Causal Identification for Complex Functional Longitudinal Studies

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Abstract

Real-time monitoring in modern medical research introduces functional longitudinal data, characterized by continuous-time measurements of outcomes, treatments, and confounders. This complexity leads to uncountably infinite treatment-confounder feedbacks, which traditional causal inference methodologies cannot handle. Inspired by the coarsened data framework, we adopt stochastic process theory, measure theory, and net convergence to propose a nonparametric causal identification framework. This framework generalizes classical g-computation, inverse probability weighting, and doubly robust formulas, accommodating time-varying outcomes subject to mortality and censoring for functional longitudinal data. We examine our framework through Monte Carlo simulations. Our approach addresses significant gaps in current methodologies, providing a solution for functional longitudinal data and paving the way for future estimation work in this domain.

1 Introduction

The advent of real-time monitoring technologies in healthcare has led to the continuous-time measurement of outcomes, treatments, and confounders, which we term “functional longitudinal data.” For example, the Medical Information Mart for Intensive Care IV (MIMIC-IV) (Johnson et al. 2023) is a freely accessible electronic health record (EHR) database that records ICU care data, including physiological measurements, laboratory values, medication administration, and clinical events. Another example is Continuous Glucose Monitoring (CGM) (Rodbard 2016, Klonoff et al. 2017), an increasingly adopted technology for insulin-requiring patients that provides insights into glycemic fluctuations. CGM offers a real-time, high-resolution stream of data, capturing the intricate fluctuations in interstitial fluid glucose levels every few minutes.

These examples illustrate the recent prevalence of functional longitudinal data, highlighting the necessity of a causal framework, as understanding treatment effects is of paramount interest in these settings. However, there is a great lack of investigation of causal inference at the intersection of longitudinal data and functional data. Even identifying causal parameters of interest through observed data becomes highly nontrivial in this setting, due to the issue of uncountably infinite treatment-confounder feedbacks (Hernán & Robins 2020) within functional longitudinal data.

To bridge this gap, we aim to propose a novel identification framework for functional longitudinal data with time-varying outcomes subject to mortality and censoring, who enjoys the nonparametric property, making it more flexible and adaptable to various datasets.

We first define a causal quantity representing the mean of counterfactual outcomes under an idealized randomized world. To connect the observed data distribution to this idealized world, inspired by the coarsened data framework (Heitjan & Rubin 1991) and through the application
of continuous-time stochastic process theory and measure theory, we upgrade classical causal assumptions to accommodate functional longitudinal data nonparametrically. These together resolve the issue of uncountably infinite treatment-confounder feedbacks (Hernán & Robins 2020) for functional longitudinal data. We generalize the well-known g-computation formula, inverse probability weighting formula, and double robust formula. We examine our identification framework through Monte Carlo simulations.

The paper is organized as follows. In Section 2 we present a literature review of related work. In Section 3, we define the notation and parameters of interest. Then we propose identification assumptions and generalize the well-known g-computation, inverse probability weighting, and double robust formulas. Additionally, we prove that our identification is nonparametric. We conduct Monte Carlo simulations to examine our framework in Section 4. Section 5 discusses future directions. While this paper builds a population-level framework with numerical results, it does not explore estimation or associated inference, which is beyond the scope of this study and left for future research.

2 Related Work

Causal Inference for Non-Functional Longitudinal Studies. Current causal frameworks for longitudinal studies are inadequate for handling “functional longitudinal data.” They were mostly designed to accommodate “regular longitudinal studies” where time moves in specific fixed intervals (Greenland & Robins 1986, Robins 1986) or “irregular longitudinal studies.” where changes in data occur randomly yet discretely (Lok 2008, Johnson & Tsiatis 2005, Røysland 2011, Hu & Hogan 2019, Rytgaard et al. 2022), allowing only finite treatment-confounder feedbacks. In Figure 1 illustrates the relation among cross sectional data, regular, irregular, and functional longitudinal data. All literature designed for non-functional longitudinal data fall short to be directly applied onto functional longitudinal data.

Causal Inference for Functional Data. Existing research on causal inference has explored the realm of functional data within observational studies, as noted in works by (Miao et al. 2020,
The data format these studies investigate is consistent with the framework of our analysis. Nonetheless, our work sets itself apart by emphasizing the temporal aspect inherent in longitudinal studies, in contrast to the primary focus on point exposure in the mentioned literature.

Existing Work for Functional Longitudinal Data. The only exceptions that investigated causal inference for functional longitudinal data are Ying (2024) and Sun & Crawford (2022). However, Ying (2024) only investigated an end-of-study outcome, neither proving the nonparametric property nor conducting any numerical investigation. On the other hand, Sun & Crawford (2022) imposed stochastic differential equations with stringent parametric assumptions. This situation highlights a significant gap in methodological advancements within the field.

3 Proposed Method

3.1 Preparation

Consider a longitudinal study spanning from time 0 to $\infty$:

- $A(t)$ and $L(t)$ are two stochastic processes denoting the treatment administered and the measured confounders, respectively, at any given time $t$. At any time, $A(t)$ and $L(t)$ could be binary, categorical, continuous, or even functional. We denote $A(t) = \{A(s) : 0 \leq s \leq t\}$ and $L(t) = \{L(s) : 0 \leq s \leq t\}$, with $\bar{A}$ and $\bar{L}$ representing the collections of treatments and confounders over the entire study.

- We are interested in an outcome of interest $Y(t)$, as a subset of $L(t)$, that is, $Y(t) \subset L(t)$.

- Let $T$ be a time-to-event endpoint, for instance, death, and $C$ be the right censoring time. Define $X = \min(T, C)$ as the censored event time and $\Delta = \mathbb{1}(T \leq C)$ the event indicator. Therefore when $\Delta = 1$, $X = T$ and when $\Delta = 0$, $X = C$.

- Write the counterfactual time-to-event endpoint $T_{\bar{a}}$ and counterfactual covariates $L_{\bar{a}}(t)$, for any $\bar{a} \in \bar{A}$, where $\bar{A}$ encompasses all possible values of $\bar{a}$. Therefore we have $X_{\bar{a}} = \min(T_{\bar{a}}, C)$ and $\Delta_{\bar{a}} = \mathbb{1}(T_{\bar{a}} < C)$. We assume that the future cannot affect the past, that is, $\mathbb{1}(T_{\bar{a}} \geq t) = \mathbb{1}(T_{\bar{a}'} \geq t)$ and $L_{\bar{a}}(t) = L_{\bar{a}'}(t)$ whenever $\bar{a}(t) = \bar{a}'(t)$. We also write $T_A = \{T_{\bar{a}}\}_{\bar{a} \in \bar{A}}$ and $\bar{L}_A = \{\bar{L}_{\bar{a}}\}_{\bar{a} \in \bar{A}}$.

- The full data are $\{\bar{A}, C, T_A, \bar{L}_A\}$ and the observed data are $\{\bar{A}, X, \Delta, \bar{L}\}$. Note that on the observed data level, $A(t)$ and $L(t)$ are not observed for $t < X$ or defined for $t \leq T$. For easier notation in this paper, we offset $A(t) = A(X)$ and $L(t) = L(X)$ for observed data whenever $t > X$. In this way, the stochastic processes $A(t)$ and $L(t)$ are well defined at any $t > 0$.

- Define $\mathcal{F}_t = \sigma\{A(s), L(s), \mathbb{1}(X \leq s), \mathbb{1}(X \leq s)\Delta : \forall s \leq t\}$ as a filtration of information observed up to time $t$. Also we write $\mathcal{F}_{t-} = \sigma(\bigcup_{0 \leq s < t}\mathcal{F}_t)$ and $\mathcal{G}_t = \sigma(\mathcal{F}_{t-}, A(t))$. We define $\mathcal{G}_{\infty+} = \mathcal{F}_\infty$. We write $\mathcal{F}_{0-}$ and $\mathcal{G}_{0-}$ as the trivial sigma algebra for convenience. Note that $X$ is a stopping time with respect to $\mathcal{F}_t$, with $\mathcal{F}_\infty = \mathcal{F}_X = \sigma\{\bar{A}, X, \Delta, \bar{L}\}$.

- To denote the distribution on the path space induced by the stochastic processes and the measure $\mathbb{P}$ on $\Omega$, we also use $\mathbb{P}(\text{d}\bar{a}|\bar{l})$ (Blattcharya & Waymire 2007, Durrett 2019, Gill & Robins 2001). Note that this is not a density function.

- Assuming the event space is Polish enables the selection of regular conditional probabilities. Conditional distribution is understood as a function over a sigma algebra combined with a path set, for instance, $\mathbb{P}(\text{d}\bar{a}|\bar{l})$, and is defined almost surely.
The total variation norm over the path space’s signed measure space is represented by \( \| \cdot \|_{TV} \).

A partition \( \Delta_K[0, \infty) \) over \([0, \infty)\) is a finite sequence of \( K + 1 \) numbers of the form \( 0 = t_0 < \cdots < t_K = \infty \). The mesh \( \| \Delta_K[0, \infty) \| \) of a partition \( \Delta_K[0, \infty) \) is defined as \( \max \{ \max_{i=0, \ldots, K}(t_{j+1} - t_j), 1/t_{K-1} \} \), representing the maximum gap length of the partition.

We are interested in learning a marginal mean of transformed potential outcomes including a time-to-event outcome and an outcome process under a user-specified treatment regime in the absence of censoring,

\[
\int_{\mathcal{A}} \mathbb{E}\{\nu(T_{\bar{a}}, \bar{Y}_{\bar{a}})\} G(d\bar{a}),
\]

where \( \nu \) is some user-specified function and \( G \) is a priori defined (signed) measure on \( \mathcal{A} \). We assume \( \mathbb{E}\{\nu(T_{\bar{a}}, \bar{Y}_{\bar{a}})\} \) is integrable against \( G \). This exploration encompasses marginal means under static treatment regimes, as discussed in various literature (Rytgaard et al. 2022, Cain et al. 2010, Young et al. 2011, Hernán & Robins 2020). This quantity can be seen as the mean of counterfactual outcomes under an idealized randomized world, where \( \bar{a} \) is randomized to follow \( G \).

### 3.2 Identification assumptions

We have defined the parameter of interest (1). Intuitively if treating treatment process \( \bar{A} \) as a selection process (Heitjan & Rubin 1991), (1) is the mean of \( \nu(T_{\bar{a}}, \bar{Y}_{\bar{a}}) \) when \( \bar{A} \) were to follow \( G \). To create such a pseudo-population, note that for any sequences of partitions \( \{\Delta_K[0, \infty)\}_{K=1}^\infty \), we have the following decomposition

\[
P(d\bar{a}d\bar{a}d\bar{a}) = \prod_{j=0}^{K-1} \mathbb{P}\{T \leq t_{j+1}|\bar{a}(t_{j+1}), 1(t_j < x \leq t_{j+1}, \delta = 0), \mathcal{F}_{t_j}\}^{1(t_j < x \leq t_{j+1}, \delta = 1)}
\]

\[
\mathbb{P}\{T > t_{j+1}|\bar{a}(t_{j+1}), 1(t_j < x \leq t_{j+1}, \delta = 0), \mathcal{F}_{t_j}\}^{1-1(t_j < x \leq t_{j+1}, \delta = 1)}
\]

\[
\mathbb{P}\{C \leq t_{j+1}|\bar{a}(t_{j+1}), \mathcal{F}_{t_j}\}^{1(t_j < x \leq t_{j+1}, \delta = 0)}
\]

\[
\mathbb{P}\{C > t_{j+1}|\bar{a}(t_{j+1}), \mathcal{F}_{t_j}\}^{1-1(t_j < x \leq t_{j+1}, \delta = 0)}
\]

\[
\mathbb{P}\{d\bar{a}(t_{j+1})|\bar{a}(t_{j+1}), \mathcal{F}_{t_j}\} \mathbb{P}\{d\bar{a}(t_{j+1})|\mathcal{F}_{t_j}\}
\]

We intervene treatment distribution at each time \( t_j \) to approximate the pseudo-population where \( \bar{A} \) were to follow \( G \) as:

\[
P_{\Delta_K[0, \infty], G}(d\bar{a}d\bar{a}d\bar{a}) = \prod_{j=0}^{K-1} \mathbb{P}\{T \leq t_{j+1}|C > t_{j+1}, \bar{a}(t_{j+1}), \mathcal{F}_{t_j}\}^{1(t_j < x \leq t_{j+1}, \delta = 1)}
\]

\[
\mathbb{P}\{T > t_{j+1}|C > t_{j+1}, \bar{a}(t_{j+1}), \mathcal{F}_{t_j}\}^{1-1(t_j < x \leq t_{j+1}, \delta = 1)}
\]

\[
\mathbb{P}\{d\bar{a}(t_{j+1})|C > t_{j+1}, \bar{a}(t_{j+1}), \mathcal{F}_{t_j}\}
\]

\[
\{1 - 1(x \leq t_{j+1}, \delta = 0)\} G\{d\bar{a}(t_{j+1})|\bar{a}(t_j)\}
\]

To eliminate confounder bias, we need to make sure there is no unmeasured confounders. We adapt the commonly known “coarsening at random” (Heitjan & Rubin 1991) assumption into:

**Assumption 1** (Full conditional randomization). The treatment assignment is independent of the all potential outcomes and covariates given history, in the sense that there exists a bounded function \( \varepsilon(t, \eta) > 0 \) with \( \int_0^\infty \varepsilon(t, \eta) dt \to 0 \) as \( \eta \to 0 \), such that for any \( t \in [0, \infty] \), \( \eta > 0 \),

\[
\sup_{\bar{a} \in \mathcal{A}} \mathbb{E}(\| \mathbb{P}\{dt_{\bar{a}}d\bar{a}|\bar{A}(t + \eta), \mathcal{F}_t\} - \mathbb{P}\{dt_{\bar{a}}d\bar{a}|\mathcal{F}_t\}\|_{TV}) < \varepsilon(t, \eta).
\]
This assumption claims that, the treatment distribution, or equally, the probability of coarsening, in a small period of time around $t$, only depends on the observed data up to time $t$ and independent of further part of counterfactuals. This assumption says in a approximating sense that there is no common cause between treatment decision between time $[t, t + \eta]$ and all future counterfactual confounders.

We also need an assumption over the censoring mechanism to eliminate the censoring bias. We consider the well-known conditionally independent censoring assumption (Tsiatis 2006, Andersen et al. 2012). Define the full data censoring time hazard function as

$$
\lambda_C(t|T, \tilde{A}, \tilde{L}) = \lim_{dt \to 0} \frac{\mathbb{P}(C \leq t + dt|C > t, T, \tilde{A}, \tilde{L})}{dt}.
$$

The following assumption requires that the full data censoring time hazard at time $t$ only depends on the observed data up to time $t$.

**Assumption 2** (Conditional independent censoring). The censoring mechanism is said to be conditionally independent if

$$
\lambda_C(t|T, \tilde{A}, \tilde{L}) = \lim_{dt \to 0} \frac{\mathbb{P}\{C \leq t + dt|C > t, T > t, \tilde{A}(t), \tilde{L}(t)\}}{dt}.
$$

Note that in order to overcome the continuous-time issue, here we impose Assumption 1 over an infinitesimal period of time. This type of idea is also adopted in Assumption 2. Note that how Assumption 2 is given on the intensity process whereas Assumption 1 is on the conditioning event. This is because one does not have intensity process for a general stochastic process.

With Assumptions 1 and 2, we are able to show that whenever $|\Delta_K[0, \infty]| \to 0$, $\mathbb{P}_{\Delta_K[0, \infty], G}$ approximates a pseudo-measure where treatment distribution are intervened by uncountable times into following $\mathbb{G}$:

**Proposition 1** (Intervenable). Under Assumptions 1 and 2, the measures $\mathbb{P}_{\Delta_K[0, \infty], G}$ converges to the same (signed) measure $\mathbb{P}_G := \mathbb{P}(dx_\alpha dl_\alpha)\mathbb{G}(d\alpha)\delta_\alpha$ in the total variation norm on the path space, regardless of the choices of partitions,

$$
\| \mathbb{P}_{\Delta_K[0, \infty], G}(dx_\alpha dl_\alpha)\mathbb{G}(d\alpha)\delta_\alpha - \mathbb{P}(dx_\alpha dl_\alpha)\mathbb{G}(d\alpha)\delta_\alpha \|_{TV} \to 0.
$$

We refer $\mathbb{P}_G$ as the target distribution. The following assumption links the observed variable with the counterfactuals.

**Assumption 3** (Full consistency). For any $t$,

$$
T = T_{\tilde{A}}, L(t) = L_{\tilde{A}}(t).
$$

The full consistency assumption links the observed outcome and the potential outcome via the treatment actually received. It says that if an individual receives the treatment $\tilde{A} = \tilde{a}$, then his/her observed outcome $Y$ matches $Y_{\tilde{a}}$.

The following assumption ensures that the observed data can identify the target distribution.

**Assumption 4** (Positivity).

$$
\mathbb{P}_G \ll \mathbb{P}.
$$

With the above assumptions, we are able to generalize the well-known identification formulas: $g$-computation, inverse probability weighting, and double robust formulas, into functional longitudinal data. Note that our assumptions can be weaker but chosen for ease to interpret.
3.3 Identification formulas

**Definition 1** (G-computation process). Under Assumptions 1 and 2, define

\[ H_G(t) = \mathbb{E}_G\{\nu(X, \bar{Y})|\mathcal{G}_t\}, \]

as a projection process, which is apparently a \( \mathbb{P}_G \)-martingale. We call \( H_G(t) \) the g-computation process. Note that

\[ H_G(\infty) = \nu(X, \bar{Y}), \quad H_G(0-) = \mathbb{E}_G\{\nu(X, \bar{Y})\}. \]

The g-computation process intuitively serves as a consecutive adjustment of the target \( \nu(X, \bar{Y}) \) from \( \infty \) to 0. It represents a mix of original conditional distributions of covariate process together with the intervened treatment process \( G \), from end of study to the beginning. Following this adjustment to the beginning of study, we have:

**Theorem 1** (G-computation formula). Under Assumptions 1, 2, 3, and 4, (1) is identified via a g-computation formula as

\[ \int \mathbb{E}\{\nu(T_\delta, \bar{Y}_\delta)|G(\delta)\} d\delta = H_G(0-). \]

**Definition 2** (Inverse probability weighting process). Under Assumptions 1 and 2, define

\[ Q_G(t) = \mathbb{E}\left\{ \frac{d\mathbb{P}_G}{d\mathbb{P}} | \mathcal{G}_t \right\}, \]

as the Radon-Nikodym derivative at any time \( t \), which is apparently a \( \mathbb{P} \)-martingale. We call \( Q_G(t) \) the inverse probability weighting process. Note that

\[ Q_G(\infty) = \mathbb{E}\left\{ \frac{d\mathbb{P}_G}{d\mathbb{P}} | \mathcal{G}_\infty \right\} = \frac{d\mathbb{P}_G}{d\mathbb{P}}, \quad Q_G(0-) = 1. \]

The IPW process intuitively serves as a continuous adjustment of the treatment process \( \bar{A} \) from 0 to \( \infty \), using which as weights one may create a pseudo population as if the whole process were to follow \( \mathbb{P}_G \). It reweights the observed data distribution \( \mathbb{P} \) into \( \mathbb{P}_G \) from the beginning of the study to the end. Following this reweighting throughout the longitudinal study, we have:

**Theorem 2** (Inverse probability weighting formula). Under Assumptions 1, 2, 3, and 4, (1) is identified via an inverse probability weighting formula as

\[ \int \mathbb{E}\{\nu(T_\delta, \bar{Y}_\delta)|G(\delta)\} d\delta = \mathbb{E}\left\{ Q_G(\infty)\nu(X, \bar{Y}) \right\}. \]

For any two \( \mathcal{G}_t \)-adapted processes \( H(t) \) and \( Q(t) \), and a partition \( \Delta_K[0, \infty] \), we write

\[ \Xi(H, Q) = \sum_{j=0}^{K} Q(t_j) \left[ \int H(t_{j+1}) G(d\bar{a}(t_{j+1})|\bar{A}(t_j)) - H(t_j) \right] + \int H(0) G(d\bar{a}(0)). \]

We also define \( \Xi(H, Q) \) as the limit of \( \Xi(\Delta_K[0, \infty])(H, Q) \) in probability whenever it exists. We have:

**Theorem 3** (Doubly robust formula). Under Assumptions 1, 2, 3, and 4, for any \( \mathcal{G}_t \)-adapted processes \( H(t) \) and \( Q(t) \) at the law where \( \Xi(H, Q) \), as the limit of \( \Xi(\Delta_K[0, \infty])(H, Q) \) in probability, exists and

\[ \lim_{|\Delta_K[0, \infty]| \to 0} \mathbb{E}\{\Xi(\Delta_K[0, \infty])(H, Q)\} = \mathbb{E}\{\Xi(H, Q)\}, \]
we have

\[ \int_{\mathcal{A}} \mathbb{E}\{\nu(T_{\bar{a}}, \bar{Y}_{\bar{a}})\} G(\text{d}\bar{a}) = \mathbb{E}\{\Xi(H, Q)\}, \]

provided that either \( H = H_G \) or \( Q = Q_G \).

As one can see, the doubly robust formula provides extra protection against possible misspecification on either the g-computation process or the IPW process.

### 3.4 No restrictions on the observed data distribution: A Nonparametric Framework

In this subsection, we demonstrate that our identification framework imposes no restrictions on the observed data. This property is advantageous for researchers and practitioners because nonparametric frameworks are flexible and require minimal assumptions, making them robust and adaptable to diverse datasets. We demonstrate this by proving that, for any given observed data distribution, we can identify a sequence of full data distributions—each satisfying Assumptions 1, 2, 3, and 4—such that their corresponding distributions on the observed data closely approximate the initial observed data distribution. That is, we write the set of all observed data distribution as \( \mathcal{P} \) and its subset satisfying Assumptions 1, 2, 3, and 4 as \( \mathcal{M} \), then we show that \( \mathcal{M} \) is a dense subset of \( \mathcal{P} \) in the total variation norm.

Up to now, we have used \( \mathbb{P} \) to represent both the distribution on the sample space and the path space. In this subsection, we use \( \mathbb{P} \) to denote the distribution on the observed data \((\bar{A}, X, \Delta, \bar{L})\) and \( \mathbb{P}^F \) to denote the distribution on the full data \((\bar{A}, C, T_{\bar{A}}, \bar{L}_{\bar{A}})\). We have

**Theorem 4.** When the path space consists of all piece-wise continuous processes, for any measure \( \mathbb{P} \) over the observed data \((\bar{A}, X, \Delta, \bar{L})\), there exists a sequence of measures \( \mathbb{P}^F_n \) over the full data \((\bar{A}, C, T_{\bar{A}}, \bar{L}_{\bar{A}})\) satisfying Assumptions 1, 2, 3, 4, whose inductions on the observed data converges to \( \mathbb{P} \) in the total variation norm.

Technically, we have not achieved full nonparametric paradigm. However, we deem that the regularity condition “the path space is piece-wise continuous processes” is general enough for practical considerations. For example, both multivariate counting processes and continuous processes like Brownian process satisfy this regularity condition. It is noteworthy that achieving this “almost nonparametric” nature is the best one can hope for. This realization was confirmed in (Gill et al. 1997, Section 9) for “coarsening at random” assumption, even though our framework exhibits certain distinctions.

### 4 Experiment result

In this section, we employ Monte Carlo simulations to empirically assess how the identification works. We decide to evaluate the performance of the g-computation formula only, for two reasons:

1. The g-computation formula is the only one that can be easily approximated through raw simulated data, whereas inverse probability weighting (and hence the doubly robust formula) cannot be directly approximated without estimation or computation;

2. On the population level, the values of the three formulas are equal. Therefore, approximating g-computation formula is sufficient for our purposes.

To that end, we need to go through 4 steps:

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1. Come up with a reasonable data generating process;

2. Compute the parameter of interest (1) (or equivalently, the left-hand side of g-computation formula in Theorem 1) according to this data generating process;

3. Simulate according to this data generating process;

4. Approximate the right-hand side of g-computation formula in Theorem 1 using the simulated data.

**Step 1:** To sharpen the focus and ease the computation, we consider a simple setting where there is no mortality or censoring \((T = C \equiv \infty)\), or other measured confounding process, except for the outcome process itself. We take glucose levels as the outcome and insulin levels as the treatment. Both glucose and insulin levels exhibit smooth, continuous changes over time. Gaussian processes are particularly well-suited for modeling such smooth and continuous temporal processes. For \(t \in [0, 1]\), consider a potential outcome process \(\bar{Y}_a(t)\) capturing potential logarithm of glucose levels, following a Gaussian process with mean process as

\[
\mathbb{E}\{\bar{Y}_a(t)\} = -a(t),
\]

and covariance process as

\[
\text{Cov}\{\bar{Y}_a(t), \bar{Y}_a(s)\} = e^{-|t-s|}, \forall t, s \in [0, 1].
\]

This ensures the joint dependence among \(\bar{Y}_a(t)\) and negative treatment effect of logarithm of insulin level \(\bar{a}\). For instance, \(\bar{Y}_a(t)\) can be log of blood glucose level. Define \(\nu(T_a, \bar{Y}_a)\) as the integral of \(\bar{Y}_a\) over time \(t \in [0, 1]\), that is,

\[
\nu(T_a, \bar{Y}_a) = \int_0^1 \bar{Y}_a(t)dt.
\]

Suppose the targeted treatment regime \(G\) is a Gaussian measure with mean process \(t - 0.5\) and jointly independent normal variables at any time points. That is, the intervened \(A\) follows a Gaussian process with a mean process

\[
\mathbb{E}\{A(t)\} = t - 0.5,
\]

and covariance process

\[
\text{Cov}\{A(t), A(s)\} = e^{-|t-s|}, \forall t, s \in [0, 1],
\]

representing an increase of insulin level, possibly due to some insulin intake.

**Step:** Then we can show that (1) (or equivalently, the left-hand side of g-computation formula in Theorem 1) equals zero, that is,

\[
\int \mathbb{E}\{\nu(T_a, \bar{Y}_a)\}G(d\bar{a}) = \int \mathbb{E}\left\{\int_0^1 \bar{Y}_a(t)dt\right\}G(\bar{a}) = \int \int_0^1 \mathbb{E}\{\bar{Y}_a(t)\}dtG(\bar{a}) = \int \int_0^1 a(t)dtG(\bar{a}) = \int_0^1 \int a(t)G(\bar{a})dt = \int_0^1 (t - 0.5)dt = 0.
\]

**Step 3:** In practice, we observe a stochastic process at finite points. We consider evenly splitting \(t \in [0, 1]\) into a grid of size \(K + 1\): \(\Delta_K[0, 1] = \{t_0 = 0, t_1 = 1/K, \ldots, t_{K-1} = (K-1)/K, t_K = 1\}\),
and for $1 \leq i \leq n$, according to $G$ specified in Step 1, we simulate i.i.d. samples $A_i(t)$ according to $G$ specified in Step 1 at $\Delta K[0,1]$ as

$$
\begin{pmatrix}
A_i(t_0) \\
A_i(t_1) \\
\vdots \\
A_i(t_{K-1}) \\
A_i(t_K)
\end{pmatrix} \sim \mathcal{N}
\begin{pmatrix}
t_0 - 0.5 \\
t_1 - 0.5 \\
\vdots \\
t_{K-1} - 0.5 \\
t_K - 0.5
\end{pmatrix},
\begin{pmatrix}
1 & e^{-|t_1-t_0|} & \cdots & e^{-|t_{K-1}-t_0|} & e^{-|t_K-t_0|} \\
e^{-|t_1-t_0|} & 1 & \cdots & e^{-|t_{K-1}-t_1|} & e^{-|t_K-t_1|} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
e^{-|t_{K-1}-t_0|} & e^{-|t_{K-1}-t_1|} & \cdots & 1 & e^{-|t_K-t_{K-1}|} \\
e^{-|t_{K-1}-t_0|} & e^{-|t_{K-1}-t_1|} & \cdots & e^{-|t_K-t_{K-1}|} & 1
\end{pmatrix}.
$$

By according to the distribution of $Y_0(t)$ specified in Step 1 and consistency, we generate $Y_i(t)$ at $\Delta K[0,1]$ as

$$
\begin{pmatrix}
Y_i(t_0) \\
Y_i(t_1) \\
\vdots \\
Y_i(t_{K-1}) \\
Y_i(t_K)
\end{pmatrix} \sim \mathcal{N}
\begin{pmatrix}
A_i(t_0) \\
A_i(t_1) \\
\vdots \\
A_i(t_{K-1}) \\
A_i(t_K)
\end{pmatrix},
\begin{pmatrix}
1 & e^{-|t_1-t_0|} & \cdots & e^{-|t_{K-1}-t_0|} & e^{-|t_K-t_0|} \\
e^{-|t_1-t_0|} & 1 & \cdots & e^{-|t_{K-1}-t_1|} & e^{-|t_K-t_1|} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
e^{-|t_{K-1}-t_0|} & e^{-|t_{K-1}-t_1|} & \cdots & 1 & e^{-|t_K-t_{K-1}|} \\
e^{-|t_{K-1}-t_0|} & e^{-|t_{K-1}-t_1|} & \cdots & e^{-|t_K-t_{K-1}|} & 1
\end{pmatrix}.
$$

**Step 4:** The integral of $Y_i(t_k)$ over $[0,1]$ is $\sum_{k=0}^{K} Y_i(t_k)/(K + 1)$. The approximate of the right-hand side of g-computation formula is $\sum_{i=1}^{n} \sum_{k=0}^{K} Y_i(t_k)/(K + 1)/n$.

We vary the grid sizes ($K = 10, 50, 250$) to examine how a denser grid improves the approximation. This approach simulates the scenario where the mesh $[\Delta K[0,1]]$ is shrunk to zero. Additionally, we vary the sample sizes ($n = 100, 500, 2500$) to explore how larger samples enhance the approximation, leveraging the law of large numbers to better approximate the right-hand side of the g-computation formula. We repeat the process $R = 10,000$ times. The resulting 10,000 approximations of $\sum_{i=1}^{n} \sum_{k=0}^{K} Y_i(t_k)/(K + 1)/n$ are presented in boxplots in Figure 2, where we append biases.

The simulation results demonstrate that the g-computation formula can adequately approximate (1) even with moderate sample and grid sizes. Increasing the sample size while keeping the grid size fixed enhances the accuracy and reduces the variance of the approximation. In contrast, increasing the grid size while keeping the sample size fixed does not consistently improve accuracy or reduce variance. However, simultaneously increasing both the sample and grid sizes significantly improves accuracy and reduces variance in the approximation.

## 5 Conclusion

In this work, we proposed on a novel theoretical framework for causal inference under functional longitudinal studies. We introduced three methodological paradigms for causal identification: the g-computation formula, inverse probability weighting formula, and doubly robust formula. This framework, noted for nonparametric foundation, substantiates and expands upon the estimand-based causal framework introduced by Ying (2024). It incorporates considerations for time-varying outcomes and addresses complexities such as death and right censoring, marking a significant advancement in the analysis of functional longitudinal data and enhancing the toolkit for causal inference in this area.

There are significant theoretical and methodological opportunities, given the limited investigation on functional longitudinal data, for the machine learning, functional data analysis and causal inference communities. To list a few, first, adapting our framework to accommodate scenarios where Assumption 1 may not hold, including contexts involving time-dependent instrumental variables and time-dependent proxies (Ying et al. 2023), warrants rigorous exploration. Second, the positivity Assumption 4 in longitudinal studies faces practical challenges due to the potential scarcity of subjects adhering to specific treatment regimes within observed populations. One might consider
using semiparametric models such as marginal structural models (Robins 1998, Roysland 2011) and structural nested models (Robins 1999, Lok 2008). Other solutions include dynamic treatment regimes (Fitzmaurice et al. 2008, Young et al. 2011, Rytgaard et al. 2022) and incremental interventions (Kennedy 2017). Third, establishing the efficiency bound for our quantity of interest by leveraging semiparametric theory, represents an engaging challenge. Fourth, partial identification using discrete-time observations is a promising direction. Finally, developing a comprehensive estimation framework remains of ultimate interest.

References

Andersen, P. K., Borgan, O., Gill, R. D. & Keiding, N. (2012), Statistical Models based on Counting Processes, Springer Science & Business Media.

Bhattacharya, R. N. & Waymire, E. C. (2007), A Basic Course in Probability Theory, Vol. 69, Springer.

Cain, L. E., Robins, J. M., Lanoy, E., Logan, R., Costagliola, D. & Hernán, M. A. (2010), ‘When to start treatment? A systematic approach to the comparison of dynamic regimes using observational data’, The international journal of biostatistics 6(2), 1–26.

Durrett, R. (2019), Probability: Theory and Examples, Vol. 49, Cambridge university press.
Fitzmaurice, G., Davidian, M., Verbeke, G. & Molenberghs, G. (2008), Longitudinal Data Analysis, CRC press.

Gill, R. D. & Robins, J. M. (2001), ‘Causal inference for complex longitudinal data: the continuous case’, The Annals of Statistics 29(2), 1785–1811.

Gill, R. D., Van Der Laan, M. J. & Robins, J. M. (1997), Coarsening at random: Characterizations, conjectures, counter-examples, in ‘Proceedings of the First Seattle Symposium in Biostatistics: Survival Analysis’, Springer, pp. 255–294.

Greenland, S. & Robins, J. M. (1986), ‘Identifiability, exchangeability, and epidemiological confounding’, International Journal of Epidemiology 15(3), 413–419.

Heitjan, D. F. & Rubin, D. B. (1991), ‘Ignorability and coarse data’, The Annals of Statistics 19(4), 2244–2253.

Hernán, M. A. & Robins, J. M. (2020), Causal Inference: What If, Boca Raton: Chapman & Hall/CRC.

Hu, L. & Hogan, J. W. (2019), ‘Causal comparative effectiveness analysis of dynamic continuous-time treatment initiation rules with sparsely measured outcomes and death’, Biometrics 75(2), 695–707.

Johnson, A. E., Bulgarelli, L., Shen, L., Gayles, A., Shammout, A., Horng, S., Pollard, T. J., Hao, S., Moody, B., Gow, B. et al. (2023), ‘Mimic-iv, a freely accessible electronic health record dataset’, Scientific data 10(1), 1.

Johnson, B. A. & Tsiatis, A. A. (2005), ‘Semiparametric inference in observational duration-response studies, with duration possibly right-censored’, Biometrika 92(3), 605–618.

Kennedy, E. H. (2017), ‘Semiparametric theory’, arXiv preprint arXiv:1709.06418.

Klonoff, D. C., Ahn, D. & Drincic, A. (2017), ‘Continuous glucose monitoring: a review of the technology and clinical use’, Diabetes Research and Clinical Practice 133, 178–192.

Lok, J. J. (2008), ‘Statistical modeling of causal effects in continuous time’, The Annals of Statistics 36(3), 1464–1507.

Miao, R., Xue, W. & Zhang, X. (2020), ‘Average treatment effect estimation in observational studies with functional covariates’, arXiv preprint arXiv:2004.06166.

Robins, J. M. (1986), ‘A new approach to causal inference in mortality studies with a sustained exposure period—application to control of the healthy worker survivor effect’, Mathematical Modelling 7(9-12), 1393–1512.

Robins, J. M. (1998), Marginal structural models, in ‘1997 Proceedings of the Section on Bayesian Statistical Science’, Alexandria, VA: American Statistical Association, pp. 1–10.

Robins, J. M. (1999), ‘Association, causation, and marginal structural models’, Synthese 121(1/2), 151–179.

Rodbard, D. (2016), ‘Continuous glucose monitoring: a review of successes, challenges, and opportunities’, Diabetes Technology & Therapeutics 18(S2), S2–3.
Røysland, K. (2011), ‘A martingale approach to continuous-time marginal structural models’, *Bernoulli* **17**(3), 895–915.

Rytgaard, H. C., Gerds, T. A. & van der Laan, M. J. (2022), ‘Continuous-time targeted minimum loss-based estimation of intervention-specific mean outcomes’, *The Annals of Statistics* **50**(5), 2469–2491.

Sun, J. & Crawford, F. W. (2022), ‘Causal identification for continuous-time stochastic processes’, *arXiv preprint arXiv:2211.15934*.

Tan, R., Huang, W., Zhang, Z. & Yin, G. (2022), ‘Causal effect of functional treatment’, *arXiv preprint arXiv:2210.00242*.

Tsiatis, A. A. (2006), *Semiparametric Theory and Missing Data*, Springer.

Ying, A. (2024), Causality for functional longitudinal data, in ‘Causal Learning and Reasoning’, PMLR, pp. 665–687.

Ying, A., Miao, W., Shi, X. & Tchetgen Tchetgen, E. J. (2023), ‘Proximal causal inference for complex longitudinal studies’, *Journal of the Royal Statistical Society Series B: Statistical Methodology* **85**(3), 684–704.

Young, J. G., Cain, L. E., Robins, J. M., O’Reilly, E. J. & Hernán, M. A. (2011), ‘Comparative effectiveness of dynamic treatment regimes: an application of the parametric g-formula’, *Statistics in Biosciences* **3**(1), 119–143.

Zhang, X., Xue, W., Wang, Q. et al. (2021), ‘Covariate balancing functional propensity score for functional treatments in cross-sectional observational studies’, *Computational Statistics & Data Analysis* **163**(C).
A Proofs

A.1 Proof of Proposition 1

We temporarily define $\mathcal{F}_{/C,t} = \sigma\{L(s), A(s), 1(T \leq s) : \forall s \leq t\}$ as the censoring free filtration and $\mathcal{F}_{a,t} = \sigma\{L_a(s), 1(T_a \leq s) : \forall s \leq t\}$ as the counterfactual filtration:

$$\|P_{\Delta_K[0,\infty], G}(dxd\delta^\alpha d\bar{a}) - P(dxa d\bar{a})G(d\bar{a})\delta_a\|_{TV}$$

$$= \left| \prod_{j=0}^{K-1} \mathcal{G}\{\bar{a}(t_{j+1})|a(t_j)\}\{1 - 1(x \leq t_{j+1}, \delta = 0)\} P\{T \leq t_{j+1}|a(t_{j+1}), \mathcal{F}_{t_j}\}\{t_j < x \leq t_{j+1}\} P\{T > t_{j+1}|a(t_{j+1}), \mathcal{F}_{t_j}\}\{t_{j+1} < x \leq t_{j+1}\} P\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \mathcal{F}_{t_j}\}$$

$$- P(dxa d\bar{a})G(d\bar{a})\delta_a\|_{TV}$$

$$\leq \left| \prod_{j=0}^{K-1} \mathcal{G}\{\bar{a}(t_{j+1})|a(t_j)\}\{1 - 1(x \leq t_{j+1}, \delta = 0)\} P\{T \leq t_{j+1}|a(t_{j+1}), \mathcal{F}_{a,t_j}\}\{t_j < x_a \leq t_{j+1}, \delta_a = 1\}$$

$$\mathcal{P}\{T \leq t_{j+1}|a(t_{j+1}), \mathcal{F}_{a,t_j}\}\{1 - 1(t_j < x_a \leq t_{j+1}, \delta_a = 1)\} P\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \mathcal{F}_{a,t_j}\}$$

$$- P(dxa d\bar{a})G(d\bar{a})\delta_a\|_{TV}$$

$$= \left| \prod_{j=0}^{K-1} \mathcal{G}\{\bar{a}(t_{j+1})|a(t_j)\}\{1 - 1(x_a \leq t_{j+1}, \delta_a = 0)\} P\{T \leq t_{j+1}|a(t_{j+1}), \mathcal{F}_{a,t_j}\}\{t_j < x_a \leq t_{j+1}, \delta_a = 1\}$$

$$\mathcal{P}\{T_a \leq t_{j+1}|C > t_{j+1}, a(t_{j+1}), \mathcal{F}_{a,t_j}\}\{1 - 1(t_j < x_a \leq t_{j+1}, \delta_a = 1)\} P\{d\bar{a}(t_{j+1})|C > t_j, a(t_{j+1}), \mathcal{F}_{a,t_j}\}$$

$$- P(T \leq t_{j+1}|a(t_{j+1}), \mathcal{F}_{a,t_j})\{t_j < x_a \leq t_{j+1}\} P(T_a > t_{j+1}|a(t_{j+1}), \mathcal{F}_{a,t_j})\{1 - 1(t_j < x_a \leq t_{j+1})\} P\{d\bar{a}(t_{j+1})|\mathcal{F}_{a,t_j}\}\|_{TV} \to 0.$$

A.2 Proof of Theorem 1

Since $P_{G}(dxd\delta^\alpha d\bar{a})$ is a limit of measures in total variation norm of

$$\|P_{\Delta_K[0,\infty], G}(dxd\delta^\alpha d\bar{a})\|_{TV},$$

whenever $|\Delta_K[t, \infty]| \to 0$, we have

$$\int f(x, \delta, a, \bar{l}) P_{\Delta_K[0,\infty], G}(dxd\delta^\alpha d\bar{a}) \to E_G\{f(X, \Delta, A, \bar{L})\},$$

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for any bounded functions $f(x, \delta, \bar{a}, \bar{l})$. Therefore, we have
\[
\left| H_G(0-) - \int \mathbb{E}\{\nu(T_{\bar{a}}, Y_{\bar{a}})\} G(d\bar{a}) \right|
\leq \left| \int \nu(x, \bar{y}) \mathbb{P}(d\bar{a}d\bar{a}) - \int \nu(x, \bar{y}) \mathbb{P}_{\Delta_K[0,\infty], \mathbb{G}}(d\bar{a}d\bar{a}) \right|
+ \left| \int \nu(x, \bar{y}) \mathbb{P}_{\Delta_K[0,\infty], \mathbb{G}}(d\bar{a}d\bar{a}) - \int \mathbb{E}\{\nu(T_{\bar{a}}, Y_{\bar{a}})\} G(d\bar{a}) \right| + o(1)
\]
\[
= \left| \int \nu(x, \bar{y}) \prod_{j=0}^{K-1} \mathbb{P}\{T \leq t_j+1|C > t_j+1, \bar{a}(t_j+1), \mathcal{F}_{t_j}\}^{1}(t_j < t_j+1, \delta = 1) \mathbb{P}\{d\bar{l}(t_j+1)|C > t_j+1, \bar{a}(t_j+1), \mathcal{F}_{t_j}\} \right.
\]
\[
\{1 - 1(x \leq t_j+1, \delta = 0)\} G\{d\bar{l}(t_j+1)|\bar{a}(t_j)\} - \int \mathbb{E}\{\nu(T_{\bar{a}}, Y_{\bar{a}})\} G(d\bar{a}) \right| + o(1),
\]
where $o(1)$ converges to zero when $\Delta_K[0,\infty] \to 0$. By Assumptions 1, 2, 3, and 4, the above term is less than or equal to
\[
\left| \int \nu(x, \bar{y}) \mathbb{P}\{T \leq t_K|\bar{a}(t_K), \mathcal{F}_{t_K-1}\}^{1}(t_K < t_K, \delta = 1) \mathbb{P}\{d\bar{l}(t_K)|\bar{a}(t_K), \mathcal{F}_{t_K-1}\} \right.
\]
\[
\{1 - 1(x \leq t_K, \delta = 0)\} G\{d\bar{l}(t_K)|\bar{a}(t_K-1)\} \prod_{j=0}^{K-1} \mathbb{P}\{T \leq t_j+1|\bar{a}(t_j+1), \mathcal{F}_{/C,t_j}\}^{1}(t_j < t_j+1, \delta = 1) \mathbb{P}\{d\bar{l}(t_j+1)|\bar{a}(t_j+1), \mathcal{F}_{/C,t_j}\} \right.
\]
\[
\{1 - 1(x \leq t_j+1, \delta = 0)\} G\{d\bar{l}(t_j+1)|\bar{a}(t_j)\} - \int \mathbb{E}\{\nu(T_{\bar{a}}, Y_{\bar{a}})\} G(d\bar{a}) \right| + o(1),
\]
which by Assumptions 1, 2, 3, and 4, equals
\[
\left| \int \nu(x, \bar{y}) \mathbb{P}\{T_{\bar{a}} \leq t_K|\bar{a}(t_K), \mathcal{F}_{t_K-1}\}^{1}(t_K < x_{\bar{a}} \leq t_K, \delta_{\bar{a}} = 1) \mathbb{P}\{d\bar{l}(t_K)|\bar{a}(t_K), \mathcal{F}_{t_K-1}\} \right.
\]
\[
\{1 - 1(x_{\bar{a}} \leq t_K, \delta_{\bar{a}} = 0)\} G\{d\bar{l}(t_K)|\bar{a}(t_K-1)\} \prod_{j=0}^{K-1} \mathbb{P}\{T \leq t_j+1|\bar{a}(t_j+1), \mathcal{F}_{/C,t_j}\}^{1}(t_j < t_j+1, \delta = 1) \mathbb{P}\{d\bar{l}(t_j+1)|\bar{a}(t_j+1), \mathcal{F}_{/C,t_j}\} \right.
\]
\[
\{1 - 1(x \leq t_j+1, \delta = 0)\} G\{d\bar{l}(t_j+1)|\bar{a}(t_j)\} - \int \mathbb{E}\{\nu(T_{\bar{a}}, Y_{\bar{a}})\} G(d\bar{a}) \right| + o(1),
\]
14
which by Assumptions 1, 2, 3, and 4, is less than or equal to
\[ \left| \int \nu(x, \bar{y}) \mathbb{P}\{T_a \leq t_K | F_{t_{K-1}}\} \right|^1_{(t_{K-1} \leq x_a \leq t_K, \delta_a = 1)} \]
\[ \mathbb{P}\{T_a > t_K | F_{t_{K-1}}\} \right|^1_{(t_{K-1} \leq x_a \leq t_K, \delta_a = 1)} \mathbb{P}\{d\bar{a}(t_K) | F_{t_{K-1}}\} \]
\[ \{1 - 1(x_a \leq t_K, \delta_a = 0)\} G\{d\bar{a}(t_K) | \bar{a}(t_{K-1})\} \prod_{j=0}^{K-1} \mathbb{P}\{T \leq t_{j+1} | \bar{a}(t_{j+1}), F_{C(t_j)}\} \right|^1_{(t_j < x \leq t_{j+1}, \delta = 1)} \]
\[ \mathbb{P}\{T > t_{j+1} | \bar{a}(t_{j+1}), F_{C(t_j)}\} \right|^1_{(t_j < x \leq t_{j+1}, \delta = 1)} \mathbb{P}\{d\bar{a}(t_{j+1}) | \bar{a}(t_{j+1}), F_{C(t_j)}\} \]
\[ \{1 - 1(x \leq t_{j+1}, \delta = 0)\} G\{d\bar{a}(t_{j+1}) | \bar{a}(t_{j})\} - \int \mathbb{E}\{\nu(T_a, Y_a)\} G(d\bar{a}) + o(1). \]

By iterating the above process for 0 ≤ j ≤ K - 2, we arrive the conclusion.

**A.3 Proof of Theorem 2**

The proof is immediate by noting that
\[ \mathbb{E}\{Q_G(\infty) \nu(X, \bar{Y})\} = \mathbb{E}_G\{\nu(X, \bar{Y})\} = \int_A \mathbb{E}\{\nu(T_a, Y_a)\} G(d\bar{a}), \]
by Theorem 1.

**A.4 Proof of Theorem 3**

We first prove the theorem when H = H_G. Indeed, as the limit and expectation can interchange, we can show that
\[ |\mathbb{E}\left\{ \mathbb{E}_{out, \Delta_K[0, \infty]}(H_G, Q) \right\} | = |\mathbb{E}\left\{ (Q(t_K)[\nu(X, \bar{Y}) - H_G(t_K)]) \right\} | + \sum_{j=1}^{K-1} |\mathbb{E}\left\{ Q(t_j) \left[ \int H_G(t_{j+1}) G\{d\bar{a}(t_{j+1}) | \bar{A}(t_j)\} - H_G(t_j) \right] \right\} | \]
\[ \leq \sum_{j=1}^{K-1} |\mathbb{E}\left\{ Q(t_j) \left[ \int H_G(t_{j+1}) G\{d\bar{a}(t_{j+1}) | \bar{A}(t_j)\} - H_G(t_j) \right] \right\} | \]
\[ \leq \sum_{j=1}^{K-1} |\mathbb{E}\left\{ Q(t_j) \left[ \int H_G(t_{j+1}) G\{d\bar{a}(t_{j+1}) | \bar{A}(t_j)\} - \mathbb{E}_G\{H_G(t_{j+1}) | \mathbb{G}_j\} \right] \right\} | \]
\[ \leq \sum_{j=0}^{K} \kappa \|H_G(t_j) Q(t_j)\|_1(t_{j+1} - t_j)\alpha \]
\[ \leq \kappa \sup_{t} \|H_G(t) Q(t)\|_1 \sum_{j=1}^{K-1} (t_{j+1} - t_j)\alpha \to 0, \]
when \(|\Delta_K[0, \infty]| \to 0\), where we have used the fact that \(H_G(t)\) is a \(P_G\)-martingale and Assumptions 1, 2.

We now proceed to the case when \(Q = Q_G\). Indeed, as the limit and expectation can interchange, we can show that

\[
|E \{\Xi_{\text{trt},\Delta_K[0,\infty]}(H, Q_G)\}| \\
= |E \left( \sum_{j=0}^{K} [Q_G(t_j)H(t_j) - Q_G(t_{j-1}) \int H(t_j)G(d\tilde{a}(t_j)|A(t_{j-1})]\right) \\
- E \left[ Q_G(0)H(0) - \int H(0)G(d\tilde{a}(0)) \right] \\
= |E \left( \sum_{j=0}^{K} [Q_G(t_j)H(t_j) - Q_G(t_{j-1}) \int H(t_j)G(d\tilde{a}(t_j)|A(t_{j-1})]\right) + 0 \\
\leq \sum_{j=0}^{K} |E \left[ Q_G(t_j)H(t_j) - Q_G(t_{j-1}) \int H(t_j)G(d\tilde{a}(t_j)|A(t_{j-1})]\right] \\
= \sum_{j=0}^{K} |E \left[ Q_G(t_{j-1})E_G\{H(t_j)|G_{t_{j-1}}\} \\
- Q_G(t_{j-1}) \int H(t_j)G(d\tilde{a}(t_j)|A(t_{j-1})]\right]P\{d\tilde{l}(t_j)|G_{t_{j-1}}\} | \\
\leq \kappa \sum_{j=0}^{K} \|H(t_{j-1})Q_G(t_{j-1})\|_1(t_j - t_{j-1})^\alpha \\
\leq \kappa \sup_t \|H(t)Q_G(t)\|_1 \sum_{j=0}^{K} (t_j - t_{j-1})^\alpha \to 0,
\]

when \(|\Delta_K[0, \infty]| \to 0\), where we have used the fact that \(Q_G(t)\) is a \(P\)-martingale and Assumptions 1, 2.

### A.5 Proof of Theorem 4

We first simplify our setting by ignoring censoring and absorbing event time \(T\) into \(\bar{L}\) as well. This is because the conditional independent censoring assumption (Assumption 2) is known to be nonparametric. Our observed data become \((\bar{A}, \bar{L})\) and full data become \((\bar{A}, \bar{L}_A)\).

We first proves that Assumption 1 does not have restrictions on the observed data. We proceed with a constructive proof. For any partition \(\Delta_K[0, \infty]\), we define a measure on the full data path space. In fact, one has the knowledge on the decomposition

\[
P(d\tilde{a}d\tilde{l}) = \prod_{j=0}^{K-1} \{P\{d\tilde{a}(t_{j+1})|A_{t_j}\} P\{d\tilde{l}(t_{j+1})|\bar{a}(t_{j+1}), \bar{A}_{t_j}\}\} \\
= \prod_{j=0}^{K-1} \{P\{d\tilde{a}(t_{j+1})|\bar{a}(t_j), \bar{I}_a(t_j)\} P\{d\tilde{l}(t_{j+1})|\bar{a}(t_{j+1}), \bar{I}_a(t_j)\}\}.
\]
Intuitively $\mathbb{P}\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \bar{I}_A(t_j)\}$ are close to $\mathbb{P}\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \bar{I}_A(t_j)\}$, whereas the other term $\mathbb{P}\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \bar{I}(t_j)\}$ is close to $\mathbb{P}\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \bar{I}_A(t_j)\}$. Therefore, one may define a measure on the full data path space by

$$
\mathbb{P}^F_{\Delta_K[0,\infty]}(d\bar{I}_A) := \prod_{j=0}^{K-1} \mathbb{P}\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \bar{I}_A(t_j)\} = \prod_{j=0}^{K-1} \mathbb{P}\{d\bar{a}(t_{j+1})|\bar{a}(t_j), \mathcal{F}_t\}.
$$

Then without loss of generality one may construct $\mathbb{P}^F_{\Delta_K[0,\infty]}(\bar{I}_A)$ by assuming joint independence among $\bar{I}_A$. One can also define

$$
\mathbb{P}^F_{\Delta_K[0,\infty]}(d\bar{a}|\bar{I}_A) := \prod_{j=0}^{K-1} \mathbb{P}\{d\bar{a}(t_{j+1})|\mathcal{F}_t\}.
$$

Then for any sequences of partitions with the mesh going to zero, one may construct a sequence of measures and show that this sequence of measures is Cauchy by a triangular inequality and Assumption 1, following a similar logic as previous proofs. Therefore the sequence converge to a measure $\mathbb{P}^F$, which is independent of the choice of partitions.

Next we need to show that $\mathbb{P}^F$ induces $\mathbb{P}$ on the observed data and $\mathbb{P}^F$ satisfies Assumption 1. The first is trivial because any $\mathbb{P}_{\Delta_K[0,\infty]}^F$ induces $\mathbb{P}$ on the observed data, then so is their limit. To prove the second, for any time $t$ and $\epsilon > 0$, one might smartly choose a partition $\Delta_K[0,\infty]$ with $\mathbb{P}_{\Delta_K[0,\infty]}^F$ close enough to $\mathbb{P}^F$ and $t, t + \eta \in \Delta_K[0,\infty]$. This can be done because the convergence point is independent of the choice of partitions. We have

\[
\begin{align*}
\mathbb{E}^F(\|\mathbb{P}^F\{d\bar{I}_A|\mathcal{F}_t\} - \mathbb{P}^F\{d\bar{I}_A|A(t + \eta), \mathcal{F}_t\}\|_{\text{TV}}) \\
\leq \mathbb{E}^F(\|\mathbb{P}^F\{d\bar{I}_A|\mathcal{F}_t\} - \mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|\mathcal{F}_t\}\|_{\text{TV}}) \\
+ \mathbb{E}^F(\|\mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|\mathcal{F}_t\} - \mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|A(t + \eta), \mathcal{F}_t\}\|_{\text{TV}}) \\
+ \mathbb{E}^F(\|\mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|A(t + \eta), \mathcal{F}_t\} - \mathbb{P}^F\{d\bar{I}_A|A(t + \eta), \mathcal{F}_t\}\|_{\text{TV}}) \\
\leq 2\|\mathbb{P}^F - \mathbb{P}_{\Delta_K[0,\infty]}^F\|_{\text{TV}} \\
+ \mathbb{E}^F(\|\mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|\mathcal{F}_t\} - \mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|A(t + \eta), \mathcal{F}_t\}\|_{\text{TV}}) \\
\leq 4\|\mathbb{P}^F - \mathbb{P}_{\Delta_K[0,\infty]}^F\|_{\text{TV}} \\
+ \mathbb{E}_{\Delta_K[0,\infty]}^F(\|\mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|\mathcal{F}_t\} - \mathbb{P}_{\Delta_K[0,\infty]}^F\{d\bar{I}_A|A(t + \eta), \mathcal{F}_t\}\|_{\text{TV}}).
\end{align*}
\]
The first term can be chosen to be sufficiently small. We rewrite the second term

\[
\mathbb{P}^{F}_{\Delta_K[0,\infty)}(\|\mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{I}_A|\mathcal{F}_t\} - \mathbb{P}^{F}_{\Delta_K[0,\infty]}\{dI_A|A(t+\eta),\mathcal{F}_t\}\|_{TV})
\]

\[
= \sup_{f:||f||_1=1} \int f(\tilde{I}_A, \{y_{\omega}\}_{\omega \in A})[\mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{I}_A|\mathcal{F}_t\} - \mathbb{P}^{F}_{\Delta_K[0,\infty]}\{dI_A|A(t+\eta),\mathcal{F}_t\}] \mathbb{P}\{d\tilde{a}(t+\eta)d\mathcal{F}_t\}
\]

\[
\leq \sup_{f:||f||_1=1} \sup_{g:||g||_1=1} \int f(\tilde{I}_A)g(\tilde{a}(t+\eta))[\mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{I}_A|\mathcal{F}_t\} - \mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{a}(t+\eta)|\mathcal{F}_t\}] \mathbb{P}\{d\mathcal{F}_t\}
\]

\[
= \sup_{f:||f||_1=1} \sup_{g:||g||_1=1} \int f(\tilde{I}_A)g(\tilde{a}(t+\eta))\mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{I}_A|\mathcal{F}_t\}
\]

\[
[\mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{a}(t+\eta)|\tilde{a}(t),\tilde{I}_A\} - \mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{a}(t+\eta)|\mathcal{F}_t\}] \mathbb{P}\{d\mathcal{F}_t\}
\]

\[
= \sup_{f:||f||_1=1} \sup_{g:||g||_1=1} \int f(\tilde{I}_A)g(\tilde{a}(t+\eta))\mathbb{P}^{F}_{\Delta_K[0,\infty]}\{d\tilde{I}_A|\mathcal{F}_t\}
\]

\[
\left[\prod_{j=0}^{K-1} \mathbb{P}\{d\tilde{a}(t+\eta)|\mathcal{F}_t\} - \mathbb{P}\{d\tilde{a}(t+\eta)|\mathcal{F}_t\}\right] \mathbb{P}\{d\mathcal{F}_t\}
\]

\[
\leq \mathbb{E}\left[\prod_{j=0}^{K-1} \mathbb{P}\{d\tilde{a}(t+\eta)|\mathcal{F}_t\} - \mathbb{P}\{d\tilde{a}(t+\eta)|\mathcal{F}_t\}\right]_{TV}
\]

\[
\leq \varepsilon(t,\eta).
\]

Assumption 3 is irrelevant here because it is imposed on the stochastic process but not the distributions.

For Assumption 4, a necessary condition for it to hold is that \(\mathbb{P}\) is well supported on the path space conditioning on any filtration. For this to happen, note that for any \(\mathbb{P}\), one can find a well-supported \(\mathbb{P}'\) satisfying Assumptions 1, 2, 3, so that \((1-\varepsilon)\mathbb{P}+\varepsilon\mathbb{P}'\) is well-supported and hence satisfies Assumption 4. Since addition will not break Assumptions 1, 2, 3, we have found \((1-\varepsilon)\mathbb{P}+\varepsilon\mathbb{P}'\) satisfying Assumptions 1, 2, 3, and 4, and approximates \(\mathbb{P}\).