A REVIEW OF GROWING INTERF ACES IN QUENCHED DISORDERED MEDIA

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Abstract

We make a review of the two principal models that allows to explain the imbibition of fluid in porous media. These models, that belong to the directed percolation depinning (DPD) universality class, where introduced simultaneously by the Tang and Leschhorn [Phys. Rev A 45, R8309 (1992)] and Buldyrev et al. [Phys. Rev. A 45, R8313 (1992)] and reviewed by Braunstein et al. [J. Phys. A 32, 1801 (1999); Phys. Rev. E 59, 4243 (1999)]. Even these models have been classified in the same universality class than the Kardar-Parisi-Zhang equation [Phys. Rev. Lett. 56, 889, (1986)] with quenched noise (QKPZ), the contributions to the growing mechanisms are quite different. The lateral contribution in the DPD models, leads to an increasing of the roughness near the criticality while in the QKPZ equation this contribution always flattens the roughness. These results suggest that the QKPZ equation does not describe properly the DPD models even when the exponents derived from this equation are similar to the one obtained from the simulations of these models. This fact is confirmed through the deduced analytical equation for the Tang and Leschhorn model. This equation has the same symmetries than the QKPZ one but its coefficients depend on the balance between the driving force and the quenched noise.

1 Introduction

In the last years there has been a growing interest in the understanding a vast variety of scale invariant phenomena occurring in nature. Experiments and observations indeed suggest that many physical systems develop correlations with power law behaviour both in space and time. However, the fact that certain structures exhibit fractal and complex properties does not tell us why this happens. Pattern formation, aggregation phenomena, biological systems, geological systems, disordered materials, clustering of matter in the universe and many fields in which scale invariance has been observed as a common and basic feature.

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A crucial point to understand is therefore the origin of the general scale-invariance of natural phenomena. This would correspond to the understanding of the origin of fractal structures from the knowledge of the microscopic physical processes at the basis of these phenomena.

In scale invariant phenomena, events and information spread over a wide range of length and time scales, so that no matter what is the size of the scale considered one always observes surprisingly rich structures. These systems, with very many degrees of freedom, are usually so complex that their large scale behaviour cannot be predicted from the microscopic dynamics. New types of collective behaviour arise and their understanding represents one of the most challenging areas in modern statistical physics.

The concepts of scaling and power law behavior was introduced to the study of critical phenomena in second order transitions. The physics of complex systems, however, is new with respect to critical phenomena. The theory of equilibrium statistical physics is strongly based on the ergodic hypothesis and scale invariance develops at the critical equilibrium between order and disorder. Reaching this equilibrium requires the fine tuning of various parameters. In usual critical phenomena the same exponents that define the onset of magnetization also describe the liquid vapor transition in water. This strong universality appears to be a characteristic of equilibrium systems. On the contrary the origin of the scale invariance in nature in the rich domain of nonequilibrium systems is not so well understood. Systems far from equilibrium, do not seem to exhibit the same degree of universality as the fractal dimension can be easily altered by simple changes in the growth process. This lack of universality is sometimes viewed as a negative element because one is forced to describe specific systems instead of a single universal model. This is why such a problem can only be investigated using many tools as computer simulations, analytical tools and suitably designed experiments. While the theoretical activity is focused mainly on Monte Carlo simulations, it is very important to understand the relations between theory and real experiments.

Fractal geometry provided the mathematical framework for the extension of these concepts to a vast variety of natural phenomena. A principal subject where fractals play an essential role is in the study of self-affine features arising in complex systems with many degrees of freedom, such as of interface growth in disordered systems including percolation properties of fluid displacement in disordered media.

All these models have been extensively studied by computer simulations. The information generated by these computer experiments brings a visual intuition as a valuable tool in science. In this respect computer simulations represent an essential method in the physics of these systems, that allow us to design theoretical experiments tested with a computer. Numerical simulations define the basic characteristics of the models and gives a useful path to their theoretical understanding.

From a mathematical point of view the problems explored are particularly difficult because they consist of iterative systems with many degrees of freedom and irreversible dynamics. Very little can be predicted a priori for systems of this complexity even when sometimes it is not very difficult to pose a model that capture the essential features of the phenomena.

While the great majority of the theoretical activity is based upon toy models it is very important to build a bridge between theory and real experiments and this is another basic
task of computer simulations. To do that, it is necessary the development of models more realist and large scale simulations which can be used also in material characterization. The consequence of this approach is the application of fractal concepts to the solution of particular experimental problems (Oil industry, disordered materials, phase nucleation, crystal growth etc.). The theoretical effort in this field can be separated into phenomenological or scaling theories and microscopic theories. At a phenomenological level scaling theory, inspired to usual critical phenomena, has been successfully used. This is the nexus to the understanding of the results of computer simulations and experiments. This method allows us to identify the relations between different properties and exponents and to focus on the essential ones. Moreover, it will be useful to gain more insight into the microscopic dynamics that evolves into the macroscopic behavior. The connection between the microscopic and macroscopic behavior has not been extensively studied.

In this work we make a review of models of growth in quenched disordered media. In Sec. 2 we introduce the dynamics and static scaling used in experiments and models of growth. We make also a discussion about the principal source of disorder. In Sec. 3 we make a short summary of the main experiments on fluid-fluid displacement. In Sec. 4 we present the most used phenomenological equation that is said to belong to the same universality class than the experiments and/or models with quenched media in special to the directed percolation depinning (DPD) models. In Sec. 5 we present the Tang and Leschhorn model. In Sec. 6 we present the Buldyrev et al. model. In Sec. 7 some comparisons between both models. In Sec. 8 we argue why the Kardar-Parisi-Zhang equation with quenched noise does not describe the DPD models even if they are said to belong to the same universality class. In Sec. 9 we deduce an analytical stochastic differential equation for one of the DPD models starting from its microscopic equation. Finally in Sec. 10 we conclude with a discussion.

2 Scaling properties of growth in quenched media

The investigation of rough surfaces and interfaces has attracted much attention, for decades, due to its importance in many fields, such as the motion of liquids in porous media, growth of bacterial colonies, crystal growth, etc. Much effort has been done in understanding the processes that induces the roughness in these fields. When a fluid wet a porous medium a nonequilibrium self-affine rough interface is generated. The interface has been characterized through scaling of the interfacial width \( w = \langle [h_i - \langle h_i \rangle]^2 \rangle^{1/2} \) with time \( t \) and lateral size \( L \). The result is the determination of two exponents \( \beta \) and \( \alpha \) called dynamical and roughness exponents respectively. The interfacial width follows

\[
\begin{align*}
    w &\sim L^\alpha & \text{if} & & t \gg L^{\alpha/\beta}, \\
    w &\sim t^\beta & \text{if} & & t \ll L^{\alpha/\beta}.
\end{align*}
\]

The crossover time between this two regimes is of the order of \( L^{\alpha/\beta} \).

The disorder affects the motion of the interface and leads to its roughness. The main disorder proposed has been the “annealed” noise that only depends on time and the “quenched” disorder
due to the inhomogeneity of the media where the moving phase is propagating. While the annealed disorder pushes forward the interface the quenched disorder brakes this advancement. This last disorder can be due to impurities, fluctuations in the capillary size or it could represent all the effects of an inhomogeneous media.

Some experiments such as the motion of liquids in porous media, where the disorder is quenched, are well described by the directed percolation depinning model. Some other experiments like the flow of fluid through a disordered media, where the disorder is also quenched, give roughness exponents scattered between 0.6 and 1.25 questioning the existence of universality, the foundation of the scaling hypothesis (1).

3 Experiments on fluid-fluid displacement

Several fluid-fluid displacement experiments that can be described in terms of simple self-affine structures were motivated by the growing interest of physicists on the dynamics of rough surfaces. The experiments are easy to perform but the process are very complex to understand. One of the experiments that was first performed in order to explain the fact that the roughness exponent $\alpha$ was not predicted by equations with thermal noise was carried out by Rubio et al. (2) in 1 + 1 dimensions. In this experiment the air is displaced by a dyed water in a porous media consisting of glass beads packed randomly into a thin horizontal cell of glass covered with Teflon$^3$. The water was injected by one of its edges wetting the porous media and the fluid-fluid interface is digitized. The roughness exponent was measured over a distance $\ell$ obtaining a value of $\alpha = 0.73 \pm 0.03$. This value does not depend on the bead diameter and the capillary number $Ca_p$ in the range that they worked. Horváth et al. (3) reanalyzed the interface of Rubio et al. and obtained $\alpha = 0.81 \pm 0.08$. This discrepancy is not clear, but it could be assign to the sensitivity of this exponent to the details of the data analysis. Also in this experiment the exponent $\beta \sim 0.65$ was obtained. Other value of $\alpha = 0.63 \pm 0.04$, where the exponent was measured in the saturation, was obtained by Buldyrev et al. (4) with aqueous fluids absorbed in many kind of papers. In these experiments, the scaling value does not depend on the temperature, humidity and kind of paper. They inferred that the evaporation is irrelevant. Horváth and Stanley (5) performed an experiment in which the evaporation was controlled by confining the paper between two transparent polymer sheets. The lower end of the cell was immersed in liquid. The shape of the interface was digitized and the mean height of the interface was calculated in real time in order to maintain it at a fixed height. This prevents any change in the velocity. They found $\beta = 0.56 \pm 0.03$.

The scattering between the exponents obtained in these experiments puts in doubt the universality of their results. Sometimes, the uncertainties could be the result of complex crossover behaviors before the asymptotic regime is reached. On the other hand, sometimes, the asymptotic regime is not reached in the experiments. Also in many experiments the interval where

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4$Ca_p \sim a^2v/\Gamma$, where $a$ is the diameter of the bead, $v$ is the velocity of the water and $\Gamma$ is the interfacial tension.
the exponents were measured is very short enough (less than two decades), in these cases the results obtained are less reliable. This does not mean that the idea of scaling behaviors must be rejected, moreover some scaling hypothesis must be reviewed. In table 1 we shows some experimental results.

4 Phenomenological equation of growth in quenched media

Motivated by the success of the Kardar-Parisi-Zang (KPZ) equation in describing the interface motion with thermal noise, it has been proposed that many interfaces in porous media are described by the quenched KPZ (QKPZ) equation

$$\frac{\partial h(x,t)}{\partial t} = F + \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x,h)$$ (3)

where $\eta(x,h)$ represent the quenched white noise in the media and $F$ is the driving force responsible of the advance of the interface. This equation predicts the existence of a depinning transition: for driving forces $F < F_c$ the interface is pinned, and for $F > F_c$ it moves with a velocity $v \sim f_{\text{red}}^\theta$ where $\theta$ is the velocity exponent and $f_{\text{red}} = (F - F_c)/F_c$ is the reduced force. Numerical studies indicated that the discrete models can be grouped into two universality classes depending on the behavior of the QKPZ nonlinearity $\lambda$. Isotropic models, i.e. models that have no growth direction determined by the random forces, have $\lambda = 0$ or $\lambda \to 0$ as $f_{\text{red}} \to 0$. The scaling exponents can be determined analytically, in one dimension, from Eq. (3) with $\lambda = 0$, obtaining $\alpha_i = 1$, $\beta_i = 0.75$ and $\theta_i = 0.33$. However, anisotropy can induce a relevant $\lambda$ at the depinning transition. The exponents characterizing this anisotropic universality can be obtained by an exacting mapping to directed percolation, obtaining $\alpha_a = 0.63$, $\beta_a = 0.63$ and $\theta_a = 0.63$. Many numerical simulations have been done from a tilted interface in order to justify if the nonlinear term of the QKPZ is present in the dynamics of the proposed models. Recently Réka et al. implemented a method that apply to numerical models and experimental data and allows to classify the data into one of the two universality classes without relying on the determination of the exponents. They assumed that the local exponents were measured is very short enough (less than two decades), in these cases the results obtained are less reliable. This does not mean that the idea of scaling behaviors must be rejected, moreover some scaling hypothesis must be reviewed. In table 1 we shows some experimental results.

Table 1: Measured exponents from experiments

| Experiments          | $\alpha$ | $\beta$ | References |
|----------------------|----------|---------|------------|
| Flux of fluid        | 0.73     | -       | \[5\]      |
|                      | 0.81     | 0.65    | \[4\]      |
|                      | 0.65 - 0.91 | -       | \[6\]      |
| Wetting paper        | 0.63 (d=1) | 0.56    | \[4, 5\]   |
|                      | 0.5 (d=2) | -       | \[7\]      |
|                      | 0.62 - 0.78 (d=1) | 0.3-0.4 | \[9\]      |
| Bacterial growth     | 0.78     | -       | \[10\]     |
| Combustion front     | 0.71     | -       | \[11\]     |
velocity has the same scaling behavior than the global velocity \( v \), then \( u(F, s) \sim |F - F_c(s)|^{\theta'} \), where \( u(F, s) \) is the averaged local velocity of the interface with slope \( s \), \( F_c(s) \) is the depinning threshold corresponding to these segments, and \( \theta' \) is the local velocity exponent. In isotropic media \( F_c(s) = F_c \) and \( \theta' = \theta \), so the ratio \( V \equiv u(F, s)/v(F) \) does not depend on the driven force. In contrast, in anisotropic media the depinning threshold, \( F_c(s) \) decreases with \( s \) and \( \theta' = 1 \). So, for anisotropic media \( V \) has a systematic dependence on \( F \). Plotting \( V \) as a function of \( s \) for the several models and for the fluid-fluid experiment they concluded that this last experiment belongs to the isotropic class. All these numerical results confirm the relevance of a nonlinear term for any model but they do not assert to prove if these models are well represented by the KPZ equation or its linear version, the quenched Edward-Wilkinson (QEW) [17].

In particular in models with quenched noise belonging to the anisotropic class the macroscopic dynamics is influenced by the coupled effect of the interface itself and the disorder as we shall see in Sec. 5.

A powerful method to derive the macroscopic behavior of these models is to attempt to write the microscopic equation for the evolution of the local height in these models. This is done via the master equation approach [18] or directly by writing the microscopic rules for the evolution of the local height [19, 21]. These two methods have been proved to be equivalent [23]. These microscopic equations that capture the mechanisms of the growth for each model must reproduce the macroscopic behavior of the relevant observable that in this case is the interface itself. Its continuum version, in a coarsed grained scale is the equation that represents its universality class. Let us to introduce the two main models describe the principal features of the experiments of fluid imbibition on paper sheet [1].

5 Growth mechanisms for the Tang and Leschhorn model

We present now the microscopic Tang and Leschhorn (TL) model. The interface growth takes place in a lattice of \( N \) points of edge \( L \) \((N = L/a)\) where \( a \) is the lattice constant. Usually in discrete models \( a \) is taken as one, so \( N = L \). We will retain the lattice constant because we will use it below.

Periodic boundary conditions are used. The quenched noise is represented by a random pinning force \( g(r) \) uniformly distributed in \([0, 1]\) assigned to every cell of the lattice. This random noise models the random distribution of the fiber of paper. For a given pressure \( p \), that represents the driving force that push the fluid into the media, the cells are divided in two groups, active (free) cells with \( g(r) \leq p \) and inactive (blocked) cells with \( g(r) > p \). Notice that \( q = 1 - p \) is the density of blocked cells. It is well known that there exist a critical density of blocked cells \( q_c = 0.539 \), what is the directed percolation threshold for an infinite lattice, above \( q_c \) the interface becomes pinned.

In this model, the growth event is defined as follow:

1. If \( h_i \) is greater than either \( h_{i-1} \) or \( h_{i+1} \) by one or more units, the height of the lower of the two columns \((i - 1)\) and \((i + 1)\) is incremented in \( a \) (in case of tie, one of the two is chosen with equal probability).
2. In the opposite case, \( h_i < \min(h_{i-1}, h_{i+1}) + 2a \), the column \( i \) advances by one unit provided that the cell to be occupied is an active cell.

3. Otherwise no growth takes place.

In this model, the time unit is defined as one growth attempt. In numerical simulations at each growth attempt the time \( t \) is increased by \( \delta t \), where \( \delta t = 1/N \).

Notice that the condition for the interface becomes pinned is that:

- \( \Delta h = \pm a, 0 \) and
- all the sites above the interface are blocked.

So, the interface becomes pinned if a cluster of inactive sites connected horizontally or diagonally expands all the lattice. This happens for \( q \geq q_c \). This define a cluster of directed percolation.

For \( q < q_c \) the interface is characterized by two correlations lengths that behave approaching to the critical point as,

\[
\begin{align*}
\xi_\parallel &\sim |q - q_c|^{-\nu_\parallel}, \\
\xi_\perp &\sim |q - q_c|^{-\nu_\perp},
\end{align*}
\]  

which are the characteristic lengths of these clusters in both directions, with \( \nu_\parallel = 1.733 \) and \( \nu_\perp = 1.097 \).

The interface is specified by a set of integer column heights \( h_i (i = 1, \ldots, N) \). During the growth, a column is selected at random with probability \( 1/N \) and compared its height with those of its neighbors. In a temporal step the height in the site \( i \) is increased by,

1. \( a \) if \( j = i + 1 \) and \( h_{i+1} \geq h_i + 2a \) and \( h_i < h_{i+2} \),
2. \( a/2 \) if \( j = i + 1 \) and \( h_{i+1} \geq h_i + 2a \) and \( h_i = h_{i+2} \),
3. \( a \) if \( j = i - 1 \) and \( h_{i-1} \geq h_i + 2a \) and \( h_i < h_{i-2} \),
4. \( a/2 \) if \( j = i - 1 \) and \( h_{i-1} \geq h_i + 2a \) and \( h_i = h_{i-2} \),
5. \( a \) if \( j = i \) and \( h_i < \min(h_{i-1}, h_{i+1}) + 2a \) and \( G_i(h_i + a) = 1 \).

Notice that these are the rules for the growth in this column. In this context this way to describe the evolution equation is equivalent to derive the first moment from the master equation. In Fig. 1 we show this five contributions to the growth of the site \( i \).

The evolution equation \([19]\) for the interface in a time step \( \delta t = 1/N \) is

\[
h_i(t + \delta t) = h_i(t) + \delta t a R_i,
\]  

with

\[
R_i = W_{i+1} + W_{i-1} + G_i(h_i') W_i,
\]
and

\begin{align*}
W_{i\pm 1} &= \Theta(h_{i\pm 1} - h_{i} - 2a)\{[1 - \Theta(h_{i} - h_{i\pm 2})] + \delta_{h_{i},h_{i\pm 2}/2}\}, \\
W_{i} &= 1 - \Theta(h_{i} - \min(h_{i-1}, h_{i+1}) - 2a).
\end{align*}

Here \( h'_{i} = h_{i} + a \) and \( \Theta(x) \) is the unit step function defined as \( \Theta(x) = 1 \) for \( x \geq 0 \) and equals to 0 otherwise. \( G_{i}(h'_{i}) \) equals 1 if the cell at the height \( h'_{i} \) is free or active (i.e. the growth may occur at the next step) or 0 if the cell is blocked or inactive. \( G_{i} \) is called the interface activity function. Taking the limit \( \delta t \to 0 \) and averaging over the lattice we obtain \( \langle h = \langle h_{i}\rangle \rangle \)

\[
\frac{dh}{dt} = \langle 1 - W_{i} \rangle + \langle G_{i}W_{i} \rangle.
\]

Here we used \( a = 1 \) as is usual in a discrete model. This equation allow us the identification of two separate contributions: the lateral \( \langle 1 - W_{i} \rangle \) and the local one \( \langle G_{i}W_{i} \rangle \).

The temporal derivative of the square interface width (DSIW) is:

\[
\frac{dw^{2}}{dt} = 2\langle (h_{i} - \langle h_{i}\rangle) R_{i} \rangle.
\]

The DSIW can also be expressed by means of local and lateral additive contributions. The lateral contribution is

\[
2\left[ \langle (1 - W_{i}) \min(h_{i-1}, h_{i+1}) \rangle - \langle 1 - W_{i} \rangle \langle h_{i} \rangle \right],
\]

and the local contribution is

\[
2\left[ \langle h_{i} G_{i} W_{i} \rangle - \langle h_{i} \rangle \langle G_{i} W_{i} \rangle \right].
\]
Figure 2: DSIW (solid line), and its lateral (○) and local (□) contributions vs ln t; for $q$ equal to 0.3 (A), 0.539 (B) and 0.6 (C).
In order to obtain the lateral contribution [Eq. (12)] we used that \( \Theta(x - y) + \Theta(y - x) - \delta_{x,y} = 1 \).

In Figure 2 we plot both contributions as a function of time for various values of \( q \). At short times, the lateral process is unimportant because \( \Delta h \) is mostly less than two. As \( t \) increases, the behavior of this contribution depends on \( q \). Notice, from Eq. (12), that the lateral contribution may be either negative or positive. The negative contribution tends to smooth out the surface. Figure 2 shows that this case dominate for small \( q \). The positive lateral contribution enhances the roughness. This last effect is very important at the critical value. At this value, the local contribution is practically constant, but the lateral contribution is very strong, enhancing the roughness. This last contribution has important duties on the power law behavior. Generally speaking, the local contribution roughen the interface while the lateral one flatten it for small \( q \), but the lateral contribution also roughen the interface when \( q \) increases. The lateral contribution is enhanced by local growth. The lateral growth may also increase the probability of local growth. This crossing interaction mechanism makes the lateral growth dominant near the criticality.

6 Growth mechanisms for the Buldyrev et al. model

The interface growth takes place under the same initial conditions as the TL model. During the growth, a column is selected at random with probability \( 1/L \) and the highest dry active cell, in the chosen column, that is nearest-neighbor to a wet cell is wetted. Afterwards, we wet all the dry cell below it. The height is increased by [21]:

1. \( a \) if \( h_i \geq \max(h_{i+1}, h_{i-1}) \) and \( G_i(h_i + a) = 1 \),

2. \( a Y_i \) if \( h_i < \max(h_{i+1}, h_{i-1}) \) and \( G_i(h_i + a Y_i) = 1 \),

where

\[
Y_i = \sum_{k=1}^{z_i} k G_i(h_i + ka) \prod_{j=k+1}^{z_i} (1 - G_i(h_i + ja))
\]

and

\[
a z_i = \max(h_{i-1}, h_{i+1}) - h_i,
\]

with \( G_i(h_i + j) \) equal to 1 if the cell is active and 0 if the cell is inactive. In Fig. 3 we represent schematically the rules. Notice that in this model the condition for the system get pinned is that a cluster of blocked sites, connected by nearest-neighbor expand all the lattice. This is achieved, in the static regime for \( q \geq q_c \) with \( q_c = 0.469 \). In this model, the time unit is defined as one growth attempt. In numerical simulations, at each growth attempt, the time \( t \) is increased by \( \delta t = 1/N \). In this way, after \( N \) growth attempts the time is increased in one unit. We consider the evolution for the height of the \( i \)-th site of the process described above. Let us denote by \( h_i(t) \) the height of the \( i \)-th generic site at time \( t \). Freezing the simulation at a given time, we compute the temporal evolution for the interface height in the next time as

\[
h_i(t + \delta t) = h_i(t) + \delta t a \{ \Theta(-z_i) G_i(h_i + 1) + [1 - \Theta(-z_i)] Y_i \},
\]
where $\Theta(x)$ was defined in Sec.5. $Y_i$ is the increase of the height in the $i$-th column due to the contribution of the nearest-lateral-neighbor. Notice that, this kind of growth occurs by wetting the active cells of the chosen site nearest-neighbor of a wet cell by lateral contact, and then eroding all the cells bellow from a wet cell. We shall call contact contribution to the term $[1 - \Theta(-z_i)]Y_i$ and local contribution to the term $\Theta(-z_i)G_i(h_i + 1)$ from Eq. (16).

Averaging over the lattice, taking $\delta t \to 0$, the evolution equation for the square interface width is

$$
\frac{dw^2}{dt} = 2\langle (h_i - \langle h_i \rangle)\Theta(-z_i)G_i \rangle + 2\langle (h_i - \langle h_i \rangle)(1 - \Theta(-z_i))Y_i \rangle .
$$

The first term of both equations can be identified as the local growth contribution, and the second term as the contact growth contribution. In the present work we focus only on the dynamical behavior for the roughness. Figure 4 shows the temporal derivative of the square interface width (DSIW) as a function of time for various values of $p$. The initial condition is $p$ in all regimes. As we expected [24], the power law holds only at the criticality. The DSIW goes asymptotically to zero at the pinning and moving phase. In Figure 5, we show the two contributions to the DSIW for different values of $p$. The local contribution $2\langle (h_i - \langle h_i \rangle)\Theta(-z_i)G_i \rangle$ to the DSIW is always positive. As $p$ decreases this contribution becomes less important, but always rough the interface. On the other hand, for $p > p_c$, the contact contribution $2\langle (h_i - \langle h_i \rangle)(1 - \Theta(-z_i))Y_i \rangle$ can take negative values, smoothing out the surface. Otherwise, for $p \leq p_c$, the contact contribution is always positive roughening the interface. One could expect that the contact contribution always smooth out the surface because it tends to widen the roughen picks. However, near the criticality, the contact growth happens mainly in lateral neighbors cells to few height terraces above the mean height. Then, this new wetted column smooth out locally, but it moves away
from the mean height increasing the roughness.

7 Comparisons between the Buldyrev et al. and the Tang and Leschhorn models

We rescue the similarities between the Buldyrev et al. and the Tang and Leschhorn models. In spite of the strong microscopic differences between their rules, the results obtained from the microscopic equation are similar. The main conclusion is that in both models the lateral (contact) term plays an important role in the power law behavior. The origin of the behavior of the lateral term near the criticality arises from a nonlinear term. This could lead to think that the QKPZ equation contains these features, but this is not the case as we shall show (see Sec. 9). Moreover, the QKPZ does not allow to explain the cross mechanism between the two contribution, as we shall see below.

8 Does the QKPZ describe the DPD models?

Let us explain first why the QKPZ equation does not take into account this coupled effect. In the QKPZ we can distinguish two contributions, the local growth $S = F + \eta(x, h)$ and the lateral one $L = \nu \partial_x^2 h + \frac{\lambda}{2} (\partial_x h)^2$. So, we can write the evolution equation for the height $h = h(x, t)$ as

$$\partial_t h = S + L.$$
Figure 5: Semi-ln plots of the different contributions to the DSIW as a function of time for the Buldyrev et al. model for different values of $p$. The circles represent the contact contribution and the squares represent the local contribution for the moving, the pinning and the critical regime, respectively.
Figure 6: DSIW as a function of time in the critical, pinning, and moving phases for $\lambda = 1$. The parameter $\mathcal{F}$ is 0.464 (solid line), 0.43 (dashed line), and 0.54 (dotted line).

Taking the derivative of the square interface width, $w^2 = \langle (h - \langle h \rangle)^2 \rangle$, its evolution equation is given by

$$\partial_t w^2 = 2\langle (h - \langle h \rangle) \partial_t h \rangle = 2\langle (h - \langle h \rangle) S \rangle + 2\langle (h - \langle h \rangle) \mathcal{L} \rangle$$

(19)

The first term can be identified as the local growth contribution, and the second term as the lateral growth contribution. The separation into these two analytical terms allows us to compare the mechanisms of growth in this model with the mechanisms in the DPD models. We have performed the direct numerical integration of Eq. (3) in one dimension in the discretized version \[25, 26\]

$$h(x, t + \Delta t) = h(x, t) + \Delta t \left\{ h(x - 1, t) + h(x + 1, t) - 2h(x, t) + \frac{\lambda}{8} \left\{ h(x + 1, t) - h(x - 1, t) \right\}^2 + \mathcal{F} + \eta(x, [h(x, t)]) \right\}$$

where $[\ldots]$ denotes the integer part and $\eta$ is uniformly distributed in $[-a, a]$, where $a = 10^{2/3}$ is selected. We choose $\Delta t = 0.01$, use the initial condition $h(x, 0) = 0$ and the periodic boundary conditions. Figure 6 shows the DSIW as a function of time for various values of $\mathcal{F}$. Here we found that the DSIW increases continuously from zero to a maximum value, at difference of the DPD models where $p$ is the initial condition in all regimes. Moreover we did no expected to recover the initial regime because the QKPZ equation is valid only in the hydrodynamic limit. Equally to the DPD models \[19, 24, 29\], the power law holds only at the criticality. The DSIW goes asymptotically to zero at the pinning and moving phase. In the asymptotic regime, the behavior of the DSIW is similar in the QKPZ equation and in the DPD models.

In Figure 7, we show the two contributions to the DSIW for different values of $\mathcal{F}$. The local contribution $2\langle (h - \langle h \rangle) S \rangle$ to the DSIW is always positive. As $\mathcal{F}$ decreases, this contribution
becomes less important but always rough the interface. On the other hand, the lateral contribution \(2\langle (h - \langle h \rangle) L \rangle\) takes negative values in all phases smoothing out the surface. This is an important difference with the DPD models where, for \(p \leq p_c\), the lateral contribution is always positive roughening the interface. So, the QKPZ does not represent exactly the dynamics of the TL model even if the exponents derived from its numerical integration are in accord with the ones obtained for these models. The cross mechanism between both contributions is not taken into account in this equation because the noise is additive. This cross mechanism is due to the fact that the quenched noise is coupled to the dynamic of the interface as was shown by Braunstein et al. [19, 20, 21]. In order to explain the origin of this coupling let us introduce the passage to the continuum of our microscopic equation.

9 Stochastic Differential Equation for the TL model

Expanding Eq. (6) to first order in \(\delta t\), the evolution for the height of this site is [25]

\[
\frac{\partial h_i}{\partial t} = \frac{a}{\tau} R_i + \eta_i ,
\]

where \(\tau = N \delta t\) is the mean lapse between successive election of any site and \(\eta_i\) is a Gaussian “thermal” noise with zero mean and covariance \(\langle \eta_i(t) \eta_j(t') \rangle = (a^2/\tau) R_{ij} \delta(t-t')\) [13]. For this model,

\[
R_i(h_{i-1}, h_i, h_{i+1}) = W_{i+1} + W_{i-1} + G_i(h_i + a) W_i ,
\]

and \(W_{i+1}, W_{i-1}\) and \(W_i\) where defined in Eq. (9).

In order to obtain a stochastic equation we need an analytic representation of \(R_i\). To do this we proceed to regularize the height defining an interpolating function [13] for the difference of height and then to expand the step function to first order in the argument as \(\Theta(x) \approx c_0 + c_1 x + \mathcal{O}(x^2)\). This can be done providing that \(x\) is smooth obtaining

\[
\frac{\partial h(x_i, t)}{\partial t} = \frac{a}{\tau} \left[ W(x_i + a) + W(x_i - a) + W(x_i) F(x_i, h(x_i) + a) \right] + \eta(x_i, t) ,
\]

with

\[
W(x + a) + W(x - a) = (c_0 - 2c_1) + 4c_1^2 (\partial_x h)^2 + a c_1 \left[ \frac{1}{2} + 4(c_0 - 2c_1) \right] \partial_x^2 h ,
\]

\[
W(x) = 1 - (c_0 - 2c_1) - 4c_1^2 (\partial_x h)^2 + \frac{1}{2} a c_1 \partial_x^2 h .
\]

Notice that the argument of \(G = \Theta(p - g(x_i, h(x_i) + a))\) is not smooth, so its expansion is meaningless. In order to recover the early time regime where the dynamics is mainly random deposition with probability \(p\) [28, 29] we must impose the condition \(c_0 = 2c_1\). The final step is
Figure 7: Semi-ln plots of the different contributions to the DSIW as a function of time for different values of $F$ and $\lambda = 1$. The circles (○) represent the local contribution, the triangles (△) represent the lateral contribution, and the squares (□) represent the total DSIW. The (a) plot shows the critical phase $F = 0.464$. The (b) plot shows the pinning phase $F = 0.43$. The (c) plot shows the moving phase $F = 0.54$. 
a coarse-grained spatial average of the variables in order to obtain smooth continuous functions at a macroscopic level. In this way, we obtain the stochastic continuous equation for this model,

\[
\frac{\partial h}{\partial t} = \mu(G) + \nu(G) \partial_x^2 h + \lambda(G) (\partial_x h)^2 + \eta(x, t),
\]

(25)

where \(G \equiv G(x, h)\)

\[
\mu(G) = G \frac{a}{\tau},
\]

\[
\nu(G) = \frac{1}{2} c_1 (1 + G) \frac{a^2}{\tau},
\]

\[
\lambda(G) = 4 c_2^2 (1 - G) \frac{a}{\tau}.
\]

Equation (25) shows that the nonlinearity arises naturally as a consequence of the microscopic model. As we approach to the critical value, in the dynamical regime, the density of active sites \(f = \langle G \rangle\) goes asymptotically to zero [20] and the coefficient of the nonlinear term becomes relevant. In these case the main responsible of the nonlinearities is the lateral contribution [see Eq. (23)]. This explain why this contribution enhances the roughness at the criticality as was predicted by Braunstein et al. [20]. Faraway above the criticality the nonlinear term becomes less relevant. In the limit \(p \to 0 \ (p \to 1)\) we recover the KPZ (EW) equation with thermal noise as was expected. In the asymptotic regime \((t \gg t^*)\) the various derivatives of the height become very small on a coarse grained scale. In this temporal regime the mean height speed (MHS) \(\langle \partial h/\partial t \rangle \to f\). For \(p \leq p_c, f \to 0\) and the MHS goes to zero while for \(p \gg p_c, f \to \text{const}\) and MHS goes to constant.

Notice that our equation is invariant under local tilting of the interface by an infinitesimal angle in the same way that the QKPZ equation (3) is [3], so the previous numerical results obtained by Amaral et al. [13], that studied the effects of the effective coefficient \(\lambda_{eff}\) from a tilted interface, are compatible with our equation. Our result is also in agreement with those of Réka et al. [16] that obtained numerically a parabolic shape of the local velocity as a function of the gradient for the DPD model near above the criticality for different reduced forces \((p/p_c - 1)\).

In the experiments, the advancement of the interface is determinated by the coupled effect of the random distribution of the capillary sizes, the surface tension and the local properties of the flow, so it is not surprising that all these effect give rise to a multiplicative noise. This multiplicative noise must be taken into account at the time to pose a model with the essential features of the experiment of surface growth in disordered media. In the TL and the Buldyrev et al. models the growing rules for the evolution of the local height are strongly coupled to the quenched noise in a multiplicative way. In both models the microscopic rules that allows the growth from an unblocked cell [19, 24] depends in some way on the local slope. In that sense this coupled effect is not taken into account in the QKPZ equation. The effect of a multiplicative noise has been proposed by Csahók et al. [25] by means of a phenomenological equation. They found a crossover between two temporal regimes with \(\beta = 0.65\) to \(\beta = 0.26\) but the value of \(\alpha \approx 0.47\) was obtained over a short range spatial scale. Indeed, the exponents are not the same.
as of the DPD models. Moreover, processes with the same exponent may not belong to the same universality class. For example, 1 + 1-dimensional lattice gas simulations of roughening of immiscible fluid-fluid interface \[30\] lead to the same exponents as the 1 + 1-dimensional KPZ \[3\] \((\beta = 1/3 \text{ and } \alpha = 1/2)\) for surface growth, but this model is completely linear, so there is no obvious mathematical relationship between these two processes.

10 Summary

The microscopic equations for the models with quenched noise, treated in this work, allows to gain a more profound insight on the principal mechanism of growth. In special, the separation into the lateral and the local growth contributions allows to explain the great interplay between them. Obviously, the continuous equation that represents the local growth of these processes must take into account both mechanisms in addition of the symmetries allowed by the model, but this is not enough to reproduce the interplay between both mechanisms. The QKPZ equation contains a lateral and a local contribution but the quenched noise is additive. Notice that a relevant common feature of the two models treated in this work is that the rules for growth in active sites depend strongly on the slope. So, it is not surprising that this lead to a coupled effect of the quenched noise to the dynamics. This coupled effect is not taken into account in any equation with additive noise. Nevertheless, the QKPZ equation gives the same scaling exponents that these models, moreover it is not clear why. Our results suggest that the QKPZ equation does not describe properly the dynamics of the DPD models even if the exponents are similar. Our Langevin equation for the TL model reflects this coupled effect through its coefficient noise dependence. The equation obtained has the same terms than the QKPZ but its coefficient depends on the competition between the driving force and the quenched noise. In that sense, our equation is multiplicative in the noise.

Finally, we conclude that the classification of these models in universality classes is not fully developed. The derivation of the continuous equation from the microscopic dynamics is a powerful method that allows to associate them in a non ambiguous way.

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