Mean Ergodic Shadowing

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Abstract
We introduce and study a new variant of shadowing namely mean ergodic shadowing. We establish relationship of this variant with several other variants of shadowing. We show that a minimal system with shadowing cannot have mean ergodic shadowing. We give a necessary and sufficient condition for an orbital limit function to have mean ergodic shadowing property.

Keywords Shadowing · Ergodic shadowing · Sub-shadowing · Average shadowing · Minimality

Mathematics Subject Classification 37B05

1 Introduction
Throughout this paper, a dynamical system is a pair \((X, f)\), where \(X\) is a compact metric space unless otherwise stated (To avoid technical difficulties, we assume that \(X\) contains at least two points) and \(f : X \to X\) is a continuous map. The concept of shadowing was originated from Anosov closing lemma and because of its rich consequences, shadowing plays an important role in topological dynamics. It is believed that the study of certain variants or generalizations of a particular interesting concept give new insights. This belief gave the impetus to study several variants of shadowing including ergodic shadowing (Fakhari and Ghane 2010), sub-shadowing (Dastjerdi and Hosseini 2010) and average shadowing (Wu et al. 2016). These variants share a

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common motivation of studying the behaviour of a dynamical system by exploiting the closeness of approximate orbits and exact orbits. The similarities and differences between shadowing and each of these variants help to understand better the behaviour of a system with shadowing. Another important concept in topological dynamics is minimality. In a minimal system, the orbit of individual point is dense in the phase space. In (Fedeli and Donne 2009; Sharma 2015), authors investigated relationship between the closeness of approximate orbits and exact orbits. The similarities and differences.

2 Preliminaries

Let \((X,d)\) be a compact metric space and let \(f\) be a continuous map on \(X\). Let \(\mathbb{N}\) be the set of all non-negative integers. The density and upper density of \(E \subset \mathbb{N}\) is defined respectively by \(d(E) = \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \#(E \cap [0,n-1])\) and \(\overline{d}(E) = \lim sup_{n \to \infty} \#(E \cap [0,n-1])\), where \(\#(B)\) denotes the cardinality of \(B \subset \mathbb{N}\) and \([0,n-1] = \{0, 1, 2, 3, \ldots, n-1\}\).

Let \(\xi = \{x_i\}_{i \in \mathbb{N}}\) be a sequence in \(X\) and \(\epsilon, \delta\) be positive real numbers. Then,

(i) \(\xi\) is said to be a \(\delta\)-pseudo orbit if \(d(f(x_i), x_{i+1}) < \delta\) for all \(i \in \mathbb{N}\) and \(\xi\) is said to be \(\epsilon\)-shadowed by some \(x \in X\) if \(d(f^i(x), x_i) < \epsilon\) for all \(i \in \mathbb{N}\). \(f\) is said to have shadowing if for every \(\epsilon > 0\), there is \(\delta > 0\) such that every \(\delta\)-pseudo orbit \(\xi\) is \(\epsilon\)-shadowed by some point in \(X\).

(ii) \(\xi\) is said to be a \(\delta\)-ergodic pseudo orbit if \(d(f(x_i), x_{i+1}) < \delta\) for all \(i \in \mathbb{N}\) except a set of density zero. \(\xi\) is said to be \(\epsilon\)-ergodic shadowed by some \(x \in X\) if \(d(f^i(x), x_i) < \epsilon\) for all \(i \in \mathbb{N}\) except a set of density zero. \(\xi\) is said to be \(\epsilon\)-ergodic shadowed with index \(\alpha\) by some \(x \in X\) if \(\overline{d}(B(x, \xi, \epsilon)) > \alpha\), where \(B(x, \xi, \epsilon) = \{i \in \mathbb{N} \mid d(f^i(x), x_i) < \epsilon\}\). \(f\) is said to have ergodic shadowing (Fakhari and Ghane 2010) if for every \(\epsilon > 0\), there is \(\delta > 0\) such that every \(\delta\)-ergodic pseudo orbit \(\xi\) is \(\epsilon\)-ergodic shadowed by some point in \(X\). \(f\) is said to have \(M^\alpha\)-shadowing (Wu et al. 2016) if for every \(\epsilon > 0\), there is \(\delta > 0\) such that every \(\delta\)-ergodic pseudo orbit is \(\epsilon\)-ergodic shadowed with index \(\alpha\), by some point in \(X\). So, \(d\)-shadowing (Dastjerdi and Hosseini 2010) is a special case for \(\alpha = \frac{1}{2}\). \(f\) is said to have \(d\)-shadowing (Dastjerdi and Hosseini 2010) if for every \(\epsilon > 0\), there is \(\delta > 0\) such that for every \(\delta\)-ergodic pseudo orbit \(\xi = \{x_i\}_{i \in \mathbb{N}}\) there exists \(x \in X\) such that \(d(B(x, \xi, \epsilon)) > 0\).

(iii) \(\xi\) is said to be a \(\delta\)-average pseudo orbit if there is \(N \in \mathbb{N}\) such that for each \(n \geq N\) and any non-negative integer \(k\), \(\lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta\). \(\xi\) is said to be an asymptotic average pseudo orbit if \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0\). \(\xi\) is said to be a \(\delta\)-asymptotic average pseudo orbit if \(\lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) < \delta\). \(\xi\) is said to be
\( \epsilon \)-shadowed in average by some \( x \in X \) if \( \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), x_i) < \epsilon \). \( \xi \) is said to be asymptotically shadowed in average by some \( x \in X \) if \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), x_i) = 0 \). \( f \) is said to have average shadowing if for every \( \epsilon > 0 \), there is \( \delta > 0 \) such that every \( \delta \)-average pseudo orbit is \( \epsilon \)-shadowed in average by some point in \( X \). \( f \) is said to have asymptotic average shadowing if every asymptotic average pseudo orbit is asymptotically shadowed in average by some point in \( X \). \( f \) is said to have weak asymptotic average shadowing if for every \( \epsilon > 0 \), any asymptotic average pseudo orbit is \( \epsilon \)-shadowed in average by some point in \( X \).

We define \( N(U, V) = \{ n \in \mathbb{N}^+ \mid f^n(U) \cap V \neq \emptyset \} \) for two non-empty open sets \( U, V \subset X \) and \( N(x, U) = \{ n \in \mathbb{N}^+ \mid f^n(x) \in U \} \) for non-empty open set \( U \subset X \) containing \( x \). \( f \) is called transitive if \( N(U, V) \neq \emptyset \) and totally transitive if \( f^k \) is transitive for each \( k \in \mathbb{N}^+ \). A point \( x \in X \) is called syndetically recurrent if \( N(x, U) \) is syndetic for every neighborhood \( U \) of \( x \). If \( M(g) \) denote the set of all syndetically recurrent points of \( g \), then note that \( M(f) = M(f^k) \) for each \( k \in \mathbb{N}^+ \). A closed set \( L \) is minimal for \( f \) if \( L = \Omega_f(x) \) for some \( x \in M(f) \). If the phase space \( X \) is minimal, then we say that \( f \) is minimal. If \( f \) is minimal, then all points of the phase space are syndetically recurrent. But all points syndetically recurrent does not imply that the system is minimal. A finite \( \delta \)-chain is a finite set of points \( \{x_0, x_1, x_2, x_3, \ldots, x_n\} \) such that \( d(f(x_i), x_{i+1}) < \delta \) for all \( 0 \leq i \leq (n - 1) \). We say that \( f \) is chain transitive if for any \( \epsilon > 0 \) and any pair of points \( x, y \in X \) there is a finite \( \epsilon \)-chain from \( x \) to \( y \) (i.e. \( x_0 = x \) and \( x_n = y \)) and that \( f \) is totally chain transitive if \( f^k \) is chain transitive for each \( k \in \mathbb{N}^+ \).

## 3 Mean Ergodic Shadowing and Other Variants

In this section, we introduce a new variant of shadowing called mean ergodic shadowing and study its properties and its relationships with other variants of shadowing.

**Definition 3.1** Let \( f : X \to X \) be a continuous map on a compact metric space \( X \). Then \( f \) is said to have mean ergodic shadowing if for every \( \epsilon > 0 \), there is \( \delta > 0 \) such that every \( \delta \)-ergodic pseudo orbit can be \( \epsilon \)-shadowed in average by some point in \( X \).

**Lemma 3.2** Let \( f : X \to X \) be a continuous map on a compact metric space \( X \). Then, \( f \) has mean ergodic shadowing if and only if for every \( \epsilon > 0 \), there is \( \delta > 0 \) such that every \( \delta \)-ergodic pseudo orbit is \( \epsilon \)-shadowed except on a set of upper density \( \epsilon \).

**Proof** Suppose that \( f \) has mean ergodic shadowing. For every \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( \xi = \{x_i\} \in \mathbb{N} \) is a \( \delta \)-ergodic pseudo orbit, then there is \( x \in X \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), x_i) < \epsilon^2.
\]

If \( E = \{ i \in \mathbb{N} \mid d(f^i(x), x_i) \geq \epsilon \} \), then

\[
\epsilon^2 > \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), x_i) \geq \limsup_{n \to \infty} \frac{1}{n} (\epsilon ||\{0, n - 1\} \cap E||) = \epsilon \overline{d}(E),
\]

This implies that \( \overline{d}(E) < \epsilon \).
Conversely, fix $\epsilon > 0$ and choose $\eta < \frac{\epsilon}{\text{diam}(X)+1}$. Let $\delta > 0$ be such that if $\xi = \{x_i\}_{i \in \mathbb{N}}$ is a $\delta$-ergodic pseudo orbit, then $\xi$ is $\eta$-shadowed except on a set of upper density less than $\epsilon$, by some point $x$ in $X$. Thus, if $E = \{i \in \mathbb{N} \mid d(f^i(x), x_i) \geq \eta\}$, then $\overline{d}(E) < \eta$ and
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), x_i) \\
\leq \limsup_{n \to \infty} \frac{1}{n} (\text{diam}(X)\#([0, n - 1] \cap E) + \eta n) \\
\leq \text{diam}(X)\overline{d}(E) + \eta < \epsilon.
\]
This implies that $f$ has mean ergodic shadowing. □

In view of the above lemma, the following definition of mean ergodic shadowing is useful in our study.

**Definition 3.3** Let $f : X \to X$ be a continuous map on a compact metric space $X$. Then $f$ has mean ergodic shadowing if for every $\epsilon > 0$, there is $\delta > 0$ such that if $\xi = \{x_i\}_{i \in \mathbb{N}}$ is a $\delta$-ergodic pseudo orbit, there is $x \in X$ such that $\overline{d}(B^\epsilon(x, \xi, \epsilon)) < \epsilon$, where $B^\epsilon(x, \xi, \epsilon) = \{i \in \mathbb{N} \mid d(f^i(x), x_i) \geq \epsilon\}$.

It follows from Definition 3.3 that ergodic shadowing implies mean ergodic shadowing. Further, it is immediate from the identity $\overline{d}(A) = 1 - \overline{d}(A^c)$ that mean ergodic shadowing implies $d$-shadowing.

**Theorem 3.4** Let $f$ and $g$ be two continuous maps on $(X, d_1)$ and $(Y, d_2)$, respectively. If $h : X \to Y$ is a homeomorphism, then $g = h \circ f \circ h^{-1}$ has mean ergodic shadowing if and only if $f$ has mean ergodic shadowing.

**Proof** Suppose that $f$ has mean ergodic shadowing. Let $\epsilon > 0$ and $0 < \epsilon' < \epsilon$ be given for $\epsilon$ by uniform continuity of $h^{-1}$, i.e. $d_1(x, y) < \epsilon'$ implies $d_2(h(x), h(y)) < \epsilon$. Further, let $\delta' > 0$ be such that every $\delta'$-ergodic pseudo orbit of $f$ is $\epsilon'$-shadowed except on a set of upper density less than $\epsilon'$ and let $\delta > 0$ be given for $\delta'$ by uniform continuity of $h^{-1}$, i.e. $d_2(x, y) < \delta$ implies $d_1(h^{-1}(x), h^{-1}(y)) < \delta'$.

Let $\xi = \{x_i\}_{i \in \mathbb{N}}$ be a $\delta$-ergodic pseudo orbit for $g = h \circ f \circ h^{-1}$, i.e. $d(E) = 0$, where
\[
E = \{i \in \mathbb{N} \mid d_2((h \circ f \circ h^{-1})(x_i), x_{i+1}) \geq \delta\} \\
\supset \{i \in \mathbb{N} \mid d_1(f(h^{-1}(x_i)), h^{-1}(x_{i+1})) \geq \delta'\}.
\]
This shows that $\{h^{-1}(x_i)\}_{i \in \mathbb{N}}$ is a $\delta'$-ergodic pseudo orbit for $f$. So by the mean ergodic shadowing of $f$, there is $x \in X$ such that $\overline{d}(E') < \epsilon'$, where
\[
E' = \{i \in \mathbb{N} \mid d_1(f^i(x), h^{-1}(x_i)) \geq \epsilon'\} \\
\supset \{i \in \mathbb{N} \mid d_2(h(f^i(x)), x_i) \geq \epsilon\} \\
= \{i \in \mathbb{N} \mid d_2((h \circ f^i \circ h^{-1})(h(x)), x_i) \geq \epsilon\} \\
= \{i \in \mathbb{N} \mid d_2(g^i(h(x)), x_i) \geq \epsilon\}.
\]
If $E'' = \{i \in \mathbb{N} \mid d_2(g^i(h(x)), x_i) \geq \epsilon\}$, we have $\overline{d}(E'') < \epsilon$ because $\overline{d}(E') < \epsilon' < \epsilon$. Thus, $g$ has mean ergodic shadowing. The converse holds because $h$ is a homeomorphism. □
Theorem 3.5 Let $f$ and $g$ be two continuous maps on $(X, d_1)$ and $(Y, d_2)$, respectively. If $f$ and $g$ has mean ergodic shadowing, then $f \times g : X \times Y \to X \times Y$ has mean ergodic shadowing.

Proof Here, $(X \times Y, d)$ is a compact metric space, where $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ be given by $d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$. Let us fix $\epsilon > 0$. Let $\delta_1$ and $\delta_2 > 0$ be given for $\frac{\epsilon}{2}$ by mean ergodic shadowing of $f$ and $g$, respectively. Let $\delta = \min\{\delta_1, \delta_2\}$ and $(\{x_i \times y_i\}_{i \in \mathbb{N}})$ be a $\delta$-ergodic pseudo orbit for $f \times g$. Then $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are $\delta_1$ and $\delta_2$ ergodic pseudo orbit for $f$ and $g$, respectively. So, there is $x \in X$ and $y \in Y$ such that $d(A) < \frac{\epsilon}{2}$ and $d(B) < \frac{\epsilon}{2}$, where $A = \{i \in \mathbb{N} \mid d_1(f^i(x), x_i) \geq \frac{\epsilon}{2}\}$ and $B = \{i \in \mathbb{N} \mid d_2(g^i(y), y_i) \geq \frac{\epsilon}{2}\}$. Then it is clear that $d(\{i \in \mathbb{N} \mid d(f^i(x \times y), (x_i \times y_i)) \geq \epsilon\}) < \epsilon$. This completes our proof. \hfill \Box

Theorem 3.6 If $f$ has mean ergodic shadowing then for each $k \in \mathbb{N}^+$, $f^k$ has mean ergodic shadowing.

Proof Let $\epsilon > 0$ and $k > 1$ be a fixed integer. Let $\delta > 0$ be given for $\frac{\epsilon}{k}$ by mean ergodic shadowing of $f$. Let $\{x_0, x_1, \ldots, x_{m_0}, x_{m_0+1}, x_{m_0+2}, \ldots, x_{m_1}, x_{m_1+1}, \ldots\}$ be a $\delta$-ergodic pseudo orbit for $f^k$, in which $d(f(x_{m_i}), x_{m_i+1}) \geq \delta$ for all $i \in \mathbb{N}$ and $d(f(x, y)) < \delta$ for any other pair of consecutive points $x$ and $y$ in the sequence. By locating the finite sequence $(f(x_1), f^2(x_1), \ldots, f^{k-1}(x_1))$ between $x_i$ and $x_{i+1}$ for all $i \in \mathbb{N}$, we obtain the $\delta$-ergodic pseudo orbit $\{y_i\}_{i \in \mathbb{N}} = \{x_0, f(x_0), f^2(x_0), \ldots, f^{k-1}(x_0), x_1, f(x_1), f^2(x_1), \ldots, f^{k-1}(x_1), x_2, \ldots\}$ for $f$.

By mean ergodic shadowing of $f$, there is $y \in X$ such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), y_i) < \frac{\epsilon}{k}.
\]
Therefore,
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d((f^k)^i(y), x_i)
\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{ik}(y), y_{ik})
\leq k \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{ki+j}(y), y_{ki+j})
\leq k \epsilon.
\]
Thus, we conclude that $f^k$ has mean ergodic shadowing for each $k \in \mathbb{N}^+$. \hfill \Box

Example 3.7 Let $f : [0, 1] \to [0, 1]$ be given by $f(x) = 1 - x$. Since $f$ is an isometry, it cannot have shadowing. We prove that $f$ cannot have mean ergodic shadowing either. Let $\epsilon < \frac{1}{3}$ and set
\[
a_0 = \{0, 1\}, a_1 = \{0, 1, 1, 0\}, a_2 = \{0, 1, 0, 1, 1, 0, 1, 0\}, \ldots,
\]
\[
a_n = \{0, 1, 0, 1, \ldots, 0, 1 \text{ n-times}, 1, 0, \ldots, 1, 0, 1, 0 \text{ n-times}\}, \ldots
\]
(0,1) repeated
(1,0) repeated

Note that $\{a_0 \lor a_1 \lor a_2 \lor \cdots \lor a_n \lor \cdots\}$ is a $\delta$-ergodic pseudo orbit for any $\delta > 0$, where $\lor$ denotes the concatenation operation, for example, $a_0 \lor a_1 = \{0, 1, 0, 1, 1, 0\}$. But it cannot be $\epsilon$-shadowed except on a set of upper density less than $\epsilon$ due to the following reason.

Observe that a tracing point $x$ must come from the $\epsilon$-neighborhood of either zero or one. If it comes from a neighborhood of zero, then the orbit cannot trace the $n$-times
repeated \{1, 0\} in the second half of each of \(a_n\). If it comes from a neighborhood of one, then the orbit cannot trace the \(n\)-times repeated \{0, 1\} in the first half of each of \(a_n\).

**Proposition 3.8** If \(f\) has mean ergodic shadowing, then it has \(M^\alpha\)-shadowing for each \(\alpha \in (0, 1)\).

**Proof** Fix \(\alpha \in (0, 1)\) and choose \(\epsilon_0 > 0\) such that \(\alpha < 1 - \epsilon_0\). For this \(\epsilon_0\) there exists \(\delta_0 > 0\) by mean ergodic shadowing of \(f\), such that every \(\delta_0\)-ergodic pseudo orbit \(\xi = \{x_i\}_{i \in \mathbb{N}}\) satisfies \(d(\overline{B}(x, \xi, \epsilon_0)) \leq 1 - \epsilon_0 > \alpha\) for some \(x \in X\). If \(\epsilon > \epsilon_0\), then choose the same \(\delta_0\) associated with \(\epsilon_0\) so that every \(\delta_0\)-ergodic pseudo orbit \(\xi = \{x_i\}_{i \in \mathbb{N}}\) satisfies \(d(\overline{B}(x, \xi, \epsilon)) > 1 - \epsilon > 1 - \epsilon_0 > \alpha\). If \(\epsilon < \epsilon_0\), then by mean ergodic shadowing there exists \(\delta > 0\) such that every \(\delta\)-ergodic pseudo orbit \(\xi = \{x_i\}_{i \in \mathbb{N}}\) satisfies \(d(\overline{B}(x, \xi, \epsilon)) > 1 - \epsilon > 1 - \epsilon_0 > \alpha\). Since \(\alpha\) was chosen arbitrarily, we are through. \(\Box\)

**Proposition 3.9** If \(f\) has weak asymptotic average shadowing, then it has mean ergodic shadowing.

**Proof** In view of Theorem 4.3 (Wu et al. 2016), it is sufficient to show that a \(\delta\)-ergodic pseudo orbit \(\{x_i\}_{i \in \mathbb{N}}\) is a \(\delta\)-asymptotic average pseudo orbit.

Indeed if \(E = \{i \in \mathbb{N} \mid d(f(x_i), x_{i+1}) \geq \delta\}\), then \(d(E) = 0\) and hence, \(\overline{d}(E) < \eta\) where \(\eta < \frac{diam(X)}{\delta}\). Therefore,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) \\
\leq \lim_{n \to \infty} \frac{1}{n} (diam(X)\#([0, n - 1] \cap E) + \eta n) \\
\leq diam(X)\overline{d}(E) + \eta < \delta. 
\]

The following result holds because it is easy to observe that asymptotic average shadowing implies weak asymptotic average shadowing.

**Corollary 3.10** If \(f\) has asymptotic average shadowing, it has mean ergodic shadowing.

**Example 3.11** Let \(f : [0, 1] \to [0, 1]\) be a continuous monotone map with the property: \(f(x) > x\) if and only if \(x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]\). By Example 5.4 (Gu 2007), \(f\) has asymptotic average shadowing but does not have shadowing property. Therefore Corollary 3.10 implies that \(f\) has mean ergodic shadowing but does not possess shadowing property.

By Theorem 5.5 (Wu et al. 2016), we know that weak asymptotic average shadowing and average shadowing are equivalent notions. Therefore, we have the following result.

**Corollary 3.12** If \(f\) has average shadowing, then it has mean ergodic shadowing.

The following result holds because it is proved in (Garg and Das 2016) that almost average shadowing implies average shadowing.

**Corollary 3.13** If \(f\) has almost average shadowing, then it has mean ergodic shadowing.
Theorem 3.14 Let $f : X \rightarrow X$ be a continuous map on a compact metric space $X$. If $f$ has shadowing, then the following are equivalent:

(i) $f$ has mean ergodic shadowing,
(ii) $f$ has average shadowing,
(iii) $f$ is totally transitive,

Proof It follows from Proposition 3.8 and Theorem 2.2 (Dastjerdi and Hosseini 2010) and Theorem 3.6 that if $f$ has mean ergodic shadowing, then it is totally chain transitive. Since $f$ has shadowing, $f^k$ has shadowing (Aoki and Hiraide 1994). Therefore, to show (i)$\Rightarrow$(iii) it is enough to show that chain transitivity implies transitivity. Suppose that $f$ is chain transitive. Let $U, V \subset X$ be a pair of non-empty open sets and $x \in U, y \in V$. Let $\epsilon > 0$ be such that $B(x, \epsilon) \subset U$ and $B(y, \epsilon) \subset V$. Let $\delta > 0$ be given for $\epsilon$ by shadowing of $f$. Since $f$ is chain transitive, for any sufficiently large $n > 0$ there is a finite set \{ $x = x_0, x_1, \ldots, x_n = y$ \} which can be extended to a $\delta$-pseudo orbit \{ $x = x_0, x_1, \ldots, x_n = y, f(y), f^2(y), \ldots$ \}. By shadowing of $f$ there is $w \in X$ such that $d(w, x) < \epsilon$ and $d(f^n(w), y) < \epsilon$, i.e. $w \in B(x, \epsilon) \subset U$ and $f^n(w) \in B(y, \epsilon) \subset V$ which implies $f^n(U) \cap V \neq \emptyset$. Thus, $f$ is transitive. The implication (iii)$\Rightarrow$(ii) is a consequence of Theorem 3.4 (Garg and Das 2016). The implication (ii)$\Rightarrow$(i) follows from Proposition 3.12 without the presence of shadowing. $\Box$

So, mean ergodic shadowing of a continuous map with shadowing is equivalent to one and hence, all the dynamical notions listed in Theorem 3.8 (Kulczycky et al. 2014) and Theorem 3.4 (Garg and Das 2016).

Example 3.15 Let $X = C_1 \cup C_2$, where $C_1$ and $C_2$ be disjoint circles in euclidean plane and $f$ be the map sending a point $\theta$ on $C_1$ to $2\theta$ on $C_2$ and vice-versa. Then, $f^2$ is not chain transitive and hence, not totally chain transitive. So by Theorem 3.14, it cannot have mean ergodic shadowing.

Example 3.16 The period doubling map $f : S^1 \rightarrow S^1$ given by $f(\theta) = 2\theta$ has shadowing and it is totally transitive. So, by Theorem 3.14, it has mean ergodic shadowing.

Example 3.17 For any compact metric space $X$, the shift map $\sigma : X^\mathbb{N} \rightarrow X^\mathbb{N}$ is totally transitive and has shadowing. So, by Theorem 3.14, it has mean ergodic shadowing.

Since finite shadowing is equivalent to shadowing on relatively compact metric spaces, therefore Lemma 3.2 (Fakhari and Ghane 2010) is true for continuous maps on such spaces. But the following example shows that the converse is not true.

Example 3.18 Let $X = \bigcup_{n \in \mathbb{Z}} \{(n) \times [0, \frac{1}{2^n}]\} \subset \mathbb{R}^2$ be the subspace of $\mathbb{R}^2$. Let $f$ be the homeomorphism on $X$ given by

$$f(k, x) = \begin{cases} (k + 1, \frac{1}{2}x) & \text{for } k \geq 0 \\ (k + 1, 2x) & \text{for } k < 0 \end{cases}$$
Every $\delta$-pseudo orbit $\{x_i\}_{i \in \mathbb{N}}$ can be transformed into a $\delta$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ by taking $x_i = f^i(x_0)$ for all $i \leq -1$. Therefore, using Example 7 (Das et al. 2013) we conclude that $f$ has shadowing. We now show that $f$ does not have mean ergodic shadowing which in turn, implies that $f$ does not have ergodic shadowing.

Let $\delta > 0$ be given and let $M$ be a subset of $\mathbb{N}$ such that $d(M) = 0$. Let $x_i \in \{i\} \times [0, \frac{1}{2^n}]$ for $i \in \mathbb{N} \setminus M$ and $x_i \in \{0\} \times [0, 1]$ for $i \in M$. One can find such a sequence with additional property that $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{N} \setminus M$. Since $M$ has density zero, $\{x_i\}_{i \in \mathbb{N}}$ is a $\delta$-ergodic pseudo orbit. This $\delta$-ergodic pseudo orbit cannot be $\epsilon$-ergodic shadowed because for any $x \in X$, $f^i(x)$ increases along $x$-axis as $i$ goes to $\infty$.

Recall from (Wu et al. 2016) that factors onto a minimal equicontinuous system cannot have $\mathcal{A}$-shadows and hence, by Proposition 3.8 such a system cannot have mean ergodic shadowing. In particular, factor of a dyadic adding machine cannot have mean ergodic shadowing which is also evident from the following result.

**Theorem 3.19** If $f$ has shadowing and mean ergodic shadowing, it cannot be minimal.

**Proof** If possible, suppose that $f$ has both shadowing and mean ergodic shadowing. Let $x, y \in X$ with $x \neq y$ and $0 < \epsilon < \frac{d(x, y)}{2}$. Let $\delta \in (0, \frac{\epsilon}{2})$ be given for $\frac{\epsilon}{2}$ by shadowing of $f$. Since $f$ has shadowing and mean ergodic shadowing, by Proposition 3.14 it is totally transitive. So, there is $k \geq 1$ such that $f^k(B(x, \delta)) \cap B(x, \delta) \neq \emptyset$, i.e. there is $z \in B(x, \delta)$ such that $f^k(z) \in B(x, \delta)$. Then, $\{z, f(z), f^2(z), \ldots, f^{k-1}(z), z, f(z), f^2(z), \ldots, f^{k-1}(z), \ldots\}$ is a $\delta$-pseudo orbit for $f$ and hence, there is $w \in X$ such that $d(f^{kn}(w), z) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$. By Zorn’s lemma there is a syndetically recurrent point $p \in \overline{O_f(w)}$ for $f^k$. Observe that $d(f^{kn}(p), z) \leq \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$ and therefore, $d(x, f^{kn}(p)) \leq d(x, z) + d(z, f^{kn}(p)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus, $f^{kn}(p) \in B(x, \epsilon)$ and since $d(x, y) > 2\epsilon$, $y \notin \overline{O_{f^k}(p)}$. Thus, $f^k$ is not minimal. As $f^k$ is transitive, there is $q \in X$ such that $\overline{O_{f^k}(q)} = X$. Since $f^k$ is not minimal, $q \notin M(f^k)$, and hence, $q \notin M(f)$. Since every point of a minimal system is syndetically recurrent, $f$ cannot be minimal. \hfill $\Box$

Then by Lemma 3.2 (Fakhari and Ghane 2010), a map with ergodic shadowing cannot be minimal. In contrast, the following result shows that syndetically recurrent points are present everywhere in the phase space of such a system.

**Theorem 3.20** If $f$ has ergodic shadowing, then $\overline{M(f)} = X$.

**Proof** It is similar to the proof given in Theorem 3.2 (Garg and Das 2016) using Lemma 3.1, 3.2 of (Fakhari and Ghane 2010). \hfill $\Box$

Then Proposition 3.8 and Theorem 7.2 (Wu et al. 2016) imply that a map with ergodic shadowing is totally syndetically transitive.

**Theorem 3.21** A minimal system with mean ergodic shadowing is weakly mixing.

**Proof** It follows from Theorem 7.10 (Wu et al. 2016) and Proposition 3.8. \hfill $\Box$
Definition 3.22 Let \( f : X \to X \) be a continuous map on a metric space (not necessarily compact). Then \( f \) is said to be stretching map if there exists constant \( k > 1 \) such that \( d(f(x), f(y)) \geq kd(x, y) \) for all \( x \neq y \) in \( X \).

Proposition 3.23 A stretching map on a metric space (not necessarily compact) cannot possess mean ergodic shadowing.

Proof Suppose that \( f \) has mean ergodic shadowing. Let \( x, y \in X \) with \( x \neq y \) and \( \epsilon \in (0, \frac{1}{2}) \) be given. Let \( \delta > 0 \) be given for \( \frac{\epsilon}{2} \) by mean ergodic shadowing of \( f \).

Let \( m_0 = -1 \) and choose an increasing sequence \( M = \{m_i\}_{i \in \mathbb{N} \setminus \{0\}} \) of natural numbers with \( d(M) = 0 \) such that \( \mathbb{N} \) is partitioned into two subsets \( M_1 \) and \( M_2 \) satisfying

\[
M_1 = \{m_0 + 1, 1, 2, \ldots, m_1\} \cup \{m_2 + 1, m_2 + 2, \ldots, m_3\} \cup \{m_4 + 1, m_4 + 2, \ldots, m_5\} \cup \cdots \quad \text{and} \quad M_2 = \mathbb{N} \setminus M_1 \quad \text{with} \quad d(M_1) = d(M_2) = \frac{1}{2}
\]

For each \( j \in \mathbb{N} \), define

\[
w_i = \begin{cases} 
  f^i(x) & \text{for } m_{2j} + 1 \leq i \leq m_{2j+1} \\
  f^i(y) & \text{for } m_{2j+1} + 1 \leq i \leq m_{2j+2}
\end{cases}
\]

Since \( d(M) = 0 \), therefore \( \{w_i\}_{i \in \mathbb{N}} \) is a \( \delta \)-ergodic pseudo orbit. So, by mean ergodic shadowing there is \( z \in X \) such that \( d(f^i(z), w_i) < \epsilon \) except a set of upper density less than \( \epsilon \), which implies the following:

- There are infinitely many \( j \in \mathbb{N} \) for which there is \( i_j \in \{m_{2j} + 1, m_{2j} + 2, \ldots, m_{2j+1}\} \) such that \( d(f^{i_j}(z), w_{i_j}) < \epsilon \).
- There are infinitely many \( j \in \mathbb{N} \) for which there is \( i_j \in \{m_{2j+1} + 1, m_{2j+1} + 2, \ldots, m_{2j+2}\} \) such that \( d(f^{i_j}(z), w_{i_j}) < \epsilon \).

Thus, there exist \( \{k_i\}_{i \in \mathbb{N}} \) and \( \{n_i\}_{i \in \mathbb{N}} \) such that \( d(f^{k_i}(z), w_{k_i}) < \epsilon \) and \( d(f^{n_i}(z), w_{n_i}) < \epsilon \), where \( w_{k_i} = f^{k_i}(x) \) and \( w_{n_i} = f^{n_i}(y) \). This shows that \( f \) is not a stretching map.

\( \Box \)

Example 3.24 Let \( A : \mathbb{C}^n \to \mathbb{C}^n \) be a linear map having eigenvalues with modulus greater than one. Then \( A \) has shadowing property. Since \( A \) is a stretching map, Proposition 3.23 shows that \( A \) cannot possess mean ergodic shadowing.

4 Mean Ergodic Shadowing and Orbital Convergence

In this section, we study the behaviour of mean ergodic shadowing under orbital convergence of sequence of maps with mean ergodic shadowing.

Define \( MESh(f, x, \epsilon) = \{\delta > 0\} \) every \( \delta \)-ergodic pseudo orbit for \( f \) through \( x \) is \( \epsilon \)-shadowed in average by some point in \( X \).
The following notion of convergence is stronger than uniform convergence. Therefore, the limit function $f$ turns out to be continuous.

**Definition 4.1** Let $X$ be a compact metric space and for each $n \in \mathbb{N}$, $f_n : X \to X$ be continuous maps. Then $\{f_n\}_{n \in \mathbb{N}}$ is said to be orbitally convergent to $f : X \to X$ if for every $\epsilon > 0$ there exists a natural number $N$ such that $d(f_n(x), f^k(x)) < \epsilon$ for all $n \geq N$, all $x \in X$ and all $k \in \mathbb{N}$.

**Theorem 4.2** Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of continuous maps on a compact metric space $X$. If $\{f_n\}$ is orbitally converging to $f$, then $f$ has mean ergodic shadowing if and only if $\bigcup_{m \geq 1} \bigcap_{n \geq m} MESH(f_n, x, \epsilon) \neq \emptyset$ for all $x \in X$ and for every $\epsilon > 0$.

**Proof** Suppose that $f$ has the mean ergodic shadowing. Then, we want to show that $\bigcup_{m \geq 1} \bigcap_{n \geq m} MESH(f_n, x, \epsilon) \neq \emptyset$ for all $x \in X$. Let us fix $x \in X$ and $\epsilon > 0$. Further, let $0 < \delta < \frac{\epsilon}{2}$ be given for $\frac{\epsilon}{2}$ by the mean ergodic shadowing of $f$. So, every $\delta$-ergodic pseudo orbit through $x$ is $\frac{\epsilon}{2}$-shadowed in average by some point in $X$. Since $\{f_n\}$ is orbitally converging to $f$, we choose $M \geq 1$ such that $d(f_m^k(z), f^k(z)) < \frac{\epsilon}{2}$ for each $m \geq M$, for all $z \in X$ and for each $k \in \mathbb{N}^+$. For a fixed $m \geq M$, let us consider a $\frac{\delta}{2}$-ergodic pseudo orbit $\gamma = \{x_i\}_{i \in \mathbb{N}}$ for $f_m$ through $x$. Let $N \subseteq \mathbb{N}$ with $d(N) = 0$ be such that $d(f_m(x_i), x_{i+1}) \leq \frac{\delta}{2}$ for all $i \in \mathbb{N} \setminus N$. Then $d(f(x_i), x_{i+1}) \leq d(f(x_i), f_m(x_i)) + d(f_m(x_i), x_{i+1}) < \frac{\delta}{2} + \frac{\epsilon}{2} = \delta$ for all $i \in \mathbb{N} \setminus N$. Thus, $\gamma$ is a $\delta$-ergodic pseudo orbit for $f$ through $x$. So, there exists $y \in X$ such that $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \frac{\epsilon}{2}$.

Therefore,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_m^i(y), f^i(y)) + \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, every $\frac{\delta}{2}$-ergodic pseudo orbit for $f_m$ through $x$ is $\epsilon$-shadowed in average by some point in $X$. Since $m \geq M$ was arbitrary, we conclude that $\frac{\delta}{2} \in \bigcap_{n \geq m} MESH(f_n, x, \epsilon)$. Since $x$ and $\epsilon$ was arbitrary, we have $\bigcup_{m \geq 1} \bigcap_{n \geq m} MESH(f_n, x, \epsilon) \neq \emptyset$ for all $x \in X$ and for every $\epsilon > 0$. This proves the direct implication.

To prove the reverse implication, let us assume that $\bigcup_{m \geq 1} \bigcap_{n \geq m} MESH(f_n, x, \epsilon) \neq \emptyset$ for given $\epsilon > 0$ and given $x \in X$. We choose $M$ such that $\bigcap_{n \geq M} MESH(f_n, x, \frac{\epsilon}{2}) \neq \emptyset$. Therefore, there exists $0 < \delta_x < \epsilon$ such that for each $m \geq M$, every $\delta_x$-ergodic pseudo orbit for $f_m$ through $x$ can be $\frac{\epsilon}{2}$-shadowed in average by some point in $X$. Since $\{f_n\}$ converges orbitally to $f$, we can choose $N \geq M$ such that $d(f_m^k(z), f^k(z)) < \frac{\delta_x}{2}$ for each $m \geq N$, all $z \in X$ and for each $k \in \mathbb{N}^+$. Let $\gamma = \{x_i\}_{i \in \mathbb{N}}$ be a $\frac{\delta}{2}$-ergodic pseudo orbit for $f$ through $x$, i.e., there exists $K \subseteq \mathbb{N}$ with $d(K) = 0$ such that $d(f(x_i), x_{i+1}) < \frac{\delta_x}{2}$ for all $i \in \mathbb{N} \setminus K$. Then $d(f_N(x_i), x_{i+1}) \leq d(f_N(x_i), f(x_i)) + d(f(x_i), x_{i+1}) < \frac{\delta_x}{2} + \frac{\epsilon}{2} = \delta_x$ for all $i \in \mathbb{N} \setminus K$. Thus, $\gamma$ is a $\delta_x$-ergodic pseudo orbit for $f_N$ through $x$ and hence, can be $\frac{\epsilon}{2}$-shadowed in average by some point in $X$. So, there exists $y \in X$ such that $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_N^i(y), x_i) < \frac{\epsilon}{2}$.
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) \\
\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), f_N^i(z)) + \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_N^i(y), x_i) \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Thus, for every \( \epsilon > 0 \) there exists \( \delta_x > 0 \) such that every \( \delta_x \)-ergodic pseudo orbit through \( x \) is \( \epsilon \)-shadowed in average by some point in \( X \). This would imply that for every \( \epsilon > 0 \) there exists \( \delta_x > 0 \) such that every \( \delta_x \)-ergodic pseudo orbit through \( B[x, \delta_x] \) is \( \epsilon \)-shadowed in average by some point in \( X \). If not, suppose there exists \( \epsilon > 0 \) for which there does not exist any \( \delta > 0 \) such that every \( \delta \)-ergodic pseudo orbit is \( \epsilon \)-shadowed in average by some point in \( X \). Thus for each \( k \geq 1 \), we can choose \( \frac{1}{k} \)-ergodic pseudo orbit \( \eta^k = \{\eta^k_n\}_{n \in \mathbb{N}} \) with \( \eta^k_0 \in B[x, \frac{1}{k}] \) such that \( \eta^k \) cannot be \( \epsilon \)-shadowed in average by some point in \( X \). By compactness of \( X \) we choose \( K \) large enough so that \( d(f(\eta^k_0), f(x)) \leq \frac{\delta_x}{2} \) and \( \frac{1}{K} \leq \frac{\delta_x}{2} \). Let \( N \subset \mathbb{N} \) with \( \overline{d}(N) = 0 \) such that \( d(f(\eta^k_i), \eta^k_{i+1}) \leq \frac{1}{K} \) for all \( i \in N \setminus N \).

Define \( Y = \{y_n\}_{n \in \mathbb{N}} \) by

\[
y_n = \begin{cases} 
  x & \text{if } n = 0 \\
  \eta^K_n & \text{if } n \geq 1
\end{cases}
\]

For \( i \in \mathbb{N}^+ \setminus N \), \( d(f(y_i), y_{i+1}) = d(f(\eta^K_i), \eta^K_{i+1}) \leq \frac{1}{K} \leq \frac{\delta_x}{2} < \delta_x \).

Thus, \( Y = \{y_n\}_{n \in \mathbb{N}} \) is a \( \delta_x \)-ergodic pseudo orbit for \( f \) through \( x \). So, we can choose \( z \in X \) such that \( \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), y_i) < \epsilon \). Then,

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), \eta^K_i) = \limsup_{n \to \infty} \frac{1}{n} d(z, \eta^K_0) + \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), \eta^K_i) < 0 + \epsilon = \epsilon.
\]

It follows that \( \{\eta^K_i\}_{i \in \mathbb{N}} \) is \( \epsilon \)-shadowed in average by some point in \( X \). This contradiction makes the assumption false.

Now, for each \( x \in X \) and every \( \epsilon > 0 \), there exists \( \delta_x > 0 \) such that every \( \delta_x \)-ergodic pseudo orbit through \( B[x, \delta_x] \) is \( \epsilon \)-shadowed in average by some point in \( X \). By compactness of \( X \), we have \( X = \bigcup_{i=1}^m [x_i, \delta_i] \), where \( \delta_i = \delta_{x_i} \) for \( 1 \leq i \leq m \). Choose \( \delta = \min_{1 \leq i \leq m} \{\delta_i\} \) and let \( \{y_n\}_{n \in \mathbb{N}} \) be a \( \delta \)-ergodic pseudo orbit for \( f \). Clearly, \( y_0 \in B[x_i, \delta_i] \) for some \( 1 \leq i \leq n \). Thus, \( \{y_n\}_{n \in \mathbb{N}} \) is a \( \delta_i \)-ergodic pseudo orbit for \( f \) through \( B[x_i, \delta_i] \). So, one can find \( y \in X \) such that \( \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), y_i) < \epsilon \). This completes a proof of the fact that \( f \) has the mean ergodic shadowing property.

\[\square\]

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