NON-PARALLEL ESSENTIAL SURFACES IN KNOT COMPLEMENTS

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Abstract. We show that if a knot or link has \( n \) thin levels when put in thin position then its exterior contains a collection of \( n \) disjoint, non-parallel, planar, meridional, essential surfaces. A corollary is that there are at least \( n/3 \) tetrahedra in any triangulation of the complement of such a knot.

Keywords: thin position, essential surface, knot invariant

1. Introduction

In 1987 D. Gabai introduced the concept of thin position for knots \([\text{Gab}87]\) to solve the Property \( R \) conjecture. Since then it has been used to solve many important questions in 3-manifold topology \([\text{GL}89], \text{ST}93, \text{Tho}94\). More recently thin position has become an object of study in itself \([\text{Tho}97], \text{HK}97, \text{RS}02, \text{Wu}\).

Thin levels are particular spheres which appear in a thin presentation of a knot or link. In \([\text{Tho}97]\) A. Thompson shows that if the exterior of a knot or link \( K \) does not contain any planar, meridional, essential surfaces then thin position for \( K \) has no thin levels. The idea of the proof is to show that the property of being non-trivial, possessed by a thin level, cannot disappear after any number of compressions. Therefore, compressing any thin level as much as possible yields a planar, meridional, essential surface.

In this paper we show that the property of being non-parallel, possessed by a collection of thin levels, cannot disappear after any number of compressions (as long as the compressions are chosen in a suitably nice way). Hence, compressing all thin levels as much as possible yields at least as many non-parallel essential surfaces as the number of thin levels. As a corollary we show that there are at least \( n/3 \) tetrahedra in any triangulation of the complement of a knot or link with \( n \) thin levels.

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2. Thin Position

Suppose \( K \subset S^3 \) is an arbitrary knot or link with no trivial components and \( h \) is some standard height function on \( S^3 \) (so that for each \( p \in (0, 1), h^{-1}(p) \) is a 2-sphere), which is a Morse function when restricted to \( K \). Let \( \{q'_j\} \) denote the critical values of \( h \) restricted to \( K \), let \( q_j \) be some point in the interval \((q'_j, q'_{j+1})\), and let

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\[ S_j = h^{-1}(q_j). \] The following terminology is standard in thin position arguments (see [Gab87]).

The width of \( K \) is defined to be the quantity \( \sum_j |K \cap S_j| \). A knot is said to be in thin position if \( h \) is chosen so that the width of \( K \) is minimal (see [Gab87]). If \( j \) is such that \( |K \cap S_j| < |K \cap S_{j-1}| \) and \( |K \cap S_j| < |K \cap S_{j+1}| \) then we say the surface \( S_j \) is a thin level of \( K \). In other words, a thin level is one which appears just above a maximum of \( K \) and just below a minimum.

**Definition 2.1.** Suppose \( K \) is a knot or link in \( S^3 \), \( h \) is the standard height function, and \( \gamma \) is a 1-manifold in the exterior of \( K \). We say \( \gamma \) is horizontal (with respect to \( K \)) if it is contained in a thin level of \( K \). If \( \gamma \) has endpoints on distinct levels of \( h \) then we say it is vertical if its interior has no critical points (with respect to \( h \)). If \( \gamma \) has endpoints on the same level of \( h \) then we say it is \( U \)-shaped if its interior has exactly one maximum or minimum. Finally, \( \gamma \) is simple if it is vertical or \( U \)-shaped.

**Lemma 2.2.** Suppose \( K_1 \) and \( K_2 \) are isotopic knots or links in \( S^3 \) which agree on some 1-manifold, \( \alpha \). \( K_1 \) is not in thin position if any of the following hold:

1. \( K_2 \setminus \alpha \) is horizontal with respect to \( K_1 \).
2. \( K_2 \setminus \alpha \) is vertical, but \( K_1 \setminus \alpha \) is not.
3. \( K_2 \setminus \alpha \) is \( U \)-shaped, but \( K_1 \setminus \alpha \) is not, and the minimum (maximum) of \( K_2 \setminus \alpha \) is at least as high (low) as the minimum (maximum) of \( K_1 \setminus \alpha \).

The horizontal case is proved in [BS03]. The proofs in the vertical and \( U \)-shaped cases are similar.

### 3. Heegaard Splittings

A compression body \( W \) is a 3-manifold which can be obtained by starting with some surface \( F \) (not necessarily closed or connected), forming the product \( F \times I \), attaching some number of 2-handles to \( F \times \{1\} \), and capping off all remaining 2-sphere boundary components with 3-balls. The surface \( F \times \{0\} \) is referred to as \( \partial_+ W \). The surface \( \partial_- W \) is defined by the equation \( \partial W = \partial_+ W \cup (\partial F \times I) \cup \partial_- W \). A compression body is non-trivial if it is not a product.

A surface \( F \) in a 3-manifold \( M \) is a Heegaard splitting of \( M \) if \( F \) separates \( M \) into two compression bodies, \( W \) and \( W' \), such that \( F = \partial_+ W = \partial_- W' \). We define \( \partial_- M = \partial_- W \cup \partial_- W' \).

We now present the crucial example of a Heegaard splitting for our purposes. Let \( T_1 \) and \( T_2 \) denote consecutive thin levels of a knot or link \( K \). Let \( N \) be the submanifold of \( S^3 \) cobounded by \( T_1 \) and \( T_2 \). Let \( S \) be a level 2-sphere which separates the maxima of \( K \) in \( N \) from the minima. Then \( S \setminus K \) is a Heegaard splitting of \( N \setminus K \) and \( \partial_-(N \setminus K) = T_1 \setminus K \cup T_2 \setminus K \).

For the proof of Lemma 4.7 we will need the following result:
Lemma 3.1. (Haken [Hak68]) Let $S$ be a Heegaard splitting of a 3-manifold $N$. If $D$ is a compressing disk for $\partial_- N$ then there is a compressing disk $E$ such that $\partial E = \partial D$ and $E \cap S$ is a loop.

4. Compressing sequences

In this section we prove a few preliminary lemmas concerning sequences of compressions of the thin levels of a knot or link.

Definition 4.1. Suppose $K$ is a knot or link in $S^3$. A sphere in $S^3$ is trivial if it is disjoint from $K$, or bounds a ball which contains a single, unknotted arc of $K$.

Definition 4.2. Suppose $K$ is a knot or link in $S^3$ and $\Sigma$ denotes the union of the thin levels of $K$. A compressing sequence for $K$ is a sequence of surfaces $\{\Sigma_i\}_{i=0}^n$ such that $\Sigma_0 = \Sigma$ and $\Sigma_i$ is obtained from $\Sigma_{i-1}$ by a compression in the complement of $K$.

Definition 4.3. The compressing sequence $\{\Sigma_i\}_{i=0}^n$ is maximal if $\Sigma_n$ is incompressible in the complement of $K$.

Lemma 4.4. Suppose $K$ is a knot or link in thin position. If $\{\Sigma_i\}_{i=0}^n$ is a maximal compressing sequence for $K$ then each component of $\Sigma_n$ is essential, i.e. incompressible and non-trivial.

Proof. First we show that each component of $\Sigma_n$ is incompressible in the complement of $K$. Suppose $D$ is a compressing disk for some such component. Then by a standard innermost disk argument we can isotope $D$ so that it intersects every component of $\Sigma_n \setminus K$ in essential loops. Let $\alpha$ denote such a loop which is innermost on $D$. Then $\alpha$ bounds a subdisk of $D$ which is a compressing disk for $\Sigma_n \setminus K$. This contradicts the maximality of $\{\Sigma_i\}_{i=0}^n$.

The remainder of the argument is essentially one of Thompson’s from [Tho94]. We recall this here. The only remaining possibility is that all components of $\Sigma_n$ are trivial. Let $S$ be an innermost component of $\Sigma_n$. Let $D$ be a disk in $S^3$ such that $\partial D = \delta \cup \gamma$, where $D \cap K = \delta$ and $D \cap S = \gamma$.

Note that $S$ is the result of compressing some thin level $T$ of $K$ some number of times. It follows that $S \cap T$ is connected and we may assume that $\gamma \subset S \cap T$. Hence, we can use $D$ to guide an isotopy of $\delta$ which, in the end, is horizontal with respect to $K$. This contradicts Lemma 2.2.

Lemma 4.5. Suppose $K$ is a knot or link in thin position. Let $\{\Sigma_i\}_{i=0}^n$ be a compressing sequence for $K$. Let $N_0$ denote the closure of some component of $S^3 \setminus \Sigma_0$. If some component $K'$ of $K \cap N_0$ is U-shaped then for all $i$ both points of $\partial K'$ lie on the same component of $\Sigma_i$.

Proof. Without loss of generality, assume that $K'$ has a minimum. Then we may isotope $K'$, preserving the width of $K'$, so that any other minimum on any other component of $K \cap N_0$ is below the minimum of $K'$. 

\[\square\]
Now, suppose the lemma is false. Let $m$ denote the largest integer such that both points of $\partial K'$ lie on the same component of $\Sigma_m$. Then $\Sigma_{m+1}$ is obtained from $\Sigma_m$ by compressing along a disk $D$ whose boundary separates the boundary points of $K'$. Let $N$ denote the closure of the component of $S^3 \setminus \Sigma_m$ which contains $K'$. So $\partial D \subset \partial N$. Note that the interior of $D$ must lie outside of $N$. Otherwise, $D$ would have to intersect $K'$, as $\partial D$ separates $\partial K'$. Let $B$ be either of the balls bounded by $D \cup \partial N$ whose interior lies outside of $N$.

As the component $F$ of $\partial N$ that contains $\partial K'$ is obtained by compressing a thin level, $T$, it must be the case that $F \cap T$ is connected. Hence, we may choose a horizontal arc $\gamma$ in $F \cap T$ which connects the endpoints of $K'$. Furthermore, we may choose such an arc so that it meets $D$ in precisely one point of $\partial D$.

We now perform the isotopy depicted in Figure 1, which can be described as follows:

1. Shrink $B$ to a small ball $B'$ at the end of $K'$.
2. Contract $K'$, pulling $B'$ along with it.
3. Push $B'$ along the arc $\gamma$.
4. Inflate $B'$ back to $B$.

As in Figure 1 this isotopy will affect the other arcs of $K \cap N_0$ which meet $B$. Let $S$ denote a level 2-sphere which is just below the minimum of $K'$. Note that Steps 2 and 3 of the isotopy take place entirely in the region between $T$ and $S$. Recall that the minimum of $K'$ is above all other minima of the components of $K \cap N_0$. Hence, the subarcs of $K$ that are between $T$ and $S$ and which meet $B$ are vertical. One such arc is depicted in Figure 1. Note that after the isotopy this arc is still vertical. Hence, as in Case 1 of Lemma 2.2 we have reached a contradiction because we have made a non-horizontal subarc of $K$ into a horizontal one while preserving the width everywhere else. □

**Definition 4.6.** The compressing sequence $\{\Sigma_i\}_{i=0}^n$ is good if for each $i$, each component $N$ of $S^3 - \Sigma_i$, and each component $K'$ of $K \cap N$ there is a simple arc in $N$ connecting the endpoints of $K'$.

**Lemma 4.7.** Suppose $K$ is a knot or link in $S^3$. Then there exists a good maximal compressing sequence for $K$.

**Proof.** To establish the lemma we will define a much more rigid compressing sequence called a Haken sequence and prove that every Haken sequence is good. We will then show that a Haken sequence of maximal length is a maximal compressing sequence.

We say a compressing sequence $\{\Sigma_i\}_{i=0}^n$ is Haken if for each $i$ each component $N$ of $S^3 - \Sigma_i$ contains a 2-sphere $S$ such that

1. $S \setminus K$ is a Heegaard splitting of $N \setminus K$.
2. The surface $S$ is obtained from a level surface of $h$ by some sequence of compressions.
3. Every component of $(K \cap N) \setminus S$ is simple.
Note that a compressing sequence with a single element, by definition consisting of the thin levels of $K$, is a Haken sequence. Between any two thin levels there is a level $S$ which separates the maxima of $K$ from the minima which has the desired properties.

We now show that every Haken sequence is good. Let $\{\Sigma_i\}_{i=0}^n$ denote a Haken sequence. Choose some $i$ and let $N$ denote the closure of a component of $S^3 - \Sigma_i$. Let $K'$ denote a component of $K \cap N$. To show that our sequence is good we must produce a simple arc in $N$ connecting the endpoints of $K'$. Since our sequence is Haken there is a 2-sphere $S$ in $N$ such that $S \setminus K$ is a Heegaard splitting for $N \setminus K$.

Let $K_1$ and $K_2$ denote the closure of the subarcs of $K' \setminus S$ which contain the points of $\partial K'$. Since $S$ was obtained from a level surface of $h$ by some sequence of compressions
we may choose a horizontal arc $\alpha$ in $S$ which connects $K_1 \cap S$ to $K_2 \cap S$ (see Figure 2).

Since $\{\Sigma_i\}$ is a Haken sequence the arcs $K_1$ and $K_2$ must be vertical. Since the arc $\alpha$ is horizontal we may perturb the arc $\gamma = K_1 \cup \alpha \cup K_2$ to be simple (again, see Figure 2).

We have now shown that that every Haken sequence is good. What remains is to show is that there is a maximal compressing sequence which is a Haken sequence. Again note that for every knot or link $K$ there exists at least one Haken sequence, namely the sequence with one element consisting of the thin levels of $K$.

We now assume that $\{\Sigma_i\}_{i=0}^n$ is a Haken sequence of maximal length such that some component $T$ of $\Sigma_n$ is compressible in the complement of $K$ (i.e. $\{\Sigma_i\}_{i=0}^n$ is not a maximal compressing sequence). Let $D$ be a compressing disk for $T$ in the complement of $K$. By an innermost disk argument, we may assume that all loops of $D \cap \Sigma_n$ are essential on $\Sigma_n \setminus K$. Let $D'$ be the subdisk of $D$ bounded by an innermost such loop. Let $N$ denote the closure of the component of $S^3 \setminus \Sigma_n$ which contains $D'$. Then $\partial D'$ is a compressing disk for $\partial N$, in the complement of $K$.

As $\{\Sigma_i\}_{i=0}^n$ is a Haken sequence there is a 2-sphere $S$ in $N$ such that $S \setminus K$ is a Heegaard splitting for $N \setminus K$. By Lemma 3.1 there is a compressing disk $E$ for $\partial N$, in the complement of $K$, such that $\partial E = \partial D'$ and $E \cap S$ is a simple closed curve, $\delta$. We now compress $S$ along the subdisk of $E$ bounded by $\delta$ to obtain the spheres $S'$ and $S''$ and compress $\Sigma_n$ along $E$ to obtain $\Sigma_{n+1}$ (see Figure 3). Note that $\{\Sigma_i\}_{i=0}^{n+1}$ is also a Haken sequence, contradicting the maximality of the length of our original choice.

\[\Box\]

**Lemma 4.8.** Suppose $K$ is a knot or link in thin position. Let $\{\Sigma_i\}_{i=0}^n$ be a good compressing sequence for $K$. Let $N$ be the closure of some component of $S^3 \setminus \Sigma_n$ and let $S$ be a component of $\partial N$. Let $K'$ denote the components of $K \cap N$ which meet
If all of the components of $K'$ are parallel and connect distinct components of $\partial N$ then each component of $K'$ is vertical.

**Proof.** To prove the lemma we show that the components of $K'$ must be simple. There are then two possibilities: either they are $U$-shaped or vertical. The former is ruled out by Lemma 4.5 and the latter is the desired conclusion.

By way of contradiction, assume the components of $K'$ are not simple. As $\{\Sigma_i\}_{i=0}^n$ is good we may choose a collection of parallel simple arcs $\overline{K'}$ in $N$ such that for each component $\alpha' \subset K'$ there is a component $\alpha' \subset \overline{K}$ with $\partial \alpha' = \partial \alpha'$. We now show that $K$ is isotopic to a knot or link $\overline{K}$ which contains $\overline{K'}$, such that $K \setminus K' = \overline{K} \setminus \overline{K'}$. This then contradicts Lemma 2.2.

Let $B$ be the ball bounded by $S$ on the side opposite $N$. Note that only one endpoint of each arc component of $K'$ meets $B$, since each such arc connects distinct components of $\partial N$. The isotopy is illustrated in Figure 4 in the case where $\partial K'$ is contained in a single thin level $T$. The steps are the same in the case where the components of $K'$ connect distinct thin levels. They are as follows:

1. Shrink $B$ to a small ball $B'$ at the end of $K'$.
2. Contract the arcs of $K'$, pulling $B'$ along with it.
3. Push $B'$ along the arcs of $\overline{K}$.
4. Inflate $B'$ back to $B$.

Note that in the case that $\overline{K'}$ is $U$-shaped we may do a further width-preserving isotopy to make the minima (maxima) of $\overline{K'}$ appear above (below) the minima (maxima) of $K'$. This is necessary to appeal to Case 3 of Lemma 2.2. □
5. The Main Theorem.

In this section we prove our main theorem.

Theorem 5.1. If a knot or link has \( n \) thin levels when put in thin position then its exterior contains a collection of \( n \) disjoint, non-parallel, planar, meridional, essential surfaces.

Proof. Let \( K \) be a knot or link in thin position with \( n \) thin levels. By Lemma 4.7 we may choose a good maximal compressing sequence \( \{ \Sigma_i \}_{i=0}^n \) for \( K \).

Lemma 4.4 implies that the elements of \( \Sigma_n \) are essential in the complement of \( K \). Let \( \mathcal{S} \) denote a collection of spheres in \( S^3 \) such that
(1) every element of $\Sigma_n$ is parallel, in the complement of $K$, to an element of $\mathcal{S}$ and

(2) no two elements of $\mathcal{S}$ are parallel in the complement of $K$.

Our goal is to show that $\mathcal{S}$ has at least $n$ elements. Let $\Gamma$ denote the dual graph of $\mathcal{S}$ in $S^3$. $\Gamma$ is then a tree, whose edges correspond to elements of $\mathcal{S}$. As the number of vertices minus the number of edges of any tree is 1, it suffices to show that $\Gamma$ has at least $n + 1$ vertices.

Note that it is implicit in the assumption that $K$ is in thin position that we have fixed a height function, $h$, on $S^3$. Let $x$ and $y$ denote two critical points of $K$ with respect to $h$, which are separated by a thin level of $K$. Then $x$ and $y$ are separated by an element, $T$, of $\Sigma_n$. We claim that $x$ and $y$ are also separated by an element of $\mathcal{S}$.

Suppose this is not the case and let $M$ denote the component of $S^3 \setminus \mathcal{S}$ which contains $x$ and $y$, so that $T \subset M$. As $T$ is not an element of $\mathcal{S}$, it must be parallel (in the complement of $K$) to some element of $\mathcal{S}$, and hence, to some component, $S$, of $\partial M$. Either $x$ or $y$ lies between $T$ and $S$. Assume the former. Since $T$ and $S$ are parallel in the complement of $K$, and $x$ is a point of $K$ which lies between them, $x$ must lie on a subarc, $\alpha$, of $K$ which connects $T$ to $S$. But Lemma 4.8 implies that $\alpha$ is vertical, contradicting the fact that it contains the critical point, $x$.

We conclude by noting that our assumption that $K$ had $n$ thin levels implies that there is a collection of $n + 1$ critical points of $K$ such that any two are separated by a thin level. The above argument then shows that each of these points must lie in a distinct component of $S^3 \setminus \mathcal{S}$, implying that $\Gamma$ has at least $n + 1$ vertices. $\square$

**Corollary 5.2.** Let $K$ be a knot or link which has a thin presentation with $n$ thin levels. Let $t$ be the smallest number of tetrahedra necessary to triangulate the complement of $K$. Then $t \geq \frac{n}{3}$.

**Proof.** In [Bac] we give an improvement over the classical Kneser-Haken Finiteness Theorem [Kne29, Hak68] and show that in closed manifolds the size of any collection of pairwise disjoint, closed, essential, 2-sided surfaces is at most twice the number of tetrahedra, $|T|$. Although we do not explicitly state a result there for manifolds (and surfaces) with boundary, the same proof shows that if $M$ is a 3-manifold with non-empty boundary and $\mathcal{S}$ is a collection of properly embedded, pairwise disjoint, 2-sided, incompressible and boundary incompressible surfaces then $2|\mathcal{S}| \leq g + 6|T|$, where $g$ is the maximum number of twisted $I$-bundles that can disjointly embed in $M$. In the complement of the knot or link $K$ in $S^3$ we have $g = 0$, so $2|\mathcal{S}| \leq 6t$, or $|\mathcal{S}| \leq 3t$. Now, Theorem 5.1 says there exists such a collection $\mathcal{S}$ such that $n \leq |\mathcal{S}|$. Hence, $n \leq 3t$. $\square$

**References**

[Bac] D. Bachman. A note on Kneser-Haken finiteness. to appear in *Proceedings of the American Mathematical Society.*
D. Bachman and S. Schleimer. Thin position for tangles. *J. of Knot Theory and its Ramifications*, 12(1):117–122, 2003.

D Gabai. Foliations and the topology of three-manifolds iii. *J. Diff. Geom.*, 26:479–536, 1987.

C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2:371–415, 1989.

Wolfgang Haken. Some results on surfaces in 3-manifolds. In *Studies in Modern Topology*, pages 39–98. Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1968.

Daniel J. Heath and Tsuyoshi Kobayashi. Essential tangle decomposition from thin position of a link. *Pacific J. Math.*, 179(1):101–117, 1997.

H. Kneser. Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. *Jahresbericht der Deut. Math. Verein*, 28:248–260, 1929.

Yo’av Rieck and Eric Sedgwick. Thin position for a connected sum of small knots. *Algebr. Geom. Topol.*, 2:297–309 (electronic), 2002.

M. Scharlemann and A. Thompson. Heegaard splittings of (surface) x I are standard. *Math. Ann.*, 295:549–564, 1993.

A. Thompson. Thin position and the recognition problem for the 3-sphere. *Math. Research Letters*, 1:613–630, 1994.

A. Thompson. Thin position and bridge number for knots in the 3-sphere. *Topology*, 36:505–507, 1997.

Y.-Q. Wu. Thin position and essential planar surfaces. preprint.

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