Theory of synchrotron radiation: II. 
Backreaction in ensembles of particles

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Abstract

The standard calculations of the synchrotron emission from charged particles in magnetic fields does not apply when the energy losses of the particles are so severe that their energy is appreciably degraded during one Larmor rotation. In these conditions, the intensity and spectrum of the emitted radiation depend on the observation time $T_{\text{obs}}$: the standard result is recovered only in the limit $T_{\text{obs}} \ll T_{\text{loss}}$, where $T_{\text{loss}}$ is the time for synchrotron losses. In this case the effects of the radiation backreaction cannot be detected by the observer. We calculate the emitted power of the radiation in the most general case, naturally including both the cases in which the backreaction is relevant and the standard case, where the usual result is recovered. Finally we propose several scenarios of astrophysical interest in which the effects of the backreaction cannot and should not be ignored.

1 Introduction

As pointed out in \cite{1} (hereafter paper I), some aspects of the synchrotron emission of high energy particles did not receive proper attention in the literature, mainly because the standard treatment proved to be valid over most of the energy range of interest for astrophysical applications. Recently, this

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range of interest changed considerably, mainly due to the discovery of radiation processes at ultra-high energies. Nevertheless the important corrections to synchrotron emission in this regime have been ignored and the standard calculations have been adopted.

In paper I we explored the generalization of the calculations of the synchrotron emission from an ensemble of particles radiating coherently. We considered there the cases of bunches of particles both in the monoenergetic case and in the case of a spectrum of particles, and for each we established the criteria for the synchrotron emission to be coherent.

In the present paper we explore an effect that becomes relevant at sufficiently high energies, when the energy lost by a particle during one Larmor gyration becomes comparable with the energy of the particle itself. We call this effect backreaction, with may be a slightly improper term. We find that in this regime there are important corrections to the spectra of the emitted radiation in numerous scenarios currently discussed in the literature, confirming but extending previous findings of Ref. [2], where the case of a monoenergetic distribution of radiating electrons was considered. In this paper we also study the situation of an ensemble of particles with arbitrary energy spectra and find an approximate analytical solution of the problem of the synchrotron backreaction, that may be useful to estimate the magnitude of the effect. The formalism used here, introduced in paper I and briefly summarized in this paper, allows us to take naturally into account possible coherence effects in the synchrotron radiation, together with the backreaction. Cases of astrophysical interest in which the effects of the backreaction are supposed to play an important role will be discussed.

The paper is structured as follows: in section 2 we describe the backreaction and the effects that can be expected on simple basis. In section 3 we calculate the spectrum of the radiation emitted by monoenergetic particles. In section 4 we generalize the calculation to the case of a spectrum of radiating particles and we present analytical approximations that allow us to estimate the effects of the backreaction without (or before) being involved in the detailed calculations. We present our conclusions in section 5.

2 The backreaction

The rate of energy losses of a particle with mass $m$, Lorentz factor $\gamma$ and charge $q$ in a magnetic field $B$ can be written in the well known form

$$\frac{dE}{dt} = \frac{2q^4}{3m^2c^3} B^2 \gamma^2 = \frac{2q^4}{3m^4c^7} B^2 E^2,$$  \hspace{1cm} (1)
where for simplicity we assumed that the electron moves perpendicular to the direction of the magnetic field. The time for a Larmor rotation can be easily calculated as \( \tau_B = \frac{2\pi E}{qBc} \). The (often) hidden assumption of the standard calculations of synchrotron radiation is that the energy lost by a particle in the time \( \tau_B \) is negligible compared with the particle energy \( E \). This condition is fulfilled when

\[
\frac{\tau_B}{\tau_R} \ll 1 \rightarrow E \ll \left( \frac{3m^4c^8}{4\pi^3q^3B} \right)^{1/2}
\]  

(2)

where \( \tau_R \) is the typical time of energy losses of the particle, defined by the expression

\[
\tau_R^{-1} = \frac{1}{E} \frac{dE}{dt} = \omega_K \gamma = \frac{2q^4B^2}{3m^4c^7} E .
\]  

(3)

For energies larger than the value found in eq. (2), the standard calculations fail and a new approach is needed. In the present paper we introduce this novel approach and show that the difference between the predicted spectra and the standard ones may be extremely important and in general cannot be ignored.

Note that from equation (2) the region where the backreaction becomes important is numerically given by

\[
E \gg 1.9 \times 10^4 B_{\text{Gauss}}^{-1/2} \text{ GeV } \tag{4}
\]

for electrons, and by

\[
E \gg 6.5 \times 10^{10} B_{\text{Gauss}}^{-1/2} \text{ GeV } \tag{5}
\]

for protons. Here the magnetic field \( B_{\text{Gauss}} \) is in Gauss. A visual picture of the region in the plane \( E - B \) where the backreaction is relevant is provided in fig. 1 for both electrons and protons: the backreaction should be taken into account above the solid lines reported in fig. 1.

3 Spectrum of monoenergetic particles

In this section we compute the spectrum radiated by \( Z \) particles, all having the same initial Lorentz factor \( \gamma_0 \), as detected by an observer during an observation time \( T_{\text{obs}} \) (this calculation was also carried out with some differences in the approach, in [2]). This situation, as shown in [2] is equivalent to calculate
the power radiated by a single particle with Lorentz factor $\gamma_0$, as detected by $Z$ observers during the observation time. Although trivial for the standard scenario, this is not trivial for the situation of interest here.

We also demonstrate that for observation times smaller than the time for appreciable losses, the standard result is recovered.

A nice didactical approach illustrated in [2] allows to show that the result of the backreaction, that should be derived by the solution of the Lorentz-Dirac equation, in the end can be determined in the simple way of taking into account the time dependence of the Lorentz factor of the radiating particle.

Based on eq. (1), we can write the time dependence of the Lorentz factor of a particle subject to synchrotron energy losses in the following form

$$\gamma(t) = \frac{\gamma_0}{1 + \omega_R t}$$

where $\omega_R = \omega_K \gamma_0 = \tau_R^{-1}$ conveniently defines the time scale for the energy losses of a particle. In [2] it was shown that this expression keeps its validity also in case of strong backreaction, at least as long as quantum effects remain negligible. These effects enter the calculation only when the typical energy of the radiated photon becomes comparable to the energy of the particle, so that quantum recoil needs to be accounted for. We will not be involved here in
these extreme cases, although we believe that some interesting physics could be learned there. To our knowledge, no paper exists in the literature treating the problem of the synchrotron emission with both the backreaction and the quantum recoil at the same time.

Based on the formalism illustrated in detail in paper I, we associate to each particle a phase factor $\alpha$. Using the usual relativistic beaming condition, implying that significant synchrotron emission is confined in a narrow cone of aperture $\sim 1/\gamma$, we deduce that a given particle illuminates the observer at those retarded times that are related to the phase $\alpha$ by the equation

$$t\omega_B(t) - \alpha \approx 0$$

(7)

where $\omega_B(t) = \omega_0(1 + \omega_Rt) = \omega_L\gamma_0^{-1}(1 + \omega_Rt)$ is the (time varying) Larmor frequency of the particle ($\omega_L$ is the usual cyclotron frequency). From the condition eq. (7) the $i$-th particle of the ensemble emits its synchrotron burst at the time (we retain here for obvious reasons only the positive solutions of eq. (7))

$$t_i = \frac{1}{2\omega_R}[(1 + 4\frac{\omega_R}{\omega_0}\alpha_i)^{1/2} - 1].$$

(8)

From eq. (8) it is easy to derive the Lorentz factor of the particle at the moment of the burst:

$$\gamma_i = \gamma(\alpha_i) = \frac{2\gamma_0}{1 + (1 + 4\frac{\omega_R}{\omega_0}\alpha_i)^{1/2}}.$$  

(9)

Eq. (9) will be used to rewrite the synchrotron spectrum as the spectrum of an ensemble of particles each characterized by a Lorentz factor $\gamma_i$ and a phase $\alpha_i$.

In paper I we have discussed the differences in synchrotron emission produced by an ensemble of particles in a coherent and an incoherent configuration of phases. In the present paper we will only be concerned with the case of incoherent radiation, so that the energy radiated per unit frequency and unit solid angle in the directions perpendicular and parallel to the magnetic field can be written respectively as follows:

$$\frac{d^2W_\perp}{d\omega d\Omega} = \frac{q^2}{4\pi c} \sum_{k=1}^Z \left(\frac{\omega}{\omega_L}\right)^2 \frac{\theta_k^4}{\gamma_k^2} K_{2/3}(\eta_k),$$

(10)
and

\[
\frac{d^2 W}{d\omega d\Omega} = \frac{q^2}{4\pi c} \sum_{k=1}^{Z} \left( \frac{\omega}{\omega_L} \right)^2 \theta_k^2 \theta_k^2 K_{1/3}^2(\eta_k),
\]

(11)

where we defined the usual quantities [3,4]:

\[
\theta_k = (1 + \theta_k^2 \gamma_k^2)^{1/2} \quad \eta_k = \frac{\omega}{3\omega_L \gamma_k^2} \theta_k^3,
\]

(12)

and \(\theta = \pi/2 - \theta_z\) with \(\theta_z\) the zenith angle of the versor \(\hat{n}\) that points to the observer.

The quantity that is most interesting from a physical point of view is the power (per unit frequency) radiated by the particles. To obtain this quantity, as we have discussed in paper I, one has to divide the energy per unit frequency over the observation time \(T_{\text{obs}}\). The total power radiated per unit frequency by the \(Z\) particles in a period \(T_{\text{obs}}\) is\(^3\)

\[
\frac{dP}{d\omega} = \frac{\sqrt{3}q^2}{c} \frac{1}{T_{\text{obs}}} \sum_{k=1}^{Z} \frac{\omega}{3\omega_L \gamma_k} \int_{x_k}^{\infty} d\xi K_{5/3}(\xi)
\]

(13)

where \(x_k = \omega/(3\omega_L \gamma_k^2)\).

Consider now the summation over the phases \(\alpha_i\), we may consider these phases homogeneously distributed between \((0, \alpha_M)\), where \(\alpha_M\) is the maximum value of \(\alpha\) such that particles can illuminate the observer during the observation time \(T_{\text{obs}}\) [recall eq. (8)]:

\[
\alpha_M = \omega_L \gamma_0^{-1} T_{\text{obs}} (T_{\text{obs}} \omega_R + 2).
\]

(14)

Introducing the phase density \(\rho(\alpha) = Z/(2\pi)\), we may pass from a discrete sum to an integral by the substitution

\[
\sum_{k=1}^{Z} \rightarrow \frac{Z}{2\pi} \int_{0}^{\alpha_M} d\alpha.
\]

(15)

In conclusion, the power radiated per unit frequency by an ensemble of \(Z\) particles (all with the same initial Lorentz factor) in the backreaction regime

\(^3\) Here we have used the standard computation of the solid angle integration.
The spectrum found in eq. (16) has some general features that it is worth to investigate. First, let us check that in the limit $T_{\text{obs}} \ll \omega^{-1}$ the spectrum in eq. (16) converges to the standard synchrotron spectrum. This condition guarantees that for small observation times there is no appreciable difference between the well known synchrotron spectrum and the predicted one. It is useful to introduce the variable $\Lambda = \frac{4\omega R}{\omega T_{\text{obs}}}(\omega R T_{\text{obs}} + 2)$ and rewrite the integral over $\alpha$ as an integral over $y = \frac{4\omega R}{\omega_0} \alpha$, so that the power radiated per unit frequency reads

$$\frac{dP}{d\omega} = \frac{\sqrt{3}q^2 Z}{c} \frac{\omega_0}{2\pi} \frac{\omega}{3\omega_L \gamma(\alpha)} \int_{x(\alpha)}^{\infty} d\xi K_{5/3}(\xi).$$

(16)

where we defined

$$F(y) = \gamma(y) x(y) \int_{x(y)}^{\infty} d\xi K_{5/3}(\xi).$$

Now we compute the limit $\Lambda \to 0$ of eq. (17) [note that this corresponds to evaluate the limit for $\omega R T_{\text{obs}} \ll 1$]:

$$\lim_{\Lambda \to 0} \frac{\omega_0}{8\sqrt{1 + \frac{1}{4}\Lambda} - 1} \int_{0}^{\Lambda} dy F(y) = \frac{\omega_L}{\gamma_0} F(0).$$

(18)

Substituting this expression in (16) the standard synchrotron power per unit frequency by an ensemble of $Z$ particles all with the same Lorentz factor is readily recovered:

$$\frac{dP}{d\omega} = Z \frac{\sqrt{3}q^2}{c} \frac{\omega L}{2\pi} x_0 \int_{x_0}^{\infty} d\xi K_{5/3}(\xi).$$

(19)

where $x_0 = \omega/(3\omega_L \gamma_0^2)$.

It is now useful, mainly for practical purposes, to derive analytical approximations for the spectrum of the radiation. We assume first to be in the regime where there are severe modifications due to the backreaction, that
is \((\omega_R T_{\text{obs}} \geq 1)\). To perform the calculation we will adopt the following rough approximation of the Bessel function:

\[
x \int_0^\infty \frac{d\xi}{\xi} K_{5/3}(\xi) = 2^{2/3} \Gamma \left( \frac{2}{3} \right) \begin{cases} x^{1/3} & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}
\]

Although certainly not sophisticated, this approximation allows to treat analytically expressions that would otherwise be only of numerical access. The condition \(x(\alpha) \leq 1\), using equation (9) and the definition of \(x\), implies that

\[
\alpha \leq \gamma_0 \frac{\omega_0}{\omega_R} \left( \frac{\omega}{3\omega_L} \right)^{-1} \left[ \gamma_0 - \left( \frac{\omega}{3\omega_L} \right)^{1/2} \right] = \alpha_0
\]

Therefore, to use the approximation of the Bessel function, we have to perform the integration over \(\alpha\) between \((0, \tilde{\alpha})\) where \(\tilde{\alpha} = \min\{\alpha_M, \alpha_0\}\). The condition \(\alpha_0 = \alpha_M\) determines the frequency where there is a change in the slope of the spectrum of the emitted radiation. This identifies two frequency regimes, the low frequency one \((\tilde{\alpha} = \alpha_M)\) and the high frequency one \((\tilde{\alpha} = \alpha_0)\). The separation between the two regimes occurs at

\[
\omega = 3\omega_L \gamma_0^2 Y(T_{\text{obs}})
\]

where

\[
Y(T_{\text{obs}}) = \left[ \sqrt{1 + 4(\omega_R T_{\text{obs}})^2 + 8(\omega_R T_{\text{obs}}) - 1} \right]^2
\]

and, using the condition of strong backreaction \(\omega_R T_{\text{obs}} \geq 1\), we may approximate

\[
Y(T_{\text{obs}}) \simeq \frac{1}{(\omega_R T_{\text{obs}})^2}.
\]

In the high frequency regime \(3\omega_L \gamma_0^2 Y(T_{\text{obs}}) < \omega < 3\omega_L \gamma_0^2\) the integration over \(\alpha\) has to be performed in the range \((0, \alpha_0)\).

Within the approximations adopted here, we get the following expression for the radiated power:

- **Low frequency**
  \(\omega < 3\omega_L \gamma_0^2 Y(T_{\text{obs}})\)

\[
\frac{dP}{d\omega} = \frac{\sqrt{3} \tilde{q}^2}{c} \frac{Z}{2\pi} \frac{3}{5} 2^{2/3} \Gamma \left( \frac{2}{3} \right) \omega_L (\omega_R T_{\text{obs}})^{2/3} \left( \frac{\omega}{3\omega_L \gamma_0^2} \right)^{1/3}
\]
• High frequency

\[3\omega_L\gamma_0^2 Y(T_{\text{obs}}) < \omega < 3\omega_L\gamma_0^2\]

\[
\frac{dP}{d\omega} = \frac{\sqrt{3}q^2 Z}{c} 2^{2/3} \pi^{2/3} 1^{(2)} \frac{\omega_L}{\omega R T_{\text{obs}}} \left( \frac{\omega}{3\omega_L\gamma_0^2} \right)^{-1/2}.
\]

Eqs. (23,24) are very interesting and allow a simple interpretation of the back-reaction regime. In the low frequency regime the spectrum of the synchrotron radiation has the same slope as in the standard case \((\omega^{1/3})\); on the other hand, moving to high frequency, the spectrum has a new power law behaviour of the type \(\omega^{-1/2}\). Moreover, in the backreaction regime, the maximum of the spectrum is located at a frequency that depends on the observation time according to the following expression:

\[
\omega_{cr}(T_{\text{obs}}) = 3\omega_L\gamma_0^2 Y(T_{\text{obs}}) \simeq \frac{3\omega_L\gamma_0^2}{(\omega R T_{\text{obs}})^2} = \frac{3\omega_L}{(\omega K T_{\text{obs}})^2}.
\]

In other words the maximum gradually moves toward lower frequencies when the observation time is increased and its position does not depend on \(\gamma_0\): the integrated power becomes gradually richer in its low frequency component.

We plotted our results for the radiation spectra in fig. 2 for three cases as indicated: 1) standard case; 2) \(\Lambda = 10^4\) and 3) \(\Lambda = 10^6\). Larger values of \(\Lambda\) correspond to higher levels of backreaction, so that the spectra are gradually peaked at lower frequencies when the observation time increases. For the two cases of backreaction, we also plot our analytical approximations for the low and high frequency regimes. At low frequency the agreement with the detailed calculation is excellent. At higher frequencies clearly the rough approximation adopted for the Bessel functions gives a poorer but still acceptable agreement, very useful for practical estimates. In particular the slopes predicted by the analytical calculations reproduce very well the ones obtained in the detailed calculations.

Note that the plot in fig. 2 is made in such a way that can be applied equally well to electrons and protons as radiating particles and for any value of the magnetic field. All these parameters in fact enter the definition of \(\omega_L\).

4 Synchrotron radiation from particles with a power law spectrum

We assume here to have a spectrum of charged particles in the form

\[N(\gamma_0) = N_0\gamma_0^{-p}\]
Fig. 2. Power spectrum of the radiation emitted by monoenergetic particles for the standard case (no backreaction) and for the cases $\Lambda = 10^4$ and $\Lambda = 10^6$ (The dashed lines represent the results of the analytical approximation illustrated in the text).

where $p > 1$ is a spectral index. For most astrophysical applications this is the relevant case, therefore we explore here in detail what are our predictions for the spectrum of the synchrotron emission. Note that $N(\gamma_0)$ is here what is usually called the equilibrium spectrum of the radiating particles, as derived from a transport equation, usually including energy losses. This spectrum is typically steeper than the injection spectrum. The equilibrium spectrum is here taken to be time independent, meaning that there is a continuous replenishment of the particles at all energies, requiring the source to be active for the all duration of the phenomenon. The case of sources bursting on time scales much shorter than the observation time will not be considered here, but can be easily recovered by generalizing the discussion below.

Starting from eq. (16) we can generalize the calculations of the previous section simply by the substitution $Z \to N(\gamma_0)$ and introducing an integration over the energy spectrum of the particles. The power radiated will be

$$\frac{dP}{d\omega} = \frac{\sqrt{3}q^2}{c} \frac{1}{2\pi T_{\text{obs}}} \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} N_0(\gamma_0 - \omega)^{\alpha - 1} \frac{\omega}{3\omega_L(\gamma(\alpha))} \int_{\infty}^{\infty} d\xi K_{5/3}(\xi)$$

(27)

where $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$ are the minimum and maximum values allowed for the Lorentz factor, fixed by the particular physical system under consideration.
Using the same approximation of the Bessel function introduced in the previous section we will work out from equation (27) some interesting analytical results. Let us first assume that, fixing $T_{\text{obs}}$, for any $\gamma_0$ inside the interval $(\gamma_{\text{min}}, \gamma_{\text{max}})$ the system is in the backreaction regime $\omega R T_{\text{obs}} \geq 1$. Using the high and low frequency results of the previous section (cfr. eqs. (24) and (23)) we may calculate the integral over $\gamma_0$.

Let us start from the high frequency regime, $3\omega_L \gamma_0^2 Y(T_{\text{obs}}) < \omega < 3\omega_L \gamma_0^2$. In this case, fixing the frequency $\omega$ from the condition $\omega < 3\omega_L \gamma_0^2$ one obtains

$$\gamma_0 > \left( \frac{\omega}{3\omega_L} \right)^{1/2} = \tilde{\gamma}$$

therefore, for consistency, the integral over $\gamma_0$ has to be calculated between $\tilde{\gamma}$ and $\gamma_{\text{max}}$. In this case (recalling eq. (24)) the power radiated per unit frequency is

$$\frac{dP}{d\omega} = A \frac{\omega L}{\omega K T_{\text{obs}}} \left( \frac{\omega}{3\omega_L} \right)^{-1/2} \gamma_{\text{max}} \int_{\tilde{\gamma}}^{\gamma_{\text{max}}} d\gamma_0 \gamma_0^{-p}$$

(28)

where $A$ is a numerical factor defined as

$$A = \frac{\sqrt{3} q^2}{2 \pi c} \frac{1}{5} \frac{2^{2/3} \Gamma \left( \frac{2}{3} \right)}{N_0}.$$

Assuming that $\gamma_{\text{max}} \gg \tilde{\gamma}$ we may consider the integral in (28) from $(\tilde{\gamma}, \infty)$ obtaining the high frequency power emitted per unit frequency

$$\left( \frac{dP}{d\omega} \right)_{\text{high}}^{\text{back}} = A \omega L \frac{1}{\omega K T_{\text{obs}}} \frac{1}{p - 1} \left( \frac{\omega}{3\omega_L} \right)^{-p/2}.$$

(29)

Let us now evaluate the power emitted in the low frequency regime $\omega < 3\omega_L Y(T_{\text{obs}})$. Recalling that we are considering the case in which the backreaction is always important by construction, for any $\gamma_0 \in (\gamma_{\text{min}}, \gamma_{\text{max}})$, one can approximate $Y(T_{\text{obs}}) \simeq 1/(\omega R T_{\text{obs}})^2$ (cfr. eq. (22)) and therefore

$$\omega < \frac{3\omega_L}{(\omega K T_{\text{obs}})^2}.$$

In this case there is no condition on the lower extreme in the integral over $\gamma_0$, so that, assuming $\gamma_{\text{min}} \ll \gamma_{\text{max}}$, the low frequency power emitted per unit frequency can be written as

$$\left( \frac{dP}{d\omega} \right)_{\text{low}}^{\text{back}} = A \omega L (\omega K T_{\text{obs}})^{2/3} \gamma_{\text{min}}^{-(p-1)} \frac{1}{p - 1} \left( \frac{\omega}{3\omega_L} \right)^{1/3}.$$

(30)
It is instructive to study the frequency corresponding to the change from the low to the high frequency regime. As we have pointed out before [eq. (22)], this frequency depends on the observation time according to the following expression:

$$\omega_{cr} = 3\omega L\gamma_0^2 Y(T_{obs}) \simeq \frac{3\omega L}{(\omega K T_{obs})^2} .$$

(31)

This expression should be compared with the critical frequency defined within the standard theory of synchrotron emission for the particles with the minimum Lorentz factor, $$\omega_{syn} = 3\omega L\gamma_{min}^2$$. Since we assumed that, for any $$\gamma_0$$, we are in the backreaction regime ($$\omega_K \gamma_{min} T_{obs} \geq 1$$), we have

$$\omega_{cr} \ll \omega_{syn}$$

therefore, the change in the behaviour of the power radiated per unit frequency occurs in the low frequency regime.

We are now ready to discuss a more realistic situation, for which there is a value $$\gamma_c$$ with $$\gamma_{min} < \gamma_c < \gamma_{max}$$ that divides, for any fixed observation time, the standard regime from the backreaction one. This value of $$\gamma_c$$ is simply given by

$$\gamma_c = \frac{1}{\omega_K T_{obs}} .$$

In the situation for which $$\gamma_c \in (\gamma_{min}, \gamma_{max})$$ we can divide the spectrum in two regions: one for frequencies $$\omega < 3\omega L\gamma_c^2$$, in which backreaction is not important, and the other, for $$\omega > 3\omega L\gamma_c^2$$, in which the backreaction effects need to be included.

In the situation in which $$\gamma_c \in (\gamma_{min}, \gamma_{max})$$ the spectrum radiated by particles is

- At high frequencies, $$\omega > 3\omega L\gamma_c^2$$:

$$\frac{dP}{d\omega} = \left(\frac{dP}{d\omega}\right)^{back}_{\text{high}}$$

(32)

- At low frequencies, $$\omega < 3\omega L\gamma_c^2$$:

$$\frac{dP}{d\omega} = \left(\frac{dP}{d\omega}\right)^{syn} + \left(\frac{dP}{d\omega}\right)^{back}_{\text{low}}$$

(33)

where the first term is the standard synchrotron spectrum.
The standard synchrotron spectrum (integrated between $\gamma_{\text{min}}$ and $\gamma_c$) has the following form:

$$
\left( \frac{dP}{d\omega} \right)^{\text{syn}} = N_0 \sqrt{3} q^2 \frac{\omega_L}{c} 2^{2/3} \Gamma \left( \frac{2}{3} \right) \frac{3}{3p - 1} \left[ \left( \frac{\omega}{3\omega_L} \right)^{-\frac{p-1}{2}} - \left( \frac{\omega}{3\omega_L} \right)^{\frac{1}{3} - p + \frac{3}{4}} \gamma_c^{-p + \frac{3}{4}} \right]
$$

(34)

while the backreaction dominated spectrum is described by eqs. (29) and (30) with $\gamma_{\text{min}} = \gamma_c$. Let us now compare the two components of the low frequency spectrum. For this purpose we evaluate explicitly eq. (33)

$$
\left( \frac{dP}{d\omega} \right)^{\text{back}}_{\text{low}} + \left( \frac{dP}{d\omega} \right)^{\text{syn}} = C \cdot \left\{ \frac{3}{3p - 1} \left( \frac{\omega}{3\omega_L} \right)^{-\frac{p-1}{2}} + (\omega_K T_{\text{obs}})^{p-\frac{1}{3}} \left( \frac{\omega}{3\omega_L} \right)^{\frac{1}{3} \left[ \frac{3}{5} \frac{1}{p - 1} - \frac{3}{3p - 1} \right]} \right\}
$$

(35)

where $C$ is a numerical factor

$$
C = N_0 \sqrt{3} q^2 \frac{\omega_L}{c} 2^{2/3} \Gamma \left( \frac{2}{3} \right)
$$

and we have used the relation $\gamma_c = 1/(\omega_K T_{\text{obs}})$.

It is easy to see that at frequencies $\omega \ll \omega_c$ the low frequency spectrum is well described by the standard synchrotron emission (i.e. the second term in (35) is always negligible). Moreover, this is also rigorously true in the special case $p = 2$, as it is easy to show by explicitly calculating the second term in eq. (35).

In conclusion, in the most interesting case in which radiation reaction is efficient only for $\gamma > \gamma_c > \gamma_{\text{min}}$, the power radiated by the ensemble of particles with a fixed (power law) distribution of Lorentz factors is described at low frequency ($\omega < 3\omega_L \gamma_c^2$) by the standard synchrotron spectrum (34) and at high frequency ($\omega > 3\omega_L \gamma_c^2$) by the backreaction modified spectrum in eq. (29).

In fig. 3 we plot for illustrative purposes the results of a specific case: high energy protons with a spectrum $\propto E^{-2}$ and high energy cutoff at $10^{21}$ eV radiating by synchrotron emission in magnetic fields of order 100 Gauss (solid lines) and 1000 Gauss (dashed lines). The curves are obtained for three choices of the observation time $T_{\text{obs}}$. The rightmost curves in both cases correspond to the standard synchrotron emission (small observation time) while moving leftward we plot the cases $T_{\text{obs}} = 10^3$ s and $T_{\text{obs}} = 10^4$ s. The effect of the
Fig. 3. Power spectrum of the radiation emitted by protons with a power law spectrum \( N(\gamma_0) = N_0\gamma_0^{-p} \) for the standard case \((T_{\text{obs}} \to 0)\) and for \(T_{\text{obs}} = 10^3\) s and \(T_{\text{obs}} = 10^4\) s (moving leftward). The two sets of curves are for \(B = 100\) Gauss (solid lines) and \(B = 1000\) Gauss (dashed lines).

increasing backreaction is clear from these curves. This specific case has been chosen because the situation is similar to the one proposed in [5,6] to explain the TeV emission from BL Lac as synchrotron emission of ultra high energy protons (in these papers the standard synchrotron emission was adopted).

5 Conclusions

We calculated the spectrum of the synchrotron emission from a system of charged particles in the most general case in which the observation time is arbitrary compared to the time for the energy losses of the particles. We first calculated the effect for the case of a particle with Lorentz factor \(\gamma_0\).

Our findings on the monoenergetic case can be summarized as follows:

\(i\) There is a critical frequency \(\omega_{cr}\) such that the spectrum of the radiation at \(\omega \leq \omega_{cr}\) is the usual synchrotron spectrum \(\propto \omega^{1/3}\), while for \(\omega \geq \omega_{cr}\) the backreaction affects the spectrum changing it to \(\propto \omega^{-1/2}\).

\(ii\) The critical frequency \(\omega_{cr}\) depends on the observation time but it turns out to be independent of the Lorentz factor of the radiating particles \(\gamma_0\). The
dependence on these parameters is as found in eq. (25), so that increasing the observation time, power is moved to gradually lower frequencies.

iii) The standard limit is recovered when the observation time is much smaller than the time for losses $\omega_R T_{obs} \ll 1$.

The more realistic case investigated in this paper is that of a power law spectrum of particles $N(\gamma_0) \propto \gamma_0^{-p}$. In general what happens is that there may be a critical Lorentz factor $\gamma_c$, such that particles with $\gamma_0 \geq \gamma_c$ are affected by the backreaction while the particles with $\gamma_0 \leq \gamma_c$ behave in the standard way. The spectrum of the radiation in this case is a superposition of different components. Our findings can be summarized as follows:

a) the spectrum of the radiation can be divided into a low frequency one and a high frequency one, with the separation occurring at the frequency $\omega_{cr} = 3\omega_L/(\omega_K T_{obs})^2$.

b) the particles with Lorentz factor $\gamma_0 \leq \gamma_c$ radiate the standard synchrotron radiation whose spectrum is $\propto \omega^{-(p-1)/2}$.

c) the particles with $\gamma_0 \geq \gamma_c$ do radiate in regime of backreaction and affect both the low frequency and the high frequency regime.

d) the low frequency spectrum radiated by particles with $\gamma_0 \geq \gamma_c$ is $\propto \omega^{1/3}$ for $\omega \ll \omega_{cr}$, but it is always dominated, in the same frequency range, by the standard synchrotron radiation.

e) the high frequency radiation radiated by particles with $\gamma_0 \geq \gamma_c$ is $\propto \omega^{-p/2}$. Therefore it represents a suppression of the radiation compared to the case of standard synchrotron radiation. Note that increasing the observation time, while the slopes at low and high frequency remain unchanged, the boundary between the two regimes moves toward lower frequencies, so that, as a consequence, the height of the spectrum at high frequencies becomes increasingly lower.

f) At fixed observation time (which is obviously decided by the observer) the critical frequency only depends on the magnetic field in the production region. This is a very important point: in the standard synchrotron emission, it is in general not possible to extract the magnetic field from a measurement of the synchrotron flux because there is degeneracy between the number of radiating particles and the strength of the magnetic field. In the backreaction case however, the position of the change in slope uniquely defines the magnetic field, so that the measurement can in principle be used to directly infer the strength of the magnetic field.

The crucial question is whether there are situations in which the conditions
for the backreaction to be relevant are fulfilled. The answer can be found in fig. 1, where we plotted the regions of interest for both electrons and protons in the plane $B - E$. At each magnetic field there corresponds a range of energies for which the backreaction is important. We immediately see that for conditions typical in the Galaxy, $B \sim 1 \mu G$ only electrons with $E \gtrsim 10^7$ GeV feel the effects of the backreaction. On the other hand in magnetic fields which are typical of neutron stars, $B \sim 10^{10} - 10^{12}$ Gauss, electrons with energies in excess of MeV-GeV already need to be accounted for in the frame of a backreaction approach. In the same environment, protons with energies larger than $10^5 - 10^6$ GeV also radiate in a regime in which backreaction is important.

Some applications of the calculations and results illustrated in this paper will be presented in forthcoming papers. They include: the synchrotron emission from ultra-high energy electron-positron pairs generated as a result of the decay of super-heavy relics [7] in the Galaxy [8] or in the decay of the $Z^0$ resonance produced in $\nu\bar{\nu}$ annihilation ($Z$-burst model [9,10]). In this case we have magnetic fields $B \sim 1 \mu G$ and energies in excess of $10^{10}$ GeV, so that the synchrotron radiation is strongly affected by the backreaction.

A more conventional application concerns the TeV gamma ray emission from BL Lac objects. In [5,6] a proton synchrotron model was proposed in which the TeV emission is the result of the synchrotron emission of ultra high energy protons ($E \sim 10^{19}$ eV) in a magnetic field of order $\sim 100$ Gauss. We estimate that with these parameters and for observation times of the order of the duration of the observed flares (a few hours) the backreaction affects visibly the TeV gamma ray spectra [11].

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