Completely representable neat reducts

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Abstract. For an ordinal \( \alpha \), \( \text{PEA}_\alpha \) denotes the class of polyadic equality algebras of dimension \( \alpha \). We show that for several classes of algebras that are reducts of \( \text{PEA}_\omega \) whose signature contains all substitutions and finite cylindrifiers, if \( \mathcal{B} \) is in such a class, and \( \mathcal{B} \) is atomic, then for all \( n < \omega \), \( \text{Nr}_n \mathcal{B} \) is completely representable as a \( \text{PEA}_n \). Conversely, we show that for any \( 2 < n < \omega \), and any variety \( V \), between diagonal free cylindric algebras and quasipolyadic equality algebras of dimension \( n \), the class of completely representable algebras in \( V \) is not elementary.

1 Introduction

Relation algebras \( \mathcal{RA} \)s and cylindric algebras of dimension \( \alpha \), \( \alpha \) any ordinal \( \mathcal{CA}_\alpha \) are introduced by Tarski. Both are varieties that are axiomatized by a relatively simple schema of equations. Relation algebras are abstractions of algebras whose universe consists of binary relations, with top element an equivalence relation, and Boolean operations of union and complementation and extra operations of composition and forming converses. Such algebras are called representable relation algebras of dimension \( \alpha \), in symbols \( \mathcal{RRA} \). In both cases equality is represented by the identity relation. The last class, when the top elements are disjoint unions of cartesian squares of dimension \( \alpha \) is called the class of representable polyadic algebras of dimension \( \alpha \), and is denoted by \( \mathcal{RCA}_\alpha \). Unless otherwise explicitly indicated, let \( 2 < n < \omega \). The classes \( \mathcal{RRA} \) and \( \mathcal{RCA}_n \) are not finitely axiomatizable, In particular \( \mathcal{RRA} \subseteq \mathcal{RA} \) and similarly \( \mathcal{RCA}_n \subseteq \mathcal{CA}_n \). Polyadic algebras were introduced by Halmos [5] to provide an algebraic reflection of the study of first order logic without equality. Later the algebras were enriched by diagonal elements to permit the discussion of equality. That the notion is indeed an adequate reflection of first order logic was demonstrated by Halmos’ representation theorem for locally finite polyadic algebras (with and without equality). Daigneault and Monk proved a strong extension of Halmos’ theorem, namely that, every polyadic algebra of infinite dimension (without equality) is representable [3].

In the realm of representable algebras, there are several types of representations. Ordinary representations are just isomorphisms from Boolean algebras with operators to a more concrete structure (having the same signature) whose elements are sets endowed with set-theoretic operations like intersection and complementation. Complete representations, on the other hand, are representations that preserve arbitrary conjunctions whenever defined. More generally consider the following question: Given an algebra and a set of meets, is there a representation that carries this set of meets to set theoretic intersections? A complete representation would thus be one that preserves all existing meets (finite of course and infinite). Here we are assuming that our semantics is specified by set algebras, with the concrete Boolean operation of intersection among its basic operations. When the algebra in question is countable, and we have only countably many
meets; this is an algebraic version of an omitting types theorem; the representation omits the given set meets or non-principal types. When the algebra in question is atomic, then a representation omitting the non-principal type consisting of co-atoms, turns out to be a complete representation. This follows from the following result due to Hirsch and Hodkinson: A Boolean algebra $A$ has a complete representation $f : A \rightarrow \langle \wp(X), \cup, \cap, \sim, \emptyset, X \rangle$ ($f$ is a 1-1 homomorphism and $X$ a set) $\iff A$ atomic and $\bigcup_{x \in \text{At}_A} f(x) = X$, where $\text{At}_A$ is the set of atoms of $A$.

On the face of it, the notion of complete representations seems to be strikingly a second order one. This intuition is confirmed in [8] where it is proved that the classes of completely representable cylindric algebras of dimension at least three and that of relation algebras are not elementary. These results were proved by Hirsch and Hodkinson using so-called rainbow algebras [8]; in this paper we present entirely different proofs for all such results and some more closely related ones using so called Monk-like algebras. Our proof depends essentially on some form of an infinite combinatorial version of Ramsey’s Theorem. But running to such conclusions—concerning (non-)first order definability—can be reckless and far too hasty; for in other non-trivial cases the notion of complete representations turns not to be a genuinely second order one; it is definable in first order logic. The class of completely representable Boolean algebras is elementary; it simply coincides with the atomic ones. A far less trivial example is the class of completely representable infinite dimensional polyadic algebras; it coincides with the class of atomic, completely additive algebras. It is not hard to show that, like atomicity, complete additivity can indeed be defined in first order logic [27]. This is not true for the class $\text{PEA}_\alpha$ of polyadic algebras with equality of dimension $\alpha$. However, we will show that if $A \in \text{PEA}_\alpha$ is atomic, then all of its finite dimensional neat reducts are not only representable, but completely representable. So from one atomic algebra one obtains a plethora of completely representable ones, at least one for each finite dimension.

For some odd reason, historically the underlying intuition of the notion of complete representability progressed in a different direction. The correlation of (the first order property of) atomicity to complete representations has caused a lot of confusion in the past. It was mistakenly thought for a while, among algebraic logicians, that atomic representable relation and cylindric algebras are completely representable, an error attributed to Lyndon and now referred to as Lyndon’s error. But in retrospect, one can safely say by gathering and scrutinizing recent results that the first order definability of the the notion of complete representations heavily depends on the algebras required to be completely represented. In other words, the (possibly slippery) notion of ‘complete representations’ needs a context to be fixed one way or another, and it is surely unwise to declare a verdict without a careful and thorough investigation of the specific situation at hand. Let $\text{CRRA}$ denote the class of completely representable RAS, and $\text{CRCA}_n$ denote the class of completely representable $\text{CA}_n$s. It is shown that the classes $\text{CRRA}$ and $\text{CRCA}_n$ are not elementary, reproving a result of Hirsch and Hodkinson, and we fo further by showing that $\text{CRRA}$ is not even closed under $\equiv_{\infty, \omega}$.

In [19] it is proved that for any pair of infinite ordinals $\alpha < \beta$, the class $\text{Nr}_\alpha \text{CA}_\beta$ is not elementary. A different model theoretic proof for finite $\alpha$ is given in [25, Theorem 5.4.1]. This result is extended to many cylindric like algebras like Halmos’ polyadic algebras with and without equality, and Pinter’s substitution algebras in [20, 22]. The class $\text{CRCA}_n$ is proved not be elementary by Hirsch and Hodkinson in [8]. Neat embeddings and complete representations are linked in [26, Theorem 5.3.6] where it is shown that $\text{CRCA}_n$ coincides with the class $\text{S}_c \text{Nr}_n \text{CA}_\omega$ on atomic algebra having countably many atoms.
In [29] it is proved that this characterization does not generalize to atomic algebras having uncountably many atoms. Such counterexamples are used to violate metalogical theorems such as [26, Theorem 3.2.9-10] involving the celebrated Orey-Henkin omitting types theorem for finite variable fragment of $L_{\omega,\omega}$.

2 Preliminaries

We follow the notation of [1] which is in conformity with the notation in the monograph [7]. In particular, for any pair of ordinal $\alpha < \beta$, $CA_\alpha$ stands for the class of cylindric algebras of dimension $\alpha$, $RCA_\alpha$ denotes the class of representable $CA_\alpha$s and $Nr_\alpha CA_\beta(\subseteq CA_\alpha)$ denotes the class of $\alpha$–neat reducts of $CA_\beta$s. The last class is studied extensively in the chapter [25] of [1] as a key notion in the representation theory of cylindric algebras. The notion of neat reducts and the related one of neat embeddings are both important in algebraic logic for the simple reason that both notions are very much tied to the notion of representability, via the so–called neat embedding theorem of Henkin’s which says that (for any ordinal $\alpha$), we have $RCA_\alpha = S Nr_\alpha CA_{\alpha + \omega}$, where $S$ stands for the operation of forming subalgebras.

Definition 2.1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in CA_\beta$. Then the $\alpha$–neat reduct of $\mathfrak{B}$, in symbols $\mathfrak{N}r_\alpha \mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \setminus \alpha$, and restricting the universe to the set $\mathfrak{N}r_\alpha B = \{ x \in \mathfrak{B} : \{ i \in \beta : c_i x \neq x \} \subseteq \alpha \}$.

It is straightforward to check that $\mathfrak{N}r_\alpha \mathfrak{B} \in CA_\alpha$. Let $\alpha < \beta$ be ordinals. If $\mathfrak{A} \in CA_\alpha$ and $\mathfrak{A} \subseteq \mathfrak{N}r_\alpha \mathfrak{B}$, with $\mathfrak{B} \in CA_\beta$, then we say that $\mathfrak{A}$ neatly embeds in $\mathfrak{B}$, and that $\mathfrak{B}$ is a $\beta$–dilation of $\mathfrak{A}$, or simply a dilation of $\mathfrak{A}$ if $\beta$ is clear from context. For $K \subseteq CA_\beta$, we write $Nr_\alpha K$ for the class $\{ \mathfrak{N}r_\alpha \mathfrak{B} : \mathfrak{B} \in K \}$.

Fix $2 < n < \omega$. Following [7], $CS_n$ denotes the class of cylindric set algebras of dimension $n$, and $GS_n$ denotes the class of generalized set algebra of dimension $n$; $\mathfrak{C} \in GS_n$ if $\mathfrak{C}$ has top element $V$ a disjoint union of cartesian squares, that is $V = \bigcup_{i \in I} t^n U_i$, $I$ is a non-empty indexing set, $U_i \neq \emptyset$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$. The operations of $\mathfrak{C}$ are defined like in cylindric set algebras of dimension $n$ relativized to $V$. $CRCA_n$ denotes the class of completely representable $CA_n$s.

Definition 2.2. An algebra $\mathfrak{A} \in CRCA_n \iff$ there exists $\mathfrak{C} \in GS_n$, and an isomorphism $f : \mathfrak{A} \rightarrow \mathfrak{C}$ such that for all $X \subseteq \mathfrak{A}$, $f(\sum X) = \bigcup_{x \in X} f(x)$, whenever $\sum X$ exists in $\mathfrak{A}$. In this case, we say that $\mathfrak{A}$ is completely representable via $f$.

It is known that $\mathfrak{A}$ is completely representable via $f : \mathfrak{A} \rightarrow \mathfrak{C}$, where $\mathfrak{C} \in GS_n$ has top element $V$ say $\iff \mathfrak{A}$ is atomic and $f$ is atomic in the sense that $f(\sum \text{At } \mathfrak{A}) = \bigcup_{x \in \text{At } \mathfrak{A}} f(x) = V$ [8]. $S_c$ denotes the operation of forming complete subalgebras. The next lemma tells us that the notions of atomicity and complete representation of an algebra are inherited by complete (hence dense) subalgebras.

Lemma 2.3. Let $n < \omega$, and $\mathfrak{D}$ be a Boolean algebra. Assume that $\mathfrak{A} \subseteq_c \mathfrak{D}$. If $\mathfrak{D}$ is atomic, then $\mathfrak{A}$ is atomic [9, Lemma 2.16]. If $\mathfrak{D} \in CA_n$ is completely representable, then so is $\mathfrak{A}$.

Proof. Let everything be as in the hypothesis of the first part. We show that $\mathfrak{A}$ is atomic. Let $a \in A$ be non–zero. Then since $\mathfrak{D}$ is atomic, there exists an atom $d \in D$, such that
$d \leq a$. Let $F = \{x \in A : x \geq d\}$. Then $F$ is an ultrafilter of $\mathfrak{A}$. It is clear that $F$ is a filter. To prove maximality, assume that $c \in A$ and $c \notin F$, then $-c \cdot d \neq 0$, so $0 \neq -c \cdot d \leq d$, hence $-c \cdot d = d$, because $d$ is an atom in $\mathfrak{A}$, thus $d \leq -c$, and we get by definition that $-c \in F$. We have shown that $F$ is an ultrafilter. We now show that $F$ is a principal ultrafilter in $\mathfrak{A}$, that is, it is generated by an atom. Assume for contradiction that it is not, so that $\prod^{\mathfrak{A}} F$ exists, because $F$ is an ultrafilter and $\prod^{\mathfrak{A}} F = 0$, because it is non–principal. But $\mathfrak{A} \subseteq \mathfrak{D}$, so we obtain $\prod^{\mathfrak{A}} F = \prod^{\mathfrak{D}} F = 0$. This contradicts that $0 < d \leq x$ for all $x \in F$. Thus $\prod^{\mathfrak{A}} F = a$, $a'$ is an atom in $\mathfrak{A}$, $a' \in F$ and $a' \leq a$, because $a \in F$. We have proved the first required. Let $\mathfrak{A} \subseteq \mathfrak{D}$ and assume that $\mathfrak{D}$ is completely representable. We will show that $\mathfrak{A}$ is completely representable. Let $f : \mathfrak{D} \to \wp(V)$ be a complete representation of $\mathfrak{D}$. We claim that $g = f \upharpoonright \mathfrak{A}$ is a complete representation of $\mathfrak{A}$. Let $X \subseteq \mathfrak{A}$ be such that $\sum^{\mathfrak{A}} X = 1$. Then by $\mathfrak{A} \subseteq \mathfrak{D}$, we have $\sum^{\mathfrak{D}} X = 1$. Furthermore, for all $x \in X(\subseteq \mathfrak{A})$ we have $f(x) = g(x)$, so that $\bigcup_{x \in X} g(x) = \bigcup_{x \in X} f(x) = V$, since $f$ is a complete representation, and we are done. \[\square\]

Though the class $S_n \text{Nr}_n \mathcal{C}A_\omega$ and the class $\mathcal{C}RRA$ coincide on algebras having countably many atoms, in [24] it is shown that the condition of countability cannot be omitted: There is an atomic $\mathfrak{A} \in \text{Nr}_n \mathcal{C}A_\omega$ with uncountably many atoms such that $\mathfrak{A}$ is not completely representable. But the $\mathfrak{C} \in \mathcal{C}A_\omega$ for which $\mathfrak{A} = \text{Nr}_n \mathfrak{C}$ is atomless.

In what follows we address complete representability of a given algebra in connection to the existence of an $\omega$–dilation of this algebra that is atomic. We shall deal with many classes of cylindric–like algebras for which the neat reduct operator can be defined. In particular, for such classes, and regardless of atomicity, we can (and will) talk about an $\omega$–dilation of a given algebra. For an ordinal $\alpha$, let $\text{PA}_\alpha(\text{PEA}_n)$ denote the class of $\alpha$–dimensional polyadic (equality) algebras as defined in [7, Definition 5.4.1].

**Definition 2.4.** Let $\alpha$ be an ordinal. By a *polyadic algebra* of dimension $\alpha$, or a $\text{PA}_\alpha$ for short, we understand an algebra of the form

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\Gamma, s_{\tau}\rangle_{\Gamma \subseteq \alpha, \tau \in {}^\alpha \alpha}$$

where $c_\Gamma (\Gamma \subseteq \alpha)$ and $s_{\tau}$ ($\tau \in {}^\alpha \alpha$) are unary operations on $A$, such that postulates below hold for $x, y \in A$, $\tau, \sigma \in {}^\alpha \alpha$ and $\Gamma, \Delta \subseteq \alpha$

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra
2. $c_\Gamma \cdot 0 = 0$
3. $x \leq c_\Gamma x$
4. $c_\Gamma (x \cdot c_\Gamma y) = c_\Gamma x \cdot c_\Gamma y$
5. $c_\Gamma c_{\Delta} x = c_{\Gamma \cup \Delta} x$
6. $s_{\tau}$ is a boolean endomorphism
7. $s_{\Gamma \cdot d} x = x$
8. $s_{\sigma \circ \tau} = s_{\tau} \circ s_{\sigma}$
9. if $\sigma \upharpoonright (\alpha \sim \Gamma) = \tau \upharpoonright (\alpha \sim \Gamma)$, then $s_{\sigma} c_\Gamma x = s_{\tau} c_\Gamma x$
10. If $\tau^{-1} \Gamma = \Delta$ and $\tau \upharpoonright \Delta$ is one to one, then $c_{\Gamma} s_{\tau} x = s_{\tau} c_{\Delta} x$. 


Definition 2.5. Let $\alpha$ be an ordinal. By a polyadic equality algebra of dimension $\alpha$, or a $\text{PEA}_\alpha$ for short, we understand an algebra of the form

$$\mathfrak{A} = \langle A, +, - , 0, 1, c_\Gamma, s_\tau, d_{ij} \rangle_{\Gamma \subseteq \alpha, \tau \in \alpha, i,j < \alpha}$$

where $c_\Gamma$ ($\Gamma \subseteq \alpha$) and $s_\tau$ ($\tau \in \alpha$) are unary operations on $A$, and $d_{ij}$ are constants in the signature, such the reduct obtained by deleting these $d_{ij}$'s ($i,j \in \alpha$), is a $\text{PA}_\alpha$ and the equations below hold for $x \in A$, $\tau \in \alpha$ and and $i,j \in \alpha$

1. $d_{ii} = 1$,
2. $x \cdot d_{ij} \leq s_{[i][j]}x$,
3. $s_\tau d_{ij} = d_{\tau(i),\tau(j)}$.

We will sometimes add superscripts to cylindrifiers and substitutions indicating the algebra they are evaluated in. The class of representable algebras is defined via set-theoretic operations on sets of $\alpha$-ary sequences. Let $U$ be a set. For $\Gamma \subseteq \alpha$ and $\tau \in \alpha$, we set

$$c_\Gamma X = \{ s \in U : \exists t \in X , \forall j /\in \Gamma , t(j) = s(j) \}$$

$$s_\tau X = \{ s \in U : s \circ \tau \in X \}.$$  

$$D_{ij} = \{ s \in U : s_i = s_j \}.$$  

For a set $X$, let $\mathfrak{B}(X)$ be the boolean set algebra $(\wp(X), \cup, \cap, \sim)$. The class of representable polyadic algebras, or $\text{RPA}_\alpha$ for short, is defined by

$$SP\{ \langle \mathfrak{B}(U), c_\Gamma, s_\tau \rangle_{\Gamma \subseteq \alpha, \tau \in \alpha} : U \text{ a set } \}.$$  

The class of representable polyadic equality algebras, or $\text{RPEA}_\alpha$ for short, is defined by

$$SP\{ \langle \mathfrak{B}(U), c_\Gamma, s_\tau, D_{ij} \rangle_{\Gamma \subseteq \alpha, \tau \in \alpha} : U \text{ a set } \}.$$  

Here $SP$ denotes the operation of forming subdirect products. It is straightforward to show that $\text{RPA}_\alpha \subseteq \text{PA}_\alpha$. Daigneault and Monk [3] proved that for $\alpha \geq \omega$ the converse inclusion also holds, that is $\text{RPA}_\alpha = \text{PA}_\alpha$. This is a completeness theorem for certain infinitary extensions of first order logic without equality [15]. Let $\mathfrak{A}$ be a polyadic algebra and $f : \mathfrak{A} \rightarrow \wp(U)$ be a representation of $\mathfrak{A}$. If $s \in X$, we let

$$f^{-1}(s) = \{ a \in \mathfrak{A} : s \in f(a) \}.$$  

An atomic representation $f : \mathfrak{A} \rightarrow \wp(U)$ is a representation such that for each $s \in V$, the ultrafilter $f^{-1}(s)$ is principal. A complete representation of $\mathfrak{A}$ is a representation $f$ satisfying

$$f(\prod X) = \bigcap f[X]$$

whenever $X \subseteq \mathfrak{A}$ and $\prod X$ is defined.

A completely additive boolean algebra with operators is one for which all extra non-boolean operations preserve arbitrary joins.

Lemma 2.6. Let $\mathfrak{A} \in \text{PA}_\alpha$. A representation $f$ of $\mathfrak{A}$ is atomic if and only if it is complete. If $\mathfrak{A}$ has a complete representation, then it is atomic and is completely additive.
**Proof.** The first part is like \cite{3}. For the second part, we note that \( \text{PA}_\alpha \) is a discriminator variety with discriminator term \( c(\alpha) \). And so because all algebras in \( \text{PA}_\alpha \) are semi-simple, it suffices to show that if \( \mathfrak{A} \) is simple, \( X \subseteq A \), is such that \( \sum X = 1 \), and there exists an injection \( f : \mathfrak{A} \to \varphi(\alpha) \), such that \( \bigcup_{x \in X} f(x) = V \), then for any \( \tau \in \alpha \), we have \( \sum s_\tau X = 1 \). So assume that this does not happen for some \( \tau \in \alpha \). Then there is a \( y \in \mathfrak{A}, y < 1 \), and \( s_\tau x \leq y \) for all \( x \in X \). Now

\[
1 = s_\tau \left( \bigcup_{x \in X} f(x) \right) = \bigcup_{x \in X} s_\tau f(x) = \bigcup_{x \in X} f(s_\tau x).
\]

(Here we are using that \( s_\tau \) distributes over union.) Let \( z \in X \), then \( s_\tau z \leq y < 1 \), and so \( f(s_\tau z) \leq f(y) < 1 \), since \( f \) is injective, it cannot be the case that \( f(y) = 1 \). Hence, we have

\[
1 = \bigcup_{x \in X} f(s_\tau x) \leq f(y) < 1
\]

which is a contradiction, and we are done.

Let \( \alpha \) be an infinite ordinal. By \cite{27}, it is proved that the the condition of atomicity and complete additivity of a \( \text{PA}_\alpha \) is not only necessary for completely representability but also sufficient. But for \( \text{PEA}_\alpha \) the situation is totally different. Not only not every atomic \( \text{PEA}_\alpha \) is completely representable; there are examples of atomic \( \text{PEA}_\alpha \)s that are not representable at all. The aim of this paper is that if we pass to finite neat reducts of any atomic \( \text{PEA}_\alpha \) possibly non representable, we recover complete representability.

### 2.1 Classes between \( \text{Df}_n \) and \( \text{QEA}_n \)

We shall have the occasion to deal with (in addition to \( \text{CAs} \)), the following cylindric–like algebras \cite{1}: \( \text{Df} \) short for diagonal free cylindric algebras, \( \text{Sc} \) short for Pinter’s substitution algebras, \( \text{QA} \) (\( \text{QEA} \)) short for quasi–polyadic (equality) algebras. For \( K \) any of these classes and \( \alpha \) any ordinal, we write \( K_\alpha \) for variety of \( \alpha \)-dimensional \( K \) algebras which can be axiomatized by a finite schema of equations, and \( \text{RK}_\alpha \) for the class of representable \( K_\alpha \)s, which happens to be a variety too (that cannot be axiomatized by a finite schema of equations for \( \alpha > 2 \) unless \( K = \text{PA} \) and \( \alpha \geq \omega \)). The standard reference for all the classes of algebras mentioned previously is \cite{7}. We recall the concrete versions of such algebras. Let \( \tau : \alpha \to \alpha \) and \( X \subseteq \alpha U \), then

\[
S_\tau X = \{ s \in \alpha U : s \circ \tau \in X \}.
\]

For \( i, j \in \alpha, [i|j] \) is the replacement on \( \alpha \) that sends \( i \) to \( j \) and is the identity map on \( \alpha \sim \{i\} \) while \( [i,j] \) is the transposition on \( \alpha \) that interchanges \( i \) and \( j \).

- A *diagonal free cylindric set algebra of dimension* \( \alpha \) is an algebra of the form \( \langle \mathcal{B}(\alpha U), C_i \rangle_{i,j<\alpha} \).
- A *Pinter’s substitution et algebra of dimension* \( \alpha \) is an algebra of the form \( \langle \mathcal{B}(\alpha U), C_i, S_{[i|j]} \rangle_{i,j<\alpha} \).
- A *quasi-polyadic set algebra of dimension* \( \alpha \) is an algebra of the form \( \langle \mathcal{B}(\alpha U), C_i, S_{[i|j]}, S_{[i,j]} \rangle_{i,j<\alpha} \).
- A *quasi-polyadic equality set algebra* is an algebra of the form \( \langle \mathcal{B}(\alpha U), C_i, S_{[i|j]}, S_{[i,j]}, D_{(i)} \rangle_{i,j<\alpha} \).
For a BAO, $\mathfrak{A}$ say, for any ordinal $\alpha$, $\mathfrak{H}_{\text{sc}}\mathfrak{A}$ denotes the cylindric reduct of $\mathfrak{A}$ if it has one, $\mathfrak{H}_{\text{d}}\mathfrak{A}$ denotes the Sc reduct of $\mathfrak{A}$ if it has one, and $\mathfrak{H}_{\text{d}}\mathfrak{A}$ denotes the reduct of $\mathfrak{A}$ obtained by discarding all the operations except for cylindrifications. If $\mathfrak{A}$ is any of the above classes, it is always the case that $\mathfrak{H}_{\text{d}}\mathfrak{A} \in \mathfrak{D}_\alpha$. If $\mathfrak{A} \in \mathfrak{C}_\alpha$, then $\mathfrak{H}_{\text{d}}\mathfrak{A} \in \mathfrak{S}_\alpha$, and if $\mathfrak{A} \in \mathfrak{Q}_{\text{EA}}$, then $\mathfrak{H}_{\text{d}}\mathfrak{A} \in \mathfrak{C}_\alpha$. Roughly speaking for an ordinal $\alpha$, $\mathfrak{C}_\alpha$s are not expansions of $\mathfrak{S}_\alpha$s, but they are definitionally equivalent to expansions of $\mathfrak{S}_\alpha$s, because the $s^i_j$s are term definable in $\mathfrak{C}_\alpha$s by $s^i_j(x) = c_i(x \cdot d^i_j)$ ($i, j < \alpha$). This operation reflects algebraically the substitution of the variable $v_j$ for $v_i$ in a formula such that the substitution is free; this can be always done by reindexing bounded variables. In such situation, we say that $\mathfrak{S}_\alpha$s are generalized reducts of $\mathfrak{C}_\alpha$s. However, $\mathfrak{C}_\alpha$s and $\mathfrak{Q}_{\text{EA}}$ are (real )reducts of $\mathfrak{Q}_{\text{EA}}$s, (in the universal algebraic sense) simply obtained by discarding the operations in their signature not in the signature of their common expansion $\mathfrak{Q}_{\text{EA}}$. We give a finite approximate equational axiomatization of the concrete algebras defined above, which are the prime source of inspiration for these axiomatizations introduced to capture representability. However, like for $\mathfrak{C}_\alpha$s, this works only for certain special cases like the locally finite algebras, but does not generalize much further, cf Proposition 2.10.

**Definition 2.7. Substitution Algebra, $\mathfrak{S}_\alpha$** [17].

Let $\alpha$ be an ordinal. By a substitution algebra of dimension $\alpha$, briefly an $\mathfrak{S}_\alpha$, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s^i_j \rangle_{i, j < \alpha}$$

where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra, $c_i, s^i_j$ are unary operations on $\mathfrak{A}$ (for $i, j < \alpha$) satisfying the following equations for all $i, j, k, l < \alpha$:

1. $c_i 0 = 0$, $x \leq c_i x$, $c_i(x \cdot c_i y) = c_i x \cdot c_i y$, and $c_i c_j x = c_j c_i x$,
2. $s^i_j x = x$,
3. $s^i_j$ is a boolean endomorphisms,
4. $s^i_j c_i x = c_i x$,
5. $c_i s^i_j x = s^i_j x$ whenever $i \neq j$,
6. $s^i_j c_i x = c_k s^i_j x$, whenever $k \notin \{i, j\}$,
7. $c_i s^i_j x = c_j s^i_j x$,
8. $s^i_j s^i_k x = s^i_k s^i_j x$, whenever $\{|i, j, k, l| = 4$,
9. $s^i_j s^i_k x = s^i_k s^i_j x$. 

Figure 1: Non-Boolean operators for the classes
Quasipolyadic algebra, QEA \[18\].

A quasipolyadic algebra of dimension \(\alpha\), briefly a \(\text{QA}_\alpha\), is an algebra
\[
\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s^i_j, s_{[i,j]} \rangle_{i,j<\alpha}
\]
where the reduct to \(\text{Sc}_\alpha\) is a substitution algebra (it satisfies \([11]–[19]\) above) and additionally it satisfies the following equations for all \(i, j, k < \alpha\):
1. \(s^i_j(x) = x\); and \(s_{[i,j]}(x) = s_{[j,i]}\),
2. \(s^i_j(s^i_k x) = s^i_{k,j} x\) if \(|\{i,j,k\}| = 3\).
3. \(s^i_j s^j_k x = s^i_j x\).

Quasipolyadic equality algebra, QEA \[18\].

A quasipolyadic equality algebra of dimension \(\alpha\), briefly a \(\text{QEA}_\alpha\) is an algebra
\[
\mathfrak{B} = \langle \mathfrak{A}, d_{ij} \rangle_{i,j<\alpha}
\]
where \(\mathfrak{A}\) is a \(\text{QA}_\alpha\) (i.e. it satisfies all the equations above), \(d_{ij}\) is a constant and the following equations hold, for all \(i, j, k < \alpha\):
1. \(s^i_j d_{ij} = 1\),
2. \(x \cdot d_{ij} \leq s^i_j x\).

Definition 2.8. Let \(\alpha\) be an ordinal. We say that a variety \(V\) is a variety between \(\text{Df}_\alpha\) and \(\text{QEA}_\alpha\) if the signature of \(V\) expands that of \(\text{Df}_\alpha\) and is contained in the signature of \(\text{QEA}_\alpha\). Furthermore, any equation formulated in the signature of \(\text{Df}_\alpha\) that holds in \(V\) also holds in \(\text{Sc}_\alpha\) and all equations that hold in \(V\) holds in \(\text{QEA}_\alpha\).

Proper examples include \(\text{Sc}\), \(\text{CA}_\alpha\) and \(\text{QA}_\alpha\) (meaning strictly between). Analogously we can define varieties between \(\text{Sc}_\alpha\) and \(\text{CA}_\alpha\) or \(\text{QA}_\alpha\) and \(\text{QEA}_\alpha\), and more generally between a class \(K\) of \(\text{BAOs}\) and a generalized reduct of it. Notions like neat reducts generalize verbatim to such algebras, namely, to \(\text{Dfs}\) and \(\text{QEAs}\), and in any variety in between. This stems from the observation that for any pair of ordinals \(\alpha < \beta\), \(\mathfrak{A} \in \text{QEA}_\beta\) and any non-Boolean extra operation in the signature of \(\text{QEA}_\beta\), \(f\) say, if \(x \in \mathfrak{A}\) and \(\Delta x \subseteq \alpha\), then \(\Delta(f(x)) \subseteq \alpha\). Here \(\Delta x = \{i \in \beta : c_i x \neq x\}\) is referred as the dimension set of \(x\); it reflects algebraically the essentially free variables occurring in a formula \(\phi\). A variable is essentially free in a formula \(\Psi \iff \text{it is free in every formula equivalent to } \Psi\). Therefore given a variety \(V\) between \(\text{Sc}_\beta\) and \(\text{QEA}_\beta\), if \(\mathfrak{B} \in V\) then the algebra \(\text{Nr}_\alpha \mathfrak{B}\) having universe \(\{x \in \mathfrak{B} : \Delta x \subseteq \alpha\}\) is closed under all operations in the signature of \(V\).

Definition 2.9. Let \(2 < n < \omega\). For a variety \(V\) between \(\text{Df}_n\) and \(\text{QEA}_n\), a \(V\) set algebra is a subalgebra of an algebra, having the same signature as \(V\), of the form \(\langle \mathfrak{B}(n U), f^U_i \rangle\), say, where \(f^U_i\) is identical to the interpretation of \(f_i\) in the class of quasipolyadic equality set algebras. Let \(\mathfrak{A}\) be an algebra having the same signature of \(V\); then \(\mathfrak{A}\) is a representable \(V\) algebra, or simply representable \(\iff \mathfrak{A}\) is isomorphic to a subdirect product of \(V\) set algebras. We write \(RV\) the class of representable \(V\) algebras

\[\text{It can well happen that a variable is free in a formula that is equivalent to another formula in which this same variable is not free.}\]
It can be proved that the class RV, as defined above, is also closed under $H$, so that it is a variety. This can be proved using the same argument to show that RCA$_n$ is a variety, cf. Corollary [7, 3.1.77]. Take $\mathfrak{A} \in RV$, an ideal $J$ of $\mathfrak{A}$, then show that $\mathfrak{A}/J$ is in RV. Ideals in BAOs are defined as follows. We consider only BAOs with extra unary non-Boolean operators to simplify notation. If $\mathfrak{A}$ is a BAO, then $J \subseteq \mathfrak{A}$ is an ideal in $J$ if is a Boolean ideal and for any extra non-Boolean operator $f$, say, in the signature of BAO, and $x \in \mathfrak{A}$, $f(x) \in \mathfrak{A}$; the quotient algebra $\mathfrak{A}/J$ is defined the usual way since ideals defined in this way correspond to congruence relations defined on $\mathfrak{A}$.

**Theorem 2.10.** Let $2 < n < \omega$. Let $V$ be a variety between $Df_n$ and $QEA_n$. Then RV is not a finitely axiomatizable variety.

**Proof.** In [14] a sequence $\langle \mathfrak{A}_i : i \in \omega \rangle$ of algebras is constructed such that $\mathfrak{A}_i \in QEA_n$ and $\mathfrak{A}_n \not\subseteq Df_n$, but $\prod_{i \in \omega} \mathfrak{A}_i/F \in RQEA_n$ for any non-principal ultrafilter on $\omega$. An application of Los’ Theorem, taking the ultraproduct of $V$ reduct of the $\mathfrak{A}_i$s, finishes the proof. In more detail, let $\mathfrak{A}_\omega$ denote restricting the signature to that of $V$. Then $\mathfrak{A}_\omega \in RV$ and $\mathfrak{A}_\omega \Pi_{i \in \tau} (\mathfrak{A}_i/F) \in RV$.

The last result generalizes to infinite dimensions replacing finite axiomatization by axiomatized by a finite schema [7, 12]. We consider relation algebras as algebras of the form $\mathcal{R} = \langle R, +, -, 1', \sim, \cdot, c \rangle$, where $(R, +, -, 1')$ is a Boolean algebra $1' \in R$, $\sim$ is a unary operation and $\cdot$ is a binary operation. A relation algebra is representable $\iff$ it is isomorphic to a subalgebra of the form $\langle \wp(X), \cup, \cap, \sim, \cdot, c, \alpha, \alpha \rangle$, where $X$ is an equivalence relation, $1'$ is interpreted as the identity relation, $\sim$ is the operation of forming converses, and; is interpreted as composition of relations. Following standard notation, $(\mathcal{R})RA$ denotes the class of (representable) relation algebras. The class RA is a discriminator variety that is finitely axiomatizable, cf. [9, Definition 3.8, Theorems 3.19]. We let CRRA and LRRA, denote the classes of completely representable RAs, and its elementary closure, namely, the class of RAs satisfying the Lyndon conditions as defined in [9, §11.3.2], respectively. Complete representability of RAs is defined like the CA case. All of the above classes of algebras are instances of BAOs. The action of the non–Boolean operators in a completely additive (where operators distribute over arbitrary joins componentwise) atomic BAO, is determined by their behavior over the atoms, and this in turn is encoded by the atom structure of the algebra.

### 3 Complete representability via atomic dilations

We recall from [7, Definition 5.4.16], the notion of neat reducts of polyadic algebras. We shall be dealing with infinite dimensional such algebras. Because infinite cylindrification is allowed, the definition of neat reducts is different from the CA case. We define the neat reduct operator only PAs; the PEA case is entirely analogous considering diagonal elements.

**Definition 3.1.** Let $J \subseteq \beta$ and $\mathfrak{A} = \langle A, +, -, 0, 1, c_\beta, s_\beta \rangle_{\tau \subseteq \beta, \tau \in \beta}$ be a PA$_\beta$. Let $\text{Nr}_J \mathfrak{B} = \{ a \in A : c_{\beta \in J} a = a \}$. Then $\text{Nr}_J \mathfrak{B} = \langle \text{Nr}_J \mathfrak{B}, +, -, 0, 1, c_{\beta}, s'_{\tau} \rangle_{\tau \subseteq J, \tau \in \alpha}$ where $s'_{\tau} = s_\tau$. Here $\tau = \tau \cup \text{Id}_{\beta \in J}$. The structure $\text{Nr}_J \mathfrak{B}$ is an algebra, called the $J$-compression of $\mathfrak{B}$. When $J = \alpha$, $\alpha$ an ordinal, then $\text{Nr}_\alpha \mathfrak{B} \in \text{PA}_\alpha$ and it is called the neat $\alpha$ reduct of $\mathfrak{B}$.
Assume that $\mathcal{B} \in \text{PEA}_\beta$ for some infinite ordinal $\beta$. Then for $n < \omega$, $\text{Nr}_n \mathcal{B} \subseteq \text{Nr}_n \text{qea} \mathcal{B}$, where $\text{qea}$ denotes the quasi–polyadic reduct of $\mathcal{B}$, obtained by discarding infinitary substitutions and the definition of the neat reduct operator $\text{Nr}_n$ here is like the CA case not involving infinitary cylinders. Indeed, if $x \in \text{Nr}_n \mathcal{B}$, then $c_{\beta \setminus n} x = x$, so for any $i \in \beta \setminus n$, $c_i x \leq c_{\beta \setminus n} x = x \leq c_i x$, hence $c_i x = x$. However, the converse might not be true. If $x \in \text{Nr}_n \text{qea} \mathcal{B}$, then $c_i x = x$ for all $i \in \beta \setminus n$, but this does not imply that $c_{\beta \setminus n} x = x$; it can happen that $c_{\beta \setminus n} x > x = c_n x$ (for all $i \in \beta \setminus n$). We will show in a moment that if $\mathcal{C} \in \text{PEA}_\omega$ is atomic and $n < \omega$, then both $\text{Nr}_n \text{qea} \mathcal{C}$ and $\text{Nr}_n \mathcal{C}$ are completely representable $\text{PEA}_n$s. This gives a plethora of completely representable $\text{PEA}_n$s whose CA reducts are (of course) also completely representable.

For $\alpha \geq \omega$, we let $\text{CPA}_\alpha$ ($\text{CPEA}_\alpha$) denote the reduct of $\text{PA}_\alpha$ ($\text{PEA}_\alpha$) whose signature is obtained from that of $\text{PA}_\alpha$ ($\text{PEA}_\alpha$) by discarding all infinitary cylinders, and its axiomatization is that of $\text{PA}_\alpha$ ($\text{PEA}_\alpha$) restricted to the new signature. $\text{QA}_\alpha$ ($\text{QEA}_\alpha$) denotes the class of quasi–polyadic (equality) algebras obtained by restricting the signature and axiomatization of $\text{PA}_\alpha$ ($\text{PEA}_\alpha$) to only finite substitutions and cylinders. So here the signature does not contain infinitary substitutions, the $s_n$s are defined only for those maps $\tau : \alpha \rightarrow \alpha$ that move only finitely many points. With cylinders defined only on finitely many indices, the neat reduct operator $\text{Nr}$ for $\text{QA}_\alpha$, $\text{QEA}_\alpha$, $\text{CPA}_\alpha$ and $\text{CPEA}_\alpha$ is defined analogous to the CA case. We present analogous positive results typically of the form: If $\mathcal{K}$ is any of the classes defined above (like CPEA, CPA, QEA, QA), $\mathcal{D} \in \mathcal{K}_\omega$ is atomic, $n < \omega$, then (under certain conditions on $\mathcal{D}$) $\text{Nr}_n \mathcal{D}$ is completely representable. The ‘certain conditions’ will be formulated only for the dilation $\mathcal{D}$ and will not depend on $n$. For example for PEA, mere atomicity of $\mathcal{D}$ will suffice, for PA we will need complete additivity of $\mathcal{D}$ too.

We need a crucial lemma. But first a definition:

**Definition 3.2.** A transformation system is a quadruple of the form $(\mathfrak{A}, I, G, S)$ where $\mathfrak{A}$ is an algebra of any signature, $I$ is a non–empty set (we will only be concerned with infinite sets), $G$ is a subsemigroup of $(I, \circ)$ (the operation $\circ$ denotes composition of maps) and $S$ is a homomorphism from $G$ to the semigroup of endomorphisms of $\mathfrak{A}$. Elements of $G$ are called transformations.

The next lemma says that, roughly, in the presence of all substitution operators in the infinite dimensional case, one can form dilations in any higher dimension.

**Lemma 3.3.** Let $\alpha$ be an infinite ordinal and $\mathcal{K} \in \{\text{PA}, \text{PEA}\}$. Let $\mathcal{D} \in \mathcal{K}_\alpha$. Then for any ordinal $n > \alpha$, there exists $\mathcal{B} \in \mathcal{K}_n$ such that $\mathcal{D} = \text{Nr}_n \mathcal{B}$. Furthermore, if $\mathcal{D}$ is atomic (complete), then $\mathcal{B}$ can be chosen to be atomic (complete). An entirely analogous result holds for CPA and CPEA replacing the operator $\text{Nr}$ by the neat reduct operator $\text{Nr}$.

**Proof.** Let $\mathcal{K} \in \{\text{PA}, \text{PEA}, \text{CPA}, \text{CPEA}\}$. Assume that $\mathcal{D} \in \mathcal{K}_\alpha$ and that $n > \alpha$. If $|\alpha| = |n|$, then one fixes a bijection $\rho : n \rightarrow \alpha$, and defines the $n$-dimensional dilation of the diagonal free reduct of $\mathcal{D}$, having the same universe as $\mathcal{D}$, by re-shuffling the operations of $\mathcal{D}$ along $\rho$ [3]. Then one defines diagonal elements in the $n$-dimensional dilation of the diagonal free reduct of $\mathcal{D}$, by using the diagonal elements in $\mathcal{D}$ [7] Theorem 5.4.17. Now assume that $|n| > |\alpha|$. Let $\text{End}(\mathcal{D})$ be the semigroup of Boolean endomorphisms on $\mathcal{D}$. Then the map $S : \alpha \alpha \rightarrow \text{End}(\mathfrak{A})$ defined via $\tau \mapsto s_\tau$ is a homomorphism of semigroups. The operation on both semigroups is composition of maps, so that $(\mathcal{D}, \alpha, \alpha, S)$ is a transformation system. For any set $X$, let $F(\alpha X, \mathfrak{A})$ be the set of all maps from $\alpha X$ to $\mathfrak{A}$ endowed with Boolean operations defined pointwise and for $\tau \in \alpha \alpha$ and $f \in F(\alpha X, \mathfrak{A})$,
put \( s_p f(x) = f(x \circ \tau) \). This turns \( F(\alpha X, \mathfrak{A}) \) to a transformation system as well. The map \( H : \mathfrak{A} \to F(\alpha, \mathfrak{A}) \) defined by \( H(p)(x) = s_p x \) is easily checked to be an embedding of transformation systems. Assume that \( \beta \geq \alpha \). Then \( K : F(\alpha, \mathfrak{A}) \to F(\beta, \mathfrak{A}) \) defined by \( K(f)(x) = f(x \circ \alpha) \) is an embedding, too. These facts are fairly straightforward to establish [3] Theorems 3.1, 3.2]. Call \( F(\beta, \mathfrak{A}) \) a minimal functional dilation of \( F(\alpha, \mathfrak{A}) \).

Elements of the big algebra, or the (cylindrifier free) functional dilation, are of form \( s_p x \), \( p \in F(\alpha, \mathfrak{A}) \) where \( \sigma \upharpoonright \alpha \) is injective [3] Theorems 4.3-4.4]. Let \( \mathfrak{B}^{\text{c},d} = F(\alpha, \mathfrak{D}) \). Let \( \rho \) be any permutation such that \( \rho \circ \sigma(\alpha) \subseteq \sigma(\alpha) \). For the \( \text{PA} \) case one defines cylindrifiers on \( \mathfrak{B}^{\text{c},d} \) by setting for each \( \Gamma \subseteq n \):

\[
c_{(\Gamma)} s_{\sigma}^{(\mathfrak{B}^{\text{c},d})} p = s_{\rho^{-1}}^{(\mathfrak{B}^{\text{c},d})} c_{\rho(\Gamma) \cap \sigma(\alpha)}^{\mathfrak{D}} s_{(\rho \circ \sigma)(\alpha)}^{\mathfrak{D}} p.
\]

For the cases CPA case, one defines cylindrifiers on \( \mathfrak{B}^{\text{c},d} \) by restricting \( \Gamma \) to singletons, setting for each \( i \in n \):

\[
c_{i} s_{\sigma}^{(\mathfrak{B}^{\text{c},d})} p = s_{\rho^{-1}}^{(\mathfrak{B}^{\text{c},d})} c_{\rho(i) \cap \sigma(\alpha)}^{\mathfrak{D}} s_{(\rho \circ \sigma)(\alpha)}^{\mathfrak{D}} p.
\]

In both cases, the definition is sound, that is, it is independent of \( \rho, \sigma, p \); furthermore, it agrees with the old cylindrifiers in \( \mathfrak{D} \). Denote the resulting algebra by \( \mathfrak{B}^{\text{c},d} \).

When \( \mathfrak{D} \in \text{PA}_\alpha \), identifying algebras with their transformation systems we get that \( \mathfrak{D} \cong N_r \mathfrak{B}^{\text{c},d} \), via the isomorphism \( H \) defined for \( f \in \mathfrak{D} \) and \( x \in \alpha \alpha \) by, \( H(f)(x) = f(y) \) where \( y \in \alpha \alpha \) and \( x \upharpoonright \alpha = y \). [3] Theorem 3.10]. In [3] Theorems 4.3, 4.4] it is shown that \( H(\mathfrak{D}) = N_r \mathfrak{B}^{\text{c},d} \) where \( \mathfrak{B}^{\text{c},d} = \{ s_{\sigma}^{(\mathfrak{B}^{\text{c},d})} p : p \in \mathfrak{D} : \sigma \upharpoonright \alpha \) is injective \}. When \( \mathfrak{D} \in \text{CPA}_\alpha \), identifying \( \mathfrak{D} \) with \( H(\mathfrak{D}) \), where \( H \) is defined like in the \( \text{PA} \) case, we get that \( \mathfrak{D} \subseteq N_r \mathfrak{B}^{\text{c},d} \) with \( \mathfrak{B}^{\text{c},d} = \{ s_{\sigma}^{(\mathfrak{B}^{\text{c},d})} p : p \in \mathfrak{D} : \sigma \upharpoonright \alpha \) is injective \}. We show that \( N_r \mathfrak{B}^{\text{c},d} \subseteq \mathfrak{D} \), so that \( \mathfrak{D} = N_r \mathfrak{B}^{\text{c},d} \). Let \( x \in N_r \mathfrak{B}^{\text{c},d} \). Then there exist \( y \in \mathfrak{D} \) and \( \sigma : \beta \to \beta \) with \( \sigma \upharpoonright \alpha \) injective, such that \( x = s_{\sigma}^{(\mathfrak{B}^{\text{c},d})} y \). Choose any \( \tau : \beta \to \beta \) such that \( \tau(i) = i \) for all \( i \in \alpha \) and \( (\tau \circ \sigma)(i) \in \alpha \) for all \( i \in \alpha \). Such a \( \tau \) clearly exists. Since \( \Delta x \subseteq \alpha \), and \( \tau \) fixes \( \alpha \) pointwise, we have \( s_{\tau}^{(\mathfrak{B}^{\text{c},d})} x = x \). Then \( x = s_{\tau}^{(\mathfrak{B}^{\text{c},d})} x = s_{\tau}^{(\mathfrak{B}^{\text{c},d})} s_{\sigma}^{(\mathfrak{B}^{\text{c},d})} y = s_{\tau \circ \sigma}^{(\mathfrak{B}^{\text{c},d})} y = s_{\tau \circ \sigma | \alpha}^{\mathfrak{D}} y \in \mathfrak{D} \). In all cases, having at hand \( \mathfrak{B}^{\text{c},d} \), for all \( i < j < n \), the diagonal element \( d_{ij} \) (in \( \mathfrak{B}^{\text{c},d} \)) can be defined, using the diagonal elements in \( \mathfrak{D} \), as in [7] Theorem 5.4.17], obtaining the expanded required structure \( \mathfrak{B} \). The expanded structure \( \mathfrak{B} \) has Boolean reduct isomorphic to \( F(\alpha, \mathfrak{D}) \). In particular, \( \mathfrak{B} \) is atomic (complete) if \( \mathfrak{D} \) is atomic (complete), because a product of an atomic (complete) Boolean algebras is atomic (complete).

The proof of the following lemma follows from the definitions.

**Lemma 3.4.** If \( \mathfrak{A}, \mathfrak{B} \) and \( \mathfrak{C} \) are Boolean algebras, such that \( \mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C} \), \( \mathfrak{B} \subseteq \mathfrak{C} \) and \( \mathfrak{A} \subseteq \mathfrak{C} \), then \( \mathfrak{A} \subseteq \mathfrak{B} \).

For simplicity of notation, if \( \beta \geq \omega \), \( \mathfrak{B} \in \text{PA}_\beta \text{(PEA}_\beta \), and \( n < \omega \), then we write \( N_r \mathfrak{B} \) for \( N_r N_{\alpha \alpha} \mathfrak{B} (N_r N_{\alpha \alpha} \mathfrak{B}) \), where \( N_{\alpha \alpha} \) denotes ‘quasi-polyadic reduct’.

In this section, we understand complete representability for \( \alpha \)-dimensional algebras, \( \alpha \) any ordinal, in the classical sense with respect to generalized cartesian \( \alpha \)-dimensional spaces.

It is shown in [27], that for any infinite ordinal \( \alpha \), if \( \mathfrak{A} \in \text{PA}_\alpha \) is atomic and completely additive, then it is completely representable. From this it can be concluded that for any \( n < \omega \), any complete subalgebra of \( N_r \mathfrak{D} \) is completely representable using lemma [23] because \( N_r \mathfrak{D} \subseteq \mathfrak{D} \) (as can be easily distilled from the next proof). The
result in [27] holds for CPA’s, cf. theorem 3.7 but it does not hold for PEA_{\omega}s and CPEA_{\omega}s. It is not hard to construct atomic algebras in the last two classes that are not even representable, let alone completely representable. But for such (non–representable) algebras the \( n \)-neat reduct, for any \( n < \omega \), will be completely representable as proved next (in theorems 3.5 and 3.6):

**Theorem 3.5.** If \( 2 < n < \omega \) and \( \mathcal{D} \in \text{PEA}_n \) is atomic, then any complete subalgebra of \( \text{Nr}_n \mathcal{D} \) is completely representable. In particular, \( \text{Nr}_n \mathcal{D} \) is completely representable.

*Proof.* We identify notationally set algebras with their domain. Assume that \( \mathfrak{A} \subseteq \text{Nr}_n \mathcal{D} \), where \( \mathcal{D} \in \text{PEA}_n \) is atomic. We want to completely represent \( \mathfrak{A} \). Let \( c \in \mathfrak{A} \) be non–zero. We will find a homomorphism \( f : \mathfrak{A} \to \varphi^\omega(nU) \) such that \( f(c) \neq 0 \), and \( \bigcup_{y \in Y} f(y) = nU \), whenever \( Y \subseteq \mathfrak{A} \) satisfies \( \sum^\mathfrak{A} Y = 1 \). Assume for the moment (to be proved in a while) that \( \mathfrak{A} \subseteq \mathcal{D} \). Then by lemma 2.3 \( \mathfrak{A} \) is atomic, because \( \mathcal{D} \). For brevity, let \( X = \text{At} \mathfrak{A} \).

Let \( m \) be the local degree of \( \mathcal{D} \), \( c \) its effective cardinality and let \( \beta \) be any cardinal such that \( \beta \geq c \) and \( \sum_{\delta \leq m} \beta^\delta = \beta \); such notions are defined in [3] [27]. We can assume by lemma 3.3 that \( \mathcal{D} = \text{Nr}_\omega \mathcal{B} \), with \( \mathcal{B} \in \text{PEA}_\beta \). For any ordinal \( \mu \in \beta \), and \( \tau \in \mu^\beta \), write \( \tau^+ \) for \( \tau \cup \text{Id}_{\beta \mu}(\in \beta^\beta \). Consider the following family of joins evaluated in \( \mathcal{B} \), where \( p \in \mathcal{D} \), \( \Gamma \subseteq \beta \) and \( \tau \in \omega^\beta \): \((*)\) \( c_{(\Gamma)p} = \sum^\mathcal{B} \{ s_{\tau} + p : \tau \in \omega^\beta, \; \tau \upharpoonright \omega \downarrow \Gamma = \text{Id} \}, \) and \((**): \sum_{\tau \in \omega^\beta} X = 1 \). The first family of joins exists \([3] \text{ Proof of Theorem 6.1}, 27\), and the second exists, because \( \sum^\mathcal{B} X = \sum_{\delta < \omega} \sum^\mathcal{B} X = 1 \) and \( \tau^+ \) is completely additive, since \( \mathcal{B} \in \text{PEA}_\beta \). The last equality of suprema follows from the fact that \( \mathcal{D} = \text{Nr}_\omega \mathcal{B} \subseteq c \mathcal{B} \) and the first from the fact that \( \mathfrak{A} \subseteq \mathcal{D} \). We prove the former, the latter is exactly the same replacing \( \omega \) and \( \beta \), by \( n \) and \( \omega \), respectively, proving that \( \text{Nr}_n \mathcal{D} \subseteq \mathcal{D} \), hence \( \mathfrak{A} \subseteq \mathcal{D} \). We prove that \( \text{Nr}_\omega \mathcal{B} \subseteq c \mathcal{B} \). Assume that \( S \subseteq \mathcal{D} \) and \( \sum^\mathcal{D} S = 1 \), and for contradiction, that there exists \( d \in \mathcal{B} \) such that \( s \leq d < 1 \) for all \( s \in S \). Let \( J = \Delta d \downarrow \omega \) and take \( t = -c_{(J)}(-d) \in \mathcal{D} \). Then \( c_{(\beta \omega)} t = c_{(\beta \omega)}(-c_{(J)}(-d)) = c_{(\beta \omega)} - c_{(\beta \omega)} c_{(J)}(-d) = c_{(\beta \omega)} - c_{(\beta \omega)} c_{(J)}(-d) = -c_{(\beta \omega)} c_{(J)}(-d) = -c_{(J)}(-d) \). We have proved that \( t \in \mathcal{D} \).

We now show that \( s \leq t < 1 \) for all \( s \in S \), which contradicts \( \sum^\mathcal{D} S = 1 \). If \( s \in S \), we show that \( s \leq t \). By \( s \leq d \), we have \( s \cdot -d = 0 \). Hence by \( c_{(J)} s = s \), we get \( 0 = c_{(J)}(s \cdot -d) = s \cdot c_{(J)}(-d) \), so \( s \leq -c_{(J)}(-d) \). It follows that \( s \leq t \) as required. Assume for contradiction that \( 1 = -c_{(J)}(-d) \). Then \( c_{(J)}(-d) = 0 \), so \( -d = 0 \) which contradicts that \( d < 1 \). We have proved that \( \sum^\mathcal{B} S = 1 \), so \( \mathcal{D} \subseteq c \mathcal{B} \). Let \( F \) be any Boolean ultrafilter of \( \mathcal{B} \) generated by an atom below \( a \). We show that \( F \) will preserve the family of joins in \((*) \) and \((**) \). We use a simple topological argument used by the author in [27]. One forms nowhere dense sets in the Stone space of \( \mathcal{B} \) corresponding to the aforementioned family of joins as follows: The Stone space of (the Boolean reduct of) \( \mathcal{B} \) has underlying set, the set of all Boolean ultrafilters of \( \mathcal{B} \). For \( b \in \mathcal{B} \), let \( N_b \) be the clopen set \( \{ F \in S : b \in F \} \). The required nowhere dense sets are defined for \( \Gamma \subseteq \beta \), \( p \in \mathcal{D} \) and \( \tau \in \omega^\beta \) via: \( A_{\Gamma,p} = N_{c_{(\Gamma)p}} \setminus \bigcup_{\tau \upharpoonright \omega \downarrow \beta} N_{s_{\tau} + p} \), and \( A_{\tau} = S \setminus \bigcup_{x \in X} N_{s_{\tau} + x} \). The principal ultrafilters are isolated points in the Stone topology, so they lie outside the nowhere dense sets defined above. Hence any such ultrafilter preserve the joins in \((*) \) and \((**) \). Fix a principal ultrafilter \( F \) preserving \((*) \) and \((**) \) with \( a \in F \). For \( i, j \in \beta \), set \( iEj \iff \delta_{ij}^a \in F \). Then by the equational properties of diagonal elements and properties of filters, it is easy to show that \( E \) is an equivalence relation on \( \beta \). Define \( f : \mathfrak{A} \to \varphi^\omega(\beta/E) \), via \( x \mapsto \{ \bar{t} \in \omega^\beta(\beta/E) : \delta_{ij}^a \in F \}, \) where \( i/E = t(i) \), \( (i < n) \) and \( t \in \omega^\beta \). We show that \( f \) is a well–defined homomorphism (from \((*) \)) and that \( f \) is complete such that \( f(c) \neq 0 \). The last follows by observing that \( Id \in f(c) \). Let \( V = \beta^\beta,J \). To show
that $f$ is well defined, it suffices to show that for all $\sigma, \tau \in V$, if $(\tau(i), \sigma(i)) \in E$ for all $i \in \beta$, then for any $x \in A$, $s_\tau x \in F \iff s_\sigma x \in F$. We proceed by induction on $|\{i \in \beta : \tau(i) \neq \sigma(i)\}| < \omega$. If $J = \{i \in \beta : \tau(i) \neq \sigma(i)\}$ is empty, the result is obvious. Otherwise assume that $k \in J$. We introduce a helpful piece of notation. For $\eta \in V$, let $\eta(k \mapsto l)$ stand for the $\eta'$ that is the same as $\eta$ except that $\eta'(k) = l$. Now take any $\lambda \in \{\eta \in \beta : (\sigma)^{-1}\{\eta\} = (\tau)^{-1}\{\eta\}\} \setminus \Delta x$. Recall that $\Delta x = \{i \in \beta : c_i x \neq x\}$ and that $\beta \setminus \Delta x$ is infinite because $\Delta x \subseteq n$, so such a $\lambda$ exists. Now we freely use properties of substitutions for cylindric algebras. We have by [7, 1.11.11(i)(iv)] (a) $s_\sigma x = s_{\lambda k}^1 s_{\sigma(k \mapsto \lambda)} x$, and (b) $s_{\lambda k}^1 (d_{\lambda, \sigma k} \cdot s_\sigma x) = d_{\tau k, \sigma k} s_{\lambda k} s_\sigma x$, and (c) $s_{\lambda k}^1 (d_{\lambda, \sigma k} \cdot s_{\sigma(k \mapsto \lambda)} x) = d_{\tau k, \sigma k} \cdot s_{\lambda k} s_{\sigma(k \mapsto \lambda)} x$, and finally (d) $d_{\lambda, \sigma k} \cdot s_{\lambda k}^1 s_{\sigma(k \mapsto \lambda)} x = d_{\lambda, \sigma k} \cdot s_{\sigma(k \mapsto \lambda)} x$. Then by (b), (a), (d) and (c), we get,

$$
\begin{align*}
d_{\tau k, \sigma k} \cdot s_\sigma x &= s_{\lambda k}^1 (d_{\lambda, \sigma k} \cdot s_\sigma x) \\
&= s_{\lambda k}^1 (d_{\lambda, \sigma k} \cdot s_{\lambda k} s_{\sigma(k \mapsto \lambda)} x) \\
&= s_{\lambda k}^1 (d_{\lambda, \sigma k} \cdot s_{\sigma(k \mapsto \lambda)} x) \\
&= d_{\tau k, \sigma k} \cdot s_{\sigma(k \mapsto \tau k)} x.
\end{align*}
$$

But $F$ is a filter and $(\tau k, \sigma k) \in E$, we conclude that

$$
s_\sigma x \in F \iff s_{\sigma(k \mapsto \tau k)} x \in F.
$$

The conclusion follows from the induction hypothesis. We have proved that $f$ is well defined. We now check that $f$ is a homomorphism, i.e. it preserves the operations. For $\sigma \in \beta$, recall that $\sigma^+$ denotes $\sigma \cup Id_{\beta \setminus n} \in \beta(Id)$.

- **Boolean operations:** Since $F$ is maximal we have $\bar{\sigma} \in f(x + y) \iff s_{\sigma^+} (x + y) \in F \iff s_\sigma + x + s_\sigma + y \in F \iff s_\sigma + x$ or $s_\sigma + y \in F \iff \bar{\sigma} \in f(x) \cup f(y)$. We now check complementation.

$$
\bar{\sigma} \in f(\neg x) \iff s_{\sigma^+} (\neg x) \in F \iff -s_\sigma + x \in F \iff s_\sigma + x \notin F \iff \bar{\sigma} \notin f(x).
$$

- **Diagonal elements:** Let $k, l < n$. Then we have: $\sigma \in f d_{kl} \iff s_\sigma \cdot d_{kl} \in F \iff d_{\sigma k, \sigma l} \in F \iff (\sigma k, \sigma l) \in E \iff \sigma k / E = \sigma l / E \iff \bar{\sigma}(k) = \bar{\sigma}(l) \iff \sigma \in d_{kl}$.

- **Cylindrifications:** Let $k < n$ and $a \in A$. Let $\bar{\sigma} \in c_k f(a)$. Then for some $\lambda \in \beta$, we have $\bar{\sigma}(k \mapsto \lambda / E) \in f(a)$ hence $s_{\sigma^+} (k \mapsto \lambda) a \in F$. It follows from the inclusion $a \subseteq c_k a$ that $s_{\sigma^+} (k \mapsto \lambda) c_k a \in F$, so $s_\sigma + c_k a \in F$. Thus $c_k f(a) \subseteq f(c_k a)$.

We prove the other more difficult inclusion that uses the condition (*) of eliminating cylindrifiers. Let $a \in A$ and $k < n$. Let $\bar{\sigma}' \in f c_k a$ and let $\sigma = \sigma' \cup Id_{\beta \setminus n}$. Then $s_{\sigma^+} c_k a = s_{\sigma'} c_k a \in F$. Pick $\lambda \in \{\eta \in \beta : (\sigma)^{-1}\{\eta\} \setminus \Delta a\}$, such a $\lambda$ exists because $\Delta a$ is finite, and $|\{i \in \beta : (\sigma(i) \neq i\}| < \omega$. Let $\tau = \sigma \upharpoonright n \setminus \{k, \lambda\} \cup \{(k, \lambda), (\lambda, k)\}$. Then (in $\mathcal{F}$):

$$
c_{\lambda} s_{\tau} a = s_{\tau} c_k a = s_{\sigma} c_k a \in F.
$$

By the construction of $F$, there is some $u(\notin \Delta(s_{\sigma} a))$ such that $s_{\lambda} s_{\tau} a \in F$, so $s_{\sigma(k \mapsto u)} a \in F$. Hence $\sigma(k \mapsto u) \in f(a)$, from which we get that $\bar{\sigma}' \in c_k f(a)$.

- **Substitutions:** Direct since substitution operations are Boolean endomorphisms.
We show that the non-zero homomorphism $f$ is an atomic, hence, a complete representation. By construction, for every $s \in n(\beta/E)$, there exists $x \in X (= \text{At} \mathfrak{A})$, such that $s^{\mathfrak{B}}_{s \cap Id_{\beta \setminus c}} x \in F$, from which we get the required, namely, that $U_{x \in X} f(x) = n(\beta/E)$. The complete representability of $\text{Nr}_n \mathfrak{D}$ follows from lemmata 2.3 3.1 by observing that $\text{Nr}_n \mathfrak{D} \subseteq_c \mathfrak{D}$, hence $\text{Nr}_n \mathfrak{D} \subseteq_c \text{Nr}_n \mathfrak{D}$. □

For CPAEs, we have a slightly weaker result:

**Theorem 3.6.** If $n < \omega$ and $\mathfrak{D} \in \text{CPEA}_\omega$ is atomic, then any complete subalgebra of $\text{Nr}_n \mathfrak{C}m \text{At} \mathfrak{D}$ is completely representable. In particular, if $\mathfrak{D}$ is complete and atomic, then $\text{Nr}_n \mathfrak{D}$ is completely representable.

**Proof.** Let $\mathfrak{D} \in \text{CPEA}_\omega$ be atomic. Let $\mathfrak{D}^* = \mathfrak{C}m \text{At} \mathfrak{D}$. Then $\mathfrak{D}^*$ is complete and atomic and $\text{Nr}_n \mathfrak{D}^* \subseteq_c \mathfrak{D}^*$. To prove the last $\subseteq_c$, assume for contradiction that there is some $S \subseteq \text{Nr}_n \mathfrak{D}^*$, $\sum_{n}^{\text{Nr}_n \mathfrak{D}^*} S = 1$, and there exists $d \in \mathfrak{D}^*$ such that $s \leq d < 1$ for all $s \in S$. Take $t = -\sum_{n}^{(\beta/E)}(-c_s) d$. This infimum is well defined because $\mathfrak{D}^*$ is complete. Like in the previous proof it can be proved that $c_t = t$ for all $i \in \omega \setminus n$, hence $t \in \text{Nr}_n \mathfrak{D}^*$ and that $s \leq t < 1$ for all $s \in S$, which contradicts that $\sum_{n}^{\text{Nr}_n \mathfrak{D}^*} S = 1$. Let $\beta$ be a regular cardinal $\beta > |\mathfrak{D}^*|$ and by lemma 3.3 let $\mathfrak{B} \in \text{CPEA}_\beta$ be complete and atomic such that $\mathfrak{D}^* = \text{Nr}_n \mathfrak{B}$. Then we have the following chain of complete embeddings: $\text{Nr}_n \mathfrak{D}^* \subseteq_c \mathfrak{D}^* = \text{Nr}_n \mathfrak{B} \subseteq_c \mathfrak{B}$; the last $\subseteq_c$ follows like above using that $\mathfrak{B}$ is complete. From the first $\subseteq_c$, since $\mathfrak{D}^*$ is atomic, we get by lemma 2.3 that $\text{Nr}_n \mathfrak{D}^*$ is atomic. Let $X = \text{At} \text{Nr}_n \mathfrak{D}^*$. Then also from the first $\subseteq_c$, we get that $\sum_{n}^{\mathfrak{D}^*} X = 1$, so $\sum_{n}^{\mathfrak{D}^*} X = 1$ because $\mathfrak{D}^* \subseteq_c \mathfrak{B}$. For $k < \beta$, $x \in \mathfrak{D}^*$ and $\tau \in c_\beta$, the following joins hold in $\mathfrak{B}$: (*) $c_k x = \sum_{l \in \beta} s^{\mathfrak{B}}_{k, l} x$ and (**) $\sum_{l \in \beta} s^{\mathfrak{B}}_{k, l} X = 1$, where $\tau^+ = \tau \cup Id_{\beta \setminus c(\beta)}$. The join (**) holds, because $s^{\mathfrak{B}}_{k, \tau^+}$ is completely additive, since $\mathfrak{B}$ is completely additive. To prove (*), fix $k < \beta$. Then for all $l \in \beta$, we have $s^{\mathfrak{B}}_{k, l} x \leq c_k x$. Conversely, assume that $y \in \mathfrak{B}$ is an upper bound for $\{s^{\mathfrak{B}}_{k, l} x : l \in \beta\}$. Let $l \in \beta \setminus (\Delta x \cup \Delta y)$; such an $l$ exists, because $|\Delta x| < \beta$, $|\Delta y| < \beta$ and $\beta$ is regular. Hence, we get that $c_k x = x$ and $c_k y = y$. But then $c_k s^{\mathfrak{B}}_{k, l} x \leq y$, and so $c_k x \leq y$. We have proved that (*) hold. Let $\mathfrak{A} = \text{Nr}_n \mathfrak{D}^*$. Let $a \in \mathfrak{A}$ be non–zero. We want to find a complete representation $f : \mathfrak{A} \to \varphi(\mathfrak{V})(V$ a unit of a $G_{\mathfrak{B}n}$, i.e a disjoint union of cartesian spaces) such that $f(a) \neq 0$. Let $F$ be any Boolean ultrafilter of $\mathfrak{B}$ generated by an atom below $a$. Then, like in the proof of theorem 3.3, $F$ will preserve the family of joins in (*) and (**). Next we proceed exactly like in the proof of theorem 3.3. For $i, j \in \beta$, set $iEj \iff d^{\mathfrak{B}}_{ij} \in F$. Then $E$ is an equivalence relation on $\beta$. Define $f : \mathfrak{A} \to \varphi(n(\beta/E))$, via $x \mapsto \{t \in n(\beta/E) : s^{\mathfrak{B}}_{l, t} x \in F\}$, where $\mathfrak{V}(t(i/E)) = t(i)$ and $t \in n \beta$. Then $f$ is well–defined, a homomorphism (from(*)) and atomic (from(**)). Also $f(a) \neq 0$ because $a \in F$, so $Id_f(a)$ is complete.

**Theorem 3.7.** If $\mathfrak{D} \in \text{CPA}_\omega$ is atomic and completely additive, then it is completely representable

**Proof.** Replace $\mathfrak{D}$ by its Dedekind-MacNeille completion $\mathfrak{D}^* = \mathfrak{C}m \text{At} \mathfrak{D}$. Then $\mathfrak{D}$ is completely representable $\iff \mathfrak{D}^*$ is completely representable and furthermore $\mathfrak{D}^*$ is complete. It suffices thus to show that $\mathfrak{D}^*$ is completely representable. One forms an atomic complete dilation $\mathfrak{B}$ of $\mathfrak{D}^*$ to a regular cardinal $\beta > |\mathfrak{D}^*|$ exactly as in lemma 3.3. For $\tau \in \omega \beta$, let $\tau^+ = \tau \cup Id_{\beta \setminus c\beta}$. Then like before $\mathfrak{D}^* \subseteq_c \mathfrak{B}$ and so the following family of joins hold in $\mathfrak{B}$: For all $i < \beta$, $b \in \mathfrak{B}$ $c_i b = \sum_{l \in \beta} s^{\mathfrak{B}}_{l, i} X$ and for all $\tau \in \omega \beta$, $\sum_{l \in \beta} s^\tau X = 1$. Let $a \in \mathfrak{D}^*$ be non zero. Take any ultrafilter $F$ in the Stone space of $\mathfrak{B}$ generated by an
atom below $a$. Then $f: \mathcal{D}^* \to \varphi(\omega \beta)$ defined via $d \mapsto \{\tau \in \omega \beta : s^\varphi_{\tau} d \in F\}$ is a complete representation of $\mathcal{D}^*$ such that $f(a) \neq 0$.

If the dilations are in $\text{QEA}_\omega$ (an $\omega$ dimensional quasi–polyadic equality algebra) we have a weaker result. We do not know whether the result proved for $\text{PEA}_\omega$ holds when the $\omega$–dilation is an atomic $\text{QEA}_\omega$. Entirely analogous results hold if we replace $\text{QEA}_\omega$ by $\text{QA}_\omega$.

**Theorem 3.8.** Let $n < \omega$. Let $\mathcal{D} \in \text{QEA}_\omega$ be atomic. Assume that for all $x \in \mathcal{D}$ for all $k < \omega$, $c_k x = \sum_{\beta \in \omega} s^\beta_j x$. If $\mathcal{A} \subseteq \text{Nr}_n \mathcal{D}$ such that $\mathcal{A} \subseteq_c \mathcal{D}$ (this is stronger than $\mathcal{A} \subseteq_c \text{Nr}_n \mathcal{D}$), then $\mathcal{A}$ is completely representable.

**Proof.** First observe that $\mathcal{A}$ is atomic, because $\mathcal{D}$ is atomic and $\mathcal{A} \subseteq_c \mathcal{D}$. Accordingly, let $X = \text{At}\mathcal{A}$. Let $a \in \mathcal{A}$ be non-zero. Like before, one finds a principal ultrafilter $F$ such that $a \in F$ and $F$ preserves the family of joins $c_i x = \sum_\beta s^\beta j x$ and $\sum_\beta s^\beta X = 1$, where $\tau : \omega \to \omega$ is a finite transformation; that is $\{i \in \omega : \tau(i) \neq i\} < \omega$. The first family of joins exists by assumption, the second exists, since $\sum_\beta X = 1$ by $\mathcal{A} \subseteq_c \mathcal{D}$ and the $s,s$ are completely additive. Any principal ultrafilter $F$ generated by an atom below $a$ will do, as shown in the previous proof. Again as before, the selected $F$ gives the required complete representation $f$ of $\mathcal{A}$.

The following example shows that the existence of the joins in theorem 3.8 is not necessary.

**Example 3.9.** Let $\mathcal{D} \in \text{QEA}_\omega$ be the full weak set algebra with top element $\omega \omega^0$, where $0$ is the constant 0 sequence. Then, it is easy to show that for any $n < \omega$, $\text{Nr}_n \mathcal{A}$ is completely representable. Let $X = \{0\} \in \mathcal{D}$. Then for all $i \in \omega$, we have $s^0_i X = X$. But $(1,0,\ldots) \in c_0 X$, so that $\sum_{i \in \omega} s^0_i X = X \neq c_0 X$. Hence the joins in theorem 3.8 do not hold.

Now fix $1 < n < \omega$ and let $\mathcal{D}$ be as in the previous example. If we take $\mathcal{D}' = \mathcal{D}^\omega \text{Nr}_n \mathcal{D}$, then $\mathcal{D}'$ of course will still be a weak set algebra, and it will be locally finite, so that $c_i x = \sum_\beta s^\beta_j x$ for all $i < j < \omega$. However, $\mathcal{D}'$ will be atomless as we proceed to show. Assume for contradiction that it is not, and let $x \in \mathcal{D}'$ be an atom. Choose $k,l \in \omega$ with $k \neq l$ and $c_k x = x$, this is possible since $\omega \setminus \Delta x$ is infinite. Then $c_k(x \cdot d_{kl}) = x$, so $x \cdot d_{kl} \neq 0$. But $x$ is an atom, so $x \leq d_{kl}$. This gives that $\Delta x = 0$, and by [7 Theorem 1.3.19] $x \leq -c_k - d_{kl}$. It is also easy to see that $(c_k - d_{kl})^\omega = \omega \omega^0$, from which we conclude that $x = 0$, which is a contradiction. For an ordinal $\alpha$, we let $\text{Gwsq}_\alpha$ denote the class of $\text{QEA}_\alpha$s whose cylindric reduct is a $\text{Gwsq}_\alpha$, and the quasi–polyadic operations of substitutions defined like in quasi–polyadic equality set algebras relativized to $V$. That is, if $\mathcal{A} \in \text{Gwsq}_\alpha$, then $\text{Nd}_\alpha \mathcal{A} \in \text{Gwsq}_\alpha$ with top element $V$ say, and for $X \in \mathcal{A}$, and $i < j < \alpha$, $S_{[i,j]} X = \{s \in V : s \circ [i,j] \in X\}$.

**Theorem 3.10.** Let $\alpha$ be an infinite ordinal.

1. If $\mathcal{D} \in \text{PEA}_{\alpha+\omega}$ is atomic, then any complete subalgebra of $\text{Nr}_\alpha \mathcal{D}$ is completely representable with respect to $\text{Gwsq}_\alpha$.

2. If $\mathcal{D} \in \text{CPEA}_{\alpha+\omega}$, then any complete subalgebra of $\text{Nd}_\alpha \text{Nd}_q \text{g} \text{e}_q \text{mAt} \mathcal{D}$ is completely representable with respect to $\text{Gwsq}_\alpha$. In particular, if $\mathcal{D}$ is complete, then $\text{Nd}_\alpha \text{Nd}_q \text{g} \text{e}_q \text{mAt} \mathcal{D}$ is completely representable.
Proof. The proof is like when $\alpha = n < \omega$. Let $\mathfrak{A} \subseteq_c \mathfrak{N}_{\alpha} \mathfrak{D}$. We want to completely represent $\mathfrak{A}$ with respect to a $G_{\omega} \mathfrak{s}\mathfrak{o}_\alpha$. Given a non-zero $c \in \mathfrak{A}$, one dilates $\mathfrak{D}$ to $\mathfrak{B} \in \mathfrak{P}E\mathfrak{A}_\beta$, where $\beta$ is as specified in theorem 3.5 and finds a principal ultrafilter $F$ generated by an atom below $c$ preserving the set of joins (*) and (**) as stipulated in the proof of theorem 3.5 replacing once more set algebras by weak set algebras.

Theorem 3.6 replaces $\mathfrak{B}$ in theorem 3.5, replacing in these joins $\omega_c$ an atom below $c$. The proof is like the case when $CPEA$ required representation using $F$, like before, but using the weak space not 'a cartesian square'. The map establishing the complete representation, is defined like before, but using the weak space $G_{\omega} \mathfrak{s}\mathfrak{o}_\alpha$ At $\alpha$ like before, but using the weak space $\mathfrak{D}$ generated by a partial map $\hat{\imath} : \mathfrak{A} \to \varphi(\alpha/\beta/E)^{(\text{Id})}$, via $x \mapsto \{ \hat{\imath} \in \alpha(\beta/E)^{(\text{Id})} : s_{\alpha^1,\text{Id}_\beta}^{\beta} x \in F \}$, where $\hat{\imath}(i/E) = t(i)$ ($i < \alpha$) and $t \in \alpha \beta$. The CPEA case is entirely analogous. The proof is like the case when $\alpha = n < \omega$, dealt with in theorem 3.6 replacing once more set algebras by weak set algebras.

4 Finite dimensional algebras

This section is devoted to showing that several classes of completely representable algebras (of relations) are not elementary. We need some preparing to do. From now on, unless otherwise indicated, $n$ is fixed to be a finite ordinal $> 2$. Let $i < n$. For $n$–ary sequences $\bar{x}$ and $\bar{y}$, we write $\bar{x} \equiv_i \bar{y} \iff \bar{y}(j) = \bar{x}(j)$ for all $j \neq i$. For $i,j < n$ the replacement $[i/j]$ is the map that is like the identity on $n$, except that $i$ is mapped to $j$ and the transposition $[i,j]$ is the like the identity on $n$, except that $i$ is swapped with $j$.

Definition 4.1. Let $m$ be a finite ordinal $> 0$. An $s$ word is a finite string of substitutions $(s_i^j) (i,j < m)$, a $c$ word is a finite string of cylindrifications $(c_i), i < m$; an $sc$ word $w$, is a finite string of both, namely, of substitutions and cylindrifications. An $sc$ word induces a partial map $\hat{w} : m \to m$:

- $\hat{\epsilon} = \text{Id}$,
- $\hat{w}_j^i = \hat{w} \circ [i/j]$,
- $\hat{wc}_i = \hat{w} \upharpoonright (m \setminus \{i\})$.

If $\bar{a} \in <^{m-1}m$, we write $s_{\bar{a}}$, or $s_{a_0...a_{k-1}:}$ where $k = |\bar{a}|$, for an arbitrary chosen $sc$ word $w$ such that $\hat{w} = \bar{a}$. Such a $w$ exists by [9] Definition 5.23 Lemma 13.29.

From now on, unless otherwise indicated, $n$ is fixed to be a finite ordinal $> 2$.

Definition 4.2. (1) Let $K_n$ be any variety between $\mathfrak{S}c_n$ and $\mathfrak{QEA}_n$. Assume that $\mathfrak{A} \subseteq K_n$ is atomic and that $m,k \leq \omega$. The atomic game $G^m_k(\text{At}\mathfrak{A})$, or simply $G^m_k$, is the game played on atomic networks of $\mathfrak{A}$ using $m$ nodes and having $k$ rounds [10] Definition 3.3.2], where $\forall$ is offered only one move, namely, a cylindrifier move: Suppose that we are at round $t > 0$. Then $\forall$ picks a previously played network $N_i$ $(\text{nodes}(N_i) \subseteq m), i < n, a \in \text{At}\mathfrak{A}, \bar{x} \in ^n\text{nodes}(N_i)$, such that $N_i(\bar{x}) \leq c_i a$. For her response, $\exists$ has to deliver a network $M$ such that $\text{nodes}(M) \subseteq m, M \equiv_i N$, and there is $\bar{y} \in ^n\text{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$.

We write $G_k(\text{At}\mathfrak{A})$, or simply $G_k$, for $G^m_k(\text{At}\mathfrak{A})$ if $m \geq \omega$.

(2) The $\omega$–rounded game $G^m(\text{At}\mathfrak{A})$ or simply $G^m$ is like the game $G^m_\omega(\text{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the $m$ nodes in play.
Observe that for $k, m \leq \omega$, the games $G^m_k(\mathbb{At} \mathfrak{A})$ and $G^m(\mathbb{At} \mathfrak{A})$ depend on the signature of $\mathfrak{A}$.

**Definition 4.3.** Fix $2 < n < m$. Assume that $\mathfrak{C} \in \mathbb{CA}_m$, $\mathfrak{A} \subseteq \mathfrak{N}_n \mathfrak{C}$ is an atomic $\mathbb{CA}_n$ and $N$ is an $\mathfrak{A}$-network with nodes($N$) $\subseteq m$. Define $N^+ \in \mathfrak{C}$ by (with notation as introduced in Definition 2.1):

$$N^+ = \prod_{i_0, \ldots, i_{n-1} \in \text{nodes}(N)} s_{i_0, \ldots, i_{n-1}} N(i_0, \ldots, i_{n-1}).$$

For a network $N$ and function $\theta$, the network $N\theta$ is the complete labelled graph with nodes $\theta^{-1}(\text{nodes}(N)) = \{ x \in \text{dom}(\theta) : \theta(x) \in \text{nodes}(N) \}$, and labelling defined by

$$(N\theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})),$$

for $i_0, \ldots, i_{n-1} \in \theta^{-1}(\text{nodes}(N))$.

For a class $\mathbf{K}$ of BAOs, we denote by $\mathbf{K}^{\text{ad}}$ the class of completely additive algebras in $\mathbf{K}$.

**Lemma 4.4.** Let $2 < n < \omega$, and assume that $m > n$. Let $\mathbf{K}$ be any variety between $\mathbb{SC}_n$ and $\mathbb{QEA}_n$. If $\mathfrak{A} \in \mathbb{SC}_n \mathfrak{N}_n \mathbf{K}^{\text{ad}}$ is atomic, then $\exists$ has a winning strategy in $G^m(\mathbb{At} \mathfrak{A})$. If $\mathfrak{A} \in \mathbf{K}$, and $\mathfrak{A}$ has a complete $m$-square representation then $\exists$ has a winning strategy in $G^m(\mathbb{At} \mathfrak{A})$.

**Proof.** We give the proof for $\mathbb{CA}$s italicizing the part where additivity is used. The stipulated additivity condition when considering only $\mathbb{CA}$s is superfluous since it holds anyway. The proof lifts ideas in [6, Lemmata 29, 26, 27] formulated for relation algebras to $\mathbb{CA}_n$. Fix $2 < n < m$. Assume that $\mathfrak{C} \in \mathbb{CA}_m$, $\mathfrak{A} \subseteq \mathfrak{N}_n \mathfrak{C}$ is an atomic $\mathbb{CA}_n$. Then the following hold:

1: for all $x \in \mathfrak{C} \setminus \{0\}$ and all $i_0, \ldots, i_{n-1} < m$, there is $a \in \mathbb{At} \mathfrak{A}$, such that $s_{i_0, \ldots, i_{n-1}} a \cdot x \neq 0$.

2: for any $x \in \mathfrak{C} \setminus \{0\}$ and any finite set $I \subseteq m$, there is a network $N$ such that nodes$(N) = I$ and $x \cdot N^+ \neq 0$, with notation as in Definition 4.3. Furthermore, for any networks $M, N$ if $M^+ \cdot N^+ \neq 0$, then $M|_{\text{nodes}(M) \cap \text{nodes}(N)} = N|_{\text{nodes}(M) \cap \text{nodes}(N)}$.

3: if $\theta$ is any partial, finite map $m \rightarrow m$ and if nodes$(N)$ is a proper subset of $m$, then $N^+ \neq 0 \rightarrow (N\theta)^+ \neq 0$. If $i \not\in \text{nodes}(N)$, then $c_i N^+ = N^+$.

Since $\mathfrak{A} \subseteq c \mathfrak{N}_n \mathfrak{C}$, then $\sum \mathbb{E} \mathfrak{At} \mathfrak{A} = 1$. For (1), $s_j$ is a completely additive operator (any $i, j < m$), hence $s_{i_0, \ldots, i_{n-1}}$ is, too. So $\sum_{a \in \mathbb{At} \mathfrak{A}} s_{i_0, \ldots, i_{n-1}} a = s_{i_0, \ldots, i_{n-1}} 1 = 1$ for any $i_0, \ldots, i_{n-1} < m$. Let $x \in \mathfrak{C} \setminus \{0\}$. Assume for contradiction that $s_{i_0, \ldots, i_{n-1}} a \cdot x = 0$ for all $a \in \mathbb{At} \mathfrak{A}$. Then $1 - x$ will be an upper bound for $\{ s_{i_0, \ldots, i_{n-1}} a : a \in \mathbb{At} \mathfrak{A} \}$. But this is impossible because $\sum s_{i_0, \ldots, i_{n-1}} a = 1$.

To prove the first part of (2), we repeatedly use (1). We define the edge labelling of $N$ one edge at a time. Initially, no hyperedges are labelled. Suppose $E \subseteq \text{nodes}(N) \times \text{nodes}(N)$ is the set of labelled hyperedges of $N$ (initially $E = \emptyset$) and $x \cdot \prod_{c \in E} s_c N(c) \neq 0$. Pick $d$ such that $d \not\in E$. Then by (1) there is $a \in \mathbb{At} \mathfrak{A}$ such that $x \cdot \prod_{c \in E} s_c N(c) \cdot s_d a = 0$. Include the hyperedge $d$ in $E$. We keep on doing this until eventually all hyperedges will be labelled, so we obtain a completely labelled graph $N$ with $N^+ \neq 0$. It is easily checked that $N$ is a network. For the second part of (2), we proceed contrapositively. Assume that there is $\bar{c} \in \text{nodes}(M) \cap \text{nodes}(N)$ such...
that \( M(\bar{c}) \neq N(\bar{c}) \). Since edges are labelled by atoms, we have \( M(\bar{c}) \cdot N(\bar{c}) = 0 \), so
\[
0 = s_{\bar{c}}0 = s_{\bar{c}}M(\bar{c}) \cdot s_{\bar{c}}N(\bar{c}) \geq M^+ \cdot N^+.
\]
A piece of notation. For \( i < m \), let \( Id_{-i} \) be the partial map \( \{(k, k) : k \in m \setminus \{i\}\} \). For the first part of (3) (cf. \[9\] Lemma 13.29) using the notation in op.cit, since there is \( k \in m \setminus \text{nodes}(N) \), \( \theta \) can be expressed as a product \( \sigma_0 \sigma_1 \cdots \sigma_t \) of maps such that, for \( s \leq t \), we have either \( \sigma_s = Id_{-i} \) for some \( i < m \) or \( \sigma_s = [i/j] \) for some \( i, j < m \) and where \( i \notin \text{nodes}(N \sigma_0 \cdots \sigma_{s-1}) \). But clearly \((NId_{-j})^+ \geq N^+ \) and if \( i \notin \text{nodes}(N) \) and \( j \in \text{nodes}(N) \), then \( N^+ \neq 0 \rightarrow (N[i/j])^+ \neq 0 \). The required now follows. The last part is straightforward.

Using the above proven facts, we are now ready to show that \( \exists \) has a winning strategy in \( G^m \). She can always play a network \( N \) with \( \text{nodes}(N) \subseteq m \), such that \( N^+ \neq 0 \).

In the initial round, let \( \forall \) play \( a \in \text{At}\mathfrak{A} \). \( \exists \) plays a network \( N \) with \( N(0, \ldots, n - 1) = a \).

Then \( N^+ = a \neq 0 \). Recall that here \( \forall \) is offered only one (cylindrifier) move. At a later stage, suppose \( \forall \) plays the cylindrifier move, which we denote by \((N, \langle f_0, \ldots, f_{n-2}, k, b, l \rangle)\). He picks a previously played network \( N_1 \), \( f_i \in \text{nodes}(N), l < n, k \notin \{f_i : i < n - 2\} \), such that \( b \leq c_1 N(f_0, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_{n-2}) \) and \( N^+ \neq 0 \). Let \( a = (f_0, \ldots, f_{i-1}, k, f_{i+1}, \ldots, f_{n-2}) \).

Then by second part of (3) we have that \( c_1 N^+ \cdot s_b b \neq 0 \) and so by first part of (2), there is a network \( M \) such that \( M^+ \cdot c_1 N^+ \cdot s_b b \neq 0 \). Hence \( M(a, f_0, \ldots, f_{i-1}, k, f_{i+1}, \ldots, f_{n-2}, b) = b, \text{nodes}(M) = \text{nodes}(N) \cup \{k\} \), and \( M^+ \neq 0 \), so this property is maintained.

Assume that \( \mathfrak{A} \) is an atomic \( \text{CA}_n \), having a complete \( m \)-square representation. We will show that \( \exists \) has a winning strategy in \( G^m_{\omega}(\text{At}\mathfrak{A}) \). Let \( \text{Mo} \) be a complete \( m \)-square representation of \( \mathfrak{A} \). One constructs the \( m \)-dimensional atomic dilation \( \mathfrak{D} \) using \( L^{\omega}_{\text{At}, \omega} \) formulas from the complete \( m \)-square representation as the algebra with universe \( C^m(M) \) and operations induced by clique guarded semantics. For each \( \bar{a} \in \mathcal{D}^2 \), define \[9\] Definition 13.22 a labelled hypergraph \( N_0 \) with nodes \( m \), and \( N_0(\bar{x}) \) when \( |\bar{x}| = n \), is the unique atom of \( \mathfrak{A} \) containing the tuple of length \( m > n \), \((a_{x_0}, \ldots, a_{x_1}, \ldots, x_{a_{n-1}}, a_{x_0}, \ldots, a_{x_n})\). It is clear that if \( s \in \mathcal{D}^2 \) and \( i, j < n \), then \( s \cdot [i, j] \in \mathcal{D}^2 \). By \[9\] Lemma 13.24 \( N_0 \) is a network. Let \( H \) be the symmetric closure of \( \{N_0 : \bar{a} \in \mathcal{D}^2\} \), that is \( \{N\theta : \theta : m \rightarrow m, N \in H\} \). Then \( H \) is an \( m \)-dimensional basis. Now \( \exists \) can win \( G^m_{\omega} \) by always playing a subnetwork of a network in the constructed \( H \). In round 0, when \( \forall \) plays the atom \( a \in \mathfrak{A} \), \( \exists \) chooses \( N \in H \) with \( N(0, 1, \ldots, n - 1) = a \) and plays \( N \uparrow n \). In round \( t > 0 \), inductively if the current network is \( N_{t-1} \subseteq M \in H \), then no matter how \( \forall \) defines \( N \), we have \( N \subseteq M \) and \( |N| < m \), so there is \( z < m \), with \( z \notin \text{nodes}(N) \). Assume that \( \forall \) picks \( x_0, \ldots, x_{n-1} \in \text{nodes}(N) \), \( a \in \text{At}\mathfrak{A} \) and \( i < n \) such that \( N(x_0, \ldots, x_{n-1}) \leq c_i a \), so \( M(x_0, \ldots, x_{n-1}) \leq c_i a \), and hence (by the properties of \( H \)), there is \( M' \equiv M \) and \( M'(x_0, \ldots, z, \ldots, x_{n-1}) = a \), with \( z \) in the \( i \)th place. Now \( \exists \) responds with the restriction of \( M' \) to \( \text{nodes}(N) \cup \{z\} \).

\( \Box \)

In the next Theorem \( \text{LCA}_n \) denotes the class of atomic \( \text{CA}_n \)'s whose atom structures satisfy the Lyndon condition as defined in \[10\]. It is known that \( \text{LCA}_n \) is \( n \) elementary class admitting no finite first order axiomatization; furthermore \( \text{LCA}_n = \text{EICRCA}_n \).

**Theorem 4.5.** Let \( \kappa \) be an infinite cardinal. Then there exists a \( \mathfrak{C} \in \text{QEA}_\omega \) such that for all \( 2 < n < \omega \), \( |N_{\mathfrak{C}}\mathfrak{C}| = 2^n \), \( N_{\mathfrak{C}}\mathfrak{C} \in \text{LQEA}_n \), but \( \mathcal{M}_{\kappa} N_{\mathfrak{C}}\mathfrak{C} \) is not completely representable. cannot be omitted.

**Proof.** One uses the ideas in \[24\] replacing \( \omega \) and \( \omega_1 \) by \( \kappa \) and \( 2^\kappa \), respectively, constructing \( \mathfrak{C} \) from a relation algebra. The resulting (new) relation algebra \( \mathfrak{A} \) has an \( \omega \).
dimensional amalgamation class $S$, cf. [24] Lemma 3]. Using the notation in [24] Lemma 6], let $\mathfrak{C}$ be the subalgebra of $\mathfrak{C}a(S)$ generated by $X'$; the latter is defined just before the lemma. Then $\mathfrak{A} = \mathfrak{A} \mathfrak{a}(\mathfrak{C})$, cf. [24], Lemmata 6, 7], but $\mathfrak{A}$ has no complete representation [24] Lemma 2]. Then $\mathfrak{A}r_n \mathfrak{C} \ (2 < n < \omega)$ is atomic, but has no complete representation. By Lemma 4.4, $\mathfrak{A}$ has a winning strategy in $G_\omega(\mathfrak{A}r_n \mathfrak{C})$, hence she has a winning strategy in $G_\omega(\mathfrak{A}r_n \mathfrak{C})$, a fortiori in $G_k(\mathfrak{A}r_n \mathfrak{C})$ for all $k \in \omega$, hence by coding the winning strategy’s of the $G_k$’s in first order sentences, we get that $\mathfrak{A}r_n \mathfrak{C}$ satisfies these first order sentences which are precisely (by definition) the Lyndon conditions. We use the following uncountable version of Ramsey’s theorem due to Erdos and Rado: If $r \geq 2$ is finite, $k$ an infinite cardinal, then $\exp_r(k) + (k^+)^{r+1}$ where $\exp_0(k) = k$ and inductively $\exp_{r+1}(k) = 2^{\exp_r(k)}$. The above partition symbol describes the following statement. If $f$ is a coloring of the $r + 1$ element subsets of a set of cardinality $\exp_r(k)$ in many colors, then there is a homogeneous set of cardinality $k^+$ (a set, all whose $r + 1$ element subsets get the same $f$-value). Let $\kappa$ be the given cardinal. We use a variation a simplified more basic version of a rainbow construction where only the two predominant colours, namely, the reds and blues are available. The algebra $\mathfrak{C}$ will be constructed from a relation algebra possessing an $\omega$-dimensional cylindric basis. To define the relation algebra we specify its atoms and the forbidden triples of atoms. The atoms are $\text{id}$, $g_0^i : i < 2^\kappa$ and $r_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of $(\text{id}, x, y)$ for $x \neq y$, $(r_j, r_j, r_j)$ for $1 \leq j < \kappa$ and $(g_0^i, g_0^{i'}, g_0^{i''})$ for $i, i', i'' < 2^\kappa$. Write $g_0$ for $(g_0^i : i < 2^\kappa)$ and $r_+$ for $(r_j : 1 \leq j < \kappa)$. Call this atom structure $\alpha$. Consider the term algebra $\mathfrak{A}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{A}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are atomic relation algebras sharing the same atom structure, so that $\mathfrak{At} \mathfrak{A} = \mathfrak{At} \mathfrak{B}$, then $\mathfrak{A}$ is completely representable $\iff$ $\mathfrak{B}$ is completely representable.

Assume for contradiction that $\mathfrak{A}$ has a complete representation $\text{Mo} = r_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in \text{Mo}$ such that $\text{Mo} \models g_0^i(x, z_i) \land r_1(z_i, y)$. Let $Z = \{ z_i : i < 2^\kappa \}$. Within $Z$, each edge is labelled by one of the $\kappa$ atoms in $r_+$. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $\text{Mo} \models r_j(z^1, z^2) \land r_j(z^2, z^3) \land r_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles). Let $S$ be the set of all atomic $\mathfrak{A}$-networks $N$ with nodes $\omega$ such that $\{ r_i : 1 \leq i < \kappa : r_i$ is the label of an edge in $N \}$ is finite. Then it is straightforward to show $S$ is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv_i L \equiv_j N$, witness [24] Definition 12.8 for notation. Now let $X$ be the set of finite $\mathfrak{A}$-networks $N$ with nodes $\leq \kappa$ such that:

1. each edge of $N$ is either (a) an atom of $\mathfrak{A}$ or (b) a cofinite subset of $r_+ = \{ r_j : 1 \leq j < \kappa \}$ or (c) a cofinite subset of $g_0 = \{ g_0^i : i < 2^\kappa \}$ and

2. $N$ is ‘triangle-closed’, i.e. for all $l, m, n \in \text{nodes}(N)$ we have $N(l, n) \leq N(l, m) \wedge N(m, n)$. That means if an edge $(l, m)$ is labelled by $\text{id}$ then $N(l, n) = N(m, n)$ and if $N(l, m), N(m, n) \leq g_0$ then $N(l, n) \cdot g_0 = 0$ and if $N(l, m), N(m, n) = r_j$ (some $1 \leq j < \omega$) then $N(l, n) \cdot r_j = 0$.

For $N \in X$ let $\hat{N} \in \mathfrak{C}a(S)$ be defined by

$$\{ L \in S : L(m, n) \leq N(m, n) \text{ for } m, n \in \text{nodes}(N) \}.$$
For \( i \in \omega \), let \( N \downharpoonright_i \) be the subgraph of \( N \) obtained by deleting the node \( i \). Then if \( N \in X \), \( i < \omega \) then \( c_i N = N \downharpoonright_i \). The inclusion \( c_i N \subseteq (N \downharpoonright_i) \) is clear. Conversely, let \( L \in (N \downharpoonright_i) \). We seek \( M \equiv_i L \) with \( M \in \tilde{N} \). This will prove that \( L \in c_i \tilde{N} \), as required. Since \( L \in S \) the set \( T = \{ t_i \notin L \} \) is infinite. Let \( T \) be the disjoint union of two infinite sets \( Y \cup Y' \), say. To define the \( \omega \)-network \( M \) we must define the labels of all edges involving the node \( i \) (other labels are given by \( M \equiv_i L \)). We define these labels by enumerating the edges and labeling them one at a time. So let \( j < i < \kappa \). Suppose \( j \in \text{nodes}(N) \). We must choose \( M(i, j) \leq N(i, j) \). If \( N(i, j) \) is an atom then of course \( M(i, j) = N(i, j) \). Since \( N \) is finite, this defines only finitely many labels of \( M \). If \( N(i, j) \) is a cofinite subset of \( g_0 \) then we let \( M(i, j) \) be an arbitrary atom in \( N(i, j) \). And if \( N(i, j) \) is a cofinite subset of \( r_+ \) then let \( M(i, j) \) be an element of \( N(i, j) \cap Y \) which has not been used as the label of any edge of \( M \) which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If \( j \notin \text{nodes}(N) \) then we can let \( M(i, j) = r_k \) for some \( 1 \leq k < \kappa \) such that no edge of \( M \) has already been labelled by \( r_k \). It is not hard to check that each triangle of \( M \) is consistent (we have avoided all monochromatic triangles) and clearly \( M \in \tilde{N} \) and \( M \equiv_i L \). The labeling avoided all but finitely many elements of \( Y' \), so \( M \in S \). So \( (N \downharpoonright_i) \subseteq c_i \tilde{N} \).

Now let \( \tilde{X} = \{ \tilde{N} : N \in X \} \subseteq \mathcal{C}a(S) \). Then we claim that the subalgebra of \( \mathcal{C}a(S) \) generated by \( \tilde{X} \) is simply obtained from \( \tilde{X} \) by closing under finite unions. Clearly all these finite unions are generated by \( \tilde{X} \). We must show that the set of finite unions of \( \tilde{X} \) is closed under all cylindric operations. Closure under unions is given. For \( \tilde{N} \in X \) we have \( \tilde{N} = \bigcup_{m,n \in \text{nodes}(N)} \tilde{N}_{mn} \), where \( N_{mn} \) is a network with nodes \( \{ m, n \} \) and labeling \( N_{mn}(m, n) = N(m, n) \). \( N_{mn} \) may not belong to \( X \) but it is equivalent to a union of at most finitely many members of \( \tilde{X} \). The diagonal \( d_{ij} \in \mathcal{C}a(S) \) is equal to \( \tilde{N} \) where \( N \) is a network with nodes \( \{ i, j \} \) and labeling \( N(i, j) = \text{id} \). Closure under cylindrification is given. Let \( C \) be the subalgebra of \( \mathcal{C}a(S) \) generated by \( \tilde{X} \). Then \( A = \mathcal{R}a(C) \). To see why, each element of \( A \) is a union of a finite number of atoms, possibly a co–finite subset of \( g_0 \) and possibly a co–finite subset of \( r_+ \). Clearly \( A \subseteq \mathcal{R}a(C) \). Conversely, each element \( z \in \mathcal{R}a(C) \) is a finite union \( \bigcup_{F \in F} \tilde{N} \), for some finite subset \( F \) of \( X \), satisfying \( c_i z = z \), for \( i > 1 \). Let \( i_0, \ldots, i_k \) be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in \( F \). Then \( c_{i_0, \ldots, i_k} z = \bigcup_{F' \in F} c_{i_0} \ldots c_{i_k} \tilde{N} = \bigcup_{F' \in F} (\tilde{N} \downharpoonright_{\{0, 1\}}) \in \mathcal{A} \). So \( \mathcal{R}a(C) \subseteq \mathcal{A} \). \( \mathcal{A} \) is relation algebra reduct of \( C \in \mathcal{C}A_\omega \) but has no complete representation. But in fact \( C \) is in \( \text{QEA}_\omega \). Let \( n > 2 \). Let \( B = \mathcal{R}r_n \mathcal{C} \). Then \( B \in N_r \text{QEA}_\omega \) is atomic, but even its \( \text{Df} \) reduct has no complete representation for plainly a complete representation of \( \mathcal{R}d_r B \) induces one of \( B \) hence one for \( \mathcal{A} \). In fact, because \( B \) is generated by its two dimensional elements, and its dimension is at least three, its \( \text{Df} \) reduct is not completely representable. [13] Proposition 4.10. It remains to show that the \( \omega \)–dilation \( C \) is atomless. For any \( N \in X \), we can add an extra node extending \( N \) to \( M \) such that \( \emptyset \subseteq M' \subseteq N' \), so that \( N' \) cannot be an atom in \( C \). □

**Lemma 4.6.** Let \( 2 < n < \omega \). If \( \mathcal{A} \) is atomic and \( \mathcal{A} \in N_r \text{QEA}_\omega \) then \( \mathcal{A} \in \text{LQEA}_n \). An entirely analogous result holds for relation algebras upon replacing \( N_r \text{CA}_\omega \) by \( \mathcal{R}a \text{CA}_\omega \).

**Proof.** Assume that \( \mathcal{A} \) is as in the hypothesis. Being in the class \( N_r \text{QEA}_\omega \subseteq (\text{S}, c \text{N}_r \text{QEA}_\omega) \). By Lemma 4.4 3 has a winning strategy in \( \text{G}^\omega \text{At} \mathcal{A} \). Since infinitely many nodes are used (and reuse), hence she has a winning strategy in the usual \( \omega \) rounded usual atomic \( G_\omega \text{At} \mathcal{A} \) without the need to reuse th nodes in play, a fortiori she has a winning strategy in the \( k \) rounded atomic game \( G_k(\text{At} \mathcal{A}) \) for all \( k \in \omega \). By definition, \( \mathcal{A} \in \text{LQEA}_n \). □
In the previous construction used in Proposition 4.5 and the previous Lemma \( \mathfrak{A} \in \mathbb{RA}_{\omega} \) and \( \mathfrak{B} \in \mathbb{NR}_{n,\omega} \) satisfy the Lyndon conditions, but are not completely representable. Thus:

**Corollary 4.7.** Let \( 2 < n < \omega \). Then the classes \( \mathbb{CRRA} \) and for any variety \( V \) between \( \mathbb{DF}_{n} \) and \( \mathbb{QEA}_{n} \), \( \mathbb{CRV} \) is not elementary.

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