Checking Properties within Fairness and Behavior Abstractions

ULRICH ULTES-NITSCHE
University of Southampton
and
PIERRE WOLPER
University of Liège

This paper is motivated by the fact that verifying liveness properties under a fairness condition is often problematic, especially when abstraction is used. It shows that using a more abstract notion than truth under fairness, specifically the concept of a property being satisfied within fairness can lead to interesting possibilities. Technically, it is first established that deciding satisfaction within fairness is a PSPACE-complete problem and it is shown that properties satisfied within fairness can always be satisfied by some fair implementation. Thereafter, the interaction between behavior abstraction and satisfaction within fairness is studied and it is proved that satisfaction of properties within fairness can be verified on behavior abstractions, if the abstraction homomorphism is weakly continuation-closed.

Categories and Subject Descriptors: D.2.4 [Software Engineering]: Software/Program Verification—Model checking; F.3.1 [Logics and Meanings of Programs]: Specifying and Verifying and Reasoning about Programs—Mechanical verification

General Terms: Theory, Verification

Additional Key Words and Phrases: Relative liveness properties, satisfaction within fairness, behavior abstraction, weakly continuation-closed homomorphisms

1. INTRODUCTION

To be able to verify liveness properties of a system [Alpern and Schneider 1985], it is almost always necessary to include a fairness hypothesis in the system description [Francez 1986]. Indeed, introducing a fairness hypothesis makes it possible to ignore behaviors that correspond to extreme execution scenarios and that, in any case,
would not occur in any reasonable implementation. Even though this intuition is clear, making fairness precise is somewhat more complicated: should one be “weakly” or “strongly” fair, “transition” or “process” fair, or isn’t “justice” or even “compassion” what fairness should really be [Manna and Pnueli 1992]? Of course, there is a rational way of choosing which fairness notion is adequate for a given problem by considering the nature of the model being used and making reasonable assumptions about how it might be implemented, but it remains that this choice is crucial and delicate.

Furthermore, introducing a fairness hypothesis often makes the verification process somewhat more problematic. This is especially true when abstraction is used. Indeed, since after moving to the abstract level one deals with a reduced set of observables, it can become impossible to express correctly the fairness hypothesis under which the system is correct. This makes one wish for a more general and abstract notion of truth under fairness that would contribute to simplifying verification, especially in the context of abstraction. Intuitively, the notion to be formalized is that of a property being true provided one is given “some control” over the choices made during infinite executions. In other words, one wants to characterize the properties that can be made true by “some fair implementation” of the system.

In this paper, we show that the concept of a property being satisfied within fairness is a suitable abstraction of truth under fairness that lends itself easily to verification in the context of abstraction by using the techniques of [Nitsche and Ochsenschläger 1996; Nitsche and Wolper 1997; Ochsenschläger 1994; Ochsenschläger 1995]. The idea of satisfaction within fairness is to re-interpret the notion of relative liveness properties as a satisfaction relation. Relative liveness properties are liveness properties within the universe of behaviors of the system. Their definition is a relativized version of the definition of liveness: every prefix of a behavior of the system can be extended to an infinite behavior that satisfies the property. This concept and the dual notion of relative safety property were introduced in [Henzinger 1992] as a means of clarifying the shift from liveness to safety when timing constraints are introduced in a system. It can also be traced to the notion of machine-closed property [Abadi and Lamport 1988; Abadi and Lamport 1990; Alur and Henzinger 1997].

Here we make a different use of the concept. In fact, we interpret relative liveness as a satisfaction relation for properties represented by temporal logic formulas [Emerson 1990; Pnueli 1977]. Notice that for a property to be satisfied within fairness does correspond, in the desired abstract sense, to the property being satisfied under fairness. Indeed, in crude terms, the system almost satisfies properties that are satisfied within fairness: it just needs the “help of some fairness” (remember that every prefix of a behavior of the system can be extended to an infinite behavior that satisfies the property). Furthermore, we show that for $\omega$-regular systems and properties, deciding satisfaction within fairness is a PSPACE-complete problem. This and the fact that, in a reasonable sense, properties satisfied within fairness can be satisfied by some fair implementation are first indications of the usefulness of this concept for verification.

This usefulness is even more apparent when considering abstraction. Indeed, satisfaction within fairness enables us to circumvent the fact that truth under fairness
is usually not preserved by abstraction mappings. Precisely, we consider abstractions defined by language homomorphisms in the context of systems described by \( \omega \)-languages. We prove that whether a property is satisfied within fairness can be reliably checked on the abstract system, provided that the homomorphism is weakly continuation-closed. Weakly continuation-closed homomorphisms were introduced in [Ochsenschläger 1992] (see also [Ochsenschläger 1994]) where they are called simple homomorphisms. For homomorphisms, being weakly continuation-closed essentially means that they are faithful with respect to the continuation of a word within a language, i.e., the image of the continuation is the continuation of the image of the word in the image of the language. We show that weakly continuation-closed homomorphisms preserve exactly properties satisfied within fairness.

2. INTRODUCTORY EXAMPLES

To motivate the definitions we present later on, we start with a small example of a concurrent reactive system. Consider the system described as a Petri net in Figure 1.

![Fig. 1. A small system](image)

It is a server that, after having received a request, can send a result or a rejection to its client, depending on whether the resource it manages has been freed or locked. The possible behaviors of the system are represented by the finite-state system shown in Figure 2 (the reachability graph of the Petri net). The initial state is shaded grey, a convention we will also use in subsequent state diagrams.

![Fig. 2. The behaviors of the small system](image)

ACM Transactions in Computational Logic, Vol. TBD, No. TBD, TBD.
From Figure 2, it is easy to see that our system does not satisfy the propositional linear time temporal logic [Emerson 1990; Pnueli 1977] property $\Box\Diamond(result)$. Indeed, $lock \cdot (request \cdot no \cdot reject)^\omega$ is a computation of the system that does not satisfy $\Box\Diamond(result)$. Nevertheless, it is clear that what is missing for the property $\Box\Diamond(result)$ to be true is a fairness hypothesis on the system executions. The notion of a property being satisfied within fairness captures this: $\Box\Diamond(result)$ is satisfied within fairness by the set of behaviors described by Figure 2 (see Definition 4.12/4.1).

Figure 3 gives a finite-state diagram describing the behaviors of a system similar to the one of Figure 1 but containing an error: in Figure 3, if the resource is locked, there is no possibility to free it again. There is also another difference, namely that in Figure 3 a request can also be rejected when the resource is available, but the motivation for this is linked to our subsequent discussion of abstraction. The point to notice now, is that no notion of fairness can make $\Box\Diamond(result)$ true of the new system and that the notion satisfaction within fairness captures this again: $\Box\Diamond(result)$ is not satisfied within fairness by the set of behaviors described in Figure 3.

Let us now consider abstraction. Imagine we are only interested in the actions request, result, and reject. We thus consider an abstraction homomorphism that maps all other actions to the empty word. If we apply this homomorphism to the labeled transition system of Figure 2, we obtain after reduction the transition diagram of Figure 4. The property $\Box\Diamond(result)$ is satisfied within fairness by the behaviors described in Figure 4.

Can we conclude from there that it is also satisfied within fairness by the behaviors described by Figure 3? Not without caution since Figure 3 is also obtained by

---

**Fig. 3.** The behaviors of the small system with an error

**Fig. 4.** An abstract version of the small system.
abstracting from Figure 3. What distinguishes the two abstractions is the nature of the homomorphism. In the case of Figure 2 the homomorphism preserves properties satisfied within fairness, whereas it does not do so in the case of Figure 3. In Section 3 we will elaborate on this and show that one can conclude that properties satisfied within fairness by the abstract system also hold on the concrete system, precisely when the homomorphism is weakly continuation-closed.

3. PRELIMINARIES

For defining our concepts, we need several notions from language theory [Berstel 1979; Eilenberg 1974; Harrison 1978; Thomas 1990]. Let \( L \subseteq \Sigma^* \) be a language and let \( L_\omega \subseteq \Sigma^\omega \) be an \( \omega \)-language.

**Definition 3.1.** The left quotient of \( L \) by a word \( w \in \Sigma^* \) is defined by \( \text{cont}(w, L) = \{v \in \Sigma^* \mid vw \in L\} \). The left quotient of \( L_\omega \) by \( w \in \Sigma^* \) is similarly defined by \( \text{cont}(w, L_\omega) = \{x \in \Sigma^\omega \mid wx \in L_\omega\} \).

The left quotient describes the possible continuations of a word in a language. When considering system behaviors, it describes “what can happen after \( w \) has happened”. Therefore we denote the left quotient of \( L \) by \( w \) by \( \text{cont}(w, L) \), “the set of continuations of \( w \) in \( L \)”, instead of the notation \( w^{-1}(L) \) common in language theory.

The notation \( \text{pre}(L) \) designates the set of prefixes of words in \( L \). A language \( L \) is called \( \text{prefix-closed} \) if and only if \( L = \text{pre}(L) \). For an \( \omega \)-word \( x \), \( \text{pre}(x) \) designates the set of all finite prefixes of \( x \) and, for an \( \omega \)-language \( L_\omega \), \( \text{pre}(L_\omega) \) designates the set of all finite prefixes of \( \omega \)-words in \( L_\omega \). The Eilenberg-limit of a language \( L \) is the set \( \text{lim}(L) = \{x \in \Sigma^\omega \mid \exists^\omega w \in \text{pre}(x) : w \in L\} \). Here, “\( \exists^\omega \)” abbreviates: “there exist infinitely many different \( ... \)”. For a word \( w \) and an \( \omega \)-word \( x \), we denote their \( n \)th letter by \( w_n \) and \( x_n \) respectively. Finally, the notation \( x_{(\ldots, \ldots)} \), \( n \in \mathbb{N} \), represents the suffix \( x_nx_{n+1}x_{n+2}\ldots \) of an \( \omega \)-word \( x \in \Sigma^\omega \) starting with the \( n \)th letter of \( x \).

To describe properties, we use propositional linear-time temporal logic (PLTL) [Emerson 1991; Pnueli 1977]. PLTL-formulas are defined with respect to a set \( \text{AP} \) of atomic propositions. All atomic propositions and the proposition \( \text{true} \) are PLTL-formulas. If \( \xi \) and \( \zeta \) are PLTL-formulas, then so are \( \neg(\xi) \), \( \xi \land \zeta \), \( \xi \lor \zeta \), \( \xi \Rightarrow \zeta \), \( \xi \iff \zeta \), \( \Diamond(\xi) \equiv [(\xi) \Rightarrow (\xi)] \), \( \Box(\xi) \equiv [\Diamond(\xi) \Rightarrow (\xi)] \), \( \Diamond(\xi) \equiv (\text{true}) \Diamond(\xi) \), \( \Box(\xi) \equiv (\neg(\Diamond(\xi))) \), \( \Diamond(\xi) \equiv (\neg(\Box(\xi))) \), and \( \Box(\xi) \equiv (\neg(\Diamond(\xi))) \).

PLTL-formulas are interpreted over infinite sequences of truth values for the atomic propositions, i.e., over functions of the type \( \mathbb{N} \to 2^{\text{AP}} \) or, equivalently over \( \omega \)-words defined on the alphabet \( 2^{\text{AP}} \). For convenience, we will also interpret PLTL formulas over infinite words defined on an arbitrary alphabet \( \Sigma \) with the help of a labeling function \( \lambda : \Sigma \to 2^{\text{AP}} \). The semantics of a PLTL formula with respect to
an infinite word \( x \in \Sigma^\omega \) and a labeling function \( \lambda : \Sigma \to 2^{AP} \) is then the following. (Read “\( \models \)” as “satisfies.”)

\[
x, \lambda \models \text{true}.
\]

If \( \eta \) is an atomic proposition, then \( x, \lambda \models \eta \) if and only if \( \eta \in \lambda(x_1) \).

If \( \eta = \lnot(\xi) \), then \( x, \lambda \models \eta \) if and only if it is not the case that \( x, \lambda \models \xi \).

If \( \eta = (\xi) \land (\zeta) \), then \( x, \lambda \models \eta \) if and only if \( x, \lambda \models \xi \) and \( x, \lambda \models \zeta \).

If \( \eta = \text{true} \), then \( x, \lambda \models \eta \) if and only if \( x \in \text{pre}(\omega) \) and \( x, \lambda \models \xi \).

If \( \eta = \text{false} \), then \( x, \lambda \models \eta \) if and only if \( x \in \text{pre}(\omega) \).

\( x, \lambda \models \xi \) and, for all \( j < i \), \( x(j, \ldots), \lambda \models \xi \).

The meaning of the other operators can be derived from their definition. We will write \( L_\omega, \lambda \models \eta \) if and only if \( x, \lambda \models \eta \), for all \( x \in L_\omega \).

**Definition 3.2.** A property \( \mathcal{P} \) over an alphabet \( \Sigma \) is a subset of \( \Sigma^\omega \). An \( \omega \)-language \( L_\omega \subseteq \Sigma^\omega \) satisfies \( \mathcal{P} \) if and only if \( L_\omega \subseteq \mathcal{P} \). For an alphabet \( \Sigma \) and a labeling function \( \lambda : \Sigma \to 2^{AP} \), the property represented by a PLTL-formula \( \eta \) over \( AP \) is the set \( L_\eta = \{ x \in \Sigma^\omega \mid x, \lambda \models \eta \} \).

4. RELATIVE LIVENESS AND SAFETY

In this section, we review the definition of relative liveness properties of an \( \omega \)-language, as well as their counterpart relative safety properties. Based on the notion of a relative liveness property, we will define the satisfaction of properties within fairness. Let \( L_\omega \subseteq \Sigma^\omega \) be an \( \omega \)-language representing the behavior of a system and let \( \mathcal{P} \subseteq \Sigma^\omega \) be a property.

**Definition 4.1.** A property \( \mathcal{P} \) is a relative liveness property of \( L_\omega \) (we write this already as a satisfaction relation: \( L_\omega \models \mathcal{P} \)) if and only if \( \forall x \in \text{cont}(w, L_\omega) : x \in \text{pre}(L_\omega) \) : \( \exists x \in \text{cont}(w, L_\omega) : wx \in \mathcal{P} \).

**Definition 4.2.** A property \( \mathcal{P} \) is a relative safety property of \( L_\omega \) if and only if \( \forall x \in L_\omega \cap \mathcal{P} \), if for all \( x \notin \mathcal{P} \), then \( \exists w \in \text{pre}(x) : \forall z \in \text{cont}(w, L_\omega) : wz \notin \mathcal{P} \).

**Remark 4.3.** If \( L_\omega = \Sigma^\omega \), then the definitions of relative liveness and relative safety become exactly the definitions of liveness and safety given in [Alpern and Schneider 1987].

To prove the decidability of relative liveness and safety for regular \( \omega \)-languages, we use the following characterizations of these properties.

**Lemma 4.4.** \( \mathcal{P} \) is a relative liveness property of \( L_\omega \) if and only if \( \text{pre}(L_\omega) = \text{pre}(L_\omega \cap \mathcal{P}) \).

**Proof.** By definition, \( L_\omega \models \mathcal{P} \) if and only if, for all \( w \in \text{pre}(L_\omega) \), there exists \( x \in \text{cont}(w, L_\omega) \) such that \( wx \in \mathcal{P} \). Hence we have \( w \in \text{pre}(L_\omega \cap \mathcal{P}) \), for all \( w \in \text{pre}(L_\omega) \). This is equivalent to \( \text{pre}(L_\omega) \subseteq \text{pre}(L_\omega \cap \mathcal{P}) \). On the other hand, \( \text{pre}(L_\omega \cap \mathcal{P}) \subseteq \text{pre}(L_\omega) \), and thus \( \text{pre}(L_\omega) = \text{pre}(L_\omega \cap \mathcal{P}) \).

If \( \text{pre}(L_\omega) = \text{pre}(L_\omega \cap \mathcal{P}) \), then \( w \in \text{pre}(L_\omega \cap \mathcal{P}) \), for all \( w \in \text{pre}(L_\omega) \). Therefore, for all \( w \in \text{pre}(L_\omega) \), there exists an \( x \in \text{cont}(w, L_\omega) \) such that \( wx \in \mathcal{P} \) and hence \( \mathcal{P} \) is a relative liveness property of \( L_\omega \).
LEMMA 4.5. $\mathcal{P}$ is a relative safety property of $L_\omega$ if and only if

$$L_\omega \cap \text{lim}(\text{pre}(L_\omega \cap \mathcal{P})) \subseteq \mathcal{P}.$$ 

PROOF. By definition, $\mathcal{P}$ is a relative safety property of $L_\omega$ if and only if

$$\forall x \in L_\omega : (x \notin \mathcal{P} \Rightarrow (\exists w \in \text{pre}(x) : \forall z \in \text{cont}(w, L_\omega) : wz \notin \mathcal{P})).$$

By taking the counterpositive of the implication this is equivalent to

$$\forall x \in L_\omega : (\forall w \in \text{pre}(x) : \exists z \in \text{cont}(w, L_\omega) : wz \in \mathcal{P}) \Rightarrow x \in \mathcal{P}.$$

The part $(\forall w \in \text{pre}(x) : \exists z \in \text{cont}(w, L_\omega) : wz \in \mathcal{P})$ is equivalent to the condition $\text{pre}(x) \subseteq \text{pre}(L_\omega \cap \mathcal{P})$. Thus, $\mathcal{P}$ is a relative safety property of $L_\omega$ if and only if

$$\forall x \in L_\omega : (\text{pre}(x) \subseteq \text{pre}(L_\omega \cap \mathcal{P}) \Rightarrow x \in \mathcal{P}).$$

All $\omega$-words $x$ in $L_\omega$ such that $\text{pre}(x) \subseteq \text{pre}(L_\omega \cap \mathcal{P})$ can be represented by the set $L_\omega \cap \text{lim}(\text{pre}(L_\omega \cap \mathcal{P}))$. Thus, $\mathcal{P}$ is a relative safety property of $L_\omega$ if and only if $L_\omega \cap \text{lim}(\text{pre}(L_\omega \cap \mathcal{P})) \subseteq \mathcal{P}$. □

THEOREM 4.6. Given an $\omega$-regular language $L_\omega$ and an $\omega$-regular property $\mathcal{P}$ given by nondeterministic Büchi automata or PLTL formulas, determining if $\mathcal{P}$ is a relative liveness or safety property is decidable and is a PSPACE-complete problem.

PROOF. The characterizations given by Lemma 4.4 and Lemma 4.5 reduce the problem to questions decidable in PSPACE [Thomas 1990; Garey and Johnson 1974] (notice that for PLTL formulas one can build in PSPACE a automaton for the formula and for its complement [Vardi and Wolper 1994]). Hardness can be established by a reduction from regular language inclusion [Garey and Johnson 1974]. □

Note that Lemma 4.4 provides the link between relative liveness and machine closure. Indeed, recall the following definition [Abadi and Lamport 1988; Abadi and Lamport 1990; Alur and Henzinger 1993].

Definition 4.7. Let $\Lambda \subseteq L_\omega \subseteq \Sigma^\omega$, for an alphabet $\Sigma$. $(L_\omega, \Lambda)$ is called a machine closed live structure if and only if $\text{pre}(L_\omega) \subseteq \text{pre}(\Lambda)$.

We thus have that $\mathcal{P} \subseteq \Sigma^\omega$ is a relative liveness property of $L_\omega$ if and only if $(L_\omega, \mathcal{P} \cap L_\omega)$ is a machine closed live structure (see Lemma 4.4).

General properties can always be represented as the intersection of a liveness and a safety property [Alpern and Schneider 1983]. As given precisely below, the relativized version of this result is that a property holds for an $\omega$-language if it is both a relative liveness and a relative safety property of the language.

THEOREM 4.8. An $\omega$-language $L_\omega$ satisfies a property $\mathcal{P}$ ($L_\omega \subseteq \mathcal{P}$) if and only if $\mathcal{P}$ is a relative safety and a relative liveness property of $L_\omega$.

PROOF. If $L_\omega \subseteq \mathcal{P}$, then, trivially, $\mathcal{P}$ is a relative safety and a relative liveness property of $L_\omega$.

If $\mathcal{P}$ is a relative safety property of $L_\omega$, then $L_\omega \cap \text{lim}(\text{pre}(L_\omega \cap \mathcal{P})) \subseteq \mathcal{P}$ (Lemma 4.5). If, additionally, $\mathcal{P}$ is a relative liveness property of $L_\omega$, then, by Lemma 4.4, $\text{pre}(L_\omega) = \text{pre}(L_\omega \cap \mathcal{P})$. Therefore, we can replace $\text{pre}(L_\omega \cap \mathcal{P})$...
by \( \text{pre}(L_\omega) \) in the safety condition and obtain \( L_\omega \cap \text{lim}(\text{pre}(L_\omega)) \subseteq \mathcal{P} \). Because \( L_\omega \cap \text{lim}(\text{pre}(L_\omega)) = L_\omega \), we finally obtain \( L_\omega \subseteq \mathcal{P} \). \( \square \)

As shown in [Henzinger 1992], relative liveness and safety properties also have an elegant definition within the Cantor topology, i.e., the topological space over \( \Sigma^\omega \) compatible with the following metric [Eilenberg 1974] (For topological notions see Kelley 1955.)

**Definition 4.9.** Let \( \text{common}(x, y) \) designate the longest common prefix of two \( \omega \)-words \( x \) and \( y \) in \( \Sigma^\omega \). We define the metric \( d(x, y) \) by

\[
\forall x, y \in \Sigma^\omega, x \neq y : d(x, y) = \frac{1}{|\text{common}(x, y)| + 1}
\]

\[
\forall x \in \Sigma^\omega : d(x, x) = 0.
\]

**Lemma 4.10.** A property \( \mathcal{P} \) is a relative liveness property of an \( \omega \)-language \( L_\omega \) if and only if \( L_\omega \cap \mathcal{P} \) is a dense set in \( L_\omega \).

**Proof.** Let \( L_\omega \models \mathcal{P} \), and let \( x \in L_\omega \). Then \( \text{pre}(L_\omega) = \text{pre}(L_\omega \cap \mathcal{P}) \). Thus, \( \text{pre}(x) \subseteq \text{pre}(L_\omega \cap \mathcal{P}) \), and we have \( \forall w \in \text{pre}(x) : \exists y \in L_\omega \cap \mathcal{P} : w \in \text{pre}(y) \). We get, for all \( x \in L_\omega \) and all \( \varepsilon > 0 \) \( \varepsilon \) is related to \( \frac{1}{|w|+1} \), that there is a \( y \in L_\omega \cap \mathcal{P} \) such that \( d(x, y) < \varepsilon \). So \( L_\omega \cap \mathcal{P} \) is a dense set in \( L_\omega \).

Let \( L_\omega \cap \mathcal{P} \) be a dense set in \( L_\omega \). Then, for all \( x \in L_\omega \) and all \( \varepsilon > 0 \), there exists \( y \in L_\omega \cap \mathcal{P} \) such that \( d(x, y) < \varepsilon \). Let \( x \) be in \( L_\omega \), let \( w \) be in \( \text{pre}(x) \) and let \( \varepsilon = \frac{1}{|w|+1} \). Because \( L_\omega \cap \mathcal{P} \) is a dense set in \( L_\omega \), there exists \( y \in L_\omega \cap \mathcal{P} \) such that \( w \in \text{pre}(y) \). Thus \( \text{pre}(L_\omega) \subseteq \text{pre}(L_\omega \cap \mathcal{P}) \). Because \( \text{pre}(L_\omega \cap \mathcal{P}) \subseteq \text{pre}(L_\omega) \), we have \( \text{pre}(L_\omega) = \text{pre}(L_\omega \cap \mathcal{P}) \). By Lemma 4.4, \( \mathcal{P} \) is a relative liveness property of \( L_\omega \). \( \square \)

**Lemma 4.11.** A property \( \mathcal{P} \) is a relative safety property of an \( \omega \)-language \( L_\omega \) if and only if \( L_\omega \cap \mathcal{P} \) is a closed set in \( L_\omega \).

**Proof.** \( \mathcal{P} \) is a relative safety property of \( L_\omega \) if and only if

\[
\forall x \in L_\omega : ( x \notin \mathcal{P} \Rightarrow ( \exists w \in \text{pre}(x) : \forall z \in \text{cont}(w, L_\omega) : wz \notin \mathcal{P} ) ) .
\]

If \( \overline{\mathcal{P}} \) is the complement of \( \mathcal{P} \) with respect to \( L_\omega \), i.e., \( \overline{\mathcal{P}} = L_\omega \cap (\Sigma^\omega \setminus \mathcal{P}) \), which is equivalent to \( \overline{\mathcal{P}} = L_\omega \setminus (L_\omega \cap \mathcal{P}) \), then \( \mathcal{P} \) is a relative safety property of \( L_\omega \) if and only if \( \forall x \in L_\omega : ( x \in \overline{\mathcal{P}} \Rightarrow ( \exists w \in \text{pre}(x) : \forall z \in \text{cont}(w, L_\omega) : wz \notin \mathcal{P} ) ) . \) If we define this condition topologically, then \( \mathcal{P} \) is a relative safety property of \( L_\omega \) if and only if \( \forall x \in \overline{\mathcal{P}} : \exists \varepsilon > 0 : \forall y \in L_\omega : d(x, y) < \varepsilon \Rightarrow y \in \overline{\mathcal{P}} \). Thus, \( \mathcal{P} \) is a relative safety property of \( L_\omega \) if and only if \( \overline{\mathcal{P}} \) is an open set in \( L_\omega \). Because \( \overline{\mathcal{P}} = L_\omega \setminus (L_\omega \cap \mathcal{P}) \) is the complement of \( L_\omega \cap \mathcal{P} \) with respect to \( L_\omega \), we finally obtain that \( \mathcal{P} \) is a relative safety property of \( L_\omega \) if and only if \( L_\omega \cap \mathcal{P} \) is a closed set in \( L_\omega \). \( \square \)

Relative safety having been introduced to complete the picture around relative liveness, we will now use relative liveness as a satisfaction relation, calling it satisfaction **within fairness**.

**Definition 4.12.** We say that \( L_\omega \) satisfies \( \mathcal{P} \) **within fairness** if and only if \( L_\omega \models \mathcal{P} \).
Checking Properties within Fairness and Behavior Abstractions

We have chosen the phrase “within fairness” to stress the fact that for a property satisfied “within fairness” to be fully satisfied, the only missing element is a form of fairness condition on the set of behaviors being considered. Note that since a safety property never requires a fairness condition, a safety property satisfied within fairness by a set of behaviors is also fully satisfied by that set of behaviors. To prove this, recall the definition of a safety property ([Alpern and Schneider 1985], adapted to our notation):

**Definition 4.13.** Property \( \mathcal{P} \subseteq \Sigma^\omega \) is called a safety property if and only if, for all \( x \in \Sigma^\omega \), \( x \not\in \mathcal{P} \) implies \( \exists w \in \text{pre}(x) : \forall y \in \Sigma^\omega : wy \not\in \mathcal{P} \).

We then have the following.

**Lemma 4.14.** If \( \mathcal{P} \) is a safety property, then \( L_\omega \models_{RL} \mathcal{P} \) if and only if \( L_\omega \models \mathcal{P} \).

**Proof.** Let \( L_\omega \models_{RL} \mathcal{P} \), i.e. \( \pre(L_\omega) = \pre(L_\omega \cap \mathcal{P}) \). Assume \( L_\omega \not\models \mathcal{P} \). Let \( x \in L_\omega \) such that \( x \not\in \mathcal{P} \). Because \( \mathcal{P} \) is a safety property, there exists \( w \in \pre(x) \) such that \( \forall y \in \Sigma^\omega : wy \not\in \mathcal{P} \). So \( w \) is not a prefix of an \( \omega \)-word in \( \mathcal{P} \) and thus it is not in \( \pre(L_\omega \cap \mathcal{P}) \). Since \( w \) is in \( \pre(L_\omega) \) we have that \( \pre(L_\omega) \neq \pre(L_\omega \cap \mathcal{P}) \) which contradicts \( L_\omega \models_{RL} \mathcal{P} \). So \( L_\omega \models \mathcal{P} \) must hold.

If \( L_\omega \models \mathcal{P} \), then \( L_\omega \models_{RL} \mathcal{P} \) follows immediately. \( \square \)

5. IMPLEMENTING SYSTEMS THAT SATISFY PROPERTIES WITHIN FAIRNESS

If a property is satisfied by a set of behaviors within fairness, our expectation is that a fair implementation of this set of behaviors will satisfy the property in the classical sense. Unfortunately, this is not true for every implementation, even if one assumes strong fairness. As an example, consider the set of behaviors \( \{a,b\}^\omega \). It is not sufficient to impose strong fairness on the minimal automaton representing \( \{a,b\}^\omega \) in order to satisfy all properties that are satisfied within fairness by \( \{a,b\}^\omega \). For instance, \( \ Diamond(a \land (\Diamond a)) \) would not be satisfied, even though it is satisfied within fairness by \( \{a,b\}^\omega \). The reason for this is that, even if fairness is used, more state information needs to be kept in order to be able to satisfy the property \( \Diamond(a \land (\Diamond a)) \).

However, it is always possible to add sufficient state information to a system in order to turn properties that are satisfied within fairness into properties that are satisfied in the classical sense under fairness. The following theorem makes this precise.

**Theorem 5.1.** Let \( L_\omega \) be a limit closed finite-state set of behaviors (one accepted by a finite state automaton without acceptance conditions, i.e. by a finite-state labeled transition system) and let \( \mathcal{P} \) be an \( \omega \)-regular property. Then, if \( \mathcal{P} \) is satisfied within fairness by \( L_\omega \), there exists a finite-state labeled transition system \( A \) such that the \( \omega \)-language accepted by \( A \) is \( L_\omega \) and all strongly fair computations in \( A \) satisfy \( \mathcal{P} \).

**Proof.** Since \( \mathcal{P} \) is satisfied by \( L_\omega \) within fairness, by Lemma 4.14 we have that \( \pre(L_\omega) = \pre(L_\omega \cap \mathcal{P}) \). Furthermore, since \( L_\omega \) is limit closed we have that \( L_\omega = \lim(\pre(L_\omega)) \) and hence

\[
L_\omega = \lim(\pre(L_\omega \cap \mathcal{P})).
\] (1)

Consider thus a reduced Büchi automaton \( A \) accepting \( L_\omega \cap \mathcal{P} \) (by reduced we mean that states from which no \( \omega \)-word can be accepted have been eliminated).
The finite-state labeled transition system $A$ we are trying to construct is $A$ with its acceptance condition removed. Indeed, by equation (14), $A$ accepts $L_\omega$. Furthermore, all strongly fair infinite computations of $A$ will go infinitely often through a former accepting state of $A$ and thus will satisfy $P$. □

The theorem we have just proved gives an interesting insight into properties satisfied within fairness. They are the properties that fairness makes true of the system, but possibly at the cost of adding state information to the system implementation in a noninterfering way, i.e. without altering the set of limit-closed behaviors of the system.

6. BEHAVIOR ABSTRACTIONS

We now turn to the problem of verifying a system using abstraction. We consider finite-state labeled transition systems (i.e. without acceptance conditions). Hence the finite-word languages accepted by the systems we consider are the prefix-closed regular languages, and the $\omega$-languages they accept are the Eilenberg-limits of prefix-closed regular languages.

We consider abstractions that hide or rename the actions of our systems. Precisely, we consider abstraction homomorphisms that are extensions of alphabetic language homomorphisms to mappings on finite and infinite words as defined below.

**Definition 6.1.** Let $h : \Sigma \to (\Sigma^* \cup \{\varepsilon\})$ be a total function ($\varepsilon$ designates the empty word) and let $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$. Then, the abstraction homomorphism generated by $h$ is the extension of $h$ to a mapping $h : \Sigma^\infty \to \Sigma^\infty$ defined as follows. For all words $w = w_1 w_2 w_3 \ldots w_n \in \Sigma^*$, $n \in \mathbb{N}$, we define $h(w) = h(w_1)h(w_2)h(w_3)\ldots h(w_n)$. For all $\omega$-words $x = x_1 x_2 x_3 \ldots \in \Sigma^\omega$, we define $h(x) = h(x_1)h(x_2)h(x_3)\ldots$, if $\lim(h(pre(x))) \neq \emptyset$. Otherwise, if $\lim(h(pre(x))) = \emptyset$, then $h(x)$ is undefined.

This leads naturally to the following definition of the abstract behavior of a system under an abstraction homomorphism.

**Definition 6.2.** Let $S$ be a system whose behaviors are the limit $\lim(L)$ of a prefix-closed regular language $L$. Then, the abstract behavior of $S$ with respect to the abstraction homomorphism $h$ is $\lim(h(L))$.

Our goal is to prove properties of the behaviors $\lim(L)$ of a system $S$ by only considering the abstract behaviors $\lim(h(L))$ for some abstraction homomorphisms $h$. More specifically, we are interested in the preservation of properties satisfied within fairness under the abstraction homomorphism.

Essential information about the properties that are satisfied within fairness by $\lim(L)$ is contained in the sets $\text{cont}(w, L)$, for $w \in L$. At the abstract level, we obviously have access to $\text{cont}(h(w), h(L))$, but we really need $h(\text{cont}(w, L))$ in order to ensure that properties satisfied within fairness by the abstraction will also be satisfied within fairness by the concrete system in a corresponding way. Thus, we need to investigate the relation between the sets $h(\text{cont}(w, L))$ and $\text{cont}(h(w), h(L))$ and find conditions under which $\text{cont}(h(w), h(L))$ can be used instead of $h(\text{cont}(w, L))$.

In general, $h(\text{cont}(w, L))$ is a proper subset of $\text{cont}(h(w), h(L))$. In order to obtain sufficient information about $h(\text{cont}(w, L))$ from $\text{cont}(h(w), h(L))$, one would
be tempted to require equality of the two sets. Those homomorphisms are *continuation closed*, since computing the continuation or the abstraction first, both have the same result. However, this is stronger than needed. Indeed, since we are dealing with satisfaction within fairness, we will show that it is sufficient that the behaviors in $\text{cont}(h(w), h(L))$ “eventually” become behaviors in $h(\text{cont}(w, L))$. This condition is the one called *simplicity* of an abstraction homomorphism in [Ochsendrager 1994]. We will use a name that is more intuitive with respect to their definition and call them *weakly contiunation-closed* homomorphisms. Their exact definition is the following.

**Definition 6.3.** An abstraction homomorphism $h : \Sigma^\infty \rightarrow \Sigma'^\infty$ is *weakly continuation-closed* for a language $L \subseteq \Sigma^*$ and a word $w \in L$ if and only if there exists $u \in \text{cont}(h(w), h(L))$ such that $\text{cont}(u, \text{cont}(h(w), h(L))) = \text{cont}(u, h(\text{cont}(w, L)))$. The homomorphism $h$ is *weakly continuation-closed* for $L$ if and only if it is for all words $w \in L$.

Theorem 8.4 will show that this definition indeed meets all the requirements we have informally described above. More details about weakly continuation-closed homomorphisms can be found in [Ochsendrager 1994].

### 7. PRESERVATION OF LINEAR PROPERTIES

Before turning to the preservation of properties satisfied within fairness by weakly continuation-closed homomorphisms, we need some general results about abstraction homomorphisms and properties. The problem we address is that the properties true of the abstracted system and of the concrete system can rarely be identical. Indeed, one needs to take into account the fact that the abstraction can rename or hide symbols. Our goal here is to define a transformation on properties that mirrors this.

We consider properties defined by PLTL formulas (see Section 3). In order to make the definition of property transformations easier and to make the interpretation of formulas over words more direct (remember that we are dealing with systems represented by sets of infinite words), we define some normal forms for PLTL formulas.

A first restriction is to consider only positive normal form formulas.

**Definition 7.1.** A PLTL-formula $\eta$ is in *positive normal form* if and only if the scope of all negations is a single atomic proposition.

Now we turn to the problem of interpreting formulas over words. Our generic way of doing this (see Section 3) is to use a mapping $\lambda : \Sigma \rightarrow 2^{\text{AP}}$ from the alphabet $\Sigma$ of the word to the subsets of the atomic propositions $\text{AP}$ of the formula. However, in this context, it is quite natural to consider the elements of $\Sigma$ directly as atomic propositions, which implies that one is using a mapping $\lambda_{\Sigma}$ such that $\forall a \in \Sigma : \lambda_{\Sigma}(a) = \{a\}$. We define a normal form that corresponds to this.

**Definition 7.2.** Let $\Sigma$ be an alphabet. We say that a PLTL formula $\eta$ is in $\Sigma$-normal form if and only if it is in positive normal form and all its atomic propositions are in $\Sigma$ (i.e. $\text{AP} \subseteq \Sigma$).

For an alphabet $\Sigma$, the *canonical $\Sigma$-labeling function* $\lambda_{\Sigma} : \Sigma \rightarrow 2^{\Sigma}$ is the one such that $\forall a \in \Sigma : \lambda_{\Sigma}(a) = \{a\}$. 

ACM Transactions in Computational Logic, Vol. TBD, No. TBD, TBD TBD.
Note that using $\Sigma$-normal form formulas is not really restrictive. Indeed, for any PLTL-formulas $\eta$ over a set $AP$ of atomic proposition and any labeling function $\lambda : \Sigma \rightarrow 2^{AP}$, there exists a PLTL-formula $\eta'$ in $\Sigma$-normal form such that, for all $x \in \Sigma^\omega$, $x, \lambda \models \eta$ if and only if $x, \lambda_x \models \eta'$.

We now turn to the interaction between properties and abstraction homomorphisms. Consider an abstraction homomorphism $h : \Sigma^\omega \rightarrow \Sigma^\omega$ and assume we have established a ($\Sigma'$-normal form) property $\eta$ of the abstract version $L'_\omega \subseteq \Sigma^\omega$ of a system obtained under this homomorphism. Of what system can we say that the property is true on the concrete level? One would expect $h^{-1}(L'_\omega)$. However, this is a language on $\Sigma$ on which we cannot directly interpret $\eta$. One could modify $\eta$ to take this into account, but it is simpler to modify the labeling function.

**Definition 7.3.** For alphabets $\Sigma$ and $\Sigma'$, and for an abstraction homomorphism $h : \Sigma^\omega \rightarrow \Sigma'^\omega$, the canonical $h_{\Sigma'}$-labeling function $\lambda_{h_{\Sigma'}} : \Sigma \rightarrow 2^{\Sigma' \cup \{\varepsilon\}}$ is the one such that such that $\forall a \in \Sigma : \lambda_{h_{\Sigma'}}(a) = \{h(a)\}$.

Notice that this labeling function maps some letters to the proposition $\varepsilon$ which stands for the empty word. So, we can’t expect a formula $\eta$ true of the abstract system $L'_\omega$ to be true of $h^{-1}(L'_\omega)$, even using the mapping $\lambda_{h_{\Sigma'}}$. Indeed, this mapping takes care of the fact that letters are renamed, but does not take care of the fact that $\varepsilon$ is the empty word. What is needed is to ignore the empty word in the evaluation of the formula. This is handled by transforming the formula $\eta$ from $\Sigma'$-normal form to $\Sigma' \cup \varepsilon$-normal form as follows.

**Definition 7.4.** Let $\eta$ be a PLTL-formula in $\Sigma'$-normal form. We define recursively a mapping $T(\eta)$ that yields a formula in $\Sigma' \cup \varepsilon$-normal form (see Figure 5; $\hat{b}$ designates binary boolean operators: $\hat{b} \in \{\land, \lor, \Rightarrow, \Leftrightarrow\}$).

\[
T(\eta) = \begin{cases} 
  \text{true}, & \text{if } \eta = \text{true}, \\
  \neg(\text{true}), & \text{if } \eta = \neg(\text{true}), \\
  a, & \text{if } \eta = a \in \Sigma', \\
  (\neg(a)) \land (\neg(\varepsilon)), & \text{if } \eta = \neg(a) \text{ and } a \in \Sigma', \\
  (T(\xi)) \hat{b}(T(\zeta)), & \text{if } \eta = (\xi) \hat{b}(\zeta), \\
  ((\varepsilon) \lor (T(\xi))) U (T(\zeta)), & \text{if } \eta = (\xi) U (\zeta), \\
  (T(\xi)) B (T(\zeta)), & \text{if } \eta = (\xi) B (\zeta), \\
  \Diamond(T(\xi)), & \text{if } \eta = \Diamond(\xi), \\
  (\neg(\varepsilon)) \land (\Diamond(\varepsilon) U (T(\xi)))) \lor (T(\xi)), & \text{if } \eta = \Box(\xi), \\
  \Box((\varepsilon) \lor (T(\xi))), & \text{if } \eta = \Box(\xi), \\
  (\varepsilon) U ((\neg(\varepsilon)) \land (\Diamond(\varepsilon) U (T(\xi))))), & \text{if } \eta = \Diamond(\xi).
\end{cases}
\]

Fig. 5. The syntactical transformation of PLTL.
As defined, the mapping \( T \) does not modify pure Boolean formulas (not including any temporal operator). However, a pure Boolean formula \( \eta \) should be mapped to 
\[
(\varepsilon)\mathcal{U}(N(\eta)) \text{ where } N \text{ replaces all subformulas } \neg(a) \text{ of a PLTL-formula such that } a \text{ is an atomic proposition by } (\neg(a)) \land (\neg(\varepsilon)).
\]
We thus extend \( T \) into a mapping \( R \) such that \( R(\eta) = T(\eta) \) with all maximal pure Boolean subformulas \( \xi_b \) replaced by 
\[
(\varepsilon)\mathcal{U}(N(\xi_b)).
\]

We can now give a statement relating a property true on an abstraction of a system to a property true at the concrete level

**Lemma 7.5.** Let \( L_\omega \subseteq \Sigma^\omega \), let \( \eta \) be a PLTL-formula in \( \Sigma' \)-normal form, and let 
\[
h : \Sigma^\omega \rightarrow \Sigma^\omega
\]
be an abstraction homomorphism. Then 
\[
L'_\omega, \lambda_{\Sigma'} \models \eta \iff h^{-1}(L'_\omega), \lambda_{h_{\Sigma'}} \models R(\eta).
\]

The proof of Lemma 7.5 consist of two lemmas handling boolean formulas and purely temporal formulas respectively.

**Lemma 7.6.** Let \( h : \Sigma^\omega \rightarrow \Sigma^\omega \) be an abstraction homomorphism. Let \( x' \in \Sigma^\omega \) be an abstract computation and let \( x \in h^{-1}(x') \). Let \( \eta \) be a boolean formula in \( \Sigma' \)-normal form. Then 
\[
x', \lambda_{\Sigma'} \models \eta \text{ if and only if } x, \lambda_{h_{\Sigma'}} \models (\varepsilon)\mathcal{U}(N(\eta)).
\]

**Proof.** Let \( i \in N \) such that \( h(x_i) = x'_1 \) and, for all \( j < i \), \( h(x_j) = \varepsilon \). We have, for all atomic propositions \( a \in \Sigma' \), that \( x', \lambda_{\Sigma'} \models a \) if and only if \( x_{(i...)}), \lambda_{h_{\Sigma'}} \models a, \) and thus \( x', \lambda_{\Sigma'} \models \neg(a) \) if and only if \( x_{(i...)}), \lambda_{h_{\Sigma'}} \models \neg(a) \). Because \( h(x_i) \neq \varepsilon \), we have \( x', \lambda_{\Sigma'} \models \neg(a) \) if and only if \( x_{(i...)}), \lambda_{h_{\Sigma'}} \models (\neg(a)) \land (\neg(\varepsilon)) \). According to the semantics of boolean connectives we obtain 
\[
x', \lambda_{\Sigma'} \models \eta \text{ if and only if } x_{(i...)}), \lambda_{h_{\Sigma'}} \models (\varepsilon)\mathcal{U}(N(\eta)).
\]

**Lemma 7.7.** Let \( h : \Sigma^\omega \rightarrow \Sigma^\omega \) be an abstraction homomorphism. Let \( x' \in \Sigma^\omega \) be an abstract computation and let \( x \in h^{-1}(x') \). Let \( \eta \) be a PLTL-formula in \( \Sigma' \)-normal form such that all atomic propositions are in the scope of a temporal operator (we call these formulas purely temporal). Then 
\[
x', \lambda_{\Sigma'} \models \eta \text{ if and only if } x, \lambda_{h_{\Sigma'}} \models T(\eta).
\]

**Remark 7.8.** Lemma 7.7 is not surprising, because \( T(\eta) \) takes care of subwords of \( \omega \)-words in \( h^{-1}(x') \) that \( h \) takes to \( \varepsilon \), not changing the general structure of \( \eta \). However, because many cases need to be distinguished, the proof of Lemma 7.7 is quite lengthy.

**Proof.** The proof is by induction on the structure of \( \eta \). If \( \eta \) contains exactly one temporal operator that quantifies over all atomic propositions in \( \eta \) (the induction’s basis), then all proper subformulas \( \xi \) of \( \eta \) are boolean formulas and hence \( T(\xi) = N(\xi) \).

By Lemma 7.4 and since \( T(\xi) = N(\xi) \), for all proper subformulas \( \xi \) of \( \eta \) and all \( x \in h^{-1}(x') \) we have \( x', \lambda_{\Sigma'} \models \xi \) if and only if \( x, \lambda_{h_{\Sigma'}} \models (\varepsilon)\mathcal{U}(T(\xi)) \). Therefore, if \( h(x_1) \neq \varepsilon \), \( x', \lambda_{\Sigma'} \models \xi \) if and only if \( x, \lambda_{h_{\Sigma'}} \models T(\xi) \). We use this equivalence to...
prove the induction’s basis. Because all atomic propositions of \( \eta \) are in the scope of the only temporal operator, we need not prove the induction’s basis for boolean connectives.

\[ \eta = (\xi) \mathcal{U}(\zeta): x', \lambda_{\Sigma'} \models (\xi) \mathcal{U}(\zeta) \] if and only if there exists \( i \in \mathbb{N} \) such that \( x'_{(i...)} \lambda_{\Sigma'} \models \zeta \) and \( x'_{[i]} \lambda_{\Sigma'} \models \xi \), for all \( j < i \). This is equivalent to the existence of a \( k \in \mathbb{N} \) such that \( x'_{(k...)} \lambda_{h_{\Sigma'}} \models T(\zeta) \) and \( x'_{(l...)} \lambda_{h_{\Sigma'}} \models T(\xi) \), for all \( l < k \) such that \( h(x_l) \neq \varepsilon \). Thus, \( x', \lambda_{\Sigma'} \models (\xi) \mathcal{U}(\zeta) \) if and only if \( x, \lambda_{h_{\Sigma'}} \models (\varepsilon) \vee (T(\xi)) \mathcal{U}(T(\zeta)). \)

\[ \eta = (\xi) \mathcal{B}(\zeta): x', \lambda_{\Sigma'} \models (\xi) \mathcal{B}(\zeta) \] if and only if there exists no \( i \in \mathbb{N} \) such that \( x'_{(i...)} \lambda_{\Sigma'} \models \zeta \) or there exists an \( i \in \mathbb{N} \) and a \( j < i \) such that \( x'_{(i...)} \lambda_{\Sigma'} \models \zeta \) and \( x'_{(j...)} \lambda_{\Sigma'} \models \xi \), for all \( k < i \). This is equivalent to: There exists no \( i \in \mathbb{N} \) such that \( x'_{(i...)} \lambda_{h_{\Sigma'}} \models T(\zeta) \) or there exists an \( i \in \mathbb{N} \) and \( m < l \) such that \( x'_{(i...)} \lambda_{h_{\Sigma'}} \models T(\zeta) \) and \( x'_{(m...)} \lambda_{h_{\Sigma'}} \models T(\xi) \), for all \( n < l \). Therefore, \( x', \lambda_{\Sigma'} \models (\xi) \mathcal{B}(\zeta) \) if and only if \( x, \lambda_{h_{\Sigma'}} \models (T(\xi)) \mathcal{B}(T(\zeta)). \)

\[ \eta = (\diamond \xi): x', \lambda_{\Sigma'} \models (\diamond \xi) \] if and only if \( x'_{(i...)} \lambda_{\Sigma'} \models \xi \), for all \( i \in \mathbb{N} \). This is equivalent to: For all \( j \in \mathbb{N} \) with \( h(x_j) \neq \varepsilon \) we have \( x'_{(j...)} \lambda_{h_{\Sigma'}} \models T(\xi) \). Since \( h(x_i) = x' \) there are infinitely many different \( j \in \mathbb{N} \) with \( h(x_j) \neq \varepsilon \) and consequently \( x', \lambda_{\Sigma'} \models (\diamond \xi) \) and only if \( x, \lambda_{h_{\Sigma'}} \models (\diamond (\varepsilon) \vee (T(\xi))). \)

Thus, this last step finishes the proof of the induction’s basis. In the inductive step, the proper subformulas of \( \eta \) need not necessarily satisfy the preconditions of the lemma, because they can contain atomic propositions that are not in the scope of a temporal operator (of the subformula). Hence, in general, a subformula \( \xi \) of \( \eta \) is the boolean combination of boolean formulars \( \xi_b \) and purely temporal formulars \( \xi_t \). By induction, we have \( x', \lambda_{\Sigma'} \models \xi_t \) if and only if \( x, \lambda_{h_{\Sigma'}} \models T(\xi_t) \). According to Lemma 7, \( x', \lambda_{\Sigma'} \models \xi_b \) if and only if \( x, \lambda_{h_{\Sigma'}} \models (\varepsilon) \mathcal{U}(T(\xi_b)) \). Thus, \( x', \lambda_{\Sigma'} \models \xi_b \) if and only if \( x, \lambda_{h_{\Sigma'}} \models (\varepsilon) \mathcal{U}(T(\xi_b)) \). Hence, if \( h(x_i) \neq \varepsilon \), then \( x', \lambda_{\Sigma'} \models \xi_b \) if and only if \( x, \lambda_{h_{\Sigma'}} \models T(\xi_b) \). Therefore, for all subformulas \( \xi \) of \( \eta \), we have: if \( h(x_i) \neq \varepsilon \), then \( x', \lambda_{\Sigma'} \models \xi \) if and only if \( x, \lambda_{h_{\Sigma'}} \models T(\xi) \). We use this condition as our induction’s hypothesis.

\[ \eta = (\xi) \mathcal{B}(\zeta): \text{Because of the lemma’s preconditions, } \xi \text{ and } \zeta \text{ must be purely temporal subformulas of } \eta, \text{ for a binary boolean connective } \mathcal{B}. \text{ Then, by induction and the semantics of boolean connectives, } x', \lambda_{\Sigma'} \models (\xi) \mathcal{B}(\zeta) \text{ if and only if } x, \lambda_{h_{\Sigma'}} \models (T(\xi)) \mathcal{B}(T(\zeta)). \]

\[ \eta = (\xi) \mathcal{U}(\zeta): x', \lambda_{\Sigma'} \models (\xi) \mathcal{U}(\zeta) \text{ if and only if there exists } i \in \mathbb{N} \text{ such that } x'_{(i...)} \lambda_{\Sigma'} \models \zeta \text{ and, for all } j < i, x'_{(j...)} \lambda_{\Sigma'} \models \xi. \text{ By induction, this is equivalent to the existence of } k \in \mathbb{N} \text{ such that } x'_{(k...)} \lambda_{h_{\Sigma'}} \models T(\zeta) \text{ and, for all } l < k \text{ we have } x'_{(l...)} \lambda_{h_{\Sigma'}} \models T(\xi) \text{ or } h(x_l) = \varepsilon. \text{ Therefore, } x', \lambda_{\Sigma'} \models (\xi) \mathcal{U}(\zeta) \text{ if and only if } x, \lambda_{h_{\Sigma'}} \models ((\varepsilon) \vee (T(\xi))) \mathcal{U}(T(\zeta)). \]
η = (ξ) B (ξ): x′, λΣ′ |= (ξ) B (ξ) if and only if there exists no i ∈ ℕ such that 
\[ x′_{(i...)} , λΣ′ |= ζ \] or there exists an i ∈ ℕ and a j < i such that 
\[ x′_{(i...)} , λΣ′ |= ζ, x′_{(j...)} , λΣ′ |= ξ \] and \[ x′_{(k...)} , λΣ′ \neq ζ, \] for all k < i. By induction, this is equivalent to:
There exists no l ∈ ℕ such that \[ x_{(l...)} , λh_{ΣΣ′} |= T(ξ) \] or there exists an l ∈ ℕ and an 
m < l such that \[ x_{(l...)} , λh_{ΣΣ′} |= T(ξ), x_{(m...)} , λh_{ΣΣ′} |= T(ξ), \] and \[ x_{(n...)} , λh \neq T(ξ), \] for all n < l. Therefore, 
\[ x′, λΣ′ |= (ξ) B (ξ) \] if and only if \[ x, λh_{ΣΣ′} |= (T(ξ)) B (T(ξ)). \]

By induction, this is equivalent to the existence of j ∈ ℕ such that 
\[ x_{(j...)} , λh_{ΣΣ′} |= T(ξ). \] Hence, \[ x′, λΣ′ |= \bigcirc (ξ) \] if and only if 
\[ x, λh_{ΣΣ′} |= \bigcirc (T(ξ)). \]

Lemma 8.1 appears to be rather trivial. But, in fact, it neither holds for regular 
languages that are not prefix-closed nor for prefix-closed languages that are not 
regular. The languages \[ a^i \cdot b \text{ and } \text{pre}\{\{b^i \cdot a^i \mid i ∈ ℕ\}\} \] reveal this observation for 
the homomorphism defined by \[ h(a) = a \text{ and } h(b) = ε. \] To prove the lemma, we use 
König’s Lemma in a suitable version (Hoogeboom and Rozenberg 1986, Lemma 
3.3.):

**LEMMA 8.2 KÖNIG’S LEMMA.** Let \( R ⊆ E × E \) be a relation — \( E \) is an arbitrary 
set — and let, for all \( n ∈ ℕ, E_n \) be a finite nonempty subset of \( E \) such that 
\( \bigcup_{n ∈ ℕ} E_n \) is infinite and to each \( e ∈ E_{n+1} \) there exists an \( f ∈ E_n \) such that 
\( (f, e) ∈ R. \) Then there exists an infinite sequence \( (e_n)_{n ∈ ℕ} \) in \( E \) such that 
\( e_n ∈ E_n \) and \( (e_n, e_{n+1}) ∈ R \) for all \( n ∈ ℕ. \)

**PROOF OF LEMMA 8.1.** “\( \text{lim}(h(L)) \subseteq h(\text{lim}(L)) \)”:: We assume \( \text{lim}(h(L)) \neq \emptyset \) (otherwise the condition holds trivially).

If \( x \) is an \( ω \)-word in \( h(L) \), then \( \text{pre}(x) \subseteq h(L) \) (remember that \( L \) and 
\( R \) are)
therefore \( h(L) \) are prefix-closed). Let \( w^n \) be the prefix of \( x \) of length \( n \).\(^1\) \((w^n)_{n \in \mathbb{N}}\)

is then the sequence of all prefixes of \( x \) and thus generates \( x \) as its limit.

To each of the \( w^n \) we construct a set \( U_n \) of minimal inverse images of \( w^n \). Let \( U_n \) be the set of all words \( u \) in \( h^{-1}(w^n) \cap L \), such that there is no shorter word \( v \) in \( h^{-1}(w^n) \cap L \) \( \)with \( \text{cont}(u, L) = \text{cont}(v, L) \). We define

\[
U_n = \{ u \in h^{-1}(w^n) \cap L \mid \exists v \in h^{-1}(w^n) \cap L : |u| > |v| \land \text{cont}(u, L) = \text{cont}(v, L) \}. 
\]

Because all \( w^n \) are in \( h(L) \) there must be a \( u \in L \) such that \( h(u) = w^n \) to each \( w^n \). Consequently, \( U_n \) is not empty, for all \( n \in \mathbb{N} \).

Let \( u \in U_n \). For all \( v \in U_n \) such that \( \text{cont}(u, L) = \text{cont}(v, L) \), we have \(|v| = |u|\) by definition of \( U_n \). Because the set \( \{ \text{cont}(t, L) \mid t \in \Sigma^* \} \) is finite (its cardinality corresponds to the number of states in the minimal automaton accepting \( L \)), we obtain: \( U_n \) is a finite set, for all \( n \in \mathbb{N} \).

Because \( U_n \cap U_m = \emptyset \) if \( n \neq m \) and all \( U_n \) are nonempty sets, we observe that \( \bigcup_{n \in \mathbb{N}} U_n \) is an infinite set.

By \( \prec \) we denote the proper prefix relation; i.e. for all \( u \), \( v \in \Sigma^* \), \( u \prec v \) if and only if \( u \neq v \) and \( u \in \text{pre}(v) \). We show: For all \( n \in \mathbb{N} \) and all \( v \in U_{n+1} \), there exists a word \( u \in U_n \) such that \( u \prec v \). Let \( v \) be in \( U_{n+1} \) and let \( u \) be in \( \text{pre}(v) \) such that \( h(u) = w^n \). Hence \( u \prec v \). Because \( L \) is prefix-closed, \( u \) is in \( L \) and thus \( u \in h^{-1}(w^n) \cap L \). The remainder of \( v \) after \( u \) we call \( v' \); i.e. \( v = u v' \). We assume that \( u \) is not in \( U_n \) and show a contradiction.

If \( u \notin U_n \), then there must be a word \( u' \in h^{-1}(w^n) \cap L \) such that \(|u'| < |u|\) and \( \text{cont}(u, L) = \text{cont}(u', L) \). Because \( u' \) is in \( h^{-1}(w^n) \cap L \), we have \( h(u' v') = w^{n+1} \). Because \( \text{cont}(u, L) = \text{cont}(u', L) \), we obtain \( u' v' \in L \) and \( \text{cont}(v, L) = \text{cont}((u' v'), L) \). So \( u' v' \) is in \( h^{-1}(w^{n+1}) \cap L \), \( \text{cont}((u' v'), L) = \text{cont}(v, L) \) and \(|u' v'| < |v|\). Therefore \( v \notin U_{n+1} \), which contradicts the choice of \( v \).

Hence all preconditions to apply König’s Lemma are satisfied by the sets \( U_n, n \in \mathbb{N} \), and thus there exists an infinite sequence \((u^n)_{n \in \mathbb{N}}\) of words in \( L \) such that \( u^n \in U_n \) and \( u^n \prec u^{n+1} \), for all \( n \in \mathbb{N} \). The sequence \((u^n)_{n \in \mathbb{N}}\) uniquely generates some \( y \in \text{lim}(L) \) and, because \( h(u^n) = w^n \), for all \( n \in \mathbb{N} \), we obtain \( h(y) = x \). So, for all \( x \in \text{lim}(h(L)) \), there exists a \( y \in \text{lim}(L) \) such that \( x = h(y) \). Thus \( \text{lim}(h(L)) \subseteq \text{lim}(h(L)) \).

“\( h(\text{lim}(L)) \subseteq \text{lim}(h(L)) \)”: Let \( h(\text{lim}(L)) \neq \emptyset \). Let \( x \) be in \( \text{lim}(L) \), such that \( h(x) \) is defined. Because \( L \) is prefix-closed, all \( u \in \text{pre}(x) \) are in \( L \). So, for all \( u \in \text{pre}(x) \), \( h(u) \) is in \( \text{pre}(h(x)) \). Because \( h(x) \) is defined, there are infinitely many different \( h(u) \) in \( \text{pre}(h(x)) \), for \( u \in \text{pre}(x) \subseteq L \). Thus \( h(x) \) is in \( \text{lim}(h(L)) \), and we obtain \( h(\text{lim}(L)) \subseteq \text{lim}(h(L)) \). \( \square \)

Using Lemma 8.1, we can now prove a result relating a property satisfied within fairness by \( \text{lim}(h(L)) \) to a property satisfied within fairness by \( \text{lim}(L) \).

**Theorem 8.3.** Let \( L \subseteq \Sigma^* \) be a prefix-closed regular language, let \( h : \Sigma^\infty \to \Sigma'^\infty \) be an abstraction homomorphism such that \( h \) is weakly continuation-closed on \( L \) and \( h(L) \) does not contain maximal word\(^2\), and let \( \eta \) be a PLTL-formula in \( \Sigma' \)-

---

\(^1\)The notation \( w^n \) should not be confused with the \( n \)th power of \( w \) (\( n \) is just an index).

\(^2\)Maximal words in \( h(L) \) are words that are not a proper prefix of another word in \( h(L) \). We will lift the restriction to maximal-word-free abstractions in the next section.
normal form. Then
\[
\lim(h(L)), \lambda_{\Sigma^*} \models_{RL} \eta \text{ if and only if } \lim(L), \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta).
\]

This theorem is will be a consequence of the following two lemmas (Lemma 8.1 and Lemma 8.3).

**Lemma 8.4.** Let \( L \subseteq \Sigma^* \) be a prefix-closed regular language, let \( h : \Sigma^* \rightarrow \Sigma_{RL}^* \) be an abstraction homomorphism such that \( h \) is weakly continuation-closed on \( L \) and \( h(L) \) does not contain maximal words, and let \( \eta \) be a PLTL-formula in \( \Sigma^*\)-normal form. We have that
\[
\lim(h(L)), \lambda_{\Sigma^*} \models_{RL} \eta \implies \lim(L), \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta).
\]

**Proof.** We assume that \( \lim(h(L)), \lambda_{\Sigma^*} \models_{RL} \eta \) and derive \( \lim(L), \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta) \).

By definition \( \lim(h(L)), \lambda_{\Sigma^*} \models_{RL} \eta \) and \( R(\eta) \) if for all \( u \in L \), there exists some \( x \in \text{cont}(u, \lim(h(L))) \) such that \( ux, \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta) \). Consider thus an arbitrary \( u \in L \). Because \( h \) is weakly continuation-closed on \( L \), there exists \( v \in \text{cont}(h(u), h(L)) \) such that
\[
\text{cont}(v, h(\text{cont}(u, L))) = \text{cont}(v, \text{cont}(h(u), h(L))) = \text{cont}(h(u)v, h(L)).
\]

As \( \lim(h(L)), \lambda_{\Sigma^*} \models_{RL} \eta \), we get \( \forall r \in \text{pre}(\lim(h(L))) : \exists s \in \text{cont}(r, \lim(h(L))) : rs, \lambda_{\Sigma^*} \models \eta \), and in particular, by substituting \( h(u)v \) for \( r \), there exists some \( y \in \text{cont}(h(u)v, \lim(h(L))) = \text{cont}(h(u)v, h(L)) \) such that
\[
h(u)v, \lambda_{\Sigma^*} \models \eta.
\]

Given equation (1) this is equivalent to
\[
y \in \lim(\text{cont}(v, h(\text{cont}(u, L)))) = \text{cont}(v, \lim(h(\text{cont}(u, L)))).
\]

Thus we know that \( vy \) is in \( \lim(h(\text{cont}(u, L))) \), which, in view of Lemma 8.1, is equivalent to
\[
vy \in h(\lim(\text{cont}(u, L))).
\]

So, there exists \( x \in \lim(\text{cont}(u, L)) \) such that
\[
h(x) = vy.
\]

Viewing \( vy \) as a single word \( z \), we have shown that for all \( u \in L \), there exists \( x \in \lim(\text{cont}(u, L)) \) and \( z \in \text{cont}(h(u), \lim(h(L))) \) such that \( h(x) = z \) (because of equation (1)) and \( h(u)z, \lambda_{\Sigma^*} \models \eta \) (because of equation (2)).

Consider now the language \( \hat{L} = \text{pre}(ux) \) of prefixes of \( ux \). Clearly, \( \lim(\hat{L}) = \{ ux \} \) and \( \lim(h(\hat{L})) = \{ h(u)z \} \).

Because \( h(u)z, \lambda_{\Sigma^*} \models \eta \), we have \( \lim(h(\hat{L})), \lambda_{\Sigma^*} \models \eta \). Using Lemma 7.4 and given that \( \lim(\hat{L}) \subseteq h^{-1}(\lim(h(L))) \), we obtain \( \lim(L), \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta) \), or \( ux, \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta) \). We have thus shown that for all \( u \in L \), there exists \( x \in \text{cont}(u, \lim(L)) \), such that \( ux, \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta) \). Hence we have shown that \( \lim(L), \lambda_{\Sigma_{RL}^*} \models_{RL} R(\eta) \). 

\( \square \)
As discussed in Section 9.4 using an example, Lemma 8.4 does not hold, if we do not require the abstraction homomorphism to be weakly continuation-closed.

**Lemma 8.5.** Let $L \subseteq \Sigma^*$ be a prefix-closed regular language. Let $h : \Sigma^\infty \rightarrow \Sigma^\infty$ be an abstraction homomorphism such that $h(L)$ does not contain maximal words. Let $\eta$ be a PLTL-formula in $\Sigma'$-normal form. Then

$$\lim(L), \lambda_{\Sigma^{\infty}} \models_{RL} R(\eta) \implies \lim(h(L)), \lambda_{\Sigma^{\infty}} \models_{RL} \eta.$$  

**Proof.** We assume that $\lim(L), \lambda_{\Sigma^{\infty}} \models_{RL} R(\eta)$ and show that $\lim(h(L)), \lambda_{\Sigma^{\infty}} \models_{RL} \eta$.

Let $w' \in \text{pre}(\lim(h(L)))$, let $w \in \text{pre}(\lim(L)) \cap h^{-1}(w')$, and let $x \in \text{cont}(w, \lim(L))$ such that $wx, \lambda_{\Sigma^{\infty}} \models \eta$.

If $h(wx)$ is defined, then, by Lemma 7.5, $h(wx), \lambda_{\Sigma^{\infty}} \models \eta$. Therefore, there exists an $x' = h(x) \in \text{cont}(w', \lim(h(L)))$ such that $w'x', \lambda_{\Sigma^{\infty}} \models \eta$.

If $h(wx)$ is undefined, then there is a prefix $v$ of $wx$ such that $h(\text{cont}(v, \text{pre}(wx))) = \{\varepsilon\}$. (In fact, there are infinitely many of these prefixes $v$.) Then, by definition of $R$ and $\lambda_{\Sigma^{\infty}}$, we have, for all $y \in \Sigma^\infty$, that $vy, \lambda_{\Sigma^{\infty}} \models R(\eta)$.

If there exists $y \in \Sigma^\infty$ such that $h(y) \in \text{cont}(h(v), \lim(h(L)))$, then let $x'$ be the only $\omega$-word in $\text{cont}(w', \{h(vy)\})$. $x'$ is in $\text{cont}(w', \lim(h(L)))$. So by Lemma 7.5, $w'x', \lambda_{\Sigma^{\infty}} \models \eta$.

If there exists no $y \in \Sigma^\infty$ such that $h(y) \in \text{cont}(h(v), \lim(h(L)))$, then $h(L)$ contains maximal words, which contradicts the theorem’s preconditions.

So, for all $w' \in \text{pre}(\lim(h(L)))$, there exists an $x' \in \text{cont}(w', \lim(h(L)))$ such that $w'x', \lambda_{\Sigma^{\infty}} \models \eta$. Thus $\lim(h(L)), \lambda_{\Sigma^{\infty}} \models_{RL} \eta$. 

We discuss in the next section how we can extend Theorem 8.3 to deal with maximal words.

**9. IMPROVING THE RESULTS**

If a language $L \subseteq \Sigma^*$ contains maximal words, i.e. words that have no continuation in $L$, then $\lim(L)$ contains no information about them: if $w$ is a maximal word in $L$, then $w \not\in \text{pre}(\lim(L))$. To avoid this loss of information we extend maximal words by dummy-letters. Formally, we define satisfaction within fairness on $L$ itself instead of $\lim(L)$.

**Definition 9.1.** Let $L \subseteq \Sigma^*$. Let $\# \not\subseteq \Sigma$. We define the set of maximal words of $L$ by $\max(L) = \{w \in L \mid \text{cont}(w, L) = \{\varepsilon\}\}$. We define the extension of $L$ to be $\text{xtd}(L) = L \cup \max(L) \cup \{\#\}^*.$

If $L$ is a regular language, then the construction of an automaton accepting $\text{xtd}(L)$ is easy: for all accepting states in a reduced deterministic automaton for $L$ that have no outgoing transition, add a self-loop labelled with $\#$ to that state. Then the resulting automaton accepts $\text{xtd}(L)$.

**Definition 9.2.** Let $L \subseteq \Sigma^*$, let $\eta$ be a PLTL-formula, and let $\lambda : \Sigma \rightarrow 2^{AP}$ be a labelling function. $L$ satisfies $\eta$ within fairness with respect to $\lambda$ (written: “$L, \lambda_{\text{RL}} \models \eta$”) if and only if $\lim(\text{xtd}(L)), \lambda_{\text{RL}} \models \eta$.

**Definition 9.3.** Let $\Sigma$ be an alphabet. A PLTL-formula is in extended $\Sigma$-normal form if and only if it is in positive normal form (Definition 7.3), $\Sigma \cup \{\varepsilon\}$ is its set

ACM Transactions in Computational Logic, Vol. TBD, No. TBD, TBD TBD.
of atomic propositions, and it contains the atomic proposition $\varepsilon$ only in the form $\square(\varepsilon)$ ("all actions are hidden by the abstraction").

**Definition 9.4.** Let $\lambda : \Sigma \rightarrow 2^{AP}$ be a labelling function for an alphabet $\Sigma$ and a set of atomic propositions $AP$.

We define the $\varepsilon$-extension of $\lambda$ to be the function $\lambda^\varepsilon : \Sigma \cup \{\#\} \rightarrow 2^{AP \cup \{\varepsilon\}}$ such that $\lambda^\varepsilon(a) = \lambda(a)$, for all $a \in \Sigma$, and $\lambda^\varepsilon(\#) = \{\varepsilon\}$.

We define the $\#$-extension of $\lambda$ to be the function $\lambda^\# : \Sigma \cup \{\#\} \rightarrow 2^{AP \cup \{\#\}}$ such that $\lambda^\#(a) = \lambda(a)$, for all $a \in \Sigma$, and $\lambda^\#(\#) = \{\#\}$.

**Theorem 9.5.** Let $h : \Sigma^\infty \rightarrow \Sigma^\infty$ be a weakly continuation-closed homomorphism on the prefix-closed regular language $L \subseteq \Sigma^*$. Let $\eta$ be a PLTL-formula in extended $\Sigma'$-normalform. Then

$$L, \lambda^\varepsilon_h R(\eta) \text{ if and only if } h(L), \lambda^\varepsilon_h R(\eta).$$

**Proof.** Let the extension of $L$ with respect to empty abstract suffixes be the language $\text{xt}d_h(L) = L \cup \{w \in L \mid h(\text{cont}(w, L)) = \{\varepsilon\} \cdot \{\#\}^.*

Let $h' : (\Sigma \cup \{\#\})^\infty \rightarrow (\Sigma' \cup \{\#\})^\infty$ be the abstraction homomorphism defined by $h'(a) = h(a)$, for all $a \in \Sigma$, and $h'(\#) = \#$. Because $h$ is weakly continuation-closed on $L$, $h'$ is weakly continuation-closed on $\text{xt}d_h(L)$ and $\text{xt}d_h(L) = \text{xt}d(h(L))$. The latter equality holds, because $h$ being weakly continuation-closed on $L$ implies for all $w \in L$, $\text{cont}(h(w), h(L)) = \{\varepsilon\}$ if $h(\text{cont}(w, L)) = \{\varepsilon\}$ [Ochsenschläger 1992].

Because $h'(\text{xt}d_h(L)) = \text{xt}d(h(L))$, $h'(\text{xt}d_h(L))$ does not contain maximal words.

Let $\eta'$ be the PLTL-formula that we obtain by replacing the atomic proposition $\varepsilon$ in $\eta$ by a new atomic proposition $\#$. We have

- $\lim(h'(\text{xt}d_h(L))), \lambda^\varepsilon_{\Sigma'} R(\eta')$ if and only if $\lim(h'(\text{xt}d_h(L))), \lambda^\varepsilon_{\Sigma'} R(\eta'),$
- $\lim(\text{xt}d_h(L)), \lambda^\#_h R(\eta')$ if and only if $\lim(\text{xt}d_h(L)), \lambda^\#_h R(\eta'),$
- $\lim(\text{xt}d_h(L)), \lambda^\#_h R(\eta')$ if and only if $\lim(\text{xt}d_h(L)), \lambda^\#_h R(\eta').$

Additionally, by Theorem 5.3, we have that $\lim(h'(\text{xt}d_h(L))), \lambda^\varepsilon_{\Sigma'} R(\eta')$ if and only if $\lim(\text{xt}d_h(L)), \lambda^\#_h R(\eta')$. According to the above established equivalences and $h'(\text{xt}d_h(L)) = \text{xt}d(h(L))$, we finally obtain $L, \lambda^\varepsilon_{\Sigma'} R(\eta)$ if and only if $h(L), \lambda^\varepsilon_{\Sigma'} R(\eta)$.

If the above result is not restricted to PLTL properties but extended to all possible $\omega$-languages as properties, one can also show that weak continuation-closure of a homomorphism is not only a sufficient but also a necessary condition for an abstraction to preserve properties satisfied within fairness [Nitsche 1998b, Nitsche and Ochsenschläger 1996].

10. CONCLUSION

We have introduced satisfaction within fairness as a satisfaction relation with an inherent abstract notion of fairness. It is defined in terms of relative liveness properties [Alur and Henzinger 1995, Henzinger 1992], lifted from a property classification to a satisfaction relation [Nitsche and Ochsenschläger 1996, Nitsche and Wolper 1997]. Besides exploring the basic properties of the relation — including exploring its dual, relative safety — we have motivated its definition by considering a small but typical introductory example of a distributed system.
We have established the link from satisfaction within fairness to the usual satisfaction of linear-time properties under fairness by showing that, to a regular system behavior satisfying a linear-time property within fairness, a finite-state implementation can always be found that satisfies the property under strong fairness. As the this finite-state implementation is usually significantly bigger (many more states) than the most compact finite-state implementation of the behavior, satisfaction within fairness offers a way of dealing with linear-time satisfaction under fairness using more compact behavior representations.

Since, however, state-spaces of realistic systems are far too large to effectively be constructed, we have looked at behavior abstractions to decrease the size of the state space. Behavior abstraction is, compared to abstract interpretation, a relatively primitive but by that easy-to-apply approach to tackle state-space explosion. The two concepts in behavior abstractions are action renaming and hiding. These concepts can be defined in terms of language homomorphisms extended to operate on \( \omega \)-languages. In particular action renaming alters patterns of events in computations of a system. To handle these alterations on the level of linear-time temporal logic model-checking, we use a syntactic transformation of PLTL-formulas. We show that an abstract computation of the system satisfies a PLTL-formula if and only if the concrete computation that results in the abstract one satisfies the syntactically transformed formula.

As discussed in the context of the motivating example mentioned above, it turns out that behaviors abstractions are in general too imprecise to preserve properties satisfied within fairness. Here, preservation refers to a property being true on the abstract level implying a corresponding property (the syntactically transformed one) being true on the concrete level. Elaborating on this we give a condition for abstraction homomorphisms that guarantees the preservation of properties satisfied within fairness by the abstraction. The condition that abstraction homomorphisms must satisfy is weak continuation-closure [Ochsenh"{a}lter 1992]. The initial preservation result we establish for weakly continuation-closed abstractions and properties satisfied within fairness only holds for behaviors in which no computation is finite (no maximal words in the language representing the behavior). We have extended the result to capture also behaviors that contain terminating computations.

For practical purposes [Nitsche 1998a], it is essential to be able to obtain a representation of the abstract behavior of a system without an exhaustive construction of the concrete one. It appears promising to tackle this problem by applying partial-order reduction. The aim is to construct a (partial-order) reduced state-space that results in the same abstract state-space as the concrete state-space would. In addition, it must be possible to check weak continuation-closure of the abstraction on the concrete state-space by only considering the partial-order reduced one. A first major result in that direction is presented in [Ultes-Nitsche and St James 2000], where the persistent-set selective search [Godefroid and Wolper 1993] partial-order technique is applied in the context of the abstractions presented in this paper. The efficient construction of abstract state spaces beyond [Ultes-Nitsche and St James 2000], as well as efficiently checking weak continuation-closure will be topics for further study.
REFERENCES

Abadi, M. and Lamport, L. 1988. The existence of refinement mappings. SRC Report 29, DEC System Research Center. July.

Abadi, M. and Lamport, L. 1990. Composing specifications. SRC Report 66, DEC System Research Center. October.

Alferen, B. and Schneider, F. B. 1985. Defining liveness. Information Processing Letters 21, 4 (October), 181–185.

Alur, R. and Henzinger, T. A. 1995. Local liveness for compositional modeling of fair reactive systems. In Computer Aided Verification (CAV) ’95, P. Wolper, Ed. Lecture Notes in Computer Science, vol. 939. Springer, 166–179.

Bersche, J. 1979. Transductions and Context-Free Languages, first ed. Studienbücher Informatik. Teubner Verlag, Stuttgart.

Eilenberg, S. 1974. Automata, Languages and Machines. Vol. A. Academic Press, New York.

Emerson, E. A. 1990. Temporal and modal logic. See van Leeuwen [1990], 995–1072.

Francez, N. 1986. Fairness, first ed. Springer Verlag, New York.

Garey, M. R. and Johnson, D. S. 1979. Computers and Intractability. A Guide to the Theory of NP-Completeness. W.H. Freeman and Co., New York.

Godefroid, P. and Wolper, P. 1993. Using partial orders for the efficient verification of deadlock freedom and safety properties. Formal Methods in System Design 2, 2 (April), 149–164.

Harrison, M. A. 1978. Introduction to Formal Language Theory, first ed. Addison-Wesley, Reading, Mass.

Henzinger, T. A. 1992. Sooner is safer than later. Information Processing Letters 43, 135–141.

Hoogeboom, H. and Rozenberg, G. 1986. Infinitary languages: Basic theory and applications to concurrent systems. In Current Trends in Concurrency, J. de Bakker, W.-P. de Roever, and G. Rozenberg, Eds. Lecture Notes in Computer Science, vol. 224. Springer Verlag, 266–342.

Kelley, J. L. 1955. General Topology. Van Nostrand, Princeton.

Manna, Z. and Pnueli, A. 1992. The Temporal Logic of Reactive and Concurrent Systems—Specification, first ed. Springer Verlag, New York.

Nitsche, U. 1994. Propositional linear temporal logic and language homomorphisms. In Proceedings of the 3rd International Symposium on Logical Foundations of Computer Science (LFCS’94), A. Nerode and Y. V. Matiyasevich, Eds. Lecture Notes in Computer Science, vol. 813. Springer Verlag, Saint Petersburg, Russia, 265–277.

Nitsche, U. 1998a. Application of formal verification and behaviour abstraction to the service interaction problem in intelligent networks. Journal of Systems and Software 40, 3 (March), 227–248.

Nitsche, U. 1998b. Verification of Co-Operating Systems and Behaviour Abstraction. GMD Research Series, vol. 7. GMD, Sankt Augustin, Germany. Publication of PhD thesis. ISBN: 3-88457-331-4.

Nitsche, U. and Ochsenschläger, P. 1996. Approximately satisfied properties of systems and simple language homomorphisms. Information Processing Letters 60, 201–206.

Nitsche, U. and Wolper, P. 1997. Relative liveness and behavior abstraction (extended abstract). In Proceedings of the 16th ACM Symposium on Principles of Distributed Computing (PODC’97). Santa Barbara, CA, 45–52.

Ochsenschläger, P. 1992. Verifikation kooperierender Systeme mittels schlichter Homomorphismen. Arbeitspapiere der GMD 688, Gesellschaft für Mathematik und Datenverarbeitung (GMD), Darmstadt. Oktober.

Ochsenschläger, P. 1994. Verification of cooperating systems by simple homomorphisms using the product net machine. In Workshop: Algorithmen und Werkzeuge für Petrinetze, J. Desel, A. Oberweis, and W. Reisig, Eds. Humboldt Universität Berlin, 48–53.

Ochsenschläger, P. 1995. Compositional verification of cooperating systems using simple homomorphisms. In Workshop: Algorithmen und Werkzeuge für Petrinetze, J. Desel, H. Fleischhacker, A. Oberweis, and M. Sonnenschein, Eds. Universität Oldenburg, 8–13.

ACM Transactions in Computational Logic, Vol. TBD, No. TBD, TBD TBD.
Pnueli, A. 1977. The temporal logic of programs. In Proceedings of the 18th Annual IEEE Symposium on Foundations of Computer Science. 46–57.
Thomas, W. 1990. Automata on infinite objects. See van Leeuwen [1990], 133–191.
Ultes-Nitsche, U. and St James, S. 2000. Weakly continuation-closed abstractions can be defined on trace reductions. In Proceedings of the International Workshop on Verification and Computational Logic (VCL2000), M. Leuschel, A. Podelski, C. Ramakrishnan, and U. Ultes-Nitsche, Eds. University of Southampton, 11 pages.
van Leeuwen, J., Ed. 1990. Formal Models and Semantics. Handbook of Theoretical Computer Science, vol. B. Elsevier.
Vardi, M. Y. and Wolper, P. 1994. Reasoning about infinite computations. Information and Computation 115, 1 (November), 1–37.
Wolper, P. and Godefroid, P. 1993. Partial-order methods for temporal verification. In CONCUR’93, E. Best, Ed. Lecture Notes in Computer Science, vol. 715. Springer Verlag, 233–246.