Electromagnetic Corrections to the decays $\eta \to 3\pi$

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Abstract

We calculate the electromagnetic corrections to the decays $\eta \to 3\pi$ in next to leading order in the chiral expansion. We find that the corrections are small in accordance with Sutherland’s theorem and modify neither rate nor the Dalitz plot distributions noticeably.

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1 Introduction

In the isospin limit $m_u = m_d$ and $e = 0$, the decay $\eta \to 3\pi$ is forbidden. Consequently, the decay amplitude receives contributions from the QCD isospin violating interaction

$$\mathcal{H}_{\text{QCD}}(x) = \frac{(m_d - m_u)}{2} \langle \bar{d}d - \bar{u}u \rangle(x), \quad (1.1)$$

and from the electromagnetic interactions

$$\mathcal{H}_{\text{QED}}(x) = -\frac{1}{2}e^2 \int dy D^{\mu\nu}(x - y) T(j_\mu(x)j_\nu(y)), \quad (1.2)$$

where $D^{\mu\nu}(x - y)$ is the photon propagator. Using soft pion techniques, it has been shown that the electromagnetic contribution is much too small to account for the experimentally observed rate \[1, 2\]. Much work has been devoted to study the effects of the QCD contribution, equation (1.1). Gasser and Leutwyler \[3\] carried out the one-loop calculation within chiral perturbation theory \[4, 5, 6, 7\]. Although they found large corrections to the rate \[3\], their result failed to reproduce the experimental rates \[8\]. Recently, the complete unitary corrections were evaluated \[9\]. Despite considerable uncertainties, also these corrections are not quite sufficient to account for the observed rate, if the usually assumed value of $(m_u - m_d)$ \[10\] is taken. Instead, a somewhat smaller up-quark mass is required.

In this situation, it is of interest to reconsider the electromagnetic corrections. Strictly speaking, they may be divided into the indirect ones, affecting the parameters such as $F_\pi$ which enter the calculation and the direct ones, specific for the process under consideration. In the following we will be concerned with the latter only. Corrections to $F_\pi$ were calculated previously \[11\] and change its value by about 1 \%, leading to a noteworthy increase of the eta decay rate by about 4 \%.

Within the framework of chiral perturbation theory, electromagnetic correction can also be described by a series of effective operators of increasing power in momentum or masses of the mesons \[12\]. Sutherland’s theorem \[1\] states that the first correction of order $e^2p^0$ vanishes where $p$ is a typical momentum. Thus, the electromagnetic corrections are at most of order $e^2p^2$. If the decay amplitude is assumed to be linear in the Mandelstam variables, these corrections are further suppressed, i.e. $p^2$ is of order $M_\pi^2$ rather than $M_\eta^2$ \[2, 13\].

A similar result (Dashen’s theorem \[14\]) for the electromagnetic mass differences states that they are equal for the pions and for the Kaons to leading order in the chiral expansion. Recently, it was argued that this equality could be violated \[15\] (see, however \[16\]), and one might expect then that also Sutherland’s theorem receives larger corrections than previously thought which could enhance the decay rate of the $\eta$. In this article we determine the $O(e^2p^2)$ electromagnetic corrections to the decay amplitudes $\eta \to 3\pi$. We show that they can be safely neglected compared

\[3\] The Dalitz plot distribution does not get equally large corrections from higher orders in the chiral expansion. It is therefore possibly more sensitive to electromagnetic corrections.
to the QCD isospin violating contributions, both for the rates and the Dalitz plot
distribution. The paper is organized as follows: in sections 2 and 3 we review the
chiral lagrangian for electromagnetic interactions up to order $p^2 \alpha$. In section 4 we
calculate the electromagnetic corrections to $\eta \to 3\pi$ to one-loop chiral perturbation
theory. The numerical results are given in section 5 and section 6 contains our
conclusions.

2 Effective low-energy Lagrangian

To lowest order in Chiral Pertubation Theory (ChPT), including the electromagnetic
interactions, the Lagrangian $\mathcal{L}_2$ is given by [5, 7, 17]

$$
\mathcal{L}_2 = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\xi}{2} (\partial_\mu a_\mu)^2 \\
+ \frac{F_0^2}{4} (\nabla_\mu U \nabla_\mu U^\dagger) + \frac{F_0^2}{4} (U_\chi^\dagger + \chi U^\dagger) + C \langle Q_R U Q_L U^\dagger \rangle.
$$

Here, $f_{\mu\nu}$ is the field strength tensor of the photon field $a_\mu$

$$
f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{2.4}
$$

The parameter $\xi$ is the gauge fixing parameter, which is set to $\xi = 1$ henceforth.
The pseudoscalar meson fields are contained in the usual way in the matrix $U$, and
the field $\chi$ incorporates the coupling of the pseudoscalar mesons to the scalar and
pseudoscalar currents $s$ and $p$

$$
\chi = 2B_0 (s + ip) = 2B_0 M + \ldots \tag{2.5}
$$

where the quark mass-matrix is

$$
M = \begin{pmatrix}
  m_u & 0 & 0 \\
  0 & m_d & 0 \\
  0 & 0 & m_s
\end{pmatrix} \tag{2.6}
$$

The covariant derivative $\nabla_{\mu} U$ defines the coupling of the pseudoscalar mesons to
the photon field $a_\mu$, the external vector current $V_{\mu}$, and the external axial current
$A_{\mu}$

$$
\nabla_{\mu} U = \partial_{\mu} U - i (V_{\mu} + Q_R a_{\mu} + A_{\mu}) U - i U (V_{\mu} + Q_L a_{\mu} - A_{\mu}). \tag{2.7}
$$

The two spurios fields $Q_R$ and $Q_L$ are introduced to construct a $SU(3)_L \otimes SU(3)_R$
invariant Lagrangian. In the later calculation they will be fixed by $Q_R = Q_L = Q$, where $Q$
is the charge matrix of the three light quarks

$$
Q = \frac{e}{3} \begin{pmatrix}
  2 & 0 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & -1
\end{pmatrix}. \tag{2.8}
$$
The constant $F_0$ is up to order $O(m_q)$ the pion decay constant.

The low energy constant $C$ determines the electromagnetic part of the masses $M_{\pi^\pm}, M_{K^\pm}$, of the charged pion and Kaon in the chiral limit

$$C\langle QUQU^\dagger \rangle = -\frac{2e^2C}{F^2}(\pi^+\pi^- + K^+K^-) + ....$$

Clearly, $C\langle QUQU^\dagger \rangle$ contributes equally to the square of the masses of $\pi^\pm$ and $K^\pm$, in agreement with Dashen’s theorem [14].

Since the external currents $V^t$ and $A^\mu$ are of order $O(p)$, where $p$ is their external momentum, the product $QA^\mu$ must have dimension $O(p)$. In order to formally maintain consistent chiral counting, it is convenient to set [12]

$$e \sim O(p), \quad a^\mu \sim 1,$$  

such that $L_2$ is $O(p^2)$.

\section{Effective Lagrangian to order $O(p^4)$}

At order $O(p^4)$ the generating functional has three different types of contributions:

- An explicit local action of order $O(p^4)$
- The one-loop graphs associated with the lowest order Lagrangian $L_2$.
- The anomaly is of order $O(p^4)$: its contributions will not be discussed here. For a discussion, the reader is referred to the literature [3, 4].

Since we are not concerned here with the strong interactions, [3], we only give the electromagnetic terms. The most general chiral and Lorentz invariant, $P$ and $C$ symmetric Lagrangian of order $O(p^4)$ is [12]

$$L_4 = K_1F_0^2\langle \nabla^\mu U^\dagger \nabla_\mu U \rangle \langle Q^2 \rangle + K_2F_0^2\langle \nabla^\mu U^\dagger \nabla_\mu U \rangle \langle QUQU^\dagger \rangle$$

$$+ K_3F_0^2\left(\langle \nabla^\mu U^\dagger QU \rangle \langle \nabla_\mu U^\dagger QU \rangle + \langle \nabla^\mu QUQU^\dagger \rangle \langle \nabla_\mu QUQU^\dagger \rangle \right)$$

$$+ K_4F_0^2\langle \nabla^\mu U^\dagger QU \rangle \langle \nabla_\mu QUQU^\dagger \rangle + K_5F_0^2\left\langle \left\{ \nabla^\mu U^\dagger, \nabla_\mu U \right\} Q^2 \right.$$  

$$+ K_6F_0^2\langle \nabla^\mu U^\dagger \nabla_\mu UQU^\dagger QU + \nabla^\mu UQU^\dagger QUQU^\dagger \rangle$$

$$+ K_7F_0^2\langle \chi^\dagger U + U^\dagger \chi \rangle \langle Q^2 \rangle + K_8F_0^2\langle \chi^\dagger U + U^\dagger \chi \rangle \langle QUQU^\dagger \rangle$$

$$+ K_9F_0^2\langle \chi^\dagger U + U^\dagger \chi \rangle Q^2 + \langle \chi U^\dagger + U \chi^\dagger \rangle Q^2$$

$$+ K_{10}F_0^2\langle \chi^\dagger U + U^\dagger \chi \rangle QUQU^\dagger QU + \langle \chi U^\dagger + U \chi^\dagger \rangle QUQU^\dagger$$

$$K_{11}F_0^2\langle \chi^\dagger U - U^\dagger \chi \rangle QUQU^\dagger QU + \langle \chi U^\dagger - U \chi^\dagger \rangle QUQU^\dagger$$

$$K_{12}F_0^2\langle \nabla^\mu U^\dagger \left[ \nabla_\mu Q^R, Q \right] U + \nabla^\mu U \left[ \nabla_\mu Q^L, Q \right] U^\dagger \rangle$$

$$+ K_{13}F_0^2\langle \nabla^\mu Q^R U^\dagger \nabla_\mu Q^L U^\dagger \rangle + K_{14}F_0^2\langle \nabla^\mu Q^R U^\dagger \nabla_\mu Q^R U + \nabla^\mu Q^L \nabla_\mu Q^L \rangle$$

$$+ K_{15}F_0^4\langle QUQU^\dagger \rangle^2 + K_{16}F_0^4\langle QUQU^\dagger \rangle \langle Q^2 \rangle + K_{17}F_0^4\langle Q^2 \rangle^2,$$  

(3.11)
where $F^R_{\mu\nu}$ and $F^L_{\mu\nu}$ are the field strength tensors of $F^R_\mu$ and of $F^L_\mu$ respectively

\begin{align*}
F^R_\mu &= V_\mu + Q^R a_\mu + A_\mu \\
F^L_\mu &= V_\mu + Q^L a_\mu - A_\mu \\
F^I_{\mu\nu} &= \partial_\mu F^I_\nu - \partial_\nu F^I_\mu - i \left[ F^I_\mu, F^I_\nu \right], \quad I = R, L
\end{align*}

(3.12)

and the covariant derivative $\nabla_\mu Q^I$ is defined as

\begin{align*}
\nabla_\mu Q^I &= \partial_\mu Q^I + i \left[ Q^I, F^I_\mu \right], \quad I = R, L.
\end{align*}

(3.13)

Under a chiral transformation the covariant derivative $\nabla_\mu Q^I$ transforms like the corresponding $Q^I$

\begin{align*}
\nabla_\mu Q^I &\rightarrow V_I \nabla_\mu Q^I V_I^{-1} \quad I = R, L \\
V_{R,L} &\in SU(3)_{R,L}.
\end{align*}

(3.14)

We have set $Q^R = Q^L = Q$. Only in the covariant derivative $\nabla_\mu Q^I$ have we kept the index $I = R, L$ in order to make explicit the proper chiral transformation. The coupling constants $K_1 \ldots K_{17}$ are not determined by chiral symmetry. While in principle calculable from the fundamental lagrangian of QCD and QED, we consider them as phenomenological constants. $K_1, K_7, K_{16}$ are electromagnetic corrections to $F^2_0, B_0$ and $C$ respectively.

In the next step, the loops generated by the Lagrangian $L_2$ are calculated. They lead to terms proportional to $p^4$. Their divergencies are cancelled by infinite counterterms in the constants $K_i$. The procedure is standard and will not displayed here in detail [12]. Using the usual dimensional regularization scheme with scale $\mu$, one finds that the ultraviolet divergencies can be absorbed in the coupling constants $K_i$ with the following renormalization of the low-energy coupling constants [12]:

\begin{align*}
K_i &= K_i^\gamma(\mu) + \Sigma_i \lambda \\
\lambda &= \frac{\mu^{d-4}}{16\pi^2} \left( \frac{1}{d-4} - \frac{1}{2} \left[ \ln(4\pi) + \Gamma'(1) + 1 \right] \right)
\end{align*}

(3.15)
\[ \Sigma_1 = \frac{3}{4} \quad \Sigma_2 = Z \quad \Sigma_3 = -\frac{3}{4} \]
\[ \Sigma_4 = 2Z \quad \Sigma_5 = -\frac{9}{4} \quad \Sigma_6 = \frac{3}{2}Z \]
\[ \Sigma_7 = 0 \quad \Sigma_8 = Z \quad \Sigma_9 = -\frac{1}{4} \]
\[ \Sigma_{10} = \frac{3}{2}Z + \frac{1}{4} \quad \Sigma_{11} = \frac{1}{8} \quad \Sigma_{12} = \frac{1}{8} \]  
\[ \Sigma_{13} = 0 \quad \Sigma_{14} = 0 \quad \Sigma_{15} = 20Z^2 + 3Z + \frac{3}{2} \]
\[ \Sigma_{16} = -4Z^2 - \frac{3}{2}Z - 3 \quad \Sigma_{17} = 2Z^2 - \frac{3}{2}Z + \frac{3}{2} \]
\[ Z = \frac{C}{\mu_0} \]

Here, the \( K^r_i(\mu) \) are the renormalized couplings at the scale \( \mu \).

The scaling behaviour of \( K^r_i(\mu) \) is determined by the requirement
\[ \frac{dK_i}{d\mu} = 0 \] (3.17)
which implies for \( K^r_i(\mu) \) in the limit \( d = 4 \)
\[ K^r_i(\mu) = K^r_i(\mu_0) - \frac{\Sigma_i}{16\pi^2} \ln\left(\frac{\mu}{\mu_0}\right). \] (3.18)

4 Electromagnetic contributions to \( \eta \to 3\pi \)

The electromagnetic contributions to the decay \( \eta \to 3\pi \) have been considered long
ago \([4, 13]\). The transition amplitude is suppressed due to a soft-pion theorem \([4]\).

At next-to-leading order in a low energy expansion, there is a further suppression
factor \( m^2_\pi/m^2_K \), if linear dependence on the energy of the odd pion is assumed. This
led to the conclusion that the electromagnetic interactions alone fail completely to
explain the observed rate.

Here the problem is reinvestigated in the framework of effective chiral lagrangians,
as reviewed in section 2. We calculate the corrections of order \( e^2p^2 \), where \( p^2 \) is a
generic low energy momentum. We work in the isospin limit, as corrections of order
\( e^2(m_d - m_u) \) are very small.

As usual, we define the decay amplitude \( A \) by
\[ <\pi^0\pi^+\pi^-|\eta> = i \left(2\pi^4\right) \delta^4(P_f - P_i)A(s, t, u), \] (4.19)
with the Mandelstam variables
\[ s = (p_\eta - p_{\pi^0})^2 \quad t = (p_\eta - p_{\pi^+})^2 \quad u = (p_\eta - p_{\pi^-})^2. \] (4.20)
Due to G-parity, the three pions emerge in a $I = 1$ state. Hence, the decay amplitude into three neutral pions, $\bar{A}(s, t, u)$, is also determined by $A(s, t, u)$

$$<\pi^0\pi^0\pi^0 \text{out} | \eta> = i (2\pi^4) \delta^4(P_f - P_i) \bar{A}(s, t, u)$$

$$\bar{A}(s, t, u) = A(s, t, u) + A(t, u, s) + A(u, s, t). \quad (4.21)$$

Charge conjugation invariance requires $A(s, t, u)$ to be symmetric with respect to interchanges of $t$ and $u$.

The electromagnetic contributions to $A(s, t, u)$ may be written as

$$A_{\text{QED}}(s, t, u) = e^2 f(\Lambda, p^2, m_u, m_d, m_s, \ldots) \quad (4.22)$$

where $\Lambda$ is the scale of the theory and $p^2$ stands for any of the variables $s, t, u$. Keeping ratios $p^2:m_u:m_d:m_s$ fixed, $f$ can be expanded according to

$$f = f_0 + f_1 + \ldots \quad (4.23)$$

$f_0$ is of order $p^0$, $f_1$ of order $p^2$ etc. The lowest order term $f_0$ vanishes, as was shown by Sutherland [1]. As noted, $f_1$ is also suppressed, if the energy dependence of the amplitude is linear. This follows from the soft-pion theorem [1]

$$\lim_{p_0 \to 0} A_{\text{QED}}^{+-0} = \lim_{p_0 \to 0} A_{\text{QED}}^{000} = O(e^2 \hat{m}). \quad (4.24)$$

Hence, for a linear amplitude

$$A_{\text{QED}}^{+-0} = a + b \cdot s, \quad (4.25)$$

it follows that at the soft-pion point $s = M_\eta^2 + 3M_\pi^2$, and $t = u = 0$

$$A_{\text{QED}}^{+-0}(p_0 = 0) = a + b(M_\eta^2 + 3M_\pi^2) \quad (4.26)$$

$$A_{\text{QED}}^{000}(p_0 = 0) = 3a + b(M_\eta^2 + 3M_\pi^2), \quad (4.27)$$

and therefore

$$a = 0, \quad b = O\left(\frac{\hat{m}}{m_s}\right). \quad (4.28)$$

In chiral perturbation theory the term $f_1$ can be calculated in a systematic manner. The assumption (4.25) is seen to be violated by non-local terms arising from one-loop graphs. However, only Kaons appear in the loop, therefore the resulting amplitude is smooth throughout the physical region. Equation (4.24) is then still a good approximation and the $\frac{\hat{m}}{m_s}$ suppression is effectively at work. This is the main reason why the next-to-leading order term $f_1$ is small.

The relevant diagrams are shown in figure 1. The graphs a) and b) are the non-local unitary and the tadpole graphs respectively, and c) contains the direct and pole graphs.
Figure 1: Graphs contributing to the decay $\eta \rightarrow 3\pi$. The filled circles denote vertices of order $O(e^2)$ and the filled boxes those of order $O(e^2p^2)$.

The electromagnetic vertex in these diagrams is given by the leading order term in eq. (2.3) and by eq. (3.11). There are no graphs with pions in the loop as they are in addition suppressed by the $\pi^0 - \eta$ mixing angle $\varepsilon$, which is of the order

$$O(\varepsilon) \approx 10^{-2} = O\left(\frac{m_u - m_d}{m_s}\right).$$
Also, the mass difference \((M_{K^\pm}^2 - M_{K^0}^2)_{QCD}\) contributes only to the order \(O\left(\frac{m_u-m_d}{m_s}\right)\) and can be neglected.

We then obtain for the unitary corrections
\[
U(s,t,u) = \frac{C}{12\sqrt{3}F_0^6} \left[ 3s(3s - 4M_K^2) - \frac{1}{3}C_{kk}(s) + 6(3s - 4M_K^2)J_{kk}(s) \right. \\
- 3(3t - 4M_K^2)J_{kk}(t) - 3(3u - 4M_K^2)J_{kk}(u) \\
\left. - (5s - 4M_K^2)J_{kk}(0) \right], \tag{4.29}
\]
for the tadpoles
\[
T(s) = - \frac{C}{\sqrt{3}F_0^6}[(s - s_0)(\frac{9}{4} + \frac{2M^2}{M^2 - M^2}) \\
+ \frac{2M^2}{3} - \frac{5s}{12} + \frac{M^2}{3}]J_{kk}(0), \tag{4.30}
\]
and finally for the sum of the pole and direct graphs
\[
D(s) = - \frac{4M^2}{9\sqrt{3}F_0^2} \left[ 1 + \frac{3(s - s_0)}{M^2 - M^2} \right] \times \\
\{ -3K_3 + \frac{3}{2}K_4 + K_5 + K_6 - K_9 - K_{10} \}. \tag{4.31}
\]

We have introduced the following functions \(C_{kk}\) and \(J_{kk}\):
\[
\frac{1}{i}C_{kk}(s) = - \frac{1}{16\pi^2} \frac{1}{M_K^2} \frac{2}{\sqrt{\Delta}} \arctan \frac{\bar{s}}{\sqrt{\Delta}} \\
J_{kk}(s) = \frac{1}{16\pi^2} \left[ 2 - \frac{2\sqrt{\Delta}}{\bar{s}} \arctan \frac{\bar{s}}{\sqrt{\Delta}} \right] + J_{kk}(0) \\
J_{kk}(0) = -2\lambda - 2k + O(d-4) \tag{4.32}
\]

with
\[
\bar{s} = \frac{s}{M_K^2}, \quad \Delta = \bar{s}(4 - \bar{s}) \\
3s_0 = s + t + u = m^2 + 3m^2, \tag{4.33}
\]
and
\[
\lambda = \frac{\mu^{d-4}}{16\pi^2} \left\{ -\frac{1}{d-4} \left[ \ln(4\pi) + \Gamma'(1) + 1 \right] \right\} \\
k = \frac{1}{32\pi^2} \left\{ \ln \frac{M_K^2}{\mu^2} + 1 \right\}. \tag{4.34}
\]

With equation (3.13) and the following definition:
\[
\tilde{J}_{kk}(s) = J_{kk}(s) - J_{kk}(0) \tag{4.35}
\]
we obtain the finite result

\[
A_{\text{QED}}(s, t, u) = e^2(U(s, t, u) + T(s) + D(s))
\]

\[
= \frac{Ce^2}{\sqrt{3}F_0^2} \left[ \frac{1}{4} s(3s - 4M_K^2)\frac{1}{i}C_{kk}(s) + \frac{1}{2} (3s - 4M_K^2)\bar{J}_{kk}(s) \right.
\]

\[
- \frac{1}{4} (3t - 4M_K^2)\bar{J}_{kk}(t) - \frac{1}{4} (3u - 4M_K^2)\bar{J}_{kk}(u) \right]
\]

\[
+ \frac{4Ce^2M^2}{3\sqrt{3}F_0^2} \left( k - \frac{F_0^4}{3C}K^r(\mu) \right) [1 + \frac{3(s - s_0)}{M^2_{\eta} - M^2_{\pi}}] \quad (4.36)
\]

with

\[
K^r(\mu) = -3K^r_5(\mu) + \frac{3}{2} K^r_4(\mu) + K^r_{5}(\mu)
\]

\[
+ K^r_6(\mu) - K^r_{10}(\mu). \quad (4.37)
\]

Some remarks concerning the structure of this result are in order:

1. The scale dependence of \(k\) is cancelled by the running of \(\bar{K}^r(\mu)\). Using equations (3.16) and (3.18) one gets

\[
\bar{K}^r(\mu) = \bar{K}^r(\mu_0) - \frac{3C}{F_0^4} \ln \left( \frac{\mu^2}{\mu_0^2} \right). \quad (4.38)
\]

2. As expected, the amplitude vanishes in the soft pion point

\[
s \rightarrow M^2_{\eta}, \quad t, u \rightarrow 0 \quad (4.39)
\]

if the lowest order relation \(3m_{\eta}^2 = 4m_K^2\) is used. This is in accord with Sutherland’s theorem which predicts a vanishing amplitude if the pion mass is sent to zero.

3. The final result (4.36) consists of a nonlocal piece (first two lines) and a polynomial to first order in \(s\) (last line). The polynomial part is in proportion to the current algebra amplitude of strong interactions. It exhibits the \(M^2_{\pi}\) suppression as implied by Sutherland theorem. Since the counterterm \(\bar{K}^r\) enters only this part of the amplitude, it’s contribution is suppressed too. The nonlocal piece circumvents this suppression as it is clearly not linear in \(s\). However, since only Kaon loops contribute, this part is kinematically suppressed.

4. The rate is to a large extent fixed by the amplitude at the center of the Dalitz plot. We therefore expand (4.36) according to

\[
A_{\text{QED}}(s, t, u) = a_0 + a_1(s - s_0) + a_2(s - s_0)^2 + a_3(t - u)^2. \quad (4.40)
\]
Explicitly, the constant term is given by

\[
a_0 = \frac{C e^2}{3\sqrt{3}F_0^6} \left\{ \frac{3}{4}s_0(3s_0 - 4M_K^2) \frac{1}{i} C_{kk}(s_0) + 4M^2_{\pi} \left( k - \frac{F_0^4}{3C} K^r(\mu) \right) \right\},
\]

(4.41)

Note that at \( s = s_0 \) the remaining nonlocal piece in proportion to \( C_{kk} \) is numerically less suppressed than for instance \( J_{kk} \).

5. The linear slope \( a_1 \) is given in terms of the same counterterm \( \bar{K}^r \) as is the amplitude at the center of the physical region. Thus, if isospin breaking in the quark masses is switched off, the electromagnetic rate would be fixed in terms of the linear slope. The quadratic coefficients \( a_2, a_3 \) get contributions only from the nonlocal part of \( A_{\text{QED}} \). They are given in terms of the known coupling constant \( C \).

5 Numerical Results

As mentioned, the one-loop result of Gasser and Leutwyler yields a rate of 167 eV, much below the experimental value. Although the unitary corrections are available, we will use the one-loop result as reference. This is inessential, since the electromagnetic corrections are very small.

We set \( F_0 \) equal to the pion decay constant \( F_\pi \), whose experimental value is \( F_\pi = 92.4 \) MeV if electromagnetic corrections are included. The low energy constant \( C \) can be determined by using the underlying vector and axial-vector resonances. Using the Weinberg sum rules, one obtains

\[
C = \frac{3}{32\pi^2} M_\rho^2 F_\rho^2 \ln\left( \frac{F_\rho^2}{F_\rho^2 - F_\pi^2} \right),
\]

(5.42)

where \( M_\rho \) is the mass of the \( \rho \)-meson, \( M_\rho = 770 \) MeV and \( F_\rho \) is its decay constant, \( F_\rho = 154 \) MeV. Numerically,

\[
\frac{C}{F_\pi^4} = 0.84.
\]

(5.43)

The values of the coupling constants \( K_i^r \) are so far unknown. The ‘naive’ chiral counting law gives

\[
| K_i^r | = \frac{1}{16\pi^2} \approx 6.3 \times 10^{-3},
\]

(5.44)

while an estimate for \( K_8^r \) yields

\[
K_8^r(\mu = 0.5 GeV) = (-1.0 \pm 1.7) \times 10^{-3},
\]

(5.45)

consistent with the naive rule (5.44).

We conservatively use eq. (5.44) and vary the absolute value of \( \bar{K}^r \) between the bounds

\[
| \bar{K}^r | \leq \frac{8.5}{16\pi^2}.
\]

(5.46)
To get a feeling of the numerical size of the electromagnetic corrections, we expand the amplitude around the center of the Dalitz plot and compare it to the ChPT one-loop result for the strong interactions. Writing the Taylor expansion of the sum of these amplitudes as

\[
A_{\text{QCD}}(s, t, u) + A_{\text{QED}}(s, t, u) = a_0 + a_1(s - s_0) + a_2(s - s_0)^2 + a_3(t - u)^2 \quad (5.47)
\]

we find

\[
a_0 = a_0^{\text{QCD}} + a_0^{\text{n.l}} + a_0^{\text{l}} = -(0.17 + 0.02i) + 0.14 \times 10^{-2} \pm 0.30 \times 10^{-2}, \quad (5.48)
\]

where \(a_0^{\text{QCD}}\) denotes the strong one-loop value, \(a_0^{\text{n.l}}\) the contribution from the non-local terms in eq.(4.36), and \(a_0^{\text{l}}\) that of the local terms which depend on the low energy coupling constants \(K_i\). The variation of \(a_0^{\text{l}}\) are due to the changing value of \(\bar{K}_r\). Similarly, we obtain

\[
a_1 = a_1^{\text{QCD}} + a_1^{\text{n.l}} + a_1^{\text{l}} = -(2.25 + 0.75i) + 0.62 \times 10^{-3} \pm 0.30 \times 10^{-1} \quad [\text{GeV}^{-2}], \quad (5.49)
\]

\[
a_2 = a_2^{\text{QCD}} + a_2^{\text{n.l}} = (1.33 - 0.91i) - 0.25 \times 10^{-1} \quad [\text{GeV}^{-4}], \quad (5.50)
\]

\[
a_3 = a_3^{\text{QCD}} + a_3^{\text{n.l}} = -(0.93 - 0.41i) - 0.85 \times 10^{-2} \quad [\text{GeV}^{-4}]. \quad (5.51)
\]

Here, we used everywhere the ‘new’ value of \(F_\pi\) \(^{[11]}\); the other relevant factors for the QCD amplitude are taken from ref \([3]\). From this we see that for the extreme values of \(\bar{K}_r\) the local contribution dominates over the non-local contribution. Both are, however, small compared to the ChPT one-loop result. We therefore expect no significant changes from QED to both, the decay rates as well as the Dalitz plot distributions for the decays \(\eta \rightarrow \pi^0\pi^+\pi^-\) and \(\eta \rightarrow 3\pi^0\).

The decay rate \(\Gamma_{\eta \rightarrow \pi^0\pi^+\pi^-}\) is obtained by evaluating the phase space integral \(\int\)

\[
d\Gamma_{\eta \rightarrow \pi^0\pi^+\pi^-} = \frac{(2\pi)^4}{M_\eta} \delta(P_i - P_f) \left| A_{\text{QCD}}(s, t, u) + A_{\text{QED}}(s, t, u) \right|^2 d\mu,
\]

\[
d\mu = \frac{d^3p_{\pi^0}}{(2\pi)^4 2p_{\pi^0}^0} \frac{d^3p_{\pi^+}}{(2\pi)^4 2p_{\pi^+}^0} \frac{d^3p_{\pi^-}}{(2\pi)^4 2p_{\pi^-}^0}.
\]

We get for the integrated rate

\[
\Gamma_{\eta \rightarrow \pi^0\pi^+\pi^-} = 165 \pm 5\text{eV} \quad (5.53)
\]

where the uncertainty is due to our lack of knowledge on \(\bar{K}_r\), and for the ratio \(r\)

\[
1.42 \leq r = \frac{\Gamma_{\eta \rightarrow 3\pi^0}}{\Gamma_{\eta \rightarrow \pi^0\pi^+\pi^-}} \leq 1.43 \quad (5.54)
\]

\(^4\)Note that the phase space is very small and thus very sensitive to the mass difference \(M_{\pi^\pm} - M_{\pi^0\pi^+\pi^-}\).
In both cases electromagnetism gives, as expected, small corrections to the results obtained by [3], where the purely strong effects have been considered

\[ \Gamma_{QCD}^{\eta \rightarrow \pi^0 \pi^+ \pi^-} = 167\text{eV} \, \quad (5.55) \]

Even though QED can give rise to a correction of up to 4 per cent to the decay rate \( \Gamma_{\eta \rightarrow \pi^0 \pi^+ \pi^-} \), the Dalitz plot distribution remains unchanged. This is due to the fact that the non-local contributions are small and that the functional dependence of the local term in eq. (4.36) on the Mandelstamm variable \( s \) and that of the current algebra amplitude are the same. Since the Dalitz plot distribution is a measure of the \( s \)-dependence of the amplitude \( A(s, t, u) \), the \( K^r(\mu_0) \) contribution only leads to a different normalization of the distribution, but there is no significant change of the slopes.

### 6 Conclusions

In this paper we have calculated the direct electromagnetic corrections to the decay \( \eta \rightarrow 3\pi \). Using the general framework of chiral perturbation theory, we determine the first nontrivial term of order \( e^2p^2 \). This includes mesonic loops and a combination of counterterms introduced earlier [12]. Unlike for meson mass differences, the corrections are tiny, and amount to at most two percent of the ChPT one-loop amplitude at the center of the decay region. The Dalitz plot parameters are also rigid with respect to electromagnetic corrections; from the results in section 5 we see that both linear and quadratic slope parameters are practically unaffected by the electromagnetic corrections. This is clear from the form of the amplitude: First, the nonlocal part which circumvents the Bell-Sutherland suppression arises from Kaon loops, which are kinematically strongly suppressed. The larger (by a factor of 13) pion loops multiply the small quantity \( (m_d - m_u) \) and can be neglected. Furthermore, the polynomial part is suppressed by the small factor \( M^2_\pi / M^2_K \). This includes in particular the contributions from the coupling constants \( K_i \). Thus, the dependence on these little known constants is weak; varying them between the rather large values \( \pm \frac{8.5}{10^{10}} \) results only in a variation of less than two percent relative to the one-loop ChPT amplitude of the strong interactions.

Thus, our result confirms the usual picture, namely that the large decay rate requires rather large isospin violations in the strong interactions. Moreover, any precise measurement of the \( \eta \rightarrow 3\pi \) Dalitz plot yields important information on the strong interactions without contamination from the electromagnetic ones.

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There are other, indirect, electromagnetic corrections which influence the fundamental parameters, such as \( F_\pi \). They are known [11] and change the normalization of the QCD amplitude noticeably, resulting in a 8% increase of the rate.
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