Affine Lie Algebras, String Junctions and 7-Branes

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Abstract

We consider the realization of affine ADE Lie algebras as string junctions on mutually non-local 7-branes in Type IIB string theory. The existence of the affine algebra is signaled by the presence of the imaginary root junction δ, which is realized as a string encircling the 7-brane configuration. The level k of an affine representation partially constrains the asymptotic (p, q) charges of string junctions departing the configuration. The junction intersection form reproduces the full affine inner product, plus terms in the asymptotic charges.
1 Introduction

The compactification of F-Theory on elliptically fibered K3 manifolds has proven to be a fruitful testing ground for non-Abelian gauge symmetry enhancement. The singular fibers of the K3 are organized according to the Kodaira classification, which associates to each an ADE Lie algebra. From the IIB point of view the situation is a compactification on $S^2$, with 24 mutually non-local 7-branes. In this perspective the singularities are coinciding 7-branes and the gauge vectors are strings or string junctions with support on these branes. How the Lie-algebraic properties of the junctions arise from the intersection of the corresponding holomorphic curves in K3 has been extensively investigated [1, 2].

The Kodaira classification includes singularities corresponding to $A$-, $D$- and $E$-type Lie algebras. Since these exhaust the singularities on elliptic K3, only 7-branes corresponding to these algebras can collapse to a point. However, it is natural to wonder about other configurations of 7-branes. The Lie-algebraic weight vector associated to a string junction on a configuration of 7-branes is well-defined regardless of whether those branes can coalesce. If the root system of an algebra arises on a set of branes, the associated junctions must organize themselves into representations of that algebra, regardless of whether the gauge bosons can be massless. Hence one is led to wonder about the possibility of realizing other Lie algebras on 7-brane configurations.

Affine Lie algebras have played a prominent role in string theory. Because the worldsheet currents of the heterotic string define an affine algebra, massive stringy excitations organize themselves according to affine representations. Generalized Kac-Moody algebras have been used to investigate spaces of BPS states [3]. Furthermore, it is now well-known that there exists a 6D non-critical string carrying an $E_8$ current algebra, related to zero-size instantons [4, 5]. This non-critical string is tensionless at certain points in moduli space, leading to the possibility of an infinite tower of massless states organized according to an affine representation in the theory compactified to five or four dimensions [6, 7, 8].

Since the F-theory compactification is dual to the heterotic string, the obvious question to ask is whether affine algebras can arise on 7-branes. We find that indeed they do. Certain collections of 7-branes admit a junction $\delta$ which is a string loop of some $(p, q)$ charge winding entirely around the configuration; this situation was examined in a particular case by [9]. This plays the role of the imaginary root, the distinguishing element of the affine root system. We find that the level $k$ of the affine representation associated to a set of junctions...
constrains one linear combination of the asymptotic charges \((p, q)\). The junction intersection form includes the affine inner product, inducing the affine Cartan matrix on the junctions without asymptotic charge (hereafter, “uncharged” junctions) and including the “light-cone” product of the level and grade. Decoupling a single brane leaves behind a familiar \(ADE\) configuration.

Affine representations are all infinite-dimensional. This is due to the presence of arbitrarily large numbers of \(\delta\) contributions. Since the affine singularity cannot be realized in K3, however, all but a finite number of junctions (corresponding to the associated finite algebra) must remain massive. The necessity of ordinary finite Lie algebra representations organizing themselves into affine representations in the presence of an additional brane gives information about the representations that can arise on the worldvolume of a probe D3-brane. The 4D \(E_n\) theories associated to the non-critical string were investigated from the point of view of a 3-brane/7-brane system in [10].

Section 2 reviews basic properties of affine Lie algebras. Section 3 details the way in which affine algebras arise on 7-branes. Section 4 explores these ideas in the simplest example of \(\widetilde{su}(2)\) and the more complicated case of \(\widetilde{E}_8\) as well. Section 5 contains some concluding remarks. We do not consider the issue of which junction of an equivalence class is the BPS representative, which has been addressed elsewhere [11, 12].

## 2 Review of Affine Lie Algebras

In this section we briefly review the characteristics of affine Lie algebras that are relevant to our investigation. For the definitive mathematical treatment one should consult [13], while discussions aimed at physicists can be found in [14].

Lie algebras are determined entirely by their Cartan matrices by the Chevalley-Serre construction. An \(r \times r\) matrix \(A_{ij}\) defines a Lie algebra \(g\) by directly specifying the brackets of a subset of the generators, the \(3r\) Chevalley generators \(\{H^i, E^\pm_i\}, i = 1 \ldots r\):

\[
[H^i, H^j] = 0, \quad [H^i, E^\pm_j] = \pm A_{ji}E^\pm_j, \quad [E^+_i, E^-_j] = \delta^{ij}H^j. \tag{1}
\]

The remaining elements of the algebra are obtained from successive commutators of the Chevalley generators, modulo the Serre relations:

\[
(\text{ad } E^\pm_i)^{1-A_{ji}}E^\pm_j = 0, \quad i \neq j. \tag{2}
\]
A finite, semisimple Lie algebra is obtained when the matrix $A$ has integral entries and satisfies certain restrictions:

$$A_{ii} = 2; \quad A_{ij} \leq 0, \ i \neq j; \quad A_{ij} = 0 \leftrightarrow A_{ji} = 0; \quad \det A > 0. \quad (3)$$

An affine Cartan matrix is obtained by allowing $\det A = 0$, while requiring that removing the row and column associated to any one element leaves a semisimple Cartan matrix. Hence the affine Lie algebras are simple generalizations of finite Lie algebras from this point of view.

The degeneracy of its Cartan matrix is responsible for all the affine algebra’s unusual properties. In particular, the algebra and its root system are infinite. The degeneracy results in the existence of a linear combination of roots that has vanishing inner product with all roots, the “imaginary” root $\delta$:

$$\delta = \alpha_0 + \sum_{i=1}^{r} c^i \alpha_i, \quad (4)$$

where the Coxeter labels $c^i$ are the expansion coefficients of the highest root $\theta$ of $g$ in the basis $\{\alpha_i\}$, $\theta = \sum_{i=1}^{r} c^i \alpha_i$. The existence of $\delta$ satisfying $(\delta, \alpha_i) = 0$ is the distinguishing characteristic of an affine root system.

Because the junction realizations of Lie algebras originate with the Cartan matrix, we have emphasized understanding how the affine Lie algebra is a simple generalization of a finite algebra. We now examine the construction of an affine Lie algebra as a centrally extended loop algebra with derivation, which is the usual presentation of an affine algebra and is most useful for understanding weight vectors and their inner product.

A loop algebra is a map from the circle $S^1$ to a Lie algebra $g$. If $\{T^a\}$ are a basis for $g$, the loop algebra has basis $\{T_n^a\} \equiv \{T^a \otimes z^n\}$. This loop algebra can be given a nontrivial central extension by adding the generator $K$, satisfying $[K, T_n^a] = 0$, so that the brackets are

$$[T_n^a, T_m^b] = f^{abc} T_{n+m}^c + n \delta_{n+m,0} \delta^{ab} K, \quad (5)$$

where we have diagonalized the Killing form of $g$. Additionally one includes the derivation $D$ (sometimes called $L_0$) which measures the grade $n$:

$$[D, T_n^a] = n T_n^a, \quad [D, K] = 0. \quad (6)$$

\footnote{We will be dealing exclusively with simply laced algebras, and so will not distinguish between Coxeter labels and dual Coxeter labels.}
These generators and their brackets define the affine algebra \( \hat{g} \). Notice that the elements with \( n = 0 \) generate a subalgebra isomorphic to \( g \):

\[
[T^a_0, T^b_0] = f^{abc} T^c_0 ,
\]

(7)
called the horizontal subalgebra of \( \hat{g} \); since \( \hat{g}_{\text{hor}} \cong g \), we shall use them interchangeably.

Using a Cartan-Weyl basis for \( \hat{g} \), the algebra is generated by \( \{ H^i_n, E^\alpha_n, K, D \} \). The Cartan generators can be chosen to be \( \{ H^i_0, K, D \} \), and their eigenvalues will characterize a weight vector for any state:

\[
\lambda = (\lambda, k, n) ,
\]

(8)
where \( \lambda \) is a horizontal weight, \( k \) is the eigenvalue of \( K \) and is called the level, and \( n \) is the grade.

The roots are the weights of the adjoint representation. They are \( (\pi, 0, n) \) corresponding to the generators \( \{ E^\alpha_n \} \) and \( (0, 0, n) \) corresponding to the generators \( \{ H^i_n \} \); note that since \( K \) is central all roots have vanishing level. Simple roots are chosen such that an arbitrary root can be expressed as a linear combination with coefficients of definite sign. We choose the set

\[
\alpha_i = (\alpha_i, 0, 0) , \quad \alpha_0 = (-\theta, 0, 1) ,
\]

(9)
where \( \alpha_i \) are the simple roots of \( g \) and \( \theta \) is the highest root of \( g \). Additionally, we define the imaginary root \( \delta \):

\[
\delta \equiv (0, 0, 1) ; \quad \delta = \alpha_0 + \theta = \alpha_0 + \sum_{i=1}^r c^i \alpha_i ,
\]

(10)
where \( r \) is the rank of \( g \). Note this is the same \( \delta \) we encountered before.

As with simple Lie algebras, an inner product is induced on the space of weights by means of the Killing form. This inner product is interesting for its “light-cone” structure on the level and grade of weights:

\[
(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2) + k_1 n_2 + k_2 n_1 .
\]

(11)
The inner product of the simple roots gives the affine Cartan matrix. From this point of view \( \delta \) has vanishing inner product with all roots precisely because they have \( k = 0 \).
As a consequence of $a_i = (\lambda, \alpha_i)$ and $k = (\lambda, \delta)$ for some weight $\lambda$ and the definition (10), we have

$$k = a_0 + \sum_{i=1}^{r} c^i a_i,$$

(12)

and thus the $\{a_i\}, i = 0 \ldots r$ determine $k$, or conversely $k$ and $\lambda$ determine $a_0$.

The adjoint is not a highest weight representation. Highest weight representations do exist, and are all infinite-dimensional. They are formed by beginning with a highest weight and subtracting simple roots according to the values of the Dynkin labels as usual. Due to the degeneracy of the Cartan matrix, there is no end to the process, and the grade $n$ never stops decreasing — $\delta$ can always be subtracted from some weight without leaving the representation. For the “integrable” highest weight representations, however, there will be a finite (reducible) representation of the horizontal subalgebra at each grade. Thus the highest weight representations have a pyramid structure. The level $k$ is a constant for an entire representation, as the simple roots have $k = 0$. The condition for integrability is just that for the highest weight all Dynkin labels, $a_0 \ldots a_r$, must be non-negative integers. This implies that the horizontal piece is a horizontal highest weight, and that the level is bounded below

$$k \geq \sum_{i=1}^{8} c^i a_i \geq 0.$$ 

(13)

The grade of the highest weight vector is conventionally normalized to $n_0 = 0$.

3 Generalities of affine algebras on branes

String junctions with support on configurations of mutually nonlocal 7-branes fill out representations of a Lie algebra determined by the 7-branes. A given 7-brane is characterized by the NSNS and RR charges $[p, q]$ of the string which can end on it. Each 7-brane will influence a junction $J$ through a combination of two processes: segments of $J$ may have a string prong ending on the 7-brane, or they may undergo a monodromy in crossing the 7-brane’s branch cut. Both situations transfer a certain amount of $(p, q)$ charge to the relevant segment of the junction, and can be deformed into one another via a Hanany-Witten effect [15]. Both are taken into account by the invariant charge $Q^a(J)$, the integer multiple of $[p_a, q_a]$ added to the junction $J$ by the $a^{th}$ 7-brane in toto [2]. Since the $\{Q^a\}$ are integral, the space
of inequivalent junctions is a lattice. The Lie-algebraic weight vector $\bar{\lambda}$ associated to $J$ is determined by the invariant charges.

An inner product exists on the space of junctions, arising from the intersection of complex curves in M/F-Theory which reduce to the junctions in IIB [16, 17]. The junctions corresponding to gauge bosons in the 8D worldvolume begin and end on the 7-branes and so carry no asymptotic $(p, q)$ charge to infinity. This constrains the adjoint junctions to live in a sublattice of codimension two. The simple root states $\{\alpha_i\}$ can be chosen to span this sublattice, and their mutual intersection form produces (minus) the Cartan matrix of the appropriate Lie algebra. The Dynkin labels characterizing the weight vector associated to a junction $J$ are given by $a_i = -\langle J, \alpha_i \rangle$.

Since the Lie algebras on 7-branes are generated by the Cartan matrices, an affine Lie algebra should arise when the uncharged junctions produce an affine Cartan matrix in their intersection form. Thus despite the complications of the affine algebra, it will appear when just a single brane is added appropriately to an $ADE$ configuration.

Once the simple roots for the affine algebra are identified, the entire root system follows. The roots for the horizontal part can be found as they were in the finite case [2]. Due to the degeneracy of the Cartan matrix, the imaginary root junction

$$\delta = \alpha_0 + \sum_{i=1}^{r} c_i \alpha_i.$$  \hspace{1cm} (14)

has vanishing intersection with all simple roots, and thus with all roots, including itself. Thus the junctions of self-intersection $(-2)$ are precisely the horizontal roots $\alpha$ plus arbitrary factors of $\delta$, $\{\alpha + n \delta \mid n \in \mathbb{Z}\}$, corresponding to the generators $\{E_n\}$. Affine algebras also have roots with zero self-intersection, the $\{n \delta \mid n \in \mathbb{Z}/\{0\}\}$, corresponding to $\{H_i\}$.

The imaginary root junction $\delta$ has interesting properties. In the case of finite algebras on branes, it was not possible to have a nonzero junction that had vanishing intersection with simple root junctions and zero $(p, q)$ charge. However in the affine case, $\delta$ possesses just these traits. In all the cases examined it can be presented as a loop of string that encircles the 7-brane configuration. Naturally this has no asymptotic charge, and the fact that it has no Dynkin labels is intuitive since it does not intersect any of the simple roots, which all lie “within” it (see figures in the next section). Thus it seems that the condition that a brane configuration admit an affine algebra can be restated as the condition that the monodromy matrix admit a nontrivial eigenvector, so that there exists a $(p, q)$ charge preserved winding
around the branes.

Although the uncharged junctions have degenerate intersection form, the entire junction lattice is generically nondegenerate for affine Lie algebras. As a consequence, the affine Cartan matrix cannot be block diagonal within the junction lattice intersection form, as it was for the finite case. There is an inevitable “mixing” between the uncharged and charged sectors, which manifests itself in the association of the level $k$ with asymptotic charge, as we now explore.

The level $k$ should remain the same from junction to junction in a given representation. In fact, it turns out to be a linear function of the asymptotic charges, $k = k(p, q)$, which of course also are constant over a given representation, as the simple roots are uncharged. Let us argue that this is reasonable. For finite Lie algebras, an arbitrary junction could be characterized completely by its $r$ Dynkin labels $\{a_i\}$ and $p, q$; the intersection form block diagonalized into an uncharged “Lie algebra” part of codimension two, and a charged part characterized by $(p, q)$. For the affine case there are still two more invariant charges than there are Dynkin labels; however, the junction $\delta$ has vanishing $a_i$ and $(p, q)$. There is thus an additional integer $\tilde{n}$ which changes when $\delta$ is subtracted from a junction. We now have $(r + 4)$ labels $\{a_i, p, q, \tilde{n}\}$, $i = 0 \ldots r$ characterizing a junction $J$, while the junction lattice is only $(r + 3)$-dimensional. Since $\tilde{n}$ is independent of the others, there must be a relation among the $\{a_i, p, q\}$. However, $p$ and $q$ are constant over a representation, while the $\{a_i\}$ change from weight to weight. The only Lie-algebraic quantity that could be related to the asymptotic charges without putting constraints on the possible weight vectors is the level $k$, which is also a constant over a representation. Thus it is not too surprising that explicit computation confirms $k = k(p, q)$ in each case.

In summary, the level of the affine weight vector associated to a junction is determined by the asymptotic charges; equivalently, fixing the level of a representation puts one constraint on the asymptotic charges that representation can possess. After fixing an affine weight vector, one is still free to choose the other linear combination of $(p, q)$, as well as $\tilde{n}$.

We will employ a basis in which a junction $J$ may be expanded as

$$J = \sum_{i=1}^{r} a_i \Omega^i + k \Omega^0 + \tilde{n} \delta + \sigma \Sigma. \quad (15)$$

The junctions $\{\Omega^i, \Omega^0\}$, $i = 1 \ldots r$ are dual to the set $\{\alpha_i, \delta\}$, $i = 1 \ldots r$, and thus their coefficients are the $\{a_i\}$ and the level $k$. Of these only $\Omega^0$ has asymptotic charge, as it must
since it determines \( k \). Both \( \delta \) and \( \Sigma \) are orthogonal to all simple roots. \( \delta \) we have already discussed and is uncharged, while \( \Sigma \) must have nonzero asymptotic charge; in fact its \((p, q)\) charge is the same as that of the loop of string realizing \( \delta \). (Of course \( \delta \) has no asymptotic charge, it merely carries charge as it winds around the branes.) We constrain it to satisfy \((\delta, \Sigma) = 0\). The coefficient \( \sigma \) determines the \((p, q)\) charges not already specified by \( k \).

A basis including more familiar fundamental weight junctions \( \{\omega^i\} \), \( i = 0 \ldots r \) dual to all simple root junctions \( \{\alpha_i\} \), \( i = 0 \ldots r \) is also possible, but proves less convenient, as we shall discuss further in the next section.

As the self-intersection \((J, J)\) has been used as a powerful tool for analyzing the spectra of 3-brane worldvolume theories \[10\], it is useful to consider it for the affine case. Thus we are interested in the mutual intersections of \( \{\Omega^i, \Omega^0, \delta, \Sigma\} \). The \( \{\Omega^i\} \) are orthogonal to \( \Omega^0 \), \( \delta \) and \( \Sigma \), and thus the intersection form block diagonalizes in this basis. In the upper block, \((\Omega^i, \Omega^j) = -A^{ij} \) is minus the inverse Cartan matrix of the finite algebra \( g \). In the lower block, \((\Omega^0, \delta) = -1 \) is guaranteed since they are dual. Recalling \((\delta, \Sigma) = (\delta, \delta) = 0\), we find that \( \tilde{n} \) will appear only in a linear term with \( k \). The self-intersection of a junction is then

\[
(J, J) = -(\bar{\lambda}, \bar{\lambda})_{\text{finite}} - 2 \tilde{n} k + f(p, q),
\]

where \( \bar{\lambda} \) is the horizontal weight vector, and \( f(p, q) \) is a quadratic form in \((p, q)\) obtained from reexpressing \( k \) and \( \sigma \) in the \((\Sigma, \Sigma)\), \((\Omega^0, \Omega^0)\) and \((\Omega^0, \Sigma)\) terms, which vary from algebra to algebra.

The grade \( n \) of a weight vector is the number of imaginary roots added to the highest weight, and since \( \tilde{n} \) is the coefficient of the imaginary root junction, \( n \) will differ from \( \tilde{n} \) by at most a representation-dependent constant, \( \tilde{n}_0 = \tilde{n} - n \). We thus see that the self-intersection includes the affine inner product,

\[
(J, J) = - (\lambda, \lambda)_{\text{affine}} - 2 \tilde{n}_0 k(p, q) + f(p, q),
\]

and that the remaining contribution is entirely a function of the \((p, q)\) charges.

There will always be (at least) one brane which, when it is decoupled, will leave behind the configuration associated to the finite algebra \( g \). In the basis above, we can require that only \( \delta \) has support on this brane. As a result, the junctions which survive a decoupling from the affine algebra \( \hat{g} \) to the horizontal algebra \( g \) are precisely those with \( \tilde{n} = 0 \). Thus the self-intersection formula returns to the finite expression, and consequently \( f(p, q) \) is the same quantity that appears in the finite case.
The power of the affine algebra is that junctions must organize themselves into affine representations. In specifying a highest weight representation, one must choose \( k \) and a finite highest weight vector \( \lambda_0 \). One linear combination of the asymptotic charges is now determined by the level. The other combination of charges is still free to be chosen (modulo conjugacy constraints, as we will discuss shortly), as is the integer \( \hat{n}_0 \), which can be anything as long as the resulting junction has a good self-intersection, \((J, J) \geq -2 + \gcd(p, q)\). The grade of a weight vector in the representation will then be \( n = \hat{n} - \hat{n}_0 \).

The affine Weyl group preserves the affine inner product. If a weight is in a representation, so is its entire Weyl orbit. Since the affine inner product appears as the only contribution of the weight vector to the self-intersection, entire Weyl orbits will be forbidden or allowed collectively. Thus the self-intersection bound will be consistent with the structure of affine representations.

As is well-known, \( k > 0 \) is necessary for affine highest weight representations. For \( k < 0 \) we will instead find lowest weight representations, which terminate below at an antidominant weight vector with all Dynkin labels negative, and continue forever in the direction of increasing grade. This is in agreement with our expectations that changing all the signs of the invariant charges (and thus the weight vectors) of a representation should give us an identical or conjugate representation, and is consistent with self-intersection as well, which for \( k < 0 \) will forbid any affine weight vector with sufficiently low grade. Representations with \( k = 0 \) will be infinite in both directions; the adjoint is an example of this. In this case self-intersection is independent of the grade entirely.

Affine Lie algebras inherit unchanged the conjugacy restrictions of the corresponding finite algebra. As discussed in [2], when a Lie algebra has different conjugacy classes, some possible values of \((p, q)\) for a representation are excluded depending on the conjugacy class. This is a result of the fundamental weight junctions \( \Omega^i \) (\( \omega^i \) in [3]) not being proper, while requiring that the junctions in the representation itself are proper. In general if there are \( d \) conjugacy classes, \( p, q \) or a linear combination will be restricted to a certain value \((\text{mod } d)\). This remains in the affine case because the “new” junction in the fundamental weight basis is \( \delta \), which is always proper, as it is an integral combination of simple roots.

In Table 1, we present a list of \( A-, B- \) and \( C- \) brane configurations on which affine Lie algebras arise. The entire affine \( E_n \) series can be realized for \( n \leq 8 \), with \( \widehat{E}_5 = so(10) \), \( \widehat{E}_4 = su(5) \), \( \widehat{E}_3 = su(3) \times su(2) \), \( \widehat{E}_2 = su(2) \times u(1) \), \( \widehat{E}_1 = su(2) \). Some can be realized in more than one
Brane configuration | Affine algebra | \( k = k(p, q) \) | Brane decoupled
--- | --- | --- | ---
\((\text{A}^{n-1})\text{BCBC}\) | \(\tilde{E}_n\) | \(k = -q\) | \(b_2\)
\((\text{A}^8)\text{BCC}\) | \(E_9 \equiv \tilde{E}_8\) | \(k = -p + 3q\) | \(a_8\)
\((\text{A}^6)\text{BC}^3\) | \(\tilde{E}_7\) | \(k = -p + 2q\) | \(c_1\)
\((\text{A}^4)\text{B}^2\text{C}^2\) | \(\text{so}(10)\) | \(k = -p + 2q\) | \(b_1\)
\((\text{A}^8)\text{BCB}\) | \(\text{so}(16)\) | \(k = -p + q\) | \(b_2\)
\((\text{A}^8)\text{CBC}\) | \(\text{su}(8) \times \text{su}(2)\) | \(k = -p + 7p\) | \(b\)
\((\text{A}^4)\text{B}(\text{A}^4)\text{CC}\) | \(\text{su}(8) \times \text{su}(2)\) | \(k = -3p + 5q\) | \(b\)

Table 1: Brane configurations giving rise to affine Lie algebras, including the relation between the level \(k\) and the asymptotic charges \((p, q)\) and the brane which gives the finite algebra upon decoupling.

way. Additionally there are a few other affine algebras of horizontal rank 8. Perhaps there are other possibilities. Definitions for A-, B- and C-branes and their monodromies can be found in [2, 11]; the associated \([p, q]\) charges are \([1,0]\), \([1,-1]\) and \([1,1]\) resepctively.

Other, more complicated algebras can appear on 7-branes as well. In particular, the brane configuration \((\text{A}^{n-1})\text{BCC}\) realizes the \(E_n\) series, including not only \(E_9 \equiv \tilde{E}_8\) but the hyperbolic algebra \(E_{10}\) and all the rest. We will not discuss these other algebras in this paper.

### 4 Examples

#### 4.1 Affine \(\text{su}(2)\)

We now proceed to illustrate these ideas with a discussion of the brane realization of the simplest affine Lie algebra, \(\tilde{\text{su}}(2)\), which appears on the brane configuration \(\text{BCBC}\).

There are four invariant charges, \((Q^1_B, Q^1_C, Q^2_B, Q^2_C)\). The intersection form in this basis is

\[
\begin{pmatrix}
-1 & 1 & 0 & 1 \\
1 & -1 & -1 & 0 \\
0 & -1 & -1 & 1 \\
1 & 0 & 1 & -1
\end{pmatrix}.
\]  

The conditions for uncharged junctions are \(Q^2_B = -Q^1_B\), \(Q^2_C = -Q^1_C\). The natural choice for
Figure 1: The Dynkin diagram for the $\widehat{su}(2)$ algebra, where the double line with arrows indicates an angle of 180° between roots of equal length, and the BCBC brane configuration including simple roots $\alpha_0$, $\alpha_1$ and the imaginary root $\delta$.

A basis for the uncharged junctions is the set of simple roots

$$\alpha_0 = b_1 - b_2, \quad \alpha_1 = c_1 - c_2,$$

having the intersection form

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

which is indeed the negative of the $\widehat{su}(2)$ Cartan matrix.

The Coxeter labels for $su(n)$ algebras are all 1, and hence $\delta = \alpha_0 + \alpha_1$. Indeed we can confirm that $(\delta, \alpha_i) = 0$. The monodromy matrix for these branes is

$$\begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$$

which admits the eigenvector $(1, 0)$; $\delta$ can be presented as a $(1, 0)$ string winding around the BCBC configuration.

Taking the intersection $(\delta, J)$ for an arbitrary junction $J$, we have

$$a_0 + a_1 = k = Q_B^1 - Q_C^1 + Q_B^2 - Q_C^2 = -q,$$

the desired relation between the level and the asymptotic charges. Specifying the affine weight vector by $a_0, a_1$ thus fixes $q$ as well, leaving $p$ and $\tilde{n}$ to be determined.

One would like to expand an arbitrary junction in a basis such that the Dynkin labels, charges and $\tilde{n}$ are the expansion coefficients. Before we discuss our preferred basis $\{\Omega^i, \delta, \Sigma\}$, let us explore the more natural place to start, namely junctions $\{\omega^0, \omega^1\}$, dual to $\{\alpha_0, \alpha_1\}$ and satisfying the usual inner products of fundamental weights: $(\omega^1, \omega^1) = -1/2, (\omega^0, \omega^0) = (\omega^0, \omega^1) = 0$. One would then add to these $\delta$ and a charged junction $\Sigma$, both orthogonal to simple roots:

$$J = a_0 \omega^0 + a_1 \omega^1 + \tilde{n} \delta + \sigma \Sigma.$$
Notice that since the level is \( k = -q \), \( \sigma \) is a combination of asymptotic charges including \( p \).

There are many choices of invariant charges for such junctions, one of which is

\[
\begin{align*}
\Omega^0 &= \left( -\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, -\frac{1}{4} \right), \\
\Omega^1 &= \left( -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{3}{4} \right), \\
\delta &= (1, 1, -1, -1), \\
\Sigma &= \left( \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2} \right).
\end{align*}
\] (23)

There are disadvantages to this presentation. In particular, \( \tilde{n} = \frac{3}{4}Q^1_B - \frac{1}{4}(Q^1_C + Q^2_B + Q^2_C) \), which generically is not an integer, a situation which is not disastrous but is inconvenient.

Additionally, we are interested in the condition \( Q^2_B = 0 \) for a junction surviving the decoupling of a \( B \)-brane to the non-affine \( su(2) \), \( BCC \) configuration, which in this basis has the arcane form \( \tilde{n} = \frac{1}{4}(p + 2k) \). No other realization of the \( \{\omega^i\} \) improves the situation. In the \( \{\Omega^i\} \) basis, however, \( \tilde{n} \) is integral and the condition for decoupling is simple.

We take the junctions \( \{\Omega^0, \Omega^1\} \) to be dual to \( \{\delta, \alpha_1\} \). As in (17), a junction is given by

\[
J = a_1 \Omega^1 + k \Omega^0 + \tilde{n} \delta + p \Sigma,
\] (24)

where \( k \) now appears because \( (\Omega^0, \delta) = -1 \), and \( \sigma = p \) in this basis. The basis junctions are

\[
\begin{align*}
\Omega^0 &= \left( 0, \frac{1}{2}, 0, -\frac{1}{2} \right), \\
\Omega^1 &= \left( \frac{1}{2}, -\frac{1}{4}, 0, -\frac{1}{4} \right), \\
\delta &= (1, 1, -1, -1), \\
\Sigma &= \left( \frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4} \right).
\end{align*}
\] (25)

We now have the simple relation \( \tilde{n} = -Q^2_B \), which is always an integer for proper junctions, and implies \( J \) survives decoupling precisely when \( \tilde{n}(J) = 0 \). This is very convenient, as the integer \( \tilde{n} \) measures directly the affine support of a given junction.

Self-intersection in this basis is

\[
(J, J) = -\frac{1}{2}(a_1)^2 - \frac{7}{8}k^2 - 2\tilde{n}k - \frac{1}{4}kp + \frac{1}{8}p^2,
\] (26)

reproducing the form (10).

To find the junction realization of a given representation, we specify the representation by the level \( k \) and the Dynkin label \( a_1 \) of the highest weight. \( q = -k \) is now determined. There
are two conjugacy classes in \( su(2) \), given by \( a_1 \pmod{2} \). Requiring an arbitrary junction to be proper results in the condition

\[
\frac{1}{2}(p + q) = a_1 \pmod{2},
\]

(27)

where junctions with \( p + q \) odd cannot be realized at all. Lastly one may pick any \( \tilde{n}_0 \) for the highest weight as long as \((J, J) \geq -2 + \gcd(p, q)\). Other weights are found by subtracting \( \alpha_i \), and \( n = \tilde{n} - \tilde{n}_0 \).

Consider for example the representations at \( k = 1 \). The level fixes \( q = -1 \). We can have \( (a_0, a_1) \) either \((1, 0)\) or \((0, 1)\) for the highest weight. For \((1, 0)\), \( p - 1 = 0 \pmod{4} \). We can for example choose \( p = 1 \) in which case \( f(p = 1, q = -1) = -1 \). Also \( \gcd(p = 1, q = -1) = 1 \), and so finally we find \((J_0, J_0) = -2\tilde{n}_0 - 1 \) and \( \tilde{n}_0 \leq 0 \). We could choose \( \tilde{n}_0 = 0 \), in which case \( n = \tilde{n} \) and the only junction in the entire representation to survive decoupling to finite \( su(2) \) is the \( su(2) \) singlet at \( \tilde{n} = 0 \), which has \( Q_B^1 = 1 \), others zero. For other choices of \( \tilde{n}_0 \), no junctions satisfy \( \tilde{n} = 0 \) and the entire representation fails to survive decoupling.

A representation with \( k = 1 \) and highest weight \((0, 1)\) requires \( p - 1 = 2 \pmod{4} \). Let us this time pick \( p = -5 \), which gives \( f(p = -5, q = -1) = 7/2 \) and \( \tilde{n} \leq 2 \). Choosing \( \tilde{n}_0 = 2 \) means that the junctions which survive decoupling exist at \( n = -2 \) in this representation; the finite \( su(2) \) content is a spin-3/2 and a spin-1/2 \cite{18}. Other choices of \( \tilde{n}_0 \) produce other finite pieces upon decoupling.

### 4.2 Affine E8

Having seen how the minimal affine algebra \( \hat{su}(2) \) is realized on branes, we now turn to explore as a final example \( \hat{E}_8 \). The Dynkin diagram is displayed in Figure 2.

The finite \( E_8 \) algebra can be realized on the brane configuration \((A^7)BCC\) \cite{1}, and we follow the conventions for simple roots of \cite{2}. The affine Lie algebra \( \hat{E}_8 \) can be realized in more than...
The $\hat{E}_8$ algebra realized on the brane configuration $(A^7)BCBC$, with the simple roots $\alpha_i$ and imaginary root $\delta$ indicated.

one way. The finite $su(2)$ and its affine counterpart $\hat{su}(2)$ are part of a series covering all the exceptional algebras where the finite $E_n$ algebra is realized on $(A^{n-1})BCC$ and $\hat{E}_n$ is realized on $(A^{n-1})BCBC$, with the quadratic form $f(p, q) = \frac{1}{9-n}(p^2 - (n-3)pq + (2n-9)q^2)$. Hence $\hat{E}_8$ appears on $(A^7)BCBC$. Additionally, if one tries to extend the finite series beyond $E_8$, one gets to $E_9$ which is simply $\hat{E}_8$ again; the algebra thus also appears on $(A^8)BCC$. In all cases the asymptotic charge quadratic form is $f(p, q) = p^2 - 5pq + 7q^2$.

While $f(p, q)$ coincides for both realizations of $\hat{E}_8$, since both must reduce to the same $E_8$ upon the appropriate decoupling, the appearance of $\delta$ is different. For the entire $(A^{n-1})BCBC$ series, $\delta = b_1 + c_1 - b_2 - c_2$, which can be realized as a $(1, 0)$ string winding around the entire configuration; since it is insensitive to $A$-brane monodromies the appearance of new $A$-branes does not disturb it. As a result $k = -q$ for all these configurations, including the $(A^7)BCBC \hat{E}_8$. \{\alpha_i\}, $i = 1 \ldots 8$ are the same as the finite case, and $\alpha_0$ is then

$$\alpha_0 = \delta - \theta = - \sum_{i=1}^7 a_i - a_7 + 5b_1 + 3c_1 - b_2 + c_2,$$

and the $\hat{E}_8$ Cartan matrix is reproduced. The simple root junctions and the imaginary root $\delta$ on this brane configuration are shown realized as simple strings in Figure 3.

On the other hand, the monodromy of the $(A^8)BCC$ branes admits the eigenvector $(3, 1)$, and there is a $\delta$ junction realized by a $(3, 1)$ string winding around the branes. This junction
Figure 4: The $\hat{E}_8$ algebra realized on the brane configuration $(A^8)BCC$, with the simple roots $\alpha_i$ and imaginary root $\delta$ indicated.

coincides with that obtained by defining $\alpha_0 = a_7 - a_8$ and using

$$\delta = \alpha_0 + \theta = -\sum_{i=1}^{8} a_i + 4b + 2c_1 + 2c_2.$$  \hspace{1cm} (29)

Calculating $k = -(J, \delta)$ for an arbitrary $J$, we arrive at $k = -p + 3q$. The junctions for this configuration are displayed in Figure 4.

In both cases, we can expand a junction in the decoupling basis

$$J = \sum_{i=1}^{8} a_i \Omega^i + k \Omega^0 + \tilde{n} \delta + \sigma \Sigma,$$  \hspace{1cm} (30)

and $(J, J)$ is given by (18) with $f(p, q)$ given above. In both cases $\tilde{n} = 0$ gives the condition for junctions to survive decoupling; of course, $k$ is different for each. For $(A^7)BCBC$ we have $\sigma = p$, while for $(A^8)BCC$ we choose $\sigma = q$.

$E_8$ has no conjugacy classes and thus neither does $\hat{E}_8$.

Let us explore an example of a representation. Consider junctions of charge $(-1, -1)$ in the $(A^7)BCBC$ presentation. The charge quadratic form is $f(p = q = -1) = 3$. The level is fixed at $k = 1$. There is only one affine representation at this level, with highest weight $a_0 = 1, a_i = 0, i \neq 0$. For this weight $(J, J) = -2 \tilde{n}_0 + 3$, implying $\tilde{n}_0 \leq 2$.

In [19], worldvolume theories for D3-branes in the vicinity of 7-brane configurations were considered. It was seen that the 3875 representation must be in the spectrum of the finite $E_8$ algebra theory, as a result of consistency with the known $D_4$ spectrum, which is just Seiberg-Witten theory with $N_f = 4$ flavors. In the affine case, choose $\tilde{n}_0 = 2$. The junctions
that survive decoupling to the finite case are then at grade \( n = -2 \) of this representation, which has \( E_8 \) content [38]:

\[
\text{3875} \oplus \text{248} \oplus \text{1}
\]  

We then know that the 248 and 1 must be in the \( E_8 \) spectrum as well, since the 3875 which is known to be present must lift to some complete affine representation, and as this is the only possibility must be accompanied by the 248 and 1. Other choices of \( \tilde{n}_0 \) lead to other representations, with smaller or no surviving horizontal content. The 3875 does not appear and so makes no statement about the existence of these representations.

A similar process occurs for the (A8)BCC theory, where the \((-1, -1)\) charges will be at level \( k = -2 \). There the 3875 will lift to the \( n = 0 \) grade of a lowest weight representation, and it is the only horizontal representation at that grade.

Before concluding, we note that one could inquire about the brane configuration (A8)BCBC, which would be “affine \( E_9 \)”. This algebra is the combination of two \( so(8) \) singularities as studied by Imamura [9] and cannot be realized as a singularity on K3 without destroying the triviality of the canonical bundle. It can be thought of as finite \( E_8 \) with two different imaginary root junctions, \( \delta_1 \) and \( \delta_2 \), which have vanishing intersection with the \( E_8 \) simple roots, themselves and each other. Both can arise because the monodromy matrix is unity and thus any \((p, q)\) string can wind around the configuration. The uncharged junction sublattice has dimension 10, and so it cannot be any affine algebra, which have rank \( \leq 9 \); but the Cartan matrix is degenerate, so neither is it hyperbolic (in particular it is not \( E_{10} \)). It is not clear what kind of Lie algebra this is.

5 Conclusions

We have considered the realization of affine Lie algebras as string junctions on configurations of 7-branes. We have discussed how Lie algebras arise from the intersection form of the holomorphic curves associated to the junctions in the M/F-theory picture, when this form contains the appropriate Cartan matrix. When an affine Cartan matrix is realized, the junction associated to the imaginary root manifests itself as a loop of string whose charges are a nontrivial eigenvector of the branes’ monodromy matrix. The level of the affine algebra is equal to a linear combination of the asymptotic charges, and thus is assured to be a constant in a given representation.
The intersection form includes the full affine inner product. Unlike the inner product of finite Lie algebras, the affine version is of indefinite signature. It arises naturally as part of the junction intersection form, which has indefinite signature for both finite and affine cases. In the finite case the positive and negative eigenvalues are segregated into “Lie algebra” and “asymptotic charge” blocks. In the affine case however, the affine Cartan matrix is degenerate and so cannot be block diagonal within the intersection form; the off-diagonal elements mix the positive and negative eigenvalues and this results in the appearance of the affine inner product, as well as combining the Lie algebra and charge sectors so as to fix the relation between \( k \) and \((p, q)\).

The brane configuration associated to an affine algebra cannot coalesce to a single point on K3, but junctions must nonetheless fill out representations of the affine algebra. These representations are infinite-dimensional, but only a finite number of junctions can be massless. It has been only partially known which representations of finite algebras actually appear as BPS states in 3-brane worldvolume field theories; the structure of the affine representations and the known existence of a few finite representations requires the presence of many more.

We have laid the Lie-algebraic groundwork for understanding affine algebras on 7-branes. Many applications remain to be examined. A more complete study of which representations are required in the spectra of the 4D, \( E_n \) field theories still remains to be done. Assuming these theories to be the same as those obtained by compactifying the 6D non-critical string on a torus, a fuller geometrical investigation of the duality between these two configurations would be interesting. This could conceivably involve the \((A^8)BCBC\) brane configuration which characterizes a \( B_9 \) del Pezzo surface, or \( \frac{1}{2}K3 \). The non-critical string theory and its compactifications are still poorly understood, but the prospects for investigating them are becoming brighter.

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