ON SCATTERING BY A CYLINDRICAL TRAP IN CRITICAL CASE

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ABSTRACT. We consider a two-dimensional analogue of Helmholtz resonator with walls of finite thickness in the critical case when there exists an eigenfrequency equaling to the limit of poles generated by both the bounded component of the resonator and by the narrow connecting channel. Under assumption that the limit eigenfrequency is simple one of the bounded component, asymptotics of two poles converging to this eigenfrequency are constructed by using the method of matching asymptotic expansions. The explicit formulas for the leading terms of asymptotics for poles and for the solution of the scattering problem are obtained.

We study the case where the cross-section of the cylindrical scattering object is asymptotically homeomorphic to ring. Perturbed domain is formed by both the components of the limiting cross-section exterior (the bounded domain \( \Omega^\text{in} \) and unbounded domain \( \Omega^\text{ex} \)) and the narrow channel \( \kappa_\varepsilon \) connecting them and having the ”diameter” of order \( \varepsilon \ll 1 \). The corresponding mathematical model (both for perturbed and limits problem) is described by the Neumann boundary value problem for the Helmholtz equation. It is also known ([1]) that the analytic continuation of the perturbed solution in this case (in contrast to the case of the Dirichlet boundary condition ([2])) has two series of poles with small imaginary parts. The limiting set for the first series is a set \( \Sigma^\text{in} \) of eigenfrequencies (square roots of the eigenvalues) of the Neumann boundary value problem for \( -\Delta \) in \( \Omega^\text{in} \) (the limit internal problem). The limiting set for the second series is \( \Sigma^\text{ch} = \{m\pi/h\}_{m=1}^\infty \), \( h \) is the length of the connecting channel. Brown, Hislop and Martinez [3] considered two situations assuming that a limiting frequency \( k_0 \) is a simple eigenfrequency belonging to \( \Sigma^\text{in} \) and separated from \( \Sigma^\text{ch} \) or, on the contrary, \( k_0 \in \Sigma^\text{ch} \setminus \Sigma^\text{in} \). For these situations they showed that the analytic continuation of the Green function for the perturbed problem has the only simple pole converging to \( k_0 \) as \( \varepsilon \to 0 \) and there is only one generalized eigenfunction associated with this pole. In [4] the explicit formulae for the leading terms of both these characteristics and peaks for solution to the scattering problem for real frequencies \( k \) close to \( k_0 \) were obtained employing the method of matched asymptotic expansions [5–7]. These formulae implied that for both cases the solution of the perturbed problem differ from one of the limiting problem at a quantity \( O(1) \) in the resonator exterior (i.e. outside bounded component and connecting channel). Inside the bounded component (trap) the behavior of the perturbed solution was really different for these cases, for frequencies close to \( k_0 \in \Sigma^\text{in} \setminus \Sigma^\text{ch} \) it was of order \( O(\varepsilon^{-1}) \), while for frequencies close to \( \Sigma^\text{ch} \setminus \Sigma^\text{in} \) it was bounded.

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In this paper we analyze a critical case assuming that the limiting frequency $k_0$ is a simple eigenfrequency of interior limiting problem and, at the same time, belong to $\Sigma^{ch}$. The results of this work were announced in [8].

§1. DESCRIPTION OF THE PROBLEM, PRELIMINARY NOTES AND FORMULATION OF THE RESULTS

Let $\Omega^{in}$ and $\Omega$ be simply connected bounded domains in $\mathbb{R}^2$, $\overline{\Omega^{in}} \subset \Omega$, $\Omega^{ex} = \mathbb{R}^2 \backslash \overline{\Omega}$, $\partial \Omega^{in(ex)} \in C^\infty$, $x = (x_1, x_2)$. The domains $\Omega^{in}$ and $\Omega^{ex}$ are supposed to coincide with the half-plane $x_2 > 0$ in the neighborhood of the origin and the half-plane $x_2 < -h$ in the neighborhood of a point $x^0 = (0, -h)$, respectively. We postulate that the interval $(-h, 0)$ lying on the axis $Ox_2$ not to contain the points from $\Omega^{in} \cup \Omega^{ex}$. The domains $\Omega^{in}$ and $\Omega^{ex}$ are the interior and exterior of the resonator

$$\Omega_\varepsilon = \Omega^{in} \cup \Omega^{ex} \cup \chi_\varepsilon,$$

respectively, where $\chi_\varepsilon = (\varepsilon \omega_-, \varepsilon \omega_+) \times [-h, 0]$ is the connecting channel, $\omega_- < \omega_+$ are arbitrary constants.

It is known that the scattering of both the $H$-polarized fields on an ideal conductive cylinder with cross-section $\Omega_\varepsilon$, and the plane acoustic waves on an ideal rigid cylinder with cross-section $\Omega_\varepsilon$ are described by the solution of a boundary value problem

$$(\Delta + k^2)u_\varepsilon = F, \quad x \in \Omega_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0, \quad x \in \partial \Omega_\varepsilon, \quad (1.1)$$

$$u_\varepsilon = O \left( \frac{1}{r^{1/2}} \right), \quad \frac{\partial u_\varepsilon}{\partial r} - ik u_\varepsilon = o \left( \frac{1}{r^{1/2}} \right), \quad r \to \infty, \quad (1.2)$$

where $r = |x|$, $\nu$ is the outward normal, $k > 0$, and $F$ is a square integrable function with finite support in $\Omega^{ex}$. For the acoustic scattering, $u_\varepsilon$ is the potential of the velocity. For $H$-polarization $F$ denotes the third component of the vector $-\text{rot} \mathbf{j}$ in the case where the current vector $\mathbf{j}$ is perpendicular to a generatrix and electromagnetic field is of the form $\mathbf{H}_\varepsilon = \{0, 0, u_\varepsilon\}$, $\mathbf{E}_\varepsilon = -ik^{-1}(\text{rot} \mathbf{H}_\varepsilon - \mathbf{j})$. Throughout in what follows by interior (exterior) limiting problem we mean Neumann value problem for the Helmholtz equation in $\Omega^{in}$ (in $\Omega^{ex}$).

It is known (see, for instance, [9]) that for positive $k$ the boundary value problem (1.1), (1.2) and the exterior limiting problem are unique solvable, and their Green functions admit analytic continuations in a complex plane with a cut along the negative real semi-axis, which, for fixed $\varepsilon$, have discrete sets of poles $\Sigma_\varepsilon$ and $\Sigma^{ex}$, respectively, lying below the real axis. As it was mentioned above, it was proved in [1] that in each small neighborhood of a nonzero element $k_0 \in \Sigma = \Sigma^{in} \cup \Sigma^{ex} \cup \Sigma^{ch}$, there exists pole $\tau_\varepsilon \in \Sigma_\varepsilon$, for $\varepsilon$ enough small, and, visa versa, if a compact $K$ is separated from $\Sigma$, then $K \cap \Sigma_\varepsilon = \emptyset$, for all $\varepsilon$ enough small.

In cases, when $k_0 \in \Sigma^{in} \setminus \Sigma^{ch}$, $\Sigma^{in}$ is the set of simple nonzero eigenfrequencies of the interior limiting problem, and when $k_0 \in \Sigma^{ch} \setminus \Sigma^{in}$, there exists one pole (in each case) converging to $k_0$ ([4]). Since we consider solutions of (1.1), (1.2) for $k > 0$, and the pole $\tau_\varepsilon$ is complex, in both cases the solutions are most perturbed when $k = k(\varepsilon) = \Re \tau_\varepsilon + O(\Im \tau_\varepsilon)$. We will call such positive frequencies as the peak regime. Let $S(t)$ and $S^{ex}(t)$ be the disks of radius $t$ with their centers at the origin.
an at $x^{(0)}$, respectively, $G^{in}(x,y,k)$ ($G^{ex}(x,y,k)$) be the Green's function of the interior (exterior) limiting problem, $u^{ex}(x;k)$ be the solution of the exterior limiting problem, and $\psi$ be the eigenfunction of the interior limiting problem, associated with a simple eigenfrequency $k_0 \in \Sigma_1^0$ and normalized in $L_2(\Omega^{in})$. In [4] it was shown that at the peak regime the leading terms of asymptotics of the perturbed boundary value problem (1.1), (1.2) reads as follows:

$$u_\varepsilon(x;k(\varepsilon)) \sim \frac{1}{\varepsilon} A_1 \psi(x), \quad x \in \Omega_1^{in} \setminus S(\varepsilon^{1/2}),$$  \hfill (1.3)

$$u_\varepsilon(x;k(\varepsilon)) \sim A_2 G^{ex}(x,x^{(0)},k_0) + u^{ex}(x;k_0), \quad x \in \left(\Omega_2^{ex} \setminus S^{ex}(\varepsilon^{1/2})\right) \cap S(R)$$  \hfill (1.4)

for $k(\varepsilon) \to k_0 \in \Sigma_1^{in} \setminus \Sigma^{ch}$ and

$$u_\varepsilon(x;k(\varepsilon)) \sim B_1 G^{in}(x,0,k_0), \quad x \in \Omega_1^{in} \setminus S(\varepsilon^{1/2}),$$  \hfill (1.5)

$$u_\varepsilon(x;k(\varepsilon)) \sim B_2 G^{ex}(x,x^{(0)},k_0) + u^{ex}(x;k_0), \quad x \in \left(\Omega_1^{ex} \setminus S^{ex}(\varepsilon^{1/2})\right) \cap S(R)$$  \hfill (1.6)

for $k(\varepsilon) \to k_0 \in \Sigma^{ch} \setminus \Sigma^{in}$, where $A_j$ and $B_j$ are some constants calculated explicitly and $R > 0$ is an arbitrary number.

From (1.4) and (1.6) it follows that at peak regimes in both cases the solutions of perturbed problem differs from the solution of the exterior limiting problem at $O(1)$ in $\Omega^{ex}$. Exactly this difference was observed by Rayleigh for classical Helmholtz resonator (which is a sphere with the small connecting opening) in [10]. We call this effect as the exterior resonance. On the other hand, it follows from (1.3) and (1.5) that for peak frequencies the solutions to perturbed problem distinguish essentially each from other in $\Omega^{in}$. For $k(\varepsilon) \to k_0 \in \Sigma_1^{ch} \setminus \Sigma^{in}$, the solution is bounded and, for $k(\varepsilon) \to k_0 \in \Sigma_1^{in} \setminus \Sigma^{ch}$, it increases as $O(\varepsilon^{-1})$. We call the effect (1.3) as the interior resonance. Note, for the three-dimensional resonator with walls of finite thickness, in [11] the resonance is mean exactly in this sense, and the asymptotics of solutions, for $k(\varepsilon) \to k_0 \in \Sigma_1^{ch} \setminus \Sigma^{in}$ and $k(\varepsilon) \to k_0 \in \Sigma_1^{in} \setminus \Sigma^{ch}$, were constructed in [12], [13]. Since the difference between (1.3) and (1.5) is great enough, it is interesting to consider the case $k_0 \in \Sigma_1^{in} \cap \Sigma^{ch}$. Note that this critical case suggested to be analyzed is not very degenerated, because, for the fixed "main" trap $\Omega^{in}$, such situation can be easily achieved by a corresponding change for the length of the connecting channel (see definition of $\Sigma^{ch}$).

Hereafter we will employ the notations: $\omega$ is an interval $(\omega_-, \omega_+)$, $|\omega| = \omega_+ - \omega_-$, $x^* = (x_1, -x_2)$,

$$\sigma = \lim_{R \to \infty} \int_{\partial S(R)} |G^{ex}(x,x^{(0)},k_0)|^2 ds.$$  

The main goal of this work is to prove the following statement.

**Theorem 1.1.** Let $k_0 = \frac{m\pi}{h} \in \Sigma_1^{in} \cap \Sigma^{ch}$. Then

a) there exist two poles $\tau_\varepsilon^{(n)} \in \Sigma_\varepsilon$ ($n = 1, 2$), converging to $k_0$ and having asymptotics:

$$\tau_\varepsilon^{(n)} = k_0 + \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \varepsilon^{i/2} \ln^j \varepsilon^{(n)},$$  \hfill (1.7)
\[
\tau^{(n)}_{1,0} = (-1)^n \psi(0) \left( \left| \omega \right| \frac{1}{2h} \right)^{1/2}, \quad \tau^{(n)}_{2,1} = (-1)^n \frac{4k_0}{\pi \psi(0)} \left( \left| \omega \right| \frac{1}{2h} \right)^{1/2}, \quad (1.8)
\]

\[
\text{Im} \tau^{(n)}_{2,0} = -\frac{1}{2} \left| \omega \right| k_0^2 \sigma;
\]

b) for \( k \) close to \( k_0 \) the solution of (1.1), (1.2) and its analytic continuation can be represented as:
\[
u_\varepsilon(x; k) = -\sum_{n=1}^2 \frac{\Psi^{(n)}_\varepsilon(x)}{\left( \tau^{(n)}_\varepsilon \right)^2} - k^2 \int_{\mathbb{R}^2} \Psi^{(n)}_\varepsilon(y) F(y) dy + \tilde{u}_\varepsilon(x; k) \quad (1.9)
\]

where \( \tilde{u}_\varepsilon \) is a holomorphic on \( k \) function. If \( \text{supp} F \subset \Omega^{ex} \), then in \( \Omega^{ex} \) \( \tilde{u}_\varepsilon \) converges to the solution of the exterior limiting problem in \( L_2_{,loc}(\Omega^{ex}) \) (i.e., on any compact set from \( \Omega^{ex} \)) and it converge to zero in \( \Omega^{in} \cup \kappa_{\varepsilon} \) in \( L_2 \) norm;

c) for \( \Psi^{(n)}_\varepsilon \) the following asymptotics hold in \( L_2_{,loc}(\mathbb{R}^2) \):
\[
\Psi^{(n)}_\varepsilon(x) = \frac{(-1)^n}{\sqrt{2}} \psi(x) + o(1), \quad x \in \Omega^{in} \setminus S(\varepsilon^{1/2}),
\]
\[
\Psi^{(n)}_\varepsilon(x) = \frac{(-1)^n}{\sqrt{2}} \psi(0)(1 + o(1)), \quad x \in S(2\varepsilon^{1/2}),
\]
\[
\Psi^{(n)}_\varepsilon(x) = \frac{1}{\varepsilon^{1/2}} \left( \frac{1}{h|\omega|} \right)^{1/2} (\sin(k_0 x_2) + o(1)), \quad x \in \kappa_{\varepsilon} \setminus \left( S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2}) \right),
\]
\[
\Psi^{(n)}_\varepsilon(x) = \varepsilon^{1/2} \left( \frac{4k_0}{\pi |\omega|} \right)^{1/2} \left( \ln \varepsilon + \frac{\pi}{|\omega|} X \left( \frac{|x-x(0)|}{\varepsilon} \right) + o(1) \right),
\]
\[
\Psi^{(n)}_\varepsilon(x) = \varepsilon^{1/2} \left( \frac{-1}{h|\omega|} \right)^{1/2} G^{ex}(x, x(0), k_0) + o(1), \quad x \in \Omega^{ex} \setminus S^{ex}(\varepsilon^{1/2})
\]

where \( X \) is the function defined in Lemma 2.2.

I arises from (1.7) and (1.8) that in the case considered, the two peak regimes of the solution of the boundary value problem (1.1), (1.2) "having the same rights" reads as follows:
\[
k = k(\varepsilon) = k_0 + \varepsilon^{1/2} \tau^{(n)}_{1,0} + \varepsilon \ln \varepsilon \tau^{(n)}_{2,1} + \varepsilon^2 (t + o(1)) \quad (1.10)
\]

where \( t \) is any real number. Substituting (1.10) and the asymptotics from item c) of the theorem into (1.9), we obtain that, for such \( k \) the solution of the scattering problem obeys asymptotics
\[
u_\varepsilon(x; k) \sim \frac{1}{\varepsilon^{1/2}} c_F^{(n)}(t) \psi(x), \quad x \in \Omega^{in} \setminus S(\varepsilon^{1/2}),
\]
\[
u_\varepsilon(x; k) \sim \frac{1}{\varepsilon^{1/2}} c_F^{(n)}(t) \psi(0), \quad x \in S(2\varepsilon^{1/2}),
\]
\[
u_\varepsilon(x; k) \sim \frac{1}{\varepsilon} (-1)^n c_F^{(n)}(t) \left( \frac{2}{h|\omega|} \right)^{1/2} \sin(k_0 x_2), \quad x \in \kappa_{\varepsilon} \setminus \left( S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2}) \right),
\]
\[ u_\varepsilon(x; k) \sim c_F^{(n)}(t)(-1)^{m+n+1} \frac{k_0}{\pi} \frac{(2|\omega|)}{h} \left( \frac{2|\omega|}{h} \right)^{1/2} \left( \ln \varepsilon + \frac{\pi x}{|\omega|} X \left( \frac{x-x(0)}{\varepsilon} \right) \right), \]

\[ x \in S^{ex}(2\varepsilon^{-1/2}), \]

\[ u_\varepsilon(x; k) \sim c_F^{(n)}(t)(-1)^{m+n}k_0 \left( \frac{2|\omega|}{h} \right)^{1/2} G^{ex}(x, x(0), k) + u^{ex}(x; k), \]

\[ x \in \Omega^{ex}\setminus S^{ex}(\varepsilon^{1/2}), \]

\[ c_F^{(n)}(t) = \frac{(-1)^{m+n+1}}{2(t-\tau_{2,0}^{(n)})} \left( \frac{|\omega|}{2h} \right)^{1/2} u^{ex}(x(0); k_0). \]

These formulas show that an interior resonance takes place in \( \Omega^{in} \) in both cases but it differs from (1.3) at order.

§ 2. CONSTRUCTION OF ASYMPOTICS

Denote

\[ R_t^{in(ex),i,j}(D_y) = \sum_{q=0}^{t} a_{i,j,q}^{in(ex)} \frac{\partial^q}{\partial y_1^q}, \]

\[ \psi_\varepsilon^{in(ex)}(x, k) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \varepsilon^{i/2} \ln^j \varepsilon (k_0^2 - k^2) R_t^{in(ex),i,j}(D_y) G^{in(ex)}(x, x_0^{in(ex)}, k), \]

where \( x_0^{in} \) is the origin, \( x_0^{ex} = x(0), \) \( a_{i,j,q}^{in(ex)} \) are constants, \([N]\) is the integral part of a number \( N. \) The coefficients of \( \psi_\varepsilon^{in(ex)}(x, k) \) are analytical with respect to \( k \) in some complex neighborhood of the point \( k_0, \) satisfy the equation \((\Delta + k^2)U = 0 \) in \( \Omega^{in(ex)} \) and the boundary condition \( \partial U/\partial \nu = 0 \) on \( \partial \Omega^{in(ex)} \setminus \{x_0^{in(ex)}\}. \) For positive \( k, \) the coefficients of series \( \psi_\varepsilon^{ex}(x, k) \) also satisfy the radiation condition (1.2). Therefore, outside the connecting channel and small neighborhoods of its edges, following to [4], we seek the complete asymptotics of the "eigenfunctions" \( \Psi_\varepsilon^{(n)} \) as

\[ \psi_\varepsilon^{(n)}(x) = \psi_\varepsilon^{in(ex)}(x, \tau_\varepsilon^{(n)}), \quad x \in \Omega^{in(ex)}\setminus S^{in(ex)}(\varepsilon^{1/2}), \quad (2.1) \]

where \( S^{in}(t) = S(t), \) and \( R_t^{in(ex),i,j} \) in the definition of \( \psi_\varepsilon^{in(ex)} \) depend on \( n. \)

Remark 2.1. The function \( \Psi_\varepsilon^{(n)} \) in (1.9) is not normalized, by \( \psi_\varepsilon^{(n)} \) in (2.1) we mean the function equalling \( \Psi_\varepsilon^{(n)} \) up to scalar factor \( \alpha_n(\varepsilon). \) In § 3, it will be shown that \( \alpha_n(\varepsilon) = 1 + o(1) \) as \( \varepsilon \to 0. \)

Remark 2.2. The constructions of asymptotics for and for associated generalized eigenfunctions are identical. Because of this and in order not to overload the notations in the text by the index of correspondence to a concrete pole ("n") we will omit this index where it will be possible (including the notations introduced above).

In a small neighborhood of the connecting channel, asymptotics of \( \psi_\varepsilon^{(n)} \) are constructed in the form:

\[ \psi_\varepsilon(x) = \sum_{i=-1}^{\infty} \sum_{j=0}^{i} \varepsilon^{i/2} \ln^j \varepsilon w_{i,j}(x_2), \quad x \in \mathbb{R}_\varepsilon^\ast(S^{ex}(\varepsilon^{1/2}) \cup S^{in}(\varepsilon^{1/2})), \quad (2.2) \]
\[ \psi_{\varepsilon}(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{i} \varepsilon^{i/2} \ln^{j} \varepsilon v_{i,j}^{\text{in}(\varepsilon)} \left( \frac{x^{\text{in}(\varepsilon)}}{\varepsilon} \right), \quad x \in S^{\text{in}(\varepsilon)}(2\varepsilon^{1/2}), \tag{2.3} \]

where \( x^{\text{in}} = x \), \( x^{\text{ex}} = (x - x^{(0)})^{*} \), \( \beta_{\text{in}} = 1 \), \( \beta_{\text{ex}} = 0 \). Let us clarify the form of the leading terms in (2.1) and (2.2). The formal limit of \( \psi_{\varepsilon}(x, k) \) as \( \varepsilon \to 0 \) and \( k \to k_{0} \) implies that

\[ \psi_{\varepsilon}(x, k) \to R_{0}^{\text{in},0,0}(0) \psi(x). \tag{2.4} \]

On the other hand, in [4] it was shown that for \( k_{0} \in \Sigma_{1}^{\text{in}} \setminus \Sigma_{\text{ch}}^{\text{ch}} \), the generalized eigenfunction converges to the eigenfunction \( \psi \) (continued by zero outside \( \Omega^{\text{in}} \)), and for \( k_{0} \in \Sigma_{\text{ch}} \setminus \Sigma_{\text{in}}^{\text{in}} \), ”in principle” it is represented by

\[ \frac{1}{\varepsilon^{1/2}} \left( \frac{2}{|\omega|h} \right)^{1/2} \sin(k_{0}t) \tag{2.5} \]

in \( \kappa_{\varepsilon} \) and by zero outside \( \kappa_{\varepsilon} \). Therefore, for \( k_{0} \in \Sigma_{1}^{\text{in}} \cap \Sigma_{\text{ch}}^{\text{ch}} \), it is naturally to expect that the leading term of the generalized eigenfunction is a ”linear” combination of \( \psi(0) \) and (2.5). By the latter we arrive at the leading terms indicated in (2.1), (2.2), and, in particular an equality

\[ w_{-1,0}(t) = b_{-1,0} \sin(k_{0}t). \tag{2.6} \]

Moreover, \( \psi \) being normalized, and the norm of (2.5) in \( L_{2}(\kappa_{\varepsilon}) \) equaling one, too, by (2.4), (2.6) and by an assumption (that will be justified below) the conserving of the normalization for the leading term of the generalized eigenfunction’s asymptotics, for the case considered in the paper, we obtain the equality

\[ \left( R_{0}^{\text{in},0,0}(0) \psi(0) \right)^{2} + \frac{b_{-1,0}^{2}|\omega|h}{2} = 1. \tag{2.7} \]

The boundary value problems for the coefficients of the series (2.3) are derived by a standard substitution ([4,7]) the series (1.7), (2.3) into (1.1) for \( F = 0 \), and by passing to the ”interior” variable \( \xi = x^{\text{in}(\varepsilon)} \varepsilon^{-1} + 1 \):

\[ \Delta v_{i,j} = -k_{0}^{2} v_{i-2,j} - \sum_{q=1}^{i-4} \sum_{t=0}^{q-1} \lambda_{q,t} v_{i-q-4,j-t}, \quad \xi \in \gamma_{\omega}, \quad \frac{\partial v_{i,j}}{\partial \nu} = 0, \quad \xi \in \partial \gamma_{\omega}, \tag{2.8} \]

where \( \lambda_{q,t} \) denotes the coefficients of the series \( \lambda_{\varepsilon} = \tau_{\varepsilon}^{2} - k_{0}^{2} \) in front of \( \varepsilon^{q/2} \ln^{t} \varepsilon \),

\[ \gamma_{\omega} = (\omega \times (-\infty, 0]) \cup \{ \xi : \xi_{2} > 0 \}, \]

and the upper indexes for \( v_{q,t}^{\text{in}(\varepsilon)} \) are omitted.

Since the coefficients of the series (2.2) depends on \( x_{2} \) only (but considered as functions defined on \( \kappa_{\varepsilon} \)), then substituting the series (1.7) and (2.2) in (1.1), for \( F = 0 \), we obtain the ordinary differential equations for \( w_{j,i}^{*} \):

\[ w_{i,j}^{*}(x_{2}) + k_{0}^{2} w_{i,j}(x_{2}) + \sum_{q=1}^{i+1} \sum_{t=0}^{q-1} \lambda_{q,t} w_{i-q,j-t}(x_{2}) = 0, \quad -h < x_{2} < 0, \]
whose solutions are functions

$$w_{i,j}(x_2) = -\frac{1}{k_0} \sum_{q=1}^{i+1} \sum_{t=0}^{q-1} \lambda_{q,t} \int_{-h}^{x_2} \sin \left( k_0 (x_2 - t) \right) w_{i,q,j-t}(t) \, dt + b_{i,j} \cos(k_0 x_2), \quad j \geq 0,$$

(2.9)

where $b_{i,j}$ are arbitrary constants. Hereafter, the coefficients of the series (2.2) are chosen in accordance with (2.9). One can see, that in this case

$$w_{0,0}(t) = \tau_{1,0} k_0 b_{-1,0} \left( \frac{t \cos(k_0 t) + h \cos(k_0 t) - \frac{1}{k_0} \sin(k_0 t)}{k_0} \right) + b_{0,0} \cos(k_0 t). \quad (2.10)$$

Let $\rho = |\xi|$, $P_j(\xi)$ be homogeneous polynomials of order $j$, and $T_j(\xi)$ be homogeneous functions of order $j$ represented as $P_{j+2q}(\xi) \rho^{-2q}$ for some integer $q \geq 1$ and satisfying the boundary condition $\partial T_j(\xi)/\partial \xi_2 = 0$ as $\xi_2 = 0$, $\xi \neq 0$. Denote by $\tilde{A}_j$ the set of series of the form

$$T(\xi) = \sum_{q=-\infty}^{j} T_q(\xi) + \ln \rho \sum_{n=0}^{j} P_q(\xi).$$

Next, we indicate by $w(\varepsilon(x))$ the series (2.2). Let us define the "re-extension" operator $K_{q}^{\text{in}(\varepsilon)}$ for the summation $U(x, \varepsilon)$ of the form $w(\varepsilon(x))$ and $\psi^{\text{in}(\varepsilon)}(x, \tau_\varepsilon)$, where $\tau_\varepsilon$ is an arbitrary function with the asymptotics (1.7), in the following standard way ([7]). We expand the coefficients of $U(x, \varepsilon)$ in powers of $|x^{\text{in}(\varepsilon)}| \to 0$ and pass to the variables $\xi = x^{\text{in}(\varepsilon)} \varepsilon^{-1}$ (if $U(x, \varepsilon) = \psi^{\text{in}(\varepsilon)}(x, \tau_\varepsilon)$, the function $\tau_\varepsilon$ is replaced by its asymptotics series). In the double series obtained, we take the sum of terms $\varepsilon^j \ln^i \varepsilon \Phi(\xi)$ for $j \leq q$. Exactly this sum is denoted by $K_{q}^{\text{in}(\varepsilon)}(U(x, \varepsilon))$.

We indicate

$$g^{\text{in}}(k) = \lim_{x \to 0} \left( G^{\text{in}}(x, 0, k) + \frac{1}{\pi} \ln r + \frac{\psi^{2}(0)}{k^2 - k_0^2} \right),$$

$$g^{\text{ex}}(k) = \lim_{x \to x_0} \left( G^{\text{ex}}(x, x_0, k) + \frac{1}{\pi} \ln |x - x_0| \right),$$

and by $(\rho, \theta)$ we denote the polar coordinates. The definitions of $\psi^{\text{in}(\varepsilon)}$, $w(\varepsilon)$, $K_{q}^{\text{in}(\varepsilon)}$, the equalities (2.6) and (2.10) and the asymptotics of the Green functions and of their derivatives (see, for instance, [14]) lead as to

**Lemma 2.1.** Let $\tau_\varepsilon$ be an arbitrary function with asymptotics (1.7). Then for any integer $N \geq 0$ the equalities

$$K_{N/2}^{\text{in}(\varepsilon)}(\psi^{\text{in}(\varepsilon)}(x, \tau_\varepsilon)) = \sum_{i=1}^{N} \sum_{j=0}^{\beta_{\text{in}(\varepsilon)}} \varepsilon^{i/2} \ln^{j} \varepsilon V^{\text{in}(\varepsilon)}_{i,j}(\xi),$$

$$K_{N/2}^{\text{in}(\varepsilon)}(w(\varepsilon)) = \sum_{i=0}^{N} \sum_{j=0}^{\beta_{\text{in}(\varepsilon)}} \varepsilon^{i/2} \ln^{j} \varepsilon W^{\text{in}(\varepsilon)}_{i,j}(\xi_2),$$
hold, where \( V_{i,j}^{\text{in(ex)}} \in \tilde{A}_{[(i-j)/2]} \), and \( W_{i,j}^{\text{in(ex)}} \) are polynomials \([(i+j)/2]-\text{th}\) order.

The representation

\[
V_{0,0}^{\text{in(ex)}}(\xi) = R_{0}^{\text{in,0,0},0,0}\psi^{2}(0),
\]

\[
V_{1,0}^{\text{in(ex)}}(\xi) = \beta_{\text{in(ex)}}R_{0}^{\text{in(ex),1,0},0,0}\psi^{2}(0)
\]

\[
+ \frac{2}{\pi}k_{0}\tau_{1,0}\left(R_{0}^{\text{in(ex),0,0}}\left(\ln \rho - \pi g^{\text{in(ex)}}(k_{0})\right) - \Pi_{0,0}\right),
\]

\[
V_{k,k}^{\text{in(ex)}}(\xi) = \frac{2k_{0}}{\pi} \sum_{t=0}^{k-1} R_{0}^{\text{in(ex),t,t}}\tau_{k-t,k-t-1}, \quad k \geq 1,
\]

\[
V_{k,j}^{\text{in(ex)}}(\xi) = \tilde{V}_{k,j}^{\text{in(ex)}}(\xi) + \beta_{\text{in(ex)}}a_{k,j,0}^{\text{in(ex)}}\psi^{2}(0)
\]

\[
+ \frac{2}{\pi}k_{0}\left(\tau_{1,0}\left(a_{k-1,j,0}^{\text{in(ex)}}\left(\ln \rho - \pi g^{\text{in(ex)}}(k_{0})\right) - \Pi_{k-1,j}\right)
\]

\[
+ \tau_{k,j}\left(R_{0}^{\text{in(ex),0,0}}\left(\ln \rho - \pi g^{\text{in(ex)}}(k_{0})\right) - \Pi_{0,0}\right)\right), \quad k > 1, \quad k > j,
\]

\[
\Pi_{i,j} = \sum_{t=1}^{\infty} a_{2t+i,j,t}^{\text{in(ex)}}(t-1)!\cos(t\theta)\frac{\rho^{t}}{t},
\]

\[
\tilde{V}_{2,0}^{\text{in(ex)}}(\xi) = R_{0}^{\text{in,0,0}}\psi(0)\psi_{1}(0)\xi_{1}, \quad \tilde{V}_{2,0}^{\text{ex}}(\xi) = \tilde{V}_{2,1}^{\text{in(ex)}}(\xi) = 0,
\]

are valid, where \( \tilde{V}_{i,j}^{\text{in(ex)}} \) are independent on \( \tau_{q,s}, \Pi_{q-1,s} \) and \( a_{q-1,s,0}^{\text{in(ex)}} \) for \( q \geq i \) and \( s \geq j \).

\[
W_{i,i}^{\text{in(ex)}}(\xi_{2}) = \tau_{i+1,i}b_{-1,0}h + b_{i,i}, \quad W_{i,i}^{\text{ex}}(\xi_{2}) = (-1)^{m}b_{i,i},
\]

\[
W_{k,j}^{\text{in(ex)}}(\xi_{2}) = \tilde{W}_{k,j}^{\text{in(ex)}}(\xi_{2}) + b_{-1,0}\left(\frac{1}{2}\tau_{k,j}\tau_{1,0} + \tau_{k+1,j}\right) + b_{k,j},
\]

\[
W_{k,j}^{\text{ex}}(\xi_{2}) = \tilde{W}_{k,j}^{\text{ex}}(\xi_{2}) + (-1)^{m}b_{k,j}, \quad k > j,
\]

\[
\tilde{W}_{1,0}^{\text{in(ex)}}(\xi_{2}) = b_{-1,0}k_{0}\xi_{2}, \quad \tilde{W}_{1,0}^{\text{ex}}(\xi_{2}) = (-1)^{m+1}b_{-1,0}k_{0}\xi_{2},
\]

\[
\tilde{W}_{k+1,k}^{\text{in(ex)}}(\xi_{2}) = \alpha_{k}^{\text{in(ex)}}, \quad \tilde{W}_{k+1,k}^{\text{ex}}(\xi_{2}) = \alpha_{k}^{\text{ex}}, \quad k \geq 1, \quad \tilde{W}_{2,0}^{\text{in(ex)}}(\xi_{2}) = 0,
\]

where \( \tilde{W}_{i,j}^{\text{in(ex)}} \) are independent on \( \tau_{q,s} \) and \( b_{q,s} \), for \( q \geq i \) and \( s \geq j \), and \( \alpha_{k}^{\text{in(ex)}} \) are some constants.

The series \( V_{i,j}^{\text{in(ex)}} \) (and, hence, the series \( \tilde{V}_{i,j}^{\text{in(ex)}} \in \tilde{A}_{[(i-j)/2]} \)) are formal asymptotic solutions to the boundary problem (2.8) for \( \rho \to \infty, \xi_{2} \geq 0 \), where the functions \( v_{q,t} \) are replaced by \( \tilde{V}_{q,t}^{\text{in(ex)}} \).

If \( W_{0,0}^{\text{ex}} \equiv 0 \), then \( W_{i,j}^{\text{in(ex)}} \) (therefore, and \( \tilde{W}_{i,j}^{\text{in(ex)}} \)) are formal asymptotic solutions to the boundary problem (2.8) for \( \rho \to \infty, \xi_{2} < 0 \), where the functions \( v_{q,t} \) are replaced by \( W_{q,t}^{\text{in(ex)}} \).

In order to match the series (2.1)–(2.3), it is sufficient to show the existence of the solutions to the boundary value problems (2.8), whose asymptotics at infinity are \( V_{i,j}^{\text{in(ex)}} \) for \( \xi_{2} \geq 0 \) and \( W_{i,j}^{\text{in(ex)}} \) for \( \xi_{2} < 0 \). Let \( 0 = \mu_{0} < \mu_{1} \leq \mu_{2} \leq ... \) be the
eigenfrequencies of the Neumann problem for the operator $-d^2/dt^2$ in the interval $\omega$, $\beta_q(t)$ be the associated eigenfunctions normalized in $L_2(\omega)$. We will employ the symbol $\tilde{B}_{q,n}$ for the set of series

$$H(\xi) = R_q^{(0)}(\xi_2) + \sum_{j=1}^{\infty} R_q^{(j)}(\xi_2) \beta_j(\xi_1) \exp\{\mu_j \xi_2\},$$

where $R_q^{(m)}(t)$ are polynomials of order $i$. For negative $n$, $\tilde{B}_{q,n}$ denotes the set of polynomials of order $q$. We set $\tilde{B} = \cup_{n,q} \tilde{B}_{q,n}$. We denote by $A_n$ the set of functions from $C^\infty(\gamma_\omega) \cap W_2^1,loc(\gamma_\omega)$ satisfying the homogeneous Neumann boundary condition on $\partial \gamma_\omega$ and having differentiable asymptotics from $\tilde{A}_m$ and $\tilde{B}$ at infinity for $\xi_2 \geq 0$ and $\xi_2 < 0$, respectively. Matching of the series will be proved by using Lemma 2.1 and the following statement proved in [4].

**Lemma 2.2.** Let $f \in A_N$, and the series $V \in \tilde{A}_{N+2}$ be the formal asymptotic solution of the equation $\Delta V = f$ for $\rho \to \infty$, $\xi_2 \geq 0$, and the polynomials $W(\xi_2)$ satisfy the equation $\Delta W = f + o(1)$ (or the equation $W'' = f + o(1)$, what is the same) for $\rho \to \infty$ and $\xi_2 < 0$.

Then, there exists a function $v \in A_{N+2}$ that is the solution of the boundary value problem

$$\Delta v = f, \quad \xi \in \gamma_\omega, \quad \frac{\partial v}{\partial \nu} = 0, \quad \xi \in \partial \gamma_\omega \quad (2.11)$$

and has the following differentiable asymptotics as $\rho \to \infty$

$$v(\xi) = V(\xi) + 0 \ln \rho + \sum_{j=1}^{\infty} j \frac{\cos(j \theta)}{\rho^j}, \quad \xi_2 \geq 0,$$

$$v(\xi) = W(\xi_2) + q_0 + O(\xi_2^M \exp\{\mu_1 \xi_2\}), \quad \xi_2 < 0,$$

where $M \geq 0$, $q_0$ and $c_i$ are some numbers.

There exist functions $X \in A_0$ and $Y \in A_1$ that is harmonic in $\gamma_\omega$ and have the following differentiable asymptotics at infinity:

$$X(\xi) = \xi_2 + q_\omega + O(\exp\{\mu_1 \xi_2\}), \quad \xi_2 \leq 0,$$

$$X(\xi) = c_\omega \ln \rho + \sum_{j=1}^{\infty} c_j^+ \frac{\cos(j \theta)}{\rho^j}, \quad \xi_2 \geq 0,$$

$$Y(\xi) = \xi_1 + c_\omega^+ \ln \rho + \sum_{j=1}^{\infty} b_j \frac{\cos(j \theta)}{\rho^j}, \quad \xi_2 \geq 0,$$

$$Y(\xi) = q_\omega + O(\exp\{\mu_1 \xi_2\}), \quad \xi_2 < 0.$$

For $\omega_- = -\omega_+$, taking in account evenness, one can see that the constants $q_\omega$ and $c_\omega$ equal zero. Thus, it is clear that, in the general (nonsymmetric) case:

$$c_\omega = 0, \quad q_\omega = \frac{\omega_+ + \omega_-}{2}.$$ 

It is easy to establish the equalities $c_\omega = \pi^{-1}|\omega|$, $q_\omega = \pi^{-1}|\omega| \left(\ln \left(2|\omega|\pi^{-1}\right) - 1\right)$ by using conformal mapping of a strip onto $\gamma_\omega$ (see, for instance, [4,15]).

We denote by $v^{in(ex)}_\varepsilon(x^{in(ex)}/\varepsilon)$ the series (2.3), and by $v^{in(ex)}_{\varepsilon,N}(\xi)$ their partial sum and come to prove the key statement of the present work.
Theorem 2.1. There exist a function \( \tau_\varepsilon \) with asymptotics (1.7) and series (2.1)–(2.3) such that the coefficients \( v_{i,j}^{in(\varepsilon)} \in A_{[(i-j)/2]} \) are the solutions of the recurrent boundary value problems (2.8), the coefficients \( w_{i,j} \) are defined by (2.6), (2.9), and for any integer \( N \geq 0 \) the following differentiable asymptotic equalities hold:

\[
K_{N/2}^{in(\varepsilon)}(w_\varepsilon(x)) = v_{N/2}^{in(\varepsilon)}(\xi) + O(\varepsilon^{N/2} \xi_2^M \exp\{\mu_1 \xi_2\}), \quad \xi_2 < 0, \tag{2.12}
\]

\[
K_{N/2}^{in(\varepsilon)}(\psi_\varepsilon^{in(\varepsilon)}(x, \tau_\varepsilon)) = v_{N/2}^{in(\varepsilon)}(\xi), \quad \xi_2 \geq 0, \quad \rho \to \infty, \tag{2.13}
\]

and for the coefficients of these series the following representations are true:

\[
b_{0,0} = 0, \quad v_{0,0}^{in} = R_0^{in,0,0} \psi^2(0), \quad R_0^{ex,0,0} = (-1)^{m+1} R_0^{in,0,0},
\]

\[
b_{-1,0} = \left(\frac{1}{h|x|}\right)^{1/2}, \quad R_0^{in,0,0} = \pm \frac{1}{\psi(0)} \left(\frac{1}{2}\right)^{1/2}, \quad \tau_{1,0} = \pm \psi(0) \left(\frac{|\omega|}{2h}\right)^{1/2},
\]

\[
v_{1,1}^{ex} = \frac{2k_0 \tau_{1,0}}{\pi} R_0^{ex,0,0}, \quad \tau_{2,1} = \frac{4k_0}{b_{-1,0} h \pi} R_0^{in,0,0}, \tag{2.14}
\]

\[
\tau_{2,0} = \frac{\tau_{1,0}}{\tau_{1,0} b_{-1,0} h + R_0^{in,0,0} \psi^2(0)} \times \left( b_{-1,0} \left(2k_0 q_\omega - \frac{1}{2} h \tau_{2,1}^2\right) - 2k_0 \tau_{1,0} R_0^{in,0,0} \left(g^{in}(k_0) + g^{ex}(k_0)\right) \right)
\]

Proof. Setting \( b_{0,0} = 0 \), due to Lemma 2.1, we achieve the equalities (2.12), (2.13) for the index ”ex” as \( N = 0 \). Similarly, taking \( v_0^{ex} \) as in (2.14) and putting the additional condition

\[
R_0^{in,0,0} \psi^2(0) = \tau_{1,0} h b_{-1,0}, \tag{2.15}
\]

where \( R_0^{in,0,0}, \tau_{1,0} \) and \( b_{-1,0} \) are some constants unknown yet but satisfying (2.7), due to Lemma 2.1 (and also due to value of \( b_{0,0} \) defined above) we get the equalities (2.12), (2.13) for the index ”in” as \( N = 0 \). This was the ”zero step” of the matching procedure.

At the next step, by virtue of definition \( \tilde{W}_{1,0}^{in,ex} \) \( V_{1,0}^{in,ex} \), we put

\[
v_{1,0}^{in} = b_{-1,0} k_0 X + A_{1,0}^{in},
\]

\[
v_{1,0}^{ex} = (-1)^{m+1} b_{-1,0} k_0 X + A_{1,0}^{ex},
\]

where \( A_{1,0}^{in(ex)} \) are some constants unknown yet. It is easy to see that these functions are the solutions of the boundary value problem (2.8). Setting the (power) asymptotics of the functions \( v_{1,0}^{in(\varepsilon)} \) as \( \rho \to \infty \) and \( \xi_2 < 0 \) equal to \( W_{1,0}^{in(\varepsilon)} \), due to Lemma 2.1 we obtain the equations

\[
(-1)^{m+1} b_{-1,0} k_0 q_\omega + A_{1,0}^{ex} = (-1)^{m} b_{1,0},
\]

\[
b_{-1,0} k_0 q_\omega + A_{1,0}^{in} = b_{-1,0} \left(\frac{1}{2} \tau_{1,0}^2 + \tau_{2,0}^2\right) + b_{1,0}, \tag{2.16}
\]

Similarly, setting the asymptotics of the functions \( v_{1,0}^{in(\varepsilon)} \) as \( \rho \to \infty \) and \( \xi_2 > 0 \) equal to the series \( V_{1,0}^{in(\varepsilon)} \) up to the terms \( O(1) \) inclusive, we obtain the equations

\[
b_{-1,0} k_0 c_\omega = \frac{2}{\pi} k_0 \tau_{1,0} R_0^{in,0,0}, \tag{2.17}
\]
\[
(-1)^{m+1}b_{-1,0}k_0c_\omega = \frac{2}{\pi} k_0 \tau_{1,0} R_0^{ex,0,0},
\]

\[
R_0^{in,0,0} \psi^2(0) - 2k_0 \tau_{1,0} R_0^{in,0,0} g^{in}(k_0) = A_1^{in},
\]

\[
-2k_0 \tau_{1,0} R_0^{ex,0,0} g^{ex}(k_0) = A_1^{ex},
\]

where \( R_0^{in,0,0} \) is one more unknown constant. From (2.17) and (2.18) we get the value (2.14) for \( R_0^{ex,0,0} \), while (2.20) determines the constant \( A_1^{ex} \). Solving the system of the equations (2.7), (2.15), (2.17) we get the formulae (2.14) for \( b_{-1,0} \) and \( \tau_{1,0} \). We stress the calculating the latter quantities finally determines the constants \( R_0^{ex,0,0}, A_1^{ex} \),

\[
b_{1,0} = k_0 \left( 2\tau_{1,0} R_0^{in,0,0} g^{ex}(k_0) - b_{-1,0} q_{\omega} \right)
\]

and the function \( v_{1,0}^{ex} \). Also, the function \( v_0^{in} \) is determined up to the additive term \( A_1^{in} \), which satisfies equation (2.17). Moreover, setting the asymptotics of the functions \( v_0^{in} \) as \( \rho \to \infty \) and \( \xi_2 > 0 \) equal to the series \( V_1^{in(ex)} \) for the other degree, we determine \( \Pi_0^{in(ex)} \) (i.e., the coefficients of the higher derivatives for the polynomials \( R_i^{in(ex),2i,0} \)). And, finally, putting

\[
v_1^{in} = \frac{2k_0}{\pi} R_0^{in,0,0} \tau_{1,0},
\]

and defining \( v_{1,1}^{ex} \) in accordance with (2.14), and setting the "asymptotics" of these functions equal to the "series" \( V_1^{in(ex)} \) and to the "polynomials" \( W_{1,1}^{in(ex)} \) as \( \xi_2 > 0 \) and \( \xi_2 < 0 \), respectively, we obtain the equalities (2.12) and (2.13) for \( N = 1 \), and get the value (2.14) for \( \tau_{2,1} \) (simultaneously determining \( b_{1,1} \)). This was the first step of the matching procedure, which in addition to the equalities (2.12), (2.13) as \( N = 1 \) gives two equations (2.16) and (2.19) to three constants \( R_0^{in,1,0}, A_1^{in} \) and \( \tau_{2,0} \) unknown yet.

In the second step, by \( \bar{V}_{2,0}^{in(ex)} \) and \( \bar{W}_{2,0}^{in(ex)} \), we determine the solutions \( v_2^{in(ex)} \) of the boundary value problem (2.8) as

\[
v_2^{in} = R_0^{in,0,0} \bar{\psi}(0) \bar{\psi}_{x}(0) Y + A_2^{in}, \quad v_2^{ex} = A_2^{ex},
\]

where \( A_2^{in(ex)} \) are some constants. Setting the asymptotics of the functions \( v_2^{in(ex)} \) at infinity as \( \xi_2 < 0 \) equal to \( W_2^{in(ex)} \), we obtain the equalities:

\[
A_2^{in} + R_0^{in,0,0} \bar{\psi}(0) \bar{\psi}_{x}(0) q_{-1,0} = \frac{1}{2} \tau_{2,0} \tau_{1,0} + \tau_{3,0} + b_{2,0},
\]

\[
A_2^{ex} = (-1)^m b_{2,0}.
\]

Similarly, setting the asymptotics of the functions \( v_2^{in(ex)} \) at infinity as \( \xi_2 > 0 \) equal to \( \bar{V}_{2,0}^{in(ex)} \) up to terms \( O(1) \), we get the following equalities:

\[
\tau_{1,0} R_0^{in,1,0} + \tau_{2,0} R_0^{in,0,0} = 0,
\]
\[ \tau_{1,0}R_{0}^{ex,1,0} + \tau_{2,0}R_{0}^{ex,0,0} = 0, \quad (2.25) \]

\[ -2k_{0}g^{ex}(k_{0}) \left( \tau_{1,0}R_{0}^{ex,1,0} + \tau_{2,0}R_{0}^{ex,0,0} \right) = A_{2,0}^{ex}, \quad (2.26) \]

\[ a_{2,0,0}^{in}(0) - 2k_{0}g^{in}(k_{0}) \left( \tau_{1,0}R_{0}^{in,1,0} + \tau_{2,0}R_{0}^{in,0,0} \right) = A_{2,0}^{in}. \quad (2.27) \]

From (2.25) and (2.26) we obtain \( A_{2,0}^{ex} \) that, due to (2.23), determines \( b_{2,0} \). Furthermore, solving the system of the equations (2.16), (2.19), (2.24) (bearing in mind the equality (2.21), too), we determine \( \tau_{2,0} \) in accordance with (2.14), and also get the constants \( R_{0}^{in,1,0} \) and \( A_{1,0}^{in} \). The coefficients \( R_{0}^{in,1,0} \) and \( \tau_{2,0} \) having been determined, first, from (2.26) we obtain \( R_{0}^{ex,1,0} \), and, second, setting the asymptotics of the functions \( v_{2,0}^{in(ex)} \) at infinity as \( \xi_{2} > 0 \) equal to \( V_{2,0}^{in(ex)} \) (for the other terms) we define all coefficients of the series \( \Pi_{1,0}^{in(ex)} \) (i.e., the leading coefficients of the differential polynomials \( R_{i}^{in(ex),1+i,0} \) as \( i \geq 1 \)). The equations (2.22), (2.27) are analogue of the equations (2.16) and (2.19) for determining (in the next step) the constants \( A_{2,0}^{in}, \tau_{3,0} \) and \( a_{2,0,0}^{in} \).

The subsequent proof is carried out by induction. Before beginning the \((N,k)\)-th step, where \( k \) is a degree of \( \ln \varepsilon \), all \( \Pi_{q-2,s}^{in(ex)}, \tilde{W}_{q,s}^{in(ex)}, \tilde{V}_{q,s}^{in(ex)} \), \( a_{q-1,s,0}^{in(ex)}, \tau_{q-1,s,0}, v_{q-2,s}^{ex}, v_{q-1,s}^{ex} \) as \( q \leq N \) and \( s \leq k \) have been determined, and the functions \( v_{N-1,s}^{ex} \) are determined up to additive terms \( A_{N-1,s}^{in(ex)} \) meeting the following equations (analogue of (2.22) and (2.27)):

\[ A_{N-1,s}^{in} + B_{N-1,s}^{in} = b_{-1,0}h\tau_{N,s}, \quad (2.28) \]

\[ a_{N-1,s,0}^{in}(0) + C_{N-1,s}^{in} = A_{N-1,s}^{in}, \quad s \leq N - 1, \]

where \( B_{N-1,s}^{in} \) and \( C_{N-1,s}^{in} \) are some completely defined numbers. In the \((N,s)\)-th step, where \( s < N \), by \( \tilde{W}_{N,s}^{ex}, \tilde{V}_{N,s}^{ex} \), we determine \( v_{N,s}^{ex} \) as

\[ v_{N,s}^{ex} = \tilde{v}_{N,s}^{ex} + A_{N,s}^{in(ex)}, \]

where \( A_{N,s}^{in(ex)} \) are some undetermined constants and \( \tilde{v}_{N,s}^{in(ex)} \) are the solutions of the boundary value problems (2.8) with the asymptotics:

\[ \tilde{v}_{N,s}^{in(ex)} = \tilde{W}_{N,s}^{in(ex)} + D_{N,s}^{in(ex)} + o(1), \quad \xi_{2} < 0, \]

\[ \tilde{v}_{N,s}^{in(ex)} = \tilde{V}_{N,s}^{in(ex)} + E_{N,s}^{in(ex)} \ln \rho + F_{N,s}^{in(ex)} + O(\rho^{-1}), \quad \xi_{2} > 0, \]

where \( D_{N,s}^{in(ex)}, E_{N,s}^{in(ex)} \) and \( F_{N,s}^{in(ex)} \) are also completely defined constants. The existence the such functions follows from the statement of Lemmas 2.1 and 2.2. Setting the asymptotics of the functions \( v_{N,s}^{in(ex)} \) as \( \rho \to \infty \) and \( \xi < 0 \) equal to the polynomials \( W_{N,s}^{in(ex)} \), we obtain two equations:

\[ A_{N,s}^{in} + D_{N,s}^{in} = b_{-1,0}h \left( \frac{1}{2} \tau_{N,s}^{2} + \tau_{N+1,s} \right) + b_{N,s}, \quad (2.29) \]

\[ A_{N,s}^{ex} + D_{N,s}^{ex} = (-1)^{m}b_{N,s}. \quad (2.30) \]
Similarly, setting the asymptotics of the functions \( v^{in(ex)}_{N,s} \) at infinity as \( \xi_2 > 0 \) equal to the series \( V^{in(ex)}_{N,s} \) up to the terms \( O(1) \), we get the equalities:

\[
\frac{2k_0}{\pi} \left( \tau_{1,0} a^{in}_{N-1,s,0} + \tau_{N,s} R^{in,0}_{0} \right) = E^{in}_{N,s}, \tag{2.31}
\]

\[
\frac{2k_0}{\pi} \left( \tau_{1,0} a^{ex}_{N-1,s,0} + \tau_{N,s} R^{ex,0}_{0} \right) = E^{ex}_{N,s}, \tag{2.32}
\]

\[
- 2k_0 g^{ex}(k_0) \left( \tau_{1,0} a^{ex}_{N-1,s,0} + \tau_{N,s} R^{ex,0}_{0} \right) = A^{ex}_{N,s}, \tag{2.33}
\]

\[
a^{in}_{N,s,0} \psi^2(0) - 2k_0 g^{in}(k_0) \left( \tau_{1,0} a^{in}_{N-1,s,0} + \tau_{N,s} R^{in,0}_{0} \right) = A^{in}_{N,s}. \tag{2.34}
\]

From (2.32) and (2.33) we obtain \( A^{ex}_{N,s} \) what, due to (2.30), defines \( b_{N,s} \). Furthermore, solving the system of the equations (2.28), (2.32), we get \( \tau_{N,s}, a^{in}_{N-1,s,0} \) and \( A^{in}_{N-1,s} \). The coefficients \( a^{in}_{N-1,s,0} \) and \( \tau_{N,s} \) having been determined, first, from (2.33) we obtain \( a^{ex}_{N,s,0} \), and, second, setting the asymptotics of the functions \( v^{in(ex)}_{N,s} \) at infinity as \( \xi_2 > 0 \) equal to the series \( V^{in(ex)}_{N,s} \) (for other terms), we get all coefficients of the series \( \Pi^{in(ex)}_{N-1,s} \). The equations (2.29), (2.34) are analogue of (2.28) for the \((N + 1, s)\)-th step.

The same procedure repeats in the \((N, s + 1)\)-th step (if \( s + 1 < N \)). In the \((N,N)\)-th step the situation is simpler. In this case, Lemma 2.1 implies that

\[
v^{in}_{N,N} \equiv \frac{2k_0}{\pi} \sum_{t=0}^{N-1} R^{in,t,t}_{0} \tau_{k-t,k-t-1} = \tau_{N+1,N} b_{-1,0} h + b_{N,N}, \tag{2.35}
\]

\[
v^{ex}_{N,N} \equiv \frac{2k_0}{\pi} \sum_{t=0}^{N-1} R^{ex,t,t}_{0} \tau_{k-t,k-t-1} = (-1)^m b_{N,N}.
\]

Solving (2.35), we obtain \( b_{N,N} \) \( \tau_{N+1,N} \). Theorem is proved.

We stress that in Theorem 2.1 it is constructed two asymptotic series corresponding to \( \tau_{1,0} = (-1)^n \psi(0)(|\omega|/2h)^{1/2}, n = 1, 2 \) (or \( R^{in,0}_{0} = (-1)^n \psi^{-1} - 2^{-1/2}, \) which is the same). Moreover, from the formulae (2.14) it follows (in the formal level, for now) the formulas (1.8) for \( \tau_{1,0} \) and \( \tau_{2,1} \), and the equality

\[
\tau_{2,0}^{(n)} = \frac{1}{2} |\omega| \left( \frac{2k_0}{\pi} \left( \ln \left( \frac{2|\omega|}{\pi} \right) - 1 \right) - \frac{1}{4} \psi^2(0) - k_0 \left( g^{in}(k_0) + g^{ex}(k_0) \right) \right).
\]

Employing this equality and taking in account that \( \text{Im} g^{in}(k_0) = 0, \text{Im} g^{ex}(k_0) = k_0 \sigma \) (see, for instance, [14,16]), we obtain the formula (1.8) for \( \text{Im} \tau_{2,0} \). Finally, from (2.14) it follows that the leading terms of the series (2.1)–(2.3) has the form indicated for the functions \( \Psi^{(n)}_{\epsilon} \) in the statement c) of Theorem 1.1 in the corresponding domains.

Denote

\[
\psi^{(n)}_{\epsilon,N}(x, k) = x^{N/2} \sum_{i=0}^{N} \sum_{j=0}^{i} \epsilon^{t/2} \ln \epsilon(k_0^2 - k^2) R^{in,i-j}_{[(i-j)/2]}(D_y) G^{in}(x, x^{in}_0, k)
\]
analytic continuation of the solution of the boundary value problem (1.1), (1.2): There exist 
Lemma 3.1. \( \varepsilon \) as \( t > 2 \) and vanishing as \( t < 0 \), and the index \( n = 1, 2 \) in the left 
side corresponds as two series of the asymptotics and it is omitted in the right side 
for sake of brevity. From Theorem 2.1 by standard way (see, for instance, [17]) it 
follows

**Corollary.** Let the asymptotics of the function \( \tau_\varepsilon^{(n)} \) and the series (2.1)–(2.3) 
satisfy the statements Theorem 2.1. Then 
(a) \( \psi^{(n)}_{\varepsilon,N}(x,k) \in C^\infty(\Omega_\varepsilon) \) is a holomorphic function from \( W^1_{2,\text{loc}}(\Omega_\varepsilon) \) which, for 
\( \text{Im} \ k \geq 0 \), satisfies (1.2); 
(b) \( \psi^{(n)}_{\varepsilon,N}(x,k) \) is the solution of (1.1), where \( F(x,k) = F^{(n)}_{\varepsilon,N}(x,k) \) is a holomorphic 
function from \( L^2(\mathbb{R}^2) \), supp \( F_{\varepsilon,N} \subset \mathcal{K}_\varepsilon \cup S^{\text{in}}(2\varepsilon^{-1/2}) \cup S^{\text{ex}}(2\varepsilon^{-1/2}) \) and 
\[ \| F^{(n)}_{\varepsilon,N}(\cdot, \tau^{(n)}_\varepsilon(\cdot)) \|_{L^2(\mathbb{R}^2)} \leq C_N \varepsilon^{-N_1}, \]
where \( N_1 \) increases unboundedly with \( N \).

From the explicit form (2.14) of the leading terms of the asymptotics also it 
follows that 
\[ \| \psi^{(n)}_{\varepsilon,N}(\cdot, \tau^{(n)}_\varepsilon) \|_{L^2(\Omega_\varepsilon \cap S(T))} \to 1, \quad \int_{\Omega_\varepsilon \cap S(T)} \psi^{(1)}_{\varepsilon,N}(x, \tau^{(1)}_\varepsilon) \psi^{(2)}_{\varepsilon,N}(x, \tau^{(2)}_\varepsilon) \, dx \to 0 \] 
(2.36)
as \( \varepsilon \to 0 \) for each \( T \) sufficiently large.

The formal construction of the asymptotics is finished.

§3. Justification of the asymptotics

From [3] it follows

**Lemma 3.1.** There exist 
(a) not more than two poles \( \tau^{(n)}_\varepsilon \) converging to \( k_0 \in \Sigma^{\text{in}}_\varepsilon \cap \Sigma^{\text{ex}} \) as \( \varepsilon \to 0 \); 
(b) if \( \tau^{(1)}_\varepsilon \neq \tau^{(2)}_\varepsilon \), then, for each pole, there exist only one generalized eigenfunction.

In its turn, following [1,16] and using Lemma 3.1, it is easy to prove the following 
statement.

**Lemma 3.2.** Let \( k_0 \in \Sigma^{\text{in}}_\varepsilon \cap \Sigma^{\text{ex}} , F \in L^2(\mathbb{R}^2) \) and supp \( F \subset S(R) \). Then 
(a) for small \( \varepsilon \) and \( k \) close to \( k_0 \), the following uniform estimate holds for the 
analytic continuation of the solution of the boundary value problem (1.1), (1.2): 
\[ \| u_\varepsilon \|_{L^2(S(T))} \leq \frac{C(R,T)}{|(\tau^{(1)}_\varepsilon - k)(\tau^{(2)}_\varepsilon - k)|} \| F \|_{L^2(\mathbb{R}^2)}, \]
which is independent of that poles \(\tau^{(n)}_\varepsilon\) coincide or not;

(b) if \(\tau^{(1)}_\varepsilon \neq \tau^{(2)}_\varepsilon\), then the statement (b) of the theorem 2.1 is true, and, for small \(\varepsilon\) and \(k \in K\) close to \(k_0\), the following uniform estimate holds

\[
\|\tilde{u}_\varepsilon\|_{L_2(S(T))} \leq C(R, T)\|F\|_{L_2(\mathbb{R}^2)};
\]

(c) if \(\tau^{(1)}_\varepsilon \neq \tau^{(2)}_\varepsilon\) and for \(T > 0\) sufficiently large

\[
\frac{1}{\|\Psi^{(1)}_\varepsilon\|_{L_2(\Omega_\varepsilon \cap S(T))}\|\Psi^{(2)}_\varepsilon\|_{L_2(\Omega_\varepsilon \cap S(T))}} \int_{\Omega_\varepsilon \cap S(T)} \Psi^{(1)}_\varepsilon \Psi^{(2)}_\varepsilon \, dx \to 0, \quad \varepsilon \to 0,
\]

then \(\|\Psi^{(n)}_\varepsilon\|_{L_2(\Omega_\varepsilon \cap S(R))} \to 1\) as \(\varepsilon \to 0\).

Proof of Theorem 1.1. Validity of the statement a) follows from the statements (a) of Lemma 3.2, Corollary of Theorem 2.1 and the arbitrary choice of \(N\). Since, \(\tau^{(1)}_\varepsilon \neq \tau^{(2)}_\varepsilon\) validity of the statement b) follows from the statement (b) of Lemma 3.2. Furthermore, from the statements (b) of Lemma 3.2, Corollary of Theorem 2.1 and the arbitrary choice of \(N\), it follows that in representation (1.9):

\[
\Psi^{(n)}_\varepsilon(x) = \alpha_n(\varepsilon)\psi^{(n)}_\varepsilon(x), \quad (3.1)
\]

where \(\psi^{(n)}_\varepsilon\) has the asymptotics (2.1)–(2.3), whose coefficients satisfy the statements of Theorem 2.1, and \(\alpha_n(\varepsilon)\) is a some scalar normalizing multiplier. In their turn, from (2.36) and the statement (c) of Lemma 3.2, it follows that

\[
\alpha_n(\varepsilon) = 1 + o(1), \quad \varepsilon \to 0. \quad (3.2)
\]

Due to (3.1), (3.2) and (2.14), we obtain validity of the statement c) of Theorem 1.1. Theorem is proved.

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