A note on families of hyperelliptic curves

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Abstract. We give a stack-theoretic proof for some results on families of hyperelliptic curves.

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1. Introduction. Let $k$ be a field and $g$ be an integer such that $\text{char}(k) \neq 2$ and $g \geq 2$. All schemes that we consider are of finite type over $k$.

Any family $\mathcal{F} \to S$ of smooth genus $g$ hyperelliptic curves is a double cover of a conic bundle $\mathcal{C} \to S$ branched at a Cartier divisor $D$ finite and étale of degree $2g+2$ over the base $S$ (see [11]). Conversely, starting with a family $(\mathcal{C} \to S, D)$ as above, one can ask what are the obstructions to the existence of a corresponding family of hyperelliptic curves $\mathcal{F} \to S$ and how many such families does there exist.

The classical theory of double covers immediately gives the answer to this question in terms of the functions on $\mathcal{C}$ and its Picard group $\text{Pic}(\mathcal{C})$.

In Theorem 3.1 we give a different answer to these questions in terms of the geometry of the base $S$. Our proof is completely stack-theoretic and uses the fact that the stack $\mathcal{H}_g$ of hyperelliptic curves is a $\mu_2$-gerbe over the stack $\mathcal{D}_{2g+2}$ of conic bundles endowed with an effective Cartier divisor finite and étale of degree $2g+2$, and the fact that both these stacks have an explicit description as quotient stacks (see [2] and [6]).

As an application of the Theorem 3.1, we give a proof of two classical facts on families of hyperelliptic curves.

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In Proposition 4.7, we prove that there exists a tautological family of hyperelliptic curves over a non-empty open subset of the coarse moduli space \( H^g \) if and only if \( g \) is odd. Moreover, we give a different proof of [6, Theorem 3.12], stating that such a family never exists over the open subset \( H^g_0 \) corresponding to curves without extra-automorphisms apart from the hyperelliptic involution (this is in contrast with the fact that a tautological family exists over the open subset \( M^g_0 \subset M^g \) of general curves of genus \( g \geq 3 \) without automorphisms). From this result and the rationality of \( H^g \) (see [4] and [9]), we deduce that the stack \( H^g \) is rational if and only if \( g \) is odd (Corollary 4.9).

In Proposition 4.11, we give a different (and for us simpler) proof of a result of Mestrano–Ramanan ([12]), stating that a global \( g_2^1 \) for a family of hyperelliptic curves exists only in the case \( g \) even.

2. Notations. By \( H^g \), \( D_{2g+2} \), and \( H_g \) denote the stack of families of genus \( g \) smooth hyperelliptic curves, the stack of conic bundles together with an effective Cartier divisor finite and \( \acute{e}tale \) of degree \( 2g+2 \) over the base, and the common coarse moduli space of the two stacks above, respectively.

Recall that given a \( k \)-scheme \( X \) and a \( k \)-group scheme \( G \) acting on \( X \), the quotient stack, denoted as \( \left[ X/G \right] \), is the category fibered in groupoids over the category of \( k \)-schemes, whose fiber over a \( k \)-scheme \( S \) is the groupoid whose objects are \( G \)-torsors \( E \to S \) endowed with a \( G \)-equivariant morphism \( E \to X \) and whose arrows are isomorphisms of the above objects. In the particular case where \( X = \text{Spec}(k) \), we get the classifying stack of \( G \), denoted with \( BG \), whose fiber over \( S \) is the groupoid of \( G \)-torsors \( E \to S \).

The stacks \( H^g \) and \( D_{2g+2} \) admit the following description as quotient stacks (see [2, Corollary 4.7] and [6, Proposition 3.4]):

\[
\begin{align*}
H^g &= \left[ \mathbb{A}_{sm}(2,2g+2)/\left( GL_2/\mu_{g+1} \right) \right], \\
D_{2g+2} &= \left[ \mathbb{A}_{sm}(2,2g+2)/\left( GL_2/PGL_2 \right) \right] = \left[ \mathbb{B}_{sm}(2,2g+2)/PGL_2 \right], \\
H_g &= \mathbb{B}_{sm}(2,2g+2)/PGL_2,
\end{align*}
\]

where \( A_{sm}(2,2g+2) \) is the linear space of degree \( 2g+2 \) binary forms without multiple roots, \( B_{sm}(2,2g+2) \) is the projectivization of \( A_{sm}(2,2g+2) \), and \( GL_2 \) acts on \( A_{sm}(2,2g+2) \) by the formula \( A \cdot f(x) = f(A^{-1} \cdot x) \).

We briefly recall the notion of the rigidification of a stack (see [1, Section 5.1]). Let \( X \) be an algebraic stack over \( k \) (even though everything can be extended to a general base scheme), \( H \) a commutative \( k \)-group scheme and assume that for every object \( \xi \in X(T) \) there is an embedding \( H_T \subset \text{Aut}_T(\xi) \) compatible with pullbacks. Then there is an algebraic stack \( X^H \) (called the rigidification of \( X \) along \( H \)) together with a smooth morphism of algebraic stacks \( \phi : X \to X^H \) uniquely determined by the properties: