Asymptotic formula for $q$-Derivative of $q$-Durrmeyer Operators

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Abstract

In the manuscript, Voronovskaja type asymptotic formula for function having $q$-derivative of $q$-Durrmeyer operators and $q$-Durrmeyer-Stancu operators are discussed.

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1. Introduction

The classical Bernstein-Durrmeyer operators $D_n$ introduced by Durrmeyer[3] associate with each function $f$ integrable on the interval $[0, 1]$, the polynomial

$$D_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{1} p_{n,k}(t) f(t) dt, \quad x \in [0, 1], \quad (1.1)$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

These operators been studied by Derriennic[2] and many others. Last 30 years, the application of $q$-calculus in filed of approximation theory is active area of research. In 1987, the $q$-analogues of Bernstein operators was introduced by Lupas[10], Gupta and Hapeing[6] introduced $q$-generalization of the operators $(1.1)$ as

$$D_{n,q}(f; x) = [n + 1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q; x) \int_{0}^{1} f(t) p_{n,k}(q; qt) d_q t, \quad (1.2)$$

where $p_{n,k}(q; x) = \binom{n}{k}_q x^k (1-x)^{n-k}_q$.

The Rate of convergence of the operators $(1.2)$ was discussed by Gupta et al.[5, 19], local approximation, global approximation and simultaneous approximation properties of these operators by Finta and Gupta[4], estimation of moments and King type approximation was elaborated by Gupta and Sharma[7]. In 2014, Mishra and Patel[12, 14] talk about Stancu generalization, Voronovskaja type asymptotic formula and various other approximation
properties of the \( q \)-Durrmeyer-Stancu operators. We have the notation of \( q \)-calculus as given in \([8, 17]\). Here, in this manuscript we establish Voronovskaja type asymptotic formula for function having \( q \)-derivative.

2. Estimation of moments and Asymptotic formula

In the sequel, we shall need the following auxiliary results:

**Theorem 1.** \([2]\) If \( m \)-th \((m > 0, m \in \mathbb{N})\) order moments of operator \([1, 2]\) is defined as

\[
D^{q}_{n,m}(x) = D_{n,q}(t^{m}, x) = [n+1]_{q} \sum_{k=0}^{n} q^{-k}p_{n,k}(q; x) \int_{0}^{1} p_{n,k}(q; qt)t^{m}d_{q}t, \quad x \in [0, 1],
\]

then \( D^{q}_{n,0}(x) = 1 \) and for \( n > m + 2 \), we have following recurrence relation,

\[
[n + m + 2]_{q}D^{q}_{n,m+1}(x) = ([m + 1]_{q} + q^{m+1}x[n]_{q})D^{q}_{n,m}(x) + x(1-x)q^{m+1}D^{q}(D^{q}_{n,m}(x)).
\]

To establish asymptotic formula for function having \( q \)-derivative, it is necessary to compute moments of first to fourth degree. Using above Theorem one can have first, second, third and fourth order moments.

**Lemma 1.** For all \( x \in [0, 1] \), \( n = 1, 2, \ldots \) and \( 0 < q < 1 \), we have

- \( D_{n,q}(1, x) = 1 \);
- \( D_{n,q}(t, x) = \frac{1 + qx[n]_{q}}{[n + 2]_{q}} \);
- \( D_{n,q}(t^{2}, x) = \frac{q^{3}x^{2}[n]_{q}[n - 1]_{q} + (1 + q)^{2}q[x]_{q} + 1 + q}{[n + 3]_{q}[n + 2]_{q}} \);
- \( D_{n,q}(t^{3}, x) = \frac{4^{3}x^{3}[n]_{q}[n - 1]_{q}[n - 2]_{q} + x^{2}q^{3}[n]_{q}[n - 1]_{q} [1 + q + 2q^{2} + 3q^{3} + 2q^{4}]}{[n + 4]_{q}[n + 3]_{q}[n + 2]_{q}} + \frac{xq[2]_{q}[n]_{q} [1 + 2q + 3q^{2} + 2q^{3} + q^{4} + 3]_{q}[2]_{q}}{[n + 4]_{q}[n + 3]_{q}[n + 2]_{q}} \);
- \( D_{n,q}(t^{4}, x) = \frac{q^{15}x^{4}[n]_{q}[n - 1]_{q}[n - 2]_{q}[n - 3]_{q} + q^{8}x^{3}[n]_{q}[n - 1]_{q}[n - 2]_{q} [1 + 2q + 2q^{2} + 3q^{3} + 4q^{4} + 3q^{5} + q^{6}]}{[n + 5]_{q}[n + 4]_{q}[n + 3]_{q}[n + 2]_{q}} + \frac{q^{8}x^{2}[n]_{q}[n - 1]_{q} [1 + 2q + 4q^{2} + 8q^{3} + 12q^{4} + 14q^{5} + 13q^{6} + 10q^{7} + 6q^{8} + 2q^{9}]}{[n + 5]_{q}[n + 4]_{q}[n + 3]_{q}[n + 2]_{q}} + \frac{xq[2]_{q}[n]_{q} [1 + 3q + 6q^{2} + 9q^{3} + 10q^{4} + 9q^{5} + 6q^{6} + 3q^{7} + q^{8} + q^{9}]}{[n + 5]_{q}[n + 4]_{q}[n + 3]_{q}[n + 2]_{q}} \].

**Lemma 2.** For all \( x \in [0, 1] \), \( n = 1, 2, \ldots \) and \( 0 < q < 1 \), we have

- \( D_{n,q}((t-x)_{q}, x) = \frac{1 - (1 + q^{n+1})x}{[n + 2]_{q}} \);
- \( D_{n,q}((t-x)^{2}_{q}, x) = \frac{q^{2}x^{2}(1 + q^{n})(q^{n+1}[2]_{q} - [n]_{q}) + x(1 + q)(q^{2}[n]_{q} - 1 - q^{n+2}) + 1 + q}{[n + 3]_{q}[n + 2]_{q}} \).
\[
D_{n,q} \left( (t - x)^3_q, x \right) = q^2 x \left\{ \frac{q^5[nq][n-1]_q[n-2]_q - q[3]_q[nq][n-1]_q[n+4]_q + [n+4]_q[n+3]_q[n]_q - q[n+4]_q[n+3]_q[n+2]_q}{[n+2]_q[n+3]_q[n+4]_q} \right\}
+ q x \left\{ \frac{q^2[n]_q[n-1]_q(1 + q + 2q^2 + 3q^3 + 4q^4 - 2q^3 - 4q^4 + 3q^5 + q^6)}{[n+2]_q[n+3]_q[n+4]_q} \right\} + x \left\{ \frac{q[2]_q[n]_q(1 + 2q + 3q^2 + 4q^3 + 5q^4)}{[n+2]_q[n+3]_q[n+4]_q} \right\}.
\]

\[
D_{n,q} \left( (t - x)^4_q, x \right) = x^4 q^4 \left\{ \frac{q^4[1]_q[n]_q[n-1]_q[n-2]_q[n-3]_q - q^4[4]_q[n]_q[n-1]_q[n-2]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q} + \frac{(5)_q + q^2}{[n+3]_q[n+2]_q} \left\{ \frac{[4]_q[n]_q[n-1]_q}{[n+4]_q[n+3]_q[n+2]_q} \right\} \right\}
+ x^3 q^2 \left\{ \frac{q^3[n]_q[n-1]_q(1 + 2q + 3q^2 + 4q^3 + 5q^4)}{[n+4]_q[n+3]_q[n+2]_q} + \frac{(1 + q)_q(5)_q + q^2}{[n+3]_q[n+2]_q} \left\{ \frac{[4]_q[3]_q[n]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q} \right\} \right\}
+ x \left\{ \frac{q[2]_q[n]_q(1 + 3q + 6q^2 + 9q^3 + 10q^4 + 9q^5 + 6q^6 + 3q^7 + q^8)}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q} \right\}.
\]

**Proof:** To prove this Lemma, we use linear properties of \( q \)-Durrmeyer operators.

\[
D_{n,q}(t - x)_q = D_{n,q}(t, x) - xD_{n,q}(1, x) = \frac{1 + qx[n]_q - x}{[n+2]_q} = \frac{1 + x(q + 2 + ... + q^n - 1 - q - 2q - ... - q^n - q^{n+1})}{[n+2]_q} = 1 - \frac{(1 + q^{n+1})x}{[n+2]_q}.
\]

Using identities \((t - x)^2_q = t^2 - [2]_q xt + qx^2\), we get

\[
D_{n,q}((t - x)^2_q, x) = D_{n,q}(t^2, x) - [2]_q x D_{n,q}(t, x) + qx^2 D_{n,q}(1, x)
= \frac{q^3 x^2[n]_q[n-1]_q + (1 + q^2)qx[n]_q + 1 + q}{[n+3]_q[n+2]_q} - [2]_q x \left\{ \frac{1 + qx[n]_q}{[n+2]_q} \right\} + qx^2
= \frac{q^3 x^2[n]_q[n-1]_q + (1 + q^2)qx[n]_q + 1 + q - [2]_q x [2]_q [n+3]_q[n]_q + qx^2 [n+3]_q[n]_q + qx^2 [n+3]_q[n+2]_q}{[n+3]_q[n+2]_q}
= \frac{q^2 x^2(n-1)_q - [2]_q [n+3]_q[n]_q + [n+3]_q[n+2]_q \right\} + x \left\{ (1 + q^2)q[n]_q - [2]_q [n+3]_q \right\} + 1 + q
= \frac{q^2 x^2(1 + q^n)(q^{n+1} [2]_q - [n]_q) + x(1 + q)(q^2[n]_q - 1 - q^{n+2}) + 1 + q}{[n+3]_q[n+2]_q}.
\]
Notice that \((t - x)^3 = t^3 - [3]_q x^2 + q[2]_q x^2 t - q^3 x^3\),

\[
D_{n,q} ((t - x)^3)_q, x) = \frac{D_{n,q} (t^3, x) - [3]_q x D_{n,q} (t^2, x) + q[2]_q x^2 D_{n,q} (t, x) - q^3 x^3}{[n + 4]_q[n + 3]_q[n + 2]_q}
\]

Finally, using identities \((t - x)^3 = t^4 - [4]_q x t^3 + q ([5]_q + q^2) x^2 t^2 - q^3 x^3 [4]_q t + q^6 x^4\), we get

\[
D_{n,q} ((t - x)^4)_q, x) = \frac{D_{n,q} (t^4, x) - [4]_q x D_{n,q} (t^3, x) + q ([5]_q + q^2) x^2 D_{n,q} (t^2, x) - q^3 x^3 [4]_q D_{n,q} (t, x) + q^6 x^4}{[n + 5]_q[n + 4]_q[n + 3]_q[n + 2]_q}
\]
That is for any 

for 0, there exists a such that

\[
\theta_q(x; t) = \begin{cases} 
  \frac{f(t) - f(x) - D_q f(x)(t - x) - \frac{1}{2} q D_q^2 f(x)(t - x)^2}{(t - x)^2 Q} & \text{if } x \neq t \\
  0 & \text{if } x = t.
\end{cases}
\]  

We know that for \( n \) large enough

\[
\lim_{t \to x} \theta_q(x; t) = 0.
\]  

That is for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
|\theta_q(x; t)| \leq \epsilon.
\]  

Theorem 2. Let \( f \) bounded and integrable on the interval \([0, 1] \) and \((q_n)\) denote a sequence such that \( 0 < q_n < 1 \) and \( q_n \to 1 \) as \( n \to \infty \). Then we have for a point \( x \in (0, 1) \)

\[
\lim_{n \to \infty} [n]_q, D_{n,q_n}(f; x) - f(x)] = (1 - 2x) \lim_{n \to \infty} D_{q_n} f(x) + x(1 - x) \lim_{n \to \infty} D^2_{q_n} f(x).
\]  

Proof: By \( q-Taylor \) formula \([1]\) for \( f \), we have

\[
f(t) = f(x) + D_q f(x)(t - x) + \frac{1}{2} q D^2_q f(x)(t - x)^2 + \theta_q(x; t)(t - x)^2_q,
\]  

for \( 0 < q < 1 \), where

(2.1)
for |t - x| < δ and n sufficiently large. Using (2.1), we can write

\[ D_{n,q_n}(f; x) - f(x) = D_{q_n} f(x) D_{n,q_n}( (t - x)^2; x) + \frac{D_{q_n}^2 f(x)}{2|q_n|} D_{n,q_n}( (t - x)^4; x) + E_n^q(x), \]

where

\[ E_n^q(x) = [n + 1]_q \sum_{k=0}^{\infty} q^{-k} p_{nk}(q; x) \int_0^1 \theta_q(x; t)p_{nk}(q; qt) (t - x)^2 d_q t. \]

By Lemma 2 we have

\[ \lim_{n \to \infty} [n]_{q_n} D_{n,q_n}( (t - x)_q; x) = (1 - 2x) \]

and

\[ \lim_{n \to \infty} [n]_{q_n} D_{n,q_n}( (t - x)^2_q; x) = 2x(1 - x). \]

In order to complete the proof of the theorem, it is sufficient to show that \( \lim_{n \to \infty} [n]_{q_n} E_n^q(x) = 0 \). We proceed as follows:

Let

\[ P_{n,1}^q(x) = [n]_{q_n} \sum_{k=0}^{\infty} q^{-k} p_{nk}(q; x) \int_0^1 \theta_q(x; t)p_{nk}(q; qt) (t - x)^2 \chi_x(t) d_q t, \]

and

\[ P_{n,2}^q(x) = [n]_{q_n} \sum_{k=0}^{\infty} q^{-k} p_{nk}(q; x) \int_0^1 \theta_q(x; t)p_{nk}(q; qt) (t - x)^2 (1 - \chi_x(t)) d_q t, \]

so that

\[ [n]_{q_n} E_n^q(x) = P_{n,1}^q(x) + P_{n,2}^q(x), \]

where \( \chi_x(t) \) is the characteristic function of the interval \( \{t : |t - x| < \delta\} \).

It follows from (2.1)

\[ P_{n,1}^q(x) = 2\epsilon x(1 - x) \]

as \( n \to \infty \).

If |t - x| ≥ δ, then |\theta_q(x; t)| ≤ \( M/\delta^2 (t - x)^2 \), where \( M > 0 \) is a constant. Since

\[ (t - x)^2 = (t - q^2 x + q^2 x - x) (t - q^3 x + q^3 x - x) \]

\[ = (t - q^2 x) (t - q^3 x) + x(q^2 - 1) (t - q^2 x) + x(q^2 - 1) (t - q^3 x) + x^2 (q^2 - 1)(q^2 - q^3) + x^2 (q^2 - 1)(q^3 - 1), \]

we have

\[ |P_{n,2}^q(x)| \leq \frac{M}{\delta^2} \left[ [n]_{q_n} D_{n,q_n}( (t - x)^4_q; x) + x(2 - q^2 - q^3)[n]_{q_n} D_{n,q_n}( (t - x)^3_q; x) \right. \]

\[ \quad \left. + x^2 (q^2 - 1)^2 [n]_{q_n} D_{n,q_n}^{\alpha,\beta}( (t - x)^2_q; x) \right]. \]

Using Lemma 2 we have

\[ D_{n,q_n}( (t - x)^3_q; x) \leq \frac{C_m}{[n]_{q_n}^3}, \quad D_{n,q_n}( (t - x)^2_q; x) \leq \frac{C_m}{[n]_{q_n}^2}, \quad \text{and} \quad D_{n,q_n}( (t - x)_q; x) \leq \frac{C_m}{[n]_{q_n}}. \]
we have the desired result.

**Corollary 1.** Let $f$ bounded and integrable on the interval $[0,1]$ and $(q_n)$ denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the first and second derivative $f'(x)$ and $f''(x)$ exist at a point $x \in (0,1)$. Then we have for a point $x \in (0,1)$

$$\lim_{n \to \infty} [n]_{q_n}[D_{n,q_n}(f; x) - f(x)] = (1 - 2x)f'(x) + x(1 - x)f''(x).$$

3. **Asymptotic formula for Durrmeyer-Stancu Operators**

In year 1968, Stancu [16] generalized Bernstein operators and discussed its approximation properties. After that numbers of researchers give Stancu type generalization of several operators on finite and infinite intervals, we refer to the papers [13, 11, 15, 8, 18]. As mentioned in the introduction Stancu generalization of $q$-Durrmeyer operators [12] was discussed by Mishra and Patel [12], which is defined as follows: for $0 \leq \alpha \leq \beta$,

$$D_{n,q}^{\alpha,\beta} = [n + 1]_q \sum_{k=0}^{n} q^{-k} p_{nk}(q; x) \int_{0}^{1} f\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right) p_{nk}(q; qt) dq dt,$$

where $p_{nk}(q; x)$ as same as defined in [1, 2].

**Lemma 3.** We have $D_{n,q}^{\alpha,\beta}(1; x) = 1$, $D_{n,q}^{\alpha,\beta}(t; x) = \frac{[n]_q + \alpha [n + 2]_q + q x [n]_q^2}{[n + 2]_q ([n]_q + \beta)}$, $D_{n,q}^{\alpha,\beta}(t^2; x) = q^3 [n]_q^3 ([n]_q - 1) x^2 + \left((q + 1 + q^2 + 2 q \alpha^2) [n]_q^3 + 2 [n]_q [3]_q [n]_q^2\right)$

$$\frac{([n]_q + \beta)^2 [n + 2]_q [n + 3]_q}{([n]_q + \beta)^2 [n + 2]_q [n + 3]_q} x$$

$$+ \frac{\alpha^2}{([n]_q + \beta)^2} + \frac{(1 + q + 2 q \alpha^2) [n]_q^2 + 2 [n]_q [3]_q [n]_q}{([n]_q + \beta)^2 [n + 2]_q [n + 3]_q}.$$

**Lemma 4.** We have

$$D_{n,q}^{\alpha,\beta}(t - x, x) = \left(\frac{q [n]_q^2}{[n + 2]_q ([n]_q + \beta)} - 1\right) x + \frac{[n]_q + \alpha [n + 2]_q}{[n + 2]_q ([n]_q + \beta)},$$

$$D_{n,q}^{\alpha,\beta}(t - x^2; x) = q^4 [n]_q^4 - q^3 [n]_q^3 - 2 q^n [n]_q [3]_q [n]_q + \frac{\alpha^2}{([n]_q + \beta)^2 [n + 2]_q [n + 3]_q} x^2$$

$$+ \frac{(1 + q)^2 [n]_q^3 + 2 [n]_q [3]_q [n]_q + (2 [n]_q + 2 \alpha [n + 2]_q) [n + 3]_q [n]_q + \beta}{([n]_q + \beta)^2 [n + 2]_q [n + 3]_q} x$$

$$+ \frac{(1 + q) [n]_q^2 + 2 \alpha [n]_q [3]_q}{([n]_q + \beta)^2 [n + 2]_q [n + 3]_q}.$$

**Remark 1.** For all $m \in \mathbb{N} \cup \{0\}, 0 \leq \alpha \leq \beta$; we have the following recursive relation for the images of the monomials $t^m$ under $D_{n,q}^{\alpha,\beta}(t^m; x)$ in terms of $D_{n,q}(t^j; x); j = 0, 1, 2, \ldots, m$, as

$$D_{n,q}^{\alpha,\beta}(t^m; x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^m \alpha^{m-j}}{([n]_q + \beta)^m} D_{n,q}(t^j; x).$$
Theorem 3. Let \( f \) bounded and integrable on the interval \([0,1]\) and \((q_n)\) denote a sequence such that \(0 < q_n < 1\) and \(q_n \to 1\) as \(n \to \infty\). Then we have for a point \(x \in (0,1)\)

\[
\lim_{n \to \infty} [n]_{q_n} [ D_{n,q_n}^{\alpha,\beta} (f; x) - f(x) ] = (1 + \alpha - (2 + \beta)x) \lim_{n \to \infty} D_{q_n} f(x) + x(1-x) \lim_{n \to \infty} D_{q_n}^2 f(x).
\]

The proof of the above lemma follows along the lines of Theorem 2, using Lemma 4 and remark 1; thus, we omit the details.

Corollary 2. Let \( f \) bounded and integrable on the interval \([0,1]\) and \((q_n)\) denote a sequence such that \(0 < q_n < 1\) and \(q_n \to 1\) as \(n \to \infty\). Suppose that the first and second derivative \(f'(x)\) and \(f''(x)\) exist at a point \(x \in (0,1)\). Then we have for a point \(x \in (0,1)\)

\[
\lim_{n \to \infty} [n]_{q_n} [ D_{n,q_n}^{\alpha,\beta} (f; x) - f(x) ] = (1 + \alpha - (2 + \beta)x)f'(x) + x(1-x)f''(x).
\]

Remark 2. Theorem 2 and Theorem 3, gives asymptotic formula for \(q\)-Durrmeyer operators and \(q\)-Durrmeyer-Stancu operators respectively. If \(f\) has first and second derivative, then \(\lim_{n \to \infty} D_{q_n} f(x) = f'(x)\) and \(\lim_{n \to \infty} D_{q_n}^2 f(x) = f''(x)\). We archived results of Mishra and Patel [12, Theorem 5], which is mention in corollary 2. So presented results are more general results then exists ones.

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