A formula for Beilinson’s regulator map on $K_1$ of a fibration of curves having a totally degenerate semistable fiber

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1 Introduction

Let $X$ be a projective nonsingular variety over the complex numberfield $\mathbb{C}$. Let $H^i_{\mathbb{A}}(X, \mathbb{Z}(j))$ denote the motivic cohomology group. It is known that $H^i_{\mathbb{A}}(X, \mathbb{Q}(j))$ is isomorphic to Quillen’s $K$-group $K_{2j-i}(X)^{(j)}$. By the theory of higher Chern classes, we have the Beilinson regulator map (higher Chern class map)

$$\text{reg}_{i,j} : H^i_{\mathbb{A}}(X, \mathbb{Z}(j)) \rightarrow H^i_{\mathbb{A}}(X, \mathbb{Z}(j))$$

to the Deligne-Beilinson cohomology group ([Schneider]). The purpose of this paper is to provide a systematic method for computations of the regulator map in case that $(i, j) = (3, 2)$ (namely $K_1$) and $X$ a fibration of curves having a totally degenerate semistable fiber. Here we mean by a fibration of curves a surjective morphism $f : X \rightarrow C$ with a section $e : C \rightarrow X$ where $X$ (resp. $C$) is a projective smooth surface (resp. curve). A fiber is called semistable if it is reduced and a formal neighborhood of each singular point is isomorphic to “$xy = 0$”. A fiber is called totally degenerate if each component is a rational curve.

The cup-product pairing gives rise to a map $\mathbb{C}^\times \otimes \text{Pic}(X) \cong \mathbb{C}^\times \otimes H^2_{\mathbb{A}}(X, \mathbb{Z}(1)) \rightarrow H^3_{\mathbb{A}}(X, \mathbb{Z}(2))$. Its image is called the decomposable part, and the cokernel is called the indecomposable part. The decomposable part does not affect serious difficulty, while the indecomposable part plays the central role in the study of $H^3_{\mathbb{A}}(X, \mathbb{Z}(2))$. According to [Gordon-Lewis], we call an element $\xi \in H^3_{\mathbb{A}}(X, \mathbb{Z}(2))$ regulator indecomposable if $\text{reg}_{3,2}(\xi)$ does not lie in the image of $\mathbb{C}^\times \otimes \text{NS}(X)$. Obviously regulator indecomposable elements are indecomposable. The converse is also true if the Beilinson-Hodge conjecture for $K_2$ is true. Lewis and Gordon constructed regulator indecomposable elements in case $X$ is a product of ‘general’ elliptic curves ([Gordon-Lewis] Theorem 1). There are lots of other related works, though I don’t catch up all of them. On the other hand, in case that $X$ is defined over a number field, the question is more difficult, and as far as I know there are only a few of such examples (e.g. [Ramakrishnan] §12).

In this paper we give a new method for computation of the composition of the maps

$$H^3_{\mathbb{A},D}(X, \mathbb{Q}(2)) \rightarrow H^3_{\mathbb{A}}(X, \mathbb{Q}(2)) \rightarrow H^3_{\mathbb{A}}(X, \mathbb{Q}(2))$$

where $D = f^{-1}(P)$ is a singular fiber, $H^3_{\mathbb{A},D}(X, \mathbb{Q}(2))$ denotes the motivic cohomology supported on $D$ (Theorem 4.1). As an application we give a number of examples of regulator indecomposable elements ([5]). One of the technical key points is to use certain rational 2-forms “$\Lambda(X)_{\text{rat}}$” which will be introduced in §3.4. Our method has an advantage in explicit computations and it also works in case that the base field is a number field. I hope that it will bring a new progress in the study of Beilinson’s conjecture on special values of $L$-functions ([RSC]), though I don’t have a result in this direction so far.

This paper is organized as follows. [2] is a quick review of $H^3_{\mathbb{A}}(X, \mathbb{Q}(2))$ and Beilinson regulator. [3] is a preparation to state and prove the main theorem (Theorem 4.1) in §4. In §5 we give examples of regulator indecomposable elements for certain elliptic surfaces defined over $\mathbb{Q}$ with arbitrary large $h^{2,0}$ (Cor.5.8). [6] is an appendix providing how to compute the Gauss-Manin connection for a hyper elliptic fibrations.
Acknowledgment. A rough idea was inspired during my visit to the University of Alberta in September 2012, especially when I discussed the paper [CDKL] with Professor James Lewis. I would like to express special thanks to him. I'd also like to thank the university members for their hospitality.

2 Real Regulator map on $H^3_{\text{dR}}(X, \mathbb{Q}(2))$

For a variety $X$ over a field $K$ of characteristic zero, we denote by $H^i_{\text{dR}}(X) = H^i_{\text{dR}}(X/K)$ (resp. $H^i_{\text{dR}}(X) = H^i_{\text{dR}}(X/K)$) the de Rham cohomology (resp. de Rham homology) cf. [Hartshorne]. When $K = \mathbb{C}$ we denote by $H^i_{\text{dR}}(X, \mathbb{Q}) = H^i_{\text{dR}}(X(\mathbb{C}), \mathbb{Q})$ (resp. $H^i_{\text{dR}}(X, \mathbb{Q})$) the Betti cohomology (resp. Betti homology).

2.1 $H^3_{\text{dR}}(X, \mathbb{Q}(2))$ and indecomposable parts

Let $H^i_{\text{dR},Z}(X, \mathbb{Q}(j))$ denotes the motivic cohomology of a smooth variety $X$ supported on a closed subscheme $Z \subset X$. They fit into the localization exact sequence

$$
\cdots \rightarrow H^i_{\text{dR},Z}(X, \mathbb{Q}(j)) \rightarrow H^i_{\text{dR}}(X, \mathbb{Q}(j)) \rightarrow H^i_{\text{dR}}(X \setminus Z, \mathbb{Q}(j)) \rightarrow \cdots
$$

Of particular interest to us is the case $(i, j) = (3, 2)$. Let us describe $H^3_{\text{dR}}(X, \mathbb{Q}(2))$ explicitly.

Let $Z_i(X) = Z_{\dim X - i}(X)$ be the free abelian group of irreducible subvarieties of dimension $i$. We denote by $\eta_V$ the field of rational functions on an integral scheme $V$. Let $Z \subset X$ be an irreducible divisor, and $\tilde{Z} \to Z$ the normalization. Let $j : \tilde{Z} \to Z \hookrightarrow X$ be the composition. Then we define $\text{Div}_Z(f) := j_* \text{Div}_{\tilde{Z}}(f) \in Z^2(X)$ the push-forward of the Weil divisor on $\tilde{Z}$ by $j$. Let

$$
\partial_1 : \bigoplus_{\operatorname{codim} Z = 1} \eta_{\tilde{Z}}^\times \longrightarrow Z^2(X), \quad [f, Z] \mapsto \text{Div}_Z(f)
$$

be a homomorphism where we write

$$
[f, Z] := (\cdots, 1, f, 1, \cdots) \in \bigoplus_{\operatorname{codim} Z = 1} \eta_{\tilde{Z}}^\times, \quad (f \text{ is placed in the Z-component}).
$$

Let $Z = \sum_{i=1}^r Z_i$ be a divisor with $Z_i$ irreducible. Then, for a smooth open set $Z^o \subset Z$ we have

$$
H^3_{\text{dR},Z}(X, \mathbb{Q}(2)) \xrightarrow{\text{can}} H^3_{\text{dR},Z^o}(X^o, \mathbb{Q}(2)) \xrightarrow{\sim} \mathcal{O}(Z^o)^\times \otimes \mathbb{Q} \quad (2.1)
$$

where $X^o := X \setminus (Z \setminus Z^o)$, and this induces a canonical isomorphism

$$
H^3_{\text{dR},Z}(X, \mathbb{Q}(2)) \cong \text{Ker} \left[ \bigoplus_{i=1}^r \eta_{Z_i}^\times \xrightarrow{\partial_1} Z^2(X) \right] \otimes \mathbb{Q}. \quad (2.2)
$$

Let

$$
\partial_2 : K^M_2(\eta_X) \longrightarrow \bigoplus_{\operatorname{codim} Z = 1} \eta_{Z}^\times
$$

\[ \partial_2 \{ f, g \} = \sum_{\text{codim} Z = 1} \left( (-1)^{\text{ord}_Z(f) \text{ord}_Z(g)} \frac{f^{\text{ord}_Z(g)}}{g^{\text{ord}_Z(f)}} \right)^{Z, Z} \]

be the tame symbol. Then the natural map \( H^3_{\text{mot}}(X, \mathbb{Q}(2)) \to H^3_{\text{mot}}(X, \mathbb{Q}(2)) \) together with (2.2) induces the isomorphism

\[
H^3_{\text{mot}}(X, \mathbb{Q}(2)) \cong \left( \frac{\text{Ker}(\bigoplus \eta^0_Z \xrightarrow{\partial_1} Z^2(X))}{\text{Im}(K^M_2(X) \xrightarrow{\partial_2} \bigoplus \eta^0_Z)} \right) \otimes \mathbb{Q}. \tag{2.3}
\]

Let \( K \) be the base field of \( X \) and \( L/K \) a finite extension. Write \( X_L := X \times_K L \). Then there is the obvious map

\[
L^\times \otimes Z^1(X_L) \to H^3_{\text{mot}}(X_L, \mathbb{Q}(2)), \quad \lambda \otimes D \mapsto [\lambda, D].
\]

Let \( N_{L/K} : H^3_{\text{mot}}(X_L, \mathbb{Q}(2)) \to H^3_{\text{mot}}(X, \mathbb{Q}(2)) \) be the norm map on motivic cohomology. Then we put

\[
H^3_{\text{mot}}(X, \mathbb{Q}(2))_{\text{dec}} := \sum_{[L:K]<\infty} N_{L/K}(\text{Im}(L^\times \otimes Z^1(X_L) \to H^3_{\text{mot}}(X_L, \mathbb{Q}(2))))
\]

and call it the decomposable part. We put

\[
H^3_{\text{mot}}(X, \mathbb{Q}(2))_{\text{ind}} := H^3_{\text{mot}}(X, \mathbb{Q}(2))/H^3_{\text{mot}}(X, \mathbb{Q}(2))_{\text{dec}}
\]

and call it the indecomposable part. The indecomposable part plays the central role in the study of \( H^3_{\text{mot}}(X, \mathbb{Q}(2)) \).

### 2.2 Beilinson regulator on indecomposable parts

Let \( X \) be a smooth projective variety over \( \mathbb{C} \). By the theory of universal Chern class, there is the Beilinson regulator map

\[
\text{reg} : H^3_{\text{mot}}(X, \mathbb{Q}(2)) \to H^3_{\mathcal{G}}(X, \mathbb{Q}(2)) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))) \tag{2.4}
\]

to the Deligne-Beilinson cohomology group, which is isomorphic to the Yoneda extension group of mixed Hodge structures where \( H^2(X, \mathbb{Q}(2)) = (H^2_B(X, \mathbb{Q}(2)), F^\bullet H^2_{\text{dR}}(X)) \) denotes the Hodge structure (of weight \(-2\)). We here write down the regulator map (2.4) in terms of extension of mixed Hodge structure.

Let \( Z = \sum_{i=1}^r Z_i \subset X \) be a divisor. There is also the Beilinson regulator map on \( H^3_{\text{mot}, Z}(X, \mathbb{Q}(2)) \), which we denote by \( \text{reg}_{Z} \). Let us consider a commutative diagram

\[
\begin{array}{cccccc}
H^3_{\text{mot}}(X, \mathbb{Q}(2)) & \xrightarrow{\text{reg}} & H^3_{\text{mot}, X}(X, \mathbb{Q}(2)) & \xrightarrow{\text{reg}_Z} & H^3_{\text{mot}, Z}(X^0, \mathbb{Q}(2)) & \xrightarrow{\otimes \text{dlog}} & \mathcal{O}(Z^0)^\times \otimes \mathbb{Q} \tag{2.5} \\
\downarrow & & & & & & \\
H^3_{\mathcal{G}}(X, \mathbb{Q}(2)) & \xrightarrow{\text{reg}_{Z}} & H^3_{\mathcal{G}, Z}(X, \mathbb{Q}(2)) & \xrightarrow{\text{reg}_{Z}} & H^3_{\mathcal{G}, Z}(X^0, \mathbb{Q}(2)) & \xrightarrow{\otimes 1} & H^1_{\mathcal{G}}(Z^0, \mathbb{Q}(1)).
\end{array}
\]
Here the commutativity of the right square follows from the Riemann-Roch theorem without denominator (Gillet), and the others follow from the functoriality of the regulator map. The middle and right squares in (2.5) and the isomorphism (2.1) induce a map

$$\text{Ker} \left[ \bigoplus_{i=1}^{r} \eta^{\infty}_{Z_i} \xrightarrow{\partial} Z^2(X) \right] \rightarrow H^3_{\text{Z}}(X, \mathbb{Q}(2)) \cap H^{0,0}, \quad \xi \mapsto \nu_{\xi}, \quad (2.6)$$

where we write $M \cap H^{0,0} := \text{Hom}_{\text{MHS}}(\mathbb{Q}, M)$. This is characterized by

$$\nu_{\xi}|_{Z^0} = \left( \frac{df_i}{f} \right) \in H^3_{Z^0}(X^0, \mathbb{Q}(2)) \cong H^1(Z^0, Z(1)), \quad \xi = \sum_{i=1}^{r} [f_i, Z_i].$$

Let $\langle Z_i \rangle \subset H^2(X, \mathbb{Q}(2))$ denotes the subgroup generated by the cycle classes of $Z_i$. Then the map $\iota$ in (2.5) induces a commutative diagram

$$\xymatrix{ H^3_{\text{Z}, Z}(X, \mathbb{Q}(2)) \ar[r] \ar[d]_{\iota} & H^3_{\text{Z}}(X, \mathbb{Q}(2)) \cap H^{0,0} \ar[d]_{\delta} \ar[r]_{\text{reg}} & H^3_{\text{Z}}(X, \mathbb{Q}(2)) \ar[r] \ar[d] & \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H^2(X, \mathbb{Q}(2))/\langle Z_i \rangle) \ar[d]_{\text{reg}} \ar[l]_{\text{reg}} }$$

where $\delta$ is the connecting homomorphism arising from the localization exact sequence

$$\cdots \rightarrow H^j_{\text{Z}}(X, \mathbb{Q}(2)) \rightarrow H^j(X, \mathbb{Q}(2)) \rightarrow H^j(X - Z, \mathbb{Q}(2)) \rightarrow \cdots.$$

Summing up the above, we have the following theorem.

**Theorem 2.1 (e.g. [AS] 11.2)** The following diagram is commutative

$$\xymatrix{ \text{Ker} \left[ \bigoplus_{i=1}^{r} \eta^{\infty}_{Z_i} \xrightarrow{\partial} Z^2(X) \right] \ar[r]_{\cong} & H^3_{\text{Z}, Z}(X, \mathbb{Q}(2)) \ar[r]_{\text{reg}}_{\cong} & H^3_{\text{Z}}(X, \mathbb{Q}(2)) \cap H^{0,0} \ar[d]_{\delta} \ar[r]_{\text{reg}} \ar[l]_{\text{reg}} & \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H^2(X, \mathbb{Q}(2))/\langle Z_i \rangle). \ar[l]_{\text{reg}} }$$

For the later use, we write down $\delta$ more explicitly. Let $n = \dim X$. Under the isomorphism of Poincare-Lefschetz duality, the map $\delta$ coincides with the connecting homomorphism arising from the exact sequence

$$\cdots \rightarrow H_i(X, \mathbb{Q}(2 - n)) \rightarrow H_i(X, Z; \mathbb{Q}(2 - n)) \rightarrow H_i(Z, \mathbb{Q}(2 - n)) \rightarrow \cdots.$$

Put $H^i_{\text{dR}}(X)_Z := \text{Ker}[H^i_{\text{dR}}(X) \rightarrow H^i_{\text{dR}}(Z)]$ and

$$M := H^i_{2n-2}(X, \mathbb{Q}(2 - n))/H^i_{2n-2}(Z, \mathbb{Q}(2 - n)) \cong H^2(X, \mathbb{Q}(2))/\langle Z_i \rangle.$$

5
Then there is the natural isomorphism \( M \otimes Q C \cong \text{Hom}(H_{\text{dR}}^{2n-2}(X), C) \) and it induces

\[
\text{Ext}^1_{\text{MHS}}(Q, M) \cong \text{Coker}[H_{2n-2}(X, Q(2 - n)) \xrightarrow{\Phi} \text{Hom}(F^{n-1}H_{\text{dR}}^{2n-2}(X), C)]
\] (2.8)

where \( \Phi \) is defined as follows

\[
\Phi(\Delta) = \begin{bmatrix} \omega \mapsto \int_{\Delta} \omega \end{bmatrix}, \quad \omega \in F^{n-1}H_{\text{dR}}^{2n-2}(X).
\]

Let \( \omega_{X,Z} \in F^{n-1}H_{\text{dR}}^{2n-2}(X, Z) \) denotes a lifting of \( \omega \in F^{n-1}H_{\text{dR}}^{2n-2}(X) \) via the surjective map \( H_{\text{dR}}^{2n-2}(X, Z) \rightarrow H_{\text{dR}}^{2n-2}(X)_Z \). Let \( \gamma \in H_{2n-3}(Z, Q(2 - n)) \cap H^{0,0} \cong H_{2}(X, Q(2)) \cap H^{0,0} \). Let \( \Gamma \in H_{2n-2}(X, Z; Q(2 - n)) \) be an arbitrary element such that \( \partial(\Gamma) = \gamma \). Then we have

\[
\delta(\gamma) = \begin{bmatrix} \omega \mapsto \int_{\Gamma} \omega_{X,D} \end{bmatrix}
\]

under the isomorphism (2.8).

**Proposition 2.2** Let \( Z = \sum_{i=1}^{r} Z_i \) be a divisor. Then

\[
\text{reg}_Z : H^{3}_{B,Z}(X, Q(2)) \longrightarrow H^{3}_{Z}(X, Q(2)) \cap H^{0,0} \cong H_{2n-3}(Z, Q(2 - n)) \cap H^{0,0}
\]

is surjective.

**Proof.** Let \( Z^o = \cup Z^o_i \subset Z \) be a regular locus, and put \( \Sigma := Z \setminus Z^o \) and \( X^o := X \setminus \Sigma \). Let \( Q\Sigma \subset Z^2(X) \) be the subgroup generated by components of \( \Sigma \) of codimension 2. Then we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^{3}_{B,Z}(X, Q(2)) & \longrightarrow & \mathcal{O}(Z^o)^{\times} \otimes Q & \xrightarrow{\partial_1} & Q\Sigma \\
& \text{reg}_Z & \downarrow \text{dlog} & & \downarrow \text{dlog} & & \\
0 & \longrightarrow & H^{3}_{Z}(X, Q(2)) \cap H^{0,0} & \longrightarrow & H^{3}_{Z^o}(X^o, Q(2)) \cap H^{0,0} & \longrightarrow & H^{4}_{\Sigma}(X, Q(2)) \\
& & & & & \xrightarrow{\cong} & \\
& & & & & & H^{1}(Z^o, Q(1)) \cap H^{0,0}
\end{array}
\]

with exact rows. Now the assertion follows from the surjectivity of dlog.

Put

\[
H^{2}_{B}(X)_{\text{ind}} := H^{2}_{B}(X, Q(1))/\text{NS}(X) \otimes Q, \quad H^{2}_{\text{dR}}(X)_{\text{ind}} := H^{2}_{\text{dR}}(X)/\text{NS}(X) \otimes C,
\]

\[
H^{2}(X)_{\text{ind}} := (H^{2}_{B}(X)_{\text{ind}}, F^*H^{2}_{\text{dR}}(X)_{\text{ind}}) (= \text{a Hodge structure of weight 0}).
\]
Then the Beilinson regulator map \((2.4)\) yields a commutative diagram
\[
\begin{array}{c}
\begin{array}{c}
0 \\
H^3_\#(X, \mathbb{Q}(2))_{\text{dec}} \\
H^3_\#(X, \mathbb{Q}(2)) \\
H^3_\#(X, \mathbb{Q}(2))_{\text{ind}}
\end{array}
\end{array}
\longrightarrow
\begin{array}{c}
\begin{array}{c}
\Ext^1_{\text{MHS}}(\mathbb{Q}, \text{NS}(X) \otimes \mathbb{Q}(1))) \\
\Ext^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))) \\
\Ext^1_{\text{MHS}}(\mathbb{Q}, H^2(X)_{\text{ind}} \otimes \mathbb{Q}(1))
\end{array}
\end{array}
\end{array}
\] (2.10)

The top arrow is simply written by "log", namely the composition
\[
\mathbb{C}^\times \otimes \text{Pic}(X) \to H^3_\#(X, \mathbb{Q}(2))_{\text{dec}} \to \Ext^1_{\text{MHS}}(\mathbb{Q}, \text{NS}(X) \otimes \mathbb{Q}(1))) \cong \mathbb{C}/\mathbb{Q}(1) \otimes \text{NS}(X)
\]
is given by \(\lambda \otimes Z \mapsto \log(\lambda) \otimes Z\). The bottom arrow, the regulator map on indecomposable parts, is the main research subject of this paper.

The \textit{real regulator map} is the composition of \(\text{reg}\) and the canonical map
\[
\Ext^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))) \longrightarrow \Ext^1_{\text{R-MHS}}(\mathbb{R}, H^2(X, \mathbb{R}(2)))
\]
to the extension group of real mixed Hodge structures, which we denote by \(\text{reg}_{\text{R}}:\)
\[
\text{reg}_{\text{R}} : H^3_\#(X, \mathbb{Q}(2)) \longrightarrow \Ext^1_{\text{R-MHS}}(\mathbb{R}, H^2(X, \mathbb{R}(2))) \cong H^2_B(X, \mathbb{R}(1)) \cap H^{1,1}. \tag{2.11}
\]
This also induces the map
\[
\text{reg}_{\text{R}} : H^3_\#(X, \mathbb{Q}(2))_{\text{ind}} \to \Ext^1_{\text{R-MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1)) \cong (H^2_B(X)_{\text{ind}} \otimes \mathbb{R}) \cap H^{1,1} \tag{2.12}
\]
on the indecomposable part (we use the same symbol since there will not be confusion).

\subsection{2.3 Q-structure on determinant of \(H^3_\#(X/\mathbb{R}, \mathbb{R}(2))\)}

Suppose that \(X\) is a projective smooth variety over \(\mathbb{Q}\). Write \(X_{\mathbb{C}} := X \times_{\mathbb{Q}} \mathbb{C}\). The \textit{infinite Frobenius map} \(F_\infty\) is defined to be the anti-holomorphic map on \(X(\mathbb{C}) = \text{Mor}_{\mathbb{Q}}(\text{Spec}\mathbb{C}, X)\) induced from the complex conjugation on \(\text{Spec}\mathbb{C}\). For a subring \(A \subset \mathbb{R}\), the infinite Frobenius map acts on the Deligne-Beilinson complex \(A_X(j)_\#\) in a canonical way, so that we have the involution on \(H^*_X(A_X, A(j))\), which we denote by the same notation \(F_\infty\). We define
\[
H^*_X(X/\mathbb{R}, A(j)) := H^*_X(X_{\mathbb{C}}, A(j))^{F_\infty = 1}
\]
the fixed part by \( F_\infty \). We call it the \textit{real Deligne-Beilinson cohomology}. Since the action of \( F_\infty \) is compatible via the Beilinson regulator map, we have

\[
\text{reg}_R : H^3_{\text{dR}}(X, \mathbb{Q}(2)) \longrightarrow H_{\omega} := \text{Ext}^1_{\text{\underline{MHS}}}(\mathbb{R}, H^2(X, \mathbb{R}(2)))^{F_\infty = 1} \cong \frac{H^2_B(X, \mathbb{R}(1))^{F_\infty = 1}}{F^2 H^2_{\text{dR}}(X/\mathbb{R})}, \tag{2.13}
\]

and

\[
\text{reg}_R : H^3_{\text{dR}}(X, \mathbb{Q}(2)) \longrightarrow H_{\omega, \text{ind}} := \text{Ext}^1_{\text{\underline{MHS}}}(\mathbb{R}, H^2(X, \text{ind}(\mathbb{Q}(1)))^{F_\infty = 1} \cong \frac{[H^2_B(X, \text{ind}(\mathbb{Q}(1)))^{F_\infty = 1}}{F^2 H^2_{\text{dR}}(X/\mathbb{R})}. \tag{2.15}
\]

There are the canonical \( \mathbb{Q} \)-structures \( e_Q \) and \( e_{\text{ind}, \mathbb{Q}} \) on the determinant vector spaces \( \text{det} \, H_{\omega} \) and \( \text{det} \, H_{\omega, \text{ind}} \): \( \mathbb{R} \cdot e_Q = \text{det} \, H_{\omega} \), \( \mathbb{R} \cdot e_{\text{ind}, \mathbb{Q}} = \text{det} \, H_{\omega, \text{ind}} \).

Here we recall the definition. The isomorphisms (2.14) and (2.16) induce

\[
\text{det} \, H_{\omega} \cong \text{det} [H^2_B(X, \mathbb{R}(1))^{F_\infty = 1}] \otimes [\text{det} \, F^2 H^2_{\text{dR}}(X/\mathbb{R})]^{-1}, \tag{2.17}
\]

and

\[
\text{det} \, H_{\omega, \text{ind}} \cong \text{det} [(H^2_B(X, \text{ind}(\mathbb{Q}(1)))^{F_\infty = 1}] \otimes [\text{det} \, F^2 H^2_{\text{dR}}(X/\mathbb{R})]^{-1}. \tag{2.18}
\]

The right hand sides of (2.17) and (2.18) have the \( \mathbb{Q} \)-structures induced from the \( \mathbb{Q} \)-structures

\[
H^2_B(X, \mathbb{Q}(1))^{F_\infty = 1}, \quad H^2_B(X, \text{ind})^{F_\infty = 1}, \quad F^2 H^2_{\text{dR}}(X/\mathbb{Q}).
\]

The \( \mathbb{Q} \)-structures \( e_Q \) and \( e_{\text{ind}, \mathbb{Q}} \) are defined to be the corresponding one:

\[
\mathbb{Q} \cdot e_Q \cong \text{det} [H^2_B(X, \mathbb{Q}(1))^{F_\infty = 1}] \otimes [\text{det} \, F^2 H^2_{\text{dR}}(X/\mathbb{Q})]^{-1}, \tag{2.19}
\]

\[
\mathbb{Q} \cdot e_{\text{ind}, \mathbb{Q}} \cong \text{det} [H^2_B(X, \text{ind})^{F_\infty = 1}] \otimes [\text{det} \, F^2 H^2_{\text{dR}}(X/\mathbb{Q})]^{-1}. \tag{2.20}
\]

2.4 \( e_{\text{false}}^Q \) and \( e_{\text{false}}^{\text{ind}, \mathbb{Q}} \)

We introduce other \( \mathbb{Q} \)-structures \( e_{\text{false}}^Q \) and \( e_{\text{false}}^{\text{ind}, \mathbb{Q}} \) on \( \text{det} \, H_{\omega} \) and \( \text{det} \, H_{\omega, \text{ind}} \). For simplicity, we assume \( \dim X = 2 \). Put

\[
H_2(X, \mathbb{Q})_{\text{ind}} := H_2(X, \mathbb{Q})/(\text{NS}(X) \otimes \mathbb{Q}(1)) \cong H^2_B(X, \text{ind} \otimes \mathbb{Q}(1),
\]

\[
H^2_{\text{dR}}(X/\mathbb{Q})_{\text{ind}} := \text{Coim}(H^2_{\text{dR}}(X/\mathbb{Q}) \longrightarrow H^2_{\text{dR}}(X/\mathbb{C})/(\text{NS}(X) \otimes \mathbb{C})).
\]

Note that \( H^2_{\text{dR}}(X/\mathbb{Q})_{\text{ind}} \otimes \mathbb{C} \cong H^2_{\text{dR}}(X/\mathbb{C})/(\text{NS}(X) \otimes \mathbb{C}) \). There are exact sequences

\[
0 \longrightarrow H^2(X, \mathbb{R})^{F_\infty = 1} \longrightarrow \text{Hom}(F^1 H^2_{\text{dR}}(X/\mathbb{Q}), \mathbb{R}) \longrightarrow H_{\omega} \longrightarrow 0 \tag{2.21}
\]

\[
0 \longrightarrow H^2(X, \mathbb{R})^{\text{ind} = 1} \longrightarrow \text{Hom}(F^1 H^2_{\text{dR}}(X/\mathbb{Q})_{\text{ind}}, \mathbb{R}) \longrightarrow H_{\omega, \text{ind}} \longrightarrow 0. \tag{2.22}
\]
under the canonical isomorphisms
\[ H^2_B(X_C, \mathbb{C}) \cong H^2_{\text{dr}}(X/\mathbb{C}), \quad H^2_B(X_C, \mathbb{Q}(2)) \cong H_2(X_C, \mathbb{Q}). \tag{2.23} \]
Then we define \( e^\text{false}_Q \) and \( e^\text{false}_{\text{ind}, \mathbb{Q}} \) the \( \mathbb{Q} \)-structures induced from
\[ H_2(X_C, \mathbb{Q})^{F_\infty=1}, \quad H_2(X_C, \mathbb{R})^{\text{ind}, \mathbb{Q}} = H^2_{\text{dr}}(X/\mathbb{Q}), \quad H^2_{\text{dr}}(X/\mathbb{Q})^{\text{ind}}. \]
Hence we have
\[ Q \cdot e^\text{false}_Q \cong [\det H_2(X_C, \mathbb{Q})^{F_\infty=1}]^{-1} \otimes [\det F^1 H^2_{\text{dr}}(X/\mathbb{Q})]^{-1}, \tag{2.24} \]
\[ Q \cdot e^\text{false}_{\text{ind}, \mathbb{Q}} \cong [\det H_2(X_C, \mathbb{Q})^{\text{ind}, \mathbb{Q}}]^{-1} \otimes [\det F^1 H^2_{\text{dr}}(X/\mathbb{Q})^{\text{ind}}]^{-1}. \tag{2.25} \]

**Proposition 2.3** \textbf{Put}
\[ r := \dim H_2(X_C, \mathbb{Q})^{F_\infty=1} = \dim H^2_B(X_C, \mathbb{Q}(1))^{F_\infty=-1}, \]
\[ s := \dim H_2(X_C, \mathbb{Q})^{\text{ind}, \mathbb{Q}} = \dim H^2_B(X_C)^{\text{ind}, \mathbb{Q}} = r - \dim \text{NS}(X_C)^{F_\infty=-1}. \]
Write
\[ H_B := H^2_B(X_C, \mathbb{Q}(1)), \quad H_B, \text{ind} := H^2_B(X_C)^{\text{ind}}, \]
\[ F^* H_{\text{dr}} := F^* H^2_{\text{dr}}(X/\mathbb{Q}), \quad F^* H_{\text{dr}, \text{ind}} := F^* H^2_{\text{dr}}(X/\mathbb{Q})^{\text{ind}} \]
simply. Then
\[ Q \cdot e^\text{false}_Q = Q \cdot e_Q \otimes Q(-r) \otimes H_{\text{dr}} \otimes [\det H_B]^{-1}, \]
\[ Q \cdot e^\text{false}_{\text{ind}, \mathbb{Q}} = Q \cdot e_{\text{ind}, \mathbb{Q}} \otimes Q(-s) \otimes H_{\text{dr}, \text{ind}} \otimes [\det H_{B, \text{ind}}]^{-1}, \]
where we mean
\[ \det H_{\text{dr}} \otimes [\det H_B]^{-1} \subset \det H^2_{\text{dr}}(X/\mathbb{C}) \otimes [\det H^2_B(X_C, \mathbb{C})]^{-1} \cong \mathbb{C}, \quad \text{etc.} \]

**Proof.** By the Poincare duality, one has
\[ \det F^1 H_{\text{dr}} = [\det H_{\text{dr}}]^{-1} \otimes \det F^2 H_{\text{dr}}, \quad \det F^1 H_{\text{dr}, \text{ind}} = [\det H_{\text{dr}, \text{ind}}]^{-1} \otimes \det F^2 H_{\text{dr}}. \]
Moreover one has
\[ \det[H_2(X_C, \mathbb{Q})^{F_\infty=1}] = \det[H^2_B(X_C, \mathbb{Q}(2))^{F_\infty=1}] = \det[H^2_B(X_C)^{F_\infty=-1} \otimes \mathbb{Q}(1)] = \mathbb{Q}(r) \otimes \det H^2_{B, \text{ind}}^{F_\infty=1} \]
and
\[ \det[H_2(X_C, \mathbb{Q})^{\text{ind}, \mathbb{Q}}]^{F_\infty=1} = \det[H^2_{B, \text{ind}}^{F_\infty=-1} \otimes \mathbb{Q}(1)] = \mathbb{Q}(s) \otimes \det H^2_{B, \text{ind}}^{F_\infty=1}. \]
Therefore we have
\[ Q \cdot e^\text{false}_Q \otimes e^{-1}_Q = Q(-r) \otimes [\det H^2_B]^{-1} \otimes [\det H^2_{B, \text{ind}}]^{-1} \otimes [\det H_{\text{dr}}] \]
\[ = Q(-r) \otimes [\det H_B]^{-1} \otimes [\det H_{\text{dr}}] \]
by (2.19) and (2.24), and
\[ Q \cdot e^\text{false}_{\text{ind}, \mathbb{Q}} \otimes e^{-1}_{\text{ind}, \mathbb{Q}} = Q(-s) \otimes [\det H^2_B]^{-1} \otimes [\det H^2_{B, \text{ind}}]^{-1} \otimes [\det H_{\text{dr}, \text{ind}}] \]
\[ = Q(-s) \otimes [\det H_B]^{-1} \otimes [\det H_{\text{dr}, \text{ind}}] \]
by (2.20) and (2.25). This completes the proof. \( \square \)
Remark 2.4 The Poincare duality implies
\[
(\det H_B)^{\otimes 2} \cong H_B^1(X_C, \mathbb{Q}(2))^\otimes \cong H^4_{\text{dR}}(X/\mathbb{Q})^\otimes \cong (\det H_{\text{dR}})^{\otimes 2},
\]
and
\[
(\det H_{B, \text{ind}})^{\otimes 2} \cong H_B^4(X_C, \mathbb{Q}(2))^\otimes \cong H^4_{\text{dR}}(X/\mathbb{Q})^\otimes \cong (\det H_{\text{dR, ind}})^{\otimes 2}.
\]
Therefore \((\det H_{\text{dR}} \otimes [\det H_B]^{-1})\) and \((\det H_{\text{dR, ind}} \otimes [\det H_{B, \text{ind}}]^{-1})\) are contained in \(\sqrt{\mathbb{Q}^r}\) (possibly rational numbers).

3 Cohomology of Fibration of curves and rational 2-forms

3.1 Notation

Let \(X\) (resp. \(C\)) is a projective smooth surface (resp. curve) over \(K\) a field of characteristic 0, and let \(f : X \rightarrow C\) be a surjective morphism with a section \(e : C \rightarrow X\). The general fiber \(X_t := f^{-1}(t)\) is a projective smooth curve of genus \(g > 0\). Throughout this section, we use the following notation.

- Write \(X_K = X \times_K \overline{K}\). Define \(\text{NF}(X_K) \subset \text{NS}(X_K)\) to be the subgroup of the Neron-Severi group generated by the section \(e(C)\) and fibral divisors (i.e. irreducible components of singular fibers).
- \(\text{NF}_{\text{dR}}(X) := H^2_{\text{dR}}(X) \cap (\text{NF}(X_K) \otimes \mathbb{Z}\overline{K}) \subset H^2_{\text{dR}}(X_K/\overline{K}).\)
- For a Zariski open set \(S \subset C\) and \(V := f^{-1}(S)\), we put
  \[
  H^2_{\text{dR}}(V)_0 := \text{Ker}[H^2_{\text{dR}}(V) \rightarrow \prod_{s \in S} H^2_{\text{dR}}(f^{-1}(s)) \times H^2_{\text{dR}}(e(S))]\]
  where the arrow is the restriction map. Note \(H^2_{\text{dR}}(e(S)) = 0\) unless \(S = C\). Note also that “\(f^{-1}(s)\)” suffices to run over only singular fibers and one smooth fiber.
- \(\text{NF}_{\text{dR}}(V) := \text{Im}[\text{NF}_{\text{dR}}(X) \rightarrow H^2_{\text{dR}}(V)].\)

Remark 3.1 The intersection pairing \(\text{NF}(X_K) \otimes \text{NF}(X_K) \rightarrow \mathbb{Q}\) is non-degenerate. This follows from Zariski’s lemma ([BPV] III (8.2)).

Remark 3.2 \(\text{NF}_{\text{dR}}(X) \otimes_K \overline{K} = \text{NF}(X_K) \otimes \mathbb{Z}\overline{K} = \text{NF}_{\text{dR}}(X_K)\) in \(H^2_{\text{dR}}(X_K/\overline{K})\), and hence \(\text{NF}_{\text{dR}}(V) \otimes_K \overline{K} = \text{NF}_{\text{dR}}(V_K)\). This is proven by using [AEC] II Lemma 5.8.1.

Remark 3.3 Let \(\text{NF}_{\text{dR}}(X)^\perp\) denotes the orthogonal complements of \(\text{NF}_{\text{dR}}(X)\) in \(H^2_{\text{dR}}(X)\) with respect to the cup-product pairing. Then \(\text{NF}_{\text{dR}}(X)^\perp = H^2_{\text{dR}}(X)_0\) by definition, and hence \(\text{NF}_{\text{dR}}(X) \oplus H^2_{\text{dR}}(X)_0 = H^2_{\text{dR}}(X)\).
Proposition 3.4 Let $S^o \subset C$ be a Zariski open set such that $U^o := f^{-1}(S^o) \to S^o$ is smooth and the map
\[ \nabla: f_*\Omega^1_{U^o/S^o} \to \Omega^1_{S^o} \otimes R^1f_*\Theta_{U^o} \] (3.1)
induced from the Gauss-Manin connection is bijective. Assume $S^o \neq \emptyset$ (this is true if $f$ has a totally degenerate semistable fiber by Lem. 3.7). Then the following hold.

1) Let $S \subset C$ be an arbitrary Zariski open set and put $V = f^{-1}(S)$. Then $H^2_{\text{dR}}(V)_0 \oplus \text{NF}_{\text{dR}}(V) = H^2_{\text{dR}}(V)$. If $V \neq X$, then we also have $H^2_{\text{dR}}(V)_0 = \text{Im}[\Gamma(V, \Omega^2_V) \to \text{H}^2_{\text{dR}}(V)]$.

2) Let $S_1 \supset S_2$ and $V_i = f^{-1}(S_i)$. Then there is an exact sequence
\[ 0 \to H^2_{\text{dR}}(V_1)_0 \to H^2_{\text{dR}}(V_2) \to \bigoplus_{s \in S_1 - S_2} H^1_{\text{dR}}(f^{-1}(s)). \]

Proof. We may assume $K = \overline{\mathbb{K}}$ by Rem. 3.2. We first prove (1). In case $V = X$, this follows from Remark 3.3. Assume $V \neq X$. We consider a spectral sequence
\[ E_1^{pq} = H^q(V, \Omega^p_V) \Longrightarrow H^{p+q}_{\text{dR}}(V). \]
Since $S$ is affine by the assumption, $E_1^{pq} = H^q(V, \Omega^p_V) = \Gamma(S, R^p f_* \Omega^p_V) = 0$ unless $p \leq 2$ and $q \leq 1$, so that we have
\begin{align*}
E_2^{20} &= E_\infty^{20} = \text{Im}\Gamma(V, \Omega^2_V), & E_2^{11} = E_\infty^{11}, & E_2^{02} = 0, \\
0 &\to \text{Im}\Gamma(V, \Omega^2_V) \to H^2_{\text{dR}}(V) \to E_\infty^{11} \to 0. \quad (3.2)
\end{align*}

Lemma 3.5 The composition of maps
\[ \text{NF}_{\text{dR}}(V) \to H^2_{\text{dR}}(V) \to E_\infty^{11} \]
is surjective.

Proof. Let $Q^o := S \cap S^o$ and $j: V^o := f^{-1}(Q^o) \to V$ be the open immersion. Consider a commutative diagram
\[
\begin{array}{ccc}
\Gamma(V, j_*\Omega^1_{V^o}/\Omega^1_V) & \xrightarrow{\delta} & H^1(V, \Omega^1_V) \\
\downarrow d & & \downarrow d^o \\
H^1(V, \Omega^1_V) & \xrightarrow{j^*} & H^1(V^o, \Omega^1_{V^o})
\end{array}
\] (exact)

Let $x \in \text{Ker} d$. Then $j^*(x) \in \text{Ker} d^o$. We first see that the kernel of $d^o$ is one-dimensional, generated by the cycle class $[e(C)]$. Indeed, since $\nabla$ (3.1) is bijective, one has $R^1 f_*\Omega^1_{V^o} \xrightarrow{\cong} R^1 f_*\Omega^1_{V^o/Q^o}$, and this is generated by the cycle class of $e(C)$ as $\mathcal{O}_{Q^o}$-module. Then one can
Thus it is one-dimensional over $K$. This means $\ker d^0$ is generated by the cycle class $[e(C)]$. Thus $x' := x - c[e(C)]$ for some $c \in K$ is contained in $\ker(j^*) = \text{Im} \delta$. However, as is well-known, the image of $\delta$ is generated by the cycle classes of the irreducible components of $V - V^0$. This shows that $x$ is a linear combination of the cycle classes of $e(C)$ and fibral divisors. Since $\ker(d) \to E^{11}_2 = E^{11}_\infty$ is surjective, we are done. \hfill \Box

We turn to the proof of (1). The composition of maps

$$\text{NF}_{\text{dR}}(V) \to H^2_{\text{dR}}(V) \to \prod_{s \in S} H^2_{\text{dR}}(f^{-1}(s)) \quad (3.3)$$

is given by intersection pairing, and hence is injective by Zariski’s lemma ([BPV] III (8.2)). Moreover since the composition

$$\Gamma(V, \Omega^2_{\text{dR}}) \to H^2_{\text{dR}}(V) \to \prod_{s \in S} H^2_{\text{dR}}(f^{-1}(s))$$

is obviously zero, the second arrow in $\text{(3.3)}$ factors through $E^{11}_2 = E^{11}_\infty$ (cf. $\text{(3.2)}$). Summing up this and Lem. $\text{[3.5]}$, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \text{Im} \Gamma(V, \Omega^2_{\text{dR}}) & \to & H^2_{\text{dR}}(V/K) & \to & E^{11}_\infty & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{NF}_{\text{dR}}(V) & \to & \prod_{s \in S} H^2_{\text{dR}}(f^{-1}(s)) & & & & & & (3.4)
\end{array}
$$

with an exact row. This shows (1).

Next we show (2). Let $\langle f^{-1}(s) \rangle_{s \in S_1 - S_2}$ denotes the $K$-submodule of $H^2_{\text{dR}}(V_1)$ generated by the cycle classes of components of $f^{-1}(s)$ for $s \in S_1 - S_2$. By (1) we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & 0 & \to & \text{NF}_{\text{dR}}(V_1) & \to & \text{NF}_{\text{dR}}(V_2) & & \\
\downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & \langle f^{-1}(s) \rangle_{s \in S_1 - S_2} & \to & H^2_{\text{dR}}(V_1) & \to & H^2_{\text{dR}}(V_2) & \oplus_{s \in S_1 - S_2} H^1_{\text{dR}}(f^{-1}(s)) & \\
\downarrow & & & & \downarrow & & \downarrow & & \\
H^2_{\text{dR}}(V_1)_0 & \to & H^2_{\text{dR}}(V_2)_0 & \to & 0 & & & & 0
\end{array}
$$

with exact row and columns. The map $a$ is surjective and $\ker(a)$ is onto $\langle f^{-1}(s) \rangle_{s \in S_1 - S_2}$. Now the desired assertion follows from the snake lemma. \hfill \Box
3.2 Deligne’s canonical extension

Let $j : S \hookrightarrow C$ be a Zariski open set such that $U = f^{-1}(S) \to S$ is smooth. Put $T := C - S$ and $D := f^{-1}(T)$. By taking the embedded resolution of singularities if necessary, we can assume that $D_{\text{red}}$ is a NCD. We then consider the de Rham cohomology groups

$$H^q_{\text{dR}}(U) = H^q_{\text{zar}}(U, \Omega^\bullet_U) \cong H^q_{\text{zar}}(X, \Omega^\bullet_X(\log D))$$

with the Hodge filtration

$$F^p H^q_{\text{dR}}(U) := \text{Im}[H^q(X, \Omega^{\geq p}_X(\log D)) \hookrightarrow H^q(X, \Omega^\bullet_X(\log D))].$$

Define a sheaf $\Omega^1_{X/C}(\log D)$ by the exact sequence

$$0 \to f^*\Omega^1_C(\log T) \to \Omega^1_X(\log D) \to \Omega^1_{X/C}(\log D) \to 0.$$

This is a locally free sheaf of rank one. Let $\mathcal{H}_e := R^1 f_* \Omega^\bullet_{X/C}(\log D)$ be Deligne’s canonical extension and

$$\mathcal{H}_{e,1} := f_* \Omega^1_{X/C}(\log D), \quad \mathcal{H}_{e,0,1} := \mathcal{H}_e / \mathcal{H}_{e,1} \cong R^1 f_* \mathcal{O}_X$$

the Hodge filtration (cf. Appendix §6.3). The Gauss-Manin connection

$$\nabla : \mathcal{H}_e \to \Omega^1_C(\log T) \otimes \mathcal{H}_e$$

is defined to be the connecting homomorphism arising from an exact sequence

$$0 \to f^*\Omega^1_C(\log T) \otimes \Omega^{\leq 1}_{X/C}(\log D) \to \Omega^\bullet_X(\log D) \to \Omega^\bullet_{X/C}(\log D) \to 0$$

(see Appendix §6.2 for a remark on sign.) Note that $\mathcal{H}_e$ is characterized as a subbundle of $j_* \mathcal{H}$ such that the eigenvalues of $\text{Res}(\nabla)$ are in $[0, 1)$. In particular it does not depend on the choice of $D$. Write

$$H^2_{\text{dR}}(C, \mathcal{H}_e) := H^2_{\text{zar}}(C, \mathcal{H}_e \to \Omega^1_C(\log T) \otimes \mathcal{H}_e).$$

Theorem 3.6 (cf. [Steenbrink-Zucker] §5) There is the natural isomorphism

$$H^1_{\text{dR}}(C, \mathcal{H}_e) \cong H^2_{\text{dR}}(U)_0. \quad (3.8)$$

Moreover the Hodge filtration corresponds in the following way.

$$F^1 H^2_{\text{dR}}(U)_0 \cong H^1_{\text{zar}}(C, \mathcal{H}_{e,1} \to \Omega^1_C(\log T) \otimes \mathcal{H}_{e,1}) \quad (3.9)$$

$$F^2 H^2_{\text{dR}}(U)_0 \cong H^0_{\text{zar}}(C, \Omega^1_C(\log T) \otimes \mathcal{H}_{e,1}) \quad (3.10)$$

$$\text{Gr}_F H^2_{\text{dR}}(U)_0 \cong H^1_{\text{zar}}(C, \mathcal{H}_{e,0,1}) \quad (3.11)$$

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Proof. The exact sequence (3.7) gives rise to a spectral sequence
\[ E_2^{pq} = H_{\text{dR}}^p(C, R^q f_* \Omega^\bullet_{X/C}(\log D)) \implies H_{\text{dR}}^{p+q}(U). \]
This yields
\[ 0 \rightarrow H_1^\text{dR}(C, \mathcal{H}_e) \rightarrow H_2^\text{dR}(U) \rightarrow H_0^\text{dR}(C, R^2 f_* \Omega^\bullet_{X/C}(\log D)) \rightarrow 0. \]
Since the last term is one-dimensional, isomorphic to \( H_1^\text{dR}(X_t) \), we have (3.8).

(3.7) induces an exact sequence
\[ 0 \rightarrow f^* \Omega^1_C(\log T) \otimes \Omega^{1-2p-1}_{X/C}(\log D) \rightarrow \Omega^\bullet_{X/C}(\log D) \rightarrow \Omega^{\geq p}_{X/C}(\log D) \rightarrow 0 \]
and this yields
\[ H^1_{\text{zar}}(C, R^1 f_* \omega^\bullet_{X/C} \rightarrow \Omega^1_C(\log T) \otimes R^1 f_* \omega^{\geq p-1}_{X/C}) \]
\[ H^2_{\text{zar}}(X, \omega^{\geq p}_{X/C} \rightarrow f^* \Omega^1_C(\log T) \otimes \omega^{\geq p-1}_{X/C}) \xrightarrow{\cong} H^2_{\text{zar}}(X, \Omega^{\geq p}_{X/C}(\log D)) \]
where \( \omega^\bullet_{X/C} := \Omega^\bullet_{X/C}(\log D) \). Now (3.9), (3.10) and (3.11) easily follow from this. \( \square \)

Lemma 3.7 Let
\[ \nabla : \mathcal{H}^{1,0}_e \rightarrow \Omega^1_C(\log T) \otimes \mathcal{H}^{0,1}_e \]  (3.12)
be the \( \mathcal{O}_e \)-linear map induced from the Gauss-Manin connection (3.6). Let \( f^{-1}(P) \) be a fiber over \( P \in T \). If \( f^{-1}(P) \) is semistable, then \( \nabla \) is bijective on a neighborhood of \( P \) if and only if \( f^{-1}(P) \) is totally degenerate. If there is a component of \( f^{-1}(P) \) which is not rational, then \( \nabla \) is not surjective on a neighborhood of \( P \).

Proof. Since the both sides of (3.12) are locally free sheaves of the same rank, the bijectivity of (3.12) is equivalent to the surjectivity of it. By Nakayama’s lemma, it is also equivalent to the surjectivity modulo the maximal ideal at \( P \). We may assume \( K = \mathbb{C} \). Let \( t \in \mathcal{O}_{C,P} \) be the uniformizer and write \( \mathbb{C}_P = \mathcal{O}_{C,P}/t \mathcal{O}_{C,P} \). There is an isomorphism \( \mathcal{H}_e \otimes \mathbb{C}_P \cong H_2^\text{dR}(X_t) \) where \( X_t \) is a smooth fiber (which is “close to \( f^{-1}(P) \)”) ([Steenbrink] (2.18)). Moreover \( \text{Res}_P(\nabla) : H_2^\text{dR}(X_t) \rightarrow H_2^\text{dR}(X_t) \) coincides with the log monodromy operator \( N \) such that the eigenvalues of \( N \) are in \( (0, 1) \) ([Steenbrink] (2.21)). Let \( \hat{F}^* \) be the filtration on \( H_2^\text{dR}(X_t) \) induced from (3.5). Then \( \nabla (3.12) \) is surjective on a neighborhood of \( P \) if and only if the map
\[ \nabla : \hat{F}^1 H_1^\text{dR}(X_t) \rightarrow \text{Gr}_F^0 H_2^\text{dR}(X_t) \]
induced from \( N \) is surjective, or equivalently injective.

Assume that \( f^{-1}(P) \) is a semistable fiber. Then \( \hat{F}^* \) is the limiting Hodge filtration due to Steenbrink ([Steenbrink]). One has \( \text{Ker}(N) = F^1 H_1^\text{dR}(f^{-1}(P)) \) by the local invariant cycle theorem ([Steenbrink] (5.12)). Therefore \( \text{Ker}(N) \neq 0 \) if and only if \( f^{-1}(P) \) is a totally degenerate curve.

There is an obvious inclusion \( F^1 H_1^\text{dR}(f^{-1}(P)) \hookrightarrow \text{Ker}(N) \cap \hat{F}^1 \subset \text{Ker}(N) \) without the assumption that \( f^{-1}(P) \) is a semistable fiber. Therefore if \( f^{-1}(P) \) contains a non-rational curve, then \( \text{Ker}(N) \neq 0 \). \( \square \)
Remark 3.8 In case of elliptic fibration, it follows from Thm. \( \text{6.4 and 6.5} \) that \( \nabla \) is bijective if and only if either of the following conditions holds.

(i) \( f^{-1}(P) \) is a (non-smooth) semistable fiber (i.e. multiplicative),

(ii) \( f^{-1}(P) \) is additive and

\[
\text{Res}_P \left( \frac{t(2g_2dg_3 - 3g_3dg_2)}{\Delta} \right) \neq 0, \quad \Delta := g_2^3 - 27g_3^2
\]

where \( t \in \mathcal{O}_{C,P} \) is a uniformizer and \( y^2 = 4x^3 - g_2x - g_3 \) is the minimal Weierstrass equation of \( f \) over a neighborhood of \( P \).

However, it seems difficult to give a complete criterion of the bijectivity of \( \nabla \) in case \( g > 1 \).

### 3.3 Relative cohomology

For a smooth manifold \( M \), we denote by \( \mathcal{A}^q(M) \) the space of smooth differential \( q \)-forms on \( M \) with coefficients in \( \mathbb{C} \).

Let \( f : X \to C \) be a fibration of curves over \( \mathbb{C} \). Let \( S \subset C \) be an arbitrary Zariski open set, and put \( V := f^{-1}(S) \). Let \( D \subset V \) be a fiber. Let \( \rho : \tilde{D} \to D \) be the normalization and \( \Sigma \subset D \) the set of singular points. Let \( s : \tilde{\Sigma} := \rho^{-1}(\Sigma) \hookrightarrow \tilde{D} \) be the inclusion. There is the exact sequence

\[
0 \to \mathcal{O}_D \xrightarrow{\rho^*} \mathcal{O}_{\tilde{D}} \xrightarrow{s^*} \mathbb{C}_{\tilde{\Sigma}} / \mathbb{C}_\Sigma \to 0
\]

where \( \mathbb{C}_{\tilde{\Sigma}} = \text{Maps}(\tilde{\Sigma}, \mathbb{C}) = \text{Hom}(\mathbb{Z}\tilde{\Sigma}, \mathbb{C}) \), \( \rho^* \) and \( s^* \) are the pull-back. We define \( \mathcal{A}^\bullet(D) \) to be the mapping fiber of \( s^* : \mathcal{A}^\bullet(\tilde{D}) \to \mathbb{C}_{\tilde{\Sigma}} / \mathbb{C}_\Sigma \):

\[
\mathcal{A}^0(\tilde{D}) \xrightarrow{s^* \oplus d} \mathbb{C}_{\tilde{\Sigma}} / \mathbb{C}_\Sigma \oplus \mathcal{A}^1(\tilde{D}) \xrightarrow{0 \oplus d} \mathcal{A}^2(\tilde{D})
\]

where the first term is placed in degree 0. Then

\[
H^q_{\text{dR}}(D) = H^q(\mathcal{A}^\bullet(D))
\]

is the de Rham cohomology of \( D \), which fits into the exact sequence

\[
\ldots \to H^0_{\text{dR}}(\tilde{D}) \to \mathbb{C}_{\tilde{\Sigma}} / \mathbb{C}_\Sigma \to H^1_{\text{dR}}(D) \to H^1_{\text{dR}}(\widetilde{D}) \to \ldots
\]

There is the natural pairing

\[
H_1(D, \mathbb{Z}) \otimes H^1_{\text{dR}}(D) \to \mathbb{C}, \quad \gamma \otimes z \mapsto \int_\gamma z := \int_\gamma \eta - c(\partial^{-1}(\gamma))
\]

where \( z = (c, \eta) \in \mathbb{C}_{\tilde{\Sigma}} / \mathbb{C}_\Sigma \oplus \mathcal{A}^1(\tilde{D}) \) with \( d\eta = 0 \) and \( \partial \) denotes the boundary of homology cycles.
We define $\mathcal{A}^\bullet(V, D)$ to be the mapping fiber of $j^*: \mathcal{A}^\bullet(V) \to \mathcal{A}^\bullet(D)$ the pull-back of $j: D \hookrightarrow V$:

$$\mathcal{A}^0(V) \xrightarrow{\mathcal{D}_0} \mathcal{A}^0(\tilde{D}) \oplus \mathcal{A}^1(V) \xrightarrow{\mathcal{D}_1} \mathcal{C}_E/C_\Sigma \oplus \mathcal{A}^1(\tilde{D}) \oplus \mathcal{A}^2(V) \xrightarrow{\mathcal{D}_2} \cdots$$

where

$$\mathcal{D}_0 = j^* \oplus d, \quad \mathcal{D}_1 = \begin{pmatrix} - (s^* \oplus d) & j^* \cr d & \cr \end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix} - (0 \oplus d) & j^* \cr d & \cr \end{pmatrix}, \cdots$$

Then

$$H^q_{\text{dR}}(V, D) = H^q(\mathcal{A}^\bullet(V, D))$$

is the de Rham cohomology which fits into the exact sequence

$$\cdots \to H^{q-1}_{\text{dR}}(D) \to H^q_{\text{dR}}(V, D) \to H^q_{\text{dR}}(V) \to H^q_{\text{dR}}(D) \to \cdots.$$  

(3.15)

An element of $H^2_{\text{dR}}(V, D)$ is described by $z = (c, \eta, \omega) \in \mathcal{C}_E/C_\Sigma \oplus \mathcal{A}^1(\tilde{D}) \oplus \mathcal{A}^2(V)$ with $j^*\omega = d\eta$ and $d\omega = 0$ which are subject to relations $(s^* f, df, 0) = 0$ and $(0, j^* \theta, d\theta) = 0$ for $f \in \mathcal{A}^0(\tilde{D}_0)$ and $\theta \in \mathcal{A}^1(V)$. The natural pairing

$$H_2(V, D; \mathbb{Z}) \otimes H^2_{\text{dR}}(V, D) \to \mathbb{C}, \quad \Gamma \otimes z \mapsto \int_\Gamma z$$

is given by

$$\int_\Gamma z := \int_\Gamma \omega - \int_{\partial \Gamma} (c, \eta) = \int_\Gamma \omega - \int_{\partial \Gamma} \eta + c(\rho^{-1}(\partial \Gamma)).$$

(3.17)

**Lemma 3.9** Put

$$H^2_{\text{dR}}(V, D)_0 := \text{Ker}[H^2_{\text{dR}}(V, D) \to \prod_{s \in S} H^2_{\text{dR}}(j^{-1}(s)) \times H^2_{\text{dR}}(e(S))],$$

(3.18)

and hence there is an exact sequence

$$F^1H^1_{\text{dR}}(D) \to F^1H^2_{\text{dR}}(V, D)_0 \to F^1H^2_{\text{dR}}(V)_0 \to 0.$$  

(3.19)

For $\omega \in F^1H^2_{\text{dR}}(V)_0$ let $\omega_{V,D} \in F^1H^2_{\text{dR}}(V, D)_0$ be a lifting. Let $\Gamma \in H_2(V, D; \mathbb{Q})$. If $\gamma := \partial \Gamma \in H^1_0(D, \mathbb{Q})$ belongs to the Hodge $(0, 0)$-part, then $\int_\Gamma \omega_{V,D}$ does not depend on the choice of the lifting $\omega_{V,D}$.

**Proof.** Let $\omega'_{V,D}$ be another lifting, then $\omega'_{V,D} - \omega_{V,D}$ belongs to $F^1H^1_{\text{dR}}(D)$. Therefore by (3.17) the assertion follows from the fact that the pairing

$$F^0H^1(D, \mathbb{C}) \otimes F^1H^1_{\text{dR}}(D) \to \mathbb{C}$$

induced from (3.13) is zero. □
3.4 \( \Lambda(U)_{\text{rat}} \) and \( \Lambda(X)_{\text{rat}} \)

Let \( j : S \hookrightarrow C \) be a Zariski open set such that \( U = f^{-1}(S) \to S \) is smooth. Put \( T := C - S \) and \( D := f^{-1}(T) \). We denote by \( \mathcal{H}_e \) Deligne’s canonical extension as in \([3.2]\). Let \( C^o \subset C \) be the maximal open set such that \((3.12)\) is bijective on \( C^o \). We assume \( C^o \neq \emptyset \). By Lemma \([3.7]\) if \( f \) has a totally degenerate semistable fiber, then \( P \in C^o \) and hence \( C^o \neq \emptyset \). Put \( X^o := f^{-1}(C^o) \). We first introduce two spaces of rational 2-forms

\[
\Lambda^2(U)_{\text{rat}} \subset \Lambda^1(U)_{\text{rat}} \subset \Gamma(C^o, \Omega^1_C(\log T) \otimes \mathcal{H}^{1,0}_e) \subset \Gamma(U \cap X^o, \Omega^2_{X^o}).
\]

Define

\[
\Lambda^2(U)_{\text{rat}} := \text{Im}[\Gamma(C, \Omega^1_C(\log T) \otimes \mathcal{H}^{1,0}_e) \to \Gamma(C^o, \Omega^1_C(\log T) \otimes \mathcal{H}^{1,0}_e)] \cong F^2H^2_{dR}(U).
\]

We define \( \Lambda^1(U)_{\text{rat}} \) in the following way. Let us consider a diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \Omega^1_C(\log T) \otimes \mathcal{H}^{1,0}_e|_{C^o} \\
\downarrow & & \downarrow \\
\mathcal{H}^{1,0}_e|_{C^o} \rightarrow & \Omega^1_C(\log T) \otimes \mathcal{H}_e|_{C^o} \\
\downarrow & \cong & \downarrow \\
\mathcal{H}^{1,0}_e|_{C^o} \rightarrow & \Omega^1_C(\log T) \otimes \mathcal{H}^{0,1}_e|_{C^o} \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

Since \( \nabla \) is bijective by definition of \( C^o \), one has an isomorphism

\[
H^{1}_{\text{zar}}(C^o, \mathcal{H}^{1,0}_e) \to \Omega^1_C(\log T) \otimes \mathcal{H}_e \cong \Gamma(C^o, \Omega^1_C(\log T) \otimes \mathcal{H}^{1,0}_e). \tag{3.21}
\]

We define \( \Lambda^1(U)_{\text{rat}} \) to be the image of the composition of the following maps

\[
H^{1}_{\text{zar}}(C, \mathcal{H}^{1,0}_e) \to \Omega^1_C(\log T) \otimes \mathcal{H}_e \to H^{1}_{\text{zar}}(C^o, \mathcal{H}^{1,0}_e) \to \Omega^1_C(\log T) \otimes \mathcal{H}_e \tag{3.22}
\]

\[
\cong \Gamma(C^o, \Omega^1_C(\log T) \otimes \mathcal{H}^{1,0}_e). \tag{3.23}
\]

**Proposition 3.10**

\[
F^1H^2_{dR}(U) \cong H^{1}_{\text{zar}}(C, \mathcal{H}^{1,0}_e) \to \Omega^1_C(\log T) \otimes \mathcal{H}_e \cong \Lambda^1(U)_{\text{rat}}.
\]

**Proof.** The first isomorphism is due to Thm\([5.6]\). To show the second, it is enough to show the injectivity of \((3.22)\). However this follows from the fact that \( F^1H^2_{dR}(U) \to F^1H^2(U \cap X^o) \) is injective by Prop.\([3.4](2)\). \(\square\)
Let $\text{Res}_D : H^2_{\text{dR}}(U) \rightarrow H^1_{\text{dR}}(D)$ be the residue map along $D$. We define

$$\Lambda^i(X)_{\text{rat}} := \Lambda^i(U)_{\text{rat}} \cap \text{Ker}(\text{Res}_D).$$

(3.24)

By definition one has

$$\Lambda^i(X)_{\text{rat}} \subset \Gamma(C^o, \Omega^1_C((\log T) \otimes \mathcal{H}^{1,0}_e)) \cap \text{Ker}(\text{Res}_D) = \Gamma(X^o, \Omega^2_{X^o}).$$

(3.25)

Moreover $\Lambda^i(X)_{\text{rat}}$ does not depend on the choice of $U$.

**Proposition 3.11** $\Lambda^i(X)_{\text{rat}} \cong F^iH^2_{\text{dR}}(X)_0$. In particular, it is stable under a birational transformation $X' \rightarrow X$.

*Proof.* Prop. [3.4](2) and the definition of $\Lambda^i(X)_{\text{rat}}$ give rise to a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \Lambda^i(X)_{\text{rat}} & \rightarrow & \Lambda^i(U)_{\text{rat}} & \rightarrow & H^i_{\text{dR}}(D) \\
0 & \rightarrow & F^iH^2_{\text{dR}}(X)_0 & \rightarrow & F^iH^2_{\text{dR}}(U)_0 & \rightarrow & H^i_{\text{dR}}(D)
\end{array}
$$

with exact rows. Now the assertion follows from Prop. [3.10](2).

The following theorem is one of the technical key results, will be used in the proof of the main theorem (see Thm. [4.1](2)).

**Theorem 3.12** Let $D_0 := f^{-1}(P)$ be a fiber contained in $X^o$. For $\omega \in \Lambda^1(X)_{\text{rat}}$, let $\tilde{\omega} = (0, 0, \omega) \in H^2_{\text{dR}}(X^o, D_0)$ be the cohomology class in terms of the RHS of (3.14). Then $\tilde{\omega}$ belongs to $F^1H^2_{\text{dR}}(X^o, D_0)$ where $F^*$ denotes the Hodge filtration.

To prove Theorem [3.12](2) we may assume that $(D_0)_{\text{red}}$ and $D_{\text{red}}$ are NCD’s. Let $j : D_0 \rightarrow X$ and $\rho : \tilde{D}_0 \rightarrow D_0$ the normalization. Let $\Sigma \subset D_0$ be the singular locus and put $s : \Sigma := \rho^{-1}(\Sigma) \subset \tilde{D}_0$.

We use the Cech cocycles to describe the de Rham cohomology groups. Let us denote by $(\check{C}^*(\mathcal{F}), \delta) = (\check{C}^*(X, \mathcal{F}), \delta)$ the Cech complex of a Zariski sheaf $\mathcal{F}$. Let

$$\begin{array}{cccc}
\check{C}^1(\mathcal{O}_X) \times \check{C}^0(\Omega^1_X) & \overset{\partial_1}{\rightarrow} & \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^1_X) \times \check{C}^0(\Omega^2_X) & \overset{\partial_2}{\rightarrow} & \check{C}^3(\mathcal{O}_X) \times \check{C}^2(\Omega^1_X) \times \check{C}^1(\Omega^2_X) \\
\end{array}
$$

$$\begin{array}{c}
\partial_1 = \begin{pmatrix}
\delta & -d \\
\delta & \delta & -d \\
\end{pmatrix},
\partial_2 = \begin{pmatrix}
\delta & d & -d \\
\delta & \delta & -d \\
\end{pmatrix}.
\end{array}
$$

Then the cohomology of the middle term of the above complex gives $H^2_{\text{dR}}(X)$. In the same way we obtain the description of $H^2_{\text{dR}}(U) = H^2(X, \Omega^*_{X_0}(log D))$. Let

$$\begin{array}{cccc}
\check{C}^1(\mathcal{O}_V) \times \check{C}^0(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_V) & \overset{\partial_1}{\rightarrow} & \check{C}^2(\mathcal{O}_V) \times \check{C}^1(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_V) \times \check{C}^0(\mathcal{O}_\Sigma/\mathcal{O}_{\Sigma} \oplus \Omega^1_{\tilde{D}_0} \oplus \Omega^2_V) & \overset{\partial_2}{\rightarrow} & \check{C}^3(\mathcal{O}_V) \times \check{C}^2(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_V) \times \check{C}^1(\mathcal{O}_\Sigma/\mathcal{O}_{\Sigma} \oplus \Omega^1_{\tilde{D}_0} \oplus \Omega^2_V)
\end{array}
$$
Then the cohomology of the middle term of the above complex gives $H^2_{\text{dR}}(X, D_0)$. Note that

$$\hat{\omega} = (0) \times (0, 0) \times (0, \omega) \in H^2_{\text{dR}}(X^0, D_0)$$

in terms of the Cech cocycles.

**Lemma 3.13** Let $z = (0) \times (\alpha_{ij}) \times (\beta_i) \in \text{Ker}(\mathcal{D}_2)$ be a Cech cocycle, and $[z] \in F^1H^2_{\text{dR}}(X)$ its cohomology class $[z] \in F^1H^2_{\text{dR}}(X)$. Assume $[z] \in \text{Ker}[H^2_{\text{dR}}(X) \rightarrow H^2_{\text{dR}}(D_0)] \cong H^2_{\text{dR}}(\bar{D}_0)$, so that there is $$(\epsilon_i) \in \check{C}^0(\Omega^1_{D_0})$$ such that $\alpha_{ij} \big|_{\bar{D}_0} = \epsilon_j - \epsilon_i$. Put

$$z_{X, D_0} := (0) \times (0, \alpha_{ij}) \times (0, \epsilon_i, \beta_i) \in \text{Ker}(\mathcal{D}_4).$$

Then we have $[z_{X, D_0}] \in F^1H^2_{\text{dR}}(X, D_0)$ and this is a lifting of $[z]$ via the map $F^1H^2_{\text{dR}}(X, D_0) \rightarrow F^1H^2_{\text{dR}}(X)$.

**Proof.** Obvious from the definition of Hodge filtration. □

We turn to the proof of Theorem 3.14. Let

$$(\alpha_{ij}) \times (\beta_i) \in \check{C}^1(\mathcal{H}^{1,0}_e) \times \check{C}^0(\Omega^1_{C}(\log T) \otimes \mathcal{H}_e)$$

be a corresponding Cech cocycle to $\omega \in \Lambda^1(X)_{\text{rat}}$, and this defines

$$z := (0) \times (\eta_{ij}) \times (\pi_i) \in \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^1_X(\log D)) \times \check{C}^0(\Omega^2_X(\log D))$$

in a natural way. Since $[z] \in F^1H^2_{\text{dR}}(U)$ lies in the image of $F^1H^2_{\text{dR}}(X)_0$, there is a Cech cocycle $w = (0) \times (*) \times (*) \in \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^1_X) \times \check{C}^0(\Omega^2_X)$ such that $[w] \in F^1H^2_{\text{dR}}(X)_0$ and $[w]|_U = [z]$ in $H^2_{\text{dR}}(U)$. Since the map $H^2(X, \Omega^{\bullet+1}_X(\log D)) \rightarrow H^2(X, \Omega^{\bullet+1}_X(\log D))$ is injective, we see that there is $\tilde{y} = (0) \times (\tilde{\nu}_i) \in \check{C}^1(\mathcal{O}_X) \times \check{C}^0(\Omega^1_X(\log D))$ such that

$$w = z - \mathcal{D}_1(\tilde{y}) = (0) \times (\eta_{ij} - (\tilde{\nu}_j - \tilde{\nu}_i)) \times (\pi_i - d\tilde{\nu}_i)$$

(3.26)

and this belongs to $\check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^1_X) \times \check{C}^0(\Omega^2_X)$. Therefore, by Lemma 3.13 there is $(\epsilon_i) \in \check{C}^0(\Omega^1_{D_0})$ such that $\epsilon_j - \epsilon_i = \eta_{ij} - (\tilde{\nu}_j - \tilde{\nu}_i)|_{\bar{D}_0}$, and

$$z_{X, D_0} := (0) \times (0, \eta_{ij} - (\tilde{\nu}_j - \tilde{\nu}_i)) \times (0, \epsilon_i, \pi_i - d\tilde{\nu}_i)$$

(3.27)

defines a lifting $[z_{X, D_0}] \in F^1H^2_{\text{dR}}(X, D_0)$ of $\omega \in \Lambda^1(X)_{\text{rat}}$. We want to show that $z_{X, D_0} \equiv \hat{\omega} = (0) \times (0, 0) \times (0, \omega)$ in $H^2_{\text{dR}}(X, D_0)$ modulo the image of $F^1H^2_{\text{dR}}(D_0)$. To do this it is enough to show

$$z_{X, D_0}|_V = \hat{\omega}|_V \in H^2_{\text{dR}}(V, D_0)/\text{Im}F^1H^2_{\text{dR}}(D_0)$$

(3.28)

\footnote{One cannot directly apply Lemma 3.13 to a Cech cocycle $(0) \times (0) \times (\omega) \in \check{C}^2(\mathcal{O}_{X^0}) \times \check{C}^1(\Omega^1_{X^0}) \times \check{C}^0(\Omega^2_{X^0})$ to show $\hat{\omega} \in F^1H^2_{\text{dR}}(X^0, D_0)$ because $X^0$ is not complete.}
for a (sufficiently small) neighborhood $V$ of $D_0$.

By the definition of the locus $C_v$, there is $y_0 \in \tilde{C}^0(\mathcal{H}_e|_{C_v})$ such that

$$(0) \times (\omega) = (\alpha_{ij}) \times (\beta_i) - \mathcal{D}_0(y_0)$$

where $\mathcal{D}_0 : \tilde{C}^0(\mathcal{H}_e) \to \tilde{C}^1(\mathcal{H}_e^1) \times \tilde{C}^0(\Omega^1_{\mathcal{H}_e}(\log T) \otimes \mathcal{H}_e)$. This means that there is $y = (0) \times (\nu_i) \in \tilde{C}^1(\mathcal{O}_{X^0} \times \tilde{C}^0(\Omega^1_{X^0}(\log D)))$ such that

$$z|_{X^0} - \mathcal{D}_1(y) = (0) \times (\eta_{ij}|_{X^0} - (\nu_j - \nu_i)) \times (\pi_i|_{X^0} - d\nu_i) = (0) \times (0) \times (\omega). \quad (3.29)$$

Therefore we have

$$z_{X,D_0}|_{X^0} = (0) \times (0, (\nu_j - \nu_j) - (\nu_i - \nu_i)) \times (0, \epsilon_i, \omega + d\nu_i - d\nu_i) \quad (3.30)$$

with

$$((\nu_j - \nu_j) - (\nu_i - \nu_i))|_{D_0} = \epsilon_j - \epsilon_i.$$

We note that $\nu_i$ and $\nu_i$ have at most log pole along $D_0$.

**Lemma 3.14** Let $V$ be a (sufficiently small) neighborhood $D_0$. Let $t \in \mathcal{O}_{C,P}$ be a uniformizer at $P$. Then there is a constant $c$ such that

$$\theta_i := \tilde{\nu}_i|_V - \nu_i|_V - c \frac{dt}{t}$$

has no log pole along $D_0$.

**Proof.** There is the exact sequence

$$0 \rightarrow \Omega^1_V \rightarrow \Omega^1_V(\log D_0) \rightarrow \mathcal{O}_{D_0} \rightarrow 0.$$

Since $((\nu_j - \nu_j) - (\nu_i - \nu_i))$ has log pole along $D_0$, one has $\text{Res}(\nu_i - \nu_i) = \text{Res}(\nu_j - \nu_j)$ and hence it defines

$$e := (\text{Res}(\nu_i - \nu_i))_i \in \text{Ker}[\tilde{C}^0(\mathcal{O}_{D_0}) \rightarrow \tilde{C}^1(\mathcal{O}_{D_0})] = H^0(\mathcal{O}_{D_0}).$$

Put $e' := ((\nu_j - \nu_j) - (\nu_i - \nu_i)) \in \tilde{C}^1(\Omega^1_V)$. Then the cohomology class $[e'] \in H^1(V, \Omega^1_V)$ is the image of $e$ via the connecting homomorphism $H^0(\mathcal{O}_{D_0}) \rightarrow H^1(\Omega^1_V)$. On the other hand, it follows from (3.30) that the class $[e']|_{D_0} \in H^1(\Omega^1_{D_0}) \cong H^2_{\mathbb{A}^1}(D_0)$ coincides with the image of $z_{X^0,D_0}|_V$ via the composition of maps $H^2_{\mathbb{A}^1}(V, D_0) \rightarrow H^2_{\mathbb{A}^1}(V) \rightarrow H^2_{\mathbb{A}^1}(D_0)$, and hence the image of $\omega|_V$ via $H^2_{\mathbb{A}^1}(V) \rightarrow H^2_{\mathbb{A}^1}(D_0)$. However this is obviously zero. Thus we have

$$e \in \text{Ker}[H^0(\mathcal{O}_{D_0}) \rightarrow H^2_{\mathbb{A}^1}(D_0)] = \langle \text{Res}(\frac{dt}{t}) \rangle \cong K$$

where the middle equality follows from Zariski’s lemma ([BPV] III (8.2)). This means that there is a constant $c$ such that

$$\theta_i := \nu_i|_V - \tilde{\nu}_i|_V - c \frac{dt}{t}$$

has no log pole along $D_0$. □
Let us prove \((3.28)\). By Lemma \(3.14\) one can put \(\varepsilon_i := \theta_i|_{\tilde{D}_0}\). Hence we have from \((3.30)\)

\[
    z_{x,D_0}|_V = (0) \times (0, -\theta_j - \theta_i) \times (0, -\theta_i|_{\tilde{D}_0}, \omega|_V - d\theta_i)
    \equiv (0) \times (0, 0) \times (0, 0, \omega|_V) \mod \text{Im} D_3
\]

as required. This completes the proof of Thm. \(3.12\).

### 3.5 Lefschetz thimbles

Suppose \(K = \mathbb{C}\). Let \(S \subset C\) be an arbitrary Zariski open set, and put \(\overline{U} := f^{-1}(S)\). Put

\[
    \text{NF}^B(\overline{U}) := \text{Im}[\bigoplus_{s \in S} H_2(f^{-1}(s), \mathbb{Z}) \oplus H_2(e(S), \mathbb{Z}) \to H_2(\overline{U}, \mathbb{Z})],
\]

\[
    H_2(\overline{U}, \mathbb{Z})_0 := H_2(\overline{U}, \mathbb{Z})/\text{NF}^B(\overline{U}).
\]

Note \(H_2(e(S), \mathbb{Z}) = 0\) unless \(S = C\). By definition \(H^2_{\text{dR}}(\overline{U})_0 \cong \text{Hom}(H_2(\overline{U}, \mathbb{Q}), \mathbb{C})\).

Let \(S \subset \overline{S}\) be a Zariski open set such that \(U := f^{-1}(S) \to S\) is smooth. We put \(T := \overline{S} - S, D := f^{-1}(T)\). Take a path \(\gamma : [0, 1] \to \overline{S}(\mathbb{C}), t \mapsto \gamma_t\) such that \(\gamma_t \in S(\mathbb{C})\) for \(t \neq 0, 1\). Take a cycle \(\varepsilon \in H_1(f^{-1}(\gamma_t), \mathbb{Z})\) for some (fixed) \(t_0 \in [0, 1]\). Then it extends to a flat section \(\varepsilon_t \in H_1(f^{-1}(\gamma_t), \mathbb{Z})\) over \(t \in [0, 1]\) in a unique way. Let \(\Gamma(\varepsilon, \gamma)\) be the fibration over the path \(\gamma\) whose fiber is \(\varepsilon_t\).

Then

\[
    \Gamma(\varepsilon, \gamma) \in H_2(\overline{U}, f^{-1}(\gamma_0) + f^{-1}(\gamma_1); \mathbb{Z}), \quad \text{with} \quad \partial(\Gamma(\varepsilon, \gamma)) = \varepsilon_1 - \varepsilon_0
\]

where \(\partial : H_2(\overline{U}, f^{-1}(\gamma_0) + f^{-1}(\gamma_1); \mathbb{Z}) \to H_1(f^{-1}(\gamma_0) + f^{-1}(\gamma_1), \mathbb{Z})\) is the boundary map. The homology cycle \(\Gamma(\varepsilon, \gamma)\) is called a Lefschetz thimble. Define \(E(U, D; \mathbb{Z}) \subset H_2(\overline{U}, D; \mathbb{Z})\) the subgroup generated by the Lefschetz thimbles \(\Gamma(\varepsilon, \gamma)\) such that the initial and terminal points of \(\gamma\) lie in \(T\) (hence \(\partial\Gamma(\varepsilon, \gamma) \subset D\)). Define \(E(\overline{U}, \mathbb{Z})\) by an exact sequence

\[
    0 \to E(\overline{U}, \mathbb{Z}) \to E(\overline{U}, D; \mathbb{Z}) \to H_1(D, \mathbb{Z}).
\]

Write \(E(U, D; \mathbb{Q}) := E(U, D; \mathbb{Z}) \otimes \mathbb{Q}\) etc.
**Proposition 3.15** Assume that $f$ contains a totally degenerate semistable fiber. Then we have
\[
\begin{array}{c}
0 \longrightarrow E(U, \mathbb{Q}) \longrightarrow E(U, D; \mathbb{Q}) \longrightarrow H_1(D, \mathbb{Q}) \longrightarrow 0
\end{array}
\] (3.31)
where $H_2(U, D; \mathbb{Q})_0 := H_2(U, D; \mathbb{Q})/\text{Im} N^B(U) = H_2(U, D; \mathbb{Q})/H_2(e(\mathcal{S}), \mathbb{Q})$.

**Lemma 3.16** The fixed part $H^1(f^{-1}(s), \mathbb{Q})_{\pi_1(S, s)}$ is trivial. Moreover we have $H^1(U, \mathbb{Q}) = H^1(\mathcal{S}, \mathbb{Q})$. In particular $H^1(X, \mathbb{Q}) = H^1(C, \mathbb{Q})$.

**Proof.** Let $f^{-1}(P)$ be a totally degenerate semistable fiber, and $N$ the log monodromy on $H^1(f^{-1}(s), \mathbb{Q})$ around $P$. Then the inclusion
\[H^1(f^{-1}(s), \mathbb{Q})_{\pi_1(S, s)} = \Gamma(S, R^1f_*, \mathbb{Q}) \hookrightarrow \text{Ker}(N) \cong H^1(f^{-1}(P), \mathbb{Q})\]
preserves the mixed Hodge structure. The LHS is of weight one, while the RHS is of weight zero as $f^{-1}(P)$ is totally degenerate. Therefore the inclusion must be zero, which means $H^1(f^{-1}(s), \mathbb{Q})_{\pi_1(S, s)} = 0$. Now it is easy to show $H^1(U, \mathbb{Q}) = H^1(S, \mathbb{Q})$ by using the Leray spectral sequence for $f : U \to S$. The equality $H^1(U, \mathbb{Q}) = H^1(\mathcal{S}, \mathbb{Q})$ follows from this and a commutative diagram
\[
\begin{array}{ccc}
0 = H^1_2(U, \mathbb{Q}) & \longrightarrow & H^1(U, \mathbb{Q}) \\
\downarrow & & \downarrow \cong \\
0 = H^1(S, \mathbb{Q}) & \longrightarrow & H^1(S, \mathbb{Q}) \\
\end{array}
\]
\[
\begin{array}{ccc}
H^1_0(T, \mathbb{Q}) & \longrightarrow & H^1_0(T, \mathbb{Q})
\end{array}
\]

**Lemma 3.17** The sequence
\[
H_2(U, \mathbb{Q}) \longrightarrow H_2(U, D; \mathbb{Q})_0 \longrightarrow H_1(D, \mathbb{Q}) \longrightarrow 0
\] (3.32)
is exact.

**Proof.** The surjectivity of $\partial$ is immediate from the fact that the composition $H_1(D, \mathbb{Q}) \to H_1(U, \mathbb{Q}) \cong H_1(\mathcal{S}, \mathbb{Q})$ is zero. Let us show
\[
\text{Im}(H_2(U, \mathbb{Q}) \longrightarrow H_2(U, D; \mathbb{Q})_0) = \text{Im}(H_2(U, \mathbb{Q}) \longrightarrow H_2(U, D; \mathbb{Q})_0).
\] (3.33)
Write \[H^2(D, \mathbb{Q})_0 := \text{Coker}[H_2(e(\mathcal{S}), \mathbb{Q}) \to H^2_0(U, \mathbb{Q}) \cong H^2(D, \mathbb{Q})] \]
Consider a diagram

\[
\begin{array}{cccccc}
H_2(D, \mathbb{Q}) & \xrightarrow{a} & H_2(U, \mathbb{Q})_0 & \xrightarrow{b} & H^2(D, \mathbb{Q})_0 & \xrightarrow{c} & H_1(U, \mathbb{Q})\\
H_2(U, \mathbb{Q}) & \xrightarrow{d} & H_2(U, \mathbb{Q})_0 & \xrightarrow{e} & H^2(D, \mathbb{Q})_0 & \xrightarrow{f} & H_1(U, \mathbb{Q})
\end{array}
\]

with exact row and column. Hence it is enough to show \( \text{Im}(ba) = \text{Im}(b) \) or equivalently \( \dim \text{Coker}(ba) = \dim \text{Coker}(b)(= \dim \text{Ker}(c)) \). Since \( ba \) is given by the intersection pairing, Zariski’s lemma ([BPV] III (8.2)) shows that \( \dim \text{Coker}(ba) = \dim H_0(T) \) if \( S \neq C \) and \( = \dim H_0(T) - 1 \) if \( S = C \). On the other hand,

\[
\text{Ker}[H_1(U, \mathbb{Q}) \to H_1(U, \mathbb{Q})] \cong \text{Ker}[H_1(S, \mathbb{Q}) \to H_1(S, \mathbb{Q})] \cong \text{Coker}[H_2(S) \to H^0(T)]
\]

where we used Lemma 3.16 in the first isomorphism. So we are done. \( \square \)

**Lemma 3.18** Let \( f^{-1}(P) \) be a totally degenerate semistable fiber. Let \( \text{Ev}_P \subset H_1(f^{-1}(s), \mathbb{Q}) \) be the subspace generated by the vanishing cycles as \( s \to P \). Then we have

\[
\mathbb{Q}[\pi_1(S, s)](\text{Ev}_P) = H_1(f^{-1}(s), \mathbb{Q}).
\]

**Proof.** Put \( V = \mathbb{Q}[\pi_1(S, s)](\text{Ev}_P) \). By Deligne’s semisimplicity theorem ([HodgeII] 4.2.6) there is an complementary space \( V' \subset H_1(f^{-1}(s), \mathbb{Q}) \) which is stable under the action of \( \pi_1(S, s) \). Let \( N \) be the log monodromy around \( P \). Since \( \text{Im}(N) = \text{Ev}_P \) one has \( NV' \subset V' \cap \text{Ev}_P = 0 \). On the other hand the composition of maps \( V' \hookrightarrow H_1(f^{-1}(s), \mathbb{Q})/\text{Ev}_P \xrightarrow{N} \text{Ev}_P \) is injective and its image is \( NV' \). Therefore we have \( V' = 0 \). \( \square \)

**Proof of Prop. 3.15** Let \( \mathcal{L} \) be the local system on \( S^0(\mathbb{C}) \) whose fiber is \( H_1(f^{-1}(s), \mathbb{Q}) \). Then the image of \( H_2(U, \mathbb{Q}) \in H_2(U, D; \mathbb{Q})_0 \) coincides with that of \( H_1(S, \mathcal{L}) \). The homology group \( H_1(S, \mathcal{L}) \) is generated by Lefschetz thimbles \( \Gamma(\epsilon, \gamma) \) such that the initial and terminal points of \( \gamma \) are the same in \( S \) and \( \partial \Gamma(\epsilon, \gamma) = 0 \). Take an arbitrary path \( \delta \) such that the initial point lies in \( T \) and the terminal point is that of \( \gamma \). Put \( \tilde{\gamma} = \delta \cdot \gamma \cdot \delta^{-1} \). Then \( \Gamma(\epsilon, \tilde{\gamma}) \in \mathcal{E}(U, D; \mathbb{Z}) \) and the image of it in \( H_2(U, D; \mathbb{Q}) \) coincides with that of \( \Gamma(\epsilon, \gamma) \). This means that there is some subgroup \( \mathcal{E}(U, D; \mathbb{Q})' \subset \mathcal{E}(U, D; \mathbb{Z}) \) such that the image of \( \mathcal{E}(U, D; \mathbb{Q})' \) in \( H_2(U, D; \mathbb{Q}) \) coincides with that of \( H_2(U, \mathbb{Q}) \).

Next we show that the boundary map \( \partial : \mathcal{E}(U, D; \mathbb{Q}) \to H_1(D, \mathbb{Q}) \) is surjective. Let \( f^{-1}(P) \) be a totally degenerate semistable fiber and \( \text{Ev}_P \subset H_1(f^{-1}(s), \mathbb{Q}) \) the space of the vanishing cycles. By Lemma 3.18 for any \( \nu \in H_1(f^{-1}(s), \mathbb{Q}) \) there is a sum of Lefschetz thimbles \( \Gamma = \sum \Gamma(\epsilon, \gamma) \) with \( \gamma \in \pi_1(S, s) \) such that \( \partial \Gamma = \nu \)-(vanishing cycle). By adding a path from \( s \) to a point \( s_0 \in T \) and a path from \( s \) to a point \( P \) to \( \Gamma \), one has a thimble \( \Gamma' \in \mathcal{E}(U, D; \mathbb{Q}) \) such that \( \partial \Gamma' = \nu \in H_1(f^{-1}(s_0), \mathbb{Q}) \). This means that \( \mathcal{E}(U, D; \mathbb{Q}) \to H_1(D, \mathbb{Q}) \) is surjective.
There remains to show the injectivity of \( E(U, D; Q) \to H_2(U, D; Q)_0 \). This is trivial unless \( \overline{S} = C \). In case \( \overline{S} = C \), this follows from the following fact. The composition \( E(U, D; Q) \to H_2(U, D; Q) \to H_2(C, T; Q) \cong Q \) is zero, while the composition \( H_2(e(C)) \to H_2(U, D; Q) \to H_2(C, T; Q) \cong Q \) is bijective. Q.E.D.

4 A formula for Regulator on \( K_1 \) of a fibration of curves

The following is the main theorem of this paper.

**Theorem 4.1** Let \( f : X \to C \) be a fibration of curves over \( \mathbb{C} \) as in §3.1. Suppose that \( f \) has a totally degenerate semistable fiber. Let \( C^o \subset C \) be the maximal open set such that \( \nabla (3.12) \) is bijective on \( C^o \). Put \( X^o = f^{-1}(C^o) \). Let \( \Phi : E(X^o, Q) \cong H_2(X^o, Q)/NF^B(X^o) \to \text{Hom}(\Lambda^1(X)_{\text{rat}}, \mathbb{C}) \)

be the map of period integral defined by

\[
\Phi(\Delta) = \left[ \omega \mapsto \int_{\Delta} \omega \right], \quad \omega \in \Lambda^1(X)_{\text{rat}}
\]

where \( \Lambda^1(X)_{\text{rat}} \) and \( E(X^o, Q) \) are as in §3.4 and §3.5 respectively, and the isomorphism is due to Prop.3.15.

(1) There is an isomorphism

\[
\text{Ext}^1(Q, H^2(X, Q(2))/NF(X)) \cong \text{Coker}(\Phi).
\]

(2) Let \( D = \sum_i f^{-1}(P_i) \) be a union of singular fibers which are contained in \( X^o \). Let \( \xi \in H^3_{\text{et}, D}(X, Q(2)) \) be an arbitrary element, and put \( \gamma := \text{reg}_D(\xi) \in H^1_B(D, Q) \). Fix \( \Gamma \in E(X^o, D; Q) \) such that \( \partial(\Gamma) = \gamma \).

Then

\[
\text{reg}(\xi) = \left[ \omega \mapsto \int_{\Gamma} \omega \right] \in \text{Coker}(\Phi).
\]

**Proof.** Since \( \Lambda^1(X)_{\text{rat}} \cong F^1H^3_{\text{et}}(X)_0 \) by Prop. 3.11, the period map \( \Phi \) factors through \( H_2(X, Q)/NF^B(X) \). Then (1) follows from the fact that \( H_2(X^o, Q) \to H_2(X, Q)/NF^B(X) \) is surjective. We show (2). Let \( \delta \) be the composition of maps

\[
\begin{align*}
\text{Hom}_{\text{MHS}}(Q, H_1(D, Q)) & \to \text{Ext}^1_{\text{MHS}}(Q, H_2(X, Q)/H_2(D, Q)) \quad (4.1) \\
& \to \text{Ext}^1_{\text{MHS}}(Q, H_2(X, Q)/NF^B(X)) \quad (4.2) \\
& \cong \text{Ext}^1_{\text{MHS}}(Q, H^2(X, Q(2))/NF(X)) \quad (4.3)
\end{align*}
\]

\footnote{Note that, since \( \Lambda^1(X)_{\text{rat}} \subset F(X^o, \Omega^2_X) \) (3.25), one can \textit{a priori} define \( \int_{\Delta} \omega \) only for \( \Delta \in H_2(X^o, Q) \).}
where (4.1) is the connecting homomorphism arising from the exact sequence
\[ 0 \rightarrow H_2(X, \mathbb{Q})/H_2(D, \mathbb{Q}) \rightarrow H_2(X, D; \mathbb{Q}) \rightarrow H_1(D, \mathbb{Q}) \rightarrow 0. \]

Then one has \( \text{reg}(\xi) = \delta(\text{reg}_D(\xi)) = \delta(\gamma) \) by Theorem 2.1. To compute \( \delta(\gamma) \), we consider a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H_2(X, \mathbb{Q})/H_2(D, \mathbb{Q}) & \rightarrow & H_2(X, D; \mathbb{Q}) & \rightarrow & H_1(D, \mathbb{Q}) & \rightarrow & 0 \\
\text{surj.} & & \text{surj.} & & \text{surj.} & & \text{surj.} & & \\
0 & \rightarrow & H_2(X, \mathbb{Q})/\text{NF}^B(X) & \rightarrow & H_2(X, D; \mathbb{Q}) & \rightarrow & H_1(D, \mathbb{Q}) & \rightarrow & 0 \\
\end{array}
\]

Then it is enough to describe the extension data of the bottom row in (4.4). For \( \omega \in \Lambda^1(X)_{\text{rat}} \), let \( \omega_{X^0, D} \in F^1H^2_{\text{dR}}(X^0, D) \) denotes a lifting. Then we have
\[
\text{reg}(\xi) = \left[ \omega \mapsto \int_{\Gamma} \omega_{X^0, D} \right] \in \text{Hom}(\Lambda^1(X)_{\text{rat}}, \mathbb{C})/\text{Im}\Phi
\]
by (2.9). On the other hand, by Theorem 3.12 and (3.17), we have
\[
\int_{\Gamma} \omega_{X^0, D} = \int_{\Gamma} \omega.
\]
This competes the proof. \( \square \)

5 Example : Elliptic fibration with \( \mu_l \)-action

Let \( F \subset \mathbb{R} \) be a subfield. We consider two polynomials \( g_2(t), g_3(t) \in F[t] \) which satisfy the following (however see Remark 5.4).

- **(E1)** \( \Delta := g_2^3 - 27g_3^2 = ct^a(1 - t)^b \) for some \( a, b \in \mathbb{Z}_{\geq 1} \) and \( c \in \mathbb{R}_{>0} \),
- **(E2)** \( 2g_2g_3 - 3g_2'g_3 = cl' t^a(1 - t)^b' \) for some \( a', b' \in \mathbb{Z}_{\geq 0} \) and \( c' \neq 0 \),
- **(E3)** \( g_2(0), g_2(1) > 0 \) and \( g_3(0)g_3(1) < 0 \).
- **(E4)** \( g_2(t) \geq 0 \) for \( 0 \leq t \leq 1 \).

Let \( l \geq 1 \) and \( \kappa \in F^\times \). We discuss an elliptic fibration
\[
f : X = X_l \rightarrow \mathbb{P}^1, \quad f^{-1}(t) : \kappa y^2 = 4x^3 - g_2(t)x - g_3(t')
\]
defined over \( F \). In what follows, we take \( X \) to be minimal, i.e. there is no exceptional curve in a fiber. There is the section \( e : \mathbb{P}^1 \rightarrow X \) of “infinity”. Let \( \zeta_l \) be a \( l \)-th root of unity, and
Let $\sigma$ be an automorphism of $X_C = X \times_F \mathbb{C}$ given by $(x, y, t) \mapsto (x, y, \zeta t)$. Put $D := f^{-1}(1)$ a multiplicative fiber of type $I_b$. Let us choose $\kappa$ such that $D = f^{-1}(1)$ is split multiplicative over $F$, or equivalently
\[
\sqrt{-6kg_3(1)} \in F^\times.
\]
Then one has an element $\xi \in H^3_{\text{mot}, D}(X, \mathbb{Q}(2))$ such that $\gamma := \text{reg}_D(\xi) \in H^2(D(C), \mathbb{Q})$ is a generator (cf. Prop. (2.2)). This is uniquely determined modulo the decomposable part. We are going to compute the real regulator
\[
\text{reg}_{\mathbb{R}}(\xi) \in \text{Ext}^1_{\mathbb{R} \text{-MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1))^{F_\infty = 1}, \quad H^2(X)_{\text{ind}} := H^2(X, \mathbb{Q}(1))/\text{NS}(X)
\]
where $F_\infty$ is the infinite Frobenius.

### 5.1 Computation of $\Lambda(X)_{\text{rat}}$

The elliptic fibration (5.1) is smooth outside $t = \infty, \zeta^i, (0 \leq i \leq l - 1)$ by (E1). It follows from (E3) that $f^{-1}(0)$ is a semistable fiber of type $I_{al}$, and $f^{-1}(\zeta^i)$ is of type $I_b$ (cf. Tate’s algorithm, [Silverman] IV). Let $\nu_\infty$ be the number of irreducible components of $f^{-1}(\infty)$, and $\varepsilon_\infty$ the Kodaira index of $f^{-1}(\infty)$:

| $f^{-1}(\infty)$ | 0 | b | 2 | 3 | 4 | b + 6 | 10 | 9 | 8 |
|-------------------|---|---|---|---|---|-------|----|---|---|
| smooth $I_b$      | II | III | IV | $I_b$ | II* | III* | IV* |    |   |

Note that
\[
\varepsilon_\infty = \begin{cases} 
\nu_\infty - 1 & \text{smooth} \\
\nu_\infty & \text{multiplicative} \\
\nu_\infty + 1 & \text{additive.}
\end{cases} \tag{5.2}
\]

As is well-known, we have

\[
\begin{cases} 
K_X \cong f^* \mathcal{O}_{\mathbb{P}^1}(\frac{al+bl+\varepsilon_\infty}{12} - 2) \quad \text{(Kodaira’s canonical bundle formula)} \\
h^{20}(X) := \dim H^0(\Omega_X^2) = \frac{al+bl+\varepsilon_\infty}{12} - 1 \\
b^2 := \text{rank} H^2(X) = al + bl + \varepsilon_\infty - 2 \\
\rho_f := \text{rank}_{\mathbb{Q}}(H^2(X)) = al + (b - 1)t + \nu_\infty.
\end{cases} \tag{5.3}
\]

**Lemma 5.1** Let $s = t^{-1}$ and $k \geq 0$ be the minimal integer such that both of $\bar{g}_2(s) := s^{4k}g_2(s^{-1})$ and $\bar{g}_3(s) := s^{6k}g_3(s^{-1})$ have no pole. This is equivalent to saying that $k$ is the integer such that
\[
a'y_1^2 = 4x_1^3 - \bar{g}_2(s)x_1 - \bar{g}_3(s), \quad x_1 := s^{2k}x, \quad y_1 = s^{3k}y
\]
is the minimal Weierstrass equation of $X$ over $s = 0$ ($t = \infty$). Then
\[
\Gamma(X, \Omega_X^2) = \langle t^{i-1} dt \frac{dx}{y} \mid 1 \leq i \leq k - 1 \rangle.
\]
In particular, $h^{20} = k - 1$.\]
Proof. \( f_*K_X \cong \mathcal{O}_{\mathbb{P}^1}(\frac{at+bl+e}{12} - 2) \) is a locally free sheaf of rank one. This has a free basis \( dt \frac{dx}{y} \) on \( \mathbb{P}^1 \setminus \{\infty\} \) and \( ds \frac{dx_1}{y_1} \) on a neighborhood of \( s = 0 \) \((t = \infty)\). Then the assertion follows from \( t^{-i} dt \frac{dx}{y} = -s^{k-i-1} ds \frac{dx_1}{y_1} \).

\[ \square \]

Proposition 5.2 Suppose that \( l \) is a prime number and \( h^{20}(X) > 0 \). Then \( \dim H^2(X)_{\text{ind}} = l - 1 \) and \( \text{NF}(X_C) \otimes \mathbb{Q} = \text{NS}(X_C) \otimes \mathbb{Q} \) (hence \( H^2(X)_{0} \cong H^2(X)_{\text{ind}} \)). Moreover \( f^{-1}(\infty) \) is an additive fiber.

Proof. By (5.2) and (5.3), we have \( b^2 - \rho_f = l - 2 + (\varepsilon_\infty - \nu_\infty) \leq l - 1 \). On the other hand, \( \sigma \) acts on \( H^2(X_C, \mathbb{Q})/\text{NF}(X_C) \) and it has an eigenvalue \( \zeta_i \) since \( dt dx/y \in \Gamma(X, \Omega_X^2) \) by Lemma 5.1. Since \( l \) is a prime number, the characteristic polynomial of \( \sigma \) must be divided by \( 1 + x + x^2 + \cdots + x^{l-1} \), and hence its degree is at least \( l - 1 \). This implies \( b^2 - \rho_f \geq l - 1 \). Hence we have \( b^2 - \rho_f = l - 1 \) and \( \varepsilon_\infty - \nu_\infty = 1 \). This implies that \( f^{-1}(\infty) \) is an additive fiber by (5.2). Let \( \rho := \text{rankNS}(X_C) \). Obviously \( \rho \geq \rho_f \). Since \( \sigma \) acts on \( H^2(X_C, \mathbb{Q})/\text{NS}(X_C) \) as well, the same argument yields \( b^2 - \rho \geq l - 1 \). We thus have \( \rho \leq \rho_f \) and hence \( \rho = \rho_f \). \( \square \)

Proposition 5.3 Suppose that \( l \) is a prime number and \( h^{20}(X) > 0 \). Then

\[ \Lambda^2(X)_{\text{rat}} = \langle t^{-i} dt \frac{dx}{y} \mid 1 \leq i \leq h^{20} \rangle \cong F^2 H_{dR}(X), \]

\[ \Lambda^1(X)_{\text{rat}} = \langle t^{-i} dt \frac{dx}{y} \mid 1 \leq i \leq l - 1 - h^{20} \rangle \cong F^1 H_{dR}(X)_{\text{ind}}. \]

Proof. The former was shown in Lemma 5.1. We show the latter. Let \( s = t^{-1} \) and \( k, x_1 = s^{2k} \) and \( y_1 = s^{3k} y \) as in Lemma 5.1. Put \( T := \{0, \infty, \zeta_i \mid 0 \leq i \leq l - 1\} \subset \mathbb{P}^1 \) and \( U := f^{-1}(S) \to S := \mathbb{P}^1 - T \). Let \( \mathcal{H} \) be Deligne’s canonical extension of \( \mathcal{H} = R^1 f_* \mathcal{O}_{U/F} \). Let \( \omega, \omega^* \in \Gamma(\mathbb{P}^1 \setminus T, \mathcal{H}) \) be as in Lemma 6.1. They give a free basis of \( \mathcal{H} \) over \( \mathbb{P}^1 \setminus \{\infty\} \) by Theorem 6.5. Since \( f^{-1}(\infty) \) is an additive fiber, \( \{t^{-1} \omega, t^{-k} \omega^*\} \) is a free basis on a neighborhood of \( \infty \). We thus have

\[ \mathcal{H}_{e}^{1,0} = \mathcal{O}_{\mathbb{P}^1}(k - 1), \quad \mathcal{H}_{e}^{0,1} = \mathcal{O}_{\mathbb{P}^1}(-k), \]

(5.4)

\[ \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^2(\log T) \otimes \mathcal{H}) = \left\langle \frac{t^{-1} dt}{t(1 - t^i)} \otimes \omega, \frac{t^{-1} dt}{t(1 - t^j)} \otimes \omega^* \mid 1 \leq i \leq l + k, 1 \leq j \leq l - k + 1 \right\rangle \]

(5.5)

\[ \Gamma(\mathbb{P}^1, \mathcal{H}_{e}^{1,0}) = \langle t^{-i} \frac{dx}{y} \mid 1 \leq i \leq k \rangle \]

(5.6)

and

\[ H_{\text{zar}}^1(\mathbb{P}^1, \mathcal{H}_{e}^{1,0}) \to \Omega_{\mathbb{P}^1}^1(\log T) \otimes \mathcal{H}) \cong \text{Coker}[\Gamma(\mathbb{P}^1, \mathcal{H}_{e}^{1,0}) \to \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log T) \otimes \mathcal{H})]. \]
By Theorem 6.4, we have

$$\mathcal{H}_e^{1,0} \longrightarrow \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathcal{H}_e^{0,1}, \quad \omega \mapsto \frac{3(2g_2g_3' - 3g_3g_3')}{4\Delta} dt \otimes \omega^*.$$  \hspace{1cm} (5.7)

By (E1) and (E2) we have

$$\frac{1}{2t^2} g_2 \frac{dj}{g_3} = \frac{2g_2g_3' - 3g_3g_3'}{\Delta} dt = \frac{dt}{t^{a-a'}(1-t)^{b-b'}} \times \text{(const.)}, \quad j := \frac{1728g_2^3}{\Delta}.$$

By (E3), we have $a - a' = b - b' = 1$:

$$\frac{3(2g_2g_3' - 3g_3g_3')}{4\Delta} dt = \frac{dt}{t(1-t)} \times \text{(const.)}. \hspace{1cm} (5.8)$$

This shows that (5.7) is bijective on $\mathbb{P}^1 \setminus \{\infty\}$. Let

$$\phi : H^1_{\text{rat}}(\mathbb{P}^1, \mathcal{H}_e^{1,0}) \rightarrow \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathcal{H}_e$$

be the composition of (3.22) and (3.23). Then by (5.8) and (6.14), we have

$$\phi \left( \frac{t^{l-1}dt}{t(1-t)} \otimes \omega^* \right) = \left( -t^{l-1} \frac{d\Delta}{12\Delta} + (j-1)t^{l-2}dt \right) \frac{dx}{y} \times \text{(const.)}
= h(t) \times \frac{dt}{t(1-t)} \frac{dx}{y}$$

with $\deg h(t) \leq l + j - 1$. Hence

$$\Lambda^1(U)_{\text{rat}} := \text{Im}(\phi) \subset \left\{ \frac{t^{j-1}dt}{t(1-t)} \frac{dx}{y} \mid 1 \leq j \leq 2l - k + 1 \right\}.$$

This yields

$$\Lambda^1(X)_{\text{rat}} \subset \left\{ t^{j-1}dt \frac{dx}{y} \mid 1 \leq j \leq l - k = l - 1 - h^{20} \right\}.$$  \hspace{1cm} (Remark 5.4)

Since $\dim \Lambda^1(X)_{\text{rat}} = \dim F^1H^2_{\text{DR}}(X)_{0} = l - 1 - h^{20}$ by Prop. 5.2, the equality holds in the above. This is the desired assertion.

Remark 5.4 Since $\dim H^2(X)_{\text{ind}} \cap H^{1,1} = l - 1 - 2h^{2,0} \geq 0$, one has $l - 1 - ((al + bl + c)/6 - 2) \geq 0$ for any large prime number $l$. This implies $a + b \leq 6$, together with $a' = a - 1$ and $b' = b - 1$ by (5.8). Then by case-by-case analysis based on $(a, b)$, one can show that there are only the following pairs of $(g_2, g_3)$ satisfying (E1),. . .,(E4), up to the equivalence $(g_2, g_3) \sim (h^4g_2, h^6g_3)$ or $(g_2(t), g_3(t)) \sim (g_2(1-t), g_3(1-t))$.

(i) $(g_2, g_3) = (3, 1 - 2t), (a, b) = (1, 1)$

(ii) $(g_2, g_3) = (12 - 9t, 8 - 9t), (a, b) = (2, 1)$

(iii) $(g_2, g_3) = (27 - 24t, -8t^2 + 36t - 27), (a, b) = (3, 1)$

(iv) $(g_2, g_3) = (3(t - 16t + 16), (t - 2)(t^2 + 32t - 32)), (a, b) = (4, 1)$

(v) $(g_2, g_3) = (12(t^2 - t + 1), 4(t - 2)(t + 1)(2t - 1)), (a, b) = (2, 2)$.
5.2 Computation of Lefschetz thimbles: Cycles $\Delta$ and $\Gamma$

Let $\delta_0$ (resp. $\delta_1$) be the homology cycle in $H_1(f^{-1}(t), \mathbb{Z})$ which vanishes as $t \to 0$ (resp. $t \to 1$). Define $\Delta$ and $\Gamma$ to be fibrations over the segment $[0, 1] \subset \mathbb{P}^1(\mathbb{C})$ whose fibers are the vanishing cycles $\delta_1$ and $\delta_0$ respectively.

$$\Delta \in H_2(\overline{U}, f^{-1}(0); \mathbb{Z}), \quad \Gamma \in H_2(\overline{U}, f^{-1}(1); \mathbb{Z}).$$

The boundary $\partial \Delta$ (resp. $\partial \Gamma$) is a generator of the homology group $H_1(f^{-1}(0), \mathbb{Z})$ (resp. $H_1(f^{-1}(1), \mathbb{Z})$).

\begin{itemize}
    \item Figure of $\Delta$
    \item Figure of $\Gamma$
\end{itemize}

Lemma 5.5 Suppose that $l$ is a prime number and $h^{20}(X) > 0$. Then $H_2(X, \mathbb{Q})/NF^B(X)$ has a basis $\{\sigma_i^* \Delta - \sigma_i^{*+1} \Delta\}_{0 \leq i \leq l-2}$. 

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Proof. Let $V \subset H_2(X, \mathbb{Q})/\text{NF}^B(X)$ be the subgroup generated by the image of $\{\sigma_i^j \Delta - \sigma_i^j \Delta \}_{i<j}$. We want to show $V = H_2(X, \mathbb{Q})/\text{NF}^B(X)$. Since $H_2(X, \mathbb{Q})/\text{NF}^B(X)$ is an irreducible $\mathbb{Q}[\sigma]$-module (by the proof of Prop. 5.2) and $V$ is stable under the action of $\mathbb{Q}[\sigma]$, it is enough to show $V \neq 0$. By Prop. 5.3, it is enough to show

$$\int_{\Delta - \sigma \Delta} t^{i-1} dt \frac{dx}{y} = (1 - \zeta^i) \int_{\Delta} t^{i-1} dt \frac{dx}{y} \neq 0.$$  

Due to (E1), (E3) and (E4) there exist 3-distinct real roots $r_1(t), r_2(t), r_3(t)$ of $4x^3 - g_2(t)x - g_3(t)$ for $0 < t < 1$. Let them satisfy $r_1(t) > r_2(t) > r_3(t)$ (resp. $r_1(t) < r_2(t) < r_3(t)$) if $\kappa > 0$ (resp. $\kappa < 0$). Then

$$\int_{\Delta} t^{j-1} dt \frac{dx}{y} = 2 \sqrt{-1} \int_0^1 t^{j-1} dt \int_{r_1(t)}^{r_2(t)} \frac{dx}{\sqrt{-\kappa^{-1}(4x^3 - g_2(t)x - g_3(t))}} \in \mathbb{R}_{>0}.$$  

Similarly to (5.9), we have

$$\int_{\Gamma} t^{j-1} dt \frac{dx}{y} = 2 \int_0^1 t^{j-1} dt \int_{r_2(t)}^{r_3(t)} \frac{dx}{\sqrt{-\kappa^{-1}(4x^3 - g_2(t)x - g_3(t))}} \in \mathbb{R}_{>0}. (5.10)$$

Lemma 5.6 Let $F_\infty$ denotes the infinite Frobenius. Then

$$F_\infty(\Delta) = -\Delta, \quad F_\infty(\Gamma) = \Gamma.$$  

If $l$ is a prime number and $h^{20}(X) > 0$, then the fixed part $(H_2^B(X, \mathbb{Q})/\text{NF}^B(X))^{F_\infty=1}$ has a basis

$$\sigma_i^j \Delta - \sigma_i^j \Delta, \quad 1 \leq i \leq \frac{l-1}{2}.$$  

Proof. We show $F_\infty(\Delta) = -\Delta$. Let $\delta_1 \in H_1(f^{-1}(t), \mathbb{Z})$ ($0 < t < 1$) be the vanishing cycle as $t \to 1$. Then it is enough to show $F_\infty(\delta_1) = -\delta_1$. We keep the notation in the proof of Lemma 5.5. Fix $0 < t < 1$ and $x \in [r_1(t), r_2(t)]$. Then $4x^3 - g_1(t)x - g_3(t) \leq 0$ if $\kappa > 0$ and $\geq 0$ if $\kappa < 0$. Therefore $y$ takes values in purely imaginary numbers, so that $F_\infty(x, y) = (x, -y)$. This means $F_\infty \delta_1 = -\delta_1$. In the same way we have $F_\infty \delta_0 = \delta_0$ where $\delta_0$ denotes the vanishing cycle as $t \to 0$. This implies $F_\infty(\Gamma) = \Gamma$ as well. The last assertion follows from this and $F_\infty \sigma = \sigma^{-1} F_\infty$ together with Lemma 5.5. \hfill \square

5.3 Regulator indecomposable elements

Theorem 5.7 Suppose that $l$ is a prime number and $h^{20}(X) > 0$. Put $h = \dim F^1 H^2(X)_{\text{ind}}$ and $\zeta = \exp(2\pi i/l)$. Let

$$A = \left(\zeta^{pq} - \zeta^{-pq}\right) \int_{\Delta} t^{p-1} dt \frac{dx}{y} \quad 1 \leq p \leq h, 1 \leq q \leq (l-1)/2.$$  

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be an $h \times (l-1)/2$-matrix (the entries are real numbers by (5.9)). Then
\[
\text{Ext}^1_{\text{MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1))^{F=1} \cong \text{Coker}[A : \mathbb{R}^{(l-1)/2} \to \mathbb{R}^h]
\]
and we have
\[
\text{reg}_\mathbb{R}(\xi) = \pm \left( \int_\Gamma \frac{dt}{y}, \ldots, \int_{\Gamma} t^{h-1} dt \frac{dx}{y} \right) \in \mathbb{R}^h / \text{Im} A
\]
under the above isomorphism.

**Proof.** The first assertion is obtained by applying Prop. 5.3 and Lemma 5.6 to Theorem 4.1 (1). The second assertion follows from Theorem 4.1 (2). \hfill \square

**Corollary 5.8** Suppose that $l$ is a prime number and $h^{20}(X) > 0$. Then we have
\[
\text{reg}_\mathbb{R}(\xi) \neq 0 \in \text{Ext}^1_{\text{MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1))^{F=1}.
\]
In particular $\xi$ is real regulator indecomposable.

**Proof.** Put
\[
I_p := \int_\Delta t^{p-1} dt \frac{dx}{y}, \quad J_p := \int_\Gamma t^{p-1} dt \frac{dx}{y}.
\]
Then
\[
\text{reg}_\mathbb{R}(\xi_D) \neq 0 \in \text{Ext}^1_{\text{MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1))^{F=1}
\]
if and only if the rank of a matrix
\[
\begin{pmatrix}
(\zeta - \zeta^{-1}) I_1 & (\zeta^2 - \zeta^{-2}) I_1 & \cdots & (\zeta^{(l-1)/2} - \zeta^{-(l-1)/2}) I_1 & J_1 \\
(\zeta^2 - \zeta^{-2}) I_2 & (\zeta^4 - \zeta^{-4}) I_2 & \cdots & (\zeta^{l-1} - \zeta^{-(l-1)}) I_2 & J_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
(\zeta^h - \zeta^{-h}) I_h & (\zeta^{2h} - \zeta^{-2h}) I_h & \cdots & (\zeta^{h(l-1)/2} - \zeta^{-h(l-1)/2}) I_h & J_h
\end{pmatrix}
\]
(5.11)
is maximal. It is enough to show that
\[
\text{det}
\begin{pmatrix}
(\zeta - \zeta^{-1}) & (\zeta^2 - \zeta^{-2}) & \cdots & (\zeta^{(l-1)/2} - \zeta^{-(l-1)/2}) & J_1 / I_1 \\
(\zeta^2 - \zeta^{-2}) & (\zeta^4 - \zeta^{-4}) & \cdots & (\zeta^{l-1} - \zeta^{-(l-1)}) & J_2 / I_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
(\zeta^k - \zeta^{-k}) & (\zeta^{2k} - \zeta^{-2k}) & \cdots & (\zeta^{k(l-1)/2} - \zeta^{-k(l-1)/2}) & J_k / I_k
\end{pmatrix}
\]
(5.12)
is nonzero where $k = (l+1)/2$. Since the sum of the $(k-1)$-th row and $k$-th row is
\[
(0, \cdots, 0, J_{k-1}/I_{k-1} + J_k/I_k),
\]
one has
\[
\begin{align*}
(5.12) &= (J_{k-1}/I_{k-1} + J_k/I_k) \times \text{det}(\zeta^{pq} - \zeta^{-pq})_{1 \leq p, q \leq (l-1)/2} \\
&= (J_{k-1}/I_{k-1} + J_k/I_k) \times \sqrt{(-l)^{(l-1)/2}}.
\end{align*}
\]
Since $J_p/I_p \in \mathbb{R}_{>0}$ by (5.9) and (5.10), this is not zero. \hfill \square
5.4 Explicit computation of regulator

We show more on computation of real regulator in the following case

\[ X = X/Q : -12y^2 = 4x^3 - g_2(t)x - g_3(t), \]
\[ (g_2, g_3) = (12(9 - 8t), -8(8t^2 - 36t + 27)) \]

an elliptic surface defined over \( \mathbb{Q} \) and

\[ \xi = \left[ \frac{y - (x + 1)}{y + (x + 1)}, D \right] \in H^3_m(X, \mathbb{Q}(2)). \]

Here \( D := f^{-1}(1) \) is a multiplicative fiber of type \( I_1 \), and it splits over \( \mathbb{Q} \). One can show that if \( l \geq 5 \) is a prime number, then \( \xi \) is integral in the sense of \([\text{Scholl}]\), namely, it lies in the image of the motivic cohomology group of a regular proper flat model \( X \) over \( \text{Spec} \mathbb{Z} \). When \( l = 1 \), \( X \) is the universal elliptic curve over \( X_1(3) \). (However, if \( l > 1 \) it is no longer a universal elliptic curve for congruence subgroup.) Let \( q = \exp(2\pi iz) \) and

\[
E_{3a}(z) := 1 - 9 \sum_{n=1}^{\infty} \left( \sum_{k|n} \left( \frac{k}{3} \right) k^2 \right) q^n,
\]
\[
E_{3b}(z) := \sum_{n=1}^{\infty} \left( \sum_{k|n} \left( \frac{n/k}{3} \right) k^2 \right) q^n
\]

be the Eisenstein series of weight 3 for \( \Gamma_1(3) \), where \( \left( \frac{k}{3} \right) \) denotes the Legendre symbol. Then

\[ t^l = \frac{E_{3a}}{E_{3a} + 27E_{3b}} \]

and

\[
t^l \frac{dt \ dx}{t \ y} = -27E_{3b} \frac{du \ dq}{u \ q}, \quad \frac{t^{l-1} \ dt \ dx}{t^l - 1 \ y} = E_{3a} \frac{du \ dq}{u \ q}
\]

where \( "du/u" \) denotes the canonical invariant 1-form of the Tate curve around the cusp \( z = i\infty \) (\( t = 1 \)). Therefore we have

\[
\int_{\Delta} t^{l-1} \frac{dt \ dx}{y} = \frac{-27}{l} \times (2\pi i)^2 \int_0^{i\infty} t^l E_{3b}(z) \ dz \quad (5.13)
\]
\[
\int_{\Gamma} t^{l-1} \frac{dt \ dx}{y} = \frac{-27}{l} \times (2\pi i)^2 \int_0^{i\infty} t^l E_{3b}(z) \ dz. \quad (5.14)
\]

On the other hand there are formulas

\[
\frac{E_{3a}}{E_{3a} + 27E_{3b}} \left( \frac{-1}{3z} \right) = \frac{27E_{3b}}{E_{3a} + 27E_{3b}}(z), \quad 27E_{3b} \left( \frac{-1}{3z} \right) = 3\sqrt{3}iz^3E_{3a}(z) \quad (5.15)
\]

on the Eisenstein series. Applying (5.15) to (5.13) and (5.14), we have the following theorem.
**Theorem 5.9** Put \( c := \exp(-2\pi/\sqrt{3}) = 0.026579933 \ldots \). Define rational numbers \( a_n(j) \) and \( b_n(j) \) by

\[
E_{3b} \left( \frac{E_{3a}}{E_{3a} + 27E_{3b}} \right)^{j/l} = \sum_{n=1}^{\infty} a_n(j) q^n \\
= q + \left( 3 - 27^j/l \right) q^2 + \left( 9 - \frac{81j}{2l} + \frac{729}{2} \left( \frac{j}{l} \right)^2 \right) q^3 + \cdots ,
\]

\[
E_{3a} \left( \frac{E_{3b}}{q(E_{3a} + 27E_{3b})} \right)^{j/l} = \sum_{n=0}^{\infty} b_n(j) q^n \\
= 1 + \left( -9 - 15^j/l \right) q + \left( 27 + \frac{387j}{2} + \frac{225}{2} \left( \frac{j}{l} \right)^2 \right) q^2 + \cdots .
\]

Put

\[
I(j) = \sum_{n=1}^{\infty} \frac{a_n(j)}{n} c^n + 3^{3j/l-3} \sum_{n=0}^{\infty} b_n(j) \left( \frac{1}{n+j/l} + \frac{\sqrt{3}}{2\pi(n+j/l)^2} \right) c^n+j/l \\
J(j) = \sum_{n=1}^{\infty} a_n(j) \left( \frac{2\pi}{\sqrt{3n}} + \frac{1}{n^2} \right) c^n + 2\pi \cdot 3^{3j/l-7/2} \sum_{n=0}^{\infty} b_n(j) c^n+j/l.
\]

Then we have

\[
\int_{\Delta} \theta^{j-1} dt \frac{dx}{y} = \frac{54\pi i}{l} I(j), \quad \int_{\Gamma} \theta^{j-1} dt \frac{dx}{y} = -\frac{27}{l} J(j)
\]

for \( 1 \leq j \leq l-1 \).

This is useful since the series \( I(j) \) and \( J(j) \) converge rapidly!

**Example 5.10** Suppose \( l = 5 \). Then \( X \) is a K3 surface. By Thm 5.9 one has

| \( j \) | \( I(j) \) | \( J(j) \) |
|---|---|---|
| 1 | 0.42745977255318 | 0.717696894965804 |
| 2 | 0.151180954233147 | 0.377159120670032 |
| 3 | 0.0871841692346256 | 0.261572572611421 |
| 4 | 0.0603840144077692 | 0.202670503662525 |

\[
\text{Ext}^1_{\mathbb{R} \text{-MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1))_{F_{\infty}=1} \cong \text{Coker} (\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^3).
\]

Since this is 1-dimensional, this has the canonical base \( e_{\text{ind},Q} \) (up to \( \mathbb{Q}^\times \)) and a different base \( e_{\text{ind},Q}^{\text{false}} \). With respect to \( e_{\text{ind},Q}^{\text{false}} \), one has

\[
\text{reg}_{\mathbb{R}}(\xi_D) = \pi^2 \begin{vmatrix} i(\zeta - \zeta^{-1}) I(1) & i(\zeta^2 - \zeta^{-2}) I(1) & J(1) \\ i(\zeta^2 - \zeta^{-2}) I(2) & i(\zeta^4 - \zeta^{-4}) I(2) & J(2) \\ i(\zeta^3 - \zeta^{-3}) I(3) & i(\zeta^6 - \zeta^{-6}) I(3) & J(3) \end{vmatrix} \mod \mathbb{Q}^\times (\zeta := \exp(2\pi i/5)).
\]

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Since \( s = (l - 1)/2 = 2 \) and \( \det \mathcal{H}^2_{dR}(X/\mathbb{Q})_{\text{ind}} \otimes [\det \mathcal{H}^2_B(X_C)_{\text{ind}}]^{-1} = \sqrt{5} \), one has \( e^{false}_{\text{ind}, \mathbb{Q}} = (2\pi \sqrt{-1})^{-2} \sqrt{5} e_{\text{ind}, \mathbb{Q}} = \sqrt{5} \pi^{-2} e_{\text{ind}, \mathbb{Q}} \mod \mathbb{Q}^{\times} \) by Prop. 2.3. Hence

\[
\text{reg}_R(\xi_D) = \sqrt{5} \pi^{-2} \cdot \pi^2 \begin{vmatrix}
\frac{i(\zeta - \zeta^{-1})I(1)}{I(1)} & \frac{i(\zeta^2 - \zeta^{-2})I(1)}{I(1)} & \frac{i(\zeta^3 - \zeta^{-3})I(1)}{I(1)} & J(1) \\
\frac{i(\zeta^2 - \zeta^{-2})I(2)}{I(2)} & \frac{i(\zeta^4 - \zeta^{-4})I(2)}{I(2)} & \frac{i(\zeta^6 - \zeta^{-6})I(2)}{I(2)} & J(2) \\
\frac{i(\zeta^3 - \zeta^{-3})I(3)}{I(3)} & \frac{i(\zeta^6 - \zeta^{-6})I(3)}{I(3)} & \frac{i(\zeta^9 - \zeta^{-9})I(3)}{I(3)} & J(3) \\
\frac{i(\zeta^4 - \zeta^{-4})I(4)}{I(4)} & \frac{i(\zeta^8 - \zeta^{-8})I(4)}{I(4)} & \frac{i(\zeta^{12} - \zeta^{-12})I(4)}{I(4)} & J(4)
\end{vmatrix} = -5\sqrt{5}I(1)I(2)I(3) \left( \frac{J(2)}{I(2)} + \frac{J(3)}{I(3)} \right) = 0.346139631939354 \mod \mathbb{Q}^{\times}
\]

with respect to \( e_{\text{ind}, \mathbb{Q}} \).

**Example 5.11** Suppose \( l = 7 \). Then \( h^{20}(X) = h^{02}(X) = 2, h^{11}(X) = 30 \).

\[
\begin{array}{|c|c|c|}
\hline
& I(j) & J(j) \\
\hline
j = 1 & 0.740059830730164 & 0.987994510350351 \\
\hline
j = 2 & 0.24646699651114 & 0.51401702238944 \\
\hline
j = 3 & 0.137265313181901 & 0.354195498081428 \\
\hline
j = 4 & 0.0929578147374374 & 0.2732367697671921 \\
\hline
j = 5 & 0.0696363855176379 & 0.224004116344261 \\
\hline
j = 6 & 0.0554349861351089 & 0.19073921727221 \\
\hline
\end{array}
\]

\[
\text{Ext}^1_{\text{R-MHS}}(\mathbb{R}, \mathcal{H}^2(X)_{\text{ind}} \otimes \mathbb{R}(1))^{F_{\mathbb{Q}} = 1} \cong \text{Coker}(\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^4).
\]

Since \( s = (l - 1)/2 = 3 \) and \( \det \mathcal{H}^2_{dR}(X/\mathbb{Q})_{\text{ind}} \otimes [\det \mathcal{H}^2_B(X_C)_{\text{ind}}]^{-1} = \sqrt{-7} \), one has

\[
e^{false}_{\text{ind}, \mathbb{Q}} = (2\pi \sqrt{-1})^{-3} \sqrt{-7} e_{\text{ind}, \mathbb{Q}} = \sqrt{7} \pi^{-3} e_{\text{ind}, \mathbb{Q}} \mod \mathbb{Q}^{\times}, \text{ and}
\]

\[
\text{reg}_R(\xi_D) = \sqrt{7} \pi^{-3} \cdot \pi^3 \begin{vmatrix}
\frac{i(\zeta - \zeta^{-1})I(1)}{I(1)} & \frac{i(\zeta^2 - \zeta^{-2})I(1)}{I(1)} & \frac{i(\zeta^3 - \zeta^{-3})I(1)}{I(1)} & J(1) \\
\frac{i(\zeta^2 - \zeta^{-2})I(2)}{I(2)} & \frac{i(\zeta^4 - \zeta^{-4})I(2)}{I(2)} & \frac{i(\zeta^6 - \zeta^{-6})I(2)}{I(2)} & J(2) \\
\frac{i(\zeta^3 - \zeta^{-3})I(3)}{I(3)} & \frac{i(\zeta^6 - \zeta^{-6})I(3)}{I(3)} & \frac{i(\zeta^9 - \zeta^{-9})I(3)}{I(3)} & J(3) \\
\frac{i(\zeta^4 - \zeta^{-4})I(4)}{I(4)} & \frac{i(\zeta^8 - \zeta^{-8})I(4)}{I(4)} & \frac{i(\zeta^{12} - \zeta^{-12})I(4)}{I(4)} & J(4)
\end{vmatrix} = 49I(1)I(2)I(3)I(4) \left( \frac{J(3)}{I(3)} + \frac{J(4)}{I(4)} \right) = 0.629487860860585 \mod \mathbb{Q}^{\times} (\zeta := \exp(2\pi i/7))
\]

with respect to the canonical \( \mathbb{Q} \)-structure \( e_{\text{ind}, \mathbb{Q}} \).

**Remark 5.12** According to the Beilinson conjecture, \( \text{reg}_R(\xi_D) \) in Example 5.10 or 5.11 is expected to be the value of the \( L \)-function \( L'(h^2(X)_{\text{ind}}, 1) \) ([Schneider]).

### 6 Appendix: Gauss-Manin connection for a hyperelliptic fibration

We work over a field \( K \) of characteristic zero. For a smooth scheme \( Y \) over \( T \), we denote by \( \Omega^q_{Y/T} = \lambda_{\sigma}, \Omega^1_{Y/T} \) the sheaf of relative differential \( q \)-forms on \( Y \) over \( T \). If \( T = \text{Spec} K \), we simply write \( \Omega^q_Y = \Omega^q_{Y/K} \).
In this section, we discuss the Gauss-Manin connection
\[ \nabla : R^1 f_* \Omega^\bullet_{U/S} \to \Omega^1_S \otimes R^1 f_* \Omega^\bullet_{U/S} \]
for \( U/S \) a smooth proper family of hyperelliptic curves. This is defined to be the connecting homomorphism
\[ R^1 f_* \Omega^\bullet_{U/S} \to R^2 f_* (f^* \Omega^1_S \otimes \Omega^\bullet_{U/S}) \cong \Omega^1_S \otimes R^1 f_* \Omega^\bullet_{U/S} \quad (6.1) \]
which arises from an exact sequence
\[ 0 \to f^* \Omega^1_S \otimes \Omega^\bullet_{U/S} \to \bar{\Omega}^\bullet_U \to \Omega^\bullet_{U/S} \to 0, \quad \bar{\Omega}^\bullet_U := \Omega^\bullet_U / \text{Im}(f^* \Omega^2_S \otimes \Omega^\bullet_{U/S}) \]
(cf. [Hartshorne] Ch.III, §4). Here the first isomorphism in (6.1) is the projection formula, and the second one is due to the identification \( R^q f_* \Omega^\bullet_{U/S} \cong R^{q-1} f_* \Omega^\bullet_{U/S} \) with which we should be careful about “sign”. Indeed the differential of the complex \( \Omega^\bullet_{U/S} \) is 
\[ -d \]
so that we need to arrange the sign to make an isomorphism between \( R^q f_* \Omega^\bullet_{U/S} \) and \( R^{q+1} f_* \Omega^\bullet_{U/S} \).

We make it by a commutative diagram
\[ \begin{array}{ccc}
\mathcal{O}_U & \xrightarrow{-d} & \Omega^1_{U/S} \\
\downarrow \text{id} & & \downarrow \text{id} \\
\mathcal{O}_U & \xrightarrow{-d} & \Omega^1_{U/S}
\end{array} \quad (6.2) \]

Then \( \nabla \) satisfies the usual Leibniz rule
\[ \nabla(gx) = dg \otimes x + g \nabla(x), \quad x \in \Gamma(S, R^1 f_* \Omega^\bullet_{U/S}), \ g \in \mathcal{O}(S). \]

6.1 Family of hyperelliptic curves

Let \( S \) be an irreducible affine smooth variety over \( K \). Let \( f(x) \in \mathcal{O}_S(S)[x] \) be a polynomial of degree \( 2g + 1 \) or \( 2g + 2 \) which has no multiple roots over any geometric points \( \bar{x} \in S \). Then it defines a smooth family of hyperelliptic curves \( f : U \to S \) defined by the Weierstrass equation \( y^2 = f(x) \). To be more precise, let \( z = 1/x, \ u = y/x^{g+1} \) and put \( g(z) = z^{2g+2} f(1/z) \).

Let
\[ U_0 = \text{Spec} \mathcal{O}_S(S)[x, y]/(y^2 - f(x)), \quad U_\infty = \text{Spec} \mathcal{O}_S(S)[z, u]/(u^2 - g(z)). \]

Then \( U \) is obtained by gluing \( U_0 \) and \( U_\infty \) via identification \( z = 1/x, \ u = y/x^{g+1} \). We assume that there is a section \( e : S \to U \).

\[ x^{i-1} \frac{dx}{y} = -z^{g-i} \frac{dz}{u}, \quad \frac{y}{x^i} = \frac{u}{z^{g+1-i}}, \quad 1 \leq i \leq g. \quad (6.3) \]
We shall compute the Gauss-Manin connection
\[ \nabla : H^1_{\text{dR}}(U/S) \longrightarrow \Omega^1_S \otimes H^1_{\text{dR}}(U/S), \quad H^q_{\text{dR}}(U/S) := H^q_{\text{zar}}(U, \Omega^\bullet_{U/S}) \] (6.4)
(we use the same symbol “\( \Omega^1_S \)” for \( \Gamma(S, \Omega^1_S) \) since it will be clear from the context which is meant). To do this, we describe the de Rham cohomology in terms of the Cech complex. Write
\[ \tilde{\mathcal{C}}^0(U) := \Gamma(U, \mathcal{F}) \oplus \Gamma(U^\infty, \mathcal{F}) \]
for a (Zariski) sheaf \( \mathcal{F} \). Then the double complex
\[ \tilde{\mathcal{C}}^0(U) \quad \delta \quad \tilde{\mathcal{C}}^1(U) \]
\[ \delta \quad \delta \]
\[ \tilde{\mathcal{C}}^1(U) \quad \delta \quad \tilde{\mathcal{C}}^2(U) \]
gives rise to the total complex
\[ \tilde{\mathcal{C}}^\bullet(U/S) : \tilde{\mathcal{C}}^0(U) \rightarrow \tilde{\mathcal{C}}^1(U) \rightarrow \tilde{\mathcal{C}}^2(U) \rightarrow \cdots \]
of \( R \)-modules starting from degree 0, and the cohomology of it is the de Rham cohomology
\[ H^q_{\text{dR}}(U/S) = H^q(\tilde{\mathcal{C}}^\bullet(U/S)), \quad q \geq 0. \]
Elements of \( H^1_{\text{dR}}(U/S) \) are represented by cocycles
\[ (f) \times (x_0, x_\infty) \quad \text{with} \quad df = x_0 - x_\infty. \]

**Lemma 6.1** Suppose
\[ f(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad a_i \in \mathcal{O}(S) \]
with \( n = 2g + 1 \) or \( 2g + 2 \). Put
\[ \omega_i := (0) \times \left( \frac{x^{i-1} dx}{y}, -\frac{z^{g-i} dz}{u} \right), \quad (6.5) \]
\[ \omega_i^* := \left( \frac{y}{x^i} \right) \times \left( \sum_{m \geq i} (m/2 - i) a_m x^{m-i} \right) \left( \frac{dx}{y}, \sum_{m \leq i} (m/2 - i) a_m z^{g-m+i} \right) \left( \frac{dz}{u} \right) \quad (6.6) \]
for \( 1 \leq i \leq g \). Then they give a basis of \( H^1_{\text{dR}}(U/S) \). Moreover \( (6.5) \) span the image of
\[ \Gamma(U, \Omega^1_{U/S}) \hookrightarrow H^1_{\text{dR}}(U/S). \]
**Proof.** Exercise. \( \square \)

**Lemma 6.2** There are the following equivalence relations.
\[ (x^i y^j) \times (0, 0) \equiv (0) \times (-d(x^i y^j), 0) \mod \text{Im} \tilde{\mathcal{C}}^0(\mathcal{O}_U), \]
\[ (z^i u^j) \times (0, 0) \equiv (0) \times (0, d(z^i u^j)) \mod \text{Im} \tilde{\mathcal{C}}^0(\mathcal{O}_U). \]
**Proof.** Straightforward from the definition. \( \square \)
\section{Computation of Gauss-Manin connection}

Let us compute $\nabla(\omega_i)$ and $\nabla(\omega_i^*)$. Recall that there is the exact sequence
\begin{equation}
0 \rightarrow \check{C}^i(f^*\Omega_S^1 \otimes \Omega_U^{-1}) \rightarrow \check{C}^i(\Omega_U^i) \rightarrow \check{C}^i(\Omega_U^i) \rightarrow 0 \tag{6.7}
\end{equation}
and it gives rise to the connecting homomorphism
\[\delta : H^1_{\text{dR}}(U/S) = H^1(\check{C}^i(\Omega_U^i)) \rightarrow H^2(U, f^*\Omega_S^1 \otimes \Omega_U^{-1}) = H^2(\check{C}^i(f^*\Omega_S^1 \otimes \Omega_U^i)).\]

Recall the isomorphism
\[
\check{C}^i(f^*\Omega_S^1 \otimes \Omega_U^{-1}) \cong H_\Omega^1(S) \otimes \check{C}^i(\Omega_U^i)
\]
induced from (6.2). It induces the isomorphism
\[\iota : H^2(U, f^*\Omega_S^1 \otimes \Omega_U^{-1}) \cong H_\Omega^1(S) \otimes H^1_{\text{dR}}(U/S).\]

By definition we have $\nabla = i\delta$ the Gauss-Manin connection (6.4). Let us write down the maps $\delta$ and $\iota$ in terms of Cech cocycles. The differential operator $\partial$ on the total complex of the middle term of (6.7) is given as follows
\[
\partial : \check{C}^1(\Omega_U) \times \check{C}^0(\Omega_U^i) \rightarrow \check{C}^1(\Omega_U^i) \times \check{C}^0(\Omega_U^i),
\]
\[(\alpha) \times (\beta_0, \beta_\infty) \mapsto (-d\alpha + \beta_0 - \beta_\infty) \times (d\beta_0, d\beta_\infty).
\]
We denote a lifting of $(z_0, z_\infty) \in \check{C}^0(\Omega_U^1) \otimes \Omega_U^i)$ by $(\hat{z}_0, \hat{z}_\infty) \in \check{C}^0(\Omega_U^i)$. Then for $(\alpha) \times (z_0, z_\infty) \in H^1_{\text{dR}}(U/S)$ one has
\begin{align*}
(\alpha) \times (z_0, z_\infty) \xrightarrow{\delta} \partial((\alpha) \times (\hat{z}_0, \hat{z}_\infty)) & = (-d\alpha + \hat{z}_0 - \hat{z}_\infty) \times (d\hat{z}_0, d\hat{z}_\infty) \tag{6.8} \\
& \in \check{C}^1(f^*\Omega_S^1) \times \check{C}^0(f^*\Omega_S^1 \otimes \Omega_U^i). \tag{6.9}
\end{align*}

The isomorphism $\iota$ is given by
\[
(gdt) \times (dt \wedge z_0, dt \wedge z_\infty) \xrightarrow{\iota} dt \otimes [(-g) \times (z_0, z_\infty)] \tag{6.11}
\]
(the “sign” appears in the above due to (5.2)).

To compute $\nabla(\omega_i)$ and $\nabla(\omega_i^*)$ for the basis in Lemma 6.1, there remains to compute lifting of $dx/y$ and $dz/u$.

\textbf{Lemma 6.3} Let $A, B \in \mathcal{O}(S)[x]$ and $C, D \in \mathcal{O}(S)[z]$ satisfy
\[
Af + B\frac{\partial f}{\partial x} = 1, \quad Cg + D\frac{\partial g}{\partial z} = 1.
\]

Put differential 1-forms
\[
\frac{dx}{y} := \frac{Af + Bdf}{y} = Aydx + 2Bdy \in \Gamma(U_0, \Omega^1_U),
\]

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\[
\frac{\widehat{dz}}{u} := \frac{Cgdz + Ddg}{u} = Cudz + 2Ddu \in \Gamma(U_\infty, \Omega^1_U).
\]

Then
\[
x^i \frac{\widehat{dx}}{y} \in \Gamma(U_0, \Omega^1_{U/S}), \quad z^i \frac{\widehat{dz}}{u} \in \Gamma(U_\infty, \Omega^1_{U/S})
\]
are liftings of \(x^i dx/y \in \Gamma(U_0, \Omega^1_{U/S})\) and \(z^i dz/u \in \Gamma(U_\infty, \Omega^1_{U/S})\) respectively.

**Proof.** Straightforward. \(\square\)

By using the liftings in Lemma 6.3, one can compute the map \(\delta\). With use of Lemma 6.2, one finally obtains the connection matrix of \(\nabla\).

Here is an explicit formula in case of elliptic fibration (the proof is left to the reader).

**Theorem 6.4** Let \(S\) be a smooth affine curve and \(f : U \to S\) a projective smooth family of elliptic curves whose affine form is given by a Weierstrass equation \(y^2 = 4x^3 - g_2x - g_3\) with \(\Delta := g_3^2 - 27g_2^3 \in O_S(S)^{\times}\). Suppose that \(\Omega^1_S\) is a free \(O_S\)-module with a base \(dt \in \Gamma(S, \Omega^1_S)\).

For \(f \in O_S(S)\), we define \(f' \in O_S(S)\) by \(df = f' dt\). Let

\[
\omega := (0) \times \left(\frac{dx}{y}, -\frac{dz}{u}\right) \quad (6.12)
\]

\[
\omega^* := \left(\frac{y}{x} \times \left(\frac{2xdx}{y}, \frac{g_2z + 2g_3z^2}{2u}\right)\right) \quad (6.13)
\]

be elements in \(H^1_{dR}(U/S)\). Then we have

\[
\nabla (\omega) = \left(\frac{3(2g_2g'_3 - 3g'_2g_3)}{4\Delta} dt \otimes \omega^* - \frac{\Delta'}{12\Delta} dt \otimes \omega\right) \in \Omega^1_S \otimes H^1_{dR}(U/S), \quad (6.14)
\]

\[
\nabla (\omega^*) = \left(\frac{\Delta'}{12\Delta} dt \otimes \omega^* - \frac{g_2(2g_2g'_3 - 3g'_2g_3)}{4\Delta} dt \otimes \omega\right) \in \Omega^1_S \otimes H^1_{dR}(U/S). \quad (6.15)
\]

### 6.3 Deligne’s canonical extension and the limiting Hodge filtration

Let \(S\) be a smooth curve over \(C\) and \((\mathcal{H}, \nabla)\) a vector bundle with integrable connection over \(S^* := S - \{P\}\). Let \(j : S^* \to S\). Then there is unique subbundle \(\mathcal{H}_e \subset j_* \mathcal{H}\) satisfying the following conditions (cf. [Zucker] (17)).

- The connection extends to have log pole, \(\nabla : \mathcal{H}_e \to \Omega^1_S(\log P) \otimes \mathcal{H}_e\),
- each eigenvalue \(\alpha\) of \(\text{Res}_P(\nabla)\) satisfies \(0 \leq \text{Re}(\alpha) < 1\).

The extended bundle \((\mathcal{H}_e, \nabla)\) is called **Deligne’s canonical extension**. The inclusion map

\[
[\mathcal{H}_e \xrightarrow{\nabla} \Omega^1_S(\log P) \otimes \mathcal{H}_e] \to [j_* \mathcal{H} \xrightarrow{\nabla} \Omega^1_S \otimes j_* \mathcal{H}]
\]

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is a quasi-isomorphism of complexes of sheaves. Besides \(\exp(-2\pi i \text{Res}_P(\nabla))\) coincides with the monodromy operator on \(H_C = \text{Ker}(\nabla \text{ann})\) around \(P\) (cf. [Steenbrink], (2.21)).

Let \((H_Z, \mathcal{H}, F^\bullet, \nabla)\) be a polarized VHS on \(S\). Then the eigenvalues of \(\text{Res}_P(\nabla)\) are in \(\mathbb{Q}\) ([Schmid] (4.5)). Moreover by the nilpotent orbit theorem ([Schmid] (4.9)), one can show that \(\hat{F}^\bullet := \mathcal{H}_e \cap j_\ast F^\bullet\) are subbundles of \(\mathcal{H}_e\) (e.g. [Saito1] (2.2)). \(\hat{F}^\bullet\) and the \(V\)-filtration define the \textit{limiting Hodge filtration} on \(H_e \otimes \mathbb{C}_P\) (note that \(\hat{F}^\bullet \otimes \mathbb{C}_P\) does not necessarily coincide with the limiting Hodge filtration unless the monodromy is unipotent. See [Saito2] (3.5) for the detail).

If \((H_Z, \mathcal{H}, F^\bullet, \nabla)\) is a VHS arising from a projective flat family \(f : X \to S\) such that \(f\) is smooth over \(S^\ast\) and \(D_{\text{red}} := (f^{-1}(0))_{\text{red}}\) is a NCD, then one has

\[
\mathcal{H}_e \cong R^1 f_\ast \Omega_X^1(\log D), \quad \hat{F}^1 \cong R^1 f_\ast \Omega_X^1(\log D)
\]  

([Zucker] p.130, Corollary). Put \(\overline{\mathcal{H}}_e := \text{Coker}[\text{Res}_P(\nabla) : \mathcal{H}_e \otimes \mathbb{C}_P \to \mathcal{H}_e \otimes \mathbb{C}_P]\) and let

\[
\text{Res}_P^\mathcal{H} : H^1_{\text{dR}}(S, \mathcal{H}_e) = H^1_{\text{zar}}(S, \mathcal{H}_e \to \Omega^1_S(\log P) \otimes \mathcal{H}_e) \longrightarrow \overline{\mathcal{H}}_e
\]

be the map induced from a commutative diagram

\[
\begin{diagram}
\mathcal{H}_e & \rto & 0 \\
\Omega^1_S(\log P) \otimes \mathcal{H}_e & \rrto_{\text{Res}_P} \dto \rto \end{diagram}
\]

If \((H_Z, \mathcal{H}, F^\bullet, \nabla)\) is the case (6.16), then (6.17) is compatible with the residue map

\[
\text{Res}_D : H^{q+1}_{\text{dR}}(X - D) \longrightarrow H^{2\dim X - q - 2}_{\text{dR}}(D)
\]

under the natural maps \(H^1_{\text{dR}}(S, \mathcal{H}_e) \to H^{q+1}_{\text{dR}}(X - D)\) and \(\overline{\mathcal{H}}_e \to H^{2\dim X - q - 2}_{\text{dR}}(D)\).

We have seen how to compute a connection matrix of the Gauss-Manin connection for a family of hyperelliptic curves. Once we have it, we can get \((\mathcal{H}_e, \hat{F}^\bullet)\) automatically. In case of an elliptic fibration, they are simply given as follows.

**Theorem 6.5** Let \(f : U \to S := \text{Spec}\mathbb{C}[[t]]\) be an elliptic fibration defined by a minimal Weierstrass equation \(y^2 = 4x^3 - g_2x - g_3\) with \(g_2, g_3 \in \mathbb{C}[[t]], \Delta := g_2^3 - 27g_3^2 \neq 0\). Then \(\mathcal{H}_e\) has a basis \(\{\omega, \omega^*\}\) (resp. \(\{t\omega, \omega^*\}\)) if \(f\) has a semistable or smooth reduction (resp. additive reduction).

Since we have Theorem 6.4, we can show the above by case-by-case analysis based on \((\text{ord}(g_2), \text{ord}(g_3))\). The detail is left to the reader.

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