Higher-order non-Hermitian skin effect

Kohei Kawabata,1,‡ Masatoshi Sato,2,† and Ken Shiozaki2,*

1Department of Physics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
2Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
(Dated: August 18, 2020)

The non-Hermitian skin effect is a unique feature of non-Hermitian systems, in which an extensive number of boundary modes appear under the open boundary conditions. Here, we discover higher-order counterparts of the non-Hermitian skin effect that exhibit new boundary physics. In two-dimensional systems with the system size \(L \times L\), while the conventional (first-order) skin effect accompanies \(O(L^2)\) skin modes, the second-order skin effect accompanies \(O(L)\) corner skin modes. This also contrasts with Hermitian second-order topological insulators, in which only \(O(1)\) corner zero modes appear. Moreover, in the third-order skin effect in three dimensions, \(O(L)\) corner skin modes appear from all \(O(L^3)\) modes. We demonstrate that the higher-order skin effect originates from intrinsic non-Hermitian topology protected by spatial symmetry. We also show that it accompanies the modification of the non-Bloch band theory in higher dimensions.

I. INTRODUCTION

Topology plays an important role in characterization of phases of matter [1–3]. The central principle of topological phases is the bulk-boundary correspondence: the boundaries host anomalous gapless modes arising from the bulk topology under the open boundary conditions. The number of these boundary modes is \(O(L^{d-1})\) in \(d\)-dimensional systems with the system size \(L^d\). For example, \(O(1)\) zero-energy modes appear at the two ends of a chiral-symmetric chain such as the Su-Schrieffer-Heeger model [4], and \(O(L)\) chiral (helical) modes appear at the edges of a quantum Hall (quantum spin Hall) insulator [5, 6].

Recently, higher-order counterparts of topological phases were revealed and investigated extensively [7–19]. Higher-order topological phases are protected by spatial symmetry such as inversion, mirror, and rotation symmetry. Importantly, the nature of the bulk-boundary correspondence is changed. In two dimensions, second-order topology leads to \(O(1)\) zero modes localized at the corners, which is sharply contrasted with \(O(L)\) corner zero modes accompanied by first-order topology. Similarly, in three dimensions, third-order topology leads to \(O(1)\) zero modes localized at the corners, instead of \(O(L^2)\) boundary modes in first-order topological insulators. Higher-order topology was observed in various experiments [20–28], and may lead to unique phenomena and functionalities due to its boundary physics.

Topological phases and their boundary physics are enriched also by non-Hermiticity [29–30]. In general, non-Hermiticity arises from nonconservation of energy or particles and ubiquitously appears, for example, in nonequilibrium open systems [31, 32]. The interplay of topology and non-Hermiticity gives rise to new physics both in theory [33–91] and experiments [92–108]. One of the unique features of non-Hermitian systems is the non-Hermitian skin effect. This is the extreme sensitivity of non-Hermitian systems to boundary conditions, and an extensive number of boundary modes appear under the open boundary conditions. In particular, an extensive number of [i.e., \(O(L)\)] skin modes appear in one dimension, which is impossible in Hermitian systems. Although the skin effect invalidates the conventional Bloch band theory, researchers formulated a non-Bloch band theory that works even under arbitrary boundary conditions [46, 67]. Moreover, the skin effect was found to originate from intrinsic non-Hermitian topology [72, 80]. Symmetry further enriches the skin effect and gives rise to new types of them originating from symmetry-protected non-Hermitian topology.

Despite the rich physics of non-Hermitian topological systems, little research has hitherto addressed non-Hermitian topological phenomena in higher dimensions. In particular, the non-Hermitian skin effect has been investigated mainly in one dimension, and has remained largely unknown in higher dimensions. Similarly, the non-Bloch band theory in Refs. [46, 67] is applicable only to one dimension, and its validity in higher dimensions has been unclear.

In this work, we discover higher-order counterparts of the non-Hermitian skin effect. They give rise to new types of boundary modes as a result of higher-order non-Hermitian topology (Fig. 1). In two-dimensional systems with the system size \(L \times L\) and open boundaries along both directions, the conventional skin effect accompanies \(O(L^2)\) skin modes at arbitrary boundaries [Fig. 1(c)]. In the second-order skin effect, by contrast, \(O(L)\) skin modes appear at the corners [Fig. 1(d)]. This is also distinct from Hermitian second-order topological insulators, in which only \(O(1)\) corner modes appear as a result of Hermitian topology [Fig. 1(b)]. We demonstrate that the higher-order skin effect cannot be described by the conventional non-Bloch band theory, which implies its inevitable modification in higher dimensions.

Notably, the higher-order non-Hermitian skin effect is
II. FIRST-ORDER NON-HERMITIAN SKIN EFFECT

We begin with reviewing the conventional non-Hermitian skin effect that has the first-order nature. It accompanies the emergence of an extensive number of skin modes localized at arbitrary boundaries; $O(L^2)$ skin modes appear in $d$ dimensions. Such anomalous boundary modes are unique to non-Hermitian systems and originate from intrinsic non-Hermitian topology. This sharply contrasts with Hermitian systems, in which the bulk is insensitive to boundary conditions, and there appear up to $O(L^{d-1})$ boundary modes under the open boundary conditions.

A. Hatano-Nelson model

A prototypical model that exhibits the first-order skin effect is the Hatano-Nelson model [109]:

$$\hat{H}_{HN} = \sum_n \left[ (t - g) \hat{c}^\dagger_n \hat{c}_{n+1} + (t + g) \hat{c}^\dagger_{n+1} \hat{c}_n \right],$$

(1)

where $t, g \in \mathbb{R}$ are the hopping amplitudes, and $\hat{c}_n (\hat{c}^\dagger_n)$ annihilates (creates) a particle on site $n$. We assume $t \geq g \geq 0$ for simplicity. The corresponding Bloch Hamiltonian reads

$$H_{HN}(k) = (t - g) e^{-ik} + (t + g) e^{ik} = 2t \cos k + 2ig \sin k.$$  

(2)

Under the periodic boundary conditions, the system is described by $H_{HN}(k)$ with real wavenumbers $k \in [0, 2\pi)$. The spectrum forms a loop in the complex-energy plane, and the eigenstates are delocalized through the bulk.

Under the open boundary conditions, by contrast, the system is no longer described by $H_{HN}(k)$. To understand this, let us consider the following similarity transformation (imaginary gauge transformation [110]):

$$\hat{V}_r^{-1} \hat{c}_i \hat{V}_r = r^{-i} \hat{c}_i, \quad \hat{V}^{-1}_r \hat{c}_i^\dagger \hat{V}_r = r^{i} \hat{c}_i^\dagger$$

(3)

for $r \in (0, \infty)$. The Hamiltonian $\hat{H}_{HN}$ transforms into

$$\hat{V}^{-1}_r \hat{H}_{HN} \hat{V}_r = \sum_n \left[ r (t - g) \hat{c}^\dagger_{n+1} \hat{c}_n + \frac{t + g}{r} \hat{c}^\dagger_n \hat{c}_{n+1} \right].$$

(4)

In particular, using $r_x := \sqrt{(t + g)/(t - g)}$, we have

$$\hat{V}^{-1}_r \hat{H}_{HN} \hat{V}_{r_x} = \sqrt{t^2 - g^2} \sum_n \left( \hat{c}^\dagger_{n+1} \hat{c}_n + \hat{c}^\dagger_n \hat{c}_{n+1} \right),$$

(5)

which is Hermitian. Importantly, this transformation does not change the spectrum since it is a similarity transformation under the open boundary conditions. Hence, the non-Hermitian Hamiltonian $\hat{H}_{HN}$ with open
boundaries has the same spectrum as the Hermitian Hamiltonian $\hat{V}_r^{-1} \hat{H}_{HN} \hat{V}_r$, which is given as
\[
E(k) = 2\sqrt{t^2 - g^2 \cos k}, \quad k \in [0, 2\pi).
\] (6)
The spectrum lies on the real axis in the complex-energy plane. Since $\hat{V}_r^{-1} \hat{H}_{HN} \hat{V}_r$ hosts delocalized eigenstates, all the eigenstates of $\hat{H}_{HN}$ are localized at the left edge as $\sim e^{-\eta/\ell}$ with the localization length $\xi = (\log r_x)^{-1}$. Clearly, the spectrum and the eigenstates of the bulk are dramatically sensitive to the boundary conditions, which is impossible in Hermitian systems. This is the non-Hermitian skin effect in the Hatano-Nelson model.

In a similar manner, the skin effect generally occurs in non-Hermitian systems. In $d$ dimensions, the skin modes can appear at arbitrary boundaries including edges and corners. Still, $O(L^d)$ skin modes usually accompany the first-order skin effect. However, different types of skin effects can occur in the presence of symmetry or in higher dimensions. In particular, $O(L^{d-1}) [O(L^{d-2})]$ skin modes accompany the second-order (third-order) skin effect, which we focus on in this work.

### B. Non-Hermitian topology

The skin effect originates from intrinsic non-Hermitian topology [79, 80]. In one dimension, the topological invariant is given as a winding number $W(E) \in \mathbb{Z}$ defined for complex energy $E \in \mathbb{C}$ and the Bloch Hamiltonian $H(k)$ [45, 60]:
\[
W(E) := \int_0^{2\pi} \frac{dk}{2\pi i} \log \det [H(k) - E].
\] (7)
This topological invariant is well defined as long as the spectrum of $H(k)$ does not cross given $E$ [i.e., $H(k)$ is point-gapped in terms of a reference point $E$ 45, 60]. If $W(E)$ is nonzero, the skin effect occurs; otherwise, no skin effect occurs.

The non-Hermitian topology of $H(k)$ can also be understood on the basis of the extended Hermitian Hamiltonian
\[
\tilde{H}(k,E) := \begin{pmatrix} 0 & H(k) - E \\ H^\dagger(k) - E^* & 0 \end{pmatrix}.
\] (8)
By construction, this respects chiral symmetry
\[
\sigma_z \tilde{H}(k,E) \sigma_z^{-1} = -\tilde{H}(k,E)
\] (9)
with a Pauli matrix $\sigma_z$. If the non-Hermitian Hamiltonian $H(k)$ is topologically nontrivial for $E$ and the skin effect occurs, the extended Hermitian Hamiltonian $\tilde{H}(k, E)$ is also topologically nontrivial and hosts boundary states under the open boundary conditions.

For the Hatano-Nelson model, we have $W(E) = \text{sgn}(g)$ as long as $E$ is inside the loop described by Eq. [2]. The extended Hermitian Hamiltonian in Eq. [8] is similar to the Su-Schrieffer-Heeger model [1]. The skin modes in the Hatano-Nelson model correspond to a pair of zero modes in the Su-Schrieffer-Heeger model.

Importantly, the topological invariant $W(E)$ is intrinsic to non-Hermitian systems [45, 60]. In fact, without symmetry protection, no topological invariant is well defined in Hermitian systems in one dimension [1-3]. Such intrinsic non-Hermitian topology is the origin of the skin effect, which is also an intrinsic non-Hermitian topological phenomenon. This sharply contrasts with topologically-protected boundary states in Hermitian systems.

In the presence of symmetry, different types of topological invariants can be defined, and consequently, different types of skin effects can occur. In one-dimensional systems with symplectic reciprocity, for example, the $\mathbb{Z}_2$ topological invariant is well defined [80], although the $\mathbb{Z}$ invariant in Eq. [7] vanishes. In contrast to the conventional skin effect, the nontrivial $\mathbb{Z}_2$ topological invariant leads to the reciprocal skin effect. There, some skin modes are localized at one end and other skin modes are localized at the other end.

In higher dimensions, $W(E)$ can still be well defined as a weak topological invariant. In two dimensions, for example, $W(E)$ can be obtained for $H(k_x, k_y)$ with each $k_x$ or $k_y$ [see Eq. [29] for details]. In contrast to the weak topological invariants, strong topological invariants in higher dimensions can result in different types of skin effects that are unique to higher-dimensional systems. The higher-order skin effect is a new type of skin effects that is characterized by spatial-symmetry-protected higher-order topology, as discussed in the subsequent sections.

### C. Non-Bloch band theory

Because of the non-Hermitian skin effect, the conventional Bloch band theory is not generally applicable in non-Hermitian systems. In fact, the Bloch band theory works only under the periodic boundary conditions, and the skin effect invalidates it under the open boundary conditions. To overcome this difficulty, recent works have developed a non-Bloch band theory that works even under the open boundary conditions [46, 67].

The non-Bloch band theory is formulated as follows. Let $\beta_i$'s ($i = 1, 2, \cdots, 2M$: $|\beta_1| \leq |\beta_2| \leq \cdots \leq |\beta_{2M}|$) be the solutions to the characteristic equation $\det [\tilde{H}(E) - E] = 0$ for a given eigenenergy $E \in \mathbb{C}$. Here, the bulk Hamiltonian $H(\beta)$ is obtained by replacing $k$ with $\beta := e^{ik}$ for the Bloch Hamiltonian $H(k)$. Then, the bulk bands are formed by $H(\beta)$ with the trajectory of $\beta_M$ and $\beta_{M+1}$ satisfying
\[
|\beta_M| = |\beta_{M+1}|.
\] (10)
For example, in the Hatano-Nelson model, the bulk Hamiltonian reads
\[
H(\beta) = (t - g) \beta^{-1} + (t + g) \beta.
\] (11)
The characteristic equation \( \det[H(\beta) - E] = 0 \) forms the quadratic equation
\[
(t + g) \beta^2 - E \beta + t - g = 0. \tag{12}
\]
Since \( \beta_1 \) and \( \beta_2 \) are the two solutions to this quadratic equation, we have
\[
\beta_1 + \beta_2 = \frac{E}{t + g}, \quad \beta_1 \beta_2 = \frac{t - g}{t + g}. \tag{13}
\]
Then, the condition \ref{eq:det} leads to
\[
|\beta_1| = |\beta_2| = \sqrt{\frac{t - g}{t + g}} = r_x^{-1}, \tag{14}
\]
which reproduces the skin modes in Sec. \ref{sec:skin_modes}.

Notably, the above non-Bloch band theory can break down in the presence of symmetry. For example, it is modified in the symplectic class \ref{symplectic_class}, which accounts for the \( \mathbb{Z}_2 \) reciprocal skin effect \ref{reciprocal_skin_effect}. Furthermore, the non-Bloch band theory is not directly applicable if the open boundary conditions are imposed more than one direction. Thus, the non-Bloch band theory can be modified in higher dimensions. In the following, we demonstrate that such modification in higher dimensions indeed arises and gives rise to the higher-order skin effect.

## III. second-order non-Hermitian skin effect

In the conventional non-Hermitian skin effect discussed in the preceding section, an extensive number of eigenstates are localized at boundaries. More precisely, \( O(L^d) \) skin modes appear in \( d \)-dimensional systems with the system size \( L^d \). In the higher-order non-Hermitian skin effect, by contrast, most of the eigenstates remain delocalized and form bulk bands. Still, a part of the eigenstates exhibit the skin effect. In the second-order skin effect in two dimensions, which we focus on in the present section, \( O(L^2) \) bulk modes and \( O(L) \) corner skin modes simultaneously appear in a two-dimensional system with the system size \( L \times L \). This also contrasts with Hermitian second-order topological insulators, in which \( O(1) \) corner modes appear.

In Sec. \ref{sec:non-Hermitian_model} we introduce a non-Hermitian model in two dimensions that exhibits the second-order skin effect \cite{non-Hermitian_model}. This model is systematically constructed on the basis of a Hermitian second-order topological insulator \cite{Hermitian_insulator}. The spectra and the wavefunctions of this system are investigated in Sec. \ref{sec:spectra_wavefunctions}. Then, in Sec. \ref{sec:topological_origin} we identify the topological origin of the second-order non-Hermitian skin effect as the Wess-Zumino term. This topological invariant is protected by four-fold-rotation-type symmetry in Eqs. \ref{rotation_symmetry} and \ref{rotation_matrix}. Remarkably, the second-order skin effect requires the modification of the non-Bloch band theory, as demonstrated in Sec. \ref{sec:non-Bloch_band_theory}.

### A. Model and symmetry

We provide a model that exhibits the second-order non-Hermitian skin effect. The Bloch Hamiltonian reads
\[
H(k) = -i(\gamma + \lambda \cos k_x) + \lambda (\sin k_x) \sigma_z
+ (\gamma + \lambda \cos k_y) \sigma_y + \lambda (\sin k_y) \sigma_x, \tag{15}
\]
where \( \gamma \) and \( \lambda \) are real parameters, and \( \sigma_i \)'s \((i = x, y, z)\) are Pauli matrices. As discussed in Sec. \ref{sec:Hatano-Heeger_model}, the Hatano-Heeger model is closely related to the Su-Schrieffer-Heeger model. Similarly, this model is constructed on the basis of a Hermitian second-order topological insulator. In fact, the extended Hermitian Hamiltonian is given as
\[
\hat{H}_{BBH}(k) = \begin{pmatrix} 0 & H(k) \\ H^\dagger(k) & 0 \end{pmatrix}
= (\gamma + \lambda \cos k_x) \tau_y + \lambda (\sin k_x) \sigma_z \tau_x
+ (\gamma + \lambda \cos k_y) \sigma_y \tau_x + \lambda (\sin k_y) \sigma_x \tau_x, \tag{16}
\]
where \( \tau_i \)'s \((i = x, y, z)\) are Pauli matrices that describe the additional degrees of freedom. This Hermitian Hamiltonian is a prototypical model of a second-order topological insulator that was first introduced by Benalcazar, Bernevig, and Hughes \cite{Benalcazar}. There, no midgap modes appear under the open boundary conditions solely along one direction. Nevertheless, under the open boundary conditions along both directions, zero-energy modes appear at the corners for \( |\gamma/\lambda| < 1 \).

Spatial symmetry plays a crucial role in the second-order topological phase of \( \hat{H}_{BBH}(k) \) and the second-order non-Hermitian skin effect of \( H(k) \). First, both \( \hat{H}_{BBH}(k) \) and \( H(k) \) respect spatial-inversion (parity) symmetry:
\[
\sigma_y \hat{H}_{BBH}(k) \sigma_y^{-1} = \hat{H}_{BBH}(-k), \tag{17}
\]
\[
\sigma_y H(k) \sigma_y^{-1} = H(-k). \tag{18}
\]
Inversion symmetry vanishes the first-order skin effect in \( H(k) \), as shown in Sec. \ref{sec:inversion_symmetry}. In addition, \( \hat{H}_{BBH}(k) \) respects mirror symmetry:
\[
(\sigma_z \tau_y) \hat{H}_{BBH}(k_x, k_y) (\sigma_z \tau_y)^{-1} = \hat{H}_{BBH}(-k_x, k_y), \tag{19}
\]
\[
(\sigma_z \tau_y) \hat{H}_{BBH}(k_x, k_y) (\sigma_z \tau_y)^{-1} = \hat{H}_{BBH}(k_x, -k_y). \tag{20}
\]
Correspondingly, \( H(k) \) respects
\[
\sigma_z H^\dagger(k_x, k_y) \sigma_z^{-1} = -H(-k_x, k_y), \tag{21}
\]
\[
\sigma_x H^\dagger(k_x, k_y) \sigma_x^{-1} = -H(k_x, -k_y). \tag{22}
\]
Furthermore, \( \hat{H}_{BBH}(k) \) respects four-fold-rotation symmetry:
\[
\mathcal{R}_4 \hat{H}_{BBH}(k_x, k_y) \mathcal{R}_4^{-1} = \hat{H}_{BBH}(-k_y, k_x), \tag{23}
\]
where \( \mathcal{R}_4 \) is a unitary matrix given as
\[
\mathcal{R}_4 = \begin{pmatrix} 0 & -i \sigma_y \\ 1 & 0 \end{pmatrix}. \tag{24}
\]
5

FIG. 2. Second-order non-Hermitian skin effect. The complex spectra of the non-Hermitian model in two dimensions [Eq. (15)] are shown for 30 × 30 sites. The parameters are given as $\lambda = 1.0$, as well as $(a_1, a_2, a_3, a_4) \gamma = 0.5$, $(b_1, b_2, b_3, b_4) \gamma = 1.0$, or $(c_1, c_2, c_3, c_4) \gamma = 1.5$. The open boundary conditions are imposed along none of the directions for $(a_1, b_1, c_1)$, only along the $x$ direction for $(a_2, b_2, c_2)$, only along the $y$ direction for $(a_3, b_3, c_3)$, and along all the directions for $(a_4, b_4, c_4)$. The spectra for the periodic boundary conditions are shown as the grey regions, while the spectra for the open boundary conditions are shown as the red dots. For $|\gamma| < 1$, the corner skin modes appear under the open boundary conditions along all the directions, as shown in (a4). The spectrum of these corner skin modes is given as $E = -i \gamma (1 + e^{i\theta})$ with $\theta \in [0, 2\pi)$.

Correspondingly, $H(k)$ respects

$$-i \sigma_y H^\dagger(k_x, k_y) = H(-k_y, k_x)$$

(25)

This rotation-type symmetry protects the second-order skin effect, as shown in Sec. III C.

B. Corner skin modes

We numerically obtain the complex spectrum of the non-Hermitian model under various boundary conditions, as shown in Fig. 2. Under the periodic boundary conditions, eigenstates are delocalized through the bulk and form two bands [Fig. 2(a1, b1, c1)]; the bulk spectrum is given as

$$E(k) = \pm \sqrt{\lambda^2 \sin^2 k_x + (\gamma + \lambda \cos k_y)^2 + \lambda^2 \sin^2 k_y - i (\gamma + \lambda \cos k_x)}.$$
tions solely along the x direction [Fig. 2(a2, b2, c2)] or solely along the y direction [Fig. 2(a3, b3, c3)], no skin effect occurs, in general. This corresponds to the absence of zero modes in $H_{BHH}$ under these boundary conditions.

Under the open boundary conditions in both directions, by contrast, skin modes appear for $|\gamma| < |\lambda|$ [Fig. 2(a4)]. These skin modes are not included in the bulk spectrum and localized at the boundaries. In particular, the skin modes are localized at the corners, while the other bulk modes are delocalized (Fig. 3). From all the $2L^2$ eigenstates, the number of the corner skin modes is $2L$, while the number of the delocalized bulk modes is $2L(L-1)$. Notably, the skin spectrum forms a loop in the complex-energy plane even under the open boundary conditions, which is forbidden for the conventional skin effect.

This model can be solved also in an analytical manner (see Appendix A for details). In particular, for sufficiently large $L$, the spectrum of the corner skin modes is given as

$$E = -i\gamma\left(1 + e^{i\theta}\right), \quad \theta \in [0, 2\pi),$$

and their localization lengths $\xi_x$ and $\xi_y$ along the x and y directions are given as

$$\xi_x = \xi_y = \left(\log |\lambda/\gamma|\right)^{-1}.$$  \hspace{1cm} (28)

These analytical results are compatible with the numerical results.

The corner skin modes are a new type of boundary modes unique to non-Hermitian systems in higher dimensions. They are distinct from both $O(L^2)$ skin modes in the conventional skin effect and $O(1)$ corner modes in Hermitian second-order topological insulators. We call this new type of the skin effect the second-order skin effect. It originates from second-order non-Hermitian topology protected by spatial symmetry, as shown in the next section.

### C. Wess-Zumino term

The second-order non-Hermitian skin effect originates from the $\mathbb{Z}_2$-quantized Wess-Zumino (WZ) term introduced shortly. As discussed in Sec. 114, in two dimensions, we can define the one-dimensional winding numbers

$$W_\mu = \int_0^{2\pi} \frac{dk_\mu}{2\pi} \frac{\partial}{\partial k_\mu} \log \det [H (k_x, k_y)], \quad \mu = x, y,$$

along the x and y directions, respectively. As shown below, given a non-Hermitian Hamiltonian $H (k_x, k_y)$ which is invertible and has no one-dimensional winding numbers $W_x = W_y = 0$, we can define a geometric quantity $WZ [H]$ called the WZ term which takes a value in the circle $[0, 1)$. The absence of the one-dimensional winding numbers ensures the existence of a smooth path of invertible Hamiltonians $H (k_x, k_y, t)$ from the original one

$$H (k_x, k_y, t = 0) = H (k_x, k_y)$$

H (k_x, k_y, t = 1) = H_{const}$$

at the end. The WZ term is defined by

$$WZ [H] = \frac{1}{24\pi^2} \int_{T^2 \times [0,1]} \text{tr} \left[H^{-1}dH\right]^3,$$

where $T^2$ denotes the two-dimensional torus of the Brillouin zone. While the WZ term is a real number, it is not quantized in the absence of symmetry.

Although the extension $H (k_x, k_y) \rightarrow H (k_x, k_y, t)$ is not unique, the difference $WZ [H] - WZ [H']$ between the two extensions $H (k_x, k_y, t)$ and $H' (k_x, k_y, t)$ is nothing but the integer-valued three-dimensional winding number of the third homotopy class $\tau_3 (GL_N (\mathbb{R})) = \mathbb{Z}$, where $N \geq 2$ is the dimension of the matrix $H (k_x, k_y)$. Thus, the WZ term in Eq. (32) does not depend on extensions of $H (k_x, k_y)$ as a quantity in the circle $[0, 1)$. It is a two-dimensional analog of the Berry-phase formula of the electric polarization.

Spatial symmetry can quantize the WZ term, similarly to the quantization of the electric polarization due to spatial-inversion symmetry. Here, we focus on the following four-fold-rotation-type symmetry:

$$UH^1 (k_x, k_y) V^{-1} = H (-k_x, k_y), \quad (UV)^2 = 1,$$

where $U$ and $V$ are unitary matrices that are, in general, independent of each other. The two-dimensional model in Eq. (15) respects this symmetry with $U = -i\gamma y$ and $V = 1$ [i.e., Eq. (25)]. We show that this rotation-type symmetry indeed quantizes the WZ term to the $\mathbb{Z}_2$ values

$$WZ [H] \in \left\{0, \frac{1}{2}\right\}.$$  \hspace{1cm} (34)

Given an extension $H (k_x, k_y) \rightarrow H (k_x, k_y, t)$ for $t \in [0, 1]$, we introduce a different extension by

$$H' (k_x, k_y, t) := UH^1 (k_y, -k_x, t) V^{-1}, \quad t \in [0, 1].$$  \hspace{1cm} (35)

Thanks to rotation-type symmetry in Eq. (33), $H' (k_x, k_y, t)$ at $t = 0$ coincides with the original Hamiltonian:

$$H' (k_x, k_y, t = 0) = H (k_x, k_y).$$

In addition, in a straightforward manner, we can show

$$WZ [H'] = -WZ [H],$$

and

$$2WZ [H] = WZ [H] - WZ [H'].$$  \hspace{1cm} (38)
The right-hand side of this equation gives the integer-valued three-dimensional winding number, which proves that the WZ term $WZ[H]$ is quantized to the $\mathbb{Z}_2$ value. For our model in Eq. (15), the WZ term takes the nontrivial value $WZ[H] = 1/2$ for $|\gamma/\lambda| < 1$. Thus, the $\mathbb{Z}_2$-quantized WZ term is a meaningful topological invariant of two-dimensional non-Hermitian systems, as long as four-fold-rotation-type symmetry in Eq. (43) is respected.

In general, the WZ term is quantized to the $\mathbb{Z}_2$ value when either rotation-type symmetry

$$UH^\dagger(k) V^{-1} = H(c_n k)$$

or reflection symmetry

$$UH(k) V^{-1} = H(mk)$$

is respected, where $k \mapsto c_n k$ is an $n$-fold rotation and $k \mapsto mk$ is a reflection on an axis. It can be proven in the same way as four-fold-rotation-type symmetry in Eq. (43).

It should also be noted that four-fold-rotation-type symmetry in Eq. (43) vanishes the one-dimensional winding numbers in Eq. (29). In fact, we have

$$W_x = \int_0^{2\pi} \frac{dk_x}{2\pi} \partial \log \det \left[H^\dagger(-k_y, k_x)\right] = -W_y,$$

and on the other hand, we have

$$W_y = \int_0^{2\pi} \frac{dk_y}{2\pi} \partial \log \det \left[H^\dagger(-k_y, k_x)\right] = W_x.$$

These equations result in

$$W_x = W_y = 0.$$  \hspace{1cm} (43)

The quantization of the WZ term is closely related to the corner skin effect. This can be understood in view of the topological invariant in momentum space and the adiabatic parameter by Teo and Kane [112]. Let us consider the extended Hermitian Hamiltonian $\tilde{H}(k_x, k_y, s)$ with the parameter $s \in S^1$ characterizing the spatial modulation of the microscopic Hamiltonian far from the point defect. The defect zero modes of $\tilde{H}(k_x, k_y, s)$ are detected by the three-dimensional winding number $W_3$, which is in turn given as the winding of the WZ term

$$W_3 = \int_0^1 ds \frac{d}{ds} WZ[H(s)].$$

For nonzero $W_3$, there appear $W_3$ zero modes localized at the point defect in $\tilde{H}(k_x, k_y, s)$. In a similar manner to the Hatano-Nelson model, these zero modes accompany the skin modes at the same defect in the original non-Hermitian Hamiltonian $H(k_x, k_y, s)$.

In the following, we show that the nonzero WZ term $WZ[H]$ leads to the presence of the corner zero modes in the extended Hermitian Hamiltonian, and consequently, the presence of the corner skin modes in the original non-Hermitian Hamiltonian. Let us impose the open boundary conditions along both $x$ and $y$ directions. Near the edges, no zero modes appear because of the vanishing one-dimensional winding numbers in Eq. (29), allowing us to consider adiabatic changes of the microscopic Hamiltonian near the edges into a slowly-varying Hamiltonian while keeping the topological phase. In doing so, we can define a family of Hamiltonians $\hat{H}(k_x, k_y, s)$ for each edge such that $\hat{H}(k_x, k_y, s = 0)$ is the Hamiltonian deep inside the bulk and that $\hat{H}(k_x, k_y, s = 1)$ is outside the finite system. For example, $\hat{H}(k_x, k_y, s = 1)$ can be chosen as the vacuum Hamiltonian $H_{vac}$. Let the families of the edge Hamiltonians be $\hat{H}_1(k_x, k_y, s)$, $\hat{H}_2(k_x, k_y, s)$, $\hat{H}_3(k_x, k_y, s)$, and $\hat{H}_4(k_x, k_y, s)$ for the left, right, up, and down edges, respectively. We assume that the edge Hamiltonians, as well as the bulk Hamiltonian, enjoy the four-fold-rotation-type symmetry, meaning that they are related to each other in the four-fold-symmetric way. For example, the up-edge Hamiltonian is related to the right-edge one by

$$\hat{H}_{u}(k_x, k_y, s) = UH^\dagger(k_x, k_y, s) V^{-1}$$

for the off-diagonal parts.

Then, the changes in the WZ terms

$$\Delta WZ_{\nu} := \int_0^1 ds \frac{d}{ds} WZ[H_{\nu}(s)], \quad \nu \in \{l, r, u, d\},$$

from the bulk to the vacuum for the four edges satisfy

$$\Delta WZ_l = - \Delta WZ_u = \Delta WZ_r = - \Delta WZ_d.$$  \hspace{1cm} (47)

Here, the vacuum Hamiltonian $H_{vac}$ is assumed to be in common for all the edges. This structure gives a constraint on the three-dimensional winding numbers in Eq. (14) of the four corners: $W_3$ of the upper-right corner is given as

$$W_3 = \Delta WZ_r - \Delta WZ_u = 2\Delta WZ_r = -2WZ[H].$$  \hspace{1cm} (48)

FIG. 4. Wess-Zumino (WZ) term and corner zero modes. The number of the zero modes in the extended Hermitian Hamiltonian $\tilde{H}(k_x, k_y)$ with four-fold-rotation symmetry in Eq. (43) is shown as the even and odd integers at each corner. (a) and (b) correspond to the trivial and nontrivial WZ terms, respectively.
modulus 2. This implies that if the quantized WZ term of the bulk is nontrivial (i.e., WZ [H] = 1/2), the threedimensional winding number W 3 of the four corners should be odd, especially nonzero, and hence the extended Hermitian Hamiltonian should host zero modes localized at the corners. See Fig. 4 for possible quartets of the numbers of the corner zero modes accompanied by the trivial and nontrivial bulk WZ terms. Since the presence of the zero modes in the extended Hermitian Hamiltonian leads to the skin effect in the non-Hermitian Hamiltonian [60], the bulk WZ term leads to the corner skin effect.

D. Non-Bloch band theory

While symmetry can protect skin effects, it can also vanish skin effects. A prime example is spatial-inversion (parity) symmetry:

$$\mathcal{P} H (k) \mathcal{P}^{-1} = H (-k)$$  \hspace{1cm} (49)$$

with a unitary matrix $\mathcal{P}$. Our model with the corner skin modes also respects this symmetry with $\mathcal{P} = \sigma_y$ [i.e., Eq. (16)]. In one dimension, no skin effect occurs in the presence of inversion symmetry [60]. This is compatible with vanishing winding number in Eq. (4) in the presence of inversion symmetry.

The absence of the skin effect in one dimension can be shown on the basis of Eq. (10), which is the salient result of inversion symmetry. We begin with the characteristic equation

$$\det [H(\beta) - E] = 0.$$  \hspace{1cm} (50)$$

In terms of $H(\beta)$, inversion symmetry in Eq. (49) imposes

$$\mathcal{P} H (\beta) \mathcal{P}^{-1} = H (\beta^{-1})$$  \hspace{1cm} (51)$$

and hence leads to

$$\det [H(\beta^{-1}) - E] = 0.$$  \hspace{1cm} (52)$$

This equation implies that $\beta^{-1}$ is another solution to the characteristic equation if $\beta$ is a solution. Because of the assumption $|\beta_1| \leq |\beta_2| \leq \cdots |\beta_{2M}|$, we then have

$$\beta_{2M-i+1} = \beta_i^{-1} \quad (i = 1, 2, \cdots, M).$$  \hspace{1cm} (53)$$

Now, using Eq. (10), we finally have

$$|\beta_M| = |\beta_{M+1}| = 1,$$  \hspace{1cm} (54)$$

showing that continuum bands are formed by delocalized eigenstates.

Importantly, the above discussion is not directly applicable in higher dimensions. This is because the non-Bloch band theory in Refs. [46, 67], especially Eq. (10), is inapplicable under the open boundary conditions along more than one direction. Remarkably, the higher-order skin effect requires modification of the non-Bloch band theory in higher dimensions. In fact, if Eq. (10) were valid even in higher dimensions, no skin effect could arise in inversion-invariant systems in higher dimensions in a similar manner to the above discussion. However, this would contradict the emergence of the corner skin effect in our two-dimensional model with inversion symmetry. Hence, the non-Bloch band theory is indeed modified in higher dimensions.

Nevertheless, it is naturally expected that Eq. (10) is valid for an extensive number of eigenstates even in higher dimensions. This is consistent with the delocalization of the $O(L^2)$ bulk modes in our model. On the other hand, the $O(L)$ corner skin modes cannot be described by the current non-Bloch band theory. It is thus important to develop a non-Bloch band theory in higher dimensions in a general manner, which we leave for future work.

IV. THIRD-ORDER NON-HERMITIAN SKIN EFFECT

The non-Hermitian skin effect can even have the third-order nature in three dimensions. In the third-order non-Hermitian skin effect, $O(L)$ corner skin modes emerge from all the $O(L^3)$ eigenstates. We provide a model that exhibits the third-order skin effect. The Bloch Hamiltonian reads

$$H (k) = \iota \lambda (\sin k_y) \sigma_x + \iota (\gamma + \lambda \cos k_y) \sigma_y$$

$$\quad + \iota \lambda (\sin k_x) \sigma_z + (\gamma + \lambda \cos k_x) \tau_z$$

$$\quad + \iota \lambda (\sin k_z) \tau_y + (\gamma + \lambda \cos k_z) \tau_x,$$  \hspace{1cm} (55)$$

where $\gamma$ and $\lambda$ are real parameters, and $\sigma_i$’s and $\tau_i$’s $(i = x, y, z)$ are Pauli matrices. The extended Hermitian Hamiltonian reads

$$\tilde{H}_{BBH} (k) = \begin{pmatrix} 0 & H (k) \\ H^\dagger (k) & 0 \end{pmatrix}$$

$$= -\lambda (\sin k_x) \rho_y \sigma_x - (\gamma + \lambda \cos k_y) \rho_y \sigma_y$$

$$\quad - \lambda (\sin k_z) \rho_y \sigma_z + (\gamma + \lambda \cos k_x) \rho_z \tau_z$$

$$\quad + \lambda (\sin k_z) \rho_x \tau_y + (\gamma + \lambda \cos k_z) \rho_y \tau_x,$$  \hspace{1cm} (56)$$

where $\rho_i$’s $(i = x, y, z)$ are Pauli matrices that account for the additional degrees of freedom. Similarly to the second-order topological insulator, $\tilde{H}_{BBH} (k)$ is a prototypical example of a third-order topological insulator that was first proposed by Bencalazar, Bernevig, and Hughes [2]. It can exhibit zero-energy modes localized at the corners under the open boundary conditions along all the three directions, although no boundary modes appear under other boundary conditions.

The Hermitian model $\tilde{H}_{BBH} (k)$ respects inversion symmetry:

$$(\rho_y \sigma_y \tau_y) \tilde{H}_{BBH} (k) (\rho_y \sigma_y \tau_y)^{-1} = \tilde{H}_{BBH} (-k).$$  \hspace{1cm} (57)$$
FIG. 5. Third-order non-Hermitian skin effect. The complex spectra of the non-Hermitian model in three dimensions [Eq. (55)] are shown for $10 \times 10 \times 10$ sites. The parameters are given as $\lambda = 1.0$, as well as $(a1, a2, a3, a4) \gamma = 0.5$ or $(b1, b2, b3, b4) \gamma = 1.5$. The open boundary conditions are imposed along none of the directions for $(a1, b1)$, only along the $x$ direction for $(a2, b2)$, only along the $x$ and $y$ directions for $(a3, b3)$, and along all the directions for $(a4, b4)$. The spectra for the periodic boundary conditions are shown as the grey regions, while the spectra for the open boundary conditions are shown as the red dots. The corner skin modes appear under the open boundary conditions along all the directions, as shown in (a4).

Correspondingly, $H(k)$ respects
$$\sigma_y H^\dagger(k) \sigma_y^{-1} = -H(-k).$$

Moreover, $\tilde{H}_{BBH}(k)$ respects mirror symmetry:
$$\rho_x \sigma_z \tilde{H}_{BBH}(k_x, k_y, k_z) \rho_x \sigma_z$$
$$= \tilde{H}_{BBH}(k_x, k_y, k_z),$$

$$\rho_z \sigma_z \tilde{H}_{BBH}(k_x, k_y, k_z) \rho_z \sigma_z$$
$$= \tilde{H}_{BBH}(k_x, -k_y, k_z),$$

$$\rho_y \tau_y \tilde{H}_{BBH}(k_x, k_y, k_z) \rho_y \tau_y$$
$$= \tilde{H}_{BBH}(k_x, k_y, -k_z).$$

Correspondingly, $H(k)$ respects
$$\sigma_z H^\dagger(k_x, k_y, k_z) \sigma_z^{-1} = H(-k_x, k_y, k_z),$$

$$\sigma_x H^\dagger(k_x, k_y, k_z) \sigma_x^{-1} = H(k_x, k_y, k_z),$$

$$\tau_y H^\dagger(k_x, k_y, k_z) \tau_y^{-1} = -H(k_x, k_y, -k_z).$$

Such spatial symmetry plays a crucial role in the third-order skin effect.

The third-order topological insulator $\tilde{H}_{BBH}(k)$ exhibits zero-energy corner modes for $|\gamma/\lambda| < 1$. Correspondingly, corner skin modes can appear in the non-Hermitian model $H(k)$ in Eq. (55) with open boundaries along all the directions. In Fig. 5 we show the numerically obtained spectra for various boundary conditions. Under the periodic boundary conditions, no skin effect occurs, and all the eigenstates are delocalized through the bulk. The bulk forms four bands and their spectrum...
is given as
\[
E (k) = \pm \sqrt{(\gamma + \lambda \cos k_x)^2 + (\gamma + \lambda \cos k_z)^2 + \lambda^2 \sin^2 k_x + \gamma^2 \sin^2 k_y + \lambda^2 \sin^2 k_y}, \tag{65}
\]

Under the open boundary conditions along all the directions, an extensive number of the eigenstates remain delocalized and form the bulk bands [Fig. 3(b)]. However, some of the eigenstates exhibit the skin effect and are localized at the four corners [Figs. 3(a4) and 3(a)].

In the conventional skin effect, there appear \( O (L^4) \) skin modes in a three-dimensional system with the system size \( L \times L \times L \); in the third-order skin effect, by contrast, only \( O (L) \) skin modes appear at the corners. This also contrasts with zero-energy corner modes in Hermitian third-order topological insulators, the number of which is \( O (1) \).

Thus, the third-order non-Hermitian skin effect gives rise to a new type of boundary states in three dimensions.

V. CONCLUSION

In this work, we have discovered the higher-order non-Hermitian skin effect. It leads to new types of boundary physics, which may further give rise to new non-Hermitian topological phenomena. In two dimensions, the second-order skin effect accompanies \( O (L) \) corner skin modes in contrast to \( O (L^2) \) skin modes in the conventional (first-order) skin effect. This also contrasts with \( O (1) \) corner zero modes in Hermitian second-order topological insulators. Similarly, in three dimensions, the third-order skin effect accompanies \( O (L) \) corner skin modes in contrast to \( O (L^3) \) skin modes in the conventional (first-order) skin effect and \( O (1) \) corner zero modes in Hermitian third-order topological insulators. These higher-order skin effects originate from intrinsic non-Hermitian topology protected by spatial symmetry. Furthermore, they imply the modification of the conventional non-Bloch band theory \([46, 67]\) in higher dimensions.

It merits further research to develop a non-Bloch band theory that works even in higher dimensions. Furthermore, the higher-order skin effect is a new non-Hermitian phenomenon that originates from spatial symmetry. It is also worthwhile to further explore unique phenomena and functionalities that arise from the interplay of non-Hermiticity and spatial symmetry.

ACKNOWLEDGMENT

The authors thank Shin Hayashi, Nobuyuki Okuma, and Mayuko Yamashita for helpful discussions. This work was supported by JST CREST Grant No. JP-MJCR19T2. K.K. was supported by KAKENHI Grant No. JP19J21927 from the Japan Society for the Promotion of Science (JSPS). M.S. was supported by KAKENHI Grant No. JP20H00131 from the JSPS. K.S. was supported by PRESTO, JST (Grant No. JPMJPR18L4).

Note added. — After completion of this work, we became aware of a recent related work \([113]\).

Appendix A: Exact corner skin modes

We exactly solve the non-Hermitian Hamiltonian in Eq. (15) with open boundaries. In particular, we obtain the corner skin modes in an analytical manner. Let an eigenenergy be \( E \in \mathbb{C} \), and the component of the corresponding eigenstate at the lattice site \((m, n) \in [1, L]^2\) be \( \tilde{\psi} (m, n) \in \mathbb{C} \). Because of periodicity and symmetry of the bulk, \( \tilde{\psi} (m, n) \) can be described as

\[
\tilde{\psi} (m, n) = \beta_x^{m} \beta_y^{n} \tilde{v}_{+} + \beta_x^{m} \beta_y^{1-n} \tilde{v}_{-} + \beta_x^{L+1-m} \beta_y^{n} \tilde{v}_{-} + \beta_x^{L+1-m} \beta_y^{1-n} \tilde{v}_{+}, \tag{A1}
\]

with \( \beta_x, \beta_y \in \mathbb{C} \) and \( \tilde{v}_{\pm} \in \mathbb{C}^2 \). The normalization of \( \tilde{\psi} (m, n) \) requires \( |\beta_x| \leq 1 \) and \( |\beta_y| \leq 1 \).

In the bulk, the Schrödinger equation reads

\[
M \tilde{\psi} (m, n) + T_{x+} \tilde{\psi} (m, n-1) + T_{x-} \tilde{\psi} (m, n+1) + T_{y+} \tilde{\psi} (m+1, n) + T_{y-} \tilde{\psi} (m, n+1) = E \tilde{\psi} (m, n), \tag{A2}
\]

with

\[
M := -i \gamma + \gamma \sigma_y, \tag{A3}
\]
\[
T_{x\pm} := i \lambda (1 \pm \sigma_x), \tag{A4}
\]
\[
T_{y\pm} := \lambda (\sigma_y \pm i \sigma_x), \tag{A5}
\]

With Eq. (A1), the bulk equation leads to

\[
H (\beta_x^{1}, \beta_y^{1}) \tilde{v}_{\pm} = E \tilde{v}_{\pm}, \tag{A6}
\]

where \( H (\beta_x, \beta_y) \) is the bulk Hamiltonian

\[
H (\beta_x, \beta_y) = -i \begin{pmatrix}
\gamma + \lambda \beta_x & \gamma + \lambda \beta_y \\
- \gamma - \lambda \beta_y & \gamma + \lambda \beta_x
\end{pmatrix}. \tag{A7}
\]

At the boundaries, on the other hand, the Schrödinger equation reads

\[
T_{x+} \tilde{\psi} (0, n) = 0, \tag{A8}
\]
\[
T_{x-} \tilde{\psi} (L, n) = 0, \tag{A9}
\]
\[
T_{y+} \tilde{\psi} (m, 0) = 0, \tag{A10}
\]
\[
T_{y-} \tilde{\psi} (m, L) = 0 \tag{A11}
\]

with \( m, n = 1, 2, \ldots, L \). With Eq. (A1), these boundary equations reduce to

\[
T_{x+} \begin{pmatrix}
\tilde{v}_{+} + \beta_x^{L+1} \tilde{v}_{-}
\end{pmatrix} = T_{x+} \begin{pmatrix}
\tilde{v}_{-} + \beta_x^{L+1} \tilde{v}_{-}
\end{pmatrix} = 0, \tag{A12}
\]
\[
T_{x-} \begin{pmatrix}
\tilde{v}_{+} - \beta_x^{L+1} \tilde{v}_{-}
\end{pmatrix} = T_{x-} \begin{pmatrix}
\tilde{v}_{+} - \beta_x^{L+1} \tilde{v}_{-}
\end{pmatrix} = 0, \tag{A13}
\]
\[
T_{y+} \begin{pmatrix}
\tilde{v}_{+} + \beta_y^{L+1} \tilde{v}_{+}
\end{pmatrix} = T_{y+} \begin{pmatrix}
\tilde{v}_{+} + \beta_y^{L+1} \tilde{v}_{+}
\end{pmatrix} = 0, \tag{A14}
\]
\[
T_{y+} \begin{pmatrix}
\tilde{v}_{+} - \beta_y^{L+1} \tilde{v}_{+}
\end{pmatrix} = T_{y-} \begin{pmatrix}
\tilde{v}_{-} - \beta_y^{L+1} \tilde{v}_{+}
\end{pmatrix} = 0. \tag{A15}
\]
Now, we express \( \vec{v}_{\pm\pm} \) as \( \vec{v}_{\pm\pm} = (a_{\pm\pm} b_{\pm\pm})^T \). Then, these boundary equations reduce to
\[
\begin{align*}
\beta_x^{L+1} a_{++} + a_{--} &= \beta_x^{L+1} a_{+-} + a_{-+} = 0 \quad \text{(A16)} \\
\beta_x^{L+1} a_{++} + a_{--} &= \beta_x^{L+1} a_{+-} + a_{-+} = 0 \quad \text{(A17)} \\
a_{++} + \beta_y^{L+1} a_{-+} &= a_{+-} + \beta_y^{L+1} a_{-+} = 0 \quad \text{(A18)} \\
\beta_y^{L+1} b_{++} + b_{--} &= \beta_y^{L+1} b_{+-} + b_{-+} = 0, \quad \text{(A19)}
\end{align*}
\]
which are further simplified to
\[
\begin{align*}
\frac{a_{+-}}{a_{++}} &= \left(\frac{b_{--}}{b_{++}}\right)^{-1} = -\beta_y^{L-1}, \quad \text{(A20)} \\
\frac{a_{++}}{a_{+-}} &= \left(\frac{b_{++}}{b_{--}}\right)^{-1} = -\beta_x^{L+1}, \quad \text{(A21)} \\
\frac{a_{--}}{a_{+-}} &= \left(\frac{b_{--}}{b_{++}}\right)^{-1} = \beta_x^{L+1} \beta_y^{L-1}. \quad \text{(A22)}
\end{align*}
\]

Meanwhile, since \( \vec{v}_{++} \) (\( \vec{v}_{--} \)) is an eigenstate of \( H(\beta_x, \beta_y) \) \( \left[H(\beta_x^{1}, \beta_y^{1})\right] \) from Eq. \( \text{(A6)} \), we have
\[
\begin{align*}
(\gamma + \lambda \beta_x - iE) a_{++} + (\gamma + \lambda \beta_y) b_{++} &= 0, \quad \text{(A23)} \\
(\gamma + \lambda \beta_y - iE) a_{++} + (\gamma + \lambda \beta_y) b_{++} &= 0. \quad \text{(A24)}
\end{align*}
\]

Using Eq. \( \text{(A21)} \), we have
\[
\begin{align*}
\left(\begin{array}{c}
\gamma + \lambda \beta_x - iE \\
\beta_x^{L+1} \left(\gamma + \lambda \beta_y - iE\right)
\end{array}\right)
\left(\begin{array}{c}
\gamma + \lambda \beta_y
\end{array}\right)
\left(\begin{array}{c}
\frac{a_{++}}{b_{++}}
\end{array}\right) = 0. \quad \text{(A25)}
\end{align*}
\]
To have a nontrivial solution \( (a_{++} b_{++}) \neq 0 \), the determinant of the coefficient matrix should vanish, which results in
\[
\beta_y = -\frac{\gamma}{\lambda}, \quad \text{(A26)}
\]
or
\[
\frac{iE - \gamma}{\lambda} = \frac{\beta_x^L - \beta_y^{L+1}}{\beta_x^{L+1} - \beta_y^{L-1}}. \quad \text{(A27)}
\]

Similarly, since \( \vec{v}_{++} \) (\( \vec{v}_{--} \)) is an eigenstate of \( H(\beta_x, \beta_y) \) \( \left[H(\beta_x^{1}, \beta_y^{1})\right] \) from Eq. \( \text{(A6)} \), we have
\[
E = -i (\gamma + \lambda \beta_x) \quad \text{(A28)}
\]
or
\[
-\frac{\gamma}{\lambda} = \frac{\beta_x^L - \beta_y^{L+1}}{\beta_x^{L+1} \beta_y^{L-1}}. \quad \text{(A29)}
\]

Furthermore, since \( \vec{v}_{++} \) (\( \vec{v}_{--} \)) is an eigenstate of \( H(\beta_x, \beta_y) \) \( \left[H(\beta_x^{1}, \beta_y^{1})\right] \) from Eq. \( \text{(A6)} \), we have
\[
\begin{align*}
\left(\begin{array}{c}
\gamma + \lambda \beta_x - iE \\
\beta_x^{2(L+1)} \left(\gamma + \lambda \beta_y - iE\right)
\end{array}\right)
\left(\begin{array}{c}
\gamma + \lambda \beta_y
\end{array}\right)
\left(\begin{array}{c}
\frac{a_{++}}{b_{++}}
\end{array}\right) = 0,
\end{align*}
\]
resulting in
\[
\beta_x^{2(L+1)} \left(\gamma + \lambda \beta_y\right)^2 = -\beta_y^{2(L+1)} \left(\gamma + \lambda \beta_x - iE\right)^2. \quad \text{(A31)}
\]

Importantly, we need
\[
|\beta_x| = |\beta_y| \quad \text{(A32)}
\]
so that the above equation will hold for sufficiently large \( L \).

The corner skin modes are described by Eq. \( \text{(A28)} \). If Eq. \( \text{(A27)} \) holds in addition to Eq. \( \text{(A28)} \), we have \( \beta_x = \pm 1 \). Then, we also have \( E = -i (\gamma \pm \lambda) \) and \( \beta_y = -\lambda/\gamma, -\gamma/\lambda \), which further leads to \( |\gamma| = |\lambda| \) from Eq. \( \text{(A32)} \). Hence, we have Eq. \( \text{(A26)} \) as long as Eq. \( \text{(A28)} \) and \( |\gamma| \neq |\lambda| \) hold. Because of the normalization condition \( |\beta_y| < 1 \), we need
\[
\left|\frac{\gamma}{\lambda}\right| < 1. \quad \text{(A33)}
\]

Equations \( \text{(A26)} \) and \( \text{(A28)} \) lead to
\[
\begin{align*}
H(\beta_x, \beta_y) - E &= -i \left(\begin{array}{c}
0 \\
\lambda^2 - \gamma^2
\end{array}\right) \gamma - \lambda \left(\begin{array}{c}
0 \\
\lambda^2 - \gamma^2
\end{array}\right) \gamma \quad \text{(A34)}
\end{align*}
\]
\[
\begin{align*}
H(\beta_x^{2(L+1)}, \beta_y^{2(L+1)}) - E &= i \left(\begin{array}{c}
0 \\
\lambda (\beta_x - \beta_x^{-1}) \left(\begin{array}{c}
\lambda^2 - \gamma^2
\end{array}\right) \gamma
\end{array}\right) \gamma \quad \text{(A35)}
\end{align*}
\]

Since \( (a_{++} b_{++}) \) \( \sim (\beta_x^{2(L+1)} a_{++} \beta_y^{2(L+1)} b_{++}) \) are eigenstates of \( H(\beta_x, \beta_y) \) and \( H(\beta_x^{-1}, \beta_y^{-1}) \), respectively, we have
\[
\left(\begin{array}{c}
\beta_x^{2(L+1)} \lambda (\beta_x - \beta_x^{-1}) \left(\begin{array}{c}
\lambda^2 - \gamma^2
\end{array}\right) \gamma
\end{array}\right) \frac{a_{++}}{b_{++}} = 0, \quad \text{(A36)}
\]
which leads to
\[
(\beta_x - \beta_x^{-1})^2 \beta_x \beta_y \frac{a_{++}}{b_{++}} = -\left(\frac{\lambda^2 - \gamma^2}{\lambda \gamma}\right)^2. \quad \text{(A37)}
\]

To have this equation for sufficiently large \( L \), we need \( |\beta_x/\beta_y| = 1 \), i.e., Eq. \( \text{(A32)} \). Furthermore, the phase of \( \beta_x \) is quantized by this equation. Since we have \( \beta_x = |\beta_x| \neq 1 \), the eigenstates are localized at the corners, and the skin effect occurs. The spectrum of these corner skin modes is given as
\[
E = -i \gamma \left(1 + e^{i\theta}\right), \quad \theta \in [0, 2\pi), \quad \text{(A38)}
\]
and their number is \( 2L \).

On the other hand, the eigenstates described by Eq. \( \text{(A29)} \) are delocalized through the bulk. With \( \beta_y = e^{i k_y} \), Eq. \( \text{(A29)} \) reduces to
\[
-\frac{\gamma}{\lambda} = \sin(k_y L) / \sin(k_y (L + 1)), \quad \text{(A39)}
\]
which quantizes the wavenumber \( k_y \in [0, 2\pi) \). In fact, we have \( L \) real solutions in \( k_y \in [0, \pi] \) for \( |\gamma/\lambda| > 1 \); all the \( 2L^2 \) eigenstates do not exhibit the skin effect and delocalized through the bulk. For \( |\gamma/\lambda| < 1 \), on the other hand, we have \( (L - 1) \) real solutions in \( k_y \in [0, \pi] \); the corresponding \( 2L (L - 1) \) eigenstates are delocalized, while the other \( 2L \) eigenstates are the corner skin modes.
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