ON STRONGLY NIP ORDERED FIELDS AND DEFINABLE
CONVEX VALUATIONS

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29 November 2018

ABSTRACT. We investigate what henselian valuations on ordered fields
are definable in the language of ordered rings. This leads towards a
systematic study of the following two classes of structures: ordered fields
which are dense in their real closure and ordered abelian groups which
are dense in their divisible hull. Some results have connections to recent
conjectures on definability of henselian valuations in strongly NIP fields.

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1. INTRODUCTION

Let $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ be the language of rings, $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$ the
language of ordered rings and $\mathcal{L}_{og} = \{+, 0, <\}$ the language of ordered
groups. Throughout this note, we will abbreviate the $\mathcal{L}_r$-structure of a
field $(K, +, -, \cdot, 0, 1)$ simply by $K$, the $\mathcal{L}_{or}$-structure of an ordered field
$(K, +, -, \cdot, 0, 1, <)$ by $(K, <)$ and the $\mathcal{L}_{og}$-structure of an ordered group
$(G, +, 0, <)$ by $G$.

The following conjecture is due to Shelah–Hasson (see \cite{21, 3, 8}):

**Conjecture 1.1.** Let $K$ be an infinite strongly NIP field. Then $K$ is
either real closed, or $K$ is algebraically closed, or $K$ admits a non-trivial
$\mathcal{L}_r$-definable\footnote{Throughout this work definable always means definable with parameters.} henselian valuation.
By adapting the characterisation of dp-minimal fields in [15], Halevi, Hasson and Jahnke [8] obtain a conjectural classification of strongly NIP fields in the language $\mathcal{L}_r$ which is equivalent to Conjecture 1.1 (cf. [8, Conjecture 1.3]).

Note that in ordered fields, henselian valuations are always convex by the following fact.

**Fact 1.2.** (See [16, Lemma 2.1].) Let $(K, <)$ be an ordered field and $v$ a henselian valuation on $K$. Then $v$ is convex on $(K, <)$.

Since in ordered fields convex valuations are well-understood, we specialise Conjecture 1.1 to ordered fields and enhance it as follows.

**Conjecture 1.3.** Let $(K, <)$ be a strongly NIP ordered field. Then $K$ is either real closed, or $K$ admits an $\mathcal{L}_{or}$-definable non-trivial henselian valuation.

Conjecture 1.3 motivates the study of $\mathcal{L}_{or}$-definable non-trivial henselian valuations on a given ordered field. Moreover, we can reformulate Conjecture 1.3 in terms of the model theoretically well-studied class of almost real closed fields (cf. [1]).

**Conjecture 1.4.** Any strongly NIP ordered field is almost real closed.

We start our study of $\mathcal{L}_{or}$-definable henselian valuations in ordered fields in Section 3 with a special focus on ordered Hahn fields. Theorem 3.20 shows that henselian valuations are $\mathcal{L}_{or}$-definable if the value group is non-divisible but dense in its divisible hull or the residue field is not real closed but dense in its real closure. In Section 3 we will show that these two density properties are preserved under elementary equivalence. In Section 5 we will study in particular some algebraic and valuation theoretic properties of ordered abelian groups which are dense in their divisible hull. Our investigation of strongly NIP ordered fields starts in Section 6. In Section 7 we direct our focus towards dp-minimal ordered fields, which are a subclass of strongly NIP ordered fields. Finally, in Section 8 we will conclude that Conjecture 1.3 and Conjecture 1.4 are equivalent and make some further observations. By this, we establish a conjectural classification of strongly NIP ordered fields.

## 2. Preliminaries

All notions on valued fields and groups can be found in [14, 4] and all notions on strongly NIP theories in [22].

Let $K$ be a field and $v$ a valuation on $K$. We denote the **valuation ring** of $v$ in $K$ by $\mathcal{O}_v$, the **valuation ideal**, i.e. the maximal ideal of $\mathcal{O}_v$, by $\mathcal{M}_v$, the **ordered value group** by $vK$ and the **residue field** $\mathcal{O}_v / \mathcal{M}_v$ by $K_v$. For $a \in \mathcal{O}_v$ we also denote $a + \mathcal{M}_v$ by $\overline{a}$. For an ordered field $(K, <)$ a valuation is called **convex** (in $(K, <)$) if the valuation ring $\mathcal{O}_v$ is a convex
subset of \( K \). In this case, the relation \( \bar{a} < \bar{b} \iff \neg \varphi(a, b) \) defines an order relation on \( K \).

Let \( \mathcal{L}_{vf} = \mathcal{L}_r \cup \{O_v\} \) be the language of valued fields, where \( O_v \) stands for a unary predicate. Let \( (K,O_v) \) be a valued field. An atomic formula of the form \( v(t_1) \geq v(t_2) \), where \( t_1 \) and \( t_2 \) are \( \mathcal{L}_r \)-terms, stands for the \( \mathcal{L}_{vf} \)-formula \( t_1 = t_2 = 0 \lor (t_2 \neq 0 \land O_v(t_1/t_2)) \). Thus, by abuse of notation, we also denote the \( \mathcal{L}_{vf} \)-structure \( (K,O_v,v) \) by \( (K,v) \). Similarly, we also call \( (K,<,v) \) an ordered valued field.

We say that a valuation \( v \) is \( \mathcal{L} \)-definable for some language \( \mathcal{L} \in \{ \mathcal{L}_r, \mathcal{L}_{or} \} \) if its valuation ring is an \( \mathcal{L} \)-definable subset of \( K \).

Let \( K \) be a field and \( v,w \) be valuations on \( K \). We write \( v \leq w \) if and only if \( O_v \supseteq O_w \). In this case, we say that \( w \) is finer than \( v \) and that \( v \) is coarser than \( w \). Note that \( \leq \) defines an order relation on the set of convex valuations of an ordered field.

Let \( G \) be an ordered abelian group and let \( v \) be a valuation on \( G \). We denote the ordered value set of \( G \) under \( v \) by \( vG \). For any \( \gamma \in vG \) we denote the archimedean component of \( G \) corresponding to \( \gamma \) by \( B_\gamma = A_\gamma/A_\gamma^\gamma \), where \( A_\gamma = \{ x \in G \mid v(x) \geq \gamma \} \) and \( A_\gamma^\gamma = \{ x \in G \mid v(x) > \gamma \} \).

For an ordered field \( (K,<) \) we call two elements \( a,b \in K \) archimedean equivalent (in symbols \( a \sim b \)) if there is some \( n \in \mathbb{N} \) such that \( |a| < n|b| \) and \( |b| < n|a| \). Let \( G = \{ [a] \mid a \in K^\times \} \) the set of archimedean equivalence classes of \( K^\times \). Equipped with addition \( [a] + [b] = [ab] \) and the ordering \( [a] < [b] \iff a < b \land |b| < |a| \), the set \( G \) becomes an ordered abelian group. Then \( v : K^\times \to G \) defines a convex valuation on \( K \). This is called the natural valuation on \( K \). In a similar manner, we can define the natural valuation via archimedean equivalence classes on ordered groups.

Let \( (k,<) \) be an ordered field and \( G \) an ordered abelian group. We denote the ordered Hahn field with coefficients in \( k \) and exponents in \( G \) by \( k((G)) \). We denote an element \( s \in k((G)) \) by \( s = \sum_{g \in G} s_g t^g \), where \( s_g = s(g) \) and \( t^g \) is the characteristic function on \( G \) mapping \( g \) to 1 and everything else to 0. The ordering on \( k((G)) \) is given by \( s > 0 := s(\min \text{supps}) > 0 \), where \( \text{supps} = \{ g \in G \mid s(g) \neq 0 \} \) is the support of \( s \). Let \( v_{\min} \) be the valuation on \( k((G)) \) given by \( v_{\min}(s) = \min \text{supps} \) for \( s \neq 0 \). Note that \( v_{\min} \) is convex and henselian. Note further that if \( k \) is archimedean, then \( v_{\min} \) coincides with the natural valuation.

Let \( \mathcal{L} \) be a language and \( T \) an \( \mathcal{L} \)-theory. We fix a monster model \( M \) of \( T \). Let \( \varphi(\bar{x};y) \) be an \( \mathcal{L} \)-formula. We say that \( \varphi \) has the independence property (IP) if there are \( (a_i)_{i \in \omega} \) and \( (b_j)_{j \in \omega} \) in \( M \) such that \( M \models \varphi(a_i; b_j) \) if and only if \( i \in J \). We say that the theory \( T \) has IP if there is some formula \( \varphi \) which has IP. If \( T \) does not have IP, it is called NIP (not the independence property). For an \( \mathcal{L} \)-structure \( \mathcal{N} \), we also say that \( \mathcal{N} \) is NIP if its complete theory \( \text{Th}(\mathcal{N}) \) is NIP.

Let \( A \subseteq M \) be a set of parameters, \( \Delta \) a set of \( \mathcal{L} \)-formulas and \( (J,<) \) a linearly ordered set. A sequence \( S = (a_j \mid j \in J) \) in \( M \) is \( \Delta \)-indiscernible over \( A \) if for every \( k \in \mathbb{N} \), any increasing tuples \( i_1 < \ldots < i_k \) and
Let $p$ be a partial $n$-type over a set $A \subseteq M$. We define the dp-rank of $p$ over $A$ as follows: Let $\kappa$ be a cardinal. The dp-rank of $p$ over $A$ is less than $\kappa$ (in symbols, $\text{dp-rk}(p,A) < \kappa$) if for every family $(S_t | t < \kappa)$ of mutually indiscernible sequences over $A$ and any $b \in M^n$ realising $p$ in $M$, there is some $t < \kappa$ such that $S_t$ is indiscernible over $A \cup \{b_1, \ldots, b_n\}$. The theory $T$ is called **strongly NIP** if it is NIP and $\text{dp-rk}(\{x = x\}, \emptyset) < \aleph_0$, where $\{x = x\}$ is the partial 1-type over $\emptyset$ only consisting of the formula $x = x$. Again, we call an $\mathcal{L}$-structure $\mathcal{N}$ strongly NIP (respectively dp-minimal) if $\text{Th}(\mathcal{N})$ is strongly NIP (respectively dp-minimal).

### 3. Definable Convex Valuations

In this section we will investigate what convex valuations are $\mathcal{L}_{or}$-definable in ordered fields. Our main focus will lie on $\mathcal{L}_{or}$-definable henselian valuations in ordered Hahn fields.

We will repeatedly use the Ax–Kochen–Ershov principle for ordered fields (cf. [5, Corollary 4.2(iii)]).

**Fact 3.1** (Ax–Kochen–Ershov principle). Let $(K, <, v)$ and $(L, <, w)$ be two henselian valued ordered fields. Then $(Kv, <) \equiv (Lw, <)$ and $vK \equiv wL$ if and only if $(K, <, v) \equiv (L, <, w)$.

There is a vast collection of results giving conditions on $\mathcal{L}_{r}$-definability of henselian valuations in pure fields, many of which are from recent years (see e.g. [1, 9, 10, 13, 18]). A survey on $\mathcal{L}_{r}$-definability of henselian valuations is given in [6]. We will start by giving a brief account of definability results of specific henselian valuations which are applicable to the ordered field case.

An ordered abelian group $G$ is called **regular** if every quotient by a non-zero convex subgroup is divisible (cf. [6, p. 137]). Equivalently, $G$ is regular if and only if for any prime $p$, every infinite convex subset of $G$ contains a $p$-divisible element (cf. [9, p. 14]). In particular, any $\mathbb{Z}$-group (i.e. any ordered group which is elementarily equivalent to $\mathbb{Z}$, cf. [4, p. 159]) and any archimedean group are regular.

**Fact 3.2.** (See [9, Theorem 4].) Let $K$ be a field and $v$ a henselian valuation on $K$. Suppose that $vK$ is regular and non-divisible. Then $v$ is parameter-free $\mathcal{L}_r$-definable in $K$.

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2There are also several results in the literature which give conditions on $\mathcal{L}_r$-definability of some not specified henselian valuation in a henselian valued field. Since we are interested in specific henselian valuations, this question is not addressed here.
A field $K$ is called **hilbertian** if there exists an elementary extension $K \preceq L$ and an element $a \in L \setminus K$ such that $K(a)$ is relatively algebraically closed in $L$ (cf. [19, p. 231 f.]). In particular, any number field and any infinite finitely generated field are hilbertian.

**Fact 3.3.** (See [10, Corollary 3.3].) Let $K$ be a field and $v$ a henselian valuation on $K$. Suppose that $Kv$ is hilbertian. Then $v$ is parameter-free $L_r$-definable in $K$.

Next we will introduce the notion of almost real closed fields (cf. [1]) adapted to our context of ordered fields.

**Definition 3.4.** Let $(K, \prec)$ be an ordered field, $G$ an ordered abelian group and $v$ a henselian valuation on $K$. We call $K$ an **almost real closed field** (with respect to $v$ and $G$) if $Kv$ is real closed and $vK = G$.

**Remark 3.5.** In [1], almost real closed fields are defined as pure fields which admit a henselian valuation with real closed residue field. However, any such field admits an ordering, which is due to the Baer–Krull Representation Theorem (cf. [4, p. 37 f.]). We consider almost real closed fields as ordered fields with a fixed order.

**Fact 3.6.** (See [1, Proposition 2.9].) Let $(K, \prec)$ be an almost real closed field. Then any convex valuation on $(K, \prec)$ is henselian.

[1, Proposition 2.8] implies that the class of almost real closed fields in the language $L_r$ is closed under elementary equivalence. We can easily deduce that this also holds in the language $L_{or}$.

**Proposition 3.7.** Let $(K, \prec)$ be an almost real closed field and let $(L, \prec) \equiv (K, \prec)$. Then $(L, \prec)$ is an almost real closed field.

**Proof.** Since $L \equiv K$, we obtain by [1, Proposition 2.8] that $L$ admits a henselian valuation $v$ such that $Lv$ is real closed. Hence, $(L, \prec)$ is almost real closed. □

**Corollary 3.8.** Let $(K, \prec)$ be an ordered field. Then $(K, \prec)$ is almost real closed if and only if $(K, \prec) \equiv (\mathbb{R}((G)), \prec)$ for some ordered abelian group $G$.

**Proof.** The forward direction follows from the Ax–Kochen–Ershov principle. The backward direction is a consequence of Proposition 3.7. □

Let $(K, \prec)$ be an ordered field. We denote by $V(K)$ the set of all henselian valuations on $K$ with real closed residue field and by $v_1$ the maximum of $V(K)$, i.e. the finest valuation in $V(K)$. This exists by [1, Proposition 2.1].

Let $p$ be a prime number. A valuation $v$ on $K$ is called **$p$-Kummer henselian** if Hensel’s Lemma holds for polynomials of the form $x^p - a$ for $a \in O_v$. A field $L$ is called **$p$-euclidean** if $L = \pm L^p$. Let $V_p(K)$ be the set of all $p$-Kummer henselian valuations of $K$ with $p$-euclidean residue field. Denote by $v_p$ the minimum of $V_p(K)$ (cf. [1, p. 1126]).
Fact 3.9. (See [1, Theorem 4.4].) Let \((K, <)\) be an almost real closed field and \(v\) a henselian valuation on \(K\). Then \(v\) is \(L^r\)-definable in \(K\) if and only if \(vK\) is \(L \log\)-definable in \(v_1 K\) and \(v \leq v_p\) for some prime \(p\).

We will now turn to \(L^\text{or}\)-definability of convex valuations. For an ordered Hahn field \(k(G)\), let \(v\) be the valuation given by \(v_\text{min}(s) = \min \text{supp}(s)\) for any \(s \in k(G)\).

Proposition 3.10. Let \((K, <)\) be an almost real closed field with respect to a henselian valuation \(v\). Suppose that \(v\) is \(L^\text{or}\)-definable in \(K\). Then \(v\) is the only \(L^\text{or}\)-definable convex valuation in \(K\) with real closed residue field.

Proof. By the Ax–Kochen–Ershov principle, we have
\[
(K, <, v) \equiv (R((vK)), <, v_\text{min}).
\]
Since \(v\) is \(L^\text{or}\)-definable in \(K\), there exists an \(L^\text{or}\)-formula \(\varphi(x, y)\) such that
\[
K \models \exists y \forall x (\varphi(x, y) \leftrightarrow v(x) \geq 0).
\]
By elementary equivalence, there exists \(b \in R((vK))\) such that
\[
R((vK)) \models \forall x (\varphi(x, b) \leftrightarrow v_\text{min}(x) \geq 0).
\]
Hence, \(v_\text{min}\) is \(L^\text{or}\)-definable in \(R((vK))\). Note that \(v_\text{min}\) is the finest convex valuation in \(R((vK))\). Hence, by elementary equivalence, \(v\) is the finest \(L^\text{or}\)-definable convex valuation on \(K\).

Let \(v'\) be some \(L^\text{or}\)-definable convex and thus, by Fact 3.6, henselian valuation \(v'\) on \(K\) such that \(Kv'\) is real closed. Arguing as above, \(v'\) is the finest \(L^\text{or}\)-definable convex valuation on \(K\). This gives us \(v' = v\), as required.

Remark 3.11. For an almost real closed field \((K, <)\), let \(v_0\) be the minimum of \(V(K)\), i.e. the coarsest henselian valuation with real closed residue field. By the remarks in [1, p. 1147 f.], \(v_0\) is the only possible \(L^r\)-definable henselian valuation in \(K\). Moreover, \(v_0\) is \(L^r\)-definable if and only if there is a prime \(p\) such that \(v_0 K\) has no non-trivial convex \(p\)-divisible subgroups.

If the ordering on an almost real closed field \((K, <)\) is \(L^\text{or}\)-definable for some \(v \in V(K)\), we obtain a complete characterisation of \(L^\text{or}\)-definable convex valuations in \(K\).

Lemma 3.12. Let \((K, <)\) be an ordered field and let \(v\) be a henselian valuation on \(K\) such that \(Kv\) is root closed for positive elements and \(vK\) is \(2\)-divisible. Then the ordering \(<\) is parameter-free \(L^\text{or}\)-definable in \(K\).

In particular, if \(v\) is \(L^r\)-definable in \(K\), then any \(L^\text{or}\)-definable subset of \(K\) is already \(L^r\)-definable.

Proof. Let \(k = Kv\) and \(G = vK\). Consider the \(L^\text{or}\)-formula \(\varphi(x)\) given by
\[
x = 0 \lor \exists y (v(x - y^2) > v(x)).
\]
We will show that for any \( a \in k((G)) \), the formula \( \varphi(a) \) holds if and only if \( a \geq 0 \). Let \( a = a_g t^g + s \in k((G))^\times \), where \( a_g \in k^\times \), \( s \in k((G^c g)) \) and \( g = v_{\text{min}}(a) \).

Suppose that \( \varphi(a) \) holds. Then there exists \( y \in K^\times \) such that \( v_{\text{min}}(x - y^2) > g \). Hence, \( a_g = y_g^2 > 0 \), where \( y_g \) is the coefficient of the monomial \( t^g \) in \( y \). Thus, \( a > 0 \).

Now suppose that \( a > 0 \). Let \( y = \sqrt{a_g} g^{1/2} \). Then \( v_{\text{min}}(a - y^2) = v_{\text{min}}(s) > g = v_{\text{min}}(a) \).

By the Ax–Kochen–Ershov principle, \( (K, <, v) \equiv (K v((vK)), <, v_{\text{min}}) \). Hence, we obtain \( K \models \forall x (x \geq 0 \leftrightarrow \varphi(x)) \).

**Proposition 3.13.** Let \( (K, <) \) be an almost real closed field with respect to an \( L_r \)-definable valuation \( v \) and a 2-divisible ordered abelian group \( G \). Let \( w \) be a valuation on \( K \). If \( w \) is \( L_{\text{or}} \)-definable then it is \( L_r \)-definable.

**Proof.** Since \( vK \) is real closed, it is root closed for positive elements. By Lemma 3.12 any \( L_{\text{or}} \)-definable valuation on \( K \) is already \( L_r \)-definable. □

**Remark 3.14.** Recall that by Remark 3.11 an almost real closed field admits at most one \( L_r \)-definable henselian valuation with real closed residue field, namely the coarsest such valuation \( v_0 \). We obtain the following characterisation of \( L_{\text{or}} \)-definable convex valuations in certain almost real closed fields: Let \( (K, <) \) be an almost real closed field. Suppose that the value group \( v_0 K \) is 2-divisible and for some prime \( p \), it has no non-trivial \( p \)-divisible subgroup. Let \( v \) be a convex valuation on \( K \). By Fact 3.13 \( v \) is henselian. Thus, by Proposition 3.13 and Fact 3.9 \( v \) is \( L_{\text{or}} \)-definable in \( K \) if and only if \( vK \) is \( L_{\text{log}} \)-definable in \( v_1 K \) and \( v \leq v_p \) for some prime \( p \).

We are now going to apply the construction method of convex \( L_{\text{or}} \)-definable valuations from [14, Proposition 6.5] to ordered Hahn fields.

**Fact 3.15.** Let \( (K, <) \) be an ordered field. Then at least one of the following holds.

1. \( K \) is dense in its real closure.
2. \( K \) admits an \( L_{\text{or}} \)-definable non-trivial convex valuation.

**Remark 3.16.** In [14] it is not investigated whether the two cases in Fact 3.15 are exclusive. More precisely, we can ask whether there is an ordered field which is dense in its real closure and still admits an \( L_{\text{or}} \)-definable non-trivial convex valuation.

We will summarise the construction procedure of an \( L_{\text{or}} \)-definable non-trivial convex valuation ring of an ordered field which is not dense in its real closure given in [14, p. 163 f.]. For an ordered field \( (K, <) \), we denote its real closure by \( K^{\text{tc}} \) and its topological closure in \( K^{\text{tc}} \) under the order topology by \( \text{cl}(K) \). Likewise, for an ordered abelian group \( G \), we denote its divisible

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3. [14, Proposition 6.5] only states that \( K \) admits a definable non-trivial valuation, but the proof indeed gives a construction method for a definable non-trivial convex valuation.
hull by $G^{\text{div}}$ and its topological closure in $G^{\text{div}}$ under the order topology by $\text{cl}(G)$.

**Construction 3.17.** Let $(K, <)$ be an ordered field. Suppose that $K$ is not dense in $R = K^{\text{rc}}$. Let $s \in R \setminus \text{cl}(K)$. Set $D_s := \{z \in K \mid z < s\}$ and $A_s := \{x \in K^{\geq 0} \mid x + D_s \subseteq D_s\}$. Set $O_s := \{x \in K \mid |x|A_s \subseteq A_s\}$. Then $O_s$ is a definable non-trivial convex valuation ring of $K$.

**Proposition 3.18.** Let $k$ be an ordered field and $G \neq 0$ an ordered abelian group. Then $k((G))$ is closed in its real closure.

**Proof.** Let $K = k((G))$ and $R = K^{\text{rc}}$. We need to show that $R \setminus K$ is open in $R$. If $K$ is real closed, then $R \setminus K = \emptyset$. Hence, assume that $K$ is not real closed.

Let $s \in R \setminus K$. Then $s$ is of the form $s = s_1 + at^{g_0} + s_2$ for some $s_1 \in k((G^{<g_0}))$, $s_2 \in k^{\text{rc}}(\left((G^{\text{div}})^{>g_0}\right))$ and $at^{g_0} \notin K$. In other words, $at^{g_0}$ is the monomial of $s$ of least exponent which is not contained in $K$. Let $g_1 \in G^{>g_0}$. Then the open interval

$$I = (s - t^{g_1}, s + t^{g_1}) \subseteq R$$

contains $s$. However, any element in $I$ contains a monomial of the form $at^{g_0}$ and is thus not contained in $K$. Hence, $s$ is contained in an open neighbourhood in $R \setminus K$, as required.

Let $G, H$ be ordered abelian groups such that $G \subseteq H$. We say that $G$ has a **left-sided limit point** $g_0$ in $H$ if for any $g_1 \in H$ with $g_1 > 0$ the intersection of $(g_0 - g_1, g_0)$ with $G$ is non-empty. Similarly, $g_0$ is a **right-sided limit point** if for any $g_1 \in H$ with $g_1 > 0$ the intersection of $(g_0, g_0 + g_1)$ with $G$ is non-empty. A **one-sided limit point** is a point which is either a left-sided or a right-sided limit point. A **limit point** is a point which is both a left- and a right-sided limit point. We use a similar notion for (left-, right- and one-sided) limit points of an extension of ordered fields.

**Lemma 3.19.** Let $k$ be an ordered field and $G \neq 0$ an ordered abelian group. Suppose that at least one of the following holds.

1. $G$ is discretely ordered.
2. $G$ is non-divisible and has a one-sided limit point in $G^{\text{div}} \setminus G$.
3. $k$ is not real closed and has a one-sided limit point in $k^{\text{rc}} \setminus k$.

Then $v_{\text{min}}$ is $\mathcal{L}_{\text{or}}$-definable in $k((G))$. Moreover, in the cases (1) and (2), $v_{\text{min}}$ is definable by an $\mathcal{L}_{\text{or}}$-formula with one parameter.

**Proof.** Let $K = k((G))$ and $v = v_{\text{min}}$ on $K$. Note that $K$ is not real closed, as in each case either $k$ is not real closed or $G$ is non-divisible. We will apply Construction 3.17 with some simplifications to define $k((G^{\geq 0}))$ in $K$. Proposition 3.18 shows that we can apply the construction procedure to any element in $s \in R \setminus K$.

First suppose that $G$ is non-divisible. Let $g_0 \in G^{\text{div}} \setminus G$ and $s = t^{g_0}$. Consider the $\mathcal{L}_{\text{or}}$-definable set $D'_s = \{x \in K^{\geq 0} \mid x < t^{g_0}\}$. Since $g_0 \in G^{\text{div}}$,
there is some \( h \in G \) and \( N \in \mathbb{N} \) such that \( g_0 = \frac{h}{N} \). Thus, the set \( D'_s \) is defined by the \( \mathcal{L}_{or} \)-formula with one parameter
\[
x \geq 0 \wedge x^N < t^h.
\]
Note that for any \( x \in K^{\geq 0} \), we have \( x \in D'_s \) if and only if \( v(x) > g_0 \). Thus, \( D'_s = k((G^{g_0}))^{\geq 0} \). Let \( \mathcal{O}_s = \{ x \in K \mid |x|D'_s \subseteq D'_s \} \). Note that this set is \( \mathcal{L}_{or} \)-definable with one parameter. By definition, \( \mathcal{O}_s \) contains exactly those elements in \( K \) such that for any \( y \in K^{\geq 0} \) with \( v(y) > g_0 \) we have
\[
(3.1) \quad v(x) + v(y) = v(xy) > g_0.
\]
In particular, for any \( x \in K \) with \( v(x) \geq 0 \), condition (3.1) holds. Thus, \( k((G^{g_0})) \subseteq \mathcal{O}_s \). To show the other set inclusion, we will make a case distinction, also specifying the element \( g_0 \) for the densely ordered case.

Suppose that \( G \) is discretely ordered. Let \( g_1 \in G \) be the least element greater than \( g_0 \) and let \( g_2 \in G \) be the least element greater than \( g_0 - g_1 \). Then \( g_2 + g_1 \) is the least element greater than \( g_0 \). By choice of \( g_1 \), this gives us \( g_2 + g_1 = g_1 \) and thus \( g_2 = 0 \). Let \( x \in \mathcal{O}_s \). Since \( v^g \in D'_s \), we have \( v(xt^{g_1}) = v(x) + g_1 > g_0 \). Hence, \( v(x) > g_0 - g_1 \). By choice of \( g_2 \) as the least element greater than \( g_0 - g_1 \), we obtain \( v(x) \geq g_2 = 0 \). This implies \( \mathcal{O}_s \subseteq k((G^{g_0})) \), as required.

Suppose that \( G \) has a one-sided limit point in \( G^{div} \setminus G \). In this case, we choose \( g_0 \in G^{div} \setminus G \) such that \( g_0 \) is a one-sided limit point of \( G \). We may assume that \( g_0 \) is a right-sided limit point, as otherwise we can replace it by \( -g_0 \). Let \( x \in K \setminus k((G^{g_0})) \), i.e. \( v(x) < 0 \). Since \( g_0 \) is a right-sided limit point of \( G \) in \( G^{div} \), the interval \( (g_0, g_0 - v(x)) \subseteq G^{div} \) contains some element \( g_1 \in G \). Thus, \( g_1 > g_0 \) but \( v(x) + v(t^{g_1}) = v(x) + g_1 < g_0 \). This shows that \( x \) does not fulfill condition (3.1), whence \( x \notin \mathcal{O}_s \). We thus obtain \( \mathcal{O}_s \subseteq k((G^{g_0})) \).

Now suppose that \( k \) is not real closed and has a one-sided limit point in \( k^{rc} \setminus k \). Let \( a \in k^{rc} \setminus k \) be a one-sided limit point of \( k \) in \( k^{rc} \setminus k \). We may assume that \( a \) is a left-sided limit point, as otherwise we can replace it by \( -a \). Then \( D'_a = \{ x \in K \mid a - 1 < x < a \} \) consists exactly of the elements of the form \( b + r \), where \( b \in k \) such that \( a - 1 < b < a \) and \( r \in k((G^{g_0})) \). In other words, \( D'_a = I + k((G^{g_0})) \), where \( I \) is the convex set \( (a-1,a) \) in \( k \). Note that \( I \) is non-empty, as \( a \) is a left-sided limit point of \( k \). Let \( A'_a \) be the \( \mathcal{L}_{or} \)-definable set \( \{ x \in K^{\geq 0} \mid x + D'_a \subseteq D'_a \} \). Since \( k((G^{g_0})) \) is closed under addition, we have \( k((G^{g_0})) + D'_a \subseteq D'_a \). Thus, \( k((G^{g_0}))^{\geq 0} \subseteq A'_a \). For the other inclusion, let \( x \in K^{\geq 0} \setminus k((G^{g_0})) \), i.e. \( v(x) \leq 0 \) and \( x \geq 0 \). If \( v(x) > 0 \), then \( x + b \notin D'_a \) for any \( b \in I \). Thus, \( x \notin A'_a \). Suppose that \( v(x) = 0 \). Then \( x \) is of the form \( c + r \) with \( c \in k^{g_0} \) and \( r \in k((G^{g_0})) \). If \( c \geq 1 \), then \( x + b \notin D'_a \) for any \( b \in I \), whence \( x \notin A'_a \). If \( c < 1 \), let \( b \in k \cap (a-c,a) \), which exists, as \( a \) is a left-sided limit point of \( k \). Then \( x + b = (c + b) + r > a + r \). Thus, \( x + b \notin D'_a \) and \( x \notin A'_a \). Hence, we have shown that \( A'_a \subseteq k((G^{g_0}))^{\geq 0} \).
Now \((-A'_a \cup A'_a) = k((G^{>0}))\) is the maximal ideal of the valuation ring \(k((G^{>0}))\). Thus, the valuation ring \(k((G^{>0})) = \{ x \in K \mid x(-A'_a \cup A'_a) \subseteq (-A'_a \cup A'_a) \}\) is \(\mathcal{L}_{\text{or}}\)-definable.

**Theorem 3.20.** Let \((K, <)\) be a non-archimedean ordered field which is not real closed and let \(v\) be a henselian valuation on \(K\). Suppose that at least one of the following holds.

1. \(vK\) is discretely ordered.
2. \(vK\) is non-divisible but dense in \(vK^{\text{div}}\).
3. \(Kv\) is not real closed but dense in \(Kv^{\text{rc}}\).

Then \(v\) is \(\mathcal{L}_{\text{or}}\)-definable in \(K\).

**Proof.** By the Ax–Kochen–Ershov principle, \((K, <, v) \equiv (Kv((vK)), <, v_{\text{min}})\). Note that if \(vK\) is dense in \(vK^{\text{div}}\), then any point in \(vK^{\text{div}}\) is a limit point of \(vK\). The similar statement holds for \(Kv\) in \(Kv^{\text{rc}}\).

Thus, by Lemma 3.19 in each case there is an \(\mathcal{L}_{\text{or}}\)-formula \(\varphi(x, y)\) such that

\[
Kv((vK)) = \exists y \forall x \ (\varphi(x, y) \leftrightarrow v_{\text{min}}(x) \geq 0).
\]

By elementary equivalence, there is a parameter tuple \(\underline{b} \in K\) such that \(\varphi(x, \underline{b})\) defines \(v\) in \(K\). 

**Remark 3.21.**

1. Let \((k, <)\) be an archimedean ordered field. Then \(\mathbb{Q} \subseteq k \subseteq k^{\text{rc}} \subseteq \mathbb{R}\). Since \(\mathbb{Q}\) is dense in \(\mathbb{R}\), also \(k\) is dense in \(k^{\text{rc}}\). Thus, Theorem 3.20 holds for all ordered fields \((K, <)\) which are not real closed such that \(vK\) is archimedean.

2. In Theorem 3.20 we generalised the case that \(vK\) is non-divisible but regular in the setting of \(\mathcal{L}_{\text{or}}\)-definability. Indeed, any such group is already either discretely ordered or dense in its divisible hull (see Proposition 5.1).

4. Density in Definable Closure

In Section 3 we considered ordered abelian groups which are dense in their divisible hull and ordered fields which are dense in their real closure. Note that the theory of divisible ordered abelian groups and the theory of real closed fields share several model theoretic properties, such as completeness, \(\omega\)-minimality and quantifier elimination. This motivates a model theoretic study of these classes of structures.

For a structure \(\mathcal{M}\) and a subset \(A \subseteq M\), denote the definable closure of \(A\) in \(\mathcal{M}\) by \(\text{dcl}(A; \mathcal{M})\).

**Lemma 4.1.** Let \(\mathcal{L}\) be a language expanding \(\mathcal{L}_{\text{og}}\) and let \(\mathcal{M} = (M, +, 0, <, \ldots)\) and \(\mathcal{N} = (N, +, 0, <, \ldots)\) be ordered \(\mathcal{L}\)-structures such that \((M, +, 0, <)\) is a non-trivial ordered abelian group and \(\mathcal{M} \equiv \mathcal{N}\). Suppose that there exists a complete \(\omega\)-minimal \(\mathcal{L}\)-theory \(T \supseteq T_{\text{dlog}}\) admitting quantifier elimination such that there are \(\mathcal{M}', \mathcal{N}' \models T\) with \(M \subseteq M', N \subseteq N'\), \(\text{dcl}(M; \mathcal{M'}) = \text{dcl}(N; \mathcal{N'}) = A\).
By elementary equivalence, we obtain $\alpha$. Now there are unique $x, y \in M$ such that $\alpha < \beta$ and $\alpha$ and $\beta$ produce the same cut on $M$, i.e.

$$\{x \in M \mid x < \alpha\} = \{x \in M \mid x < \beta\}.$$ 

Since $M' = \text{dcl}(M; \mathcal{M}')$, there are $\mathcal{L}$-formulas $\varphi(x, y)$ and $\psi(x, y)$, each defining a 0-definable function from $(M')^m$ to $M'$ for some $m \in \mathbb{N}$, such that for some $a \in M$ we have

$$\mathcal{M}' \models \varphi(a, \alpha) \land \psi(a, \beta).$$

Let $f$ and $g$ be the functions corresponding to $\varphi$ and $\psi$ respectively. We may assume that for any $x \in (M')^m \setminus \{(0, \ldots, 0)\}$, we have $f(x) \neq g(x)$, as otherwise we can replace $g$ by the 0-definable function $g'$ given by

$$g'(x) = \begin{cases} 
  g(x), & \text{if } g(x) \neq f(x), \\
  -g(x), & \text{if } g(x) = f(x) \neq 0, \\
  \max\{|x_1|, \ldots, |x_m|\}, & \text{if } g(x) = f(x) = 0.
\end{cases}$$

Now let $\varphi'(x, z)$ be given by $z < f(x)$ and let $\psi'(x, z)$ be given by $z < g(x)$. By quantifier elimination in $T$, we may take $\varphi''$ and $\psi''$ quantifier-free such that they are equivalent to $\varphi'$ and $\psi'$ respectively. Note that for any $b \in M'$ we have $\mathcal{M}' \models \varphi''(a, b)$ if and only if $b < \alpha$, and $\mathcal{M}' \models \psi''(a, b)$ if and only if $b < \beta$. Hence,

$$\mathcal{M} \models \forall z (\varphi''(a, z) \leftrightarrow \psi''(a, z)).$$

We need to make a case distinction to obtain some element $a' \in N$ such that $f(a') \neq g(a')$.

**Case (1):** $a = 0$. Then

$$\mathcal{M} \models \forall z (\varphi''(0, z) \leftrightarrow \psi''(0, z)).$$

By elementary equivalence, we obtain

$$\mathcal{N} \models \forall z (\varphi''(0, z) \leftrightarrow \psi''(0, z)).$$

Now there are unique $\alpha', \beta' \in N'$ such that

$$\mathcal{N}' \models f(0) = \alpha' \land g(0) = \beta'.$$

Since $\mathcal{M}' \models f(0) \neq g(0)$, we obtain by completeness of $T$ that $\mathcal{N}' \models f(0) \neq g(0)$ and thus that $\alpha' \neq \beta'$.

**Case (2):** $a \neq 0$. Then

$$\mathcal{M} \models \exists x (x \neq 0 \land \forall z (\varphi''(x, z) \leftrightarrow \psi''(x, z))).$$

By elementary equivalence, we obtain

$$\mathcal{N} \models \exists x (x \neq 0 \land \forall z (\varphi''(x, z) \leftrightarrow \psi''(x, z))).$$

Let $a' \in N$ with $a' \neq 0$ such that

$$\mathcal{N} \models \forall z (\varphi''(a', z) \leftrightarrow \psi''(a', z)).$$
By assumptions on $f$ and $g$, there are unique $\alpha', \beta' \in \mathbb{N}'$ such that $\alpha' \neq \beta'$ and

$$\mathbb{N}' \models f(a') = \alpha' \land g(a') = \beta'.$$

This completes the case distinction.

For any $b' \in \mathbb{N}$ we have $b' < \alpha'$ if and only if $\mathbb{N}' \models f(a') = \alpha'$. By definition of $\phi'$, this holds if and only if $\mathbb{N}' \models \phi'(a', b')$. Again, by quantifier elimination, this is equivalent to $\mathbb{N}' \models \psi''(a', b')$. Since $\psi''$ is quantifier-free, this holds if and only if $\mathbb{N} \models \psi''(a', b')$. Similarly, we obtain that for any $b' \in \mathbb{N}$ we have $b' < \beta'$ if and only if $\mathbb{N} \models \psi''(a', b')$.

Hence, we obtain that for any $b' \in \mathbb{N}$ we have $b' < \alpha'$ if and only if $b' < \beta'$.

This shows that $\alpha'$ and $\beta'$ produce the same cut in $\mathbb{N}$, and hence, that $\mathbb{N}$ is also not dense in $\mathbb{N}'$. □

Since the theory of divisible ordered abelian groups $T_{doag}$ and the theory of real closed fields $T_{rcf}$ are complete, o-minimal and admit quantifier elimination, we obtain that for ordered abelian groups density in the divisible hull and for ordered fields density in the real closure are preserved by elementary equivalence.

**Theorem 4.2.** (1) Let $G$ and $H$ be ordered abelian group such that $G \equiv H$. Then $G$ is dense in its divisible hull if and only if $H$ is dense in its divisible hull.

(2) Let $(K, <)$ and $(L, <)$ be ordered fields such that $(K, <) \equiv (L, <)$. Then $K$ is dense in its real closure if and only if $L$ is dense in its real closure.

**Proof.** This follows immediately from Lemma 4.1, noting that for a non-trivial ordered abelian group $G$, the definable closure of $G$ in $G_{\text{div}}$ is $G_{\text{div}}$, and similarly for an ordered field $(K, <)$, the definable closure of $K$ in $(K_{\text{rc}}, <)$ coincides with $K_{\text{rc}}$. □

**Remark 4.3.** Recall that any archimedean ordered field is dense in its real closure. In Proposition 4.18 we have shown that for an ordered field $k$ and an ordered abelian group $G \neq 0$, the ordered Hahn field $k((G))$ is closed in its real closure. In particular, if $k((G))$ is not real closed, it is not dense in its real closure. Theorem 4.2 thus shows that any non-archimedean Hahn field which is not real closed does not have an archimedean model. More generally, any ordered field which is not dense in its real closure does not have an archimedean model.

**Proposition 4.4.** Let $G$ be an archimedean densely ordered abelian group. Then $G$ is dense in its divisible hull.

**Proof.** If $G = 0$, there is nothing to show. Thus, suppose that $G \neq 0$. Let $g, h \in G$ with $0 < g < h$ and $N \in \mathbb{N}_{\geq 2}$. We need to find some $c \in G$ such that

$$\frac{g}{N} < c < \frac{h}{N}.$$
Since $G$ is densely ordered, there are $c_1, \ldots, c_N \in G$ such that $0 < c_1 < \ldots < c_N < h - g$. Let $c' = \min \{ c_{i+1} - c_i \mid i = 0, \ldots, N \}$, where $c_0 = 0$ and $c_{N+1} = h - g$. Then $Nc' \leq c_N < h - g$. Since $G$ is archimedean ordered, there is a minimal $m \in \mathbb{N}$ such that $g < mc'$. Since $c' < \frac{h - g}{N}$, we also have by minimality of $m$ that
\[
g < mc' < (m + 1)c < \ldots < (m + N - 1)c' < h.
\]
Now let $\ell \in \{0, \ldots, N - 1\}$ such that $m + \ell$ is divisible by $N$, and let $c = \frac{m + \ell}{N}c'$. Then $h < Nc < g$, whence $\frac{h}{N} < c < \frac{h}{N}$. \qed

**Corollary 4.5.** (1) Let $G$ be a densely ordered abelian group such that $\text{Th}(G)$ has an archimedean model. Then $G$ is dense in $G^{\text{div}}$.

(2) Let $(K, <)$ be an ordered field such that $\text{Th}(K, <)$ has an archimedean model. Then $K$ is dense in $K^{\text{rc}}$.

**Proof.** By Proposition 4.4, any archimedean densely ordered group is dense in its divisible hull. Thus, by Theorem 4.2 (1), any densely ordered abelian group whose theory has an archimedean model is dense in its divisible hull.

By Remark 4.3 (2) and Theorem 4.2 (2) we can argue similarly to show the ordered field case. \qed

In Example 5.4 we will see a non-archimedean densely ordered group which is not dense in its divisible hull.

Moreover, in Example 5.3 we will see an ordered abelian group which is non-archimedean and dense in its divisible hull but has no archimedean model. We can ask whether there exists an analogue for ordered fields.

**Question 4.6.** Is there a non-archimedean ordered field which is dense in its real closure but has no archimedean model?

5. Density in Divisible Hull

We now want to specifically study ordered abelian groups which are dense in their divisible hull.

**Proposition 5.1.** Let $G$ be a regular densely ordered abelian group. Then $G$ is dense in $G^{\text{div}}$.

**Proof.** If $G$ is divisible, the conclusion in trivial. Suppose that $G$ is non-divisible. By regularity, for any prime $p$, any infinite convex subset of $G$ contains a $p$-divisible element. Let $a, b \in G$ and $N \in \mathbb{N}$ with $0 < \frac{a}{N} < \frac{b}{N}$. We need to find some $c \in G$ such that $\frac{a}{N} < c < \frac{b}{N}$. Let $N = p_1 \ldots p_m$ be the prime factorisation of $N$. Since $G$ is densely ordered, the interval $(a, b)$ in $G$ contains infinitely many elements. By regularity, there are $p_1$-divisible $d_1, \ldots, d_m \in G$ such that $a < d_1 < \ldots < d_m < b$. We obtain
\[
\frac{a}{p_1} < \frac{d_1}{p_1} < \ldots < \frac{d_m}{p_1} < \frac{b}{p_1}.
\]
Again, by regularity, there are $p_2$-divisible $d'_1, \ldots, d'_{m-1} \in G$ such that
\[
\frac{a}{p_1} < \frac{d_1}{p_1} < \frac{d'_1}{p_1} < \ldots < \frac{d'_{m-1}}{p_1} < \frac{b}{p_1}.
\]
Thus,
\[
\frac{a}{p_1p_2} < \frac{d'_1}{p_2} < \ldots < \frac{d'_{m-1}}{p_2} < \frac{b}{p_1p_2}.
\]
Continuing this procedure, we finally obtain some $c \in G$ with the required property. \qed

The proof of Proposition 5.1 directly verifies that any regular densely ordered abelian group is dense in its divisible hull. Note that this also follows from the fact that any regularly ordered group has an archimedean model.

**Fact 5.2.** (See [20, p. 236].) Let $G$ be a regular ordered abelian group. Then there exists an archimedean ordered abelian group $H$ such that $G \cong H$.

By Fact 5.2 any regular densely ordered abelian group $G$ has a densely ordered archimedean model $H$. By Proposition 4.4 $H$ is dense in its divisible hull, and by Theorem 4.2 this property transfers to $G$. Moreover, we can construct a non-archimedean ordered abelian group which is dense in its divisible hull and has no archimedean model.

**Example 5.3.** Let $A = \left\{ \frac{a}{n} \mid a \in \mathbb{Z}, n \in \mathbb{N}_0 \right\}$. Note that $A$ is dense in its divisible hull $\mathbb{Q}$. Set $G = \mathbb{Q} \oplus A$ ordered lexicographically. Then $G$ is dense in $G^{\text{div}} = \mathbb{Q} \oplus \mathbb{Q}$. However, $G$ is not regular, as there is no 3-divisible element between $(0,0)$ and $(0,1)$ in $G$. Hence, $G$ has no archimedean model, as any archimedean group is regular.

In order to show that $G$ is dense in $G^{\text{div}}$, one needs to verify that every point in $G^{\text{div}}$ is a limit point of $G$. In certain cases we can show that 0 is a limit point of $G$. Note that this does in general not suffice to show density, as the following example will show.

**Example 5.4.** Let $A$ be as in Example 5.3. Then the group $G = A \oplus \mathbb{Q}$, ordered lexicographically, is densely ordered, as both $A$ and $\mathbb{Q}$ are densely ordered. It has divisible hull $G^{\text{div}} = \mathbb{Q} \oplus \mathbb{Q}$. Let $(a,b) \in G^{\text{div}}$ with $(a,b) > (0,0)$. If $a = 0$, then for $(0,b/2) \in G$ we have $(0,0) < (0,b/2) < (0,b)$. If $a > 0$, then for $(0,1) \in G$ we have $(0,0) < (0,1) < (a,b)$. Hence, $(0,0)$ is a limit point of $G$. However, $G$ is not dense in $G^{\text{div}}$, as for instance there is no element in $G$ lying between $\left( \frac{1}{3}, 0 \right)$ and $\left( \frac{1}{3}, 1 \right)$.

**Proposition 5.5.** Let $G$ be an densely ordered abelian group such that for some $n \in \mathbb{N}_{\geq 2}$ we have $[G : nG] < \infty$. Then 0 is a limit point of $G$ in $G^{\text{div}}$.

**Proof.** If $G$ is divisible, the conclusion is trivial. Let $G$ be non-divisible and let $m \in \mathbb{N}$ such that $[G : nG] = m$. Let $a \in G^{\text{div}}$ with $a > 0$. We need to find $b \in G$ such that $0 < b < a$. 
Since $a$ is in the divisible hull of $G$, there are some $N \in \mathbb{N}$ and $c_0 \in G$ such that $Na = c_0$. We will construct a decreasing sequence of positive elements $c_i \in G$ such that for any $i \geq 0$ we have $c_{i+1} < \frac{c_i}{n}$. In particular, for $\ell$ with $n^\ell > N$,

$$0 < c_\ell < \frac{c_0}{n^\ell} < \frac{c_0}{N} = a.$$

Assume that $c_i$ is already constructed. Since $G$ is densely ordered, there are $d_1, \ldots, d_m \in G$ such that $0 < d_1 < \ldots < d_m < c_i$. Let $d = \min\{d_{j+1} - d_j \mid j = 0, \ldots, m\}$, where $d_0 = 0$ and $d_m + 1 = c_i$. Then $0 < d < 2d < \ldots < md < c_i$. Let $q \leq m$ be the order of $d + nG$ in $G/nG$. Then $qd \in nG$, i.e. $qd$ is $n$-divisible. Set $c_{i+1} = \frac{qd}{n}$. Then $0 < c_{i+1} = \frac{qd}{n} < \frac{c_0}{n}$, as required. \(\square\)

For a definition of an immediate extension of ordered abelian groups, we refer the reader to [17, p. 3].

**Remark 5.6.** Let $G$ be an ordered abelian group and $v$ the natural valuation on $G^{\text{div}}$. Then the extension $G \subseteq G^{\text{div}}$ is immediate if and only if all archimedean components of $G$ are divisible. This is due to the fact that in either case the divisible hull of a particular archimedean component of $G$ is equal to the corresponding archimedean component of $G^{\text{div}}$.

**Proposition 5.7.** Let $G \subseteq H$ be an extension of ordered abelian groups such that $H$ is densely ordered. Let $v$ be the natural valuation on $H$. Suppose that the extension $G \subseteq H$ is immediate. Then $0$ is a limit point of $G$ in $H$.

**Proof.** Let $a \in H$ with $a > 0$. We need to find some $b \in G$ such that $0 < b < a$.

Suppose that $v(a)$ is not maximal in $vG$. Choose $b \in G^{>0}$ such that $v(b) > v(a)$. Then $0 < b < a$.

Now suppose that $\gamma = v(a)$ is maximal in $vG$. Consider the archimedean component $B_\gamma$ of $G$. Since $G \subseteq H$ is immediate, $B_\gamma$ is an archimedean component of both $G$ and $H$. In particular, $B_\gamma$ is densely ordered. Let $b \in G$ such that $b + A^\gamma < a + A^\gamma$. Then in particular $0 < b < a$. \(\square\)

**Corollary 5.8.** Let $G$ be an ordered abelian group and let $v$ be the natural valuation on $G^{\text{div}}$. Suppose that the extension $G \subseteq G^{\text{div}}$ is immediate. Then $0$ is a limit point of $G$ in $G^{\text{div}}$.

**Proof.** Apply Proposition 5.7 to $H = G^{\text{div}}$. \(\square\)

**Proposition 5.9.** Let $G$ and $H$ be ordered abelian groups such that $H$ is densely ordered and $G \subseteq H$. Let $v$ be the natural valuation on $H$. Suppose that the extension $G \subseteq H$ is immediate and that $vG \subseteq \omega$. Then $G$ is dense in $H$.

**Proof.** Let $a, b \in H$ with $0 < a < b$. Let $v(a - b) = n$. We will construct $c' \in G$ such that we have $v(b - c') = n$ or $v(a - c') = n$.

If $n = 0$, we can simply set $c' = 0$.

Suppose that $n \geq 1$. We will construct $c_0, \ldots, c_{n-1}$ such that for $c' = c_0 + \ldots + c_{n-1}$ we have $v(b - c') = n$ or $v(a - c') = n$. Set $a_0 = a$ and
Note that \( a_0 - b_0 \in A^0 \), where \( A^0 = \{ x \in H \mid v(x) > 0 \} \). Since the extension \( G \subseteq H \) is immediate, there is some element \( c_0 \in G \) such that \( c_0 + A^0 = a_0 + A^0 = b_0 + A^0 \). Let \( a_1 = a_0 - c_0 \) and \( b_1 = b_0 - c_0 \). Note that \( v(a_0 - c_0) \geq 1 \) and \( v(b_0 - c_0) \geq 1 \). We can repeat this step \( n - 1 \) times to obtain a sequences \( (c_i) \in G \) and \( (a_i), (b_i) \in H \) for \( i = 1, \ldots, n - 1 \) such that for any \( i \) we have \( a_i = a_{i - 1} - c_{i - 1}, b_i = b_{i - 1} - c_{i - 1} \) and \( v(a_i - c_i), v(a_i - c_i) \geq i + 1 \).

Set \( c' = c_0 + \ldots + c_{n - 1} \). Then \( a - c' = a_0 - c_0 - \ldots - c_{n - 1} = a_{n - 1} - c_{n - 1} \). For this we have \( v(a - c') \geq n \). Similarly, \( v(b - c') \geq n \). If both \( v(a - c') > n \) and \( v(b - c') > n \), then \( v(a - b) = v((a - c') - (b - c')) \geq n + 1 \), a contradiction. Hence, either \( v(a - c') = n \) or \( v(b - c') = n \).

We now need to find some \( c \in G \) with \( a - c' < c < b - c' \), as then \( a < c + c' < b \), as required.

Since either \( v(a - c') = n \) or \( v(b - c') = n \), we have \( a - c' + A^n < b - c' + A^n \). Since \( H \) is densely ordered, also the archimedean component \( B_n \) of \( G \) and \( H \) is densely ordered. Let \( c \in G \) such that \( a - c' + A^n < c + A^n < b - c' + A^n \). Then \( a - c' < c < b - c' \).

**Corollary 5.10.** Let \( G \) be an ordered abelian group and let \( v \) be the natural valuation on \( G^{\text{div}} \). Suppose that the extension \( G \subseteq G^{\text{div}} \) is immediate and that \( vG \subseteq \omega \). Then \( G \) is dense in \( G^{\text{div}} \).

**Proof.** Apply Proposition 5.9 to \( H = G^{\text{div}} \).

Note that without the condition \( vG \subseteq \omega \) in Corollary 5.10 the conclusion that \( G \) is dense in \( G^{\text{div}} \) does not hold in general, as the following example will show.

**Example 5.11.** Let \( H \) be the Hahn product

\[
H = \bigoplus_{\gamma \in \omega + 1} \mathbb{Q} = \{ s : \omega + 1 \to \mathbb{Q} \mid \text{supps is well-ordered} \}.
\]

We express elements \( s \) of \( H \) by \( s = \sum_{\gamma \in \omega + 1} s_\gamma \mathbbm{1}_\gamma \), where \( s_\gamma = s(\gamma) \) and \( \mathbbm{1}_\gamma \) is the characteristic function of \( \gamma \) to 1 and everything else to 0. Note that \( H \) is an ordered abelian group under pointwise addition and the order relation \( s > 0 \Leftrightarrow s(\min \text{supps}) > 0 \). Let \( H' \) be the Hahn sum

\[
H' = \bigoplus_{\gamma \in \omega + 1} \mathbb{Q} \subseteq H,
\]

i.e. the subgroup of all elements of \( H \) with finite support. Note that \( H' \subseteq H \) is an immediate extension under the natural valuation (cf. [17, p. 3]). It follows that for any ordered abelian groups \( G_1 \) and \( G_2 \) with \( H' \subseteq G_1 \subseteq G_2 \subseteq H \), also the extension \( G_1 \subseteq G_2 \) is immediate.

Let \( G \subseteq H \) be given by

\[
G = H' + a\mathbb{Z}, \text{ where } a = \sum_{\gamma \in \omega} \mathbbm{1}_\gamma.
\]

Now \( G^{\text{div}} = H' + a\mathbb{Q} \subseteq H \), and the extension \( G \subseteq G^{\text{div}} \) is immediate. Let \( c = \frac{1}{2}a + \frac{3}{5}\mathbb{1}_2 \) and \( d = \frac{1}{2}a + \frac{2}{5}\mathbb{1}_2 \). Then \( c, d \in G^{\text{div}} \) with \( 0 < c < d \). However,
there is no element in \(G\) strictly between \(c\) and \(d\). Thus \(G\) is not dense in \(G^{\text{div}}\).

**Proposition 5.12.** Let \(G\) and \(H\) be ordered abelian groups such that \(G \subseteq H\). Let \(v\) be the natural valuation on \(H\). Suppose that \(vG\) has no last element and \(G\) is dense in \(H\). Then the extension \(G \subseteq H\) is immediate.

**Proof.** In order to show that \(G \subseteq H\) is immediate, we need to show that for any \(a \in H\) there exists \(b \in G\) such that \(v(a - b) > v(a)\). Let \(a \in H\) with \(a > 0\). Since \(vG\) has no last element, there is some \(c \in G^{>0}\) such that \(v(c) > v(a)\). By density of \(G\) in \(H\) there is some \(b \in G\) such that \(a - c < b < a + c\). We obtain \(v(a - b) \geq v(c) > v(a)\), as required. \(\square\)

**Corollary 5.13.** Let \(G\) be an ordered abelian group and let \(v\) be the natural valuation on \(G^{\text{div}}\). Suppose that \(vG\) has no last element and \(G\) is dense in \(G^{\text{div}}\). Then the extension \(G \subseteq G^{\text{div}}\) is immediate.

**Proof.** Apply Proposition 5.12 to \(H = G^{\text{div}}\). \(\square\)

Corollary 5.13 does not hold in general in the case where \(vG\) has a last element, as the following example will show.

**Example 5.14.** Let \(A\) be as in Example 5.3. Then the group \(G = \mathbb{Q} \oplus A\) ordered lexicographically has divisible hull \(G^{\text{div}} = \mathbb{Q} \oplus \mathbb{Q}\). Moreover, \(G\) is dense in \(G^{\text{div}}\), as \(A\) is dense in \(\mathbb{Q}\). However, the extension is not immediate, as the archimedean components do not coincide.

**Remark 5.15.** Both in Corollary 5.10 and Corollary 5.13 we imposed a condition on the value set of \(G\) under the natural valuation. Moreover, we provided counterexamples when this condition is not satisfied. Note that there is exactly one case in which the condition on the value set of \(G\) in Corollary 5.10 and Corollary 5.13 are both satisfied, namely when \(vG \cong \omega\). We obtain the following: Let \(G\) be an ordered abelian group and \(vG \cong \omega\). Then \(G\) is dense in \(G^{\text{div}}\) if and only if the extension \(G \subseteq G^{\text{div}}\) is immediate.

Let \(G\) be an ordered abelian group. A valuation \(w\) on an ordered abelian group \(G\) is **convex** if for any \(g_1, g_2 \in G\) with \(0 < g_1 \leq g_2\), we have \(w(g_1) \geq w(g_2)\). Let \(v\) be the natural valuation on \(G\) and let \(\Gamma = vG\). There is a one-to-one correspondence between non-trivial convex subgroups of \(G\) and final segments of \(\Gamma\) (see [17, p. 50 f.]). Namely, for any non-trivial convex subgroup \(H \subseteq G\), the set \(v(H)\) is a non-empty final segment of \(\Gamma\). Vice versa, any final segment \(\Delta\) of \(\Gamma\) can be associated with the convex subgroup \(\{g \in G \mid v(g) \in \Delta\} \cup \{0\}\). For a final segment \(\Delta \subseteq \Gamma\), set \(\Gamma_\Delta = (\Gamma \setminus \Delta) \cup \{\delta\}\), where \(\Gamma_\Delta\) is ordered by \(\delta > \Gamma\) and \(\Gamma \setminus \Delta\) inheriting the order from \(\Gamma\). Then

\[
v_\Delta : G \to \Gamma_\Delta, g \mapsto \begin{cases} v(g), & \text{if } v(g) \notin \Delta, \\ \delta, & \text{if } v(g) \in \Delta. \end{cases}
\]

defines a convex valuation on \(G\). In particular, any non-trivial convex subgroup of \(G\) induces a convex valuation on \(G\).
In Section 3, we addressed the question what convex valuations are $L_{or}$-definable in ordered fields. In analogy to Construction 1.17 we will show that for any ordered abelian group which is not dense in its real closure, there exists a proper non-trivial convex $L_{og}$-definable subgroup, which, by the observation above, induces a convex valuation on $G$.

**Proposition 5.16.** Let $G$ be an ordered abelian group which is not dense in its divisible hull. Suppose that $0$ is a limit point of $G$ in $G^{div}$. Then $G$ has a proper non-trivial convex subgroup which is $L_{og}$-definable with one parameter.

**Proof.** Let $g_0 \in G^{div} \setminus cl(G)$ with $g_0 > 0$. Then there is some $g_1 \in G$ with $g_1 > 0$ and $N \in \mathbb{N}$ such that $g_0 = \frac{g_1}{N}$. Consider the set

$$D = \{g \in G^\geq 0 \mid g < g_0\} = \{g \in G^\geq 0 \mid Ng < g_1\}.$$ 

This set is $L_{og}$-definable with the parameter $g_1$. Let

$$A = \{g \in G^\geq 0 \mid g + D \subseteq D\}.$$ 

Again, $A$ is $L_{og}$-definable with the parameter $g_1$. Note that $A$ is convex. Obviously, $A \neq G^\geq 0$, as for any $g \in D$ and $g_2 \in G^\geq 0$ with $g_2 > g_0 - g$ we have $g + g_2 > g_0$ and thus $g_2 \notin A$. Assume that $A = 0$. Then for any $g_2 \in G^\geq 0$ there is some $g_3 \in G^\geq 0$ with $g_3 < g_0$ such that $g_2 + g_3 > g_0$ and thus $g_0 < g_2 + g_3 < g_2 + g_0$. Hence, for any $g_2 \in G^\geq 0$ we can find some $g_4 \in G$ such that $g_0 < g_4 < g_0 + g_2$. Since $0$ is a limit point of $G$ in $G^{div}$, for any $g_5 \in G^{div}$ with $g_5 > 0$, there exists some $g_2 \in G$ such that $0 < g_2 < g_5$. Thus, for any $g_5 \in G^{div}$ with $g_5 > 0$ there exists some $g_4 \in G$ such that $g_0 < g_4 < g_0 + g_5$. This shows that $g_0$ is a limit point of $G$ in $G^{div}$, contradicting the choice of $g_0 \notin cl(G)$. Hence, $A \neq 0$.

Now for any $a, b \in A$ with $0 < a < b$, we have $(a + b) + D \subseteq a + D \subseteq D$, whence $a + b \in A$. Moreover, $0 < b - a < b$ and $-b < a - b < 0$. Thus, by convexity, $b - a \in A$ and $a - b \in -A$. Similarly, for any $a, b \in -A$ with $a < b < 0$, we have $a \pm b \in -A$ and $b - a \in A$. This shows that $H = -A \cup A$ is closed under addition and thus a convex $L_{og}$-definable subgroup of $G$. Since $0 \neq A \neq G^\geq 0$, we also have that $H$ is a proper non-trivial subgroup of $G$. 

### 6. Strongly NIP Ordered Fields

In this section we will study the class of strongly NIP ordered field in the light of Conjecture 1.3. We start by adapting [7, Proposition 5.2] in the case that the residue field is not separably closed.

**Proposition 6.1.** Let $(K, <)$ be a strongly NIP ordered field and let $v$ be a henselian valuation on $K$. Then also $(Kv, <)$ and $vK$ are strongly NIP.

**Proof.** Since $Kv$ is an ordered field, it is not separably closed. Thus, by [11, Theorem A], $v$ is definable in the Shelah expansion $(K, <)^{Sh}$ (cf. [11],...
Section 2) of \((K,<)^{Sh}\) is also strongly NIP, whence \((K,<,v)\) is strongly NIP. Hence, also \((Kv,<)\) and \(vK\) are strongly NIP. \(\square\)

**Remark 6.2.** By applying Proposition 6.1 to \(v_{min}\), we obtain the following: Let \((k,<)\) be an archimedean field and let \(G\) be an ordered abelian group. Suppose that the ordered Hahn field \((k((G)),<)\) is strongly NIP. Then \((k,<)\) and \(G\) are strongly NIP.

The next result is obtained from a slight adjustment of the proof of [8, Fact 1.8].

**Proposition 6.3.** Let \((K,<)\) be a strongly NIP ordered field which not real closed and almost real closed with respect to a valuation \(v\). Then \((K,\leq)\) and \(G\) are strongly NIP.

**Proof.** By Proposition 6.1, \(vK = G\) is strongly NIP. Since \(K\) is not real closed, \(G\) is non-divisible. By [7, Proposition 5.5], any henselian valuation with non-divisible value group on a strongly NIP field has an \(L_{r}\)-definable henselian coarsening. Hence, there is an \(L_{r}\)-definable henselian coarsening \(u\) of \(v\). By Fact 6.2, \(u\) is convex on \((K,<)\). \(\square\)

Proposition 6.3 can be strengthened in the case that \(vK\) is discretely ordered or dense in its divisible hull. In this case, \(v\) is \(L_{or}\)-definable by Theorem 3.20. An explicit dp-minimal and thus strongly NIP discretely ordered abelian group is given in Example 7.2. For an example of a non-divisible strongly NIP densely ordered abelian group, we use the characterisation of strongly NIP ordered abelian groups from [7, Theorem 1].

**Fact 6.4.** Let \(G\) be an ordered abelian group. Then the following are equivalent:

1. \(G\) is strongly NIP.
2. \(G\) is elementarily equivalent to a lexicographic sum of ordered abelian groups \(\bigoplus_{i \in I} G_i\), where for every prime \(p\),

\[|\{i \in I \mid pG \neq G\}| < \infty,\]

and for any \(i \in I\)

\[|\{p \text{ prime} \mid [G_i : pG_i] = \infty\}| < \infty.\]

**Example 6.5.**

1. Let

\[B = \left\{ \frac{a}{p_1 \ldots p_m} \mid a \in \mathbb{Z}, i \in \mathbb{N} \text{ and } p_1, \ldots, p_i \geq 3 \text{ are prime} \right\}.\]

\(B\) is \(p\)-divisible for any prime \(p \geq 3\). Thus, \(|\{p \text{ prime} \mid [G_i : pG_i] = \infty\}| = 1\). By Fact 6.4, \(B\) is strongly NIP. Moreover, it dense in its divisible hull \(\mathbb{Q}\) but not divisible, as \(\frac{1}{2} \notin B\).

2. Let \(G = B \oplus B\) ordered lexicographically. By Fact 6.4, \(G\) is strongly NIP. However, since there is no element in \(G\) between \((\frac{1}{2},0)\) and \((\frac{1}{2},1)\) in \(G^{\text{div}} = \mathbb{Q} \oplus \mathbb{Q}\), we have that \(G\) is not dense in \(G^{\text{div}}\).
Next, we adapt [8, Lemma 1.9] to the context of ordered fields.

**Proposition 6.6.** Assume that any strongly NIP ordered field is either real closed or admits a non-trivial henselian valuation. Let \((K, <)\) be a strongly NIP ordered field. Then \((K, <)\) is almost real closed with respect to the canonical valuation.

**Proof.** Let \((K, <)\) be a strongly NIP ordered field. If \(K\) is real closed, we can take the trivial valuation. Otherwise, by assumption, the set of non-trivial henselian valuations on \(K\) is non-empty. Let \(v\) be the canonical valuation on \(K\), i.e. the finest henselian valuation on \(K\).

By Proposition \[6.1\], \((Kv, <)\) is strongly NIP. Note that \(Kv\) cannot admit a non-trivial henselian valuation, as otherwise this would induce a non-trivial henselian valuation on \(K\) finer than \(v\). Hence, by assumption, \(Kv\) must be real closed. \(
\)

We will now exploit that the backward direction of [8, Conjecture 1.3] can be proved unconditionally (cf. [8, p. 2]).

**Fact 6.7.** Let \(K\) be a perfect field. Suppose that there exists a henselian valuation \(v\) on \(K\) such that the following hold:

1. \(v\) is defectless.
2. The residue field \(Kv\) is either an algebraically closed field of characteristic \(p\) or elementarily equivalent to a local field of characteristic 0.
3. The ordered value group \(vK\) is strongly NIP.
4. If \(\text{char}(Kv) = p \neq \text{char}(K)\), then \([-v(p), v(p)] \subseteq pvK\).

Then \(K\) is strongly NIP.

**Proposition 6.8.** Let \(G\) be a strongly NIP ordered abelian group. Then \((\mathbb{R}((G)), <)\) is strongly NIP.

**Proof.** If \(K = \mathbb{R}((G))\) is real closed then we are done.

Otherwise, let \(v\) be the natural valuation on \(K\). We will first verify that \(v\) satisfies conditions (1)–(4) of Fact \[6.7\]. Condition (4) is trivially satisfied; (2) and (3) hold by assumption. The valuation \(v\) is defectless if every finite extension \((L, v)\) over \((K, v)\) is defectless. Since this always holds in the characteristic 0 case, (1) is satisfied.

Now \(K\) is ac-valued with angular component map \(\text{ac} : K \to \mathbb{R}\) given by \(\text{ac}(s) = s(v(s))\) for \(s \neq 0\) and \(\text{ac}(0) = 0\) (see [2, Section 5.4 f.]). Following the argument of [8, p. 2], we obtain that \((K, v, \text{ac})\) is a strongly NIP ac-valued field. Since \(\mathbb{R}\) is closed under square roots for positive element, for any \(a \in K\) we have \(a \geq 0\) if and only if the following holds in \(K\):

\[\exists y \ y^2 = \text{ac}(a)\]

Hence, the order relation \(<\) is definable in \((K, v, \text{ac})\). We obtain that \((K, <)\) is strongly NIP. \(\Box\)
Proposition 6.9. Let \((K, <)\) be an almost real closed with respect to a valuation \(v\) and a strongly NIP ordered group \(G\). Then \((K, <)\) is strongly NIP.

Proof. If \(v\) is trivial, then \(K = Kv\) and we are done. Otherwise, by the Ax–Kochen–Ershov principle for ordered fields, we have \((K, <) \equiv (\mathbb{R}((vK)), <)\). By Proposition 6.8, \((\mathbb{R}((vK)), <)\) is strongly NIP. Hence, by elementary equivalence, \((K, <)\) is strongly NIP.

Remark 6.10. We obtain from Proposition 6.9 and Proposition 6.1 the following characterisation of strongly NIP almost real closed fields: Let \((K, <)\) be an almost real closed field with respect to some ordered abelian group \((G, <)\). Then \((K, <)\) is strongly NIP if and only if \((G, <)\) is strongly NIP.

We end this section by giving a class of examples of strongly NIP ordered fields.

Lemma 6.11. Let \((k, <)\) be an almost real closed field and \(G\) an ordered abelian group. Then \((k\langle G \rangle, <)\) is almost real closed.

Proof. Let \(v\) be a henselian valuation on \(k\) such that \(kv\) is real closed. By the Ax–Kochen–Ershov principle, we have \((k\langle G \rangle, <, v_{\text{min}}) \equiv (\mathbb{R}((vk\langle G \rangle), <, v_{\text{min}}))\). Now \((\mathbb{R}((vk\langle G \rangle), <) \equiv (\mathbb{R}((G \oplus kv), <)),\) where \(G \oplus kv\) is the ordered abelian group \(G \oplus kv\) ordered lexicographically.

Proposition 6.12. Let \(G\) and \(H\) be strongly NIP ordered abelian groups. Let \((K, <)\) be an almost real closed field with respect to \(G\). Then \((K\langle G \rangle, <)\) is a strongly NIP ordered field.

Proof. As in the prove of Lemma 6.11 we have that \((K\langle G \rangle, <) \equiv (\mathbb{R}(G \oplus H), <)\). Since \(G\) and \(H\) are strongly NIP, also \(G \oplus H\) is strongly NIP by Fact 6.4. Hence, by Remark 6.10 also \((K\langle G \rangle, <)\) is strongly NIP.

7. dp-minimal Ordered Fields

A special class of strongly NIP ordered fields are dp-minimal ordered fields. These are fully classified in [14]. In this section we will show that Conjecture [14] holds for dp-minimal fields.

An ordered group \((G, <)\) is called non-singular if \(G/pG\) is finite for all prime numbers \(p\).

Fact 7.1. (See [14], Proposition 5.1.) An \(\aleph_1\)-saturated ordered abelian group \(G\) is dp-minimal if and only if it is non-singular.

Example 7.2. Let \(G\) be an \(\aleph_1\)-saturated extension of the ordered abelian group \(\mathbb{Z}\). Then \(G\) is a dp-minimal ordered abelian group which is discretely ordered.

Fact 7.3. (See [14], Theorem 6.2.) An ordered field \((K, <)\) is dp-minimal if and only if there exists a non-singular ordered abelian group \(G\) such that \((K, <) \equiv (\mathbb{R}(G), <)\).
We immediately obtain the following.

**Corollary 7.4.** Let \((K, <)\) be an ordered field. Suppose that \(K\) is almost real closed with respect to a dp-minimal ordered abelian group. Then \((K, <)\) is dp-minimal.

**Proposition 7.5.** Any dp-minimal ordered field is almost real closed.

*Proof.* Let \((K, <)\) be a dp-minimal ordered field. By Fact 7.3, \((K, <) \equiv (\mathbb{R}(\langle G \rangle), <)\) for some non-singular ordered abelian group \(G\). Note that \((\mathbb{R}(\langle G \rangle), <)\) is almost real closed. Thus, by Proposition 3.7, also \((K, <)\) is almost real closed. \(\square\)

**Corollary 7.6.** Let \((K, <)\) be a dp-minimal archimedean field. Then \(K\) is real closed.

*Proof.* The only archimedean almost real closed fields are the archimedean real closed fields. Thus, by Proposition 7.5, any archimedean dp-minimal ordered field is real closed. \(\square\)

**Question 7.7.** Let \((K, <)\) be a strongly NIP archimedean ordered field. Is \(K\) necessarily real closed?

---

8. Concluding Observations

Recall our two main conjectures.

**Conjecture 1.3.** Let \((K, <)\) be a strongly NIP ordered field. Then \(K\) is either real closed, or \(K\) admits an \(L_{\omega_1}\)-definable non-trivial henselian valuation.

**Conjecture 1.4.** Any strongly NIP ordered field is almost real closed.

In this final section, we will show that Conjecture 1.3 and Conjecture 1.4 are equivalent.

**Remark 8.1.**

1. Any archimedean field does not admit a non-trivial convex valuation, as \(\mathbb{Z}\) must be contained in any valuation ring. Conjecture 1.3 for archimedean fields thus states that any strongly NIP archimedean ordered field is real closed.
2. An ordered field is real closed if and only if it is o-minimal. Hence, for any real closed field \(K\), if \(\mathcal{O} \subseteq K\) is a definable convex ring, its endpoints must lie in \(K \cup \{\pm \infty\}\). This implies that any definable convex valuation ring must already contain \(K\), i.e. is trivial. Thus, the two cases in the consequence of Conjecture 1.3 are exclusive.
3. A full characterisation of strongly NIP ordered groups in terms of algebraic properties is given in [7, Theorem 1]. Thus, Conjecture 1.4 and Proposition 6.9 would give us a full characterisation of strongly NIP ordered fields in terms of algebraic properties of the value group and the residue field under a henselian valuation.
(4) The model theory of almost real closed fields is treated in [1]. It is known that any almost real closed field is NIP as an ordered field (see [12, p. 1]).

By Remark 8.1, in order to prove Conjecture 1.3, one has to show that a strongly NIP non-archimedean ordered field which is not real closed admits an $L_{or}$-definable non-trivial henselian valuation.

**Theorem 8.2.** Conjecture 1.3 and Conjecture 1.4 are equivalent.

**Proof.** Let $(K, <)$ be a strongly NIP ordered field which is not real closed.

Assume Conjecture 1.4. Then $(K, <)$ admits a henselian valuation $v$. By Proposition 6.3, it also admits an $L_{or}$-definable henselian valuation.

Now assume Conjecture 1.3. Let $(K, <)$ be strongly NIP ordered field. By Conjecture 1.3 and Proposition 6.6, $K$ is almost real closed with respect to the canonical valuation $v$. □

**Remark 8.3.** We now obtain the following conjectural classification of strongly NIP ordered fields which is equivalent to Conjecture 1.4: An ordered field is strongly NIP if and only if it is almost real closed with respect to a strongly NIP ordered abelian group. Indeed, by Remark 6.10, any real closed field with respect to a strongly NIP ordered abelian group is strongly NIP; conversely, if one assumes Conjecture 1.4, then any strongly NIP ordered field is almost real closed with respect to some ordered abelian group, which is necessarily strongly NIP by Proposition 6.1.

We conclude with a remark concerning density in real closure.

**Remark 8.4.** Note that any almost real closed field which is not real closed cannot be dense in its real closure. This is due to the fact that dense extensions of ordered valued fields are immediate (cf. [17, Lemma 1.31]). Thus, if Conjecture 1.4 is true, then, in particular, a strongly NIP ordered field which is not real closed cannot be dense in its real closure. Note that any dp-minimal ordered field which is dense in its real closure is real closed. Indeed, by Fact 7.3 and Theorem 4.2, any dp-minimal ordered field which is dense in its real closure is elementarily equivalent to a non-archimedean ordered Hahn field which is dense in its real closure. However, by Remark 4.3, the only ordered Hahn fields which are dense in their real closure are the real closed ones.

**Question 8.5.** Let $(K, <)$ be a strongly NIP ordered field which is dense in its real closure. Is $(K, <)$ real closed?

**Acknowledgements.** The first author was supported by a doctoral scholarship of Studienstiftung des deutschen Volkes as well as of Carl-Zeiss-Stiftung.
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