Loop Adaptive Subdivision Surface

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Abstract. Subdivision surface has a good advantage in constructing 3D model of complex geometric shape. The subdivided surface is closer to the real model, which is beneficial to the generation of tool path in machining, and has a broad development prospect in the field of automatic machining. With the subdivision progresses, mesh patch number increased and huge amount of data that is inconvenient for computer storage and operation. The distance formula between the mesh vertex of and the limit position of the vertex is derived in this paper. Given the arbitrary precision threshold, the k value is calculated to determine the subdivision depth, and the adaptive error control is proposed, then the subdivision problem of complex surface model is solved. The example simulation and theoretical analysis results show that the proposed method determines the depth in subdivision of the surface model in any precision range, avoids the data redundancy caused by the unlimited subdivision of the surface model. The new mesh provides convenience for the subsequent study of tool path planning.

1. Introduction

Surface subdivision is an important foundation of digital machining and manufacturing. It plays a key role in the process of generating tool path and geometric modeling in NC machining of complex parts. In the past few years, surface models were mostly represented by parametric surfaces, such as NURBS free form surfaces, which can be easily obtained by subdivision. Compared with parametric surface model, the subdivision of complex surface model is an iterative process, and there is no specific analytical expression, so it becomes difficult to calculate in the process of subdivision. How to make better use of surface information for subdivision approximation and depth estimation is a key problem for surface subdivision and improving surface precision.

Wu,J.H. and Liu,W.J used vertex flatness as the threshold of adaptive subdivision, and proposed an adaptive subdivision algorithm for triangular meshes[1]. Amresh and Farin adopted the dihedral angle between adjacent triangles as the threshold to participate in the next subdivision, and also proposed the watershed segmentation method to divide the surface into different regions, then determined the regions that needed to participate in the next subdivision[2]. Zhang,Z and Feng,Y[3] proposed that the Loop local subdivision method should be used to deal with the mesh model before the rotation tool path was generated. Chen,T.T and Zhao,G[4] regarded the chord length error of triangle as Loop adaptive subdivision threshold, and achieved the purpose of adaptive subdivision by interpolation. Zhou,H and Zhou,L.S proposed the algorithm of generating triangular mesh model by fitting. The control vertices of fitting subdivision surface were solved by modifying the constraint of the control grid vertex, optimizing the shape of the mesh and local adaptive subdivision to constructing the subdivision mesh model[5]. Xu,J.T and Liu,W.J mentioned that triangular mesh was a linear
approximation of the initial model. Because of the loss of the model precision, it will be difficult to generate accurate tool path. How to subdivide the triangle mesh and improve the accuracy of the model is also a key issue of tool path.

This paper discussed the problems of precision loss, difficulty in calculating subdivision depth and the amount of data in subdivision surface, and gave the calculation formula of subdivision depth under arbitrary precision threshold, which has successfully solved the precision determination of surface mesh model and the calculation of subdivision depth. Under the condition that the mesh surface is smooth and the surface subdivision depth is reasonable, an adaptive subdivision algorithm with error control is proposed for Loop adaptive subdivision surface machining by setting precision threshold \( \epsilon \), which satisfies the user’s requirement for arbitrary refinement.

2. Loop subdivision surface

In the actual machining, the precision and quality of surface are required. The Loop subdivision model based on triangular meshes can maintain \( C^2 \) continuity at rules and \( C^1 \) continuity at singular points. The Loop subdivision surface model can ensure the topological relationship between the triangles and improve the efficiency of tool path generation. Loop subdivision is an approximate face splitting mode, which is shrinking in the subdivision process, the number of triangular patches increases at a speed of 4 times, so it is also called the limit surface.

2.1 Subdivision rule

Loop subdivision is a split mode based on triangle faces. It is first proposed by Charles Loop of the United States in 1987. The basic idea is to insert new vertices on each side of the triangle surface, and connect the new vertices together in turn to form four small triangles. The number of triangles will increase fourfold after subdivision, as shown in figure 1.

According to the geometric rules of the Loop mode, the calculation mode of the new vertex \( V_e \) (singular points) and \( V_v \) (even points) inserted inside the mesh surface is shown in figure 2.

The method of calculating the interior singularity \( V_e \) after subdivision:

\[
V_e = \frac{1}{8} (3V_i + V_2 + 3V_3 + V_4)
\]  

(1)

\( V_1, V_3 \) are the two vertex of the edge, and the two triangles that share this edge are \((V_1, V_2, V_3)\) and \((V_1, V_3, V_4)\). The method of calculating the interior even \( V_v \) after subdivision:

\[
V_v = (1 - n\beta_n)V + \beta_n \sum_{j=0}^{n-1} V_j
\]  

(2)

The adjacent points of the interior vertex \( V \) are \( V_0, V_1, ..., V_n \), where \( n \) is the order of the vertex \( V \) neighborhood, \( V \) is the old vertex corresponding to vertex \( V_i \), and the \( \beta_n \) is defined as:
\[ \beta_n = \begin{cases} \frac{3}{16}, & (n = 3) \\ \frac{1}{n} \left( \frac{5}{8} - \left( \frac{3}{4} + \frac{1}{4} \cos \frac{2\pi}{n} \right) \right), & (n > 3) \end{cases} \]  

(3)

The calculation mode of the boundary singular points and even points after subdivision once is shown in figure 3, where boundary singularity and boundary Even Point is defined as:

\[ V_E = \frac{1}{2}(V_1 + V_2) \]  

(4)

\[ V_1 = \frac{6}{8}V' + \frac{1}{8}(V_0 + V_1) \]  

(5)

Figure 3. Boundary singularity, even point calculation template diagram

In order to make the surface smooth at the boundary, Zorin method is used to modify the weight of the boundary point. The singularity of the interior edge of the boundary end is shown in figure 4, where

\[ \gamma_1 = \frac{1}{4}, \gamma_2 = \frac{1}{2} \]  

(6)

or

\[ \gamma_1 = \frac{1}{2} - \frac{1}{4} \cos \frac{2\pi}{n-1}, \gamma_2 = \frac{1}{4} + \frac{1}{4} \cos \frac{2\pi}{n-1} \]  

(7)

When \( \beta_n \) satisfies \( \frac{1}{4} \left( 1 - \cos \frac{2\pi}{n} \right) < \frac{1}{n} \beta_n < \frac{1}{4} \left( 1 + \cos \frac{2\pi}{n} \right) \), Loop surface is first order smooth. Through the subdivision rule, the limit surface is \( C^2 \) continuous, and \( C^1 \) is continuous at the singular point.

2.2 Calculation of vertex position of limit surface

Given the initial control mesh \( M^0 \), after loop mode subdivision, get the mesh sequences \( M^0, M^1, M^k \). The limit of a mesh sequence is called a subdivision surface, also called limit surface. Due to the subdivision surface has no specific analytic formula, the local subdivision matrix is analyzed to calculate the geometric properties of the subdivision surface. The limit position of the vertex by analyzing the subdivision matrix composed of the 1-neighborhood of the vertex \( V \). The initial control triangle mesh is \( M_0 \). After \( k \) times subdivision, the subset of the control mesh \( M_k \) is represented as \( M^k = (V^k, V_0^k, V_1^k, \ldots, V_n^k) \). Write equation (1) and equation (2) as a matrix expression of

\[ M^{k+1} = SM^k \]  

(8)

The matrix \( S \) is called the subdivision matrix after the \( k \) times subdivision of the control mesh \( M_k \). Combined with the subdivision rules introduced in section 1.1, the subdivision matrix is represented as
The expression of subdivision matrix shows that $S$ has the largest eigenvalue 1 and the repetition is 1, and its corresponding left eigenvector is $l_0 = \left[ \frac{3}{8\beta_n} + n \right]^\top \left[ \frac{3}{8\beta_n}, 1, 1, 1, \ldots, 1 \right]$, the control vertex $V$ converges to $V^\infty$. Let the 1-neighborhood of the control vertex $V$ be $\Psi^1 = \{V_0, V_1, \ldots, V_{n-1}\}$, and the vertex position on the limit subdivision surface is represented as

$$V^\infty = l_0 \Psi^1 = \left[ \frac{3}{8\beta_n} + n \right]^{-1} \left[ \frac{3}{3+8n\beta_n}, V + V_0 + V_1 + \cdots + V_{n-1} \right] = \frac{3}{3+8n\beta_n} V + \frac{8\beta_n}{3+8n\beta_n} \sum_{j=0}^{n-1} V_j$$  \(10\)

3. Adaptive subdivision based on error control

When the surface mesh model is subdivided by loop, the number of triangle faces is too much, which has a serious impact on the storage space and operation time, and the subsequent tool path planning is more difficult. Therefore, it is necessary to control the depth of subdivision and the number of triangulated surfaces by adopting appropriate adaptive subdivision method. The problem encountered in the process of subdivision is how many times a model needs to be subdivided to meet the specified accuracy threshold $\varepsilon$. Most of the previous researches on this issue are complicated. For example, three different error values are given to compare with the calculated maximum error in reference [9], an optimal error value is sought as the basis of the subdivision surface model. It is necessary to calculate the maximum difference between the upper and lower bounds of the coordinate components of the mesh, but the coordinate components must be parameterized. Although the maximum distance between the subdivision surface and its linear approximation is estimated by this method, the calculation steps are complicated.

3.1 Subdivision surface depth estimation

The maximum distance between the mesh vertex and its limit position is derived through the subdivision rules and the calculation of the limit position in the first section, and seeking subdivision times to satisfy precision to control subdivision depth. The convergence speed of each triangular patch is different in the process of subdivision, which leads to different distances between each vertex on the control mesh to its corresponding limit position. Therefore, the maximum distance can be calculated and compared it with the given precision threshold $\varepsilon$ to determine whether the subdivided mesh surface attain the precision requirement. This reduces the depth of subdivision, saves time and satisfies the needs of different users for surface accuracy and smoothness. Let $v^i_k$ be the vertex of the model after subdividing $k$ times. $k$ is 0 to represent the vertex on the initial control mesh, $v^\infty_i$ is the position of $v_i$ on the limit surface, and $\Pi_k$ is the set of vertex after the $k$ times subdivision. $d^i_k = \|v^\infty_i - v^i_k\|$ is the distance between $v^i_k$ and $v^\infty_i$. The maximum distance $d^i_{\text{max}}$ between the vertex and the limit position on the control mesh after $k$ times subdivisions is defined as

$$d^i_{\text{max}} = \max_{v^i_k \in \Pi_k} \|v^\infty_i - v^i_k\|$$  \(11\)

The limit position of the vertex is given by section 1.2, bring formula (10) into formula (11), we can get

\begin{equation}
\begin{bmatrix}
8 - 8n\beta_n & 8\beta_n & 8\beta_n & \cdots & 8\beta_n \\
3 & 3 & 1 & \cdots & 0 & 1 \\
3 & 1 & 3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
3 & 0 & 0 & \cdots & 3 & 1 \\
3 & 1 & 0 & \cdots & 1 & 3 \\
\end{bmatrix}
\end{equation}

\[(9)\]
\[ d_{\max}^k = \max_{v_i \in I^k} \left\| 8n \beta_n \left( \frac{1}{n} \sum_{j=0}^{n-1} v_j^i - v_i^k \right) \right\| \] (12)

Similarly,

\[ d_{\max}^{k+1} = \max_{v_i \in I^{k+1}} \left\| 8n \beta_n \left( \frac{1}{n} \sum_{j=0}^{n-1} v_j^{k+1} - v_i^{k+1} \right) \right\| \] (13)

Based on the convex hull property of the subdivision surface, the singular points on the same layer are smaller than their corresponding limited positions, so bring the even point calculation formula (2) in Loop subdivision mode to \( v_i^{k+1} \) in formula (13), as shown in figure 5, then

\[ d_{\max}^{k+1} = \max_{v_i \in I^{k+1}} \left\| \frac{5}{8} - n \beta_n \right\| 8n \beta_n \left( \frac{1}{n} \sum_{j=0}^{n-1} v_j^{k+1} - v_i^{k+1} \right) \right\| \] (14)

\[ \text{Figure 5. The 1-neighborhood subdivision of vertex } v \text{ before and after subdivision} \]

Assume \( n>3 \), \( \beta_n = \frac{1}{n} \left( \frac{5}{8} - \frac{3}{8} + \frac{1}{4} \cos \frac{2\pi}{n} \right)^2 \) is known by Loop subdivision rules, then

\[ \frac{5}{8} - n \beta_n = \left( \frac{3}{8} + \frac{1}{4} \cos \frac{2\pi}{n} \right)^2 \] (15)

Obviously \( \frac{5}{8} - n \beta_n \geq 0 \), combining formula (12), formula (14) satisfies the following inequality

\[ d_{\max}^{k+1} \leq \left( \frac{5}{8} - n \beta_n \right) d_{\max}^k \] (16)

Based on recursion, the following

\[ d_{\max}^{k+1} \leq \left( \frac{5}{8} - n \beta_n \right)^k d_{\max}^0 \] (17)

Let \( \varepsilon \) be the given accuracy error, and \( d_{\max}^{k+1} \leq \left( \frac{5}{8} - n \beta_n \right)^k d_{\max}^0 \leq \varepsilon \), after solving inequality

\[ k \geq \log_{\frac{5}{8} - n \beta_n} \sigma \] (18)

In formula (18), \( \rho = \left( \frac{5}{8} - n \beta_n \right)^{-1}, \sigma = \frac{d_{\max}^0}{\varepsilon} \). Given the arbitrary accuracy error \( \varepsilon \) and the initial control mesh model, the minimum number of subdivisions that can be calculated for the distance \( d_{\max}^k \leq \varepsilon \) between the subdivided mesh surface and the limit subdivision surface is

\[ k = \left\lfloor \log_{\frac{5}{8} - n \beta_n} \sigma \right\rfloor \] (19)

3.2 Adaptive subdivision algorithm step
The algorithm uses the distance $d_{\text{max}}^k$ between the vertex of the control mesh and the limit position of the vertex as the threshold for adaptive subdivision. In the subdivision process, the adaptive subdivision depth is controlled by setting the accuracy error $\varepsilon$. The error control adaptive subdivision algorithm is specifically described as follows:

Input: initial triangle mesh $M_0$.
Output: the subdivided triangular mesh $M_k$.

Step.1. Traverse the triangular mesh surface according to the subdivision rule and formula (10) respectively to calculate the vertex of the fine initial mesh and the limit position of the vertex.

Step.2. Traverse the even points inside the mesh, calculate the distance $d_i^k$ between the vertices of the triangle and its extreme position according to equation (12), and search for the maximum distance $d_{\text{max}}^k$.

Step.3. The precision error $\varepsilon$ is used to determine whether the mesh continues to subdivide, the minimum number of subdivision $k$ that satisfies this error is calculated by using equation (19).

Step.4. The mesh model is subdivided $k$ times to get a new triangular mesh model.

Step.5. Check whether there are any degraded points and triangle patches in the new mesh model, and remove them if there are any.

Step.6. Outputting the mesh model which meets the requirements, end.

Algorithm flow chart shown in figure 6:

![Algorithm Flow Chart](image-url)

**Figure 6. Error controlled adaptive subdivision algorithm flow chart**

4. Algorithm Verification And Analysis

4.1 Algorithm Example

In order to verify the effectiveness of the adaptive control based on error control proposed in this paper, the following figure 7 is the subdivision process of the face model. When the accuracy $\varepsilon = 0.05$, the adaptive subdivision algorithm using error control achieves the accuracy requirements after the face model is subdivided twice, and do not need to be further subdivision. The initial model is selected as
2500 vertices and 4802 triangular patches. After the first subdivision, the number of triangular patches is 19208 and the number of vertices is 9801. After the second subdivision, the number of triangular patches is 76,832 and vertices is 38,809.

![Initial model](image1) ![Subdivision one time](image2) ![Subdivision two times](image3) ![Local magnification of (c)](image4)

Figure 7. Adaptive subdivision face model diagram

From figure 7, with the increase of subdivision times, the amount of model data increases gradually, and the surface of the model tends to be smooth and the accuracy increases. Further verification by a computer, the results of subdividing twice and subdividing three times can be obtained. The smoothness and flatness of the surface are not much different. However, the amount of data and computer time need to subdivide three times is much larger than that of subdivision twice. Using this algorithm, after setting the precision value, then calculating the depth of subdivision can effectively avoid the above problems. The adaptive subdivision algorithm based on error control can quickly complete the surface subdivision process. The adaptive subdivision of the surface model saves storage space and running time, which facilitates the subsequent generation of a reasonable tool path.

Figure 8 is the experimental result of the subdivision of the mesh model, figure 8(b) is the result of the subdivision of the initial model when the given accuracy threshold $\varepsilon = 0.06$ ($k = 2$) and $\varepsilon = 0.007$ ($k = 3$). From figure (8), the smoothness of the subdivided surface model is different when given different accuracy. Selecting the appropriate precision threshold within a certain range can attain the needs of different users for the surface model. Using the method proposed in this paper, under the premise of ensuring the smoothness of the mesh model surface, given arbitrary accuracy $\varepsilon$, the depth of subdivision of the model is calculated from the formula (19), and the surface mesh model with different accuracy can be obtained.

![Initial mesh model](image5) ![Subdivision ε = 0.06](image6) ![Subdivision ε = 0.007](image7)

Figure 8. Mesh Model Adaptive Subdivision Surfaces with Different Accuracy

4.2 Subdivision Depth Analysis

By the formula (19),

$$k = \left| \log_\rho \sigma \right| = \left| \log_\rho \frac{d^0_{\text{max}}}{\varepsilon} \right| = \left| \log_\rho d^0_{\text{max}} - \log_\rho \varepsilon \right|$$

(20)

Combined with the previous section 2.1, $\rho > 1$ and $0 < \varepsilon \leq d^0_{\text{max}} \leq 1$, get $k = \left| \log_\rho d^0_{\text{max}} \right| + \left| \log_\rho \varepsilon \right|$. $\left| \log_\rho d^0_{\text{max}} \right|$ is regarded as constant, The relationship between subdivision depth and arbitrary precision $\varepsilon$ can be obtained from the above relation, as shown in figure 9.
From the above analysis, when the accuracy value is given, the least number of subdivision times of the surface mesh model can be calculated by formula (19). The subdivision depth of the face model shown in figure 7 is obtained under the given arbitrary accuracy. The experimental results show that when the given precision is less than 0.02, the face surface mesh model needs to be subdivided four times. Considering the factors such as large growth series and slowing run time, 3 times of subdivision is the best. For any mesh model, the relationship between precision \( \varepsilon \) and subdivision depth can be found by subdivision depth estimation formula (19), the depth of the mesh model needs to be subdivided and the precision range of the mesh model can be determined.

5. Conclusion
An adaptive subdivision algorithm for error control is proposed, after the simulation test according to the algorithm, the following conclusions are obtained:

(1). The algorithm can effectively control the subdivision depth of complex surfaces, according to the given arbitrary accuracy threshold \( \varepsilon \) and the depth of the surface subdivision obtained by calculation, the tessellation model required by the user can be directly simulated in the computer, which saves time and improves efficiency.

(2). Under the relationship between the subdivision depth and the set accuracy, considering the factors such as running time and data amount, the optimal subdivision times and the appropriate accuracy range can be determined.

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Reference
[1] Wu J.H., Liu W. J., and Wang T.C. (2006) Adaptive Subdivision for Triangle Mesh. Computer Engineering, 32:14-16.
[2] Amresh, A., Farin, G., and Razdan, A. (2003) Adaptive subdivision schemes for triangular meshes. 2002:319-327.
[3] Zhang, Z. X., et al. (2016) Exploratory study of spiral NC tool path generation on triangular mesh based on local subdivision. International Journal of Advanced Manufacturing Technology 83: 835-845.
[4] Chen, T.T., and Zhao, G. (2015) Loop Subdivision Surface Finishing Tool Trajectory Generation. Journal of Beijing University of Aeronautics and Astronautics 41: 663-668.
[5] Zhou, H., and Zhou, L. S. (2007) Subdivision Surface Fitting of Triangular Mesh Model. Journal of Nanjing University of Aeronautics and Astronautics 39: 258-262.
[6] Xu, J. t., et al. (2010) Simultaneous Residue Cutter Trajectory Generation Method Based on mesh Surface Model. Journal of Mechanical Engineering 46: 193-198.
[7] Loop, C. Masters Thesis University of Utah Department of Mathematics. (1987) Smooth Subdivision Surfaces Based on Triangles. http://www.microsoft.com/en-us/research/publication/smooth-subdivision-surfaces-based-on-triangles/.
[8] Huang, Q. (2014) Research on Adaptive Tool Path Planning Based on Loop Subdivision. Diss. Hunan University.
[9] Wu, X. B., and Peters, J. (2005) An Accurate Error Measure for Adaptive Subdivision Surfaces. In: International Conference on Shape Modeling and Applications IEEE Computer Society. Cambridge. pp. 51-56.