Analysis of the Stability and Hopf Bifurcation of a Three-Dimensional System with Delays

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Abstract. We propose a model of three-dimensional autonomous system with delays. We explore the dynamical behavior of the proposed autonomous system by examining bifurcation diagrams, Lyapunov exponents, equilibrium and stability, and the influence of time delay on Hopf bifurcation. A bifurcation theory is used to analyze and detail the problem. In addition, the explicit algorithm that determines the direction of Hopf bifurcation, along with the stability of bifurcating periodic, has been established. Also, there are specific operating conditions that must be met in order to achieve Hopf bifurcation. In the proposed autonomous system, we analyze the procedures for designing chaotic based systems including parameter selection, discretization of the results, as well as exploring the changing regularity of the bifurcation value. A series of numerical simulations is presented to illustrate the analytical results.

Keywords: Bifurcation, Periodic Solution, Chaotic Attractor, Delay, Numerical simulations

1. Introduction

An autonomous dynamical systems are described by at least three coupled ordinary nonlinear differential equations, that chaos occurs in multiple nonlinear system when numerically computed. As these different values of parameters, variations may occur in the quantitative figure of the solutions for given parameter values. Hopf bifurcation occurs as a entangle conjugate couple of eigenvalues of the linearized system at a certain point becomes purely imaginary. These systems have a unique dynamical appearance, Hopf bifurcation can only occur in systems of dimension two or higher. Lately, researchers started bifurcation analysis and numerical computer simulation for these systems, e.g [1, 2], [3, 4], [5, 6], references therein. In particular, chaos theory has become practicable tools in facilitation secure optical communication devices and secure data encryption.

Over the past one decade, chaos applications have been used in information processing, nonlinear circuits, and etc. Zhang et al studied Hopf bifurcation with
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chaos entanglement function. The authors reviewed bifurcations of nonlinear dynamical systems in the new chaotic system [7, 8], [9, 10], and [11]. The existence of autonomous systems are extensively encountered in many fields such as engineering and chemistry. Delays are inevitable in autonomous activities; everything takes time, with that we proposed a delayed in Hopf bifurcation in three-dimensional based on chaos entanglement function and the results show there are four requirements that are needed to achieve chaos [12], Kutorzi and Yufeng. The stability problem of the dynamic systems with delays is of more intrigue as part of this study.

In this paper, the priority issue is the stability of dynamic systems with delays. We introduced a time delayed in Hopf bifurcation in three-dimensional based on chaos entanglement function. For autonomous models, chaotic behavior has been developed to displays dynamic behaviors. The systems are considered non-linear because of the multiple feedback between the components of the system. In addition we present a specific range of the operating conditions that are needed to achieve Hopf bifurcation.

The outline of the present paper is organized as follows. In section two and three, we foremost described the characteristics of chaotic three dimensional autonomous system. Significantly, the chaotic dynamics are all established when chaotic dynamics are effectively stabilize, explored in details, and the characteristics are obtained by employing varying subsystems. Furthermore, we study the the algorithm for establishing the direction of hopf bifurcation and as well investigated the stability of bifurcating periodic. In section four, detailed numerical simulations are presented. Finally, we present our findings and conclusion in section five.

2. The Proposed Three-Dimensional Autonomous

Consider 2 linear subsystems: Two-dimensional nonlinear autonomous system and one-dimensional.

\[
\dot{x} = ax - by \\
\dot{y} = cx + ay. 
\] (1)

According to [2] where he extend the model to explore the dynamical properties.

\[
\dot{z} = dz 
\] (2)

Where \((x, y, z)\) are steady variable. When \(a < 0, c < 0\) and \(d < 0\) are both the stable subsystems. Now consider time delay into the system (1) & (2), the proposed three-order time delayed system could be derived thus and so:

\[
\dot{x} = ax - by(t - \tau) \\
\dot{y} = cx + ay \\
\dot{z} = dz(t - \tau)
\] (3)

wherein \(\tau \in R^+\) is the integral time delay in system (3).
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2.1. Stability of Fixed Point

According to system (3), equilibria should satisfy

\[ a x - b y(t - \tau) = 0, \]
\[ c x + a y = 0, \]
\[ d z(t - \tau) = 0 \]

We obtain system (4) has equilibria at the fixed point \( E^\star = (0, 0, 0) \). Additionally, linearizing system (4) around \( E^\star = (0, 0, 0) \) expressed as

\[
J = \begin{pmatrix}
a & -b e^{-\lambda \tau} & 0 \\
c & a & 0 \\
0 & 0 & d e^{-\lambda \tau}
\end{pmatrix}
\]

This leads to the associated characteristic equation

\[
D(\lambda, \tau) = \det(\lambda E - C_1 - C_2 e^{-\lambda \tau})
\]
\[
\lambda^3 + f_1 \lambda^2 + f_2 \lambda + f_3 + (f_4 \lambda + f_5) e^{-\lambda \tau} = 0
\]

wherein

\[
\{ f_1 = a^2 + a + d, f_2 = a^4 + 2a^2 d, f_3 = a^2(1 + d^2), f_4 = -bc, f_5 = a^2 bc \} \quad (8)
\]

The manifold case of (7) with \( \tau = 0 \), the characteristic equation reduces to

\[
\lambda^3 + f_1 \lambda^2 + f_2 \lambda + f_3 + (f_4 \lambda + f_5) = 0
\]

(9)

The roots of (9) have negative real parts for every nonnegative integer and the positive equilibrium solution with restriction of time delay is asymptotically stable. With the help of Routh-Hurwitz criterion determinant structure all roots of (9) is stable if and only if the following hypothesis holds:

\[
(H1) \{ f_1 > 0, f_2 + f_4 > 0, f_3 + f_5 > 0, \quad and \quad f_1(f_2 + f_4) - (f_3 + f_5) > 0 \} \quad (10)
\]

Theorem 1. If (10) is satisfied, then the equilibrium \( E^\star \) of system (3) implies that appropriate parameters are chosen to admits asymptotic stability of the system for \( \tau = 0 \).

Now we review the effect of the distribution of characteristic roots or delay \( \tau \) when \( \tau > 0 \). Hopf bifurcation will occur if at least one eigenvalue has positive real parts, unstable and have negative real parts. Assume that \( i \omega(\omega > 0) \) is a root of (7). Then \( \omega \) must satisfy any nonnegative integer in the following nonlinear equation group

\[
-\omega_i^3 - f_1 \omega^2 + f_2 \omega_i + f_3 + (f_4 \omega_i + f_5) \cos(\omega \tau) - \sin(\omega \tau)i = 0
\]

thus

\[
-\omega_i^3 - f_1 \omega^2 + f_2 \omega_i + f_3 + f_4 \omega \cos(\omega \tau)i + f_4 \omega \sin(\omega \tau) + f_5 \cos(\omega \tau) - f_5 \sin(\omega \tau)i
\]
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which implies

\[
\begin{align*}
\omega^3 - f_2 \omega &= f_4 \cos \omega (\omega \tau) - f_5 \sin (\omega \tau) \\
1 + \omega^2 - f_3 &= f_4 \omega \sin (\omega \tau) + f_5 \cos (\omega \tau)
\end{align*}
\]

By adding both squares, thus have

\[
\omega^6 + (f_1^2 - 2f_2)f_\alpha^4 + (f_2^2 - 2f_1f_3 - f_4^3)\omega^2 + f_3^2 - f_5^2 = 0
\]

For convenience, let \(\alpha = \omega^2\), then, (14) can be rewritten as

\[
R(\alpha) = \alpha^3 + X\alpha^2 + Y\alpha + Z = 0
\]

where

\[
X = f_1^2 - 2f_2, Y = f_2^2 - 2f_1f_3 - f_4^3, Z = f_3^2 - f_5^2
\]

State the following conditions:

\[\text{(H2)} \quad X > 0 \quad \text{and} \quad Z > 0\]

holds, then it implies that the positive equilibrium of \(E^*\) is usually asymptotically stable for any \(\tau > 0\) and that time delayed system (3) has no periodic solution.

\[\text{(H3)} \quad \alpha > 0 \quad R(\alpha) \leq 0, \quad \text{where} \quad \alpha = (-X + \sqrt{X_0^2 - 3Y_0})/3\]

According to (H2), (14) has no positive root, denoted any positive (14) by \(\alpha_0\); as well, \(\pm \omega_0\)

\[
\lambda^3 + f_{1,0} \lambda^2 + f_{2,0} \lambda + f_{3,0} + (f_{4,0} \lambda + f_{5,0})e^{-\lambda \tau} = 0
\]

which implies that

\[
\begin{align*}
\sin \omega^* \tau &= \frac{(f_{1,0}f_{4,0} - f_{5,0})\omega^{*3} + (f_{2,0}f_{5,0} - f_{3,0}f_{4,0})\omega^*}{f_{2,0,0}^2 \omega^{*2} + f_{5,0}^2} \cong f_s \\
\cos \omega^* \tau &= \frac{f_{4,0} \omega^{*4} + (f_{1,0}f_{5,0} - f_{2,0}f_{4,0})\omega^{*2} - f_{3,0}f_{5,0}}{f_{2,0,0}^2 \omega^{*2} + f_{5,0}^2} \cong f_c
\end{align*}
\]

Thus, we denote

\[
\tau_j^0 = \begin{cases}
\frac{1}{\omega_0} \{2\pi - \arccos f_c + 2j\pi\}, & f_s \geq 0, \\
\frac{1}{\omega_0} \{\arccos f_c + 2j\pi\}, & f_c < 0
\end{cases}
\]

where \(j = 0, 1, 2, 3, \ldots\), then \(\pm \omega_0\) is a pair of purely imaginary roots of (17) with \(\tau_0 = \tau_0^0 = \lim_{j \to +\infty}\). Therefore, we define \(\tau_0 = \tau_0^0 = \min \{\tau_0^j\}\).

\[\text{(H4)} \quad R'(\alpha_0) \neq 0,\]

thereafter

\[
\frac{d(Re\lambda(\tau))}{d\tau} |_{\tau = \tau_0^j, \omega = \omega_0} \neq 0.
\]
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Differentiating (7) with respect to $\tau$, consequently

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2f_{1,0}\lambda + f_{2,0})e^{\lambda\tau} + f_{4,0}}{\lambda(f_{4,0} + f_{5,0})} - \frac{\tau}{\lambda}$$

(22)

As a result

$$\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} |_{\tau - \tau^*_0, \omega = \omega_0} = \frac{\omega_0^2 R'(\alpha^*)}{(f_{4,0}\omega_0^2)^2 + (f_{5,0}\omega_0)^2} \neq 0.$$  

(23)

We applied normal form hypothesis and center manifold theorem by Hassard [14]. Accordingly, we have the following result on the analysis of existence of Hopf bifurcation:

**Theorem 2.** Assume that conditions $(H1), (H2), (H3), \text{and } (H4)$ hold. Then, the following is considered.

- The Hopf bifurcation occurs at $\tau = \tau_0$ and the equilibrium solution is unstable for $\tau > \tau_0$
- When $\tau \in [0, \tau_0)$, equilibrium solution $E^*$ of the time delayed system (3) are locally asymptotically stable

3. Direction and Stability of the Hopf Bifurcation with Delay

In said section, we investigate explicit algorithm for establishing the direction of hopf bifurcation and the stability of bifurcating periodic solution. Based on the discussion in the last section, we derived some conditions under which the system (3) undergoes a Hopf bifurcation. Without the loss of generality, we considered a pair of purely imaginary eigenvalues of system (3); $A_0 = \{\lambda_1, \lambda_{2,3}\}$ same as the delayed system; $\Lambda_0 = \{\omega_0i\tau^*, -\omega_0i\tau^*\}$.

By applying methods from normal form and centre manifold reduction theory [14], we to proof more detailed information of Hopf bifurcation. Denote $\tau_0^*$ by $\tau^*$ and rescaling time $\tau \rightarrow t/\tau$.

According to system (7), equilibria should satisfy

$$ax - by = 0,$$

$$cx + ay = 0,$$

$$dz = 0.$$  

(24)

As has already been mentioned in this section, we are able to apply more detailed conditions under which system undergoes Hopf bifurcation at $E^*$; that is the four-parameter family of differential equations (1), (2). We will also consider the direction of Hopf bifurcation at the equilibrium and stability of the bifurcating periodic solutions by using the normal form theory and center manifold for functional differential equations.

Normalizing the delay $\tau$ by scaling $t \rightarrow \tau$ and establish the new parameter $\mu = \tau - \tau^*$, then, the new Hopf bifurcation value is $\mu = 0$. The system (3) is transformed into

$$\dot{U}(t) = L_{\tau^*}(U_t) + F(U_t, \mu)$$

(25)
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Hence $\mu = (x, y, z)^T \in \mathbb{R}^3$, $L_\mu : C \rightarrow \mathbb{R}^3$ is the functional differential equation and $f : \mathbb{R} \rightarrow \mathbb{R}^3$ is the nonlinear part which are given, respectively, as a result of

$$L_\mu(\varphi) = (\tau^*) \times \begin{bmatrix} a & 0 & 0 \\ c & a & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \end{bmatrix}$$ (26)

and

$$L_\mu(\varphi) = (\tau^*) \times \begin{bmatrix} a\varphi_1(0) + b\varphi_2(-1) \\ c\varphi_1(0) + a\varphi_2(0) \\ d\varphi_3(-1) \end{bmatrix} + h.o.t. \quad (27)$$

for $\varphi(\varphi_1, \varphi_2, \varphi_3)^T \in C$. Then linearized system (25) at origin is

$$\dot{U}(t) = L_{\tau^*}(U_t) \quad (29)$$

Clearly, from the discussion in section 2, the characteristic equation of (7) has a pair of purely imaginary roots $\Lambda_0 = \{\omega_0i\tau^*, -\omega_0i\tau^*\}$, we assume that it has the effect of delay $\tau$; easily cognise that for $\tau = \tau^*$ Let $\mathcal{H} := C([-1, 0], \mathbb{R}^3)$, following this, we consider the following functional differential equation on $\mathcal{H}$:

$$\frac{d}{dt}z = L_{\tau^*}(z_t) \quad (30)$$

Using Riesz representation theorem, there exists a $3 \times 3$ matrix function $\eta(\theta, \tau)$ of bounded variation for $\theta \in [-1, 0]$. It is obvious $L(\tau^*)$ is a continuous linear function mapping $C([-1, 0], \mathbb{R}^3)$ consequently

$$L(\tau^*) \varphi = \int_{-1}^{0} [d\eta(\theta, \tau)] \varphi(\theta), \quad \varphi \in C^1([-1, 0], \mathbb{R}^3) \quad (31)$$

For $\varphi \in C^1([-1, 0], \mathbb{R}^3)$, determine

$$A(\tau) \varphi(\theta) = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(s, \tau) \varphi(s), & \theta = 0 \end{cases} \quad (32)$$

In fact, we may take

$$L := L_\mu : C([-1, 0], \mathbb{R}^3) \rightarrow \mathbb{R}^3 \quad (33)$$

By the Riesz representation theorem,

$$L(\tau^*)(\varphi) = \int_{-1}^{0} d\eta(\theta) \varphi(\theta) \quad (34)$$
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where

$$\eta(\theta, \tau^*) = \tau^* \begin{pmatrix} a & 0 & 0 \\ c & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta) - \tau^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1),$$  \hspace{1cm} (35)

with \( \delta(\cdot) \) being the Dirac function, if we know the solution for the \( \delta(\cdot) \) function, then we can solve it for every single value. For \( \varphi = (\varphi_1, \varphi_2) \in C \), determine

$$A(\mu)\varphi(\theta) = \begin{cases} \frac{d\varphi}{d\theta}(\theta), & \theta \in [-1, 0) , \\ \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta), & \theta = 0 \end{cases}$$  \hspace{1cm} (36)

and

$$R(\mu)\varphi = \begin{cases} 0, & \text{for } \theta \in [-1, 0) , \\ f(\mu, \varphi) & \text{for } \theta = 0 \end{cases}$$  \hspace{1cm} (37)

Thus, the system (3) could be transformed into functional differential equation as follows:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t,$$  \hspace{1cm} (38)

where \( u_t(\theta) = u(t + \theta), \quad \theta \in [-1, 0] \).

Whereas, for \((\Psi_1, \Psi_2) \in C([0, 1], \mathbb{R}^3)\), let us review the operator

$$A^* : C([0, 1], \mathbb{R}^3) \rightarrow \mathbb{R}^3,$$  \hspace{1cm} (39)

defined by

$$A^*\Psi(s) = \begin{cases} \frac{d\Psi}{ds}(s), & \text{for } s \in (0, 1] \\ -\int_{-1}^{0} \Psi(-s)d\eta(s), & \text{for } s = 0, \end{cases}$$  \hspace{1cm} (40)

and for \( \varphi \in C([-1, 0], \mathbb{R}^3) \) and \( \Psi \in C([0, 1], \mathbb{R}^3) \) we depict a bilinear inner product:

$$\langle \Psi, \phi \rangle = \langle \Psi(0), \phi(0) \rangle - \int_{-1}^{0} \int_{\xi=0}^{\theta} \Psi(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi = \langle \Psi(0), \phi(0) \rangle + \tau^* \int_{-1}^{0} \Psi(\theta + 1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi(\theta)d\theta,$$  \hspace{1cm} (41)

wherein \( \eta(\theta) = \eta(\theta, 0) \), and \( A(0) \) and \( A^* \) are adjoint operators.

Presuppose that \( P(\theta) \) and \( P^*(s) \) are generalized eigenvectors of \( A(\tau^*) \) and \( A^* \) corresponding to \( \Lambda_0 = \{i\omega_0\tau^*, -i\omega_0\tau^*\} \), respectively; then, \( P^*(s) \) is the adjoint of \( P \) and \( \dim P = \dim P^* = 2 \). Where \( \varphi \in C \) and \( \Psi \in \mathcal{C}^* = C([0, 1], \mathbb{R}^3) \). By a direct calculation, the lemma follows immediately.
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Lemma 4. Let

\[ \xi = -\frac{a}{\omega_0i + b}, \zeta = -\frac{c}{\omega_0i + a}, \xi^* = \frac{1}{\omega_0i + c}, \zeta^* = \frac{d}{\omega_0i + a} \]

then a basis of \( P \) with \( \Lambda_0 \) and

\[ p_1(\theta) = e^{-\omega_0ir\theta}(1, \xi, \zeta)^T, p_2(\theta) = \tilde{p}_1(\theta), \quad \theta \in [-1, 0] \]

is a basis of \( P^* \)

\[ q_1^*(s) = e^{-\omega_0ir^*s}(1, \xi^*, \zeta^*)^T, q_2(s) = \tilde{q}_1(s), \quad s \in [-1, 0] \]

Now using condition \( \langle q_1(s), q(\theta) \rangle = 1 \), leads to have

Let \( \Phi = (\Phi_1, \Phi_2), \Psi^* = (\Psi_1^*, \Psi_2^*)^T \), where

\[ \Phi_1(\theta) = \frac{p_1(\theta) + p_2(\theta)}{2}, \Phi_2(\theta) = \frac{p_1(\theta) - p_2(\theta)}{2i}, \quad \theta \in [-1, 0], \]

and

\[ \Psi_1^*(s) = \frac{q_1(s) + q_2(s)}{2}, \Psi_2^*(s) = \frac{q_1(s) - q_2(s)}{2i}, \quad s \in [-1, 0], \]

From (41) leads to obtain \((\Psi_1^*, \Phi_2), (\Psi_1, \Phi_2)\), and further notice

\[ (q_1, p_1) = (\Psi_1^*, \Phi_1) - (\Psi_2^*, \Phi_2) + [(\Psi_1^*, \Phi_2) - (\Psi_2^*, \Phi_1)]i; \]

that is, leads to

\[ (q_1, p_1) = 1 + \xi \xi^* + \zeta \zeta^* - \zeta \zeta^* \tau^* e^{-\omega_0ir^*} := D^*. \]

Consequently, we have

\[ (\Psi_1^*, \Phi_1) - (\Psi_2^*, \Phi_2) = Re\{D^*\}, (\Psi_1^*, \Phi_2) - (\Psi_2^*, \Phi_1) = Im\{D^*\}, \]

Now, we define \((\Psi^*, \Phi) = (\Psi_m^*, \Phi_n)(m, n = 1, 2)\) and construct a new basis for \( P^* \)
by \( \Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1} \Psi^* \). In addition, we define \( f_0 = (\xi_0^1, \xi_0^2, \xi_0^3) \), where

\[ \xi_0^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \xi_0^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \xi_0^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

Let \( c \cdot f_0 \) be defined by

\[ c \cdot f_0 = c_1 \xi_0^1 + c_2 \xi_0^2 + c_3 \xi_0^3 \]

for \( c = (c_1, c_2, c_3)^T \), and \( c_j \in \mathbb{R} (j = 1, 2, 3) \). Then, the center space of linear equation (29) is determined by \( P_{CN3} \), where

\[ P_{CN3} \varphi = \Phi(\Psi, \langle \varphi, f_0 \rangle) \cdot f_0, \quad \varphi \in \mathbb{Z}, \]

and \( C = P_{CN3} \oplus P_3; P_3 \) we denote the complementary subspace of \( P_{CN3} \). If

\[ A_{\tau} \psi(\theta) = \psi(\theta) + X_0(\theta)[L(\tau^*)(\varphi(\theta) - \varphi(0))], \quad \varphi \in B\mathbb{Z}, \]

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where \( X_0 : [-1, 0] \to B(X, X) \) and \( \begin{cases} 0, & \theta < 0 \\ I, & \theta = 0 \end{cases} \). Then \( A_\tau \) is the infinitesimal generator induced by the solutions of (25) and (29) hence could be written as the abstract differential equation operator:

\[
\dot{U}_t = A_\tau U_t + X_0 F(U_t, \mu), 
\]

where \( h(x_1, x_2) \in P_2, h(0, 0, 0) = Db(0, 0, 0) = 0 \). Especially plug the solution of (25) on the center manifold is determined by

\[
U^*_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_0 + h(x_1, x_2, 0). 
\]

Let \( z = x_1 - x_2 \), by stating that \( p_1 = \Phi_1 + \Phi_2 \), then (56) leads to rewrite as

\[
U^*_t = \frac{1}{2} \Phi \begin{pmatrix} z + \bar{z} \\ (z - \bar{z})i \end{pmatrix} \cdot f_0 + W(z, \bar{z}) = \frac{1}{2} (p_1 z + \bar{p}_1 \bar{z}) \cdot f_0 + W(z, \bar{z}), 
\]

where \( W(z, \bar{z}) = h((z + \bar{z})/2, -(z - \bar{z})/2i, 0) \). Furthermore from [17], \( z \) satisfies

\[
\dot{z} = \omega_0 i \tau^* z + g(z, \bar{z}), 
\]

where

\[
g(z, \bar{z}) = \Psi_1(0) - \Psi_2 i(0) \langle F(U^*_t, 0), f_0 \rangle. 
\]

Let

\[
W(z, \bar{z}) = W_{20} \frac{z^2}{2} + W_{11} z \bar{z} + W_{02} \frac{\bar{z}^2}{2} + .... 
\]

and

\[
g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + .... 
\]

By (57), it can be easily to compute

\[
\langle F(U^*_t, 0), f_0 \rangle = \frac{\tau^* z^2}{4} \begin{pmatrix} 1 \\ \xi 0 \end{pmatrix} + \frac{\tau^* \bar{z} z}{4} \begin{pmatrix} 2 \\ \xi + \bar{\xi} 0 \end{pmatrix} + \frac{\tau^* \bar{z}^2}{4} \begin{pmatrix} 1 \\ \xi^2 0 \end{pmatrix} + .... 
\]

\[
+ \frac{\tau^* z^2}{4} \begin{pmatrix} \langle 4\omega_{11}^{(1)}(0) + 2\omega_{20}^{(1)}(0), 1 \rangle \\ \langle 2\omega_{11}^{(2)}(0) + \omega_{20}^{(2)}(0) + 2\xi\omega_{11}^{(1)}(0) + \bar{\xi}\omega_{20}^{(0)}(0), 1 \rangle 0 \end{pmatrix} + ...., 
\]
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where \( \langle W_{ij}^{(n)}(\theta), 1 \rangle = (1/\pi) \int_{\bar{\Psi}} W_{ij}^{(n)}(\theta)(x)dx \), \( i + j = 2, n = 1, 2, 3 \). Let \( (\Psi_1, \Psi_2, \Psi_3) = \Psi_1(0) - \Psi_2(0) \). This leads to the following expressions:

\[
g_{20} = \frac{\tau^*}{2} [\Psi_1 + \alpha_2 \Psi_2],
\]
\[
g_{11} = \frac{\tau^*}{4} [2\Psi_1 + (\alpha + \bar{\alpha})\Psi_2],
\]
\[
g_{21} = \frac{\tau^*}{2} [4\omega_1^{(1)}(0) + 2\omega_2^{(1)}(0), 1]\Psi_1 + (2\omega_1^{(2)}(0) + \alpha\omega_2^{(1)}(0) + \bar{\alpha}\omega_2^{(1)}(0), 1)\Psi_2].
\]

(63)

Since \( W_{20}(\theta) \) and \( W_{11}(\theta) \) appear in \( g_{21} \), we should compute them. According to (60) we obtain

\[
\dot{W}(z, \bar{z}) = W_{20}z \ddot{z} + W_{11}(\ddot{z} \bar{z} + \bar{z} \ddot{z}) + W_{02}\dddot{z} + \ldots
\]

and

\[
A_{rr}.W = A_{rr}.W_{20}\frac{z^2}{2} + A_{rr}.W_{11}z \ddot{z} + A_{rr}.W_{02}\dddot{z} + \ldots
\]

(65)

Along with it, [17], \( \dot{W}(z(t), \bar{z}(t)) \) also satisfies

\[
\dot{W} = A_{rr}.W + H(z, \bar{z}),
\]

(66)

where

\[
H(z, \bar{z}) = H_{20}\frac{z^2}{2} + H_{11}z \ddot{z} + H_{02}\dddot{z} + \ldots
\]

(67)

with \( H_{ij} \in P_{3} \) and \( i + j = 2 \). Which lead to (57),(64),(65), and (66), we have

\[
\begin{align*}
(2\omega_0i\tau^* - A_{rr}) & W_{20} = H_{20}, \\
-A_{rr}.W_{11} & = H_{11}.
\end{align*}
\]

(68)

Noticing that \( A_{rr} \) comprises two purely imaginary characteristic roots, (66) has the unique solution so as

\[
\begin{align*}
W_{20} &= (2\omega_0i\tau^* - A_{rr})^{-1}H_{20}, \\
W_{11} &= -A_{rr}^{-1}H_{11}.
\end{align*}
\]

(69)

For \( \theta \in [-1, 0] \), it follows from (67)

\[
H(z, \bar{z}) = -\Phi(\theta)\Psi(\theta)\langle F(U^*_0, 0), f_0 \rangle \cdot f_0
\]

\[
= -\left( \frac{p_1(\theta) + p_2(\theta)}{2}, \frac{p_1(\theta) - p_2(\theta)}{2i}, (\Psi_1(0), \Psi_2(0))^T \right) \langle F(U^*_0, 0), f_0 \rangle \cdot f_0
\]

\[
= \frac{1}{2}[p_1(\theta)(\Psi_1(0) - \Psi_2(0)) + p_2(\theta)(\Psi_1(0) + \Psi_2(0))] \times \langle F(U^*_0, 0), f_0 \rangle \cdot f_0
\]

\[
= \frac{1}{4}[g_{20}p_1(\theta) + g_{02}p_2(\theta)]z^2 \cdot f_0 - \frac{1}{2}[g_{11}p_1(\theta) + g_{12}p_2(\theta)]z \ddot{z} \cdot f_0 + \ldots
\]

(70)
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Therefore, $\theta \in [-1, 0]$, leads to

$$H_{20}(\theta) = -\frac{1}{2}[g_{20}p_{1}(\theta) + \bar{g}_{02}p_{2}(\theta)] \cdot f_{0}, \quad (71)$$

$$H_{11}(\theta) = \frac{1}{2}[g_{11}p_{1}(\theta) + \bar{g}_{11}p_{2}(\theta)] \cdot f_{0}, \quad (72)$$

$$H(z, \bar{z})(0) = F(U_{1}^{*}, 0) - \Phi(\Psi, \{F(U_{1}^{*}, 0); f_{0}\}) \cdot f_{0}, \quad (73)$$

$$H_{20}(0) = \frac{\tau^{*}}{2} \left( \begin{array}{c} 1 \\ \xi \\ 0 \end{array} \right) - \frac{1}{2}[g_{20}p_{1}(0) + \bar{g}_{02}p_{2}(0)] \cdot f_{0}, \quad (74)$$

and

$$H_{11}(0) = \frac{\tau^{*}}{4} \left( \begin{array}{c} 2 \\ \xi + \bar{\xi} \\ 0 \end{array} \right) - \frac{1}{2}[g_{11}p_{1}(0) + \bar{g}_{11}p_{2}(0)] \cdot f_{0}. \quad (75)$$

With the combination definition of $A_{\tau^{*}}$, leads to obtain from (69) that

$$\dot{W}_{20} = 2\omega_{0}i\tau^{*}W_{20}(\theta) + \frac{g_{20}p_{1}(0) + \bar{g}_{02}p_{2}(0)}{2} \cdot f_{0}, \quad - \leq \theta < 0. \quad (76)$$

Since $p_{1}(\theta) = p_{1}(0)e^{\omega_{0}i\tau^{*}}, -1 \leq \theta \leq 0$, leads to

$$W_{20}(\theta) = i\left[ \frac{g_{20}}{\omega_{0}\tau^{*}}p_{1}(\theta) + \frac{\bar{g}_{02}}{3\omega_{0}\tau^{*}}p_{2}(\theta) \right] \cdot f_{0} + E e^{2\omega_{0}i\tau^{*}\theta}, \quad (77)$$

where

$$E = W_{20}(0) = \frac{i}{2} \left[ \frac{g_{20}}{\omega_{0}\tau^{*}}p_{1}(0) + \frac{\bar{g}_{02}}{3\omega_{0}\tau^{*}}p_{2}(0) \right] \cdot f_{0} \quad (78)$$

From the definition of $A_{\tau^{*}}$ and combine with (69) and (78), leads to

$$2\omega_{0}i\tau^{*} \left[ \frac{g_{20}}{2\omega_{0}\tau^{*}}p_{1}(0) \cdot f_{0} + \frac{\bar{g}_{02}}{6\omega_{0}\tau^{*}}p_{2}(0) \cdot f_{0} + E \right]$$

$$- L(\tau^{*}) \left[ \frac{g_{20}}{2\omega_{0}\tau^{*}}p_{1}(\theta) \cdot f_{0} + \frac{\bar{g}_{02}}{6\omega_{0}\tau^{*}}p_{2}(0) \cdot f_{0} + E e^{2\omega_{0}i\tau^{*}\theta} \right]$$

$$= \frac{\tau^{*}}{2} \left( \begin{array}{c} 1 \\ \xi \\ 0 \end{array} \right) - \frac{1}{2}[g_{20}p_{1}(0) + \bar{g}_{02}p_{2}(0)] \cdot f_{0}. \quad (79)$$

Noticing that

$$\begin{cases} L(\tau^{*})[p_{1}(\theta) \cdot f_{0}] = \omega_{0}i\tau^{*}p_{1}(0) \cdot f_{0}, \\ L(\tau^{*})[p_{2}(\theta) \cdot f_{0}] = -\omega_{0}i\tau^{*}p_{2}(0) \cdot f_{0}. \end{cases} \quad (80)$$

this leads to

$$2\omega_{0}i\tau^{*}E - \tau^{*}D \Delta E - L(\tau^{*})(E e^{2\omega_{0}i\tau^{*}\theta}) = \frac{\tau^{*}}{2} \left( \begin{array}{c} 1 \\ \xi \\ 0 \end{array} \right) \quad (81)$$
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From the previous formulas, this leads to easily obtain

\[ E = \frac{1}{2} \begin{pmatrix} 2\omega_0 i + a & e^{-2\omega_0 i\tau^*} & 0 \\ c & 2\omega_0 i + a & 0 \\ 0 & 0 & 2\omega_0 i + de^{-2\omega_0 i\tau^*} \end{pmatrix}^{-1} \times \begin{pmatrix} 1 \\ \xi \\ 0 \end{pmatrix} \quad (82) \]

By the same method, leads to

\[ \dot{W}_{11}(\theta) = \frac{1}{2} \left[ g_{11}p_1(\theta) + \bar{g}_{11}p_2(\theta) \right] \cdot f_0, \quad -\leq \theta \leq 0, \quad (83) \]

and

\[ W_{11}(\theta) = \frac{i}{2\omega_0 i\tau^*} \left[ -g_{11}p_1(\theta) + \bar{g}_{11}p_2(\theta) \right] \cdot f_0 + F. \quad (84) \]

By the same method, this leads to obtain

\[ F = \frac{1}{4} \begin{pmatrix} a & -b & 0 \\ c & c & 0 \\ 0 & 0 & d \end{pmatrix}^{-1} \times \begin{pmatrix} 2 \\ \xi + \bar{\xi} \\ 0 \end{pmatrix} \quad (85) \]

Hence, \( g_{21} \) could be represented explicitly.

Each \( g_{21} \) is appropriately stated, accordingly, allows us to compute the following results:

\[ C_1(0) = \frac{i}{2\omega_0 i\tau^*} [g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}] + \frac{g_{21}}{2}. \quad (86) \]

\[ \sigma_2 = \frac{\text{Re}(C_1(0))}{\text{Re}(\lambda'(\tau^*))} \quad (87) \]

\[ \beta = 2\text{Re}(C_1(0)) \quad (88) \]

\[ T_2 = \frac{\text{Im}(C_1(0)) + \sigma_2\text{Im}(\lambda'(\tau^*))}{\omega_0 i\tau^*} \quad (89) \]

The general result of Hopf bifurcation theory (see [14]), we recognize that the parameters in (86) demonstrate the properties of Hopf bifurcation which we can describe specifically:- \( \beta_2 \) determines the stability of the bifurcating periodic solutions: the bifurcating periodic solution are stable(unstable) if \( \beta_2 < 0(> 0) \); \( \sigma_2 \) determines the directions of the Hopf bifurcation: if \( \sigma > 0(< 0) \), then the direction of the Hopf bifurcation is forward(backward), that is the bifurcating periodic solutions exist when \( \tau > \tau_0(< \tau_0) \); and \( T_2 \) determines the period of the bifurcation periodic solutions: the period increase(decrease), if \( T_2 > 0(T_2 < 0) \).

**Theorem 3** System (3) has the following pointcaré normal form:

\[ \dot{\omega} = \omega_0 i\tau^* \omega + C_1(0)|\omega|^2 + 0(|\omega|^5), \quad (90) \]
4. Numerical Results

In the present section, we make some numerical simulations with help of MATLAB to support our analytical results. We choose the following parameter values: \( a = -2, b = 6, c = 3 \) and \( d = -4 \). By a direct calculation, we obtain that system (3) is locally asymptotically stable at equilibrium for all \( \tau \geq 0 \); see Figure 1.

Through discussions in section 3, results in \( \tau_0 = 2.005, \tau_1 = 10.984, \tau_2 = 19.935, \cdots \) and \( ReC_0^0(0) = -521.13, ReC_0^1(0) = -2853.88, ReC_0^2(0) = -5186.33, \cdots, \) which, together with equations (86)-(89), implies \( \sigma_2 > 0 \) and \( \beta_2 < 0 \). Therefore, we recognize that the bifurcated periodic solutions are locally asymptotically stable on the center manifold. Hence, equilibrium \( E_0 \) is stable when \( \tau < \tau_0 \). When \( \tau \) passes through the critical value \( \tau_0 \), \( E_0 \) loses its stability and a Hopf bifurcation occurs. Furthermore, the formulae in section 3, Theorem 1, it follows that system (3) has a stable center manifold in the neighbourhood of equilibrium \( \tau_0 = 5.85 \). Since \( \sigma_2 > 0 \) and \( \beta_2 > 0 \), the direction of the Hopf bifurcation is \( \tau > \tau_0 \), and these bifurcating periodic solutions from \( E_0 \) at \( \tau_0 \) are stable, as shown in Figure 1. Figure 4 shows their time trajectory planes.

![Figure 1. Time history and phase diagram \( \tau_0 = 2.005 \)](image-url)

5. Conclusion

Our paper proposes a delayed model of three-dimensional autonomous. We determine the direction of hopf bifurcation with the center manifold theorem and the stability of bifurcating periods with respect to the normal form theory. In numerical simulations we report the bifurcating periodic solution is stable or unstable at some bifurcation points, depending on the choice of parameters. The time history and phase diagram of the time-delayed system (3) are shown for a range of parameters. The results could help in appreciating the role of fluctuations and interpreting autonomous systems. Numerical findings are presented to illustrate the analytical results and gives understanding for the direction of Hopf bifurcation of the model. The study will assist in understanding
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Figure 2. Time history and phase diagram of system

Figure 3. Time history and phase diagram of system

Figure 4. Time trajectories of system and phase diagram

the importance of nonlinear autonomous schemes and explain concepts of fluctuation phenomena.
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