Another approach to local cohomology problem in abelian lattice gauge theories

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Abstract:
A new technique is proposed to classify a topological field in abelian lattice gauge theories. We perform the classification by regarding the topological field as a local composite field of the gauge field tensor instead of the vector potential associated to an admissible gauge field. Our method reproduces the result obtained by the ordinary method in the infinite four-dimensional lattice and can be extended to arbitrary higher dimensions. It also works in the direct cohomological analysis on a finite lattice.

Keywords: Lattice Gauge Theory, Chiral Symmetry, the Ginsparg-Wilson relation.
1. Introduction

The classification of topological fields on the lattice is an important and interesting question. In the ordinary compact formulation of lattice gauge theories, it is not possible to express the topology of gauge fields because any link variables can be deformed to unity continuously. However if one considers the class of lattice gauge fields which satisfy the admissibility condition

\[ \| 1 - U(x, \mu)U(x + \hat{\mu}, \nu)U^{-1}(x + \hat{\nu}, \mu)U^{-1}(x, \nu) \| < \epsilon, \quad (1.1) \]

where \( \epsilon \) is sufficiently small positive number, then the nontrivial topology of lattice gauge fields emerges and a topological invariant can be defined as a smooth, local, gauge-invariant function of link variables even when lattice spacing is finite.\[1, 2, 3\] The density of the invariant is a topological field in the sense that its local variation with respect to the gauge field sums up to zero:

\[ \sum_x \delta q(x) = 0. \quad (1.2) \]

As a consequence of the topological nature plus the locality and the gauge-invariance, such topological fields may be classified by the Chern characters modulo trivial divergence term as in the continuum theory. And this local cohomology problem is closely related to the question how to achieve the exact cancellation of gauge anomalies in chiral lattice gauge theories.\[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\]

In the case of abelian gauge group, it has been shown by Lüscher\[3\] that any topological field on the four-dimensional infinite lattice is classified uniquely in the following form:

\[ q(x) = \alpha + \beta_{\mu\nu} F_{\mu\nu}(x) + \gamma \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x)F_{\rho\sigma}(x + \hat{\mu} + \hat{\nu}) + \partial^*_{\mu} k_{\mu}(x), \quad (1.3) \]

where \( F_{\mu\nu}(x) \) is the field tensor, \( \alpha, \beta_{\mu\nu} \) and \( \gamma \) are constants independent of the gauge fields and \( k_{\mu}(x) \) is a gauge invariant local current.\[2\] This result can be extended to arbitrary higher dimensions.\[10\] In the proof of the result, in the ordinary approach, a vector potential is introduced for each admissible gauge field so that it represents the original link variables and the field tensor as

\[ e^{iA_{\mu}(x)} = U(x, \mu), \quad |A_{\mu}(x)| \leq \pi (1 + 8 ||x||), \quad (1.4) \]

\[ \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x) = F_{\mu\nu}(x), \quad (1.5) \]

and the topological field is regarded as a gauge-invariant function of the vector potential.

The aim of this paper is to point out that the introduction of the vector potential is not actually necessary to prove the result eq. (1.3). We will show that it is indeed possible to give a proof by regarding the topological field as a gauge-invariant, local, function of

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1 Through out this paper, we set the lattice spacing unity, \( a = 1 \).

2 In the formulation of abelian chiral gauge theories using the Ginsparg-Wilson fermions\[19, 20, 21, 22, 23, 24\], the gauge anomaly is identical to the chiral anomaly, which is given by Jacobian of the exact chiral transformation.\[25, 26, 27, 28, 29, 30\] It satisfies the index theorem and its local density is a topological field. Then using the result eq. (1.3), one can establish the exact cancellation of the gauge anomaly.\[3\]
linear independent variables of the field tensor. Our new technique can be easily extended to higher dimensional cases. It also works in the direct cohomological analysis on the finite-volume lattice, which is formulated for practical uses.\cite{7,18}

The new method based on the field tensor may have some advantages over the ordinary method using the vector potential. In fact, we can simplify the cohomological analysis by reducing several steps of it, as we will see below. Moreover, for non-abelian theories, a vector-potential-representation of the link variables and the field tensor can be rather complex because of the non-linear nature of the field tensor. If we can do without introducing the vector potential, the cohomological analysis would be simpler.

This paper is organized as follows. In section 2, we recall the definition of admissible abelian lattice gauge fields and the notion of local composite fields of gauge fields. Section 3 is devoted to the description of the basic idea of our new method based on the field tensor. The complete analysis of the topological fields is given in the next sections 4 and 5. We conclude in section 6.

2. Admissible gauge fields and locality

In this section we give a definition of admissible abelian gauge fields on the infinite lattice of dimension $n$. We then recall the notion of local composite field of the admissible gauge fields.

2.1 Admissible abelian gauge fields

We consider the compact formulation of abelian lattice gauge theory. Gauge fields are represented by the link variables which take values in the gauge group $U(1)$,

$$U(x, \mu) \in U(1), \quad x \in \mathbb{Z}^n, \quad \mu = 1, \ldots, n,$$

and gauge transformations are defined to vary the link variables as

$$U(x, \mu) \rightarrow \Lambda(x)U(x, \mu)\Lambda(x + \hat{\mu})^{-1}, \quad \Lambda(x) \in U(1),$$

where $\Lambda(x)$ is a gauge transformation function. The field tensor may be defined through the plaquette variables as

$$F_{\mu\nu}(x) = -i\ln P_{\mu\nu}(x), \quad -\pi < F_{\mu\nu}(x) \leq \pi,$$

$$P_{\mu\nu}(x) = U(x, \mu)U(x + \hat{\mu}, \nu)U(x + \hat{\nu}, \mu)^{-1}U(x, \nu)^{-1}.$$

We then impose the so-called admissibility condition given by\textsuperscript{3}

$$|F_{\mu\nu}(x)| < \frac{\pi}{3} \quad \text{for all} \quad x, \mu, \nu.$$

In general, the field tensor satisfies a lattice counterpart of the Bianchi identity,

$$e^{i(\partial_{\mu}F_{\nu\rho}(x) + \partial_{\nu}F_{\rho\mu}(x) + \partial_{\rho}F_{\mu\nu}(x))} = 1 \quad \text{for} \quad \mu, \nu, \rho = 1, \ldots, n,$$

\textsuperscript{3}In two-dimensions, we adopt the following admissibility condition: $|F_{\mu\nu}(x)| < \pi$
where $\partial_\mu$ denotes the forward nearest-neighbor difference operator. This implies that the exterior difference of the field tensor is equal to an integer multiple of $2\pi$. However, for admissible gauge fields, it vanishes identically,

$$\partial_\mu F_{\nu\rho}(x) + \partial_\nu F_{\rho\mu}(x) + \partial_\rho F_{\mu\nu}(x) = 0 \quad \text{for} \quad \mu, \nu, \rho = 1, \ldots, n.$$  \hspace{1cm} (2.7)

Namely, the usual Bianchi identity holds exactly.

### 2.2 Local composite field of admissible gauge fields

Under the admissibility condition, the link variables are not entirely independent. Then the locality with respect to the gauge field is not quite trivial. Here we adopt the definition of local composite fields of the admissible gauge fields, which is given in [5].

**Local composite fields in the infinite lattice:**

We refer a composite field $\phi(x)$ on the infinite lattice as local if $\phi(x)$ is given by the series expansion,

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k(x),$$  \hspace{1cm} (2.8)

where $\phi_k(x)$ are strictly local fields with a localization range proportional to $k$ and the derivatives $\phi_k(x; y_1, \nu_1; \cdots; y_m, \nu_m)$ with respect to the gauge field variables $U(y_1, \nu_1), \cdots, U(y_m, \nu_m)$ exist and are bounded by

$$|\phi_k(x; y_1, \nu_1; \cdots; y_m, \nu_m)| \leq c_m k^{p_m} e^{-\theta k},$$  \hspace{1cm} (2.9)

where the constants $c_m, p_m > 0$ and $\theta > 0$ are independent of the gauge field.

From this definition, it is obvious that the infinite sum eq. (2.8) is well-defined and the derivatives of $\phi(x)$ satisfy the following locality property,

$$|\phi(x; y_1, \nu_1; \cdots; y_m, \nu_m)| \leq d_m (1 + \|x - z\|^{p_m}) e^{-\theta \|x - z\|},$$  \hspace{1cm} (2.10)

where $z$ is chosen from $y_1, \cdots, y_m$ so that $\|x - z\|$ is the maximum. The constant $d_m > 0$ is independent of the gauge field.

### 3. New approach to local cohomology problem

In this section, we introduce a new technique in which the topological field on the lattice is regarded as a local composite field in terms of the linear independent components of the field tensor. We first explain how to choose the linear independent components of the field tensor. We also argue that the locality property of the topological field is maintained through this change of variables. Then we describe the basic idea of the cohomological analysis based directly on the field tensor. The complete analysis of the topological field is given in the next sections.
3.1 Field tensor as independent variables

If a composite field of link variables is gauge-invariant, it can be regarded as a function of the field tensor instead of the link variables. But, the components of the field tensor are not linear independent due to the Bianchi identity eq. (2.7), and in order to treat the field tensor as the independent variables, we must specify a set of the linear independent components of the field tensor. A simple way to do this is to use an appropriate complete gauge fixing. Because, in the course of solving the gauge-fixing condition, a certain set of the independent components of the field tensor is automatically selected and the gauge-fixed link variables are represented as a product of these independent plaquette variables.

For our purpose, we adopt the complete axial gauge with the reference point set to the origin. In this gauge, the link variables are fixed as follows:

\[ U^g(x, \mu)|_{x_1=\cdots=x_{\mu-1}=0} = 1 \quad \text{for } \mu = 1, \ldots, n. \]  

(3.1)

The gauge transformation function of this complete axial gauge is given by the product of the link variables on the path \((0, x)_p\) which starts from the origin and ends at \(x\) along which the coordinate \(x_i\) increases (or decreases) in the descending order \(i = n, n-1, \ldots, 1:\)

\[ \Lambda(x) = \prod_{U \in (0,x)_p} U(z, \mu). \]  

(3.2)

Non-trivial components of the gauge-fixed link variables are obtained by solving

\[ U^g(x + \hat{\mu}, \nu)U^g(x, \nu)^{-1} = \exp\{iF_{\mu\nu}(x)\} \quad \text{for } \nu > \mu \]  

(3.3)

in the order \(\mu = n - 1, \ldots, 1\). Then these variables are expressed by the product of the plaquette variables which are specified by the way to solve the gauge-fixing condition. Explicitly they are given as

\[ U^g(x, \mu) = \prod_{F \in S(x_0, x; \mu)} \exp(iF_{\rho\sigma}(z)) \quad \text{for all } x, \mu, \]  

(3.4)

where \(S(x_0, x; \mu)\) is a set of variables of the field tensor which is contained in the minimal surface with the boundary \((x_0, x)_p + (x, x + \hat{\mu})_p - (x + \hat{\mu}, x_0)_p\). The components of the field tensor which appears in the above formula must be independent degrees of freedom of the gauge field. The other components of the field tensor are to be computed from the gauge-fixed link variables, or using the Bianchi identity. Note also that this choice of the independent components of the field tensor depends on the reference point of the complete axial gauge. Thus we can specify the set of the independent components of the field tensor with a reference point \(x_0\) as

\[ \mathcal{F}_{x_0} \equiv \{ F_{\rho\sigma}(z) \mid F_{\rho\sigma}(z) \in S(x_0, x, \mu), x \in \mathbb{Z}^n, \mu = 1, \ldots, n \} \]  

(3.5)

\[ = \{ F_{\rho\sigma}(z) \mid z_1=x_{\rho 1}, \ldots, z_{\mu-1}=x_{\mu-1}, \rho, \sigma = 1, \ldots, n, \sigma > \rho, \}. \]  

(3.6)
3.2 Locality property in terms of the field tensor

From the discussion in the previous subsection, we can see that any gauge-invariant composite field $\phi(x)$ of link variables can be regarded as a composite field of the independent components of the field tensor,

$$\phi(x)[U] = \phi(x)[U^g] = \phi(x)[F \in \mathcal{F}_x].$$

(3.7)

We next argue that in this change of variables the locality property of the composite field can be maintained by the suitable choice of the reference point.

Let $\phi(x)$ be a gauge-invariant local composite field of the link variables. Then we can see that the derivatives with respect to the gauge-fixed link variables satisfy eq. (2.10). Next we consider the derivatives of $\phi(x)$ with respect to the independent components of the field tensor with the reference point $x$, that is, $\mathcal{F}_x$. According to eq. (3.4), the variation of the component of the field tensor at some point $y$ generates many variations of the gauge-fixed link variables. But these variations occur at the points which are further than $y$ from $x$ in the taxi driver distance. Then, by using the chain rule, we can show that the derivatives $\phi(x; y_1, \mu_1, \nu_1; \cdots; y_m, \mu_m, \nu_m)$ with respect to the field tensor $F_{\mu_1\nu_1}(y_1), \cdots, F_{\mu_m\nu_m}(y_m) \in \mathcal{F}_x$ satisfy the bound

$$|\phi(x; y_1, \mu_1, \nu_1; \cdots; y_m, \mu_m, \nu_m)| \leq d'_m (1 + \|x - z\|)^q \theta_{\mu\nu}(x, y) e^{-\theta\|x - z\|},$$

(3.8)

where $z$ is chosen from $y_1, \cdots, y_n$ so that $\|x - z\|$ is maximal and the constants $d'_m, q'_m > 0$ independent of the gauge field.

Conversely, let us assume that $\phi(x)$ has the locality property with respect to the independent components of the field tensor, eq. (3.8). Since the field tensor is a local field of the link variables, it immediately follows that the composite field has the locality property with respect to the link variables, eq. (2.10). Thus the locality property of a gauge-invariant local composite field does not depend on the choice of the sets of the independent variables.

3.3 Field tensor-based cohomological analysis

We now describe how to perform the cohomological analysis of topological fields with the field tensor. Let us consider a topological field $q(x)$, which is a gauge-invariant local composite field of admissible gauge fields and satisfy the property,

$$\sum_{x \in \mathbb{Z}^n} \delta q(x) = 0,$$

(3.9)

under the local variation of the link variables. We regard the value of $q(x)$ at the point $x$ as the local function of the linear independent components of the field tensor in $\mathcal{F}_x$. Scaling those components by the parameter $t \in [0, 1]$ and differentiating and integrating $q(x)$ with respect to the parameter $t$, we obtain

$$q(x) = \alpha + \frac{1}{2} \sum_{y \in \mathbb{Z}^n} \theta_{\mu\nu}(x, y) F_{\mu\nu}(y), \quad \theta_{\mu\nu}(x, y) = \int_0^1 dt \theta_{\mu\nu}(x, y) \bigg|_{F \to tF},$$

(3.10)
where $\alpha$ is a constant that is independent of the gauge field and $\hat{\theta}_{\mu\nu}(x, y)$ is given by
\[
\hat{\theta}_{\mu\nu}(x, y) = \begin{cases} 
\frac{\partial q(x)}{\partial F_{\mu\nu}(y)} & \text{for } F_{\mu\nu}(y) \in \mathcal{F}_x, \\
0 & \text{for } F_{\mu\nu}(y) \notin \mathcal{F}_x.
\end{cases}
\] (3.11)

Note that the dependent components of the field tensor appear in eq. (3.10). These terms do not actually contribute to the topological field because the coefficients $\hat{\theta}_{\mu\nu}(x, y)$ of the dependent components vanish identically. But for later convenience we include them as dummy variables and the summation over the indices $y, \mu, \nu$ are taken for all possible values.

In order to derive the constraint on the bi-local field $\theta_{\mu\nu}(x, y)$, which follows from the topological property of $q(x)$, we consider an infinitesimal variation of the link variables,
\[
\delta q U(x, \mu) = i\eta_{\mu}(x) U(x, \mu).
\] (3.12)

This causes the variations of both the independent and dependent components of the field tensor as
\[
\delta q F_{\mu\nu}(y) = \partial_{\mu} \eta_{\nu}(y) - \partial_{\nu} \eta_{\mu}(y).
\] (3.13)

Then the topological field $q(x)$ changes as
\[
\delta q q(x) = \sum_{y \in \mathbb{Z}^n} \hat{\theta}_{\mu\nu}(x, y) \delta q F_{\mu\nu}(y).
\] (3.14)

By using eq.(3.13) and the integration over $t$, the topological property of $q(x)$ leads to the following constraint,
\[
\sum_{x \in \mathbb{Z}^n} \theta_{\mu\nu}(x, y) \hat{\theta}^\mu_{\nu} = 0,
\] (3.15)

where $\hat{\theta}^\mu_{\nu}$ is the backward nearest-neighbor difference operator with respect to $y$.

We note that the above relation is exactly same as that obtained in the course of the original cohomological analysis.\cite{5} Once we obtain this relation, we can immediately see that the same argument as in the original analysis using the Poincaré lemma applies and it leads to the same first-step result, lemma 6.1. in \cite{5}. The second-step result, lemma 6.2 in \cite{5}, can be also obtained by this method. Thus we can see that our method reproduces the result of the original cohomological analysis in \cite{5} and that the cohomological classification can be achieved without introducing the vector potential representation of admissible gauge fields.

In the following sections, we perform systematic cohomological analysis based on the direct use of the field tensor for topological fields in arbitrary n-dimensional infinite lattices and finite-volume lattices.

4. Field tensor-based analysis in n-dimensional infinite lattice

4.1 A lemma for the difference operators

In the following cohomological analysis, we require a version of the Poincaré lemma on difference operators. For this purpose, we first introduce the differential forms defined on
the n-dimensional lattice. A general \( k \)-form is defined through

\[
    f(x) = \frac{1}{k!} f_{\mu_1 \ldots \mu_k}(x) dx_{\mu_1} \cdots dx_{\mu_k},
\]

where \( f_{\mu_1 \ldots \mu_k}(x) \) is totally anti-symmetric field and \( dx_{\mu_1}, \ldots, dx_{\mu_k} \) satisfy the Grassmann algebra. We denote a linear space of \( k \)-forms by \( \Omega_k \). For \( k < 0 \) or \( k > n \), \( \Omega_k = \emptyset \).

An exterior difference operator \( d : \Omega_k \rightarrow \Omega_{k+1} \) and a divergence operator \( d^* : \Omega_k \rightarrow \Omega_{k-1} \) may be defined by

\[
    df(x) = \frac{1}{k!} \partial_\mu f_{\mu_1 \ldots \mu_k}(x) dx_\mu dx_{\mu_1} \cdots dx_{\mu_k},
\]
\[
    d^*f(x) = \frac{1}{(k-1)!} \partial_\mu^* f_{\mu_2 \ldots \mu_k}(x) dx_{\mu_2} \cdots dx_{\mu_k},
\]

where \( \partial_\mu \) and \( \partial_\mu^* \) denote the forward and backward nearest-neighbor difference operators, respectively. We then introduce more general class of difference operators, \( L : \Omega_l \rightarrow \Omega_k \), which are defined by

\[
    Lf(x) = \frac{1}{k!!} dx_{\mu_1} \cdots dx_{\mu_k} \sum_{y \in \mathbb{Z}^n} L_{\mu_1 \ldots \mu_k, \nu_1 \ldots \nu_l}(x, y) f_{\nu_1 \ldots \nu_l}(y).
\]

We assume that \( L_{\mu_1 \ldots \mu_k, \nu_1 \ldots \nu_l}(x, y) \) has a compact support including \( x \) with respect to \( y \), and vice versa. We also define simple multiplication operations by using the following abbreviations of the summation of the lattice index: \( \sum \): \( \Omega_l \rightarrow \Omega_k \) stands for the total summation of the kernel function \( L_{\mu_1 \ldots \mu_k, \nu_1 \ldots \nu_l}(x, y) \) over \( x \),

\[
    \sum_1 Lf(x) = \frac{1}{k!!} dx_{\mu_1} \cdots dx_{\mu_k} \sum_{z \in \mathbb{Z}^n} L_{\mu_1 \ldots \mu_k, \nu_1 \ldots \nu_l}(z, x) f_{\nu_1 \ldots \nu_l}(x).
\]

It is also convenient to introduce the \( \circ \) product : \( \Omega_p \otimes \Omega_k \rightarrow \Omega_{p-k} \) defined by

\[
    \theta \circ f(x) = \frac{1}{(p-k)!} dx_{\mu_1} \cdots dx_{\mu_{p-k}} \times \theta_{\mu_1 \ldots \mu_{p-k}, \nu_1 \ldots \nu_k}(\bar{x} - \hat{\nu}_1 \ldots - \hat{\nu}_k) f_{\nu_1 \ldots \nu_k}(\bar{x} - \hat{\nu}_1 \ldots - \hat{\nu}_k),
\]

where \( \bar{x} = x + \hat{1} + \cdots + \hat{n} \), first introduced by Suzuki.\(^4\) This product have the crucial property,

\[
    d^* \theta \circ = (d^* \theta) \circ + (-1)^{p-k-1} \theta \circ d,
\]

when acting on \( k \)-forms.

In the cohomological analysis of the topological field, we often encounter a difference operator \( L \) which satisfies \( d^* Ld = 0 \). The question how to express such a difference operator has been solved originally by Fujiwara et al.\(^10\). Here we state the result as a lemma in terms of difference operators using the \( \circ \) product. The proof is given in appendix\(^{[A]}\).

\(^4\)x is shifted to \( \bar{x} \) so that the expression of the Chern characters in the final result assumes the form given in eq. \((4.20)\).
Lemma 4.1

Let $L : \Omega_l \rightarrow \Omega_k$ ($l \neq 0$) be a difference operator which satisfies

$$d^* L d = 0 \quad \text{and} \quad \sum_1 L d = 0 \quad \text{if} \quad k = 0 \quad \text{(4.8)}$$

Then there exist a form $\theta \in \Omega_{k+l}$ that satisfies $d^* \theta = 0$ and two difference operators, $Q_1 : \Omega_{l+1} \rightarrow \Omega_k$ and $Q_2 : \Omega_l \rightarrow \Omega_{k+1}$ such that

$$L = \theta \circ + Q_1 d + d^* Q_2. \quad \text{(4.9)}$$

If the coefficient of $L$ is local with respect to $x$ or $y$ in the sense defined in subsect. 3.2, then the coefficients of $Q_1$ and $Q_2$ are also local with respect to $x$ or $y$ and $\theta$ is a strict local field. Moreover, if $L$ satisfies the following locality condition instead of the compact support,

$$|L_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_l}(x, y)| < c(1 + \|x - y\|^p) e^{-\|x - y\|/\varrho}, \quad \text{(4.10)}$$

where $c, p$ and $\varrho$ are constants independent of the gauge fields, then $Q_1$ and $Q_2$ satisfy

$$|Q_i_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_l}(x, y)| < c_i(1 + \|x - y\|^p) e^{-\|x - y\|/\varrho} \quad \text{for} \quad i = 1, 2, \quad \text{(4.11)}$$

where $c_i$ are constants independent of the gauge fields, and $\theta$ is a local field in the sense defined in subsect. 3.2.

4.2 Field tensor-based analysis of topological fields on the infinite lattice

We first establish the following lemma through the field-tensor based method.

Lemma 4.2

A gauge invariant local field $\phi(x) \in \Omega_p$ which satisfies

$$d^* \phi(x) = 0 \quad \text{and} \quad \sum_{x \in \mathbb{Z}^n} \delta \phi(x) = 0 \quad \text{if} \quad p = 0, \quad \text{(4.12)}$$

where $\delta$ is any local variation of the admissible gauge fields. Then there exist two gauge invariant local fields, $\theta(x) \in \Omega_{p+2}$ which satisfies $d^* \theta = 0$ and $\omega(x) \in \Omega_{p+1}$, such that

$$\phi(x) = c + \theta \circ F(x) + d^* \omega(x), \quad \text{(4.13)}$$

where $c$ is a constant which is independent of the gauge field.

Proof: The gauge invariant local field $\phi(x)$ is regarded as a function of the independent components of the field tensors, $F_{\mu\nu}(y) \in \mathfrak{F}_x$. We now rescale the independent variables by a continuous parameter $t \in [0, 1]$, differentiate $\phi(x)$ by $t$ and integrate it over the region $[0, 1]$. Then we obtain

$$\phi(x) = c + LF(x), \quad L = \int_0^1 dt \left. \frac{d}{dt} \right|_{F \rightarrow tF}, \quad \text{(4.14)}$$
where \( c = \phi(x)|_{F=0} \) is a constant and a gauge invariant difference operator \( \hat{L} : \Omega_2 \rightarrow \Omega_p \) is defined by

\[
\hat{L}_{\mu\nu}(x, y) = \begin{cases} 
\frac{\partial\phi(x)}{\partial F_{\mu\nu}(y)} & \text{for } F_{\mu\nu}(y) \in \mathcal{F}, \\
0 & \text{otherwise},
\end{cases}
\] (4.15)

The field tensor transforms as

\[
\delta_\eta F = d\eta
\] (4.16)

under any local variation of the admissible gauge fields, eq. (3.12). By the assumption of the lemma, \( L \) has the following property,

\[
d^*Ld = 0 \text{ and } \sum_1^Ld = 0 \text{ if } p = 0.
\] (4.17)

These are the premises of the lemma 4.1(b) and therefore we have

\[
L = \theta \circ + Q_1d + d^*Q_2,
\] (4.18)

where \( Q_1 : \Omega_3 \rightarrow \Omega_p \) and \( Q_2 : \Omega_2 \rightarrow \Omega_{p+1} \) are gauge invariant difference operators and \( \theta \in \Omega_{p+2} \) is a gauge invariant field. Using eq. (4.18) and the Bianchi identity, we obtain

\[
\phi(x) = c + \theta \circ F(x) + d^*\omega(x),
\] (4.19)

where \( \omega = Q_2F \) and \( d^*\theta = 0 \). \( \square \)

Now let us consider a gauge invariant, local and topological field \( q(x) \). We apply the above lemma to \( q(x) \) by regarding it as \( \phi(x) \in \Omega_0 \). In this step, we obtain a gauge-invariant, local field \( \theta \in \Omega_2 \) which satisfies \( d^*\theta = 0 \). This condition for \( \theta \) is just same as the premise of the lemma. Next we apply the lemma to \( \theta \). Repeating this and using eq. (4.7) and the Bianchi identity, we finally obtain the desired result.

\[
q(x) = \sum_{k=0}^{[n/2]} \left( \cdots \left( (c^{(k)} \circ F) \circ F \right) \circ F \right) \circ F(x) + d^*\omega(x),
\] (4.20)

where in the \( k \)-th term of the summation the number of \( \circ \) product counts \( k \). Explicitly,

\[
q(x) = \sum_{k=0}^{[n/2]} \left( \frac{1}{2} \right)^k c^{(k)}_{\mu_1, \ldots, \mu_{2k}} F_{\mu_1\mu_2}(x) F_{\mu_3\mu_4}(x + \hat{\mu}_1 + \hat{\mu}_2) \times \cdots \times F_{\mu_{2k-1}\mu_{2k}}(x + \hat{\mu}_1 + \cdots + \hat{\mu}_{2k-2}) + d^*\omega(x).
\] (4.21)

5. Field tensor-based analysis in \( n \)-dimensional finite lattice

5.1 Admissible gauge fields on the finite lattice

We define the admissible abelian gauge fields on the finite \( n \)-dimensional lattice by imposing the periodic boundary condition,

\[
U(x, \mu) = U(x + \hat{L}\nu, \mu) \quad \text{for} \quad x \in \mathbb{Z}^n, \ \mu, \nu = 1, \ldots, n.
\] (5.1)
Independent degrees of freedom of the gauge fields are restricted in the finite box,
\[ \Gamma_n = \left\{ x \left| \frac{L}{2} \leq x_\mu < \frac{L}{2} - 1, \text{ for } \mu = 1, \ldots, n \right\} \right. \]
where \( L \) is a lattice size. The space of such gauge fields is divided into the connected subspaces labeled by the magnetic flux numbers,
\[ m_{\mu\nu} = \frac{1}{2\pi} \sum_{s,t=0}^{L-1} F_{\mu\nu}(x + \hat{\mu} + \hat{\nu}). \]

Each subspace is isomorphic to a space of the product of \( U(1) \) times a contractible space.
\[ U[m] \cong U(1)^N \times \text{[a contractible space]}, \]
where \( N = d + L^d - 1 \), the number of non-contractible Wilson loops plus the number of gauge transformations which are independent each other. In fact, the link variables of the gauge field in \( U[m] \) are uniquely represented as
\[ U_{[m]}(x, \mu) = \Lambda(x) U_{[w]}(x, \mu) \Lambda^{-1}(x + \hat{\mu}) V_{[m]}(x, \mu) \mathcal{U}(x, \mu). \]

\( V_{[m]}(x, \mu) \) is the instanton configuration chosen as the representative of \( U[m] \),
\[ V_{[m]}(x, \mu) = \exp \left\{ -\frac{2\pi i}{L^2} \left[ L\delta_{\tilde{x}_\mu,L-1} \sum_{\nu > \mu} m_{\mu\nu} \tilde{x}_\nu + \sum_{\nu < \mu} m_{\mu\nu} \tilde{x}_\nu \right] \right\}, \]
where \( \tilde{x}_\mu = x_\mu \mod L \). \( \mathcal{U}(x, \mu) \) is to reproduce the field tensor subtracted by the contribution from the instanton:
\[ \bar{F}_{\mu\nu}(x) = F_{\mu\nu}(x) - 2\pi \frac{m_{\mu\nu}}{L^2}. \]
\( \mathcal{U}(x, \mu) \) can be chosen in the complete axial gauge as
\[ \mathcal{U}(x, \mu) = \prod_{\nu = \mu + 1, \ldots, n} \exp \left( -i \delta_{x_\mu,L/2-1} \sum_{y_\nu=0}^{L/2-1} \bar{F}_{\mu\nu}(y_\nu, y_\nu) \right) \times \prod_{\bar{F} \in S(0,x;\mu)} \exp(i \bar{F}_{\rho\sigma}(z)) \]
for \( x \in \Gamma_n \)
where \( (y_\mu, y_\nu) = (0, ..., 0, y_\mu, x_{\mu+1}, ..., x_{\nu-1}, y_\nu, 0, ..., 0) \) and \( \sum' \) is defined by
\[ \sum_{i=0}^{x_i-1} f(x) = \begin{cases} \sum_{i=0}^{x_i-1} f(x) & (x_i \geq 1) \\
0 & (x_i = 0) \\
\sum_{i=x_i}^{-x_i} (-1)^{x_i} f(x) & (x_i \leq -1) \end{cases}. \]

\( U_{[w]}(x, \mu) \) is responsible for the \( d \) non-contractible Wilson loops and defined by
\[ U_{[w]}(x, \mu) = \begin{cases} w_\mu & \text{for } x_\mu = 0 \mod L, \\
0 & \text{otherwise,} \end{cases} \]
where \( w_\mu \in U(1) \).

The components of the subtracted field tensor, \( \bar{F}_{\mu\nu}(x) \), which appears in the above expression are the linear independent components of the field tensor and can be used as the independent degrees of freedom of the admissible gauge field in each magnetic sector.
5.2 Local composite field of admissible gauge fields on the finite lattice

In the finite lattice, the notion of the local composite field of the admissible gauge field should be reconsidered. Here we adopt the definition of the local composite fields in the finite-volume lattice given in [17].

**Local composite fields in the finite lattice:**
If \( \phi(x) \) is a local composite field in the finite lattice, there exist two local composite fields defined on the infinite lattice \( \phi_\infty(x) \) and \( \Delta \phi_\infty(x) \) such that
\[
\phi(x) = \phi_\infty(x) + \Delta \phi_\infty(x), \quad |\Delta \phi_\infty(x)| < cL^q e^{-L/2q},
\]
where the gauge fields in \( \phi_\infty \) and \( \Delta \phi_\infty \) are periodic. We also assume that \( \phi(x) \) is expressed by the series expansion,
\[
\phi(x) = \sum_{k=1}^{\infty} \phi_k(x),
\]
where \( \phi_k(x) \) and their derivatives \( \phi_k(x; y_1, \nu_1; \cdots; y_m, \nu_m) \) with respect to the periodic gauge field variables \( U(y_1, \nu_1), \cdots, U(y_m, \nu_m) \) satisfy
\[
|\phi_k(x; y_1, \nu_1; \cdots; y_m, \nu_m)| = \begin{cases} 
0 & \text{for } 2k < \max_{z=y_1, \cdots, y_m} \|x - z\| \\
\leq c_m k^p e^{-\theta k} & \text{otherwise}
\end{cases}
\]
with \( m \geq 1 \) and the constants \( c_m, p_m > 0 \) and \( \theta > 0 \) independent of the gauge field.

This locality property holds true for a gauge-invariant local composite field on the finite lattice even when the field tensor is chosen as the independent variables.

5.3 Modified lemma for the difference operators on the finite lattice

In the finite-volume lattice, the difference operators are introduced in the same way as in the infinite lattice given by eqs. (4.4) and (4.5), except that they are periodic with respect to \( x, y \) and the summation ranges for \( x, y \) are restricted to \( \Gamma_n \). Then the lemma corresponding to the Lemma 4.1 can be formulated by including finite-volume correction terms. The proof is given in appendix B.

**Lemma 5.3**
Let \( L : \Omega_l \rightarrow \Omega_k (l \neq 0) \) be a difference operator which satisfies
\[
d^* L d = 0 \quad \text{and} \quad \sum_1 L d = 0 \quad \text{if} \quad k = 0.
\]
Then there exist a form \( \theta \in \Omega_{k+l} \) and three difference operators \( Q_0 : \Omega_{l+1} \rightarrow \Omega_k, Q_1 : \Omega_l \rightarrow \Omega_{k+1} \) and \( \Delta L : \Omega_l \rightarrow \Omega_k \) such that
\[
L = \theta \circ + Q_1 d + d^* Q_2 + \Delta L,
\]
where \( d^* \theta = 0 \) and the coefficients of \( \Delta L \) is a linear combination of those of \( L \) with \( \|x - y\| \geq L/2 \).
When $L$ is a gauge invariant operator, three operators $Q_1, Q_2$ and $\Delta L$ and a field $\theta$ are gauge invariant. If the coefficients of $L$ are local with respect to $x$ or $y$ in terms of subsect.5.2 and satisfy the following condition,

$$|L_{\mu_1\ldots\mu_k,\nu_1\ldots\nu_l}(x, y)| < c(1 + \|x - y\|^p)e^{-\|x - y\|/\varepsilon} \quad \text{for } x - y \in \Gamma_n,$$

(5.15)

where $c, p$ and $\varepsilon$ are constants independent of the gauge fields, then the coefficients of $Q_1$ and $Q_2$ are local with respect to $x$ or $y$ and satisfy

$$|Q_{i\mu_1\ldots\mu_k,\nu_1\ldots\nu_l}(x, y)| < c_i(1 + \|x - y\|^p)e^{-\|x - y\|/\varepsilon} \quad \text{for } x - y \in \Gamma_n, \ i = 1, 2,$$

(5.16)

where $c_i$ are constants independent of the gauge fields, and $\theta$ is a local field. The coefficients of $\Delta L$ satisfy

$$|\Delta L_{\mu_1\ldots\mu_k,\nu_1\ldots\nu_l}(x, y)| < c'L^p e^{-L/2\varepsilon}.$$  

(5.17)

where $c'$ is a constant independent of the gauge fields.

### 5.4 Field tensor-based analysis of topological fields on the finite lattice

We first establish the following lemma through the field tensor-based method.

**Lemma 5.4**

A gauge invariant local form $\phi \in \Omega_p$ which satisfies

$$d^*\phi(x) = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} \delta\phi(x) = 0 \quad \text{if } p = 0.$$  

(5.18)

Then there exist three gauge invariant local fields, $\theta \in \Omega_{p+2}$ which satisfies $d^*\theta = 0$, $\omega \in \Omega_{p+1}$ and $\Delta\phi \in \Omega_p$, such that

$$\phi(x) = c_{[m,w]}(x) + \theta \circ \bar{F}(x) + d^*\omega(x) + \Delta\phi(x),$$

(5.19)

where $c_{[m,w]}(x)$ is $\phi(x)$ for the gauge field $V_{[m]}(x, \mu)U_{[w]}(x, \mu)$. The coefficients of $\Delta\phi$ satisfy

$$|\Delta\phi_{\mu_1\ldots\mu_k}(x)| \leq \kappa L^\sigma e^{-L/2\varepsilon},$$

where $\sigma$ and $\kappa$ are constants independent of the gauge field.

**Proof:** We rescale the independent components of the subtracted field tensor $\bar{F}_{\mu\nu}(x) \in \tilde{\mathfrak{F}}_x$ by the continuous parameter $t \in [0, 1]$, differentiate $\phi(x)$ by $t$ and integrate it over the region $[0, 1]$. Then we obtain

$$\phi(x) = c_{[m,w]} + LF(x),$$

(5.20)

where $c_{[m,w]} = \phi(x)|_{F=0}$ and $L$ is a gauge invariant operator which satisfies the premise of the lemma 5.3. Then we obtain a solution to the constraint on $L$ with a finite-volume correction term.

$$L = \theta \circ + Q_1 d + d^*Q_2 + \Delta L,$$

(5.21)

where $|\Delta L_{\mu_1\ldots\mu_k,\nu_1\ldots\nu_l}(x, y)| < \kappa' L^\sigma e^{-L/2\varepsilon}$ with constants $\sigma'$ and $\kappa'$ independent of the gauge fields. Then it follows

$$\phi = c_{[m,w]} + \theta \circ \bar{F} + d^*\omega + \Delta\phi,$$

(5.22)
where $\Delta \phi = \Delta L \bar{F}$. □

Now by the several uses of the lemma 5.4, we can show that the topological field $q(x)$ is expressed as

$$q = \sum_{k=0}^{[n/2]} (\cdots ((c^{(k)}_{[m,w]} \circ \bar{F}) \circ \bar{F}) \cdots \circ \bar{F}) \circ \bar{F} + d^* \omega + \Delta q,$$

where $c^{(k)}_{[m,w]}(x)$ depends on the magnetic fluxes and the non-contractible Wilson loops and $|\Delta q| \leq c L^q e^{-L/2\eta}$. The topological properties of $q(x)$ implies $\sum_x \Delta q(x) = 0$ and such a form is always expressed as an exact form, $\Delta q = d^* \Delta \omega$. Then, without violating the locality property of the current $\bar{\omega}$ on the finite lattice, we can redefine $\bar{\omega}$ by including $\Delta \bar{\omega}$.

As the final step, we rewrite the above result in terms of the original field tensor $F = \bar{F} + \bar{m}$ where $\bar{m} = 2\pi m_{\mu\nu}/L^2 \, dx_\mu dx_\nu$. For this purpose, we note the following relation between the coefficients $c^{(k)}_{[m,w]}(x)$ in eq. (5.23) and the coefficients $c^{(k)}$ of the result in the infinite lattice, eq. (4.20):

$$\left| c^{(l)}_{[m,w]}(x) - \sum_{r=1}^{[n/2]} r C_{r-l} \left( \cdots ((c^{(r)} \circ \bar{m}) \circ \bar{m}) \cdots \circ \bar{m} \right) \right| \leq \kappa L^{q_1} e^{-L/2\eta}.$$

This relation is obtained by using the decomposition, $q(x) = q_\infty(x) + \Delta q_\infty(x)$, and by comparing in detail the algebraic construction of these coefficients. The details of the proof of the four-dimensional case have been given in [17]. Using this relation and $F = \bar{F} + \bar{m}$, we can rewrite $q(x)$ and obtain the final result

$$q(x) = \sum_{k=0}^{[n/2]} (\cdots ((c^{(k)} \circ \bar{F}) \circ \bar{F}) \cdots \circ \bar{F}) \circ \bar{F}(x) + d^* \omega(x).$$

\section{6. Conclusion}

We have shown that the cohomological classification of the topological fields in abelian lattice gauge theories can be achieved by regarding the topological field as a gauge-invariant, local, function of linear independent variables of the field tensor. This new method allows us to simplify the cohomological analysis by reducing several steps both in the infinite lattice and the finite lattice. This result will be useful in the practical numerical implementation of chiral lattice gauge theories. The application to other cases such as the local cohomology problem in non-abelian lattice gauge theories is an open question. We leave this question for future study.

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A. Poincaré lemma

Lemmas in terms of differential forms: infinite lattice

In the contractible space, any closed form is always exact. This fact is well-known as the Poincaré lemma in the continuum theory and can be also formulated on the infinite lattice. The lemma can be also formulated for the general class of difference operators, $L : \Omega_l \to \Omega_k$, defined in the previous section, keeping locality property of the operators. Here we simply quote the result.

Lemma 4.1(a) (Poincaré lemma)

Let $L$ be a difference operator $: \Omega_l \to \Omega_k$. Then

$$dL = 0 \quad \Rightarrow \quad L = \delta_{kn} \sum_1 L + dK,$$

$$Ld = 0 \quad \Rightarrow \quad L = \delta_{l0} \sum_2 L + Kd,$$

$$d^*L = 0 \quad \Rightarrow \quad L = \delta_{k0} \sum_1 L + d^*K,$$

$$Ld^* = 0 \quad \Rightarrow \quad L = \delta_{ln} \sum_2 L + Kd^*,$$

where $K$ is difference operator constructed from $L$. An important point to note is that $\sum_1, \sum_2 L$ and $K$ are gauge invariant if $L$ is gauge invariant. If a coefficient of the operator $L$ is local with respect to $x$ or $y$, then one of the solution $K$ is local with respect to $x$ or $y$. Moreover, instead of the compact support if the coefficient satisfies

$$|L_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_l}(x, y)| < c_1(1 + \|x - y\|^p)e^{-\|x - y\|/\varrho},$$

where $c_1, p$ and $\varrho$ are constants independent of the gauge fields, then the coefficient of $K$ satisfies

$$|K_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_l}(x, y)| < c_2(1 + \|x - y\|^p)e^{-\|x - y\|/\varrho},$$

where $c_2$ is a constant independent of the gauge fields.

The Poincaré lemma concerns the case of $d^*L = 0$ or $Ld = 0$. However, in the cohomological analysis of the topological field, we often encounter a difference operator $L$ which satisfies $d^*Ld = 0$. Then it turns out to be useful to formulate a lemma which tells us how to express such a difference operator. Here we give a proof of the lemma using the differential forms and the $\circ$ product.

Lemma 4.1(b)

Let $L : \Omega_l(l \neq 0) \to \Omega_k$ be a difference operator which satisfies

$$d^*Ld = 0 \quad \text{and} \quad \sum_1 Ld = 0 \quad \text{if} \quad k = 0$$

Then there exist a form $\theta \in \Omega_{k+l}$ that satisfies $d^*\theta = 0$ and two difference operators, $Q_0 : \Omega_{l+1} \to \Omega_k$ and $Q_1 : \Omega_l \to \Omega_{k+1}$ such that

$$L = \theta \circ + Q_0d + d^*Q_1.$$
If the coefficient of $L$ is local with respect to $x$ or $y$ and satisfies

$$|L_{\mu_1\ldots\mu_k,\nu_1\ldots\nu_l}(x, y)| < c(1 + \|x - y\|^p)e^{-\|x - y\|/\epsilon}, \quad (A.9)$$

where $c, p$ and $\epsilon$ are constants independent of the gauge fields, then the coefficients of $Q_1$ and $Q_2$ are also local with respect to $x$ or $y$ and satisfy

$$|Q_{i \mu_1\ldots\mu_k,\nu_1\ldots\nu_l}(x, y)| < c_i(1 + \|x - y\|^p)e^{-\|x - y\|/\epsilon}, \quad (A.10)$$

where $c_i (i = 1, 2)$ are constants independent of the gauge fields, and $\theta$ is a local field.

**Proof:** Here we consider the case of $k + l \leq n$, but it is easy to extend the case of $k + l > n$. Applying the Poincaré lemma repeatedly, the following chain is obtained,

\[
\begin{align*}
L^{(0)} d &= d^* L^{(1)}, \quad (A.11) \\
L^{(1)} d &= d^* L^{(2)}, \quad (A.12) \\
&\vdots \\
L^{(l-2)} d &= d^* L^{(l-1)}, \quad (A.13) \\
L^{(l-1)} d &= d^* L^{(l)}, \quad (A.14) \\
L^{(l)} d &= 0, \quad (A.15)
\end{align*}
\]

where $L = L^{(0)}$ and $L^{(m)} : \Omega_{l-m} \rightarrow \Omega_{k+m}$. Then the Poincaré lemma allow us to conclude that there exists a difference operator $Q^{(l)} : \Omega_1 \rightarrow \Omega_{k+l}$ such that

\[
L^{(l)} = \sum_2 L + B^{(l)} d, \quad (A.16)
\]

and we may express $\sum_2 L = \theta \circ \theta \in \Omega_{k+l}$. By acting eq.\(A.14\) on a constant, we can assert that $d^* \theta = 0$. Then using eq.\(4.7\), eq.\(A.14\) is rewritten as follows:

\[
(L^{(l-1)} - \theta \circ -d^* Q^{(l)}) d = 0. \quad (A.17)
\]

where an extra sign in the second equation of eq.\(4.7\) is absorbed into a redefinition of $\theta$. Again the Poincaré lemma can be used to obtain

\[
L^{(l-1)} = \theta \circ +d^* Q^{(l)} + Q^{(l-1)} d, \quad (A.18)
\]

where $Q^{(l-1)} : \Omega_2 \rightarrow \Omega_{k+l-1}$. Repeating the same process, we finally obtain

\[
L^{(0)} = \theta \circ +d^* Q^{(1)} + Q^{(0)} d, \quad (A.19)
\]

where $d^* \theta = 0$, $Q^{(1)} : \Omega_1 \rightarrow \Omega_{k+1}$ and $Q^{(0)} : \Omega_{l+1} \rightarrow \Omega_k$. For $k + l > n$ we have $\theta = 0$. It is obvious that the statement with respect to the locality satisfies because the solutions of the Poincaré lemma always have same locality properties. \hfill $\Box$
B. Modified Poincaré lemmas in terms of differential forms: finite lattice

On n-dimensional torus in the continuum, a closed form can not be always expressed as an exact form globally because the cohomology group of the space is non-trivial. Then, if one consider the finite periodic lattice with the linear extent \( L \), the Poincaré lemma formulated in the previous section does not hold anymore in general. However, we can show that the lattice counterpart of the corollary of de Rham theorem holds true and moreover, for a sufficiently large (finite) lattice, a modified version of the lemma holds with a small correction term suppressed exponentially in lattice size. These lemmas can substitute the Poincaré lemma in the infinite lattice and allows us to perform the cohomological analysis directly on the finite lattice.\(^{[17]}\) Here we simply quote those results.

**Lemma 5.3(a) (Modified Poincaré lemma)**

Let \( L : \Omega_l \to \Omega_k \) be a difference operator which satisfies

\[
dL = 0 \quad s.t. \quad L = \delta_{kn} \sum_1 L + dK + \Delta L, \tag{B.1}
\]
\[
Ld = 0 \quad s.t. \quad L = \delta_{00} \sum_2 L + Kd + \Delta L, \tag{B.2}
\]
\[
d^*L = 0 \quad s.t. \quad L = \delta_{k0} \sum_1 L + d^*K + \Delta L, \tag{B.3}
\]
\[
Ld^* = 0 \quad s.t. \quad L = \delta_{ln} \sum_2 L + Kd^* + \Delta L, \tag{B.4}
\]

where \( K, \sum_{1,2} L \) and \( \Delta L \) are difference operators. Moreover \( \Delta L \) is constructed from a linear summation of \( L \) where \( \left\| x - y \right\| \geq L/2 \). \( \Delta L = 0 \) when the terms \( \sum_{1,2} L \) exist.

**Lemma 5.3(b) (Corollary of de Rham theorem)**

Let \( L : \Omega_l \to \Omega_k \) be a difference operator which satisfies

\[
dL = 0, \sum_1 L = 0 \quad s.t. \quad L = dK, \tag{B.5}
\]
\[
Ld = 0, \sum_2 L = 0 \quad s.t. \quad L = Kd, \tag{B.6}
\]
\[
d^*L = 0, \sum_1 L = 0 \quad s.t. \quad L = d^*K, \tag{B.7}
\]
\[
Ld^* = 0, \sum_2 L = 0 \quad s.t. \quad L = Kd^*, \tag{B.8}
\]

where \( K \) is a difference operator constructed from \( L \).

For lemma 5.3 (a) and (b), if the coefficient of \( L \) is local with respect to \( x \) or \( y \) and satisfies

\[
\left| L_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_l}(x, y) \right| < c_1 (1 + \| x - y \|^{p})e^{-\| x - y \|/\varrho}, \tag{B.9}
\]

where \( c_1, p \) and \( \varrho \) are constants independent of the gauge fields, then the coefficients of \( K \) is also local with respect to \( x \) or \( y \) and satisfies

\[
\left| K_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_l}(x, y) \right| < c_2 (1 + \| x - y \|^{p})e^{-\| x - y \|/\varrho}, \tag{B.10}
\]
and the coefficients of $\Delta L$ satisfies
\[ |\Delta L_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_l}(x, y)| < c_3 L^p e^{-L/2} \] (B.11)
where $c_2$ and $c_3$ and constants independent of the gauge fields.

A lemma which corresponds to Lemma 4.1(b) can be formulated in the infinite lattice.

**Lemma 5.3(c)**

Let $L : \Omega_l \rightarrow \Omega_k$ be a difference operator which satisfies
\[ d^* L d = 0 \text{ and } \sum L d = 0 \text{ if } k = 0. \] (B.12)
Then there exist a form $\theta \in \Omega_{k+l}$ and three difference operators $Q_0 : \Omega_{l+1} \rightarrow \Omega_k$, $Q_1 : \Omega_l \rightarrow \Omega_{k+1}$ and $\Delta L : \Omega_l \rightarrow \Omega_k$ such that
\[ L = \theta \circ + Q_1 d + d^* Q_2 + \Delta L, \] (B.13)
where $d^* \theta = 0$ and $\Delta L$ is constructed from a linear summation of $L$ where $\|x - y\| \geq L/2$.

For lemma 5.3(c), if $L$ satisfies the conditions which are mentioned below lemma 5.3(b), then $K$ and $\Delta L$ also satisfy the conditions mentioned there and $\theta$ is a local field.

**Proof:** The strategy of the proof is almost same as the case of lemma 4.1(b). However the existence of the finite size correction leads us to extra difficulties. If the difference operator $L^{(m)}$ satisfies the lemma’s assumption, $d^* L^{(m)} d = 0$, applying the modified Poincaré lemma eq.(B.3) we conclude that
\[ L^{(m)} d = d^* \bar{L}^{(m+1)} + \Delta \bar{L}^{(m+1)}. \] (B.14)
Now we have to redefine the operator $\bar{L}^{(m+1)}$ appropriately in order to use the lemma’s assumption repeatedly. Anyway, multiplying $d$ to eq.(B.14) from right we obtain
\[ d^* \bar{L}^{(m+1)} d = -\Delta \bar{L}^{(m+1)} d. \] (B.15)
It is obvious that the operator $\Delta \bar{L}^{(m+1)} d$ satisfies the premise of the corollary of de Rham theorem. We can change the right hand side of eq.(B.15) in the form as $d^* K d$, however, a simple application of the lemma (B.7) seems to lead that $d$ vanishes instead of appearing $d^*$. It relates how to construct a solution of the Poincaré lemma. In general, when we construct a typical solution of the Poincaré lemma, for example, for the case of $d^* L(x, y) = 0$, the solution $K(x, y)$ is obtained by integrating $L(x, y)$ over $x$. To preserve the locality, $y$ should be chosen as the reference point that corresponds an initial point of the integral. It causes the fact that the equation $d^* (L d) = 0$ can not be solved as the form $L d = d^* K d$ without violating the locality property. However since the operator $\Delta \bar{L}^{(m+1)} d$ is exponentially small, it is not necessary to take care of the locality property and the reference point can
be taken arbitrary. Therefore the difference operator $\Delta L^{(m+1)}d$ can be expressed as the following form,

$$\Delta \bar{L}^{(m+1)}d = d^* \Delta K^{(m+1)}d.$$  (B.16)

Then we introduce new difference operators $L^{(m+1)}$ and $\Delta R^{(m+1)}$ as

$$L^{(m+1)} = \bar{L}^{(m+1)} + \Delta K^{(m+1)}, \quad \Delta R^{(m+1)} = \Delta \bar{L}^{(m+1)} - d^* \Delta K^{(m+1)}.$$  (B.17)

Thus, instead of eq.(B.14), we obtain the following equations

$$L^{(m)}d = d^* L^{(m+1)} + \Delta R^{(m+1)}, \quad d^* L^{(m+1)}d = 0.$$  (B.18)

We can write down the chain for the case of $k + l \leq n$.

$$L^{(0)}d = d^* L^{(1)} + \Delta R^{(1)}, \quad d^* L^{(1)}d = 0,$$
$$L^{(1)}d = d^* L^{(2)} + \Delta R^{(2)}, \quad d^* L^{(2)}d = 0,$$
$$\vdots$$
$$L^{(l-2)}d = d^* L^{(l-1)} + \Delta R^{(l-1)}, \quad d^* L^{(l-1)}d = 0,$$
$$L^{(l-1)}d = d^* L^{(l)} + \Delta R^{(l)}, \quad \sum_{2} d^* L^{(l)} = 0,$$
$$L^{(l)}d = 0.$$  (B.23)

Applying the modified Poincaré lemma eq.(B.2) to the last line of the chain, we get the same equation to the case of the infinite volume lattice,

$$L^{(l)} = \theta \circ + Q^{(l)}d$$  (B.24)

where $\theta \in \Omega_{k+l}$. For $\Delta R^{(l)}$, we apply the corollary of de Rham theorem and get

$$\Delta R^{(l)} = \Delta B^{(l)}d.$$  (B.25)

Acting eq.(B.22) on a constant, $\theta$ satisfies $d^* \theta = 0$. Therefore eq.(B.22) can be expressed in the form,

$$(L^{(l-1)} - \theta \circ - d^* Q^{(l)} - \Delta B^{(l)})d = 0.$$  (B.26)

where we redefine $(-1)^{k+l} \theta$ to $\theta$. Again we apply the modified Poincaré lemma eq.(B.2) and redefine $\Delta B^{(l)}$ and $\Delta L^{(l-1)}$, which is an exponentially small correction that arises from eq.(B.2), to $\Delta L^{(l-1)}$. Then we obtain

$$L^{(l-1)} = \theta \circ + d^* Q^{(l)} + Q^{(l-1)}d + \Delta L^{(l-1)}.$$  (B.27)

Thus we carry out above procedure repeatedly and finally obtain

$$L^{(0)} = \theta \circ + d^* Q^{(1)} + Q^{(0)}d + \Delta L^{(0)}.$$  (B.28)

where $d^* \theta = 0$. For the case of $k + l > n$, the chain stops at $n - k - 1$ line and we have vanishing $\theta$. □
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