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Polarized $e-p$ elastic scattering in the collider frame

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Double polarization elastic $\vec{e}-\vec{p}$ cross sections and asymmetries are considered in collider kinematics. Covariant expressions are derived for the general situation involving crossed beams; these are checked against the well-known results obtained when the proton is at rest. Results are given using modern models for the proton electromagnetic form factors for kinematics of interest in e-p colliders such as the Electron-Ion Collider facility which is in its planning stage. In context, parity-violating elastic $e-p$ scattering is compared and contrasted with these double-polarization (parity-conserving) results.

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I. INTRODUCTION

The main focus of this paper is the development of the formalism for polarized $\vec{e}-\vec{p}$ elastic scattering in the collider frame, stimulated by interest in e-p colliders such as the Electron-Ion Collider (EIC) facility which is in its planning stage. At least two things motivate such a study. (i) One has the possibility of measuring the electromagnetic form factors of the proton in unusual kinematics, that is, with high-energy colliding beams of polarized electrons and protons, in contrast to the usual situation with polarized electrons scattering from protons at rest with either the target proton polarized or when the recoiling final-state proton’s polarization is measured. Specifically, we shall see that the role played by $2\gamma$ corrections to the dominantly one-photon-exchange diagram (the only one we consider in detail in this work) quite differs for collider kinematics. (ii) One may use the reasonably well-known double-polarization asymmetry to determine the product of the electron and proton polarizations, $p_e p_p$, when the focus is placed on other reactions (e.g., deep inelastic scattering).

The basic kinematical formalism is presented in Sec. II for both collinear and crossed beams. This is followed, in Sec. III, by developments of the leptonic (electron) and hadronic (proton) tensors needed in discussing parity-conserving double-polarized $\vec{e}-\vec{p}$ elastic scattering. Here a general frame is considered and all quantities are kept completely covariant so that any situation can be explored, including the special case where the proton is at rest (the “rest frame”) and where the formalism is well known. In Sec. IV the leptonic and hadronic tensors are contracted to yield the invariant matrix element required in constructing the polarized cross sections and asymmetries, while in Sec. V expressions for the latter are presented.

Our approach in this study has been to develop the formalism in detail and thereby to bring out clearly the particular roles played by the different proton form factors. We shall see that the fact that the proton’s Pauli form factor is nonzero alters the character of the asymmetry from the simpler answer obtained when colliding point Dirac particles. We retain both the electron and proton masses (i.e., we do not invoke the extreme relativistic limit) so any choice of kinematics can be explored. In the case of the electron this is not needed except when scattering at very small angles; however, for the proton it is critical to retain the mass terms if one wishes to be able to go to the rest frame. Since the formalism is only a little more involved when keeping all mass terms than to drop them as is often done, we retain them throughout this study.

We shall see that the asymmetries are rather small (the reasons for this will be explained later) and, accordingly, in Sec. VI, we also briefly consider parity-violating elastic $\vec{e}-\vec{p}$ scattering in collider kinematics. We shall see that this single-polarization asymmetry is only about one order of magnitude smaller than the double-polarization parity-conserving asymmetries.

In Sec. VII results are presented for two choices of kinematics that may be relevant for a future EIC facility. The asymmetries (PC and PV) are all given, as is the figure of merit and thereby the anticipated fractional uncertainty expected given specific collider luminosities and polarizations. The computer code BRASIL2011 has been developed to handle any kinematical situation and can be obtained by anyone interested in exploring other conditions that may be relevant when planning for a future e-p collider. Finally, our conclusions are presented in Sec. VIII.

II. BASIC COLLIDER-FRAME KINEMATICS

A. Collinear beams

The coordinate frame used in this work is shown in Fig. 1; we start with collinear kinematics and then in the

\[1\]The C++ computer code BRASIL2011 that yields all of the kinematic variables, cross sections, asymmetries, and figures of merit may be obtained by contacting c.sofiatti@gmail.com.
following subsection generalize to the situation where the electron and proton beams are crossed. Here an electron with four-momentum \( K'^\mu \) is incident from the left and a proton with \( P'^\mu \) enters from the right. The final state has an electron with \( K'^\mu \) scattered at an angle \( \theta_q \) and a proton with \( P'^\mu \) scattered at an angle \( \theta_p \), as shown. The four-momentum transfer \( Q'^\mu = K'^\mu - K''^\mu = P'^\mu - P''^\mu \) makes an angle \( \theta_q \) with respect to the beam axis. The four-momenta in the problem are thus the following: \( K'^\mu = (\epsilon', \mathbf{k}') \), \( K''^\mu = (\epsilon, \mathbf{k}) \), \( P'^\mu = (E', \mathbf{p}') \), \( P''^\mu = (E, \mathbf{p}) \), and \( Q'^\mu = (\omega, \mathbf{q}) \) with \( \mathbf{k} = ku_3 \) and \( \mathbf{p} = -pu_3 \). Since the electron and proton are both on-shell, one can write their energies in terms of their three-momenta: \( \epsilon = \sqrt{k^2 + m_e^2}, E = \sqrt{p^2 + m_p^2} \), and \( E' = \sqrt{p'^2 + m_p^2} \).

We assume that the variables used to specify the kinematics are \( (k, p, \epsilon, \theta) \) and then through the equations above the energies of the incident particles are also given. It proves useful to define the total four-momentum
\[
P'^\mu \equiv K'^\mu + P'^\mu \equiv (E_{\text{tot}}, P_{\text{tot}}) \]
which constitutes an equation for \( k' \). Solving it, one has
\[
k' = \frac{1}{a} \left[ b + E_{\text{tot}} \sqrt{\xi^2 - m_e^2 (m_p^2 + p_{\text{tot}}^2 \sin^2 \theta_e^2)} \right]
\]
with
\[
a = E_{\text{tot}}^2 - p_{\text{tot}}^2 \cos^2 \theta_e - s + p_{\text{tot}}^2 \sin^2 \theta_e \]
\[
b = m_p^2 - m_e^2 + 2\xi + p_{\text{tot}}^2 \sin^2 \theta_e \geq (m_p + m_e)^2 > 0 \]
and knowing \( k' \) one can use the equations given above to determine \( \epsilon', P', E', \) and \( \theta_p \). Since the argument of the square root in Eq. (12) must be non-negative one has
\[
\xi \geq m_e \sqrt{m_p^2 + p_{\text{tot}}^2 \sin^2 \theta_e}. \tag{16}
\]
The four-momentum transfer is also now specified:
\[
\begin{align*}
\omega &= \epsilon - \epsilon', \\
q &= \sqrt{k^2 + k'^2 - 2kk' \cos \theta_e}, \\
\sin \theta_q &= \frac{k'}{q} \sin \theta_e, \\
\cos \theta_q &= \frac{1}{q} (k - k' \cos \theta_e). \tag{10}
\end{align*}
\]
From Eqs. (17) and (18), together with the energy-momentum relationships above, one has
\[
Q'^2 = -2(\epsilon \epsilon' - kk' \cos \theta_e - m_e^2),
\]

Thus, the four-momentum transfer is spacelike.\(^2\)

To conclude this brief discussion of the basic collinear kinematics, it is instructive to express the kinematic variables above in the proton rest frame. For any four-vector in the collinear frame the corresponding quantities in the proton rest frame may be found by boosting in the \( u_3 \) direction by \( \beta_p \equiv p/E \) with \( \gamma_p = E/m_p = [1 - \beta_p^2]^{-1/2} \). In particular, the proton in its rest frame of course has \( P'^\mu = (m_p, 0, 0, 0) \) while the incident electron has three-momentum and energy given by
\[
\begin{align*}
k_{\text{rest}} &= \gamma_p (k + \beta_p e), \\
\epsilon_{\text{rest}} &= \gamma_p (\epsilon + \beta_p e). \tag{24}
\end{align*}
\]
For the scattered electron one has
\[
\begin{align*}
\epsilon'_{\text{rest}} &= \gamma_p (\epsilon' + \beta_p k' \cos \theta_e), \\
k'_{\text{rest}} &= \sqrt{(\epsilon'_{\text{rest}})^2 - m_e^2}, \\
\sin \theta'_{\text{rest}} &= \frac{k'}{k'_{\text{rest}}} \sin \theta'_e, \\
\cos \theta'_{\text{rest}} &= \frac{\gamma_p (k' \cos \theta_e + \beta_p \epsilon')}{{k'_{\text{rest}}}}. \tag{28}
\end{align*}
\]

\(^2\)In the conventions employed in this and other work on which these studies are based the four-vector conventions outlined in the Appendix are adopted and, consequently, \( Q'^2 \) is negative when spacelike.

FIG. 1. (Color online) Electron-proton elastic scattering in collider kinematics.
TABLE I. Selected kinematics and rest-frame variables.

|                  | Kinematics I | Kinematics II |
|------------------|--------------|--------------|
| $k$ (GeV/c)      | 10           | 2            |
| $p$ (GeV/c)      | 250          | 50           |
| $k_{\text{rest}}$ (GeV/c) | 5329     | 213.2        |
| $\theta_{\text{rest}}$ (deg) at $\theta_1 = 1^\circ$ | 0.00188 | 0.00938 |
| $\tan(\frac{\theta_{\text{rest}}}{2})$ at $\theta_1 = 1^\circ$ | $1.638 \times 10^{-5}$ | $8.187 \times 10^{-5}$ |
| $1 - \hat{E}$ at $\theta_1 = 1^\circ$ | $5.410 \times 10^{-10}$ | $1.341 \times 10^{-8}$ |
| $\theta_{\text{rest}}$ (deg) at $\theta_1 = 5^\circ$ | 0.00939 | 0.0469 |
| $\tan(\frac{\theta_{\text{rest}}}{2})$ at $\theta_1 = 5^\circ$ | $8.193 \times 10^{-5}$ | $4.096 \times 10^{-4}$ |
| $1 - \hat{E}$ at $\theta_1 = 5^\circ$ | $1.633 \times 10^{-8}$ | $3.385 \times 10^{-7}$ |

To get some feeling for the extreme nature of the kinematics typically of interest (see Sec. VII), we consider two choices for the kinematics: kinematics I, $k = 10$ GeV/c with $p = 250$ GeV/c, and kinematics II, $k = 2$ GeV/c with $p = 50$ GeV/c (see also Table I in Sec. VII). For the former high-energy case we have $\beta_p \approx 1$, $\gamma_p \approx 250$, so $k_{\text{rest}} \approx 500 \approx 5$ TeV/c and $\theta_{\text{rest}} \approx \sin \theta_{\text{rest}} \approx \sin \theta_e/500$. This means that for this choice of kinematics and, for instance, for $1^\circ (5^\circ)$ scattering in the collider frame the equivalent rest frame (for example, for fixed target measurements) has an incident electron beam of 5.3 TeV/c scattering at 0.002° (0.01°). For the lower-energy case (ii) we have again for $1^\circ (5^\circ)$ scattering in the collider frame that the equivalent rest frame has an incident electron beam of 213 GeV/c with scattering angle 0.009° (0.047°). Any of the other kinematic variables above may be related to their rest-frame equivalents in a similar manner.

B. Crossed beams

In this subsection we provide the extensions that are necessary when the electron and proton beams are not collinear but are crossed. The kinematics for this are shown in Fig. 2. The electron is assumed to be incident along the $u_1$ axis as before; however, now the proton beam is assumed to have its momentum $p$ directed along the $-u_3$ axis where the $(1', 2', 3')$ system is rotated from the $(1,2,3)$ through an angle $\chi$ as shown.

Clearly,

\begin{align*}
    u_1 &= \cos \chi u_1 - \sin \chi u_3, \\
    u_3 &= \sin \chi u_1 + \cos \chi u_3.
\end{align*}

Let us repeat the kinematics developments of the previous subsection, now working in the $(1', 2', 3')$ system where we have $\mathbf{k} = k(\sin \chi u_1 + \cos \chi u_3)$ and $\mathbf{p} = -p u_3$. $E_{\text{tot}}$ in Eq. (4) is as before; however, now we have

\[ p_{\text{tot}} = k \sin \chi u_1 + (k \cos \chi - p)u_3, \]

yielding

\[ p_{\text{tot}}^2 = k^2 + p^2 - 2kp \cos \chi \]

and, therefore,

\[ s = p_{\text{tot}}^2 - E_{\text{tot}}^2 = m_e^2 + m_p^2 + 2\xi, \]

where

\[ \xi = E + kp \cos \chi; \]

cf. Eq. (7). The extension of Eq. (8) is

\[ p' = \sqrt{k^2 + p_{\text{tot}}^2 - 2k'[k \cos \chi - p \cos(\theta_e + \chi)]} \]

and of Eqs. (9) and (10) are

\[ \sin(\theta_p + \chi) = \frac{1}{p'} [k' \sin(\theta_e + \chi) - k \sin \chi], \]

\[ \cos(\theta_p + \chi) = \frac{1}{p'} [k' \cos(\theta_e + \chi) - (k \cos \chi - p)], \]

from which the angle $\theta_p$ may be found by taking the inverse sine and cosine. The analog of Eq. (11) is

\[ E_{\text{tot}}' - k' p_{\text{cross}} = m_e^2 + \xi, \]

where for convenience we have defined

\[ p_{\text{cross}} \equiv k \cos \theta_e - p \cos(\theta_e + \chi), \]

\[ p_{\text{cross}}^i \equiv k \sin \theta_e - p \sin(\theta_e + \chi). \]

One then has that

\[ k' = \frac{1}{\alpha} \left[ \mathbb{b} + E_{\text{tot}} \sqrt{\xi^2 - m_e^2 (m_p^2 + [p_{\text{cross}}^i]^2)} \right] \]

with

\[ \mathbb{a} \equiv E_{\text{tot}}^2 - [p_{\text{cross}}^i]^2 = s + [p_{\text{cross}}^i]^2 \]

\[ = m_e^2 + m_p^2 + 2\xi + [p_{\text{cross}}^i]^2 \]

\[ = \mathbb{b} (m_e^2 + \xi) p_{\text{cross}}^i \]

The rest of the developments go through as before in the collinear case.

III. LEPTONIC AND HADRONIC TENSORS

For the leptonic (here electron) tensor one has

\[ \eta_{\mu \nu} = \frac{1}{2} [\eta_{\mu \nu}^{\text{unpol}} + \eta_{\mu \nu}^{\text{pol}}] , \]

where, following standard developments [2], the unpolarized tensor (symmetric under $\mu \leftrightarrow \nu$) is given by

\[ 2m_e^2 \eta_{\mu \nu}^{\text{unpol}} = \frac{1}{2} Q^2 \left( g_{\mu \nu} - \frac{Q_\mu Q_\nu}{Q^2} \right) + 2 R_{\mu \nu}, \]
with
\[ R_\mu \equiv \frac{1}{2} (K_\mu + K'_\mu). \] (47)

The polarized tensor (antisymmetric under \( \mu \leftrightarrow \nu \)) is given by
\[ 2m^2 S_\mu^\alpha_{\nu}^\beta = i \epsilon_{\mu\nu\alpha\beta} (m_\mu S_\nu^\alpha) Q^\beta, \] (48)
where it can be shown [2] that the general spin four-vector is given by
\[ m_\mu S_\nu^\alpha = h_\epsilon e \left( \beta_\epsilon \cos \gamma_\epsilon, \cos \gamma_\epsilon \mu_\epsilon u^\mu + \frac{1}{\gamma_\epsilon} \sin \gamma_\epsilon \mu_\epsilon u^\mu \right). \] (49)

Here \( u^\mu \) is a unit vector pointing along \( k \) and \( u^\mu_\perp \) is transverse to this direction. As usual, one has \( \beta_\epsilon = k/\epsilon \) and \( \gamma_\epsilon = \epsilon/m_\epsilon = [1 - \beta_\epsilon^2]^{-1/2} \). Also, the factor \( h_\epsilon = \pm 1 \) is introduced simply to make it easy to switch the electron’s polarization from along the beam direction to opposite the beam direction. From this equation one sees that transverse polarizations are suppressed by the relativistic \( \gamma \) factor and so, henceforth, we consider only longitudinally polarized incident electrons:
\[ m_\mu S_\nu^\alpha = h_\epsilon e (\beta_\epsilon, u^\mu_\parallel). \] (50)

This yields only three distinct cases for the polarized electron tensor. Since the tensor is antisymmetric under \( \mu \leftrightarrow \nu \) we can restrict our attention to cases where \( \mu < \nu \), the others being given by using the antisymmetry. The nonzero cases are then:
\[ 2m^2 
\]
\[ \bar{\nu}_\mu \equiv \left\{ \begin{array}{ll}
\frac{1}{2} Q^\mu & \mu \nu = 12, \\
\bar{\nu}_\mu & \mu \nu = 02, \\
\frac{1}{2} Q^\mu & \mu \nu = 23,
\end{array} \right. \] (52)

One can verify that \( Q^\mu \bar{\nu}_{\mu\nu} = 0 \), as should be the case.

Again restricting our attention to collinear beams at first, the proton’s polarization four-vector is similar to the one for the electron in Eq. (49), namely
\[ m_\mu S_\nu^\alpha = h_p E \left( \beta_p \cos \gamma_p, \cos \gamma_p \mu_p u^\mu + \frac{1}{\gamma_p} \sin \gamma_p \mu_p u^\mu \right). \] (53)

In the case of the proton we must remember that it is moving to the left and thus, with \( L \) for longitudinal, \( S \) for sideways, and \( N \) for normal, one has
\[ u^\mu_L = -u^\mu_3, \] (54)
\[ u^\mu_S = \cos \eta_p u^\mu_3 + \sin \eta_p u^\mu_\parallel, \] (55)

where \( u^\mu_\parallel = -u^\mu_1 \) and \( u^\mu_\parallel = u^\mu_2 \); see Fig. 3. We shall specify the proton polarization by choosing to be along the \( L, S, \) and \( N \) directions and then any general case (for instance, when treating crossed beams; see below) can be decomposed into components along these orthogonal directions. It is worthwhile to reiterate that the conventions used here have +\( L \) polarization when it points in the \( -u^\mu_3 \) direction, +\( S \) polarization when it points in the \( -u^\mu_1 \) direction, and +\( N \) polarization when it points in the \( +u^\mu_2 \) direction. One finds that
\[ S_\mu^\alpha_L = h_p \gamma_p (\beta_p, -u_3), \] (56)
\[ S_\mu^\alpha_S = h_p (0, -u_1), \] (57)
\[ S_\mu^\alpha_N = h_p (0, -u_2). \] (58)

Now, for the hadronic (here proton) tensor one has the analogs of the leptonic tensors:
\[ W^{\mu\nu} = \frac{1}{2} [W^{\mu\nu}_{\text{unpol}} + W^{\mu\nu}_{\text{pol}}], \] (59)
where the symmetric unpolarized tensor is given by
\[ W^{\mu\nu}_{\text{unpol}} = -W_1 \left( \theta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) + \frac{1}{m_p^2} W_2 T^\mu T^\nu, \] (60)
with
\[ T^\mu \equiv \frac{1}{2} (P^\mu + P^\nu). \] (61)

The invariant functions \( W_{1,2} \) (functions only of \( Q^2 \)) are given as usual in terms of the proton’s electromagnetic Sachs form factors by
\[ W_1 = \tau (G^p_M)^2, \] (62)
\[ W_2 = \frac{1}{1 + \tau} \left[ (G^p_E)^2 + \tau (G^p_M)^2 \right], \] (63)
where \( \tau \equiv |Q^2|/4m_p^2 \gg 0 \). The antisymmetric polarized proton tensor is given by
\[ W^{\mu\nu}_{\text{pol}} = \frac{-i}{m_p} G^p_M \left[ (G^p_m)^2 + \frac{F^p_{12}}{m_p} (\epsilon^{\mu\nu\alpha\beta} T^\alpha T^\beta Q^\beta) + \frac{F^p_{22}}{m_p} (\epsilon^{\mu\nu\alpha\beta} T^\alpha T^\beta Q^\beta) \right] + \frac{F^p_{12}}{m_p} (\epsilon^{\mu\nu\alpha\beta} T^\alpha T^\beta Q^\beta) \] (64)
\[ \equiv -ih_p \gamma_p \frac{1}{m_p} G^p M Z^{\mu\nu}, \] (65)

with
\[ Z^{\mu\nu} \equiv G^p M Z^{\mu\nu}_1 + \frac{1}{m_p^2} F^p_{12} Z^{\mu\nu}_2 \] (66)
\[ Z^{\mu\nu}_1 \equiv h_p \epsilon^{\mu\nu\alpha\beta} S^\alpha Q^\beta, \] (67)
\[ Z^{\mu\nu}_2 \equiv h_p (\epsilon^{\mu\nu\alpha\beta} T^\alpha T^\beta Q^\beta). \] (68)

One can verify that \( Q^\mu Z^{\mu\nu} = 0 \) as should be the case; in fact, \( Q^\mu Z^{\mu\nu}_1 = Q^\mu Z^{\mu\nu}_2 = 0 \). For reasons that will become clear in the following section, in these expressions it proves useful to
use a mixture of the Sachs magnetic form factor with the Pauli form factor; the Sachs and Dirac/Pauli form factors are related in the familiar way:

\[ G_E^p = F_1^p - \tau F_2^p, \quad G_M^p = F_1^p + F_2^p, \]

\[ F_1^p = \frac{1}{1 + \tau} [G_E^p + \tau G_M^p], \quad F_2^p = \frac{1}{1 + \tau} [G_M^p - G_E^p]. \]  

\( \text{(69)} \)

IV. CONTRACTIONS OF LEPTONIC AND HADRONIC TENSORS

We start by obtaining the contraction of the two symmetric unpolarized tensors, namely

\[ X^{\text{unpol}} \equiv \left[ 2m_e^2 \eta_{\mu\nu}^{\text{unpol}} \right] \times \{ W^{\mu\nu} \} \]

\( \equiv -W_1 \left( \frac{3}{2} Q^2 + 2R^2 \right) \]

\[ + \frac{1}{m_p^2} W_2 \left( \frac{1}{2} Q^2 R^2 + 2(RT)^2 \right). \]  

\( \text{(70)} \)

One has \( \frac{1}{2} Q^2 + 2R^2 = Q^2 + 2m_e^2 \) and \( T^2 = m_p^2 - \frac{1}{4} Q^2 \) and, using the fact that \( RT = KP + \frac{1}{4} Q^2 = \xi + \frac{1}{4} Q^2 \),

\[ \text{one, therefore, has} \]

\[ \frac{1}{2} Q^2 R^2 + 2(RT)^2 = 2(\xi^2 + \frac{1}{4} Q^2 \xi + \frac{1}{4} m_p^2 Q^2). \]  

\( \text{(71)} \)

Defining dimensionless variables \( \lambda \equiv \omega/2m_e \) and \( \kappa \equiv q/2m_p \), where then \( \tau = \kappa^2 - \lambda^2 \), as usual, and defining

\[ \bar{\epsilon} \equiv \xi / m_p. \]

\( \text{(72)} \)

one then has

\[ X^{\text{unpol}} \equiv -W_1 \left( Q^2 + 2m_e^2 \right) + 2m_p^2 W_2 [\bar{\epsilon}^2 - 2\tau \bar{\epsilon} - \tau]. \]  

\( \text{(73)} \)

To connect with standard notation let us use the following

\[ V_0 \equiv 4m_p^2 \bar{\epsilon}^2 - 2\tau \bar{\epsilon} - \tau \]

\[ = 4m_p^2 \left[ \left( \frac{1}{2m_p^2} \left( (\epsilon + \epsilon') E + (k + k' \cos \theta_q) p \right) \right)^2 \right. \]

\[ - \tau (1 + \tau) \right] \]

\( \text{(74)} \)

\[ \tan^2 \theta_q'/2 \equiv \frac{Q^2}{V_0} = \frac{\tau}{\bar{\epsilon}^2 - 2\tau \bar{\epsilon} - \tau} \]

\( \text{(75)} \)

and then

\[ X^{\text{unpol}} = \frac{2m_p^2 \tau}{\tan^2 \theta_q'/2} F^2(\tau, \theta_q) \]  

\( \text{(76)} \)

where, defining

\[ E' \equiv \left[ 1 + 2(1 + \tau) \tan^2 \theta_q'/2 \left( 1 + \frac{2m_p^2}{Q^2} \right) \right]^{-1}. \]

\( \text{(77)} \)

one has for the (squared) e-p scattering form factor

\[ F^2(\tau, \theta_q) = W_2 + 2W_1 \tan^2 \theta_q'/2 \left( 1 + \frac{2m_p^2}{Q^2} \right) \]

\[ = \frac{1}{(1 + \tau) E} \left( E'(G_E^p)^2 + \tau (G_M^p)^2 \right). \]  

\( \text{(78)} \)

It is useful at this point to check this general result for the special case of the laboratory frame; there \( p = 0 \) and so \( E = m_p \); also \( \lambda_{\text{lab}} = \tau \) and \( \kappa_{\text{lab}} = \sqrt{\tau(1 + \tau)} \). This implies that \( \bar{\epsilon}_{\text{lab}} = m_p \epsilon \) and thus \( \bar{\epsilon}_{\text{lab}} = \epsilon / m_p \) and then in the laboratory system one can show that \( V_0_{\text{lab}} = v_0 = (\epsilon + \epsilon')^2 - q^2 \), the usual answer [2], and one then has

\[ X^{\text{unpol}} \equiv \frac{1}{\epsilon} v_0 F^2(\tau, \theta_q). \]  

\( \text{(79)} \)

Furthermore, in the electron extreme relativistic limit (ERL) where the electron’s mass may be neglected with respect to its momentum, \( \theta_q' = \theta_e \), \( E' \to E \) (the usual so-called virtual photon longitudinal polarization) and the expression above becomes proportional to \( E(G_E^p)^2 + \tau (G_M^p)^2 \), the familiar answer.

For the contraction of the two antisymmetric tensors we have from the expressions above

\[ X^{\text{pol}} \equiv \left[ 2m_e^2 \eta_{\mu\nu}^{\text{pol}} \right] \times \{ W^{\mu\nu} \} \]

\[ = -h_p \gamma_p \frac{1}{m_p} G_{\mu\nu}^p Z_{\mu\nu}, \]  

\( \text{(80)} \)

where \( Z^{\mu\nu} \) may be decomposed into \( Z_1^{\mu\nu} \) and \( Z_2^{\mu\nu} \) as in Eq. (66). Since the choice of longitudinal electron polarization led to only the components \( \mu \nu = 12, \ 02, \text{ and } 23 \) (together with their opposites, which, using the antisymmetry, leads to an overall factor of 2 if only this order is retained), we have only three cases to consider. Furthermore, note that all cases here have either \( \mu \) or \( \nu \) equal to 2 and so the proton polarization cannot have component 2. Since this is the only component for N polarization [see Eq. (58)] we find that the proton’s polarization (in one-photon-exchange approximation) cannot be normal, as expected. For the tensors of type 1 the results are the following:

\[ [Z_1^{\mu\nu}]_x = \left\{ \begin{array}{lc} \omega + \beta_p q \cos \theta_q = -\frac{1}{2} \frac{Q^2}{E}, & \mu \nu = 12, \\
-q \sin \theta_q, & \mu \nu = 02, \\
-\beta_p q \sin \theta_q, & \mu \nu = 23, \end{array} \right. \]  

\( \text{(81)} \)

and

\[ [Z_1^{\mu\nu}]_y = \left\{ \begin{array}{lc} 0, & \mu \nu = 12, \\
-\frac{1}{\gamma_p} q \cos \theta_q, & \mu \nu = 02, \\
0, & \mu \nu = 23, \end{array} \right. \]  

\( \text{(82)} \)

with no allowed N components. For the tensors of type 2 one has

\[ h_p \gamma_p Z_2^{\mu\nu} = T^\mu Z_0, \]

\( \text{(83)} \)

with \( \mu = 0, \ 1, \text{ or } 3 \) and

\[ Z_0 \equiv \epsilon \frac{2}{\omega} \beta e' \left[ S_{\mu}^p T_{\mu} Q_{e'}. \right] \]

\( \text{(84)} \)
yielding

\[ h_p^L[Z_0] = -m_p q \sin \theta_q, \]
\[ h_p^S[Z_0] = -[\rho \omega + E q \cos \theta_q], \]
\[ [Z_0]_N = 0. \]

The polarized contraction in Eq. (85) is then given by

\[ X^{\text{pol}} = -h_p h_p 4E G^M G^E \]
\[ \times \text{pol}, \]

where

\[ C^{\text{pol}} = \frac{1}{3\hbar p} \sqrt{\eta_{\mu \nu} Z^{\mu \nu}} = G^M C^M_1 + F^E C^E, \]
\[ = \frac{1}{3\hbar p} \left( \epsilon_{12} Z^{12} + \epsilon_{02} Z^{02} + \epsilon_{23} Z^{23} \right) \]

with \( \eta_{\mu \nu} \) from Eq. (52) and \( Z^{\mu \nu} \) from the developments given above, with the subscripts 1 and 2 referring to the two contributions in \( Z^{\mu \nu} \). The extra factor of 2 in Eq. (94) comes from using the antisymmetry together with only one order of the indices \( \mu \nu \). Again employing dimensionless variables, one has

\[ C^{\text{pol}}_1 = -\frac{t^2 + k^2}{\gamma_p} \sin^2 \theta_q, \]
\[ C^{\text{pol}}_2 = \kappa \sin \theta_q (\lambda + \beta \kappa \cos \theta_q) \sqrt{\epsilon^2 - (m_e/m_p)^2} \]
\[ - (\beta \kappa + \kappa \cos \theta_q) \frac{\epsilon}{p}, \]
\[ C^{\text{pol}}_3 = \frac{\kappa}{\gamma_p} \sin \theta_q \beta \kappa \lambda + \kappa \cos \theta_q, \]
\[ C^{\text{pol}}_4 = \kappa \sin \theta_q (\beta \kappa + \kappa \cos \theta_q), \]

where the result in Eq. (96) is obtained using the fact that

\[ k = E \sqrt{\epsilon^2 - (m_e/m_p)^2} - \frac{p \epsilon}{p}, \]
\[ \epsilon = E \sqrt{\epsilon^2 - (m_e/m_p)^2}. \]

It can be shown that \( C^{\text{pol}}_1, C^{\text{pol}}_2, \) and \( C^{\text{pol}}_3, C^{\text{pol}}_4 \) are all of order unity when \( \gamma_p \to \infty \), whereas \( C^{\text{pol}}_2 \) goes as \( 1/\gamma_p \) in that limit. Furthermore, if one sets \( F_1^1 = 1 \) and \( F_1^2 = 0 \), then, through Eqs. (69), one has \( G^E = G^M \), and thus no terms of type 2 contribute (no terms involving \( C^{\text{pol}}_2 \) or \( C^{\text{pol}}_3 \)). This special case makes the proton current take on the same form as that of a point Dirac particle like the electron. Accordingly, one sees that the only surviving contribution in the extreme relativistic limit for collisions of point Dirac particles is the one involving \( C^{\text{pol}}_1 \), as expected. This also underlines the fact that the sideways contribution in the general case for ultrarelativistic protons arises because of the anomalous magnetic moment, i.e., the parts of the current involving \( F_2 \).

Finally, using the relationship for the Pauli form factor in terms of Sachs form factors Eqs. (69) and defining

\[ C^M = \frac{1}{1 + \tau} [1 + \tau C^{\text{pol}}_1 + C^{\text{pol}}_2], \]
\[ C^E = \frac{1}{1 + \tau} \left[ - C^{\text{pol}}_2 \right], \]

one can then write

\[ C^{\text{pol}}_M = C^{\text{pol}}_E G^P_1 + C^{\text{pol}}_E G^P_2. \]

From above we, therefore, have that

\[ C^{\text{pol}}_M = -\frac{1}{\gamma_p} \left[ \tau (1 + \tau) \right] \sin \theta_q \sin \theta_q, \]
\[ C^{\text{pol}}_S = \frac{1}{1 + \tau} \kappa \sin \theta_q (1 + \tau) (\lambda + \beta \kappa \cos \theta_q) \]
\[ \sqrt{\epsilon^2 - (m_e/m_p)^2 - \tau (\beta \kappa + \kappa \cos \theta_q) \epsilon}, \]
\[ C^{\text{pol}}_L = -\frac{1}{\gamma_p} \kappa \sin \theta_q (1 + \tau) \lambda \sin \theta_q, \]
\[ C^{\text{pol}}_S = -\frac{1}{1 + \tau} \kappa \sin \theta_q (\lambda + \kappa \cos \theta_q) \sin \theta_q. \]

Next, we will want to check these results by going to the laboratory frame where \( p = 0, E = m_p, \beta = 0, \gamma_p = 1, \epsilon = \epsilon/m_p, \sqrt{\epsilon^2 - (m_e/m_p)^2} = k/m_p, \lambda = \tau, \) and \( \kappa = \sqrt{(1 + \tau)^2 - 1} \). One has

\[ C^{\text{pol}}_M = -\tau^2 \left( 1 + \frac{\epsilon}{m_p} \sin^2 \theta_q \right), \]
\[ C^{\text{pol}}_S = \tau \frac{\epsilon}{m_p} \sin \theta_q \sin \theta_q, \]
\[ C^{\text{pol}}_L = -\tau \frac{\epsilon}{m_p} \sin \theta_q \cos \theta_q, \]
\[ C^{\text{pol}}_S = -\tau \frac{\epsilon}{m_p} \sin \theta_q \cos \theta_q. \]

In the laboratory system the polarizations of the proton are usually specified with respect to the \((L', S', N')\) coordinate system as shown in Fig. 4. Rotating to this system one has

\[ C^{\text{pol}}_{XL} = -\cos \theta_q C^{\text{pol}}_{XL} + \sin \theta_q C^{\text{pol}}_{XS}, \]
\[ C^{\text{pol}}_{XS} = \cos \theta_q C^{\text{pol}}_{XS}, \]
\[ C^{\text{pol}}_{YL} = \tau \cos \theta_q + \sqrt{(1 + \tau)^2 - 1} \frac{k}{m_p} \sin^2 \theta_q, \]
\[ C^{\text{pol}}_{YS} = 0, \]
\[ C^{\text{pol}}_{ES} = \tau \sin \theta_q. \]

FIG. 4. (Color online) Proton polarizations \((L', S', N')\) in the laboratory system where \( p = 0 \).
where the zeros in Eqs. (115) and (116) are expected from the familiar laboratory frame analysis.

V. THE CROSS SECTION AND POLARIZATION ASYMMETRY

Finally, in these developments of the formalism, we now want to obtain the cross section and polarization asymmetry in the collider frame, together with their laboratory frame and extreme relativistic limits. We begin with the unpolarized cross section in the general collider frame. First, the flux factor to be used in applying the Feynman rules must now be generalized (see Ref. [3], Eqs. (7.41) and (B.1)): where in the laboratory frame one has the multiplicative factor $1/\beta_{e}\gamma_{e}$ and one now has the replacement

$$1/\beta_{e}\gamma_{e} \rightarrow \gamma_{p} \gamma_{p}(\beta_{e} + \beta_{p}),$$

where both factors of $\beta$ are to be taken as positive. Clearly, the laboratory frame result emerges when $\beta_{p} \rightarrow 0$ and $\gamma_{p} \rightarrow 1$. The recoil factor can be shown to generalize to

$$F_{\text{rec}} = 1 + \frac{e k' - e'(k - p) \cos \theta_{e}}{E k},$$

and, using $V_{0}$ given in Eq. (76), one has

$$\frac{d\sigma}{d\Omega_{e}}_{\text{unpol,collider}} = \sigma_{M}^{\text{collider}}(F_{\text{rec}})^{-1} F^{2}(\tau, \theta_{e})$$

with the square of the form factor from Eq. (81) and the generalized Mott cross section given by

$$\sigma_{M}^{\text{collider}} = \left(\frac{\alpha}{Q^{2}}\right)^{2} k'\frac{V_{0}}{k} \frac{\beta_{e} 1}{\beta_{e} + \beta_{p} \gamma_{p}}.$$  

These results can be checked by going to the laboratory frame and shown to agree with the familiar answers.

Various limiting cases may be straightforwardly obtained. First, in the ERL, one has

$$F_{\text{ERL}}^{\text{unpol,collider}} = 1 + \beta_{p} \cos \theta_{e},$$

$$\epsilon_{\text{ERL}} = \frac{k}{m_{p}} \gamma_{p} 1 + \beta_{p},$$

$$\sigma_{M}^{\text{collider,ERL}} = \left(\frac{\alpha}{Q^{2}}\right)^{2} k'\frac{V_{0}}{k} \frac{1}{1 + \beta_{p} \gamma_{p}}.$$  

where $V_{\text{ERL}}$ may be obtained using Eq. (76) and therefore $\sigma_{M}^{\text{collider,ERL}}$ using Eq. (121).

With both beams polarized the cross section has two terms, one ($\Sigma$) containing no dependence on the polarizations and one ($\Delta$) containing only terms where both beams are polarized, the latter being proportional to the product $h_{e}h_{p}$:

$$\frac{d\sigma}{d\Omega_{e}}_{\text{pol,collider}} = \gamma + h_{e}h_{p}\Delta.$$  

By flipping the spins one can form the polarization asymmetry:

$$[A]_{\text{pol,collider}} = \frac{[\sigma_{M}^{\text{collider}}(F_{\text{rec}})^{-1} F^{2}(\tau, \theta_{e})]}{[\sigma_{M}^{\text{collider}}(F_{\text{rec}})^{-1} F^{2}(\tau, \theta_{e})]}.$$  

Using the developments above we immediately have that

$$[A]_{\text{unpol,collider}} = \frac{h_{e}h_{p}X_{\text{pol}}^{\text{unpol}}}{X_{\text{pol}}^{\text{unpol}}} ,$$

where $X_{\text{pol}}^{\text{unpol}}$ is given in Eq. (79) and $X_{\text{pol}}^{\text{unpol}}$ is defined in Eq. (85). Substituting for these and expressing $C_{\text{pol}}$ in terms of $G_{M}^{p}$ and $G_{E}^{p}$ using Eq. (103) we find that

$$[A]_{\text{pol,collider}} = \frac{-N}{F^{2}(\tau, \theta_{e})},$$

where the numerator is given by

$$N = \frac{2}{\tau} \gamma_{p} \tan^{2} \theta_{e}/2 [G_{M}^{p}(C_{\text{pol}}^{M} G_{M}^{p} + C_{\text{pol}}^{E} G_{E}^{p})]$$

$$= N_{M}(G_{M}^{p})^{2} + N_{E} G_{E}^{p} G_{M}^{p}$$

and where $C_{\text{pol}}^{M,E}$ are given in Eqs. (104)–(107) for the two types of proton polarization, $L$ and $S$. Again, for reference, the denominator in Eq. (128) is given in Eq. (81) and $\tan^{2} \theta_{e}/2$ is given in Eq. (78).

As a first check, let us go to the laboratory frame and consider the $L'$ and $S'$ polarizations discussed above. We have from Eq. (114) that the $L'$ part of the numerator in Eq. (129) in the laboratory frame where $\theta_{e}' \rightarrow \theta_{e}$ is given by

$$-\frac{2}{\tau} \tan^{2} \tilde{\theta}_{e}/2 [C_{\text{pol}}^{M,L'}(G_{M}^{p})^{2}$$

$$= -2 \tan^{2} \tilde{\theta}_{e}/2$$

$$\times \left[ \tau \cos \theta_{q} + \sqrt{\tau(1 + \tau)} \frac{k}{m_{p}} \sin^{2} \theta_{q} \right] (G_{M}^{p})^{2}$$

$$\equiv V_{T} W_{L'}^{T}.$$  

using a notation where

$$W_{L'}^{T} = -2\tau (G_{M}^{p})^{2}.$$  

This implies that

$$V_{T} = \frac{1}{\tau} \tan^{2} \tilde{\theta}_{e}/2 \left[ \tau \cos \theta_{q} + \sqrt{\tau(1 + \tau)} \frac{k}{m_{p}} \sin^{2} \theta_{q} \right],$$

$$= \frac{1}{\tau} \tan^{2} \tilde{\theta}_{e}/2 \left[ \frac{\epsilon + e'}{q} \right] \left[ 1 - \frac{2m_{q}^{2} q^{2}}{\epsilon(\epsilon + e')(Q)} \right],$$

which agrees with the standard notation (e.g., see Ref. [2]). The $S'$ part of the numerator in Eq. (128) in the laboratory frame is found similarly:

$$-\frac{2}{\tau} \tan^{2} \tilde{\theta}_{e}/2 [C_{\text{pol}}^{E,S'}(G_{E}^{p})^{2} G_{M}^{p}$$

$$= 2 \tan^{2} \tilde{\theta}_{e}/2 \frac{\epsilon}{m_{p}} \sin \theta_{q} G_{E}^{p} G_{M}^{p}$$

$$\equiv V_{T} W_{S'}^{T}.$$  

again using the notation where

$$W_{S'}^{T} = 2\sqrt{2}\tau(1 + \tau) G_{E}^{p} G_{M}^{p}.$$  

---

4Here we consider only parity-conserving $e$-$p$ scattering; for the parity-violating situation where only the electron beam is assumed to be polarized, see Sec. IV.
yielding

\[ V_{TL'} = -\frac{1}{\sqrt{2}} \frac{\tau}{k^2} \epsilon k' \sin \theta_e, \tag{139} \]

which agrees with the standard notation (e.g., see Ref. [2]).

Finally, for the crossed beams situation, one must extend the discussion of the leptonic and hadronic tensors in Sec. III. In this case it is more convenient to work in the (1, 2, 3) system. The leptonic tensor will be as before; however, the hadronic tensor will now be assumed to have L and S polarizations with respect to the rotated frame. That is, we will assume that L or S polarizations occur when the proton’s polarization lies along the directions

\[ u_{L, \text{crossed}} = -u_{L'} = \sin \chi u_1 - \cos \chi u_3 \tag{140} \]

\[ u_{S, \text{crossed}} = -u_{v'} = -\cos \chi u_1 - \sin \chi u_3. \tag{141} \]

Accordingly, the asymmetries in the crossed beams situation are simply linear combinations of the collinear ones that we derived in Sec. V:

\[ [A]_{\text{collinear}}^{\text{pol}} \big|_{L, \text{collinear}} = -\sin \chi [A]_{\text{collinear}}^{\text{pol}} \big|_{S, \text{collinear}} + \cos \chi [A]_{\text{collinear}}^{\text{pol}} \big|_{L, \text{collinear}}, \tag{142} \]

\[ [A]_{\text{collinear}}^{\text{pol}} \big|_{S, \text{collinear}} = \cos \chi [A]_{\text{collinear}}^{\text{pol}} \big|_{L, \text{collinear}} + \sin \chi [A]_{\text{collinear}}^{\text{pol}} \big|_{L, \text{collinear}}. \tag{143} \]

It is straightforward to verify that all of the collinear results obtained above are recovered when the crossing angle \( \chi \) is set to zero.

### VI. PARITY-VIOLATING ELASTIC ELECTRON SCATTERING

While our main focus in the present work is placed on the double-polarization reaction \( \bar{p}(\bar{e}, e)p \) where parity-conserving (PC) asymmetries occur, it is interesting also to consider in context parity-violating (PV) single-polarization \( p(e, e)p \) scattering. The general structure of PV elastic electron scattering from the proton is illustrated in Fig. 5. Two diagrams are relevant: the usual photon exchange diagram which is parity conserving and a \( Z^0 \) exchange diagram which has both polar vector (V) and axial vector (A) contributions. To obtain the total cross section one must take the sum of these two diagrams and compute the square of the absolute value of that sum, leading to terms from the square of the photon exchange diagrams and compute the square of the absolute value of that in Eq. (145) is

\[ \chi_{\mu\nu}^{(a)} \equiv i \epsilon_{\mu\nu\alpha\beta} Q^a R^\beta, \tag{145} \]

which is antisymmetric under interchange of \( \mu \) and \( \nu \). Now, since the weak interaction has both vector and axial vector contributions, a second tensor must be considered:

\[ \chi_{\mu\nu}^{(s)} \equiv \frac{1}{2} Q^2 \left( g_{\mu\nu} - \frac{Q_{\mu} Q_{\nu}}{Q^2} \right) + 2 R_{\mu} R_{\nu}, \tag{144} \]

which is symmetric under interchange of \( \mu \) and \( \nu \). The leptonic tensor for the helicity-difference matrix elements may be written

\[ \chi_{\mu\nu}^{(PV, \text{hel. diff.})} = a_A \chi_{\mu\nu}^{(a)} + a_V \chi_{\mu\nu}^{(s)}, \tag{146} \]

where the standard model electroweak couplings are

\[ a_V = -(1 - 4 \sin^2 \theta_W) \simeq -0.092, \tag{147} \]

\[ a_A = -1, \tag{148} \]

using for the weak mixing angle the value \( \sin^2 \theta_W \simeq 0.227 \).

The hadronic (proton) tensor is similar: The symmetric tensor is the analog of the unpolarized tensor given in Eq. (60)

\[ \widetilde{W}_{\mu\nu}^{(s)} = -\widetilde{W}_1 \left( g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) + \frac{1}{m_p^2} \widetilde{W}_2 T^\mu T^\nu, \tag{149} \]

which has the same form as the unpolarized proton tensor but with different structure functions \( \widetilde{W}_{1,2} \), to be discussed below; these are differentiated by the tildes in Eq. (149). Likewise, the antisymmetric proton tensor that is the analog of that in Eq. (145) is

\[ \widetilde{W}_{\mu\nu}^{(a)} = \frac{i}{m_p^2} \widetilde{W}_3 \epsilon^\mu \epsilon^{\nu \alpha \beta} Q_{\alpha \beta}, \tag{150} \]

with an additional structure function \( \widetilde{W}_3 \).
The structure functions (functions only of \( \tau \)) occurring above in Eq. (149) are the analogs of the familiar \( W_{1,2} \) in PC elastic e-p scattering given above (see Sec. III), except that the PV analogs involve interferences between the \( \gamma \) and \( Z^0 \) diagrams and so contain products of EM form factors and their weak neutral current counterparts (indicated with tildes; see [4]):

\[
\tilde{W}_1 = \tau G^p_M \tilde{G}^p_M
\]
\[
\tilde{W}_2 = \frac{1}{1 + \tau} [G^p_E \tilde{G}^p_E + \tau G^p_M \tilde{G}^p_M].
\]

These are all of polar vector type and go with the leptonic axial vector term (i.e., proportional to \( a_A \)) to make the VA interference. The antisymmetric case has a polar vector but an interference of one polar and one axial vector contribution for the proton to make the VA interference (see Ref. [4]):

\[
\tilde{W}_3 = -\frac{1}{2} G^p_A \tilde{G}^A_M.
\]

Finally, the weak neutral current form factors in the standard model are the following:

\[
\tilde{G}_E^p = \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W) G_E^p - G^n_E - G_{E(0)} \right],
\]
\[
\tilde{G}_M^p = \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W) G_M^p - G^n_M - G_{M(0)} \right],
\]
\[
\tilde{G}_A^p = \frac{1}{2} \left[ G_A^{(1)} - G_A^{(0)} \right],
\]

where now the neutron’s electromagnetic form factors \( G_{E,M}^n \) enter, there are potential strangeness form factors \( G_{E,M,A}^{(s)} \), and the axial vector, isosvector form factor \( G_A^{(1)} \) (that, for instance, enters in \( n \rightarrow p \beta \)-decay) occurs in Eq. (156).

When contracting these leptonic and hadronic tensors, of course, only the symmetric or antisymmetric contributions contract with one another and no cross terms (symmetric times antisymmetric) can occur. The results are as follows: for the contraction of the symmetric tensors we have

\[
\chi_{\mu \nu}^{(s)} \tilde{W}_{s (a)} = -Q^2 \tilde{W}_1 + 2m^2_p [\tilde{e}^2 - 2\tilde{e} (\tilde{e} - \tau)] \tilde{W}_2
\]

\[
= \frac{1}{2} V_0 (\tilde{W}_2 + 2\tilde{W}_1 \tan^2 \theta'_/2),
\]

where \( \tilde{e} \) is defined via Eq. (74) and the last equality comes from using Eq. (76); these developments completely parallel the unpolarized contraction discussed in Sec. IV. Note that the result here again involves the definition of an effective angle, i.e., Eq. (78). For the contraction of the antisymmetric tensors the result is

\[
\chi_{\mu \nu}^{(a)} \tilde{W}_{a (a)} = -8m^2_p \tau (\tilde{e} - \tau) \tilde{W}_3
\]

\[
= \frac{1}{2} V_0 \frac{\tau (\tilde{e} - \tau)}{\tilde{e}^2 - 2\tilde{e} (\tilde{e} - \tau)} (-2 \tilde{W}_3).
\]

These results are for a general frame and thus may be used directly in the collider frame.

Using the explicit results for the ERL, the PV asymmetry may be written

\[
\mathcal{A}_\text{PV} = \frac{A_0}{D_{\text{PV}}} N_{\text{PV}},
\]

where

\[
A_{\text{PV}} = \frac{G |Q^2|}{2 \pi \alpha \sqrt{2}} = \frac{\sqrt{2G \pi \alpha}}{\pi} \simeq 6.33423 \times 10^{-4} \tau,
\]

with \( G \) the Fermi weak coupling constant. The denominator in Eq. (161) comes from the helicity sum cross section, namely the PC elastic electron scattering cross section discussed above. Omitting factors that are common to the numerator it may be written

\[
D_{\text{PV}} = \mathcal{E}' (G_E^p)^2 + \tau (G_M^p)^2.
\]

Finally, the expressions above may be checked (see Ref. [4]) by going to the laboratory frame, where for the ERL, \( \mathcal{E}' \rightarrow \mathcal{E} \).

VII. TYPICAL RESULTS FOR E-P SCATTERING IN COLLIDER KINEMATICS

In presenting results in this section we have made two choices for specific kinematics, a high-energy choice that may be typical of a future EIC facility and a somewhat lower-energy one where it may be possible to make measurements of the proton EM form factors at a level of precision that is interesting (see below). The two choices are listed in Table I along with rest-frame variables corresponding to typical collider-frame scattering angles, \( \theta_e \) and \( 5^\circ \) (see the discussions below).

For use in assessing the feasibility of performing asymmetry measurements one frequently employs the so-called figure of merit (FOM):

\[
\mathcal{F} \equiv \left[ \frac{d\sigma}{d\Omega_e} \right]_{\text{pol.collision}} \left( \left[ \mathcal{A}_\text{E}^{\text{pol.collision}} \right]^2 \right).
\]

Moreover, to incorporate the fact that the solid angle goes as \( \sin \theta_e \), it is also appropriate to consider the product \( \mathcal{F} \sin \theta_e \). Given luminosity \( L \), electron polarization \( p_e \), proton polarization \( p_p \), run time \( T \), and an averaged FOM times solid angle

\[
\mathcal{F}_{\text{avg}} \Delta \Omega_e \equiv \int_{0}^{\theta_e} \mathcal{F} \sin \theta_e \sin \theta_e d \sin \theta_e.
\]

| Table II. Assumed experimental conditions. |
|-----------------------------------------|
| \( L = 10^{34} \text{ cm}^{-2} \text{s}^{-1} \) |
| \( p_e = 90\% \) |
| \( p_p = 70\% \) |
| \( T = 1000 \text{ h} \) |
| \( \theta_0 = 90\° \) |
where \( \theta_0 \) is whatever one wishes to choose for the lower limit of integration over a scattering angle (written in the integrand above as \( \theta_e' \) to avoid confusion with \( \theta_e \) which has been used above and has a different meaning) and where the full \( 2\pi \) integration in azimuthal angle has been assumed, the fractional uncertainty in the \( L \) or \( S \) asymmetry is given by

\[
f = \frac{(p_e p_p \sqrt{L T \Delta \Omega_o})^{-1}}{\sin \theta_e}.
\]  

(167)

For the results to follow we have assumed the conditions listed in Table II. For simplicity the lower limit chosen for the averaging of the FOM has been fixed to \( \theta_0 = 0^\circ \) for the results shown in the figures; the time assumed is relatively short. In any practical situation these two parameters will need to be adjusted to find the optimal choices. The present study is not intended to be more than a preliminary exploration of typical results and so no attempt has been made to optimize the choices here. On the other hand, a general program, BRASIL2011, has been developed to study \( e^- p \) scattering in collider kinematics. A comprehensive description and sample input/output are included in the code package. All of the parameters above can be chosen by the user. In addition, the program provides the option of choosing between two models for computing the nucleon form factors, namely a simple model which gives a reasonable starting point for \( e^- p \) studies and the vector-meson dominance plus pQCD GKex model which yields good agreement with most of the World form factor data (see Ref. [6] and references cited therein for details). Alternatively, it is straightforward for any user to provide other form factor representations. In the following we show selected results using this code with the GKex form factors.

In Fig. 6 the scattered electron’s three-momentum \( k' \) (left axis) and \(|Q^2|\) (right axis) are shown as functions of the electron scattering angle for kinematics I; the range of angles chosen here is dictated by where the FOM is significant, as seen later in Fig. 8. Clearly, \( k' \) does not vary significantly over the chosen range of angles, less than 75 MeV/c of about 10 GeV/c. For the same range of angles the scattered proton three-momentum goes from 250 GeV/c at \( \theta_e = 0^\circ \) down by only about 75 MeV/c at \( 10^\circ \) while scattering through an angular range of \( \theta_p = 0^\circ \) at \( \theta_e = 0^\circ \) to \( \theta_p = 0.4^\circ \) at \( \theta_e = 10^\circ \). It is clear from our initial exploratory studies that energy resolution alone is rather unlikely to be enough to separate elastic events from inelastic ones (the code BRASIL2011 contains some kinematic developments where the final hadronic state can be

![FIG. 6. (Color online) Kinematics I: Scattered electron three-momentum \( k' \) and four-momentum transfer squared \(|Q^2|\) versus collider-frame electron scattering angle \( \theta_e \).](image-url)

![FIG. 8. (Color online) Kinematics I: \( F \sin \theta_e \) versus electron scattering angle \( \theta_e \).](image-url)

![FIG. 7. (Color online) Kinematics I: Double-polarization asymmetries [\( A_{pol}^{collider} \) for \( L \) and \( S \) proton polarizations versus electron scattering angle \( \theta_e \). Also shown is the parity-violating e-p elastic asymmetry \( A_{PV} \) for the same kinematics.](image-url)

![FIG. 9. (Color online) Kinematics I: Fractional uncertainty \( f \) [see Eq. (167) for polarizations \( L \) and \( S \) versus electron scattering angle \( \theta_e \).](image-url)
taken to have invariant mass \( W \neq m_p \) and some final-state particle identification will likely be needed to isolate elastic scattering in practical situations.

In Fig. 7 the asymmetries are shown for the same range of electron scattering angles used above. The \( \vec{e}\vec{p}L \) polarization is larger in magnitude than for the \( S \) case, although the two are comparable. The latter changes sign at roughly \( \theta_e = 1.3^\circ \). In the region where the FOM peaks (roughly \( \theta_e \sim 2^\circ \); see Fig. 8) the \( L \) asymmetry is a few times \( 10^{-5} \). One might be surprised that this is so low. The reason is clear on examination of the rest-frame variables in Table I:

\[
\tan(\theta_{\text{rest}}/2) \text{ goes from } 1.6 \times 10^{-5} \text{ at } \theta_e = 1^\circ \text{ to } 8.2 \times 10^{-5} \text{ at } \theta_e = 5^\circ. 
\]

Since the asymmetries are proportional either to the generalized “Rosenbluth” factor \( V_T \) for \( L \) polarization or to \( V_{T\perp} \) for \( S \) polarization (see Sec. V) and these both have an overall factor \( \tan(\theta_{\text{rest}}/2) \) in the rest frame [2], the kinematical factors are small. In other words, the scattering in the equivalent rest frame occurs at such small angles that the double-polarization asymmetries are suppressed. For comparison the parity-violating asymmetry \( A_{\text{PV}} \) is also shown in Fig. 7. In this case, even though the weak interaction is involved, some of the contributions occurring in the ratio forming the asymmetry (see Sec. VI) are not suppressed by similar factors. In the same notation used above the Rosenbluth factors \( V_L \) and \( V_T \) occur for the VV hadronic contributions [those involving \( G_{E,M}^p \hat{G}_{E,M}^p \) in Eq. (164)] and these do not contain overall factors of \( \tan(\theta_{\text{rest}}/2) \). On the other hand, the VA interference in Eq. (164) (involving \( G_M^p \hat{G}_M^p \)) has a factor \( \sqrt{1-\vec{E}^2} \) and so is strongly suppressed (see Table I).

Another observation from Table I is that the smallness of \( 1 - \mathcal{E} \), going from \( 5.4 \times 10^{-10} \) at \( \theta_e = 1^\circ \) to \( 1.6 \times 10^{-5} \) at \( \theta_e = 5^\circ \), implies something very different about any potential \( 2\gamma \) corrections to the dominantly \( 1\gamma \) diagram. Namely, from treatments of the former (e.g., see [7]) one expects the \( 2\gamma \) contributions to vanish when \( \mathcal{E} \to 1 \), which is surely the case for the rest-frame-equivalent conditions studied here.

Figure 9 shows the fractional uncertainty obtained using Eq. (167) and implies that in a time \( T = 1000 \text{ h} \) one could achieve roughly a 15–20% determination of the \( L \) polarization asymmetry. Since the fractional uncertainty goes as \( 1/\sqrt{T} \), clearly with longer run times even higher precision can be obtained.

For kinematics II, the results are shown in Figs. 10–13. Now the angular range is larger (\( \theta_e \) up to \( 20^\circ \)), again as dictated by the significance of the FOM (see Fig. 12 below). The scattered proton’s three-momentum \( k' \) in this case varies from 50 GeV/c at \( \theta_e = 0^\circ \) to 15 MeV/c less than this at \( 10^\circ \), while the proton’s scattering angle \( \theta_p \) goes from \( 0^\circ \) up to \( 0.4^\circ \) over

FIG. 10. (Color online) As for Fig. 6 but now for kinematics II.

FIG. 11. (Color online) As for Fig. 7 but now for kinematics II.

FIG. 12. (Color online) As for Fig. 8 but now for kinematics II.

FIG. 13. (Color online) As for Fig. 9 but now for kinematics II.
the same range. The asymmetries are shown in Fig. 11: these are significantly larger than the case for kinematics I and now lie typically two or more orders of magnitude above the PV asymmetry. Figure 12 shows the FOM and indicates that for the \( L \) polarization case a peak occurs for \( \theta_E \) between 8° and 9°. This yields the fractional uncertainty displayed in Fig. 13, clearly showing that a 1–2% determination of the asymmetries may be possible, given the assumed luminosity, polarizations, and run time.

**VIII. CONCLUSIONS**

In this study elastic \( \vec{e} \cdot \vec{p} \) cross sections and asymmetries have been considered in collider kinematics. The formalism has been developed directly using the electron and proton tensors and working in a general frame; since the resulting expressions are covariant it is easy to evaluate the results in any chosen frame, including the system where the proton is at rest and where the asymmetries and cross sections are well known, thereby providing a sensitive check on the formalism. In contrast, parity-violating elastic \( \vec{e} \cdot \vec{p} \) scattering has also been explored in collider kinematics to be able to compare and contrast the resulting asymmetry with the double-polarization (parity-conserving) results.

Several observations and conclusions can be made from these studies:

(i) The double-polarization asymmetries are relatively small, since the effective rest-frame electron scattering angle is very small for typical collider kinematics and since both the \( L \) and \( S \) asymmetries in the rest frame have an overall factor of \( \tan(\theta_E^{\text{rest}}/2) \). In contrast, the PV asymmetry has contributions that do not contain this factor and therefore survive when the electron scattering angle becomes very small. Indeed, for some kinematical situations the PV asymmetry is only about one order of magnitude smaller than the PC asymmetries.

(ii) The \( L \) and \( S \) asymmetries, while small, are still sufficiently large that it may prove possible to make relatively high-precision measurements of them with a future EIC facility as it is presently envisioned. If, with such a facility, it proves possible to use beams that have a large dynamical range, including lower energies and beams that are more symmetrical in energy (the EIC designs being considered are typically asymmetric with the proton beam being much higher in energy than the electron beam), then very likely quite high-precision determinations of the double-polarization asymmetries can be made.

(iii) Given that the double-polarization asymmetries can be measured with sufficient precision two paths may be followed: (i) the proton EM form factors themselves may be studied, and, given that we know these form factors reasonably well from fixed-target experiments, (ii) the asymmetries may be used to determine the product of the electron and proton polarizations when the focus is on other e-p reactions, including studies of deep inelastic scattering (DIS).

(iv) With respect to point (1) above, the collider kinematics are very unusual in that the effective rest-frame kinematics typically occur at very large energies and very small angles. This means that the virtual photon longitudinal polarization \( \gamma \) is extremely close to unity where it is predicted that \( 2\gamma \) corrections to the dominantly \( 1\gamma \) diagram go to zero.

(v) Specifics of how the proton EM form factors enter the asymmetries are interesting: If the anomalous magnetic or Pauli form factor \( F_2^p \) were zero, as is the case for a point Dirac particle, then the \( L/S \) structure of the asymmetries would differ markedly. The fact that \( F_2^p \neq 0 \) leads to clear signatures in the double-polarization asymmetries.

(vi) Finally, while this work is a theoretical study and has been focused on the formalism plus presentations of a few typical results, some initial exploration has been made of the issues that will probably confront any practical experiment. In particular, it is very unlikely for high-energy collider kinematics that energy resolution alone will be capable of isolating elastic e-p scattering from inelastic scattering. Instead, one will have to detect both the scattered electron and specify the final hadronic state including the elastic events where only the scattered proton occurs. This is not the point of the present study, although corresponding code is included in the present study, although corresponding code is included in BRASIL2011 and is available for others involved in design studies for a future EIC facility.

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**APPENDIX**

The conventions of Bjorken and Drell [3] are used throughout together with the following notation: four-vectors are written with capital letters

\[
A^\mu = (A^0, A^1, A^2, A^3) = (A^0, \mathbf{a}), \quad (A1)
\]

\[
A_\mu = (A^0, -A^1, -A^2, -A^3) = (A^0, -\mathbf{a}), \quad (A2)
\]

whereas three-vectors are written with bold lowercase letters and their magnitudes with normal lowercase letters, \( a = |\mathbf{a}| \). The scalar product of two four-vectors \( A \) and \( B \) is given by

\[
A \cdot B = A^\mu B_\mu = A^0 B^0 - \mathbf{a} \cdot \mathbf{b}, \quad (A3)
\]

where the summation convention is employed, namely repeated Greek indices are summed (\( \mu = 0, 1, 2, 3 \)). Thus one has for the scalar product of a four-vector with itself

\[
A^2 = (A^0)^2 - a^2. \quad (A4)
\]

Throughout we use \( h = c = 1 \).
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