New Turán exponents for two extremal hypergraph problems

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Abstract

An r-uniform hypergraph is called t-cancellative if for any t+2 distinct edges \( A_1, \ldots, A_t, B, C \), it holds that \( (\bigcup_{i=1}^{t} A_i) \cup B \neq (\bigcup_{i=1}^{t} A_i) \cup C \). It is called t-union-free if for any two distinct subsets \( A, B \), each consisting of at most t edges, it holds that \( \cup_{A \in A} A \neq \cup_{B \in B} B \). Let \( C_t(n, r) \) (resp. \( U_t(n, r) \)) denote the maximum number of edges of a t-cancellative (resp. t-union-free) r-uniform hypergraph on n vertices. Among other results, we show that for fixed \( r \geq 3, t \geq 3 \) and \( n \to \infty \),

\[
\Omega(n^{\frac{2r}{r+1} + \frac{2r}{r+1} (\text{mod} \, t+2)}) = C_t(n, r) = O(n^{\frac{r}{r+1}}) \quad \text{and} \quad \Omega(n^{\frac{r}{r+1}}) = U_t(n, r) = O(n^{\frac{r}{r+1}}),
\]

thereby significantly narrowing the gap between the previously known lower and upper bounds. In particular, we determine the Turán exponent of \( C_t(n, r) \) when \( 2 \mid t \) and \( (t/2 + 1) \mid r \), and of \( U_t(n, r) \) when \( (t - 1) \mid r \).

The main tool used in proving the two lower bounds is a novel connection between these problems and sparse hypergraphs.

1 Introduction

A hypergraph \( \mathcal{H} \) on vertex set \([n] := \{1, \ldots, n\}\) is simply a family of distinct subsets of \([n]\), called edges of \( \mathcal{H} \). If each edge is of fixed size \( r \), then \( \mathcal{H} \) is said to be r-uniform or an r-graph. For a positive integer \( t \), \( \mathcal{H} \) is called t-cancellative if for any \( t + 2 \) distinct edges \( A_1, \ldots, A_t, B, C \in \mathcal{H} \), it holds that

\[
(\cup_{i=1}^{t} A_i) \cup B \neq (\cup_{i=1}^{t} A_i) \cup C.
\]

Furthermore, \( \mathcal{H} \) is called t-union-free if for any two distinct subsets of edges \( A, B \subseteq \mathcal{H} \), with \( 1 \leq |A|, |B| \leq t \), it holds that

\[
\cup_{A \in A} A \neq \cup_{B \in B} B.
\]

In this paper we study the maximum size (maximum number of edges) of an r-graph that is either t-cancellative or t-union-free.

To the best of our knowledge (see also [9, 12]), the study of 1-cancellative and 2-union-free hypergraphs (not necessarily uniform) were initiated by Erdős and Katona [14] and Erdős and Moser [8], respectively. Notice that the question of 1-union-free is trivial, since any hypergraph is 1-union-free. The study of the closely related problem of weakly union-free families dates back to the work of Erdős [5] in 1938. Later, the general questions of t-cancellative and t-union-free hypergraphs were first considered by Füredi [12] (for 2-cancellative hypergraphs, see also [16]) and Kautz and Singleton [15], respectively.

Not much is known about these problems for r-graphs and \( t \geq 3 \), besides this work. In fact, most of the known results were derived by translating results from the extensive literature on cover-free hypergraphs to results on cancellative and union-free hypergraphs. Next, we define cover-free hypergraphs and present the simple observations that facilitate the translation of the results.

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An r-graph $\mathcal{H}$ is called $t$-cover-free by Erdős, Frankl and Füredi [6] (see also [15] for a different terminology), if for any $t+1$ distinct edges $A_1, \ldots, A_t, B \in \mathcal{H}$, it holds that

$$B \nsubseteq \bigcup_{i=1}^{t} A_i.$$  

In the literature, the following observations are well-known.

(a) If $\mathcal{H}$ is $(t+1)$-cover-free then it is also $t$-cancellative; if $\mathcal{H}$ is $t$-cancellative then by removing from it at most $1 + \lfloor \frac{t}{2} \rfloor$ edges one gets a subhypergraph of $\mathcal{H}$, that is, $\lfloor \frac{t}{2} \rfloor$-cover-free;

(b) If $\mathcal{H}$ is $t$-cover-free then it is also $t$-union-free; if $\mathcal{H}$ is $t$-union-free then it is also $(t-1)$-cover-free.

These observations follow straightforwardly from the definitions above (see Theorem 3.2 of [12] for the proof of the second claim of (a)).

Below we assume that $r, t$ are fixed integers and $n$ tends to infinity. Let $F_t(n, r), C_t(n, r)$ and $U_t(n, r)$ denote the maximum size of $t$-cover-free, $t$-cancellative and $t$-union-free $r$-graphs on $n$ vertices, respectively. Frankl and Füredi [11] showed that for fixed $r, t$, it holds that

$$F_t(n, r) = (\gamma(r, t) + o(1)) n^{\left\lceil \frac{t}{r} \right\rceil},$$  

(1)

where $\gamma(r, t)$ is a constant depending only on $r, t$ and $o(1) \to 0$ as $n \to \infty$. Then by observations (a), (b), and [11] (see (10) in [13] and (4.2) in [12]) we have

$$\Omega(n^{\left\lceil \frac{t}{r} \right\rceil}) = C_t(n, r) = O(n^{\left\lceil \frac{r}{t} \right\rceil}) \quad \text{and} \quad \Omega(n^{\frac{1}{r}}) = U_t(n, r) = O(n^{\frac{r}{t}}).$$  

(2)

In this paper we only study the exponents of $n$ in $C_t(n, r)$ and $U_t(n, r)$. If $C_t(n, r) = \Theta(n^a)$ for some fixed number $a > 0$, then it is called the Turán exponent of $C_t(n, r)$. The Turán exponent of $U_t(n, r)$ is defined similarly. Clearly, in [2], the upper and lower bounds on the exponents of $n$ are far from being tight, and in this paper we take another step towards bridging them.

In the literature, there are only a handful of results that improve on [2]. For cancellative hypergraphs, it is known that $\frac{\log r}{r} \binom{n}{r} < C_1(n, r) \leq \frac{r}{r-1} \binom{n}{r}$ (see [9] for the upper bound and [15] for the lower bound), and [12] showed that $\Omega(n^{\frac{2}{r}}) = C_2(n, r) = O(n^{\frac{2}{r}})$ and $n^{2-o(1)} < C_2(n, 3) = O(n^2)$. For union-free hypergraphs, it is known that $U_2(n, r) = \Theta(n^{\frac{4r}{3r-1}/2})$ [10] and $n^{2-o(1)} < U_r(n, r) = O(n^r)$ [13].

By the discussion above, it is clear that the Turán exponents of $C_t(n, r)$ are known for $t = 1$ and $t = 2$, $2 \mid r$, and the Turán exponents of $U_t(n, r)$ are only known for $t = 2$. In this paper we present new constructions which considerably narrow the gaps between the upper and lower bounds of [2], as stated next.

### 1.1 $t$-cancellative r-graphs

**Theorem 1.1.** For fixed integers $r \geq 3, t \geq 3$ and $n \to \infty$, it holds that

$$\Omega(n^{\left\lceil \frac{2r}{t+1} \right\rceil}) = C_t(n, r) = O(n^{\left\lceil \frac{2r}{t+1} \right\rceil}).$$  

(3)

Moreover, if $\gcd(2r - \left\lfloor \frac{2r-t-1}{t+1} \right\rfloor, t+1) = 1$, then

$$C_t(n, r) = \Omega(n^{\left\lceil \frac{2r}{t+1} + \frac{r \left( \text{mod} \ t+1 \right)}{t+1} \right\rceil} (\log n)^{t+1}).$$  

Note that by [3] for even $t$ and $r$ divisible by $t/2 + 1$ the Turán exponent of $C_t(n, r)$ is known. We speculate that for any $t, r \geq 3$ the upper bound of [3] can be further improved to $C_t(n, r) = O(n^{\left\lceil \frac{2r}{t+1} \right\rceil})$. In particular, it would be interesting to determine whether $C_3(n, 5) = O(n^2)$. If our guess is correct, then the lower bound of [3] gives the Turán exponent of $C_t(n, r)$ for all $r, t$ satisfying $(t+2) \mid 2r$. 


For \( (t + 2) \nmid 2r \), it may be a difficult problem to determine the asymptotic order of \( C_t(n, r) \). For example, Füredi [12] conjectured that (see Conjecture 12.1 of [12])
\[
n^{k+1-o(1)} < C_2(n, 2k + 1) = o(n^{k+1})
\]
for all fixed \( k \geq 1 \). Moreover, he showed that for \( k = 1 \), the conjectured upper bound \( C_2(n, 3) = o(n^2) \) is closely related to a celebrated and longstanding conjecture of Brown, Erdős and Sós [4] in extremal graph theory (see Section 2 below for the details). We are able to verify the lower bound part of Füredi’s conjecture.

**Theorem 1.2.** For any fixed integer \( k \geq 1 \) and \( n \to \infty \), it holds that \( C_2(n, 2k + 1) > n^{k+1-o(1)} \).

### 1.2 \( t \)-union-free \( r \)-graphs

**Theorem 1.3.** For fixed integers \( r \geq 3, t \geq 3 \) and \( n \to \infty \), we have that
\[
\Omega(n^{\frac{2t}{r-1}}) = U_t(n, r) = O(n^{\frac{1}{r-1}}).
\]
Moreover, if \( \gcd(r, t-1) = 1 \), then \( U_t(n, r) = \Omega(n^{\frac{2t}{r-1}}(\log n)^{\frac{2t}{r-1}}) \).

Note that the upper bound was proved already in [2], whereas the lower bound \( U_t(n, r) = \Omega(n^{\frac{2t}{r-1}}) \), although not explicitly stated, was implied by an earlier work of Blackburn (see Theorem 5 in [4]), on a problem in multimedia fingerprinting. The main novelty of Theorem 1.2 compared to [3] is the application of sparse hypergraphs in proving this result (which gives a simpler proof), and the improved lower bound of \( U_t(n, r) = \Omega(n^{\frac{2t}{r-1}}(\log n)^{\frac{2t}{r-1}}) \).

Clearly, (5) gives the Turán exponent of \( U_t(n, r) \) when \( (t - 1) \nmid r \). Similarly, for \( (t - 1) \nmid r \), the determination of the correct order of \( U_t(n, r) \) may be difficult. Indeed, Füredi and Ruszinkó [13] conjectured that
\[
U_r(n, r) = o(n^2)
\]
for all fixed \( r \geq 3 \), where the conjectured upper bound is also related to the aforementioned conjecture of [4].

Recall that as mentioned previously, prior to this paper and [3], for \( t \geq 3 \) the only known lower bound of \( C_t(n, r) \) (resp. \( U_t(n, r) \)) was derived from that of \( F_t(n, r) \), using [11] and observation (a) (resp. (b)). Surprisingly, we show that sparse hypergraphs, as introduced in the next section, can be used as a unified tool to construct \( t \)-cancellative and \( t \)-union-free \( r \)-graphs (see Lemmas 2.3, 2.4 and 2.5 below for details), thereby providing much better lower bounds for \( C_t(n, r) \) and \( U_t(n, r) \).

The remaining part of this paper is organised as follows. In Section 2 we introduce sparse hypergraphs and in Section 3 we show how they can be used to prove the lower bounds in Theorems 1.1, 1.2 and 1.3. We defer the proof of the upper bound in Theorem 1.2 to Section 4.

**Notations.** We will use the standard Bachmann-Landau notations \( \Omega(\cdot), \Theta(\cdot), O(\cdot) \) and \( o(\cdot) \), whenever the constants are not important. All logarithms are of base 2.

## 2 Constructions based on sparse hypergraphs

For integers \( v \geq r + 1, e \geq 2 \), an \( r \)-graph is called \( \mathcal{G}_r(v, e) \)-free if the union of any \( e \) distinct edges contains at least \( v + 1 \) vertices. Such \( r \)-graphs are also called sparse due to the sparsity of its edges, i.e., any set of \( v \) vertices spans less than \( e \) edges. Let \( f_r(n, v, e) \) denote the maximum number of edges of a \( \mathcal{G}_r(v, e) \)-free \( r \)-graph on \( n \) vertices. Brown, Erdős and Sós [4] showed that
\[
\Omega(n^{\frac{e-1}{r-1}}) = f_r(n, v, e) = O(n^{\frac{e-1}{r-1}}),
\]
where the lower bound was proved by a probabilistic construction using the alteration method (see, e.g. [2]). Indeed, a slightly more sophisticated probabilistic analysis leads to the following stronger lower bound.
Lemma 2.1 (see Lemma 1.7 of [13] and Proposition 6 of [17]). Let $s \geq 1, r \geq 3$ and $(v_i, e_i)$ for $1 \leq i \leq s$ be pairs of integers satisfying $v_i \geq r + 1, e_i \geq 2$.

(a) Let $h := \min \{\frac{e_{i} - 1}{v_i - 1} : 1 \leq i \leq s\}$. Then there exists an $r$-graph with $\Omega(n^h)$ edges that is simultaneously $\mathcal{G}_r(v_i, e_i)$-free for each $1 \leq i \leq s$.

(b) Suppose further that $e_1 \geq 3, \text{gcd}(e_1 - 1, e_1 r - v_1) = 1$ and $\frac{e_{i} - 1}{v_i - 1} < \frac{e_{i+1} - 1}{v_{i+1} - 1}$ for each $2 \leq i \leq s$.

Then there exists an $r$-graph with $\Omega(n \frac{e_{i} - 1}{v_i - 1} (\log n)^{\frac{1}{v_i - 1}})$ edges that is simultaneously $\mathcal{G}_r(v_i, e_i)$-free for each $1 \leq i \leq s$.

In the literature, the following celebrated conjecture is well-known (see, e.g. [3] and [1]).

Conjecture 2.2. For all fixed integers $r > k \geq 2, c \geq 3$ and $n \to \infty$, it holds that $f_r(n, cr - (e - 1)k + 1, e) = o(n^k)$.

In [12] and [13] it was shown that $C_2(n, 3) = \Theta(f_3(n, 7, 4))$ and $U_r(n, r) = O(f_r(n, r^2 - r + 1, r + 1))$, respectively. Indeed, those two bounds and Conjecture 2.2 motivated the two conjectured upper bounds in [3] and [1]. It is to our surprise that sparse hypergraphs are even more closely related to $t$-cancellative and $t$-union-free $r$-graphs, than what was known before. In particular, we prove that $t$-cancellative and $t$-union-free $r$-graphs can both be constructed by sparse hypergraphs, as stated in the following three lemmas.

Lemma 2.3. Let $H$ be an $r$-graph that is simultaneously $\mathcal{G}_r(tr + \lceil \frac{2r-t-1}{t+2} \rceil, t+2)$-free and $\mathcal{G}_r(2r - \lceil \frac{2r-t-1}{t+2} \rceil - 1, 2)$-free, then it must be $t$-cancellative.

Lemma 2.4. If a $(2k+1)$-graph is $\mathcal{G}_r(4k + 2, 3)$-free, then it is also $2$-cancillative.

Lemma 2.5. If an $r$-partite $r$-graph $H$ is simultaneously $\mathcal{G}_r(tr - r, t)$-free and $\mathcal{G}_r(tr, 2t)$-free, then it is also $t$-union-free.

Using the above lemmas in concert with Lemma 2.1 one can easily deduce the lower bounds in Theorems 1.1, 1.2 and 1.3 as detailed below.

2.1 Proofs of Theorems 1.1, 1.2 and 1.3

Proof of Theorem 1.1. The proof of the upper bound is postponed to Section 3. For the first lower bound we invoke Lemma 2.1 (a). Indeed, since for $r \geq 3, t \geq 3$, $\frac{2r - \lceil \frac{2r-t-1}{t+2} \rceil}{t+1} \leq \lceil \frac{2r-t-1}{t+2} \rceil + 1$ and $\frac{2r - \lceil \frac{2r-t-1}{t+2} \rceil}{t+1} = \lceil \frac{2r}{t+2} \rceil + \frac{2r \text{ mod } (t+2)}{t+1} := h$, then there exists an $r$-graph with $\Omega(n^h)$ edges that is simultaneously $\mathcal{G}_r(tr + \lceil \frac{2r-t-1}{t+2} \rceil, t+2)$-free and $\mathcal{G}_r(2r - \lceil \frac{2r-t-1}{t+2} \rceil - 1, 2)$-free. Then, by Lemma 2.3 this $r$-graph is $t$-cancellative.

For the second lower bound we invoke Lemma 2.1 (b). Indeed, if $\text{gcd}(2r - \lceil \frac{2r-t-1}{t+2} \rceil, t + 1) = 1$, then it is not hard to see that $t + 2 \mid 2r - t - 1$ and hence $\frac{2r - \lceil \frac{2r-t-1}{t+2} \rceil}{t+1} < \lceil \frac{2r-t-1}{t+2} \rceil + 1$. Lastly, we invoke again Lemma 2.3.

Proof of Theorem 1.2. It was shown by Alon and Shapira (see Theorem 1 of [1]) that for any fixed $k \geq 1$, $f_{2k+1}(n, 4k + 2, 3) > n^{k+1-o(1)}$, where $o(1) \to 0$ as $n \to \infty$. The result follows by invoking Lemma 2.4.

To prove Theorem 1.3 we need a well-known result of Erdős and Kleitman [7].
**Lemma 2.6** (see Theorem 1 of [7]). Any $r$-graph $\mathcal{H}$ contains an $r$-partite subhypergraph $\mathcal{H}^* \subseteq \mathcal{H}$ with at least $|\mathcal{H}^*| \geq \frac{k^2}{t^2} |\mathcal{H}|$ edges.

**Proof of Theorem 1.3** For the first lower bound, observe that for $r, t \geq 3$, $\frac{t^2}{r-1} < \frac{r^2}{t-1}$, so by Lemma 2.1 (a) and Lemma 2.2 there exists an $r$-partite $r$-graph with $\Omega(n^{\frac{r^2}{t-1}})$ edges that is simultaneously $\mathcal{G}_r(r(t-r, t)-free$ and $\mathcal{G}_r(2r-t, t)-free$. Then by Lemma 2.3 this $r$-graph is $t$-union free. The second lower bound is proved similarly by invoking Lemma 2.1 (b), Lemma 2.6 and Lemma 2.5.

The proofs of Lemmas 2.3, 2.4 and 2.5 are presented in the next section.

3 Proofs of Lemmas 2.3, 2.4 and 2.5

3.1 Proof of Lemma 2.3

We prove the following claim which is slightly more general than Lemma 2.3. The proof of the latter follows by setting $x = \lceil \frac{2r-t-1}{t+2} \rceil$.

**Claim 3.1.** Let $\mathcal{H}$ be an $r$-graph that is simultaneously $\mathcal{G}_r(tr+x, t+2)$-free and $\mathcal{G}_r(2r-x-1, 2)$-free for some integer $0 \leq x \leq r-1$, then $\mathcal{H}$ is $r$-cancellative.

**Proof.** Assume towards contradiction that there exist four distinct edges $A_1, \ldots, A_t, B, C$ of $\mathcal{H}$ such that $(\cup_{i=1}^t A_i) \cup B = (\cup_{i=1}^t A_i) \cup C$, hence

$$\left(B \cup C\right) \setminus (B \cap C) \subseteq \cup_{i=1}^t A_i. \quad (7)$$

Furthermore, since $\mathcal{H}$ is $\mathcal{G}_r(2r-x-1, 2)$-free then

$$|B \cap C| \leq x. \quad (8)$$

Therefore

$$|\left(\cup_{i=1}^t A_i\right) \cup B \cup C| = |\cup_{i=1}^t A_i| + |B \cup C| - |\left(\cup_{i=1}^t A_i\right) \cap (B \cup C)|$$

$$\leq tr + |B \cup C| - |\left(\cup_{i=1}^t A_i\right) \cap (B \cup C)|$$

$$\leq tr + |B \cup C| - |B \cup C| - |(B \cap C)|$$

$$\leq tr + x, \quad (9)$$

where (7) and (10) follow from (7) and (8) respectively. The contradiction follows since $\mathcal{H}$ is $\mathcal{G}_r(tr+x, t+2)$-free.

3.2 Proof of Lemma 2.4

Let $\mathcal{H}$ be a $(2k+1)$-graph that is $\mathcal{G}_{2k+1}(4k+2, 3)$-free, we show next that $\mathcal{H}$ is also 2-cancellative. Suppose for contradiction that there exist four distinct edges $A_1, A_2, B, C \in \mathcal{H}$ such that $A_1 \cup A_2 \cup B = A_1 \cup A_2 \cup C$. Let $|B \cap C| = x$ for some integer $0 \leq x \leq 2k$. It is easy to see that $|\left((B \cup C) \setminus (B \cap C)\right)| = 2 \times (2k+1-x)$ and $|\left((B \cup C) \setminus (B \cap C)\right)| \subseteq A_1 \cup A_2$. Therefore, by the pigeonhole principle, there exists $i \in \{1, 2\}$ such that $|A_i \cap \left((B \cup C) \setminus (B \cap C)\right)| \geq 2k+1-x$, which implies that $|A_i \setminus (B \cup C)| \leq x$. Then

$$|A_i \cup B \cup C| = |B \cup C| + |A_i \setminus (B \cup C)| = 2(2k+1) - x + |A_i \setminus (B \cup C)| \leq 4k+2,$$

which violates the assumption that $\mathcal{H}$ is $\mathcal{G}_{2k+1}(4k+2, 3)$-free.
3.3 Proof of Lemma 2.5

The following fact is easy to verify.

Fact 3.2. If an \( r \)-graph \( H \) with at least \( t \) edges is \( \mathcal{G}_r(tr-r,t) \)-free, then it is also \( \mathcal{G}_r(sr-r,s) \)-free for any \( 1 \leq s \leq t \).

We need one more result to prove Lemma 2.5.

Lemma 3.3. Let \( H \) be an \( r \)-partite \( r \)-graph with vertex parts \( V_1, \ldots, V_r \). If \( H \) is \( \mathcal{G}_r(tr-r,t) \)-free, then for any \( t \) distinct edges \( A_1, \ldots, A_t \) of \( H \), there exists \( 1 \leq i \leq r \), such that the \( t \) vertices \( A_1 \cap V_i, \ldots, A_t \cap V_i \) are all distinct.

Proof. Suppose for contradiction that the \( t \) distinct edges \( A_1, \ldots, A_t \in H \) violate the assertion of the lemma. Then, for each \( 1 \leq i \leq r \), \( |\cup_j (A_j \cap V_i)| \leq t-1 \), hence \( |\cup_j A_j| = \sum_{i=1}^r |\cup_j (A_j \cap V_i)| \leq tr-r \), and we arrive at a contradiction.

Proof of Lemma 2.5. Suppose for contradiction that \( H \) is not \( t \)-union-free. Then there exist distinct subsets \( A, B \subseteq H, 1 \leq |A|, |B| \leq t \), such that \( \cup_{A \in A} A = \cup_{B \in B} B \).

It is easy to see that \( |A| = |B| = t \). Indeed, assume that \( |A| = s \leq t-1 \) and let \( A = \{A_1, \ldots, A_s\} \). Since \( B \subseteq \cup_{i=1}^s A_i \) for \( B \in B \), then

\[
|\cup_{i=1}^s A_i \cup B| = |\cup_{i=1}^s A_i| \leq sr,
\]

which by Fact 3.2 violates the \( \mathcal{G}_r(sr, s+1) \)-freeness of \( H \).

It is also not hard to show that \( A \cap B \neq \emptyset \), since otherwise \( A \cup B \) consists of \( 2t \) distinct edges of \( H \) that satisfy

\[
|\cup_{A \in A} A \cup (\cup_{B \in B} B)| = |\cup_{A \in A} A| \leq tr,
\]

violating the \( \mathcal{G}_r(tr, 2t) \)-freeness of \( H \). Therefore we may assume that \( |A \cap B| = i \geq 1 \).

Suppose that we have

\[
A := \{C_1, \ldots, C_i, A_{i+1}, \ldots, A_t\} \quad \text{and} \quad B := \{C_1, \ldots, C_i, B_{i+1}, \ldots, B_t\},
\]

where \( C_1, \ldots, C_i, A_{i+1}, \ldots, A_t, B_{i+1}, \ldots, B_t \) are \( 2t-i \) distinct edges of \( H \). By applying Lemma 3.3 to the \( t \) edges in \( A' := A \cup \{B_{i+1}\} \setminus \{C_1\} \) we conclude that there exists \( 1 \leq j \leq r \) such that the \( t \) elements \( A \cap V_j, A \in A' \) are pairwise distinct.

Let \( c_l = C_l \cap V_j \), for \( 1 \leq l \leq i \) and \( a_l = A_l \cap V_j, \) for \( i < l \leq t \), and note that \( b_{i+1}, c_2, \ldots, c_i, a_{i+1}, \ldots, a_t \) are pairwise distinct vertices. By assumption

\[
(\cup_{A \in A} A) \cap V_j = \{c_1, \ldots, c_i, a_{i+1}, \ldots, a_t\} = \{c_1, \ldots, c_i, b_{i+1}, \ldots, b_t\} = (\cup_{B \in B} B) \cap V_j.
\]

Hence, by (11) and the pairwise distinctness of \( b_{i+1}, c_2, \ldots, c_i, a_{i+1}, \ldots, a_t \), it is clear that \( b_{i+1} = c_1 \) and therefore also \( c_1, c_2, a_{i+1}, \ldots, a_t \) are pairwise distinct too.

We conclude that the leftmost set of (11) consists of exactly \( t \) distinct elements, whereas the rightmost set consists of at most \( t-1 \) distinct elements, and we arrive at a contradiction.

4 Proof of Theorem 1.1, the upper bound

We will need several auxiliary lemmas before presenting the proof that \( C_t(n,r) = O(n^\lceil \frac{r}{r+2} \rceil) \).

The following fact is easy to verify.

Fact 4.1. A \( t \)-cancellative \( r \)-graph with at least \( t + 2 \) edges is also \( s \)-cancellative for any \( 1 \leq s \leq t \).
For an $r$-graph $\mathcal{H} \subseteq \binom{[n]}{r}$ and a subset $T \subseteq [n]$ with $1 \leq |T| \leq r$, the codegree of $T$ with respect to $\mathcal{H}$, denoted as $\text{deg}_\mathcal{H}(T)$, is the number of edges of $\mathcal{H}$ that contain $T$, i.e., $\text{deg}_\mathcal{H}(T) = |\{A \in \mathcal{H} : T \subseteq A\}|$.

**Lemma 4.2.** For integers $s \geq 1$ and $1 \leq k \leq r$, any $r$-graph $\mathcal{H}$ contains a subhypergraph with at least $\max\{|\mathcal{H}| - s\binom{n}{k}, 0\}$ edges, such that each $k$-subset has codegree either zero or at least $s+1$.

*Proof.* Let us successively remove from $\mathcal{H}$ the edges that contain at least one $k$-subset of codegree at most $s$. Let $A_i$ be the $i$-th removed edge from $\mathcal{H}$, and $T_i$ be some $k$-subset of codegree at most $s$ contained in $A_i$. We say that $T_i$ is responsible for $A_i$. During the edge removal process, the codegree (with respect to the $r$-graph after the removal of the edges $A_1, A_2, \ldots$) of any $k$-subset can only decrease, therefore each of the $\binom{n}{k}$ $k$-subsets can be responsible for the removal of at most $s$ edges. Hence, the process will terminate after at most $s\binom{n}{k}$ edge removals. Clearly, the resulting $r$-graph is either empty or has at least $|\mathcal{H}| - s\binom{n}{k}$ edges, and it satisfies the assertion on the codegrees. □

**Lemma 4.3.** For any positive integer $t$, $\mathcal{C}_t(n, r) = O(n^{\lceil \frac{r-1}{t+1} \rceil})$.

*Proof.* It is sufficient to prove the lemma under the assumption that $(t+1) \mid r$. Indeed, assuming that $\mathcal{C}_t(n, (t+1)k) = O(n^k)$, then for $r$ not divisible by $t+1$ write $r = (t_1 + 1)(k - 1) + y$, with $1 \leq y \leq t_1$ an integer. Then,

$$\mathcal{C}_t(n, r) \leq \mathcal{C}_t(n + t + 1 - y, r + t + 1 - y) = \mathcal{C}_t(n + t + 1 - y, (t+1)k) = O(n^k) = O(n^{\lceil \frac{r-1}{t+1} \rceil}),$$

as needed.

Next we show that $\mathcal{C}_t(n, (t+1)k) = O(n^k)$. Let $\mathcal{H}$ be any $2t$-cancellative $(t+1)k$-graph defined on $n$ vertices. We proceed to show that $|\mathcal{H}| \leq 2\binom{n}{k} + 2t + 1$. Assume towards contradiction that $|\mathcal{H}| \geq \binom{n}{k} + 2t + 2$, then by applying Lemma 4.2 with $s = 2$ there exists a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ with at least $|\mathcal{H}'| \geq 2t + 2$ edges, such that no $k$-subset has codegree equals to one or two.

Let $C \in \mathcal{H}'$ be an arbitrary edge and $X \subseteq C$ an arbitrary $k$-subset. By the codegree assertion, there exist and edge $D \neq C$ that contains $X$. Let $X_1, \ldots, X_t$ and $Y_1, \ldots, Y_t$ be arbitrary partitions of $C \setminus X$ and $D \setminus X$ respectively, to pairwise disjoint $k$-subsets. Again, by the codegree assertion, for each $i$ there exist edges $A_i, B_i \in \mathcal{H}' \setminus \{C, D\}$ such that $X_i \subseteq A_i$ and $Y_i \subseteq B_i$. Consequently,

$$A_1 \cup \cdots \cup A_t \cup B_1 \cdots \cup B_t \cup C = A_1 \cup \cdots \cup A_t \cup B_1 \cdots \cup B_t \cup D,$$

and we arrive at a contradiction to the fact that $\mathcal{H}$ is $2t$-cancellative combined with Fact 4.1. □

**Proof of Theorem 1.1, the upper bound.** For even $t$, the result follows from Lemma 4.3. For odd $t$

$$\mathcal{C}_t(n, r) \leq \mathcal{C}_{t-1}(n, r) = O(n^{\lfloor \frac{r-1}{t+1} \rfloor}) = O(n^{\lfloor \frac{r-2}{t+1} \rfloor}).$$

□

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