Thermodynamic relations and fluctuations with physical quantities in the Tsallis statistics

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Abstract

The thermodynamic relations in the Tsallis statistics were discussed with physical temperature and physical pressure. An additive entropic variable related to the Tsallis entropy was introduced by assuming the form of the first law of the thermodynamics. The fluctuations in the Tsallis statistics were derived with physical quantities with the help of the introduced entropic variable. It was shown that these fluctuations are related to heat capacities and compressibilities, as shown in the conventional thermodynamics. The fluctuation of the Tsallis entropy was also given through the fluctuation of the entropic variable.

Keywords: Thermodynamic relation, Fluctuation, Heat capacity, Compressibility, Tsallis statistics

1. Introduction

The statistics which show power-like distributions have been interested in many branches of science. One of them is the Tsallis statistics which is an possible extension of the Boltzmann-Gibbs statistics, and the statistics has been applied in various fields [1–2]. The entropy called Tsallis entropy and escort average are employed in this statistics, and the probability distribution is obtained in the maximum entropy principle (MEP). The relations between thermodynamic quantities such as internal energy and entropy, have been discussed. The statistics may describe the phenomena which show power-like distributions.

The physical temperature and the physical pressure were introduced with Tsallis entropy [3–9]. In the Boltzmann-Gibbs statistics, the inverse temperature is given by the partial derivative of the entropy with respect to the internal energy. In the Tsallis statistics, the physical temperature was introduced in the similar way, though the inverse temperature-like parameter appears as a Lagrange multiplier in MEP. The physical temperature seems to be an appropriate variable to describe the system [10, 11].

An entropic variable as a function of the Tsallis entropy was introduced by considering the Legendre transform structure in the Tsallis statistics [5]. It is considered that the
Legendre transform structure is an essential ingredient \cite{12}. In contrast, it is rarely noted that the Legendre transform structure may be unnecessary in the unconventional statistics \cite{13}. It was also shown that the Legendre transform structure is robust against the choice of entropy and the definition of mean value \cite{12, 14}. Therefore, it may be better to introduce the entropic variable without using the Legendre transform structure explicitly. The entropic variable is useful in the description of the thermodynamics.

It is considered that Tsallis-type distributions are related to fluctuations. The Tsallis-type distribution which is described with the $q$-exponential function is often used to describe the phenomena, and the distribution was obtained by assuming that the heat capacity of the environment is exactly constant \cite{15}. This distribution was also obtained for the system with fluctuation \cite{16}. For example, the Tsallis distribution in heavy ion collisions was derived for the system with fluctuating temperature. The entropic parameter $q$ of the Tsallis statistics is related to the heat capacity \cite{17}.

In this paper, we reconsider thermodynamic relations, and attempt to find the expressions of the fluctuations in the Tsallis statistics. In section 2, we reconsider the thermodynamic relations with physical temperature and physical pressure in the Tsallis statistics. In section 3, the fluctuations are discussed in the Tsallis statistics with the introduced entropic variable and physical quantities. The last section is assigned for conclusions.

2. Thermodynamic relations with physical temperature and physical pressure

We treat a system and an environment. The system and the environment are labeled with the superscripts $(S)$ and $(E)$, respectively. The total system constructed from the system and the environment is labeled by the superscript $(S + E)$.

Firstly, we find the relation among the internal energy $U_q$, the physical temperature $T_{ph}$, the entropic variable $X_q$, the physical pressure $P_{ph}$, and the volume $V$. The entropic variable $X_q$ was already introduced in the reference \cite{5}. This variable $X_q$ is given below in this paper. The following discussion is based on the discussion given in the references \cite{4} and \cite{5}.

The Tsallis entropy $S_q(U_q, V)$ with entropic parameter $q$ satisfies the following relation:

$$S_q^{(S+E)} = S_q^{(S)} + S_q^{(E)} + (1 - q)S_q^{(S)}S_q^{(E)}.$$

The additivity of the internal energy is assumed:

$$U_q^{(S+E)} = U_q^{(S)} + U_q^{(E)}.$$

The total volume $V^{(S+E)}$ is the sum of the $V^{(S)}$ and $V^{(E)}$:

$$V_q^{(S+E)} = V_q^{(S)} + V_q^{(E)}.$$

The maximum entropy principle requires $\delta S_q^{(S+E)} = 0$, and the total internal energy and the total volume satisfy $\delta U_q^{(S+E)} = 0$ and $\delta V^{(S+E)} = 0$. With these requirements,
we define the physical temperature \( T_{\text{ph}} \) and the physical pressure \( P_{\text{ph}} \):

\[
\frac{1}{T_{\text{ph}}^{(S)}} := \frac{1}{1 + (1 - q)S_{q}^{(S)}} \left( \frac{\partial S_{q}^{(S)}}{\partial U_{q}^{(S)}} \right)_{V^{(S)}},
\]

\[
\frac{1}{T_{\text{ph}}^{(E)}} := \frac{1}{1 + (1 - q)S_{q}^{(E)}} \left( \frac{\partial S_{q}^{(E)}}{\partial U_{q}^{(E)}} \right)_{V^{(E)}},
\]

\[
P_{\text{ph}}^{(S)} := \frac{1}{1 + (1 - q)S_{q}^{(S)}} \left( \frac{\partial S_{q}^{(S)}}{\partial V_{q}^{(S)}} \right)_{U_{q}^{(S)}},
\]

\[
P_{\text{ph}}^{(E)} := \frac{1}{1 + (1 - q)S_{q}^{(E)}} \left( \frac{\partial S_{q}^{(E)}}{\partial V_{q}^{(E)}} \right)_{U_{q}^{(E)}}.
\]

We have the relations \( T_{\text{ph}}^{(S)} = T_{\text{ph}}^{(E)} \) and \( P_{\text{ph}}^{(S)} = P_{\text{ph}}^{(E)} \) with these definitions.

The differential of the Tsallis entropy is

\[
dS_{q} = \left( \frac{\partial S_{q}^{(S)}}{\partial U_{q}^{(S)}} \right)_{V^{(S)}} dU_{q}^{(S)} + \left( \frac{\partial S_{q}^{(S)}}{\partial V_{q}^{(S)}} \right)_{U_{q}^{(S)}} dV^{(S)}. \tag{5}
\]

We have the following relation by using Eqs. (4a) and (4c):

\[
dU_{q}^{(S)} = \left( \frac{T_{\text{ph}}^{(S)}}{1 + (1 - q)S_{q}^{(S)}} \right) dS_{q} - P_{\text{ph}} dV^{(S)}. \tag{6}
\]

This is the first law of the thermodynamics in the Tsallis statistics. Here, we introduce an entropic variable \( X_{q} \) by requiring that Eq. (6) has the following form in this paper:

\[
dU_{q}^{(S)} = T_{\text{ph}}^{(S)} dX_{q}^{(S)} - P_{\text{ph}} dV^{(S)}. \tag{7}
\]

This requirement is satisfied by defining \( X_{q} \) as

\[
X_{q}^{(S)} = \frac{1}{1 - q} \ln \left( 1 + (1 - q)S_{q}^{(S)} \right). \tag{8}
\]

From Eq. (7), it is natural to define the heat transfer \( d'Q_{q}^{(S)} \) as follows:

\[
d'Q_{q}^{(S)} := T_{\text{ph}}^{(S)} dX_{q}^{(S)}. \tag{9}
\]

The alternative definition of heat transfer is given as \( T_{\text{ph}}^{(S)} dS_{q}^{(S)} \) by using the temperature \( T^{(S)} \) which is the inverse of the Lagrange multiplier \[18\]. It may be worth to mention that the relation, \( T_{\text{ph}}^{(S)} dX_{q}^{(S)} = T^{(S)} dS_{q}^{(S)} \), is easily shown. Similar relation between the heat transfer in the incomplete non-extensive statistics and that in the Rényi statistics was shown in the reference \[19\].

As pointed by some researchers \[3, 20\], the introduced entropic variable \( X_{q}^{(S)} \) and \( X_{q}^{(E)} \) are additive:

\[
X_{q}^{(S+E)} = X_{q}^{(S)} + X_{q}^{(E)}. \tag{10}
\]
This property is easily shown from the pseudo-additivity of the Tsallis entropy. The pseudo-additivity of the Tsallis entropy $S_q^{(S)}$ is mapped to the additivity of the entropic variable $X_q^{(S)}$.

Equation (7) indicates that the variables $T_{ph}^{(S)}$ and $X_q^{(S)}$ are a Legendre pair. Therefore, the free energy $F_q^{(S)}$ in terms of $T_{ph}^{(S)}$ is naturally introduced by using the Legendre transformation of $U_q^{(S)}$ [4]:

$$F_q^{(S)} := U_q^{(S)} - T_{ph}^{(S)} X_q^{(S)}.$$  \hspace{1cm} (11)

We here note another definition of the free energy $\tilde{F}^{(S)}$ [18] which is given by

$$\tilde{F}^{(S)} = U_q^{(S)} - T^{(S)} S_q^{(S)}.$$  \hspace{1cm} (12)

3. Fluctuations in the system contacted to the Bath in the Tsallis statistics

The Tsallis entropy is given by [1, 21, 22]

$$S_q^{(S)} = \ln_q W_q^{(S)},$$  \hspace{1cm} (12a)

$$S_q^{(E)} = \ln_q W_q^{(E)},$$  \hspace{1cm} (12b)

$$S_q^{(S+E)} = \ln_q W_q^{(S+E)},$$  \hspace{1cm} (12c)

where $W$ represents the number of states and $\ln_q x$ is the $q$-logarithm function. As for the number of states, we assume that the system $S$ and the environment $E$ are independent:

$$W_q^{(S+E)} = W_q^{(S)} W_q^{(E)}.$$  \hspace{1cm} (13)

In such a case, the Tsallis entropy has the pseudo-additivity which is shown from Eqs. (12a), (12b), and (12c):

$$S_q^{(S+E)} = S_q^{(S)} + S_q^{(E)} + (1 - q) S_q^{(S)} S_q^{(E)}.$$  \hspace{1cm} (14)

From the definition of the entropic variable $X_q^{(S+E)}$, $X_q^{(S+E)}$ has the following relation.

$$W_q^{(S+E)} = \exp(X_q^{(S+E)}).$$  \hspace{1cm} (15)

We note calculations to clarify the procedure, though the following procedure is standard in the conventional statistics. We introduce the quantity $\Delta X_q^{(S+E)}$ which is the deviation from the equilibrium value $X_q^{(S+E)}$ of the isolated system $S + E$, and introduce the probability $P_r(\Delta X_q^{(S+E)})$ which is the probability of the occurrence of $\Delta X_q^{(S+E)}$.

The probability $P_r(\Delta X_q^{(S+E)})$ is given by

$$P_r(\Delta X_q^{(S+E)}) = \frac{\exp(X_q^{(S+E)} + \Delta X_q^{(S+E)})}{\sum_{\Delta X_q^{(S+E)}} \exp(X_q^{(S+E)} + \Delta X_q^{(S+E)})} = \frac{\exp(\Delta X_q^{(S+E)})}{\sum_{\Delta X_q^{(S+E)}} \exp(\Delta X_q^{(S+E)})}.$$  \hspace{1cm} (16)

Therefore, we focus on $\Delta X_q^{(S+E)}$.  \hspace{1cm} (4)
The quantity $\Delta X_q^{(S+E)}$ is given by

$$\Delta X_q^{(S+E)} = \Delta X_q^{(S)} + \Delta X_q^{(E)}. \quad (17)$$

The quantity $\Delta X_q^{(S)}$ is expanded as follows:

$$\Delta X_q^{(S)}(U_q^{(S)}, V^{(S)}) = \left( \frac{1}{T_{ph}^{(S)}} \right) (\Delta U_q^{(S)}) + \left( \frac{P_{ph}^{(S)}}{T_{ph}^{(S)}} \right) (\Delta V^{(S)})$$

$$+ \frac{1}{2} \left( \Delta \left( \frac{1}{T_{ph}^{(S)}} \right) \right) (\Delta U_q^{(S)}) + \frac{1}{2} \left( \Delta \left( \frac{P_{ph}^{(S)}}{T_{ph}^{(S)}} \right) \right) (\Delta V^{(S)}) + O(\Delta^3). \quad (18)$$

The last term, $O(\Delta^3)$, represents $(\Delta U_q^{(S)})^i (\Delta V^{(S)})^j$ terms $(i + j \geq 3)$. In the same way, we obtain $\Delta X_q^{(E)}$:

$$\Delta X_q^{(E)}(U_q^{(E)}, V^{(E)}) = \left( \frac{1}{T_{ph}^{(E)}} \right) (\Delta U_q^{(E)}) + \left( \frac{P_{ph}^{(E)}}{T_{ph}^{(E)}} \right) (\Delta V^{(E)})$$

$$+ \frac{1}{2} \left( \Delta \left( \frac{1}{T_{ph}^{(E)}} \right) \right) (\Delta U_q^{(E)}) + \frac{1}{2} \left( \Delta \left( \frac{P_{ph}^{(E)}}{T_{ph}^{(E)}} \right) \right) (\Delta V^{(E)}) + O(\Delta^3). \quad (19)$$

Hereafter, we treat the case that the environment is the bath with $\Delta T_{ph}^{(B)} = \Delta P_{ph}^{(B)} = 0$. We attach the superscript $(B)$ for the bath instead of $(E)$. For the bath, from Eq. (19), the quantity $\Delta X_q^{(B)}$ is given by

$$\Delta X_q^{(B)}(U_q^{(B)}, V^{(B)}) = \left( \frac{1}{T_{ph}^{(B)}} \right) (\Delta U_q^{(B)}) + \left( \frac{P_{ph}^{(B)}}{T_{ph}^{(B)}} \right) (\Delta V^{(B)}) + O(\Delta^3). \quad (20)$$

The deviation $\Delta X_q^{(S+B)}$ with $\Delta U_q^{(S+B)} = \Delta V^{(S+B)} = 0$ is given by

$$\Delta X_q^{(S+B)} = \Delta X_q^{(S)}(U_q^{(S)}, V^{(S)}) + \Delta X_q^{(B)}(U_q^{(B)}, V^{(B)})$$

$$= \frac{1}{2} \left( \Delta \left( \frac{1}{T_{ph}^{(S)}} \right) \right) (\Delta U_q^{(S)}) + \frac{1}{2} \left( \Delta \left( \frac{P_{ph}^{(S)}}{T_{ph}^{(S)}} \right) \right) (\Delta V^{(S)}) + O(\Delta^3). \quad (21)$$

We expand $\Delta U_q^{(S)}(X_q^{(S)}, V^{(S)})$ in order to represent the right-hand side of Eq. (21) with the variables, $X_q^{(S)}$ and $V^{(S)}$:

$$\Delta U_q^{(S)}(X_q^{(S)}, V^{(S)}) = \tilde{T}_{ph}^{(S)} \Delta X_q^{(S)} - \tilde{P}_{ph}^{(S)} \Delta V^{(S)} + O(\Delta^2), \quad (22)$$
where $\tilde{T}^{(S)}_{ph}$ and $\tilde{P}^{(S)}_{ph}$ are defined by

\begin{align}
\tilde{T}^{(S)}_{ph} &:= \left( \frac{\partial U^{(S)}_q}{\partial X^{(S)}_q} \right)_{V^{(S)}} , \\
\tilde{P}^{(S)}_{ph} &:= - \left( \frac{\partial U^{(S)}_q}{\partial V^{(S)}} \right)_{X^{(S)}} .
\end{align}

Substituting Eq. (22) into Eq. (21) with $T^{(S)}_{ph} = \tilde{T}^{(S)}_{ph}$ and $P^{(S)}_{ph} = \tilde{P}^{(S)}_{ph}$, we have

\begin{equation}
\Delta X^{(S+B)}_q = - \frac{1}{2T^{(S)}_{ph}} \left[ (\Delta T^{(S)}_{ph})(\Delta X^{(S)}_q) - (\Delta P^{(S)}_{ph})(\Delta V^{(S)}_q) \right] + O(\Delta^3). \quad (24)
\end{equation}

As a result, the probability $P_r(\Delta X^{(S+B)}_q)$ is approximately given by

\begin{align}
P_r(\Delta X^{(S+B)}_q) &= N^{-1} \exp \left( - \frac{1}{2T^{(S)}_{ph}} \left[ (\Delta T^{(S)}_{ph})(\Delta X^{(S)}_q) - (\Delta P^{(S)}_{ph})(\Delta V^{(S)}_q) \right] \right) , \quad (25a) \\
N &= \int_D d(\Delta T^{(S)}_{ph})d(\Delta V^{(S)}_q) P_r(\Delta X^{(S+B)}_q) , \quad (25b)
\end{align}

where the notation $D$ represents the appropriate region of the integral. This region comes from the restrictions of the parameters. For example, the volume of the system is not less than zero.

It is possible to calculate the fluctuations with the probability. For example, the fluctuation of the temperature is given by

\begin{equation}
\langle (\Delta T^{(S)}_{ph})^2 \rangle = N^{-1} \int_D d(\Delta T^{(S)}_{ph})d(\Delta V^{(S)}_q) P_r(\Delta X^{(S+B)}_q) (\Delta T^{(S)}_{ph})^2 . \quad (26)
\end{equation}

As we obtain the fluctuations in the conventional thermodynamics, we have

\begin{align}
\langle (\Delta T^{(S)}_{ph})^2 \rangle &\sim \left( \frac{T^{(S)}_{ph}}{C_{qV}} \right)^2 , \quad (27a) \\
\langle (\Delta V^{(S)}_q)^2 \rangle &\sim T^{(S)}_{ph} \kappa_T^{(S)} V^{(S)} , \quad (27b) \\
\langle (\Delta X^{(S)}_q)^2 \rangle &\sim C_{qP} , \quad (27c) \\
\langle (\Delta P^{(S)}_{ph})^2 \rangle &\sim \left( \frac{T^{(S)}_{ph}}{V^{(S)}\kappa_X^{(S)}} \right)^2 , \quad (27d)
\end{align}

where $C_{qV}$ is the heat capacity at constant volume, $C_{qP}$ is the heat capacity at constant (physical) pressure, $\kappa_T^{(S)}$ is the isothermal compressibility, $\kappa_X^{(S)}$ is the adiabatic compressibility, $V^{(S)}$ is the volume of the system. The heat capacities, $C_{qV}$ and $C_{qP}$, are given
by

\[ C_{qV} = T_{ph} \left( \frac{\partial X_q}{\partial T_{ph}} \right)_V = \left( \frac{\partial U_q}{\partial T_{ph}} \right)_V, \]  
\[ C_{qP} = T_{ph} \left( \frac{\partial X_q}{\partial T_{ph}} \right)_{P_{ph}}. \]  

Equations (28a) and (28b)

The compressibilities, \( \kappa_T \) and \( \kappa_X \), are given by

\[ \kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P_{ph}} \right)_{T_{ph}}, \]  
\[ \kappa_X = -\frac{1}{V} \left( \frac{\partial V}{\partial P_{ph}} \right)_{X_q}. \]  

These are the same relations given in the conventional thermodynamics.

It is possible to obtain the relative fluctuation \( \langle (\Delta S_q/S_q)^2 \rangle \) for \( q \neq 1 \) from Eq. (27a) by using the relation between \( X_q^{(S)} \) and \( S_q^{(S)} \), Eq. (8), when \( S_q^{(S)} \) is large enough. The fluctuation \( \langle (\Delta X_q^{(S)})^2 \rangle \) is represented with \( \langle (\Delta S_q^{(S)})^2 \rangle \):

\[ \langle (\Delta X_q^{(S)})^2 \rangle = \left( \frac{1}{1 + (1-q)S_q^{(S)}} \right)^2 \langle (\Delta S_q^{(S)})^2 \rangle + O(\langle (\Delta S_q^{(S)})^3 \rangle). \]  

That is

\[ \left( \frac{1}{1 + (1-q)S_q^{(S)}} \right)^2 \langle (\Delta S_q^{(S)})^2 \rangle \sim C_qP. \]  

For sufficiently large \( S_q^{(S)} \) with \( q \neq 1 \), we have

\[ \langle (\Delta S_q^{(S)}/S_q^{(S)})^2 \rangle \sim (1-q)^2 C_qP. \]  

The heat capacity \( C_qP \) gives the relative fluctuation of the Tsallis entropy \( \langle (\Delta S_q^{(S)}/S_q^{(S)})^2 \rangle \) for sufficiently large value of \( S_q^{(S)} \) with \( q \neq 1 \).

4. Conclusions

In this paper, we reconsidered the thermodynamic relations in the Tsallis statistics by assuming the form of the first law with physical quantities. We also calculated the fluctuations of thermodynamic quantities by using the entropic variable defined with the Tsallis entropy.

The first law was naturally described with the physical temperature \( T_{ph} \) and the physical pressure \( P_{ph} \) by introducing the entropic variable \( X_q \) conjugate to the physical temperature. The requirement that the first law has the form \( dU_q = T_{ph} dX_q - P_{ph} dV \) suggests the form of \( X_q \) as a function of the Tsallis entropy \( S_q \).
We obtained the expressions of the fluctuations of the physical quantities: the physical temperature $T_{ph}$, the physical pressure $P_{ph}$, the volume $V$, and the entropic variable $X_q$. The expressions of fluctuations in the Tsallis statistics are similar to those in the conventional statistics. The expression of the fluctuation of $X_q$ gives that of the Tsallis entropy $S_q$.

The introduced entropic variable $X_q$ can be constructed when the thermodynamic quantities such as the heat capacities are observed, and the Tsallis entropy $S_q$ is obtained by using the equation: $S_q = (\exp((1-q)X_q) - 1)/(1-q)$. The fluctuation of the Tsallis entropy is obtained with the heat capacity at constant (physical) pressure $C_qP$ when the Tsallis entropy $S_q$ is given.

The calculations of the fluctuations are based on the expression, $W = \exp(X_q)$, where $X_q$ is the entropic variable and $W$ is the number of states. That is, the calculations depend on the property of the exponential function: $\exp(x + \Delta x) = \exp(x)\exp(\Delta x)$. The calculations also depend on the division between the system and the environment represented as $W^{(S+E)} = W^{(S)}W^{(E)}$, where $W^{(S+E)}$, $W^{(S)}$, and $W^{(E)}$ are the number of states for the whole system, that for the system, and that for the environment. Therefore, the results in the present study will be modified in the case of $W^{(S+E)} \neq W^{(S)}W^{(E)}$.

In this paper, the thermodynamic relations were reconsidered with the physical quantities in the Tsallis statistics, and the fluctuations of the physical variables were obtained. The results given in this paper will be helpful to study the phenomena described with the statistics which give power-like distributions.

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