THE THEORY OF PRIME ENDS
AND SPATIAL MAPPINGS

DENIS KOVTONYUK AND VLADIMIR RYAZANOV

January 21, 2014

Abstract

It is given a canonical representation of prime ends in regular spatial domains and, on this basis, it is studied the boundary behavior of the so-called lower $Q$-homeomorphisms that are the natural generalization of the quasiconformal mappings. In particular, it is found a series of effective conditions on the function $Q(x)$ for a homeomorphic extension of the given mappings by prime ends in domains with regular boundaries. The developed theory is applied, in particular, to mappings of the classes of Sobolev and Orlicz–Sobolev and also to finitely bilipschitz mappings that a far-reaching extension of the well-known classes of isometric and quasiisometric mappings.

2010 Mathematics Subject Classification: Primary 30C85, 30D40, 31A15, 31A20, 31A25, 31B25. Secondary 37E30.

1 Introduction

The problem of the boundary behavior is one of the central topics of the theory of quasiconformal mappings and their generalizations. During the last years they intensively studied various classes of mappings with finite distortion in a natural way generalizing conformal, quasiconformal and quasiregular mappings, see many references in the monographs [8] and [20]. In this case, as it was earlier, the main geometric approach in the modern mapping theory is the method of moduli, see, e.g., the monographs [8], [20], [24], [29], [39], [40] and [41].

From the point of view of the theory of conformal mappings, it was unsatisfactory to consider the individual points of the boundary of a simply connected domain as the primitive constituents of the boundary. Indeed, if correspondingly to the Riemann theorem such a domain is mapped conformally onto the
unit disk, then the points of the unit circumference correspond to the so-called prime ends of the domain.

The term "prime end" originated from Caratheodory [2] who initiated the systematic study of the structure of the boundary of a simply connected domain. His approach was topological and dealt with concepts subdomains, crosscuts etc. that are defined with reference to the given domain. The problem arisen under his approach to show that prime ends are preserved under conformal mappings was just solved by one of Caratheodory’s fundamental theorems.

Lindelöf [19] circumvented this difficulty by defining prime ends of a domain with reference to the conformal map of the unit disk onto the domain; namely in terms of the set of indetermination or cluster set. However, his method does not obviate an explicit analysis of the topological situation in the domain itself.

Two other schemes for the definition of prime ends deserve brief mention. Mazurkiewicz [22] introduced a metric \( \rho(\pi(z_1, z_2)) \) that is equivalent to the euclidean metric in a domain in the sense that \( \rho(\pi(z_j, z_0)) \to 0 \) if and only if \( |z_j - z_0| \to 0 \) for any sequence \( \{z_j\} \) of points of the domain. The boundary of the domain with respect to \( \rho(\pi) \), i.e. the complement of the domain with respect to its \( \rho(\pi) \)-completion, is a space that can be identified with the set of prime ends of Caratheodory.

Finally, Ursell and Young [38] to introduce the prime ends of a domain have used the notion of an equivalence class of paths that converge to the boundary of the domain. For the history of the question, see also [1], [4] and [23] and further references therein.

In what follows, we use in \( \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\} \) the spherical (chordal) metric \( h(x, y) = |\pi(x) - \pi(y)| \) where \( \pi \) is the stereographic projection of \( \overline{\mathbb{R}^n} \) onto the sphere \( S^n(\frac{1}{2}e_{n+1}, \frac{1}{2}) \) in \( \mathbb{R}^{n+1} \), i.e.

\[
h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y, \quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}.
\]

The quantity

\[
h(E) = \sup_{x, y \in E} h(x, y)
\]

is said to be spherical (chordal) diameter of a set \( E \subset \overline{\mathbb{R}^n} \).
Let \( \omega \) be an open set in \( \mathbb{R}^k \), \( k = 1, \ldots, n - 1 \). A (continuous) mapping \( \sigma : \omega \to \overline{\mathbb{R}^n} \) is called a \( k \)-dimensional surface in \( \mathbb{R}^n \). An \((n-1)\)-dimensional surface \( \sigma \) in \( \mathbb{R}^n \) is called also a surface. A surface \( \sigma : \omega \to D \) is called a Jordan surface in \( D \) if \( \sigma(z_1) \neq \sigma(z_2) \) whenever \( z_1 \neq z_2 \). Later on, we sometimes use \( \sigma \) to denote the whole image \( \sigma(\omega) \subseteq \mathbb{R}^n \) under the mapping \( \sigma \), and \( \sigma \) instead of \( \sigma(\omega) \) in \( \mathbb{R}^n \) and \( \partial \sigma \) instead of \( \sigma(\omega) \setminus \sigma(\omega) \). A Jordan surface \( \sigma \) in \( D \) is called a cut of \( D \) if \( \sigma \) splits \( D \), i.e. \( D \setminus \sigma \) has more than one component, \( \partial \sigma \cap \partial D = \emptyset \) and \( \partial \sigma \cap \partial D \neq \emptyset \).

A sequence \( \sigma_1, \ldots, \sigma_m, \ldots \) of cuts of \( D \) is called a chain if:

(i) \( \sigma_i \cap \sigma_j = \emptyset \) for every \( i \neq j, i, j = 1, 2, \ldots \);

(ii) \( \sigma_{m-1} \) and \( \sigma_{m+1} \) are contained in different components of \( D \setminus \sigma_m \) for every \( m > 1 \);

(iii) \( \cap d_m = \emptyset \) where \( d_m \) is a component of \( D \setminus \sigma_m \) containing \( \sigma_{m+1} \).

Finally, we will call a chain of cuts \( \{ \sigma_m \} \) regular if

(iv) \( h(\sigma_m) \to 0 \) as \( m \to \infty \).

Correspondingly to the definition, a chain of cuts \( \{ \sigma_m \} \) is determined by a chain of domains \( d_m \subset D \) such that \( \partial d_m \cap D \subseteq \sigma_m \) and \( d_1 \supset d_2 \supset \cdots \supset d_m \supset \cdots \). Two chains of cuts \( \{ \sigma_m \} \) and \( \{ \sigma'_k \} \) are called equivalent if, for every \( m = 1, 2, \ldots \), the domain \( d_m \) contains all domains \( d'_k \) except a finite number and, for every \( k = 1, 2, \ldots \), the domain \( d'_k \) contains all domains \( d_m \) except a finite number, too. An end \( K \) of the domain \( D \) is an equivalence class of chains of cuts of \( D \).

Let \( K \) be an end of a domain \( D \) in \( \overline{\mathbb{R}^n} \) and \( \{ \sigma_m \} \) and \( \{ \sigma'_m \} \) be two chains in \( K \) and \( d_m \) and \( d'_m \) be domains corresponding to \( \sigma_m \) and \( \sigma'_m \), respectively. Then

\[
\bigcap_{m=1}^{\infty} d_m \subseteq \bigcap_{m=1}^{\infty} d'_m \subset \bigcap_{m=1}^{\infty} \overline{d_m}
\]

and, thus,

\[
\bigcap_{m=1}^{\infty} \overline{d_m} = \bigcap_{m=1}^{\infty} \overline{d'_m}.
\]
i.e. the set
\[ I(K) = \bigcap_{m=1}^{\infty} \partial d_m \]
depends only on \( K \) but not on a choice of its chain of cuts \( \{\sigma_m\} \). The set \( I(K) \) is called the \textbf{impression of the end} \( K \). It is well-known that \( I(K) \) is a continuum, i.e. a connected compact set, see, e.g., I(9.12) in [42]. Moreover, in view of the conditions (ii) and (iii), we obtain that
\[ I(K) = \bigcap_{m=1}^{\infty} (\partial d_m \cap \partial D) = \partial D \cap \bigcap_{m=1}^{\infty} \partial d_m. \]

Thus, we come to the following conclusion.

\textbf{Proposition 1.1.} For every end \( K \) of a domain \( D \) in \( \mathbb{R}^n \),
\[ I(K) \subseteq \partial D. \tag{1.1} \]

Following [23], we say that \( K \) is a \textbf{prime end} if \( K \) contains a chain of cuts \( \{\sigma_m\} \) such that
\[ \lim_{m \to \infty} M(\Delta(C, \sigma_m; D)) = 0 \tag{1.2} \]
for a continuum \( C \) in \( D \) where \( \Delta(C, \sigma_m; D) \) is the collection of all paths connecting the sets \( C \) and \( \sigma_m \) in \( D \) and \( M \) denotes its modulus, see the next section.

If an end \( K \) contains at least one regular chain, then \( K \) will call \textbf{regular}. As it will easy follow from Lemma 3.1, every regular end is a prime end.

\section{On lower \( Q \)-homeomorphisms}

The class of lower \( Q \)-homeomorphisms was introduced in the paper [14], see also the monograph [20], and was motivated by the ring definition of quasiconformal mappings of Gehring, see [6]. The theory of lower \( Q \)-homeomorphisms has found interesting applications to the theory of the Beltrami equations in the plane and to the theory of mappings of the classes of Sobolev and Orlich-Sobolev in the space, see, e.g., [11], [12], [16], [17], [18], [20] and [31].
Let $\omega$ be an open set in $\mathbb{R}^k$, $k = 1, \ldots, n - 1$. Recall that a (continuous) mapping $S : \omega \to \mathbb{R}^n$ is called a $k$-dimensional surface $S$ in $\mathbb{R}^n$. The number of preimages

$$N(S, y) = \text{card } S^{-1}(y) = \text{card } \{x \in \omega : S(x) = y\}, \ y \in \mathbb{R}^n \quad (2.1)$$

is said to be a \textbf{multiplicity function} of the surface $S$. It is known that the multiplicity function is lower semicontinuous, i.e.,

$$N(S, y) \geq \liminf_{m \to \infty} N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \ldots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$, see, e.g., [26], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure $H^k$, see, e.g., [35], p. 52.

Recall that a $k$-dimensional Hausdorff area in $\mathbb{R}^n$ (or simply \textbf{area}) associated with a surface $S : \omega \to \mathbb{R}^n$ is given by

$$A_S(B) = A_S^k(B) := \int_B N(S, y) \, dH^k y \quad (2.2)$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set that is measurable with respect to $H^k$ in $\mathbb{R}^n$, cf. 3.2.1 in [5] and 9.2 in [20].

If $\varrho : \mathbb{R}^n \to \mathbb{R}^+ \to 0, \infty$ is a Borel function, then its \textbf{integral over} $S$ is defined by the equality

$$\int_S \varrho \, dA := \int_{\mathbb{R}^n} \varrho(y) \, N(S, y) \, dH^k y. \quad (2.3)$$

Given a family $\Gamma$ of $k$-dimensional surfaces $S$, a Borel function $\varrho : \mathbb{R}^n \to [0, \infty]$ is called \textbf{admissible} for $\Gamma$, abbr. $\varrho \in \text{adm } \Gamma$, if

$$\int_S \varrho^k \, dA \geq 1 \quad (2.4)$$

for every $S \in \Gamma$. The \textbf{modulus} of $\Gamma$ is the quantity

$$M(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \varrho^n(x) \, dm(x). \quad (2.5)$$
We also say that a Lebesgue measurable function \( \varrho : \mathbb{R}^n \to [0, \infty] \) is **extensively admissible** for a family \( \Gamma \) of \( k \)-dimensional surfaces \( S \) in \( \mathbb{R}^n \), abbr. \( \varrho \in \text{ext adm} \Gamma \), if a subfamily of all surfaces \( S \) in \( \Gamma \), for which (2.4) fails, has the modulus zero.

Given domains \( D \) and \( D' \) in \( \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{ \infty \} \), \( n \geq 2 \), \( x_0 \in \overline{D} \setminus \{ \infty \} \), and a measurable function \( Q : \overline{\mathbb{R}^n} \to (0, \infty) \), we say that a homeomorphism \( f : D \to D' \) is a **lower \( Q \)-homeomorphism at the point** \( x_0 \) if

\[
M(f \Sigma_\varepsilon) \geq \inf_{\varrho \in \text{ext adm} \Sigma_\varepsilon} \int_{D \cap R_\varepsilon} \frac{\varrho^n(x)}{Q(x)} \, dm(x)
\]  

(2.6)

for every ring \( R_\varepsilon = \{ x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0 \} \), \( \varepsilon \in (0, \varepsilon_0) \), \( \varepsilon_0 \in (0, d_0) \), where \( d_0 = \sup_{x \in D} |x - x_0| \), and \( \Sigma_\varepsilon \) denotes the family of all intersections of the spheres \( S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \} \), \( r \in (\varepsilon, \varepsilon_0) \), with \( D \). This notion can be extended to the case \( x_0 = \infty \in \overline{D} \) by applying the inversion \( T \) with respect to the unit sphere in \( \mathbb{R}^n \), \( T(x) = x/|x|^2 \), \( T(\infty) = 0 \), \( T(0) = \infty \). Namely, a homeomorphism \( f : D \to D' \) is said to be a **lower \( Q \)-homeomorphism at** \( \infty \in \overline{D} \) if \( F = f \circ T \) is a lower \( Q_* \)-homeomorphism with \( Q_* = Q \circ T \) at 0.

We also say that a homeomorphism \( f : D \to \overline{\mathbb{R}^n} \) is a **lower \( Q \)-homeomorphism in** \( D \) if \( f \) is a lower \( Q \)-homeomorphism at every point \( x_0 \in \overline{D} \).

Recall the criterion for homeomorphisms in \( \mathbb{R}^n \) to be lower \( Q \)-homeomorphisms, see Theorem 2.1 in [14] or Theorem 9.2 in [20].

**Proposition 2.1.** Let \( D \) and \( D' \) be domains in \( \overline{\mathbb{R}^n} \), \( n \geq 2 \), let \( x_0 \in \overline{D} \setminus \{ \infty \} \), and \( Q : D \to (0, \infty) \) be a measurable function. A homeomorphism \( f : D \to D' \) is a lower \( Q \)-homeomorphism at \( x_0 \) if and only if

\[
M(f \Sigma_\varepsilon) \geq \int_{\varepsilon}^{\varepsilon_0} \frac{dr}{\|Q\|_{n-1}(x_0, r)} \quad \forall \, \varepsilon \in (0, \varepsilon_0), \, \varepsilon_0 \in (0, d_0),
\]  

(2.7)

where

\[
\|Q\|_{n-1}(x_0, r) = \left( \int_{D(x_0, r)} Q^{n-1}(x) \, dA \right)^{\frac{1}{n-1}}
\]  

(2.8)

is the \( L_{n-1} \)-norm of \( Q \) over \( D(x_0, r) = \{ x \in D : |x - x_0| = r \} = D \cap S(x_0, r) \).
3 On canonical representation of ends of spatial domains

**Lemma 3.1.** Every regular end $K$ of a domain $D$ in $\mathbb{R}^n$ includes a chain of cuts $\sigma_m$ lying on the spheres $S_m$ centered at a point $x_0 \in \partial D$ with hordal radii $\rho_m \to 0$ as $m \to \infty$. Every regular end $K$ of a bounded domain $D$ in $\mathbb{R}^n$ includes a chain of cuts $\sigma_m$ lying on the spheres $S_m$ centered at a point $x_0 \in \partial D$ with euclidean radii $r_m \to 0$ as $m \to \infty$.

**Proof.** We restrict ourselves to the case of a domain $D$ in $\mathbb{R}^n$ with the hordal metric. The second case is similar.

Let $\{\sigma_m\}$ be a chain of cuts in the end $P$ and $x_m$ a sequence of points in $\sigma_m$. Without loss of generality we may assume that $x_m \to x_0 \in \partial D$ as $m \to \infty$ because $\mathbb{R}^n$ is a compact metric space. Then $\rho_m := h(x_0, \sigma_m) \to 0$ because $h(\sigma_m) \to 0$ as $m \to \infty$. Furthermore,

$$\rho_m^+ := H(x_0, \sigma_m) = \sup_{x \in \sigma_m} h(x, x_0) = \sup_{x \in \sigma_m} h(x, x_0)$$

is the Hausdorff distance between the compact sets $\{x_0\}$ and $\sigma_m$ in $\mathbb{R}^n$. By the condition (i) in the definition of an end, we may assume without loss of generality that $\rho_m > 0$ and $\rho_{m+1}^+ < \rho_m^-$ for all $m = 1, 2, \ldots$

Set

$$\delta_m = \Delta_m \setminus d_{m+1}$$

where $\Delta_m = S_m \cap d_m$ and

$$S_m = \{ x \in \mathbb{R}^n : h(x_0, x) = \frac{1}{2}(\rho_m^- + \rho_{m+1}^+) \}.$$

It is clear that $\Delta_m$ and $\delta_m$ are relatively closed in $d_m$.

Note that $d_{m+1}$ is contained in one of the components of the open set $d_m \setminus \delta_m$. Indeed, assume that there is a pair of points $x_1$ and $x_2 \in d_{m+1}$ in different components $\Omega_1$ and $\Omega_2$ of $d_m \setminus \delta_m$. Then $x_1$ and $x_2$ can be joined by a continuous curve $\gamma : [0, 1] \to d_{m+1}$. However, $d_{m+1}$, and hence $\gamma$, does not intersect $\delta_m$ by the construction and, consequently, $[0, 1] = \bigcup_{k=1}^{\infty} \omega_k$ where $\omega_k = \gamma^{-1}(\Omega_k)$, $\Omega_k$ is enumeration of components $d_m \setminus \delta_m$. But $\omega_k$ are open in $[0, 1]$ because $\Omega_k$ are open and $\gamma$ is continuous. The later contradicts to the connectivity of $[0, 1]$.
because $\omega_1 \neq \emptyset$ and $\omega_2 \neq \emptyset$ and, moreover, $\omega_i$ and $\omega_j$ are mutually disjoint whenever $i \neq j$.

Let $d^*_m$ be a component of $d_m \setminus \delta_m$ containing $d_{m+1}$. Then by the construction $d_{m+1} \subseteq d^*_m \subseteq d_m$. It remains to show that $\partial d^*_m \setminus \partial D \subseteq \delta_m$. First, it is clear that $\partial d^*_m \setminus \partial D \subseteq \delta_m \cup \sigma_m$ because every point in $d_m \setminus \delta_m$ belongs either to $d^*_m$ or to other component of $d_m \setminus \delta_m$ and hence not to the boundary of $d^*_m$ in view of the relative closeness of $\delta_m$ in $d_m$. Thus, it is sufficient to prove that $\sigma_m \cap \partial d^*_m \setminus \partial D \neq \emptyset$.

Let us assume that there is a point $x_\ast \in \sigma_m$ in $d^*_m \setminus \partial D$. Then there is a point $y_\ast \in d^*_m$ which is close enough to $\sigma_m$ with $h(x_0, y_\ast) > \frac{1}{2} (\rho_m^- + \rho_{m+1}^+)$ because $h(x_0, y_\ast) \geq \rho_m^-$ and $\rho_{m+1}^+ < \rho_m^-$. On the other hand, there is a point $z_\ast \in d_{m+1}$ which is close enough to $\sigma_{m+1}$ such that $h(x_0, z_\ast) < \frac{1}{2} (\rho_m^- + \rho_{m+1}^+)$. However, the points $z_\ast$ and $y_\ast$ can be joined by a continuous curve $\gamma : [0, 1] \to d^*_{m+1}$. Note that the sets $\gamma^{-1}(d^*_m \setminus \overline{d_{m+1}})$ consists of a countable collection of open disjoint intervals of $[0, 1]$ and the interval $(t_0, 1]$ with $t_0 \in (0, 1)$ and $z_0 = \gamma(t_0) \in \sigma_{m+1}$. Thus,

$$h(x_0, z_0) < \frac{1}{2} (\rho_m^- + \rho_{m+1}^+)$$

because $h(x_0, z_0) \leq \rho_{m+1}^+$ and $\rho_{m+1}^+ < \rho_m^-$. Now, by the continuity of the function $\varphi(t) = h(x_0, \gamma(t))$, there is $\tau_0 \in (t_0, 1)$ such that $h(x_0, y_0) = \frac{1}{2} (\rho_m^- + \rho_{m+1}^+)$ where $y_0 = \gamma(\tau_0) \in d^*_m$ by the choice of $\gamma$. The contradiction disproves the above assumption and, thus, the proof is complete. $\square$

Later on, given a domain $D$ in $\mathbb{R}^n$, $n \geq 2$, we say that a sequence of points $x_k \in D$, $k = 1, 2, \ldots$, converges to its end $K$ if, for every chain $\{\sigma_m\}$ in $K$ and every domain $d_m$, all points $x_k$ except a finite collection belong to $d_m$. 


4 On regular domains

Recall first of all the following topological notion. A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be **locally connected at a point** $x_0 \in \partial D$ if, for every neighborhood $U$ of the point $x_0$, there is a neighborhood $V \subseteq U$ of $x_0$ such that $V \cap D$ is connected. Note that every Jordan domain $D$ in $\mathbb{R}^n$ is locally connected at each point of $\partial D$, see, e.g., [43], p. 66.

Following [13] and [14], see also [20] and [30], we say that $\partial D$ is **weakly flat at a point** $x_0 \in \partial D$ if, for every neighborhood $U$ of the point $x_0$ and every number $P > 0$, there is a neighborhood $V \subseteq U$ of $x_0$ such that

$$M(\Delta(E, F, D)) \geq P$$

(4.1)

for all continua $E$ and $F$ in $D$ intersecting $\partial U$ and $\partial V$. Here and later on, $\Delta(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ connecting $E$ and $F$ in $D$, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$. We say that the boundary $\partial D$ is **weakly flat** if it is weakly flat at every point in $\partial D$.

We also say that a **point** $x_0 \in \partial D$ is **strongly accessible** if, for every neighborhood $U$ of the point $x_0$, there exist a compactum $E$ in $D$, a neighborhood $V \subset U$ of $x_0$ and a number $\delta > 0$ such that

$$M(\Delta(E, F, D)) \geq \delta$$

(4.2)

for all continua $F$ in $D$ intersecting $\partial U$ and $\partial V$. We say that the boundary $\partial D$ is **strongly accessible** if every point $x_0 \in \partial D$ is strongly accessible.

**Remark 4.1.** Here, in the definitions of strongly accessible and weakly flat boundaries, we may take as neighborhoods $U$ and $V$ of a point $x_0$ only balls (closed or open) centered at $x_0$ or only neighborhoods of $x_0$ in another fundamental system of neighborhoods of $x_0$. These conceptions can also be extended in a natural way to the case of $\mathbb{R}^n$ and $x_0 = \infty$. Then we must use the corresponding neighborhoods of $\infty$.

It is easy to see that if a domain $D$ in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then the point $x_0$ is strongly accessible from $D$. Moreover, it was proved by us that if a domain $D$ in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then $D$ is locally connected at $x_0$, see, e.g., Lemma 5.1 in [14] or Lemma 3.15 in [20].
By the classical geometric definition of Väisälä, see, e.g., 13.1 in [40], a homeomorphism \( f \) between domains \( D \) and \( D' \) in \( \mathbb{R}^n \), \( n \geq 2 \), is \( K \)-quasiconformal, abbr. \( K \)-qc mapping, if

\[
M(\Gamma)/K \leq M(f\Gamma) \leq K M(\Gamma)
\]

for every path family \( \Gamma \) in \( D \). A homeomorphism \( f : D \to D' \) is called quasi-conformal, abbr. \( qc \), if \( f \) is \( K \)-quasiconformal for some \( K \in [1, \infty) \), i.e., if the distortion of the moduli of path families under the mapping \( f \) is bounded.

We say that the boundary of a domain \( D \) in \( \mathbb{R}^n \) is locally quasiconformal if every point \( x_0 \in \partial D \) has a neighborhood \( U \) that can be mapped by a quaisconformal mapping \( \varphi \) onto the unit ball \( B^n \subset \mathbb{R}^n \) in such a way that \( \varphi(\partial D \cap U) \) is the intersection of \( B^n \) with a coordinate hyperplane. Note that a locally quasiconformal boundary is weakly flat directly by definitions.

In the mapping theory and in the theory of differential equations, it is often applied the so-called Lipschitz domains whose boundaries are locally quasiconformal.

Recall first that a map \( \varphi : X \to Y \) between metric spaces \( X \) and \( Y \) is said to be Lipschitz provided \( \text{dist}(\varphi(x_1), \varphi(x_2)) \leq M \cdot \text{dist}(x_1, x_2) \) for some \( M < \infty \) and for all \( x_1 \) and \( x_2 \in X \). The map \( \varphi \) is called bi-Lipschitz if, in addition, \( M^* \text{dist}(x_1, x_2) \leq \text{dist}(\varphi(x_1), \varphi(x_2)) \) for some \( M^* > 0 \) and for all \( x_1 \) and \( x_2 \in X \). Later on, \( X \) and \( Y \) are subsets of \( \mathbb{R}^n \) with the Euclidean distance.

It is said that a domain \( D \) in \( \mathbb{R}^n \) is Lipschitz if every point \( x_0 \in \partial D \) has a neighborhood \( U \) that can be mapped by a bi-Lipschitz homeomorphism \( \varphi \) onto the unit ball \( B^n \subset \mathbb{R}^n \) in such a way that \( \varphi(\partial D \cap U) \) is the intersection of \( B^n \) with the a coordinate hyperplane and \( f(x_0) = 0 \), see, e.g., [24]. Note that bi-Lipschitz homeomorphisms are quasiconformal and hence the Lipschitz domains have locally quasiconformal boundaries.

We call a bounded domain \( D \) in \( \mathbb{R}^n \) regular if \( D \) can be mapped by a quasiconformal mapping onto a domain with locally quasiconformal boundary.

It is clear that each regular domain is finitely connected because under every homeomorphism between domains \( D \) and \( D' \) in \( \mathbb{R}^n \), \( n \geq 2 \), there is a natural
one-to-one correspondence between components of the boundaries \( \partial D \) and \( \partial D' \), see, e.g., Lemma 5.3 in [9] or Lemma 6.5 in [20]. Note also that each finitely connected domain in the plane whose boundary has no one degenerate component can be mapped by a conformal mapping onto some domain bounded by a finite collection of mutually disjoint circles and hence it is a regular domain, see, e.g., Theorem V.6.2 in [7].

As it follows from Theorem 5.1 in [23], each prime end of a regular domain in \( \mathbb{R}^n \), \( n \geq 2 \), is regular. Combining this fact with Lemma 3.1 above, we obtain the following statement.

**Lemma 4.1.** Each prime end \( P \) of a regular domain \( D \) in \( \mathbb{R}^n \), \( n \geq 2 \), contains a chain of cuts \( \sigma_m \) lying on spheres \( S_m \) with center at a point \( x_0 \in \partial D \) and with euclidean radii \( r_m \to 0 \) as \( m \to \infty \).

**Remark 4.2.** As it follows from Theorem 4.1 in [23], under a quasiconformal mapping \( g \) of a domain \( D_0 \) with a locally quasiconformal boundary onto a domain \( D \) in \( \mathbb{R}^n \), \( n \geq 2 \), there is a natural one-to-one correspondence between points of \( \partial D_0 \) and prime ends of the domain \( D \) and, moreover, the cluster sets \( C(g, b) \), \( b \in \partial D_0 \), coincide with the impression \( I(P) \) of the corresponding prime ends \( P \) in \( D \).

If \( \overline{D}_p \) is the completion of a regular domain \( D \) with its prime ends and \( g_0 \) is a quasiconformal mapping of a domain \( D_0 \) with a locally quasiconformal boundary onto \( D \), then it is natural to determine in \( \overline{D}_p \) a metric \( \rho_0(p_1, p_2) = |\tilde{g}_0^{-1}(p_1) - \tilde{g}_0^{-1}(p_2)| \) where \( \tilde{g}_0 \) is the extension of \( g_0 \) to \( \overline{D}_0 \) mentioned above.

If \( g_* \) is another quasiconformal mapping of a domain \( D_* \) with a locally quasiconformal boundary onto the domain \( D \), then the corresponding metric \( \rho_*(p_1, p_2) = |\tilde{g}_*^{-1}(p_1) - \tilde{g}_*^{-1}(p_2)| \) generates the same convergence and, consequently, the same topology in \( \overline{D}_p \) as the metric \( \rho_0 \) because \( g_0 \circ g_*^{-1} \) is a quasiconformal mapping between the domains \( D_* \) and \( D_0 \) that by Theorem 4.1 in [23] is extended to a homeomorphism between \( \overline{D}_* \) and \( \overline{D}_0 \).

Later on, we will call the given topology in the space \( \overline{D}_p \) the **topology of prime ends** and mean the continuity of mappings \( F : \overline{D}_p \to \overline{D}_p \) just with respect to this topology.
5 On extension of direct mappings

Lemma 5.1. Let $D$ and $D'$ be regular domains in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to D'$ be a lower $Q$-homeomorphism. If

$$\delta(x_0) \int_0^\delta dr \frac{dr}{||Q||_{n-1}(x_0, r)} = \infty \quad \forall x_0 \in \partial D$$

(5.1)

for some $\delta(x_0) < d(x_0) = \sup_{x \in D} |x - x_0|$ where

$$||Q||_{n-1}(x_0, r) = \left( \int_{D \cap S(x_0, r)} Q^{n-1} dA \right)^{\frac{1}{n-1}}.$$

then $f$ can be extended to a continuous mapping of $\overline{D}_p$ onto $\overline{D'}_p$.

Proof. In view of Remark 4.2 with no loss of generality we may assume that the domain $D'$ has locally quasiconformal boundary and $\overline{D'}_p = \overline{D'}$. Again by Remark 4.2 namely by metrizability of spaces $\overline{D}_p$ and $\overline{D'}_p$, it suffices to prove that, for each prime end $P$ of the domain $D$, the cluster set

$$L = C(P, f) := \left\{ y \in \mathbb{R}^n : y = \lim_{k \to \infty} f(x_k), x_k \to P, x_k \in D \right\}$$

consists of a single point $y_0 \in \partial D'$.

Note that $L \neq \emptyset$ by compactness of the set $\overline{D'}$, and it is a subset of $\partial D'$, see, e.g., Proposition 2.5 in [30] or Proposition 13.5 in [20]. Let us assume that there exist at least two points $y_0$ and $z_0$ in $L$. Set $U = B(y_0, r_0)$ where $0 < r_0 < |y_0 - z_0|$.

Let $x_0 \in I(P) \subseteq \partial D$ and let $\sigma_k$, $k = 1, 2, \ldots$, be a chain of cuts of $D$, lying on spheres $S_k = S(x_0, r_k)$ from Lemma 4.1 with the associated domains $D_k$, $k = 1, 2, \ldots$. Then there exist points $y_k$ and $z_k$ in the domains $D'_k = f(D_k)$ such that $|y_0 - y_k| < r_0$ and $|y_0 - z_k| > r_0$ and, moreover, $y_k \to y_0$ and $z_k \to z_0$ as $k \to \infty$. Let $C_k$ a continuous curves joining $y_k$ and $z_k$ in $D'_k$. Note that by the construction $\partial U \cap C_k \neq \emptyset$.

By the condition of strong accessibility of the point $y_0$, see Remark 4.1 there is a continuum $E \subset D'$ and a number $\delta > 0$ such that

$$M(\Delta(E, C_k; D')) \geq \delta.$$
for all large enough $k$.

Without loss of generality, we may assume that the latter condition holds for all $k = 1, 2, \ldots$. Note that $C = f^{-1}(E)$ is a compact subset of $D$ and hence $\varepsilon_0 = \text{dist}(x_0, C) > 0$. Again, with no loss of generality, we may assume that $r_k < \varepsilon_0$ for all $k = 1, 2, \ldots$.

Let $\Gamma_m$ be a family of all continuous curves in $D \setminus D_m$ joining the sphere $S_0 = S(x_0, \varepsilon_0)$ and $\sigma_m$, $m = 1, 2, \ldots$. Note that by the construction $C_k \subset D_k' \subset D_m'$ for all $m \leq k$ and, thus, by the principle of minorization $M(f(\Gamma_m)) \geq \delta$ for all $m = 1, 2, \ldots$.

On the other hand, the quantity $M(f(\Gamma_m))$ is equal to the capacity of the condenser in $D'$ with facings $D_m' \setminus f(D \setminus B_0)$ where $B_0 = B(x_0, \varepsilon_0)$, see, e.g., [37]. Thus, by the principle of minorization and Theorem 3.13 in [44]

$$M(f(\Gamma_m)) \leq \frac{1}{M^{n-1}(f(\Sigma_m))}$$

where $\Sigma_m$ is the collection of all intersections of the domain $D$ and the spheres $S(x_0, \rho)$, $\rho \in (r_m, \varepsilon_0)$, because $f(\Sigma_m) \subset \Sigma(f(S_m), f(S_0))$ where $\Sigma(f(S_m), f(S_0))$ consists of all closed subsets of $D'$ separating $f(S_m)$ and $f(S_0)$. Finally, by the condition (5.1) we obtain that $M(f(\Gamma_m)) \to 0$ as $m \to \infty$.

The obtained contradiction disproves the assumption that the cluster set $C(P, f)$ consists of more than one point. □

6 On extension of inverse mappings

Lemma 6.1. Let $D$ and $D'$ be regular domains in $\mathbb{R}^n$, $n \geq 2$, $P_1$ and $P_2$ be different prime ends of the domain $D$, $f$ be a lower $Q$-homeomorphism of the domain $D$ onto the domain $D'$, and let $\sigma_m$, $m = 1, 2, \ldots$, be a chain of cuts of the prime end $P_1$ from Lemma 4.1 lying on spheres $S(z_1, r_m)$, $z_1 \in I(P_1)$, with associated domains $D_m$. Suppose that the function $Q$ is integrable in the degree $n - 1$ over the surfaces

$$D(r) = \{x \in D : |x - z_1| = r\} = D \cap S(z_1, r)$$

(6.1)

for a set $E$ of numbers $r \in (0, d)$ of a positive linear measure where $d = r_{m_0}$ and where $m_0$ is a minimal number such that the domain $D_{m_0}$ does not contain
sequences of points converging to $P_2$. If $\partial D'$ is weakly flat, then

$$C(P_1, f) \cap C(P_2, f) = \emptyset.$$  \hfill (6.2)

Note that in view of metrizability of the completion $\overline{D}_p$ of the domain $D$ with prime ends, see Remark 4.2, the number $m_0$ in Lemma 6.1 always exists.

Proof. Let us choose $\varepsilon \in (0, d)$ such that $E_0 := \{r \in E : r \in (\varepsilon, d)\}$ has a positive linear measure. Such a choice is possible in view of subadditivity of the linear measure and the exhaustion $E = \bigcup E_m$ where $E_m = \{r \in E : r \in (1/m, d)\}$, $m = 1, 2, \ldots$. Note that by Proposition 2.1

$$M(f(\Sigma_\varepsilon)) > 0$$  \hfill (6.3)

where $\Sigma_\varepsilon$ is the family of all surfaces $D(r), r \in (\varepsilon, d)$, from (6.1).

Let us assume that $C_1 \cap C_2 \neq \emptyset$ where $C_i = C(P_i, f), i = 1, 2$. By the construction there is $m_1 > m_0$ such that $\sigma_{m_1}$ lies on the sphere $S(z_1, r_{m_1})$ with $r_{m_1} < \varepsilon$. Let $D_0 = D_{m_1}$ and $D_* \subset D \setminus D_{m_0}$ be a domain associated with a chain of cuts of the prime end $P_2$. Let $y_0 \in C_1 \cap C_2$. Choose $r_0 > 0$ such that $S(y_0, r_0) \cap f(D_0) \neq \emptyset$ and $S(y_0, r_0) \cap f(D_*) \neq \emptyset$.

Set $\Gamma = \Gamma(D_0, D_*, D)$. Correspondingly (6.3), by the principle of minorization and Theorem 3.13 in [44],

$$M(f(\Gamma)) \leq \frac{1}{M^{n-1}(f(\Sigma_\varepsilon))} < \infty.$$  \hfill (6.4)

Let $M_0 > M(f(\Gamma))$ be a finite number. By the condition $\partial D'$ is weakly flat and hence there is $r_* \in (0, r_0)$ such that

$$M(\Delta(E, F; D')) \geq M_0$$

for all continua $E$ and $F$ in $D'$ intersecting the spheres $S(y_0, r_0)$ and $S(y_0, r_*)$. However, these spheres can be joined by continuous curves $c_1$ and $c_2$ in the domains $f(D_0)$ and $f(D_*)$ and, in particular, for these curves

$$M_0 \leq M(\Delta(c_1, c_2; D')) \leq M(f(\Gamma)).$$  \hfill (6.5)

The obtained contradiction disproves the assumption that $C_1 \cap C_2 \neq \emptyset$. \qed
Theorem 6.1. Let $D$ and $D'$ be regular domains in $\mathbb{R}^n$, $n \geq 2$. If $f$ is a lower $Q$-homeomorphism $D$ onto $D'$ with $Q \in L^{n-1}(D)$, then $f^{-1}$ can be extended to a continuous mapping of $\overline{D}_p$ onto $\overline{D'}_p$.

Proof. By the Fubini theorem, see, e.g., [35], the set
\[
E(x_0) = \{ r \in (0, d(x_0)) : Q|_{D(x_0, r)} \in L^{n-1}(D(x_0, r)) \} \quad \forall \ x_0 \in \partial D,
\]
where $d(x_0) = \sup_{x \in D} |x - x_0|$ and $D(x_0, r) = D \cap S(x_0, r)$, has a positive linear measure because $Q \in L^{n-1}(D)$. By Remark 4.2 without loss of generality we may assume that the domain $D'$ has a weakly flat boundary. Thus, the conclusion of the theorem is obtained from Lemma 6.1 arguing by contradiction and taking into account the metrizability of the spaces $\overline{D}_p$ and $\overline{D'}_p$ correspondingly to Remark 4.2. 

Similarly, combining Lemma 6.1 above and Lemma 9.2 in [14], see also Lemma 9.6 in [20], we obtain the following statement.

Theorem 6.2. Let $D$ and $D'$ be regular domains in $\mathbb{R}^n$, $n \geq 2$. If $f : D \to D'$ is a lower $Q$-homeomorphism with condition (5.1), then $f^{-1}$ can be extended to a continuous mapping of $\overline{D'}_p$ onto $\overline{D}_p$.

7 On homeomorphic extension to the boundary

Combining Lemma 5.1 and Theorem 6.2, we obtain the next conclusion.

Theorem 7.1. Let $D$ and $D'$ be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a lower $Q$-homeomorphism with
\[
\delta(x_0) \int_0^{\delta(x_0)} \frac{dr}{||Q||_{n-1}(x_0, r)} = \infty \quad \forall \ x_0 \in \partial D
\]
for some $\delta(x_0) \in (0, d(x_0))$ where $d(x_0) = \sup_{x \in D} |x - x_0|$ and
\[
||Q||_{n-1}(x_0, r) = \left( \int_{D \cap S(x_0, r)} Q^{n-1}(x) \, dA \right)^{\frac{1}{n-1}}.
\]
Then \( f \) can be extended to a homeomorphism of \( \overline{D_p} \) onto \( \overline{D_p'} \).

**Corollary 7.1.** In particular, the conclusion of Theorem 7.1 holds if

\[
q_{x_0}(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right) \quad \forall \ x_0 \in \partial D
\] (7.2)

as \( r \to 0 \) where \( q_{x_0}(r) \) is the mean integral value of \( Q^{n-1} \) over the sphere \( |x - x_0| = r \).

Using Lemma 2.2 in [32], see also Lemma 7.4 in [20], by Theorem 7.1 we obtain the following general lemma that, in turn, makes possible to obtain new criteria in a great number.

**Lemma 7.1.** Let \( D \) and \( D' \) be regular domains in \( \mathbb{R}^n \), \( n \geq 2 \), and let \( f : D \to D' \) be a lower \( Q \)-homeomorphism. Suppose that

\[
\int_{D(x_0, \varepsilon)} Q^{n-1}(x) \cdot \psi^n(|x - x_0|) \, dm(x) = o\left(I_{x_0}(\varepsilon)\right) \quad \forall x_0 \in \partial D
\] (7.3)

as \( \varepsilon \to 0 \) where \( D(x_0, \varepsilon) = \{x \in D : \varepsilon < |x - x_0| < \varepsilon_0\} \) for \( \varepsilon_0 = \varepsilon(x_0) > 0 \) and where \( \psi_{x_0, \varepsilon}(t) : (0, \infty) \to [0, \infty], \varepsilon \in (0, \varepsilon_0) \), is a two-parameter family of measurable functions such that

\[
0 < I_{x_0}(\varepsilon) = \int_{\varepsilon}^{\varepsilon_0} \psi_{x_0, \varepsilon}(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).
\]

Then \( f \) can be extended to a homeomorphism of \( \overline{D_p} \) onto \( \overline{D_p'} \).

**Remark 7.1.** Note that (7.3) holds, in particular, if

\[
\int_{B(x_0, \varepsilon_0)} Q^{n-1}(x) \cdot \psi^n(|x - x_0|) \, dm(x) < \infty \quad \forall x_0 \in \partial D
\] (7.4)

where \( B(x_0, \varepsilon_0) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon_0\} \) for some \( \varepsilon_0 = \varepsilon(x_0) > 0 \) and where \( \psi(t) : (0, \infty) \to [0, \infty] \) is a measurable function such that \( I_{x_0}(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \). In other words, for the extendability of \( f \) to a homeomorphism of \( \overline{D_p} \) onto \( \overline{D_p'} \), it suffices the integrals in (7.4) to be convergent for some nonnegative function \( \psi(t) \) that is locally integrable on \( (0, \varepsilon_0] \) but it has a non-integrable singularity at zero.
Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 1$. Recall that a real valued function $\varphi \in L^1_{\text{loc}}(D)$ is said to be of **bounded mean oscillation** in $D$, abbr. $\varphi \in \text{BMO}(D)$ or simply $\varphi \in \text{BMO}$, if
\[
\|\varphi\|_* = \sup_{B \subset D} \int_B |\varphi(z) - \varphi_B| \, dm(z) < \infty
\]  
where the supremum is taken over all balls $B$ in $D$ and
\[
\varphi_B = \int_B \varphi(z) \, dm(z) = \frac{1}{|B|} \int_B \varphi(z) \, dm(z)
\]  
is the mean value of the function $\varphi$ over $B$. Note that $L^\infty(D) \subset \text{BMO}(D) \subset L^p_{\text{loc}}(D)$ for all $1 \leq p < \infty$, see, e.g., [28].

A function $\varphi$ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi \in \text{VMO}$, if the supremum in (7.5) taken over all balls $B$ in $D$ with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [36]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see, e.g., [3], [10], [21], [25] and [27].

Following [9], we say that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, has **finite mean oscillation** at a point $x_0$, write $\varphi \in \text{FMO}(x_0)$, if $\varphi \in L^1_{\text{loc}}$ and
\[
\lim_{\varepsilon \to 0} \int_{B(x_0,\varepsilon)} |\varphi(x) - \tilde{\varphi}_\varepsilon| \, dm(x) < \infty
\]  
where $\tilde{\varphi}_\varepsilon$ denotes the mean integral value of the function $\varphi$ over the ball $B(x_0,\varepsilon)$. We also write $\varphi \in \text{FMO}(D)$ or simply $\varphi \in \text{FMO}$ by context if this property holds at every point $x_0 \in D$. Clearly that BMO $\subset \text{FMO}$. By definition $\text{FMO} \subset L^1_{\text{loc}}$ but FMO is not a subset of $L^p_{\text{loc}}$ for any $p > 1$, see [20]. Thus, the class FMO is essentially more wide than BMO$_{\text{loc}}$.

Choosing in Lemma 7.1 $\psi(t) := \frac{1}{t \log 1/t}$ and applying Corollary 2.3 on FMO in [9], see also Corollary 6.3 in [20], we obtain the next result.

**Theorem 7.2.** Let $D$ and $D'$ be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a lower $Q$-homeomorphism. If $Q^{n-1}(x)$ has finite mean oscillation at every point $x_0 \in \partial D$, then $f$ can be extended to a homeomorphism of $\overline{D_p}$ onto $\overline{D'_p}$.
Corollary 7.2. In particular, the conclusion of Theorem 7.2 holds if
\[ \lim_{\varepsilon \to 0} \int_{B(x_0,\varepsilon)} Q^{n-1}(x) \, dm(x) < \infty \quad \forall \, x_0 \in \partial D \] (7.8)

Recall that a point \( x_0 \) is called a \textbf{Lebesgue point} of a function \( \varphi : D \to \mathbb{R} \) if \( \varphi \) is integrable in a neighborhood of \( x_0 \) and
\[ \lim_{\varepsilon \to 0} \int_{B(x_0,\varepsilon)} |\varphi(x) - \varphi(x_0)| \, dm(x) = 0 . \] (7.9)

Corollary 7.3. The conclusion of Theorem 7.2 holds if every point \( x_0 \in \partial D \) is a Lebesgue point of the function \( Q : \mathbb{R}^n \to (0, \infty) \).

The next statement also follows from Lemma 7.1 under the choice \( \psi(t) = 1/t \).

Theorem 7.3. Let \( D \) and \( D' \) be regular domains in \( \mathbb{R}^n \), \( n \geq 2 \), and \( f : D \to D' \) be a lower \( Q \)-homeomorphism. If, for some \( \varepsilon_0 = \varepsilon(x_0) > 0 \), as \( \varepsilon \to 0 \)
\[ \int_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \frac{dm(x)}{|x-x_0|^n} = o \left( \left[ \log \frac{1}{\varepsilon} \right]^n \right) \quad \forall \, x_0 \in \partial D , \] (7.10)
then \( f \) can be extended to a homeomorphism of \( \overline{D_p} \) onto \( \overline{D'_p} \).

Remark 7.2. Choosing in Lemma 7.1 the function \( \psi(t) = 1/(t \log 1/t) \) instead of \( \psi(t) = 1/t \), (7.10) can be replaced by the more weak condition
\[ \int_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \frac{dm(x)}{|x-x_0|^n \log |x-x_0|} = o \left( \left[ \log \log \frac{1}{\varepsilon} \right]^n \right) \] (7.11)
and (7.2) by the condition
\[ q_{x_0}(r) = o \left( \left[ \log \log \frac{1}{r} \log \log \frac{1}{r} \right]^{n-1} \right) . \] (7.12)

Of course, we could give here the whole scale of the corresponding condition of the logarithmic type using suitable functions \( \psi(t) \).

Theorem 7.1 has a magnitude of other fine consequences, for instance:
Theorem 7.4. Let $D$ and $D'$ be regular domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a lower $Q$-homeomorphism with
\[
\int_D \Phi(Q^{n-1}(x)) \, dm(x) < \infty \quad (7.13)
\]
for a nondecreasing convex function $\Phi : [0, \infty] \to [0, \infty]$ such that, for some $\delta > \Phi(0)$,
\[
\int_\delta^\infty \frac{d\tau}{\tau^{[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}}} = \infty \quad (7.14)
\]
Then $f$ can be extended to a homeomorphism of $\overline{D}_p$ onto $\overline{D'}_p$.

Indeed, by Theorem 3.1 and Corollary 3.2 in [34], (7.13) and (7.14) imply (7.1) and, thus, Theorem 7.4 is a direct consequence of Theorem 7.1.

Corollary 7.4. In particular, the conclusion of Theorem 7.2 holds if
\[
\int_D e^{\alpha Q^{n-1}(x)} \, dm(x) < \infty \quad (7.15)
\]
for some $\alpha > 0$.

Remark 7.3. Note that the condition (7.14) is not only sufficient but also necessary for a continuous extension to the boundary of the mappings $f$ with integral restrictions of the form (7.13), see, e.g., Theorem 5.1 and Remark 5.1 in [15].

Moreover, by Theorem 2.1 in [34], see also Proposition 2.3 in [33], (7.14) is equivalent to every of the conditions from the following series:
\[
\int_\delta^\infty H_{n-1}'(t) \, \frac{dt}{t} = \infty \quad , \quad \delta > 0 \quad (7.16)
\]
\[
\int_\delta^\infty \frac{dH_{n-1}(t)}{t} = \infty \quad , \quad \delta > 0 \quad (7.17)
\]
\[
\int_\delta^\infty H_{n-1}(t) \, \frac{dt}{t^2} = \infty \quad , \quad \delta > 0 \quad (7.18)
\]
\[ \int_0^\Delta H_{n-1} \left( \frac{1}{t} \right) \, dt = \infty \, , \quad \Delta > 0 \, , \quad (7.19) \]

\[ \int_{\delta_*}^\infty \frac{d\eta}{H_{n-1}^{-1}(\eta)} = \infty \, , \quad \delta_* > H_{n-1}(+0) \, , \quad (7.20) \]

\[ \int_{\delta_*}^\infty \frac{d\tau}{\tau \Phi_{n-1}^{-1}(\tau)} = \infty \, , \quad \delta_* > \Phi(+0) \, , \quad (7.21) \]

where

\[ H_{n-1}(t) = \log \Phi_{n-1}(t) \, , \quad \Phi_{n-1}(t) = \Phi \left( t^{n-1} \right) \, . \quad (7.22) \]

Here, in (7.16) and (7.17), we complete the definition of integrals by \( \infty \) if \( \Phi_{n-1}(t) = \infty \), correspondingly, \( H_{n-1}(t) = \infty \), for all \( t \geq T \in \mathbb{R}^+ \). The integral in (7.17) is understood as the Lebesgue–Stieltjes integral and the integrals in (7.16) and (7.18)–(7.21) as the ordinary Lebesgue integrals.

It is necessary to give one more explanation. From the right hand sides in the conditions (7.16)–(7.21) we have in mind \( +\infty \). If \( \Phi_{n-1}(t) = 0 \) for \( t \in [0, t_*] \), then \( H_{n-1}(t) = -\infty \) for \( t \in [0, t_*] \) and we complete the definition \( H'_{n-1}(t) = 0 \) for \( t \in [0, t_*] \). Note, the conditions (7.17) and (7.18) exclude that \( t_* \) belongs to the interval of integrability because in the contrary case the left hand sides in (7.17) and (7.18) are either equal to \( -\infty \) or indeterminate. Hence we may assume in (7.16)–(7.19) that \( \delta > t_0 \), correspondingly, \( \Delta < 1/t_0 \) where \( t_0 := \sup_{\Phi_{n-1}(t)=0} \, t, \quad t_0 = 0 \) if \( \Phi_{n-1}(0) > 0 \).

The most interesting of the above conditions is (7.18) that can be rewritten in the following form:

\[ \int_\delta^\infty \log \Phi(t) \, \frac{dt}{t^{n'}} = \infty \quad (7.23) \]

where \( \frac{1}{n} + \frac{1}{n'} = 1 \), i.e. \( n' = 2 \) for \( n = 2 \), \( n' \) is strictly decreasing in \( n \) and \( n' = n/(n-1) \to 1 \) as \( n \to \infty \).

The theory of the boundary behavior for the lower \( Q \)-homeomorphisms developed here will find its applications, in particular, to mappings in classes of
Sobolev and Orlicz-Sobolev and also to finitely bilipschitz mappings that a far-reaching extension of the well-known classes of isometric and quasiisometric mappings, see, e.g., [11], [12], [16], [17], [18], [20] and [31].

**Remark 7.4.** It was already noted above that each finitely connected domain in the finite plane, whose boundary has no single point component, is regular. The question whether this fact is valid in the spaces $\mathbb{R}^n$, $n \geq 3$, remains open.

**References**

[1] Adamowicza T., Björna A., Björna J., Shanmugalingamb N. Prime ends for domains in metric spaces // Advances in Mathematics. – 2013. – 238. – P. 459 – 505.

[2] Caratheodory C. Über die Begrenzung der einfachzusammenhängender Gebiete // Math. Ann. – 1913. – 73. – P. 323 – 370.

[3] Chiarenza F., Frasca M., Longo P. $W^{2,p}$-solvability of the Dirichlet problem for non-divergence elliptic equations with VMO coefficients // Trans. Amer. Math. Soc. – 1993. – 336, no. 2. – P. 841–853.

[4] Collingwood E. F., Lohwator A.J. The Theory of Cluster Sets. – Cambridge Tracts in Math. and Math. Physics 56. – Cambridge: Cambridge Univ. Press, 1966.

[5] Federer H. Geometric Measure Theory. – Berlin: Springer-Verlag, 1969.

[6] Gehring F.W. Rings and quasiconformal mappings in space // Trans. Amer. Math. Soc. – 1962. – 103, no. 3. – P. 353 – 393.

[7] Goluzin G. M. Geometric Theory of Functions of a Complex Variable. – Transl. of Math. Monographs 26. – Providence: AMS, 1969.

[8] Gutlyanskii V., Ryazanov V., Srebro U., Yakubov E. The Beltrami Equations: A Geometric Approach. – Developments in Math. 26. – New York etc.: Springer, 2012.

[9] Ignat’ev A., Ryazanov V. Finite mean oscillation in the mapping theory // Ukr. Mat. Vis. – 2005. – 2, no. 3. – P. 395–417, 443 [in Russian]; transl. in Ukr. Math. Bull. – 2005. – 2, no. 3. – P. 403–424, 443.

[10] Iwaniec T., Sbordone C. Riesz transforms and elliptic PDEs with VMO coefficients // J. Anal. Math. – 1998. – 74. – P. 183–212.

[11] Kovtonyuk D., Petkov I., Ryazanov V. On the boundary behaviour of solutions to the Beltrami equations // Complex Variables and Elliptic Equations. – 2013. – 58, no. 5. – P. 647 – 663.
[12] Kovtonyuk D.A., Petkov I.V., Ryazanov V.I., Salimov R.R. Boundary behavior and Dirichlet problem for Beltrami equations // Algebra and Analysis. – 2013. – 25, № 4. – P. 101-124 [in Russian].

[13] Kovtonyuk D.A., Ryazanov V.I. On boundaries of space domains // Proc. Inst. Appl. Math. & Mech. NAS of Ukraine. – 2006. – 13. – P. 110 – 120.

[14] Kovtonyuk D.A., Ryazanov V.I. Toward the theory of lower $Q$-homeomorphisms // Ukr. Mat. Visn. – 2008. – 5, № 2. – P. 159 – 184 [in Russian]; transl. in English by AMS in Ukrainian Math. Bull. – 2008. – 5, № 2. – P. 157 – 181.

[15] Kovtonyuk D., Ryazanov V. On the boundary behavior of generalized quasi-isometries // J. Anal. Math. – 2011. – 115. – P. 103-119.

[16] Kovtonyuk D.A., Ryazanov V.I., Salimov R.R., Sevost’yanov E.A. On mappings in the Orlicz-Sobolev classes // Ann. Univ. Bucharest, Ser. Math - 2012. - 3(LXI), no. 1. - P. 67-78.

[17] Kovtonyuk D.A., Ryazanov V.I., Salimov R.R., Sevost’yanov E.A. Toward the theory of classes of Orlicz–Sobolev // Algebra and Analysis. – 2013. – 25, № 6. – P. 49 – 101 [in Russian].

[18] Kovtonyuk D.A., Salimov R.R., Sevost’yanov E.A. (ed. Ryazanov V.I.) Toward the Theory of Mappings in Classes of Sobolev and Orlicz–Sobolev. – Kiev: Naukova dumka, 2013 [in Russian].

[19] Lindelöf E. Sur un principe general de l’analyse et ses applications a la theorie de la representation conforme // Acta Soci. Sci. Fenn. – 1915. – 46, no. 4. – P. 1 – 35.

[20] Martio O., Ryazanov V., Srebro U., Yakubov E. Moduli in Modern Mapping Theory. – Springer Monographs in Mathematics. – New York etc.: Springer, 2009.

[21] Martio O., Ryazanov V., Vuorinen M. BMO and Injectivity of Space Quasiregular Mappings // Math. Nachr. – 1999. – 205. – P. 149–161.

[22] Mazurkiewicz S. Über die Definition der Primenden // Fund. Math. – 1936. – 26. – P. 272 – 279.

[23] Nåkki R. Prime ends and quasiconformal mappings // J. Anal. Math. – 1979. – 35. – P. 13 – 40.

[24] Ohtsuka M. Extremal Length and Precise Functions. – Tokyo: Gakkotosho Co., 2003.

[25] Palagachev D.K. Quasilinear elliptic equations with VMO coefficients // Trans. Amer. Math. Soc. – 1995. – 347, no. 7. – P. 2481–2493.

[26] Rado T., Reichelderfer P.V. Continuous Transformations in Analysis. – Berlin etc.: Springer, 1955.
[27] RAGUSA M.A. Elliptic boundary value problem in vanishing mean oscillation hypothesis // Comment. Math. Univ. Carolin. – 1999. – 40, no. 4. – P. 651–663.

[28] REIMANN H.M., RYCHENER T. Funktionen Beschränkter Mittlerer Oscillation. – Lecture Notes in Math. 487. – Berlin etc.: Springer-Verlag, 1975.

[29] RICKMAN S. Quasiregular Mappings. – Berlin etc.: Springer-Verlag, 1993.

[30] RYAZANOV V.I., SALIMOV R.R. Weakly flat spaces and boundaries in the mapping theory // Ukr. Mat. Vis. – 2007. – 4, № 2. – P. 199 – 234 [in Russian]; transl. in Ukrainian Math. Bull. – 2007. – 4, № 2. – P. 199 – 233.

[31] RYAZANOV V., SALIMOV R., SREBRO U., YAKUBOV E. On Boundary Value Problems for the Beltrami Equations // Contemporary Math. – 2013. – 591. – P. 211-242.

[32] RYAZANOV V., SEVOST’YANOV E. Equicontinuous classes of ring Q-homeomorphisms // Sibirsk. Math. Zh. – 2007. – 48, № 6. – P. 1361–1376 [in Russian]; transl. in Siberian Math. J. – 2007. – 48, № 6. – P. 1093–1105.

[33] RYAZANOV V., SEVOST’YANOV E. Equicontinuity of mappings quasiconformal in the mean // Ann. Acad. Sci. Fenn. – 2011. – 36. – P. 231 – 244.

[34] RYAZANOV V., SREBRO U., YAKUBOV E. On integral conditions in the mapping theory // Ukrainian Math. Bull. – 2010. – 7, № 1. – P. 73-87.

[35] SAKS S. Theory of the Integral. – New York: Dover Publications Inc., 1964.

[36] SARASON D. Functions of vanishing mean oscillation // Trans. Amer. Math. Soc. – 1975. – 207. – (1975), P. 391–405.

[37] SHELYK V.A. On the equality between p-capacity and p-modulus // Sibirsk. Mat. Zh. – 1993. – 34, № 6. – P. 216 – 221 [in Russian]; transl. in Siberian Math. J. – 1993. – 34, № 6. – P. 1196 – 1200.

[38] URSELL H.D., YOUNG L.C. Remarks on the theory of prime ends // Memoirs of the AMS. – 1951. – 3. – 29 pp.

[39] VASIL’EV A. Moduli of Families of Curves for Conformal and Quasiconformal Mappings. – Lecture Notes in Math. 1788. – Berlin etc.: Springer-Verlag, 2002.

[40] VÄISÄLÄ J. Lectures on n-Dimensional Quasiconformal Mappings. – Lecture Notes in Math. 229. – Berlin etc.: Springer-Verlag, 1971.

[41] VUORINEN M. Conformal Geometry and Quasiregular Mappings. – Lecture Notes in Math. 1319. – Berlin etc.: Springer-Verlag, 1988.

[42] WHYBURN G.TH. Analytic Topology. – Providence: AMS, 1942.

[43] WILDER R.L. Topology of Manifolds. – New York: AMS, 1949.
[44] Ziemer W.P. Extremal length and conformal capacity // Trans. Amer. Math. Soc. – 1967. – 126, no. 3. – P. 460 – 473.

Denis Kovtonyuk and Vladimir Ryazanov,
Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine,
74 Roze Luxemburg Str., Donetsk, 83114, Ukraine,
vl.ryazanov1@gmail.com