On canonically polarized smooth 3-folds satisfying the Noether equality

Yifan Chen Yong Hu

Abstract

We completely classify canonically polarized smooth 3-folds $X$ with $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$. New examples of 3-folds satisfying the properties above are constructed.

1 Introduction

Throughout the article, we work over the complex number field $\mathbb{C}$.

Let $S$ be a smooth minimal projective surface of general type. We have the classical Noether inequality $K_S^2 \geq 2p_g(S) - 4$ (cf. [15]). In [10] Horikawa classified surfaces on the Noether line.

Let $X$ be a projective 3-fold of general type. There have been many works dedicated to prove the 3-dimensional version of the Noether inequality (cf. [12], [5], [6], [2] and [4]). In [4], the inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ is proved under the assumption that the 3-fold $X$ is Gorenstein minimal. This inequality is sharp according to M. Kobayashi’s examples (cf. [12]).

In this article, we study the 3-folds on the Noether line $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$. We restrict our attention to the situation where $X$ is a smooth projective 3-fold with ample canonical divisor $K_X$. We construct examples satisfying these properties.

Theorem 1.1. Let $(e, a)$ be a pair of integers such that $a \geq e \geq 3$; or $1 \leq e \leq 2$, $a \geq e + 1$; or $e = 0$, $a \geq 2$. Then there are smooth 3-folds $X$ with $K_X^3 = 8a - 4e - 6$ and $p_g(X) = 6a - 3e - 2$. Moreover, the canonical divisor $K_X$ is ample and the canonical image of $X$ is the image of the embedding of the Hirzebruch surface $\Sigma_e$ into $\mathbb{P}_{p_g(X)}$ induced by the linear system $|s + (3a - e - 2)l|$. We recall that the surface $\Sigma_e$ is the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ over $\mathbb{P}^1$. Here $l$ stands for a fiber of the natural ruling $\Sigma_e \to \mathbb{P}^1$ and $s$ stands for the section corresponding to the projection of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$ to the second summand. It is clear that the 3-folds $X$ are not isomorphic for different pairs $(e, a)$. Our construction in Section 2 essentially follows the same method as [12] (3.2)-(3.5)] and all these 3-folds are finite double covers of certain $\mathbb{P}^1$-bundles over the Hirzebruch surfaces. But we do construct more examples than [12]. Actually, the 3-folds there correspond to the pairs $(e, a)$ with $a = e \geq 3$. Also observe that $p_g(X) \geq 7$ for any $X$ in Theorem 1.1. But see Remark 2.3 for examples of 3-folds with $K_X^3 = 2$ and $p_g = 4$, having exactly one singularity and canonically fibred by curves.

We also classify smooth 3-folds on the Noether line under the assumption that $K_X$ is ample and $p_g(X) \geq 7$. Assertions (a) and (b) in the following theorem characterize the base locus of $|K_X|$. They are the key points in the classification.

Theorem 1.2 (cf. Main Theorem (3.1) in [12]). Let $X$ be a smooth 3-fold with $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$. Assume that $K_X$ is ample.

(a) The base locus of $|K_X|$ consists of a smooth rational curve $\Gamma$. 


(b) Let \( \pi : Y \to X \) be the blowup along \( \Gamma \) and let \( E_0 \) be the exceptional divisor. Then \( |\pi^*K_X - E_0| \) is base point free and it induces a fibration \( \phi : Y \to \Sigma \), where \( \Sigma \) is a surface in \( \mathbb{P}^{s_0}(X^{-1}) \) with \( \deg \Sigma = p_g(X) - 2 \).

(c) Let \( C \) be the general fiber of \( \phi \). Then \( g(C) = 2 \) and \( \pi^*K_X \cdot C = 1 \).

(d) The restriction \( \phi|_{E_0} : E_0 \to \Sigma \) is birational.

Finally, we are able to give a complete classification.

**Theorem 1.3.** Let \( X \) be a smooth 3-fold as in the Theorem 1.2. Then the canonical image \( \Sigma \) of \( X \) is smooth and \( X \) is one of the smooth 3-folds constructed in Section 2.

In generality, one would like to classify Gorenstein minimal 3-folds on the Noether line. However, this problem does not seem possible to resolve with the methods and the techniques of the present article.

## 2 Examples

In this section, we construct canonically polarized smooth 3-folds with \( K_X^2 = \frac{4}{3}p_g(X) - \frac{40}{3} \) and \( p_g(X) \geq 7 \). Our construction starts from the Hirzbruch surfaces and certain \( \mathbb{P}^1 \)-bundles over them (cf. [8, p. 162, Proposition 7.11-7.12; p. 253, Exercise 8.4; Chapter V, Section 2]).

We denote by \( \Sigma_e \) the Hirzbruch surface \( \mathbb{P}^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \) for \( e \geq 0 \). Denote by \( l \) the fiber of the natural ruling \( \mathbb{P}^1 \to \Sigma_e \) and by \( s \) the section corresponding to the projection of \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \) to the second summand \( \mathcal{O}_{\mathbb{P}^1}(-e) \). Then

\[
s^2 = -e, \quad \mathcal{O}_{\Sigma_e}(s) = \mathcal{O}_{\Sigma_e}(1), \quad \text{Pic}(\Sigma_e) = \mathbb{Z}s \oplus \mathbb{Z}l \quad \text{and} \quad K_{\Sigma_e} = -2s - (e + 2)l \tag{2.1}
\]

The curve \( s \) is called the negative section when \( e > 0 \).

Let \( E \) be a locally free sheaf of rank 2 as an extension

\[
0 \to \mathcal{O}_{\Sigma_e} \to E \to \mathcal{O}_{\Sigma_e}(-2s - 2al) \to 0 \tag{2.2}
\]

where \( a \) is an integer, and let \( P := \mathbb{P}_{\Sigma_e}(E) \). Denote by \( \tau : P \to \Sigma_e \) the natural projection and by \( E \) the section of \( \tau \) corresponding to the quotient morphism \( E \to \mathcal{O}_{\Sigma_e}(-2s - 2al) \) in (2.2). By abuse of notation, we identify \( \text{Pic}(E) \) with \( \mathbb{Z}s \oplus \mathbb{Z}l \) via \( \tau|_E : E \cong \Sigma_e \).

**Lemma 2.1.** Keep the same notation as above. Then

(a) \( \mathcal{O}_P(E) = \mathcal{O}_P(1), \quad K_P = \tau^*(-4s - (2a + e + 2)l) - 2E \) and \( \mathcal{O}_E(E) = \mathcal{O}_E(-2s - 2al) \);

(b) the exact sequence (2.2) splits if \( a \geq e \);

(c) the linear system \( |E + \tau^*(2s + 2al)| \) is base point free if \( a \geq e \).

**Proof.** The first two formulae of (a) are standard. Since \( K_E = -2s - (e + 2)l \), the third one follows from the second one by the adjunction formula.

For (b), it suffices to show \( H^1(\Sigma_e, \mathcal{O}_{\Sigma_e}(2s + 2al)) = 0 \). According to [8, p. 371, Lemma 2.4 and p. 162, Proposition 7.11],

\[
H^1(\Sigma_e, \mathcal{O}_{\Sigma_e}(2s + 2al)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a)) \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a - e)) \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a - 2e)),
\]

which is 0 since \( a \geq e \).

If \( a \geq e \), then \( |2s + 2al| \) is base point free by [8, p. 380, Corollary 2.8] and so is \( \tau^*(2s + 2al) \). From the third formula in (a), we have \( \mathcal{O}_P(E + \tau^*(2s + 2al)|_E = \mathcal{O}_E \). So we have the following exact sequence

\[
0 \to \mathcal{O}_P(\tau^*(2s + 2al)) \to \mathcal{O}_P(E + \tau^*(2s + 2al)) \to \mathcal{O}_E \to 0.
\]

Since \( R^1\tau_*\mathcal{O}_P = 0 \),
for \( i \geq 1 \), we have \( H^1(P, \mathcal{O}_P(\tau^*(2s + 2al))) = H^1(\Sigma, \mathcal{O}_\Sigma(2s + 2al)) = 0 \). So the trace of \( |E + \tau^*(2s + 2al)| \) on \( E \) is base point free. This completes the proof of (c).

In the remaining of this section, we fix a pair \((e, a)\) as in the assumption of Theorem 1.1. We use the theory of double covers to construct 3-folds and prove Theorem 1.1.

Let \( T \) be any smooth effective divisor in \([5E + \tau^*(10s + 10al)]\). Such \( T \) exists by Lemma 2.1(d) and the Bertini theorem. Then \( T \cap E = \emptyset \) since \( \mathcal{O}_T(T|_E) = \mathcal{O}_E \) (see the proof of Lemma 2.1(c)). We also have

\[
E + T \sim 2L, \quad \text{where} \quad L = 3E + \tau^*(5s + 5a) \tag{2.3}
\]

Therefore there is a finite smooth double cover \( \psi: Y \to P \) branched along \( E + T\) such that \( \psi_* \mathcal{O}_Y = \mathcal{O}_Y \oplus \mathcal{L}' \). Set \( E_0 := \psi^{-1}(E) \). Then \( \psi^*E = 2E_0 \). We identify \( \text{Pic}(E_0) \) with \( \mathbb{Z}s \oplus \mathbb{Z}l \) via \( \tau|_E \circ \psi|_{E_0} : E_0 \cong \mathbb{Z} \cong E \).

**Proposition 2.2.** Let \( H := K_Y - E_0 \). Then

(a) \( K_Y = \psi^*(K_P + \mathcal{L}) \) and \( p_g(Y) = p_g(P) + h^0(P, \mathcal{O}_P(K_P + \mathcal{L})) = h^0(P, \mathcal{O}_P(K_P + \mathcal{L})) \) since \( p_g(P) = 0 \). Note that \( K_P + \mathcal{L} = E + \tau^*(s + (3a - e - 2)l) \) by the second formula of Lemma 2.1(a) and (2.3). The formula for \( K_Y \) follows. By Lemma 2.1(a)-(b), we have \( \tau_* \mathcal{O}_P(K_P + \mathcal{L}) = \mathcal{E} \otimes \mathcal{O}_\Sigma(s + (3a - e - 2)l) = \mathcal{O}_\Sigma(s + (3a - e - 2)l) \oplus \mathcal{O}_\Sigma(-s + (a - e - 2)l) \). As in the proof of Lemma 2.1(b), a similar calculation yields \( h^0(\Sigma, \mathcal{O}_\Sigma(s + (3a - e - 2)l)) = 6a - 3e - 2 \) and \( h^0(\Sigma, \mathcal{O}_\Sigma(-s + (a - e - 2)l)) = 0 \). Therefore \( p_g(Y) = h^0(P, \tau_* \mathcal{O}_P(K_P + \mathcal{L})) = 6a - 3e - 2 \).

Let \( M := \psi^*(s + (3a - e - 2)l) \). Then \( H = M + E_0 \), \( K_Y = M + 2E_0 \) and \( |M|_{E_0} = s + (3a - e - 2)l \). So \( E_0|_{E_0} = \frac{1}{2}(K_{E_0} - M|_{E_0}) = -s - al \) by the adjunction formula and then \( H|_{E_0} = (2a - e - 2)l \). Therefore \( H^2E_0 = 0 \), \( HME_0 = H|_{E_0}M|_{E_0} = 2a - e - 2 \) and \( M^2E_0 = (|M|_{E_0})^2 = 6a - 3e - 4 \). It follows that \( H^3 = 8a - 4e - 6 \).

Now assume \( H.C \leq 0 \) for an irreducible curve \( C \). If \( \tau_C(C) \) is a point, then \( \tau.C = \frac{1}{2} \psi^*E.C > 0 \) since \( E \) is a section of \( \tau \). So \( \tau_C(C) \) is a curve. Since \( s + (3a - e - 2) \) is very ample by [3] p. 380, Corollary 2.18, \( M.C > 0 \) and thus \( E_0C = (H - M)C < 0 \). Therefore \( C \) is contained in \( E_0 \). Since \( H|_{E_0} = (2a - e - 2)l \) and \( 2a - e - 2 > 0 \), we conclude that \( H.C = 0 \) and \( C \) is a fiber of the ruling of \( E_0 \). This completes the proof of (b) and (c).

Because both \( H \) and \( M \) are nef and \( H^3 > 0 \), \( 3H - K_Y = H + M \) is nef and big. Since \( M|_{E_0} \) is very ample, it follows by (c) that \( (3H - K_Y)C > 0 \) for any curve \( C \). Assume \( (3H - K_Y)^2S = (H + M)^2S = 0 \) for some irreducible surface \( S \). Then \( H^2S = HMS - M^2S = 0 \) and thus \( ME_0S = M(H - M)S = 0 \). Because \( M|_{E_0} \) is very ample, we conclude \( S \cap E_0 = \emptyset \). Then the equality \( HMS = 0 \) yields a contradiction to (c), since \( |M| \) is base point free. We have shown \( (3H - K_Y)^2S > 0 \) for any irreducible surface \( S \). Therefore \( 3H - K_Y \) is ample.

By Proposition 2.2(c)-d, the base point free theorem [11] Theorem 3-1-1 implies that the image \( X \) of the morphism \( \pi: Y \to \mathbb{P}(H^0(Y, \mathcal{O}_Y(mH)))^* \) for \( m >> 0 \) can be identified with the blowdown along the ruling of \( E_0 \). Therefore \( X \) is a 3-fold birational to \( Y \).

We now show that \( X \) satisfies the properties in Theorem 1.1. Observe that \( (3H - K_Y) + tK_Y|_{E_0} = 3l + (t - 1)(-s + (a - e - 2)l) \) for any \( t \in \mathbb{R} \). Hence \( \max\{t \in \mathbb{R}|3H - K_Y + tK_Y \text{ is nef}\} = 1 \).
Since \( \mathcal{O}_{E_0}(E_0) \cong \mathcal{O}_{E_0}(-s - ad) \), \( X \) is smooth by [14, Theorem (3.3) (3.3.1)]. Also \( K_X \) is ample by the base point free theorem. From the construction, we have \( \pi^* K_X \cong H \) and thus 
\[ K_X^3 = H^3 = 8a - 4e - 6. \] 
Since \( X \) is birational to \( Y \), \( p_0(X) = p_0(Y) = 6a - 3e - 2. \) We have seen that \( H = E_0 + \psi^* \tau^*(s + (3a - e - 2)l) \) in the proof of Proposition 2.2. Because the movable part of \( |\pi^* K_X| = |H| \) is \( \psi^* \tau^*(s + (3a - e - 2)l) \), we see that the base locus \( |K_X| \) consists of the smooth rational curve \( \pi(E_0) \) and the canonical image of \( X \) is the image of the embedding of \( \Sigma_e \) into \( \mathbb{P}p_0(X)^{-1} \) induced by \( |s + (3a - e - 2)l| \).

\textbf{Remark 2.3.} If \((c, a) = (0, 1)\), the construction of \( Y \) still works and Proposition 2.2 still holds except the assertion (c). Indeed, \( |H|_{E_0} = 0 \) in this case and \( |mH| \) \((m >> 0)\) contracts exactly the whole divisor \( E_0 \). The 3-fold \( X \) obtained by the base point free theorem has ample canonical divisor and it is still on the Noether equality. Indeed, \( K_X^3 = 2 \) and \( p_0(X) = 4. \) But \( X \) is no longer smooth. It has exactly one singularity (cf. [14, Theorem (3.3) (3.3.3)]). Moreover, \( |K_X| \) has this singularity as base locus and the canonical image of \( X \) is a smooth quadric.

3 Base locus of the canonical linear systems

We prove Theorem 1.2 in this section. Throughout this section, we denote by \( X \) a smooth projective 3-fold with ample canonical divisor \( K_X \).

3.1 Setting

Write \( [K_X] = [M] + Z \), where \( [M] \) is the movable part of \( [K_X] \) and \( Z \) is the fixed part of \( [K_X] \). By Hironaka’s Theorem (cf. [9]), we can take successive blowups

\[ \pi: Y = X_{n+1} \xrightarrow{\pi_n} X_n \rightarrow \cdots \rightarrow X_{i+1} \xrightarrow{\pi_i} X_i \rightarrow \cdots \rightarrow X_1 \xrightarrow{\pi_1} X_0 = X \]

such that

(1) \( \pi_i \) is a single blowup along a smooth center \( W_i \) on \( X_i \) for \( 0 \leq i \leq n; \)

(2) \( W_i \) is contained in the base locus of the movable part of \( \left| (\pi_0 \circ \pi_1 \circ \cdots \circ \pi_{i-1})^*M \right|; \)

(3) the movable part of \( \pi^* [M] \) has no base points.

In particular, \( W_i \) is a reduced closed point or a smooth projective curve on \( X_i \) for each \( i \). We may choose \( \pi \) such that the number \( n + 1 \) of blowups is minimal. On the smooth 3-fold \( Y \), we denote by \( E_i \) the strict transform of the exceptional divisor of \( \pi_i \) for each \( i \). Then we have

\[ K_Y = \pi^* K_X + \sum_{i=0}^{n} a_i E_i, \quad \pi^* [M] = M + \sum_{i=0}^{n} b_i E_i, \quad (3.1) \]

where \( a_i \) and \( b_i \) are positive integers for \( 0 \leq i \leq n \), and \( [M] \) is the movable part of \( \pi^* [M] \).

\textbf{Lemma 3.1.} For \( 0 \leq i \leq n \), we have \( a_i \leq 2b_i \).

(a) If \( a_k = b_k = 1 \) for some \( k \) such that \( 0 \leq k \leq n \), then \( W_k \) is a curve of \( X_k \) and the general member of \( [M] \) is smooth at a general point of \( \pi_0 \circ \cdots \circ \pi_{k-2} \circ \pi_{k-1}(W_k) \).

(b) If \( a_k = 2b_k \) for some \( k \) such that \( 0 \leq k \leq n \), then \( W_k \) is a point of \( X_k \) and the general member of \( [M] \) is smooth at \( \pi_0 \circ \cdots \circ \pi_{k-2} \circ \pi_{k-1}(W_k) \).

The assertion \( a_i \leq 2b_i \) is exactly [5, Lemma 4.2]. What really involved here is the assertion (b).
Proof. We introduce some notation. For any \(0 \leq j < i \leq n\), let \(E_j^{(i)} (\subset X_i)\) be the strict transform of the exceptional divisor of \(\pi_j\) and let \(E_j^{(n+1)} := E_j\). According to the definition of \(a_i\) and \(b_i\), for any \(0 \leq i \leq n\), we have

\[
K_{X_{i+1}} = (\pi_0 \circ \pi_1 \circ \cdots \circ \pi_i)^* K_X + \sum_{j=0}^i a_j E_j^{(i+1)},
\]

\[
(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_i)^*(\mathcal{M}) = M_{i+1} + \sum_{j=0}^i b_j E_j^{(i+1)}, \tag{3.2}
\]

where \(|M_{i+1}|\) is the movable part of \(|K_{X_{i+1}}|\).

Considering the single blowup \(\pi_k : X_{k+1} \to X_k\) for \(1 \leq k \leq n\), we have for \(j < k\),

\[
\pi_k^* E_j^{(k)} = E_j^{(k+1)} + r_j E_k^{(k+1)}, \quad \text{where} \quad r_j \in \mathbb{Z} \text{ and } r_j \geq 0, \quad \text{and } r_j > 0 \text{ if and only if } W_k \subseteq E_j^{(k)},
\]

\[
K_{X_{k+1}} = \pi_k^* K_{X_k} + a_k^* E_k^{(k+1)}, \quad \text{where} \quad a_k^* = 1 \text{ if } W_k \text{ is a curve or } a_k^* = 2 \text{ if } W_k \text{ is a point},
\]

\[
\pi_k^* M_k = M_{k+1} + b_k^* E_k^{(k+1)}, \quad \text{where} \quad b_k^* \geq 1 \text{ since } W_k \text{ is contained in the base locus of } |M_k|.
\]

Comparing these formulae with (3.2) when \(i = k\) and \(i = k - 1\), we obtain

\[
a_k = a_k^* + \sum_{j=0}^{k-1} r_j a_j, \quad b_k = b_k^* + \sum_{j=0}^{k-1} r_j b_j \text{ for } 1 \leq k \leq n \tag{3.3}
\]

Using (3.3), we easily conclude that \(a_i \leq 2b_i\) for \(0 \leq i \leq n\) by induction on \(i\).

For (a), the case \(k = 0\) is trivial. If \(k > 0\), by (3.3), we have \(a_k^* = 1\), \(b_k^* = 1\) and \(r_j = 0\) for any \(j < k\). Therefore \(W_k\) is a smooth curve of \(X_k\), the general member of \(|M_k|\) is smooth at a general point of \(W_k\) and \(W_k \not\subseteq E_j^{(k)}\) for \(j < k\). Assertion (a) follows.

We proceed to prove (b) by induction on \(k\). Assume \(a_0 = 2b_0\). Note that \(a_0 = 1\) or \(2\) by property (1). Therefore \(a_0 = 2\) and \(b_0 = 1\). It follows that \(W_0\) is a point of \(X\) and the general member of \(|M|\) is smooth at \(W_0\).

Now assume \(a_k = 2b_k\) for some \(k > 0\). Since \(a_j \leq 2b_j\) for any \(j < k\), by (3.3), we have

\[
a_k^* = 2, \quad b_k^* = 1 \quad \text{and} \quad a_j = 2b_j \text{ if } j < k \text{ and } r_j > 0.
\]

Therefore \(W_k\) is a smooth point of \(X_k\) and the general member of \(M_k\) is smooth at \(W_k\). If \(r_j = 0\) for any \(j < k\), then the point \(W_k \not\subseteq E_j^{(k)}\) for any \(j < k\) and thus the second statement of (b) clearly holds. If \(r_j > 0\) for some \(j < k\), then \(a_j = 2b_j\) and \(\pi_0 \circ \cdots \pi_{k-2} \circ \pi_{k-1}(W_k) = \pi_0 \circ \cdots \pi_{j-2} \circ \pi_{j-1}(W_j)\). Assertion (b) follows by induction.

Denote by \(\phi_k X\) the canonical map of \(X\) and by \(\Sigma\) the image of \(\phi_k X\). Let \(\phi\) be the morphism induced by the linear system \(|M|\). Then \(\phi = \phi_k X \circ \pi\). Let \(Y \xrightarrow{f} B \xrightarrow{\delta} \Sigma\) be the Stein factorization of \(\phi\). We have the following commutative diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & B \\
\downarrow \pi & & \downarrow \delta \\
X & \xrightarrow{\phi_k X} & \Sigma
\end{array}
\]

Note that \(B\) is normal. We have the following known results: if \(\dim B = 3\), then \(K_X^3 \geq 2p_g(X) - 6\) (cf. [12] Main Theorem); if \(\dim B = 2\), then \(K_X^3 \geq \left\lceil \frac{2}{3}(g(C) - 1) \right\rceil (p_g(X) - 2)\) where \(g(C)\) is the genus of a general fiber \(C\) of \(f\) (cf. [4] Theorem 4.1 (ii)); if \(\dim B = 1\), then \(K_X^3 \geq \frac{5}{3}p_g(X) - 2\) (cf. [4] Theorem 4.1 (iii)) and [2] Theorem 4.1].
3.2 Some basic results

From now on, we assume further that $X$ is on the Noether line $K_X^3 = \frac{1}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$.

**Lemma 3.2.** Keep the same notation as above.

(a) The morphism $f$ is a fibration over the normal surface $B$.

(b) Let $C$ be a general fiber of $f$. Then $C$ is a smooth curve of genus 2 with $\pi^*K_X.C = 1$.

(c) Let $d_\Sigma$ be the degree of $\Sigma$ in $\mathbb{P}^{p_g(X)-1}$. Then $d_\Sigma \geq p_g(X) - 2$ and $M^2 \equiv d_\Sigma C$.

(d) The morphism $\delta$ is birational.

Here the symbol $\equiv$ in (c) stands for numerical equivalence.

**Proof.** From the discussion at the end of the last subsection, we see that $B$ is a normal surface and $g(C) = 2$. In particular, we have $M^2 \equiv d_\Sigma \cdot \deg \delta \cdot C$.

Because $\Sigma$ is non-degenerate, we have $d_\Sigma \geq p_g(X) - 2$. Since both $\pi^*K_X$ and $M$ are nef, we conclude that

$$K_X^3 \geq \pi^*K_X M^2 = d_{\Sigma} \cdot \deg \delta \cdot \pi^*K_X.C \geq (p_g(X) - 2) \deg \delta \cdot \pi^*K_X.C.$$ 

If $\pi^*K_X.C \geq 2$ or $\deg \delta \geq 2$, then $K_X^3 \geq 2p_g(X) - 4$, a contradiction to the Noether equality or the assumption $p_g(X) \geq 7$. Therefore $\pi^*K_X.C = 1$, $\deg \delta = 1$ and thus $M^2 \equiv d_\Sigma C$. \hfill \Box

**Lemma 3.3.** (a) There exists a unique integer $l$ such that $0 \leq l \leq n$, $E_lC = 1$, $a_l = 1$ and $E_lC = 0$ for any $i \neq l$.

(b) Moreover, $b_l = 1$ and $\pi^*Z.C = 0$.

**Proof.** Since $g(C) = 2$, we have $K_Y.C = 2$ by the adjunction formula. Note that $K_X = M + Z$ and $MC = 0$. According to (3.1) and Lemma 3.2 (b), we have $\sum_{i=0}^{n} a_i E_i.C = 1$ and $(\pi^*Z + \sum_{i=1}^{n} b_i E_i)C = 1$. Then (a) and (b) follow by the fact that $a_i$ and $b_i$ are positive integers for any $i$. \hfill \Box

In what follows, we shall figure out the geometric information hidden in the Noether equality $K_X^3 = \frac{1}{3}p_g(X) - \frac{10}{3}$. For this purpose, we use the techniques in the proof of [3, Theorem 4.3]. Note that $K_X^3 = (\pi^*K_X)^3 = (\pi^*K_X)^2(M + \sum_{i=0}^{n} b_i E_i + \pi^*Z) = (\pi^*K_X)^2M + K_X^3Z$. We aim to bound $(\pi^*K_X)^2M$ from below.

Let $S$ be a general member of $|M|$. According to Bertini's Theorem, $S$ is smooth. Also $S$ is of general type because so is $X$. By abuse of notation, we still denote by $C$ the general fiber of the fibration $f|_S$. Clearly, for a general $S$, $E_i|_S$ is effective for any $i$. Moreover, according to Lemma 3.3 we have

$$\sum_{i=0}^{n} a_i E_i|_S = \Gamma + E_V, \sum_{i=0}^{n} b_i E_i|_S + \pi^*Z|_S = \Gamma + E_V'$$

where $\Gamma$ is the unique irreducible component of $E_i|_S$ such that $\Gamma C = 1$, $E_V$ and $E_V'$ are effective divisors contained in the fibers of $f|_S$, and

$$E_V = \sum_{i \neq l} a_i E_i|_S + (E_l|_S - \Gamma), E_V' = \sum_{i \neq l} b_i E_i|_S + (E_l|_S - \Gamma).$$

**Lemma 3.4.** In the above setting, we have
(a) \( Z = 0 \);

(b) the divisors \( E_V \) and \( E'_V \) satisfy the following properties: \( \text{Supp} E_V = \text{Supp} E'_V \), \( 2E'_V - E_V \geq 0 \) and \( (2E'_V - E_V)\Gamma = 0 \);

(c) \( \delta: B \to \Sigma \) is an isomorphism and \( \Sigma \) is a normal rational surface with \( d_\Sigma = p_g(X) - 2 \);

(d) \( \Gamma \) is a smooth rational curve and \( \pi^*K_X|_S \cdot \Gamma = \frac{1}{3} p_g(X) - \frac{4}{3} \); 

(e) \( \pi^*K_X|_S . E_V = \pi^*K_X|_S . E'_V = 0 \).

**Proof.** From \([3.3]\), we have \( \text{Supp} E_V = \text{Supp} E'_V \). Since \( a_i \leq 2b_i \) by Lemma \([3.1]\), \( 2E'_V - E_V \) is an effective divisor. Because \( \Gamma \) is a section of \( f|_S \) and \( 2E'_V - E_V \) is contained in fibers of \( f|_S \), we obtain

\[
\Gamma \cdot (2E'_V - E_V) \geq 0 \quad (3.6)
\]

The adjunction formula yields

\[
K_S \Gamma + \Gamma^2 = 2p_a(\Gamma) - 2 \geq -2 \quad (3.7)
\]

Note that \( K_Y|_S = \pi^*K_X|_S + \Gamma + E_V \) and \( \pi^*K_X|_S = M|_S + \Gamma + E'_V \), by \([3.1]\) and \([3.4]\). By \([3.6]\), one has

\[
-2 \leq (K_S + \Gamma)\Gamma = (K_Y + M)|_S \Gamma + \Gamma^2 \\
= (\pi^*K_X|_S + M|_S + E_V + 2\Gamma)\Gamma \\
\leq (\pi^*K_X|_S + M|_S + 2E'_V + 2\Gamma)\Gamma \\
= (3\pi^*K_X|_S - M|_S)\Gamma \\
= 3\pi^*K_X|_S \cdot \Gamma - d_\Sigma. 
\]

The last equality holds by \( M^2 \equiv d_\Sigma C \) (see Lemma \([3.2]\)(c)). Also \( d_\Sigma \geq p_g(X) - 2 \), we obtain

\[
\pi^*K_X|_S \cdot \Gamma \geq \frac{1}{3}(d_\Sigma - 2) \geq \frac{1}{3}(p_g(X) - 4) \quad (3.9)
\]

Finally, we have

\[
K_X^3 = (\pi^*K_X)^3 = (\pi^*K_X|_S)^2 + K_X^2 Z \\
= \pi^*K_X|_S . M|_S + \pi^*K_X|_S . \Gamma + \pi^*K_X|_S . E'_V + K_X^2 Z \\
\geq d_\Sigma + \frac{1}{3} p_g(X) - \frac{4}{3} + \pi^*K_X|_S . E'_V + K_X^2 Z \\
\geq \frac{4}{3} p_g(X) - \frac{10}{3} \quad (3.10)
\]

By assumption, we see that all the equalities in the inequalities \([3.6]-[3.10]\) hold.

Since \( K_X \) is ample, (a) follows by the equality \( K^3 X Z = 0 \) in \([3.10]\). And then (b) follows by the equality in \([3.6]\). For (c), we conclude \( d_\Sigma = p_g(X) - 2 \) from \([3.9]\). This implies that \( \Sigma \) is of minimal degree. Since \( \Sigma \) is non-degenerate, \( \Sigma \) is a normal rational surface (cf. [II Exercises IV. 18. 4]). Because \( B \) is also normal and \( \delta \) is finite by Stein factorization, \( \delta \) is an isomorphism by Lemma \([3.2]\)(d). Assertion (d) follows by the equalities in \([3.7]\) and \([3.9]\). Because \( \pi^*K_X|_S \) is nef and \( \text{Supp} E_V = \text{Supp} E'_V \), (e) follows by the equality in \([3.10] \). \( \square \)

**Corollary 3.5.** The base locus of \( |K_X| \) consists of a unique irreducible curve \( \Gamma = \pi(E_i) \).
See Lemma \[3.3\] for the property of \(E_l\).

**Proof.** We remark that \(|K_X|\) has base locus because \(|K_X|\) is ample and the canonical image of \(X\) is a surface.

Note that \(\pi(\Gamma)\) is an irreducible curve by Lemma \[3.4\] (d). Since \(\Gamma \leq E_l|_S\), we have \(\pi(\Gamma) = \pi(E_l)\). Let \(\Gamma' := \pi(\Gamma) = \pi(E_l)\). Then \(\Gamma\) is contained in the base locus of \(|K_X|\).

Assume that \(\Gamma'\) is an irreducible curve contained in the base locus of \(|K_X|\) and \(\Gamma' \neq \Gamma\). Then from the construction of \(\pi\), there exists an irreducible curve \(\Gamma''\) contained in some \(E_l|_S\) such that \(\Gamma'' = \pi(\Gamma')\). Since \(\Gamma'' \neq \Gamma\), we have \(i \neq l\) and thus \(\Gamma''\) is contained in \(\text{Supp} E_V\). Hence \(K_X\Gamma'' = 0\) by Lemma \[3.4\] (e). Therefore \(\Gamma'' = 0\) since \(K_X\) is ample, a contradiction.

Suppose \(p\) is an isolated base point of \(|K_X|\) and \(p \notin \Gamma\). We can write \(E_l' = A + B\), where \(A\) and \(B\) are effective divisors such that \(\pi(\text{Supp} A) = p\) and \(p \notin \text{Supp}(B)\). Then \(A \cdot B = 0\) and \(A \cdot B = 0\) since \(p \notin \Gamma\). Because \(E_l'\) is contained in the fiber of \(f|_S\), we have \(M|_S, A = 0\). Since \(\pi^* K_X|_S = M|_S + \Gamma + E_l' = M|_S + \Gamma + A + B\) by \[3.1\] and \[3.4\], we conclude from Lemma \[3.4\] (e) that \(A^2 = 0\). But \(\pi^* K_X|_S\) is nef and big, this contradicts the algebraic index theorem. Therefore the base locus of \(|K_X|\) consists of the irreducible curve \(\Gamma\).

\[\square\]

3.3 Proof of Theorem 1.2

We are able to complete the proof of Theorem 1.2. Let \(X\) be a smooth 3-fold as in the assumption of Theorem 1.2. We stick to the same notation as above.

Recall that the base locus of \(|K_X|\) consists of an irreducible curve \(\Gamma\) (see Corollary \[3.5\]). So a general member \(S \in |M|, S := \pi(S)\) is a divisor in \(|K_X|\). According to Bertini’s theorem andLemma \[3.1\] we may choose \(S \in |M|\) such that

1. \(S\) is smooth outside \(\Gamma\);
2. \(S\) is smooth at a general point of \(\Gamma\);
3. \(S\) is smooth at the point \(\pi_0 \circ \cdots \circ \pi_{k-1}(W_k)\) for those \(k\) such that \(a_k = 2b_k\).

In particular, \(S\) is a normal surface. Denote by \(\sigma = \pi|_S : S \to S\). Note that \(\sigma\) is isomorphic at the points outside \(\Gamma \cup \text{Supp} E_V\) (see \[3.4\] and \[3.5\]) for the definition of \(E_V\) and \(E_l'\).

Let \(q\) be any point of \(\overline{\sigma} = S\) such that \(\sigma^{-1}\) is not defined at \(q\). Then \(q \in \Gamma\) by the choice of \(S\). According to Zariski’s main theorem, \(\pi^{-1}(q)\) is a connected curve. Since \(q \in \Gamma\), there is an irreducible component \(Q\) of \(\pi^{-1}(q)\) such that \(Q\Gamma\) is not defined at \(q\). Note that \(Q \leq E_V\). Then Lemma \[3.4\] (e) yields \(Q \leq E_{l*}\) for some \(k \neq l\) and \(a_k = 2b_k\) (see Lemma \[3.3\] for the property of the integer \(l\)). Therefore \(S\) is smooth at \(q\) by the choice of \(S\).

We have indeed shown that \(S\) is smooth. Therefore \(\sigma\) is compositions of blowups at smooth points and the support of the exceptional divisors of \(\sigma\) is \(\text{Supp} E_V\). Recall that \(f|_S\) is the fibration induced by \(|M|\), and \(C\) is a general fiber of \(f|_S\), and \(E_V\) is contained in the fibers of \(f|_S\). Hence there is a base point free pencil \(\mathcal{C}\) of curves such that \(\pi^* (\mathcal{C}) = C\) is a hyperplane section of \(S\). Since \(S\) is smooth, \(\mathcal{C}\) is a smooth rational curve. Therefore \(|M|\) is linearly equivalent to \(d_2 C\). Also \(\pi^* (K_X|_S) = \pi^* K_X|_S = M|_S + \Gamma + E_V = d_2 C + \Gamma + E_V\). Hence we obtain \(K_X|_S = d_2 C + \Gamma\).

Let \(\pi' : X' \to X\) be the blowup of \(X\) along the curve \(\Gamma\). We have

\[K_{X'} = \pi'^* (K_X) + E', \pi'^* (K_X) = M' + E',\]

8
where $E'$ is the exceptional divisor of the blowup and $|M'|$ is the movable part of $\pi^*(K_X)$. To prove (b), it suffices to prove that $|M'|$ is base point free (recall that we assume that the number $n+1$ of the blowups is minimal in the choice of the morphism $\pi: Y \to X$).

Let $S'$ be the strict transform of $S$ and let $\sigma' := \pi'|_{S'}: S' \to \overline{S}$. Since $\overline{S}$ is smooth along $E'$, $S'$ is smooth, $\pi^*\overline{S} = S' + E'$ and $\sigma'$ is an isomorphism. Moreover, $E'|_{S'}$ is a smooth rational curve and $\sigma'^*(\Gamma) = E'|_{S'}$.

It suffices to show that the trace of $|M'|$ on $S'$ is base point free since $S' \in |M'|$. Note that $M'|_{S'} = \pi^*(K_X)|_{S'} - E'|_{S'} = \sigma'^*(K_X|_{\overline{S}}) - \sigma'^*(\Gamma)$. We have seen $K_X|_{\overline{S}} = d\overline{C} + \Gamma$. Therefore $M'|_{S'} = \sigma'^*(d\overline{C})$. Because $|C|$ is base point free and $\dim |M'||_{S'} = p_g(X) - 2 = d_{\Sigma}$ we conclude that $|M'||_{S'}$ is base point free and complete the proof of (b).

Assertions (c) and (d) follow by Lemma 3.2 (b)-(c) and Lemma 3.4 (c).

## 4 Classification

The whole section is devoted to prove Theorem 1.3. We stick to the same notation in Theorem 1.2. In particular, the left triangle of the following diagram

\[ Y \xrightarrow{\pi} \Sigma_e \]

\[ X \xrightarrow{\phi_{K_X}} \Sigma \xrightarrow{pp_s(X)-1} Y \]

is commutative. And we have

\[ K_Y = \pi^*K_X + E_0, \quad \pi^*K_X = M + E_0 \]

(4.2)

where $|M|$ is base point free and $\phi$ is induced by $|M|$. Also $\phi|_{E_0}: E_0 \to \Sigma$ is a birational morphism.

According to Theorem 1.2 (c), $\deg \Sigma = p_g(X) - 2 \geq 5$. So $\Sigma$ is obtained from a Hizhebruch surface $\Sigma_e$ for some $e \geq 0$ via the birational morphism $r: \Sigma_e \to \Sigma$ induced by the linear system $|s + (e + k)|$, where $k$ is a nonnegative integer such that

\[ p_g(X) = 2k + e + 2 \text{ and } \deg \Sigma = 2k + e \]

(4.3)

(see Section 2 for the notation for the Hizhebruch surfaces). More precisely, we have two possibilities as follows (cf. [1] Exercises IV.18 4) or [8], p. 380, Corollary 2.19).

1. If $k \geq 1$, then $r$ is an isomorphism and $\Sigma$ is indeed smooth. In this case, $\phi|_{E_0}: E_0 \to \Sigma$ is indeed an isomorphism since $E_0$ is also a Hizhebruch surface. If $e > 0$, then the ruling $|l|$ of $\Sigma_e$ coincides with $\pi|_{E_0}: E_0 \to \Gamma$ via $r^{-1}\phi|_{E_0}$ because $\Sigma_e$ has a unique ruling. We may assume this still holds when $e = 0$ by possibly exchanging the two rulings of $\Sigma_0$.

2. If $k = 0$, then $e \geq 5$ and $\Sigma$ is a cone over a rational normal curve. Moreover, $r$ contracts exactly the negative section $s$ and $v := r(s)$ is the vertex of the cone $\Sigma$.

To prove Theorem 1.3 we shall exclude the case (2). The following lemma allows us to treat both cases in a unified way.

**Lemma 4.1.** The rational map $\overline{\phi} = r^{-1}\phi$ is indeed a morphism and $\overline{\phi}|_{E_0}: E_0 \to \Sigma_e$ is an isomorphism.
Proof. The lemma is nontrivial only for $k = 0$. In this case, since $\rho(E_0) = 2$ and $\rho(\Sigma) = 1$, the birational morphism $\phi|_{E_0} : E_0 \to \Sigma$ contracts exactly the negative section $s_{E_0}$ of the ruling $\pi|_{E_0} : E_0 \to \Pi$ and maps any fiber $l_{E_0}$ of this ruling to a line on $\Sigma$. Since $\phi|_{E_0}$ is induced by $|M|_{E_0}$, we conclude that $M|_{E_0} = s_{E_0} + eL_{E_0}$.

Assume by contradiction that $r^{-1}\phi$ is not a morphism. Then the locus where $r^{-1}\phi$ is not defined is contained in $\phi^{-1}(v)$. Let $\hat{\pi} : \hat{Y} \to Y$ be the resolution of the indeterminacy of $r^{-1}\phi$ and let $\hat{\phi} : \hat{Y} \to \Sigma_\epsilon$ be the induced morphism such that $\phi\hat{\pi} = r\hat{\phi}$:

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi} & \Sigma_e \\
\downarrow \hat{\pi} & & \downarrow r \\
\hat{Y} & \xrightarrow{\hat{\phi}} & \Sigma \\
\end{array}
$$

$s = r^{-1}(v)$

We may assume that $\hat{\pi}$ is compositions of blowups along smooth centers and that the number of blowups is minimal.

From the commutative diagram, we have

$$\hat{\pi}^* M = \hat{\pi}^* \phi^* \mathcal{O}_\Sigma(1) = \hat{\phi}^* r^* \mathcal{O}_\Sigma(1) = \hat{\phi}^*(s + eL) = \Delta + e\hat{\Delta},$$

where $\Delta := \hat{\phi}^* s$, $\hat{\Delta} := \hat{\phi}^* t$ and $|\hat{L}|$ is a base point free pencil of divisors. It follows that $M = \Delta + eL$, where $\Delta := \hat{\pi}_* \hat{\Delta}$, $L := \hat{\pi}_* \hat{\Delta}$ and $|L|$ has no fixed part.

Recall that $C$ is a general fiber of $\phi$ and $E_0 C = 1$. Since $MC = 0$, we have $\Delta C = 0 = LC$. Then $\Delta \not\geq E_0$ and $L \not\geq E_0$ since $E_0 C = 1$.

We now show that $|L|$ is a base point free pencil. Note that $e + 2 = h^0(Y, \mathcal{O}_Y(M)) \geq h^0(Y, \mathcal{O}_Y(eL)) = e + 1$. Therefore $\dim |L| = 1$. Since $\Delta|_{E_0} + eL|_{E_0} = M|_{E_0} \simeq s_{E_0} + eL_{E_0}$ and $\Delta \not\geq E_0$, we have $\Delta|_{E_0} > 0$ and thus $\Delta > 0$. Moreover, from the commutativity of the diagram above and the definition of $\Delta$, we see that $\phi|_{E_0}$ contracts exactly the curve $s_{E_0}$, $\Delta|_{E_0} = bs_{E_0}$ for some integer $b \geq 1$. Then $e^2 \pi^* K_X \cdot L^2 = (M + E_0)(M - \Delta)^2 = M|_{E_0} - 2M|_{E_0} \Delta|_{E_0} + \Delta|_{E_0}^2 = e(1 - b^2)$. Since $K_X$ is ample and $|L|$ has no fixed part, we obtain $\pi^* K_X \cdot L^2 = 0$, $b = 1$ and thus $L|_{E_0} = \frac{1}{e}(M - \Delta)|_{E_0} = l_{E_0}$. Then the trace of the pencil $|L|$ on $E_0$ is $|l_{E_0}|$ since $L \not\geq E_0$. This implies the base locus of $|L|$ is disjoint from $E_0$. If the base locus of $|L|$ is nonempty, it is of pure dimension 1 since $Y$ is a smooth 3-fold and $\dim |L| = 1$. Hence the ampleness of $K_X$ and $\pi^*K_X \cdot L^2 = 0$ imply that $|L|$ is base point free.

Because both $|L|$ and $\hat{L}$ are base point free and $L = \hat{\pi}_* \hat{L}$, we have $\hat{\pi}^* L = \hat{L}$. Let $\hat{\pi}_1 : \hat{Y} \to \hat{Y}_1$ be the last blowup in $\hat{\pi}$ and let $\hat{E}$ be the exceptional divisor of $\hat{\pi}_1$. Let $F$ be any curve of $\hat{E}$ contracted by $\hat{\pi}_1$. Then $\hat{L} F = \hat{\pi}^* L_{\hat{E}} = 0$. On one hand, since $\hat{L} = \hat{\phi}^*(l)$, $\hat{\phi}(F)$ is contained in one of the fiber of ruling induced by $|l|$. On the other hand, $r^{-1}\phi$ is defined outside $\phi^{-1}(v)$, so $r\hat{\phi}(F) = \hat{\phi}(F) = v$ and thus $\hat{\phi}(F)$ is contained in $r^{-1}(v) = s$. Therefore $\hat{\phi}(F)$ is a point in $\Sigma_\epsilon$.

This means $\phi$ factors though $\hat{Y}_1$. But this contradicts the choice of $\hat{\pi}$.

Hence $\hat{\phi}$ is indeed a morphism. Its restriction $\hat{\phi}|_{E_0} : E_0 \to \Sigma_\epsilon$ is birational because so is $\phi|_{E_0}$.

Then it is an isomorphism because both $E_0$ and $\Sigma_\epsilon$ are Hizebruch surfaces. \hfill \Box

From now on, we denote by $j$ the inverse of the isomorphism $\phi|_{E_0} : E_0 \to \Sigma_\epsilon$. By abuse of notation, we identify $\text{Pic}(E_0)$ with $\text{Pic}(\Sigma_\epsilon) = \mathbb{Z}l \oplus \mathbb{Z}s$. Since $M = \phi^* \mathcal{O}_\Sigma(1)$ and $r^* \mathcal{O}_\Sigma(1) = s + (e + k)l$, we have

$$M = \phi^*(s + (e + k)l), M|_{E_0} = s + (e + k)l$$

(4.4)

Since $r\phi = \phi$, we still denote by $C$ the general fiber of $\phi$. 

10
Lemma 4.2. Every fiber of $\overline{\sigma}$ is 1-dimensional, reduced and irreducible. In particular, $\overline{\sigma}$ is flat.

Proof. Assume that $\Phi$ is a 2-dimensional irreducible component of a fiber of $\overline{\sigma}$. Then either $E_0 \cap \Phi = \emptyset$ or $\dim E_0 \cap \Phi = 1$ since $Y$ is smooth. Because $E_0$ is a section of $\overline{\sigma}$, we have $E_0 \cap \Phi = \emptyset$. Therefore $(\pi^*K_X)^2 \cdot \Phi = (M + E_0)^2 \cdot \Phi = 0$ according to (1.2). Since $K_X$ is ample, it follows that $\dim \pi(\Phi) \leq 1$. Then $\pi(\Phi) = \Gamma$ and $\Phi = E_0$, a contradiction. Hence every fiber of $\overline{\sigma}$ is 1-dimensional and $\overline{\sigma}$ is flat (cf. [13, p. 179, Theorem 23.1 and Corollary]).

Because $K_X$ is ample and $\pi^*K_X \cdot C = 1$ by Theorem 1.2 (c), if $\sigma$ has reducible fibers, then some irreducible component of some reducible fiber is contained in $E_0$. This contradicts that $E_0$ is a section of $\overline{\sigma}$. Therefore any fiber is irreducible. Also $\pi^*K_X \cdot C = 1$ implies that any fiber is reduced. □

Lemma 4.3. Let $\mathcal{E} := \overline{\sigma}_*\mathcal{O}_Y(2E_0)$.

(a) Then $\mathcal{E}$ is a locally free sheaf of rank 2 and the natural morphism $\overline{\sigma}^*\mathcal{E} \rightarrow \mathcal{O}_Y(2E_0)$ is surjective.

(b) Let $P := \mathbb{P}_{\Sigma_\epsilon}(\mathcal{E})$ and let $\tau: P \rightarrow \Sigma_\epsilon$ be the natural projection. Then the $\Sigma_\epsilon$-morphism $\psi: Y \rightarrow P$ associated to the surjective morphism $\overline{\sigma}^*\mathcal{E} \rightarrow \mathcal{O}_Y(2E_0)$ is finite of degree 2.

(c) Let $E = \psi(E_0)$. Then $E$ is a section of $\tau$ and it is an irreducible connected component of the branch divisor of $\psi$.

(d) The section $E$ of $\tau$ corresponds to the surjective morphism $\mathcal{E} = j^*\overline{\sigma}^*\mathcal{E} \rightarrow j^*\mathcal{O}_Y(2E_0)$, whose kernel is $\mathcal{O}_{\Sigma_\epsilon}$.

Proof. Let $C$ be any fiber of $\overline{\sigma}$. Then $p_\alpha(C) = 2$ by Theorem 1.2 (c). Because $\overline{\sigma}$ is a flat morphism between smooth varieties, $C$ is Gorenstein. Since $C$ is reduced and irreducible by the previous lemma, $|\omega_C|$ is base point free by [3, Theorem 3.3]. We have $K_C = K_Y|_{C} = 2E_0|_{C}$ from [4, 2]. So $\mathcal{O}_Y(2E_0)|_{C}$ is generated by global sections and $h^0(C, \mathcal{O}_Y(2E_0)|_{C}) = p_\alpha(C) = 2$. Then (a) follows by Grauert’s Theorem.

For (b), we have the following diagram such that $\tau \psi = \overline{\sigma}$.

Note the the restriction $\psi|_{C}$ is indeed the canonical map of $C$, which is a finite morphism of degree 2 to the projective line $\mathbb{P}^1$. This proves (b).

Because $E_0$ is a section of $\overline{\sigma}$, $E$ is a section of $\tau$. Moreover, since $E_0|_{C}$ consists one point and $K_C = 2E_0|_{C}$, we conclude that $E_0|_{C}$ is a ramification point of the canonical morphism of $C$. Now (c) follows by the fact that the branch locus of $\psi$ is smooth of pure dimension 1.

Note that $E = \psi(E_0) = \psi j(\Sigma_\epsilon)$. By the construction of $\psi$, $\psi j$ corresponds the pullback of the surjective morphism $\overline{\sigma}^*\mathcal{E} \rightarrow \mathcal{O}_Y(2E_0)$ by $j^*$, which is $\mathcal{E} = j^*\overline{\sigma}^*\mathcal{E} \rightarrow j^*\mathcal{O}_Y(2E_0)$. Denote by $K$ its kernel. Then $\mathcal{O}_P(E) \otimes \tau^*K = \mathcal{O}_P(1)$ (see the proof of [8, p. 371, Proposition 2.6]). Applying $(\psi j)^*$ to this equality, since $\psi^*E = 2E_0|_{C}$ by (c), $\tau(\psi j) = \text{id}_{\Sigma_\epsilon}$ and $(\psi j)^*\mathcal{O}_P(1) = j^*\mathcal{O}_Y(2E_0)$, we conclude that $K = \mathcal{O}_{\Sigma_\epsilon}$. □

Let $D$ be the branch divisor of the double cover $\psi: Y \rightarrow P$. Then

$$D \sim 2\mathcal{L} \text{ and } K_Y = \psi^*(K_P + 3\mathcal{L}) \quad (4.5)$$
for some \( \mathcal{L} \in \text{Pic}(P) \). Since \( P = \mathbb{P}_{\Sigma_a} (\mathcal{E}) \) with \( \mathcal{E} \) as an extension
\[
0 \to \mathcal{O}_{\Sigma_a} \to \mathcal{E} \to j^* \mathcal{O}_Y(2E_0) \to 0
\] we have \( \text{Pic}(P) = \mathbb{Z}E \oplus \tau^* \text{Pic}(\Sigma_a) \),
\[
\mathcal{O}_P(E) = \mathcal{O}_P(1) \text{ and } K_P = \tau^*(K_{\Sigma_a} + j^* \mathcal{O}_Y(2E_0)) - 2E
\] We shall determine \( j^* \mathcal{O}_Y(2E_0) \) and \( \mathcal{L} \) in terms of \( \text{Pic}(\Sigma_a) \) and \( \mathcal{O}_P(E) \).

**Lemma 4.4.** We have
\[
j^* \mathcal{O}_Y(2E_0) \cong \mathcal{O}_{\Sigma_a}(-2s - 2a) \text{ and } \mathcal{L} = 3E + \tau^*(5s + 5a),
\] where \( a \) is an integer such that
\[
k = 3a - 2e - 2
\] Proof. By \([4.2], [4.4]\) and the adjunction formula, we have \( \mathcal{O}_{E_0}(E_0) = \frac{1}{2}(K_{\mathcal{E}_0} - M_{E_0}) = -s - al \) with integer \( a = \frac{1}{3}(k + 2e + 2) \). Since \( j : \Sigma_a \to Y \) factors through \( E_0, j^* \mathcal{O}_Y(2E_0) \cong \mathcal{O}_{\Sigma_a}(-2s - 2a) \).

Since \( \psi^* E = 2E_0 \), according to \([4.2] \text{ and } [4.4] \), \( K_Y = \psi^* \tau^*(s + (e + k)l) + E \). On the other hand, by \([4.5] \text{ and } [4.7] \), \( K_Y = \psi^* \tau^*(-4s - (e + 2a + 2)l - 2E + \mathcal{L}) \). It follows that \( \psi^* (L_0) = \mathcal{O}_Y \), where \( L_0 = L - (3E + \tau^*(5s + 5a)) \). Note that \( \psi_* \mathcal{O}_Y = \mathcal{O}_P \oplus \mathcal{L}' \). The projection formula yields \( L_0 \oplus (L_0 \otimes \mathcal{L}') = \mathcal{O}_P \oplus \mathcal{L}' \). It is clear that \( H^0(P, L') = 0 \) and \( H^0(P, L_0 \otimes \mathcal{L}') = 0 \).

We obtain \( L_0 = \mathcal{O}_P \) and the required formula for \( L \).

**Proposition 4.5.** The pair \((e, a)\) with a defined by \([4.9]\) satisfies the assumption of Theorem 1.4.

Proof. Let \( \omega_{Y/\Sigma_a} = K_Y - \delta \mathcal{O}_{\Sigma_a} \) and let \( C \) be any fiber of \( \delta \). Then \( \omega_{Y/\Sigma_a}|_C = 2E_0|_C = K_C \) by \([4.2]\). Therefore \( \mathcal{O}_{\omega_{Y/\Sigma_a}} \) is a locally free sheaf of rank 2 by Grauert’s theorem. It follows that \( \mathcal{O}_{\omega_{Y/\Sigma_a}} \) is semi-positive by \([16] \text{ Theorem III and (1.3) Remark (iii)} \).

We have seen that \( K_Y = \psi^* \tau^*(s + (3a - e - 2)l) + E \) and \( \mathcal{O}_P(E) = \mathcal{O}_P(1) \). Also \( \psi_* \mathcal{O}_Y = \mathcal{O}_P \oplus \mathcal{L}' \). Applying the projection formula to \( \psi \) and then to \( \tau \), we obtain \( \mathcal{O}_Y(3s + 3al) \). It has \( \mathcal{O}_{\Sigma_a}(s + al) \) as a quotient by \([4.6] \text{ and } [4.8]\). So \( a - e = \deg \mathcal{O}_{\Sigma_a}(s + al) \geq 0 \) by the semi-positivity of \( \mathcal{O}_{\omega_{Y/\Sigma_a}} \).

Note that \( \mathcal{p}_y(X) = 6a - 3e - 2 \) by \([4.3] \text{ and } [4.9]\). Since \( \mathcal{p}_y(X) \geq 7 \) by the assumption of Theorem 1.3 it is clear that the pair \((e, a)\) satisfies the assumption of Theorem 1.1.

From this proposition, we see that \( k = 3a - 2e - 2 \geq 1 \) and thus \( \Sigma \) is smooth by the discussion at the beginning of this section. Also the exact sequence \([4.10] \text{ splits by Lemma 2.1 (b)} \). Moreover, we have seen \( \mathcal{p}_y(X) = 6a - 3e - 2 \) in the proof of the proposition and thus \( K_X = 8a - 6e - 4 \) by the Noether equality. Comparing \([4.5], [4.6] \text{ and } [4.8] \), with \([2.2], [2.3] \), we conclude that \( X \) is one of the 3-folds constructed in Section 2 and complete the proof of Theorem 1.3.

**Acknowledgement.** Both authors would like to thank Meng Chen for many valuable suggestions and helpful discussions. The first author thanks Xiaotao Sun for all the support during the preparation of the paper. The first author was partially supported by the China Postdoctoral Science Foundation (Grant No.: 2013MS541062) and the second author is partially supported by the National Natural Science Foundation of China (Grant No.: 11171068).
References

[1] A. Beauville, *Complex Algebraic Surfaces*, Cambridge Univ. Press (1983) (translation of *Surfaces algébriques complexes*, Astérisque 54 Soc. Math. France, Paris (1978)).

[2] F. Catanese, M. Chen and D.-Q. Zhang, *The Noether inequality for smooth minimal 3-folds*, Math. Res. Lett. 13 (2006), no. 4, 653-666.

[3] F. Catanese, M. Franciosi, K. Hulek and M. Reid, *Embeddings of curves and surfaces*, Nagoya Math. J. 154 (1999), 185–220.

[4] J. A. Chen and M. Chen, *The Noether inequality for Gorenstein minimal 3-folds*, Comm. Anal. Geom. 23 (2015). ArXiv: 1310.7709.

[5] M. Chen, *Inequalities of Noether type for 3-folds of general type*, J. Math. Soc. Japan 56 (2004), 1131-1155.

[6] M. Chen, *Minimal threefolds of small slope and the Noether inequality for canonically polarized threefolds*, Math. Res. Lett. 11 (2004), 833-852.

[7] T. Fujita, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan 30 (4), 779-794 (1978).

[8] R. Hartshorne, *Algebraic Geometry*, Springer Verlag, Berlin, etc. (1977).

[9] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, I, Ann. of Math. (2), 79 (1964), 109-203; II, ibid., 205-326.

[10] E. Horikawa, *Algebraic surfaces of general type with small $c_1^2$*, I, Ann. Math. 104 (1976), 357-387.

[11] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. 10 (1987), 283-360.

[12] M. Kobayashi, *On Noether’s inequality for threefolds*, J. Math. Soc. Japan 44 (1992), no. 1, 145-156.

[13] H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, Cambridge New York 1986.

[14] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) 116 (1982), no. 1, 133–176.

[15] M. Noether, *Zur Theorie des eindeutigen Entsprechens algebraischer Ge-bilde*, Math. Ann. 2 (1870), 293-316; 8 (1875), 495-533.

[16] E. Viehweg, *Weak Positivity and the Additivity of the Kodaira dimension for certain fibre spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), 329–353, Adv. Stud. Pure Math., 1, North-Holland, Amsterdam, 1983.

Authors’ Addresses:

Yifan Chen,
School of Mathematics and Systems Science, Beijing University of Aeronautics and Astronautics,
Xueyuan Road No. 37, Beijing 100191, P. R. China
Email: chenyifan1984@gmail.com

Yong Hu,
School of Mathematical Sciences, Fudan University,
Shanghai 200433, P. R. China
Email: 11110180002@fudan.edu.cn

13