Entanglement preparation and work extraction with a truly quantum Maxwell demon

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The information of a quantum system acquired by a Maxwell demon can be used for either work extraction or entanglement preparation. We study these two tasks by using a thermal qubit, in which a demon obtains her information from measurements on the environment of the qubit. The allowed entanglement, between the qubit and an auxiliary system, is enhanced by the information. And, the increment is find to be equivalent to the extractable work. The Maxwell demon is called to be quantum by Beyer et al. [Phys. Rev. Lett 123, 250606 (2019)] if there is quantum steering from the environment to the qubit. In this case, the postmeasured states of the qubit, after the measurements on its environment, cannot be simulated by an objective local statistical ensemble. We present a upper bound of extractable work, and equivalently of the allowed entanglement, for unsteerable demons, considering two measurements inducing two orthogonal changes of the Bloch vector of the qubit.

I. INTRODUCTION

Thermodynamics and information are connected through the Maxwell demon, which was revealed by the model of Szilárd engine [1]. With the rapid development of the quantum information science, the studies of the relationship between thermodynamics and information has been recently extended to the quantum regime. Quantum correlations in quantum systems[2–4] are important subjects in such cross field. Some researchers study the influence of quantum thermodynamics on quantum entanglement in quantum system[5, 6], and others focus on the quantum correlations in quantum thermal machines[7].

Choosing a quantum system in a thermal state to act as a quantum machine, one can utilize it both to extract work and to prepare quantum correlations. But the amounts of extractable work and allowed correlations are suppressed by thermodynamic properties of the machine. When a Maxwell demon measures the quantum state, it extracts the microstate information and enhances the purity of the state. Observers can make use of the information to extract work from the postmeasured state or prepare quantum correlations.

The demon can obtain the information by measuring the environment coupled with the quantum system. This indirect measureme does not change the energy of the system. In general, the demon can affect the postmeasured states of the system by choosing different measurements on the state of environment. This is a unique feature of quantum mechanics and termed as steering by Schrödinger[8]. In 2007, Wiseman et al. [9] defined this phenomenon as a quantum correlation by using a quantum information task. Suppose that Alice (Maxwell demon) and Bob share an entangled state \( \rho_{se} \). Alice can prepare Bob’s system into different states by choosing her local measurement. She can convince him that she has such an ability if and only if the unnormalized postmeasured states cannot be described by a local-hidden-state (LHS) model. In this case, , the state \( \rho_{se} \) is said to have the quantum demon of steering from Alice to Bob. Beyer et al. defines Alice in this situation as a quantum demon[10]. Under this framework, we reconsider the problem of using a quantum Maxwell demon to extract work and prepare entanglement.

In this work, we consider the extractable work and allowed entanglement from a thermal qubit, whose environment is modeled as another qubit. Let the environment qubit belongs to Alice (Maxwell demon) and the system qubit belongs to Bob. We assume that, Bob obtain information of the system with aid of Alice’s measurement on the environment, in the steering procedure defined in [9] as shown in figure. 1. He chose a measurement \( \vec{n} \cdot \vec{\sigma} \) and asked Alice to measure the state of the environment. Bob can be aware of the state of his qubit \( \{ \rho_{e}^+, \rho_{e}^- \} \) based on Alice’s measurement results. He couples the qubit to another quantum system to extract work or create en-

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tangle. Different with the original work by Beyer et al. [10], we provide no limitation on Bob’s transformation. We find an equivalence between the work and entanglement, and derive their upper bounds for un-steering demon.

II. FRAMEWORK

We assume that an observer has a system qubit $S$ with Hamiltonian $H_S$. The Hamiltonian of $S$ can be written as

$$H_S = -\frac{\omega_0 \sigma_z}{2},$$  \hspace{1cm} (1)

where $\sigma_{x,y,z}$ are the Pauli matrices. For an arbitrary state

$$\rho_S = \frac{1}{2} (1 + \vec{r} \cdot \vec{\sigma}),$$  \hspace{1cm} (2)

the observer want to use $\rho_S$ to extract work, who need bring in another system $\pi$ with Hamiltonian $H_\pi$. Hamiltonian of the global system $H = H_S \otimes 1 + 1 \otimes H_\pi$. Extracting work, one need to perform a unitary operation $U_{\pi \pi}$ on the global state. While $[U_{\pi \pi}, H] = 0$, the energy reduction of system $S$ is equal to the energy increase of system $\pi$. Beyer et al. [10] proved that it’s possible to replace the global unitary transformation $U_{\pi \pi}$ with the local unitary transformation $U_S$ and apply it to $\rho_S[10]$. It’s appropriate to choose $U_S$ that makes

$$U_S \rho_S U_S^\dagger = \frac{1}{2} (1 + |\vec{r}| \sigma_z).$$  \hspace{1cm} (3)

The work can be extracted is

$$\delta W = \text{Tr} \left[ H_S \left( \rho_S - U_S \rho_S U_S^\dagger \right) \right]$$
$$= \frac{\omega_0}{2} \left( -r_z + |\vec{r}| \right).$$  \hspace{1cm} (4)

In quantum thermodynamics, we not only study the thermodynamic properties of quantum systems, but also the quantum correlations in quantum systems[6]. Next think over how observer makes the maximum entanglement with another state. First, we consider a general global state $\rho_{sc} = \rho_S \otimes \rho_c$. It’s proved that when $\rho_c$ is a pure state, observer may prepare more entanglement. Therefore, we suppose the global state is

$$\rho_{sc} = \rho_S \otimes |1\rangle\langle1|. $$  \hspace{1cm} (5)

The rank of $\rho_{sc}$ is 2 and there is a global unitary transformation

$$\mathcal{U} = |\phi_+\rangle\langle\phi_01| + |\phi_0\rangle\langle\phi_10| + |01\rangle\langle\phi_00| + |10\rangle\langle\phi_11|$$  \hspace{1cm} (6)
such that $\rho_{sc}$ becomes[5, 11]

$$\rho_{sc}' = \frac{1 + |\vec{r}|}{2} |\phi_+\rangle\langle\phi_+| + \frac{1 - |\vec{r}|}{2} |10\rangle\langle10|. $$  \hspace{1cm} (7)

In Eq.(6), $|\phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ are eigenvectors of $\rho_S$. The global unitary transformation $\mathcal{U}$ does not change the eigenvalue of the global state $\rho_{sc}$ and the form of the maximum entangled state must be $\rho_{sc}'$. Obviously, the maximum entanglement that can prepare is

$$C = \frac{1 + |\vec{r}|}{2}. $$  \hspace{1cm} (8)

From the above, we can find that the result of extract work and prepare entanglement are related to $\rho_S$. The $\vec{r}$ in Eq.(4) and Eq.(8) is the Bloch vector of $\rho_S$, which represents the confusion degree of $\rho_S$. When someone gets information about $\rho_S$, the entropy of $\rho_S$ will decrease, and the modulus of the Bloch vector of $\rho_S$ will increase. In other words, by obtaining information from $\rho_S$, observer can use the information to extract more work and prepare more entanglement.

III. STEERING THERMAL MACHINE

Bob is an observer who has a quantum system $S$ which state reads

$$\rho_{th} = \frac{1 + \eta}{2} |1\rangle\langle 1 | + \frac{1 - \eta}{2} |0\rangle\langle 0 |. $$  \hspace{1cm} (9)

The model in this paper uses indirect measurement to obtain $\rho_{th}$’s information. At this time we introduce an auxiliary system $\varepsilon$ to couple with the quantum system $S$ of Bob. Alice (Maxwell demon) can do any operation on the auxiliary system $\varepsilon$. The coupled system’s state is given by

$$\rho_{sc} = \frac{1}{4} (1 - \eta \sigma_z \otimes 1 + 1 \otimes \sigma_x \cdot \vec{b} + T_{ij} \sigma_i \otimes \sigma_j). $$  \hspace{1cm} (10)

The process of extracting work can be described as Bob tells Alice to measure her qubit in $\vec{b} \cdot \vec{\sigma}$. Alice will get two measurement results $\{+1, -1\}$ and tell Bob the measurement results through classic communication. Alice prepares a lot of $\rho_{sc}$ ensemble, for different measurements, Bob will get many different decompositions $D = \{p_k; p_k\}$ of $\rho_S$. For each $p_k$ in $D$, Alice will provide a suitable $U_k$ to Bob for transfer the energy of system $S$ to auxiliary system $\varepsilon$. The specific $\rho_k$ can be calculated in the following way

$$\rho_{sc}^\varepsilon = \text{Tr}_\varepsilon \left[ \frac{1}{2} (1 + \vec{n} \cdot \vec{\sigma}) \rho_{sc} \right]$$
$$= \frac{1 \pm \vec{n} \cdot \vec{b}}{4} \left( 1 - \frac{\eta \sigma_z \mp T\vec{n} \cdot \vec{\sigma}}{1 \pm \vec{n} \cdot \vec{b}} \right). $$  \hspace{1cm} (11)
By the way, the Bloch vectors of $\rho_+^a$ and $\rho_-^a$ are

$$r_\pm^a = \frac{-\eta \vec{k} \pm T \vec{n}}{1 \pm \vec{n} \cdot \vec{b}}.$$  \hspace{1cm} (12)

Through Eq.(2) to Eq.(4), we directly obtain the final extractable work

$$\delta W = \text{Tr} \left[ H_S \left( \rho_S - \frac{1 + \vec{n} \cdot \vec{b}}{4} (\mathbb{1} + \sigma_r) \right) \right]$$

$$- \frac{1 - \vec{n} \cdot \vec{b}}{4} \left( \mathbb{1} + \sigma_r \right) \right) \right]$$

$$= \omega_0 \left( \frac{1}{4} + 1 \right) \left( -\eta \vec{k} + T \vec{n} \right)$$

$$+ | - \eta \vec{k} - T \vec{n} \rangle \langle . \hspace{1cm} (13)$$

The entanglement can be prepared from a state like $\rho_{\text{LHS}} \otimes |1\rangle\langle 1|$ is $\frac{1 - \eta}{2}$. As mentioned earlier, the preparable entanglement is

$$\delta C = \frac{\eta}{2} + \frac{1}{4} \left( -\eta \vec{k} + T \vec{n} \right)$$

$$+ | - \eta \vec{k} - T \vec{n} \rangle \langle \hspace{1cm} (14)$$

When Bob asks Alice to measure two operations, the result can be expressed as

$$\delta W = \frac{\omega_0}{8} \left( 4 \eta + | - \eta \vec{k} + T \vec{n} | + | - \eta \vec{k} - T \vec{n} |$$

$$+ | - \eta \vec{k} + T \vec{n} | + | - \eta \vec{k} - T \vec{n} | \right)$$

$$\delta C = \frac{1}{8} \left( 4 \eta + | - \eta \vec{k} + T \vec{n} | + | - \eta \vec{k} - T \vec{n} |$$

$$+ | - \eta \vec{k} + T \vec{n} | + | - \eta \vec{k} - T \vec{n} | \right) \hspace{1cm} (15)$$

Then we introduce a particular state

$$\rho = p \rho_{\text{qu}} + (1 - p) \rho_{\text{el}}$$

which is obtained by mixing a entangled pure state and a separable state. In Eq.(16),

$$\rho_{\text{qu}} = \sqrt{\frac{1 + \eta}{2}} | 1 \rangle \langle 1 | + \sqrt{\frac{1 - \eta}{2}} | 0 \rangle \langle 0 | \hspace{1cm} (16)$$

and

$$\rho_{\text{el}} = \frac{1 + \eta}{2} | 1 \rangle \langle 1 | \otimes | - \rangle \langle - | + \frac{1 - \eta}{2} | 0 \rangle \langle 0 | \otimes | + \rangle \langle + | . \hspace{1cm} (17)$$

| $-$ and $|$ + are the two eigenstates of $\sigma_z$, which eigenvalues are $-1$ and $+1$. For this special state, Alice measures $\sigma_x$ and $\sigma_z$, and use the first formula in Eq.(15) to calculate the extractable work. At the same time, Alice use the work extraction scheme given by Beyer et al. to calculate the extractable work of this state[10]. For each postmeasured state, Alice will choose the best unitary transformation from the four unitary transformations given in

$$\rho_{\text{LHS}}^{\text{LHS}} = \int \omega(\vec{\lambda}) \frac{1}{4} \left[ 1 + a f(\vec{n}, \vec{\lambda}) \right] (1 + \vec{\lambda} \cdot \vec{\sigma}) d\vec{\lambda} \hspace{1cm} (18)$$

The $\vec{\lambda}$ represents a classical (hidden) variable with a distribution $\omega(\vec{\lambda})$. The hidden state $\rho_{\lambda} = \frac{1}{2} (1 + \vec{\lambda} \cdot \vec{\sigma})$ is a state of Bob’s system depending on $\vec{\lambda}$, and

$$P(a|\vec{n}, \vec{\lambda}) = \frac{1}{2} \left[ 1 + a f(\vec{n}, \vec{\lambda}) \right]$$

is the probability of outcome $a$ under the condition of $\vec{n}$ and $\vec{\lambda}$. If there exists a LHS model satisfying

$$\rho_{\text{LHS}}^{\text{LHS}} = \rho_{\text{LHS}}^{a|\vec{n}}$$

for all measurements, the results of Alice’s measurements can be simulated by a LHS strategy without any entan-

FIG. 2: The amount of extractable work in the two schemes. The solid lines are calculated by using our present scheme, and the dashed lines are for the scheme of Beyer et al. [10]. The three sets of results from top to bottom are obtained when $\eta = -0.2$, $\eta = -0.5$, and $\eta = -0.08$. Beyer et al.’s article to extract work.

The extractable work in figure 2 does not increase with the increase of $p$. When $p = 0$ and $p = 1$, Alice measures $\sigma_x$ or $\sigma_z$ for the state $\rho$, separately. Bob will get two pure states which Bloch vector is 1. Simultaneously, Bob can extract the maximum work. When $p$ is between 0 and 1, Alice measures $\sigma_x$ and $\sigma_z$. Bob’s pure state will become a mixed state, so that the work that can be extracted is reduced. It can be seen from Figure 2 that the work extraction scheme in this article is better. Because we do not limit the unitary operation of extracting work.

IV. UPPER BOUNDS FOR LHS MODELS

At last, we focus on whether the process of extract work and prepare entanglement is steering or not. In other words, it’s considering whether Alice is a classical demon or a quantum demon. It’s convenient to use the language of the LHS models to describe our task. A LHS model of a two-qubit system is defined as[12]

$$\rho_{\text{LHS}}^{a|\vec{n}} = \int \omega(\vec{\lambda}) \frac{1}{4} \left[ 1 + a f(\vec{n}, \vec{\lambda}) \right] (1 + \vec{\lambda} \cdot \vec{\sigma}) d\vec{\lambda} \hspace{1cm} (17)$$

The $\vec{\lambda}$ represents a classical (hidden) variable with a distribution $\omega(\vec{\lambda})$. The hidden state $\rho_{\lambda} = \frac{1}{2} (1 + \vec{\lambda} \cdot \vec{\sigma})$ is a state of Bob’s system depending on $\vec{\lambda}$, and

$$P(a|\vec{n}, \vec{\lambda}) = \frac{1}{2} \left[ 1 + a f(\vec{n}, \vec{\lambda}) \right]$$

is the probability of outcome $a$ under the condition of $\vec{n}$ and $\vec{\lambda}$. If there exists a LHS model satisfying

$$\rho_{\text{LHS}}^{a|\vec{n}} = \rho_{\text{LHS}}^{a|\vec{n}}$$

for all measurements, the results of Alice’s measurements can be simulated by a LHS strategy without any entan-
If we define the average of Bloch vector as
\[
\int \omega(\vec{\lambda}) d\vec{\lambda} = 1
\]
\[
\int \omega(\vec{\lambda}) \vec{\lambda} d\vec{\lambda} = -\eta \vec{k}
\]
\[
\int \omega(\vec{\lambda}) f(\vec{n}, \vec{\lambda}) d\vec{\lambda} = \vec{n} \cdot \vec{b}
\]
\[
\int \omega(\vec{\lambda}) f(\vec{n}, \vec{\lambda}) \vec{\lambda} d\vec{\lambda} = T\vec{n}.
\]
From Eq.(15), their value is only related to the change \(\vec{\lambda}\) and \(\vec{e}\) and \(\vec{e}\) are convenient for calculations. The next thing to do is to find the upper bound that Bob can extract work and prepare entanglement under the above conditions. We define the average of Bloch vector as
\[
\tau = \frac{1}{4} \left( | - \eta \vec{k} + \alpha \vec{e}_1 | + | - \eta \vec{k} - \alpha \vec{e}_1 | + | - \eta \vec{k} + \beta \vec{e}_2 | + | - \eta \vec{k} - \beta \vec{e}_2 | \right) .
\]
Through simple calculations we get
\[
\tau \leq \frac{1}{2} \left( \sqrt{\eta^2 + \alpha^2} + \sqrt{\eta^2 + \beta^2} \right) .
\]
The requirement to get the maximum value of the above formula is that \(\vec{k} \perp \vec{e}_1\) and \(\vec{k} \perp \vec{e}_2\). Our goal is to make \(\alpha\) and \(\beta\) large. While presume that \(f(\vec{n}, \vec{\lambda}) = sgn(\vec{\lambda} \cdot \vec{e}_1)\) and \(f(\vec{n}, \vec{\lambda}) = sgn(\vec{\lambda} \cdot \vec{e}_2)\), we found that it’s a necessary condition for Eq.(22) to get the maximum value.
\[
\int \omega(\vec{\lambda}) f(\vec{n}, \vec{\lambda}) (\vec{\lambda} \cdot \vec{e}_1) d\vec{\lambda} = \alpha
\]
\[
\int \omega(\vec{\lambda}) f(\vec{n}, \vec{\lambda}) (\vec{\lambda} \cdot \vec{e}_2) d\vec{\lambda} = \beta.
\]
If \(\omega(\vec{\lambda})\) is symmetric about \(\vec{e}_1\) and \(\vec{e}_2\), \(\alpha\) and \(\beta\) will be larger. To find the maximum value of \(\alpha\) and \(\beta\), the integral of the quarter sphere needs to be premeditated. If we want to maximize \(\alpha\) and \(\beta\), \(\vec{q}\) must be in the same plane as \(\vec{e}_1\) and \(\vec{e}_2\).
\[
\int \frac{1}{4} \omega(\vec{\lambda}) \vec{\lambda} d\vec{\lambda} = \frac{\vec{q}}{4}
\]
\[
\alpha = 4 \frac{\vec{q}}{4} \cdot \vec{e}_1 = \cos \theta, \beta = -4 \frac{\vec{q}}{4} \cdot \vec{e}_2 = \sin \theta.
\]

The result can be written as
\[
\delta W = \frac{\omega_0}{2} \left( \eta + \sqrt{\eta^2 + \frac{1}{2}} \right)
\]
\[
\delta C = \frac{1}{2} \left( \eta + \sqrt{\eta^2 + \frac{1}{2}} \right).
\]
\(\tau \leq 1\) is a rough requirement for Eq.(20). The final result is rewritten as
\[
\delta W = \frac{\omega_0}{2} \left( \eta + \text{Min}\{\sqrt{\eta^2 + \frac{1}{2}}\} \right)
\]
\[
\delta C = \frac{1}{2} \left( \eta + \text{Min}\{\sqrt{\eta^2 + \frac{1}{2}}\} \right).
\]
In figure 4, we compare the upper bound of the extractable work in Eq.(26) with the work Bob extracts from \(\rho\) by measuring \(\sigma_x\) and \(\sigma_y\). While \(p = 0.7\), \(p = 0.8\), \(p = 0.9\) and \(\eta\) is large, the work Bob extracts can exceed the upper bound. As the quantum correlation of \(\rho\) becomes more and more obvious, the part where the extractable work exceeds the upper bound becomes larger and larger.
V. SUMMARY

A Maxwell demon can acquire information of a quantum system from measurements on its environment, which is naturally connected to the task of steering. The information can be used for either work extraction or entanglement preparation. We find an equivalence between the work and entanglement, and derive their upper bounds for unsteering demon. Here we provide no limitation on Bob’s transformation. This work establishes the relationship between quantum steering and entanglement on the thermodynamic framework. As we just consider a very simple model, it is a natural question to reveal more abundant properties between quantum steering and other quantum correlations by using the steering machine.

Acknowledgments

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