ROTA’S CLASSIFICATION PROBLEM, REWRITING SYSTEMS AND
GRÖBNER-SHIRSHOV BASES

XING GAO AND LI GUO

ABSTRACT. In this paper we revisit Rota’s Classification Problem on classifying algebraic identities
for linear operator. We reformulate Rota’s Classification Problem in the contexts of rewriting
systems and Gröbner-Shirshov bases, through which Rota’s Classification Problem amounts to the
classification of operators, given by their defining operator identities, that give convergent rewriting
systems or Gröbner-Shirshov bases. Relationship is established between the reformulations in
terms of rewriting systems and that of Gröbner-Shirshov bases. We provide an effective condition
that gives Gröbner-Shirshov operators and obtain a new class of Gröbner-Shirshov operators.

CONTENTS

1. Introduction
1.1. Motivation
1.2. Rota’s Classification Problem in special cases
1.3. Rota’s Classification Problem in the general case
2. Reformulations of Rota’s Classification Problem
2.1. Operated polynomial identities
2.2. Rota’s Classification Problem via rewriting systems
2.3. Rota’s Classification Problem via Gröbner-Shirshov bases
3. Relationship between reformulations of Rota’s Classification Problem
3.1. Term-rewriting systems
3.2. Gröbner-Shirshov OPIs and convergent OPIs
4. A sufficient condition for Gröbner-Shirshov OPIs
4.1. Statement of the main theorem and examples
4.2. The proof of Theorem 4.1
References

1. INTRODUCTION

1.1. Motivation. Motivated by the important roles played by various linear operators in the study
of mathematics through their actions on objects, Rota [16] posed the problem of

finding all possible algebraic identities that can be satisfied by a linear operator on an algebra,

henceforth called Rota’s Classification Problem.

Operator identities that were interested to Rota included

\[
\begin{align*}
\text{Endomorphism operator} & \quad d(xy) = d(x)d(y), \\
\text{Differential operator} & \quad d(xy) = d(x)y + xd(y), \\
\text{Average operator} & \quad P(x)P(y) = P(xP(y)),
\end{align*}
\]

Date: January 19, 2016.
2010 Mathematics Subject Classification. 16W99, 13P10, 16S15, 12H05, 08A70 16S20 16R99.
Key words and phrases. Rota’s Classification Problem, linear operators, operator identities, Gröbner-Shirshov
bases, term rewriting systems, normal forms, free objects.
Inverse average operator \( P(x)P(y) = P(P(x)y), \)

(Rota-)Baxter operator of weight \( \lambda \) \( P(x)P(y) = P(xP(y) + P(xy) + \lambda xy), \)

where \( \lambda \) is a fixed constant,

Reynolds operator \( P(x)P(y) = P(xP(y) + P(xy) - P(x)P(y)). \)

After Rota posed his problem, more operators have appeared, such as

Differential operator of weight \( \lambda \) \( d(xy) = d(x)y + xd(y) + \lambda d(x)d(y), \)

where \( \lambda \) is a fixed constant,

Nijenhuis operator \( P(x)P(y) = P(xP(y) + P(xy) - P(xy)), \)

Leroux’s TD operator \( P(x)P(y) = P(xP(y) + P(xy) - P(1)y). \)

The pivotal roles played by the endomorphisms (such as in Galois theory) and derivations (such as in calculus) are well-known. Their abstractions have led to the concepts of difference algebra and differential algebra respectively. The other operators also found applications in a broad range of pure and applied mathematics, including combinatorics, probability and mathematical physics \([2, 3, 4, 5, 15, 17, 19, 20, 23, 25, 26]\). See \([3, 16]\) for further references.

These sustained interests in linear operators that satisfy special operator identities warrant a systematic study of Rota’s Classification Problem, leading to the articles \([17, 16]\). There are multiple benefits in such study, on the one hand to find a uniform approach to these various existing operators and on the other hand to understand the nature of these operators, namely what distinguish them from a randomly taken operator identity. The latter also sheds light on possible new operator identities that might arise in mathematics and its applications.

1.2. Rota’s Classification Problem in special cases. There are two stages in the recent approach to Rota’s Classification Problem. The first stage is to establish an algebraic framework in which to consider algebraic identities satisfied by a linear operator in Rota’s Classification Problem. As a prototype, we recall that an algebraic identity satisfied by an algebra is an element in a noncommutative polynomial algebra, as a realization of a free (associative) algebra, leading to the extensive study of polynomial identity (PI) rings \([17, 24, 27]\). Since there is an operator involved in an algebraic identity in Rota’s Classification Problem, we take an algebraic identity satisfied by an operator to be an element in a free object in the category of algebras with an operator, or operated algebras, whose origin can be tracked back to Kurosh \([18]\). In \([17]\), such a free object is realized in the form of polynomials in variables together with their formal derivations, amenable to serve as operated polynomial identities (OPIs) for an algebra with operators.

In this sense, all the operators list above are defined by OPIs. This naturally leads to the second stage in our understanding of Rota’s Classification Problem: what distinguishes the OPIs satisfied by these operators from the OPIs defined by arbitrary elements from the operated polynomial algebras? This is a key difference between PI algebras and OPI algebras. In the study of the former, not much difference is made among the elements in the polynomial algebras. This is apparent not the case for elements from the operated polynomial algebras, hence Rota’s Classification Problem. In other words, Rota apparently asked to identify special OPIs that are worth of further study, as in the case of the OPIs in the above lists. As a hint for what to look for in these “good” OPIs, we pay special attention that Rota’s Classification Problem asks for linear operators defined on an algebra, which in his context means an associative algebra. Therefore, such a “good” OPIs should satisfy certain compatibility condition with the associativity of the algebra that the operator acts on.
In order to make sense of this compatibility for arbitrary OPIs, we first tested two classes of OPIs which, despite their special forms, are general enough to cover all the operators considered above, except the Reynolds operator. The two classes of operators are called the differential type operators and Rota-Baxter type operators, for their resemblance to the differential operator and the Rota-Baxter operator respectively.

As the initial step, differential type operators, the easier of the two classes of operators, were studied in [16], revealing that, the seemingly vague and specialized problem of Rota can be casted in completely general setups. First of all, it was showed that, the somehow ad hoc properties defining differential type operators turn out to be equivalence to the convergence of the rewriting systems defined by these operators. Second, these properties are also equivalent to the existence of a generalization of the Gröbner basis, called the Gröbner-Shirshov basis, for the ideals defined by these OPIs, giving rise to an explicit construction of the free objects in the category of the algebras satisfying the OPIs. These equivalences suggest intimate connection from Rota’s Classification Problem to rewriting systems and Gröbner bases.

To obtain more evidence for this speculation, the class of Rota-Baxter type operators was studied in [13]. It is encouraging to see that, despite the much more challenging nature of Rota-Baxter operators, the same connections can be established from them to convergent rewriting systems on the one hand and to Gröbner-Shirshov bases on the other.

1.3. Rota’s Classification Problem in the general case. The success in characterizing these two important classes of operators in terms of general properties in rewriting systems and ideal generators motivates us to understand Rota’s Classification Problem in the context of these general properties for OPIs, rather than by certain special forms such as being of the differential type or Rota-Baxter type. We carry out this approach in this paper.

We give, in Section 2, two formulations of Rota’s Classification Problem for desirable systems of operator identities, one in terms of convergent rewriting systems and one in terms of Gröbner-Shirshov bases. When one monomial in an operated identity is chosen as the leading term, the identity gives a rewriting rule. Our first formulation of Rota’s Classification Problem is to find OPIs for which one rewriting system obtained this way is convergent (Problem 2.13).

An important and effective way to determine the convergency of a rewriting system is the method of Gröbner bases in the case of commutative algebras, or Gröbner-Shirshov bases in general. Thus our second formulation of Rota’s Classification Problem is to find systems of OPIs that are Gröbner-Shirshov bases of the operated ideals that these systems generate, leading to the concepts of Gröbner-Shirshov and potentially Gröbner-Shirshov systems of OPIs, and the corresponding Gröbner-Shirshov and potentially Gröbner-Shirshov operators (Problem 2.26).

In Section 3, we establish the relationship between the two reformulations of Rota’s Classification Problem, by showing that a Gröbner-Shirshov system of OPIs gives a convergent system (Theorem 3.16). The interplay between the two systems proves to be fruitful. For example, it is not hard to show that the OPIs for the two-sided averaging operator is not convergent and hence not Gröbner-Shirshov (Corollary 3.17); while from showing that it is potentially Gröbner-Shirshov we conclude that it is potentially convergent (Remark 3.18).

This conceptual approach allows us to obtain an effective criterion to obtain Gröbner-Shirshov operators (Theorem 4.1), including not only the two previously known differential type and Rota-Baxter type operators, but also the modified Rota-Baxter operator [11] with motivation from modified classical Yang-Baxter equation on Lie algebras [28]. As an application, using the composition-diamond lemma, we obtain the free objects in the category of modified Rota-Baxter algebras.
Finally we define the monoid whose elements are called since induces a monoid embedding has been obtained. Then we define together with a map (resp. \( U \)) such that

\[
\text{Definition 2.1.} \quad \text{An operated monoid (resp. \textit{operated k-algebra}) is a monoid (resp. k-algebra) } U \text{ together with a map (resp. k-linear map) } P_U : U \to U. \text{ A morphism from an operated monoid (resp. k-algebra) } (U, P_U) \text{ to an operated monoid (resp. k-algebra) } (V, P_V) \text{ is a monoid (resp. k-algebra, resp. k-module) homomorphism } f : U \to V \text{ such that } f \circ P_U = P_V \circ f.
\]

Let \( X \) be a given set. We will construct the free operated monoid over \( X \). The construction proceeds via the finite stages \( \mathcal{M}_n(X) \) recursively defined as follows. The initial stage is \( \mathcal{M}_0(X) := M(X) \text{ and } \mathcal{M}_1(X) := M(X \cup [\mathcal{M}_0(X)]) \), where \( [\mathcal{M}_0(X)] := \{ [u] \mid u \in \mathcal{M}_0(X) \} \) is a disjoint copy of \( \mathcal{M}_0(X) \). The inclusion \( X \hookrightarrow X \cup [\mathcal{M}_0(X)] \) induces a monomorphism

\[
i_0 : \mathcal{M}_0(X) = M(X) \hookrightarrow \mathcal{M}_1(X) = M(X \cup [\mathcal{M}_0(X)])
\]

of monoids through which we identify \( \mathcal{M}_0(X) \) with its image in \( \mathcal{M}_1(X) \).

For \( n \geq 2 \), assume inductively that \( \mathcal{M}_{n-1}(X) \) has been defined and the embedding

\[
i_{n-2,n-1} : \mathcal{M}_{n-2}(X) \hookrightarrow \mathcal{M}_{n-1}(X)
\]

has been obtained. Then we define

\[
\mathcal{M}_n(X) := M(X \cup [\mathcal{M}_{n-1}(X)])
\]

Since \( \mathcal{M}_{n-1}(X) = M(X \cup [\mathcal{M}_{n-2}(X)]) \) is a free monoid, the injection

\[
[\mathcal{M}_{n-2}(X)] \hookrightarrow [\mathcal{M}_{n-1}(X)]
\]

induces a monoid embedding

\[
\mathcal{M}_{n-1}(X) = M(X \cup [\mathcal{M}_{n-2}(X)]) \hookrightarrow \mathcal{M}_n(X) = M(X \cup [\mathcal{M}_{n-1}(X)])
\]

Finally we define the monoid

\[
\mathcal{M}(X) := \bigcup_{n \geq 0} \mathcal{M}_n(X),
\]

whose elements are called \textbf{bracketed words} or \textbf{bracketed monomials} on \( X \).
Lemma 2.2. ([13], Corollary 3.7) Let \( i_X : X \to \mathcal{M}(X) \) and \( j_X : \mathcal{M}(X) \to k\mathcal{M}(X) \) be the natural embeddings. Then, with the notations above,

(a) the triple \( (\mathcal{M}(X), \lfloor \cdot \rfloor, i_X) \) is the free operated monoid on \( X \); and

(b) the triple \( (k\mathcal{M}(X), \theta, j_X \circ i_X) \) is the free operated \( k \)-algebra on \( X \).

Definition 2.3. Let \( \phi(x_1, \ldots, x_k) \in k\mathcal{M}(X) \) with \( k \geq 1 \) and \( x_1, \ldots, x_k \in X \). We call \( \phi(x_1, \ldots, x_k) = 0 \) (or simply \( \phi(x_1, \ldots, x_k) \)) an **operated polynomial identity (OPI)**.

Let \( \phi = \phi(x_1, \ldots, x_k) \in k\mathcal{M}(X) \) be an OPI. For any operated algebra \( (R, P) \) and any map \( \theta : x_i \mapsto r_i, i = 1, \ldots, k \), using the universal property of \( k\mathcal{M}(x_1, \ldots, x_k) \) as a free operated algebra on \( \{x_1, \ldots, x_k\} \), there is a unique morphism \( \theta : k\mathcal{M}(x_1, \ldots, x_k) \to R \) of operated algebras that extends the map \( \theta \). We use the notation

\[
\phi(r_1, \ldots, r_k) := \theta(\phi(x_1, \ldots, x_k))
\]

for the corresponding evaluation or substitution of \( \phi(x_1, \ldots, x_k) \) at the point \((r_1, \ldots, r_k)\). Informally, this is the element of \( R \) obtained from \( \phi \) upon replacing every \( x_i \) by \( r_i \), \( 1 \leq i \leq k \) and the operator \( \lfloor \cdot \rfloor \) by \( P \).

Definition 2.4. With the above notations, we say that \( \phi(x_1, \ldots, x_k) = 0 \) (or simply \( \phi(x_1, \ldots, x_k) \)) is an **OPI satisfied by** \( (R, P) \) if

\[
\phi(r_1, \ldots, r_k) = 0 \quad \text{for all } r_1, \ldots, r_k \in R.
\]

In this case, we call \( (R, P) \) (or simply \( R \)) a **\( \phi \)-algebra** and \( P \) a **\( \phi \)-operator**. More generally, For a subset \( \Phi \subseteq k\mathcal{M}(X) \), we call \( R \) (resp. \( P \)) a **\( \Phi \)-algebra** (resp. **\( \Phi \)-operator**) if \( R \) (resp. \( P \)) is a \( \phi \)-algebra (resp. \( \phi \)-operator) for each \( \phi \in \Phi \).

For example, when \( \phi = [x_1 x_2] - [x_1] x_2 - x_1 [x_2] \) (resp. \( \phi = [x_1] x_2 - [x_1][x_2] - [\lfloor x_1 \rfloor x_2] - \lambda [x_1 x_2] \)), a \( \phi \)-algebra is simply a differential algebra (resp. a Rota-Baxter algebra of weight \( \lambda \)). When \( \phi = x_1 x_2 - x_2 x_1 \), a \( \phi \)-algebra is a commutative algebra.

For \( S \subseteq R \), the **operated ideal** \( \text{Id}(S) \) of \( R \) generated by \( S \) is defined to be the smallest operated ideal of \( R \) containing \( S \). For \( \Phi \subseteq k\mathcal{M}(X) \) and a set \( Z \), let \( S_{\Phi}(Z) \subseteq k\mathcal{M}(Z) \) denote the substitution set

\[
S_{\Phi}(Z) := \{ \phi(u_1, \ldots, u_k) \mid u_1, \ldots, u_k \in \mathcal{M}(Z), \phi(x_1, \ldots, x_k) \in \Phi \}.
\]

The following well-known result exhibits the existence of a free \( \Phi \)-algebra whose explicit construction will be explored in this paper.

Proposition 2.5. ([13], Proposition 1.3.6) Let \( X \) be a set and \( \Phi \subseteq k\mathcal{M}(X) \) a system of OPIs. Then for a set \( Z \), the quotient operated algebra \( k\mathcal{M}(Z)/\text{Id}(S_{\Phi}(Z)) \) is the free \( \Phi \)-algebra on \( Z \).

2.2. Rota’s Classification Problem via rewriting systems. As preparation, we recall concepts on term-rewriting systems from [11,13].

**Definition 2.6.** Let \( V \) be a \( k \)-space with a given \( k \)-basis \( W \).

(a) For \( f = \sum_{w \in W} c_w w \in V \) with \( c_w \in k \), the **support** \( \text{Supp}(f) \) of \( f \) is the set \( \{w \in W \mid c_w \neq 0 \} \).

As convention, we take \( \text{Supp}(0) = \emptyset \).
Let \( f, g \in V \). We use \( f + g \) to indicate the property that \( \text{Supp}(f) \cap \text{Supp}(g) = \emptyset \). If this is the case, we say \( f + g \) is a direct sum of \( f \) and \( g \), and use \( f + g \) also for the sum \( f + g \).

(c) For \( f \in V \) and \( w \in \text{Supp}(f) \) with the coefficient \( c_w \), write \( R_w(f) := c_w w - f \in V \). So \( f = c_w w + (-R_w(f)) \).

**Definition 2.7.** Let \( V \) be a \( k \)-space with a \( k \)-basis \( W \).

(a) A term-rewriting system \( \Pi \) on \( V \) with respect to \( W \) is a binary relation \( \Pi \subseteq W \times V \). An element \( (t, v) \in \Pi \) is called a (term-)rewriting rule of \( \Pi \), denoted by \( t \rightarrow v \).

(b) The term-rewriting system \( \Pi \) is called simple with respect to \( W \) if \( t + v \) for all \( t \rightarrow v \in \Pi \).

(c) If \( f = c_t + (-R_t(f)) \in V \), using the rewriting rule \( t \rightarrow v \), we get a new element \( g := c_t v - R_t(f) \in V \), called a one-step rewriting of \( f \) and denoted \( f \rightarrow \Pi g \) or \( f \overset{(t,v)}{\rightarrow} \Pi g \).

(d) The reflexive-transitive closure of \( \rightarrow \Pi \) (as a binary relation on \( V \)) is denoted by \( \rightarrow^* \Pi \) and, if \( f \rightarrow^* \Pi g \), we say \( f \) rewrites to \( g \) with respect to \( \Pi \).

(e) Two elements \( f, g \in V \) are joinable if there exists \( h \in V \) such that \( f \overset{*}{\rightarrow} \Pi h \) and \( g \overset{*}{\rightarrow} \Pi h \); we denote this by \( f \downarrow^* \Pi g \).

(f) An element \( f \in V \) is a normal form if no more rules from \( \Pi \) can apply, more precisely, if \( \text{Supp}(f) \cap \text{Dom}(\Pi) = \emptyset \) where \( \text{Dom}(\Pi) \) is the domain of \( \Pi \subseteq W \times V \).

The crucial point of Item [1] in Definition 2.7 is that, in order to apply a rewriting rule \( t \rightarrow v \) to \( f \), one must firstly express \( f \) as the direct sum \( f = c_t + (-R_t(f)) \). The following definitions are adapted from abstract rewriting systems [1, 2].

**Definition 2.8.** A term-rewriting system \( \Pi \) on \( V \) is called

(a) terminating if there is no infinite chain of one-step rewriting

\[
\ldots \rightarrow^\Pi f_0 \rightarrow^\Pi f_1 \rightarrow^\Pi f_2 \cdots
\]

(b) confluent (resp. locally confluent) if every fork (resp. local fork) is joinable.

(c) convergent if it is both terminating and confluent.

Given a system of OPIs, we can associate it with a term-rewriting system. For this, we need the following concept.

**Definition 2.9.** Let \( Z \) be a set, \( \star \) a symbol not in \( Z \) and \( Z^\star = Z \cup \{\star\} \).

(a) By a \( \star \)-bracketed word on \( Z \), we mean any bracketed word in \( \mathcal{M}(Z^\star) \) with exactly one occurrence of \( \star \), counting multiplicities. The set of all \( \star \)-bracketed words on \( Z \) is denoted by \( \mathcal{M}^\star(Z) \).

(b) For \( q \in \mathcal{M}^\star(Z) \) and \( u \in \mathcal{M}(Z) \), we define \( q|_{\star \rightarrow u} \) to be the bracketed word on \( Z \) obtained by replacing the symbol \( \star \) in \( q \) by \( u \).

(c) For \( q \in \mathcal{M}^\star(Z) \) and \( s = \sum_i c_i u_i \in k\mathcal{M}(Z) \), where \( c_i \in k \) and \( u_i \in \mathcal{M}(Z) \), we define

\[
q|_s := \sum_i c_i q|_{u_i}
\]

(d) A bracketed word \( u \in \mathcal{M}(Z) \) is a subword of another bracketed word \( w \in \mathcal{M}(Z) \) if \( w = q|_u \) for some \( q \in \mathcal{M}^\star(Z) \).

More generally, let \( \star_1, \ldots, \star_k \) be distinct symbols not in \( Z \) and set \( Z^{\star_k} := Z \cup \{\star_1, \ldots, \star_k\}, k \geq 1 \).

(e) We define an \( (\star_1, \ldots, \star_k) \)-bracket word on \( Z \) to be an expression in \( \mathcal{M}(Z^{\star_k}) \) with exactly one occurrence of each of \( \star_i \), \( 1 \leq i \leq k \). The set of all \( (\star_1, \ldots, \star_k) \)-bracket words on \( Z \) is denoted by \( \mathcal{M}^{\star_k}(Z) \).
For \( q \in \mathcal{M}^*(Z) \) and \( u_1, \ldots, u_k \in k\mathcal{M}^*(Z) \), we define
\[
q_{|u_1,\ldots,u_k} := q_{\star_1 \leftarrow u_1, \ldots, \star_k \leftarrow u_k}
\]
to be the element of \( k\mathcal{M}(Z) \) obtained from \( q \) when the letter \( \star_i \), \( 1 \leq i \leq k \), in \( q \) is replaced by \( u_i \).

**Definition 2.10.** Let \( Z \) be a set and \( S \subseteq k\mathcal{M}(Z) \).

(a) Let \( s \in k\mathcal{M}(Z) \) and fix a monomial \( \overline{s} \) of \( s \), called an orientation of \( s \). The monicization of \( s \) with respect to \( \overline{s} \) is replacing \( s \) by its quotient over the coefficient of \( \overline{s} \), making \( s \) monic if \( \overline{s} \) is taken as the leading term. When this is done for each \( s \) in a subset \( S \) of \( k\mathcal{M}(Z) \), then we call \( S \) monicized with respect to the orientation \( \overline{S} := \{ \overline{s} | s \in S \} \).

(b) Let \( S \subseteq k\mathcal{M}(Z) \) with a given orientation \( \overline{S} := \{ \overline{s} | s \in S \} \). We can write \( s = \overline{s} + (-R(s)) \).

Define a term-rewriting system on \( k\mathcal{M}(Z) \) by
\[
\Pi_{\overline{S}} := \Pi_{\overline{S}, \overline{S}}(Z) := \{ q_{\mid_{\overline{s}}} \rightarrow q_{\mid_{[R(s)]}} | s \in S, \ q \in \mathcal{M}^*(Z) \} \subseteq \mathcal{M}(Z) \times k\mathcal{M}(Z).
\]

We call \( \Pi_{\overline{S}} \) the term-rewriting system associated to \( S \) with respect to \( \overline{S} = \{ \overline{s} | s \in S \} \).

(c) Let \( \Phi \subseteq k\mathcal{M}(X) \) be a system of OPIs. For a set \( Z \), let
\[
\overline{\Phi}(Z) := \{ \phi_{\mid_{\overline{s}}} | \phi_{\mid_{\overline{s}}} \in S_{\Phi}(Z) \},
\]
be an orientation of the set \( S_{\Phi}(Z) \) in Eq. (1). We call the resulting rewriting system
\[
\Pi_{\overline{S_{\Phi}(Z)}} := \Pi_{\overline{S_{\Phi}(Z)}, \overline{S_{\Phi}(Z)}} := \{ q_{\mid_{\overline{s}}} \rightarrow q_{\mid_{[R(s)]}} | s \in \mathcal{M}^*(Z), \phi_{\mid_{\overline{s}}} \in S_{\Phi}(Z) \} \subseteq \mathcal{M}(Z) \times k\mathcal{M}(Z)
\]
the term-rewriting system with respect to \( \overline{S_{\Phi}(Z)} \). In particular, if \( \Phi = \{ \phi \} \), we get a term-rewriting system associated to \( \Phi \) with respect to \( \overline{S_{\Phi}(Z)} := \{ \phi_{\mid_{\overline{s}}} | \phi_{\mid_{\overline{s}}} \in S_{\Phi}(Z) \} \)
\[
\Pi_{\overline{S_{\Phi}(Z)}} := \Pi_{\overline{S_{\Phi}(Z)}, \overline{S_{\Phi}(Z)}} := \{ q_{\mid_{\overline{s}}} \rightarrow q_{\mid_{[R(s)]}} | s \in \mathcal{M}^*(Z), \phi_{\mid_{\overline{s}}} \in S_{\Phi}(Z) \} \subseteq \mathcal{M}(Z) \times k\mathcal{M}(Z).
\]

For notational clarify, we will often abbreviate \( \rightarrow_{\Pi_{\overline{S_{\Phi}(Z)}}} \) (resp. \( \rightarrow_{\Pi_{\overline{S_{\Phi}(Z)}}}^\ast \), resp. \( \downarrow_{\Pi_{\overline{S_{\Phi}(Z)}}} \) to \( \rightarrow_{\Phi} \) (resp. \( \rightarrow_{\Phi}^\ast \), resp. \( \downarrow_{\Phi} \)).

**Definition 2.11.** Let \( X \) be a set and \( \Phi \subseteq k\mathcal{M}(X) \) a system of OPIs. Let \( Z \) be a set and \( \Pi_{\overline{S_{\Phi}(Z)}} = \Pi_{\overline{S_{\Phi}(Z)}, S_{\Phi}(Z)} \) a term-rewriting system with respect to an orientation \( \overline{S_{\Phi}(Z)} \) of \( S_{\Phi}(Z) \).

(a) We call \( \Phi \) convergent on \( Z \) with respect to \( \overline{S_{\Phi}(Z)} \) if \( \Pi_{\overline{S_{\Phi}(Z)}} = \Pi_{\overline{S_{\Phi}(Z)}, S_{\Phi}(Z)} \) is convergent.

(b) We call \( \Phi \) potentially convergent on \( Z \) with respect to \( \overline{S_{\Phi}(Z)} \) if, there is a superset \( \Phi' \subseteq k\mathcal{M}(Z) \) of \( \Phi \) with \( \text{Id}(S_{\Phi}(Z)) = \text{Id}(S_{\Phi}(Z)) \) and an orientation \( \overline{S_{\Phi'}(Z)} := \{ \phi_{\mid_{\overline{s}}} | \phi_{\mid_{\overline{s}}} \in S_{\Phi'}(Z) \} \) containing \( \overline{S_{\Phi}(Z)} \), such that \( \Pi_{\overline{S_{\Phi'}(Z)}, \overline{S_{\Phi}(Z)}} \) is convergent.

**Definition 2.12.** Let \( X \) be a set, and let \( \Phi \subseteq k\mathcal{M}(X) \) be a system of OPIs.

(a) We call \( \Phi \) convergent (resp. potentially convergent) if, for each set \( Z \), there is an orientation \( \overline{S_{\Phi}(Z)} \) such that \( \Phi \) is convergent (resp. potentially convergent) on \( Z \) with respect to \( \overline{S_{\Phi}(Z)} \).

(b) A \( \Phi \)-operator \( P \) is called convergent (resp. potentially convergent) if \( \Phi \) is so.

We can now interpret Rota’s Classification Problem in terms of rewriting systems.

**Problem 2.13.** (Rota’s Classification Problem via rewriting systems) Determine all convergent and potentially convergent systems of OPIs.
The well-known (two-sided) averaging operator $P$ (see [22] for example) satisfies

$$P(x_1)P(x_2) = P(P(x_1)x_2) = P(x_1P(x_2))$$

and hence is defined by the system of OPIs

\begin{align*}
\phi_1(x_1, x_2) &= [x_1][x_2] - [x_1]_1x_2], \\
\phi_2(x_1, x_2) &= [x_1][x_2] - [x_1]_1x_2].
\end{align*}

\(2\)

**Proposition 2.14.** The system of OPIs for the (two-sided) averaging operator is not convergent.

As we will see in Remark 3.18, this system of OPIs is potentially convergent.

**Proof.** Let $Z = \{z_1, z_2\}$, $w = [[z_1]z_2]\in \mathcal{W}(Z)$ and $\Phi = \{\phi_1, \phi_2\}$. Write

\[
\phi_1 = \phi_1(z_1, z_2) \text{ and } \phi_2 = \phi_2(z_1, z_2).
\]

According to the choice of orientations $\tilde{\phi}_1$ and $\tilde{\phi}_2$ of $\phi_1$ and $\phi_2$, we have the following four cases.

**Case 1.** $\tilde{\phi}_1 = [z_1][z_2]$ and $\tilde{\phi}_2 = [z_1][z_2]$. Then Eq. (2) induces two rewriting rules

$$[z_1][z_2]\rightarrow_\phi_1 [z_1][z_2] \text{ and } [z_1][z_2]\rightarrow_\phi_2 [z_1][z_2].$$

We have

$$w = [[z_1]z_2] \rightarrow_\phi_1 [z_1][z_2] \text{ and } w = [[z_1]z_2] \rightarrow_\phi_2 [z_1][z_2].$$

Since $[[z_1]z_2]$ and $[z_1][z_2]$ are different normal forms, $\Pi_{S\phi}(Z)$ is not confluent.

**Case 2.** $\tilde{\phi}_1 = [z_1][z_2]$ and $\tilde{\phi}_2 = [z_1][z_2]$. Then Eq. (2) induces two rewriting rules

$$[z_1][z_2]\rightarrow_\phi_1 [z_1][z_2] \text{ and } [z_1][z_2]\rightarrow_\phi_2 [z_1][z_2].$$

We have

$$w = [[z_1]z_2] \rightarrow_\phi_1 [z_1][z_2] \text{ and } w = [[z_1]z_2] \rightarrow_\phi_2 [z_1][z_2].$$

Again, since $ [[z_1]z_2]$ and $[z_1][z_2]$ are different normal forms, $\Pi_{S\phi}(Z)$ is not confluent.

**Case 3.** $\tilde{\phi}_1 = [z_1][z_2]$ and $\tilde{\phi}_2 = [z_1][z_2]$. Then Eq. (2) induces two rewriting rules

$$[[z_1]z_2]\rightarrow_\phi_1 [z_1][z_2] \text{ and } [z_1][z_2]\rightarrow_\phi_2 [z_1][z_2].$$

We have

$$w = [[z_1]z_2] \rightarrow_\phi_1 [z_1][z_2] \text{ and } w = [[z_1]z_2] \rightarrow_\phi_2 [z_1][z_2].$$

Since $[[z_1]z_2]$ and $[z_1][z_2]$ are different normal forms, $\Pi_{S\phi}(Z)$ is not confluent.

**Case 4.** $\tilde{\phi}_1 = [z_1][z_2]$ and $\tilde{\phi}_2 = [z_1][z_2]$. Then Eq. (2) induces two rewriting rules

$$[[z_1]z_2]\rightarrow_\phi_1 [z_1][z_2] \text{ and } [z_1][z_2]\rightarrow_\phi_2 [z_1][z_2].$$

We have

$$w = [[z_1]z_2] \rightarrow_\phi_1 [z_1][z_2] \text{ and } w = [[z_1]z_2] \rightarrow_\phi_2 [z_1][z_2].$$

Again, since $[[z_1]z_2]$ and $[z_1][z_2]$ are different normal forms, $\Pi_{S\phi}(Z)$ is not confluent.

In summary, for the set $Z = \{z_1, z_2\}$, there is no $\Pi_{S\phi}(Z)$ such that $\Phi$ is confluent on $Z$ with respect to $S\phi(Z)$. So $\Phi$ is not convergent.
2.3. Rota’s Classification Problem via Gröbner-Shirshov bases. In this subsection, we give the definitions of Gröbner-Shirshov and potentially Gröbner-Shirshov systems of OPIs. Let us first recall some background on Gröbner-Shirshov bases. See [8, 16] for further details.

**Definition 2.15.** Let \( Z \) be a set, \( \leq \) a linear order on \( \mathcal{M}(Z) \) and \( f \in k\mathcal{M}(Z) \).

(a) Let \( f \notin k \). The leading monomial of \( f \), denoted by \( \overline{f} \), is the largest monomial appearing in \( f \). The leading coefficient of \( f \), denoted by \( c_f \), is the coefficient of \( \overline{f} \) in \( f \). We call \( f \) monic with respect to \( \leq \) if \( c_f = 1 \).

(b) If \( f \in k \) (including the case \( f = 0 \)), we define the leading monomial of \( f \) to be 1 and the leading coefficient of \( f \) to be \( c_f = f \).

(c) A subset \( S \subseteq k\mathcal{M}(Z) \) is called monicized with respect to \( \leq \) if each element of \( S \) is replaced by its quotient over the coefficient of its leading monomial, and hence is monic.

**Definition 2.16.** Let \( Z \) be a set. A monomial order on \( \mathcal{M}(Z) \) is a well-order \( \leq \) on \( \mathcal{M}(Z) \) such that

\[
(3) \quad u < v \implies q|_u < q|_v \quad \text{for all } u, v \in \mathcal{M}(Z) \text{ and } q \in \mathcal{M}^+(Z).
\]

We denote \( u < v \) if \( u \leq v \) but \( u \neq v \).

Since \( \leq \) is a well-order, it follows from Eq. (3) that \( 1 \leq u \) and \( u < \lfloor u \rfloor \) for all \( u \in \mathcal{M}(Z) \).

**Remark 2.17.** If there is a linear order \( \leq \) on \( \mathcal{M}(Z) \), then in Definition 2.14, we can take \( \overline{s} \) as the leading monomial \( \overline{s} \) of \( s \) with respect to \( \leq \). We call \( \Pi_S = \Pi_S^{\overline{\mathcal{M}}} = \{ q|_{\overline{s}} \rightarrow q|_{R(s)} \mid s \in S, \ q \in \mathcal{M}^+(Z) \} \subseteq \mathcal{M}(Z) \times k\mathcal{M}(Z) \) the term-rewriting system from \( \leq \).

Let \( f \in \mathcal{M}(Z) \) with \( f \neq 1 \). Then \( f \) can be uniquely written as a product \( f_1 \cdots f_n \), where \( n \leq 1 \) and \( f_i \in Z \cup \{\mathcal{M}(Z)\} \) for \( 1 \leq i \leq n \). We call \( n \) the breadth of \( f \), denoted by \( |f| \). If \( f = 1 \), we define \( |f| = 0 \).

**Definition 2.18.** Let \( \leq \) be a monomial order on \( \mathcal{M}(Z) \) and \( f, g \in k\mathcal{M}(Z) \) be monic.

(a) If there are \( w, u, v \in \mathcal{M}(Z) \) such that \( w = \overline{f}u = v\overline{g} \) with \( \max\{|\overline{f}|, |\overline{g}|\} < |w| < |\overline{f}| + |\overline{g}| \), we call

\[
(f, g)^w_{uv} := fu - vg
\]

the intersection composition of \( f \) and \( g \) with respect to \( w \).

(b) If there are \( w \in \mathcal{M}(Z) \) and \( q \in \mathcal{M}^+(Z) \) such that \( w = \overline{f} = q_{\overline{f}} \), we call

\[
(f, g)^q_w := f - q|_g
\]

the including composition of \( f \) and \( g \) with respect to \( w \).

**Definition 2.19.** Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathcal{M}(Z) \).

(a) An element \( f \in k\mathcal{M}(Z) \) is called trivial modulo \( (S, w) \) if

\[
f = \sum c_iq_i|_{\overline{s}_i} \text{ with } q_i|_{\overline{s}_i} < w, \text{ where } c_i \in k, q_i \in \mathcal{M}^+(Z), s_i \in S.
\]

(b) Let \( S \subseteq k\mathcal{M}(Z) \). Then \( S \) is called a Gröbner-Shirshov basis in \( k\mathcal{M}(Z) \) with respect to \( \leq \) if, for all pairs \( f, g \in S \) monicized with respect to \( \leq \), every intersection composition of the form \( (f, g)^w_{uv} \) is trivial modulo \( (S, w) \), and every including composition of the form \( (f, g)^q_w \) is trivial modulo \( (S, w) \).
Theorem 2.20. (Composition-Diamond Lemma [3, 16]) Let \( Z \) be a set, \( \leq \) a monomial order on \( \mathfrak{M}(Z) \) and \( S \subseteq k\mathfrak{M}(Z) \). Then the following conditions are equivalent.

(a) \( S \) is a Gröbner-Shirshov basis in \( k\mathfrak{M}(Z) \).

(b) Let \( \eta: k\mathfrak{M}(Z) \rightarrow k\mathfrak{M}(Z)/\text{Id}(S) \) be the quotient homomorphism of \( k \)-spaces. Denote

\[
\text{Irr}(S) := \mathfrak{M}(Z) \setminus \{q|_{\eta} \mid s \in S\}.
\]

As a \( k \)-space, \( k\mathfrak{M}(Z) = k\text{Irr}(S) \oplus \text{Id}(S) \) and \( \eta(\text{Irr}(S)) \) is a \( k \)-basis of \( k\mathfrak{M}(Z)/\text{Id}(S) \).

Definition 2.21. Let \( X \) be a set and \( \Phi \subseteq k\mathfrak{M}(X) \) a system of OPIs. Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathfrak{M}(Z) \).

(a) We call \( \Phi \) Gröbner-Shirshov on \( Z \) with respect to \( \leq \) if \( S_{\phi}(Z) \) is a Gröbner-Shirshov basis in \( k\mathfrak{M}(Z) \) with respect to \( \leq \).

(b) We call \( \Phi \) potentially Gröbner-Shirshov on \( Z \) with respect to \( \leq \) if there is a superset \( \Phi' \subseteq k\mathfrak{M}(X) \) of \( \Phi \) such that \( \text{Id}(S_{\phi}(Z)) = \text{Id}(S_{\phi'}(Z)) \) and \( \Phi' \) is Gröbner-Shirshov on \( Z \) with respect to \( \leq \).

Definition 2.22. Let \( X \) be a set and \( \Phi \subseteq k\mathfrak{M}(X) \) a system of OPIs.

(a) We call \( \Phi \) Gröbner-Shirshov (resp. potentially Gröbner-Shirshov) if, for each set \( Z \), there is a monomial order \( \leq \) on \( \mathfrak{M}(Z) \) such that \( \Phi \) is Gröbner-Shirshov (resp. potentially Gröbner-Shirshov) on \( Z \) with respect to \( \leq \).

(b) A \( \Phi \)-operator \( P \) is called Gröbner-Shirshov (resp. potentially Gröbner-Shirshov) if \( \Phi \) is.

Example 2.23. A differential type OPI [13], defining a differential type operator \( d = [\ ] \), is

\[
\phi = \phi(x_1, x_2) = [x_1, x_2] - N(x_1, x_2),
\]

with \( N(x_1, x_2) \in k\mathfrak{M}(x_1, x_2) \) satisfying certain conditions, to be recalled in Example 4.4. By [16, Theorem 5.7], \( S_{\phi}(Z) \) is a Gröbner-Shirshov basis of \( \text{Id}(S_{\phi}(Z)) \) with respect to a monomial order. Hence a differential type OPI is Gröbner-Shirshov. This fact will be proved directly in Example 4.4.

Example 2.24. A Rota-Baxter type OPI [13], defining a Rota-Baxter type operator, is

\[
\phi = \phi(x_1, x_2) = [x_1][x_2] - [B(x_1, x_2)],
\]

with \( B(x_1, x_2) \in k\mathfrak{M}(x_1, x_2) \) satisfying certain conditions detailed in Example 4.5. It was shown in [13, Corollary 3.13, Theorem 4.9], and again in Example 4.5, that \( S_{\phi}(Z) \) is a Gröbner-Shirshov basis of \( \text{Id}(S_{\phi}(Z)) \) with respect to a monomial order. Hence a Rota-Baxter type OPI is Gröbner-Shirshov.

Example 2.25. As shown below in Theorem 4.6, a modified Rota-Baxter type OPI is Gröbner-Shirshov. By [13, Theorems 2.41, 3.10], the system of (two-sided) averaging OPIs defined in Eq. (2) is potentially Gröbner-Shirshov.
We now propose another reformulation of Rota’s Classification Problem.

**Problem 2.26. (Rota’s Classification Problem via Gröbner-Shirshov bases)** Determine all Gröbner-Shirshov and potentially Gröbner-Shirshov systems of OPIs.

### 3. Relationship between reformulations of Rota’s Classification Problem

In this section, we establish the relationship between reformulations of Rota’s Classification Problem.

#### 3.1. Term-rewriting systems

We recall some basic results from [13] for term-rewriting systems. We will need the following Newman’s lemma on rewriting systems.

**Lemma 3.1.** ([13], Lemma 2.7.2) A terminating rewriting system is confluent if and only if it is locally confluent.

The next results will also be used later.

**Lemma 3.2.** ([13], Proposition 2.18, Theorem 2.20) Let $V$ be a $k$-space with a given $k$-basis $W$, and let $\Pi$ be a simple term-rewriting system on $V$ with respect to $W$.

(a) $(f - g) \to_{\Pi} 0$ implies $f \downarrow_{\Pi} g$ for all $f, g \in V$.

(b) If $\Pi$ is confluent, then for all $f, g, h \in V$,

$$f \downarrow_{\Pi} g, \ g \downarrow_{\Pi} h \implies f \downarrow_{\Pi} h.$$  

(c) If $\Pi$ is confluent, then

$$f \downarrow_{\Pi} g, \ f' \downarrow_{\Pi} g' \implies (f + f') \downarrow_{\Pi} (g + g') \quad \forall f, g, f', g' \in V.$$  

(d) If $\Pi$ is confluent, then, for all $m \geq 1$ and $f_1, \ldots, f_m, g_1, \ldots, g_m \in V$,

$$f_i \downarrow_{\Pi} g_i \quad (1 \leq i \leq m), \text{ and } \sum_{i=1}^{m} g_i = 0 \implies \left(\sum_{i=1}^{m} f_i\right) \to_{\Pi} 0.$$  

**Remark 3.3.** If $\Pi$ is confluent and $f \downarrow_{\Pi} g$, together with the fact $-g \downarrow_{\Pi} -g$, we get $f - g \to_{\Pi} 0$ by Lemma 3.2(d).

The following is a stronger condition than locally confluence.

**Definition 3.4.** Let $V$ be a $k$-spaces with a $k$-basis $W$ and let $\Pi$ be a simple term-rewriting system on $V$ with respect to $W$.

(a) A local base-fork is a fork $(kt \to_{\Pi} kv_1, kt \to_{\Pi} kv_2)$, where $k \in k^*$ and $t \to v_1, t \to v_2 \in \Pi$.

The term-rewriting system $\Pi$ is locally base-confluent if for every local base-fork $(kt \to_{\Pi} kv_1, kt \to_{\Pi} kv_2)$, we have $k(v_1 - v_2) \to_{\Pi} 0$.

(b) We say that $\Pi$ is compatible with a linear order $\leq$ on $W$ if $\overline{v} < t$ for each $t \to v \in \Pi$.

**Lemma 3.5.** ([13], Lemma 2.22) Let $V$ be a $k$-space with a $k$-basis $W$ and let $\Pi$ be a term-rewriting system on $V$ which is compatible with a well order $\leq$ on $W$. If $\Pi$ is locally base-confluent, then it is locally confluent.

**Lemma 3.6.** Let $V$ be a $k$-space with a given $k$-basis $W$, and let $\Pi$ be a simple term-rewriting system on $V$ with respect to $W$. Let $f, g \in V$. If $f \to_{\Pi} g$, then $kf \to_{\Pi} kg$ for any $k \in k$. 

Proof. If \( f = g \) or \( k = 0 \), then \( kf = kg \) and \( kf \rightarrow^*_{\Pi} kg \). Suppose \( f \neq g \) and \( k \neq 0 \). Let \( n \geq 1 \) be the minimum step that \( f \) rewrites to \( g \) and

\[
f := f_0 \rightarrow_{\Pi} f_1 \rightarrow_{\Pi} \cdots \rightarrow_{\Pi} f_n := g.
\]

We prove the result by induction on \( n \). If \( n = 1 \), we may write

\[
f = ct + (-R_t(f)) \quad \text{and} \quad g = cv - R_t(f),
\]

where \( c \in k^\times \) and \( t \rightarrow v \in \Pi \). Then

\[
 kf = kct + (-kR_t(f)) \quad \text{and} \quad kg = kcv - kR_t(f).
\]

Since \( k, c \neq 0 \) and \( k \) is a field by our hypothesis, \( kc \neq 0 \) and so \( kf \rightarrow^*_{\Pi} kg \). Assume that the result is true for \( n \leq m \) and consider the case of \( n = m + 1 \). Then by the induction hypothesis, we have \( kf_0 \rightarrow^*_{\Pi} kf_1 \) and \( kf_1 \rightarrow^*_{\Pi} kf_n \). Hence by the transitivity of \( \rightarrow^*_{\Pi} \) we have \( kf = kf_0 \rightarrow^*_{\Pi} kf_n = kg \), as required.

\[\square\]

The following concepts are adapted from general abstract rewriting systems [\[5\], Definition 1.1.6].

**Definition 3.7.** Let \( V \) be a \( k \)-spaces with a \( k \)-basis \( W \) and let \( \Pi \) be a simple term-rewriting system on \( V \) with respect to \( W \). Let \( Y \subseteq W \) and \( \Pi_{kY} \subseteq Y \times kY \). We call \( \Pi_{kY} \) a **sub-term-rewriting system** of \( \Pi \) on \( kY \) with respect to \( Y \), denoted by \( \Pi_{kY} \subseteq \Pi \), if

(a) \( \Pi_{kY} \) is the restriction of \( \Pi \), i.e., for any \( f, g \in kY \), \( f \rightarrow_{\Pi_{kY}} g \Leftrightarrow f \rightarrow_{\Pi} g \).

(b) \( kY \) is closed under \( \Pi \), i.e., for any \( f \in kY \) and any \( g \in V \), \( f \rightarrow_{\Pi} g \) implies \( g \in kY \).

The following result characterizes the sub-term-rewriting system when \( \Pi_{kY} = \Pi \cap (Y \times kY) \).

**Proposition 3.8.** Let \( V \) be a \( k \)-space with a \( k \)-basis \( W \) and let \( \Pi \) be a simple term-rewriting system on \( V \) with respect to \( W \). Let \( Y \subseteq W \) and \( \Pi_{kY} := \Pi \cap (Y \times kY) \). Then \( \Pi_{kY} \) is a sub-term-rewriting system of \( \Pi \) on \( kY \) with respect to \( Y \) if and only if \( kY \) is closed under \( \Pi \) in the sense of Definition 3.7.

**Proof.** (\( \Rightarrow \)) This direction follows from Definition 3.7.

(\( \Leftarrow \)) With Item (\ref{case}) of Definition 3.7 being our hypothesis, we only need to show that Item (\ref{case}) is valid, that is, \( \Pi_{kY} \) is the restriction of \( \Pi \) to \( kY \). Let \( f, g \in kY \) with \( f \rightarrow_{\Pi_{kY}} g \). Since \( \Pi_{kY} = \Pi \cap (Y \times kY) \subseteq \Pi \), we have \( f \rightarrow_{\Pi} g \). Conversely, suppose \( f \rightarrow_{\Pi} g \). Write \( f = ct + f_1 \) and \( g = cv + f_1 \), where \( c \in k^\times \), \( t \in Y \), \( v \in V \), \( f_1 \in kY \) and \( t \rightarrow v \in \Pi \). Since \( g \in kY \) and \( f_1 \in kY \), we have \( cv \in kY \). Since \( W \) is a \( k \)-basis of \( V \) and \( Y \subseteq W \), we may write

\[
v = \sum_i c_i y_i + \sum_j d_j x_j, \quad \text{where} \quad c_i, d_j \in k, y_i \in Y, x_j \in W \setminus Y.
\]

Then

\[
cv = \sum_i c c_i y_i + \sum_j c d_j x_j \in kY \quad \text{and} \quad \sum_j c d_j x_j \in kY,
\]

and so \( cd_j = 0 \) for each \( j \). Since \( k \) is a field and \( c \neq 0 \), we get \( d_j = 0 \) for each \( j \), that is, \( v \in kY \). Thus \( t \rightarrow v \in \Pi_{kY} \) and so \( ct + f_1 \rightarrow_{\Pi_{kY}} cv + f_1 \), as required.

\[\square\]

The term-rewriting system \( \Pi_{S_{\mu}(Z)} \) from a monomial order is simple. To show this, we need the following fact.

**Lemma 3.9.** Let \( Z \) be a set and \( \leq \) a monomial order on \( \mathcal{M}(Z) \). If \( q|_u = q|_v \) with \( q \in \mathcal{M}^*(Z) \) and \( u, v \in \mathcal{M}(Z) \), then \( u = v \).
Proof. We prove the result by induction on the order of \(q|_u \geq u\). For the initial step, we have \(q|_u = u\). So \(q = \star\) and \(u = q|_u = q|_0 = v\). For the induction step, depending on the first symbol occurring in \(q\) is a variable in \(Z\), or a \(\star\), or a bracket, we have the following cases to consider.

**Case 1.** \(q = xp\) for some \(x \in Z\) and \(p \in \mathfrak{M}^*(Z)\). Then

\[
xp|_u = q|_u = q|_0 = xp|_0,
\]

and so \(p|_u = p|_0\). Since \(\leq\) is a monomial order, we have \(q|_u > p|_u\). By the induction hypothesis and \(p|_u = p|_0\), we have \(u = v\).

**Case 2.** \(q = \star w\) and \(w \in \mathfrak{M}(Z)\). Then \(uw = q|_u = q|_0 = vw\) and so \(u = v\).

**Case 3.** The first symbol in \(q\) is a bracket. In this case, we have two subcases.

**Case 3.1.** \(q = [p]w\) for some \(p \in \mathfrak{M}^*(Z)\) and \(w \in \mathfrak{M}(Z)\). Then

\[
[p|_u]w = q|_u = q|_0 = [p|_0]w
\]

and so \(p|_u = p|_0\). Since \(\leq\) is a monomial order, we have \(q|_u > p|_u\). By the induction hypothesis and \(p|_u = p|_0\), we get \(u = v\).

**Case 3.2.** \(q = [w]p\) for some \(w \in \mathfrak{M}(Z)\) and \(p \in \mathfrak{M}^*(Z)\). Then

\[
[w]p|_u = q|_u = q|_0 = [w]p|_0.
\]

Thus \(p|_u = p|_0\). Again since \(\leq\) is a monomial order, we get \(q|_u > p|_u\). By the induction hypothesis and \(p|_u = p|_0\), we obtain \(u = v\). This completes the proof. \(\square\)

**Lemma 3.10.** Let \(Z\) be a set and \(\leq\) a monomial order on \(\mathfrak{M}(Z)\). The \(\Pi_{S,\leq}(Z)\) from \(\leq\) is a simple term-rewriting system on \(k\mathfrak{M}(Z)\).

**Proof.** We only need to show that \(q|_{\phi(u)} + q|_{\phi(u)}x\) for any \(q \in \mathfrak{M}^*(Z)\) and \(\phi(u) \in S_{\phi}(Z)\). If \(R(\phi(u)) = 0\), there is nothing to prove. Suppose \(R(\phi(u)) \neq 0\) and write

\[
R(\phi(u)) = \sum_{i=1}^{m} c_i R_i^{\phi(u)},
\]

where \(c_i \in k^\times\) and \(R_i^{\phi(u)}\), \(1 \leq i \leq m\), are mutually distinct monomials of \(R(\phi(u))\). If \(q|_{\phi(u)} + q|_{\phi(u)}x\) fails, from Definition 3.4, there is some \(1 \leq i \leq m\) such that \(q|_{\phi(u)} = q|_{R_i^{\phi(u)}}\). So \(\phi(u) = R_i^{\phi(u)}\) by Lemma 3.3, contradicting \(\phi(u) > R_i^{\phi(u)}\). \(\square\)

The following result gives a sufficient condition for terminating.

**Lemma 3.11.** \(\Pi_{S,\leq}(Z)\) is terminating.

**3.2. Gröbner-Shirshov OPIs and convergent OPIs.** In this subsection, we study the relationship between a Gröbner-Shirshov system of OPIs and a convergent system of OPIs. In terms of \(\star\)-bracketed words, the operated ideals in \(k\mathfrak{M}(Z)\) can be characterized [15, 16] as follows.

**Lemma 3.12.** \(\Pi_{S,\leq}(Z)\) is terminating.

**Lemma 3.13.** Let \(Z\) be a set, and let \(\leq\) be a linear order on \(\mathfrak{M}(Z)\). Let \(S \subseteq k\mathfrak{M}(Z)\) be monicized with respect to \(\leq\), and let \(\Pi_S\) be the term-rewriting system from \(\leq\). If \(f \rightarrow_{\Pi_S} g\) for \(f, g \in k\mathfrak{M}(Z)\), then \(f - g \in \text{Id}(S)\).
**Proof.** If \( f = g \), then \( f - g = 0 \in \text{Id}(S) \). Suppose \( f \neq g \). Let \( n \geq 1 \) be the minimum number such that \( f \) rewrites to \( g \) by \( n \) steps. We prove the result by induction on \( n \). If \( n = 1 \), then \( f \rightarrow_{\Pi S} g \). Write

\[
f = cq_{s}f_1 + f_1 \rightarrow_{\Pi S} cq_{\Pi(s)} + f_1 = g,
\]

where \( c \in k^\times \), \( s \in S \) and \( f_1 \in k\mathcal{M}(Z) \). Then

\[
f - g = cq_{s} - cq_{\Pi(s)} = cq_{\Pi(s)} = cq_s \in \text{Id}(S).
\]

Assume that the result is true for \( n = m + 1 \) and consider the case of \( n = m + 1 \geq 2 \). Then we have \( f \rightarrow_{\Pi S} h \rightarrow_{\Pi S} g \) for some \( f \neq h \in k\mathcal{M}(Z) \). By the induction hypothesis, \( f - h \in \text{Id}(S) \) and \( h - g \in \text{Id}(S) \). Thus \( f - g \in \text{Id}(S) \), as required. \( \square \)

**Lemma 3.14.** Let \( Z \) be a set, and let \( \leq \) be a linear order on \( \mathcal{M}(Z) \). Let \( S \subseteq k\mathcal{M}(Z) \) be monicized with respect to \( \leq \), and let \( \Pi S \) be the term-rewriting system from \( \leq \).

(a) If \( \Pi S \) is confluent, then \( u \in \text{Id}(S) \) if and only if \( u \rightarrow_{\Pi S} 0 \).

(b) If \( \Pi S \) is confluent, then \( \text{Id}(S) \cap k\mathcal{Irr}(S) = 0 \).

(c) If \( \Pi S \) is terminating and \( \text{Id}(S) \cap k\mathcal{Irr}(S) = 0 \), then \( \Pi S \) is confluent.

(d) If \( \Pi S \) is terminating, then \( k\mathcal{M}(Z) = \text{Id}(S) + k\mathcal{Irr}(S) \), where \( \mathcal{Irr}(S) = \mathcal{M}(Z) \setminus \{cq_s \mid s \in S \} \).

**Proof.** Note that \( k\mathcal{Irr}(S) \) is precisely the set of normal forms for \( \Pi S \).

1. If \( u \rightarrow_{\Pi S} 0 \), then \( u \in \text{Id}(S) \) from Lemma 5.13. Conversely, let \( u \in \text{Id}(S) \). By Eq. (5), we have

\[
u = \sum_{i=1}^{k} c_i q_i s_i, \text{ where } c_i \in k^\times, s_i \in S, q_i \in \mathcal{M}^*(Z), 1 \leq i \leq k.
\]

For each \( s_i = \sum_{j} (-R(s_i)) \) with \( 1 \leq i \leq k \), we have

\[
c_i q_i s_i = c_i q_i - c_i q_i R(s_i) \rightarrow_{\Pi S} c_i q_i R(s_i) - c_i q_i R(s_i) = 0 \text{ and so } c_i q_i s_i \rightarrow_{\Pi S} 0.
\]

Since \( \Pi S \) is confluent, by Lemma 5.2, we have \( u = \sum_{i=1}^{k} c_i q_i s_i \rightarrow_{\Pi S} 0 \).

2. Suppose \( \text{Id}(S) \cap k\mathcal{Irr}(S) \neq 0 \). Let \( 0 \neq w \in \text{Id}(S) \cap k\mathcal{Irr}(S) \). Since \( w \in k\mathcal{Irr}(S) \), \( w \) is in normal form. On the other hand, from \( w \in \text{Id}(S) \) and Item 1, we have \( w \rightarrow_{\Pi S} 0 \). So \( w \) has two normal forms \( w \) and \( 0 \), contradicting that \( \Pi S \) is confluent.

3. Suppose to the contrary that \( \Pi S \) is not confluent. Since \( \Pi S \) is terminating, there is \( w \in k\mathcal{M}(Z) \) such that \( w \) has two distinct normal forms, say \( u \) and \( v \). Thus \( u, v \in k\mathcal{Irr}(S) \) and so \( u - v \in k\mathcal{Irr}(S) \). From Lemma 5.13, \( w - u \in \text{Id}(S) \) and \( w - v \in \text{Id}(S) \). Hence \( 0 \neq u - v \in \text{Id}(S) \cap k\mathcal{Irr}(S) \), a contradiction.

4. Let \( w \in k\mathcal{M}(Z) \). Since \( \Pi S \) is terminating, there is \( u \in k\mathcal{Irr}(S) \) such that \( w \rightarrow_{\Pi} u \). From Lemma 5.13, we have \( w - u \in \text{Id}(S) \) and so \( w \in \text{Id}(S) + k\mathcal{Irr}(S) \). \( \square \)

**Theorem 3.15.** Let \( Z \) be a set, and let \( \leq \) be a monomial order on \( \mathcal{M}(Z) \). Let \( S \subseteq k\mathcal{M}(Z) \) be monicized with respect to \( \leq \), and let \( \Pi S \) be the term-rewriting system from \( \leq \). Then the following statements are equivalent.

(a) \( \Pi S \) is convergent.

(b) \( \Pi S \) is confluent.

(c) \( \text{Id}(S) \cap k\mathcal{Irr}(S) = 0 \).

(d) \( \text{Id}(S) + k\mathcal{Irr}(S) = k\mathcal{M}(Z) \).

(e) \( S \) is a Gröbner-Shirshov basis in \( k\mathcal{M}(Z) \) with respect to \( \leq \).
3.14 we give the statement of the theorem and show that previously known examples \( \varphi \) in Section 4.2 prove that the modified Rota-Baxter OPI is Gröbner-Shirshov. The proof of the theorem is given of Gröbner-Shirshov OPIs can be easily verified by this theorem. As another application, we for

Clearly, Item (4) implies Item (1). The converse employs Item (2) in Lemma 3.14. Finally, the equivalence of Item (1) and Item (4) is obtained by Theorem 2.21. □

Now we are ready to give the relationship between the reformulations of Rota’s Classification Problem.

**Theorem 3.16.** Let \( \Phi \subseteq \mathbb{k}\mathfrak{m}(X) \) be a system of OPIs.

(a) For any set \( Z \) and any monomial order \( \leq \) on \( \mathfrak{m}(Z) \), \( \Phi \) is Gröbner-Shirshov on \( Z \) with respect to \( \leq \) if and only if \( \Phi \) is convergent on \( Z \) with respect to the orientation \( S_\Phi(Z) := \{ \phi(u) | \phi(u) \in S_\Phi(Z) \} \) from \( \leq \).

(b) If \( \Phi \) is Gröbner-Shirshov, then \( \Phi \) is convergent.

(c) If \( \Phi \) is potentially Gröbner-Shirshov, then \( \Phi \) is potentially convergent.

**Proof.** (4) Item (4) follows from applying Theorem 3.15 to \( S = S_\Phi(Z) \).

(1) Suppose that \( \Phi \) is Gröbner-Shirshov. By Definition 2.22, for any set \( Z \), there is a monomial order \( \leq \) on \( \mathfrak{m}(Z) \) such that \( \Phi \) is Gröbner-Shirshov on \( Z \) with respect to \( \leq \). By Item (4), \( \Phi \) is convergent on \( Z \) with respect to the orientation \( S_\Phi(Z) \) from \( \leq \) and so is convergent.

(2) Suppose \( \Phi \) is potentially Gröbner-Shirshov. From Definition 2.22, for any set \( Z \), there is a monomial order on \( \mathfrak{m}(Z) \) such that \( \Phi \) is potentially Gröbner-Shirshov on \( Z \) with respect to \( \leq \). By Definition 2.21, there is a superset \( \Phi' \subseteq \mathbb{k}\mathfrak{m}(X) \) of \( \Phi \) such that \( \text{Id}(S_\Phi(Z)) = \text{Id}(S_{\Phi'}(Z)) \) and \( \Phi' \) is Gröbner-Shirshov on \( Z \) with respect to \( \leq \). In view of Item (2), \( \Phi' \) is convergent on \( Z \) with respect to the orientation \( S_{\Phi'}(Z) \) from \( \leq \). Hence \( \Phi \) is potentially convergent. □

**Corollary 3.17.** Let \( \Phi \) be the system of (two-sided) averaging OPIs defined in Eq. (3). Then \( \Phi \) is not Gröbner-Shirshov.

**Proof.** By Proposition 2.14, \( \Phi \) is not convergent. From Theorem 3.16 (1), \( \Phi \) is not Gröbner-Shirshov. □

**Remark 3.18.** By [12, Theorems 2.41, 3.10], the system of averaging OPIs \( \Phi \) in Corollary 3.17 can be extended to a set of OPIs that is Gröbner-Shirshov. Thus \( \Phi \) is potentially Gröbner-Shirshov and hence is potentially convergent.

4. A sufficient condition for Gröbner-Shirshov OPIs

In this section, we provide a sufficient condition for an OPI to be Gröbner-Shirshov. In Section 4.1 we give the statement of the theorem and show that previously known examples of Gröbner-Shirshov OPIs can be easily verified by this theorem. As another application, we prove that the modified Rota-Baxter OPI is Gröbner-Shirshov. The proof of the theorem is given in Section 4.2.

4.1. Statement of the main theorem and examples. Like the differential operator and Rota-Baxter operator, many operators are defined by a single OPI. In this subsection, we consider a single OPI \( \phi \) and supply a method to prove that \( \phi \) is Gröbner-Shirshov.

Let \( \phi = \phi(x_1, \ldots, x_k) = \frac{d}{dx} - R(\phi) \in \mathbb{k}\mathfrak{m}(X) \) be an OPI. In the rest of this paper, we write \( \phi(x) \) for \( \phi(x_1, \ldots, x_k) \) in short. We call \( \phi(x) \) multiple linear (or totally linear) if \( \phi(x) \) is linear in each variable \( x_i \), \( 1 \leq i \leq k \). Let \( Z \) be a set. We say that an element \( f \in \mathbb{k}\mathfrak{m}(Z) \) is in \( \phi \)-normal form if no monomial of \( f \) contains any subword of the form \( \phi(u) \) with \( u \in \mathfrak{m}(Z)^k \).
Theorem 4.1. Let \( \phi(x) \in k\mathcal{M}(X) \) be a multi-linear OPI such that \( R(\phi(x)) \) is in \( \phi \)-normal form. Suppose that, for any set \( Z \), there is a monomial order \( \preceq \) on \( \mathcal{M}(Z) \), such that the following two conditions hold:

(a) if \( \phi(u), \phi(v) \in S_\phi(Z) \) are such that \( \overline{\phi(u)} = ab \) and \( \overline{\phi(v)} = bc \) for some \( a, b, c \in \mathcal{M}(Z) \) and \( u, v \in \mathcal{M}(Z)^k \), then \( R(\phi(u))c \downarrow_\phi aR(\phi(v)) \), where \( \Pi_\phi := \Pi_{S_\phi(Z)} \) is the term-rewriting system from \( \preceq \).

(b) if \( \phi(u) = q_1 \) for some \( \star \neq q \in \mathcal{M}^*(Z) \) and \( u, v \in \mathcal{M}(Z)^k \), then \( \overline{\phi(v)} \) is a subword of some \( u_i, 1 \leq i \leq k \).

Then \( \phi(x) \) is Gröbner-Shirshov, as is its defined operator.

We postpone the proof of Theorem 4.1 to Section 4.2 and first give some remarks and examples.

Remark 4.2. Condition (5) is a necessary condition for \( \phi(x) \) to be a Gröbner-Shirshov OPI. Indeed, let \( \phi(u), \phi(v) \in S_\phi(Z) \) with \( \overline{\phi(u)} = ab \) and \( \overline{\phi(v)} = bc \) for some \( a, b, c \in \mathcal{M}(Z) \). Since \( \phi(x) \) is Gröbner-Shirshov, \( S_\phi(Z) \) is a Gröbner-Shirshov basis by Definition 2.22. By Theorem 3.15, the term-rewriting system \( \Pi_\phi = \Pi_{S_\phi(Z)} \) from \( \preceq \) is confluent. So for the local fork

\[
\begin{align*}
(abc &= \phi(u)c \rightarrow_\phi R(\phi(u))c, \ abc = a\phi(v) \rightarrow_\phi aR(\phi(v))),
\end{align*}
\]

we have \( R(\phi(u))c \downarrow_\phi aR(\phi(v)) \).

Remark 4.3. As a counter-example of condition (5), consider \( \phi(x) = [\{x\}], q = [\star], u = [x] \) and \( v = x \). Then

\[
\overline{\phi(u)} = [\{u\}] = [[[x]]] = q_{[[x]]} = q_{[\phi(u)]}.
\]

But \( \overline{\phi(v)} = [[x]] \) is not a subword of \( u = [x] \).

However, Item (5) is not a necessary condition for \( \phi(x) \) to be a Gröbner-Shirshov OPI. For example, let \( \preceq \) be a monomial order on \( \mathcal{M}(Z) \) and \( \phi(x) = [[x]] \). Then we have a term-rewriting system from \( \preceq \)

\[
\Pi_{S_\phi(Z)} = \{q_{[[u]]} \rightarrow 0 \mid q \in \mathcal{M}^*(Z), u \in \mathcal{M}(Z)\},
\]

which is confluent. By Theorem 3.15, \( \phi(x) \) is a Gröbner-Shirshov OPI. But as explained just above, \( \phi(x) \) does not satisfy condition (5).

Example 4.4. (Differential type OPI) A differential type OPI [16], defining a differential type operator, is

\[
\phi(x_1, x_2) = [x_1, x_2] - N(x_1, x_2),
\]

where

(a) \( N(x_1, x_2) \) is multi-linear in \( x_1 \) and \( x_2 \);

(b) \( N(x_1, x_2) \) is in \( \phi(x_1, x_2) \)-normal form;

(c) For any set \( Z \) and \( u, v, w \in \mathcal{M}(Z) \setminus \{1\} \),

\[
N(uv, w) - N(u, vw) \rightarrow_\phi 0.
\]

We verify that, with respect the monomial order \( \preceq \) defined in [16], \( \phi(x_1, x_2) \) satisfies the conditions (5) and (7) in Theorem 4.1 and therefore is a Gröbner-Shirshov OPI. This gives another proof of [16, Theorem 5.7]. We begin with verifying the first condition. Let \( \Pi_\phi = \Pi_{S_\phi(Z)} \) be the term-rewriting system from \( \preceq \). Note that

\[
\overline{\phi(u_1, u_2)} = [u_1 u_2] \text{ and } R(\phi(u_1, u_2)) = N(u_1, u_2) \text{ for } u_1, u_2 \in \mathcal{M}(Z).
\]
Let $\phi(u_1, u_2)$ and $\phi(v_1, v_2)$ be in $S_\phi(Z)$ such that
\[
\phi(u_1, u_2) = ab \quad \text{and} \quad \phi(v_1, v_2) = bc \quad \text{for some} \quad u_1, u_2, v_1, v_2 \in \mathfrak{M}(Z) \setminus \{1\}, a, b, c \in \mathfrak{M}(Z).
\]
Then
\[
\phi(u_1, u_2) = [u_1u_2] = ab \quad \text{and} \quad \phi(v_1, v_2) = [v_1v_2] = bc.
\]
So
\[
(7) \quad a = c = 1 \quad \text{and} \quad b = [u_1u_2] = [v_1v_2].
\]
Note that
\[
\phi(u_1, u_2) = [u_1u_2] \to_\phi N(u_1, u_2) = R(\phi(u_1, u_2)) = R(\phi(u_1, u_2))c,
\]
\[
\phi(v_1, v_2) = [v_1v_2] \to_\phi N(v_1, v_2) = R(\phi(v_1, v_2)) = aR(\phi(v_1, v_2)).
\]
From Eq. (7), we have $u_1u_2 = v_1v_2$. If $u_1 = v_1$, then $u_2 = v_2$ and $\phi(u_1, u_2) = \phi(v_1, v_2)$. So
\[
R(\phi(u_1, u_2)) \downarrow_\phi R(\phi(v_1, v_2)), \quad R(\phi(u_1, u_2))c \downarrow_\phi aR(\phi(v_1, v_2)),
\]
by the fact that $a = c = 1$ in Eq. (7). Suppose $u_1 \neq v_1$. Since $u_1u_2 = v_1v_2$, either $u_1 = v_1v$ or $v_1 = u_1v$ for some $v \in \mathfrak{M}(Z) \setminus \{1\}$. In the former case, we have $u_1u_2 = v_1v_2 = v_1v$ and so $v_1u_2 = v_2$. From Eqs. (6) and (3),
\[
R(\phi(u_1, u_2))c - aR(\phi(v_1, v_2)) = N(u_1, u_2) - N(v_1, v_2) = N(v_1, v_2) - N(v_1, v_2) \to_\phi 0.
\]
Using Lemma 3.2(2),
\[
R(\phi(u_1, u_2))c \downarrow_\phi aR(\phi(v_1, v_2)).
\]
In the latter case of $v_1 = u_1v$, we get $u_2 = vv_2$ and
\[
aR(\phi(v_1, v_2)) - R(\phi(u_1, u_2))c = N(v_1, v_2) - N(u_1, u_2) = N(u_1, v_2) - N(u_1, v_2) \to_\phi 0.
\]
So
\[
aR(\phi(v_1, v_2)) \downarrow_\phi R(\phi(u_1, u_2))c.
\]
To verify condition (3) in Theorem 4.1, let
\[
[u_1u_2] = \phi(u_1, u_2) = q_{\phi(u_1, u_2)} = q_{[v_1v_2]}
\]
for some $\star \neq q \in \mathfrak{M}^*(Z)$ and $u_1, u_2, v_1, v_2 \in \mathfrak{M}(Z) \setminus \{1\}$. Since $q \neq \star, [u_1u_2] \neq [v_1v_2]$ and so $[v_1v_2]$ is a subword of $u_1u_2$. Since the breadth of $[v_1v_2]$ is 1, $[v_1v_2]$ is a subword of $u_1$ or $u_2$, as needed.

**Example 4.5. (Rota-Baxter type OPI)** A Rota-Baxter type OPI [13], defining a Rota-Baxter type operator, is
\[
\phi(x_1, x_2) = [x_1][x_2] - [B(x_1, x_2)],
\]
where $B(x_1, x_2)$ satisfies
\begin{enumerate}
  \item[(a)] $B(x_1, x_2)$ is multi-linear in $x_1$ and $x_2$;
  \item[(b)] $B(x_1, x_2)$ is in $\phi(x_1, x_2)$-normal form;
  \item[(c)] The term-rewriting system $\Pi_{S_\phi(Z)}$ is terminating;
  \item[(d)] For any set $Z$ and $u, v, w \in \mathfrak{M}(Z)$,
\end{enumerate}
\[
(9) \quad B(B(u, v), w) - B(u, B(v, w)) \to_\phi 0.
\]
We show that $\phi(x_1, x_2)$ satisfies the two conditions in Theorem 4.1 with respect the monomial order $\leq_{db}$ defined in [13] and therefore is a Gröbner-Shirshov OPI. This gives another proof of [13, Theorem 4.9]. Let $\Pi_\phi = \Pi_{S_\phi(Z)}$ be the term-rewriting system from $\leq_{db}$. To verify condition (3) in Theorem 4.1, note that
\[
\phi(u_1, u_2) = [u_1][u_2] \quad \text{and} \quad R(\phi(u_1, u_2)) = [B(u_1, u_2)] \quad \text{for} \quad u_1, u_2 \in \mathcal{M}(Z).
\]
Let $\phi(u_1, u_2)$ and $\phi(v_1, v_2)$ be in $S_\phi(Z)$ such that
\[
\phi(u_1, u_2) = ab \quad \text{and} \quad \phi(v_1, v_2) = bc \quad \text{for some} \quad u_1, u_2, v_1, v_2, a, b, c \in \mathcal{M}(Z).
\]
Then
\[
\phi(u_1, u_2) = [u_1][u_2] = ab \quad \text{and} \quad \phi(v_1, v_2) = [v_1][v_2] = bc.
\]
Thus
\[
(10) \quad a = [u_1], b = [u_2], c = [v_2] \quad \text{and} \quad u_2 = v_1.
\]
So
\[
R(\phi(u_1, u_2))c = [B(u_1, u_2)][v_2] \rightarrow_\phi [B(B(u_1, u_2), v_2)],
\]
\[
aR(\phi(v_1, v_2)) = [u_1][B(u_2, v_2)] \rightarrow_\phi [B(u_1, B(u_2, v_2))].
\]
It follows from Eq. (9) that
\[
R(\phi(u_1, u_2))c - aR(\phi(v_1, v_2)) = [B(B(u_1, u_2), v_2) - B(u_1, B(u_2, v_2))] \rightarrow_\phi 0.
\]
By Lemma 4.2[4], we have
\[
R(\phi(u_1, u_2))c \downarrow_\phi aR(\phi(v_1, v_2)).
\]
Hence condition (3) in Theorem 4.1 holds. For condition (7) in Theorem 4.1, let
\[
[u_1][u_2] = \phi(u_1, u_2) = q_{\phi(v_1, v_2)} = q_{[v_1][v_2]}
\]
for some $\star \neq q \in \mathcal{M}^*(Z)$ and $u_1, u_2, v_1, v_2 \in \mathcal{M}(Z)$. Since $q \neq \star$, $[u_1][u_2] \neq [v_1][v_2]$ and so $[v_1][v_2]$ is a subword of $[u_1]$ or $[u_2]$. Since the breadth of $[u_1]$ is 1 and the breadth of $[v_1][v_2]$ is 2, $[u_1] \neq [v_1][v_2]$. Similarly, $[u_2] \neq [v_1][v_2]$. Hence $[v_1][v_2]$ is a subword of $u_1$ or $u_2$, as required.

We finally give an application to an OPI that has been been considered in the context of Rota’s Classification Problem before. The modified Rota-Baxter OPI of weight $\lambda$ is
\[
\phi(x_1, x_2) = [x_1][x_2] - [x_1][x_2] - [x_1][x_2] - \lambda x_1 x_2, \quad \text{where} \quad \lambda \in k.
\]
When $\lambda = -\mu^2$, this gives [17]
\[
P(x_1)P(x_2) = P(x_1)P(x_2) + P(P(x_1)x_2) - \mu^2 x_1 x_2,
\]
as an associative analog of the modified classical Yang-Baxter equation on Lie algebras [28]. Note the subtle difference between this operator and the Rota-Baxter operator.

**Theorem 4.6.** The modified Rota-Baxter OPI is Gröbner-Shirshov.

**Proof.** For the proof, we verify that the OPI satisfies the conditions in Theorem 4.1 for the monomial order $\leq_{db}$ defined in [13]. Let $\Pi_\phi = \Pi_{S_\phi(Z)}$ be the term-rewriting system from $\leq_{db}$. With the order, we have
\[
\phi(u_1, u_2) = [u_1][u_2] \quad \text{and} \quad R(\phi(u_1, u_2)) = [u_1][u_2] + [u_1][u_2] + \lambda u_1 u_2 \quad \text{for} \quad u_1, u_2 \in \mathcal{M}(Z).
\]
Since \( \phi(u_1, u_2) \) has the same leading monomial as the one for Rota-Baxter type operators, by the same argument as for Example 4.3, condition (3) in Theorem 4.1 holds. Now we show that condition (7) is also fulfilled. With notations in Example 4.3 and from Eq. (10), we have
\[
R(\phi(u_1, u_2))c = ([u_1][u_2]] + [[u_1]u_2] + \lambda u_1 u_2[2]]
\]
and
\[
aR(\phi(v_1, v_2)) = [u_1]R(\phi(u_2, v_2)) = [u_1]([u_2][v_2]] + [[u_2]v_2] + \lambda u_2 v_2).
\]
On the one hand, we have
\[
R(\phi(u_1, u_2))c
= [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda u_1 u_2[2] = [u_1][u_2][v_2] + (|[u_1]u_2][v_2] + \lambda u_1 u_2[v_2])
\]
\[
\rightarrow [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda u_1 u_2[2] + [u_1][u_2][v_2] + \lambda u_1 u_2[v_2]
= [u_1][u_2][v_2] + (|[u_1]u_2][v_2] + [u_1][u_2][v_2] + \lambda u_1 u_2[v_2] + \lambda u_1 u_2[v_2])
\]
\[
\rightarrow [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda u_1 u_2[2] + [u_1][u_2][v_2] + \lambda u_1 u_2[v_2]
+ [u_1][u_2][v_2] + \lambda u_1 u_2[v_2] + \lambda u_1 u_2[v_2]
\]
\[
\rightarrow [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda u_1 u_2[2] + [u_1][u_2][v_2] + \lambda u_1 u_2[v_2]
+ \lambda u_1 u_2[v_2] + [u_1][u_2][v_2] + \lambda u_1 u_2[v_2]
\]
On the other hand, we have
\[
aR(\phi(v_1, v_2))
= [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2 = [u_1][u_2][v_2] + ([u_1][u_2][v_2] + \lambda [u_1]u_2v_2)
\]
\[
\rightarrow [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2 + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2
= [u_1][u_2][v_2] + ([u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2 + \lambda [u_1]u_2v_2)
\]
\[
\rightarrow [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2 + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2
+ [u_1][u_2][v_2] + \lambda [u_1]u_2v_2 + \lambda [u_1]u_2v_2)
\]
\[
\rightarrow [u_1][u_2][v_2] + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2 + [u_1][u_2][v_2] + \lambda [u_1]u_2v_2
+ [u_1][u_2][v_2] + \lambda [u_1]u_2v_2 + \lambda [u_1]u_2v_2.
\]
Hence
\[
R(\phi(a, b))c \downarrow aR(\phi(b, c))
\]
and so condition (7) is verified. This completes the proof. \(\square\)

As a consequence, we obtain a construction of free modified Rota-Baxter algebras. For a set \( Z \), denote
\[
\mathcal{R}(Z) := \mathcal{M}(Z) \setminus \{q|_{[u_1][v]} | u, v \in \mathcal{M}(Z)\} = \mathcal{M}(Z) \setminus \{q|_{s} | s \in S_\phi(Z)\} =: \text{Irr}(S_\phi(Z)),
\]
where \( S_\phi(Z) \) is defined in Eq. (11).
Corollary 4.7. Let $Z$ be a set. We have the following module isomorphism
\[ k\mathfrak{m}(Z)/\operatorname{Id}(S_{\phi}(Z)) \cong k\mathfrak{m}(Z). \]

More precisely,
\[ k\mathfrak{m}(Z) = \operatorname{Id}(S_{\phi}(Z)) \oplus k\mathfrak{m}(Z). \]

Proof. This follows from Theorems 2.20 and 1.6. □

4.2. The proof of Theorem 4.1. Before starting the proof of Theorem 4.1, we recall the following concepts [29].

Definition 4.8. Let $Z$ be a set. The particular location of the subword $u$ in the word $w$ under the substitution $q|_u$ is called the placement of $u$ in $w$ by $q$, denoted by $(u, q)$ for distinction.

A subword $u$ may appear at multiple locations (and hence have distinct placements using distinct $q$’s) in a bracketed word $w$. For example, there are two placements of $x$ in $w = x[x] \in \mathfrak{m}(x)$, given by $(x, q_1)$ and $(x, q_2)$ where $q_1 = x[x]$ and $q_2 = x[x]$.

Definition 4.9. Let $Z$ be a set and $w \in \mathfrak{m}(Z)$ such that
\[ q_1|_{u_1} = w = q_2|_{u_2} \text{ for some } u_1, u_2 \in \mathfrak{m}(Z), q_1, q_2 \in \mathfrak{m}^*(Z). \]

The two placements $(u_1, q_1)$ and $(u_2, q_2)$ are called

(a) separated if there exist $p \in \mathfrak{m}^* \setminus \{1\}(Z)$ and $a, b \in \mathfrak{m}(Z)$ such that $q_1|_{\star_1} = p|_{\star_1, b}$, $q_2|_{\star_2} = p|_{a, \star_2}$, and $w = p|_{a, b};$

(b) nested if there exists $q \in \mathfrak{m}^*(Z)$ such that either $q_2 = q_1|_q$ or $q_1 = q_2|_q;$

(c) intersecting if there exist $q \in \mathfrak{m}^*(Z)$ and $a, b, c \in \mathfrak{m}(Z) \setminus \{1\}$ such that $w = q_{abc}$ and either
   (i) $q_1 = q|_{a*}$ and $q_2 = q|_{a*}$; or
   (ii) $q_1 = q|_{a*}$ and $q_2 = q|_{a*}$.

Proposition 4.10. ([29, Theorem 4.11]) Let $Z$ be a set and $w \in \mathfrak{m}(Z)$. Any two placements $(u_1, q_1)$ and $(u_2, q_2)$ in $w$ are either separated or nested or intersecting.

Now we are ready for the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\phi = \phi(x)$ and $\Pi_{\phi}$ the term-rewriting system from $\leq$. We prove the result by showing that $\phi$ is Gröbner-Shirshov with respect to $\leq$. By Theorem 3.16 (2), it suffices to prove that $\phi$ is convergent on $Z$ with respect to the orientation from $\leq$, that is, $\Pi_{\phi}$ is convergent by Definition 3.13 (2).

Since $\leq$ is a monomial order on $\mathfrak{m}(Z)$, $\Pi_{\phi}$ is terminating by Lemma 3.11. From Lemma 3.1, we are left to show that $\Pi_{\phi}$ is locally confluent. Since
\[ R(\phi(u)) < \phi(u) \quad \text{and} \quad q|_{R(\phi(u))} < q|_{\phi(u)} \quad \text{for } q \in \mathfrak{m}^*(Z), \phi(u) \in S_{\phi}(Z), \]
$\Pi_{\phi}$ is compatible with $\leq$. Using Lemma 3.5, it suffices to show $\Pi_{\phi}$ is locally base-confluent, that is, for any local base-fork $(dw \rightarrow \phi dv_1, dw \rightarrow \phi dv_2)$, we have $dv_1 - dv_2 \rightarrow \phi 0$. Suppose to the contrary that $\Pi_{\phi}$ is not locally base-confluent. Then $\mathcal{C} \neq \emptyset$, where
\[ \mathcal{C} = \left\{ w \in \mathfrak{m}(Z) \mid \begin{array}{c} \text{there is a local fork base-fork } (dw \rightarrow \phi dv_1, dw \rightarrow \phi dv_2) \\ \text{for some } d \in k^\times \text{ such that } dv_1 - dv_2 \not\rightarrow \phi 0 \end{array} \right\}. \]
Since \( \leq \) is a well-order, \( \mathbb{C} \) has the least element with respect to \( \leq \), say \( w \). Thus there are some \( q_1, q_2 \in \mathcal{M}(Z) \), \( u, v \in \mathcal{M}(Z) \) and \( d \in \mathbb{R}^\times \) such that
\[
\begin{aligned}
w &= q_1_{|\mathcal{R}(u)} = q_2_{|\mathcal{R}(v)} \in \mathcal{M}(Z), \quad dw \rightarrow_\phi dq_1_{|\mathcal{R}(u)} , \\
dw \rightarrow_\phi dq_2_{|\mathcal{R}(v)} , \quad \text{and} \quad dq_1_{|\mathcal{R}(u)} - dq_2_{|\mathcal{R}(v)} \not\rightarrow_\phi 0 .
\end{aligned}
\]

Let
\[
Y := \{ u \in \mathcal{M}(Z) \mid u < w \} \quad \text{and} \quad \Pi_{kY} = \Pi_\phi \cap (Y \times kY).
\]
Since \( \leq \) is a monomial order, we have
\[
\begin{aligned}
q_1_{|\mathcal{R}(u)} = q_1_{|\mathcal{R}(v)} < q_1_{|\mathcal{R}(u)} = w , \\
nq_2_{|\mathcal{R}(v)} = q_2_{|\mathcal{R}(v)} < q_2_{|\mathcal{R}(v)} = w
\end{aligned}
\]
and
\[
q_1_{|\mathcal{R}(u)} , q_2_{|\mathcal{R}(v)} \in kY.
\]
So \( Y \neq \emptyset \). For any \( f \rightarrow_\phi g \) with \( f \in kY \), since \( \leq \) is compatible with \( \Pi_\phi \), we get \( g \leq f \) and so \( g \in kY \). Thus \( \Pi_{kY} \) is closed under \( \Pi_\phi \). By Proposition 5.8, we conclude that \( \Pi_{kY} \leq \Pi_\phi \) is a sub-term-rewriting system of \( \Pi_\phi \). For any local base-fork \( (ey \rightarrow_{\Pi_{kY}} ev_1, ey \rightarrow_{\Pi_{kY}} ev_2) \) of \( \Pi_{kY} \) with \( e \in k^\times \), \( y \in Y \) and \( v_1, v_2 \in kY \), it induces a local base-fork \( (ey \rightarrow_\phi ev_1, ey \rightarrow_\phi ev_2) \) of \( \Pi_\phi \). Since \( y \in Y \), we have \( y < w \) and \( y \notin \mathbb{C} \) by the minimality of \( w \). So \( ev_1 - ev_2 \not\rightarrow_\phi 0 \) by the definition of \( \mathbb{C} \). Since \( \Pi_{kY} \leq \Pi_\phi \) and \( ev_1 - ev_2 \in kY \), we have \( ev_1 - ev_2 \not\rightarrow_{\Pi_{kY}} 0 \). Thus \( \Pi_{kY} \) is locally base-confluent and so is confluent by Lemmas 3.1 and 3.3.

Since \( \phi(x) \) is multi-linear, we may write
\[
R(\phi(x)) = \sum_{i=1}^m r_i p_i_{|x} := \sum_{i=1}^m r_i p_i_{|x_1, \ldots, x_k},
\]
where \( r_i \in k^\times \), \( p_i \in \mathcal{M}^k(Z) \) and \( p_i_{|x_1, \ldots, x_k} \), \( 1 \leq t \leq m \), are mutually distinct monomials. Then
\[
\begin{aligned}
R(\phi(u)) &= \sum_{i=1}^m r_i p_i_{|u} := \sum_{i=1}^m r_i p_i_{|u_1, \ldots, u_k} , \\
R(\phi(v)) &= \sum_{i=1}^m r_i p_i_{|v} := \sum_{i=1}^m r_i p_i_{|v_1, \ldots, v_k} ,
\end{aligned}
\]
and by Eq. (13),
\[
q_1_{|p_i_{|u}} , q_2_{|p_i_{|v}} \in kY \quad \text{for} \quad 1 \leq t \leq m.
\]
By Proposition 4.10, these two placements \( (\phi(u), q_1) \) and \( (\phi(v), q_2) \) in \( w \) have three possible relative locations.

**Case I: Separate placements.** By Definition 4.9 there exists \( p \in \mathcal{M}^{*1:*2}(Z) \) such that
\[
q_1_{|*_1} = p|_{*_1, \phi(u)} \quad \text{and} \quad q_2_{|*_2} = p|_{*_2, \phi(v)}.
\]
So
\[
\begin{aligned}
kY \ni q_1_{|R(\phi(u))} = p|_{R(\phi(u)), \phi(u)} = \sum_{i=1}^m r_i p_i_{|p_i_{|u}, \phi(u)} , \\
\end{aligned}
\]
where the last step employs Eq. (14). For each \( 1 \leq t \leq m \),
\[
p|_{p_i_{|u}, \phi(u)} \rightarrow_{\Pi_{kY}} p|_{p_i_{|u}, R(\phi(u))} \quad \text{and so} \quad p|_{p_i_{|u}, \phi(u)} \downarrow_{\Pi_{kY}} p|_{p_i_{|u}, R(\phi(u))}.
\]
Since $\Pi_{kY}$ is confluent,

\begin{equation}
q_1|_{R(\phi(u))} = \sum_{t=1}^{m} r_t p_{[p_{\phi(u)}, R(\phi(v))]} \downarrow \Pi_{kY} \sum_{t=1}^{m} r_t s p_{[p_{\phi(u)}, R(\phi(v))]} = \sum_{t,s=1}^{m} r_t r_s p_{[p_{\phi(u)}, p_{\phi(v)}]},
\end{equation}

where the first equation follows from Eq. (17), the confluence step from Lemmas 3.3 and 3.2 [1], and the next equation from Eq. (17). On the other hand,

\begin{equation}
q_2|_{R(\phi(v))} = p_{[p_{\phi(u)}, R(\phi(v))]} \downarrow \Pi_{kY} \sum_{s=1}^{m} r_s p_{[p_{\phi(u)}, p_{\phi(v)}]} = \sum_{s=1}^{m} r_s r_s p_{[p_{\phi(u)}, p_{\phi(v)}]}.
\end{equation}

Since $\Pi_{kY}$ is confluent, by Lemma 3.2 [1], Eqs. (17) and (18) we obtain

\[ q_1|_{R(\phi(u))} \downarrow \Pi_{kY} q_2|_{R(\phi(v))}. \]

Then it follows from Eq. (13) and Remark 3.3 [1] that

\[ q_1|_{R(\phi(u))} - q_2|_{R(\phi(v))} \rightarrow^{*}_{\Pi_{kY}} 0. \]

By $\Pi_{kY} \subseteq \Pi_{\phi}$ being a sub-term-rewriting system and Lemma 3.4 [1], we have

\[ q_1|_{R(\phi(u))} - q_2|_{R(\phi(v))} \rightarrow^{*}_{\phi} 0 \quad \text{and} \quad dq_1|_{R(\phi(u))} - dq_2|_{R(\phi(v))} \rightarrow^{*}_{\phi} 0, \]

contradicting Eq. (17).

**Case II: Intersecting placements.** By the symmetry of (i) and (ii) in Item (1) of Definition 4.3 [1], we may assume that Item (1) (i) holds and hence $q_1 \neq q_2$. So there exist $q \in \mathcal{M}^*(Z)$ and $a, b, c \in \mathcal{M}(Z) \setminus \{1\}$ such that $w = q_{abc}, q_1 = q_{acl}$ and $q_2 = q_{acl}$. Then

\[ q_1|_{R(\phi(u))} = q|_{R(\phi(u))c} \quad \text{and} \quad q_2|_{R(\phi(v))} = q|_{aR(\phi(v))}. \]

So from Eq. (12),

\[ q|_{R(\phi(u))c} = q_1|_{R(\phi(u))} < w \quad \text{and} \quad q|_{aR(\phi(v))} = q_2|_{R(\phi(v))} < w. \]

This implies that

\[ q|_{R(\phi(u))}, q|_{aR(\phi(v))} \in kY \quad \text{and} \quad R(\phi(u))c, aR(\phi(v)) \in kY. \]

Together with $R(\phi(u))c \downarrow_{\phi} aR(\phi(v))$ and Theorem 7.1 [1], we have

\[ R(\phi(u))c \downarrow_{\Pi_{kY}} aR(\phi(v)) \quad \text{and} \quad R(\phi(u))c - aR(\phi(v)) \rightarrow^{*}_{\Pi_{kY}} 0, \]

where the last step employs the fact that $\Pi_{kY} \subseteq \Pi_{\phi}$ is confluent and Remark 3.3 [1]. Thus

\[ q_1|_{R(\phi(u))c} - q|_{aR(\phi(v))} = q|_{R(\phi(u))c - aR(\phi(v))} \rightarrow^{*}_{\Pi_{kY}} 0, \]

that is, $q_1|_{R(\phi(u))} - q_2|_{R(\phi(v))} \rightarrow^{*}_{\Pi_{kY}} 0$.

By $\Pi_{kY} \subseteq \Pi_{\phi}$ and Lemma 3.4 [1],

\[ q_1|_{R(\phi(u))} - q_2|_{R(\phi(v))} \rightarrow^{*}_{\phi} 0 \quad \text{and} \quad dq_1|_{R(\phi(u))} - dq_2|_{R(\phi(v))} \rightarrow^{*}_{\phi} 0, \]

contradicting Eq. (17).

**Case III: Nested placements.** By symmetry, we may suppose that there is $q \in \mathcal{M}^*(Z)$ such that $q_1|_{\phi(u)} = q_2$. Let us first consider $q = \ast$. Then $q_1 = q_2$. Since $q_1|_{\phi(u)} = q_2|_{\phi(v)}$, by Lemma 3.9 [1], we get $\phi(u) = \phi(v)$. Then $a = c = 1$ and $b = \phi(u) = \phi(v)$ in Theorem 4.1 [1], we have

\[ R(\phi(u)) \downarrow_{\phi} R(\phi(v)), q_1|_{R(\phi(u))} \downarrow_{\phi} q_2|_{R(\phi(v))} \quad \text{and} \quad q_1|_{R(\phi(u))} \downarrow_{\Pi_{kY}} q_2|_{R(\phi(v))}, \]
where the second confluence follows from \( \phi_1 = \phi_2 \) and the last confluence from Eq. (13). Since 
\( \Pi_{kr} \) is confluent, it follows from Remark 3.3 and Lemma 3.6 that
\[
q_1|_{\phi(1)} - q_2|_{\phi(2)} \xrightarrow{\ast} \Pi_{kr} 0 \quad \text{and} \quad dq_1|_{\phi(1)} - dq_2|_{\phi(2)} \xrightarrow{\ast} \Pi_{kr} 0,
\]
contradicting Eq. (14).

Consider next \( q \neq \phi_0 \). So \( q_1 \neq q_2 \) and \( \phi(u) \neq \phi(v) \). From
\[
q_1|_{\phi(u)} = q_2|_{\phi(u)} = q_1|_{\phi(v)},
\]
we have \( \phi(u) = q'_1|_{\phi(v)} \) by Lemma 3.9. Using Theorem 4.1, there are some \( u_i \) with \( 1 \leq i \leq k \) and
\( q' \in \mathcal{R}^*(Z) \) such that \( u_i = q'_i \). Write
\[
(19) \quad \phi(u) = p|_{u_1, \ldots, u_k} \quad \text{for some} \quad p \in \mathcal{R}^*(Z).
\]
Then
\[
q_1|_{\phi(v)} = p|_{u_1, \ldots, u_k} = q_1 p|_{u_1, \ldots, u_k, q'|_{\phi(v)}, u_{i+1}, \ldots, u_k} = (p|_{u_1, \ldots, u_k, q'|_{\phi(v)}, u_{i+1}, \ldots, u_k}|_{\phi(v)}).
\]
This implies that
\[
q = p|_{u_1, \ldots, u_i, q', u_{i+1}, \ldots, u_k} = \phi(u_1, \ldots, u_{i-1}, q', u_{i+1}, \ldots, u_k),
\]
where the second step employs Eq. (15). From Eq. (14), we may write
\[
(20) \quad q|_{\phi(1)} = \sum_{i=1}^m r_i q_1|_{p_i|_{\phi_1}},
\]
where
\[
(22) \quad u = (u_1, \ldots, u_k) = (u_1, \ldots, u_{i-1}, q'|_{\phi(v)}, u_{i+1}, \ldots, u_k).
\]
Write
\[
(23) \quad u' = (u_1, \ldots, u_{i-1}, q'|_{\phi(1)}, u_{i+1}, \ldots, u_k) \quad \text{and} \quad u'_s = (u_1, \ldots, u_{i-1}, q'|_{p_s|_{\phi_1}}, u_{i+1}, \ldots, u_k)
\]
for \( 1 \leq s \leq m \). Then
\[
(24) \quad q_1|_{p_s|_{\phi_1}} \xrightarrow{\Pi_{kr}} q_1|_{p_s|_{\phi_1}} = \sum_{s=1}^m r_s q_1|_{p_s|_{\phi_1}}, \quad \text{for} \quad 1 \leq t \leq m,
\]
where the first rewriting step follows from Eqs. (15) and (22), and the equation from Eq. (14). This implies that
\[
(25) \quad q_2|_{\phi(0)} = q_1|_{q_2|_{\phi(0)}} = \sum_{s=1}^m r_s q_1|_{p_s|_{\phi_1}}, \quad \text{for} \quad 1 \leq t \leq m.
\]
By Lemmas 3.3 and 3.2,}
\[
(24) \quad \sum_{t=1}^m r_t q_1|_{p_t|_{\phi_1}} \xrightarrow{\Pi_{kr}} \sum_{t,s=1}^m r_t r_s q_1|_{p_s|_{\phi_1}}.
\]
On the other hand,
where the first equation follows from $q_2 = q_1|_q$, the second from Eq. (14), the third from Eq. (20) and the fourth from Eq. (23). Since $\preceq$ is a monomial order and $p_{s \leq \phi(v)}$, we have

$$q'|_{p_{s \leq \phi(v)}} < q'|_{\phi(u')} < \phi(u)$$ for $1 \leq s \leq m,$

where the second inequality employs Eqs (22) and (23). This implies that

$$q_1|_{\phi(u')} < q_1|_{\phi(u)} = w$$ and $q_1|_{\phi(u')} \in kY$ for $1 \leq s \leq m.$

So

$$q_1|_{\phi(u')} \rightarrow_{\Pi_kY} q_1|_{R(\phi(u'))} = \sum_{t=1}^{m} r_t q_1|_{p_{s \leq \phi(v)}}$$ for $1 \leq s \leq m$

and

$$q_1|_{\phi(u')} \downarrow_{\Pi_kY} \sum_{t=1}^{m} r_t q_1|_{p_{s \leq \phi(v)}}$$ for $1 \leq s \leq m.$

Again applying Lemmas 3.6 and 3.2(b), we have

$$(26) \sum_{s=1}^{m} r_s q_1|_{\phi(u')} \downarrow_{\Pi_kY} \sum_{t=1}^{m} r_t q_1|_{p_{s \leq \phi(v)}}.$$

Since $\Pi_kY$ is confluent, by Lemma 3.2(b), Eqs. (24) and (26) we obtain

$$\sum_{t=1}^{m} r_t q_1|_{p_{s \leq \phi(v)}} \downarrow_{\Pi_kY} \sum_{s=1}^{m} r_s q_1|_{\phi(u')}.$$

Then Remark 3.3 yields,

$$\sum_{t=1}^{m} r_t q_1|_{p_{s \leq \phi(v)}} - \sum_{s=1}^{m} r_s q_1|_{\phi(u')} \rightarrow_{\Pi_kY} 0.$$

By Eqs. (21) and (25), this is equivalent to

$$q_1|_{R(\phi(u))} - q_2|_{R(\phi(v))} \rightarrow_{\Pi_kY} 0.$$

Hence from Lemma 3.6 and $\Pi_kY \subseteq \Pi_\phi$, we conclude

$$dq_1|_{R(\phi(u))} - dq_2|_{R(\phi(v))} \rightarrow_{\Pi_kY} 0$$ and $dq_1|_{R(\phi(u))} - dq_2|_{R(\phi(v))} \rightarrow_{\phi} 0,$

contradicting Eq. (11).

Thus $C \neq \emptyset$ leads to contradiction in all possible cases. This completes the proof of Theorem 4.1. □

**Acknowledgements:** This work was supported by the and the National Science Foundation of US (Grant No. DMS 1001855) and National Natural Science Foundation of China (Grant No. 11201201, 11371177 and 11371178).
References

[1] F. Baader and T. Nipkow, 1998. Term Rewriting and All That, Cambridge U. P., Cambridge.

[2] C. B. O. Bellier, L. Guo and X. Ni, Splitting of operations, Manin products and Rota-Baxter operators, Int. Math. Res. Not. IMRN. (2013), 485–524.

[3] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, Pacific J. Math. 10 (1960), 731–742.

[4] G. M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978), 178–218.

[5] Terese, Term rewriting systems, Cambridge University Press, 2003.

[6] L. A. Bokut, Y. Chen and J. Qiu, Gröbner-Shirshov bases for associative algebras with multiple operators and free Rota-Baxter algebras, J. Pure Appl. Algebra 214 (2010), 89–110.

[7] J. Cariñena, J. Grabowski and G. Marmo, Quantum bi-Hamiltonian systems, Internat. J. Modern Phys. A 15, (2000), 4797–4810.

[8] P. M. Cohn, Further Algebra and Applications, Springer, second edition 2003.

[9] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, Comm. Math. Phys., 210 (2000), 249–273.

[10] V. Drensky and E. Fromanek, Polynomial Identity Rings, Birkhäuser, 2004.

[11] K. Ebrahimi-Fard, Loday-Type Algebras and the Rota-Baxter Relation, Lett. Math. Phys., 61 (2002), 139–147.

[12] X. Gao and T. Zhang, Averaging algebras, rewriting systems and Gröbner-Shirshov bases, arXiv:1601.00533.

[13] X. Gao, L. Guo, W. Sit and S. Zheng, Rota-Baxter type operators, rewriting systems and Gröbner-Shirshov bases, J. Symbolic Computation, to appear, arXiv:1412.8055v1.

[14] L. Guo, Operated semigroups, Motzkin paths and rooted trees, J. Algebraic Combinatorics 29 (2009), 35–62.

[15] L. Guo, An Introduction to Rota-Baxter Algebra, International Press (US) and Higher Education Press (China), 2012.

[16] L. Guo, W. Sit and R. Zhang, Differentail Type Operators and Gröbner-Shirshov Bases, J. Symb. Comput. 52 (2013), 97–123.

[17] E. Kolchin, Differential algebraic groups, Academic Press, Inc., Orlando, FL, 1985.

[18] A. G. Kurosh, Free sums of multiple operator algebras, Siberian Math. J. 1 (1960), 62–70 (in Russian).

[19] J. B. Miller, Averaging and Reynolds operators on Banach algebra I, Representation by derivation and antiderivations, J. Math. Anal. Appl. 14 (1966), 527–548.

[20] A. Nijenhuis, X_n-lg-forming sets of eigenvectors. Indag. Math. 13 (1951), 200–212.

[21] E. Ohlebusch, Advanced topics in term rewriting, Springer, New York, 2002.

[22] J. Pei and L. Guo, Averaging algebras, Schröder numbers, rooted trees and operads, J. Algebraic Combinatorics 42 (2015), 73–109.

[23] M. van der Put and M. Singer, Galois Theory of Linear Differential Equations, Grundlehren der mathematischen Wissenschaften, 328, Springer, 2003.

[24] C. Procesi, Rings with polynomial identities Pure Appl. Math., 17 (1973), Marcel Dekker, Inc., New York.

[25] O. Reynolds, On the dynamic theory of incompressible viscous fluids and the determination of the criterion, Phil. Trans. Roy. Soc. A 136 (1895), 123–164.

[26] G. C. Rota, Baxter operators, an introduction, In: “Gian-Carlo Rota on Combinatorics, Introductory papers and commentaries”, Joseph P.S. Kung, Editor, Birkhäuser, Boston, 1995.

[27] L. H. Rowen, Polynomial identities in ring theory. Pure Appl. Math., 84 (1980) Academic Press, Inc.

[28] M. A. Semenov-Tian-Shansky, What is a classical r-matrix?, Funct. Ana. Appl., 17 (1983), 259–272.

[29] S. Zheng and L. Guo, Relative locations of subwords in free operated semigroups and Motzkin words, Frontier Math. 10 (2015), 1243–1261.
School of Mathematics and Statistics, Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou, 730000, P.R. China
E-mail address: gaoxing@lzu.edu.cn

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, USA
E-mail address: liguo@rutgers.edu