Many-body interaction in fast soliton collisions

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Abstract

We study \(n\)-pulse interaction in fast collisions of \(N\) solitons of the cubic nonlinear Schrödinger (NLS) equation in the presence of generic weak nonlinear loss. We develop a reduced model that yields the contribution of \(n\)-pulse interaction to the amplitude shift for collisions in the presence of weak \((2m+1)\)-order loss, for any \(n\) and \(m\). We first employ the reduced model and numerical solution of the perturbed NLS equation to analyze soliton collisions in the presence of septic loss \((m = 3)\). Our calculations show that three-pulse interaction gives the dominant contribution to the collision-induced amplitude shift already in a full-overlap four-soliton collision, and that the amplitude shift strongly depends on the initial soliton positions. We then extend these results for a generic weak nonlinear loss of the form \(G(|\psi|^2)\psi\), where \(\psi\) is the physical field and \(G\) is a Taylor polynomial of degree \(m_c\). Considering \(m_c = 3\), as an example, we show that three-pulse interaction gives the dominant contribution to the amplitude shift in a six-soliton collision, despite the presence of low-order loss. Our study quantitatively demonstrates that \(n\)-pulse interaction with high \(n\) values plays a key role in fast collisions of NLS solitons in the presence of generic nonlinear loss. Moreover, the scalings of \(n\)-pulse interaction effects with \(n\) and \(m\) and the strong dependence on initial soliton positions lead to complex collision dynamics, which is very different from the one observed in fast NLS soliton collisions in the presence of cubic loss.

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I. INTRODUCTION

The problem of predicting the dynamic evolution of $N$ physical interacting objects or quantities, commonly known as the $N$-body problem, is an important subject of research in science and engineering. The study of this problem plays a key role in many fields, including celestial mechanics [1, 2], nuclear physics, solid-state physics, and molecular physics [3]. In many cases, the dynamics of the $N$ objects is governed by a force, which is a sum over two-body forces. This is the situation in celestial mechanics [1, 2] and in other systems [3], and it has been discussed extensively in the literature. A different but equally interesting dynamic scenario emerges when the $N$-body dynamics is determined by a force involving $n$-body interaction with $n \geq 3$ [4]. Indeed, $n$-body forces with $n \geq 3$ have been employed in a variety of problems including van der Waals interaction between atoms [5], interaction between nucleons in atomic nuclei [6–9], and in cold atomic gases in optical lattices [10–12].

A fundamental question in these studies concerns the physical mechanisms responsible for the emergence of $n$-body interaction with a given $n$ value. A second important question revolves around the dependence of the interaction strength on $n$ and on the other physical parameters. In the current study we investigate a new class of $N$-body problems, in which $n$-body forces play a dominant role. More specifically, we study the role of $n$-body interaction in fast collisions between $N$ solitons of the cubic nonlinear Schrödinger (NLS) equation in the presence of generic weak nonlinear loss. In this case the solitons experience significant collision-induced amplitude shifts, and important questions arise regarding the role of $n$-pulse interaction in the process, and the dependence of the amplitude shift and the $n$-pulse interaction on the physical parameters.

The NLS equation is one of the most widely used nonlinear wave models in the physical sciences. It was successfully employed to describe a large variety of physical systems, including water waves [13, 14], Bose-Einstein condensates [15, 16], pulse propagation in optical waveguides [17, 18], and nonlinear waves in plasma [19–21]. The most ubiquitous solutions of the NLS equation are the fundamental solitons. The dynamics of fundamental solitons in these systems can be affected by loss, which is often nonlinear [22]. Nonlinear loss arises in optical waveguides due to gain/loss saturation or multiphoton absorption [23]. It is also quite common in certain Bose-Einstein condensates [24, 25] and in many other systems that are described by the complex Ginzburg-Landau equation [26]. It is therefore important to
study the impact of nonlinear loss on the propagation of fundamental NLS solitons.

The main effect of weak nonlinear loss on the propagation of a single NLS soliton is a continuous decrease in the soliton’s energy. This single-pulse amplitude shift is qualitatively similar to the one due to linear loss, and can be calculated in a straightforward manner by employing the standard adiabatic perturbation theory. Nonlinear loss also strongly affects the collisions of NLS solitons, by causing an additional decrease of soliton amplitudes. The character of this collision-induced amplitude shift was recently studied for fast soliton collisions in the presence of cubic and quintic loss [27, 28]. The results of these studies indicate that the amplitude dynamics in soliton collisions in the presence of generic nonlinear loss might be quite complicated due to $n$-pulse interaction effects. More specifically, in Ref. [27] it was shown that the total collision-induced amplitude shift in a fast three-soliton collision in the presence of cubic loss is given by a sum over amplitude shifts due to two-pulse interaction, i.e., the contribution to the amplitude shift from three-pulse interaction is negligible. In contrast, in Ref. [28] it was found that three-pulse interaction enhances the amplitude shift in a fast three-soliton collision in the presence of quintic loss by a factor of 1.38.

The results of Ref. [28] suggest that $n$-pulse interaction with $n \geq 3$ might play an important role in fast NLS soliton collisions in the presence of generic or high-order nonlinear loss. Despite of this fact, a systematic analytic or numerical study of the role of $n$-pulse interaction in these collisions is still missing. In the current study we address this important problem. For this purpose, we first develop a general reduced model for amplitude dynamics, which allows us to calculate the contribution of $n$-pulse interaction to the amplitude shift for collisions in the presence of weak $(2m+1)$-order loss, for any $n$ and $m$. We then use the reduced model and numerical solution of the perturbed NLS equation to analyze soliton collisions in the presence of septic loss ($m = 3$). Our calculations show that three-pulse interaction gives the dominant contribution to the collision-induced amplitude shift already in a full-overlap four-soliton collision. Furthermore, we find that the amplitude shift strongly depends on the initial soliton positions with a pronounced maximum in the case of a full-overlap collision. We then generalize these results for generic weak nonlinear loss of the form $G(|\psi|^2)\psi$, where $\psi$ is the physical field and $G$ is a Taylor polynomial of degree $m_c$. We consider $m_c = 3$, as an example. That is, we take into account the effects of linear, cubic, quintic, and septic loss on the collision. We show that in this case three-pulse interaction gives the dominant contribution to the amplitude shift in a six-soliton collision, despite the
presence of linear and cubic loss. Our study uncovers a new type of $n$-body interaction involving fast collisions of NLS solitons, and demonstrates that this interaction plays a key role in collisions in the presence of generic nonlinear loss. Moreover, the scalings of $n$-pulse interaction effects with $n$ and $m$ and the strong dependence on initial positions lead to complex collision dynamics. This dynamics is very different from the one encountered in fast $N$-soliton collisions in the presence of weak cubic loss, where the total collision-induced amplitude shift is a sum over amplitude shifts due to two-pulse interaction [27].

The rest of the paper is organized as follows. In Sec. II, we obtain the reduced model for amplitude dynamics in a fast $N$-soliton collision in the presence of weak nonlinear loss. We then employ the model to calculate the total collision-induced amplitude shift and the contribution from $n$-soliton interaction. In Sec. III, we analyze in detail the predictions of the reduced model for the amplitude shifts in four-soliton and six-soliton collisions. In addition, we compare the analytic predictions with results of numerical simulations with the perturbed NLS equation. In Sec. IV we present our conclusions. Appendix A is devoted to the derivation of the equation for the collision-induced change in the soliton’s envelope due to $n$-pulse interaction in a fast $N$-soliton collision.

II. AMPLITUDE DYNAMICS IN $N$-SOLITON COLLISIONS

Consider propagation of soliton pulses of the cubic NLS equation in the presence of generic weak nonlinear loss $L(\psi)$, where $\psi$ is the physical field. In the context of propagation of light through optical waveguides, for example, $\psi$ is proportional to the envelope of the electric field. Assume that $L(\psi)$ can be approximated by $G(|\psi|^2)\psi$, where $G$ is a Taylor polynomial of degree $m_c$. Thus, we can write:

$$L(\psi) \simeq G(|\psi|^2)\psi = \sum_{m=0}^{m_c} \epsilon_{2m+1} |\psi|^{2m} \psi,$$

where $0 \leq \epsilon_{2m+1} \ll 1$ for $m \geq 0$. We refer to the $m$th summand on the right hand side of Eq. (1) as $(2m+1)$-order loss and remark that this term is often associated with $(m+1)$-photon absorption [23]. Under the aforementioned assumption on the loss, the dynamics of the pulses is governed by:

$$i\partial_z \psi + \partial_t^2 \psi + 2|\psi|^2 \psi = -i \sum_{m=0}^{m_c} \epsilon_{2m+1} |\psi|^{2m} \psi.$$

(2)
Here we adopt the notation used in nonlinear optics, in which $z$ is propagation distance and $t$ is time. The fundamental soliton solution of the unperturbed NLS equation with central frequency $\beta_j$ is

$$\psi_j(t, z) = \eta_j \frac{\exp(i\chi_j)}{\cosh(x_j)},$$

where $x_j = \eta_j(t - y_j - 2\beta_j z)$, $\chi_j = \alpha_j + \beta_j(t - y_j) + (\eta_j^2 - \beta_j^2) z$, and $\eta_j$, $y_j$, and $\alpha_j$ are the soliton amplitude, position, and phase, respectively.

The effects of the nonlinear loss on single pulse propagation can be calculated by employing the standard adiabatic perturbation theory [17]. This perturbative calculation yields the following expression for the rate of change of the soliton amplitude

$$\frac{d\eta_j(z)}{dz} = - \sum_{m=0}^{\infty} \epsilon_{2m+1} a_{2m+1} \eta_j^{2m+1}(z),$$

where $a_{2m+1} = (2^{m+1} m!/((2m+1)!!)$. The $z$ dependence of the soliton amplitude is obtained by integration of Eq. (4).

Let us discuss the calculation of the effects of weak nonlinear loss on a fast collision between $N$ NLS solitons. The solitons are identified by the index $j$, where $1 \leq j \leq N$. Since we deal with a fast collision, $|\beta_j - \beta_k| \gg 1$ for any $j \neq k$. The only other assumption of our calculation is that $0 \leq \epsilon_{2m+1} \ll 1$ for $m \geq 0$. Under these assumptions, we can employ a generalization of the perturbation technique, developed in Ref. [29], and successfully applied for studying fast two-soliton and three-soliton collisions in different setups [27–34]. Note that the generalized technique in the current paper is more complicated than the one used in Refs. [27–34]. We therefore provide a brief outline of the main steps in the generalized calculation. (1) We first consider the effects of $(2m+1)$-order loss, and calculate the contribution of $n$-soliton interaction with $n \leq m + 1$ to the collision-induced amplitude shift, for a given $n$-soliton combination [35]. (2) We then add the contributions coming from all possible $n$-soliton combinations. This sum is the total contribution of $n$-pulse interaction to the amplitude shift in a fast collision in the presence of $(2m+1)$-order loss. (3) Summing the amplitude shifts calculated in (2) over all relevant $m$ values, $1 \leq m \leq m_c$, we obtain the total contribution of $n$-pulse interaction to the amplitude shift in a collision in the presence of generic nonlinear loss. (4) The total collision-induced amplitude shift is obtained by summing the amplitude shifts in (3) over all possible $n$-values, $2 \leq n \leq m + 1$. 

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5
Following this procedure, we first calculate the collision-induced change in the amplitude of the \(j\)th soliton due to \((2m+1)\)-order loss. The dynamics is determined by the following perturbed NLS equation

\[
i\partial_t \psi + \partial_x^2 \psi + 2|\psi|^2 \psi = -i\epsilon_{2m+1}|\psi|^{2m}\psi. \tag{5}\]

We start by considering the amplitude shift of the collision-induced effects for the \(m\)th pulse solution of Eq. \((5)\) in the form

\[
\psi_j(x) = \psi_j(t) \exp(i\chi_j), \quad \phi_j(t) = \Phi_j(x) \exp(i\chi_j),
\]

where \(\psi_j\) is the \(j\)th single-soliton solution of Eq. \((5)\) with \(0 < \epsilon_{2m+1} \ll 1\), \(\phi_k\) describes collision-induced effects for the \(k\)th soliton, and the ellipsis represents higher-order terms. We then substitute \(\psi_j\) along with \(\psi_j(t, z) = \Psi_j(x_j) \exp(i\chi_j), \phi_j(t, z) = \Phi_j(x_j) \exp(i\chi_j), \psi_{l,j'}(t, z) = \Psi_{l,j'}(x_{l,j'}) \exp(i\chi_{l,j'}),\) and \(\phi_{l,j'}(t, z) = \Phi_{l,j'}(x_{l,j'}) \exp(i\chi_{l,j'})\) for \(j' = 1, \ldots, n-1\), into Eq. \((5)\). Since the frequency difference for each soliton pair is large, we can employ the resonant approximation, and neglect terms with rapid oscillations with respect to \(z\). Under this approximation, Eq. \((5)\) decomposes into a system of equations for the evolution of \(\Phi_j\) and the \(\Phi_{l,j'}\). [See, for example, Refs. \([27, 28]\), for a discussion of the cases \(n = 2\) and \(n = 3\) for \(m = 1\) and \(m = 2\)]. The system of equations is solved by expanding \(\Phi_j\) and each of the \(\Phi_{l,j'}\) in a perturbation series with respect to \(\epsilon_{2m+1}\) and \(1/|\beta_{l,j'} - \beta_j|\). We focus attention on \(\Phi_j\) and comment that the equations for the \(\Phi_{l,j'}\) are obtained in a similar manner. The only collision-induced effect in order \(1/|\beta_{l,j'} - \beta_j|\) is a phase shift \(\Delta\alpha_j = 4 \sum_{j' = 1}^{n-1} \eta_{l,j'} / |\beta_{l,j'} - \beta_j|\), which already exists in the unperturbed collision \([36]\). Thus, we find that the main effect of \((2m+1)\)-order loss on the collision is of order \(\epsilon_{2m+1}/|\beta_{l,j'} - \beta_j|\). We denote the corresponding term in the expansion of \(\Phi_j\) by \(\Phi_{j2}^{(1m)}\), where the first subscript stands for the soliton index, the second subscript indicates the combined order with respect to both \(\epsilon_{2m+1}\) and \(1/|\beta_{l,j'} - \beta_j|\), and the superscripts represent the order in \(\epsilon_{2m+1}\) and the order of the nonlinear loss, respectively. Furthermore, the contribution to \(\Phi_{j2}^{(1m)}\) from \(n\)-soliton interaction with the \(l_1, l_2, \ldots, l_{n-1}\) solitons is denoted by \(\Phi_{j2(l_1 \ldots l_{n-1})}^{(1mn)}\). In Appendix A, we show that the latter contribution satisfies:

\[
\partial_z \Phi_{j2(l_1 \ldots l_{n-1})}^{(1mn)} = -\epsilon_{2m+1} \sum_{k_{l_1} = 1}^{m-(n-2)} \sum_{k_{l_2} = 1}^{m-k_{l_1}-(n-3)} \cdots \sum_{k_{l_{n-1}} = 1}^{m-s_{n-2}} \frac{m!(m+1)!}{(k_{l_1}! \cdots k_{l_{n-1}})!^2} \times [(m+1-s_{n-1})!(m-s_{n-1})!]^{-1} |\Psi_{l_1}|^{2k_{l_1}} \cdots |\Psi_{l_{n-1}}|^{2k_{l_{n-1}}} |\Psi_j|^{2m-2s_{n-1}} \Psi_j, \tag{6}\]

6
where \( s_n = \sum_{j=1}^{n} k_{l_j} \). Note that all terms in the sum on the right hand side of Eq. (3) contain the products \( |\Psi_{l_1}|^{2k_{l_1}} \ldots |\Psi_{l_{n-1}}|^{2k_{l_{n-1}}} |\Psi_j|^{2k_j} \Psi_j \), where \( k_{l_1} + \ldots + k_{l_{n-1}} + k_j = m \), and \( 1 \leq k_{l_j} \leq m - (n - 2) \) for \( 1 \leq j' \leq n - 1 \). Therefore, the largest value of \( n \) that can induce non-vanishing effects is obtained by setting \( k_j = 0 \) and \( k_{l_j} = 1 \) for \( 1 \leq j' \leq n - 1 \). This yields \( n_{\text{max}} = m + 1 \) for the maximum value of \( n \).

Next, we obtain the equation for the rate of change of the \( j \)th soliton’s amplitude due to \( n \)-pulse interaction with the \( l_1, l_2, \ldots, l_{n-1} \) solitons. For this purpose, we first expand both sides of Eq. (6) with respect to the eigenmodes of the linear operator \( \hat{L} \) describing small perturbations about the fundamental NLS soliton \([27–31]\). We then project the two expansions onto the eigenmode \( f_0(x_j) = \text{sech}(x_j)(1,-1)^T \) and integrate over \( x_j \). This calculation yields the following equation for the rate of change of the amplitude:

\[
\frac{d\eta_{j(l_1 \ldots l_{n-1})}^{(mn)}}{dz} = -\epsilon_{2m+1} \sum_{k_{l_1}=1}^{m-(n-2)} \ldots \sum_{k_{l_{n-1}}=1}^{m-s_{n-2}} \frac{m!(m+1)!\eta_{l_1}^{2k_{l_1}} \ldots \eta_{l_{n-1}}^{2k_{l_{n-1}}} \eta_j^{2m-2s_{n-1}+1}}{(k_{l_1}! \ldots k_{l_{n-1}}!)^2(m+1-s_{n-1})!(m-s_{n-1})!} \\
\times \int_{-\infty}^{\infty} dx_j [\cosh(x_{l_1})]^{-2k_{l_1}} \ldots [\cosh(x_{l_{n-1}})]^{-2k_{l_{n-1}}} [\cosh(x_j)]^{-(2m-2s_{n-1}+2)}. \tag{7}
\]

We now proceed to the second calculation step, in which we obtain the total rate of change in the \( j \)th soliton’s amplitude due to \( n \)-pulse interaction in a fast \( N \)-soliton collision in the presence of \((2m+1)\)-order loss. For this purpose, we sum Eq. (7) over all \( n \)-soliton combinations \((j, l_1, \ldots, l_{n-1})\), where \( 1 \leq l_j' \leq N \), \( l_j \neq j \), and \( 1 \leq j' \leq n - 1 \). Thus, the total rate of change of the amplitude due to \( n \)-pulse interaction is:

\[
\frac{d\eta_j^{(nm)}}{dz} = \sum_{l_1=1}^{N} \sum_{l_2=l_1+1}^{N} \ldots \sum_{l_{n-1}=l_{n-2}+1}^{N} \prod_{j'=1}^{n-1} \left( 1 - \delta_{j_j'} \right) \frac{d\eta_{j(l_1 \ldots l_{n-1})}^{(mn)}}{dz}, \tag{8}
\]

where \( \delta_{jk} \) is the Kronecker delta function. The total rate of change in the \( j \)th soliton’s amplitude in an \( N \)-soliton collision in the presence of the generic nonlinear loss due to \( n \)-soliton interaction is calculated by summing both sides of Eq. (8) over \( m \) for \( n-1 \leq m \leq m_c \). This yields:

\[
\frac{d\eta_j^{(n)}}{dz} = \sum_{m=n-1}^{m_c} \frac{d\eta_j^{(mn)}}{dz}. \tag{9}
\]

To obtain the total rate of change of the amplitude in the collision, we sum Eq. (9) over \( n \) for \( 2 \leq n \leq m_c + 1 \), and also take into account the effects of single-pulse propagation, as
described by Eq. (4). We arrive at the following equation:

$$\frac{d\eta_j}{dz} = \sum_{n=2}^{m_c+1} \frac{d\eta_j^{(n)}}{dz} - \sum_{m=0}^{m_c} \epsilon_{2m+1} a_{2m+1} \eta_j^{2m+1},$$

for \( j = 1, \ldots, N \). Equations (7)-(10) provide a complete description of the collision-induced amplitude dynamics under the assumptions of a fast collision and weak loss.

Important insight into the effects of \( n \)-pulse interaction on \( N \)-soliton collisions is obtained by studying full-overlap collisions, i.e., collisions in which the envelopes of all \( N \) solitons overlap at a certain distance \( z_c \). More specifically, we would like to calculate the total collision-induced amplitude shift \( \Delta \eta_j \) in these collisions, and compare it with the contributions of \( n \)-pulse interaction to the amplitude shift \( \Delta \eta_j^{(n)} \), for \( n = 2, \ldots, m_c + 1 \). For this purpose, we consider first a full-overlap \( N \)-soliton collision in the presence of \((2m+1)\)-order loss. The rate of change in the \( j \)th soliton’s amplitude due to \( n \)-pulse interaction with solitons with indexes \( l_1, l_2, \ldots, l_{n-1} \), where \( 1 \leq l_j \leq N \) and \( l_j \neq j \) for \( 1 \leq j \leq n - 1 \), is given by Eq. (7). In a fast full-overlap collision in the presence of weak \((2m+1)\)-order loss, the main contribution to the amplitude shift comes from the close vicinity of the collision point \( z_c \). Therefore, an approximate expression for the contribution of \( n \)-pulse interaction to the amplitude shift can be obtained by integrating Eq. (7) over \( z \) from \(-\infty\) to \( \infty \), while taking the amplitude values on the right hand side of the equation as constants [37]: \( \eta_k = \eta_k(z_c^-) \).

Employing these steps, we arrive at

$$\Delta \eta_j^{(mn)}(l_1, \ldots, l_{n-1}) = -\epsilon_{2m+1} \sum_{k_{l_1}=1}^{m-(n-2)} \cdots \sum_{k_{l_{n-1}}=1}^{m-s_{n-2}} \frac{m!(m+1)! \eta_{l_1}^{2k_{l_1}} \cdots \eta_{l_{n-1}}^{2k_{l_{n-1}}} \eta_j^{2m-2s_{n-1}+1}}{(k_{l_1}! \cdots k_{l_{n-1}}!)^2 (m+1-s_{n-1})!(m-s_{n-1})!}$$

$$\times \int_{-\infty}^{\infty} dx_j [\cosh(x_j)]^{-(2m-2s_{n-1}+2)} \int_{-\infty}^{\infty} dz \left[ \cosh(x_{l_1}) \right]^{-2k_{l_1}} \cdots \left[ \cosh(x_{l_{n-1}}) \right]^{-2k_{l_{n-1}}} \left[ \cosh(x_j) \right]^{-2k_{l_{n-1}}},$$

Equation (11)

The total contribution of \( n \)-pulse interaction to the amplitude shift in a fast full-overlap \( N \)-soliton collision in the presence of \((2m+1)\)-order loss is obtained by summing Eq. (11) over all \( n \)-soliton combinations \((j, l_1, \ldots, l_{n-1})\):

$$\Delta \eta_j^{(mn)} = \sum_{l_1=1}^N \sum_{l_2=l_1+1}^N \cdots \sum_{l_{n-1}=l_{n-2}+1}^N \prod_{j'=1}^{n-1} (1 - \delta_{jj'}) \Delta \eta_j^{(mn)}(l_1, \ldots, l_{n-1}).$$

Equation (12)

Summation of Eq. (12) over \( m \) yields the total contribution of \( n \)-pulse interaction to the amplitude shift in a full-overlap collision in the presence of the generic nonlinear loss:

$$\Delta \eta_j^{(n)} = \sum_{m=n-1}^{m_c} \Delta \eta_j^{(mn)}.$$

Equation (13)
Thus, the approximate expression for the total amplitude shift in a fast full-overlap collision is

\[ \Delta \eta_j = \sum_{n=2}^{m_c+1} \Delta \eta_j^{(n)} . \]  

(14)

Note that since Eqs. (7)-(10) and Eqs. (11)-(14) are independent of the soliton phases, the total collision-induced amplitude shift and the contribution of \( n \)-soliton interaction are expected to be phase-insensitive.

III. ANALYSIS AND SIMULATIONS OF FOUR-SOLITON AND SIX-SOLITON COLLISIONS

Let us demonstrate the implications of Eqs. (7)-(10) and Eqs. (11)-(14) on collision-induced amplitude dynamics in specific setups. We start by analyzing the effects of fast full-overlap \( N \)-soliton collisions in the presence of \((2m + 1)\)-order loss, where the dynamics is described by Eq. (5). Since the cases \( m = 1 \) and \( m = 2 \) were already studied by us in Refs. \[27, 28\], we first focus attention on collisions in the presence of septic loss \((m = 3)\).

As we show below, the analysis of this case is sufficient for uncovering the main scaling properties and the importance of \( n \)-soliton interaction in soliton collisions in the presence of high-order loss. For concreteness, we consider four-soliton and six-soliton collisions with soliton frequencies, \( \beta_1 = 0, \beta_2 = -\Delta \beta, \beta_3 = \Delta \beta, \beta_4 = 2\Delta \beta \) for \( N = 4 \), and \( \beta_1 = 0, \beta_2 = -2\Delta \beta, \beta_3 = -\Delta \beta, \beta_4 = \Delta \beta, \beta_5 = 2\Delta \beta, \beta_6 = 3\Delta \beta \) for \( N = 6 \). Note that this choice corresponds, for example, to the one used in optical waveguide links employing wavelength-division-multiplexing \[38\]. The initial amplitudes and phases are \( \eta_j(0) = 1 \) and \( \alpha_j(0) = 0 \) for \( 1 \leq j \leq N \), respectively. The initial positions are \( y_0(1) = 0, y_2(0) = 20, y_3(0) = -20, y_4(0) = -40 \) for \( N = 4 \), and \( y_0(1) = 0, y_2(0) = 40, y_3(0) = 20, y_4(0) = -20, y_5(0) = -40, y_6(0) = -60 \) for \( N = 6 \). Thus, the solitons are well separated before the collision. In addition, the final propagation distance \( z_f \) is assumed to be large enough, so that the solitons are well separated after the collision. The value of the septic loss coefficient is taken as \( \epsilon_7 = 0.002 \).

Figure \[1\] shows the \( \Delta \beta \)-dependence of the total collision-induced amplitude shift in four-pulse and six-pulse collisions, for the \( j = 1 (\beta_j = 0) \) soliton. Both the prediction of Eqs. (11)-(14) and the result obtained by numerical solution of Eq. (5) are presented. The figure also shows the analytic prediction for the contributions of two-, three-, and four-soliton
interaction to the amplitude shift, $\Delta \eta_1^{(2)}$, $\Delta \eta_1^{(3)}$, and $\Delta \eta_1^{(4)}$, respectively. The agreement between the analytic prediction and the numerical simulations is very good for $\Delta \beta \geq 15$, where the perturbation description is expected to hold. Moreover, our calculations show that the dominant contribution to the total amplitude shift in a four-soliton collision comes from three-soliton interaction. The contribution from four-soliton interaction increases from 15.9% in a four-soliton collision to 39.4% in a six-soliton collision. Consequently, in a six-soliton collision the effects of three-pulse and four-pulse interaction are both important, while those of two-pulse interaction are relatively small (about 9.6%). Further numerical simulations of fast full-overlap four-soliton collisions show that the total collision-induced amplitude shift is insensitive to the initial phases of the solitons, in agreement with the analytic prediction of Eqs. (11)-(14). Based on these observations we conclude that phase-insensitive $n$-pulse interaction with high $n$ values, satisfying $2 < n \leq m + 1$, plays a crucial role in fast full-overlap $N$-soliton collisions in the presence of $(2m + 1)$-order loss.

We now turn to analyze more generic fast $N$-soliton collisions, in which the solitons’ envelopes do not completely overlap. Based on Eq. (7), the contribution of $n$-pulse interaction to the total amplitude shift should strongly depend on the degree of soliton overlap during the collision, for $n \geq 3$, $m \geq 2$, and $N \geq 3$. Consequently, the total collision-induced amplitude shift might strongly depend on the initial soliton positions in this case. We therefore focus our attention on this dependence. We consider, as an example, a four-soliton collision in the presence of septic loss with $\epsilon_7 = 0.02$, where the soliton frequencies are $\beta_1 = 0$, $\beta_2 = -10$, $\beta_3 = 10$, and $\beta_4 = 20$. The initial amplitudes and phases are $\eta_j(0) = 1$ and $\alpha_j(0) = 0$ for $1 \leq j \leq 4$. The initial positions are $y_0(0) = 0$, $y_2(0) = 20$, $y_4(0) = -40$, and $-39 \leq y_3(0) \leq -1$. That is, the initial position of the $j = 3$ soliton is varied, while the initial positions of the other solitons are not changed. Notice that in this setup, the four-soliton collision is not a full-overlap collision, except for at $y_3(0) = -20$. As a result, Eqs. (11)-(14) cannot be employed to analyze the collision-induced amplitude dynamics, and Eqs. (7)-(10) should be used instead. We therefore solve Eqs. (7)-(10) with the aforementioned initial parameter values for $0 \leq z \leq z_f$, where $z_f = 6$, and plot the final amplitudes $\eta_j(z_f)$ vs $y_3(0)$. The curves are shown in Fig. (2) along with the curves obtained by numerical solution of Eq. (5). The agreement between the analytic prediction and the simulations result is good. As can be seen, each $\eta_j(z_f)$-vs-$y_3(0)$ curve has a pronounced minimum at $y_3(0) = -20$, i.e., at the initial position value of the $j = 3$ soliton corresponding to a full-overlap collision. Thus,
FIG. 1: (Color online) The total collision-induced amplitude shift of the $j = 1$ soliton $\Delta \eta_1$ vs frequency difference $\Delta \beta$ in a full-overlap four-soliton collision (a) and in a full-overlap six-soliton collision (b) in the presence of septic loss with coefficient $\epsilon_7 = 0.002$. The solid black line is the analytic prediction of Eqs. (11)-(14) and the squares represent the result of numerical simulations with Eq. (5). The dotted red, dashed blue, and dashed-dotted green lines correspond to the contributions of two-, three-, and four-soliton interaction to the amplitude shift, $\Delta \eta_{1(2)}$, $\Delta \eta_{1(3)}$, and $\Delta \eta_{1(4)}$, respectively.

A strong dependence of the collision-induced amplitude shift on the initial soliton positions is observed already in a four-soliton collision in the presence of septic loss. This means that the collision-induced amplitude dynamics in fast $N$-soliton collisions in the presence of weak generic loss can be quite complex due to the dominance of contributions from $n$-pulse interaction with high $n$-values. This behavior is sharply different from the one encountered in fast $N$-soliton collisions in the presence of weak cubic loss. In the latter case, the total
collision-induced amplitude shift is a sum over contributions from two-pulse interaction, and the collision can be accurately viewed as consisting of a collection of pointwise two-soliton collisions [27].

The analysis of the effects of \((2m+1)\)-order loss on \(N\)-soliton collisions is very valuable, since it explains the importance of \(n\)-pulse interaction and uncovers the scaling laws for this interaction. However, in most systems one has to take into account the impact of the low-order loss terms, whose presence can enhance the effects of two-pulse interaction. It is therefore important to take into account all the relevant loss terms when analyzing collision-induced dynamics in the presence of generic loss. We now turn to address this aspect of the problem, by considering the effects of generic weak nonlinear loss of the form (11) on fast \(N\)-soliton collisions. For concreteness, we assume \(m_c = 3\) and loss coefficients \(\epsilon_1 = 0.002\), \(\epsilon_3 = 0.004\), \(\epsilon_5 = 0.006\), and \(\epsilon_7 = 0.001\). We also assume full-overlap collisions, but emphasize that the analysis can be extended to treat the general case by the same method described in the preceding paragraph. We consider four-soliton and six-soliton collisions with the same pulse parameters used for full-overlap collisions in the presence of septic loss. Figure 3 shows the \(\Delta \beta\) dependence of the total collision-induced amplitude shift in four-soliton
and six-soliton collisions for the $j = 1$ soliton, as obtained by Eqs. (11)-(14). The result obtained by numerical solution of Eq. (2) and the analytic predictions for the contributions of two-, three-, and four-soliton interaction, $\Delta \eta_1^{(2)}$, $\Delta \eta_1^{(3)}$, and $\Delta \eta_1^{(4)}$, are also shown. We observe that in four-soliton collisions, $\Delta \eta_1^{(2)}$ is comparable to $\Delta \eta_1^{(3)}$, while $\Delta \eta_1^{(4)}$ is much smaller. That is, the inclusion of the low-order loss terms does lead to an enhancement of the fractional contribution of two-pulse interaction to the amplitude shift. In contrast, in six-soliton collisions, $\Delta \eta_1^{(3)}$ (53.2%) is significantly larger than $\Delta \eta_1^{(2)}$ (22.2%), while $\Delta \eta_1^{(4)}$ (24.6%) is comparable to $\Delta \eta_1^{(2)}$. Based on the latter observation we conclude that when the low-order loss coefficients $\epsilon_1$ and $\epsilon_3$ are comparable in magnitude to the higher-order loss coefficients, the contributions to the amplitude shift from $n$-pulse interaction with $n \geq 3$ can be much larger than the one coming from two-pulse interaction.

IV. CONCLUSIONS

In summary, we studied $n$-pulse interaction in fast collisions of $N$ solitons of the cubic NLS equation in the presence of generic weak nonlinear loss, which can be approximated by the series (1). Due to the presence of nonlinear loss, the solitons experience collision-induced amplitude shifts that are strongly enhanced by $n$-pulse interaction. We first developed a general reduced model that allowed us to calculate the contribution of $n$-pulse interaction to the amplitude shift in fast $N$-soliton collisions in the presence of $(2m+1)$-order loss, for any $n$ and $m$. We then used the reduced model and numerical simulations with the perturbed NLS equation to analyze four-soliton and six-soliton collisions in the presence of septic loss ($m = 3$). Our calculations showed that three-pulse interaction gives the dominant contribution to the collision-induced amplitude shift already in a full-overlap four-soliton collision, while in a full-overlap six-soliton collision, both three-pulse and four-pulse interaction are important. Furthermore, we found that the collision-induced amplitude shift has a strong dependence on the initial soliton positions, with a pronounced maximum in the case of a full-overlap collision. We then generalized these results by considering $N$-soliton collisions in the presence of generic weak nonlinear loss of the form (1) with $m_c = 3$. Our analytic calculations and numerical simulations showed that three-pulse interaction gives the dominant contribution to the amplitude shift in a full-overlap six-soliton collision, despite the presence of linear and cubic loss. All the collision-induced effects were found
FIG. 3: (Color online) The total collision-induced amplitude shift of the \( j = 1 \) soliton \( \Delta \eta_1 \) vs frequency difference \( \Delta \beta \) in a full-overlap four-soliton collision (a) and in a full-overlap six-soliton collision (b) in the presence of generic nonlinear loss of the form (1) with \( m_c = 3 \) and loss coefficients \( \epsilon_1 = 0.002, \epsilon_3 = 0.004, \epsilon_5 = 0.006, \) and \( \epsilon_7 = 0.001 \). The solid black line is the analytic prediction of Eqs. (11)-(14) and the squares correspond to the result of numerical simulations with Eq. (2). The dotted red, dashed blue, and dashed-dotted green lines represent the contributions of two-, three-, and four-soliton interaction to the amplitude shift, \( \Delta \eta_1^{(2)} \), \( \Delta \eta_1^{(3)} \), and \( \Delta \eta_1^{(4)} \), respectively.

to be insensitive to the soliton phases for fast collisions. Based on these observations we conclude that phase-insensitive \( n \)-pulse interaction with high \( n \) values plays a key role in fast collisions of NLS solitons in the presence of generic weak nonlinear loss. The complex scalings of \( n \)-pulse interaction effects with \( n \) and \( m \) and the strong dependence on initial soliton positions lead to complex collision dynamics. This dynamics is very different from the one observed in fast collisions of \( N \) NLS solitons in the presence of weak cubic loss, where
the total collision-induced amplitude shift is a sum over amplitude shifts due to two-pulse interaction [27].

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Appendix A: Derivation of Eq. (6)

In this Appendix, we derive Eq. (6) for the collision-induced change in the envelope of a soliton due to \( n \)-pulse interaction in a fast \( N \)-soliton collision in the presence of weak \( (2m+1) \)-order loss. More specifically, we consider the change in the envelope of the \( j \)th soliton induced by \( n \)-pulse interaction with solitons with indexes \( l_1, l_2, \ldots, l_{n-1} \), where \( 1 \leq l_{j'} \leq N \) and \( l_{j'} \neq j \) for \( 1 \leq j' \leq n-1 \). The derivation is based on a generalization of the perturbation procedure developed in Ref. [29]. Following this procedure, we look for a solution of Eq. (5) in the form

\[
\psi_n = \psi_j + \phi_j + \sum_{j'=1}^{n-1} \left[ \psi_{l_{j'}} + \phi_{l_{j'}} \right] + \ldots
\]

where \( \psi_k \) is the \( k \)th single-soliton solution of Eq. (5) with \( 0 < \epsilon_{2m+1} \ll 1 \), \( \phi_k \) describes collision-induced effects for the \( k \)th soliton, and the ellipsis represents higher-order terms. We then substitute \( \psi_n \) along with \( \psi_j(t, z) = \Psi_j(x_j) \exp(i\chi_j) \), \( \phi_j(t, z) = \Phi_j(x_j) \exp(i\chi_j) \), \( \psi_{l_{j'}}(t, z) = \Psi_{l_{j'}}(x_{l_{j'}}) \exp(i\chi_{l_{j'}}) \), and \( \phi_{l_{j'}}(t, z) = \Phi_{l_{j'}}(x_{l_{j'}}) \exp(i\chi_{l_{j'}}) \) for \( j' = 1, \ldots, n-1 \), into Eq. (5). Next, we use the resonant approximation, and neglect terms with rapid oscillations with respect to \( z \). We find that the main effect of \( (2m+1) \)-order loss on the envelope of the \( j \)th soliton is of order \( \epsilon_{2m+1} / |\beta_{l_{j'}} - \beta_j| \).

We denote this collision-induced change in the envelope by \( \Phi^{(1m)}_{l_{j'}} \), and the contribution to this change from \( n \)-soliton interaction with the \( l_1, l_2, \ldots, l_{n-1} \) solitons by \( \Phi^{(1mn)}_{j2(l_1 \ldots l_{n-1})} \). Within the resonant approximation, the phase factor of terms contributing to changes in the \( j \)th soliton’s envelope must be equal to \( \chi_j \). Consequently, these terms must be proportional to:

\[
|\Psi_{l_1}^{2k_1} \ldots |\Psi_{l_{n-1}}^{2k_{n-1}} |\Psi_j^{2k_j} \Psi_j|
\]

where \( k_1 + \cdots + k_{n-1} + k_j = m \), and \( 1 \leq k_{l_{j'}} \leq m - (n-2) \) for \( 1 \leq j' \leq n-1 \). Summing over all possible contributions of this form, we obtain the
following evolution equation for $\Phi^{(1mn)}_{j2(l_1...l_{n-1})}$:

$$
\partial_z \Phi^{(1mn)}_{j2(l_1...l_{n-1})} = -\epsilon_{2m+1} \sum_{k_1=1}^{m-(n-2)} \sum_{k_1=1}^{m-k_1-(n-3)} \sum_{k_{n-1}=1}^{m-s_{n-2}} b_k 
\times |\Psi_{l_1}|^{2k_1} \cdots |\Psi_{l_{n-1}}|^{2k_{n-1}} |\Psi_j|^{2m-2s_{n-1}} \Psi_j,
$$

(A1)

where $s_n = \sum_{j'=1}^n k_{l_{j'}}$, $b_k$ are constants, and $k = (k_1, k_2, \ldots, k_{l_{n-1}})$.

To calculate the expansion coefficients $b_k$, we first note that

$$
|\Psi|^{2m} \Psi = \left( \psi_j + \sum_{j'=1}^{n-1} \Psi_{l_{j'}} \right)^{m+1} \left( \psi_j^* + \sum_{j'=1}^{n-1} \Psi_{l_{j'}}^* \right)^m.
$$

(A2)

Employing the multinomial expansion formula for the two terms on the right hand side of Eq. (A2), we obtain:

$$
\left( \psi_j + \sum_{j'=1}^{n-1} \Psi_{l_{j'}} \right)^{m+1} = \sum_{k_1=0}^{m+1} \cdots \sum_{k_{n-1}=0}^{m+1} \frac{(m+1)!}{(k_1! \cdots k_{n-1}!)(m+1-s_{n-1})!}
\times \Psi_{l_1}^{k_1} \cdots \Psi_{l_{n-1}}^{k_{n-1}} \psi_j^{m+1-s_{n-1}},
$$

(A3)

and

$$
\left( \psi_j^* + \sum_{j'=1}^{n-1} \Psi_{l_{j'}}^* \right)^m = \sum_{k_1=0}^{m} \cdots \sum_{k_{n-1}=0}^{m} \frac{m!}{(k_1! \cdots k_{n-1}!)(m-s_{n-1})!}
\times \Psi_{l_1}^{*k_1} \cdots \Psi_{l_{n-1}}^{*k_{n-1}} \psi_j^{*m-s_{n-1}}.
$$

(A4)

Combining Eqs. (A2)-(A4), we find that the expansion coefficients $b_k$ are given by:

$$
b_k = \frac{m!(m+1)!}{(k_1! \cdots k_{n-1}!)^2(m+1-s_{n-1})!(m-s_{n-1})!}.
$$

(A5)

Substituting this relation into Eq. (A2), we arrive at Eq. (6).

[1] A.E. Roy, Orbital Motion (Institute of Physics, Bristol, 2005).

[2] Y. Hagihara, Celestial Mechanics I: Dynamical Principles and Transformation Theory (MIT Press, Cambridge, MA, 1970).

[3] D.J. Thouless, The Quantum Mechanics of Many-Body Systems (Academic, New York, 1972).
Here we use the term $n$-body interaction (or force) to describe an interaction (or a force) that does not exist in a system with $k$ objects, where $2 \leq k \leq n - 1$, but does appear in an $n$-body system.

[5] B.M. Axilrod and E. Teller, J. Chem. Phys. 11, 299 (1943).
[6] J. Fujita and H. Miyazawa, Prog. Theor. Phys. 17, 360 (1957).
[7] B.A. Loiseau and Y. Nogami, Nucl. Phys. B 2, 470 (1967).
[8] H. Witala, W. Glöckle, D. Hüber, J. Golak, and H. Kamada, Phys. Rev. Lett. 81, 1183 (1998).
[9] E. Epelbaum, A. Nogga, W. Glöckle, H. Kamada, U.G. Meissner, and H. Witala, Phys. Rev. C 66, 064001 (2002).
[10] A.J. Daley, J.M. Taylor, S. Diehl, M. Baranov, and P. Zoller, Phys. Rev. Lett. 102, 040402 (2009).
[11] J. Honer, H. Weimer, T. Pfau, and H.P. Büchler, Phys. Rev. Lett. 105, 160404 (2010).
[12] Y. Liang and H. Guo, J. Phys. B 45, 175303 (2012).
[13] S. Novikov, S.V. Manakov, L.P. Pitaevskii, and V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method (Plenum, New York, 1984).
[14] A.C. Newell, Solitons in Mathematics and Physics (SIAM, Philadelphia, 1985).
[15] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[16] R. Carretero-González, D.J. Frantzeskakis, and P.G. Kevrekidis, Nonlinearity 21, R139 (2008).
[17] A. Hasegawa and Y. Kodama, Solitons in Optical Communications (Clarendon, Oxford, 1995).
[18] G.P. Agrawal, Nonlinear Fiber Optics (Academic, San Diego, CA, 2001).
[19] N. Asano, T. Taniuti, and N. Yajima, J. Math. Phys. 10, 2020 (1969)
[20] Y.H. Ichikawa, T. Imamura, and T. Taniuti, J. Phys. Soc. Jpn. 33, 189 (1972).
[21] W. Horton and Y.H. Ichikawa, Chaos and Structure in Nonlinear Plasmas (World Scientific, Singapore, 1996).
[22] Y.S. Kivshar and B.A. Malomed, Rev. Mod. Phys. 61, 763 (1989).
[23] R.W. Boyd, Nonlinear Optics (Academic, San Diego, CA, 2008).
[24] E.A. Burt, R.W. Ghrist, C.J. Myatt, M.J. Holland, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. 79, 337 (1997).
[25] K.M. Mertes, J.W. Merrill, R. Carretero-González, D.J. Frantzeskakis, P.G. Kevrekidis, and D.S. Hall, Phys. Rev. Lett. 99, 190402 (2007).
[26] I.S. Aranson and L. Kramer, Rev. Mod. Phys. 74, 99 (2002).
[27] A. Peleg, Q.M. Nguyen, and Y. Chung, Phys. Rev. A 82, 053830 (2010).
[28] A. Peleg and Y. Chung, Phys. Rev. A 85, 063828 (2012).
[29] A. Peleg, M. Chertkov, and I. Gabitov, Phys. Rev. E 68, 026605 (2003).
[30] J. Soneson and A. Peleg, Physica D 195, 123 (2004).
[31] Y. Chung and A. Peleg, Nonlinearity 18, 1555 (2005).
[32] Y. Chung and A. Peleg, Phys. Rev. A 77, 063835 (2008).
[33] Q.M. Nguyen and A. Peleg, J. Opt. Soc. Am. B 27, 1985 (2010).
[34] A. Peleg and Y. Chung, Opt. Commun. 285, 1429 (2012).
[35] Note that terms with higher $n$ values do not contribute due to fast oscillations with respect to $z$. See, for example, [27].
[36] V.E. Zakharov and A.B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys. JETP 34, 62 (1972)].
[37] See Refs. [27, 28], for an application of this approximation for the cases $m = 1$ and $m = 2$, respectively.
[38] L.F. Mollenauer and P.V. Mamyshev, IEEE J. Quantum Electron. 34, 2089 (1998).