Uniform-Price Mechanism Design for a Large Population of Dynamic Agents

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Abstract—This paper focuses on the coordination of a large population of dynamic agents with private information over multiple periods. Each agent maximizes the individual utility, while the coordinator determines the market rule to achieve group objectives. The coordination problem is formulated as a dynamic mechanism design problem. A mechanism is proposed based on the competitive equilibrium of the large population game. We derive the conditions for the general nonlinear dynamic systems under which the proposed mechanism is incentive compatible and can implement the social choice function in \( \epsilon \)-Nash equilibrium. In addition, we show that for linear quadratic problems with bounded parameters, the proposed mechanism can maximize the social welfare subject to a total resource constraint in \( \epsilon \)-dominant strategy equilibrium.

I. INTRODUCTION

The coordination problem of a large number of agents has attracted extensive research attentions over recent years due to its significance in many domains [1], [2], [3]. It typically involves coordinating a group of self-interested agents to achieve group objectives. Based on the information available to the coordinator, the coordination problem can be categorized in two types: the complete information problem and the incomplete information problem. When the coordinator has complete information, the problem can be cast as a Stackelberg game [4], [5], [6] or reverse Stackelberg game [7], [8], [9]. Due to the bilevel nature of the problem, aside from the linear quadratic case, it is rather computationally intensive to solve the Stackelberg solution for a large-scale dynamic Stackelberg games in general form. When the coordinator has incomplete information, there are various ways to achieve the group objective. One branch of work realizes efficient energy allocation by iteratively exchanging information with individual agents [10], [11], [12] until it converges to the social optimal solution. This approach may require considerable communication resources. In addition, in order to achieve the desired aggregate response, the individual needs to bid truthfully during the iterations. However, it is hard to guarantee that the self-interested agents do not bid strategically for his own benefit. Another branch of works learns the individual utilities based on the past agent responses [13], [14]. This approach approximates the agent response to coordination signals in least-square manner using a group of basis functions. The performance of such method critically depends on the selection of basis functions, and it is hard to reliably induce the desired aggregate response in a systematic way.

Another important paradigm that deals with private information is called mechanism design. In a mechanism design problem, the coordinator needs to provide proper incentives to align the individual preferences with the social choice, i.e., the strategic behaviors of the self-interested agents result in an outcome that achieves the desired group objectives. Many mechanisms have been proposed to implement the optimal social choice, [1], [3], [15], [16], [17], [18]. Some of these works consider the mechanism design problem in dynamic setting [19], [20], [21], [22], [23], [24]. Although these existing works guarantee incentive compatibility and realize efficient resource allocation, they typically lead to discriminatory pricing schemes: the unit price of the mechanism is different for different agents. Such pricing scheme is not applicable in many applications, including the existing electricity market.

Uniform-price mechanisms draw extensive research attentions due to its wide applicability in engineering problems. Many works study the design of incentive mechanisms for price-taking agents [2], [25], [26] who do not anticipate the effect of his bid on the market clearing price. While intuitively the effect of individual agent actions on market price may vanish if the population grows large, the price-taker assumption requires a more rigorous justification. In the case where the agents are not treated as price-takers, the interactions between agent decisions make it challenging to even analyze the game solutions of a given mechanism. One branch of works studies the decentralized convergence to the solution of some given mechanisms using mean-field approximations [27], [28], [29], [30], [31], assuming that the agents are indifferent to a change in his utility up to a constant. While such method proposes a way to efficiently compute the game solutions, it does not involve group objectives. In contrast, another branch of works designs mechanisms that take group objectives into account [11], [32], [33]. In these works, proper pricing schemes are proposed to motivate agents to achieve or approximately achieve socially optimal resource allocations. However, most of these works focus on static mechanism design, where system dynamics are not involved.

In this paper we consider the coordination of a large population of dynamic agents to achieve group objectives. Each individual agent is self-interested and strategically chooses control actions to maximize his utility. In the meanwhile, the coordinator needs to provide proper incentives (typically via pricing) for individual agents to achieve certain group objectives, i.e., maximize the social welfare. This problem poses several challenges: first, we focus on uniform-price
mechanisms, where the agents face a single-valued unit price. Standard mechanism design approach such as VCG mechanism is not directly applicable in this scenario. Second, to identify the optimal mechanism that maximizes the social welfare, a subproblem is to determine the solution of the game induced by a given mechanism. This problem is challenging, especially when large-scale coupled dynamic systems are involved. Existing works mostly focus on mean field control [27], [28], [29], [30], [31], which may not be optimal. A more general framework that addresses other scenarios are suspiciously missing.

The key contribution of this work lies in the development of a mechanism that addresses the aforementioned challenges. In this paper, we solve the coordination problem using the mechanism design approach. The self-interested agents are modeled as strategic individual utility maximizers, while the group objective is encoded in the social choice function, which is to maximize the social welfare. A uniform-price mechanism is proposed. We show that when the effect of individual bid on the market clearing price is bounded, the proposed mechanism is incentive compatible for general nonlinear dynamic systems. We also argue that when the social choice function is uniform-price implementable, the proposed mechanism can implement the social choice function in ε-Nash equilibrium. Furthermore, for linear quadratic problems, stronger results can be obtained: as long as the system parameters are properly bounded, the proposed mechanism can maximize the social welfare in ε-dominant strategy equilibrium.

The rest of the paper proceeds as follows. The coordination problem is formulated as a dynamic mechanism design problem in Section II. A uniform-price mechanism is proposed in Section III. In Section IV a special case is studied: the linear quadratic problem. We show that the proposed mechanism maximizes the social welfare subject to a total resource constraint in ε-dominant strategy equilibrium.

II. PROBLEM FORMULATION

In this section we formulate the coordination problem as a mechanism design problem. Each agent is modeled as an individual utility maximizer who chooses a bid to optimize individual utility over a planning horizon. In the meanwhile, the coordinator needs to design the message space and market clearing rules (mechanism) to motivate the self-interested agents to achieve group objectives.

A. Individual Utility Maximization

Consider a group of N agents with a planning horizon of length K. Let \( x_k^i \) and \( a_k^i \) denote the system state and control action of the \( i \)-th agent at \( k \)-th period, respectively. The system state and control are typically bounded. Therefore, we consider the following discrete time dynamic systems with constraint:

\[
\begin{align*}
    x_{k+1}^i &= f_i(x_k^i, a_k^i; \theta_k^i) \\
    a_k^i &\in \Omega_k^i,
\end{align*}
\]

where \( \theta_k^i \in \Theta^i \) denotes the private information of agent \( i \).

In the mechanism design context, the private information is typically referred to as the agent type. In the individual decision problem, each agent acts strategically to maximize the individual utility, which is the valuation minus the payment. The agent valuation function \( V_k^i : X_k^i \times \Omega_k^i \rightarrow R \) is assumed to be stage-additive. It evaluates the level of comfort the agent experiences when receiving an allocation decision \( a_k^i \in \Omega_k^i \) at state \( x_k^i \in X_k^i \). Let \( p_k \) denote the unit price of the allocation at time \( k \), then the utility of the \( i \)-th agent can be denoted as follows:

\[
\begin{align*}
    K \quad \sum_{k=1}^{K} U_k^i(x_k^i, a_k^i, p_k; \theta_k^i) &= \sum_{k=1}^{K} V_k^i(x_k^i, a_k^i; \theta_k^i) - p_k a_k^i \quad (2)
\end{align*}
\]

Since the agents are self-interested, each agent would choose control actions to maximize his utility subject to the dynamic constraint (1).

Remark 1: In the above formulation, the unit price \( p_k \) is the same to different agents. This is called the non-discriminatory pricing. The non-discriminatory pricing scheme is compatible with many existing markets, and is easy to implement. On the other hand, many existing works focus on VCG-based mechanism design [3], [34], [35], which typically leads to discriminatory pricing. The uniform-price mechanism design problem (especially the dynamic case) is not well studied yet.

B. The Mechanism Design Problem

Since the agents have private information, the coordinator asks each agent to submit a bid denoted as \( r^i = (r_1^i, \ldots, r_K^i) \). This bid must be in the message space, \( M^i \), specified by the coordinator. The coordinator then collects these bids and determines the market outcome accordingly, which consists of the allocation decision \( a = (a_1, \ldots, a_K) \) and the unit price \( p = (p_1, \ldots, p_K) \), where \( a_k = (a_1^k, \ldots, a_K^k) \). This market clearing process maps the agent bids to the market outcome, which can be modeled as an outcome function \( g : M^1 \times \cdots \times M^N \rightarrow \Omega \times P \), where \( \Omega \) denotes all feasible resource allocations and \( P \) denotes all feasible unit prices. A mechanism consists of the message space and the outcome function, whose formal definition is given below:

Definition 1: A mechanism \( \Gamma = (M, g(\cdot)) \) consists of a collection of message spaces \( M \) and an outcome function \( g(\cdot) \), where \( M = M^1 \times \cdots \times M^N \).

Based on the definition, the outcome function can be written as \( g(r) = (a_1, \ldots, a_N, p) \). For notation convenience, we denote \( g_k^i(a) = a_k^i \) and \( g_k^i(r) = p_k \). In our problem, when a mechanism is given, each agent is faced with the following utility maximization problem:

\[
\begin{align*}
    \max_{r^i \in M^i} \quad & \sum_{k=1}^{K} U_k^i(x_k^i, g_k^i(r_k), g_k^i(p_k); \theta_k^i) \\
    \text{s.t.} \quad & x_{k+1}^i = f_i(x_k^i, g_k^i(r_k); \theta_k^i) \\
    & a_k^i \in \Omega_k^i
\end{align*}
\]

Note that in the utility maximization problem (3), since \( r \) is the vector consisting of all agent bids, each agent’s utility
depends on the actions of other agents. Therefore, this is a game problem induced by the mechanism \( \Gamma \). We denote this game as \( G(\Gamma) \).

The goal of this paper is to find the mechanism that achieves the group objective. The group objective can be encoded in a social choice function \( \phi: \Theta^1 \times \cdots \times \Theta^N \rightarrow \Omega \) that maps the agent type to a feasible resource allocation. Here we consider the social choice function that maximizes the social welfare subject to a total resource constraint:

\[
\phi(\theta) = \arg \max_{\theta} \sum_{i=1}^{N} \sum_{k=1}^{K} V_i^k(x_i^k, a_i^k; \theta_i^k) + \sum_{k=1}^{K} \sigma_k \left( \sum_{i=1}^{N} a_i^k \right)
\]

s.t. \[
\begin{align*}
& x_{i+1}^k = f_i(x_i^k, a_i^k; \theta_i^k) \\
& a_i^k \in \Omega_k^i \\
& \sum_{i=1}^{N} a_i^k \leq D_k \quad \forall k
\end{align*}
\]

In the above definition, \( D_k \) is the total resource constraint, and \( \sigma_k(\cdot) \) is a function denoting the cost for the coordinator to procure certain amount of resources at time \( k \). The value of the social choice function indicates the socially desired resource allocation. On the other hand, the individual agent is self-interested. When a mechanism is given, he chooses the bid according to his type to maximize his own individual utility. This decision making process can be captured by a bidding strategy \( m_i^k: \Theta \rightarrow M_i^k \). In our case, \( m_i^1(\theta_i) = (m_i^1(\theta_i^1), \ldots, m_i^N(\theta_i^N)) \) is the game solution to \( G(\Gamma) \), which depends on the mechanism. Different mechanisms result in different resource allocations. Therefore, to achieve the group objective, the coordinator needs to find a mechanism whose resulting allocation plan coincides with the value of the social choice function.

**Definition 2:** The mechanism \( \Gamma \) implements the social choice function \( \phi(\cdot) \) if there exists an equilibrium strategy profile \( (m_i^1(\cdot), \ldots, m_i^N(\cdot)) \) of the game \( G(\Gamma) \) such that

\[
g_a(m_i^1(\theta_i), \ldots, m_i^N(\theta_i^N)) = \phi(\theta), \quad \forall \theta \in \Theta.
\]  

where \( \Theta = \Theta^1 \times \cdots \times \Theta^N \).

To this point, the problem of this paper can be stated as follows: design a uniform-price mechanism to implement the social choice function \( \phi(\cdot) \).

**III. The Uniform-price Mechanism**

In this section, we first propose a mechanism based on the competitive equilibrium of the market. Then, we show that the proposed mechanism can implement the social choice function when each agent strategically chooses bid to maximize individual utilities. Due to revelation principle, we focus on the direct mechanism, where the message space is the space of agent types, i.e., \( M^i = \Theta^i \) for \( \forall i \).

**A. The Proposed Mechanism**

To introduce our proposed mechanism, let us first consider a price-response problem, where each agent takes price as given and chooses optimal control \( a^i \) to maximize the individual utility. In the price-response problem, the control strategy of each agent depends on the price. Therefore, we have the following:

\[
\mu^i(p, r^i) = \arg \max_{\theta} \sum_{k=1}^{K} U_k^i(x_k^i, a_k^i, p_k^i, r_k^i) 
\]

s.t.

\[
\begin{align*}
& x_{k+1}^i = f_i(x_k^i, a_k^i, r_k^i) \\
& a_k^i \in \Omega_k^i \\
& r_k^i \leq D_k
\end{align*}
\]

In the context of economics, the price-allocation pair \( (p, \mu^1(p, r^1), \ldots, \mu^N(p, r^N)) \) is defined as the competitive equilibrium [35]. Here we propose a mechanism \( \Gamma_c \) based on the competitive equilibrium concept to solve the mechanism design problem of this paper.

**Definition 3:** The proposed mechanism \( \Gamma_c \) is a direct mechanism with the following outcome function:

\[
\begin{align*}
g_p(r) &= \arg \max_{\theta} \sum_{k=1}^{K} \sum_{i=1}^{N} V_i^k(x_i^*, a_i^*, r_i^*) \\
& \quad + \sum_{k=1}^{K} \sigma_k \left( \sum_{i=1}^{N} a_i^* \right)
\]

s.t.

\[
\begin{align*}
& x_{i+1}^* = f_i(x_i^*, a_i^*; r_i^*) \\
& a_i^* = \mu^i(p, r^i) \\
& \sum_{i=1}^{N} a_i^* \leq D_k, \quad \forall k
\end{align*}
\]

where \( g_p^a(r) = (g_{1, a}^p(r), \ldots, g_{K, a}^p(r)) \) and \( g_p(r) = (g_{1, p}(r), \ldots, g_{K, p}(r)) \). In the above problem, each price corresponds to a resource allocation in the price-response problem [5]. The proposed mechanism clears the market with a price that leads to the socially optimal resource allocations.

**Remark 2:** Note that although the mechanism is defined based on the price-taker game [5], we do not assume that agents are price-taker. On the contrary, in our problem we consider each agent to be price anticipative. Therefore, the resulting outcome of the game may differ from the competitive equilibrium [11], [32]. On the other hand, we conjecture that the effect of the bids of an individual agent on the market price will diminish as the population size grows. In this case the agent behavior should be very similar to price-taker agents when the game has a large population. The rest of this section provides rigorous justification for this conjecture.

**B. Properties of the Proposed Mechanism**

This subsection discusses some properties of the proposed mechanism.

In section II, we have denoted \( m^1(\theta) \) as the solution to the induced game \( G(\Gamma) \). As a matter of fact, there are several solution concepts for a game, including Nash equilibrium, Bayesian Nash equilibrium, dominant strategy equilibrium, etc [35]. In this paper we choose the \( \epsilon \)-Nash equilibrium solution concept, which assumes that each agent is indifferent to a change in his utility up to a constant \( \epsilon \). Different from Nash equilibrium, which is rather hard to characterize for a given mechanism, the \( \epsilon \)-Nash equilibrium concept enriches the class of mechanisms that we can select from. In addition, a
utility of $0.999\$ may have no difference from a utility of $1\$ to real agents. Therefore, this assumption is reasonable in many practical applications. If we let 
$$f_k(r) = \left( g_{k,a}(r), g_{k,p}(r) \right),$$
and plug in the system dynamics (1) into the individualize utility maximization problem (3), then the stage utility of the $i$th agent at time $k$ can be written as a function of $f_k(r)$ only (neglecting the dependence on initial state). Using these notations, the formal definition of the $\epsilon$-equilibrium is given as below:

**Definition 4:** Given a game $G(\Gamma)$, a bidding collection $r^\ast = (r_1^\ast, \ldots, r_N^\ast)$ is an $\epsilon$-equilibrium of the game if for any $i = 1, \ldots, N$, we have:
\[
\sum_{k=1}^{K} U_k^i(f_k(r^\ast); \theta_k^i) \geq \sum_{k=1}^{K} U_k^i(f_k(r^\ast); \theta_k^i) - \epsilon
\]  
for all $r^i \in M^i$. For a given direct mechanism, when bidding truthfully ($r^i = \theta^i$) is an $\epsilon$-equilibrium of the induced game, we say the mechanism is incentive compatible in $\epsilon$-equilibrium. Under certain conditions, we can show that the proposed mechanism is incentive compatible. To present our result, we first introduce some regularity conditions.

**Definition 5:** We say a function $\varphi: R^k \rightarrow R$ is Lipschitz continuous with constant $C$ in $L_\infty$ norm for $x \in X$, if there exists $C > 0$ such that:
\[
|\varphi(x_1) - \varphi(x_2)| \leq C||x_1 - x_2||_\infty, \forall x_1 \in X, x_2 \in X
\]  

Throughout the rest of the paper, whenever we say a function is Lipschitz continuous with constant $C$, we mean the function is Lipschitz continuous with constant $C$ in $L_\infty$ norm for its parameters in the corresponding space. The incentive compatibility result of the proposed mechanism can be summarized as the following proposition:

**Proposition 1:** Assume that $U_k^i$ is Lipschitz continuous with respect to both $a_k^i$ and $p_k^i$ with constant $C_1$ and $C_2$, respectively. Assume $\mu_k^i$ is Lipschitz continuous with respect to $p$ with constant $C_3$. Furthermore, assume that there exists $\epsilon_1 > 0$ for the proposed mechanism $\Gamma_c$ such that:
\[
|g_{k,p}(r^i, \theta^{-i}) - g_{k,p}(\theta^i, \theta^{-i})| \leq \epsilon_1, \quad \forall r^i \in M^i,
\]  
then $\Gamma_c$ is incentive compatible in $\epsilon$-equilibrium where $\epsilon = K(C_1C_3 + C_2)\epsilon_1$. 

**Proof:** Let $r^i$ denotes the bids of the $i$th agent. Let $p^\ast = g_p(r)$. According to the definition of the proposed mechanism $\Gamma_c$, for each agent $i$, we have the following:
\[
\sum_{k=1}^{K} U_k^i(\mu_k^i(p^\ast, r^i), p^\ast; r^i) \geq \sum_{k=1}^{K} U_k^i(\mu_k^i(p^\ast, \tilde{r}^i), p^\ast; r^i)
\]  
for all $\tilde{r}^i \in M^i$. Due to the Lipschitz continuity and the assumption (5) of the proposition, it can be verified that the following inequality holds:
\[
\sum_{k=1}^{K} U_k^i(\mu_k^i(g_p(r, \tilde{r}^i), g_p(r); r^i) + \epsilon \geq \sum_{k=1}^{K} U_k^i(\mu_k^i(g_p(\tilde{r}^i, r^{-i}), g_k,p(\tilde{r}^i, r^{-i}); r^i) - \epsilon
\]  
for all $\tilde{r}^i \in M^i$. Therefore, the proposed mechanism guarantees that the agent bid $r$ satisfies the above condition (11).

On the other hand, the $i$th agent seeks to find a bid $r_i$ such that:
\[
\sum_{k=1}^{K} U_k^i(\mu_k^i(g_p(r^i, \theta^{-i}), \theta^i), g_k,p(\tilde{r}^i, r^{-i}); \theta^i) \geq \sum_{k=1}^{K} U_k^i(\mu_k^i(g_p(\tilde{r}^i, \theta^{-i}), \theta^i), g_k,p(\tilde{r}^i, r^{-i}); \theta^i) - \epsilon
\]  
for all $\tilde{r}^i \in M^i$. Letting $r_i = \theta_i$ exactly satisfies this objective. This completes the proof.

**Remark 3:** We comment that in general it is rather challenging to characterize the solution to a large-scale dynamic game. Our result borrows some principle ideas from mean-field approximations [27], [28], [29], [30], [31]. However, these works are mostly based on mean-field control, which is not optimal for the desired social choice. In our problem, the optimal mechanism is much more complicated than mean-field control, and therefore much more difficult to analyze.

In our paper, since the mechanism is designed based on the competitive equilibrium, which deviates from the true agent behavior, incentive compatibility does not necessarily guarantee the price objective can be achieved. To account for this, we first introduce the following definition before proceeding to the key result of this paper.

**Definition 6:** Given a resource allocation $a$, we say $a$ is uniform-price implementable if there exists a price $p \in P$ such that $a^i = \mu^i(p, \theta^i)$ for all $i = 1, \ldots, N$. 
Based on the definition, if a resource allocation $a$ is uniform-price implementable by $p$, then $(a, p)$ is a competitive equilibrium of the game. Now we can present the key result of this paper:

**Proposition 2:** Assume that the social choice function $\phi(\theta)$ is uniform-price implementable, and the proposed mechanism $\Gamma_c$ is incentive compatible, then $\Gamma_c$ cannot implement the social choice function $\phi(\theta)$ in $\epsilon$-Nash equilibrium.

**Proof:** On one hand, based on Definition 3, the proposed mechanism $\Gamma_c$ renders an optimal resource allocation among all uniform-price implementable allocations. Therefore, the resulting social welfare is no less than that of the desired social choice function $\phi(\theta)$.

On the other hand, the social choice function is the team problem for the proposed mechanism $\Gamma$ [36], [37]. Therefore, it provides an upper bound for attainable welfare for $\Gamma$. This completes the proof. ■

### IV. THE LINEAR QUADRATIC CASE

In this section, we derive the conditions under which the proposed mechanism can achieve the group objective in linear quadratic problems.

Based on Proposition 2 to achieve the group objective, the following conditions need to be verified: first, the social choice function should be uniform-price implementable; second, the proposed mechanism $\Gamma_c$ needs to be incentive compatible. In the rest of this section, we will introduce the problem setup and check these two conditions in this setup.

#### A. Problem Setup

Let us consider the following linear dynamical system for the $i$th agent:

$$x_{k+1}^i = A^i x_k^i + B_k^i a_k^i$$

(13)

where $x_k^i \in R$. We assume that $0 < A \leq A_i \leq \bar{A}$ and $B \leq B_i \leq \bar{B} < 0$ for all $i = 1, \ldots, N$. This linear model captures various dynamic assets, including batteries, PEVs, thermostatically controlled loads (with proper control strategy), among others.

Let $(d_1^i, \ldots, d_K^i)$ be the desired state trajectory for agent $i$. Consider a valuation function of quadratic form that penalizes the deviation from this desired sequence. The individual utility can be written as follows:

$$\sum_{k=1}^{K} U_k^i(x_k^i, a_k^i, p_k; \theta_k^i) = \sum_{k=1}^{K} \beta_k^i (x_k^i - d_k^i)^2 - p_k a_k^i$$

(14)

where $\theta_k^i = (A^i, B^i, C^i, \beta^i, d_k)$. Note that $B^i$ denotes a vector, and the same applies to other time-varying parameters.

In this setup, the social choice function has the following form:

$$\phi_{LQ}(\theta) = \arg \max_a \sum_{i=1}^{N} \sum_{k=1}^{K} \beta_k^i (x_k^i - d_k^i)^2 + \sum_{k=1}^{K} \sigma_k \left( \sum_{i=1}^{N} a_k^i \right)$$

(15)

where $\beta_k^i < 0$ for all $i$ and $k$. For sake of analysis, we assume that the function $\sigma_k$ has a linear structure, i.e., $\sigma_k(\sum_{i=1}^{N} a_k^i) = p_k^w (\sum_{i=1}^{N} a_k^i)$. In electricity market, $p_k^w$ is the wholesale energy price.

#### B. Uniform-price Implementability

In this subsection we show that the social choice function $\phi_{LQ}(\cdot)$ is uniform-price implementable.

**Proposition 3:** The social choice function $\phi_{LQ}(\cdot)$ is uniform-price implementable, i.e., for any report $r$, there exists a price $p^r$ such that $\phi_{LQ}(r) = \mu^i(p^r, r^i)$ for all $i = 1, \ldots, N$, where $\phi_{LQ}(r)$ is the allocation to the $i$th agent.

**Proof:** First, under a given price $p^r$, the optimality conditions for the price-response problem (5) are as follows:

$$- \sum_{k=1}^{K} \frac{\partial V_k^i(x_k^i)}{\partial a_k^i} \bigg|_{a_k^i = \mu_k^i(p^r, r^i)} + p_k^w = 0$$

(16)

On the other hand, the social welfare maximization problem (15) is convex. The necessary and sufficient optimality condition is that there exists a unique multiplier $\xi^i$, such that:

$$- \sum_{k=1}^{K} \frac{\partial V_k^i(x_k^i)}{\partial a_k^i} \bigg|_{a_k^i = \mu_k^i(p^r, r^i)} + p_k^w + \xi_k = 0$$

(17)

where $\phi_{LQ}^i(k)$ is the allocation to the $i$th agent at time $k$, and $\xi_k$ is non-negative and satisfies $\xi_k(\sum_{i=1}^{N} \phi_{LQ}^i(k) - D) = 0$ for all $k = 1, \ldots, K$. Let $\tilde{p}_k^w = p_k^w + \xi_k$. It can be verified that $\phi_{LQ}(r) = \mu^i(p^r, r^i)$ is a solution to (16). Since the price-response problem is convex, optimality condition (16) is both necessary and sufficient. This indicates that $\tilde{p}_k^w$ can implement the social choice function, which completes the proof. ■

The result of Proposition 3 shows that there exists a price to implement the social choice function $\phi_{LQ}(\cdot)$. In addition to this, we can also derive the conditions for this price, which can be summarized as the following corollary:

**Corollary 1:** The price $p^r$ satisfies $\phi_{LQ}^i(r) = \mu^i(p^r, r^i)$ for all $i = 1, \ldots, N$ if and only if the following condition holds for all $k = 1, \ldots, K$:

$$\begin{cases} \tilde{p}_k^w = p_k^w & \text{if } \sum_{i=1}^{N} \mu_k^i(p^r, r^i) < D_k \\ \tilde{p}_k^w \geq p_k^w & \text{if } \sum_{i=1}^{N} \mu_k^i(p^r, r^i) = D_k \end{cases}$$

(18)

This result can be directly derived from the proof of Proposition 3. We comment that the result holds when $\sigma_k(\cdot)$ is nonlinear. In this case, $p_k^w$ is the marginal cost for obtaining certain amount of resources.

#### C. Incentive Compatibility

Now we verify the incentive compatibility of the proposed mechanism based on Proposition 1. First, we observe that under the linear quadratic setup, utility function $U_k^i$ is Lipschitz continuous with respect to both $a_k^i$ and $p_k^w$ for a bounded domain. To verify the Lipschitz continuity of the
function $\mu^i$, we observe that the analytic expression of $\mu^i$ can be derived for the linear quadratic case:

$$
\begin{align*}
\mu^i_k(p^i, r^i) &= \theta^i_k p^i_k + \theta^i_{k-1} p^i_{k-1} - \theta^i_{k+1} p^i_{k+1}, \quad \forall k \neq 1 \\
\mu^i_1(p^i, r^i) &= \theta^i_1 p^i_1 - \theta^i_2 p^i_2
\end{align*}
$$

where $\theta^i_k = \frac{A^i_k}{2\varphi^i_k B^i_k}$, $\theta^i_2 = -\frac{A^i_2}{2\varphi^i_2 B^i_2}$, and the coefficients for $k \neq 1$ are defined as follows:

$$
\begin{align*}
\theta^i_k &= \frac{1}{2\varphi^i_k B^i_k} + \frac{A^i_k A^i_k}{A^i_k}, \\
\theta^i_{k-1} &= -\frac{2\varphi^i_{k-1} B^i_{k-1} B^i_k}{A^i_k}, \\
\theta^i_{k+1} &= \frac{2\varphi^i_{k+1} B^i_{k+1}}{A^i_k}
\end{align*}
$$

(19)

Since the function $\mu^i$ is linear with respect to $p^i$, it is Lipschitz continuous. In addition to Lipschitz continuity, we can also show that condition (5) holds for the linear quadratic case.

**Proposition 4:** In the linear quadratic setup, there exists $C_4 > 0$ (not depending on $N$) for the proposed mechanism $\Gamma_c$ such that:

$$
|g_{k,p}(r^i, \theta^{-i}) - g_{k,p}(\theta^i, \theta^{-i})| \leq C_4/N, \quad \forall r^i \in M^i, \quad (20)
$$

**Proof:** To prove this result, we need to study how agent bid affects the price. The price and the agent bids satisfy (18). This optimality condition can be transformed into the following equation $\nu(p, r^i) = 0$, where $p$ is the root of the mapping $\nu: P \times M^i \rightarrow P$, and this mapping can be defined as follows:

$$
\nu_k(p, r^i) = \begin{cases}
\alpha_k^1(\sum_{i=1}^N \mu^i_k(p, r^i) - D_k), & \text{if } p_k \geq \sigma'(D_k) \\
\alpha_k^2(p_k - \sigma'(\sum_{i=1}^N \mu^i_k(p, r^i))), & \text{otherwise}
\end{cases}
$$

(21)

where $\nu_k$ is the $k$th element of vector $\nu$.

Let $J_{\nu_p}$ and $J_{\nu_r}$ be the Jacobian matrix of $\nu$ with respect to $p$ and $r^i$, respectively. If $J_{\nu_p}$ is invertible, then $P$ is a function of $r^i$ (if not, $r^i$ does not affect $p$, and the result of the proposition holds). Let $J_{\nu_r}$ be the Jacobian matrix of $p$ with respect to $r^i$ and we have $J_{\nu_r} = -(J_{\nu_p})^{-1}J_{\nu_r}$, whenever $J_{\nu_r}$ exists. Based on (19), since the report $r^i$ is bounded, there exists $C_5 > 0$ and $C_6 > 0$ such that $C_5 N \leq \frac{\partial \nu_k}{\partial p_k} \leq C_6 N$ for all $k$ and $k'$ where $p_k \geq \sigma'(D)$. Due to the particular structure of $J_{\nu_r}$, we can derive the analytic form for $(J_{\nu_p})^{-1}$ and verify that there exists $C_7$ not depending on $N$ such that $\frac{\partial \nu_k}{\partial r^i} \leq C_7/N$. Therefore, as $r^i$ is bounded, there exists $m > 0$ not depending on $N$, such that:

$$
|g_{k,p}(r^i, \theta^{-i}) - g_{k,p}(\theta^i, \theta^{-i})| \leq \sum_{l=1}^m \frac{\partial \nu_k}{\partial r^i} d r^i \leq \frac{L m C_7}{N}
$$

(22)

In this case, $C_4 = \frac{L m C_7}{N}$, which completes the proof.

To this point, all the conditions for Proposition 1 are verified. Therefore, the proposed mechanism is incentive compatible. Furthermore, since the social choice function $\phi_{LQ}$ is uniform-price implementable, we conclude that the mechanism $\Gamma_c$ implements the social choice function $\phi_{LQ}$ in $\epsilon$-Nash equilibrium.

**Remark 4:** We comment that in the linear quadratic case, the proposed mechanism actually implements the social choice function in $\epsilon$-dominant strategy equilibrium. This solution concept is stronger than $\epsilon$-Nash equilibrium. It indicates that if agents are indifferent to an $\epsilon$-perturbation in his utility, then the equilibrium strategy (truthful bidding) is the optimal strategy regardless of other agents’ actions. This solution is rather robust, since the agents do not need to know other agents’ actions when making a decision.

V. CONCLUSION

This paper presents a dynamic mechanism design approach for the coordinator of large population agents with private information. A mechanism is proposed to motivate self-interested agents to achieve group objectives. We show that the proposed mechanism is incentive compatible, and it implements the desired social choice function in $\epsilon$-Nash equilibrium. We also show that for linear quadratic problems, the conditions can be found under which the proposed mechanism maximizes the social welfare in $\epsilon$-dominant strategy equilibrium. Future work includes developing the mechanism design framework for more general dynamic models, and applying the results to practical applications such as thermostatically controlled loads, plug-in electric vehicles, dyers, washers, among others.

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