Monochromatic $k$-connected Subgraphs in 2-edge-colored Complete Graphs

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Abstract

Bollobás and Gyárfás conjectured that for any $k, n \in \mathbb{Z}^+$ with $n > 4(k - 1)$, every 2-edge-coloring of the complete graph on $n$ vertices leads to a $k$-connected monochromatic subgraph with at least $n - 2k + 2$ vertices. We find a counterexample with $n = 5k - 2 \lceil \sqrt{2k - 1} \rceil - 3$, thus disproving the conjecture, and we show the conjecture is true for $n \geq 5k - \min\{\sqrt{4k - 2} + 3, 0.5k + 4\}$.

Keywords: Connectivity, Monochromatic

1. Introduction

Ramsey theory is one of the most important research areas in combinatorics. For any given integers $s, t$, the Ramsey number $R(s, t)$ is the smallest integer $n$ such that for any 2-edge-colored (red/blue) $K_n$, there must exist a red $K_s$ or a blue $K_t$. In 1930, Ramsey [17] proved the existence of Ramsey numbers. However, estimating Ramsey numbers is known to be notoriously challenging.

There are many variations of the original Ramsey problem, including the one considering highly-connected subgraphs instead of cliques. A graph is $k$-connected if and only if it has more than $k$ vertices and does not have a vertex cut of size at most $k - 1$. Let $r_c(k)$ denote the smallest integer such that every $c$-edge-colored complete graph on $r_c(k)$ vertices must contain a $k$-connected monochromatic subgraph. In 1983, Matula [16] proved $2c(k - 1) + 1 \leq r_c(k) < (10/3)c(k - 1) + 1$. Moreover, for 2-edge-coloring, Matula [16] improved the upper bound to $r_2(k) < (3 + \sqrt{11}/3)(k - 1) + 1$. However, Matula’s result does not have any restriction on the order of the $k$-connected monochromatic subgraph. Bollobás and Gyárfás [1] proposed the following conjecture:

Conjecture 1.1. Let $k, n$ be positive integers. For $n > 4(k - 1)$, every 2-edge-colored $K_n$ contains a $k$-connected monochromatic subgraph with at least $n - 2k + 2$ vertices.

Note that the statement is not true for $n \leq 4(k - 1)$ by Matula’s result [16]. (Also see [1].) Moreover, no matter how large $n$ is, $n - 2k + 2$ is the best possible bound for the order of the $k$-connected subgraph by the example $B(n, k)$ in [1]. Besides proposing
the conjecture, Bollobás and Gyárfás verified the conjecture for $k \leq 2$, and showed it is sufficient to prove the conjecture holds for $4k - 3 \leq n < 7k - 5$. Liu, Morris, and Prince [13] verified the conjecture for $k = 3$, and proved it for $n \geq 13k - 15$. Later, Fujita and Magnant [2] improved the bound to $n > 6.5(k - 1)$. Recently, Luczak [14] claimed the proof of the conjecture. However, a gap has been found in the proof and not yet fixed [11]. (Also see [15].) Many of the ideas in our note were inspired by Luczak’s proof, although we do not agree with his conclusion.

Bollobás and Gyárfás’ conjecture could be generalized to multicolored graphs. (See [12], [7], [9], and [10].) Besides, there are some other approaches to force large highly connected subgraphs. For example, Fujita, Liu, and Sarkar [4], [5] proved the existence of large highly connected monochromatic subgraphs with given independence number. The characterization of 2-edge-colored $K_n$ with no large $k$-connected monochromatic subgraphs has also been studied. (See [8].)

The main result of this paper is that we show Conjecture 1.1 fails for $n = 5k - 2\lceil\sqrt{2k - 1}\rceil - 3$. On the other hand, we verify the conjecture for larger $n$.

**Theorem 1.2.**

1. For every $k \in \mathbb{Z}^+$, let $n = 5k - 2\lceil\sqrt{2k - 1}\rceil - 3$. There exists a 2-edge-colored $K_n$, such that there is no $k$-connected monochromatic subgraph, which contains at least $n - 2k + 2$ vertices.

2. Let $k, n \in \mathbb{Z}^+$. If $n \geq 5k - \sqrt{4k - 2} - 3$ and $n \geq 5k - 0.5k - 4$, then for any 2-edge-colored $K_n$, there exists a $k$-connected monochromatic subgraph, which contains at least $n - 2k + 2$ vertices.

Note that when $k \neq 3, 5, 7$, we will always have $\lfloor\sqrt{4k - 2}\rfloor + 3 \leq \lfloor0.5k\rfloor + 4$. Besides, since $\lfloor\sqrt{4k - 2}\rfloor + 3 \geq 4$ and $\lfloor0.5k\rfloor + 4 \geq 4$ for all $k \in \mathbb{Z}^+$, the statement always holds for $n \geq 5k - 4$.

Moreover, our result improves the bounds for some other related problems. For example, since every $k$-connected graph has minimum degree at least $k$, Theorem 1.2(2) leads to the following corollary:

**Corollary 1.3.** If $n \geq 5k - \sqrt{4k - 2} - 3$ and $n \geq 5k - 0.5k - 4$, then for any 2-edge-colored $K_n$, there exists a monochromatic subgraph with minimum degree at least $k$, which contains at least $n - 2k + 2$ vertices.

This problem concerning monochromatic large subgraphs with a specified minimum degree in edge-colored graphs has been studied by Caro and Yuster [2]. By applying their conclusion on 2-edge-colored complete graphs, the corollary holds when $n \geq 7k + 4$, which could be covered by our result.

In Section 2, we will give some definitions and lemmas related to connectivity. In Section 3, we will prove Theorem 1.2(1), and in Section 4, we will prove Theorem 1.2(2). The rest of this section will be devoted to terminologies and notations. We follow the notations and terminologies for graphs from [3].

Given a graph $G$ and an edge-coloring of $G$ with colors red and blue, let $R$ be the spanning subgraph induced by red edges, and $B$ be the spanning subgraph induced by blue edges.

We use $\delta(G)$ to denote the minimum degree of $G$. Given vertex sets $V_1, V_2 \subseteq V(G)$, let $e_G(V_1, V_2)$ be the number of edges with one endpoint in $V_1$ and the other endpoint in $V_2$. For $S \subseteq V(G)$, we use $N_G(S)$ to denote the vertex set $\{v : v \notin S, \exists u \in S, uv \in E(G)\}$, and
2. Graphs without large \( k \)-connected subgraphs

In this section, we will introduce a decomposition for graphs with no \( k \)-connected subgraphs of large order.

Definition 2.1. Let \( k \in \mathbb{Z}^+ \), \( f(k) \) be a non-negative function on \( k \). Let \( G \) be a graph on \( n \) vertices, where \( n \geq f(k) + k \). We define an \(( f(k), k)\)-decomposition of \( G \) to be a sequence of triples \( \{(A_i, C_i, D_i)\} \), \( i \in [1, l] \), such that

1. \( V(G) \) is a disjoint union of \( A_1, C_1, D_1 \)
2. \( C_i \cup D_i \) is a disjoint union of \( A_{i+1}, C_{i+1}, D_{i+1} \), \( i \in [1, l-1] \)
3. \( |C_i| \leq k - 1, 1 \in [1, l] \)
4. \( 1 \leq |A_i| \leq |D_i| \), and there is no edge between \( A_i \) and \( D_i \), \( i \in [1, l] \)
5. \( |C_i| + |D_i| \geq n - f(k) \), \( i \in [1, l-1] \)
6. \( |C_i| + |D_i| < n - f(k) \)

By (1) and (2) of Definition 2.1 we have:

Proposition 2.2. \( V(G) \) is a disjoint union of \( A_1, \ldots, A_i, C_i, D_i \) for any \( i \in [1, l] \).

We can also find a partition for the edges in \( G \) with respect to the decomposition:

Proposition 2.3. \( E(G) \) is a disjoint union of \( E^{AA}, E^{AC}, E^i \), where \( E^{AA} \) contains all edges with both endpoints in \( A_i \), for some \( i \in [1, l] \), \( E^{AC} \) contains all edges with one endpoint in \( A_i \), and the other in \( C_i \) for some \( i \in [1, l] \), and \( E^i \) contains all edges with both endpoints in \( C_i \cup D_i \).

Proof. Let \( e = uv \in E(G) \). Suppose there exists \( i \in [1, l] \), such that \( \{u, v\} \cap A_i \neq \emptyset \), we take the smallest such \( i \). By symmetry, we may assume \( u \in A_i \), then by (4) of Definition 2.1 either \( v \in A_i \) or \( v \in C_i \). Thus \( e \in E^{AA} \) or \( e \in E^{AC} \). Suppose there does not exist such \( i \), then by Proposition 2.2 \( u, v \in C_i \cup D_i \), hence \( e \in E^i \).

Lemma 2.4. Let \( k \in \mathbb{Z}^+ \), \( f(k) \) be a non-negative function on \( k \). Let \( G \) be a graph on \( n \) vertices with \( n \geq f(k) + k \). If \( G \) does not have a \( k \)-connected subgraph with at least \( n - f(k) \) vertices, then \( G \) has an \(( f(k), k)\)-decomposition.

Proof. Let \( G_0 = G \). Since \( f(k) \) is non-negative, \( |G_0| = n \geq n - f(k) \). We repeat the following steps until \( |G_i| < n - f(k) \).

1. Let \( C_{i+1} \) be a cut of \( G_i \) of size at most \( k - 1 \). Since \( |G_i| \geq n - f(k) \geq k \), there must exist one such cut.
2. Let \( A_{i+1} \) be the vertex set of smallest component of \( G_i - C_{i+1} \), and \( D_{i+1} = V(G_i) \setminus (A_{i+1} \cup C_{i+1}) \).
3. Let \( G_{i+1} \) be the subgraph of \( G_i \) induced by \( C_{i+1} \cup D_{i+1} \).
The sequence of triples generated by the above procedure is an \((f(k), k)\)-decomposition of \(G\).

**Definition 2.5.** We say an \((f(k), k)\)-decomposition is **strong** if \(|A_i| + |C_i| < n - f(k)\), for any \(i \in [1, l]\).

**Lemma 2.6.** Let \(k \in \mathbb{Z}^+\), \(f(k)\) be a non-negative function on \(k\). Let \(G\) be a graph on \(n\) vertices, where \(n \geq f(k) + k\). If \(G\) has a strong \((f(k), k)\)-decomposition, then \(G\) does not have a \(k\)-connected subgraph with at least \(n - f(k)\) vertices.

**Proof.** Let \(\{(A_i, C_i, D_i)\}, i \in [1, l]\) be a strong \((f(k), k)\)-decomposition of \(G\). Suppose \(G\) has a \(k\)-connected subgraph \(H\) such that \(|H| \geq n - f(k)\). Let \(i^*\) be the smallest \(i\) such that \(A_i \cap V(H) \neq \emptyset\). Note that by Proposition 2.2 and (6) of Definition 2.1, such \(i^*\) must exist. Then \(H\) must be a subgraph of \(G(A_{i^*} \cup C_{i^*} \cup D_{i^*})\). We claim \(V(H) \cap D_{i^*} = \emptyset\). Otherwise by (3) and (4) of Definition 2.1, \(V(H) \cap C_{i^*}\) is a cut of \(H\) of size at most \(k - 1\), which is a contradiction to the connectivity of \(H\). Thus \(H\) must be a subgraph of \(G(A_{i^*} \cup C_{i^*})\). However since the decomposition is strong, \(|H| \leq |A_{i^*}| + |C_{i^*}| < n - f(k)\). We conclude that \(G\) does not have a \(k\)-connected subgraph with at least \(n - f(k)\) vertices.

**3. The counterexample**

In this section, we will demonstrate and verify the counterexample in Theorem 2.2 (1).

**Proof of Theorem 2.2 (1).** Let \(\tau = \lceil \sqrt{2k - 1} \rceil\), then \(n = 5k - 2\tau - 3\). We may assume \(n \geq 4k - 3\), otherwise Theorem 2.2 (1) holds by the example \(B(k)\) given in \([1]\). (Also see [16]) Thus we assume \(\tau \in (0, 0.5k]\), as \(5k - 2\tau - 3 \geq 4k - 3\).

Let \(V(G) = A_1 \cup A_2 \cup \cdots \cup A_{k+1} \cup C_{k+1} \cup D_{k+1}\), where \(A_i = \{a_i\}\) for \(i \in [1, k]\), \(A_{k+1} = \{a_1^{k-1}\}, C_{k+1} = \{c_1, \ldots, c_{k-1}\}\), and \(D_{k+1} = \{d_1, \ldots, d_{2k-2}\}\). Thus \(n = |V(G)| = \sum_{i=1}^{k+1} |A_i| + |C_{k+1}| + |D_{k+1}| = k + (k-1) + (k-1) + (2k-2) = 5k - 2\tau - 3\).

Next we will set \(C_i\) and \(D_i\) for \(i \in [1, k]\). Let \(ALU = \{a_1^1, \ldots, a_1^k\}\), and \(DLU = \{d_1^1, \ldots, d_1^\tau\}\). Let \(S = \{s_1\}\) be the sequence \(1, \ldots, \tau, 1, \ldots, \tau, \ldots, 1, \ldots, \tau\) (repeat \(\tau\) times). Since \(\tau = \lceil \sqrt{2k - 1} \rceil\), the length of \(S\) is at least \(\tau^2 = \lceil \sqrt{2k - 1} \rceil^2 \geq 2k - 1\). Note that there is no \(i \in [1, k-1]\) such that \(s_{2i-1} = s_{2i}\). We set \(C_i = (ALU \setminus \{a_1^{2i-1}, a_1^{2i}\}) \cup \{c_i\} \cup DLU\) for \(i \in [1, k-1]\), \(C_k = (ALU \setminus \{a_1^{2k-1}\}) \cup DLU\), \(D_1 = V(G) \setminus (A_1 \cup C_1)\), and \(D_i = C_{i-1} \cup D_{i-1} \setminus (A_i \cup C_i)\) for \(i \in [2, k]\).

We color all edges between \(A_i\) and \(D_i\) blue for all \(i \in [1, k+1]\), and all the other edges red. We use \(R\) (resp. \(B\)) to denote the spanning subgraph induced by red (resp. blue) edges.

**Claim 3.1.** \(R\) does not have a \(k\)-connected subgraph with at least \(n - 2k + 2\) vertices.

**Proof.** We prove this by verifying \(\{(A_i, C_i, D_i)\}, i \in [1, l]\) is a strong \((2k-2, k)\)-decomposition of \(R\). Note that in our example, \(l = k + 1\).

By the construction, (1)(2)(5)(6) of Definition 2.1 hold. We will verify 2.1 (3) and (4) for our example.

(3) \(|C_i| \leq k - 1, i \in [1, l]\)

For \(i \in [1, k-1]\), \(|C_i| = |ALU| - 2 + 1 + |DLU| = \tau - 2 + 1 + (k-\tau) = k - 1\).

(|\(C_k| = |ALU| - 1 + |DLU| = \tau - 1 + (k-\tau) = k - 1, |C_{k+1}| = \{|c_1, \ldots, c_{k-1}\}| = k - 1\).

(4) \(1 \leq |A_i| \leq |D_i|\), and there is no edge between \(A_i\) and \(D_i\), \(i \in [1, l]\)
For $i \in [1,k]$, $1 = |A_i| \leq |D_i|$. $1 \leq |A_{k+1}| = (k-1) \leq 2k - 2\tau - 1 = |D_{k+1}|$ as $\tau \in (0,0.5k]$. All edges between $A_i$ and $D_i$ are blue. Thus $R$ does not contain any edge between $A_i$ and $D_i$ for all $i \in [1,l]$.

Moreover, since $n = 5k - 2\tau - 3 \geq 4k - 3$, $|A_i| + |C_i| = 1 + (k-1) = k < n - 2k + 2$ for $i \in [1,k]$, and $|A_{k+1}| + |C_{k+1}| = (k-1) + (k-1) = 2k - 2 < n - 2k + 2$. Thus $|A_i| + |C_i| < n - (2k - 2)$ for any $i \in [1,l]$. The decomposition is strong.

Thus by Lemma 2.6, $R$ does not have a $k$-connected subgraph with at least $n - 2k + 2$ vertices.

\[\square\]

**Claim 3.2.** $B$ does not have a $k$-connected subgraph with at least $n - 2k + 2$ vertices.

**Proof.** We prove the claim by finding a sequence of vertices $u_1,u_2,\ldots,u_{2k-1}$ such that $e_B(\{u_x\}, V(B) \setminus \{u_1,\ldots,u_x\}) \leq k - 1$. That is, there is no subgraph $H$ of $B$ with $\delta(H) \geq k$ and order at least $n - 2k + 2$.

We claim that $\{u_x\} = c_1^1,\ldots,c_{k-1}^1, d_1^1,\ldots,d_{k-\tau}^1, a_1^1,\ldots,a_{\tau}^1$ is a sequence that satisfies the requirement. Note that the sequence contains $(k-1) + (k-\tau) + \tau = 2k - 1$ vertices.

1. $1 \leq x \leq k - 1$.
   For the first $k - 1$ vertices in the sequence, since $C_{k+1}$ has no neighbor in $A_{k+1} \cup C_{k+1} \cup D_{k+1}$ and $c_i^1 \in C_i$, we have
   $$e_B(\{u_x\}, V(B) \setminus \{u_1,\ldots,u_x\}) = |N_B(c_i^1)| = |\{a_1,\ldots,a_k\} \setminus \{a_x\}| = k - 1.$$  

2. $k \leq x \leq k - 1 + (k-\tau)$.
   The next $k - \tau$ vertices are all in the set $DLU$. They are not adjacent to any of the $a_1,\ldots,a_k$, since $DLU \subseteq C_i$ for $i \in [1,k]$. Moreover, all edges with both endpoints in $D_{k+1}$ are red. Thus, take $x' = x - (k - 1)$, we have
   $$e_B(\{u_x\}, V(B) \setminus \{u_1,\ldots,u_x\}) = |N_B-C_{k+1}(d_i')| = |A_{k+1}| = k - 1.$$  

3. $k + (k - \tau) \leq x \leq 2k - 1$.
   The last $\tau$ vertices are all in $ALU$. According to the definition of $C_i$ for $i \in [1,k]$, every vertex in $ALU$ is adjacent to at most $\tau$ of $a_1,\ldots,a_k$. Indeed, let $a_i^1 \in ALU$, where $j \in [1,\tau]$. Then $a_i^1$ is adjacent to $a_i$ in $B$ if and only if $j \in \{s_{2i-1},s_{2i}\}$ for $i \in [1,k-1]$, and $a_i^1$ is adjacent to $a_k$ in $B$ if and only if $j = s_{2k-1}$. Since $j$ occurs $\tau$ times in $S$, at most $\tau$ of $a_1,\ldots,a_k$ are adjacent to $a_i^1$. Moreover, all edges with both endpoints in $A_{k+1}$ are red. Thus, take $x'' = x - (k - 1) - (k - \tau)$, we have
   $$e_B(\{u_x\}, V(B) \setminus \{u_1,\ldots,u_x\}) = |N_B-C_{k+1}(d_i'')| |D_{k+1} \setminus DLU| + \tau = (2k - 2\tau - 1) - (k - \tau) + \tau = k - 1.$$  

Thus we conclude that $B$ does not have a subgraph $H$ with at least $n - 2k + 2$ vertices with $\delta(H) \geq k$. Hence, $B$ does not have a $k$-connected subgraph of order at least $n - 2k + 2$. \[\square\]

By Claim 3.1 and 3.2, the 2-coloring we proposed contains no $k$-connected monochromatic subgraph of order at least $n - 2k + 2$, which completes the proof of Theorem 1.2(1). \[\square\]
4. Proof of Theorem 1.2 (2)

For \( k, n \in \mathbb{Z}^+ \), let \( \lambda = \min\{ \lfloor \sqrt{4k-2} \rfloor + 3, \lfloor 0.5k \rfloor + 4 \} \). We will prove the statement for \( n \geq 5k - \lambda \). Since \( 4k - 2 \) is not a perfect square for any \( k \in \mathbb{Z}^+ \), we will always have \( \lambda < \sqrt{4k-2} + 3 \). Thus, we claim that \( 4k - \lambda^2 + 6\lambda - 11 > 0 \), as \( \lambda > -\sqrt{4k-2} + 3 \) for all \( k \in \mathbb{Z}^+ \). Moreover, we will always have \( n \geq 4k - 3 \), since \( \lfloor 0.5k \rfloor + 4 \leq k + 3 \) for all \( k \in \mathbb{Z}^+ \).

Suppose there exists a 2-edge-colored \( K_n \), such that there is no \( k \)-connected monochromatic subgraph with at least \( n - 2k + 2 \) vertices. Let \( R \) be the red graph, and \( B \) be the blue graph. We may assume \( E(R) \) is maximized.

Since \( R \) does not have a \( k \)-connected subgraph with at least \( n - 2k + 2 \) vertices, by Lemma 2.3, \( R \) must have a \((2k-2,k)\)-decomposition \( \{ (A_i, C_i, D_i) \} \), \( i \in [1,l] \). By Definition 2.1 (4), for any \( i \in [1,l] \), there is no edge between \( A_i \) and \( D_i \) in \( R \). Thus \( (A_i, D_i) \) is complete in \( B \) for any \( i \in [1,l] \).

**Proposition 4.1.** Suppose exists \( i \in [1,l] \) such that \( B(C_i \cup D_i) \) has a \( k \)-connected subgraph \( H \) of order at least \( 2k - 1 \), then \( B(A_i \cup V(H)) \) is \( k \)-connected.

**Proof.** Since \( H \) is a subgraph of \( B(C_i \cup D_i) \) and \( |C_i| \leq k - 1 \), we have \( |V(H) \cap D_i| = |V(H)| - |V(H) \cap C_i| \geq (2k - 1) - (k - 1) = k \). Since \( [A_i, V(H) \cap D_i] \) is complete in \( B(A_i \cup V(H)) \), \( B(A_i \cup V(H)) \) is \( k \)-connected.

By Definition 2.1 (2), \( C_i \cup D_i = A_{i+1} \cup C_{i+1} \cup D_{i+1}, i \in [1,l-1] \). We will have the following corollary if we apply Proposition 4.1 recursively:

**Corollary 4.2.** Suppose exists \( i \in [1,l] \) such that \( B(C_i \cup D_i) \) has a \( k \)-connected subgraph \( H \) of order at least \( 2k - 1 \), then \( B(A_1 \cup A_2 \cup \cdots \cup A_i \cup V(H)) \) is \( k \)-connected.

**Claim 4.3.** \( |A_i| \leq k - 1, \forall i \in [1,l] \).

**Proof.** Suppose exists \( i \) such that \( |A_i| \geq k \). By (4) of Definition 2.1, \( |D_i| \geq |A_i| \geq k \). Since \( [A_i, D_i] \) is complete in \( B \), we have \( B(A_i \cup D_i) \) is \( k \)-connected. If \( i = 1 \), \( B(A_1 \cup D_1) \) is a \( k \)-connected subgraph of \( B(C_{i-1} \cup D_{i-1}) \). Moreover, \( |B(A_1 \cup D_1)| = |A_1| + |D_1| \geq k + k > 2k - 1 \). Thus by applying Corollary 4.2 on \((i-1)\) and \( H = B(A_1 \cup D_1) \), we have \( B(A_1 \cup A_2 \cup \cdots \cup A_i \cup D_i) \) is \( k \)-connected. However, by Proposition 4.1 and (3) of Definition 2.1, \( |A_1 \cup A_2 \cup \cdots \cup A_i \cup D_1| = |V(G)| - |C_i| \geq n - (k - 1) \geq n - 2k + 2 \), a contradiction.

Combining (5)(6) of Definition 2.1 and Proposition 2.2 we have the following corollary:

**Corollary 4.4.** \( 2k - 1 \leq \sum_{i=1}^{l} |A_i| \leq 3k - 3 \).

**Proof.** By Definition 2.1 (6) and Proposition 2.2, \( \sum_{i=1}^{l} |A_i| = n - (|C_i| + |D_i|) \geq n - (n - 2k + 1) = 2k - 1 \). By Definition 2.1 (5) and Proposition 2.2, \( \sum_{i=1}^{l} |A_i| = (\sum_{i=1}^{l-1} |A_i|) + |A_l| = n - (|C_{i-1}| + |D_{i-1}|) + |A_l| \leq n - (n - 2k + 2) + (k - 1) = 3k - 3 \).

By the maximality of \( E(R) \), we have:

**Claim 4.5.** \( A_i \) is complete in \( R \) for any \( i \in [1,l] \).

**Proof.** Suppose there exists \( i^* \in [1,l] \), and \( u, v \in A_{i^*} \), such that \( uv \notin E(R) \). Consider \( R' = R + uv \). Since there does not exist \( i' \in [1,l] \) such that \( uv \) is an edge between \( A_{i'} \) and \( D_{i'} \), \( \{ (A_i, C_i, D_i) \}, i \in [1,l] \) is also a \((2k-2,k)\)-decomposition of \( R' \). Moreover, by Claim
and (3) of Definition 2.1, for any \( i \in [1, l] \), \(|A_i| + |C_i| \leq (k - 1) + (k - 1) \leq n - 2k + 2\) since \( n \geq 4k - 3 \). Hence, the decomposition is strong, and by Lemma 2.6, \( R' \) does not have a \( k \)-connected subgraph with at least \( n - 2k + 2 \) vertices, a contradiction to the maximality of \( E(R) \). Thus \( A_i \) is complete in \( R \) for any \( i \in [1, l] \).

Similarly we can prove:

Claim 4.6. \( D_i \) is complete in \( R \).

Claim 4.7. \( C_i \) is complete in \( R \).

Claim 4.8. \([A_i, C_i] \) is complete in \( R \) for any \( i \in [1, l] \).

Claim 4.9. \([D_i, C_i] \) is complete in \( R \).

According to the above 5 claims, we conclude that

Claim 4.10. \( E(B) \) is the union of all edges between \( A_i \) and \( D_i \) for all \( i \in [1, l] \), and all other edges are in \( E(R) \).

Since \( B \) does not have a \( k \)-connected subgraph with at least \( n - 2k + 2 \) vertices, by Lemma 2.4, \( B \) must have a \((2k - 2, k)\)-decomposition \( \{ (U_x, S_x, T_x) \}, x \in [1, l_B] \). For convenience, we use \( U \) to denote \( U_1 \cup U_2 \cup \cdots \cup U_{l_B} \), and \( U \) to denote \( V(G) \setminus U \). Note that by Definition 2.1 (3), \(|S_x| \leq k - 1 \). Moreover, by a similar proof of Claim 4.3, we will have

Claim 4.11. \(|U_x| \leq k - 1 \).

As a corollary, we have

Corollary 4.12. \( 2k - 1 \leq |U| \leq 3k - 3 \).

We will complete the proof by counting the total number of edges between \( U_x \) and \( S_x \). For every \( i \in [1, l] \), we use \( X_i \) to denote the set of integers \( x \in [1, l_B] \) such that \( A_i \cap U_x \neq \emptyset \) and \( D_i \cap U_x \neq \emptyset \), and \( U_i \) to denote \( \cup_{x \in X_i} U_x \). We use \( E^{UL}_B \) (resp. \( E^{UR}_B \)) to denote the set of all blue (resp. red) edges between \( U_x \) and \( S_x \) for all \( x \in [1, l_B] \). Thus

\[
|E^{UL}_B| + |E^{UR}_B| \leq \sum_{x=1}^{l_B} |U_x||S_x| \leq (k - 1) \sum_{x=1}^{l_B} |U_x| = (k - 1)|U|.
\]

Next, we will show that \(|E^{UL}_B| + |E^{UR}_B| > (k - 1)|U|\).

The blue graph, by Claim 4.10 contains all edges between \( A_i \) and \( D_i \) for \( i \in [1, l] \). On the other hand, by Proposition 2.3 we can classify the blue edges into 3 types: edges with both endpoints in \( \overline{T} \) (the \( E^l \) type), edges with both endpoints in \( U_x \) for some \( x \in [1, l_B] \) (the \( E^{AA} \) type), and those in \( E^{UL}_B \) which has one endpoint in \( U_x \) and the other in \( S_x \) for some \( x \in [1, l_B] \) (the \( E^{AC} \) type). Thus,

\[
|E^{UL}_B| = \sum_{i=1}^{l} \sum_{x=1}^{l_B} (|A_i \cap U_x||D_i \cap S_x| + |A_i \cap S_x||D_i \cap U_x|)
= \sum_{i=1}^{l} (|A_i||D_i| - |A_i \cap \overline{U}||D_i \cap U| - \sum_{x \in X_i} |A_i \cap U_x||D_i \cap U_x|).
\]
We will first estimate the lower bound for $\sum_{i=1}^{l} (|A_i||D_i| - |A_i \cap \overline{U}||D_i \cap \overline{U}|)$.

\[
\sum_{i=1}^{l} (|A_i||D_i| - |A_i \cap \overline{U}||D_i \cap \overline{U}|) \\
\geq \sum_{i=1}^{l} |A_i|(n - |C_i|) - \sum_{j=1}^{i} |A_j| - \sum_{i=1}^{l} |A_i \cap \overline{U}|(n - |U| - \sum_{j=1}^{i} |A_j \cap \overline{U}|) \\
\geq (4k - \lambda + 1)(\sum_{i=1}^{l} |A_i|) - (5k - \lambda - |U|)(\sum_{i=1}^{l} |A_i \cap \overline{U}|) \\
- \frac{1}{2} (\sum_{i=1}^{l} |A_i|^2 + \sum_{i=1}^{l} |A_i \cap \overline{U}|^2) - \frac{1}{2} \sum_{i=1}^{l} (|A_i| + |A_i \cap \overline{U}|)(|A_i| - |A_i \cap \overline{U}|) \\
= (3k - \lambda + 2)(\sum_{i=1}^{l} |A_i|) - (4k - \lambda + 1 - |U|)(\sum_{i=1}^{l} |A_i \cap \overline{U}|) \\
+ (k - 1)(\sum_{i=1}^{l} |A_i| - \sum_{i=1}^{l} |A_i \cap \overline{U}|) - \frac{1}{2} (\sum_{i=1}^{l} |A_i|^2 + \sum_{i=1}^{l} |A_i \cap \overline{U}|^2) \\
- \frac{1}{2} \sum_{i=1}^{l} ((|A_i \cap U| + |A_i \cap \overline{U}|) + |A_i \cap U|)(|A_i \cap U|) \\
= -\frac{1}{2} ((3k - \lambda + 2) - \sum_{i=1}^{l} |A_i|^2 + \frac{1}{2} (3k - \lambda + 2)^2 \\
+ \frac{1}{2} (4k - \lambda + 1 - |U|) - \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2 - \frac{1}{2} (4k - \lambda + 1 - |U|)^2 \\
+ (k - 1)(\sum_{i=1}^{l} |A_i \cap U|) - \frac{1}{2} \sum_{i=1}^{l} (|A_i \cap U| + 2|A_i \cap \overline{U}|)(|A_i \cap U|) \\
\geq -\frac{1}{2} ((3k - \lambda + 2) - (2k - 1)^2 + \frac{1}{2} (3k - \lambda + 2)^2 \\
- \frac{1}{2} (3k - \lambda + 2 - |U|)^2 - (k - 1)(3k - \lambda + 2 - |U|) - \frac{1}{2} (k - 1)^2 \\
\]
\[ + \sum_{i=1}^{l} ((k-1) - |A_i \cap \overline{U}|)(|A_i \cap U|) - \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2 \]

\[ \geq -\frac{1}{2}((3k - \lambda + 2) - (2k - 1))^2 + \frac{1}{2}(3k - \lambda + 2)^2 \]

\[ - \frac{1}{2}((3k - \lambda + 2) - (2k - 1))^2 - (k-1)(3k - \lambda + 2) - \frac{1}{2}(k-1)^2 \]

\[ + (k-1)|U| + \sum_{i=1}^{l} ((k-1) - |A_i \cap \overline{U}|)(|A_i \cap U|) - \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2 \]

\[ = - (k-\lambda + 3)^2 + \frac{1}{2}(3k - \lambda + 2)^2 - (k-1)(3k - \lambda + 2) - \frac{1}{2}(k-1)^2 \]

\[ + (k-1)|U| + \sum_{i=1}^{l} ((k-1) - |A_i \cap \overline{U}|)(|A_i \cap U|) - \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2 \]

\[ \geq \frac{1}{2}(4k - \lambda^2 + 6\lambda - 11) + (k-1)|U| \]

\[ + \sum_{i=1}^{l} ((k-1) - |A_i \cap \overline{U}|)(|A_i \cap U|) - \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2 \]

\[ \geq (k-1)|U| + \sum_{i=1}^{l} ((k-1) - |A_i \cap \overline{U}|)(|A_i \cap U|) - \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2 \]

Notes: ① By Proposition 2, and |C_t \cap \overline{U}| is omitted. ② n \geq 5k - \lambda, |A_t| \geq |A_t \cap \overline{U}|, and |C_t| \leq k - 1. ③ \lambda \leq 0.5k + 4. ④ 2k - 1 \leq \sum_{i=1}^{l} |A_i| \leq 3k - 3. ⑤ (1/2)((4k - \lambda + 1 - |U|) - \sum_{i=1}^{l} |A_i \cap \overline{U}|)^2 is omitted. ⑥ 2k - 1 \leq |U| \leq 3k - 3. ⑦ 4k - \lambda^2 + 6\lambda - 11 > 0.

Next, we will find an upper bound for the summation of |A_i \cap U_x||D_i \cap U_x|. We remark that for every \( i \in [1, l] \), \( X_i \) is the set of integers \( x \in [1, l_B] \) such that \( A_i \cap U_x \neq \emptyset \) and \( D_i \cap U_x \neq \emptyset \). Moreover, \( \overline{U}^i = \bigcup_{x \in X_i} U_x \). By Claim 4.11 \, \big| D_i \cap U_x \big| \leq \left| U_x \right| - \big| A_i \cap U_x \big| \leq (k-1) - \big| A_i \cap U_x \big| \). Thus,

\[ \sum_{x_1}^{l} \sum_{x_2}^{l} |A_i \cap U_x||D_i \cap U_x| \leq \sum_{i=1}^{l} (k-1)|A_i \cap \overline{U}^i| - \frac{1}{2} \sum_{x_1}^{l} \sum_{x_2}^{l} |A_i \cap U_x|^2 \]

In the red graph, for \( i \in [1, l] \), suppose there exists \( x \in X_i \). By the choice of \( X_i \), \( D_i \cap U_x \neq \emptyset \). Since \( |A_i, D_i| \) is complete in blue, we must have \( A_i \cap \overline{U} \subseteq S_x \), and \( A_i \cap U_x' \subseteq S_x \) for all \( x' \in X_i \), \( x' > x \). Since \( A_i \) is complete in red, all edges between \( A_i \cap U_x \) and \( A_i \cap S_x \) are red. Thus

\[ |E_R^U| \geq \sum_{i=1}^{l} |A_i \cap \overline{U}| |A_i \cap \overline{U}^i| + \sum_{x_1, x_2 \in X_i} \sum_{x_2 < x_2} |A_i \cap U_{x_1}| |A_i \cap U_{x_2}| \]

\[ = \sum_{i=1}^{l} |A_i \cap \overline{U}| |A_i \cap \overline{U}^i| + \frac{1}{2} \sum_{i=1}^{l} |A_i \cap \overline{U}^i|^2 - \frac{1}{2} \sum_{i=1}^{l} \sum_{x \in X_i} |A_i \cap U_x|^2 \]
Finally, by summing up the 2 bounds which add up to $|E_B^U|$, and the bound for $|E_R^U|$, we will have

$$|E_B^U| + |E_R^U|$$

$$>(k - 1)|U| + \sum_{i=1}^{l}((k - 1) - |A_i \cap \overline{U}|)(|A_i \cap U|) - \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2$$

$$- \sum_{i=1}^{l} (k - 1)|A_i \cap \overline{U}^i| + \sum_{i=1}^{l} \sum_{x \in X_i} |A_i \cap U_x|^2$$

$$+ \sum_{i=1}^{l} |A_i \cap \overline{U}| |A_i \cap \overline{U}^i| + \frac{1}{2} \sum_{i=1}^{l} |A_i \cap \overline{U}^i|^2 - \frac{1}{2} \sum_{i=1}^{l} \sum_{x \in X_i} |A_i \cap U_x|^2$$

$$=(k - 1)|U| + \sum_{i=1}^{l} ((k - 1) - |A_i \cap \overline{U}|)(|A_i \cap U| - |A_i \cap \overline{U}^i|)$$

$$- \frac{1}{2} \sum_{i=1}^{l} |A_i \cap U|^2 - \sum_{i=1}^{l} |A_i \cap \overline{U}^i|^2 + \frac{1}{2} \sum_{i=1}^{l} \sum_{x \in X_i} |A_i \cap U_x|^2$$

$\textcircled{5} \geq (k - 1)|U| + \sum_{i=1}^{l} ((k - 1) - |A_i \cap \overline{U}|)(|A_i \cap U| - |A_i \cap \overline{U}^i|)$$

$$- \frac{1}{2} \sum_{i=1}^{l} (|A_i \cap U| + |A_i \cap \overline{U}^i|)(|A_i \cap U| - |A_i \cap \overline{U}^i|)$$

$$\geq (k - 1)|U| + \sum_{i=1}^{l} ((k - 1) - |A_i \cap \overline{U}|)(|A_i \cap U| - |A_i \cap \overline{U}^i|)$$

$$- \frac{1}{2} \sum_{i=1}^{l} (|A_i \cap U| + |A_i \cap \overline{U}|)(|A_i \cap U| - |A_i \cap \overline{U}^i|)$$

$$= (k - 1)|U| + \sum_{i=1}^{l} ((k - 1) - |A_i \cap \overline{U}| - |A_i \cap U|)(|A_i \cap U| - |A_i \cap \overline{U}^i|)$$

$\textcircled{6} \geq (k - 1)|U|$

Notes: $\textcircled{5} (1/2) \sum_{i=1}^{l} \sum_{x \in X_i} |A_i \cap U_x|^2$ is omitted. $\textcircled{6} |A_i \cap \overline{U}| + |A_i \cap U| = |A_i| \leq k - 1.$

which is a contradiction to $|E_B^U| + |E_R^U| \leq (k - 1)|U|$. Thus, $G$ must have a $k$-connected monochromatic subgraph with at least $n - 2k + 2$ vertices.

5. Conclusion

In this paper, we presented a counterexample of Bollobás and Gyárfás’ conjecture with $n = 5k - 2[\sqrt{2k-1}] - 3$. We also verified the conjecture for $n \geq 5k - \lambda$, where $\lambda =$
min\{\lfloor \sqrt{k - 2} \rfloor + 3, \lfloor 0.5k \rfloor + 4\}. Remark \(\lambda = \lfloor \sqrt{4k - 2} \rfloor + 3\) when \(k \neq 3,5,\) or \(7\). However, there is still a \(\Theta(\sqrt{k})\) gap between the two bounds. We have found some examples, which are very different from the counterexample we raised in section 3, and are not covered by the inequality in section 4. However, none of them could serve as a counterexample to the statement. Thus, we conjecture that:

**Conjecture 5.1.** Let \(k, n \in \mathbb{Z}^+\). If \(n > 5k - \lfloor 2\sqrt{2k - 1} \rfloor - 3\), then for any 2-edge-colored \(K_n\), there exists a \(k\)-connected monochromatic subgraph, which contains at least \(n - 2k + 2\) vertices.

More generally, consider the statement “ For \(k, n \in \mathbb{Z}^+\) with \(n \geq g(k)\), every 2-edge-colored \(K_n\) must contain a \(k\)-connected monochromatic subgraph with at least \(n - f(k)\) vertices”. For a given \(f(k) \geq 2k - 2\), what is the minimum \(g(k)\) for the statement to be true? Note that when \(f(k) \leq 2k - 1\), the example \(B(n,k)\) in [1] can always serve as a counterexample of the statement. On the other hand, if \(g(k) \in [4k - 3, 5k - 4]\) is fixed, what is the correlated \(f(k)\)? In other words, given the number of vertices \(n\), what is the order of the largest \(k\)-connected monochromatic subgraph we can guarantee in a 2-edge-colored \(K_n\)?

Furthermore, there are some open problems related to Bollobás and Gyárfás’ conjecture, such as the multicoloring version of the conjecture, and forcing large highly connected subgraphs with given independence number. We believe the decomposition we introduced in this paper could also be applied to improve the results of those topics.

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