ON EMBEDDINGS OF HALF-CUBE GRAPHS IN HALF-SPIN GRASSMANNIANS

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Abstract. We investigate isometric embeddings of the $m$-dimensional half-cube graph $\frac{1}{2}H_m$ in the half-spin Grassmannians associated with a polar space of type $D_n$ with $n \geq m \geq 4$. We establish the following: if a such embedding can be extended to an embedding of the hypercube graph $H_m$ in the dual polar space (the latter embedding is not assumed to be isometric) and $m$ is even then the image is an apartment in a parabolic subspace.

1. Introduction

In the present paper we continue to discussing the problem of metric characterization of apartments in building Grassmannians [11, 12]. This problem is motivated by the well-known metric characterization of the intersections of apartments with the chamber sets of Tits buildings (see, for example, [2, p. 90]) and closely related with Cooperstein–Kasikova–Shult results [3].

By [13], a building is a simplicial complex $\Delta$ containing a family of subcomplexes called apartments and satisfying some axioms. In particular, all apartments are isomorphic to a certain Coxeter complex (the simplicial complex associated with a Coxeter system) which defines the type of our building. Maximal simplices of $\Delta$ are called chambers; they have the same cardinality $n$ called the rank of $\Delta$. Two chambers are said to be adjacent if their intersection consists of $n - 1$ vertices. The vertex set of $\Delta$ can be labeled by the nodes of the diagram corresponding to the Coxeter system associated with $\Delta$; such labeling is unique up to a permutation on the set of nodes. All vertices corresponding to the same node form a Grassmannian. So, the vertex set of $\Delta$ is decomposed in precisely $n$ distinct Grassmannians. The intersections of apartments of $\Delta$ with a Grassmannian of $\Delta$ are called apartments in this Grassmannian.

Let $G$ be one of the Grassmannians associated with $\Delta$. Two distinct vertices $a, b \in G$ are adjacent if there exists a simplex $P \in \Delta$ such that $P \cup \{a\}$ and $P \cup \{b\}$ both are chambers. Denote by $\Gamma$ the associated Grassmann graph, i.e. the graph whose vertex set is $G$ and whose edges are pairs of adjacent vertices. Let $A$ be an apartment of $G$ and let $\Gamma_A$ be the restriction of the graph $\Gamma$ to $A$. We want to distinguish all cases such that the image of every isometric embedding of $\Gamma_A$ in $\Gamma$ is an apartment of $G$ (in some cases this does not hold [3, 11]).

A building is spherical if the associated Coxeter system is finite. By [13], there are precisely the following seven types of irreducible thick spherical buildings of rank $\geq 3$: three classical types $A_n, B_n = C_n, D_n$ and four exceptional types $F_4, E_i, i = 6, 7, 8$.

Every building of type $A_{n-1}$ is the flag complex of a certain $n$-dimensional vector space $V$ (over a division ring). The Grassmannians of this building are the usual
Grassmannians $G_k(V)$, $k \in \{1, \ldots, n-1\}$. Two elements of $G_k(V)$ are adjacent if their intersection is $(k-1)$-dimensional. The associated Grassmann graph is denoted by $\Gamma_k(V)$. The case when $k = 1, n-1$ is trivial — any two distinct vertices of $\Gamma_k(V)$ are adjacent. Every apartment of $G_k(V)$ is defined by a certain base of $V$: it consists of all $k$-dimensional subspaces spanned by subsets of this base. All apartments of $G_k(V)$ are the images of some isometric embeddings of the Johnson graph $J(n, k)$ in $\Gamma_k(V)$. However, the image of every isometric embedding of $J(n, k)$ in $\Gamma_k(V)$ is an apartment of $G_k(V)$ if and only if $n = 2k$. This follows from the classification of isometric embeddings of Johnson graphs $J(l, m)$, $1 < m < l - 1$ in the Grassmann graph $\Gamma_k(V)$, $1 < k < n - 1$ given in [11].

All buildings of types $C_n$ and $D_n$ are defined by polar spaces. Every $C_n$-building is the flag complex formed by singular subspaces of a rank $n$ polar space. The Grassmannians of this building are the associated polar Grassmannians and the Grassmannian consisting of maximal singular subspaces is called the dual polar space. By [12], apartments in the dual polar space can be characterized as the images of isometric embeddings of the $n$-dimensional hypercube graph $H_n$ in the corresponding Grassmann graph.

Every $D_n$-building can be obtained from a polar space of type $D_n$ (this construction is known as the oriflamme complex). The Grassmannians of such building are some of the polar Grassmannians and so-called half-spin Grassmannians. Apartments of the half-spin Grassmannians are the images of isometric embeddings of the half-cube graph $\frac{1}{2}H_n$ in the associated Grassmann graphs.

In this paper we investigate isometric embeddings of $\frac{1}{2}H_m$ in the half-spin Grassmannians associated with a polar space of type $D_n$ with $n \geq m \geq 4$. Our main result states that if a such embedding can be extended to an embedding of $H_m$ in the dual polar space (the latter embedding is not assumed to be isometric) and $m$ is even then the image is an apartment in a parabolic subspace and we get an apartment of the Grassmannian if $n = m$.

Note that in [4] apartments of usual Grassmannians, dual polar spaces and half-spin Grassmannians were characterized in terms of independent subsets of the corresponding Grassmann spaces. Some more general results can be found in [8, 9].

2. Preliminaries

2.1. Graphs. We recall a few definitions from graph theory.

The distance between two vertices in a connected graph is the smallest number $i$ such that there exists a path of length $i$ (a path consisting of $i$ edges) connecting these vertices. A path connecting two vertices is said to be a geodesic if the number of edges in this path is equal to the distance between the vertices. The maximum of all distances between vertices in a graph is called the diameter of the graph. In the case when the diameter is finite, two vertices are said to be opposite if the distance between them is equal to the diameter.

A subset in the vertex set of a graph is called a clique if any two distinct elements of this subset are adjacent vertices. Using Zorn lemma, we can show that maximal cliques exist and every clique is contained in a certain maximal clique.

An injective mapping of the vertex set of a graph $\Gamma$ to the vertex set of a graph $\Gamma'$ is an embedding of $\Gamma$ in $\Gamma'$ if two vertices of $\Gamma$ are adjacent only in the case when their images are adjacent vertices of $\Gamma'$. We will use the following property of embeddings: if the distance between two vertices of $\Gamma$ is equal to 2 then the
distance between the images of these vertices also is equal to 2. An embedding is said to be *isometric* if it preserves the distance between vertices. We refer [3] Part III for the general theory of isometric embeddings.

2.2. Partial linear spaces. Let \( P \) be a non-empty set whose elements will be called *points*. Let also \( \mathcal{L} \) be a family of proper subsets of \( P \). Elements of this family will be called *lines*. We say that two or more points are *collinear* if there is a line containing all of them. Now, assume that the pair \( \Pi = (P, \mathcal{L}) \) is a *partial linear space*, i.e. the following axioms hold:

- every line contains at least two points and every point belongs to a line;
- for any distinct collinear points \( p, q \in P \) there is precisely one line containing them, this line will be denoted by \( p \perp q \).

We say that \( S \subset P \) is a *subspace* of \( \Pi \) if for any distinct collinear points \( p, q \in S \) the line \( p \perp q \) is contained in \( S \). A subspace is said to be *singular* if any two distinct points of the subspace are collinear (by the definition, the empty set and a single point are singular subspaces). Using Zorn lemma, we establish the existence of maximal singular subspaces and the fact that every singular subspace is contained in a certain maximal singular subspace.

For every subset \( X \subset P \) the minimal subspace containing \( X \), i.e. the intersection of all subspaces containing \( X \), is called *spanned* by \( X \) and denoted by \( \langle X \rangle \). We say that \( X \) is *independent* if the subspace \( \langle X \rangle \) can not be spanned by a proper subset of \( X \).

Let \( S \) be a subspace of \( \Pi \) (possible \( S = P \)). An independent subset \( X \subset S \) is a *base* of \( S \) if \( \langle X \rangle = S \). The *dimension* of \( S \) is the smallest cardinality \( \alpha \) such that \( S \) has a base of cardinality \( \alpha + 1 \). The dimension of the empty set and a single point is equal to \(-1\) and \(0\) (respectively), lines are 1-dimensional subspaces. A 2-dimensional singular subspace is called a *plane*.

Two partial linear spaces \( \Pi = (P, \mathcal{L}) \) and \( \Pi' = (P', \mathcal{L}') \) are *isomorphic* if there exists a bijection \( f : P \rightarrow P' \) such that \( f(\mathcal{L}) = \mathcal{L}' \). This bijection is called a *collineation* of \( \Pi \) to \( \Pi' \).

2.3. Polar spaces. Following F. Buekenhout and E. E. Shult [1], we define a *polar space* as a partial linear space \( \Pi = (P, \mathcal{L}) \) satisfying the following axioms:

- every line contains at least three points,
- there is no point collinear with all points,
- if \( p \in P \) and \( L \in \mathcal{L} \) then \( p \) is collinear with one or all points of the line \( L \),
- any flag formed by singular subspaces is finite.

If our polar space \( \Pi \) contains a singular subspace whose dimension is not less than 2 then all maximal singular subspaces of \( \Pi \) are projective spaces of the same dimension \( n \geq 2 \) and the number \( n + 1 \) is called the *rank* of \( \Pi \).

The collinearity relation of \( \Pi \) will be denoted by \( \perp \). For points \( p, q \in P \) we write \( p \perp q \) if \( p \) is collinear with \( q \) and \( p \not\perp q \) otherwise. Moreover, if \( X, Y \subset P \) then \( X \perp Y \) means that every point of \( X \) is collinear with all points of \( Y \). If \( X \perp X \) then the subspace \( \langle X \rangle \) is singular.

**Lemma 2.1.** The following assertions are fulfilled:

1. If \( p \in P \) and \( X \subset P \) then \( p \perp X \) implies that \( p \perp \langle X \rangle \),
2. If \( p \in P \) and \( S \) is a maximal singular subspace of \( \Pi \) then \( p \perp S \) implies that \( p \in S \).
Proof. See, for example, [10] Section 4.1.

For every polar space of rank $n$ one of the following two possibilities is realized:

- type $C_n$ — every $(n - 2)$-dimensional singular subspace is contained in at least three maximal singular subspaces,
- type $D_n$ — every $(n - 2)$-dimensional singular subspace is contained in precisely two maximal singular subspaces.

2.4. Dual polar spaces. Let $\Pi = (P, \mathcal{L})$ be a polar space of rank $n$. For every $k \in \{0, 1, \ldots, n - 1\}$ we denote by $\mathcal{G}_k(\Pi)$ the polar Grassmannian consisting of all $k$-dimensional singular subspaces of $\Pi$; in particular, $\mathcal{G}_{n-1}(\Pi)$ is formed by maximal singular subspaces. The associated Grassmann graph is denoted by $\Gamma_k(\Pi)$.

Two distinct elements of $\mathcal{G}_{n-1}(\Pi)$ (vertices of $\Gamma_{n-1}(\Pi)$) are adjacent if their intersection is $(n - 2)$-dimensional. The graph $\Gamma_{n-1}(\Pi)$ is connected and we denote by $d(S, U)$ the distance between $S, U \in \mathcal{G}_{n-1}(\Pi)$; this distance is equal to

\[n - 1 - \dim(S \cap U)\]

The diameter of $\Gamma_{n-1}(\Pi)$ is $n$ and two vertices of $\Gamma_{n-1}(\Pi)$ are opposite if and only if they are disjoint subspaces.

Lemma 2.2 (Lemma 2 in [12]). If $X_1, \ldots, X_i$ is a geodesic in $\Gamma_{n-1}(\Pi)$ then

\[X_1 \cap X_i \subset X_j\]

for every $j \in \{1, \ldots, i\}$.

Let $M$ be an $m$-dimensional singular subspace of $\Pi$. For every natural $k > m$ we denote by $[M]_k$ the set of all elements of $\mathcal{G}_k(\Pi)$ containing $M$.

If $m = n - 2$ then $[M]_{n-1}$ is called a line of $\mathcal{G}_{n-1}(\Pi)$. The Grassmannian $\mathcal{G}_{n-1}(\Pi)$ together with the set of all such lines is a partial linear space; it is called the dual polar space associated with $\Pi$. Two distinct points of this space are collinear if and only if they are adjacent vertices of the Grassmann graph $\Gamma_{n-1}(\Pi)$.

Suppose that $m < n - 2$. Then $[M]_{n-1}$ is a non-singular subspace of the dual polar space. Subspaces of such type are called parabolic [4]. Now, we show that the parabolic subspace $[M]_{n-1}$ is isomorphic to the dual polar space of a rank $n - m - 1$ polar space.

Let us consider $[M]_{m+1}$. A subset of $[M]_{m+1}$ is called a line if there exists $N \in [M]_{m+2}$ such that this subset is formed by all elements of $[M]_{m+1}$ contained in $N$. Then $[M]_{m+1}$ together with the set of all such lines is a polar space of rank $n - m - 1$. If $\mathcal{S}$ is a maximal singular subspace of this polar space then there exists $S \in [M]_{n-1}$ such that $\mathcal{S}$ consists of all elements of $[M]_{m+1}$ contained in $S$. Therefore, maximal singular subspaces of $[M]_{m+1}$ can be identified with elements of $[M]_{n-1}$. An easy verification shows that this correspondence is a collineation between the parabolic subspace $[M]_{n-1}$ and the dual polar space of the polar space $[M]_{m+1}$.

Note that $[M]_{m+1}$ is a polar space of type $D_{n-m-1}$ if $\Pi$ is of type $D_n$.

2.5. Half-spin Grassmannians. Let $\Pi = (P, \mathcal{L})$ be a polar space of type $D_n$. In this case the Grassmannian $\mathcal{G}_{n-1}(\Pi)$ can be uniquely decomposed in the sum of two disjoint subsets, we denote them by $\mathcal{G}_+(\Pi)$ and $\mathcal{G}_-(\Pi)$, such that the distance between any two elements of $\mathcal{G}_\delta(\Pi), \delta \in \{+, -\}$ (in the Grassmann graph $\Gamma_{n-1}(\Pi)$) is even and the distance between any $S \in \mathcal{G}_\delta(\Pi)$ and $U \in \mathcal{G}_{-\delta}(\Pi)$ is odd (we write...
equal to $n$. Their intersection is  a single point. In the case when $n$ is odd, the diameter is $\frac{n-1}{2}$ and two vertices are opposite if and only if their intersection is a single point.

**Remark 2.1.** If $n = 2$ then any two distinct elements of $G_3(\Pi)$ are disjoint lines and any two distinct vertices of $\Gamma_3(\Pi)$ are adjacent.

Let $M$ be an $m$-dimensional singular subspace of $\Pi$ and $m \leq n - 2$. We define $[M]_\delta := G_\delta(\Pi) \cap [M]_{n-1}$. If $m = n - 3$ then this subset is said to be a line of $G_\delta(\Pi)$. The half-spin Grassmannian $G_\delta(\Pi)$ together with the set of all such lines is a partial linear space; it is called the half-spin Grassmann space associated with $G_\delta(\Pi)$. Two distinct points of this space are collinear if and only if they are adjacent vertices of $\Gamma_\delta(\Pi)$.

Let $m < n - 3$. Then $[M]_\delta$ is a subspace of the half-spin Grassmann space, this subspace is non-singular if $m < n - 4$. As in the case of dual polar spaces, subspaces of such type are called parabolic [4].

The polar space $[M]_{m+1}$ is of type $D_{n-m-1}$. The natural collineation between $[M]_{n-1}$ and the dual polar space of $[M]_{m+1}$ (it was considered in the previous subsection) induces a collineation between $[M]_\delta$ and one of the half-spin Grassmann spaces of $[M]_{m+1}$.

**Remark 2.2.** If $n = 3$ then $\Pi$ is isomorphic to the index two Grassmann space of a 4-dimensional vector space and the half-spin Grassmann spaces of $\Pi$ are the associated projective and dual projective spaces [10] Subsection 4.3.3]. In the case when $n = 4$, the half-spin Grassmann spaces of $\Pi$ are polar spaces of type $D_4$ [10] Subsection 4.5.2].

**Lemma 2.3.** If $X, Y \in G_\delta(\Pi)$ are adjacent and $Z \in G_\delta(\Pi)$ is adjacent with both $X, Y$ then $Z$ is adjacent with every element of the line joining $X$ and $Y$; in other words, for any $X \in G_\delta(\Pi)$ all vertices of $\Gamma_\delta(\Pi)$ adjacent with $X$ form a subspace in the half-spin Grassmann space.

**Proof.** See [10] Proposition 4.18].

If $n = 3$ then any two distinct vertices of the graph $\Gamma_\delta(\Pi)$ are adjacent, see Remark 2.2. In the case when $n \geq 4$, there are precisely the following two types of maximal cliques of $\Gamma_\delta(\Pi)$ [10] Subsection 4.5.2]:

- the star $[M]_\delta, M \in G_{n-4}(\Pi)$,
- the special subspace $[U]_\delta, U \in G_{n-4}(\Pi)$ formed by all elements of $G_\delta(\Pi)$ intersecting $U$ in $(n - 2)$-dimensional subspaces, i.e. all vertices of $\Gamma_{n-1}(\Pi)$ adjacent with $U$.

**Remark 2.3.** The class of maximal cliques of $\Gamma_\delta(\Pi)$ coincides with the class of maximal singular subspaces of the associated half-spin Grassmann space.
Lemma 2.4. If \( n \geq 4 \) then the intersection of two distinct maximal cliques of \( \Gamma_\delta(\Pi) \) is the empty set, a point, a line, or a plane. This intersection is a plane if and only if one of the cliques is a star \([S]\delta \) and the other is a special subspace \([U]\delta \) such that \( S \subset U \).

Proof. See Exercises 4.9 – 4.11 in [10]. \( \square \)

3. Apartments

3.1. Apartments in dual polar spaces. Let \( \Pi = (P, \mathcal{L}) \) be a polar space of rank \( n \). Apartments in the associated polar Grassmannians are defined by frames of \( \Pi \). Recall that a subset \( \{p_1, \ldots, p_{2n}\} \subseteq P \) is a frame if for every \( i \in \{1, \ldots, 2n\} \) there exists unique \( \sigma(i) \in \{1, \ldots, 2n\} \) such that \( p_i \not\perp p_{\sigma(i)} \). Frames are independent subsets and any \( k \) mutually collinear points in a frame span a \((k - 1)\)-dimensional singular subspace.

Let \( B = \{p_1, \ldots, p_{2n}\} \) be a frame of \( \Pi \). The associated apartment \( \mathcal{A} \subset \mathcal{G}_{n-1}(\Pi) \) is formed by all maximal singular subspaces spanned by subsets of \( B \), these are the subspaces of type \( \langle p_{i_1}, \ldots, p_{i_m} \rangle \) such that

\[ \{i_1, \ldots, i_n\} \cap \{\sigma(i_1), \ldots, \sigma(i_n)\} = \emptyset. \]

Every element of \( \mathcal{A} \) contains precisely one of the points \( p_i \) or \( p_{\sigma(i)} \) for each \( i \). It is easy to check that \( \mathcal{A} \) is the image of an isometric embedding of the \( n \)-dimensional hypercube graph \( H_n \) in \( \Gamma_{n-1}(\Pi) \).

Now consider an \((n - m - 1)\)-dimensional singular subspace \( M \) and a frame \( B \) of \( \Pi \) such that \( M \) is spanned by a subset of \( B \). The intersection of the associated apartment of \( \mathcal{G}_{n-1}(\Pi) \) with the parabolic subspace \([M]_{n-1}\) is called an apartment in this parabolic subspace. Our parabolic subspace can be identified with the dual polar space of the rank \( m \) polar space \([M]_{n-m}\) (Subsection 2.4) and the natural collineation between these spaces establishes a one-to-one correspondence between their apartments. All apartments of \([M]_{n-1}\) are the images of isometric embeddings of \( H_m \) in \( \Gamma_{n-1}(\Pi) \).

Theorem 3.1 (M. Pankov [12]). The image of every isometric embedding of the \( m \)-dimensional hypercube graph \( H_m, m \leq n \) in the Grassmann graph \( \Gamma_{n-1}(\Pi) \) is an apartment in a parabolic subspace \([M]_{n-1}\), where \( M \) is an \((n - m - 1)\)-dimensional singular subspace of \( \Pi \). In particular, the image of every isometric embedding of \( H_n \) in \( \Gamma_{n-1}(\Pi) \) is an apartment of \( \mathcal{G}_{n-1}(\Pi) \).

3.2. Apartments of half-spin Grassmannians and half-cube graphs. From this moment we will suppose that \( \Pi = (P, \mathcal{L}) \) is a polar space of type \( D_n \) and \( n \geq 4 \).

Let \( B \) be a frame of \( \Pi \). For each \( k \in \{0, 1, \ldots, n - 1\} \) we denote by \( \mathcal{A}_k \) the associated apartment of \( \mathcal{G}_k(\Pi) \). The intersection

\[ \mathcal{A}_\delta := \mathcal{A}_{n-1} \cap \mathcal{G}_\delta(\Pi), \quad \delta \in \{+, -\} \]

is the apartment of the half-spin Grassmannian \( \mathcal{G}_\delta(\Pi) \) associated with the frame \( B \). It is the image of an isometric embedding of the \( n \)-dimensional half-cube graph \( \frac{1}{2}H_n \) in \( \Gamma_{\delta}(\Pi) \).

So, the half-cube graph \( \frac{1}{2}H_n \) can be identified with the restriction of \( \Gamma_{\delta}(\Pi) \) to the apartment \( \mathcal{A}_\delta \). This graph has precisely the following two types of maximal cliques (see, for example, [6, 7]):

- the star \( \mathcal{A}_\delta \cap [S]_\delta, S \in \mathcal{A}_{n-4} \), which consists of 4 vertices;
• the special subset $A_{\delta} \cap [U]_{\delta}$, $U \in A_{-\delta}$ which consists of $n$ vertices.

If $n$ is even then the diameter of $\frac{1}{2}H_n$ is equal to $\frac{n}{2}$ and for every vertex there is unique opposite vertex.

**Lemma 3.1.** If $n$ is even and $v, w$ are opposite vertices of $\frac{1}{2}H_n$ then every vertex of $\frac{1}{2}H_n$ belongs to a geodesic connecting $v$ and $w$.

**Proof.** Easy verification. □

In the case when $n$ is odd, the diameter of $\frac{1}{2}H_n$ is equal to $\frac{n-1}{2}$ and for every vertex there are precisely $n$ opposite vertices.

Now, let $M$ be an $(n-m-1)$-dimensional singular subspace of $\Pi$ and let $B$ be a frame of $\Pi$ such that $M$ is spanned by a subset of $B$. The intersection of the associated apartment of $G_{\delta}(\Pi)$ with the parabolic subspace $[M]_{\delta}$ is an apartment of this parabolic subspace. Recall that $[M]_{n-m}$ is a polar space of type $D_n$ and there is natural collineation of $[M]_{\delta}$ to one of the half-spin Grassmann spaces of $[M]_{n-m}$. This collineation establishes a one-to-one correspondence between apartments. All apartments of $[M]_{\delta}$ are the images of isometric embeddings of $\frac{1}{2}H_m$ in $\Gamma_{\delta}(\Pi)$.

In the next section we will need the following lemmas.

**Lemma 3.2.** For any three distinct mutually adjacent vertices $v, w, u$ of $\frac{1}{2}H_n$ there exists a vertex adjacent with $v, w$ and non-adjacent with $u$.

**Proof.** An easy verification shows that the intersection of all maximal cliques of $\frac{1}{2}H_n$ containing both $v$ and $w$ coincides with the set $\{v, w\}$. Thus there exists a maximal clique which contains $v, w$ and does not contain $u$. Such clique contains a vertex non-adjacent with $u$. □

**Lemma 3.3.** The following assertions are fulfilled:

1. The intersection of two distinct maximal cliques of $\frac{1}{2}H_n$ contains at most three vertices; this intersection contains precisely three vertices if and only if one of the cliques is a star and the other is a special subset.

2. Every clique of $\frac{1}{2}H_n$ formed by three distinct vertices is contained in precisely one star and precisely one special subset.

3. For any four distinct mutually adjacent vertices $v, w, u, s$ of $\frac{1}{2}H_n$ there is a vertex adjacent with $v, w, u$ and non-adjacent with $s$.

**Proof.** (1). Easy verification.

(2). It is not difficult to see that any 3-element clique is contained in a star and a special subset. The first statement guarantees that these star and special subset are unique.

(3). By (2), $\{v, w, u\}$ is the intersection of two maximal cliques. One of these cliques does not contain $s$ and it has a vertex non-adjacent with $s$. □

4. Extensions of embeddings

Let $f$ be an embedding of $\frac{1}{2}H_m$, $m \geq 4$ in $\Gamma_{\delta}(\Pi)$, $\delta \in \{+, -\}$. The embedding is not assumed to be isometric and its image will be denoted by $I$.

**Lemma 4.1.** The images of any three distinct mutually adjacent vertices of $\frac{1}{2}H_m$ form an independent subset in the half-spin Grassmann space.
Proof. Suppose that three distinct vertices $v, w, u$ of $\frac{1}{2}H_m$ form a clique and $f(u)$ belongs to the line joining $f(v)$ and $f(w)$. By Lemma 2.3 every element of $I$ adjacent with both $f(v)$ and $f(w)$ is adjacent with $f(u)$. Since $f$ is an embedding, this implies that every vertex of $\frac{1}{2}H_m$ adjacent with $v$ and $w$ is adjacent with $u$. The latter contradicts Lemma 3.2.

Lemma 4.2. The images of any four distinct mutually adjacent vertices of $\frac{1}{2}H_m$ form an independent subset in the half-spin Grassmann space.

Proof. Let $v, w, u, s$ be four distinct vertices of $\frac{1}{2}H_m$ which form a clique, but the set consisting of their images is not independent. Then, by the previous lemma, $f(s)$ belongs to the plane spanned by $f(v), f(w), f(u)$. Lemma 2.3 guarantees that every element of $I$ adjacent with all $f(v), f(w), f(u)$ is adjacent with $f(s)$. Then every vertex of $\frac{1}{2}H_m$ adjacent with $v, w, u$ is adjacent with $s$. By (3) of Lemma 3.3 this is impossible.

The image of every maximal clique of $\frac{1}{2}H_m$ is a clique in $\Gamma_3(\Pi)$ and it is contained in a certain maximal clique of $\Gamma_3(\Pi)$ (a star or a special subspace). The images of distinct maximal cliques of $\frac{1}{2}H_m$ are contained in distinct maximal cliques of $\Gamma_3(\Pi)$ (otherwise, there exist adjacent elements of $I$ whose pre-images are not adjacent).

Lemma 4.3. The image of every maximal clique of $\frac{1}{2}H_m$ is contained in precisely one maximal clique of $\Gamma_3(\Pi)$.

Proof. Suppose that the image of a certain maximal clique of $\frac{1}{2}H_m$ is contained in two distinct maximal cliques of $\Gamma_3(\Pi)$. By Lemma 2.4 the intersection of these cliques is contained in a plane. Every maximal clique of $\frac{1}{2}H_m$ contains at least four vertices. Hence there are four distinct mutually adjacent vertices of $\frac{1}{2}H_m$ whose images belong to a plane, but this contradicts Lemma 4.2.

Proposition 4.1. For every embedding of $\frac{1}{2}H_m$, $m \geq 4$ in $\Gamma_3(\Pi)$, $\delta \in \{+, -\}$ one of the following possibilities is realized:

(A) the image of every star is contained in a star and the image of every special subset is contained in a special subspace,

(B) the images of stars are contained in special subspaces and the images of special subsets are contained in stars.

Proof. Let $f$, as above, be an embedding of $\frac{1}{2}H_m$ in $\Gamma_3(\Pi)$. Suppose that $\mathcal{X}_0$ is a special subset whose image is contained in a special subspace. We take any star $\mathcal{Y}$ such that $|\mathcal{X}_0 \cap \mathcal{Y}| = 3$ (see Lemma 3.3). By Lemma 4.1 the set $f(\mathcal{X}_0) \cap f(\mathcal{Y})$ spans a plane. Since $f(\mathcal{X}_0)$ is contained in a special subspace, Lemma 2.3 implies that the maximal clique of $\Gamma_3(\Pi)$ containing $f(\mathcal{Y})$ is a star. The same arguments show that for any special subset $\mathcal{X}$ satisfying $|\mathcal{X} \cap \mathcal{Y}| = 3$ the image $f(\mathcal{X})$ is contained in a special subspace.

Now, let $\mathcal{X}$ be an arbitrary special subset. There exists a sequence $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_{2k} = \mathcal{X}$ such that $|\mathcal{X}_{i-1} \cap \mathcal{X}_i| = 3$ for all $i \in \{1, \ldots, 2k\}$ and $\mathcal{X}_i$ is a special subset or a star if $i$ is even or odd, respectively. By the arguments given above, $f(\mathcal{X})$ is contained in a special subspace. Similarly, we establish that the image of every star is a subset of a star.

Using the same arguments, we show that the case (B) is realized if the image of a certain special subset is contained in a star.
An embedding of $\frac{1}{2}H_m$ in $\Gamma_\delta(\Pi)$ will be called extendible if it can be extended to an embedding of $H_m$ in $\Gamma_{n-1}(\Pi)$.

**Proposition 4.2.** An embedding of $\frac{1}{2}H_m$ in $\Gamma_\delta(\Pi)$ is extendible if and only if it satisfies (A).

**Proof.** We take any polar space of type $D_m$. Let $A$ be an apartment in the corresponding dual polar space. Denote by $A_+$ and $A_-$ the associated apartments in the half-spin Grassmannians. Recall that $A$ is the disjoint sum of $A_+$ and $A_-$. Let $f : A_+ \to G_\delta(\Pi)$ be an embedding of $\frac{1}{2}H_m$ in $\Gamma_\delta(\Pi)$.

Suppose that $f$ is extendible. The associated extension also will be denoted by $f$. Then $f$ transfers every special subset $A_+ \cap [U]_+$, $U \in A_-$ to a subset of $[f(U)]_{\delta}$ which implies (A).

Now, suppose that $f$ satisfies (A). For every $X \in A_-$ there exists unique $X' \in G_{-\delta}(\Pi)$ such that $f(A_+ \cap [X]_+) \subset [X']_{\delta}$ and we define $f(X) := X'$. The mapping $f : A \to G_{n-1}(\Pi)$ is injective. Show that this is an embedding of $H_m$ in $\Gamma_{n-1}(\Pi)$.

Since $f(A_+) \subset G_{\delta}(\Pi)$ and $f(A_-) \subset G_{-\delta}(\Pi)$, for any two vertices of $f(A)$ adjacent in $\Gamma_{n-1}(\Pi)$ one of these vertices belongs to $f(A_+)$ and the other is an element of $f(A_-)$. It is clear that $X \in f(A_+)$ and $Y \in f(A_-)$ are adjacent vertices of $\Gamma_{n-1}(\Pi)$ if and only if $X \in [Y]_{\delta}$. The latter is possible only in the case when $f^{-1}(X) \in [f^{-1}(Y)]_+$, i.e. $f^{-1}(X)$ and $f^{-1}(Y)$ are adjacent vertices of $H_m$.

**Remark 4.1.** Suppose that $f$ is an embedding of $\frac{1}{2}H_4$ in $\Gamma_\delta(\Pi)$ satisfying (B). We take any automorphism $h$ of $\frac{1}{2}H_4$ changing the type of every maximal clique (it is easy to check that such automorphisms exist). Then $fh$ is an embedding of $\frac{1}{2}H_4$ in $\Gamma_\delta(\Pi)$ satisfying (A), i.e. an extendible embedding. The embeddings $f$ and $fh$ have the same image.

By Remark 4.1 non-extendible embeddings of $\frac{1}{2}H_4$ exist, but they can be reduced to extendible embeddings. The following question is an open problem: are there non-extendible embeddings or non-extendible isometric embeddings of $\frac{1}{2}H_m$, $m > 4$?

5. **Main result**

**Theorem 5.1.** If $m \geq 4$ is even then the image of every extendible isometric embedding of the half-cube graph $\frac{1}{2}H_m$ in the half-spin Grassmann graph $\Gamma_\delta(\Pi)$ is an apartment in a parabolic subspace $[M]_{\delta}$, where $M$ is an $(n-m-1)$-dimensional singular subspace of $\Pi$. In the case when $n$ is even, the image of every extendible isometric embedding of $\frac{1}{2}H_n$ in $\Gamma_\delta(\Pi)$ is an apartment of $G_\delta(\Pi)$.

Theorem 5.1 together with Remark 4.1 give the following.

**Corollary 5.1.** The image of every isometric embedding of $\frac{1}{2}H_4$ in $\Gamma_\delta(\Pi)$ is an apartment in a parabolic subspace $[M]_{\delta}$, where $M$ is an $(n-5)$-dimensional singular subspace of $\Pi$. In the case when $n = 4$, the image of every isometric embedding of $\frac{1}{2}H_4$ in $\Gamma_\delta(\Pi)$ is an apartment of $G_\delta(\Pi)$.
Remark 5.1. Our proof of Theorem 5.1 is a modification of the proof of Theorem 3.1 given in [12]. We are not able to show that for every extendible isometric embedding of $\frac{1}{2}H_m$ in $\Gamma_\delta(\Pi)$ the associated embedding of $H_m$ in $\Gamma_{n-1}(\Pi)$ is isometric. If this is possible then the required result follows from Theorem 3.1.

6. Proof of Theorem 5.1

Throughout the section we suppose that $m$ is even and not less than 4.

Lemma 6.1. The image of every isometric embedding of $\frac{1}{2}H_m$ in $\Gamma_\delta(\Pi)$ is contained in a parabolic subspace $[M]_\delta$, where $M$ is an $(n-m-1)$-dimensional singular subspace of $\Pi$.

Proof. Let $f$ be an isometric embedding of $\frac{1}{2}H_m$ in $\Gamma_\delta(\Pi)$. We take any geodesic $v_0, v_1, \ldots, v_{m/2}$ in $\frac{1}{2}H_m$. The distance between $f(v_0)$ and $f(v_{m/2})$ in $\Gamma_\delta(\Pi)$ is equal to $\frac{m}{2}$. Hence the distance between these vertices in $\Gamma_{n-1}(\Pi)$ is $m$ and

$$f(v_0), f(v_1), \ldots, f(v_{m/2})$$

can be extended to a geodesic of $\Gamma_{n-1}(\Pi)$. Then the singular subspace

$$M := f(v_0) \cap f(v_{m/2})$$

is $(n-m-1)$-dimensional. By Lemma 2.2 every $f(v_i)$ contains $M$, i.e. all $f(v_i)$ belong to $[M]_\delta$. This gives the claim, since every vertex of $\frac{1}{2}H_m$ is on a geodesic connecting $v_0$ with $v_{m/2}$ (Lemma 3.1). \hfill $\square$

If $M$ is an $(n-m-1)$-dimensional singular subspace of $\Pi$ then $[M]_{n-m}$ is a polar space of type $D_m$ and there is the natural collineation of $[M]_\delta$ to one of the half-spin Grassmann spaces of this polar space; this collineation establishes a one-to-one correspondence between apartments. Therefore, by Lemma 6.1 it is sufficient to prove Theorem 5.1 only in the case when $m = n$.

So, let $m = n$. Let also $\{p_1, \ldots, p_{2n}\}$ be a frame of $\Pi$. Denote by $A$ and $A_+, A_-$ the associated apartments in the Grassmannians $G_{n-1}(\Pi)$ and $G_+(\Pi), G_-(\Pi)$, respectively. Let also $f : A \to G_{n-1}(\Pi)$ be an embedding of $H_n$ in $\Gamma_{n-1}(\Pi)$ such that the restriction of $f$ to $A_+$ is an isometric embedding of $\frac{1}{2}H_n$ in $\Gamma_\delta(\Pi)$, $\delta \in \{+, -\}$; in other words, $f|_{A_+}$ is an extendible isometric embedding of $\frac{1}{2}H_n$ in $\Gamma_\delta(\Pi)$.

For every $i \in \{1, \ldots, 2n\}$ we define

$$A_i := A \cap \{p_i\}_{n-1}.$$

This is an apartment in the parabolic subspace $[p_i]_{n-1}$ and the restriction of $\Gamma_{n-1}(\Pi)$ to $A_i$ is isomorphic to $H_{n-1}$.

Recall that the distance in the Grassmann graphs $\Gamma_{n-1}(\Pi)$ and $\Gamma_\delta(\Pi)$ is denoted by $d$ and $d_\delta$, respectively.

Lemma 6.2. If $X, Y \in A_+$ then

$$(1) \quad d(X, Y) = d(f(X), f(Y)).$$

Proof. Since the restriction of $f$ to $A_+$ is an isometric embedding of $\frac{1}{2}H_n$ in $\Gamma_\delta(\Pi)$, we have

$$d(X, Y)/2 = d_+(X, Y) = d_\delta(f(X), f(Y)) = d(f(X), f(Y))/2$$

which implies (1). \hfill $\square$
Lemma 6.3. The restriction of \( f \) to every \( A_i \) is an isometric embedding of \( H_{n-1} \) in \( \Gamma_{n-1}(\Pi) \).

Proof. We need to show that (1) holds for all \( X, Y \in A_i \). It follows from Lemma 6.2 if \( X, Y \) both belong to \( A_+ \).

Consider the case when \( X \in A_+ \) and \( Y \in A_- \). Let

\[
(2) \quad X = X_0, X_1, \ldots, X_k = Y
\]

be a geodesic of \( \Gamma_{n-1}(\Pi) \) contained in \( A \). Since \( n \) is even, \( d(X, Y) = k < n \) and there exists \( X_{k+1} \in A_+ \) such that

\[
X_0, X_1, \ldots, X_k, X_{k+1}
\]

is a geodesic of \( \Gamma_{n-1}(\Pi) \). We have \( X_0, X_{k+1} \in A_+ \) and, by Lemma 6.2

\[
d(f(X_0), f(X_{k+1})) = k + 1.
\]

Therefore,

\[
f(X_0), f(X_1), \ldots, f(X_k), f(X_{k+1})
\]

is a geodesic in \( \Gamma_{n-1}(\Pi) \). The latter guarantees that \( d(f(X_0), f(X_k)) \) is equal to \( k \) and we get (1).

Now, suppose that \( X, Y \in A_- \). As above, let (2) be a geodesic of \( \Gamma_{n-1}(\Pi) \) contained in \( A \). Since \( X, Y \in A_- \) both contain the point \( p_i \) (they are elements of \( A_i \)), the intersection of \( X \) and \( Y \) contains a line and

\[
d(X, Y) \leq n - 2.
\]

Then (2) can be extended to a geodesic

\[
(3) \quad X_{-1}, X_0, X_1, \ldots, X_k, X_{k+1}
\]

of \( \Gamma_{n-1}(\Pi) \), where \( X_{-1} \) and \( X_{k+1} \) belong to \( A_+ \). By Lemma 6.2 the image of (3) is a geodesic of \( \Gamma_{n-1}(\Pi) \) which implies (1).

Remark 6.1. Lemma 6.3 shows that (1) holds in the case when \( d(X, Y) < n \). We can not prove (1) if \( X, Y \in A_- \) and \( d(X, Y) = n \).

Theorem 6.1 and Lemma 6.3 imply the existence of points \( q_1, \ldots, q_{2n} \) such that each \( f(A_i) \) is an apartment in the parabolic subspace \( \langle q_i \rangle_{n-1} \). Note that

\[
(4) \quad q_i \perp q_j \quad \text{if} \quad j \neq \sigma(i)
\]

(if \( j \neq \sigma(i) \) then there is \( X \in A \) containing \( p_i, p_j \) and \( f(X) \) contains \( q_i, q_j \)).

Lemma 6.4. If \( p_{i_1}, \ldots, p_{i_n} \) span an element of \( A_- \) then \( q_{i_1}, \ldots, q_{i_n} \) form an independent subset and

\[
f(\langle p_{i_1}, \ldots, p_{i_n} \rangle) = \langle q_{i_1}, \ldots, q_{i_n} \rangle.
\]

Proof. Suppose that \( q_{i_n} \) belongs to the singular subspace \( \langle q_{i_1}, \ldots, q_{i_{n-1}} \rangle \). Let \( X \) and \( Y \) be the maximal singular subspaces spanned by

\[
p_{i_1}, \ldots, p_{i_{n-1}}, p_{\sigma(i_n)} \quad \text{and} \quad p_{\sigma(i_1)}, \ldots, p_{\sigma(i_{n-1})}, p_{i_n},
\]

respectively. These are disjoint elements of \( A_+ \) and Lemma 6.2 shows that

\[
f(X) \cap f(Y) = \emptyset.
\]

On the other hand,

\[
\langle q_{i_1}, \ldots, q_{i_{n-1}} \rangle \subset f(X) \quad \text{and} \quad q_{i_n} \in f(Y)
\]
which contradicts \( q_i \in \langle q_i, \ldots, q_{i_n-1} \rangle \). So, \( \langle q_i, \ldots, q_{i_n} \rangle \) is a maximal singular subspace of \( \Pi \). Since \( f(\langle p_{i_1}, \ldots, p_{i_n} \rangle) \) contains \( q_{i_1}, \ldots, q_{i_n} \), we get the required equality. \( \square \)

**Lemma 6.5.** \( q_i \not\perp q_{\sigma(i)} \) for every \( i \in \{1, \ldots, 2n\} \).

**Proof.** We choose \( X, Y \in \mathcal{A} \) such that
\[
X = \langle p_i, p_k, p_{i_1}, \ldots, p_{i_n-2} \rangle \quad \text{and} \quad Y = \langle p_{\sigma(i)}, p_{\sigma(k)}, p_{i_1}, \ldots, p_{i_n-2} \rangle.
\]
It is clear that \( d(X, Y) = 2 \). Since \( f \) is an embedding of \( H_n \) in \( \Gamma_{n-1}(\Pi) \),
\[
d(f(X), f(Y)) = 2
\]
(see Subsection 2.1). By Lemma 6.4
\[
f(X) = \langle q_i, q_{i_1}, \ldots, q_{i_n-2} \rangle, \quad f(Y) = \langle q_{\sigma(i)}, q_{\sigma(k)}, q_{i_1}, \ldots, q_{i_n-2} \rangle
\]
and (5) guarantees that
\[
f(X) \cap f(Y) = \langle q_{i_1}, \ldots, q_{i_{n-2}} \rangle.
\]
Now, assume that \( q_i \perp q_{\sigma(i)} \). Then (4) and the first part of Lemma 2.1 imply that \( q_i \perp f(Y) \). Since \( f(Y) \) is a maximal singular subspace, we get \( q_i \in f(Y) \) (by the second part of Lemma 2.1). So, \( q_i \) belongs to \( f(X) \cap f(Y) \) which is impossible (because \( q_i, q_{i_1}, \ldots, q_{i_n-2} \) form an independent subset). \( \square \)

It follows from Lemma 6.5 and (4) that \( q_1, \ldots, q_{2n} \) form a frame. Then any subset of \( \{q_1, \ldots, q_{2n}\} \) is independent and, as in the proof of Lemma 6.4 we establish that every element of \( f(\mathcal{A}) \) is spanned by a subset of this frame.

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