A Free Boundary Problem in the Theory of the Stars*

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Abstract

We investigate numerically models of the static spherically symmetric boson-fermion stars in the scalar-tensor theory of gravity with massive dilaton field. The proper mathematical model of such stars is interpreted as a nonlinear two-parametric eigenvalue problem with unknown internal boundary. To solve this problem the Continuous Analogue of Newton Method is used.

Keywords: mixed fermion-boson stars, scalar-tensor theory of gravity, massive dilaton field, two-parametric nonlinear eigenvalue problem, Continuous Analog of the Newton Method.

1 Main model

Boson stars are gravitationally bound macroscopic quantum states made up of scalar bosons. They differ from the usual fermionic stars in that they are only prevented from collapsing gravitationally by the Heisenberg uncertainty principle. For a self-interacting boson field the mass of the boson star, even for small values of the coupling constant, turns out to be of the order of Chandrasekhar’s mass when the boson mass is similar to proton mass. Thus, the boson stars arise as possible candidates for non-baryonic dark matter in the universe and, consequently, as a possible solution of one of the outstanding problems in modern astrophysics: the problem of non-luminous matter in the universe.

Most of the stars are of primordial origin, being formed from an original gas of fermions and bosons in the early universe. That is why it should be expected that most stars are a mixture of both fermions and bosons in different proportions.

Boson-fermion stars are also a good model to understand more about the nature of strong gravitational fields not only in general relativity but also in the other theories of gravity.

The most natural and promising generalizations of general relativity are the scalar-tensor theories of gravity [2]. In these theories the gravity is mediated not only by a tensor field (the metric of space-time) but also by a scalar field (the dilaton). These dilatonic theories of gravity contain arbitrary functions of the scalar field that determine the gravitational “constant” as a dynamical variable and the strength of the coupling between the scalar field and matter. It should be stressed that specific

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scalar-tensor theories of gravity arise naturally as a low energy limit of the string theory [3] which is the most promising modern model of unification of all fundamental physical interactions.

Boson stars in the scalar-tensor theories of gravity with massless dilaton have been widely investigated recently both numerically and analytically [4] (and references therein). Mixed boson-fermion stars in scalar tensor theories of gravity however have not been investigated so far in contrast to general relativistic case where boson-fermion stars have been investigated [5].

We will consider a static and spherically symmetric mixed boson-fermion star in asymptotic flat space-time. Then the main system of differential equations can be written in the following form [6]:

\[
\begin{align*}
\frac{d\lambda}{dr} &= F_1 = \frac{1 - \exp(\lambda)}{r} + r \left\{ \exp(\lambda) \left[ \frac{F}{T_0^0} + \frac{B}{T_0^0} + \frac{1}{2} \gamma^2 V(\varphi) \right] + \left( \frac{d\varphi}{dr} \right)^2 \right\}, \\
\frac{d\nu}{dr} &= F_2 = -\frac{1 - \exp(\lambda)}{r} - r \left\{ \exp(\lambda) \left[ \frac{F}{T_1^1} + \frac{B}{T_1^1} + \frac{1}{2} \gamma^2 V(\varphi) \right] - \left( \frac{d\varphi}{dr} \right)^2 \right\}, \\
\frac{d^2\varphi}{dr^2} &= F_3 = -\frac{2}{r} \frac{d\varphi}{dr} + \frac{1}{2} (F_1 - F_2) \frac{d\varphi}{dr} + \frac{1}{2} \exp(\lambda) \left[ \alpha(\varphi) (\frac{F}{T} + \frac{B}{T}) + \frac{1}{2} \gamma^2 V'(\varphi) \right], \\
\frac{d^2\sigma}{dr^2} &= F_4 = -\frac{2}{r} \frac{d\sigma}{dr} + \left[ \frac{1}{2} (F_1 - F_2) - 2 \alpha(\varphi) \frac{d\varphi}{dr} \right] \frac{d\sigma}{dr} - \sigma \exp(\lambda) \left[ \Omega^2 \exp(-\nu) + 4 \sigma A^2(\varphi) W'(\sigma) \right], \\
\frac{d\mu}{dr} &= F_5 = -\frac{g(\mu) + f(\mu)}{f'(\mu)} \left[ \frac{1}{2} F_2 + \alpha(\varphi) \frac{d\varphi}{dr} \right].
\end{align*}
\]

Here the metric functions \(\lambda(r)\) and \(\nu(r)\), the dilaton field \(\varphi(r)\), the densities of the bosonic \(\sigma(r)\) and the fermionic \(\mu(r)\) matter are unknown functions of the radial coordinate \(r \in [0, R_s] \cup (R_s, \infty)\), where \(R_s\) is the unknown radius of the star. The quantity \(\Omega\) (the time-frequency of the bosonic field) is an unknown parameter as well. The quantities \(T_n^0, T_n^1, n = 0, 1\) (the diagonal components of the energy-momentum tensors), \(\frac{F}{T}, \frac{B}{T}\) (the corresponding traces), \(\alpha(\varphi), A(\varphi), V(\varphi), W(\sigma), f(\mu), \) and \(g(\mu)\) are given functions of their arguments, and \(\gamma\) is a given constant (see below).

Having in mind the physical assumptions, we have to set the following boundary conditions at \(r = 0\):

\[
\lambda(0) = \frac{d\varphi}{dr}(0) = \frac{d\sigma}{dr}(0) = 0, \quad \sigma(0) = \sigma_c, \quad \mu(0) = \mu_c, \quad \mu(R_s) = 0,
\]

where \(\sigma_c\) and \(\mu_c\) are the values of density of, respectively, the bosonic and fermionic matter at the star’s centre \(r = 0\). The first three conditions in (2) guarantee the nonsingularity of the metrics and the functions \(\lambda(r), \varphi(r), \sigma(r)\) at the star’s centre.

In the external domain \(r > R_s\) there is no fermionic matter, i.e., one can formally suppose that the corresponding density \(\mu(r) \equiv 0\) if \(x \geq R_s\). The fermionic part \(T_n^0\) of the energy-momentum tensor vanishes also identically and, thus, the last equation (with respect to the function \(\mu(r)\)) in (1) can be removed from the system.

As it is required by the asymptotic flatness of space-time, the boundary conditions at the infinity are:

\[
\nu(\infty) = \varphi(\infty) = \sigma(\infty) = 0 \quad \text{where} \quad (\cdot)(\infty) \stackrel{\text{def}}{=} \lim_{r \to \infty} (\cdot)(r).
\]

We seek for a solution \([\lambda(r), \nu(r), \varphi(r), \sigma(r), \mu(r), (R_s, \Omega)]\) subjected to the nonlinear system (1), satisfying the boundary conditions (2) and (3). At that, we assume the function \(\mu(r)\) is continuous in the interval \([0, R_s]\), while the functions \(\lambda(r), \nu(r)\) are continuous and the functions \(\varphi(r), \sigma(r)\) are smooth in the whole interval \([0, \infty)\), including the unknown internal boundary \(r = R_s\).

The formulated boundary value problem (BVP) is a two-parametric eigenvalue problem with respect to the quantities \(R_s\) and \(\Omega\).

Let us emphasize that a number of methods for solving the free-boundary problems are considered in detail in [7].

Here we aim at applying the new solving method to the above formulated problem. This method differs from the one proposed in [8] and for the governing field equations written in the forms (1) it possesses certain advantages.
2 Method of solution

At first we scale the variable $r$ using the Landau transformation $x = r/R_s$, $x \in [0, 1] \cup (1, \infty)$

For given values of the parameters $R_s$ and $\Omega$ the independent solving of the system (1) in the internal domain $x \in [0, 1]$ (inside the star) requires seven boundary conditions. At the same time, we have at disposal only six conditions of the kind (3). In order to complete the problem, we set additionally one more parametric condition (the value of one from among the functions $\lambda_i(x), \nu_i(x), \varphi_i(x), \varphi'_i(x), \sigma_i(x)$, or $\sigma'_i(x)$) at the point $x = 1$). Let us set for example

$$\varphi_i(1) = \varphi_s, \quad \varphi_s - \text{parameter.} \quad (4)$$

Obviously, the solution $y_i \equiv \{\lambda_i, \nu_i, \varphi_i, \sigma_i, \mu\}$ in the internal domain $x \in [0, 1]$ depends not only on the variable $x$, but it is a function of the three parameters $R_s, \Omega, \varphi_s$ as well.

Concerning the solution $y_e \equiv \{\lambda_e, \nu_e, \varphi_e, \sigma_e\}$ in the external domain $x > 1$ six boundary conditions are indispensable for the solution of the equations (1). At the same time, only the three boundary conditions (3) are given. Let us consider that the solution in the internal domain

Concerning the solution (1) we assign to two arbitrary functions from among $\lambda(x), \nu(x), \varphi'(x), \sigma(x)$, and $\sigma'(x)$, for example $\lambda_e(1) = \lambda_1(1); \nu_e(1) = \nu_s; \sigma_e(1) = \sigma_1(1).

Let us suppose that the internal and external solutions are known. Generally speaking, for given arbitrary values of the parameters $R_s, \Omega$, and $\varphi_s$ the continuity conditions with respect to the functions $\nu(x), \varphi'(x)$, and $\sigma'(x)$ at the point $x = 1$ are not satisfied. We choose the parameters $R_s, \Omega$, and $\varphi_s$ in such manner that the continuity conditions for the mentioned functions to be held,

$$\nu_s(1, R_s, \Omega, \varphi_s) - \nu_i(1, R_s, \Omega, \varphi_s) = 0,$$

$$\varphi'_s(1, R_s, \Omega, \varphi_s) - \varphi'_i(1, R_s, \Omega, \varphi_s) = 0,$$

$$\sigma'_s(1, R_s, \Omega, \varphi_s) - \sigma'_i(1, R_s, \Omega, \varphi_s) = 0. \quad (5)$$

These conditions should be interpreted as three nonlinear algebraic equations in regard to the unknown quantities $R_s, \Omega$ and $\varphi_s$.

So, we have to solve the nonlinear BVP (1) - (3) in conjunction with the algebraic system (5). The traditional technology for solving such problems, based on the methods like shooting (3), has some well known disadvantages.

In this paper we use the Continuous Analogue of Newton Method (3) for solving the above nonlinear spectral problem which proposes a common treatment of both differential and algebraic problems. Detailed description of the algorithm can be found in (3).

3 Some numerical results

For the purpose of illustrating we will shortly discuss some results obtained from numerical experiments.

In the present article we consider concrete scalar-tensor model with functions (see Section 1)

$$A(\varphi) = \exp \left( \frac{\varphi}{\sqrt{\phi}} \right), \quad V(\varphi) = \frac{3}{2} [1 - A^2(\varphi)]^2, \quad W(\sigma) = -\frac{1}{2} \left( \sigma^2 + \frac{1}{2} \Lambda \sigma^4 \right),$$

$$f(\mu) = \frac{1}{8} \left[ (2 \mu - 3) \sqrt{\mu + \mu^2} + 3 \ln \left( \sqrt{\mu + \sqrt{1 + \mu}} \right) \right],$$

$$g(\mu) = \frac{1}{8} \left[ (6 \mu + 3) \sqrt{\mu + \mu^2} - 3 \ln \left( \sqrt{\mu + \sqrt{1 + \mu}} \right) \right].$$

The quantities $b$ and $\Lambda$ are given parameters. The functions $f(\mu)$ and $g(\mu)$ represent in parametric form the equation of state of the non-interacting neutron gas, while the function $W(\sigma)$ describes the boson field with quadratic self-interaction.

The behaviour of the calculated eigenfunctions $\sigma(x), \varphi(x), \nu(x)$, and $\mu(x)$ is typical for wide range of the parameters and was discussed in detail in (3).

From a physical point of view, it is important to know the mass of the boson-fermion star and the total number of particles (bosons and fermions) making up the star.
Parameters: $\Lambda = 0$, $\gamma = 0.1$, $b = 1$, $\sigma_c = 0.002$

Figure 1: The star mass $M$ and the rest fermion mass $M_{RF}$ as functions of the central value $\mu_c$.

The dimensionless star mass can be calculated by the formula:

$$M = \int_0^\infty r^2 \left[ \frac{B}{T_0^0} + \frac{F}{T_0^0} + \exp(-\lambda) \left( \frac{d\varphi}{dr} \right)^2 + \frac{\gamma^2}{2} V(\varphi) \right] dr.$$

The dimensionless rest mass of the bosons (total number of bosons times the boson mass) and the dimensionless rest mass of the fermions are given by:

$$M_{RB} = \Omega \int_0^\infty r^2 A^2(\varphi) \exp\left( \frac{\lambda - \nu}{2} \right) \sigma^2 dr, \quad M_{RF} = b \int_0^\infty r^2 A^3(\varphi) \exp\left( \frac{\lambda}{2} \right) n(\mu) dr,$$

where $n(\mu)$ is the density of the fermions. In the case we have $n(\mu) = \mu^2(x)$.

The dependencies of the star mass $M$ and the rest mass of fermions $M_{RF}$ on the central value $\mu_c$ of the function $\mu(x)$ are shown in the configuration diagram on Fig. 1 for $\Lambda = 0$, $\gamma = 0.1$, $b = 1$ and $\sigma_c = 0.002$. It should be pointed that for such small central value $\sigma_c$ we actually have a “pure” fermionic star. On the figure, it is seen that from small enough values of $\mu_c$, in the vicinity of the the peak the rest mass is greater than the total mass of the star, which means that the star is potentially stable.

On Fig. 2 the binding energy of the star $E_b = M - M_{RB} - M_{RF}$ is drawn against the rest mass of fermions $M_{RF}$ for $\Lambda = 0$, $\gamma = 0.1$, $b = 1$ and $\sigma_c = 0.002$. Fig. 2 is actually a bifurcation diagram. By increasing the central value of the function $\mu(x)$ a cusp appears. The presence of the cusp shows that the stability of the star changes - one perturbation mode develops instability. Beyond the cusp, the star is unstable and may collapse, eventually forming a black hole. The corresponding physical results for pure boson stars are considered in our recent paper [3].

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Figure 2: The binding energy $E$ versus the rest fermion mass $M_{RF}$.

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