TILTING MODULES AND DOMINANT DIMENSION WITH RESPECT TO INJECTIVE MODULES

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Abstract. In this paper, we study a relationship between tilting modules with finite projective dimension and dominant dimension with respect to injective modules as a generalisation of results of Crawley-Boevey–Sauter, Nguyen–Reiten–Todorov–Zhu and Pressland–Sauter. Moreover, we give characterisations of n-almost Auslander–Gorenstein algebras and n-almost Auslander algebras by the existence of tilting modules. As an application, we describe a sufficient condition of 1-almost Auslander algebras to be strongly quasi-hereditary by comparing such tilting modules and characteristic tilting modules.

1. Introduction

Tilting theory gives a universal method to construct derived equivalences and is considered as one of the effective tools in the study of many areas of mathematics (e.g., the representation theories of finite dimensional algebras, finite groups and algebraic groups, algebraic geometry, and algebraic topology). In this theory, the notion of tilting modules plays a crucial role. More precisely, the endomorphism algebras of tilting modules are derived equivalent to the original algebra. Hence it is important to give a construction of tilting modules for a given algebra.

In this paper, we study a relationship between tilting modules with finite projective dimension and dominant dimension with respect to injective modules, which is one of generalisations of results in Crawley-Boevey–Sauter [CBS], Nguyen–Reiten–Todorov–Zhu [NRTZ] and Pressland–Sauter [PrSa]. Crawley-Boevey–Sauter gave a new characterisation of artin algebras with global dimension at most two to be Auslander algebras by the existence of certain tilting modules. As a refinement of their result, Nguyen–Reiten–Todorov–Zhu showed the following theorem.

Theorem 1.1 ([CBS, Lemma 1.1] and [NRTZ, Theorem 3.3.4]). Let \( A \) be an artin algebra and \( I \) a maximal projective-injective direct summand of \( A \). Let \( C := \text{Fac}_1(I) \cap \text{Sub}^1(I) \). Then \( \text{domdim} A \geq 2 \) if and only if there exists a unique basic tilting module such that its projective dimension is exactly one and it is contained in \( C \).

Moreover, Pressland–Sauter studied the existence of tilting modules with finite projective dimension generated and cogenerated by projective-injective modules over an artin algebra with positive dominant dimension. As an application, they characterised higher minimal Auslander–Gorenstein algebras in terms of such tilting modules.

Our aim of this paper is to give a generalisation of their results from the aspect of dominant dimension. Namely, let \( A \) be an artin algebra and \( I \) a direct sum of all pairwise non-isomorphic injective modules with projective dimension at most one. Then we write...
\[ I \text{-domdim} A \geq n + 1 \] if \( A \) has a minimal injective coresolution \( 0 \to A \to I^0 \to I^1 \to \cdots \to I^n \to \cdots \) with \( I^0, I^1, \ldots, I^n \in \text{add} I \). The following theorem is one of our main results.

**Theorem 1.2** (Theorem 3.1). Let \( A \) be an artin algebra and \( I \) an injective module with projective dimension at most one. For an integer \( 1 \leq i \leq n + 1 \), let \( C_i := \text{Fac}_i(I) \cap \text{Sub}^{n+1-i}(I) \). Then \( I \text{-domdim} A \geq n + 1 \) if and only if there exists a unique basic tilting module such that its projective dimension is exactly \( d \) and it is contained in \( C_d \) for some integer \( 1 \leq d \leq n + 1 \).

In this paper, we introduce the weaker notions of \( n \)-minimal Auslander–Gorenstein algebras and \( n \)-Auslander algebras. Namely, we call an algebra \( A \) an \( n \)-almost (minimal) Auslander–Gorenstein algebra (respectively, \( n \)-almost Auslander algebra) if it satisfies

\[ \text{id} A \leq n + 1 \leq I \text{-domdim} A \] (respectively, \( \text{gldim} A \leq n + 1 \leq I \text{-domdim} A \)).

Note that if \( \text{add} I = \text{proj} A \cap \text{inj} A \), then they coincide with an \( n \)-minimal Auslander–Gorenstein algebra and an \( n \)-Auslander algebra respectively. As an application of Theorem 1.2, we give characterisations of \( n \)-almost Auslander–Gorenstein algebras and \( n \)-almost Auslander algebras.

**Theorem 1.3** (Theorem 3.14). Let \( A \) be an artin algebra which is not a 1-Iwanaga–Gorenstein algebra. Then the following statements are equivalent.

1. \( A \) is an \( n \)-almost Auslander–Gorenstein algebra.
2. There exists a unique basic tilting module such that
   a. its projective dimension is exactly \( d \),
   b. it is contained in \( C_d \), and
   c. it is cotilting with injective dimension exactly \( n + 1 - d \)
   for some integer \( 1 \leq d \leq n + 1 \).

   If in addition we assume \( \text{gldim} A < \infty \), then the following statement is also equivalent.

3. \( A \) is an \( n \)-almost Auslander algebra.

Moreover, we study a relationship between 1-almost Auslander algebras and strongly quasi-hereditary algebras which are a special class of quasi-hereditary algebras. Quasi-hereditary algebras arose from the representation theory of complex Lie algebras and algebraic groups. One of the important properties of quasi-hereditary algebras is the existence of tilting modules, called characteristic tilting modules, by Ringel [R1]. Recall that strongly quasi-hereditary algebras are defined as quasi-hereditary algebras whose standard modules have projective dimension at most one and costandard modules have injective dimension at most one. It is known that if an artin algebra is strongly quasi-hereditary, then its global dimension is at most two [R2, Proposition A.2]. However, the converse does not hold in general. By focusing on connection between the tilting modules in Theorem 1.3 and characteristic tilting modules, we give a sufficient condition of 1-almost Auslander algebras to be strongly quasi-hereditary algebras.

**Theorem 1.4** (Theorem 5.3 and Corollary 5.7). Let \( A \) be a 1-almost Auslander algebra. Let \( T^1 \) be the tilting module with projective dimension exactly one in Theorem 1.3 and \( T \) a characteristic tilting module of \( A \). If \( T \) coincides with \( T^1 \), then \( A \) is a strongly quasi-hereditary algebra. Moreover, if \( A \) is an Auslander algebra, then the converse also holds.

**Notation.** Throughout this paper, \( A \) is a non-semisimple artin algebra and \( D \) is its Matlis dual. We denote by \( \text{gldim} A \) the global dimension of \( A \) and \( \text{domdim} A \) the dominant dimension of \( A \). We write \( \text{mod} A \) for the category of finitely generated right \( A \)-modules and
proj } A \) (respectively, \( \text{inj} A \)) for the full subcategory of \( \text{mod} A \) consisting of finitely generated projective (respectively, injective) \( A \)-modules. For \( M \in \text{mod} A \), we denote by \( \text{add} M \) the full subcategory of \( \text{mod} A \) whose objects are direct summands of finite direct sums of \( M \). We denote by \( \text{pd} M \) (respectively, \( \text{id} M \)) the projective (respectively, injective) dimension of \( M \).

2. Preliminaries

In this section, we recall the notions of dominant dimension with respect to injective modules and tilting modules with finite projective dimension.

2.1. Dominant dimension with respect to injective modules. In this subsection, we recall the definition of dominant dimension with respect to injective modules (see \([11]\) and \([12]\) for details). Throughout this paper, the following notation is convenient.

**Definition 2.1.** Fix an integer \( n \geq 0 \). Let \( A \) be an artin algebra and \( Q \) an \( A \)-module.

1. We define \( \text{Sub}^{n+1}(Q) \) to be the full subcategory of \( \text{mod} A \) whose object \( X \) has an exact sequence

\[
0 \to X \xrightarrow{f^0} Q^0 \xrightarrow{f^1} Q^1 \to \cdots \xrightarrow{f^n} Q^n
\]

such that \( Q^i \in \text{add} Q \) for each \( 0 \leq i \leq n \). Set \( X^0 := X \) and \( X^i := \text{Cok} f^{i-1} \) for each \( 1 \leq i \leq n+1 \). Let \( \text{Sub}^0(Q) := \text{mod} A \). Moreover, we write \( Q \)-codim \( X \leq n \) if \( f^n \) is an epimorphism.

2. We define \( \text{Fac}^{n+1}_n(Q) \) to be the full subcategory of \( \text{mod} A \) whose object \( Y \) has an exact sequence

\[
Q_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} Q_1 \xrightarrow{f_0} Q_0 \xrightarrow{f_0} Y \to 0
\]

such that \( Q_i \in \text{add} Q \) for each \( 0 \leq i \leq n \). Set \( Y_0 := Y \) and \( Y_i := \text{Ker} f_{i-1} \) for each \( 1 \leq i \leq n+1 \). Let \( \text{Fac}^0(Q) := \text{mod} A \). Moreover, we write \( Q \)-dim \( X \leq n \) if \( f_n \) is a monomorphism.

We collect some properties on \( \text{Sub}^{n+1}(Q) \). Dually, we have similar results for \( \text{Fac}^{n+1}_n(Q) \).

**Proposition 2.2.** Let \( A \) be an artin algebra and \( Q \) an \( A \)-module. Then we have the following statements.

1. If \( m, n \) are integers with \( m \geq n \), then \( \text{Sub}^m(Q) \subseteq \text{Sub}^n(Q) \).

2. Fix an integer \( n \geq 0 \). Assume that \( X \in \text{Sub}^{n+1}(Q) \) and \( \text{Ext}^i_A(X, Q) = 0 \) for all \( i \geq 1 \). Then \( \text{Ext}^j_A(X^{n+1}, Q) = 0 \) for each \( 1 \leq j \leq n+1 \) if and only if the inclusion \( \nu^i : X^i \to Q^i \) is a left \( \text{add} Q \)-approximation for each \( 0 \leq i \leq n \). In this case, the induced complex \( \text{Hom}_A(Q^m, Q) \to \cdots \to \text{Hom}_A(X, Q) \to 0 \) is exact.

To prove Proposition 2.2(2), we need the following lemma, which is frequently used in this paper.

**Lemma 2.3** (see \([11]\) Lemma 1.1). Let \( Q \) be an \( A \)-module. For an exact sequence

\[
0 \to X^0 \xrightarrow{f^0} Y^0 \xrightarrow{f^1} Y^1 \to \cdots \xrightarrow{f^{n-1}} Y^{n-1} \to X^n \to 0
\]

and \( X^i := \text{Cok} f^{i-1} \) for each \( 1 \leq i \leq n \), the following statements hold.
Let \( X \) of Definition 2.6.

\[ I \text{ isomorphic indecomposable direct summands of } M \]

If \( \text{Ext}^i_{A}(Q, X^i) \neq 0 \) for all \( k \geq 1 \), then we have

\[ \text{Ext}^i_{A}(Q, X^d) \cong \text{Ext}^{i+1}_{A}(Q, X^{d-1}) \cong \cdots \cong \text{Ext}^{j+d}_{A}(Q, X^0) \]

for all \( j \geq 1 \). In particular, if one of \( \text{pd} Q \leq d \) and \( \text{id} X^0 \leq d \) is satisfied, then \( \text{Ext}^i_{A}(Q, X^d) = 0 \).

(2) If \( \text{Ext}^i_{A}(Y^i, Q) = 0 \) for all \( k \geq 1 \), then we have

\[ \text{Ext}^i_{A}(X^0, Q) \cong \text{Ext}^{i+1}_{A}(X^1, Q) \cong \cdots \cong \text{Ext}^{j+d}_{A}(X^d, Q) \]

for all \( j \geq 1 \). In particular, if one of \( \text{id} Q \leq d \) and \( \text{pd} X^d \leq d \) is satisfied, then \( \text{Ext}^i_{A}(X^0, Q) = 0 \).

**Proof of Proposition 2.2.** (1) This is clear.

(2) Note that \( \ell' \) is a left \( \text{add} Q \)-approximation of \( X^i \) if and only if \( \text{Ext}^i_{A}(X^{i+1}, Q) = 0 \).

By Lemma 2.3(2), we obtain the following isomorphisms

\[ \text{Ext}^i_{A}(X^{i+1}, Q) \cong \text{Ext}^{i+1}_{A}(X^{i+2}, Q) \cong \cdots \cong \text{Ext}^{n+1-i}_{A}(X^{n+1}, Q). \]

Hence the assertion follows. \( \square \)

Now, we introduce the following central notion of this paper.

**Definition 2.4.** Let \( I \) be an injective \( A \)-module and \( X \) an \( A \)-module. Then we write \( I \cdot \text{domdim}(X) \geq n+1 \) if \( X \in \text{Sub}^{n+1}(I) \). In this case, we say that the dominant dimension of \( X \) with respect to \( I \) is at least \( n+1 \).

If \( \text{add} I = \text{proj} A \cdot \text{inj} A \), then we have \( I \cdot \text{domdim} X = \text{domdim} X \) for each \( X \in \text{mod} A \). Let \( I \) be an injective \( A \)-module with projective dimension at most \( l-1 \). Then \( I \cdot \text{domdim} A \geq n \) is called that \( A \) satisfies the \((l, n)\)-condition in \([P1]\) and \([P2]\).

**Remark 2.5.** The notion of dominant dimension with respect to an injective module \( I \) is not always left–right symmetry. Namely, there exists an example of an artin algebra \( A \) which satisfies \( I \cdot \text{domdim} A \geq n+1 \) but \( A^{\text{op}} \) not (see, for example, \([P1]\ \text{Remark 2.1.1(2)})

2.2. Tilting theory. In this subsection, we recall the definition and basic properties of tilting modules.

**Definition 2.6.** Fix an integer \( d \geq 0 \) and let \( T, C \) be \( A \)-modules.

(1) We call \( T \) a pretilting module (respectively, tilting module) if it satisfies (T1) and (T2) (respectively, (T1), (T2), and (T3)):

(T1) \( \text{pd} T < \infty \);

(T2) \( \text{Ext}^i_{A}(T, T) = 0 \) holds for all \( i \geq 1 \);

(T3) \( T \cdot \text{codim} A < \infty \).

Moreover, a tilting module \( T \) is called a \( d \)-tilting module if \( \text{pd} T = d \).

(2) We call \( C \) a precotilting module (respectively, cotilting module) if it satisfies (C1) and (C2) (respectively, (C1), (C2), and (C3)):

(C1) \( \text{id} C < \infty \);

(C2) \( \text{Ext}^i_{A}(C, C) = 0 \) holds for all \( i \geq 1 \);

(C3) \( C \cdot \text{dim} D A < \infty \).

Moreover, a cotilting module \( C \) is called a \( d \)-cotilting module if \( \text{id} C = d \).

We collect well-known results for tilting modules. We denote by \( \text{tilt} A \) the set of isomorphism classes of basic tilting \( A \)-modules. For \( M, M' \in \text{mod} A \), we write \( M \succeq M' \) if \( \text{Ext}^i_{A}(M, M') = 0 \) for all \( i \geq 1 \). We denote by \(|M|\) the number of its pairwise non-isomorphic indecomposable direct summands of \( M \).
Proposition 2.7. The following statements hold.

1. \([\text{RS HU1}]\) \(\geq\) gives a partial order on \(\text{tilt} A\). Moreover, if \(T \unlhd T'\) in \(\text{tilt} A\), then \(\text{pd} T \leq \text{pd} T'\) holds.

2. \((\text{Mi Theorem 1.19})\) If \(T\) is a tilting \(A\)-module, then we have \(|T| = |A|\).

3. \((\text{Mi Theorem 1.4} \text{ and } \text{H Lemma III.2.2})\) Let \(T\) be an \(A\)-module with \(\text{pd} T < \infty\) and \(\text{Ext}^i_A(T, T) = 0\) for all \(i \geq 1\). Assume \(T\)-codim \(A < \infty\). Then we have \(T\)-codim \(A = \text{pd} T\).

Next we recall the notion of (left) mutations of tilting modules with finite projective dimension (see \(\text{RS HU2 CHU}\) for details). Let \(T = X \oplus U\) be an \(A\)-module and \(X\) a non-zero \(A\)-module. Take a minimal left \(\text{add} U\)-approximation \(f : X \to U\) of \(X\). We call \(\mu_X (T) := \text{Cok} f \oplus U\) a mutation of \(T\) with respect to \(X\).

Proposition 2.8 (\(\text{CHU}\)). Let \(T = X \oplus U\) be an \(A\)-module with \(X \in \text{Sub}^1(U)\) and \(T' := \mu_X (T)\). Then the following statements hold.

1. If \(T\) is a tilting \(A\)-module, then so is \(T'\). Moreover, we have \(T \succ T'\).

2. Assume that \(T\) is a \(d\)-tilting \(A\)-module which has an exact sequence

\[
0 \to A \xrightarrow{f^d} T^0 \xrightarrow{f^1} T^1 \to \cdots \to T^{d-2} \xrightarrow{f^{d-1}} T^{d-1} \xrightarrow{f^d} T^d \to 0
\]

(2.8.1)

such that \(T^i \in \text{add} T\) for each \(0 \leq i \leq d\) and \(f^j\) is left minimal for each \(0 \leq j \leq d - 1\). If \(\text{add} X \cap \text{add} T^d \neq \{0\}\), then \(T'\) is a \((d+1)\)-tilting \(A\)-module.

For the convenience of readers, we give a proof of Proposition 2.8. We need the following lemma.

Lemma 2.9 (see \(\text{ASS Proposition A.4.7}\)). Let \(0 \to X \to Y \to Z \to 0\) be an exact sequence in \(\text{mod} A\). Then we have \(\text{pd} Z \leq \max\{\text{pd} X + 1, \text{pd} Y\}\) and the equality holds if \(\text{pd} X \neq \text{pd} Y\).

Proof of Proposition 2.8. (1) We check only the condition (T3). Let \(i\) be the maximum integer with respect to \(\text{add} X \cap \text{add} T^i \neq \{0\}\). Then we have \(T^{i+1} \in \text{add} U\) and decompose \(T^i\) as \(T^i = X' \oplus U^i\), where \(X' \in \text{add} X\) and \(U^i \in \text{add} U\). Take a minimal left \(\text{add} U\)-approximation \(\varphi' : X' \to U^i\). Then \(\varphi'\) is a monomorphism by \(X' \in \text{Sub}^1(U)\). Since \(\varphi'\) is a left \(\text{add} U\)-approximation and \(T^{i+1} \in \text{add} U\), there exist \(\alpha : U' \to T^{i+1}\) and \(\beta : Y' \to T^{i+2}\) such that the following diagram is commutative

\[
\begin{array}{c}
0 \rightarrow X' \xrightarrow{\varphi'} U' \xrightarrow{\psi'} Y' \xrightarrow{\psi'} 0 \\
T^{i-1} \xrightarrow{f^i} X' \oplus U^i \xrightarrow{f^{i+1}} T^{i+1} \xrightarrow{f^{i+2}} T^{i+2} \xrightarrow{f^{i+3}} T^{i+3} \\
\end{array}
\]

where \(f^i = \begin{bmatrix} f_{X'X}^i \\ 0 \end{bmatrix}\) and \(f^{i+1} = \begin{bmatrix} f_{X'X}^{i+1} \\ f_{U^iU}^{i+1} \end{bmatrix}\). Then we have an exact sequence

\[
T^{i-2} \xrightarrow{[f_{X'X}^{i-1}]} X' \oplus T^{i-1} \rightarrow U' \oplus X' \oplus U^i \xrightarrow{[\varphi' \alpha f_{X'X}^{i+1} f_{U^iU}^{i+1}]} Y' \oplus T^{i+1} \xrightarrow{[\beta f^{i+2}]} T^{i+2}.
\]

Thus we have the following exact sequence:

\[
\begin{array}{c}
\cdots \rightarrow T^{i-2} \xrightarrow{f^{i-1}} T^{i-1} \rightarrow U' \oplus U^i \xrightarrow{[\varphi' \alpha f_{X'X}^{i+1}]} Y' \oplus T^{i+1} \xrightarrow{[\beta f^{i+2}]} T^{i+2} \rightarrow \cdots
\end{array}
\]
Repeating this process, we have the desired exact sequence, and hence $T'$-codim $A < \infty$.

(2) It is enough to show $\text{pd} T' = d + 1$. By $X \in \text{Sub}^1(U)$, there exists an exact sequence $0 \to X \to \overline{U} \to Y \to 0$ with $\overline{U} \in \text{add} U$. By Lemma 2.9 we have $\text{pd} Y \leq \max\{\text{pd} X + 1, \text{pd} \overline{U}\} \leq d + 1$. On the other hand, by (1) and Proposition 2.7(1), we have $d = \text{pd} T \leq \text{pd} T'$. Hence we obtain that $\text{pd} T' \in \{d, d + 1\}$. By our assumption, we can decompose $T^d$ as $T^d = X' \oplus U^d$, where $X' \in \text{add} X$ and $U^d \in \text{add} U$. Applying the same argument in (1), we have an exact sequence

$$0 \to A \to T'^0 \to \cdots \to T'^{d-1} \mathop{\longrightarrow}^{\varphi^d} U' \oplus U^d \mathop{\longrightarrow}^{[\psi' \ 0]} Y' \to 0,$$

where $T'^j \in \text{add} T'$ for each $0 \leq j \leq d - 1$. Since $\psi'$ is a non-split epimorphism and $Y' \neq 0$, we have $\text{pd} T' = T'$-codim $A = d + 1$ by Proposition 2.7(3).

$$\square$$

3. Main results

In this section, we study a relationship between tilting modules with finite projective dimension and dominant dimension with respect to injective modules. Namely, the following theorem is a main result of this paper.

**Theorem 3.1.** Fix an integer $n \geq 0$. Let $A$ be an artin algebra and $I$ an injective $A$-module with $\text{pd} I \leq 1$. Then the following statements are equivalent.

1. $I$-domdim $A \geq n + 1$.
2. There exists $0 \leq m \leq n$ such that, for each $1 \leq d \leq m + 1$, $\text{Fac}_d(I) \cap \text{Sub}^{n+1-d}(I)$ admits a $d$-tilting $A$-module.
3. There exists an integer $1 \leq d \leq n + 1$ such that $\text{Fac}_d(I) \cap \text{Sub}^{n+1-d}(I)$ admits a $d$-tilting $A$-module.

Clearly (2)$\Rightarrow$(3) holds. In Subsections 3.1 and 3.2 we prove Theorem 3.1(1)$\Rightarrow$(2) and (3)$\Rightarrow$(1) respectively. In Subsection 3.3, as an application, we give characterisations of almost Auslander–Gorenstein algebras and almost Auslander algebras by using tilting modules in Theorem 3.1.

3.1. The proof of Theorem 3.1(1)$\Rightarrow$(2). We describe the following proposition which plays crucial role in the proof.

**Proposition 3.2.** Fix an integer $m \geq 0$. Let $Q$ be an $A$-module with $\text{Ext}_A^1(Q, Q) = 0$ and $\text{pd} Q \leq 1$. Assume that $A$ has an exact sequence

$$0 \to A \mathop{\longrightarrow}^{f_0} Q^0 \mathop{\longrightarrow}^{f_1} Q^1 \to \cdots \mathop{\longrightarrow}^{f_m} Q^m$$

such that non-zero $Q^i \in \text{add} Q$ and the inclusion $i^i : \text{im} f^i \to Q^i$ is a minimal left $\text{add} Q$-approximation for each $0 \leq i \leq m$. Let $A^d := \text{im} f^d$ and $T^d := T_Q^d := A^d \oplus Q$ for each $0 \leq d \leq m + 1$. Then the following statements hold.

1. $T^1$ is a $1$-tilting $A$-module.
2. If $m \geq 1$ holds, then $T^d$ is a mutation of $T^{d-1}$ with respect to $A^d$ for each $2 \leq d \leq m + 1$. In particular, $T^d$ is a $d$-tilting $A$-module.

We denote by $T^d$ the basic module of $T^d$ for each $0 \leq d \leq m + 1$. Note that $T^0$ is tilting if and only if $Q \in \text{proj} A$. 
Proof. For each $1 \leq d \leq m + 1$, we have an exact sequence

$$0 \to A^{d-1} \xrightarrow{\iota^{d-1}} Q^{d-1} \to A^d \to 0 \quad (3.2.2)$$

such that $\iota^{d-1}$ is a minimal left $\add Q$-approximation.

(1) We check that $T := T^1$ satisfies the conditions (T1), (T2), and (T3) in Definition 2.6.

(T1) Applying Lemma 2.3 to (3.2.2), we have $\pd A^1 \leq \max\{\pd A^0 + 1, \pd Q^0\} \leq 1$, and hence $\pd T \leq 1$.

(T2) We prove $\Ext^1_A (T, T) = 0$. Clearly we obtain

$$\Ext^1_A (T, T) \cong \Ext^1_A (Q, Q) \oplus \Ext^1_A (Q, A^1) \oplus \Ext^1_A (A^1, Q) \oplus \Ext^1_A (A^1, A^1) \cong \Ext^1_A (Q, Q) \oplus \Ext^1_A (A^1, Q) \oplus \Ext^1_A (A^1, A^1).$$

First we show $\Ext^1_A (Q, A^1) = 0$. This follows from Lemma 2.3 (1). Secondly, we claim $\Ext^1_A (A^1, Q) = 0$. Applying $\Hom_A (\cdot, Q)$ to (3.2.2), we obtain an exact sequence

$$\Hom_A (Q^0, Q) \xrightarrow{\Hom(\iota^0, Q)} \Hom_A (A, Q) \to \Ext^1_A (A^1, Q) \to \Ext^1_A (Q^0, Q) = 0.$$

Hence the claim follows from that $\Hom(\iota^0, Q)$ is an epimorphism. Finally, we prove $\Ext^1_A (A^1, A^1) = 0$. Applying $\Hom_A (A^1, \cdot)$ to (3.2.2) gives an exact sequence

$$\Ext^1_A (A^1, Q^0) \to \Ext^1_A (A^1, A^1) \to \Ext^1_A (A^1, A),$$

where the left-side hand vanishes by the second claim and the right-hand side vanishes by $\pd A^1 \leq 1$. Hence the assertion follows.

(T3) This is clear by (3.2.2).

(2) This follows from Proposition 2.8. □

In Proposition 3.2, let $Q$ be a maximal projective-injective direct summand of $A$. Then $T^d = A^d \oplus Q$ coincides with the tilting module which is shown in [CBS, NRTZ, PrSa]. Thus Proposition 3.2 can be regarded as one of generalisations of their results.

Now we are ready to prove Theorem 3.1(1)⇒(2).

Proof of Theorem 3.1. (1)⇒(2): Let $I$ be an injective module with $\pd I \leq 1$. If $A$ is self-injective, then there is nothing to prove. We assume that $A$ is not self-injective.

By (d) $A \geq 1$ and $I$-domdim $A \geq n + 1$, there exists a minimal injective coresolution of $A$

$$0 \to A \xrightarrow{I^0} I^0 \xrightarrow{I^1} I^1 \to \cdots \xrightarrow{I^m} I^m \to \cdots$$

such that non-zero $I^0, I^1, \cdots, I^m \in \add I$ for some integer $m \leq n$. By Proposition 3.2, $T^d$ is $d$-tilting for each $1 \leq d \leq m + 1$ because the inclusion $\iota : \Im f \to I^i$ is a minimal left $\add I$-approximation. Moreover, $T^d$ is clearly contained in $\Fac_d (I) \cap \Sub (n+1-d) (I)$. This finishes the proof. □

In the following, we give an example of Proposition 3.2.

Example 3.3. Let $A$ be the algebra defined by the quiver

$$\begin{array}{ccc}
1 & \xrightarrow{a} & 2 \\
\beta & \downarrow & 3 \\
3 & \xleftarrow{c} & 4 \\
\gamma & \downarrow & \phi \\
5 & \xleftarrow{d} & \end{array}$$
with relations $\alpha \gamma - \beta \delta$, $\epsilon \varphi$ and $\varphi \gamma$. Let $Q := P(1) \oplus X \oplus P(1)/P(3) \oplus P(5)$, where $X := \text{Cok}(P(2) \to P(1) \oplus P(5))$. Then $Q$ is not injective with $\text{pd} Q \leq 1$ and $\text{Ext}^1_A(Q,Q) = 0$. We can check that $A$ has an exact sequence

$$0 \to A \to P(1)^{\oplus 4} \oplus P(5)^{\oplus 2} \to X^{\oplus 2} \oplus (P(1)/P(3))^{\oplus 2}.$$ 

Then we obtain that $T^1 = P(1)/P(4) \oplus Q$ is a 1-tilting $A$-module and $T^2 = I(2) \oplus Q$ is a 2-tilting $A$-module.

If we do not assume $\text{pd} Q \leq 1$, then Proposition 3.2 does not necessarily hold as the following example shows.

**Example 3.4.** Let $A$ be the algebra defined by the quiver

$$
1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4
$$

with relations $\beta \alpha$, $\gamma \delta$ and $\delta \alpha$. Then $A$ has a minimal injective coresolution

$$0 \to A \to I(2)^{\oplus 2} \oplus I(4)^{\oplus 2} \to I(2) \oplus I(3) \oplus I(4) \to I(1) \oplus I(3) \oplus I(4) \to I(1) \to 0. $$

Setting $Q = I(2) \oplus I(3) \oplus I(4)$, we have $\text{pd} Q = 2$ and $A \in \text{Sub}^2(Q)$. Then we have $T^1 = S(2) \oplus S(4) \oplus Q$. However, we obtain that $\text{Ext}^1_A(T^1, T^1) \neq 0$ since $\text{Ext}^1_A(I(2), S(2)) \cong \text{Hom}_A(P(2), S(2)) \neq 0$. Therefore, in this case, we can not obtain tilting modules by the similar construction of Proposition 3.2.

As an application of Proposition 3.2, we give the minimum element in

$$\text{tilt}_d^Q A := \{ T \in \text{tilt} A \mid \text{pd} T \leq d \text{ and } T \succeq Q \},$$

which is an analog of [IZ, Theorem 3.4(2)].

**Corollary 3.5.** Keep the notation in Proposition 3.2. For each $1 \leq d \leq m + 1$, the $d$-tilting $A$-module $T^d$ is the minimum element in $\text{tilt}_d^Q A$.

**Proof.** Let $T^d$ be the basic module of the $d$-tilting module $A^d \oplus Q$. We claim that $\text{Ext}^1_A(T, T^d) = 0$ for each tilting $A$-module $T \in \text{tilt}_d^Q A$ and all integers $i > 0$. Namely, it is enough to show that $\text{Ext}^1_A(T, A^d) = 0$. This follows from Lemma 2.3(1). Hence we have the assertion. \hfill \Box

We give some remark on Bongartz completion.

**Remark 3.6.** Let $Q$ be a basic $A$-module with $\text{Ext}^1_A(Q, Q) = 0$ and $\text{pd} Q \leq 1$. Namely, it is partial tilting. Thus there exists an $A$-module $X$ such that $T := X \oplus Q$ is a basic tilting module with projective dimension at most one by [B]. Note that $T^1$ is not always isomorphic to $T$. For example, assume $A$ is a path algebra of Dynkin type of $A$ with linear orientation and take $Q$ a maximal projective-injective direct summand of $A$. Then we have $T = A$ and $T^1 = DA$.

## 3.2. The proof of Theorem 3.1(3)⇒(1). We need the following lemma.

**Lemma 3.7.** Let $I$ be an injective $A$-module with $\text{pd} I \leq 1$. Fix an integer $d \geq 1$ and let $\{ X_j \}_{j \in J}$ be the set of all pairwise non-isomorphic indecomposable $A$-modules in $\text{Fac}_d(I)$ such that $\text{pd} X_j \leq d$. Assume that $\text{Fac}_d(I)$ admits a basic tilting $A$-module $T$ with $\text{pd} T \leq d$. Then the following statements hold.

...
(1) If \( J' \) is a finite subset of \( J \), then \( X_{J'} \oplus T \) is tilting, where \( X_{J'} := \oplus_{j \in J'} X_j \).

(2) \( J' \) is a finite set.

(3) \( T \) is isomorphic to \( X_J \). In particular, \( \text{Fac}_d(I) \) has a unique basic tilting module with projective dimension at most \( d \) if it exists.

**Proof.** (1) The conditions (T1) and (T3) clearly hold. The condition (T2) follows from Lemma 2.3(1).

(2) Suppose that \( J' \) is any finite subset of \( J \) with \( |A| < |X_{J'}| \). Since \( T \) and \( X_{J'} \oplus T \) are tilting \( A \)-modules, we have \( X_{J'} \in \text{add} T \), and hence \( |X_{J'}| < |T| \), a contradiction.

(3) By definition, we have \( T \in \text{add} X_J \). Since \( X_J \oplus T \) is also tilting by (1), we have the assertion. \( \square \)

We give a remark on uniqueness of \( d \)-tilting modules.

**Remark 3.8.** By Lemma 3.7 the basic \( d \)-tilting module constructed in Theorem 3.1(2) and (3) uniquely exists in \( \text{Fac}_d(I) \cap \text{Sub}^{n+1-d}(I) \). Moreover, for each \( 1 \leq d \leq n \), since \( \mathbb{T}^d \) is contained in \( \text{Sub}^1(I) \), all indecomposable injective \( A \)-modules in \( \text{Fac}_d(I) \) have projective dimension at most one. Hence we can decompose \( \mathbb{T}^d \) as \( \mathbb{T}^d = X^d \oplus I \), where \( X^d \) is a maximal direct summand of \( \mathbb{T}^d \) which contains no non-zero injective modules as a direct summand.

Now we complete the proof of Theorem 3.1.

**Proof of Theorem 3.1 (3)$$\Rightarrow$$ (1):** By our assumption, there exists a basic \( d \)-tilting module \( T \in \text{Fac}_d(I) \cap \text{Sub}^{n+1-d}(I) \) for some integer \( 1 \leq d \leq n+1 \). Since \( T \)-codim \( A = d \) holds, we have an exact sequence

\[
0 \to A \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1 \to \cdots \xrightarrow{f^{d-1}} T^{d-1} \xrightarrow{f^d} T^d \to 0
\]

with \( T^i \in \text{add} T \). We set \( A^0 := A \) and \( A^i := \text{Cok} f^{i-1} \) for \( 1 \leq i \leq d \). Without loss of generality, we can assume that the inclusion \( \iota^i : A^i \to T^i \) is a minimal left \( \text{add} T \)-approximation by Proposition 2.2(2).

In the following, we claim \( T^i \in \text{add} I \) for all \( 0 \leq i \leq d-1 \) by induction on \( i \). By Lemma 3.7(3), we obtain \( T^i \cong X_J \). Then we can decompose \( T^i \) as \( T^i = X^i \oplus I^i \) and \( \iota^i = (\iota^i_X, \iota^i_I) : A^i \to X^i \oplus I^i \), where \( X^i \) is a maximal direct summand of \( T^i \) which contains no non-zero injective modules as a direct summand. Namely, we prove \( X^i = 0 \) for each \( 0 \leq i \leq d-1 \). Suppose to the contrary that \( X^i \neq 0 \). Note that by definition \( X^i \notin \text{add} I \). Then \( X^i \in \text{Fac}_d(I) \) gives an exact sequence

\[
I^i_{d-1} \xrightarrow{g^i_{d-1}} \cdots \xrightarrow{g^i_1} I^i_0 \xrightarrow{g^i_0} X^i \to 0
\]

with non-zero \( I^i_j \in \text{add} I \). Let \( X^i_j := \text{Ker} g^i_j \). Applying \( \text{Hom}_A(A^i, -) \) to an exact sequence

\[
0 \to X^i_1 \to I^i_0 \to X^i \to 0,
\]

we obtain an exact sequence

\[
\text{Hom}_A(A^i, I^i_0) \to \text{Hom}_A(A^i, X^i) \to \text{Ext}_A^1(A^i, X^i).
\]

Then we claim \( \text{Ext}_A^1(A^i, X^i_1) = 0 \). Indeed, if it is true, then there exists \( h^i : A^i \to I^i_0 \) such that \( \iota^i_X = g^i_0 h^i \). Hence \( (h^i, \iota^i_I) : A^i \to I^i_0 \oplus I^i \) is a left \( \text{add} I \)-approximation of \( A^i \). By the minimality of \( \iota^i \), this implies \( X^i \in \text{add} I \), a contradiction. Thus we have \( X^i = 0 \).

In the following, we show \( \text{Ext}_A^1(A^i, X^i_1) = 0 \). If \( i = 0 \), then \( A^0 = A \) is projective. Hence the claim follows and we have \( X^0 = 0 \). For \( i \geq 1 \), we proceed by induction. Applying
Lemma 2.3(1) to (3.8.1), we have
\[
\text{Ext}^1_A(A^i, X^i_1) \cong \text{Ext}^{i+1}_A(A^i, X^i_{i+1}).
\]
By induction hypothesis, we have \(X^j = 0\), that is, \(T^j = I^j\) for each \(0 \leq j \leq i-1\). Thus we obtain \(\text{pd} A^i \leq i\) by repeating Lemma 2.9. This implies \(\text{Ext}^{i+1}_A(A^i, X^i_{i+1}) = 0\), and hence \(\text{Ext}^1_A(A^i, X^i_1) = 0\). Therefore we have \(X^i = 0\) and moreover \(T^i \in \text{add} I\). Namely, we obtain an exact sequence
\[
0 \to A \to I^0 \to \cdots \to I^{d-1} \to T^d \to 0.
\]
On the other hand, by \(T^d \in \text{Sub}^{n+1-d}(I)\), there exists an exact sequence
\[
0 \to T^d \to I^d \to \cdots \to I^n.
\]
Composing two exact sequences, we have the following exact sequence:
\[
0 \to A \to I^0 \to \cdots \to I^{d-1} \to I^d \to \cdots \to I^n.
\]
This finishes the proof. \(\square\)

3.3. Tilting modules and almost Auslander–Gorenstein algebras. In this subsection, we give characterisations of \(n\)-almost Auslander–Gorenstein algebras and \(n\)-almost Auslander algebras by the existence of the tilting modules in Theorem 3.1. We start this subsection with giving the definition of almost Auslander–Gorenstein algebras and almost Auslander algebras.

Definition 3.9. Fix an integer \(n \geq 0\). Let \(A\) be an artin algebra and \(I\) a direct sum of all pairwise non-isomorphic indecomposable injective \(A\)-modules with projective dimension at most one.

(1) We call \(A\) an \(n\)-almost minimal Auslander–Gorenstein algebra if it satisfies
\[
\text{id} A \leq n + 1 \leq I^\text{-domdim} A.
\]

(2) We call \(A\) an \(n\)-almost Auslander algebra if it satisfies
\[
\text{gldim} A \leq n + 1 \leq I^\text{-domdim} A.
\]

Throughout this paper, for brevity we omit the word “minimal” in \(n\)-almost minimal Auslander–Gorenstein algebras. Here are some examples of \(n\)-almost Auslander–Gorenstein algebras and \(n\)-almost Auslander algebras.

Example 3.10. (1) Clearly \(n\)-minimal Auslander–Gorenstein algebras (respectively, \(n\)-Auslander algebras) are \(n\)-almost Auslander–Gorenstein algebras (respectively, \(n\)-almost Auslander algebras).

(2) All 1-Iwanaga–Gorenstein algebras (respectively, hereditary algebras) are \(n\)-almost Auslander–Gorenstein algebras (respectively, \(n\)-almost Auslander algebras) for each \(n \geq 0\).

(3) Let \(A\) be an artin algebra and \(\mathcal{F}\) a faithful torsion-free class on \(\text{mod} A\). Assume that \(\mathcal{F}\) has an additive generator \(M\). Then the endomorphism algebra \(\text{End}_A(M)\) is a 1-almost Auslander algebra by [11, Theorem 2.1].

We give concrete examples of \(n\)-almost Auslander–Gorenstein algebras and \(n\)-almost Auslander algebras.
Example 3.11. Let $n \geq 4$ be an integer and $A$ the algebra defined by the quiver

\[
\begin{array}{cccc}
\beta_1 & \beta_2 & \ldots & \beta_{n-1} & \beta_n \\
1 & a_1 & a_2 & \cdots & a_{n-2} & n-1 \quad n
\end{array}
\]

with relations $\alpha_i \alpha_{i+1}$ $(1 \leq i \leq n-3)$, $\beta_i \alpha_i - \alpha_i \beta_{i+1}$ $(1 \leq i \leq n-2)$, $\beta_i^2$ $(1 \leq i \leq n)$ and $\beta_n \alpha_{n-1} - \alpha_{n-1} \beta_{n-1}$. Clearly we obtain $\text{gldim} A = \infty$. Let $I$ be a direct sum of all indecomposable injective $A$-modules with projective dimension at most one. Then we have $I = P(2) \oplus P(3) \oplus \cdots \oplus P(n)$. We can check that $I$ has a minimal injective coreolution

\[
0 \to A \to I^0 \to I^1 \to I^2 \to \cdots \to I^{n-2} \to 0,
\]

where $I^0 := I(2) \oplus \cdots \oplus I(n-2) \oplus I(n-1)^{\oplus 2}$, $I^1 := I(n-2)^{\oplus 2} \oplus I(n)^{\oplus 2}$, $I^2 := I(n-3)^{\oplus 2}$, $\ldots$, and $I^{n-2} := I(1)^{\oplus 2}$. Thus $A$ is an $(n-2)$-almost Auslander–Gorenstein algebra which is not an $(n-2)$-almost Auslander algebra.

Next $A'$ is the factor algebra $A/I'$, where $I'$ is a two-sided ideal of $A$ generated by $\beta_i$ $(1 \leq i \leq n)$. Then we can easily check that $A'$ is an $(n-2)$-almost Auslander algebra.

In the following, we give some properties on $n$-almost Auslander–Gorenstein algebras and $n$-almost Auslander algebras. An $n$-almost Auslander–Gorenstein algebra is not always left–right symmetry (see Remark 2.5 and Example 3.11). We have the following proposition, which is an analog of $n$-minimal Auslander–Gorenstein algebras and $n$-Auslander algebras.

Proposition 3.12. Let $A$ be an artin algebra. Then we have the following statements.

1. Fix an integer $n \geq 0$. Then $n$-almost Auslander algebras coincide with $n$-almost Auslander–Gorenstein algebras with finite global dimension.
2. If $A$ is an $n$-almost Auslander–Gorenstein algebra, then either $A$ is a 1-Iwanaga–Gorenstein algebra or $\text{id} A = n + 1 = I$ - domdim $A$ holds.
3. If $A$ is an $n$-almost Auslander algebra, then either $A$ is a hereditary algebra and $\text{gldim} A = n + 1 = I$ - domdim $A$ holds.

To show Proposition 3.12 we need the following lemma.

Lemma 3.13. Let $A$ be an artin algebra with $\text{id} A = m + 1$ and $I$ an injective $A$-module with $\text{pd} I \leq l$. Let $0 \to A \to I^0 \to I^1 \to \cdots \to I^{m+1} \to 0$ be a minimal injective coreolution of $A$. If $I' \in \text{add} I^{m+1}$, then $\text{pd} I' \geq m + 1$. Moreover, if $\text{gldim} A < \infty$, then we have the following statements.

1. $\text{id} A = \text{gldim} A$.
2. $\text{inj} A = \text{add} (I^0 \oplus I^1 \oplus \cdots \oplus I^{m+1})$.
3. Assume $I$ - domdim $A = m + 1$. Then the projective dimension of each injective $A$-module is one of $0, 1, \ldots, l$ and $m + 1$. In particular, for an injective $A$-module $I'$, we obtain that $\text{pd} I' = m + 1$ if and only if $I' \in \text{add} I^{m+1}$.

Proof. Let $I' \in \text{add} I^{m+1}$ be an indecomposable module with simple socle $S$. Then we have $\text{Hom}_A(S, I^{m+1}) \neq 0$. Suppose to the contrary $\text{pd} I' \leq m$. Applying $\text{Hom}_A(-, A)$ to an exact sequence $0 \to S \to I' \to I'/S \to 0$, we have

\[
\text{Ext}^{m+1}_A(I', A) \to \text{Ext}^{m+1}_A(S, A) \to \text{Ext}^{m+2}_A(I'/S, A).
\]

Since $\text{id} A = m + 1$ holds, the right-hand side vanishes. On the other hand, we have the left-hand side vanishes by $\text{pd} I' \leq m$. Thus we have $\text{Hom}_A(S, I^{m+1}) \cong \text{Ext}^{m+1}_A(S, A) = 0$, a contradiction.
(1) and (2) are well-known results (see [ARS, Lemma VI.5.5(b)]). Moreover, (3) follows from (2).

Now we are ready to prove Proposition 3.12.

**Proof of Proposition 3.12.**
(1) This follows from Lemma 3.13(1)
(2) If $A$ is not a 1-Iwanaga–Gorenstein algebra, then the last term of the minimal injective coresolution of $A$ has injective dimension at least two by Lemma 3.13. (3) This follows from (1) and (2).

The following theorem is a main result of this subsection.

**Theorem 3.14.** Fix an integer $n \geq 0$. Let $A$ be an artin algebra which is not a 1-Iwanaga–Gorenstein algebra and $I$ a direct sum of all indecomposable injective $A$-modules with projective dimension at most one. Then the following statements are equivalent.

1. $A$ is an $n$-almost Auslander–Gorenstein algebra.
2. There exists a unique basic $d$-tilting $A$-module $T \in \text{Fac}_d(I) \cap \text{Sub}^{n+1-d}(I)$ which is an $(n+1-d)$-cotilting $A$-module for some integer $1 \leq d \leq n+1$.

If in addition we assume $\text{gldim} A < \infty$, then the following statement is also equivalent.

3. $A$ is an $n$-almost Auslander algebra.

To show Theorem 3.14, we observe a minimal injective coresolution of $A$. First we state the following useful lemma without proof.

**Lemma 3.15.** Let $Q$ be an $A$-module with $\text{Ext}^1_A(Q,Q) = 0$. Let $0 \to X \overset{f}{\to} Z \overset{g}{\to} Y \to 0$ be a non-split exact sequence. Then the following conditions are equivalent.

1. $X$ is indecomposable, $f$ is a minimal left $\text{add}Q$-approximation of $X$, and $\text{Ext}^1_A(Q,X)$ vanishes.
2. $Y$ is indecomposable, $g$ is a minimal right $\text{add}Q$-approximation of $Y$, and $\text{Ext}^1_A(Y,Q)$ vanishes.

In the following, we assume that $A$ has a minimal injective coresolution of $A$

$$0 \to A \overset{f_0}{\to} I_0 \overset{f_1}{\to} I_1 \to \cdots \overset{f_n}{\to} I_n \to \cdots. \ (3.15.1)$$

Let $A^i := \text{Cok} f^{i-1}$ for each $i \geq 1$. Remark that we do not necessarily assume $\text{id} A < \infty$.

Let $I$ be a direct sum of all indecomposable injective $A$-modules with projective dimension at most one. Then we have the following proposition.

**Proposition 3.16.** Assume $I$-domdim $A \geq n+1$. Fix an integer $1 \leq d \leq n$. Let $X \in \text{add}A^d$ be a non-injective $A$-module. Then there exists an exact sequence

$$0 \to X \to I_X^d \to I_X^{d+1} \to \cdots \overset{f^n}{\to} I_X^n \to A_X^{n+1} \to 0$$

with $I_X^j \in \text{add}I^j$ for each $d \leq j \leq n$ and $A_X^{n+1} \in \text{add}A_X^{n+1}$. Moreover, we have the following statements.

1. If $X$ is indecomposable, then so is $A_X^{n+1}$.
2. Let $X, X' \in \text{add}A^d$ be non-injective $A$-modules. Then $X \not\cong X'$ if and only if $A_X^{n+1} \not\cong A_X^{n+1}$.
3. For each $A$-module $Y \in \text{add}A^{n+1}$, there uniquely exists an $A$-module $X$ such that $Y \cong A_X^{n+1}$. 


In particular, the map $X \mapsto A^{n+1}_X$ gives a bijection from the set of isomorphism classes of indecomposable non-injective direct summands of $A^d$ to the set of isomorphism classes of indecomposable direct summands of $A^{n+1}$.

Proof. For simplicity, we assume that $X \in \text{add} A^d$ is an indecomposable non-injective $A$-module. Let $\iota : X \to I^d_X$ be the injective hull of $X$. In particular, $I^d_X \in \text{add} I^d$. Then $\iota$ is clearly a minimal left $\text{add} I$-approximation of $X$. Moreover, by Lemma 2.3(1), we have $\text{Ext}^1_A(I, X) = 0$. Hence $A^{d+1}_X := \text{Cok} \iota \in \text{add} A^{d+1}$ is an indecomposable $A$-module satisfying $A^{d+1}_X \notin \text{add} I$ by Lemma 3.15. If $d = n$, then there is nothing to prove. Assume $d < n$. Then clearly $A^{d+1}_X$ is not injective. Repeating this process, we have the desired exact sequence and $A^{n+1}_X$ is indecomposable. Hence (1) holds. Moreover, (2) follows from uniqueness of a minimal left/right $\text{add} I$-approximation. Finally we show (3). By the construction of the exact sequence, we obtain $A^{n+1} = \oplus_X A^{n+1}_X$, where $X$ runs over all pairwise non-isomorphic indecomposable non-injective direct summands of $A^{d+1}$. This finishes the proof. \hfill $\Box$

As an application of Proposition 3.16, we have an injective coresolution of $\mathcal{T}^d$.

Corollary 3.17. Let $A$ be an $n$-almost Auslander–Gorenstein algebra which is not a 1-Iwanaga–Gorenstein algebra. Then $\mathcal{T}^d$ has an injective coresolution

$$0 \to \mathcal{T}^d \to J^d \oplus I \to \cdots \to J^{n-1} \to J^n \oplus I \to DA \to 0$$

with $J^j \in \text{add} I^l$ for each $d \leq j \leq n + 1$. In particular, $\mathcal{T}^d$ is $(n + 1 - d)$-cotilting.

Proof. If $d = n+1$, then we have $\mathcal{T}^{n+1} = DA$, and hence there is nothing to prove. Assume $d \leq n$. Let $X$ be a direct sum of all pairwise non-isomorphic indecomposable non-injective direct summand of $A^d$. By Proposition 3.16 we have an exact sequence

$$0 \to X \to I^d_X \to I^{d+1}_X \to \cdots \to I^n_X \to A^{n+1}_X \to 0.$$

Then $A^{n+1}_X$ is injective. Letting $J^j := I^j_X$, we obtain the desired injective coresolution because $\mathcal{T}^d = X \oplus I$ and $\mathcal{T}^{n+1} = DA = A^{n+1}_X \oplus I$ hold. \hfill $\Box$

Now we are ready to prove Theorem 3.14.

Proof of Theorem 3.14. (1)$\Rightarrow$(2): This follows from Theorem 3.11(1)$\Rightarrow$(3) and Corollary 3.17.

(2)$\Rightarrow$(1): By Theorem 3.11(3)$\Rightarrow$(1), we have $I$-domdim $\geq n + 1$. Thus it is enough to show $\text{id} A \leq n + 1$. In the proof of Theorem 3.11(3)$\Rightarrow$(1), we have the exact sequence $3.8.2$:

$$0 \to A \to I^0 \to \cdots \to I^{d-1} \xrightarrow{f^d} I^d \to \cdots \to I^{n-1} \xrightarrow{f^n} I^n \to \text{Cok} f^n \to 0$$

with $I^j \in \text{add} I$ for each $0 \leq j \leq n$ and $\text{Im} f^d \in \text{add} \mathcal{T}^d$. Since $\text{add}(\text{Im} f^d \oplus I) = \text{add} \mathcal{T}^d$ holds, we have $\text{id} \text{Im} f^d = \text{id} \mathcal{T}^d = n + 1 - d$, and hence $\text{Cok} f^n$ is injective. Therefore the exact sequence above gives a minimal injective coresolution of $A$. Namely, we obtain $\text{id} A = n + 1$.

In the following, we assume $\text{gldim} A < \infty$. Then (1)$\Leftrightarrow$(3) follows from Proposition 3.12(1). This finishes the proof. \hfill $\Box$

We can recover Crawley-Boevey–Sauter’s result.
Corollary 3.18 ([CBS Lemma 1.1]). Let $A$ be an artin algebra with $\text{gldim } A = 2$. Then $A$ is an Auslander algebra if and only if there exists a unique basic 1-tilting and 1-cotilting $A$-module $T \in \text{Fac}_1(Q) \cap \text{Sub}^1(Q)$, where $\text{add } Q \in \text{proj } A \cap \text{inj } A$.

Proof. Keep the notation in Theorem 3.14. First we show the “if” part. By Theorem 3.1, $\text{domdim } A = Q \text{-domdim } A \geq 2$ holds. Next we show the “only if” part. Since $A$ is an Auslander algebra, we have $I = Q$ by Lemma 3.13(3). Hence this follows from Theorem 3.14(1)$\Rightarrow$(2). □

4. The endomorphism algebras of the $d$-tilting modules

In this section, we study the endomorphism algebra $B^d := \text{End}_A(T^d)$ of the $d$-tilting module $T^d$ over an $n$-almost Auslander algebra $A$. Throughout this section, $I$ is a direct sum of all pairwise non-isomorphic indecomposable injective $A$-modules with projective dimension at most one and $A$ is an $n$-almost Auslander algebra. If $A$ is a 0-almost Auslander algebra, or equivalently, a hereditary algebra, then $T^d \cong DA$ and $B^d \cong A$. In the following, we always assume $n \geq 1$, that is, $\text{id } A = n + 1 = I \text{-domdim } A$.

We start this section with observing the projective dimension of $\text{Hom}_A(T^d, I')$ for an injective module $I'$. Let $\nu$ be the Nakayama functor of $\text{mod } A$.

Lemma 4.1. Let $I'$ be an indecomposable injective $A$-module. Then we have

$$\text{pd } \text{Hom}_A(T^d, I') \leq \begin{cases} 0 & (I' \in \text{add } I) \\ n + 1 - d & (I' \notin \text{add } I) \end{cases}$$

Proof. If $I' \in \text{add } I$, then $\text{Hom}_A(T^d, I')$ is a projective $B^d$-module, and hence the assertion holds. In the following, we assume $I' \notin \text{add } I$. Then we have $I' \in \text{add } I^{n+1}$ by Lemma 3.13(3). By Proposition 3.16 we have an exact sequence

$$0 \rightarrow X \rightarrow I_{n-d} \rightarrow \cdots \rightarrow I_0 \rightarrow I' \rightarrow 0$$

with $I_i \in \text{add } I$ and $X \in \text{add } T^d$. Applying $\text{Hom}_A(T^d, -)$, we have a projective resolution of $\text{Hom}_A(T^d, I')$

$$0 \rightarrow \text{Hom}_A(T^d, X) \rightarrow \text{Hom}_A(T^d, I_{n-d}) \rightarrow \cdots \rightarrow \text{Hom}_A(T^d, I_0) \rightarrow \text{Hom}_A(T^d, I') \rightarrow 0,$$

by the dual statement of Proposition 2.2(2). Thus the proof is complete. □

By Lemma 4.1, we give an upper bound for global dimension of $B^d$.

Proposition 4.2. Fix an integer $n \geq 1$. Assume that $A$ is an $n$-almost Auslander algebra which is not hereditary. Let

$$0 \rightarrow P^d_0 \rightarrow \cdots \rightarrow P^d_1 \rightarrow P^d_0 \rightarrow T^d \rightarrow 0$$

be a minimal projective resolution of $T^d$. Then the following statements hold.

1. $\text{gldim } B^d \leq \text{gldim } A$.
2. If $\nu P^d_1 \in \text{add } I$, then $\text{gldim } B^1 = n$.

Proof. (1) Applying $\text{Hom}_A(-, T^d)$ to the exact sequence (4.2.1), we have an exact sequence

$$0 \rightarrow \text{Hom}_A(T^d, T^d) \rightarrow \text{Hom}_A(P^d_0, T^d) \rightarrow \cdots \rightarrow \text{Hom}_A(P^d_1, T^d) \rightarrow 0$$
by Proposition 2.22. By Serre duality, we have \( \text{Hom}_A(P_i^{\text{op}}, T^d) \cong D \text{Hom}_A(T^d, \nu^P_i) \). Applying \( D \), we have an exact sequence
\[
0 \to \text{Hom}_A(T^d, \nu^P_i) \to \cdots \to \text{Hom}_A(T^d, \nu^P_0) \to DB \to 0.
\]
By Lemma 2.9 we have
\[
\text{pd } DB \leq \max \{ \text{pd } \text{Hom}_A(T^d, \nu^P_i) + i \mid i \in \{0, 1, \ldots, d\} \}.
\]
Due to [H, Proposition III.3.4], we have
\[
\text{gldim } A - \text{gldim } B^d \leq \text{pd } T^d.
\] Thus we have the assertion by Lemma 4.1.

(2) Since \( \text{pd } DB \leq \max \{ \text{pd } \text{Hom}_A(T^d, \nu^P_0) \} \) holds, we have
\[
\text{pd } DB \leq \max \{1, \text{pd } \text{Hom}_A(T^d, \nu^P_0)\} \leq n
\]
by Lemma 4.1. Thus we have \( \text{gldim } B^1 \leq n \). The assertion follows from (4.2.2).

In the rest of this section, we give a sufficient condition of \( B^\text{op} \) to be an \( n \)-almost Auslander algebra again, where \( B := B^1 = \text{End}_A(T^1) \).

Definition 4.3. Fix an integer \( n \geq 1 \). Assume that \( A \) is an \( n \)-almost Auslander algebra. Let \( I \) be a direct sum of all indecomposable injective \( A \)-modules with projective dimension at most one. We define \( D \) to be the full subcategory of \( \text{Fac}_1(I) \cap \text{Sub}^1(I) \) consisting of \( A \)-modules \( X \) with \( \text{id } \text{Hom}_A(T^1, X) \leq 1 \).

For each \( X \in \text{Fac}_1(I) \), we have \( \text{id } \text{Hom}_A(T^1, X) \leq 1 + \text{id } X \) by [ASS, VI.7.20]. Thus we have \( \text{add } I \subset D \). The following result is a main result of this section.

Theorem 4.4. Fix an integer \( n \geq 1 \). Let \( A \) be an \( n \)-almost Auslander algebra and \( B := \text{End}_A(T^1) \). Let \( I^0 \) be a direct sum of all pairwise non-isomorphic indecomposable injective \( B^\text{op} \)-modules. Then we have \( I^0 \)-domdim \( B^\text{op} \geq n \). In particular, \( B^\text{op} \) is either an \( (n-1) \)-almost Auslander algebra or an \( n \)-almost Auslander algebra. Moreover if \( \text{add } T^1 \subset D \) and \( \text{gldim } B = n + 1 \), then \( B^\text{op} \) is an \( n \)-almost Auslander algebra.

To prove Theorem 4.4, we need the following lemma.

Lemma 4.5. Keep the notation in Theorem 4.4. Then the following statements hold for each \( P \in \text{proj } A \).

1. If \( \nu P \in \text{add } I \), then \( \text{Hom}_A(P, T^1) \) is an injective \( B^\text{op} \)-module with projective dimension at most one.
2. If \( \nu P \notin \text{add } I \), then \( \text{Hom}_A(P, T^1) \in \text{Sub}^n(I^0) \). Moreover if \( \text{add } T^1 \subset D \), then we have \( \text{Hom}_A(P, T^1) \in \text{Sub}^n(I^0) \).

Proof. (1) By \( \nu P \in \text{add } I \), we obtain that \( \text{Hom}_A(P, T^1) \cong D \text{Hom}_A(T^1, \nu P) \) is injective. Since \( T^1 \) is tilting, there exists an exact sequence \( 0 \to P \to T^0 \to T^1 \to 0 \) with \( T^0, T^1 \in \text{add } T^1 \). Applying \( \text{Hom}_A(-, T^1) \) to the exact sequence above, we have an exact sequence \( 0 \to \text{Hom}_A(T^1, T^1) \to \text{Hom}_A(T^0, T^1) \to \text{Hom}_A(P, T^1) \to 0 \).

Therefore the assertion follows from \( \text{Hom}_A(T^1, T^1), \text{Hom}_A(T^0, T^1) \in \text{proj } B^\text{op} \).

(2) Since \( \nu P \notin \text{add } I \), there exists an indecomposable direct summand \( I' \) of \( \nu P \) such that \( I' \) is a direct summand of \( I^{n+1} \) by Lemma 3.13(3). Applying Proposition 3.16, we have an exact sequence
\[
0 \to X \to I_{n-1} \to \cdots \to I_0 \to I' \to 0
\]
with \( \lambda \in \text{add}I \) and \( X \in \text{add}T^1 \). Applying \( \text{DHom}_A(T^1, -) \) to the exact sequence above and using Serre duality, we have an exact sequence

\[
0 \to \text{Hom}_A(P', T^1) \to \text{Hom}_A(P_0, T^1) \to \cdots \to \text{Hom}_A(P_{n-1}, T^1) \to \text{DHom}_A(T^1, X) \to 0,
\]

where \( I' = \nu P' \) and \( I_i = \nu P_i \) for each \( 0 \leq i \leq n-1 \). By (1), \( \text{Hom}_A(P_1, T^1) \) is injective with projective dimension at most one. Since \( P' \in \text{add}P \) holds, we have \( \text{Hom}_A(P, T^1) \in \text{Sub}^n(I^0) \). Moreover, if \( X \in D \), then \( \text{pd} \text{DHom}_A(T^1, X) \leq 1 \), and hence \( \text{Hom}_A(P, T^1) \in \text{Sub}^{n+1}(I^0) \).

Now we are ready to show Theorem 4.4.

**Proof of Theorem 4.4.** By \( \text{pd} T^1 = 1 \), there is a minimal projective resolution \( 0 \to P_1 \to P_0 \to T^1 \to 0 \). Applying \( \text{Hom}_A(-, T^1) \) to the projective resolution above, we have an exact sequence

\[
0 \to \text{Hom}_A(T^1, T^1) \to \text{Hom}_A(P_0, T^1) \to \text{Hom}_A(P_1, T^1) \to 0.
\]

This exact sequence gives \( \text{Hom}_A(T^1, T^1) \in \text{Sub}^n(I^0) \) by Lemma 4.5(2). Hence we have \( I^0 \)-domdim \( B^{\text{op}} \geq n \).

In the following, we assume \( \text{gl.dim} B^{\text{op}} = n + 1 \). By Proposition 4.2(2), we have \( \nu P_1 \notin \text{add}I \). If \( \nu P_0 \in \text{add}I \), then we have the assertion by Lemma 4.5(1). On the other hand, if \( \nu P_0 \notin \text{add}I \), then the assertion follows from Lemma 4.5(2). \( \square \)

5. **Almost Auslander Algebras and Strongly Quasi-hereditary Algebras**

In this section, we study a relationship between 1-almost Auslander algebras and strongly quasi-hereditary algebras. We start with recalling the definition of strongly quasi-hereditary algebras (see [R2] and [T]) for details). We fix a complete set \( \{ S(\lambda) \mid \lambda \in \Lambda \} \) of representatives of isomorphism classes of simple \( A \)-modules and denote by \( P(\lambda) \) the projective cover of \( S(\lambda) \). Let \( \leq \) be a partial order on \( \Lambda \). For each \( \lambda \in \Lambda \), we denote by \( \Delta(\lambda) \) the standard \( A \)-module (i.e., it is a maximal factor module of \( P(\lambda) \) whose composition factors have the form \( S(\mu) \) for some \( \mu \leq \lambda \)). Dually, we define the costandard module \( \nabla(\lambda) \) for each \( \lambda \in \Lambda \). Let \( \mathcal{F}(\Delta) \) be the full subcategory of \( \text{mod}A \) whose objects are the modules which have a \( \Delta \)-filtration. For \( M \in \mathcal{F}(\Delta) \), we denote by \( (M : \Delta(\lambda)) \) the filtration multiplicity of \( \Delta(\lambda) \), which does not depend on the choice of \( \Delta \)-filtrations.

A pair \( (A, \leq) \) (or simply \( A \)) is called a quasi-hereditary algebra if for each \( \lambda \in \Lambda \) there exists an exact sequence

\[
0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0
\]

satisfying the following conditions:

- \( K(\lambda) \in \mathcal{F}(\Delta) \);
- if \( (K(\lambda) : \Delta(\mu)) \neq 0 \), then \( \lambda < \mu \).

It is well known that all quasi-hereditary algebras have finite global dimension (see [PaSc] Theorem 4.3). By [R1] Theorem 5], a quasi-hereditary algebra \( A \) has a basic tilting and cotilting \( A \)-module \( T \), which is a direct sum of all Ext-injective objects in \( \mathcal{F}(\Delta) \). We call \( T \) a characteristic tilting module.

**Definition 5.1.** ([R2] Proposition A.1). Let \( (A, \leq) \) be a quasi-hereditary algebra and \( T \) its characteristic tilting module.

(1) A pair \( (A, \leq) \) (or simply \( A \)) is called a right-strongly quasi-hereditary algebra if it satisfies one of the following equivalent conditions.

(a) \( \text{pd} \Delta(\lambda) \leq 1 \) for each \( \lambda \in \Lambda \).
(b) \( \text{pd} \, X \leq 1 \) for each \( X \in \mathcal{F}(\Delta) \).
(c) \( \text{pd} \, T \leq 1 \).

Dually, we define a left-strongly quasi-hereditary algebra.

(2) A pair \((A, \leq)\) (or simply \(A\)) is called a **strongly quasi-hereditary algebra** if it is both right-strongly quasi-hereditary and left-strongly quasi-hereditary.

Ringel showed if \(A\) is strongly quasi-hereditary, then its global dimension is at most two (see [R2, Proposition A.2]). However, the converse does not hold in general. On the other hand, if \(\text{gldim} \, A \leq 2\), then there exists a partial order \(\leq\) on \(\Lambda\) such that \((A, \leq)\) is a right-strongly quasi-hereditary algebra but not necessarily strongly quasi-hereditary (see [T1, Theorems 4.1 and 4.6]). Then we have the following question.

**Question 5.2.** Assume that \(\text{gldim} \, A \leq 2\) and then \((A, \leq)\) is right-strongly quasi-hereditary. When is \((A, \leq)\) a strongly quasi-hereditary algebra?

In [T1] and [T2], the author gave a complete answer to the question when \(A\) is an Auslander algebra or an Auslander–Dlab–Ringel algebra. In the following, we give a partial answer for 1-almost Auslander algebras. We assume that \(A\) is a 1-almost Auslander algebra. Let \(I\) be a direct sum of all pairwise non-isomorphic indecomposable injective \(A\)-modules with \(\text{pd} \, J \leq 1\) and \(T^1\) the basic 1-tilting module. By \(\text{gldim} \, A \leq 2\), we can take a right-strongly quasi-hereditary algebra \((A, \leq)\) and let \(T\) be its characteristic tilting module.

The following theorem is a main result of this section.

**Theorem 5.3.** Keep the notation above. Consider the following conditions:

1. \((A, \leq)\) is a strongly quasi-hereditary algebra,
2. \(T \cong T^1\),
3. \(P(T) \in \text{add} \, I\), where \(P(T)\) is the projective cover of \(T\).

Then (3) \(\Rightarrow\) (2) \(\Rightarrow\) (1) holds. Moreover if \(I\) is projective, then (1) \(\Rightarrow\) (3) holds.

First we give an observation for 0-almost Auslander algebras (or equivalently, hereditary algebras).

**Example 5.4.**

1. Any 0-almost Auslander algebra is always a strongly quasi-hereditary algebra since all standard modules have projective dimension at most one and all costandard modules have injective dimension at most one.
2. If \(A\) is a right-strongly (respectively, left-strongly) quasi-hereditary algebra with \(T \cong DA\) (respectively, \(T \cong A\)), then \(A\) is a 0-almost Auslander algebra. Indeed, since \(A\) is a right-strongly quasi-hereditary algebra, we have \(\text{pd} \, DA \leq 1\), and hence \(\text{pd} \, DA \leq 1\). Hence the assertion follows from Lemma 3.13(1).

To prove Theorem 5.3, we need the following lemma.

**Lemma 5.5.** The following statements hold.

1. Let \(I\) be an injective \(A\)-module. Assume that \(A \in \text{Sub}^2(I)\). If \(\text{pd} \, X \leq 1\), then the injective hull \(I(X)\) is in \(\text{add} \, I\). In particular, \(X \in \text{Sub}^1(I)\).
2. Let \(P\) be an projective \(A\)-module. Assume that \(DA \in \text{Fac}_3(P)\). If \(\text{id} \, Y \leq 1\), then the projective cover \(P(Y)\) is in \(\text{add} \, P\). In particular, \(Y \in \text{Fac}_1(P)\).

**Proof.** We only prove (1); the proof of (2) is similar. By \(\text{pd} \, X \leq 1\), we obtain a minimal projective resolution

\[
0 \to P_1 \overset{\rho_1}{\to} P_0 \overset{\rho_0}{\to} X \to 0.
\]
Let \( \iota^i : P_i \to I(P_i) \) be the injective hull of \( P_i \) for each \( i \in \{0,1\} \). Then we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & P_1 & \xrightarrow{\rho_1} & P_0 & \xrightarrow{\rho_0} & X & \rightarrow & 0 \\
\downarrow{\iota^1} & & \downarrow{\iota^0} & & \downarrow{\iota} & & \\
0 & \rightarrow & I(P_1) & \xrightarrow{\rho_1'} & I(P_0) & \xrightarrow{\rho_0'} & X' & \rightarrow & 0
\end{array}
\]

By the Snake lemma, there exists a monomorphism \( \gamma : \text{Ker}\ i \to \text{Cok}\ \iota^1 \). Since \( A \in \text{Sub}^2(I) \), \( \text{Cok}\ \iota^1 \) is embedded into some \( I' \in \text{add}\ I \). By composing it and \( \gamma \), we have a monomorphism \( \varphi : \text{Ker}\ i \to I' \). Let \( \mu : \text{Ker}\ i \to I(\text{Ker}\ i) \) be the injective hull of \( \text{Ker}\ i \). Then there exists a split monomorphism \( g : I(\text{Ker}\ i) \to I' \) such that \( g\mu = \varphi \). Hence \( I(\text{Ker}\ i) \in \text{add}\ I \).

Moreover, since \( \rho_1' \) is also splitting, we have \( X' \in \text{add}\ I \). We define a morphism \( \psi : X \to X' \oplus I(\text{Ker}\ i) \) as

\[
\psi(x) = \begin{cases} 
\mu(x) & (x \in \text{Ker}\ i) \\
\iota(x) & (x \notin \text{Ker}\ i).
\end{cases}
\]

Then \( \psi \) is a monomorphism. Hence \( I(X) \in \text{add}\ I \). \( \blacksquare \)

Now we are ready to prove Theorem \( \ref{theorem:main} \).

**Proof of Theorem \( \ref{theorem:main} \).** \((3) \Rightarrow (2)\): Since \( A \) is right strongly quasi-hereditary, we have \( \text{pd}\ T \leq 1 \). By Lemma \( \ref{lemma:projective} (1) \), we obtain \( T \in \text{Sub}^1(I) \). On the other hand, \( P(T) \in \text{add}\ I \) implies \( T \in \text{Fac}^1(I) \). Hence \( T \in \text{Fac}^1(I) \cap \text{Sub}^1(I) \). Thus \( T \cong T^1 \) by Lemma \( \ref{lemma:main} (3) \).

\((2) \Rightarrow (1)\): Note that \( T^1 \) is a 1-cotilting module by Theorem \( \ref{theorem:main} (2) \). Since \( \text{id}\ T = \text{id}\ T^1 \leq 1 \) holds, \( A \) is a left-strongly quasi-hereditary algebra. Hence the assertion holds.

In the following, we assume that \( I \) is projective.

\((1) \Rightarrow (3)\): Since \( A \) is left-strongly quasi-hereditary, we have \( \text{id}\ T \leq 1 \). By Lemma \( \ref{lemma:main} (2) \), we obtain the projective cover \( P(T) \) is in \( \text{add}\ I \). \( \blacksquare \)

If we do not assume that \( I \) is projective, then \((1) \Rightarrow (2)\) is not always satisfied as the following example shows.

**Example 5.6.**

1. Let \( A \) be a 0-almost hereditary algebra. Then we have \( T^1 = DA \).

   On the other hand, by Example \( \ref{example:almost} (1) \), \( A \) is a strongly quasi-hereditary algebra with characteristic tilting module \( T \). If \( T \cong A \), then we have \( T \neq T^1 \). For example, when \( A \) is the path algebra of \( 1 \to 2 \to 3 \) with partial order \( \{3 < 2 < 1\} \), we have \( T \cong A \).

2. Let \( A \) be the algebra defined by the quiver

\[
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
\gamma & \searrow & \swarrow \beta \\
3 & \xrightarrow{\delta} & 4
\end{array}
\]

with a relation \( \alpha\beta - \gamma\delta \). Then we obtain \( I = I(2) \oplus I(3) \oplus I(4) \), which is not projective. Moreover, \( \text{gldim}\ A = 2 = \text{I-domdim}\ A \) holds. Indeed, \( A \) has a minimal injective coresolution

\[
0 \to A \to I(4)^{\oplus 4} \to I(2)^{\oplus 2} \oplus I(3)^{\oplus 2} \to I(1) \to 0.
\]
Therefore we have $T^1 = I(4)/S(4) \oplus I(2) \oplus I(3) \oplus I(4)$. On the other hand, $A$ is a strongly quasi-hereditary algebra with respect to $\{2 < 3 < 1 < 4\}$ and the characteristic tilting module $T = I(4)/S(4) \oplus S(2) \oplus S(3) \oplus I(4)$.

Let $A$ be an artin algebra with $	ext{gldim} A = 2$. Then $A$ is an Auslander algebra if and only if $I$-domdim $\geq 2$ and there exists no indecomposable injective $A$-module with projective dimension exactly one. Indeed, the “only if” part follows from Lemma 3.13(3) and the “if” part is clear. Hence, as an application of Theorem 5.3, we have the following corollary.

**Corollary 5.7.** Let $A$ be an Auslander algebra. Then the following statements are equivalent.

1. $A$ is a strongly quasi-hereditary algebra.
2. $T \cong T^1$.
3. $P(T) \in \text{add} I$, where $P(T)$ is the projective cover of $T$.
4. $\text{End}_A(I)$ is a Nakayama algebra.

**Proof.** (1)$\implies$(2)$\implies$(3): This follows from Theorem 5.3.

(1)$\implies$(4): This follows from [T1, Theorem 4.6].

As an application, we give the following proposition, which is a generalisation of [DR2, § 7] and [E].

**Proposition 5.8.** Let $A$ be an Auslander algebra and $eA$ a maximal projective-injective direct summand of $A$. If $A$ is strongly quasi-hereditary, then $\text{mod}(A/eA)$ is equivalent to $\mathcal{F}(\Delta)/\text{add} T^1$.

In the rest of this section, we give a proof of Proposition 5.8 following the strategy of DR2.

**Lemma 5.9** (DR2, Theorem 3). Let $A$ be a strongly quasi-hereditary algebra and $T$ a characteristic tilting module of $A$. Then we have an equivalence $\mathcal{F}(\Delta)/\text{add} T \simeq \mathcal{H}(T)$, where $\mathcal{H}(T) := \{Y \in \text{mod} A \mid \text{Hom}_A(T, Y) = \{0\}\}$.

For $M, N \in \text{mod} A$, we denote by $\text{Tr}_N M$ the trace of $N$ in $M$ (i.e., it is the submodule of $M$ generated by all homomorphic images of $N$ in $M$).

**Lemma 5.10** (DR2, Theorem 4). Assume that $A$ is a quasi-hereditary algebra and every projective cover of costandard module is injective. Then we have $\mathcal{H}(T) = \text{mod} A / \text{Tr}_T A$.

**Lemma 5.11.** Let $A$ be a left-strongly quasi-hereditary Auslander algebra and $T$ a characteristic tilting module of $A$. Then $\text{Tr}_T M = \text{Tr}_{P(T)} M$ holds for each $M \in \text{mod} A$.

**Proof.** Since $A$ is a left-strongly quasi-hereditary algebra, $\text{id} T \leq 1$. Since $A$ is an Auslander algebra, $DA$ is in $\text{Fac}_2(Q)$, where $\text{add} Q = \text{proj} A \cap \text{inj} A$. By Lemma 5.3(2), the projective cover $P(T)$ of $T$ is in $\text{add} Q$. Thus we have each indecomposable direct summand of $P(T)$ is a direct summand of $T$, and hence we have the assertion.

We are ready to prove Proposition 5.8.

**Proof of Proposition 5.8.** We may assume that $A$ is a basic algebra. By lemma 5.9, we have $\mathcal{F}(\Delta)/\text{add} T \simeq \mathcal{H}(T)$. Since $A$ is a left-strongly quasi-hereditary algebra, the injective dimension of each costandard module is at most one. By Lemma 5.3(2), every projective cover of costandard module is injective since $A$ is an Auslander algebra. Thus we obtain that $\mathcal{H}(T) = \text{mod} A / \text{Tr}_T A$ by Lemma 5.10. By Lemma 5.11, $\text{Tr}_T A = \text{Tr}_{P(T)} A$ and we can easily check that $\text{Tr}_{P(T)} A = AeA$. Therefore, we have the assertion by Corollary 5.7.
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References

[ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras. Vol. 1*, London Mathematical Society Student Texts 65, Cambridge university press (2006).

[ARS] M. Auslander, I. Reiten, S. O. Smals, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, 36, Cambridge University Press, Cambridge, 1995.

[B] K. Bongartz, *Tilted algebras*, Representations of algebras (Puebla, 1980), pp. 26–38, Lecture Notes in Math., 903, Springer, Berlin-New York, 1981.

[CX] H. Chen, C. Xi, *Dominant dimension, derived equivalences and tilting modules*, Israel J. Math. 215 (2016), no. 1, 349–395.

[CHU] F. Coelho, D. Happel, L. Unger, *Complements to partial tilting modules*, J. Algebra 170 (1994), no. 1, 184–205.

[CBS] W. Crawley-Boevey, J. Sauter, *On quiver Grassmannians and orbit closures for representation-finite algebras*, Math. Z. 285 (2017), 367–395.

[DR1] V. Dlab and C. M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. 33 (1989), no. 2, 280–291.

[DR2] V. Dlab, C. M. Ringel, *The module theoretical approach to quasi-hereditary algebras*, Representations of algebras and related topics (Kyoto, 1990), 200–224, London Math. Soc. Lecture Note Ser., 168, Cambridge Univ. Press, Cambridge, 1992.

[E] Ö. Eiriksson, *From submodule categories to the stable Auslander algebra*, J. Algebra 486 (2017), 98–118.

[H] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[HU1] D. Happel, L. Unger, *On the quiver of tilting modules*, J. Algebra 284 (2005), no. 2, 857–868.

[HU2] D. Happel, L. Unger, *On a partial order of tilting modules*, Algebr. Represent. Theory 8 (2005), no. 2, 147–156.

[I1] O. Iyama, *The relationship between homological properties and representation theoretic realization of Artin algebras*, Trans. Amer. Math. Soc. 357 (2005), no. 2, 709–734.

[I2] O. Iyama, *r-Categories III: Auslander orders and Auslander–Reiten quivers*, Algebr. Represent. Theory 8 (2005), no. 5, 601–619.

[I3] O. Iyama, X. Zhang, *Tilting modules over Auslander–Gorenstein algebras*, to appear in Pacific J. Math. [arXiv:1801.04738v2].

[Mi] Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Z. 193 (1986), no. 1, 113–146.

[NRTZ] V. C. Nguyen, I. Reiten, G. Todorov, S. Zhu, *Dominant dimension and tilting modules*, to appear in Math. Z. [arXiv:1706.00475].

[PaSc] B. Parshall, L. Scott, *Derived categories, quasi-hereditary algebras and algebraic groups*, Proceedings of Ottawa-Moosonee Workshop in Algebra, Carleton Univ. Notes, no. 3, (1988).

[PrSa] M. Pressland, J. Sauter, *Special tilting modules for algebras with positive dominant dimension*, [arXiv:1705.03867].

[R1] C. Riedtmann, A. Schofield, *On a simplicial complex associated with tilting modules*, Comment. Math. Helv. 66 (1991), no. 1, 70–78.

[R2] C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. 208 (1991), no. 2, 209–223.

[R3] C. M. Ringel, *Iyama’s finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra 214 (2010), no. 9, 1687–1692.

[T1] M. Tsukamoto, *Strongly quasi-hereditary algebras and rejective subcategories*, to appear in Nagoya Math. J. [arXiv:1705.00279].

[T2] M. Tsukamoto, *On an upper bound for the global dimension of Auslander–Dlab–Ringel algebras*, Arch. Math. (Basel) 112 (2019), no. 1, 41–51.
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