Birkhoff normal form for the periodic Toda lattice

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To Percy Deift at the occasion of his 60th birthday

Abstract

In this paper we compute the Birkhoff normal form of the periodic Toda lattice up to order four. As an application, we verify that Kolmogorov’s nondegeneracy condition in the KAM theorem holds almost everywhere in phase space.

1 Introduction

Consider the periodic Toda lattice with period $N$ ($N \geq 2$),

$$\dot{q}_n = \partial_{p_n} H, \quad \dot{p}_n = -\partial_{q_n} H$$

for $n \in \mathbb{Z}$, where the (real) coordinates $(q_n, p_n)_{n \in \mathbb{Z}}$ satisfy $(q_n, p_n) = (q_{n+N}, p_{n+N})$ for any $n \in \mathbb{Z}$ and the Hamiltonian $H$ is given by

$$H = \frac{1}{2} \sum_{n=1}^{N} p_n^2 + \sum_{n=1}^{N} V(q_n - q_{n+1})$$

with potential $V(x) = \gamma^2 e^{\delta x} + V_1 x + V_2$ and $\gamma, \delta, V_1, V_2 \in \mathbb{R}$ constants. The Toda lattice has been introduced by Toda \cite{Toda} and studied extensively in the sequel. It is an FPU lattice, i.e. a Hamiltonian system of particles in one space dimension with nearest neighbor interaction. Models of this type have been studied by Fermi-Pasta-Ulam [FPU]. In numerical experiments they found recurrent features for the lattices they considered. Despite an enormous effort from the physics and mathematics community, some of these numerical experiments still defy an explanation. For a recent account of the fascinating history of the FPU problem, see e.g. \cite{Deift}. At least in the case of the periodic Toda

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lattice, the recurrent features can be fully accounted for. In fact, Flaschka [2], Hénon [3], and Manakov [5] independently proved that the periodic Toda lattice is integrable. In this paper, we show that (small) Hamiltonian perturbations of the Toda lattice have many - large and small - quasi-periodic solutions as well. To this end, we put the periodic Toda lattice into Birkhoff normal form up to order 4 and then show that Kolmogorov’s nondegeneracy condition of the KAM theorem holds almost everywhere in phase space - see e.g. [7], Appendix G, for the notion of Birkhoff normal form up to order m.

Returning to the Toda Hamiltonian, let us first note that w.l.o.g. we can assume that $V_1 = V_2 = 0$. When expressed in the canonical coordinates $(\delta q_j, \frac{1}{2} p_j)_{1 \leq j \leq N}$ the Hamiltonian $H$ is, up to a scaling factor $\delta^{-2}$, of the form

$$H_{\text{Toda}} = \frac{1}{2} \sum_{n=1}^{N} p_{n}^{2} + \alpha^{2} \sum_{n=1}^{N} e^{q_{n} - q_{n+1}},$$

where $\alpha^{2} = (\gamma \delta)^{2}$. Moreover notice that the total momentum $\sum_{n=1}^{N} p_{n}$ is conserved; hence the motion of the center of mass $\frac{1}{N} \sum_{n=1}^{N} q_{n}$ is linear and therefore unbounded. However, it turns out that the orbits of the system relative to the center of mass all lie on tori, generically of dimension $N - 1$. To describe these orbits, it suffices to consider the relative coordinates $v_{n} := q_{n+1} - q_{n}$ ($1 \leq n \leq N - 1$) and their conjugate ones, $u_{n} := n \beta - \sum_{j=1}^{n} p_{k}$ ($1 \leq n \leq N - 1$), where $\beta = \frac{1}{N} \sum_{j=1}^{N} p_{n}$. The corresponding phase space is then $\mathbb{R}^{2N - 2}$.

In terms of the variables $(v_{k}, u_{k})_{1 \leq k \leq N - 1}$, $H_{\text{Toda}}$ is given by

$$H_{\beta, \alpha} = \frac{N \beta^{2}}{2} \frac{1}{2} \sum_{n=1}^{N} \left( u_{n} + \sum_{l=1}^{n-2} (u_{l} - u_{l+1})^{2} + u_{n-2}^{2} \right) + \alpha^{2} \left( \sum_{k=1}^{N-1} e^{-v_{k}} + e^{-\sum_{k=1}^{N-1} v_{k}} \right),$$

with parameters $\alpha$ and $\beta$ - see section 2 for more details.

The main result of this paper is the following

**Theorem 1.1.** For any fixed $\beta \in \mathbb{R}$, $\alpha > 0$, and $N \geq 2$, the periodic Toda lattice admits a Birkhoff normal form. More precisely, there are (globally defined) canonical coordinates $(x_{k}, y_{k})_{1 \leq k \leq N - 1}$ so that $H_{\beta, \alpha}$, when expressed in these coordinates, takes the form $H(\beta, \alpha)(I) := \frac{N \beta^{2}}{2} + H_{\alpha}(I)$, where $H_{\alpha}(I)$ is a real analytic function of the action variables $I_{k} = (x_{k}^{2} + y_{k}^{2})/2$ ($1 \leq k \leq N - 1$). Moreover, near $I = 0$, $H_{\alpha}(I)$ has an expansion of the form

$$N \alpha^{2} + \alpha \sum_{k=1}^{N-1} s_{k} I_{k} + \frac{1}{4N} \sum_{k=1}^{N-1} I_{k}^{2} + O(I^{3}),$$

with $s_{k} = 2 \sin \frac{k \pi}{N}$ for $1 \leq k \leq N - 1$.

**Remark 1.2.** In particular, Theorem 1.1 says that near 0, the Toda lattice can be viewed as a system of $(N - 1)$ nonlinear harmonic oscillators which are uncoupled up to order 4. The system is resonant at 0 with resonance lattice $\mathcal{R} := \{(j_{1}, \ldots, j_{N-1}) \in \mathbb{Z}^{N-1} | \sum_{j=1}^{N-1} j_{s_{j}} = 0\}$ of dimension at least $\lfloor \frac{N-1}{2} \rfloor$, as $s_{k} - s_{N-k} = 0$ for any $1 \leq k \leq \lfloor \frac{N-1}{2} \rfloor$. 
Corollary 1.3. Let $\alpha > 0$ and $\beta \in \mathbb{R}$ be arbitrary. Then the Hessian of $\mathcal{H}_{\beta, \alpha}(I)$ at $I = 0$ is given by

$$d^2_H|_{I=0} = \frac{1}{2N} Id_{N-1}.$$ 

In particular, the frequency map $I \mapsto \nabla_I \mathcal{H}_{\beta, \alpha}$ is nondegenerate at $I = 0$ and hence, by analyticity, nondegenerate on an open dense subset of $(\mathbb{R}_{\geq 0})^{N-1}$.

Theorem 1.1 and Corollary 1.3 allow to apply the KAM-theorem (cf e.g. [7, 9]) to the Toda lattice on an open dense subset of the phase space. Moreover, as the Hessian of $\mathcal{H}_{\beta, \alpha}$ at $I = 0$ is positive definite and hence $\mathcal{H}_{\beta, \alpha}$ is strictly convex near $I = 0$, one can also apply Nekhoroshev’s theorem (cf e.g. [10]) to the Toda lattice near the equilibrium point.

Using the method of proof of this paper we computed in [6] the Birkhoff normal form up to order 4 for any FPU chain and applied these computations to improve on results of Rink [11] with regard to the verification of Kolmogorov’s condition near the equilibrium point for these systems.

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2 Relative coordinates

To prove the integrability of the Toda lattice, Flaschka introduced in [2] the (noncanonical) coordinates

$$b_n := -p_n \in \mathbb{R}, \quad a_n := \alpha e^{\frac{1}{2}(q_n + q_{n+1})} \in \mathbb{R}_{>0} \quad (n \in \mathbb{Z}).$$  \hfill (4)

They are coordinates which describe the motion of the Toda lattice relative to the center of mass. In these coordinates, the Hamiltonian $H_{\text{Toda}}$ takes the simple form $H = \frac{1}{2} \sum_{n=1}^{N} b_n^2 + \sum_{n=1}^{N} a_n^2$, and the equations of motion are

$$\begin{align*}
\dot{b}_n &= a_n^2 - a_{n-1}^2 \quad (n \in \mathbb{Z}),
\dot{a}_n &= \frac{1}{2a_n}(b_{n+1} - b_n).
\end{align*}$$ \hfill (5)

Note that $(b_{n+N}, a_{n+N}) = (b_n, a_n)$ for any $n \in \mathbb{Z}$, and $\prod_{n=1}^{N} a_n = \alpha^N$. Hence we can identify the sequences $(b_n)_{n \in \mathbb{Z}}$ and $(a_n)_{n \in \mathbb{Z}}$ with the vectors $(b_n)_{1 \leq n \leq N} \in \mathbb{R}^N$ and $(a_n)_{1 \leq n \leq N} \in \mathbb{R}^N_{>0}$. The phase space of the system (5) is then given by

$$\mathcal{M} := \mathbb{R}^N \times \mathbb{R}^N_{>0},$$

and it turns out that (5) is a Hamiltonian system with respect to the non-standard Poisson structure $J \equiv J_{b,a}$, defined at a point $(b,a) = (b_n, a_n)_{1 \leq n \leq N}$ by

$$J = \begin{pmatrix}
0 & A \\
-A^T & 0
\end{pmatrix}$$
where $A$ is the $b$-independent $N \times N$-matrix

$$A = \frac{1}{2} \begin{pmatrix} a_1 & 0 & \ldots & 0 & -a_N \\ -a_1 & a_2 & 0 & \ddots & 0 \\ 0 & -a_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -a_{N-1} & a_N \end{pmatrix}.$$  \hspace{1cm} (6)

The Poisson bracket corresponding to $J$ is then given by

$$\{F, G\}_J(b, a) = \langle (\nabla_b F, \nabla_a F), J(\nabla_b G, \nabla_a G) \rangle_{\mathbb{R}^{2N}}$$

$$= \langle \nabla_b F, A \nabla_a G \rangle - \langle \nabla_a F, A^T \nabla_b G \rangle_{\mathbb{R}^N}$$ \hspace{1cm} (7)

where $F, G \in C^1(M)$ and where $\nabla_b$ and $\nabla_a$ denote the gradients with respect to $b = (b_1, \ldots, b_N)$ and $a = (a_1, \ldots, a_N)$, respectively. Therefore, equations (6) can alternatively be written as $b_n = \{b_n, H\}_J$, $a_n = \{a_n, H\}_J$ ($1 \leq n \leq N$).

Since the matrix $A$, defined by (6), has rank $N - 1$, the Poisson structure $J$ is degenerate. It admits the two Casimir functions\(^2\)

$$C_1 := -\frac{1}{N} \sum_{n=1}^{N} b_n \quad \text{and} \quad C_2 := \left( \prod_{n=1}^{N} a_n \right)^{\frac{1}{N}}.$$ \hspace{1cm} (8)

The gradients $\nabla_{b,a} C_i = (\nabla_b C_i, \nabla_a C_i)$ $(i = 1, 2)$ are given by

$$\nabla_b C_1 = -\frac{1}{N} (1, \ldots, 1), \quad \nabla_a C_1 = 0,$$ \hspace{1cm} (9)

$$\nabla_b C_2 = 0, \quad \nabla_a C_2 = \frac{C_2}{N} \left( \frac{1}{a_1}, \ldots, \frac{1}{a_N} \right).$$ \hspace{1cm} (10)

They are linearly independent at each point $(b, a) \in M$.

Let $M_{\beta, \alpha} := \{(b, a) \in M : (C_1, C_2) = (\beta, \alpha)\}$ denote the level set of $(C_1, C_2)$ at $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$. As $C_1$ and $C_2$ are real analytic on $M$ and the gradients $\nabla_{b,a} C_1$ and $\nabla_{b,a} C_2$ are linearly independent everywhere on $M$, the sets $M_{\beta, \alpha}$ are real analytic submanifolds of $M$ of (real) codimension two. Furthermore the Poisson structure $J$, restricted to $M_{\beta, \alpha}$, becomes nondegenerate everywhere on $M_{\beta, \alpha}$ and therefore induces a symplectic structure on $M_{\beta, \alpha}$. In this way, we obtain a symplectic foliation of $M$ with $M_{\beta, \alpha}$ being the symplectic leaves. We denote by $H_{\beta, \alpha}$ the restriction of the Hamiltonian $H_{\text{Toda}}$ to $M_{\beta, \alpha}$.

Besides Flaschka’s coordinates of the Toda lattice we will also need to consider relative coordinates. Introduce $(v_1, \ldots, v_N) \in \mathbb{R}^N$ given by $v_i := q_{i+1} - q_i$ for $1 \leq i \leq N - 1$ and $v_N := \frac{1}{N} \sum_{i=1}^{N} q_i$. Then $v = Mq$ is a linear change of the

\(^2\)A smooth function $C : M \to \mathbb{R}$ is a Casimir function for $J$ if $\{C, \cdot\}_J \equiv 0$.\]
position coordinates where $M$ is the $N \times N$-matrix

$$
M = \begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \ddots & 1 \\
N^{-1} & \cdots & \cdots & 1 & N^{-1}
\end{pmatrix}.
$$

The variables $u = (u_1, \ldots, u_N)$ conjugate to $v = (v_1, \ldots, v_N)$ are then given by $u = (M^T)^{-1} p$. The matrix $(M^T)^{-1}$ can be computed to be

$$
(M^T)^{-1} = \frac{1}{N} \begin{pmatrix}
1 & \cdots & \cdots & 1 \\
2 & \cdots & \cdots & 2 \\
\vdots & \ddots & \ddots & \ddots \\
N & \cdots & \cdots & N
\end{pmatrix} - \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0
\end{pmatrix}.
$$

For any $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$ the variables $(v_k, u_k)_{1 \leq k \leq N-1}$ are canonically conjugate variables on $\mathcal{M}_{\beta, \alpha}$. We want to express the Hamiltonian $H_{\beta, \alpha}$ in terms of the coordinates $(v, u) = (v_k, u_k)_{1 \leq k \leq N-1}$. Note that on $\mathcal{M}_{\beta, \alpha}$, $u_k = k\beta - \sum_{j=1}^{k} p_j$ for $1 \leq k \leq N - 1$ and $u_N = N \beta$. Hence $p_1 = -u_1 + \beta$, $p_N = u_{N-1} + \beta$, and $p_k = (u_{k-1} - u_k) + \beta$ for $2 \leq k \leq N - 1$, and thus

$$
\frac{1}{2} \sum_{j=1}^{N} p_j^2 = \frac{N \beta^2}{2} + \frac{1}{2} \left( u_1^2 + (u_1 - u_2)^2 + \cdots + (u_{N-2} - u_{N-1})^2 + u_{N-1}^2 \right).
$$

Moreover, using that $q_N - q_{N+1} = q_N - q_1 = \sum_{k=1}^{N-1} (q_{k+1} - q_k)$ one gets

$$
\sum_{j=1}^{N} e^{q_j - q_{j+1}} = \sum_{k=1}^{N-1} e^{-q_k + \sum_{k=1}^{N-1} q_k}.
$$

Combining the two expressions displayed above yields

$$
H_{\beta, \alpha} = \frac{N \beta^2}{2} - \frac{1}{2} \left( u_1^2 + \sum_{l=1}^{N-2} (u_l - u_{l+1})^2 + u_{N-1}^2 \right) + \alpha^2 \left( \sum_{k=1}^{N-1} e^{-q_k} + e^{\sum_{k=1}^{N-1} q_k} \right),
$$

(11)

The coordinates $(v_k, u_k)_{1 \leq k \leq N-1}$ (as well as $u_N$, but not $v_N$) can easily be expressed in terms of $(b_j, a_j)_{1 \leq j \leq N}$,

$$
v_k = 2 \log \frac{\alpha}{a_k}, \quad u_k = - \frac{k}{N} \sum_{j=1}^{N} b_j + \sum_{j=1}^{k} b_j \quad (1 \leq k \leq N - 1)
$$

(12)

Note that $u_k$ depends linearly on the $b$-coordinates and is independent of the $a$-coordinates. On the other hand, $v_k$ depends only on the $a$-coordinates. The
partial derivatives of the \( v_k \)'s at \( a = \alpha 1_N \), where \( 1_N = (1, \ldots, 1) \in \mathbb{R}^N \), can be computed to be

\[
\frac{\partial v_k}{\partial a_j} = -\frac{2}{\alpha} \delta_{kj} + \frac{2}{N\alpha}.
\] 

(13)

3 Birkhoff coordinates

Let us recall from [4] and [5] our results concerning global Birkhoff coordinates on the phase space \( \mathcal{M} \). As a model space, we introduced the space \( \mathcal{P} := \mathbb{R}^{2N-2} \times \mathbb{R} \times \mathbb{R}_{>0} \), foliated by \( \mathcal{P}_{\beta,\alpha} := \mathbb{R}^{2N-2} \times \{\beta\} \times \{\alpha\} \) which are endowed with the standard symplectic structure. Denote by \( J_0 \) the degenerate Poisson structure on \( \mathcal{P} \) having \( \mathcal{P}_{\beta,\alpha} \) as its symplectic leaves and the coordinates \( \beta \) and \( \alpha \) as its Casimirs.

In [4] we have proved that the Toda lattice admits global Birkhoff coordinates.

**Theorem 3.1.** There exists a map

\[
\Phi : (\mathcal{M}, J) \to (\mathcal{P}, J_0)
\]

\[
(b, a) \mapsto ((x_k, y_k)_{1 \leq k \leq N-1}, C_1, C_2)
\]

with the following properties:

- \( \Phi \) is a real analytic diffeomorphism.
- \( \Phi \) is canonical, i.e. it preserves the Poisson brackets. In particular, the symplectic foliation of \( \mathcal{M} \) by \( \mathcal{M}_{\beta,\alpha} \) is trivial.
- The coordinates \( (x_k, y_k)_{1 \leq k \leq N-1}, C_1, C_2 \) are global Birkhoff coordinates for the periodic Toda lattice, i.e. the transformed Toda Hamiltonian \( H^* = H \circ \Phi^{-1} \) is a function of the actions \( (I_k)_{1 \leq k \leq N-1} \) and \( C_1, C_2 \) alone, where

\[
I_k = \frac{1}{2}(x_k^2 + y_k^2)
\]

for any \( 1 \leq k \leq N - 1 \). (In the sequel we will denote \( H^* \) by \( H \) as well.)

As an immediate consequence of Theorem 3.1 one gets

**Corollary 3.2.** For every \( \beta \in \mathbb{R} \) and \( \alpha > 0 \),

\[
\Phi(\mathcal{M}_{\beta,\alpha}) = \mathcal{P}_{\beta,\alpha},
\]

and \( \Phi|_{\mathcal{M}_{\beta,\alpha}} : \mathcal{M}_{\beta,\alpha} \to \mathcal{P}_{\beta,\alpha} \) is a symplectomorphism. In particular, the moment map

\[
\mathcal{M}_{\beta,\alpha} \to \mathbb{R}^{N-1}_{\geq 0}, \quad (b, a) \mapsto (I_k(\Phi(b, a)))_{1 \leq k \leq N-1}
\]

is onto.

We wish to analyze the Birkhoff map \( \Phi \) and its restriction to the leaves \( \mathcal{M}_{\beta,\alpha} \) near the equilibrium points \( (b, a) = (\beta 1_N, \alpha 1_N) \), where \( 1_N = (1, \ldots, 1) \in \mathbb{R}^N \). Introduce for \( k \in \mathbb{Z} \) with \( 1 \leq |k| \leq N - 1 \) the abbreviation

\[
\lambda_k := \left| \sin \frac{k\pi}{N} \right|^2.
\]
To compute the Jacobian of $\Phi^{-1}$ at the points $(0_{N-1}, 0_{N-1}, \beta, \alpha)$ we first compute the Jacobian of $\Phi$. It turns out that the formulas are easier to express in complex notation. In [5] we have shown the following

**Lemma 3.3.** For any $\beta \in \mathbb{R}, \alpha > 0$, and $1 \leq k \leq N-1$, the gradient

$$\nabla_{b,a}(x_k + iy_k) = \frac{1}{\sqrt{2\alpha N}} \frac{1}{\lambda_k} \begin{cases} e^{2\pi i (j-1)k/N} & (1 \leq j \leq N) \\ -2e^{i(2j-1)k/N} & (N + 1 \leq j \leq 2N) \end{cases}$$

(14)

and

$$\nabla_{b,a}C_1 = -\frac{1}{N} (1_N, 0_N), \quad \nabla_{b,a}C_2 = \frac{1}{N} (0_N, 1_N).$$

(15)

According to the formulas (14) and (15), the Jacobian $d_{b,a}\Phi$ of $\Phi$ at $(b, a) = (\beta 1_N, \alpha 1_N)$ is given by the $(2N \times 2N)$-matrix

$$
\begin{pmatrix}
\left( \frac{1}{\sqrt{2\alpha N}} \frac{1}{\lambda_k} \cos \left( \frac{(2j-2)\pi k}{N} \right) \right)_{kj} & \left( \frac{-2}{\sqrt{2\alpha N}} \frac{1}{\lambda_k} \cos \left( \frac{(2j-1)\pi k}{N} \right) \right)_{kj} \\
\left( \frac{1}{\sqrt{2\alpha N}} \frac{1}{\lambda_k} \sin \left( \frac{(2j-2)\pi k}{N} \right) \right)_{kj} & \left( \frac{-2}{\sqrt{2\alpha N}} \frac{1}{\lambda_k} \sin \left( \frac{(2j-1)\pi k}{N} \right) \right)_{kj} \\
-N^{-1} \ldots & \ldots \ldots \ldots \ldots \\
0 \ldots & \ldots \ldots \ldots \ldots \\
N^{-1} \ldots & \ldots \ldots \ldots \ldots \\
0 \ldots & \ldots \ldots \ldots \ldots \\
\end{pmatrix}
$$

(16)

where in each of the four $(N-1) \times N$-submatrices the row and column indices $k$ and $j$ run from 1 to $N-1$ and 1 to $N$, respectively. In order to compute its inverse, we note that (16) can be written as the product $\Delta_2 \cdot P \cdot \Delta_1$ of the diagonal matrices

$$\Delta_1 := \text{diag} (1_N, -2 \cdot 1_N)$$

and

$$\Delta_2 := \frac{1}{\sqrt{2\alpha}} \text{diag} \left( \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_{N-1}}, \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_{N-1}}, \sqrt{\frac{2\alpha}{N}} \cdot \sqrt{\frac{\alpha}{2N}} \right)$$

with the orthogonal $2N \times 2N$ matrix

$$P := \frac{1}{\sqrt{N}} \begin{pmatrix} P^{(1)} & P^{(2)} \\ P^{(3)} & P^{(4)} \end{pmatrix}$$

(17)

where for $1 \leq i \leq 4$, the submatrices $(P^{(i)}_{kj})$ are $(N-1) \times N$-matrices given by

$P^{(1)} := \left( \cos \left( \frac{(2j-2)\pi k}{N} \right) \right)_{kj}, P^{(2)} := \left( \cos \left( \frac{(2j-1)\pi k}{N} \right) \right)_{kj}, P^{(3)} := \left( \sin \left( \frac{(2j-2)\pi k}{N} \right) \right)_{kj}, P^{(4)} := \left( \sin \left( \frac{(2j-1)\pi k}{N} \right) \right)_{kj}.$
The inverse \((d_{b,a}\Phi)^{-1}\) = \(\Delta_1^{-1}P^T\Delta_2^{-1}\) at \((b, a) = (\beta \cdot 1_N, \alpha \cdot 1_N)\) can then easily computed to be

\[
\begin{pmatrix}
(\sqrt{\frac{2\alpha}{N}} \lambda_k \cos (\frac{(2j-2)\pi k}{N})_{jk} & (\sqrt{\frac{2\alpha}{N}} \lambda_k \sin (\frac{(2j-2)\pi k}{N})_{jk} \\
-\sqrt{\frac{2\beta}{N}} \lambda_k \cos (\frac{(2j-1)\pi k}{N})_{jk} & -\sqrt{\frac{2\beta}{N}} \lambda_k \sin (\frac{(2j-1)\pi k}{N})_{jk}
\end{pmatrix}
\]

again with \(1 \leq j \leq N\) and \(1 \leq k \leq N - 1\), but with \(k\) and \(j\) now being column and row indices, respectively.

The row vectors in the \(2N \times 2N\)-matrix \((16)\) are the gradients \(\nabla_{b,a}x_k, \nabla_{b,a}y_k\) \((1 \leq k \leq N - 1)\) and \(\nabla_{b,a}C_i\) \((i = 1, 2)\). In the sequel we will consider the restrictions of the coordinates \((x_k, y_k)_{1 \leq k \leq N - 1}\) to the symplectic leaves \(M_{\beta,\alpha}\). In order to obtain the orthogonal projections of the gradients \(\nabla_{b,a}x_k, \nabla_{b,a}y_k, \nabla_{b,a}C_i\) onto the tangent space \(T_{b,a}M_{\beta,\alpha}\) of the leaf \(M_{\beta,\alpha}\) at \((b, a)\). Note that \(T_{b,a}M_{\beta,\alpha}(\cong \mathbb{R}^{2(N-1)})\) is the orthogonal complement of \(\text{span}(\nabla_{b,a}C_1, \nabla_{b,a}C_2)\). In view of the formulas \((17)\) for the gradients of \(C_1\) and \(C_2\) one gets

\[T_{b,a}M_{\beta,\alpha} = \{ (\xi, \eta) \in \mathbb{R}^{2N} | \sum_{j=1}^N \xi_j = 0, \sum_{j=1}^N \frac{\eta_j}{a_j} = 0 \}. \quad (18)\]

For \((b, a) = (\beta 1_N, \alpha 1_N)\), the conditions in the set in \((15)\) simply are \(\sum_{j=1}^N \xi_j = 0\) and \(\sum_{j=1}^N \eta_j = 0\). By formula \((14)\), one sees that for \(1 \leq k \leq N - 1\),

\[
\sum_{j=1}^N \frac{\partial}{\partial y_j}(x_k + iy_k) = \frac{1}{\sqrt{2\alpha N}} \frac{1}{\lambda_k} \sum_{j=1}^N e^{2\pi i(j-1)k/N} = 0,
\]

\[
\sum_{j=1}^N \frac{\partial}{\partial a_j}(x_k + iy_k) = \frac{2}{\sqrt{2\alpha N}} \frac{1}{\lambda_k} e^{i\pi k/N} \sum_{j=1}^N e^{2\pi i(j-1)k/N} = 0.
\]

Hence for \((b, a) = (\beta 1_N, \alpha 1_N)\) the gradients \(\nabla_{b,a}x_k, \nabla_{b,a}y_k\) are contained in \(T_{b,a}M_{\beta,\alpha}\) for any \(1 \leq k \leq N - 1\), and there is no need to take projections.

### 4 Linearized Birkhoff Coordinates

For any given values of the Casimir functions, \(C_1 = \beta, C_2 = \alpha\), we wish to compute the first few coefficients of the Birkhoff normal form of the Toda Hamiltonian near the elliptic fixed point \((x, y) = (0, 0)\). We recall that these coefficients are essentially unique, i.e. do not depend on the choice of the Birkhoff coordinates. A possible way of proceeding is to substitute \((b, a) = \Phi^{-1}(x, y, \beta, \alpha)\)
into the expression for the Toda Hamiltonian \( H = \frac{1}{2} \sum_{j=1}^{N} b_j^2 + \sum_{j=1}^{N} a_j^2 \) and then expand \( \Phi^{-1} (x, y, \beta, \alpha) \) at \( (x, y) = (0, 0) \). However it seems difficult to explicitly compute the terms of this expansion, except the first ones which we have computed in the last section - see formula (17). We proceed differently and choose as the starting point of our computations the canonical coordinates \( v_k, u_k \) introduced in section 2 rather than the non-canonical variables \( b_j, a_j \). When expressed in these coordinates, the Toda Hamiltonian takes the form \( H_{\beta, \alpha} = \frac{N^2}{2} + H_u + \alpha^2 H_v \) where by (11),

\[
H_u = \frac{1}{2} \left( u_1^2 + \sum_{l=1}^{N-2} (u_l - u_{l+1})^2 + u_{N-1}^2 \right),
\]

\[
H_v = \sum_{l=1}^{N-1} e^{-v_l} + e^{\sum_{i=1}^{N-1} v_l}.
\]

Note that the Taylor expansion of \( H_{\beta, \alpha} \) at \( (v, u) = (0_{N-1}, 0_{N-1}) \) is not in Birkhoff normal form up to order 2. In a first step we therefore want to choose a linear canonical transformation \( (\xi, \eta) \mapsto (v_k, u_k)_{1 \leq k \leq N-1} \) so that when expressed in the new variables \( (\xi, \eta) = (\xi_k, \eta_k)_{1 \leq k \leq N-1} \), the Toda Hamiltonian is in Birkhoff normal form up to order 2. Consider the composition

\[
\Omega_{\beta, \alpha} : \mathbb{R}^{2N-2}(\cong \mathcal{P}_{\beta, \alpha}) \rightarrow \mathcal{M}_{\beta, \alpha} \rightarrow \mathbb{R}^{2N-2}
\]

\[
(x_k, y_k)_{1 \leq k \leq N-1} \mapsto (b_j, a_j)_{1 \leq j \leq N} \mapsto (v_k, u_k)_{1 \leq k \leq N-1}
\]

of the inverse of the Birkhoff map \( (\Phi|_{\mathcal{M}_{\beta, \alpha}})^{-1} : \mathbb{R}^{2N-2} \rightarrow \mathcal{M}_{\beta, \alpha} \) with the coordinate transformation defined in (12). Then \( \Omega_{\beta, \alpha} \) is a canonical real analytic transformation as both \( (x, y) \) and \( (v, u) \) are canonical coordiantes for \( \mathcal{M}_{\beta, \alpha} \). Its Jacobian

\[
R_{\beta, \alpha} : \mathbb{R}^{2N-2} \rightarrow \mathbb{R}^{2N-2}, \quad (\xi, \eta) \mapsto (v, u) = d_{x,y} \Omega_{\beta, \alpha}|_{(x,y)=(0,0)}(\xi, \eta)
\]

at \( (x, y) = (0, 0) \) is a linear transformation with the desired properties. We will compute \( R_{\beta, \alpha} \) as a composition of the Jacobian of \( (x, y, \beta, \alpha) \mapsto (b, a) \) at \( (x, y) = (0, 0) \) (with \( \beta, \alpha \) fixed) and the one of \( (b, a) \mapsto (v, u) \) at \( (b, a) = (\beta 1_N, \alpha 1_N) \).

It is convenient to use complex notation for \( \xi_k, \eta_k \) \( (1 \leq k \leq N-1) \),

\[
\zeta_k := \frac{1}{\sqrt{2}} (\xi_k - i\eta_k), \quad \zeta_{-k} := \frac{1}{\sqrt{2}} (\xi_k + i\eta_k),
\]

where the sign in the definition of \( \zeta_k \) is chosen so that \( d\zeta_k \wedge d\zeta_{-k} = id\xi_k \wedge d\eta_k \). The vector \( \zeta = (\zeta_k)_{1 \leq |k| \leq N-1} \) is an element in the space

\[
\mathbb{Z} := \{ \zeta = (\zeta_k)_{1 \leq |k| \leq N-1} \in \mathbb{C}^{2N-2} : \zeta_{-k} = \zeta_{-k} \forall 1 \leq k \leq N-1 \}.
\]

The components of \( \zeta \) satisfy the identity

\[
e^{i\pi jk/N} \zeta_k + e^{-i\pi jk/N} \zeta_{-k} = \sqrt{2} \left( \cos \left( \frac{j\pi k}{N} \right) \zeta_k + \sin \left( \frac{j\pi k}{N} \right) \eta_k \right).
\]
As for any \(1 \leq k \leq N - 1\), \(u_k\) is a linear function of the \(b_j\)'s by \(12\) one has
\[
u_k = \sum_{j=1}^N b_j(\zeta) + \sum_{j=1}^k b_j(\zeta) \quad (1 \leq k \leq N - 1)
\]
where \(b_j(\zeta) (1 \leq j \leq N)\) can be computed using formula \(24\) and the Jacobian of \((\Phi|_{M,\beta,\alpha})^{-1}\) at \((x, y) = (0, 0)\) obtained in \(17\) to get, for \(1 \leq j \leq N\),
\[
u_j(\zeta) = \sqrt{\alpha/N} \sum_{1 \leq |k| \leq N-1} \lambda_k \lambda_{k'} \left( \sum_{l=0}^{N-1} e^{2\pi il(k+k')/N} \zeta_k \zeta_{k'} \right).
\]
Using that \(\sum_{l=0}^{N-1} e^{2\pi ilk/N} = N\delta_{k0}\) for any \(0 \leq |k| \leq N - 1\), one gets an expression which is quadratic in the \(\zeta\)-variables,
\[
u_k(\zeta) = \alpha \sum_{k=1}^{N-1} \lambda_k^2 \zeta_k \zeta_{-k} - k.
\]
Now let us turn to \(H_v\) given by formula \(20\). As \((b, a) = (\beta 1_N, \alpha 1_N)\) we have by \(13\),
\[
u_k = -\frac{2}{M} a_k(\zeta) + \frac{2}{N} \sum_{j=1}^N a_j(\zeta)
\]
where \( a_j(\zeta) (1 \leq j \leq N) \) can be computed using formula \( 24 \) and the Jacobian \( d_{x,y,\beta,\alpha} \Phi^{-1} \) at \((x, y) = (0, 0)\) obtained in \( 17 \) to get
\[
a_j(\zeta) = -\frac{1}{2} \sqrt{\frac{\alpha}{N}} \sum_{k=1}^{N-1} \lambda_k \left( e^{i\pi(2j-1)k/N} \zeta_k + e^{-i\pi(2j-1)k/N} \zeta_{-k} \right)
\]
\[
= -\frac{1}{2} \sqrt{\frac{\alpha}{N}} \sum_{1 \leq |k| \leq N-1} \lambda_k e^{i\pi(2j-1)k/N} \zeta_k.
\]
Again we have that \( \sum_{j=1}^{N} a_j(\zeta) = 0 \) as for any \( 1 \leq |k| \leq N - 1, \)
\[
\sum_{j=1}^{N} e^{i\pi(2j-1)k/N} = e^{-i\pi k/N} \left( \sum_{j=1}^{N} e^{2\pi i j k/N} \right) = 0.
\]
Hence, for \( 1 \leq l \leq N - 1, \)
\[
v_l = \frac{1}{\sqrt{\alpha N}} \sum_{1 \leq |k| \leq N-1} \lambda_k e^{2\pi i k/N} e^{-i\pi k/N} \zeta_k.
\]
Define \( v_0 \) by the expression on the right hand side of \( 29 \) evaluated at \( l = 0. \)
Note that
\[
\sum_{l=0}^{N-1} v_l = 0. \sum_{l=0}^{N-1} e^{2\pi i l k/N} = N \delta_{k0} \text{ for any } 0 \leq |k| \leq N - 1.
\]
The terms of third and fourth order in \( H_v \) are treated similarly.

Combining the above computations leads to
Proposition 4.1. Let \( \beta \in \mathbb{R} \) and \( \alpha > 0 \). Then
\[
H_{\beta, \alpha} \circ R_{\beta, \alpha}(\zeta) = G_0 + \alpha G_2 + \sqrt{\alpha} G_3 + G_4 + O(\zeta^5)
\]
(32)
where \( R_{\beta, \alpha} \) is the linear canonical transformation introduced in (22) and \( G_i \) (\( 0 \leq i \leq 4 \)) are given by
\[
G_0 := \frac{N \beta^2}{2} + N\alpha^2,
\]
(33)
\[
G_2 := 2 \sum_{k=1}^{N-1} \lambda_k^2 \zeta_k \zeta_{-k},
\]
(34)
\[
G_3 := -\frac{1}{6\sqrt{N}} \sum_{1 \leq |k|, |k'|, |k''| \leq N-1, k+k'+k'' \equiv 0 \mod N} (-1)^{(k+k''/N) \lambda_k \lambda_{k'} \lambda_{k''} \zeta_k \zeta_{k'} \zeta_{k''},}
\]
(35)
\[
G_4 := \frac{1}{24N} \sum_{1 \leq |k|, |k'|, |k''|, |k'''| \leq N-1, k+k'+k''+k''' \equiv 0 \mod N} (-1)^{(k+k''+k''/N) \lambda_k \lambda_{k'} \lambda_{k''} \lambda_{k'''}, \zeta_k \zeta_{k'} \zeta_{k''} \zeta_{k'''}}.
\]
(36)

Note that \( H_{\beta, \alpha} \circ R_{\beta, \alpha}(\zeta) \) depends on \( \beta \) only through the constant term \( \frac{N \beta^2}{2} \) and that it is in Birkhoff normal form up to order 2.

To finish this section let us express the Birkhoff coordinates \((x, y)\) in terms of the coordinates \((\xi, \eta)\) near the origin. The two coordinate systems are related by
\[
(x, y) = (\Omega_{\beta, \alpha})^{-1} d_{0,0} \Omega_{\beta, \alpha}(\xi, \eta)
\]
where we used that \( R_{\beta, \alpha} = d_{0,0} \Omega_{\beta, \alpha} \). Hence
\[
(x_k, y_k) = (\xi_k, \eta_k) + O(||(\xi, \eta)||^2) \quad \forall 1 \leq k \leq N - 1
\]
(37)
and
\[
I_k = \frac{x_k^2 + y_k^2}{2} = \frac{\xi_k^2 + \eta_k^2}{2} + O(||(\xi, \eta)||^3) \quad \forall 1 \leq k \leq N - 1.
\]

Denote by \( C^{\omega}_{\beta, \alpha} \) the Poisson algebra of \( H_{\beta, \alpha} \circ R_{\beta, \alpha} \), i.e. the space of germs of real analytic functions \( F(\xi, \eta) = \sum_{\gamma + |\delta| \geq 2} f_{\gamma, \delta} \xi^\gamma \eta^\delta \) such that \( \{ F, I_k \} = 0 \) for any \( 1 \leq k \leq N - 1 \). In view of (37) we say that \( C^{\omega}_{\beta, \alpha} \) is non-resonant. The following result is then well known (see e.g. [7], Appendix G).

Corollary 4.2. For any \( \beta \in \mathbb{R} \), \( \alpha > 0 \), and \( m \geq 3 \), there exists a (germ of a) real analytic canonical transformation of the form “Id + higher order terms” given by \( X_{F_m}^t |_{t=1} \), where \( X_{F_m}^t \) is the flow of the Hamiltonian vector field associated to the Hamiltonian
\[
F_m(\xi, \eta) = \sum_{3 \leq |\gamma| + |\delta| \leq m} f_{\gamma, \delta} \xi^\gamma \eta^\delta,
\]
such that \( F \circ X_{F_m}^t |_{t=1} \) is in Birkhoff normal form up to order \( m \) for any \( F \) in \( C^{\omega}_{\beta, \alpha} \).
5 Proof of Theorem 1.1

We now transform the Hamiltonian $H_{\beta,\alpha} \circ R_{\beta,\alpha}(\zeta)$ into its Birkhoff normal form up to order 4 by a standard procedure - see e.g. section 14 in [7]. The phase space $Z$, defined in (23), is endowed with the Poisson bracket

\[ \{F, G\} = \sum_{1 \leq |k| \leq N-1} \sigma_k \frac{\partial F}{\partial \zeta_k} \frac{\partial G}{\partial \zeta_{-k}}, \]

where $\sigma_k = \text{sgn}(k)$ is the sign of $k$. The Hamiltonian vector field $X_F$ associated to the Hamiltonian $F$ is then given by $X_F = \sum_{1 \leq |k| \leq N-1} \sigma_k \frac{\partial H}{\partial \zeta_k} \frac{\partial}{\partial \zeta_{-k}}$. With a first canonical transformation we want to eliminate the third order term $G_3$ in the expansion (32) of $H_{\beta,\alpha} \circ R_{\beta,\alpha}(\zeta)$. We construct such a canonical transformation on the phase space $Z$ as the time-1-map $\Psi_1 = X_t^F|_{t=1}$ of the flow $X_t^F$ of a real analytic Hamiltonian $F = \alpha^{-1/2}F_3$ which is a homogeneous polynomial in $\zeta_k$ ($1 \leq |k| \leq N-1$) of degree 3 and solves the following homological equation

\[ \{\alpha G_2, F\} + \alpha \frac{\partial F}{\partial \zeta_3} = 0. \] (38)

To simplify notation we momentarily write $H$ instead of $H_{\beta,\alpha} \circ R_{\beta,\alpha}$. Assuming for the moment that (38) can be solved and that $X_t^F$ is defined for $0 \leq t \leq 1$ in some neighbourhood of the origin in $Z$ we can use Taylor’s formula to expand $H \circ X_t^F$ around $t = 0$,

\[ H \circ X_t^F = H \circ X_0^F + \int_0^t \frac{d}{ds}(H \circ X_s^F)ds = H + \int_0^t \{H, F\} \circ X_s^F ds = H + t \{H, F\} + \int_0^t (t-s)\{\{H, F\}, F\} \circ X_s^F ds. \] (39)

When evaluating this expression at $t = 1$, one gets

\[ H \circ \Psi_1 = G_0 + \alpha G_2 + \{\alpha G_2, F\} + \int_0^1 (1-t)\{\{\alpha G_2, F\}, F\} \circ X_t^F dt + \sqrt{\alpha} G_3 + \int_0^1 \{\sqrt{\alpha} G_3, F\} \circ X_t^F dt + G_4 + O(\zeta^5). \]

Using that $\{\alpha G_2, F\} + \sqrt{\alpha} G_3 = 0$, the latter expression simplifies and we get

\[ H \circ \Psi_1 = G_0 + \alpha G_2 + \int_0^1 t \{\sqrt{\alpha} G_3, F\} \circ X_t^F dt + G_4 + O(\zeta^5). \]

Integrating by parts once more and taking into account that $F = \alpha^{-1/2}F_3$ is homogeneous of degree 3 one obtains, in view of (39),

\[ H_{\beta,\alpha} \circ R_{\beta,\alpha} \circ \Psi_1 = G_0 + \alpha G_2 + G_4 + \frac{1}{2} \{G_3, F_3\} + O(\zeta^5). \] (40)
Note that \( \{G_3, F_3\} \) is homogeneous of order 4. Hence our first step is achieved. It remains to solve \( G_3, F_3 \).

By Corollary 4.2 with \( m = 3 \) there exists a polynomial homogeneous of degree 3,

\[
W_3 = \sum_{1 \leq |k|, |k'| , |k''| \leq N-1} W_{kk'k''}^{(3)} \zeta_k \zeta_{k'} \zeta_{k''},
\]

(with \( W_{kk'k''}^{(3)} \) invariant under permutations of \( k, k', k'' \)) so that the time-1-map \( X_W|_{t=1} \) of the flow \( X_W^t \) corresponding to the Hamiltonian \( W = \alpha^{-1/2} W_3 \) brings any Hamiltonian in \( G_{\beta, \alpha} \) into Birkhoff normal form up to order 3. In particular, the identity \( \{G_2, W_3\} + G_3 = 0 \) is satisfied. Note that

\[
\{G_2, W_3\} = i \sum_{1 \leq |k| \leq N-1} 2\sigma_k \nu_k^2 \zeta_k \frac{\partial W_3}{\partial \zeta_k} = -i \sum_{1 \leq |k|, |k'|, |k''| \leq N-1} (s_k + s_{k'} + s_{k''}) W_{kk'k''}^{(3)} \zeta_k \zeta_{k'} \zeta_{k''}
\]

as \( s_k = 2 \sin \frac{k\pi}{N} = 2\sigma_k \lambda_k^2 \), and it follows that

\[
(s_k + s_{k'} + s_{k''}) i W_{kk'k''}^{(3)} = G_{kk'k''}^{(3)}
\]

where \( G_{kk'k''}^{(3)} \) are the coefficients of \( G_3 \) defined by (35).

\[
G_{kk'k''}^{(3)} = \begin{cases} 
\frac{-1}{(6\sqrt{N})(-1)^{(k+k'+k'')/N}} \lambda_k \lambda_{k'} \lambda_{k''} & \text{for } k+k'+k'' \equiv 0 \text{ mod } N \\
0 & \text{otherwise.}
\end{cases}
\]

As \( G_{kk'k''}^{(3)} \neq 0 \) if \( k + k' + k'' \equiv 0 \text{ mod } N \) it follows that for any triple \( (k, k', k'') \) with \( k + k' + k'' \equiv 0 \text{ mod } N \),

\[
s_k + s_{k'} + s_{k''} \neq 0
\]

and

\[
i W_{kk'k''}^{(3)} = \frac{G_{kk'k''}^{(3)}}{s_k + s_{k'} + s_{k''}}.
\]

Now define \( F_3 = \sum_{1 \leq |k|, |k'| , |k''| \leq N-1} F_{kk'k''}^{(3)} \zeta_k \zeta_{k'} \zeta_{k''} \) and \( F = \alpha^{-1/2} F_3 \) by

\[
F_{kk'k''}^{(3)} = \begin{cases} 
W_{kk'k''}^{(3)} & \text{for } k + k' + k'' \equiv 0 \text{ mod } N \\
0 & \text{otherwise.}
\end{cases}
\]

Then clearly \( \{G_2, F_3\} + G_3 = 0 \) and hence by the considerations above \( \Psi_3 := X^t_{\beta, \alpha}[\cdot] = 1 \) has the property that \( H_{\beta, \alpha} \circ R_{\beta, \alpha} \circ \Psi_1 \) satisfies identity (10).

Now let us investigate the 4th order term \( G_4 + \frac{1}{2} \{G_3, F_3\} \) in (10). We decompose this sum into its contribution to the Birkhoff normal form and the

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3See (13) (cf. also (6)) for a direct proof of this nonresonance condition.
rest, to be transformed away in a moment. Let us first compute \( \{G_3,F_3\} \) in a more explicit form. By (35) and (42),

\[
\{G_3,F_3\} = \sum_{1 \leq |k| \leq N-1} \sigma_k \frac{\partial G_3}{\partial k} \frac{\partial F_3}{\partial \zeta_k}
\]

\[
= \frac{1}{36N} \sum_{1 \leq |k| \leq N-1} \sigma_k \left( \sum_{1 \leq |l'|,|m'| \leq N-1, l'+m' \equiv k \pmod{N}} 3 (-1)^r \lambda_k \lambda_{l'} \lambda_{m'} \zeta_l \zeta_{l'} \zeta_m \zeta_{m'} \right)
\]

\[
\cdot \left( \sum_{1 \leq |l'|,|m'| \leq N-1, l'+m' \equiv k \pmod{N}} (-1)^{s_k} \lambda_{l'} \lambda_{m'} \lambda_l \lambda_m \zeta_{l'} \zeta_{l'} \zeta_m \zeta_{m'} \right)
\]

\[
= \frac{1}{8N} \sum_{1 \leq |k| \leq N-1} \sum_{1 \leq |l'|,|m'| \leq N-1, l'+m' \equiv k \pmod{N}} \frac{(-1)^{l+m+m'}}{1 + (s_{l'} + s_{m'})/s_k} \zeta_{l'} \zeta_{l'} \zeta_m \zeta_{m'}
\]

where for the latter equality we used again that \( 2\sigma_k \lambda_k^2 = s_k \). Setting

\[
\varepsilon_{l+m+m'} = \frac{l + m + l' + m'}{N}
\]

and using that \( s_{-k} = -s_k \) one then gets

\[
\{G_3,F_3\} = \frac{1}{8N} \sum_{1 \leq |k| \leq N-1} \sum_{l+m \equiv k \pmod{N}} \frac{(-1)^{l+m+m'}}{1 + (s_{l'} + s_{m'})/s_k} \zeta_{l'} \zeta_{l'} \zeta_m \zeta_{m'}
\]

\[
= \frac{1}{8N} \sum_{k=1}^{N-1} \frac{1}{l+m \equiv k \pmod{N}} (-1)^{l+m+m'} \frac{\lambda_l \lambda_m \lambda_{l'} \lambda_{m'} \zeta_{l'} \zeta_{l'} \zeta_m \zeta_{m'}}{1 + (s_{l'} + s_{m'})/s_k}
\]

\[
+ \frac{1}{8N} \sum_{k=1}^{N-1} \frac{1}{l+m \equiv k \pmod{N}} (-1)^{l+m+m'} \frac{\lambda_{l'} \lambda_{m'} \lambda_l \lambda_m \zeta_{l'} \zeta_{l'} \zeta_m \zeta_{m'}}{1 - (s_{l'} + s_{m'})/s_k}
\]

\[
= \frac{1}{8N} \sum_{k=1}^{N-1} \frac{1}{l+m \equiv k \pmod{N}} \left( \frac{1}{1 + (s_{l'} + s_{m'})/s_k} + \frac{1}{1 - (s_{l'} + s_{m'})/s_k} \right)
\]

\[
\cdot (-1)^{l+m+m'} \lambda_l \lambda_m \lambda_{l'} \lambda_{m'} \zeta_{l'} \zeta_{l'} \zeta_m \zeta_{m'}
\]

where \( \ldots \) stand for \( 1 \leq |l'|,|m'|,|l'|,|m'| \leq N-1 \). Note that for \( k = l' + m' + r' N \) with \( 1 \leq k \leq N-1 \) and \( r' \in \mathbb{Z} \) we have

\[
s_k = |s_{l'}|.
\]

Hence

\[
\frac{1}{2} \{G_3,F_3\} = \frac{1}{24N} \sum_{l+m+l'+m' \equiv 0 \pmod{N}} \varepsilon_{l+m+m'} (-1)^{l+m+m'} \lambda_l \lambda_m \lambda_{l'} \lambda_{m'} \zeta_{l'} \zeta_{l'} \zeta_m \zeta_{m'}
\]

(43)
Lemma 5.1. The normal form part of $H_c(47)$ for which in addition $c_{klm'} \neq 0 \bmod N$.

We now decompose (45) into its contribution $π_c$ and the rest.

Lemma 5.1. The normal form part of $G_4 + \frac{1}{2} \{G_3, F_3\}$ is given by

$$π_N \left( G_4 + \frac{1}{2} \{G_3, F_3\} \right) = \frac{1}{4N} \sum_{l=1}^{N-1} |ζ_l|^4. \quad (46)$$

Proof. The indices $k, k', k''$ of the terms in (45) contributing to the normal form satisfy the condition

$$\{k, k', k''\} = \{l, -l, m, -m\} \quad (47)$$

with $1 \leq l \leq m \leq N - 1$. In the case $l = m$, $\{l, -l, l, -l\}$ is viewed as a set-like object whose two elements $l$ and $-l$ each have multiplicity two.

We investigate $π_N(G_4)$ and $π_N(\frac{1}{2} \{G_3, F_3\})$ separately. Let us start with $G_4$. We distinguish the cases $l = m$ and $l \neq m$ in (47). For $l = m$, there are (4) = 6 distinct combinations of indices $(k, k', k'', k''')$ satisfying (47), whereas for $l \neq m$, there are $4l = 24$ (automatically distinct) permutations of $(l, m, -l, -m)$. Hence we have

$$π_N(G_4) = \frac{1}{24N} \left( 6 \sum_{l=1}^{N-1} λ_l^4 |ζ_l|^4 + 24 \sum_{1 \leq l < m \leq N-1} λ_l^2 λ_m^2 |ζ_l|^2 |ζ_m|^2 \right)$$

$$= \frac{1}{4N} \left( \sum_{l=1}^{N-1} λ_l^4 |ζ_l|^4 + 4 \sum_{1 \leq l < m \leq N-1} λ_l^2 λ_m^2 |ζ_l|^2 |ζ_m|^2 \right). \quad (48)$$

Now let us compute $π_N(\frac{1}{2} \{G_3, F_3\})$. We have to single out the matches of (47) for which in addition $c_{klml'} \neq 0$, i.e.

$$k + k' \neq 0 \bmod N, \ k + k' + k'' + k''' \equiv 0 \bmod N$$

To keep the formulas as simple as possible we have not symmetrized the coefficients $c_{klm'l'}$.

\[\text{where}^{4}\]

$$c_{lmml'} = \begin{cases} \frac{\lambda_l}{N} \left( \frac{1}{1 + λ_l^2 (|l| + |m|)} \right) \left( \frac{1}{1 - 1 + λ_l^2 (|l| + |m|)} \right) & \text{if } 1 \leq |l|, |m|, |l'|, |m'| \leq N - 1 \text{ and } \frac{l + m}{2} \notin \frac{N}{2} \bmod N, \ l + m + l' + m' \equiv 0 \bmod N \\ 0 & \text{otherwise} \end{cases} \quad (44)$$

Combined with formula (30) for $G_4$, the sum $G_4 + \frac{1}{2} \{G_3, F_3\}$ equals

$$\frac{1}{24N} \sum_{1 \leq |k|, |k'|, |k''|, |k'''| \leq N - 1} \sum_{k + k' + k'' + k''' = 0 \bmod N} (-1)^{c_{kkk''k'''}} (1 + c_{kkk''k'''}) \cdot λ_k λ_{k'} λ_{k''} λ_{k'''} ζ_k ζ_{k'} ζ_{k''} ζ_{k'''}. \quad (45)$$

We now decompose (15) into its contribution $π_N(G_4 + \frac{1}{2} \{G_3, F_3\})$ to the Birkhoff normal form of $H_{β, α} = R_{β, α}$ and the rest.

5 Proof of Theorem ??
To fulfill (47), there are therefore the two possibilities

\[ k + k'' = 0 \quad \text{or} \quad k + k''' = 0 \]

(49)

In both cases, we have \( s_{k''} + s_{k'''}' = -(s_k + s_{k'}) \), and therefore (44) reduces to

\[ c_{kk',k',k''} = \frac{-3|s_{k+k'}|}{|s_{k+k'}| + s_k + s_{k'}}. \]

(50)

Note that (50) remains valid for \( k + k' = N \), since in this case \( s_{k+k'} = 0 \) and \( s_k + s_{k'} > 0 \) as \( k \) and \( k' \) must satisfy \( 1 \leq k, k' \leq N - 1 \), but not for \( k + k' = 0 \), since in this case \( |s_{k+k'}| + s_k + s_{k'} = 0 \).

We first compute the diagonal part of \( \pi_N \left( \frac{1}{4} \{G_3, F_3\} \right) \). In this case, the two possibilities in (49) coincide and the solutions are

\[ (k, k', k'', k''') = \begin{cases} (l, l, -l, -l) \\ (-l, -l, l, l) \end{cases}. \]

(51)

The sum of the coefficients \( c_{kk',k',k''} \) for the two cases listed in (51) is

\[ c_{l,l,-l,-l} + c_{-l,-l,l,l} = -3|s_{2l}| \left( \frac{1}{|s_{2l}| + 2s_l} + \frac{1}{|s_{2l}| - 2s_l} \right) = -\frac{6s_{2l}^2}{s_{2l}^2 - 4s_l^2} = 6 \cot \frac{2\pi}{N}. \]

We now turn to the off-diagonal part of \( \pi_N \left( \frac{1}{4} \{G_3, F_3\} \right) \). The matches of (47), (49) for given \( \{l, m\} \subseteq \{1, \ldots, N-1\} \) with \( l < m \), \( (k, k') = (\pm l, \pm m) \) and \( (k'', k''') = (\pm l, \pm m) \) are

\[ (k, k', k'', k''') = \begin{cases} (l, m, -l, -m) \\ (l, -m, -l, m) \\ (-l, -m, l, m) \end{cases}. \]

(52)

The remaining matches are obtained from (52) by permuting the first and second or the third and fourth columns on the right hand side of (52), bringing the total of all matches to 16 = 4 · 4. Note that by formula (51), these permutations leave the value of the coefficients \( c_{kk',k',k''} \) invariant. Taking the sum of the coefficients \( c_{kk',k',k''} \) for all the matches, we obtain from (50)

\[ 4(c_{l,m,-l,-m} + c_{l,-m,-l,m} + c_{-l,m,-l,-m} + c_{-l,-m,l,l}) \]

\[ = -12 \left( \frac{|s_{l+m}|}{|s_{l+m}| + s_l + s_m} + \frac{|s_{l-m}|}{|s_{l-m}| + s_l - s_m} + \frac{|s_{l-m}|}{s_{l-m} + s_l - s_m} + \frac{|s_{l+m}|}{s_{l+m} - s_l - s_m} \right) \]

\[ = -24 \left( \frac{s_{l-m}^2}{s_{l-m}^2 - (s_l - s_m)^2} + \frac{s_{l+m}^2}{s_{l+m}^2 - (s_l + s_m)^2} \right) \]

\[ = -24 \left( \frac{2s_{l-m}^2s_{l+m}^2 - s_{l-m}^2(s_l + s_m)^2 - s_{l+m}^2(s_l - s_m)^2}{s_{l-m}^2s_{l+m}^2 + (s_l - s_m)^2(s_l + s_m)^2 - s_{l-m}^2s_{l+m}^2} \right) \]

\[ = -24. \]
since \( s_{l-m}^2 s_{l+m}^2 = (s_l - s_m)^2 (s_l + s_m)^2 \). Collecting terms, we thus have

\[
\pi_N \left( \frac{1}{2} \{G_3, F_3\} \right) = \frac{1}{24N} \sum_{l=1}^{N-1} 6 \cos \frac{\pi l}{N} |\zeta|^4 - 24 \sum_{1 \leq l < m \leq N-1} \lambda^2_{l,m} |\zeta_l|^2 |\zeta_m|^2 \]

Adding up (44) and (49), we obtain (40).

In a next step we want to remove \( (1 - \pi_N) \left( G_4 + \frac{1}{2} \{G_3, F_3\} \right) \) from the Hamiltonian by a second coordinate transformation \( \Psi_2 \). In view of Corollary 4.2 with \( m = 4 \), there exists a canonical transformation \( \Psi_2 \) of the form “Id + higher order terms” so that for any Hamiltonian \( F \) in the Poisson algebra \( \Psi_1^* C^\infty_\beta,\alpha \), \( F \circ \Psi_2 \) is in Birkhoff normal form up to order 4. We have proved the following

**Proposition 5.2.** Let \( \beta \in \mathbb{R} \) and \( \alpha > 0 \) be given. The real analytic symplectic coordinate transformation \( \zeta = \Xi(z) = \Psi_1 \circ \Psi_2(z) \) defined in a neighborhood of the origin in \( \mathbb{C} \), transforms the Hamiltonian \( H_{\beta,\alpha} \circ R_{\beta,\alpha} \) into its Birkhoff normal form up to order 4. More precisely,

\[
H_{\beta,\alpha} \circ R_{\beta,\alpha} \circ \Xi = G_0 + \alpha G_2 + \pi_N \left( G_4 + \frac{1}{2} \{G_3, F_3\} \right) + O(z^5), \tag{54}
\]

with \( G_0, G_2 \) and \( \pi_N(G_4 + \frac{1}{2} \{G_3, F_3\}) \) defined by (45), (51), and (40).

**Proof of Theorem 1.1** The map \( \Omega_{\beta,\alpha} : \mathcal{T}_{\beta,\alpha} \rightarrow \mathbb{R}^{2N-2} \) introduced in (21) is a canonical real analytic diffeomorphism so that \( H_{\beta,\alpha} \circ \Omega_{\beta,\alpha} = (H_{\beta,\alpha} \circ R_{\beta,\alpha}) \circ \Omega_{-1} \circ \Omega_{\beta,\alpha} \) is in Birkhoff normal form. By the definition (22), the map \( R_{-1} \circ \Omega_{\beta,\alpha} \) is canonical and, near the origin, of the form “Id + higher order terms”. The canonical transformation \( \Xi \) of Proposition 5.2 is also of the form “Id + higher order terms” and brings \( H_{\beta,\alpha} \circ R_{\beta,\alpha} \) locally around the origin into Birkhoff normal form up to order 4. Hence the transformations \( R_{-1} \circ \Omega_{\beta,\alpha} \) and \( \Xi \) differ, near the origin, by a transformation of the form “Id + higher order terms”. By the uniqueness of the Birkhoff normal form, the expansions of the Toda Hamiltonian near the origin, when expressed in these two coordinate systems, coincide up to order 4. Hence Proposition 5.2 provides us with the Taylor series expansion of \( H_{\beta,\alpha}(I) = \frac{N\beta^2}{2} + H_\alpha(I) \) in terms of the actions

\[
I = (I_k)_{1 \leq k \leq N-1}, \quad I_k = \frac{x_k^2 + y_k^2}{2}
\]

up to order 2. In view of (45), (51), and (40) one has

\[
H_\alpha(I) = N \alpha^2 + \alpha \sum_{k=1}^{N-1} s_k I_k + \frac{1}{4N} \sum_{k=1}^{N-1} f_k^2 + O(f^3).
\]

This proves Theorem 1.1. \( \square \)
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