A Biform Game Model with the Shapley Allocation Functions

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Abstract: We define the mixed strategy form of the characteristic function of the biform games and build the Shapley allocation function (SAF) on each mixed strategy profile in the second stage of the biform games. SAF provides a more detailed and accurate picture of the fairness of the strategic contribution and reflects the degree of the players’ further choices of strategies. SAF can guarantee the existence of Nash equilibrium in the first stage of the non-cooperative games. The existence and uniqueness of SAF on each mixed strategy profile overcome the defect that the core may be an empty set and provide a fair allocation method when the core element is not unique. Moreover, SAF can be used as an important reference or substitute for the core with the confidence index.

Keywords: biform games; Shapley allocation function; mixed strategy; Nash equilibrium; properties

1. Introduction

To provide a theory for addressing the questions arising from property rights and the nature of the firm, Grossman and Hart [1] started with a non-cooperative second-stage game, from which they then defined an associated cooperative game. Hart and Moore [2] used this idea in a multi-asset, multi-individual economy to study how changes in ownership affect the incentives of non-owners of assets (employees) as well as the incentives of owner/managers. Following this idea, Stuart [3] and Brandenburger and Stuart [4] proposed a general framework called the biform game, which has been applied to several problems in supply chain management (for instance [5]). The biform game is a hybrid non-cooperative-cooperative game model with two stages designed to formalize the notion of business strategy as making moves to try to shape the competitive environment in a favorable way. The first stage involves players choosing strategies in a non-cooperative state; however, the consequences of these strategic moves are not utilities (at least not directly), but the formation of a competitive environment for the second stage. The second stage of the cooperative game involves an analysis of how to allocate and how much utility is allocated.

Brandenburger and Stuart [4] considered the core (It is not a normal core [6], it is only defined on a pure strategy combination (see Lemma 5.1 of Brandenburger and Stuart [4])) as the solution of the cooperative second stage, and the pure strategy Nash equilibrium as the solution of the induced non-cooperative game. To obtain the existence of the pure strategy Nash equilibrium, they assumed that each cooperative game in the second stage has a nonempty core on each pure strategy profile, and the biform game satisfies adding up (AU) (only considered the contribution of players to grand coalition), no externalities (NE), and no coordination (NC). However, the AU on each strategy profile is not necessarily satisfied, such as the strategy selection and surplus division in supply chain management [5]. As the solution concept for the surplus division, Feess et al. [5] used the Shapley value and introduced assumptions that ensure that the Shapley value is in the core. The NE is also not necessarily satisfied with practical problems because a player chooses a different strategy, which usually affects the utility of the coalition that does not include him/her.
The allocation methods of the cooperative game in the second stage may encounter two problems: Problem 1 is that the general model of the biform game proposed by Brandenburger and Stuart [4] uses the core as the allocation vector, but the core may not be unique or the core may be an empty set (see Example 2.1 and Section 6.1 of Brandenburger and Stuart [4]). Problem 2 is that the induced non-cooperative game generated in the second stage of cooperative games may not have pure strategy Nash equilibria, whether it is caused by the Shapley value or the core.

For problem 1, following the idea in Hart and Moore [2] and Feess et al. [5], we use the Shapley value to say how players who jointly create some value might agree to divide it. The existence and uniqueness of the Shapley allocation function (SAF) on each mixed strategy profile can naturally avoid the defect that the core may be not unique or an empty set. Further, corresponding to the AU condition given by Brandenburger and Stuart, as the definition of SAF determines that the sum of the utilities of all players on each mixed strategy profile is equal to the utility of the grand coalition on this mixed strategy profile, we do not necessarily define the AU condition.

For problem 2, we define a mixed strategy form of the Shapley value and build an allocation function (SAF) on each mixed strategy profile in the second stage. Unlike the model \((S^1, \cdots, S^n; V; \alpha^1, \cdots, \alpha^n)\) in Brandenburger and Stuart [4], the biform model here is independent of the confidence index denoted by \((X^1, \cdots, X^n; V; \Phi^1, \cdots, \Phi^n)\) (for a detailed explanation of the confidence index, please refer to Brandenburger and Stuart [4]), as the uniqueness of SAF on each mixed strategy profile uniquely determines the allocation vector for players. The biform game \((X^1, \cdots, X^n; V; \Phi^1, \cdots, \Phi^n)\) with the mixed strategy form of the Shapley value ensures the existence of Nash equilibria in non-cooperative games. Besides, a new form of expected utility is proposed by the mixed strategy form of the Shapley value which provides a survey for efficiently determining the allocation vector for players. The mixed strategy form of the Shapley value is not only the basis of utility generation and allocation but also has a practical meaning (see Section 2.2). In comparison with the pure strategy form, the mixed strategy form concerns how much a player chooses a strategy further (for example, the amounts of investment to choose or the level of service to provide).

Our main contributions can be summarized as follows. First, we use the Shapley value to build the allocation functions in the second stage of cooperative games for the general model of the biform game proposed by Brandenburger and Stuart [4]. We define a mixed strategy form of the Shapley value and build an allocation function (SAF) on each mixed strategy profile in the second stage. This result can ensure the existence of Nash equilibria in the non-cooperative game of the first stage. Such Nash equilibrium provides an important basis for how much allocation value each player gets. Second, the existence and uniqueness of SAF on each mixed strategy profile overcome two defects of the core: that is, it may be an empty set or its elements are not unique. As shown in Section 5.2, where the core is empty on the strategy combination \((b_1, b_2, a_3)\), while SAF can be used to calculate the allocation value that the player should get. When the element of the core is not unique, Brandenburger and Stuart [4] used the confidence index to allow/facilitate the calculation of the second-stage expected value reflected in the first-stage problem. SAF provides a more detailed and accurate description of the contribution of the strategy, thus making it fairer (see Examples 4 and 5). By applying SAF, players do not need to spend more effort to determine the values of confidence indices. Third, Brandenburger and Stuart [4] used the core and the confidence index to analyze the various cooperative games that result from the players’ strategic choices. In fact, the SAF can be an important reference for the value of the confidence index or a substitute for the core (see Example 6). Fourth, when the players invest in their assets (that is, pure strategies, see Example 1), our model is an extension of the model determined by the Formula (4) of Hart et al. [2]. Fifth, we investigate the properties of SAF which satisfies efficiency, anonymity, dummy player property, and additivity (see Section 4). These properties are the important guarantee of fairness. Sixth, we provide the conditions that a Nash equilibrium of the biform game is an
efficient solution (see Proposition 2). We define the consistent preference of individual and grand coalition (IGCP) to ensure that each efficient solution is Nash equilibrium.

The Shapley value [7] always gives a determinate outcome. Hart and Moore [2] used the Shapley value to model the agreements among players who jointly create some value. Marchi and Cohen [8] proposed a convex biform game, which makes Shapley value in a single point nonempty core. Ghadimi et al. [9] construct a three-person cooperative game in addition to their non-cooperative approach, which is also solved by Shapley value. A more complicated application of Shapley value is that Feess et al. [5] used it to calculate the revenue of each firm (related to the surplus of a supply chain and the return on this investment), then used it to subtract the investment cost to get the payoff of each firm, and then applied Nash equilibrium to analyze the integration among firms based on the payoffs. Li et al. [10] applied the Shapley value to calculate the players’ profits in the E-commerce game and Software firm game. Fiala [11] proposed a profit allocation mechanism in supply chains by using the method of biform game, the cooperation part only focused on coalition formations by resource capacity constraints and profit-sharing, the profit-sharing is based on Shapley value concept. Nan et al. [12] proposed the Nash equilibrium solution based on Shapley value concept. Brandenburger and Stuart [4] proposed that the core is empty or the core allocation is not unique.

None of the above applications of Shapley value involves mixed strategy, while this paper combines Shapley value with the mixed strategy to obtain SAF, thus serving the biform games.

Using the core to analyze the benefit allocation among players, some studies have given various examples to demonstrate the outcome. Stuart [13] studied competitive inventory decisions and established a non-procedural model of price competition to give the core of the related cooperative game. Ryall et al. [14] used the core to provide an attractive solution concept for agency competition. For more applications of the core, please refer to Plambeck et al. [15], Summerfield et al. [16], Fandel et al. [17], and González et al. [18].

The rest of the paper is organized as follows. Section 2 establishes the biform game model \((X_1, \ldots, X_n; V; \Phi_1, \ldots, \Phi_n)\) and shows the meaning of the mixed strategy. Section 3 presents the existences of Nash equilibrium and efficient solution of the biform game \((X_1, \ldots, X_n; V; \Phi_1, \ldots, \Phi_n)\). Section 4 studies the SAF properties. Section 5 compares the advantages and application ranges of the single point core [4], CIS value [19], a-CIS value [20], and SAF. Section 6 concludes.

2. The Biform Game \((X_1, \ldots, X_n; V; \Phi_1, \ldots, \Phi_n)\)

2.1. Model

Let \(N = \{1, \ldots, n\}\) be the set of players, \(2^N\) denote the set of coalitions (i.e., subsets) of \(N\), for each \(S \in 2^N\), \(|S|\) denote the number of players in \(S\). For each \(i \in N\), \(A_i = \{c_{i1}, \ldots, c_{im}\}\) is \(i\)'s pure strategy set, \(A = \prod_{i \in N} A_i\) is the pure strategy combination set of all players. Corresponding to \(A_i\), \(X_i = \{x_i = (x_{i1}, \ldots, x_{im}) | x_{ij} \geq 0, \sum_{j=1}^{m_i} x_{ij} = 1\} \subseteq \mathbb{R}^{m_i}\), \(i\)'s a mixed strategy set, \(X = \prod_{i \in N} X_i\) is the strategy profile set of all players. For each \(i \in N\), let \(-i = N \setminus \{i\}\), then \(A_{-i} = \prod_{j \in -i} A_j\), \(A_{-i} A = (A_i, A_{-i})\), \(X_{-i} = \prod_{j \in -i} X_j\), \(x = (x_1, \ldots, x_n) = (x_i, x_{-i}) \in X = (X_i, X_{-i})\). Let \(B_i = \{S_{i1}, \ldots, S_{ik}\}\) be the set of coalitions that player \(i\) can choose (i.e., \(i \in S_{ij}\)), where \(S_{ij} \in 2^N, j = 1, \ldots, k_i(k_i = 2^n - 1)\). For instance, if \(N = \{1, 2, 3\}\), then \(B_1 = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}, B_2 = \{\{1\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}, B_3 = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}\).

The \(n\)-person biform game [4] is a collection \((A_1, \ldots, A_n; V; a_1, \ldots, a_n)\), where \(V\) is a map from \(A\) to the set of maps from \(2^N\) to the reals, with \(V(c)(\emptyset) = 0\) for every \(c = (c_{11}, \ldots, c_{nn}) \in A\). For each \(i \in N\), the number \(a_i(0 \leq a_i \leq 1)\) is player \(i\)'s confidence
index, roughly speaking, it indicates how well player \( i \) anticipates doing in the resulting cooperative games.

For each \( c \in A \), the Shapley value for player \( i \in N \) is defined as

\[
\Phi_i(c) = \sum_{S_{ij} \in B_i} p_{ij}[V(c)(S_{ij}) - V(c)(S_{ij}\setminus\{i\})]
\]

(1)

where \( p_{ij} = \frac{(|S_{ij}|-1)!}{n!|S_{ij}\setminus\{i\}|!} \), the meaning of \( p_{ij} \) below is the same. It is easy to understand that \( \Phi_i(c) \) is the average marginal contribution of player \( i \) to all coalitions in \( B_i \) on \( c \).

In Section 1, we have presented some researchers to the application of Shapley value in biform games, which is an important basis for players in the pursuit of fair allocations. When the core allocation is not unique, different confidence indices give different induced non-cooperative games, this may increase the difficulty for players to choose a strategy. In contrast, for each \( c \in A \), all \( \Phi_i(c) \) form a unique allocation vector \( (\Phi_1(c), \ldots, \Phi_n(c)) \), thereby obtaining a unique induced non-cooperative game. Thus, the definition of \( \Phi_i(c) \) determines that the utility allocations of players are independent of the confidence index (Please refer to Sections 4 and 6.4 of Brandenburger and Stuart [4] for the explanation of induced non-cooperative game and confidence index.).

To ensure the existence of Nash equilibrium of the induced non-cooperative game, in the first stage of a biform game, we consider that players are each trying to use mixed strategies (pure strategy is a special form of mixed strategy) to select the best game for themselves, where by “game” is meant the subsequent (second-stage) game of value. Equally, they can also try to change the game with mixed strategy, if we define one of the themselves, where by “game” is meant the subsequent (second-stage) game of value.

For the strategy profile \( x \in X \) generated by all players selecting one of their own mixed strategies, the expected characteristic function of coalition \( S \in 2^N \) is defined as

\[
V_S(x) = \sum_{t_1=1}^{m_1} \cdots \sum_{t_n=1}^{m_n} V((c_{1t_1}, \ldots, c_{nt_n}))(S) \prod_{i=1}^{n} x_{it_i}(c_{it_i}),
\]

(2)

where the pure strategy \( c_{it_i} \in A_i \), \( x_{it_i}(c_{it_i}) \) is the probability that player \( i \) selects the pure strategy \( c_{it_i} \), and \( V((c_{1t_1}, \ldots, c_{nt_n}))(S) \) is the characteristic function of \( S \) on \( (c_{1t_1}, \ldots, c_{nt_n}) \in A \). When the strategy profile \( x \) corresponds to a pure strategy combination \( c \), \( V_S(x) = V(c)(S) \).

In the second stage of the biform game, for each \( x \in X \), the Shapley allocation function (SAF) for player \( i \in N \) is defined as

\[
\Phi_i(x) = \sum_{S_{ij} \in B_i} p_{ij}[V_{S_{ij}}(x) - V_{S_{ij}\setminus\{i\}}(x)],
\]

(3)

where \( V_{S_{ij}}(x) - V_{S_{ij}\setminus\{i\}}(x) \) is the expected marginal contribution of the player \( i \) to coalition \( S_{ij} \) on \( x \), and \( p_{ij} \) is the probability of player \( i \) entering the coalition \( S_{ij} \). Thus, \( \Phi_i(x) \) is the expected marginal contribution of player \( i \) to all coalitions in \( B_i \) on \( x \), which reflects the fairness of utility allocation to player \( i \). The \( \Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x)) \) from \( X \) to \( R^n \) is a vector-valued function. For the meaning of the Formula (3), please refer to the Formula (4) of Hart and Moore [2].

**Definition 1.** An n-person biform game (This game is to define the allocation method in the case that the characteristic function values of all coalitions on each pure strategy combination are known, which is different from the normal form game that studies the allocation method by knowing the utilities of all players on each pure strategy combination.) in which players’ utilities are determined by SAF is defined as a collection

\[(X_1, \ldots, X_n; V; \Phi_1, \ldots, \Phi_n)\]
where \( \Phi_i \) is defined by Formulas (2) and (3), \( V \) is a map from \( A \) to the set of maps from \( 2^N \) to the reals, with \( V(c)(\emptyset) = 0 \), and \( X_i \) is the mixed strategy set of player \( i \in N \).

We refer to Definition 4.2, Remark 4.1, and Remark 4.2 of Brandenburger and Stuart [4] to give the following definition and remark.

**Definition 2.** Let \( V_S : X \to R \) be the expected characteristic function of a biform game \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\), for any \( S_1, S_2 \in 2^N, S_1 \cap S_2 = \emptyset \). Then

1. \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) is said to be superadditive if \( V_{S_1 \cup S_2}(x) \geq V_{S_1}(x) + V_{S_2}(x) \) for any \( x \in X \);
2. \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) is said to be subadditive if \( V_{S_1 \cup S_2}(x) \leq V_{S_1}(x) + V_{S_2}(x) \) for any \( x \in X \);
3. \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) is said to be additive if \( V_{S_1 \cup S_2}(x) = V_{S_1}(x) + V_{S_2}(x) \) for any \( x \in X \);

**Remark 1.** The n-person biform game \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) becomes an n-person normal form non-cooperative game, if it is additive.

2.2. The Examples of Mixed Strategy

**Example 1.** Consider a 2-person biform game \( A \), player 1’s pure strategies are \( a_1, b_1, c_1 \), and player 2’s pure strategies are \( a_2, b_2, c_2 \). Table 1 shows the coalitions’ utilities generated in the first stage.

| \( S \) | \( (a_1, a_2) \) | \( (a_1, b_2) \) | \( (a_1, c_2) \) | \( (b_1, b_2) \) | \( (b_1, c_2) \) | \( (c_1, a_2) \) | \( (c_1, b_2) \) | \( (c_1, c_2) \) |
|---|---|---|---|---|---|---|---|---|
| \( \{1\} \) | 5 | 9 | 0 | 9 | 5 | 1 | 1 | 1 |
| \( \{2\} \) | 9 | 5 | 1 | 5 | 9 | 1 | 0 | 1 |
| \( \{1, 2\} \) | 16 | 16 | 2 | 16 | 16 | 2 | 2 | 2 |

In the first stage, it is easy to see that players will all rationally select the mixed strategy \((\frac{1}{2}, \frac{1}{2}, 0)\), then yield a strategy profile \( x = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0) \), and \( \Phi_1(x) = \Phi_2(x) = 8 \). If Formula (1) is used to allocate utility, then the corresponding optimal pure strategy combinations are \((a_1, a_2)\), \((a_1, b_2)\), \((b_1, a_2)\), \((b_1, b_2)\). On any one of these combinations, the difference between players’ utilities is 4; this will lead to unfairness. Contrary to this, SAF allocation method \((\Phi_1(x) = \Phi_2(x) = 8)\) fairly treats the contribution of these four pure strategy combinations to all coalitions, which is acceptable to the players. This reflects the importance of the mixed strategy \((\frac{1}{2}, \frac{1}{2}, 0)\) in the utility allocation.

For each player \( i \in N \), the coordinates of mixed strategy \( x_i \) can be the allocation proportions of player i’s resources (such as capital and labor) in his/her pure strategies. For instance, using Table 1 to consider such a game. Player 1’s pure strategies \( a_1, b_1, \) and \( c_1 \) are project 1, project 2, and project 3, respectively, and his/her resource is funds (the quantity is \( q \)). Then, \( x_1 = (\frac{1}{2}, \frac{1}{2}, 0) \) means that the proportion of funds invested in projects 1, 2 is both \( \frac{1}{2} \) (the quantity is \( \frac{1}{2} q \)), and the proportion of funds invested in project 3 is 0 (the quantity is 0). While player 2’s pure strategies \( a_2, b_2, \) and \( c_2 \) are also project 1, project 2 and project 3, respectively, but his/her resources are labor force. Then, \( x_2 = (\frac{1}{2}, \frac{1}{2}, 0) \) means that the proportion of labor force invested in projects 1, 2 is both \( \frac{1}{2} \), and the proportion of labor force invested in project 3 is 0.

Hart et al. [2] took the action \( x_i \in [0, \bar{x}_i] \) of agent \( i \) to be a pure investment in human capital, where \( x_i \geq 0 \). The practical meaning of our mixed strategy is similar to the \( x_i \). We can match the above projects with the human capital and on-the-job training of Hart et al., which are pure strategies, and match the above allocation proportions with their pure investments \( x_i \). Then, based on the Shapley value, compared with the model determined by their Formula (4) \((B_i(a|x) \equiv \sum_{S \in \mathcal{S}} p(S)[v(S, a(S)|x) - v(S\{i\}, a(S\{i\})|x)])\), where \( p(S) = \frac{(s-1)(1-s)!}{n!} \), our model is an extension of their model.
From the above example, the mixed strategy is an important basis for players to strive for the fair utility, we call this the “fair meaning” of mixed strategy. We now use the game B in Table 2 to illustrate the “cooperation meaning” of mixed strategy. The symbolic representation of pure strategy for players is the same as that in Table 1.

Table 2. Coalitions’ utilities of the 2-person biform game B.

|                | (a₁, a₂) | (a₁, b₂) | (a₁, c₂) | (b₁, a₂) | (b₁, b₂) | (b₁, c₂) | (c₁, a₂) | (c₁, b₂) | (c₁, c₂) |
|----------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| {1}            | 2        | 2        | 1        | 3        | 1        | 1        | 1        | 1        | 2        |
| {2}            | 9        | 1        | 1        | 1        | 2        | 1        | 1        | 1        | 2        |
| {1, 2}         | 12       | 4        | 2        | 4        | 2        | 2        | 4        | 4        |          |

Example 2. In Table 2, on the strategy combination \((a₁, a₂)\), we have \(Φ_1((a₁, a₂)) = 2.5\) and \(Φ_2((a₁, a₂)) = 9.5\) by Formula (1), the utility allocation \((2.5, 9.5)\) only reflects the fairness on pure strategy combination \((a₁, a₂)\), but does not show the influence of other pure strategy combinations. On the pure strategy combinations \((a₁, c₂)\), \((b₁, c₂)\), \((c₁, a₂)\), and \((c₁, c₂)\), the utilities of player 1 and player 2 are equal, on the pure strategy combinations \((a₁, b₂)\) and \((b₁, a₂)\), the utility of player 1 is more than that of player 2. Obviously, player 2 expects to cooperate with player 1 to obtain the pure strategy combination \((a₁, a₂)\), thereby allocating utility 9.5. However, it is very likely that player 1 will threaten player 2 with strategy \(b₁\) in order to gain more utility from grand coalition, that is, player 1 has the intention to select pure strategy \(b₁\). In this case, let \(x = (1 - p, p, 0, 1, 0, 0)\) \((0 < p ≤ 1)\), then \(Φ₁(x) = 2.5 + 0.5p\), \(Φ₂(x) = 9.5 - 8p\), \(Φ₁(x) + Φ₂(x) = 12 - 8p\), it is good for player 1 and bad for player 2 and grand coalition. Therefore, player 1 will select the mixed strategy \((1 - p, p, 0)\) to rationally ask player 2 to make concession from the utility 9.5. This reflects the “cooperation meaning” of the mixed strategy to player 1.

Note that the 2-person biform games in Tables 1 and 2 are superadditive.

3. The Nash Equilibrium and Efficient Solution

3.1. Definitions and Existences

A profile of strategies \(x^* = (x^*_1, \ldots, x^*_n) \in X\) in a biform game \((X₁, \ldots, X_n; V; Φ₁, \ldots, Φ_n)\) is said to be (mixed strategy) Nash equilibrium if it is a (mixed strategy) Nash equilibrium of the induced non-cooperative game. That is, a strategy profile \(x^* \in X\) is a Nash equilibrium of an \(n\)-person biform game \((X₁, \ldots, X_n; V; Φ₁, \ldots, Φ_n)\) if and only if for every \(i \in N\),

\[
Φ₁(x^*_i, x^*_{-i}) = \max_{z_i \in X_i} Φ_i(z_i, x^*_{-i}).
\]

The non-cooperative solution concept of Nash equilibrium extends naturally to the game \((X₁, \ldots, X_n; V; Φ₁, \ldots, Φ_n)\), each player selects a mixed strategy that maximizes one’s utility, given the strategic selections of the other players.

Referring to Nash [21], we immediately obtain the following existence theorem for Nash equilibria.

Theorem 1. Given a biform game \((X₁, \ldots, X_n; V; Φ₁, \ldots, Φ_n)\), for each \(i \in N\), if \(Φ_i(x)\) is continuous on \(X\). Then \((X₁, \ldots, X_n; V; Φ₁, \ldots, Φ_n)\) has at least one Nash equilibrium.

Denote by \(E(Φ)\) the set of Nash equilibria. We stipulate that the minimum utility \(d_i\) that player \(i\) can accept in a biform game \((X₁, \ldots, X_n; V; Φ₁, \ldots, Φ_n)\) is:

\[
d_i = \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} Φ_i(x_i, x_{-i}).
\]

Proposition 1. (Individual rationality of Nash equilibrium) For each \(i \in N\), \(Φ_i(x^*) \geq d_i\) for any \(x^* \in E(Φ)\).
Proof. For any \( x^* \in E(\Phi) \), by the definition of Nash equilibrium of \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\), we have
\[
\Phi_i(x^*) = \max_{x_i \in X_i} \Phi_i(x_i, x^*_{-i}) \geq \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} \Phi_i(x_i, x_{-i}) = d_i
\]
for every \( i \in N \). This completes the proof. \( \square \)

A strategy profile \( x^* = (x^*_1, \cdots, x^*_n) \in X \) is said to be an efficient solution (Please refer to Propositions 5.1 and 5.2 of Brandenburger and Stuart [4] for the efficient solution. On it, the grand coalition gets the most utility.) of an \( n \)-person biform game \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) if
\[
V_N(x^*) = \sum_{i=1}^{n} \Phi_i(x^*) = \max_{x \in X} \sum_{i=1}^{n} \Phi_i(x).
\]
That is, \( V_N(x^*) \geq V_N(x) \) for any \( x \in X \).

From \( X \) is a nonempty compact and convex set and \( \Phi_i(x) \) is continuous on \( X \), it is easy to know that \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) has at least one efficient solution.

Remark 2.
1. If \( x^* \in X \) is an efficient solution of \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\), then there is always
\[
\sum_{i \in N} \Phi_i(x^*) = V_N(x^*) \geq V_N(z_i, x^*_{-i}) = \sum_{i \in N} \Phi_i(z_i, x^*_{-i}), \forall z_i \in X_i.
\]
2. Denote by \( V(N) \) the maximum utility created by grand coalition. If \( x^* \) is an efficient solution of \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\), then
\[
V(N) = V_N(x^*) = \sum_{i \in N} \Phi_i(x^*).
\]

Definition 3. A biform game \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) satisfies no coordination (NC) if for each \( i = 1, \cdots, n, x_i, y_i \in X_i \), and \( x_{-i}, y_{-i} \in X_{-i} \), \( V_N(x_i, x_{-i}) \geq V_N(y_i, x_{-i}) \) if and only if \( V_N(x_i, y_{-i}) > V_N(y_i, y_{-i}) \).

Corresponding to the AU condition given by Brandenburger and Stuart, as the definition of SAF determines that the sum of the utilities of all players on each mixed strategy profile is equal to the utility of the grand coalition on this mixed strategy profile, we do not necessarily define the AU condition here, and this definition of SAF also determines that we do not need to define NE condition here.

Proposition 2. Consider a biform game \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) satisfies the following conditions:
1. NC condition;
2. A Nash equilibrium \( x^* \in X \) satisfies \( V_N(x^*) \geq V_N(y_i, x^*_{-i}) \) for any \( y_i \in X_i \).

Then, the Nash equilibrium \( x^* \in X \) is efficient.

Proof. For any \( y \in X \), write
\[
V_N(x^*) - V_N(y) = V_N(x^*_1, y_2, \cdots, y_n) - V_N(y) + V_N(x^*_1, x^*_2, y_3, \cdots, y_n) - V_N(x^*_1, y_2, \cdots, y_n) + \cdots + V_N(x^*) - V_N(x^*_1, \cdots, x^*_{n-1}, y_n).
\]

The condition (2) immediately gives that the last pair of terms on the right-hand side of this equation is non-negative. The condition (1) implies that the first pair of the
terms is non-negative is equivalent to \( V_N(x_i^*, x_{i-1}^*) - V_N(y_1, x_{i-1}^*) \) being non-negative, and the condition (2) shows that \( V_N(x_i^*, x_{i-1}^*) - V_N(y_1, x_{i-1}^*) \) is non-negative, so the first pair of the terms is non-negative. The condition (1) implies that the second pair of the terms is non-negative is equivalent to \( V_N(x_i^2, x_{i-2}^2) - V_N(y_2, x_{i-2}^2) \) being non-negative, and the condition (2) gives that \( V_N(x_i^2, x_{i-2}^2) - V_N(y_2, x_{i-2}^2) \) is non-negative, so the second pair of the terms is non-negative. Similarly, we can get that

\[
V_N(x_i^3, x_{i-3}^3) - V_N(y_3, x_{i-3}^3), \ldots, V_N(x_{n-1}^*, x_{(n-1)}^*) - V_N(y_{n-1}, x_{(n-1)}^*)
\]

are non-negative, thus, the third to \((n-1)\)th pairs of the terms are non-negative. Therefore, \( V_N(x^*) - V_N(y) > 0 \), that is, \( x^* \) is efficient. This completes the proof. \( \square \)

For each \( i \in N \), for any \( x, y \in X \), we define player \( i \)'s preference for strategy profiles \( x, y \) as \( x \preceq y \) if and only if \( \Phi_i(x) \geq \Phi_i(y) \). For any \( x, y \in X \), we define the grand coalition's preference for strategy profiles \( x, y \) as \( x \preceq y \) if and only if \( V_N(x) \geq V_N(y) \).

**Definition 4.** For any \( x, y \in X \), if every player \( i \in N \) and the grand coalition have the same preferences for \( x \) and \( y \), we say that the biform game \( (X_1, \ldots, X_n; V; \Phi_1, \ldots, \Phi_n) \) satisfies the consistent preference of individual and grand coalition, abbreviated as IGCP condition.

**Proposition 3.** Consider a biform game \( (X_1, \ldots, X_n; V; \Phi_1, \ldots, \Phi_n) \) satisfies the IGCP condition. Then, if a strategy profile \( x^* \in X \) is an efficient solution, it is also a Nash equilibrium.

**Proof.** Assume that \( x^* \in X \) is not a Nash equilibrium, then there exists at least a player \( i \in N \) and a \( z_i \in X_i \) such that \( \Phi_i(x_i^*, x_{-i}^*) < \Phi_i(z_i, x_{-i}^*) \). This contradicts the IGCP condition. This completes the proof. \( \square \)

Let \( T(\Phi) \) denote the set of efficient solutions of the biform game \( (X_1, \ldots, X_n; V; \Phi_1, \ldots, \Phi_n) \) satisfying IGCP condition, then \( T(\Phi) \subseteq E(\Phi) \).

### 3.2. Examples

**Example 3.** Consider a 2-person biform game \( C \), player 1's pure strategies are \( a_1, b_1 \), and player 2's pure strategies are \( a_2, b_2 \). Table 3 shows the coalitions' utilities on each pure strategy profile.

|       | \( (a_1, a_2) \) | \( (a_1, b_2) \) | \( (b_1, a_2) \) | \( (b_1, b_2) \) |
|-------|-----------------|-----------------|-----------------|-----------------|
| \{1\} | 1               | 1               | 1               | 0               |
| \{2\} | 1               | 1               | 1               | 0               |
| \{1,2\}| 3               | 2               | 2               | 1               |

**Table 3. Coalitions' utilities of the 2-person biform game C.**

By Formula (1),

\[
\Phi_1((a_1, a_2)) = \Phi_2((a_1, a_2)) = 1.5, \Phi_1((a_1, b_2)) = \Phi_2((a_1, b_2)) = 1, \\
\Phi_1((b_1, a_2)) = \Phi_2((b_1, a_2)) = 1, \Phi_1((b_1, b_2)) = \Phi_2((b_1, b_2)) = 0.5.
\]

It is easy to obtain that the biform game \( C \) satisfies the IGCP condition, and \( (a_1, a_2) \) is both an efficient solution and a Nash equilibrium.

Because each non-cooperative game is actually an additive biform game, IGCP condition can be used in non-cooperative games to judge whether the Nash equilibrium is efficient. For example, the Prisoners Dilemma in Table 4.
Table 4. A prisoners dilemma.

|        | \((a_1, a_2)\) | \((a_1, b_2)\) | \((b_1, a_2)\) | \((b_1, b_2)\) |
|--------|----------------|----------------|----------------|----------------|
| \(\{1\}\) | 3              | 0              | 4              | 1              |
| \(\{2\}\) | 3              | 4              | 0              | 1              |
| \(\{1, 2\}\) | 6              | 4              | 4              | 2              |

As

\[
\Phi_1((a_1, a_2)) = \Phi_2((a_1, a_2)) = 3, \Phi_1((a_1, b_2)) = 0, \Phi_2((a_1, b_2)) = 4, \\
\Phi_1((b_1, a_2)) = 4, \Phi_2((b_1, a_2)) = 0, \Phi_1((b_1, b_2)) = \Phi_2((b_1, b_2)) = 1.
\]

We can easily get that the Prisoners Dilemma does not meet the IGCP condition.

4. The Properties of SAF

Property 1. (Efficiency) If \(x^* \in T(\Phi)\), then \(V(N) = \sum_{i=1}^{n} \Phi_i(x^*)\).

This property is immediately obtained from Remark 2 (2).

The set of \(n!\) permutations of all players in set \(N\) is denoted by \(\Pi(N)\), for each permutation \(\sigma \in \Pi(N)\), denote \(\sigma = \sigma(1), \cdots, \sigma(n) = \sigma(i), \sigma(-i)\).

Property 2. (Anonymity) For each \(x \in X\) and \(\sigma \in \Pi(N)\), \(\Phi_{\sigma(i)}(x_{\sigma(i)}, x_{\sigma(-i)}) = \Phi_i(x_{i}, x_{-i})\).

This property means that the SAF of player \(i\) is only related to strategy \(x = (x_i, x_{-i})\), that is, the utility allocated to player \(i\) determined by strategy \(x\)'s contribution to coalitions in \(B_i\), regardless of who he/she is. Therefore, the renumbering of players has no effect on SAF.

Property 3. (Additivity) Let \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) and \((X_1, \cdots, X_n; \bar{V}; \Phi_1, \cdots, \Phi_n)\) be any two independent biform games, then for each \(i \in N\), \((\Phi_i + \Phi_i)(x) = \Phi_i(x) + \Phi_i(x)\) for any \(x \in X\).

**Proof.** For any \(c \in A\), by \((X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)\) and \((X_1, \cdots, X_n; \bar{V}; \Phi_1, \cdots, \Phi_n)\) are independent of each other, we have

\[
(V + \bar{V})(c)(S_{ij}) = V(c)(S_{ij}) + \bar{V}(c)(S_{ij}).
\]

Then, for any \(x \in X\), according to Formula (2),

\[
(V + \bar{V})S_{ij}(x) = V_{S_{ij}}(x) + \bar{V}_{S_{ij}}(x).
\]

Therefore, for each \(i \in N\),

\[
(\Phi_i + \Phi_i)(x) = \sum_{S_{ij} \in B_i} p_{ij}[V + \bar{V}]S_{ij}(x) - (V + \bar{V})S_{ij \setminus \{i\}}(x) \\
= \sum_{S_{ij} \in B_i} p_{ij}[V_{S_{ij}}(x) - V_{S_{ij \setminus \{i\}}}(x)] + \sum_{S_{ij} \in B_i} p_{ij}[\bar{V}_{S_{ij}}(x) - \bar{V}_{S_{ij \setminus \{i\}}}(x)] \\
= \Phi_i(x) + \Phi_i(x)
\]

for any \(x \in X\). This completes the proof. \(\Box\)

For each \(x \in X\), player \(i \in N\) is said to be a dummy player if \(V_{S \cup \{i\}}(x) = V_S(x) + V_{\{i\}}(x)\) for any \(S \in 2^N \setminus \{i\}\).

Property 4. (Dummy player property) If player \(i \in N\) is a dummy player, then \(\Phi_i(x) = V_{\{i\}}(x)\).
Proof. As player \(i \in N\) is a dummy player, we have

\[
\Phi_i(x) = \sum_{S_j \in B_i} p_{ij} [V_{S_j}(x) - V_{S_j \setminus \{i\}}(x)] \\
= \sum_{S_j \in B_i} p_{ij} V_{\{i\}}(x) - V_{\{i\}}(x)
\]

for every \(i \in N\). This completes the proof. \(\square\)

This property indicates that the dummy player \(i\) cannot obtain additional utility from the grand coalition.

**Theorem 2.** For each \(i \in N\) and \(x^* \in T(\Phi) \subseteq E(\Phi)\), SAF satisfies efficiency, anonymity, dummy player property, and additivity.

5. Comparisons of the SAF, CIS Value, \(\alpha\)-CIS Value, and Core

5.1. Examples

**Example 4.** Figure 7 of Brandenburger and Stuart [4] depicts a coordination game associated with each pure strategy profile (see Table 5). Where there are three players 1, 2, and 3, each with two strategies, labeled No and Yes, the three pure strategies of each pure strategy combination in Table 5 are owned by players 1, 2 and 3 in turn, the utilities of all one-player coalitions are taken to be zero, we did not enter the utilities of these coalitions in Table 5.

| Coalition          | (No, No, No) | (No, Yes, No) | (Yes, No, No) | (Yes, Yes, No) |
|--------------------|--------------|---------------|---------------|---------------|
| \(\{1, 2\}\)     | 4            | 3             | 3             | 6             |
| \(\{1, 3\}\)     | 4            | 4             | 3             | 3             |
| \(\{2, 3\}\)     | 4            | 3             | 4             | 3             |
| \(\{1, 2, 3\}\) | 6            | 5             | 5             | 6             |

| Coalition          | (No, No, Yes) | (No, Yes, Yes) | (Yes, No, Yes) | (Yes, Yes, Yes) |
|--------------------|--------------|---------------|---------------|---------------|
| \(\{1, 2\}\)     | 4            | 3             | 3             | 6             |
| \(\{1, 3\}\)     | 3            | 3             | 6             | 6             |
| \(\{2, 3\}\)     | 3            | 6             | 3             | 3             |
| \(\{1, 2, 3\}\) | 5            | 6             | 6             | 9             |

We use the Formula (1) to yield a 3-person-induced non-cooperative game \(D\) in Table 6. This game has two pure strategy Nash equilibria: \(e_1 = (\text{No}, \text{No}, \text{No})\) and \(e_2 = (\text{Yes}, \text{Yes}, \text{Yes})\). Clearly, Nash equilibrium \(e_1\) is inefficient, while \(e_2\) is efficient. In Figure 8 (induced by core) of Brandenburger and Stuart [4] (see Table 7), the utility of player 3 on the pure strategy combination (Yes, No, No) is 0. However, the value calculated by Formula (1) is 1 (see Table 6), which reflects the contribution fairness of player 3’s strategy "No" to the coalitions that include him/her.

**Table 5.** Coalitions’ utilities of a coordination game.

| Player 2 × Player 3 | Player 1 | Player 2 | Player 3 |
|---------------------|----------|----------|----------|
|                     | No, No   | Yes, No  | No, Yes  | Yes, Yes |
| Player 1            |          |          |          |
| No                  | 2, 2, 2  | \(\frac{11}{13}, \frac{8}{13}, \frac{11}{13}\) | \(\frac{11}{13}, \frac{11}{13}, \frac{8}{13}\) |
| Yes                 | \(\frac{8}{13}, \frac{11}{13}, \frac{11}{13}\) | \(\frac{11}{13}, \frac{11}{13}, \frac{8}{13}\) |

**Table 6.** The 3-person-induced non-cooperative game D.

| Player 2 × Player 3 | Player 1 | Player 2 | Player 3 |
|---------------------|----------|----------|----------|
|                     | No       | Yes, No  | No, Yes  | Yes, Yes |
| Player 1            |          |          |          |
| No                  | (2, 2, 2)| (2, 1, 2)| (2, 2, 1)| (0, 3, 3) |
| Yes                 | (1, 2, 2)| (3, 3, 0)| (3, 0, 3)| (3, 3, 3) |

**Table 7.** The 3-person-induced non-cooperative game E.
Example 5. Given a 3-person biform game \( E = (X_1, X_2, X_3; V; \Phi_1, \Phi_2, \Phi_3) \), the pure strategy sets are \( A_i = \{a_i, b_i\} \), \( i = 1, 2, 3 \), the mixed strategy sets are \( X_i = \{(x_{i1}, x_{i2})|x_{i1} + x_{i2} = 1, 0 \leq x_{i1}, x_{i2} \leq 1\} \), \( i = 1, 2, 3 \). The utilities (The utility of these coalitions comes from the literature of Nan et al. [12]) of each coalition on all pure strategy combinations in the first stage are shown in Table 8.

Table 8. Coalitions’ utilities of the 3-person biform game \( E \).

| \((a_1, a_2, a_3)\) | \((a_1, b_2, a_3)\) | \((a_1, b_2, b_3)\) | \((a_1, b_2, a_3)\) | \((b_1, a_2, a_3)\) | \((b_1, b_2, a_3)\) | \((b_1, b_2, b_3)\) | \((b_1, b_2, b_3)\) |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 1                   | 1                   | 1                   | 1                   | 1                   | 1                   | 1                   | 1                   |
| 2                   | 2                   | 2                   | 2                   | 2                   | 2                   | 2                   | 2                   |
| 3                   | 3                   | 3                   | 3                   | 3                   | 3                   | 3                   | 3                   |
| (1,2)               | 3                   | 5                   | 4                   | 3                   | 4                   | 5                   | 4                   |
| (1,3)               | 8                   | 8                   | 9                   | 9                   | 5                   | 8                   | 8                   |

In the second stage, an induced non-cooperative game \( F \) in Table 9 is yielded by Formula (1). Table 8 shows that the Nash equilibrium of \( D \) is only \((0, 1, 1, 0, 1, 0)\) (the pure strategy form is \((b_1, a_2, a_3)\)). Clearly, it is efficient. However, in Example 1 of Nan et al. [12], there are two pure strategy Nash equilibria: \((b_1, a_2, a_3)\) and \((a_1, b_2, b_3)\), as they use CIS value (The CIS value is a solution that concerns the worth of the individuals and the grand coalition. It first gives every player their individual worth, and then allocates the remaining worth of the grand coalition equally among all players, i.e., for any \( c \in A \), \( CIS_i(c)(N) = V(c)(\{i\}) + \frac{1}{n} \sum \{V(c)(N) - V(c)(\{j\})\} \), for all \( i \in N \)) to calculate the utilities of players, while we use SAF. Our method embodies the fairness of the strategies’ contribution to all coalitions, and describes the role of strategies in more detail.

Table 9. The 3-person-induced non-cooperative game \( F \).

| Player 2 \( \times \) Player 3 | \( a_2, a_3 \) | \( b_2, a_3 \) | \( a_2, b_3 \) | \( b_2, b_3 \) |
|------------------------------|----------------|----------------|----------------|----------------|
| Player 1                     | \( a_1 \)      | \( a_1 \)      | \( a_1 \)      | \( a_1 \)      |
| \( a_1 \)                    | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) |
| \( b_1 \)                    | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) | \( \frac{10}{3}, \frac{10}{3}, \frac{10}{3} \) |

5.2. Comparisons

In Figure 8 (see Table 7) of Brandenburger and Stuart [4], the core of \( V(c) \) consists of a single point, in which each player \( i \) receives \( V(c)(N) - V(c)(N\setminus\{i\}) \) and \( \sum_{i=1}^{n} \{V(c)(N) - V(c)(N\setminus\{i\})\} = V(c)(N) \). It is easy to see that the applicability of this allocation method is not extensive, since in general, the core is not a single point set or even an empty set. For instance, the cores in Table 10 (Table 10 is generated from Table 8), where \( y_1, y_2, \) and \( y_3 \) represent the utility of players 1, 2, and 3 in the core, respectively, and the symbol \( \emptyset \) represents that there is no core solution on the corresponding pure strategy combination. Example 4 shows that the allocation methods of SAF and CIS value are feasible.

Table 10. The cores of the 3-person biform game \( E \).

| \((a_1, a_2, a_3)\) | \((a_1, b_2, a_3)\) | \((a_1, a_2, b_3)\) | \((a_1, b_2, b_3)\) |
|---------------------|---------------------|---------------------|---------------------|
| 1 \( \leq y_1 \leq 3 \) | 1 \( \leq y_1 \leq 3 \) | 1 \( \leq y_1 \leq 4 \) | 1 \( \leq y_1 \leq 3 \) |
| 2 \( \leq y_2 \leq 5 \) | 2 \( \leq y_2 \leq 3 \) | 2 \( \leq y_2 \leq 5 \) | 2 \( \leq y_2 \leq 6 \) |
| 2 \( \leq y_3 \leq 5 \) | 2 \( \leq y_3 \leq 5 \) | 2 \( \leq y_2 \leq 5 \) | 2 \( \leq y_2 \leq 6 \) |
| \((b_1, a_2, a_3)\) | \((b_1, b_2, a_3)\) | \((b_1, a_2, b_3)\) | \((b_1, b_2, b_3)\) |
| 1 \( \leq y_1 \leq 5 \) | \( \emptyset \) | 1 \( \leq y_1 \leq 4 \) | 1 \( \leq y_1 \leq 3 \) |
| 1 \( \leq y_2 \leq 5 \) | 1 \( \leq y_2 \leq 3 \) | 1 \( \leq y_1 \leq 4 \) | 1 \( \leq y_2 \leq 4 \) |
| 3 \( \leq y_3 \leq 6 \) | 3 \( \leq y_2 \leq 5 \) | 3 \( \leq y_3 \leq 5 \) | 3 \( \leq y_3 \leq 6 \) |
Comparing the allocation methods of the single point core, CIS value, and SAF, we get the following results:

1. On the strategy profiles of a biform game, when the cores are not single point sets or there are empty sets in them, both CIS value and SAF can be used to obtain the Nash equilibrium of the biform game.

2. Compared with the single point core and CIS value, SAF more accurately describes the contributions of strategies and gives more fair allocation vectors (see Examples 4 and 5).

3. CIS value and SAF have nothing to do with the confidence index. In a biform game, the players need to agree with the allocation methods of CIS value and SAF instead of considering the confidence index. The advantages of SAF increase the recognition of the players.

Egalitarianism and marginalism are two major thoughts in economic allocation problems. The SAF is a marginal solution and the CIS value is an equal solution. However, the CIS value ignores the differences among individuals so that the players may have no incentives to produce more. Referring to Nan et al. [12], the $\alpha$-CIS value (For a given $\alpha \in [0, 1]$ and any $c \in A$, the $\alpha$-CIS value is $CIS\alpha_i(c)(V) = \alpha V(c)(\{i\}) + \frac{1}{n}[V(c)(N) - \alpha \sum_{j=1}^{n} V(c)(\{j\})]$, for all $i \in N$. Obviously, CIS value is a special form of $\alpha$-CIS value.) [20] can be introduced into a biform game, which is a convex combination of the equal division solution and the CIS value. It can reconcile the two major economic allocation thoughts: egalitarianism and marginalism.

The single point core, CIS value, and $\alpha$-CIS value only concern the worth of individuals and the grand coalition as the grand coalition sometimes forms directly from singletons, without other intermediate coalitions forming. In particular, the $\alpha$-CIS value is more reasonable than the CIS value in the situation that the individual worths are relatively large so that the sum of them exceeds the worth of the grand coalition. In the case of intermediate coalitions, SAF is a good choice for players because SAF is fairer and more accurate than the single point core CIS and $\alpha$-CIS values.

Nan et al. [12] obtained the existence of the efficient solution when the characteristic function of the biform game $(S^1, \cdots, S^n; V; \alpha^1, \cdots, \alpha^n)$ satisfies the independence (ID) condition and NC condition. Corresponding to Shapley value, Xu et al. [20] axiomatized the $\alpha$-CIS value in the cooperative game $(N, v)$, we investigated the properties of SAF in Section 4. These axiomatization and property research focus on the fair apportionment of the coalitions’ utilities to each player in cooperative games.

The single point core, CIS value, $\alpha$-CIS value, and SAF achieve corresponding game solutions through non-cooperative methods. Specifically, $\alpha$-CIS uses a punishment mechanism (see Mechanism 5.1 of Xu et al. [20]), while the single point core, CIS value, and SAF use Nash equilibrium.

The following is a comparison of Shapley value and core, where the allocations determined by the core applies the confidence index. Referring to Example 2.1 of Brandenburger and Stuart [4], we give a Branded Ingredient Game as follows.

**Example 6.** The player 1 is a Supplier, players 2 and 3 are Firms 1 and 2, respectively. Supplier’s strategies are “status-quo” and “branded-ingredient”, abbreviated as s and b, respectively. The strategies of Firm 1 are “acception” and “rejection”, abbreviated as a and r, respectively. Firm 2’s strategy is “acception” (a). The utilities of the coalitions are given by Table 11. The confidence index of player 1 is $\alpha^1$. 

Table 11. The coalitions’ utilities of the branded ingredient game.

|               | (s, a, a) | (s, r, a) | (b, a, a) | (b, r, a) |
|---------------|-----------|-----------|-----------|-----------|
| {1}           | 2         | 0         | 5         | 0         |
| {2}           | 0         | 0         | 0         | 0         |
| {3}           | 0         | 0         | 0         | 0         |
| {1, 2}        | 8         | 0         | 7         | 0         |
| {1, 3}        | 2         | 0         | 5         | 0         |
| {2, 3}        | 0         | 0         | 0         | 0         |
| {1, 2, 3}     | 8         | 0         | 7         | 0         |

On the strategy profiles (s, a, a) = c^{3} and (b, a, a) = c^{2}, the ranges of core allocation for the player 1 are [2, 8] and [5, 7], respectively. The player 1 selects strategy b if a^{3}7 + (1 − a^{1})5 > a^{1}8 + (1 − a^{1})2, or a^{1} < 3/4, so the player 1 selects this strategy unless he/she is very optimistic. By Formula (1), Φ_{1}(c^{3}) = 5, Φ_{2}(c^{3}) = 3, Φ_{1}(c^{2}) = 6, Φ_{2}(c^{2}) = 1. The SAF results show that player 2 prefers strategy profile c^{3}, which implies player 1 should select strategy s. If not, player 2 may choose strategy r so that everyone has no benefit, because player 2 is more indifferent to the allocation value of 0 or 1 than player 1 is to the allocation value of 0 or 6. Thus, player 1’s selection determined by SAF is consistent with the result of confidence index judgment. Therefore, SAF can be an important reference for “residual” bargaining problem with confidence index, or it can be used as a substitute for core allocation with confidence index.

6. Conclusions

In the present paper, we define a Shapley allocation function (SAF) on each mixed strategy profile for biform games. In comparison with the pure strategy form, the mixed strategy form concerns how much a player chooses a strategy further. A new form of expected utility is proposed by SAF which provides a method to determine the allocation vector efficiently. We derive four main results: First, the biform game model with SAF ensures the existence of Nash equilibria in the non-cooperative game stage, and SAF describes the contribution of players’ strategies more detailed than CIS value and a-value. Second, the function property of SAF determines the uniqueness of utility allocation, making utility allocation independent of the confidence index. Which avoids the case that the core does not exist or the allocation vectors in the core are not unique. Third, the SAF can be an important reference for the value of the confidence index, or it can be used as a substitute for core allocation with confidence index because the different values of the confidence index may increase the complexity of strategy selection. Fourth, Proposition 2 provides the conditions that a Nash equilibrium of the biform game is an efficient solution. Proposition 3 shows that when the biform game satisfies the IGCP condition, each efficient solution of the game is Nash equilibrium. Fifth, on each efficient solution, SAF satisfies the properties of Shapley value [7], that is, efficiency, anonymity property, dummy player property, and additivity. Based on our assumptions, each Nash equilibrium satisfies individual rationality. Sixth, Our model is an extension of the model determined by Hart’s Formula (4) [2].

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