INTEGRAL EQUATIONS ON COMPACT CR MANIFOLDS

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Abstract. Assume that $M$ is a CR compact manifold without boundary and CR Yamabe invariant $Y(M)$ is positive. Here, we devote to study a class of sharp Hardy-Littlewood-Sobolev inequality as follows

$$
\left| \int_M \int_M \left[ G_{\theta}^{\alpha}(\eta) \frac{Q-\alpha}{Q-2} f(\xi) g(\eta) dV_\theta(\xi) dV_\theta(\eta) \right] \leq Y_\alpha(M) \| f \|_{L^2(\mathcal{O})} \| g \|_{L^2(\mathcal{O})},
$$

where $G_{\theta}^{\alpha}(\eta)$ is the Green function of CR conformal Laplacian $L_\theta = b_n \Delta_b + R$, $Y_\alpha(M)$ is sharp constant, $\Delta_b$ is Sublaplacian and $R$ is Tanaka-Webster scalar curvature. For the diagonal case $f = g$, we prove that $Y_\alpha(M) \geq Y_\alpha(S^{2n+1})$ (the unit complex sphere of $\mathbb{C}^{n+1}$) and $Y_\alpha(M)$ can be attained if $Y_\alpha(M) > Y_\alpha(S^{2n+1})$. So, we got the existence of the Euler-Lagrange equations

$$
\phi^{\frac{Q-\alpha}{Q-2}} = \int_M \left[ G_{\theta}^{\alpha}(\eta) \frac{Q-\alpha}{Q-2} \phi(\eta) \right] dV_\theta, \quad 0 < \alpha < Q.
$$

Moreover, we prove that the solution of (1) is $\Gamma_\alpha(M)$. Particular, if $\alpha = 2$, the previous extremal problem is closely related to the CR Yamabe problem. Hence, we can study the CR Yamabe problem by integral equations.

1. Introduction. CR geometry, the abstract models of real hypersurfaces in complex manifolds, has attracted much attention in the past decades. Noticing that there is a far-reaching analogy between conformal and CR geometry, such as model space, scalar curvature, Sublaplacian and Yamabe equation etc., many interesting and profound results on CR geometry were obtained, see [2, 5, 6, 8–14, 16, 18–24, 27, 28, 34, 35] and the references therein. Inspired by the idea of [7, 16, 17, 36], we want to study the curvature problem of CR geometry from the point of integral curvature equation. Following, involved notations can be found in the Section 2.

Let $(M, J, \theta)$ be a compact pseudohermitian manifold without boundary. Under the transformation $\tilde{\theta} = \phi \frac{4}{\Delta_b} \theta$ with $\phi \in C^\infty(M)$ and $\phi > 0$, Tanaka-Webster scalar curvatures $R$ and $\tilde{R}$, corresponding to $\theta$ and $\tilde{\theta}$ respectively, satisfy

$$
b_n \Delta_b \phi + R \phi = \tilde{R} \phi^{\frac{Q+2}{Q-2}},
$$

where $L_\theta = b_n \Delta_b + R$ is the CR conformal Laplacian related to $\theta$ and $b_n = 2 + \frac{2}{n}$. For a given constant curvature $\tilde{R}$, the existence of (1.1) is known as CR Yamabe

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problem, which was introduced by Jerison and Lee in [20]. There, they studied the CR Yamabe invariant

$$\mathcal{Y}(M) = \inf \{ A_\theta(\phi) : B_\theta(\phi) = 1 \}$$

with

$$A_\theta(\phi) = \int_M (b_\mu d\phi^\mu + R\phi^2) \, dV_\theta, \quad B_\theta(\phi) = \int_M |\phi|^p \, dV_\theta, \quad dV_\theta = \theta \wedge d\theta^n,$$

and proved that $\mathcal{Y}(M) \leq \mathcal{Y}(\mathbb{S}^{2n+1})$ and the infimum can be attained if $\mathcal{Y}(M) < \mathcal{Y}(\mathbb{S}^{2n+1})$. It is well known that the case of $\mathcal{Y}(M) \leq 0$ is easy to deal. While for the positive case, it is complicated.

If $\mathcal{Y}(M) > 0$, then the first eigenvalue $\lambda_1(\mathcal{L}_\theta) > 0$ and then $\mathcal{L}_\theta$ is invertible. Furthermore, for any $\xi \in N$, there exists a Green function $G^\theta_\xi(\eta)$ of $\mathcal{L}_\theta$ such that the solution of (1.1) satisfies

$$\phi(\xi) = \int_M G^\theta_\xi(\eta) \tilde{R}(\eta) \phi(\eta) \frac{\eta + \xi}{\eta + \xi} \, dV_\theta. \tag{1.2}$$

So, scalar curvature $\tilde{R}$ can also be defined implicitly by the integral equation (1.2).

Noting that $G^\theta_\xi(\eta) = \phi^{-1}(\xi) \phi^{-1}(\eta) G^\theta_\xi(\eta)$, we easily get

$$\int_M G^\theta_\xi(\eta) \tilde{R}(\eta) u(\eta) \, dV_\theta = \phi^{-1} \int_M G^\theta_\xi(\eta) \tilde{R}(\eta) \phi \frac{\eta + \xi}{\eta + \xi} u(\eta) \, dV_\theta.$$

Moreover, we can study a class of more general CR conformal integral equation as in [36]. Specifically, define the operator

$$I_{M, \theta, \alpha}(u) = \int_M [G^\theta_\xi(\eta)]^{\frac{2n}{2n-\alpha}} u(\eta) \, dV_\theta \quad \text{with} \quad \alpha \neq Q, \tag{1.3}$$

and we can prove that, under the transformation $\tilde{\theta} = \phi^{\frac{1}{2n-\alpha}} \theta$,

$$I_{M, \tilde{\theta}, \alpha}(u) = \phi^{-\frac{2n}{2n-\alpha}} I_{M, \theta, \alpha}(\phi^{\frac{1}{2n-\alpha}} u). \tag{1.4}$$

If take $u \equiv C$ and let $\varphi(\xi) = \phi \frac{\eta + \xi}{\eta + \xi}$, then we have the following generalizing curvature equation, up to a constant multiplier,

$$\varphi(\xi) \frac{2n}{2n-\alpha} = \int_M [G^\theta_\xi(\eta)]^{\frac{2n}{2n-\alpha}} \varphi(\eta) \, dV_\theta, \quad \alpha \neq Q. \tag{1.5}$$

As pointed by Zhu in [36], on $\mathbb{S}^n$ integral curvature equations are equivalent to the classical curvature quation if $\alpha$ is strictly less than dimension; while for the case $\alpha$ strictly greater than dimension, they are not equivalent and integral curvature equation has some advantages. So, it is interesting and valuable to study the integral curvature equation (1.5).

Moreover, if $G^\theta_\xi(\eta) = G^\theta_\eta(\xi)^1$, we note that (1.5) is the Euler-Lagrange equation of the extremal problem

$$\mathcal{Y}_\alpha(M) = \sup_{f \in L^{(2n)/(Q+\alpha)}(\{0\})(M)} \left\{ \frac{\int_M \int_M [G^\theta_\xi(\eta)]^{\frac{2n}{2n-\alpha}} f(\xi) f(\eta) dV_\theta(\xi) dV_\theta(\eta)}{\|f\|^{(2n)/(Q+\alpha)}(M)} \right\}, \tag{1.6}$$

$^1$Recently, I proved that $G^\theta_\xi(\eta) = G^\theta_\eta(\xi)$ holds on compact CR manifolds. In fact, since $\mathcal{L}_\theta$ is symmetric and $\mathcal{Y}(M) > 0$, there exist countable eigenvalues $\{\lambda_k\}_{k=1}^{+\infty}$ satisfying $0 < \lambda_1 \leq \lambda_1 \leq \cdots$ and $\lambda_k \to +\infty$ as $k \to +\infty$, and a sequence eigenfunctions $\{\varphi_k(\xi)\}_{k=1}^{+\infty}$, which satisfy $\mathcal{L}_\theta \varphi_k = \lambda_k \varphi_k$ and form an orthonormal basis of $L^2(M)$. So, $G^\theta_\xi(\eta) = \sum_{k=1}^{+\infty} \varphi(\xi) \varphi(\eta)$. 
which is closely related to a class of Hardy-Littlewood-Sobolev inequalities with kernel $[G_{\xi}^\alpha(\eta)]^{\frac{Q-\alpha}{Q-2}}$. Namely, for any $f, g \in L^{2Q/(Q+\alpha)}(M)$ with $0 < \alpha < Q$, there exists some positive constant $C(\alpha, M)$ such that

$$\left| \int_M \int_M [G_{\xi}^\alpha(\eta)]^{\frac{Q-\alpha}{Q-2}} f(\xi)g(\eta)dV_\theta(\xi)dV_\phi(\eta) \right| \leq C(\alpha, M)\|f\|_{L^{\frac{2Q}{Q-\alpha}}(M)}\|g\|_{L^{\frac{2Q}{Q-\alpha}}(M)}.$$  

(1.7)

In fact, by the parametrix method, we know by [11] that $(G_{\xi}^\alpha(\eta))^{\frac{Q-\alpha}{Q-2}}$ holds, then $\rho(\xi, \eta) \to 0$. So, (1.7) holds by a similar argument with Theorem 15.11 of [11].

In this paper, we will mainly devoted to study the extremal problem (1.6) by Hardy-Littlewood-Sobolev inequalities and will prove the following results.

**Theorem 1.1.** Firstly, $\mathcal{Y}_\alpha(M) \geq \mathcal{Y}_\alpha(\mathbb{S}^{2n+1})$. Moreover, if the strict inequality holds, then $\mathcal{Y}_\alpha(M)$ is attained.

Because of the hypoellipticity of operator $\mathcal{L}$ (in fact $\mathcal{L}$ satisfies the Hörmander condition [18]), we know that the Green function $G_{\xi}^\alpha(\eta)$ is $C^\infty$ if $\xi \neq \eta$. Moreover, using CR normal coordinates at $\xi$ and the classical method of parametrix, we can construct the Green function as (without loss of generality, we take the coefficient of singular part as one)

$$G_{\xi}(\eta) = \rho^{-2n} + w(\xi, \eta),$$

where $w$ is the regular part. Particular, if $M$ is locally CR conformal flat, then $w$ satisfies $\Delta_b w = 0$ in some neighbourhood of $\xi$. Therefore, $w$ is $C^\infty$ in this neighbourhood because of the hypoellipticity of $\Delta_b$. If $n = 1$, Cheng, Malchiodi and Yang [5] proved that $w \in C^{1,\gamma}(M)$ for any $\gamma \in (0, 1)$. In the sequel, we always assume that $w(\xi, \eta) \in C^1(M \times M)$. Then, we can rewrite $G_{\xi}(\eta)$ as

$$G_{\xi}(\eta) = \rho^{-2n} + A(\xi) + O(\rho) \quad \text{with} \quad A(\xi) = w(\xi, \xi).$$  

(1.8)

On the other hand, on a locally CR conformal flat manifold, we note that $\rho(\xi, \eta) = \rho(\eta, \xi)$ holds on some neighbourhood of diagonal of $M \times M$. So, we give a sufficient condition for the strict inequality of Theorem 1.1.

**Proposition 1.2.** Assume that $M$ is a locally CR conformal flat manifold with $\mathcal{Y}(M) > 0$. If there exists some point $\xi_0 \in M$ such that $A(\xi_0) = w(\xi_0, \xi_0) > 0$, then $\mathcal{Y}_\alpha(M) > \mathcal{Y}_\alpha(\mathbb{S}^{2n+1})$.

The paper is organized as follows. In Section 2, we introduce the definition of CR manifold, some notations and some known results. Section 3 is mainly devoted to the first part of Theorem 1.1, namely, the estimation $\mathcal{Y}_\alpha(M) \geq \mathcal{Y}_\alpha(\mathbb{S}^{2n+1})$. For the discussion of the second part of Theorem 1.1, we will adopt the blowup analysis. So, we will study the subcritical case of Hardy-Littlewood-Sobolev inequality in the Section 4. Then, in Section 5, we complete the proof of Theorem 1.1 and discuss the condition of strict inequality, namely Proposition 1.2. For completeness, we study the CR conformality of operator (1.3) in the Appendix A.

2. Preliminary. In this section, we will introduce some notations and some known facts. The details can be seen in [2, 8, 11, 12, 19–24, 27, 28, 34, 35] and references therein.
2.1. CR manifold and CR Yamabe problem. A CR manifold is a real oriented $C^\infty$ manifold $M$ of dimension $2n + 1$, $n = 1, 2, \cdots$, together with a subbundle $T_{1,0}$ of the complex tangent bundle $TM$ satisfying:

(a) $\dim \mathbb{C} T_{1,0} = n$,
(b) $T_{1,0} \cap T_{0,1} = \{0\}$ with $T_{0,1} = T_{1,0}$,
(c) $T_{1,0}$ satisfies the formal Frobenius condition $[T_{1,0}, T_{1,0}] \subset T_{1,0}$.

Denote by $Q = 2n + 2$ the homogeneous dimension.

An almost CR structure on $M$ is a pair $(H(M), J)$, where $H(M) = \text{Re}(T_{1,0} + T_{0,1})$ is a subbundle of rank $2n$ of $T(M)$ and $J : H(M) \to H(M)$, given by $J(V + W) = \sqrt{-1}(V - W)$ for $V \in T_{1,0}$, is an almost complex structure on $H(M)$.

Since $M$ is orientable, then $H(M)$ is oriented by its complex structure. So, there always exists a global nonvanishing 1-form $\theta$ which annihilates exactly $H(M)$ and for which there exists a natural volume form $dV_\theta = \theta \wedge d\theta^n$. Any such $\theta$ is called a pseudo-hermitian structure on $M$.

Associated with each $\theta$, Levi form $L_\theta$ is defined on $H(M)$ as

$$L_\theta(V, W) = \langle d\theta, V \wedge JW \rangle = d\theta(V, JW), \quad V, W \in G.$$ 

By complex linearity, we can extend $L_\theta$ to $CH(M)$ and induce a hermitian form on $T_{1,0}$ as

$$L_\theta(V, \overline{W}) = -\sqrt{-1}(d\theta, V \wedge \overline{W}) = -\sqrt{-1}d\theta(V, \overline{W}), \quad V, W \in T_{1,0}.$$ 

It is easy to see that Levi form is CR invariant. Namely, if $\theta$ is replaced by $\tilde{\theta} = f\theta$, $L_\theta$ changes conformally by $L_{\tilde{\theta}} = fL_\theta$. We say $M$ is nondegenerate if the Levi form is nondegenerate at every point, and say $M$ is strictly pseudoconvex if the form is positive definite everywhere. In this paper, we always assume that $M$ is strictly pseudoconvex.

Based on the Levi form $L_\theta$, we can take a local unitary frame $\{T_\alpha : \alpha = 1, \cdots, n\}$ for $T_{1,0}(M)$. Then, there is a natural second order differential operator, namely the Sublaplacian $\Delta_b$, which is defined on the function $u$ as

$$\Delta_b u = -(u_{\alpha,\bar{\alpha}} + u_{\bar{\alpha},\alpha}). \quad (2.1)$$

Under the transformation $\tilde{\theta} = \phi^{Q/2} \theta$ with $\phi \in C^\infty(M)$ and $\phi > 0$, the CR conformal Laplacian $L_\tilde{\theta} = b_n \Delta_b + R$ satisfies

$$L_{\tilde{\theta}}(\phi^{-1} u) = \phi^{-\frac{Q+2}{Q-2}} L_\theta(u), \quad (2.2)$$

where $R$ is the Tanaka-Webster scalar curvatures and $b_n = \frac{2Q}{Q-2}$. Take $u = \phi$, we have the prescribed curvature equation (1.1). Furthermore, for given constant curvature $R$, the existence of (1.1) is known as CR Yamabe problem, which was introduced by Jerison and Lee, see [19, 20].

2.2. Heisenberg group $\mathbb{H}^n$, complex sphere $\mathbb{S}^{2n+1}$ and the Cayley transform. Heisenberg group $\mathbb{H}^n$ is $\mathbb{C}^n \times \mathbb{R} = \{u = (z, t) : z = (z_1, \cdots, z_n) = (x_1 + \sqrt{-1}y_1, \cdots, x_n + \sqrt{-1}y_n, t) \in \mathbb{C}^n, t \in \mathbb{R}\}$ with the multiplication law

$$(z, t)(z', t') = (z + z', t + t' + 2Im(z \cdot \overline{z'})),$$

where $z \cdot \overline{z'} = \sum_{j=1}^{n} z_j \overline{z'_j}$. A class of natural norm function is given by $|u| = (|z|^4 + t^2)^{1/4}$ and the distance between points $u$ and $v$ is defined by $d(u, v) = |v^{-1} u|$. Take
a set of basis $Z_j = \frac{\partial}{\partial z_i} + \sqrt{-1} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$, $j = 1, 2, \ldots, n$, which spans $T_{1,0}$ of $\mathbb{H}^n$. A natural 1-form is

$$\theta_0 = dt + \sqrt{-1} \sum_{j=1}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j),$$

and the corresponding Levi form can be specified as

$$L_{\theta_0}(Z_j, \bar{Z}_j) = \delta_{jk}, \quad j, k = 1, 2, \ldots, n.$$ The Sublaplacian is defined with respect to $\{T_j, j = 1, 2, \ldots, n\}$ as

$$\Delta_H = -\sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j). \quad (2.3)$$

The sphere $S^{2n+1}$ is $\{ \xi = (\xi_1, \ldots, \xi_{n+1} : \xi \cdot \bar{\xi} = \sum_{j=1}^{n+1} |\xi_j|^2 = 1 \} \subset \mathbb{C}^{n+1}$ and the subspace $T_{1,0} \subset CT(S^{2n+1})$ is spanned by

$$T_j = \frac{\partial}{\partial \xi_j} - \bar{\xi}_j R, \quad R = \sum_{k=1}^{n+1} \xi_k \frac{\partial}{\partial \xi_k}, \quad j = 1, \ldots, n + 1.$$ A natural 1-form is

$$\theta_S = \sqrt{-1} \sum_{j=1}^{n+1} (\xi_j d\bar{\xi}_j - \bar{\xi}_j d\xi_j)$$

and the Sublaplacian is

$$\Delta_S = -\sum_{j=1}^{n+1} (T_j T_j + \bar{T}_j \bar{T}_j). \quad (2.4)$$

On the sphere the distance function is defined as $d(\zeta, \eta)^2 = 2 |1 - \zeta \cdot \bar{\eta}|$.

Cayley transform $C : \mathbb{H}^n \to S^{2n+1} \setminus \overline{(0, \ldots, 0, -1)}$ and its reverse are defined as

$$C(z, t) = \left( \frac{2z}{1 + |z|^2 + it}, \frac{1 - |z|^2 - it}{1 + |z|^2 + it} \right),$$

$$C^{-1}(\xi) = \left( \frac{\xi_1}{1 + \xi_{n+1}}, \frac{\xi_2}{1 + \xi_{n+1}}, \ldots, \frac{\xi_n}{1 + \xi_{n+1}}, \frac{1 - \xi_{n+1}}{1 + \xi_{n+1}} \right),$$

respectively. The Jacobian of the Cayley transform is

$$J_C(z, t) = \frac{2^{2n+1}}{(1 + |z|^2)^2 + t^2)^n},$$

which implies that

$$\int_{\mathbb{H}^n} f dV_0 = 2^{2n} n! \int_{\mathbb{H}^n} f du = \int_{S^{2n+1}} F dV_S = 2^{2n+1} n! \int_{S^{2n+1}} F d\xi, \quad (2.5)$$

where $f = (F \circ C)(2|J_C|)$, $dV_0 = \theta_0 \wedge d\theta_0 \wedge \cdots \wedge d\theta_0$, $dV_S = \theta_S \wedge d\theta_S \wedge \cdots \wedge d\theta_S$, $du = dzdt = dzdydt$ is the Haar measure on $\mathbb{H}^n$ and $d\xi$ is the Euclidean volume element of $S^{2n+1}$. Moreover, through the Cayley transform, we have the following relations between two distance functions

$$d(\zeta, \eta) = d(u, v) \left( \frac{4}{1 + |\zeta|^2 + t^2} \right)^{1/4} \left( \frac{4}{1 + |\zeta'|^2 + t'^2} \right)^{1/4},$$

where $\zeta = C(u)$, $\eta = C(v)$, $u = (z, t)$ and $v = (z', t')$.

Adopting the above notations, we can rewrite the sharp Hardy-Littlewood-Sobolev inequalities on $\mathbb{H}^n$ and $S^{2n+1}$ (see Frank and Lieb’s result [12]) as
Then for any $f, g \in L^p(\mathbb{H}^n)$,
\[
\left| \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{f(u)g(v)}{d(u,v)^{Q-\alpha}} dV(u)dV(v) \right| \leq D_H \|f\|_{L^p(\mathbb{H}^n,dV_0)} \|g\|_{L^p(\mathbb{H}^n,dV_0)}
\]  
(2.7)
where
\[
D_H := 2\pi^{\frac{n}{2}} \frac{n!\Gamma(\alpha/2)}{\Gamma^2((Q+\alpha)/4)} = \mathcal{Y}_\alpha(\mathbb{H}^n).
\]  
(2.8)
And equality holds if and only if
\[
f(u) = c_1g(u) = c_2H(\delta_r(a^{-1}u)),
\]  
(2.9)
for some $c_1, c_2 \in \mathbb{C}$, $r > 0$ and $a \in \mathbb{H}^n$ (unless $f \equiv 0$ or $g \equiv 0$). Here $H$ is defined as
\[
H(u) = H(z,t) = ((1 + |z|^2)^2 + t^2)^{-\alpha(Q+\alpha)/4}.
\]  
(2.10)

**Theorem 2.2** (Sharp HLS inequality on $\mathbb{S}^{2n+1}$). For $0 < \alpha < Q$ and $p = \frac{2Q}{Q+\alpha}$. Then for any $f, g \in L^p(\mathbb{S}^{2n+1})$,
\[
\left| \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{f(\xi)g(\eta)}{d(\xi,\eta)^{Q-\alpha}} dV_\xi dV_\eta \right| \leq D_S \|f\|_{L^p(\mathbb{S}^{2n+1},dV)} \|g\|_{L^p(\mathbb{S}^{2n+1},dV)}
\]  
(2.11)
where
\[
D_S := D_H = \mathcal{Y}_\alpha(\mathbb{S}^{2n+1}).
\]  
(2.12)
And equality holds if and only if
\[
f(\xi) = c_1|1 - \bar{\xi} \cdot \xi|^{-\alpha(Q+\alpha)/2}, \quad g(\xi) = c_2|1 - \bar{\xi} \cdot \xi|^{-\alpha(Q+\alpha)/2}
\]  
(2.13)
for some $c_1, c_2 \in \mathbb{C}$ and some $\xi \in \mathbb{C}^{n+1}$ with $|\xi| < 1$ (unless $f \equiv 0$ or $g \equiv 0$).

### 2.3. Folland-Stein normal coordinates (see [11, 20]).

On some open set $V \subset M$, take a set of pseudohermitian frame $\{W_1, \cdots, W_n\}$. Then, $\{W_i, \bar{W}_i, T; i = 1, \cdots, n\}$ forms a local frame, where $T$ is determined by $\Theta(T) = 1$ and $d\theta(T, X) = 0$ for all $X \in TM$. As the Theorem 4.3 and Remark 4.4 of [20], we can summarize the result of Folland-Stein normal coordinates in the following theorem.

**Theorem 2.3** (Theorem 4.3 of [20]). There is a neighbourhood $\Omega \subset M \times M$ of the diagonal and a $C^\infty$ mapping $\Theta: \Omega \to \mathbb{H}^n$ satisfying:

(a) $\Theta(\xi, \eta) = -\Theta(\eta, \xi) = \Theta(\eta, \xi)^{-1} \in \mathbb{H}^n$. (In particular, $\Theta(\xi, \xi) = 0$.)

(b) Denote $\Theta_\xi(\eta) = \Theta(\xi, \eta)$. $\Theta_\xi$ is thus a diffeomorphism of a neighbourhood $\Omega_\xi$ of $\xi$ onto a neighbourhood of the origin in $\mathbb{H}^n$. Denote by $u = (z, t) = \Theta(\xi, \eta)$ the coordinates of $\mathbb{H}^n$. Denote by $\mathbb{C}^{\infty}$, $k = 1, 2, \cdots$, a $C^\infty$ function $f$ of $\xi$ and $y$ such that for each compact set $K \subset V$ there is a constant $C_K$, with $|f(\xi, y)| \leq C_K |y|^k$ (Heisenberg norm) for $\xi \in K$. Then,
\[
(\Theta^{-1}_\xi)^* = \theta + O^1 dt + \sum_{j=1}^{2n}(O^2 dz_j + O^2 d\bar{z}_j),
\]  
\[
(\Theta^{-1}_\xi)^*(\theta \wedge d\theta^n) = (1 + O^1) \theta_0 \wedge d\theta^n_0.
\]

(c)
\[
\Theta_\xi W_j = Z_j + O^1 \mathcal{E}(\partial_z) + O^2 \mathcal{E}(\partial_{\bar{z}}), \quad \Theta_\xi T = \frac{\partial}{\partial \theta} + O^1 \mathcal{E}(\partial_z, \partial_\theta),
\]
\[
\Theta_\xi \Delta_b = \Delta_H + \mathcal{E}(\partial_z) + O^1 \mathcal{E}(\partial_\theta, \partial_z) + O^2 \mathcal{E}(\partial_z \partial_{\bar{z}}) + O^3 \mathcal{E}(\partial_{\bar{z}}^2),
\]
in which $O^k$ indicates an operator involving linear combinations of the indicated derivatives with coefficients in $O^k$, and we have used $\partial_\xi$ to denote any of the derivatives $\partial/\partial z_j$, $\partial/\partial \bar{z}_j$. (The uniformity with respect to $\xi$ of bounds on functions in $O^k$ is not stated explicitly [11], but follows immediately from the fact that the coefficients are $C^\infty$.)

**Theorem 2.4** (Remark 4.4 of [20]). Let $T^\delta(z,t) = (\delta^{-1}z,\delta^{-2}t)$, $K \subset \subset V$, and let r be fixed. With the notation of Theorem 2.3 and $B_r = \{u \in \mathbb{H}^n : |u| \leq r\}$, then $T^\delta \circ \Theta(\Omega_\xi) \supset B_r$ for sufficiently small $\delta$ and all $\xi \in K$. Moreover, for $\xi \in K$ and $u \in B_r$,

$$(T^\delta \circ \Theta(\xi)^{-1})^* \theta = \delta^2(1 + \delta O^1)\theta_0,$$

$$(T^\delta \circ \Theta(\xi)^{-1})^*(\theta \wedge d\theta^n) = \delta^{2n+2}(1 + \delta O^1)\theta_0 \wedge d\theta^n,$$

$$(T^\delta \circ \Theta(\xi))_* W_j = \delta^{-1}(Z_j + \delta O^1\mathcal{E}(\partial_z) + \delta^2 O^2\mathcal{E}(\partial_\xi)),$$

$$(T^\delta \circ \Theta(\xi))_* \Delta_\delta = \delta^{-2}(\Delta_H + \delta(\partial_z) + \delta O^1\mathcal{E}(\partial_z, \partial_z^2) + \delta^2 \mathcal{E}(\partial_z \partial_\xi) + \delta^3 O^3\mathcal{E}(\partial_\xi^2)).$$

(Here $O^k$ may depend also on $\delta$, but its derivatives are bounded by multiples of the frame constants, uniformly as $\delta \to 0$. Recall that $T^\delta_* Z_j = \delta^{-1}Z_j$, and $((T^\delta)^{-1})^* \theta_0 = \delta^{-2}\theta_0$.)

**2.4. Function spaces (see [11, 20]).** Let $U$ be a relatively compact open subset of a normal coordinate neighbourhood $\Omega_\xi \subset M$ and take $(W_1, \ldots, W_n)$ be a pseudo-hermitian frame on $U$. Let $X_j = \text{Re}W_j$ and $X_{j+n} = \text{Im}W_j$ for $j = 1, \ldots, n$. Denote $X^\alpha = X_{\alpha_1} \cdots X_{\alpha_k}$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$, each $\alpha_j$ an integer $1 \leq \alpha_j \leq 2n$, and denote $l(\alpha) = k$.

The Folland-Stein space $S^p_k(U)$ is defined as the completion of $C^\infty_0(U)$ with respect to the norm

$$\|f\|_{S^p_k(U)} = \sup_{l(\alpha) \leq k} \|X^\alpha f\|_{L^p(U)}, \quad \text{with} \quad \|X^\alpha f\|_{L^p(U)}^p = \int_U |X^\alpha f|^p dV_0.$$  

On the other hand, Folland and Stein also defined H"{o}lder spaces $\Gamma_\beta(U)$ as follows. Take $\rho(\xi, \eta) = |\Theta(\xi, \eta)|$ (Heisenberg norm) as the natural distance function on $U$. Then, for $0 < \beta < 1$ define

$$\Gamma_\beta(U) = \{f \in C^0(\overline{U}) : |f(\eta) - f(\xi)| \leq C\rho(\xi, \eta)^\beta\}$$

with norm

$$\|f\|_{\Gamma_\beta(U)} = \sup_{\xi \in U} |f(\xi)| + \sup_{\xi, \eta \in U} \frac{|f(\eta) - f(\xi)|}{\rho(\xi, \eta)^\beta}.$$  

For $\beta = 1$ define

$$\Gamma_1(U) = \{f \in C^0(\overline{U}) : |f(\eta) + f(\hat{\eta}) - 2f(\xi)| \leq C\rho(\xi, \eta)\}$$

with norm

$$\|f\|_{\Gamma_1(U)} = \sup_{\xi \in U} |f(\xi)| + \sup_{\xi, \eta \in U} \frac{|f(\eta) + f(\hat{\eta}) - 2f(\xi)|}{\rho(\xi, \eta)},$$

where $\hat{\eta} = \Theta^{-1}(\bar{\Theta}(\xi))$. For $\beta = k + \beta'$ with $k = 1, 2, \cdots$ and $0 < \beta' \leq 1$, define

$$\Gamma_\beta(U) = \{f \in C^0(\overline{U}) : X^\alpha f \in \Gamma_{\beta'}(U) \text{ for } l(\alpha) \leq k\}$$

and we have used $\partial_\xi$ to denote any of the derivatives $\partial/\partial z_j$, $\partial/\partial \bar{z}_j$. (The uniformity with respect to $\xi$ of bounds on functions in $O^k$ is not stated explicitly [11], but follows immediately from the fact that the coefficients are $C^\infty$.)
with norm
\[ \|f\|_{r_\beta(U)} = \begin{cases} 
\sup_{\xi \in U} |f(\xi)| + \sup_{\xi,\eta \in U \atop l(\alpha) \leq k} \frac{|X^\alpha f(\eta) - X^\alpha f(\xi)|}{\rho(\xi,\eta)^{\beta'}} & 0 < \beta' < 1, \\
\sup_{\xi \in U} |f(\xi)| + \sup_{\xi,\eta \in U \atop l(\alpha) \leq k} \frac{|X^\alpha f(\eta) + X^\alpha f(\xi)|}{\rho(\xi,\eta)^{\beta}}, & \beta' = 1.
\end{cases} \]

Fix the local coordinates of \( U \) by \( u = (z,t) = \Theta_\xi \) for some given point \( \xi \in U \). Then, for \( 0 < \beta < 1 \), the standard Hölder space \( \Lambda_\beta(U) \) is
\[ \Lambda_\beta(U) = \{ f \in C^0(\overline{U}) : |f(u) - f(v)| \leq C\|u - v\|^\beta \} \]
with norm
\[ \|f\|_{\Lambda_\beta(U)} = \sup_{u \in U} |f(\xi)| + \sup_{u,v \in U} \frac{|f(u) - f(v)|}{\|u - v\|^\beta}. \]

While for \( \beta \geq 1 \), \( \Lambda_\beta(U) \) can be defined similarly. Furthermore, we have

**Proposition 2.5** (Theorem 20.1 of [11] & Proposition 5.7 of [20]). \( \Gamma_\beta \subset \Lambda_{\beta/2}(\text{loc}) \) for \( 0 < \beta < \infty \) and there exists some positive constant \( C \) such that \( \|f\|_{\Lambda_{\beta/2}(U)} \leq C\|f\|_{r_\beta(U)} \) for any \( f \in C_0(U) \).

Now for a compact strictly pseudoconvex pseudohermitian manifold \( M \), choose a finite open covering \( U_1, \ldots, U_m \) for which each \( U_j \) has the properties of \( U \) above. Choose a \( C^\infty \) partition of unity \( \phi_i \) subordinate to this covering, and define
\[ S^p_\beta(M) = \{ f \in L^1(M) : \phi_i f \in S^p_\beta(U) \text{ for all } i \}, \]
\[ \Gamma_\beta(M) = \{ f \in C^0(M) : \phi_i f \in \Gamma_\beta(U_j) \text{ for all } i \}. \]

Following, for convenience, denote \( p_\alpha = \frac{2Q}{Q - a} \) and \( q_\alpha = \frac{2Q}{Q + a} \).

**3. Estimation of the sharp constant.**

**Proposition 3.1.**
\[ Y_\alpha(M) \geq D_H. \]

**Proof.** Since \((G^\alpha_\xi(\eta))_{\xi,\eta} \sim \rho(\xi,\eta)^{a-Q} \) as \( \rho(\xi,\eta) \to 0 \), then for any small enough \( \delta > 0 \), there exists a neighbourhood \( V \) of the diagonal of \( M \times M \) such that
\[ (1 - \delta)\rho(\xi,\eta)^{a-Q} \leq (G^\alpha_\xi(\eta))_{\xi,\eta} \leq (1 + \delta)\rho(\xi,\eta)^{a-Q}. \]
(3.1)

Recall that \( f(u) = H(u) \) is an extremal function to the sharp HLS inequality in Theorem 2.1, as well as its conformal equivalent class:
\[ f_\epsilon(u) = \epsilon^{-\frac{a}{2 - a}} H(\delta_{\epsilon^{-1}}(u)), \quad \forall \epsilon > 0. \]
(3.2)
Thus
\[ \|I_\alpha f\|_{L^{p_\alpha}(\mathbb{H}^n)} = \|I_\alpha f_\epsilon\|_{L^{p_\alpha}(\mathbb{H}^n)}, \quad \|f\|_{L^{q_\alpha}(\mathbb{H}^n)} = \|f_\epsilon\|_{L^{q_\alpha}(\mathbb{H}^n)}, \]
and \( f_\epsilon(u) \) satisfies integral equation
\[ f_\epsilon^{\frac{2-a}{2}}(u) = B \int_{\mathbb{H}^n} \frac{f_\epsilon(v)}{|u - v|^{2-a}} \, dV_0(v), \]
(3.3)
where \( B \) is a positive constant.

Let \( \Sigma_R = \{ u = (z,t) \in \mathbb{H}^n : |z| < R, |t| < R^2 \} \) be a cylindrical set, where \( R \) is a fixed constant to be determined later, and take a test function \( g(u) \in L^{q_\alpha}(\mathbb{H}^n) \) as
\[ g(u) = \begin{cases} 
f_\epsilon(u), & u \in \Sigma_R(\xi), \\
0, & u \in \Sigma_R^c = \mathbb{H}^n \setminus \Sigma_R.
\end{cases} \]
Then,
\[
\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{g(u)g(v)}{|u^{-1}v|^{Q-\alpha}} dV_0(u)dV_0(v) = \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{f_\epsilon(u)f_\epsilon(v)}{|u^{-1}v|^{Q-\alpha}} dV_0(u)dV_0(v)
\]

\[-2 \int_{\mathbb{H}^n} \int_{\Sigma_\mathbb{H}^n} \frac{f_\epsilon(u)f_\epsilon(v)}{|u^{-1}v|^{Q-\alpha}} dV_0(u)dV_0(v) + \int_{\Sigma_\mathbb{H}^n} \int_{\Sigma_\mathbb{H}^n} \frac{f_\epsilon(u)f_\epsilon(v)}{|u^{-1}v|^{Q-\alpha}} dV_0(u)dV_0(v) \]

\[= D_H \|f_\epsilon\|_{L^{\infty}(\mathbb{H}^n)}^2 - I_1 + I_2,
\]

where

\[I_1 := 2 \int_{\mathbb{H}^n} \int_{\Sigma_\mathbb{H}^n} \frac{f_\epsilon(u)f_\epsilon(v)}{|u^{-1}v|^{Q-\alpha}} dV_0(u)dV_0(v)
\]

\[I_2 := \int_{\Sigma_\mathbb{H}^n} \int_{\Sigma_\mathbb{H}^n} \frac{f_\epsilon(u)f_\epsilon(v)}{|u^{-1}v|^{Q-\alpha}} dV_0(u)dV_0(v).
\]

With (3.3), we have

\[I_1 = C \int_{\Sigma_\mathbb{H}^n} f_\epsilon^{2Q/(Q+\alpha)}(u)dV_0(u) = O\left(\frac{R}{\epsilon}\right)^{-Q}, \quad \text{as} \quad \epsilon \to 0.
\]

For $I_2$, by HLS inequality (2.7), we have

\[I_2 \leq D_H \|f_\epsilon\|_{L^{\infty}(\Sigma_\mathbb{H}^n)}^2 = O\left(\frac{R}{\epsilon}\right)^{-Q} \quad \text{as} \quad \epsilon \to 0.
\]

Hence, for small enough $\epsilon$, we have

\[
\int_{\Omega} \int_{\Omega} g(u)g(u)|u^{-1}v|^{\alpha-Q} dV_0(u)dV_0(v) \geq \frac{D_H}{\|g\|_{L^{\infty}(\Omega)}^2} \geq D_H - O\left(\frac{R}{\epsilon}\right)^{-Q}.
\]

For any given point $\xi \in M$, there exists a neighbourhood $V_\xi \subset V$ such that Theorem 2.3 hold. So, choose $R$ small enough such that $\Sigma_R \subset \Theta_\xi(V_\xi)$ and $(\Theta_\xi^{-1})^*(dV_\delta) = (1 + O^1)dV_\delta$. Let

\[
\Phi(\eta) = \begin{cases} f_\epsilon(\Theta_\xi(\eta)), & \text{in} \quad \Theta_\xi^{-1}(\Sigma_R), \\ 0, & \text{in} \quad M \setminus \Theta_\xi^{-1}(\Sigma_R). \end{cases}
\]

Then,

\[
\int_M |\Phi(\eta)|^{2Q/(Q+\alpha)} dV_\delta(\eta) = \int_{\Sigma_R} |f_\epsilon(u)|^{2Q/(Q+\alpha)}(1 + O^1) dV_\delta,
\]

\[
\int_{M \times M} \Phi(\eta)\Phi(\zeta)(G_\xi^\delta(\eta))^{\frac{Q}{Q+\alpha}} dV_\delta(\eta)dV_\delta(\zeta)
\]

\[\geq (1 - \delta) \int_{M \times M} \Phi(\eta)\Phi(\zeta)\rho(\eta, \zeta)^{\alpha-Q} dV_\delta(\eta)dV_\delta(\zeta)
\]

\[= (1 - \delta) \int_{\Sigma_R \times \Sigma_R} f_\epsilon(u)f_\epsilon(v)|u^{-1}v|^{\alpha-Q}(1 + O^1)^2 dV_\delta(u)dV_\delta(v),
\]

and

\[
\mathcal{Y}_\alpha^*(M) \geq \frac{\int_{M \times M} \Phi(\eta)\Phi(\zeta)(G_\xi^\delta(\eta))^{\frac{Q}{Q+\alpha}} dV_\delta(\eta)dV_\delta(\zeta)}{\|\Phi\|_{L^{2Q/(Q+\alpha)}(M)}^2}
\]

\[\geq \frac{(1 - \delta) \int_{\Sigma_R \times \Sigma_R} f_\epsilon(u)f_\epsilon(v)|u^{-1}v|^{\alpha-Q}(1 + O^1)^2 dV_\delta(u)dV_\delta(v)}{(1 + O^1)^{(Q+\alpha)/Q}\|f_\epsilon\|_{L^{2Q/(Q+\alpha)}(\Sigma_R)}^2}
\]
\[ \geq (1 - \delta)(1 + O^1)^{2-(Q+\alpha)/Q}(D_H - O(\frac{R}{\epsilon})^{-Q}). \]

Sending \( \epsilon \) to 0 and then letting \( R, \delta \) approach to zero, we obtain the estimate. \( \square \)

4. Subcritical HLS inequalities on compact CR manifold.

4.1. Subcritical HLS inequalities and their extremal function.

**Proposition 4.1** (Young’s inequality). Let \( X \) and \( Y \) are measurable spaces, and let the kernel function \( K : X \times Y \rightarrow R \) be a measurable function satisfying
\[ \int_X |K(x,y)|^r dx \leq C^r \quad \text{and} \quad \int_Y |K(x,y)|^s dy \leq C^s, \]
where \( C \) is some positive constant and \( r \geq 1 \). Then, for any \( f \in L^p(Y) \) with \( 1 - 1/r \leq 1/p \leq 1 \), the integral operator
\[ Af(x) = \int_Y K(x,y)f(y)dy \]
satisfies \( \|Af\|_{L^q(X)} \leq C\|f\|_{L^p(Y)} \), where \( 1 \leq p \leq q \) and \( \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 \).

**Proof.** For the case \( r = 1 \), the result reduces to the case of Lemma 15.2 of [11].

If \( r > 1 \), then \( 1 \leq p \leq \frac{r}{r-1} \). When \( p = 1 \), by Minkowski’s inequality,
\[ \left( \int_X |Af(x)|^r dx \right)^{1/r} \leq \int_Y \left( \int_X |K(x,y)f(y)|^r dx \right)^{1/r} dy \leq C\|f\|_{L^1(Y)}. \]

While for \( p > 1 \), noting \( (1 - \frac{1}{p}) + (1 - \frac{1}{r}) + \frac{1}{q} = 1 \), we have
\[ |Af(x)| \leq \int_Y |K(x,y)|^{r-1/p}|K(x,y)|^{1/q}|f(y)|^{p/q}|f(y)|^{p(1-1/r)}dy \]
\[ \leq \left( \int_Y |K(x,y)|^{r} dy \right)^{1-1/p} \left( \int_Y |K(x,y)|^{p/q} dy \right)^{1/q} \left( \int_Y |f(y)|^{p/q} dy \right)^{1-1/r}, \]
and then
\[ \left( \int_X |Af(x)|^q dx \right)^{1/q} \]
\[ \leq \left( \int_Y |K(x,y)|^{r} dy \right)^{1-1/p} \|f\|_{L^p(Y)}^{p(1-1/r)} \left( \int_X \int_Y |K(x,y)|^{r} |f(y)|^{p} dy dx \right)^{1/q} \]
\[ = \left( \int_Y |K(x,y)|^{r} dy \right)^{1/r} \|f\|_{L^p(Y)} \leq C\|f\|_{L^p(Y)}. \]

Since \( (G^\alpha_\xi(\eta))^{\frac{Q}{Q-\alpha}} \sim \rho(\xi, \eta)^{\alpha - Q} \) as \( \rho(\xi, \eta) \rightarrow 0 \) and \( (G^\alpha_\xi(\eta))^{\frac{Q}{Q-\alpha}} \in C^\infty(M \setminus \{\xi = \eta\}) \), then \( (G^\alpha_\xi(\eta))^{\frac{Q}{Q-\alpha}} \leq C\rho(\xi, \eta)^{\alpha - Q} \) and \( (G^\alpha_\xi(\eta))^{\frac{Q}{Q-\alpha}} \) satisfies
\[ \int_M |K(\xi, \eta)|^r dV_\alpha(\xi) \leq C \quad \text{and} \quad \int_M |K(\xi, \eta)|^r dV_\alpha(\eta) \leq C, \quad \forall 1 \leq r < \frac{Q}{Q - \alpha}. \]

So, we take \((G^\alpha_\xi(\eta))^{\frac{Q}{Q-\alpha}}\) as the kernel of Proposition 4.1 and have the following subcritical HLS inequalities.
Proposition 4.2. There exists a positive constant $C$ such that for any $f \in L^p(M)$ with $1 < p < \frac{Q}{\alpha}$, one has

$$\|Af(\xi)\|_{L^q(M)} \leq C\|f\|_{L^p(M)},$$

where $q > 1$ and $\frac{1}{q} > \frac{1}{p} - \frac{\alpha}{Q}$. Moreover, operator $A$ is compact for any $q$ satisfying $q > 1$ and $\frac{1}{q} > \frac{1}{p} - \frac{\alpha}{Q}$, namely, for any bounded sequence $\{f_j\}_{j=1}^{+\infty} \subset L^p(M)$, there exists a subsequence of $\{Af_j\}_{j=1}^{+\infty}$ which converges in $L^q(M)$.

Proof. Obviously, it is sufficient to prove the compactness of the operator $A$ with kernel $K(\xi,\eta) = \rho(\xi,\eta)^{n-Q}$.

Since the sequence $\{f_j\}$ is bounded in $L^p(M)$, then there exists a subsequence (still denoted by $\{f_j\}$) and a function $f \in L^p(M)$ such that

$$f_j \rightharpoonup f \text{ weakly in } L^p(M). \tag{4.2}$$

For $K(\xi,\eta)$, we decompose the integral operator as

$$\int_M K(\xi,\eta)f_j(\eta)d\nu(\eta) \triangleq \int_M K^s(\xi,\eta)f_j(\eta)d\nu(\eta) + \int_M K_s(\xi,\eta)f_j(\eta)d\nu(\eta),$$

where

$$K^s(\xi,\eta) = \begin{cases} K(\xi,\eta), & \text{if } \rho(\xi,\eta) > s, \\ 0 & \text{otherwise} \end{cases}$$

and $K_s(\xi,\eta) = K(\xi,\eta) - K^s(\xi,\eta), s > 0$ will be chosen later. Noting that, for any $\xi$, $\int_M |K^s(\xi,\eta)|^{\frac{Q}{Q-\alpha}}d\nu(\eta) \leq C_0$, where $C_0$ is independent on $\xi$, we know that $\int_M K^s(\xi,\eta)f_j(\eta)d\nu(\eta)$ converges pointwise to $\int_M K^s(\xi,\eta)f(\eta)d\nu(\eta)$ and $\int_M K_s(\xi,\eta)f_j(\eta)d\nu(\eta)$ is bounded uniformly. So, it is deduced by the dominated convergence that

$$\int_M K^s(\xi,\eta)f_j(\eta)d\nu(\eta) \rightarrow \int_M K^s(\xi,\eta)f(\eta)d\nu(\eta), \text{ in } L^q(M). \tag{4.3}$$

Next, we analyze the convergence of $\int_M K_s(\xi,\eta)f_j(\eta)d\nu(\eta)$ by Young’s inequality (Proposition 4.1). Take $r$ satisfying $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1$. Then, $r < \frac{Q}{Q-\alpha}$ and

$$\left(\int_M |K_s(\xi,\eta)|^r d\nu(\eta)\right)^{1/r} \leq Cs^\beta$$

with $\beta = Q(\frac{1}{r} - \frac{Q-\alpha}{Q})$. By the Young’s inequality, we have

$$\|\int_M K_s(\xi,\eta)(f_j(\eta) - f(\eta))d\nu(\eta)\|_{L^q(M)} \leq \|f_j - f\|_{L^p(M)} \left(\int_M |K_s(\xi,\eta)|^r d\nu(\eta)\right)^{1/r} \leq Cs^\beta. \tag{4.4}$$

By now, through choosing first $s$ small and then $j$ large, we deduce by (4.3) and (4.4) that

$$\int_M K(\xi,\eta)f_j(\eta)d\nu(\eta) \rightarrow \int_M K(\xi,\eta)f(\eta)d\nu(\eta), \text{ in } L^q(M).$$

So, we complete the proof. \qed
Define the extremal problem as
\[
D_{M,p,q} := \sup_{f \in L^p(M) \setminus \{0\}} \frac{\|Af\|_{L^q(M)}}{\|f\|_{L^p(M)}}
\]
\[
= \sup_{f \in L^p(M) \setminus \{0\}, g \in L^q(M)} \frac{\int_M f(\xi) g(\xi) (G^p_\xi(\eta))^{\frac{q}{q-2}} \eta \cdot dV_\theta(\eta)}{\|f\|_{L^p(M)} \|g\|_{L^q(M)}},
\]
where $1 < p < \frac{Q}{\alpha}$, $1 > \frac{1}{q} > \frac{1}{p} - \frac{q}{Q}$ and $q' = \frac{q}{q-1}$. Obviously, we know that $D_{M,p,q} < +\infty$ because of Proposition 4.2. Moreover, we have

**Theorem 4.3.** $D_{M,p,q}$ can be attained, namely, there exists a nonnegative function $f \in L^p(M)$ satisfying $\|f\|_{L^p(M)} = 1$ and $D_{M,p,q} = \|Af\|_{L^q(M)}$.

**Proof.** Without loss of generality, choose a nonnegative maximizing sequence $\{f_j\}_{j=1}^{+\infty} \subset L^p(M)$ such that $\|f_j\|_{L^p(M)} = 1$ and
\[
\lim_{j \to +\infty} \|Af_j\|_{L^q(M)} = D_{M,p,q}.
\]
Combining the boundedness of the sequence $\{f_j\}$ in $L^p(M)$ and the compactness result (see Proposition 4.2), there exists a subsequence (still denoted by $\{f_j\}$) and a function $f \in L^p(M)$ such that
\[
f_j \rightharpoonup f \quad \text{weakly in} \quad L^p(M),
\]
\[
Af_j \to Af \quad \text{strongly in} \quad L^q(M).
\]
Thus $\|f\|_{L^p(M)} \leq \liminf_{j \to +\infty} \|f_j\|_{L^p(M)}$ and
\[
D_{M,p,q} = \lim_{j \to +\infty} \frac{\|Af_j\|_{L^q(M)}}{\|f_j\|_{L^p(M)}} \leq \frac{\|Af\|_{L^q(M)}}{\|f\|_{L^p(M)}}.
\]
So, $f$ is a maximizer. \hfill \Box

4.2. Subcritical HLS inequalities for the diagonal case. To discuss the extremal problem related to $\mathcal{V}_\alpha(M)$, we need only to consider the diagonal case. Hence, corresponding to Theorem 4.3, we have

**Theorem 4.4.** For any $f \in L^p(M)$ with $\frac{2Q}{Q+p\alpha} < p < 2$, it holds
\[
|\int_M f(\xi) f(\eta) (G^p_\xi(\eta))^{\frac{q}{q-2}} \eta \cdot dV_\theta(\eta)| \leq D_{M,p} \|f\|_{L^p(M)},
\]
where the sharp constant
\[
D_{M,p} := \sup_{f \in L^p(M) \setminus \{0\}} \frac{\int_M f(\xi) f(\eta) (G^p_\xi(\eta))^{\frac{q}{q-2}} \eta \cdot dV_\theta(\eta)}{\|f\|_{L^p(M)}}
\]
can be attained by some nonnegative function $f_p \in L^p(M)$ satisfying $\|f_p\|_{L^p(M)} = 1$ and
\[
D_{M,p} = \int_M f(\xi) f(\eta) (G^p_\xi(\eta))^{\frac{q}{q-2}} \eta \cdot dV_\theta(\eta).
\]

**Remark 4.5.** A direct computation deduces that the extremal function $f_p$ satisfies the Euler-Lagrange equation
\[
2D_{M,p} f_p^{p-1}(\xi) = \int_M f(\eta) (G^p_\xi(\eta))^{\frac{q}{q-2}} \eta \cdot dV_\theta(\eta) + \int_M f(\eta) (G^p_\eta(\xi))^{\frac{q}{q-2}} \eta \cdot dV_\theta(\eta). \tag{4.6}
\]
Theorem 5.1. If Sharp HLS inequalities on compact CR manifold.

Lemma 5.2. Proposition 4.6

Proof. By Lemma 4.7, it is sufficient to prove that \( \Gamma \) uniformly bounded in \( \Gamma \).

Lemma 4.7. If \( f \in L^\infty(M) \), then \( A f \in \Gamma \).

Proof. Because of the compactness of \( M \), it is sufficient to prove that, for any \( \xi \in M \), Lemma 4.7 holds on the neighbourhood \( V_\xi \). Hence, without loss of generality, we restrict variable \( \xi \) on a neighbourhood \( V_\xi,0 \) for some point \( \xi_0 \in M \).

Using the Folland-Stein normal coordinates, we can complete the proof by a similar process of the second part of Lemma 4.3 of [16]. For concise, we omit the details.

Lemma 4.8. If \( g(\xi) \in L^p(M), 2 \leq p < \frac{2Q}{Q+\alpha}, \) satisfies (4.7), then \( g \in L^\infty(M) \).

The proof is a repetition line by line of the first part of Lemma 4.3 of [16]. So, it is omitted.

5. Sharp HLS inequalities on compact CR manifold.

Theorem 5.1. If \( \mathcal{Y}_\alpha(M) > D_H \), then \( \mathcal{Y}_\alpha(M) \) is attained by some function \( f \in \Gamma_\alpha(M) \).

Following, we will investigate the limitation of the sequence of solutions \( \{f_p\} \subset \Gamma_\alpha(M) \) of (4.6), and then complete the proof of Theorem 5.1 by compactness.

First, it is routine to prove

Lemma 5.2. \( D_{M,p} \to \mathcal{Y}_\alpha(M) \) as \( p \to \left( \frac{2Q}{Q+\alpha} \right)^+ \).

Proposition 5.3. If \( D_{M,p} \geq D_H + \epsilon \), then the positive solutions \( \{f_p\} \frac{Q}{Q+\alpha} \leq p < 2 \) is uniformly bounded in \( \Gamma_\alpha(M) \).

Proof. By Lemma 4.7, it is sufficient to prove that \( \{f_p\} \frac{Q}{Q+\alpha} \leq p < 2 \) is uniformly bounded in \( L^\infty(M) \). Following, we will prove it by contradiction. Suppose not.

Then \( f_p(\xi_p) \to +\infty \) as \( p \to \left( \frac{2Q}{Q+\alpha} \right)^+ \), where \( f_p(\xi_p) = \max_{\xi \in M} f_p(\xi) \).

Let \( \Theta_{\xi_p} \) be normal coordinates. We can assume that there is a fixed neighbourhood \( U = B_r(0) \) of the origin in \( \mathbb{H}^n \) contained in the image of \( \Theta_{\xi_p} \) for all \( p \), and for each \( p \) we will use \( \Theta_{\xi_p} \) to identify \( U \) with a neighbourhood of \( \xi_p \) with coordinates \( (z,t) = \Theta_{\xi_p} \).

For any \( u = (z,t) = \Theta_{\xi_p}(\xi) \in U \), we have

\[
2D_{M,p}f_p(\Theta_{\xi_p}^{-1}(u))p^{-1} := 2D_{M,p}f_p(\xi)p^{-1}
\]

\[
= \int_M f_p(\eta) (G_\Theta^p(\eta)) \frac{Q}{Q+\alpha} dV_\Theta(\eta) + \int_M f_p(\eta) (G_\Theta^p(\xi)) \frac{Q}{Q+\alpha} dV_\Theta(\eta)
\]

\[
= 2 \int_{\Theta_{\xi_p}^{-1}(U)} f_p(\eta) \rho(\xi,\eta) \frac{Q}{Q+\alpha} dV_\Theta(\eta)
\]
\[
+ \int_{E_p^{-1}(U)} f_p(\eta)E(\xi, \eta)dV_0(\eta) + \int_{E_p^{-1}(U)} f_p(\eta)E(\eta, \xi)dV_0(\eta)
+ \int_{M \setminus E_p^{-1}(U)} f_p(\eta)(G^0_\xi(\eta))^{\frac{Q-\alpha}{r}}dV_0(\eta) + \int_{M} f_p(\eta)(G^0_\eta(\xi))^{\frac{Q-\alpha}{r}}dV_0(\eta)
= 2 \int_{U} f_p(\Theta^{-1}_\xi(v))\rho(u, v)^{\alpha-Q}(1 + O^1)dV_0(v) + I + II
= 2(1 + O^1) \int_{U} f_p(\Theta^{-1}_\xi(v))\rho(u, v)^{\alpha-Q}dV_0(v) + I + II,
\]
where \( O^1 \to 0 \) as \( r \to 0 \),
\[
I := \int_{E_p^{-1}(U)} f_p(\eta)E(\xi, \eta)dV_0(\eta) + \int_{E_p^{-1}(U)} f_p(\eta)E(\eta, \xi)dV_0(\eta),
II := \int_{M \setminus E_p^{-1}(U)} f_p(\eta)(G^0_\xi(\eta))^{\frac{Q-\alpha}{r}}dV_0(\eta) + \int_{M} f_p(\eta)(G^0_\eta(\xi))^{\frac{Q-\alpha}{r}}dV_0(\eta)
\]
and
\[
E(\xi, \eta) = (G^0_\eta(\xi))^{\frac{Q-\alpha}{r}} - \rho(\xi, \eta)^{\alpha-Q} \leq C\rho(\xi, \eta)^{\alpha-2}
\]
for any \( \xi, \eta \in \Theta^{-1}_\xi(U) \).

Take \( \mu_p = \frac{2p}{p-2}(\xi_0) \) and \( g_p(u) = \frac{\alpha-\alpha}{r} f_p(\Theta^{-1}_\xi(\delta_{\mu_p}(u))) \), \( u \in B_{r/\mu_p}(0) \) with \( \delta_{\mu_p}(u) = (\mu_p^\gamma, \mu_p^\gamma) \). Then, \( g_p(u) \) satisfies
\[
2D_{M,p}g_p(u)^{p-1} = 2(1 + O^1) \int_{B_{r/\mu_p}(0)} g_p(v)\rho(u, v)^{\alpha-Q}dV_0(v) + I' + II',
\]
and \( g_p(0) = 1, g_p(u) \in (0, 1] \), where
\[
I' = \mu_p^{\frac{\alpha(p-1)}{p}} \times I \quad \text{and} \quad II' = \mu_p^{\frac{\alpha(p-1)}{p}} \times II \quad \text{with} \quad \xi = \Theta^{-1}_\xi(\delta_{\mu_p}(u)).
\]
Since
\[
I' \leq \mu_p^{\frac{\alpha(p-1)}{p}} \cdot C \int_{U} f_p(\Theta^{-1}_\xi(v))\rho(\delta_{\mu_p}(u), v)^{\alpha-2}dV_0(v)
= C\mu_p^{\frac{Q-2}{r}} \int_{B_{r/\mu_p}(0)} g_p(v)\rho(u, v)^{\alpha-2}dV_0(v)
\leq C\mu_p^{\frac{Q-2}{r}} \left( \frac{r}{\mu_p} \right)^{\alpha-2 + \frac{2}{p}} \left\| g_p \right\|_{L^p(B_{r/\mu_p}(0))}
\leq C\mu_p^{\frac{Q-2}{r}} \left( \frac{r}{\mu_p} \right)^{\alpha-2 + \frac{2}{p}} \mu_p^{\frac{\alpha(p-1)}{p}} \left\| f_p \right\|_{L^p(M)}
\]
with \( p^* = \frac{p}{p-1} \), we have \( I' \to 0 \) as \( p \to \left( \frac{2Q}{Q+\alpha} \right)^+ \). Now, we assume further that \( \xi \in \Theta^{-1}_\xi(B_{r/2}(0)) \). Then \( (G^0_\xi(\eta))^{\frac{Q-\alpha}{r}} \) is uniformly bounded on \( M \setminus \Theta^{-1}_\xi(U) \). Hence, on \( \Theta^{-1}_\xi(B_{r/2}(0)) \), \( I' \to 0 \) as \( p \to \left( \frac{2Q}{Q+\alpha} \right)^+ \).

On the other hand, for large \( R > 0 \) and \( u \in B_{R/2}(0) \), we have
\[
\int_{B_{r/\mu_p}(0) \setminus B_R(0)} g_p(v)\rho(u, v)^{\alpha-Q}dV_0(v)
\]
we have

Thus, therefore, we give a partial answer.

5.1. Discussion of the condition of strict inequality.

For a given compact locally CR conformal flat manifold \( M \), we know whether the strict inequality of Theorem 5.1 holds for every CR compact manifold \( M \) with \( \mathcal{Y}(M) > 0 \), which is not CR conformally to complex sphere \( (\mathbb{S}^{2n+1}, \theta_S) \). Following, we give a partial answer.

Proposition 5.4. For a given compact locally CR conformal flat manifold \( M \) with \( \mathcal{Y}(M) > 0 \), if there exists some point \( \xi_0 \in M \) such that \( A(\xi_0) = w(\xi_0, \xi_0) > 0 \) (see the definition of \( A(\xi) \) in (1.8)), then \( \mathcal{Y}_\alpha(M) > \mathcal{Y}_\alpha(\mathbb{S}^{2n+1}) \).
Proof. By the assumption of \( w(\xi, \eta) \in C^1(M \times M) \) and \( A(\xi_0) > 0 \), there exists a small neighbourhood \( V_{\xi_0} \subset M \) and some positive constant \( A_0 \) such that \( A(\xi) > A_0 > 0 \). Moreover, on \( V_{\xi_0} \times V_{\xi_0} \), there exists some positive constant \( C \) such that
\[
\left( G^\theta_{\xi}(\eta) \right)^{\frac{Q-\alpha}{Q-2}} \geq \rho(\xi, \eta)^{\alpha-Q} + C \rho^{\alpha-2}.
\]
Then by a similar argument with Proposition 2.9 of [17], we can complete the proof.

Remark 5.5. In [5], Cheng, Malchiodi and Yang have proved a class of positive mass theorem in three dimensional CR geometry. Namely, under the assumptions of \( \mathcal{Y}(M) > 0 \) and the non-negativity of the CR Paneitz operator, they proved that, if \( M \) is not CR equivalent to \( S^3 \) (endowed with its standard CR structure), then \( A(\xi) > 0 \). Therefore, our main result holds for three dimensional compact CR manifold without boundary. While for other cases, it is still an open question as far as we know.

Appendix A. CR conformality. For completeness, we give the CR conformality of operator \( I_{M, \theta, \alpha} \) defined in (1.3).

Proposition A.1. Let \( \tilde{\theta} = \phi^{\frac{1}{Q-2}} \theta \). Then
\[
G^\tilde{\theta}_{\xi}(\eta) = \phi^{-1}(\xi)\phi^{-1}(\eta)G^\theta_{\xi}(\eta). \tag{A.1}
\]

Proof. It is sufficient to prove that, for any \( u \in C^2(M) \),
\[
\int_M G^\tilde{\theta}_{\xi}(\eta)u(\eta) \, dV_{\tilde{\theta}} = \int_M \phi^{-1}(\xi)\phi^{-1}(\eta)G^\theta_{\xi}(\eta)u(\eta) \, dV_{\theta}. \tag{A.2}
\]
In fact, by (2.2), we have
\[
L_{\tilde{\theta}} \left[ \int_M \phi^{-1}(\xi)\phi^{-1}(\eta)G^\theta_{\xi}(\eta)u(\eta) \, dV_{\theta} \right] = \phi^{-\frac{Q-\alpha}{Q-2}}(\xi) L_\theta \left[ \int_M \phi^{-1}(\eta)G^\theta_{\xi}(\eta)u(\eta) \, dV_{\theta} \right] = \phi^{-\frac{Q-\alpha}{Q-2}}(\xi) L_\theta \left[ \int_M \phi^{-1}(\eta)G^\theta_{\xi}(\eta)u(\eta) \phi(\eta) \frac{2Q}{Q-2}(\eta) \, dV_{\theta} \right] = u(\xi).
\]
So, we complete the proof by the property of Green function.

Proposition A.2. For positive constant \( \alpha \neq Q \), we have
\[
I_{M, \tilde{\theta}, \alpha}(u) = \phi^{-\frac{Q-\alpha}{Q-2}} I_{M, \theta, \alpha}(\phi^{\frac{Q-\alpha}{Q-2}} u). \tag{A.3}
\]

Proof.
\[
I_{M, \tilde{\theta}, \alpha}(u) = \int_M \left[ G^\tilde{\theta}_{\xi}(\eta) \right]^\frac{Q-\alpha}{Q-2} u(\eta) \, dV_{\tilde{\theta}} = \int_M \left[ G^\theta_{\xi}(\eta) \right]^\frac{Q-\alpha}{Q-2} \phi(\eta) \frac{Q-\alpha}{Q-2} u(\eta) \, dV_{\theta}.
\]

Remark A.3. When \( \alpha = 2 \), (1.4) is
\[
I_{N, \tilde{\theta}, 2}(u) \phi(\xi) = \int_N G^\tilde{\theta}_{\xi}(\eta)u(\eta) \phi(\eta) \frac{Q+2}{Q-2} \theta \wedge d\theta^\alpha. \tag{A.4}
\]
For given nonnegative function $u$ and taking $\phi(\xi) = \frac{\Phi(\xi)}{I_{N,\tilde{\theta},2}(u)}$, then (A.4) becomes

$$\Phi(\xi) = \int_N G^0_{\xi}(\eta) \tilde{R}(\eta) \Phi(\eta) \frac{Q+2}{Q-2} \theta \wedge d\theta^n$$

with $\tilde{R}(\eta) = u(\eta) I_{N,\tilde{\theta},2}(u) \frac{Q+2}{Q-2}$, which is the integral type curvature equation (1.2). Particular, case $u \equiv \text{constant}$ is the CR Yamabe problem.

Similarly, (A.3) is corresponding to a class of general integral equation

$$\Phi(\xi) \frac{Q-\alpha}{Q-2} = \int_N G^0_{\xi}(\eta) \tilde{R}(\eta) \Phi(\eta) \frac{Q+2}{Q-2} \theta \wedge d\theta^n.$$

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