NON-GAUSSIAN WAVES IN ŠEBA’S BILLIARD

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Abstract. The Šeba billiard, a rectangular torus with a point scatterer, is a popular model to study the transition between integrability and chaos in quantum systems. Whereas such billiards are classically essentially integrable, they may display features such as quantum ergodicity [11] which are usually associated with quantum systems whose classical dynamics is chaotic. Šeba proposed that the eigenfunctions of toral point scatterers should also satisfy Berry’s random wave conjecture, which implies that the value distribution of the eigenfunctions ought to be Gaussian.

We prove a conjecture of Keating, Marklof and Winn who suggested that Šeba billiards with irrational aspect ratio violate the random wave conjecture. More precisely, in the case of diophantine tori, we construct a subsequence of the set of new eigenfunctions having even/even symmetry, of essentially full density, and show that its fourth moment is not consistent with a Gaussian value distribution. In fact, given any set Λ interlacing with the set of unperturbed eigenvalues, we show non-Gaussian value distribution of the Green’s functions $G_λ$, for $λ$ in an essentially full density subsequence of Λ.

1. Introduction

Šeba’s billiard, a rectangular billiard $M$ with irrational aspect ratio and a Dirac mass placed in its interior, is a popular model in the field of Quantum Chaos to investigate the transition between chaos and integrability in quantum systems. The model was originally proposed by Petr Šeba in 1990 [14] and has since attracted much attention in the literature [6, 4, 5, 2, 3, 10, 13, 17, 11, 15, 9, 12]. Although, the Dirac mass only affects a measure zero subset of trajectories in phase space and thus has essentially no effect on the classical dynamics, Šeba argued that the wave functions of the associated quantized billiard may display similar features as quantum systems which are classically chaotic.

In particular, Šeba conjectured that the wave functions should obey Berry’s random wave model, i.e. be well approximated by a superposition of monochromatic random waves as the eigenvalue tends to infinity. A consequence of this conjecture is that the moments of the eigenfunctions should...
converge to the Gaussian moments in the limit as the eigenvalue tends to infinity (cf. [1], p. 240, eqs. (78-80)). In particular, denoting an $L^2$-normalized (real) wave function with eigenvalue $\lambda$ by $\psi_\lambda$, one expects that the fourth moment of $\psi_\lambda$ (possibly after excluding a zero density subsequence of exceptional eigenvalues) converge to the corresponding Gaussian moment as $\lambda \to \infty$, namely that
\[
E(\psi^4_\lambda) = \int_{\mathcal{M}} \psi^4_\lambda d\mu \to 3,
\]
where $d\mu = d\mu_{\text{Leb.}}/\text{vol}(\mathcal{M})$ denotes the normalized Lebesgue measure.

Šeba calculated the value distribution for high energy wave functions and found seemingly strong numerical evidence for a Gaussian value distribution in line with Berry’s predictions. Later Keating, Marklof and Winn cast doubt on Šeba’s conjecture when they showed that quantum star graphs, a model known to be similar in behaviour to Šeba’s billiard, did indeed violate the random wave model [10, 3].

In this paper we put this matter to rest by showing that for a Šeba billiard with diophantine aspect ratio (a condition that holds generically), the fourth moment of the eigenfunction cannot tend to a Gaussian. In fact we can find a subsequence of arbitrarily high density such that the moment stays bounded away from the Gaussian moment as the eigenvalue tends to infinity. In fact, our results are valid for any sequence of numbers which interlace with the Laplace eigenvalues, in particular for the new eigenvalues of the both the weak and the strong coupling quantizations of the Šeba billiard. The former arises from von Neumann’s theory of self-adjoint extensions, whereas the latter, investigated numerically in Šeba’s paper, uses a different renormalization which is considered more physically relevant (cf. [16] for a detailed discussion of weak and strong coupling quantizations.)

1.1. Background. Before we state the results, let us recall the mathematical definition of Šeba’s billiard. In this paper we will mainly focus on periodic boundary conditions (the case of Dirichlet boundary conditions is treated in the Appendix) and thus deal with a flat 2-torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathcal{L}$, where $\mathcal{L} = \mathbb{Z}(a, 0) \oplus \mathbb{Z}(0, 1/a)$ for some $a > 0$ such that $a^4$ is a diophantine number (cf. [7, Ch. 2.8]). The formal Schrödinger operator associated with a Dirac mass placed at the point $x_0 \in \mathbb{T}^2$ is given by
\[
-\Delta + \alpha \delta_{x_0}.
\]
This formal operator may be associated with a one-parameter family of self-adjoint extensions of the restricted positive Laplacian $-\Delta|_{C^\infty(\mathbb{T}^2 - \{x_0\})}$. For the details of this theory we refer the reader to the introduction and appendix of the paper [13]. We adopt the notation of this paper and refer to the self-adjoint extensions as $-\Delta_\varphi$, where $\varphi \in (-\pi, \pi)$ is the extension parameter.

One of the key features of the spectral theory of the operator $-\Delta_\varphi$ is that it represents a rank-one perturbation of the Laplacian. That is, for
each Laplace eigenspace the perturbation “tears off” a new eigenvalue, and
the spectrum of $-\Delta_0$ therefore consists of two parts: the “old” and the
“new” eigenvalues. The multiplicity of each old eigenvalue is reduced by
one and the corresponding eigenspace is just the co-dimension one subspace
of Laplace eigenfunctions which vanish at $x_0$. This part of the spectrum is
therefore not affected by the presence of the Dirac mass. On the other hand,
the new part of the spectrum “feels” the presence of the scatterer and the
4th moment of these “new eigenfunctions” will be the focus of this paper.

The new eigenvalues interlace with the old Laplace eigenvalues and the
associated eigenfunctions are just Green’s functions which have the following
$L^2$-expansion:

$$G_\lambda(x, x_0) = \sum_{\xi \in \mathcal{L}} \frac{e^{i\langle \xi, x-x_0 \rangle}}{|\xi|^2 - \lambda},$$

with the following formula for the 2nd moment:

$$\int_{\mathbb{T}^2} |G_\lambda(x, x_0)|^2\,d\mu(x) = \sum_{\xi \in \mathcal{L}} \frac{1}{(|\xi|^2 - \lambda)^2},$$

where $d\mu(x) = dx/(4\pi^2)$ denotes the normalized Lebesgue measure on $\mathbb{T}^2$.

The set of new eigenvalues can be determined as the solutions of a spectral
equation [13]. There is in fact another quantization condition — known as
a strong coupling quantization — which is considered more relevant in the
physics literature and requires a renormalization of the self-adjoint extension
parameter $\varphi$ as the eigenvalue $\lambda$ increases. This leads to a different spectral
equation, but as our results will in fact hold for any sequence which interlaces
with the unperturbed eigenvalues we will not dwell on this matter (details
can be found [16, 15].)

1.2. Results. Let us denote by $g_\lambda = G_\lambda/\|G_\lambda\|_2$ the $L^2$-normalized new
eigenfunctions. The following theorem is our main result and shows that
the fourth moment of eigenfunctions of Šeba’s billiard is not Gaussian, in
particular that the value distribution of the wave functions is not consistent
with a Gaussian distribution in the limit as the eigenvalue $\lambda$ tends to infinity
— a contradiction to Berry’s random wave model.

**Theorem 1.1.** Consider a 2-torus with diophantine aspect ratio, and let
$\Lambda \subset \mathbb{R}$ denote any subset interlacing with the set of unperturbed eigenvalues.
Given $\epsilon \in (0, 1)$ there exists a subsequence of $\Lambda$, of relative density $1 - \epsilon$, and
a constant $C_\epsilon > 0$ such that for $\lambda$ tending to infinity along said subsequence
we have

$$1 - o(1) \leq \mathbb{E}(g_\lambda^4) \leq 3 - C_\epsilon + o(1).$$

**Remark.** As the torus is homogenous we may place the scatterer at $0$ and it
is then natural to desymmetrize with respect to odd/even-ness vis-a-vis horizontal and vertical reflections. The set of new eigenfunctions is then exactly
the eigenspaces having even/even-invariance. Thus, within this symmetry
class, essentially all of the eigenfunctions have non-Gaussian fourth moments.

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2. Approximating the 4th moment

2.1. \( L^4 \) convergence. Let \( \mathcal{L} \) be an irrational rectangular unimodular lattice and consider the 2-torus \( \mathbb{T}^2 = \mathbb{R}^2 / 2\pi \mathcal{L} \). Fix \( \lambda > 0 \) a new eigenvalue. We define \( c_\lambda(\xi) := (|\xi|^2 - \lambda)^{-1} \) and we take \( L = L(\lambda) \) to be an increasing function such that \( L \to +\infty \) as \( \lambda \to +\infty \).

The following expansion for the Green’s function holds in the \( L^2 \)-sense:

\[
G_\lambda(x) := G_\lambda(x, 0) = \sum_{\xi \in \mathcal{L}} c_\lambda(\xi) e^{i \xi \cdot x}
\]

(without loss of generality we may assume that \( x_0 = 0 \).) Our aim is, first of all, to show that this expansion also holds in the \( L^4 \)-sense. We thus introduce the truncated Green’s function

\[
G_T^\lambda(x) = \sum_{\xi \in \mathcal{L}, |\xi| \leq T} c_\lambda(\xi) e^{i \xi \cdot x}, \quad T = T(\lambda) := 10 \lambda^{1/2},
\]

and show that \( G_T^\lambda \) converges in \( L^4(\mathbb{T}^2) \), as \( T \to \infty \).

We will achieve this by showing that \( G_T^\lambda \) is Cauchy in \( L^4(\mathbb{T}^2) \), in particular we will bound the \( L^4 \)-norm of the difference \( G_T^{2\lambda} - G_T^\lambda \). Letting

\[
A(T) := \{v \in \mathcal{L} : |v| \in [T, 2T]\}
\]

we then have (recall \( T = 10 \lambda^{1/2} \), and thus \( c_\lambda(v) > 0 \) for \( v \in A(T) \))

\[
\int_{\mathbb{T}^2} |G_T^{2\lambda}(x) - G_T^\lambda(x)|^4 d\mu(x) = \sum_{v_1, v_2, v_3, v_4 \in A(T) : \sum_{i=1}^4 v_i = 0} \prod_{i=1}^4 c_\lambda(v_i)
\]

\[
\ll \sum_{v_1, v_2, v_3, v_4 \in A(T) : \sum_{i=1}^4 v_i = 0} \frac{1}{|v_1|^2 |v_2|^2 |v_3|^2 |v_4|^2}
\]

\[
\ll \frac{1}{T^8} \cdot |\{v_1, v_2, v_3, v_4 \in A(T) : \sum_{i=1}^4 v_i = 0\}|
\]

\[\text{(2.2)}\]

\[\text{1We denote by } f \ll g \text{ that there exists a constant } C > 0 \text{ s.t. } f \leq C g.\]
and, since $v_4 = -\sum_{i=1}^{3} v_i$, we find that the number of 4-tuples is at most $|A(T)|^3 \ll (T^2)^3$, and thus the above is $\ll \frac{1}{T^2}$. Hence
\[ \|G^{2T}_{\lambda} - G^T_{\lambda}\|_4 \ll T^{-1/2} \]
and, similarly,
\[ \|G^{2k+1}_{\lambda} - G^{2k}_{\lambda}\|_4 \ll 2^{-k/2}T^{-1/2} \]
which implies for any integers $p > q > 0$
\[ \|G^{2pT}_{\lambda} - G^{2qT}_{\lambda}\|_4 \ll T^{-1/2} \sum_{k=q}^{p-1} 2^{-k/2} \ll T^{-1/2} 2^{-q/2}. \]

This then implies, by a telescopic summation, that $(G^{2T}_{\lambda})_q$ is a Cauchy sequence and therefore converges to a limit in $L^4$ as $q \to \infty$. An argument similar to the one used above shows that if $\tilde{T} \in [2^k T, 2^{k+1}T]$ then $\|G^{2kT}_{\lambda} - G^T_{\lambda}\|_4 \ll T^{-1/2} 2^{-k/2}$, and thus $(G^T_{\lambda})_T$ is also a Cauchy sequence.

In particular, we have
\begin{equation}
\|G^T_{\lambda}\|_4^2 = \sum_{v_1, v_2, v_3 \in L} c_{\lambda}(v_1) c_{\lambda}(v_2) c_{\lambda}(v_3) c_{\lambda}(v_1 + v_2 - v_3). \tag{2.3}
\end{equation}

2.2. Further truncations. Let $A(\lambda, L)$ denote the annulus
\[ A(\lambda, L) := \{ v \in L : |v|^2 \in [\lambda - L, \lambda + L] \} \]

We introduce the Green’s function truncated to lattice points inside the annulus $A(\lambda, L)$
\begin{equation}
G_{\lambda,L}(x) = \sum_{\xi \in A(\lambda, L)} c_{\lambda}(\xi)e^{i\xi \cdot x}, \quad c_{\lambda}(\xi) := \frac{1}{|\xi|^2 - \lambda}. \tag{2.4}
\end{equation}

We have the following lemma which shows that $G_{\lambda,L}$ approximates $G_{\lambda}$ in $L^4(T^2)$ as $\lambda \to \infty$ if $L$ is any growing function of $\lambda$ which tends to infinity.

Lemma 2.1. Let $L = L(\lambda) \geq 10$ be an increasing function that tends to infinity with $\lambda$. There exists a full density subsequence of new eigenvalues such that, for all $\lambda$ in said subsequence, we have
\[ \|G_{\lambda} - G_{\lambda,L}\|_4 \ll L^{-1/2+o(1)}. \]

Proof. Let $A_+ = A_+(\lambda, L)$ denote the set $\{ v \in L : |v|^2 > \lambda + L \}$ and by $A_- = A_-(\lambda, L)$ the disk $\{ v \in L : |v|^2 < \lambda - L \}$. We begin by noting that
\[ \|G_{\lambda} - G_{\lambda,L}\|_4^4 = \sum_{v_1, \ldots, v_4 \in A_+ \cup A_-} \prod_{i=1}^{4} \frac{1}{|v_i|^2 - \lambda}; \]
writing $G_{\lambda} - G_{\lambda,L} = \sum_{v \in A_+} + \sum_{v \in A_-}$ and using the $L^4$ triangle inequality we can treat large and small $v$ separately. We begin by showing that
\[ \sum_{v_1, \ldots, v_4 \in A_+^{(\lambda,L)}; \sum_{i=1}^{4} v_i = 0} \prod_{i=1}^{4} \frac{1}{|v_i|^2 - \lambda} \]
is small (for most $\lambda \in \Lambda$) given that $L$ tends to infinity as $\lambda$ grows. Up to a bounded combinatorial factor, we may after reordering terms assume that $|v_{i+1}|^2 - \lambda \geq |v_i|^2 - \lambda > 0$ for $i = 1, 2, 3$, hence $|v_4|^2 - \lambda \geq \prod_{i=1}^3 (|v_i|^2 - \lambda)^{1/3}$; on noting that $v_4$ is determined by $v_1, v_2, v_3$, it is enough to show that

$$\prod_{i=1}^3 \left( \sum_{|v_i|^2 \geq \lambda + 1} \frac{1}{(|v_i|^2 - \lambda)^{4/3}} \right) = o(1).$$

In particular, it is enough to show that $\sum_{|v|^2 \geq \lambda + 1} \frac{1}{(|v|^2 - \lambda)^{4/3}} = o(1)$; which in turn reduces to showing that

$$\sum_{2\lambda \geq |v|^2 \geq \lambda + 1} \frac{1}{(|v|^2 - \lambda)^{4/3}} = o(1)$$

(to see this, use Weyl’s law and partial summation to bound the contribution from $v$ such that $|v|^2 > 2\lambda$.)

Now, given an integer $k \geq 0$, let $M(k)$ denote the number of unperturbed eigenvalues in the interval $[k, k + 1)$, or equivalently, the number of lattice points $v$ such that $|v|^2 \in [k, k + 1)$. We consider the sum over all $\lambda \in (T/2, T) \cap \Lambda$, and show that dyadic means of $\sum_{2\lambda \geq |v|^2 \geq \lambda + 1} \frac{1}{(|v|^2 - \lambda)^{4/3}}$ are small for $T$ large. More precisely,

$$\sum_{\lambda \in \Lambda \cap (T/2, T) \atop 2\lambda \geq |v|^2 \geq \lambda + 1} \frac{1}{(|v|^2 - \lambda)^{4/3}} \ll \sum_{l < T} M(l) \sum_{\lambda \geq k \geq L} \frac{M(l + k)}{k^{4/3}}$$

which, using the same argument as in the proof of [12, Lemma 3] is

$$\ll \sum_{\lambda \geq k \geq L} \frac{1}{k^{4/3}} \sum_{l < T} M(l) M(k + l) \ll \sum_{\lambda \geq k \geq L} \frac{1}{k^{4/3}} \sum_{l < T} M(l)^2 \ll L^{-1/3} T.$$

Hence, using Chebychev’s inequality, for most $\lambda \in \Lambda \cap (T/2, T)$ we find that

$$\sum_{v \in \mathbb{L} : |v|^2 \geq \lambda + 1} \frac{1}{(|v|^2 - \lambda)^{4/3}} \ll L^{-1/3 + o(1)}.$$

A similar argument shows that, for most $\lambda \in \Lambda \cap (T/2, T),$

$$\sum_{v \in \mathbb{L} : |v|^2 \leq \lambda - L} \frac{1}{(|v|^2 - \lambda)^{4/3}} \ll L^{-1/3 + o(1)}$$

and hence the $L^4$ norm is $\ll L^{-1/12 + o(1)}$.

\[ \Box \]

3. Proof of Theorem 1.1

One finds (cf. the $L^2$-expansion of the Green’s function [2.1], or see [13, eq. (3.22)]) that

$$\|G_\Lambda\|_2^2 = \int_{T^2} |G_\Lambda|^2 d\mu = \sum_{\xi \in \mathbb{L}} \frac{1}{(|\xi|^2 - \lambda)^2} = \sum_{n \in \mathbb{N}} \frac{r(n)}{(n - \lambda)^2},$$

where $r(n)$ is the number of representations of $n$ as the sum of two squares. The above bounds imply that

$$\|G_\Lambda\|_2^2 \ll \sum_{n \in \mathbb{N}} \frac{r(n)}{(n - \lambda)^2} \ll L^{-1/2 + o(1)}$$

and hence

$$\|G_\Lambda\|_2 \ll L^{-1/4 + o(1)}.$$
where \( r(n) \) is the multiplicity of the Laplace eigenvalue \( n \) and
\[
\mathcal{N} = \{ n_0 = 0 < n_1 < n_2 < \cdots \}
\]
denotes the set of distinct (unperturbed) Laplace eigenvalues.

Also (cf. (2.3)),
\[
\| G_\lambda \|_4^4 = \int_{T^2} |G_\lambda|^4 d\mu = \sum_{\xi_1, \xi_2 = \eta_1, \eta_2} \frac{1}{(|\xi_1|^2 - \lambda)(|\xi_2|^2 - \lambda)(|\eta_1|^2 - \lambda)(|\eta_2|^2 - \lambda)}.
\]
(3.2)

3.1. The sequence \( \Lambda_g \). We recall some useful results from sections 6 and 7 of [13]. Let \( \theta < 1/3 \) denote the best known exponent in the error term for the circle problem for a rectangular lattice [8]. In fact, we will only need \( \theta < 1/2 \), just a bit beyond the trivial geometric estimate. Adopting the notation of [13] we let \( \delta \in (0, 2/3(1/2 - \theta)) \) and define
\[
S(\lambda) = \bigcup_{0 \neq \zeta \in \mathcal{L}} S_\zeta,
\]
where we define \( S_\zeta \) for any \( \zeta \in \mathcal{L} \setminus \{0\} \) as the set of solutions to a certain diophantine inequality (cf. eq. (6.1) in [13]), namely
\[
S_\zeta := \{ \eta \in \mathcal{L} \mid |\langle \eta, \zeta \rangle| \leq |\eta|^{2\delta} \}.
\]

We will show that the subset of “good” eigenvalues
\[
\Lambda_g := \{ \lambda \in \Lambda \mid A(\lambda, \lambda^\delta) \cap S(\lambda) = \emptyset \}
\]
is of full density in \( \Lambda \) (recall that \(|\{ \lambda \in \Lambda : \lambda \leq X\}| \sim X\)), by showing that
\[
\{ \lambda \in \Lambda \setminus \Lambda_g \mid \lambda \leq X \} \ll X^{1-\delta_0}
\]
for \( \delta_0 = \frac{1}{2} - \theta - \frac{3}{2} \delta > 0 \). To see this, observe that the complement of \( \Lambda_g \), i.e. the set of “bad” elements which we denote by \( \Lambda_b \), is of the form
\[
\Lambda_b = \Lambda \setminus \Lambda_g = \bigcup_{0 \neq \zeta \in \mathcal{L}} B_\zeta
\]
where \( B_\zeta = \{ \lambda \in \Lambda \mid A(\lambda, \lambda^\delta) \cap S_\zeta \neq \emptyset \} \). Here we used
\[
\lambda \notin \Lambda_g \iff A(\lambda, \lambda^\delta) \cap S(\lambda) \neq \emptyset \iff \bigcup_{0 \neq \zeta \in \mathcal{L}} (A(\lambda, \lambda^\delta) \cap S_\zeta) \neq \emptyset
\]
which is equivalent to
\[
\lambda \in \bigcup_{0 \neq \zeta \in \mathcal{L}} B_\zeta.
\]
We recall the bound (6.4) in [13], namely that, for fixed $\zeta \in \mathcal{L}$,
\[ \left| \{ \lambda \in B_\zeta \mid \lambda \leq X \} \right| \leq \frac{X^{1/2+\theta+\delta}}{|\zeta|}. \]
(Note that in the proof of (6.4), the only property used regarding the location of the $\lambda$’s is the interlacing property. Further, the lower bound $\theta/2 > \delta$, stated at the beginning of [13], Section 6, is not used in order to prove (6.4).) We may now apply this bound to get an estimate on the number of bad eigenvalues $\lambda \leq X$. Note that we are summing over lattice vectors $\zeta \in \mathcal{L}$ which are not too large, i.e. $|\zeta| < \lambda^{\delta/2} \leq X^{\delta/2}$, and we find that
\[ \left| \{ \lambda \in \Lambda_b \mid \lambda \leq X \} \right| \leq \frac{1}{|\zeta|} \left( \sum_{0 \neq \zeta \in \mathcal{L}} \frac{1}{|\zeta|} \ll X^{1/2+3\delta/2} = X^{1-\delta_0} \right) \]
where $\delta_0 = \frac{1}{2} - \theta - \frac{3}{2} \delta > 0$ (we stress that only the condition $0 < \delta < \frac{2}{3}(\frac{1}{2} - \theta)$ is required).

3.2. Diagonal solutions. We begin with the following Lemma which shows that if $\lambda \in \Lambda_g$, then $A(\lambda, L)$ contains only lattice points that are reasonably well-spaced. Recall that $a^2$ is the aspect ratio of the lattice $\mathcal{L}$.

**Lemma 3.1.** Let $1 \leq \lambda \in \Lambda_g$ and put $L = L(\lambda) := \frac{1}{20} \min(a, 1/a)\lambda^{\delta/2}$. If $\xi$ and $\eta$ are two distinct lattice points belonging to $A(\lambda, L)$, then $|\xi - \eta| \geq \lambda^{\delta/2}$.

**Proof.** To see this, put $\beta = \eta - \xi$ and suppose for contradiction that $|\beta| = |\eta - \xi| < \lambda^{\delta/2}$. As $\lambda \in \Lambda_g$, and $\xi \in A(\lambda, L)$ we find that
\[ ||\xi|^2 - \lambda| = ||\eta - \beta| - \lambda| < L \]
and after multiplying out we obtain
\[ ||\eta|^2 - \lambda + |\beta|^2 - 2(\eta, \beta) | < L. \]
Now, since $|\beta| < \lambda^{\delta/2}$ and $\eta \in A(\lambda, L)$, it follows
\[ 2|\eta, \beta| - ||\eta|^2 - \lambda + |\beta|^2| \leq ||\eta|^2 - \lambda| + |\beta|^2 + L < 2L + \lambda^\delta \]
and, since our assumption implies $L < \frac{1}{3}\lambda^{\delta/2}$,
\[ |\eta, \beta| < L + \frac{1}{3}\lambda^{\delta/2} + \frac{1}{2}\lambda^\delta < \frac{3}{4}\lambda^\delta \leq (\frac{3}{4})^{1-\delta}|\eta|^{2\delta} \leq |\eta|^{2\delta}, \]
where we used $\lambda \leq ||\eta|^2 + L < ||\eta|^2 + \frac{1}{2}\lambda$ and therefore $\lambda \leq \frac{4}{3}|\eta|^2$.

This shows that $A(\lambda, \lambda^\delta) \cap S_{\beta} \neq \emptyset$, for some $\beta \neq 0$ such that $|\beta| < \lambda^{\delta/2}$, which in turn implies $A(\lambda, \lambda^\delta) \cap S \neq \emptyset$, contradicting that $\lambda \in \Lambda_g$. So it follows that $|\beta| \geq \lambda^{\delta/2}$.

The following key Lemma will be used in the computation of the fourth moment.
Lemma 3.2. Let $\lambda \in \Lambda_g, \lambda^{\delta/2} > 2$ and put $L = L(\lambda) := \frac{1}{20} \min(a, 1/a) \lambda^{\delta/2}$.

For $\xi, \eta \in A(\lambda, L)$ distinct, the equation

(3.3) \[ \xi - \eta = \eta' - \xi', \quad \xi', \eta' \in A(\lambda, L) \]

has only the trivial solutions

(3.4) \[ (\xi', \eta') = \begin{cases} (\eta, \xi) \\ (-\xi, -\eta) \end{cases} \]

Proof. We define the annulus centered at $\omega \in \mathbb{R}^2$ by

$A(\omega) = A(\omega, L) = \{ x \in \mathbb{R}^2 \mid |x - \omega|^2 - \lambda < L \}$

and denote $A = A(0), B = A \cap \mathcal{L}$. Let $\eta, \xi \in B$ and denote $\beta = \eta - \xi$.

We consider the set

(3.5) \[ S(\beta) = \{ (\eta', \xi') \in B \times B \mid \eta' - \xi' = \beta \} \]

and prove that

$S(\beta) = \{(\eta, \xi), (-\xi, -\eta)\}$.

First of all we have from Lemma 3.1 that $|\xi - \eta|, |\xi + \eta| \geq \lambda^{\delta/2}$. Also note that any element $(\eta', \xi')$ of $S(\beta)$ satisfies

$\lambda - L < |\eta'|^2 < \lambda + L$

and

$\lambda - L < |\xi|^2 = |\eta' - \beta|^2 < \lambda + L$

and thus $\eta'$ is constrained to lie in $A \cap A(\beta) \cap \mathcal{L}$. Rotate $A$ around the origin such that $\beta$ is horizontal.

We next show that the intersection of the two annuli cannot have a single connected component. To see this let $R = \sqrt{\lambda + L}, r = \sqrt{\lambda - L}$ and note that the case of a single connected component implies the inequality

$\sqrt{\lambda - L} = r \leq \frac{1}{2} |\beta|.$

Suppose, for a contradiction, that this inequality holds. Then

$\frac{1}{4} |\beta|^2 + \frac{1}{4} |\xi + \eta|^2 = \frac{1}{4} |\xi - \eta|^2 + \frac{1}{4} |\xi + \eta|^2 = \frac{1}{2} (|\xi|^2 + |\eta|^2) \leq R^2 = \lambda + L.$

These two inequalities imply, on recalling our assumption $L = \frac{1}{20} \min(a, 1/a) \lambda^{\delta/2}$

$\frac{1}{4} |\eta + \xi|^2 \leq \lambda - L - \frac{1}{4} |\beta|^2 \leq 2L < \frac{1}{2} \lambda^{\delta/2}$

and thus $|\eta + \xi| < \sqrt{2} \lambda^{\delta/4}$. But, as we saw above, our assumption $\lambda \in \Lambda_g$ implies $|\eta + \xi| \geq \lambda^{\delta/2}$, which contradicts the assumption $\lambda^{\delta/2} > 2$.

The case of two connected components. By the above argument, the set

$A \cap A(\beta) =: D(\eta) \cup D(-\xi)$
is thus the union of two approximate parallelograms containing $\eta$ and $-\xi$ respectively (cf. Figure 1.)

**Finding the solutions.** We introduce coordinates $x$, $y$ such that the annulus $\mathcal{A}$ is centered at $(x, y) = (0, 0)$ and $\mathcal{A}(\beta)$ is centered at $(x, y) = (|\beta|, 0)$. We compute the coordinates of the vertices $\omega_1, \omega_2, \nu_1, \nu_2$ of $\mathcal{D}(\eta)$ in order to calculate the distances $h = |\omega_1 - \omega_2|$ and $w = |\nu_1 - \nu_2|$ (cf. Figure 1).

![Figure 1. The intersection of the two annuli $\mathcal{A}(0)$ and $\mathcal{A}(\beta)$. In order to calculate the diameter of the approximate parallelogram $\mathcal{D}(\eta)$ with the vertices $\omega_1, \omega_2, \nu_1, \nu_2$ we have applied a rotation and introduced cartesian coordinates $x, y$ such that $\beta = (0, |\beta|)$ in these new coordinates.](image)

We aim for a bound on the diameter of $\mathcal{D}$ which is smaller than the minimal distance between two lattice points, so that $\mathcal{D}$ may contain at most one lattice point. To this end, we observe that $\mathcal{D} \subset \mathcal{R} = [x_-, x_+] \times [y_r, y_R]$, where $x_-, x_+$ are the $x$-coordinates of the points $\nu_1, \nu_2$ and $y_r, y_R$ are the $y$-coordinates of the points $\omega_1, \omega_2$. We then bound the diameter of $\mathcal{R}$.

By solving the equations

$$x^2 + y^2 = r_1^2 \quad (x - |\beta|)^2 + y^2 = r_2^2$$

for the cases $r_1 = r, R$ and $r_2 = r, R$, we obtain

$$\omega_1 = \left(\frac{1}{2}|\beta|, y_r\right), \quad \omega_2 = \left(\frac{1}{2}|\beta|, y_R\right)$$
where $y_r = \sqrt{r^2 - \frac{1}{4} |\beta|^2}$ and $y_R = \sqrt{R^2 - \frac{1}{4} |\beta|^2}$. It follows that
\[
h = |\omega_1 - \omega_2| = y_R - y_r = \sqrt{R^2 - \frac{1}{4} |\beta|^2} - \sqrt{r^2 - \frac{1}{4} |\beta|^2}
\]
and therefore (recall $R = \sqrt{\lambda + L}$ and $r = \sqrt{\lambda - L}$)
\[
h = \frac{R^2 - r^2}{\sqrt{R^2 - \frac{1}{4} |\beta|^2} + \sqrt{r^2 - \frac{1}{4} |\beta|^2}} = \frac{2L}{\sqrt{\lambda + L - \frac{1}{4} |\beta|^2} + \sqrt{\lambda - L - \frac{1}{4} |\beta|^2}}.
\]
Furthermore, by symmetry we have
\[
\nu_1 = (x-, y_\nu), \quad \nu_2 = (x+, y_\nu)
\]
for some $y_\nu > 0$ and $x_\pm = \frac{1}{2} |\beta| \pm \Delta_\nu$ for some $\Delta_\nu > 0$. We then have
\[
x_+ - x_- = |\nu_1 - \nu_2| = 2\Delta_\nu.
\]
In order to determine $\Delta_\nu$ we solve the system of equations
\[
x_-^2 + y_\nu^2 = r^2, \quad x_+^2 + y_\nu^2 = R^2
\]
which implies
\[
x_+^2 - x_-^2 = R^2 - r^2.
\]
It follows that $2|\beta|\Delta_\nu = R^2 - r^2 = 2L$. In summary, using that $|\beta| = |\eta - \xi| \geq \lambda^{\delta/2}$, we find that
\[
h = \frac{2L}{\sqrt{\lambda + L - \frac{1}{4} |\beta|^2} + \sqrt{\lambda - L - \frac{1}{4} |\beta|^2}} \quad \text{and} \quad w = \frac{2L}{|\beta|} < \frac{L}{\lambda^{\delta/2}},
\]
respectively. Now, since $0 < L < \frac{1}{4\sqrt{2}} \min(a, 1/a) \lambda^{\delta/2}$, it follows that $w < \min(a, 1/a)/\sqrt{2}$ and
\[
h \leq \frac{2L}{\sqrt{R^2 - \frac{1}{4} |\beta|^2}} \leq \frac{4L}{\lambda^{\delta/2}} < \frac{\min(a, 1/a)}{\sqrt{2}}
\]
since $\frac{1}{4} |\beta|^2 + \frac{1}{4} |\xi + \eta|^2 = \frac{1}{2} (|\xi|^2 + |\eta|^2) \leq R^2$ and $|\xi + \eta| \geq \lambda^{\delta/2}$.

Hence $\text{diam } \mathcal{D}(\eta) \leq \text{diam } \mathcal{R}(\eta) = \sup_{x,y \in \mathcal{R}(\eta)} |x - y| \leq \sqrt{2} \max\{w, h\} < \min(a, 1/a)$ and, therefore, $\eta$ is the only lattice point in $\mathcal{D}(\eta)$.

By symmetry it follows that $\mathcal{D}(-\xi)$ also contains only the lattice point $-\xi$. This proves the claim. \hfill \Box

### 3.3. Evaluating the fourth moment

Recall the truncated Green’s function
\[
G_{\lambda, L}(x) = \sum_{\xi \in A(\lambda, L)} c_\lambda(\xi) e^{i \xi \cdot x}, \quad c_\lambda(\xi) = \frac{1}{|\xi|^2 - \lambda}.
\]
We evaluate the $L^4$-norm of the truncated Green’s function in terms of its $L^2$-norm.
Lemma 3.3. Let $\lambda \in \Lambda_g$ and and put $L = L(\lambda) := \frac{1}{20} \min(a, 1/a)\lambda^{5/2}$.

**Theorem.**

$$
\mathbb{E} \left( \frac{G_{\lambda,L}^4}{\|G_{\lambda,L}\|_2^4} \right) = 3 - 2 \sum_{\xi \in A(\lambda,L)} c_{\lambda}(\xi)^4 \|G_{\lambda,L}\|_2^4.
$$

**Proof.** Let

$$
a_\xi = \begin{cases} 
\frac{1}{|\xi|^2 - \lambda}, & \text{if } ||\xi|^2 - \lambda| < L \\
0, & \text{otherwise}; 
\end{cases}
$$

clearly $a_\xi = a_{-\xi}$. Now

$$
\|G_{\lambda,L}\|_2^4 = \sum_{\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{Z}^2} a_{\xi_1}a_{\xi_2}a_{\eta_1}a_{\eta_2}
$$

$$
= \sum_{0=\xi_1 - \eta_1 = \eta_2 - \xi_2} a_{\xi_1}a_{\xi_2}a_{\eta_1}a_{\eta_2} + \sum_{\beta \neq 0} a_{\xi_1}a_{\xi_2}a_{\eta_1}a_{\eta_2}.
$$

(3.7)

The first sum can be rewritten as

$$
\sum_{\xi_1, \xi_2} a_{\xi_1}^2 a_{\xi_2}^2 = \|G_{\lambda,L}\|_2^4.
$$

With regard to the second sum let us consider the solutions of the equation

$$
\eta_2 - \xi_2 = \beta
$$

where

$$
0 \neq \beta = \xi_1 - \eta_1
$$

and

$$
\xi_1, \xi_2, \eta_1, \eta_2 \in A(\lambda, L).
$$

Our assumption that $\lambda \in \Lambda_g$, together with Lemma 3.4, implies that the only solutions are of the form

$$
(\xi_2, \eta_2) = \begin{cases} 
(\eta_1, \xi_1) \\
(-\xi_1, -\eta_1). 
\end{cases}
$$

(3.9)

Hence, we can rewrite the second sum as

$$
2 \sum_{\xi_1, \eta_1, \xi_1 \neq \eta_1} a_{\eta_1}^2 a_{\xi_1}^2 = 2 \|G_{\lambda,L}\|_2^4 - 2 \sum_{\xi} a_{\xi}^4.
$$

(3.10)

The result follows. \square

We have the following Lemma which shows that the 4th moment cannot be Gaussian, unless the Laplace spectrum has unbounded multiplicities.

---

\[\text{Recall that } \mathbb{E}(f) = \int_{\mathbb{R}} f(x) \mu(x).\]
Lemma 3.4. Given $\epsilon \in (0, 1)$ there exists a subset of $\Lambda$, of density $1 - \epsilon$, and a constant $C_\epsilon > 0$ such that for all sufficiently large $\lambda$ in said subsequence, we have
\[
\frac{\sum_{\xi \in A(\lambda, L)} c_\lambda(\xi)^4}{\|G_{\lambda, L}\|_2^4} \geq C_\epsilon.
\]

Proof. We claim that there exists a subsequence of $\Lambda$ of the form $\{\lambda_m\}_{m \in \mathcal{N}'}$, where $\mathcal{N}'$ is of density $1 - \epsilon$ in $\mathcal{N}$, such that a positive proportion of the $L^2$-norm is captured by a finite set of frequencies in the sense that for $I_m := \mathcal{N} \cap [m_--3, m_+ + 3]$, we have
\[
\sum_{n} \frac{r_\mathcal{L}(n)}{(n - \lambda_m)^2} \ll \epsilon \sum_{n \in I_m} \frac{r_\mathcal{L}(n)}{(n - \lambda_m)^2}
\]
and, as $m \to \infty$ along this subsequence, that $|I_m|$ remains bounded.

Let us explain the construction in more detail. In view of the remarks after Lemma 4.2 in [12] we may construct a subsequence $\mathcal{N}''$ of density $1 - \epsilon$ such that for $m \in \mathcal{N}''$ we have
\[
\sum_{|n-m| > 3} \frac{r(n)}{(n-m)^2} \leq F_\epsilon,
\]
\[
\# \{0 < |n-m| \leq 3\} \leq E_\epsilon
\]
and
\[
|m - \lambda_m| \leq G_\epsilon,
\]
for some numbers $E_\epsilon, F_\epsilon, G_\epsilon > 0$.

We then have
\[
(3.11) \quad \sum_{n} \frac{r(n)}{(n - \lambda_m)^2} = \sum_{n \in I_m} \frac{r(n)}{(n - \lambda_m)^2} + \sum_{n \notin I_m} \frac{r(n)}{(n - \lambda_m)^2} < \sum_{n \in I_m} \frac{r(n)}{(n - \lambda_m)^2} + \sum_{|n-m_-| > 3} \frac{r(n)}{(n-m_-)^2} + \sum_{|n-m| > 3} \frac{r(n)}{(n-m)^2}
\]
where we used the inequalities
\[
\sum_{n \notin m_-} \frac{r(n)}{(n - \lambda_m)^2} \ll \sum_{n \in m_-} \frac{r(n)}{(n - \lambda_m)^2}
\]
and
\[
\sum_{|n-m| > 3} \frac{r(n)}{(n - \lambda_m)^2} \ll \sum_{n > m} \frac{r(n)}{(n-m)^2}.
\]

So we may define a subsequence $\mathcal{N}' \subset \mathcal{N}''$ (of density at least $1 - 2\epsilon$) consisting of those $m \in \mathcal{N}''$ such that also $m_- \in \mathcal{N}''$ holds. For $m \in \mathcal{N}'$ we
have
\[ \sum_{n} \frac{r(n)}{(n - \lambda m)^2} < \sum_{n \in I_m} \frac{r(n)}{(n - \lambda m)^2} + 2F_\epsilon \]

The term corresponding to the choice \( n = m \) in the short sum (i.e., the sum over elements \( n \in I_m \)) is bounded below by \( 1/G_\epsilon^2 \), so on multiplying by \( F_\epsilon G_\epsilon^2 \) we get
\[ F_\epsilon \leq F_\epsilon G_\epsilon^2 \sum_{n \in I_m} \frac{r(n)}{(n - \lambda m)^2} \]

and the number of terms in the short sum is bounded by \( 2E_\epsilon \).

So we have
\[ \sum_{n} \frac{r(n)}{(n - \lambda m)^2} \leq (1 + 2F_\epsilon G_\epsilon^2) \sum_{n \in I_m} \frac{r(n)}{(n - \lambda m)^2} \]

and we note that \( \#\{n \in I_m\} \leq 2E_\epsilon \). This implies
\[ (3.12) \]
\[ \left( \sum_{|n - \lambda m| \leq L} \frac{r(n)}{(n - \lambda m)^2} \right)^2 \leq \left( \sum_{n} \frac{r(n)}{(n - \lambda m)^2} \right)^2 \leq \epsilon \left( \sum_{n \in I_m} \frac{r(n)}{(n - \lambda m)^2} \right)^2 \]
\[ \ll \epsilon \left| \{n \in I_m\} \right| \sum_{n \in I_m} \frac{r(n)^2}{(n - \lambda m)^4} \ll \epsilon \sum_{|n - \lambda m| \leq L} \frac{r(n)}{(n - \lambda m)^2} = \sum_{\xi \in A(\lambda, L)} c_{\lambda m}(\xi)^4 \]

where we used Cauchy-Schwarz and the fact that the multiplicities \( r_L(n) \) are bounded (as the aspect ratio of \( L \) is irrational.) \( \square \)

It is a simple consequence of the Lemma above that if the multiplicities in the unperturbed Laplace spectrum are bounded, as is the case for Šebá’s billiard in the irrational aspect ratio case, then one can construct an essentially full density subsequence of new eigenvalues such that the 4th moment does not converge to the Gaussian 4th moment, as the eigenvalue tends to infinity.

**Corollary 3.5.** Denote by \( g_{\lambda, L} \) the \( L^2 \)-normalized, truncated Green’s function on an irrational torus with Diophantine aspect ratio, and put \( L = L(\lambda) := \frac{a}{20} \min(a, 1/a) \lambda^{3/2} \). For any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) and a subsequence of \( \Lambda \), of density \( 1 - \epsilon \), such that
\[ 1 \leq \mathbb{E}(g_{\lambda, L}^4) \leq 3 - 2C_\epsilon \]
as \( \lambda \to \infty \) along said subsequence.

**Proof.** We recall that there exists a full density subsequence \( \Lambda_g \) such that for \( \lambda \in \Lambda_g \) we have
\[ \frac{\|G_{\lambda, L}\|^4}{\|G_{\lambda, L}\|^2} = 3 - 2 \frac{\sum_{\xi \in A(\lambda, L)} c_{\lambda}(\xi)^4}{\|G_{\lambda, L}\|^2} \].
We also note
\[ \sum_{\xi \in A(\lambda, L)} c_{\lambda}(\xi)^4 \leq \left( \sum_{\xi \in A(\lambda, L)} c_{\lambda}(\xi)^2 \right)^2 = \| G_{\lambda, L} \|^4_2. \]

At the same time Lemma \ref{lem:3.4} shows that for any \( \epsilon > 0 \) there exists \( C_{\epsilon} > 0 \) and a subsequence of density \( 1 - \epsilon \) such that
\[ 1 \geq \frac{\sum_{\xi \in A(\lambda, L)} c_{\lambda}(\xi)}{\| G_{\lambda, L} \|^2} \geq C_{\epsilon}. \]

More precisely, if we take \( \lambda \) belonging to the intersection of the two subsequences (a subsequence of density at least \( 1 - 2\epsilon \)) we have
\[ 1 \leq E(g_{\lambda, L}^4) \leq 3 - 2C_{\epsilon}. \]

\[ \square \]

In order to conclude the proof of the theorem we need the following approximation.

**Lemma 3.6.** Let \( L = L(\lambda) := \frac{1}{20} \min(a, 1/a)\lambda^{5/2} \). There exists a subsequence \( \{\lambda_j\}_k \) of \( \Lambda = \{\lambda_j\}_j \) of density at least \( 1 - \epsilon \) s. t.
\[ \lim_{k \to \infty} \left| \frac{\| G_{\lambda_k, L} \|^4_4}{\| G_{\lambda_k, L} \|^2_2} - \frac{\| G_{\lambda_k} \|^4_4}{\| G_{\lambda_k} \|^2_2} \right| = 0. \]

**Proof.** Recall \( E(g_{\lambda, L}^4) = \| G_{\lambda_k, L} \|^4_4/\| G_{\lambda_k, L} \|^2_2 \).

There is a subsequence \( \Lambda_1 \subset \Lambda \) (cf. Corollary \ref{cor:3.5}) of density at least \( 1 - \epsilon \) s. t. for \( \lambda \in \Lambda_1 \) we have that \( \| G_{\lambda, L} \|^4_4/\| G_{\lambda, L} \|^2_2 \) is bounded from both above and below. Moreover, there is another subsequence \( \Lambda_2 \subset \Lambda \) of density at least \( 1 - \epsilon \) s. t. for \( \lambda \in \Lambda_2 \) we have \( \| G_{\lambda_k} \|^2 \gg 1 \) (by the same argument as in \cite{12}, taking \( G = \epsilon^{-1} \) on p. 16).

Let us denote \( \Lambda_1 \cap \Lambda_2 = \{\lambda_j\}_{k=0}^{+\infty} \) which is a subsequence of density at least \( 1 - 2\epsilon \).

It is sufficient to show that
\[ \left| \frac{\| G_{\lambda_k, L} \|^4_4}{\| G_{\lambda_k, L} \|^2_2} - \frac{\| G_{\lambda_k} \|^4_4}{\| G_{\lambda_k} \|^2_2} \right| \to 0. \]

We have
\[
\begin{align*}
\left| \frac{\| G_{\lambda_k, L} \|^4_4}{\| G_{\lambda_k, L} \|^2_2} - \frac{\| G_{\lambda_k} \|^4_4}{\| G_{\lambda_k} \|^2_2} \right| &\leq \| G_{\lambda_k} \|_2^{-1} \left| \frac{\| G_{\lambda_k} \|^4_4}{\| G_{\lambda_k} \|^2_2} - \frac{\| G_{\lambda_k} \|_2 - \| G_{\lambda_k} \|_2}{\| G_{\lambda_k} \|^2_2} \right| \\
\text{(3.13)} &\leq \| G_{\lambda_k} \|_2^{-1} \left| \frac{\| G_{\lambda_k} \|^4_4}{\| G_{\lambda_k} \|^2_2} - \frac{\| G_{\lambda_k} \|^4_4}{\| G_{\lambda_k} \|^2_2} \right| + \| G_{\lambda_k} \|_2^{-1} \left| \frac{\| G_{\lambda_k, L} \|^4_4}{\| G_{\lambda_k, L} \|^2_2} - \frac{\| G_{\lambda_k, L} \|^4_4}{\| G_{\lambda_k, L} \|^2_2} \right|
\end{align*}
\]

where we used \( \| G_{\lambda_k} \|_2 \gg 1 \), as well as Corollary \ref{cor:3.5}, and finally the reverse triangle inequality \( \| |f|_p - \| g\|_p \| \leq \| f - g\|_p \) for \( p = 2, 4 \).
We thus find that the right hand side of (3.13) is

\[ \ll \varepsilon \| G_{\lambda_{jk}} - G_{\lambda_{jk},L} \|_4 + \| G_{\lambda_{jk}} - G_{\lambda_{jk},L} \|_2 \ll L^{-1/12 + o(1)} + L^{-1/2 + o(1)} \rightarrow 0, \text{ as } \lambda \rightarrow \infty, \]

where we used Lemma 2.1 and that

\[ \| G_{\lambda_{jk}} - G_{\lambda_{jk},L} \|_2^2 = \sum_{\xi \in L : |\xi|^2 - \lambda_{jk} \geq 2} \frac{1}{(|\xi|^2 - \lambda_{jk})^2} \ll L^{-1 + o(1)}, \]

which follows from an argument similar to the one used to deduce (2.5) and (2.6). \( \Box \)

If we take \( \lambda \) belonging to the subsequence of Lemma 3.6, then we have for \( \lambda \) sufficiently large

\[ \| g_{\lambda} \|_4^4 - \| g_{\lambda,L} \|_4^4 = o(1) \]

and

\[ \| g_{\lambda,L} \|_4^4 \in [1, 3 - 2C_\varepsilon]. \]

Hence, it follows (recall \( E(g_{\lambda}^4) = \| g_{\lambda} \|_4^4 \))

\[ 1 + o(1) \leq E(g_{\lambda}^4) \leq 3 - 2C_\varepsilon + o(1). \]

Appendix A. Dirichlet boundary conditions

In [14] Šeba discussed irrational aspect ratio rectangles with Dirichlet boundary conditions rather than rectangular tori. In particular, this means that the wave functions and the spectrum depend on the position of the scatterer. We briefly discuss here how our results can easily be extended to this setting.

Let us denote the position of the scatterer by \( y \). Denote \( L^+ = \{ \xi \in L \mid \xi_1, \xi_2 > 0 \} \). The new eigenfunctions are then of the form

\[ G_\lambda(x) = \sum_{\xi \in L^+} c_\lambda(\xi) \psi_\xi(y) \psi_\xi(x), \]

where \( \psi_\xi(x) = \sin(2\pi \xi_1 x_1) \sin(2\pi \xi_2 x_2) \). We note that the summation can easily be written over \( L \):

\[ G_\lambda(x) = -\frac{1}{4} \sum_{\xi \in L} c_\lambda(\xi) \psi_\xi(y) \chi(\xi) e(\xi \cdot x), \]

where \( \chi(\xi) = \text{sgn}(\xi_1) \text{sgn}(\xi_2) \).

In order to prove the analogue of Theorem 1.1 we require analogues of the argument for the \( L^4 \)-convergence in section 2.1, as well as the Lemmas 2.1, 3.3, 3.4 and 3.6.

The arguments of section 2.1 and Lemma 2.1 work analogously because of the bound \( |\psi_\xi(y)| \leq 1 \).
The proof of Lemma 3.3 works exactly the same way, as it only depends on the structure of the set of lattice points in the annulus \( A(\lambda, L) \). In the case of Dirichlet boundary conditions it yields

\[
E \left( \frac{G_{\lambda,L}^4}{\|G_{\lambda,L}\|^4} \right) = 3 - 2 \sum_{\xi \in A(\lambda, L)} c_\lambda(\xi) \psi_\xi(y)^4 \frac{\psi_\xi(y)}{\|G_{\lambda,L}\|^4}
\]

and

\[
\|G_{\lambda,L}\|^2 = \sum_{\xi \in A(\lambda, L)} c_\lambda(\xi)^2 \psi_\xi(y)^2
\]

The analogue of Lemma 3.4 can then be readily obtained by replacing \( r_L(n) \) with the function

\[
r_L(n, y) = \sum_{|\xi|^2 = n} \psi_\xi(y)^2 \leq r_L(n),
\]

provided we can construct a (large density) subsequence of \( \Lambda \) such that

\[
\sum_{n \in I_m} r_L(n) \frac{n - \lambda_m}{(n - \lambda_m)^4} \ll \sum_{n \in I_m} r_L(n, y) \frac{n - \lambda_m}{(n - \lambda_m)^4}.
\]

To do this, we define the “bad” set of eigenvalues

\[
B = \{ \lambda_k \in \Lambda' \mid \exists n \in \mathcal{N} \cap I_k : |\psi_\xi(y)| < \delta, |\xi|^2 = n \}
\]

where \( \Lambda' \) denotes the subsequence of eigenvalues such that \#\{\( n \in I_m \)\} remains bounded. For \( \epsilon > 0 \) we may construct \( \Lambda' \) of density at least \( 1 - \epsilon \) such that \#\{\( n \in I_m \)\} \( \leq N(\epsilon) \).

We can now estimate the cardinality of the bad set, because for each \( n \in \mathcal{N} \) such that \( |\psi_\xi(y)| < \delta \) for \( |\xi|^2 = n \) there exists only a finite number \( K_\epsilon \) of \( \lambda_k \in \Lambda' \) with \( n \in I_k \). At the same time

\[
\# \{ n \in \mathcal{N} : n \leq T, |\psi_\xi(y)| < \delta, |\xi|^2 = n \} = O(\delta T)
\]

so that \( |B| = O(\delta TK_\epsilon) \) and we can make \( \delta \) small enough in terms of \( \epsilon \) such that the subsequence of bad eigenvalues is of density less than \( \epsilon \). So excluding the bad eigenvalues we obtain a subsequence of density at least \( 1 - 2\epsilon \).

The proof of Lemma 3.6 however, requires a lower bound for \( \|G_{\lambda,L}\|_2 \). In fact, it was already pointed out in the appendix of \[12\] that for a generic position \( y \), in the sense that the coordinates \( y_1, y_2 \) are irrational, there exists a subsequence of Laplace eigenvalues of arbitrarily high density such that for \( |\xi|^2 = n \) we have \( \lim \inf_{n \to \infty} |\psi_\xi(y)| > 0 \). This yields the lower bound \( \|G_{\lambda,L}\|_2 \gg \epsilon \).

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