Open Orbits and Augmentations of Dynkin Diagrams

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Abstract

Given any representation \( V \) of a complex linear reductive Lie group \( G_0 \), we show that a larger semi-simple Lie group \( G \) with

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus V \oplus V^* \oplus \cdots ,
\]

exists when \( V \) has a finite number of \( G_0 \)-orbits together with a few exceptions corresponding to a twisted version of it. In particular, \( V \) admits an open \( G_0 \)-orbit. Furthermore, this corresponds to an augmentation of the Dynkin diagram of \( \mathfrak{g}_0 \).

The representation theory of \( \mathfrak{g} \) should be useful in describing the geometry of manifolds with stable forms as studied by Hitchin.

1 Introduction

Each fundamental representation \( \Lambda^k \mathbb{R}^n \) of \( GL_n(\mathbb{R}) \) corresponds to the node labelled by \( k \) in the following Dynkin diagram \( \Gamma \) of \( GL_n(\mathbb{R}) \)

\[
\circ_1 \circ_2 \cdots \circ_{n-2} \circ_{n-1}.
\]

It is interesting to observe that \( \Lambda^k \mathbb{R}^n \) has an open orbit precisely when we can form a new Dynkin diagram by attaching a new node to \( \Gamma \) at the place labelled by \( k \). Furthermore, the simple Lie algebra \( \mathfrak{g} \) corresponding to this new Dynkin diagram can be built from \( \mathfrak{gl}_n \) and \( \Lambda^k \mathbb{R}^n \) and it is of the form

\[
\mathfrak{g} = \mathfrak{gl}_n \oplus \Lambda^k \mathbb{R}^n \oplus (\Lambda^k \mathbb{R}^n)^* \oplus \cdots .
\]

In this paper, we show that this phenomenon holds true in general. Given any complex linear reductive Lie group \( G_0 \) and any irreducible representation \( V \) of it. One could try to form a larger semi-simple Lie group \( G \), or equivalently a Lie algebra \( \mathfrak{g} \), of the form

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus V \oplus V^* \oplus \cdots ,
\]

as a Lie algebra with a \( \mathbb{Z} \)-gradation.
The main result of this paper shows that such a Lie algebra \( \mathfrak{g} \) exists precisely when the number of \( \mathcal{G}_0 \)-orbits in \( V \) is finite. Moreover, the Dynkin diagram of \( \mathfrak{g} \) is an augmentation of the Dynkin diagram of \( \mathfrak{g}_0 \) in the same way as in the \( GL_n \) case, or a twisted version of it. Furthermore, the length of the \( \mathbb{Z} \)-gradation can be easily read off from the Kac diagram \([17]\) and it is at most six. In particular, \( V \) has an open orbit. Irreducible representations which admit open orbits are completely classified (\([14]\), also see Tables 3 and 4 for those which admits a finite number of orbits). We see that all cases except one have a finite number of orbits. More precisely, there are one series of such representations, namely \((GL_2 \times SL_{2m+1}, \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m+1}), m \geq 4\), which have infinite number of orbits. In fact the failure of having a finite number of orbits in these representations is related to the fact that \( PGL_1 \) does not act \( m \)-transitively on \( \mathbb{P}^1 \) for \( m \geq 4 \). In Section 9 we will discuss their orbit structures in detail.

**Remark 1:** After the completion of the preliminary version of this article, we are informed by Landsberg and later by Rubenthaler that most of our results are already scattered around in Vinberg \([23]\), Rubenthaler \([20]\), \([21]\) and Kac \([12]\), and are closely related to Landsberg recent joint works with Manivel and Robles respectively. More specifically, Theorem 4.1 is similar to Lemma 1.3 in \([12]\) which has been established in Vinberg \([23]\), and Theorem 3.2 was first proved in \([20]\). On the other hand, Landsberg and Manivel \([16]\) provide a geometric description via projective geometry for the minuscule representations, which are subclasses of representations possessing open orbits. Landsberg also informed us that his recent joint work with Robles extends the geometric description in \([16]\) to the general case.

Let us consider the \( GL_n \)-case in greater details. There is a classical result about the irreducible representations of \( GL_n(\mathbb{R}) \): The fundamental representations \( \Lambda^k \mathbb{R}^n (k \leq \frac{n}{2}) \) of \( GL_n(\mathbb{R}) \) has an open orbit if and only if \( (n, k) \) lies in one of the following classes:

(i) \( n \geq 2, k = 1 \);

(ii) \( n \geq 4, k = 2 \);

(iii) \( n = 6, 7, 8, k = 3 \).

Due to the isomorphism \( \Lambda^k \mathbb{R}^n \cong \Lambda^{n-k}(\mathbb{R}^n)^* \) as \( GL_n(\mathbb{R}) \) representations, we can confine to the cases where \( k \leq \frac{n}{2} \). One observation is that such configurations can be reinterpreted as follows: Starting from a Dynkin diagram of type \( A_{n-1} \)

\[
\circ_1 \quad \circ_2 \quad \cdots \quad \circ_{n-2} \quad \circ_{n-1}
\]

one tries to add an extra node to obtain another simply-laced Dynkin diagram of one higher rank. According to the classification of reduced root systems \([9]\), we have a full list of possibilities:
(i) $A_{n-1} \rightarrow A_n$

(ii) $A_{n-1} \rightarrow D_n$

(iii) $A_{n-1} \rightarrow E_n$ for $n = 6, 7, 8$

In other words, they are obtained from attaching the extra node to the "$k^{th}$ node" of the original diagram. Then the possible pairs of $(n, k)$ coincide with the list given in \(1.1\).

The complete dictionary between the existence of open orbits in $\Lambda^k \mathbb{R}^n$ and the simply-laced extensions of Dynkin diagram of type $A_{n-1}$ at the "$k^{th}$ node" suggests a representation-theoretic explanation of this phenomenon. This will constitute the main content of this paper.

In the case of the fundamental representation $\Lambda^k (\mathbb{R}^n)^*$ of $GL_n (\mathbb{R})$, an element $\omega \in \Lambda^k (\mathbb{R}^n)^*$ which lies in an open orbit is called a stable form. Clearly this notion is independent of the choice of coordinates and hence it can be defined on any smooth manifolds. Hitchin \[6\] studied closed differential forms with such properties. These stable forms have the advantage that they are stable under deformations and are the critical points of the associated volume functionals in their respective cohomology classes, which can be treated as a nonlinear version of the Hodge theory. The symmetry group $\text{Aut}(\mathbb{R}^n, \omega)$ of a stable form $\omega$ is just the isotropy subgroup of $GL_n (\mathbb{R})$ at $\omega$. For example, when $n$ is even the geometry of stable two forms is the symplectic geometry and $\text{Aut}(\mathbb{R}^n, \omega) = \text{Sp}(n, \mathbb{R})$. Moreover, the $E_n$ cases are related to the exceptional geometries, these geometries are essential to mathematical physics, especially in developing mathematical models for string theory, these are studied by Witten \[18\] and his collaborators. For instance, the geometry of stable 3-forms on 7-manifolds are known as the $G_2$-geometry which is an essential ingredient in the $M$-theory. In general, given a representation $V$ of a linear reductive group $G_0$, we try to construct a new semisimple Lie algebra

$$\mathfrak{g} = \mathfrak{g}_0 \oplus V \oplus V^* \oplus \cdots.$$
But this is possible only when there is an open orbit, or infinitesimally, we have \( g_0 = \text{Aut}(V, \omega) \oplus V \) for some \( \omega \in V \). It suggests that the representation theory of \( g \) should be a useful tool to study the geometry of manifolds with stable forms.

Roughly speaking, we have established the following one-to-one correspondence:

\[
\begin{align*}
\{ & \text{Irreducible prehomogeneous} \quad & \text{Augmentations of} \\
& \quad \text{vector spaces of parabolic type} & \text{Dynkin diagrams} \}
\end{align*}
\]

The term ”prehomogeneous vector spaces (PVS)” was first introduced by M. Sato in 1961, since then a lot of results concerning these objects have been established, in particular the classification of irreducible prehomogeneous vector spaces was completed in [14], we will discuss them in Section 2. According to a result of Richardson [19], one source of irreducible prehomogeneous vector spaces is obtained by considering a parabolic subgroup of a complex semisimple Lie group, from which we obtain a representation of its Levi factor on the vector space \( u/[u,u] \), where \( u \) is the nilpotent radical of the corresponding parabolic subalgebra. Those irreducible prehomogeneous vector spaces from this origin are said to be of parabolic type. Indeed, they lies in a subclass of irreducible prehomogeneous vector spaces which consists of a finite number of orbits. However, they fail to occupy the whole subclass with a few exceptions which fall into the class of prehomogeneous vector spaces of twisted affine type. We will justify our terminology in Section 8.

Now let's sketch our approach and state our main results. Let \( G \) be a connected complex semisimple Lie group with Lie algebra \( g \). Upon choosing a Cartan subalgebra \( h \), there associates a root system \( \Delta \) of \( g \). Then we arbitrarily pick up a system of simple roots \( \Pi = \{ \alpha_0, \alpha_1, \ldots, \alpha_\ell \} \) and define \( c \in h \) to be the unique element such that \( \alpha_0(c) = 1 \) and \( \alpha_i(c) = 0 \) for all \( 1 \leq i \leq \ell \). Set \( g_i \) to be the eigenspace of \( \text{ad} \ c \) in \( g \) with eigenvalue \( i \), so that we obtain a \( \mathbb{Z} \)-gradation

\[
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i.
\]

It follows immediately that \( g_0 \) is a regular reductive subalgebra of \( g \) as the centralizer of \( c \) and \( g \) is called an ambient Lie algebra containing \( g_0 \). Let \( G_0 = Z_G(c)^0 \) be the closed connected subgroup of \( G \) with Lie algebra \( g_0 \). It is then clear that all \( g_i \) are invariant under \( G_0 \) and thus are its representations. Note that under the Killing form of \( g \), we can identify \( g_i \) as the dual space of \( \mathfrak{g}_{-i} \) for all \( i \neq 0 \) and that such identification is \( G_0 \)-equivariant, i.e. they are dual as \( G_0 \)-representations. Let \( \mathfrak{g}_0^{ss} := [g_0, g_0] \) denote the semisimple part of \( g_0 \). Then the inclusion \( \mathfrak{g}_0^{ss} \subset g \) induces a corresponding inclusion of their Dynkin diagrams \( \Gamma(g_0^{ss}) \subset \Gamma(g) \). Our main result is that to every connected augmentation of Dynkin diagrams, there exists a unique irreducible reduced PVS with an extra data called connecting multiplicities to be defined in Section 3. More precisely, we have the following theorem.

\[\text{(See Appendix A for its definition.)}\]
Theorem 1.1. Let $G, G_0, \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be defined as in the above paragraph, we have:

(i) For $i \neq 0$, $\mathfrak{g}_i$ are weight multiplicity free irreducible representations of $G_0$ with a finite number of orbits; in particular, it implies that $(G_0, \mathfrak{g}_i)$ are prehomogeneous vector spaces.\footnote{See Definition \ref{def:homogeneous}}

(ii) Every augmentation of Dynkin diagrams can be realized by a connected complex semisimple Lie group $G$ with a suitable choice of simple root $\alpha_0$, i.e. it can be expressed in the form $(\Gamma(g), \Gamma(g_{ss}^0))$.\footnote{See Definition \ref{def:prehomogeneous}}

(iii) There is an one-to-one correspondence

\[
\begin{align*}
\text{Isogeny classes of} & \quad \text{Simply-laced augmentations of} \\
\text{simply-laced irreducible} & \quad \text{Dynkin diagrams} \\
\text{prehomogeneous vector spaces} & \quad \text{with finite number of orbits}
\end{align*}
\]

Namely, we associate to a simply-laced augmentation of Dynkin diagrams $(\Gamma(g), \Gamma(g_{ss}^0))$ the simply-laced prehomogeneous vector space $(G_0, \mathfrak{g}_{-1})$.

(iv) There is an one-to-one correspondence

\[
\begin{align*}
\text{Irreducible prehomogeneous} & \quad \text{Connected augmentations of} \\
\text{vector spaces of} & \quad \text{Dynkin diagrams} \\
\text{parabolic type together} & \quad \text{with their connecting multiplicities}
\end{align*}
\]

Explicitly, we assign to each connected augmentation of Dynkin diagrams $(\Gamma(g), \Gamma(g_{ss}^0))$ the irreducible reduced prehomogeneous vector space $(G_0, \mathfrak{g}_{-1})$

with the corresponding connecting multiplicities $\nu(\mathfrak{g}, \mathfrak{g}_{-1})$.

Now we illustrate how to apply Theorem \ref{thm:main} to our motivating question at the beginning. The first observation is that given the representation $\Lambda^k \mathbb{R}^n$ of $GL_n(\mathbb{R})$, we can complexify it to a representation of $GL_n(\mathbb{C})$ and then restricted to $SL_n(\mathbb{C})$. By taking differentials, we obtain the representation $\Lambda^k \mathbb{C}^n$ of $sl_n(\mathbb{C})$. Except the trivial cases where $k = 0$ or $n$, for all other cases included in \ref{table:1}, $\Lambda^k \mathbb{C}^n$ are corresponding to the $k^{th}$ or $(n-k)^{th}$ fundamental weight of $sl_n(\mathbb{C})$ which is exactly corresponding to the $k^{th}$ or $(n-k)^{th}$ node of its Dynkin diagram. It provides one possible linkage between the two sets of objects.

From Table \ref{table:1} we see that the representation $\Lambda^k \mathbb{C}^n$ always exists as the $(-1)$-graded component and it turns out to be the case in general. Then by Theorem \ref{thm:main} $(GL_n(\mathbb{C}), \Lambda^k \mathbb{C}^n)$ have open orbits exactly when $(n, k)$ are as listed in \ref{table:1}. Finally by a theorem of Whitney, we successfully translate the result
Table 1: Graded pieces of semisimple Lie algebras associated to \((GL_n, \Lambda^k C^n)\)

\[
\mathfrak{g} = \bigoplus_{i=-3}^{3} \mathfrak{g}_i, \quad \mathfrak{g}_{-1} \cong \mathfrak{g}_1
\]

| \mathfrak{g}         | \mathfrak{g}_0 | \mathfrak{g}_{-1} | \mathfrak{g}_{-2} | \mathfrak{g}_{-3} |
|----------------------|---------------|-------------------|-------------------|-------------------|
| \mathfrak{sl}_2 \times \mathfrak{sl}_n | \mathfrak{sl}_n | \mathbb{C}        | 0                 | 0                 |
| \mathfrak{sl}_{n+1} | \mathfrak{sl}_n | \mathbb{C}^n      | 0                 | 0                 |
| \mathfrak{so}_{2n} | \mathfrak{sl}_n | \Lambda^2 \mathbb{C}^n | 0          | 0                 |
| \mathfrak{e}_6      | \mathfrak{sl}_6 | \Lambda^3 \mathbb{C}^6 | \Lambda^6 \mathbb{C}^6 | 0                 |
| \mathfrak{e}_7      | \mathfrak{sl}_7 | \Lambda^3 \mathbb{C}^7 | \Lambda^6 \mathbb{C}^7 | 0                 |
| \mathfrak{e}_8      | \mathfrak{sl}_8 | \Lambda^3 \mathbb{C}^8 | \Lambda^6 \mathbb{C}^8 | \mathbb{C}^8 \otimes \Lambda^8 \mathbb{C}^8 |

back to the real cases when the corresponding complex representation of its real form is of real type; in particular it is always the cases for split real forms.

Let us briefly describe the content of the paper. In Sections 2 and 3 we will set up the general framework and the terminology used throughout this paper. The main result on the finiteness of orbits will be established in Section 4. Then the termination of \(\mathbb{Z}\)-gradations will be discussed in Section 5. After that, we will give an explicit construction of generic elements in the simply-laced cases in Section 6. The proof of Theorem 1.1 will be completed in Section 7. Sections 8 and 9 are devoted to the discussion of the two exceptional cases in our construction. Finally the first two appendices present the basics of \(\mathbb{Z}\)-gradations and algebraic groups, and the tables are contained in the last appendix.

## 2 Weight Multiplicity Free Representations and Prehomogeneous Vector Spaces

In this section, we will introduce two notions in representation theory which have been well understood for a long time, namely that of weight multiplicity free representations and prehomogeneous vector spaces. Both objects have been completely classified and proved to be useful in many branches of mathematics. Here we will use them to give a necessary condition for the existence of augmentation of Dynkin diagrams.

**Definition 2.1.** Let \(\mathfrak{g}\) be a complex semisimple Lie algebra. A representation \(V\) of \(\mathfrak{g}\) is said to be weight multiplicity free if every weight space is one dimensional.

The classification of irreducible weight multiplicity free representations of complex simple Lie algebras can be found in [7]. The complete list is as follows:

\(^{4}\)Complete classification of real forms of irreducible prehomogeneous vector spaces of parabolic type is obtained in [21].
(i) $A_\ell (\mathfrak{sl}_{\ell+1}\mathbb{C})$:
   (a) The fundamental representations $\Lambda^m \mathbb{C}^{\ell+1}$ with highest weight $\omega_m$, for $m = 1, \ldots, \ell$.
   (b) The symmetric tensor powers $S^m \mathbb{C}^{\ell+1}$ and $S^m (\mathbb{C}^{\ell+1})^*$ with highest weights $m\omega_1$ and $m\omega_\ell$, for $m \in \mathbb{Z}_{\geq 0}$.

(ii) $B_\ell (\mathfrak{so}_{2\ell+1}\mathbb{C})$:
   (a) The standard representation $\mathbb{C}^{2\ell+1}$ with highest weight $\omega_1$.
   (b) The spin representation $S$ with highest weight $\omega_\ell$.

(iii) $C_\ell (\mathfrak{sp}_{2\ell}\mathbb{C})$:
   (a) The standard representation $\mathbb{C}^{2\ell}$ with highest weight $\omega_1$.
   (b) When $\ell = 2$ or 3, the last fundamental representation, $\Lambda^2_{\text{prim}} \mathbb{C}^4$ and $\Lambda^3_{\text{prim}} \mathbb{C}^6$ respectively, with highest weight $\omega_\ell$.

(iv) $D_\ell (\mathfrak{so}_{2\ell}\mathbb{C})$:
   (a) The standard representation $\mathbb{C}^{2\ell}$ with highest weight $\omega_1$.
   (b) The two half-spin representations $S^+$ and $S^-$ with highest weights $\omega_{\ell-1}$ and $\omega_\ell$ respectively.

(v) $E_\ell (\ell = 6, 7, 8)$:
   (a) The two 27-dimensional representations of $E_6$ with highest weights $\omega_1$ and $\omega_6$.
   (b) The 56-dimensional representations of $E_7$ with highest weight $\omega_7$.
   (c) There are no weight multiplicity free representations for $E_8$.

(vi) $F_4$: There are no weight multiplicity free representations for $F_4$.

(vii) $G_2$: The 7-dimensional representation of $G_2$ with highest weight $\omega_2$.

Here the numbering of the fundamental weights $\omega_i$ are adopted to that of Bourbaki [2] (also see Table 3).

**Proposition 2.1.** Let $\mathfrak{g} = a_1 \times \cdots \times a_k$ be a decomposition of a semisimple Lie algebra $\mathfrak{g}$ into simple ideals $a_i, i = 1, \ldots, k$, and suppose that $V_i$ is a finite dimensional representation of $a_i$ for each $i$. Then $V_1 \otimes \cdots \otimes V_k$ is a weight multiplicity free representation of $\mathfrak{g}$ if and only if all the $V_i$’s are weight multiplicity free.

**Proposition 2.2.** Let $V$ be a weight multiplicity free representation of a complex semisimple Lie algebra $\mathfrak{g}$. If all weights of $V$ are congruent modulo the root lattice $\Lambda$ of $\mathfrak{g}$, then $V$ is irreducible.
Proof. Fix a Cartan subalgebra $\mathfrak{h}$ and a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ so that $\Lambda = \mathbb{Z}\langle \alpha_1, \ldots, \alpha_\ell \rangle$, and let $H_i$ be the unique element in $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}]$ with $\alpha_i(H_i) = 2$. It suffices to show that the highest weight of $V$ is unique. Suppose on the contrary that $\lambda$ and $\mu$ are two distinct highest weights of $V$. Since $\lambda$ and $\mu$ are congruent modulo $\Lambda$, $\lambda - \mu = \alpha$ for some $\alpha \in \Lambda$. Now by separating the positive and negative parts of $\alpha$ as a linear combination of $\alpha_1, \ldots, \alpha_\ell$, we can find two disjoint subset $I, J$ of $\{\alpha_1, \ldots, \alpha_\ell\}$ and positive integers $n_i, m_j$ for every $i \in I, j \in J$, such that

$$
\eta := \lambda - \sum_{i \in I} n_i \alpha_i = \mu - \sum_{j \in J} m_j \alpha_j.
$$

Note that for each $i \notin J$,

$$
\eta(H_i) = \mu(H_i) - \sum_{j \in J} m_j \alpha_j(H_i) \geq 0
$$

since $\alpha_j(H_i) \leq 0$ for all $j \neq i$. Similarly, we have for each $j \in I$,

$$
\eta(H_j) = \lambda(H_j) - \sum_{i \in I} n_i \alpha_i(H_j) \geq 0.
$$

As $I$ and $J$ are disjoint, we conclude that $\eta(H_i) \geq 0$ for all $i = 1, \ldots, \ell$ which implies that $\eta$ is a dominant integral weight in $\mathfrak{h}^\ast$. It follows that $\eta$ must be a weight of $V$ with multiplicity at least two since each of the highest weight submodules of $V$ of weights $\lambda$ and $\mu$ contributes at least one dimension to the weight space $V_\eta$. But this contradicts that $V$ is weight multiplicity free. \qed

Definition 2.2. Let $G$ be a complex reductive algebraic group and $V$ a rational representation of $G$. $(G, V)$ is called a prehomogeneous vector space if there exists a dense $G$-orbit in $V$.

According to Proposition B.1 in Appendix B, the dense orbit must be open. In other words, prehomogeneous vector spaces are just representations with exactly one open orbit. All irreducible prehomogeneous vector spaces have been classified in [14].

Proposition 2.3. Given a representation $V$ of a linear connected algebraic group $G$. Then the following conditions are equivalent:

(i) $(G, V)$ is a prehomogeneous vector space.

(ii) There exists a vector $v \in V$ such that $\dim G_v = \dim G - \dim V$, where $G_v = \{g \in G | g \cdot v = v\}$.

(iii) There exists a vector $v \in V$ such that $\mathfrak{g} \cdot v = V$.

Theorem 2.4 (Richardson [19]). Let $G$ be a connected complex semisimple Lie group, and let $P$ be a parabolic subgroup of $G$ with Levi decomposition $P = LU$, where $L$ is its Levi factor and $U$ its unipotent radical. If $\mathfrak{u} = \text{Lie}(U)$, then $(L, \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}])$ is a prehomogeneous vector space.
Definition 2.3. Let \((G_0, V)\) be a prehomogeneous vector space, where \(G_0\) is a connected complex reductive Lie group of the form \(GL_1 \times G_{0}^{ss}\) for some complex semisimple Lie group \(G_{0}^{ss}\).

(i) \((G_0, V)\) is said to of parabolic type if it can be obtained from a connected complex semisimple Lie group \(G\) in the sense of Theorem 2.4.

(ii) \((G_0, V)\) is said to be reduced if its dimension is minimal over all prehomogeneous vector spaces which are castling equivalent\(^5\) to it.

(iii) \((G_0, V)\) is said to be irreducible if \(V\) is irreducible as a \(G_0\)-representation.

3 Augmentations of Dynkin Diagrams

It is well-known that every complex semisimple Lie algebra \(g\) admits a unique Dynkin diagram \(\Gamma(g)\) determined by the associated Cartan matrix, up to permutations of numbering of its entries. We also know that \(\Gamma(g)\) is connected if and only if \(g\) is simple. In this section, we would like to study when one can add an extra node to a given Dynkin diagram with corresponding relations of the edges attaching to the node so that it remains a Dynkin diagram of some semisimple Lie algebra of higher rank. In other words, we want to study all pairs of Dynkin diagrams \((\Gamma, \Gamma_0)\) consisting of a Dynkin diagram \(\Gamma\) and a subdiagram \(\Gamma_0\) obtained by removing a single node and all edges attached to it.

Definition 3.1. An augmentation of Dynkin diagrams is a pair \((\Gamma, \Gamma_0)\) of Dynkin diagrams such that \(\Gamma_0\) is a subdiagram of \(\Gamma\) obtained by removing exactly one node and all the edges connected to it.

Note that subdiagrams of a Dynkin diagram correspond exactly to the principal minors of the corresponding Cartan matrix. Therefore any subdiagram of a Dynkin diagram is also a Dynkin diagram and the above definition makes sense. To represent an augmentation of Dynkin diagrams \((\Gamma, \Gamma_0)\) diagrammatically, we will use the Dynkin diagram \(\Gamma\) with a painted node indicating the omitted node in \(\Gamma_0\).

Starting with a semisimple Lie algebra \(g\), our approach is to give a realization of \(\Gamma_0\) as a subsystem of simple roots of a semisimple subalgebra of \(g\) through a \(\mathbb{Z}\)-gradation of \(g\), and we will associate to it a collection of irreducible representations which detect the validity of such pair.

First of all, let’s set up some notations. Let \(G\) be a connected complex semisimple Lie group with Lie algebra \(g\), and \(\mathfrak{h}\) be a Cartan subalgebra of \(g\). Then we have a root space decomposition of \(g\) with respect to \(\mathfrak{h}\)

\[ g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} g_{\alpha} \]

\(^5\)(\(GL_n \times G, C^n \otimes V\)) \(\cong\) \((GL_{n-m} \times G, C^{n-m} \otimes V^*)\) where dim \(V = m < n\) induces an equivalence relation on the set of PVS, which is called the castling equivalence.
where $\Delta$ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Assume that $\text{rank}(\mathfrak{g}) = \ell + 1$, and let $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\} \subset \Delta$ be a system of simple roots. For each $i \in \mathbb{Z}$, set $\Delta_i = (\mathbb{Z}\langle \alpha_1, \ldots, \alpha_\ell \rangle + i\alpha_0) \cap \Delta$ so that

$$
\Delta = \bigsqcup_{i \in \mathbb{Z}} \Delta_i.
$$

Now for each $i \neq 0$, denote $\mathfrak{g}_i = \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha$, and define $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha$. Then it is easy to verify that $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$-gradation. From Proposition A.1 in Appendix A, $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$ and thus $\mathfrak{g}_0^{ss}$ is a semisimple subalgebra of rank $\ell$.

Let $c$ be the element in $\mathfrak{h}$ such that $\alpha_0(c) = 1$ and $\alpha_i(c) = 0$ for all $1 \leq i \leq \ell$. Then we can write $\mathfrak{h} = \mathbb{C}c \oplus \mathfrak{t}$ where $\mathfrak{t}$ is the orthogonal complement of $\mathbb{C}c$ in $\mathfrak{h}$ with respect to the Killing form of $\mathfrak{g}$. It is clear that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_0^{ss}$ and the corresponding root system $\Delta_0 = \{\alpha|\alpha \in \Delta_0\}$ has a subsystem of simple roots given by $\tilde{\Pi}_0 = \{\alpha_i|\alpha \in \Delta\}$.

If we identify $\tilde{\Pi}_0$ with $\Pi_0 = \{\alpha_1, \ldots, \alpha_\ell\}$, we see that $\langle (\Gamma(\mathfrak{g}), \Gamma(\mathfrak{g}_0^{ss})) \rangle$ is an augmentation of Dynkin diagrams. As long as only augmentations of Dynkin diagrams are concerned, the choices of the Cartan subalgebra $\mathfrak{h}$ and the system of simple roots $\Pi$ are inessential, and we will fix $\mathfrak{h}$ and $\Pi$ once and for all in the remaining part of this paper.

The element $c$ constructed above plays an important role in the structure of the $\mathbb{Z}$-gradation of $\mathfrak{g}$, for instance, we have $\mathfrak{g}_i = \{X \in \mathfrak{g}||c, X| = iX\}$ for all $i \in \mathbb{Z}$; in particular, $\mathfrak{g}_0$ is the centralizer of $c$ in $\mathfrak{g}$ with center $\mathfrak{z}_\mathfrak{g}_0 = \mathbb{C}c$.

**Lemma 3.1.** Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be the $\mathbb{Z}$-gradation constructed as in the above discussion, and let $\langle \cdot , \cdot \rangle_\mathfrak{h}$ and $\langle \cdot , \cdot \rangle_\mathfrak{t}$ be the Cartan products on $\mathfrak{h}^{*}$ and $\mathfrak{t}^{*}$ respectively. Then $\langle \alpha , \beta \rangle_\mathfrak{h} = \langle \alpha |_1 , \beta |_1 \rangle_\mathfrak{t}$ for all $\alpha , \beta \in \Delta$.

**Proof.** Let $H_\beta$ be the unique element in $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \subset \mathfrak{t}$ such that $\beta(H_\beta) = 2$. Since $\mathfrak{g}_\beta = (\mathfrak{g}_0^{ss})|_{\beta |_1}$, $H_\beta$ is also the unique element in $[(\mathfrak{g}_0^{ss})|_{\beta |_1}, (\mathfrak{g}_0^{ss})|_{-\beta |_1}]$ such that $\beta|_1(H_\beta) = 2$. It follows that

$$
\langle \alpha |_1 , \beta |_1 \rangle_\mathfrak{t} = \alpha|_1(H_\beta) = \alpha(H_\beta) = \langle \alpha , \beta \rangle_\mathfrak{h}.
$$

\[\square\]

**Theorem 3.2.** Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be defined as above.

(i) For each $k \neq 0$, $\mathfrak{g}_k$ is an irreducible weight multiplicity free representation of $\mathfrak{g}_0^{ss}$.

(ii) $-\alpha_0|_1$ is the highest weight of $\mathfrak{g}_{-1}$ as a $\mathfrak{g}_0^{ss}$-representation.

\[6\] The theorem was first proved in [20]. See also Remark 1.
Proof. (i) Note that from \( g_k = \bigoplus_{\alpha \in \Delta_k} g_{\alpha} \), we have each root space \( g_{\alpha} \) in \( g_k \) is contained in the weight space \( g_k \) as a \( g_{ss}^0 \)-module of weight \( \alpha |_t \). Now for any two distinct \( \alpha, \beta \in \Delta_k \), we have \( \alpha - \beta \in \mathbb{Z} \langle \alpha_1, \ldots, \alpha_\ell \rangle \) and thus \( \alpha |_t - \beta |_t \) is a nonzero element in the root lattice \( \Lambda_0 \) generated by \( \Delta_0 \). Hence distinct root spaces in \( g_k \) lie in different weight spaces of \([g_0, g_0]\); in other words, \( g_k = \bigoplus_{\alpha \in \Delta_k} g_{\alpha} \) is precisely the weight space decomposition as \( g_{ss}^0 \)-module. Now since each root space is one dimensional, we conclude that \( g_k \) is a weight multiplicity free representation of \( g_{ss}^0 \). Finally as all weights are congruent to each other modulo the root lattice \( \Lambda_0 \), \( g_k \) is irreducible according to Proposition 2.2.

(ii) Note that from the proof of (i), we have the set of weights of \( g_{-1} \) being the restriction of the elements in \( \Delta_{-1} \) to \( t \). Now for each \( \alpha \in \Delta_{-1} \),

\[
\alpha = -\alpha_0 - \sum_{i=1}^{\ell} n_i \alpha_i
\]

for some non-negative integers \( n_i (i = 1, \ldots, \ell) \). It follows that

\[
-\alpha_0 - \alpha = \sum_{i=1}^{\ell} n_i \alpha_i
\]

is positive in the lexicographical ordering. Thus \( -\alpha_0 |_t \) is the highest weight of \( g_{-1} \) as a \( g_{ss}^0 \)-representation.

\[ \square \]

Up to now, we have established (i) and (ii) of Theorem 1.1. From Theorem 3.2, we observe that the \( g_{ss}^0 \)-representation \( g_{-1} \) imposes severe constraint on the possible augmentations of Dynkin diagrams as it gives a finite list of possible weights \( -\alpha_0 |_t \). Assuming the existence of such augmentation of Dynkin diagrams, in virtue of Lemma 3.1, \( g_{-1} \) determines the Cartan matrix of the possible \( g \) up to the choice of the values \( \langle \alpha_i, \alpha_0 \rangle_h \) for \( 1 \leq i \leq \ell \). But according to the properties of Cartan matrices, the only ambiguity happens for those \( i \) where \( \langle \alpha_0, \alpha_i \rangle_h = -1 \). Therefore, if we restrict to only simply-laced simple Lie algebra \( g \), \( g_{-1} \) determines completely the structure of \( g \).

An alternative method to remove the ambiguity is to associate the missing vector of integers \(( -\langle \alpha_1, \alpha_0 \rangle_h , \ldots, -\langle \alpha_\ell, \alpha_0 \rangle_h )\) to the representation \( g_{-1} \). Since every Dynkin diagram does not have any cycles, the extra node can only connected to each component of \( \Gamma(g_{ss}^0) \) at no more than one node \( \alpha_i \), and only those \( \langle \alpha_i, \alpha_0 \rangle_h \) give us information. Hence we define the notion of connecting multiplicities to capture this piece of information.

**Definition 3.2.** Let \( g = \bigoplus_{i \in \mathbb{Z}} g_i \) be defined as above. Suppose that \( g_{ss}^0 = a_1 \times \)
\[\cdots \times a_r \text{ is the decomposition of } g_0^{a_r} \text{ into simple ideals. The } \mathbb{Z}\text{-valued vector } \nu(g, g_{-1}) = (a_1, \ldots, a_r) \text{ with } a_i = \max_{\alpha \in \Gamma(a_i)} -\langle \alpha, \alpha_0 \rangle_h,\]
is called the connecting multiplicities of \(g_0^{a_r}\)-representation \(g_{-1}\) with respect to \(g\).

**Remark 2:** To avoid ambiguity in defining \(\nu(g, g_{-1})\), we adopt the convention that the simple ideals \(a_i\) are lined up in alphabetical order according to their Lie types and among those with the same Lie type we write the one with smaller rank in front. If it happens that some of them are exactly the same, we simply put the values \(a_i\) in descending order.

Clearly, elements in \(\nu(g, g_{-1})\) takes values only from \(\{0, 1, 2, 3\}\) and that 2, 3 cannot appear twice or at the same time. Also \(g\) is simple if and only if \(\nu(g, g_{-1})\) does not contain 0.

The following proposition captures some important direct consequences from the Cartan matrix of \(g\).

**Proposition 3.3.** Let \(g = \bigoplus_{i \in \mathbb{Z}} g_i\) be defined as above, \(\omega\) be the highest weight of \(g_{-1}\) as a \(g_0^{a_r}\)-representation and \(A\) be the Cartan matrix of \(g\) with respect to the system of simple roots \(\Pi = \{\alpha_0, \ldots, \alpha_{\ell}\}\). Suppose \(\nu(g, g_{-1}) = (a_1, \ldots, a_r)\) and \(a_i = \langle \alpha_i, \alpha_0 \rangle_h\).

(i) \(a_i = 0\) if and only if \(\langle \omega, \alpha_i \rangle_t = 0\).

(ii) For all \(a_i \neq 0\), we have

\[\frac{a_i(H_{\alpha_i}, H_{\alpha_i})}{\omega(H_{\alpha_i})}\]

is a nonzero constant independent of \(i\).

(iii) All principal minors of \(A\) are positive definite.

In fact, we will show in Section 7 that those conditions in Proposition 3.3 are the only conditions required to construct back the ambient Lie algebra \(g\). As a result, we abstractly define the connecting multiplicities of an arbitrary irreducible representation.

**Definition 3.3.** Let \(V\) be an irreducible representation of a semisimple Lie algebra \(g\) with highest weight \(\omega\). A \(\mathbb{Z}\)-valued vector \(\nu = (a_1, \ldots, a_r)\) is called the connecting multiplicities of \(V\) if the following conditions are satisfied: There is a system of simple roots \(\Pi = \{\alpha_1, \ldots, \alpha_{\ell}\}\) so that

(i) \(a_i = 0\) if and only if \(\langle \omega, \alpha_i \rangle = 0\).

(ii) For all \(a_i \neq 0\), we have

\[\frac{a_i(H_{\alpha_i}, H_{\alpha_i})}{\omega(H_{\alpha_i})}\]

is a nonzero constant independent of \(i\).
(iii) All principal minors of

\[
A = \begin{pmatrix}
2 & -\langle \omega, \alpha_1 \rangle & \cdots & -\langle \omega, \alpha_r \rangle & 0 & \cdots & 0 \\
-a_1 & \langle \alpha_1, \alpha_1 \rangle & \cdots & \langle \alpha_1, \alpha_r \rangle & \langle \alpha_1, \alpha_{r+1} \rangle & \cdots & \langle \alpha_1, \alpha_\ell \rangle \\
\vdots & & & & & & \\
-a_r & & \cdots & & & & \\
0 & & \vdots & & \vdots & & \\
\vdots & & & & & & \\
0 & \langle \alpha_\ell, \alpha_1 \rangle & \cdots & \cdots & \langle \alpha_\ell, \alpha_\ell \rangle
\end{pmatrix}
\]

are positive definite.

Remark 3: From the definition of \( \nu \), \( A \) is a Cartan matrix.

There is one further possible reduction of the problem, namely it suffices to consider simple Lie algebra \( g \). The reason is that only the connected component of \( \Gamma(g) \) containing \( \alpha_0 \) is sensitive to our \( \mathbb{Z} \)-gradation of \( g \). Note that the root system of \( g \) decomposes into irreducible subsystems which are mutually orthogonal to each other, each of which corresponds to a connected component of \( \Gamma(g) \). Thus the root spaces \( g_\alpha \) are contained in \( g_0 \) for those \( \alpha \) lying in an irreducible subsystem not containing \( \alpha_0 \), and they act trivially on each \( g_k (k \neq 0) \).

Therefore, if \( \Gamma_0(g) \) denotes the connected component of \( \Gamma(g) \) containing \( \alpha_0 \) and \( \Gamma_0(g_0^\alpha) \) is the subdiagram of \( \Gamma(g_0^\alpha) \) obtained by deleting the nodes lying in the connected components not containing \( \alpha_0 \), then \( (\Gamma_0(g), \Gamma_0(g_0^\alpha)) \) is still an augmentation of Dynkin diagrams, and all the nonzero graded pieces \( g_k \) remain unchange.

In the next section, we will show that such \( g_k \) are prehomogeneous vector spaces with respect to a closed connected reductive algebraic group \( G_0 \) corresponding to the Lie algebra \( g_0 \).

4 Orbit Finiteness and Prehomogeneity

Up to now, we have shown that \( g_k \) for \( k \neq 0 \) are irreducible weight multiplicity free representations of \( g_0^\alpha \). By taking into account of the action of the closed connected subgroup \( G_0 \) of \( G \) with Lie algebra \( g_0 \), we succeed in showing that \( (G_0, g_k) \) are prehomogeneous vector spaces for all \( k \neq 0 \). Essentially the proof will be separated into two steps: 1) To establish an orbit finiteness statement of \( G_0 \) on \( g_k \); 2) to show that there is exactly one open orbit in \( g_k \), which is the restriction of a nilpotent orbit in \( g \) onto \( g_k \).

Theorem 4.1. Let \( G \) be a connected complex semisimple Lie group with Lie algebra \( g \). Suppose that \( g \) has a \( \mathbb{Z} \)-gradation \( g = \bigoplus_{i \in \mathbb{Z}} g_i \). Let \( G_0 \) be the closed connected subgroup of \( G \) with Lie algebra \( g_0 \). Then for each \( k \neq 0 \), the action of \( G_0 \) on \( g_k \) has a finite number of orbits.

\footnote{The theorem was first proved in \cite{23}. See also Remark 1.}
Before going into the proof, we need a simple lemma.

**Lemma 4.2.** Under the same conditions as in Theorem 5.1, for each $k 
eq 0$, every element in $\mathfrak{g}_k$ is nilpotent in $\mathfrak{g}$.

**Proof.** Pick any $X \in \mathfrak{g}_k$, we have, for all $i \in \mathbb{Z}$, $ad^m_X(\mathfrak{g}_k) \subset \mathfrak{g}_{i+mk}$. Since $\mathfrak{g}$ is finite dimensional and $k \neq 0$, $\mathfrak{g}_{i+mk} = 0$ for sufficiently large $m$. Hence $ad^m_X = 0$ and $X$ is nilpotent in $\mathfrak{g}$. □

**Proof of Theorem 5.1.** First note that $[\mathfrak{g}_0, \mathfrak{g}_k] \subset \mathfrak{g}_k$, and hence $\mathfrak{g}_k$ is $G_0$-invariant. Let $\mathfrak{g}_k := \text{Ad}(G) \cdot \mathfrak{g}_k$ be the $G$-saturation of $\mathfrak{g}_k$ in $\mathfrak{g}$. By Lemma 4.2, every element in $\mathfrak{g}_k$ is nilpotent in $\mathfrak{g}$, so that the same is true for $\mathfrak{g}_k$. In other words, $\mathfrak{g}_k$ is a union of nilpotent $G$-orbits in $\mathfrak{g}$, which must be finite since there are only finite number of nilpotent $G$-orbits in $\mathfrak{g}$ [3]. It remains to show that for every $G$-orbit $O$ in $\tilde{\mathfrak{g}}_k$, $O \cap \mathfrak{g}_k$ splits into a finite number of $G_0$-orbits in $\mathfrak{g}_k$.

By the definition of $\tilde{\mathfrak{g}}_k$, there exists $X \in \mathfrak{g}_k$ such that $O = \text{Ad}(G) \cdot X$, so that $O \cap \mathfrak{g}_k \neq \emptyset$. Now for any $X' \in O \cap \mathfrak{g}_k$,

$$T_{X'}(O \cap \mathfrak{g}_k) \subset [\mathfrak{g}, X'] \cap \mathfrak{g}_k = [\mathfrak{g}_0, X'] = T_{X'}(\text{Ad}(G_0) \cdot X').$$

But on the other hand, we have $O \cap \mathfrak{g}_k \supset \text{Ad}(G_0) \cdot X'$ since $\mathfrak{g}_k$ is $G_0$-invariant. It follows that $T_{X'}(O \cap \mathfrak{g}_k) = T_{X'}(\text{Ad}(G_0) \cdot X')$, thus $X'$ is a nonsingular point of $O \cap \mathfrak{g}_k$ and $\text{Ad}(G_0) \cdot X'$ is open in $O \cap \mathfrak{g}_k$. As $X' \in O \cap \mathfrak{g}_k$ is arbitrary, $\text{Ad}(G_0) \cdot X'$ is also closed in $O \cap \mathfrak{g}_k$ for its complement is a union of such orbits. Thus the $G_0$-orbits in $O \cap \mathfrak{g}_k$ are precisely all the connected components of $O \cap \mathfrak{g}_k$ and so $O \cap \mathfrak{g}_k$ being a smooth manifold can possess only finite number of $G_0$-orbits. □

**Remark 4:** The analogous statement of Theorem 4.1 for $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ holds true as long as $\mathfrak{g}_k, k \neq 0$, are contained in the nilpotent cone of $\mathfrak{g}$. The line of proof runs exactly the same except one must replace Lemma 4.2 by the above assumption.

**Theorem 4.3.** Under same conditions as in Theorem 4.1, $\mathfrak{g}_k$ has a unique open $G_0$-orbit of the form $O_k \cap \mathfrak{g}_k$, where $O_k$ is the unique open $G$-orbit in $\tilde{\mathfrak{g}}_k$. In particular, $(G_0, \mathfrak{g}_k)$ is a prehomogeneous vector space for every $k \neq 0$.

**Proof.** Note that $\mathfrak{g}_k$ is irreducible as an affine variety, which forces all open $G_0$-orbits in $\mathfrak{g}_k$ to be dense and thus coincide. Therefore, $(G_0, \mathfrak{g}_k)$ is trivially a prehomogeneous vector space in virtue of Theorem 4.1. This proves the second statement. To establish the first assertion, we need the following two lemmas:

**Lemma 4.4.** With the same notations as in Theorem 5.1, there is a unique nilpotent $G$-orbit in $\mathfrak{g}_k = \text{Ad}(G) \cdot \mathfrak{g}_k$ which is open in $\mathfrak{g}_k$ for every $k \neq 0$.

**Proof.** Suppose on the contrary that there are two such nilpotent $G$-orbits $O', O''$. Then $O' \cap \mathfrak{g}_k$ and $O'' \cap \mathfrak{g}_k$ are nonempty and open in $\mathfrak{g}_k$. As $\mathfrak{g}_k$ is an affine space, and hence irreducible, $O' \cap \mathfrak{g}_k$ and $O'' \cap \mathfrak{g}_k$ are dense in $\mathfrak{g}_k$. It follows that $O'$ intersects with $O''$ nontrivially, which forces $O' = O''$. □
Lemma 4.5. Let $O_k$ be the unique nilpotent $G$-orbit contained in $\tilde{g}_k$ obtained in Lemma 4.4. Then $O_k \cap g_k$ is connected.

Proof. Suppose that there exist two nonempty proper open subsets $U_1, U_2$ of $O_k \cap g_k$ such that

$$O_k \cap g_k = U_1 \cup U_2 \quad \text{and} \quad U_1 \cap U_2 = \emptyset.$$ 

Then by definition, we can find two nonempty open subsets $\Omega_1, \Omega_2$ of $g_k$ such that

$$U_i = \Omega_i \cap O_k \cap g_k, \quad i = 1, 2.$$ 

Since $\Omega_1, \Omega_2$ and $O_k \cap g_k$ are nonempty open subsets of $g_k$, all of them are dense in $g_k$. Therefore,

$$U_1 \cap U_2 = \Omega_1 \cap \Omega_2 \cap O_k \cap g_k \neq \emptyset,$$

which contradicts our assumption. 

By Lemmas 4.4 and 4.5, $O_k \cap g_k$ is an open dense connected subset of $g_k$. Referring to the proof of Theorem 4.1, we see that $O_k \cap g_k$ is a smooth manifold and the $G_0$-orbits of $g_k$ inside $O_k \cap g_k$ are precisely its connected components, which must be $O_k \cap g_k$ itself. Thus $O_k \cap g_k$ is an open dense $G_0$-orbit in $g_k$. 

In fact, from Proposition 2.3, we know that prehomogeneity is an infinitesimal notion determined only by the action of the Lie algebra $g_0$, so that it depends only on the Lie type of the complex semisimple Lie group $G$ and that of the reductive subalgebra $G_0$. This suggests a reason why this notion should be related to augmentations of Dynkin diagrams, which capture exactly the Lie types.

Finally, we close this section with a result concerning the corresponding action of the real forms of $G_0$.

Theorem 4.6. Let $(G_0)_R$ be a real form of $G_0$. Regarding $g_i \ (i \neq 0)$ as a complex representation of $(G_0)_R$ if it is of real type, then the $(G_0)_R$ action on $(g_i)_R$ has a finite number of orbits. In particular, if $(G_0)_R$ is the split form of $G_0$, then $g_i \ (i \neq 0)$ is always of real type and the corresponding real representation $(g_i)_R$ consists of a finite number of orbits.

Proof. It is a direct consequence of Theorem 4.1 and Theorem 4.6.

5 Termination of $\mathbb{Z}$-Gradings

Recall that upon choosing a simple root $\alpha_0 \in \Pi$, we have constructed a $\mathbb{Z}$-gradation

$$g = \bigoplus_{i \in \mathbb{Z}} g_i.$$
Since $\mathfrak{g}$ is finite dimensional, there exists a positive integer $n$ such that $\mathfrak{g}_i = 0$ for all $|i| > n$. In virtue of Proposition A.1, $\mathfrak{g}_{-i}$ is naturally identified with $\mathfrak{g}_i^*$ as a $\mathfrak{g}_0$-representation using the Killing form of $\mathfrak{g}$. In other words, we have

$$\mathfrak{g} = \bigoplus_{i=-n}^{n} \mathfrak{g}_i,$$

where $\dim \mathfrak{g}_n = \dim \mathfrak{g}_{-n} \neq 0$. We call $n$ the order of $\mathfrak{g}$ with respect to $\alpha_0$ or the order of the $\mathbb{Z}$-gradation. In fact, there is an easy algorithm to compute the order $n$. We will consider only the case in which $\mathfrak{g}$ is simple, the general case follows by considering the simple ideal containing the root space $\mathfrak{g}_{\alpha_0}$.

From now on, suppose that $\mathfrak{g}$ is a complex simple Lie algebra. Let $\tilde{\alpha} \in \Delta$ be the highest root of $\mathfrak{g}$. Then

$$\tilde{\alpha} = \sum_{i=0}^{\ell} n_i \alpha_i$$

for some positive integers $n_i, i = 0, \ldots, \ell$.

**Proposition 5.1.** With the above notations, the order of $\mathfrak{g}$ with respect to $\alpha_0$ is $n_0$.

**Proof.** By definition, we have $\tilde{\alpha} \in \Delta_{n_0}$ and that for every root

$$\alpha = \sum_{i=0}^{\ell} m_i \alpha_i \in \Delta$$

where $m_i$ are non-negative integers for $i = 0, \ldots, \ell$, we have $m_i \leq n_i$ for all $i = 0, \ldots, \ell$. In particular, $n_0$ is the greatest integer $n$ for which $\Delta_{n} \neq \emptyset$. \hfill $\square$

Indeed, in each simple case, we can write down the highest root explicitly. Table 2 shows the Dynkin diagrams of all simple complex Lie algebras with each node labelled by the coefficient of the corresponding simple root in the highest root, which is just the order with respect to the corresponding simple root according to Proposition 5.1.

From Table 2 we see immediately that the only possible orders are $1 \leq n_0 \leq 6$. Indeed, for those $\mathfrak{g}$ with order $n_0 > 1$ with respect to $\alpha_0$, we can find a semisimple regular subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ containing $\mathfrak{g}_0$ with the same rank as $\mathfrak{g}$ such that the corresponding $\mathbb{Z}$-gradation has order 1 and that the original $\mathbb{Z}$-gradation factors as a $\mathbb{Z}_{n_0}$-gradation over $\tilde{\mathfrak{g}}$.

Note that given a system of simple roots $\Pi = \{\alpha_0, \ldots, \alpha_\ell\}$, we have an extended system of simple roots $\tilde{\Pi} = \Pi \cup \{-\tilde{\alpha}\}$ by adjoining the lowest root. $-\tilde{\alpha}$ to it. Set $\Pi_i = \tilde{\Pi} \setminus \{\alpha_i\}$ and $\Delta(i) = \mathbb{Z} \langle \Pi_i \rangle \cap \Delta$ for $i = 0, \ldots, \ell$. Then it is known that $\Delta(i)$ forms a reduced root system of $\Delta$ which corresponds to a semisimple subalgebra of $\mathfrak{g}$ of the same rank with a system of simple roots given by $\Pi_i$. Besides, we have the following result concerning the maximal regular reductive subalgebras of $\mathfrak{g}$ which is a direct consequence of a result by Borel-de Siebenthal [1] on the maximal closed subroot systems:
Table 2: Dynkin diagrams with nodes labelled by the orders with respect to the corresponding simple roots.

| Diagram | Diagram |
|---------|---------|
| $A_n$   | ![Diagram $A_n$](image)
| $B_n$   | ![Diagram $B_n$](image)
| $C_n$   | ![Diagram $C_n$](image)
| $D_n$   | ![Diagram $D_n$](image)
| $E_6$   | ![Diagram $E_6$](image)
| $E_7$   | ![Diagram $E_7$](image)
| $E_8$   | ![Diagram $E_8$](image)
| $F_4$   | ![Diagram $F_4$](image)
| $G_2$   | ![Diagram $G_2$](image) |
**Theorem 5.2.** Let $\mathfrak{g}$ be the simple Lie algebra defined above and let $\tilde{\alpha} = \sum_{i=0}^{\ell} n_i \alpha_i$ be the highest root with respect to the simple root system $\Pi = \{ \alpha_0, \ldots, \alpha_\ell \}$. Then all maximal regular reductive subalgebras can be obtained in one of the following ways:

(i) when $n_i$ is a prime number, the regular semisimple subalgebra is $$\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(i)} \mathfrak{g}_{\alpha}$$
with root system given by $\Delta(i)$;

(ii) when $n_i = 1$, the regular reductive subalgebra is $$\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0(i)} \mathfrak{g}_{\alpha},$$
where $\Delta_0(i) = \mathbb{Z} \langle \Pi \setminus \{ \alpha_i \} \rangle \cap \Delta$.

For a detailed proof see Goto and Grosshans [5]. The main idea is that every maximal subroot system is generated by an element $\lambda \in \mathfrak{h}^*$ in the sense of $\{ \alpha \in \Delta | \langle \lambda, \alpha \rangle \in \mathbb{Z} \}$. But the choice of such $\lambda$ is invariant under the affine Weyl group $W_{\text{aff}}$, which can be assumed to lie in the closure of the fundamental alcove. Finally by explicit case-by-case computations, we obtain the above result.

There is a useful criterion for a regular subalgebra being reductive:

**Proposition 5.3.** Let $\mathfrak{g} = \mathfrak{k} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$ be a regular subalgebra of a semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$, where $\mathfrak{k}$ is a subspace of $\mathfrak{h}$ and $\Phi \subset \Delta$ as a subroot system. Then $\mathfrak{g}$ is reductive if and only if $\Phi$ is closed and symmetric (i.e. $(\Phi + \Phi) \cap \Delta \subset \Phi$ and $\Phi = -\Phi$) and $\text{span}\{ H_{\alpha} | \alpha \in \Phi \} \subset \mathfrak{k}$.

**Proposition 5.4.** Let $\mathfrak{g} = \bigoplus_{i=-n}^{n} \mathfrak{g}_i$ be the $\mathbb{Z}$-gradation defined above with $\mathfrak{g}_n \neq 0$ and $m$ a positive integer dividing $n$. Then $$\mathfrak{g}(m)_0 := \bigoplus_{i \equiv 0 \text{ (mod } m)} \mathfrak{g}_i$$
is a regular semisimple subalgebra of $\mathfrak{g}$ of the same rank containing $\mathfrak{g}_0$. Moreover, $\mathfrak{g}(n)_0$ is a maximal regular reductive subalgebra of $\mathfrak{g}$ whose root system is isomorphic to $\Delta(0) = \mathbb{Z} \langle \Pi_0 \rangle \cap \Delta$ and its system of simple roots is $\Pi_0$.

**Proof.** Note that the set of roots in $\Delta$ occurring in $\mathfrak{g}(m)_0$ is $$\bigcup_{k \in \mathbb{Z}} \Delta_{km},$$
which is closed and symmetric. Thus, by Proposition 5.3, \( g^{(m)}_0 \) is a regular reductive subalgebra of \( g \). As \( g_0 \subset g^{(m)}_0 \), \( 3g^{(m)}_0 \subset 3g_0 = C \). But \( c \) acts non-trivially on \( g_m \) implies that \( 3g^{(m)}_0 = 0 \). Hence \( g^{(m)}_0 \) is a semisimple subalgebra of the same rank.

To verify the second statement is equivalent to show that

\[
g^{(n)}_0 = h \oplus \bigoplus_{\alpha \in \Delta(0)} g_{\alpha},
\]

which reduces to verify that \( \Delta(0) = \Delta_{-n} \cup \Delta_0 \cup \Delta_n \). Note that \( \tilde{\alpha} \in \Delta_n \), we have

\[
\Delta_n = (\mathbb{Z} \langle \alpha_1, \ldots, \alpha_\ell \rangle + \tilde{\alpha}) \cap \Delta;
\]

similarly, we have \( \Delta_{-n} = (\mathbb{Z} \langle \alpha_1, \ldots, \alpha_\ell \rangle - \tilde{\alpha}) \cap \Delta \).

Therefore,

\[
\Delta(0) = \mathbb{Z} \langle \alpha_1, \ldots, \alpha_\ell, \tilde{\alpha} \rangle \cap \Delta
= \bigcup_{i=-1}^{1} ((\mathbb{Z} \langle \alpha_1, \ldots, \alpha_\ell \rangle + i\tilde{\alpha}) \cap \Delta)
= \Delta_{-n} \cup \Delta_0 \cup \Delta_n.
\]

We have the following characterization of the \( \mathbb{Z} \)-gradation \( g = \bigoplus_{i=-n}^{n} g_i \), when 

\[ n = 1 \text{ or } n \text{ is a prime number.} \]

**Theorem 5.5.** Let \( g = \bigoplus_{i=-n}^{n} g_i \) be a \( \mathbb{Z} \)-gradation of \( g \) as constructed above with \( g_n \neq 0 \).

(i) If \( n = 1 \), then \( g_0 \) is a maximal reductive subalgebra of \( g \) with an one-dimensional center;

(ii) if \( n \) is a prime number, then \( g^{(n)}_0 \) is a maximal semisimple subalgebra of \( g \) of the same rank for which \( g_0 \) lies in \( g^{(n)}_0 \) as a maximal reductive subalgebra with a one-dimensional center.

**Proof.** By Proposition 5.4, we have \( g_0 \) in (i) being the second case of Theorem 5.2 and \( g^{(n)}_0 \) in (ii) being the first case of Theorem 5.2.

**Proposition 5.6.** Under the same conditions as in Proposition 5.4, we have

\[
g = \bigoplus_{j \in \mathbb{Z}_m} g^{(j)}
\]

is a \( \mathbb{Z}_m \)-gradation of \( g \), where \( g^{(j)} = \bigoplus_{i \equiv j \pmod{m}} g_i \).

**Proof.** For \( j_1, j_2 \in \mathbb{Z}_m \) and \( i_s \equiv j_s \pmod{m} \), \( s = 1, 2 \), we have \( i_1 + i_2 \equiv j_1 + j_2 \pmod{m} \) and \( [g_{i_1}, g_{i_2}] \subset g_{i_1+i_2} \subset g^{(j_1+j_2)} \). Hence

\[
[g^{(j_1)}, g^{(j_2)}] \subset g^{(j_1+j_2)}.
\]
6 Explicit Construction of Generic Elements in Simply-laced Cases

Throughout this section, \( \mathfrak{g} \) is assumed to be simply-laced, i.e. the corresponding Dynkin diagram \( \Gamma(\mathfrak{g}) \) consists of single edges only. Assume that the nodes of \( \Gamma(\mathfrak{g}) \) are indexed by a system of simple roots \( \Pi = \{ \alpha_0, \ldots, \alpha_\ell \} \). In this case, the corresponding Cartan matrix \((\langle \alpha_i, \alpha_j \rangle_h)_{i,j=0,\ldots,\ell} \) is completely determined by their restrictions onto \( t \), namely \( \langle \alpha_i, \alpha_j \rangle_h = \langle \alpha_i|_t, \alpha_j|_t \rangle_t \) for \( j \neq 0, i = 0, \ldots, \ell \), and \( \langle \alpha_0, \alpha_0 \rangle_h = \langle \alpha_0|_t, \alpha_0|_t \rangle_t \) according to Lemma 3.1 and the fact that all roots have the same length in a simply-laced semisimple Lie algebra.

Now we denote \( W_0 \) as the Weyl group of \( \mathfrak{g}_0^\ast \) generated by the reflections in \( t^\ast \) along \( \{ \alpha_1|_t, \ldots, \alpha_\ell|_t \} \). Let \( \bar{W}_0 \) be the subgroup of the Weyl group \( W \) of \( \mathfrak{g} \) generated by reflections along \( \{ \alpha_1, \ldots, \alpha_\ell \} \). As every element in \( \bar{W}_0 \) preserves the subspace \( t^\ast \), the natural restriction map induces an isomorphism from \( \bar{W}_0 \) to \( W_0 \), denoted by \( \bar{w} \mapsto w \). Also, we see that \( \bar{W}_0 \) stabilizes each \( \Delta_t \).

Let \( (H_\alpha, X_\alpha, Y_\alpha) \) be a standard \( \mathfrak{sl}_2 \)-triple corresponding to \( \alpha \in \Delta^+ \), which will be fixed once and for all throughout the whole section. In the following, we will give an explicit construction of generic elements in \( \mathfrak{g}_1 \) and \( \mathfrak{g}_{-1} \) as \( G_0 \) representations.

Lemma 6.1. \( \bar{w}(\alpha)|_t = w(\alpha)|_t \) for all \( \bar{w} \in \bar{W}_0, \alpha \in \Delta \). In particular, we have \( |\bar{W}_0 : \alpha_0| = |W_0 : \alpha_0| \).

Proof. For all \( j = 1, \ldots, \ell \), by Lemma 5.1 we have
\[
(\bar{w}(\alpha)|_t, \alpha_j|_t)_t = (\bar{w}(\alpha), \alpha_j)_h
= (\alpha, \bar{w}^{-1}(\alpha_j))_h
= (\alpha|_t, \bar{w}^{-1}(\alpha_j)|_t)_t
= (\alpha|_t, w^{-1}(\alpha_j|_t))_t
= (w(\alpha)|_t, \alpha_j|_t)_t.
\]
Since \( \{ \alpha_1, \ldots, \alpha_\ell \} \) form a basis of \( t^\ast \), we conclude that \( \bar{w}(\alpha)|_t = w(\alpha)|_t \).

Lemma 6.2. For \( k \neq 0 \),
\[
\sum_{\alpha \in \bar{W}_0 \cdot \alpha_0} \langle \alpha, \alpha_k \rangle_h = 0.
\]

Proof. Note that
\[
\sum_{\alpha \in \bar{W}_0 \cdot \alpha_0} \langle \alpha, \alpha_k \rangle_h = \sum_{\alpha \in \bar{W}_0 \cdot \alpha_0} \langle \alpha|_t, \alpha_k|_t \rangle_t \quad \text{(Lemma 5.1)}
= \sum_{\beta \in \bar{W}_0 \cdot \alpha_0|_t} \langle \beta, \alpha_k|_t \rangle_t \quad \text{(Lemma 6.1)}
= \left\langle \sum_{\beta \in \bar{W}_0 \cdot \alpha_0|_t} \beta, \alpha_k|_t \right\rangle_t.
\]
Here $\sum_{\beta \in W_0 \cdot \alpha_0 \mid t} \beta$ is just the sum of all extremal weights of $g_1$ as a $g_0^\ast \ast$ representation. As the set of extremal weights is symmetric about the origin, we have $\sum_{\beta \in W_0 \cdot \alpha_0 \mid t} \beta = 0$ and thus $\sum_{\alpha \in W_0 \cdot \alpha_0} \langle \alpha, \alpha \rangle_h = 0.$

**Lemma 6.3.**

$$\sum_{\alpha \in W_0 \cdot \alpha_0} \langle \alpha, \alpha_0 \rangle_h = 2 |W_0 \cdot \alpha_0 \mid t| \cdot \frac{\|\alpha'_0\|^2}{\|\alpha_0\|^2},$$

where $\alpha'_0$ is the orthogonal projection of $\alpha_0$ to $(\mathbb{C}c)^\ast$, i.e. $\alpha_0 = \alpha_0 \mid t + \alpha'_0$ with $\langle \alpha_0 \mid t, \alpha'_0 \rangle_h = 0.$

**Proof.** First note that $\tilde{W}_0 \cdot \alpha_0 \subset \Delta_1$ so that every element in $\tilde{W}_0 \cdot \alpha_0$ has the same orthogonal projection $\alpha'_0$ onto $(\mathbb{C}c)^\ast$. Then by applying the above two lemmas, we obtain

$$\sum_{\alpha \in W_0 \cdot \alpha_0} \alpha = \sum_{\alpha \in W_0 \cdot \alpha_0} (\alpha \mid t + \alpha'_0)$$

$$= \sum_{\beta \in W_0 \cdot \alpha_0 \mid t} \beta + \sum_{\alpha \in W_0 \cdot \alpha_0} \alpha'_0$$

$$= |\tilde{W}_0 \cdot \alpha_0 | \cdot \alpha'_0$$

$$= |W_0 \cdot \alpha_0 \mid t | \cdot \alpha'_0.$$ 

It follows that

$$\sum_{\alpha \in W_0 \cdot \alpha_0} \langle \alpha, \alpha_0 \rangle_h = |W_0 \cdot \alpha_0 \mid t | \langle \alpha'_0, \alpha_0 \rangle_h$$

$$= 2 |W_0 \cdot \alpha_0 \mid t | \cdot \frac{\langle \alpha'_0, \alpha_0 \mid t + \alpha'_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle}$$

$$= 2 |W_0 \cdot \alpha_0 \mid t | \cdot \frac{\|\alpha'_0\|^2}{\|\alpha_0\|^2}.$$

**Theorem 6.4.** Let $g$ be a simply-laced semisimple Lie algebra as defined above. If $(G_0, g_{-1})$ is a regular prehomogeneous space, then there exist $X \in g_1, Y \in g_{-1}$ such that $[X, Y] = c.$
Proof. For \( k = 0, \ldots, \ell \), we have

\[
\left( H_{\alpha_k}, \left[ \sum_{\alpha \in W_0 \cdot \alpha_0} X_\alpha, \sum_{\beta \in W_0 \cdot \alpha_0} Y_\beta \right] \right) = \left( \sum_{\alpha \in W_0 \cdot \alpha_0} \alpha (H_{\alpha_k}) X_\alpha, \sum_{\alpha \in W_0 \cdot \alpha_0} Y_\alpha \right)
\]

\[
= \sum_{\alpha \in W_0 \cdot \alpha_0} \langle \alpha, \alpha_k \rangle_h \left( X_\alpha, \sum_{\alpha \in W_0 \cdot \alpha_0} Y_\alpha \right)
\]

\[
= \frac{1}{2} \sum_{\alpha \in W_0 \cdot \alpha_0} \langle \alpha, \alpha_k \rangle_h (H_\alpha, H_\alpha)
\]

\[
= \frac{1}{2} \sum_{\alpha \in W_0 \cdot \alpha_0} \langle \alpha, \alpha_k \rangle_h (H_\alpha_0, H_\alpha_0)
\]

\[
= \begin{cases} 
0, & \text{if } k \neq 0 \text{ (Lemma 6.2),} \\
\frac{1}{|W_0 \cdot \alpha_0|_1} \cdot \frac{\langle \alpha_0 \rangle_0 \cdot \langle \alpha_0 \rangle_0}{\| \alpha_0 \|_2} \cdot (H_\alpha_0, H_\alpha_0), & \text{if } k = 0 \text{ (Lemma 6.3).}
\end{cases}
\]

Set

\[
X = \frac{1}{\sqrt{2|W_0 \cdot \alpha_0|_1}} \cdot \frac{\| \alpha_0 \|_0}{\| \alpha_0 \|_2} \sum_{\alpha \in W_0 \cdot \alpha_0} X_\alpha, \quad Y = \frac{1}{\sqrt{2|W_0 \cdot \alpha_0|_1}} \cdot \frac{\| \alpha_0 \|_0}{\| \alpha_0 \|_2} \sum_{\alpha \in W_0 \cdot \alpha_0} Y_\alpha.
\]

Then \( X \in \mathfrak{g}_1, \ Y \in \mathfrak{g}_{-1} \) and that

\[
\alpha_k([X, Y]) = \frac{2 (H_{\alpha_k}, [X, Y])}{(H_{\alpha_k}, H_{\alpha_k})} = \begin{cases} 
0, & \text{if } k \neq 0, \\
1, & \text{if } k = 0.
\end{cases}
\]

Note that the regularity condition is to ensure that \([X, Y]\) lies in \( \mathfrak{h} \) as in this case

\[
\left[ \sum_{\alpha \in W_0 \cdot \alpha_0} X_\alpha, \sum_{\beta \in W_0 \cdot \alpha_0} Y_\beta \right] = \sum_{\alpha \in W_0 \cdot \alpha_0} [X_\alpha, Y_\alpha].
\]

It follows that \([X, Y] = c\). \( \square \)
Corollary 6.5. Let \( g \) be a simply-laced semisimple Lie algebra as defined above and \( X \in g_1 \). If there exists \( Y \in g_{-1} \) such that \([X,Y] = c\), then \([X,g_0] = g_1\), i.e. \( X \) is a generic element of \((G_0,g_1)\).

Proof. First note that \( \frac{1}{2}c, X, Y \) form a standard set of generators for an \( \mathfrak{sl}_2 \) subalgebra \( a \) of \( g \). Then from \( \mathfrak{sl}_2 \) theory, if we decompose \( g \) into irreducible \( a \) representations, there are no weight spaces with weight 1 and that \( g_1 \) is the direct sum of all weight spaces of weight 2. Hence, we have \([X,g_0] = g_1\). The stabilizer \((g_0)_X \) is reductive as it is the centralizer of \( a \).

7 The Ambient Lie Algebras of Parabolic PVS’s

With the effort of the previous sections, we can already conclude Theorem 1.1(iii) on the class of simply-laced Lie algebras. The general situation is more complicated as there can be more than one ambient Lie algebra \( g \) associated to an irreducible prehomogeneous vector space \((G_0,V)\). For instance, if we consider the prehomogeneous vector space \((GL_2,\mathbb{C}^2)\), we can choose \( g \) to be either \( sl_3 \), \( g_2 \).

In this section, we will finish the proof of Theorem 1.1 by showing that given an irreducible parabolic PVS \((G_0,V,\nu)\) with connecting multiplicities there exists exactly one ambient Lie algebra \( g \) containing \( g_0 \) for which \((\Gamma(g),\Gamma(g_0^{ss}))\) maps to \((G_0,V,\nu(g,V))\) under the correspondence set up in Theorem 1.1(iv).

The main result we used here is the Serre’s Theorem which states that given a Cartan matrix \( A = (a_{ij}) \) of rank \( \ell \) there is a semisimple Lie algebra with \( 3\ell \) generators \( \{H_i, X_i, Y_i\}, i = 1,\ldots, \ell \), satisfying

\[
\begin{align*}
[H_i,H_j] &= 0 \\
[X_i,Y_j] &= H_i, [X_i,Y_j] = 0 \text{ if } i \neq j \\
[H_i,X_j] &= a_{ji}X_j, [H_i,Y_j] = -a_{ji}Y_j \\
(adX_i)^{-a_{ji}+1}(X_j) &= 0 \\
(adY_i)^{-a_{ji}+1}(Y_j) &= 0
\end{align*}
\]

unique up to isomorphism.

Suppose \( g_0^{ss} \) is of rank \( \ell \) and by choosing a Cartan subalgebra \( t \) as usual, we obtain a corresponding root system \( \Delta_0 \). Finally, we fix a system of simple roots \( \Pi_0 = \{\alpha_1,\ldots,\alpha_\ell\} \) of \( \Delta_0 \). To each simple root \( \alpha_i \), we already have \( \{H_i, X_i, Y_i\} \) satisfying relations in (7.2)-(7.6). Let \( \mathfrak{h} = \mathfrak{g}_0 \oplus t \). To construct \( g \) it suffices to find \( H_0, X_0, Y_0 \) which are compatible with other \( H_i, X_i, Y_i \).

Let \( \nu = (a_1,\ldots,a_r) \) and \( \omega \) be the highest weight of the irreducible representation \((\pi, V)\) of \( g_0^{ss} \). Without loss of generality, we can assume that all \( a_i \neq 0 \)

---

8This corollary is first proved in [15].
and that the matrix
\[
A = \begin{pmatrix}
2 & -\langle \omega, \alpha_1 \rangle & \cdots & -\langle \omega, \alpha_r \rangle & 0 & \cdots & 0 \\
-a_1 & \langle \alpha_1, \alpha_1 \rangle & \cdots & \langle \alpha_1, \alpha_r \rangle & \langle \alpha_1, \alpha_{r+1} \rangle & \cdots & \langle \alpha_1, \alpha_\ell \rangle \\
\vdots \\
-\alpha_r & \vdots & \vdots \\
0 & \vdots & \vdots \\
0 & \langle \alpha_\ell, \alpha_1 \rangle & \cdots & \cdots & \langle \alpha_\ell, \alpha_\ell \rangle 
\end{pmatrix}
\]
is a Cartan matrix, so that we have
\[
a_i(H, H) = K
\]
for some fixed nonzero constant $K$. Then there exists a unique element $H \in \mathfrak{t}$ such that
\[
a_i(H) = \begin{cases} 
-a_i & \text{if } i = 1, \ldots, r, \\
0 & \text{otherwise.}
\end{cases}
\]
Pick any nonzero $X_0 \in \mathfrak{v}^* - \omega$, we can find a unique $c \in \mathfrak{g}_0$ such that
\[
\pi^*(H + c)X_0 = 2X_0.
\]
Let $\kappa$ be the unique $G_0$-invariant nondegenerate bilinear form on $\mathfrak{g}_0$ extending the Killing form of $\mathfrak{g}_0^{ss}$ and satisfies
\[
\kappa(c, c) = K - (H, H).
\]
Note that $\kappa(c, \mathfrak{g}_0^{ss}) = 0$ is automatic from the invariance property of $\kappa$, it follows that
\[
\kappa(H_0, H_0) = \kappa(H, H) + \kappa(c, c) = K.
\]
Choose $Y_0 \in \mathfrak{v}_\omega$ such that $X_0(Y_0) = -\frac{\kappa}{2}$. Formally, we can now impose conditions (7.2)-(7.6) to $\{H_i, X_i, Y_i\}_{i=0}^{\ell}$, whence we obtain a semisimple Lie algebra $\mathfrak{g}$ by applying the Serre’s theorem. Since the last $3\ell$ generators $\{H_i, X_i, Y_i\}_{i=1}^{\ell}$ are also generators for $\mathfrak{g}_0^{ss}$, we obtain an embedding $\mathfrak{g}_0^{ss} \subset \mathfrak{g}$, and that $\mathfrak{g}_0 = \mathbb{C}H_0 \oplus \mathfrak{g}_0^{ss} \subset \mathfrak{g}$. The remaining task is to construct the bracket relations between elements in $\mathfrak{g}_0, \mathfrak{v}$ and $\mathfrak{v}^*$ which coincide with that abstractly defined in terms of the generators of $\mathfrak{g}$. The obvious choice of defining the bracket on $\mathfrak{g}_0 \times \mathfrak{v}$ and $\mathfrak{g}_0 \times \mathfrak{v}^*$ is
\[
[Z, v] = \pi(Z)v, \quad [Z, f] = \pi^*(Z)f
\]
for all $Z \in \mathfrak{g}_0, v \in \mathfrak{v}, f \in \mathfrak{v}^*$. Direct checking shows that (7.2)-(7.6) are satisfied except the equality $[X_0, Y_0] = H_0$ has not yet been established.
Since $X_0$ is a lowest weight vector of $(\pi^*,V^*)$, for all $i = 1, \ldots, \ell$,

$$(\pi^*(Y_i)X_0)Y_0 = 0 = \kappa(H_0,Y_i).$$

Similarly, as $Y_0$ is a highest weight vector of $(\pi,V)$, for all $i = 1, \ldots, \ell$,

$$X_0(\pi(X_i))Y_0 = 0 = \kappa(H_0,X_i).$$

Clearly, for $i = r+1, \ldots, \ell$, $a_i = \omega(H_i)$, and

$$X_0(\pi(X_i))Y_0 = \omega(H_i)X_0(Y_0) = 0 = \kappa(H_0,H_i).$$

For $i = 1, \ldots, r$,

$$X_0(\pi(X_i))Y_0 = \omega(H_i)X_0(Y_0) = -\omega(H_i)\frac{K}{2} = -a_i\kappa(H_i,H_i) = \frac{\alpha_i(H)(H_i,H_i)}{2} = (H,H_i) = \kappa(H_0,H_i).$$

All together, we get

$$\kappa(H_0,Z) = X_0(\pi(Z)Y_0) = -(\pi^*(Z)X_0)Y_0 = \kappa(H_0,Z) = \kappa(H_0,Z) = \kappa(H_0, Z) = (7.7)$$

for all $Z \in \mathfrak{g}_0$.

From $\pi : \mathfrak{g}_0 \to \text{End}(V) = V^* \otimes V$, we get the moment map

$$\mu_\pi : V^* \times V \to \mathfrak{g}_0^*.$$ 

By identifying $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ through $\kappa$, we get a bilinear map

$$\phi : V^* \times V \to \mathfrak{g}_0.$$ 

Explicitly, given $v \in V$, $f \in V^*$, $\phi(f,v)$ is the unique element such that

$$\kappa(\phi(f,v),Z) = f(\pi(Z)v) = -(\pi^*(Z)f)v$$

for all $Z \in \mathfrak{g}_0$.

In view of (7.7), we have $\phi(X_0,Y_0) = H_0$. Thus this map coincides with the bracket structure constructed on $\mathfrak{g}$. In other words, we have $V$ and $V^*$ embedded into $\mathfrak{g}$ with the bracket between elements of $V$ and $V^*$ given by the map $\phi$. In particular, we have $X_0$ being a root vector corresponding to a root $\tilde{\alpha}_0$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\tilde{\alpha}_i \in \mathfrak{h}^*$ be the extension of $\alpha_i \in \mathfrak{t}^*$ by setting $\tilde{\alpha}_i(c) = 0$. Then we can easily see that $\Pi = \{\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_\ell\}$ is a system of simple roots to $\mathfrak{g}$ with $\tilde{\alpha}_0|_t = -\omega, \tilde{\alpha}_i|_t = \alpha_i$ for $i = 1, \ldots, \ell$, and the corresponding Cartan matrix is given by $A$. This complete the proof of Theorem 1.1.
8 PVS’s of Twisted Affine Type

As we have mentioned in the introduction, there are some examples of pre-homogeneous vector spaces consisting of finitely many orbits which are not of parabolic type. Among the irreducible reduced ones, there are six exceptional cases as listed in Table 4. We will briefly explain these structures and will find a unified way of constructing them.

Observe that we have the following grading:

\[
\begin{align*}
\mathfrak{so}_{10} &= \mathfrak{gl}_1 \times \mathfrak{g}_2 \oplus \mathbb{C}^7 \oplus (\mathbb{C}^7)^* \oplus \mathbb{C} \oplus (\mathbb{C}^*)^* \\
E_6 &= \mathfrak{gl}_2 \times (\mathbb{C}^2 \otimes \mathbb{C}^7) \oplus (\mathbb{C}^2 \otimes (\mathbb{C}^7)^*) \oplus (\mathbb{C}^2 \otimes (\mathbb{C}^7)^*)^* \oplus (\mathbb{C}^2 \otimes (\mathbb{C}^7)^*)^* \\
\mathfrak{so}_{12} &= \mathfrak{sl}_2 \times \mathfrak{so}_7 \oplus (\mathbb{C}^2 \otimes S) \oplus (\mathbb{C}^2 \otimes S)^* \oplus (\mathbb{C}^2 \otimes S)^* \\
E_7 &= \mathfrak{gl}_3 \times \mathfrak{so}_7 \oplus (\mathbb{C}^3 \otimes S) \oplus (\mathbb{C}^3 \otimes S)^* \oplus (\mathbb{C}^3 \otimes S)^* \\
\mathfrak{e}_6 &= \mathfrak{gl}_1 \times \mathfrak{so}_9 \oplus \mathbb{C}^9 \oplus (\mathbb{C} \oplus S) \oplus (\mathbb{C} \oplus S)^* \\
\mathfrak{e}_7 &= \mathfrak{gl}_1 \times \mathfrak{so}_{11} \oplus \mathbb{C}^{11} \oplus (\mathbb{C} \oplus S) \oplus (\mathbb{C} \oplus S)^* \oplus \mathbb{C} \oplus \mathbb{C}^*
\end{align*}
\]

They are obtained from successive \(\mathbb{Z}\)-gradations and then further decomposed by an outer automorphism of the 0th graded reductive subalgebra. For example, \((\mathfrak{gl}_1 \times \mathfrak{g}_2, \mathbb{C} \otimes \mathbb{C}^7)\) can be obtained by first considering the \(\mathbb{Z}\)-gradation associated to the augmentation of Dynkin diagrams \((D_5, D_4)\) and then further decompose the gradation into irreducible representations of the fixed point subalgebra \(\mathfrak{g}_2\) of \(\mathfrak{so}_8\) by an outer automorphism induced from the triality of the Dynkin diagram \(D_4\). Other cases can be done similarly by a suitable reduction of their Dynkin diagrams to the one possessing a nontrivial outer automorphism and then decompose the gradation by the fixed point subalgebra obtained from the corresponding outer automorphism. The advantage of doing this is that the irreducible subrepresentations contained in any nonzero components of the \(\mathbb{Z}\)-gradations still lie in the nilpotent cone of the original ambient semisimple Lie algebra, so that our previous arguments in Section 4 are still valid in these cases according to Remark 4 after the proof of Theorem 4.1. Collectively speaking, the six cases above can be obtained from the twisted affine diagrams

\[
\begin{align*}
E_6^{(3)} &= \bullet \quad \circ \quad \circ \quad \circ \\
E_6^{(2)} &= \circ \quad \bullet \quad \circ \quad \circ \quad \circ \quad \circ \\
E_7^{(2)} &= \bullet \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ 
\end{align*}
\]

by deleting the painted node. For example, in the case of \((\mathfrak{gl}_1 \times \mathfrak{g}_2, \mathbb{C} \otimes \mathbb{C}^7)\), the naive way to associate the twisted affine diagram is the construct an
augmentation of the Dynkin diagram $G_2$ by adjoining the lowest weight of $\mathbb{C} \otimes \mathbb{C}^7$ to the corresponding system of simple roots. It also works for the cases $(GL_2 \times Spin_7, \mathbb{C}^2 \otimes S)$ and $(GL_1 \times Spin_9, \mathbb{C} \otimes S)$. In fact, these pieces of information give rise to $\mathbb{Z}_m$-gradations instead of $\mathbb{Z}$-gradations since they can be treated as the fixed point algebra of appropriate outer automorphisms of a regular subalgebra of the ambient Lie algebra, and then the corresponding branching of the adjoint representations yields the above decompositions. These $\mathbb{Z}_m$-gradations possess an extra symmetry between the graded pieces which allow us to extend the symmetry group to $GL_2 \times G_2, GL_3 \times Spin_7$ and $GL_1 \times Spin_{11}$ respectively in the remaining three cases. From this point of view, it is reasonable to call them the prehomogeneous vector spaces of twisted affine type.

9 Orbit Structure of $(GL_2 \times SL_{2m+1}, \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m+1})$

In this section, we will examine the orbit structure of the exceptional series $(GL_2 \times SL_{2m+1}, \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m+1}), m \geq 4$, of irreducible reduced PVS's consisting of an infinite number of orbits. Basically, the reason of having an infinite number of orbits is due to the absence of an open orbit in $(GL_2 \times SL_{2m}, \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m})$. At the same time, the construction given below also explains why it is not the case when $m \leq 3$.

First, we decompose $\mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m+1}$ into two two parts:

$$\mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m+1} = \{(1, 0) \otimes \omega_1 + (0, 1) \otimes \omega_2 | \omega_1, \omega_2 \in \Lambda^2 \mathbb{C}^{2m+1}\} = U_1 \cup U_2$$

where

$$U_1 = \{(1, 0) \otimes \omega_1 + (0, 1) \otimes \omega_2 | \omega_1, \omega_2 \in \Lambda^2 V \text{ for some } 2m \cdot \dim \ell V \subset \mathbb{C}^{2m+1}\},$$

$$U_2 = \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m+1} - U_1.$$

Note that for any $(A, g) \in GL_2 \times SL_{2m+1}, A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), (1, 0) \otimes \omega_1 + (0, 1) \otimes \omega_2 \in \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m+1}$, we have

$$(A, g) \cdot [(1, 0) \otimes \omega_1 + (0, 1) \otimes \omega_2] = (a, c) \otimes g\omega_1 + (b, d) \otimes g\omega_2 = (1, 0) (ag\omega_1 + bg\omega_2) + (0, 1) (cg\omega_1 + dg\omega_2).$$

So if both $\omega_1, \omega_2 \in \Lambda^2 V$ for some $V \subset \mathbb{C}^{2m+1}$, $g\omega_1, g\omega_2 \in gV$ and $(1, 0) \otimes (ag\omega_1 + bg\omega_2) + (0, 1) \otimes (cg\omega_1 + dg\omega_2) \in U_1$. It follows that $U_1$ and $U_2$ are $GL_2 \times SL_{2m+1}$-invariant subsets.

We fix the standard $\mathbb{C}^{2m}$ as generated by the first $2m$ coordinate vectors $e^1, \ldots, e^{2m}$ of $\mathbb{C}^{2m+1}$. We see that any $GL_2 \times SL_{2m+1}$-orbit in $U_1$ intersects $\Lambda^2 \mathbb{C}^{2m}$ nontrivially as a $GL_2 \times SL_{2m}$-orbit in $\Lambda^2 \mathbb{C}^{2m}$. In other words, we have an one-to-one correspondence between the $GL_2 \times SL_{2m+1}$-orbits in $U_1$ and the $GL_2 \times SL_{2m}$-orbits in $\Lambda^2 \mathbb{C}^{2m}$.
To see that $\Lambda^2 \mathbb{C}^{2m}$ has infinitely many $GL_2 \times SL_{2m}$-orbits, we attach to each $(1, 0) \otimes \omega_1 + (0, 1) \otimes \omega_2$ a two parameter family of top exterior forms
\[(\lambda \omega_1 + \mu \omega_2)^m = f_{(\omega_1, \omega_2)}(\lambda, \mu) e^1 \wedge \cdots \wedge e^{2m},\]
where $f_{(\omega_1, \omega_2)}$ is a homogeneous polynomial of degree $m$ in $\lambda, \mu$. It is easy to check that such polynomials satisfies
\[f_{A(\omega_1, \omega_2)}(\lambda, \mu) = f_{(\omega_1, \omega_2)}((\lambda, \mu)A)\]
for any $(A, g) \in GL_2 \times SL_{2m}$. Thus we obtain a map
\[
\Phi : \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m} / GL_2 \times SL_{2m} \rightarrow S^m \mathbb{C}^2 / GL_2, \tag{9.8}
\]
where $S^m \mathbb{C}^2$ is identified with the space of homogeneous polynomials of degree $m$.

**Proposition 9.1.** The map $\Phi$ defined in (9.8) is surjective.

**Proof.** Pick any nonzero homogeneous polynomial $f(\lambda, \mu)$ of degree $m$, there exists $p_1 = [\lambda_1 : \mu_1], \ldots, p_m = [\lambda_m : \mu_m] \in \mathbb{C}P^1$ unique up to reordering such that
\[f(\lambda, \mu) = \prod_{i=1}^m (\mu_i \lambda - \lambda_i \mu)\]
for suitable representatives of homogeneous coordinates $\lambda_i, \mu_i$. Then by setting $\omega_1 = \sqrt{\frac{1}{m!}}(\mu_1 e^1 \wedge e^2 + \cdots + \mu_m e^{2m-1} \wedge e^{2m})$ and $\omega_2 = -\sqrt{\frac{1}{m!}}(\lambda_1 e^1 \wedge e^2 + \cdots + \lambda_m e^{2m-1} \wedge e^{2m})$, we have
\[(\lambda \omega_1 + \mu \omega_2)^m = \frac{1}{m!}[(\mu_1 \lambda - \lambda_1 \mu)e^1 \wedge e^2 + \cdots + (\mu_m \lambda - \lambda_m \mu)e^{2m-1} \wedge e^{2m}]^m
= \prod_{i=1}^m (\mu_i \lambda - \lambda_i \mu)e^1 \wedge \cdots \wedge e^{2m}
= f(\lambda, \mu)e^1 \wedge \cdots \wedge e^{2m}.
\]
Thus $f_{(\omega_1, \omega_2)} = f$ and $\Phi$ is surjective. \[\square\]

**Corollary 9.2.** $U_1$ consists of infinitely many orbits for $m \geq 4$.

**Proof.** Note that
\[S^m \mathbb{C}^2 / GL_2 = (S^m \mathbb{C}^2 - \{0\}) / GL_2 = \overline{\mathcal{M}_{0,m}},\]
where $\mathcal{M}_{0,m}$ is the moduli space of $m$-points in $\mathbb{P}^1$, which is infinite iff $m \geq 4$ according to the fact that $PGL_1$ acts 3-transitively on $\mathbb{P}^1$. By Proposition 9.1, $\mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m} / GL_2 \times SL_{2m}$ is infinite as $\Phi$ is surjective. The result then follows from the one-to-one correspondence between $U_1 / GL_2 \times SL_{2m+1}$ and $\mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^{2m} / GL_2 \times SL_{2m}$ established above. \[\square\]
In particular, it forces that the open orbit of $C^2 \otimes \Lambda^2 C^{2m+1}$ lies in $U_2$, and with a little bit more effort, we see that $U_2$ actually forms a single orbit. The reason is that under the action of $SL_2$ every element $(1, 0) \otimes \omega_1 + (0, 1) \otimes \omega_2 \in U_2$ can be conjugated so that $\omega_1, \omega_2$ are of rank $m$, and that all those full rank elements are inside the same $GL_2 \times SL_{2m+1}$-orbit.

In fact, outside of $\Phi^{-1}(0)$,

$$\Phi : (C^2 \otimes \Lambda^2 C^{2m} - \Phi^{-1}(0)) / GL_2 \times SL_{2m} \rightarrow (S^m C^2 - \{0\}) / GL_2$$

is a $m : 1$ branched cover of projective varieties. In particular, when $m \leq 3$, there are only finite number of orbits upstairs outside the central fibre $\Phi^{-1}(0)$, while $\Phi^{-1}(0)$ can be identified with $(C^2 \otimes \Lambda^2 C^{2m-1}) / GL_2 \times SL_{2m-1}$. So by backward induction, we see that $C^2 \otimes \Lambda^2 C^{2m-1}$ has a finite number of $GL_2 \times SL_{2m-1}$-orbits for $m \leq 3$. The result is summerized in the following theorem.

**Theorem 9.3.** $(C^2 \otimes \Lambda^2 C^{2m+1}, GL_2 \times SL_{2m+1})$ has an open orbit for all $m \geq 1$, and it consists of finite number of orbits if and only if $m \geq 4$.  

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Appendix

A  $\mathbb{Z}$-Gradations of Semisimple Lie Algebras

This section is devoted to the generalities of $\mathbb{Z}_m$-gradations of semisimple Lie algebras which were encountered when we discussed augmentation of Dynkin diagrams. For the sake of completeness, we have included the proofs of some standard results which can be found in [24].

**Definition A.1.** Let $\mathfrak{g}$ be a Lie algebra and $m \in \mathbb{Z}_{\geq 0}$. A $\mathbb{Z}_m$-gradation of $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$$

of $\mathfrak{g}$ into vector subspaces $\mathfrak{g}_i$ $(i \in \mathbb{Z}_m)$ satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}_m$.

Given a $\mathbb{Z}_m$-gradation $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ of $\mathfrak{g}$, we see that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and that every $\mathfrak{g}_k$ $(k \neq 0)$ is a $\mathfrak{g}_0$ representation through the adjoint action. Especially, when $\mathfrak{g}$ is complex semisimple and $m = 0$ (i.e. $\mathbb{Z}_m = \mathbb{Z}$), there is a more detailed description about the $\mathbb{Z}$-gradation.

**Proposition A.1.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a $\mathbb{Z}$-gradation $\bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, and let $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ denote the Killing form of $\mathfrak{g}$. Then

(i) $\kappa(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ whenever $i + j \neq 0$.

(ii) $\kappa|_{\mathfrak{g}_i \times \mathfrak{g}_{-i}}$ is nondegenerate for all $i \in \mathbb{Z}$; in particular, it implies that $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$.

**Proof.**

(i) Pick any $X \in \mathfrak{g}_i, Y \in \mathfrak{g}_j$, then for every $k \in \mathbb{Z}$

$$(\text{ad}_X \circ \text{ad}_Y)^m(\mathfrak{g}_k) \subset \mathfrak{g}_{k+m(i+j)}.$$

Since $i + j \neq 0$, for sufficiently large $m$, $\mathfrak{g}_{k+m(i+j)} = 0$. Hence $\text{ad}_X \circ \text{ad}_Y$ is nilpotent and $\kappa(X, Y) = \text{Tr}(\text{ad}_X \circ \text{ad}_Y) = 0$.

(ii) For every nonzero $X \in \mathfrak{g}_i$, there exists an $Y \in \mathfrak{g}$ such that $\kappa(X, Y) \neq 0$ as $\kappa$ is nondegenerate on $\mathfrak{g}$. Now let $Y_{-j}$ be the component of $Y$ in $\mathfrak{g}_j$ for $j \in \mathbb{Z}$. Then in view of (i),

$$\kappa(X, Y) = \sum_{j \in \mathbb{Z}} \kappa(X, Y_j) = \kappa(X, Y_{-i}) \neq 0.$$

Thus $\kappa|_{\mathfrak{g}_i \times \mathfrak{g}_{-i}}$ is nondegenerate.

\[\square\]
B Basic Facts about Algebraic Groups

Let $V$ be a complex $G$-variety, i.e. a complex algebraic variety with a continuous group homomorphism $\pi$ from $G$ to the group of biregular morphisms $\mathbb{C}[V]^*$ on $V$. For each $x \in V$, we can form the orbit $G \cdot x$ and consider the orbit map

$$\pi_x : G \to G \cdot x.$$

**Proposition B.1.** Let $G, V, \pi, \pi_x$ be defined as above.

(i) For each $x \in V$, the orbit closure $\overline{G \cdot x}$ is a subvariety of $V$. Moreover, if $V$ is affine, then so is $G \cdot x$.

(ii) $G \cdot x$ is open in $G \cdot x$. In particular, there is a natural structure of smooth algebraic variety on $G \cdot x$.

(iii) The orbit map $\pi_x$ is a surjective morphism of varieties.

Note that the identity component $G^0$ of $G$ is connected, which is equivalent to $G^0$ being irreducible. We have, for every $x \in V$, $G^0 \cdot x = \pi_x(G^0)$ is irreducible, as $\pi_x$ is a surjective morphism. In general, we can write $G \cdot x$ as a finite union of $G^0$-orbits in $G$; these $G^0$-orbits are both connected and irreducible components of $G \cdot x$.

**Proposition B.2.** Let $G_x := \{ g \in G | g \cdot x = x \}$ denote the stabilizer (also called isotropy subgroup) of $x \in V$. Then $G_x$ is a closed subgroup of $G$ and $\pi_x$ induces an isomorphism

$$\pi_x : G/G_x \to G \cdot x.$$

In particular, we have

$$\dim G \cdot x = \dim G - \dim G_x.$$

**Corollary B.3.** For every $x \in V$, $G \cdot x$ is a smooth equidimensional algebraic variety. More precisely, all irreducible components of $G \cdot x$ are smooth subvarieties having the same dimension $\dim G - \dim G \cdot x$.

Note that $\overline{G \cdot x}$ is clearly stable under the action of $G$, hence it is a union of $G$-orbits. In particular, this enables us to define a partial ordering on the set of $G$-orbits.

**Definition B.1.** For any pair of elements $x, y \in V$, we say that $G \cdot y$ is less than $G \cdot x$, denoted by $G \cdot y \prec G \cdot x$, if $G \cdot y \subseteq \overline{G \cdot x}$. This yields partial ordering, called the closure ordering, on the set $G \backslash V$ of $G$-orbits in $V$.

Now let $V$ be defined over $\mathbb{R}$, so that the set of $\mathbb{R}$-rational points $V_\mathbb{R}$ is a variety over $\mathbb{R}$. We have the following fundamental result of Whitney [25]:

**Theorem B.4.** Let $V$ be a complex algebraic variety defined over $\mathbb{R}$. Then the set of $\mathbb{R}$-rational points $V_\mathbb{R}$ of $V$ decomposes into a finite number of connected components.
Corollary B.5. Let $G$ be a connected complex algebraic group defined over $\mathbb{R}$. Then $G_{\mathbb{R}}$ has a finite number of connected components.

Theorem B.6. Let $G$ be a complex reductive algebraic group defined over $\mathbb{R}$ and $V$ be a representation of $G$ with finite number of $G$-orbits whose restriction to $G_{\mathbb{R}}$ is of real type. Then the real representation $V_{\mathbb{R}}$ of $G_{\mathbb{R}}$ has a finite number of $G_{\mathbb{R}}$-orbits. In particular, $V_{\mathbb{R}}$ has an open $G_{\mathbb{R}}$-orbit.

Proof. Note that every $G$-orbit in $V$ is also stable under $G_{\mathbb{R}}$. So it suffices to show that every $G$-orbit intersects $V_{\mathbb{R}}$ with a finite number of $G_{\mathbb{R}}$-orbits. Now fix any $G$-orbit $O$ in $V$. Then for every $v \in O \cap V_{\mathbb{R}}$,

$$T_v(O \cap V_{\mathbb{R}}) \subset g \cdot v \cap V_{\mathbb{R}} = g_{\mathbb{R}} = T_v(G_{\mathbb{R}} \cdot v).$$

On the other hand, we have $O \cap V_{\mathbb{R}} \supset G_{\mathbb{R}} \cdot v$. Thus $T_v(O \cap V_{\mathbb{R}}) = T_v(G_{\mathbb{R}} \cdot v)$.

It follows that $G_{\mathbb{R}} \cdot v$ is open in $O \cap V_{\mathbb{R}}$ and is also closed in $O \cap V_{\mathbb{R}}$ since its complement is union of such $G_{\mathbb{R}}$-orbits. Hence we conclude that $G_{\mathbb{R}} \cdot v$ is a union of connected components of $O \cap V_{\mathbb{R}}$. Now by Theorem B.4, number of connected components of $O \cap V_{\mathbb{R}}$ must be finite. As the number of connected components of $O \cap V_{\mathbb{R}}$ must exceed the number of $G_{\mathbb{R}}$-orbits in $O \cap V_{\mathbb{R}}$, there can only have finite number of $G_{\mathbb{R}}$-orbits in $O \cap V_{\mathbb{R}}$.  \qed
### C Tables

Table 3: Table for irreducible PVS of parabolic type.

| $G$ | $\alpha_0$ | $(G_0, V)$ | $\nu(g, V)$ |
|-----|------------|-------------|-------------|
| $A_n$ | $1$ | $(GL_n, \mathbb{C}^n)$ | $(1)$ |
|       | $k$ | $(GL_k \times SL_{n-k}, \mathbb{C}^k \otimes (\mathbb{C}^{n-k})^*)$ | $2 \leq k \leq \frac{n-2}{2}$ | $(1,1)$ |
| $B_n$ | $1$ | $(GL_1 \times SO_{2n-1}, \mathbb{C} \otimes \mathbb{C}^{2n-1})$ | $(1)$ |
|       | $k$ | $(GL_k \times SO_{2n-2k+1}, \mathbb{C}^k \otimes \mathbb{C}^{2n-2k+1})$ | $2 \leq k \leq n-1$ | $(1,1)$ |
|       | $n$ | $(GL_{n-1} \times SL_2, \mathbb{C}^{n-1} \otimes \mathbb{C}^2)$ | $(1,2)$ |
|       | | $(GL_n, S^2\mathbb{C}^n)$ | $(1)$ |
| $C_n$ | $1$ | $(GL_1 \times Sp_{n-1}, \mathbb{C} \otimes \mathbb{C}^{2n-2})$ | $(1)$ |
|       | $k$ | $(GL_k \times Sp_{n-k}, \mathbb{C}^k \otimes \mathbb{C}^{2n-2k})$ | $2 \leq k \leq n-2$ | $(1,1)$ |
|       | $n-1$ | $(GL_{n-1} \times SL_2, \mathbb{C}^{n-2} \otimes \mathbb{C}^2)$ | $(1,2)$ |
|       | $n$ | $(GL_n, \Lambda^2\mathbb{C}^n)$ | $(1)$ |
| $D_n$ | $1$ | $(GL_1 \times SO_{2n-2}, \mathbb{C} \otimes \mathbb{C}^{2n-2})$ | $(1)$ |
|       | $k$ | $(GL_k \times SO_{2n-2k}, \mathbb{C}^k \otimes \mathbb{C}^{2n-2k})$ | $2 \leq k \leq n-2$ | $(1,1)$ |
|       | $n$ | $(GL_n, \Lambda^2\mathbb{C}^n)$ | $(1)$ |
| $E_6$ | \(2\) 1 \(\circ\) 3 \(\circ\) 4 \(\circ\) 5 \(\circ\) 6 \(\circ\) 2 \(\circ\) 1 \(\circ\) 3 \(\circ\) 4 \(\circ\) 5 \(\circ\) 6 \(\circ\) 2 \(\circ\) 1 \(\circ\) 3 \(\circ\) 4 \(\circ\) 5 \(\circ\) 6 \(\circ\) 7 | \(\begin{align*}
(\text{GL}_1 \times \text{Spin}_{10}, \mathbb{C} \otimes S^+) \\
(\text{GL}_2 \times \text{SL}_5, \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^5) \\
(\text{GL}_2 \times \text{SL}_3, \mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3) \\
(\text{GL}_2 \times \text{SL}_7, \Lambda^3 \mathbb{C}^6) \\
(\text{GL}_6, \Lambda^3 \mathbb{C}^6)
\end{align*}\) | \(1\) |
| \(\begin{align*}
(\text{GL}_1 \times \text{Spin}_{12}, \mathbb{C} \otimes S^+) \\
(\text{GL}_2 \times \text{SL}_6, \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^6) \\
(\text{GL}_2 \times \text{SL}_3 \times \text{SL}_4, \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4) \\
(\text{GL}_3 \times \text{SL}_5, \mathbb{C}^3 \otimes \Lambda^2 \mathbb{C}^5) \\
(\text{GL}_2 \times \text{Spin}_{10}, \mathbb{C}^2 \otimes S^+) \\
(\text{GL}_1 \times E_6, \mathbb{C} \otimes \mathbb{C}^27) \\
(\text{GL}_7, \Lambda^3 \mathbb{C}^7)
\end{align*}\) | \(1\) |
| \(\begin{align*}
(\text{GL}_1 \times \text{Spin}_{14}, \mathbb{C} \otimes S^+) \\
(\text{GL}_2 \times \text{SL}_7, \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^7) \\
(\text{GL}_2 \times \text{SL}_3 \times \text{SL}_5, \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5) \\
(\text{GL}_4 \times \text{SL}_5, \mathbb{C}^4 \otimes \Lambda^2 \mathbb{C}^9) \\
(\text{GL}_3 \times \text{Spin}_{10}, \mathbb{C}^3 \otimes S^+) \\
(\text{GL}_2 \times E_6, \mathbb{C}^2 \otimes \mathbb{C}^27) \\
(\text{GL}_1 \times E_7, \mathbb{C} \otimes \mathbb{C}^56) \\
(\text{GL}_8, \Lambda^3 \mathbb{C}^8)
\end{align*}\) | \(1\) |
| \(\begin{align*}
(\text{GL}_1 \times \text{Spin}_7, \mathbb{C} \otimes S) \\
(\text{GL}_2 \times \text{SL}_3, \mathbb{C}^2 \otimes \mathbb{C}^3) \\
(\text{GL}_2 \times \text{SL}_3, \mathbb{C}^2 \otimes S^2 \mathbb{C}^3) \\
(\text{GL}_1 \times \text{Sp}_3, \mathbb{C} \otimes \Lambda_3 \mathbb{C}^6)
\end{align*}\) | \(1\) |
| \(\begin{align*}
(\text{GL}_2, \mathbb{C}^2) \\
(\text{GL}_2, \mathbb{C}^2)
\end{align*}\) | \(3\) |
Table 4: Table for twisted affine type.

| \( g \) | \((G_0, V)\) |
|--------|----------------|
| \( \mathfrak{so}_{10} \) | \((GL_1 \times G_2, \mathbb{C} \otimes \mathbb{C}^7)\) |
| \( E_6 \) | \((GL_2 \times G_2, \mathbb{C}^2 \otimes \mathbb{C}^7)\) |
| \( \mathfrak{so}_{12} \) | \((GL_2 \times \text{Spin}_7, \mathbb{C}^2 \otimes S)\) |
| \( E_7 \) | \((GL_3 \times \text{Spin}_7, \mathbb{C}^3 \otimes S)\) |
| \( E_6 \) | \((GL_1 \times \text{Spin}_9, \mathbb{C} \otimes S)\) |
| \( E_7 \) | \((GL_1 \times \text{Spin}_{11}, \mathbb{C} \otimes S)\) |

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