STATE SPACE FORMULAS FOR STABLE RATIONAL MATRIX SOLUTIONS OF A LEECH PROBLEM

A.E. FRAZHO, S. TER HORST, AND M.A. KAASHOEK

Abstract. Given stable rational matrix functions \(G\) and \(K\), a procedure is presented to compute a stable rational matrix solution \(X\) to the Leech problem associated with \(G\) and \(K\), that is, \(G(z)X(z) = K(z)\) and \(\sup_{|z| \leq 1} \|X(z)\| \leq 1\). The solution is given in the form of a state space realization, where the matrices involved in this realization are computed from state space realizations of the data functions \(G\) and \(K\).

1. Introduction

Throughout this paper \(G\) and \(K\) are stable rational complex-valued matrix functions of sizes \(m \times p\) and \(m \times q\), respectively. Here stable means that \(G\) and \(K\) have no poles in the closed unit disc \(|z| \leq 1\). In particular, \(G\) and \(K\) are matrix-valued \(H^\infty\) functions on the open unit disc \(D\). For simplicity we write \(G \in RH^\infty_{m \times p}\) and \(K \in RH^\infty_{m \times q}\), where \(R\) stands for rational. We say that a \(p \times q\) matrix-valued \(H^\infty\) function \(X\) is a contractive analytic solution to \(GX = K\) if

\[
G(z)X(z) = K(z) \quad (z \in D) \quad \text{and} \quad \|X\|_\infty = \sup_{z \in D} \|X(z)\| \leq 1.
\]

Leech’s theorem (see [19, page 107] or [10, Section VIII.6]) tells us that there exists an \(X \in H^\infty_{p \times q}\) such that (1.1) holds if and only if

\[
T_GT_G^* - T_KT_K^* \text{ is nonnegative.}
\]

Here \(T_G : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^m)\) and \(T_K : \ell^2_+(\mathbb{C}^q) \to \ell^2_+(\mathbb{C}^m)\) are the (block) Toeplitz operators defined by \(G\) and \(K\) respectively. The positivity condition (1.2) is also equivalent to the requirement that the map

\[
L(z, \lambda) = \frac{G(\lambda)G(z)^* - K(\lambda)K(z)^*}{1 - \lambda z} \quad (z, \lambda \in D)
\]

is a positive kernel in the sense of Aronszajn [2], that is, (again see [19, page 107]) that for all finite sequences \(z_1, \ldots, z_r \in D\) and \(x_1, \ldots, x_r \in \mathbb{C}^m\), where \(r\) is an arbitrary positive integer, we have

\[
\sum_{j,k=1}^r \frac{\langle (G(z_j)G(z_j)^* - K(z_k)K(z_j)^*)x_j, x_k \rangle}{(1 - \bar{z}_j z_k)} \geq 0.
\]

The special case of Leech’s theorem with \(q = m\) and \(K\) identically equal to the \(m \times m\) identity matrix \(I_m\) is part of the corona theorem, which is due to Carlson [7].

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for $m = 1$, and Fuhrmann [15], for arbitrary $m$. An algorithm to produce rational solutions to the corona problem with $m = 1$ and polynomial data functions is given in [21]. For an engineering perspective on the corona problem and its applications in signal processing see [21] [23] and the references therein.

When $G$ and $K$ are rational, it is known (see [22] or [17]) that condition (1.2) is also necessary and sufficient for the existence of stable rational matrix solutions of (1.1). In the present paper we derive a state space formula for a rational matrix solution whose McMillan degree is at most equal to the McMillan degree of $[G,K]$. Along the way, we obtain a self contained proof of the existence of a rational matrix solution.

The fact that $G$ and $K$ are stable rational matrix functions implies that the function $[G(z) \quad K(z)]$ is also a stable rational matrix function and hence, as is well-known from mathematical systems theory (see, e.g., Chapter 1 of [8] or Chapter 4 in [4]), admits a minimal state space realization of the following form:

\begin{align}
(G(z) \quad K(z)) &= [D_1 \quad D_2] + zC(I_n - zA)^{-1}[B_1 \quad B_2].
\end{align}

Here $I_n$ is the $n \times n$ identity matrix, $A$ is a square matrix of order $n$, and $B_1, B_2, C, D_1$ and $D_2$ are matrices of appropriate sizes. Moreover, $A$ is a stable matrix, that is, $A$ has all its eigenvalues in the open unit disc $\mathbb{D}$. In what follows we denote by $W_{\text{obs}}$ the observability operator defined by the pair $\{C, A\}$, and for $j = 1, 2$ we denote by $P_j$ the controllability Gramian of the pair $\{A, B_j\}$, that is

\begin{align}
W_{\text{obs}} &= \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots
\end{bmatrix}
\quad \text{and} \quad P_j = \sum_{\nu=0}^{\infty} A^\nu B_j D_j^* A^\nu (j = 1, 2).
\end{align}

Note that $W_{\text{obs}}$ is an operator mapping $\mathbb{C}^n$ into $\ell_1^n(\mathbb{C}^m)$, and $P_1$ and $P_2$ are $n \times n$ matrices that satisfy the Stein equations

\begin{align*}
P_1 &= AP_1 A^* + B_1 B_1^* \quad \text{and} \quad P_2 = AP_2 A^* + B_2 B_2^*.
\end{align*}

Minimality means there exists no realization as in (1.5) with ‘state operator’ $A$ a matrix of smaller size than the one in the given realization. Our first main result is the following theorem.

**Theorem 1.1.** Let $G \in \mathcal{RH}_{m \times p}^\infty$ and $K \in \mathcal{RH}_{m \times q}^\infty$ be given by the minimal realization (1.5). Assume that $T_G T_G^* - T_K T_K^* \geq 0$. Then there exists a function $F \in \mathcal{RH}_{m \times r}^\infty$, for some $r \leq m$, of the form

\begin{align}
F(z) &= D_3 + zC(I_n - A)^{-1}B_3,
\end{align}

such that the following holds:

(i) $T_G T_G^* - T_K T_K^* - T_F T_F^* = W_{\text{obs}}(P_3 + P_2 - P_1)W_{\text{obs}}^*$, where $P_3$ is the controllability Gramian of the pair $\{A, B_3\}$;

(ii) $P_3 + P_2 - P_1$ is nonnegative.

In particular, $T_G T_G^* - T_K T_K^* - T_F T_F^*$ is nonnegative and has rank at most $n$, and

\begin{align}
G(e^{it})G(e^{it})^* - K(e^{it})K(e^{it})^* - F(e^{it})F(e^{it})^* = 0 \quad (t \in [0, 2\pi]).
\end{align}

We see the above theorem as the state space version of the rational matrix analogue of Theorem 0.1 in [17]. Furthermore, to construct the function $F$ in (1.7)
we follow the method of proof given in Section 2 of [17], specifying each step in an appropriate state space setting, and using the fact that

\[(1.9) \quad \text{Im} H_G + \text{Im} H_K = \text{Im} [H_G \ H_K] = \text{Im} W_{obs},\]

where $H_G$ and $H_K$ are the Hankel operators defined by $G$ and $K$, respectively. In the construction of $F$ an important role is played by the rational $m \times m$ matrix function $R$ defined by

\[(1.10) \quad R(z) = G(z)G(z^{-1})^* - K(z)K(z^{-1})^*.\]

Using (1.4) one sees that the positivity condition (1.2) implies that $R$ is nonnegative on the unit circle, and hence $R$ admits an outer spectral factor $\Phi$, that is, $\Phi$ is an outer function in $\mathcal{RH}_{r \times m}$, for some $r \leq m$, such that $R(z) = \Phi(z^{-1})^* \Phi(z)$. The construction of $F$ is then done in three steps:

1. Construct a state space realization for the outer spectral factor $\Phi$.
2. Put $M_{\Phi} = \{ f \in \ell_2^\infty(C^r) \mid T_{\Phi}^* f \in \text{Im} W_{obs} \}$, which is a backward shift invariant subspace of $\ell_2^\infty(C^r)$, and construct a state space realization for the 2-sided inner function $\Theta$ determined by $\text{Ker} T_{\Theta}^* = M_{\Phi}$.
3. Put $F = \Phi^* \Theta$, and compute a state space realization for $F$.

The explicit constructions of state space realizations for $\Phi$, $\Theta$ and $F$ are given in Section 2.

As soon as Theorem 1.1 is proved we can use the “lurking isometry” approach to Leech’s theorem from Ball-Trent [3] to derive stable rational matrix solutions to the Leech problem [15]. The next theorem is our second main result.

**Theorem 1.2.** Let $G \in \mathcal{RH}_{m \times p}^\infty$ and $K \in \mathcal{RH}_{m \times q}^\infty$ be given by the minimal realization (1.3), and let $F \in \mathcal{RH}_{m \times r}^\infty$ be as in Theorem 1.1. Let $Y$ be the solution of the Stein equation

\[Y = A^*YA + C^*C, \quad \text{that is,} \quad Y = \sum_{r=0}^{\infty} (A^*)^r C^r C A^r,\]

set $\Upsilon = (P_3 + P_2 - P_1)^{1/2}$, and let

\[(1.11) \quad U = \left[ \begin{array}{c} \alpha \\ \beta_1 \\ \beta_2 \\ \gamma \\ \delta_1 \\ \delta_2 \end{array} \right] : \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^q \\ \mathbb{C}^r \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^q \end{bmatrix} \]

be defined by

\[(1.12) \quad U = \left[ \begin{array}{ccc} \Upsilon \Upsilon & \Upsilon \Upsilon B_1 & \Upsilon \Upsilon B_2 \\ B_1^* B_1 & D_1^* D_1 + B_1^* B_1 & D_1^* D_2 + B_1^* Y B_2 \\ \Upsilon \Upsilon B_3 & D_2^* C \Upsilon + B_1^* \Upsilon \Upsilon A \Upsilon & D_2^* D_3 + B_1^* Y B_3 \end{array} \right]^+.\]

Here the superindex $^+$ means that we take the Moore-Penrose generalized inverse of the matrix involved. Then $U$ is a partial isometry and the following conditions hold:
(i) the function $X$ defined on $\mathbb{D}$ by
\begin{equation}
X(z) = \delta_1 + z\gamma (I - z\alpha)^{-1}\beta_1
\end{equation}
is a $p \times q$ stable contractive rational matrix solution to the Leech problem
\begin{equation}
\text{(1.1)}
\end{equation}:
(ii) the function $\Psi$ defined on $\mathbb{D}$ by
\begin{equation}
\Psi(z) = \delta_2 + z\gamma (I - z\alpha)^{-1}\beta_2
\end{equation}
is a $p \times r$ stable rational matrix function, $\|\Psi\|_{\infty} \leq 1$, and $\Psi$ satisfies the equation $G(z)\Psi(z) = F(z)$.

As we shall see, the proof of the above theorem uses the fact that item (i) in Theorem 1.1 yields the identity:
\begin{equation}
\lambda\bar{z}\Lambda(\lambda)\Lambda(z)^* + G(\lambda)G(z)^* = \\
\Lambda(\lambda)\Lambda(z)^* + K(\lambda)K(z)^* + F(\lambda)F(z)^* \quad (z, \lambda \in \mathbb{D}),
\end{equation}
where $\Lambda(z) = C(I_n - zA)^{-1}(P_3 + P_2 - P_1)^{1/2}$. This allows one to construct a partial isometry $U$ such that
\begin{equation}
[z\Lambda(z) \quad G(z)] U = [\Lambda(z) \quad K(z) \quad F(z)] .
\end{equation}
In fact, we will show that the matrix $U$ defined by (1.12) has these properties.

**Remark 1.3.** It can happen (cf., [17, Theorem 3.2]) that the $m \times m$ rational matrix function $R$ defined by (1.10) is identically equal to zero. For instance, take
\begin{equation}
G(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad K(z) = z.
\end{equation}
If $R$ is identically equal to zero, then items (i) and (ii) in Theorem 1.1 hold true with the function $F$ identically equal to zero and $P_3 = 0$. Furthermore, Theorem 1.2 holds with $\mathbb{C}^r$ being replaced by $\mathbb{C}^0 = \{0\}$ and setting $\Upsilon = (P_2 - P_1)^{1/2}$. See Theorem 3.2 below for further details.

**Remark 1.4.** If the rational matrix function $R$ defined by (1.10) is not identically equal to zero, Theorem 1.1 tells us that one can reduce the problem to the case where $R$ is identically equal to zero without increasing the complexity of the problem. More precisely, Theorem 1.1 shows that there exists $F \in \mathcal{RH}_{m \times r}^\infty$ of the form (1.7) such that condition (1.2) holds with $[K \quad F]$ in place of $K$, the realization
\begin{equation}
\begin{bmatrix} G(z) & K(z) & F(z) \end{bmatrix} = [D_1 \quad D_2 \quad D_3] + zC(I_n - zA)^{-1} [B_1 \quad B_2 \quad B_3]
\end{equation}
is minimal, and the rational matrix function defined by (1.10) with $[K \quad F]$ in place of $K$ is identically equal to zero.

The paper consists of six sections including the present introduction. In the second section we construct the function $F$ following the three steps listed above. This is done in a somewhat more general setting, not using $G$ and $K$, but only an $m \times m$ rational matrix function $R$ which has no pole on the unit circle $T$ and whose values on $T$ are nonnegative. In Section 3 we prove Theorem 1.1. The proof of Theorem 1.2 is given in Section 4. In Section 5 we specify the results for the case when on the unit circle the values of the function $R$ defined by (1.10) are strictly positive. In the final section we illustrate the main theorems on an example.
Some terminology and notation. We conclude this introduction with some terminology and notation that will be used throughout the paper. Given a subspace $\mathcal{U}$ of a Hilbert space $\mathcal{Y}$ we denote by $E_\mathcal{U}$ the canonical embedding of $\mathcal{U}$ into $\mathcal{Y}$. Note that $E_\mathcal{U}^*$ is the orthogonal projection of $\mathcal{Y}$ onto $\mathcal{U}$ viewed as an operator from $\mathcal{Y}$ to $\mathcal{U}$. Thus the orthogonal projection of $\mathcal{Y}$ onto $\mathcal{U}$ viewed as an operator on $\mathcal{Y}$ is given by $E_\mathcal{U}E_\mathcal{U}^*$. The latter operator will also be denoted by $P_\mathcal{U}$. For any positive integer $k$ we write $E$ for the canonical embedding of $\ell^2_k(\mathbb{C}^k)$ onto the first coordinate space of $\ell^2_k(\mathbb{C}^k)$, that is, $E^* = [I_k \ 0 \ 0 \ \cdots]$. Here $\ell^2_k(\mathbb{C}^k)$ denotes the Hilbert space of unilateral square summable sequences of vectors in $\mathbb{C}^k$.

Let $T$ be a bounded linear operator from the Hilbert space $\mathcal{U}$ into the Hilbert space $\mathcal{Y}$, and assume that $T$ has a closed range. Then $T^+$ denotes the Moore-Penrose generalized inverse of $T$, that is, $T^+$ is the unique operator from $\mathcal{Y}$ into $\mathcal{U}$ such that $T^+T = P_{\mathcal{U}}T^*$ and $TT^+ = P_{\mathcal{U}}T$. If $T$ is a Hilbert space operator on $\mathcal{U}$, i.e., from $\mathcal{U}$ into $\mathcal{U}$, then $T$ is called nonnegative in case $(Tu, u) \geq 0$ for all $u \in \mathcal{U}$, and strictly positive if $T$ is nonnegative and invertible. We will use the notation $T \geq 0$ to indicate that $T$ is nonnegative.

For a rational matrix function $\Omega$ we define $\Omega^*(z) = \Omega(\bar{z}^{-1})^*$. If $\Omega$ has no poles on the unit circle $\mathbb{T}$, then $\Omega^*(\zeta) = \Omega(\zeta)^*$ for any $\zeta \in \mathbb{T}$. If $\Omega$ is a $k \times l$ matrix function with entries in $L^\infty$ on the unit circle $\mathbb{T}$, i.e., $\Omega$ is measurable and essentially bounded on $\mathbb{T}$, then $T_\Omega$ is the Toeplitz operator defined by

$$T_\Omega = \begin{bmatrix} \Omega_0 & \Omega_{-1} & \Omega_{-2} & \cdots \\ \Omega_1 & \Omega_0 & \Omega_{-1} & \cdots \\ \Omega_2 & \Omega_1 & \Omega_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+^{\mathbb{C}^l} \to \ell^2_+^{\mathbb{C}^k}. \tag{1.17}$$

Here $\ldots, \Omega_{-1}, \Omega_0, \Omega_1, \ldots$ are the (block) Fourier coefficients of $\Omega$. The function $\Omega$ is in $H^\infty_{k \times l}$ if and only if $T_\Omega$ is a (block) lower triangular Toeplitz matrix. By $H_\Omega$ we denote the block Hankel operator determined by the block Fourier coefficients $\Omega_1, \Omega_2, \ldots$, that is,

$$H_\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & \cdots \\ \Omega_2 & \Omega_3 & \Omega_4 & \cdots \\ \Omega_3 & \Omega_4 & \Omega_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+^{\mathbb{C}^l} \to \ell^2_+^{\mathbb{C}^k}. \tag{1.18}$$

Now assume $\Omega \in \mathfrak{RH}^\infty_{k \times l}$. In that case, $\Omega$ admits a state space realization of the form

$$\Omega(z) = D + zC(I_n - zA)^{-1}B, \tag{1.19}$$

with $A$ a stable $n \times n$ matrix, and $B$, $C$, and $D$ matrices of appropriate size. The integer $n$ is referred to as the state dimension. The observability operator $W_{\text{obs}}$ and controllability operator $W_{\text{con}}$ defined by the pairs $\{C, A\}$ and $\{A, B\}$, respectively, are defined by

$$W_{\text{obs}} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell^2_+^{\mathbb{C}^k}, \quad W_{\text{con}} = \begin{bmatrix} B^* \\ B^*A^* \\ B^*A^{*2} \\ \vdots \end{bmatrix} : \ell^2_+^{\mathbb{C}^l} \to \mathbb{C}^n.$$
Moreover, the observability Gramian $P$ and controllability Gramian $Q$ are the $n \times n$ matrices given by

$$P = W_{\text{obs}}^* W_{\text{obs}} = \sum_{\nu=0}^{\infty} A^{\nu} C^* C A^{\nu},$$

$$Q = W_{\text{con}}^* W_{\text{con}} = \sum_{\nu=0}^{\infty} A^{\nu} B B^* A^{\nu}.$$

The pair $\{C, A\}$ (or the realization (1.19)) is called observable in case $P$ is strictly positive, or equivalently, if $\text{Ker} W_{\text{obs}} = \{0\}$, and the pair $\{A, B\}$ (or the realization (1.19)) is called controllable in case $Q$ is strictly positive, or equivalently, if $\text{Im} W_{\text{obs}} = C^k$. It is well known that the realization (1.19) is minimal, i.e., there is no state space realization of $\Omega$ with smaller state dimension, if and only if the realization (1.19) is observable and controllable. Finally, note that, given the realization (1.19), we have $H_\Omega = W_{\text{obs}} W_{\text{con}}$, and hence $H_\Omega^* H_\Omega = W_{\text{obs}} Q W_{\text{obs}}^*$.

2. State space formulas for the outer spectral factor and related functions

In this section $R$ is a non-zero $m \times m$ rational matrix function with no pole on the unit circle $T$. We assume that $R(\zeta)$ is hermitian for each $\zeta \in T$, and hence $R$ admits a state space realization of the following form:

$$(2.1) \quad R(z) = zC(I_n - zA)^{-1}\Gamma + R_0 + \Gamma^*(zI_n - A^*)^{-1}C^*.$$

Here $I_n$ is the $n \times n$ identity matrix, and $A$ is a stable $n \times n$ matrix, i.e., all the eigenvalues of $A$ are in the open unit disc $D$. In the sequel $W_{\text{obs}}$ denotes the observability operator defined by the pair $\{C, A\}$, that is, $W_{\text{obs}}$ is the map from $C^n$ into $\ell_2(C^m)$ given by the first identity in (1.6).

Throughout this section we shall assume that $R(\zeta)$ is a nonnegative matrix for each $\zeta \in T$. At this level of generality we shall carry out the three steps of the procedure outlined in the introduction, leading to the construction of a function $F$ with the properties stated in Theorem 1.1.

Step 1: The outer spectral factor $\Phi$. The assumption that $R(\zeta) \succeq 0$ on $T$ implies (see [19, Section 6.8]) that the Toeplitz operator $T_R$ is a nonnegative operator and $R$ admits an outer spectral factor $\Phi$, that is, $\Phi$ is in $\mathcal{RH}_{r \times m}^{\infty}$, for some $r \leq m$, such that

$$(2.2) \quad R(z) = \Phi^*(z)\Phi(z)$$

and the range of the Toeplitz operator $T_\Phi$ is a dense set in $\ell_2^2(C^r)$. Recall (see the final paragraph of Section I) that $\Omega^*(z) = \Omega(\bar{z}^{-1})^*$ for any rational matrix function $\Omega$. The outer spectral factor $\Phi$ is unique up to a unitary constant operator on the left, that is, if $\Psi$ is another outer function satisfying $R(z) = \Psi^*(z)\Psi(z)$, then $\Phi(z) = U\Psi(z)$ where $U$ is a constant unitary operator; see [20] [12] for further details.

The following theorem shows how a state space realization of $\Phi$ can be constructed from the state space realization of $R$. It does not require the pair $\{C, A\}$ to be observable.
**Theorem 2.1.** Let $R$ be as in (2.1). Assume that $R(\zeta) \geq 0$ for each $\zeta \in \mathbb{T}$, and let $\Phi \in \mathcal{RH}^\infty_{r,m}$ be an outer spectral factor of $R$. Put

$$X_\Phi = \{ x \in \mathbb{C}^n \mid W_{obs}x \in \text{Im} T_\Phi^* \}.$$  

Then $X_\Phi$ is invariant under $A$, the space $\sqrt{\nu=0} A^\nu \mathbb{C}^m$ is contained in $X_\Phi$, and there exists a $r \times n$ matrix $C_\Phi$ such that

$$
\Phi(z) = \Phi(0) + zC_\Phi(I_n - zA)^{-1}\Gamma;
$$

$$W_{obs}x = T_\Phi^* W_{obs}x, \quad x \in X_\Phi.
$$

Here $W_{\Phi,obs} = [ C^*_\Phi \quad A^* C^*_\Phi \quad A^{*2} C^*_\Phi \quad \cdots ]^*$ which is the observability operator defined by the pair $\{ C_\Phi, A \}$. Moreover, $C_\Phi|_{X_\Phi}$ is uniquely determined by (2.5). Furthermore, defining $Q_\Phi$ to be the observability Gramian of the pair $\{ C_\Phi, A \}$, that is, $Q_\Phi = \sum_{\nu=0}^{\infty} (A^\nu)^* C^*_\Phi C_\Phi A^\nu$, we have

$$\Phi(0)^* C_\Phi x = Cx - \Gamma^* Q_\Phi Ax \quad (x \in X_\Phi),$$

$$\Phi(0)^* \Phi(0) = R_0 - \Gamma^* Q_\Phi \Gamma. $$

In particular, $R_0 - \Gamma^* Q_\Phi \Gamma$ is nonnegative.

Although (2.5) only determines $C_\Phi$ uniquely on $X_\Phi$, the fact that $X_\Phi$ is invariant under $A$ and $\Gamma \mathbb{C}^m \subset X_\Phi$ implies that we can define $C_\Phi$ on the orthogonal complement of $X_\Phi$ arbitrarily, without violating (2.4)–(2.7).

In Section 5 we shall further specify Theorem 2.1 for the case when the values of $R$ on the unit circle are strictly positive. As we shall see, in that case $C_\Phi$ is uniquely determined, and hence so is the observability Gramian $Q_\Phi$, and $Q_\Phi$ appears as the stabilizing solution of a certain algebraic Riccati equation.

**Proof of Theorem 2.1.** We split the proof into four parts.

**Part 1.** In this part we show that $X_\Phi$ is invariant under $A$ and that $X_\Phi$ contains $\sqrt{\nu=0} A^\nu \mathbb{C}^m$.

Take $x \in X_\Phi$. Then there exists a $f \in \ell^2_1(\mathbb{C}^r)$ such that $W_{obs}x = T_\Phi^* f$. Let $S_m$ and $S_r$ be the (block) forward shifts on $\ell^2_1(\mathbb{C}^m)$ and $\ell^2_1(\mathbb{C}^r)$, respectively. Since $T_\Phi$ is an analytic Toeplitz operator, $S_r T_\Phi = T_\Phi S_m$, and hence

$$W_{obs}Ax = S_m^* W_{obs}x = S_m^* T_\Phi^* f = T_\Phi^* S_r^* f \in \text{Im} T_\Phi^*.$$ 

If follows that $Ax \in X_\Phi$, and thus $X_\Phi$ is invariant under $A$.

To prove the second statement, given the invariance of $X_\Phi$ under $A$, it suffices to show that $\Gamma$ maps $\mathbb{C}^m$ into $X_\Phi$. To accomplish this, let $R_n$ and $\Phi_n$, $n = 0, 1, 2, \ldots$, be the $n$-th Fourier coefficients of $R$ and $\Phi$, respectively. The fact that $R = \Phi^* \Phi$ implies that

$$R_j = \Phi_0^* \Phi_j + \Phi_1^* \Phi_{j+1} + \Phi_2^* \Phi_{j+2} + \cdots, \quad j = 0, 1, 2, \ldots,$$

If follows that

$$W_{obs} \Gamma u = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \end{bmatrix} u = T_\Phi^* \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{bmatrix} u, \quad u \in \mathbb{C}^m.$$ 

This proves that $\Gamma u \in X_\Phi$ for each $u \in \mathbb{C}^m$. 

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Part 2. In this part we define $C_\Phi$, derive (2.10), and prove the uniqueness statement.

Take $x \in X_\Phi$. Then there exists a $f \in \ell^2_+(C^r)$ such that $T_\Phi f = W_{obs}x$. Since $T_\Phi$ is one-to-one, the vector $f$ is uniquely determined by $x$, and hence there exists a unique linear map $W$ from $X_\Phi$ into $\ell^2_+(C^r)$ such that

$$(2.10) \quad T_\Phi Wx = W_{obs}x \quad (x \in X_\Phi).$$

We use $W$ to define $C_\Phi$ as follows:

$$(2.11) \quad C_\Phi = E^*WE_{X_\Phi} : \mathbb{C}^n \to C^r.$$

Here $E : C^r \to \ell^2_+(C^r)$ and $E_{X_\Phi} : X_\Phi \to \mathbb{C}^n$ are the embedding operators defined in the final paragraph of Section 3.

Next we prove (2.5). Using the canonical embedding of $X_\Phi$ into $\mathbb{C}^n$ we can rewrite (2.10) as $T_\Phi W = W_{obs}E_{X_\Phi}$. Recall that $S_rT_\Phi = T_\Phi S_m$. Thus

$$T_\Phi S_m^* W = S_m^* T_\Phi^* W = S_m^* W_{obs}E_{X_\Phi} = W_{obs}AE_{X_\Phi} = T_\Phi^* WAE_{X_\Phi}.$$ 

The fact that $T_\Phi^*$ is one-to-one implies that $S_m^* W = WAE_{X_\Phi}$. Since $W$ maps $X_\Phi$ into $\ell^2_+(C^r)$, the operator $W$ admits a matrix representation of the form:

$$W = \begin{bmatrix} Y_0^* & Y_1^* & Y_2^* & \cdots \end{bmatrix} : X_\Phi \to \ell^2_+(C^r).$$

Notice that $E^* W = Y_0$, where $E$ is as in (2.11). Using $S_m^* W = WAE_{X_\Phi}$ for any integer $j \geq 1$, we have

$$Y_j = E^* S_m^j W = E^* W A^j E_{X_\Phi} = Y_0 A^j E_{X_\Phi}.$$ 

Therefore $W$ admits a representation of the form

$$\begin{bmatrix} Y_0 \\ Y_0 A E_{X_\Phi} \\ Y_0 A^2 E_{X_\Phi} \\ \vdots \end{bmatrix} : X_\Phi \to \ell^2_+(C^r).$$

Thus by (2.11) we have $C_\Phi = Y_0 E_{X_\Phi}^*: \mathbb{C}^n \to C^r$. Using the fact that $X_\Phi$ is an invariant subspace for $A$, we see that

$$(2.12) \quad Wx = W_{\Phi, obs}x \quad (x \in X_\Phi).$$

Since $T_\Phi W = W_{obs}E_{X_\Phi}$, the identity (2.12) yields (2.5). Finally, because $W$ is uniquely determined by (2.12), the operator $Y_0 = C_\Phi |_{X_\Phi}$ is uniquely determined as well.

Part 3. In this part we prove (2.4). From the first part of the proof we know that $\text{Im} \Gamma$ is contained in $X_\Phi$. Thus the identity (2.5) yields $T_\Phi W_{\Phi, obs} \Gamma = W_{obs} \Gamma$. But then we can use (2.9) to show that

$$T_\Phi W_{\Phi, obs} \Gamma = W_{obs} \Gamma = T_\Phi^* \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \vdots \end{bmatrix}.$$ 

Since $T_\Phi$ is one to one, we see that $W_{\Phi, obs} \Gamma = \text{col} [\Phi_j]_{j=0}^\infty$. Hence $C_\Phi A^{-1} \Gamma = \Phi_j$ for $j = 1, 2, \ldots$. The latter is equivalent to (2.4).
PART 4. In this part we prove (2.6) and (2.7). To establish (2.6), note that $Q_\Phi = W_{\Phi, obs}^* W_{\Phi, obs}$. Using (2.5), we see that each $x \in X_\Phi$ we have

$$C x = E^* W_{\Phi, obs} x = E^* T_\Phi^* W_{\Phi, obs} x$$

$$= \begin{bmatrix} \Phi_0^* & \Phi_1^* & \Phi_2^* & \Phi_3^* & \cdots \end{bmatrix} \begin{bmatrix} C_\Phi \\ W_{\Phi, obs} A \end{bmatrix} x$$

$$= \Phi_0^* C_\Phi x + \Gamma^* W_{\Phi, obs}^* W_{\Phi, obs} A x \quad \text{[because of (2.9)]}$$

$$= \Phi_0^* C_\Phi x + \Gamma^* Q_\Phi A x \quad (x \in X_\Phi).$$

This proves (2.6). For $j = 0$ the identity (2.8) yields

$$R_0 = \Phi_0^* \Phi_0 + \sum_{k=1}^{\infty} \Phi_k^* \Phi_k$$

$$= \Phi_0^* \Phi_0 + \sum_{k=1}^{\infty} \Gamma^* A^k C_\Phi^* C_\Phi A^k \Gamma = \Phi_0^* \Phi_0 + \Gamma^* Q_\Phi \Gamma.$$

Therefore (2.7) holds. \hfill \Box

**Corollary 2.2.** Let $R$ be as in (2.1). Assume $R(\zeta) \geq 0$ for each $\zeta \in \mathbb{T}$, and let $\Phi$ be an outer spectral factor of $R$ given by the state space realization (2.4). Define $X_\Phi$ as in (2.3). Set $A_\circ = P_{X_\Phi} A |_{X_\Phi}$ on $X_\Phi$ and $C_\circ = C_\Phi |_{X_\Phi}$, and let $W_{\circ, obs}$ denote the observability operator defined by $\{C_\circ, A_\circ\}$. Then:

(i) If the pair $\{C, A\}$ is observable, then the operator $W_{\Phi, obs}|_{X_\Phi}$ is one-to-one on $X_\Phi$, $W_{\Phi, obs}|_{X_\Phi} = W_{\circ, obs}$, and the pair $\{C_\circ, A_\circ\}$ is observable.

(ii) If the pair $\{A, \Gamma\}$ is controllable, then $X_\Phi = \mathbb{C}^m$.

In particular, if $\{C, A\}$ is observable and $\{A, \Gamma\}$ is controllable, then the state space realization (2.4) of $\Phi$ is minimal.

**Proof.** We start with claim (i). Assume $\{C, A\}$ is an observable pair. Then $W_{\circ, obs}$ is one-to-one. Hence, by (2.3), we find that $W_{\Phi, obs}|_{X_\Phi}$ is one-to-one on $X_\Phi$. Since $X_\Phi$ is invariant under $A$, it follows that $W_{\Phi, obs}|_{X_\Phi} = W_{\circ, obs}$. Hence $W_{\circ, obs}$ is one-to-one, and thus the pair $\{C_\circ, A_\circ\}$ is observable.

Claim (ii) follows directly from the fact that $\{A, \Gamma\}$ being controllable is equivalent to $\bigvee_{n=0}^{\infty} A^\nu \Gamma \mathbb{C}^m \subset X_\Phi$, which by the inclusion $\bigvee_{n=0}^{\infty} A^\nu \Gamma \mathbb{C}^m \subset X_\Phi$, derived in Theorem 2.1, implies $X_\Phi = \mathbb{C}^m$. In particular, in that case $A_\circ = A$ and $C_\circ = C_\Phi$. Thus if $\{C, A\}$ is observable and $\{A, \Gamma\}$ is controllable, then $\{C_\circ, A_\circ\}$ is observable as well. Hence the realization (2.4) of $\Phi$ is minimal, as claimed. \hfill \Box

Step 2: The two-sided inner function $\Theta$. Let $R$ be given by (2.1). Assume that $R(\zeta) \geq 0$ for each $\zeta \in \mathbb{T}$, and let $\Phi$ be the outer spectral factor of $R$ defined by (2.4). We define $\mathcal{M}_\Phi$ to be the subspace of $\ell_+^2(\mathbb{C}^r)$ given by

$$\mathcal{M}_\Phi = \{ f \in \ell_+^2(\mathbb{C}^r) \mid T_\Phi^* f \in \text{Im} W_{\circ, obs}\},$$

in line with the definition of $\mathcal{M}_\Phi$ in the second step of the procedure outlined in the introduction. Note that $\mathcal{M}_\Phi$ is invariant under the backward shift $S_\Phi^*$, since for each $f \in \mathcal{M}_\Phi$ we have

$$T_\Phi^* S_\Phi^* f = S_\Phi^* T_\Phi^* f \in S_\Phi^* \text{Im} W_{\circ, obs} \subset \text{Im} W_{\circ, obs},$$

using the fact that $\text{Im} W_{\circ, obs}$ is invariant under $S_\Phi^*$. By the Beurling-Lax theorem, there exists a two-sided inner function $\Theta \in \mathcal{RH}_{r,x}^\infty$ such that $\mathcal{M}_\Phi = \text{Ker} T_\Phi^*$. In
the sequel we shall refer to Θ as the inner function determined by \( M_Φ \). Before deriving a state space realization for Θ, in Proposition 2.4 below, we first prove an alternative formula for the space \( M_Φ \).

**Lemma 2.3.** Let \( W_{Φ,obs} \) be the observability operator defined by the pair \( \{C_Φ, A\} \). Then the space \( M_Φ \) in (2.14) is also given by
\[
M_Φ = W_{Φ,obs}X_Φ.
\]

If, in addition, the pair \( \{C, A\} \) is observable, then \( M_Φ = \text{Im} W_{o,obs} \), where \( W_{o,obs} \) is the observability operator defined by the pair \( \{C_o, A_o\} \) given in Corollary 2.2.

**Proof.** The identity \( T_Φ^*W_{Φ,obs}x = W_{obs}x \) for \( x \) in \( X_Φ \) in (2.5) implies that \( M_Φ \supseteq W_{Φ,obs}X_Φ \). On the other hand, if \( f \in M_Φ \), then \( T_Φ^*f = W_{obs}x \) for some \( x \) in \( \mathbb{C}^n \), and thus, \( x \) must be in \( X_Φ \). The identities (2.11) and (2.12) show that \( f = W_{Φ,obs}x \) and hence \( M_Φ \subset W_{Φ,obs}X_Φ \). Therefore \( M_Φ = W_{Φ,obs}X_Φ \). The identity \( M_Φ = \text{Im} W_{o,obs} \) for the case that \( \{C, A\} \) is observable now follows directly from Corollary 2.2 part (i).

\[\square\]

**Proposition 2.4.** Assume the pair \( \{C, A\} \) is observable. Let \( Θ \in \mathcal{RH}_r(\mathbb{C}^n) \) be the two-sided inner function determined by \( M_Φ \). Then \( Θ \) admits a state space realization of the form:
\[
Θ(z) = D_Φ + zC_Φ(J_n - zA)^{-1}B_Φ.
\]
Here \( C_Φ \) is as in (2.3), and \( B_Φ \) and \( D_Φ \) are matrices of sizes \( n \times r \) and \( r \times r \), respectively, satisfying the following two identities:
\[
(B_Φ^*Q_ΦA + D_ΦC_Φ)E_{X_Φ} = 0 \quad \text{and} \quad B_Φ^*Q_ΦB_Φ + D_ΦD_Φ = I_r.
\]
Finally, \( Q_Φ \) is the observability Gramian corresponding to the pair \( \{C_Φ, A\} \).

**Proof.** To prove the proposition we apply Theorem III.7.2 in [11]. Define \( A_o \) and \( B_o \) as in Corollary 2.2 and let \( W_{o,obs} \) and \( Q_o \) be the observability operator, respectively observability Gramian, defined by the pair \( \{C_o, A_o\} \). Note that, by Lemma 2.3
\[
\text{Im} W_{o,obs} = W_{Φ,obs}X_Φ = M_Φ = \text{Ker} T_Φ^*.
\]
Since the pair \( \{C, A\} \) is observable, the same holds true for the pair \( \{C_o, A_o\} \), by Corollary 2.2. Hence \( Q_o \) is strictly positive, and according to [11] Theorem III.7.2, see also [18] Lemma 3.2], there exist linear maps \( B_o : \mathbb{C}^r \to \mathcal{X}_Φ \) and \( D_o : \mathbb{C}^r \to \mathbb{C}^r \) such that
\[
Θ(z) = D_o + zC_o(I_{X_Φ} - zA_o)^{-1}B_o,
\]
\[
\begin{bmatrix}
A_o^* & C_o^* \\
B_o^* & D_o^*
\end{bmatrix}
\begin{bmatrix}
Q_o & 0 \\
0 & I_r
\end{bmatrix}
\begin{bmatrix}
A_o & B_o \\
C_o & D_o
\end{bmatrix}
= \begin{bmatrix}
Q_o & 0 \\
0 & I_r
\end{bmatrix}.
\]

Now put \( D_Φ = D_o \) and \( B_Φ = E_{X_Φ}B_o \), where \( E_{X_Φ} \) is the canonical embedding of \( X_Φ \) into \( \mathbb{C}^n \). Then (2.15) and (2.17) are satisfied. To see this we first note that the definitions of \( A_o \) and \( C_o \) yield
\[
E_{X_Φ}A_o = AE_{X_Φ} \quad \text{and} \quad C_o = C_ΦE_{X_Φ}.
\]
The first identity implies that
\[
E_{X_Φ}((I_{X_Φ} - zA_o) = E_{X_Φ} - zE_{X_Φ}A_o = E_{X_Φ} - zAE_{X_Φ} = (I_n - zA)E_{X_Φ}.
\]
This yields $(I_n - zA)^{-1}E_{X_0} = E_{X_0}(I_{X_0} - zA_0)^{-1}$. Recall that $B_\Theta = E_{X_0}B_0$. It follows that

$$(I_n - zA)^{-1}B_\Theta = (I_n - zA)^{-1}E_{X_0}B_0 = E_{X_0}(I_{X_0} - zA_0)^{-1}B_0.$$  

Since $D_\Theta = D_0$ and $C_\Phi = C_0E_{X_0}$ we see that (2.17) implies (2.15).

Next we prove (2.16). To do this note that (2.18) yields the following two identities (see also [11 Lemma III.7.3]):

$$B_0^*Q_0A_0 + D_0^*C_0 = 0 \quad \text{and} \quad B_0^*Q_0B_0 + D_0^*D_0 = I_r.$$  

Using $Q_0 = E_{X_0}^*Q_0E_{X_0}$, $B_\Theta = E_{X_0}B_0$, and the first identity in (2.19) we obtain

$$B_0^*Q_0A_0 = B_0^*E_{X_0}Q_0E_{X_0}A_0 = B_0^*Q_0AE_{X_0}.$$  

Similarly, using the second identity in (2.19), we get $D_0^*C_0 = D_0^*C_0E_{X_0}$. It follows that

$$(B_0^*Q_0A + D_0^*C_0)E_{X_0} = B_0^*Q_0AE_{X_0} + D_0^*C_0E_{X_0} = B_0^*Q_0A_0 + D_0^*C_0 = 0.$$  

This proves the first identity in (2.16). The second identity in (2.16) follows from $B_\Theta = E_{X_0}B_0$, $D_\Theta = D_0$, and $Q_0 = E_{X_0}^*Q_0E_{X_0}$. Indeed,

$$B_0^*Q_0B_0 + D_0^*D_0 = B_0^*E_{X_0}Q_0E_{X_0}B_0 + D_0^*D_0 = B_0^*Q_0B_0 + D_0^*D_0 = I_r.$$  

\[ \square \]

**Lemma 2.5.** Assume the pair $\{C, A\}$ is observable. In that case the linear map

$$\Omega_\Phi = E_{X_0}^*Q_0E_{X_0} : X_0 \to X_0 \text{ is invertible.}$$

Furthermore, the orthogonal projection of $\ell_+^2(\mathbb{C}^r)$ mapping $\ell_+^2(\mathbb{C}^r)$ onto the finite dimensional space $M_\Phi$ is given by

$$P_{M_\Phi} = W_{\Phi, obs}^*\Delta W_{\Phi, obs}^*,$$

where $\Delta = E_{X_0}\Omega_\Phi^{-1}E_{X_0}^* : \mathbb{C}^n \to \mathbb{C}^n$.

**Proof.** Note that $Q_\Phi = W_{\Phi, obs}^*W_{\Phi, obs}$. Since by assumption $\{C, A\}$ is observable, Corollary [2.2 part (i), shows that $W_{\Phi, obs}$ is one-to-one on $X_0$. The latter is equivalent to $\Omega_\Phi$ being invertible.

Next, let $\Lambda : X_0 \to \ell_+^2(\mathbb{C}^r)$ be the map defined by $\Lambda = W_{\Phi, obs}E_{X_0}$. Then $\Lambda$ is one-to-one and its range is closed and equals $M_\Phi$. Hence the orthogonal projection onto $M_\Phi$ is given by $\Lambda^*(\Lambda^*\Lambda)^{-1}\Lambda$ which yields (2.22).

\[ \square \]

**Corollary 2.6.** The linear map $\Delta$ on $\mathbb{C}^n$ defined in the second part of (2.22) is equal to the controllability Gramian of the pair $\{A, B_\Theta\}$, where $B_\Theta$ is as in (2.15).

**Proof.** We shall freely use the notation introduced in the proof of Proposition [2.2].

Since $Q_0$ is invertible, the identity (2.18) implies that

$$\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} Q_0^{-1} & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} A_0^* & C_0^* \\ B_0^* & D_0^* \end{bmatrix} = \begin{bmatrix} Q_0^{-1} & 0 \\ 0 & I_r \end{bmatrix}.$$  

In particular, we have $A_0Q_0^{-1}A_0^* + B_0B_0^* = Q_0^{-1}$. In other words

$$Q_0^{-1} - A_0Q_0^{-1}A_0^* = B_0B_0^*.$$
Now recall that \( B_\Theta = E_{X_\Phi} B_0 \) and \( A E_{X_\Phi} = A_0 E_{X_\Phi} \). It follows that
\[
B_\Theta B_\Theta^* = E_{X_\Phi} B_0 B_0^* E_{X_\Phi} = E_{X_\Phi} Q_0^{-1} E_{X_\Phi}^* - E_{X_\Phi} A_0 Q_0^{-1} A_0^* E_{X_\Phi}^* - A E_{X_\Phi} Q_0^{-1} E_{X_\Phi} A^*.
\]
(2.24)

Since \( W_{\Phi, \text{obs}} = W_{\Phi, \text{obs}} E_{X_\Phi} \), we see that
\[
Q_0 = W_{\Phi, \text{obs}} W_{\Phi, \text{obs}} = E_{X_\Phi} W_{\Phi, \text{obs}} W_{\Phi, \text{obs}} E_{X_\Phi} = E_{X_\Phi} Q_0 E_{X_\Phi}.
\]

It follows that \( Q_0 = \Omega_\Phi \), where \( \Omega_\Phi \) is defined by (2.21). But then (2.24) and the definition of \( \Delta \) in (2.22) yield
\[
B_\Theta B_\Theta^* = E_{X_\Phi} Q_0^{-1} E_{X_\Phi}^* - A E_{X_\Phi} Q_0^{-1} E_{X_\Phi} A^* = \Delta - A \Delta A^*.
\]

This proves that \( \Delta \) is the controllability Gramian of the pair \( \{A, B_\Theta^*\} \).

\( \square \)

Lemma 2.7. Let \( \Theta \in H_\infty^{r \times r} \) be the two-sided inner function determined by \( \mathcal{M}_\Phi \). Let \( \mathcal{M}_\Phi \) be the controllability operator defined by the pair \( \{A, \Gamma\} \), and \( \mathcal{X}_\text{con} \subset \mathcal{X}_\Phi \), by Theorem 2.1.

Then
\[
\text{Im } H_\Phi = \text{Ker } T_\Phi^* \iff \mathcal{X}_\text{con} = \bigvee_{\nu \geq 0} A^\nu \mathcal{X}_\Phi.
\]

(2.25)

Proof. Set \( \mathcal{X}_\text{con} = \bigvee_{\nu \geq 0} A^\nu \mathcal{X}_\Phi \), \( \mathcal{X}_\text{con} = \text{Im } W_{\text{con}} \), where \( W_{\text{con}} \) is the controllability operator defined by the pair \( \{A, \Gamma\} \), and \( \mathcal{X}_\text{con} \subset \mathcal{X}_\Phi \), by Theorem 2.1. Then
\[
\text{Im } H_\Phi = \text{Im } W_{\Phi, \text{obs}} W_{\text{con}} = W_{\Phi, \text{obs}} \mathcal{X}_\text{con} \subset W_{\Phi, \text{obs}} \mathcal{X}_\Phi = \mathcal{M}_\Phi = \text{Ker } T_\Phi^*.\]

The last but one identity follows from (2.24). Moreover, since \( W_{\Phi, \text{obs}} |_{\mathcal{X}_\Phi} \) is one-to-one, by Corollary 2.2 the above inclusion \( W_{\Phi, \text{obs}} \mathcal{X}_\text{con} \subset W_{\Phi, \text{obs}} \mathcal{X}_\Phi \) turns into an identity if and only if \( \mathcal{X}_\text{con} = \mathcal{X}_\Phi \). Hence (2.25) holds.

Step 3: The function \( F \). The final step in the procedure asks for a state space realization for the function \( F \) given by \( F = \Phi^* \Theta \). The following proposition provides such a realization.

Proposition 2.8. Let \( \Phi \in \mathcal{RH}_\infty^{m \times r} \) be the outer spectral factor of the function \( R \) given by (2.4), and let \( \Theta \in \mathcal{RH}_\infty^{r \times r} \) be the two-sided inner function determined by \( \mathcal{M}_\Phi \). Assume the pair \( \{C, A\} \) is observable. Then the function \( F = \Phi^* \Theta \) belongs to \( \mathcal{RH}_\infty^{m \times r} \), and \( F \) admits the following state space realization:
\[
F(z) = \Phi(0)^* D_\Theta + \Gamma^* Q_\Phi B_\Theta + z C(I_n - z A)^{-1} B_\Theta.
\]
(2.26)

Here \( Q_\Phi \) is the observability Gramian of the pair \( \{C_\Phi, A\} \), and \( B_\Theta \) and \( D_\Theta \) are as in (2.15). Furthermore,
\[
T_R = T_F T_F^* + W_{\text{obs}} \Delta W_{\text{obs}}^*,
\]
(2.27)

where \( \Delta \) is the linear map on \( \mathbb{C}^n \) defined in the second part of (2.22) or, equivalently, \( \Delta \) is the controllability Gramian of the pair \( \{A, B_\Theta\} \).

Proof. Since \( Q_\Phi \) is the observability Gramian of the pair \( \{C_\Phi, A\} \), we have, \( Q_\Phi - A^* Q_\Phi A = C_\Phi^* C_\Phi \), and hence
\[
z C_\Phi^* C_\Phi = (z I_n - A^*) Q_\Phi + A^* Q_\Phi (I_n - z A) \quad (z \in \mathbb{C}).
\]
(2.28)
To get (2.26) we use the state space formulas (2.4) and (2.15) which represent \( \Phi \) and \( \Theta \), respectively. This yields:

\[
F(z) = \Phi^*(z)\Theta(z)
\]

\[
= \Phi(0)^*D_\Theta + \Gamma^*(zI_n - A^*)^{-1}C_\Phi^*D_\Theta + \alpha(z) + z\Phi(0)^*C_\Phi(I_n - zA)^{-1}B_\Theta
\]

where

\[
\alpha(z) = \Gamma^*(zI_n - A^*)^{-1}(zC_\Phi C_\Phi)(I_n - zA)^{-1}B_\Theta.
\]

The identity (2.28) then shows that

\[
\alpha(z) = \Gamma^*(zI_n - A^*)^{-1}(zI_n - A^*)Q_\Phi + A^*Q_\Phi(I_n - zA)\]

\[
= \Gamma^*Q_\Phi(I_n - zA)^{-1}B_\Theta + \Gamma^*(zI_n - A^*)^{-1}A^*Q_\Phi B_\Theta
\]

\[
= \Gamma^*Q_\Phi B_\Theta + z\Gamma^*Q_\Phi A(I_n - zA)^{-1}B_\Theta + \Gamma^*(zI_n - A^*)^{-1}A^*Q_\Phi B_\Theta.
\]

It follows that

\[
F(z) = \Phi(0)^*D_\Theta + \Gamma^*Q_\Phi B_\Theta + z\Phi(0)^*C_\Phi + \Gamma^*Q_\Phi A(I_n - zA)^{-1}B_\Theta.
\]

Recall that \( B_\Theta = E_{\mathcal{X}_\Phi}B_\circ \), where \( B_\circ \) is as in (2.18). In particular, \( B_\Theta \) maps \( \mathbb{C}^r \) into \( \mathcal{X}_\Phi \). But \( \mathcal{X}_\Phi \) is invariant under \( A \). Therefore \( (I_n - zA)^{-1}B_\circ \) maps \( \mathbb{C}^r \) into \( \mathcal{X}_\Phi \). In other words, for each \( u \in \mathbb{C}^r \) the vector \( x = (I_n - zA)^{-1}B_\circ u \) belongs to \( \mathcal{X}_\Phi \). Hence \( (\Phi(0)^*C_\Phi + \Gamma^*Q_\Phi A)x = Cx \) by (2.6), and it follows that (2.30) is equal to \(+zC(I_n - zA)^{-1}B_\Theta\).

Next we show that the term in (2.29) is zero. To accomplish this, note that (2.10) shows that \((D_\Theta C_\Phi + B_\Theta Q_\Phi A^*)x = 0\) for each \( x \in \mathcal{X}_\Phi \). Since \( \text{Im } \Gamma \subset \mathcal{X}_\Phi \) and the space \( \mathcal{X}_\Phi \) is invariant under \( A \), we have the inclusion \( \text{Im } (I_n - zA)^{-1}\Gamma \subset \mathcal{X}_\Phi \), and thus \((D_\Theta C_\Phi + B_\Theta Q_\Phi A^*)(I_n - zA)^{-1}\Gamma \) is identically zero. Taking the adjoint shows that the term in (2.29) is zero. Summarizing we see that (2.29) is proved.

It remains to prove (2.27). To do this note that \( T^*_\Phi T_\Theta = T^*_{\Phi,\Theta} = T_F \). It follows that

\[
(2.31) \quad T_R = T_{\Phi,\Theta} = T^*_\Phi T_\Theta = T^*_\Phi T_\Theta T^*_\Phi T_\Theta + T^*_\Phi (I - T_\Theta T^*_\Theta) T_\Phi
\]

\[
(2.32) \quad = T_F T^*_F + T^*_\Phi (I - T_\Theta T^*_\Theta) T_\Phi.
\]

Recall that \( \mathcal{M}_\Phi = \text{Ker } T^*_\Theta \), and hence \( P_{\mathcal{M}_\Phi} \) is the orthogonal projection on \( \ell_2^2(\mathbb{C}^r) \) mapping \( \ell_2^2(\mathbb{C}^r) \) onto \( \text{Ker } T^*_\Theta \). Since \( \Theta \) is inner, \( T_\Theta \) is an isometry, and hence the orthogonal projection on \( \ell_2^2(\mathbb{C}^r) \) mapping \( \ell_2^2(\mathbb{C}^r) \) onto \( \text{Ker } T^*_\Theta \) is equal to \( I - T_\Theta T^*_\Theta \), that is,

\[
(2.33) \quad P_{\mathcal{M}_\Phi} = I - T_\Theta T^*_\Theta.
\]

The latter identity, together with (2.32) and (2.22), shows that

\[
(2.34) \quad T_R = T_F T^*_F + T^*_\Phi P_{\mathcal{M}_\Phi} T_\Phi = T_F T^*_F + T^*_\Phi W_{\Phi,obs} \Delta W^*_\Phi,obs T_\Phi.
\]

According to the definition of \( \Delta \) in the second part of (2.22) the operator \( \Delta \) maps \( \mathcal{X}_\Phi \) into itself and is zero on \( \mathcal{C}^n \ominus \mathcal{X}_\Phi \). But then (2.10) tells us that \( T^*_\Phi W_{\Phi,obs} \Delta W^*_\Phi,obs T_\Phi = W_{\Phi,obs} \Delta W^*_\Phi,obs \) which completes the proof of (2.27).
Let $F = \Phi^* \Theta$ be the rational matrix function defined in the preceding proposition. Since $\Theta$ is 2-sided inner, $\Theta \Theta^* = \Theta^* \Theta$ is identically equal to the $r \times r$ identity matrix. It follows that

$$R = \Phi^* \Phi = \Phi^* \Theta \Theta^* \Phi = FF^* \quad \text{and} \quad \Phi = \Theta F^*.$$  

The first identity in (2.35) shows that $F$ appears as left spectral factor of $R$. The second identity tells us that $F$ and $\Theta$ appear as the factors in a Douglas-Shapiro-Shields factorization of $\Phi$.

Recall, e.g., from [9] or Sections 4.7 and 4.8 in [12], that a Douglas-Shapiro-Shields (DSS) factorization of a function $\Phi \in H^\infty_{r \times m}$ is a factorization $\Phi = \Theta F^*$ with $\Theta$ a two-sided inner function in $H^\infty_{r \times r}$ and $F$ a function in $H^\infty_{m \times r}$. A DSS factorization $\Phi = \Theta_c F^*_c$ of $\Phi$ is called canonical if the only common right inner factor between $\Theta_c$ and $F_c$ is a unitary constant $r \times r$ matrix. Moreover, any DSS factorization $\Phi = \Theta F^*$ admits a decomposition of the form $\Theta = \Theta_c \Theta_1$ and $F = F_c \Theta_1$ where $\Phi = \Theta_c F^*_c$ is the canonical factorization and $\Theta_1$ is an inner function. Finally, it is noted that $\Phi = \Theta F^*$ is a canonical factorization if and only if $\text{Im} H_\Phi = \text{Ker} T^{\phi}_{\Theta_1}$.

Now let $\Phi = \Theta F^*$ be our DSS factorization where $\text{Ker} T^{\phi}_{\Theta_1} = W_{\Phi, \text{obs}} \mathcal{X}_\Phi$. Then we have

$$W_{\Phi, \text{obs}} \mathcal{X}_\Phi = \text{Ker} T^{\phi}_{\Theta_1} = \text{Ker} T^{\phi}_{\Theta_{1, \text{obs}}} = \text{Ker} T^{\phi}_{\Theta_1} \oplus (T^{\phi}_{\Theta_1} \text{Ker} T^{\phi}_{\Theta_1})$$

$$= \text{Im} H_\Phi \oplus (T^{\phi}_{\Theta_1} \text{Ker} T^{\phi}_{\Theta_1}) \supset \text{Im} H_\Phi.$$  

Hence $W_{\Phi, \text{obs}} \mathcal{X}_\Phi = \text{Im} H_\Phi$ if and only if $\text{Ker} T^{\phi}_{\Theta_1} = \{0\}$, or equivalently, $\Theta_1$ is a unitary constant. In other words, $W_{\Phi, \text{obs}} \mathcal{X}_\Phi = \text{Im} H_\Phi$ if and only if $\Phi = \Theta F^*$ is a canonical factorization. If the pair $\{C, A\}$ is observable, then $W_{\Phi, \text{obs}}$ is one to one. In this case, the dimension of $\mathcal{X}_\Phi$ equals the rank of $H_\Phi$ (or equivalently the McMillan degree of $\Phi$) if and only if $\Phi = \Theta F^*$ is a canonical factorization. Thus Lemma 2.7 yields the following result.

**Corollary 2.9.** Let $R$ be as in (2.31) with $\{C, A\}$ observable and $R(\zeta) \geq 0$ for each $\zeta \in \mathbb{T}$. Then the DSS factorization $\Phi = \Theta F^*$ of the outer spectral factor $\Phi$ of $R$, with $\Theta$ and $F$ as in Propositions 2.4 and 2.8 respectively, is canonical if and only if

$$\mathcal{X}_\Phi = \nu \geq 0 A^\ast \Gamma \mathbb{C}^m.$$  

In particular, $\Phi = \Theta F^*$ is canonical in case the pair $\{A, \Gamma\}$ is controllable.

We conclude this section with an observation that will be useful in the next section, and which is still valid at the level of generality considered in the present section.

**Lemma 2.10.** Set $\mathcal{M} = \mathcal{M}_\Phi = W_{\Phi, \text{obs}} \mathcal{X}_\Phi$ and $\mathcal{N} = \text{Im} W_{\text{obs}}$, and consider the orthogonal direct sum decompositions

$$\ell^+_2(C^r) = \mathcal{M} \oplus \mathcal{M}^\perp \quad \text{and} \quad \ell^+_2(C^m) = \mathcal{N} \oplus \mathcal{N}^\perp.$$  

Then, with respect to these decompositions, $T^{\phi}_{\Phi}$ has a $2 \times 2$ matrix representation of the form

$$T^{\phi}_{\Phi} = \begin{bmatrix} T_{11} & T_{12}^* \\ 0 & T_{22}^* \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{N} \\ \mathcal{N}^\perp \end{bmatrix}.$$
with $T_{22}^*$ one-to-one. Furthermore, the function $F \in H_{m\times r}^\infty$, defined in Proposition 2.8 satisfies

$$T_F T_F^* = \begin{bmatrix} I_N & T_{21}^* \\ 0 & T_{22}^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I_{M^\perp} & I_{21} \end{bmatrix} \begin{bmatrix} I_N & 0 \\ T_{21} & T_{22} \end{bmatrix},$$

and the third factor in the right hand side of (2.37), that is,

$$\left[ \begin{array}{cc} I_N & 0 \\ T_{21} & T_{22} \end{array} \right] : \left[ \begin{array}{c} \mathcal{N} \\ \mathcal{N}^\perp \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathcal{N} \\ \mathcal{M}^\perp \end{array} \right]$$

has dense range. Here $I_N$ and $I_{M^\perp}$ denote the identity operators on $\mathcal{N}$, respectively $\mathcal{M}^\perp$.

**Proof.** The form of the $2 \times 2$ matrix representation of $T_F^*$ is obvious from the fact that $T_F^*$ maps $\mathcal{M}$ into $\mathcal{N}$. The formula for $T_F T_F^*$ follows from

$$T_F T_F^* = T_F^* (I - P_{\mathcal{M}}) T_F = \begin{bmatrix} T_{21} T_{21} & T_{21} T_{22} \\ T_{22} T_{21} & T_{22} T_{22} \end{bmatrix}$$

$$= \begin{bmatrix} T_{21} & T_{22} \\ T_{22} & T_{21} \end{bmatrix}$$

$$= \begin{bmatrix} I_N & 0 \\ 0 & I_{M^\perp} \end{bmatrix} \begin{bmatrix} I_N & 0 \\ T_{21} & T_{22} \end{bmatrix}.$$

We prove that $T_{22}^*$ is one-to-one. Note that the fact the (2.38) has dense range is a direct consequence of this. To see that $T_{22}^*$ is one-to-one, take $f \in \mathcal{M}^\perp$ and assume that $T_{22}^* f = 0$. This implies that $T_{22}^* f \in \mathcal{N} = \text{Im} W_{\text{obs}}$. But then (2.13) tells us that $f \in \mathcal{M} = \mathcal{M}$, and $f$ must be zero. Therefore $T_{22}^*$ is one-to-one, as claimed. \(\square\)

### 3. Proof of Theorem 1.1

Throughout this section $G$ and $K$ are stable rational matrix functions, $G \in \mathcal{RH}_{m\times p}^\infty$ and $K \in \mathcal{RH}_{m\times q}^\infty$, and we assume that $[G(z) \quad K(z)]$ is given by a stable state space realization of the following form:

$$[G(z) \quad K(z)] = [D_1 \quad D_2] + z C (I_n - z A)^{-1} [B_1 \quad B_2].$$

In particular, $A$ is a stable matrix. Note that (3.1) is equivalent to the following two realizations:

$$G(z) = D_1 + z C (I_n - z A)^{-1} B_1,$$

$$K(z) = D_2 + z C (I_n - z A)^{-1} B_2.$$

The following lemma will allow us to apply the results of the previous section.

**Lemma 3.1.** Let $G \in \mathcal{RH}_{m\times p}^\infty$ and $K \in \mathcal{RH}_{m\times q}^\infty$, and put

$$R(z) = G(z) G^*(z) - K(z) K^*(z).$$

Assume (3.2) and (3.3) are stable realizations, and let $P_j$ be the controllability Gramian for the pair $\{A, B_j\}$ for $j = 1, 2$. Then $R$ is an $m \times m$ rational matrix
function with no pole on \( T \), and \( R \) admits the following state space realizations:

\[
R(z) = zC(I_n - zA)^{-1} \Gamma + R_0 + \Gamma^*(zI_n - A^*)^{-1} C^*,
\]

where

\[
R_0 = D_1 D_1^* - D_2 D_2^* + C(P_1 - P_2) C^*, \quad \Gamma = B_1 D_1^* - B_2 D_2^* + A(P_1 - P_2) C^*.
\]

**Proof.** The result follows from Lemma 3.1 in [12]. Indeed, applying the latter lemma to \( GG^* \) yields

\[
G(z)G^*(z) = zC(I_n - zA)^{-1} \Gamma_1 + D_1 D_1^* + CP_1 C^* + \Gamma_1^*(zI_n - A^*)^{-1} C^*, \quad \text{where} \quad \Gamma_1 = D_1 B_1^* + AP_1 C^*.
\]

In a similar way one obtains

\[
K(z)K^*(z) = zC(I_n - zA)^{-1} \Gamma_2 + D_2 D_2^* + CP_2 C^* + \Gamma_2^*(zI_n - A^*)^{-1} C^*, \quad \text{where} \quad \Gamma_2 = B_2 D_2^* + AP_2 C^*.
\]

Together these two realizations yield (3.5). \( \square \)

**Theorem 3.2.** Let \((G, K)\) be a realization of \([G, K]\) which is minimal, that is, both observable and controllable. Then \( T_GT_G^* - T_KT_K^* \) is nonnegative if and only if the following two conditions hold:

(i) The rational matrix function \( R \) defined by (3.4) has nonnegative values on \( T \) or, equivalently, \( R \) has an outer spectral factor \( \Phi \) belonging to \( \mathfrak{R} H_{r \times m}^\infty \) for some \( r \leq m \).

(ii) The operator \( \Delta + P_2 - P_1 \) is nonnegative. Here \( P_1 \) and \( P_2 \) are the controllability Gramians corresponding to the pairs \( \{A, B_1\} \) and \( \{A, B_2\} \), respectively, and \( \Delta \) is the linear map defined by the second part of (2.22).

In the special case when the function \( R \) defined by (3.4) is identically zero item (i) is automatically fulfilled (with \( r = 0 \)) and item (ii) holds with \( \Delta = 0 \).

**Proof.** As noted in the introduction the condition \( T_GT_G^* - T_KT_K^* \) is nonnegative implies that the function \( R \) defined in (3.4) is nonnegative on \( T \), or equivalently, \( R \) has an outer spectral factor, \( \Phi \) say, which belongs to \( \mathfrak{R} H_{r \times m}^\infty \). Therefore in what follows we shall assume that condition (i) is fulfilled. Since we assume that (i) holds, it remains to prove that \( T_GT_G^* - T_KT_K^* \geq 0 \) if and only if \( \Delta + P_2 - P_1 \geq 0 \).

A classical identity for Toeplitz and Hankel operators (see, e.g., [9] Proposition 2.14, or [10] Section XXIII.4) yields

\[
T_GT_G^* = T_GT_G^* + H_G H_G^* \quad \text{and} \quad T_KT_K^* = T_KT_K^* + H_K H_K^*.
\]

Since \( R = GG^* - KK^* \), we have \( T_R = T_GT_G^* - T_KT_K^* \). Using the two identities in (3.8) we see that

\[
T_GT_G^* - T_KT_K^* = T_R - (H_G H_G^* - H_K H_K^*).
\]

Recall that \( H_G H_G^* = W_{obs} P_1 W_{obs}^* \) and \( H_K H_K^* = W_{obs} P_2 W_{obs}^* \). Thus (3.9) can be rewritten as

\[
T_GT_G^* - T_KT_K^* = T_R + W_{obs}(P_2 - P_1) W_{obs}^*.
\]

We shall first attend to the case where \( R \) is identically zero. As observed in Remark 1.3 this is not a trivial case. When \( R \) is identically zero, identity (3.10)
implies that $T_G T_G^* - T_K T_K^* \geq 0$ if and only if $W_{obs}(P_2 - P_1)W_{obs}^*$ is nonnegative. But $W_{obs}$ is one-to-one and hence $W_{obs}^*$ has dense range. It follows that

$$W_{obs}(P_2 - P_1)W_{obs}^* \geq 0 \iff P_2 - P_1 \geq 0.$$  

This proves the theorem for the case when $R \equiv 0$.

Next assume that $R$ is not identically zero. This allows us to apply the results of the previous section. Using (2.27), the identity (3.10) can be rewritten as

$$W_{obs}(\Delta + P_2 - P_1)W_{obs}^* \equiv 0 \iff \frac{\partial}{\partial t^3} W_{obs} E_{\mathcal{N}} E_{\mathcal{N}}^*.$$  

Using the previous identity and (2.37) we obtain

$$T_G T_G^* - T_K T_K^* = \begin{bmatrix} I_N & T_{21}^* \\ 0 & T_{22}^* \end{bmatrix} E_N W_{obs}(\Delta + P_2 - P_1)W_{obs}^* E_N^* \begin{bmatrix} I_N \\ 0 \end{bmatrix} \begin{bmatrix} I_N & 0 \\ T_{21} & T_{22} \end{bmatrix}.$$  

By Lemma 2.10 the third factor on the right hand side has dense range, and consequently, the first factor on the right hand side has a trivial kernel. It follows that

$$W_{obs}(\Delta + P_2 - P_1)W_{obs}^* \geq 0 \iff \Delta + P_2 - P_1 \geq 0.$$  

The second equivalence follows from the fact that $W_{obs}^*$ has dense range. We conclude (assuming item (i) holds) that the operator $T_G T_G^* - T_K T_K^*$ is nonnegative if and only if item (ii) is satisfied. \hfill $\square$

**Proof of Theorem 1.1.** Assume that $[G \ K]$ is given by (3.1) and that the right hand side of (3.4) is a minimal realization. Define $\Phi$ by (3.5), where $R_0$ and $\Gamma$ are given by (3.6) and (3.7), respectively. The fact that $T_G T_G^* - T_K T_K^*$ is nonnegative implies that $R$ admits an outer spectral factorization, $\Phi$ say, which belongs to $\mathcal{RH}^\infty_{n \times \infty}$. Using this $\Phi$, one constructs $\Theta$ and $F$ as in Section 2. We claim that $F$ has the desired properties. Indeed, (2.30) shows that $F$ is of the form (1.7) with

$$B_3 = B_\Theta \quad \text{and} \quad D_3 = \Phi(0)^* D_\Theta + \Gamma^* Q_8 B_\Theta.$$  

Furthermore, by Corollary 2.6 in this case the controllability Gramian of the pair $\{A, B_3\} = \{A, B_\Theta\}$ is equal to the matrix $\Delta$ in (2.22). In other words

$$P_3 = \Delta \quad \text{and} \quad \Delta + P_2 - P_1 = P_3 + P_2 - P_1.$$  

But then (3.11) shows that item (i) in Theorem 1.1 is satisfied. Finally, the fact that $T_G T_G^* - T_K T_K^*$ is nonnegative implies that $\Delta + P_2 - P_1 = P_3 + P_2 - P_1$ is nonnegative, which proves item (ii).

It remains to prove the final statements in Theorem 1.1. From (3.11) it follows that

$$\text{rank}(T_G T_G^* - T_K T_K^* - T_F T_F^*) = \text{rank} W_{obs}(P_3 + P_2 - P_1)W_{obs}^* \leq n.$$
Since $G$, $H$, and $F$ are rational matrix functions, the corresponding Hankel operators have finite rank. Hence, using

$$T_{(GG^*-KK^*-FF^*)} = (T_GT_G^* - T_KT_K^* - T_FT_F^*) - (H_GH_G^* - HKH_K^* - HFH_F^*),$$

it follows that rank$(T_GT_G^* - T_KT_K^* - T_FT_F^*)$ is finite, implies that the rank of the Toeplitz operator $T_{GG^*-KK^*-FF^*}$ is finite. This can only happen when the function $GG^* - KK^* - FF^*$ is zero; cf., [17] Theorem 3.2.

\section*{4. Proof of Theorem 1.2}

Let $G \in \mathcal{RH}_{m \times p}^\infty$ and $K \in \mathcal{RH}_{m \times q}^\infty$ be stable rational matrix functions, and assume that $[G(z)\ K(z)]$ is given by the minimal realization (1.5). Furthermore, assume that the positivity condition (1.2) is satisfied. Then, by Theorem 1.1 there exists a $F \in \mathcal{RH}_{m \times r}^\infty$, for some $r \leq m$, such that $F$ admits a realization of the form (1.7) and conditions (i), (ii) in Theorem 1.1 are satisfied.

\textbf{Lemma 4.1.} Condition (i) in Theorem 1.1 implies that (1.15) holds.

\textbf{Proof.} For each $z$ in the open unit disc $\mathbb{D}$, let $\varphi_z$ be the operator defined by

\begin{equation}
\varphi_z = [I_v \ zI_v \ z^2I_v \ \cdots ]^*: \mathbb{C}^v \to \ell^2_+(\mathbb{C}^v).
\end{equation}

Here $v$ is an arbitrary positive integer, the value of which will be clear from the context.

Notice that

$$T_G^*\varphi_z = \varphi_z G(z)^*, \quad T_K^*\varphi_z = \varphi_z K(z)^*, \quad T_F^*\varphi_z = \varphi_z F(z)^*,$$

$$\varphi_z^*W_{\text{obs}} = C(I - zA)^{-1}, \quad \varphi_\lambda^*\varphi_z = \frac{1}{1 - \lambda z}I.$$

It follows that for each $z$ and $\lambda$ in $\mathbb{D}$ we have

\begin{equation}
\varphi_\lambda^*(T_GT_G^* - T_KT_K^* - T_FT_F^*)\varphi_z = \frac{G(\lambda)G(z)^* - K(\lambda)K(z)^* - F(\lambda)F(z)^*}{1 - \lambda z},
\end{equation}

$$\varphi_\lambda^*(W_{\text{obs}}(P_3 + P_2 - P_1))W_{\text{obs}}^*\varphi_z = C(I_n - \lambda A)^{-1}(P_3 + P_2 - P_1)(I_n - \bar{z}A^*)_\lambda^{-1} C^*.$$

But then condition (i) in Theorem 1.1 implies that

$$G(\lambda)G(z)^* - K(\lambda)K(z)^* - F(\lambda)F(z)^* = (1 - \lambda z)C(I_n - \lambda A)^{-1}(P_3 + P_2 - P_1)(I_n - \bar{z}A^*)_\lambda^{-1} C^* \quad (z, \lambda \in \mathbb{D}).$$

Recall that $\Lambda(z) = C(I_n - zA)^{-1}(P_3 + P_2 - P_1)^{1/2}$. Hence the preceding identity is just the same as the identity (1.15). \hfill $\square$

\textbf{Proof of Theorem 1.2.} Recall that

$$G(z) = D_1 + zC(I_n - zA)^{-1}B_1, \quad K(z) = D_2 + zC(I_n - zA)^{-1}B_2,$$

$$F(z) = D_3 + zC(I_n - zA)^{-1}B_3,$$

$$\Lambda(z) = C(I_n - zA)^{-1}\Upsilon, \quad \text{where} \quad \Upsilon = (P_3 + P_2 - P_1)^{1/2}.$$
Next put 

\[ M(z) = \begin{bmatrix} z\Lambda(z) & G(z) \end{bmatrix}, \quad N(z) = \begin{bmatrix} \Lambda(z) & K(z) & F(z) \end{bmatrix}. \]

Using the state space realizations of \( G, K, F, \) and \( \Lambda \) given above we see that \( M \) and \( N \) admit the following realizations:

\[
M(z) = \begin{bmatrix} 0 & D_1 \\ \end{bmatrix} + zC(I_n - zA)^{-1} \begin{bmatrix} T & B_1 \end{bmatrix}, \\
N(z) = \begin{bmatrix} CT & D_2 & D_3 \end{bmatrix} + zC(I_n - zA)^{-1} \begin{bmatrix} AT & B_2 & B_3 \end{bmatrix}.
\]

Furthermore, the identity (4.5) tells us that 

\[
M(\lambda)M(z)^* = N(\lambda)N(z)^* \quad (z, \lambda \in \mathbb{D}).
\]

This allows us to apply Lemma 4.2 below. It follows that the linear operator

\[
\text{admit the following realizations:}
\]

\[
N(z) = \begin{bmatrix} C(T) & D_2 & D_3 \end{bmatrix} + zC(I_n - zA)^{-1} \begin{bmatrix} A(T) & B_2 & B_3 \end{bmatrix}.
\]

The first identity implies that \( \Lambda(z) = G(z)\gamma(I - z\alpha)^{-1} \). Using this expressing for \( \Lambda(z) \) in the other two identities yields

\[
G(z)(\delta_1 + z\gamma(I - z\alpha)^{-1})\delta_1 = F(z) \quad \text{and} \quad G(z)(\delta_2 + z\gamma(I - z\alpha)^{-1})\beta_2.
\]

Since \( U \) is a contraction, it follows from the bounded real lemma in systems theory or the Sz.-Nagy-Foias model theory in operator theory (see also Theorem 5.2 in [1]) that the matrix function \( \begin{bmatrix} X & \Psi \end{bmatrix} \), with \( X \) and \( \Psi \) defined as in (4.1) and (4.2), respectively, satisfies \( \| [X \Psi] \|_\infty \leq 1 \), in particular, \( X \) is a rational contractive function on \( \mathbb{D} \). Furthermore, the first identity in (4.3) implies that \( X \) satisfies the Leech equation \( GX = K \). In the same way, using the second identity in (4.3), one shows that the function \( \Psi \) in (4.4) has the desired properties.

In the next lemma \( M \) and \( N \) are stable rational matrix functions, \( M \in \mathcal{RH}_{m \times k}^\infty \) and \( N \in \mathcal{RH}_{m \times \ell}^\infty \). We assume that \( M \) and \( N \) are given by the stable realizations:

\[
\begin{align*}
M(z) &= D_M + zC(I_n - zA)^{-1}B_M, \\
N(z) &= D_N + zC(I_n - zA)^{-1}B_N.
\end{align*}
\]

In particular, \( A \) is stable.

**Lemma 4.2.** Let \( M \in \mathcal{RH}_{m \times k}^\infty \) and \( N \in \mathcal{RH}_{m \times \ell}^\infty \) be given by the stable realizations (4.4) and (4.5), respectively, and let \( W_{obs} \) be the observability operator defined by the pair \( \{ C, A \} \). Put \( Y = W^*_{obs}W_{obs} \). If

\[
M(\lambda)M(z)^* = N(\lambda)N(z)^* \quad (z, \lambda \in \mathbb{D}),
\]

then the \( k \times \ell \) matrix \( U = (D^*_M D_M + B_M^* Y B_M)^*(D^*_M D_N + B_M^* Y B_N) \) is a partial isometry and \( M(z)U = N(z) \) for all \( z \) in \( \mathbb{D} \).

**Proof.** Let \( \Omega_M \) and \( \Omega_N \) be the operators defined by

\[
\Omega_M = \begin{bmatrix} D_M \\ W_{obs} B_M \end{bmatrix} : \ell_2^k(\mathbb{C}^m), \quad \Omega_N = \begin{bmatrix} D_N \\ W_{obs} B_N \end{bmatrix} : \ell_2^\ell(\mathbb{C}^m).
\]
For each \( z \) in the open unit disc \( \mathbb{D} \), let \( \varphi_z \) be the operator defined by (4.1). Then \( M(z)^* = \Omega_M^* \varphi_z \) and \( N(z)^* = \Omega_N^* \varphi_z \) for all \( z \) in \( \mathbb{D} \). Thus for \( \lambda \) and \( z \) in \( \mathbb{D} \), with \( v \) and \( w \) in \( \mathbb{C}^m \), we have
\[
\langle \Omega_M \Omega_M^* \varphi_z w, \varphi_z v \rangle = \langle \Omega_M^* \varphi_z w, \Omega_M^* \varphi_z v \rangle = \langle M(\lambda)^* w, M(z)^* v \rangle = \langle w, M(\lambda)M(z)^* v \rangle
\]
and
\[
\langle \Omega_N \Omega_N^* \varphi_z w, \varphi_z v \rangle = \langle \Omega_N^* \varphi_z w, \Omega_N^* \varphi_z v \rangle = \langle N(\lambda)^* w, N(z)^* v \rangle = \langle w, N(\lambda)N(z)^* v \rangle.
\]
Because \( \{ \varphi_z \mathbb{C}^m \mid z \in \mathbb{D} \} \) spans a dense set in \( \ell_+^2(\mathbb{C}^m) \), we see that condition (4.6) implies that
\[
(4.7) \quad \Omega_M \Omega_M^* = \Omega_N \Omega_N^*.
\]
It follows that there exists a unique partial isometry \( U \) mapping \( \mathbb{C}^k \) into \( \mathbb{C}^k \) with initial space \( \text{Im} \Omega_N \) and final space \( \text{Im} \Omega_M \) such that \( \Omega_M U = \Omega_N \). In fact this unique isometry \( U \) is given by \( (\Omega_M \Omega_M^*) \dagger \Omega_M \Omega_N \), where \( (\Omega_M \Omega_M^*) \dagger \) stands for the Moore-Penrose inverse of the finite dimensional selfadjoint operator \( \Omega_M \Omega_M^* \).

Finally, using \( Y = W_{obs}^* W_{obs} \) and the definitions of \( \Omega_M \) and \( \Omega_N \) in the beginning of the proof, we obtain
\[
\Omega_M^* \Omega_M = \begin{bmatrix} D_M^* & B_M^* W_{obs}^* \end{bmatrix} \begin{bmatrix} D_M & W_{obs} B_M \end{bmatrix} = D_M^* D_M + B_M^* W_{obs} B_M = D_M^* D_M + B_M^* Y B_M.
\]
and
\[
\Omega_N^* \Omega_N = \begin{bmatrix} D_N^* & B_N^* W_{obs}^* \end{bmatrix} \begin{bmatrix} D_N & W_{obs} B_N \end{bmatrix} = D_N^* D_N + B_N^* W_{obs} B_N = D_N^* D_N + B_N^* Y B_N.
\]
Thus \( U = (D_M^* D_M + B_M^* Y B_M)^\dagger (D_N^* D_N + B_N^* Y B_N) \) as desired. \( \square \)

5. The strictly positive case

We begin by specifying Theorem 2.1 for the case when the values of \( R \) on the unit circle are strictly positive. If \( \Xi \) is an invertible operator on a Hilbert space, then \( \Xi^{-1} \) denotes the adjoint of \( \Xi \).

**Proposition 5.1.** Let \( R \) be as in (2.1). Assume that \( R(\zeta) \) is strictly positive for each \( \zeta \in \mathbb{T} \), and let \( \Phi \in \mathcal{RH}_{\infty}^\infty \) be an outer spectral factor of \( R \), as in Theorem 2.1. Then \( T_\Phi \) is invertible, and
\[
\chi_\Phi = \mathbb{C}^n, \quad C_\Phi = E^* T_\Phi^{-1} W_{obs}, \quad W_{\Phi, obs} = T_\Phi^{-1} W_{obs}.
\]
Here \( E \) is the embedding of \( \mathbb{C}^r \) onto the first coordinate space of \( \ell_+^2(\mathbb{C}) \). The observability Gramian \( Q_\Phi \) determined by the pair \( \{ C_\Phi, A \} \) is also given by \( Q_\Phi = W_{obs}^* T_R^{-1} W_{obs} \), the matrix \( R_0 - \Gamma^* Q_\Phi \Gamma \) is strictly positive, and
\[
(5.2) \quad C_\Phi = \Phi(0)(R_0 - \Gamma^* Q_\Phi \Gamma)^{-1} (C - \Gamma^* Q_\Phi A).
\]
Finally, in this case, we may assume without loss of generality that \( \Phi(0) \) is given by
\[
(5.3) \quad \Phi(0) = (R_0 - \Gamma^* Q_\Phi \Gamma)^{1/2}.
\]
Proof. Since \( R(\zeta) \) is strictly positive for each \( \zeta \in T \), the outer factor \( \Phi \) is an invertible outer factor, which is equivalent to \( T_\Phi \) being invertible. In particular, \( T_\Phi \) is surjective. Thus for each \( x \in \mathbb{C}^n \) the vector \( W_{\text{obs}}x \) belongs to \( \text{Im} \, T_\Phi \). This shows that the space \( X_\Phi \) is equal to the full space \( \mathbb{C}^n \). The two other identities in (5.1) then follow from (2.5). Next, one computes that
\[
Q_\Phi = W_{\Phi, \text{obs}}^* W_{\Phi, \text{obs}} = W_{\text{obs}} T_\Phi^{-1} T_{\Phi}^* W_{\text{obs}} = W_{\text{obs}}^* (T_\Phi T_{\Phi})^{-1} W_{\text{obs}} = W_{\text{obs}}^* T_R^{-1} W_{\text{obs}}.
\]
This proves \( Q_\Phi = W_{\text{obs}}^* T_R^{-1} W_{\text{obs}} \). Since \( r = m \) and \( \Phi(0) \) is invertible, the identity (2.7) shows that \( R_0 - \Gamma^* Q_\Phi \Gamma \) is strictly positive. Similarly, the identity (2.6) with (5.3) yields (5.2).

It remains to prove the final statement. From (2.7) and the fact that \( \Phi(0) \) is invertible it follows that the polar decomposition of \( \Phi(0) \) is given by \( \Phi(0) = U (R_0 - \Gamma^* Q_\Phi \Gamma)^{1/2} \), where \( U \) is unitary. Recall that \( \Phi \) is uniquely determined up to a unitary matrix from the left. Thus without loss of generality we may replace \( \Phi \) by \( U^{-1} \Phi \), and then (5.3) holds. \( \square \)

The results listed in the above proposition also follow from Theorem 1.1 in [13]; cf., Section 3 in [14]. To be more specific let \( R \) be as in (2.1), and consider the associate algebraic Riccati equation
\[
Q = A^* Q A + (C - \Gamma^* Q A)^* (R_0 - \Gamma^* Q \Gamma)^{-1} (C - \Gamma^* Q A).
\]
An \( n \times n \) matrix \( Q \) is called a stabilizing solution to this algebraic Riccati equation if
\begin{enumerate}
  \item \( Q \) is a solution to (5.4),
  \item \( R_0 - \Gamma^* Q \Gamma \) is strictly positive,
  \item the matrix \( A - \Gamma(R_0 - \Gamma^* Q \Gamma)^{-1} (C - \Gamma^* Q A) \) is stable.
\end{enumerate}
It turns out that if the algebraic Riccati equation (5.4) admits a stabilizing solution \( Q \), then this solution is nonnegative and unique. By the symmetric version of Theorem 1.1 in [12] (see Section 14.7 in [5] or Sections 10.2 and 10.2 in [12]) we know that the following are equivalent:
\begin{enumerate}
  \item The values of the function \( R \) on \( T \) are strictly positive.
  \item The function \( R \) admits an invertible outer spectral factor \( \Phi \), i.e., the outer spectral factor \( \Phi \) is square and \( T_\Phi \) is invertible.
  \item The algebraic Riccati equation (5.4) admits a stabilizing solution \( Q \).
\end{enumerate}
Moreover, in this case, the following holds:
\begin{enumerate}
  \item The invertible outer spectral factor \( \Phi \) of \( R \) is given by
    \[
    \Phi(z) = \Phi(0) + z C_0 (I_n - zA)^{-1} \Gamma,
    \]
    where
    \[
    \Phi(0) = (R_0 - \Gamma^* Q \Gamma)^{1/2},
    \]
    \[
    C_0 = \Phi(0) (R_0 - \Gamma^* Q \Gamma)^{-1} (C - \Gamma^* Q A).
    \]
  \item The unique stabilizing solution \( Q \) to (5.4) is given by
    \[
    Q = W_{\text{obs}}^* T_R^{-1} W_{\text{obs}}.
    \]
Finally, if in addition \( \{C, A\} \) is observable, then \( W_{\text{obs}} \) is one to one, and thus, \( Q \) is strictly positive.

From Proposition \( 5.1 \) above we know that \( Q_{\Phi} = W_{\text{obs}}^* T_R^{-1} W_{\text{obs}} \). But then \( (5.7) \) shows that the stabilizing solution \( Q \) of the Riccati equation \( (5.4) \) coincides with the observability Gramian \( Q_{\Phi} \). Furthermore, \( C_{\Phi} = C_0 \) and the outer spectral factor \( \Phi \) in Proposition \( 5.1 \) is equal to the outer spectral factor \( \Phi \) given by \( (5.5) \). Finally, assuming \( \{C, A\} \) is observable and using the first identity in \( (5.1) \), we conclude from \( (2.21) \) that \( \Omega_{\Phi} = Q_{\Phi} \), and hence \( (2.22) \) tells us that \( \Delta = Q_{\Phi}^{-1} \).

Applied to the Leech problem \( (1.1) \) the above results yield the following algorithm to compute a solution when \( R \) admits an invertible outer spectral factor. This algorithm can be easily programmed in Matlab.

**Procedure 5.2.** Let \( G \in \mathcal{RH}_{m \times p}^\infty \) and \( K \in \mathcal{RH}_{m \times q}^\infty \) be given by the minimal realization \( (1.5) \). Consider the algebraic Riccati equation \( (5.4) \) where \( R_0 \) and \( \Gamma \) are now given by \( (3.6) \) and \( (3.7) \), respectively.

(i) Assume that there exists a stabilizing solution \( Q \) to the algebraic Riccati equation \( (5.4) \), or equivalently, the values of \( R \) on the unit circle are strictly positive.

(ii) Then there exists a stable rational matrix solution \( X \) to the Leech problem \( (1.1) \) if and only if \( Q^{-1} \geq P_1 - P_2 \). Therefore in what follows we assume that \( Q^{-1} \geq P_1 - P_2 \).

If (i) and (ii) hold, then such a solution \( X \) can be computed by the following steps:

- Let \( \Phi(0) \) and \( C_{\Phi} \) be the matrices defined by
  \[
  \Phi(0) = (R_0 - \Gamma^* Q \Gamma)^{1/2} \quad \text{and} \quad C_{\Phi} = \Phi(0)(R_0 - \Gamma^* Q \Gamma)^{-1}(C - \Gamma^* Q A).
  \]

- Find matrices \( B_{\Theta} \) and \( D_{\Theta} \) such that
  \[
  \begin{bmatrix}
  A^* & C_{\Phi}^*
  \\
  B_{\Theta}^* & D_{\Theta}^*
  \end{bmatrix}
  \begin{bmatrix}
  Q & 0 \\
  0 & I_r
  \end{bmatrix}
  \begin{bmatrix}
  A & B_{\Theta} \\
  C_{\Phi} & D_{\Theta}
  \end{bmatrix}
  =
  \begin{bmatrix}
  Q & 0 \\
  0 & I_r
  \end{bmatrix}.
  \]

- Set \( P_3 = Q^{-1} \) and \( B_3 = B_{\Theta} \), and put
  \[
  D_3 = \Phi(0)^* D_{\Theta} + \Gamma^* Q B_{\Theta}.
  \]

- Use Theorem \( 1.2 \) to compute \( U \) in \( (1.11) \). Then a stable rational matrix solution \( X \) to \( (1.1) \) is given by \( X(z) = \delta_1 + z \gamma (I - z \alpha)^{-1} \beta_1 \), as in \( (1.13) \).

- The function \( F(z) = D_3 + z C (I_n - A)^{-1} B_3 \) satisfies items (i) and (ii) of Theorem \( 1.1 \).

- Finally, \( \Psi(z) = \delta_2 + z \gamma (I - z \alpha)^{-1} \beta_2 \) is a stable rational matrix function satisfying \( G \Psi = F \) and \( \| \Psi \|_{\infty} \leq 1 \); see Theorem \( 1.2 \).

6. Example

To gain some further insight into the solution obtained by the algorithm described by Procedure \( 5.2 \), let us consider the simple case when

\[
(6.1) \quad G(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad K(z) = \frac{z}{2}.
\]

Let \( \tau \) be any function in \( H^\infty \) satisfying \( \| \tau \|_{\infty} \leq 1 \). One can easily see that

\[
(6.2) \quad X(z) = \frac{z}{2 \sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\sqrt{3}}{2 \sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tau(z), \quad |z| < 1,
\]
is a solution to the corresponding Leech problem (1.1). In fact, all possible solutions are obtained in this way. Note that the problem has infinitely many stable rational solutions.

Here we will see that our algorithm yields the particular solution $X$ in (6.2) with $\tau$ identically equal to zero, that is,

$$X(z) = \frac{z}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (6.3)$$

(It turns out that $X$ in (6.3) is also the minimal $H^\infty$ and the minimal $H^2$ solution to $GX = K$.) For $G$ and $K$ in (6.1), a state space realization for $[G \ K]$ is given by (1.3) where

$$A = 0, \ C = 1, \ B_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \ D_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}, \ B_2 = \frac{1}{2}, \ D_2 = 0. \quad (6.4)$$

With this choice the realization of $[G \ K]$ is minimal. The controllability Gramians in (1.9) are given by $P_1 = 0$ and $P_2 = 1/4$. The function $R$ in (1.10) is defined by $R(z) = G(z)G(1/\bar{z})^* - K(z)K(1/\bar{z})^*$ and $\Gamma = 0$. Hence $\Phi(z) = \sqrt{3}/2$, the subspace $X_\Phi = \mathbb{C}$, and $C_\Phi = 2/\sqrt{3}$; see (2.6). The inner function $\Theta$ is given by $\Theta(z) = z$, and $B_3 = B_\Theta = \sqrt{3}/2$, while $D_\Theta = 0$. Moreover, $F(z) = \Phi(1/\bar{z})^*\Theta(z) = z\sqrt{3}/2$ and $D_3 = 0$. The controllability Gramian $P_3$ of the pair $\{A, B_3\}$ is given by $P_3 = 3/4$. Therefore $P_3 + P_2 - P_1 = 1$ and $\Upsilon = 1$. According to item (i) in Theorem 1.1, the operator $T_G T_G^*-T_K T_K^*$ is nonnegative.

Now we can use (1.12) to compute a contractive solution to $GX = K$. In this case, the observability Gramian $Y$ for the pair $\{C, A\}$ is given by $Y = 1$, and $U = V^+ V_1$ where

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{on } \mathbb{C}^3 \quad \text{and} \quad V_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} \quad \text{on } \mathbb{C}^3.$$

Note that $V$ is an orthogonal projection, and thus $V^+ = V$. A simple calculation shows that

$$U = V^+ V_1 = VV_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}. \quad (6.5)$$

Hence

$$\alpha = 0, \quad \gamma = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad \beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{\sqrt{3}}{2}, \quad \delta_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \delta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (6.6)$$

Therefore

$$X(z) = \delta_1 + z\gamma(I_1 - z\alpha)^{-1}\beta_1 = \frac{z}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a stable rational matrix solution to the Leech problem (1.1) with $G$ and $K$ as in (6.1). Finally,

$$\Psi(z) = \delta_2 + z\gamma(I_1 - z\alpha)^{-1}\beta_2 = \frac{z\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a stable rational matrix solution to the Leech problem (1.1) with $G$ and $K$ as in (6.1).
is a contractive stable rational matrix solution to $G\Psi = F$.

**Remark 6.1.** The example presented in this section is of a special kind. Recall that $R(z) \equiv 3/4$, and thus $R$ is strictly positive on $T$. Hence in constructing a rational solution to the Leech problem we could have used the procedure described in Procedure 5.2 to get the solution $X$. Note that in this case, given the data (6.4) and the equalities $R_0 = 3/4$ and $\Gamma = 0$, the Riccati equation (5.4) reduces to $Q = 4/3$. The procedure outlined in Procedure 5.2 then yields the same solution $X$ as the one obtained above.

Another special feature of the above example is the fact that $P_2 - P_1 = 1/4$ is positive. This implies that for any stable rational function $F$ such that $F(z)F(\bar{z}^{-1})^* = R(z) = 3/4$, not only the one constructed above, the operator $T_GT_G^* - T_KT_K^* - T_FT_F^*$ is non-negative. This fact follows from the following variant of (3.11):

$$T_GT_G^* - T_KT_K^* - T_FT_F^* = HFH_F^* + W_{obs}(P_2 - P_1)W_{obs}^*.$$ 

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