Discrete \( \mathcal{RP} \) groups with a parabolic generator

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March 13, 2022

Abstract

We deal with two-generator subgroups of \( \text{PSL}(2, \mathbb{C}) \) with real traces
of both generators and their commutator. We give discreteness criteria
for these groups when at least one of the generators is parabolic. We also
present a list of the corresponding orbifolds.

Mathematics Subject Classification (2000): Primary: 30F40;
Secondary: 20H10, 22E40, 57M60, 57S30.

Key words: Kleinian group, discrete group, hyperbolic orbifold.

1 Introduction

A two-generator subgroup \( \Gamma = \langle f, g \rangle \) of \( \text{PSL}(2, \mathbb{C}) \) is determined up to conjugacy
by its parameters \( \beta(f) = \text{tr}^2 f - 4, \beta(g) = \text{tr}^2 g - 4, \) and \( \gamma(f, g) = \text{tr}[f, g] - 2 \)
whenever \( \gamma(f, g) \neq 0 \) [7].

We are concerned with the class of \( \mathcal{RP} \) groups (two-generator groups with
real parameters):

\[ \mathcal{RP} = \{ \Gamma | \Gamma = \langle f, g \rangle \text{ for some } f, g \in \text{PSL}(2, \mathbb{C}) \text{ with } \beta(f), \beta(g), \gamma(f, g) \in \mathbb{R} \} \]

Since discreteness conditions for elementary, Fuchsian, and NEC groups are
available, we consider only the non-elementary \( \mathcal{RP} \) groups \( \Gamma = \langle f, g \rangle \) without
invariant planes. In this paper, we deal with the case of \( f \) elliptic, parabolic, or
hyperbolic (see the definitions in the beginning of Section 2) and \( g \) parabolic.
We give criteria for discreteness of these groups (Theorem 2.3) and for each
discrete \( \Gamma \) we obtain a presentation and the corresponding Kleinian orbifold
\( Q(\Gamma) \) (Theorem 3.1). Theorem 2.3 appeared in the PhD thesis of the first
author [8], but its proof was not published yet.

The groups generated by two parabolic elements were under study earlier.
In an unpublished preprint [12], Parker gave a discreteness criterion for an
arbitrary group generated by two parabolic elements. However, to apply this
criterion a simple condition needs to be checked for all elements of the group.
In fact, the author uses this criterion to give a nice discreteness condition for

*The authors were partially supported by Gettysburg College Research and Professional
Development Grant, 2003–2004.
Fuchsian groups (Proposition 3.1) and \( \mathcal{RP} \) groups (Proposition 3.2). The latter proposition and our Corollary 2.5 for the case \( \beta(f) = 0 \) imply each other.

Adams [1] gives some necessary condition for discreteness of a two parabolic generator non-elementary subgroup of \( \text{PSL}(2, \mathbb{C}) \). He proves existence of universal upper bounds on the “length” of each of the parabolic generators and on the “distance” between them (the lengths and distances are measured in a canonical choice of cusp boundaries). Using Thurston’s Orbifold Theorem, he shows also that a non-elementary orientable finite volume hyperbolic 3-manifold \( M \) has fundamental group generated by two parabolic if and only if \( M \) is the compliment of a two-bridge link in \( S^3 \) that is not a 2-braid. Agol [2] generalized Adams’ argument to classify all two parabolic generator \( 3 \)-orbifolds.

2 Discreteness criteria

Recall that an element \( f \in \text{PSL}(2, \mathbb{C}) \) with real \( \beta(f) \) is elliptic, parabolic, hyperbolic, or \( \pi \)-loxodromic according to whether \( \beta(f) \in [-4, 0) \), \( \beta(f) = 0 \), \( \beta(f) \in (0, +\infty) \), or \( \beta(f) \in (-\infty, -4) \). If \( \beta(f) \notin [-4, +\infty) \), then \( f \) is called strictly loxodromic. Among all strictly loxodromic elements, only \( \pi \)-loxodromics have real \( \beta(f) \).

Every element \( f \in \text{PSL}(2, \mathbb{C}) \) with real \( \beta(f) \) has invariant planes. The following lemma characterizes the non-elementary \( \mathcal{RP} \) groups in terms of the invariant planes of generators.

**Lemma 2.1.** Let \( \Gamma = \langle f, g \rangle \) be a non-elementary subgroup of \( \text{PSL}(2, \mathbb{C}) \) with real \( \beta(f) \) and \( \beta(g) \). Then \( \gamma(f, g) \) is real if and only if either

1. \( f \) and \( g \) have a common invariant plane or
2. each of the generators \( f \) and \( g \) has an invariant plane orthogonal to all invariant planes of the other generator.

**Proof** follows from Theorems 1–3 of [3].

**Remark 2.2.** Clearly, all non-elementary \( \mathcal{RP} \) groups without invariant plane satisfy the condition (2) of Lemma 2.1. Theorem 2.3 below characterizes all such discrete groups in case when one of the generators is parabolic and the other has real trace.

An elliptic element \( f \) of order \( n \) is said to be non-primitive if \( f \) is a rotation through \( 2\pi k/n \), where \( k \) and \( n \) are coprime \( (1 < k < n/2) \). If \( f \) is a rotation through \( 2\pi/n \), then \( f \) is called primitive.

It is easy to see that if \( f \) is a non-primitive elliptic element of order \( n \), then there exists an integer \( r \geq 2 \) such that \( f^r \) is a primitive elliptic element of the same order \( n \). It is clear that \( \langle f, g \rangle = \langle f^r, g \rangle \). Therefore, we assume without loss of generality that the elliptic generator is primitive (see also Remark 2.6).

**Theorem 2.3.** Let \( f \in \text{PSL}(2, \mathbb{C}) \) be a hyperbolic, parabolic, or primitive elliptic element of order \( n \geq 3 \), let \( g \in \text{PSL}(2, \mathbb{C}) \) be a parabolic element, and let \( \Gamma = \langle f, g \rangle \) be a non-elementary \( \mathcal{RP} \) group without invariant plane. Then:
(1) there exists an element \( h \in \text{PSL}(2, \mathbb{C}) \) such that \( h^2 = fgf^{-1}g^{-1} \) and \( (hg)^2 = 1 \);

(2) \( \Gamma = \langle f, g \rangle \) is discrete if and only if \( h \) is a hyperbolic, parabolic, or primitive elliptic element of order \( p \geq 3 \).

Proof. We start with construction of a reflection group \( \Gamma^* \) containing \( \Gamma \) as a subgroup of finite index. Such a group is discrete if and only if so is \( \Gamma \). Then we find discreteness criteria for \( \Gamma^* \) and rewrite them as simple conditions on the generators of \( \Gamma \).

1. Construction of \( \Gamma^* \) and a polyhedron \( \mathcal{T} \) bounded by the planes of reflections of \( \Gamma^* \). Let \( f \) and \( g \) be as in the statement of the theorem. Since \( \Gamma = \langle f, g \rangle \) is non-elementary, \( f \) and \( g \) have no fixed point in common. Let \( Q \in \partial \mathbb{H}^3 \) be the fixed point of \( g \) and let \( \zeta \) be the invariant plane of \( f \) that passes through \( Q \).

Since \( \Gamma = \langle f, g \rangle \) is a non-elementary \( \mathcal{RP} \) group without invariant plane, it follows from Lemma 2.1 that there is an invariant plane \( \eta \) of \( g \) orthogonal to all invariant planes of \( f \) and, in particular, to \( \zeta \). Note that if \( f \) is elliptic, the axis of \( f \) lies in \( \eta \); and if \( f \) is hyperbolic, the axis of \( f \) is orthogonal to \( \eta \).

We see that there exists planes \( \sigma \) and \( \tau \) so that \( f = R_\sigma R_\eta \) and \( g = R_\tau R_\zeta \) (we denote the reflection in a plane \( \kappa \) by \( R_\kappa \)). Clearly, \( \tau \) and \( \zeta \) are parallel and meet at a point \( Q \in \partial \mathbb{H}^3 \). Since \( \eta \) is an invariant plane of \( g \), \( \eta \) is orthogonal to both \( \tau \) and \( \zeta \). Similarly, since \( \zeta \) is an invariant plane of \( f \), \( \zeta \) is orthogonal to both \( \sigma \) and \( \eta \).

If \( f \) is elliptic then the planes \( \eta \) and \( \sigma \) intersect at an angle of \( \pi/n \); the line of their intersection is the axis of \( f \) (see Figure 1(a)). If \( f \) is parabolic then \( \eta \) and \( \sigma \) are parallel and meet at a point \( P \in \partial \mathbb{H}^3 \) (see Figure 1(b)); moreover, \( \zeta \) passes through \( P \). If \( f \) is hyperbolic, the planes \( \eta \) and \( \sigma \) are disjoint, the axis of \( f \) is orthogonal to both \( \eta \) and \( \sigma \) (see Figure 1(c)).

Consider half-turns \( e = R_\zeta R_\eta = R_\eta R_\zeta, e_f = R_\sigma R_\zeta, \) and \( e_g = R_\tau R_\eta. \) Then we have

\[
\begin{align*}
    f &= R_\sigma R_\eta = (R_\sigma R_\zeta)(R_\zeta R_\eta) = e_f e \\
    g &= R_\tau R_\zeta = (R_\tau R_\eta)(R_\eta R_\zeta) = e_g e.
\end{align*}
\]

We define two finite index extensions of the group \( \Gamma = \langle f, g \rangle \) as follows: \( \overline{\Gamma} = \langle f, g, e \rangle \) and \( \Gamma^* = \langle f, g, e, R_\eta \rangle \).

It is easy to see that \( \overline{\Gamma} = \Gamma \cup \Gamma e \). If \( e \in \Gamma \) then \( \overline{\Gamma} = \Gamma \) and if \( e \notin \Gamma \) then \( \Gamma \) is a subgroup of index 2 in \( \overline{\Gamma} \). As we will see, both possibilities are realized. Since, moreover, \( \overline{\Gamma} \) is the orientation preserving subgroup of index 2 in \( \Gamma^* \), the groups \( \Gamma, \overline{\Gamma}, \) and \( \Gamma^* \) are either all discrete or all non-discrete.

It is clear that \( \Gamma^* = \langle R_\eta, R_\zeta, R_\sigma, R_\tau \rangle \).

Consider the infinite volume polyhedron \( \mathcal{T} \) bounded by the planes \( \eta, \zeta, \sigma, \) and \( \tau \). As we have seen before, \( \eta \) and \( \zeta \), \( \eta \) and \( \tau \), and \( \zeta \) and \( \sigma \) are orthogonal; \( \zeta \) and \( \tau \) are parallel; \( \eta \) and \( \sigma \) either intersect at an angle of \( \pi/n \), or are parallel or disjoint depending on the type of \( f \). The planes \( \sigma \) and \( \tau \) may either intersect,
Figure 1: Fundamental polyhedra for $\Gamma^*$

(a) $f$ is elliptic

(b) $f$ is parabolic

(c) $f$ is hyperbolic
or be parallel or disjoint (this depends on the type of $fgf^{-1}g^{-1}$ as we will show in part 2 of the proof).

If $\sigma$ and $\tau$ intersect, then the dihedral angle of $\mathcal{T}$ between these planes (denote it by $\pi/p$ with $p$ not necessarily an integer) is acute. Indeed, there exists a hyperbolic plane $\kappa_1$ orthogonal to $\zeta, \sigma,$ and $\tau$; such a plane passes through $Q$. The planes $\zeta, \sigma,$ and $\tau$ cut off a hyperbolic triangle $\Delta$ with angles $0, \pi/2,$ and $\pi/p$ from $\kappa_1$ ($\Delta$ is shaded in Figures $1(a)$–$1(c)$). Therefore, $\pi/p < \pi/2$. We keep the notation $\pi/p$ taking $p = \infty$ or $\infty$ for the cases of parallel and disjoint $\sigma$ and $\tau$, respectively. (We regard $\infty > x > \infty$, $\infty/x = x/\infty = 0$, $\infty/x = \infty$, and $\infty/x = \infty$ for every positive real $x$.)

Similarly, if $\eta, \sigma,$ and $\tau$ do not have a common point in $\mathbb{H}^3 \cup \partial \mathbb{H}^3$, there exists a hyperbolic plane $\kappa_2$ orthogonal to them. The planes $\kappa_1$ and $\kappa_2$ cut off a finite volume polyhedron $\mathcal{T}$ from $\mathcal{T}$. By the Andreev theorem, $\mathcal{T}$ exists in hyperbolic space for all $p > 2$. Therefore, so does $\mathcal{T}$. In fact, it is $\mathcal{T}$ what is drawn in Figure 1; moreover, it is drawn under assumption that $p < \infty$ and, for Figure 1(a), $1/p + 1/n > 1/2$.

2. Existence of $h$ and a sufficient discreteness condition for $\Gamma$. It is clear that if

$$p \text{ is an integer } (p > 2), \infty, \text{ or } \infty,$$

then $\mathcal{T}$ and reflections $R_\eta, R_\zeta, R_\sigma,$ and $R_\tau$ satisfy the hypotheses of the Poincaré theorem $[5]$. $\Gamma^*$ is discrete, and $\mathcal{T}$ is its fundamental polyhedron.

Now we rewrite the condition (2.1) via conditions on some elements of $\text{PSL}(2, \mathbb{C})$.

Let us prove that there is a unique $h$ in $\text{PSL}(2, \mathbb{C})$ that satisfies both $h^2 = [f, g]$ and $(hg)^2 = 1$; moreover, $h = R_\sigma R_\tau$.

Since $\langle f, g \rangle$ is a non-elementary $\mathcal{RP}$ group, $[f, g]$ is parabolic, or hyperbolic, or elliptic. If $[f, g]$ is parabolic then it has only one square root; if $[f, g]$ is hyperbolic or elliptic, then $[f, g]$ has exactly two square roots $h$ and $\overline{h}$ in $\text{PSL}(2, \mathbb{C})$. Namely, if $[f, g]$ is hyperbolic, then one of the roots is hyperbolic and the other is $\pi$-loxodromic. If $[f, g]$ is elliptic, then $h$ and $\overline{h}$ are both elliptic.

Let us show that if we take $h = R_\sigma R_\tau$ then $h^2 = [f, g]$ and $(hg)^2 = 1$ hold. Indeed,

$$h^2 = (R_\sigma R_\tau)^2 = (R_\sigma R_\zeta R_\tau)^2 = (e_f g^{-1})^2 = (e_f e e g)^2 = (e_f e g) (e e g) (e_f) (ee g) = fg f^{-1} g^{-1}.$$  

Moreover, $hg = (R_\sigma R_\tau)(R_\sigma R_\zeta) = R_\sigma R_\tau = e_f$. So, $(hg)^2 = e_f^2 = 1$.

Let us explain now what $\overline{h}$ is. If $[f, g]$ is hyperbolic, then $\overline{h}$ is $\pi$-loxodromic with the same axis and translation length as $h$. If $[f, g]$ is elliptic, then $\overline{h}$ is elliptic with the same axis as $h$ and with rotation angle $(\pi - 2\pi/p)$, while $h$ is a rotation through $2\pi/p$ in the opposite direction. It is clear that in both cases $(\overline{h}g)^2 \neq 1$.

This means that the only element $h$ that satisfies both $h^2 = [f, g]$ and $(hg)^2 = 1$ can be written as $h = R_\sigma R_\tau$. Thus, part (1) of Theorem 2.3 is proved.
The element \( h = R_\sigma R_\tau \) is a primitive elliptic element of order \( p \geq 3 \) if and only if the dihedral angle of \( \mathcal{T} \) at the edge \( \sigma \cap \tau \) is equal to \( \pi/p \), \( p \in \mathbb{Z} \); \( h \) is parabolic (hyperbolic) if and only if \( \sigma \) and \( \tau \) are parallel (disjoint, respectively).

Therefore, we have proved that the condition (2.1) is equivalent to the condition that

\[ h \text{ is a hyperbolic, parabolic, or primitive elliptic element of order } p > 2. \] (2.2)

So, (2.2) implies that \( \Gamma \) is discrete.

3. **The sufficient condition (2.3) is also a necessary condition.** Now suppose that \( \Gamma \) is discrete but (2.2) fails. This means that \( h \) is a non-primitive elliptic element of finite order, i.e., \( p = q/k > 2 \), where \( q \) and \( k \) are coprime, \( k \geq 2 \). Let us show that this is impossible.

Consider the hyperbolic plane \( \kappa_1 \) that is orthogonal to the planes \( \zeta, \sigma, \) and \( \tau \). Since \( \langle \epsilon_f, g \rangle \subset \Gamma \) keeps \( \kappa_1 \) invariant and preserves orientation of \( \kappa_1 \), \( \langle \epsilon_f, g \rangle \) acts on \( \kappa_1 \) as a subgroup of \( \text{PSL}(2, \mathbb{R}) \). However, \( \langle \epsilon_f, g \rangle \) is not discrete if \( h = \epsilon_f g^{-1} \) is non-primitive elliptic, which is a rotation through \( 2k\pi/q \) in our case, by [10] or [11]. Theorem 2.3 is proved.

**Remark 2.4.** For the benefit of the reader, we give a description of \( \mathcal{T} \) in the upper half-space model of hyperbolic 3-space \( \mathbb{H}^3 = \{(z,t) : z \in \mathbb{C}, t > 0 \} \) with the Poincaré metric \( ds^2 = \left| dz \right|^2 + dt^2 / t^2 \).

It suffices to assume that in the proof of Theorem 2.3 \( \Gamma = \langle f, g \rangle \) is normalized so that \( g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and the fixed points of \( f \) are \( z_0 \) and \(-z_0 \), \( z_0 \in \mathbb{C} \).

(If \( f \) is parabolic, then the only fixed point of \( f \) is \( z_0 = 0 \).)

Suppose that \( f \) is elliptic. Since \( \Gamma \) is a non-elementary \( \mathbb{RP} \) group without invariant plane, both fixed points of \( f \) lie in an invariant plane of \( g \) by Lemma 2.1. Taking into account the fact that every invariant plane of \( g \) is given by \( \{(z,t) \in \mathbb{H}^3 : \text{Im} z = \text{const} \} \), we conclude that \( z_0 = x_0 \) is real.

Analogously, if \( f \) is hyperbolic, the fixed points of \( f \) are symmetric to each other with respect to an invariant plane of \( g \) and hence \( z_0 = iy_0 \), \( y_0 \in \mathbb{R} \).

So we have normalized \( \Gamma \) so that \( g(\infty) = \infty, g(0) = 1, f(z_0) = z_0, f(-z_0) = -z_0 \), where \( z_0 \) equals to \( x_0, 0 \), or \( iy_0 \) (\( x_0 \) and \( y_0 \) are some non-zero real numbers), if \( f \) is elliptic, parabolic, or hyperbolic, respectively.

Then the planes \( \eta, \zeta, \sigma, \) and \( \tau \) in the proof are given by

\[
\eta = \{(z,t) \in \mathbb{H}^3 : \text{Im} z = 0 \}; \\
\zeta = \{(z,t) \in \mathbb{H}^3 : \text{Re} z = 0 \}; \\
\sigma = \{(z,t) \in \mathbb{H}^3 : |z - iy_0|^2 + t^2 = r^2 \}, \ y_0 \in \mathbb{R}\{0\}, \ r^2 = y_0^2 + z_0^2; \\
\tau = \{(z,t) \in \mathbb{H}^3 : \text{Re} z = 1/2 \}.
\]

Moreover, \( \kappa_1 = \{(z,t) \in \mathbb{H}^3 : \text{Im} z = y_0 \} \) is the plane that plays a key role in part 3 of the proof. It is clear that this plane is orthogonal to all \( \zeta, \sigma, \) and \( \tau \).
Corollary 2.5. Let \( f, g \in \text{PSL}(2, \mathbb{C}) \), \( \beta(f) \in [0, +\infty) \) or \( \beta(f) = -4\sin^2(\pi/n) \), \( n \in \mathbb{Z}, n \geq 3 \), and let \( \beta(g) = 0 \). Suppose that \( \gamma(f, g) < 0 \). Then \( \Gamma = \langle f, g \rangle \) is discrete if and only if one of the following holds:

1. \( \gamma(f, g) \in (-\infty; -4] \);
2. \( \gamma(f, g) = -4\cos^2(\pi/p), p \in \mathbb{Z}, p \geq 3 \).

Proof. Since \( \langle f, g \rangle \) is an \( \mathcal{RP} \) group, \( \beta(g) = 0 \), and \( f \) is not \( \pi \)-loxodromic, \( \gamma \) is a non-elementary group without invariant plane if and only if \( \gamma(f, g) < 0 \) [9, Theorem 4]. So it is clear that the hypotheses of Corollary 2.5 are equivalent to those of Theorem 2.3.

Therefore, to prove the corollary it suffices to rewrite part (2) of Theorem 2.3 in terms of \( \gamma(f, g) \). Since \( \gamma(f, g) = \text{tr}[f, g] - 2 \) and \( [f, g] = h^2 \), it is not difficult to find \( \gamma(f, g) \).

The element \( h \) is hyperbolic if and only if the planes \( \sigma \) and \( \tau \) are disjoint. (We denote planes as in the proof of Theorem 2.3.) Let \( d \) be the hyperbolic distance between them. Since \( [f, g] = h^2 = (R_\sigma R_\tau)^2 \),

\[
\gamma(f, g) = \text{tr}[f, g] - 2 = -2\cosh(2d) - 2 < -4
\]

we must take \( \text{tr}[f, g] \) to be negative, because \( \gamma(f, g) \) is negative by assumption.

The element \( h \) is parabolic if and only if \( [f, g] \) is parabolic, that is, \( \text{tr}[f, g] = -2 \) (\( \text{tr}[f, g] = 2 \) would give \( \gamma(f, g) = 0 \) which is impossible in our case). Hence, \( \gamma(f, g) = \text{tr}[f, g] - 2 = -4 \).

Thus, \( h \) is hyperbolic or parabolic if and only if \( \gamma(f, g) \in (-\infty, -4] \), and part 1 of Corollary 2.5 is proved.

Now suppose that \( h \) is an elliptic element with rotation angle \( \varphi \), where \( \varphi/2 = \pi/p \) is the dihedral angle of \( T \) between \( \sigma \) and \( \tau \). Then \( [f, g] = h^2 \) is also elliptic with rotation angle \( 2\varphi \). Since \( \text{tr}[f, g] \) is well-defined (does not depend on the choice of representatives for \( f \) and \( g \) in \( \text{SL}(2, \mathbb{C}) \)) we can determine which formula, \( \text{tr}[f, g] = +2\cos\varphi \) or \( \text{tr}[f, g] = -2\cos\varphi \), is correct. The easiest way to do this is by using the continuity of \( \text{tr}[f, g] \) as a function of \( \varphi \) and the limit condition \( \text{tr}[f, g] \to -2 \) as \( \varphi \to 0 \). So we must take \( \text{tr}[f, g] = -2\cos\varphi \) where \( \varphi < \pi \) is the doubled dihedral angle of \( T \).

Vice versa, if \( \text{tr}[f, g] \) is given, we can use the formula \( \text{tr}[f, g] = -2\cos\varphi \), \( \varphi < \pi \), to determine the rotation angle \( \varphi \) of the element \( h \) from Theorem 2.3.

Thus, \( h \) is a primitive elliptic element of order \( p \), that is \( \varphi = 2\pi/p \), if and only if

\[
\gamma(f, g) = \text{tr}[f, g] - 2 = -2\cos(2\pi/p) - 2 = -4\cos^2(\pi/p), p \in \mathbb{Z}.
\]

Corollary 2.5 is proved.

Remark 2.6. For simplicity, in the statement of Corollary 2.5 the elliptic generator \( f \) is assumed to be primitive. If \( f \) is non-primitive elliptic then Corollary 2.5 still can be used to verify whether \( \Gamma \) is discrete, but first we must replace the triple \( (\beta(f), \beta(g), \gamma(f, g)) \), where \( \beta(f) = -4\sin^2(q\pi/n), (q, n) = 1, n \in \mathbb{Z} \).

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1 < q < n/2, with the new triple \((\tilde{\beta}, \beta(g), \tilde{\gamma})\), where \(\tilde{\beta} = -4\sin^2(\pi/n)\) and \(\tilde{\gamma} = (\beta(\beta(f))\gamma(f, g).\) The new triple corresponds to the same group by Gehring and Martin \cite{4} (cf. \cite{8} Remark 2, p. 262).

3 Kleinian orbifolds with a parabolic generator

Let \(\Gamma\) be a non-elementary Kleinian group. Denote by \(\Omega(\Gamma)\) the discontinuity set of \(\Gamma\). Following \cite{4}, we say that the Kleinian orbifold \(Q(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma\) is an orientable 3-orbifold with a complete hyperbolic structure on its interior \(\mathbb{H}^3/\Gamma\) and a conformal structure on its boundary \(\Omega(\Gamma)/\Gamma\).

We use the following notations:

- \(GT[n, m; q] = \langle f, g \mid f^n = g^m = [f, g]^q = 1 \rangle\), where \(n, m, q \in \{2, 3, \ldots\} \cup \{\infty, \infty\}\). If some relation has power \(\infty\), then we simply remove this relation from the presentation. Further, if some relation has power \(\infty\) and we keep it, we get a Kleinian group presentation. To obtain an abstract group presentation, we need to remove this relation as well.

  For example, the group \(GT[n, \infty; \infty]\) has Kleinian group presentation \(\langle f, g \mid f^n = g^\infty = 1 \rangle\) and is isomorphic to \(\mathbb{Z}_n * \mathbb{Z}\).

- \(Tet[n, m; q] = \langle x, y, z \mid x^2 = y^2 = z^n = (xy^{-1})^m = (yz^{-1})^2 = (zx^{-1})^q = 1 \rangle\), where \(n, m, q \in \mathbb{Z} \cup \{\infty, \infty\}\). For finite \(n\), \(m\), and \(q\), this group is a tetrahedron group generated by rotations in edges of a face of an orthoscheme with the Coxeter diagram \(\circ - o - o - m\). Note that \(Tet[n, m; q] \cong Tet[m, n; q]\).

**Theorem 3.1.** Let \(\Gamma = \langle f, g \rangle\) be a non-elementary discrete \(\mathbb{RP}\) group without invariant plane. Let \(\beta(g) = 0\) and let \(\beta(f) = -4\sin^2(\pi/n)\), \(n \in \mathbb{Z}\), \(n \geq 3\), or \(\beta(f) \in [0, +\infty)\). Put \(n = \infty\) for \(\beta(f) = 0\) and \(n = \infty\) for \(\beta(f) \in (0, +\infty)\).

1. If \(\gamma(f, g) \in (-\infty; -4)\), then \(\Gamma\) is isomorphic to \(GT[n, \infty; \infty]\).
2. If \(\gamma(f, g) = -4\), then \(\Gamma\) is isomorphic to \(GT[n, \infty; \infty]\).
3. If \(\gamma(f, g) = -4\cos^2(\pi/p)\), \((p, 2) = 2, p \geq 4\), then \(\Gamma\) is isomorphic to \(GT[n, \infty; p/2]\).
4. If \(\gamma(f, g) = -4\cos^2(\pi/p)\), \((p, 2) = 1, p \geq 3\), then \(\Gamma\) is isomorphic to \(Tet[n, \infty; p]\).

**Remark 3.2.** For all \(\Gamma\) from Theorem 3.1 the orbifolds \(Q(\Gamma)\) are shown in Figures 3 and 4, where the singular sets and boundaries of \(Q(\Gamma)\) are drawn. The indices on edges correspond to the orders of cone points, indices 2 are omitted. Each \(Q(\Gamma)\) is embedded in \(S^3 = \mathbb{R}^3 \cup \{\infty\}\) so that \(\infty\) is a non-singular interior point of \(Q(\Gamma)\).

Since we obtain an orbifold by gluing faces of a fundamental polyhedron for the group action on \(\mathbb{H}^3\), not only topological, but also metric structure is
uncovered (the lengths of singular geodesics, the structure of cusps, etc.). In fact, since all fixed points of parabolic elements of $\Gamma$ belong to the limit set $\Lambda(\Gamma)$, they have no images in $Q(\Gamma)$. For example, in Figure 2(b), the boundary of $Q(\Gamma)$ is the union of two thrice-punctured 2-spheres.

The fat vertices in Figures 3 and 4 are either singular points of the orbifold, or punctures, or correspond to removed open balls. In the first case the vertex corresponds to the image of a point of $H^3$, in the second it corresponds to the image of a point of the limit set $\Lambda(\Gamma)$ and, therefore, does not belong to the orbifold, and in the last case the fat vertex corresponds to a boundary component of $\Omega(\Gamma)/\Gamma$. The type of a fat vertex depends on the indices at the edges incident to it. For example, in Figure 4(a), the fat vertex is an interior singular point of the orbifold if $1/2 + 1/n + 1/p > 1$, is a puncture if $1/2 + 1/n + 1/p = 1$, and is a removed open ball if $1/2 + 1/n + 1/p < 1$.

Proof of Theorem 3.1 We give a proof only for the case $\beta(f) = -4\sin^2(\pi/n)$, $n \in \mathbb{Z}$.

All parameters for the discrete groups in the statement of Theorem 3.1 are described in Corollary 2.5. We will obtain a presentation for each case by using the Poincaré polyhedron theorem.

We start with construction of a fundamental polyhedron and a presentation for the group $\tilde{\Gamma}$ defined in the proof of Theorem 2.3. Since $\tilde{\Gamma}$ is the orientation preserving index 2 subgroup in $\Gamma^*$ and $T$ is a fundamental polyhedron for $\Gamma^*$, a fundamental polyhedron $P$ for $\tilde{\Gamma}$ consists of two copies of $T$ (see Figure 2(a)).

By applying the Poincaré polyhedron theorem to $P$ and face pairing transformations $e, e_g, f$, we get

$$\tilde{\Gamma} = \langle e, e_g, f \mid e^2 = e_g^2 = f^n = (e_g e)\infty = (fe)^2 = (fe)^p = 1 \rangle,$$

where $p$ is an integer, $\infty$, or $\infty$. Since $g = e_g e$,

$$\tilde{\Gamma} = \langle f, g, e \mid f^n = g^\infty = e^2 = (fe)^2 = (ge)^2 = (fg)^p = 1 \rangle.$$

Note that if $p \geq 3$ is odd, then using the relations $(fe)^2 = (ge)^2 = 1$, from $(fg)^p = 1$ we obtain $e = (fgf^{-1}g^{-1})^{(p-1)/2}fg$. Hence, in this case $\tilde{\Gamma} = \Gamma$ and $\Gamma \cong Tet[n, \infty; p]$. Identifying the faces of $P$, we get the orbifold $\mathbb{H}^3/\Gamma$ shown in Figure 4(a).

If $p \geq 4$ is even, $\infty$, or $\infty$, then $\Gamma$ is a subgroup of index 2 in $\tilde{\Gamma}$. To see this we apply the Poincaré theorem to a polyhedron consisting of four copies of $T$ (see Figure 2(b)). Then

$$\Gamma = \langle f, g \mid f^n = g^\infty = (fgf^{-1}g^{-1})^{p/2} = 1 \rangle.$$

The orbifold $Q(\Gamma)$ for finite $q = p/2$ is shown in Figure 1(a). For the case of parallel $\sigma$ and $\tau$ ($p = \infty$), $Q(\Gamma)$ is shown in Figure 1(b), and for the case of disjoint $\sigma$ and $\tau$, $Q(\Gamma)$ is shown in Figure 1(c).

The cases when $f$ is parabolic or hyperbolic have similar proofs and we leave them to the reader. \qed
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\[ \pi_{1}^{orb}(Q) \cong \langle f, g \mid f^{n} = g^{\infty} = [f, g]^{q} = 1 \rangle, \; n \geq 3, \; q \geq 2 \]

\[ q < \infty \quad q = \infty \quad q = \infty \]

(1-a) \hspace{1cm} (1-b) \hspace{1cm} (1-c)

\[ \pi_{1}^{orb}(Q) \cong \langle f, g \mid f^{\infty} = g^{\infty} = [f, g]^{q} = 1 \rangle, \; q \geq 2 \]

\[ q < \infty \quad q = \infty \quad q = \infty \]

(2-a) \hspace{1cm} (2-b) \hspace{1cm} (2-c)

\[ \pi_{1}^{orb}(Q) \cong \langle f, g \mid g^{\infty} = [f, g]^{q} = 1 \rangle, \; q \geq 2 \]

\[ q < \infty \quad q = \infty \quad q = \infty \]

(3-a) \hspace{1cm} (3-b) \hspace{1cm} (3-c)

Figure 3: Orbifolds \( Q \) with fundamental group \( GT[n, \infty; q] \)
$\pi_1^{orb}(Q) \cong \langle f, g, e \mid f^n = g^\infty = e^2 = (fe)^2 = (ge)^2 = (gfe)^p = 1 \rangle, \ p \geq 3$

- $f$ is elliptic
  - $n < \infty$

- $f$ is parabolic
  - $n = \infty$

- $f$ is hyperbolic
  - $n = \infty$

Figure 4: Orbifolds $Q$ with fundamental group $Tet[n, \infty; p]$

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