THE NONLINEAR SCHRÖDINGER EQUATIONS WITH HARMONIC POTENTIAL IN MODULATION SPACES

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Abstract. We study nonlinear Schrödinger equation associated to harmonic oscillator $H = -\Delta + |x|^2$ in modulation spaces $M^{p,q}$. When $F(u) = (|x|^{-\gamma} \ast |u|^2)u$, we prove global well-posedness for (NLSH) in modulation spaces $M^{p,p}(\mathbb{R}^d)$ ($1 \leq p < \frac{2d}{d+\gamma}$, $0 < \gamma < \min\{2, \frac{d}{2}\}$).

When $F(u) = (K \ast |u|^{2k})u$ with $K \in \mathcal{FL}^q$ (Fourier-Lebesgue spaces) or $M^{\infty,1}$ (Sjöstrand’s class) or $M^{1,\infty}$, some local and global well-posedness for (NLSH) are obtained in some modulation spaces. As a consequence, we can get local and global well-posedness for (NLSH) in a function spaces which are larger than usual $L^p$–Sobolev spaces.

1. Introduction. We study Cauchy problem for the nonlinear Schrödinger equation with the harmonic oscillator $H = -\Delta + |x|^2$:

$$i\partial_t u(t,x) - Hu(t,x) = F(u), \quad u(0,x) = u_0(x),$$

where $u : \mathbb{R}_t \times \mathbb{R}^d_x \rightarrow \mathbb{C}$, $u_0 : \mathbb{R}^d_x \rightarrow \mathbb{C}$, $\Delta = \sum_1^d \frac{\partial^2}{\partial x_i^2}$, and $F : \mathbb{C} \rightarrow \mathbb{C}$ is a nonlinearity.

Mainly we consider nonlinearity of the Hartree and power type. Specifically, we study (1) with the Hartree type nonlinearity

$$F(u) = (K \ast |u|^{2k})u,$$

where $*$ denotes the convolution in $\mathbb{R}^d$, $k \in \mathbb{N}$, and $K$ is of the following type:

$$K(x) = \frac{\lambda}{|x|^\gamma}, \quad (\lambda \in \mathbb{R}, \gamma > 0, x \in \mathbb{R}^d),$$

$$K \in \mathcal{FL}^q(\mathbb{R}^d) \quad (1 \leq q \leq \infty),$$

$$K \in M^{\infty,1}(\mathbb{R}^d),$$

$$K \in M^{1,\infty}(\mathbb{R}^d),$$

where $\mathcal{FL}^q$ is a Fourier-Lebesgue space, and $M^{\infty,1}(\mathbb{R}^d) \supset \mathcal{FL}^1(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d) \subset M^{1,\infty}(\mathbb{R}^d)$ are modulation spaces (see Definition 2.1 below). The homogeneous kernel of the form (3) is known as Hartree potential. The kernel of the form (5) is sometimes called Sjöstrand class (particular modulation space). We also study

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(1) when $F$ is real entire and $F(0) = 0$ (see Definition 4.4 below). In this case power-type nonlinearity

$$F(u) = \pm |u|^{2k}u \ (k \in \mathbb{N})$$

is covered, and in particular, when $F(u) = -|u|^2u$, equation (1) is the well-known Gross-Pitaevskii equation.

The harmonic oscillator (also known as Hermite operator) $H$ is a fundamental operator in quantum physics and in analysis [30]. Equation (1) models Bose-Einstein condensates with attractive interparticle interactions under a magnetic trap [8, 27, 36, 18]. The isotropic harmonic potential $|x|^2$ describes a magnetic field whose role is to confine the movement of particles [8, 27, 32]. A class of NLS with a “nonlocal” nonlinearity that we call “Hartree type” occurs in the modeling of quantum semiconductor devices (see [12] and the references therein, cf. [9]). It is known that the free Schrödinger propagator $e^{it\Delta}$ is bounded on Lebesgue spaces $L^p(\mathbb{R}^d)$ if and only if $p = 2$ (see e.g., [29, Exercise 2.35]). Hence, we cannot expect to solve Schrödinger equation on $L^p$-spaces. This has inspired to study in other function spaces (e.g., modulation spaces $M^{p,q}$) arising in harmonic analysis. In fact, in contrast to $L^p$-spaces, in [34, 2] it is proved that the Schrödinger propagator $e^{it(-\Delta)^\frac{\alpha}{2}} \ (0 \leq \alpha \leq 2)$ is bounded on $M^{p,q}(\mathbb{R}^d)$ for all $1 \leq p, q \leq \infty$. Local well-posedness results of the corresponding nonlinear equations, with nonlinearity of power-type, or more generic real entire, were obtained in [3, 6]. And global well-posedness results, with nonlinearity of power or Hartree type, were obtained in [33, 4, 34]. We refer to excellent survey [26] and the reference therein for details.

Coming back to harmonic oscillator $H$, we note that well-posedness for (1) with Hartree and power type nonlinearity were obtained in the energy space in [11, 10, 12, 13, 9, 36, 27]. On the other hand, many authors have studied Schrödinger equations with generic potential, i.e., $i\partial_t u - (-\Delta)^{\alpha/2}u + V(t,x)u$, with $V$ real valued, $0 < \alpha \leq 2$. In [22, Theorems 1.1 and 1.2] it is proved that when $\alpha = 2, V \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$ with quadratic growth (or sub quadratic) in the sense that $\partial^\beta_\xi V(t,x) \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ for $|\beta| \geq 2$ (or $\partial^\beta_\xi^2 V(t,x) \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ for $|\beta| \geq 1$) the corresponding propagator is bounded on $M^{p,p}(\mathbb{R}^d)$ (or $M^{p,q}(\mathbb{R}^d)$), $1 \leq p, q \leq \infty$ (cf. [7, 21, 14] and Proposition 2.4 below). Cordero-Nocola-Rodino [15] have generalized this result when the potential $V(t,x) = \sum_{\ell=0}^s V_\ell(t,x)$ with certain derivatives of $V_\ell$ in Sjöstrand class $M^{\infty,1}(\mathbb{R}^d)$ (see also [16, Theorem 1.5], [24]). As an application, they obtained [15, Theorem 6.1] local well-posedness for the corresponding nonlinear equation with power type (real entire) nonlinearity. However, it seems there is no result concerning global well-posedness of equation (1) in modulation spaces so far. We also note that Cauchy data in modulation spaces are rougher (see Proposition 1 below) than $L^p$–Sobolev spaces (see definition in Subsection 2.1 below).

Taking these considerations into account, we are inspired to study (1) in modulation spaces. Specifically, we have following theorem.

**Theorem 1.1.** Let $F(u)$ and $K$ be given by (2) and (3) respectively with $k = 1$, and $0 < \gamma < \min\{2, d/2\}, d \in \mathbb{N}$. Assume that $u_0 \in M^{p,p}(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+\gamma}$. Then there exists a unique global solution of (1) such that

$$u \in C([0,\infty), M^{p,p}(\mathbb{R}^d)) \cap L^{8/\gamma}_{lo}([0,\infty), L^{4d/(2d-\gamma)}(\mathbb{R}^d)).$$

Since $L^p$–Sobolev spaces $L^p_{lo}(\mathbb{R}^d) \subseteq M^{p,p}(\mathbb{R}^d)$ for $s \geq d(\frac{2}{p} - 1), 1 < p \leq 2$ and $L^4_{lo}(\mathbb{R}^d) \subseteq M^{1,1}(\mathbb{R}^d)$ for $s > d$ (see Proposition 1 below). Theorem 1.1 reveals that we can get the global well-posedness for (1) with Cauchy data rougher than
\(L^p\)–Sobolev spaces. We do not know whether the range of \(p\) in Theorem 1.1 is sharp or not for the Hartree potential. However, if we take potential from Sjöstrand’s class, the range of \(p\) can be improved. Since modulation spaces enlarges as their exponents are increasing (see Lemma 2.2 (1) below), we can solve (1) with Cauchy data in relatively more low regularity spaces. Specifically, we have following theorem.

**Theorem 1.2.** Let \(F(u)\) be given by (2) with \(k = 1\). Assume that \(K\) is given by (5) and \(u_0 \in M^{p,p}(\mathbb{R}^d)\) \((1 \leq p \leq 2)\). Then there exists a unique global solution of (1) such that

\[ u \in C([0, \infty), M^{p,p}(\mathbb{R}^d)). \]

Now we note that formally the solution of (1) satisfies (see for e.g., [12, 36, 9]) the conservation of mass:

\[ \|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad (t \in \mathbb{R}^+), \]

and exploiting this mass conservation law, below Theorem 2.4, and techniques from time-frequency analysis, we prove global existence results (above Theorems 1.1 and 1.2) for (1) with Hartree type nonlinearity.

**Remark 1.** Consider equation (1) with generic potential, that is,

\[ i\partial_t u + \Delta u + V_2(t,x)u + V_1(t,x)u + V_0(t,x)u = F(u), u(x,0) = u_0(x) \quad (7) \]

where \(\partial^2_\beta V_2(t,\cdot) \in M^{\infty,1}(\mathbb{R}^d)\) for \(|\beta| = 2\), \(\partial^2_\beta V_1(t,\cdot) \in M^{\infty,1}(\mathbb{R}^d)\) for \(|\beta| = 1\), \(V_0 \in M^{\infty,1}(\mathbb{R}^d)\), with \(V_2, V_1\) real valued. Taking [15, Theorem 1.1 and Section 2.3] into account, the method of proof of Theorems 1.1 and 1.2 may further be applied to (7) to obtain global well-posedness in some modulation spaces.

**Theorem 1.3.** Let \(F(u)\) be given by (2).

1. Assume that \(K\) is given by (4) and \(u_0 \in M^{1,1}(\mathbb{R}^d)\). Then there exists \(T^* = T^*(\|u_0\|_{M^{1,1}})\) such that (1) has a unique solution \(u \in C([0,T^*), M^{1,1}(\mathbb{R}^d))\).

2. Assume that \(K\) is given by (4) with \(1 < q < r \leq 2\), \(k = 1\) and \(u_0 \in M^{\frac{2r}{r-1},\frac{2r}{2r-1}}(\mathbb{R}^d)\). Then there exists \(T^* = T^*(\|u_0\|_{M^{\frac{2r}{r-1},\frac{2r}{2r-1}}})\) such that (1) has a unique solution \(u \in C([0,T^*), M^{\frac{2r}{r-1},\frac{2r}{2r-1}}(\mathbb{R}^d))\).

3. Assume that \(K\) is given by (6) and \(u_0 \in M^{1,1}(\mathbb{R}^d)\). Then there exists \(T^* = T^*(\|u_0\|_{M^{1,1}})\) such that (1) has a unique solution \(u \in C([0,T^*), M^{1,1}(\mathbb{R}^d))\).

**Remark 2.** The analogue of Theorem 1.3 may further be obtained for equation (7) in some modulation spaces.

Finally, for the power type nonlinearity, we recall following local wellposedness result and point out that it remains an open problem to get the global existence (cf. [26, p. 280]) of (1) in modulation spaces.

**Theorem 1.4.** Let \(F\) is real entire and \(F(0) = 0\). Assume that \(u_0 \in M^{1,1}(\mathbb{R}^d)\). Then there exists \(T^* = T^*(\|u_0\|_{M^{1,1}})\) such that (1) has a unique solution \(u \in C([0,T^*), M^{1,1}(\mathbb{R}^d))\).

Now we briefly mention the mathematical literature. Carles-Mauser-Stimming [13] and Cao-Carles [9] have studied well-posedness for (1) with Hartree type nonlinearity. Zhang [36] and Carles [11, 10, 12] have studied well-posedness for (1) with power type nonlinearity. We would like to mention that so far all previous authors have studied (1) in the energy space

\[ \Sigma = \{ f \in S'(\mathbb{R}^d) : \|f\|_\Sigma := \|f\|_{L^2} + \|\nabla f\|_{L^2} + \|xf\|_{L^2} < \infty \}. \]
Finally, we note that the existence of solution for (1) is shown under very low regularity assumption for the initial data. Specifically, to see what typical Cauchy data Theorems 1.1, 1.2, and 1.4 can handle, see Proposition 1 and Lemma 2.2 (2) below. Theorems 1.1, 1.2,1.3, and 1.4 highlight that modulation spaces are a good alternative as compared to Sobolev and Besov spaces to study equation (1). And we hope that our results will be useful for the further study (e.g., stability and scattering theory) of equation (1).

This paper is organized as follows. In Section 2, we introduce notations and preliminaries which will be used in the sequel. In particular, in Subsections 2.1 and 2.2, we introduce $L_p^r$—Sobolev spaces and modulation spaces (and their properties) respectively. In Subsection 2.3, we review boundedness of Schrödinger propagator associated with harmonic potential on modulation spaces. As an application of Strichartz’s estimates and conservation of mass, we obtain global well-posedness for (1) in $L^2(\mathbb{R}^d)$ in Section 3. We shall see this will turn out to be one of the main tools to obtain global well-posedness in modulation spaces. In Subsections 4.1, 4.2, 4.3 and 4.4, we prove Theorems 1.1, 1.2, 1.4 and 1.3 respectively.

2. Preliminaries.

2.1. Notations. The notation $A \lesssim B$ means $A \leq cB$ for some constant $c > 0$. The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space $A_1$ into $A_2$. If $\alpha = (\alpha_1, ... , \alpha_d) \in \mathbb{N}^d$ is a multi-index, we set $|\alpha| = \sum_{j=1}^{d} \alpha_j, \alpha! = \prod_{j=1}^{d} \alpha_j!$. If $z = (z_1, ..., z_d) \in \mathbb{C}^d$, we put $z^\alpha = \prod_{j=1}^{d} z_{j}^{\alpha_j}$. The characteristic function of a set $E \subset \mathbb{R}^d$ is $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. Let $I \subset \mathbb{R}$ be an interval. Then the norm of the space-time Lebesgue space $L^p(I,L^q(\mathbb{R}^d))$ is defined by

$$\|u\|_{L^p(I,L^q(\mathbb{R}^d))} = \|u\|_{L^p_t L^q_x} = \left(\int_{I} \|u(t)\|_{L^q_x}^p \, dt \right)^{1/p}.$$  

If there is no confusion, we simply write

$$\|u\|_{L^p(I,L^q)} = \|u\|_{L^p_t L^q_x} = \|u\|_{L^p_{t,x}}.$$  

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$ (with its usual topology), and the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. For $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we put $x \cdot y = \sum_{i=1}^{d} x_i y_i$. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, dx, \ \xi \in \mathbb{R}^d.$$  

Then $\mathcal{F}$ is a bijection and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x) = \check{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi, \ x \in \mathbb{R}^d.$$  

The Fourier transform can be uniquely extended to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. The Fourier-Lebesgue spaces $\mathcal{F}L^p(\mathbb{R}^d)$ is defined by

$$\mathcal{F}L^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \hat{f} \in L^p(\mathbb{R}^d) \right\}.$$  

The $\mathcal{F}L^p(\mathbb{R}^d)$—norm is denoted by

$$\|f\|_{\mathcal{F}L^p} = \|\hat{f}\|_{L^p} \ (f \in \mathcal{F}L^p(\mathbb{R}^d)).$$
The standard Sobolev spaces $W^{s,p}({\mathbb R}^d)$ $(1 < p < \infty, s \geq 0)$ have a different character according to whether $s$ is integer or not. Namely, for $s$ integer, they consist of $L^p-$functions with derivatives in $L^p$ up to order $s$, hence coincide with the $L^p_s -$Sobolev spaces (also known as Bessel potential spaces), defined for $s \in \mathbb R$ by

$$L^p_s({\mathbb R}^d) = \{ f \in S'({\mathbb R}^d) : \mathcal{F}^{-1}[(\cdot)^s \mathcal{F}(f)] \in L^p({\mathbb R}^d) \},$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{s/2}$ ($\xi \in {\mathbb R}^d$). Note that $L^p_{s_1}({\mathbb R}^d) \hookrightarrow L^p_{s_2}({\mathbb R}^d)$ if $s_2 \leq s_1$.

2.2. Modulation spaces. Feichtinger [19] introduced a class of Banach spaces, the so-called modulation spaces, which allow a measurement of space variable and Fourier transform variable of a function or distribution on $\mathbb R^d$ simultaneously using the short-time Fourier transform (STFT). The STFT of a function $f$ with respect to a window function $g \in \mathcal{S}(\mathbb R^d)$ is defined by

$$V_g f(x,y) = (2\pi)^{-d/2} \int_{\mathbb R^d} f(t)g(t-x)e^{-iy \cdot t} \, dt, \quad (x,y) \in \mathbb R^{2d}$$

whenever the integral exists. For $x,y \in \mathbb R^d$ the translation operator $T_x$ and the modulation operator $M_y$ are defined by $T_x f(t) = f(t-x)$ and $M_y f(t) = e^{iy \cdot t} f(t)$. In terms of these operators the STFT may be expressed as

$$V_g f(x,y) = \langle f, M_y T_x g \rangle$$

where $\langle f,g \rangle$ denotes the inner product for $L^2$ functions, or the action of the tempered distribution $f$ on the Schwartz class function $g$. Thus $V : (f,g) \rightarrow V_g(f)$ extends to a bilinear form on $S'({\mathbb R}^d) \times \mathcal{S}(\mathbb R^d)$ and $V_g(f)$ defines a uniformly continuous function on $\mathbb R^d \times \mathbb R^d$ whenever $f \in S'({\mathbb R}^d)$ and $g \in \mathcal{S}(\mathbb R^d)$.

**Definition 2.1** (modulation spaces). Let $1 \leq p,q \leq \infty$, and $0 \neq g \in \mathcal{S}(\mathbb R^d)$. The modulation space $M^{p,q}({\mathbb R}^d)$ is defined to be the space of all tempered distributions $f$ for which the following norm is finite:

$$||f||_{M^{p,q}} = \left( \int_{\mathbb R^d} \left( \int_{\mathbb R^d} |V_g f(x,y)|^p \, dx \right)^{q/p} \, dy \right)^{1/q},$$

for $1 \leq p,q < \infty$. If $p$ or $q$ is infinite, $||f||_{M^{p,q}}$ is defined by replacing the corresponding integral by the essential supremum.

**Remark 3.** We note following

1. The definition of the modulation space given above, is independent of the choice of the particular window function. See [20, Proposition 11.3.2(c), p.233], [35].
2. For a pair of functions $f$ and the Gaussian $\Phi_0(\xi) = \pi^{-d/2} e^{-1/2|\xi|^2}$, the Fourier-Wigner transform of $f$ and $\Phi_0$ is defined by

$$F(x,y) := \langle \pi (x+iy)f, \Phi_0 \rangle = \int_{\mathbb R^d} e^{i(x \cdot \xi + \frac{1}{2} y \cdot \xi)} f(\xi + y) \Phi_0(\xi) \, d\xi.$$

We say $f \in M^{p,q}({\mathbb R}^d)$ if $||F(x,y)||_{L^p_\xi} < \infty$.

Next we justify Remark 3(2): we shall see how the Fourier-Wigner transform and the STFT are related. We consider the Heisenberg group $\mathbb H^d = \mathbb C^d \times \mathbb R$ with the group law

$$(z,t)(w,s) = \left( z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \bar{w}) \right).$$
Let \( \pi \) be the Schrödinger representation of the Heisenberg group which is realized on \( L^2(\mathbb{R}^d) \) and explicitly given by

\[
\pi(x, y, t) \phi(\xi) = e^{it(\xi x + \frac{1}{2} x y)} \phi(\xi + y)
\]

where \( x, y \in \mathbb{R}^d, t \in \mathbb{R}, \phi \in L^2(\mathbb{R}^d) \). The Fourier-Wigner transform of two functions \( f, g \in L^2(\mathbb{R}^d) \) is defined by

\[
W_g f(x, y) = (2\pi)^{-d/2} \langle \pi(x, y, 0) f, g \rangle.
\]

We recall polarised Heisenberg group \( \mathbb{H}^d_{pol} \) which is just \( \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \) with the group law

\[
(x, y, t)(x', y', t') = (x + x', y + y', t + t' + x' \cdot y)
\]

and the representation \( \rho(x, y, t) \) acting on \( L^2(\mathbb{R}^d) \) is given by

\[
\rho(x, y, t) \phi(\xi) = e^{it\xi y} \phi(\xi + y), \quad \phi \in L^2(\mathbb{R}^d).
\]

We now write the Fourier-Wigner transform in terms of the STFT: Specifically, we introduced in Remark 3(2) and Definition 2.1 is essentially the same. This useful identity (8) reveals that the definition of modulation spaces we have introduced in Remark 3(2) and Definition 2.1 is essentially the same.

The following basic properties of modulation spaces are well-known and for the proof we refer to [20, 19, 35].

**Lemma 2.2.** Let \( p, q, p_1, q_1 \in [1, \infty] \). Then

1. \( M^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2, q_2}(\mathbb{R}^d) \) whenever \( p_1 \leq p_2, q_1 \leq q_2 \).
2. \( M^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{q_2, q_2}(\mathbb{R}^d) \) holds for \( q_1 \leq \min\{p, p'\} \) and \( q_2 \geq \max\{p, p'\} \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \).
3. \( M^{\min\{p', 2\}, p}(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^p(\mathbb{R}^d) \hookrightarrow M^{\max\{p', 2\}, p}(\mathbb{R}^d), \frac{1}{p} + \frac{1}{p'} = 1 \).
4. \( S(\mathbb{R}^d) \) is dense in \( M^{p, q}(\mathbb{R}^d) \) if \( p \) and \( q < \infty \).
5. The Fourier transform \( \mathcal{F} : M^{p, p}(\mathbb{R}^d) \rightarrow M^{p, p}(\mathbb{R}^d) \) is an isomorphism.
6. The space \( M^{p, q}(\mathbb{R}^d) \) is a Banach space.
7. The space \( M^{p, q}(\mathbb{R}^d) \) is invariant under complex conjugation.

**Proof.** For the proof of statements (1), (2), (3) and (4), see [20, Theorem 12.2.2], [31, Proposition 1.7], [17, Corollary 1.1] and [20, Proposition 11.3.4] respectively. The proof for the statement (5) can be derived from the fundamental identity of time-frequency analysis:

\[
V_g f(x, w) = e^{-ix \cdot w} V_{\hat{x}} \hat{f}(w, -x),
\]

which is easy to obtain. The proof of the statement (7) is trivial, indeed, we have \( \|f\|_{M^{p, q}} = \|\hat{f}\|_{M^{p, q}} \).

**Theorem 2.3** (Algebra property). Let \( p, q, p_i, q_i \in [1, \infty] \) (\( i = 0, 1, 2 \)).

1. If \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0} \) and \( \frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q_0} \), then

\[
M^{p_1, q_1}(\mathbb{R}^d) \cdot M^{p_2, q_2}(\mathbb{R}^d) \hookrightarrow M^{p_0, q_0}(\mathbb{R}^d)
\]

with norm inequality \( \|fg\|_{M^{p_0, q_0}} \lesssim \|f\|_{M^{p_1, q_1}} \|g\|_{M^{p_2, q_2}} \). In particular, the space \( M^{p, q}(\mathbb{R}^d) \) is a pointwise \( \mathcal{F}L^1(\mathbb{R}^d) \)-module, that is, it satisfies

\[
\|fg\|_{M^{p, q}} \lesssim \|f\|_{\mathcal{F}L^1} \|g\|_{M^{p, q}}.
\]
Proposition 1 (examples). The following are true:

1. \( L^p_s(\mathbb{R}^d) \to M^{p,p}(\mathbb{R}^d) \) for \( s \geq d\left(\frac{2}{p} - 1\right) \) and \( 1 < p \leq 2 \).
2. \( L^p_s(\mathbb{R}^d) \to M^{r,1}(\mathbb{R}^d) \) for \( s > d, 1 \leq p, q, r \leq \infty \), and \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \).
3. Define

\[
    f(x) = \sum_{k \neq 0} |k|^{-\frac{d}{p} - 1} e^{ik \cdot x} e^{-|x|^2} \text{ in } S'(\mathbb{R}^d).
\]

Then \( f \in M^{p,q}(\mathbb{R}^d) \) for \( 1 \leq p, q \leq \infty, (p,q) \neq (1,\infty), (\infty,1) \).

Proof. For the proof of statements (1), (2), see [23, Theorem 3.1], [25, Theorem 3.4]. For the proof of statement 3, see [28, Lemma 3.8].

For further relations between modulation, Sobolev, and Besov spaces, we refer to [26, Theorems 3.6 and 3.8] and [28, 25, 31, 23, 17, 34, 33, 35] and the references therein.

2.3. Hermite and special Hermite functions. The spectral decomposition of \( H = -\Delta + |x|^2 \) is given by the Hermite expansion. Let \( \Phi_\alpha(x) \), \( \alpha \in \mathbb{N}^d \) be the normalized Hermite functions which are products of one dimensional Hermite functions. More precisely, \( \Phi_\alpha(x) = \prod_{x_j} h_{\alpha_j}(x_j) \) where

\[
    h_k(x) = (\sqrt{\pi} 2^k k!)^{-1/2} (-1)^k e^{\frac{1}{2} x^2} \frac{d^k}{dx^k} e^{-x^2}.
\]
The Hermite functions $\Phi_\alpha$ are eigenfunctions of $H$ with eigenvalues $(2|\alpha|+d)$ where $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Moreover, they form an orthonormal basis for $L^2(\mathbb{R}^d)$. The spectral decomposition of $H$ is then written as

$$H = \sum_{k=0}^{\infty} (2k + d) P_k, \quad P_k f(x) = \sum_{|\alpha| = k} \langle f, \Phi_\alpha \rangle \Phi_\alpha$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^d)$. Given a function $m$ defined and bounded on the set of all natural numbers we can use the spectral theorem to define $m(H)$. The action of $m(H)$ on a function $f$ is given by

$$m(H)f = \sum_{\alpha \in \mathbb{N}^d} m(2|\alpha| + d) \langle f, \Phi_\alpha \rangle \Phi_\alpha = \sum_{k=0}^{\infty} m(2k + d) P_k f. \quad (9)$$

This operator $m(H)$ is bounded on $L^2(\mathbb{R}^d)$. This follows immediately from the Plancherel theorem for the Hermite expansions as $m$ is bounded. On the other hand, the mere boundedness of $m$ is not sufficient to imply the $L^p$ boundedness of $m(H)$ for $p \neq 2$ (see [30]).

In the sequel, we make use of some properties of special Hermite functions $\Phi_{\alpha,\beta}$ which are defined as follows. We recall (8) and define

$$\Phi_{\alpha,\beta}(z) = (2\pi)^{-d/2} (\pi(z) \Phi_\alpha, \Phi_\beta). \quad (10)$$

Then it is well known that these so called special Hermite functions form an orthonormal basis for $L^2(\mathbb{C}^d)$. In particular, we have ([30, Theorem 1.3.5])

$$\Phi_{\alpha,0}(z) = (2\pi)^{-d/2} (\alpha!)^{-1/2} \left( \frac{i}{\sqrt{2}} \right)^{|\alpha|} e^{-\frac{1}{4}|z|^2}. \quad (11)$$

We define Schrödinger propagator associated to harmonic oscillator $m(H) = e^{itH}$, denoted by $U(t)$, by equation (9) with $m(n) = e^{itn}$ $(n \in \mathbb{N}, t \in \mathbb{R})$. Next proposition says that $U(t)$ is uniformly bounded on $M^{p,p}(\mathbb{R}^d)$. Specifically, we have

**Theorem 2.4 ([7]).** The Schrödinger propagator $m(H) = U(t) = e^{itH}$ is bounded on $M^{p,p}(\mathbb{R}^d)$ $(t \in \mathbb{R})$ for all $1 \leq p < \infty$. In fact, we have $\|e^{itH} f\|_{M^{p,p}} \leq \|f\|_{M^{p,p}}$.

**Proof.** Let $f \in S(\mathbb{R}^d)$. Then we have the Hermite expansion of $f$ as follows:

$$f = \sum_{\alpha \in \mathbb{N}^d} \langle f, \Phi_\alpha \rangle \Phi_\alpha. \quad (12)$$

Now using (12) and (10), we obtain

$$\langle \pi(z) f, \Phi_0 \rangle = \sum_{\alpha \in \mathbb{N}^d} \langle f, \Phi_\alpha \rangle \langle \pi(z) \Phi_\alpha, \Phi_0 \rangle$$

$$= \sum_{\alpha \in \mathbb{N}^d} \langle f, \Phi_\alpha \rangle \Phi_{\alpha,0}(z). \quad (13)$$

Since $\{\Phi_\alpha\}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$, (13) gives

$$\langle \pi(z)m(H) f, \Phi_0 \rangle = \sum_{\alpha \in \mathbb{N}^d} \langle m(H) f, \Phi_\alpha \rangle \Phi_{\alpha,0}(z)$$

$$= \sum_{\alpha \in \mathbb{N}^d} m(2|\alpha| + d) \langle f, \Phi_\alpha \rangle \Phi_{\alpha,0}(z).$$
Therefore, for $m(H) = e^{itH}$, we have

\[
\langle \zeta e^{itH} f, \Phi_0 \rangle = e^{itd} \sum_{\alpha \in \mathbb{N}^d} e^{2it|\alpha|} \langle f, \Phi_\alpha \rangle \Phi_{\alpha,0}(z)
\]

\[
= e^{i\alpha (2\pi)^{-d/2}} \sum_{\alpha \in \mathbb{N}^d} e^{2it|\alpha|} \langle f, \Phi_\alpha \rangle (\alpha!)^{-1/2} \left( \frac{i}{\sqrt{\alpha}} \right)^{|\alpha|} z^\alpha e^{-\frac{1}{4}|z|^2}
\]

\[
= e^{i\alpha (2\pi)^{-d/2}} \sum_{\alpha \in \mathbb{N}^d} C_\alpha(f) e^{2it|\alpha|} z^\alpha e^{-\frac{1}{4}|z|^2}
\]

where $C_\alpha(f) := \langle f, \Phi_\alpha \rangle (\alpha!)^{-1/2} \left( \frac{i}{\sqrt{\alpha}} \right)^{|\alpha|}$. In view of (8) and (14), we have

\[
\|e^{itH} f\|_{M^{p,p}}^p = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{C}^d} \left| \sum_{\alpha \in \mathbb{N}^d} C_\alpha(f) e^{2it|\alpha|} z^\alpha e^{-\frac{1}{4}|z|^2} \right|^p dz.
\]

By using polar coordinates $z_j = r_j e^{i\theta_j}$, $r_j := |z_j| \in [0, \infty)$, $z_j \in \mathbb{C}$ and $\theta_j \in [0, 2\pi)$, we get

\[
z^\alpha = r^\alpha e^{i\alpha \cdot \theta} \quad \text{and} \quad dz = r_1 r_2 \cdots r_d dr d\theta
\]

where $r = (r_1, \ldots, r_d)$, $\theta = (\theta_1, \ldots, \theta_d)$, $dr = dr_1 \cdots dr_d$, $d\theta = d\theta_1 \cdots d\theta_d$, $|r| = \sqrt{\sum_{j=1}^d r_j^2}$.

By writing the integral over $\mathbb{C}^d = \mathbb{R}^{2d}$ in polar coordinates in each time-frequency pair and using (16), we have

\[
\int_{\mathbb{C}^d} \left| \sum_{\alpha \in \mathbb{N}^d} C_\alpha(f) e^{2it|\alpha|} z^\alpha e^{-\frac{1}{4}|z|^2} \right|^p dz
\]

\[
= \prod_{j=1}^d \int_{\mathbb{R}^+} \int_{[0, 2\pi]} \left| \sum_{\alpha \in \mathbb{N}^d} C_\alpha(f) r^\alpha e^{i\sum_{j=1}^d (2t-\theta_j)\alpha_j} e^{-\frac{1}{4}|r|^2} \right|^p r_j dr_j d\theta_j.
\]

By a simple change of variable $(\theta_j - 2t) \rightarrow \theta_j$, we obtain

\[
\prod_{j=1}^d \int_{\mathbb{R}^+} \int_{[0, 2\pi]} \left| \sum_{\alpha \in \mathbb{N}^d} C_\alpha(f) r^\alpha e^{i\sum_{j=1}^d (2t-\theta_j)\alpha_j} e^{-\frac{1}{4}|r|^2} \right|^p r_j dr_j d\theta_j
\]

\[
= \prod_{j=1}^d \int_{\mathbb{R}^+} \int_{[0, 2\pi]} \left| \sum_{\alpha \in \mathbb{N}^d} C_\alpha(f) r^\alpha e^{-i\theta_j} e^{-\frac{1}{4}|r|^2} \right|^p r_j dr_j d\theta_j.
\]

Combining (15), (17), (18), and Lemma 2.2(4), we have $\|e^{itH} f\|_{M^{p,p}} = \|f\|_{M^{p,p}}$ for $f \in M^{p,p}(\mathbb{R}^d)$.

\[\square\]

**Remark 4.** In view of (13) and (14), in Theorem 2.4, we cannot expect to replace $M^{p,p}$ norm by $M^{p,q}$ norm. See also [21].

3. **Global wellposedness in $L^2(\mathbb{R}^d)$.** In this section we prove global well-posedness for (1) with Cauchy data in $L^2(\mathbb{R}^d)$. To this end, we need Strichartz estimates, and hence we recall it.
Definition 3.1. A pair \((q, r)\) is admissible if \(2 \leq r < \frac{2d}{d-2}\) \((2 \leq r \leq \infty \text{ if } d = 1 \text{ and } 2 \leq r < \infty \text{ if } d = 2)\)

\[
\frac{2}{q} = d \left(\frac{1}{2} - \frac{1}{r}\right).
\]

Proposition 2 ([11]). Let \(\phi \in L^2(\mathbb{R}^d)\) and

\[
DF(t, x) := U(t)\phi(x) + \int_0^t U(t - \tau)F(\tau, x)d\tau.
\]

Then for any time slab \(I\) and admissible pairs \((p_i, q_i)(i = 1, 2)\), there exists \(C = C(|I|, q_1)\) (constant) such that for all intervals \(I \ni 0\),

\[
\|D(F)\|_{L^{p_1, q_1}} \leq C\|\phi\|_{L^2} + C\|F\|_{L^{p_2, q_2}}, \forall F \in L^{p_2}(I, L^{q_2})
\]

where \(p'_i\) and \(q'_i\) are Hölder conjugates of \(p_i\) and \(q_i\) respectively.

We also need to work with the convolution with the Hartree potential \(|x|^{-\gamma}\), so for the convenience of reader we recall:

Proposition 3 (Hardy-Littlewood-Sobolev inequality). Assume that \(0 < \gamma < d\) and \(1 < p < q < \infty\) with \(\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{q}\). Then the map \(f \mapsto |x|^{-\gamma} * f\) is bounded from \(L^p(\mathbb{R}^d)\) to \(L^q(\mathbb{R}^d)\):

\[
\||x|^{-\gamma} * f\|_{L^q} \leq C_{d, \gamma, p}\|f\|_{L^p}.
\]

Proposition 4. Let \(F(u)\) and \(K\) be given by (2) and (3) respectively with \(k = 1\), and \(0 < \gamma < \min\{2, d\}, d \in \mathbb{N}\). If \(u_0 \in L^2(\mathbb{R}^d)\), then (1) has a unique global solution

\[
u \in C([0, \infty), L^2(\mathbb{R}^d)) \cap L^{\frac{4d}{d-4}}([0, \infty), L^{\frac{4d}{d-4}}(\mathbb{R}^d)).
\]

In addition, its \(L^2\)-norm is conserved,

\[
\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \forall t \in \mathbb{R},
\]

and for all admissible pairs \((p, q)\), \(u \in L^p_{loc}(\mathbb{R}, L^q(\mathbb{R}^d))\).

Proof. By Duhamel’s formula, we write (1) as

\[
u(t) = U(t)u_0 - i \int_0^t U(t - \tau)(K * |u|^2)u(\tau)d\tau := \Phi(u)(t).
\]

We introduce the space

\[
Y(T) = \{\phi \in C([0, T], L^2(\mathbb{R}^d)) : \|\phi\|_{L^{\infty}([0, T], L^2)} \leq 2\|u_0\|_{L^2}, \|\phi\|_{L^\frac{8}{\gamma}([0, T], L^{\frac{4d}{d-4-\gamma}})} \leq \|u_0\|_{L^2}\}
\]

and the distance

\[
d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{L^\frac{8}{\gamma}([0, T], L^{\frac{4d}{d-4-\gamma}})}.
\]

Then \((Y, d)\) is a complete metric space. Now we show that \(\Phi\) takes \(Y(T)\) to \(Y(T)\) for some \(T > 0\). We put

\[
q = \frac{8}{\gamma}, \quad r = \frac{4d}{2d - \gamma}.
\]

Note that \((q, r)\) is admissible and

\[
\frac{1}{q'} = \frac{4 - \gamma}{4} + \frac{1}{q}, \quad \frac{1}{r'} = \frac{\gamma}{2d} + \frac{1}{r}.
\]
Let \((q, r) \in \{(q, r), (\infty, 2)\}\). By Proposition 2 and Hölder inequality, we have
\[
\|\Phi(u)\|_{L_t^q, x^r} \lesssim \|u_0\|_{L^2} + \|\mathcal{K} \ast |u|^2 u\|_{L_t^{q'}, x^{r'}}
\]
\[
\lesssim \|u_0\|_{L^2} + \|\mathcal{K} \ast |u|^2\|_{L_t^{q'}, x^{r'}} \lesssim \|u\|_{L_t^{q}, x^{r}}.
\]
Since \(0 < \gamma < \min\{2, d\}\), by Proposition 3, we have
\[
\|K \ast |u|^2\|_{L_t^{q}, x^{r}}^{\frac{4}{q} + \frac{2d}{r}} = \left\|\left(\|K \ast |u|^2\|_{L_t^{q}, x^{r}}^{\frac{2}{q}}\right)^{\frac{4}{q} + \frac{2d}{r}}\right\|_{L_t^4} \lesssim \|u\|_{L_t^{q}, x^{r}}^{\frac{4}{q} + \frac{2d}{r}} \lesssim \|u\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}} \lesssim T^{1 - \frac{2}{q}} \|u\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}}.
\]
(In the last inequality we have used inclusion relation for the \(L^p\) spaces on finite measure spaces: \(\|\cdot\|_{L^p(X)} \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \cdot \|\cdot\|_{L^q(X)}\) if measure of \(X: \mu(X) < \infty, 0 < p < q < \infty\).) Thus, we have
\[
\|\Phi(u)\|_{L_t^q, x^r} \lesssim \|u_0\|_{L^2} + \mathcal{T}^{1 - \frac{2}{q}} \|u\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}}.
\]
This shows that \(\Phi\) maps \(Y(T)\) to \(Y(T)\). Next, we show \(\Phi\) is a contraction. For this, as calculations performed before, first we note that
\[
\|(K \ast |v|^2)(v - w)\|_{L_t^{q'}, x^{r'}} \lesssim T^{1 - \frac{2}{q'}} \|v\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}} \|v - w\|_{L_t^{q}, x^{r}}.
\]
Put \(\delta = \frac{8}{4 - \gamma}\) and notice that
\[
\left(\frac{1}{q} - \frac{1}{2} + \frac{1}{\delta} = \frac{1}{r} + \frac{\gamma}{2d} + \frac{1}{2d} - \frac{1}{2} = \frac{\gamma}{2d} + \frac{1}{q}.
\]
Now using Hölder inequality, we obtain
\[
\|(K \ast (|v|^2 - |w|^2))w\|_{L_t^{q'}, x^{r'}} \lesssim \|K \ast (|v|^2 - |w|^2)\|_{L_t^{q'}, x^{r'}} \lesssim \left\|\left(\|K \ast (v(v - w))\|_{L_t^{q'}, x^{r'}} + \|K \ast \bar{w}(v - w)\|_{L_t^{q'}, x^{r'}}\right) \|w\|_{L_t^{q}, x^{r}} \lesssim \left(\|v\|_{L_t^{q}, x^{r}} \|w\|_{L_t^{q'}, x^{r'}} + \|w\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}}\right) \|v - w\|_{L_t^{q}, x^{r}} \lesssim T^{1 - \frac{2}{q'}} \left(\|v\|_{L_t^{q}, x^{r}} \|w\|_{L_t^{q'}, x^{r'}} + \|w\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}}\right) \|v - w\|_{L_t^{q}, x^{r}}.
\]
In view of the identity
\[
(K \ast |v|^2)v - (K \ast |w|^2)w = (K \ast |v|^2)(v - w) + (K \ast (|v|^2 - |w|^2))w,
\]
(19), and (20) gives
\[
\|\Phi(v) - \Phi(w)\|_{L_t^{q}, x^{r}} \lesssim \|(K \ast |v|^2)(v - w)\|_{L_t^{q'}, x^{r'}} + \|(K \ast (|v|^2 - |w|^2))w\|_{L_t^{q'}, x^{r'}} \lesssim T^{1 - \frac{2}{q'}} \left(\|v\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}} + \|v\|_{L_t^{q}, x^{r}} \|w\|_{L_t^{q'}, x^{r'}} + \|w\|_{L_t^{q}, x^{r}} ^{\frac{4}{q} + \frac{2d}{r}}\right) \|v - w\|_{L_t^{q}, x^{r}}.
\]
Thus \(\Phi\) is a contraction form \(Y(T)\) to \(Y(T)\) provided that \(T\) is sufficiently small (notice that \(T\) only depends on the \(L^2\)-norm of the initial data, as in any subcritical
regime). Then there exists a unique \( u \in Y(T) \) solving (1). The global existence of the solution (1) follows from the conservation of the \( L^2 \)-norm of \( u \). The last property of the proposition then follows from the Strichartz estimates applied with an arbitrary admissible pair on the left hand side and the same pairs as above on the right hand side. \( \square \)

4. Proof of the main results.

4.1. Global well-posedness in \( M^{p,q} \) for the Hartree potential. In this section, we shall prove Theorem 1.1. For convenience of the reader, we recall

**Proposition 5** ([1]). Let \( d \geq 1 \), \( 0 < \gamma < d \) and \( \lambda \in \mathbb{R} \). There exists \( C = C(d,\gamma) \) such that the Fourier transform of \( K \) defined by (3) is

\[
\hat{K}(\xi) = \frac{\lambda C}{|\xi|^{d-\gamma}}.
\]

We start with decomposing the Fourier transform of Hartree potential into Lebesgue spaces: indeed, in view of Proposition 5, we have

\[
\hat{K} = k_1 + k_2 \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d),
\]

where \( k_1 := \chi_{\{|\xi| \leq 1\}} \hat{K} \in L^p(\mathbb{R}^d) \) for all \( p \in [\frac{d}{d-\gamma}, \infty) \) and \( k_2 := \chi_{\{|\xi| > 1\}} \hat{K} \in L^q(\mathbb{R}^d) \) for all \( q \in (\frac{d}{d-\gamma}, \infty) \).

**Lemma 4.1** ([5]). Let \( K \) be given by (3) with \( \lambda \in \mathbb{R} \), and \( 0 < \gamma < d \), and \( 1 \leq p \leq 2 \), \( 1 \leq q < \frac{2d}{d-\gamma} \). Then for any \( f \in M^{p,q}(\mathbb{R}^d) \), we have

\[
\|(K*|f|^2)f\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}}^3.
\]

**Proof.** By Theorem 2.3 (1) and (22), we have

\[
\|(K*|f|^2)f\|_{M^{p,q}} \lesssim \|K*|f|^2\|_{L^1} \|f\|_{M^{p,q}}
\lesssim \left(\|k_1|f|^2\|_{L^1} + \|k_2|f|^2\|_{L^q}\right) \|f\|_{M^{p,q}}.
\]

Note that

\[
\|k_1|f|^2\|_{L^1} \lesssim \|k_1\|_{L^{d}} \|\hat{|f|^2}\|_{L^{\infty}} \lesssim \|\hat{|f|^2}\|_{L^{q}} = \|f\|_{L^p}^2 \lesssim \|f\|_{M^{p,q}}^2.
\]

Let \( 1 < \frac{d}{d-\gamma} < r < \frac{2}{d-\gamma} < 2, \frac{1}{r} + \frac{1}{r'} = 1 \). Note that \( \frac{1}{r} + 1 = \frac{1}{r_1} + \frac{1}{r_2} \), where \( r_1 := \frac{2r}{2-r} \in [1,2] \), and \( r_1' \in [2,\infty) \) where \( \frac{1}{r_1} + \frac{1}{r_1'} = 1 \). Now using Young’s inequality for convolution, Lemma 2.2 (3), Lemma 2.2 (1), and Lemma 2.2 (7), we obtain

\[
\|k_2|f|^2\|_{L^1} \lesssim \|k_2\|_{L^r} \|\hat{|f|^2}\|_{L^{r'}}
\lesssim \|\hat{f} \hat{\hat{f}} \|_{L^{r'}} \lesssim \|\hat{f}\|_{L^{r_1'}} \|\hat{\hat{f}}\|_{L^{r_1}}
\lesssim \|f\|_{M^{\min(r_1',2),r_1}}^2 \lesssim \|f\|_{M^{p,q}}^3.
\]

Since \( f : [\frac{d}{d-\gamma}, \infty] \to \mathbb{R} \), \( f(r) = \frac{2r}{2-r} \) is a decreasing function, by Lemma 2.2(1), we have

\[
\|k_2|f|^2\|_{L^1} \lesssim \|f\|_{M^{p,q}}^3.
\]

Combining (24), (25), and (26), we obtain (23). \( \square \)

**Lemma 4.2** ([5]). Let \( 0 < \gamma < d \), and \( 1 \leq p \leq 2, 1 \leq q < \frac{2d}{d-\gamma} \). For any \( f,g \in M^{p,q}(\mathbb{R}^d) \), we have

\[
\|(K*|f|^2)f - (K*|g|^2)g\|_{M^{p,q}} \lesssim \left(\|f\|_{M^{p,q}}^3 + \|f\|_{M^{p,q}} \|g\|_{M^{p,q}} + \|g\|_{M^{p,q}}^3\right) \|f - g\|_{M^{p,q}}.
\]
Proof. By exploiting the ideas from the proof of Lemma 4.1, we obtain
\[
\| (K \ast |f|^2)(g - f) \|_{M^{p,q}} \lesssim \| K \ast |f|^2 \|_{L^1} \| g - f \|_{M^{p,q}}
\]
\[
\lesssim \left( \| k_1 |f|^2 \|_{L^1} + \| k_2 |f|^2 \|_{L^1} \right) \| g - f \|_{M^{p,q}}
\]
\[
\lesssim \| f \|_{M^{p,q}}^2 \| g - f \|_{M^{p,q}}. \tag{27}
\]
Let \( 1 < \frac{d}{d-\gamma} < t \leq 2, \frac{1}{t} + \frac{1}{p} = 1 \). We note that
\[
\| (K \ast (|f|^2 - |g|^2))g \|_{M^{p,q}} \lesssim \| K \ast (|f|^2 - |g|^2) \|_{L^1} \| g \|_{M^{p,q}}
\]
\[
\lesssim \left( \| k_1 \|_{L^1} \| |f|^2 - |g|^2 \|_{L^\infty} + \| k_2 \|_{L^1} \| |f|^2 - |g|^2 \|_{L^{t'_1}} \right) \| g \|_{M^{p,q}}
\]
\[
\lesssim \left( \| |f|^2 - |g|^2 \|_{L^\infty} + \| |f|^2 - |g|^2 \|_{L^{t'_1}} \right) \| g \|_{M^{p,q}}. \tag{28}
\]
Note that \( \frac{1}{t_1} + 1 = \frac{1}{t_2} + 1 \), where \( t_1 = t_2 := \frac{2t}{2t-1} \in [1, 2] \), and \( \frac{1}{t_1} + \frac{1}{t_1'} = 1 \). Now using Young’s inequality for convolution, and exploiting ideas performed as in the proof of Lemma 4.1, we obtain
\[
\| |f|^2 - |g|^2 \|_{L^{t'_1}} \lesssim \| (f - g)^2 \|_{L^{t'_1}} + \| g \|_{L^{t'_1}}
\]
\[
\lesssim \| (f - g) \|_{L^{t'_1}} \| g \|_{L^{t'_1}} \lesssim \| (f - g) \|_{L^{t'_1}} + \| g \|_{L^{t'_1}} + 1 = \frac{2}{t_1} \| g \|_{L^{t'_1}} \lesssim \| g \|_{M^{p,q}}.
\]
in the above arguments: \( t \) can tend to \( \frac{d}{d-\gamma} + \), then the corresponding \( t_2 \) can tend to \( \frac{2d}{d-\gamma} + \), which then accommodates \( q \). By Hausdorff and Schwartz inequalities and Lemma 2.2(2), we have
\[
\| |f|^2 - |g|^2 \|_{L^\infty} \lesssim \| (f - g)^2 \|_{L^1} \lesssim \| f - g \|_{L^1} + \| f - g \|_{L^1}
\]
\[
\lesssim \| f \|_{L^2} + \| g \|_{L^2} \lesssim \| f \|_{M^{p,q}} + \| g \|_{M^{p,q}} \| f - g \|_{M^{p,q}}.
\]
Using these, (28) gives
\[
\| (K \ast (|f|^2 - |g|^2))g \|_{M^{p,q}} \lesssim \| (f \|_{M^{p,q}} + \| g \|_{M^{p,q}}) \| g \|_{M^{p,q}} \| f - g \|_{M^{p,q}}. \tag{29}
\]
Now taking the identity (21) into our account, (27) and (29) gives the desired inequality.

Proof of Theorem 1.1. By Duhamel’s formula, we note that (1) can be written in the equivalent form
\[
u(t, \tau) = U(t)u_0 - i \int_0^t U(t - \tau) [ (K \ast |u|^2(\tau))u(\tau) ] \, d\tau := \mathcal{F}(u), \tag{30}
\]
where \( U(t) = e^{itH} \). We first prove the local existence on \([0, T] \) for some \( T > 0 \). By Minkowski’s inequality for integrals, Theorem 2.4, and Lemma 4.1, we obtain
\[
\left\| \int_0^t U(t - \tau) [(K \ast |u|^2(\tau))u(\tau)] \, d\tau \right\|_{M^{p,q}} \leq T \| (K \ast |u|^2(t))u(t) \|_{C([0,t], M^{p,q})}
\]
\[
\leq T \| u(t) \|_{C([0,T], M^{p,q})}^3.
\]
By Theorem 2.4, and using above inequality, we have
\[ \| J u \|_{C([0,T],M^{p,p})} \leq \| u_0 \|_{M^{p,p}} + cT \| u \|^{3}_{M^{p,p}}, \]
for some universal constant c. For \( M > 0 \), put
\[ B_{T,M} = \{ u \in C([0,T], M^{p,p}(\mathbb{R}^d)) : \| u \|_{C([0,T],M^{p,p})} \leq M \}, \]
which is the closed ball of radius M, and centered at the origin in \( C([0,T],M^{p,p}(\mathbb{R}^d)) \). Next, we show that the mapping \( J \) takes \( B_{T,M} \) into itself for suitable choice of \( M \) and small \( T > 0 \). Indeed, if we let, \( M = 2\| u_0 \|_{M^{p,p}} \) and \( u \in B_{T,M} \), it follows that
\[ \| J u \|_{C([0,T],M^{p,p})} \leq \frac{M}{2} + cTM^3. \]
We choose a \( T \) such that \( cTM^2 \leq 1/2 \), that is, \( T \leq \frac{1}{2c} \). For some universal constant \( c \).

By Theorem 2.4, and using above inequality, we have
\[ \| J u \|_{C([0,T],M^{p,p})} \leq \| u_0 \|_{M^{p,p}} + cT \| u \|^{3}_{M^{p,p}}. \]
that is, \( J u \in B_{T,M} \). By Lemma 4.2, and the arguments as before, we obtain
\[ \| J u - J v \|_{C([0,T],M^{p,p})} \leq \frac{1}{2} \| u - v \|_{C([0,T],M^{p,p})}. \]
Therefore, using the Banach’s contraction mapping principle, we conclude that \( J \) has a fixed point in \( B_{T,M} \) which is a solution of (30).

Taking Proposition 4 into account, to prove Theorem 1.1, it suffices to prove that the modulation space norm \( \| u \|_{M^{p,p}} \) cannot become unbounded in finite time. In view of (22) and to use the Hausdorff-Young inequality we let \( 1 < \frac{d}{2 - \gamma} < q \leq 2 \), and we obtain
\[ \| u(t) \|_{M^{p,p}} \leq \| u_0 \|_{M^{p,p}} + \int_0^t \| K * |u(\tau)|^2 \|_{L^1} \| u(\tau) \|_{M^{p,p}} d\tau \]
\[ \leq \| u_0 \|_{M^{p,p}} + \int_0^t \left( \| k_1 \|_{L^1} \| u(\tau) \|_{L^2}^2 + \| k_2 \|_{L^q} \| u(\tau) \|_{L^q}^2 \right) \| u(\tau) \|_{M^{p,p}} d\tau \]
\[ \leq \| u_0 \|_{M^{p,p}} + \int_0^t \left( \| u(\tau) \|_{L^2}^2 + \| u(\tau) \|_{L^q}^2 \| u(\tau) \|_{L^q} \right) \| u(\tau) \|_{M^{p,p}} d\tau \]
\[ \leq \| u_0 \|_{M^{p,p}} + \int_0^t \| u(\tau) \|_{M^{p,p}} d\tau + \int_0^t \| u(\tau) \|_{L^q}^2 \| u(\tau) \|_{L^q} d\tau, \]
where we have used Theorem 2.3, Hölder’s inequality, and the conservation of the \( L^2 \)-norm of \( u \).

We note that the requirement on \( q \) can be fulfilled if and only if \( 0 < \gamma < d/2 \). To apply Proposition 4, we let \( \beta > 1 \) and \( (2\beta, 2q) \) is admissible, that is, \( \frac{2}{2q} = d \left( \frac{1}{2} - \frac{1}{q} \right) \) such that \( \frac{1}{2} \beta = \frac{2}{2q} \left( 1 - \frac{1}{q} \right) < 1 \). This is possible provided \( \frac{q - 1}{q} < \frac{d}{4} \) if and only if \( \gamma < 2 \). Using the Hölder’s inequality for the last integral, we obtain
\[ \| u(t) \|_{M^{p,p}} \leq \| u_0 \|_{M^{p,p}} + \int_0^t \| u(\tau) \|_{M^{p,p}} d\tau + \| u(\tau) \|_{L^q}^2 \| u(\tau) \|_{L^q} d\tau, \]
where $\beta'$ is the Hölder conjugate exponent of $\beta$. Put
\[
h(t) := \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{MP,p}.
\]
For a given $T > 0$, $h$ satisfies an estimate of the form
\[
h(t) \lesssim \|u_0\|_{MP,p} + \int_0^t h(\tau) d\tau + C_0(T) \left( \int_0^t h(\tau)^{\beta'} d\tau \right)^{\frac{1}{\beta'}},
\]
provided that $0 \leq t \leq T$, and where we have used the fact that $\beta'$ is finite. Using the Hölder's inequality we infer that
\[
h(t) \lesssim \|u_0\|_{MP,p} + C_1(T) \left( \int_0^t h(\tau)^{\beta'} d\tau \right)^{\frac{1}{\beta'}}.
\]
Raising the above estimate to the power $\beta'$, we find that
\[
h(t)^{\beta'} \lesssim C_2(T) \left( 1 + \int_0^t h(\tau)^{\beta'} d\tau \right).
\]
In view of Gronwall inequality, one may conclude that $h \in L^{\infty}([0,T])$. Since $T > 0$ is arbitrary, $h \in L^{\infty}_{loc}(\mathbb{R})$, and the proof of Theorem 1.1 follows.

4.2. Global well-posedness in $MP,p$ for the potential in $M^{\infty,1}(\mathbb{R}^d)$. In this section we prove Theorem 1.2.

Lemma 4.3. Let $K \in M^{\infty,1}(\mathbb{R}^d)$, and $1 \leq p, q \leq 2$. For $f \in MP,p(\mathbb{R}^d)$, we have
\[
\|(K * |f|^2)f\|_{MP,q} \lesssim \|f\|_{MP,q}^3,
\]
and
\[
\|(K * |f|^2)f - (K * |g|^2)g\|_{MP,q} \lesssim (\|f\|_{MP,q}^2 + \|f\|_{MP,q} \|g\|_{MP,q} + \|g\|_{MP,q}^2) \|f - g\|_{MP,q}.
\]
Proof. We note that $L^1 \subset M^{1,\infty}$ (see Lemma 2.2 (2)) and using Theorem 2.3, we have
\[
\|(K * |f|^2)f\|_{MP,q} \lesssim \|K * |f|^2\|_{M^{\infty,1}} \|f\|_{MP,q} \lesssim \|f\|_{L^2}^2 \|f\|_{MP,q} \lesssim \|f\|_{MP,q}^3.
\]
This proves the first inequality. In view of (21), Theorem 2.3, and exploiting ideas from the first inequality gives the second inequality.

Proof of Theorem 1.2. We note that (1) can be written in the equivalent form
\[
u(\cdot, t) = U(t)u_0 - i \int_0^t U(t - \tau) [(K * |u|^2(\tau))u(\tau)] d\tau := J(u), \quad (31)
\]
where $U(t) = e^{itH}$. We first prove the local existence on $[0,T)$ for some $T > 0$. By Minkowski's inequality for integrals, Theorem 2.4, and Lemma 4.3, we obtain
\[
\left\| \int_0^t U(t - \tau) [(K * |u|^2(\tau))u(\tau)] d\tau \right\|_{MP,p} \leq cT \|u(t)\|_{MP,p}^3,
\]
for some universal constant $c$. By Theorem 2.4 and the above inequality, we have
\[
\|Ju\|_{C([0,T],MP,p)} \leq \|u_0\|_{MP,p} + cT \|u\|_{MP,p}^3.
\]
For $M > 0$, put
\[
B_{T,M} = \{ u \in C([0,T],MP,p(\mathbb{R}^d)) : \|u\|_{C([0,T],MP,p)} \leq M \},
\]
which is the closed ball of radius $M$, and centered at the origin in $C([0,T], M^{p,p}(\mathbb{R}^d))$. Next, we show that the mapping $J$ takes $B_{T,M}$ into itself for suitable choice of $M$ and small $T > 0$. Indeed, if we let, $M = 2\|u_0\|_{M^{p,p}}$ and $u \in B_{T,M}$, it follows that

$$\|Ju\|_{C([0,T], M^{p,p})} \leq \frac{M}{2} + cTM^3.$$  

We choose a $T$ such that $cTM^2 \leq 1/2$, that is, $T \leq \tilde{T}(\|u_0\|_{M^{p,p}}, d)$ and as a consequence we have

$$\|Ju\|_{C([0,T], M^{p,p})} \leq \frac{M}{2} + \frac{M}{2} = M,$$

that is, $Ju \in B_{T,M}$. By Lemma 4.3, and the arguments as before, we obtain

$$\|Ju - Jv\|_{C([0,T], M^{p,p})} \leq \frac{1}{2}\|u - v\|_{C([0,T], M^{p,p})}.$$  

Therefore, using Banach’s contraction mapping principle, we conclude that $J$ has a fixed point in $B_{T,M}$ which is a solution of (1).

Indeed, the solution constructed before is global in time: in view of the conservation of $L^2$ norm, Theorem 2.3, and Lemma 2.2, we have

$$\|u(t)\|_{M^{p,p}} \lesssim \|u_0\|_{M^{p,p}} + \int_0^t \|K\|_{M^{\infty,1}} \|u(\tau)\|_{M^{p,p}} d\tau$$

and by Gronwall’s inequality, we conclude that $\|u(t)\|_{M^{p,q}}$ remains bounded on finite time intervals. This completes the proof. 

4.3. Local well-posedness in $M^{1,1}$ for power type non linearity. In this subsection we prove Theorem 1.4. We start by recalling following

**Definition 4.4.** A complex valued function $F$ defined on the plane $\mathbb{R}^2$ is said to be real entire, if $F$ has the power series expansion

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn} s^m t^n \quad (32)$$

that converges absolutely for every $(s,t) \in \mathbb{R}^2$.

**Notations.**

1. For $u : \mathbb{R}^d \to \mathbb{C}$, we put $u = u_1 + iu_2$, where $u_1, u_2 : \mathbb{R}^d \to \mathbb{R}$, and write

$$F(u) = F(u_1, u_2),$$

where $F : \mathbb{R}^2 \to \mathbb{C}$ is real entire on $\mathbb{R}^2$ with $F(0) = 0$.

2. If $F$ is real entire function given by (32), then we denote by $\tilde{F}$ the function given by the power series expansion

$$\tilde{F}(s,t) = \sum_{m,n=0}^{\infty} |a_{mn}| s^m t^n.$$
Note that $\tilde{F}$ is real entire if $F$ is real entire. Moreover, as a function on $[0, \infty) \times [0, \infty)$, it is monotonically increasing with respect to each of the variables $s$ and $t$.

**Proposition 6** ([6]). Let $F$ be real entire, $F(0) = 0$ and $1 \leq p \leq \infty$. Then

1. $\|F(u)\|_{MP^{1.3}} \leq \tilde{F}(\|u\|_{MP^{1.3}}, \|v\|_{MP^{1.3}})$.
2. For $u, v \in MP^{1.3}(\mathbb{R}^d)$, we have

$$\|F(u_1, u_2) - F(v_1, v_2)\|_{MP^{1.3}} \leq \|u - v\|_{MP^{1.3}} \left(\|u\|_{MP^{1.3}} + \|v\|_{MP^{1.3}}\right).$$

**Proof of Theorem 1.4.** Equation (1) can be written in the equivalent form

$$u(\cdot, t) = U(t)u_0 - i \int_0^t U(t - \tau)[F(u(\cdot, \tau))] d\tau := J(u),$$

where $U(t) = e^{itH}$. We show that the mapping $J$ has a unique fixed point in an appropriate functions space, for small $t$. For this, we consider the Banach space $X_T = C([0, T], M^{1.2}(\mathbb{R}))$, with norm

$$\|u\|_{X_T} = \sup_{t \in [0, T]} \|u(\cdot, t)\|_{M^{1.2}}, \ (u \in X_T).$$

Note that if $u \in X_T$, then $u(\cdot, t) \in M^{1.2}(\mathbb{R}^d)$ for each $t \in [0, T]$. Now Proposition 6 gives $F(u(\cdot, t)) \in M^{1.2}(\mathbb{R}^d)$ and we have

$$\|F(u(\cdot, t))\|_{M^{1.2}} \leq \tilde{F}(\|u(\cdot, t)\|_{M^{1.2}}, \|u(\cdot, t)\|_{M^{1.2}}) \leq \tilde{F}(\|u\|_{X_T}, \|u\|_{X_T}),$$

where the last inequality follows from the fact that $\tilde{F}$ is monotonically increasing on $[0, \infty) \times [0, \infty)$ with respect to each of its variables. Now an application of Minkowski's inequality for integrals and the estimate (34) and Theorem 2.4, yields

$$\left\| \int_0^t U(t - \tau)[F(u(\cdot, \tau))] d\tau \right\|_{M^{1.2}} \leq \int_0^t \|U(t - \tau)[F(u(\cdot, \tau))]\|_{M^{1.2}} d\tau \leq T \tilde{F}(\|u\|_{X_T}, \|u\|_{X_T})$$

for $0 \leq t \leq T$. Using above estimates, we see that

$$\|J(u)\|_{X_T} \leq \|u_0\|_{M^{1.2}} + T \tilde{F}(\|u\|_{X_T}, \|u\|_{X_T}) \leq \|u_0\|_{M^{1.2}} + T \|u\|_{X_T} G(\|u\|_{X_T})$$

where $G$ is a real analytic function on $[0, \infty)$ such that $\tilde{F}(x, x) = x G(x)$. This factorization follows from the fact that the constant term in the power series expansion for $\tilde{F}$ is zero, (i.e., $\tilde{F}(0, 0) = 0$). We also note that $G$ is increasing on $[0, \infty)$.

For $M > 0$, put $X_{T, M} = \{u \in X_T : \|u\|_{X_T} \leq M\}$, which is the closed ball of radius $M$, and centered at the origin in $X_T$. We claim that

$$J : X_{T, M} \to X_{T, M},$$

for suitable choice of $M$ and small $T > 0$. Let $C_1 \geq 1$, and putting $M = 2C_1 \|u_0\|_{M^{1.2}}$ from (36) we see that for $u \in X_{T, M}$ and $T \leq 1$

$$\|J(u)\|_{X_T} \leq \frac{M}{2} + T C_1 MG(M) \leq M$$

for $T \leq T_1 := \frac{1}{2C_1 MG(M)}$. Thus $J : X_{T, M} \to X_{T, M}$, for $M = 2C_1 \|u_0\|_{M^{1.2}}$, and all $T \leq T_1$, hence the claim.
Now we show that \( J \) satisfies the contraction estimate
\[
\|J(u) - J(v)\|_{X_T} \leq \frac{1}{2} \|u - v\|_{X_T}
\]  
(38)
on \( X_{T,M} \) if \( T \) is sufficiently small. By Theorem 2.4, we see that
\[
\|J(u(\cdot,t)) - J(v(\cdot,t))\|_{M^{1,1}} \leq \int_0^T \|F(u(\cdot,\tau)) - F(v(\cdot,\tau))\|_{M^{1,1}} \, d\tau.
\]  
(39)
By Proposition 6 this is at most
\[
C \int_0^T \|u - v\|_{M^{1,1}} \left[ \left( \partial_x F + \partial_y F \right) (\|u\|_{M^{1,1}} + \|v\|_{M^{1,1}}, \|u\|_{M^{1,1}} + \|v\|_{M^{1,1}}) \right] \, d\tau,
\]for some constant \( C > 0 \). Now taking supremum over all \( t \in [0,T] \), we see that
\[
\|J(u) - J(v)\|_{X_T} \leq C T \|u - v\|_{X_{T,M}} \left( \|u\|_{X_T} + \|v\|_{X_T}, \|u\|_{X_T} + \|v\|_{X_T} \right).
\]  
(40)
Now if \( u \) and \( v \) are in \( X_{T,M} \), the RHS of (40) is at most
\[
C T \|u - v\|_{X_{T,M}} \left( \|u\|_{X_T} + \|v\|_{X_T} \right) \leq \frac{\|u - v\|_{X_T}}{2}
\]  
(41)
for all \( T \leq T_2 := \left[ 2C \left( \partial_x F + \partial_y F \right) (2M, 2M) \right]^{-1} \). Thus from (41), we see that the estimate (38) holds for all \( T < T_2 \). Now choosing \( T^1 = \min\{T_1, T_2\} \), so that both the inequalities (37) and (38) are valid for \( T < T^1 \). Hence for such a choice of \( T \), \( J \) is a contraction on the Banach space \( X_{T,M} \) and hence has a unique fixed point in \( X_{T,M} \), by the Banach’s contraction mapping principle. Thus we conclude that \( J \) has a unique fixed point in \( X_{T,M} \) which is a solution of (1) on \( [0,T] \) for any \( T < T^1 \). Note that \( T^1 \) depends on \( \|u_0\|_{M^{1,1}} \).

The arguments above also give the solution for the initial data corresponding to any given time \( t_0 \), on an interval \([t_0, t_0 + T^1]\) where \( T^1 \) is given by the same formula with \( \|u(0)\|_{M^{1,1}} \) replaced by \( \|u(t_0)\|_{M^{1,1}} \). In other words, the dependence of the length of the interval of existence on the initial time \( t_0 \), is only through the norm \( \|u(t_0)\|_{M^{1,1}} \). Thus if the solution exists on \( [0,T^1] \) and if \( \|u(T^1)\|_{M^{1,1}} < \infty \), the above arguments can be carried out again for the initial value problem with the new initial data \( u(T^1) \) to extend the solution to the larger interval \([0,T^\ast]\). This procedure can be continued and hence we get a solution on maximal interval \([0,T^\ast]\). This completes the proof. \( \square \)

4.4. Local well-posedness in \( M^{p,q} \) with potentials in \( \mathcal{F}L^q \) and \( M^{1,\infty} \). In this section, we prove Theorem 1.3. We start by following elementary

**Proposition 7.** If \( 0 < p < q < r \leq \infty \), then \( L^q(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) + L^r(\mathbb{R}^d) \), that is, each \( f \in L^1(\mathbb{R}^d) \) is the sum of function in \( L^p(\mathbb{R}^d) \) and a function in \( L^r(\mathbb{R}^d) \).

**Proof.** If \( f \in L^q(\mathbb{R}^d) \), let \( E = \{x : |f(x)| > 1\} \) and set \( g = f\chi_E \) and \( h = f\chi_{E^c} \). Then \( |g|^p = |f|^p\chi_E \leq |f|^q \chi_E \), so \( g \in L^p(\mathbb{R}^d) \), and \( |h|^r = |f|^r \chi_{E^c} \leq |f|^q \chi_{E^c} \), so \( h \in L^r(\mathbb{R}^d) \). For \( r = \infty \), we have \( \|h\|_{L^\infty} \leq 1 \) (this follows as \( E^c = \{x : |f(x)| \leq 1\} \) and \( h = f\chi_{E^c} \)). \( \square \)

**Lemma 4.5.** Let \( K \in \mathcal{F}L^q(\mathbb{R}^d) \), \( k \in \mathbb{N} \), and \( 1 \leq p \leq 2 \). Then

1. \( \|(K * |f|^2)^k\|_{M^{1,1}} \leq \|f\|_{M^{1,1}}^{2k+1} \) for \( f \in M^{1,1}(\mathbb{R}^d) \) and \( 1 \leq q \leq \infty \).
2. \( \|(K * |f|^2)\|_{M^{p,\frac{2}{p}}} \leq \|f\|_{M^{p,\frac{2}{p}}}^3 \) for \( f \in M^{p,1}(\mathbb{R}^d) \) and \( 1 < q < r \leq 2 \).
Note that Lemma 2.2 (1), and Lemma 2.2 (7), we obtain

\[ \|K * |f|^{2k}f\|_{M^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1} \quad \text{for } f \in M^{p,1}(\mathbb{R}^d) \ (1 \leq p \leq \infty) \text{ and } K \in M^{1,\infty}(\mathbb{R}^d) \supset L^1(\mathbb{R}^d). \]

Proof. Let \( K \in F L^q(\mathbb{R}^d) (1 < q \leq \infty) \). Then by Proposition 7, we get \( k_1 \in L^1(\mathbb{R}^d) \) and \( k_2 \in L^\infty(\mathbb{R}^d) \) so that

\[ \tilde{K} = k_1 + k_2. \] (42)

By Theorem 2.3, (42), Hölder’s inequality, Lemma 2.2(2), Lemma 2.2(5), and Lemma 2.2(7), we obtain

\[ \|(K * |f|^{2k}f)\|_{M^{p,1}} \lesssim \|K * |f|^{2k}\|_{F L^q} \|f\|_{M^{p,1}} \lesssim \|\tilde{K} |f|^{2k}\|_{L^q} \|f\|_{M^{p,1}} \lesssim \|	ilde{K} |f|^{2k}\|_{L^q} \|f\|_{M^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1}. \]

Let \( K \in F L^1(\mathbb{R}^d) \). Then

\[ \|(K * |f|^{2k}f)\|_{M^{p,1}} \lesssim \|K * |f|^{2k}\|_{F L^1} \|f\|_{M^{p,1}} \lesssim \|	ilde{K} |f|^{2k}\|_{L^1} \|f\|_{M^{p,1}} \lesssim \|	ilde{K} |f|^{2k}\|_{L^1} \|f\|_{M^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1}. \]

Let \( K \in F L^\infty(\mathbb{R}^d) \). Then

\[ \|(K * |f|^{2k}f)\|_{M^{p,1}} \lesssim \|K * |f|^{2k}\|_{F L^\infty} \|f\|_{M^{p,1}} \lesssim \|	ilde{K} |f|^{2k}\|_{L^\infty} \|f\|_{M^{p,1}} \lesssim \|	ilde{K} |f|^{2k}\|_{L^\infty} \|f\|_{M^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1}. \]

This completes the proof of statement (1). For statement (2), we note that by Proposition 7, we get \( k_1 \in L^1(\mathbb{R}^d) \) and \( k_2 \in L^r(\mathbb{R}^d) \ (1 < q < r \leq 2) \) so that

\[ \tilde{K} = k_1 + k_2. \] (43)

Let \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = 1 \) and so \( \frac{1}{r} + 1 = \frac{1}{r_1} + \frac{1}{r_2} \), where \( r_1 = r_2 := \frac{2r}{2r-1} \in [1, 2] \), and \( r_1' = 2 \) where \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = 1 \). Now using Young’s inequality for convolution, Lemma 2.2 (3), Lemma 2.2 (1), and Lemma 2.2 (7), we obtain

\[ \|f\|_{\tilde{M}^{p,1}}^{2k+1} \lesssim \|f\|_{\tilde{M}^{p,1}} \|f\|_{M^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}} \|f\|_{\tilde{M}^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1}. \]

Note that \( \|k_1 f\|_{L^1} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1} \) (see (25)). For statement (3), we note that by Theorem 2.3, we have

\[ \|(K * |f|^{2k})f\|_{M^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1} \|f\|_{M^{p,1}} \lesssim \|f\|_{\tilde{M}^{p,1}}^{2k+1}. \]

\( \square \)
Lemma 4.6. The following statements are true.
1. For $K \in FL^q(\mathbb{R}^d)$ ($1 \leq q \leq \infty$), $k \in \mathbb{N}$ and $f \in \mathcal{M}^{1,1}(\mathbb{R}^d)$, we have
$$\| (K \ast |f|^{2k})f - (K \ast |g|^{2k})g \|_{\mathcal{M}^{1,1}} \lesssim \left(\|f\|^{2k}_{\mathcal{M}^{1,1}} + \|f\|^{2k}_{\mathcal{M}^{1,1}} + \cdots + \|g\|^{2k-1}_{\mathcal{M}^{1,1}} \|f\|_{\mathcal{M}^{1,1}} + \|g\|^{2k}_{\mathcal{M}^{1,1}} \right) \|f - g\|_{\mathcal{M}^{1,1}}.$$

2. For $K \in FL^q(\mathbb{R}^d)$ ($1 < q \leq 2$), and $f \in \mathcal{M}^{p, \frac{2}{p-1}}(\mathbb{R}^d)$, we have
$$\| (K \ast |f|^2)f - (K \ast |g|^2)g \|_{\mathcal{M}^{p, \frac{2}{p-1}}} \lesssim \left(\|f\|^2_{\mathcal{M}^{p, \frac{2}{p-1}}} + \|f\|_{\mathcal{M}^{p, \frac{2}{p-1}}} \|g\|_{\mathcal{M}^{p, \frac{2}{p-1}}}, \|g\|^2_{\mathcal{M}^{p, \frac{2}{p-1}}} \right) \|f - g\|_{\mathcal{M}^{p, \frac{2}{p-1}}}.$$

3. For $K \in \mathcal{M}^{1, \infty}(\mathbb{R}^d)$, $k \in \mathbb{N}$ and $f \in \mathcal{M}^{p,1}(\mathbb{R}^d)$ ($1 \leq p \leq \infty$), we have
$$\| (K \ast |f|^{2k})f - (K \ast |g|^{2k})g \|_{\mathcal{M}^{p,1}} \lesssim \left(\|f\|^{2k}_{\mathcal{M}^{p,1}} + \|f\|^{2k-1}_{\mathcal{M}^{p,1}} \|f\|_{\mathcal{M}^{p,1}} + \cdots + \|g\|^{2k-1}_{\mathcal{M}^{p,1}} \|f\|_{\mathcal{M}^{p,1}} + \|g\|^{2k}_{\mathcal{M}^{p,1}} \right) \|f - g\|_{\mathcal{M}^{p,1}}.$$

Proof. We notice the identities
$$(K \ast |f|^{2k})f - (K \ast |g|^{2k})g = (K \ast |f|^2)(f - g) + (K \ast (|f|^{2k} - |g|^{2k}))(f - g)$$
and
$$x^k - y^k = (x - y) \sum_{n=0}^{k-1} x^{k-1-n} y^n \quad (x, y \geq 0, k \in \mathbb{N}).$$

Now exploiting ideas from Lemma 4.5, Lemma 4.2, and in view of the above identities, Lemma 2.2, and Proposition 2.3, the proofs can be produced. We omit the details.

Proof of Theorem 1.3. In view of Theorem 2.4, Lemmas 4.5 and 4.6, the Banach contraction principle gives the desired result. The details are omitted.

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