Interpretation of Feynman´s formalism of quantum mechanics in terms of probabilities of paths

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Abstract

Feynman´s path integrals formalism for non-relativistic quantum mechanics is revisited. A comparison is made with the cases of light propagation (Huygens principle) and Brownian motion. The difficulties for a physical model behind Feynman´s formalism are pointed out. It is proposed a reformulation where the transition probability from one space-time point to another one is the sum of probabilities of the possible paths. The Born approximation for scattering is derived within the formalism, which suggests an interpretation in terms of particles, without the need of Born´s assumption that the modulus squared of the wavefunction is a probability density.

Keywords: path integrals, Born approximation, models of quantum physics, Huygens principle, Brownian motion

1 Feynman formulation of quantum mechanics

The path integral formulation of quantum mechanics was introduced by Feynman in 1948[1]. In his book on the subject[2] the formalism is shown to be a straightforward consequence of the superposition principle. In fact, let us assume that there is a source of particles at point $x = x_0, y = 0$, a screen with two slits at $x = \pm a, y = b$, and the particles may be detected at any point in the plane $y = c$. (For simplicity I ignore here the third coordinate, $z$). If a
particle leaves the source at time \( t = 0 \), crosses a slit at \( t_1 \) and arrives at the detector at \( t_2 \), the probability, \( P(x) \), of reaching a point with coordinate \( x \) in the screen \( y = c \) is proportional to the modulus squared of a “probability amplitude”, and the latter is the sum of two amplitudes, that is

\[
P(x) \propto |A(x_0, 0 | x, t_2)|^2, \quad A(x_0, 0 | x, t_2) = A(x_0, 0 | a, t_1)A(a, t_1 | x, t_2) + A(x_0, 0 | -a, t_1)A(-a, t_1 | x, t_2).
\]  

(1)

Now we consider the case of having many screens at positions \( y_1, y_2, \ldots \), where the particle may arrive at times \( t_1, t_2, \ldots \), respectively, and that every screen possesses many slits, at positions \( x_1, x_2, \ldots \). In this case the amplitude eq.(1) is replaced by

\[
A(x_0, 0 | x, t) = \sum_j \sum_k \cdots \sum_q A(x_0, 0 | x_j, t_1)A(x_j, t_1 | x_k, t_2)\ldots A(x_q, t_n | x, t).
\]

(2)

Here the set of positions \( \{x_0, x_j, x_k, x_q, x\} \) may be called a path, so that the amplitude \( A(x_0, 0 | x, t) \) is a sum of amplitudes, every one corresponding to one possible path. If the number of slits in every screen increases indefinitely, at the end there will be no screen at all. Then the discrete values \( x_1, x_k, \ldots \) become continuous and the sums become integrals, that is

\[
A(x_0, 0 | x, t) = \int dx_1 \int dx_{n-1}A(x_0, 0 | x_1, t_1)\ldots A(x_{n-1}, t_{n-1} | x, t).
\]

(3)

The time intervals may be chosen identical, that is \( t_{j+1} - t_j = \varepsilon \), with \( \varepsilon \) as small as desired. In the limit \( \varepsilon \to 0 \), \( A(x_0, 0 | x, t) \) becomes an integral of path amplitudes.

The derivation shows that eq.(3) is not specific of Feynman’s formulation of quantum mechanics, but it may be valid for any field fulfilling a superposition property, e.g. classical optics. In this case, however, \( P(x) \) of eq.(1) would not be a probability but a light intensity, as will be illustrated below with Huygen’s principle. Also, an expression formally similar to eq.(2) may be used in the study of random motion when the probability that a particle goes from one position to another one in some time interval is the sum of probabilities corresponding to different paths, a method pioneered by Norbert Wiener in the study of Brownian motion. This will be illustrated below with a derivation of the diffusion law via a path integral. In this case the probability density is given directly, rather than via the modulus squared of
an amplitude. More generally, a path integrals formulation might be associated to the evolution of any physical system if it is governed by a linear partial differential equation.

What is specific of Feynman’s formulation of (non-relativistic) quantum mechanics is the choice of \( A \) to be the exponential of \( \frac{i}{\hbar} \) times the (classical) Lagrangian, \( L \), of the particle’s motion, an idea taken from Dirac\(^3\). For instance in the case of one-dimensional motion in a potential \( V(x) \) we have

\[
L = \frac{1}{2} m x^2 - V(x)
\]

whence

\[
A(x_{j-1}, t_{j-1} \mid x_j, t_j) = \sqrt{\frac{m}{2\pi i\hbar \varepsilon}} \exp\left(\frac{i\varepsilon}{\hbar} L\right),
\]

(4)

where \( m \) is the mass of the particle. (This expression differs from the original one of Feynman\(^2\) because I have substituted \( \frac{1}{2} [V(x_{j-1}) + V(x_j)] \) for \( V[(x_{j-1} + x_j)/2] \) for later convenience. Also I ignore the terms \( V(x_0) \) and \( V(x) \), so that the potential enters \( n - 1 \) times in eq. (3), as it should. Both formulations agree in the limit \( \varepsilon \to 0 \).

It is common to take the continuous limit and write eq. (2) in the form

\[
A(x_0, 0 \mid x, t) = \int \mathcal{D}(\text{paths}) \exp\left[ \frac{i}{\hbar} \int_0^t dt L(x, \dot{x}) \right]
= \int \mathcal{D}(\text{paths}) \exp\left[ \frac{i}{\hbar} \int_0^t dt \left( \frac{1}{2} m \dot{x}^2 - V(x) \right) \right].
\]

(5)

This symbolic equation actually means the limit \( \varepsilon \to 0 \) of eq. (3) with \( A \) given by eq. (4). The amplitude eq. (5) may be calculated explicitly for simple potentials\(^2\).

The amplitude \( A(x_0, 0 \mid x, t) \) is named the “propagator”. The interesting property is that the propagator allows getting the wavefunction at time \( t \) from the wavefunction at time 0, that is\(^2\)

\[
\psi(x, t) = \int dx_0 \psi(x_0, t) A(x_0, 0 \mid x, t).
\]

(6)

Hence it follows that the propagator fulfils the Schrödinger equation with the initial condition

\[
A(x_0, 0 \mid x, 0) = \delta(x - x_0),
\]

3
where $\delta(x)$ is Dirac’s delta. Thus the propagator is the Green’s function of the Schrödinger equation.

As is well known the path integrals formulation may be generalized to 3 dimensions, to many-particles quantum mechanics and also to relativistic field theory. It has an extremely important role in modern theoretical physics, both because it is well adapted to derive general properties, e. g. symmetries, and due to the relevance for actual calculations, as shown by the use of Feynman graphs in covariant perturbation theory\cite{4}. However dealing with formal and calculational aspects lies out of the scope of this article, which is devoted to the physical interpretation of the Feynman formalism in (non-relativistic) quantum mechanics. In particular I will present below a possible interpretation of the path integrals formalism, but before I will revisit the use of a formula similar to eq.(3) in classical optics and diffusion theory in order to make a comparison with the Feynman formalism.

## 2 Huygens principle

Historically the first formulation of a physical theory in terms of “path integrals” goes back to Christiaan Huygens, more than three centuries ago. In fact Huygens proposed that light are waves, in opposition to his contemporary Isaac Newton who supported a corpuscular theory. He was able to explain the straight line propagation and other properties of light from his celebrated principle. Huygens principle states that light propagation may be understood as if every point where light arrives becomes the source of a spherical wave, and the waves coming from different points are able to interfere. In practice this implies that from each point of a given wavefront at time $t$ spherical wavelets originate. Thus Huygens principle may be formalized stating that the light arriving at time $t$ at a point $\mathbf{r}$ may be calculated from the three-dimensional generalization of eq.(3) with appropriate transition amplitudes $A (\mathbf{r}_j, t_j \mid \mathbf{r}_{j+1}, t_{j+1})$. At a difference with the cases of Feynman’s formalism, eq.(5), or diffusion (see below eq.(13)), where the velocities of the particles may have any value, light travels in vacuum with a fixed velocity, $c$, which puts the constraint $|\mathbf{r}_{j+1} - \mathbf{r}_j| = c (t_{j+1} - t_j)$. Hence the transition amplitude should be of the form

$$A (\mathbf{r}_j, t_j \mid \mathbf{r}_{j+1}, t_{j+1}) = f (\mathbf{r}_j \mid \mathbf{r}_{j+1}) \delta (|\mathbf{r}_{j+1} - \mathbf{r}_j| - c (t_{j+1} - t_j)). \quad (7)$$
Light consists of transverse waves (that is the electric and magnetic fields are perpendicular to the direction of propagation) which implies that the function $f$ should depend also on the angle between the electric field and the $\mathbf{r}_j - \mathbf{r}_{j+1}$ vectors. I shall avoid this complication, irrelevant for our purposes, considering in the following longitudinal waves, e.g. sound propagation in air.

For monocromatic sound waves in air the function $f$ of eq.(7) is specially simple, namely

$$f(r_j | r_{j+1}) = \frac{\exp(ik|\mathbf{r}_{j+1} - \mathbf{r}_j|)}{2\pi i |\mathbf{r}_{j+1} - \mathbf{r}_j|}.$$  \hspace{1cm} (8)

where the denominator takes into account that the intensity from a point source decreases as the inverse of the distance squared. I stress that here the use of complex amplitudes, i.e. the introduction of the imaginary unit number $i$, has no deep meaning, it is just a convenient mathematical procedure to simplify the calculations. Actually the wave amplitudes might be always represented by real numbers (indeed the light amplitude may be taken to be the electric field of the radiation and the amplitude of sound waves in air to be the excess pressure).

In practice it is most frequent to use very simple “paths” consisting of just three points, namely \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}\}. For instance a typical problem solved with that choice is the diffraction by a small hole (see any book of electromagnetic theory, e.g. [5]).

### 3 Diffusion

The path integral formulation is most intuitive in the case of diffusion. Let us consider the well known problem of the random walk in one dimension. It consists of a particle that travels a distance $\lambda$, either towards the left or towards the right with equal probability, the step taking a time $\varepsilon$. During the next time interval $\varepsilon$ the particle travels again a distance $\lambda$ either to the left or to the right with equal probability, and so on. The problem is to find the probability that the particle is at a given distance, say $x = l\lambda$, of the origin after $n$ steps, $n$ being a natural number and $l$ an integer, $|l| \leq n$. The
problem might be stated by means of a sum of paths as follows

\[ P(x_0, 0 \mid x, t) = \sum_j \ldots \sum_q P(x_0, 0 \mid x_j, \varepsilon)P(x_j, \varepsilon \mid x_k, 2\varepsilon) \ldots P(x_q, (n-1)\varepsilon \mid x, t), \]  

(9)

where the set of positions \( \{x_0, x_j, x_k, \ldots x_q, x\} \), represents a path. Remember that \( |x_j - x_0| = |x_k - x_j| = \ldots = \lambda \) and that the probabilities for every step are

\[ P(x_s, m\varepsilon \mid x_r, (m+1)\varepsilon) = \frac{1}{2} \text{ if } x_r - x_s = \lambda, P = \frac{1}{2} \text{ if } x_r - x_s = -\lambda. \]  

(10)

The solution may be found as follows. In order that the particle arrives at a distance \( l\lambda \) from the origin after \( n \) steps, it is necessary and sufficient that \( k \equiv (n+l)/2 \) steps are towards the right and \( n-k = (n-l)/2 \) towards the left (assuming that the positive direction is to the right.) The set of positions may be called the path of the particle. The probability of a given path is \((1/2)^n\) and the number of paths leading from 0 to \( l\lambda \) is the combinatorial number \( \binom{n}{k} \). As a consequence the probability of reaching \( l\lambda \) in \( n \) steps is

\[ P_{mL} = \binom{n}{k} \left(\frac{1}{2}\right)^n = \frac{n!}{k!(n-k)!} \left(\frac{1}{2}\right)^n = \frac{n!}{((n+l)/2)!\left((n-l)/2\right)!} \left(\frac{1}{2}\right)^n. \]

If \( n \) and \( m \) are very large, we may approximate the factorials by the Stirling formula, that is

\[ \log (k!) \simeq \frac{1}{2} \log (2\pi k) + k \log k - k. \]

and we get, ignoring terms which do not depend on \( l \),

\[ \log P_x \simeq \text{const.} - \frac{n+l}{2} \log \left(\frac{n+l}{2}\right) - \frac{n-l}{2} \log \left(\frac{n-l}{2}\right). \]

Hence I obtain

\[ P_x \propto (n+l)^{n+l/2} (n-l)^{n-l/2} \propto \left(1 - \frac{l^2}{n^2}\right)^{n/2} \left(\frac{n-l}{n+l}\right)^{m/2} \simeq \exp \left(-\frac{l^2}{2n}\right), \]

where I have taken into account that in most paths \( l << n \) for very large \( n \). In terms of the total time, \( t \equiv n\varepsilon \), and the final position \( x \equiv n\lambda \), this may be written as a probability density which properly normalized reads

\[ \rho(x_0, 0 \mid x, t) = \sqrt{\frac{1}{4\pi t}} \exp \left(-\frac{x^2}{4Dt}\right), \]  

(11)
the parameter $D \equiv \lambda^2/\varepsilon$ being the diffusion constant. For us the relevant conclusion is that the diffusion probability may be calculated as a sum of the probabilities corresponding to all paths leaving the origin at the initial time and arriving at the point $x$ at time $t$. The calculation is rather simple because all paths have the same probability.

We may also derive the diffusion probability assuming that in every step of duration $\varepsilon$ the particle may travel any distance, $\Delta x$, with a probability proportional to $\exp\left(-\frac{\Delta x^2}{2D\varepsilon}\right)$. Then the probability of reaching $x$ will be

$$P \propto \int dx_1 \int dx_2 \ldots \int dx_{n-1} \exp \left[-\frac{1}{4D} \sum_{k=0}^{n-1} \frac{(x_{k+1} - x_k)^2}{t_{m+1} - t_m}\right], x_n = x, \quad (12)$$

In the limit of large $n$, but fixed $t = n\varepsilon$, this may be written substituting integrals for the sums, that is

$$P \propto \int D(paths) \exp \left(\frac{1}{4D} \int_0^t dt \Delta x^2\right), \quad (13)$$

which is formally similar to the Feynman path integral eq.(5). There are however two fundamental differences. Firstly in the diffusion case eq.(13) gives directly the probability, rather than the amplitude, and secondly the quantity inside the exponential is real, so that the sum involves positive probabilities, rather than complex numbers. The integrals in eq.(12) are rather simple and the result is the probability density eq.(11), which is the Green’s function of the diffusion equation

$$\frac{\partial}{\partial t} \rho(0, 0 | x, t) = D \frac{\partial^2}{\partial x^2} \rho(0, 0 | x, t).$$

A technique similar to the one illustrated here for diffusion may be applied to any process where the probability of going from an initial state to a final one is the sum of the probabilities of the different paths. This is the case for any random motion, but the fact that the transition probability in a step does not depend on the previous positions is not general. If it is fulfilled the stochastic process is called Markovian. This property holds true in eq.(9) where the transition probabilities in different steps are statistically independent of each other.
4 Physical interpretation of path integrals in quantum mechanics

It is sometimes stated that Feynman’s path integral formalism provides an intuitive picture of the quantum mechanical evolution. I do not agree. The formalism of non-relativistic quantum mechanics is counterintuitive for two reasons. Firstly because Feynman paths are just sets of disconnected points, rather than (continuous) paths in the usual sense of the word. Secondly because in an intuitive picture the probability of travel of a particle between two points should involve a sum of probabilities not amplitudes. I comment on these two shortcomings in the following.

The alleged path represented by the successive positions \( \{x_0, x_1, x_2, \ldots, x_n, x\} \) is actually a set of points, each one separated from the previous one by a long distance. Indeed the position \( x_{j-1} \) may have values in the interval \((-\infty, \infty)\), which is the interval of integration, all values having the same weight according to the Riemann measure of the integral (see eq.\( \text{(3)} \)). Similarly the position \( x_j \) may have values in the interval \((-\infty, \infty)\) all values having the same weight. Thus the step \( u_j = x_j - x_{j-1} \) between two positions may be arbitrarily large. Indeed its mean squared value, \( \langle u_j^2 \rangle \), diverges if calculated with the Riemann measure. The counterintuitive character is enhanced by the fact that the (indefinitely long) step \( u_j \) takes place in an infinitesimal time interval \( \varepsilon \). Thus Feynman’s eq.\( \text{(3)} \) should be seen as a purely mathematical construction, an useful calculational tool where the physical meaning appears only in the final result of the calculation.

In sharp contrast Huygens’ path integrals, eqs.\( \text{(7)} \), are continuous. In fact the quantities \(|r_{j+1} - r_j|\) are never too large due to the denominator in eq.\( \text{(8)} \) and, above all, they decrease to zero when the time interval, \( t_{j+1} - t_j \), goes to zero. Thus the sum or integral involved is over continuous paths. A similar thing happens in the diffusion problem defined by eqs.\( \text{(9)} \) and \( \text{(10)} \). In this case every path is continuous although non-differentiable. The same is true if we define the path via eq.\( \text{(12)} \) where the probability that the mean velocity in a step, \( (x_{k+1} - x_k)/(t_{m+1} - t_m) \), surpasses a value \( K \) goes exponentially to zero when \( K \to \infty \).

From a physical rather than formal point of view, Feynman’s path integral is more similar to Huygen’s principle of classical optics than to the diffusion problem. Indeed the use of an amplitude suggests a wave picture although the fact that the modulus squared of the amplitude is a probability,
rather than an intensity, gives a particle appearance. However the particle picture is misleading. In fact, in spite of the frequent use of the expression “probability amplitude” in quantum theory, I am unable to give a real meaning to these two words. In contrast the interpretation of Feynman’s path integrals for radiation (“photons”) is more intuitive. It might be considered as an elaboration of Huygens principle of classical optics. As said above the main conceptual difference is that in classical optics the amplitude squared is an intensity (power per unit area) whilst in Feynman’s formalism it is assumed a probability density (of having a photon at a given point of space-time). But there is no real problem in assuming also in the latter formalism the intensity interpretation if we suppose that, at the time of detection, the probability is proportional to the intensity arriving at the detector, with some peculiarities\[6\].

5 Transition probability as a sum of probabilities of the paths

In the following I propose a possible intuitive interpretation of Feynman’s formalism in the case of particles. It leads to the probability as a sum of path’s probabilities[\?] . This method might provide a physical model of the motion of quantum particles, becoming an alternative to the popular de Broglie-Bohm theory (or Bohmian mechanics).

I start considering an integral of pairs of paths, starting from eq. (5), as follows

\[
P(x_0, 0 \mid x, t) \propto |A(x_0, 0 \mid x, t)|^2
= \int \mathcal{D}(\text{pathpairs}) \exp \left[ \frac{i}{\hbar} \int_0^t dt \left( \frac{1}{2} m \dot{x}^2 - V(x) \right) \right]
\times \exp \left[ -\frac{i}{\hbar} \int_0^t dt \left( \frac{1}{2} m \dot{y}^2 - V(y) \right) \right].
\]  (14)

After that I introduce the new variables \(x + y \equiv 2z, x - y \equiv w\) and integrate over \(w\), thus getting an integral over paths, every one in terms of the variable \(z\). The interpretation of the quantum motion as a kind of random motion would be possible if the probability of every path is never negative. However a general proof that this is the case will not be made in this paper (see arXiv: quant-ph, 1603.02215), but I shall study only a few particular cases where
the positivity holds true. Firstly I shall consider the case of free propagation, that is $V = 0$. Thus I get

$$P(x_0, 0 \mid x, t) \propto \lim_{\varepsilon \to 0} \left( \frac{m}{2\pi \hbar \varepsilon} \right)^n \int dx_1 ... \int dx_{n-1} \int dy_1 ... \int dy_{n-1} \times \prod_{j=1}^{n} \exp \left[ \frac{i m^2}{2\hbar \varepsilon} (x_j - x_{j-1})^2 \right] \exp \left[ -\frac{i m^2}{2\hbar \varepsilon} (y_j - y_{j-1})^2 \right].$$

With the change of variables above mentioned, this becomes after some algebra

$$P(x_0, 0 \mid x, t) \propto \lim_{\varepsilon \to 0} \left( \frac{m}{2\pi \hbar \varepsilon} \right)^n \int dz_1 ... \int dz_{n-1} \int dw_1 ... \int dw_{n-1} \times \prod_{j=1}^{n-1} \exp \left[ \frac{i m^2}{\hbar \varepsilon} w_j (z_{j-1} - 2z_j + z_{j+1}) \right],$$

(15)

where I have taken into account that $w_0 = w_n = 0$. Hence, as $z_0 = x_0, z_n = x$, I get

$$P(x_0, 0 \mid x, t) \propto \lim_{\varepsilon \to 0} \frac{m}{2\pi \hbar \varepsilon} \int dz_1 ... \int dz_{n-1} \prod_{j=1}^{n-1} \delta (z_{j+1} - 2z_j + z_{j-1}).$$

(16)

If the integrals in the variables $z_j$ are performed it is easy to check that the result agrees with the standard one[2]. Eq.(16) should be interpreted as a propagation with constant velocity, $v = (x - x_0) / t$. In fact the Dirac deltas $\delta (z_{j+1} - 2z_j + z_{j-1})$ imply $(z_{j+1} - 2z_j + z_{j-1}) / \varepsilon^2 = 0$, that is nil acceleration.

Now I will generalize the procedure to three dimensions and to the motion in the presence of a potential $V(r)$. In this case, steps similar to those leading to eq.(14) give

$$P(r_0, 0 \mid r, t) \propto \lim_{\varepsilon \to 0} \frac{m}{2\pi \hbar \varepsilon} (2\pi)^{3-3n} \int dr_1 ... \int dr_{n-1} \int dv_1 ... \int dv_{n-1} \times \prod_{j=1}^{n-1} \exp \left[ iv_j \cdot s_j + \frac{i \varepsilon}{\hbar} V \left( r_j - \frac{\hbar \varepsilon v_j}{m} \right) - \frac{i \varepsilon}{\hbar} V \left( r_j + \frac{\hbar \varepsilon v_j}{m} \right) \right].$$

(17)

where

$$v_j \equiv \frac{m}{2\hbar \varepsilon} w_j, s_j \equiv r_{j-1} - 2r_j + r_{j+1}$$
Performing the integrals in \( v_j \) is not possible without a knowledge of potential, \( V(r) \), but to lowest order in \( \hbar \) it is simple. It is obtained

\[
P(r_0, 0 | r, t) \propto \lim_{\varepsilon \to 0} \frac{m}{2\pi \hbar \varepsilon} \int dr_1 \ldots \int dr_{n-1} \prod_{j=1}^{n-1} \delta^3 \left( s_j + \frac{2\varepsilon^2}{m} \nabla V(r_j) \right), \quad (18)
\]

which corresponds to a motion fulfilling at every time

\[
m \frac{r_{j-1} - 2r_j + r_{j+1}}{\varepsilon^2} = -\nabla V(r_j),
\]

that is the classical equation of motion. Thus we get the classical limit of quantum mechanics.

It is interesting that the same result, eq. \((18)\), is obtained if the potential, \( V(r_j) \), is at most quadratic in the coordinates, which should be interpreted saying that in linear problems the quantum particle follows the classical path. The typical example is the harmonic oscillator. This is the reason why quantum mechanics of linear systems looks semiclassical. In this case all quantum effects come from the fact that the initial wave function cannot be localized in a too small region due to the Heisenberg uncertainty principle, a constraint which does not appear directly in our path integrals formalism. We see that some information is lost in passing from the (Feynman’s) sum of amplitudes of paths to our sum of probabilities of paths, like eq. \((18)\).

6 The Born approximation

With the formalism here presented it is easy to derive Born’s approximation for scattering as follows. I consider an experiment where a particle leaves a source placed at position \( r_0 \) at time \( t = 0 \), it moves freely (i.e., in the absence of forces) until about time \( t_k \), where the particle enters a region with a potential. Then the particle is deflected by the potential \( V(r) \) and, after leaving the interaction region, it may move freely towards a detector placed at the point \( r \). In order to calculate the probability of arrival I will start from eq. \((17)\) and make a calculation involving some approximations resting upon the fact that the potential is weak in some sense to be specified, this being the basic hypothesis of the Born approximation. Firstly I restrict the potential to act on the particle only once, at time \( t_k \). Thus we get from
\[ P \propto \lim_{\varepsilon \to 0} \varepsilon^{-1} \int dr_1 ... \int dr_{n-1} \int d\nu_1 ... \int d\nu_{n-1} \prod_{j=1}^{k-1} \exp (i\nu_j \cdot s_j) \]
\[ \times \exp \left[ i\varepsilon \nu_k \cdot s_k + \frac{i\varepsilon}{\hbar} V \left( r_k - \frac{\hbar \varepsilon \nu_k}{m} \right) - \frac{i\varepsilon}{\hbar} V \left( r_k + \frac{\hbar \varepsilon \nu_k}{m} \right) \right] \prod_{l=k+1}^{n-1} \exp (i\nu_l \cdot s_l), \]

where I have removed a constant factor, irrelevant because only the relative probability is calculated (hence the use of the proportionality symbol, \( \propto \)). For later convenience I will write the exponential involving \( \nu_k \) in the form

\[ \exp (i\varepsilon \nu_k \cdot s_k) + \exp (i\varepsilon \nu_k \cdot s_k) \left\{ \exp \left[ \frac{i\varepsilon}{\hbar} V \left( r_k - \frac{\hbar \varepsilon \nu_k}{m} \right) - \frac{i\varepsilon}{\hbar} V \left( r_k + \frac{\hbar \varepsilon \nu_k}{m} \right) \right] - 1 \right\}, \]

so that eq. (19) becomes

\[ P \propto \lim_{\varepsilon \to 0} \varepsilon^{-1} \int dr_1 ... \int dr_{n-1} \int d\nu_1 ... \int d\nu_{n-1} \prod_{j=1}^{k-1} \exp (i\nu_j \cdot s_j) \]
\[ + \lim_{\varepsilon \to 0} \varepsilon^{-1} \int dr_1 ... \int dr_{n-1} \int d\nu_1 ... \int d\nu_{n-1} \prod_{j=1}^{k-1} \exp (i\nu_j \cdot s_j) \exp (i\varepsilon \nu_k \cdot s_k) \]
\[ \times \left\{ \exp \left[ \frac{i\varepsilon}{\hbar} V \left( r_k - \frac{\hbar \varepsilon \nu_k}{m} \right) - \frac{i\varepsilon}{\hbar} V \left( r_k + \frac{\hbar \varepsilon \nu_k}{m} \right) \right] - 1 \right\} \prod_{l=k+1}^{n-1} \exp (i\nu_l \cdot s_l). \]

The first term takes into account the possibility that the particle goes from the source to the detector in a straight line, without crossing the target. In the experimental practice this possibility is eliminated using appropriate collimators, and I shall ignore this term.

In the following I shall study the second term of eq. (20). The integrals in all the variables \( \nu_j \) and \( \nu_l \) are trivial and give Dirac's deltas, which implies

\[ s_j \equiv r_{j-1} - 2r_j + r_{j+1} = 0, s_j \equiv r_{l-1} - 2r_l + r_{l+1} = 0, \]

for all \( j \) and \( l \). This means that the particle moves in a straight line with constant velocity, \( \nu_{\text{in}} \), from \( r_0 \) to \( r_k \) and again with constant velocity, \( \nu_{\text{out}} \), from \( r_k \) to \( \mathbf{r} \). Thus we have

\[ \nu_{\text{in}} \equiv \frac{r_k - r_0}{k\varepsilon} = \frac{r_k - r_{k-1}}{\varepsilon}, \nu_{\text{out}} \equiv \frac{\mathbf{r} - r_k}{(n-k)\varepsilon} = \frac{r_{k+1} - r_k}{\varepsilon}, \]
\[ 12 \]
which implies
\[ s_k \equiv r_{k-1} - 2r_k + r_{k+1} = \varepsilon (v_{\text{out}} - v_{\text{in}}) \equiv \varepsilon \Delta v. \]

Our aim is to get the probability that the final velocity (in the detector) is \( v_{\text{out}} \) for given initial velocity (in the source) \( v_{\text{in}} \). This will be the sum of probabilities of the possible paths with the velocity change \( \Delta v \). Each path is defined by the initial and final positions plus a point \( r_k \) in the target. However the region where the potential \( V(r_k) \) is relevant (the target) is microscopic, whilst the distances from source to target, \( |r_k - r_0| \), and from target to detector, \( |r - r_k| \), are both macroscopic. Hence \( \Delta v \) may be defined without a precise knowledge of the actual positions of the particle in the source and the detector. Therefore an accurate value of \( r_0 \) and \( r \) are not required. The integrals in \( r_k \) and \( v_k \), eq. (19), that is
\[
\int dr_k \int dv_k \exp \left( i \varepsilon v_k \cdot s_k \right) \left\{ \exp \left[ \frac{i \varepsilon}{\hbar} V \left( r_k - \frac{\hbar \varepsilon v_k}{m} \right) \right] - \frac{i \varepsilon}{\hbar} V \left( r_k + \frac{\hbar \varepsilon v_k}{m} \right) \right\} - 1 \right\}, \tag{22}
\]
will be calculated as follows.

In order to simplify the notation, in the derivation I will use units \( \hbar = \varepsilon = m = 1 \) and remove the subindices \( k \) in eq. (22). The dimensional quantities will be restored at the end. As said above the basis of the Born approximation is the assumption that the potential is weak, which justifies to approximate the last factor of eq. (22) by
\[
\exp \left[ iV(r - v) - iV(r + v) \right] - 1 \approx iV(r - v) - iV(r + v) + \frac{1}{2} [iV(r - v) - iV(r + v)]^2 \approx iV(r - v) - iV(r + v) - \frac{1}{2} [V(r - v)]^2 - \frac{1}{2} [V(r + v)]^2 + V(r - v) V(r + v). \tag{23}
\]
Now I will show that only the latter term gives a relevant contribution. In fact we get
\[
\int dr \int dv \exp (i \varepsilon v \cdot \Delta v) V(r - v) = \int dv \exp [i \varepsilon v \cdot \Delta v] \int dx V(x),
\]
where I have introduced the new variable \( x = r - v \). Similarly with the change \( y = r + v \) I obtain
\[
\int dr \int dv \exp (i \varepsilon v \cdot \Delta v) V(r + v) = \int dv \exp [i \varepsilon v \cdot \Delta v] \int dy V(y).
\]
We see that the contributions of the former two terms of eq. (23) cancel out. The contribution of the next term is
\[-\frac{1}{2} \int dr \int dv \exp (i \varepsilon v \cdot \Delta v) [V (r - v)]^2 = \int dv \exp [i \varepsilon v \cdot \Delta v] \int dx [V (x)]^2,
\]
and the integral in \(v\) is zero except if \(\Delta v = 0\), which corresponds to a motion in straight line from the source to the detector, and the same is true for the contribution of the term with \([V (r + v)]^2\). I shall ignore these terms for the same reasons why I ignored the first term in eq. (20). I conclude that in the expansion of the exponential eq. (23) the term giving the leading contribution is the following
\[P \propto \int dr \int dv \exp (i \varepsilon v \cdot \Delta v) V (r - v) V (r + v)\]
\[= \frac{1}{2} \int dx \int dy \exp [i \varepsilon (x - y) \cdot \Delta v] V (x) V (y)\]
\[= \frac{1}{2} \left[ \int dx \exp (i \varepsilon x \cdot \Delta v) V (x) \right]^2. \quad (24)\]

After including the dimensional quantities \(\hbar\) and \(m\), we get the probability
\[P(r_0, 0 | r, t) \propto \left[ \int dx \exp (i \varepsilon x \cdot \Delta k) V (x) \right]^2, \Delta k \equiv \frac{m}{\hbar} \Delta v. \quad (25)\]

The result of the calculation agrees with the standard Born approximation for the scattering of a particle by a potential. However the interpretation is here quite different. The usual derivation is made in terms of waves and after that it is necessary to make an additional assumption, quite strange for waves, namely to interpret the amplitude squared as a probability (this is the celebrated Born interpretation of the wavefunction). In sharp contrast, the obvious interpretation in our derivation is that a particle travels with constant velocity from the source to the target, it is deflected by the potential and again it travels with constant velocity to the detector. The expression eq. (25) gives precisely the (positive) probability of a deflection with the change of velocity \(\Delta v = v_{out} - v_{in}\). It is not necessary to pass from a wave to a particle picture by introducing the probabilistic (Born) interpretation of a wavefunction as an additional hypothesis.
7 Discussion

The interpretation of the Feynman path integrals formulation of (non-relativistic) quantum mechanics as giving the probabilities of the different paths of a particle in going from a space-time point to another one is appealing, but it presents at least three difficulties.

Firstly it is not guaranteed that, after integrating in the variables $v_j$, the transition probability, eq.(17), becomes a sum of (positive) probabilities of the different paths, defined by the points $\{r_0, r_1, ... r_{n-1}, r\}$. I have shown that this is the case for the free motion and for the motion in potentials quadratic in the coordinates, but it may be not true for other potentials. However it might hold true for all physical (sufficiently smooth) potentials, but this is a speculation. This problem is not solved in this paper.

Secondly the probabilistic and nonlocal action of the potential on the particle shown in eq.(24) is certainly strange. It strongly departs from the classical (deterministic and local) action governed by Newton’s second law. In relation with this objection I may argue that, although strange, the action is compatible with the “ontology” of classical physics, that is particles are (localized) corpuscles. The nonlocal character of the action might be due to some random motion superimposed to the smooth path, hidden in the effective law of force, eq.(25). On the other hand non-relativistic quantum mechanics is an approximation of a relativistic field theory, whence the formalism here studied may be also an approximation to a more accurate theory.

Thirdly the probability of the travel from one point of space-time to another one does not cover all the richness of quantum mechanics, in particular it does not contain the Heisenberg uncertainty relations.

In summary it is not obvious whether the development of Feynman’s formalism here presented possesses a physical interest or it is just a mathematical exercise devoid of any physical meaning. In any case it illustrates the fact that a direct interpretation of the quantum formalism may not be the only one possible. For instance the interpretation of Feynman path integrals in quantum mechanics should not mean that “the particle goes via all possible paths simultaneously” as sometimes stated.
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