Possible Wormhole Solutions in (4+1) Gravity

A. G. Agnese\textsuperscript{1} A. P. Billyard\textsuperscript{2} H. Liu\textsuperscript{3} P. S. Wesson \textsuperscript{4}

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\textsuperscript{1}Dipartimento di Fisica, Università di Genova, Instituto Nazionale di Fisica Nucleare, Sezione di Genova, Italy
\textsuperscript{2}Dept. Physics, Dalhousie Univ., Halifax, NS, Canada
\textsuperscript{3}Dept. Physics, Univ. Waterloo, Waterloo, ON, Canada
\textsuperscript{4}Dept. Physics, Univ. Waterloo, Waterloo, ON, Canada
Abstract

We extend previous analyses of soliton solutions in $(4+1)$ gravity to new ranges of their defining parameters. The geometry, as studied using invariants, has the topology of wormholes found in $(3+1)$ gravity. In the induced-matter picture, the fluid does not satisfy the strong energy conditions, but its gravitational mass is positive. We infer the possible existence of $(4+1)$ wormholes which, compared to their $(3+1)$ counterparts, are less exotic.
1 Introduction

Multi-dimensional theories of gravity have been studied in great detail since 1921 [1, 2, 3]. For extensive reviews of these theories, we refer the reader to papers by Duff [4] and by Overduin and Wesson [5]. In five dimensions, there is considerable literature, both in cosmology and astrophysics. Specific to the latter there is a class of static, spherically symmetric solutions which are parameterized by three constants: $M, \epsilon, \kappa$, (the second two of which are related to one another). These solutions are the analogues to the four-dimensional Schwarzschild solution, allowed by the non-applicability of Birkhoff’s theorem in (4+1) dimensions. These “soliton” solutions were first extensively studied by Gross and Perry [6] and by Davidson and Owen [7], although the solutions were known previous to them. These solutions have been studied within the induced-matter theory of Wesson [8, 9], in which the five-dimensional vacuum solutions give rise to four-dimensional static, spherically symmetric solutions with a radiation-like matter field.

In a recent paper [10], the definition of the (three-dimensional) spatial origin to these solitons was discussed, as were the definitions of mass. The assumptions $\epsilon > 0$ and $\kappa > 0$ were used in order for the induced four-dimensional gravitational mass, pressure and density to be positive. Below, we relax previous assumptions and consider $\epsilon < 0$ and $\kappa < 0$, finding that while the induced matter does not satisfy the strong energy condition, its gravitational mass is still positive. We also study the geometry of the (4+1) spacetime using invariants, finding it has the topology typical of wormholes [11, 12, 13]. We infer the possible existence of (4+1) wormholes which, compared to their (3+1) counterparts, are in some ways less “exotic”.
2 Five-Dimensional Soliton Solutions

Following Davidson and Owen's notation [7], the line element of the solitons can be written

\[ ds^2 = T^2(\rho)dt^2 - S^2(\rho)(d\rho^2 + \rho^2 d\Omega^2) - \Phi^2(\rho)d\psi^2, \]  

where

\[ T(\rho) = \left(\frac{\rho - M/2}{\rho + M/2}\right)^{\epsilon k}, \]  \hspace{1cm} (2a)

\[ S(\rho) = \left(1 - \frac{M^2}{4\rho^2}\right)\left(\frac{\rho + M/2}{\rho - M/2}\right)^{\epsilon(k-1)}, \]  \hspace{1cm} (2b)

\[ \Phi(\rho) = \left(\frac{\rho + M/2}{\rho - M/2}\right)^\epsilon. \]  \hspace{1cm} (2c)

Here \( d\Omega \equiv d\theta^2 + \sin^2\theta d\phi^2 \) as usual, \( M \) is a constant with units of mass, and we set \( c = G = 1 \).

The two constants \( \epsilon \) and \( \kappa \) are related by \( \epsilon = \pm 1/\sqrt{\kappa^2 - \kappa + 1} \), but we leave both explicit for algebraic convenience. In [10] it was shown that for \( 0 < \kappa < \infty \) and \( \epsilon > 0 \) these solutions represent naked singularities with an origin located at \( \rho = \frac{1}{2}M \). In the limiting case \( (\epsilon, \kappa, \epsilon\kappa) \to (0, \infty, 1) \), which is the “Schwarzschild” limit, a black hole is obtained with its origin located at \( \rho = -\frac{1}{2}M \). These results were obtained from examining at which radii do surface areas vanish, and from examining divergences in the Kretchmann scalar

\[ R_{ABCD}R^{ABCD} = \left[ \rho^4 - 2\epsilon(\kappa - 1)(2 + \epsilon^2\kappa)\rho^3 \left(\frac{M}{2}\right) + 2(3 - \epsilon^4\kappa^2)\rho^2 \left(\frac{M}{2}\right)^2 \right. \]
\[ \left. -2\epsilon(\kappa - 1)(2 + \epsilon^2\kappa)\rho \left(\frac{M}{2}\right)^3 + \left(\frac{M}{2}\right)^4 \right] \cdot \frac{48M^2\rho^6}{(\rho^2 - M^2)^8} \left(\frac{\rho - M}{2}\right)^{4\epsilon(k-1)}. \]  

(3)
Here, large Latin indices run 01234, and below we will use small Greek indices that run 0123.

In Billyard, Kalligas and Wesson [10] a restriction of $\epsilon > 0$ and $\kappa > 0$ was made based on induced-matter arguments. However, the gravitational and inertial mass (defined in five dimensions) are

\[
M_{\text{grav}} = \epsilon \kappa M, \\
M_{\text{inert}} = \epsilon (\kappa - \frac{1}{2}) M.
\]

These only require $\epsilon \kappa > 0$ and $\epsilon (\kappa - 1/2) > 0$. For $\epsilon > 0$, this sets $\kappa > \frac{1}{2}$, but for $\epsilon < 0$ one may have $k < 0$ and both masses will still remain positive. In fact, the only qualitative difference the latter makes to (1) is that $\Phi(\rho)$ is inverted. It is this latter range for $(\epsilon, \kappa)$ which was not previously considered and which leads to possible wormhole solutions.

The existence of these is not apparent in the $\rho$ coordinates of (1), but will become so if we consider an $r-$coordinate give by

\[
r = \rho \left(1 - \frac{M^2}{4\rho^2}\right) \left(\frac{\rho + M/2}{\rho - M/2}\right)^{\epsilon(k-1)}.
\]

The nature of this transformation is shown in Figure 1. The curve which ends at $\rho = \frac{1}{2} M$ is representative of the transformation for $\epsilon > 0$, $\kappa > 0$ for which $-1 \leq \epsilon(\kappa - 1) < 1$. The curve which diverges at $\rho = 0$ is the limiting Schwarzschild curve. The solid curve represents the transformation (1) for which $\epsilon < 0$, $\kappa < 0$ such that $1 < \epsilon(\kappa - 1) < 2/\sqrt{3}$. In this regime,
the minimum of \( r \) is located at \((\rho_T, r_T)\) and is at

\[
\rho_T = \frac{1}{2} M \left[ \epsilon(\kappa - 1) + \sqrt{\epsilon^2(\kappa - 1)^2 - 1} \right],
\]

\[\text{(5a)}\]

\[
r_T = M \left[ \frac{\epsilon(\kappa + 1) + 1}{\epsilon(\kappa + 1) - 1} \right]^{1/2} \epsilon^2 - \frac{1}{2}.\]

\[\text{(5b)}\]

The location of this minimum ranges from \( \rho_T \approx 0.9M, r_T \approx 2.6M \) for \( \kappa = -1/2 \), to \((\frac{1}{2}M, 2M)\) for \( \kappa \to -\infty \). Under transformation (4), the isotropic spacetime of (1) becomes

\[
ds^2 = A^{\kappa} dt^2 - \frac{A}{B^2} dr^2 - r^2 d\Omega^2 - A^{-\epsilon} d\psi^2,
\]

\[\text{(6)}\]

where

\[
A = 1 - \frac{2M}{R},
\]

\[\text{(7a)}\]

\[
B = 1 - \frac{[1 + \epsilon(k - 1)] M}{R},
\]

\[\text{(7b)}\]

\[
R \equiv \rho \left( 1 + \frac{M}{2\rho} \right)^{2}.
\]

\[\text{(7c)}\]

These “quasi-curvature” coordinates are useful in calculations such as surface areas of spheres at fixed radii: e.g., \( \mathcal{A} \equiv 4\pi r^2 \). It is apparent from (3) that there is a divergence in the metric (the \( g_{rr} \) term) at \( R = [\epsilon(\kappa - 1) + 1]M \), where \( \rho = \rho_T \) and \( r = r_T \). This is a coordinate artifact; the other metric components neither diverge nor vanish at this point, and the Kretschmann scalar (3) remains well behaved. This suggests that this is the throat of a wormhole. If so, an observer travelling from \( r > r_T \) towards the throat would reach \( r = r_T \) in a finite time (\( g_{tt} \) is well defined) and would then proceed to travel in \( r > r_T \) in the other spacetime. The
The surface area of the throat is determined by the values of $\epsilon$ and $\kappa$, namely,

$$A = 4\pi M^2 \left[ \frac{\epsilon(\kappa + 1) + 1}{\epsilon(\kappa + 1) - 1} \right]^{(\kappa - 1) + 1}.$$

For $-\infty < \kappa < 0$, there is still a curvature singularity present at $\rho = \frac{1}{2} M$, so only one of the spacetimes is asymptotically flat ($\rho > \rho_T$), whilst the other has a curvature singularity at $r = \infty$, ($\rho = \frac{1}{2} M$).

### 3 Induced Matter

In the induced-matter approach to $(4 + 1)$ gravity [5, 8, 9, 10], the field equations in terms of the five-dimensional Ricci tensor are

$$R_{AB} = 0,$$

but the first ten components are rewritten as the four-dimensional Einstein equations with an effective or induced energy-momentum tensor given by

$$8\pi T_{\alpha\beta} = \frac{\Phi_{\alpha\beta}}{\Phi} - \frac{\pm 1}{2\Phi^2} \left\{ \frac{\Phi g_{\alpha\beta}}{\Phi} - g_{\alpha\beta} + g_{\mu\nu} g_{\alpha\beta}^{*} + g_{\mu\nu}^{*} g_{\beta\alpha}^{*} - \frac{g_{\mu\nu} g_{\alpha\beta}^{*} g_{\alpha\beta}^{*}}{2} + \frac{g_{\alpha\beta}^{*}}{4} \left[ g_{\mu\nu}^{*} g_{\mu\nu}^{*} + \left( g_{\mu\nu}^{*} g_{\mu\nu}^{*} \right)^2 \right] \right\}.$$

Here the extra metric component is $g_{44} = g_{\psi\psi} = \pm \Phi^2$ ($g_{4\alpha} = 0$), a semicolon represents the usual $(3+1)$ covariant derivative, $\Phi_{\alpha} \equiv \partial \Phi / \partial x^{\alpha}$ and an overstar represents $\partial / \partial \psi$. The
soliton solutions \((1), (2)\) satisfy \((9)\), and its associated fluid has matter properties given by \((10)\). For \((1)\) it was found \([8]\) that the induced matter was a bath of radiation, satisfying 
\[
\mu = 3p,
\]
where \(\mu\) is the energy density and \(p\) is the average pressure.

As previously mentioned, the assumption of \(\epsilon > 0, \kappa > 0\) was used because in the induced-matter scenario the effective four-dimensional mass-energy density, averaged pressure and induced gravitational mass are respectively \([8, 9]\),
\[
\begin{align*}
8\pi\mu &= \frac{\epsilon^2 \kappa M^2}{\rho^4 \left(1 - \frac{M^2}{4\rho^2}\right)^4} \left(\frac{\rho - \frac{M^2}{2}}{\rho + \frac{M^2}{2}}\right)^{2\epsilon(\kappa-1)}, \quad (11a) \\
8\pi p &= \frac{1}{3} \mu, \quad (11b) \\
M_{grav} &= \epsilon \kappa \left(\frac{\rho - \frac{M^2}{2}}{\rho + \frac{M^2}{2}}\right)^{\epsilon} M. \quad (11c)
\end{align*}
\]

Clearly, the strong energy condition, \(\mu + 3p > 0\), is satisfied only for \(\kappa > 0\) and so one further assumes \(\epsilon > 0\) so that \(M_{grav} > 0\). However, even if \(\kappa < 0\) and \(\epsilon < 0\) so that the induced matter does not satisfy the strong energy condition, the induced gravitational mass is still positive. Violation of the strong energy condition is not new, and is always encountered when considering wormholes in conventional four-dimensional theories (see for example \([11, 12, 13]\)).

With the induced matter, we are in a position to calculate the tension of the wormhole’s throat, which is the negative value of the radial pressure \([12]\). The pressure \((11b)\) is obtained from the average of the three pressures
\[
8\pi p_r \equiv 8\pi T_{rr} = \frac{8\pi \mu}{\epsilon \kappa} \left[\frac{M}{2\rho}\right] \left[\left(\frac{2\rho}{M}\right)^2 - \epsilon(\kappa - 2) \frac{2\rho}{M} + 1\right], \quad (12a)
\]
\[ 8\pi p_\theta \equiv 8\pi T^\theta_\theta = -\frac{8\pi \mu}{2\epsilon \kappa} \left[ \frac{M}{2\rho} \right] \left[ \left( \frac{2\rho}{M} \right)^2 - 2\epsilon (\kappa - 1) \frac{2\rho}{M} + 1 \right], \quad (12b) \]

\[ 8\pi p_\phi = 8\pi p_\theta. \quad (12c) \]

Now, for \( \kappa < 0 \), \( p_r \) remains negative throughout \( \frac{1}{2} M < \rho < \infty \), whereas the transverse pressures \( p_\theta \) and \( p_\phi \) are negative for \( \rho < \rho_T \), zero at the throat, and positive for \( \rho > \rho_T \).

Through some algebra, it may be verified that at the throat, the tension is

\[ \tau \equiv -p_r = -\mu > 0, \quad (13) \]

where \( 0 < \epsilon \kappa < 1 \). We conclude this calculation with the note that regular four-dimensional wormholes typically have \( \tau > \mu \) \([12]\), which is not found here.

Whilst we are considering the induced matter of \([\Pi]\), let us examine the scalars \( I_1 \equiv R^\alpha_\alpha \), \( I_2 \equiv R^\alpha_\beta R^\alpha_\beta \) and \( I_3 \equiv R^\alpha_\beta_\gamma_\delta R^\alpha_\beta_\gamma_\delta \) on the four-dimensional hypersurfaces where the induced matter is defined. This will enable us to ascertain where four-dimensional singularities occur. Because the induced matter is radiation-like, then \( I_1 \equiv 0 \). The other two scalars are given by

\[ I_2 = \frac{6\epsilon^2 M^2}{R^6} A^{2[\epsilon(\kappa - 1)-1]} \left[ 1 + \frac{D_1 M}{RA} + \frac{D_2 M^2}{R^2 A^2} \right], \quad (14) \]

\[ I_3 = \frac{48 M^2}{R^6} A^{2[\epsilon(\kappa - 1)-1]} \left[ D_3 + \frac{D_4 M}{RA} + \frac{D_5 M^2}{R^2 A^2} \right], \quad (15) \]

where

\[ D_1 = 2 - \frac{2}{3} \epsilon \kappa + 3\epsilon, \quad (16a) \]
\[ D_2 = \frac{1}{6}[\epsilon^2 \kappa^2 + 2(\epsilon(k - 1) - 1)^2 + (\epsilon(k - 2) - 2)^2], \]  \tag{16b} \\
\[ D_3 = 1 - \frac{1}{2}\epsilon^2, \]  \tag{16c} \\
\[ D_4 = -\frac{1}{3}[a^2(\epsilon(3\kappa - 2) - 3) + (\epsilon(\kappa - 1) - 1)^3 - (\epsilon(\kappa - 1) - 1)], \]  \tag{16d} \\
\[ D_5 = \frac{1}{12}[a^2(\epsilon(2\kappa - 1) - 2)^2 + (\epsilon(\kappa - 1) - 1)^4 + 2(\epsilon^2 \kappa^2 + 1)(\epsilon(\kappa - 1) - 1)^2]. \]  \tag{16e} \\

For the range \( \epsilon < 0 \) and \( \kappa < 0 \), excluding the Schwarzschild limit, it may be easily verified that these scalars are divergent at \( \rho = \frac{1}{2}M \) and are well defined at the throat of the wormhole, \( \rho = \rho_T \).

## 4 Discussion and Conclusion

By relaxing restrictions on the parameters \( \epsilon \) and \( \kappa \), we find that the static, spherically symmetric solutions in five dimensions can be interpreted as wormholes. That is, there are solutions where there is a bridge between two spacetimes; one is asymptotically flat and the other containing a curvature singularity at spatial infinity. In the induced matter scenario, the induced mass-energy density and pressure violate the strong energy condition, as do the matter sources for wormholes in four dimensions, yet the gravitational mass remains positive. However, unlike the solitons (i.e., equation (1) with \( \epsilon > 0, \kappa > 0 \)) this induced gravitational mass diverges in the limit \( \rho \to \frac{1}{2}M \) (\( r \to \infty \)), and so may be considered the source of the singularity found there. We were able to calculate the tension in the throat and, unlike four-dimensional wormholes, the magnitude of the tension is less than that of the mass-energy density. In the asymptotically flat universe, the transverse (angular) pressures are positive...
while they are negative in the asymptotically singular spacetime. In both space-times, the radial pressure is negative. We can envision an observer travelling from the asymptotically flat space-time into the asymptotically singular space-time and encountering shells of matter (radiation or ultra-relativistic particles) whose density and pressure steadily increase with radial distance and eventually diverge at infinity. Although the matter seems exotic in four dimensions, the five-dimensional spacetime (on both sides of the throat) is that of a vacuum and so the energy conditions are (trivially) satisfied. Therefore, we find that Kaluza-Klein theory in the context of induced matter theory can help alleviate concerns about the existence of “exotic” matter [11] in four dimensions; the matter observed in four dimensions is indeed derived from a five-dimensional theory where the energy conditions are satisfied.

In the limiting Schwarzschild case, we note in passing that the wormhole becomes a black hole once again, due to the vanishing component of $g_{tt}$; the time it would take for an infalling object to reach the throat, as measured by an observer in the asymptotically flat spacetime, would be infinite. Within the wormhole class, however, the time would remain finite (i.e., $g_{tt}(\rho_T) \neq 0$).

As is quite evident, the parameter $\kappa$ crucially determines what type of manifold is described by (1): soliton ($\kappa < 0$), wormhole ($\kappa > 0$) or black hole ($\kappa \rightarrow \pm \infty$), and we note here the physical interpretation of this constant. Because of the definition of the gravitational mass, $M_{\text{grav}} = \epsilon \kappa M$ (this is defined at large $\rho$, see [11]), then $\kappa$ is a function of the ration of the gravitational mass to the constant $M$. However, there is also another interpretation. As was recently examined, there is a formal equivalence between a five-dimensional vacuum solution which is independent of the extra coordinate and a four-dimensional vacuum theory
of gravity minimally coupled to a scalar field, \( \varphi \) (and also to a four-dimensional vacuum scalar-tensor theory) [14]. In this formal equivalence, the scalar field goes as \( \varphi \propto \ln(g_{44}) \) and so, the constant \( \epsilon \) is this field’s strength. Following the syntax of Agnese and La Camera [15], the field’s strength is given by \( \sigma/M \), where \( \sigma \) is a scalar charge (constant). For the wormhole solutions, the scalar charge is negative whilst for the soliton solutions it is positive. Because of the constraint \( \epsilon^2(\kappa^2 - \kappa + 1) = 1 \), this scalar charge constant is related to the gravitational mass constant and to the constant \( M \) by

\[
M^2 = M_{\text{grav}}^2 - \sigma M_{\text{grav}} + \sigma^2.
\]

Finally, because of this formal equivalence, there are similar wormhole solutions both in vacuum theories of general relativity coupled to a scalar field and in vacuum scalar-tensor theories, the transformations between all three are not singular at the throat of these wormholes. However, it should be stressed that although there is a mathematical equivalence, the theories are distinct from each other physically, and the wormholes discussed here arise from the non-trivial curvature of the extra dimension, the matter properties are geometric in origin.

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References

[1] T. H. Kaluza, Per. Akad. Phys. Math. K1 54, 966 (1921).
Figure 1: Transformation between quasi-curvature and isotropic co-ordinates
Isotropic coordinate, $\rho$

Quasi-curvature coordinate, $r$

- $\varepsilon(\kappa-1) > 1$
- $\varepsilon(\kappa-1) = 1$
- $\varepsilon(\kappa-1) < 1$