DATA SECURITY EQUALS GRAPH CONNECTIVITY

MING-YANG KAO†

Abstract. To protect sensitive information in a cross tabulated table, it is a common practice to suppress some of the cells in the table. This paper investigates four levels of data security of a two-dimensional table concerning the effectiveness of this practice. These four levels of data security protect the information contained in, respectively, individual cells, individual rows and columns, several rows or columns as a whole, and a table as a whole. The paper presents efficient algorithms and NP-completeness results for testing and achieving these four levels of data security. All these complexity results are obtained by means of fundamental equivalences between the four levels of data security of a table and four types of connectivity of a graph constructed from that table.

Key words. statistical tables, linear algebra, graph theory, mixed graphs, strong connectivity, bipartite-(k+1)-connectivity, bipartite-completeness.

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1. Introduction. Cross tabulated tables are used in a wide variety of documents to organize and exhibit information. The values of sensitive cells in such tables are routinely suppressed to conceal sensitive information. There are two fundamental issues concerning the effectiveness of this practice. One is whether an adversary can deduce significant information about the suppressed cells from the published data of a table. The other is how a table maker can suppress a small number of cells in addition to the sensitive ones so that the resulting table does not leak significant information.

This paper investigates how to protect the information in a two-dimensional table that publishes three types of data (see [16] for examples): (1) the values of all cells except a set of sensitive ones, which are suppressed, (2) an upper bound and a lower bound for each cell, and (3) all row sums and column sums of the complete set of cells. The cells may have real or integer values. They may have different bounds, and the bounds may be finite or infinite. The upper bound of a cell should be strictly greater than its lower bound; otherwise, the value of that cell is immediately known even if that cell is suppressed. The cells that are not suppressed also have upper and lower bounds. These bounds are necessary because some of the unsuppressed cells may later be suppressed to protect the information in the sensitive cells.

The focus of this paper is on how to protect the type of information defined here. A bounded feasible assignment to a table is an assignment of values to the suppressed cells such that each row or column adds up to its published sum and the bounds of the suppressed cells are all satisfied. A linear combination of the suppressed cells is a linear invariant if it has the same value at all bounded feasible assignments (see [16] for examples). Intuitively, the information contained in a linear invariant is unprotected because its value can be uniquely deduced from the published data. Five classes of linear invariants are of special significance. A positive invariant is one whose coefficients are all nonnegative with at least one coefficient being positive. A unitary invariant is one whose coefficients are all nonnegative with at least one coefficient being positive. A sum invariant is one whose coefficients are all nonnegative with at least one coefficient being positive.
whose coefficients are +1 or 0. A rectangular sum invariant is one that sums over all suppressed cells shared by a set of rows and a set of columns. An invariant cell is a suppressed cell that forms a linear invariant all by itself.

Four levels of data security of a table are discussed in this paper. To motivate the discussion, suppose that a given table tabulates the quantities of several products made by different factories. A row represents a factory, a column records the quantities of a product, and a cell contains the quantity of a product made by a factory. Level 1 protects the suppressed cells individually. A factory wishes to conceal the quantity of a particular product. Naturally, that quantity should be suppressed and its precise value should not be uniquely determined from the published data of the table. Thus, a suppressed cell is protected if it is not an invariant cell. Level 2 protects a row (or column) as a whole. After all suppressed cells are protected, an adversary may still be able to obtain useful information by combining the suppressed cells. If a factory wishes to protect the information about the quantities of all its products as a whole, it must ensure that no information of a sensitive type can be extracted by combining the suppressed cells in the row representing that factory. Hence, a row is protected if there is no linear invariant of a desired type that combines the suppressed cells in that row. Level 3 protects a set of \( k \) rows (or \( k \) columns) as a whole. Suppose that a company owns \( k \) factories. It wishes to conceal aggregate information about all its factories, not just the information about each individual factory. It should require that no information of an important type may be derived by combining the suppressed cells in the \( k \) rows for its \( k \) factories. Thus a set of \( k \) rows is protected if there is no linear invariant of a desired type that combines the suppressed cells in those rows. Level 4 protects the given table as a whole. Further suppose that the above company owns all the factories tabulated in the table. It wishes to protect aggregate information about all its factories and all their products. It stipulates that only trivial information may be found among a desired class of combinations of the suppressed cells. Thus, a table is protected if it has no linear invariant of a desired type that combines its suppressed cells.

The key contribution of this paper is to establish that the latter three levels of data security of a table are equivalent to three types of connectivity of a graph called the suppressed graph of that table. Previously, Gusfield showed that the first level of data security is equivalent to a certain type of connectivity of the suppressed graph \([12]\). The paper further uses these fundamental equivalences to obtain three sets of complexity results. Firstly, the second and the fourth level of data security of a table can be tested in optimal linear time and that the third level can be tested in polynomial time. Previously, Gusfield showed how to find all invariant cells of a table and test for its first level of data security in optimal linear time \([12]\). Secondly, for each of the four levels of data security it is an NP-complete problem to compute and suppress the minimum number of additional cells in a table in order to achieve the desired level of data security. Thirdly, for a large and practical class of tables, the above optimal suppression problem for the second and the fourth level of data security can be solved in optimal linear time. For the first level of data security, Gusfield showed that the optimal suppression problem can be solved in optimal linear time \([11]\). For the third level of data security the optimal suppression problem remains open.

We review basics of graphs and tables in \(\S 2\), discuss the four levels of data security in \(\S 3\) through \(\S 5\), and compare them in \(\S 6\).

2. Preliminaries. Every graph in this paper is a mixed graph, i.e., it may contain both undirected and directed edges with at most one edge between two ver-
vertices. Let $T$ be a table. The suppressed graph $\mathcal{H} = (A,B,E)$ and the total graph $\mathcal{H}' = (A,B,E')$ of $T$ are the bipartite graphs constructed here (see Figure 1 for an example). For each row (respectively, column) of $T$, there is a vertex in $A$ (respectively, $B$); this vertex is called a row (respectively, column) vertex. For each cell $x$ at row $i$ and column $j$, there is an edge $e \in E'$ between the vertices of row $i$ and column $j$. If the value of $x$ is strictly between its lower and upper bounds, then $e$ is undirected. Otherwise, if the value equals the lower (respectively, upper) bound, then $e$ points from the row endpoint to the column endpoint (respectively, from the column endpoint to the row endpoint). $E$ consists of the edges corresponding to the suppressed cells. Note that $\mathcal{H}$ is a subgraph of $\mathcal{H}'$ and $\mathcal{H}'$ is complete (i.e., for all $u \in A$ and $v \in B$, $E'$ has exactly one edge between $u$ and $v$). Also, given an arbitrary complete bipartite graph and a subgraph on the same vertices, it takes only linear time to construct a table with these two graphs as its total and suppressed graphs.

A traversable cycle or path is one that can be traversed along its edge directions. A direction-blind cycle or path is one that can be traversed if its edge directions are disregarded; we often omit the word direction-blind for brevity. A graph is connected if each pair of vertices are in a path. A connected component is a maximal connected subgraph. A nonsingleton connected component is one with two or more vertices. A graph is strongly connected if each pair of vertices are in a traversable cycle. A strong component is a maximal strongly connected subgraph.

The effective area of a linear invariant $F$ of $T$, denoted by $EA(F)$, is the set of suppressed cells in the nonzero terms of $F$. $EA(F)$ is also regarded as a set of edges in $\mathcal{H}$. $F$ is nonzero if $EA(F) \neq \emptyset$. $F$ is minimal if it is nonzero and $T$ has no nonzero linear invariant whose effective area is a proper subset of $EA(F)$. Note that given a minimal linear invariant $F$, if $F'$ is a nonzero linear invariant with $EA(F') \subseteq EA(F)$, then $F'$ is also minimal and is a multiple of $F$. Thus, a minimal linear invariant is unique up to a multiplicative factor with respect to its effective area.

An edge set of a graph is an edge cut if its removal disconnects a connected component. An edge cut is minimal if no proper subset of it is an edge cut.

**Fact 1.** Let $Z$ be an edge set of a strong component $\mathcal{H}'$ of $\mathcal{H}$. $Z$ is a minimal edge cut of $\mathcal{H}'$ if and only if $\mathcal{H}' - Z$ has exactly two connected components, say, $\mathcal{H}_1$ and $\mathcal{H}_2$, and each edge of $Z$ is between $\mathcal{H}_1$ and $\mathcal{H}_2$.

Assume that $Z$ is a minimal edge cut of $\mathcal{H}'$. $Z$ is bipartite if the endpoints of $Z$ in $\mathcal{H}'$ are all row vertices or all column vertices. An edge set of $\mathcal{H}$ is a (respectively, bipartite) basic set if it consists of an edge not in any strong component of $\mathcal{H}$ or is a (respectively, bipartite) minimal edge cut of some strong component.

**Theorem 2.1 ([4]).**

1. A linear invariant of $T$ is minimal if and only if its effective area is a basic set of $\mathcal{H}$. Also, for each basic set $Z$ of $\mathcal{H}$, there is a minimal linear invariant $F$ of $T$ with $EA(F) = Z$.

2. Every minimal linear invariant is a multiple of a unitary invariant. Furthermore, a minimal linear invariant $F$ of $T$ is a multiple of a sum invariant if and only if $EA(F)$ is a bipartite basic set of $\mathcal{H}$.

3. For each nonzero linear invariant $F$ of $T$, there exist unitary linear invariants $F_1, \ldots, F_k$ of $T$ such that $F = \sum_{i=1}^{k} c_i F_i$ for some $c_i > 0$, $EA(F) = \bigcup_{i=1}^{k} EA(F_i)$, and for each $F_i$ and each $e \in EA(F_i)$, the coefficients of $e$ in $F$ and $F_i$ are either both positive or both negative.

Remark. A referee has indicated that a different proof for Theorem 2.1 from that in [4] can be constructed by means of conformal vector decomposition [13, 15].
3. Protection of a cell. A suppressed cell of $T$ is *protected* if it is not an invariant cell.

3.1. Cell protection and bridge-freeness. A graph is *bridge-free* if it has no edge cut consisting of a single edge.

**Theorem 3.1 ([12]).**

1. A suppressed cell of $T$ is protected if and only if it is an edge in an edge-simple traversable cycle of $H$.
2. The suppressed cells of $T$ are all protected if and only if each connected component of $H$ is strongly connected and bridge-free.

**Corollary 3.2 ([12]).** Given $H$, the unprotected cells of $T$ can be found in $O(|H|)$ time.

3.2. Optimal suppression problems for cell protection. The problem below is concerned with suppressing the minimum number of additional cells in $T$ such that the original and the new suppressed cells in the resulting table are all protected.

**Problem 1 (Protection of All Cells).**

- **Input:** $T$ and an integer $p \geq 0$.
- **Output:** Is there a set $P$ consisting of at most $p$ published cells of $T$ such that all suppressed cells are protected in the table formed by $T$ with the cells in $P$ also suppressed?

Problem 1 can be reformulated as the graph augmentation problem below.

**Problem 2.**

- **Input:** A complete bipartite graph $H'$, a subgraph $H$, and an integer $p \geq 0$.
- **Output:** Is there a set $P$ of at most $p$ edges in $H' - H$ such that each connected component of $H \cup P$ is strongly connected and bridge-free?

**Lemma 3.3.** Problems 1 and 2 can be reduced to each other in linear time.

**Proof.** The proof follows from Theorem 3.1(2). □

The next problem is NP-complete [10]. It is used here to prove that Problems 1 and 2 are hard.

**Problem 3 (Hitting Set).**

- **Input:** A finite set $S$, a nonempty set $W \subseteq 2^S$, and an integer $h \geq 0$.
- **Output:** Is there a subset $S'$ of $S$ such that $|S'| \leq h$ and $S'$ contains at least one element in each set in $W$?

**Theorem 3.4.** Problems 1 and 3 are NP-complete.

**Proof.** Problems 1 and 3 are both in NP. To prove their completeness, by Lemma 3.3, it suffices to reduce Problem 3 to Problem 1.

Given an instance $S = \{s_1, \ldots, s_\alpha\}$, $W = \{S_1, \ldots, S_\beta\}$, $h$ of Problem 3, an instance $H' = (A, B, E')$, $H = (A, B, E)$, $p$ of Problem 1 is constructed as follows:

- **Rule 1:** Let $A = \{a_0, a_1, \ldots, a_\alpha\}$. The vertices $a_1, \ldots, a_\alpha$ correspond to $s_1, \ldots, s_\alpha$, but $a_0$ corresponds to no $s_i$.

- **Rule 2:** Let $B = \{b_0, b_1, \ldots, b_\beta\}$. The vertices $b_1, \ldots, b_\beta$ correspond to $S_1, \ldots, S_\beta$ of $S$, but $b_0$ corresponds to no $S_j$.

- **Rule 3:** Let $E'$ consist of the following edges:
  1. The edge between $a_0$ and $b_0$ is $b_0 \to a_0$.
  2. For all $j$ with $1 \leq j \leq \beta$, the edge between $a_0$ and $b_j$ is $a_0 \to b_j$.
  3. For all $i$ with $1 \leq i \leq \alpha$, the edge between $a_i$ and $b_0$ is $a_i \to b_0$.
  4. For each $s_i$ and each $S_j$, if $s_i \in S_j$, then the edge between $a_i$ and $b_j$ is $b_j \to a_i$; otherwise it is $a_i \to b_j$.

- **Rule 4:** Let $E = \{b_0 \to a_0\} \cup \{a_0 \to b_1, \ldots, a_0 \to b_\beta\}$. 


• Rule 5: Let $p = h + \beta$.

The above construction can be easily computed in polynomial time. The next two claims show that it is indeed a desired reduction from Problem 3 to Problem 2.

Claim 1. If some $S' \subseteq S$ with $|S'| \leq h$ has at least one element in each $S_j$, then some $P \subseteq E' - E$ consists of at most $p$ edges such that every connected component of $H \cup P$ is strongly connected and bridge-free.

To prove this claim, observe that for each $S_j$, some $s_{i_j} \in S' \cap S_j$ exists. By Rule 3(4), $P_1 = \{b_1 \rightarrow a_{i_1}, \ldots, b_\beta \rightarrow a_{i_\beta}\}$ exists. By Rule 3(3), $P_2 = \{a_{i_1} \rightarrow b_0, \ldots, a_{i_\beta} \rightarrow b_0\}$ exists. Let $P = P_1 \cup P_2$. Note that $P_1$ consists of $\beta$ edges. $P_2$ consists of at most $|S'|$ edges. Thus $P$ has at most $p = \beta + h$ edges. For all $j$ with $1 \leq j \leq \beta$, the edges $b_0 \rightarrow a_{i_1}, a_{i_1} \rightarrow b_j, b_j \rightarrow a_{i_j}, a_{i_j} \rightarrow b_0$ form a vertex-simple traversable cycle. Because $E \cup P$ consists of the edges in these cycles, every connected component of $H \cup P$ is strongly connected and bridge-free. This finishes the proof of Claim 1.

Claim 2. If some $P \subseteq E' - E$ consists of at most $p$ edges such that every connected component of $H \cup P$ is strongly connected and bridge-free, then some $S' \subseteq S$ with $|S'| \leq h$ has at least one element in each $S_j$.

To prove this claim, observe that for all $j$ with $1 \leq j \leq \beta$, by Rule 4, $E$ contains $a_0 \rightarrow b_j$, but no edge pointing from $b_j$. Because every connected component of $H \cup P$ is strongly connected, $P$ contains an edge $b_j \rightarrow a_i$ for some $i_j$. By Rule 3(4), $s_{i_j} \in S_j$. Let $S' = \{s_{i_1}, \ldots, s_{i_\beta}\}$. Note that $P$ contains $b_1 \rightarrow a_{i_1}, \ldots, b_\beta \rightarrow a_{i_\beta}$ but $E$ contains no edges pointing from $\{a_{i_1}, \ldots, a_{i_\beta}\}$. Because every connected component of $H \cup P$ is strongly connected, $P$ must also contain at least one edge pointing from each vertex in $\{a_{i_1}, \ldots, a_{i_\beta}\}$. Thus $P$ contains at least $|S'| + \beta$ edges. Then $|S'| \leq h$ because $|P| \leq \beta + h$. This finishes the proof of Claim 2 and thus that of Theorem 3.4.

The next two problems are optimization versions of Problems 3 and 4 for undirected graphs and tables whose total graphs are undirected.

Problem 4 (Protection of All Cells).

- Input: The suppressed graph of a table $T$ whose total graph is undirected.
- Output: A set $P$ consisting of the smallest number of published cells of $T$ such that all suppressed cells are protected in the table formed by $T$ with the cells in $P$ also suppressed.

Problem 5.

- Input: A bipartite undirected graph $H = (A, B, E)$.
- Output: A set $P$ consisting of the smallest number of undirected edges between $A$ and $B$ but not in $E$ such that every connected component of $(A, B, E \cup P)$ is bridge-free.

Note that Problem 5 needs not specify $H'$ because it is undirected and thus is unique for $H$. Similarly, $H \cup P$ is always strongly connected.

Lemma 3.5. Problems 3 and 4 can be reduced to each other in linear time.

Proof. The proof is similar to that of Lemma 3.3.

Theorem 3.6 (\cite{11}). Problem 3 is solvable in linear time; thus so is Problem 4.

4. Protection of rows and columns. This section discusses the data security of a table at Levels 2 and 3 in a unified framework. Let $EA(R)$ denote the set of suppressed cells in a row or column $R$. Let $\overline{R} = \sum_{e \in EA(R)} e$. Let $R_1, \ldots, R_k$ be $k$ rows or $k$ columns of $T$, but no mixed case. For Level 3 data security, $\{R_1, \ldots, R_k\}$ is protected with respect to the linear invariants (respectively, the positive invariants, the unitary invariants, the sum invariants, or the rectangular sum invariants) if the conditions below hold:
1. Each linear invariant (respectively, positive invariant, unitary invariant, sum invariant, or rectangular sum invariant) $F$ of $T$ with $EA(F) \subseteq \cup_{i=1}^{k} EA(R_i)$ is a linear combination of $\overline{R}_1, \ldots , \overline{R}_k$.

2. No suppressed cell of $R_1, \ldots , R_k$ is an invariant cell.

Level 2 data security is a special case of Level 3 with $k = 1$ and its definitions can be simplified. A row or column $R$ is protected with respect to the linear invariants (respectively, the positive invariants, the unitary invariants, or the sum invariants) if the conditions below hold:

1. Each linear invariant (respectively, positive invariant, unitary invariant, or sum invariant) $F$ with $EA(F) \subseteq EA(R)$ is a multiple of $\overline{R}$.

2. No suppressed cell in $R$ is an invariant cell.

We do not explicitly consider the protection of $R$ with respect to the rectangular sum invariants because for $k = 1$ these invariants are the same as the sum invariants. Also, the five types of invariants here are implicitly considered for cell protection because a linear invariant with exactly one nonzero term is essentially an invariant cell.

The two conditions in the definitions are based on technical considerations. No matter how many cells in $T$ are suppressed, $\overline{R}_1, \ldots , \overline{R}_k$ and their linear combinations are always linear invariants. Thus the first condition gives the best possible protection for $R_1, \ldots , R_k$ as a whole. If $R_i$ has either no suppressed cell or at least two, the first condition implies the second one; otherwise, the first condition holds trivially but the only suppressed cell in $R_i$ is an invariant. The second condition is adopted to avoid this undesirable situation.

These definitions also require that $R_1, \ldots , R_k$ be all rows or all columns. In these two pure cases, $EA(R_1), \ldots , EA(R_k)$ are pairwise disjoint. Therefore, a linear combination of $\overline{R}_1, \ldots , \overline{R}_k$ has a very simple structure and encodes essentially the same information as do $\overline{R}_1, \ldots , \overline{R}_k$. In contrast, if at least one $R_i$ is a row and at least one $R_j$ is a column, then a linear combination of $\overline{R}_1, \ldots , \overline{R}_k$ may have a very complex structure and may encode very different information from that contained in $\overline{R}_1, \ldots , \overline{R}_k$. Furthermore, unlike in the two pure cases, these definitions do not seem to have useful characterizations in the mixed case.

The importance of the first four types of invariants considered in the definitions is evident. The fifth type, a rectangular sum invariant, is motivated by a popular technique for protecting information in a table. Let $e$ be an invariant cell at row $i$ and column $j$. To protect $e$, row $i$ can be split into several rows, and column $j$ into several columns. Correspondingly, $e$ is split into four or more cells. Then enough of these refined cells can be suppressed to ensure that each suppressed refined cell is protected. However, the sum of the suppressed refined cells of $e$ is a rectangular sum invariant. This property can be used to uniquely determine the value of $e$. Thus the consideration of rectangular sum invariants renders this refinement approach useless at the third level of data security.

4.1. Equivalence of $k$ row-column protection. This section shows that the five definitions of $k$ row-column protection are all equivalent.

Lemma 4.1. Every sum minimal invariant is rectangular.

Proof. Let $F$ be a sum minimal invariant of $T$. If $EA(F)$ consists of an edge not in any strong component of $H$, then $F$ is trivially rectangular. Otherwise, by Theorem 2.3 $EA(F)$ is a bipartite minimal cut set of a strong component $H'$ of $H$. By Fact 1, $H' = EA(F)$ has two connected components $H'_1$ and $H'_2$. Let $U_1$ and $U_2$ be the sets of endpoints of $EA(F)$ in $H'_1$ and $H'_2$, respectively. By the bipartiteness of $EA(F)$, without loss of generality the vertices in $U_1$ are rows in $T$ and those in $U_2$
are columns. Then $F$ is rectangular because $EA(F)$ consists of the edges between $U_1$ and $U_2$ in $\mathcal{H}$. 

**Lemma 4.2.** If no $EA(R_i)$ is empty, the statements below are equivalent:
1. Every positive invariant $F$ with $EA(F) \subseteq \bigcup_{i=1}^{k} EA(R_i)$ is a linear combination of $\overline{R}_1, \ldots, \overline{R}_k$.
2. Every sum invariant $F$ with $EA(F) \subseteq \bigcup_{i=1}^{k} EA(R_i)$ is a linear combination of $\overline{R}_1, \ldots, \overline{R}_k$.
3. Every rectangular sum invariant $F$ with $EA(F) \subseteq \bigcup_{i=1}^{k} EA(R_i)$ is a linear combination of $\overline{R}_1, \ldots, \overline{R}_k$.
4. $\overline{R}_1, \ldots, \overline{R}_k$ are the only sum minimal invariants of $\mathcal{T}$ whose effective areas are subsets of $\bigcup_{i=1}^{k} EA(R_i)$.

**Proof.** The directions 1 $\Rightarrow$ 2 $\Rightarrow$ 3 are straightforward. The direction 4 $\Rightarrow$ 1 follows from the fact that by Statement 4 $\overline{R}_1, \ldots, \overline{R}_k$ are the only factors in the decomposition in Theorem 2.1(3) for a positive invariant $F$ with $EA(F) \subseteq \bigcup_{i=1}^{k} EA(R_i)$. To prove 3 $\Rightarrow$ 4, note that because $\overline{R}_j$ is a positive invariant for all $R_j$, by Theorem 2.1(3) there is a sum minimal invariant $F$ with $EA(F) \subseteq EA(\overline{R}_j)$. Since $F$ is also rectangular, by Statement 3, $F = \sum_{i=1}^{k} c_i \overline{R}_i$ for some $c_i$. Because $\overline{R}_1, \ldots, \overline{R}_k$ share no variable, by the minimality of $F$ and coefficient comparison $\overline{R}_j$ equals $F$ and thus is a sum minimal invariant. To prove the desired uniqueness of $\overline{R}_1, \ldots, \overline{R}_k$, let $F'$ be a sum minimal invariant with $EA(F') \subseteq \bigcup_{i=1}^{k} EA(R_i)$. By Lemma 3.1, $F'$ is rectangular. By Statement 3, $F' = \sum_{i=1}^{k} c'_i \overline{R}_i$ for some $c'_i$. Because $F'$ is nonzero, some $c'_i \neq 0$. Because $\overline{R}_1, \ldots, \overline{R}_k$ do not share variables, $EA(\overline{R}_i) \subseteq EA(F')$. Then, $F' = \overline{R}_i$ by coefficient comparison and the minimality of $F'$. 

**Lemma 4.3.** If no $EA(R_i)$ is empty, the statements below are equivalent:
1. Every linear invariant of $\mathcal{T}$ whose effective area is a subset of $\bigcup_{i=1}^{k} EA(R_i)$ is a linear combination of $\overline{R}_1, \ldots, \overline{R}_k$.
2. Every unitary invariant whose effective area is a subset of $\bigcup_{i=1}^{k} EA(R_i)$ is a linear combination of $\overline{R}_1, \ldots, \overline{R}_k$.
3. $\overline{R}_1, \ldots, \overline{R}_k$ and their nonzero multiples are the only minimal linear invariants of $\mathcal{T}$ whose effective areas are subsets of $\bigcup_{i=1}^{k} EA(R_i)$.

**Proof.** The proof is similar to that of Lemma 4.2.

**Lemma 4.4.** $\{R_1, \ldots, R_k\}$ is protected with respect to the positive invariants (respectively, the linear invariants) if and only if the following statements hold:
1. For each strong component $D$ of $\mathcal{H}$ and each vertex $R_i$ contained in $D$, the component $D$ contains all edges incident to $R_i$ in $\mathcal{H}$.
2. The nonempty sets among $EA(R_1), \ldots, EA(R_k)$ are the only bipartite minimal edge cuts (respectively, the only minimal edge cuts) of the strong components of $\mathcal{H}$ among the subsets of $\bigcup_{i=1}^{k} EA(R_i)$.
3. Each vertex $R_i$ is either isolated or incident to two or more edges in $\mathcal{H}$.

**Proof.** The proof of the lemma for the positive invariants and that for the general invariants are similar; only the former is detailed here. For the direction $\Rightarrow$, Statement 3 follows from the second condition of the definition of $\{R_1, \ldots, R_k\}$ being protected. Then, Statements 1 and 2 follow by Statements 1 and 2, Theorem 2.1, and Lemma 4.3. For the direction $\Leftarrow$, by Statements 1 and 2, Theorem 2.1 and Lemma 4.3, the first condition of $\{R_1, \ldots, R_k\}$ being protected is satisfied. The second condition then follows from Statement 3. 

A set of vertices in a connected graph is a **vertex cut** if its removal disconnects the graph.

**Fact 2.** If each $EA(R_i)$ is included in the strong component of $\mathcal{H}$ that contains
$R_i$, then the following statements are equivalent:

1. Among the subsets of $\bigcup_{i=1}^{k} EA(R_i)$, the nonempty sets $EA(R_i)$ are the only minimal edge cuts of the strong components of $G$.
2. Among the subsets of $\bigcup_{i=1}^{k} EA(R_i)$, the nonempty sets $EA(R_i)$ are the only bipartite minimal edge cuts of the strong components of $G$.
3. $\{R_1, \ldots, R_k\}$ includes no vertex cut of any strong component of $G$.

**Theorem 4.5.** The five definitions of a set of $k$ rows or $k$ columns being protected are all equivalent.

**Proof.** If some $EA(R_i) = \emptyset$, then $\{R_1, \ldots, R_k\}$ is protected if and only if $\{R_1, \ldots, R_k\} - \{R_i\}$ is protected. Thus without loss of generality assume that no $EA(R_i)$ is empty. Then, by Lemma 4.2 the protection definitions with respect to the positive, sum, and rectangular invariants are all equivalent. Similarly, by Lemma 4.3, those with respect to the general and unitary invariants are also equivalent. This theorem then follows directly from Lemma 4.2 and Fact 4.

**4.2. $k$ Row-column protection and bipartite-($k+1$)-connectivity.** A connected bipartite graph $G = (X, Y, I)$ is bipartite-($k+1$)-connected if $|X| \geq k+1$, $|Y| \geq k+1$, and neither $X$ nor $Y$ includes a vertex cut of at most $k$ vertices. $G$ is $(k+1)$-connected if $|X \cup Y| \geq k+1$ and there is no vertex cut of at most $k$ vertices.

**Lemma 4.6.** $\{R_1, \ldots, R_k\}$ is protected if and only if the statements below hold:

1. For each strong component $D$ of $G$ and each vertex $R_i \in D$, $D$ contains all the edges incident to $R_i$ in $G$.
2. $\{R_1, \ldots, R_k\}$ includes no vertex cut of any strong component of $G$.
3. Each $R_i$ is either isolated or incident to two or more edges in $G$.

**Proof.** This lemma follows from Theorem 4.5, Lemma 4.4 and Fact 5.

**Theorem 4.7.** Every set of at most $k$ rows or $k$ columns of $T$ is protected if and only if every nonsingleton connected component of $G$ is strongly connected and bipartite-($k+1$)-connected.

**Proof.** This theorem follows directly from Lemma 4.6.

**Corollary 4.8.**

1. Given $G$ and $\{R_1, \ldots, R_k\}$, whether $\{R_1, \ldots, R_k\}$ is protected can be determined in $O(|G|)$ time.
2. Given $G$ and $k$, whether $T$ has any unprotected set of at most $k$ rows or $k$ columns can be answered in $O(k^3 n^2)$ time, where $n$ is the number vertices in $G$.

**Proof.** Statement 1 follows from Lemma 4.6 in a straightforward manner using linear-time algorithms for connectivity and strong connectivity 5. Statement 2 follows from Theorem 4.7. The key step is to test the bipartite-($k+1$)-connectivity of $G$ within the stated time bound. We first construct two auxiliary graphs $G_A$ and $G_B$. For each vertex $u \in A$, replace $u$ with $k+1$ copies in $G_A$. For each $u \in A$ and each edge $e$ in $B$ between $u$ and a vertex $v \in B$, replace $e$ with $k+1$ copies between $v$ and the $k+1$ copies of $u$ in $G_A$. $G_B$ is obtained by exchanging $A$ and $B$ in the construction. Because $G$ is connected and each vertex in $A$ is duplicated $k+1$ times, $G_A$ has a vertex cut $U$ of at most $k$ vertices if and only if $U$ is a subset of $B$ and is a vertex cut of $G$. A symmetrical statement for $B$ also holds. Thus $G$ is bipartite-($k+1$)-connected if and only if both $G_A$ and $G_B$ are ($k+1$)-connected. This corollary then follows from the fact 2, 7 that the ($k+1$)-connectivity of an $m$-vertex graph can tested in $O(k^2 m^2)$ time if $k \leq \sqrt{m}$.

**Corollary 4.9.** Given $G$, it takes $O(|G|)$ time to find the unprotected rows and columns of $T$ and decide whether all individual rows and columns of $T$ are protected.
4.3. Optimal suppression problems for $k$ row-column protection.

**Problem 6 (Protection of All Sets).**
- **Input:** $T$ and two integers $k > 0$ and $p \geq 0$.
- **Output:** Is there a set $P$ consisting of at most $p$ published cells of $T$ such that every set of at most $k$ rows or $k$ columns is protected in the table formed by $T$ with the cells in $P$ also suppressed?

Problem 6 can be reformulated as the following graph augmentation problem.

**Problem 7.**
- **Input:** A complete bipartite graph $H'$, a subgraph $H$, and integers $k > 0$ and $p \geq 0$.
- **Output:** Is there a set $P$ of at most $p$ edges in $H' - H$ such that each nonsingleton connected component of $H \cup P$ is strongly connected and bipartite-$(k+1)$-connected?

**Lemma 4.10.** Problems 6 and 7 can be reduced to each other in linear time.

**Proof.** The proof follows from Theorem 4.7.

**Theorem 4.11.** For $k = 1$, Problems 6 and 7 are NP-complete. Thus, both problems are NP-complete for general $k$.

**Proof.** Problems 6 and 7 are both in NP. To prove their completeness for $k = 1$, by Lemma 4.10 it suffices to reduce Problem 6 to Problem 7 with $k = 1$. Given an instance $S = \{s_1, \ldots, s_a\}$, $W = \{S_1, \ldots, S_\beta\}$, $h$ of Problem 6, let $H' = (A, B, E')$, $H = (A, B, E)$, $p$ be the instance constructed for Theorem 3.4. The next two claims show that this transformation is indeed a desired reduction.

**Claim 3.** If some $S' \subseteq S$ with $|S'| \leq h$ has at least one element in each $S_j$, then some $P \subseteq E' - E$ consists of at most $p$ edges such that every nonsingleton connected component of $H \cup P$ is strongly connected and bipartite-2-connected.

To prove this claim, observe that for each $S_j$, some $s_{ij} \in S' \cap S_j$ exists. Let $P_1 = \{b_1 \rightarrow a_{i_1}, \ldots, b_\beta \rightarrow a_{i_\beta}\}$, which exists by Rule 3(4) of the construction of $H'$, $H$, and $p$. By Rule 3(3), $P_2 = \{a_{i_1} \rightarrow b_0, \ldots, a_{i_\beta} \rightarrow b_0\}$ exists. Let $P = P_1 \cup P_2$. Note that $P_1$ consists of $\beta$ edges. $P_2$ consists of at most $|S'|$ edges. Thus $P$ has at most $p = \beta + h$ edges. For all $j$ with $1 \leq j \leq \beta$, the edges $b_0 \rightarrow a_0, a_0 \rightarrow b_j, b_j \rightarrow a_{ij}, a_{ij} \rightarrow b_0$ form a vertex-simple traversable cycle. These cycles all go through $b_0 \rightarrow a_0$ and form the only nonsingleton connected component of $H \cup P$. This component is clearly strongly connected and bipartite-connected. This finishes the proof of Claim 3.

**Claim 4.** If some $P \subseteq E' - E$ consists of at most $p$ edges such that every nonsingleton connected component of $H \cup P$ is strongly connected and bipartite-2-connected, then some $S' \subseteq S$ with $|S'| \leq h$ has at least one element in each $S_j$.

The proof of this claim is the same as that of Claim 2 and uses only the componentwise strong connectivity of $H \cup P$. This finishes the proof of Theorem 4.11.

The next two problems are variants of Problems 6 and 7.

**Problem 8 (Protection of All Sets).**
- **Input:** The suppressed graph of a table $T$ whose total graph is undirected, and a positive integer $k$.
- **Output:** A set $P$ consisting of the smallest number of published cells of $T$ such that every set of at most $k$ rows or $k$ columns is protected in the table formed by $T$ with the cells in $P$ also suppressed.

**Problem 9.**
Input: A bipartite undirected graph $\mathcal{H} = (A, B, E)$ and a positive integer $k$.

Output: A set $P$ consisting of the smallest number of undirected edges between $A$ and $B$ but not in $E$ such that every nonsingleton connected component of $(A, B, E \cup P)$ is bipartite-$(k+1)$-connected.

**Lemma 4.12.** Problems 3 and 3 can be reduced to each other in linear time.

**Proof.** The proof is similar to that of Lemma 4.10. \(\square\)

**Theorem 4.13.** For $k = 1$, Problem 3 can be solved in linear time.

**Theorem 4.14.** For $k = 1$, Problem 3 can be solved in linear time.

**Proof.** The proof follows from Lemma 4.12 and Theorem 4.13. \(\square\)

### 5. Protection of a Table

Let $R_1, \ldots, R_n$ be the rows and columns of $T$. $T$ is **protected** with respect to the positive invariants (respectively, the sum invariants, or the rectangular sum invariants) if it holds the conditions below:

1. Every positive invariant (respectively, nonzero sum invariant, or nonzero rectangular sum invariant) of $T$ is a positive linear combination of $\overline{R}_1, \ldots, \overline{R}_n$, where a positive linear combination is one that has no negative coefficients and at least one positive coefficient.

2. $T$ has no invariant cell.

These definitions allow only positive linear combinations, because general linear combinations of $\overline{R}_1, \ldots, \overline{R}_n$ generate all linear invariants and leave nothing for protection. This restriction excludes the protection with respect to the general linear invariants.

As a result, the protection with respect to the unitary invariants are also not considered, because by Theorem 2.3, these invariants have the same structures as the general linear invariants do.

**Theorem 5.1.** The three definitions of a table being protected are all equivalent.

**Proof.** Because a protected table has no invariant cells, each row or column has either no suppressed cell or at least two suppressed cells. It suffices to prove that if $T$ holds this condition, then the statements below are equivalent:

1. Every positive invariant is a positive linear combination of $\overline{R}_1, \ldots, \overline{R}_n$.

2. Every nonzero sum invariant is a positive linear combination of $\overline{R}_1, \ldots, \overline{R}_n$.

3. Every nonzero rectangular sum invariant of $T$ is a positive linear combination of $\overline{R}_1, \ldots, \overline{R}_n$.

4. The nonzero linear invariants among $\overline{R}_1, \ldots, \overline{R}_n$ are the only sum minimal invariants of $T$.

The directions $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are straightforward. The direction $4 \Rightarrow 1$ follows from Theorem 2.2(3). To prove the direction $3 \Rightarrow 4$, note that for each $R_i$ with $EA(R_i) \neq \emptyset$, $\overline{R}_i$ is a nonzero sum invariant. By Theorem 2.2 there is a sum minimal invariant $F$ with $EA(F) \subseteq EA(\overline{R}_i)$. $F$ is also rectangular. By Statement 3, $F = \sum_{i=1}^{k} c_i \overline{R}_i$ where $c_i \geq 0$. By coefficient comparison there is some $c_h > 0$. Because $c_i \geq 0$, $\emptyset \neq EA(\overline{R}_h) \subseteq EA(F) \subseteq EA(\overline{R}_j)$. Then $R_h = R_j$ because two distinct $R_i$ cannot share more than one cell and each nonempty $EA(R_i)$ contains at least two cells. Thus $\overline{R}_j$ equals $F$ and is a sum minimal invariant. To prove the desired uniqueness of $\overline{R}_1, \ldots, \overline{R}_n$, let $F'$ be a sum minimal invariant with $EA(F') \subseteq \cup_{i=1}^{k} EA(R_i)$. By Lemma 1.1 $F'$ is rectangular. By Statement 3, $F' = \sum_{i=1}^{k} c'_i \overline{R}_i$ where $c'_i \geq 0$. By coefficient comparison there is some $c'_j > 0$ with $EA(\overline{R}_j) \neq \emptyset$. Because $c'_j \geq 0$, $EA(\overline{R}_j) \subseteq EA(F')$. Then $F' = \overline{R}_j$ by coefficient comparison and the minimality of $F'$. \(\square\)

### 5.1. Table Protection and Bipartite-Completeness

A graph $G = (X, Y, I)$ is **bipartite-complete** if it is complete, $|X| \geq 2$ and $|Y| \geq 2$. 
FACT 3. Let \( u_1, \ldots, u_q \) be the vertices in \( G \). Let \( EA(u_i) \) be the set of edges incident to \( u_i \). Then \( G \) is bipartite-complete if and only if it is bridge-free and has more than one vertex, and the sets \( EA(u_i) \) are its only bipartite minimal edge cuts.

THEOREM 5.2. \( T \) is protected if and only if each non-singleton connected component of \( H \) is strongly connected and bipartite-complete.

Proof. By Fact 3 it suffices to prove that the following statements are equivalent: 1. \( T \) is protected. 2. The nonzero invariants among \( R_1, \ldots, R_n \) are the only sum minimal invariants of \( T \). Also each \( R_i \) contains either no suppressed cell or at least two suppressed cells. 3. Each connected component of \( H \) is strongly connected and bridge-free. Also the nonempty sets among \( EA(R_1), \ldots, EA(R_n) \) are the only bipartite minimal edge cuts of the strong components of \( H \).

The equivalence 1 \( \iff \) 2 follows from the proof of Theorem 5.1. The equivalence 2 \( \iff \) 3 follows from Theorems 2.1 and 3.1. \( \square \)

COROLLARY 5.3. Given \( H \), it takes linear time in the size of \( H \) to determine whether \( T \) is protected.

Proof. This is an immediate corollary of Theorem 5.2. \( \square \)

5.2. Optimal suppression problems for table protection.

PROBLEM 10 (Protection of a Table).
- Input: \( T \) and a nonnegative integer \( p \).
- Output: Is there a set \( P \) consisting of at most \( p \) published cells of \( T \) such that the table formed by \( T \) with the cells in \( P \) also suppressed is protected?

PROBLEM 11.
- Input: A complete bipartite graph \( H' \), a subgraph \( H \), and an integer \( p \geq 0 \).
- Output: Is there a set \( P \) of at most \( p \) edges in \( H' \) such that each non-singleton connected component \( H \cup P \) is strongly connected and bipartite-complete?

LEMMA 5.4. Problems [10] and [11] can be reduced to each other in linear time.

Proof. The proof follows from Theorem 5.2. \( \square \)

THEOREM 5.5. Problems [10] and [11] are \( NP \)-complete.

Proof. Problems [10] and [11] are both in \( NP \). To prove their completeness, by Lemma 5.4 it suffices to reduce Problem [10] to Problem [11]. Given an instance \( S = \{s_1, \ldots, s_n\}, W = \{s_1, \ldots, s_j\} \), of Problem [10] let \( H' = (A, B, E'), H = (A, B, E), p \) be the instance constructed for Theorem 3.4 with the modification below:
- Rule 5′: Let \( p = (\beta + 1) \cdot h \).

This construction can be computed in polynomial time. The next two claims show that it is a desired reduction from Problem [10] to Problem [11].

CLAIM 5. If some \( S' \subseteq S \) with \(|S'| \leq h \) has at least one element in each \( S_j \), then some \( P \subseteq E' - E \) consists of at most \( p \) edges such that every non-singleton connected component of \( H \cup P \) is strongly connected and bipartite-complete.

To prove this claim, observe that for each \( S_j \), some \( s_i \in S' \cap S_j \) exists. Let \( A' = \{a_i, \ldots, a_\beta\} \). Let \( B' = \{b_1, \ldots, b_\beta\}. \) Let \( P_1 \) be the set of edges in \( E' \) from \( B' \) to \( A' \). Let \( P_2 \) be the set of edges in \( E' \) from \( A' \) to \( b_0 \). Let \( P = P_1 \cup P_2 \). Note that \( P \) has at most \( p = (\beta + 1) \cdot h \) edges because \( A' \) has at most \(|S'| \leq h \) vertices. For each \( j \) with \( 1 \leq j \leq \beta \), the edge \( b_j \rightarrow a_i \) is in \( P_1 \) by Rule 3(4) of the construction of \( H' \), \( H \), and \( p \). Also, \( b_0 \rightarrow a_0 \), \( a_0 \rightarrow b_j \), and \( a_i \rightarrow b_0 \) are in \( H \cup P \). These four edges form a vertex-simple traversable cycle. These cycles form the only non-singleton connected component in \( H \cup P \). Because these cycles all go through \( a_0 \), this component is...
strongly connected. By the choice of $P$, this component is bipartite-complete. This finishes the proof of Claim 6.

Claim 6. If some $P \subseteq E' - E$ consists of at most $p$ edges such that every non-singleton connected component of $H \cup P$ is strongly connected and bipartite-complete, then some $S' \subseteq S$ with $|S'| \leq h$ has at least one element in each $S_j$.

To prove this claim, observe that because every connected component of $H \cup P$ is strongly connected, for each $j$ with $1 \leq j \leq \beta$, the set $P$ contains some edges $b_j \rightarrow a_{ij}$ and $a_{ij} \rightarrow b_j$. Then $i_j \neq 0$ and $s_{ij}$ exists in $S_j$ by Rule 3 of the construction of $H'$, $H$, and $p$. Let $S' = \{s_{i_1}, \ldots, s_{i_\beta}\}$. Let $D$ be the connected component of $H \cup P$ that contains $a_0$. Then $D$ also contains $a_{i_1}, \ldots, a_{i_\beta}$ and $b_0, \ldots, b_\beta$. By the completeness of $D$, the set $P$ has at least $(\beta + 1)|S'|$ edges. Thus $|S'| \leq h$ because $|P| \leq p = (\beta + 1)h$. This finishes the proof of Claim 6 and thus that of Theorem 5.5.

The next two problems are variants of Problems 10 and 11.

Problem 12 (Protection of a Table).
- Input: The suppressed graph $H$ of a table $T$ whose total graph is undirected.
- Output: A set $P$ consisting of the smallest number of published cells of $T$ such that the table formed by $T$ with the cells in $P$ also suppressed is protected.

Problem 13.
- Input: A bipartite undirected graph $H = (A, B, E)$.
- Output: A set $P$ consisting of the smallest number of undirected edges between $A$ and $B$ but not in $E$ such that every non-singleton connected component of $(A, B, E \cup P)$ is bipartite-complete.

Lemma 5.6. Problems 12 and 13 can be reduced to each other in linear time.

Proof. The proof is similar to that of Lemma 5.4.

Theorem 5.7 ([15]). Problem 13 can be solved in optimal $O(|H| + p)$ time, where $p$ is the output size.

Proof. This theorem follows from Lemma 5.6 and Theorem 5.5.

6. Discussions. The relationship between the data security of $T$ and the connectivity of $H$ are summarized and compared below.

| Levels of Data Security | Degrees of Graph Connectivity |
|-------------------------|-----------------------------|
| all cells               | strongly connected, bridge-free |
| all rows and columns    | strongly connected, bipartite-2-connected |
| all sets of $k$ rows or $k$ columns | strongly connected, bipartite-$(k+1)$-connected |
| the whole table         | strongly connected, bipartite-complete |

Lemma 6.1. Let $R$ be a row or column of $T$. Let $k$ be the smallest number of row vertices or column vertices in any non-singleton connected component of $H$.

1. If $R$ is protected, then every suppressed cell in $R$ is also protected.
2. If a set of $k$ rows or $k$ columns of $T$ is protected, then every subset of that set is also protected.
3. If $T$ is protected, then every set of $k-1$ rows or $k-1$ columns is also protected.

Note that the converses of the above statements are all false.

Proof. Statements 1 and 2 are straightforward. Statement 3 follows from Theorems 4.7 and 5.2.
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The number in a table cell is its value. A cell with a box is suppressed. The lower and upper bounds of the cells are 0 and 9. The graph is the suppressed graph of the table. Vertex \( R_p \) corresponds to row \( p \), and vertex \( C_q \) to column \( q \).

**Fig. 1.1. A Table and Its Suppressed Graph.**