PSEUDOHOLOMORPHIC CURVES IN NEARLY KAHLER CP$^3$

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Abstract. We study pseudoholomorphic curves in the nearly Kahler CP$^3$. It is shown that a class of curves called null-torsion are in one to one correspondence with the integrals of a holomorphic contact system on the usual Kahler CP$^3$ studied by Bryant. Browing Bryant’s result we get plenty of such curves. Rational curves are shown to be either vertical or horizontal or null-torsion. Both horizontal and null-torsion curves can be understood by mentioned Bryant’s results while vertical curves are just fibers of the twistor fibration.

1. Introduction

The most interesting almost Hermitian manifolds are perhaps the nearly Kahler manifolds. An almost Hermitian manifold is nearly Kahler provided its almost complex structure $J$ satisfies $\nabla_X(J)X = 0$ for any vector field $X$ where $\nabla$ is the Levi-Civita connection. Although $J$ is nonintegrable, many aspects of Kahler geometry generalize to nearly Kahler manifold. For example, a generalized Hermitian-Yang-Mills theory was developed in [3]. In this paper we are interested in another aspect, the pseudoholomorphic curves in nearly Kahler manifolds. Pseudoholomorphic curves in $S^6$ has been studied by Bryant in [2]. He showed that every Riemann surface appears in $S^6$ as a null-torsion pseudoholomorphic curve with an arbitrarily large ramification degree.

In this paper we study the next interesting case - curves in the nearly Kahler 6 manifold CP$^3$. It is well known that there is a twistor fibration of the Kahler 3—projective space, denoted by $\mathbb{CP}^3$, over $S^4$ with the fibers $\mathbb{CP}^1$. The nearly Kahler CP$^3$ is defined by reversing the almost complex structure on the fibers. For each pseudoholomorphic curve $X : M^2 \to \mathbb{CP}^3$ we construct holomorphic sections $I_1$, $I_2$ and $II$ of three holomorphic line bundles over $M$. We call those curves with $II \equiv 0$ null-torsion. This is a well behaved condition. In fact, it turns out that null-torsion curves are in one to one correspondence with the holomorphic integrals of a holomorphic contact structure on the usual CP$^3$. The latter has been thoroughly studied in [1]. In particular, a Weierstrass formula was derived for such curves. Translating these results by this correspondence we get plenty of existence for null-torsion curves. If $M = S^2$ (these are called rational), it can be proved that $X$ necessarily falls into three categories: (1) vertical (2) horizontal (3) null-torsion. Both (2) and (3) can be reduced to the integrals of the holomorphic contact system mentioned before. Therefore we get a complete understanding of rational curves.

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2. Structure Equations, Projective Spaces and the Flag Manifold

In this section we collect some facts needed in next section and formulate them in terms of the moving frame. Let $\mathbf{H}$ denote the real division algebra of quaternions. An element of $\mathbf{H}$ can be written uniquely as $q = z + jw$ where $z, w \in \mathbf{C}$ and $j\mathbf{H}$ satisfies

$$j^2 = -1, \quad zj = j\bar{z}$$

for all $z \in \mathbf{C}$. In this way we regard $\mathbf{C}$ as subalgebra of $\mathbf{H}$ and give $\mathbf{H}$ the structure of a complex vector space by letting $\mathbf{C}$ act on the right. We let $\mathbf{H}^2$ denote the space of pairs $(q_1, q_2)$ where $q_i \in \mathbf{H}$. We will make $\mathbf{H}^2$ into a quarternion vector space by letting $\mathbf{H}$ act on the right

$$(q_1, q_2)q = (q_1q, q_2q).$$

This automatically makes $\mathbf{H}^2$ into a complex vector space of dimension 4. In fact, regarding $\mathbf{C}^4$ as the space of 4--tuples $(z_1, z_2, z_3, z_4)$ we make the explicit identification

$$(z_1, z_2, z_3, z_4) (z_1 + jz_2, z_3 + jz_4).$$

This specific isomorphism is the one we will always mean when we write $\mathbf{C}^4 = \mathbf{H}^2$. If $v \in \mathbf{H}^2 \setminus \{(0,0)\}$ is given, let $v \mathbf{C}$ and $v \mathbf{H}$ denote respectively the complex line and the quarternion line spanned by $v$. The assigment $v \mathbf{C} \rightarrow v \mathbf{H}$ is a well defined mapping $T : \mathbf{CP}^3 \rightarrow \mathbf{HP}^1$. The fibres of $T$ are $\mathbf{CP}^1$s. So we have a fibration

$$\begin{array}{c}
\mathbf{CP}^1 \\ \downarrow \\
\mathbf{HP}^1
\end{array}$$

This is the famous twistor fibration. In order to study its geometry more thoroughly, we will now introduce the structure equations of $\mathbf{H}^2$. First we endow $\mathbf{H}^2$ with a quarternion inner product $\langle, \rangle : \mathbf{H}^2 \times \mathbf{H}^2 \rightarrow \mathbf{H}$ defined by

$$\langle (q_1, q_2), (p_1, p_2) \rangle = \bar{q}_1p_1 + \bar{q}_2p_2.$$

We have identities

$$\langle v, wq \rangle = \langle v, w \rangle q, \quad \bar{\langle v, w \rangle} = \langle w, v \rangle, \quad \langle vq, w \rangle = \bar{q} \langle v, w \rangle.$$

Moreover, $Re (\langle, \rangle)$ is a positive definite inner product which gives $\mathbf{H}^2$ the structure of a Euclidean space $\mathbf{E}^8$. Let $\mathfrak{g}$ denote the space of pairs $f = (e_1, e_2)$ with $e_i \in \mathbf{H}^2$ satisfying

$$\langle e_1, c_1 \rangle = \langle e_2, c_2 \rangle = 1, \quad \langle e_1, e_2 \rangle = 0.$$

We regard $e_i(f)$ as functions on $\mathfrak{g}$ with values in $\mathbf{H}^2$. Clearly $e_1(\mathfrak{g}) = S^7 \subset \mathbf{E}^8 = \mathbf{H}^2$. It is well known that $\mathfrak{g}$ maybe canonically identified with $Sp(2)$ up to a left translation in $Sp(2)$. There are unique quaternion-valued 1-forms $\{\phi^a_b\}$ so that

$$de_a = e_b \phi^b_a,$$

(2.2)

$$d\phi^a_b + \phi^a_c \wedge \phi^c_b = 0,$$

(2.3)

and

$$\phi^a_a + \bar{\phi}^a_a = 0.$$

(2.4)

We have two canonical maps $C_1 : \mathfrak{g} \rightarrow \mathbf{CP}^3$ and $C_2 : \mathfrak{g} \rightarrow \mathbf{CP}^1$ by sending $f \in \mathfrak{g}$ to the complex lines spanned by $e_1(f)$ and $e_2(f)$ respectively. Recall that we have denoted the Kahler projective space by $\mathbf{CP}^3$ and the nearly Kahler one by $\mathbf{CP}^1$.
whose structure will be explicitly described below. We are mainly interested in \( \mathbb{CP}^3 \). However, \( \mathbb{CP}^3 \) will play an important role. We now write structure equations for \( C_1 \) and \( C_2 \). First we immediately see that \( C_1 \) gives \( \mathbb{F} \) the structure of an \( S^1 \times S^3 \) bundle over \( \mathbb{CP}^3 \) where we have identified \( S^1 \) with the unit complex numbers and \( S^3 \) with the unit quaternions. The action is given by

\[
f(z, q) = (e_1, e_2)(z, q) = (e_1 z, e_2 q),
\]

where \( z \in S^1 \) and \( q \in S^3 \). If we set

\[
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix} = \begin{bmatrix}
i\rho_1 + j\omega_1 \\
\omega_1 \\
i\rho_2 + j\omega_2
\end{bmatrix}
\]

where \( \rho_1 \) and \( \rho_2 \) are real 1-forms while \( \omega_1, \omega_2, \omega_3 \) and \( \tau \) are complex valued, we may rewrite one part of the structure equation (2.3) relative to the \( S^1 \times S^3 \) structure on \( \mathbb{CP}^3 \) as

\[
d\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} = -\begin{pmatrix}
i(\rho_2 - \rho_1) & -\tau & 0 \\
\tau & -i(\rho_1 + \rho_2) & 0 \\
0 & 0 & 2i\rho_1
\end{pmatrix} \wedge \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} + \begin{pmatrix}
\omega_2 \wedge \omega_3 \\
\omega_3 \wedge \omega_1 \\
\omega_1 \wedge \omega_2
\end{pmatrix}.
\]

This in particular defines a nearly Kahler structure on \( \mathbb{CP}^3 \) by requiring \( \omega_1, \omega_2, \omega_3 \) and \( \tau \) to be of type \((1, 0)\) (note that this almost complex structure is nonintegrable, thus different from the usual integrable one). We denote

\[
\begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} = \begin{pmatrix}
i(\rho_2 - \rho_1) & -\tau \\
\tau & -i(\rho_1 + \rho_2)
\end{pmatrix}
\]

and \( k_{33} = 2i\rho_1 \) in the usual notation of a connection. Then the other part of the structure equation (2.3) may be written as the curvature of this nearly Kahler structure

\[
d\begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} + \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} \wedge \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} = \begin{pmatrix}
\omega_1 \wedge \omega_1 - \omega_3 \wedge \omega_3 & \omega_1 \wedge \omega_2 \\
\omega_2 \wedge \omega_1 & \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3
\end{pmatrix},
\]

as well as

\[
dk_{33} = -(\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 - 2\omega_3 \wedge \omega_3).
\]

In an exactly analogous fashion, \( C_2 \) gives \( \mathbb{F} \) a structure of an \( S^1 \times S^3 \) bundle over \( \mathbb{CP}^3 \) with the action now given by

\[
(e_1, e_2)(q, z) = (e_1 q, e_2 z),
\]

where \( z \in S^1 \) and \( q \in S^3 \). However, \( \omega_1, \omega_2, \kappa_{12} \) and their complex conjugates become semibasic and \( \omega_3 \) is not. The usual Kahler structure on \( \mathbb{CP}^3 \) is defined by requiring \( \frac{\omega_1}{\sqrt{2}}, \frac{\omega_2}{\sqrt{2}} \) and \( \kappa_{21} \) to be of type \((1, 0)\) and unitary. Relative to this Kahler structure, we may rewrite part of the structure equations as

\[
d\begin{pmatrix}
\frac{\omega_1}{\sqrt{2}} \\
\frac{\omega_2}{\sqrt{2}} \\
\kappa_{21}
\end{pmatrix} = -\begin{pmatrix}
-\kappa_{11} & \omega_3 & -\frac{\omega_1}{\sqrt{2}} \\
0 & \kappa_{22} & \frac{\omega_2}{\sqrt{2}} \\
\frac{\omega_1}{\sqrt{2}} & -\frac{\omega_2}{\sqrt{2}} & \kappa_{21} - \kappa_{11}
\end{pmatrix}
\]

We will also need some properties of the flag manifold \( \mathbb{F} = \mathbb{F}/(U(1) \times U(1)) \). Equivalently \( \mathbb{F} \) consists of pairs of complex lines \( ([e_1], [e_2]) \) with \( (e_1, e_2) = 0 \). Of course \( \mathbb{F} \) defines a natural \( S^1 \times S^1 \) structure on \( \mathbb{F} \) for which the forms \( \omega_1, \omega_2, \omega_3, \kappa_{21} \)
and their complex conjugates are semibasic. Moreover, we have a double fibration of $\mathbf{F}_1$ over the two projective spaces:

$$
\begin{array}{c}
\mathbb{C}P^3 \\
\downarrow \\
\mathbf{F}_1 \\
\downarrow \\
\mathbb{C}P^3
\end{array}
$$

We denote the first fibration by $\Pi_1$ and the second fibration by $\Pi_2$. Explicitly $\Pi_a$ ($a = 1, 2$) sends $([e_1], [e_2]) \in \mathbf{F}_1$ to the complex line $[\epsilon_a]$. By requiring $\overline{\omega_1}, \omega_2, \omega_3, \kappa_2 j$ to be complex linear we define an almost complex structure on $\mathbf{F}_1$. It is easy to check from the structure equations that this almost complex structure is integrable and $\Pi_2$ is thus a holomorphic projection.

Finally there are various complex vector bundles associated with $\mathfrak{g}$ which will be important. First on $\mathbb{C}P^3$, there are two obvious complex bundles, the tautological bundle $\epsilon$ and the trivial rank 4 bundle $\mathbb{C}^4$. We view $\epsilon$ as the subbundle of $\mathbb{C}^4$ spanned by $e_1$ and denote the quotient bundle by $Q$. Using the obvious Hermitian product, we identify $Q$ as a subbundle of $\mathbb{C}^4$ locally spanned by $e_{1j}$, $e_2$ and $e_{2j}$. Note that $e_{1j}$ itself spans a well-defined line bundle, which is isomorphic to $\epsilon^*$. Denote the quotient $Q/\epsilon^*$ by $\tilde{Q}$ which, again, may be regarded as a subbundle of $\mathbb{C}^4$ locally spanned by $e_2, e_{2j}$. We write $T\mathbb{C}P^3$ the complex tangent bundle of $\mathbb{C}P^3$. The $S^1 \times S^3$ determines a splitting

$$
T\mathbb{C}P^3 = \mathcal{H} \oplus \mathcal{V},
$$

where $\mathcal{H}$ has rank 2 and $\mathcal{V}$ has rank 1. We call $\mathcal{H}$ the horizontal part and $\mathcal{V}$ the vertical part relative to the fibration $\mathfrak{g}$. One can show $\mathcal{V}$ is isomorphic to $\epsilon^*$ as a Hermitian line bundle by locally identifying $\frac{1}{\sqrt{2}} e_1 \otimes e_1$ with the complex tangent vector dual to the $(1, 0)$ form $\omega_1$, denoted by $f_1$. Similarly $\mathcal{H}$ is isomorphic to $\epsilon^* \otimes \tilde{Q}$ with $\frac{1}{\sqrt{2}} e_1^* \otimes e_2$ identified with the tangent vector $f_2$ dual to $\omega_2$ and $\frac{1}{\sqrt{2}} e_1^* \otimes e_{2j}$ identified with $f_3$ dual to $\omega_3$. Pulled back to $\mathbf{F}_1$ by $\Pi_1$, the bundles $\tilde{Q}$ and $\mathcal{H}$ splits as

$$
\Pi_1^* \tilde{Q} = \tilde{\epsilon} \oplus \tilde{\epsilon}^*,
$$

where $\tilde{\epsilon}$ is locally spanned by $e_2$ and $\tilde{\epsilon}^*$ denotes its dual, locally spanned by $e_{2j}$. Of course $\Pi_1^* \mathcal{H}$ splits correspondingly as

$$
\Pi_1^* \mathcal{H} = \epsilon^* \otimes \tilde{\epsilon} \oplus \epsilon^* \otimes \tilde{\epsilon}^*.
$$

Similar constructions apply to $\mathbb{C}P^3$. We will only point out differences and some relations. The tautological bundle on $\mathbb{C}P^3$ becomes $\tilde{\epsilon}$ when pulled back to $\mathbf{F}_1$. The complex tangent bundle also splits as a sum of a vertical part $\mathcal{V}$ and a horizontal part $\mathcal{H}$. The vertical part is isomorphic to $(\tilde{\epsilon}^*)^2$ compared with $\mathbb{C}P^3$ case because of the reversed almost complex structure. The horizontal part, when pulled back by $\Pi_2$ to $\mathbf{F}_1$ is isomorphic to $\tilde{\epsilon}^* \oplus \tilde{\epsilon} \oplus \tilde{\epsilon}^* \otimes \epsilon$. Note this splitting shares a common factor with $\Pi_1^* \mathcal{H}$ which will become important later. In the various isomorphisms, we no longer need the $\frac{1}{\sqrt{2}}$ to make them Hermitian. Moreover, since the complex structure on $\mathbb{C}P^3$ is integrable, many of these bundles have holomorphic structures. Among them the dual of the vertical tangent bundle of $\mathbb{C}P^3$, which we denote by $\mathcal{V}^*$ is particularly important. Locally $\mathcal{V}^*$ is spanned by $\kappa_{21}$ as a subbundle of the
complex cotangent bundle of $\mathbb{C}P^3$. We have the following result due to R. Bryant.

**Lemma 2.1.** The bundle $V^*$ is isomorphic to $\mathbb{C}^2$ as a Hermitian holomorphic line bundle. Moreover, it induces a holomorphic contact structure on $\mathbb{C}P^3$.

The integrals of this holomorphic contact system was thoroughly investigated in [1] (see Section 3).

3. **Pseudoholomorphic Curves in $\mathbb{C}P^3$**

Let $M^2$ be a connected Riemann surface. A map $X : M^2 \to \mathbb{C}P^3$ is called a pseudoholomorphic curve if $X$ is nonconstant and the differential of $X$ commutes with the almost complex structures. We let $x : \mathcal{F}_X \to M^2$, $\nu_X \to M^2$ and $\mathcal{H}_X \to M^2$ be the pull back bundles of $\mathcal{F}$, $\nu$ and $\mathcal{H}$ respectively. Moreover, it induces a holomorphic contact structure on $\mathbb{C}P^3$.

Thus for instance, we have

$$\mathcal{F}_X = \{ (x,f) \in M^2 \times \mathcal{F}|X(x) = C_1(f) \}.$$

Of course, $\mathcal{F}_X$ is an $S^1 \times S^3$ bundle over $M^2$ and $\nu_X$ and $\mathcal{H}_X$ are Hermitian complex bundles of rank 1 and 2 respectively. Moreover, the natural map $\mathcal{F}_X \to \mathcal{F}$ pulls back various quantities on $\mathcal{F}$, which we still denote by the same letters. For example, $f_1$, $f_2$ now denote functions on $\mathcal{F}_X$ valued in $\mathcal{H}_X$. The structure equations (2.5), (2.6) and (2.7) still hold, on $\mathcal{F}_X$ now. Also for functions and sections with domains in $M^2$, we will pull these back up via $x^*$ to $\mathcal{F}_X$. For example, any section $s : M^2 \to \mathcal{H}_X$ can be written in the form $s = f_1s_1 + f_2s_2$ where $s_i$ are complex functions on $\mathcal{F}_X$. Using this convention, the pullback of $\kappa$ induces connections on $\mathcal{H}_X$ and $\nu_X$ compatible with the Hermitian structures. Namely $\nabla : \Gamma(\mathcal{H}_X) \to \Gamma(\mathcal{H}_X \otimes T^*M^2)$ is given by

$$\nabla(f_is_i) = f_i \otimes (ds_i + \kappa_{ij}s_j).$$

Since we are working over a Riemann surface, it is well-known that there are unique holomorphic structures on $\mathcal{H}_X$ and $\nu_X$ compatible with these connections. From now on we will regard these two bundles as holomorphic Hermitian vector bundles over $M^2$.

Another thing to notice is that $\{\omega_i\}$ are semi-basic with respect to $x : \mathcal{F}_X \to M^2$. Moreover, they are of type $(1,0)$ since $dX$ is complex linear. Set

$$I_1 = f_1 \otimes \omega_1 + f_2 \otimes \omega_2, I_2 = f_3 \otimes \omega_3.$$

It is clear that $I_1$ and $I_2$ are well defined sections of $\mathcal{H}_X \otimes T^*M^2$ and $\nu \otimes T^*M^2$ respectively where $T^*M^2$ is the holomorphic line bundle of $(1,0)$ forms on $M^2$.

**Lemma 3.1.** The sections $I_1$ and $I_2$ are holomorphic. Moreover, $I_1$ and $I_2$ only vanish at isolated points unless $X(M^2)$ is horizontal (when $I_2$ vanishes identically) or vertical (when $I_1$ vanishes identically and thus $X(M^2)$ is an open set of a fiber $\mathbb{C}P^1$ in $\mathbb{C}P^3$).

**Proof.** We only show $I_1$ is holomorphic and leave $I_2$ for the reader. Choose a uniformizing parameter $z$ on a neighborhood of $x_0 \in M$. In a neighborhood of $x^{-1}(x_0)$, there exist functions $a_i$ so that $\omega_i = a_idz$. It follows that $\omega_i \wedge \omega_j = 0$, so we have $da_i = -\kappa_{ij} \wedge \omega_j$. This translates to $(da_i + \kappa_{ij}a_j) \wedge dz = 0$ so there exists $b_i$ so that

$$da_i + \kappa_{ij}a_j = b_idz.$$
Thus, when we compute \( \bar{\partial} I_1 \) we have
\[
\bar{\partial} I_1 = (\nabla (f_1 a_i) \otimes dz)^{0,1} \\
= f_i \otimes dz \otimes (da_i + \kappa \bar{a}_j)^{0,1} \\
= f_i \otimes dz \otimes (b_i dz)^{0,1} \\
= 0,
\]
so \( I_1 \) is holomorphic. Moreover, by complex analysis, if \( I_1 \) or \( I_2 \) vanishes at a sequence of points with an accumulation, the section has to be identically 0 since \( M^2 \) is connected.

\[ \square \]

Remark 3.2. It is clear that \( I_1 \) and \( I_2 \) are just horizontal and vertical parts of the evaluation map \( X_*(TM) \to T_X \).

We will call a curve with \( I_1 = 0 \) (\( I_2 = 0 \)) vertical (horizontal). Of course vertical curves are just the fibers \( \mathbb{CP}^1 \) of \( T \). To study horizontal curves it does no harm to reverse the almost complex structure on the fiber of \( T \). This new complex structure is integrable and actually equivalent to the usual complex structure on the 3 projective space. The horizontal bundle \( \mathcal{H} \) turns out to be a holomorphic contact structure under the usual complex structure. The integral curves of this contact system are thoroughly described in [1]. We therefore have a good understanding of horizontal pseudoholomorphic curves in \( \mathbb{CP}^3 \).

We now assume both \( I_1 \) is not identically 0. There exists a holomorphic line bundle \( L \subset H \) so that \( I_1 \) is a nonzero section of \( L \otimes T^*M \). We let \( R_1 \) be the ramification divisor of \( I_1 \). That is, \( R_1 = \sum_{p: I_1(p) = 0} \text{ord}_p(I_1)p \).

\( R_1 \) is obviously effective, and we have \( L = TM \otimes [R_1] \).

Similarly if \( I_2 \) does not vanish identically let \( R_2 \) be the ramification divisor of \( I_2 \). Then \( R_2 \) is effective and \( V_X = TM \otimes [R_2] \).

Now we adapt frames in accordance with the general theory. We let \( \mathfrak{S}^{(1)}_X \) be the subbundle of pairs \((x, f)\) with \( f_2 \in L_x \). Then \( \mathfrak{S}^{(1)}_X \) is a \( U(1) \times U(1) \) bundle over \( M \). The cononical connection on \( L \) is described as follows: If \( s : M \to L \) is a section, then \( s = f_2 s_2 \) for some function \( s_2 \) on \( \mathfrak{S}^{(1)}_X \). Then
\[
\nabla s = f_2 \otimes (ds_2 + \kappa \bar{s}_2).
\]
Similarly the quotient bundle \( N_X = \mathcal{H}_X / L \) has a natural holomorphic Hermitian structure. Let \( (f_1) : \mathfrak{S}^{(1)}_X \to N_X \) be the function \( f_1 \) followed by the projection \( \mathcal{H}_X \to N_X \). If \( s : M \to N_X \) is any section, then \( s = (f_1) s_1 \) for \( s_1 \) on \( \mathfrak{S}^{(1)}_X \) and we have
\[
\nabla s = (f_1) \otimes (ds_1 + \kappa \bar{s}_1).
\]
Note since \( I_1 \) has values in \( L \otimes T^*M \), we must have \( \omega_1 = 0 \) on \( \mathfrak{S}^{(1)}_X \). If we differential this using structures \( \mathfrak{S}^{(1)}_X \) we have
\[
d\omega_1 = -\kappa_1 \bar{\tau} \wedge \omega_2 = 0.
\]
It follows that \( \kappa_{12} \) is of type \((1,0)\).
Lemma 3.3. Let \( \mathbf{II} = (f_1) \otimes f_2 \otimes \kappa_{12} \) where \( f_2 \) is the dual of \( f_2 \). Then \( \mathbf{II} \) is a holomorphic section of \( \mathcal{N}_X \otimes \mathcal{L}^* \otimes \mathcal{T}^* \).

Proof. Since \( \kappa_{12} \) is of type \((1, 0)\), there exists \( b \) locally such that \( \kappa_{12} = bdz \). The structure equations \((2.6)\) pulled back to \( \mathcal{F}^{(1)} \) gives \( d\kappa_{12} = -\kappa_{11} \wedge \kappa_{12} - \kappa_{12} \wedge \kappa_{22} + \omega_1 \wedge \bar{\omega}_2 = -(\kappa_{11} - \kappa_{22}) \wedge \kappa_{12} \). This translates into \( (db + (\kappa_{11} - \kappa_{22})b) \wedge dz = 0 \).

The rest follows exactly as in Lemma 3.1. \( \square \)

We say a curve has null-torsion if \( \mathbf{II} = 0 \). Since \( \wedge^2 \mathcal{H} \otimes \mathcal{V} \cong \mathbb{C} \) we have

\[
\mathcal{N}_X \otimes \mathcal{L} \otimes \mathcal{V} \cong \mathbb{C}.
\]

If \( \mathbf{II} \) is not identically 0, we define the planar divisor by

\[
P = \sum_{p: \mathbf{II}(p) = 0} \text{ord}_p(\mathbf{II})p.
\]

In this case, we have

\[
\mathcal{N}_X = [P] \otimes \mathcal{L} \otimes \mathcal{T}M.
\]

Theorem 3.4. Let \( M = \mathbb{CP}^1 \). Then any complex curve \( X : M \to \mathbb{CP}^3 \) either is one of the vertical fibers or horizontal or has null-torsion.

Proof. Assume both \( \mathbf{I}_1 \) and \( \mathbf{I}_2 \) are not identically 0. We must show that \( \mathbf{II} \) vanishes identically. If not, we have, for \( R_1, R_2, P \geq 0 \),

\[
\mathcal{V}_X = [R_2] \otimes \mathcal{T}M, L = [R_1] \otimes \mathcal{T}M, \mathcal{N}_X = [P] \otimes \mathcal{L} \otimes \mathcal{T}M,
\]

which implies, since \( \mathcal{N}_X \otimes \mathcal{L} \otimes \mathcal{V} \cong \mathbb{C} \),

\[
(\mathcal{T}M)^3 \otimes [2R_1 + P + R_2] \cong \mathbb{C},
\]

thus \( \text{deg}TM \leq 0 \), but \( \text{deg}TM = 2 \) when \( M = \mathbb{CP}^1 \). \( \square \)

Remark 3.5. The computation in this theorem actually shows that if \( M^2 \) has genus \( g \), then any pseudoholomorphic curve \( X : M \to \mathbb{CP}^3 \) with none of \( \mathbf{I}_1, \mathbf{I}_2 \) and \( \mathbf{II} \) vanishing identically must satisfy

\[
6(g - 1) = 2\text{deg}(R_1) + \text{deg}(R_2) + \text{deg}(P).
\]

This puts severe restrictions on the bundles \( \mathcal{L}, \mathcal{V}_X \) and \( \mathcal{N}_X \). For example, if \( g = 1 \), so that \( M \) is elliptic, then a pseudoholomorphic curve \( X : M \to \mathbb{CP}^3 \) must satisfy \( R_1 = R_2 = P = 0 \), so that \( \mathcal{V}_X = \mathcal{T}M, L = \mathcal{T}M \) and \( \mathcal{N}_X = (\mathcal{T}M)^2 \).

If the pseudoholomorphic curve \( X : M^2 \to \mathbb{CP}^3 \) has \( \mathbf{I}_1 \neq 0 \), we have a lift of \( X \) to a map \( \hat{X} : M^2 \to \mathbb{F}l \) defined by \( x \mapsto (X(x), N_X(x) \otimes X(x)) \). Some clarification may be necessary. The bundle \( N_X \) can be viewed canonically as a subbundle of \( X^* (\mathcal{Q} \otimes \mathcal{C}) \subset \mathcal{C}^4 \otimes X^*(\mathcal{C}_*) \). By tensoring with \( X^* \mathcal{C} \) and canonically identifying \( \mathcal{C} \otimes \mathcal{C}_* = \mathbb{C}^4 \) we see that \( N_X(x) \otimes X^*(\mathcal{C}(x)) = N_X(x) \otimes X(x) \) is a complex line in \( \mathbb{C}^4 \). It is easy to see that this line is Hermitian orthogonal to \( X(x) \subset \mathcal{C} \) and \( X(x)j \subset \mathcal{C}_* \) and thus \( \hat{X} \) is well-defined. Moreover, \( X \) has null torsion iff \( X^*(\kappa_{21}) = 0 \). Composed with \( \Pi_2 : \mathbb{F}l \to \mathbb{C}^3, \hat{X} \) induces a map \( Y = \Pi_2 \circ \hat{X} : M^2 \to \mathbb{CP}^3 \).

Theorem 3.6. The assignment \( X \mapsto Y \) establishes a \( 1 \to 1 \) correspondence between null-torsion pseudoholomorphic curves in \( \mathbb{CP}^3 \) and nonconstant holomorphic integrals of the holomorphic contact system \( \mathcal{V}^* \) on \( \mathbb{CP}^3 \).
Proof. It is clear from the structure equations that $Y$ is an integral of $V^*$ if $X$ has null torsion. Conversely, if $Y : M^2 \to \mathbb{C}P^3$ is a nonconstant holomorphic integral of $V^*$, there exists a unique line bundle $\mathcal{L} \subset \mathbb{H} \subset T\mathbb{C}P^3$ which contains $Y_* T M$. We lift $Y$ to a map $\hat{Y} : M^2 \to \mathbb{F}^1$ by $x \mapsto ((\mathbb{H}/\mathcal{L})(x) \otimes Y(x), Y(x))$. We define the corresponding map $X = \Pi_1 \circ \hat{Y} : M \to \mathbb{C}P^3$. It is clear from the structure equations that such an $X$ has null-torsion. We next show that if we start with null-torsion curve $X : M^2 \to \mathbb{C}P^3$ and run the procedure $X \to Y \to X$ of the above constructions, we arrive at the original curve. In fact the frame adaptations we made before shows we can arrange $\{e_a\}$ so that $\Pi_1^* L_X(x)$ is spanned by $\frac{1}{\sqrt{2}} e_2j \otimes e_1^*$ and $\Pi_1^* N_X(x)$ is spanned by $\frac{1}{\sqrt{2}} e_2 \otimes e_1^*$. Thus by definition $Y(x) = [e_2]$. Since $\omega^1 = 0$, $\Pi_2^* \mathcal{L}$ is spanned by $j e_1$ from the structure equations. Therefore $\Pi_2^* (\mathbb{H}/\mathcal{L}) = [e_1]$ from which we see $\Pi_1 (Y(x)) = X(x)$. We omit the proof that if we start with $Y$ and run the procedure of constructions $Y \to X \to Y$ we get $Y$ back. \qed

As mentioned before, a powerful construction the integrals of the holomorphic contact system $V^*$ was provided in [1] (see Section 3). Of course, there are corresponding results about null-torsion pseudoholomorphic curves in $\mathbb{C}P^3$. We leave most of translation work for the reader and only mention some consequences.

**Theorem 3.7.** Let $M$ be a compact Riemann surface. There always exists a pseudoholomorphic embedding $M \to \mathbb{C}P^3$ with null torsion.

This is the translation of Theorem G in [1].

A horizontal pseudoholomorphic curve $X : M^2 \to \mathbb{C}P^3$ with null torsion corresponds to $Y : M^2 \to \mathbb{C}P^3$ which is superminimal with both positive spin and negative spin in the sense of Theorem C in [1]. Thus $M^2$ must be rational. Combining this with Theorem 2.3, we have

**Corollary 3.8.** There exist pseudoholomorphic curves which are neither vertical nor horizontal.

A rational pseudoholomorphic curve is either vertical or horizontal or has null torsion. Both horizontal and null-torsion curves are reduced to integrals of the holomorphic contact system $V^*$ by Theorem 2.2. By the result in [1], Section 2, such an integral represents a lift of a minimal 2-sphere in $S^4$. Thus the space of nonvertical rational curves in $\mathbb{C}P^3$ can be regarded as the union of 2 copies of the space of minimal 2-spheres in $S^4$. These two copies have a nonempty intersection, corresponding to geodesic 2-spheres.

**References**

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