An Asymptotic Series for an Integral

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Abstract

We obtain an asymptotic series $\sum_{j=0}^{\infty} \frac{I_j}{n^j}$ for the integral $\int_0^1 [x^n + (1 - x)^n]^{\frac{1}{n}} dx$ as $n \to \infty$, and compute $I_j$ in terms of alternating (or “colored”) multiple zeta value. We also show that $I_j$ is a rational polynomial the ordinary zeta values, and give explicit formulas for $j \leq 12$. As a byproduct, we obtain precise results about the convergence of norms of random variables and their moments. We study $\| (U, 1-U) \|_n$ as $n$ tends to infinity and we also discuss $\| (U_1, U_2, \ldots, U_r) \|_n$ for standard uniformly distributed random variables.

1 Introduction

Let

$$I(n) = \int_0^1 [x^n + (1 - x)^n]^{\frac{1}{n}} dx.$$  \hspace{1cm} (1)

We shall obtain an asymptotic series

$$I(n) = I_0 + \frac{I_1}{n} + \frac{I_2}{n^2} + \frac{I_3}{n^3} + \cdots$$

This integral has been discussed in [9] (together with a different problem proposed by M.D. Ward). Therein, it as treated by a different approach using Euler sums and polylogarithms, leading to the first few terms $I_0$ up $I_7$ in terms of multiple zeta values.
Here, we give a complete expansion of $I(n)$. The coefficients $I_k$ can be written in terms of alternating or “colored” multiple zeta values. The multiple zeta values are defined by

$$\zeta(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}$$

for positive integers $i_1, \ldots, i_k$ with $i_1 > 1$. This notation can be extended to alternating or “colored” multiple zeta values by putting a bar over those exponents with an associated sign in the numerator, as in

$$\zeta(\bar{3}, \bar{1}, 1) = \sum_{n_1 > n_2 > n_3 \geq 1} \frac{(-1)^{n_1 + n_2}}{n_1^3 n_2 n_3}.$$ 

Note that $\zeta(a_1, a_2, \ldots, a_k)$ converges unless $a_1$ is an unbarred 1. We have $\zeta(\bar{1}) = -\log 2$ and $\zeta(\bar{n}) = (2^{1-n} - 1)\zeta(n)$ for $n \geq 2$. Alternating multiple zeta values have been extensively studied, and some identities for them are established in [2]. Our formula for $I_k, k \geq 2$, can be stated as

$$I_k = \frac{(-1)^k}{2} \sum_{j=2}^{k} E_{2\lfloor \frac{j}{2} \rfloor + 1}(0) \zeta(\bar{j}, 1, \ldots, 1),$$

(2)

where $E_n$ is the $n$th Euler polynomial. But in fact the right-hand side of Eq. (2) can always be rewritten as a rational polynomial in the ordinary zeta values $\zeta(i), i \geq 2$. This follows from an identity of Kölblig [8] that relates alternating multiple zeta values $\zeta(\bar{n}, 1, \ldots, 1)$ and multiple zeta values $\zeta(n, 1, \ldots, 1)$.

After our main result, we interpret the integral $I(n)$ as the expected value of a certain random variable $Z_n$, defined in terms of the $n$th norm of the random vector $(U, 1 - U)$. Here, $U$ denotes a standard uniformly distributed random variable. We complement our analysis of $I(n) = E(Z_n)$ by studying the positive real moments $E(Z_n^r)$ in terms of (alternating) multiple zeta values, as $n$ tends to infinity. Moreover, we also discuss as a counterpart the $n$th norm of the random vector $(U_1, U_2, \ldots, U_r)$ for $r \geq 2$ and derive its moments in terms of multiple zeta values and related sums.
2 Main result: a complete expansion of $I(n)$

Because of the symmetry around $x = \frac{1}{2}$ in (1), one can write

$$I(n) = 2 \int_0^{\frac{1}{2}} \left[ x^n + (1 - x)^n \right]^\frac{1}{2} dx = 2 \int_0^{\frac{1}{2}} (1 - x) \left[ 1 + \left( \frac{x}{1 - x} \right)^n \right]^\frac{1}{2} dx.$$ 

Now let $u = \frac{x}{1 - x}$, or $x = \frac{u}{1 + u}$. Then $dx = \frac{du}{(1 + u)^2}$, and we have

$$I(n) = 2 \int_0^{1} \left( 1 - \frac{u}{1 + u} \right) \left( 1 + u^n \right)^\frac{1}{2} \frac{du}{(1 + u)^2} = 2 \int_0^{1} \left( 1 + u^n \right)^\frac{1}{2} \frac{du}{(1 + u)^3}.$$ 

Writing $(1 + u^n)^\frac{1}{2}$ as $\exp\left( \frac{1}{n} \log(1 + u^n) \right)$ and expanding the exponential in series, we have

$$I(n) = 2 \int_0^{1} \left( 1 + \frac{1}{k!} \left( \frac{1}{n} \log(1 + u^n) \right)^k \right) \frac{du}{(1 + u)^3}.$$ 

Now we can write (see [5, p. 351])

$$(\log(1 + x))^k = k! \sum_{m=1}^{\infty} \frac{x^m}{m!} s(m, k), \tag{3}$$

where the $s(m, k)$ are (signed) Stirling numbers of the first kind. Hence

$$I(n) = 2 \int_0^{1} \frac{du}{(1 + u)^3} + 2 \sum_{k=1}^{\infty} \int_0^{1} k! \sum_{m=1}^{\infty} \frac{u^m s(m, k)}{m! n^k k!} \frac{du}{(1 + u)^3}$$

$$= \frac{3}{4} + 2 \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{m=1}^{\infty} \frac{s(m, k)}{m!} \int_0^{1} \frac{u^m}{(1 + u)^3} du.$$ 

If we let $\zeta_r(i_1, \ldots, i_k)$ denote the truncated multiple zeta value

$$\zeta_r(i_1, \ldots, i_k) = \sum_{r \geq n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}},$$

then we have the following relation, which is well-known although perhaps not in this notation (cf. [1]).
Lemma 1. For positive integers \( m \geq k \),
\[
s(m, k) = (-1)^{m-k}(m-1)!\zeta_{m-1}(\{1\}_{k-1}),
\]
where \( \{1\}_m \) means 1 repeated \( m \) times.

Proof. From the relation
\[
x(x-1)\cdots(x-n+1) = \sum_{n=0}^{\infty} s(n,k) x^n
\]
it follows that \( s(n,k) = (-1)^{n-k}e_{n-k}(1,2,\ldots,n-1) \), were \( e_j \) is the \( j \)th elementary symmetric function. Divide by \((n-1)!\) to get
\[
s(n,k) = (-1)^{n-k}e_{n-k}(1,2,\ldots,n-1) = (-1)^{n-k}e_{k-1}(1,\frac{1}{2},\ldots,\frac{1}{n-1}),
\]
and the conclusion follows since evidently \( \zeta_{n-1}(\{1\}_{n-k}) = e_{k-1}(1,\frac{1}{2},\ldots,\frac{1}{n-1}) \).

Thus
\[
I(n) = \frac{3}{4} + 2 \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \int_0^1 \frac{u^{mn}}{(1+u)^3} \, du. \quad (4)
\]

If we write
\[
\int_0^1 \frac{u^r}{(1+u)^3} \, du = \sum_{j=1}^{\infty} \frac{\beta_j}{r^j},
\]
then the \( \beta_j \) can be computed explicitly as follows.

Lemma 2.
\[
\beta_j = \frac{(-1)^j}{4} (E_{j+1}(-1) + E_{j+2}(-1)),
\]
where the \( E_j \) are Euler polynomials.

Proof. Making the change of variable \( u = e^{-t} \), we have
\[
\int_0^1 \frac{u^r}{(1+u)^3} \, du = \int_0^{\infty} \frac{e^{-t}}{(1+e^{-t})^3} e^{-rt} dt.
\]
By direct computation
\[
\frac{e^{-t}}{(1 + e^{-t})^3} = \frac{1}{4} \left[ \frac{d^2}{dt^2} \left( \frac{2e^t}{1 + e^{-t}} \right) - \frac{d}{dt} \left( \frac{2e^t}{1 + e^{-t}} \right) \right].
\]
The generating function of the Euler polynomials is defined by
\[
\mathcal{E}(t, x) = \frac{2e^{tx}}{1 + e^t} = \sum_{j \geq 0} E_j(x) \frac{t^j}{j!}. \tag{5}
\]
Differentiating \( \mathcal{E}(-t, -1) \) gives
\[
\frac{d}{dt} \left( \frac{2e^t}{1 + e^{-t}} \right) = -\sum_{n=0}^{\infty} (-1)^n E_{n+1}(-1) \frac{t^n}{n!}
\]
and
\[
\frac{d^2}{dt^2} \left( \frac{2e^t}{1 + e^{-t}} \right) = \sum_{n=0}^{\infty} (-1)^n E_{n+2}(-1) \frac{t^n}{n!}.
\]
Hence
\[
\int_0^\infty \frac{e^{-t}}{(1 + e^{-t})^3} e^{-rt} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{4} \left( E_{n+2}(-1) + E_{n+1}(-1) \right) \int_0^\infty \frac{t^n}{n!} e^{-rt} dt
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{4} \left( E_{n+2}(-1) + E_{n+1}(-1) \right) \frac{1}{r^{n+1}},
\]
from which the conclusion follows. \( \square \)

The well-known identity
\[
E_n(x) + E_n(x + 1) = 2x^n \tag{6}
\]
gives \( E_n(-1) = 2(-1)^n - E_n(0) \), so that
\[
E_{j+1}(-1) + E_{j+2}(-1) = -E_{j+1}(0) - E_{j+2}(0).
\]
But \( E_n(0) = 0 \) for \( n \) even, so we have
\[
\beta_j = \begin{cases} 
\frac{1}{4} E_{j+2}(0), & \text{if } j \text{ is odd}, \\
-\frac{1}{4} E_{j+1}(0), & \text{if } j \text{ is even},
\end{cases}
\]
or more succinctly \( \beta_j = (-1)^{j+1} \frac{1}{4} E_{2j+1}(0) \). If we set \( a_n = \frac{1}{2} E_{2n+1}(0) \), then 
\[ 2(-1)^{j-1} \beta_j = a_{\lfloor \frac{j+1}{2} \rfloor}. \]

The \( a_n \) can be written in terms of Bernoulli numbers as

\[ a_n = \frac{(1 - 2^{2n+2}) B_{2n+2}}{2n + 2}, \]

and we also have the exponential generating function

\[ \sum_{n=0}^{\infty} a_n \frac{t^{2n+1}}{(2n+1)!} = -\frac{1}{2} \tanh \frac{t}{2}. \]

The first few \( a_j \) are

\[ a_0 = -\frac{1}{4}, \quad a_1 = \frac{1}{8}, \quad a_2 = -\frac{1}{4}, \quad a_3 = \frac{17}{16}, \quad a_4 = -\frac{31}{4}, \quad a_5 = \frac{691}{8}, \quad a_6 = -\frac{5461}{4}. \]

Using Eq. (4) we can write

\[ I(n) = \frac{3}{4} + 2 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{m-k} \zeta_{m-1}(\{1\}_{k-1}) \beta_{j-1} \frac{1}{n^j k m^{j+1}} \]

\[ = \frac{3}{4} + 2 \sum_{p=2}^{\infty} \frac{1}{n^p} \sum_{m=1}^{\infty} \sum_{k=1}^{p-1} (-1)^m \zeta_{m-1}(\{1\}_{k-1}) \beta_{p-k-1} \frac{1}{m^{p-k+1}} \]

\[ = \frac{3}{4} + 2 \sum_{p=2}^{\infty} \frac{1}{n^p} \sum_{k=1}^{p-1} (-1)^k \beta_{p-k-1} \zeta(p - k + 1, \{1\}_{k-1}), \]

from which we see that \( I_0 = \frac{3}{4}, I_1 = 0 \), and

\[ I_p = 2 \sum_{k=1}^{p-1} (-1)^k \beta_{p-k-1} \zeta(p - k + 1, \{1\}_{k-1}) = 2 \sum_{j=2}^{p} (-1)^{p-j-1} \beta_{j-2} \zeta(\bar{j}, \{1\}_{p-j}) \]

for \( p \geq 2 \). We have proved the following result.

**Theorem 1.** For \( p \geq 2 \),

\[ I_p = (-1)^p \sum_{j=2}^{p} a_{\lfloor \frac{j+1}{2} \rfloor} \zeta(\bar{j}, \{1\}_{p-j}), \]

where \( a_n = \frac{1}{2} E_{2n+1}(0) = (1 - 2^{2n+2}) B_{2n+2}/(2n + 2) \).
The first two cases are as follows.

\[ I_2 = a_0 \zeta(2) = \frac{1}{8} \zeta(2) \]
\[ I_3 = -a_0 \zeta(\bar{2},1) - a_1 \zeta(\bar{3}) = \frac{1}{4} \cdot \frac{\zeta(3)}{8} + \frac{1}{8} \cdot \frac{3}{4} \zeta(3) = \frac{1}{8} \zeta(3). \]

In all further computations, expressions for alternating multiple zeta values are simplified using the Multiple Zeta Value Data Mine [3]. By Theorem 1,

\[ I_4 = a_0 \zeta(\bar{2},1,1) + a_1 \zeta(\bar{3},1) + a_2 \zeta(\bar{4}) = -\frac{1}{4} \zeta(\bar{2},1,1) + \frac{1}{8} \zeta(\bar{3},1) + \frac{1}{8} \zeta(4), \]

and since \( \zeta(4) = -\frac{7}{8} \zeta(4) \), \( \zeta(\bar{2},1,1) = -\frac{1}{16} \zeta(4) + \frac{1}{2} \zeta(3,1) \), this implies \( I_4 = -\frac{5}{48} \zeta(4) \). Similarly,

\[ I_5 = -a_0 \zeta(\bar{2},1,1,1) - a_1 \zeta(\bar{3},1,1) - a_1 \zeta(\bar{4},1) - a_2 \zeta(\bar{5}) = \]
\[ \frac{1}{4} \zeta(\bar{2},1,1,1) - \frac{1}{8} \zeta(\bar{3},1,1) - \frac{1}{8} \zeta(\bar{4},1) + \frac{1}{4} \zeta(5). \]

Now \( \zeta(5) = -\frac{15}{16} \zeta(5) \), and from [3]

\[ \zeta(\bar{4},1) = -\frac{29}{32} \zeta(5) + \frac{1}{2} \zeta(2) \zeta(3) \]
\[ \zeta(\bar{2},\{1\}_{\bar{3}}) = \frac{31}{64} \zeta(5) - \frac{1}{4} \zeta(2) \zeta(3) + \frac{1}{2} \zeta(\bar{3},1,1), \]

giving the result \( I_5 = -\frac{1}{8} \zeta(2) \zeta(3). \)
Here, without further details, are $I_j$ for $j = 6, 7, 8, 9, 10, 11, 12$.

\[ I_6 = \frac{83}{256} \zeta(6) - \frac{1}{16} \zeta(3)^2 \]
\[ I_7 = \frac{3}{16} \zeta(7) + \frac{27}{64} \zeta(3) \zeta(4) + \frac{3}{16} \zeta(2) \zeta(5) \]
\[ I_8 = -\frac{2533}{1536} \zeta(8) + \frac{3}{16} \zeta(3) \zeta(5) + \frac{5}{32} \zeta(2) \zeta(3)^2 \]
\[ I_9 = -\frac{5}{6} \zeta(9) - \frac{289}{128} \zeta(3) \zeta(6) - \frac{135}{64} \zeta(4) \zeta(5) - \frac{9}{8} \zeta(2) \zeta(7) + \frac{5}{96} \zeta(3)^3 \]
\[ I_{10} = \frac{293937}{20480} \zeta(10) - \frac{87}{32} \zeta(3) \zeta(7) - \frac{9}{16} \zeta(5)^2 - \frac{81}{64} \zeta(3)^2 \zeta(4) - \frac{21}{16} \zeta(2) \zeta(3) \zeta(5) \]
\[ I_{11} = \frac{63}{8} \zeta(11) + \frac{58007}{3072} \zeta(3) \zeta(8) + \frac{5187}{256} \zeta(5) \zeta(6) + \frac{135}{8} \zeta(4) \zeta(7) + \frac{115}{12} \zeta(2) \zeta(9) \]
\[ - \frac{13}{48} \zeta(2) \zeta(3)^3 - \frac{21}{32} \zeta(3)^2 \zeta(5) \]
\[ I_{12} = -\frac{2095281645}{11321344} \zeta(12) + \frac{115}{12} \zeta(3) \zeta(9) + \frac{81}{8} \zeta(5) \zeta(7) + \frac{5765}{512} \zeta(3)^2 \zeta(6) \]
\[ + \frac{1323}{64} \zeta(3) \zeta(4) \zeta(5) + \frac{45}{4} \zeta(2) \zeta(3) \zeta(7) + \frac{45}{8} \zeta(2) \zeta(5)^2 - \frac{13}{192} \zeta(3)^4 \]

In fact, the $I_n$ are always rational polynomials in the ordinary zeta values $\zeta(i)$, in consequence of the following result.

**Theorem 2.** For $p \geq 2$,

\[ I_p = \frac{(-1)^p}{2} \sum_{k=1}^{p-1} (-1)^k \zeta(k + 1, \{1\}_{p-k-1}) \sum_{j=0}^{k-1} \binom{k-1}{j} a_{p-1-j}. \]

The proof makes use of an identity of Kölbig [8], which is phrased in terms of the integral

\[ S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1}(t) \log^p(1 - zt)}{t} dt. \]

But $S_{n,p}(z)$ can be written as a multiple zeta value if $z = 1$, and as an alternating multiple zeta value if $z = -1$. The key is the following result.

**Lemma 3.** If $|z| \leq 1$, then

\[ \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1}(t) \log^p(1 - zt)}{t} dt = \sum_{j_1 > j_2 > \cdots > j_p \geq 1} \frac{z^{j_1}}{j_1^{n+1} j_2 \cdots j_p}. \]

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Proof. Since
\[ \log(1 - zt) = -\sum_{i \geq 1} \frac{z^i t^i}{i} \quad \text{and} \quad \int_0^1 t^{n-1} \log^n(t)dt = \frac{(n-1)!}{n^n}, \]
we have
\[ \int_0^1 \frac{\log^{n-1}(t) \log^n(1 - zt)}{t} dt \]
\[ = (-1)^{p} \sum_{i_1 = 1}^{\infty} \sum_{i_2 = 1}^{\infty} \cdots \sum_{i_p = 1}^{\infty} \int_0^1 \frac{z^{i_1 + \cdots + i_p} t^{i_1 + \cdots + i_p - 1} \log^{n-1}(t)}{i_1 i_2 \cdots i_p} dt \]
\[ = (-1)^{p} \sum_{i_1 = 1}^{\infty} \sum_{i_2 = 1}^{\infty} \cdots \sum_{i_p = 1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{i_1 i_2 \cdots i_p (i_1 + \cdots + i_p)^n}. \]
By [6, Lemma 4.3], this is
\[ (-1)^{p} \sum_{j_1 > j_2 > \cdots > j_p \geq 1} \frac{(-1)^{n-1}(n-1)!z^{j_i}}{j_1^{n-1} j_2 \cdots j_p} \]
and the conclusion follows. \qed

It then follows from definitions that
\[ S_{n,p}(1) = \zeta(n+1, \{1\}_{p-1}) \quad \text{and} \quad S_{n,p}(-1) = \zeta(n+1, \{1\}_{p-1}). \]
In [8] Kölbıg refers to \( S_{n,p}(1) \) as \( s_{n,p} \) and \( S_{n,p}(-1) \) as \( (-1)^{p} \sigma_{n,p} \); the result we need is [8, Theorem 3], which reads
\[ \sum_{j=1}^{n} \binom{n+p-j-1}{p-1} \sigma_{j,n+p-j} + \sum_{j=1}^{p} \binom{n+p-j-1}{n-1} \sigma_{j,n+p-j} = s_{n,p}. \quad (7) \]

Proof of Theorem 2. Note that we can rewrite Theorem 1 as
\[ I_p = \sum_{i=1}^{p-1} (-1)^i a_{\lfloor \frac{i}{2} \rfloor} \sigma_{i,p-i} \]
and Eq. (7) as
\[ \sum_{i=1}^{p-1} \left( \binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) \sigma_{i,p-i} = s_{j,p-j}. \]
If we can find $\rho_j$ so that

$$
\sum_{j=1}^{p-1} \rho_j s_{j,p-j} = \sum_{j=1}^{p-1} \sum_{i=1}^{p-1} \rho_j \left( \binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) \sigma_{i,p-i}
$$

$$
= \sum_{i=1}^{p-1} \sigma_{i,p-i} \sum_{j=1}^{p-1} \rho_j \left( \binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) = I_p,
$$

i.e.,

$$
\sum_{j=1}^{p-1} \rho_j \left( \binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) = (-1)^i a_{1|\frac{p}{2}},
$$

(8)

for $i = 1, 2, \ldots, p - 1$, then $I_p$ can be written in terms of the $s_{m,n}$. Now Eqs. (8) can be written

$$
\sum_{j=1}^{p-i} \rho_{p-j} \left( \binom{p-i-1}{j-1} \right) + \sum_{j=1}^{p-i} \rho_j \left( \binom{p-i-1}{j-1} \right) = (-1)^i a_{1|\frac{p}{2}}, 1 \leq i \leq p - 1,
$$

and if we make the condition $\rho_{p-j} = \rho_j$, this becomes

$$
\sum_{j=1}^{p-i} \rho_j \left( \binom{p-i-1}{j-1} \right) = \frac{(-1)^i}{2} a_{1|\frac{p}{2}}, 1 \leq i \leq p - 1,
$$

(9)

Restrict the system (9) to the last $\lfloor \frac{p}{2} \rfloor$ equations ($i = \lfloor \frac{p+1}{2} \rfloor, \ldots, p - 1$) and use binomial inversion to get

$$
\rho_k = \frac{(-1)^{p+k}}{2} \sum_{j=0}^{k-1} a_{1|\frac{p-j-1}{2}} \binom{k-1}{j}, 1 \leq k \leq \lfloor \frac{p}{2} \rfloor.
$$

(10)

We claim that $\rho_k$ so defined, if the definition is extended to $1 \leq k \leq p - 1$, is also a solution of the first $\lfloor \frac{p-1}{2} \rfloor$ equations of (9). The conclusion then follows.

To prove the claim, it is enough to show that the extension of Eqn. (10) to $1 \leq k \leq p - 1$ is consistent with the condition $\rho_{p-k} = \rho_k$, i.e., that

$$
\frac{(-1)^k}{2} \sum_{j=0}^{p-k-1} a_{1|\frac{p-j-1}{2}} \binom{p-k-1}{j} = \frac{(-1)^{p+k}}{2} \sum_{j=0}^{k-1} a_{1|\frac{p-j-1}{2}} \binom{k-1}{j},
$$

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or, using the definition of $a_n$,

$$\sum_{j=0}^{p-k-1} E_{2\left\lfloor \frac{p-j-1}{2} \right\rfloor + 1}(0) \binom{p-k-1}{j} = (-1)^p \sum_{j=0}^{k-1} E_{2\left\lfloor \frac{p-j-1}{2} \right\rfloor + 1}(0) \binom{k-1}{j}.$$

By considering the cases $p$ odd and $p$ even, we see this can be written

$$\sum_{j=0}^{p-k} E_{p-j}(0) \binom{p-k}{j} = (-1)^p \sum_{j=0}^{k} E_{p-j}(0) \binom{k}{j}.$$

The result then follows from taking $n = p - k$ in Lemma 4 below.

**Lemma 4.** For nonnegative integers $n, k$,

$$\sum_{j=0}^{n} E_{k+j}(0) \binom{n}{j} = (-1)^{n+k} \sum_{j=0}^{k} E_{n+j}(0) \binom{k}{j}.$$

**Proof.** Start with

$$\sum_{j=0}^{n} E_j(0) \binom{n}{j} = -E_n(0)$$

which follows from setting $x = 0$ in the identity (6). Since $E_n(0) = 0$ for $n$ even, we can write this as

$$\sum_{j=0}^{n} E_j(0) \binom{n}{j} = (-1)^n E_n(0),$$

which is the case $k = 0$ of the conclusion. We can then use it as the base
case of a proof of the conclusion by induction on $k$. We have

$$(-1)^{n+k+1} \sum_{j=0}^{k+1} E_{n+j}(0) \binom{k+1}{j} =$$

$$(-1)^{n+k+1} \left[ \sum_{j=1}^{k+1} E_{n+j}(0) \binom{k}{j-1} + \sum_{j=0}^{k} E_{n+j}(0) \binom{k}{j} \right] =$$

$$(-1)^{n+k+1} \left[ \sum_{j=0}^{k} E_{n+1+j}(0) \binom{k}{j} + \sum_{j=0}^{k} E_{n+j}(0) \binom{k}{j} \right] =$$

$$\sum_{j=0}^{n+1} E_{k+j}(0) \binom{n+1}{j} - \sum_{j=0}^{n} E_{k+j}(0) \binom{n}{j} = \sum_{j=1}^{n+1} E_{k+j}(0) \binom{n}{j-1} = \sum_{j=0}^{n} E_{k+1+j}(0) \binom{n}{j}.$$ 

\[ \square \]

Corollary 1. For $p \geq 2$, $I_p$ is a rational polynomial in the the $\zeta(i)$.

Proof. For any positive integers $n, m$ the multiple zeta value $\zeta(n+1, \{1\}_m)$ is a rational polynomial in the $\zeta(i)$, as follows from [2, Eq. (10)]. Then Theorem 2 implies the conclusion. \[ \square \]

3 Applications: convergence of norms

Let $U = \text{Uniform}[0, 1]$ denote a standard uniformly distributed random variable. Furthermore, for positive real $n$ we define random variables $Z_n$ by

$$Z_n = \|(U, 1-U)\|_n = (U^n + (1-U)^n)^{\frac{1}{n}}.$$ 

From the theory of norms we expect that the limit $Z_\infty$ exists and

$$Z_\infty = \|(U, 1-U)\|_\infty = \max\{U, 1-U\}.$$ 

It is known that $\max\{U, 1-U\} = \text{Uniform}[\frac{1}{2}, 1]$. It turns out that our previous considerations allow to refine this intuition. The integral $I(n)$ treated in detail before is exactly the expected value of $Z_n$. In the following we give asymptotic expansion of all positive real moments of $Z_n$. 12
Theorem 3. The random variable $Z_n$, defined in terms of $U = \text{Uniform}[0, 1]$, converges for $n \to \infty$ in distribution and with convergence of all integer moments,

$$Z_n = \left(U^n + (1 - U)^n\right)^{\frac{1}{n}} \to Z_\infty = \max\{U, 1 - U\},$$

For positive integer $s \geq 1$ we have

$$E(Z_n^s) = \frac{2(1 - \frac{1}{2^{s+1}})}{s + 1} + \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s+1} \sum_{j=1}^{s+1} \gamma_{s+1,j} (-1)^{j-1} E_{p-k+j-1}(0) \zeta(p + 1 - k, \{1\}_{k-1}),$$

where the values $\gamma_{s+1,j}$ are given by

$$\gamma_{s+1,j} = \left(\frac{-1}{j!}\right)^{s+1-j} \frac{(-1)^{j-1} \zeta(s, \{1\}_{j-1})}{s!}.$$

For arbitrary positive real $s > 0$ we have

$$E(Z_n^s) = \frac{2(1 - \frac{1}{2^{s+1}})}{s + 1} + \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s+1} \zeta(p + 1 - k, \{1\}_{k-1})$$

$$\times \sum_{\ell=1}^{p-k} (s + 1)^\ell B_{p-k,\ell}(E_1(0), \ldots, E_{p-k-\ell+1}(0)),$$

where $B_{n,k}(x_1, \ldots, x_{n+1-k})$ denote the Bell polynomials.

A first by product of our moment expansions is a rate of convergence.

Corollary 2. The distribution functions $F_n(x) = \mathbb{P}\{Z_n \leq x\}$ and $F_\infty(x) = \mathbb{P}\{Z_\infty \leq x\}$ satisfy

$$\sup_{x \in \mathbb{R}} |F_n(x) - F_\infty(x)| \leq \frac{C}{n}.$$

We also can directly strengthen to almost-sure convergence.

Corollary 3. The random variable $Z_n = \left(U^n + (1 - U)^n\right)^{\frac{1}{n}}$ converges almost surely to $Z_\infty = \max\{U, 1 - U\}$.

Remark 1. We obtain in a similar way moment convergence of random variables

$$Z_n = \left(B^n + (1 - B)^n\right)^{\frac{1}{n}},$$

with $B$ denoting a $\text{Beta}(\alpha, \beta)$ distributed random variable with real $\alpha, \beta > 0$, generalizing our results above (case $\alpha = \beta = 1$).
We note that
\[ E(Z^n_s) = \int_{\Omega} \left( (U^n + (1-U)^n)^{\frac{s}{n}} \right) dP = \int_0^1 \left( x^n + (1-x)^n \right)^{\frac{s}{n}} dx. \]

Proceeding as before we use the symmetry of the integrand.
\[ E(Z^n_s) = 2 \int_0^1 (1-x)^s \left[ 1 + \left( \frac{x}{1-x} \right)^n \right]^{\frac{s}{n}} dx. \]

Substituting again \( u = \frac{x}{1-x} \), or \( x = \frac{u}{1+u} \), leads to
\[ E(Z^n_s) = 2 \int_0^1 \left( 1 - \frac{u}{1+u} \right)^s (1+u^n)^{\frac{s}{n}} \frac{du}{(1+u)^2} = 2 \int_0^1 (1+u^n)^{\frac{s}{n}} \frac{du}{(1+u)^3}. \]

Writing \((1+u^n)^{\frac{s}{n}}\) as \( e^{\left( \frac{s}{n} \log(1+u^n) \right)} \) and expanding the exponential in series, we have
\[ E(Z^n_s) = 2 \int_0^1 \left( 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{s}{n} \log(1+u^n) \right)^k \right) \frac{du}{(1+u)^{s+2}}. \]

As before,
\[ E(Z^n_s) = 2 \int_0^1 \frac{du}{(1+u)^{s+2}} + 2 \sum_{k=1}^{\infty} \int_0^1 k! \sum_{m=1}^{\infty} \frac{u^m s(m,k) s^k}{m! n^k k!} \frac{du}{(1+u)^{s+2}} \]
\[ = \frac{2(1 - \frac{1}{s+1})}{s+1} + \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}k-1)}{m} \int_0^1 \frac{2u^m}{(1+u)^{s+2}} du. \]

It remains to expand the integral into powers of \( n \). Make the substitution \( u = e^{-t} \) and then integrate by parts:
\[ \int_0^1 \frac{2u^m}{(1+u)^{s+2}} du = \int_0^\infty \frac{2e^{-t}}{(1+e^{-t})^{s+2}} e^{-mnt} dt = \]
\[ - \frac{1}{2^s(s+1)} + \frac{nm}{s+1} \int_0^\infty \frac{2}{(1+e^{-t})^{s+1}} e^{-mnt} dt. \]

We adapt the previous result for \( s = 1 \) using derivative polynomials. Changing the sign of the variable \( t \) in (5) and evaluation at \( x = 0 \) gives
\[ E(-t, 0) = \frac{2}{1+e^{-t}} = \sum_{j \geq 0} (-1)^j E_j(0) \frac{t^j}{j!}. \]
Thus, for our base function we choose the logistic function
\[
f(t) = \frac{1}{2} \mathcal{E}(-t, 0) = \frac{1}{1 + e^{-t}}.
\]

**Lemma 5** (Derivative polynomials - logistic function). For positive integer \( r \) the derivative \( f_r(z) := \frac{d^{r-1}}{dt^{r-1}} f(t) \) can be written as a polynomial in \( f \):
\[
f_r(z) = \sum_{j=1}^{r} c_{r,j} \cdot f(t)^j = \sum_{j=1}^{r} \frac{c_{r,j}}{(1 + e^{-t})^j}.
\]
The numbers \( c_{r,j} \) are explicitly given by
\[
(-1)^{j-1}(j-1)! \left\{ \frac{r}{j} \right\},
\]
where \( \left\{ \frac{n}{k} \right\} \) is the number of ways to partition \( \{1, 2, \ldots, n\} \) into \( k \) nonempty subsets (Stirling number of the second kind). In particular, \( c_{r,1} = 1 \) and \( c_{r,r} = (r-1)!(r-1)^{-1} \).

**Proof.** In [7] a general theory of derivative polynomials is developed: if \( f \) is a function such that \( f'(t) = P(f(t)) \) for a polynomial function \( P \), then evidently \( f^{(n)}(t) = P_n(f(t)) \) for polynomials \( P_n \), and if we let
\[
F(x, t) = \sum_{n \geq 0} \frac{t^n}{n!} P_n(x)
\]
then [7, Theorem 1] gives
\[
F(x, t) = f(f^{-1}(x) + t).
\] (11)
In the case \( f(t) = (1 + e^{-t})^{-1} \), Eq. (11) gives
\[
\sum_{n \geq 0} \frac{t^n}{n!} P_n(x) = \frac{x}{x + (1 - x)e^{-t}} = \frac{xe^t}{1 + x(e^t - 1)} = xe^t \sum_{m=0}^{\infty} (-1)^m x^m (e^t - 1)^m.
\]
Using the identity
\[
(e^t - 1)^m = m! \sum_{p \geq m} \left\{ \frac{p}{m} \right\} \frac{t^p}{p!}.
\]

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this becomes
\[
\sum_{n \geq 0} \frac{t^n}{n!} P_n(x) = x e^t \sum_{m=0}^{\infty} (-1)^m x^m m! \sum_{p \geq m} \left\{ \begin{array}{c} p \\ m \end{array} \right\} \frac{t^p}{p!} = \\
\sum_{q=0}^{\infty} \frac{t^q}{q!} \sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{m=0}^{p} (-1)^m x^{m+1} m! \left\{ \begin{array}{c} p \\ m \end{array} \right\}.
\]

Extract the coefficient of $t^n/n!$ on both sides to get
\[
P_n(x) = \sum_{p=0}^{n} \binom{n}{p} (-1)^m x^{m+1} m! \left\{ \begin{array}{c} p \\ m \end{array} \right\} = \sum_{m=0}^{n} (-1)^m x^{m+1} m! \sum_{p=m}^{n} \binom{n}{p} \left\{ \begin{array}{c} p \\ m \end{array} \right\} = \sum_{m=0}^{n} (-1)^m x^{m+1} m! \left\{ \begin{array}{c} n+1 \\ m+1 \end{array} \right\}.
\]

where we used the identity [5, Eq. (6.15)] in the last step. The conclusion then follows.

Henceforth $c_{r,j}$ denotes the coefficients of the derivative polynomials discussed above.

**Lemma 6.** Define $\gamma_{s+1,r}$ as the solutions of the triangular linear system of equations
\[
\begin{pmatrix}
  c_{1,1} & c_{2,1} & \cdots & c_{s+1,1} \\
  0 & c_{2,2} & \cdots & c_{s+1,2} \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & c_{s+1,s+1}
\end{pmatrix} \cdot \begin{pmatrix}
  \gamma_{s+1,1} \\
  \gamma_{s+1,2} \\
  \vdots \\
  \gamma_{s+1,s+1}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  1
\end{pmatrix}.
\]

Then, $\gamma_{s+1,r}$ is given by
\[
\gamma_{s+1,r} = \frac{(-1)^{r-1} \binom{s+1}{r}}{s!} = (-1)^{r-1} \zeta_s(\{1\}_{r-1}).
\]

and
\[
\frac{2}{(1 + e^{-t})^{s+1}} = \sum_{k \geq 0} \frac{t^k}{k!} \sum_{j=1}^{s+1} (-1)^{k+j-1} \gamma_{s+1,j} E_{k+j-1}(0).
\]
Proof. The system of linear equations can be expressed as
\[
\sum_{r=j}^{s+1} c_{r,j} \gamma_{s+1,r} = \delta_{j,s+1}, \quad 1 \leq j \leq s + 1.
\]
Hence,
\[
(-1)^{j-1} (j - 1)! \sum_{r=j}^{s+1} \binom{r}{j} \gamma_{s+1,r} = \delta_{j,s+1}.
\]
By the inversion relationships between Stirling numbers we directly observe that
\[
\gamma_{s+1,r} = \frac{(-1)^{r-1} \binom{s+1}{r}}{s!}.
\]
By Lemma 1 we obtain the second expression.

Remark 2. The generalized Euler polynomials \(E_n^{(r)}(x), r \in \mathbb{N}\), are defined by the generating function
\[
E_r(t, x) = \left(\frac{2}{1 + e^t}\right)^r e^{xt} = \sum_{k \geq 0} E_k^{(r)}(x) \frac{t^k}{k!},
\]
see [12]. The result above implies the formula
\[
E_k^{(r)}(0) = 2^{r-1} \sum_{j=1}^{r} (-1)^{j-1} \gamma_{r,j} E_{k+j-1}(0),
\]
also leading to a new formula for \(E_k^{(r)}(x)\). Cf.
\[
E_k^{(r)}(0) = \frac{2^{r-1}}{(r-1)!} \sum_{j=0}^{r} s(r, j)(-1)^{r+j} E_{k+j-1}(0)
\]
which follows from [10] and gives an alternative derivation of the \(\gamma_{r,j}\).
Proof. By our previous result
\[
\frac{1}{(1 + e^{-t})^{s+1}} = \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} f(t),
\]
where \( f(t) = \frac{1}{2} \mathcal{E}(-t, 0) = \frac{1}{1 + e^{-t}} \). Then

\[
\frac{2}{(1 + e^{-t})^{s+1}} = \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} 2f(t)
\]

\[
= \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} \sum_{k \geq 0} (-1)^k E_k(0) \frac{t^k}{k!}
\]

\[
= \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{k \geq 0} (-1)^{k+j-1} E_{k+j-1}(0) \frac{t^k}{k!}.
\]

Lemma 6 implies that

\[
\int_0^\infty \frac{2}{(1 + e^{-t})^{s+1}} e^{-nt} dt = \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{k \geq 0} (-1)^{k+j-1} E_{k+j-1}(0) \frac{1}{m^{k+1} n^{k+1}}.
\]

Furthermore

\[
\mathbb{E}(Z_n^s) = \frac{2(1 - \frac{1}{2^s s+1})}{s+1} + \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \times \left( -\frac{1}{2^s(s+1)} + \frac{1}{s+1} \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{\ell \geq 0} (-1)^{\ell+j} E_{\ell+j-1}(0) \frac{1}{m^{\ell} n^{\ell}} \right).
\]

Setting \( t = 0 \) in Lemma 6 we get

\[
\frac{1}{2^s} = \sum_{j=1}^{s+1} (-1)^{j-1} \gamma_{s+1,j} E_{j-1}(0).
\]
Consequently, the first summand cancels and we get

\[ \mathbb{E}(Z_n) = \frac{2(1 - \frac{1}{s+1})}{s+1} + \]

\[ \sum_{k=1}^{\infty} s^k \sum_{m=1}^{\infty} (-1)^m \zeta_{m-1} \frac{(1)_{k-1}}{m(s+1)} \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{\ell \geq 1} (-1)^{\ell+j-1} E_{\ell+j-1}(0) \frac{1}{m!n!} \]

by changing the order of summation.

Concerning arbitrary positive real \( s > 0 \) we have to proceed in a slightly different way. Let \( B_{n,k}(x_1, \ldots, x_{n-k+1}) \) denote the \( k \)th Bell polynomial defined by

\[ B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{n-k+1 \leq j, j \geq 0 \sum_{j=1}^{s+1} \gamma_{s+1,j} (-1)^{j-1} E_{p-k-j-1}(0) \zeta(p-k+1, \{1\}_{k-1}) \]

We have

\[ \frac{2}{(1 + e^{-t})^{s+1}} = (\mathcal{E}(-t, 0))^{s+1} = (1 + (\mathcal{E}(-t, 0) - 1))^{s+1} = \]

\[ \sum_{j \geq 0} \frac{(s+1)^j}{j!} \mathcal{E}(-t, 0)^j - 1 = \sum_{j \geq 0} \sum_{\ell=1}^{j} \frac{(s+1)^j B_{j,\ell}(E_1(0), \ldots, E_{j-\ell+1}(0))}{j!} (-1)^j t^j. \]

Consequently,

\[ \int_0^{\infty} \frac{2}{(1 + e^{-t})^{s+1}} e^{-mnt} dt = \sum_{j \geq 0} (-1)^j \frac{\sum_{\ell=1}^{j} (s+1)^j B_{j,\ell}(E_1(0), \ldots, E_{j-\ell+1}(0))}{(mn)^{j+1}}. \]
Finally,

\[ \mathbb{E}(Z_n^\ell) = \frac{2(1 - \frac{1}{2^s})}{s + 1} + \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \]

\[ \times \frac{1}{s + 1} \sum_{j \geq 1} (-1)^j \sum_{t=1}^{j} (s + 1)^{\ell} B_{j,k}(E_1(0), \ldots, E_{j-\ell+1}(0)) \]

\[ = \frac{2(1 - \frac{1}{2^s})}{s + 1} + \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s + 1} \zeta(p + 1 - k, \{1\}_{k-1}) \]

\[ \times \sum_{t=1}^{p-k} (s + 1)^{\ell} B_{p-k,\ell}(E_1(0), \ldots, E_{p-\ell+1}(0)). \]

**Proof of Corollary 2.** We use the general version of the Berry-Esseen inequality [4]:

\[ \sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq c_1 \int_{-T}^{T} \left| \frac{\phi_F(t) - \phi_G(t)}{t} \right| dt + c_2 \sup_{x \in \mathbb{R}} (G(x + \frac{1}{T}) - G(x)). \]

From our moment expansion

\[ \mathbb{E}(Z_n^\ell) = \mathbb{E}(Z_\infty^\ell) + \mathcal{O}\left( \frac{s\zeta(2)}{2s^2n^2} \right), \]

we obtain for the characteristic functions \( \phi_n(t) = \mathbb{E}(e^{itZ_n}) \) and \( \phi_\infty(t) = \mathbb{E}(e^{itZ_\infty}) \)

\[ \frac{|\phi_n(t) - \phi_\infty(t)|}{|t|} \leq C_1 \frac{1}{n^2}. \]

Choosing \( T = n \) this gives a \( \frac{1}{n} \) bound for the integral. We get \( \sup_{x \in \mathbb{R}} (G(x + \frac{1}{T}) - G(x)) \leq \frac{C_2}{n} \) leading to the stated result. \( \square \)

**Proof of Corollary 3.** By the Markov inequality we have

\[ \mathbb{P}\{|Z_n - Z_\infty| > \frac{1}{\ell}\} \leq \ell^2 \mathbb{E}((Z_n - Z_\infty)^2) = \ell^2 (\mathbb{E}(Z_n^2) + \mathbb{E}(Z_\infty^2) - 2\mathbb{E}(Z_nZ_\infty)). \]

The random variables \( Z_n \) and \( Z_\infty \) are defined in terms of the same uniform distribution and we readily obtain the expansion of

\[ \mathbb{E}(Z_nZ_\infty) = \int_0^1 (x^n + (1-x)^n)^{\frac{1}{n}} \cdot \max\{x, 1-x\} dx = \frac{2}{3} \cdot \frac{7}{8} + \mathcal{O}(\frac{1}{n^2}) \]
leading to $\mathbb{P}\{|Z_n - Z_\infty| > \frac{1}{\ell}\} \leq C \cdot \frac{\ell^2}{n^2}$. Let

$$E_{n,\ell} = \left\{ \omega \in \Omega : |Z_n - Z_\infty| > \frac{1}{\ell} \right\}, \ n \in \mathbb{N}, \ \ell > 0.$$ \hspace{1cm}

We have

$$\sum_{n \geq 1} \mathbb{P}\{E_{n,\ell}\} \leq \sum_{n \geq 1} \frac{C\ell^2}{n^2} < \infty.$$

Let $E_\ell = \limsup E_{n,\ell}$. By the Borel-Cantelli Lemma we have $\mathbb{P}(E_\ell) = 0$ for any $\ell > 0$, giving the stated result. \hfill \square

### 3.1 Independent uniformly distributed random variables

Let $U_j$ denote mutually independent standard uniformly distributed random variables, $1 \leq j \leq r$ with $r \geq 2$. Further, let $U$ denote the random vector $U = (U_1, \ldots, U_r)$.

Let $Z_n$ be defined as

$$Z_n = \|U\|_n = (U_1^n + U_2^n + \cdots + U_r^n)^\frac{1}{n}.$$ \hspace{1cm}

A folklore result states that any order statistic for uniform distributions is Beta-distributed. In particular,

$$Z_\infty = \|U\|_\infty = B(r, 1).$$ \hspace{1cm}

We are interested in the asymptotics of $Z_n$ as $n \to \infty$ and derive asymptotics of the moments

$$I_s = \mathbb{E}(Z_n^s) = \int_{[0,1]^r} \left( x_1^n + \cdots + x_r^n \right)^\frac{s}{n} d(x_1, \ldots, x_r).$$ \hspace{1cm}

The special case $r = 2$, $Z_n = (U_1^n + U_2^n)^\frac{1}{n}$ is the direct counterpart of our earlier results for $(U^n + (1 - U)^n)^\frac{1}{n}$. Our asymptotic series involves for $r \geq 2$ multiple zeta values. Interestingly, for $r \geq 3$ variants of multiple zeta values and Euler sums appear. Let $\zeta^*_r(i_1, \ldots, i_k)$ denote the truncated multiple zeta star value

$$\zeta^*_r(i_1, \ldots, i_k) = \sum_{r \geq n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}, \hspace{1cm} \text{(13)}$$
and $\zeta^*_r(i_1, \ldots, i_k; x_1, \ldots, x_k)$ denote the truncated weighted multiple zeta star value
\[
\zeta^*_r(i_1, \ldots, i_k; x_1, \ldots, x_k) = \sum_{r \geq n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.
\] (14)

Then $\zeta^*_r(i_1, \ldots, i_k; \{1\}_k)$ is the ordinary zeta value $\zeta^*_r(i_1, \ldots, i_k)$, and
\[
\zeta^*_r(\{1\}_k; \{1\}_{k-1}, 2) = \sum_{r \geq n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{2^{n_k}}{n_1 n_2 \cdots n_k} = \sum_{n_1=1}^{r} \frac{1}{n_1} \sum_{n_2=1}^{n_1} \frac{1}{n_2} \cdots \sum_{n_k=1}^{n_{k-1}} \frac{2^{n_k}}{n_k}.
\]

**Theorem 4.** The random variable $Z_n = \|U\|_n$ converges to $Z_{\infty} = B(r, 1)$ with convergence of all positive integer moments.

\[
\mathbb{E}(Z_n^\ast) = \frac{r}{r - 1 + s} \left(1 - \frac{s(r - 1)}{n^2} \zeta(2) + O\left(\frac{1}{n^3}\right)\right).
\]

In particular, for $r = 2$ and $Z_n = (U_1^2 + U_2^2)^{1/2}$ we have the exact representation

\[
\mathbb{E}(Z_n^\ast) = \frac{2}{1 + s} \left(1 + \sum_{p \geq 2} \frac{(-1)^p}{n^p} \sum_{\ell=0}^{p-2} s^{p-\ell-1} \left(-\zeta(\ell + 2, \{1\}_{p-\ell-2})\right)\right).
\]

For $r = 3$ we have the exact representation

\[
\mathbb{E}(Z_n^\ast) = \frac{3}{2 + s} \left(1 + \sum_{p \geq 2} \frac{2(-1)^p}{n^p} \sum_{\ell=0}^{p-2} s^{p-\ell-1} \left(-\zeta(\ell + 2, \{1\}_{p-\ell-2})\right)\right) + \frac{3}{2 + s} \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1 + \ell_2}}{n^{\ell_1 + \ell_2 + 2}}
\]

\[
\times \sum_{m=1}^{\infty} \frac{(-1)^{m-k} \zeta_{m-1}(\{1\}_{k-1}) \left(\zeta_m^\ast(\{1\}_{\ell_1+2-i}; \{1\}_{\ell_1+1-i}, 2) - \zeta_m^\ast(\{1\}_{\ell_1+2-i}) - \frac{1}{m^{\ell_1+1+i}}\right)}{m^{\ell_1+1+i}}
\]

\[
+ \sum_{m=1}^{\infty} \frac{(-1)^{m-k} \zeta_{m-1}(\{1\}_{k-1}) \left(\zeta_m^\ast(\{1\}_{\ell_2+2-i}; \{1\}_{\ell_2+1-i}, 2) - \zeta_m^\ast(\{1\}_{\ell_2+2-i}) - \frac{1}{m^{\ell_2+1+i}}\right)}{m^{\ell_2+1+i}}.
\]

**3.2 Exact representations**

First, we decompose the hypercube into $r$ parts according to the maximum of the $x_i$:

\[
[0, 1]^r = \bigcup_{i=1}^{r} \{x_i \in [0, 1], \ 0 \leq x_j \leq x_i, \ j \in \{1, \ldots, r\} \setminus \{i\}\}.
\]

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These parts are not disjoint, but their intersection is of measure zero. By the symmetry of \( I_s \) we get

\[
I_s = r \int_0^1 \left( \int_{[0,x_r]^{r-1}} (x_1^n + x_2^n + \cdots + x_r^n) \hat{z} \, d(x_1, \ldots, x_{r-1}) \right) \, dx_r.
\]

We use the substitution \( x_j = x_r u_j, \, dx_j = x_r \, du_j \) to obtain

\[
I_s = r \int_0^1 x_r^{r-1} \left( \int_{[0,1]^{r-1}} (x^n_r u_1^n + x^n_r u_2^n + \cdots + x^n_{r-1} u_{r-1}^n + 1) \hat{z} \, du \right) \, dx_r.
\]

This implies that the integrals can be separated:

\[
I_s = r \int_0^1 x_r^{r-1+s} \, dx_r \cdot \int_{[0,1]^{r-1}} (1 + u_1^n + \cdots + u_{r-1}^n) \hat{z} \, du
\]
\[
= \frac{r}{r-1+s} \cdot \int_{[0,1]^{r-1}} (1 + u_1^n + \cdots + u_{r-1}^n) \hat{z} \, du.
\]

In order to derive an asymptotic expansion of the remaining integral we use the \( \exp - \log \) representation:

\[
(1+u_1^n+\cdots+u_{r-1}^n) \hat{z} = \exp \left( \frac{s}{n} \ln(1+u_1^n+\cdots+u_{r-1}^n) \right) = 1 + \sum_{k=1}^\infty \frac{s^k}{k!} \ln^k(1+u_1^n+\cdots+u_{r-1}^n).
\]

Using Eq. (3), this implies

\[
I_s = \frac{r}{r-1+s} \cdot \left( 1 + \sum_{k=1}^\infty \int_{[0,1]^{r-1}} \sum_{m=1}^\infty \frac{s^k \xi(m,k)}{n^k m!} (u_1^n + \cdots + u_{r-1}^n)^m \, du \right),
\]

where (as above) \( \xi(m,k) \) denotes the signed Stirling numbers of the first kind. Then using Lemma 1, we have

\[
I_s = \frac{r}{r-1+s} \cdot \left( 1 + \sum_{k=1}^\infty \frac{s^k (-1)^k}{n^k} \sum_{m=1}^\infty \frac{(-1)^m \zeta(m-1)}{m} \int_{[0,1]^{r-1}} (u_1^n + \cdots + u_{r-1}^n)^m \, du \right).
\]

In order to evaluate the remaining integral we substitute \( u_j = e^{-t_j} \) and obtain

\[
\int_{[0,1]^{r-1}} (u_1^n + \cdots + u_{r-1}^n)^m \, du = \int_{[0,\infty]^{r-1}} e^{-t_1-\cdots-t_{r-1}} (e^{-t_1 n} + \cdots + e^{-t_{r-1} n})^m \, dt.
\]
We expand the exponentials and use the multinomial theorem. By the symmetry of the integrand and the fact
\[
\int_0^\infty u^p e^{-ku} du = \frac{p!}{k^{p+1}}
\]
we obtain
\[
\int_{\{0, \infty\}}^{r-1} e^{-t_1-\cdots-t_r} (e^{-t_1 n} + \cdots + e^{-t_r n})^m dt
\]
\[
= \sum_{a=1}^{r-1} \binom{r-1}{a} \sum_{j_1+\cdots+j_a=m} \sum_{j_1, \ldots, j_a \geq 1} \frac{(-1)^{j+1+\cdots+j_a}}{n^{j_1+\cdots+j_a+a} j_1^{j_1+1} \cdots j_a^{j_a+1}}.
\]
For \( r = 2 \) there is only a single summand and we get
\[
\int_{\{0, \infty\}} e^{-t_1}  e^{-t_2} dt = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(nm)^{\ell+1}}.
\]
Changing summation gives the desired result. For \( r = 3 \) we get
\[
\int_{\{0, \infty\}^2} e^{-t_1}  e^{-t_2}  e^{-t_3} dt(1, t_2) =
\]
\[
2 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(nm)^{\ell+1}} + \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2} j_1^{j_1+1} \cdots j_a^{j_a+1}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{j_1+1} (m-j)^{j_2+1}}.
\]
In order to simplify the arising sums we use a classical partial fraction decomposition, which appears already in [11],
\[
\frac{1}{j^a (m-j)^b} = \sum_{i=1}^a \frac{(i+b-2)}{(b-1)} \frac{m^{i+b-1} j^{a+1-i}}{a} + \sum_{i=1}^b \frac{(i+a-2)}{a-1} \frac{m^{i+a-1} (m-j)^{b+1-i}}{a+1}.
\]
Thus,
\[
\sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2} j_1^{j_1+1} \cdots j_a^{j_a+1}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{j_1+1} (m-j)^{j_2+1}}
\]
\[
= \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2} \sum_{i=1}^{\ell_1+1} \frac{m}{i^{i+1}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_1+1} (m-j)^{\ell_2+1}} + \sum_{i=1}^{\ell_2+1} \frac{(i+\ell_2-1)}{\ell_2} \frac{m^{i+\ell_2} (m-j)^{\ell_2+1-i}}{m^{i+\ell_1} j^{\ell_1+1} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_2+2-i}}}}.
\]
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Lemma 7. For positive integers \( r, m \) we have
\[
\sum_{j=1}^{m} \binom{m}{j} \frac{1}{j^r} = \zeta_m^*({\{1\}_r}; {\{1\}_{r-1}, 2}) - \zeta_m^*({\{1\}_r}).
\]

Proof. We use induction with respect to \( r \). For \( r = 1 \) we have
\[
\sum_{j=1}^{m} \binom{m}{j} \frac{1}{j} = \int_0^1 \frac{(1+t)^m - 1}{t} dt = \int_1^2 \frac{t^m - 1}{t-1} dt = \int_1^2 (t^{m-1} + t^{m-2} + \cdots + t + 1) dt = \sum_{k=1}^{m} \frac{2^k}{k} - H_m = \zeta_m^*({1}; 2) - \zeta_m^*({1}).
\]
Assuming the result for \( r - 1 \),
\[
\sum_{j=1}^{m} \binom{m}{j} \frac{1}{j^r} = \sum_{j=1}^{m} \sum_{k=1}^{m} \binom{k-1}{j-1} \frac{1}{j^r} = \sum_{k=1}^{m} \sum_{j=1}^{k} \binom{k-1}{j-1} \frac{1}{j^r} = \sum_{k=1}^{m} \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{j^{r-1}}
\]
\[
= \sum_{k=1}^{m} \frac{1}{k} \left( \zeta_k^*({\{1\}_{r-1}}; {\{1\}_{r-2}, 2}) - \zeta_k^*({\{1\}_{r-1}}) \right) = \zeta_m^*({\{1\}_r}; {\{1\}_{r-1}, 2}) - \zeta_m^*({\{1\}_r}).
\]

This gives
\[
\sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_1+1}(m-j)^{\ell_2+1}}
\]
\[
= \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}} \sum_{i=1}^{\ell_1+1} \binom{i+\ell_2-1}{\ell_2} \left( \zeta_m^*({\{1\}_{\ell_1+2-i}}; {\{1\}_{\ell_1+1-i}, 2}) - \zeta_m^*({\{1\}_{\ell_1+2-i}}) - \frac{1}{m^{\ell_1+2-i}} \right)
\]
\[
+ \sum_{i=1}^{\ell_2+1} \frac{(-1)^{\ell_1}}{m^{\ell_1+\ell_1}} \left( \zeta_m^*({\{1\}_{\ell_2+2-i}}; {\{1\}_{\ell_2+1-i}, 2}) - \zeta_m^*({\{1\}_{\ell_2+2-i}}) - \frac{1}{m^{\ell_2+2-i}} \right).
\]

4 Outlook and Acknowledgments

It seems that similar phenomena appear when discussing random variables
\( Z_n = \| (X_1, \ldots, X_n) \|_n \), where the \( X_i \) are i.i.d. random variables.
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