Q-data and Representation Theory of Untwisted Quantum Affine Algebras

Ryo Fujita¹, Se-jin Oh²

¹ Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, Université de Paris, Bâtiment Sophie Germain, 75013 Paris, France. E-mail: ryo.fujita@imj-prg.fr
² Ewha Womans University Seoul, 52 Ewhayeodae-gil, Daehyeon-dong, Seodaemun-gu, Seoul, South Korea. E-mail: sejin092@gmail.com
URL: https://sites.google.com/site/mathsejinoh/

Received: 6 July 2020 / Accepted: 30 January 2021
Published online: 24 March 2021 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract: For a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, we introduce the notion of Q-datum, which generalizes the notion of a Dynkin quiver with a height function from the viewpoint of Weyl group combinatorics. Using this notion, we develop a unified theory describing the twisted Auslander–Reiten quivers and the twisted adapted classes introduced in Oh and Suh (J Algebra 535(1):53–132, 2019) with an appropriate notion of the generalized Coxeter elements. As a consequence, we obtain a combinatorial formula expressing the inverse of the quantum Cartan matrix of $\mathfrak{g}$, which generalizes the result of Hernandez and Leclerc (J Reine Angew Math 701:77–126, 2015) in the simply-laced case. We also find several applications of our combinatorial theory of Q-data to the finite-dimensional representation theory of the untwisted quantum affine algebra of $\mathfrak{g}$. In particular, in terms of Q-data and the inverse of the quantum Cartan matrix, (i) we give an alternative description of the block decomposition results due to Chari and Moura (Int Math Res Not 5:257–298, 2005) and Kashiwara et al. (Block decomposition for quantum affine algebras by the associated simply-laced root system, 2020. arXiv:2003.03265), (ii) we present a unified (partially conjectural) formula of the denominators of the normalized $R$-matrices between all the Kirillov–Reshetikhin modules, and (iii) we compute the invariants $\Lambda(V, W)$ and $\Lambda^\infty(V, W)$ introduced in Kashiwara et al. (Compos Math 156(5):1039–1077, 2020) for each pair of simple modules $V$ and $W$. 

Contents

1. Introduction ................................. 1352
2. Dynkin Quivers, Adapted Classes and Associated Auslander–Reiten Quivers 1355
3. Q-data, Twisted Adapted Classes and Twisted Auslander–Reiten Quivers 1361

Ryo Fujita was supported in part by Grant-in-Aid for JSPS Research Fellow JP18J10669, by JSPS Grant-in-Aid for Scientific Research (B) JP19H01782, (A) JP17H01086 and also by JSPS Overseas Research Fellowships (during the revision). Se-jin Oh was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2019R1A2C4069647).
1. Introduction

1.1. For a complex finite-dimensional simple Lie algebra \( \mathfrak{g} \), let \( U'_q(\hat{\mathfrak{g}}) \) denote its untwisted quantum affine algebra, where \( q \) is a generic quantization parameter. Originally, it was introduced in the middle of the 80s in the context of solvable lattice models in order to give a systematic way to construct R-matrices, i.e., (trigonometric) solutions for the quantum Yang–Baxter equation. In this paper, we are interested in the rigid monoidal abelian category \( \mathcal{C} \) formed by finite-dimensional modules over \( U'_q(\hat{\mathfrak{g}}) \). The structure of the category \( \mathcal{C} \) is quite rich. In particular, it is neither semisimple as an abelian category nor braided as a monoidal category. Due to its rich structure, it has been intensively studied in fruitful connections with many research areas of mathematics and physics including quantum groups, statistical mechanics, cluster algebras, dynamical systems, geometry of quiver varieties etc (see for example [AK97,CP94,FR99,HL15,HL16,IKT12,Nak01]).

Our main motivation is to obtain a better understanding of the structure of the category \( \mathcal{C} \). Here we remark that the quantum Cartan matrix \( C(z) \) of \( \mathfrak{g} \) (see Sect. 4.1 for its definition), or rather its inverse \( \tilde{C}(z) := C(z)^{-1} \), has been playing an important role as a combinatorial key ingredient in the study of the category \( \mathcal{C} \). For instance, it appeared in:

- the factorization formula of the universal R-matrix due to [KT94,Dam98],
- the study of the \( q \)-characters (natural quantum affine analog of the usual characters) of modules in \( \mathcal{C} \) and the deformed \( \mathcal{W} \)-algebras due to e.g. [FR99,FM01],
- the construction of the quantum Grothendieck ring \( K_t(\mathcal{C}) \), which is a \( t \)-deformation of the usual Grothendieck ring \( K(\mathcal{C}) \), due to [Nak04b,VV03,Her04].

In this paper, we particularly focus on this combinatorial ingredient. Roughly speaking, we establish a unified formula computing the inverse \( \tilde{C}(z) \) of the quantum Cartan matrix and apply it to the study of the category \( \mathcal{C} \) especially for non-simply-laced cases.

1.2. When \( \mathfrak{g} \) is of simply-laced type (i.e., of types ADE), the category \( \mathcal{C} \) has an intimate connection with the representation theory of a Dynkin quiver \( Q \) of the same type as \( \mathfrak{g} \). Such a remarkable connection was initially established in the seminal paper [HL15] by D. Hernandez and B. Leclerc. In that paper, it was proved that the quantum Grothendieck ring \( K_t(\mathcal{C}^0) \) of a “skeleton” rigid monoidal subcategory \( \mathcal{C}^0 \subset \mathcal{C} \) is isomorphic to the derived Hall algebra \( DH(Q) \) of the category Rep(\( Q \)) of representations of a Dynkin quiver \( Q \). Under this isomorphism, the cohomological degree shift \([1]\) in the derived category \( D^b(\text{Rep}(Q)) \) corresponds to the right dual functor \( \mathcal{D} \) on \( \mathcal{C}^0 \). In addition, restricting the isomorphism \( K_t(\mathcal{C}^0) \simeq DH(Q) \) to the subalgebras corresponding to the heart Rep(\( Q \)) \( D^b(\text{Rep}(Q)) \) of the standard \( t \)-structure yields an isomorphism

\[
K_t(\mathcal{C}_Q) \simeq U^{+}_t(\mathfrak{g}),
\]

where \( \mathcal{C}_Q \subset \mathcal{C}^0 \) is a certain monoidal subcategory defined by using the Auslander–Reiten (AR) quiver \( \Gamma_Q \) of the category Rep(\( Q \)), and \( U^{+}_t(\mathfrak{g}) \) is the positive half of the quantized enveloping algebra \( U_t(\mathfrak{g}) \) of the finite-dimensional Lie algebra \( \mathfrak{g} \). (Strictly
speaking, the definition of the subcategory $\mathcal{C}_Q$ and hence the isomorphism (1.1) actually depend not only on the quiver $Q$ but also on the choice of a height function $\xi$ on it.

To establish these isomorphisms, Hernandez–Leclerc [HL15] gave a homological interpretation of the inverse $\overset{\sim}{C}(z)$ of the quantum Cartan matrix in terms of the derived category $D^b(\text{Rep}(Q))$. This homological interpretation implies that $\overset{\sim}{C}(z)$ can be computed from the AR quiver $\Gamma_Q$. Moreover, it also yields a combinatorial formula expressing $\overset{\sim}{C}(z)$ in terms of the action of the Coxeter element $\tau_Q$ adapted to $Q$ on the set of roots of $\mathfrak{g}$. From these results, we can deduce some useful properties of the coefficients of $\overset{\sim}{C}(z)$ such as periodicity and positivity.

1.3. We aim to extend the above results by [HL15] to general $\mathfrak{g}$ including non-simply-laced cases. Now let $\mathfrak{g}$ be another simple Lie algebra of simply-laced type whose Dynkin diagram $\Delta$ is related to that of $\mathfrak{g}$ via the folding with respect to a graph automorphism $\sigma$ on $\Delta$. When $\mathfrak{g}$ is of simply-laced type, we understand that $\sigma$ is the identity and $\mathfrak{g} = \mathfrak{g}$.

Recall that the AR quiver $\Gamma_Q$ has another combinatorial meaning (due to [Béd99]) as the Hasse quiver of the convex partial ordering among the positive roots arising from the commutation class of reduced words for the longest element of the Weyl group adapted to $Q$. Generalizing this combinatorial aspect of the AR quivers, U. R. Suh and the second named author [OS19d] defined the twisted AR quivers to be the Hasse quivers of the convex ordering among positive roots of $\mathfrak{g}$ (not $\mathfrak{g}$) arising from another kind of commutation classes called $\sigma$-adapted classes (or twisted adapted classes). With this notion, they introduced a nice subcategory of $\mathcal{C}_Q$ for $\mathfrak{g}$ attached to each $\sigma$-adapted class, which serves as a generalization of Hernandez–Leclerc’s subcategory $\mathcal{C}_Q$ in the simply-laced case.

Having this construction, it is natural to expect that we can compute the inverse $\overset{\sim}{C}(z)$ of the quantum Cartan matrix of general $\mathfrak{g}$ from the twisted AR quivers. In the former half of this paper, we realize this expectation.

In order to make our formulation unified, we introduce a new combinatorial notion of a $Q$-datum. By definition, a $Q$-datum for $\mathfrak{g}$ is a triple $Q = (\Delta, \sigma, \xi)$, where $(\Delta, \sigma)$ is as above and $\xi: \Delta_0 \to \mathbb{Z}$ is a function on the vertex set $\Delta_0$, which we call a height function on $(\Delta, \sigma)$, subject to certain axioms. When $\mathfrak{g}$ is of simply-laced type ($\sigma = \text{id}$), a height function $\xi$ on $(\Delta, \text{id})$ is the same as a usual height function on a unique Dynkin quiver $Q$. In this sense, the notion of a $Q$-datum is a combinatorial generalization of the notion of a Dynkin quiver with a height function.

For a $Q$-datum $Q$ for $\mathfrak{g}$, we can define the notion of $Q$-adaptedness for reduced expressions of elements in the Weyl group $W$ of $\mathfrak{g}$ as an analog of the $Q$-adaptedness to a usual quiver $Q$. Then we prove that the set $[Q]$ of reduced words of the longest element of $W$ which are adapted to $Q$ forms a $\sigma$-adapted class and that the assignment $Q \mapsto [Q]$ gives a bijection between the set of $Q$-data for $\mathfrak{g}$ (up to adding constants to height functions) and the set of $\sigma$-adapted classes. Moreover, we introduce the generalized Coxeter element $\tau_Q$ as a specific element of $W \rtimes \langle \sigma \rangle$ and construct the corresponding twisted AR quiver $\Gamma_Q$ by using $\tau_Q$ only. When $\sigma = \text{id}$, they respectively coincide with the usual Coxeter element $\tau_Q$ and the AR quiver $\Gamma_Q$ in 1.2. Thus the notion of $Q$-datum yields a unified description of the $\sigma$-adapted classes and their twisted AR quivers.

By means of these gadgets, we obtain the desired combinatorial formula of $\overset{\sim}{C}(z)$ for general $\mathfrak{g}$ in terms of the action of the generalized Coxeter element $\tau_Q$ on the set $R$ of roots of $\mathfrak{g}$ (Theorem 4.8), which generalizes the above formula by [HL15] in 1.2. From this formula, we deduce several properties of the coefficients of $\overset{\sim}{C}(z)$ for general $\mathfrak{g}$ such as periodicity and positivity. These results finally imply that $\overset{\sim}{C}(z)$ can be computed
from the twisted AR quiver \( \Gamma_Q \) associated with an arbitrary \( Q \). We will carry out such a computation for all non-simply-laced \( g \) with a specific choice of \( Q \) (Sect. 4.3).

1.4. In the latter half of this paper, we pursue several applications of the above combinatorial theory of Q-data to the study of the category \( \mathcal{C} \).

For each Q-datum \( Q \) for \( g \), we define a certain monoidal subcategory \( \mathcal{C}_Q \) of the category \( \mathcal{C} \) (Sect. 5.4). From the construction, it coincides with the subcategory introduced in [OS19c] (mentioned above in 1.3) attached to the \( \sigma \)-adapted class \([\bar{Q}]\). Then we observe that the duals of simple objects in \( \mathcal{C}_Q \), i.e., the set \( \{ \bar{\mathcal{G}}^k(V) \mid V \in \text{Irr} \mathcal{C}_Q, k \in \mathbb{Z} \} \) monoidally generates the whole category \( \mathcal{C}_Q \). In this sense, our subcategory \( \mathcal{C}_Q \) plays a role of a “heart” of the category \( \mathcal{C}_Q \), just as the subcategory \( \mathcal{C}_Q \) does in the simple-laced case.

In addition, we can assign to each simple module \( V \) in \( \mathcal{C}_Q \) an element \( \text{wt}_Q(V) \) of the root lattice \( Q = \mathbb{Z}R \) of the simply-laced Lie algebra \( g \), which we call the Q-weight of \( V \) (Definition 5.10). It turns out that the assignment \( V \mapsto \text{wt}_Q(V) \) is comparable with the elliptic character considered in [CM05] and therefore we obtain a block decomposition of the category \( \mathcal{C}_Q \) labeled by \( Q \). The resulting block decomposition is in turn the same as the one recently obtained in [KKKOP20]. Thus the notion of Q-weight connects these two known block decomposition results.

1.5. We can also apply our combinatorial theory of Q-data and the inverse of the quantum Cartan matrix to the study of R-matrices and related invariants.

For any pair \((V, W)\) of simple modules in \( \mathcal{C} \), we have the normalized R-matrix \( R_{V,W}^{\text{norm}}(z) \), which is a unique non-trivial intertwining operator between the tensor products \( V \otimes W \) and \( W \otimes V \) depending rationally on the spectral parameter \( z \), whose singularity strongly reflects the structure of these tensor product modules. For example, assuming one of the simple modules is real (i.e., its tensor square is again simple), we have \( V \otimes W \cong V \otimes W \) if and only if both \( R_{V,W}^{\text{norm}}(z) \) and \( R_{W,V}^{\text{norm}}(z) \) have no poles at \( z = 1 \). Thus to compute the denominator \( d_{V,W}(z) \) of \( R_{V,W}^{\text{norm}}(z) \) is of fundamental importance in the study of the monoidal category \( \mathcal{C} \).

When \( g \) is of simply-laced type, the present authors recently discovered in [Oh18, Fuj19] that the denominators between the fundamental modules are closely related to the AR quiver \( \Gamma_Q \) for a Dynkin quiver \( Q \). In particular, we find that these denominators are expressed in a unified way in terms of the inverse \( \tilde{\mathcal{C}}(z) \) of the Cartan matrix as the first named author proved in [Fuj19] by using the geometry of quiver varieties. In this paper, motivated by this result, we present a conjectural unified denominator formula between all the Kirillov–Reshetikhin (KR) modules (Conjecture 6.7). Recall that the KR modules form a family of simple modules in \( \mathcal{C} \) whose \( q \)-characters satisfy the remarkable functional relations called \( T \)-systems [Nak03,Her06] and hence play an important role in the theory of monoidal categorifications of cluster algebras [HL10, HL16, KKKOP20]. Our conjectural KR denominator formula is also expressed in terms of \( \tilde{\mathcal{C}}(z) \). We check that it holds at least for types ABCD by comparing the explicit case-by-case computations of the denominators obtained in [OS19b] with the explicit computations of \( \tilde{\mathcal{C}}(z) \) obtained in this paper.

Finally, we compute the \( \mathbb{Z} \)-valued invariants \( \Lambda(V, W) \) and \( \Lambda^\infty(V, W) \) in terms of \( \tilde{\mathcal{C}}(z) \) and the \( Q \)-weights (Sect. 6.4). These invariants are recently introduced by M. Kashiwara, M. Kim, E. Park and the second named author [KKKOP20] to give a monoidal categorification of a cluster algebra. They are defined from the denominator \( d_{V,W}(z) \) and also the universal coefficient
$a_{V, W}(z)$, i.e., the ratio of the universal $R$-matrix $R_{V, W}^{\text{univ}}(z)$ and the normalized $R$-matrix $R_{V, W}^{\text{norm}}(z)$. Based on a formula of $a_{V, W}(z)$ due to Frenkel–Reshetikhin [FR99], we observe that

\[
\Lambda(V, W) = \mathcal{N}(V, W) + \deg_{z=1} d_{V, W}(z),
\]

\[
\Lambda^\infty(V, W) = -(\text{wt}_Q(V), \text{wt}_Q(W))
\]

hold for any simple modules $V, W$ in $\mathcal{C}$. Here $(\cdot, \cdot)$ is the standard symmetric bilinear form on the root lattice $Q$, and $\mathcal{N}(V, W)$ is a skew-symmetric $\mathbb{Z}$-valued form defined as a certain signed sum of coefficients of $\tilde{C}(z)$, which was introduced by Hernandez [Her04] in his algebraic construction of the quantum Grothendieck ring $K_t(\mathcal{C}_0)$. Thus the invariants $\Lambda(V, W)$ and $\Lambda^\infty(V, W)$ are also closely related to Q-data and $\tilde{C}(z)$.

1.6. We expect that our results in this paper have some nice applications in the study of the quantum Grothendieck ring $K_t(\mathcal{C}_0)$. In particular, we hope to generalize the results of [HL15] (especially the isomorphism (1.1)) to the case of general $g$ and investigate an expected quantum cluster algebra structure in the quantum Grothendieck ring using our combinatorial theory of Q-data. We will come back to these problems in forthcoming papers.

1.7. Organization This paper is organized as follows. In Sect. 2, we give a quick review of the classical notion of the usual Dynkin quiver with height function, associated AR quivers and adapted commutation classes in Weyl groups. In Sect. 3, to generalize the story in Sect. 2, we introduce the notion of Q-datum and develop a unified theory for twisted AR quivers and generalized Coxeter elements. In Sect. 4, we study the inverse $\tilde{C}(z)$ of the quantum Cartan matrix in relation with Q-data. Up to this part, our exposition is a pure combinatorics. In Sect. 5, after reviewing some basic facts on the finite-dimensional representation theory of the untwisted quantum affine algebras, we revisit the block decomposition results of the category $\mathcal{C}_0$ with the notion of $Q$-weights. In the final Sect. 6, we present a conjectural unified KR denominator formula and compute the invariants $\Lambda(V, W)$ and $\Lambda^\infty(V, W)$ in terms of $\tilde{C}(z)$ and the $Q$-weights. Appendix A is a list of case-by-case formulae of denominators of the normalized $R$-matrices between KR modules known at this moment.

Convention. Throughout this paper, we keep the following conventions.

1. For a statement $\mathcal{P}$, we set $\delta(\mathcal{P})$ to be 1 or 0 according that $\mathcal{P}$ is true or not. In particular, we set $\delta_{i,j} := \delta(i = j)$ (Kronecker’s delta).
2. For an abelian category $\mathcal{C}$, we denote by $\text{Irr} \mathcal{C}$ the set of simple objects (up to isomorphisms) and by $K(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$. The class of an object $X \in \mathcal{C}$ is denoted by $[X] \in K(\mathcal{C})$. If moreover the category $\mathcal{C}$ is monoidal with a bi-exact tensor product $\otimes$, $K(\mathcal{C})$ is endowed with a ring structure by $[X] \cdot [Y] = [X \otimes Y]$, which we call the Grothendieck ring of $\mathcal{C}$.

2. Dynkin Quivers, Adapted Classes and Associated Auslander–Reiten Quivers

In this section, we briefly recall the classical notion of Dynkin quivers of type ADE and their height functions, which give a useful description of the associated Auslander–Reiten quiver and the adapted commuting classes in the Weyl group.
2.1. Basic notation  Let \( g \) be a finite-dimensional complex Lie algebra of type ADE of rank \( n \). We denote by \( \Delta \) the Dynkin diagram of \( g \) (simply-laced) and by \( \Delta_0 \) the set of vertices of \( \Delta \). For \( i, j \in \Delta_0 \), we write \( i \sim j \) if \( i \) is adjacent to \( j \) in the Dynkin diagram \( \Delta \).

Let \( P = \bigoplus_{i \in \Delta_0} \mathbb{Z} \sigma_i \) denote the weight lattice of \( g \), where \( \sigma_i \) is the \( i \)-th fundamental weight. Let \( \alpha_i^j = 2 \sigma_j - \sum_{j \sim i} \sigma_j \) be the \( i \)-th simple root and \( Q = \bigoplus_{i \in \Delta_0} \mathbb{Z} \alpha_i \subseteq P \) be the root lattice. We set \( P^+ := \sum_{i \in \Delta_0} \mathbb{Z}_{\geq 0} \sigma_i \) and \( Q^+ := \sum_{i \in \Delta_0} \mathbb{Z}_{\geq 0} \alpha_i \). Let \( (, ) : P \times P \rightarrow \mathbb{Q} \) denote the symmetric bilinear form determined by \( (\sigma_i, \alpha_j) = \delta_{i,j} \) for \( i, j \in \Delta_0 \). The Weyl group \( W \) of \( g \) is defined as a subgroup of \( \mathrm{Aut}(P) \) generated by the simple reflections \( \{ s_i \}_{i \in \Delta_0} \) defined by \( s_i(\lambda) = \lambda - (\lambda, \alpha_i) \alpha_i \) for \( \lambda \in P \). The set of roots is defined by \( R := W \cdot \{ \alpha_i \}_{i \in \Delta_0} \). We have the decomposition \( R = R^+ \sqcup R^- \), where \( R^+ := R \cap Q^+ \) is the set of positive roots and \( R^- := -R^+ \) is the set of negative roots.

For an element \( w \in W \), a sequence \( (i_1, \ldots, i_l) \) of elements of \( \Delta_0 \) is called a reduced word for \( w \) if it satisfies \( w = s_{i_1} \cdots s_{i_l} \) and \( l \) is the smallest among all sequences with this property. The length of a reduced word of \( w \) is called the length of \( w \), denoted by \( \ell(w) \). It is well-known that there exists a unique element \( w_0 \in W \) with the largest length \( \ell(w_0) = N := |R^+| \). We define an involution \( i \mapsto i^* \) on the set \( \Delta_0 \) by the relation \( w_0(\alpha_i) = -\alpha_{i^*} \).

2.2. Dynkin quivers and height functions  By definition, a Dynkin quiver \( Q \) of type \( g \) is a quiver whose underlying graph is equal to the Dynkin diagram \( \Delta \). A height function on \( Q \) is a function \( \xi : \Delta_0 \rightarrow \mathbb{Z} \) satisfying the condition that

\[ \xi_i = \xi_j + 1 \quad \text{if} \quad Q \text{ has an arrow } i \rightarrow j, \]

where we set \( \xi_i := \xi(i) \) for simplicity. Note that a height function on \( Q \) is determined uniquely up to constant functions.

Example 2.1. Here are two examples of Dynkin quivers with height functions in type \( A_3 \) and type \( D_4 \) respectively. We put the values of height functions near the vertices.

\[
Q^{(1)} = \begin{pmatrix} 1 \rightarrow 2 \leftarrow 3 \leftarrow 2 \rightarrow 3 \end{pmatrix}, \quad Q^{(2)} = \begin{pmatrix} 1 \rightarrow 2 \leftarrow 0 \rightarrow 3 \leftarrow 0 \rightarrow 3 \end{pmatrix}.
\]

We fix a function \( \varepsilon : \Delta_0 \rightarrow \{0, 1\} \) such that \( \varepsilon_i \neq \varepsilon_j \) when \( i \sim j \). We refer to such a function \( \varepsilon \) as a parity function on \( \Delta_0 \). Since the Dynkin diagram \( \Delta \) is a connected tree, there are only two choices of parity functions and their difference is not essential. Adding a constant if necessary, we can make a given height function \( \xi \) satisfy the condition

\[ \xi_i - \varepsilon_i \in 2\mathbb{Z} \quad \text{for each } i \in \Delta_0. \]

Thus, without loss of generality, we require that a height function \( \xi \) always satisfy the parity condition (2.2) in the rest of this section.

We say that a vertex \( i \in \Delta_0 \) is a source of \( Q \) if every arrow in \( Q \) incident to \( i \) has \( i \) as its tail. In terms of the height function \( \xi \), a vertex \( i \) is a source of \( Q \) if and only if we have \( \xi_i > \xi_j \) for any \( j \sim i \). Let \( i \in \Delta_0 \) be a source of \( Q \). We denote by \( s_iQ \) the new Dynkin quiver of the same type \( g \) obtained from \( Q \) by reversing all arrows incident to \( i \).

Moreover, for a height function \( \xi \) on \( Q \), we can define the height function \( s_i \xi \) on \( s_iQ \) by

\[ (s_i \xi)_j := \xi_j - 2\delta_{i,j} \quad \text{for each } j \in \Delta_0. \]
We say that a sequence \((i_1, \ldots, i_l)\) of elements of \(\Delta_0\) is adapted to \(Q\) if
\[
i_k \text{ is a source of } s_{i_{k-1}}s_{i_{k-2}}\cdots s_{i_1}Q \text{ for all } 1 \leq k \leq l.
\]

Recall that a Coxeter element \(\tau\) of \(W\) is a product of the form \(\tau = s_{i_1}\cdots s_{i_n}\) such that \(\{i_1, \ldots, i_n\} = \Delta_0\). All the Coxeter elements are conjugate in \(W\) and the order of Coxeter elements is called the (dual) Coxeter number of \(\mathfrak{g}\) and denoted by \(h^\vee\). Note that we have the relation \(nh^\vee = 2N\).

For each Dynkin quiver \(Q\), there exists a unique Coxeter element \(\tau_Q\), all of whose reduced words are adapted to \(Q\). For example, a sequence \((i_1, \ldots, i_n)\) satisfying \(\Delta_0 = \{i_1, \ldots, i_n\}\) and \(\xi_{i_1} \geq \cdots \geq \xi_{i_n}\) for a height function \(\xi\) on \(Q\) gives a reduced word for \(\tau_Q\). Conversely, for each Coxeter element \(\tau\), there exists a unique Dynkin quiver \(Q\) such that all reduced words for \(\tau\) are adapted to \(Q\).

### 2.3. Associated Auslander–Reiten quivers

For a Dynkin quiver \(Q\), we denote by \(\text{Rep}(Q)\) the category of finite-dimensional representations of \(Q\) over the field \(\mathbb{C}\) of complex numbers. In this subsection, we recall a combinatorial description of the Auslander–Reiten (AR) quiver of the category \(\text{Rep}(Q)\) and that of the derived category \(\mathcal{D}_Q := D^b(\text{Rep}(Q))\) by using a height function \(\xi\) on \(Q\). By the definition, the vertex set of the AR quiver of \(\text{Rep}(Q)\) (resp. \(\mathcal{D}_Q\)) is the set of isomorphism classes of indecomposable objects, denoted by \(\text{Ind Rep}(Q)\) (resp. \(\text{Ind } \mathcal{D}_Q\)). By the fundamental result of Gabriel [Gab72], we have the canonical bijection \(R^+ \cong \text{Ind Rep}(Q)\) which associates each positive root \(\alpha \in R^+\) with the class of an indecomposable representation \(M_Q(\alpha) \in \text{Rep}(Q)\) whose dimension vector is \(\alpha\). Furthermore we have a canonical bijection \(\hat{R}^+ := R^+ \times \mathbb{Z} \cong \text{Ind } \mathcal{D}_Q\) which associates each element \((\alpha, k) \in \hat{R}^+\) with a stalk complex \(M_Q(\alpha)[k] \in \mathcal{D}_Q\). Here we naturally identify the abelian category \(\text{Rep}(Q)\) with the heart of the standard \(R\)-structure of \(\mathcal{D}_Q\) and denote by \([k]\) the cohomological degree shift by \(k\).

Using the fixed parity function \(\epsilon\) in (2.2), we define the repetition quiver associated with \(\Delta\) to be the quiver \(\hat{\Delta}\) whose vertex set \(\hat{\Delta}_0\) and arrow set \(\hat{\Delta}_1\) are given by
\[
\hat{\Delta}_0 := \{(i, p) \in \Delta_0 \times \mathbb{Z} \mid p - \epsilon_i \in 2\mathbb{Z}\},
\]
\[
\hat{\Delta}_1 := \{(i, p) \rightarrow (j, p+1) \mid (i, p) \in \hat{\Delta}_0, \ j \sim i\}.
\]

**Example 2.2.** Here are some examples of the repetition quiver \(\hat{\Delta}\).
1. When \(\mathfrak{g}\) is of type \(A_5\), the repetition quiver \(\hat{\Delta}\) is depicted as:

![Repetition Quiver A5](image)

2. When \(\mathfrak{g}\) is of type \(D_4\), the repetition quiver \(\hat{\Delta}\) is depicted as:

![Repetition Quiver D4](image)

It was shown by Happel [Hap87] that the AR quiver of \(\mathcal{D}_Q\) is isomorphic to the repetition quiver \(\hat{\Delta}\). An explicit underlying bijection \(\phi_Q : \hat{\Delta}_0 \rightarrow \text{Ind } \mathcal{D}_Q\) between the vertex sets is given by
\[
\phi_Q(i, p) := \tau^{(\xi_i - p)/2}M_Q(Y_i^Q),
\]
where \( \tau \) denotes the AR translation of \( D_Q \) and we set

\[
\gamma_i^Q := (1 - \tau Q) \omega_i
\]

for each \( i \in \Delta_0 \). Here \( \tau_Q \in \mathcal{W} \) is the Coxeter element adapted to \( Q \). Note that the representation \( M_Q(\gamma_i^Q) \in \Rep(Q) \) is an injective hull of the simple representation associated with \( i \in \Delta_0 \) since \( \gamma_i^Q \) is a sum of all the simple roots \( \alpha_j \) such that there exists an oriented path from \( j \) to \( i \) in \( Q \).

In what follows, we regard \( \phi_Q \) as a bijection \( \phi_Q : \Delta_0 \to \hat{R}^+ \) via the above canonical bijection \( D_Q \cong \hat{R}^+ \). Then it has the following recursive description ([HL15, §2.1]):

1. \( \phi_Q(i, \xi_i) = (\gamma_i^Q, 0) \) for each \( i \in \Delta_0 \).
2. If \( \phi_Q(i, p) = (\beta, k) \), we have

\[
\phi_Q(i, p \pm 2) = \begin{cases} 
(\tau_Q^{1}(\beta), k) & \text{if } \tau_Q^{1}(\beta) \in \mathbb{R}^+, \\
(\tau_Q^{1}(\beta), k \pm 1) & \text{if } \tau_Q^{1}(\beta) \in \mathbb{R}^-.
\end{cases}
\]

In particular, we have \( \tau_Q^{(\xi_i - p)/2} \gamma_i^Q = (-1)^k \beta \) if \( \phi_Q(i, p) = (\beta, k) \).

The repetition quiver \( \hat{\Delta} \) satisfies the additive property: For \( i \in \Delta_0 \) and \( l \in \mathbb{Z} \), we have

\[
\tau_Q^l(\gamma_i^Q) + \tau_Q^{l+1}(\gamma_i^Q) = \sum_{j \sim i} \tau_Q^{l+(\xi_j - \xi_i)}/2(\gamma_j^Q).
\] (2.4)

This is because we have the following mesh in the AR quiver of \( D_Q \):

\[
\tau^{l+(\xi_j - \xi_i)}/2 M_Q(\gamma_j^Q) \rightarrow \cdots \rightarrow \tau^{l} M_Q(\gamma_j^Q)
\]

where \( \{j, \ldots, j'\} = \{j \in \Delta_0 \mid j \sim i\} \), which corresponds to an Auslander–Reiten triangle in \( D_Q \).

Let \( \Gamma_Q \) be the full subquiver of \( \hat{\Delta} \) whose vertex set \( (\Gamma_Q)_0 \) is given by \( \phi_Q^{-1}(\mathbb{R}^+ \times \{0\}) \). We define the bijection \( \phi_{Q, 0} : (\Gamma_Q)_0 \to \mathbb{R}^+ \) by \( \phi_Q(i, p) = (\phi_{Q, 0}(i, p), 0) \) for \( (i, p) \in (\Gamma_Q)_0 \). For \( \beta \in \mathbb{R}^+ \), we call \( (i, p) = \phi_Q^{-1}(\beta) \) the coordinate of \( \beta \) in \( \Gamma_Q \).

In what follows, we identify the vertex set \( (\Gamma_Q)_0 \) with the set \( \mathbb{R}^+ \) of positive roots via the bijection \( \phi_{Q, 0} \). By construction, the quiver \( \Gamma_Q \) is isomorphic to the AR quiver of \( \Rep(Q) \) under the canonical bijection \( \mathbb{R}^+ \cong \Ind\Rep(Q) \). In addition, we have the following explicit characterization of \( \Gamma_Q \) in \( \Delta \):

\[
\Gamma_Q = \phi_Q^{-1}(\mathbb{R}^+ \times \{0\}) = \{(i, \xi_i - 2k) \in \hat{\Delta}_0 \mid 0 \leq 2k < h^\vee + \xi_i - \xi_i^+\}. \tag{2.5}
\]

See [Gab80, §6.5]. For \( d \in \mathbb{Z}_{>0} \), we say that a subset \( S \subset \mathbb{Z} \) is a \( d \)-segment if \( S = \{l + kd \mid k = 0, \ldots, r\} \) for some \( l \in \mathbb{Z} \) and \( r \in \mathbb{Z}_{\geq 0} \). The characterization (2.5) shows the 2-segment property of \( \Gamma_Q \).
Example 2.3. For the Dynkin quivers $Q^{(1)}$ and $Q^{(2)}$ in Example 2.1, the corresponding AR quivers $\Gamma_{Q^{(1)}}$ and $\Gamma_{Q^{(2)}}$ can be depicted as follows. Here $[a, b] := \epsilon_a - \epsilon_{b+1}$ and $(a, \pm b) := \epsilon_a \pm \epsilon_b$ denote the positive roots defined as in Sects. 4.3.1 and 4.3.2 below.

\[
\begin{array}{cccccc}
(i, p) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
1 & 2 & & & & & & \\
2 & & 4 & & & & & \\
3 & & & 1, 4 & & & & \\
4 & & & & 2, 5 & & & \\
5 & & & & & 3, 5 & & \\
\hline
(i, p) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
1 & 2 & & & & & & \\
2 & & 1, -2 & & & & & \\
3 & & & 1, -4 & & & & \\
4 & & & & 2, -4 & & & \\
\end{array}
\]

\[
\Gamma_{Q^{(1)}} = \begin{array}{cccccc}
1 & 2 & 4 & 3 & 0 & 1 & 2 & 3 \\
1 & [4] & [1, 5] & [2, 3] & & & & \\
2 & [1, 4] & [2, 5] & [3] & & & & \\
3 & [2] & [3, 4] & [5] & & & & \\
4 & [1] & & & & & & \\
\hline
1 & 2 & 4 & 3 & 0 & 1 & 2 & 3 \\
1 & (1, -2) & (2, -3) & (2, 3) & & & & \\
2 & (1, -4) & (2, 4) & (3, -4) & & & & \\
3 & & & & & & & \\
4 & & & & & & & \\
\end{array}
\]

For a source $i \in \Delta_0$ of $Q$, the quiver $\Gamma_{s_i Q}$ can be obtained from $\Gamma_Q$ in the following way:

1. A positive root $\beta \in \mathbb{R}^+ \setminus \{\alpha_i\}$ is located at the coordinate $(j, p)$ in $\Gamma_{s_i Q}$, if $s_i \beta$ is located at the coordinate $(j, p)$ in $\Gamma_Q$.
2. The simple root $\alpha_i$ is located at the coordinate $(i^*, \xi_i - h^\vee)$ in $\Gamma_{s_i Q}$, while $\alpha_i = \gamma_i^Q$ was located at the coordinate $(i, \xi_i)$ in $\Gamma_Q$.

2.4. Generalities on commutation classes Two sequences $i$ and $i'$ of elements of $\Delta_0$ are said to be commutation equivalent if $i'$ is obtained from $i$ by applying a sequence of operations which transform some adjacent components $(i, j)$ such that $i \sim_j i'$. This defines an equivalence relation. We refer to the equivalent class containing $i$ as the commutation class of $i$ and denote it by $[i]$. Note that the set of all the reduced words of an element $w \in W$ is divided into a disjoint union of commutation classes. In this paper, we mainly consider the commutation classes of reduced words for the longest element $w_0$.

For a reduced word $i = (i_1, \ldots, i_N)$ for the longest element $w_0 \in W$, we have $\mathbb{R}^+ = \{\beta^k | 1 \leq k \leq N\}$, where $\beta^k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_i)$. Thus the reduced word $i$ defines a total order $<_i$ on $\mathbb{R}^+$, namely we write $\beta^k <_i \beta^l$ if $k < l$. Note that the order $<_i$ is convex in the sense that if $\alpha, \beta, \alpha + \beta \in \mathbb{R}^+$, we have either $\alpha <_i \alpha + \beta <_i \beta$ or $\beta <_i \alpha + \beta <_i \alpha$. For a commutation class $[i]$ of reduced words for $w_0$, we define the convex partial order $\leq_{[i]}$ on $\mathbb{R}^+$ so that

$$\alpha \leq_{[i]} \beta \quad \text{if and only if} \quad \alpha <_{i'} \beta \text{ for all } i' \in [i].$$

For a commutation class $[i]$ of reduced words of $w_0$ and a positive root $\alpha \in \mathbb{R}^+$, we define the $[i]$-residue of $\alpha$, denoted by $\text{res}^{[i]}(\alpha)$, to be $i_k \in \Delta_0$ if we have $\alpha = \beta^k_i$ with $i = (i_1, \ldots, i_N)$. Note that this is well-defined, i.e. if we have $\alpha = \beta^k_{i'}$ for another $i' = (i'_1, \ldots, i'_N) \in [i]$, then $i'_k = i_k$.

In [OS19c], U. R. Suh and the second named author defined the combinatorial Auslander–Reiten quiver $\Upsilon_i$ for each reduced word $i = (i_1, \ldots, i_N)$ for $w_0$. By definition, it is a quiver whose vertex set is $\mathbb{R}^+$ and we have an arrow in $\Upsilon_i$ from $\beta^k_i$ to
reflection functor

and only if \( i_j' \in \{ i_j, i_k \} \). The quiver \( \Upsilon_i \) satisfies the following nice properties. We say that a total ordering \( R^i = \{ \beta_1, \beta_2, \ldots, \beta_N \} \) is a compatible reading of \( \Upsilon_i \) if we have \( k < l \) whenever there is an arrow \( \beta_l \to \beta_k \) in \( \Upsilon_i \).

**Theorem 2.4** ([OS19c]). For a commutation class \([i]\) for \( w_0 \), we have the followings:

1. If \( i' \in [i] \), then \( \Upsilon_i = \Upsilon_{i'} \). Hence \( \Upsilon_{[i]} \) is well-defined.
2. For \( \alpha, \beta \in R^i \), we have \( \alpha \leq_{[i]} \beta \) if and only if there exists a path from \( \beta \) to \( \alpha \) in \( \Upsilon_{[i]} \).

In other words, the quiver \( \Upsilon_{[i]} \) is a Hasse quiver of the partial ordering \( \leq_{[i]} \).
3. For \( \alpha, \beta \in R^i \), we have \( (\alpha, \beta) = 0 \) if they are not comparable with respect to \( \leq_{[i]} \).
4. A sequence \( i' = (i_1', \ldots, i_N') \in \Delta_0^N \) belongs to the commutation class \([i]\) if and only if there is a compatible reading \( R^i = \{ \beta_1, \ldots, \beta_N \} \) of \( \Upsilon_{[i]} \) such that \( i_k' = \res ([i]) (\beta_k) \) for all \( 1 \leq k \leq N \).

For a reduced word \( i = (i_1, i_2, \ldots, i_N) \) of \( w_0 \), the sequence \( i' = (i_2, \ldots, i_N, i_1') \) is also a reduced word of \( w_0 \) and \([i] \neq [i']\). This operation is referred to as a (combinatorial) reflection functor and we write \( i' = r_{i_1} i \). We have the induced operation on commutation classes (i.e. \( r_{i_1} [i] := [r_{i_1} i] \) is well-defined). The relations \([i] \sim [i']\) for \( i \in \Delta_0 \) generate an equivalence relation, called the reflection equivalent relation \( \sim \), on the set of commutation classes \([i]\) of reduced words for \( w_0 \). For a given reduced word \( i \) of \( w_0 \), the family of commutation classes \([i]\) is called an \( r \)-cluster point.

2.5. Adapted commutation classes Let \( Q \) be a Dynkin quiver of type \( g \). It is well-known that the set of all reduced words of \( w_0 \) adapted to \( Q \) forms a single commutation class \([Q]\) of \( w_0 \), and \([Q] = [Q']\) if and only if \( Q = Q' \) (see [Béd99]). Furthermore, if \( i \) is a source of \( Q \), we have \( r_i [Q] = [s_i Q] \). Since any Dynkin quiver \( Q' \) of type \( g \) can be obtained from a given \( Q \) by a sequence of source reflections as \( Q' = s_{i_1} \cdots s_{i_m} Q \), we have \([Q'] \sim [Q]\). Therefore the set \([Q] \mid Q \) is a Dynkin quiver of type \( g \) forms a single reflection equivalent class \([\Delta]\) called the adapted \( r \)-cluster point. We call a commutation class in \([\Delta]\) an adapted class. By the above discussion, we have a canonical bijection

\[
\{ \text{Dynkin quivers of type } g \} \leftrightarrow [\Delta]
\]

which associates a Dynkin quiver \( Q \) with the adapted class \([Q]\).

**Theorem 2.5** ([Béd99]). Let \( Q \) be a Dynkin quiver of type \( g \).

1. If \( \beta \in R^i \) is located at the coordinate \((i, p)\) in \( \Gamma_Q \), we have \( \res_{[Q]} (\beta) = i \).
2. We have \( \Upsilon_{[Q]} = \Gamma_Q \).

2.6. Remark So far we have defined a height function \( \xi \) on a given Dynkin quiver \( Q \) to be a function \( \xi : \Delta_0 \to \mathbb{Z} \) satisfying the condition (2.1). Conversely, suppose that a function \( \xi : \Delta_0 \to \mathbb{Z} \) satisfies

\[
|\xi_i - \xi_j| = 1 \quad \text{if } i \sim j.
\]

Then it defines a Dynkin quiver \( Q \) of type \( g \) such that there is an arrow \( i \to j \) in \( Q \) if and only if \( i \sim j \) and \( \xi_i > \xi_j \). The function \( \xi \) gives a height function on this Dynkin quiver \( Q \).
Table 1. Foldings and related numerical data.

| g     | σ   | r  | g  | h ∨ | N   |
|-------|-----|----|----|-----|-----|
| A_n   | id  | 1  | A_n| n + 1| n(n + 1)/2 |
| D_n   | id  | 1  | D_n| 2n - 2| n(n - 1) |
| E_6,7,8 | id  | 1  | E_6,7,8| 12, 18, 30| 36, 63, 120 |
| A_{2n-1} | ∨  | 2  | B_n| 2n - 1| n(2n - 1) |
| D_{n+1} | ∨  | 2  | C_n| n + 1| n(n + 1) |
| E_6   | ∨   | 2  | F_4| 9 | 36 |
| D_4   | ∨, ∨^2 | 3  | G_2| 4 | 12 |

Since a height function \( \xi \) is determined uniquely up to constant from the Dynkin quiver \( Q \), we can rewrite the canonical bijection (2.6) as:

\[
\{ \text{functions } \xi : \Delta_0 \to \mathbb{Z} \text{ satisfying (2.2) and (2.7)} \}/2\mathbb{Z} \overset{1:1}{\longleftrightarrow} [\Delta]. \quad (2.8)
\]

In the next section, we generalize this bijection for the twisted adapted classes, which are other special kinds of commutation classes for \( w_0 \in W \).

3. Q-data, Twisted Adapted Classes and Twisted Auslander–Reiten Quivers

In this section, we first recall the twisted adapted class associated with a pair \((\Delta, \sigma)\) of Dynkin diagram \( \Delta \) of type ADE and a diagram automorphism \( \sigma \) on it. Then we introduce the notion of a Q-datum to develop a unified theory of twisted AR quivers and generalized twisted Coxeter elements.

3.1. Basic notation

Let \( g \) be a finite-dimensional complex simple Lie algebra of type ADE and \( \Delta \) be its Dynkin diagram (simply-laced). We keep the notation in Sect. 2.1 except for \( n \) and \( h^\vee \), which will denote respectively the rank and the dual Coxeter number of another simple Lie algebra \( g \) defined below. Also we mainly use the symbols \( i, j, \ldots \) for denoting the vertices in \( \Delta_0 \) in order to save the symbols \( i, j, \ldots \) for denoting the Dynkin indices of \( g \).

Let \( \sigma \) be an automorphism of \( \Delta \) satisfying the condition

there is no index \( i \in \Delta_0 \) such that \( i \sim \sigma(i) \). \quad (3.1)

Such a pair \((\Delta, \sigma)\) is classified in Table 1, where the automorphisms \( \vee \) and \( \widetilde{\vee} \) are defined as follows:
Hereafter we set \( r := |\sigma| \in \{1, 2, 3\} \). Given such a pair \((\Delta, \sigma)\), we denote by \( I \) the set of \( \sigma \)-orbits of \( \Delta_0 \). We write the natural quotient map \( \Delta_0 \to I \) by \( i \mapsto \bar{i} \). We set \( n := |I| \). For each \( i \in I \), we define
\[
d_i := |i| \in \{1, r\}.
\]
Note that we have \( d_i = 1 \) if and only if \( i \in \Delta_0 \) is fixed by \( \sigma \). Also we define
\[
r_i := r/d_i \in \{1, r\}.
\]
For \( i, j \in I \), we write \( i \sim j \) if there are \( i, j \in \Delta_0 \) satisfying \( \bar{i} = i, \bar{j} = j \) and \( i \sim j \). We attach an integer \( c_{ij} \) to each pair \((i, j) \in I \) by
\[
c_{ij} := \begin{cases} 
2 & \text{if } i = j; \\
-[d_j/d_i] & \text{if } i \sim j; \\
0 & \text{otherwise}.
\end{cases}
\]
The resulting matrix \( C = (c_{ij})_{i,j \in I} \) gives the Cartan matrix of a complex simple Lie algebra \( g \). Conversely, every complex simple Lie algebra \( g \) is obtained in this way from the unique pair \((\Delta, \sigma)\) determined as in Table 1. Note that \( g \) is not simply-laced if and only if \( \sigma \) is non-trivial. If we set \( D := \text{diag}(d_i \mid i \in I) \), the product \( DC \) becomes symmetric:
\[
d_i c_{ij} = d_j c_{ji} \quad \text{for any } i, j \in I.
\]
Write

$$C^{-1} = (\tilde{c}_{ij})_{i,j \in I} \in GL_I(Q).$$

Since $DC^{-1} = D \cdot (DC)^{-1} \cdot D$ is symmetric, we also have

$$d_i \tilde{c}_{ij} = d_j \tilde{c}_{ji} \quad \text{for any } i, j \in I. \quad (3.4)$$

Let $h^\vee$ denote the dual Coxeter number of the simple Lie algebra $g$. For each pair $(\Delta, \sigma)$, we have the equality

$$nr h^\vee = 2N, \quad (3.5)$$

which can be checked easily from Table 1.

**Remark 3.1.** Let $(\Delta, \sigma)$ be as above. Recall the involution $i \mapsto i^*$ on the set $\Delta_0$ from Sect. 2.1. Under the assumption $\sigma \neq id$, we have $* = id$ if $h^\vee$ is even, and $* = \sigma$ if $h^\vee$ is odd. Note also that $r = 2$ whenever $h^\vee$ is odd. In particular, the involution $*$ on $\Delta_0$ induces an involution on the set $I$, for which we use the same notation $i \mapsto i^*$. The latter involution is trivial if $g$ is not simply-laced.

### 3.2. Twisted adapted classes

Note that the diagram automorphism $\sigma$ naturally gives an element of $\text{Aut}(P)$ which we denote by the same symbol $\sigma$. Namely we define $\sigma(\sigma_i) := \sigma(\sigma_i)$ for each $i \in \Delta_0$. Then it is immediate that $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ and $\sigma s_i \sigma^{-1} = s_{\sigma(i)}$ for each $i \in \Delta_0$.

For a sequence $(\tau_1, \ldots, \tau_n) \in (\Delta_0)^n$ such that $\{\tau_1, \ldots, \tau_n\} = I$, the element $\tau = s_{\tau_1} \cdots s_{\tau_n} \in W \sigma$ is called a $\sigma$-Coxeter element (or a twisted Coxeter element) [Spr74, CP96b]. Note that if $\sigma = id$, a $\sigma$-Coxeter element is the same as a usual Coxeter element.

**Theorem 3.2** ([OS19d]). Assume $\sigma \neq id$. Note that we have $rh^\vee \in 2\mathbb{Z}$ in this case.

1. For a $\sigma$-Coxeter element $\tau = s_{\tau_1} \cdots s_{\tau_n} \in W \sigma$, we have

$$\tau^{rh^\vee/2} = w_0 \sigma^{rh^\vee/2} = -1.$$ 

In particular, it defines a reduced word

$$\iota(\tau) := (\tau_1, \ldots, \tau_n, \sigma(\tau_1), \ldots, \sigma(\tau_n), \ldots, \sigma^{rh^\vee/2-1}(\tau_1), \ldots, \sigma^{rh^\vee/2-1}(\tau_n))$$

for the longest element $w_0$.

2. The commutation class $[\iota(\tau)]$ depends only on the element $\tau$.

3. For any two $\sigma$-Coxeter elements $\tau_1, \tau_2 \in W \sigma$, we have $[\iota(\tau_1)] = [\iota(\tau_2)]$ if and only if $\tau_1 = \tau_2$. On the other hand, we have $[\iota(\tau_1)] \sim [\iota(\tau_2)]$ in any cases.\(^1\)

**Proof.** See [OS19d, Section 3]. Note that the fact $w_0 \sigma^{rh^\vee/2} = -1$ follows from Remark 3.1 above. □

**Definition 3.3.** Assume $\sigma \neq id$. Thanks to Theorem 3.2, there exists a unique $r$-cluster point containing all the commutation classes $[\iota(\tau)]$ arising from $\sigma$-Coxeter elements $\tau$. We call it the $\sigma$-adapted $r$-cluster point (or the twisted adapted $r$-cluster point) and denote it by $[\Delta^r]$. We refer to a commutation class belonging to $[\Delta^r]$ as a $\sigma$-adapted class (or a twisted adapted class).

\(^1\) This latter assertion goes back to a classical result due to Springer [Spr74, Lemma 7.5].
Convention. When $\sigma = \text{id}$, we understand $[\Delta^\sigma] = [\Delta]$ so that an id-adapted class is the same as a usual adapted class. For a usual Coxeter element $\tau \in W$, we define $[i(\tau)] := [Q] \in [\Delta]$, where $Q$ is the unique Dynkin quiver to which $\tau$ is adapted, i.e. $\tau_Q = \tau$.

Remark 3.4. By definition, we have an inclusion $\{[i(\tau)] \mid \tau \text{ is a } \sigma\text{-Coxeter element}\} \subset [\Delta^\sigma]$ for any $(\Delta, \sigma)$. However the opposite inclusion is not always true. It fails precisely when $(g, \sigma) = (A_{2n-1}, \vee)$ with $n > 2$, or $(g, \sigma) = (E_6, \vee)$. See Corollary 3.32 below.

3.3. Q-datum In what follows, we fix a pair $(\Delta, \sigma)$ of Dynkin diagram $\Delta$ of a Lie algebra $g$ of type ADE and an automorphism $\sigma$ on $\Delta$ satisfying the condition (3.1), or equivalently, a complex simple Lie algebra $g$. Recall that we set $I := \hat{\Lambda}_0$.

Definition 3.5. A function $\xi: \Delta_0 \to \mathbb{Z}$ is called a height function on $(\Delta, \sigma)$ if the following two conditions are satisfied.

(H1) For any $i, j \in \Delta_0$ such that $i \sim j$ and $d_i = d_j$, we have $|\xi_i - \xi_j| = d_i = d_j$.
(H2) For any $i, j \in I$ with $i \sim j$ and $d_i < d_j$, there exists a unique element $j^0 \in j$ such that $|\xi_i - \xi_{j^0}| = 1$ and $\xi_{\sigma^l(j^0)} = \xi_{j^0} - 2l$ for any $0 \leq l < r$, where $i \in j$ is the unique element. (Note that we have $d_i = 1$ and $d_j = r$ in this case.)

We refer to a triple $Q = (\Delta, \sigma, \xi)$ as a Q-datum for $g$.

The condition (H2) can be seen as a local condition at a “branching point” $i = \{i\}$, while the condition (H1) is a local condition at a “non-branching point”.

Example 3.6. To illustrate the condition (H2), we depict two examples of Q-data for $g$ of type $G_2$:

\[
\begin{array}{c}
\xi_{\sigma(j^0)} = 2 \\
\xi_j^0 = 4 \\
\xi_i = 5 \\
\xi_{\sigma^2(j^0)} = 0
\end{array}
\quad
\begin{array}{c}
\xi_{\sigma(j^0)} = 2 \\
\xi_j^0 = 4 \\
\xi_i = 3 \\
\xi_{\sigma^2(j^0)} = 0
\end{array}
\]

Note that an arbitrary Q-datum of type $G_2$ is identical to one of these two cases up to adding a constant and permuting the vertices.

Example 3.7. Here are two examples of Q-data for $g$ of type $B_3$. Note that the corresponding pair is $(g, \sigma) = (A_5, \vee)$. We put the values of height functions above the vertices.

\[
Q^{(1)} = \left( \begin{array}{cccc}
6 & 4 & 7 & 6 \\
\circ & \circ & \circ & \circ
\end{array} \right),
Q^{(2)} = \left( \begin{array}{cccc}
2 & 4 & 5 & 6 & 8 \\
\circ & \circ & \circ & \circ & \circ
\end{array} \right).
\]

Here $i \to j$ implies $\xi_i > \xi_j$.

Remark 3.8. When $\sigma = \text{id}$, the condition (H2) becomes empty. Hence a height function $\xi$ on $(\Delta, \text{id})$ is nothing but a function $\xi$ satisfying (2.7). Thus, in view of Sect. 2.6, the notion of Q-datum for a simply-laced $g$ is equivalent to the notion of Dynkin quiver $Q$ with a height function.

We can deduce the following properties of the height function $\xi$, easily from its definition and the classification of $(\Delta, \sigma)$ in Table 1.
Lemma 3.9. Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum and $i \in I$ be an index.

1. For any $i, i' \in i$, we have $\xi_i \equiv \xi_{i'} \pmod{2}$.
2. For any $i \in I$ and $l \in \mathbb{Z}$, we have $\xi_{\sigma^l(i)} \equiv \xi_i - 2l \pmod{2d_i}$.
3. For any $j \in I$ and $i \in I$ with $i \sim j$, we have $\xi_i \equiv \xi_j + \min(d_i, d_j)$ (mod $2\min(d_i, d_j)$).

In what follows, we fix a $\Delta_0$-tuple $\epsilon = (\epsilon_i)_{i \in \Delta_0}$ of integers with $0 \leq \epsilon_i < d_i$ for each $i \in \Delta_0$ such that the three conditions in Lemma 3.9 are satisfied with $\xi_i$ therein replaced by $\epsilon_i$. We refer to such a tuple $\epsilon$ as a $\sigma$-parity function on $\Delta$. Note that there are only $2r$ choices of $\sigma$-parity functions and their differences will not affect the results in this paper. Adding a constant if necessary, we can make a given height function $\xi$ satisfy the condition:

(H3) We have $\xi_i - \epsilon_i \in 2d_i\mathbb{Z}$ for each $i \in \Delta_0$.

In what follows, we require a height function $\xi: \Delta_0 \rightarrow \mathbb{Z}$ always satisfies the condition (H3) together with (H1) and (H2) without loss of generality.

Definition 3.10. Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum for $\mathfrak{g}$. A vertex $i \in \Delta_0$ is called a source of $Q$ if we have $\xi_i > \xi_j$ for any $j \in \Delta_0$ with $i \sim j$.

The following lemma is immediate from the definition.

Lemma 3.11. Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum and $i \in \Delta_0$ be an index. We define a function $s_i \xi: \Delta_0 \rightarrow \mathbb{Z}$ by the rule

$$ (s_i \xi)_j := \xi_j - \delta_{i,j} \times 2d_j. \quad (3.6) $$

Then $s_i \xi$ defines a height function on $(\Delta, \sigma)$ if and only if $i$ is a source of $Q$.

Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum and $i \in \Delta_0$ be a source of $Q$. Then we define a new Q-datum $s_i Q$ to be the triple $(\Delta, \sigma, s_i \xi)$.

Definition 3.12. Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum for $\mathfrak{g}$. We say that a sequence $(t_1, \ldots, t_l)$ of elements of $\Delta_0$ is adapted to $Q$ if

$$ t_k \text{ is a source of } s_{t_{k-1}}s_{t_{k-2}} \cdots s_{t_1} Q \text{ for all } 1 \leq k \leq l. $$

When $\sigma = \text{id}$, this is equivalent to the usual notion of adaptedness for a Dynkin quiver.

3.4. Repetition quiver and compatible readings

Definition 3.13. Recall that we have fixed a $\sigma$-parity function $\epsilon$ on $\Delta$. The repetition quiver associated with $(\Delta, \sigma)$ is the quiver $\hat{\Delta}^\sigma$ whose vertex set $\hat{\Delta}^\sigma_0$ and arrow set $\hat{\Delta}^\sigma_1$ are given by

$$ \hat{\Delta}^\sigma_0 := \{(i, p) \in \Delta_0 \times \mathbb{Z} \mid p - \epsilon_i \in 2d_i\mathbb{Z}\}, $$

$$ \hat{\Delta}^\sigma_1 := \{(i, p) \rightarrow (j, s) \mid (i, p), (j, s) \in \hat{\Delta}^\sigma_0, j \sim i, s - p = \min(d_i, d_j)\}. $$

Example 3.14. Here are some examples of the repetition quiver.
1. When \( g \) is of type \( B_3 \), we choose the pair \((A_5, \vee)\). The repetition quiver \( \widehat{\Delta}^\sigma \) is depicted as:

![Diagram of \( \widehat{\Delta}^\sigma \) for \( B_3 \)]

2. When \( g \) is of type \( C_4 \), we choose the pair \((D_5, \vee)\). The repetition quiver \( \widehat{\Delta}^\sigma \) is depicted as:

![Diagram of \( \widehat{\Delta}^\sigma \) for \( C_4 \)]

3. When \( g \) is of type \( G_2 \), we choose the pair \((G_4, \vee)\). The repetition quiver \( \widehat{\Delta}^\sigma \) is depicted as:

![Diagram of \( \widehat{\Delta}^\sigma \) for \( G_2 \)]

When \( \sigma = \text{id} \), the repetition quiver \( \widehat{\Delta}^\sigma \) is the same as the repetition quiver \( \widehat{\Delta} \) defined in Sect. 2.3. We denote by \( \pi : \widehat{\Delta}^\sigma_0 \to \Delta_0 \) the projection of the first components.

**Remark 3.15.** In [HL16], Hernandez–Leclerc introduced quivers \( \Gamma \) and \( G \) for untwisted affine types whose set of vertices can be identified with \( \widehat{\Delta}^\sigma_0 \). We remark here that, for \( \sigma = \text{id} \), the quiver \( \widehat{\Delta} \) is isomorphic to the quivers \( \Gamma \) and \( G \), when we remove the horizontal arrows \((i, r) \to (i, r + 2)\) in \( \Gamma \) and \( G \) corresponding to the AR-translations. However, for \( \sigma \neq \text{id} \), the quiver \( \widehat{\Delta}^\sigma \) is not isomorphic to any of \( \Gamma \) and \( G \) even though we remove the horizontal arrows \((i, r) \to (i, r + 2d_i)\) in \( \Gamma \) and \( G \).

**Definition 3.16.** Let \( X \subset \widehat{\Delta}^\sigma_0 \) be a finite subset.

1. We say that a total ordering \((x_1, x_2, \ldots, x_l)\) of \( X \) is a compatible reading of \( X \) if we have \( k < k' \) whenever there is an arrow \( x_{k'} \to x_k \) in the quiver \( \widehat{\Delta}^\sigma \).
2. We define the element \( w[X] \in W \) by

\[
w[X] := s_{\pi(x_1)} s_{\pi(x_2)} \cdots s_{\pi(x_l)},
\]

where \((x_1, x_2, \ldots, x_l)\) is a compatible reading of \( X \). It does not depend on the choice of compatible reading of \( X \) by Lemma 3.18 below.

**Remark 3.17.** Note that we have \( \pi(x) \not\sim \pi(x') \) if there is no oriented path between \( x \) and \( x' \) in \( \widehat{\Delta}^\sigma \).

**Lemma 3.18.** Let \((x_1, \ldots, x_l)\) and \((x'_1, \ldots, x'_l)\) be two compatible readings of a finite subset \( X \subset \widehat{\Delta}^\sigma_0 \). Then the sequences \( i := (\pi(x_1), \ldots, \pi(x_l)) \) and \( i' := (\pi(x'_1), \ldots, \pi(x'_l)) \) are commutation equivalent to each other.
Proof. Let $1 \leq k \leq l$ be the largest number such that $x_s = x_s'$ for $1 \leq s \leq k$. We shall prove the assertion by downward induction on $k$. When $k = l$, we have $(x_1, \ldots, x_l) = (x_1', \ldots, x_l')$ and nothing to prove. Assume $k < l$. Let $k + 1 < k' \leq l$ be the smallest number such that $x_{k'} = x_k$. Then there is no oriented path between $x_k'$ and $x_k'$ for all $k < t < k'$ and hence $\pi(x_k') \not\sim \pi(x_k')$ by Remark 3.17. Therefore the sequence

$$(x_1'', \ldots, x_l'') := (x_1', \ldots, x_k', x_k', x_{k+1}', x_{k+2}', \ldots, x_{l-1}', x_{l+1}', \ldots, x_l')$$

gives a compatible reading of $X$ and the sequence $i'' := (\pi(x_1''), \ldots, \pi(x_l''))$ is commutation equivalent to $i'$. Note that we have $x_s = x_s''$ for $1 \leq s \leq k + 1$ and hence $i''$ is commutation equivalent to $i'$ by the induction hypothesis. Thus $i'$ is also commutation equivalent to $i$. \hfill \Box

Lemma 3.19. Let $(t_1, \ldots, t_l)$ be a sequence of elements of $\Delta_0$. We set $m_i := |\{k \mid t_k = i, 1 \leq k \leq l\}|$ for each $i \in \Delta_0$. Take a Q-datum $Q = (\Delta, \sigma, \xi)$ and assume that

the map $\Delta_0 \rightarrow \mathbb{Z}$ given by $i \mapsto \xi_i - 2m_id_i$ defines a height function on $(\Delta, \sigma)$.

(3.7)

Then the sequence $(t_1, \ldots, t_l)$ is adapted to $Q$ if and only if there exist a compatible reading $(x_1, \ldots, x_l)$ of the subset

$$X := \{(t, \xi_t - 2kd_i) \in \hat{\Delta}_0^\sigma \mid 0 \leq k < m_t\}$$

of $\hat{\Delta}_0^\sigma$ such that we have $t_k = \pi(x_k)$ for all $1 \leq k \leq l$.

Proof. We prove the assertion by induction on $l$. First let us assume that a given sequence $(t_1, \ldots, t_l)$ is adapted to $Q$. Then $t_1$ is a source of $Q$ and the sequence $(t_2, \ldots, t_l)$ is adapted to $s_{t_1}Q$. Defining

$$X' := \{(t, (s_{t_1} \xi_t) - 2kd_i) \in \hat{\Delta}_0^\sigma \mid 0 \leq k < m_t - \delta_{t_1,t_1}\},$$

the induction hypothesis implies that there exists a compatible reading $(x_2, \ldots, x_l)$ of $X'$ such that $t_k = \pi(x_k)$ for all $2 \leq k \leq l$. Since $X = X' \cup \{(t_1, \xi_{t_1})\}$, we obtain the desired compatible reading $(x_1, x_2, \ldots, x_l)$ by setting $x_1 := (t_1, \xi_{t_1})$.

Conversely assume that there is a compatible reading $(x_1, \ldots, x_l)$ of $X$ satisfying $t_k = \pi(x_k)$ for all $1 \leq k \leq l$. By the assumption (3.7), we see that $x_1 = (t_1, \xi_{t_1})$. Let us prove that $\xi := t_1$ is a source of $Q$. To deduce a contradiction, we assume the contrary that there is a vertex $j \in \Delta_0$ such that $i \sim j$ and $\xi_i < \xi_j$. Then we have $(j, \xi_j) \not\in X$ and hence $m_j = 0$. Therefore we have

$$(\xi_j - 2m_jd_j) - (\xi_i - 2m_id_i) = (\xi_j - \xi_i) + 2m_id_i \geq \min(d_i, d_j) + 2d_i,$$

which contradicts the assumption (3.7). Thus $i = t_1$ is a source of $Q$ and hence $s_{t_1}Q$ is well-defined. Since $(x_2, \ldots, x_l)$ is a compatible reading of the set $X \setminus \{x_1\} = X'$, the sequence $(t_2, \ldots, t_l)$ is adapted to $s_{t_1}Q$ by induction hypothesis. This completes the proof. \hfill \Box

Corollary 3.20. Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum. Assume that two reduced words $i = (t_1, \ldots, t_l)$ and $i' = (t'_1, \ldots, t'_l)$ of an element $w \in \mathcal{W}$ are both adapted to $Q$. Then $i'$ is commutation equivalent to $i$.

Proof. It follows from Lemmas 3.18 and 3.19. \hfill \Box
Corollary 3.21. Let \((t_1, \cdots, t_N)\) be a reduced word of \(w_0\). Then the following statements are equivalent:

(a) For any \(i, j \in \Delta_0\) with \(i \sim j\) and \(k\) such that \(1 \leq k \leq k^+ \leq N\) and \(i = t_k\), we have

\[
-c_{ji} = \begin{cases} 
|s \mid k < s < k^+, j = \overline{t_s}| & \text{if } d_i < d_j, \\
|s \mid k < s < k^+, j = t_s| & \text{if } d_i > d_j.
\end{cases}
\]

Here \(k^+ := \min\{p \mid k < p, \ t_k = t_p\}\).

(b) The reduced word \((t_1, \cdots, t_N)\) is adapted to some \(Q\)-datum \(Q\) for \(g\).

Proof. Assume (a). For each \(i \in \Delta_0\), we set \(k(i) := \min\{1 \leq k \leq N \mid t_k = i\}\). Since \((t_1, \cdots, t_N)\) is a reduced word of the longest element \(w_0\), \(\{k(i) \mid i \in \Delta_0\}\) is well-defined as a subset of \(\{1, \ldots, N\}\) of cardinality \(|\Delta_0|\). Thanks to our assumption, there exists a height function \(\xi\) on \((\Delta, \sigma)\) uniquely up to adding an integer so that we have (i) \(\xi_i > \xi_j\) if \(i \sim j\) and \(k(i) < k(j)\), and (ii) \(k(j) < k(\sigma(j)) < \cdots < k(\sigma^{r-1}(j))\) if \(d_i < d_j\) and \(i \sim j\) (with the notation in Definition 3.5). The same assumption also enables us to prove that the sequence \((t_1, \cdots, t_N)\) is adapted to the resulting \(Q\)-datum \(Q = (\Delta, \sigma, \xi)\). This proves the implication (a) \(\Rightarrow\) (b).

The other implication (b) \(\Rightarrow\) (a) follows from Lemma 3.19 immediately. \qed

3.5. Twisted Auslander–Reiten quivers Mimicking the characterization (2.5) of the usual AR quiver \(\Gamma_Q\) for a Dynkin quiver \(Q\), we give the following definition.

Definition 3.22. Let \(Q = (\Delta, \sigma, \xi)\) be a \(Q\)-datum for \(g\). We define the twisted Auslander–Reiten quiver \(\Gamma_Q\) of \(Q\) as the full subquiver of \(\hat{\Delta}^\sigma\) whose vertex set \((\Gamma_Q)_0\) is given by

\[
(\Gamma_Q)_0 := \{(t, \bar{\xi}_t - 2d_t k) \in \hat{\Delta}^\sigma_0 \mid 0 \leq 2d_t k < r h^+ + \bar{\xi}_t - \bar{\xi}_{t^*}\}.
\] (3.8)

Example 3.23. The twisted AR quivers associated with the \(Q\)-data \(Q^{(1)}\) and \(Q^{(2)}\) for \(g\) of type \(B_3\) given in Example 3.7 above are depicted as follows:

\[
\Gamma_Q^{(1)} = \quad \Gamma_Q^{(2)}
\]

The following theorem gives a twisted analog of the canonical bijection (2.8).

Theorem 3.24 (cf. [OS19d]). For each \(Q\)-datum \(Q = (\Delta, \sigma, \xi)\) for \(g\), we denote by \([Q]\) the set of all the reduced words for the longest element \(w_0 \in W\) adapted to \(Q\). Then \([Q]\) forms a single commutation class and there is a unique isomorphism \(\Gamma_Q \cong \Upsilon_{[Q]}\) of quivers which intertwines \(\pi\) and \(\text{res}^{[Q]}\). Moreover, the assignment \(Q \mapsto [Q]\) gives a bijection

\[
\{Q\text{-datum for }g\}/2r\mathbb{Z} \leftrightarrow [\Delta^\sigma],
\]

where \(2r\mathbb{Z}\) means that we ignore constant differences between height functions.
Proof. In the case \(\sigma = \text{id}\), the assertion goes back to the previous section. Let us focus on the case \(\sigma \neq \text{id}\). We pick a compatible reading \((x_1, \ldots, x_N)\) of \((\Gamma_Q)_0\) and set \(i := (\pi(x_1), \ldots, \pi(x_N)) \in \Delta_0^N\). Here we used the fact \(|(\Gamma_Q)_0| = N = \ell(w_0)\). By construction, the commutation class \([i]\) does not depend on the choice of the compatible reading and hence we denote it by \([Q]\). Then the following results have been obtained in [OS19d, Section 4]:

- The sequence \(i\) obtained as above gives a reduced word for \(w_0\).
- The assignment \(Q \mapsto [Q]\) yields a bijection \([Q]\)-datum for \(g|/2r\mathbb{Z} \to \mathbb{Z}_1\) \([\Delta]\). \[\]
- We have a unique isomorphism \(\Gamma_Q \cong \Upsilon_{[Q]}\) of quivers which intertwines \(\pi\) and \(\text{res}_{[Q]}\).

Thanks to Lemma 3.19 and Corollary 3.20, we have \([Q] = [Q]'\), which completes the proof. \(\square\)

Let \(\phi_{Q,0}: (\Gamma_Q)_0 \to \mathbb{R}^+\) denote the underlying bijection of the isomorphism \(\Gamma_Q \cong \Upsilon_{[Q]}\) in Theorem 3.24. For \(\beta \in \mathbb{R}^+\), we call \((\iota, \rho) = \phi_{Q,0}^{-1}(\beta)\) the coordinate of \(\beta\) in \(\Gamma_Q\). Theorem 3.24 implies that if \(\beta \in \mathbb{R}^+\) is located at the coordinate \((\iota, \rho)\) in \(\Gamma_Q\), we have \(\text{res}_{[Q]}(\beta) = \iota\).

The following is a twisted analogue of the algorithm in the last paragraph of Sect. 2.3.

**Proposition 3.25** ([OS19d]). Let \(Q = (\Delta, \sigma, \xi)\) be a \(Q\)-datum. For a source \(\iota \in \Delta_0\) of \(Q\), we have \(r_1[Q] = [s_1Q]\). The bijection \(\psi_{s_1Q,0}: (\Gamma_{s_1Q})_0 \to \mathbb{R}^+\) can be obtained from the bijection \(\phi_{Q,0}: (\Gamma_Q)_0 \to \mathbb{R}^+\) in the following way:

1. A positive root \(\beta \in \mathbb{R}^+\setminus\{\alpha_t\}\) is located at the coordinate \((j, \rho)\) in \(\Gamma_{s_1Q}\) if \(s_1\beta\) is located at the coordinate \((j, \rho)\) in \(\Gamma_Q\).
2. The simple root \(\alpha_t\) is located at the coordinate \((\iota^*, \xi_t - rh^V)\) in \(\Gamma_{s_1Q}\) while \(\alpha_t\) was located at the coordinate \((\iota^*, \xi_t)\) in \(\Gamma_Q\).

**3.6. Generalizations of Coxeter element** First we introduce a generalization of the \(r\)-th power \(\tau'\) of a \(\sigma\)-twisted Coxeter element \(\tau\) for any \(Q\)-datum. Note that we have \(\sum_{\iota \in \Delta_0} r_\iota = nr\).

**Proposition 3.26.** For each \(Q\)-datum \(Q = (\Delta, \sigma, \xi)\), there exists a unique element \(\widehat{\tau}_Q \in W\) with \(\ell(\widehat{\tau}_Q) = nr\) which has a reduced word \((\iota_1, \ldots, \iota_{nr})\) adapted to \(Q\) such that

\[\{k \mid 1 \leq k \leq nr, \iota_k = \iota\} = r_\iota \quad \text{for each } \iota \in \Delta_0.\]

**Proof.** Define the subset \(X_Q \subset \Delta_0^r\) by

\[X_Q := \{(\iota, \xi_t - 2kd_\iota) \in \Delta_0^r \mid 0 \leq k < r_\iota\}.\]

Note that the condition (3.7) is satisfied when \(m_\iota = r_\iota\) because we have \(r_\iota d_\iota = r\) for any \(\iota \in \Delta_0\). Let \((x_1, \ldots, x_{nr})\) be a compatible reading of \(X_Q\). By construction, it can be extended to a compatible reading \((x_1, \ldots, x_N)\) of \((\Gamma_Q)_0\). By Theorem 3.24, the sequence \((\iota_1, \ldots, \iota_N) := (\pi(x_1), \ldots, \pi(x_N))\) gives a reduced word for \(w_0\). In particular, the subsequence \((\iota_1, \ldots, \iota_{nr}) := (\pi(x_1), \ldots, \pi(x_{nr}))\) gives a reduced word for the element \(\widehat{\tau}_Q := w[X_Q] = s_{i_1} \cdots s_{i_{nr}}\), which is adapted to \(Q\) thanks to Lemma 3.19. The uniqueness also follows from Lemma 3.19. \(\square\)
Definition 3.27. We refer to the unique element $\tilde{\tau}_Q \in W$ in Proposition 3.26 as the quasi Coxeter element associated with $Q$.

Corollary 3.28. Let $Q = (\Delta, \sigma, \xi)$ be a $Q$-datum and $i \in \Delta_0$ be a source of $Q$. Then we have

$$s_is_iQs_i = \tilde{\tau}_sQ.$$

Proof. Let $(x_1, \ldots, x_{nr})$ be a compatible reading of $X_Q$ such that $x_1 = (i, \xi_i)$. By Proposition 3.26, we get a reduced word $(t_2, \ldots, t_{nr}) := (\pi(x_2), \ldots, \pi(x_{nr}))$ for the element $s_is_iQ$, and hence $s_is_iQ = w[X']$ with $X' := X_Q \setminus \{(i, \xi_i)\}$. On the other hand, the sequence $(t_2, \ldots, t_{nr}, i)$ is adapted to $s_is_iQ$ and hence there is a compatible reading $(x_1', \ldots, x_{nr}')$ of $X_{s_iQ}$ such that $x_{nr}' = (i, \xi_i - 2r)$ by Lemma 3.19. Thus we have $s_iQs_iQ = w[X'']$ where $X'' := X_{s_iQ} \setminus \{(i, \xi_i - 2r)\}$. Since $X' = X''$, we obtain the conclusion. $\square$

Next we consider a generalization of twisted Coxeter element for any $Q$-datum $Q$.

Definition 3.29. Let $Q = (\Delta, \sigma, \xi)$ be a $Q$-datum for $g$. For each $i \in I$, we denote by $i^\circ$ an element in the $\sigma$-orbit $i$ satisfying the condition

$$\xi_i^\circ = \max\{\xi_i \mid i \in I\}.$$

By Lemma 3.9 (2), $i^\circ \in i$ is uniquely determined. The subset $I^\circ := \{i^\circ \mid i \in I\} \subset \Delta_0$ gives a section of the natural quotient map $\tau^\circ: \Delta_0 \to I$. We denote the corresponding $\sigma$-Coxeter element by $\tau^\circ_Q := w[X^\circ_Q]\sigma$, where $X^\circ_Q := \{(i, \xi_i) \mid i \in I^\circ_Q\} \subset \tilde{\Delta}_0^\sigma$.

Lemma 3.30. Let $i, j \in I$ with $i \sim j$. With the above notation, we have:

1. $|\xi_i^\circ - \xi_j^\circ| = \min(d_i, d_j)$ if $i^\circ \sim j^\circ$,
2. $\xi_i^\circ = \xi_j^\circ$ if $i^\circ \not\sim j^\circ$.

Proof. (1) is immediate from Definition 3.5. For (2), we note that the conditions $i \sim j$ and $i^\circ \not\sim j^\circ$ are satisfied only if $d_i = d_j = r = 2$. In this case, we have $\xi_i^\circ \geq \xi_\sigma(i^\circ) + 2$ and $\xi_j^\circ \geq \xi_\sigma(j^\circ) + 2$. Moreover we have $i^\circ \sim \sigma(j^\circ)$ and $j^\circ \sim \sigma(i^\circ)$. If $\xi_i^\circ > \xi_j^\circ$, we have $\xi_i^\circ > \xi_\sigma(j^\circ) + 2$, which contradicts the condition (H1): $|\xi_i^\circ - \xi_\sigma(j^\circ)| = 2$. Similarly we cannot have $\xi_i^\circ < \xi_j^\circ$. Thus we obtain $\xi_i^\circ = \xi_j^\circ$. $\square$

Proposition 3.31. A twisted adapted class $[Q] \in \llbracket \Delta^\sigma \rrbracket$ contains a reduced word arising from a $\sigma$-Coxeter element, i.e. $[Q] = [\bar{i}(\tau)]$ for some $\tau$ if and only if the corresponding $Q$-datum $Q = (\Delta, \sigma, \xi)$ satisfies the following condition:

For each $i \in I$ and $0 \leq k < d_i$, we have $\xi_{\sigma k(i^\circ)} = \xi_i^\circ - 2k$. (3.9)

Moreover, if this condition (3.9) is satisfied, we have $[Q] = [\bar{i}(\tau_Q^\circ)]$ and $(\tau_Q^\circ)^\tau = \tau_Q^\circ$.

Proof. Let $\tau = s_{t_1} \cdots s_{t_n}\sigma \in W$ be a $\sigma$-Coxeter element. Let us choose a function $\hat{\xi}^\tau: \Delta_0 \to \mathbb{Z}$ satisfying the following three conditions, which is unique up to constant:

1. $\hat{\xi}^\tau_i = \hat{\xi}^\tau_{i'} + \min(d_{ik}, d_{i'})$ if $1 \leq k < l \leq n$ and $i_k \sim i_l$,
2. $\hat{\xi}^\tau_i = \hat{\xi}^\tau_{i'}$ if $\tilde{i}_k \sim \tilde{i}_l$ and $t_k \not\sim t_l$,
3. $\hat{\xi}_{\sigma i(i_k)}^\tau = \hat{\xi}_{i_k}^\tau - 2l$ for each $1 \leq k \leq n$ and $0 \leq l < d_{ik}$.


Note that (2) occurs only when \(d_{ij} = d_{ij} = r = 2\). It is easy to see that \(\xi^\tau\) defines a height function on \((\Delta, \sigma)\). Hence we obtain a \(Q\)-datum \(Q^\tau := (\Delta, \sigma, \xi^\tau)\) satisfying the condition (3.9). By construction, we have \(I_{Q^\tau}^\sigma = \{i_k \mid 1 \leq k \leq n\}\) and \(\tau_{Q^\tau}^\sigma = \tau\). Conversely, every \(Q\)-datum \(Q\) satisfying the condition (3.9) is obtained in this way, i.e. we can realize \(Q = Q^\tau\) for a unique \(\sigma\)-Coxeter element \(\tau\) by Lemma 3.30.

It remains to show that \(\tau^\tau = \bar{\tau}_{Q^\tau}\) and \([i(\tau)] = [Q^\tau]\). In the case \(\sigma = \text{id}\), we have nothing to prove. Assume that \(\sigma \neq \text{id}\). We shall apply Lemma 3.19 to the set \(X_{Q^\tau}^\sigma = \{(i_k, \xi^\tau_{i_k}) \mid 1 \leq k \leq n\}\). Note that the condition (3.7) is satisfied for \(m_i = \delta(i \in I_{Q^\tau}^\sigma)\) since we have

\[
\xi_{i}^\tau - 2d_{ij} \times \delta(i \in I_{Q^\tau}^\sigma) = \xi_{i}^\tau - 2
\]

by (3). As a result, the sequence \((i_1, \ldots, i_n)\) is adapted to \(Q^\tau\) and \(s_{i_n} \cdots s_{i_1} \xi^\tau = \xi^\tau\).

Then the equation (10.10) also implies that the sequence \((\sigma(i_1), \ldots, \sigma(i_n))\) is adapted to \((\Delta, \sigma, \xi^\tau)\). Repeating this argument, we see that the sequence \(i(\tau)\) defined in Theorem 3.2 is adapted to \(Q^\tau\). Therefore, we obtain \(\tau^\tau = \bar{\tau}_{Q^\tau}\), and also \([i(\tau)] = [Q^\tau]\) by Theorem 3.24. \(\square\)

**Corollary 3.32.** Except for \(g = B_n\) or \(F_4\), every twisted adapted class contains a reduced word arising from a \(\sigma\)-Coxeter element, i.e. \([[\Delta^\sigma]] = \{|i(\tau)| \mid \tau \text{ is a } \sigma\text{-Coxeter element.}\}.\)

**Proof.** Unless \(g\) is of type \(B_n\) nor of type \(F_4\), every \(Q\)-datum for \(g\) satisfies the condition (3.9). Therefore Proposition 3.31 proves the assertion. \(\square\)

Let \(Q = (\Delta, \sigma, \xi)\) be a \(Q\)-datum. Associated with \(\xi\), we define another height function \(\xi^\circ: \Delta_0 \to \mathbb{Z}\) by \(\xi^\circ_{\sigma(i)} = \xi_i - 2k\) for \(i \in I_{Q}^\circ\) and \(0 \leq k < d_i\). Let \(Q^\circ := (\Delta, \sigma, \xi^\circ)\) be the corresponding \(Q\)-datum. Proposition 3.31 shows that we have \([i(\tau^\circ_{Q})] = [Q^\circ]\). Let

\[
X_{Q}^\circ := \{(\sigma(i^\circ), p) \in \hat{\Delta}_0^\circ \mid i \in I, \xi_{\sigma(i^\circ)} < p \leq \xi_{i^\circ} - 2\}.
\]

By definition, \(X_{Q}^\circ = \emptyset\) if and only if the condition (3.9) is satisfied. Observe that we have \(i \neq j\) if \(i \in I_{Q}^\circ\) and \(j \in \pi(X_{Q}^\circ)\).

Pick a compatible reading \((y_1, \ldots, y_m)\) of \(X_{Q}^\circ\), where \(m = |X_{Q}^\circ|\). By Lemma 3.19, the sequence \((j_1, \ldots, j_m) := (\pi(y_1), \ldots, \pi(y_m))\) is adapted to \(Q^\circ\) and we have

\[
Q = s_{j_m} \cdots s_{j_1} Q^\circ.
\]

Recall that we have defined \(w[X_{Q}^\circ] := s_{j_1} \cdots s_{j_m}\).

**Definition 3.33.** Under the above notation, we define the generalized \(\sigma\)-Coxeter element (or the generalized twisted Coxeter element) \(\tau_Q\) associated with \(Q\) by

\[
\tau_Q := w[X_{Q}^\circ]^{-1} \cdot \tau^\circ_Q \cdot w[X_{Q}^\circ].
\]

Note that the generalized twisted Coxeter element \(\tau_Q\) is equal to the twisted Coxeter element \(\tau^\circ_Q\) if and only if the condition (3.9) is satisfied. In particular, we have \(\tau_{Q^\circ} = \bar{\tau}_{Q}\).

**Proposition 3.34.** Let \(Q = (\Delta, \sigma, \xi)\) be a \(Q\)-datum.

1. If \(i \in \Delta_0\) is a source of \(Q\), we have \(s_i \tau_Q s_i = \tau_{s_i Q}\).
2. The order of \(\tau_Q\) is \(rh^\vee\).
3. We have \(\tau_Q^\vee = \bar{\tau}_{Q}\). Hence the order of \(\bar{\tau}_{Q}\) is \(h^\vee\).
(4) If $\sigma \neq \text{id}$, we have $\tau^{|\sigma|/2}_{\nabla} = -1$.

Proof. The assertions (2), (3) and (4) follow easily from the assertion (1), Theorem 3.2, Corollary 3.28 and Proposition 3.31. Let us prove the remaining assertion (1). First we assume $t \notin I_{Q}^{\circ}$. Then we have $X_{s_{t}Q} = X_{Q}^{\circ}$ and $X'_{s_{t}Q} = X'Q \cup \{(i, \xi_{t}-2d_{i})\}$, which imply that $\tau_{s_{t}Q} = \tau_{Q}^{\circ}$ and $w[X'_{s_{t}Q}] = w[X'Q]s_{i}$. Thus we obtain $\tau_{s_{t}Q} = s_{t}\tau_{Q}s_{i}$ as desired.

Now we assume $t \in I_{Q}^{\circ}$. Take a compatible reading $(x_{1}, \ldots, x_{n})$ (resp. $(y_{1}, \ldots, y_{m})$) of $X_{Q}^{\circ}$ (resp. $X'_{Q}$) and set $t_{k} := \pi(x_{k})$, $j_{l} := \pi(y_{l})$. Since $t$ is a source of $Q$, we may assume that $t_{1} = t$. By definition, we have

$$
\tau_{Q} = s_{j_{m}} \cdots s_{j_{l}}s_{i}s_{l_{2}} \cdots s_{l_{n}}s_{\sigma(j_{1})} \cdots s_{\sigma(j_{m})}\sigma.
$$

Since $t \neq j_{l}$ and $t_{k} \neq j_{l}$ for any $k, l$, we have

$$
\tau_{Q}s_{t} = s_{j_{m}} \cdots s_{j_{l}}s_{i}s_{l_{2}} \cdots s_{l_{n}}s_{\sigma(i)}s_{\sigma(j_{1})} \cdots s_{\sigma(j_{m})}\sigma.
$$

Assume $\xi_{\sigma(i)} = \xi_{t} - 2$. Then we have $X'_{s_{t}Q} = (X_{Q}^{\circ}(\{(i, \xi_{t})\}) \cup \{\sigma(i), \xi_{\sigma(i)}\})$ and $X'_{s_{t}Q} = X'Q$, which imply $s_{l_{2}} \cdots s_{l_{n}}s_{\sigma(i)} = \tau_{Q}^{\circ}$ and $s_{j_{l}} \cdots s_{j_{m}} = w[X'_{s_{t}Q}]$ respectively. Hence the RHS of (3.12) is equal to $\tau_{Q}$ as desired. Finally we assume $\xi_{\sigma(i)} < \xi_{t} - 2$. Note that it forces $r = d_{i} = 2$. In case $t$ is a source of $Q$, we may assume that $y_{1} = (\sigma(i), \xi_{t} - 2)$ and hence $j_{1} = \sigma(i)$. Since $\sigma(i) \neq t_{k}$ for all $2 \leq k \leq n$, the equation (3.12) is rewritten as

$$
s_{t}\tau_{Q}s_{t} = s_{j_{m}} \cdots s_{j_{l}}s_{i}s_{l_{2}} \cdots s_{l_{n}}s_{\sigma(i)}s_{\sigma(j_{1})} \cdots s_{\sigma(j_{m})}\sigma
= s_{j_{m}} \cdots s_{j_{l}}s_{i}s_{l_{2}} \cdots s_{l_{n}}s_{\sigma(j_{2})} \cdots s_{\sigma(j_{m})}\sigma.
$$

On the other hand, we have $X_{s_{t}Q} = (X_{Q}^{\circ}(\{(i, \xi_{t})\}) \cup \{(i, \xi_{t} - 4)\}$ and $X'_{s_{t}Q} = X'Q(\{(\sigma(i), \xi_{t} - 2)\})$, which imply $s_{l_{2}} \cdots s_{l_{n}}s_{i} = \tau_{Q}^{\circ}$ and $s_{j_{2}} \cdots s_{j_{m}} = w[X'_{s_{t}Q}]$ respectively. Therefore the RHS of the equation (3.13) is equal to $\tau_{Q}$ as desired. $\square$

3.7. The bijection $\phi_{Q}$ and g-additive property. In this subsection, we extend the bijection $\phi_{Q} : \hat{\Delta}_0 \to \hat{R}^{+}$ (see [HL15, Section 2.2]), induced from a Dynkin quiver $Q$ and its Coxeter element $\tau_{Q}$, to the bijection $\phi_{Q} : \hat{\Delta}_0^{\sigma} \to \hat{R}^{+}$ by using a Q-datum $Q = (\Delta, \sigma, \xi)$ and its generalized twisted Coxeter element $\tau_{Q}$. Then we will prove g-additive property, which is also well-known for $Q$ being a Dynkin quiver $Q$.

For each $t \in \Delta_{0}$, we assign a positive root by

$$
y_{t}^{Q} := (1 - \tau_{Q}^{d_{t}})\sigma_{t}.
$$

Then we define the map $\phi_{Q} : \hat{\Delta}_{0}^{\sigma} \to \hat{R}^{+}$ recursively by the following rules:

1. $\phi_{Q}(t, \xi_{t}) = (y_{t}^{Q}, 0)$ for each $t \in \Delta_{0}$.
2. If $\phi_{Q}(t, p) = (\beta, k)$, we have

$$
\phi_{Q}(t, p \pm 2d_{t}) = \begin{cases} 
(\tau_{Q}^{d_{t}}(\beta), k) & \text{if } \tau_{Q}^{d_{t}}(\beta) \in \mathbb{R}^{+}, \\
(\tau_{Q}^{d_{t}}(\beta), k \pm 1) & \text{if } \tau_{Q}^{d_{t}}(\beta) \in \mathbb{R}^{-}.
\end{cases}
$$

By definition, we have $\tau_{Q}^{(\xi_{t}-p)/2}(y_{t}^{Q}) = (-1)^{k} \beta$ if $\phi_{Q}(t, p) = (\beta, k)$.

Theorem 3.35. Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum.
1. The map $\phi_\mathcal{Q}: \mathcal{Q}_0 \to \hat{\mathcal{R}}^+$ is a bijection.

2. We have $\phi_\mathcal{Q}^{-1}(\mathcal{R}^+ \times \{0\}) = (\mathcal{Q})_0$ and $\phi_\mathcal{Q}(t, p) = (\phi_{\mathcal{Q},0}(t, p), 0)$ for each $(t, p) \in (\mathcal{Q})_0$. Here $\phi_{\mathcal{Q},0}: (\mathcal{Q})_0 \to \mathcal{R}^+$ is the underlying bijection of the isomorphism $\Gamma_{\mathcal{Q}} \cong \gamma_{\mathcal{Q}}$ in Theorem 3.24. In particular, we have

$$\phi_{\mathcal{Q},0}(t, p) = \tau^{(\xi_p - p)/2}(\gamma^p_{t, \mathcal{Q}})$$

for each $(t, p) \in (\mathcal{Q})_0$.

For a proof, we need some lemmas.

**Lemma 3.36.** Let $(t_1, \ldots, t_{nr})$ be a reduced word for the quasi Coxeter element $\tilde{r}_\mathcal{Q}$ adapted to $\mathcal{Q}$ such that $|\{k \mid \xi_k = 1\}| = r_i$ for each $i \in \Delta_0$. Then, for each $i \in \Delta_0$, we have

$$\gamma^i_{t, \mathcal{Q}} = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_i,$$

where $k$ is the smallest number such that $\xi_k = 1$. In particular, we have $\gamma^i_{t, \mathcal{Q}} = \alpha_i$ if and only if $i$ is a source of $\mathcal{Q}$.

**Proof.** In view of Lemma 3.18 and Lemma 3.19, it suffices to prove the assertion for a special reduced word $(t_1, \ldots, t_{nr})$ for $\tilde{r}_\mathcal{Q}$ satisfying the above conditions. Let us take a compatible reading $(x_1, \ldots, x_{nr})$ of the set $X_{\mathcal{Q}}$ such that the first $n$-letters $(x_1, \ldots, x_n)$ gives a compatible reading of the subset $X_{\mathcal{Q}} \subseteq X_{\mathcal{Q}}$. Then we have $\tilde{r}_\mathcal{Q} = s_{i_1} \cdots s_{i_{nr}}$ and $\tau^i_{\mathcal{Q}} = s_{i_1} \cdots s_{i_n} \sigma$, where $(t_1, \ldots, t_{nr}) = (\pi(x_1), \ldots, \pi(x_{nr}))$. Let $k$ be the smallest number such that $\xi_k = 1$. If $d_i = r$, we have $\tau^{d_i}_{\mathcal{Q}} = \tilde{r}_\mathcal{Q}$ by Proposition 3.34 (3) and $\alpha_i = t_i$ only if $l = k$. Therefore we have $\tau^{d_i}_{\mathcal{Q}} = \tau^{d_i}_{\mathcal{Q}} = \sigma_i = s_{i_1} \cdots s_{i_{k-1}} \alpha_i$ and hence $\gamma^i_{t, \mathcal{Q}} = s_{i_1} \cdots s_{i_{k-1}} \alpha_i$. If $d_i = 1$, we have $\tau^{d_i}_{\mathcal{Q}} = \tau^{d_i}_{\mathcal{Q}} = w[X_{\mathcal{Q}}]^{-1} \cdot t^{d_i}_{\mathcal{Q}} \cdot w[X_{\mathcal{Q}}]$ and $1 \leq k \leq n$. Note that $w[X_{\mathcal{Q}}] \cdot \sigma_i = \sigma_i$ and $s_i w[X_{\mathcal{Q}}] = w[X_{\mathcal{Q}}]s_i$ for any $1 \leq l \leq n$. Moreover, we have $\alpha_i = t_i$ with $1 \leq l \leq n$ only if $l = k$. Thus we have $\gamma^i_{t, \mathcal{Q}} = (1 - \tau^{d_i}_{\mathcal{Q}}) \sigma_i = s_{i_1} \cdots s_{i_{k-1}} \alpha_i$. □

**Lemma 3.37.** Let $i \in \Delta_0$ be a source of $\mathcal{Q}$. For any $(j, p) \in \Delta_0 \times \mathbb{Z}$ such that $p - \xi_j \in 2\mathbb{Z}$, we have

$$s_i \tau^{(\xi_j - p)/2}(\gamma^p_{j, \mathcal{Q}}) = \tau^{(s_i \xi_j - p)/2}(\gamma^p_{s_i, \mathcal{Q}}).$$

**Proof.** First we consider the case when $j \neq i$. By Proposition 3.34 (1) and $s_i \sigma_j = \sigma_j$, we have

$$s_i \tau^{(\xi_j - p)/2}(\gamma^p_{j, \mathcal{Q}}) = \tau^{(\xi_j - p)/2}(1 - d_i \tau^{d_i}_{\mathcal{Q}}) \sigma_j = \tau^{(s_i \xi_j - p)/2}(\gamma^p_{s_i, \mathcal{Q}}),$$

which proves the assertion. Let us consider the other case when $j = i$. Since $i$ is a source of $\mathcal{Q}$, we have $\gamma^i_{t, \mathcal{Q}} = \alpha_i = (1 - s_i) \sigma_i$ by Lemma 3.36. Thus we obtain

$$s_i \tau^{(\xi_j - p)/2}(\gamma^p_{i, \mathcal{Q}}) = \tau^{(\xi_j - p)/2}(s_i - 1) \sigma_i$$

$$= \tau^{(\xi_j - 2d_i - p)/2} \tau^{d_i}_{s_i, \mathcal{Q}} \tau^{(s_i \xi_j - p)/2}(\gamma^p_{s_i, \mathcal{Q}})$$

$$= \tau^{(\xi_j - 2d_i - p)/2} \tau^{d_i}_{s_i, \mathcal{Q}} \tau^{(s_i \xi_j - p)/2}(\gamma^p_{s_i, \mathcal{Q}})$$

$$= \tau^{(s_i \xi_j - p)/2}(\gamma^p_{s_i, \mathcal{Q}})$$
as desired. Here for the third equality we used the fact $\tau_{st, Q}^{d_t} = \sigma_t$, which holds because the element $\tau_{st, Q}^{d_t}$ has an expression without the simple reflection $s_t$.

**Lemma 3.38.** Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum. We fix an index $i \in I$. For any $t, t' \in I$ we have

$$\tau_{Q}^{(\xi_t - \xi_{t'})/2}(\gamma_{t'}^Q) = \gamma_{t'}^Q.$$  

**Proof.** First we consider the case when $[Q]$ contains a reduced word arising from a twisted Coxeter element $\tau_{Q} = s_{t_1} \cdots s_{t_n} \sigma$, or equivalently the condition (3.9) is satisfied. In this case, what we have to show is that $\gamma_{t'}^Q = \gamma_{t'}^Q$ for all $i \in I_Q$ with $d_i > 1$. This can be deduced immediately from Lemma 3.36. A general case is reduced to this special case by Lemma 3.37 since any $Q$ can be obtained from $Q^\circ$ by a suitable sequence of source reflections (cf. (3.11)). □

**Corollary 3.39.** Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum. For each $i \in \Delta_0$, we have

$$\tau_{Q}^{(rh^\vee + \xi_t - \xi_{t'})/2}(\gamma_{t'}^Q) = -\gamma_{t'}^Q.$$  

**Proof.** When $\sigma = \text{id}$, the assertion is well-known. When $\sigma \neq \text{id}$, we have $i^* \in \tilde{t}$ for any $i \in \Delta_0$ (see Remark 3.1). Thus the assertion follows from Proposition 3.34 (4) and Lemma 3.38. □

**Proof of Theorem 3.35.** Let $i \in \Delta_0$ be a source of $Q$. We shall prove the assertions for $s_i Q$ assuming that they are true for $Q$. By Lemma 3.37, we have $\phi_{s_i Q} = \tilde{s}_i \circ \phi_Q$, where $\tilde{s}_i : \mathbb{R}^+ \to \mathbb{R}^+$ is the bijection defined by

$$\tilde{s}_i(\beta, k) := \begin{cases} (s_i \beta, k) & \text{if } \beta \in \mathbb{R}^+ \setminus \{\alpha_t\}, \\ (\alpha_t, k + 1) & \text{if } \beta = \alpha_t \end{cases} \quad (3.14)$$

(see also [KKOP20]). Therefore $\phi_{s_i Q}$ is a bijection. From the assumption $\phi_{Q}^{-1}(\mathbb{R}^+ \times \{0\}) = (\Gamma_Q)_0$ and Corollary 3.39, we obtain $\phi_{Q}(i^*, \xi_t - rh^\vee) = (\alpha_t, -1)$. Therefore we have

$$\phi_{s_i Q}^{-1}(\mathbb{R}^+ \times \{0\}) = \phi_{s_i Q}^{-1}(\tilde{s}_i^{-1}(\mathbb{R}^+ \times \{0\})) = ((\Gamma_Q)_0 \setminus \{(i, \xi_t)\}) \cup \{(i^*, \xi_t - rh^\vee)\} = (\Gamma_{s_i Q})_0.$$  

Furthermore, thanks to Proposition 3.25, we obtain $\phi_{s_i Q}(j, p) = (\phi_{s_i Q}(j, p), 0)$ for any $(j, p) \in (\Gamma)_{s_i Q}$.

Note that any Q-datum $Q$ can be obtained from $Q^\circ$ by a suitable sequence of source reflections (cf. (3.11)). Since the assertions for $Q^\circ$ follow immediately from Theorem 3.24 and the definition of $T_{(\Gamma)Q^\circ}$, we have obtained the proof for general $Q$. □

As a consequence of Theorem 3.35 and Corollary 3.39, we observe the following.

**Corollary 3.40.** Let $Q = (\Delta, \sigma, \xi)$ be a Q-datum and $(i, p) \in \hat{\Delta}_0^\circ$. If $\phi_{Q}(i, p) = (\beta, k)$, we have

$$\phi_{Q}(i^*, p \oplus rh^\vee) = (\beta, k \pm 1).$$

The next statement is a generalization of the additive property (2.4).
Theorem 3.41 (g-additive property). Let $Q = (\Delta, \sigma, \xi)$ be a $Q$-datum for $g$ and $I^0 \subset \Delta_0$ be an arbitrary section of the natural quotient map $\gamma : \Delta_0 \to I$. For any $i \in \Delta_0$ and $l \in \mathbb{Z}$, we have

$$\tau^l_Q(\gamma^Q_i) + \tau^{l+d_i}_Q(\gamma^Q_i) = \sum_{j \in I^0, j \sim \gamma} \sum_{t=0}^{-c_{ji}-1} \tau^{l+(\xi_j-\xi_i+\min(d_i,d_j))/2}_Q(\gamma^Q_j).$$

Proof. Note that each summand $\tau^{l+(\xi_j-\xi_i+\min(d_i,d_j))/2}_Q(\gamma^Q_j)$ does not depend on the choice of the representative $j \in \mathcal{J}$ thanks to Lemma 3.38. This verifies the second equality and it suffices to prove the first equality (3.15) for a special choice of $I^0$. Moreover, Lemma 3.37 reduces the situation to the case when $Q$ satisfies the condition (3.9) (i.e. $\tau^l_Q = \tau^0_Q$) and the vertex $i$ is a source of $Q$.

The LHS of (3.15) is computed as

$$\tau^l_Q(\gamma^Q_i) + \tau^{l+d_i}_Q(\gamma^Q_i) = \tau^l_Q(1 - \tau^{d_i}_Q)\sigma_i + \tau^{l+d_i}_Q(1 - \tau^{d_i}_Q)\sigma_i = \tau^l_Q(1 - \tau^{d_i}_Q)(1 + \tau^{d_i}_Q)\sigma_i = \sum_{j \in \Delta_0, j \sim i} \tau^l_Q(1 - \tau^{d_i}_Q)\sigma_j. \quad (3.16)$$

Here for the last equality we used the fact

$$(1 + \tau^{d_i}_Q)\sigma_i = 2\sigma_i - \alpha_i = \sum_{j \in \Delta_0, j \sim i} \sigma_j,$$

which holds because $i$ is assumed to be a source of $Q$.

Now we assume that $d_i = r$. Noting that $1 - \tau^{d_i}_Q = (1 - \tau^{d_i}_Q)\sum_{t=0}^{-c_{ji}-1} \tau^l_Q$ in this case, we have $(1 - \tau^{d_i}_Q)\sigma_j = \sum_{t=0}^{-c_{ji}-1} \tau^l_Q(\gamma^Q_j)$ for any $j \sim i$. On the other hand, we see $\xi_i - \xi_j = \min(d_i, d_j)$ for any $j \in \Delta_0$ with $j \sim i$ since $i$ is a source of $Q$ and $d_i = r$. This completes a proof of (3.15) in this case.

Finally we consider the case $d_i = 1$. Note that, for any $j \in I$, we have

$$(1 - \tau^l_Q)\sum_{j \in I} \sigma_j = \gamma^Q_j$$

by Lemma 3.36 and the assumption $\tau^l_Q = \tau^0_Q$. Applying this to the equation (3.16), we have

$$\tau^l_Q(\gamma^Q_i) + \tau^{l+d_i}_Q(\gamma^Q_i) = \sum_{j \in \mathcal{J}_Q, j \sim i} \tau^l_Q(\gamma^Q_j).$$

On the other hand, we have $c_{ji} = -1$ and $\xi_i - \xi_j = \min(d_i, d_j)$ for any $j \in \mathcal{J}_Q$ with $j \sim i$ under our assumption. This completes a proof of (3.15). \qed
3.8. Folding twisted AR quivers Let \( \epsilon : I \to \{0, 1\} \) be the function given by \( \epsilon_i \equiv \epsilon_r \pmod{2} \) for any \( i \in I \). Note that this is well-defined by Lemma 3.9. We define the infinite set \( \widehat{I} \) by
\[
\widehat{I} := \{(i, p) \in I \times \mathbb{Z} \mid p - \epsilon_i \in 2\mathbb{Z}\}. 
\] (3.17)

Restricting the map \( \Delta_0 \times \mathbb{Z} \to I \times \mathbb{Z} \) given by \( (i, p) \mapsto (i, p) \), we obtain the folding map \( f : \Delta_0^e \to \widehat{I} \). By Lemma 3.9, the map \( f \) is a bijection.

Let \( Q = (\Delta, \sigma, \xi) \) be a Q-datum. By composing, we obtain a bijection
\[
\begin{align*}
\hat{\phi}_Q := \phi_Q \circ (f^{-1}) : \widehat{I} & \to \mathbb{R}^+, \\
\end{align*}
\]
which satisfies
\[
\tau_Q^{(i_p - p)/2}(\gamma_i^Q) = (-1)^k \alpha \quad \text{if} \quad \hat{\phi}_Q(i, p) = (\alpha, k) 
\]
for any \( (i, p) \in \widehat{I} \) and \( i \in I \). Note that the LHS of (3.18) does not depend on the choice of \( i \in I \) thanks to Lemma 3.38.

Following [OS19d], we define the folded AR quiver of \( Q \) to be the quiver \( \widehat{\Gamma}_Q \) whose vertex set is
\[
\widehat{\Gamma}_Q := f((\Gamma_Q)_0) = \{(i, p) \in \widehat{I} \mid (i, p) \in (\Gamma_Q)_0\}
\]
and such that the restriction \( f_Q := f|_{(\Gamma_Q)_0} : (\Gamma_Q)_0 \to \widehat{\Gamma}_Q \) induces an isomorphism \( \Gamma_Q \simeq \widehat{\Gamma}_Q \) of quivers. We have the bijection \( \hat{\phi}_{Q,0} := \phi_{Q,0} \circ (f_Q^{-1}) : \widehat{\Gamma}_Q \to \mathbb{R}^+ \), where \( \phi_{Q,0} : (\Gamma_Q)_0 \to \mathbb{R}^+ \) is the bijection introduced in Sect. 3.5. Thanks to Theorem 3.35 and (3.18), we have \( \hat{\phi}_Q(i, p) = (\hat{\phi}_{Q,0}(i, p), 0) \) and \( \hat{\phi}_{Q,0}(i, p) = \tau_Q^{(i_p - p)/2}(\gamma_i^Q) \) for any \( (i, p) \in \widehat{\Gamma}_Q \) and \( i \in I \).

Example 3.42. The folded AR quivers associated with the Q-data \( Q^{(1)} \) and \( Q^{(2)} \) for \( \mathfrak{g} \) of type \( \mathfrak{B}_3 \) given in Example 3.7 above are depicted as follows. Compare with the twisted AR quivers \( \Gamma_Q^{(1)} \) and \( \Gamma_Q^{(2)} \) depicted in Example 3.23.

\[
\begin{align*}
\widehat{\Gamma}_Q^{(1)} &= \\
\widehat{\Gamma}_Q^{(2)} &=
\end{align*}
\]

Remark 3.43. Unlike the (twisted) AR quiver \( \Gamma_Q \), the folded AR quiver \( \widehat{\Gamma}_Q \) may not satisfy the 2-segment property. See \( \widehat{\Gamma}_Q^{(2)} \) in Example 3.42 above. Proposition 3.31 tells us that the folded AR quiver \( \widehat{\Gamma}_Q \) satisfies the 2-segment property if and only if \( [Q] = [i(\tau)] \) for some twisted Coxeter element \( \tau \), or equivalently the condition (3.9) is satisfied. When \( \sigma \neq \text{id} \), we can see the 2-segment property as
\[
\widehat{\Gamma}_Q = \{(i, \xi_k - 2k) \in I \times \mathbb{Z} \mid k \in \mathbb{Z}, 0 \leq k < rh^\vee/2\}
\]
under the condition (3.9).

Proposition 3.44. Let \( Q = (\Delta, \sigma, \xi) \) be a Q-datum and \( (i, p) \in \widehat{I} \). If \( \hat{\phi}_Q(i, p) = (\alpha, k) \), we have
\[
\hat{\phi}_Q(i^*, p \mp rh^\vee) = (\alpha, k \pm 1).
\]

Proof. This is just a re-expression of Corollary 3.40. \( \Box \)
4. Inverse of Quantum Cartan Matrices

In this section, we show that the inverse of the quantum Cartan matrix of \( \mathfrak{g} \) can be computed by using the generalized twisted Coxeter element \( \tau_\mathcal{Q} \) associated with a Q-datum \( \mathcal{Q} \) for \( \mathfrak{g} \). Let us keep the notation in Sect. 3. Note that the results in this section can be understood as generalizations of the results for symmetric affine types in [HL15, §2.5] to all untwisted affine types.

4.1. Quantum Cartan matrix  Let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra and \( C = (c_{ij})_{i,j \in I} \) denote its Cartan matrix as in Sect. 3.1.

**Definition 4.1.** Let \( z \) be an indeterminate. The quantum Cartan matrix of \( \mathfrak{g} \) is the \( \mathbb{Z}[z^{\pm 1}] \)-valued \((I \times I)\)-matrix \( C(z) = (C_{ij}(z))_{i,j \in I} \) defined by

\[
C_{ij}(z) = \begin{cases} z^{d_i} + z^{-d_j} & \text{if } i = j, \\ [c_{ij}]_z & \text{if } i \neq j. \end{cases}
\]  

(4.1)

where \([k]_z\) denotes the quantum integer \([k]_z := \frac{z^k - z^{-k}}{z - z^{-1}} \in \mathbb{Z}[z^{\pm 1}]\) for each \( k \in \mathbb{Z}\). Note that, for \( i \sim j \), we have \([c_{ij}]_z = -[r]_z\) (resp. \(-1\)) if \( d_j > d_i \) (resp. \( d_j \leq d_i \)).

We set \( D(z) := \text{diag} ([d_i]_z \mid i \in I) \). The following property is easy to see from the definitions.

**Lemma 4.2.** We have \( D(z)C(z) = ([d_i]_z c_{ij})_{i,j \in I} \). In particular, it is symmetric (cf. (3.3)).

Note that we have \( C(z)|_{z=1} = C \in GL_I(\mathbb{Q}) \). We regard \( C(z) \) as an element of \( GL_I(\mathbb{Q}(z)) \) and denote its inverse by \( \tilde{C}(z) = (\tilde{C}_{ij}(z))_{i,j \in I} \). Let

\[
\tilde{C}_{ij}(z) = \sum_{u \in \mathbb{Z}} \tilde{c}_{ij}(u) z^u
\]

denote the formal Laurent expansion of the \((i, j)\)-entry \( \tilde{C}_{ij}(z) \) at \( z = 0 \).

**Lemma 4.3.** For any \( i, j \in I \) and \( u \in \mathbb{Z} \), we have \( \tilde{c}_{ij}(u) \in \mathbb{Z} \). Moreover, we have

1. \( \tilde{c}_{ij}(u) = 0 \) if \( u < d_i \),
2. \( \tilde{c}_{ij}(d_i) = \delta_{ij} \).

**Proof.** Set \( E(z) = (E_{ij}(z))_{i,j \in I} := C(z) \cdot z^D \), where \( z^D := \text{diag}(z^{d_i} \mid i \in I) \). Since \( \tilde{C}(z) = z^D \cdot E(z)^{-1} \), it suffices to show that \( E_{ij}(z) \in \delta_{ij} + z\mathbb{Z}[z] \) for each \( i, j \in I \). If \( i = j \), we have \( E_{ii}(z) = 1 + z^{2d_i} \). If \( i \neq j \), we have \( -E_{ij}(z) = z^{d_j} \sum_{u=0}^{c_{ij}-1} z^{c_{ij}+1+2u} \). Therefore it is enough to show \( d_j + c_{ij} + 1 > 0 \), or equivalently \( d_j \geq -c_{ij} = [d_j/d_i] \) for \( i \sim j \). The last condition is now obvious. \( \Box \)

**Lemma 4.4.** The integers \( \{\tilde{c}_{ij}(u) \mid i, j \in I, u \in \mathbb{Z}\} \) enjoy the following properties:

1. We have \( \tilde{c}_{ii}(u) = \tilde{c}_{ai}(a_{ij})(u) \) for any automorphism \( a \) of the Dynkin diagram of \( \mathfrak{g} \).
2. For any \( i, j \in I \) and \( u \in \mathbb{Z} \), we have

\[
\tilde{c}_{ij}(u) = \begin{cases} \tilde{c}_{ij}(u) & \text{if } d_i = d_j, \\ \tilde{c}_{ij}(u+1) + \tilde{c}_{ij}(u-1) & \text{if } (d_i, d_j) = (1, 2), \\ \tilde{c}_{ji}(u+2) + \tilde{c}_{ji}(u) + \tilde{c}_{ji}(u-2) & \text{if } (d_i, d_j) = (1, 3). \end{cases}
\]
(3) For any $i, j \in I$ and $u \in \mathbb{Z}$, we have
\[
\tilde{c}_{ij}(u + d_i) - \tilde{c}_{ij}(u - d_i) = \tilde{c}_{ji}(u + d_j) - \tilde{c}_{ji}(u - d_j).
\]

**Proof.** The assertion (1) is immediate from the definition. Let us prove (2) and (3). Since the product $D(z)C(z)$ is symmetric by Lemma 4.2, $D(z)C(z)^{-1} = D(z) \cdot (D(z)C(z))^{-1}$. $D(z)$ is symmetric as well. In other words, for each $i, j \in I$, we have
\[
\frac{z^{d_i} - z^{-d_i}}{z - z^{-1}} \sum_{u \in \mathbb{Z}} \tilde{c}_{ij}(u)z^u = \frac{z^{d_j} - z^{-d_j}}{z - z^{-1}} \sum_{u \in \mathbb{Z}} \tilde{c}_{ji}(u)z^u.
\]
Comparing the coefficients of $z^u$ in both sides of (4.2), we obtain the assertion (2) for the cases $(d_i, d_j) = (1, 1), (1, 2), (1, 3)$. When $(d_i, d_j) = (2, 2)$, we obtain
\[
\tilde{c}_{ij}(u + 1) + \tilde{c}_{ij}(u - 1) = \tilde{c}_{ji}(u + 1) + \tilde{c}_{ji}(u - 1).
\]
Then the assertion $\tilde{c}_{ij}(u) = \tilde{c}_{ji}(u)$ can be proved by induction on $u$ since we know $\tilde{c}_{ij}(u) = \tilde{c}_{ji}(u) = 0$ for $u \leq 0$ thanks to Lemma 4.3. The case $(d_i, d_j) = (3, 3)$ is proved in the same way. Similarly, after multiplying $z - z^{-1}$ to the both sides of (4.2), we obtain the assertion (3) by similarly comparing the coefficients of $z^u$. $\Box$

4.2. A combinatorial formula Recall the symmetric bilinear form $(\langle \cdot, \cdot \rangle) : P \times P \to \mathbb{Q}$ determined by $(\sigma_i, \alpha_j) = \delta_{i,j}$ for $i, j \in \Delta_0$. This is invariant under the action of $W \rtimes \langle \sigma \rangle$.

**Definition 4.5.** Let $Q = (\Delta, \sigma, \xi)$ be a $Q$-datum for $g$. For each $i, j \in \Delta_0$, we define a function $\eta_{ij}^Q : \mathbb{Z} \to \mathbb{Z}$ by
\[
\eta_{ij}^Q(u) := \begin{cases} 
(\sigma_i, \tau_{Q}^{(u+\xi_j-\xi_i-d_i)/2}(\gamma_j^Q)) & \text{if } u + \xi_j - \xi_i - d_i \in 2\mathbb{Z}, \\
0 & \text{otherwise.}
\end{cases}
\]

**Lemma 4.6.** Let $Q'$ be another $Q$-datum for $g$ and $i' \in \bar{i}, j' \in \bar{j}$. Then we have $\eta_{i'j'}^{Q'} = \eta_{ij}^Q$.

**Proof.** We only have to consider the case when $u + \xi_j - \xi_i - d_i \in 2\mathbb{Z}$. The equality $\eta_{ij}^Q = \eta_{ij'}^{Q'}$ for $j' \in \bar{j}$ follows from Lemma 3.38. To prove the equality $\eta_{i'j}^{Q'} = \eta_{ij}^Q$ for any $Q'$, it suffices to verify the equality $\eta_{ij}^Q = \eta_{ij}^{Q_k}$, where $k \in \Delta_0$ is a source of $Q$. When $k \neq i$, we have $s_k \sigma_i = \sigma_i$ and $(s_k \xi_i)_t = \xi_i$. Therefore using Lemma 3.37, we obtain
\[
\eta_{i'j}^{Q'}(u) = (\sigma_i, s_k \tau_{Q}^{(u+\xi_j-\xi_i-d_i)/2}(\gamma_j^Q)) = (\sigma_i, \tau_{Q_k}^{(u+(s_k \xi)_j-(s_k \xi)_i-d_i)/2}(\gamma_j^{Q_k})) = \eta_{ij}^{Q_k}(u).
\]
When $k = i$, we have $s_i \tau_{Q}^{d_i}(\sigma_i = \sigma_i$ since the element $s_i \tau_{Q}^{d_i}$ has an expression without the simple reflection $s_i$. Using Lemma 3.37 again, we obtain
\[
\eta_{i'j}^{Q'}(u) = (s_i \tau_{Q}^{d_i}(\sigma_i, s_i \tau_{Q}^{(u+\xi_j-\xi_i+d_i)/2}(\gamma_j^Q)) = (\sigma_i, \tau_{Q_k}^{(u+(s_i \xi)_j-(s_i \xi)_i-d_i)/2}(\gamma_j^{Q_k})) = \eta_{i'j}^{Q_k}(u).
\]
Finally, let us verify the equality $\eta_{i'j}^{Q} = \eta_{i'j'}^{Q'}$ for $i' \in \bar{i}$. By the independence of the choice of $Q$, we may assume that $\tau_{Q}^{d_i}$ is a twisted Coxeter element, or equivalently $Q$ satisfies
the condition (3.9). Note that we have $\tau^l_Q \sigma_t = \sigma^{\iota(l)}_t$ for any $t \in I^\circ_Q$ and $0 \leq l < d_t$. Therefore we obtain

$$\eta^Q_{ij}(u) = (\tau^l_Q \sigma_t, \tau^Q_{(2l+u+\varepsilon_j-\varepsilon_i+d_t)/2}(y^Q_j)) = (\sigma^{\iota(l)}_t, \tau^{(u+\varepsilon_j-\varepsilon^{\iota(l)}+d_t)/2}_Q(y^Q_j)) = \eta^Q_{\iota(l)j}(u)$$

as desired.  

By Lemma 4.6, the following notation is well-defined.

**Definition 4.7.** For each $i, j \in I$, we define

$$\eta_{ij} := \eta^Q_{ij},$$

where $Q$ is a $Q$-datum for $\mathfrak{g}$ and $i \in I, j \in J$.

The following statement is the main theorem of this subsection, which gives a generalization of [HL15, Proposition 2.1].

**Theorem 4.8.** For each $i, j \in I$ and $u \in \mathbb{Z}_{\geq 0}$, we have $\tilde{c}_{ij}(u) = \eta_{ij}(u)$. In other words, we have

$$\tilde{c}_{ij}(u) = \begin{cases} (\sigma_t, \tau^Q_{(u+\varepsilon_j-\varepsilon_i-d_t)/2}(y^Q_j)) & \text{if } u + \varepsilon_j - \varepsilon_i - d_t \in 2\mathbb{Z}, \\ 0 & \text{otherwise} \end{cases}$$

for any $Q$-datum $Q = (\Delta, \sigma, \varepsilon)$ for $\mathfrak{g}$ and $u \in \mathbb{Z}_{\geq 0}, i \in I, j \in J$.

For a proof of Theorem 4.8, we need a lemma.

**Lemma 4.9.** The functions $\{\eta_{ij} : \mathbb{Z} \rightarrow \mathbb{Z} | i, j \in I\}$ enjoy the following properties:

(1) $\eta_{ij}(u + rh^\vee) = -\eta_{ij}(u)$ for any $u \in \mathbb{Z}$.
(2) For any $u \in \mathbb{Z}$, we have

$$\eta_{ij}(u - d_j) + \eta_{ij}(u + d_j) = \sum_{k \sim j} \sum_{l=0}^{-c_{kj}-1} \eta_{ik}(u + c_{kj} + 1 + 2l).$$

(3) For any $u \in \mathbb{Z}$, we have

$$-\eta_{ij}(-u) = \begin{cases} \eta_{ji}(u) & \text{if } d_i = d_j, \\ \eta_{ji}(u+1) + \eta_{ji}(u-1) & \text{if } (d_i, d_j) = (1, 2), \\ \eta_{ji}(u+2) + \eta_{ji}(u) + \eta_{ji}(u-2) & \text{if } (d_i, d_j) = (1, 3). \end{cases}$$

(4) $\eta_{ij}(u) = 0$ if $|u| < d_i - \delta_{ij}$.
(5) $\eta_{ii}(\pm d_i) = \pm 1$.

**Proof.** When $\sigma \neq \text{id}$, the property (1) is a consequence of Proposition 3.34 (4). When $\sigma = \text{id}$, it is proved in [Fuj19, Lemma 3.7]. The property (2) follows from the $\mathfrak{g}$-additive property (Theorem 3.41) if we note that $d_j + c_{kj} + 1 = d_j - [d_j/d_k] + 1 = \min(d_j, d_k)$ for any $j, k \in I$. 

Let us prove the property (3). We choose a Q-datum \( Q = (\Delta, \sigma, \xi) \) for \( g \) and \( i \in i, j \in j \). Let \( u \in \mathbb{Z} \) satisfy \( u + \xi_j - \xi_i - d_i \in 2\mathbb{Z} \). We compute

\[
-\eta_{ij}(-u) = (\omega_i, \tau_Q^{(u+\xi_j-\xi_i-d_i)/2}(\tau_Q^{d_j} - 1)\omega_j) = (\omega_Q, \tau_Q^{(u+\xi_i-\xi_j+d_i)/2}(\tau_Q^{d_j} - 1)\omega_i, \omega_j) = (\omega_Q, \tau_Q^{(u+d_j-d_i)+\xi_i-\xi_j-d_j)/2}(1-\tau_Q^{d_j})\omega_i).
\]

Under the assumption \( d_i \leq d_j \), we have \( 1 - \tau_Q^{d_j} = (\sum_{l=0}^{d_j-d_i} \tau_Q^{l})(1 - \tau_Q^{d_j}) \). Combining with the above computation, we obtain (3).

Finally we shall prove the property (4) and (5). Fix \( i, j \in I \) and choose a Q-datum \( Q = (\Delta, \sigma, \xi) \) for \( g \) satisfying the condition (3.9) and such that \( (a) j^\circ \) is a source of \( Q \), \( (b) \xi_{ij} := \xi_{j^\circ} - \xi_{i^\circ} \in [0,1) \). The condition (a) implies \( y_{j^\circ} = \alpha_{j^\circ} \) by Lemma 3.36. For any \( u \in \mathbb{Z} \) such that \( u + \xi_{ij} - d_i \in 2\mathbb{Z} \), we have \( \eta_{ij}(u) = (\tau_Q^{d_i-\xi_{ij}-u}/2)(\omega_{i^\circ}), \alpha_{j^\circ}) \).

Combining with the fact that \( \tau_Q^{l}(\omega_{i^\circ}) = \omega_{i^\circ}^{\alpha_{j^\circ}} \) for \( 0 \leq l < d_i \), we obtain the desired equalities (4) and (5) under the assumption \( 0 \leq u \leq d_i \). The other case when \( -d_i \leq u < 0 \) follows from this case and the property (3). \( \square \)

**Proof of Theorem 4.8.** Setting \( H_{ij}(z) := \sum_{u \geq 0} \eta_{ij}(u)z^u \in \mathbb{Z}[z] \) for each \( i, j \in I \), we have to show that

\[
\sum_{k \in I} H_{ik}(z)C_{kj}(z) = \delta_{i,j}. \tag{4.3}
\]

We denote by \( x_{ij}(u) \) the coefficient of \( z^u \) in the LHS of (4.3) for each \( u \in \mathbb{Z} \). Then the equality (4.3) is equivalent to \( x_{ij}(u) = \delta_{i,j}\delta_{u,0} \). By Lemma 4.9 (4), we can write \( H_{ij}(z) = \sum_{u > -d_i} \eta_{ij}(u)z^u \) for any \( i, j \in I \). Therefore we have

\[
x_{ij}(u + d_j) = \eta_{ij}(u) + \eta_{ij}(u + 2d_j) - \sum_{k \sim j} \sum_{l=0}^{-c_{kj}-1} \eta_{ik}(u + d_j + c_{kj} + 1 + 2l) = 0
\]

for any \( u > -d_i \) thanks to Lemma 4.9 (2). On the other hand, Lemma 4.9 (4) also tells us that

\[
H_{ik}(z)C_{kj}(z) \in \begin{cases} z^{d_i+\delta(i \neq k)+c_{kj}+1}\mathbb{Z}[z] & \text{if } k \neq j, \\ z^{d_i+\delta(i \neq j)-d_j}\mathbb{Z}[z] & \text{if } k = j, \end{cases}
\]

which implies \( x_{ij}(u) = 0 \) if \( u < d_i - d_j + \delta(i \neq j) \). Therefore it is enough to show \( x_{ij}(u) = \delta_{i,j}\delta_{u,0} \) only for \( d_i - d_j + \delta(i \neq j) \leq u \leq d_j - d_i \). When \( d_i \geq d_j \), we only have to consider the case \( i = j \) and \( u = 0 \). In this case, we have

\[
x_{ii}(0) = \eta_{ii}(d_i) - \sum_{k \sim i} \sum_{l=0}^{-c_{ki}-1} \eta_{ik}(u + c_{ki} + 1 + 2l) = -\eta_{ii}(-d_i) = 1
\]

as desired thanks to Lemma 4.9 (2) and (5).

In the remaining case \( (d_i, d_j) = (1, r) \) with \( r > 1 \), we have to verify \( x_{ij}(u) = 0 \) for \( 2r - r < u < r - 1 \). First we assume that \( r = 2 \). We observe \( x_{ij}(0) = \eta_{ij}(2) - \eta_{ii}(1) \times \delta(i \sim j) \) using Lemma 4.9 (4). Let us choose a Q-datum \( Q = (\Delta, \sigma, \xi) \) satisfying
Corollary 4.10. The integers \( \tilde{c}_{ij}(u) \mid i, j \in I, \ u \in \mathbb{Z} \) enjoy the following properties:

1. \( \tilde{c}_{ij}(u + rh^\vee) = -\tilde{c}_{ij}^*(u) \) for \( u \geq 0 \).
2. \( \tilde{c}_{ij}(u + 2rh^\vee) = \tilde{c}_{ij}(u) \) for \( u \geq 0 \).
3. \( \tilde{c}_{ij}(rh^\vee - u) = \tilde{c}_{ij}^*(u) \) for \( 0 \leq u \leq rh^\vee \).
4. \( \tilde{c}_{ij}(2rh^\vee - u) = -\tilde{c}_{ij}(u) \) for \( 0 \leq u \leq 2rh^\vee \).
5. \( \tilde{c}_{ij}(u) = 0 \) if \( |u - krh^\vee| \leq d_i - \delta_{ij} \) for some \( k \in \mathbb{Z}_{\geq 0} \).
6. \( \tilde{c}_{ij}(u) \geq 0 \) for \( 0 \leq u \leq rh^\vee \).
7. \( \tilde{c}_{ij}(u) \leq 0 \) for \( rh^\vee \leq u \leq 2rh^\vee \).

Proof. The property (1) follows from Theorem 4.8 and Lemma 4.9 (1). The property (2) is a consequence of the property (1). Let us verify the property (3). For the case \( d_i \leq d_j \), we compute:

\[
\tilde{c}_{ij}(rh^\vee - u) = -\eta_{ij}^*(-u) \quad \text{(by Theorem 4.8 and Lemma 4.9 (1))}
\]

\[
= \sum_{l=0}^{d_i-d_j} \eta_{j+i}(u + d_i - d_j + 2l) \quad \text{(by Lemma 4.9 (3))}
\]

\[
= \sum_{l=0}^{d_j-d_i} \tilde{c}_{j+i}^*(u + d_i - d_j + 2l) \quad \text{(by Lemma 4.9 (4) and Theorem 4.8)}
\]

\[
= \tilde{c}_{ij}^*(u) \quad \text{(by Lemma 4.4 (2))}
\]

The other case \( d_i > d_j \) can be proved by induction on \( u \) using Lemma 4.4 (2) and Lemma 4.9 (3). The property (4) is immediate from the properties (1) and (3). The property (5) is a consequence of Theorem 4.8 and Lemma 4.9 (1), (4). To verify the property (6), it is enough to show \( \tilde{c}_{ij}(u) \geq 0 \) for \( d_i \leq u \leq rh^\vee - d_i \) thanks to (5). For the case \( \sigma = \text{id} \), see [Fuj19, Lemma 3.7] for instance. Here we focus on the case \( \sigma \neq \text{id} \). Although our discussion here is quite similar to that for the case \( \sigma = \text{id} \). We take a Q-datum \( Q = (\Delta, \sigma, \xi) \) for \( q \) satisfying the condition (3.9) and such that \( \xi_{ij} := \xi_{j^\circ} - \xi_{i^\circ} \in [0, 1] \). By the parity reason, we may assume \( u + \xi_{ij} - d_i \in 2\mathbb{Z} \). By Remark 3.43, we see that \( \tau_Q^{(u+\xi_{ij}-d_i)/2}(\gamma_{j^\circ}^Q) \in \mathbb{R}^+ \) whenever \( 0 \leq u + \xi_{ij} - d_i \leq rh^\vee \).

This last condition is always satisfied under our assumption \( d_i \leq u \leq rh^\vee - d_i \). Since \( (\sigma rh^\vee, \alpha) \geq 0 \) holds if \( \alpha \in \mathbb{R}^+ \), we obtain \( \tilde{c}_{ij}(u) = (\sigma rh^\vee, \tau_Q^{(u+\xi_{ij}-d_i)/2}(\gamma_{j^\circ}^Q)) \geq 0 \) as desired. The last property (7) follows from (1) and (6).
Theorem 4.8 tells us that one can compute the explicit values of the integers $\tilde{c}_{ij}(u)$ from any (folded) AR quiver $\tilde{\Gamma}_Q$. In this subsection, we carry out such case-by-case computations for all non-simply-laced $\mathfrak{g}$. The results are used in Sect. 6 to deduce a unified denominator formula for the normalized R-matrices. We note that a list of explicit values of $\tilde{C}_{ij}(z)$ for all $\mathfrak{g}$ appeared also in [GTL17, Appendix A] without a proof.

Remark 4.11. Assume $\mathfrak{g}$ is not simply-laced, or equivalently $\sigma \neq \text{id}$. To compute all the values of $\{\tilde{c}_{ij}(u) \mid i, j \in I, u \in \mathbb{Z}\}$, it is enough to compute

$$(1 + z^{rh^\vee})\tilde{C}_{ij}(z) = \sum_{u=0}^{rh^\vee-1} \tilde{c}_{ij}(u)z^u$$

for each $i, j \in I$ thanks to Corollary 4.10 (1).

4.3.1. Type $B_n$. Let $\mathfrak{g}$ be of type $B_n$. The corresponding pair in Table 1 is $(\mathfrak{g}, \sigma) = (A_{2n-1}, \vee)$. We use the labeling $\Delta_0 = \{1, \ldots, 2n-1\}$, $I = \{1, \ldots, n\}$ as in (3.2a). The involution $\vee$ on $\Delta$ is given by $i^\vee = 2n - i$ for $1 \leq i \leq 2n - 1$, and we set $\tilde{i} = i$ for $1 \leq i \leq n$. Note that we have $(d_1, \ldots, d_n) = (2, \ldots, 2, 1)$ and $rh^\vee = 4n - 2$.

Theorem 4.12. For $i, j \in I = \{1, 2, \ldots, n\}$, the closed formula of $\tilde{C}_{ij}(z)$ is given as follows:

$$(1 + z^{4n-2})\tilde{C}_{ij}(z) = \begin{cases} \sum_{s=1}^{\min(i, j)} (z^2(i-j)+2s+1) + z^{2(2n-i-j-2s+1)} & \text{if } 1 \leq i, j < n, \\ \sum_{s=1}^{i} z^{2n-2i-3+4s} & \text{if } i < n, j = n, \\ \sum_{s=1}^{j} (z^{2n-2j-4+4s} + z^{2n-2j+2+4s}) & \text{if } i = n, j < n, \\ \sum_{s=1}^{n} z^{4s-3} & \text{if } i = j = n. \end{cases}$$

(4.4)

To prove Theorem 4.12, let us choose the height function $\xi : \Delta_0 \to \mathbb{Z}$ on $(\Delta, \vee)$ given by

$$\xi_i = \begin{cases} -2i + 2 & \text{if } 1 \leq i < n, \\ -2n + 1 & \text{if } i = n, \\ 2i - 4n - 4 & \text{if } n < i \leq 2n - 1. \end{cases}$$

Clearly, the corresponding Q-datum $Q := (\Delta, \vee, \xi)$ satisfies the condition (3.9) and $I_Q^\circ = \{1, 2, \ldots, n\}$. Thus the corresponding twisted Coxeter element $\tau_Q$ is given by $\tau_Q = s_1s_2 \cdots s_n \vee$ and $\tilde{\tau}_Q = \tau_Q^2 = s_1s_2 \cdots s_{n-1}s_n s_{2n-1} \cdots s_{n+1}s_n$. Let us realize the root lattice $\mathfrak{Q}$ inside the lattice $\bigoplus_{i=1}^{2n} \mathbb{Z}\alpha_i$ in a standard way as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i < 2n$. Then we have $R^+ = \{[k, l] \mid 1 \leq k \leq l < 2n\}$, where $[k, l] := \epsilon_k - \epsilon_{l+1}$. We simplify $[i] := [i, i] = \alpha_i$ for $1 \leq i < 2n$. Using Lemma 3.36, we have $\gamma_i^Q = [1, i]$ for $1 \leq i \leq n$.

By a direct computation, we can check the following.
Lemma 4.13. With the above notation, we have:

\[ \tau_Q^{2^l-2}(\gamma_i Q) = \begin{cases} 
[t, t+i-1] & \text{if } 1 \leq i < n \text{ and } 1 \leq t \leq n - i, \\
[t, 3n-t-i] & \text{if } 1 \leq i < n \text{ and } n - i < t \leq n, \\
[t, n] & \text{if } i = n \text{ and } 1 \leq t \leq n, 
\end{cases} \]

\[ \tau_Q^{2^l-1}(\gamma_i Q) = \begin{cases} 
[2n-t+i+1, 2n-t] & \text{if } 1 \leq i < n \text{ and } 1 \leq t \leq n - i, \\
[-n+i+t+1, 3n-t-i-1] & \text{if } 1 \leq i < n \text{ and } n - i < t < n, \\
[n+1, 2n-t] & \text{if } i = n \text{ and } 1 \leq t < n. 
\end{cases} \]

In view of (3.18), Lemma 4.13 enables us to depict the folded AR quiver \( \tilde{\Gamma}_Q \).

Example 4.14. When \( n = 4 \), the folded AR quiver \( \tilde{\Gamma}_Q \) can be depicted as follows:

\[ S_{ij}(1) := \{ t \mid 0 \leq t < 2n-1, (\varpi_j, \tau_Q^{t}(\gamma_j Q)) = 1 \} \]

is given by:

\[ S_{ij}(1) = \begin{cases} 
\{2k\}_{k=0}^{j-1} \cup \{2n - 2j - 1 + 2k\}_{k=0}^{j-1} & \text{if } i \leq j < n, \\
\{2i - 2j + 2k\}_{k=0}^{j-1} \cup \{2n - 2j - 1 + 2k\}_{k=0}^{j-1} & \text{if } j < i \leq n, \\
\{2k\}_{k=0}^{j-1} & \text{if } i \leq j = n. 
\end{cases} \]  \( (4.5) \)

Proof. First we observe that

\[ (\varpi_i, [k, l]) = \begin{cases} 
1 & \text{if } k \leq i \leq l, \\
0 & \text{otherwise}, 
\end{cases} \]  \( (4.6) \)

for any \( 1 \leq i < 2n \) and \( 1 \leq k \leq l < 2n \). Using this observation and Lemma 4.13, we can easily check that the RHS of (4.5) is included in \( S_{ij}(1) \). To see that this inclusion is actually an equality, we can use the fact \( \sum_{\alpha \in \mathbb{R}^+(\varpi_i, \alpha)} = 2n_i - i^2 \) for any \( 1 \leq i < 2n \), which also follows from (4.6). \( \square \)

Proof of Theorem 4.12. Let \( 1 \leq i, j \leq n \). For \( t \in S_{ij}(1) \), let us compute \( s \) by the equation \( t = (s + \xi_j - \xi_i - d_i)/2 \) using Lemma 4.15. Then one can easily check that \( s \) coincides with the exponents appearing in the RHS of in (4.4). Hence our assertion follows from Theorem 4.8. \( \square \)

4.3.2. Type \( G_n \) Let \( g \) be of type \( G_n \). The corresponding pair in Table 1 is \( (g, \sigma) = (D_{n+1}, \vee) \). We use the labeling \( \Delta_0 = \{1, \ldots, n+1\}, I = \{1, \ldots, n\} \) as in (3.2b). The involution \( \vee \) on \( \Delta \) is given by

\[ k^\vee = \begin{cases} 
k & \text{if } k \leq n - 1, \\
n + 1 & \text{if } k = n, \\
n & \text{if } k = n + 1. 
\end{cases} \]

We set \( \tilde{k} = k \) for \( 1 \leq k \leq n \). Note that \( (d_1, \ldots, d_n) = (1, \ldots, 1, 2) \) and \( rh^\vee = 2n + 2 \).
Theorem 4.16. For $i, j \in I = \{1, 2, \ldots, n\}$, the closed formula of $\widetilde{C}_{ij}(z)$ is given as follows:

$$(1 + z^{2n+2})\widetilde{C}_{ij}(z) = \begin{cases} 
\sum_{s=1}^{\min(i, j)} (z|j|+2s-1 + z^{2n-i-j+2s+1}) & \text{if } i, j < n, \\
\sum_{s=1}^{i} (z^{n-i}+2s + z^{n+1-i+2s+1}) & \text{if } i < n, j = n, \\
\sum_{s=1}^{j} z^{j-s} & \text{if } i = n, j < n, \\
\sum_{s=1}^{n} z^{2s} & \text{if } i = j = n.
\end{cases}$$

To prove Theorem 4.16, we choose the height function $\xi: \Delta_0 \to \mathbb{Z}$ on $(\Delta, \vee)$ given by

$$\xi_i = \begin{cases} 
-i + 1 & \text{if } 1 \leq i \leq n, \\
-n - 1 & \text{if } i = n + 1.
\end{cases}$$

Clearly, the corresponding Q-datum $Q := (\Delta, \vee, \xi)$ satisfies the condition (3.9) and $I_Q^{\circ} = \{1, 2, \ldots, n\}$. Thus the corresponding twisted Coxeter element $\tau_Q$ is given by $\tau_Q = s_1s_2\cdots s_n \vee$ and $\tilde{\tau}_Q = \tau_Q^{\circ} = s_1s_2\cdots s_{n-1}s_n\vee s_1s_2\cdots s_{n-1}s_{n+1}$. Let us realize the root lattice $Q$ inside the lattice $\bigoplus_{i=1}^{n+1} \mathbb{Z} e_i$ in a standard way as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n$ and $\alpha_{n+1} = \epsilon_n + \epsilon_{n+1}$. Then we have $R^+ = \{(k, \pm l) \mid 1 \leq k < l \leq n + 1\}$, where $\langle k, \pm l \rangle := \epsilon_k \pm \epsilon_l$. Using Lemma 3.36, we have $\gamma_i^Q = (1, -(i+1))$ for $1 \leq i \leq n$.

By a direct computation, we can check the following.

Lemma 4.17. With the above notation, for each $1 \leq i \leq n$, we have:

$$\tau_Q^{-1}(\gamma_i^Q) = \begin{cases} 
\{t, -(i + 1)\} & \text{if } 1 \leq t \leq n + 1 - i, \\
\{t + i - n - 1, t\} & \text{if } n + 1 - i < t \leq n + 1.
\end{cases}$$

In view of (3.18), Lemma 4.17 enables us to depict the folded AR quiver $\hat{\Gamma}_Q$.

Example 4.18. When $n = 3$, the folded AR quiver $\hat{\Gamma}_Q$ can be depicted as follows:

(\begin{array}{cccccccc}
(\cdot \setminus p) & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & (1, 4) & & (3, -4) & (2, -3) & (1, -2) \\
2 & (2, 4) & (1, 3) & (2, -4) & (1, -3) \\
3 & (3, 4) & (2, 3) & & (1, 2) & (1, -4)
\end{array})

Lemma 4.19. For $1 \leq i, j \leq n$ and $k \in \mathbb{Z}_{\geq 0}$, we define $S_{ij}(k) := \{t \mid 0 \leq t < 2n - 1, (\omega_i, \tau_Q^t(\gamma_j^Q)) = k\}$ and $S_{ij}(\geq k) := \bigcup_{l \geq k} S_{ij}(l)$. Then the followings hold:

1. For $i \leq j < n$, we have

$$S_{ij}(\geq 1) = \{k\}_{k=0}^{i-1} \cup \{n - j + 1 + k\}_{k=0}^{i-1},$$
$$S_{ij}(2) = \{k\}_{k=0}^{i-1} \cap \{n - j + 1 + k\}_{k=0}^{i-1}.$$
2. For \( j < i < n \), we have
\[
S_{ij}(\geq 1) = \{i - j + k\}_{k=0}^{l-1} \cup \{n - j + 1 + k\}_{k=0}^{j-1},
\]
\[
S_{ij}(2) = \{i - j + k\}_{k=0}^{l-1} \cap \{n - j + 1 + k\}_{k=0}^{j-1}.
\]

3. For \( i < j = n \), we have \( S_{ij}(\geq 1) = \{k\}_{k=0}^{j} \) and \( S_{ij}(2) = \{k\}_{k=1}^{j-1} \).

4. For \( j < i = n \), we have \( S_{ij}(\geq 1) = S_{ij}(1) = \{n - j + k\}_{k=0}^{j-1} \).

Proof. First we observe that
\[
(\sigma_i, (k, -l)) = \begin{cases} 1 & \text{if } k \leq i < l, \\ 0 & \text{otherwise}, \end{cases}
\]
and
\[
(\sigma_i, (k, l)) = \begin{cases} 2 & \text{if } l \leq i \leq n - 1 \\ 1 & \text{if } k \leq i < l \text{ or } i \in \{n, n + 1\}, \\ 0 & \text{otherwise}, \end{cases}
\]
for any \( 1 \leq i \leq n + 1 \) and \( 1 \leq k < l \leq n + 1 \). Using this observation and Lemma 4.17, we can easily check that the LHS includes the RHS for each equation in (1)–(4). To see that these inclusions are actually equalities, we can use the fact
\[
\sum_{\alpha \in \mathbb{R}^+} (\sigma_i, \alpha) = \begin{cases} 2ni - l + 3i & \text{if } i < n, \\ (n^2 + n)/2 & \text{if } i = n, \end{cases}
\]
which also follows from (4.8). \( \square \)

Proof of Theorem 4.16. Let \( 1 \leq i, j \leq n \). For \( t \in S_{ij}(\geq 1) \), let us compute \( s \) by the equation \( t = (s + \xi_j - \xi_i - d_i)/2 \) using Lemma 4.19. Then one can easily check that \( s \) coincides with the exponents appearing in the RHS of in (4.7). Hence our assertion follows from Theorem 4.8. \( \square \)

4.3.3. Type \( F_4 \). Let \( \mathfrak{g} \) be of type \( F_4 \). The corresponding pair in Table 1 is \( (\mathfrak{g}, \sigma) = (\mathfrak{E}_6, \vee) \). We use the labeling \( \Delta_0 = \{1, 2, 3, 4, 5, 6\} \), \( I = \{1, 2, 3, 4\} \) as in (3.2c). The involution \( \vee \) on \( \Delta \) is given by \( 1^\vee = 6, 2^\vee = 1, 3^\vee = 5, 4^\vee = 3, 5^\vee = 2, 6^\vee = 4 \) and we set \( 1 = 1 \), \( 2 = 3 \), \( 3 = 5 \), \( 4 = 2 \). Note that \( (d_1, d_2, d_3, d_4) = (2, 2, 1, 1) \) and \( rh^\vee = 18 \).

The following is an example of the folded AR quiver of a Q-datum for \( \mathfrak{g} \). Here \( (a_1a_2a_3a_4a_5a_6) \) denotes the positive root \( \sum_{i=1}^{6} a_i \alpha_i \in \mathbb{R}^+ \) and the Q-datum can be read from the folded AR quiver.

Using Theorem 4.8, we can compute the explicit values of \( \tilde{C}_{ij}(z) \) as follows:
\[
\tilde{C}_{11}(z) = (z^2 + z^8 + z^{10} + z^{16})/(1 + z^{18}),
\]
Therefore it is easy to compute its inverse \( \widetilde{C}(z) \) as:

\[
\widetilde{C}(z) = \frac{1}{z^4 - 1 + z^{-4}} \left( \frac{z + z^{-1}}{z^2 + 1 + z^{-2}} \right) \left( \frac{1}{z^3 + z^{-3}} \right)
\]

By a comparison, we can check that Theorem 4.8 also holds in this case. Thus the proof of Theorem 4.8 is completed.
5. Representations of Untwisted Quantum Affine Algebras

In this section, we recall some basic facts on the finite-dimensional representation theory of untwisted quantum affine algebras. We also introduce the notion of $Q$-weights and observe its relation to the block decomposition of the finite-dimensional module category established by [CM05, KKOP20a]. Let us keep the notation from the previous sections.

5.1. Untwisted quantum affine algebras

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra as before. We denote by $\widehat{\mathfrak{g}}$ the untwisted affine Lie algebra of $\mathfrak{g}$. The Cartan matrix $C = (c_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is extended to the generalized Cartan matrix $C_{af} = (c_{ij})_{i,j \in I_{af}}$ of $\widehat{\mathfrak{g}}$, where $I_{af} := I \cup \{0\}$. We set $d_0 := r$ and $r_0 := r/d_0 = 1$.

Let $q$ be an indeterminate. We denote by $k := \overline{\mathbb{Q}(q)}$ the algebraic closure of the field $\mathbb{Q}(q)$ inside the ambient field $\bigcup_{m \in \mathbb{Z}_{\geq 1}} \overline{\mathbb{Q}(q^{1/m})}$. For each $i \in I_{af}$, we set $q_i := q^{d_i/r} = q^{1/r_i} \in k^\times$. We also use the notation $q_s := q^{1/r}$, where $s$ stands for “short”.

For $a, b \in \mathbb{Z}_{\geq 0}$ with $b \leq a$ and $i \in I_{af}$, we set

$$[a]_i! := \prod_{k=1}^{a} [k]_{q_i}, \quad \left[\begin{array}{c} a \\ b \end{array}\right]_i := \frac{[a]_i!}{[a-b]_i! [b]_i!}.$$

**Definition 5.1.** We define the quantum affine algebra $U_q' (\widehat{\mathfrak{g}})$ (without the degree operator) to be the $k$-algebra given by the set of generators $\{e_i, f_i, K_i^{\pm 1} | i \in I_{af}\}$ satisfying the following relations:

- $K_i K_i^{-1} = 1 = K_i^{-1} K_i, K_i K_j = K_j K_i$ for $i, j \in I_{af}$,
- $K_i e_j K_i^{-1} = q_i^{c_{ij}} e_j, K_i f_j K_i^{-1} = q_i^{-c_{ij}} f_j$ for $i, j \in I_{af}$,
- $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I_{af}$,
- $\sum_{k=0}^{1-c_{ij}} (-1)^k \left[\begin{array}{c} 1 - c_{ij} \\ k \end{array}\right]_i e_i^{1-c_{ij}-k} e_j e_i^k = \sum_{k=0}^{1-c_{ij}} (-1)^k \left[\begin{array}{c} 1 - c_{ij} \\ k \end{array}\right]_i f_i^{1-c_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

We denote by $U_q^+ (\widehat{\mathfrak{g}})$ (resp. $U_q^- (\widehat{\mathfrak{g}})$) the $k$-subalgebra of $U_q' (\widehat{\mathfrak{g}})$ generated by $e_i$ (resp. $f_i$) for $i \in I_{af}$. The algebra $U_q' (\widehat{\mathfrak{g}})$ is equipped with a structure of Hopf algebra over $k$ whose coproduct is given by

$$e_i \mapsto e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad f_i \mapsto f_i \otimes 1 + K_i \otimes f_i, \quad K_i \mapsto K_i \otimes K_i,$$

for $i \in I_{af}$. We denote by $\mathcal{C}$ the category of finite-dimensional $U_q' (\widehat{\mathfrak{g}})$-modules of type 1, i.e. finite-dimensional modules on which the element $K_i$ acts as a diagonalizable linear operator whose eigenvalues belong to the set $\{q_i^k | k \in \mathbb{Z}\}$ for each $i \in I_{af}$. It is well-known that the description of general finite-dimensional $U_q' (\widehat{\mathfrak{g}})$-modules are essentially reduced to that of finite-dimensional $U_q' (\widehat{\mathfrak{g}})$-modules of type 1. The $k$-linear abelian category $\mathcal{C}$ becomes a rigid monoidal category by the above Hopf algebra structure of $U_q' (\widehat{\mathfrak{g}})$. We denote the right dual (resp. left dual) of an object $V \in \mathcal{C}$ by $^*V$ (resp. $V^*$).
For \( k \in \mathbb{Z} \) and a module \( V \) in \( \mathcal{C} \), we define
\[
\mathcal{D}^k(V) := \begin{cases} 
\left( \cdots \left( \left( V^* \right)^\ast \cdots \right)^\ast \right)^{(k)}-\text{times} & \text{if } k < 0, \\
\left( \cdots \left( \left( V^* \right)^\ast \cdots \right)^\ast \right)^{(k)}-\text{times} & \text{if } k \geq 0.
\end{cases}
\]

The category \( \mathcal{C} \) is neither semisimple as an abelian category, nor braided as a monoidal category. We say that two objects \( V, W \in \mathcal{C} \) mutually commute if we have \( V \otimes W \cong W \otimes V \) as \( U_q(\mathfrak{g}) \)-modules. We also say that a simple object \( V \in \mathcal{C} \) is real if the tensor square \( V \otimes V \) remains simple.

Recall the central element \( q^c := \prod_{i \in \mathcal{I}_d} K_{ai}^{a_i} \in U_q(\mathfrak{g}) \), where the positive integers \((a_i)_{i \in \mathcal{I}_d}\) are defined as in [Kac90, Chapter 4, Table Aff1]. It is well known that for any module \( V \in \mathcal{C} \), the element \( q^c \) acts by the identity. The quotient algebra \( U_q(\mathfrak{g})/\langle q^c - 1 \rangle \) is isomorphic to the quantum loop algebra \( U_q(Lg) \) defined below (see [Bec94, CP94, Dam98, Dam12, Dam15]). Via this isomorphism \( U_q(\mathfrak{g})/\langle q^c - 1 \rangle \cong U_q(Lg) \), we can identify the category \( \mathcal{C} \) with the category of finite-dimensional \( U_q(Lg) \)-modules of type 1.

**Definition 5.2.** The quantum loop algebra \( U_q(Lg) \) is the \( k \)-algebra given by the set of generators
\[
\{k_i^{\pm 1} | i \in \mathcal{I}\} \cup \{x_{i,k}^{\pm} | i \in \mathcal{I}, k \in \mathbb{Z}\} \cup \{h_{i,l} | i \in \mathcal{I}, l \in \mathbb{Z}\setminus\{0\}\}
\]
satisfying the following relations:

- \( k_i k_i^{-1} = k_i^{-1} k_i, k_i k_j = k_j k_i \) for \( i, j \in \mathcal{I} \),
- \( k_i x_{j,k}^{\pm} k_i^{-1} = q_i^{\pm c_{ij}} x_{j,k}^{\pm} \) for \( i, j \in \mathcal{I} \) and \( k \in \mathbb{Z} \),
- \([k_i, h_{j,l}] = [h_{i,l}, h_{j,m}] = 0 \) for \( i, j \in \mathcal{I} \) and \( l, m \in \mathbb{Z}\setminus\{0\} \),
- \([x_i^\pm(z), x_j^\pm(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}} (\delta(z/w)\phi_i^+(w) - \delta(w/z)\phi_i^-(z)) \) for \( i, j \in \mathcal{I} \),
- \((q_i^{\pm c_{ij}} z - w)x_i^\pm(z)x_j^\pm(w) = (z - q_i^{\pm c_{ij}} w)x_j^\pm(w)x_i^\pm(z) \) for \( i, j \in \mathcal{I} \),
- \((q_i^{\pm c_{ij}} z - w)\phi_i^\pm(x_j^\pm(w)) = (z - q_i^{\pm c_{ij}} w)x_j^\pm(w)\phi_i^\pm(z) \) for \( i, j \in \mathcal{I} \) and \( \varepsilon \in \{+, -\} \),
- \[
\sum_{g \in \mathcal{S}_{1-c_{ij}}} \sum_{k=0}^{1-c_{ij}} (-1)^k \left[ \frac{1 - c_{ij}}{k} \right] x_i^\pm(z_{g(1)}) \cdots x_i^\pm(z_{g(k)}) x_j^\pm(w) x_i^\pm(z_{g(k+1)})
\]
\[
\cdots x_i^\pm(z_{g(1-c_{ij})}) = 0 \text{ for } i, j \in \mathcal{I} \text{ with } i \sim j,
\]
where \( \delta(z), x_i^\pm(z), \phi_i^\pm(z) \) are formal power series defined by
\[
\delta(z) := \sum_{l=-\infty}^{\infty} z^l, x_i^\pm(z) := \sum_{l=-\infty}^{\infty} x_{i,k}^\pm z^k, \phi_i^\pm(z) := k_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{l=1}^{\infty} h_{i,\pm l} z^{\pm l} \right).
\]

**Remark 5.3.** Under the isomorphism \( U_q(\mathfrak{g})/\langle q^c - 1 \rangle \cong U_q(Lg) \), the generators \( e_i, f_i, K_i \) correspond to \( x_i^+, x_i^-, k_i \) respectively for each \( i \in \mathcal{I} \). Strictly speaking, the isomorphism depends on the choice of a function \( o: I \to \{\pm 1\} \) such that \( o(i) \neq o(j) \) if \( i \sim j \). In what follows, we choose and fix such a function \( o \), although the results in this paper will not depend on this choice.
Let \( z \) be an indeterminate. For any object \( V \in \mathcal{C} \), we can consider its affinization \( V_z \). This is the \( k[z^\pm] \)-module \( V_z := V \otimes k[z^\pm] \) equipped with a structure of left \( U_q(L_\mathfrak{g}) \)-module by

\[
  k_i(v \otimes f(z)) = (k_i v) \otimes f(z), \quad x_{i,k}^\pm(v \otimes f(z)) = (x_{i,k}^\pm v) \otimes z^k f(z),
\]

\[
  h_{i,l}(v \otimes f(z)) = (h_{i,l} v) \otimes z^l f(z),
\]

for any \( v \in V \) and \( f(z) \in k[z^\pm] \). For a non-zero scalar \( a \in k^\times \), we set \( V_a := V_z/(z - a)V_z \), which is an object of \( \mathcal{C} \). The assignment \( V \mapsto V_a \) defines a monoidal self-equivalence \( T_a \) of \( \mathcal{C} \).

\[5.2. \text{ Simple modules and } q\text{-characters} \]

A complete classification of the simple objects of \( \mathcal{C} \) up to isomorphism was given by Chari–Pressley (see [CP95] or [CP94, Chapter 12]) in terms of the so-called Drinfeld polynomials. Let \( \mathcal{D}^+ := (1 + z k[z])^I \) denote the multiplicative monoid consisting of \( I \)-tuples of polynomials with constant terms 1. This is a commutative monoid freely generated by the elements \( \pi_i, a := ((1 - az)^{b_{ij}})_{j \in I} \) for \( i \in I, \ a \in k^\times \).

**Theorem 5.4 ([CP95]).** For each \( P = (P_i(z))_{i \in I} \in \mathcal{D}^+ \), there exists a simple object \( L(P) \in \mathcal{C} \) with a unique line \( k v \subset L(P) \) such that, in \( L(P)[z, z^{-1}] \), we have

\[
  x_i^+(z) v = 0, \quad \phi_i^\pm(z) v = q_i^{\deg P_i(z)} \left[ \frac{P_i(q_i^{-1} z)}{P_i(z)} \right]^\pm v \quad \text{for each } i \in I,
\]

where \( (f(z))^\pm \) denotes the formal Laurent expansion of the rational function \( f(z) \in k(z) \) at \( z^\pm = 0 \). The correspondence \( P(z) \mapsto L(P(z)) \) gives a bijection between \( \mathcal{D}^+ \) and the set \( \text{Irr } \mathcal{C} \) of isomorphism classes of simple objects of \( \mathcal{C} \).

We call the vector \( v \) the \( \ell \)-highest weight vector of \( L(P) \), which is unique up to multiplication by \( k^\times \). By the above characterization, we have \( L(P)_a \cong L(P_a) \) with \( P_a := (P_i(az))_{i \in I} \).

For \( (i, a) \in I \times k^\times \) and \( l \in \mathbb{Z}_{\geq 0} \), we set

\[
  \pi_{l,a}^{(j)} := \prod_{k=0}^{l-1} \pi_{i_aq_i^2k} \in \mathcal{D}^+.
\]

The corresponding simple modules \( L(\pi_{l,a}^{(j)}) \) are called Kirillov–Reshetikhin (KR) modules. In particular, when \( l = 1 \), the modules \( L(\pi_{i,a}) \) are called fundamental modules.

Next we recall the notion of \( q\text{-characters} \) introduced by Frenkel–Reshetikhin [FR99]. Let \( \mathcal{D} \) denote the multiplicative group of \( I \)-tuples of rational functions \( \Psi = (\Psi_i(z))_{i \in I} \in k(z)^I \) satisfying \( \Psi_i(1) = 1 \) for all \( i \in I \). Note that \( \mathcal{D} \) is naturally identified with the Grothendieck group of \( \mathcal{D}^+ \). By [FR99, Proposition 1], for any object \( V \in \mathcal{C} \), we have a decomposition

\[
  V = \bigoplus_{\Psi \in \mathcal{D}} V_\Psi,
\]

where \( V_\Psi \) is the subspace of \( V \) on which the coefficient of \( z^k \) in the series \( \phi_i^\pm(z) - q_i^{\deg \Psi_i(z)} \left[ \Psi_i(q_i^{-1} z)/\Psi_i(z) \right]^\pm \) acts nilpotently for every \( k \in \mathbb{Z}_{\geq 0} \). Here we define
deg\( (f(z)/g(z)) := \deg f(z) - \deg g(z) \) for \( f(z) \in \mathbb{k}[z], g(z) \in \mathbb{k}[z]\setminus\{0\} \). Then the \( q \)-character \( \chi_q(V) \) of \( V \) is defined to be
\[
\chi_q(V) := \sum_{\Psi \in \mathcal{D}} \dim_k(V_{\Psi})[\Psi],
\]
which is an element of the group ring \( \mathbb{Z}[\mathcal{D}] = \bigoplus_{\Psi \in \mathcal{D}} \mathbb{Z}[\Psi] \). It is known that the assignment \( V \mapsto \chi_q(V) \) defines an injective ring homomorphism \( \chi_q : K(\mathcal{C}) \hookrightarrow \mathbb{Z}[\mathcal{D}] \), where \( K(\mathcal{C}) \) is the Grothendieck ring of \( \mathcal{C} \) [FR99, Corollary 2].

Finally, we recall the behaviour of the duality functors \( \hat{\mathcal{D}}^{\pm} \). Denote by \( \mathbb{D} \) the automorphism of the group \( \mathcal{D} \) given by \( \mathbb{D}(\pi_{i,a}) := \pi_{i,a}q^{-1} \) for all \( i \in I \) and \( a \in \mathbb{k}^\times \). It naturally extends to a ring automorphism of \( \mathbb{Z}[\mathcal{D}] \).

**Proposition 5.5** (cf. [CP96a, Proposition 5.1(b)], [AK97, (A.5)], [FM01, Corollary 6.10]) For each \( P \in \mathcal{D}^+ \) and \( k \in \mathbb{Z} \), we have
\[
\hat{\mathcal{D}}^k(L(P)) \cong L(\mathbb{D}^k(P))
\]
as \( U_q(\hat{\mathfrak{g}}) \)-modules. Moreover, for any \( V \in \mathcal{C} \), we have
\[
\chi_q(\hat{\mathcal{D}}^k(V)) = \mathbb{D}^k(\chi_q(V)).
\]

### 5.3. The Hernandez–Leclerc category \( \mathcal{C}^0 \)

Let us consider the Laurent polynomial ring
\[
\mathcal{Y} = \mathbb{Z}[Y_{i,p}^{\pm 1} \mid (i, p) \in \hat{T}],
\]
where the set \( \hat{T} \) is defined as in Sect. 3.8. We denote by \( \mathcal{M}^+ \) (resp. \( \mathcal{M} \)) the set of all monomials (resp. Laurent monomials) in the variables \( Y_{i,p} \). Note that \( \mathcal{M}^+ \) is a commutative monoid freely generated by the set \( \{Y_{i,p} \mid (i, p) \in \hat{T}\} \), and \( \mathcal{M} \) is the Grothendieck group of \( \mathcal{M}^+ \). In what follows, we regard \( \mathcal{M} \) as a subgroup of \( \mathcal{D} \) via the injective homomorphism \( \mathcal{M} \hookrightarrow \mathcal{D} \) given by \( Y_{i,p} \mapsto \pi_{i,a}q^b \).

**Definition 6.6** ([HL10,HL16,KKO19]). The category \( \mathcal{C}^0 \) is defined to be the Serre subcategory of \( \mathcal{C} \) such that \( \text{Irr} \mathcal{C}^0 = \{L(m) \mid m \in \mathcal{M}^+\} \).

**Remark 5.7.** The category \( \mathcal{C}^0 \) was introduced in [HL10, Section 3.7] and [HL16] for untwisted affine types, and was extended to other affine types in [KKO19] by using the denominator formulas between fundamental modules. The descriptions for \( \mathcal{C}^0 \) given in [HL10,HL16,KKO19] are different, but coincide for untwisted affine types since the denominator formulas are meaningful only for the pair of fundamental modules \( L(Y_{i,p}) \) and \( L(Y_{j,s}) \) when \( (i, p), (j, s) \in \hat{T} \) (see Sect. 6 below).

The subcategory \( \mathcal{C}^0 \) captures an essential part of the monoidal category \( \mathcal{C} \) in the following sense.

**Theorem 5.8** (cf. [HL10, Section 3.7], [HL16, Lemma 3.8]). The followings hold.

1. The full subcategory \( \mathcal{C}^0 \subset \mathcal{C} \) is closed under the tensor product \( \otimes \) and duals \( \hat{\mathcal{D}}^{\pm 1} \). Moreover, the \( q \)-character homomorphism \( \chi_q \) restricts to the injective ring homomorphism
\[
\chi_q : K(\mathcal{C}^0) \hookrightarrow \mathcal{Y} = \mathbb{Z}[\mathcal{M}].
\]
2. Any simple object $V \in \text{Irr} \mathcal{C}$ decomposes as

$$V \cong (V_1)_{a_1} \otimes \cdots \otimes (V_m)_{a_m}$$

for some $m \in \mathbb{Z}_{\geq 1}$ and $\{(V_k, a_k) \in \text{Irr} \mathcal{C}^0 \times \mathbb{K}^\times \mid 1 \leq k \leq m\}$ such that $a_k/a_l \notin q_s^{2\mathbb{Z}}$ for $1 \leq k < l \leq m$. Moreover, in this case, the modules $\{(V_k)_{a_k} \mid 1 \leq k \leq m\}$ pairwise commute.

In the sequel, we shall focus on the category $\mathcal{C}^0$ rather than $\mathcal{C}$.

5.4. The category $\mathcal{C}_Q$ Let us fix a Q-datum $Q$ for $\mathfrak{g}$. Recall the bijection $\bar{\phi}_Q : \hat{\mathcal{I}} \rightarrow \hat{\mathbb{R}}^+$ and its restriction $\bar{\phi}_{Q,0} : \hat{\mathcal{I}}_Q \rightarrow \mathbb{R}^+$ defined in Sect. 3.8. Following [HL15,OS19d,OS19a], we set

$$V_Q(\alpha) := L(Y_{i,p}), \quad \text{if } \bar{\phi}_Q^{-1}(\alpha) = (i,p) \in \hat{\mathcal{I}}_Q$$

for each $\alpha \in \mathbb{R}^+$. Then the category $\mathcal{C}_Q$ is defined to be the smallest monoidal Serre subcategory of $\mathcal{C}_0$ containing the collection of fundamental modules $\{V_Q(\alpha) \mid \alpha \in \mathbb{R}^+\} = \{L(Y_{i,p}) \mid (i,p) \in \hat{\mathcal{I}}_Q\}$. It turns out that we have $\text{Irr} \mathcal{C}_Q = \{L(m) \mid m \in \mathcal{M}_Q^+\}$, where $\mathcal{M}_Q^+$ is the set of all the monomials in the variables $Y_{i,p}$ with $(i,p) \in \hat{\mathcal{I}}_Q$ (see [HL15, Lemma 5.8] and [HO19, Lemma 3.26]).

The category $\mathcal{C}_Q$ can be seen as a “heart” of the category $\mathcal{C}_0$ due to the following property.

**Proposition 5.9.** For any $(\alpha, k) \in \hat{\mathbb{R}}^+$, we have

$$D^k(V_Q(\alpha)) \cong L(Y_{i,p}), \quad \text{where } (i,p) = \bar{\phi}_Q^{-1}(\alpha, k).$$

In particular, the set $\{D^k(V_Q(\alpha)) \mid (\alpha, k) \in \hat{\mathbb{R}}^+\}$ forms a complete and irredundant collection of fundamental modules in $\mathcal{C}_Q^0$ up to isomorphisms.

**Proof.** It follows from Proposition 3.44. $\square$

5.5. Q-weights and block decomposition For a Laurent monomial $m \in \mathcal{M}$, we write

$$m = \prod_{(i,p) \in \hat{\mathcal{I}}} Y_{i,p}^{u_{i,p}(m)}$$

with $u_{i,p}(m) \in \mathbb{Z}$. We define the map $h : \hat{\mathbb{R}}^+ \rightarrow \mathbb{Q}$ by $h(\alpha, k) := (-1)^k \alpha$.

**Definition 5.10** (cf. [KKOP20a]). Let $Q$ be a Q-datum for $\mathfrak{g}$ and $m \in \mathcal{M}$ be a Laurent monomial. With the above notation, we define the $Q$-weight of $m$ by

$$\text{wt}_Q(m) = \sum_{(i,p) \in \hat{\mathcal{I}}} u_{i,p}(m) h(\bar{\phi}_Q(i,p)) \in \mathbb{Q}.$$  

Note that the $Q$-weights belong to the root lattice $\mathbb{Q}$ of the simply-laced Lie algebra $\mathfrak{g}$ rather than that of $\mathfrak{g}$. The assignment $m \mapsto \text{wt}_Q(m)$ defines the group homomorphism $\text{wt}_Q : \mathcal{M} \rightarrow \mathbb{Q}$ and equips the Laurent polynomial ring $\mathcal{Y} = \mathbb{Z}[\mathcal{M}]$ with a $\mathbb{Q}$-grading.
We shall see that the simple $q$-characters are homogeneous with respect to these $Q$-gradings. Let $\widehat{I}_{+(D)} := \{(i, p) \in I \times \mathbb{Z} \mid (i, p - d_i) \in \hat{I}\}$. For each $(i, p) \in \widehat{I}_{+(D)}$, following [FR99], we define the Laurent monomial $A_{i, p} \in \mathcal{M}$ by

$$A_{i, p} := Y_{i, p - d_i}^{-1} Y_{i, p + d_i} \left( \prod_{j : c_{ji} = -1} Y_{j, p}^{-1} \right) \left( \prod_{j : c_{ji} = -2} Y_{j, p - 1}^{-1} Y_{j, p + 1}^{-1} \right) \left( \prod_{j : c_{ji} = -3} Y_{j, p - 2}^{-1} Y_{j, p - 1}^{-1} Y_{j, p + 2}^{-1} \right).$$  

(5.2)

By the $g$-additive property in (3.15) and the property (3.18), one can easily check

$$\text{wt}_Q(A_{i, p}) = 0$$  

(5.3)

for any $(i, p) \in \widehat{I}_{+(D)}$ and a $Q$-datum $Q$ for $g$. Recall the following important result.

**Theorem 5.11** ([FM01]). For each dominant monomial $m \in \mathcal{M}^+$, the $q$-character of the corresponding simple module $L(m) \in \mathcal{C}^0$ is of the form

$$\chi_q(L(m)) = m \left( 1 + \sum_k M_k \right)$$

where each $M_k$ is a monomial in the variables $A_{i, p}^{-1}$ with $(i, p) \in \widehat{I}_{+(D)}$.

By (5.3) and Theorem 5.11, we immediately obtain the following.

**Proposition 5.12.** For any dominant monomials $m \in \mathcal{M}^+$ and a monomial $m'$ occurring in $\chi_q(L(m))$ and any $Q$-datum $Q$ for $g$, we have

$$\text{wt}_Q(m) = \text{wt}_Q(m').$$

In other words, the $q$-character $\chi_q(L(m))$ is a homogeneous element of $\mathcal{Y}$ with respect to the $Q$-grading given by the $Q$-weight.

**Definition 5.13.** Let $Q$ be a $Q$-datum for $g$. For a simple module $V$ in $\mathcal{C}^0$, we set

$$\text{wt}_Q(V) := \text{wt}_Q(m) \quad \text{for any } m \in \mathcal{M} \text{ occurring in } \chi_q(V).$$

This is well-defined thanks to Proposition 5.12.

Let $\mathcal{M}_A \subset \mathcal{M}$ denote the subgroup generated by the elements $A_{i, p}$ with $(i, p) \in \widehat{I}_{+(D)}$ and consider the quotient group $\mathcal{M}/\mathcal{M}_A$, which was initially introduced by [CM05].

**Proposition 5.14.** For each $Q$-datum $Q$ for $g$, the homomorphism $\text{wt}_Q : \mathcal{M} \rightarrow \mathbb{Q}$ induces an isomorphism of abelian groups

$$\mathcal{M}/\mathcal{M}_A \simeq \mathbb{Q}.$$  

(5.4)

**Proof.** By Remark 5.3, the map $\text{wt}_Q$ induces a surjective homomorphism $\text{wt}_Q : \mathcal{M}/\mathcal{M}_A \rightarrow \mathbb{Q}$. Note that the quotient group $\mathcal{M}/\mathcal{M}_A$ is generated by the classes of the elements $\{Y_{i, \xi_i} \mid i \in \Delta_0\}$. On the other hand, we know that their images $\{g^Q_t = \text{wt}_Q(Y_{i, \xi_i}) \mid t \in \Delta_0\}$ form a free basis of $\mathbb{Q}$ by Lemma 3.36. Therefore $\text{wt}_Q$ gives an isomorphism $\mathcal{M}/\mathcal{M}_A \simeq \mathbb{Q}$.  \[\square\]
Under the isomorphism (5.4) $\text{wt}_Q : \mathcal{M}/\mathcal{M}_A \simeq \mathcal{Q}$, the element $\text{wt}_Q(V)$ is identical to the elliptic character of the simple module $V$ in the sense of [CM05]. Therefore, we can re-express the main result of [CM05] in terms of the $\mathcal{Q}$-weight as follows.

**Theorem 5.15** ([CM05, Theorem 8.3]). Fix a $\mathcal{Q}$-datum $\mathcal{Q}$ for $\mathfrak{g}$. For each $\alpha \in \mathcal{Q}$ let $\mathcal{C}^0_\alpha$ denote the Serre subcategory of $\mathcal{C}^0$ generated by all the simple modules $V$ satisfying $\text{wt}_Q(V) = \alpha$. Then we have a block decomposition

$$\mathcal{C}^0 = \bigoplus_{\alpha \in \mathcal{Q}} \mathcal{C}^0_\alpha.$$

In particular, for any indecomposable module $V \in \mathcal{C}^0$, its $q$-character $\chi_q(V) \in \mathcal{Y}$ is homogeneous with respect to $\text{wt}_Q$ and hence $\text{wt}_Q(V)$ is well-defined.

**Remark 5.16.** The block decomposition by the $\mathcal{Q}$-weight in Theorem 5.15 turns out to be the same as the block decomposition recently obtained in [KKOP20a], which investigates all quantum affine algebras. See Remark 6.17 below. In this sense, our description connects the result of [CM05] with that of [KKOP20a].

**Lemma 5.17.** Let $\mathcal{Q}$ be a $\mathcal{Q}$-datum for $\mathfrak{g}$. For any indecomposable module $V \in \mathcal{C}^0$ and $k \in \mathbb{Z}$, we have

$$\text{wt}_Q(D^k V) = (-1)^k \text{wt}_Q(V).$$

**Proof.** It follows from Proposition 5.5 and Proposition 3.44. \qed

5.6. **The skew-symmetric pairing $\mathcal{N}$** In this subsection, we recall a $\mathbb{Z}$-valued skew-symmetric pairing $\mathcal{N}$ on the group $\mathcal{M}$ introduced by Hernandez [Her04] in his construction of the quantum Grothendieck ring (see Remark 5.20 below). Here we relate it with the $\mathcal{Q}$-weight defined in the last subsection.

**Definition 5.18** ([Her04]). Recall the notation (5.1). We define the group bi-homomorphism $\mathcal{N} : \mathcal{M} \times \mathcal{M} \to \mathbb{Z}$ by setting

$$\mathcal{N}(m; m') := \sum_{(i, p); (j, s) \in \hat{I}} u_{i, p}(m)u_{j, s}(m')\mathcal{N}(i, p; j, s),$$

for any $m, m' \in \mathcal{M}$, where

$$\mathcal{N}(i, p; j, s) := \tilde{c}_{ij}(p - s - d_i) - \tilde{c}_{ij}(p - s + d_i) - \tilde{c}_{ij}(s - p - d_i) + \tilde{c}_{ij}(s - p + d_i).$$

For simple modules $V, W$ in $\mathcal{C}^0$, we set

$$\mathcal{N}(V, W) := \mathcal{N}(m; m') \quad \text{if } V \cong L(m), W \cong L(m') \text{ with } m, m' \in \mathcal{M}^+. $$

**Remark 5.19.** For any $(i, p), (j, s) \in \hat{I}$, we have

$$\mathcal{N}(i, p; j, s) = -\mathcal{N}(j, s; i, p)$$

by Lemma 4.4 (3). Therefore the pairing $\mathcal{N}$ is skew-symmetric. Moreover, we have

$$\mathcal{N}(i, p; j, s) = \tilde{c}_{ij}(p - s - d_i) - \tilde{c}_{ij}(p - s + d_i) \quad \text{if } p - s \geq \delta_{i, j},$$

by Lemma 4.3.
Remark 5.20. Let \( t \) be an indeterminate with a formal square root \( t^{1/2} \). The skew-symmetric pairing \( \mathcal{N} \) was introduced by Hernandez [Her04] to define the quantum Grothendieck ring \( K_t(\mathcal{C}) \), which is a \( t \)-deformation of the Grothendieck ring \( K(\mathcal{C}) \). The ring \( K_t(\mathcal{C}) \) is constructed as a \( \mathbb{Z}[t^{\pm 1}] \)-subalgebra of the quantum torus \( (\mathcal{Y}_t, \ast) \) that is \( \mathcal{Y}_t := \mathbb{Z}[t^{\pm 1}][\mathcal{M}] \cong \mathbb{Z}[t^{\pm 1/2}] \otimes \mathbb{Z} \mathcal{Y} \) with the multiplication \( \ast \) given by

\[
m \ast m' = t^{\mathcal{N}(m;m')} m' \ast m := t^{\mathcal{N}(m;m')/2} m m'
\]

for \( m, m' \in \mathcal{M} \). It satisfies that \( \text{ev}_{t=1}(K_t(\mathcal{C})) = \chi_q(K(\mathcal{C})) \), where \( \text{ev}_{t=1} : \mathcal{Y}_t \to \mathcal{Y} \) is the evaluation map at \( t = 1 \).

For each simple module \( L(m) \in \mathcal{C} \) with \( m \in \mathcal{M}^+ \), Hernandez [Her04] further constructed its \((q, t)\)-character \( [L(m)]_t \) as an element of \( K_t(\mathcal{C}) \), which contains \( m \) as the leading term, by means of an analog of Kazhdan–Lusztig algorithm. It was conjectured that \( \text{ev}_{t=1}([V]_t) = \chi_q(V) \) holds for any simple module \( V \) in \( \mathcal{C} \) ([Her04, Conjecture 7.3]). Note that we have

\[
[V]_t \ast [W]_t = t^{\mathcal{N}(V,W)} [W]_t \ast [V]_t
\]

in \( K_t(\mathcal{C}) \) if \([V]_t \) and \([W]_t \) commute up to a power of \( t^{\pm 1/2} \).

Proposition 5.21. Let \((i, p), (j, s) \in \widehat{T} \). If \( \bar{\phi}_Q(i, p) = (\alpha, k) \) and \( \bar{\phi}_Q(j, s) = (\beta, l) \), we have

\[
\mathcal{N}(i, p; j, s) = (-1)^{\delta(p \geq s) + kl} \delta((\alpha, k) \neq (\beta, l))(\alpha, \beta).
\]

Proof. Since the both sides of the desired equality are skew-symmetric, we only have to consider the case \((i, p) \neq (j, s) \) and \( p \geq s \). Take \( i \in I \) and \( j \in J \). By Theorem 4.8 and (5.7), we have

\[
\mathcal{N}(i, p; j, s) = -(\varpi_i, \tau_Q^{(p-s+\xi_j-\xi_i)/2}(\gamma_j^Q) - \tau_Q^{(p-s+\xi_j-\xi_i)/2-d_i}(\gamma_j^Q)) \nonumber
\]

\[
= -(\tau_Q^{(\xi_i-p)/2}(1 - \tau_Q^{d_i})(\varpi_i, \tau_Q^{(\xi_j-s)/2}(\gamma_j^Q)) \nonumber
\]

\[
= -(\tau_Q^{(\xi_i-p)/2}(\gamma_i^Q), \tau_Q^{(\xi_j-s)/2}(\gamma_j^Q)) \nonumber
\]

\[
= (-1)^{1+kl}(\alpha, \beta),
\]

where the last equality follows from (3.18). This completes the proof. \( \square \)

Corollary 5.22. Let \( m, m' \in \mathcal{M} \) such that

\[
\min\{p \mid \exists i \in I, u_{i,p}(m) \neq 0\} > \max\{p \mid \exists i \in I, u_{i,p}(m') \neq 0\}.
\]

Then, for any \( Q \)-datum \( Q \) for \( g \), we have

\[
\mathcal{N}(m; m') = -(\text{wt}_Q(m), \text{wt}_Q(m')).
\]

6. R-matrices and Related Invariants

In this section, we present a conjectural unified formulae of the denominators of the normalized R-matrices between all the Kirillov–Reshetikhin modules. We also compute the \( \Lambda \)-invariants introduced by [KKOP20d] in terms of the \( Q \)-weights and the skew-symmetric pairing \( \mathcal{N} \) discussed in Sect. 5.

Convention. Throughout this section, we keep the following convention:

1. For \( f(z), g(z) \in \mathbb{k}(\mathbb{z}) \), we write \( f(z) \equiv g(z) \) if \( f(z)/g(z) \in \mathbb{k}[\mathbb{z}^{\pm 1}]^\times \).
2. For \( f(z) \in \mathbb{k}(\mathbb{z}) \) and \( a \in \mathbb{k} \), we denote by \( \text{zero}_{z=a}(f(z)) \) the order of zero of \( f(z) \) at \( z = a \).
6.1. R-matrices In this subsection, we recall the notion of R-matrices between $U'_q(\hat{\mathfrak g})$-modules together with their denominators and universal coefficients following [Kas02, Section 8] and [AK97, Appendix A]. Choose a basis $\{E_v\}_v$ of $U'_q(\hat{\mathfrak g})$ and a basis $\{F_v\}_v$ of $U_q^- (\hat{\mathfrak g})$ (with $v$ running over a certain index set) which are dual to each other with respect to a suitable pairing between $U_q^+(\hat{\mathfrak g})$ and $U_q^- (\hat{\mathfrak g})$. For $U_q(L_{\mathfrak g})$-modules $V$ and $W$ of type $\mathfrak g$, the universal R-matrix defines a $U_q(L_{\mathfrak g})$-linear homomorphism $V \otimes W \to W \otimes V$ by

$$R_{V,W}^{\text{univ}}(v \otimes w) = q^{\langle \lambda, \mu \rangle} \sum_v E_v w \otimes F_v v$$

provided that the infinite sum has a meaning, where $v \in V$, $w \in W$ and $\lambda = (\lambda_i)_{i \in I}$, $\mu = (\mu_i)_{i \in I} \in \mathbb{Z}^I$ are such that $K_i v = q^{\lambda_i} v$, $K_i w = q^{\mu_i} w$ for $i \in I$. Here we set $\langle \lambda, \mu \rangle : = \sum_{i,j \in I} d_{ij} \lambda_i \mu_j$. Note that we have $\langle \lambda, \mu \rangle = \langle \mu, \lambda \rangle$ by (3.4).

Recall the affinization $W_z$ of an object $W \in \mathcal{C}$ defined in Sect. 5.1. For any objects $V, W \in \mathcal{C}$, it is known that $R_{V,W}^{\text{univ}}$ converges in the $z$-adic topology and induces a $U_q(L_{\mathfrak g}) \otimes \mathbb{k}(z)$-linear isomorphism

$$R_{V,W}^{\text{univ}} : (V \otimes W_z) \otimes \mathbb{k}[z^{\pm 1}] \mathbb{k}(z) \sim (W_z \otimes V) \otimes \mathbb{k}[z^{\pm 1}] \mathbb{k}(z).$$

Moreover, if $V, W$ are simple modules with $\ell$-highest weight vectors $v \in V$, $w \in W$, there exists a unique element $a_{V,W}(z) \in \mathbb{k}[[z]]^\times$ such that

$$R_{V,W}^{\text{univ}}(v \otimes w_z) = a_{V,W}(z)(w_z \otimes v),$$

where $w_z := w \otimes 1 \in W_z$. Then $R_{V,W}^{\text{norm}} := a_{V,W}(z)^{-1} R_{V,W}^{\text{univ}} |_{(V \otimes W_z) \otimes \mathbb{k}[z^{\pm 1}] \mathbb{k}(z)}$ induces a unique $U_q(L_{\mathfrak g}) \otimes \mathbb{k}(z)$-linear isomorphism

$$R_{V,W}^{\text{norm}} : (V \otimes W_z) \otimes \mathbb{k}[z^{\pm 1}] \mathbb{k}(z) \sim (W_z \otimes V) \otimes \mathbb{k}[z^{\pm 1}] \mathbb{k}(z)$$

satisfying $R_{V,W}^{\text{norm}}(v \otimes w_z) = w_z \otimes v$. We call $a_{V,W}(z)$ (resp. $R_{V,W}^{\text{norm}}$) the universal coefficient (resp. the normalized R-matrix) between $V$ and $W$.

Let $d_{V,W}(z) \in \mathbb{k}[z]$ be a monic polynomial of the smallest degree such that the image of $d_{V,W}(z) R_{V,W}^{\text{norm}}$ is contained in $W_z \otimes V$. We call $d_{V,W}(z)$ the denominator of $R_{V,W}^{\text{norm}}$.

The singularities of the normalized R-matrices strongly reflect the structure of the tensor product modules. For example, we have the following proposition. Here we say that a simple module $V$ in $\mathcal{C}$ is real if its tensor square $V \otimes V$ is again simple. For instance, every Kirillov–Reshetikhin module is known to be real (see e.g. [FH15, proof of Theorem 4.11]).

**Proposition 6.1** ([KKKO15, Section 3.2]). Let $V, W \in \text{Irr} \mathcal{C}$ such that at least one of them is real. Then the following three conditions are mutually equivalent:

- The tensor product $V \otimes W$ is a simple $U'_q(\hat{\mathfrak g})$-module.
- The objects $V$ and $W$ mutually commute.
- We have $d_{V,W}(1) \cdot d_{W,V}(1) \neq 0$.

The orders of zeros of the denominators $d_{V,W}(z)$ also play an important role especially in the theory of monoidal categorification of cluster algebras and the construction of the generalized quantum affine Schur–Weyl duality functors. See [KKOP20d, KKK18] for more details.

The following properties are well-known.
Lemma 6.2 (cf. [AK97, Appendix A]). Let $V, W$ be two simple modules in $C$ and $a, b \in k^\times$. We have

\begin{enumerate}
\item $a_{V_a, W_b}(z) = d_{V, W}(z/b)z$, and $d_{V_a, W_b}(z) = d_{V, W}(z/b)z$.
\item $a_{V, W}(z) = a_{V, W}(z) = d_{V, W}(z) = d_{V, W}(z)$.
\end{enumerate}

6.2. A conjectural unified KR denominator formula In this subsection, we give a conjectural unified formula expressing the denominators of the normalized R-matrices between all the KR modules.

For each $i \in I$ and $l \in \mathbb{Z}_{\geq 1}$, we write

$$V_l^{(i)} := L\left(\pi_l^{(i)}, q_i^{l-1}\right), \quad \text{where } \pi_l^{(i)} = \pi_{l, q_i^{l-1}} = \pi_{l, q_i^{l-1}} = \cdots = \pi_{l, q_i^{l-1}}.$$ 

Note that every KR module can be obtained as $(V_l^{(i)})_a$ for a suitable $a \in k^\times$. When $l = 1$, it gives a fundamental module $V_1^{(i)} = L(\pi_i, 1)$. In view of 6.2 (1), it is enough to consider the denominators

$$d_{i, j}^m(z) := d_{V_l^{(i)}, V_m^{(j)}}(z) \quad (6.1)$$

for $i, j \in I$ and $l, m \in \mathbb{Z}_{\geq 1}$. We also write $d_{i, j}^m(z) := d_{i, j}^m(z)$ and $d_{i, j}(z) := d_{i, j}(z)$ for simplicity.

These denominators $d_{i, j}(z)$ have been computed by [DO94, AK97, KKK15, Oh15, OS19a, Fuj19] and $d_{i, j}^m(z)$ have been computed in many cases by [OS19b].

In Appendix A, we give a list of all the formulae of $d_{i, j}^m(z)$ which are currently known.

Remark 6.3. We have some remarks on the convention. Our fundamental module $V_1^{(i)}$ is slightly different from the fundamental module $V(\pi_i)$ in the references e.g., [AK97, Kas02, KKP20d], which possesses a global basis. Indeed, it was shown by Nakajima [Nak04a, Section 3.1] that we have

$$V(\pi_i) = L(\pi_i, a_i) \equiv (V_1^{(i)})_{a_i}, \quad \text{with } a_i := -o(i)(-1)^{h} q^{-h}.$$ 

(6.2)

where $h$ is the Coxeter number of $\mathfrak{g}$ and $o : I \rightarrow \{\pm 1\}$ is as in Remark 5.3.

Similarly, our KR modules $V_l^{(i)}$ is slightly different from the KR modules $V(i_l)$ appearing in [OS19b]. Indeed, we have

$$V(i_l) = L\left(\pi_i^{(i_l)}, q_i^{l-1}\right) = (V_l^{(i)})_{(-1)^{l-1} a_i}.$$ 

(6.3)

For $i, j \in I$, we have $a_i/a_j = o(i)/o(j) = (-1)^{d(i, j)}$, where $d(i, j)$ denotes the distance between $i$ and $j$ in the Dynkin diagram of $\mathfrak{g}$. Thus, Lemma 6.2 together with (6.3) implies

$$d_{i, j}^m(z) = d_{V(i_l), V(j_m)}((-1)^{d(i, j)+l+m} z) \quad (6.4)$$

for each $i, j \in I$ and $l, m \in \mathbb{Z}_{\geq 1}$. One should notice that the same symbol $d_{i, j}^m(z)$ is used in [OS19b] for denoting $d_{V(i_l), V(j_m)}(z)$, which is different from our convention. The case-by-case denominator formulae listed in Appendix A have been rewritten from the original formulae of $d_{V(i_l), V(j_m)}(z)$ by using (6.4). They turn out to be obtained simply by forgetting signs appearing in the original ones.
Lemma 6.4. For any $i, j \in I$, we have

$$d_{i,j}(z) = d_{j,i}(z) = d_{i^*, j^*}(z).$$

Proof. By [AK97, (A.7)], we know $d_{V(\sigma i), V(\sigma j)}(z) = d_{V(\sigma j), V(\sigma i)}(z)$. Combining with Lemma 6.2 (1), we get $d_{i,j}((a_j/a_i)z) = d_{j,i}((a_i/a_j)z)$. Since $a_j/a_i = o(i)o(j) = a_i/a_j$, we obtain the equality $d_{i,j}(z) = d_{j,i}(z)$. The other equality $d_{i,j}(z) = d_{i^*, j^*}(z)$ follows from Lemma 6.2 (2) and Proposition 5.5. $\square$

Note that the case-by-case formulae of $d_{i,l^m}(z)$ listed in Appendix A are expressed in a symmetric way under the exchange $l \leftrightarrow m$. In what follows, we obtain more unified formulae expressed in terms of the integers $\tilde{c}_{ij}(u)$, which however break such a symmetry. First, we observe the following. Recall the notation $q_s := q^{1/r}$ and $r_i = r/d_i$.

Proposition 6.5. For $i, j \in I$, we have

$$d_{r_i, j}(z) = \frac{r}{h} \prod_{u=0}^{r_h} (z - q_s^{u+r} \tilde{c}_{ij}(u)). \quad (6.5)$$

Proof. When $g$ is simply-laced, this is [Fuj19, Theorem 2.10] (see (A.1)). When $g$ is not simply-laced, we can check the assertion directly by comparing the known explicit formulae of $d_{r_i, j}(z)$ as in Appendix A with the explicit values of $\tilde{C}(z)$ computed in Sect. 4.3. $\square$

Remark 6.6. After the authors first uploaded this paper to arXiv, the second named author and T. Scrimshaw computed all the denominators $d_{r_i, j}(z)$ of type $\mathbf{F}_4$, and proved that Proposition 6.5 holds also for $\mathbf{F}_4$. The result will be dealt with in the near future (see Appendix A below).

Now we propose the following more general formula.

Conjecture 6.7 (A unified KR denominator formula). Let $i, j \in I$ and $l, m \in \mathbb{Z}_{\geq 1}$ such that $ld_i \geq md_j$. Then we have

$$d_{l^i, j^m}(z) = d_{j^m, i^l}(z) = \prod_{k=0}^{m-1} \prod_{u=0}^{r_h} \left(z - q_s^{u+ld_i+(2k-m+1)d_j} \tilde{c}_{ij}(u)\right) \quad (6.6)$$

with the following three exceptions:

(\text{EX1}) $g$ is of type $\mathbf{C}_n$, $d_i = d_j = 1$ and $l = m = 1$,

(\text{EX2}) $g$ is of type $\mathbf{F}_4$, $d_i = d_j = 1$ and $l = m = 1$,

(\text{EX3}) $g$ is of type $\mathbf{G}_2$, $d_i = d_j = 1$ and $l = m \in \{1, 2\}$.

Remark 6.8. Note that the three exceptions (EX1), (EX2), (EX3) occur only if $ld_i = md_j < r$. Even in these exceptional cases, the denominator $d_{l^i, j^m}(z) = d_{j^m, i^l}(z)$ always divides the RHS of (6.6).

As an evidence for Conjecture 6.7, we give a partial result here.

Theorem 6.9. Conjecture 6.7 holds when $g$ is neither of type $\mathbf{E}_{6,7,8}$ nor of type $\mathbf{F}_4$. 
Proof. Here we give a case-by-case proof based on Proposition 6.5 and the results of [OS19b]. Since the equality $d_{i^*,jm}(z) = d_{jm,i^*}(z)$ is already understood by [OS19b, Lemma 2.2] (see also (6.4) above), we only have to prove that $d_{i^*,jm}(z)$ is equal to the RHS of (6.6). We shall provide a detailed proof only for type $C_n$, since the other cases are quite similar or easier.

Now we suppose $g$ is of type $C_n$. We use the labeling $I = \{1, \ldots, n\}$ as in (3.2b). In the case $1 \leq i, j < n$, we have $d_i = d_j = 1$ and $r_i = 2$. The condition $ld_i \geq md_j$ implies $l \geq m$. When $(l, m) = (2, 1)$, we have

$$d_{i^*,jm}(z) = \prod_{u=0}^{2h\vee} (z - q_s^{u+2}) \tilde{c}_{ij}(u) = \prod_{u=1}^{\min(i,j)} (z - q_s^{(i-j)+1+2u})(z - q_s^{2n+3-i-j+2u})$$

by Proposition 6.5 and (A.6). For general $l, m$ with $l \geq m$ and $l \geq 2$, we have

$$d_{i^*,jm}(z) = \prod_{k=0}^{m-1} \prod_{u=1}^{\min(i,j)} (z - q_s^{(i-j)+(l-m)+2(u+k)})(z - q_s^{2n+2-i-j+(l-m)+2(k+u)})$$

(by (A.6))

$$= \prod_{k=0}^{m-1} \prod_{u=0}^{\min(i,j)} (z - q_s^{(i-j)+1+2u-1+l+2k-m})(z - q_s^{2n+3-i-j+2u-1+l+2k-m})$$

$$= \prod_{k=0}^{m-1} (z - q_s^{u+2-1+l+2k-m}) \tilde{c}_{ij}(u)$$

(by (6.7))

as desired.

In the case $i < j = n$, we have $d_i = 1$ and $r_i = d_j = 2$. The condition $ld_i \geq md_j$ implies $l \geq 2m$. When $(l, m) = (2, 1)$, we have

$$d_{i^*,jn}(z) = \prod_{u=0}^{2h\vee} (z - q_s^{u+2}) \tilde{c}_{ia}(u) = \prod_{k=0}^{i} (z - q_s^{n+1-i+2k+2u})$$

by Proposition 6.5 and (A.7). For general $l, m$ with $l \geq 2m$, we have

$$d_{i^*,jn}(z) = \prod_{k=0}^{m-1} \prod_{u=1}^{i} (z - q_s^{n+1-i+(l-2m)+2k+2u})$$

(by (A.7))

$$= \prod_{k=0}^{m-1} \prod_{u=0}^{1} (z - q_s^{n+1-i+2k'+2u}) \tilde{c}_{ia}(u)$$

(replace $k$ with $2k + k'$)

(by (6.8))

$$= \prod_{k=0}^{m-1} (z - q_s^{u+2+l+4k-2m}) \tilde{c}_{ia}(u)$$

(by (6.8))

$$= \prod_{k=0}^{m-1} (z - q_s^{u+l+2(2k-m+1)}) \tilde{c}_{ia}(u),$$
as desired.

In the case \( i = n > j \), we have \( d_i = 2 \) and \( r_i = d_j = 1 \). The condition \( ld_i \geq md_j \) implies \( 2l \geq m \). When \( (l, m) = (1, 1) \), we have

\[
d_{n, j}(z) = \prod_{u=0}^{2h^\vee} (z - q_s^{u+2} \tilde{\gamma}_{nj}(u)) = \prod_{u=1}^{j} (z - q_s^{n+2-j+2u})
\]

by Proposition 6.5 and (A.7). For general \( l, m \) with \( 2l \geq m \), we have

\[
d_{n', j'}(z) = \prod_{k=0}^{m-1} \prod_{u=1}^{j} (z - q_s^{n+1-j+(2l-m)+2k+2u}) \quad \text{(by (A.7))}
\]

\[
= \prod_{k=0}^{m-1} \prod_{u=0}^{j} (z - q_s^{n+2-j+2u-1+2l+2k-m})
\]

\[
= \prod_{k=0}^{m-1} \prod_{u=0}^{2h^\vee} (z - q_s^{u+2} - 1+2l+2k-m) \tilde{\gamma}_{nj}(u) \quad \text{(by (6.9))}
\]

\[
= \prod_{k=0}^{m-1} \prod_{u=0}^{2h^\vee} (z - q_s^{u+2l+(2k-m+1)} \tilde{\gamma}_{nj}(u)),
\]

as desired.

In the case \( i = j = n \), we have \( d_i = d_j = 2 \) and \( r_i = 1 \). The condition \( ld_i \geq md_j \) implies \( l \geq m \). When \( (l, m) = (1, 1) \), we have

\[
d_{n, n}(z) = \prod_{u=0}^{2h^\vee} (z - q_s^{u+2} \tilde{\gamma}_{nn}(u)) = \prod_{u=1}^{n} (z - q_s^{2+2u})
\]

by Proposition 6.5 and (A.8). For general \( l, m \) with \( l \geq m \), we have

\[
d_{n', n''}(z) = \prod_{k=0}^{m-1} \prod_{u=1}^{n} (z - q_s^{2u+2(2l-2m)+4k}) \quad \text{(by (A.8))}
\]

\[
= \prod_{k=0}^{m-1} \prod_{u=0}^{2h^\vee} (z - q_s^{u+2+2l+4k-2m} \tilde{\gamma}_{nj}(u)) \quad \text{(by (6.10))}
\]

\[
= \prod_{k=0}^{m-1} \prod_{u=0}^{2h^\vee} (z - q_s^{u+2l+2(2k-m+1)} \tilde{\gamma}_{nj}(u)),
\]

as desired. \( \square \)

6.3. A universal coefficient formula In this subsection, we briefly recall the formula computing the universal coefficients \( a_{V, W}(z) \in k[[z]]^\times \) for all \( V, W \in \text{Irr } \mathcal{C} \) due to Frenkel–Reshetikhin [FR99]. First we notice the following bi-multiplicativity of the universal coefficients.
Lemma 6.10 ([FR99, Proposition 5]). For $P_1, P_2, P_3 \in \mathcal{D}^+$, we have

$$a_{L(P_1), L(P_3)}(z) = a_{L(P_1), L(P_2)}(z) \cdot a_{L(P_2), L(P_3)}(z),$$

$$a_{L(P_1), L(P_2), L(P_3)}(z) = a_{L(P_1), L(P_2)}(z) \cdot a_{L(P_1), L(P_3)}(z).$$

In view of Lemma 6.10 and Lemma 6.2 (1), it suffices to consider the fundamental case

$$a_{i,j}(z) := a_{\nu_i, \nu_j, \nu}(z) = a_{L(x_{i,1}), L(x_{j,1})}(z)$$

for $i, j \in I$. The following simple formula was obtained by Frenkel–Reshetikhin (see the discussion in Section 4.3 of [FR99]). It is based on a factorization formula of the universal R-matrix established by Khoroshkin–Tolstoy [KT94] and also by Damiani [Dam98].

Theorem 6.11 ([FR99, Section 4.3]). For $i, j \in I$, we have

$$a_{i,j}(z) = q_{ij}^{2rh^\vee} \prod_{u=0}^{2rh^\vee} \left( \frac{[u - d_i]}{[u + d_i]} \right) \tilde{c}_{ij}(u) = q_{ij}^{2rh^\vee} \prod_{u=0}^{\infty} \left( 1 - q_s^{u-d_i} z \right) \tilde{c}_{ij}(u), \quad (6.11)$$

where, for each $m \in \mathbb{Z}$, we set

$$[m] := (q_s^m z, q^{2h^\vee})_\infty = \prod_{k=0}^{\infty} (1 - q_s^{m+2krh^\vee} z) \in \mathbb{k}[\lbrack z \rbrack]^\times.$$

Note that the second equality in (6.11) follows from the $2rh^\vee$-periodicity of $\tilde{c}_{ij}(u)$ (Corollary 4.10 (2)).

6.4. Computation of $\Lambda$-invariants Let us recall the definitions of the degree functions introduced in [KKOP20d]. First we define the subgroup $\mathcal{G} \subset \mathbb{k}(z)^\times$ by

$$\mathcal{G} := \left\{ cz^m \prod_{a \in \mathbb{k}^\times} \varphi(a z)^{\mu_a} \bigg| c \in \mathbb{k}^\times, m \in \mathbb{Z}, \mu_a \in \mathbb{Z} \text{ vanishes except for finitely many } a's. \right\},$$

where we set $\varphi(z) := (z; q^{2h^\vee})_\infty = \prod_{k=0}^{\infty} (1 - q^{2k h^\vee} z) \in \mathbb{k}[\lbrack z \rbrack]^\times$. Note that $\mathbb{k}(z)^\times \subset \mathcal{G}$. Then we define the group homomorphisms $\text{Deg}: \mathcal{G} \rightarrow \mathbb{Z}$ and $\text{Deg}^\infty: \mathcal{G} \rightarrow \mathbb{Z}$ by

$$\text{Deg}(f(z)) := \sum_{a \in q^{2kh^\vee} \{ k \in \mathbb{Z} \leq 0 \}} \mu_a - \sum_{a \in q^{2kh^\vee} \{ k \in \mathbb{Z} > 0 \}} \mu_a,$$

$$\text{Deg}^\infty(f(z)) := \sum_{a \in q^{2kh^\vee} \{ k \in \mathbb{Z} \}} \mu_a$$

for $f(z) = cz^m \prod_{a \in \mathbb{k}^\times} \varphi(a z)^{\mu_a} \in \mathcal{G}$. For a rational function $f(z) \in \mathbb{k}(z)$, it follows that $\text{Deg}(f(z)) = 2 \text{zero}_{z=1}(f(z))$ and $\text{Deg}^\infty(f(z)) = 0$ (see [KKOP20d, Lemma 3.4]).

By Lemma 6.10 and Theorem 6.11, we see that the universal coefficient $a_{\nu_1, \nu_2}(z)$ belongs to $\mathcal{G}$ for any $\nu, \nu_1 \in \text{Irr} \mathcal{G}$. Thus it makes sense to consider $\text{Deg}(a_{\nu_1, \nu_2}(z))$ and
Definition 6.14. For an ordered pair \((6.15)\) and \((6.16)\) follow from \((6.14)\) and Lemma 6.10.

\[\Lambda(V, W) := \text{Deg}(d_{V,W}(z)/a_{V,W}(z)) = 2\text{zero}_{z=1}(d_{V,W}(z)) - \text{Deg}(a_{V,W}(z)),\]  
(6.12)

\[\Lambda^\infty(V, W) := \text{Deg}^\infty(d_{V,W}(z)/a_{V,W}(z)) = -\text{Deg}^\infty(a_{V,W}(z)).\]  
(6.13)

In the reminder of this section, we compute \(\text{Deg}(a_{V,W}(z))\) and \(\text{Deg}^\infty(a_{V,W}(z))\) in terms of the skew-symmetric pairing \(\mathcal{N}\) and the \(\mathcal{Q}\)-weight studied in the previous section.

Theorem 6.12. For any \((i, p), (j, s) \in \hat{I}\), we have

\[\text{Deg}(a_{L(Y_i,p),L(Y_j,s)}(z)) = -\mathcal{N}(i, p; j, s).\]  
(6.14)

In particular, for any \(V, W \in \text{Irr} \mathcal{C}^0\), we have

\[\text{Deg}(a_{V,W}(z)) = -\mathcal{N}(V, W).\]  
(6.15)

We need a lemma. Recall that we have defined \([m] := \varphi(q^m z)\) for each \(m \in \mathbb{Z}\).

Lemma 6.13. For any \(i, j \in I\) and \(x \in \mathbb{Z}\), we have

\[\sum_{u=0}^{2rh^\vee} \tilde{c}_{ij}(u) \cdot \text{Deg}[u - x] = \tilde{c}_{ij}(|x|) = \tilde{c}_{ij}(x) + \tilde{c}_{ij}(-x).\]

Proof. For \(x \in \mathbb{Z}\), we denote by \(\bar{x}\) the unique integer such that \(0 \leq \bar{x} < 2rh^\vee\) and \(\bar{x} - x \in 2rh^\vee\mathbb{Z}\). Under this notation, we have

\[\sum_{u=0}^{2rh^\vee} \tilde{c}_{ij}(u) \cdot \text{Deg}[u - x] = \tilde{c}_{ij}(\bar{x}) \cdot \text{Deg}[\bar{x} - x].\]

By the definition of Deg, we have

\[\text{Deg}[\bar{x} - x] = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}\]

When \(x \geq 0\), we have \(c_{ij}(\bar{x}) = c_{ij}(x)\) by Corollary 4.10 (2). On the other hand, when \(x < 0\), we have \(\tilde{c}_{ij}(\bar{x}) = -\tilde{c}_{ij}(2rh^\vee - \bar{x}) = -\tilde{c}_{ij}(-x)\) by Corollary 4.10 (4) and (2). As a result, we obtain \(\tilde{c}_{ij}(\bar{x}) \cdot \text{Deg}[\bar{x} - x] = \tilde{c}_{ij}(|x|)\), which proves the first equality of the assertion. The second equality follows from the fact that \(\tilde{c}_{ij}(x) = 0\) for \(x \leq 0\). \(\square\)

Proof of Theorem 6.12. By Theorem 6.11, we have

\[\text{Deg}(a_{L(Y_i,p),L(Y_j,s)}(z)) = \sum_{u=0}^{2rh^\vee} \tilde{c}_{ij}(u) \cdot (\text{Deg}[u + s - p - d_i] - \text{Deg}[u + s - p + d_i]).\]

Applying Lemma 6.13 above, we obtain the desired equality \((6.14)\). The latter identities \((6.15)\) and \((6.16)\) follow from \((6.14)\) and Lemma 6.10. \(\square\)

Definition 6.14. For an ordered pair \((V, W)\) of (real) simple modules in \(\mathcal{C}^0\), we say that

(i) the pair is left pre-commutative if \(\text{zero}_{z=1}(d_{V,W}(z)) = 0\),

(ii) the pair is right pre-commutative if \( \text{zero}_{z=1}(d_{W,V}(z)) = 0 \).

Note that a pair \((V, W)\) is commutative if and only if it is left and right pre-commutative thanks to Proposition 6.1.

**Corollary 6.15.** For any \( V, W \in \text{Irr} \mathcal{C}^0 \), we have
\[
\Lambda(V, W) = 2 \text{zero}_{z=1}(d_{V,W}(z)) + \mathcal{N}(V, W) \tag{6.16}
\]

In particular, for a left pre-commutative pair \((V, W)\) of simple modules in \( \mathcal{C}^0 \), we have
\[
\Lambda(V, W) = \mathcal{N}(V, W).
\]

**Theorem 6.16.** For any \((i, p), (j, s) \in \bar{T} \) and a Q-datum \( Q = (\Delta, \sigma, \xi) \) for \( \mathfrak{g} \), we have
\[
-\Lambda^\infty(L(Y_{i,p}), L(Y_{j,s})) = \text{Deg}^\infty(a_{L(Y_{i,p}),L(Y_{j,s})}(z)) = (-1)^{k+1}(\alpha, \beta), \tag{6.17}
\]

where \( \bar{\phi}_Q(i, p) = (\alpha, k) \) and \( \bar{\phi}_Q(j, s) = (\beta, l) \). In particular, for any \( V, W \in \text{Irr} \mathcal{C}^0 \), we have
\[
\Lambda^\infty(V, W) = -(\text{wt}_Q(V), \text{wt}_Q(W)). \tag{6.18}
\]

**Proof.** By Theorem 6.11, we have
\[
\text{Deg}^\infty(a_{L(Y_{i,p}),L(Y_{j,s})}(z)) = \sum_{u=0}^{2rh^\vee} \tilde{c}_{ij}(u) \cdot (\text{Deg}^\infty[u+s-p-d_i] - \text{Deg}^\infty[u+s-p+d_i])
\]
\[
= \tilde{c}_{ij}(\bar{s} + p + d_i) - \tilde{c}_{ij}(\bar{s} + p - d_i),
\]
where \( \bar{x} \) denotes for each \( x \in \mathbb{Z} \) the unique integer such that \( 0 \leq \bar{x} < 2rh^\vee \) and \( \bar{x} - x \in 2rh^\vee \mathbb{Z} \) as before. Let us choose \( i, j \in \Delta_0 \) such that \( \bar{i} = i, \bar{j} = j \). By Theorem 4.8, we have
\[
\tilde{c}_{ij}(\bar{x}) = (\sigma, \tau(y_i^\Delta \bar{\xi}_i - y_j^\Delta \bar{\xi}_j, d_i/2) / (y_j^\Delta))
\]
for each \( x \in \mathbb{Z} \). By the same computation as in the proof of Proposition 5.21, we obtain
\[
\tilde{c}_{ij}(\bar{s} + p + d_i) - \tilde{c}_{ij}(\bar{s} + p - d_i) = (\tau(y_i^\Delta \bar{\xi}_i - p/2) / (y_j^\Delta), \tau(y_j^\Delta \bar{\xi}_j - s/2) / (y_j^\Delta)) = (-1)^{k+1}(\alpha, \beta),
\]
which proves (6.17). The equality (6.18) follows from (6.17) and Lemma 6.10. \( \square \)

**Remark 6.17.** Theorem 6.16 gives a new unified proof of \([\text{KKOP20a}, (5.2)]\) for all untwisted affine types, which was a crucial step in the proof of \([\text{KKOP20a}, \text{Theorem 3.6}]\). In particular, it leads to a block decomposition of the category \( \mathcal{C}^0 \) indexed by the simply-laced root lattice \( Q \) by \([\text{KKOP20a}, \text{Section 4}]\), which is comparable with Theorem 5.15 above.

**Corollary 6.18.** (cf. \([\text{KKOP20a}, (3.6)]\)) For simple modules \( V \) and \( W \) in \( \mathcal{C}^0 \), we have
\[
\mathcal{N}(\mathcal{O}^{2k}V, W) = \mathcal{N}(V, \mathcal{O}^{-2k}W) = \Lambda^\infty(V, W) \text{ for } k \gg 0.
\]
Proof. Let \( V \cong L(m) \) and \( W \cong L(m') \) with \( m, m' \in \mathcal{M}_4 \). The first equality follows from Proposition 5.5 and the fact \( D^2(Y_i, p) = Y_i, p + 2rh^\vee \). Choose \( k \in \mathbb{Z} \) large enough so that we have

\[
\min\{ p \mid \exists i \in I, u_i, p(\mathbb{D}^k m) \neq 0 \} > \max\{ p \mid \exists i \in I, u_i, p(m') \neq 0 \}.
\]

Then the second equality follows from Lemma 5.17, Corollary 5.22 and (6.18). \( \square \)

Acknowledgements. We are deeply grateful to David Hernandez for helpful discussions and for pointing out that some of the results in the initial version of this paper were already known in the literature. We also wish to thank Masaki Kashiwara, Bernhard Keller, Myungho Kim, Bernard Leclerc and Euiyong Park for stimulating discussions and comments. Finally, we would like to thank the anonymous referees for many helpful suggestions about the exposition and the appropriate references.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A. Denominator Formulae

In this appendix, we give a list of all the formulae of \( d_{l, m}^i(z) \) which are currently known. These are quoted from [Oh15,OS19a,OS19b,Fuj19]. See Remark 6.3 for our convention and Remark 6.6 for type \( F_4 \). For non-simply-laced \( g \), we use the labeling \( I = \{1, \ldots, n\} \) as in (3.2).

(**Type ADE**) For any \( i, j \in I \), we have

\[
d_{i, j}(z) = \prod_{u=0}^{h^\vee} (z - q^{u+1}c_{ij}(u)). \tag{A.1}
\]

If \( g \) is of type AD and \( l, m \in \mathbb{Z}_{\geq 1} \), we have

\[
d_{l, m}^i(z) = \prod_{t=0}^{\min(l, m) - 1} d_{i, j}(q^{-|l-m|-2t}z). \tag{A.2}
\]

(**Type B_n**) For \( 1 \leq i < n \) and \( l, m \in \mathbb{Z}_{\geq 1} \), we have

\[
d_{i, j}(z) = \prod_{u=0}^{\min(i, j)} (z - q^{i-j+2u})(z - q^{2n-i-j+1+2u}) \tag{A.3}
\]

and \( d_{l, m}^i(z) \) are given by the same formula as (A.2). For \( 1 \leq i < n \) and \( l, m \in \mathbb{Z}_{\geq 1} \), we have

\[
d_{l, m}^i(z) = d_{m, l}^i(z) = \prod_{t=0}^{\min(2l, m) - 1} \prod_{u=1}^{i} (z - q^{2n-2i-2|l-m|+4u+2t}), \tag{A.4}
\]

\[
d_{n, l}^i(z) = \prod_{t=0}^{\min(l, m) - 1} d_{n, n}(q^{-|l-m|-2t}z). \tag{A.5}
\]
(Type $\mathbb{C}_n$) For $1 \leq i, j < n$ and $l, m \in \mathbb{Z}_{\geq 1}$ with $\max(l, m) > 1$, we have

$$d_{i,j}^l(z) = \prod_{t=0}^{\min(l,m)-1} \prod_{u=1}^{\min(i,j)} (z - q_s^{i-j+l-m+2(\ell+u)})(z - q_s^{2n+2-l-j+l-m+2(\ell+u)}),$$

(A.6)

$$d_{n,n}^l(z) = d_{n,m}^l(z) = \prod_{t=0}^{\min(l,2m)-1} \prod_{u=1}^i (z - q_s^{n+1-i+l-m+2m-2l+l+2u}),$$

(A.7)

$$d_{n,m}^l(z) = \prod_{t=0}^{\min(l,m)-1} \prod_{u=1}^n (z - q_s^{2n+2-l-j+l-m+2m+4u}),$$

(A.8)

while, for $1 \leq i, j \leq n$, we have

$$d_{i,j}(z) = \prod_{u=1}^{\min(i,j,n-i,j-n)} (z - q_s^{i-j+l-m+2u}) \prod_{u=1}^{\min(i,j)} (z - q_s^{2n+2-l-j+l-m+2u}).$$

(A.9)

(Type $\mathbb{F}_4$) We have

$$d_{1,1}(z) = (z - q_s^4)(z - q_s^{10})(z - q_s^{12})(z - q_s^{18}),$$

(A.10)

$$d_{1,2}(z) = (z - q_s^5)(z - q_s^8)(z - q_s^{10})(z - q_s^{12})(z - q_s^{14})(z - q_s^{16}),$$

(A.11)

$$d_{1,3}(z) = (z - q_s^7)(z - q_s^9)(z - q_s^{13})(z - q_s^{15}),$$

(A.12)

$$d_{1,4}(z) = (z - q_s^8)(z - q_s^{14}),$$

(A.13)

$$d_{2,2}(z) = (z - q_s^4)(z - q_s^6)(z - q_s^8)^2(z - q_s^{10})^2(z - q_s^{12})^2(z - q_s^{14})^2$$

$$- (z - q_s^{16})(z - q_s^{18}),$$

(A.14)

$$d_{2,3}(z) = (z - q_s^5)(z - q_s^7)(z - q_s^9)(z - q_s^{11})^2(z - q_s^{13})(z - q_s^{15})(z - q_s^{17}),$$

(A.15)

$$d_{2,4}(z) = (z - q_s^8)(z - q_s^{14}),$$

(A.16)

$$d_{3,3}(z) = (z - q_s^2)(z - q_s^6)(z - q_s^8)(z - q_s^{10})(z - q_s^{12})^2(z - q_s^{16})(z - q_s^{18}),$$

(A.17)

$$d_{3,4}(z) = (z - q_s^3)(z - q_s^7)(z - q_s^{11})(z - q_s^{13})(z - q_s^{17}),$$

(A.18)

$$d_{4,4}(z) = (z - q_s^2)(z - q_s^8)(z - q_s^{12})(z - q_s^{18}),$$

(A.19)

$$d_{3,1}(z) = (z - q_s^5)(z - q_s^{10})(z - q_s^{12})(z - q_s^{14})^2(z - q_s^{16}),$$

(A.20)

$$d_{3,2}(z) = (z - q_s^4)(z - q_s^6)^2(z - q_s^8)^2(z - q_s^{10})^3(z - q_s^{12})^3(z - q_s^{14})^2$$

$$- (z - q_s^{16})^2(z - q_s^{18}),$$

(A.21)

$$d_{3,3}(z) = (z - q_s^3)(z - q_s^5)(z - q_s^7)(z - q_s^9)^2(z - q_s^{11})^2(z - q_s^{13})^2$$

$$- (z - q_s^{15})(z - q_s^{17})(z - q_s^{19}),$$

(A.22)

$$d_{3,4}(z) = (z - q_s^4)(z - q_s^8)(z - q_s^{10})(z - q_s^{12})(z - q_s^{14})(z - q_s^{18}),$$

(A.23)

$$d_{4,1}(z) = (z - q_s^7)(z - q_s^9)(z - q_s^{13})(z - q_s^{15}),$$

(A.24)

$$d_{4,2}(z) = (z - q_s^5)(z - q_s^9)(z - q_s^{11})^2(z - q_s^{13})(z - q_s^{15})(z - q_s^{17}),$$

(A.25)

$$d_{4,3}(z) = (z - q_s^4)(z - q_s^8)(z - q_s^{10})(z - q_s^{12})(z - q_s^{14})(z - q_s^{18}).$$

(A.26)
\[d_{q^2} A(z) = (z - q_s^3)(z - q_s^9)(z - q_s^{13})(z - q_s^{19}).\]  
(A.27)

**Type G**  
We have  
\[d_{1,1}(z) = (z - q_s^6)(z - q_s^8)(z - q_s^{10})(z - q_s^{12}), \quad d_{1,2}(z) = (z - q_s^7)(z - q_s^{11}),\]  
(A.28)

and for \(l, m \in \mathbb{Z}_{\geq 1}\), we have  
\[d_{1^l, 1^m}(z) \equiv \prod_{t=1}^{\min(l,m)-1} d_{1,1}(q_s^{-|l-m|-2t}z),\]  
(A.29)

\[d_{1^l, 2^m}(z) = d_{2^m, 1^l}(z) \equiv \prod_{t=0}^{\min(3l,m)-1} d_{1,2}(q_s^{-|3l-m|+2-2t}z).\]  
(A.30)

For \(l, m \in \mathbb{Z}_{\geq 1}\) with \((l, m) \neq (1, 1), (2, 2)\), we have  
\[d_{2^l, 2^m}(z) = \prod_{t=0}^{\min(l,m)-1} \prod_{u=1}^2 (z - q_s^{-2+|l-m|+4u+2t}) (z - q_s^{4+|l-m|+4u+2t}),\]  
(A.31)

while  
\[d_{2^2, 2^2}(z) = (z - q_s^2)(z - q_s^8)(z - q_s^{12}),\]  
(A.32)

\[d_{2^2, 2^2}(z) = (z - q_s^2)(z - q_s^4)(z - q_s^8)(z - q_s^{10})(z - q_s^{12})(z - q_s^{14}).\]  
(A.33)

**References**

[AK97] Akasaka, T., Kashiwara, M.: Finite-dimensional representations of quantum affine algebras. Publ. Res. Inst. Math. Sci. 33(5), 839–867 (1997)

[Bec94] Beck, J.: Braid group action and quantum affine algebras. Commun. Math. Phys. 165(3), 555–568 (1994)

[Béd99] Bédard, R.: On commutation classes of reduced words in Weyl groups. Eur. J. Combin. 20(6), 483–505 (1999)

[CM05] Chari, V., Moura, A.A.: Characters and blocks for finite-dimensional representations of quantum affine algebras. Int. Math. Res. Not. 5, 257–298 (2005)

[CP94] Chari, V., Pressley, A.: A Guide to Quantum Groups. Cambridge University Press, Cambridge (1994)

[CP95] Chari, Vyjayanthi, Pressley, Andrew: Quantum affine algebras and their representations. In: Representations of groups (Banff, AB, 1994), volume 16 of CMS Conf. Proc., pages 59–78. Amer. Math. Soc., Providence, RI (1995)

[CP96a] Chari, V., Pressley, A.: Minimal affinizations of representations of quantum groups: the simply laced case. J. Algebra 184(1), 1–30 (1996)

[CP96b] Chari, V., Pressley, A.: Yangians, integrable quantum systems and Dorey’s rule. Commun. Math. Phys. 181(2), 265–302 (1996)

[Dam98] Damiani, Ilaria: La \( R \)-matrice pour les algèbres quantiques de type affine non tordu. Ann. Sci. École Norm. Sup. (4) 31(4), 493–523 (1998)

[Dam12] Damiani, I.: Drinfeld realization of affine quantum algebras: the relations. Publ. Res. Inst. Math. Sci. 48(3), 661–733 (2012)

[Dam15] Damiani, I.: From the Drinfeld realization to the Drinfeld–Jimbo presentation of affine quantum algebras: injectivity. Publ. Res. Inst. Math. Sci. 51(1), 131–171 (2015)

[DO94] Date, E., Okado, M.: Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type \( A_n^{(1)} \). Int. J. Modern Phys. A 9(3), 399–417 (1994)
[FH15] Frenkel, E., Hernandez, D.: Baxter’s relations and spectra of quantum integrable models. Duke Math. J. 164(12), 2407–2460 (2015)

[FM01] Frenkel, E., Mukhin, E.: Combinatorics of $q$-characters of finite-dimensional representations of quantum affine algebras. Commun. Math. Phys. 216(1), 23–57 (2001)

[FR99] Frenkel, Edward, Reshetikhin, Nicolai: The $q$-characters of representations of quantum affine algebras and deformations of $W$-algebras. In: Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), volume 248 of Contemp. Math., pages 163–205. Amer. Math. Soc., Providence, RI (1999)

[Fuj19] Fujita, Ryo: Graded quiver varieties and singularities of normalized R-matrices for fundamental modules. Preprint, arXiv:1911.12693 (2019)

[Gab72] Gabriel, Peter: Unzerlegbare Darstellungen. I. Manuscripta Math., 6:71–103; correction, ibid. 6 (1972), 309, 1972

[Gab80] Gabriel, Peter: Auslander–Reiten sequences and representation-finite algebras. In: Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), volume 831 of Lecture Notes in Math., pages 1–71. Springer, Berlin (1980)

[GTL17] Gautam, S., Laredo, V.T.: Meromorphic tensor equivalence for Yangians and quantum loop algebras. Publ. Math. Inst. Hautes Études Sci. 125, 267–337 (2017)

[Hap87] Happel, D.: On the derived category of a finite-dimensional algebra. Comment. Math. Helv. 62(3), 339–389 (1987)

[Her04] Hernandez, D.: Algebraic approach to $q$, $t$-characters. Adv. Math. 187(1), 1–52 (2004)

[Her06] Hernandez, D.: The Kirillov–Reshetikhin conjecture and solutions of T-systems. J. Reine Angew. Math. 596, 63–87 (2006)

[HL10] Hernandez, D., Leclerc, B.: Cluster algebras and quantum affine algebras. Duke Math. J. 154(2), 265–341 (2010)

[HL15] Hernandez, D., Leclerc, B.: Quantum Grothendieck rings and derived Hall algebras. J. Reine Angew. Math. 701, 77–126 (2015)

[HL16] Hernandez, D., Leclerc, B.: A cluster algebra approach to $q$-characters of Kirillov–Reshetikhin modules. J. Eur. Math. Soc. (JEMS) 18(5), 1113–1159 (2016)

[HO19] Hernandez, D., Oya, H.: Quantum Grothendieck ring isomorphisms, cluster algebras and Kazhdan–Lusztig algorithm. Adv. Math. 347, 192–272 (2019)

[IKT12] Inoue, Rei, Kuniba, Atsuo, Takagi, Taichiro: Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry. J. Phys. A 45(7), 073001, 64 (2012)

[Kac90] Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)

[Kas02] Kashiwara, M.: On level-zero representations of quantized affine algebras. Duke Math. J. 112(1), 117–175 (2002)

[KKK15] Kang, Seok-Jin., Kashiwara, Masaki, Kim, Myungho: Symmetric quiver Hecke algebras and $R$-matrices of quantum affine algebras. II. Duke Math. J. 164(8), 1549–1602 (2015)

[KKK18] Kang, Seok-Jin., Kashiwara, Masaki, Kim, Myungho: Symmetric quiver Hecke algebras and $R$-matrices of quantum affine algebras. Invent. Math. 211(2), 591–685 (2018)

[KKKO15] Kang, Seok-Jin., Kashiwara, Masaki, Kim, Myungho, Se-jin, Oh.: Simplicity of heads and socles of tensor products. Compos. Math. 151(2), 377–396 (2015)

[KKO19] Kashiwara, Masaki, Kim, Myungho, Se-jin, Oh.: Monoidal categories of modules over quantum affine algebras of type A and B. Proc. Lond. Math. Soc. (3) 118(1), 43–77 (2019)

[KKOP20a] Kashiwara, Masaki, Kim, Myungho, Oh, Se-jin, Park, Euiyong: Block decomposition for quantum affine algebras by the associated simply-laced root system. Preprint, arXiv:2003.03265 (2020)

[KKOP20b] Kashiwara, Masaki, Kim, Myungho, Oh, Se-jin, Park, Euiyong: Braid group action on the module category of quantum affine algebras. Preprint, arXiv:2004.04939 (2020)

[KKOP20c] Kashiwara, Masaki, Kim, Myungho, Se-jin, Oh., Park, Euiyong: Categories over quantum affine algebras and monoidal categorification. Preprint, arXiv:2005.10969 (2020)

[KKOP20d] Kashiwara, Masaki, Kim, Myungho, Se-jin, Oh., Park, Euiyong: Monoidal categorification and quantum affine algebras. Compos. Math. 156(5), 1039–1077 (2020)

[KT94] Khoshtinik, S.M., Tolstoy, V.N.: Twisting of quantum (super-) algebras. In: Generalized symmetries in physics (Clausthal, 1993), pages 42–54. World Sci. Publ., River Edge, NJ (1994)

[Nak01] Nakajima, H.: Quiver varieties and finite-dimensional representations of quantum affine algebras. J. Am. Math. Soc. 14(1), 145–238 (2001)

[Nak03] Nakajima, Hiraku: $t$-analogs of $q$-characters of Kirillov–Reshetikhin modules of quantum affine algebras. Represent. Theory 7, 259–274 (2003) (electronic)

[Nak04a] Nakajima, Hiraku: Extremal weight modules of quantum affine algebras. In: Representation theory of algebraic groups and quantum groups, volume 40 of Adv. Stud. Pure Math., pages 343–369. Math. Soc. Japan, Tokyo (2004)
[Nak04b] Nakajima, Hiraku: Quiver varieties and \( t \)-anals of \( q \)-characters of quantum affine algebras. Ann. of Math. (2) 160(3), 1057–1097 (2004)

[Oh15] Oh, S.: The denominators of normalized \( R \)-matrices of types \( A_{2n-1}^{(2)} \), \( A_{2n}^{(2)} \), \( B_n^{(1)} \) and \( D_{n+1}^{(2)} \). Publ. Res. Inst. Math. Sci. 51(4), 709–744 (2015)

[Oh18] Oh, Se-jin: Auslander–Reiten quiver and representation theories related to KLR-type Schur–Weyl duality. Mathematische Zeitschrift, (Jun 2018)

[OS19a] Oh, S., Scrimshaw, T.: Categorical relations between Langlands dual quantum affine algebras: exceptional cases. Commun. Math. Phys. 368(1), 295–367 (2019)

[OS19b] Oh, Se-jin, Scrimshaw, Travis: Simplicity of tensor products of Kirillov–Reshetikhin modules: nonexceptional affine and \( G \) types. Preprint, arXiv:1910.10347, (2019)

[OS19c] Oh, Se-jin, Suh, Uhi Rinn: Combinatorial Auslander–Reiten quivers and reduced expressions. J. Korean Math. Soc. 56(2), 353–385 (2019)

[OS19d] Oh, Se-jin, Suh, Uhi Rinn: Twisted and folded Auslander–Reiten quivers and applications to the representation theory of quantum affine algebras. J. Algebra 535(1), 53–132 (2019)

[Spr74] Springer, T.A.: Regular elements of finite reflection groups. Invent. Math. 25, 159–198 (1974)

[VV03] Varagnolo, Michela, Vasserot, Eric: Perverse sheaves and quantum Grothendieck rings. In: Studies in memory of Issai Schur, pages 343–365. Springer (2003)

Communicated by Y. Kawahigashi