Multibody expansion of the local integrals of motion: how many pairs of particle-hole do we really need to describe the quasiparticles in the many-body localized phase?

Zahra Gholami, Mohsen Amini, Morteza Soltani* and Ebrahim Ghanbari-Adivi

Department of Physics, Faculty of Physics, University of Isfahan, Isfahan 81746-73441, Iran

E-mail: mo.soltani@sci.ui.ac.ir

Received 13 November 2022; revised 16 February 2023
Accepted for publication 6 March 2023
Published 17 March 2023

Abstract
The emergent integrability in a many-body localized (MBL) system can be well characterized by the existence of the complete set of local integrals of motion (LIOMs). Such exactly conserved and exponentially localized operators are often understood as quasiparticle operators which can be expanded in terms of single-particle operators dressed with different numbers of particle-hole pairs. Here, we consider a one-dimensional XXZ spin-\(\frac{1}{2}\) Heisenberg chain in the presence of a random field and try to quantify the corrections needed to be considered in the picture of quasiparticles associated with LIOMs due to the presence of particle-hole excitations. To this end, we explicitly present the multibody expansion of LIOM creation operators of the system in the MBL regime. We analytically obtain the coefficients of this expansion and discuss the effect of higher-order corrections associated with different numbers of particle-hole excitations. Our analysis shows that depending on the localization length of the system, there exist a regime in which the contributions that come from higher-order terms can break down the effective one-particle description of the LIOMs and such quasiparticles become essentially many-body-like.

Keywords: local integrals of motion, many-body localization, quasiparticle

(Some figures may appear in colour only in the online journal)
1. Introduction

One of the best examples of the breaking of ergodicity is the many-body localization (MBL) phenomenon. It is a dynamical phase of disordered interacting quantum systems which fail to thermalize and hence challenges the very foundations of quantum statistical physics [1–3]. It is known as the many-body version of the Anderson localization of non-interacting disorder systems [4] which is first predicted theoretically [3, 5, 6] and then observed experimentally mainly in cold-atom systems [7–11]. MBL is an interesting phase of the system as it exhibits many different characteristic features like the exponentially small transport coefficients, retaining the local memory of the initial state for long times, and logarithmic-in-time growth of entanglement which are of particular importance [5, 12–15]. Indeed, these unique properties of the MBL phase are due to the existence of (quasi-)local operators that are exact integrals of motion and are usually known as local integrals of motion (LIOMs) [16–20]. To date, several different methods have been used to construct a complete set of LIOMs, for instance, the construction of LIOMs via labeling the eigenstates through their corresponding LIOM eigenvalues [16, 17], by time-averaging local operators [18, 21, 22], and different methods based on the exact diagonalization techniques [23–27].

However, despite the vast progress made in developing different methods for the construction of LIOMs and the knowledge afforded by such investigations, quantifying their rich behavior is limited by a lack of explicit analytic studies of their physical properties. In this sense, an important issue is to further understand the picture of the quasiparticles associated with LIOMs [28] which was originally pointed out by Basko et al [3]. In this picture, LIOMs are quasiparticle density operators \( n_i = \hat{c}_i^{\dagger} \hat{c}_i \) where \( \hat{c}_i^{\dagger} = c_i^{\dagger} + \sum_{klm} A_{klm}^i c_k^l c_m^l + \ldots \) is a dressed version of some local products of single-particle operators \( c_i^{\dagger} \). In the absence of interaction (Anderson model), we have only the first term of this expansion which means that all the eigenstates are Slater determinants of single-particle states. However, in the presence of interaction, one needs to consider higher-order corrections from considering different numbers of pairs of particle-hole excitation in the above expansion. A natural fundamental question is then whether this expansion is perturbative. In a recent publication [29], some aspects of this question are studied and it is shown that even if the individual eigenstates of the MBL system can be well approximated by single Slater determinants, they can still show deviations from the uniquely-defined single-particle picture which is itself another urgent reason for needing a better understanding of the picture of quasiparticles.

In this work, we try to explicitly perform the above-mentioned expansion and obtain their coefficients in terms of different number of pairs of quasiparticles. We consider a disordered spin-\( \frac{1}{2} \) XXZ model on a one-dimensional lattice and focus on the effect of higher-order terms of the LIOMs expansion. For this system LIOM operators are a dressed version of some local products of the original Pauli operators and our analysis reveals that how the corrections associated with different number of particle-hole excitations change the convergence of a perturbative expansion. Our results show that the convergence of such perturbative expansion strongly depends on the characteristic localization length of the system in the MBL phase.

The rest of the paper is organized as follows. In section 2 we will draw the picture of quasiparticles associated with LIOMs precisely. For this purpose, we introduce our model first. Then, we will introduce a detailed analysis of the approach we used to obtain the explicit form of the coefficients of the expansion. Section 3 is devoted to presenting the numerical computations of the coefficients of the expansion and discussing its convergency for different
localization regimes based on the localization length of the system and finally, concluding remarks are given in section 4.

2. The quasiparticle associated with LIOMs

2.1. Model Hamiltonian

We consider a spin-1/2 XXZ chain of length L with open boundary conditions in a random magnetic field in the z-direction which can be written as:

\[ H = J \sum_{i=1}^{L-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \Delta \sigma_i^z \sigma_{i+1}^z + \sum_{i=1}^{L} h_i \sigma_i^z \]  (1)

where \( \sigma_i^{x,y,z} \) denote the Pauli operators acting on spin \( i \). We fix the exchange interaction coupling at \( J = 1 \) as our unit of energy and choose the random fields \( h_i \) independently from a random uniform distribution in \([-W, W]\). The parameter \( \Delta \) in this model determines the anisotropy in the interaction which is set to be \( J_z = J \Delta = 0 \) for a free system and, \( J_z \neq 0 \) for an interacting system. When \( J_z = 1 \), the model is known to show a phase transition at a critical disorder strength \( W = W_c \sim 7 \) from an ergodic phase to an MBL phase \([30, 31]\). For the rest of the paper, we will focus on the fully MBL side of the transition, \( W > W_c \), in which the system fails to thermalize due to the existence of LIOMs.

2.2. Expansion of the LIOMs in terms of pairs of particle-hole excitations

As it is well known, one of the most important features of the MBL regime is the existence of LIOMs which allows to describe the system with an effective phenomenological Hamiltonian as

\[ H = \sum_i \epsilon_i \tau_i^z + \sum_{ij} \epsilon_{ij} \tau_i^z \tau_j^z + \sum_{ijk} \epsilon_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots, \]  (2)

in which the emergent localized pseudo-spin operators \( \tau_i^z \) are related to physical spin operators by a local unitary transformation as

\[ \tau_i^z = U \sigma_i^z U^\dagger. \]  (3)

Coefficients \( \epsilon_{ij} \ldots \) in equation (2) with units of energy can be determined by choosing a particular set of LIOMs which, in principle, should satisfy the following conditions: (i) \( \tau_i^z \) should be (quasi-)localized and independent, (ii) they obey the Pauli spin algebra commutator relations, and by definition, (iii) they are conserved quantities \( [\tau_i^z, H] = 0 \), and should be identified in such a way that their interaction range decays rapidly.

Let us now use the above-mentioned property (ii), and write the operators \( \tau_i^z \) in terms of the pseudo-spin raising and lowering operators \( \hat{\tau}_i^\pm \) as

\[ \tau_i^z = \hat{\tau}_i^+ \hat{\tau}_i^- - \hat{\tau}_i^- \hat{\tau}_i^+. \]  (4)

Here, \( \hat{\tau}_i^+ \) is a pseudo-spin raising operator and can be expanded in terms of \( n \)-pairs of fermion-like operators, \( \tilde{\tau}_i^{+(n)} \), which belong to a sector with \( n \)-excited spins as

\[ \tilde{\tau}_i^+ = \tilde{\tau}_i^{+(1)} + \tilde{\tau}_i^{+(2)} + \ldots + \tilde{\tau}_i^{+(N)} = \sum_{n=1}^{N} \tilde{\tau}_i^{+(n)} \]  (5)
where each term shows the contribution of its corresponding sector and can be constructed from the original physical spin operators using the Jordan–Wigner transformation. Let us write out the first few terms explicitly. The first term is the fraction of the operator norm that comes from the single-particle sector (the sector with single excited spin) and can written in terms of raising operators \( \tilde{\sigma}^+ \) with coefficients \( \alpha_j \) as
\[
\tilde{\tau}_i^{+(1)} = \sum_j \alpha_j \tilde{\sigma}^+_j,
\]
in which the \( \tilde{\sigma}^+_j \) operator is related to the physical spin operators by the following Jordan–Wigner transformation
\[
\tilde{\sigma}^+_j = \left( \prod_{k<j} \sigma^z_k \right) \sigma^+_j.
\]
Likewise, the second term in equation (5) is the contribution of the two-particle sector to \( \tilde{\tau}_i^{+} \) and reads as
\[
\tilde{\tau}_i^{+(2)} = \sum_{j_1 < j_2 < j_3} \alpha_{j_1} \alpha_{j_2} \tilde{\sigma}^+_j_1 \tilde{\sigma}^+_j_2 \tilde{\sigma}^-_{j_3},
\]
which represents the fraction of the operator norm that comes from two-particle sector of the Hilbert space correspondingly.

In general, the \( k \)th term on the right-hand side of the equation (5) can be treated as
\[
\tilde{\tau}_i^{+(k)} = \sum_{j_1 < j_2 < \ldots < j_{2k-1}} \alpha_{j_1} \ldots \alpha_{j_{2k-1}} \tilde{\sigma}^+_j_1 \tilde{\sigma}^+_j_2 \ldots \tilde{\sigma}^-_{j_{2k-1}} \tilde{\sigma}^-_{j_{2k-1}},
\]
In what follows we will try to obtain the set of expansion coefficients \( \alpha \) introduced in the above equations (6), (8), and (9).

2.3. Finding the coefficients of the expansion

Before proceeding to obtain the coefficients of the expansion, it will be well to introduce the following notation in which the physical spin basis in the single-excitation sector is shown by \( |j\rangle \equiv \sigma^+_j |0\rangle \) where the reference state \( |0\rangle \) is considered as a state with all spins down and \( j \) denotes the site with flipped spin. By the same token, we can use \( |j_1 j_2 \ldots j_k\rangle \equiv \sigma^+_j \sigma^+_j \ldots \sigma^+_j |0\rangle \) to show a generic basis state in a sector with \( k \)-excitations. Similarly, we use the notation \( |j_1 j_2 \ldots j_k\rangle \equiv \tau^+_j \tau^+_j \ldots \tau^+_j |0\rangle \) to show the same thing in the LIOM basis.

Let us suppose now that we have obtained the set of LIOM operators for this model. This can be done by employing different schemes introduced for this purpose [25, 26], however, we will use the fast and accurate method in [27] which is recently discussed by our group. Now we can start the calculation of the coefficients sector by sector.

In the single-excitation sector and as long as the LIOMs are available in this sector the following relation between the real space physical spin basis and LIOM basis set holds:
\[
|\tilde{i}\rangle = \sum_j \beta_j |j\rangle,
\]
in which \( \beta_j \) are the matrix elements of the unitary transformation \( U = \sum_j |j\rangle \langle j| \). On the other hand, based on the above definition we have
\[
|\tilde{i}\rangle = \tilde{\tau}_i^+ |0\rangle = \tilde{\tau}_i^{+(1)} |0\rangle = \sum_j \alpha_j |j\rangle.
\]
where we have used (5) and (6) and the fact that the terms with $n > 1$ give zero when acting on the vacuum state $|0\rangle$, that is $\tilde{\tau}_i^{+(n>1)}|0\rangle = 0$. A comparison between equations (10) and (11) immediately gives the coefficients $\alpha_j^i$ of the single-excitation sector as

$$\alpha_j^i = \beta_j^i.$$  \hspace{1cm} (12)

Before going to the next sector it is worth mentioning that $\tilde{\tau}_i^{+(1)}$ satisfies the fermionic anticommutation relations while the higher-order terms in equation (5) does not obey the fermionic algebra.

The next sector after the single-spin-flip sector has two flipped spins (two-excitation sector) for which the following relation holds

$$\langle i_1 i_2 | \tilde{\tau}_i^{+(2)} | j_1 j_2 \rangle = \sum_{i > j} \beta_{i,j}^{j_1 j_2} | j_1 j_2 \rangle,$$  \hspace{1cm} (13)

in which we need to keep $i_2 > i_1$ and $j_2 > j_1$ to ensure the correct fermionic anticommutation relations. Like the single-excitation sector, one can write the following relation using equation (8)

$$\langle i_1 | \tilde{\tau}_i^{+(2)} | i_2 \rangle = \tilde{\tau}_i^{+(2)}|0\rangle = \left( \tilde{\tau}_i^{+(2)} + \tilde{\tau}_i^{+(1)} \right) \tilde{\tau}_i^{+(1)}|0\rangle.$$  \hspace{1cm} (14)

It is now possible to multiply the left side of the equations (13) and (14) by $\langle j_0 j_2 | j \rangle$ which results in

$$\langle j_0 j_2 | \sum_{i < j < j_0} \alpha_{j_1,j_2,j_0}^i \tilde{\tau}_i^{+} \tilde{\tau}_j^{+} | j \rangle = \beta_{j_1,j_2,j_0}^{j_0} \delta_{j,j_0},$$  \hspace{1cm} (15)

in which

$$\beta_{j_1,j_2,j_0}^{j_0} = \beta_{j_1,j_2}^{j_0} - \langle j_0 j_2 | \tilde{\tau}_i^{+(1)} \tilde{\tau}_i^{+(2)} | 0\rangle.$$  \hspace{1cm} (16)

Thus one can easily conclude that:

$$\sum_{j'} \alpha_{j_1,j_0,j}^{j_1,j_0} \beta_{j_0,j_2}^{j_2} = \beta_{j_1,j_2,j_0}^{j_0} \delta_{j,j_0},$$  \hspace{1cm} (17)

It is now possible to use the orthogonality condition

$$\sum_{i_2} \beta_{i_1,i_2}^{j_1,j_0} \beta_{i_2,i_0}^{j_2,j_0} = \delta_{j_1,j_0},$$  \hspace{1cm} (18)

to obtain the coefficients $\alpha_{j_1,j_0,j_0}^{j_1}$ as

$$\alpha_{j_1,j_0,j_0}^{j_1} = \sum_{i_2} \beta_{i_1,i_2}^{j_1,j_0} \beta_{i_2,j_0}^{j_2,j_0},$$  \hspace{1cm} (19)

which shows that in order to obtain the coefficients of the expansion $\tilde{\tau}_i^{+(2)}$ one needs only to use the information of the first and second sectors.

Similarly, for the three-particle sector, we have:

$$\langle i_1 i_2 i_3 | \tilde{\tau}_i^{+(3)} | j_1 j_2 j_3 \rangle = \sum_{j_1 < j_2 < j_3} \beta_{j_1,j_2,j_3}^{i_1,i_2,i_3} | j_1 j_2 j_3 \rangle,$$  \hspace{1cm} (20)

and on the other hand,

$$\langle i_1 i_2 i_3 | \tilde{\tau}_i^{+(3)} | 0\rangle = \tilde{\tau}_i^{+(3)} \sum_{j_1 < j_2} \beta_{j_1,j_2}^{i_1,i_2,i_3} | j_1 j_2 \rangle.$$  \hspace{1cm} (21)
According to the expansion of $\tilde{\tau}_{i_1}^+$, we conclude that
\[ \langle j_{01},j_{02},j_{03}|\tilde{\tau}_{i_1}^+|^{(3)} \sum_{j'_{12}j'_3} \beta_{j_{01},j_{02},j_{03}}^t j'_{12} j'_3 \rangle = \beta_{j_{01},j_{02},j_{03}}^t \beta_{j_{01},j_{02},j_{03}}^t \beta_{j_{01},j_{02},j_{03}}^t, \] (22)
in which
\[ \beta_{j_{01},j_{02},j_{03}}^t = \beta_{j_{01},j_{02},j_{03}}^t = \langle j_{01},j_{02},j_{03}|(\tilde{\tau}_{i_1}^+)^{(1)}(\tilde{\tau}_{i_2}^+)^{(2)}(\tilde{\tau}_{i_3}^+)^{(1)}|0\rangle, \] (23)
and results in
\[ \sum_{j_1,j_2} c_{j_{01},j_{02},j_{03},j_{04},j_{05}}^{j_{01},j_{02},j_{03}} = \beta_{j_{01},j_{02},j_{03}}^t \beta_{j_{01},j_{02},j_{03}}^t \beta_{j_{01},j_{02},j_{03}}^t \] (24)
Now it is again straightforward to use the following orthogonality condition
\[ \sum_{i,j} \beta_{i=0}^{x=x-1} \beta_{j}^{x=x-1} \beta_{i=0}^{x=x-1} = \delta_{i=0}^{x=x-1} \delta_{i=0}^{x=x-1} \] (25)
to obtain the corresponding coefficients of this sector as
\[ c_{j_{01},j_{02},j_{03},j_{04},j_{05}}^{j_{01},j_{02},j_{03}} = - \sum_{i,j} \beta_{i=0}^{x=x-1} \beta_{j}^{x=x-1} \beta_{i=0}^{x=x-1} \] (26)
Again, it is obvious that the coefficients of the expansion, up to the third order, only depend on the information of the LIOMs in the sectors with the number of excitations equal to or smaller than this order.

Finally, let us write the generic expression of the coefficients for an expansion up to an arbitrary order $L$ as
\[ c_{j_{1},\ldots,j_{L-1}}^{j_{1},\ldots,j_{L-1}} = \sum_{i_1,\ldots,i_L} \beta_{i_{1},\ldots,i_{L}}^{j_{1},\ldots,j_{L}} \beta_{i_{1},\ldots,i_{L}}^{j_{1},\ldots,j_{L}} \beta_{i_{1},\ldots,i_{L}}^{j_{1},\ldots,j_{L}}, \] (27)
where:
\[ \beta_{i_{1},\ldots,i_{L}}^{j_{1},\ldots,j_{L}} \beta_{i_{1},\ldots,i_{L}}^{j_{1},\ldots,j_{L}} = \langle j_{1},\ldots,j_{L}|(\tilde{\tau}_{i_1}^+)^{(m)}(\tilde{\tau}_{i_2}^+)^{(m)}(\tilde{\tau}_{i_3}^+)^{(m)}(\tilde{\tau}_{i_4}^+)^{(m)}(\tilde{\tau}_{i_5}^+)^{(m)}|0\rangle. \] (28)

In the next section, we will use the above-mentioned coefficients and confirm their validity by numerical computations.

3. Numerical results

To illustrate the accuracy of our theoretical analysis presented above we will now discuss some numerical results for a spin chain with different lengths $L = 7, 9, 11$ and in the strong disorder regime. The reason why we consider odd-length chains is that we are using open boundary condition and are interested in the LIOM located at the central site of the chain, therefore, to avoid the finite size effects that comes from the inequality of the lengths of the two segments on the right and left sides we take the odd-length chain cases. Since we are interested in the full MBL regime we do not need to be much worry about the small system sizes and use the following strategy. We take sufficiently large disorder intensities, $W \gg W_c$, in such a way that the localization length $\xi$ is smaller than our system size, that is $\xi \ll L$ (to determine the localization length as a function of disorder intensity, $\xi(W)$, we can use the results of [18, 27]). In what follows we consider large number of different realizations of disorder that
Figure 1. Difference between the corrections comes from different excitation sectors in the expansion of the LIOM creation operator $\tilde{\tau}^+ (N)$ truncated at different orders $N$ and its exact numerical value which is defined as $\tilde{\mathcal{E}}(N)$ in equation (29) for a spin chain of lengths $L = 7, 9, 11$ and with different localization lengths $\xi = 1, 2, 3$ correspondingly which is averaged over different disorder configurations.

depends on the system size $L$, ranging from $10^6$ to $10^4$ disorder realizations. We also, take each magnetization sector separately to compute its corresponding coefficient of the expansion and focus on the following two aspects. The first quantity is the $\tilde{\tau}^+$ defined in equation (5). We are interested to compare the resulting approximated-operator $\tilde{\tau}^+ (N)$ truncated at different orders $N$ with its exact numerical value $\tilde{\tau}^+ (\text{Exact})$. Thus, we can use the method of [27] to obtain $\tilde{\tau}^+ (\text{Exact})$ and then define the relative norm difference between this quantity and its approximated value as

$$\mathcal{E}(N) = \frac{||\tilde{\tau}^+ (N) - \tilde{\tau}^+ (\text{Exact})||_F}{||\tilde{\tau}^+ (\text{Exact})||_F},$$

(29)

where $||O||_F = \text{tr}(O^\dagger O)$ defines the Frobenius norm of an operator $O$.

Figure 1 represents the relative norm difference $\mathcal{E}$ defined in equation (29) as a function of the truncation order of the expansion $N$. It is obvious that the effects of higher-order corrections strongly depend on the order of expansion. This means that they are not necessarily a monotonically decreasing function of the order of expansion. As we can see, when the order of truncation is smaller than the localization length $\xi$ measured in the lattice spacing units $a$, ($N < (\xi/a)$), the corrections associated with higher-order become important and not negligible. In contrast, when the truncation order reaches the localization length ($N > (\xi/a)$), higher-order corrections become a monotonic decreasing function of the truncation order of the expansion.
Figure 2. The Frobenius norm of the operator $(\tilde{\tau}_i^+)^2$ which is obtained by the expansion of equation (5) versus the truncation orders $N$ for a spin chain of lengths $L = 7, 9, 11$ and with different localization lengths $\xi = 1, 2, 3$ correspondingly which is averaged over different disorder configurations.

To check the impact of the truncation order more precisely, we have also studied the behavior of $||(\tilde{\tau}_i^+)^2||_F$ which is shown in figure 2. In principal, $\tilde{\tau}_i^+$ should satisfy the fermionic algebra and hence, $(\tilde{\tau}_i^+)^2 = 0$. However, the plots of figure 2 show that the higher-order corrections induce a large deviation from their fermionic property if $N < (\xi/a)$ which means that the dressing effects of the larger number of pairs of particle-hole excitation may significantly change the picture of quasiparticles associated with LIOMs and their perturbative expansion is not always a trivial one. Despite this general trending behavior, in this case, we can see that the case of $L = 9$ with $\xi = 2$ shows an unexpected increase at $N = 3$ which seems to be the effect of finite size.

The second quantity which we are interested in is the relative Frobenius norm of the commutator of approximated-LIOM operators and the Hamiltonian which is defined as

$$\Delta(N) = \frac{||[H, \tau^+_i(N)]||_F}{||\tau^+_i(\text{Exact})||_F},$$

where $\tau^+_i(N)$ can be obtained by plugging the result of equation (5) in equation (4) and $\tau^+_i(\text{Exact})$ is the exact LIOM operator obtained using the method of [27]. As it is obvious again in figure 3 that the localization length is the most important parameter in the expansion of LIOMs. Because the contributions come from the sectors in which the number of pairs of particle-hole excitation is smaller than the localization length plays a crucial role in the construction of LIOM operators. This means that writing a perturbative expansion for the construction of
LIOMs (or corresponding quasiparticle operators) in the MBL phase is always a tricky business since the contributions come from different sectors strongly depend on the localization length of the system.

4. Summary

In conclusion, we studied the mechanism of dressing the quasiparticle operator associated with LIOMs with different numbers of pairs of particle-hole excitation in the MBL phase of a disordered spin chain. By expanding the LIOM creation operator in terms of different single-particle operators dressed by different numbers of particle-hole excitations and analytical derivation of the coefficients of the expansion, we explicitly obtained the contributions of the sectors with different excitations. We have performed an order-by-order expansion which allows identify precisely the role played by higher order corrections. Our analysis showed that the key parameter to determine the optimum order of truncation for this expansion is the localization length of the system.

Before ending this section, let us add a few comments regarding the results obtained in this study. We believe that our findings enrich the insights into the MBL context as the following. On one hand, it shows that the information needed to describe the MBL phase of the system can be extracted only by considering the sectors with the number of excitations smaller than the localization length instead of considering the whole Hilbert space. On the other hand, it may have some influences on the measurable quantities like the entanglement entropy. This is very
interesting since we know that one of the differences between the usual Anderson localized states from their interacting counterparts is the presence of logarithmic growth of the von Neumann entanglement entropy in the MBL phase [13]. This means that the presence of multi-body terms certainly changes the long-time behavior of the dynamics of the entanglement. In this regard, recently, a new experimentally feasible procedure is outlined for measuring LIOMs based on their contribution to the slow growth of the negativity which is based on the tensor network approach [32]. This needs a full analysis of the entanglement entropy according to the different orders of the truncation in the LIOM expansion which is beyond the scope of this work and will remain for future works.

**Data availability statement**

The data cannot be made publicly available upon publication because they are not available in a format that is sufficiently accessible or reusable by other researchers. The data that support the findings of this study are available upon reasonable request from the authors.

**Acknowledgments**

MA acknowledges the support of the Abdus Salam (ICTP) associateship program.

**ORCID iDs**

Mohsen Amini  
https://orcid.org/0000-0001-8448-4820

Ebrahim Ghanbari-Adivi  
https://orcid.org/0000-0003-3844-0928

**References**

[1] Abanin D A, Altman E, Bloch I and Serbyn M 2019 Many-body localization, thermalization and entanglement Rev. Mod. Phys. 91 021001

[2] Gornyi I V, Mirlin A D and Polyakov D G 2005 Interacting electrons in disordered wires: Anderson localization and low-T transport Phys. Rev. Lett. 95 206603

[3] Basko D, Aleiner I and Altshuler B 2006 Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states Ann. Phys., NY 321 1126

[4] Anderson P W 1958 Absence of diffusion in certain random lattices Phys. Rev. 109 1492

[5] Oganesyan V and Huse D A 2007 Localization of interacting fermions at high temperature Phys. Rev. B 75 155111

[6] Luitz D J, Laflorencie N and Alet F 2015 Many-body localization edge in the random-field Heisenberg chain Phys. Rev. B 91 081103(R)

[7] Schreiber M, Hodgman S S, Bordia P, Luschen H P, Fischer M H, Vosk R, Altman E, Schneider U and Bloch I 2015 Observation of many-body localization of interacting fermions in a quasirandom optical lattice Science 349 842–5

[8] Choi J-Y, Hild S, Zeiher J, Schauss P, Rubio-Abadal A, Yefsah T, Khemani V, Huse D A, Bloch I and Gross C 2016 Exploring the many-body localization transition in two dimensions Science 352 1547–52

[9] Smith J, Lee A, Richerme P, Neyenhuis B, Hess P W, Hauke P, Heyl M, Huse D A and Monroe C 2016 Many-body localization in a quantum simulator with programmable random disorder Nat. Phys. 12 907–11

[10] Roushan P et al 2017 Spectroscopic signatures of localization with interacting photons in superconducting qubits Science 358 1175–9

[11] Xu K et al 2018 Emulating many-body localization with a superconducting quantum processor Phys. Rev. Lett. 120 050507
[12] Znidaric M, Prosen T and Prelovsek P 2008 Many-body localization in the Heisenberg XXZ magnet in a random field Phys. Rev. B 77 064426
[13] Bardarson J H, Pollmann F and Moore J E 2012 Unbounded growth of entanglement in models of many-body localization Phys. Rev. Lett. 109 017202
[14] Gopalakrishnan S, Muller M, Khemani V, Knap M, Demler E and Huse D A 2015 Phys. Rev. B 92 104202
[15] Serbyn M, Papic Z and Abanin D A 2013 Universal slow growth of entanglement in interacting strongly disordered systems Phys. Rev. Lett. 110 260601
[16] Serbyn M, Papic Z and Abanin D A 2013 Local conservation laws and the structure of the many-body localized states Phys. Rev. Lett. 111 127201
[17] Huse D A, Nandkishore R and Oganesyam V 2014 Phenomenology of fully many-body-localized systems Phys. Rev. B 90 174202
[18] Chandran A, Kim I H, Vidal G and Abanin D A 2015 Constructing local integrals of motion in the many-body localized phase Phys. Rev. B 91 085425
[19] Ros V, Muller M and Scardicchio A 2015 Integrals of motion in the many-body localized phase Nucl. Phys. B 891 420
[20] Imbrie J Z, Ros V and Scardicchio A 2017 Local integrals of motion in many-body localized systems Ann. Phys., Lpz. 529 1600278
[21] Geraedts S D, Bhatt R N and Nandkishore R 2017 Emergent local integrals of motion without a complete set of localized eigenstates Phys. Rev. B 95 064204
[22] Singh H, Ware B, Vasveur R and Gopalakrishnan S 2021 Local integrals of motion and the quasiperiodic many-body localization transition Phys. Rev. B 103 L220201
[23] Rademaker L and Ortuño M 2016 Explicit local integrals of motion for the many-body localized state Phys. Rev. Lett. 116 010404
[24] O’Brien T E, Abanin D A, Vidal G and Papic Z 2016 Explicit construction of local conserved operators in disordered many-body systems Phys. Rev. B 94 144208
[25] Goihl M, Gluza M, Krumnow C and Eisert J 2018 Construction of exact constants of motion and effective models for many-body localized systems Phys. Rev. B 97 134202
[26] Peng P, Li Z, Yan H, Wei K X and Cappellaro P 2019 Comparing many-body localization lengths via nonperturbative construction of local integrals of motion Phys. Rev. B 100 214203
[27] Adami S, Amini M and Soltani M 2022 Structural properties of local integrals of motion across the many-body localization transition via a fast and efficient method for their construction Phys. Rev. B 106 054202
[28] Bera S, Martynec T, Schomerus H, Heidrich-Meisner F and Bardarson J H 2017 One-particle density matrix characterization of many-body localization Ann. Phys., Berl. 529 1600356
[29] Chen C P and Schomerus H 2021 Fock-space geometry and strong correlations in many-body localized systems Phys. Rev. B 104 205411
[30] Pal A and Huse D A 2010 Many-body localization phase transition Phys. Rev. B 82 174411
[31] Luitz D J, Laflorence N and Alet F 2015 Many-body localization edge in the random-field Heisenberg chain Phys. Rev. B 91 081103(R)
[32] Lu B, Bertoni C, Thomson S J and Eissert J 2023 Measuring out quasi-local integrals of motion from entanglement (arXiv:2301.01787)