THE NON-COMMUTATIVE SCHEME HAVING A FREE ALGEBRA AS A HOMOGENEOUS COORDINATE RING

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ABSTRACT. Let \( k \) be a field and \( TV \) the tensor algebra on a finite-dimensional \( k \)-vector space \( V \). This paper proves that the quotient category \( \text{QGr}(TV) := \text{Gr}(TV)/\text{Fdim} \) of graded \( TV \)-modules modulo those that are unions of finite dimensional modules is equivalent to the category of modules over the direct limit of matrix algebras, \( \lim_{\to} M_n(k)^{\otimes r} \). Non-commutative algebraic geometry associates to a graded algebra \( A \) a “non-commutative scheme” \( \text{Proj}_{nc} A \) that is defined implicitly by declaring that the category of “quasi-coherent sheaves” on \( \text{Proj}_{nc} A \) is \( \text{QGr} A \). When \( A \) is coherent and \( \text{gr} A \) its category of finitely presented graded modules, \( \text{qgr} A := \text{gr} A/\text{Fdim} \) is viewed as the category of “coherent sheaves” on \( \text{Proj}_{nc} A \). We show that when \( \dim V \geq 2 \), \( \text{qgr}(TV) \) has no indecomposable objects, no noetherian objects, and no simple objects. Moreover, every short exact sequence in \( \text{qgr}(TV) \) splits.

We also prove \( \text{QGr}(TV) \equiv \text{Gr} L \) where \( L \) is the Leavitt algebra on \( 2 \dim V \) generators that embeds as a dense subalgebra of the Cuntz algebra \( O_{\dim V} \).

1. INTRODUCTION

1.1. Let \( n \) be a positive integer.

Throughout this paper \( k \) is a field and

\[
R := k\langle x_0, x_1, \ldots, x_n \rangle
\]

is the free algebra on \( n + 1 \) variables with \( \mathbb{Z} \)-grading given by declaring that \( \deg x_i = 1 \) for all \( i \). This paper concerns the categories of coherent and quasi-coherent “sheaves” on the “non-commutative scheme”

\[
X^n := \text{Proj}_{nc} k\langle x_0, x_1, \ldots, x_n \rangle
\]

with “homogeneous coordinate ring” \( R \).

The “scheme” \( \text{Proj}_{nc} R \) is an imaginary object: there is no underlying topological space endowed with a sheaf of rings. Rather one declares that the category of “quasi-coherent sheaves” on \( \text{Proj}_{nc} R \) is the quotient category

\[
\text{Qcoh}(\text{Proj}_{nc} R) := \text{QGr} R := \frac{\text{Gr} R}{\text{Fdim} R}
\]

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where $\text{Gr} R$ is the category of $\mathbb{Z}$-graded left $R$-modules with degree-preserving homomorphisms and $\text{Fdim} R$ its full subcategory of direct limits of finite-dimensional modules. The imaginary space $\text{Proj}_{nc} R$ manifests itself through the category $\text{Qcoh}(\text{Proj}_{nc} R)$.

The category $\text{QGr} R$ and its full subcategory of finitely presented objects, $\text{qgr} R$, is the focus of this paper. We think of $\text{qgr} R$ as the category of “coherent sheaves” on $\text{Proj}_{nc} R$.

1.2. The notations $\text{QGr} R$ and $\text{Qcoh} X^n$ are interchangeable as are the notations $\text{qgr} R$ and $\text{coh} X^n$. The reader may adopt either so as to reinforce either an algebraic or geometric perspective.

1.3. We write $\pi^* : \text{Gr} R \to \text{QGr} R$ for the quotient functor and define $\mathcal{O} := \pi^* R$. We call $\mathcal{O}$ a structure sheaf for $\text{Proj}_{nc} R$.

For the commutative polynomial ring $k[x_0, \ldots, x_n]$, $\text{QGr} k[x_0, \ldots, x_n]$ is equivalent to $\text{Qcoh} \mathbb{P}^n$ and $\pi^* \text{ is } M \mapsto \tilde{M}$ in algebraic geometry texts and $\pi^* (k[x_0, \ldots, x_n])$ is the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$.

1.4. The twist functor on $\text{Gr} R$, denoted $M \rightsquigarrow M(n)$, is defined by $M(n) := M_{n+1}$ with the same action of $R$. The subcategory $\text{Fdim} R$ is stable under twisting so there is an induced functor on $\text{QGr} R$ that we denote by $\mathcal{F} \rightsquigarrow \mathcal{F}(n)$ and call the Serre twist.

1.5. The main results. We define the direct limit algebra

$$S := \lim_{\longrightarrow} M_{n+1}(k)^{\otimes i}$$

where the maps in the directed system are $a_1 \otimes \cdots \otimes a_i \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_i$. The ring $S$ is coherent so finitely presented $S$-modules form an abelian category.

**Theorem 1.1.** There is an equivalence of categories

$$\text{Hom}_{\text{QGr} R}(\mathcal{O}, -) : \text{QGr} R \equiv \text{Mod}_r S,$$

the category of right $S$-modules. The equivalence sends $\mathcal{O}$ to $S$, i.e., $S = \text{Hom}_{\text{QGr} R}(\mathcal{O}, \mathcal{O})$. Furthermore, the equivalence restricts to an equivalence

$$\text{qgr} R \equiv \text{mod}_r S,$$

the category of finitely presented right $S$-modules.

The ring $S$ is anti-isomorphic to itself so it doesn’t really matter whether we choose to work with left or right $S$-modules.

The key to Theorem 1.1 is the following preliminary result.

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1An object $\mathcal{M}$ in an additive category $A$ is finitely presented if $\text{Hom}_A(\mathcal{M}, -)$ commutes with direct limits; is finitely generated if whenever $\mathcal{M} = \sum M_i$ for some directed family of subobjects $M_i$ there is an index $j$ such that $\mathcal{M} = M_j$; is coherent if it is finitely presented and all its finitely generated subobjects are finitely presented.
Theorem 1.2. \( \mathcal{O} \) is a finitely generated, projective, generator in \( \text{Qcoh} \mathbb{X}^n \).

As the next result emphasizes, the categories \( \text{coh} \mathbb{X}^n \) and \( \text{Qcoh} \mathbb{X}^n \) are unlike the categories of coherent and quasi-coherent sheaves over quasi-projective schemes.

Theorem 1.3. Suppose \( n \geq 1 \).

1. There are no indecomposable objects in \( \text{coh} \mathbb{X}^n \), hence no simple objects, and therefore no noetherian objects other than 0.
2. Every short exact sequence in \( \text{coh} \mathbb{X}^n \) splits.
3. Every object in \( \text{coh} \mathbb{X}^n \) is isomorphic to a finite direct sum of various \( \mathcal{O}(i) \)s with finite multiplicities.
4. If \( \mathcal{F}, \mathcal{G} \in \text{coh} \mathbb{X}^n \) are non-zero, then \( \dim_k \text{Hom}_{\mathbb{X}^n}(\mathcal{F}, \mathcal{G}) = \infty \).
5. The Grothendieck group of the abelian category \( \text{coh} \mathbb{X}^n \) is isomorphic to \( \mathbb{Z}(n+1)^\ast \) as an additive group.

The reason \( \text{coh} \mathbb{X}^n \) and \( \text{Qcoh} \mathbb{X}^n \) behave so differently from categories of (quasi-)coherent sheaves on quasi-projective schemes is that \( \mathcal{O}(-1) \) is a non-trivial direct summand of \( \mathcal{O} \). Since \( R_{\geq 1} \) is isomorphic to \( R(-1)^{[n+1]} \),
\[
\mathcal{O} \cong \mathcal{O}(-1)^{[n+1]}.
\]

This behavior also occurs for the graded algebras \( k\langle x, y \rangle / (y^{n+1}) \) in [15].

Section 2 of [15] establishes some general results that are applied in the present paper to the free algebra: almost everything in the present paper is a rather simple consequence of those more general results.

1.6. Connection to Leavitt algebras and Cuntz algebras. Let \( L \) be the Leavitt algebra generated by the entries in the row vectors \( x = (x_0, \ldots, x_n) \) and \( x^* = (x_0^*, \ldots, x_n^*) \) subject to the relations
\[
x^* x^T = 1 \quad \text{and} \quad x^T x^* = I_{n+1},
\]
the \((n+1) \times (n+1)\) identity matrix. Give \( L \) a \( \mathbb{Z} \)-grading by \( \deg x_i = 1 \) and \( \deg x_i^* = -1 \). The connection between \( \text{QGr} R \) and \( L \) is made in the following result which is proved in section 4.

Theorem 1.4. The algebra \( L \) is strongly graded, \( L_0 \cong S \), and there is an equivalence of categories
\[
\text{QGr} R \cong \text{Gr} L \cong \text{Mod} L_0.
\]

The proof makes use of the following facts: \( L \) is the universal localization of \( R \) that inverts the homomorphism \( R(-1)^{n+1} \rightarrow R \) whose cokernel is \( R/R_{\geq 1} \); \( L \) is flat as a right \( R \)-module; if \( M \in \text{Gr} R \) and \( L \otimes_R M = 0 \), then \( M \in \text{Fdim} R \). The first of these facts is well-known; the second is proved in [1]; a version of the third for modules in \( \text{gr} R \) is proved in [1].

In [5], Cuntz defined a class of C*-algebras \( O_{n+1}, n \geq 1 \), generated by the “same” elements and relations for \( L \). The algebra \( L \) is a dense subalgebra of \( O_{n+1} \).
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2. Coherent sheaves on Proj\textsubscript{nc} TV

Let $V$ be a $k$-vector space of dimension $d = n + 1$ and

$$R := TV = k \langle x_0, \ldots, x_n \rangle$$

be the tensor algebra on $V$ with $\mathbb{Z}$-grading $R_i := V^{\otimes i}$.

2.1. Graded modules over the free algebra. A ring is left coherent if all its finitely generated left ideals are finitely presented. Since $TV$ is anti-isomorphic to itself we can dispense with the adjectives left and right when discussing properties of $TV$ like coherence.

The important property for us, indeed an equivalent characterization of coherence, is that the category of finitely presented left modules over a left coherent ring is abelian.

The following facts are well-known.

**Proposition 2.1.** Let $V$ be a vector space of finite dimension $d \geq 1$.

1. Every left ideal in $TV$ is free.
2. $TV$ is coherent and has global dimension one.
3. Every finitely generated projective $R$-module is free.
4. If $d \geq 2$, $TV$ has exponential growth: $H(R,t) = (1 - dt)^{-1}$.
5. If $d \geq 2$, $TV$ is not noetherian.
6. $R_{\geq i}$ is isomorphic to $R(-i)^{dt}$, the free $R$-module of rank $d^i$ with basis in degree $i$.

Suppose $N$ is a graded $R$-module. There is an exact sequence $L \to M \to N \to 0$ in $\text{Mod}R$ with $L$ and $M$ finitely generated if and only if there is an exact sequence $L' \to M' \to N \to 0$ in $\text{Gr}R$ with $L'$ and $M'$ generated by a finite number of homogeneous elements.

We define

$$\text{gr}R := \text{the category of finitely presented graded } R\text{-modules}.$$ 

This is an abelian category because $R$ is coherent.

**Proposition 2.2.** Let $R = TV$ and $M \in \text{QGr} R$. 

(1) M is graded-coherent if and only if for all \( i \gg 0 \),
\[
M_{\geq i} \cong R(-i)^{t_i}
\]
for some integer \( t_i \) depending on \( i \).

(2) If \( 0 \to L \to M \to N \to 0 \) is an exact sequence in \( \text{gr} \; R \), then \( 0 \to L_{\geq i} \to M_{\geq i} \to N_{\geq i} \to 0 \) splits for \( i \gg 0 \).

(3) If \( M \in \text{gr} \; R \), then \( M \) has a largest finite dimensional graded submodule.

**Proof.** (1) Suppose \( M \) is finitely presented. Then there is an exact sequence \( 0 \to F' \to F \to M \to 0 \) in \( \text{gr} \; R \) with \( F \) and \( F' \) finitely generated graded free \( R \)-modules. Since \( F', F, \) and \( M \), are finitely generated graded modules, \( F_{\geq i} \), \( F_{\geq i} \), and \( M_{\geq i} \), are generated as \( R \)-modules by \( F_{i} \), \( F_{i} \), and \( M_{i} \), respectively for all sufficiently large \( i \). But \( R_{\geq m} \cong R(-m)^{d_{m}} \) so
\[
(0 \to F' \\ F \to M \to 0)_{\geq i} = (0 \to R(-i)^{r} \to R(-i)^{s} \to M_{\geq i} \to 0)
\]
for \( i \gg 0 \). Every degree-zero homomorphism \( R(-i)^{r} \to R(-i)^{s} \) splits so \( M_{\geq i} \cong R(-i)^{s-r} \).

The converse is trivial.

(2) By (1), \( N_{\geq i} \) is free for \( i \gg 0 \), hence the splitting.

(3) Since \( R \) is a domain its only finite dimensional submodule is zero. It now follows from (1) that the only finite dimensional submodule of \( M_{\geq i} \) is the zero submodule for \( i \gg 0 \). There is therefore an integer \( n \) such that every finite dimensional submodule of \( M \) is contained in \( \sum_{j=-i}^{i} M_{j} \). But \( M \) is finitely generated so \( \sum_{j=-i}^{i} M_{j} \) has finite dimension. Hence the sum of all finite dimensional submodules of \( M \) has finite dimension, and that sum is therefore the largest finite dimensional graded submodule of \( M \). \( \square \)

2.2. We write \( \text{fdim} \; R \) for \( \text{Fdim} \; R \cap \text{gr} \; R \). Thus \( \text{fdim} \; R \) is the full subcategory of \( \text{Gr} \; R \) consisting of the finite dimensional submodules. We define the category of “coherent sheaves” on \( \mathbb{X}^{n} \) by
\[
\text{coh} \; \mathbb{X}^{n} = \text{qgr} \; R := \frac{\text{gr} \; R}{\text{fdim} \; R}.
\]
Since \( \text{Fdim} \) satisfies condition (2) of \( \text{[S]} \) Prop. A.4, p. 113 with respect to the Serre subcategory \( \text{fdim} \) of \( \text{gr} \; R \), \( \text{Fdim} \) is localizing of finite type which then allows us to apply \( \text{[S]} \) Prop. A.5, p. 113 and so conclude that \( \text{coh} \; \mathbb{X}^{n} \) consists of finitely presented objects in \( \text{Qcoh} \; \mathbb{X}^{n} \) and every object in \( \text{Qcoh} \; \mathbb{X}^{n} \) is a direct limit of objects in \( \text{coh} \; \mathbb{X}^{n} \).

**Proposition 2.3.** Every short exact sequence in \( \text{qgr} \; R \) splits.

**Proof.** By \( \text{[S]} \) Cor. 1, p. 368, every short exact sequence in \( \text{qgr} \; R \) is of the form
\[
0 \to \pi^{*} L \xrightarrow{\pi^{*} f} \pi^{*} M \xrightarrow{\pi^{*} g} \pi^{*} N \to 0
\]
where \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \) is an exact sequence in \( \text{gr} \; R \). But
\[
\pi^{*}(0 \to L \to M \to N \to 0) \cong \pi^{*}(0 \to L_{\geq i} \to M_{\geq i} \to N_{\geq i} \to 0)
\]
for all \( n \) and \( N_{\geq i} \) is free for \( i \gg 0 \) so the sequence \( 0 \to L_{\geq i} \to M_{\geq i} \to N_{\geq i} \to 0 \) splits for \( i \gg 0 \). Applying \( \pi^* \) to a split exact sequence yields a split exact sequence, whence the result. \( \square \)

**Corollary 2.4.** Let \( \mathcal{F} \in \mathfrak{gr} R \). If \( i \gg 0 \) there is an integer \( r \), depending on \( i \), such that

\[
\mathcal{F} \cong O(i)^r.
\]

**Proof.** There is an \( M \in \mathfrak{gr} R \) such that \( \mathcal{F} = \pi^* M \). But \( \dim_k M/M_{\geq i} < \infty \) so \( \pi^* M \cong \pi^* M_{\geq i} \) for all \( i \gg 0 \). Now apply Proposition 2.2(1). \( \square \)

**Corollary 2.5.** Every object in \( \mathfrak{gr} R \) is injective and projective. Furthermore, \( O \) is projective as an object in \( \mathbb{QGr} R \).

**Proof.** It follows from Proposition 2.3 that every object in \( \mathfrak{gr} TV \) is injective and projective in \( \mathfrak{gr} TV \). Since every left ideal of \( R \) is free it is a tautology that every graded left ideal that has finite codimension in \( R \) contains a free graded left ideal that has finite codimension in \( R \). Hence [15, Prop. 2.7] applies and tells us that \( O \) is projective in \( \mathbb{QGr} R \). \( \square \)

2.3. Since \( R_{\geq i} \cong R(-i)^{\oplus (\dim V)_i} \) has finite codimension in \( R \), [15] Prop. 2.7 and Thm. 2.8 may be applied to yield the next two results.

See Footnote 1 for the definition of finitely generated, finitely presented, and coherent, objects in an abelian category.

**Proposition 2.6.** Let \( A \) be a left graded-coherent ring and write \( O \) for the image of \( A \) in \( \mathbb{QGr} A \). Then

(1) \( \mathbb{QGr} A \) is a locally coherent category;
(2) the full subcategory of \( \mathbb{QGr} A \) consisting of the finitely presented objects is equivalent to \( \mathfrak{gr} A \);
(3) \( O \) is coherent: i.e., \( \text{Hom}_{\mathbb{QGr} A}(O,-) \) commutes with direct limits;
(4) \( O \) is finitely generated.

**Theorem 2.7.**

(1) \( O \) is a progenerator in \( \mathbb{QGr} R \).
(2) The functor \( \text{Hom}(O, -) \) is an equivalence from the category \( \mathbb{QGr} R \) to the category of right modules over the endomorphism ring \( \text{End}_{\mathbb{QGr} R} O \).
(3) \( \text{End}_{\mathbb{QGr} R} O \cong \lim_{\longrightarrow} \text{End}_{\mathbb{Gr} R}(R_{\geq i}) \), the direct limit of the directed system

\[
\cdots \longrightarrow \text{End}_{\mathbb{Gr} R}(R_{\geq i}) \overset{\theta_i}{\longrightarrow} \text{End}_{\mathbb{Gr} R}(R_{\geq i+1}) \longrightarrow \cdots
\]

of \( k \)-algebras in which \( \theta_i(f) = f|_{R_{\geq i+1}} \).

We will determine this direct limit in section 3.
Lemma 2.8. If \( \dim V = d \geq 1 \), then
\[
\mathcal{O} \cong \mathcal{O}(-1)^{\oplus d} \cong \mathcal{O}(-2)^{\oplus d^2} \cong \cdots
\]

Proof. From the exact sequence \( 0 \to R_{\geq 1} \to R \to k \to 0 \) we see that \( \pi^* R \cong \pi^* R_{\geq 1} \). But \( R_{\geq 1} \cong R \otimes_k V \cong R(-1)^d \) so \( \pi^* R_{\geq 1} \cong O(-1)^d \). Hence \( O = \pi^* R \cong O(-1)^d \). The result now follows by induction. \( \square \)

Lemma 2.8 implies that for every \( F \in \text{qgr} R \) and every \( r \geq 1 \),
\[
F \cong F(-r)^{\oplus d^r}.
\]

Corollary 2.9. Let \( d = \dim V \geq 1 \). If \( F \in \text{qgr} R \) is non-zero, then for all integers \( r \geq 0 \) there is an injective ring homomorphism
\[
M_{d^r}(k) \to \text{End} F.
\]

Corollary 2.10. Suppose \( \dim V \geq 2 \). Then
(1) the only noetherian object in \( \text{qgr} R \) is the zero object;
(2) there are no simple objects in \( \text{qgr} R \);
(3) there are no indecomposable objects in \( \text{qgr} R \);
(4) \( \dim \text{Hom}_{\text{QGr}}(F, G) = \infty \) for all non-zero objects \( F \) and \( G \) in \( \text{qgr} R \).

Proof. (2) By Schur’s lemma the endomorphism ring of a simple object is a division algebra but Corollary 2.9 says that endomorphism rings of objects in \( \text{coh} \mathbb{X}^n \) are never division rings when \( \dim V \geq 2 \). Hence \( \text{coh} \mathbb{X}^n \) has no simple objects. Part (1) follows immediately because a non-zero noetherian object has at least one maximal subobject and hence a simple quotient.

(3) See the remark after Lemma 2.8.

(4) Since \( \text{Hom}_{\mathbb{X}^n}(F, G) \) is an \( \text{End} G \)-\text{End} \( F \)-bimodule it is a module over the matrix algebra \( M_{d^r}(k) \) for all \( r \neq 0 \). It therefore suffices to show that \( \text{Hom}_{\mathbb{X}^n}(F, G) \) is non-zero. By Lemma 2.8 there is a monic map \( O(-1) \to O \) and an epic map \( O \to O(-1) \). Because the twist (1) is an auto-equivalence it follows (using compositions of twists of the monic and epic maps just mentioned) that there is a monic map \( O(i) \to O(j) \) whenever \( i \leq j \) and an epic map \( O(j) \to O(i) \) whenever \( j \geq i \). It now follows from Corollary 2.4 that \( \text{Hom}_{\mathbb{X}^n}(F, G) \neq 0 \). \( \square \)

3. A DIRECT LIMIT OF MATRIX ALGEBRAS

We will now show that the endomorphism ring of \( O \) is isomorphic to the ring \( S \) we are about to define.

Let \( S_i := M_{n+1}(k)^{\otimes i} \) and define
\[
(3-1) \quad \theta_i : S_i \to S_{i+1} = M_{n+1}(k) \otimes S_i \quad \text{by} \quad \theta_i(a) = 1 \otimes a.
\]
The homomorphisms \( \theta_i \) determine a directed system and we define
\[
S := \varinjlim_i S_i.
\]
We write \( \text{Mod} S \) for the category of right \( S \)-modules and \( \text{mod} S \) for its full category of finitely presented \( S \)-modules.

**Theorem 3.1.** If \( S \) is the ring above, then \( \text{End} \mathcal{O} \cong S \), the functor \( \text{Hom}_{\mathcal{O}}(\mathcal{O},-) \) is an equivalence of categories

\[
\text{Qcoh} X^n \cong \text{Mod} S
\]

sending \( \mathcal{O} \) to \( S \), and \( \text{Hom}_{\mathcal{O}}(\mathcal{O},-) \) restricts to an equivalence

\[
\text{coh} X^n \cong \text{mod} S.
\]

**Proof.** Write \( R = TV \) as in section 2. By the definition of morphisms in a quotient category,

\[
(3-2) \quad \text{End} X^n \mathcal{O} = \text{Hom}_{\text{Gr} R}(\pi^* R, \pi^* R) = \lim_{\rightarrow} \text{Hom}_{\text{Gr} R}(R', R/R'')
\]

where \( R' \) runs over all graded left ideals in \( R \) such that \( \dim_k (R/R') < \infty \) and \( R'' \) runs over all graded left ideals in \( R \) such that \( \dim_k R'' < \infty \).

By Theorem 2.7 (see [15, Sect. 2] for a fuller explanation), this reduces to

\[
\text{End}_{\text{Gr} R} \mathcal{O} = \lim_{\rightarrow} \text{Hom}_{\text{Gr} R}(R_{\geq i}, R_{\geq i}).
\]

As a left \( R \)-module, \( R_{\geq i} \cong R \otimes_k V \otimes i \) where \( V \) is placed in degree 1. There is a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_k(V \otimes i, V \otimes i) & \xrightarrow{\psi_i} & \text{Hom}_k(V \otimes i+1, V \otimes i+1) \\
\rho_i \downarrow & & \rho_{i+1} \downarrow \\
\text{Hom}_{\text{Gr} R}(R_{\geq i}, R_{\geq i}) & \xrightarrow{\theta_i} & \text{Hom}_{\text{Gr} R}(R_{\geq i+1}, R_{\geq i+1})
\end{array}
\]

in which

\[
\rho_i(f)(r \otimes v) = rf(v) \quad \text{for all } r \in R \text{ and } v \in V \otimes i
\]

and

\[
\psi_i(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_i) = a_0 \otimes f(a_1 \otimes \cdots \otimes a_i).
\]

But \( \rho_i \) is an isomorphism so

\[
\text{End} X^n \mathcal{O} \cong \lim_{\rightarrow} \text{End}_k V \otimes i \cong \lim_{\rightarrow} M_{n+1}(k) \otimes i.
\]

The proof is complete. \( \square \)

**Proposition 3.2.**

1. \( S \) is a simple ring;
2. \( S \) has no finite dimensional modules other than 0.
3. Every finitely generated left ideal of \( S \) is generated by an idempotent.
4. \( S \) is a von Neumann regular ring.
5. \( S \) is left and right coherent.
6. Every left \( S \)-module is flat.
(7) Every finitely generated left $S$-module is projective.

(8) $K_0(S) \cong \mathbb{Z}\left[\frac{1}{n+1}\right]$ via an isomorphism sending $[S]$ to 1.

**Proof.** (8) Since $K_0(-)$ commutes with direct limits and $K_0(S_i) \cong \mathbb{Z}$ with $[S_i] = n + 1$ under the isomorphism, $K_0(S)$ is the direct limit of the directed system

$$\cdots \rightarrow \mathbb{Z}^{\frac{n+1}{n+1}} \rightarrow \mathbb{Z}^{\frac{n+1}{n+1}} \rightarrow \cdots .$$

This direct limit is obviously isomorphic to $\mathbb{Z}\left[\frac{1}{n+1}\right]$. □

It seems worthwhile to determine the Grothendieck group of $\text{qgr } R$ independently of the equivalence of categories, i.e., without appealing to part (8) of Proposition 3.2. The next result does this.

**Proposition 3.3.** As an additive group, the Grothendieck group of $\text{coh } \mathbb{X}^n$ is isomorphic to $\mathbb{Z}\left[\frac{1}{n+1}\right]$ with $[O(i)] \leftrightarrow (n + 1)^i$.

**Proof.** Since $\text{fdim } R$ is a Serre subcategory of $\text{gr } R$ there is an exact sequence

$$K_0(\text{fdim } R) \rightarrow K_0(\text{gr } R) \rightarrow K_0(\text{coh } \mathbb{X}^n) \rightarrow 0.$$  

It is clear that $K_0(\text{gr } R) \cong \mathbb{Z}[t^{\pm 1}]$ via $R(-i) \leftrightarrow t^i$. From the exact sequence $0 \rightarrow R(-1)^{n+1} \rightarrow R \rightarrow k \rightarrow 0$ we obtain $[k] = [R] - (n + 1)[R(-1)] = 1 - (n + 1)t$. Modules in $\text{fdim } R$ are finite dimensional so have compositions series in which the composition factors are of the form $k(i)$ for various integers $i$. Hence, by dévissage, $K_0(\text{fdim } R) \cong K_0(\text{gr } k)$ which is isomorphic as an additive group to $\mathbb{Z}[t^{\pm 1}]$ with $k(i) \leftrightarrow t^{-i}$. The image of the map $K_0(\text{fdim } R) \rightarrow K_0(\text{gr } R)$ is therefore the ideal in $\mathbb{Z}[t^{\pm 1}]$ generated by the image of $[k]$. Therefore

$$K_0(\text{coh } \mathbb{X}^n) \cong \frac{\mathbb{Z}[t]}{(1 - (n + 1)t)} \cong \mathbb{Z}\left[\frac{1}{n+1}\right].$$

This completes the proof. □

**Corollary 3.4.** If $m \neq n$, then $\mathbb{X}^m \not\cong \mathbb{X}^n$.

It is reasonable to define $\text{rank } O(i) = (n + 1)^i$.

4. The relation to the Leavitt algebra $L(1, n+1)$ and the Cuntz algebra $O_{n+1}$

As before $R$ is the free algebra $k\langle x_0, \ldots, x_n \rangle$.

The main result in this section is that $\text{QGr } R \equiv \text{Gr } L \equiv \text{Mod } L_0$ where $L = L(1, n+1)$ is the finitely generated $\mathbb{Z}$-graded algebra defined below. It is well-known that $L_0$ is the algebra we have called $S$ in the earlier part of this paper.
4.1. The Leavitt algebra. The Leavitt algebra $L = L(1, n+1)$, first defined in [3], is the $k$-algebra generated by elements $x_0, \ldots, x_n, x_0^*, \ldots, x_n^*$ subject to the relations

$$(4-1) \quad x_i x_i^* = 1 = x_0^* x_0 + \cdots + x_n^* x_n \quad \text{and} \quad x_i x_j^* = 0 \text{ if } i \neq j.$$ 

A more meaningful definition is that $L$ is the universal localization [3, Sect. 7.2] of $R$ inverting the injective homomorphism

$$\iota: R^{n+1} \to R, \quad (r_0, \ldots, r_n) \mapsto r_0 x_0 + \cdots + r_n x_n.$$ 

Since $\iota$ is right multiplication by $(x_0, \ldots, x_n)^T$, the formal inverse of $\iota$ is right multiplication by $(x_0^*, \ldots, x_n^*)$ where

$$(x_0, \ldots, x_n)^T (x_0^*, \ldots, x_n^*) = I_{n+1} \quad \text{and} \quad (x_0^*, \ldots, x_n^*)(x_0, \ldots, x_n)^T = 1$$

and $I_{n+1}$ is the identity matrix.

4.2. The Cuntz algebra. The Cuntz algebra $O_{n+1}$ [5] is the universal $C^*$-algebra generated by elements $x_0, \ldots, x_n$ subject to the relations \[4-1\]. It is well-known that $L$ embeds in $O_{n+1}$ as a dense subalgebra. Much of the work in Cuntz’s paper [5] involves purely algebraic calculations carried out inside $L$.

4.3. One anticipates a relation between $\text{Gr}L$ and $\text{QGr}R$ because the fact that $\text{id}_L \otimes \iota$ is an isomorphism implies that

$$0 = L \otimes_R \text{coker}(\iota) = L \otimes_R (R/R_{\geq 1}).$$

It follows that $L \otimes_R -$ kills all finite dimensional graded $R$-modules, and hence all modules in $\text{Fdim}R$.

4.4. We make $L$ a $\mathbb{Z}$-graded algebra by defining $\text{deg} x_i = 1$ and $\text{deg} x_i^* = -1$ for all $i$. The canonical map $R \to L$ is a homomorphism of graded rings and is injective. It is well-known, and not hard, to show that if $r > 0$, then $L_r = x_0^r L_0$ and $L_{-r} = L_0 (x_0^*)^r$ (see, e.g., [5 Sect. 1.6]). It follows from this that $L$ is strongly graded, i.e., $L_j L_{-j} = L_0$ for all integers $j$, and therefore

$$\text{Gr}L \equiv \text{Mod}L_0$$

where the functor giving the equivalence sends a graded module $M$ to its degree-zero component $M_0$.

**Proposition 4.1** (Cuntz). [5 Prop. 1.4] $L_0 \cong S$.

4.5. For $r \geq 1$, let $X_r$ be the set of words of length $r$ in the letters $x_i$, $0 \leq i \leq n$, and $X_\infty$ the union of all $X_r$, $r \geq 0$. Cuntz [5 Lem. 1.3] shows that $L = \text{span}\{w^* w' \mid w, w' \in X_\infty\}$. 
4.6. The next result was proved in [1 Prop. 2.1] but its utility is such that it seems useful to give a more direct proof.

**Proposition 4.2.** Let \( R = k\langle x_0, \ldots, x_n \rangle \) and define \( L \) as above. The ring \( L \) is flat as a right \( R \)-module.

**Proof.** We will show \( L \) is an ascending union of finitely generated free right \( R \)-modules. Let

\[
F_r = \sum_{w \in X_r} w^* R.
\]

Suppose

\[
\sum_{w \in X_r} w^* r_w = 0
\]

for some elements \( r_w \in R \). Let \( z \in X_r \). Then \( zw^* = \delta_{w,z} \) so \( r_z = 0 \). It follows that all the \( r_w \)s are zero and the sum in (4-2) is therefore a direct sum. Because \( ww^* = 1 \), each \( w^* R \) is a free \( R \)-module. Hence \( F_r \) is free.

Since \( w^* = \sum_{i \in I} w^* x_i' x_i \), \( F_r \subset F_{r+1} \). Since \( L \) is spanned by elements \( w^* w' \), \( L \) is the ascending union of the \( F_r \)s and therefore flat. □

A version of the following result for finitely presented not-necessarily-graded modules is given in [1 Thm. 5.1]. Our proof differs in spirit from that in [1].

**Proposition 4.3.** Let \( R \) and \( L \) be as above and \( M \in \text{Gr} R \). Then \( L \otimes_R M = 0 \) if and only if \( M \in \text{Fdim} R \).

**Proof.** We observed in section 4.3 that \( L \otimes_R M = 0 \) if \( M \in \text{Fdim} R \).

To prove the converse suppose \( L \otimes_R M = 0 \). First we will show \( M \) is finite dimensional under the additional hypothesis that it is finitely presented. By Proposition 2.2, \( M_{\geq i} \) is a free \( R \)-module for \( i \gg 0 \). If \( M_{\geq i} \) is a non-zero free module, then \( L \otimes_R M \) would contain a non-zero free \( L \)-module. This does not happen because \( L \otimes_R M = 0 \) so we deduce that \( M_{\geq i} = 0 \) for \( i \gg 0 \). Since \( M \) is finitely generated it is therefore finite dimensional.

Now let \( M \) be an arbitrary graded \( R \)-module such that \( L \otimes_R M = 0 \). To prove the proposition it suffices to show that \( \dim_k Rm < \infty \) for all homogeneous \( m \in M \). Let \( m \in M \) be a homogeneous element. Since \( R \) is coherent \( M \) is a direct limit of finitely presented graded modules, say \( M = \lim_{\lambda} M_{\lambda} \) where each \( M_{\lambda} \) is finitely presented. Let \( \theta_{\lambda} : M_{\lambda} \to M \) be the canonical map and let \( m_{\lambda} \in M_{\lambda} \) be a homogeneous element such that \( \theta_{\lambda}(m_{\lambda}) = m \). Using \( \theta_{\lambda} \), there is a map \( L \otimes_R Rm_{\lambda} \to L \otimes_R Rm \); but \( L \otimes_R Rm = 0 \) so the image of \( 1 \otimes m_{\lambda} \) in \( L \otimes_R M_{\nu} \) is zero for some \( \nu \gg i \). Let \( m_{\nu} \) be the image of \( m_{\lambda} \) in \( M_{\nu} \). Then \( L \otimes_R Rm_{\nu} = 0 \). Since \( R \) is coherent and \( Rm_{\nu} \) is a finitely generated submodule of \( M_{\nu} \), \( Rm_{\nu} \) is finitely presented. The previous paragraph allows us to conclude that \( \dim_k Rm_{\nu} < \infty \). But \( Rm \) is the image of \( Rm_{\nu} \) in \( M \) so \( \dim_k Rm < \infty \). □
A version of the next result for finitely presented not-necessarily-graded modules is given in \cite[Thm. 5.1]{1}. As with the previous result, the ideas in our proof differ from those in \cite{1}.

**Theorem 4.4.** Let \( \pi^* : \text{Gr} R \longrightarrow \text{QGr} R \) be the quotient functor and let \( i^* = L \otimes_R - : \text{Gr} R \longrightarrow \text{Gr} L \). Then

\[ \text{QGr} R \cong \text{Gr} L \]

via a functor \( \alpha^* : \text{QGr} R \rightarrow \text{Gr} L \) such that \( \alpha^* \pi^* = i^* \).

**Proof.** We already know \( i^* \) is exact and vanishes on \( \text{Fdim} R \) so, by the universal property of \( \text{QGr} R \), there is a unique functor \( \alpha^* : \text{QGr} R \rightarrow \text{Gr} L \) such that \( \alpha^* \pi^* = i^* \); furthermore, \( \alpha^* \) is exact.

The forgetful functor \( i_* : \text{Gr} L \rightarrow \text{Gr} R \) is exact and right adjoint to \( i^* \). We will show that \( \pi^* i_* \) is quasi-inverse to \( \alpha^* \). A diagram will help us keep track of the data:

\[
\begin{array}{ccc}
\text{Gr} R & \xrightarrow{\pi^*} & \text{QGr} R \\
\downarrow{i_*} & & \downarrow{\alpha^*} \\
\text{Gr} L & \xleftarrow{i^*} & \text{Gr} R
\end{array}
\]

Since \( R \rightarrow L \) is a universal localization it is an epimorphism in the category of rings. The multiplication map \( L \otimes_R L \rightarrow L \) is therefore an isomorphism of \( L \)-bimodules. Thus, if \( N \in \text{Gr} L \), then

\[ i^* i_* N = L \otimes_R N = L \otimes_R (L \otimes_L N) \cong N. \]

Therefore \( \alpha^*(\pi^* i_* N) = i^* i_* \cong \text{id}_{\text{Gr} R} \).

Let \( M \in \text{Gr} R \) and consider the exact sequence

\[
0 \rightarrow \text{Tor}^1_R(L/R, M) \rightarrow M \xrightarrow{f} L \otimes_R M = i_* \pi^* M \rightarrow (L/R) \otimes_R M \rightarrow 0
\]

where \( f(m) = 1 \otimes m \). Since \( L \otimes_R L \cong L \), \( i^*(f) \) is an isomorphism. But \( i^* \) is exact so it vanishes on \( \text{Tor}^1_R(L/R, M) \) and \( (L/R) \otimes_R M \). Thus, by Proposition 4.3, both these modules are in \( \text{Fdim} R \). Therefore \( \pi^* \) vanishes on them. Hence \( \pi^*(f) \) is an isomorphism. In other words, the natural transformation \( \pi^* \rightarrow \pi^* i_* i^* \) is an isomorphism.

In particular, \( \pi^* \cong \pi^* i_* \alpha^* \pi^* \). By the universal property of \( \text{QGr} R \), there is a unique functor \( \beta^* \) such that the diagram

\[
\begin{array}{ccc}
\text{Gr} R & \xrightarrow{\pi^*} & \text{QGr} R \\
\downarrow{\pi^*} & & \downarrow{\beta^*} \\
\text{QGr} R & \xleftarrow{\beta^*} & \text{Gr} R
\end{array}
\]

commutes, i.e., \( \pi^* = \beta^* \pi^* \). That \( \beta^* \) is, of course, \( \text{id}_{\text{QGr} R} \). But \( \pi^* \cong (\pi^* i_* \alpha^*) \pi^* \) so we conclude that \( \pi^* i_* \alpha^* \cong \text{id}_{\text{QGr} R} \). This completes the proof that \( \alpha^* \) and \( \pi^* i_* \) are mutually quasi-inverse. \( \square \)
5. Remarks

5.1. Most non-commutative projective algebraic geometry to date involves non-commutative rings that are noetherian. See, for example, Artin and Zhang’s paper [2] and the survey article of Stafford-Van den Bergh [17]. Two notable exceptions are (1) the rings $A := k\langle x_1, \ldots, x_n \rangle/(f)$ where $f$ is a homogeneous quadratic element of rank $n \geq 3$ ([10], [11]) and (2) the non-commutative homogeneous coordinate rings appearing in Polishchuk’s work ([12] and [13]) on non-commutative elliptic curves or, equivalently, non-commutative 2-tori endowed with a complex structure. The significance of the first is that $\mathbb{D}^b(QGr A)$ is equivalent to the bounded derived category of representations of the generalized Kronecker quiver (i.e., that with two vertices and $n$ parallel arrows from one to the other); $\text{Proj}_{nc} A$ is viewed as a non-commutative analogue of the projective line. Polishchuk’s work provides a beautiful and deep connection between non-commutative geometry based on operator algebras and non-commutative projective algebraic geometry.

The direct limit algebra $S$ in Theorem 1.1 provides a further link. When the base field is $\mathbb{C}$ the norm-completion of $S$ belongs to an important class of $C^*$-algebras, the AF-algebras (AF=approximately finite). Under the philosophy that non-commutative $C^*$-algebras correspond to “non-commutative topological spaces” AF-algebras are often viewed as corresponding to 0-dimensional spaces; see, for example, the paragraph at the foot of page 10 of Connes’s book [4], although they also exhibit features of higher dimensional spaces. A prominent example is the AF-algebra associated to the space of Penrose tilings (see [4] and [15] for details).

5.2. A homological remark. A connected graded $k$-algebra $A$ is said to be Artin-Schelter regular of dimension $d$ if $\text{Ext}^i_A(k, A) = 0$ when $i \neq d$. Many of the proofs in non-commutative projective algebraic geometry work only for Artin-Schelter regular rings of finite Gelfand-Kirillov dimension.

The next result shows that $TV$ is far from being Artin-Schelter regular when $\dim V \geq 2$.

**Lemma 5.1.** Let $R$ be the free algebra on $d$ variables. Then there is an exact sequence

$$0 \to R^{d^2-1} \to \text{Ext}^1_R(k, R) \to k(1)^d \to 0$$

of graded right $R$-modules.

**Proof.** Applying $\text{Hom}_R(-, R)$ to a minimal resolution $0 \to R(-1)^d \to R \to k \to 0$ of left $R$-modules produces the top row in the commutative diagram
of exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}(R, R) & \longrightarrow & \text{Hom}(R(-1)^d, R) & \longrightarrow & \text{Ext}^1_R(k, R) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & R(1)^d & \longrightarrow & \text{Ext}^1_R(k, R) & \longrightarrow & 0
\end{array}
\]

in which \( f(1) = (x_1, \ldots, x_d) \) and \( x_1, \ldots, x_d \) is a basis for \( V \).

Let \( e_1 := (1,0,\ldots,0), e_2 := (0,1,0,\ldots,0), \ldots, e_d := (0,\ldots,0,1) \) be the standard basis for \( R^d \). Define the left \( R \)-module homomorphism \( h : R \rightarrow (R^d)^{\oplus d} \) by \( h(1) = (e_1, \ldots, e_d) \). Let \( g : R^d \rightarrow R(1)^d \) be the unique left \( R \)-module homomorphism such that \( g(e_i) = x_i \) for all \( i \). There is an exact sequence \( 0 \rightarrow R^d \xrightarrow{g} R(1) \rightarrow k(1) \rightarrow 0 \). Let \( (g, \ldots, g) : (R^d)^{\oplus d} \rightarrow (R(1)^d)^{\oplus d} \) be the left \( R \)-module homomorphism defined by \( (g, \ldots, g)(u_1, \ldots, u_d) = (g(u_1), \ldots, g(u_d)) \) where \( u_i \in R^d \). Then there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & (R^d)^{\oplus d} & \xrightarrow{(g, \ldots, g)} & R(1)^{\oplus d} & \longrightarrow & k(1)^{\oplus d} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ext}^1_R(k, R) & \longrightarrow & k(1)^d & \longrightarrow & 0
\end{array}
\]

in which the columns are exact. The result now follows by applying the Snake Lemma to this diagram. \( \square \)

The bottom row of (5-1) yields an exact sequence \( 0 \rightarrow O \rightarrow O(1)^d \rightarrow \mathcal{E} \rightarrow 0 \) in \( \text{coh} \mathbb{X}^{d-1} \) (to be precise, since we started with left \( R \)-modules we should be replace the bottom row of (5-1) with the analogous exact sequence of left \( R \)-modules). By the lemma, \( \mathcal{E} \cong O^{d-1} \).

5.3. \( \mathbb{X}^n \) has only the trivial closed subspaces. There is a notion of closed subspace in non-commutative algebraic geometry \cite{18} Sect. 3.3. Rosenberg \cite{14} Prop. 6.4.1, p.127] proved that closed subspaces of an affine nc-space are in natural bijection with the two-sided ideals in a coordinate ring for it. The only two-sided ideals in \( S \) are the zero ideal and \( S \) itself so the only closed subspaces of \( \mathbb{X}^n \) are the empty set and \( \mathbb{X}^n \) itself.

This is a surprise because the free algebra contains a wealth of two-sided ideals. For example, the polynomial ring on \( n+1 \) variables is a quotient
of the free algebra $k\langle x_0, \ldots, x_n \rangle$ so $\text{Qcoh}\mathbb{P}^n$ is a full subcategory of $\text{Qcoh}\mathbb{X}^n$ but coherent $\mathcal{O}_{\mathbb{P}^n}$-modules are not finitely presented as objects in $\text{Qcoh}\mathbb{X}^n$.

5.4. In [16] we extend the ideas and results in this paper to path algebras of quivers: the free algebra is replaced by a path algebra $kQ$ and the category of “quasi-coherent sheaves” on $\text{Proj}_{nc}(kQ)$ is equivalent to the category of modules over a direct limit of semisimple $k$-algebras, namely $\lim\limits_{\rightarrow} \text{End}_{kI}(kQ_1)^{\otimes n}$ where $I$ is the set of vertices and $kQ_1$ the linear span of the arrows.

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