Normalized Weyl-type ◦-product
on Kähler manifolds

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Abstract
We define a normalized Weyl-type ◦-product on general Kähler manifolds. Expanding this product perturbatively we show that the cumbersome term, which appears in a Berezin-type product, does not appear at least in the first order of \( \hbar \). This means a normalization factor, which is introduced by Reshetikhin and Takhtajan for a Berezin-type product, is unnecessary for our Weyl-type product at that order.

1 Introduction
A new kind of mathematics is thought necessary for a non-perturbative description of the string theory just as Riemannian geometry is indespensible for the description of the theory of general relativity. Non-commutative geometry is one of the strong candidate for it.

We would like to construct a non-commutative manifold with a Kähler metric from this perspective. We take deformation quantization approach, which introduces a non-commutative product into the ring of functions on a commutative manifold. Kontsevich showed how to construct a non-commutative product on Poisson manifolds, which includes Kähler manifolds [1, 2], but the effect of a metric is implicit in their construction. Dependence of a non-commutativity on a metric is clear in our construction. We give an outline of our construction in the rest part of this section.

Among other approaches we focus our attention on two different types of non-commutative products, the Weyl-type and the Berezin-type. The ordinary

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Moyal product of functions $f_1$ and $f_2$ on $C^1$ can be written in an integral representation, as

$$(f_1 \ast_M f_2)(z, \overline{z}) = \int_{C^1} f_1(w, \overline{v}) f_2(v, \overline{w}) \frac{e^{(z\overline{v} + v\overline{z} - z\overline{w} - w\overline{v})/\hbar}}{e^{(w\overline{v} + v\overline{w} - z\overline{v} - w\overline{z})/\hbar}} \frac{dv \overline{d}v \, dw \overline{d}w}{2\pi \hbar^2\bar{h}}$$

which reduces to

$$f_1 f_2 + \frac{\hbar}{2\sqrt{-1}} \{f_1, f_2\}_P + O(\hbar^2)$$

in the small $\hbar$ limit and satisfies associativity. This approach of quantization is applicable only to a flat manifold.

There are papers on Berezin-type star-product on Kähler manifolds. The Berezin star-product is defined on certain Kähler manifolds by

$$(f_1 \bullet f_2)(z, \overline{z}) = \int_{C^n} f_1(z, \overline{v}) f_2(v, \overline{z}) e^{(\Phi(z, \overline{v}) + \Phi(v, \overline{z}) - \Phi(z, \overline{z}) - \Phi(v, v))/\hbar} \, d\mu_{\hbar}(v, \overline{v})$$

where $\Phi$ is a Kähler potential of the manifold. This product also satisfies the associativity. The reduction to the Poisson bracket is achieved, however, only in the difference of two terms:

$$f_1 \bullet f_2 - f_2 \bullet f_1 = \frac{\hbar}{2\sqrt{-1}} \{f_1, f_2\}_P + O(\hbar^2)$$

Reshetikhin and Takhtajan showed does not necessarily satisfy $f \bullet 1 = 1 \bullet f = f$ for the general Kähler potentials and defined a product which satisfies the above equality.

It was shown in that there exists another possible approach

$$(f_1 \odot f_2)(z, \overline{z}) = \int f_1(w, \overline{v}) f_2(v, \overline{w}) \frac{e^{(\Phi(z, \overline{v}) + \Phi(v, \overline{z}) - \Phi(z, \overline{z}) - \Phi(v, v))/\hbar}}{\bar{h}} \, d\mu_{\hbar}(v, \overline{v}) d\mu_{\hbar}(w, \overline{w})$$

which interpolates between the Weyl-type and Berezin-type star-products. In the flat space this is the same as the Moyal product. The associativity, however, does not hold by itself, but is fulfilled in the functional integral limit of its multiple products.

Ref. is insufficient in the sense that there is no consideration to the general Kähler manifolds for which $A = \frac{1}{2} \sum_{i,j=1}^n h^{ij} \partial_i \partial_j \log \det H$ is not 0, where the matrix $H$ is a metric.

We generalize the product defined in so that it is applicable to general Kähler manifolds. Moreover we expand the product perturbatively and show that it really has a property characteristic of a Weyl-type product and that the normalization factor necessary for a Berezin-type product introduced by Reshetikhin and Takhtajan is unnecessary for a Weyl-type product at least in the first order of $\hbar$. 
2 Construction

Reshetikhin and Takhtajan showed the Berezin-type product \(1\) does not necessarily satisfy \(f \bullet 1 = 1 \bullet f = f\) for the general Kähler potentials and defined, as a product which satisfies the above equality, a normalized star-product

\[
e^{-1}_h(z, \overline{z}) ( (f_1 e_h) \bullet (f_2 e_h) ) (z, \overline{z}), \tag{3}
\]

where \(e_h\) is a normalization factor defined as \(f \bullet e_h = e_h \bullet f = f\).

As you can see, \(2\) consists of two kinds of Berezin-type products, an ordinary Berezin-type product

\[
(f_1 \bullet f_2) (z, \overline{z}) = \int_{\mathbb{C}^n} f_1(z, \overline{z}) f_2(v, \overline{v}) e^{\Phi(z, \overline{z}) + \Phi(v, \overline{v}) - \Phi(z, \overline{v}) - \Phi(v, \overline{z})} \, d\mu_h(v, \overline{v})
\]

and a new kind of Berezin-type product

\[
(f_1 \circ f_2) (z, \overline{z}) := \int_{\mathbb{C}^n} f_1(v, \overline{z}) f_2(z, \overline{v}) e^{-(\Phi(z, \overline{z}) + \Phi(v, \overline{v}) - \Phi(z, \overline{v}) - \Phi(v, \overline{z}))} \, d\mu_h(v, \overline{v}).
\]

A normalization factor is necessary for each kind of Berezin-type product, so a normalization factor is also necessary for a Weyl-type product.

We define a normalization factor \(e_h\) for a new kind of Berezin-type product \(f_1 \circ f_2\) so that \(f \circ e_h = e_h \circ f = f\).

\[
e_h = 1 - hA + O(h^2)
\]

whereas

\[
\hat{e}_h = 1 + hA + O(h^2).
\]

We define a normalized star-product for \(f_1 \circ f_2\) just as \(3\) for \(f_1 \bullet f_2\) as

\[
\hat{e}^{-1}_h(z, \overline{z}) ( (f_1 \hat{e}_h) \circ (f_2 \hat{e}_h) ) (z, \overline{z}).
\]

With two kinds of normalization factors \(e_h\) and \(\hat{e}_h\) we define a normalized Weyl-type product \(f_1 \ast f_2\) as

\[
(f_1 \ast f_2) (z, \overline{z}) := \hat{e}^{-1}_h(z, \overline{z}) \left( (e_h f_1 \hat{e}_h) \circ (e_h f_2 \hat{e}_h) \right) (z, \overline{z}) e^{-1}_h(z, \overline{z}) \tag{4}
\]

\[
= \int_{\mathbb{C}^n} f_1(w, \overline{v}) f_2(v, \overline{w}) e^{\Phi(z, \overline{z}) + \Phi(v, \overline{v}) - \Phi(z, \overline{v}) - \Phi(v, \overline{z})} \, d\mu_h(v, \overline{v})
\]

This product satisfies

\[
f \ast 1 = 1 \ast f = f
\]

for general Kähler manifolds. It is clear from \(3\) generally \(f_1 \circ f_2 \neq f_1 \ast f_2\) unless \(\hat{e}_h = e^{-1}_h\).
As non-normalized star-product $f_1 \odot f_2$, normalized star-product $f_1 * f_2$ does not satisfy the associativity in this form, but the transition to the functional integral version goes as follows.

Multi-products from both sides $(f^{(0)} *) (\cdots * (f^{(N-2)} *) (f^{(N-1)} *) f^{(N)})$ and $(\cdots ((f^{(0)} *) f^{(1)}) * f^{(2)}) * \cdots * f^{(N)})$ are respectively

\[
\left( f^{(0)} * (\cdots * (f^{(N-2)} * (f^{(N-1)} * f^{(N)})) \cdots) \right)(z, \bar{z}) \\
= \int \prod_{j=1}^{N} d\mu_h \left( z^{(j-1)}, \bar{z}^{(j)} \right) d\mu_h \left( v^{(j)}, \bar{z}^{(j-1)} \right) e^{\phi(z^{(j-1)}, \bar{z}^{(j)}; v^{(j)}, \bar{z}^{(j-1)})} e^{\hat{e}_h(z^{(j-1)}, \bar{z}^{(j-1)})} e_h(z^{(j-1)}, \bar{z}^{(j-1)})
\]

\[
\times e_h(z^{(N)}, \bar{z}^{(N)}) e_h(z^{(N)}, \bar{z}^{(N)}) \prod_{j=0}^{N} f(j) \left( z^{(j)}, \bar{z}^{(j)} \right) \left( v^{(N)} = z^{(N)}, \ v^{(0)} = z \right),
\]

\[
\left( \cdots ((f^{(0)} *) f^{(1)}) * f^{(2)}) \cdots * f^{(N)} \right)(z, \bar{z}) \\
= \int \prod_{j=1}^{N} d\mu_h \left( z^{(j)}, \bar{z}^{(j-1)} \right) d\mu_h \left( v^{(j-1)}, \bar{z}^{(j)} \right) e^{\phi(z^{(j-1)}, \bar{z}^{(j)}; v^{(j)}, \bar{z}^{(j-1)})} e^{\hat{e}_h(z^{(j-1)}, \bar{z}^{(j-1)})} e_h(z^{(j-1)}, \bar{z}^{(j-1)})
\]

\[
\times e_h(z^{(0)}, \bar{z}^{(0)}) e_h(z^{(0)}, \bar{z}^{(0)}) \prod_{j=0}^{N} f(j) \left( z^{(j)}, \bar{z}^{(j)} \right) \left( v^{(0)} = z^{(0)}, \ v^{(N)} = z \right).
\]

Therefore they have the same functional integral limit,

\[
\int \mathcal{D}\mu (z, \bar{z}) \mathcal{D}\mu (v, \bar{z}) \\
\times \exp \left[ \int d\tau \left\{ \frac{\partial \Phi (z, \bar{z})}{\partial \bar{v}} - \frac{\partial \Phi (v, \bar{z})}{\partial \bar{v}} \right\} - \frac{\partial \Phi (v, \bar{z})}{\partial v} + \frac{\partial \Phi (v, \bar{z})}{\partial v} \right] + \log e_h(z, \bar{z}) + \log \hat{e}_h(z, \bar{z}) + \log f(z, \bar{z}) \right].
\]

as in [8], where $f(z, \bar{z})$ is defined as

\[
\int d\tau \log f(z, \bar{z}) = \lim_{N \to \infty} \sum_{j=0}^{N} \frac{1}{N} \log f^{(j)}(z^{(j)}, \bar{z}^{(j)}) \quad \left( z^{(j)} - z^{(j-1)} = \frac{1}{N} \right).
\]

\[
\hat{f} = \int \mathcal{D}\mu (z, \bar{z}) \mathcal{D}\mu (v, \bar{z})
\]

From the same discussion as in [8], we define a normalized associative Weyl-type product as

\[
f_1 * f_2 = \int \mathcal{D}\mu (z, \bar{z}) \mathcal{D}\mu (v, \bar{z})
\]
one of the Berezin-type products is given by [7]:

\[
\frac{\partial \Phi(z, \overline{\tau})}{\partial \overline{v}} - \frac{\partial \Phi(v, \overline{\tau})}{\partial \overline{v}} \right) + \log \epsilon_h (z, \overline{\tau}) + \log \epsilon_h (z, \tau)
\]

where \( \tau_1 \) and \( \tau_2 \) are fixed points.

### 3 Perturbation

We expand \( f_1 \odot f_2 \) perturbatively in small \( \hbar \). The perturbative expansion of one of the Berezin-type products is given by [7]:

\[
f_1 \cdot f_2 = \pi^{-n} \det H \int_{C^n} e^{-(H y, y)} \prod_{i=1}^{n} \frac{|d y^i \wedge d \overline{y}^i|}{2} \left[ f_1 f_2 + \epsilon^2 \left( y^i \overline{y}^j (\overline{\partial}_j f_1) (\overline{\partial}_j f_2) \right) \right.
\]

\[
+ f_1 f_2 \left( -\frac{1}{4} y^i y^j y^k y^l \partial_i \overline{\partial}_k \Phi \overline{\partial}_j \Phi^* \right) \left( \overline{\partial}_l \Phi \overline{\partial}_m \Phi^* \right) + y^i \overline{y}^j \left( \partial_i \det H \right) \left( \overline{\partial}_j \det H \right) \left( \overline{\partial}_j \det H \right) \left( \overline{\partial}_j \det H \right) \left( \overline{\partial}_j \det H \right)
\]

\[
- \frac{1}{2} y^i y^j y^k y^l \left( \frac{\partial_i \Phi_{j,k} \overline{\partial}_l \det H}{\det H} + \frac{(\partial_i \det H) \overline{\partial}_l \Phi_{j,k}}{\det H} \right) \right) + O (\epsilon^3)
\]

\[
f_1 f_2 + h \left( A f_1 f_2 + \sum_{j=1}^{\infty} \hbar^j \overline{\partial}_j f_1 \overline{\partial}_j f_2 \right) + O (\hbar^2), \quad (\hbar = \epsilon^2)
\]

(5)

We apply this method of perturbation to the new Berezin-type product, \( f_1 \odot f_2 \) we have defined above:

\[
f_1 \odot f_2 = \pi^{-n} \det H \int_{C^n} e^{(H y, y)} \prod_{i=1}^{n} \frac{|d y^i \wedge d \overline{y}^i|}{2} \left[ f_1 f_2 + \epsilon^2 \left( y^i \overline{y}^j \partial_i \overline{\partial}_j f_1 \right) \right.
\]

\[
+ f_1 f_2 \left( \frac{1}{4} y^i y^j y^k y^l \partial_i \overline{\partial}_k \Phi \overline{\partial}_j \Phi^* + y^i \overline{y}^j \left( \partial_i \overline{\partial}_j \det H \right) \left( \overline{\partial}_j \det H \right) \left( \overline{\partial}_j \det H \right) \left( \overline{\partial}_j \det H \right)
\]

\[
+ y^i \overline{y}^j \left( f_1 \overline{\partial}_j f_2 \overline{\partial}_i \det H + \overline{\partial}_j f_2 \overline{\partial}_i \det H \right)
\]
\[ + f_1 f_2 \left( \frac{1}{4} y^i y^j y^k y^l y^m y^n \left( \partial_i \Phi_j \partial_k \partial_l \Phi_m \partial_n \right) + y^i y^j \frac{\det H}{(\det H)^2} \right) \]
\[ + \frac{1}{2} y^i y^j y^k y^l \left\{ \frac{\partial_i \Phi_j}{\det H} \left( \partial_l \det H \right) + \frac{\partial_i \det H}{\det H} \left( \partial_l \Phi_j \right) \right\} \right] + O(\epsilon^3) \]
\[ = f_1 f_2 - \hbar \left( A f_1 f_2 + \sum_{i,j=1}^n h^2 \partial_i f_1 \partial_j f_2 \right) + O(\hbar^2) \quad (\hbar = \epsilon^2). \tag{6} \]

As a result we get
\[ f_1 \circ f_2 = f_1 f_2 + \hbar \sum_{i,j=1}^n h^2 \left( \partial_j f_1 \partial_i f_2 - \partial_i f_1 \partial_j f_2 \right) + O(\hbar^2). \tag{7} \]

A Poisson bracket appears in the first order of \( \hbar \), which means the product \( f_1 \circ f_2 \) is really a Weyl-type product and which was not shown in \[8\].

What is surprising is the disappearance of a term \( A = \frac{1}{2} \hbar^2 \partial_i \partial_j \log \det H \) in the first order. Compare this result with the expansion of \( f_1 \bullet f_2 \) in \[8\], namely \( (\hbar) \), and of \( f_1 \circ f_2 \). The cumbersome term A, which appears in a Berezin-type product, disappears in a Weyl-type product at least in the order of \( \hbar \). This means, therefore, the normalization factor is unnecessary for the Weyl-type product at least in the first order of \( \hbar \).

In conclusion we have found that the Weyl-type product defined in \[8\] can be decomposed into two kinds of Berezin-type products and introduce a normalization factor into a Weyl-type product since a normalization factor is necessary for each kind of Berezin-type product. Note, in general, we must use \( f_1 \ast f_2 \) (\( \neq f_1 \circ f_2 \) unless \( \hbar = \epsilon^{-1} \)).

Moreover we perform a perturbative expansion of non-normalized Weyl-type product. In the first order of \( \hbar \) a Poisson bracket appears and \( A \) does not appear. Therefore the normalization factor which is necessary for a Berezin-type product is unnecessary for a Weyl-type product at least in the first order of \( \hbar \). The normalization factor is unnecessary for Weyl-type product non-perturbatively if \( \hbar = \epsilon^{-1} \) is shown.

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