E² and Gamma distributions in polygonal networks

Ran Li¹, Consuelo Ibar², Zhenru Zhou², Seyedsajad Moazzeni¹, Andrew N. Norris¹, Kenneth D. Irvine²,†, Liping Liu¹,3,‡, Hao Lin¹,‡

¹Department of Mechanical and Aerospace Engineering, Rutgers, The State University of New Jersey
²Waksman Institute and Department of Molecular Biology and Biochemistry, Rutgers, The State University of New Jersey
³Department of Mathematics, Rutgers, The State University of New Jersey

Abstract

From solar supergranulation to salt flat in Bolivia, from veins on leaves to cells on Drosophila wing discs, polygon-based networks exhibit great complexities, yet similarities and consistent patterns emerge. Based on analysis of 99 polygonal tessellations of a wide variety of physical origins, this work demonstrates the ubiquity of an exponential distribution in the squared norm of the deformation tensor, E², which directly leads to the ubiquitous presence of Gamma distributions in polygon aspect ratio as recently demonstrated by Atia et al. [Nat. Phys. 14, 613 (2018)]. In turn an analytical approach is developed to illustrate its origin. E² relates to most energy forms, and its Boltzmann-like feature allows the definition of a pseudo-temperature that promises utility in a thermodynamic ensemble framework.

I. INTRODUCTION

Polygonal networks are one of nature’s favorite ways of self-organizing: from supergranulation on the solar surface [1] to cracked dry earth [2]; from ice wedges in northern Canada [3] to the scenic Salar de Uyuni in Bolivia [4]; and from veins on leaves [5] to cells on Drosophila wing discs [6] (Fig. 1). These systems are driven by distinctive physical mechanisms, yet they share common features. Individual constituents, namely, “cells” appear to “randomize” into statistical distributions, and only interact with their immediate neighbors. On the collective level, especially in the dynamic and active systems, rich phenomena are observed, including unjamming and jamming, fluid-to-solid phase transitions, and flow and migration [7–15].

Despite the complexity and variabilities involved in these phenomena, similarity patterns emerge. One particularly interesting instance is provided recently by Atia et al. [16]. Within the context of confluent biological tissue and based on extensive experiments both in vitro and in vivo, the authors found that data on cell aspect ratio collapse and follow a normalized
Gamma distribution, implying a universal principle governing the geometric configuration and pertinent processes.

What is the basis of this universality? Does it carry beyond the biological context? This work comprises of a two-part discovery to answer these questions. In the first, we demonstrate that the Gamma distribution in polygon aspect ratio derives from an exponential distribution in the squared strain tensor norm, $E^2$. We show that such Gamma distributions do not share a similar physical origin to those in the granular packing system as suggested by Atia et al. [16, 17], but rather is an approximation of a unifying solution that we develop. More importantly, we are able to extend universality of the observations to a wide variety of systems (99 data sets in 8 groups) that include convection patterns (solar supergranulation), landforms (salt flats, on Mars, and in or near the Arctic), cracked dry earth, and biological patterns (veins on leaves, cells on Drosophila wing discs, and plated MDCK cells). (Fig. 1 and Table I.) Both exponential and Gamma distributions persist in all data examined. In the second, we tackle the origin of the $E^2$ distribution, and present a theoretical framework to accurately compute $E^2$ from vertex statistics. Importantly, $E^2$ is closely related to common definitions of system energy including all of the bulk-, perimeter-, and moment-based (known as the Quantizer) forms [7, 10, 18–21]. The Boltzmann-like feature of $E^2$ enables the definition of a pseudo-temperature which promises utility in an ensemble-based thermodynamic framework [22–25].

II. $E^2$ AND ITS RELATIONSHIP WITH ASPECT RATIO

In this first part of this work, we define $E^2$ and analytically establish its relationship with the polygon aspect ratio. We demonstrate that a $k$-Gamma distribution in the latter is derivable from an exponential distribution in the former. We begin by defining the mean-field deformation tensor, $E$. An exemplary processed image of a Drosophila wing disc 120h after egg laying (AEL) is shown in Fig. 1(h), where the color scale indicates magnitude of the cell aspect ratio. We choose as our reference frame a regular $n$-polygon centered at the origin, with vertices

$$x_j = r_0 e_j, \; e_j = [\cos(j2\pi/n), \sin(j2\pi/n)],$$

for $j = 1, \ldots, n$. This polygon is denoted by $\mathcal{P}_R$, and $r_0$ is to be determined such that $\mathcal{P}_R$ has the same area as the polygon in comparison. Figure 1(i) uses $n = 6$, a hexagon as an illustrative example. We consequently regard any $n$-sided polygon $\mathcal{P}$ with vertices $y_j$ as a “deformation” from $\mathcal{P}_R$, namely, $y_j = x_j + u_j$, where we also assume that their centroids (defined by Eq. (S6) in the Supplemental Material (SM) [27]) are aligned. The deformation is in general non-uniform, in the sense that for vertex number $n \geq 3$, a single deformation tensor of $F \in \mathbb{R}^{2 \times 2}$ cannot be identified by $y_j = Fx_j$ for all $j$. Nevertheless we can introduce an affine approximation, namely, $y_j \approx y_j' = Fx_j$. The use of a uniform deformation to approximate the local and non-uniform deformation field is effectively coarse-graining, reducing the degree of freedom from $2n$ to 4 and suppressing the fluctuations. This idea is illustrated in Fig. 1(i) (right), where $\mathcal{P}'$ is the approximate and uniformly deformed polygon. From $F$ we define the usual strain tensor and its squared Frobenius norm,
\[ E = \left( F^T F \right)^{1/2} - I \approx \left( F - I \right) + \left( F - I \right)^T / 2, \]  
\[ (2) \]

where \( I \) is the identity tensor, and the Frobenius norm is \( |E|^2 = Tr(E^T E) \). Here we will use \( |E|^2 \) and \( E^2 \) (terminology) interchangeably. The restriction to area-conserved deformation requires that \( TrE = 0 \), which is satisfied by choosing \( r_0 \) in (1). Consequently, \( E \) has a degree of freedom of 2. We pursue an analytical expression for \( E \) via minimizing the difference between \( y_j \)'s and \( y'_j \)'s, from which we obtain [27]

\[ |E|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i \cdot C_{ij} v_j, \]
\[ (3) \]

\[ C_{ij} = \frac{2}{n^2} [(e_i \cdot e_j)I + e_j \otimes e_i - e_i \otimes e_j], \]
\[ (4) \]

where \( v_j = u_j / r_0, \) \( r_0 = \frac{1}{n} \sum_{j=1}^{n} y_j \cdot e_j, \) and \( \otimes \) denotes a dyadic product.

Next we show that if \( E^2 \) follows an exponential distribution (validated below both via data and analysis), namely, \( \rho_E(|E|^2) = \beta \exp(-\beta|E|^2) \), then the aspect ratio follows a \( k \)-Gamma distribution such as shown in [16]. Here \( \rho(\cdot) \) denotes a probability density function (PDF), and \( \beta \) is similar to an inverse temperature. The aspect ratio \( a_r \) of the polygon is calculated via the second area moment, and is related to \( E^2 \) as [27]

\[ a_r^2 = (x + 1)^2 = g(|E|^2), \]
\[ (5) \]

where \( g(t) = 1 + 2t^2 + 4t + [(2t^2 + 4t + 1)^2 - 1]^{1/2} \), and for convenience we have defined a shape factor \( x \). Based on (5), a transformation \( \rho_E(|E|^2) d|E|^2 = \rho_X(x) dx \) leads to

\[ \rho_X(x) = \rho_E(|E|^2) \frac{d|E|^2}{dx} = \beta \zeta'(x) \exp(-\beta \zeta(x)), \]
\[ (6) \]

where \( \zeta(x) = g^{-1}((x + 1)^2) \), and the superscript denotes an inverse function. Equation (6) is our prediction of the distribution in the aspect ratio.

### III. DATA VALIDATE E^2 AND a_r DISTRIBUTIONS

Both distributions and their relationship are extensively validated with a total of 99 data sets spanning 8 groups summarized in Table I; detailed descriptions, data sources, and method of analysis are also presented in the SM [27]. Figure 2 and S2 use 8 representative cases from each group to demonstrate the agreement. The left column shows the PDFs of \( |E|^2 \), which are very well fitted by the exponential form, \( \exp(-\beta|E|^2) \), where \( \beta \) is extracted as a fitting parameter. [See also Fig. 4(b) where all \( |E|^2 \) profiles are presented and the value of \( \beta \) is theoretically predicted.]
The center column shows the PDFs of the shape factor, \(a_r - 1\) (symbols). The theoretical predictions per Eq. (6) are shown in dashed, and exhibit excellent agreement with data. They are generated per Eq. (6), with the single input parameter, \(\beta\), extracted from the analysis of \(|E|^2\) distribution.

Lastly in the right column, the PDFs for \(a_r - 1\) are normalized with \(\langle a_r - 1 \rangle\), where \(\langle \cdot \rangle\) denotes a mean. Both data and theoretical predictions are normalized following this practice. The dot-dashed are best fits using a \(k\)-Gamma distribution following [16],

\[
\rho_{kG}(x_1; k) = \frac{k^k}{\Gamma(k)} x_1^{k-1} \exp(-k x_1),
\]

where \(\Gamma(k)\) is the Gamma function, and \(k\) is the single fitting parameter. The agreement is evident, and the \(k\)-values are found to vary about 2–3.

Overall corroboration between theory and data is quantified by the coefficient of determination, \(R^2\), and are listed in Table I for all cases. The values are uniformly close to 1. Validity of the theoretical prediction (6) is also attested by Fig. 3(a). Here we define a pseudo temperature \(T\) as the inverse of \(\beta\), namely, \(T = \beta^{-1}\). To compare with data, a theoretical prediction is generated by using (6) to obtain the mean.

The above results validate that the \(E^2\) does follow an exponential, Boltzmann-like distribution. The universality of this distribution in all data sets, according to our theory, necessarily leads to a universality of \(k\)-Gamma distributions for the aspect ratio. That is, the validity extends beyond the confluent tissues studied in [16], to all systems we analyzed.

**IV. \(E^2\) DISTRIBUTION IS A \(\chi^2\)-DISTRIBUTION**

In the second part of this work, we demonstrate the origin of the highly regular statistical distribution in \(E^2\). We first show that due to the small degree of freedom of \(E\), we can write \(|E|^2\) as the sum of two squared entities. These entities are then shown to follow normal distributions due to machinery of the central limit theorem. The combined effects lead to an exponential distribution for \(|E|^2\) as a special case of the \(\chi^2\)-distribution. Figure 3(b) shows results comparing the pseudo-temperature computed from data with the theoretical prediction we develop below (denoted “theory”). Here the subscript \(n\) denotes a restriction to the sub-ensemble of \(n\)-gons, \(T_n = \langle |E|^2 \rangle_n\). Polygons other than \(n = 4–8\) are of statistically insignificant occurrences and not included.

The key relationship we utilize is a quadratic form to compute \(|E|^2\) given vertex displacement, \(v_j\). We concatenate \(v_j\)'s to a vector in \(\mathbb{R}^{2n}\), namely, \(\bar{v} = [v_1; \ldots; v_n] = [\bar{v}_1; \ldots; \bar{v}_{2n}]\). (We similarly define other vectors and tensors from their two-dimensional counterparts, and denote by a hat.) We then have \(|E|^2 = \bar{v} \cdot \tilde{C} \bar{v}\), where \(\tilde{C} \in \mathbb{R}^{2n \times 2n}_{\text{sym}}\) has block components \(C_{ij}\) given in (4). \(\tilde{C}\) has a single eigenvalue \(2/n\) of multiplicity 2 [27]. \(|E|^2\) can thus be reduced to a particularly simple form,
\[ E^2 = \frac{2}{n} \left( \hat{w}_1^2 + \hat{w}_2^2 \right), \]  

where \( \hat{w}_k = \hat{p}_k \cdot \hat{v} \), and \( \hat{p}_k \) is an eigenvector of \( \hat{C} \). If \( \hat{v} \) is characterized by a covariance matrix, \( \hat{\Sigma} \), then the variance of \( \hat{w}_k \) is \( \text{Var}(\hat{w}_k) = \hat{p}_k \cdot \hat{\Sigma} \hat{p}_k = \text{Tr}(\hat{p}_k \otimes \hat{p}_k \cdot \hat{\Sigma}) \) [30], and \( T_n \) is readily computed as,

\[ T_n = \left( |E^2|_n \right) = \frac{2}{n} \text{Var}(\hat{w}_1 + \hat{w}_2) = \text{Tr}(\hat{C} \hat{\Sigma}). \]  

Note we have used \( \hat{C} = \frac{2}{n} (\hat{p}_1 \otimes \hat{p}_1 + \hat{p}_2 \otimes \hat{p}_2) \), and \( \hat{v} \) has a zero mean. Equation (9) is a precise expression to compute \( T_n \) given \( \hat{\Sigma} \), and is used to generate the theoretical predictions in Fig. 3(b), main panel. The tessellation average \( T \) can be computed by taking the weighted sum of \( T_n \), namely,

\[ T = \sum_n N_n T_n / \sum_n N_n, \]  

where \( N_n \) is the number of \( n \)-gons. On the other hand, sub-ensemble temperatures are typically quantitatively similar to the tessellation temperature, as shown in the inset in Fig. 3(b).

If we further assume that \( \hat{w}_{1,2} \) follow identical normal distributions, immediately we have

\[ \rho_E(E^2) = \frac{1}{T_n} \exp \left( -|E^2|_n / T_n \right). \]  

In other words, the exponential distribution arises actually as a \( \chi^2 \)-distribution with 2 degrees of freedom. On the other hand, if the variances \( \text{Var}(\hat{w}_{1,2}) \) are not identical but quantitatively similar, which is true for all tessellations we study (Fig. S5), Eq. (10) still holds to the leading order. (This point is straightforward to prove via Taylor expansion and not shown here for brevity.) Note that even in this situation, per (9) the formula for \( T_n \) is still accurate without approximation. This provides an essential illustration of the origin of the \( E^2 \) distribution, and Eq. (10) is a main result of the current work. It remains to be shown below that \( \hat{w}_{1,2} \) distributions are indeed approximately normal and independent.

V. ASYMPTOTIC NORMALITY CONTRIBUTES TO STATISTICAL REGULARITY

Fig. 4(a) presents \( \hat{w}_{1,2} \) in the hexagon sub-ensemble \( (n = 6) \) for all 99 tessellations, whereas the cases for \( n = 5 \) and 7 are included in the SM [27]. PDFs are all normalized for comparison with standard Gaussian, \( \mathcal{N}(0, 1) \) (dark solid lines). Although the PDFs exhibit appreciable fluctuations due to the relatively small sample size in the \( n \)-sub-ensemble, the approximate normalities are evident. Quantitative similarities of \( \hat{w}_{1,2} \) are demonstrated in Fig. S5. In addition, \( \hat{w}_{1,2} \) are indeed only weakly dependent, as \( \text{Cov}(\hat{w}_1, \hat{w}_2) / \text{Var}(\hat{w}_1) = 0.078 \pm 0.046 \) for all cases, consistent with the anticipated 2 degrees of freedom. All \( E^2 \) distributions normalized by the predicted temperature \( T_n \) are shown in Fig. 4(b).
The apparent candidate to explain the resulting normality is the central limit theorem in the generalized version for dependent and identical random variables [31], noting that \( \hat{y}_k \) derives from \( \hat{u}_k \) via a linear combination. It is peculiar to note that \( \hat{u}_k \) and \( \hat{u}_k \) themselves also demonstrate approximate normality, shown in Fig. 4(c) and 4(d). The normality in \( \hat{u}_k \) is again explained by the central limit theorem. We can write \( \hat{u} \), the concatenated vector for \( u_j \)'s as \( \hat{u} = \mathbf{R}(\mathbf{y} - \langle \mathbf{y} \rangle) \), where \( \mathbf{R} = \mathbf{I} - \frac{1}{\mathbf{n}} \mathbf{e} \mathbf{e}^T \). In the absence of apparent anisotropy, components of \( \mathbf{y} - \langle \mathbf{y} \rangle \) are approximately identical, satisfying precondition of the theorem. Hence \( \hat{u}_k \) asymptotes to normality. On the other hand, we have \( \mathbf{v} = \mathbf{u}/r_0 \) and \( r_0 = (\mathbf{y} \cdot \mathbf{e})/n \). The normality in \( \mathbf{v} \) is difficult to theoretically prove. However, it is reasonable to speculate the loss of the apparent scale would create similarity to preserve or even enhance normality [see also Fig. 4(f) and 4(g)]. In addition, it is extensively confirmed by the data as shown in Fig. 4(d).

The asymptotic normality can be better illustrated via a simple Monte Carlo simulation following the schematic in Fig. 1(i), where we temporarily restrict to an isolated hexagon, and initial (centroid-uncorrected) displacements \( \hat{u}_{k,0} (k = 1, 2, \ldots, 2n) \) are prescribed according to independent, identical distributions as shown in Fig. 4(e) [27]. Two representative cases are examined, the first with a steeper than Gaussian initial descent (red), and the second non-monotonic (blue). In Figs. 4(f) and 4(g), both \( \hat{u}_k \) and \( \hat{u}_k \) already demonstrate trending toward normality, although some differences from \( N(0, 1) \) are still visible. Note that \( \hat{u}_k \) is obtained from \( \hat{u}_{k,0} \) after centroid correction, and only an arbitrary index \( k \) is shown as these distributions are expected to be identical. Subsequently, in Fig. 4(h) the normality of \( \hat{u}_{1,2} \) are well established. \( |E|^2 \) distribution quantitatively follows our theoretical prediction and is not shown. Although only two exemplary tests are presented, repeated simulations reveal the same asymptotic trend to normality and the quantitative relationships (9) and (10) always hold.

In a summary, the above exercises demonstrate that asymptotic normality is prevalent in planar tessellations, as key variables derive from linear combinations of statistically similar components. As a result \( E^2 \) distributions become highly regular due to combined normality and its low-dimensionality.

VI. PHYSICAL IMPLICATIONS

Atia et al. contemplates that the Gamma distributions in aspect ratio share a similar physical origin to granular packing systems, in which the conservation of area gives rise to the \( k \)-Gamma distribution where \( k \) is the number of “local elementary cells” [17]. However, in such a theory \( k \) is typically a large number when compared with [16] or here. This Letter indicates that it is rather an approximation of solution (6) derived above, and \( k \) is an empirical fitting parameter which is positively related to \( \beta \) [Fig. S7(a)]. This trend is fully corroborated with data from our own work and Atia et al. [16] (Fig. 3 therein), as well as predictions from a self-propelled Voronoi model in the supplemental information of the latter.
We propose that the deformation tensor $E$ is also a more fundamental quantity with evident physical meaning: it represents deformation, and hence is typically associated with energy in one form or another. Some usual possibilities can be contemplated. If the energy is bulk-elastic in nature, then any physically reasonable elastic model of a polygon must follow the form $\Delta \Psi = \mu |E|^2$ [32], where $\mu$ is the first Lamé constant [27]. On the other hand, if energy is associated with edge lengths or perimeters, such as in the case of models for 2D confluent tissues [7, 10, 21], the linear dependence on $|E|^2$ is still a formally valid approximation to leading-order [27]. Last but not least, in the Quantizer problem [18–20] the cell-wise energy functional is the moment of inertia, which is $\text{Tr}M = 2m_0(1 + |E|^2)$ in both two and three dimensions [27], and $2m_0$ is the moment of inertia of the regular reference polygon, $\mathcal{P}_R$. Thus its distribution can be computed via knowing both the area and $E^2$ distributions. These examples of constitutive relations cover a reasonably wide range of physical systems.

Above we have taken the liberty in naming a pseudo-temperature, $T$ (or $T_n$ for the subensembles). Indeed, such definition is both tempting and appropriate in the presence of a Boltzmann-like distribution. The tests by Dean and Lefèvre [25] and McNamara et al. [23] become trivial: the ratio of two overlapping exponential distributions will necessarily give another exponential distribution. We therefore name this pseudo-temperature the “$E^2$ temperature”. This temperature quantifies the overall deformation, and possibly also system energy. Not surprisingly, this temperature is very well correlated with a similar pseudo-temperature defined in Atia et al. [16], due to the direct relationship between $|E|^2$ and aspect ratio (Fig. S7b). On the other hand, the relationship between the $E^2$ temperature and “compactivity” in a granular assembly such as in [22, 23] awaits further exploration. In our on-going endeavor we attempt to incorporate this temperature into an ensemble framework.

We have thus demonstrated three main points in this work: i) An exponential distribution in $E^2$ leads to a $\chi$-Gamma distribution in the aspect ratio. In fact, $\chi$-Gamma distributions are mere approximations to a more basic solution that depends on the $E^2$ temperature. ii) $E^2$ arises as a $\chi^2$-distribution which results from both asymptotic normality and its small dimensionality, which is analogous to the small dimensionality of the volume function in granular assembly [33]. iii) $E^2$ and aspect ratio distributions as well as normality in displacements are true universal features as we have shown via both a large collection of data and theoretical derivations illustrating their origins. The strong regularity in $E^2$ and vertex displacements are “hidden patterns” revealed by this work. The mean-field strain tensor, with its clear physical and geometric meaning, is an ideal quantity connecting the conservation principles, the energy landscape, and the geometric distributions. Analysis may also extend to polytopes in three and higher dimensions.

**Supplementary Material**

Refer to Web version on PubMed Central for supplementary material.

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Fig. 1. (a)-(h) Examples of randomized polygonal networks. (a) Land cracks due to desiccation [2]; (b) Ice wedges from northern Canada [3]; (c) Veins on a *Ficus lyrata* leaf [5]; (d) Desiccation pattern of an ancient lake on Mars (HiRISE image: PSP_002140_2025 [26], credit to NASA/JPL/University of Arizona); (e) A snapshot of Salar de Uyuni in Bolivia, world’s largest salt flat [4]; (f) Supergranulation on solar surface [1]; (g) Plated MDCK cells. (h) A processed image of a developing Drosophila wing disc with the aspect ratio of cells color-mapped. (i) A regular hexagon $P_R$ (blue, with vertices $x_j$), a deformed hexagon $P$ (red, $y_j$), and a uniform deformation approximation $P'$ (black dashed outline, $y'_j$).
FIG. 2.  
Universality in strain and aspect ratio distributions. Left column shows the PDFs of $|E|^2$, 
fitted with an exponential form $\exp(-\beta|E|^2)$ to extract $\beta$. $\beta$ is then used in (6) to generate 
the theoretical curves in the middle column (dash), in comparison to the aspect ratio (symbols). 
The coefficients of determination, $R^2$ are shown. Right column: both data and theoretical 
curves from the center column are normalized and fitted with a $k$-Gamma function (7) (thick 
dashed). $k$ is extracted and shown in legends. Data are from [28],[29], and this work.
FIG. 3.  
(a) Overall correlation between $a_\rho$ and $T(\beta^{-1})$ for all 99 data sets (symbols); the theoretical prediction (dashed) is generated per (6). (b) A comparison between data and predicted temperature, $T_n$ and $T$ ("theory"). The inset shows that sub-ensemble temperature $T_n$’s are quantitatively similar to tessellation temperature, $T$. 

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FIG. 4.
Normality of key variables; * denotes normalization by its own standard deviation to compare with \( \mathcal{N}(0, 1) \) (dark lines). (a) \( \hat{w}_{1,2} \) for all tessellations (198 profiles). (b) Normalized \( |E|^2 / \text{Tr}(\hat{\Sigma}) \) follows a simple exponential distribution (99 profiles). (c) and (d) demonstrate the normality of \( \hat{u}_k \) and \( \hat{v}_k \) for \( n = 6 \), 1188 profiles each. (e) Initial displacement for two exemplary cases. (f) and (g) \( \hat{u}_k \) and \( \hat{v}_k \) asymptope toward normality. (g) Normalities of \( \hat{w}_{1,2} \) are well-established.
### TABLE I.

Summary of data for a total of $M_{\text{tot}} = 99$ tessellations.

| Type (abbreviation) | $M$ | $N$ | $R^2, |E|^2$ | $R^2, a_r - 1$ |
|---------------------|-----|-----|-----------|----------------|
| Salt Flat of Uyuni (Salt Flat) | 7 | 193 – 849 | $0.994 \pm 0.0058$ | $0.939 \pm 0.0255$ |
| Landforms on Mars (Mars) | 9 | 219 – 5826 | $0.986 \pm 0.0125$ | $0.935 \pm 0.0461$ |
| Veins on Leaves (Leaves) | 6 | 338 – 6050 | $0.994 \pm 0.0047$ | $0.936 \pm 0.0328$ |
| Landforms in the Arctic (Arctic) | 11 | 104 – 1061 | $0.982 \pm 0.0169$ | $0.902 \pm 0.0728$ |
| Supergranulation on Solar Surface (Solar) | 9 | 192 – 1645 | $0.991 \pm 0.0075$ | $0.932 \pm 0.0463$ |
| Cracked Dry Earth (Cracks) | 11 | 298 – 1596 | $0.992 \pm 0.0067$ | $0.943 \pm 0.0353$ |
| Drosophila Wing Disc (Droso) | 42 | 902 – 4205 | $0.991 \pm 0.0083$ | $0.955 \pm 0.0335$ |
| Plated MDCK Cells (MDCK) | 4 | 1148 – 2283 | $0.997 \pm 0.0012$ | $0.936 \pm 0.0157$ |

Abbreviations used in Fig. 3 are given in parentheses. $M$ is the number of data sets in each type; $N$ is the number of polygons in each set (a range is provided). $R^2$ for $|E|^2$ indicates the quality of fitting (e.g., in the left column of Fig. 2); $R^2$ for $a_r - 1$ indicates the quality of agreement between theory and data (e.g., in the middle column of Fig. 2).