TWISTING BORDERED KHOVANOV HOMOLOGY

NGUYEN D. DUONG

ABSTRACT. We describe the bordered version for totally twisted Khovanov homology. We first twist Roberts’ type \( D \) structure by adding the “vertical” type \( D \) structure which is generalized from vertical maps in twisted tangle homology. The new type \( D \) structure has distinct advantage of having the deformation retraction type \( D \) structure supported on the “spanning tree” generators. We also describe how to twist Roberts’ type \( A \) structure for the left tangle in such a way that paring our type \( A \) and type \( D \) structures will result in totally twisted Khovanov homology.

1. Introduction

Modeled from the construction of the bordered Heegaard-Floer homology given by R. Lipshitz, P. Ozsvath, D. Thurston [8], L. Roberts, in [12] and [13], described how to obtain the Khovanov homology of a link from a diagram divided into two parts, inside and outside tangles. To those tangles, we associate different types of tangle invariants in such a way that it allows to glue those two invariants to recover Khovanov homology. This construction gives us a way to study Khovanov homology of a link from its tangle-components. On a different story, by using the idea of twisted Khovanov homology [14], reduced Khovanov homology (at least for the knot-case, proved by T. Jaeger [16]) can be described neatly and explicitly in terms of spanning trees. In an effort to combine those two theories and study reduced Khovanov homology, we will twist the Roberts’ tangle invariants to get the twisted invariants for each tangle-component of a link, which recovers the reduced Khovanov homology by gluing the tangle-components. Since the underlying complexes in the ”bordered” Khovanov homology construction were based on the extension of Khovanov homology for tangles given by M. Asaeda, J. Przytycki, and A. Sikora [1], [2], we will take the approach of twisted skein homology, described by the author and L. Roberts [15].

We first describe the chain complex by recalling the setting of [12], [13]. Let \( T \) be a link diagram of link \( \mathcal{T} \), which is transverse to the \( y \)-axis. The \( y \)-axis divides \( T \) into two parts: the left tangle \( \overleftarrow{T} \) and the right one \( \overrightarrow{T} \). Two left (right) tangles will be equivalent if they are related by ambient isotopy preserving \( y \)-axis pointwise. Labeling each arc \( f \) of \( \overrightarrow{T} \) with a formal variable \( x_f \), form the polynomial ring \( \mathbb{P}_{\overrightarrow{T}} = \mathbb{Z}_2[x_f | f \in \text{arc}(\overrightarrow{T})] \). The field of fractions of \( \mathbb{P}_{\overrightarrow{T}} \) will be denoted \( \mathbb{F}_{\overrightarrow{T}} \). The same thing can be done for \( \overleftarrow{T} \) to get \( \mathbb{F}_{\overleftarrow{T}} \).
We then associate to each right (left) tangle diagram \( \overrightarrow{T} \) (respectively \( \overleftarrow{T} \)) a bigraded vector space \( \left[ \overrightarrow{T} \right] \) (respectively \( \left\langle \overleftarrow{T} \right\rangle \)) over \( F_{\overrightarrow{T}} \) (respectively \( F_{\overleftarrow{T}} \)), generated by a collection of states \((r, s)\), described as follows:

1. \( r \) is a pair \((\overrightarrow{m}, \rho)\) (respectively \((\rho, \overrightarrow{m})\)) where \( \overrightarrow{m} \) (\( \overleftarrow{m} \)) is the left (right) planar matching (the left (right) planar matching is a collection of \( n \) embedded arcs in the half left (right) plan whose boundaries are the intersection points of \( T \) and \( y\)-axis) and \( \rho \) is a resolution of \( \overrightarrow{T} \) (\( \overleftarrow{T} \)).
2. \( s \) is a decoration on each circle of \( r \) by either + or −.

For the right tangle \( \overrightarrow{T} \), in [12], Roberts defines a type D-structure \( \overrightarrow{\delta_T} \) on \( \left[ \overrightarrow{T} \right] \) over a cleaved algebra \((\mathcal{B}\Gamma_n, \mathcal{I}_n)\) such that the homotopy class of \((\left[ \overrightarrow{T} \right], \overrightarrow{\delta_T})\) is an invariant of tangle \( \overrightarrow{T} \). Inspired by the idea of twisted skein homology [13], we will define a \((0, -2)\)-bigrading ”vertical” type D-structure \( \overrightarrow{\delta_V} \) on \( \left[ \overrightarrow{T} \right] \). In the bigrading of \( \left[ \overrightarrow{T} \right] \) and \( \mathcal{B}\Gamma_n \), \( \overrightarrow{\delta_T} \) is \((1, 0)\)-map while \( \overrightarrow{\delta_V} \) is \((0, -2)\)-map. By collapsing the bigrading to the single grading \( \zeta = i - j/2 \), both \( \overrightarrow{\delta_T} \) and \( \overrightarrow{\delta_V} \) are degree 1 map. Note that this is the usual \( \delta \)-grading but we can not use delta since it overlaps the notation of a type D structure \( \overrightarrow{\delta_T} \). Consider the grading of the module and algebra as \( \zeta \)-grading, we will also prove that \( \overrightarrow{\delta_V} \) type D-commutes to \( \overrightarrow{\delta_T} \) in a sense that \( \overrightarrow{\delta_T} \circ \overrightarrow{\delta_V} = \overrightarrow{\delta_V} \circ \overrightarrow{\delta_T} \) in a type D-structure on \( \left[ \overrightarrow{T} \right] \) (see proposition 14). Furthermore, using a trick to move the weights developed by T. Jaeger and an aid of stable homotopy, we will prove the following theorem in section 9:

**Theorem 1.** The homotopy class of the type D-structure \((\left[ \overrightarrow{T} \right], \overrightarrow{\delta_T})\) is (stably) invariant under the first, second and third Reidemeister moves applied to \( \overrightarrow{T} \).

Additionally, in section 9, using a cancellation lemma, we will get the deformation retraction of \((\left[ \overrightarrow{T} \right], \overrightarrow{\delta_T})\) supported on states which do not contain any free circles (free circle is the one not crossing the \( y\)-axis). We denote the collection of those states to be \( \text{st}_n(\overrightarrow{T}) \). We also denote:

\[
\left[ \overrightarrow{CT} \right] := \text{span}_{F_{\overrightarrow{T}}} \{(r, s) \in \text{st}_n(\overrightarrow{T})\}
\]

We now describe the type D structure \( \overrightarrow{\delta_{T,n}} \) on \( \overrightarrow{T} \), which is a deformation retraction of \((\left[ \overrightarrow{T} \right], \overrightarrow{\delta_T})\). The map \( \overrightarrow{\delta_{T,n}} : \left[ \overrightarrow{T} \right] \rightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} \left[ \overrightarrow{T} \right][-1] \)
is defined by specifying its image on each generator \((r, s) \in \text{ST}_n(\tilde{T})\). We define:

\[
\delta_{T,n}(r, s) = \sum_{(r', s')} \langle (r, s), (r', s') \rangle I_{\partial(r,s)} \otimes (r', s') + \sum_{\gamma \in \text{BRIDGE}(r)} B(\gamma)
+ \sum_{C \in \text{CR}(\partial(r,s)), s(C)=+} (\tilde{\ell}_C + \tilde{w}_C \tilde{c}_C) \otimes (r, s_C)
\]

The definition of the last two term will be defined in section 4. We now need to clarify the notation of first term. We have: \((r', s') \in \text{ST}_n(\tilde{T})\) is different from \((r, s)\) at two crossings \(c_1\) and \(c_2\), where \((r, s)\) receives the value 0 at \(c_1\) and \(c_2\) and \((r', s')\) receives the value 1 at these crossings. The coefficient \(\langle (r, s), (r', s') \rangle\) is calculated from the weights assigned to the arcs. First, \(\langle (r, s), (r', s') \rangle = 0\) unless there are two crossings \(c_1\) and \(c_2\) which are resolved with the 0 resolution in \(r\) and the 1 resolution in \(r'\). Let \(r_{01}\) \((r_{10})\) be the resolution where \(c_1\) is resolved with the 0 resolution (respectively 1 resolution) and \(c_2\) resolved with the 1 resolution (respectively 0 resolution). Furthermore, if \(C\) is a free circle, let \(\tilde{w}_C = \sum_{f \in \text{ARC}(C)} x_f\). Then:

\[
\langle (r, s), (r', s') \rangle = \begin{cases} 1/\tilde{w}_{C_{10}} + 1/\tilde{w}_{C_{01}} & \text{if } r_{10} \text{ and } r_{01} \text{ have free circles } C_{10} \text{ and } C_{01} \text{ respectively} \\ 1/\tilde{w}_{C_{10}} & \text{if } r_{10} \text{ has a free circle } C_{10} \text{ but } r_{01} \text{ does not} \\ 1/\tilde{w}_{C_{01}} & \text{if } r_{01} \text{ has a free circle } C_{01} \text{ but } r_{10} \text{ does not} \\ 0 & \text{if neither } r_{10} \text{ nor } r_{01} \text{ has a free circle} \end{cases}
\]

Then, combining with the theorem 1 we have the following corollary:

**Corollary 2.** The homotopy class of type D-structure \(\langle \tilde{C}T, \tilde{\delta}_{T,n} \rangle\) is an invariant of \(\tilde{T}\).

In \([13]\), associated to \(\tilde{T}\), there exists an \(A_{\infty}\) structure \(\langle \tilde{T}, m_1, m_2 \rangle\) over \((\mathcal{B} \Gamma_n, \mathcal{I}_n)\) where \(m_1\) is a differential \(d_{APS}\) and \(m_2\) is a right action \(\langle \tilde{T}, m_1, m_2 \rangle \otimes_{\mathcal{I}_n} \mathcal{B} \Gamma_n \rightarrow \langle \tilde{T}, m_1, m_2 \rangle\). In section 10, we will define a new type A structure on \(\langle \tilde{T}, m_1, m_2 \rangle\) where \(m_1\) is obtained from \(m_1\) by adding the vertical maps in the twisted skein homology and the only difference between the actions \(m_2\) and \(m_2\) is the action of left decoration elements on \(\langle \tilde{T} \rangle\). This construction ensures that it still provides an invariant for tangle \(\tilde{T}\) in category of \(A_{\infty}\) module and that using the gluing theory described in \([8]\), we can pair type D structure \(\langle \tilde{T}, \tilde{\delta}_{T,n} \rangle\) and type A structure \(\langle \tilde{T}, m_1, m_2 \rangle\) to recover the totally twisted Khovanov homology of \(T = \tilde{T} \# \tilde{T}\).

### 1.1. The structure of this paper.

In section 2 we recall the definition of cleaved algebra \(\mathcal{B} \Gamma_n\) in \([12]\) with two minor modifications: the ground ring \(\mathbb{Z}_2\) instead of \(\mathbb{Z}\) and the requirement on the decoration of the marked circle. In section 3 we construct the expanded complex \(\tilde{T}\) and states our main result for twisted tangle homology.
It will be followed by the construction of “vertical” type $D$-structure ($\{ \overrightarrow{T} \}, \overrightarrow{\delta_T}$) in section 4. In section 5, we will prove $\overrightarrow{\delta_T} := \overrightarrow{\delta_T} + \overrightarrow{\delta_V}$ is a type $D$-structure on $\{ \overrightarrow{T} \}$. We, next, will prove that the ($\{ \overrightarrow{C\overrightarrow{T}} \}, \overrightarrow{\delta_n,T}$) is the deformation retraction of ($\{ \overrightarrow{T} \}, \overrightarrow{\delta_T}$) in section 6. The proof of theorem 1 will be described in section 9 by using a trick to move weights, described in section 7. The whole section 8 is devoted to establish the definition of stable homotopic equivalence of type $A$ (as well as type $D$) structures.

In section 10 and 11, we define the type $A$-structure on $\langle \langle \leftarrow \overrightarrow{T} \rangle \rangle$. We will prove that it is an invariant for the left tangle in section 12. The relationship of totally twisted Khovanov homology and the complex, obtained by pairing our twisted type $A$ and type $D$, will be described in section 13. In the last section 14, we will illustrate how to set up type $D$ and $A$ structures of tangles by showing the computations of two specific examples.

1.2. Acknowledgment. I would like to thank Lawrence Roberts for a lot of helpful conversations, corrections to the earlier draft of this paper and for letting me to use his latex code to recall the definitions and properties of cleaved algebra in section 2.

2. Recall the definition of The algebra from cleaved links in [12]

2.1. The algebra $B\Gamma_n$. In [12], to each linearly ordered (according to $y$-axis orientation) collection of $2n$-points $P_n=\{p_1 = (0, 1), p_2 = (0, 2), \ldots, p_{2n} = (0, 2n)\}$ in the $y$-axis, Roberts associates a bigraded algebra $B\Gamma_n (\text{over } \mathbb{Z})$ equipped with a $(1,0)$-differential $d_{\Gamma_n}$ which satisfies Leibniz identity. For our purpose, it is enough to consider $B\Gamma_n$ as a free vector space over the ground ring $\mathbb{Z}_2$ and thus, we do not need to worry about sign issues. Since $B\Gamma_n$ is described by generators and relations, we first recall the definition of its generators and then, rewrite the set of relations, ignoring the sign. We note that $B\Gamma_n$, as described in the following, is actually a subalgebra of $B\Gamma_n$ in [12].

Definition 3. [12] An $n$-decorated, cleaved link $(L, \sigma)$ is an embedding of disjoint circles in $\mathbb{R}^n$ such that:

1. Each circle is crossing the $y$-axis at at least 2 points in $P_n$
2. $L \cap \{0 \times (-\infty, \infty)\} = P_n$
3. $\sigma$ is a function which assigns the decoration on each circle either + or −

Let $\text{CIR}(L)$ be the set of circles of a cleaved link $L$. We call $p_{2n}$ the marked point and the component circle of cleaved link passing through the marked point is called the marked circle and denoted $L(\ast)$. We then denote $\mathcal{CL}_n$ the set of equivalence classes of $n$-decorated, cleaved links whose decorations on marked circles are −.

Definition 4. [12] A right (left) planar matching $M$ of $P_n$ in half right (left) plane $\mathbb{H}^\leftarrow / (\mathbb{H}^\rightarrow)$ is a proper embedding of $n$-arcs $\alpha_i : [0, 1] \rightarrow \mathbb{H}^\leftarrow / (\mathbb{H}^\rightarrow)$ such that $\alpha_i(0)$ and $\alpha_i(1)$ belong to $P_n$.  
As a result, a cleaved link \( L \) can be obtained canonically by gluing a right planar matching \( \overrightarrow{L} \) and a left one \( \overleftarrow{L} \) along \( P_n \). We denote the equivalence classes of right and left planar matching \( \text{Match}(n) \) and \( \overleftarrow{\text{Match}}(n) \) respectively.

Since a generator \( e \) of \( \mathcal{B} \Gamma_n \) can be thought as an oriented edge whose source and target vertices, denoted by \( s(e) \) and \( t(e) \), are decorated cleaved links, we need the definition of a bridge of a cleaved link.

**Definition 5.** \[^{12}\] A bridge of a cleaved link \( L \) is an embedding of \( \gamma : [0,1] \to \mathbb{R}^2 \setminus \{0 \times (-\infty, \infty)\} \) such that:

1. \( \gamma(0) \) and \( \gamma(1) \) are on distinct arcs of either \( \overrightarrow{L} \) or \( \overleftarrow{L} \)
2. \( \gamma(0,1) \cap L = \emptyset \)

The equivalence class of bridges are denoted by \( \text{Bridge}(L) \).

Corresponding to each \( (L, \sigma) \in \mathcal{CL}_n^* \), there is an idempotent \( I_{(L,\sigma)} \). \( \mathcal{B} \Gamma_n \) is freely generated over \( \mathbb{Z}_2 \) by the idempotents and the following elements, subject to the relations described below

1. For each circle \( C \in \text{cir}(L) \) where \( \sigma(C) = + \), we have a “dual” decorated cleaved link \( (L, \sigma_C) \) where \( \sigma_C(C) = - \) and \( \sigma_C(D) = \sigma(D) \) for each \( D \in \text{cir}(L) \setminus \{C\} \). There are two elements \( \overrightarrow{e_C} \) and \( \overleftarrow{e_C} \), called right and left decoration elements, whose source is \( (L, \sigma) \) and target is \( (L, \sigma_C) \). \( C \) is called the support of \( \overrightarrow{e_C} \) and \( \overleftarrow{e_C} \).
2. Let \( \gamma \in \text{Bridge}(L) \), then there is a bridge element \( e_{(\gamma; \sigma, \sigma_\gamma)} \) whose source is \( (L, \sigma) \) and target is \( (L_\gamma, \sigma_\gamma) \) where \( L_\gamma \) is obtained from \( L \) by surgering along \( \gamma \) and \( \sigma_\gamma \) is any decoration compatible with \( \sigma \) and Khovanov sign rules.

With these generators and idempotents, we have:

**Proposition 6.** \[^{12}\] \( \mathcal{B} \Gamma_n \) is finite dimensional

Furthermore, \( \mathcal{B} \Gamma_n \) can be given a bigrading, \[^{12}\]. In this paper, we collapse the bigrading by using \( \zeta(i,j) = i - j/2 \) to give a new grading on \( \mathcal{B} \Gamma_n \). On the generating elements, the new grading is specified by setting:

\[
\begin{align*}
I_{(L,\sigma)} & \rightarrow (0,0) \rightarrow 0 \\
\overrightarrow{e_C} & \rightarrow (0,-1) \rightarrow 1/2 \\
\overleftarrow{e_C} & \rightarrow (1,1) \rightarrow 1/2 \\
\overrightarrow{e_\gamma} & \rightarrow (0,-1/2) \rightarrow 1/4 \\
\overleftarrow{e_\gamma} & \rightarrow (1,1/2) \rightarrow 3/4
\end{align*}
\]

Then it provides the grading to every other elements by extending the grading on generators homomorphically.
As the next step, based on the above set of generator, there are a set of commutativity relation of generators, divided into following cases:

**Disjoint support and squared bridge relations**: We describe the set of relations in this case by using the model:

\[ e_\alpha e_{\beta'} = e_\beta e_{\alpha'} \]

One of the requirement is that \( e_\alpha \) and \( e_{\alpha'} \) are the same type of elements (decoration or bridge) and they also have same location (left or right). The similar requirement is applied for the pair \( e_\beta \) and \( e_{\beta'} \).

Let \((L, \sigma) \in \mathcal{CL}_n\) such that \( I_{(L, \sigma)} \) is the source of both \( e_\alpha \) and \( e_\beta \), we have the following cases:

1. If \( C \) and \( D \) are two distinct + circles of \((L, \sigma)\), there are two ways to obtain \((L, \sigma_{C,D})\) from \((L, \sigma)\) by changing the decoration on either \( C \) or \( D \) first and then changing the decoration on the remaining + circle. The recording algebra elements for two paths will form a relation.

2. If \( e_\alpha = e_{(\gamma, \sigma, \sigma')} \) for a bridge \( \gamma \) in \((L, \sigma)\) and \( e_\beta \) is a decoration element for \( C \in \text{cir}(L) \), with \( C \) not in the support of \( \gamma \), due to the disjoint support, there will exist \( e_{\alpha'} = e_{(\gamma, \sigma_C, \sigma'_C)} \) and \( e_{\beta'} \) which is a decoration element whose source is \((L_\gamma, s')\) and target is \((L_\gamma, s'_C)\).

3. If \( e_\alpha = e_{(\gamma, \sigma, \sigma')} \) and \( e_\beta = e_{(\eta, \sigma, \sigma'')} \) are bridge elements for distinct bridges \( \gamma \) and \( \eta \) in \((L, \sigma)\), with \( \eta \in B_d(L, \gamma) \) and \( e_{\beta'} = e_{(\eta, \sigma', \sigma'')} \) and \( e_{\alpha'} = e_{(\gamma, \sigma''', \sigma'')} \) for some decoration \( \sigma''' \) on \( L_{\gamma, \eta} \), we obtain the commutativity relation. We note that \( B_d(L, \gamma) \) stands for the set of bridges of \((L, \gamma)\) neither of whose ends is on an arc with \( \gamma \).

4. If \( e_\alpha = e_{(\gamma, \sigma, \sigma')} \) and \( e_\beta = e_{(\eta, \sigma, \sigma'')} \) are bridge elements for distinct right bridges \( \gamma \) and \( \eta \) in \((L, \sigma)\), and \( e_{\beta'} = e_{(\eta, \sigma', \sigma'')} \) and \( e_{\alpha'} = e_{(\gamma, \sigma'', \sigma'')} \), such that \( L_{\gamma, \delta} = L_{\eta, \omega} \), and some compatible decoration \( \sigma''' \), it will form a commutativity relation.

5. If \( e_\alpha = e_{(\gamma, \sigma, \sigma')} \) and \( e_\beta = e_{(\eta, \sigma, \sigma'')} \) are bridge elements for distinct left bridges in \((L, \sigma)\), with \( \eta \in B_o(L, \gamma) \), and \( e_{\beta'} = e_{(\delta, \sigma', \sigma'')} \) and \( e_{\alpha'} = e_{(\omega, \sigma'', \sigma'')} \) with \( L_{\gamma, \delta} = L_{\eta, \omega} \), and some compatible decoration \( \sigma''' \). We denote \( B_o(L, \gamma) \) the set of bridges of \((L, \gamma)\) whose one end is on the same arc at \( \gamma \) and lying on opposite side of the arc as \( \gamma \).

**Other bridge relations**: [12] Suppose \( \gamma \in \overline{\text{Br}}(L) \) and \( \eta \in B_o(L_{\gamma}, \gamma^!) \) where \( B_o(L_{\gamma}, \gamma^!) \) consists of the classes of bridges all of whose representatives intersect \( \gamma^! \), then

\[ e_{(\gamma, \sigma, \sigma')} e_{(\eta, \sigma', \sigma'')} = 0 \]

whenever \( \sigma' \) and \( \sigma'' \) are compatible decorations.
The second possibility is that suppose \( \alpha \in B_s(L, \beta) \) where \( B_s(L, \gamma) \) stands for the set of bridges whose one end is on the same arc at \( \gamma \) and lying on same side of the arc as \( \gamma \). There is a natural left bridge \( \delta \) by sliding \( \alpha \) over \( \beta \). In this case there are three paths from \( s(\gamma, \beta) \) to \( t(\gamma, \beta) \) and they will form a relation whenever the decorations are compatible:

\[
\hat{\epsilon}_\alpha \hat{\epsilon}_\beta + \hat{\epsilon}_\delta + \hat{\epsilon}_\gamma \hat{\epsilon}_\eta = 0
\]

where \( \hat{\delta} \) and \( \hat{\eta} \) are the images of \( \hat{\gamma} \) in \( L_\beta \) and in \( L_\gamma \) respectively.

Additionally, if there is a circle \( C \in \text{cir}(L) \) with \( \sigma(C) = + \), and there are elements \( e_{(\gamma, \sigma, \sigma')} \) and \( e_{(\gamma, \sigma', \sigma C)} \) for a bridge \( \gamma \in \text{Br}(L) \) then

\[
e_{(\gamma, \sigma, \sigma')} e_{(\gamma, \sigma', \sigma C)} = e_C
\]

Relations for decoration edges: When the support of \( e_C \) is not disjoint from that of \( e_{(\gamma, \sigma, \sigma')} \) the relations are different depending upon the location of \( e_C \).

1. **The relations for \( e_C \):** If \( C_1 \) and \( C_2 \) be two + circles in \( L \) and \( \gamma \) be a bridge which merges \( C_1 \) and \( C_2 \) to form a new circle \( C \), we then obtain the following relation:

\[
e_{(\gamma, \sigma, \sigma')} m_{(\gamma, \sigma C_1, \sigma C)} = e_{C_2} m_{(\gamma, \sigma C_2, \sigma C)} = m_{(\gamma, \sigma, \sigma)} e_C
\]

Similarly, if \( C \) be a + circle in \( L \) and \( \gamma \) is a bridge which divides \( C \) into \( C_1 \) and \( C_2 \) in \( \text{cir}(L) \), then we impose the relation:

\[
e_{(\gamma, \sigma, \sigma')} f_{(\gamma, \sigma C_1, \sigma C)} = f_{(\gamma, \sigma, \sigma)} e_{C_1} = f_{(\gamma, \sigma, \sigma)} e_{C_2}
\]

where \( \sigma_i \) assigns + to \( C_i \) and - to \( C_{3-i} \).

2. **The relations for \( e_C \):** Since there are two types of decoration elements, in the two above relations, if we replace the role of right decoration elements by the left ones, we obtain two following corresponding relations:

\[
\hat{e}_{C_1} m_{(\gamma, \sigma C_1, \sigma C)} + \hat{e}_{C_2} m_{(\gamma, \sigma C_2, \sigma C)} + m_{(\gamma, \sigma, \sigma)} \hat{e}_C = 0
\]

\[
\hat{e}_{C_1} f_{(\gamma, \sigma C_1, \sigma C)} + f_{(\gamma, \sigma, \sigma)} \hat{e}_{C_1} + f_{(\gamma, \sigma, \sigma)} \hat{e}_{C_2} = 0
\]

**Note:** Without the worry about the sign issue, we do not have to distinguish the relation for \( e_C \) when the merge (divide) is left or right type as in [12].

The main result about \( B\Gamma_n \) in [12] is:
**Proposition 7.** Let \((L, \sigma) \in \mathcal{CL}_n^*\) such that there is a circle \(C \in \text{cir}(L)\) with \(\sigma(C) = +\). Let \(\xi_C\) be the decoration element corresponding to \(C\). Let

\[
d_I(\xi_C) = \sum e_{(\gamma, \sigma, \sigma)} e_{(\gamma^\dagger, \sigma, \sigma_C)}
\]

where the sum is over all \(\gamma \in \hat{\text{BR}}(L)\) with \(C\) as active circle, and all decorations \(\sigma_\gamma\) which define compatible elements. Let \(d_I(e) = 0\) for every other generator \(e\) (including idempotents). Then \(d_I\) can be extended to a 1-differential on graded algebra \(B\Gamma\) which satisfies the following Leibniz identity:

\[
d_I(\alpha \beta) = (d_I(\alpha)) \beta + \alpha (d_I(\beta))
\]

\((B\Gamma, d_I)\) denotes this differential, graded algebra over \(\mathbb{Z}_2\).

Additionally, since the grading of an idempotent is 0, it’s straightforward to have:

**Corollary 8.** \((B\Gamma, d_I)\) is differential, \(\mathbb{Z}\)-graded algebra over the ground ring \(\mathcal{I}_n\)

### 3. Twisted tangle homology and its expansion

#### 3.1. Twisted tangle homology

In [15], the author and L. Roberts define a twisted Khovanov homology for tangles embedded in thickened surface by twisting the reduced Khovanov tangle chain complex, defined by M. Asaeda, J. Przytycki, A. Sikora in [2].

Let \(\overrightarrow{T} \subset \overrightarrow{\mathbb{H}} = [0, \infty) \times \mathbb{R} \times \{0\}\) be a diagram of an oriented tangle \(\overrightarrow{T} \subset [0, \infty) \times \mathbb{R}^2\), properly embedded in half space of \(\mathbb{R}^3\). The set of crossings in \(\overrightarrow{T}\) will be denoted \(\text{cr}(\overrightarrow{T})\), and the set of arcs will be denoted \(\text{arc}(\overrightarrow{T})\). The number of positive crossings will be denoted \(n_+(\overrightarrow{T})\) and the number of negative crossings will be \(n_-(\overrightarrow{T})\). We will often omit the reference to \(\overrightarrow{T}\) when the choice of diagram is clear.

As in [15], [14], we label each arc \(f \in \text{arc}(\overrightarrow{T})\) with a formal variable \(x_f\) and form the polynomial ring \(\mathbb{P}_{\overrightarrow{T}} = \mathbb{Z}_2[\{x_f | f \in \text{arc}(\overrightarrow{T})\}]\). The field of fractions of \(\mathbb{P}_{\overrightarrow{T}}\) will be denoted \(\mathbb{F}_{\overrightarrow{T}}\).

**Definition 9.** For each subset \(S \subset \text{cr}(\overrightarrow{T})\), the resolution \(\rho_S\) of \(\overrightarrow{T}\) is a collection of arcs and circles in \(\overrightarrow{\mathbb{H}}\), considered up to isotopy, found by locally replacing each crossing \(s \in \text{cr}(T)\) according to the rule:

\[
\begin{align*}
\begin{array}{c}
\text{(a)} \quad s \notin S \\
\text{(b)} \quad s \in S
\end{array}
\end{align*}
\]
The set of resolutions for $\overrightarrow{T}$ will be denoted $\text{Res}(\overrightarrow{T})$. For each resolution $\rho$, denote by $h(\rho)$ the number of elements in the corresponding subset $S \subset \text{cr}(\overrightarrow{T})$.

Given a crossing $c \in \text{cr}(\overrightarrow{T})$ and a resolution $\rho = \rho_S$ we will also use the notation $\rho(c) = 0$ for $c \not\in S$ and $\rho(c) = 1$ for $c \in S$. Thus $\rho$ will stand both for the resolution diagram and for the indicator function for the set of crossings defining the resolution. Depending on the fact that the value of $\rho$ at $c$ is either 0 or 1, the local arc $\beta$, which is showed up when we resolve crossing $c$, is called active or inactive respectively. We also denote $\rho_\beta = \rho \cup c$ if $\beta$ is active. Furthermore, let $\text{bc}(\rho)$ be the set of circles and arcs in $\rho$ while $\overrightarrow{m}_\rho$ stands for the planar matching of $\rho$, obtained by deleting all of circles in $\text{bc}(\rho)$. We also denote $\text{fcir}(\rho)$ be the set of free circles (the ones not crossing the $y$-axis) of $\rho$.

We next assign a weight to each circle (or arc) in a resolution by adding the formal variables along each circle (or arc).

**Definition 10.** Let $\rho$ be a resolution for $\overrightarrow{T}$, then for each $C \in \text{bc}(\rho)$, we define:

$$\overrightarrow{w}_C = \sum_{f \in \text{arc}(C)} x_f$$

For each resolution $\rho(\overrightarrow{T})$, let $\text{fcir}(\rho) = \{C_1, \ldots, C_k\}$. We then associate to each $C_i$ the complex $\mathcal{K}([C_i])$:

$$0 \rightarrow \mathbb{F} \rightarrow \mathbb{F} v^+ \xrightarrow{\cdot \overrightarrow{w}_C} \mathbb{F} v^- \rightarrow 0$$

where $v_{\pm}$ occur in bigradings $(0, \pm 1)$. The differential in $\mathcal{K}([C_i])$ will be denoted $\partial_{C_i}$

Then, we associate to $\rho$ the bigraded Koszul chain complex, defined as:

$$\mathcal{K}(\rho) = \mathcal{K}([C_1], \ldots, [C_k]) = \mathcal{K}([C_1]) \otimes_{\mathbb{F}[\partial]} \mathcal{K}([C_2]) \otimes_{\mathbb{F}[\partial]} \cdots \otimes_{\mathbb{F}[\partial]} \mathcal{K}([C_k])$$

The differential in $\mathcal{K}(\rho)$ is then $\partial_{\mathcal{K}(\rho)} = \sum_{i>0} \partial_{C_i}$.

As we need to shift the grading, we introduce our notation for these shifts:

**Definition 11.** Let $M = \oplus_{\vec{v} \in \mathbb{Z}^k} M_{\vec{v}}$ be a $\mathbb{Z}^k$-graded $R$-module where $R$ is a ring, then $M[\vec{w}]$ is the $\mathbb{Z}^k$-graded module with $(M[\vec{w}])_{\vec{v}} \cong M_{\vec{v} - \vec{w}}$.

Then, we define the vertical complex for the resolution $\rho$ of the tangle diagram $T$ is the complex

$$\mathcal{V}(\rho) = \mathcal{K}(\rho) [[h(\rho), h(\rho)]]$$

where the differential $\partial_{\mathcal{V}(\rho)}$ will change the bigrading by $(0, -2)$.
We now define a bigraded chain group:

\[ C_{APS}(\vec{T}) = \bigoplus_{\rho \in \text{RES}(\vec{T})} V_{\rho} \]

\( C_{APS}(\vec{T}) \) is a bigraded chain complex with a \((0, -2)\) differential:

\[ \partial_V = \bigoplus_{\rho \in \text{RES}(\vec{T})} \partial_{V(\rho)} \]

Additionally, in [2], M. Asaeda, J. Przytycki, A. Sikora defines a \((1, 0)\) differential \( d_{APS} \) on this bigraded module, satisfying:

\[ d_{APS}\partial_V + \partial_V d_{APS} = 0 \]

Therefore, by collapsing the bigrading using \( \zeta(i, j) = i - j/2 \), we can make \( d_{APS} + \partial_V \) a differential of \( C_{APS}(\vec{T}) \). Furthermore, since both \( d_{APS} \) and \( \partial_V \) are constructed to preserve the right planar matching, we can decompose:

\[ (C_{APS}, d_{APS} + \partial_V) = \bigoplus_{m \in \text{MATCH}(n)} \left( C_{APS}(\vec{T}, m), d_{APS,m} + \partial_{V,m} \right) \]

We now can describe the main theorem of [15]:

**Theorem 12.** For each planar matching \( m \in \text{MATCH}(n) \), the homology \( H_*(C_{APS}, d_{APS,m} + \partial_{V,m}) \), as a relative \( \zeta \)-graded module, is an invariant of isotopy class of \( \vec{T} \).

Following [12], we first expand the chain group:

\[ \left[ \vec{T} \right] = \bigoplus_{(L, \sigma) \in \mathcal{CL}_n^*} C_{APS}(\vec{T}, \vec{L}, \sigma) = \bigoplus_{(L, \sigma) \in \mathcal{CL}_n^*} C_{APS}(\vec{T}, L)^{i(L, \sigma)} \]

where \( i(L, \sigma) \) is computed by subtracting the number of \(-\) non-marked circles in \( L \) from the number of \(+\) circles in \( L \). Note that since \((L, \sigma)\) is a cleaved link, all of its circles cross the \( y\)-axis. Also, for a technical reason in section [13], we do not count the marked circle in the formula of \( i(L, \sigma) \) as in [12]. As we can see, a generator of \( \left[ \vec{T} \right] \) is 1-1 corresponding to a triple \((\vec{m}, \rho, s)\), where \( \vec{m} \) is a left planar matching, \( \rho \) is resolution of \( \vec{T} \) and \( s : \text{CIR}(\vec{m} \# \rho) \rightarrow \{+, -\} \), required that \( s(\vec{m} \# \rho_{\sigma}(\rho_{\sigma}(\vec{m} \# \rho))) = - \). Note that \( \text{CIR}(\vec{m} \# \rho) \) is a collection of all cleaved and free circles of \( \vec{m} \# \rho \). For a sake of simplicity, we denote \( r = m \# \rho \) and we call \((r, s)\) a state. We denote \( \partial(r, s) \) to be the cleaved link diagram, obtained by deleting the all of free circles (the circles which do not intersect the \( y\)-axis) of \( \vec{m} \# \rho \) and the decoration is induced from the decoration \( s \) on \( \vec{m} \# \rho \). Therefore:

\[ \left[ \vec{T} \right] = \bigoplus_{(L, \sigma) \in \mathcal{CL}_n^*} \bigoplus_{\partial(r, s) = (L, \sigma)} \mathbb{F}_{\vec{T}}(r, s) \]
Additionally, each state \((r,s) = (\bar{m} \# \rho, s)\) has a \(\zeta\)-grading, computed from a bigrading \((h,q)\) where \(h = h(\rho) - n_\) and \(q = h(\rho) + \frac{i(L,\sigma)}{2} + \#(+\text{ free circles}) - \#(-\text{ free circles}) + n_+ - 2n_-\).

**Note.** The whole process can be applied exactly the same to give the construction of the expanded complex \(\langle\langle \hat{T} \rangle\rangle\) associated to the left tangle \(\hat{T}\).

In the next section, we will describe a type \(D\)-structure on this underlying module \(\langle\langle \hat{T} \rangle\rangle\) respect to \(\zeta\)-grading.

### 4. Defining Type D Structure in Twisted Tangle Homology

Using the idea of twisted skein homology, in this section, we will describe a way to twist the Roberts’ type \(D\)-structure on \(\langle\langle \hat{T} \rangle\rangle\) [12]. First of all, we define a "vertical" type \(D\)-structure on \(\langle\langle \hat{T} \rangle\rangle\).

Given \(\hat{T}\) respect to \(\zeta\)-grading, we define a left \(\mathcal{I}_n\)-module map:

\[
\delta^\to_v : \langle\langle \hat{T} \rangle\rangle \to \mathcal{B}\Gamma_n \otimes \mathcal{I}_n \langle\langle \hat{T} \rangle\rangle[-1]
\]

by specifying the image of \(\delta^\to_v\) on the generators \(\xi = (r, s)\) of \(\langle\langle \hat{T} \rangle\rangle\):

\[
(\delta^\to_v(r,s) = I_{\partial(r,s)} \otimes \partial_v(r, s) + \sum_{C \in \text{cir}(\partial(r,s)), s(C) =+} \overrightarrow{w_C e_C} \otimes (r, s_C)
\]

**Note :** \(\langle\langle \hat{T} \rangle\rangle\) is a left \(\mathcal{I}_n\)-module, defined by the trivial action: \(I_{(L,\sigma)}(r, s) = (r, s)\) if \(\partial(r,s) = (L,\sigma)\) or 0 else. An implication in the definition of \(\delta^\to_v\) is that for an ease of notation, \(\mathcal{B}\Gamma_n\) here stands for \(\mathcal{B}\Gamma_n \otimes_{\mathbb{Z}_2} \mathbb{F}_{\hat{T}}\). The lasted one actually has the DGA structure over \(\mathbb{F}_{\hat{T}}\), induced by the DGA structure over \(\mathbb{Z}_2\) of \(\mathcal{B}\Gamma_n\) (this fact will be mentioned again in section 8). Also, the decoration \(s_C\) on \(r\) is obtained from \(s\) by changing the decoration on cleaved circle \(C\) from \(+\) to \(-\) and using the same decoration for other circles. Therefore, if \(\partial(r,s) = (L,\sigma)\) then \(\partial(r,s_C) = (L,\sigma_C)\).

Furthermore, the bigrading of the terms \(I_{\partial(r,s)} \otimes \partial_v\) are decreased by \((0,2)\) due to the facts that the bigrading of idempotents are \((0,0)\) and \(\partial_v\) is \((0,-2)\) differential. Therefore, \(\zeta\)-grading is increased by 1. Similarly, since the \(\zeta\)-grading of \(\overrightarrow{e_C}\) is 1/2 and \(\zeta(r,s) - \zeta(r,s_C) = 1/2[i(L,\sigma)/2 - i(L,\sigma_C)/2] = 1/2\), the \(\zeta\)-grading of terms \(\overrightarrow{e_C} \otimes (r, s_C)\) is increased by 1. As a result, \(\delta^\to_v\) is \(\zeta\)-grading preserving.

We next prove the following:
Proposition 13. $\tilde{\delta}_V$ is a type D-structure on $[T]$, which means it satisfies the following equation:

$$(\mu_{\mathcal{B}T_n} \otimes I_d)(I \otimes \tilde{\delta}_V)\tilde{\delta}_V + (d_{\mathcal{B}T_n} \otimes I_d)\tilde{\delta}_V = 0$$

Proof. Since the image $d_{\mathcal{B}T_n}$ on the idempotents or right decoration element $e_C$ equals 0, it suffices to verify that:

$$(\mu_{\mathcal{B}T_n} \otimes I_d)(I \otimes \tilde{\delta}_V)\tilde{\delta}_V(r, s) = 0$$

for each generator $\xi = (r, s)$ of $[T]$. Since the image of $\tilde{\delta}_V(r, s)$ contains states $(r, s_C)$ (with the coefficients in $\mathcal{B}T_n$), the image of $(I \otimes \tilde{\delta}_V)\tilde{\delta}_V(r, s)$ are the states $(r, s_{C_{1,2}})$, where decoration $s_{C_{1,2}}$ of $r$ is obtained from $s$ by changing the decoration on $C_1, C_2 \in \text{cir}(r)$ from + to −. Therefore, we have:

$$(\mu_{\mathcal{B}T_n} \otimes I_d)(I \otimes \tilde{\delta}_V)\tilde{\delta}_V(r, s) = \sum_{C_1, C_2 \in \text{cir}(r)} A_{(r, s_{C_{1,2}})}(r, s_{C_{1,2}})$$

We will prove that each $A_{(r, s_{C_{1,2}})} = 0$. Since the circle in $\text{cir}(r)$ is either cleaved or free, we have the following cases:

1. Both $C_1$ and $C_2$ are free circles, there are two ways to obtain $(r, s_{C_{1,2}})$ from $(r, s)$

   $$(r, s) \xrightarrow{\overline{w}_{C_1}I_{\theta(r, s)}} (r, s_{C_1}) \xrightarrow{\overline{w}_{C_2}I_{\theta(r, s)}} (r, s_{C_{1,2}})$$

   $$(r, s) \xrightarrow{\overline{w}_{C_2}I_{\theta(r, s)}} (r, s_{C_2}) \xrightarrow{\overline{w}_{C_1}I_{\theta(r, s)}} (r, s_{C_{1,2}})$$

   Therefore, $A_{(r, s_{C_{1,2}})} = (\overline{w}_{C_1}\overline{w}_{C_2} + \overline{w}_{C_2}\overline{w}_{C_1})I_{\theta(r, s)} = 0$.

2. $C_1$ is free circle and $C_2$ is cleaved circle. We have two following paths:

   $$(r, s) \xrightarrow{\overline{w}_{C_1}I_{\theta(r, s)}} (r, s_{C_1}) \xrightarrow{\overline{w}_{C_2}e_{C_2}} (r, s_{C_{1,2}})$$

   $$(r, s) \xrightarrow{\overline{w}_{C_2}e_{C_2}} (r, s_{C_2}) \xrightarrow{\overline{w}_{C_1}I_{\theta(r, s_{C_1})}} (r, s_{C_{1,2}})$$

   and therefore,

   $$A_{(r, s_{C_{1,2}})} = \overline{w}_{C_1}\overline{w}_{C_2}e_{C_1}e_{C_2}I_{\theta(r, s_{C_{1,2}})} + \overline{w}_{C_1}\overline{w}_{C_2}e_{C_1}e_{C_2}I_{\theta(r, s_{C_{1,2}})} = 0$$

3. Both $C_1$ and $C_2$ are cleaved circles. We have:

   $$(r, s) \xrightarrow{\overline{w}_{C_1}e_{C_1}} (r, s_{C_1}) \xrightarrow{\overline{w}_{C_2}e_{C_2}} (r, s_{C_{1,2}})$$

   $$(r, s) \xrightarrow{\overline{w}_{C_2}e_{C_2}} (r, s_{C_2}) \xrightarrow{\overline{w}_{C_1}e_{C_1}} (r, s_{C_{1,2}})$$

   By relation [1] defining $\mathcal{B}T_n$, we have $e_{C_2}e_{C_1} + e_{C_1}e_{C_2} = 0$. Therefore, $A_{(r, s_{C_{1,2}})} = 0$. $\Diamond$
L. P. Roberts, in [12], discovers a type $D$ structure $\overrightarrow{\delta_T}$ on the bigrading module $[\overrightarrow{T}]$, which increases the bigrading by $(1,0)$ and thus, increases the $\zeta$-grading by 1.

As the next step, we will show that $\overrightarrow{\delta_V}$ commutes to $\overrightarrow{\delta_T}$ as a type $D$-structure in following sense:

**Proposition 14.** Two types $D$-structure $\overrightarrow{\delta_V}$ and $\overrightarrow{\delta_T}$ on $[\overrightarrow{T}]$ satisfy:

$$(\mu_{BR_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_V})\overrightarrow{\delta_T} + (\mu_{BR_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_T})\overrightarrow{\delta_V} = 0$$

The proof of this proposition will be presented in the next section. Combining with the fact that $\overrightarrow{\delta_V}$ and $\overrightarrow{\delta_T}$ are type $D$-structures on $[\overrightarrow{T}]$, proposition 14 ensures that $\overrightarrow{\delta_T}$, $\overrightarrow{\delta_T}$ is a type $D$-structure on $[\overrightarrow{T}]$.

In section 9, we will prove one of the main theorems of this paper:

**Theorem 15.** The homotopy class of the type $D$-structure $([\overrightarrow{T}], \overrightarrow{\delta_T})$ is invariant under the first, second and third Reidemeister moves applied to $\overrightarrow{T}$

5. **Proof of proposition 14**

Before proving this proposition, let us shortly recall the definition of $\overrightarrow{\delta_T}$ and notations, defined in [12]. For each generator $(r, s)$ of $[\overrightarrow{T}]$: $\overrightarrow{\delta_T}(r, s) := I_{\partial(r,s)} \otimes d_{APS} + \sum_{\gamma \in \text{BRIDGE}(r)} \mathcal{B}(\gamma)$

$$+ \sum_{\gamma \in \text{DEC}(r, s)} e_C(\gamma) \otimes (r_{\gamma}, s_{\gamma}) + \sum_{C \in \text{CIR}(\partial(r, s), s(C) = +)} e_C \otimes (r, s_C)$$

where:

1. BRIDGE($r$) is the union of left bridges of $\partial(r, s)$ and the subset of active bridges of $r$, containing active bridges $\gamma$ such that the right planar matching of $r_{\gamma} = \gamma_c(r)$ is different from the one of $r$. Furthermore, $\mathcal{B}(\gamma)$ is the sum of $(r_{\gamma}, s_{\gamma}^i)$ where $s_{\gamma}^i$ is computed from Khovanov Frobenius algebra, with the coefficients corresponding bridge elements: $\partial(r, s) \rightarrow \partial(r_{\gamma}, s_{\gamma}^i)$ in $\mathcal{B} \Gamma_n$.

2. DEC($r, s$) is a collection of active bridges $\gamma$ of $r$ such that either both feet of $\gamma$ belongs to a $+$ cleaved circle $C$ of $r$ or one foot of $\gamma$ belongs to a $+$ cleaved circle $C$ of $r$ and another belong to a $-$ free circle of $r$. In both cases, $s_{\gamma}$ is computed from Khovanov Frobenius algebra with a condition that $s_{\gamma}(C_{\gamma}) = -$ where $C_{\gamma}$ is a cleaved circle of $r_{\gamma}$, obtained by surgering $C$.

**Proof of 14** It suffices to prove that for each generator $\xi = (r, s)$ of $[\overrightarrow{T}]$,

$$(\mu_{BR_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_V})\overrightarrow{\delta_T}(r, s) + (\mu_{BR_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_T})\overrightarrow{\delta_V}(r, s) = 0$$
We rewrite the left hand side:

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \delta^i_T)\delta_T(r, s) + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \delta^j_T)\delta_T(r, s) = \sum_{(r', s')} A(r', s') \otimes (r', s')$$

where $A(r', s')$ is computed by taking the sum of products $e_\alpha e_\beta$, modeled by:

$$(r, s) \xrightarrow{e_\alpha} (r_1, s_1) \xrightarrow{e_\beta} (r', s')$$

Note that $e_\alpha$ and $e_\beta$ are elements in $\mathcal{B}\Gamma_n$, corresponding to $\partial(r, s) \to \partial(r_1, s_1)$ and $\partial(r_1, s_1) \to \partial(r', s')$ respectively. Additionally, one of them is either the idempotent or right decoration element (this term comes from $\delta^i_T$) and another is either the idempotent, bridge element, right decoration element or left decoration element (this term comes from $\delta^j_T$).

Our goal is to prove that $A(r', s') = 0$. We have two cases:

**Case I :** The term coming from $\delta^i_T$ is right decoration element $e_\gamma^i$. In this case, we follow subcases:

1. The term coming from $\delta^i_T$ is the left decoration element $e_\delta^i$. Then $(r', s')$ is obtained from $(r, s)$ by changing the decoration on two cleaved circle $C, D$ of $r$ from $+$ to $−$. Therefore, there are exactly two paths from $(r, s)$ to $(r', s')$ as following:

   $$(r, s) \xrightarrow{\delta^i} (r, s_C) \xrightarrow{e_\gamma} (r', s')$$
   $$\quad (r, s) \xrightarrow{\delta^i} (r, s_D) \xrightarrow{\delta^i} (r', s')$$

   We have: $A(r', s') = w_C^i e_C^i e_D^i + e_D^i e_C^i = 0$ by relation [1] defining $\mathcal{B}\Gamma_n$.

2. The term coming from $\delta^i_T$ is either the idempotent, right decoration element $e_\gamma$ where $\gamma \in \text{BRIDGE}(r)$ such that the support of $\gamma$ is disjoint from $C$. In this case, there are again exactly two paths from $(r, s)$ to $(r', s')$. Similar to the previous case or the proof of proposition [13], using the relation [1] or [2] in the set of disjoint support and squared bridge relations, we can easily compute that $A(r', s') = 0$

3. The term from $\delta^i_T$ is bridge element $e_\gamma$ where $\gamma \in \text{BRIDGE}(r)$ and $C$ is in the support of $\gamma$. We only describe the merging case since the case when $\gamma$ is division can be handled by similar calculation. Let $C_1$ be a cleaved circle in $\partial(r, s)$, merged with $C$ by $\gamma$ to form another cleaved circle $C_2$. For the path from $(r, s)$ to $(r', s')$ to exist, $s(C_1) = +$. Then, there are three paths in $A(r', s')$:

   $$(r, s) \xrightarrow{\delta^i} (r, s_C) \xrightarrow{e_\gamma} (r', s')$$
   $$\quad (r, s) \xrightarrow{\delta^i} (r, s_{C_1}) \xrightarrow{e_\gamma} (r', s')$$
   $$\quad (r, s) \xrightarrow{e_\gamma} (r, s_\gamma) \xrightarrow{\delta^i} (r', s')$$
As the result, \( A(r', s') = \overrightarrow{w_C e_C e_\gamma} + \overrightarrow{w_C e_C e_\gamma} + e_\gamma \overrightarrow{w_C e_C e_\gamma} \). Since \( \overrightarrow{w_C} = \overrightarrow{w_c_1} + \overrightarrow{w_C} \), we can rewrite: \( A(r', s') = \overrightarrow{w_C e_C e_\gamma} + e_\gamma \overrightarrow{w_C e_C e_\gamma} + \overrightarrow{w_C (\overrightarrow{e_C e_\gamma} + e_\gamma \overrightarrow{e_C e_\gamma})} = 0 \), by relations \( 4 \) in the set of relations for decoration edges.

**Case II**: The term coming from \( \overrightarrow{\delta_V} \) is an idempotent obtained by changing decoration on + free circle \( C \) to -. We have following subcases:

1. Term of \( \overrightarrow{\delta_T} \) is from \( I \otimes d_{APS} \). In this case, we know that the product of weights of two paths from \((r, s)\) to \((r', s')\) will be canceled out by \( [15] \). Therefore, \( A(r', s') = 0 \).

2. Term from \( \overrightarrow{\delta_T} \) is a bridge element \( e_\gamma \) where \( \gamma \in \text{BRIDGE}(r) \). Since the support of \( \gamma \) is definitely disjoint from \( C \), we have only two paths from \((r, s)\) to \((r', s')\) and thus \( A(r', s') = \overrightarrow{w_C I_{\partial(r, s)} e_\gamma} + e_\gamma \overrightarrow{w_C I_{\partial(r, s)}} = 0 \).

3. Term from \( \overrightarrow{\delta_T} \) is a right decoration element, obtained by surgering \((r, s)\) along \( \gamma \in \text{DEC}(r, s) \). It is possible that the support of \( \gamma \) is disjoint or not disjoint from \( C \). When the support of \( \gamma \) is disjoint from \( C \), the proof of \( A(r, s) = 0 \) is similar as in case II.2. When the support of \( \gamma \) is not disjoint from \( C \), the situation is more interesting. Since the dividing case is similar, we here only present the proof in the case when \( \gamma \) is merging a + cleaved circle \( D \in \partial(r, s_C) \) with \( C \) to give a cleaved circle \( C_1 = \partial(r', s') \). In this case, there are exactly three paths from \((r, s)\) to \((r', s')\) as following:

\[
\begin{align*}
(r, s) & \xrightarrow{I_{\partial(r, s)}} (r_2, s_2) \xrightarrow{\overrightarrow{w_C e_C e_\gamma}} (r', s') \\
(r, s) & \xrightarrow{\overrightarrow{w_C I_{\partial(r, s)}}} (r, s_C) \xrightarrow{\overrightarrow{e_D}} (r', s') \\
(r, s) & \xrightarrow{\overrightarrow{w_C e_D}} (r, s_D) \xrightarrow{I_{\partial(r, s)} \cdot \overrightarrow{\delta}} (r', s')
\end{align*}
\]

As the result, \( A(r', s') = (\overrightarrow{w_C} + \overrightarrow{w_D} + \overrightarrow{w_D}) e_D = 0 \) since \( \overrightarrow{w_C} = \overrightarrow{w_C} + \overrightarrow{w_D} \)

6. **The deformation retraction of type D structure**

Let \( \text{ST}_{n}(T) \) be the collection of states of \( \overrightarrow{T} \), consisting of those states that do not have free circle in its resolution. In this section, we will prove that the type D structure \( ([\overrightarrow{C}], \overrightarrow{\delta_n T}) \), described in section \( [4] \) is D-homotopic equivalent to \( ([\overrightarrow{T}], \overrightarrow{\delta_T}) \).

First of all, we state the type D cancellation lemma whose proof can be found in \( [12] \).

Let \((A, d)\) be a \( \mathbb{Z} \)-graded differential algebra over ground ring \( R \) (characteristic 2). Let \( N \) be a \( \mathbb{Z} \)-graded module over \( R \). Suppose over \( R, N \) can be generated by a basis \( \{x_1, \ldots, x_n\} \) and \( a_{i,j} \in A \) such that:

\[
(11) \quad d(a_{ik}) + \sum_{j=1}^{n} a_{ij} a_{jk} = 0 \quad i, k \in \{1, \ldots, n\}
\]
We also assume that \( gr(a_{ij}) = |x_i| - |x_j| + 1 \). Then \( \delta : N \to (A \otimes_R N)[-1] \), defined by:
\[
\delta(x_i) = \sum_{j=1}^{n} a_{ij} \otimes x_j
\]

is a type D-structure on \( N \).

**Lemma 16 (Cancellation).** \([12]\) Let \( \delta \) be a D-structure on \( N \). Suppose there is a basis \( B \) whose structure coefficients satisfy \( a_{ii} = 0 \) and \( a_{12} = 1_A \). Let \( N = \text{span}_R \{ \pi_3, \ldots, \pi_n \} \). Then
\[
\overline{\delta}(\pi_i) = \sum_{j \geq 3} (a_{ij} - a_{i2} a_{1j}) \otimes \pi_j
\]
is a D-structure on \( \overline{N} \). Furthermore, the maps
\[
i : \overline{N} \to A \otimes N \\
i(\pi_i) = 1_A \otimes x_i - a_{i2} \otimes x_1
\]
\[
\pi : N \to A \otimes \overline{N} \\
\pi(x_i) = \begin{cases} 0 & i = 1 \\
\sum_{j \geq 3} a_{1j} \otimes \pi_j & i = 2 \\
1_A \otimes \pi_i & i \geq 3
\end{cases}
\]
realize \( \overline{N} \) as a deformation retraction of \( N \) with \( i \circ \pi \simeq_H \mathbb{I}_N \) using the homotopy \( H : N \to A \otimes N[-1] \)
\[
H(x_i) = \begin{cases} 1_A \otimes x_1 & i = 2 \\
0 & i \neq 2
\end{cases}
\]

**Proposition 17.** \([\overline{\mathbb{T}}, \overline{\delta}_{n,T}] \), defined in section \(1\), is a type D-structure over \( (\mathcal{B} \Gamma_n, \mathcal{I}_n) \). Additionally, \( (\mathbb{T}, \delta_{T,\bullet}) \), defined in section \(4\), is homotopic equivalent to \( (\overline{\mathbb{T}}, \overline{\delta}_{n,T}) \).

**Proof.** Let \((r, s)\) be a state of \( \mathbb{T} \) containing a free circle \( C \). Then the decoration \( s(C) \) is either + or −. Corresponding to \((r, s)\), there is a state \((r, s_1)\) of \( \mathbb{T} \), obtained by changing the decoration on \( C \) from ± to ±. We call this pair of states a mutual pair. We will use the cancellation lemma \([16]\) to cancel out the mutual pairs \((r, s)\) and \((r_1, s_1)\). Indeed, \( \delta_{T,\bullet}(r, s) = \overline{w_C I_{\partial(r,s)}} \otimes (r, s_1) + Q = \overline{w_C 1_{\mathcal{B} \Gamma_n}} \otimes (r, s_1) + Q \) where \( Q \) is a linear combination supported on states not equal to either \((r, s)\) or \((r, s_1)\). Since \( \overline{w_C 1_{\mathcal{B} \Gamma_n}} \) is invertible, we can cancel this pair to get a deformation retraction of \( (\mathbb{T}, \delta_{T,\bullet}) \) supported on \( \text{State}(T) \setminus \{(r, s), (r, s_1)\} \). We also note that the new perturbation does not alter the relationship between another mutual pair. As a result, we can cancel out all of the mutual pair of states and what we have left is a homotopic equivalent type D-structure \( \delta_n \) supported on \( \overline{\mathbb{T}} \). We also need to verify that \( \delta_n \) is the same as \( \overline{\delta}_{n,T} \). Let \((r, s), (r', s')\) be two states in \( \text{ST}_n(T) \) such that \( \langle \delta_n(r, s), (r', s') \rangle \neq 0 \). Therefore, under the action of \( \overline{\delta}_{T,\bullet} \), we have the following
transition of states in $\text{State}(T)$:

$$(r, s) = (r_0, s_0^+) \rightarrow (r_1, s_1^-) \rightarrow (r_1, s_1^+) \rightarrow \ldots \rightarrow (r_k, s_k^+) \rightarrow (r_{k+1}, s_{k+1}^-) = (r', s')$$

where each transition $(r_i, s_i^-) \rightarrow (r_i, s_i^+)$ comes from a mutual pair and we denote $C_i$ be the free circle where $s_i^-(C_i) = -$ and $s_i^+(C_i) = +$. Other transitions come from the fact that $(r_{i+1}, s_{i+1}^-)$ is in the image of $\delta_T(r_i, s_i^+)$. By denoting the number of $+$ and $-$ free circles for each state $(r, s)$ of $T$ by $J(r, s) = (J_+(r, s), J_-(r, s))$, we see that $J(r_i, s_i^+) - J(r_i, s_i^-) = (1, -1)$ for each $i \in \{1, \ldots, k\}$. Additionally, we evaluate $J_i = J(r_{i+1}, s_{i+1}^-) - J(r_i, s_i^+)$ as following cases:

1. $J_i = (0, 0)$ if the corresponding coefficient from $\partial(r_i, s_i^+) \rightarrow \partial(r_{i+1}, s_{i+1}^-)$ is $\hat{e}_C$ or bridge element.
2. If the corresponding coefficient is $\hat{e}_C$ then $J_i$ belongs to $\{(1, 0), (0, -1), (0, 0)\}$.
3. If this transition comes from $I \otimes d_{APS}$, $J_i$ belongs to $\{(-1, 0), (0, 1)\}$.

Since $(r, s), (r', s') \in \text{ST}_n(T)$, we have $J(r, s) = J(r', s') = (0, 0)$. Furthermore, we have:

$$J(r', s') - J(r, s) = \sum_{i=0}^{k} [J(r_i, s_i^+) - J(r_i, s_i^-)] + \sum_{i=0}^{k} J_i = (0, 0)$$

Therefore, $\sum_{i=0}^{k} J_i = (-k, k)$. Looking through all of possible cases of $J_i$, $k$ has to be either 0 or 1. If $k = 0$, $(r', s')$ is obtained from $(r, s)$ by either changing a decoration on a cleaved circle from $+$ to $-$ or surgering along $\gamma \in \text{BRIDGE}(r)$. In this case:

$$\langle \delta_n(r, s), (r', s') \rangle = \langle \delta_T(r, s), (r', s') \rangle$$

If $k = 1$, we need to have $\partial(r, s) = \partial(r_1, s_1^-) = \partial(r_1, s_1^+) = \partial(r', s')$. As a result, we know how to calculate $\delta_n$. We call a transition to be $I - \text{transition}$ if it is of the following form: $(r, s) \rightarrow (r_1, s_1^-) \rightarrow (r_1, s_1^+) \rightarrow (r', s')$ where the boundary of states in this transition is fixed. Then:

$$\langle \delta_n(r, s), (r', s') \rangle = \sum_{I-\text{transition}} 1/\hat{w}_C^I$$

Investigating more on $I$-transition: $(r, s) \rightarrow (r_1, s_1^-) \rightarrow (r_1, s_1^+) \rightarrow (r', s')$, we see that $(r_1, s_1^-)$ is obtained from $(r, s)$ by surgering along an active resolution bridge $\gamma_1$ of $r$. Since $(r, s) \in \text{ST}_n(T)$ and a new free circle $C$ is created in $(r_1, s_1^-)$, $\gamma_1$ has to divide a cleaved circle $C_1$ of $r$ into $C$ and another cleaved circle $C_2$ of $r_1$. Furthermore, $(r', s')$ is obtained from $(r_1, s_1^+)$ by surgering along an active resolution bridge $\gamma_2$ of $r_1$. Since $(r', s') \in \text{ST}_n(T)$, $\gamma_2$ merges $C$ to a cleaved circle $D$ of $r_1$. If $C_2 \equiv D$, there are two $I$-transitions from $(r, s)$ to $(r', s')$. If $C_2 \neq D$, there is only one $I$-transition from $(r, s)$ to $(r', s')$. Therefore, $\delta_n$ is exactly $\delta_{T,n}$. \(\diamond\)
7. TYPE $D$-HOMOTOPY EQUIVALENCE UNDER THE WEIGHTS MOVE

Following [16] and [15], we will prove that the type $D$-structure in twisted tangle homology will be preserved (up to $D$-homotopy equivalence) under weight moving. Let $T$ and $T'$ be weighted tangle diagrams of $T$ with weighted edges before and after the weight $w$ is moved along the crossing $c$ as in the following figures:

![Diagram](image)

We need to show that $(\langle \Delta^D \rangle, \delta_{T,\bullet})$ is homotopy equivalent to $(\langle \Delta^D \rangle, \delta_{T',\bullet}).$

Let $D_c : \langle \Delta^D \rangle \longrightarrow B\Gamma_n \otimes_{I_n} \langle \Delta^D \rangle$ be a $F\overline{\partial}$-linear map, defined as follows. Let $\xi = (r,s)$ be a generator of $\langle \Delta^D \rangle.$

(1) $r'$ is the resolution obtained from $r$ by surgering the inactive bridge $\gamma$ at $c$ of $r$.

(2) $s'(D) = s(D)$ for all circles $D$ not abutting $c$ and the signs on circle(s) abutting $c$ is computed by using Khovanov maps.

(3) $e \in B\Gamma_n$ will be specific element, whose source is $I_{\partial(r,s)}$ and target is $I_{\partial(r',s')}$, corresponding to $\partial(r,s) \rightarrow \partial(r',s').$ We note that since $r'$ is obtained from $r$ by surgering a bridge on the right side, $e$ is either a right bridge element or a right decoration element or an idempotent.

Additionally, if $e$ is a bridge element in $B\Gamma_n,$ we have:

$$\zeta(r',s') = h(r',s') - q(r',s')/2 = [h(r,s) - 1] - [q(r,s) - 3/2]/2 = \zeta(r,s) - 1/4$$

By similar computation, if $e$ is either an idempotent or right decoration elements, we have: $\zeta(r',s') + \zeta(e) = \zeta(r,s).$ As a result, $D_c$ is $\zeta$-grading preserving map.

**Proposition 18.** The map $\Psi : \langle \Delta^D \rangle \longrightarrow B\Gamma_n \otimes_{I_n} \langle \Delta^D \rangle$ where $\Psi(r,s) = I_{\partial(r,s)} \otimes (r,s) + w.D_c(r,s)$ is a $D$-homomorphism.

**Proof.** For an ease of notation, we let $\delta, \delta'$ stand for $\delta_{T,\bullet}, \delta_{T',\bullet}$ respectively.

It suffices to prove:

$$(\mu_{B\Gamma_n} \otimes I_d)(\mathbb{I} \otimes \delta')\Psi + (\mu_{B\Gamma_n} \otimes I_d)(\mathbb{I} \otimes \Psi)\delta + (d_{\Gamma_n} \otimes I_d)\Psi = 0$$

when applied to each $(r,s) \in \text{STATE}(\Delta^D)$.

Since the image of $\Psi$ does not have any term of the form $\mathbb{I}_C \otimes (r_1, s_1),$ the last term of (12) will be 0. Also, the map $\delta$ can be written as the sum: $\delta = D_T + D_{T,c} + E_T + E_{T,c}$
where $D_T$ and $D_{T,c}$ are the images of $\delta$ obtained by surgering either one left bridge or one active bridge at $c_1 \neq c$ and $c$ respectively. $E_T$ ($E_{T,c}$) are the images of $\delta$ obtained by changing the decoration on a circle not abutting $c$ (abutting $c$) from $+$ to $-$. We can write down the similar sum for $\delta'$.

As we can canonically identify $[\overrightarrow{T}']$ as $[\overrightarrow{T}']$, let $\delta_1$ be a $D-$structure on $[\overrightarrow{T}]$, which is precisely the same as $\delta'$ on $[\overrightarrow{T}']$. Note that: $\delta + \delta_1 = E_{T,c} + E_{T',c}$.

Let $\mathbb{I}_T : [\overrightarrow{T}] \rightarrow \mathcal{B}\Gamma_n \otimes \mathcal{I}_n [\overrightarrow{T}']$, defined by $\mathbb{I}_T(r,s) = I_{\delta(r,s)} \otimes (r, s)$. The left hand side of (12) can be written as the sum:

$$[(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta_1] + [(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta_1] + [(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta_1] + [\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta_1]$.

The sum of first two terms will be 0 since both are equal $\delta'$. Rearrange the other terms, we need to prove:

$$\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta = (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta_1$$

By using the decomposition of $\delta$ and $\delta'$, we will in turn prove the following equations:

$$\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta = (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta = (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta_1$$

Equations (14) comes from the proof of $\delta_T$ to be a type $D-$structure on $[\overrightarrow{T}]$ (for the mirror crossing at $c$) and (15) is one of case of proposition (14) (for the mirror crossing at $c$). Therefore they are true and we do not repeat the proof here. For (16), we have a remark that both terms will be 0 if $r(c) = 0$ because $E_{T,c}$ does not change the value of $r(c)$. Also, $E_{T,c}$ will be 0 unless there is a circle decorated with + in the image of $D_c$. As the result, the left term will be 0 if one of the circle(s) abutting $c$ contains marked point. In this case, $(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1})((\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}))(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{1}) \delta_1$.

For case (2), we only deal with the merging case (the dividing case is similar). Let $\pm_c$ and $\pm_f$ denote the decoration on the cleaved and free circles respectively. We have:
\[
\begin{align*}
&= (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes E_{T',c})D_c(+c+f) + (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes D_c)E_{T',c}(+c+f) \\
&= (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes E_{T',c})(I_{\partial(r,s)} \otimes +c) + (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes D_c)[(\mathcal{e}_C + \mathcal{w}_C\mathcal{e}_C) \otimes -c + f] \\
&+ (\mathcal{w}_f I_{\partial(r,s)} \otimes +c-f)] \\
&= (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I_{\partial(r,s)} \otimes ((\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C) \otimes -c)) + (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)[(\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C) \otimes I_{\partial(r,s)} \otimes -c + \mathcal{w}_f \mathcal{e}_C \otimes -c] \\
&= (\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C + \mathcal{w}_f \mathcal{e}_C \otimes -c) + (\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C) \otimes -c + \mathcal{w}_f \mathcal{e}_C \otimes -c = 0
\end{align*}
\]
Furthermore, if either \( s_c = - \) or \( s_f = - \), then both terms will be 0 and thus, in this case, \([16]\) is true.

For case (3), again, we will only prove the equality for the splitting case (the merging case is similar). Let \( \pm_1, \pm_2 \) be the decorations on cleaved circles \( C_1, C_2 \) respectively and \( \pm_c \) be the decoration on merged cleaved circle \( C \). Without confusion, we will use the same \( e_{\gamma_c} \) to be the bridge edges in \( B_{\Gamma_n} \) between idempotents (with suitable signs on cleaved circle), obtained by surgering resolution bridge \( \gamma \) at \( c \) from 1 to 0 Then we have:

\[
\begin{align*}
&= (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes E_{T',c})D_c(+1+2) + (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes D_c)E_{T',c}(+1+2) \\
&= (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes E_{T',c})(e_{\gamma_c} \otimes +c) + (\mu_{B_{\Gamma}}, \otimes \mathbb{I}_d)(I \otimes D_c)[((\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C) \otimes -1 + 2) \\
&+ ((\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C) \otimes +1 + 2)] \\
&= e_{\gamma_c} e_C + \mathcal{w}_C \mathcal{e}_C \otimes -c + (\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C) e_{\gamma_c} \otimes -c + (\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C e_{\gamma_c}) \otimes -c \\
&= (\mathcal{e}_C + \mathcal{w}_C e_{\gamma_c} + \mathcal{w}_C \mathcal{e}_C e_{\gamma_c}) \otimes -c + \mathcal{w}_C (e_{\gamma_c} e_C + \mathcal{w}_C e_{\gamma_c}) \otimes -c + \mathcal{w}_C (e_{\gamma_c} e_C + \mathcal{w}_C e_{\gamma_c}) \otimes -c = 0
\end{align*}
\]
since the first, second and third terms equal 0 by the relation \([6]\) and \([\mathbb{1}]\) defining \( B_{\Gamma_n} \) respectively.

The rest is to prove that \([17]\) is true. Again, since the proof of merging case is similar, at here, we just present the proof for the splitting case. We also can assume no circle, considered below, contains the marked point. We have three following situations:

1. All of the circles are free: The proof of \([17]\) again is similar to the following case (2) and already described in \([15]\).
2. There is at least one free circle \( F \) and one cleaved circle \( C \). Since the case where \( C \) contains marked point is the special case of what we are about to prove, we can assume \( C \) does not contain marked point. Let \( \pm_C \) and \( \pm_f \) denote the decoration on the cleaved and free circles respectively. We describe the map of right hand side \([17]\) as following:

\[
\begin{align*}
+_{c+f} & \rightarrow w[(\mathcal{e}_C \otimes -c+f) + (I_{\partial(r,s,c)} \otimes +c-f)] \\
-_{c+f} & \rightarrow w.I_{\partial(r,s,c)} \otimes -c-f \\
+_{c-f} & \rightarrow w.e_C \otimes -c-f
\end{align*}
\]
if \( r(c) = 1 \) or \( +_c \rightarrow -c \) if \( r(c) = 0 \).

On one hand, if \( r(c) = 0 \), the LHS of \([17]\) equal 0 by the definition of \( D_c \), supporting on only state \((r, s)\) whose \( r(c) = 1 \). The second term of LHS maps \(+_c\) to \((\mathcal{e}_C + \mathcal{w}_C \mathcal{e}_C) \otimes -c = 0 \). Therefore, \([17]\) is true in this case.
In case of: \[ +c_1 \otimes +c_2 \]
\[ -c_1 \otimes +c_2 \]
\[ +c_1 \otimes -c_2 \]
\[ -c_1 \otimes -c_2 \]
\[ +c \]
\[ -c \]
\[ +c_1 \otimes +c_2 \]
\[ -c_1 \otimes +c_2 \]
\[ +c_1 \otimes -c_2 \]
\[ -c_1 \otimes -c_2 \]
\[ +c \]
\[ -c \]

**Figure 1.** Illustrating how to construct a type \( D \)-morphism between the two type \( D \)-structures before and after moving weight. The algebra elements above the arrows corresponds to the change of idempotents. Attention: the diagram does not cover fully all terms of type \( D \)-structures as well as of the morphism. It basically illustrates only the last case of proposition [18]. Additionally, this picture can be modified to give the picture of other cases by changing \( e_{\gamma_i} \) and \( e_{\gamma_i}^\dagger \) to suitable algebra elements.

On another hand, if \( r(c) = 1 \), the second term of LHS of (17) is 0 and the map of the first term can be describe:

\[ +c + f \rightarrow (w.I_{\partial (r,s)} e_C \otimes - c + f) + (w.I_{\partial (r,s)} I_{\partial (r,s)} \otimes + c - f) \]
\[ -c + f \rightarrow w.I_{\partial (r,s)} I_{\partial (r,s)} \otimes - c - f \]
\[ +c - f \rightarrow w.e_C I_{\partial (r,s)} \otimes - c - f \]

By comparing the value of two sides on each generator, the image of LHS and RHS agree on each generator. As the result, we finish the proof of (17) in this case.

(3) All of the circles are cleaved circle: The proof will be similar as above when we can prove that two sides agree on every generator \( \xi = (r, s) \) of \( T \) by
using the relation \( \overrightarrow{e}_{(\gamma,\sigma,\sigma')} \overrightarrow{e}_{(\gamma',\sigma',\sigma)} = \overrightarrow{e}_{C} \). The figure 1 illustrates the proof for this case. In this figure, \( C \) is the cleaved circle of 0-resolution while the \( C_1 \) and \( C_2 \) are corresponding to the top and the bottom cleaved circles of 1-resolution. We also have: \( x_1 = \overrightarrow{e}_{C_1} + (w + b)\overrightarrow{e}_{C_1} \) and \( x_2 = \overrightarrow{e}_{C_2} + (c + a)\overrightarrow{e}_{C_2} \) while \( x_1' = \overrightarrow{e}_{C_1} + b\overrightarrow{e}_{C_1} \) and \( x_2' = \overrightarrow{e}_{C_2} + (w + a + c)\overrightarrow{e}_{C_2} \). In equation (17):

\[
\text{LHS}(+c_1+c_2) = w.e_\gamma e_{\sigma_1} \otimes -c_1 + c_2 \oplus w.e_\gamma e_{\sigma_2} \otimes +c_1 - c_2
\]

and:

\[
\text{RHS}(+c_1+c_2) = (x_1 + x_1') - c_1 + c_2 \oplus (x_2 + x_2') - c_2 + c_1
\]

Using the relation (17), it shows that the \( \text{LHS}(+c_1+c_2) = \text{RHS}(+c_1+c_2) \). Similarly, we can prove (17) is true when applied for \(-c_1+c_2\) or \(+c_1-c_2\). Therefore, \( \Psi \) is a \( D \)-homomorphism \( \diamond \)

**Proposition 19.** \(((T),\delta_T)\) is isomorphic to \(((T'),\delta_{T'})\) as type \( D \)-structure.

**Proof.** Let \( \Phi : \langle T' \rangle \rightarrow B_\Gamma_n \otimes \mathbb{I}_n \langle T \rangle \) where \( \Phi(r, s) = I_{\partial(r, s)} \otimes (r, s) + w.D_c(r, s) \) where \( D_c \) is defined identically as \( D_c \) but from \( T' \) to \( T \). As in Proposition 18, \( \Phi \) is a \( D \)-homomorphism. We will prove: \( \Phi \ast \Psi = I_{\langle T \rangle} \) and \( \Psi \ast \Phi = I_{\langle T' \rangle} \), then we can conclude \(((T),\delta_T),(T'),\delta_{T'})\) are homotopy equivalent. Indeed,

\[
\Phi \ast \Psi
\]

\[
= (\mu_{B_\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \Phi)\Psi
\]

\[
= (\mu_{B_\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_T)T + w.\left[ (\mu_{B_\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes D_c)T + (\mu_{B_\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_T)D_c + (\mu_{B_\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_T)D_c \right]
\]

We note that the last term is 0 since \( D_c \) supports on the states \((r, s)\) where \( r(c) = 1 \) while the image of \( D_c \) contains the states \((r_1, s_1)\) where \( r_1(c) = 0 \). The sum of the two middle terms is also 0 since second term is equal to the third term. Moreover, \( (\mu_{B_\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_T)T = I_{\langle T' \rangle} \). Combining all these facts, we obtain the equation: \( \Phi \ast \Psi = I_{\langle T \rangle} \). Similarly, \( \Psi \ast \Phi = I_{\langle T' \rangle} \). \( \diamond \)

8. Graded differential algebra and stable homotopy

8.1. Preliminary. The different projections of outside tangle \( \overrightarrow{T} \) can have a different number of arcs and thus, the corresponding type \( D \) structures will be vector spaces over different base fields. To relate these structures, we will need the appropriate algebraic tool: stable homotopy, whose construction is based on the idea as in section 4 of [14]. Let \( \mathbb{F} \) be a field and \( W \) be a vector space over \( \mathbb{F} \). We denote \( \mathbb{F}_W \) be the the field of rational function of \( P_W \) where \( P_W \) is the symmetric algebra of \( W \). Recall: let \( W, W' \) be 2 vector spaces over \( \mathbb{F} \) and let \( M, M' \) be 2 vector spaces over \( \mathbb{F}_W \), \( \mathbb{F}_W' \) respectively. A pair \((M, W)\) is stably isomorphic to \((M', W')\) if there is a triple
(\(W'', i, i'\)) where \(W''\) is a vector space over \(\mathbb{F}\), \(W \xrightarrow{i} W''\) and \(W' \xrightarrow{i'} W''\) are injective linear maps, such that \(M \otimes_{\mathbb{F}_W} \mathbb{F}_W'' \cong M' \otimes_{\mathbb{F}_W} \mathbb{F}_W''\) as vector spaces over \(\mathbb{F}_W''\). This relation is proved to be an equivalent relation (14). For our purpose, we always assume \(\mathbb{F}\) to be \(\mathbb{Z}_2\) and \(W\) to be a vector space over \(\mathbb{Z}_2\) unless otherwise stated. This whole section is devoted to giving a modification of this definition in such a way that it allows us to relate two \(D\) (or \(A\)) structures over two different fields. Furthermore, the complexes obtained by gluing stable homotopy type \(D\) and type \(A\) will be stably homotopic in sense of (14). To modify the definition of stable category, we first present the following lemma and upgrade the graded differential algebra over \(\mathbb{Z}_2\) to the one over \(\mathbb{F}_W\).

**Lemma 20.** Let \(\alpha_i : C_i \longrightarrow C_{i+1}\) \((i = 1, ..., n)\) be a linear map between two vector spaces over \(\mathbb{F}_W\). The injective linear map \(\phi : W \hookrightarrow \widetilde{W}\) induces a map \(\widetilde{\alpha_i} : C_i \otimes_{\phi} \mathbb{F}_W \longrightarrow C_{i+1} \otimes_{\phi} \widetilde{\mathbb{F}_W}\) for each \(i\). Then we have the identity: \(\widetilde{\alpha_n} \circ \widetilde{\alpha_{n-1}} \circ ... \circ \widetilde{\alpha_1} = (\alpha_n \circ \alpha_{n-1} \circ ... \circ \alpha_1) \otimes I_{\mathbb{F}_W}\)

**Proof.** By induction, it suffices to prove the above identity for \(n = 2\). Let \(v \in C_1\) and \(r \in \mathbb{F}_W\). Using the definition, we have:

\[
\widetilde{\alpha_2} \circ \widetilde{\alpha_1}(v \otimes r) = (\alpha_2(\alpha_1(v)) \otimes r) = \widetilde{\alpha_2}(\alpha_1(v) \otimes r) = \widetilde{\alpha_2}(\widetilde{\alpha_1}(v \otimes r))
\]

\[\Box\]

### 8.2. \(A_{\mathcal{W}, \mathcal{I}}\)-Category and stable \(A_\infty\)-homotopy.

**Definition 21.** Let \((A, \mu_1, \mu_2)\) be unital, differential, \(\mathbb{Z}\)-graded algebra (DGA) over \(\mathbb{Z}_2\). Let \(\mathcal{I}\) be subalgebra of \(A\), containing all of idempotents in degree 0 and satisfying \(\mu_1(x) = 0\) for every \(x \in \mathcal{I}\). For each vector space \(W\) over \(\mathbb{Z}_2\), let \(A_W := A \otimes_{\mathbb{Z}_2} \mathbb{F}_W\), equipped with following \(\mathbb{F}_W\)-linear maps

\[
\mu_{W,1} : A_W \longrightarrow A_W[-1]
\]

\[
\mu_{W,2} : A_W \otimes_{\mathbb{F}_W} A_W \longrightarrow A_W
\]

where \(\mu_{W,1} = \mu_1 \otimes I_{\mathbb{F}_W}\) and \(\mu_{W,2} = \mu_2 \otimes I_{\mathbb{F}_W}\) under the canonical isomorphism: \(A_W \otimes_{\mathbb{F}_W} A_W \cong A^{\otimes 2} \otimes_{\mathbb{Z}_2} \mathbb{F}_W\). We also denote \(\mathcal{I}_W\) to be \(\mathcal{I} \otimes_{\mathbb{Z}_2} \mathbb{F}_W\). Using [20], it is straightforward to prove following proposition:

**Proposition 22.** If \((A, \mu_1, \mu_2)\) is a DGA over \(\mathbb{Z}_2\) then \((A_W, \mu_{W,1}, \mu_{W,2})\) is unital, dga over \(\mathbb{F}_W\). Furthermore, \(A_W\) is an \(\mathcal{I}_W\) bimodule where both actions of \(\mathcal{I}_W\) on the right and left are commute with the differential \(\mu_{W,1}\).
We first recall the definition of $A_{\infty}$, as in \[8\].

**Definition 23.** Let $W$ be a vector space over $\mathbb{Z}_2$. $(M, \{m_i\}_{i \in \mathbb{N}})$ is a strictly unital, $A_{\infty}$-module over $(A_W, I_W)$ if:

1. $M$ is a $\mathbb{Z}$-graded $I_W$-module (and thus, its also a vector space over $\mathbb{F}_W$), characterized by the $\mathbb{F}_W$-linear map $m : M \otimes_{\mathbb{F}_W} I_W \to M$ such that $m(m(x \otimes u_1) \otimes u_2) = m(x \otimes (u_1 u_2))$ for $x \in M$ and $u_1, u_2 \in I_W$.

2. For each $i \in \mathbb{N}$, $m_i : M \otimes_{I_W} A_W^{(i-1)} \to M[i-2]$ is an $I_W$-linear map, which satisfies the following compatibility condition:

\[
0 = \sum_{i+j=n+1} m_i(m_j \otimes I^{(i-1)}) + \sum_{i+j=n+1, j<3} m_i(I^{(i-k)} \otimes \mu_{W,j} \otimes I^{(i-k-1)})
\]

**Definition 24.** Let $(M, \{m_i\})$ and $(M', \{m'_i\})$ be strictly unital $A_{\infty}$-modules over $(A_W, I_W)$. An $A_{\infty}$ homomorphism $\Psi$ is a collection of maps:

\[
\psi_i : M \otimes_{I_W} A_W^{(i-1)} \to M'[i-1]
\]

indexed by $i \in \mathbb{N}$, satisfying the compatibility conditions:

\[
0 = \sum_{i+j=n+1} m'_i(\psi_j \otimes I^{(i-1)}) + \sum_{i+j=n+1} \psi_i(m_j \otimes I^{(i-1)}) + \sum_{i+j=n+1, k>0, j<3} \psi_i(I^{(i-k)} \otimes \mu_{W,j} \otimes I^{(i-k-1)})
\]

Additionally, a homotopy $H$ between two $A_{\infty}$-morphisms $\Psi$ and $\Phi$ is a set of maps $\{h_i\}$ with $h_i : M \otimes_{I_W} A_W^{(i-1)} \to M'[i]$ such that:

\[
\psi_n + \phi_n = \sum_{i+j=n+1} m'_i(h_j \otimes I^{(i-1)}) + \sum_{i+j=n+1} h_i(m_j \otimes I^{(i-1)}) + \sum_{i+j=n+1, k>0, j<3} h_i(I^{(i-k)} \otimes \mu_{W,j} \otimes I^{(i-k-1)})
\]

For a sake of simplicity, we sometimes let $M$ stand for $(M, \{m_i\})$ when it is clear from the context.

**Proposition 25.** Let $A_{W, I}$ be a collection of strictly unital, $A_{\infty}$-modules over $(A_W, I_W)$. $\text{Mor}(M, N)$ is a set of $A_{\infty}$-homomorphism $(\Psi, \{\psi_i\})$ and the composition of $\Phi * \Psi$ is a set of maps:

\[
(\Phi * \Psi)_n = \sum_{i+j=n+1} \phi_i(\psi_j \otimes I^{(i-1)})
\]

for each $n \in \mathbb{N}$ Then $A_{W, I}$ forms a category.

**Definition 26.** Let $\varphi : W \to W'$ be a linear injection. Let $(M, \{m_i\}_{i \in \mathbb{N}})$ be an object of $A_{W, I}$ and $(\Psi, \{\psi_i\}_{i \in \mathbb{N}}) \in \text{Mor}((M, \{m_i\}), (M', \{m'_i\}))$ be a morphism of $A_{W, I}$. We define:
Proposition 28.

Aject of Proposition 27.

The proof of this Proposition follows directly from Lemma 4.

(1) \[ \mathcal{F}_\varphi(M) = M \otimes_\varphi F_{W'} \]

which is a graded \( I_{W'} \)-module, characterized by \( m'(M \otimes_\varphi F_{W'}) \otimes_{F_{W'}} I_{W'}, \rightarrow M \otimes_\varphi F_{W'} \). Here, \( m' \) is defined to be \( m \otimes I_{F_{W'}} \), under the natural identification \( (M \otimes_\varphi F_{W'}) \otimes_{F_{W'}} I_{W'} \cong (M \otimes_{F_W} I_{W'}) \otimes_\varphi F_{W'} \) as vector spaces over \( F_W \).

(2) For each \( i \in \mathbb{N} \), \( \mathcal{F}_\varphi(m_i) : \mathcal{F}_\varphi(M) \otimes_{I_{W',}} A_{W'}^{\otimes(i-1)} \rightarrow \mathcal{F}_\varphi(M)[i-2] \) is defined by:

\[ \mathcal{F}_\varphi(m_i)((m \otimes r) \otimes (a_1 \otimes r_1) \otimes \ldots \otimes (a_{i-1} \otimes r_{i-1})) = m_i(m \otimes a_1 \otimes \ldots \otimes a_{i-1}) \otimes rr_1 \ldots r_{i-1} \]

where \( m \in M \), \( a_j \in A \) and \( r, r_j \in F_{W'} \) for \( j = 1, \ldots, i-1 \).

(3) Similarly, for each \( i \in \mathbb{N} \), \( \mathcal{F}_\varphi(\psi_i) : \mathcal{F}_\varphi(M) \otimes_{I_{W',}} A_{W'}^{\otimes(i-1)} \rightarrow \mathcal{F}_\varphi(M')[i-1] \) is defined as the same manner as \( \mathcal{F}_\varphi(m_i) \)

Combining lemma 20 and the following canonical isomorphism:

\[ (M \otimes_\varphi F_{W'}) \otimes_{I_{W'}} (A_{W'} \otimes_\varphi F_{W'})^{\otimes(i-1)} \cong (M \otimes_{I_{W'}} A_{W'^{(i-1)}}) \otimes_\varphi F_{W'} \]

for \( i \in \mathbb{N} \), it is straightforward to have two following propositions:

**Proposition 27.** \( \mathcal{F}_\varphi(M, \{m_i\}) = (\mathcal{F}_\varphi(M), \{\mathcal{F}_\varphi(m_i)\}) \) is well-defined and it is an object of \( A_{W',I} \). Likewise, \( \mathcal{F}_\varphi(\Psi) \) is well-defined and belongs to \( \text{Mor}(\mathcal{F}_\varphi(M, \{m_i\}), \mathcal{F}_\varphi(N, \{n_i\})) \)

**Proposition 28.** For each injection \( \varphi : W \hookrightarrow W' \), there is functor \( \mathcal{F}_\varphi : A_{W,I} \rightarrow A_{W',I} \) defined by

\[ \mathcal{F}_\varphi(M, \{m_i\}) = (\mathcal{F}_\varphi(M), \{\mathcal{F}_\varphi(m_i)\}) \]

\[ \mathcal{F}_\varphi(\Psi, \{\psi_i\}) = (\mathcal{F}_\varphi(\Psi), \{\mathcal{F}_\varphi(\psi_i)\}) \]

Furthermore, if \( \Psi \) and \( \Phi \) are \( A_\infty \)-homotopic in \( A_{I,W} \) then \( \mathcal{F}_\varphi(\Psi) \) is \( A_\infty \)-homotopic to \( \mathcal{F}_\varphi(\Phi) \) in \( A_{I,W'} \).

We now have enough tools to relate two \( A_\infty \) structures over the different fields in our construction, by defining stable \( A_\infty \)-homotopy:

**Definition 29.** Let \( W \) and \( W' \) be two vector spaces over \( \mathbb{Z}_2 \). Let \( (M, \{m_i\}), (M', \{m'_i\}) \) be objects of \( A_{W,I} \) and \( A_{W',I} \) respectively. Then \( (M, \{m_i\}) \) is stably homotopic to \( (M', \{m'_i\}) \) if there is a triple \( (\varphi, \varphi', W'') \) where \( W'' \) is a vector space over \( \mathbb{Z}_2 \), \( \varphi : W \hookrightarrow W'' \) and \( \varphi' : W' \hookrightarrow W'' \) are linear injections, such that \( \mathcal{F}_\varphi(M, \{m_i\}) \) is \( A_\infty \)-homotopic to \( \mathcal{F}_\varphi(M', \{m'_i\}) \) in the category \( A_{W',I} \).

**Proposition 30.** \( (M, \{m_i\}, m) \) stably homotopic to \( (M', \{m'_i\}, m') \) is an equivalent relation.

The proof of this Proposition follows directly from Lemma 4.3 in [14], the fact that homotopy equivalence of \( A_\infty \)-modules is an equivalence relation and the following lemma.
Lemma 31. Using the notation as above definition, let \( \tilde{W} \) be a vector space over \( \mathbb{Z}_2 \) and \( \tilde{\varphi} : W'' \rightarrow \tilde{W} \) be an linear injection. If \((M, \{m_i\})\) is stably homotopic to \((M', \{m'_i\})\) via the triple \((\varphi, \varphi', W'')\), then \((M, \{m_i\})\) is stably homotopic to \((M', \{m'_i\})\) via \((\tilde{\varphi} \circ \varphi, \tilde{\varphi} \circ \varphi', \tilde{W})\)

Proof: By proposition \[28\], \( \mathcal{F}\tilde{\varphi} \) induces the functor from the homotopy category of \( A_{W'', I} \) to the homotopy category of \( A_{\tilde{W}, I} \). Therefore, \( \mathcal{F}\tilde{\varphi}(\mathcal{F}\varphi(M)) \) is homotopic to \( \mathcal{F}\tilde{\varphi}(\mathcal{F}\varphi'(M')) \). We, thus, will finish the proof of lemma if we can prove \( \mathcal{F}\tilde{\varphi}(\mathcal{F}\varphi)(M) \cong A_{\tilde{W}, I} \mathcal{F}\tilde{\varphi} \circ \mathcal{F}\varphi(M) \) and likewise \( \mathcal{F}\tilde{\varphi}(\mathcal{F}\varphi')(M') \cong A_{\tilde{W}, I} \mathcal{F}\tilde{\varphi} \circ \mathcal{F}\varphi'(M') \)

First of all, we have: \( \mathcal{F}\tilde{\varphi}(\mathcal{F}\varphi)(M) = M \otimes \mathcal{F}\varphi F_{\tilde{W}} \cong (M \otimes \mathcal{F}\varphi W'') \otimes \mathcal{F}\tilde{\varphi} F_{\tilde{W}} = \mathcal{F}\tilde{\varphi} \mathcal{F}\varphi(M) \). Secondly, under this identification of underlying modules, we need to prove \( \mathcal{F}\tilde{\varphi}(\mathcal{F}\varphi)(m_i) = \mathcal{F}\tilde{\varphi} \mathcal{F}\varphi(m_i) \). Indeed, using the definition \[26\] we have:

\[
\mathcal{F}\tilde{\varphi} \mathcal{F}\varphi(m_i) \left( (m \otimes r) \otimes (a_1 \otimes r_1) \otimes \ldots \otimes (a_{i-1} \otimes r_{i-1}) \right) = m_i(m \otimes a_1 \otimes \ldots \otimes a_{i-1}) \otimes \tilde{\varphi} r_1 \ldots r_{i-1}
\]

On the other hand,

\[
\mathcal{F}\tilde{\varphi} \mathcal{F}\varphi(m_i) \left( (m \otimes 1) \otimes (a_1 \otimes 1) \otimes \ldots \otimes (a_{i-1} \otimes 1) \right) \otimes \tilde{\varphi} r_1 \ldots r_{i-1} = (m_i(m \otimes a_1 \otimes \ldots \otimes a_{i-1}) \otimes 1) \otimes \tilde{\varphi} r_1 \ldots r_{i-1}
\]

8.3. \( D_W \) Category and Stable D-homotopy.

We first review the definition of type D structure and \( D_W \)-category as in [LOT]. As before, we use the subscription \( W \) when we want to emphasize the base field \( F_W \)

Definition 32. \[8.3\] Let \( N \) be a graded \( I_W \)-module. A (left) D-structure on \( N \) is a linear map:

\[
\delta : N \longrightarrow (A_W \otimes_{I_W} N)[-1]
\]

satisfying:

\[
(\mu_{W,2} \otimes 1)(1_{A_W} \otimes \delta) + (d \otimes 1_{N}) \delta = 0
\]

A morphism of D-structures \((N, \delta) \rightarrow (N', \delta')\) is a map \( \psi : N \longrightarrow A_W \otimes_{I_W} N' \) such that:

\[
(\mu_{W,2} \otimes 1)(1_{A_W} \otimes \delta') \psi + (\mu_{W,2} \otimes 1)(1_{A_W} \otimes \psi) \delta + (\mu_{W,1} \otimes 1) \psi = 0
\]

\( H : N \longrightarrow (A_W \otimes_{I_W} N')[1] \) is a homotopy of two type D-morphisms \( \psi \) and \( \phi \) if:

\[
\psi + \phi = (\mu_{W,2} \otimes 1)(1_{A_W} \otimes \delta') H + (\mu_{W,2} \otimes 1)(1_{A_W} \otimes H) \delta + (\mu_{W,1} \otimes 1) H
\]

Remark 33. Since \( N \) is a graded \( I_W \)-module, \( N \) is also a vector space over \( \mathbb{F}_W \). Also, let \( n : I_W \otimes_{F_W} N \longrightarrow N \) be a \( \mathbb{F}_W \)-linear map, characterizing the \( I_W \)-module structure of \( N \). We have: \( n(x \otimes n(y \otimes a)) = n(xy \otimes a) \) for every \( x, y \in I_W \) and \( a \in N \).
**Proposition 34.** Let $D_{W,I}$ be the collection of type $D$-structures $(N,\delta)$ over $(A_W,I_W)$. Let $\text{Mor}((N,\delta),(N',\delta'))$ be the set of type $D$-morphisms and the composition $\phi \ast \psi$ of $\psi : N \to A_W \otimes N'$ and $\phi : N' \to A_W \otimes I_W N''$ is defined to be $(\mu_{W,2} \otimes I_{N''})(I_{A_W} \otimes \phi)\psi$. Additionally, the identity morphism at $(N,\delta)$ is $I_N : N \to A_W \otimes I_W N$ given by $x \mapsto 1_{A_W} \otimes N$. Then $D_{W,I}$ forms a category.

Using the same technique as in [8.2] for each $\varphi : W \to W'$ linear injection, we will construct a functor from $D_{W,I}$-category to $D_{W',I'}$-category. Before doing that, we have the following definition.

**Definition 35.** Let $\varphi : W \hookrightarrow W'$ linear injection. Let $(N,\delta)$ be an object of $D_{W,I}$-category and $\psi \in \text{Mor}((N,\delta),(N',\delta'))$. We define:

$$\mathcal{G}_\varphi(N) := N \otimes_\varphi F_{W'}$$

which is a graded $I_W$-module, characterized by $n' : I_{W'} \otimes_{F_{W'}} (N \otimes_\varphi F_{W'}) \to N \otimes_\varphi F_{W'}$. Here, $n'$ is defined to be $n \otimes I_{F_{W'}}$ under the identification: $I_{W'} \otimes_{F_{W'}} (N \otimes_\varphi F_{W'}) \cong (I_W \otimes_{F_W} N) \otimes_\varphi F_{W'}$.

$$\mathcal{G}_\varphi(\delta) : N \otimes_\varphi F_{W'} \to A_{W'} \otimes_{I_{W'}} (N \otimes_\varphi F_{W'})[-1]$$

is defined to be $\delta \otimes I_{F_{W'}}$ under the canonical isomorphism $A_{W'} \otimes_{I_{W'}} (N \otimes_\varphi F_{W'}) \cong (A_W \otimes_{I_W} N) \otimes_\varphi F_{W'}$.

$$\mathcal{G}_\varphi(\psi) : N \otimes_\varphi F_{W'} \to A_{W'} \otimes_{I_{W'}} (N' \otimes_\varphi F_{W'})$$

is defined to be $\psi \otimes I_{F_{W'}}$ under the canonical isomorphism $A_{W'} \otimes_{I_{W'}} (N' \otimes_\varphi F_{W'}) \cong (A_W \otimes_{I_W} N') \otimes_\varphi F_{W'}$.

**Proposition 36.** Using the same notation as above definition, there exists a functor $\mathcal{G}_\varphi : D_{W,I} \to D_{W',I}$ defined by:

$$\mathcal{G}_\varphi(N,\delta) = (\mathcal{G}_\varphi(N),\mathcal{G}_\varphi(\delta))$$

and $\mathcal{G}_\varphi(\psi)$ is defined as above. Furthermore, let $\psi$ and $\phi$ belong to $\text{Mor}((N,\delta),(N',\delta'))$ in $D_{W,I}$ and if $H$ is a homotopy of $\psi$ and $\phi$, then $\mathcal{G}_\varphi(H)$ is a homotopy of $\mathcal{G}_\varphi(\psi)$ and $\mathcal{G}_\varphi(\phi)$ in $D_{W',I}$ where:

$$\mathcal{G}_\varphi(H) : N \otimes_\varphi F_{W'} \to A_{W'} \otimes_{F_{W'}} (N' \otimes_\varphi F_{W'})[1]$$

is defined to be $H \otimes I_{F_{W'}}$ under the isomorphism $A_{W'} \otimes_{F_{W'}} (N' \otimes_\varphi F_{W'}) \cong (A_W \otimes_{I_W} N') \otimes_\varphi F_{W'}$. Therefore, $\mathcal{G}_\varphi$ induces a functor from the homotopy category of $D_{W,I}$ to the homotopy category of $D_{W',I}$.

Since the proof of this proposition involves the same method as in proposition [28] by using lemma [20], we do not present it here. We can now give a definition of stable $D$-homotopy.
Definition 37. Let $W$ and $W'$ be vector spaces over $\mathbb{Z}_2$. Let $(N, \delta), (N', \delta')$ be objects of $D_{W, \mathcal{I}}$ and $D_{W', \mathcal{I}}$ respectively. Then $(N, \delta)$ is stably homotopic to $(N', \delta')$ if there is a triple $(\varphi, \varphi', W'')$ where $W''$ is a vector space over $\mathbb{Z}_2$, $\varphi : W \hookrightarrow W''$ and $\varphi' : W' \hookrightarrow W''$ are linear injections, such that $\mathcal{G}_\varphi(N, \delta)$ is $D$-homotopic (in regular sense) to $\mathcal{G}_{\varphi'}(N', \delta')$ in the category $D_{W'', \mathcal{I}}$.

Proposition 38. Stable homotopy of type $D$ structures is an equivalence relation.

Again, the proof of this proposition is similar to the proof of 30. Therefore, instead of giving the proof, we have the following remark about the property of composition of two functors, which is really useful for the next subsection.

Remark 39. Let $\varphi_1 : W \hookrightarrow W_1$ and $\varphi_2 : W_1 \hookrightarrow W_2$ be injective linear maps. Let $(N, \delta)$ be an object of $A_W$. Then $\mathcal{G}_{\varphi_2 \circ \varphi_1}(N, \delta) \simeq_D \mathcal{G}_{\varphi_2}(\mathcal{G}_{\varphi_1})(N, \delta)$.

8.4. Paring an $A_{\infty}$-structure and a type $D$-structure over different DGA.

In [8], there is the result that we can pair an object $(M, \{m_i\})$ of $A_{W, \mathcal{I}}$ and an object $(N, \delta)$ of $D_{W, \mathcal{I}}$ to form a chain complex $(M \boxtimes N, \partial^S)$ over $\mathbb{F}_W$. Additionally, if $(M, \{m_i\}) \simeq_{A_{\infty}} (M', \{m'_i\})$ in $A_{W, \mathcal{I}}$ category and $(N, \delta) \simeq_D (N', \delta')$ in $D_{W, \mathcal{I}}$ category, then $(M \boxtimes N, \partial^S)$ is chain homotopic to $(M' \boxtimes N', \partial^S)$. On the other hand, for our purpose, since we need to pair a type $A$ and a type $D$ structures over distinct differential graded algebras, we will adjust the way to pair them to get a chain complex. Furthermore, we will prove that under the change of either type $A$ or type $D$ by a stable homotopy equivalence, the resulting chain complex is preserved up to stable homotopy.

Let $W$ and $W'$ be two vector spaces over $\mathbb{Z}_2$. Let $(M, \{m_i\}), (N, \delta)$ be objects of $A_{W, \mathcal{I}}$ and $D_{W', \mathcal{I}}$ respectively. Let $W_1 := W \oplus W'$ and denote $\varphi : W \hookrightarrow W_1, \varphi' : W' \hookrightarrow W_1$ be the two canonical linear injections.

Definition 40. Define $M \boxtimes_S N$ to be the graded vector space $\mathcal{F}_{\varphi}(M) \otimes_{I_{W_1}} \mathcal{G}_{\varphi'}(N)$ over $\mathbb{F}_{W_1}$ and $\partial^S : M \boxtimes_S N \to (M \boxtimes_S N)[-1]$ to be the map:

$$\partial^S = \sum_{k=0}^{\infty} (\mathcal{F}_{\varphi}(m_{k+1}) \otimes \mathcal{G}_{\varphi'}^{(k+1)}(N)) \circ (\mathbb{I}_{\mathcal{F}_{\varphi}}(M) \otimes \Delta_k)$$

where $\Delta_k : \mathcal{G}_{\varphi'}(N) \to A_{W_1}^{\otimes k} \otimes_{I_{W_1}} \mathcal{G}_{\varphi'}(N)[-k]$ is defined by induction $\Delta_0 = \mathbb{I}_{\mathcal{G}_{\varphi'}(N)}$, $\Delta_1 = \mathcal{G}_{\varphi'}(\delta)$ and the relation: $\Delta_n = (\mathbb{I} \otimes \mathcal{G}_{\varphi'}(\delta))^{\otimes n-1} \Delta_n$.

Note that: $(M \boxtimes_S N, \partial^S)$ is defined exactly the same as the definition of $(\mathcal{F}_{\varphi}(M) \boxtimes \mathcal{G}_{\varphi'}(N), \partial^S)$ in [8] and thus, is a chain complex.

Proposition 41. Let $W$, $W'$ and $\widetilde{W}$ be three vector spaces over $\mathbb{Z}_2$. Let $(N, \delta)$ be object of $D_{\widetilde{W}, \mathcal{I}}$. Let $(M, \{m_i\}), (M', \{m'_i\})$ be objects of $A_{W, \mathcal{I}}$ and $A_{W', \mathcal{I}}$ respectively,
such that \((M, \{m_i\})\) is stable \(A\)-homotopic to \((M', \{m_i'\})\) via the triple \((\varphi, \varphi', W'')\). Then \((M \boxtimes S N, \partial_S^G)\) is stably chain homotopic to \((M' \boxtimes S N, \partial_S^G)\).

Before proving this proposition, we state the following lemma without proof since it can be proved similarly as \[28\] by using lemma \[20\].

**Lemma 42.** Let \((M, \{m_i\})\) and \((N, \delta)\) be objects of \(A_{W,I}\) and \(D_{W,I}\) categories respectively. Let \(\varphi : W \rightarrow W'\) be a linear injection. Then \(((M \boxtimes N) \otimes \varphi F_{W'}, \partial \otimes I_{F_{W'}})\) is chain isomorphic to \((F_{\varphi}(M) \boxtimes G_{\varphi}(N), \partial^G)\).

**Proof of 41.**
Define \(V := W \oplus \tilde{W}, V' := W' \oplus \tilde{W}\) and \(V'' := W'' \oplus \tilde{W}\). Let \(\tilde{\varphi} : V \rightarrow V'\) and \(\tilde{\varphi}' : V' \rightarrow V''\) be defined to be \(\varphi \oplus I_W\) and \(\varphi' \oplus I_W\). Let \(p : \tilde{W} \rightarrow W, p' : \tilde{W} \rightarrow V', \pi : W \rightarrow V, \pi' : W' \rightarrow V'\) and \(\pi'' : W'' \rightarrow V''\). We immediately have the following relations:

1. \(\tilde{\varphi} \circ p = \tilde{\varphi}' \circ p'\)
2. \(\tilde{\varphi} \circ \pi = \pi'' \circ \varphi\)
3. \(\tilde{\varphi}' \circ \pi' = \pi'' \circ \varphi'\)

We will prove that \((M \boxtimes S N, \partial_S^G)\) is stably chain homotopic to \((M' \boxtimes S N, \partial_S^G)\) via the triple \((\tilde{\varphi}, \tilde{\varphi}', V'')\). Indeed, using lemma \[42\] and the fact that underlying complex of \((M \boxtimes S N, \partial_S^G)\) is \(F_{\tilde{\varphi}}(F_{\pi}(M)) \boxtimes G_{\tilde{\varphi}}(G_{\pi}(N), \partial^G)\)

Additionally, as in the proof of lemma \[31\] we have: \(F_{\tilde{\varphi}}(F_{\pi}(M)) \approx A F_{\tilde{\varphi}\pi\pi}(M) = F_{\pi'\pi}(F_{\varphi}(M))\). Note that the second identity comes from the equation \(\tilde{\varphi} \circ \pi = \pi'' \circ \varphi\). Similarly, \(G_{\tilde{\varphi}}(G_{\pi}(N))\) is stably \(D\)-homotopic to \(G_{\tilde{\varphi}\pi\pi}(N)\). Therefore, following \[8\], \((M \boxtimes S N) \otimes_{\tilde{\varphi}} F_{V''}, \partial_S^G \otimes I_{F_{V''}}\) is chain homotopic to \((F_{\pi''}(F_{\varphi}(M)) \boxtimes G_{\tilde{\varphi}\pi\pi}(N), \partial^G)\). Likewise, we have:

\[(M' \boxtimes S N) \otimes_{\varphi'} F_{V''}, \partial_S^G \otimes I_{F_{V''}}\approx (F_{\pi''}(F_{\varphi'}(M')) \boxtimes G_{\varphi'\pi\pi}(N), \partial^G)\]

Since \((M, \{m_i\})\) is stable \(A\)-homotopic to \((M', \{m_i'\})\) in \(A_{W',I}\) category. By proposition \[28\], \(F_{\varphi'}(F_{\varphi}(M)) \approx A F_{\pi''}(F_{\varphi'}(M'))\). Furthermore, since \(\tilde{\varphi} \circ p = \tilde{\varphi}' \circ p'\), we have \(G_{\tilde{\varphi}\pi\pi}(N) \equiv G_{\varphi'\pi\pi}(N)\).

Therefore, \((F_{\pi''}(F_{\varphi}(M)) \boxtimes G_{\varphi'\pi\pi}(N), \partial^G)\) is chain homotopic to \((F_{\pi''}(F_{\varphi'}(M')) \boxtimes G_{\varphi'\pi\pi}(N), \partial^G)\) and thus, \((M \boxtimes S N) \otimes_{\tilde{\varphi}} F_{V''}, \partial_S^G \otimes I_{F_{V''}}\) is chain homotopic to \((M' \boxtimes S N) \otimes_{\varphi'} F_{V''}, \partial_S^G \otimes I_{F_{V''}}\). As a result, \((M \boxtimes S N, \partial_S^G)\) is stably chain homotopic to \((M' \boxtimes S N, \partial_S^G)\) via the triple \((\tilde{\varphi}, \tilde{\varphi}', V'')\).

The same method can be used to prove the following theorem:

**Proposition 43.** Let \(W, W'\) and \(\tilde{W}\) be two vector spaces over \(Z_2\). Let \((M, \{m_i\})\) be objects of \(A_{W,I}\). Let \((N, \delta), (N', \delta')\) be objects of \(D_{W,I}\) and \(D_{W',I}\), such that \((N, \delta)\) is
stably \(D\)-homotopic to \((N', \delta')\) via the triple \((\varphi, \varphi', W')\). Then \((M \boxtimes_S N, \partial_S^2)\) is stably chain homotopic to \((M \boxtimes_S N', \partial_S^2)\).

9. INVARIANCE UNDER Reidemeister moves

Let \(T\) be a diagram of \(\mathcal{T}\) before Reidemeister move and use \(T'\) for the diagram of \(\mathcal{T}\) after the Reidemeister move. First of all, we will use the moving weights trick, described in Section 6 to move weights off the free circle in local figures 2, 4, 5 and 6 of Reidemeister’s moves, then using the cancellation method to get a deformation of \((\mathcal{T}, \delta)\). As the next step, we will implement the algebraic tool as in Section 8 to show that \((\mathcal{T}, \delta)\) is (stably) homotopic to \((\mathcal{T'}, \delta')\).

**Attention:** In the proof of invariance under Reidemeister moves, we sometimes use the index \(T\) in \(\mathcal{B}_T\) to emphasize the dependence of the ground field \(F_T\) on the diagram \(T\).

9.1. INvariance under the first Reidemeister move. Figure 2 shows the complex for a diagram prior to and after an RI move applied to a right-handed crossing.

As usual, we can decompose \([\mathcal{T}']\) = \(V_0 \oplus V_1\) corresponding to the states where \(r(c) = 0\) or \(r(c) = 1\). Furthermore, since each state generating \(V_0\) always has a free circle \(C\) as in the local diagram, we can continue decomposing \(V_0 = (V' \otimes +c) \oplus (V' \otimes -c)\) where \(\pm_c\) is the decoration on free circle \(C\).

Let \((r, s)\) be a state generating \(V' \otimes +c\). As we can see, the only state \((r', s')\), which generates \(V_1\) and is in the image of \(\delta_T\) \((r, s)\) is the one obtained by applying \(d_{APS}\). In this case, the coefficient of \((r', s')\) in \(\mathcal{B}_T\) is \(I_{\partial(r, s)} = I_{\partial(r', s')}\). Using this fact, we can use the cancellation theorem to cancel out \(V' \otimes +c\) and \(V_1\). We have left only \(V_2 = V' \otimes -c\) with the new structure \(\delta' : V_2 \rightarrow \mathcal{B}_T \otimes V_2\) which is \(\delta' = \delta_T\) \(\otimes\) \(V_2\), plus a perturbation term.

Taking a deeper look into the cancellation process, we see that the perturbation term arises from the following diagram: \(\xi \rightarrow \xi_1 \rightarrow \xi_2 \rightarrow \xi_3\) where \(\xi \in V_2\). \(\xi_1\) is a generator of \(V_1\) which is in the image of \(\delta_T\) \((\xi)\), \(\xi_2 \in V' \otimes +c\), maps under \(\delta_T\) has the image \(\xi_3\) as above (the coefficient in this case is an idempotent) and \(\xi_3\) generates \([\mathcal{T}']\) and belongs to \(\delta_T\) \((\xi_1)\). We note that \(\xi_3\) has to be a generator of \((V' \otimes +c) \oplus V_1\). Indeed, the only possibility for \(\xi_3 \in V' \otimes -c = V_2\) is that \(\xi_3\) is obtained from \(\xi_2\) by applying the vertical map \(\delta_T\) to change the decoration on \(C\). But since we already moved the weight out of \(C\), \(\delta_T\) = 0. Therefore, \(\xi_3 \in (V' \otimes +c) \oplus V_1\) and as a result, \(\xi_3\) is canceled out by another terms in either \(V' \otimes +c\) or \(V_1\). Consequently, the perturbation term will be 0 and \(V_2, \delta' \) \(\simeq\) \([\mathcal{T}']\).

Although there is one-to-one corresponding between generators of \([\mathcal{T}']\) and of \(V_2\), we are still working over different fields \(F_T\) and \(F_{T'}\). Additionally, the local arc in \(T\) is
Figure 2. In the top row, we use the weight shift isomorphisms to move all the weights to the bottom of the diagram. Surgering the crossing $c$ in both ways gives a finer view into the complex. Regardless of whether the local arc is on a free circle or a cleaved circle, the algebra element of thickened arrow is always an invertible element of $B\Gamma_n$ (however, the recording algebra element of the dashed arrow depends upon the type of the local arc). When the complex is reduced along the thickened arrow we obtain the complex for the diagram after the RI move, but with the weight $x_A + x_n + x_B$ on the local arc.

decorated by $y_i$ while that for $V_2$ is decorated by $x_A + x_B + x_n$. To relate them, let $\tilde{\varphi} : B\Gamma_{T,n} \to B\Gamma_{T',n}$ be the map induced by the inclusion $\varphi : F_{T} \to F_{T'}$, defined by: $y_i \to x_A + x_B + x_n$ and $y_j \to x_j$ for $i \neq j$. Then, we define:

$$\delta_{T',\bullet} : \left[\overrightarrow{T}\right] \otimes_{\varphi} F_{T'} \rightarrow B\Gamma_{T',n} \otimes_{I_n} \left[\left[\overrightarrow{T}\right] \otimes_{\varphi} F_{T'}\right]$$
by specifying \( \langle \overrightarrow{\delta_{T,\star}}(r, s) \otimes 1, (r', s') \otimes 1 \rangle = \overline{\varphi}\langle \overrightarrow{\delta_{T,\star}}(r, s), (r', s') \rangle \). Following the proposition \( \text{[36]} \), \( \overrightarrow{\delta_{T,\star}} \) is a type \( D \)-structure of \( \left[ \overrightarrow{T} \right] \otimes_{\varphi} \mathbb{F}_{\overrightarrow{\phi}} \) over \( \mathcal{B}\Gamma_{T',n} \).

If we further canonically identify the generator \((r, s)\) of \( V_{2} \) with \((r, s) \otimes 1\) of \( \left[ \overrightarrow{T} \right] \otimes_{\varphi} \mathbb{F}_{\overrightarrow{\phi}} \), we have \( \langle \overrightarrow{\delta'}(r, s), (r', s') \rangle = \langle \overrightarrow{\delta_{T,\star}}((r, s) \otimes 1), (r', s') \otimes 1 \rangle \). Therefore,

\[
\left( \left[ \overrightarrow{T} \right] \otimes_{\varphi} \mathbb{F}_{\overrightarrow{\phi}} \right) \approx (V_{2}, \overrightarrow{\delta'}) \simeq \left( \left[ \overrightarrow{T} \right], \overrightarrow{\delta_{T,\star}} \right)
\]

over \( (\mathcal{B}\Gamma_{T',n}, \mathcal{L}_{T',n}) \).

Consequently, \( \left( \left[ \overrightarrow{T} \right], \overrightarrow{\delta_{T,\star}} \right) \) is (stably) homotopic to \( \left( \left[ \overrightarrow{T} \right], \overrightarrow{\delta_{T,\star}} \right) \).

Note that the invariance under a left-handed Reidemeister I move can be obtained by combining the invariance of type \( D \) structures under a right-handed RI and under Reidemeister II move.

### 9.2. Invariance under the second Reidemeister move.

As in the proof of the RI move, first, we shift all of middle weights of local diagram to the bottom as in \( \text{[3]} \). Let \( c_{1}, c_{2} \) be two crossings in local diagram. For \( i = 0, 1, j = 0, 1 \), we define \( V_{i,j} \) be the set of states where \( c_{0} \) is resolved by \( i \) and \( c_{1} \) is resolved by \( j \). We can decompose \( \left[ \overrightarrow{T} \right] = V_{00} \oplus V_{01} \oplus V_{10} \oplus V_{11} \). We further decompose \( V_{10} = (V \otimes +_{c}) \oplus (V \otimes -_{c}) \) where \( \pm_{c} \) is the decoration on the free circle \( C \). Basically, the same sort of argument as in the proof of RI move can be used to explain why we can cancel out \( V \otimes +_{c} \) and \( V_{11} \) without creating any perturbation. The reason why its true again comes from the fact that \( \overline{w_{c}} = 0 \) and thus, we do not have any map from \( V \otimes +_{c} \) to \( V \otimes -_{c} \). This can be illustrated by \( \text{[4]} \).

The next step is to cancel out \( V_{00} \) and \( V \otimes -_{c} \) by applying an isomorphism \( I \otimes d_{APS} : V_{00} \rightarrow V \otimes -_{c} \). Again, there is no perturbation to be appeared since there is no map from \( V_{01} \) to \( V_{00} \oplus (V \otimes -_{c}) \). At the end, we will be left solely with \( V_{01} \), with a new \( D \)-structure \( \delta' : V_{01} \rightarrow \mathcal{B}\Gamma_{T',n} \otimes V_{01} \) where \( \delta' = \overrightarrow{\delta_{T,\star}}|_{V_{01}} \). Note that the generators...
Figure 4. This figure illustrates the proof of invariance under the second Reidemeister move. As we can see, regardless of whether the local arcs lie on free circles or cleaved circles, the algebra elements of dashed arrows are always idempotents. Additionally, if we cancel the bottom arrow first, and then the top arrow, we introduce no new perturbation terms in the boundary map since the weight on $C$ is 0. These cancellations produce the deformation retraction of type $D$ structure.
Invariance under the third Reidemeister move. Since the proof of invariance under the third Reidemeister move is similar to as in the first and the second one, we just give the strategy of the proof and the figures to illustrate it. However, we will construct the map to relate one, we just give the strategy of the proof and the figures to illustrate it. However, we will construct the map to relate $\mathcal{B} \Gamma_{T,n}$ and $\mathcal{B} \Gamma_{T',n}$, since it is somewhat different from the $RI$ and $RII$ cases. The strategy is to follow these steps: 1) shift the weights to the bottom as in [5,6] 2) use the cancellation method exactly as in $RII$ move for the top faces of top diagrams, [6] we get the lower diagrams with the two new perturbation maps. The important point is that if we canonically identify the generators of lower diagrams in figures 5, 6, the maps from tops to the bottoms of the lower diagrams will be the same (see [12] for more detail).

Assuming the weights for the edges not showed in the picture are $z_7, ..., z_l$. We denote the type $D$-structure of the first and second lower diagram to be $(R_1, \delta_1)$ and $(R_2, \delta_2)$. Let $(S, \delta_S)$ be a type $D$-structure with the identical maps as in lower picture but with the black circle representing $u_1$, the white circle representing $u_2$ and the gray circle representing $u_3$. Caution:) $(S, \delta_S)$ is a type $D$-structure over $\mathcal{B} \Gamma_{S,n} = \mathcal{B} \Gamma_n \otimes_{\mathbb{Z}_2} \mathbb{F}_S$ where $\mathbb{F}_S$ is the field of fractions of $\mathbb{Z}_2[u_1, u_2, u_3, z_7, ..., z_l]$. We define the map $\varphi_1 : \mathcal{B} \Gamma_{S,n} \rightarrow \mathcal{B} \Gamma_{T,n}$ which is induced by the inclusion $\tilde{\varphi}_1 : \mathbb{F}_S \rightarrow \mathbb{F}_T$: $u_1 \rightarrow y_1 + y_4$, $u_2 \rightarrow y_2 + y_5$ and $u_3 \rightarrow y_3 + y_6$. As in the proof of Reidemeister $I$, this map will give us a path to relate $(S, \delta_S)$ and $(R_1, \delta_1)$ and as a result, $(S, \delta_S)$ is (stable) homotopic to $(R_1, \delta_1)$. Similarly, the map $\varphi_2 : \mathcal{B} \Gamma_{S,n} \rightarrow \mathcal{B} \Gamma_{T',n}$, which is induced by the inclusion $\varphi_2 : \mathbb{F}_S \rightarrow \mathbb{F}_{T'}$: $u_1 \rightarrow x_3 + y_4$, $u_2 \rightarrow y_1 + y_5$ and $u_3 \rightarrow y_2 + y_6$, will result in the fact that $(S, \delta_S)$ is (stable) homotopic to $(R_2, \delta_2)$. Then by Proposition 33, $(R_1, \delta_1)$ is (stable) homotopic to $(R_2, \delta_2)$. As a result, $(\mathbb{T}, \delta_{T'})$ is (stable) homotopic to $(\mathbb{T}', \delta_{T'})$.

10. Definition of type $A$ twisted structure

In [13], L. P. Roberts describes a type $A$-structure on the same underlying module $\langle \mathbb{T} \rangle$ over $\mathcal{B} \Gamma_n$ (if tensor with $\mathbb{F}_{T'}$). It is characterized by two maps: 1) $m_1$ which increases the bigrading by $(1, 0)$ and thus, $+1 \zeta$-grading map and 2) $m_2$ which preserves bigrading and thus, $\zeta$-grading preserving. For bordered twisted homology, we
Figure 5. The local picture for a diagram before an RIII move. We decompose the module along the eight possible ways of resolving the local crossings. The four resolutions with crossing $c$ resolved using a 0 resolution replicate the diagrams in the proof of RII invariance. Using the cancellation process in the top diagram, gives the diagram at the bottom. A new term in boundary map may occur from the thicker arrow in the bottom picture; however, under the identification of generators of lower diagrams in 5 and 6, it will be the same as the map of the lower diagram of 6 which is obtained by surgering a bridge at crossing $d$. 
will describe another type $A$–structure on $\langle \hat{T} \rangle$ over $\mathcal{B}\Gamma_n$ by two following $\zeta$–grading preserving maps:

$$m_1,\cdot : \langle \hat{T} \rangle \rightarrow \langle \hat{T} \rangle[-1]$$

$$m_2,\cdot : \langle \hat{T} \rangle \otimes_{\mathcal{I}_n} \mathcal{B}\Gamma_n \rightarrow \langle \hat{T} \rangle$$

Let $\xi = (r, s)$ be a generator of $\langle \hat{T} \rangle$ and $e$ be a generator of $\mathcal{B}\Gamma_n$. For $m_1,\cdot$: denote $m_1,\cdot(\xi) := m_1(\xi) + v(\xi) = d_{APS}(\xi) + v(\xi)$ where $v(\xi) = \sum_{s(C) = +, C} \hat{w}_C(r, s_C)$
Note that: since vertical map \( v \) will decrease the bigrading by \((0,2)\), \( m_{1, \bullet} \) is differential \( \zeta \)-grading preserving into \( H_{T}[-1] \).

For \( m_{2, \bullet} \), we first define the action of generator of \( \mathcal{B} \Gamma_n \):

- If \( e \neq \hat{e}_C \), then \( m_{2, \bullet}(\xi \otimes e) := m_{2}(\xi \otimes e) \)
- If \( e = \hat{e}_C \), then \( m_{2, \bullet}(\xi \otimes \hat{e}_C) := m_{2}(\xi \otimes \hat{e}_C) + \hat{w}_Cm_{2}(\xi \otimes \hat{e}_C) = m_{2}(\xi \otimes \hat{e}_C) + \hat{w}_C(r, s_C) \).

Since the bigrading of \( \hat{e}_C \) is \((1,1)\), the fact that \( m_{2, \bullet} \) is \( \zeta \)-grading preserving immediately imply \( m_{2, \bullet} \) is also \( \zeta \)-grading.

To define the action of \( \mathcal{B} \Gamma_n \), for \( p_1, p_2 \in \mathcal{B} \Gamma_n \), we impose the relation:

\[
(20) \quad m_{2, \bullet}(\xi \otimes p_1 p_2) = m_{2, \bullet}(m_{2, \bullet}(\xi \otimes p_1) \otimes p_2)
\]

For definition to be well defined, we need to prove the following Proposition:

**Proposition 44.** If two products of the generators \( p_1 \) and \( p_2 \) define the same element in \( \mathcal{B} \Gamma_n \). Then \( m_{2, \bullet}(\xi \otimes p_1) = m_{2, \bullet}(\xi \otimes p_2) \).

**Proof:** It suffices to prove \( m_{2, \bullet}(\xi \otimes p) = 0 \) if \( p \) is a relation defining \( \mathcal{B} \Gamma_n \). First of all, we recall two following facts:

- \( m_{2}(\xi \otimes p) = 0 \) as in [13]
- \( m_{2, \bullet}(\xi \otimes e) = m_{2}(\xi \otimes e) \) for every generator of \( \mathcal{B} \Gamma_n \) unless \( e = \hat{e}_C \)

Combining these two facts and using \((20)\), we have if \( p \) does not involve \( \hat{e}_C \), then \( m_{2, \bullet}(\xi \otimes p) = m_{2}(\xi \otimes p) = 0 \). On the other hand, if \( p \) involves \( \hat{e}_C \) and let \( p_1 \in \mathcal{B} \Gamma_n \) is obtained from \( p \) by substituting \( \hat{e}_C \) for each term \( \hat{e}_C \) appeared in \( p \). In this situation, we have two situations:

**Case I:** If \( p_1 \) is a relation defining \( \mathcal{B} \Gamma_n \), then we have two possibilities:
- If \( p = \hat{e}_C \hat{e}_D + \hat{e}_D \hat{e}_C \): Since \( m_{2, \bullet}(\xi \otimes \hat{e}_D) = m_{2}(\xi \otimes \hat{e}_D) + \hat{w}_Dm_{2}(\xi \otimes \hat{e}_D) \) and \( m_{2, \bullet}(\xi \otimes \hat{e}_D) = m_{2}(\xi \otimes \hat{e}_D) + \hat{w}_Dm_{2}(\xi \otimes \hat{e}_D) \), we have:

\[
m_{2, \bullet}(\xi \otimes p) = m_{2, \bullet}(m_{2, \bullet}(\xi \otimes \hat{e}_D) \otimes \hat{e}_C)
\]

**Case II:** If \( p_1 \) is not a relation defining \( \mathcal{B} \Gamma_n \). Looking through the set of relations of \( \mathcal{B} \Gamma_n \), we will see that only the relations, which involves left decoration edge and satisfies **Case II**, come from either merging two + cleaved circles or dividing a + cleaved circle by \( \gamma \in \text{Merge}(L) \) or \( \gamma \in \text{Divide}(L) \). Since the proof of the dividing case is similar, we just present the merging case. In this case, \( p = \hat{e}_Cm_{\gamma} + \hat{e}_Cm_{\gamma} + \gamma \hat{e}_C \).
We need to prove:

\[(21)\quad m_2.(\xi \otimes \tilde{e}_C, m_\gamma) + m_2.(\xi \otimes \tilde{e}_C, m_\gamma) + m_2.(\xi \otimes m_\gamma \tilde{e}_C) = 0\]

Rewrite it:

\[\iff m_2.(m_2.(\xi \otimes \tilde{e}_C, m_\gamma) + m_2.(\xi \otimes \tilde{e}_C, m_\gamma) + m_2.(\xi \otimes m_\gamma \tilde{e}_C) = 0\]

\[\iff [m_2(m_2.(\xi \otimes \tilde{e}_C, m_\gamma) + m_2.(\xi \otimes \tilde{e}_C, m_\gamma) + m_2.(\xi \otimes m_\gamma \tilde{e}_C)] + [m_2(w(\xi \otimes e_{C_1}) \otimes m_\gamma) + m_2(w(\xi \otimes e_{C_2}) \otimes m_\gamma) + m_2((\xi \otimes m_\gamma \tilde{e}_C)] = 0\]

The sum of the first bracket will be 0 since \(m_2.(\xi \otimes p) = 0\) and thus it suffices to prove:

\[m_2(w(\xi \otimes e_{C_1}) \otimes m_\gamma) + m_2(w(\xi \otimes e_{C_2}) \otimes m_\gamma) + m_2((\xi \otimes m_\gamma \tilde{e}_C)] = 0\]

Since \(\tilde{w}_{C} = \tilde{w}_{C_1} + \tilde{w}_{C_2}\), we can rewrite it:

\[\iff [m_2(\xi \otimes e_{C_1}) \otimes m_\gamma) + m_2(\xi \otimes e_{C_2}) \otimes m_\gamma) + m_2((\xi \otimes m_\gamma \tilde{e}_C)] = 0\]

The sums of the first and second brackets equal 0 since \(e_{C_1} m_\gamma + e_{C_2} m_\gamma + m_\gamma \tilde{e}_C\) are relations defining \(\mathcal{B} \Gamma_n\). [21] is true and thus, \(m_2.\) is well-defined. \(\diamond\)

11. Property of type A

In this section, we will prove that type \(A\)–structure describe in section [10] makes \(\tilde{T}\) become an \(A_\infty\)–module over the differential graded algebra \(\mathcal{B} \Gamma_n\) with \(m_n. = 0\) for \(n \geq 3\).

**Proposition 45.** For \(\xi = (r, s)\) be a generator of \(\tilde{T}\) and \(p_1, p_2 \in \mathcal{B} \Gamma_n\). The maps \(m_1.\) and \(m_2.\) satisfy the following:

1. \(m_1.(m_1.)(\xi) = 0\)
2. \(m_2.(m_1.)(\xi) \otimes p_1 + m_2.(\xi \otimes d_{\Gamma_n}(p_1)) + m_1.(m_2.)(\xi \otimes p_1) = 0\)
3. \(m_2.(\xi \otimes p_1 p_2) = m_2.(m_2.)(\xi \otimes p_1) \otimes p_2\)

**Proof:** The first and third equations come from the properties of \(m_1.\) and the construction of \(m_2.\) we have shown in section [10]. Therefore, we have to prove is the second identity. It suffices to prove the second identity when \(p_1\) is a generator of \(\mathcal{B} \Gamma_n\).

We will divide it into two cases to deal with:

**Case I:** If \(p_1 \neq \tilde{e}_C\), using the fact that \(d_{\Gamma_n}(p_1)\) does not involve any left decoration edge and the relations between \(m_1.\), \(m_2.\), and \(m_1, m_2\), we rewrite the second equation as following:

\[\iff m_2(m_1(\xi) \otimes p_1) + m_2(\xi \otimes d_{\Gamma_n}(p_1)) + m_1(m_2(\xi \otimes p_1)] + [m_2(v(\xi \otimes p_1) + v(m_2(\xi \otimes p_1))] = 0\]

Because the sum in the first bracket is 0, which is derived by the fact that with the differential \(m_1\) and action \(m_2\), \(\tilde{T}\) is an \(A_\infty\)–module over the differential graded algebra \(\mathcal{B} \Gamma_n\), its equivalent to prove that:

\[(22)\quad m_2(v(\xi \otimes p_1) + v(m_2(\xi \otimes p_1)) = 0\]
There are three possibilities for $p_1$

(1) If $p_1$ is an idempotent $I_{(L, \sigma)}$, then both terms will equal to $v(\xi)$ if $\partial(r, s) = I_{(L, \sigma)}$ and they are both 0 if $\partial(r, s) \neq I_{(L, \sigma)}$. As the result, (22) is true.

(2) If $p_1 = \overline{e_C}$, then $m_2(v(\xi) \otimes p_1) = v(m_2(\xi \otimes p_1)) = \sum_{D \in \text{free}(r)} \overline{w_D(r, s_{C, D})}$.

(3) If $p_1 = \gamma \in BR(L)$, changing the decoration on $+$ free circle to $-$ commutes the surgery along $\gamma$ since their supports are disjoint. At the end, two terms of (22) are equal.

**Case II**: If $p_1 = \overline{e_C}$, we need to prove the following:

$m_2, (m_1, \xi \otimes \overline{e_C}) + m_2, (\xi \otimes d_{r_n}(\overline{e_C})) + m_1, (m_2, (\xi \otimes \overline{e_C})) = 0$

$\iff m_2, (m_1, \xi \otimes \overline{e_C}) + m_2, (v(\xi) \otimes \overline{e_C}) + m_2, (\xi \otimes d_{r_n}(\overline{e_C})) + m_1, (m_2, (\xi \otimes \overline{e_C})) + m_2, (m_1, (m_2, (\xi \otimes \overline{e_C}))) + 0$

$\iff \sum_{\gamma} [\overline{w_C}, m_2(\xi, \overline{e_C})] + [m_2, (m_1, (\xi, \overline{e_C}))) + 0 = 0$

The sum inside the first bracket is 0 since with the differential $m_1$ and action $m_2$, $\overline{L}$ is an $A_{\infty}$-module over the differential graded algebra $\mathcal{B}$. The sum inside the second bracket is also 0 by (22). Therefore, we only need to make sure the third sum is also 0:

$$\sum_{\gamma} \overline{w_C}, m_2(\xi, \overline{e_C}) + m_2, (v(\xi) \otimes \overline{e_C}) + v(m_2(\xi \otimes \overline{e_C})) + \overline{w_C}, m_1, (m_2, (\xi \otimes \overline{e_C})) = 0$$

Since we have the following identity:

$$m_2, (m_1, \xi \otimes \overline{e_C}) = m_2, (m_1, \xi \otimes d_{r_n}(\overline{e_C})) + m_1, (m_2, (\xi \otimes \overline{e_C})) = m_1, (m_2, (\xi \otimes \overline{e_C})),$$

we can rewrite (23) as following:

$$\sum_{\gamma} m_2, (\overline{w_C}, \xi, \overline{e_C}) + m_2, (v(\xi) \otimes \overline{e_C}) + v(m_2(\xi \otimes \overline{e_C})) = 0$$

Note that: the second term contains generators $(D, \gamma)$ from changing the decoration on $+$ free circle $D$ of $\xi$ first, then resolving an active arc $\gamma$ which changes a decoration on $C_\xi$ from $+$ to $-$. On the other hand, the third term will be the sum of generators $(\gamma_1, D_1)$ coming from surgering an active arc $\gamma_1$ to change the decoration on $C_\xi$ first, then changing a $+$ free circle $D_1$ of $\xi$ to $-$. Taking the sum of the second and the third terms, the pairs $(D, \gamma)$ in the second term will be canceled out by the reverse pair $(\gamma, D)$ if $(\gamma, D)$ belongs to the third term and vice versa. However, there are two
exceptional cases when reversing a pair of the second (third) term does not belong to the third (second) term: 1) \( \widetilde{w}_D(D, \gamma) \) where \( D \) is a +free circle of \( \xi \) and \( \gamma \) has one foot on \( D \), another on \( C_\xi \) or 2) \( \widetilde{w}_{D,\gamma}(\gamma, D_\gamma) \) where \( \gamma \) is an active arc whose feet belongs \( C_\xi \) and \( D_\gamma \) is new + free circle, which is created by surgering \( \gamma \). On the other hand, the first term contains generators whose coefficients \( \widetilde{w}_C \gamma + \widetilde{w}_{C_\xi} \neq 0 \) if and only if the active bridge \( \gamma \) in the has at least one foot on \( C_\xi \). Therefore, by comparing the algebra coefficients, those generators will be canceled out by generators in two above exceptional cases. As the result, the sum of three terms is 0.

12. Type A invariance under the weight-move and Reidemeister moves

12.1. Type A invariance under the weight-move. In section 7 we proved that the type D-structure of twisted right tangle homology, described in section 4, is invariant under the weight moving by using the trick in 16 or 15. Similarly, in this section, we will prove that under the weight moving, the type A-structure of twisted left tangle homology is preserved. Additionally, based on the construction of type A and type D structures, it is not hard to define a type DA bimodule version of twisted Khovanov homology for an \( (m, n) \)-tangle. Then thank to the gluing process in 9 to pair a type A and type DA, it suffices to prove the result for the local case (as the following figure) and then the result for global case will follows immediately.

We will prove only the left case of the above figure since the proof of the right case is similar. Let \( \widehat{T} \) and \( \widehat{T}' \) be the weighted tangle before and after the weight \( w \) moved along crossing \( c \). Let \( m_1, m_2 \) and \( m_1', m_2' \) be maps defining the type A-structure of \( \langle \langle \widehat{T} \rangle \rangle \) and \( \langle \langle \widehat{T}' \rangle \rangle \) respectively, described in section 10. Note that in this case, \( m_1, = m_1' = 0 \). We will construct an \( A_\infty \)-morphism \( \Psi = \psi_1, \psi_2 \) from \( \langle \langle \widehat{T} \rangle \rangle \) to \( \langle \langle \widehat{T}' \rangle \rangle \) as following:

- \( \psi_1 : \langle \langle \widehat{T} \rangle \rangle \longrightarrow \langle \langle \widehat{T}' \rangle \rangle \) is identity map since the generators of \( \langle \langle \widehat{T} \rangle \rangle \) can be canonically identical to generators of \( \langle \langle \widehat{T}' \rangle \rangle \).
- \( \psi_2 : \langle \langle \widehat{T} \rangle \rangle \otimes_T \mathcal{B}_4 \longrightarrow \langle \langle \widehat{T}' \rangle \rangle[1] \) is firstly defined based on its image on \( \xi \otimes e \) where \( \xi = (r, s) \) is generator of \( \langle \langle \widehat{T} \rangle \rangle \) and \( e \) is a generator of \( \mathcal{B}_4 \). We define \( \psi_2(\xi \otimes e) = w.(r_\gamma, s_\gamma) \) if \( \xi \) and \( e \) satisfy the following conditions:
  1. \( \partial(r, s) = s(e), r(c) = 1 \) and \( e \) is left bridge element.
(2) \( r_\gamma \) is obtained by surgering along the inactive bridge resolution \( \gamma \) at crossing \( c \) and \( s_\gamma \) is computed by using Khovanov maps (Note that: \( r_\gamma \) is a resolution of \( T' \) under the identification \( \overline{T} \) and \( \overline{T'} \)) and \((r_\gamma, s_\gamma) = t(e)\).

Otherwise, we define \( \psi_2(\xi \otimes e) = 0 \). We also describe the definition of \( \psi_2 \) as the red thick maps of figure 7. As the next step, we will define \( \psi_2 \) for \( e \) to be an element in \( \mathcal{B}\Gamma_4 \) by imposing the following relation for each \( \xi \in \mathcal{B}\Gamma_\triangleleft \) and \( e_1, e_2 \in \mathcal{B}\Gamma_4 \):

\[
(24) \quad \psi_2(\xi \otimes e_1 e_2) = m_{2, \bullet}(\psi_2(\xi \otimes e_1) \otimes e_2) + \psi_2(m_{2, \bullet}(\xi \otimes e_1) \otimes e_2)
\]
For \( \psi_2 \) to be well-defined, we need to verify that with the relation we just imposed, 
\[ \psi_2(\xi \otimes p) = 0 \] for each relation \( p \) defining \( \mathcal{B}_4 \). Since for each \( e \) generating \( \mathcal{B}_4 \), 
\[ \psi_2(\xi \otimes e) = 0 \] unless \( e \) is a left bridge element, \( \psi_2(\xi \otimes p) \) is trivially 0 if \( p \) does not involve 
left bridge element(s). Therefore, it leaves us two cases to verify: 1) \( p \) is relation (3) of 
Squared relation and 2) \( p \) is relation (1) of Relations for decoration edges. 

With an aid of \( 7 \), the proof is straightforward. We will do one example to illustrate 
the method. For example: since we have relation: \( p = e_{\gamma_1}e_{\eta_1} + e_{\gamma_2}e_{\eta_2} = 0 \) which is 
the squared relation, we need to verify \( \psi_2(\xi \otimes e_{\gamma_1}e_{\eta_1}) = \psi_2(\xi \otimes e_{\gamma_2}e_{\eta_2}) \) where \( \xi \) is the 
right top corner state. Indeed, since \( d \) is chain map from \( (\xi \otimes e_{\gamma_1}e_{\eta_1}) \) to \( (\xi \otimes e_{\gamma_2}e_{\eta_2}) \) where \( \xi \) is the 
left bridge element, \( \psi_2(\xi \otimes e_{\gamma_1}e_{\eta_1}) \) is trivially 0 if \( e_{\gamma_1} \) acting on \( \xi \), 
then followed by the dash dotted purple curve due to the action of \( e_{\eta_1} \). Similarly, 
\( \psi_2(\xi \otimes e_{\gamma_2}e_{\eta_2}) \) is calculated by first going along the red curve under the action of \( e_{\gamma_2} \) 
and then, following by \( e_{\eta_2} \)-dashed dotted purple curve. Therefore, the result will be 
the same if we follow those paths.

**Proposition 46.** \( \Psi = \{ \psi_1, \psi_2 \} \) is an \( A_\infty \) morphism from \( \langle \overset{\mathcal{B}}{T} \rangle \) to \( \langle \overset{\mathcal{B}}{T}' \rangle \)

**Proof.** We need to verify three following conditions:

1. \( \psi_1 \) is chain map from \( (\overset{\mathcal{B}}{T}, m_{1,*}) \) to \( (\overset{\mathcal{B}}{T}', m_{1,*}') \)
2. \( m_{2,*}'(\psi_1(\xi \otimes e)) + m_{1,*}'(\psi_2(\xi \otimes e) + \psi_1(m_{2,*}(\xi \otimes e)) + \psi_2(m_{1,*}(\xi \otimes e) + \psi_2(\xi \otimes d_{\Gamma_4}(e))) = 0 \)
3. \( \psi_2(\xi \otimes d_{\Gamma_4}(e)) = \psi_2(m_{2,*}(\xi \otimes e_1) \otimes e_2) + m_{2,*}'(\psi_2(\xi \otimes e_1) \otimes e_2) \)

The first condition is trivially true since both \( m_{1,*} \) and \( m_{1,*}' \) are zero maps. The third 
one comes from the relation \( 24 \) that we impose on \( \psi_2 \). For the second condition to be verified, 
we suffices to prove that:

\[
(25) \quad \psi_2(\xi \otimes d_{\Gamma_4}(e)) = m_{2,*}'(\psi_1(\xi \otimes e)) + \psi_1(m_{2,*}(\xi \otimes e))
\]

Let \( m_{2,*}' \) be defined identically as \( m_{2,*}' \) but in \( \langle \overset{\mathcal{B}}{T}' \rangle \). We will prove that (25) is true 
when \( e \) is a (length 0 or 1) generator of \( \mathcal{B}_4 \). The case when \( e \) is arbitrary is followed 
immediately from the case when \( e \) is of length 0 or 1. If \( e \) is not left decoration 
element, the left hand side is 0 by the definition of \( d_{\Gamma_4} \). Similarly, the bright hand side is 0 
since the action of an algebra element on \( \langle \overset{\mathcal{B}}{T}' \rangle \) does not change after we 
move weight if it does not involve the left decoration element. If \( e \) is a left decoration 
edge \( \mathcal{E}_C \) where \( C \) is + cleaved circle of \( \partial(\xi) \), we have two possibilities:

1) \( C \) is the only cleaved circle of \( \partial(\xi) \) then:

\[
m_{2,*}'(\psi_1(\xi \otimes e)) + \psi_1(m_{2,*}(\xi \otimes e)) = 0
\]
because both terms in the identity are $\tilde{w}_C(r, s_C)$ (the moved weight $w$ is still on $C$ in this case)

2) $C$ is not the only one, then:

$$m'_2,\cdot (\psi_1(\xi) \otimes e) + \psi_1(m_2,\cdot (\xi \otimes e)) = w.(r, s_C)$$

On the other hand, by the definition of $d_{T_4}$ and relation 24, $\psi_2(\xi \otimes d_{T_4}(\tilde{c}_C))$ is calculated as a sum of path formed by starting at $\xi$, go along either the dashed blue curve (action of left bridge element) or thick red curve (action of $\psi_2$) and then following either the thick red curve or dashed blue curve (see 7). Note that: if $C$ is the only one cleaved circle then there are two such paths and their sum will be canceled out. Otherwise, we have only one such path and the end point of this path is $(r, s_C)$. The reason is that according to the definition:

$$\psi_2(\xi \otimes d_{T_4}(\tilde{c}_C)) = \psi_2(m_2,\cdot (\xi \otimes e) \otimes e_{\gamma}) + m'_2,\cdot (\psi_2(\xi \otimes e_{\gamma}) \otimes e_{\gamma}^\dagger)$$

and depending on $\xi(c) = 0$ or $\xi(c) = 1$, the second term or the first term in the latest sum will disappear. Furthermore, the weight $w$ comes from the chain map $\psi_2$. Therefore, $\Psi = (\psi_1, \psi_2)$ is $A_{\infty}$ morphism from $\langle \langle \leftarrow T \rangle \rangle$ to $\langle \langle \leftarrow T' \rangle \rangle$. ♦

If we define $\Phi$ identically as $\Psi$ but from $\langle \langle \leftarrow T' \rangle \rangle$ to $\langle \langle \leftarrow T \rangle \rangle$, we immediately have the following:

1) $(\Psi \ast \Phi)_1 = I_{\langle \langle T \rangle \rangle}$ since $\psi_1$ and $\phi_1$ are identity maps.

2) $(\Psi \ast \Phi)_2 = \psi_1 \circ \phi_2 + \psi_2 \circ (\phi_1 \otimes I) = 0$ since both terms are $\phi_2$ under the canonical identification of generators of $\langle \langle T \rangle \rangle$ and $\langle \langle T' \rangle \rangle$

3) $(\Psi \ast \Phi)_3 = \psi_2(\phi_2 \otimes I) = 0$ since both $\psi_2$ and $\phi_2$ support on states $\xi$ where $\xi(c) = 1$ and their images contain states whose crossing $c$ is resolved 0.

Therefore, $\Psi \ast \Phi = 1_{\langle \langle T \rangle \rangle}$. Similarly, we have $\Phi \ast \Psi = 1_{\langle \langle T' \rangle \rangle}$. As the result, $\langle \langle T \rangle \rangle, m_1,\cdot , m_2,\cdot \rangle$ is isomorphic to $\langle \langle T' \rangle \rangle, m'_1,\cdot , m'_2,\cdot \rangle$ as type $A$-structure.

12.2. Invariance under Reidemeister Moves. Due to the bimodule structure introduced in the begining of this section, we only need to prove the invariance under the Reidemeister moves as in the following figure:

```
  |   |
  |   |
  |   |
  |   |
  |   |
```

The strategy is to use the isomorphism in 12.1 to move the weights so that the weights are close to the $y$-axis. We then can apply Roberts arguments for local invariants of untwisted case (see section 5 and 6 of [13]). Let $T$ and $T'$ are the tangles before and after Reidemeister moves. We will briefly sketch his arguments for untwisted case
and how we can apply those arguments for our case as following steps:

(1) Since the weights are close to the $y$-axis, we have $m_{1,\bullet} = m_1$.

(2) Decomposing the complex of tangle $\langle \widehat{T}' \rangle$ as direct sum of summands where each summand is corresponding to the resolution of the crossings of the tangle and regardless of the choice of planar matching. There always exists a summand $V$ consisting states with a free circle. We then continue decomposing $V = V_+ \oplus V_-$, based on the decoration on the free circle.

(3) For the differential $m_1$, there are two types of isomorphisms coming either merging a decorated circle to a $+$ free circle or dividing a decorated one to get a $-$ free circle. We can use this isomorphism to cancel out 1) the summand $V_+$ and its image $d_{APS}(V_+)$ for Reidemeister move I or 2) $V_-$ and $d_{APS}(V_+)$ first and then cancel out $V_-$ and its preimage under the dividing isomorphism for Reidemeister moves II and III. After the cancellation, we get exactly the same chain complex before Reidemeister moves with the possibility that the higher order actions might appear.

(4) Roberts proves that the higher order actions actually do not show up because of two properties. The first property is the image of $m_1$ on $V_+$ is another summand of complex, which is canceled out by cancellation process. The second property is the images of higher order actions always lie on $V_+$ and it will be canceled out at the end.

(5) For our case, the same technique can be used to prove there is no higher order actions. Since we have $m_{1,\bullet} = m_1$, we definitely have the first property (Note: If we do not move weight close to the $y$-axis, the image of $m_{1,\bullet}$ on $V_+$ also contains $V_-$). Additionally, since $m_{2,\bullet}$ is different from $m_2$ only on the action of the left decoration element, the image of $m_{2,\bullet}(V_+)$ lies on $V_+$ and therefore the image of higher order actions lie on $V_+$. At the end, they all disappear when we cancel out $V_+$. The same argument can be applied for $V_-$ if needed (as in Reidemeister move II or III).

(6) After canceling out those terms, we obtain almost exactly the type $A$-structure of the tangle before doing Reidemeister moves. The only difference is again we are working over different ground rings. This issue can be dealt similar as type $D$-structure (see [9]) by using the stable equivalence relation, described in [8].

Therefore, we have the following theorem:

**Theorem 47.** Let $\widehat{T}$ be the left tangle with diagram $\overleftrightarrow{T}$. The homotopy class of $(\overleftrightarrow{T}, m_{1,\bullet}, m_{2,\bullet})$, defined as in [10], is an invariant of left tangle $\widehat{T}$.
13. Relation to Reduced Khovanov Homology by Gluing Left and Right Tangles

As described in section 1, let \( T \) be a link diagram of link \( \mathcal{T} \) which is divided by the \( y \)-axis into two parts: the left tangle \( \overrightarrow{T} \) and the right one \( \overleftarrow{T} \). Using the pairing technique as in \[8.4\] in this section, we will prove the the chain complex obtained by gluing the type \( A \)-structure \((\llbracket \overrightarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet}) \) and the type \( D \)-structure of \((\llbracket \overleftarrow{T} \rrbracket, \delta_{T,\bullet}) \) is chain isomorphic to the totally twisted Khovanov homology of \( L \).

Let \( \mathbb{F}_T \) be the field of fractions of \( \mathbb{P}_T = \mathbb{Z}_2[x_f| f \in \text{ARC}(L)] \) where \( \text{ARC}(T) \) is the collection of segments whose endpoints are either the crossings of \( T \) or the intersection points of \( T \) and the \( y \)-axis. Let \([T]\) be the graded Khovanov complex over \( \mathbb{F}_T \), equipped with a totally twisted differential \( \tilde{\partial} = \partial_{KH} + \partial_{V,T} \) where \( \partial_{KH} \) is regular Khovanov map and \( \partial_{V,T} \) is Koszul map, defined similar as \( \partial_{\psi} \) in \([3.1]\) (see \([14]\) for more detail). Recall that \([\llbracket T\rrbracket, \tilde{\partial}]\) is a link invariance.

There are natural injections \( \phi_l : \mathbb{F}_T \hookrightarrow \mathbb{F}_\overrightarrow{T} \) and \( \phi_r : \mathbb{F}_\overleftarrow{T} \hookrightarrow \mathbb{F}_T \), which are induced from the fact that \( \text{ARC}(T) \) is the disjoint union of \( \text{ARC}(\overrightarrow{T}) \) and \( \text{ARC}(\overleftarrow{T}) \). To describe the gluing complex, we first need to describe the type \( A \)-structure \((\mathcal{F}_{\phi_l}(\llbracket \overrightarrow{T} \rrbracket), \mathcal{F}_{\phi_l}(m_{1,\bullet}), \mathcal{F}_{\phi_l}(m_{2,\bullet})) \) and the type \( D \)-structure \((\mathcal{G}_{\phi_r}(\llbracket \overleftarrow{T} \rrbracket, \mathcal{G}_{\phi_r}(\delta_{T,\bullet})) \). By using formulas in \([26]\) and \([35]\) over \( \mathbb{F}_T \), the generators of \( \mathcal{F}_{\phi_l}(\llbracket \overrightarrow{T} \rrbracket) \) and \( \mathcal{G}_{\phi_r}(\llbracket \overleftarrow{T} \rrbracket) \) are identified with the generators of \( \llbracket \overrightarrow{T} \rrbracket \) (as a vector space over \( \mathbb{F}_\overrightarrow{T} \)) and \( \llbracket \overleftarrow{T} \rrbracket \) (over \( \mathbb{F}_\overleftarrow{T} \)) respectively. Additionally, under this identification, the maps \( \mathcal{F}_{\phi_l}(m_{1,\bullet}), \mathcal{F}_{\phi_l}(m_{2,\bullet}) \) and \( \mathcal{G}_{\phi_r}(\delta_{T,\bullet}) \) are identical to \( m_{1,\bullet}, m_{2,\bullet} \) and \( \delta_{T,\bullet} \) correspondingly.

Therefore, without abuse of notation, we can use \((\llbracket \overrightarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet}) \) (respectively \((\llbracket \overleftarrow{T} \rrbracket, \delta_{T,\bullet}) \)) to stand for \((\mathcal{F}_{\phi_l}(\llbracket \overrightarrow{T} \rrbracket), \mathcal{F}_{\phi_l}(m_{1,\bullet}), \mathcal{F}_{\phi_l}(m_{2,\bullet})) \) (respectively \((\mathcal{G}_{\phi_r}(\llbracket \overleftarrow{T} \rrbracket), \mathcal{G}_{\phi_r}(\delta_{T,\bullet})) \)). Keep in mind that from now to the end of this section, \( \llbracket \overrightarrow{T} \rrbracket \) and \( \llbracket \overleftarrow{T} \rrbracket \) are vector spaces over \( \mathbb{F}_T \) while \( m_{1,\bullet}, m_{2,\bullet} \) and \( \delta_{T,\bullet} \) are defined in \([10]\) and \([4]\) with an attention that the weight of a cleaved circle in type \( A \) (respectively type \( D \)) is calculated as the sum of weights of left-side (respectively right-side) arcs belonging to this circle.

The module structure of gluing complex then can be described as the following:

\[
\llbracket \overrightarrow{T} \rrbracket \otimes_S \llbracket \overleftarrow{T} \rrbracket = \llbracket \overrightarrow{T} \rrbracket \otimes_{I_n} \llbracket \overleftarrow{T} \rrbracket
\]

Additionally, the differential of this complex is given by the following formula (see \([8.4]\)):

\[
\partial_S^e(x \otimes y) = m_{1,\bullet}(x) \otimes y + (m_{2,\bullet} \otimes 1)(x \otimes \delta_{T,\bullet}(y))
\]
We now prove the gluing theorem:

**Theorem 48.** \( (\langle \langle \leftarrow \rightarrow T \rangle \rangle \boxtimes \langle \langle \leftarrow \rightarrow \rangle \rangle, \partial^S) \) is chain isomorphic to \( ([T], \tilde{\partial}) \).

**Proof.** Due to the module structures of \( \langle \langle \leftarrow \rightarrow T \rangle \rangle \) and \( \langle \langle \leftarrow \rightarrow \rangle \rangle \) as vector spaces over \( \mathbb{F}_T \) and the action of \( \mathcal{I}_n \) on them, \( \langle \langle \leftarrow \rightarrow T \rangle \rangle \boxtimes \langle \langle \leftarrow \rightarrow \rangle \rangle \) is a vector space over \( \mathbb{F}_T \) whose generators are identified with the generators of \([T]\) (see section 7 of [13]). Furthermore, this identification was also proved to be bigrading preserving and thus, it is \( \zeta \)-grading preserving. Therefore, it suffices to prove that, under this identification, \( \partial^S = \tilde{\partial} \).

Since \( m_1, \cdot = d_{APS} + \partial_V \) and \( \delta_T, \cdot = \delta_T + \delta_V \), the differential \( \partial^S \) can be decomposed as following:

\[
\partial^S(x \otimes y) = [d_{APS}(x) \otimes y + (m_2 \otimes \mathbb{I})(x \otimes \delta_T(y))] + [\partial_V(x) \otimes y + ((m_2, \cdot - m_2) \otimes \mathbb{I})(x \otimes \delta_T(y))]
\]

From [13], we know that:

\[
d_{APS} \otimes \mathbb{I} + (m_2 \otimes \mathbb{I})(\mathbb{I} \otimes \delta_T) = \partial_{KH}
\]

Therefore, we will complete the proof of the theorem if we can show:

\[
(26) \quad \partial_V(x) \otimes y + ((m_2, \cdot - m_2) \otimes \mathbb{I})(x \otimes \delta_T(y)) + (m_2, \cdot \otimes \mathbb{I})(x \otimes \delta_V(y)) = \partial_{V,T}(x \otimes y)
\]

for \( x \) and \( y \) are generators of \( \langle \leftarrow \rightarrow T \rangle \) and \( \langle \rightarrow \rangle \) such that \( I_{\partial(x)} = I_{\partial(y)} \).

We note that \( x \otimes y \) is a resolution \((r, s)\) of \([T]\), consisting of a collection of circles which belong to the following four groups:

1. Circles decorated by \(-\)
2. Left free circles decorated by \(+\)
3. Right free circles decorated by \(+\)
4. Cleaved circles decorated by \(+\)

Therefore, the right hand side of (26) can be written as following:

\[
\partial_{V,T}(x \otimes y) = \sum_{C \in (2) \cup (3) \cup (4)} w_C. (r, s_C)
\]

where \( w_C = \sum_{f \in \text{arc}(C)} x_f \). On the other hand, we have:

\[
\partial_V(x) \otimes y = \sum_{C \in (2)} w_C. (r, s_C)
\]

Additionally, according to the construction of \( m_2, \cdot \), the action \( m_2, \cdot - m_2 \) is supported only on the left decoration element. As a result:

\[
((m_2, \cdot - m_2) \otimes \mathbb{I})(x \otimes \delta_T(y)) = \sum_{C \in (4)} ((m_2, \cdot - m_2) \otimes \mathbb{I})(x \otimes \delta_C \otimes y_C)
\]
From the definition of $\delta_V$ (10), we can calculate the last term of the left hand side of 26 as following:

$$(m_2, \otimes \mathbb{I})(x \otimes \delta_V(y)) = \sum_{C \in (4)} \overline{w}_C(r, s_C) + \sum_{C \in (3)} w_C(r, s_C)$$

Rewriting the left hand side of 26 we have:

$LHS = \sum_{C \in (2)} w_C.(r, s_C) + \sum_{C \in (3)} w_C.(r, s_C) + \sum_{C \in (4)} (\overline{w}_C + \overline{w}_C).(r, s_C)$

Since $w_C = \overline{w}_C + \overline{w}_C$ for each cleaved circle, we can conclude that the equation 26 is true and thus, $\partial_S^3 = \bar{\partial}$. As a result, $([\overrightarrow{T}], \mathbb{I})$, $([\overrightarrow{T}], \partial_S^3)$ is chain isomorphic to $([\overrightarrow{T}], \bar{\partial})$.

14. Examples of type D and type A

In this section, we will illustrate the way to calculate our type $D$ and type $A$ structures in totally twisted Khovanov by showing two examples.

Example 1: Hopf link

Consider the Hopf link $T$ which is transverse to the $y$-axis at 4 points. We also label each arc of $T$ as the following picture. It divides the link $T$ into two parts: the left tangle $\overrightarrow{T}$ and the right one $\overleftarrow{T}$.

We first describe our type $D$-structure on $\overrightarrow{T}$. As shown in the following figure, $\overrightarrow{T}$ can be thought as a vector space over $\mathbb{F}_T$ (by the same argument as in section 13), generated by six elements: $\xi_1, \ldots, \xi_6$ corresponding to bottom row of the figure from left to right. Their $\zeta$-gradings are $-\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{3}{4}$ and $\frac{1}{2}$ respectively. For example:
since the bigrading of $\xi^4$ is $(-1, -5/2)$ due to $n_+(\xi^4) = 0$ and $n_-(\xi^4) = 2$, the $\zeta$-grading of $\xi^4$ is $-1 + 5/4 = 1/4$. We also denote $\{\tilde{\eta}_i\}_{i=1}^4$ and $\{\gamma^i\}_{j=1,2}$ to be left bridges and right active resolution bridges of the corresponding generators as in the figure.

\[
\begin{align*}
\delta_{T,\bullet}(\xi^1) &= e_{\tilde{\eta}_1} \otimes \xi_3 + e_{\tilde{\eta}_1} \otimes \xi_6 + [\tilde{e}_C + (x_3 + x_4)e_D] \otimes \xi_2 \\
\delta_{T,\bullet}(\xi^3) &= e_{\tilde{\eta}_2} \otimes \xi_5 + e_{\tilde{\eta}_3} \otimes \xi_2 \\
\delta_{T,\bullet}(\xi^4) &= e_{\tilde{\eta}_4} \otimes \xi_6 + [\tilde{e}_D + (x_2 + x_3)e_D] \otimes \xi_5 \\
\delta_{T,\bullet}(\xi^6) &= e_{\tilde{\eta}_4} \otimes \xi_5
\end{align*}
\]

We next describe the type $A$ structure on $\langle \xi^1 \rangle$. Similar to $\langle \xi^4 \rangle$, $\langle \xi^6 \rangle$ is a six dimensional vector space over $\mathbb{F}_T$, generated by $\{\xi_i\}_{i=1}^{10}$ where $\xi_i$ has the same boundary as $\xi_i$. We label the bridges of each generators $\xi_i$ exactly as the ones of $\xi_i$ in type $D$. For example: $\xi_i$ has a left active resolution bridge $\tilde{\eta}_i$ and right bridge $\gamma_i$. We also let $\gamma^3$ and $\gamma^4$ be the right bridges of $\xi^3$ and $\xi^4$ respectively. Additionally, the $\zeta$-grading of $\xi^1, \ldots, \xi^6$ are $-1/4$, $1/4$, $1/2$, $3/4$ and 0 respectively. Since none of generators
has free circle, the type A structure on $\langle \leftarrow T \rangle \boxtimes \langle \rightarrow T \rangle$ can be described as following:

\[
m_{1, \bullet} = 0
\]

\[
m_{2, \bullet}(\xi_1 \otimes e_{\pi_1}) = \xi_3
\]

\[
m_{2, \bullet}(\xi_1 \otimes e_{\pi_2}) = \xi_6
\]

\[
m_{2, \bullet}(\xi_1 \otimes e_C) = \xi_2
\]

\[
m_{2, \bullet}(\xi_1 \otimes e_C) = (x_7 + x_8)\xi_2
\]

\[
\text{(28)}
\]

\[
m_{2, \bullet}(\xi_3 \otimes e_{\pi_1}) = \xi_5
\]

\[
m_{2, \bullet}(\xi_4 \otimes e_{\pi_2}) = \xi_3
\]

\[
m_{2, \bullet}(\xi_4 \otimes e_D) = \xi_5
\]

\[
m_{2, \bullet}(\xi_4 \otimes e_D) = (x_6 + x_7)\xi_5
\]

\[
m_{2, \bullet}(\xi_6 \otimes e_{\pi_1}) = \xi_2
\]

\[
m_{2, \bullet}(\xi_6 \otimes e_H) = \xi_5
\]

Then $\langle \leftarrow T \rangle \boxtimes \langle \rightarrow T \rangle$ is graded vector space over $F_T$, generated by $\{\xi_i\}_{i=1,6}$ where $\xi_i = \xi_1 \otimes I_2' \xi_i$. Since $\zeta(\xi_i) = \zeta(\xi_i) + \zeta(\xi_i)$, the $\zeta$-grading of $\xi_1, \ldots, \xi_6$ are $-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}$ and $\frac{1}{2}$ respectively. Since $m_{1, \bullet} = 0$, the differential of $\langle \leftarrow T \rangle \boxtimes \langle \rightarrow T \rangle$ is:

\[
\partial^\zeta(x \otimes y) = (m_{2, \bullet} \otimes I)(x \otimes \delta_{T, \bullet}(y))
\]

Therefore, using (27) and (28) we have:

\[
\partial^\zeta(\xi_1) = \xi_3 + \xi_6 + (x_3 + x_4 + x_7 + x_8)\xi_2
\]

\[
\partial^\zeta(\xi_3) = \xi_5
\]

\[
\partial^\zeta(\xi_4) = (x_2 + x_3 + x_6 + x_7)\xi_5
\]

\[
\partial^\zeta(\xi_6) = \xi_5
\]

The above description of $\zeta$-complex $(\langle \leftarrow T \rangle \boxtimes \langle \rightarrow T \rangle, \partial^\zeta)$ is exactly the same as totally twisted Khovanov homology of Hopf link whose homology is $F_T \oplus F_T$, occurring at the $\zeta$-grading $\frac{1}{2}$.

**Example 2**: Consider the following tangle diagram $\leftarrow T$:
We label five crossings of $\overrightarrow{T}$ as in the above figure. In this example we will investigate the image of type $D$-structure $\delta_{n,T}$ on some generators of $]\overrightarrow{CT}[$. Recall: $(]\overrightarrow{CT}, \overrightarrow{\delta_{n,T}})$ is the deformation retraction of $(]\overrightarrow{T}, \overrightarrow{\delta_{T,\ast}})$, defined in the section 1 and 6. For this tangle $\overrightarrow{T}$, there are two left planar matchings $m_1$ (containing an arc which connects the mark point and the most bottom point in the $y$-axis) and $m_2$. A generator of $]\overrightarrow{CT}[$ is obtained by resolving the crossings of $\overrightarrow{T}$ by either 0 or 1 in such a way that it has no free circle, then capping it off by either $m_1$ or $m_2$ and finally, decorate the unmarked cleaved circle (it might not exist) by $\pm$. Therefore, we can encode a generator of $]\overrightarrow{CT}[$ as in the form of either $\xi^{\ast\ast\ast\ast\ast,i}$ or $\xi^{\ast\ast\ast\ast\ast,i,\pm}$ where the subscript each $\ast$ receives the value of 0, 1 and $i$ receives the value of either 1 or 2, depending on the left planar matching $m_1$ or $m_2$ we choose. Additionally, if the generator has an unmarked cleaved circle, we use the second form with specified decoration on it. Otherwise, we use the first form to represent this generator. For example, the following figure represents $\xi_{00000,1}$. 
We have:
\[(30)\]
\[
\overrightarrow{\delta_{n,T}}(\xi_{00000,1}) = [x_3 + x_6 + x_{12}]^{-1}I_{\partial(\xi_1)} \otimes \xi_{10010,1} + [x_2 + x_7 + x_{10}]^{-1}I_{\partial(\xi_1)} \otimes \xi_{01001,1}
\]
\[
+ ([x_3 + x_7 + x_{11}]^{-1} + [x_2 + x_7 + x_{10}]^{-1})I_{\partial(\xi_1)} \otimes \xi_{00101,1}
\]
\[
+ ([x_3 + x_6 + x_{12}]^{-1} + [x_3 + x_7 + x_{11}]^{-1})I_{\partial(\xi_1)} \otimes \xi_{00110,1}
\]
\[
+ e_{\overleftarrow{\gamma_1}} \otimes \xi_{01000,1,-} + e_{\overleftarrow{\gamma_2}} \otimes \xi_{01000,1,-}
\]
\[
+ e_{\overleftarrow{\eta}} \otimes \xi_{00000,2,-}
\]
where \(I_{\partial(\xi_1)} := I_{\partial(\xi_{00000,1})}\), \(\{e_{\overleftarrow{\gamma_i}}\}_{i=1,2}\) are the right bridge element of \(B\Gamma_2\), corresponding to the change of idempotents after surgering \(\xi_{00000,1}\) along the active resolution bridge at crossing \(i\) and \(\eta_1\) is the unique left bridge of left planar matching \(m_1\).

As we can see, in the right hand side of (30), the first (or the second term) comes from a generator, obtained from \(\xi_{00000,1}\) by surgering along two active resolution bridges. Depending on which crossing we resolve first, there are two ways to make this surgery. However, there is only one way which create a free circle (or equivalently, a transition as in [6]) and the second way will change the idempotents. On the other hand, both two ways to surge \(\xi_{00000,1}\) to obtain a generator of the third (or fourth) term contain free circles and as the result, we count the coefficient for both paths.

Similarly, we compute \(\overrightarrow{\delta_{n,T}}(\xi_{00000,2,+})\):
\[(31)\]
\[
\overrightarrow{\delta_{n,T}}(\xi_{00000,2,+}) = [x_3 + x_6 + x_{12}]^{-1}I_{\partial(\xi_2)} \otimes \xi_{10010,2,+} + [x_2 + x_7 + x_{10}]^{-1}I_{\partial(\xi_2)} \otimes \xi_{01001,2,+}
\]
\[
+ ([x_3 + x_7 + x_{11}]^{-1} + [x_2 + x_7 + x_{10}]^{-1})I_{\partial(\xi_2)} \otimes \xi_{00101,2,+}
\]
\[
+ ([x_3 + x_6 + x_{12}]^{-1} + [x_3 + x_7 + x_{11}]^{-1})I_{\partial(\xi_2)} \otimes \xi_{00110,2,+}
\]
\[
+ e_{\overleftarrow{\gamma_1}} \otimes \xi_{01000,2} + e_{\overleftarrow{\gamma_2}} \otimes \xi_{01000,2}
\]
\[
+ e_{\overleftarrow{\eta}} \otimes \xi_{00000,2} + [\overleftarrow{\xi_C} + (x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_{10} + x_{11} + x_{12})e_{\overleftarrow{\gamma_2}}] \otimes \xi_{00000,2,-}
\]
where \(I_{\partial(\xi_2)} = I_{\partial(\xi_{00000,2,+)\}}, \overleftarrow{\xi_2}\) is a unique left bridge of \(m_2\) and \(C\) stands for unmarked cleaved circle of \(\xi_{00000,2,+.}\) We note that the main difference between (30) and (31) is the image of \(\overrightarrow{\delta_{n,T}}\) on \(\xi_{00000,2,+}\) contains a term coming from decoration element.

**REFERENCES**

[1] M. Asaeda, J. Przytycki, A. Sikora, *Categorification of the Kauffman bracket skein module of \(I\)-bundles over surfaces*. Algebr. Geom. Topol. 4 (2004), 11771210 (electronic).

[2] M. Asaeda, J. Przytycki, A. Sikora, *Categorification of the skein module of tangles*. Primes and knots, 18, Contemp. Math., 416, Amer. Math. Soc., Providence, RI, 2006.

[3] D. Bar-Natan, *On Khovanov’s categorification of the Jones polynomial*. Alg. & Geom. Top. 2:337–370 (2002).

[4] D. Bar-Natan, *Khovanov’s homology for tangles and cobordisms*. Geom. Topol. 9:1443-1499 (2005).

[5] M. Khovanov, *A categorification of the Jones polynomial*. Duke Math. J. 101(3):359–426 (2000).
[6] M. Khovanov, A functor-valued invariant of tangles. Algebr. Geom. Topol. 2:665-741 (2002).
[7] A. D. Lauda & H. Pfeiffer, Open-closed TQFTS extend Khovanov homology from links to tangles. J. Knot Theory Ramifications 18(1)87150 (2009)
[8] R. Lipshitz, P. S. Ozsvath, & D. P. Thurston. Bordered Heegaard Floer homology: Invariance and pairing. arXiv:0810.0687
[9] R. Lipshitz, P. S. Ozsvath, & D. P. Thurston. Bimodules in bordered Heegaard Floer homology. arXiv:1003.0598 (2010)
[10] E. S. Lee, An endomorphism of the Khovanov invariant. Adv. Math. 197(2):554-586 (2005).
[11] O. Viro, Khovanov homology, its definition and ramifications. Fund. Math. 184:317–342 (2004).
[12] L. P. Roberts, A type D structure in Khovanov Homology. arXiv:1304.0463v3.
[13] L. P. Roberts, A type A structure in Khovanov Homology. arXiv:1304.0465v3.
[14] L. P. Roberts, Totally Twisted Khovanov Homology arXiv:math.GT/1109.0508
[15] N. D. Duong & L. P. Roberts, Twisted skein homology. arXiv:1209.2967v1.
[16] Jaeger, Thomas C, A remark on Roberts’ totally twisted Khovanov Homology. Journal of Knot Theory and Its Ramifications 22.06 (2013).
[17] S. Wehrli, A spanning tree model for Khovanov homology. J. Knot Theory Ramifications 17 (2008), no. 12, 1561-1574.