Numerical solution the Volterra dual integral equation of the first kind based on a method of Runge-Kutta type

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Abstract. The paper focuses on the non-classical Volterra dual equation of the first kind, to which the problem of identifying non-symmetric Volterra kernels is reduced. For this equation, we obtained difference analogs based on the application of the Runge-Kutta type method. Such equations arise in the problem of modeling nonlinear dynamical systems of black-box type, which employ the Volterra integro-power series, when the input perturbation is a vector function of time.

1. Introduction
Currently, there are many different approaches to modeling dynamic processes in technical systems. Each of the mathematical approaches has advantages and disadvantages. This paper focuses on the integro-power Volterra series

\[ y(t) = \sum_{m=1}^{N} \sum_{1 \leq i_1 \leq \ldots \leq i_m \leq p} f_{i_1 \ldots i_m}(t) \text{,} t \in [0, T], \]

\[ f_{i_1 \ldots i_m}(t) = \int_0^t \int_0^t \prod_{j=1}^{m} K_{i_j}(s_j) \, dx_j \, ds_j, \]

which are applied to describe the black-box systems. Here, input \( x(t) = (x_1(t), \ldots, x_p(t)) \) is a vector function of time, and output \( y(t) \) is a scalar function of time, \( K_{i_1 \ldots i_m} \) are Volterra kernels (symmetrical with respect to variables \( s_1, \ldots, s_m \), which correspond to the coinciding indices \( i_1, \ldots, i_m \)).

One of the strongest advantages of this tool is that the model built with it will be fast, while its significant drawback is the difficulty in identifying Volterra kernels.

Despite the non-trivial problem of identifying Volterra kernels, this tool is used in many areas of natural science. For example, it is used in the description of thermophysical [1, 2] and electrophysical [3] systems; sensorimotor mechanisms [4]; when diagnosing malfunctions of technical facilities [5], in aerodynamics [6]; when modeling biological systems [7, 8] and adaptive automatic control systems [9, 10]. It is also used to forecast non-stationary time series [11] and to filter signal and image distortions in telecommunication and visualization systems [12, 13].

2. Problem statement
Let in (1), \( N=2, p=2 \). Assume that the problem of decomposition of output \( y(t) \) into components \( f_i(t), f_{ij}(i=1,2), \) and \( f_{12}(t) \) is solved. Consider the problem of search of non-symmetrical kernel \( \hat{K}_{12} \) [14]

\[ f_{12}(t) = \int_0^t \int_0^t K_{12}(s_1, s_2) x_1(t-s_1) x_2(t-s_2) \, ds_1 \, ds_2, \quad t \in [0, T]. \]

Determine the inputs \( x_1, x_2 \) of the form (they are graphically presented in figure 1, 2)
by matching the points of discontinuity with the nodes of uniform mesh. Here $h > 0$ is a sampling interval of the output, $T = Nh$, $e(t)$ is Heaviside function.

Figure 1. Test signals $x_1(t)$ and $x_2(t)$.

Figure 2. Test signals $x_{1v}(t)$ and $x_2(t)$.

Substitution of (4), (5) in (3) yields a two-dimensional dual Volterra integral equation of the first kind

\[
\int_{t-h}^{t} \int_{t-v-h}^{t} ds_1 \int_{t-v-h}^{t} K_{12}(s_1,s_2) \; ds_2 = f(t,v),
\]

(6)

\[
\int_{t-v}^{t-v-h} \int_{t-v-h}^{t} ds_1 \int_{t-v-h}^{t} K_{12}(s_1,s_2) \; ds_2 = f(t,v),
\]

(7)

which is analytically solved as follows [15]:

\[
\overline{K}_{12}(t,v)=\sum_{i=1}^{N+1} D_2 f(t) \int_{N \in H} \left( K^{(i-1)}_{12}(t,v-h)+K^{(i-1)}_{12}(v-t-h) + K^{(i-1)}_{12}(v-t-v-h) \right),
\]

(1)

\[
\overline{K}_{12}(t,v,t)=\sum_{i=1}^{N+1} D_2 f(t) \int_{N \in H} \left( K^{(i-1)}_{12}(t-v-h)+K^{(i-1)}_{12}(v-t-h) + K^{(i-1)}_{12}(v-t,v-h) \right),
\]

(2)

\[
D_2 f(t,v)=\left( f^{(1)}_{tv} + f^{(1)}_{vv} \right), \quad D_2 f(t,v)=\left( f^{(2)}_{tv} + f^{(2)}_{vv} \right).
\]
Here \( N=\frac{T}{h}, \ k=\frac{1}{h}, \ \Delta_k=[t, v; v+h\leq t, \ kh\leq t<(k+1)h], \ \Delta_{N+1}=[t, v; v+h\leq t, \ Nh\leq t\leq T], \) where \( v\geq 0, \ \Delta_0=[t, v; G_1\cup G_2, \ v\geq 0], \ G_1=[t, v; v\leq t, \ 0\leq h\leq t], \ G_2=[t, v; t-h\leq v\leq t, \ h\leq t], \) \( N(t,v) \) is a point in the plane with Cartesian coordinates. In this case \( \kappa_{12}(s,s_2)=\kappa_{12}^{(0)}(s_1, s_2), \ \kappa_{12}(s_2)=\kappa_{12}^{(0)}(s_1, s_2), \) where \( s_1, s_2 \in [0,h], \ s \in [0,T], \) are considered to be given.

3. Numerical solutions

In [16], the authors give quadrature formulas characterized by the property of self-regularization, where mesh step is a natural parameter of regularization. These formulas include methods of right and left rectangles, which have the first order of convergence, the method of middle rectangles (the second order of convergence), and methods of a higher order, for example, the methods of the Runge-Kutta type [17]. We will use the method of the Runge-Kutta type to solve (6), (7).

To use this method, we introduce the following uniform mesh on \( h\leq t\leq T \) (in a general case the submesh for this method can be non-uniform)

\[
t_i, \ \ i=1,n, \ \ \ v_j, \ \ j=0,n-1, \ \ \ h=\frac{T}{n},
\]

\[
\tau_p, \ \ p=1,m, \ \ 0<u_1<u_2<\ldots<u_m=1,
\]

\[
\nu_{pq}, \ \ q=1,m, \ \ 0<\nu_1<\nu_2<\ldots<\nu_m=1.
\]

where \( m \) is the number of nodes on the submesh.

We will consider only the first part of equation (6), (7), because the second part is considered similarly. Further, assuming that \( t=\tau_p, \ v=\nu_{pq} \) in (6), we switch to the system of numerical equalities

\[
(1) \quad f(t_{pq}, v_{pq}) = \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} k_{12}(s_1, s_2) \ ds_2, \quad i=1,n, \quad j=0,n-1, \quad p=1,m, \quad q=1,m.
\]

Due to the continuity of the kernel, we separate the integrals so that a “whole” node is highlighted (a graphical interpretation is presented in figure 3), i.e.

\[
(1) \quad f(t_{pq}, v_{pq}) = \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} k_{12}(s_1, s_2) \ ds_2 =
\]

\[
= \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} K_{12}(s_1, s_2) \ ds_2 + \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} K_{12}(s_1, s_2) \ ds_2 =
\]

\[
= \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} K_{12}(s_1, s_2) \ ds_2 + \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} K_{12}(s_1, s_2) \ ds_2 +
\]

\[
+ \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} K_{12}(s_1, s_2) \ ds_2 + \int_{\tau_p}^{\tau_{p+1}} \int_{\nu_{pq}}^{\nu_{pq+1}} K_{12}(s_1, s_2) \ ds_2.
\]
Approximate each integral by quadrature formulas:
\[
\int_{t_i}^{t_{i+1}} \phi(s) ds = h \sum_{k=1}^{m} a_k \phi(t_k), \quad p=1, m,
\]
\[
\int_{t_{i}}^{t_{i+1}} \phi(s) ds = h \sum_{k=1}^{m} b_k \phi(t_k), \quad p=1, m
\]
with coefficients
\[
a_k = \int_{0}^{u_p} L_k(s) ds, \quad b_k = \int_{u_p}^{1} L_k(s) ds, \quad p=1, m, \quad k=1, m
\]
where
\[
L_k = \frac{\omega(t)}{(t-u_k)\omega^\prime (u_k)}, \quad k=1, \quad \omega(t) = \prod_{k=1}^{m} (t-u_k).
\]
These formulas are based on the approximation of the function \(\phi(t)\) by Lagrange interpolation polynomials of the \(m\)-th degree. To make the expression shorter, we will introduce the following notation
\[
r = \{ p-q, p>q, \}
\]
Thus, we obtain a numerical scheme for (6)
\[
\sum_{k=1}^{m} \sum_{i=1}^{m} b_k b_i K_{12} (t_{i-k, t_{i-j, 1}})^r + \sum_{k=1}^{m} \sum_{i=1}^{m} a_k a_i K_{12} (t_{i-k, t_{i-j, 1}})^r + \sum_{k=1}^{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_k b_i a_i b_i K_{12} (t_{i-k, t_{i-j, 1}})^r = f (t_{i-p}, v_{j})
\]  
(8)

here \(i=1, n, j=0, n-1\) and \(t_{i,j} = t_i - v_j\). In the case of (7), we act in the same way and obtain
\[
\sum_{k=1}^{m} \sum_{i=1}^{m} b_k b_i K_{12} (t_{i-k, t_{i-j, 1}})^r + \sum_{k=1}^{m} \sum_{i=1}^{m} a_k a_i K_{12} (t_{i-k, t_{i-j, 1}})^r + \sum_{k=1}^{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_k b_i a_i b_i K_{12} (t_{i-k, t_{i-j, 1}})^r = f (t_{i-p}, v_{j}),
\]  
(9)
where \( i = 2n \), \( j = 0, 1, 2 \) and \( t_{ij} = t_i v_j \).

Example.

Let the exact solution to (6), (7) has the following analytical form:

\[
K_{12}(s_1, s_2) = a \sin(s_1) - b \cos(s_2),
\]

Then the right-hand sides can be represented by the functions

\[
\begin{align*}
(1) \quad f(t, v) &= 2h \sin \left( \frac{h}{2} \left[ a \cos \left( \frac{t}{2} \right) - b \sin \left( \frac{t-v}{2} \right) \right] \right), \\
(2) \quad f(t, v) &= 2h \sin \left( \frac{h}{2} \left[ a \cos \left( \frac{t-v}{2} \right) - b \sin \left( \frac{t}{2} \right) \right] \right).
\end{align*}
\]

Substitute (10) in (8), (9) (let \( m = 3 \)) and find the error by the formulas

\[
\begin{align*}
\varepsilon_1 &= \max_i \left| K_{ij}^h - K(t_i, t_{ij}) \right|, \quad \varepsilon_2 = \max_i \left| K_{ij}^h - K(t_{ij}, t_i) \right|.
\end{align*}
\]

The results of the calculation of \( \varepsilon = \max \{ \varepsilon_1, \varepsilon_2 \} \) for \( a = 4, b = -1 \) are presented in Table 1.

**Table 1.** The quadrature method of middle rectangles.

| \( h \)  | \( \varepsilon \)       |
|---------|-------------------------|
| 0.25    | 0.00038642              |
| 0.125   | 0.00005431              |
| 0.0625  | 0.00000716              |

In this case, the method has a cubic rate of convergence, namely, when the mesh step is doubled, the error \( \varepsilon \) decreases by a factor of eight.

4. Application

In the future, this numerical algorithm is planned to be applied in modeling the dynamics of thermal devices being part of the power unit at the Nazarovskaya CPP. This plant is one of the largest electricity producers in Eastern Siberia and the Far East.

Figure 4 demonstrates a simplified diagram of the power unit with a capacity of 135 MW [18]. This diagram was built on the basis of [19]. It includes the following components (figure 4): D is thermal deaerator, FEP is a system of feed water pumps, OTB-1, OTB-2 a once-through boilers, HPC is high pressure cylinder, MPC is medium pressure cylinder, LPC is low pressure cylinder, BS is boiler system, C is condenser, CPS is condensate pump system, LPH a low pressure heaters, HPH a high pressure heaters.
Figure 4. Block diagram of the plant cycle.

The simulation model of this power unit was implemented with the aid of “P150” software, which represents the development of the model of the Irkutsk CHP-10 power unit [19]. The dotted line in Figure 4 shows a section chosen to study the nonlinear dynamics of pressure and temperature at the outlet of the heat exchangers. This section includes a condenser C of “80- type and a low-pressure heater LPH-1 from the LPH group.

The authors of [20] built two models. The first of them was built on the assumption that the steam flows $D_{1s}$ and $D_{2s}$, that are supplied from LPC and MPC to condenser and LPH, respectively, are constant. The input for such a model was water flow $D_w$. The second model was built on the assumption that the steam flow $D_{2s}$ and water flow $D_w$ are constant, and the input signal was steam flow $D_{1s}$. The deviations of pressure $\Delta p$ and temperature $\Delta t_1$ in condenser as well as the deviations of temperature $\Delta t_2$ in LPH-1 were considered as output signals in both models.

We plan to build a model based on the previous two as follows: the water flow $D_w$ and steam flow $D_{1s}$ will be supplied simultaneously to the outlet, whereas the stationary values will remain the same.

should be taken into account when preparing them.

5. Conclusion
The paper presents an algorithm for solving the two-dimensional Volterra dual integral equation of the first kind, which appears in the problem of identifying non-symmetric Volterra kernels. A mesh analog of the solution based on a Runge-Kutta type method is presented. The practical importance of the research related to modeling the dynamics of a large thermal facility is noted.

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