STABILITY ANALYSIS OF THE HELMHOLTZ OSCILLATOR WITH TIME VARYING MASS.

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Abstract. We study the effects of periodically time varying mass on the stability of the Helmholtz oscillator, which, when linearised, takes the form of Ince’s equation and exhibits parametric resonance. The resonance regions in the parameter space are mapped and we use the Melnikov function to demonstrate that the parametric instability does not affect the qualitative dynamics of the nonlinear system.

1. INTRODUCTION

The recent activity in the application of Nonlinear Dynamics to problems in Naval Architecture is exemplified by the papers in [13]. One particular success has been a more rational understanding of beam sea capsize and the development of a robust mathematical toolkit for the study of this problem.

Following the work of Wright and Marshfield, [20] it has become common to study the roll dynamics using a one degree of freedom nonlinear oscillator taking the form

\[ \ddot{\theta} + g(\dot{\theta}) + G_Z(\theta) = F(t). \]

Here \( \theta \) is the roll angle relative to the wave normal, \( g(\dot{\theta}) \) is the, often linear–viscous, damping function, and \( G_Z(\theta) \) (so denoted as it is usually some polynomial fit to the GZ curve) encapsulates the restoring moment.

The forcing, \( F(t) \), is often taken to be a simple sinusoidal function, see for example [17], although stochastic models of forcing are also studied, e.g. [3].

Observing that capsize is equivalent to the roll angle exceeding the angle of vanishing stability leads to the consideration of the more general dynamical problem of escape of a particle from a potential well. The Helmholtz oscillator

\[ \ddot{x} + \beta \dot{x} + x - x^2 = F \sin \omega t, \]

known to Naval–Architects as the Helmholtz–Thompson equation, provides a simple archetype to study the phenomena of escape directly. It has been central to a number of developments. Thompson’s paper [14] provides an overview of the dynamic response of the system which has been studied experimentally by Gottwald et. al. [2]. The inhibition of chaotic escape is considered in the context of (2) in [14] and a more general approach to the problem is discussed in [6]. The equation finds direct application in the study of bubble dynamics [5] and is much discussed in the Naval Architecture literature, see [14]. The engineering integrity diagram, [11] and the use of Melnikov theory to predict parameter values for which erosion of basins of attraction takes place were developed in this context. These concepts continue to find fruitful application in quantification of capsize resistance, see [12].

Providing a reasonable nonlinear dynamical theory of the stability of a damaged ship remains a considerable challenge. Compared with intact stability there are a number of dynamical differences which effect ship motion. The mass of a progressively flooding ship changes with time. The motion becomes coupled with that of water on deck or in compartments. The water itself provides non-trivial modelling problems. Masses of water lead to biases which may also vary with time.

The increased complexity means that one is forced to make a choice between sacrificing dynamical information by making simplifying assumptions or dynamical insight by building less simple comprehensive models.

When making assumptions to simplify a model enough that some theoretical analysis is possible there is a danger of assuming so much that all physical meaning is lost. The balance between finding a model that is simple enough to analyze yet still manages to capture the essence of some real physical dynamics is difficult to attain. One successful example is found in Murashige et al’s analysis of flooded ship motion, see [9] and references therein.
In this paper we choose a different approach. The dynamic phenomena listed above will have varying importance in building a comprehensive theory of damage stability. Rather than start with a physical model and make simplifying assumptions, we have chosen to put one of these effects, time varying mass, directly under the microscope. Whilst some properties of time varying mass dynamics which will influence the behaviour of any model whose mass has a periodic component are discussed, we make no claim to direct physical relevance.

The model we study is a Helmholtz oscillator with time varying mass. We use the generalised form of Newton’s Second Law,

$$F = \frac{d}{dt}(m(t)\dot{x})$$

so that an oscillator modelled as $$m\ddot{x} + g(x, \dot{x}) = F(t)$$ whose mass varies with time becomes

$$m(t)\ddot{x} + g(x, \dot{x}) + \dot{m}(t)\dot{x} = F(t).$$

Equation (2) becomes

$$\ddot{x} + \beta + \dot{m}(t)\dot{x} + \frac{x - x^2}{m(t)} = \frac{F \sin \omega_F t}{m(t)}$$

where we have divided through by $$m(t)$$.

In this paper we study mass varying sinusoidally with time and choose the simple form

$$m(t) = \delta + \gamma \sin \omega_m t.$$ 

Substituting (6) into (5) and writing $$\hat{\delta} = 1/\delta$$ (in order to simplify the perturbation calculations which follow) we have

$$\ddot{x} + \hat{\delta} \beta + \hat{\delta} \gamma \omega_m \cos \omega_m t \dot{x} + \frac{x - x^2}{\omega_m 1 + \gamma \delta \sin \omega_m t} = \hat{\delta} F \sin \omega_F t 1 + \gamma \delta \sin \omega_m t.$$

2. LINEAR STABILITY ANALYSIS OF FIXED POINTS

Our first step is to consider the linear stability of the two fixed points of the system with $$F = 0$$. We start by scaling time in order to normalise the frequency of the mass variation. Setting $$\tau = \omega_m t$$ yields

$$\ddot{x} + \frac{\hat{\delta}}{1 + \gamma \delta \sin \omega_m t} \ddot{x} + \frac{\hat{\delta}}{\omega_m 1 + \gamma \delta \sin \omega_m t} \frac{x - x^2}{1 + \gamma \delta \sin \omega_m t} = 0.$$ 

after relabelling $$t = \tau$$. Writing (8) as a system

(9a) $$\dot{x} = y$$

(9b) $$\dot{y} = -\frac{\hat{\delta}}{\omega_m^2 1 + \gamma \delta \sin \omega_m t} x - x^2 - \frac{\hat{\delta}}{1 + \gamma \delta \sin \omega_m t} \frac{\hat{\delta}}{\omega_m 1 + \gamma \delta \sin \omega_m t} y$$

we see that the two fixed points are $$(x, y) = (0, 0)$$ and $$(1, 0)$$. We know that for $$\gamma = 0$$ the origin is a stable focus or centre as $$\beta > 0$$ or $$\beta = 0$$ respectively. The point $$(1, 0)$$ is a saddle independent of the value of $$\beta$$. For both fixed points the linearised system takes the form

$$\dot{x} = A(t)x$$

where $$A(t) = A(t + 2\pi)$$ is a periodic matrix and we are in the setting of Floquet theory. For the saddle point we have

$$A_{(1,0)}(t) = \begin{pmatrix} 0 & 1 \\ \frac{\hat{\delta}}{1 + \gamma \delta \sin \omega_m t} & -\frac{\hat{\delta}}{1 + \gamma \delta \sin \omega_m t} \end{pmatrix}.$$ 

The origin gives

$$A_{(0,0)}(t) = \begin{pmatrix} 0 & 1 \\ \frac{\hat{\delta}}{1 + \gamma \delta \sin \omega_m t} & -\frac{\hat{\delta}}{1 + \gamma \delta \sin \omega_m t} \end{pmatrix}.$$
THE FIXED POINT AT (1, 0). The linear stability analysis of this fixed point does not require the full machinery of Floquet theory. The ODE linearised about this point is
\[
\ddot{x} + \frac{\beta}{\omega_m} + \hat{\delta} \gamma \cos t \dot{x} - \frac{\hat{\delta}}{1 + \gamma \hat{\delta} \sin t} x = 0.
\]
Upon expanding \( x = x_0 + \gamma x_1 + O(\gamma^2) \) and \( \hat{\delta} = \hat{\delta}_0 + \gamma \hat{\delta}_1 + O(\gamma^2) \) and collecting terms in \( \gamma \) we see that the \( O(1) \) equation is
\[
\ddot{x}_0 + \frac{\beta}{\omega_m} \dot{x}_0 - \hat{\delta}_0 x_0 = 0.
\]
The origin in this equation is stable only when \( \hat{\delta}_0 < 0 \) which is physically meaningless. We conclude that the fixed point of the nonlinear system is unstable for all \( \hat{\delta} > 0 \), regardless of the value of \( \beta \) and \( \omega_m \).

THE FIXED POINT AT THE ORIGIN. The ODE linearised about the origin is
\[
(1 + a \cos 2t) \ddot{x} + b \sin 2t \dot{x} + (c + d \cos 2t) x = 0,
\]
see [10], we also translate time \( \tau \mapsto \tau + 2\pi \). Relabelling \( \tau = t \) for notational convenience we have
\[
(1 + \hat{\delta} \gamma \cos 2t) \ddot{x} + 2 \left( \frac{\beta}{\omega_m} - \hat{\delta} \gamma \sin 2t \right) \dot{x} + 4 \frac{\delta}{\omega_m} x = 0.
\]
When \( \beta = 0 \) this is Ince’s equation with \( a = \hat{\delta} \gamma \), \( b = -2 \hat{\delta} \gamma \), \( c = 4 \frac{\delta}{\omega_m} \) and \( d = 0 \). We will consider \( \beta \neq 0 \) when the \( \beta = 0 \) dynamics are understood.

Remark. The scaling of time has no effect on the stability of the origin in the linearised system. The stable parameters for system [12] are exactly those that are stable for system [13].

As transition curves are characterised by the existence of solutions of period \( 2\pi \) and \( 4\pi \) we use harmonic balance to seek solutions of this form. Substituting \( x = \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt \) into [13], combining trigonometric terms and equating coefficients of \( \cos nt \) and \( \sin nt \) leads to an infinite set of linear homogeneous equations in the Fourier coefficients \( a_i \) and \( b_i \). The first few of these are
\[
\begin{align*}
4 \frac{\delta}{\omega_m^2} a_0 &= 0 \\
(1 + \frac{\hat{\delta} \gamma}{2} - 4 \frac{\delta}{\omega_m^2}) a_1 - \frac{3}{2} \hat{\delta} \gamma a_3 &= 0 \\
(1 - \frac{\hat{\delta} \gamma}{2} - 4 \frac{\delta}{\omega_m^2}) b_1 - \frac{3}{2} \hat{\delta} \gamma b_3 &= 0 \\
(4 \frac{\delta}{\omega_m^2} - 4) a_2 - 4 \hat{\delta} \gamma a_4 &= 0 \\
&\vdots
\end{align*}
\]
The existence of a transition curve in the parameter space is equivalent to the existence of a nontrivial solution to this system of equations. This in turn is equivalent to the vanishing of the associated determinants of the
system. This last condition says that the transition curves in the parameter space are composed of points which satisfy the equations

\[
\Delta_{a_{2i}} = \begin{pmatrix}
\frac{4 - \delta}{\omega_m} & -2\gamma & 0 & 0 & 0 & \cdots \\
0 & \frac{4 - \delta}{\omega_m} & -4\gamma & 0 & 0 & \cdots \\
0 & 0 & 4\frac{\delta}{\omega_m} - 16 & 0 & 0 & \cdots \\
0 & 0 & 0 & 4\frac{\delta}{\omega_m} - 36 & 0 & \cdots \\
0 & 0 & 0 & 0 & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
= 0,
\]

(14a)

\[
\Delta_{b_{2i}} = \begin{pmatrix}
\frac{4 - \delta}{\omega_m} & -4\gamma & 0 & 0 & 0 & \cdots \\
0 & -4\gamma & 4\frac{\delta}{\omega_m} - 16 & 0 & 0 & \cdots \\
0 & 0 & -12\gamma & 4\frac{\delta}{\omega_m} - 36 & 0 & \cdots \\
0 & 0 & 0 & -24\gamma & 4\frac{\delta}{\omega_m} - 64 & \cdots \\
0 & 0 & 0 & 0 & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
= 0,
\]

(14b)

\[
\Delta_{a_{2i+1}} = \begin{pmatrix}
\frac{4 - \delta}{\omega_m} & \frac{3}{2}\gamma & 0 & 0 & 0 & \cdots \\
-\frac{4}{\gamma} & \frac{4 - \delta}{\omega_m} & -9 & 0 & 0 & \cdots \\
0 & 0 & -15\gamma & 4\frac{\delta}{\omega_m} - 25 & 0 & \cdots \\
0 & 0 & 0 & -45\gamma & 4\frac{\delta}{\omega_m} - 49 & \cdots \\
0 & 0 & 0 & 0 & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
= 0
\]

(14c)

\[
\Delta_{b_{2i+1}} = \begin{pmatrix}
\frac{4 - \delta}{\omega_m} & -\frac{3}{2}\gamma & 0 & 0 & 0 & \cdots \\
-\frac{4}{\gamma} & \frac{4 - \delta}{\omega_m} & -9 & 0 & 0 & \cdots \\
0 & 0 & -15\gamma & 4\frac{\delta}{\omega_m} - 25 & 0 & \cdots \\
0 & 0 & 0 & -45\gamma & 4\frac{\delta}{\omega_m} - 49 & \cdots \\
0 & 0 & 0 & 0 & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
= 0
\]

(14d)

We note that

\[
\Delta_{a_{2i}} = \frac{\delta}{\omega_m^2}\Delta_{b_{2i}}.
\]

This fact has two consequences. First of all it is clear that \(\delta = 0\) is a solution of (14a) so that the first transition curve we find is

\[
\delta = 0.
\]

Secondly all other zeros of \(\Delta_{a_{2i}} = 0\) will coincide with the zeros of \(\Delta_{b_{2i}} = 0\). Upon setting \(\gamma = 0\) it is clear that these curves emanate from the points

\[
\delta = \left(\frac{2\pi \omega_m}{2}\right)^2, \ n \in \{1, 2, 3, \ldots\}.
\]

However because they coincide the regions they bound have zero width and no instability occurs. The coincidence of two transition curves and associated vanishing of a regime of instability is known as coexistence see [7] or [10] for details.

Remark. The lack of instability is not a consequence of the vanishing width of the unstable regime. Periodic solutions exist on the transition curves and those on the \(\Delta_{a_{2i}} = 0\) are even functions of \(t\). On the \(\Delta_{b_{2i}} = 0\) the solutions are odd. As odd and even solutions are linearly independent we have, on two coinciding curves, a spanning set for the vector space of solutions. A solution space spanned by bounded solutions cannot contain an unbounded solution.
In order to evaluate the solution curves of $\Delta a_{2i+1} = 0$ and $\Delta b_{2i+1} = 0$ we substitute

$\hat{\delta} = \hat{\delta}_0 + \hat{\delta}_1 \gamma + \hat{\delta}_2 \gamma^2 + \hat{\delta}_3 \gamma^3 + \hat{\delta}_4 \gamma^4 + \ldots$

into a finite, $n \times n$ where $n$ is a small integer, truncation of the determinants $\Delta a_{2i+1}$ and $\Delta b_{2i+1}$, equate to zero and collect coefficients of $\gamma$ in the usual fashion. Using Maple and determinants of order 25 $\times$ 25 we can easily write down the first few curves.

$$\hat{\delta} = \frac{\omega_m^2}{4} - \frac{\omega_m^4}{32} \gamma + O(\gamma^2)$$  \hspace{1cm} (16a)

$$\hat{\delta} = \frac{\omega_m^2}{4} + \frac{\omega_m^4}{32} \gamma + O(\gamma^2)$$  \hspace{1cm} (16b)

$$\hat{\delta} = \frac{9}{4} \omega_m^2 - \frac{16767}{4096} \omega_m^6 \gamma^2 - \frac{6561}{131072} \omega_m^8 \gamma^3 + O(\gamma^4)$$  \hspace{1cm} (16c)

$$\hat{\delta} = \frac{9}{4} \omega_m^2 + \frac{16767}{4096} \omega_m^6 \gamma^2 + \frac{6561}{131072} \omega_m^8 \gamma^3 + O(\gamma^4)$$  \hspace{1cm} (16d)

$$\hat{\delta} = \frac{25}{4} \omega_m^2 - \frac{1109375}{12288} \omega_m^6 \gamma^2 + \frac{3030048828125}{1811939328} \omega_m^{10} \gamma^4 + 2197265625 \omega_m^{12} \gamma^5 + O(\gamma^6)$$  \hspace{1cm} (16e)

$$\hat{\delta} = \frac{25}{4} \omega_m^2 - \frac{1109375}{12288} \omega_m^6 \gamma^2 + \frac{3030048828125}{1811939328} \omega_m^{10} \gamma^4 - 2197265625 \omega_m^{12} \gamma^5 + O(\gamma^6)$$  \hspace{1cm} (16f)

As the equations of the curves suggest, it is possible to prove:

**Proposition 1.** The pair of transition curves emanating from the point

$$\hat{\delta} = \left( \frac{(2k - 1) \omega_m}{2} \right)^2$$

coincide to order $\gamma^{2k-2}$. They split at order $\gamma^{2k-1}$.

The geometry of this fact is that as $\hat{\delta}$ grows the curves becomes thinner and thinner. The areas of the parameter space in which instability occurs decreases. Figure 1 illustrates the first three regions of instability, all of which appear thin on this scale. A closer look at the first region (along with the effect of $\beta > 0$, discussed below) is seen in Figure 2.

**Remark.** We have not mentioned that a mathematically rigourous treatment would have to address the divergence of the determinants in their current form. The solution to this problem involves noting that the individual coefficients of $\gamma^n$ are defined as ratios of infinite determinants. One must then consider the coefficient as the ratio of two $k \times k$ determinants, let $k \to \infty$, and show that the expression converges.
This, along with the proof of Proposition 1, are essentially exercises in algebra and we do not reproduce the calculations here.

When $\beta \neq 0$ the above analysis fails as the systems of equations for the coefficients of the Fourier series do not decouple in a nice way. We use a perturbation argument instead to illustrate the effect of $\beta$. It is necessary to assume that $\beta$ is of the same order as $\gamma$ which we consider to be a small parameter. Thus we write $\beta = \gamma \hat{\beta}$ and we are studying

$$\ddot{x} + \gamma \hat{\beta} \frac{\dot{x}}{1 + \gamma \hat{\beta} \sin \omega_m t} \dot{x} + \frac{\hat{\delta}}{\omega_m^2 (1 + \gamma \hat{\beta} \sin \omega_m t)} x = 0. \tag{17}$$

We expand $\hat{\delta} = \hat{\delta}_0 + \hat{\delta}_1 \gamma + \hat{\delta}_2 \gamma^2 + O(\gamma^3)$, $x = x_0 + x_1 \gamma + x_2 \gamma^2 + O(\gamma^3)$, substitute and collect coefficients in the usual way. Considering $\hat{\delta}_0 = \frac{\omega_m^2}{4}$ we see that

$$\hat{\delta}_1 = \pm \frac{\omega_m^2}{32} \sqrt{\omega_m^2 - 16 \hat{\beta}^2}. \tag{18}$$

Now remembering that $\gamma \hat{\beta} = \beta$ we have

$$\hat{\delta} = \frac{\omega_m^2}{4} \pm \frac{\omega_m^2}{32} \sqrt{\omega_m^2 - 16 \beta^2} + O(\gamma^2) \tag{19}$$

In Figure 2 we see that the effect of $\beta > 0$ is to lift the curves from the $\gamma = 0$ axis, increasing the amplitude of mass variation that is required to produce an instability. Figure 3 illustrates the numerical response of the system in $(x, t)$ space. The left hand graph corresponds to parameters outwith the region of parametric resonance, the right has parameters inside the region.

### 3. THE MELNIKOV FUNCTION FOR THE NONLINEAR SYSTEM

Having understood the linear dynamics of the two fixed points we now wish to consider the full nonlinear problem. We will use Melnikov Theory (see [15] or [19]) to understand which parameter values support homoclinic tangling (and therefore safe basin erosion) and compare the analytic results with numerically calculated integrity curves.

To calculate the Melnikov Function we need to consider $(\hat{\delta})$ as a nearly integrable system taking the form

$$\dot{x} = f(x) + \epsilon g(x, t).$$

We assume that $F, \gamma,$ and $\beta$ are $O(\epsilon)$ and expand

$$\frac{\hat{\delta}}{1 + \gamma \hat{\beta} \sin \omega_m t} = 1 - \gamma \hat{\beta} \sin \omega_m t + O(\gamma^2).$$

**Figure 2.** The First Region of Instability (With and Without Damping). $\omega_m = 1$
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Figure 3. Trajectories of the Linear System. The left hand solution has parameters just outside the resonant region. The right hand, just inside. In both cases the solution were plotted using the Matlab ode45 routine with initial condition $(1, 0)$.

Writing $F = \epsilon F$, $\gamma = c \gamma$ and $\beta = c \beta$ for notational convenience we have (7) in the form (19) and the Melnikov function is

$$M(t_0) = \int_{-\infty}^{\infty} \left( F \delta \sin (\omega_F (t + t_0)) y_h(t) - \delta \beta y_h^2(t) - \delta \omega_m \cos (\omega_m (t + t_0)) y_h^2(t) \right. $$

$$\left. + \gamma \delta^2 \sin (\omega_m (t + t_0)) (x_h(t) - x_h^2(t)) y_h(t) \right) dt. $$

Here

$$x_h(t) = 1 - \frac{3}{1 + \cosh \sqrt{\delta} t}, \quad y_h = \dot{x}_h$$

is the parameterised orbit homoclinic to $(1, 0)$. Using trigonometric reduction, changes of variables and residue calculus this evaluates as

$$M(t_0) = \frac{6F \omega^2 \pi \cos \omega Ft_0}{\sinh \frac{\pi \delta}{\sqrt{\delta}}} - \frac{6 \delta^2 \beta}{5} - \frac{\gamma \pi \omega^2 m \cos \omega m t_0}{5 \delta} \left( \delta^2 - \omega^4 m \right).$$

Notice that when $\delta = 1$ and $\gamma = 0$ this is identical to the Melnikov function for the standard Helmholtz oscillator, [15].

**Remark.** In cases where $\omega_m \neq \omega_F$ and in particular when $\omega_m/\omega_F \notin \mathbb{Q}$ we must rely on the generalisation of Melnikov theory to quasi–periodic perturbations due to Meyer and Sell [8] and Wiggins [18]. In this paper Melnikov Function is used as a synonym for Generalised Quasi–Periodic Melnikov Function.

The Effect of Linear Resonance on Basin Erosion. To estimate the minimal $F$ for which erosion is expected we set $M(t_0) = 0$ and solve the function for $F$, selecting $t_0$ to minimise as appropriate. The first question we wish to answer is does basin erosion differ depending on whither or not we are located within a region of parametric instability. Consider the region illustrated in Figure 2. In this case $\omega_m = 1$ and the region is centred at $\delta = 0.25$. First we fix $\beta = 0.01$, $\delta = 0.25$ and vary $\gamma$ to cross the boundary of the region. With these parameters the Melnikov estimate of $F$ looks like

$$F_M(\gamma, \omega_F) = \frac{0.0015 \sinh 2\pi - 2.25 \pi \gamma \sinh 2\pi \omega_F}{6\pi \omega_F^2 \sinh 2\pi}$$

and for fixed $\omega_F$ this is just a linear function of $\gamma$, see Figure 4.

If the presence of parametric instability had any interesting effect upon basin erosion we would expect some effect on $F_M$ as $\gamma$ moved across the stability boundary. There is, however, no cusp, blip, change in gradient or other effect in the Melnikov estimate as the parameters move across the transition curve.
The parametric resonance has no effect on the speed of homoclinic tangling and therefore no import for basin erosion. As the tangling and erosion processes are governed by the invariant manifolds of the saddle this result is unsurprising. Intuitively the homoclinic orbit does not see the effect of the local dynamics at the other fixed point. Time varying mass is, however, important in its quantitative effects on basin erosion.

**Time Varying Mass as Excitation.** Notice in Figure 4 that when is slightly greater than 0.055 the lines all cross $F_M = 0$. To understand this we consider the Melnikov Function with $F = 0$ as an estimator of $\gamma$. We have

$$M(t_0) = -\frac{6}{5} \delta^2 \beta - \gamma \frac{3 \pi \omega_m^2}{\delta} \frac{\cos \omega_m t_0}{\sinh \frac{\pi \omega_m}{\sqrt{\delta}}} \left( \delta^2 - \omega_m^4 \right).$$

The energy balance between the linear damping and the excitation provided by mass variation which yields a zero of the Melnikov Function occurs when

$$\gamma = \frac{2 \delta^2 \beta \sinh \frac{\omega_m}{\sqrt{\delta}}}{\omega_m^2 \pi \left( \omega_m^4 - \delta^2 \right) \cos \omega_m t_0}.$$  

For the parameters in Figure 4 this evaluates to $\gamma \approx 0.05681$, so for $\gamma$ above this value the mass variation alone is enough to cause basin erosion. It is worth noting that this also occurs away from regions of parametric instability. For $\omega_m = \pi$ there is no parametric resonance near $\delta = 1$, yet when $F = 0$, $\gamma_M \approx 6.467\beta$ still provides a zero of $M(0)$.

**The General Effect of Time Varying Mass.** The form of the Melnikov Function for periodic time varying mass (20) can be considered as the usual Helmholtz–Thompson Melnikov function plus a correction which takes into account the mass variation. The mass varying term is

$$-\frac{3}{5} \frac{\pi \omega_m^2}{\delta} \frac{\cos \omega_m t_0}{\sinh \frac{\omega_m}{\sqrt{\delta}}} \left( \delta^2 - \omega_m^4 \right)$$

and this changes sign at $\delta = \pm \omega_m$. This means that when $\delta = \omega_m^2$ the time varying mass does not affect the Melnikov estimate of the first homoclinic tangle. Figure 5 is the engineering integrity diagram for this case. Although there is some minor quantitative effect on the size of the initial safe basin due to $\gamma > 0$ the position
of the Dover Cliff reduction in basin area is not affected. If we set $M(t_0) = 0$ and solve for $F$ as usual we get

$$F = \frac{\delta^2 \beta}{5 \pi \omega_F^2} \sinh \frac{\pi \omega_F}{\sqrt{\delta}} + \gamma \frac{\pi \omega_m^2}{10 \pi \omega_F^2} \sinh \frac{\pi \omega_m}{\sqrt{\delta}} \cos \omega_m t_0 \left( \delta^2 - \omega_F^4 \right).$$

Assuming $\gamma > 0$ this makes it clear that when $\delta^2 > \omega_m^4$ the effect of the time varying mass is to delay basin erosion. When $\delta^2 < \omega_m^4$ the basin erosion occurs for smaller $F$. This effect is obvious in the integrity diagrams in Figures 6 and 7.

4. CONCLUSIONS

When Time Varying Mass in an oscillator is taken to be periodic the effect is similar to parametric forcing. Mathieu type resonance occurs although due to coexistence only half as many regimes of parametric resonance exist. Also as the parameter $\delta$ grows, i.e. as the mean mass of the oscillator tends to zero, the regimes of instability become thinner and thinner. Whilst the linear parametric resonance does not have a direct qualitative effect on the dynamics of basin erosion, the quantitative effects are significant and depend critically on the relationship between the mean mass and the frequency of the variation. In some cases it can be broadly considered a stabilising influence whilst in others the opposite is true.

In a realistic model of a damaged ship the assumption of (sinusoidal) periodicity for the variation of the mass due to water ingress is unlikely to be realised exactly. However it is often the case that flood water mass has some periodic component. When this occurs the effects studied in this paper are likely to be an important element of a complete picture of the dynamics.

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Figure 6. $\hat{\delta} = 1, \omega_m = 0.85$

Figure 7. $\hat{\delta} = 1, \omega_m = 1.15$

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