Inverse problem for coefficients of equations describing propagation of COVID-19 epidemic

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Abstract. The inverse problems for coefficients of ordinary differential equations describing propagation of coronavirus infection are studied. The well-known models of SEI and SEIR, and their generalization are used. Important role plays the coefficients of these equations that can be estimated by in-direct observations and depends on many factors. This approach allowed us to solve the problem with several waves of epidemic and to predict further propagation.

1. Model with variable coefficients for monitoring response of healthcare to epidemic dynamics

The SEIR model is widely used in mathematical epidemiology:

\[
\begin{align*}
\dot{S} &= -\beta(t)S(I + E), \\
\dot{E} &= \beta(t)S(I + E) - (\gamma(t) + \delta(t))E, \quad \Rightarrow I(t, \beta(t), \gamma(t), \delta(t), \nu) \\
\dot{I} &= \delta(t)E - \gamma(t)I, \\
\dot{R} &= \nu I,
\end{align*}
\]

where \(S(t)\) is the number of people who can be infected at time \(t\); \(E(t)\) is the number of infected people whom illness is not indentified yet at time \(t\), but they are able to infect surrounded people; \(I(t)\) is the number of illed people with confirmed diagnose at time \(t\); \(R(t)\) is the number of recovered people at time \(t\).

We assume that coefficients of eqs. (1) can be variable unlike the SEIR model: \(\beta(t)\) is proportional to probability of infection and it can vary due to health measures in specific country. Decrease of \(\beta(t)\) can consider response to introduced restricted measures (region 1, Fig. 1). Weakening of these measures can result in increase of \(\beta(t)\) (region 2, Fig.1). Increase of \(\gamma(t)\) can be interpreted as decrease of average time of determining of illness (region 5, Fig. 1). There is opposite effect (region 6, Fig. 1). Increase of \(\delta(t)\) reflects decrease of average time of recovering (region 3, Fig. 1). Increase of this parameter can be interpreted as decrease of \(\delta(t)\) (region 4, Fig. 1).

2. Algorithm for the inverse problem solution with variable coefficients

Determination of three unknown functions \(\beta(t), \gamma(t), \delta(t)\), based on the unique curve of data \(I_{dat}(t)\) is the ill-posed problem. Its solution can does not exist or is not unique, or is unstable to
the errors of input data. But existence of prior data can help us to find restrictions for unique solution. We will use the special regularization algorithms based on the Tikhonov method with aposteriori choice of the regularization parameter [1]. We minimize the Tikhonov functional $M^\lambda[p] = \lambda \|p\|^2 + \Phi_1^2(p)$, where $\|p\|^2 = \|\{\beta_i\}\|^2 + \|\{\gamma_i\}\|^2 + \|\{\delta_i\}\|^2$. Approximate minimization of this functional with accuracy $\varepsilon$ at the restriction set we can get the extremal $p^\lambda_\varepsilon$ that depends on the regularization parameter $\lambda > 0$. The regularization parameter is chosen by the discrepancy principle as $\lambda(\varepsilon) > 0$, so that equality takes place: $\Phi_1(p^\lambda_\varepsilon) = \Phi_{l_{\text{min}}} + \varepsilon$. Here $\Phi_{l_{\text{min}}}$ is estimation of approximate minimum value of functional $\Phi_1(p)$ at set $K_0$. In case of non-unique solution of the inverse problem it can prove that approximate parameter sets $p^\lambda_\varepsilon$ converge to the normal solution of the inverse problem at $\varepsilon \rightarrow 0$. With applied point of view we look for the solution of the inverse coefficient problem system (1) that is subjected to minimal variations and deviations. In the same time the restrictions $K_0$ takes place.

Such approach was investigated and applied for many problems. The numeric methods can be used to find out the solution [2].

3. Results
The modelling of two epidemic waves by this model was carried out for various countries. We show results for Italy, Russia, Czech and Germany in the following four Figures 2-5.

Figure 1. The parameters $\beta(t)$, $\gamma(t)$ and $\delta(t)$. 
Figure 2. The reconstructions for Italy.

Figure 3. The reconstructions for Czech.
Figure 4. The reconstructions for Russia.

Figure 5. The reconstructions for Germany.
4. Conclusions

- Models SEI and SEIR give satisfactory approximation of function. It can be noted that the SEIR model gives better extrapolation of the function in the future. It can allows us to predict the recovery dynamics. The coefficients for the SEI and SEIR models for the first and second epidemic waves are varied for different countries that reflects specific conditions for each country.

- The best accuracy of solutions of the inverse problems for two epidemic waves achieves under application the SEI model with variable coefficients and the hybrid SEIR model. The solutions of these problems based on the application of the ill-posed inversion algorithms with regularization procedure.

- It is possible to find out the posterior estimation of accuracy of the regularization solution with possible ambiguity of the solution.

- The coefficient curves depend on each country. But the common behavior of these curve is due to the following tendency: the drop of coefficients at the beginning of the first epidemic wave and the growth of coefficients at the beginning of the second wave. This analysis allows us to estimate quality of strategy of counteraction to epidemic in different countries.

References

[1] Leonov A S 2012 A posteriori accuracy estimations of solutions to ill-posed inverse problems and extra-optimal regularizing algorithms for their solution *Numerical Analysis and Applications* 5 68–83

[2] Leonov A S 2012 Extra-optimal methods for solving ill-posed problems *Journal of Inverse and Ill-Posed problems* 20 637–665