze by Fourier transform. Its spectrum is entirely continuous, consisting of all real numbers greater than or equal to the product $a_1a_2 \cdots a_n$. Moreover, by Weyl’s essential spectrum theorem (see Reed and Simon [21], Theorem XIII.14), $L$ and $\tilde{L}$ share the same essential spectrum $[a_1a_2 \cdots a_n, +\infty)$.

The tool for counting the the number of negative eigenvalues of $L$ in [18] (see Lemma 2.2) is as follows:

**Lemma 3.1.** Suppose that $L$ is a self-adjoint, 2n-th order operator of the form

\[ Lv = \tilde{L}v + Av, \]

where $\tilde{L}$ is of the form (3.2) and

\[ Av = \sum_{j=1}^{n-1} (-1)^j \frac{\partial^2}{\partial x^2} (q_j(x)\partial^2_x v) \]

with $q_j$ smooth and rapidly decaying as $x \to \pm\infty$. Then the operator $L$ has precisely $k$ negative eigenvalues where $k$ is number of roots of the $n \times n$ Wronskian
determinant of any set of \( n \) linearly independent solutions to \( Lv = 0 \), \( v(x) \to 0 \) as \( x \to -\infty \).

**Remark 2.** When \( \nu = 0 \), in this case, \( L_+ = -\partial_x^2 + \omega - \frac{3}{2}Q^2 \), we have \( n = 1 \), the Wronskian determinant reduces to \( v(x) \), where \( v(x) \) is a solution of \( L_+v = 0 \) and \( v(x) \to 0 \) as \( x \to -\infty \). We know that for some constants \( C \neq 0 \), \( v(x) = CQ'(x) = C \tanh(\sqrt{\omega}x) \text{sech}(\sqrt{\omega}x) \), which vanishes only once at \( x = 0 \), then by Lemma 3.1, we deduce that \( L_+ \) has one negative eigenvalue. Similarly, when \( \nu = 0 \), \( L_- = -\partial_x^2 + \omega - \frac{1}{2}Q^2 \), in this case, for some constants \( C \neq 0 \), \( v(x) = CQ(x) = C \text{sech}(\sqrt{\omega}x) \), which does not vanish any more, then by Lemma 3.1, we deduce that \( L_- \) has no negative eigenvalue.

However, Lemma 3.1 cannot apply to our case since the kernels of \( L_+ \), \( L_- \) do not spanned by two linearly independent functions. To overcome this, we follow the approach in [20], which study the isoinertial family of operators, this technique was used to prove stability of multi-solitons KdV [18] and BO [20] equations.

Let us define two auxiliary linear operators as follows:

\[ \begin{align*}
Mh(x) &= h'(x) + \tanh(x)h(x), & M'h = -h'(x) + \tanh(x)h(x). 
\end{align*} \tag{3.3} \]

We can easily verify that \( M \) and \( M' \) map odd functions in even functions and even functions in odd functions. Moreover, we have follow results concerning the kernel and the range of the auxiliary operators \( M \) and \( M' \) (see Lemma 5 in [20]).

**Lemma 3.2.** If \( M, M' : D(M) = D(M') = H^1 \subset L^2 \to L^2 \), are given by (3.3), then we have

1) the null space of \( M \) is spanned by \( Q \) and \( M' \) is injective;
2) \( M \) is onto and the image of \( M' \) is the subspace orthogonal to \( Q \).

Let us begin with the spectral analysis of \( L_0 \). Since \( L_0 \) is a diagonal operator, its eigenvalues are given by the eigenvalues of the operators \( L_+ \) and \( L_- \).

### 3.1. The spectrum of \( L_+ \)

Let \( Q \) be given in section 2 and then the operator \( L_+ \) reads as

\[ L_+ = -\partial_x^2 + \beta_0 - \frac{3}{2}Q^2 - \nu(\partial_x^4 + \frac{15}{8}Q^4 + \frac{5}{2}Q_x^2 + 5QQ_x\partial_x + 5QQ_{xx} + \frac{5}{2}Q^2\partial_x^2). \tag{3.4} \]

We have the following spectral properties concerning the operator \( L_+ \).

**Proposition 2.** Suppose that \( \nu \leq 0 \) and \( 1 + 2\nu \omega > 0 \). The operator \( L_+ \) in (3.4) defined on \( L^2(\mathbb{R}) \) with domain \( H^4(\mathbb{R}) \) has a unique negative eigenvalue. The eigenvalue zero is simple with associated eigenfunction \( Q' \). Moreover, the rest of the spectrum is bounded away from zero and the essential spectrum is the interval \([\beta_0, \infty)\).

**Proof.** By the exponential localization of \( Q \), the operator \( L_+ \) is a compact perturbation of the constant coefficient operator

\[ \hat{L} = -\nu\partial_x^4 - \partial_x^2 + \beta_0, \quad \beta_0 = \omega(1 + \nu \omega) > 0, \]

it is of the form \( (3.2) \), by Weyl's essential spectrum theorem, the essential spectrum of \( L_+ \) is the same as that of \( \hat{L} \), which is the interval \([\beta_0, \infty)\).

The main ingredient of the spectral analysis is that \( L_+ \) (here we take \( \omega = 1 \) to simplify the calculations) the following identity:

\[ ML_+M' = M'\hat{L}M. \tag{3.5} \]
Then the rest statement in Proposition 2 follows from the following result:

**Claim.** For \( h \in H^2_{\text{odd}} \), we have \( \langle L_+ h, h \rangle \geq 0 \) and \( \langle L_+ h, h \rangle = 0 \) if and only if \( h \) is a multiple of \( Q' = v \). Suppose that \( 1 + 2\nu \omega > 0 \). The operator \( L_+ \) in \( H^2_{\nu} \) has a unique negative eigenvalue and zero is not an eigenvalue any more.

Indeed, from Lemma 3.2, we see that \( h \in H^2 \) can be written as \( h = aQ + M'g \).

Now we consider first the case \( h \in H^2_{\text{odd}} \). In this case, of course \( a = 0 \), \( g \) is even and using (3.9) we deduce that

\[
\langle L_+ h, h \rangle = \langle L_+ M'g, M'g \rangle = \langle ML_+ M'g, g \rangle = \langle M' \tilde{L} M g, g \rangle = \langle \tilde{L} M'g, M'g \rangle.
\]

Since \( \langle \tilde{L}s, s \rangle \geq 0 \) for any \( s \in H^2 \) and \( \langle \tilde{L}s, s \rangle = 0 \) iff \( s = 0 \), we conclude that if \( h \in H^2_{\text{odd}} \) then we have \( \langle L_+ h, h \rangle \geq 0 \) and \( \langle L_+ h, h \rangle = 0 \) iff \( M'g = 0 \). Moreover, by Lemma 3.2 \( M'g = 0 \) implies that \( g \) has to be a multiple of \( Q \) and then \( h = M'g \) is also a multiple of \( Q' \).

Next we consider the case \( h \) is even, then \( h = aQ + M'g \), with \( g \) odd. In the hyperplane \( a = 0 \) we have \( \langle L_+ h, h \rangle = \langle \tilde{L} M g, M g \rangle \geq 0 \) and \( \langle L_+ h, h \rangle = 0 \) iff \( M g = 0 \), according to Lemma 3.2 since \( g \) is even, this implies that \( g = 0 \). Therefore in a hyperplane \( a = 0 \) we have \( \langle L_+ h, h \rangle > 0 \), this implies that \( L_+ \) can have at most one non-positive eigenvalue. Since \( L_+ \partial_a Q = -(1 + 2\nu \omega)Q < 0 \), we see that \( (L_+ \partial_a Q, \partial_a Q) < 0 \) and which implies that \( L_+ \) has exactly one negative eigenvalue in \( H^2_{\text{ev}} \), the claim is thus proved.

3.2. **The spectrum of \( L_- \).** The operator \( L_- \) reads as

\[
L_- = -\partial_x^2 + \beta_0 - \frac{1}{2}Q^2 - \nu(\partial_x^4 + \frac{3}{8}Q^4 - \frac{1}{2}Q^2 + 3QQ_x \partial_x + QQ_{xx} + \frac{3}{2}Q^2 \partial_x^2). \tag{3.6}
\]

We have the following spectral properties concerning the operator \( L_- \).

**Proposition 3.** Suppose that \( \nu \leq 0 \) and \( \beta_0 > 0 \). The operator \( L_- \) defined on \( L^2(\mathbb{R}) \) with domain \( H^1(\mathbb{R}) \) has no negative eigenvalue. The eigenvalue zero is simple with associated eigenfunction \( Q \). Moreover, the rest of the spectrum is bounded away from zero and the essential spectrum is the interval \([\beta_0, \infty)\).

**Proof.** The fact that the essential spectrum is \([\beta_0, \infty)\) is an easy consequence of the Weyl’s essential spectrum theorem.

It is clear that zero is an eigenvalue with eigenfunction \( Q \). Next, we show that the eigenvalue 0 is simple. Indeed, assume by contradiction that \( P(x, \nu) \) is another eigenfunction with associated value zero. Then \( P \) satisfy \( P(x, 0) = Q(x) \) (see Remark 2) and \( (P, Q)_{L^2} = 0 \), which particularly indicates that \( (Q, Q)_{L^2} = 0 \), a contradiction.

It remains to show that \( L_- \) has no negative eigenvalues. To this aim, we notice that \( L_- \) is the sum of two self-adjoint operators, that is

\[
L_-(\nu) = S_--\nu F_- := -\partial_x^2 + \omega - \frac{1}{2}Q^2 - \nu(\partial_x^4 + \omega^2 + \frac{3}{8}Q^4 - \frac{1}{2}Q^2 + 3QQ_x \partial_x + QQ_{xx} + \frac{3}{2}Q^2 \partial_x^2). \tag{3.7}
\]

We denote by \( \lambda_1(\nu) \) the lowest eigenvalue of \( L_-(\nu) \) in \( H^2 \). For \( \nu = 0 \), from Lemma 2.1 (see also Remark 2) we conclude that \( \lambda_1(0) = 0 \) and all the other eigenvalues of \( L_-(0) = S_- \) are positive. From the variational characterization of the eigenvalues for self-adjoint operators, we deduce that the eigenvalues of \( L_-(\nu) \) move to the right as \( \nu \) decreases from 0 to \(-\infty \). Therefore, \( L_- \) has no negative eigenvalues, which concludes the proof of the Proposition 3. 

\( \square \)
3.3. **The spectrum of** $L_0$. Combining Proposition 2 and 3, we have the following spectral properties of the linearized operator $L_0$.

**Theorem 3.3.** Suppose that $\nu < 0$ and $\beta_0 > 0$. The operator $L_0$ defined on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with domain $H^4(\mathbb{R}) \times H^4(\mathbb{R})$ has a unique negative eigenvalue, which is simple. The eigenvalue zero is double with associated eigenfunctions $(Q, 0)$ and $(0, Q)$. Moreover, the essential spectrum is the interval $[\beta_0, \infty)$.

3.4. **The spectrum of** $L_\alpha$. Motivated by the proofs of spectrum of $L_0$, we can get the spectrum information of $L_\alpha$ as follows:

**Theorem 3.4.** Suppose that $\nu < 0$ and $|\alpha| >> 1$ such that $-\nu \alpha^4 + \beta + \alpha^2 > 0$. The operator $L_\alpha$ defined on $L^2(\mathbb{R})$ with domain $H^4(\mathbb{R})$ has a unique negative eigenvalue, which is simple. The eigenvalue zero is double with associated eigenfunctions $Q'$ and $iQ$. Moreover, the essential spectrum is the interval $[-\nu \alpha^4 + \beta + \alpha^2, \infty)$.

**Proof.** By the exponential localization of $R$, the self-adjoint operator $L_\alpha$ corresponding to the quadratic form $H$ (viewed on $H^4(\mathbb{R}, \mathbb{R}^2)$) is a compact perturbation of the following self-adjoint operator

$$L := \begin{pmatrix} L_1 & -4 \nu \alpha \omega \partial_x + 4 \nu \alpha \partial_x^3 \\ 4 \nu \alpha \partial_x - 4 \nu \alpha \partial_x^2 & L_2 \end{pmatrix},$$

with $D(L) = H^4(\mathbb{R}) \times H^4(\mathbb{R})$, where

$$L_1 = (6 \nu \alpha^2 - 1) \partial_x^2 + (-\nu \alpha^4 + \beta + \alpha^2) - \nu \partial_x^4,$$

$$L_2 = (6 \nu \alpha^2 - 1) \partial_x^2 + (-\nu \alpha^4 + \beta + \alpha^2) - \nu \partial_x^2.$$

By Weyl’s essential spectrum Theorem, $L_\alpha$ and $L$ share the same essential spectrum $[-\nu \alpha^4 + \beta + \alpha^2, \infty)$.

Next, we show that $\text{Ker}(L_\alpha) = \text{Span}(Q', iQ)$, in view of (2.6), it suffice to show that $\text{Ker}(S''(R)) = \text{Span}(\partial_x R, iR)$. The inclusion $\supset$ is easy. Indeed, due to the invariance by space translation and phase shifts, for any $\theta, y \in \mathbb{R}$, we have

$$S'(e^{i\theta} R(\cdot + y)) = 0.$$

The result is obtained by deriving with respect to $\theta$ and $y$ at $\theta = 0, y = 0$. The reverse inclusion is more involved. In view of particular form of $L$, if for $j = 1, 2$ we denote by $m_j \pm n_j i$ the four roots of the algebra equation

$$(6 \nu \alpha^2 - 1) \xi^2 + (-\nu \alpha^4 + \beta + \alpha^2) - \nu \xi^4 + 4 \nu \alpha \omega \xi - 4 \nu \alpha \xi^3 = 0,$$

then we infer that the null space of $L_\alpha$ is spawned by functions of the following type

$$\begin{pmatrix} e^{m_j x} \cos(l_j x) \\ e^{-m_j x} \cos(l_j x) \end{pmatrix}, \begin{pmatrix} e^{m_j x} \sin(l_j x) \\ e^{-m_j x} \sin(l_j x) \end{pmatrix}, m_j, l_j \text{ depend on } \nu, \alpha, \beta, j = 1, 2.$$

(of course this set is linearly independent). Among these four vector functions, there are only two of them are $L^2 \times L^2$ integrable in the semi-infinite plane. Therefore, the null space of $S''(R)$ is spanned by at most two $L^2 \times L^2$-functions.

To prove that $L_\alpha$ defined on $L^2(\mathbb{R})$ with domain $H^4(\mathbb{R})$ has a unique simple, negative eigenvalue, we need a different argument since $L_\alpha$ is not a diagonal operator any more. We set up the eigenvalue problem in the form

$$L_\alpha \psi = -\delta^2 \psi, \text{ where } \psi = f + ig, \delta > 0.$$
We need to show that there exists an unique $\delta_0 > 0$, such that the eigenvalue problem has an unique (up to a multiplicative constant) solution $\psi$. Write the above eigenvalue problem in the form

$$\begin{cases} L_{\alpha,+} f + P g = -\delta^2 f, \\ L_{\alpha,-} g + P^* f = -\delta^2 g. \end{cases}$$

From the second equation, we can express $g = - (L_{\alpha,-} + \delta^2)^{-1} P^* f$, this is possible, since by Proposition 3, for $\alpha$ large, $L_{\alpha,-}$ is also a nonnegative operator, we have $L_{\alpha,-} + \delta^2 \geq \delta^2 \text{Id}$ and hence is invertible. Plugging this into the first equation, one has

$$(L_{\alpha,+} + \delta^2) f - P (L_{\alpha,-} + \delta^2)^{-1} P^* f = 0,$$  

which is equivalent to the eigenvalue problem $L_{\alpha} \psi = -\delta^2 \psi$. Therefore, we need only to show that there exists a unique $\delta_0 > 0$, such that the operator

$$M_\delta := L_{\alpha,+} + \delta^2 - P (L_{\alpha,-} + \delta^2)^{-1} P^*,$$

has an eigenvalue at zero for $\delta = \delta_0$ and in addition, this eigenvalue is simple. To this aim, we first claim that $M_\delta, \delta \geq 0$ are self-adjoint and in addition, for $\delta_1 \geq \delta_2 \geq 0$, we have

$$M_{\delta_1} \geq M_{\delta_2} + (\delta_1^2 - \delta_2^2) \geq M_{\delta_2}.  \tag{3.9}$$

Indeed, by the particular form of the operators $M_\delta$, it suffices to show that for all test functions $f$ and $\delta_1 \geq \delta_2$,

$$\langle - P (L_{\alpha,-} + \delta_1^2)^{-1} P^* f, f \rangle \geq \langle - P (L_{\alpha,-} + \delta_2^2)^{-1} P^* f, f \rangle.$$

This is easily seen by the Plancherels theorem, in particular, denoting $h = P^* f$, then we have

$$\langle - P (L_{\alpha,-} + \delta_1^2)^{-1} P^* f, f \rangle = - \int \frac{|\hat{h}|^2}{m(\xi) + \delta_1^2} d\xi \geq - \int \frac{|\hat{h}(\xi)|^2}{m(\xi) + \delta_2^2} d\xi = \langle - P (L_{\alpha,-} + \delta_2^2)^{-1} P^* f, f \rangle,$$

here we denote by $m(\xi)$ the symbol of the operator $L_{\alpha,-}$.

Denote

$$\lambda(\delta) = \inf \{ \lambda : \lambda \in \sigma(M_\delta) \} = \inf_{\|f\|=1} \langle M_\delta f, f \rangle.$$

It is easy to see that $\delta \to \lambda(\delta)$ is a continuous and increasing function (from $\text{3.9}$). Next, we observe that for $\delta$ large, $M_\delta$ becomes a positive operator, thus we obtain $\lambda(\delta) > 0$. For $\delta = 0$, we already know that $M_0(0') = 0$. By a direct computation, for $\nu < 0$ and $\alpha >> 1$, one has

$$\langle M_0 Q, Q \rangle = \langle L_{\alpha,+} Q - P L_{\alpha,-}^{-1} P^* Q, Q \rangle = \langle L_{\alpha,+} Q, Q \rangle - \langle L_{\alpha,-}^{-1} P^* Q, P^* Q \rangle = (6\nu \alpha^2 - 1) \int Q^4 - \nu \int \frac{3}{2} Q^6 + 10 \omega Q^3 - \frac{11}{4} Q^5 - \int \frac{|\hat{h}(\xi)|^2}{m(\xi)} d\xi = - C_1 \alpha^2 + C_2 < 0,$$

that is, $M_0$ has a negative eigenvalue. Thus, the continuous and increasing function $\delta \to \lambda(\delta)$ is negative at $\delta = 0$ and positive for large $\delta$, whence it has exactly one zero, say $\delta_0$. Thus, the eigenvalue that we are looking for is $\lambda = -\delta_0^2 < 0$.

We still need to check that this eigenvalue is simple, which by the equivalences that we have established means that we have to show that $0$ is an isolated (simple) eigenvalue of $M_{\delta_0}$. However, this is an easy consequence of the fact that $M_{\delta_0} \geq M_0 + \delta_0^2 \text{Id}$ (by $\text{3.9}$). Indeed, denote by $\phi_0$ the eigenvector for $M_0$, which corresponds
to its unique and simple negative eigenvalue. Then, $M_0|_{\sigma_0^+} \geq 0$, by the Courant maxmin principle, we deduce that
\[
\lambda_1(M_0) = \sup_{f \neq 0, \|f\| = 1} \inf_{g \perp \phi_0, \|g\| = 1} \langle M_0 g, g \rangle \geq \delta_0^2 + \inf_{g \perp \phi_0, \|g\| = 1} \langle M_0 g, g \rangle \geq \delta_0^2.
\]
Therefore, we obtain $\lambda(\delta_0) = \lambda_0(M_0) = 0$, while $\lambda_1(M_0) \geq \delta_0^2$, which concludes the proof of Theorem 3.4.

3.5. Coercivity of Hessian. To obtain our main results, we need to prove a coercivity property on the Hessian of the action $H$. There exists $\lambda \in H^2(\mathbb{R})$ such that for any $\phi \in H^2(\mathbb{R})$ satisfying the following orthogonality conditions
\[
\langle \phi, \Xi \rangle_{L^2} = \langle \phi, R \rangle_{L^2} = \langle \phi, iR \rangle_{L^2} = 0,
\]
then we have
\[
H(t, \varphi) \geq C\|\varphi\|_{L^2}^2.
\]

It turns out that $\Xi$ is hard to handle, so we need a more applicable version of 3.5. To this aim, let us consider the natural modes associated to the scaling of $H$. More precisely, for $t$ fixed, let us define $\Lambda_{\omega}R := \partial_{\omega} R$ and $\Lambda_\alpha R := \partial_\alpha R$, by (2.1), we have
\[
\begin{align*}
S''(R)\Lambda_{\omega}R &= -\Lambda_{\omega}(\omega C + \beta)R - i\Lambda_{\omega}c \partial_x R = -(2\nu \omega^2 + 2\nu \omega + 1)R - 4\nu \omega \partial_x R \\
S''(R)\Lambda_\alpha R &= -\Lambda_\alpha(\alpha C + \beta)R - i\Lambda_\alpha c \partial_x R = -(12\nu \alpha^3 - 4\nu \alpha \omega + 2\omega)R \\
&+ i(12\nu \alpha^2 - 4\nu \omega - 2)\partial_x R.
\end{align*}
\]
We now define a function $\Psi := f_1(\alpha, \omega)\Lambda_{\omega}R + f_2(\alpha, \omega)\Lambda_\alpha R$, where
\[
f_1(\alpha, \omega) = \frac{-6\nu \alpha^2 + 2\nu \omega - 1}{36\nu^2 \alpha^4 - 8\nu^2 \omega \alpha^2 - 8\nu \alpha^2 + 4\nu^2 \omega^2 - 1},
\]
and
\[
f_2(\alpha, \omega) = \frac{-2\nu \alpha}{36\nu^2 \alpha^4 - 8\nu^2 \omega \alpha^2 - 8\nu \alpha^2 + 4\nu^2 \omega^2 - 1}.
\]
Now we can check that
\[
S''(R)\Psi = -R,
\]
and
\[
\langle S''(R\Psi), \Psi \rangle = -f_1(\alpha, \omega)\Lambda_{\omega}M(R).
\]

We have the following result which gives a coercivity property of $H$:

Lemma 3.6. Suppose that $\nu < 0$ and $|\alpha| >> 1$ such that $-\nu \alpha^4 + \beta + \alpha^2 > 0$. Let $R$ be given by (1.10), and $H$ the functional defined by (2.3). There exists $C > 0$ such that for any $\varphi \in H^2(\mathbb{R})$ satisfying the following orthogonality conditions
\[
\langle \varphi, R \rangle_{L^2} = \langle \varphi, R' \rangle_{L^2} = \langle \varphi, iR \rangle_{L^2} = 0,
\]
then we have
\[ H(t, \varphi) \geq C\|\varphi\|^2_{H^2}. \]

**Proof.** The proof is standard, we include it for the sake of completeness. First notice that by (3.13) and (3.14), we have
\[ (\Psi, R)_{L^2} = -\langle S''(R)\Psi, \Psi \rangle > 0 \] (3.16)
The next step is to decompose \( \varphi \) and \( \Psi \) in \( \text{Span}(\Xi, iR, \partial_x R) \), one has
\[ \varphi = \tilde{\varphi} + m\Xi, \quad \Psi = \Psi_0 + n\Xi + a\partial_x R + biR, \quad m, n, a, b \in \mathbb{R}, \]
where
\[ \langle \tilde{\varphi}, \Xi \rangle_{L^2} = \langle \tilde{\varphi}, \partial_x R \rangle_{L^2} = \langle \tilde{\varphi}, iR \rangle_{L^2} = \langle \Psi_0, \Xi \rangle_{L^2} = \langle \Psi_0, \partial_x R \rangle_{L^2} = \langle \Psi_0, iR \rangle_{L^2} = 0. \]
In addition,
\[ (\Xi, \partial_x R)_{L^2} = (\Xi, iR)_{L^2} = 0. \]
From the above identities, we infer that
\[ \langle S''(R)\varphi, \varphi \rangle = \langle S''(R)\tilde{\varphi} - m\lambda_0^2\Xi, \tilde{\varphi} + m\Xi \rangle = \langle S''(R)\tilde{\varphi}, \tilde{\varphi} \rangle - m^2\lambda_0^2. \] (3.17)
From (3.13) and the self-adjointness of \( S''(R) \), we deduce that
\[ 0 = (\varphi, R)_{L^2} = -\langle S''(R)\varphi, \varphi \rangle = -\langle \Psi, S''(R)\varphi \rangle = -\langle \Psi_0, S''(R)\tilde{\varphi} \rangle + mn\lambda_0. \] (3.18)
On the other hand,
\[ (\Psi, R)_{L^2} = -\langle S''(R)\Psi, \Psi \rangle = -\langle S''(R)\Psi_0, \Psi_0 \rangle + \lambda_0^2n^2. \] (3.19)
Combining (3.17), (3.18) and (3.19), by Cauchy-Schwartz inequality, we obtain
\[ \langle S''(R)\varphi, \varphi \rangle \geq \frac{(S''(R)\tilde{\varphi}, \Psi_0)^2}{(S''(R)\Psi_0, \Psi_0) + (\Psi, R)_{L^2}} \geq \frac{(S''(R)\tilde{\varphi}, \tilde{\varphi})_{L^2}}{(S''(R)\Psi_0, \Psi_0) + (\Psi, R)_{L^2}} \geq \lambda\langle S''(R)\tilde{\varphi}, \tilde{\varphi} \rangle, \] (3.20)
where \( 0 < \lambda < 1 \). From (3.17), we infer that \( \langle S''(R)\tilde{\varphi}, \tilde{\varphi} \rangle \geq m^2\lambda_0^2 \). There exists \( C > 0 \), such that
\[ \langle S''(R)\varphi, \varphi \rangle \geq \lambda\langle S''(R)\tilde{\varphi}, \tilde{\varphi} \rangle \geq \frac{\lambda}{2}(S''(R)\tilde{\varphi}, \tilde{\varphi}) + \frac{\lambda}{2}m^2\lambda_0^2 \]
\[ \geq C(2\|\tilde{\varphi}\|^2_{H^2} + 2m^2\|\Psi_0\|^2_{H^2}) \geq C\|\varphi\|^2_{H^2}. \] (3.21)
The proof of Lemma 3.0 is concluded. \( \Box \)

**Remark.** By the relation (1.11), we obtain
\[ -\nu\alpha^4 + \beta + \alpha^2 = -6\nu\omega\alpha^2 + \nu\omega^2 + \omega, \] (3.22)
now we can choose \( |\alpha| \) large, say \( |\alpha| \geq \alpha_0 \) such that (3.22), \( f_{1,\alpha} \) and \( f_{2,\alpha} \) are positive.

4. **Nonlinear stability.** In this section we prove Theorem 1.1 and Corollary 1. To prove Theorem 1.1, we introduces a Lyapunov functional, at the \( H^2 \) level, which allows to describe the dynamics of small perturbations and a direct control of the corresponding instability modes. In particular, degenerate directions are controlled by using \( L^2 \) conservation law. To prove Corollary 1, we follow the approach in [17], an inductional argument will be proposed by employing the infinite conservation laws of (4NLS).
4.1. **Lyapunov function.** In this subsection we introduce a Lyapunov functional for the 4NLS, which will be well-defined at the natural $H^2$ level. Indeed, by Proposition 4.1 for any $u_0 \in H^2(\mathbb{R})$, we have global in time $H^2(\mathbb{R})$ solution $u(t)$, recall $S$ defined in (2.2) with conserved quantities $E, M, P, F$, so that for $t \in \mathbb{R}$, we have

\[
S(u(t)) = E(u(t)) + (\beta + \alpha c)M(u(t)) + cP(u(t)) - \nu F(u(t)) = S(u_0).
\]  

(4.1)

It is clear that $S(u)$ represents a real-valued conserved quantity, well-defined for $H^2$-solutions of 4NLS. Moreover, one has the following Taylor expansion.

**Lemma 4.1.** Let $\varphi \in H^2(\mathbb{R})$ be any function with sufficiently small $H^2$-norm, and $R$ be the Hasimoto soliton. Then, for all $t \in \mathbb{R}$, one has

\[
S(R + \varphi) = S(R) + \frac{1}{2} H(\varphi) + O(\|\varphi\|^3_{H^2(\mathbb{R})}),
\]  

(4.2)

where $H(\varphi)$ is defined in (2.3).

**Proof.** The proof is a direct consequence of the fact that $R$ is a critical point of the functional $S$, namely, $S'(R) = 0$. \qed

4.2. **Proof of main results.** In this subsection we give the proofs of our main results.

**Proof of Theorem 1.1.** Let $u_0 \in H^2$ be a function such that $u_0$ satisfying (1.12), let the solution, say $u(t)$, of the Cauchy problem associated to 4NLS with initial data $u_0$. From the continuity of the 4NLS flow for $H^2$ data, there exists a time $T_0 > 0$ and continuous parameters $\theta(t), y(t) \in \mathbb{R}$, defined for all $t \in [0, T_0]$, and such that the solution $u(t)$ of the Cauchy problem for 4NLS, with initial data $u_0$, satisfies

\[
\sup_{t \in [0, T_0]} \|u(t) - e^{i(\theta(t) + i \alpha \omega)}Q_\omega(x - y(t))\|_{H^2(\mathbb{R})} < 2\epsilon.
\]  

(4.3)

We want to show that $T_0 = +\infty$. To this aim, let $K > 2$ be a constant, to be fixed later. Let us suppose, by contradiction, that the maximal time of stability $T^*$, that is

\[
T^* = \sup\{T > 0, \text{for all } t \in [0, T], \text{there exists } \hat{\theta}(t), \hat{y}(t) \in \mathbb{R} \text{ such that} \}
\sup_{t \in [0, T]} \|u(t) - e^{i(\hat{\theta}(t) + i \alpha \omega)}Q_\omega(x - \hat{y}(t))\|_{H^2(\mathbb{R})} < K\epsilon,
\]  

(4.4)

is finite. By (4.3), we see easily that $T^*$ is well-defined. Our idea is to find a suitable contradiction to the assumption $T^* < +\infty$.

By taking $\epsilon_0$ smaller, if necessary, we can apply modulation theory for the solution $u(t)$.

**Lemma 4.2.** There exists $\epsilon_0 > 0$, depending only on $R$, such that for all $\epsilon \in (0, \epsilon_0)$, the following property is verified. There exist $\theta(t), y(t) \in \mathbb{R}$ defined on $[0, T^*]$, such that if we denote

\[
\Upsilon(t) = u(t) - \hat{R}(t), \quad \hat{R}(t) = e^{i\theta(t) + i \alpha \omega}Q_\omega(x - y(t)),
\]  

(4.5)

then for all $t \in [0, T^*]$, $\Upsilon$ satisfies the orthogonality conditions

\[
(\Upsilon, i\hat{R})_{L^2} = (\Upsilon, \partial_x \hat{R})_{L^2} = 0.
\]  

(4.6)

Moreover, for all $t \in [0, T^*]$, we have

\[
\|\Upsilon(t)\|_{H^2} + |\theta'(t) - \beta| + |y'(t) - c| < CK\epsilon, \quad \|\Upsilon(0)\|_{H^2} \leq C\epsilon,
\]  

(4.7)

for some constant $C > 0$, independent of $K$. 
Proof. The proof of this result relies on the Implicit Function Theorem. Indeed, let
\[ F_1(u(t, x), \theta, y) = \langle u - e^{i\theta + i\alpha x} Q_\omega(x - y), i e^{i\theta + i\alpha x} Q_\omega(x - y) \rangle, \]
\[ F_2(u(t, x), \theta, y) = \langle u - e^{i\theta + i\alpha x} Q_\omega(x - y), e^{i\theta + i\alpha x} \partial_x Q_\omega(x - y) \rangle. \]
We clearly have
\[ F_j(e^{i\theta + i\alpha x} Q_\omega(x - y), \theta, y) = 0, \quad j = 1, 2. \]
On the other hand, for \( j = 1, 2 \), we can check \( \nabla F_j \) as follows
\[
\frac{\partial F_1}{\partial \theta} \left(e^{i\theta + i\alpha x} Q_\omega(x - y), \theta, y \right) \\
= -\langle ie^{i\theta + i\alpha x} Q_\omega(x - y), ie^{i\theta + i\alpha x} Q_\omega(x - y) \rangle = -\|Q_\omega\|_{L^2}^2,
\]
\[
\frac{\partial F_1}{\partial y} \left(e^{i\theta + i\alpha x} Q_\omega(x - y), \theta, y \right) = \frac{\partial F_2}{\partial \theta} \left(e^{i\theta + i\alpha x} Q_\omega(x - y), \theta, y \right) = 0,
\]
\[
\frac{\partial F_2}{\partial y} \left(e^{i\theta + i\alpha x} Q_\omega(x - y), \theta, y \right) = \|\partial_x Q_\omega\|_{L^2}^2.
\]
Therefore, we have the desired invertibility, by the Implicit Function Theorem, we can write the decomposition \( 4.5 \) with property \( 4.6 \) in a a small \( H^2 \) neighborhood of \( \tilde{R}(t) \), for \( t \in [0, T^\ast] \).

Now we check \( 4.7 \). The first bounds are consequence of the decomposition itself and the equations satisfied by the derivatives of the parameters \( \theta, y \), after taking time derivative in \( 4.6 \) and using the invertibility property of \( \nabla F_j \). More precisely, we first write the equation verified by \( \Upsilon \). Recall that \( u \) satisfies \( 4NLS \) in \( \Omega(t) \) in \( 4NLS \) to obtain
\[
i\partial_t \Upsilon + L_N \Upsilon = \left( (\partial_t \theta - \beta) \tilde{R} + i(\partial_t y - c) \cdot Q'_\omega e^{i\theta + i\alpha x} \right) + \mathcal{N}(\Upsilon) + O(\|\Upsilon\|_{H^2}^3) \quad (4.8)
\]
where
\[
L_N \Upsilon := \partial_x^2 \Upsilon + \nu \partial_x^4 \Upsilon + \left( \frac{1}{2} |\tilde{R}|^2 \Upsilon + \text{Re}(\tilde{R} \Upsilon) \tilde{R} \right) + \nu \left( \frac{3}{8} \left( |\tilde{R}|^4 \Upsilon + 4 |\tilde{R}|^2 \text{Re}(\tilde{R} \Upsilon) \tilde{R} \right) + \frac{3}{2} \left( (\partial_x \tilde{R})^2 \Upsilon + 2 \tilde{R} \partial_x \tilde{R} \partial_x \Upsilon \right) + \left( |\partial_x \tilde{R}|^2 \Upsilon + 2 \text{Re}(\partial_x \tilde{R} \Upsilon) \tilde{R} \right) + \frac{1}{2} \left( \tilde{R}^2 \partial_x^2 \Upsilon + 2 \tilde{R} \partial_x^2 \tilde{R} \Upsilon \right) + 2 \left( \tilde{R}^2 \partial_x^2 \Upsilon + 2 \text{Re}(\tilde{R} \Upsilon) \partial_x^2 \tilde{R} \right) \right)
\]
and \( \mathcal{N}(\Upsilon) \) is the remaining nonlinear part.

Take now the scalar product of \( 4.8 \) with \( i\tilde{R}, \partial_x \tilde{R} \). By the definition of \( \tilde{R} \) and the orthogonality conditions \( 4.6 \), we obtain a differential system for the modulation equations vector \( \text{Mod}(t) := (\partial_t \theta - \beta, \partial_t y - c) \) of the form
\[
A \cdot \text{Mod}(t) = B(\Upsilon) + O(\|\Upsilon\|_{H^2}^3),
\]
where \( |B(\Upsilon)| \leq C \|\Upsilon\|_{H^2} \). As long as the modulation parameter do not vary too much and \( \|\Upsilon\|_{H^2} \) remains small, \( A \) is invertible (it is of the form \( \nabla F + \text{small} \) ) and we can deduce that
\[
|\text{Mod}(t)| \leq C \|\Upsilon\|_{H^2} + O(\|\Upsilon\|_{H^2}^3). \quad (4.9)
\]
The last bound in \( 4.7 \) is consequence of \( 1.12 \). The proof of the lemma \( 4.2 \) is completed.

Next, since \( \Upsilon(t) \) defined by \( 4.5 \) is small, by Lemma \( 4.1 \) we have
\[
S(u(t)) = S(\tilde{R}(t)) + H(\Upsilon(t)) + O(\|\Upsilon(t)\|_{H^2}^3).
\]
From (4.1) and Lemma 4.1, we obtain
\[ H(\Upsilon(t)) \leq H(\Upsilon(0)) + O(\|\Upsilon(t)\|_{H^2}^3) + O(\|\Upsilon(0)\|_{H^2}^3) \]
\[ \leq C\|\Upsilon(0)\|_{H^2}^3 + C\|\Upsilon(t)\|_{H^2}^3. \]
Since \( \Upsilon(t) \) satisfies (4.6), by Lemma 3.6, we deduce that
\[ \|\Upsilon(0)\|_{H^2}^3 \leq C\|\Upsilon(0)\|_{H^2}^3 + C(\tilde{R}(t), \Upsilon(t))_{L^2}^2 \]
\[ \leq C\epsilon^2 + CK^3\epsilon^3 + C(\tilde{R}(t), \Upsilon(t))_{L^2}^2. \]
(4.10)

Using the conservation of mass, we have
\[ \|u(t)\|_{L^2}^2 = \|\tilde{R}(t)\|_{L^2}^2 + \|\Upsilon(t)\|_{L^2}^2 + 2(\tilde{R}(t), \Upsilon(t))_{L^2} \]
\[ = \|\tilde{R}(0)\|_{L^2}^2 + \|\Upsilon(0)\|_{L^2}^2 + 2(\tilde{R}(0), \Upsilon(0))_{L^2} = \|u(0)\|_{L^2}^2, \]
then for \( t \in [0, T^*] \), we obtain
\[ \|(\tilde{R}(t), \Upsilon(t))_{L^2}\| \leq C\|(\tilde{R}(0), \Upsilon(0))_{L^2}\| + C\|\Upsilon(0)\|_{H^2} + C\|\Upsilon(t)\|_{H^2}^2 \]
\[ \leq C(\epsilon + K^2\epsilon^2). \]
Replacing this last identity in (4.10), by choosing \( K \) large, one has
\[ \|\Upsilon(t)\|_{H^2}^2 \leq C\epsilon^2(1 + K^3\epsilon + K^4\epsilon^2) \leq \frac{1}{2} K^2\epsilon^2, \]
that is,
\[ \|\Upsilon(t)\|_{H^2} \leq \frac{\sqrt{2}}{2} K\epsilon. \]

However, this last fact contradicts the definition of \( T^* \) in (4.4) and therefore the stability property (1.13) holds true. Finally, (1.14) is a direct consequence of (1.7).

**Proof of Corollary**

**Step 1.** \( H^3 \) stability.

Suppose \( e^{it\theta} e^{i\alpha x} Q_{\omega}(x - ct) \) is unstable in \( H^3(\mathbb{R}) \), then there exists \( u_{n, 0}, t_n, \delta_0 > 0 \), such that
\[ \inf_{(y, \theta) \in \mathbb{R}^2} \|u_{n, 0} - e^{i\theta} e^{i\alpha x} Q_{\omega}(x + y)\|_{H^3(\mathbb{R})} \rightarrow 0, \]
\[ \inf_{(y, \theta) \in \mathbb{R}^2} \|u_{n}(t_n) - e^{i\theta} e^{i\alpha x} Q_{\omega}(x + y)\|_{H^3(\mathbb{R})} \geq \delta_0. \]

Since we have proved the \( H^2 \) stability, we can assume that
\[ \inf_{(y, \theta) \in \mathbb{R}^2} \|\partial^3_x u_{n}(t_n) - e^{i\theta} \partial^3_x Q_{\omega, \alpha}(\cdot + y)\|_{L^2(\mathbb{R})} \geq \frac{\delta_0}{2}, \]
for \( n \) large enough. Moreover, by taking a subsequence (we still denote by \( u_n \)), we have \( u_n(t_n) \rightarrow e^{i\theta_0} e^{i\alpha x} Q_{\omega}(\cdot + y_0) \) in \( H^2 \) for some \( \theta_0, y_0 \in \mathbb{R} \), therefore, one has
\[ \partial^3_x u_{n}(t_n) \rightarrow e^{i\theta_0} \partial^3_x Q_{\omega, \alpha}(\cdot + y_0), \]
in \( L^2 \). By the conservation of \( G_3 \) in (1.9), we obtain
\[ G_3(u_n(t_n)) = G_3(u_{n, 0}) \rightarrow G_3(e^{i\theta} Q_{\omega, \alpha}). \]
Now we define
\[ \tilde{G}_3(u) := G_3(u) - \frac{1}{2} \|u''\|_{L^2}^2, \]
so that we have \( \tilde{G}_3(u_n(t_n)) \rightarrow \tilde{G}_3(e^{i\theta} Q_{\omega, \alpha}) \). Since \( \tilde{G}_3 \) is continuous in \( H^2 \) and \( u_n(t_n) \) converges to \( e^{i\theta_0} e^{i\alpha x} Q_{\omega}(x + y_0) \) in \( H^2 \), thus we deduce that
\[ \|\partial^3_x u_{n}(t_n)\|_{L^2} \rightarrow \|\partial^3_x Q_{\omega, \alpha}\|_{L^2}, \]
this last fact implies that $u_n(t_n) \to e^{iθ_n}e^{iαx}Q_ω(x + y_0)$ in $H^3$. However, which is a contradiction.

Step 2. $H^m$ stability. First we observe that $G_m = \frac{1}{2}||u^{(m)}||_{L^2}^2 + \tilde{G}_m(u)$, where $\tilde{G}_m$ is defined and continuous on $H^{m-1}$.

We can now show $H^m$ stability by induction. Suppose $e^{iθ_n}e^{iαx}Q_ω(x - ct)$ is stable in $H^{m-1}$ and unstable in $H^m$, then following the argument for the proof of $H^1$ stability, we have the $H^m$ stability from $H^{m-1}$ stability. Combing the $H^1$ stability result in [24], we obtain the $H^m$ stability for all $m \in \mathbb{Z}_+$.

Acknowledgments. The author thanks the referees for their valuable comments and suggestions.

REFERENCES

[1] J. P. Albert, Positivity properties and stability of solitary-wave solutions of model equations for long waves Commun. Partial Differential Equations, 17 (1992), 1–22.

[2] J. P. Albert and J. L. Bona, Total positivity and the stability of internal waves in stratified fluids of finite depth IMA J. Appl. Math., 46 (1991), 1–19.

[3] J. P. Albert, J. L. Bona and D. Henry, Sufficient conditions for instability of solitary-wave solutions of model equation for long waves Physica D, 24 (1987), 343–366.

[4] T. B. Benjamin, The stability of solitary waves Proc. R. Soc. London, Ser. A, 328 (1972), 153–183.

[5] J. L. Bona, On the stability theory of solitary waves Proc Roy. Soc. Lond. Ser. A, 344 (1975), 363–374.

[6] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations Commun. Math. Phys., 85 (1982), 549–561.

[7] L. S. Da Rios, On the motion of an unbounded fluid with a vortex filament of any shape, Rend. Circ. Mat. Palermo., 22 (1906), 117–135 (in Italian).

[8] Y. Fukumoto and H. K. Moffatt, Motion and expansion of a viscous vortex ring, Part I. A higher-order asymptotic formula for the velocity J. Fluid Mech., 417 (2000), 1–45.

[9] L. Greenberg, An oscillation method for fourth order, self-adjoint, two-point boundary value problems with nonlinear eigenvalues SIAM J. Math. Anal., 22 (1991), 1021–1042.

[10] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry I J. Funct. Anal., 74 (1987), 160–197.

[11] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry II J. Funct. Anal., 94 (1990), 308–348.

[12] H. Hasimoto, A soliton on a vortex filament J. Fluid Mech., 51 (1972), 477–485.

[13] S. M. Hoseini and T. R. Marchant, Solitary wave interaction for a higher-order nonlinear Schrödinger equation JMA J. Appl. Math., 72 (2007), 206–222.

[14] Z. Huo and Y. Jia, The Cauchy problem for the fourth-order nonlinear Schrödinger equation related to the vortex filament J. Differential Equations, 214 (2005), 1–35.

[15] Z. Huo and Y. Jia, A refined well-posedness for the fourth-order nonlinear Schrödinger equation related to the vortex filament Comm. Partial Differential Equations, 32 (2007), 1493–1510.

[16] J. Langer and R. Perline, Poisson geometry of the filament equation J. Nonlinear Sci., 1 (1991), 71–93.

[17] M. Maeda and J. Segata, Existence and stability of standing waves of fourth order nonlinear Schrödinger type equation related to vortex filament Funkcial. Ekvac., 54 (2011), 1–14.

[18] J. H. Maddocks and R. L. Sachs, On the stability of KdV multi-solitons Comm. Pure Appl. Math., 46 (1993), 867–901.

[19] F. Natali and A. Pastor, The Fourth-order dispersive nonlinear Schrödinger equation: Orbital stability of a standing wave SIAM J. Appl. Dyn. Syst., 14 (2015), 1326–1347.

[20] A. Neves and O. Lopes, Orbital stability of double solitons for the Benjamin-Ono equation Comm. Math. Phys., 262 (2006), 757–791.

[21] M. Reed and B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, New York, 1978.
[22] R. L. Ricca, The contributions of Da Rios and Levi-Civita to asymptotic potential theory and vortex filament dynamics, Fluid Dyn. Res., 18 (1996), 245–268.

[23] J. Segata, Well-posedness and existence of standing waves for the fourth order nonlinear Schrödinger type equation, Discrete Contin. Dyn. Syst., 27 (2010), 1093–1105.

[24] J. Segata, Orbital stability of a two parameter family of solitary waves for a fourth order nonlinear Schrödinger type equation, J. Math. Phys., 54 (2013), 061503, 6 pp.

[25] M. I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal., 16 (1985), 472–491.

[26] M. I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math., 39 (1986), 51–67.

[27] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Soviet Physics JETP, 34 (1972), 62–69.

Received December 2015; revised April 2016.

E-mail address: wangzh79@mail2.sysu.edu.cn