Spectrum Generating Algebra
and
No-Ghost Theorem
for
Fermionic Massive String

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Abstract

The covariant operator quantization of the ordinary free spinning BDH string modified by adding the supersymmetric Liouville sector is analysed in the even target space dimensions $d = 2, 4, 6, 8$. The spectrum generating algebra for this model is constructed and a general version of the no-ghost theorem is proven. A counterpart of the GSO projection leads to a family of tachyon free unitary free string theories. One of these models is equivalent to the non-critical Ramond-Neveu-Schwarz spinning string truncated in the Neveu-Schwarz sector to the tachyon free eigenspace of the fermion parity operator.
# Introduction

The relevance of the (super-)Liouville theory for a proper description of quantum string in non-critical dimensions was first pointed out in Polyakov’s celebrated papers on conformal anomaly in bosonic [1] and fermionic [2] string models. This observation inspired extensive studies of the quantum Liouville [3] and the super-Liouville theory [4]. In spite of significant progress in $D \leq 1$ non-critical strings [5] and superstrings [6] and the subsequent development of the corresponding Liouville [7] and the super-Liouville [8] conformal field theories the question whether the famous $D = 1$ barrier can be overcome remains open.

There are basically two different approaches to this problem. One of them developed by Gervais and collaborators [9] has already brought many promising results but technicalities involved make it difficult to go beyond topological models [10]. The second approach recently advocated by Polyakov [11] identifies the Liouville degree of freedom with an extra fifth curved dimension. This idea received unexpected support from the recently discovered relation between the fundamental IIB superstring in $AdS_5 \times S^5$ background and the supersymmetric Yang-Mills theory in four dimensions [12]. It is believed that this approach may lead to a plausible scenario for non-critical strings but there are still many difficult open problems [13].

Whatever an ultimate understanding of the Liouville dynamics would be it seems reasonable to analyse less ambitious and more elementary questions about the role of the Liouville degrees of freedom in the quantum mechanics of free strings. The first attempt in this direction was made long time ago by Marnelius [14]. He considered the standard string model modified by adding the Liouville sector. Assuming some general features of the quantum Liouville dynamics he was able to show [15] that the non-zero Liouville modes can be identified with the longitudinal Brower excitations of the non-critical Nambu-Goto string [16].

This modification of bosonic string has been recently reconsidered under the assumption that the bulk and boundary cosmological constants vanish [17]. This apparently drastic simplification leads however to a nontrivial free string model exhibiting many interesting features [17]. In particular the constraint eliminating the Liouville zero mode appears as a consistency condition for the variational principle. This excludes for instance the interpretation of the Liouville sector in terms of an extra target space dimension.

In the covariant quantization the Liouville sector is described by a free 2-dimensional conformal field theory of the Fégol-Fuchs type [18]. The corresponding no-ghost theorem [17] admits a family of new non-critical free bosonic quantum strings called massive strings for the properties of their spectra - all states except the tachyonic ground state are massive. One of them with the largest possible space of null states is called the critical massive string. It is completely equivalent to the non-critical Nambu-Goto string [16]. In this special case one can also develop the light-cone formulation and calculate the particle content of the model [19].

The problem addressed in the present paper is the covariant operator quantization of the fermionic counterpart of open massive string and the general no-ghost theorem for this model. The paper is organised as follows. In Section 2 the classical model of massive string is introduced. In Section 3 we present the covariant quantization of the model. In Section 4 the space of physical states is explicitly constructed in terms of DDF operators.

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3The model is critical with respect to the structure of null states rather than the dimension $D$ of the target space which may vary in the range $1 < D < 25$. 

This is our main result. It is used in Section 5 to prove a general version of the no-ghost theorem. In Section 6 we introduce a non-critical counterpart of the GSO projection and construct a family of tachyon free unitary fermionic string models.

2 Classical theory

One can introduce the fermionic massive string as a world-sheet $N = 1$ supersymmetric extension of the bosonic model. It is defined by the spinning string covariant action $[20]$ supplemented by the supersymmetric Liouville action $[21, 22, 23]$ with vanishing cosmological terms. In the superspace notation $[24]$ it takes the form:

$$S[E, \Phi, X] = -\frac{\alpha}{2\pi} \int_M d^2 z \, d^2 \theta \, E D_\alpha X^a D^\alpha X^\mu$$

$$-\frac{\beta}{2\pi} \int_M d^2 z \, d^2 \theta \, E \left( D_\alpha \Phi D^\alpha \Phi - 2i S_E \Phi \right) ,$$

where

$$X^\mu(x, \theta) = x^\mu(x) + i \vec{\theta} \psi^\mu + \frac{i}{2} \vec{\theta} D \mu \ , \ \Phi(x, \theta) = \varphi(x) + i \vec{\theta} \psi^L + \frac{i}{2} \vec{\theta} D^L$$

are the embedding and the Liouville 2-dim real scalar superfields, respectively. For the 2-dim supergravity sector we use Howe’s notations and conventions $[24]$ ($E^a_M$ denotes the supervierbein and $S_E$ is the corresponding curvature scalar superfield). In all three sectors we assume unique supersymmetric extensions $[25, 21]$ of the boundary conditions of the bosonic massive string $[17]$.

The action is invariant under superdiffeomorphisms and local Lorentz transformations. Due to the absence of cosmological terms it is also invariant with respect to a special class of superconformal transformations

$$E^a_M \rightarrow e^\Sigma E^a_M \ , \ E^a_M \rightarrow e^{\frac{\Sigma}{2}} E^a_M - \frac{i}{2} \epsilon^{\Sigma} E^a_M (\gamma_a)^{\alpha\beta} D_\beta \Sigma ,$$

with scaling superfields $\Sigma$ satisfying the equation $D_\alpha D^\alpha \Sigma = 0$. Due to this extra symmetry the supergravity sector can be completely gauged away. In the flat superconformal gauge $E^a_M = \tilde{E}^a_M$

$$\tilde{E}_m = \delta_m^a \ ; \ \tilde{E}_m^a = 0 \ ; \ \tilde{E}_\mu^a = i \theta^\lambda (\gamma^a)^{\lambda\mu} \ ; \ \tilde{E}_\mu^a = 0 ,$$

the model is given by the following system of equations

$$D_\alpha D^\alpha X^\mu = D_\alpha D^\alpha \Phi = 0 ,$$

$$\alpha (\gamma^b \gamma^a)^{\beta} D_\beta X^\mu \partial_\mu X^a + \beta (\gamma^b \gamma^a)^{\beta} (D_\beta \Phi \partial_\mu \Phi - 2 D_\beta \partial_\mu \Phi) = 0$$

$$\int_{\partial M_i} ds \, d^2 \theta \theta^\alpha n_a \gamma^\alpha D_\alpha \Phi = 0 ,$$

where $\partial M_i$ denotes the ”initial” world-sheet boundary. The origin of the last constraint is essentially the same as in the bosonic massive string $[17]$. Proceeding to the components and eliminating the auxiliary fields $D^\mu, D^L$ one has

$$\partial^a \partial_\alpha x^\mu = \gamma^a \partial_\alpha \psi^\mu = \partial^a \partial_\alpha \varphi = \gamma^a \partial_\alpha \psi^L = 0$$

$$J^b_\alpha = T_{ab} = \int_{\partial M_i} ds \, n^a \partial_\alpha \varphi = 0 ,$$
where the supersymmetry current $J^a_\alpha$ and the energy momentum tensor $T_{ab}$ are given by

$$J^a_\alpha \equiv \frac{\alpha}{\pi} \partial_\mu (\gamma^\mu \gamma^a \psi_\mu)_\alpha + \frac{\beta}{\pi} \left( \partial_\nu \varphi (\gamma^\nu \gamma^a \psi^L)_\alpha - \eta^{ab} \partial^a \psi^L_\alpha \right) ,$$

$$T_{ab} \equiv \frac{\alpha}{\pi} \left( \partial_a \varphi \partial_b \varphi - \frac{1}{2} \eta_{ab} \partial^\mu \varphi \partial_\mu \varphi - 4 \varphi \psi^L - \frac{1}{4} \eta_{ab} \psi^L \gamma^c \partial_c \psi^L \right) .$$

In the flat superconformal gauge the supersymmetric boundary conditions [23, 21] take the form

$$x^\mu (\tau, 0) = x^\mu (\tau, \pi) = 0 , \quad \varphi (\tau, 0) = \varphi (\tau, \pi) = 0 ,$$

$$\psi_+^\mu (\tau, 0) = \psi_-^\mu (\tau, 0) , \quad \psi_+^L (\tau, 0) = \psi_-^L (\tau, 0) ,$$

$$\psi_+^\mu (\tau, \pi) = (-1)^\epsilon \psi_+^{\mu} (\tau, \pi) , \quad \psi_+^L (\tau, \pi) = (-1)^\epsilon \psi_+^{L} (\tau, \pi) ,$$

where $\epsilon = 0$ corresponds to the Rammond, and $\epsilon = 1$ to the Neveu-Schwarz sector; $\psi_\pm$ denote components of 2-dim spinor in the basis of $\gamma^0 \gamma^1$ eigenvectors.

Introducing holomorphic variables

$$x^\mu \pm x^\mu = \frac{1}{\sqrt{\alpha}} \sum_{k \in \mathbb{Z}} d_k e^{-ik(\tau + \sigma)} , \quad \varphi \pm \varphi' = \frac{1}{\sqrt{\beta}} \sum_{k \in \mathbb{Z}} c_k e^{-ik(\tau + \sigma)} ,$$

$$\psi_+^\mu = \frac{1}{\sqrt{\alpha}} \sum_{r \in \mathbb{Z} + \frac{\pi}{\alpha}} b_r e^{-ir(\tau + \sigma)} , \quad \psi_+^L = \frac{1}{\sqrt{\beta}} \sum_{r \in \mathbb{Z} + \frac{\pi}{\beta}} d_r e^{-ir(\tau + \sigma)} ,$$

$$q_0^\mu = \frac{\alpha}{\pi} \int_0^\pi d\sigma \, \varphi^\mu (\sigma, 0) , \quad q_0^L = \frac{\beta}{\pi} \int_0^\pi d\sigma \, \varphi (\sigma, 0) ,$$

satisfying the canonical graded Poisson bracket relations

$$\{a^\mu, a^\nu_n\} = i m \eta^{\mu\nu} \delta_{m, -n} , \quad \{c_m, c_n\} = i m \delta_{m, -n} ,$$

$$\{a_0^\mu, q_0^\nu\} = \eta^{\mu\nu} , \quad \{c_0^\mu, q_0^L\} = 1 ,$$

$$\{b^\mu, b^\nu_r\} = i m \eta^{\mu\nu} \delta_{m, -n} , \quad \{d_m, d_n\} = i \delta_{m, -n} ,$$

one can rewrite the constraints in the following standard form

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} a_{n+m} \cdot a_{n-m} + (1 - \epsilon) a_0^2 \delta_{m, 0} + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{\pi}{2}} r b_{-r} \cdot b_{r+m}$$

$$+ \frac{1}{2} \sum_{n \in \mathbb{Z}} c_{n} c_{n+m} + 2i \sqrt{\beta} m c_0 + 2 \beta \delta_{m, 0} + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{\pi}{2}} r d_{-r} d_{r+m} \tag{2} ,$$

$$G_r = \sum_{n \in \mathbb{Z}} a_{n} \cdot b_{n+r} + \sum_{n \in \mathbb{Z}} c_{n} d_{n+r} + 4i \sqrt{\beta} r d_r \tag{3} ,$$

$$c_0 = 0 .$$

The Poincare generators are represented by

$$P^\mu = \sqrt{\alpha} a_0^\mu ,$$

$$M^{\mu\nu} = x_0^\mu P^\nu - x_0^\nu P^\mu - i \sum_{n > 0} \frac{1}{2} \left( a_{-n}^\mu a_n^\nu - a_n^\mu a_{-n}^\nu \right)$$

$$- i \sum_{r > 0} \left( b_{-r}^\mu b_r^\nu - b_r^\mu b_{-r}^\nu \right) - \epsilon i b_0^\mu b_0^\nu \tag{4} ,$$

where $x_0^\mu = \frac{1}{\sqrt{\alpha}} a_0^\mu$. 

3
3 Covariant quantization

Following standard prescriptions of covariant quantization we start with the algebra of canonical commutation and anticommutation relations

\[
[a^\mu_n, a^\nu_m] = m\eta^\mu_\nu \delta_{m,-n}, \quad [c_m, c_n] = m \delta_{m,-n},
\]

\[
[a^\mu_0, q^\nu_0] = -i\eta^\mu_\nu, \quad [c_0, q^L_0] = -i
\]

\[
\{b^\mu_r, b^\nu_s\} = r\eta^\mu_\nu \delta_{r,-s}, \quad \{d_r, d_s\} = \delta_{r,-s},
\]

supplemented by the conjugation properties

\[
(a_0^\mu)^+ = a^\mu_0, \quad (q_0^\nu)^+ = q^\nu_0, \quad (c_0)^+ = c_0, \quad (q_0^L)^+ = q^L_0,
\]

\[
(a_m^\mu)^+ = a_{-m}^\mu, \quad (b_r^\mu)^+ = b_{-r}^\mu, \quad (c_m)^+ = c_{-m}, \quad (d_r)^+ = d_{-r},
\]

where \(m \in \mathbb{Z}; r \in \mathbb{Z} + \frac{1}{2}\). Let us denote by \(F_\epsilon(p,p^L)\) the Fock space generated by the algebra of non-zero modes out of the unique vacuum state \(\Omega_\epsilon(p,p^L)\) satisfying

\[
a^\mu_m \Omega_\epsilon(p,p^L) = c_m \Omega_\epsilon(p,p^L) = 0, \quad m > 0,
\]

\[
b^\mu_r \Omega_\epsilon(p,p^L) = d_r \Omega_\epsilon(p,p^L) = 0, \quad r > 0,
\]

\[
P^\mu_\epsilon(p,p^L) = p^\mu \Omega_\epsilon(p,p^L),
\]

\[
P^L_\epsilon(p,p^L) = p^L \Omega_\epsilon(p,p^L).
\]

The space of states is a direct sum of the pseudo-Hilbert spaces \(H_\epsilon(p,p^L)\) along \((d + 1)\)-dimensional spectrum of the momentum operators \(P^\mu, P^L \equiv \sqrt{\beta} c_0\)

\[
H_\epsilon = \int d^d p d^L p \ H_\epsilon(p,p^L).
\]

In the Neveu-Schwarz sector \((\epsilon = 1)\)

\[
H_1(p,p^L) = F_1(p,p^L).
\]

In the Rammond sector \((\epsilon = 0)\) the fermionic zero modes \(b^\mu_0, d_0\) form the real Clifford algebra \(\mathcal{C}(d,1)\) corresponding to the metric of \((d,1)\) signature. If one requires a well defined fermion parity operator the zero mode sector of \(H_0(p,p^L)\) must carry an irreducible representation of the real Clifford algebra \(\mathcal{C}(d+1,1)\).

For the sake of simplicity we restrict ourselves to the even dimensions \(d = 2, 4, 6, 8\) of the target space. As we shall see in Sect.5. higher dimensions are excluded by the no-ghost theorem. Let \(D(d+2)\) be the space of an irreducible representation of the complex extension \(\mathcal{C}^C(d+2) = \mathcal{C}(d,1) \otimes \mathcal{C}\) of the real Clifford algebra \(\mathcal{C}(d+1,1)\). Note that for even \(d\) the algebra \(\mathcal{C}^C(d+2)\) is simple and there is only one such representation of complex dimension \(2^{\frac{d+2}{2}}\). \(D(d+2)\) regarded as a representation of \(\mathcal{C}(d+1,1)\) is irreducible only for \(d = 4\) [20]. For \(d = 2, 6, 8\) the complex representation \(D(d+2)\) decomposes into a direct sum of two equivalent real irreducible representations of \(\mathcal{C}(d+1,1)\)

\[
D(d+2) = S(d+1,1) \oplus S(d+1,1),
\]

where \(S(d+1,1)\) are characterised by appropriate Majorana conditions [20].
Strictly speaking the rules of covariant quantization require an irreducible representation of the real Clifford algebra \( \mathcal{C}(d + 1, 1) \). We shall however admit the complex representation \( D(d + 2) \) which leads to the following structure of \( H_0(p, p^L) \)

\[
H_0(p, p^L) = F_0(p, p^L) \otimes D(d + 2) ,
\]

and provides a unified formulation for all cases \( d = 2, 4, 6, 8 \). Other reasons for this choice will be discussed in Sect. 5.

Let us denote by \( \bar{a}_n^\mu, \bar{c}_n, \bar{b}_r^\mu, \bar{d}_r \) the operators on \( F_0(p, p^L) \) representing non-zero bosonic and fermionic modes. Using gamma matrices of the \( D(d + 2) \) representation normalised by

\[-(\Gamma^0)^2 = (\Gamma^1)^2 = \ldots = (\Gamma^{d-1})^2 = (\Gamma^L)^2 = (\Gamma^F)^2 = 1,\]

one can construct a representation of the algebra (5) on \( H_0(p, p^L) \) as follows

\[
\begin{align*}
\alpha_n^\mu &= \bar{a}_n^\mu \otimes 1 , & c_n &= \bar{c}_n \otimes 1 , & n \neq 0 , \\
b_r^\mu &= \bar{b}_r^\mu \otimes \Gamma^F , & d_r &= \bar{d}_r \otimes \Gamma^F , & r \neq 0 , \\
b_0^\mu &= 1 \otimes \frac{1}{\sqrt{2}} \Gamma^\mu , & d_0 &= 1 \otimes \frac{1}{\sqrt{2}} \Gamma^L ,
\end{align*}
\]

The fermion parity operator on \( H_0(p, p^L) \) can be defined by

\[
(-1)^F = (-1)^{\bar{F}} \otimes \Gamma^F
\]

where \( \bar{F} = \sum_{r>0} \bar{b}_{-r} \cdot \bar{b}_r + \sum_{r>0} \bar{d}_{-r} \bar{d}_r \) is the fermion number operator on \( F_0(p, p^L) \).

The hermitian scalar products in the spaces \( F_0(p, p^L) \), \( F_1(p, p^L) \), and \( D(d + 2) \) are determined by the relations (3) and the conjugation properties (3).

The constraint operators \( L_m, G_r \) on \( H_\epsilon \) are represented by normally ordered counterparts of the classical expressions (2). For the bosonic and fermionic zero modes we assume the symmetric and the antisymmetric ordering, respectively. In both sectors the quantum constraints algebra takes the form

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{8}(d + 1 + 32\beta)(m^3 - m)\delta_{m,-n} , \\
[L_m, G_r] &= (\frac{m}{2} - r)G_{m+r} , \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2}(d + 1 + 32\beta)(r^2 - \frac{1}{4})\delta_{r,-s} .
\end{align*}
\]

The subspace \( \mathcal{H}_{\epsilon}^{ph} \subset H_\epsilon \) of physical states is defined by the conditions

\[
\begin{align*}
(L_n - a_\epsilon \delta_{n,0})\Psi &= 0 , & n \geq 0 , \\
(G_r + m_0 g_0 \delta_{r,0})\Psi &= 0 , & r \geq \frac{\epsilon}{2} , \\
(P^L - p^L)\Psi &= 0 ,
\end{align*}
\]

where \( g_0 = 1 \otimes \Gamma^F \), and \( a_\epsilon, m_\epsilon, p^L \in \mathbb{R} \), are regarded as parameters of the quantum model. They are related by the condition

\[
a_\epsilon = -m_\epsilon^2 + (1 - \epsilon)\frac{1}{16}(d + 1) + 2\beta .
\]

The "mass" operator \( g_0 \) has been introduced in order to obtain the most general setting in which the conditions involving \( L_0 \), and \( G_0 \) are consistent with the algebra relation

\[
G_0^2 = L_0 - \frac{1}{16}(d + 1 + 32\beta).
\]

The normal ordering does not alter the classical expressions for the Poincare generators (4) and their algebra is represented on \( \mathcal{H}_{\epsilon}^{ph} \) without anomaly.
4 DDF construction

The DDF construction [27] for the Neveu-Schwarz model was developed long time ago by Schwarz [28], and by Brower and Friedman [29]. In this section we shall extend this technique to the fermionic massive string. For this purpose it is convenient to introduce the subspace $\mathcal{H}_\epsilon \subset H_\epsilon$ of on-shell states $\Psi$ satisfying the on-mass-shell conditions:

$$ (L_0 - a_\epsilon) \Psi = \left( \frac{1}{2a} P^\mu P_\mu + \frac{1}{123} (P^L)^2 + R + m^2_\epsilon \right) \Psi = 0 \ , \ P^L \Psi = p^L \Psi \quad (8) $$

where

$$ R = \sum_{m \geq 0} \left( a_{-m} a_m + c_{-m} c_m \right) + \sum_{r > 0} r \left( b_{-r} b_r + d_{-r} d_r \right) \ , $$

is the level operator. Since $L_0$ commutes with the Poincare generators $\mathcal{H}_\epsilon$ carries a representation of the Poincare algebra. The decomposition of $\mathcal{H}_\epsilon$ with respect to the mass coincides with the level structure

$$ \mathcal{H}_\epsilon = \bigoplus_{N \geq 0} \mathcal{H}^N_{\epsilon} \ ; \ R \mathcal{H}^N_{\epsilon} = N \mathcal{H}^N_{\epsilon} \ . \quad (9) $$

Each subspace $\mathcal{H}^N_{\epsilon}$ is a direct integral of finite dimensional subspaces $\mathcal{H}^N_{\epsilon}(p)$ with fixed on-shell momentum $p$

$$ \mathcal{H}^N_{\epsilon} = \int_{S^N_{\epsilon}} d\mu^N(p) \mathcal{H}^N_{\epsilon}(p) \ , \quad (10) $$

where $S^N_{\epsilon}$ denotes the mass shell at level $N$ determined by the condition (8), and $d\mu^N(p)$ is the Lorentz invariant measure on $S^N_{\epsilon}$.

Following [28, 29] we introduce the DDF fields

$$ \tilde{X}^\mu(\theta) = q^\mu_0 + a^\mu_0 \theta + \sum_{m \neq 0} \frac{i}{m} a^\mu_m e^{-im\theta} \ , \ \Phi(\theta) = q^L_0 + c^L_0 \theta + \sum_{m \neq 0} \frac{i}{m} c^L_m e^{-im\theta} \ , $$

$$ \tilde{P}^\mu(\theta) = \tilde{X}^{\mu'}(\theta) \ , \ \Pi(\theta) = \Phi'(\theta) \ , $$

$$ \tilde{\Psi}^\mu(\theta) = \sum_{r \in \mathbb{Z} + \frac{d}{2}} \psi^\mu_r e^{-ir\theta} \ , \ \Psi_L(\theta) = \sum_{r \in \mathbb{Z} + \frac{d}{2}} \psi^L_r e^{-ir\theta} \ , $$

where prime denotes differentiation with respect to $\theta$.

Let us fix a light-cone frame $\{e_\pm, e_1, \ldots, e_{d-2}\}$ in $d$-dimensional Minkowski space normalised by $e_\pm^2 = 0, e_+ \cdot e_- = -1$, and $e_i \cdot e_j = \delta_{ij}$ for $i, j = 1, \ldots, d-2$. We shall use the following notation for the light-cone components of a vector $V$

$$ V_\pm = e_\pm \cdot V \ , \ V^i = e_i \cdot V \ . $$

In the original construction the domain of DDF operators is restricted to states with momenta satisfying $\frac{1}{\sqrt{p^+}} p^+ \in \mathbb{Z}$. This restriction can be overcome introducing slightly modified DDF fields

$$ X_\pm(\theta) = \left( \frac{\sqrt{a}}{p^+} \right)^{\pm 1} \tilde{X}_\pm(\theta) \ , \ X^i(\theta) = \tilde{X}^i(\theta) \ , $$

$$ P_\pm(\theta) = \left( \frac{\sqrt{a}}{p^+} \right)^{\pm 1} \tilde{P}_\pm(\theta) \ , \ P^i(\theta) = \tilde{P}^i(\theta) \ , $$

$$ \Psi_\pm(\theta) = \left( \frac{\sqrt{a}}{p^+} \right)^{\pm 1} \tilde{\Psi}_\pm(\theta) \ , \ \Psi^i(\theta) = \tilde{\Psi}^i(\theta) \ . $$

The on-mass-shell conditions are not essential for the construction of DDF operators. We impose them at the beginning in order to simplify our presentation.
The rescaling above leaves unchanged all the (anti)commutation relations relevant for calculating the algebra of DDF operators. Let us note that a single light-cone frame is sufficient for a global DDF parameterisation of $\mathcal{H}_\ell$. Indeed the momenta with $p_+ = 0$ for which the construction gets singular form a zero measure subset of each mass shell $S_N^\ell$ and can be neglected.

The DDF operators for the "super-matter" sector are exactly the same as in the Neveu-Schwarz model [28, 29]. In our notation they read

$$
A_m^i = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (P^i - m\Psi^i\Psi^+) e^{imX_+} : ,
$$

$$
B_r^i = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (\Psi^i P^\frac{1}{2}_+ - P^i \Psi^+_P + \frac{1}{2} i\Psi^i\Psi^+ + \Psi^+_{-i}) e^{irX_+} : ,
$$

$$
A_m^{-} = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (P_- - m\Psi_-\Psi^+) e^{imX_+} :
$$

$$
- \frac{1}{2\pi} \int_0^{2\pi} d\theta : \left( \frac{1}{2}(mP^{-1}_+ P^i + m^2\Psi^+_P\Psi^+_{-i}) \right) e^{imX_+} : ,
$$

$$
B_r^{-} = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (\Psi^i P^\frac{1}{2}_+ - P^i \Psi^+_P + \frac{1}{2} i\Psi^i\Psi^+ + \Psi^+_{-i}) e^{irX_+} :
$$

$$
+ \frac{1}{2\pi} \int_0^{2\pi} d\theta : \left[ \frac{1}{8}(\Psi^i P^{-1}_+ P^i - \frac{5}{2}(\Psi^i P^{-1}_+ P^i) + \frac{1}{8}\Psi^i P^i\Psi^+_{-i} + \Psi^+_{-i}) e^{irX_+} : .
$$

Note that the antisymmetric ordering of fermionic zero modes is responsible for the sector dependent term in the expression for $B_r^{-}$.

In order to find appropriate DDF operators for the super-Liouville excitations one can start with the "naive" operators

$$
C_m^0 = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (\Pi - m\Psi^i\Psi^+) e^{imX_+} : ,
$$

$$
D_r^0 = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (\Psi^i_P - \Pi\Psi^+_P + \frac{1}{2} i\Psi^i\Psi^+ + \Psi^+_{-i}) e^{irX_+} : ,
$$

and calculate their (anti)commutators with the fermionic constraints

$$
\{ G_r, C_m^0 \} = -\frac{1}{2\pi} \int_0^{2\pi} d\theta r e^{ir\theta} : 4i\sqrt{\beta}\Psi^+ e^{imX_+} : ,
$$

$$
\{ G_r, D_r^0 \} = \frac{1}{2\pi} \int_0^{2\pi} d\theta r e^{ir\theta} : 4i\sqrt{\beta} \left( P^\frac{1}{2}_+ - \frac{1}{2} i\Psi^i\Psi^+ + P^\frac{1}{2}_+ \right) e^{irX_+} : .
$$

The corrections required can be deduced by comparing the expressions above with the corresponding calculations for the "naive" longitudinal DDF operators [28, 29]. In the
case of the bosonic Liouville operator \( C_m = C^0_m + \delta C_m \) the correction

\[
\delta C_m = \frac{1}{2\pi} \int_0^{2\pi} d\theta : 2 \sqrt{\beta} \left( P^{-1}_+ P'_+ + m\Psi'_+ P^{-1}_+ \right) e^{imX_+} : 
\]

is proportional to the bosonic longitudinal correction [28, 29]. The correction for the fermionic Liouville operator \( D_r = D^0_r + \delta D_r \) is not that obvious. It can be however identified as a part of the fermionic longitudinal correction derived in [28, 29]

\[
\delta D_r = -\frac{1}{2\pi} \int_0^{2\pi} d\theta : 4 \sqrt{\beta} (\Psi_+ P^{-1}_+)' P^{\frac{1}{2}}_+ e^{irX_+} : 
\]

The transverse and the Liouville DDF operators satisfy the canonical (anti)commutation relations

\[
[A^i_m, A^j_n] = m\delta^{ij}\delta_{m,-n} \quad , \quad [C_m, C_n] = m\delta_{m,-n} \quad , \\
\{B^i_r, B^j_s\} = \delta_{ij}\delta_{r,-s} \quad , \quad \{D_r, D_s\} = \delta_{r,-s} 
\]

and all cross (anti)commutators between these two families vanish.

The longitudinal DDF operators \( A^i_m, B^i_r \) do not commute with other DDF operators. Following [10] one can diagonalize the algebra introducing "shifted" longitudinal operators

\[
A^L_m = A^-_m - L_m + \frac{1}{2}\delta_{m,0} \quad , \\
B^L_r = B^-_r - G_r 
\]

where

\[
L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} A^i_{-n} A^i_{n+m} + (1 - \epsilon)\frac{d-1}{4}\delta_{m,0} + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r B^i_{-r} B^i_{r+m} + \frac{1}{2} \sum_{n \in \mathbb{Z}} C^i_{-n} C^i_{n+m} + 2i\sqrt{\beta}mC^i_m + 2\beta\delta_{m,0} + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r D_{-r} D_{r+m} \\
G_r = \frac{1}{2} \sum_{n \in \mathbb{Z}} A^i_{-n} B^i_{n+r} + \frac{1}{2} \sum_{n \in \mathbb{Z}} C^i_{-n} D^i_{n+r} + 4i\sqrt{\beta}r \delta_{r,0} 
\]

The new longitudinal operators commute with the transverse and the Liouville DDF operators and form an \( N = 1 \) superconformal algebra with the central charge \( \hat{c} = 9 - d - 32\beta \)

\[
[A^L_m, A^L_n] = (m - n)A^L_{m+n} + \frac{1}{8} (9 - d - 32\beta) (m^3 - m)\delta_{m,-n} \\
[A^L_m, B^L_r] = \left( \frac{m}{2} - r \right) B^L_{m+r} \\
\{B^L_r, B^L_s\} = 2A^L_{r+s} + \frac{1}{2} (9 - d - 32\beta) (r^2 - \frac{1}{4})\delta_{r,-s} 
\]

By construction all the DDF operators \( D_a, 2a \in \mathbb{Z} \) satisfy the following basic relations

\[
[L_m, D_a] = [G_r, D_a] = 0 \quad , \quad m \in \mathbb{Z} \quad , \quad r \in \mathbb{Z} + \frac{1}{2} \\
[R, D_a] = -a D_a \\
D_a \Omega_{\epsilon}(p) = 0 \quad , \quad a > 0 
\]
where the on-shell vacua $\Omega_\epsilon(p) \in \mathcal{H}_\epsilon^0(p)$ are defined as 0-level states with the on-shell momentum $p \in S_\epsilon^0$.

In the Neveu-Schwarz sector the space $\mathcal{H}_1^0(p)$ is 1-dimensional and the unique on-shell vacuum $\Omega_1(p)$ is physical. In the Rammond sector the space $\mathcal{H}_0^0(p)$ carries a finite dimensional representation of the Clifford algebra. The subspace $\mathcal{H}_0^0(p)^{\text{ph}} \subset \mathcal{H}_0^0(p)$ of physical vacua is determined by the condition

$$G_0 \Omega_0(p) = \frac{1}{2} \left( \frac{p_\mu}{\sqrt{\alpha}} \mu_0 + \frac{p_L}{\sqrt{\beta}} d_0 + m_0 g_0 \right) \Omega_0(p) = 0 \ .$$

We introduce a common for both sectors notation $\{\Omega^J_\epsilon(p)\}_{J \in J_\epsilon}$ for a basis in the space $\mathcal{H}_\epsilon^0(p)$ of physical vacua with a momentum $p$.

Let us consider monomials $\vartheta[D]$ of the DDF operators $D_a$ with negative indices $a < 0$, and with some fixed ordering. With the restriction $a < 0$ the ordering matters only for the longitudinal DDF operators where one can choose for instance the standard ordering of the superconformal Verma module construction. We denote by $\{\vartheta_I^{(N)}[D]\}_{I \in I^{(N)}}$ the collection of all ordered normalised monomials of the DDF operators with negative indices and with a fixed level $N$

$$\left[ R, \vartheta_I^{(N)}[D] \right] = N \vartheta_I^{(N)}[D] \ .$$

With the above notation one gets

**Lemma** For any $N > 0$ and $p \in S_\epsilon^N$, $p_+ \neq 0$ the DDF states

$$\vartheta_I^{(N)}[D] \Omega^J_\epsilon \left( p + \frac{\alpha N}{p_+} e_+ \right) \quad J \in J_\epsilon , \ I \in I^{(N)} ,$$

form a basis in the subspace $\mathcal{H}_\epsilon^N(p)^{\text{ph}} \subset \mathcal{H}_\epsilon^N(p)$ of all physical states with the momentum $p$.

The lemma can be proven using Brower’s ideas [16] in exactly the same way as in the case of the Neveu-Schwarz model [29].

**5 No-ghost theorem**

For any $p_+ \neq 0$, and $\vec{p} = p^i e_i$, let us consider the subspace $\mathcal{H}_\epsilon(p)^{\text{ph}}$ of all states generated by the DDF operators out of the physical vacua $\Omega_\epsilon(p) \in \mathcal{H}_\epsilon^0(p)^{\text{ph}}$ with the momentum

$$p = -\frac{\alpha}{p_+} (\frac{1}{2\beta} p_L^2 + m_\epsilon^2) e_+ - p_+ e_- + \vec{p} \ .$$

It follows from the algebra of DDF operators [11,13] that $\mathcal{H}_\epsilon(p)^{\text{ph}}$ has the structure of the tensor product

$$\mathcal{H}_\epsilon(p)^{\text{ph}} = \mathcal{F}^{\text{tr}} \otimes \mathcal{F}^L \otimes \mathcal{V}_\epsilon(\hat{c}, h_\epsilon) \otimes \mathcal{H}_\epsilon^0(p)^{\text{ph}} \ ,$$

where $\mathcal{F}^{\text{tr}}, \mathcal{F}^L$ denote Fock spaces generated by the algebra of the transverse and the super-Liouville DDF operators [11], and $\mathcal{V}_\epsilon(\hat{c}, h_\epsilon)$ is the superconformal Verma module generated by the Rammond-Neveu-Schwarz superconformal algebra of the “shifted” longitudinal DDF operators with the central charge $\hat{c} = 9 - d - 32\beta$ [13]. The highest weight $h_\epsilon$ of
\( \mathcal{V}_c(\hat{c}, h_c) \) is determined by the structure of the "shifted" longitudinal operator \( A^L_0 \) (12), and the on mass shell condition (8)

\[
h_c = \frac{1}{2} - (1 - \epsilon) \frac{1}{16} (d - 1) - 2\beta + m_c^2.
\]

Using the light-cone parameterisation of the mass shells \( S^N \) and the lemma of Sect.4 one can represent the subspace \( \mathcal{H}_c^{\text{ph}} \) of all physical states in the following form

\[
\mathcal{H}_c^{\text{ph}} = \int \frac{dp}{|p_+|} d^{d-2} \beta \, \mathcal{H}_c(p)^{\text{ph}} .
\]

(16)

The metric structure of each \( \mathcal{H}_c(p)^{\text{ph}} \) is completely determined by the algebra of DDF operators (11,13) along with their conjugation properties with respect to the metric structure of the original pseudo-Hilbert space \( H_c \):

\[
(A^i_m)^\dagger = A^i_{-m} , \quad (A^L_m)^\dagger = A^L_{-m} , \quad (C_m)^\dagger = C_{-m} , \quad m \in \mathbb{Z} ,
\]

\[
(B^i_r)^\dagger = B^i_{-r} , \quad (B^L_r)^\dagger = B^L_{-r} , \quad (D_r)^\dagger = D_{-r} , \quad r \in \mathbb{Z} + \frac{\epsilon}{2} .
\]

The Fock space component \( F^{\text{tr}} \otimes F^L \) of \( \mathcal{H}_c(p)^{\text{ph}} \) (15) is the same in both sectors and carries a positive definite metric. The metric structure of the superconformal Verma module \( \mathcal{V}_c(\hat{c}, h_c) \) had been extensively studied in the context of unitary highest weight representations of the Ramond-Neveu-Schwarz superconformal algebra. The necessary conditions for the absence of negative norm states in \( \mathcal{V}_c(\hat{c}, h_c) \) were derived by Friedan, Qiu, and Shenker [30, 31]. It was further proven by Goddard, Kent, and Olive [32] that these conditions are also sufficient.

In the Neveu-Schwarz sector the subspace \( \mathcal{H}_0^0(p)^{\text{ph}} \) is 1-dimensional and the decomposition (15) simplifies

\[
\mathcal{H}_0(p)^{\text{ph}} = F^{\text{tr}} \otimes F^L \otimes \mathcal{V}_1(\hat{c}, h_1) .
\]

Then the results concerning the metric structure of \( \mathcal{V}_1(\hat{c}, h_1) \) [30, 31, 32] yield the following

No-Ghost Theorem – Neveu-Schwarz sector

The space of physical states in the Neveu-Schwarz sector of the fermionic massive string is ghost free if and only if one of the following two conditions is satisfied

1. continuous series

\[
0 < \beta \leq \frac{8 - d}{32} , \quad m_1^2 \geq 2\beta - \frac{1}{2} ,
\]

2. discrete series

\[
\beta = \beta_m = \frac{8 - d}{32} + \frac{1}{4m(m + 2)} , \quad m_1^2 = m_{p,q}^2 = - \frac{d}{16} + \frac{[(m + 2)p - mq]^2}{8m(m + 2)} ,
\]

where \( m, p, q \) are integers satisfying \( 2 \leq m, 1 \leq p < m, 1 \leq q < m + 2, p - q \) even.

In the Ramond sector the structure of \( \mathcal{H}_0(p)^{\text{ph}} \) is more complicated

\[
\mathcal{H}_0(p)^{\text{ph}} = F^{\text{tr}} \otimes F^L \otimes \mathcal{V}_0(\hat{c}, h_0) \otimes \mathcal{H}_0^0(p)^{\text{ph}} .
\]
Let us assume that the necessary and sufficient conditions for the non-negative metric structure of the superconformal Verma module of the Rammond superalgebra \[30, 31, 32\] are satisfied. Then the properties of the metric structure of \( \mathcal{H}_{0}^{\text{ph}} \) are entirely determined by the properties of the metric structures of the subspaces of physical vacua \( \mathcal{H}_{0}^{\text{ph}}(p) \).

In order to find an explicit form of the scalar product in \( \mathcal{H}_{0}^{\text{ph}}(p) \) let us first analyse hermitian scalar products on the irreducible representations \( D(d + 2) \) of the complex Clifford algebras \( \mathcal{C}C(d + 2) \). There exists one and only one hermitian positive definite scalar product \( (\cdot, \cdot) \) on \( D(d + 2) \) such that all gamma matrices are isometries. We define the parity operator
\[
\theta = \begin{cases} 
  i\Gamma^{0}\Gamma^{1} \cdots \Gamma^{d-1}\Gamma^{L}\Gamma^{F} & \text{for } d = 2, 6 \\
  \Gamma^{0}\Gamma^{1} \cdots \Gamma^{d-1}\Gamma^{L}\Gamma^{F} & \text{for } d = 4, 8
\end{cases}
\]
Note that \( \theta \) is hermitian with respect to \( (\cdot, \cdot) \) and \( \theta^{2} = 1 \). The rules of covariant quantization require a hermitian scalar product on \( D(d + 2) \) for which \( \Gamma^{\mu}, \Gamma^{L}, \Gamma^{F} \) are all hermitian. A unique product with these properties is given by
\[
\langle \xi, \zeta \rangle = (\xi, \Gamma^{0}\theta\zeta), \quad \xi, \zeta \in D(d + 2).
\]
The 0-level physical states \( \psi \in \mathcal{H}_{0}^{\text{ph}} \) satisfy \( G_{0} \) constraint \([14]\). In the position representation it takes the form of the Dirac equation
\[
\left( -\frac{i}{\sqrt{2}\alpha} \Gamma^{\mu}\partial_{\mu} + \frac{p^{L}}{\sqrt{2}\beta} \Gamma^{L} + m_{0}\Gamma^{F} \right) \psi = 0.
\]
For any two solutions \( \psi(x), \phi(x) \) the vector current
\[
j^{\mu}(x) = \langle \psi(x), \Gamma^{\mu}\phi(x) \rangle
\]
is conserved, and can be used for constructing a Lorentz invariant scalar product on \( \mathcal{H}_{0}^{\text{ph}} \)
\[
\langle \psi, \phi \rangle = \int dx^{1} \cdots dx^{d-2} (\psi(x), \theta\phi(x))
\]
\[
= \int dx^{-d-2} \sqrt{-g} (\psi(x), \Gamma^{+}\Gamma^{0}\theta\phi(x))
\]
Proceeding to the momentum representation and taking into account the Lorentz invariant measure in \([13]\) one gets the scalar product in \( \mathcal{H}_{0}^{\text{ph}}(p) \)
\[
\langle \psi(p), \phi(p) \rangle_{\text{l.c.}} = (\psi(p), \Gamma^{+}\Gamma^{0}\theta\phi(p))
\]
Since it is neutral so is the metric structure of \( \mathcal{H}_{0}^{\text{ph}}(p) \), and in consequence the metric structure of the space \( \mathcal{H}_{0}^{\text{ph}} \) of all physical states. This is not necessarily a disaster for the unitarity of the model. What saves the day is a Lorentz invariant decomposition of \( \mathcal{H}_{0}^{\text{ph}} \) into a direct sum of two orthogonal components
\[
\mathcal{H}_{0}^{\text{ph}} = \mathcal{H}_{0}^{\text{ph}} + \mathcal{H}_{0}^{\text{ph}}
\]
with a positive and a ”negative” definite scalar products, respectively. In order to construct such decomposition let us consider the extension \( \Theta_{0} = \theta \otimes 1 \) of \( \theta \) from \( D(d + 2) \)
to the whole pseudo-Hilbert space \( H_0 \) of covariant quantization. Using the explicit realisation (15) one can easily verify that \( \Theta_0 \) has all the properties of the fermion parity operator. This in particular implies that \( \Theta_0 \) commutes with the Lorentz generators, and anticommutes with the Dirac operator (17). Then it follows from the representation (18) that the eigenspaces \( H_{0 \pm}^{\text{ph}} \) corresponding to \( \pm 1 \) eigenvalues of \( \Theta_0 \) yield the decomposition required.

The unitarity problem can be solved by imposing the superselection rule related to the decomposition (19). This is possible if and only if an operator \( \Theta_0 \) with the required properties can be constructed which is always the case if we start with the complex representation \( D(d+2) \). Note that for \( d < 9 \) and within the covariant quantization based on the irreducible representations \( S(d+1,1) \) of the Clifford algebra \( C(d+1,1) \) the operator \( \Theta_0 \) exists only for \( d = 4 \) and \( d = 8 \) [29]. This provides another justification for our choice made in Sect.3.

Taking into account the restrictions for possible central charges, and highest weights of the superconformal Verma module \( \mathcal{V}_0(\hat{c}, h_0) \) of the Rammond superconformal algebra one gets the following

**No-Ghost Theorem – Rammond sector**

The space \( H_{0}^{\text{ph}} \) of physical states in the Rammond sector admits a unique Lorentz invariant scalar product. This product is neutral. \( H_{0}^{\text{ph}} \) decomposes into an orthogonal direct sum

\[
H_{0}^{\text{ph}} = H_{0}^{\text{ph}+} \oplus H_{0}^{\text{ph}-}
\]

of the eigenspaces of the parity operator \( \Theta_0 \).

The eigenspaces \( H_{0 \pm}^{\text{ph}} \) are ghost free if and only if one of the following two conditions is satisfied

1. **continuous series**

   \[
   0 < \beta \leq \frac{8 - d}{32} , \quad m_0^2 \geq 2\beta - \frac{8 - d}{16} ,
   \]

2. **discrete series**

   \[
   \beta = \beta_m = \frac{8 - d}{32} + \frac{1}{4m(m+2)} , \quad m_0^2 = m_{p,q}^2 = \frac{[(m+2)p - mq]^2}{8m(m+2)} ,
   \]

   where \( m, p, q \) are integers satisfying \( 2 \leq m, 1 \leq p < m, 1 \leq q < m+2, p - q \) odd.

In all our considerations the zero mode of the Liouville momentum \( p^L \) has been regarded as a free parameter of the quantum theory. The no-ghost theorems do not impose any restriction on \( p^L \). Since there is no ordering ambiguity in the constraint \( P^L = 0 \) we shall assume \( p^L = 0 \). In this case the parameters \( m_0, m_1 \) have the interpretation of the physical masses of the corresponding ground states.

**6 Conclusions**

The results of the previous section yield a complete classification of all admissible Hilbert spaces \( \mathcal{H}_{0}^{\text{ph}}(\beta, m_\epsilon) \) of both sectors of the massive fermionic string in terms of the parameters \( \beta \), and \( m_\epsilon \). For \( \beta \) from the discrete series the total Hilbert space is given by

\[
\mathcal{H}^{\text{ph}}(m) = \bigoplus_{p-q \text{ odd}} \mathcal{H}_{0}^{\text{ph}}(\beta_m, m_{p,q}) \bigoplus_{p-q \text{ even}} \mathcal{H}_{1}^{\text{ph}}(\beta_m, m_{p,q}) ,
\]
where the sum runs over all \(1 \leq p < m, 1 \leq q < m + 2\), and \(m = 2, 3, \ldots\). The central charge \(\hat{c}\) of the superconformal algebra of the "shifted" longitudinal DDF operators

\[
\hat{c} = \hat{c}_m = 1 - \frac{8}{m(m + 2)}
\]

"measures" the number of longitudinal physical degrees of freedom. In particular for \(m = 2\), one gets \(\hat{c}_2 = 0\), and all states containing longitudinal excitations are null. The quantum theory contains one "superfunctional" physical degree of freedom less than the classical one. For large \(m\) the central charge \(\hat{c}_m\) approaches 1. The limiting case \(\hat{c} = \hat{c}_\infty = 1\) corresponds to \(\beta = \frac{1}{32}(8 - d)\) – the upper bound of the continuous series. We define

\[
\mathcal{H}^{ph}(\infty) = \mathcal{H}^{ph}_0\left(\frac{1}{32}(8 - d), 0\right) \oplus \mathcal{H}^{ph}_1\left(\frac{1}{32}(8 - d), -\frac{1}{16}d\right).
\]

In this model the space of physical states is largest possible. Both the classical, and the quantum theories contain \(d\) "superfunctional" physical degrees of freedom.

For \(\beta\) in the range \(0 < \beta < \frac{1}{32}(8 - d)\), one has \(1 < \hat{c}\), and the structure of the physical degrees of freedom of the quantum theory is essentially the same as in the model described by \(\mathcal{H}^{ph}(\infty)\). One could in principle define a continuous family of models but without any extra physical assumption the ground state masses \(m_0, m_1\) are undetermined free parameters of such construction.

Models from the family \(\left\{\mathcal{H}^{ph}(m)\right\}_{m=2}^{\infty}\) are not quite satisfactory. The spectrum of each Neveu-Schwarz sector contains tachyon, while the metric in the Rammond sector is always neutral. As we have seen in the previous section the second problem can be overcome by introducing superselection rule related to the operator \(\Theta_0\). One can try to extend this rule to the whole Hilbert space. Following the scheme of the GSO projection in the critical string [33] we decompose the space \(\mathcal{H}^{ph}_1(m)\) of the Neveu-Schwarz sector into the direct sum

\[
\mathcal{H}^{ph}_1(m) = \mathcal{H}^{ph}_1(m) \oplus \mathcal{H}^{ph}_{-1}(m)
\]

of the \(\pm 1\) eigenspaces of the fermion parity operator \(\Theta_1\)

\[
\Theta_1 = (-1)^{F+1}, \quad F = \sum_{r>0}(b_{-r} \cdot b_r + d_{-r}d_r)
\]

With this definition of \(\Theta_1\) the eigenspace \(\mathcal{H}^{ph}_1(m)\) corresponding to \(+1\) eigenvalue does not contain the tachyonic ground state. We introduce a non-critical counterpart of the GSO projection as the projection on the \(+1\) eigenspace

\[
\mathcal{H}^{ph}_+(m) = \mathcal{H}^{ph}_0(m) \oplus \mathcal{H}^{ph}_1(m)
\]

of the fermion parity operator \(\Theta = \Theta_0 \oplus \Theta_1\). It yields a family of tachyon free unitary non-critical fermionic strings. In contrast the original GSO projection [33] it does not lead to a supersymmetric spectrum. Note that there are no massless states in models with odd \(m\).

The model corresponding to beginning of the discrete series \(\beta = \beta_2 = \frac{9-d}{32}\) is especially interesting. In this case the subspace of null physical states is largest possible. For this reason it will be called the critical (fermionic) massive string. The space of "true" physical states \(\mathcal{H}^{tr}(2)\) is given by the quotient

\[
\mathcal{H}^{tr}(2) = \frac{\mathcal{H}^{ph}(2)}{\{\text{null states}\}}.
\]
As in the case of the bosonic critical massive string [13], one can show that the subspace $\mathcal{H}^{RNS} = \mathcal{H}^{RNS}_0 \oplus \mathcal{H}^{RNS}_1 \subset \mathcal{H}^{ph}(2)$ generated by the transverse $A^i, B^i$, and the longitudinal $A^-, B^-$, DDF operators is a good gauge slice for the quotient map $\mathcal{H}^{ph}(2) \rightarrow \mathcal{H}^{tr}(2)$. In consequence the even (with respect to $\Theta$) subspace $\mathcal{H}^{RNS}_+ = \mathcal{H}^{RNS}_{0+} \oplus \mathcal{H}^{RNS}_{1+}$ yields a 1-1 parameterisation of the quotient space $\mathcal{H}^{tr}_+(2)$ of the GSO projected model.

The superalgebra of the transverse and the longitudinal DDF operators of the critical massive string is by construction isomorphic with the whole DDF algebra of the non-critical Rammond-Neveu-Schwarz string [28, 29]. This implies that in the Neveu-Schwarz sector the subspace $\mathcal{H}^{RNS}_1$ is isomorphic with the tachyon free eigenspace of the fermion parity operator in the space of physical states of the non-critical RNS string. In the Rammond sector each eigenspace $\mathcal{H}^{RNS}_{0 \pm}$ of $\Theta_0$ carries the same complex representation $D(d)$ of the real Clifford algebra $C(d-1,1)$, and is therefore isomorphic with the whole space of physical states in the Rammond sector of the non-critical RNS string.

One can easily verify that the subspace $\mathcal{H}^{RNS}_+$ is stable with respect to the Poincare transformations, and carries a representation which is isomorphic with the representation of the Poincare group in the non-critical RNS string. It follows that the GSO projected fermionic critical massive string is equivalent with the non-critical RNS string truncated in the Neveu-Schwarz sector to the tachyon free eigenspace of the fermion parity operator.

We have shown that the covariant quantization of the free fermionic massive string in the even dimensions $d = 2, 4, 6, 8$ leads to a family of new tachyon free unitary fermionic strings. One of these models - the critical fermionic massive string - is closely related to the non-critical RNS string.

There are at least two interesting open problems before one may try to attack the question of the interacting theory. First of all it would be desirable to develope the light-cone formulation of the critical massive string and to calculate its spin content. In view of the equivalence discussed above it would provide the particle spectrum of the non-critical RNS string. The second problem is to analyse the superconformal field theory structure of the fermionic massive string. This would in particular clarify the status of the non-critical GSO projection proposed in this paper.

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