The main aim of this paper is to improve some known estimates for the classical Kantorovich operators. We obtain a quantitative Voronovskaya-type result in terms of the second moduli of continuity, which improves some previous results. In order to explain the nonmultiplicativity of the Kantorovich operators, we present a Chebyshev–Grüss inequality. Two Grüss–Voronovskaya theorems for Kantorovich operators are also considered.

1. Introduction

In 1930 L. V. Kantorovich [11] introduced a significant modification of the classical Bernstein operators given by

\[ K_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) \, dt. \]

Here, \( n \geq 1, \ f \in L_1[0,1], \ x \in [0,1], \) and

\[ p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n, \]

\[ p_{n,k} \equiv 0, \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n. \]

These mappings are relevant because they provide a constructive tool for approximating any function in \( L_p[0,1], \) \( 1 \leq p < \infty, \) in the \( L_p \)-norm. For \( p = \infty, \ C[0,1] \) should be used instead of \( L_\infty[0,1]. \)

Since that time, these classical Kantorovich operators attract a lot of attention but the results obtained in this field are somewhat scattered in the literature. This situation is shared by some other relevant Bernstein-type variations: Durrmeyer, genuine Bernstein–Durrmeyer, and (the last but not least) variation-diminishing Schoenberg splines.

In the present note, we first collect and improve some known estimates by giving a quite precise inequality for \( f \in C^r[0,1], \ r \in \mathbb{N} \cup \{0\}, \) a new Voronovskaya result in terms of \( \omega_2, \) and a Chebyshev–Grüss inequality that gives an explanation of their nonmultiplicativity. The last part of the paper deals with two Grüss–Voronovskaya theorems for the Kantorovich operators.

In the present paper, most estimates are presented via the moduli of smoothness of higher order. In the background, without explicitly mentioning, we always have the \( K \)-functional technique. In this sense, we were very much influenced by the work by Zygmund (see, e.g., [16]), a difficultly accessible conference contribution by Peetre [12], and also by the Dzyadyk’s book [4].

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2. Some Previous Results

In this section, we collect some results presented earlier. A strong general result was obtained by the second author and X. Zhou [10] in 1995.

Let \( \varphi(x) = \sqrt{x(1-x)} \) and let \( P(D) \) be the differential operator given by

\[
P(D)f := (\varphi^2 f')', \quad f \in C^2[0, 1].
\]

For \( f \in L_p[0, 1], \) \( 1 \leq p \leq \infty \), the functional \( K(f, t)_p \) is defined as follows:

\[
K(f, t)_p := \inf \{ \|f - g\|_p + t^2\|P(D)g\|_p : g \in C^2[0, 1] \}.
\]

By using this functional the following theorem was proved in [10]:

**Theorem 2.1.** There exists an absolute positive constant \( C \) such that, for all \( f \in L_p[0, 1], \) \( 1 \leq p \leq \infty \), the following inequalities are true:

\[
C^{-1}K(f, n^{-1/2})_p \leq \|f - K_n f\|_p \leq CK(f, n^{-1/2})_p.
\]

In addition, in order to characterize the \( K \)-functional used in Theorem 2.1, the following result was presented in [10]:

**Theorem 2.2.** The following relations are true:

\[
K(f, t)_p \sim \omega^2_\varphi(f, t)_p + t^2 E_0(f)_p, \quad 1 < p < \infty,
\]

and

\[
K(f, t)_\infty \sim \omega^2_\varphi(f, t)_\infty + \omega(f, t^2)_\infty.
\]

Here, \( \omega(f, t)_p \) is the classical modulus, \( \omega^2_\varphi(f, t)_\infty \) denotes the second-order modulus of smoothness with weight function \( \varphi \), and \( E_0(f)_p \) is the best-approximation constant of \( f \) defined by

\[
E_0(f)_p = \inf_c \|f - c\|_p.
\]

Moreover, all quantities with subscripts \( \infty \) are taken with respect to the uniform norm in \( C[0, 1] \). The following theorem by Păltănea [13] is the key to give a more explicit result in terms of the classical moduli for continuous functions. (See [8] for details.)

**Theorem 2.3 [13].** If \( L : C[0, 1] \to C[0, 1] \) is a positive linear operator, then, for \( f \in C[0, 1], \) \( x \in [0, 1] \), and each \( 0 < h \leq \frac{1}{2} \), the following inequality holds:

\[
|L(f; x) - f(x)| \leq |L(e_0; x) - 1| |f(x)| + \frac{1}{h} |L(e_1 - x; x)| \omega(f; h)
\]

\[
+ \left[ (Le_0)(x) + \frac{1}{2h^2} L((e_1 - x)^2; x) \right] \omega_2(f; h).
\]

The condition \( h \leq 1/2 \) can be eliminated for operators \( L \) reproducing linear functions.
Theorem 2.4. For all \( f \in C[0, 1] \) and all \( n \geq 4 \),

\[
\|K_n f - f\|_\infty \leq \frac{1}{2\sqrt{n}} \omega_1 \left( f; \frac{1}{\sqrt{n}} \right) + \frac{9}{8} \omega_2 \left( f; \frac{1}{\sqrt{n}} \right).
\]

This result can be extended to simultaneous approximation (see again [8]).

Theorem 2.5. Let \( r \in \mathbb{N}_0, n \geq 4 \), and \( f \in C^r[0, 1] \). Then

\[
\|D^r K_n f - D^r f\|_\infty \leq \frac{(r + 1)r}{2n} \|D^r f\|_\infty + \frac{r + 1}{2\sqrt{n}} \left( D^r f; \frac{1}{\sqrt{n}} \right) + \frac{9}{8} \omega_2 \left( D^r f; \frac{1}{\sqrt{n}} \right).
\]

3. A Quantitative Voronovskaya Result

This section has its predecessor in a hardly known booklet of Videnskij in which one can find a quantitative version of the well-known Voronovskaya theorem for the classical Bernstein operators (see [15]). This estimate was generalized and improved in [9]. The application to Kantorovich operators was made in [8]. Here, we improve it as follows:

Theorem 3.1. For \( n \geq 1 \) and \( f \in C^2[0, 1] \),

\[
\|n (K_n f - f) - \frac{1}{2} (X f')'\|_\infty \leq \frac{2}{3(n + 1)} \left( \frac{3}{4} \|f'\|_\infty + \|f''\|_\infty \right)
\]

\[
+ \frac{9}{32} \left\{ \frac{2}{\sqrt{n + 1}} \omega_1 \left( f''; \frac{1}{\sqrt{n + 1}} \right) + \omega_2 \left( f''; \frac{1}{\sqrt{n + 1}} \right) \right\},
\]

where \( X = x(1 - x) \) and \( X' = 1 - 2x \), \( x \in [0, 1] \).

Proof. It follows from [9] (Theorem 3) that

\[
K_n (f; x) - f(x) - K_n (t - x; x) f'(x) - \frac{1}{2} K_n ((e_1 - x)^2; x) f''(x)
\]

\[
\leq K_n ((e_1 - x)^2; x) \left\{ \frac{|K_n((e_1 - x)^3; x)|}{K_n((e_1 - x)^2; x)} \frac{5}{6h} \omega_1(f''; h) + \left( \frac{3}{4} + \frac{K_n((e_1 - x)^4; x)}{K_n((e_1 - x)^2; x)} \frac{1}{16h^2} \right) \omega_2(f''; h) \right\}.
\]

By using the central moments for the Kantorovich operators (up to the fourth order), namely,

\[
K_n (t - x; x) = \frac{1 - 2x}{2(n + 1)},
\]

\[
K_n ((t - x)^2; x) = \frac{1}{(n + 1)^2} \left\{ x(1 - x)(n - 1) + \frac{1}{3} \right\},
\]

\[
K_n ((t - x)^3; x) = \frac{1 - 2x}{4(n + 1)^3} \left\{ 10x(1 - x)n + 2x^2 - 2x + 1 \right\},
\]
\[ K_n((t - x)^4; x) = \frac{1}{(n + 1)^4} \left\{ 3x^2(1 - x)^2n^2 + 5x(1 - x)(1 - 2x)^2n + x^4 - 2x^3 + 2x^2 - x + \frac{1}{5} \right\}, \]

we get

\[ \frac{|K_n((t - x)^3; x)|}{K_n((t - x)^2; x)} \leq \frac{5}{2(n + 1)}, \quad \frac{|K_n((t - x)^4; x)|}{K_n((t - x)^2; x)} \leq \frac{3(n + 2)}{(n + 1)^2}. \]

Therefore, the following inequality holds:

\[ \left| K_n(f; x) - f(x) - \frac{1 - 2x}{2(n + 1)} f'(x) - \frac{1}{2} \left[ \frac{x(1 - x)(n - 1)}{(n + 1)^2} + \frac{1}{3(n + 1)^2} \right] f''(x) \right| \]

\[ \leq \left[ x(1 - x) \frac{n - 1}{(n + 1)^2} + \frac{1}{3(n + 1)^2} \right] \left\{ \frac{25}{12h(n + 1)} \omega_1(f''; h) + \left( \frac{3}{4} + \frac{3(n + 2)}{16h^2(n + 1)^2} \right) \omega_2(f''; h) \right\} \]

and, for \( h = \frac{1}{\sqrt{n + 1}} \), we obtain, after multiplying both sides by \( n \),

\[ \left| n[K_n(f; x) - f(x)] - \frac{n}{n + 1} \left( \frac{1}{2} - x \right) f'(x) \right| \]

\[ \leq \left[ x(1 - x) \frac{n(n - 1)}{(n + 1)^2} + \frac{n}{3(n + 1)^2} \right] f''(x) \]

\[ \leq \frac{9}{32} \left\{ \frac{2}{\sqrt{n + 1}} \omega_1(f''; \frac{1}{\sqrt{n + 1}}) + \omega_2(f''; \frac{1}{\sqrt{n + 1}}) \right\}. \]

We can write

\[ \left| n[K_n(f; x) - f(x)] - \frac{1 - 2x}{2} f'(x) - \frac{1}{2} x(1 - x)f''(x) \right| \]

\[ \leq \left| n[K_n(f; x) - f(x)] - \frac{n}{n + 1} \left( \frac{1}{2} - x \right) f'(x) \right| \]

\[ - \frac{1}{2} \left[ x(1 - x) \frac{n(n - 1)}{(n + 1)^2} + \frac{n}{3(n + 1)^2} \right] f''(x) \]

\[ + \left| \frac{1 - 2x}{2} \frac{1}{n + 1} f'(x) + \frac{1}{2} x(1 - x) \frac{3n + 1}{(n + 1)^2} f''(x) - \frac{n}{6(n + 1)^2} f''(x) \right| \]

\[ \leq \frac{9}{32} \left\{ \frac{2}{\sqrt{n + 1}} \omega_1(f''; \frac{1}{\sqrt{n + 1}}) + \omega_2(f''; \frac{1}{\sqrt{n + 1}}) \right\} \]

\[ + \frac{2}{3(n + 1)} \left( \frac{3}{4} \|f'\|_{\infty} + \|f''\|_{\infty} \right). \]
4. Chebyshev–Grüss Inequality for Kantorovich Operators

In 2011, Acu, Gonska, and Raşa [1] published the following Grüss-type inequality for positive linear operators reproducing constant functions (here, we present the improved form of this inequality established by Rusu in [14]):

**Theorem 4.1.** Let $H : C[a, b] \to C[a, b]$ be positive and linear. Assume that they satisfy $He_0 = e_0$. Also let

$$D(f, g; x) := H(fg; x) - H(f; x)H(g; x).$$

Then, for $f, g \in C[a, b]$ and $x \in [a, b]$ fixed, the following inequality is true:

$$|D(f, g; x)| \leq \frac{1}{4} \tilde{\omega} \left( f; 2\sqrt{H((e_1 - x)^2; x)} \right) \tilde{\omega} \left( g; 2\sqrt{H((e_1 - x)^2; x)} \right).$$

Here, $\tilde{\omega}$ is the least concave majorant of the first-order modulus $\omega_1$ given by

$$\tilde{\omega}(f; t) = \sup \left\{ \frac{(t - x)\omega_1(f; y) + (y - t)\omega_1(f; x)}{y - x} : 0 \leq x \leq t \leq y \leq b - a, x \neq y \right\}.$$

**Remark 4.1.** For an accessible proof of the equality between $\tilde{\omega}$ and a certain $K$-functional used in the proof of the theorem presented above, see [13].

Hence, the nonmultiplicativity of the Kantorovich operators can be explained as in the following theorem:

**Theorem 4.2.** For the classical Kantorovich operators $K_n : C[0, 1] \to C[0, 1]$, the following uniform inequality is true:

$$\|K_n(fg) - K_nfK_ng\|_{\infty} \leq \frac{1}{4} \tilde{\omega} \left( f; 2\sqrt{\frac{1}{2(n + 1)}} \right) \tilde{\omega} \left( g; 2\sqrt{\frac{1}{2(n + 1)}} \right), \quad n \geq 1,$$

(2)

for all $f, g \in C[0, 1]$.

**Proof.** The most precise upper bound is obtained if we use the exact representation

$$K_n \left( (t - x)^2; x \right) = \frac{1}{(n + 1)^2} \left\{ (n - 1)x(1 - x) + \frac{1}{3} \right\}.$$

Close to $x = 0, 1$, this shows the familiar endpoint improvement. For the sake of brevity, we apply the estimate

$$K_n \left( (t - x)^2; x \right) \leq \frac{1}{2(n + 1)}.$$

5. Grüss–Voronovskaya Theorems

The first Grüss–Voronovskaya theorem for the classical Bernstein operators was obtained by Gal and Gonska [5]. In Theorem 2.1 of this paper, a quantitative form was given (see also Theorem 2.5 in [5]). The other
examples presented in [5] deal with the operators reproducing linear functions; this is not the case for the Kantorovich mappings. It has been recently shown in [2] that the limit for $K_n$ is the same as in the Bernstein case, namely,

$$f'(x)g'(x)x(1-x) \text{ for } f, g \in C^2[0,1].$$

Our first quantitative version is given in the following theorem:

**Theorem 5.1.** Let $f, g \in C^2[0,1]$. Then, for each $x \in [0,1]$,

$$
\|n [K_n(fg) - K_n f \cdot K_n g] - Xf'g'\|_\infty = \begin{cases} 
  o(1), & f, g \in C^2[0,1], \\
  O\left(\frac{1}{\sqrt{n}}\right), & f, g \in C^3[0,1], \\
  O\left(\frac{1}{n}\right), & f, g \in C^4[0,1].
\end{cases}
$$

**Proof.** We proceed as in [5] by creating the first three Voronovskaya-type expressions from the difference in question plus the remaining quantities. Recall that the Voronovskaya limit for the Kantorovich operators is

$$\frac{1}{2}(Xf')' = \frac{1}{2}Xf''(x) + \frac{1}{2}Xf'(x),$$

where $X := x(1-x)$ and, hence, $X' = 1 - 2x$.

For $f, g \in C^2[0,1]$, we have

$$
K_n(fg; x) - K_n(f; x)K_n(g; x) - \frac{1}{n}Xf'(x)g'(x)
= K_n(fg; x) - (fg)(x) - \frac{1}{2n} (X(fg)')'
- f(x)\left[K_n(g; x) - g(x) - \frac{1}{2n} (Xg')'\right] - g(x)\left[K_n(f; x) - f(x) - \frac{1}{2n} (Xf')'\right]
+ [g(x) - K_n(g; x)][K_n(f; x) - f(x)]
- K_n(f; x)K_n(g; x) - \frac{1}{n}Xf'g' + (fg)(x) + \frac{1}{2n} (X(fg)')'
+ f(x)\left[K_n(g; x) - g(x) - \frac{1}{2n} (Xg')'\right] + g(x)\left[K_n(f; x) - f(x) - \frac{1}{2n} (Xf')'\right]
- [g(x) - K_n(g; x)][K_n(f; x) - f(x)].
$$

The first three lines are estimated in what follows. First, we show that the sum of the last three lines is equal to 0.
We now omit the argument $x$. Thus, we get

\[-K_n f \cdot K_n g - \frac{1}{n} X f' g' + fg + \frac{1}{2n} (X'(fg)' + X(fg)'') + fK_n g - f g - \frac{1}{2n} f(X'g' + Xg'') \]

\[\cdot gK_n f - \frac{1}{2n} g(X'f' + Xf'') - [g - K_n g][K_n f - f] - fg - \frac{1}{2n} g(X'f' + Xf'') - [g - K_n g][K_n f - f]\]

\[= -K_n f \cdot K_n g - \frac{1}{n} X f' g' + fg + \frac{1}{2n} (X'(fg)' + X(fg)'') + \frac{1}{2n} X (f''g + 2f'g' + fg'') + fK_n g - f g - \frac{1}{2n} (gX'f' + gXf'') - gK_n f + K_n g \cdot K_n f + fg - fK_n g = 0.\]

For the first two lines, we use the Voronovskaya estimate given earlier, namely,

\[
\left\| n (K_n h - h) - \frac{1}{2} (Xh')' \right\|_\infty \leq \frac{2}{3(n + 1)} \left( \frac{3}{4} \| h' \|_\infty + \| h'' \|_\infty \right)
\]

\[+ \frac{9}{32} \left\{ \frac{2}{\sqrt{n} + 1} \omega_1 \left( h'' ; \frac{1}{\sqrt{n} + 1} \right) + \omega_2 \left( h'' ; \frac{1}{\sqrt{n} + 1} \right) \right\}
\]

\[=: U(h, n).
\]

for $h \in C^2[0, 1]$. For the third line, we use Theorem 2.4 showing that

\[
\| K_n h - h \|_\infty \leq \frac{1}{2n} \| h' \|_\infty + \frac{9}{8n} \| h'' \|_\infty = \mathcal{O} \left( \frac{1}{n} \right)
\]

for $h \in C^2[0, 1]$. Gathering these inequalities, we find

\[
\left\| n [K_n (fg) - K_n f \cdot K_n g] - Xf' g' \right\|_\infty
\]

\[\leq U(fg, n) + \| f \|_\infty U(g, n) + \| g \|_\infty U(f, n) + \mathcal{O} \left( \frac{1}{n} \right)
\]

\[= \begin{cases} 
  o(1), & f, g \in C^2[0, 1], \\
  \mathcal{O} \left( \frac{1}{\sqrt{n}} \right), & f, g \in C^3[0, 1], \\
  \mathcal{O} \left( \frac{1}{n} \right), & f, g \in C^4[0, 1].
\end{cases}
\]
In what follows, we give a Grüss–Voronovskaya-type theorem for the case where \( f \) and \( g \) are only in \( C^1[0,1] \).

**Theorem 5.2.** Let \( f, g \in C^1[0,1] \) and \( n \geq 1 \). Then there is a constant \( C \) independent of \( n, f, g, \) and \( x \), such that

\[
\left\| K_n(fg) - K_n(f) \cdot K_n(g) - \frac{X}{n} f'g' \right\|_{\infty} \leq \frac{C}{n} \left\{ \omega_3\left(f', n^{-\frac{1}{6}}\right) \omega_3\left(g', n^{-\frac{1}{6}}\right) + \|f'||\infty \omega_3\left(g', n^{-\frac{1}{6}}\right) + \|g'||\infty \omega_3\left(f', n^{-\frac{1}{6}}\right) + \max \left\{ \frac{\|f'||\infty}{n^2} \omega_3\left(f', n^{-\frac{1}{6}}\right) + \frac{\|g'||\infty}{n^2} \omega_3\left(g', n^{-\frac{1}{6}}\right) \right\} \right\}.
\]

**Proof.** Let

\[
E_n(f, g; x) = K_n(fg; x) - K_n(f; x)K_n(g; x) - \frac{x(1-x)}{n} f'(x)g'(x).
\]

By \( C \) we denote a constant independent of \( n, f, g, \) and \( x \) that may change its values in the course of the proof. For fixed \( f, g \in C^1[0,1] \) and arbitrary \( u, v \in C^4[0,1] \), we obtain

\[
|E_n(f, g; x)| = |E_n(f - u + u, g - v + v; x)| \\
\leq |E_n(f - u, g - v; x)| + |E_n(u, g - v; x)| + |E_n(f - u, v; x)| + |E_n(u, v; x)|.
\]

Let \( h(x) = x, x \in [0,1] \). Applying \([1]\) (Theorem 4), we conclude that there exist \( \eta, \theta \in [0,1] \) such that

\[
K_n(fg; x) - K_n(f; x)K_n(g; x) = f'(\eta)g'(\theta)\left[K_n(h^2; x) - (K_n(h; x))^2\right] = f'(\eta)g'(\theta)\left\{ x(1-x) \left( \frac{n}{(n+1)^2} + \frac{1}{12(n+1)^2} \right) \right\}.
\]

From (3) and (5), we get

\[
|nE_n(f, g; x)| \leq \left[ x(1-x) \left( \frac{n^2}{(n+1)^2} + \frac{n}{12(n+1)^2} + x(1-x) \right) \right] \|f'||\infty \|g'||\infty \\
\leq 2 \left[ x(1-x) + \frac{1}{24(n+1)} \right] \|f'||\infty \|g'||\infty.
\]

Using Theorem 3.1, for \( f \in C^4[0,1] \), we find

\[
\left| n \left[ K_n(f; x) - f(x) \right] - \frac{1}{2} (Xf')'(x) \right| \leq C \frac{1}{n} \left( \|f'||\infty + \|f''||\infty + \|f'''||\infty + \|f^{(4)}||\infty \right).
\]

However, for \( f \in C^n[a, b] \), \( n \in \mathbb{N} \), we have (see \([6]\), Remark 2.15)

\[
\max_{0 \leq k \leq n} \left\{ \|f^{(k)}\| \right\} \leq C \max \left\{ \|f||\infty, \|f^{(n)}||\infty \right\}.
\]
Therefore,
\[
\left| n \left[ K_n(f; x) - f(x) \right] - \frac{1}{2} \left( X f' \right)'(x) \right| \leq \frac{C}{n} \max \left\{ \| f' \|_{\infty}, \| f^{(4)} \|_{\infty} \right\}.
\] (7)

For \( u, v \in C^3[0, 1] \) by using the same decomposition as in proof of Theorem 5.1, relation (7), and Theorem 2.4, we get
\[
|E_n(u, v; x)| \leq \left| K_n(uv; x) - (uv)(x) - \frac{1}{2n} (X(uv)')' \right|
+ |u(x)| \left| K_n(v; x) - v(x) - \frac{1}{2n} (Xv)'ight|
+ |v(x)| \left| K_n(u; x) - u(x) - \frac{1}{2n} (Xu)'ight|
+ |v(x) - K_n(v; x)| |K_n(u; x) - u(x)|
\leq \frac{C}{n^2} \max \left\{ \| u' \|_{\infty}, \| u^{(4)} \|_{\infty} \right\} \max \left\{ \| v' \|_{\infty}, \| v^{(4)} \|_{\infty} \right\}.
\] (8)

According to relations (4), (6) and (8), we obtain
\[
|E_n(f, g; x)| \leq \frac{2}{n} \left[ x(1 - x) + \frac{1}{24(n + 1)} \right] \left\{ \| (f - u)' \|_{\infty} (g - v)' \|_{\infty} + \| u' \|_{\infty} (g - v)' \|_{\infty}
+ \| (f - u)' \|_{\infty} \| v' \|_{\infty} \right\} + \frac{C}{n^2} \max \left\{ \| u' \|_{\infty}, \| u^{(4)} \|_{\infty} \right\} \max \left\{ \| v' \|_{\infty}, \| v^{(4)} \|_{\infty} \right\}.
\]

Using [7] (Lemma 3.1) for \( r = 1, s = 2, f_{h,3} = u \) and \( g_{h,3} = v \), for all \( h \in (0, 1] \) and \( n \in \mathbb{N} \), we find
\[
|E_n(f, g; x)| \leq \frac{C}{n} \left\{ \omega_3(f', h) \omega_3(g', h) + \frac{1}{h} \omega_1(f, h) \omega_3(g', h) + \frac{1}{h} \omega_1(g, h) \omega_3(f', h) \right\}
+ \frac{C}{n^2} \max \left\{ \frac{1}{h} \omega_1(f, h), \frac{1}{h^3} \omega_3(f', h) \right\} \max \left\{ \frac{1}{h} \omega_1(g, h), \frac{1}{h^3} \omega_3(g', h) \right\}
\leq \frac{C}{n} \left\{ \omega_3(f', h) \omega_3(g', h) + \| f' \|_{\infty} \omega_3(g', h) + \| g' \|_{\infty} \omega_3(f', h) \right\}
+ \frac{C}{n^2} \max \left\{ \| f' \|_{\infty}, \frac{1}{h^3} \omega_3(f', h) \right\} \max \left\{ \| g' \|_{\infty}, \frac{1}{h^3} \omega_3(g', h) \right\}.
\]

Choosing \( h = n^{-\frac{1}{6}} \), we obtain
\[
|E_n(f, g; x)| \leq \frac{C}{n} \left\{ \omega_3 \left( f', n^{-\frac{1}{6}} \right) \omega_3 \left( g', n^{-\frac{1}{6}} \right) \right\}.
\]
\[ + \left\| f' \right\|_{\infty} \omega_3 \left( f', n^{-\frac{1}{\alpha}} \right) + \left\| g' \right\|_{\infty} \omega_3 \left( g', n^{-\frac{1}{\delta}} \right) \]

\[ + \max \left\{ \frac{\left\| f' \right\|_{\infty}}{n^{\frac{1}{2}}}, \omega_3 \left( f', n^{-\frac{1}{\alpha}} \right) \right\} \max \left\{ \frac{\left\| g' \right\|_{\infty}}{n^{\frac{1}{2}}}, \omega_3 \left( g', n^{-\frac{1}{\delta}} \right) \right\}. \]

This implies the theorem.

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