Unimodal patterns appearing in the Kolmogorov flows at large Reynolds numbers

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Abstract
We study stability and bifurcation of stationary and time-periodic solutions of Kolmogorov’s problem for the Navier–Stokes equations in two-dimensional (2D) flat tori. Specifically we look for a unimodal solution, which is characterized by having a large, topologically simple pattern of streamlines. We present a conjecture that such simple patterns emerge in steady-states or time-periodic solutions at large Reynolds numbers, no matter what the external force may be. We confirm this conjecture by some numerical experiments. Thus the well-noted fact that a large structure appears in 2D large Reynolds number flows is reinforced in another form.

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(Some figures may appear in colour only in the online journal)
1. Introduction

We consider Kolmogorov’s problem for two-dimensional (2D) incompressible viscous fluid driven by various external forces, and compute bifurcating solutions at very large Reynolds numbers. In the previous paper [16] we set forth a conjecture that at least one unimodal steady-state exists no matter what the external force may be, if the Reynolds number is large enough. By unimodal we mean that the streamline pattern of the solution is topologically simple in the sense that one and only one pair of vortices exists in the whole domain occupied by the fluid. Examples of such flows will soon appear in what follows. Since their emergence is independent of the external driving force, this phenomenon is universal if it is confirmed to exist.

Since the conjecture in [16] was derived from a limited number of numerical experiments, it would be necessary to examine more cases for confirmation. A paper by Sasaki and others [30] considers an analogous problem for a rotating 2D sphere. They find a symmetry increasing phenomenon, which is compatible with the results in [16]. For instance, they obtain a branch of steady-states, where three pairs of vortices become two pairs as the Reynolds number is increased. But they did not find any unimodal steady-state in their numerical experiment. This made us wonder whether unimodal steady-states do not appear as universally as was claimed in [16]. Thus comes a motivation for the present work to extend our experiments to new driving forces and to improve and modify (if necessary) the original unimodal conjecture.

With this motivation, we carried out more computer experiments in the present paper. Our experiments below put some new insights and we are forced to modify our conjecture. However, a basic philosophy of the conjecture is maintained. The key idea is that we should consider not only stationary solutions but also time-periodic solutions: even if we cannot find a unimodal solution in a continuum of stationary solutions, there may exist a time-periodic solution which has a unimodal pattern. In this regard, both [16] and [30] are incomplete, since they computed steady-states only and ignored time-periodic solutions. Also, there are cases where a unimodal solution exists in a branch separated from the continuum of steady-states which contains the basic solution. Such a separated continuum of solutions was not considered in either of [16, 30]. Some of earlier results are announced in [18] together with the new version of conjecture. The situation will be fully explained below with more details.

We recently learned of a paper [11], where a phenomenon similar to ours is reported. Solutions obtained by them may be regarded as unimodal solutions. The reader may well wonder if the existence of a topologically simple, stable solution is contradictory to the occurrence of chaotic motions in high Reynolds number flows. For instance, chaotic motions in Kolmogorov flows are reported in many papers among which [9, 22] are quite recent ones. In the viewpoint of the theory of dynamical systems, one may suspect that only a small basin of attraction may be admitted for some unimodal solutions, even if they are stable. If so, the co-existence of a stable unimodal pattern and more frequently observed chaotic motions in the 2D Navier-Stokes equations is no contradiction at all. Finally we remark that unimodal solutions seem to appear only in two dimensions and the allusion in [16] that it may be related to the inverse cascade theory of 2D turbulence (see [4, 20, 21]) may not be without reason. For instance, if the Reynolds number is large, the number of vortex pairs decreases as time $t$ increases (see, e.g. [31]). Appearance of stable unimodal solutions seems to be compatible with this. The authors of [10] consider a quasi-geostrophic point vortex system and observed that a large cluster of vortices are formed by taking average over time. Their averaged solutions show a unimodal pattern similar to ours. On the other hand, [25] considers a 2D Navier-Stokes equations on a sphere with a random external forcing of small scales. They show that, after quite long time, the number of positive and negative jets (i.e. the band-like regions of...
unidirectional velocity) decreases. Eventually, flow patterns of many bands are transformed to those of only two or three bands. These phenomena may be related to ours.

The present paper consists of nine sections. We explain the basic set-up and the numerical method in section 2. Section 3 presents results in the case of one of the wave number is zero. Having analyzed these numerical data, we re-consider the definition of unimodality in section 4. Sections 5 through 8 are devoted to the cases of more complicated external forces. Concluding remarks and conjectures are discussed in section 9.

2. The Kolmogorov flows

We consider a problem which was originally proposed by Kolmogorov in 1950’s. Its history can be found in [2, 24, 26, 28]. Briefly speaking, it requires us to study the 2D Navier–Stokes equations from the viewpoint of the theory of dynamical systems. We go straight into the mathematical setting: after a suitable nondimensionalization, the problem is to study 2D flows in a flat torus $(−\pi/\alpha, \pi/\alpha) \times (−\pi, \pi)$, where $\alpha$ denotes the aspect ratio of the rectangle. This torus is denoted by $T_\alpha$. We consider the Navier–Stokes equations for incompressible viscous fluid in $T_\alpha$ with the periodic boundary conditions in both $x$ and $y$ directions. Then the stream function $\psi$ is governed by

$$\frac{\partial \Delta \psi}{\partial t} - \psi_x \Delta \psi_y + \psi_y \Delta \psi_x = \frac{1}{R} (\Delta^2 \psi + f),$$

where $f$ is derived from the external force and $R$ is the Reynolds number. Without losing generality, we consider only the external force which is inversely proportional to the Reynolds number $R$. In particular, even if we replace the right hand side of (1) by $\frac{1}{R} \Delta^2 \psi + f$, we can find a change of scales of variables so that the new equations becomes (1): if we consider asymptotic behavior as $R$ tends to infinity, our scaling (1) is most useful. See [16].

Let $\ell$ and $k$ be positive integers. If $f = \cos(\alpha \ell x + ky)$, then $\psi = -\frac{1}{(\alpha^2 + k^2)} \cos(\alpha \ell x + ky)$ annihilates the nonlinear term and satisfies (1) exactly, whatever the Reynolds number may be. We call this the basic solution, from which a lot of new interesting solutions bifurcate. Numerical computations of steady-states are carried out in [26–28] when $\alpha < 1$, $\ell = 0$ and $k = 1$. When $R$ is very large, nearly singular solutions are discovered. By near singularity we mean that a solution possesses an interior transition layer. We considered in [15] the case where $f$ is so chosen that $\psi = -\cos \alpha x - \cos y$ is a solution for all $R$. The results in [26–28] are extended by [16], where the cases of $\alpha = 0.7$, $\ell = 0$ and $k = 2, 3$ were considered. The case where $f = \cos 2y$ is considered, for $\alpha = 1$, by many authors, notably [1, 7], and interesting bifurcation phenomena are found (see also [13]). However, it seems to us that there is much room for studying solutions at very large Reynolds numbers.

We considered in [16] the cases where $f = \cos ky$ with $k = 2$ and $k = 3$. In each case we found, naturally, bifurcating steady-states which have $k$ pairs of vortices if the Reynolds number is small. But when we computed steady-states up until $R$ becomes sufficiently large, we observed that the number of the vortices decreased by mutual merges and eventually one pair of vortices survives in at least one branch of solutions. We called such a solution unimodal. Figures 5(f) and 9(d) below are typical examples. Note that here and in [16] we have not given a definition of a unimodal solution in exact terms. We will come back to this issue later. Note also that we do not say that all the steady-states become unimodal. Majority of them are not: some of them can be very complicated. We are claiming that at least one of them becomes unimodal.
Unimodal solutions appear also in the Proudman–Johnson equation, which are related to the 2D Navier–Stokes equations: see [17]. However, we believe that unimodal solutions do not appear in other equations such as reaction-diffusion equations or even in 3D Navier–Stokes equations. See [16, 29]. We expected that a unimodal solution exists universally if the Reynolds number is large enough. ‘Universal’ means that it appears irrespectively of the external force \( f \). In what follows we demonstrate by numerical evidence that a unimodal solution appears in many cases other than those considered in [16].

We represent the solution \( \psi \) as

\[
\psi(x, y) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} a(m, n) \exp(iamx + iny),
\]

where \( i = \sqrt{-1} \) and \( a(0,0) = 0 \). As in the previous studies, we restrict ourselves to those solutions which are invariant with respect to the transformation \((x, y) \rightarrow (-x, -y)\). This implies that a solution satisfies \( a(m, n) = a(-m, -n) \). Since \( \psi \) is real-valued, we have \( \bar{a}(m, n) = a(-m, -n) \). Consequently all the Fourier coefficients are real. We discretize the differential equation by the spectral method (the Fourier–Galerkin method). Accordingly, we choose two positive integers \( M \) and \( N \), and set

\[
\psi_{M,N}(x, y) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} a(m, n) \exp(iamx + iny).
\]

The resulting nonlinear equation is solved by the Newton method. In order to trace the path of steady-states in the function space we use the path-continuation method. See [12, 16] or [26] for details of the numerical method.

In our computation of steady-states, we chose \( M = N = 24 \) in the most cases of \( R \leq 5000 \). If the Reynolds number is larger, we increased \( M \) and/or \( N \), but even in the case of \( R = 100000 \), \( M = N = 50 \) is found to be accurate enough. More specifically if the relative error of the vorticity satisfies, for instance,

\[
\frac{||\Delta \psi_{28,28} - \Delta \psi_{32,32}||}{||\Delta \psi_{32,32}||} < 10^{-4},
\]

where \( || \cdot || \) denotes the \( L^2 \)-norm, then we are content with the solution \( \psi_{32,32} \). For the time-periodic solution we used \( M = N = 20 \), which seems to be a maximum resolution for our current computing environment, or less in all the cases. If the reader compare our situation with computations of nonstationary flows, where thousands by thousands of modes are used (see for instance [14]), one may well wonder whether \( 50 \times 50 \) is enough. Our steady-states, which are analytic functions and whose Fourier coefficients decay more quickly than nonstationary solutions, can be computed with much less modes. We also note that our primary target is unimodal solutions, in which, luckily, modes of high frequency are of small magnitude. In this respect, we would like to refer the reader to [23] and [32], where some of our previous results are justified by interval arithmetic; in the former paper the author proves stability of the bifurcating solutions in [26]; in the latter paper the author proves rigorously non-trivial Kolmogorov flows in [26].

In what follows \( f \) in (1) is chosen as \( \cos(a\ell'x + ky) \), where \( a \) is the aspect ratio of the torus. We divide the cases with various \((a, \ell', k)\).
3. The cases of $\ell = 0$

If $\ell = 0$, there are many researches both numerical and analytical about bifurcation of solutions. See [7, 8, 26] and the references therein. The latter two are important in that they consider many cases of Hopf bifurcations and numerical experiments on torus bifurcations (and more). This section is devoted to the cases where $\ell = 0$, $0.7 \leq \alpha \leq 1$ and $k = 4, 5, \ldots, 10$ are tested, but because of limitation of space we do not present all the details we obtained. The particular value $\alpha = 0.7$ does not matter very much. We wish to demonstrate that a unimodal solution appears even for untypical values, and for that reason we chose $\alpha = 0.7$ rather than more typical or symmetrical value such as $\alpha = 1$ or $1/2$. In fact the final unimodal state is rather robust with respect to a small change of $\alpha$.

3.1. The case of $\alpha = 0.7, \ell = 0, k = 4$

The cases of $f = \cos ky, \alpha = 0.7, k = 2, 3$ are reported in details in [16]. It is unnecessary to recall the result in [16], since quite a similar result will be seen below.

As a next step we consider the case where $f = \cos 4y, \alpha = 0.7$. In this case, the basic solution is $\psi_0 = -\frac{1}{256} \cos 4y$, whose streamlines are drawn in figure 2(a). This basic solution loses stability at $R \approx 379$, and non-trivial steady-states bifurcate. Another bifurcation of steady-states from the basic solution occur at $R \approx 434$ and also at $R \approx 545$, and possibly at other Reynolds numbers, too. But these are not important as far as unimodal solutions are concerned. Hence we consider those solutions from $R = 379$. The branch of those solutions is called the primary branch.

The primary branch has a secondary bifurcation point at $R \approx 843$. See figure 1. The solutions on this secondary branch (drawn in red color in figure 1) acquire unimodality as $R \uparrow \infty$, as we will soon show. Before that we examine the pattern formation of the solutions along the primary branch.

Remark 3.1. In the present paper, we sometimes use two colors in drawing streamlines. Sometimes we use four. If only red and blue are used, it implies that $\psi > 0$ for red and $\psi < 0$ for blue. The contour $\psi = 0$ is drawn in black. If four colors, red, yellow, green, and blue are
used, they correspond to, $\bar{\mu} > \psi > \mu/2$, $\mu/2 > \psi > 0$, $0 > \psi > \mu/2$, and $\mu/2 > \psi > \underline{\mu}$, respectively. Here $\bar{\mu} = \max_{\tau \in [0, 1]} \psi$ and $\underline{\mu} = \min_{\tau \in [0, 1]} \psi$.

All the solutions on the primary branch has the following two properties: (1) The streamlines consist of four parts, each of which is a solution (see figures 2(b) and 3) in the case of $f = \cos \psi$ and $\alpha = 0.7/4 = 0.175$; (2) The patterns have symmetry with respect to the transformation $y \mapsto y + \frac{\pi}{2}$. At very large Reynolds numbers the majority of the streamlines are vertical, in spite of the horizontally exerted external force. Compare figure 2(b) with figure 4.
This curious phenomena is observed in other situations and also in some experiments. See [5, 6, 24, 26].

As figure 1 shows, there exist four secondary solutions at $R = 2000$, which are labeled as $A$, $B$, $C$, and $D$ in figure 1. The four actually have the same pattern, since if one of them is denoted by $\psi(x, y)$, then others are $\psi(x + \pi, y)$, $\psi(x, y + \pi)$, $\psi(x + \pi, y + \pi)$. We now look at the solutions on the branch containing $A$. On the right hand side of $A$, the solutions continue to exist until $R = 100000$, where we stopped our computation. It seems to us that those solutions exist for all the Reynolds numbers. The streamlines are drawn in figure 5. There exist one and only one positive vortex and negative vortex if $R$ is large. Thus it confirms our conjecture about the unimodal solution.

Although the solution bifurcating from $R \approx 434$ and $R \approx 545$ does not show unimodality, we show two typical streamline patterns in figures 6(a) and (b). The branch which bifurcates from the basic solution at $R \approx 434$ has secondary and tertiary branches. One of these solutions is shown in figure 6(c). The streamlines of positive $\psi$ are unimodal, but those of negative $\psi$ are not. The corresponding bifurcation diagram is rather complicated and is omitted. It may
be interesting that the positive vortex in figure 6(c) is different from, but looks similar to the positive vortex in figure 5(f).

3.2. The case of $k = 5$ and $k = 10$

If $(\alpha, \epsilon', k) = (0.7, 0, 5)$, the basic solution $\psi_0 = -\frac{1}{625} \cos 5y$ loses its stability at $R \approx 910$. The primary branch loses stability at $R \approx 1631$ to a secondary branch of steady-states, see

Figure 7. Bifurcation diagram. $k = 5, 10$. The vertical axis represents $\frac{1}{2} \psi(0, 0)$. (a) $(\alpha, \epsilon', k) = (0.7, 0, 5)$. (b) $(\alpha, \epsilon', k) = (0.7, 0, 10)$.

Figure 8. Streamline patterns. $(\alpha, \epsilon', k) = (0.7, 0, 5)$. (a) $R = 2000$. (b) $R = 4000$. (c) $R = 10000$. (d) $R = 100000$.

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The existence of the unimodal solution on the secondary branch is demonstrated by figure 8. If \( \alpha = 0.7, 0, 10 \), the basic solution is \( \psi_0 = -\frac{1}{20000} \cos 10y \). It loses its stability at \( R \approx 14246 \) to a pitchfork of steady-states. The solution of the primary branch loses stability at \( R \approx 17298 \); see figure 7(b). The solutions on the secondary branch evolve, as \( R \) increases, to a unimodal solution, as is seen in figure 9. The solutions in figures 5(f), 8(d) and 9(d) are not the same but the mutual similarity would be obvious. We call such solutions unimodal.

3.3. The case of \( \alpha, \ell, k = (1, 0, 4) \)

The present case is different from that in section 3.1 only in the value of \( \alpha \). Namely, \( (\alpha, \ell, k) = (1, 0, 4) \). The basic solution is \( \psi_0 = -\frac{1}{235} \cos 4y \). As we will see below, however, there exists a marked difference in the present case. The primary bifurcation occurs at \( R \approx 397.4 \). Secondary bifurcation occurs at \( R \approx 790 \); see figure 10. The diagram looks quite similar to those in the case of \( \alpha = 0.7 \). However, we see some difference as we trace the solution branch by increasing \( R \). Two solutions on the secondary branch are drawn in figure 11. The solution at \( R = 100000 \) is not totally unimodal as is seen from its bird’s-eye view shown in figure 11(c): indeed, although the negative vortex has a monotone bottom, the positive vortex has a dent at the four corners. This solution tells us that we need a careful definition of unimodality. Although monotonicity is lost, the profile of the positive vortex in the region

**Figure 9.** Streamline patterns. \((\alpha, \ell, k) = (0.7, 0, 10)\). (a) is on the primary branch. Other are on the secondary branch. (a) \( R = 17000 \). (b) \( R = 30000 \). (c) \( R = 100000 \). (d) \( R = 700000 \).
is, for some $\delta > 0$, homeomorphic to a disk. Here a new term may be used. A solution having this property is not unimodal in literal sense, but we may at least say that the streamline pattern is topologically simple. We will discuss this issue in the next section. We looked for other bifurcating solutions (data omitted) but could not find a unimodal steady-state.

The difference between the cases of $\alpha = 0.7$ and $\alpha = 1$ is definite. The difference may be explained by the existence of a Hopf bifurcation point. (By the way Hopf bifurcations and torus bifurcations are found in Kolmogorov flows in many situations. See [8, 26].) From our data, we have the following scenario: the solution in the case of $\alpha = 1$ is stable for $R < 3710$. At $R \approx 3710$ it undergoes a Hopf bifurcation and loses the stability. Then in some sense unimodality is inherited to the stable time-periodic solution. If we restrict ourselves to steady-states only, the bifurcation diagrams for $\alpha = 0.7$ and $\alpha = 1$ are qualitatively the same. However, the solutions on the secondary branch with $R > 3710$ are no longer stable if $\alpha = 1$. We admit that a Hopf bifurcation exists in the case of $\alpha = 0.7$, too. But it occurs at much larger Reynolds number, at which the solution is already unimodal. Somehow, the unimodality is not lost once it is acquired.

**Figure 10.** The bifurcation diagram. $(\alpha, \ell, k) = (1, 0, 4)$. The vertical axis represents $\alpha(0, 1) + \alpha(1, 0)$.

**Figure 11.** The streamline patterns. $(\alpha, \ell, k) = (1, 0, 4)$. In (c), at the four corners of the rectangle there is a dent. There is no dent on the bottom. (a) $R = 900$. (b) $R = 100000$. (c) The bird’s-eye view.

$\{(x, y) : \psi(x, y) > \delta\}$ is, for some $\delta > 0$, homeomorphic to a disk.
We computed the time-periodic solutions which emanate from $R = 3710$, when $\alpha = 1$. For this computation we used a software called AUTO [3]. The time-periodic solution seems to continue to exist for all $R > 3710$. However, it loses stability at $R \approx 4216$ through a torus bifurcation. Its streamlines are drawn in figure 12. If we compare figures 12 and 13, we see that the periodic solution is oscillating about the steady-states in figure 13. Anyway, 4216 might be too small for a time-periodic solution to become unimodal. Since we do not know what happens on the doubly-periodic solutions in $R > 4216$, we cannot say anything about them. What are left to us is the unimodal-like solutions in figure 11.

For the sake of comparison and reference, we present two other cases with $\alpha = 1$ in figures 14–16. If $k = 3$ the solution is unimodal. If $k = 5$, there is a dent on both the bottom
and the peak. Dents can be better seen if we draw the stream function on the diagonal. Figure 16 shows graphs of $\psi(t, t)$ in $-\pi \leq t \leq \pi$.

Finally we would like to give a remark about whether the Reynolds number is large enough. Since we conjecture that something happens if $R$ is large enough, we should, by some independent measure, ascertain the largeness of $R$. In other words, we must be sure that a different phenomena may not appear if $R$ is increased further. If $R$ tends to infinity in the equation (1), then at least formally we may conclude that the limit function satisfies the Euler equation. Namely,

$$\lim_{R \to \infty} \psi = \psi_{\infty},$$

where $\psi_{\infty}$ satisfies the following stationary Euler equation:

$$- \frac{\partial \psi_{\infty}}{\partial x} \frac{\Delta \psi_{\infty}}{\partial y} + \frac{\partial \psi_{\infty}}{\partial y} \frac{\Delta \psi_{\infty}}{\partial x} = 0.$$
This is (at least formally) equivalent to saying that \( \psi_\infty \) and \( \Delta \psi_\infty \) are functionally dependent, or that there exists a function \( F \) such that \(- \Delta \psi_\infty = F(\psi_\infty)\). Existence of such a functional relation can be examined by plotting \( (\psi(x, y), \omega(x, y)) \) for many points \((x, y) \in \mathbb{T}_n\), where \( \omega = -\Delta \psi \) is the vorticity. Accordingly, by observing a functional relation, we may judge whether \( \psi \) has already converged to a certain limit or not.

Figure 17(a) shows the graph when \( a = 1.0, \ell = 0, k = 4, R = 100,000 \). Figure 17(b) shows the graph when \( R = 1200 \). In figure 17(a) a functional relation may be seen, although the function \( F \) is many-valued. This clearly shows that \( \psi \) is already close enough to the supposed limit \( \psi_\infty \). Therefore the patterns of streamlines we have seen in the present section will not change even if we increase \( R \) further. In contrast, no functional relation is seen in figure 17(b).

### 4. Definition of unimodality

Having seen the results in the previous section, we are now in a position to clarify the definition of unimodality. A vague definition suffices for most purposes. But we believe that the following definitions may be useful.

**Definition 4.1.** A stream function is called strongly unimodal if it has one and only one local maximum and local minimum, both of which are attained by one point.

It is called quasi-unimodal if there are positive constants \( a \) and \( b \) such that \( \{(x, y) : \psi(x, y) > a\} \) and \( \{(x, y) : \psi(x, y) < -b\} \) are connected and the following conditions are satisfied:

\[
\{(x, y) : \max_{\mathcal{T}_u} \psi > \psi(x, y) > a\} \quad \text{and} \quad \{(x, y) : \min_{\mathcal{T}_u} \psi < \psi(x, y) < -b\}
\]

are occupied by nested, closed streamlines which are homeomorphic to a circle.

We note that the majority of steady-states (such as those in figure 6) are neither strongly unimodal nor quasi-unimodal. A typical quasi-unimodal flow is that in the middle and right figures of of figure 14, while the left figure is unimodal. A quasi-unimodal solution may possibly have complicated streamlines in \( \{(x, y) : a > \psi(x, y) > -b\} \), but we ignore them. We also
remark that another notion weakly unimodal was previously introduced in [17] with different meaning. This is the reason for introducing the quasi-unimodality concept here.

Using these terminologies, we may say that even if there is not a strongly unimodal solution, there may exist a quasi-unimodal solution. There may be a characteristic better than the one above, but our definition captures important characters of ‘topologically simple’ flows.

5. The case of $(\alpha, \ell, k) = (1, 1, 1)$

Hereafter we consider some cases where $\ell \neq 0$. If $(\alpha, \ell, k) = (1, 1, 1)$, the basic flow $\psi_0 = -\frac{1}{2} \cos(x + y)$ loses stability at $R \approx 27$ by a Hopf bifurcation. No bifurcation of steady-states or a Hopf bifurcation was detected for $27 < R < 100,000$. It seems to us that no steady-state bifurcates from $\psi_0$ in any $R$. However, there exist non-trivial solutions which are separated from the basic solution. See figure 18(b). One of the two nontrivial solutions at $R = 5000$ shows unimodality: see figure 19.

The solutions in a separated branch cannot be computed by the continuation from the basic solutions. They are computed by the homotopy method from a unimodal solution in the case of $f = \cos 2y, \alpha = 0.7$. Namely we take an artificial parameter $\sigma \in [0, 1] and modify the equation to

$$\frac{\partial \Delta \psi}{\partial t} - \psi_x \Delta \psi_x + \psi_y \Delta \psi_y = \frac{1}{R} [\Delta^2 \psi + (1 - \sigma) \cos 2y + \sigma \cos(x + y)],$$

and $\alpha = (1 - \sigma) \times 0.7 + \sigma \times 1$. By starting from $\sigma = 0$ and increasing $\sigma$ until $\sigma = 1$, the path-continuation method yields a solution in the separated branch in figure 18(b). In this way we arrived at one of the non-trivial solutions of figure 18(b). From there it is easy to apply a path-continuation method and we eventually obtain the unimodal solution in figures 19(a) and (c). This unimodal solution is unstable. Unimodal solutions we obtained in [16] are stable, and we believed without much evidence that unimodal steady-states are stable. The current example shows that the situation is not that simple. As far as we have computed, stable steady-states with $R \gg 1$, if any exists, are unimodal. But the present example shows that the converse proposition is not true.

We computed, by AUTO, the branch of time-periodic solutions which bifurcate from the basic solution at $R \approx 27$. Since the computation of time-periodic solutions requires a lot of

![Figure 18](image-url)
computer resources we cannot take big $M$ or $N$. If $R$ is small, a periodic orbit can be computed even with $M = N = 7$. But if $R$ is larger, larger $M$ and $N$ are necessary. After considering these factors we settled down with $M = N = 20$ in the present situation. With this resolution we computed solutions up to the case of $R = 10,757$, where the computation requires about 60 d in a Linux workstation with 16 CPUs of Xeon CPU of 2 GHZ. This is no longer a tolerable situation and we stopped there.

Figure 20 shows the stream functions at $R = 50$ at $t = 0, T/4, T/2, 3T/4$, where $T$ is the period. There are two positive and one negative vortices, or one positive and two negative vortices, whence it is not unimodal. We plotted the same snapshots in the case of $R = 500$ in figure 21. It is oscillating with spatial unimodality. The change from multimodal to unimodal periodic orbit occurs somewhere between $R = 100$ and $R = 200$. Clear boundary is difficult to draw from the experimental data. However, it should be noted that the time-periodic solution loses stability at $R \approx 2300$. Though it is unstable for $R > 2300$, it remains being unimodal. And the streamlines keep showing unimodal pattern even for $R$ much greater than 2300. We may say this: a unimodal solution, once it has acquired its simple pattern, never loses its simplicity even if it loses stability.

6. The case of $(\alpha, \ell, \kappa) = (2, 1, 1)$

The basic solution is $\psi_0 = -\frac{1}{25} \cos(2x + y)$. It loses stability at $R \approx 176.1$ through a Hopf bifurcation. A bifurcation of steady-states from the basic solution occurs at $R \approx 286$. The primary
branch does not have a unimodal solution as is shown in figure 22. It has a secondary bifurcation at $R_{305}$, which is a subcritical pitchfork (figure 23). This branch has two turning points at $R_{141}$. After having experienced this turning point, the steady-states are stable and becomes a unimodal solutions (figure 24). The upper primary branch in figure 23 has a bifurcation point, too. But since it is obtained by $\psi(x, y) \mapsto \psi(x + \pi/2, y + \pi)$, we omit that branch in figure 23.

7. The case of $(\alpha, \epsilon, k) = (1, 1, 2)$

If $f = \cos(x + 2y)$ and $\alpha = 1$, then $\psi_0 = -\frac{1}{25} \cos(x + 2y)$ is the basic solution, the streamlines of which are plotted in figure 26(a).

Along this basic solution, a bifurcation of steady-states occurs at around $R = 88.4$. The bifurcating solutions on the primary branch (the blue broken line in figure 25) change their streamline pattern as is shown in figure 26. The streamlines of the basic solution are inclined parallel lines with inclination $-1/2$. At $R = 2000$, we find many streamlines whose
Figure 21. Periodic solution at $R = 500. (\alpha, \varepsilon, k) = (1, 1, 1)$. (a) $t = 0$. (b) $t = T/4$. (c) $t = T/2$. (d) $t = 3T/4$.

Figure 22. Streamlines of the basic solution and those on the primary branch. $(\alpha, \varepsilon, k) = (2, 1, 1)$. (a) Basic solution (b) $R = 300$. (c) $R = 1000$. 
inclinations are approximately 1, see figure 26(f). This would be an interesting contrast with the fact that vertical streamlines exists in the case of $f_k = k, 2, 3, \ldots$: compare this with figure 4(b).

Along the primary bifurcation branch, there appears a Hopf bifurcation point at $R = 292.6$ and the time-periodic solutions inherit the stability. They continue to exist stably up to $R = 357.9$, where the stability is lost. Immediately after the bifurcation point the numerical computation becomes divergent, which makes further continuation impossible. The periodic solutions do not show unimodality (figure omitted.) Currently we do not know exactly why the numerical procedure diverges.

On the primary branch of steady-states, we find a secondary bifurcation point of steady-states at around $R \approx 325$, where steady-states of different nature bifurcate. At $R \approx 2600$, which is close to the rightmost turning point, there exists one and only one negative vortex, but there are three positive vortex. One of them is definitely large and the other two are small (Figures

Figure 23. Bifurcation diagram. $(\alpha, \ell, k) = (2, 1, 1)$.

Figure 24. Streamlines of the solutions on the secondary branch—the broken line in figure 23. $(\alpha, \ell, k) = (2, 1, 1)$. (a)–(d) corresponds to the points (a)–(d) of figure 23. Since the Reynolds number is too large, the case (e) cannot be drawn in figure 23. (a) $R = 270$. (b) $R = 180$. (c) $R = 200$. (d) $R = 1000$. (e) $R = 10000$. 

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omitted.) We computed steady-states as far as we could, but no unimodal steady-state was obtained in these branches.

We therefore looked for new solutions by the homotopy method described in section 5 to successfully find a separated branch. Two solutions on the separated branch, which is drawn in Figure 25.

Figure 25. The bifurcation diagram. The vertical axis represents $\frac{1}{2}y(0, 0, (\alpha, \varepsilon, k) = (1, 1, 2)$.

Figure 26. Streamlines of the solutions of the primary branch (the blue broken line of figure 25. $(\alpha, \varepsilon, k) = (1, 1, 2)$). (a) basic sol. (b) $R = 90$. (c) $R = 110$. (d) $R = 150$. (e) $R = 300$. (f) $R = 2000$.
Figure 27. Streamlines of the solutions of the separated branch. \((\alpha, \varepsilon, k) = (1, 1, 2)\).
(a) \(R = 20000\). (b) \(R = 20000\).

Figure 28. Steady-states. \((\alpha, \varepsilon, k) = (1, 1, 3)\).
(a) Streamlines \(R = 275\). (b) Streamlines \(R = 600\). (c) Streamlines \(R = 100000\). (d) Bird’s eye view. \(R = 100000\).
red in figure 25, are shown in figure 27. One of them is unimodal. However, it should be noted that this unimodal solution is unstable.

8. The case of \((\alpha, \epsilon^2, k) = (1, 1, 3)\)

The basic solution is \(\psi_0 = -\frac{1}{100} \cos(x + 3y)\), whose streamlines are straight lines of inclination \(-1/3\). The bifurcation diagram and details of data are omitted. We just show some steady-states in figure 28. The quasi-unimodal solution at \(R = 100,000\), which lies on a secondary
bifurcation branch, is different but similar to the one in the case of $(\alpha, \ell', k) = (1, 0, 4)$. Compare it with figure 11(c).

9. Conclusion

We have computed solutions in many cases. Whether unimodal solutions are a consequence of the inverse cascade theory or not, we are not sure. We are content with the factual reports, for which others may offer a theory.

Since the magnitude of the Fourier modes may be related to the unimodal solutions, we here offer a graph which shows the distribution of the Fourier modes. See figure 29. In figure 29(a) we plotted $k$ versus $e(k)$, where $e(k)$ is defined as

$$e(k) = \sum_{k-1 < \sqrt{m^2+n^2} < k+1} a(m,n)^2.$$  

Thus low modes such as $(\pm 1,0)$ and $(0,\pm 1)$ are definitely large in absolute value. But as figure 29(b) shows, some other modes are of similar magnitude. Here $\log_{10}(||a(m,n)||/A)$ are plotted with $A = \max_{(m,n)}|a(m,n)|$. They are distinguished by the color: those of order one are drawn in black, those of order two in red, those of order three in green, etc. The solution is computed with $M = 40$, $N = 60$, but only those with $|m|, |n| < 20$ are plotted.

Now we would like to summarize our results and derive some conjectures. The strongly unimodal steady-state is not so ubiquitous as was suspected by [16]. However, quasi-unimodal solutions and unimodal time-periodic solutions together, they appear in quite many external forces. Also, the following proposition, which was conjectured by [16], was reconfirmed by the present computation. Namely, although a stable steady-state does not necessarily exist at large Reynolds number, but if it exists then its streamline pattern is unimodal. The following three remarks may now be in order:

(i) The set of steady-states may be disconnected. Accordingly path-continuations from the basic solution alone are not enough to obtain unimodal or quasi-unimodal solutions.

(ii) The time-periodic solutions which result from the Hopf bifurcation may exhibit a kind of unimodality when $R$ is increased.

(iii) Unimodal and quasi-unimodal solutions can appear only in large Reynolds numbers. Largeness depends on individual external force. In a case, $R = 500$ is enough (section 5), in another, $R = 100\,000$ is necessary ($(\alpha, \ell', k) = (0.7, 0, 10))$.

If a bold imagination is permitted, there might be a unimodal solution which is more complicated than being simply periodic in $t$. Thus there may be a solution which is spatially very simple but complicated in $t$. According to Dr. E. Sasaki, all the steady-states in [30] lose their stability by a Hopf bifurcation. Therefore it is worth looking for a unimodal time-periodic solutions in their problem.

We also would like to draw the reader’s attention to the fact that we here find some unstable unimodal solutions, while all the unimodal solutions in [16] are stable. It will be interesting to clarify whether the unimodality has any correlation with stability.

Another interesting feature of the quasi-unimodal solutions is that near the peak or bottom some of them have almost circular streamlines. Namely, they are radial near the maxima or minima. See figures 11, 15 and 28. We cannot offer an explanation for this phenomena.

The solutions have interior layers, to which singular perturbation theory may be applied [19, 28]. But we omit this analysis, since that requires more space.
Finally, since there are infinity of choices on \((\alpha, \ell, k)\), and we expect that further interesting phenomena are waiting for being explored. The reader is invited to find a new one.

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