SEMIDEFINITE PROGRAMMING BOUNDS FOR ERROR-CORRECTING CODES

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ABSTRACT. This chapter is written for the forthcoming book “A Concise Encyclopedia of Coding Theory” (CRC press), edited by W. Cary Huffman, Jon-Lark Kim, and Patrick Solé. This book will collect short but foundational articles, emphasizing definitions, examples, exhaustive references, and basic facts on the model of the Handbook of Finite Fields. The target audience of the Encyclopedia is upper level undergraduates and graduate students.

1. INTRODUCTION

Linear programming bounds belong to the most powerful and flexible methods to obtain bounds for extremal problems in coding theory. Initially, Delsarte [26] developed linear programming bounds in the algebraic framework of association schemes.

A central example in Delsarte’s theory is finding upper bounds for the parameter \( A_2(n, d) \), the largest number of codewords in a binary code of length \( n \) with minimum Hamming distance \( d \).

The application of linear programming bounds led to the best known asymptotic bounds [48]. It was realized that linear programming bounds are also applicable to finite and infinite two-point homogeneous spaces [18, Chapter 9]. These are metric spaces in which the symmetry group acts transitively on pairs of points having the same distance. So one can treat metric spaces like the \( q \)-ary Hamming space \( F_q^n \), the sphere, real/complex/quaternionic projective space, or Euclidean space [16].

In recent years, semidefinite programming bounds have been developed with two aims: to strengthen linear programming bounds and to find bounds for more general spaces. Semidefinite programs are convex optimization problems which can be solved efficiently and which are a vast generalization of linear programs. The optimization variable of a semidefinite program is a positive semidefinite matrix whereas it is a nonnegative vector for a linear program.

Schrijver [57] was the first who applied semidefinite programming bounds to improve the known upper bounds for \( A_2(n, d) \) for many parameters \( n \) and \( d \). The underlying idea is that linear programming bounds only exploit constraints involving pairs of codewords, whereas semidefinite programming bounds can exploit constraints between triples, quadruples, \ldots of codewords.

This chapter introduces semidefinite programming bounds with an emphasis on error-correcting codes. The structure of the chapter is as follows:

In Section 2 the basic theory of linear and semidefinite programming is reviewed in the framework of conic programming.

Semidefinite programming bounds can be viewed as semidefinite programming hierarchies for difficult combinatorial optimization problems. One can express the
computation of $A_2(n, d)$ as finding the independence number of an appropriate graph $G(n, d)$ and apply the Lasserre hierarchy to find upper bounds for $A_2(n, d)$. This approach is explained in Section 3.

The graph $G(n, d)$ has exponentially many vertices and the Lasserre hierarchy for $G(n, d)$ employs matrices whose rows and columns are indexed by all $t$-element subsets of $G(n, d)$, so a computation of the semidefinite programs is not directly possible. However, the graph has many symmetries and these symmetries can be exploited to substantially reduce the size of the semidefinite programs. The technique of symmetry reduction is the subject of Section 4. There, this technique is applied to the graph $G(n, d)$ and the result of Schrijver is explained.

After Schrijver’s breakthrough result, semidefinite programming bounds were developed for different settings. These developments are reviewed in Section 5.

2. CONIC PROGRAMMING

Semidefinite programming is a vast generalization of linear programming. Geometrically, both linear and semidefinite programming are concerned with minimizing or maximizing a linear functional over the intersection of a fixed convex cone with an affine subspace. In the case of linear programming the fixed convex cone is the nonnegative orthant and the resulting intersection is a polyhedron. In the case of semidefinite programming the fixed convex cone is the cone of positive semidefinite matrices and the resulting intersection is a spectrahedron. Linear and semidefinite programming belong to the field of conic programming.

Textbooks and research monographs dealing with semidefinite programming are: Wolkowicz, Saigal, and Vandenberghe (ed.) [68], Ben-Tal and Nemirovski [12], de Klerk [19], Tunçel [65], Anjos and Lasserre (ed.) [1], Gärtner and Matoušek [30], Blekherman, Parrilo and Thomas (ed.) [13], Laurent and Vallentin [42].

2.1. Conic programming and its duality theory. Conic programs are convex optimization problems. In general, conic programming deals with minimizing or maximizing a linear functional over the intersection of a fixed convex cone with an affine subspace. See Nemirovski [50] and the references therein for a detailed overview of conic programming.

Let $E$ be an $n$-dimensional real or complex vector space equipped with a real-valued inner product $\langle \cdot, \cdot \rangle_E : E \times E \to \mathbb{R}$.

**Definition 2.1.** A set $K \subseteq E$ is called a **(convex) cone** if for all $x, y \in K$ and all nonnegative numbers $\alpha, \beta \in \mathbb{R}_+$ one has $\alpha x + \beta y \in K$. A convex cone $K$ is called **pointed** if $K \cap (-K) = \{0\}$. A convex cone is called **proper**, if it is pointed, closed, and full-dimensional. The **dual cone** of a convex cone $K$ is given by

$$K^* = \{ y \in E : \langle x, y \rangle_E \geq 0 \text{ for all } x \in K \}.$$

The simplest convex cones are **finitely generated cones**; the vectors $x_1, \ldots, x_N \in E$ determine the finitely generated cone $K$ by

$$K = \text{cone}\{x_1, \ldots, x_N\} = \left\{ \sum_{i=1}^{N} \alpha_i x_i : \alpha_1, \ldots, \alpha_N \geq 0 \right\}.$$

A pointed convex cone $K \subseteq E$ determines a partial order on $E$ by

$$x \succeq_K y \text{ if and only if } x - y \in K.$$
To define a conic program, we fix the space $E$, a proper convex cone $K \subseteq E$, and an $m$-dimensional vector space $F$ with inner product $\langle \cdot, \cdot \rangle_F$.

**Definition 2.2.** A linear map $A : E \to F$ and vectors $c \in E$, $b \in F$ determine a **primal conic program**

$$p^* = \sup \{ \langle c, x \rangle_E : x \in K, Ax = b \}.$$

The corresponding **dual conic program** is

$$d^* = \inf \{ \langle b, y \rangle_F : y \in F, A^T y - c \in K^* \},$$

where $A^T : F \to E$ is the usual adjoint of $A$.

The vector $x \in E$ is the **optimization variable** of the primal, the vector $y \in F$ is the optimization variable of the dual. A vector $x$ is called **feasible** for the primal if $x \in K$ and $Ax = b$. It is called **strictly feasible** if additionally $x$ lies in the interior of $K$. It is called **optimal** if $x$ is feasible and $p^* = \langle x, c \rangle_E$. Similarly, a vector $y$ is called feasible for the dual if $A^T y - c \in K^*$, and it is called strictly feasible if $A^T y - c$ lies in the interior of $K^*$. It is called optimal if $y$ is feasible and $d^* = \langle b, y \rangle_F$.

The bipolar theorem (see for example Barvinok [11] or Simon [59]) states that $(K^*)^* = K$ when $K$ is a proper convex cone. From this it follows easily that taking the dual of the dual conic program gives a conic program which is equivalent to the primal.

Duality theory of conic programs looks at the (close) relationship between the primal and dual conic programs. In particular duality can be used to systematically find upper bounds for the primal program and lower bounds for the dual program.

**Theorem 2.3.** (Duality theorem of conic programs)

1. **weak duality**: $p^* \leq d^*$.
2. **optimality condition/complementary slackness**: Suppose that $p^* = d^*$. Let $x$ be a feasible solution for the primal and let $y$ be a feasible solution of the dual. Then $x$ is optimal for the primal and $y$ is optimal for the dual if and only if $\langle x, A^T y - c \rangle_E = 0$ holds.
3. **strong duality**: Suppose that primal and dual conic programs both have a strictly feasible solution. Then $p^* = d^*$ and both primal and dual possess an optimal solution.

### 2.2. Linear programming

To specialize conic programs to linear programs we choose $E$ to be $\mathbb{R}^n$ with standard inner product $\langle x, y \rangle_E = x^T y$. For the convex cone $K$ we choose the nonnegative orthant:

**Definition 2.4.** The **nonnegative orthant** is the following proper convex cone

$$\mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0 \}.$$  

The nonnegative orthant is self-dual, $(\mathbb{R}_+^n)^* = \mathbb{R}_+^n$. So, for a matrix $A \in \mathbb{R}^{m \times n}$, a vector $b \in \mathbb{R}^m$ and a vector $c \in \mathbb{R}^n$ we get the **primal linear program**

$$p^* = \sup \{ c^T x : x \geq 0, Ax = b \},$$

and its **dual linear program**

$$d^* = \inf \{ b^T y : y \in \mathbb{R}^m, A^T y - c \geq 0 \}.$$
Here we simply write $x \geq 0$ for the partial order $x \succeq_R 0$.

Linear programming is a well established method, which is extremely useful in theory and practice; see for example Schrijver [56], Grötschel, Lovász, and Schrijver [34] and Wright [69]. The main algorithms to solve linear programs are the simplex method, the ellipsoid method, and the interior-point method. Each one of these three algorithms has specific advantages: In practice, the simplex method and the interior-point method can solve very large instances. The simplex method allows the computation of additional information which is useful for the broader class of mixed integer linear optimization problems, where some of the optimization variables are constrained to be integers. The ellipsoid method and the interior-point method are polynomial time algorithms. The ellipsoid method is a versatile mathematical tool to prove the existence of polynomial time algorithms, especially in combinatorial optimization.

2.3. Semidefinite programming. To specialize conic programs to semidefinite programs we choose $E$ to be the $n(n+1)/2$-dimensional space $S^n$ of real symmetric $n \times n$ matrices. This space is equipped with the trace (Frobenius) inner product $\langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_T$ defined by

$$\langle X, Y \rangle_T = \text{Tr}(Y^TX) = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}Y_{ij}$$

where Tr denotes the trace of a matrix. For the convex cone $K$ we choose the cone of positive semidefinite matrices:

**Definition 2.5.** The cone of positive semidefinite matrices (or: the psd cone) is the following proper convex cone:

$$S^n_+ = \{ X \in S^n : X \text{ is positive semidefinite} \}.$$

Let us recall that a matrix $X$ is positive semidefinite if and only if for all $x \in \mathbb{R}^n$ we have $x^TXx \geq 0$. Alternatively, looking at a spectral decomposition of $X$, given by

$$X = \sum_{i=1}^{n} \lambda_i u_i u_i^T,$$

where $\lambda_i$ are the (real) eigenvalues of $X$ and $u_i$ is an orthonormal basis consisting of corresponding eigenvectors, $X$ is positive semidefinite if and only if all its eigenvalues are nonnegative: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n$. We write $X \succeq 0$ for $X \geq S^n_+ 0$.

The cone of positive semidefinite matrices is self-dual, $(S^n_+)^* = S^n_+$. So, for symmetric matrices $A_1, \ldots, A_m \in S^n$, a vector $b \in \mathbb{R}^m$ and a symmetric matrix $C \in S^n$ we get the **primal semidefinite program**

$$(1) \quad p^* = \sup \{ \langle C, X \rangle_T : X \succeq 0, \langle A_1, X \rangle_T = b_1, \ldots, \langle A_m, X \rangle_T = b_m \}.$$ 

Its dual semidefinite program is

$$d^* = \inf \left\{ b^Ty : y \in \mathbb{R}^m, \sum_{j=1}^{m} y_j A_j - C \succeq 0 \right\}.$$ 

Restricting semidefinite programs to diagonal matrices, one recovers linear programming as a special case of semidefinite programming.
Definition 2.6. The set of feasible solutions of a primal semidefinite program
\[ F = \{ X \in S^n : X \succeq 0, \langle A_j, X \rangle_T = b_j \text{ for } j = 1, \ldots, m \} \]
is called a spectrahedron.

Spectrahedra are generalizations of polyhedra. They are central objects in convex algebraic geometry; see [13].

Under mild technical assumptions one can solve semidefinite programming problems in polynomial time. The following theorem was proved in Grötschel, Lovász, and Schrijver [34] using the ellipsoid method and by de Klerk and Vallentin [20] using the interior-point method.

Theorem 2.7. Consider the primal semidefinite program (1) with rational input \( C, A_1, \ldots, A_m \), and \( b_1, \ldots, b_m \). Suppose we know a rational point \( X_0 \in F \) and positive rational numbers \( r, R \) so that \( B(X_0, r) \subseteq F \subseteq B(X_0, R) \), where \( B(X_0, r) \) is the ball of radius \( r \), centered at \( X_0 \), in the affine subspace
\[ \{ X \in S^n : \langle A_j, X \rangle_T = b_j \text{ for } j = 1, \ldots, m \}. \]

For every positive rational number \( \epsilon > 0 \) one can find in polynomial time a rational matrix \( X^* \in F \) such that
\[ (C, X^*)_T - p^* \leq \epsilon, \]
where the polynomial is in \( n, m, \log_2 R, \log_2 (1/\epsilon) \), and the bit size of the data \( X_0, C, A_1, \ldots, A_m, \) and \( b_1, \ldots, b_m \).

Sometimes—especially when dealing with invariant semidefinite programs or in the area of quantum information theory—it is convenient to work with complex Hermitian matrices instead of real symmetric matrices. A complex matrix \( X \in \mathbb{C}^{n \times n} \) is called Hermitian if \( X = X^* \), where \( X^* = X^T \) denotes the conjugate transpose of \( X \), i.e. \( X_{ij} = X_{ji}^* \). A Hermitian matrix is called positive semidefinite if for all vectors \( x \in \mathbb{C}^n \) we have \( x^* X x \geq 0 \). The space of Hermitian matrices is equipped with the real-valued inner product \( \langle X, Y \rangle_T = \text{Tr}(Y^* X) \). Now a primal complex semidefinite program is
\[
\text{sup} \{ (C, X)_T : X \succeq 0, \langle A_1, X \rangle_T = b_1, \ldots, \langle A_m, X \rangle_T = b_m \},
\]
where \( A_1, \ldots, A_m \in \mathbb{C}^{n \times n} \), and \( C \in \mathbb{C}^{n \times n} \) are given Hermitian matrices, \( b \in \mathbb{R}^m \) is a given vector and \( X \in \mathbb{C}^{n \times n} \) is the positive semidefinite Hermitian optimization variable (denoted by \( X \succeq 0 \)).

One can easily reduce complex semidefinite programming to real semidefinite programming by the following construction: A complex matrix \( X \in \mathbb{C}^{n \times n} \) defines a real matrix
\[ X' = \begin{pmatrix} \Re(X) & -\Im(X) \\ \Im(X) & \Re(X) \end{pmatrix} \in \mathbb{R}^{2n \times 2n}, \]
where \( \Re(X) \in \mathbb{R}^{n \times n} \) and \( \Im(X) \in \mathbb{R}^{n \times n} \) are the real, respectively, the imaginary parts of \( X \). Then \( X \) is Hermitian and positive semidefinite if and only if \( X' \) is symmetric and positive semidefinite.
3. Independent sets in graphs

3.1. Independence number and codes. In the following we are dealing with finite simple graphs. These are finite undirected graphs without loops and multiple edges. This means that the vertex set is a finite set and the edge set consists of (unordered) pairs of vertices.

Definition 3.1. Let \( G = (V,E) \) be simple finite graph with vertex set \( V \) and edge set \( E \). A subset of the vertices \( I \subseteq V \) is called an independent set if every pair of vertices \( x, y \in I \) is not adjacent, i.e. \( \{x,y\} \notin E \). The independence number \( \alpha(G) \) is the largest cardinality of an independent set in \( G \).

In the optimization literature, independent sets are sometimes also called stable sets, and the independence number is referred to as the stability number.

Frequently the largest number of codewords in a code with given parameters can be equivalently expressed as the independence number of a specific graph.

Example 3.2. Recall that \( A_2(n,d) \) is the largest number \( M \) of codewords in a binary code of length \( n \) with minimum Hamming distance \( d \). Consider the graph \( G(n,d) \) with vertex set \( V = \mathbb{F}_2^n \) and edge set \( E = \{ \{x,y\} : d_H(x,y) < d \} \). Then independent sets in \( G(n,d) \) are exactly binary codes \( C \) of length \( n \) with minimum Hamming distance \( d \). Furthermore, \( A_2(n,d) = \alpha(G(n,d)) \).

The graph \( G(n,d) \) can also be seen as a Cayley graph over the additive group \( \mathbb{F}_2^n \). The vertices are the group elements and two vertices \( x \) and \( y \) are adjacent if and only if their difference \( x - y \) has Hamming weight strictly less than \( d \).

Computing the independence number of a given a graph \( G \) is generally a very difficult problem. Computationally, determining even approximate solutions of \( \alpha(G) \) is an NP-hard problem; see Håstad [35].

3.2. Semidefinite programming bounds for the independence number. One possibility to systematically find stronger and stronger upper bounds for \( \alpha(G) \), which is often quite good for graphs arising in coding theory, is the Lasserre hierarchy of semidefinite programming bounds.

The Lasserre hierarchy was introduced by Lasserre in [39]. He considered the general setting of 0/1 polynomial optimization problems, and he proved that the hierarchy converges in finitely many steps using Putinar’s Positivstellensatz [53]. Shortly after, Laurent [40] gave a combinatorial proof, which we reproduce here.

The definition of the Lasserre hierarchy requires some notation. Let \( V \) be a finite set. By \( \mathcal{P}_t(V) \) we denote the set of all subsets of \( V \) of cardinality at most \( t \).

Definition 3.3. Let \( t \) be an integer with \( 0 \leq t \leq n \). A symmetric matrix \( M \in \mathbb{S}^{\mathcal{P}_t(V)} \) is called a (combinatorial) moment matrix of order \( t \) if

\[
M_{I,J} = M_{I',J'} \quad \text{whenever} \quad I \cup J = I' \cup J'.
\]

A vector \( y = (y_I) \in \mathbb{R}^{\mathcal{P}_t(V)} \) defines a combinatorial moment matrix of order \( t \) by

\[
M_t(y) \in \mathbb{S}^{\mathcal{P}_t(V)} \quad \text{with} \quad (M_t(y))_{I,J} = y_{I \cup J}.
\]

The matrix \( M_t(y) \) is called the (combinatorial) moment matrix of order \( t \) of \( y \).
Example 3.4. For \( V = \{1, 2\} \), the moment matrices of order one and order two of \( y \) have the following form:

\[
M_1(y) = \begin{pmatrix}
0 & 1 & 2 \\
\emptyset & y_0 & y_1 & y_2 \\
1 & y_1 & y_1 & y_2 \\
2 & y_2 & y_1 & y_2 \\
12 & y_2 & y_1 & y_2 \\
\end{pmatrix},
\]

\[
M_2(y) = \begin{pmatrix}
0 & 1 & 2 & 12 \\
\emptyset & y_0 & y_1 & y_2 & y_12 \\
1 & y_1 & y_1 & y_2 & y_12 \\
2 & y_2 & y_12 & y_2 & y_12 \\
12 & y_12 & y_12 & y_12 & \end{pmatrix}.
\]

Here and in the following, we simplify notation and use \( y_i \) instead of \( y_{\{i\}} \) and \( y_{12} \) instead of \( y_{\{1,2\}} \). Note that \( M_1(y) \) occurs as a principal submatrix of \( M_2(y) \).

Definition 3.5. Let \( G = (V, E) \) be a graph with \( n \) vertices. Let \( t \) be an integer with \( 1 \leq t \leq n \). The Lasserre bound of \( G \) of order \( t \) is the value of the semidefinite program

\[
\text{las}_t(G) = \max \left\{ \sum_{i \in V} y_i : y \in \mathbb{R}_+^{P_t(V)}, \ y_0 = 1, \ y_{ij} = 0 \text{ if } \{i, j\} \in E, \ M_t(y) \in S_+^{P_t(V)} \right\}.
\]

Theorem 3.6. The Lasserre bound of \( G \) of order \( t \) forms a hierarchy of stronger and stronger upper bounds for the independence number of \( G \). In particular,

\[
\alpha(G) \leq \text{las}_n(G) \leq \ldots \leq \text{las}_2(G) \leq \text{las}_1(G)
\]

holds.

Proof: To show that \( \alpha(G) \leq \text{las}_t(G) \) for every \( 1 \leq t \leq n \) we construct a feasible solution \( y \in \mathbb{R}_+^{P_t(V)} \) from any independent set \( I \) of \( G \). This feasible solution will satisfy \( |I| = \sum_{i \in V} y_i \) and the desired inequality follows. For this, we simply set \( y \) to be equal to the characteristic vector \( \chi^I \in \mathbb{R}_+^{P_t(V)} \) defined by

\[
\chi^I_J = \begin{cases} 
1 & \text{if } J \subseteq I, \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly, \( y \) satisfies the conditions \( y_0 = 1 \) and \( y_{ij} = 0 \) if \( i \) and \( j \) are adjacent. The moment matrix \( M_t(y) \) is positive semidefinite because it is a rank-one matrix of the form (note the slight abuse of notation here, \( \chi^I \) is now interpreted as a vector in \( \mathbb{R}_+^{P_t(V)} \))

\[
M_t(y) = \chi^I (\chi^I)^T \text{ where } M_t(y)_{J,J'} = y_{J \cup J'} = \chi^J \chi^{J'}_T \text{ and } \chi^I \in \mathbb{R}_+^{P_t(V)}.
\]

Since \( M_t(y) \) occurs as a principal submatrix of \( M_{t+1}(y) \), the inequality \( \text{las}_{t+1}(G) \leq \text{las}_t(G) \) follows.

One can show, using the Schur complement for block matrices,

\[
\begin{pmatrix}
A & B \\
B^T & C
\end{pmatrix} \geq 0 \iff C - B^T A^{-1} B \geq 0,
\]

that the first step of the Lasserre bound coincides with the Lovász \( \vartheta \)-number of \( G \), a famous graph parameter which Lovász [45] introduced to determine the Shannon capacity \( \Theta(C_5) \) of the cycle graph \( C_5 \). Determining the Shannon capacity of a given graph is a very difficult problem and has applications to the zero-error capacity of a noisy channel; see Shannon [58]. For instance, the value of \( \Theta(C_7) \) is currently not known.
Theorem 3.7. Let \( G = (V, E) \) be a graph. We have \( \text{las}_t(G) = \vartheta'(G) \) where \( \vartheta'(G) \) is defined as the solution of the following semidefinite program

\[
\vartheta'(G) = \max \left\{ \sum_{i,j \in V} X_{i,j} : X \in S^V, X_{i,j} \geq 0 \text{ for all } i,j \in V, \quad \text{Tr}(X) = 1, \ X_{i,j} = 0 \text{ if } \{i,j\} \in E \right\}.
\]

Technically, the parameter \( \vartheta'(G) \) is a slight variation of the original Lovász \( \vartheta \)-number as introduced in [45]. The difference is that in the definition of \( \vartheta(G) \) one omits the nonnegativity condition \( X_{i,j} \geq 0 \) for all \( i,j \in V \).

Schrijver [55] and independently McEliece, Rodemich, and Rumsey [47] realized that \( \vartheta'(G) \) is nothing other than the Delsarte Linear Programming Bound in the special case of the graph \( G = G(n,d) \), which was defined in Example 3.2. We will provide a proof of this fact in Section 4.3.

An important feature of the Lasserre bound is that it does not loose information. If the step of the hierarchy is high enough, we can exactly determine the independence number of \( G \).

Theorem 3.8. For every graph \( G \) the Lasserre bound of \( G \) of order \( t = \alpha(G) \) is exact; that means \( \text{las}_t(G) = \alpha(G) \) for every \( t \geq \alpha(G) \).

Proof: (sketch) First we show that the hierarchy becomes stationary after \( \alpha(G) \) steps. Let \( J \subseteq V \) be a set of vertices which contains an edge, \( \{i,j\} \in E \) with \( i,j \in J \). Let \( y \in \mathbb{R}^{2^{|J|}} \) be a feasible solution of \( \text{las}_t(G) \) with \( 2t \geq |J| \). Then \( y_J = 0 \), which can be seen as follows: Write \( J = J_1 \cup J_2 \) with \( |J_1|, |J_2| \leq t \) and \( \{i,j\} \subseteq J_1 \). First, consider the following \( 2 \times 2 \) principal submatrix of the positive semidefinite matrix \( M_t(y) \)

\[
\begin{pmatrix}
ij & J_1 \\
J_1 & \begin{pmatrix} y_{ij} & y_{J_1} \\ y_{J_1} & y_{J_2} \end{pmatrix} \end{pmatrix} \succeq 0 \implies y_{J_1} = 0,
\]

where we applied the constraint \( y_{ij} = 0 \). Then, consider the following \( 2 \times 2 \) principal submatrix of \( M_t(y) \)

\[
\begin{pmatrix}
J_1 & J_2 \\
J_2 & \begin{pmatrix} y_{J_1} & y_J \\ y_J & y_{J_2} \end{pmatrix} \end{pmatrix} \succeq 0 \implies y_J = 0.
\]

Hence,

\[
\text{las}_t(G) = \text{las}_{t+1}(G) = \cdots = \text{las}_n(G) \quad \text{for } t \geq \alpha(G).
\]

The next step is showing that vectors \( y \in \mathbb{R}^{P_2(V)} \), indexed by the full power set \( P_2(V) \), which determine a positive semidefinite moment matrix \( M_n(y) \), form a finitely generated cone:

\[
M_n(y) \succeq 0 \iff y \in \text{cone}\{\chi^I : I \subseteq V\},
\]

where \( \chi^I \) are the characteristic vectors. Sufficiency follows easily from \( \chi^I_{J_1,J_2} = \chi^I_{J_1} \chi^J_{J_2} \). For necessity, we first observe that the characteristic vectors form a basis of \( \mathbb{R}^{P_2(V)} \). Let \( (\psi^I)_{I \in P_2(V)} \) be its dual basis; it satisfies \( (\chi^I)^T \psi^J = \delta_{I,J} \). Let \( y \) be so that \( M_n(y) \) is positive semidefinite and write \( y \) in terms of the basis

\[
y = \sum_{I \in P_2(V)} \alpha_I \chi^I \quad \text{with } \alpha_I \in \mathbb{R}.
\]
Since $M_n(y)$ is positive semidefinite we have
\[
0 \leq (\psi^J)^T M_n(y) \psi^J = \alpha_J.
\]

Now we finish the proof. Let $y \in \mathbb{R}^{P_n(V)}$ be a feasible solution of $\text{las}_n(G)$. Then from the previous arguments we see
\[
y = \sum_{I \text{ independent}} \alpha_I \chi^I, \quad \text{with } \alpha_I \geq 0.
\]
Furthermore, the semidefinite program is normalized by
\[
1 = y_{\emptyset} = \sum_{I \text{ independent}} \alpha_I,
\]
and the objective value of $y$ equals
\[
\sum_{i \in V} y_i = \sum_{i \in V} \sum_{I \text{ indep.}} \alpha_I \chi^I(i) = \sum_{I \text{ indep.}} \alpha_I \sum_{i \in V} \chi^I(i) \leq 1 \cdot \alpha(G).
\]

\[
\square
\]

4. Symmetry reduction and matrix $\ast$-algebras

One can obtain semidefinite programming bounds for $A_2(n,d)$ by using the Lasserre bound of order $t$ for the graph $G(n,d)$, defined in Example 3.2. Since the graph $G(n,d)$ has exponentially many vertices, even computing the first step $t = 1$ amounts to solving a semidefinite program of exponential size. On the other hand, the graph $G(n,d)$ is highly symmetric and these symmetries can be used to simplify the semidefinite programs considerably.

4.1. Symmetry reduction of semidefinite programs. Symmetry reduction of semidefinite programs is easiest explained using complex semidefinite programs of the form (2). Let $\Gamma$ be a finite group and let $\pi : \Gamma \to U(\mathbb{C}^n)$ be a unitary representation of $\Gamma$; that is a group homomorphism from $\Gamma$ to the group of unitary matrices $U(\mathbb{C}^n)$. Then $\Gamma$ acts on the space of complex matrices by
\[
(g,X) \mapsto gX = \pi(g)X\pi(g)^*.
\]
A complex matrix $X$ is called $\Gamma$-invariant if $X = gX$ holds for all $g \in \Gamma$. By
\[
(\mathbb{C}^{n \times n})^\Gamma = \{X \in \mathbb{C}^{n \times n} : X = gX \text{ for all } g \in \Gamma\}
\]
we denote the set of all $\Gamma$-invariant matrices.

**Definition 4.1.** Let $\Gamma$ be a finite group. A complex semidefinite program is called $\Gamma$-invariant if for every feasible solution $X$ and every $g \in \Gamma$ the matrix $gX$ also is feasible and $\langle C, X \rangle_T = \langle C, gX \rangle_T$ holds. (Recall $\langle X, Y \rangle_T = \text{Tr}(Y^*X)$.)

Suppose that the complex semidefinite program (2) is $\Gamma$-invariant. Then we may restrict the optimization variable $X$ to be $\Gamma$-invariant without changing the supremum. In fact, if $X$ is feasible for (2), so is its $\Gamma$-average
\[
X = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} gX.
\]
Hence, (2) simplifies to
\[
p^* = \sup\{\langle C, X \rangle_T : X \succeq 0, X \in (\mathbb{C}^{n \times n})^\Gamma, \langle A_1, X \rangle_T = b_1, \ldots, \langle A_m, X \rangle_T = b_m\}.
\]

(3)
If we intersect the $\Gamma$-invariant complex matrices $(\mathbb{C}^{n \times n})^\Gamma$ with the Hermitian matrices we get a vector space having a basis $B_1, \ldots, B_N$. If we express $X$ in terms of this basis, (3) becomes

$$p^* = \sup \{ \langle C, X \rangle_T : x_1, \ldots, x_N \in \mathbb{C},$$

$$X = x_1 B_1 + \cdots + x_N B_N \succeq 0, \quad \langle A_1, X \rangle_T = b_1, \ldots, \langle A_m, X \rangle_T = b_m \};$$

(4)

So the number of optimization variables is $N$. It turns out that we can simplify (4) even more by performing a simultaneous block diagonalization of the basis $B_1, \ldots, B_N$. This is a consequence of the main structure theorem of matrix $*$-algebras.

4.2. Matrix $*$-algebras.

Definition 4.2. A linear subspace $A \subseteq \mathbb{C}^{n \times n}$ is called a matrix algebra if it is closed under matrix multiplication. It is called a matrix $*$-algebra if it is closed under taking the conjugate transpose: if $A \in A$, then $A^* \in A$.

The space of $\Gamma$-invariant matrices $(\mathbb{C}^{n \times n})^\Gamma$ is a matrix $*$-algebra. Indeed, for $\Gamma$-invariant matrices $X, Y$ and $g \in \Gamma$, we have

$$g(XY) = \pi(g)XY \pi(g)^* = (\pi(g)X \pi(g)^*)(\pi(g)Y \pi(g)^*) = (gX)(gY) = XY,$$

and

$$g(X^*) = \pi(g)X^* \pi(g)^* = (\pi(g)X \pi(g)^*)^* = (gX)^* = X^*.$$

The main structure theorem of matrix $*$-algebras—it is due to Wedderburn and it is well-known in the theory of $C^*$-algebras, where it can be also stated for the compact operators on a Hilbert space—is the following:

Theorem 4.3. Let $A \subseteq \mathbb{C}^{n \times n}$ be a matrix $*$-algebra. Then there are natural numbers $d, m_1, \ldots, m_d$ such that there is a $*$-isomorphism between $A$ and a direct sum of full matrix $*$-algebras

$$\varphi: A \to \bigoplus_{k=1}^d \mathbb{C}^{m_k \times m_k}.$$  

Here a $*$-isomorphism is a bijective linear map between two matrix $*$-algebras which respects multiplication and taking the conjugate transpose.

An elementary proof of Theorem 4.3, which also shows how to find a $*$-isomorphism $\varphi$ algorithmically, is presented in [6]. An alternative proof is given in [67, Section 3] in the framework of representation theory of finite groups; see also [66] and [4].

Now we want to apply Theorem 4.3 to block diagonalize the $\Gamma$-invariant semidefinite program (4). Let $A = (\mathbb{C}^{n \times n})^\Gamma$ be the matrix $*$-algebra of $\Gamma$-invariant matrices. Let $\varphi$ be a $*$-isomorphism as in Theorem 4.3; then $\varphi$ preserves positive semidefiniteness. Hence, (4) is equivalent to

$$p^* = \sup \{ \langle C, X \rangle_T : x_1, \ldots, x_N \in \mathbb{C},$$

$$x_1 \varphi(B_1) + \cdots + x_N \varphi(B_N) \succeq 0,$$

$$X = x_1 B_1 + \cdots + x_N B_N,$$

$$\langle A_1, X \rangle_T = b_1, \ldots, \langle A_m, X \rangle_T = b_m \};$$

(4)
Thus, instead of dealing with one (potentially big) matrix of size $n \times n$ one only has to work with $d$ (hopefully small) block diagonal matrices of size $m_1, \ldots, m_d$. This reduces the dimension from $n^2$ to $m_1^2 + \cdots + m_d^2$. Many practical semidefinite programming solvers can take advantage of this block structure and numerical calculations can become much faster. However, finding an explicit $*$-isomorphism is usually a nontrivial task, especially if one is interested in parameterized families of matrix $*$-algebras.

4.3. Example: The Delsarte Linear Programming Bound. Let us apply the symmetry reduction technique to demonstrate that the exponential size semidefinite program $\vartheta(G(n, d))$ collapses to the linear size Delsarte Linear Programming Bound.

Since the graph $G(n, d)$ is a Cayley graph over the additive group $\mathbb{F}_n^2$, the semidefinite program $\vartheta(G(n, d))$ is $\mathbb{F}_n^2$-invariant where the group is acting as permutations of the rows and columns of the matrix $X \in \mathbb{C}^{\mathbb{F}_n^2 \times \mathbb{F}_n^2}$. The graph $G(n, d)$ has even more symmetries. Its automorphism group $\text{Aut}(G(n, d))$ consists of all permutations of the $n$ coordinates $x = x_1 x_2 \cdots x_n \in \mathbb{F}_2^n$ followed by independently switching the elements of $\mathbb{F}_2$ from 0 to 1, or vice versa. So the semidefinite program $\vartheta(G(n, d))$ is $\text{Aut}(G(n, d))$-invariant. The $*$-algebra $\mathcal{B}_n$ of $\text{Aut}(G(n, d))$-invariant matrices is called the Bose-Mesner algebra (of the binary Hamming scheme). A basis $B_0, \ldots, B_n$ is given by zero-one matrices

$$(B_r)_{x,y} = \begin{cases} 1, & \text{if } d_H(x, y) = r, \\ 0, & \text{otherwise}, \end{cases}$$

with $r = 0, \ldots, n$. So, $\vartheta(G(n, d))$ in the form of (4) is the following semidefinite program in $n + 1$ variables:

$$\max \left\{ 2^n \sum_{r=0}^{n} \binom{n}{r} x_r : x_0 = \frac{1}{2^n}, x_1 = \cdots = x_{d-1} = 0, \right.$$ 

$$x_{d}, \ldots, x_n \geq 0, \sum_{r=0}^{n} x_r B_r \succeq 0 \right\}.$$ 

Finding a simultaneous block diagonalization of the $B_r$’s is easy since they pairwise commute and have a common system of eigenvectors. An orthogonal basis of eigenvectors is given by $\chi_a \in \mathbb{C}^{\mathbb{F}_2^n}$ defined componentwise by

$$(\chi_a)_x = \prod_{j=1}^{n} (-1)^{a_j x_j}.$$
Indeed,

\[
(B_r \chi_a)x = \sum_{y \in \mathbb{F}_2^n} (B_r)x,y(\chi_a)y
= \sum_{y \in \mathbb{F}_2^n} (B_r)x,y(\chi_a)y - x(\chi_a)x
= \left( \sum_{y \in \mathbb{F}_2^n, d_H(x,y) = r} (\chi_a)y \right)(\chi_a)x
= \left( \sum_{y \in \mathbb{F}_2^n, d_H(0,y) = r} (\chi_a)y \right)(\chi_a)x.
\]

The eigenvalues are given by the Krawtchouk polynomials

\[
K_r^{(n,2)}(x) = \sum_{j=0}^{r} (-1)^j \binom{n}{j} \binom{n-x}{r-j}
\]

through

\[
\sum_{y \in \mathbb{F}_2^n, d_H(0,y) = r} (\chi_a)y = K_r^{(n,2)}(d_H(0,a)).
\]

Altogether, we have the \(r\)-algebra isomorphism

\[
\varphi : B_n \to \bigoplus_{r=0}^{n} \mathbb{C},
\]

(so \(m_0 = \cdots = m_n = 1\)) defined by

\[
\varphi(B_r) = (K_r^{(n,2)}(0), K_r^{(n,2)}(1), \ldots, K_r^{(n,2)}(n)).
\]

So the semidefinite program \(\varphi'(G(n,d))\) degenerates to the following linear program

\[
\max \left\{ 2^n \sum_{r=0}^{n} \binom{n}{r} x_r : x_0 = \frac{1}{2^n}, x_1 = \cdots = x_{d-1} = 0, x_d, \ldots, x_n \geq 0, \sum_{r=0}^{n} x_r K_r^{(n,2)}(j) \geq 0 \text{ for } j = 0, \ldots, n \right\}.
\]

This is the Delsarte Linear Programming Bound.

4.4. Example: The Schrijver Semidefinite Programming Bound. To set up a stronger semidefinite programming bound one can apply the Lasserre bound directly, but also many variations are possible. These variations are crucial to be able to exploit the symmetries of the problem at hand. For instance, one can consider only “interesting” principal submatrices of the moment matrices to simplify the computation.

A rough classification for these variations can be given in terms of \(k\)-point bounds. This refers to all variations which make use of variables \(y_I\) with \(|I| \leq k\). A \(k\)-point bound is capable of using obstructions coming from the local interaction of configurations having at most \(k\) points. For instance Lovász \(\vartheta\)-number is a 2-point bound and the \(t\)-th step in Lasserre’s hierarchy is a \(2t\)-point bound. The relation between \(k\)-point bounds and Lasserre’s hierarchy was first made explicit by Laurent [41] in the case of bounds for binary codes; see also Gijswijt [31], who
discusses the symmetry reduction needed to compute $k$-point bounds for block codes, and de Laat and Vallentin [24], who consider $k$-point bounds for compact topological packing graphs.

Schrijver’s bound for binary codes [57] is a 3-point bound. Essentially, it looks at principal submatrices $M_a \in \mathbb{R}^{F_n^2 \times F_n^2}$ of the matrix $M_2(y)$ defined by

$$(M_2(y))_{b,c} = y_{[a,b,c]}$$

with $a, b, c \in F_n^2$.

The group which leaves the corresponding semidefinite program invariant is the stabilizer of a codeword in $\text{Aut}(G(n,d))$, which is the symmetric group permuting the $n$ coordinates of $F_n^2$.

The algebra $A_n \subseteq \mathbb{R}^{F_n^2 \times F_n^2}$ invariant under this group action is called the Terwilliger algebra of the binary Hamming scheme. Schrijver determined a block diagonalization of the Terwilliger algebra which we recall here.

For nonnegative integers $i, j, t$, with $t \leq i, j$ and $i + j \leq n + t$, the matrices

$$(B^t_{i,j})_{x,y} = \begin{cases} 1, & \text{if } \text{wt}_H(x) = i, \text{wt}_H(y) = j, d_H(x,y) = i + j - 2t, \\ 0, & \text{otherwise.} \end{cases}$$

form a basis of $A_n$. Hence, $\dim A_n = \binom{n+3}{3}$. The desired $\ast$-isomorphism

$$\varphi : A_n \rightarrow \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \mathbb{C}^{(n-2k+1) \times (n-2k+1)}$$

is defined as follows: Set

$$\beta^t_{i,j,k} = \sum_{u=0}^{n} (-1)^{u-t} \binom{n}{u} \binom{n-2k}{u-k} \binom{n-k-u}{i-u} \binom{n-k-u}{j-u}$$

so that

$$\varphi(B^t_{i,j}) = \left( \ldots, \left( \binom{n-2k}{i-k}^{-1/2} \beta^t_{i,j,k} \binom{n-k}{j-k}^{-1/2} \right)_{k=0,\ldots,\lfloor n/2 \rfloor} \right).$$

Schrijver determined the $\ast$-isomorphism from first principles using linear algebra. Later, Vallentin [67] used representation theory of finite groups to derive an alternative proof. Here the connection to the orthogonal Hahn and Krawtchouk polynomials becomes visible. Another constructive proof of the explicit block diagonalization of $A_n$ was given by Srinivasan [60]; see also Martin and Tanaka [46].

5. Extensions and ramifications

Explicit computations of $k$-point semidefinite programming bounds have been done in a variety of situations, in the finite and infinite setting. Table 1 gives a guide to the literature.

Semidefinite programming bounds have also been developed for generalized distances and list decoding radii of binary codes by Bachoc and Zémor [9], for permutation codes by Bogaerts and Dukes [14], for mixed binary/ternary codes by Litjens [43], for subsets of coherent configurations by Hobart [36] and Hobart and Williford [37], for ordered codes by Trinker [64], for energy minimization on $S^2$ by de Laat [21] and for spherical two-distance sets and for equiangular lines by Barg and Yu [10] and by Machado, de Laat, Oliveira, and Vallentin [23]. They have been used by Brouwer and Polak [15] to prove the uniqueness of several constant weight codes.
Table 1. Computation of $k$-point bounds.

| Problem                  | 2-point bound       | 3-point bound       | 4-point bound       |
|--------------------------|---------------------|---------------------|---------------------|
| Binary codes             | Delsarte [26]       | Schrijver [57]      | Gijswijt, Mittelmann, Schrijver [32] |
| q-ary codes              | Delsarte [26]       | Gijswijt, Schrijver, Tanaka [52] |
|                         |                     |                     | Litjens, Polak, Schrijver [44] |
| Constant weight codes   | Delsarte [26]       | Schrijver [57], Regts [54] |
| Lee codes                | Astola [2]          | Polak [51]          | Polak [52]          |
|                         | Bachoc, Chandar, Solé |
|                         | Tchamkerten [5]     |
| Grassmannian codes      | Bachoc [3]          |                     |                     |
| Projective codes        | Bachoc, Passuello, Vallentin [7] |
| Spherical codes         | Delsarte, Goethals, Seidel [27] |
|                         |                     | Bachoc, Vallentin [8] |
| Codes in $\mathbb{R}^{n-1}$ | Kabatiansky, Levenshtein [38] |
|                         |                     | Cohn, Woo [17]      |
| Sphere packings         | Cohn, Elkies [16]   |
| Binary sphere and       | de Laat, Oliveira, Vallentin [22] |
| spherical cap packings  |                     | Dostert, Guzmán, Oliveira, Vallentin [28] |
| Translative body packings |                     |                     |
| Congruent copies of a convex body | Oliveira, Vallentin [25] |

In extremal combinatorics, (weighted) vector space versions of the Erdős-Ko-Rado Theorem for cross intersecting families have been proved using semidefinite programming bounds by Suda and Tanaka [61] and by Suda, Tanaka and Tokushige [62], see also the survey by Frankl and Tokushige [29].

Another coding theory application of the symmetry reduction technique are new approaches to the Assmus-Mattson Theorem by Tanaka [63] and by Morales and Tanaka [49].

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