A note on W–algebra Realisations

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ABSTRACT

We provide a general description of realisations of W–algebras in terms of smaller W–algebras and free fields. This is based on the definition of the W–algebra as the commutant of a set of screening charges. This is conjectured to be related to partial gauge-fixings in the Hamiltonian reduction model.
1 Introduction

In this note we hope to clarify some recent results on W–algebra realisations. W–algebras are non-linear generalisations of the Virasoro algebra which arise as symmetry algebras in two-dimensional conformal field theory. The first such extension was constructed abstractly by Zamolodchikov [1], and it became clear that such algebras could be associated to any semi-simple simply-laced Lie algebra, and constructions could be found in free fields [2, 3], in GKO coset models [4], and in the fields in $\hat{g}$ where $g$ is the maximally non-compact real form of the algebra [5]. However, recently constructions have been found of W–algebras in terms of W–algebras for groups of lower rank, and free fields [6, 7]. These are based heavily on the free field construction of [3]. In this note we analyse this free field construction further enabling us to give a comprehensive set of realisations of any particular W–algebra in terms of lower rank W–algebras and free fields. Firstly we review the free field construction, and then the known realisations in terms of the Virasoro algebra and free fields. We then analyse these models using the free field construction and screening charges. We end with a short example and some discussion on extensions to the affine case and the connection with Hamiltonian reduction.

2 Free field construction of W–algebras

A Quantum W–algebra can be defined in terms of a number of basic fields $\{W^a\}$ and their operator product expansions [8]. One of these fields, denoted by $L(a)$, forms a closed operator product algebra with itself, corresponding to the Virasoro algebra. The value of the central charge $c$ of this Virasoro algebra is usually the only free parameter in the W–algebras, the other structure constants being functions of $c$. The other fields are quasi-primary fields with respect to this Virasoro algebra of given conformal weight. Some progress has been made by considering the consistency requirement in conjunction with the abstract definition to show that this imposes strict limits on the conformal spins of the other basic fields. In particular W–algebras which have a few basic fields of low conformal weight have received particular attention and it has proven possible to show that the W–algebras with fields of weights $\{2, 3\}, \{2, 3, 4\}, \{2, 4\}, \{2, 6\}$ are consistent and unique and conversely that algebras with fields of spins $\{2, 5\}, \{2, 7\}$ cannot be consistently defined for arbitrary $c$ [9–11]. However most progress has been made by considering a particular realisation in terms of free massless fields. This was the form in which the original W–algebra was first constructed [12], and then the series $WA_n, WB(0, n)$ and $WD_n$ [3]. The operator product of such bosonic massless free fields $X^i(z)$ takes the form

$$X^i(z)i\partial X^j(\zeta) = \frac{i\delta^{ij}}{z-\zeta} + \circ X^i(z)i\partial X^j(\zeta)\circ,$$

where $\circ\circ$ denotes normal ordering. In ref. [2], Fateev and Lukyanov were able to show that the $n$ fields $W^a(z)$ defined by

$$\sum_m W^m(z)(\alpha\partial)^{n+1-m} = (i\partial X\cdot h^1 + \alpha\partial)(i\partial X\cdot h^2 + \alpha\partial)\ldots(i\partial X\cdot h^{n+1} + \alpha\partial),$$

generate a closed operator product algebra, if $h^i$ are a basis of $\mathbb{R}^n$ satisfying $h^i\cdot h^j = \delta^{ij} - 1/(n+1)$. This generalises the well known construction for the Virasoro algebra in
terms of one free boson, and is known as the $WA_n$ algebra. Such closed expressions have not been found for any other series of $W$–algebras, although partial results have been found for $WD_n$ and $WB(0,n)$. At this point we should make some remarks about notation. We take the $W$–algebra $Wg$ to a ‘reductive’ $W$–algebra (in the sense of [13]) with basic fields whose conformal spin is one greater than the exponents of $g$. Consequently we reserve the names $WB_n$, $WC_n$ for $W$–algebras with purely bosonic fields of spin $\{2,...,2n\}$. These are all well defined at the classical level, and at the quantum level one must rely on the free field construction to provide a proof of existence and consistency [14].

2.1 Realisations of $WA_n$ and $WD_n$ in the Virasoro algebra and free fields.

These realisations were found by Lu et al. [7], based on the earlier work of Romans [6]. They used the explicit form of the free field construction to deduce that one can construct $Wg_n$ in terms of $Wg_{n-1}$ and one free field, for $g = A, D$. We shall simply recap for the case $g = A$. Here, the construction is given by 2. By considering the basis in which $h^1$ is given by

$$h^1 = (0,0,...,-\sqrt{n/n+1}),$$

the authors were able to show that one could express $WA_n$ in terms of $WA_{n-1}$ (constructed from the fields $X^1...X^{n-1}$ and the field $X^n$). By recursively applying this argument they showed that there is a realisation of $WA_n$ in terms of fields $X^2...X^n$ and the field $X^1$ where $X^1$ occurs only through the Virasoro algebra generated by $-(\partial X^1)^2 + \alpha \partial^2 X^1$. By applying a similar line of reasoning to the field $f(z)$ which is conjectured to generate all the fields in the $WD_n$ algebra [3], one can show that there is a construction of $WD_n$ in terms of a free scalar field and $WD_{n-1}$. Since $D_2 \equiv A_1 \oplus A_1$, the final step in this recursion leads to two Virasoro algebras. Again, since $D_3 \equiv A_3$, one can show that for $n > 2$ $WA_n$ can also be realised in terms of $(n-2)$ free fields and two Virasoro algebras of the same central charge.

They then applied these results to string theories based on $W$–symmetry. We shall now reconsider these results in the light of our new definition of a $W$–algebra.

3 Free field realisations and screening charges

Our definition of the $W$–algebra free field realisation is in terms of meromorphic conformal field theory [15]. A meromorphic conformal field theory (mcft) consists of a Hilbert space $\mathcal{H}$ and a vertex operator map from a dense subspace $\mathcal{F}$ of $\mathcal{H}$ into the space of fields. There is a distinguished state, the ‘conformal state’ $|L\rangle$, whose vertex operator is the stress-energy tensor of the theory, and whose modes form a copy of the Virasoro algebra.

The important result for us is that the vertex operator gives an isomorphism between states and local fields via $|\psi\rangle \rightarrow V(\psi,z)$; from the axioms, it is possible to deduce the uniqueness property that if $U_\phi(z)|0\rangle = e^{zL_{-1}}|\phi\rangle$, then $U_\phi(z) = V(\phi,z)$. Since the singular part of the operator product of two fields only includes fields of weight less than the sum of the two weights, to show that there is a $W$–algebra of fields of spins $\Delta^i$ in a mcft with a given Hilbert space, it is only necessary to consider the space of states up to level $2 \max(\Delta) - 1$. 

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It was quite necessary for the subsequent development of the theory of the algebras that they commuted with certain screening charges, which are (formal) operators of the form

$$Q^j = \oint \frac{dz}{2\pi i} \exp(i\beta \alpha^j \cdot X(z)),$$

(4)

where $\beta$ is a constant related to the central charge of the W-algebra, and $\alpha^j$ runs over the simple roots of the associated finite Lie algebra $g$. However, it has proved more profitable to turn this around and instead take this property as the definition of an mcft, which proves to be a W–algebra [14]. For each given W–algebra one can show that the subspace of level $2\max(\Delta) - 1$ of the Hilbert space of the W–algebra is isomorphic to the space of states annihilated by the screening charges for all but a finite set of $c$-values. Thus, we say that for all but a finite set of $c$–values, the W–algebra corresponds to the set of fields which commute with the screening charges, or by the field–state isomorphism, to the set of states annihilated by the screening charges. This is the analogue of Felder’s BRST construction of the Virasoro algebra [16], and is discussed for $WA_n$ by Mizoguchi and Nakatsu in [17] and for general $g$ by Feigin and Frenkel in [14]. This definition gives a closed W–algebra satisfying our requirements concerning field content and is reductive, being a quantum deformation of the classical case. If, for all but a finite set of $c$–values, for level less than $2\max(\Delta)$ the two Hilbert spaces $\mathcal{H}, \mathcal{H}'$ are isomorphic, and the operator products of the two mcfts associated to them are isomorphic to the same level, then we shall write

$$\mathcal{H} \sim \mathcal{H}'.$$

(5)

So, we shall take the Hilbert space $\mathcal{H}_W$ of the W–algebra $W g$ of a Lie algebra $g$ of rank $n$ as

$$\mathcal{H}_W \sim (\cap_{\alpha \in \Sigma} \ker Q^\alpha(H^{(n)})).$$

(6)

Whenever $c$ takes on of the finite set of values, one finds that there are too many states in the kernel of the $Q^j$, corresponding to the presence of extra null states in the W–algebra Verma module.

It is important to note that the states in the free-field construction which are annihilated by all the screening charges (modulo our restrictions on level and $c$ above) are isomorphic to the states in an abstract W–algebra. Thus we can factor any representation of a more complex W–algebra, which uses only those states which are annihilated by a set of screening charges, through a representation of the corresponding W–algebra at that $c$–value. The rest of this section will simply expand on this point and find some consequences. We shall also use the field – state isomorphism of meromorphic conformal field theory and restrict our attention to the states created by the W–algebra fields.

To proceed, we need some notation which we introduce now. For any vector $\mu$, the field $\mu \cdot X/|\mu|$ satisfies the free field operator product. We shall write the Hilbert space of this field as $\mathcal{H}_\mu$. If $\mu^i$ are a basis of $\mathbb{R}^n$, then the Hilbert space of $n$ free bosons can be written as

$$\mathcal{H}^{(n)} = \otimes \mathcal{H}_\mu^i.$$  

(7)

We shall in particular be interested in the cases $\mu$ a simple root $\alpha^j$, or fundamental weight $\lambda^j$ of a semi-simple Lie algebra. These satisfy

$$\alpha^j \cdot \lambda^k = \delta^{jk}(\alpha^j)^2/2.$$  

(8)
We denote the set of simple roots of $g$ by $\Sigma$. We shall also often denote the semi-simple Lie algebra whose simple roots are given by $\Sigma$ by $g(\Sigma)$, and the corresponding W–algebra by $W(\Sigma)$.

As a first case, let us consider how a single free field in the direction of a simple root $\alpha_{i_0}$ enters the expressions for the W–algebra in the free field realisation. Let us consider the basis of $\mathbb{R}^n$ given by $\{\lambda^j, j \neq i_0\} \cup \{\alpha_{i_0}\}$. Then we can write the free field Fock space as

$$H^{(n)} = H_{\alpha_{i_0}} \otimes H^\perp, \text{ where } H^\perp = \otimes_{j \neq i_0} H_{\lambda^j}. \quad (9)$$

We can now apply the results of Felder, [16] which says that for the states up to some level $2\Delta$, for all but a finite set of $c$–values,

$$\ker Q_{i_0}(H_{\alpha_{i_0}}) \quad (10)$$

is isomorphic to the space of states $H_{L_{i_0}}$ generated by the Virasoro algebra $L_{i_0}$ where

$$L_{i_0}(z) = \frac{\alpha_{i_0}}{\alpha_{i_0}} - \frac{1}{2|\alpha_{i_0}|^2}(\alpha_{i_0} \cdot \partial X(z))^2 + (\beta - 2/(\beta|\alpha_{i_0}|^2))\alpha_{i_0} \cdot \partial^2 X(z)^0. \quad (11)$$

(In fact if you look more closely at the construction we see that these are a subset of those for which the W–algebra isomorphism breaks down) From (8) we also know that

$$Q_{i_0} H^\perp = 0, \quad (12)$$

And so combining (6) and (12) we obtain

$$H_W \sim (\cap_{\alpha \in \Sigma, i \neq i_0} \ker Q_i(H^{(n)})) \cap (H_{L_{i_0}} \otimes H^\perp). \quad (13)$$

Thus we see automatically that the free bosonic field $(\alpha_{i_0} \cdot \partial X(z))/(|\alpha_{i_0}|)$ only occurs in $H_W$ via the Virasoro algebra $L_{i_0}$. This provides our first result, that the free field in the direction of any simple root appears only via its corresponding Virasoro algebra $L^j$. This generalises the results obtained by [7, 6] for the simple roots at the end of the $A_n$ Dynkin diagrams, and those corresponding to the spinor and spinor–bar representations of $D_n$ to all the simple root directions of $g$.

If we regard this as a construction in the W–algebra of an isolated simple root and $n - 1$ free fields, we can see that we can extend this result to the case of two (or more) mutually orthogonal subsets of simple roots. We start by deleting one spot from the Dynkin diagram of $g$ to leave two subsets of simple roots, $\Sigma^1, \Sigma^2$. Let the deleted spot correspond to the simple root $\alpha_{i_0}$, and we also denote the Fock space corresponding to the set of root directions $\Sigma$ by $H^\Sigma$, so that

$$H^\Sigma = \otimes_{\alpha \in \Sigma} H_{\alpha}. \quad (14)$$

Then, given we can decompose the whole Fock space as

$$H^{(n)} = H^{\Sigma^1} \otimes H_{\alpha_{i_0}} \otimes H^{\Sigma^2}. \quad (15)$$

Since

$$\alpha^j \cdot \alpha^k = 0, \quad \alpha^j \in \Sigma^1, \alpha^k \in \Sigma^2, \text{ and } \alpha^j \cdot \lambda_{i_0} = 0, \quad \alpha^k \in \Sigma^1, \alpha^j \in \Sigma^2, \quad (16)$$

we immediately see that

$$Q^j(H_{\lambda_{i_0}} \otimes H^{\Sigma^1}) = 0, \quad \text{for } \alpha^j \in \Sigma^2, \quad (17)$$
and similarly replacing 1 by 2. We then obtain

\[
\mathcal{H}_{W(\Sigma)} \sim \ker Q^0_0(\mathcal{H}^{(n)}) \cap \left( \bigcap_{\alpha^i \in \Sigma^1} \ker Q^i(\mathcal{H}^{(n)}) \right) \cap \left( \bigcap_{\alpha^i \in \Sigma^2} \ker Q^i(\mathcal{H}^{(n)}) \right) \\
= \ker Q^0_0(\mathcal{H}^{(n)}) \cap \left[ \left( \bigcap_{\alpha^i \in \Sigma^1} \ker Q^i(\mathcal{H}^{\Sigma^1}) \right) \otimes \mathcal{H}_{\lambda^0} \right] \times \left( \bigcap_{\alpha^i \in \Sigma^2} \ker Q^i(\mathcal{H}^{\Sigma^2}) \right) \\
\cap \left[ \mathcal{H}_{\lambda^0} \otimes \mathcal{H}_{\Sigma^1} \otimes \left( \bigcap_{\alpha^i \in \Sigma^2} \ker Q^i(\mathcal{H}^{\Sigma^2}) \right) \right] \\
\sim \ker Q^0_0(\mathcal{H}^{(n)}) \cap \left[ \mathcal{H}_{W(\Sigma^1)} \otimes \mathcal{H}_{\lambda^0} \otimes \mathcal{H}_{W(\Sigma^2)} \right]
\]  

(18)

Again, if we are careful about the \(c\)-values for the various \(W\)-algebras, and use the determinant formulae for the \(W\)-algebras, we can see that the \(c\)-values for which there are null states for the smaller \(W\)-algebras at levels less than \(2\Delta\) are a subset of those of the larger.

The final upshot is that the free fields in the directions spanned by the simple roots in the sets \(\Sigma^1, \Sigma^2\) only appear in the final realisation of \(W(\Sigma)\) through the \(W\)-algebras \(W(\Sigma^1)\) and \(W(\Sigma^2)\) respectively. Since the spaces \(\mathcal{H}_{W(\Sigma)}\) are isomorphic to the vacuum representations of the \(W\)-algebras \(W(\Sigma^i)\), this means that we can replace them by the abstract \(W\)-algebra, subject to the condition that these have central charges \(c^i\) given by

\[
c^i = n^i - (\beta \rho^i - 1/\beta \rho^{\vee})^2 ,
\]

(19)

where \(n^i = \text{rank}(g(\Sigma^i)), \rho^i = \sum_{\alpha^i \in \Sigma^i} \lambda^i, \rho^{\vee} = \sum_{\alpha^i \in \Sigma^i} \lambda^{\vee}i\).

This now provides us with our second result, which is that one can find a realisation of a \(W\)-algebra \(W(\Sigma)\) in the \(W\)-algebras \(W(\Sigma^1)\) for subdiagrams obtained by deleting one node, and one free field, subject to the condition that the \(c\)-values of the three \(W\)-algebras all satisfy (19) for some \(\beta\).

3.1 Example

We run through the simple example of \(WA_4\) in the free field construction and show that we can find a realisation in terms of \(WA_2, WA_1\) and one free field.

The Dynkin diagram of \(WA_4\) is simply

\[
\begin{array}{c}
\circ - \circ - \circ
\end{array}
\]

If we number the simple roots from 1 to 4 from left to right, then we shall consider deleting the root direction 3, and take a basis of \(\mathbb{R}^4\)

\[
\{\alpha^1, \alpha^2, \lambda^3, \alpha^4\}.
\]

(20)

If we consider the vectors \(h^i\) arising in the free field construction (2), these can be written in terms of the basis vectors as

\[
3h^1 = \lambda^3 + 2\alpha^1 + \alpha^2 , \quad 3h^2 = \lambda^3 - \alpha^1 + \alpha^2 , \quad 3h^3 = \lambda^3 - 2\alpha^2 - \alpha^1 , \quad 3h^4 = \alpha^4 - \lambda^3 , \quad 2h^5 = -\alpha^4 - \lambda^3.
\]

(21)  (22)
If we insert these into (2), we obtain
\[
\sum_{m} W^m(z)(\alpha \partial)^{n+1-m} = (23)
\]
\[
(W_{1,2} + \mu L_{1,2} + \mu^3 + 3\alpha \mu' + \alpha^2 \mu'') + (L_{1,2} + 3\mu^2 + 3\alpha \mu')\alpha \partial + (3\mu)(\alpha \partial)^2 + (\alpha \partial)^3) \quad (24)
\]
\[
\times \left( \left( (9/2)\mu^2 - (3\alpha/2)\mu' - L_4 \right) - 3\mu \alpha \partial + (\alpha \partial)^2 \right), \quad (25)
\]
where
\[
W_{1,2} + L_{1,2}(\alpha \partial) + (\alpha \partial)^3 = \left( i\partial X \cdot (2\alpha^1 + \alpha^2)/3 + \alpha \partial \right) \left( i\partial X \cdot (-\alpha^1 + \alpha^2)/3 + \alpha \partial \right)
\]
\[
\times \left( i\partial X \cdot (-\alpha^1 - 2\alpha^2)/3 + \alpha \partial \right), \quad (26)
\]
\[
L_4 = -(\alpha^4 \cdot \partial X/|\alpha^4|)^2 + i\alpha^4 \cdot \partial^2 X,
\]
\[
\mu = (1/3)\lambda^3 \cdot i\partial X.
\]

As can be seen, this provides a construction in \(WA_2, WA_1\) and one free field, with the central charges satisfying (19), where \(\alpha = \beta - 1/\beta\).

4 Conclusions

By repeating this procedure outlined above, we can construct a realisation of a W–algebra \(W(\Sigma)\) in the W–algebras \(W(\Sigma^i)\) for subdiagrams obtained by deleting nodes from the Dynkin diagram of the finite algebra, and one free field for each node deleted, subject to the condition that the c-values of the W–algebras all satisfy (19) for some \(\beta\).

This is very reminiscent of the classification of regular subgroups of a semisimple Lie group. In that case there is a regular subgroup for of the form
\[
G(\Sigma^1) \otimes \ldots \otimes G(\Sigma^n) \otimes U(1)^{(n-1)}, \quad (27)
\]
where \(\{\Sigma^i\}\) are the simple roots of sub-diagrams obtained by deleting \(n\) spots from the affine Dynkin diagram for \(g\). Let us now take the point of view that W–algebra is the symmetry algebra obtained by Quantum Hamiltonian reduction of a certain coadjoint orbit of the affine Lie group, by a subgroup corresponding to the nilpotent subalgebra. In refs. [18], de Groot et al. considered a similar system in which they associated an integrable hierarchy to each affine Lie algebra, choice of grading of the affine root space, and choice of regular element of a particular subspace, by a Hamiltonian reduction with respect to a nilpotent subalgebra of the affine algebra. They then found that they could find different spaces which all had the same integrals of motion by a set of partial gauge fixings of the reduction, which were given by gradings of the root space which were between the grading they first thought of and another in a certain partial ordering. The case we have here of a W–algebra can be certainly be viewed as a Hamiltonian reduction, and the grading to which it is associated is the grading inherited from the principal \(sl(2)\) embedding. This assigns grade 1 to each simple root. There is a partial gauge fixing for each grade which assign either 1 or 0 to each simple root. The occurrence of the realisations we have found can be viewed as a consequence of this property; we can partially gauge fix and so reduce our effective space to the space of the sub-W–algebra and the free fields before completing the process to arrive at the W–algebra.
One can also turn to the affine equivalent of this construction. The commutant of the screening charges associated to all the simple roots in an affine diagram now give the local conserved quantities of motion of the Affine Toda Field Theory [19]. The analysis presented here would say that one can find a construction of these local conserved quantities in terms of the W–algebras for any regular subalgebra of $g$ and the appropriate number of free fields.

If one wishes to go on to consider the more general W–algebras obtained recently [20], then one needs to have a better understanding of the quantum screening charges for these cases, which have the classical form $Q_\beta = \sum_{\alpha \in \Sigma} \int \text{Tr}(G(x)E_\alpha G(x)^{-1}E_{-\beta})$ where $G$ is a WZW field, and $\Sigma$ is a set of simple roots for the nilpotent subalgebra being gauged.

5 Acknowledgements

I would like to thank P. Bowcock for several illuminating conversations, and H.G. Kausch for discussions over a long period. This work was supported by an SERC research assistantship.

6 References

[1] A. B. Zamolodchikov, Theoretical and Mathematical Physics 65 (1985) 347.
[2] V. A. Fateev and S. L. Luk’yanov, Int. J. Mod. Phys. A3 (1988) 507.
[3] V. A. Fateev and S. L. Luk’yanov, Sov. Sci. Rev. A15 (1990) 1.
[4] F. A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, Nucl. Phys. B304 (1988) 348; Nucl. Phys. B304 (1988) 371.
[5] J. Balog, L. Fehér, P. Forgács, L. O’ Raifeartaigh and A. Wipf, Annals of Physics 203 No. 1 (1990) 76. Phys. Lett. B244 (1990) 435.
[6] L. J. Romans, Nucl. Phys. B352 (1991) 829.
[7] H. Lu, C. N. Pope, S. Schrans and K. W. Xu, The Complete Spectrum of the $W_N$ String, Texas A and M Preprint CTP TAMU–5/92 (1992). H. Lu, C. N. Pope, S. Schrans and X. J. Wang, New Realisations of W algebras and W strings, Texas A and M Preprint CTP TAMU–15/92 (1992); , Sibling and Exceptional W Strings, Texas A and M Preprint CTP TAMU–10/92 (1992).
[8] G. M. T. Watts, Phys. Lett. B245 (1990) 65.
[9] P. Bouwknegt, Phys. Lett. 207B (1988) 295.
[10] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, W–algebras with two and three Generators, Nucl. Phys. B361 (1991) 255.
[11] H. G. Kausch and G. M. T. Watts, Nucl. Phys. (1991) 740.
[12] V. Fateev and A. Zamolodchikov, Nucl. Phys. B280 [FS18] (1987) 644.
[13] P. Bowcock and G. M. T. Watts, On the Classification of Quantum W–algebras, Enrico Fermi Institute Preprint EFI–91–63 (1991), To appear in Nucl. Phys. B.
[14] B. Feigin and E. Frenkel, Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras, Research Institute in Mathematical Sciences, Kyoto, Preprint RIMS 796 (1991).
[15] P. Goddard, Meromorphic Conformal Field Theory, *in: Infinite Dimensional Lie Algebras and Lie Groups*, ed. V. G. Kac, World Scientific, 1989, CIRM-Luminy July conference on Infinite dimensional Lie Algebras and Lie Groups, Marseille 1988.

[16] G. Felder, *BRST Approach to Minimal Models*, Nucl. Phys. B317 (1989) 215.

[17] S. Mizoguchi and T. Nakatsu, *BRST Structure of the $W_3$ minimal model*, University of Tokyo Preprint UT-566 (1991).

[18] M. F. de Groot, T. J. Hollowood and J. L. Miramontes, *Generalised Drinfeld Sokolov Hierarchies I*, IASSNS-HEP-91/19; N. Burroughs, M. F. de Groot, T. J. Hollowood and J. L. Miramontes, *Generalised Drinfeld Sokolov Hierarchies II*, IASSNS-HEP-91/42; N. Burroughs, *Coadjoint orbits of the generalised $sl(2)$, $sl(3)$ KdV hierarchies*, IASSNS-HEP-91/67.

[19] B. L. Feigin and E. V. Frenkel, *Free Field resolutions and affine Toda theory*, Research Institute in Mathematical Sciences, Kyoto, Preprint RIMS–827 (1991).

[20] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, *Generalized Toda theories and $W$ algebras associated with integral gradings*, Ann. Phys. 213 (1992) 1–20; L. Fehér, *$W$-Algebras of generalized Toda theories*, Dublin preprint DIAS-STP-91-22; F. A. Bais, T. Tjin and P. van Driel, *Covariantly coupled chiral algebras*, Nucl. Phys. B357 (1991) 632.