Classical and Quantal Ternary Algebras

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Abstract

We consider several ternary algebras relevant to physics. We compare and contrast the quantal versions of the algebras, as realized through associative products of operators, with their classical counterparts, as realized through classical Nambu brackets. In some cases involving infinite algebras, we show the classical limit may be obtained by a contraction of the quantal algebra, and then explicitly realized through classical brackets. We illustrate this classical-contraction method by the Virasoro-Witt example.
I. INTRODUCTION

Ternary algebras can be realized in two ways: They may be built from antisymmetrized products of three linear operators – so-called quantal brackets – or they may be realized through a generalization of Poisson brackets in terms of multivariable Jacobians – so-called classical brackets. Both constructions were introduced in the physics literature by Nambu \[30\] after about twenty years of gestation. Lately, these algebras seem to be gaining in usefulness and importance to physics. For example, there has been some progress in constructing a 2 + 1-dimensional local quantum field theory with $SO(8)$ superconformal symmetry \[4, 5, 21, 22, 24\] as a stepping-stone to obtain a world-volume Lagrangian description for coincident M2-branes. The use of ternary algebras is crucial to the construction.

More generally, $N$-algebras involve antisymmetric operations on $N$ entities at a time. The mathematics was developed by Filippov in a paper that appeared just over a decade after Nambu’s \[17\], although Filippov does not seem to have been aware of Nambu’s earlier work \[12\]. In any case, both Nambu’s and Filippov’s work motivated and inspired other mathematical investigations of these ideas \[7, 13, 33, 34\] along with many other physics studies (see \[13, 14\] and references therein, especially \[16, 25\]), as well as much more recent work \[1, 3, 6, 10, 18, 23, 28, 29\].

Here we compare classical and quantal realizations of ternary algebras for several interesting cases relevant to physics. The quantal and classical algebras are usually not the same, but for one very special case they are: Nambu’s $su(2)$. For some infinite cases, especially the Virasoro-Witt algebra \[11\], the classical limit may be obtained initially by a contraction of the quantal algebra, and then explicitly realized through classical 3-brackets.

II. CLASSICAL AND QUANTAL BRACKETS

In this section, we define and compare the properties of various realizations of 3-brackets. These are just trilinear operations performed either on three operators or on three functions defined on a manifold. For the latter case, some further discussion of the manifold’s classical geometric structure is given in an Appendix.

A. Three specific realizations of 3-brackets

The classical Nambu bracket \[30\] based on three variables (say $x, y, z$) is the simplest to compute, in most situations. It involves the Jacobian-like determinant of partial derivatives of three functions $A, B, C$.

$$\{A, B, C\} = \frac{\partial (A, B, C)}{\partial (x, y, z)} = \varepsilon^{abc} \partial_a A \partial_b B \partial_c C, \quad (1)$$

where $a, b, c$ are implicitly summed over 1, 2, 3. The classical bracket is totally antisymmetric in the three argument functions. (The \{$,\}$ notation is here used to distinguish this classical case from the quantal bracket, and should not be confused with anti-commutation.)

The operator or “quantal” 3-bracket was originally defined \[30\] to be a totally antisymmetrized sum of trinomials,

$$[A, B, C] = ABC - BAC + CAB - ACB + BCA - CBA. \quad (2)$$
regardless of the number of underlying independent variables. This can be equivalently expressed as a sum of single operators produced with commutators of the remaining two, or as anticommutators acting on the commutators,

\[ [A, B, C] = A [B, C] + B [C, A] + C [A, B] = [B, C] A + [C, A] B + [A, B] C . \] (3)

In principle, as well as in practice, it is necessary to have some information about products, hence about anticommutators as well as commutators, to perform the actual evaluation of this quantal bracket. Relatedly, the trace of \([A, B, C]\) is non-trivial, in general.

There is yet another definition of an operator 3-bracket, introduced by Awata, Li, Minic, and Yoneya [2], which is valid for operators of trace class. It consists of re-packaging the commutators of any Lie algebra to define

\[ \langle A, B, C \rangle = [A, B] \text{ Tr}(C) + [C, A] \text{ Tr}(B) + [B, C] \text{ Tr}(A) . \] (4)

This is again a totally antisymmetric trilinear combination, but it is a singular construction for finite dimensional realizations in the sense that \(\text{Tr} \langle A, B, C \rangle = 0\). For reasons to be discussed, this ALMY bracket has properties intermediate between the classical 3-bracket and the full quantal bracket.

B. Properties of the various brackets

As mentioned already, all three are trilinear and totally antisymmetric. On the other hand, only two out of three automatically satisfy the Filippov condition [17], the so-called “FI” (also see [32, 33]). Namely,

\[ \{A, B, \{C, D, E\}\} = \{\{A, B, C\}, D, E\} + \{C, \{A, B, D\}, E\} + \{C, D, \{A, B, E\}\} , \]
\[ \langle A, B, \{C, D, E\}\rangle = \langle\langle A, B, C\rangle, D, E\rangle + \langle C, \langle A, B, D\rangle, E\rangle + \langle C, D, \langle A, B, E\rangle\rangle . \] (5)

So, for the classical and ALMY brackets, the FI is indeed an identity, hence it is a necessary condition to realize a ternary algebra in terms of either of these 3-brackets. But in general the FI does not hold for associative operator products.

\[ 0 \neq [A, B, [C, D, E]] - [[A, B, C], D, E] - [C, [A, B, D], E] - [C, D, [A, B, E]] \]
\[ \equiv \text{fi} (A, B; C, D, E) . \] (6)

That is to say, the Filippov condition, \(\text{fi} (A, B; C, D, E) = 0\), is not an operator identity. It holds only in special circumstances. In this sense the FI differs from the Jacobi identity for associative operator products, a 2-bracket-acting-on-2-bracket situation. Looking ahead, we write the Jacobi identity in the somewhat unusual form

\[ 2\varepsilon^{ij} [[A, B_i], B_j] = \varepsilon^{ij} [A, [B_i, B_j]] , \] (7)

where \(A\) is fixed but all indexed entries are implicitly summed over \(i, j = 1, 2\), hence anti-symmetrized.

Rather remarkably, all three types of 3-brackets do satisfy an identity first found by Bremner (henceforth, the “BI”), a 3-on-3-on-3 multiple bracket relation [8, 31]:

\[ \varepsilon^{i_1 \ldots i_6} \{\{A, \{B_{i_1}, B_{i_2}, B_{i_3}\}, B_{i_4}\}, B_{i_5}, B_{i_6}\} = \varepsilon^{i_1 \ldots i_6} \{\{A, B_{i_1}, B_{i_2}\}, \{B_{i_3}, B_{i_4}, B_{i_5}\}, B_{i_6}\} , \]
\[ \varepsilon^{i_1 \ldots i_6} \langle\langle A, \langle B_{i_1}, B_{i_2}, B_{i_3}\rangle, B_{i_4}\rangle, B_{i_5}, B_{i_6}\rangle = \varepsilon^{i_1 \ldots i_6} \langle\langle A, B_{i_1}, B_{i_2}\rangle, \langle B_{i_3}, B_{i_4}, B_{i_5}\rangle, B_{i_6}\rangle , \]
\[ \varepsilon^{i_1 \ldots i_6} [\langle A, [B_{i_1}, B_{i_2}, B_{i_3}]\rangle, B_{i_4}], B_{i_5}, B_{i_6}] = \varepsilon^{i_1 \ldots i_6} [\langle A, B_{i_1}, B_{i_2}\rangle, [B_{i_3}, B_{i_4}, B_{i_5}], B_{i_6}] . \] (8)
where again, $i_1, \cdots, i_6$ are implicitly summed from 1 to 6. For the classical bracket in (11), both left- and right-hand-sides of the BI actually vanish. For the ALMY bracket, the identity follows by direct calculation.

For quantal 3-brackets the BI is a consequence of associativity. If one posits an operator 3-bracket based on associative products, and it does not satisfy this identity, then one has erred. Thus the BI is a necessary condition to realize a ternary algebra in terms of operator brackets.

Finally, we note that the antisymmetrization of an operator 3-bracket acting on a second 3-bracket does not close to give a third 3-bracket, but rather a 5-bracket: $\varepsilon^{i_1 \cdots i_5} [B_{i_1}, B_{i_2}, B_{i_3}], B_{i_4}, B_{i_5}] \propto [B_1, B_2, B_3, B_4, B_5]$. The operator 5-bracket is defined in general [14] as the totally antisymmetric signed sum over all 5! distinct words $ABCDE$, etc.

III. EXAMPLES OF TERNARY ALGEBRAS.

When the 3-brackets close to yield other functions or operators of a designated set, one is considering a ternary algebra. In this event, one may define structure constants to write, say for the quantal bracket,

$$[A_i, A_j, A_k] = f_{ijk}^l A_l .$$

Similarly for the classical and ALMY brackets. For the latter two types of brackets, the FI always holds, and this implies a bilinear consistency condition on the respective structure constants. This is similar to the consistency condition imposed on the structure constants of a Lie algebra by the Jacobi identity. However, this bilinear condition does not necessarily apply to the quantal structure constants [14].

More generally, the BI gives a trilinear consistency condition on the structure constants for all three types of brackets. This trilinear condition must always be satisfied by any ternary algebra. For the classical and ALMY brackets, the BI structure constant condition is not independent of the FI condition. But for a quantal 3-algebra the BI condition is the only constraint imposed on the $f_{ijk}^l$ by associativity. It would be interesting to classify all quantal 3-algebras by constructing all solutions of the BI condition on the $f_{ijk}^l$. We leave this for the well-motivated reader to pursue. Here, we just consider various basic examples of intrinsic interest to physics.

A. Nambu’s $su(2)$

First, consider Nambu’s application to $su(2)$.

$$[L_x, L_y, L_z] \equiv L_x [L_y, L_z] + L_y [L_z, L_x] + L_z [L_x, L_y] = i \left( L_x^2 + L_y^2 + L_z^2 \right) .$$

To close the algebra, it is necessary to include the $su(2)$ Casimir. But, having done so, one may rescale by a fourth root of the Casimir

$$Q_x = \frac{L_x}{\sqrt{L^2}} , \quad Q_y = \frac{L_y}{\sqrt{L^2}} , \quad Q_z = \frac{L_z}{\sqrt{L^2}} ,$$

and define a fourth charge as that fourth root,

$$Q_t = \sqrt{L^2} .$$
Then,

$$[Q,a, Q_b, Q_c] = i \varepsilon_{abcd} Q^d.$$  \hspace{1cm} (13)

where $\varepsilon_{xyzt} = +1$ with a $[-1, -1, -1, +1]$ Lorentz signature. The usual $\varepsilon\varepsilon$ identities now imply that this example is special: The FI holds for Nambu’s $su(2)$. In fact, this is the only quantal ternary algebra that satisfies the FI, and, as we shall see, it has an explicit realization through classical 3-brackets.

Toward that end, we note that Nambu $su(2)$ has sub-3-algebras that close. They are easily found. For example,

$$Q_x, \; Q_y, \; Q_z \pm Q_t. \hspace{1cm} (14)$$

Moreover, each of these subalgebras can be realized in terms of the 3-bracket (1), as is evident if we just consider the brackets of $x\sqrt{z}, \; y\sqrt{z}, \; z$ and $x\sqrt{z}, \; y\sqrt{z}, \; x^2 + y^2$. \hspace{1cm} (15)

Thus the complete 3-algebra of all four $Q_a$ can be realized in terms of classical brackets. Having done so, it follows without calculation that the structure constants of this 3-algebra are such that the FI is satisfied identically, since the Nambu 3-brackets in (1) always obey this condition. In contrast, however, we note that (13) cannot be realized as ALMY brackets, since $\sqrt{L^2} = Q_t = -i [Q_x, Q_y, Q_z]$ is not traceless.

B. The bosonic oscillator.

The usual four charges 1, $a$, $a^\dagger$, and $N = a^\dagger a$ give the quantal ternary algebra

$$[1, N, a] = -a, \quad [1, N, a^\dagger] = a^\dagger, \quad [1, a, a^\dagger] = 1, \quad [N, a, a^\dagger] = -1 - N. \hspace{1cm} (16)$$

Three of these reduce to just commutators: $[1, N, a] = [N, a], \quad [1, N, a^\dagger] = [N, a^\dagger], \quad$ and $[1, a, a^\dagger] = [a, a^\dagger]$. This would suggest that the same algebra might also be realized as ALMY brackets, except for the fact that the operators at hand are not of trace class. In any case, the fourth relation in (16) is not so simple.

However, if we take linear combinations as

$$R_1 = N, \quad R_2 = \frac{1}{\sqrt{2}} (a^\dagger + a), \quad R_3 = \frac{1}{\sqrt{2}} i (a^\dagger - a), \quad R_4 = N + 1, \hspace{1cm} (17)$$

then we are back to a variation on Nambu’s theme for $su(2)$: In this case, $sl(2, \mathbb{R})$.

$$[R_a, R_b, R_c] = i \varepsilon_{abcd} R^d, \hspace{1cm} (18)$$

with $\varepsilon_{1234} = +1$, again with Lorentz metric to raise indices, $\eta_{ab} = [1, 1, 1, -1]$. So, what’s new here?

There are two additional bilinears, $a^2$ and $a^{12}$, whose 3-brackets give oscillator trilinears.

$$[a, a^2, a^{12}] = 2a + 2Na, \quad [a^\dagger, N, a^2] = -2a - Na, \quad [a, N, a^2] = -a^3, \quad [a^\dagger, a^2, a^{12}] = 2a^\dagger + 2a^\dagger N, \quad [a, N, a^{12}] = 2a^\dagger + a^\dagger N, \quad [a^\dagger, N, a^{12}] = a^{13}. \hspace{1cm} (19)$$
Therefore, upon closure, the ternary algebra becomes infinite, and the standard enveloping algebra for the oscillator is obtained. From Filippov’s perspective, it is perhaps disappointing that the oscillator enveloping algebra does not satisfy the FI. For example

\[-2 = \left[[a^\dagger, a^\dagger a, a^{12}], a, a^2\right] - \left[[a^\dagger, a, a^2], a^\dagger a, a^{12}\right] - \left[a^\dagger, [a^\dagger a, a, a^2], a^{12}\right] - \left[a^\dagger a, a^{12}, [a^\dagger, a, a^2]\right], \]

\[20a^\dagger = \left[[a^\dagger a, a^{12}, a^2], a, a^{12}\right] - \left[[a^\dagger a, a, a^{12}], a^{12}, a^2\right] - \left[a^\dagger a, [a^{12}, a, a^2], a^2\right] - \left[a^\dagger a, a^{12}, [a^2, a, a^{12}]\right]. \quad (20)\]

In any case, the FI does not hold in this example. But, necessarily, the associative enveloping algebra does satisfy the BI.

C. Virasoro–Witt 3-algebra.

For the oscillator there is a familiar, infinite Lie algebra contained within the enveloping algebra. Consider

\[L_n = - (a^\dagger)^n N, \quad (21)\]

for \(n \geq 0\). Commutators give the well-known Virasoro-Witt (VW) algebra.

\[[L_n, L_m] = (n - m) L_{n+m}, \quad (22)\]

for \(m, n \geq 0\). It is less well-known that the corresponding quantal 3-brackets are

\[[L_n, L_m, L_k] = 0. \quad (23)\]

Thus we have a null 3-algebra for an infinite set of non-trivial, non-commuting oscillator charges. The FI is trivially satisfied in this case, as is the BI.

More structure is evident if we slightly modify the oscillator realization of the VW algebra. For parameters \(\beta\) and \(\gamma\), define and compute,

\[L_n = -(a^\dagger)^n (N + \gamma + n\beta), \quad [L_n, L_m] = (n - m) L_{n+m}. \quad (24)\]

The parameter \(\beta\) is related to the \(sl(2, \mathbb{R})\) Casimir, \(C = \beta (1 - \beta)\). Now we find a non-null quantal 3-bracket when \(0 \neq \beta \neq 1\).

\[[L_n, L_m, L_k] = \beta (1 - \beta) (n - m) (m - k) (n - k) M_{n+m+k}, \quad (25)\]

where a second sequence of charges has been defined by

\[M_n = (a^\dagger)^n. \quad (26)\]

While the Lie algebra of the \(Ls\) and \(Ms\) is also well-known [20], their 3-algebra has been investigated only recently [11]. To close the 3-algebra, we must consider all additional 3-brackets involving the \(Ms\):

\[[M_n, L_m, L_k] = (m - k) (L_{n+m+k} + (1 - 2\beta) nM_{n+m+k}), \]

\[[M_n, M_m, L_k] = (m - n) M_{n+m+k}, \]

\[[M_n, M_m, M_k] = 0. \quad (27)\]
While the calculation is involved, the BI may be confirmed to hold for this ternary algebra. This result follows from the use of only (25) and (27), and does not make explicit use of the oscillator realization employed to obtain the 3-algebra. So this algebra is consistent with an underlying associative operator product no matter how it is realized.

The modification of the oscillator realization to include the parameter $\beta$ has led to a larger ternary algebra involving both $L$s and $M$s. But, so enlarged, the algebra as presented in (25) and (27) is cumbersome. It may be streamlined by a linear change of basis, effectively from $L_n$ and $M_n$ back to the original $L_n$ and $M_n$, as in (21) and (24). That is to say, let

$$\mathcal{L}_n \equiv L_n + (\gamma + \beta n) M_n. \quad (28)$$

Regardless of how the algebra is realized, this change of basis simplifies (25) and (27). We find a remarkably concise form for the ternary algebra.

$$[\mathcal{L}_k, \mathcal{L}_m, \mathcal{L}_n] = 0,$$
$$[M_k, M_m, M_n] = 0,$$
$$[\mathcal{L}_k, M_m, M_n] = (n - m) M_{k+m+n},$$
$$[M_k, L_m, \mathcal{L}_n] = (m - n) (\mathcal{L}_{k+m+n} - k M_{k+m+n}). \quad (29)$$

All explicit $\beta$ dependence is thereby removed from the 3-algebra in this basis, for all values of the $sl(2, \mathbb{R})$ Casimir. Now, what about the FIs?

The FIs sometimes fail. This was discussed in [11], in the original basis, but it is much more transparent in terms of (29). It is trivial to see that the Filippov condition is satisfied when only $L$s, or when only $M$s, are involved: $\fil(\mathcal{L}_p, \mathcal{L}_q; \mathcal{L}_k, \mathcal{L}_m, \mathcal{L}_n) = 0 = \fil(M_p, M_q; M_k, M_m, M_n)$. The condition is also satisfied when there are two, three, or four $M$s mixing it up with $L$s. But when one $M$ is entangled with four $L$s, the condition fails, in general:

$$\fil(\mathcal{L}_p, \mathcal{L}_q; \mathcal{L}_k, \mathcal{L}_m, \mathcal{L}_n) = (p - q) (k - m) (k + m - p - q) n M_{k+m+n+p+q},$$
$$\fil(\mathcal{L}_p, \mathcal{L}_q; \mathcal{L}_k, \mathcal{L}_m, \mathcal{L}_n) = (n - k) (k - m) (m - n) q M_{k+m+n+p+q}. \quad (30)$$

On the other hand, the BI is again seen to always hold. We stress that these results follow from the use of only (29) without explicit use of the oscillator realization.

The situation with the FIs can be remedied if we perform an Inönü-Wigner contraction [26]. This produces an algebra that satisfies the FI in all cases. The procedure is to rescale $\mathcal{L}_k, M_k \mapsto \mathcal{L}_k \equiv \lambda^{-1} \mathcal{L}_k, \mathcal{P}_k \equiv \lambda M_k$ and take the formal limit $\lambda \to \infty$. The result is just to discard the term $k M_{k+m+n}$ in the last line of (29),

$$[\mathcal{L}_k, \mathcal{L}_m, \mathcal{L}_n] = 0,$$
$$[\mathcal{P}_k, \mathcal{P}_m, \mathcal{P}_n] = 0,$$
$$[\mathcal{L}_k, \mathcal{P}_m, \mathcal{P}_n] = (n - m) \mathcal{P}_{k+m+n},$$
$$[\mathcal{P}_k, \mathcal{L}_m, \mathcal{L}_n] = (m - n) \mathcal{L}_{k+m+n}. \quad (31)$$

Remarkably, the contracted 3-algebra so obtained is invariant under the $O(2)$ transformation

$$\mathcal{L}_k, \mathcal{P}_k \mapsto \mathcal{L}_k \cos \theta + \mathcal{P}_k \sin \theta, \mathcal{P}_k \cos \theta - \mathcal{L}_k \sin \theta. \quad (32)$$
An interpretation of this symmetry, as well as the validity of the FI's, is obvious in the contracted algebra's realization as a classical 3-bracket algebra. That is,

\[
\begin{align*}
\{xe^{kz}, xe^{mz}, xe^{nz}\} &= \{ye^{kz}, ye^{mz}, ye^{nz}\} = 0, \\
\{xe^{kz}, ye^{mz}, ye^{nz}\} &= (n - m) ye^{(k+m+n)z}, \\
\{ye^{kz}, xe^{mz}, xe^{nz}\} &= (m - n) xe^{(k+m+n)z}.
\end{align*}
\tag{33}
\]

In this realization the \(O(2)\) symmetry is nothing but a rotation about the \(z\)-axis.

The results in (29)-(33) provide the whole story, so far as we know it, for the ternary VW algebra. However, for completeness, we also wish to make contact with various other results in [11]. By redefinition of the charges of the original basis, it was observed in [11] that a "classical limit" could be constructed, in which the \(sl(2,\mathbb{R})\) Casimir went to infinity, in such a way that all FI's were OK. In fact, this also just amounts to a contraction of the ternary algebra. Rescaling

\[
Q_k \equiv \frac{1}{\sqrt{\beta (1 - \beta)}} L_k, \quad R_k \equiv \sqrt{\beta (1 - \beta)} M_k,
\tag{34}
\]

substituting into (25) and (27) above, and taking the limit \(\beta \to \infty\), the resulting algebra is

\[
\begin{align*}
[Q_k, Q_m, Q_n] &= (k - m) (m - n) (k - n) R_{k+m+n}, \\
[R_k, Q_m, Q_n]_{\beta \to \infty} &= (m - n) (Q_{k+m+n} + 2i R_{k+m+n}), \\
[Q_k, R_m, R_n] &= (n - m) R_{k+m+n}, \\
[R_m, R_n, R_k] &= 0.
\end{align*}
\tag{35}
\]

For finite \(\beta\), there would be an additional \(R_{k+m+n}\) term in the second relation.

Again, the contracted algebra obeys the FI's in all cases. This also follows immediately from the fact that we may realize (33) in terms of classical 3-brackets. Explicitly we find

\[
\begin{align*}
\{(x - iky) e^{kz}, (x - imy) e^{mz}, (x - iny) e^{nz}\} &= (k - m) (m - n) (k - n) ye^{(k+m+n)} , \\
\{ye^{kz}, (x - imy) e^{mz}, (x - iny) e^{nz}\} &= (m - n) (x - i (k + m + n) y) e^{(k+m+n)} + 2i k ye^{(k+m+n)} , \\
\{(x - iky) e^{kz}, ye^{mz}, ye^{nz}\} &= (n - m) ye^{(k+m+n)} , \\
\{ye^{kz}, ye^{mz}, ye^{nz}\} &= 0.
\end{align*}
\tag{36}
\]

In the next section of the paper, we will explain in detail how (33) and (36) were first found. It is not difficult to guess one form given the other.

But suppose we just transform back to the original linear combinations to recover the classical versions of the \(L\)s. What is the effect on the algebra? To answer this, let

\[
\ell_n \equiv (x - (\gamma + n/\beta) y) e^{nz}, \quad p_n = ye^{nz}
\tag{37}
\]

We obtain

\[
\begin{align*}
\{\ell_k, \ell_m, \ell_n\} &= -\beta^2 (k - m) (k - n) (m - n) p_{k+m+n}, \\
\{\ell_k, \ell_m, p_n\} &= (k - m) (\ell_{k+m+n} - 2\beta np_{k+m+n}), \\
\{\ell_k, p_m, p_n\} &= (n - m) p_{k+m+n}, \\
p_k, p_m, p_n &= 0.
\end{align*}
\tag{38}
\]
This differs from the original, uncontracted quantal algebra (25) and (27) only in the \( \beta \)-dependent coefficients on the RHS. Namely, \(-\beta^2\) appears instead of \(\beta(1-\beta)\) and \(-2\beta\) instead of \(1-2\beta\). So, to repeat the observation made in [11], we may again identify this classical 3-algebra with the infinite \(sl(2,\mathbb{R})\) Casimir limit, \(\beta \to \pm \infty\), of the quantal algebra.

Finally, for emphasis, since the classical 3-bracket always obeys FIs, it follows that these conditions must necessarily hold true for each of the various forms of the contracted VW algebra given here.

**D. Classical 3-bracket algebra for exponentials**

Consider the infinite set of exponentials,

\[
E_a = \exp (a \cdot r) ,
\]

and compute the classical bracket,

\[
\{E_a, E_b, E_c\} = a \cdot (b \times c) \ E_{a+b+c} .
\]  

The indices here are 3-vectors, with \(\cdot\) and \(\times\) the usual dot and cross products. This infinite algebra does satisfy Filippov’s condition, since all classical brackets do, as well as the Bremner identity.

It is not known how to realize (40) as operator 3-brackets. Although, there is a quantal 4-bracket which gives this 3-bracket as a classical limit [15]. To see this, compute the operator 4-bracket \(\{\exp (a \cdot r), \exp (b \cdot r), \exp (c \cdot r), w\}\) where we assume the exponentials do not involve \(w\), and where we take \(w\) and \(x\), and also \(y\) and \(z\), to be independent canonically conjugate pairs of variables, i.e. \([w, x] = i\hbar, [y, z] = i\hbar\), but \([w, y] = 0\), etc. The result for the 4-bracket is then given directly by the commutator resolution [14, 35].

\[
\begin{align*}
\{e^{a \cdot r}, e^{b \cdot r}, e^{c \cdot r}, w\} &= 4\hbar e^{(a+b+c) \cdot r} \left( a_x \sin \left( \frac{1}{2} \hbar b_\perp \times c_\perp \right) \cos \left( \frac{1}{2} \hbar (b_\perp + c_\perp) \times a_\perp \right) \\
&\quad + b_x \sin \left( \frac{1}{2} \hbar c_\perp \times a_\perp \right) \cos \left( \frac{1}{2} \hbar (c_\perp + a_\perp) \times b_\perp \right) + c_x \sin \left( \frac{1}{2} \hbar a_\perp \times b_\perp \right) \cos \left( \frac{1}{2} \hbar (a_\perp + b_\perp) \times c_\perp \right) \right) ,
\end{align*}
\]  

(41)

where \(a = (a_x, a_y, a_z), a_\perp = (a_y, a_z), a_\perp \times b_\perp = a_y b_z - a_z b_y\), etc. In the limit \(\hbar \to 0\), this gives the anticipated classical 3-bracket,

\[
\frac{1}{2\hbar^2} \{\exp (a \cdot r), \exp (b \cdot r), \exp (c \cdot r), w\} = a \cdot (b \times c) \ \exp (a + b + c) \cdot r + O(\hbar^2) .
\]  

(42)

Before the classical limit is taken, however, (41) does not satisfy the FI: There are violations at \(O(\hbar^6)\) and beyond.

We now describe in more detail the relation between (40) and the classical realization of the VW ternary algebra (31). In fact, the classical VW is a subalgebra of (40).

This may be understood as follows. Clearly, from (40), any three exponentials with co-planar vectors will have a vanishing classical bracket. By representing all the \(\mathcal{L}\)s with a set of such co-planar exponentials, and all the \(\mathfrak{P}\)s with another set of co-planar exponentials, the first two lines of (31) will be satisfied. In general then, there are two distinct planes: One for the \(\mathcal{L}\)s and one for the \(\mathfrak{P}\)s. The remaining challenge, viewed geometrically, is to put these two distinct planes together so that the last two lines of (31) will also be satisfied. An
obvious guess is that the two planes should intersect at right angles. Another, related guess
is that the index appearing in (31) should correspond to modes along the line of intersection
of the two planes.

Therefore, to play the role of the classical $\mathcal{L}$s, take $l_k \equiv E_{\hat{x}+k\hat{z}} = \exp(x+kz)$, while for
the $p_k$s take $p_k \equiv E_{\hat{y}+k\hat{z}} = \exp(y+kz)$. Some elementary algebra then gives $(\hat{x}+m\hat{z}) \cdot
((\hat{y}+n\hat{z}) \times (\hat{y}+k\hat{z})) = k-n$ and $(\hat{y}+k\hat{z}) \cdot ((\hat{x}+m\hat{z}) \times (\hat{x}+n\hat{z})) = m-n$ as well as
$(\hat{x}+m\hat{z}) + (\hat{y}+n\hat{z}) + (\hat{y}+k\hat{z}) = (\hat{x}+\hat{y}) + \hat{y} + (k+m+n) \hat{z}$ and $(\hat{y}+k\hat{z}) + (\hat{x}+m\hat{z}) +
(\hat{x}+n\hat{z}) = (\hat{x}+\hat{y}) + \hat{x} + (k+m+n) \hat{z}$. So, modulo the common spurious vector $(\hat{x}+\hat{y})$
we have just what we need to obtain the contracted algebra from the classical brackets (41).

Now, if we incorporate the inverse of this spurious term into the definition of a modified
classical 3-bracket, as a multiplicative factor,
\[
\{A, B, C\}_{\text{mod}} = \frac{\partial (A, B, C)}{\partial (x, y, z)} e^{-(x+y)\cdot T} = \frac{\partial (A, B, C)}{\partial (x, y, z)} e^{-x-y},
\]
then we have realized on exponentials the classical, contracted VW 3-algebra.
\[
\{l_k, l_m, l_n\}_{\text{mod}} = 0, \quad \{p_k, p_m, p_n\}_{\text{mod}} = 0,
\]
\[
\{l_k, p_m, p_n\}_{\text{mod}} = (n-m) p_{k+m+n}, \quad \{p_k, l_m, l_n\}_{\text{mod}} = (m-n) l_{k+m+n}. \tag{44}
\]

But what effect does the multiplicative factor have on FIs?

It cannot obviate the FIs, because we have already verified them for the contracted
algebra. Another way to see this is to note the multiplicative factor is just the Jacobian
for the variable change $(x, y, z) \mapsto (e^x, e^y, z)$. In terms of these new exponential variables the
realization is
\[
l_k = xe^{kz}, \quad p_k = ye^{kz}, \tag{45}
\]
where these are to be acted on by unmodified classical 3-brackets for the new $x, y, z$ variables.
Thus we obtain (33) of the previous section.

We may summarize either (33) or (36) as simply the closure of functions of the form $xf(z)$
and $yg(z)$ under classical 3-brackets. A complementary algebra is given by the closure
of the classical brackets for functions of the form $\sqrt{z}f(x, y)$. This may be expressed as a
two-parameter algebra [9] if we choose $f(x, y) = \exp(ax + by)$ Again, the FI is guaranteed
to hold since only classical brackets are involved.

\section*{IV. CONCLUSIONS}

We have discussed ternary algebras and the Bremner identities which they always obey,
as well as the Filippov conditions which hold identically for classical and ALMY 3-brackets
but not for quantal brackets. We stress that the Bremner identities are universal for
ternary algebras, and would constrain inclusion of central charges and related extensions in
such algebras, while the FI conditions would provide further constraints but only in more
specialized situations. Nevertheless, we recognize the FI is simpler, when it applies, and
often useful in specific applications.

For Nambu’s $su(2)$, as well as various infinite dimensional algebras, we have provided
classical realizations, which, ipso facto, ensure FI compliance. We suspect that all ternary
algebras based on ALMY brackets can be realized as classical 3-brackets as well, but we have
not shown this. We also suspect the role of the BI in CFT operator product expansions,
including supersymmetric extensions, may be very interesting. We believe these open
questions are worthy of further investigation.
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APPENDIX A: CLASSICAL BRACKETS AND MANIFOLDS

We give here a brief overview of classical brackets defined on \( n \)-dimensional manifolds. Some features of the quantal brackets are foreshadowed in this classical context. For systematic discussions, as well as guides to the literature, see [16,19,33,34].

We consider a general Poisson bracket involving an antisymmetric, but otherwise arbitrary, 2-tensor \( \omega^{ab} \).

\[
\{A, B\} = \omega^{ab} \partial_a A \partial_b B .
\] (A1)

Repeated indices are implicitly summed from 1 to \( n \). For any \( \omega^{ab} \) this is obviously a derivation:

\[
\{A, BC\} = \{A, B\} C + B \{A, C\} .
\]

But it is more interesting for physics purposes that there are situations where the Poisson bracket realizes a Lie algebra. This is evident if we bracket \( \{A, B\} \) with \( C \) to obtain two functionally independent terms.

\[
\{C, \{A, B\}\} = \omega^{cd} \omega^{ab} \partial_d (\partial_c C \partial_a A \partial_b B) + (\partial_c C \partial_a A \partial_b B) \omega^{cd} \partial_d \omega^{ab} .
\] (A2)

The combination that constitutes the Lie-algebra mandated Jacobi “identity” similarly gives two terms.

\[
\{C, \{A, B\}\} - \{\{C, A\}, B\} - \{A, \{C, B\}\} = \Omega^{abcd} \partial_d (\partial_a A \partial_b B \partial_c C) + \Omega^{abc} (\partial_a A \partial_b B \partial_c C) ,
\] (A3)

where we have defined 4- and 3-tensors

\[
\Omega^{abcd} \equiv \omega^{cd} \omega^{ab} - \omega^{ac} \omega^{bd} + \omega^{ad} \omega^{bc} , \quad \Omega^{abc} \equiv \omega^{cd} \partial_d \omega^{ab} - \omega^{bd} \partial_d \omega^{ac} + \omega^{ad} \partial_d \omega^{bc} .
\] (A4)

Now the first term on the RHS of (A3) always vanishes by symmetry: For any \( \omega^{ab} = -\omega^{ba} \) the corresponding \( \Omega^{abcd} \) is a totally antisymmetric 4-tensor, and hence \( \Omega^{abcd} \partial_d (\partial_a A \partial_b B \partial_c C) \) is identically zero. So, for constant \( \omega^{ab} \) the Jacobi identity \( \{C, \{A, B\}\} - \{\{C, A\}, B\} - \{A, \{C, B\}\} = 0 \) is indeed an identity for Poisson brackets.

But in general, the second term on the RHS of (A3) does not vanish for non-constant 3-tensors. Hence there is a condition for the Jacobi identity to be satisfied: \( \Omega^{abc} = 0 \). When true, we are dealing with a Lie algebra on a Poisson manifold. If in addition \( n \) is even and the 2-tensor has an inverse, such that \( \omega_{ab} \omega^{bc} = \delta^c_a \), then we have a symplectic manifold, and we can construct the 2-form \( \omega = \omega_{ab} \, dx^a \wedge dx^b \). In this case the condition for the Jacobi identity to hold is easier rendered to be \( 0 = \partial_a \omega_{bc} + \partial_b \omega_{ca} + \partial_c \omega_{ab} \), or equivalently just that the 2-form is closed: \( d\omega = 0 \). While this is at first sight a generalization from the constant 3-tensor case, this is somewhat illusory. For such closed 2-forms Darboux proved the existence of local coordinates on the manifold such that \( \omega_{ab} \) is constant.

Next, we consider a general classical Nambu 3-bracket involving an arbitrary, totally antisymmetric 3-tensor \( \omega^{abc} \).

\[
\{A, B, C\} = \omega^{abc} \partial_a A \partial_b B \partial_c C .
\] (A5)
With this structure we encounter a few similarities with the Poisson bracket case, but more importantly, we also encounter some dramatic differences. Just like the Poisson bracket above, this is a derivation: \( \{A, B, CD\} = \{A, B, C\} \ D + C \ \{A, B, D\} \). Also like the Poisson bracket case, the action of one 3-bracket on another produces two independent terms.

\[
\{\{A, B, C\}, D, E\} = \omega^{abc} \omega^{\text{def}} \ \partial_f (\partial_a A \ \partial_b B \ \partial_c C \ \partial_d D \ \partial_e E) + (\partial_a A \ \partial_b B \ \partial_c C \ \partial_d D \ \partial_e E) \omega^{\text{def}} \partial_f \omega^{abc} .
\]

But here the differences arise. What is the appropriate analogue of the Jacobi identity?

In general, there is no perfect analogue of the Jacobi identity involving the action of one 3-bracket on another, or a linear combination of such. To see this we need only consider the case of constant \( \omega^{abc} \) and make use of some elementary group theory: The symmetrized product of two antisymmetric 3-tensors does not contain a totally antisymmetric 6-tensor. Indeed, with the standard partition labeling of symmetric group representations, where the sequence of integers represents the number of boxes in the rows of a Young frame, we have

\[
\{1, 1, 1\}^2_{\text{symmetric}} = \{2, 1, 1, 1\} + \{2, 2, 2\} .
\]

The antisymmetric 6-tensor is found instead in the antisymmetrized product, \( \{1, 1, 1\}^2_{\text{antisymmetric}} = \{1, 1, 1, 1, 1\} + \{2, 2, 1, 1\} \). Now, because it is the symmetrized tensor product of two \( \omega \)'s that appears in (A6), these group properties imply that any linear combination, obtained by permuting the entries in one classical 3-bracket acting on another, cannot possibly vanish without imposing some condition on \( \omega^{abc} \), and/or on the number of variables \( n \). Neither partial nor full antisymmetrizations of \( A, B, C, D, E \) in (A6) can avoid both of the tensors on the RHS of (A7), and in general \( \omega^{\text{abcdef}} (\partial_a A \ \partial_b B \ \partial_c C \ \partial_d D \ \partial_e E) \neq 0 \neq \omega^{\text{abcdef}} (\partial_a A \ \partial_b B \ \partial_c C \ \partial_d D \ \partial_e E) \). We emphasize that this is different from the Poisson bracket case, where the group theory is \( \{1, 1\}^2_{\text{symmetric}} = \{1, 1, 1\} + \{2, 2\} \) and \( \{1, 1\}^2_{\text{antisymmetric}} = \{2, 1, 1\} \), and where the linear combination of brackets in the Jacobi identity serves to single out \( \{1, 1, 1\} \) and thus eliminate the \( \omega \omega \) term for all antisymmetric 2-tensors.

Admittedly, there is one very special case where group theory does not impose an impasse for classical 3-on-3-bracket identities, namely, \( n = 3 \). When there are only three independent variables, the \( \{2, 1, 1, 1\} \) representation is absent! In this special case we obtain the FI.

\[
\{\{A, B, C\}, D, E\} = \{\{A, D, E\}, B, C\} + \{A, \{B, D, E\}, C\} + \{A, B, \{C, D, E\}\} .
\]

Of course, for this case there is only one antisymmetric 3-tensor, namely, that of Kronecker, so \( \omega^{abc} \propto \varepsilon^{abc} \). But, alas, Filippov’s condition for 3-brackets does not go willingly into higher dimensional manifolds.

More generally, if we define a classical N-bracket as \( \{A_1, A_2, \cdots, A_N\} = \omega^{a_1a_2\cdots a_N} \partial_{a_1} A_1 \partial_{a_2} A_2 \cdots \partial_{a_N} A_N \), then the even N cases are like that of the Poisson bracket, while the odd N cases are like that for the 3-bracket. When N is even, we have \( \{1^{2N}\} \subset \{1^N\}^2_{\text{symmetric}} \), so for general constant \( \omega \)'s the action of one N-bracket on another produces \( \omega \omega \) in the double bracket. When N is odd, we have \( \{1^{2N}\} \subset \{1^N\}^2_{\text{antisymmetric}} \), so for constant \( \omega \)'s the action of any permuted linear combination of one N-bracket on another does not vanish without imposing some condition on the \( \omega \)'s and/or on the number of variables. For example, if there are only N variables and \( \omega \) is the Kronecker tensor, the N-bracket version of the FI holds.
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