Type I Tobit Bayesian Additive Regression Trees for Censored Outcome Regression

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Abstract

Censoring occurs when an outcome is unobserved beyond some threshold value. Methods that do not account for censoring produce biased predictions of the unobserved outcome. This paper introduces Type I Tobit Bayesian Additive Regression Tree (TOBART-1) models for censored outcomes. Simulation results and real data applications demonstrate that TOBART-1 produces accurate predictions of censored outcomes. TOBART-1 provides posterior intervals for the conditional expectation and other quantities of interest. The error term distribution can have a large impact on the expectation of the censored outcome. Therefore, the error is flexibly modeled as a Dirichlet process mixture of normal distributions. An R package is available at https://github.com/EoghanONeill/TobitBART.

Keywords: BART, censored regression, regression trees, Bayesian nonparametrics, machine learning.

1 Introduction

Censoring occurs when, beyond some threshold value, the observed outcome is equal to the threshold instead of the true latent outcome value. For example, scientific equipment can often only make accurate measurements within a known range of outcome values, and observations outside this range are set to its limits. Often the estimand of interest is the conditional expectation or conditional average treatment effect on the outcome before censoring. Estimation of a standard regression model using data without censored values, or with censored observations set equal to threshold values, results in biased estimates. Tobit models directly model the latent outcome and censoring process (Tobin, 1958).

In this paper, we combine the Bayesian Type I Tobit model (Chib, 1992) with Bayesian Additive Regression Trees (Chipman et al., 2010). The latent outcome (before censoring) is modeled as
a sum-of-trees, which allows for nonlinear functions of covariates. The error term is modeled as a Dirichlet process mixture of normal distributions, as in fully nonparametric BART (George et al., 2019). Smooth data generating processes with sparsity are modelled by soft trees with a Dirichlet prior on splitting variable probabilities, as introduced by Linero and Yang (2018).

In simulations and applications to real data, TOBART-1 outperforms a Tobit gradient boosted tree method, Grabit (Sigrist and Hirnschall, 2019), a Tobit Gaussian Process model (Groot and Lucas, 2012), standard linear Tobit, and simple hurdle models based on standard machine learning methods. Unlike other methods, TOBART-1 accounts for model uncertainty and can nonparametrically model the error term. Posterior intervals are available for censored outcomes, uncensored outcomes, conditional expectations, and probabilities of censoring. Grabit, Gaussian Processes, and other methods rely on cross-validation for parameter tuning and are sensitive to the tuned variance of the error term, whereas TOBART-1 performs well without parameter tuning and accounts for uncertainty in the variance of the error term.

TOBART-1 with a Dirichlet process mixture of normal distributions for the error term (TOBART-1-NP) removes the restrictive normality assumption often imposed in censored outcome models. We observe that this can lead to more accurate outcome predictions in simulations with non-normally distributed errors, and in real data applications, which may involve non-normally distributed outcomes.\footnote{A Dirichlet process mixture for the error term distribution has previously been included in a censored outcome model by Kottas and Krunjajić (2009).}

A variety of methods have been proposed for nonparametric and semiparametric censored outcome models. Lewbel and Linton (2002) describe a local linear kernel estimator for the setting in which both the uncensored outcome mean function of regressors and error distribution are unknown. Fan and Gijbels (1994) describe a quantile-based local linear approximation method. Huang (2021) introduces a semiparametric method involving B-splines. Chen et al. (2005) use a local polynomial method. Other papers on the topic of semiparametric and nonparametric censored outcome regression include Cheng and Small (2021); Heuchenne and Van Keilegom (2007, 2010); Huang et al. (2019); and Oganisian et al. (2021). Gaussian Process censored outcome regression methods are applied by Groot and Lucas (2012); Cao et al. (2018); Gammelli et al. (2020, 2022) and Basson et al. (2023). Zhang et al. (2021) and Wu et al. (2018) implement censored outcome neural network methods.

A number of recent papers have considered Tobit model selection and regularization. Zhang et al. (2012) describe Focused Information Criteria based Tobit model selection and averaging.
Jacobson and Zou (2022) provide theoretical and empirical results for Tobit with a Lasso penalty and a folded concave penalty (SCAD). Müller and van de Geer (2016) and Soret et al. (2018) describe a LASSO penalized censored outcome models. Bradic and Guo (2016) study robust penalized estimators for censored outcome regression.

The Bayesian Tobit literature includes quantile regression methods (Ji et al., 2012; Yu and Stander, 2007; Alhamzawi, 2016), and Bayesian elastic net Tobit (Alhamzawi, 2020). Ji et al. (2012) account for model uncertainty by implementing Tobit quantile regression with Stochastic Search Variable Selection. However, the outcome and latent variable are modeled as linear functions of covariates. TOBART-1 provides a competing approach to the methods referenced above that does not impose linearity.

The remainder of the paper is structured as follows: In section 2 we describe the TOBART-1 model and Markov chain Monte Carlo (MCMC) implementation, section 3 contains simulation studies for prediction and treatment effect estimation with censored data, section 4 contains applications to real world data, and section 5 concludes the paper.

2 Methods

2.1 Review of Bayesian Additive Regression Trees (BART)

Suppose there are $n$ observations, and the $n \times p$ matrix of explanatory variables, $X$, has $i$th row $x_i = [x_{i1}, ..., x_{ip}]$. Following the notation of Chipman et al. (2010), let $T$ be a binary tree consisting of a set of interior node decision rules and a set of terminal nodes, and let $M = \{\mu_1, ..., \mu_b\}$ denote a set of parameter values associated with each of the $b$ terminal nodes of $T$. The interior node decision rules are binary splits of the predictor space into the sets $\{x_{is} \leq c\}$ and $\{x_{is} > c\}$ for continuous $x_s$. Each observation’s $x_i$ vector is associated with a single terminal node of $T$, and is assigned the $\mu$ value associated with this terminal node. For a given $T$ and $M$, the function $g(x_i; T, M)$ assigns a $\mu \in M$ to $x_i$.

For the standard BART model, the outcome is determined by a sum of trees,

$$Y_i = \sum_{j=1}^{m} g(x_i; T_j, M_j) + \varepsilon_i$$

where $g(x_i; T_j, M_j)$ is the output of a decision tree. $T_j$ refers to a decision tree indexed by $j = 1, ..., m$, where $m$ is the total number of trees in the model. $M_j$ is the set of terminal node parameters of $T_j$, and $\varepsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2)$. 


Prior independence is assumed across trees \( T_j \) and across terminal node means \( M_j = (\mu_{1j}, ..., \mu_{bj}) \) (where \( 1, ..., b_j \) indexes the terminal nodes of tree \( j \)). The form of the prior used by Chipman et al. (2010) is

\[
p(M_1, ..., M_m, T_1, ..., T_m, \sigma) \propto \left[ \prod_j \left[ \prod_k p(\mu_{kj} | T_j) \right] p(T_j) \right] p(\sigma) \quad \text{where} \quad \mu_{kj} | T_j \sim_i \mathcal{N}(0, \sigma^2_j)
\]

where \( \sigma_j = \frac{\beta_0}{\sqrt{\kappa}} \) and \( \kappa \) is a user-specified hyper-parameter.

Chipman et al. (2010) set a regularization prior on the tree size and shape \( p(T_j) \). The probability that a given node within a tree \( T_j \) is split into two child nodes is \( \alpha(1 + d_h)^{-\beta} \), where \( d_h \) is the depth of (internal) node \( h \) and the parameters \( \alpha \) and \( \beta \) determine the size and shape of \( T_j \) respectively. Chipman et al. (2010) use uniform priors on available splitting variables and splitting points. The model precision \( \sigma^{-2} \) has a conjugate prior distribution \( \sigma^{-2} \sim \mathcal{Ga}(\frac{v}{2}, \frac{v \lambda}{2}) \) with degrees of freedom \( v \) and scale \( \lambda \).

Samples from \( p((T_1, M_1), ..., (T_m, M_m), \sigma | y) \) can be made by a Bayesian backfitting MCMC algorithm. This algorithm involves \( m \) successive draws from \( (T_j, M_j) | (T_{(j)}, M_{(j)}), \sigma, y \) for \( j = 1, ..., m \), where \( T_{(j)}, M_{(j)} \) are the trees and parameters for all trees except the \( j^{\text{th}} \) tree, followed by a draw of \( \sigma \) from the full conditional \( \sigma | T_1, ..., T_m, M_1, ..., M_m, y \). After burn-in, the sequence of \( f^* \) draws, \( f_1^*, ..., f_Q^* \), where \( f^*(\cdot) = \sum_{j=1}^m g(\cdot; T_j^*, M_j^*) \), is an approximate sample of size \( Q \) from \( p(f | y) \).

### 2.2 Soft Trees and Sparse Splitting Rules

In addition to the standard Bayesian tree model for \( f(x_i) \) described in section 2.1, we also implement TOBART and TOBART-NP with soft trees and sparse splitting rules as described by Linero and Yang (2018). Predictions from soft trees are weighted linear combinations of all terminal node parameter values, with the weights being functions of distances between covariates and splitting points. The prediction from a single tree function is

\[
g(x_i; T_j, M_j) = \sum_{\ell=1}^{L_j} \mu_{j,\ell} \xi(x_i, T_j, \ell)
\]

\[
\xi(x_i, T_j, \ell) = \prod_{b \in A(\ell)} \left[ \begin{array}{c}
\frac{x_{jb} - C_b}{\tau_b} \\
1 - \frac{x_{jb} - C_b}{\tau_b}
\end{array} \right] \mathbb{I}(x_{jb} > C_b) \times \mathbb{I}(x_{jb} \leq C_b)
\]

where \( L_j \) is the number of leaves in the \( j^{\text{th}} \) tree, \( \mu_{j,\ell} \) is the \( \ell^{\text{th}} \) terminal node parameter of the \( j^{\text{th}} \) tree, \( A(\ell) \) denotes the set of ancestor nodes of terminal node \( \ell \). The splitting variable, splitting point, and bandwidth parameter at internal node \( b \) are denoted by \( x_{jb}, C_b, \) and \( \tau_b \) respectively. The gating function \( \zeta \) is the logistic function \( \zeta(x) = (1 + \exp(-x))^{-1} \).

Sparse splitting rules are introduced by placing a Dirichlet prior on the splitting probabilities \((s_1, \ldots, s_p) \sim \mathcal{D}(\frac{a}{1}, \ldots, \frac{a}{p})\). The parameter \( a \) controls the level of sparsity and has the prior \( \text{Beta}(0.5, 1) \). Linero and Yang (2018) demonstrate...
that soft trees allow BART to model smooth functions, and the Dirichlet prior on splitting probabilities adapts to unknown levels of sparsity to provide improved predictions on high dimensional data sets.

2.3 Type I Tobit and TOBART

2.3.1 Type I Tobit Model

The Type I Tobit model with censoring from below at \( a \) and censoring from above at \( b \) is:

\[
Y_i^* = x_i \beta + \epsilon_i, \quad \epsilon_i \sim i.i.d. N(0, \sigma^2)
\]

\[
Y_i = \begin{cases} 
  a & \text{if } Y_i^* \leq a \\
  Y_i^* & \text{if } a < Y_i^* < b \\
  b & \text{if } b \leq Y_i^* 
\end{cases}
\]

where a normal prior is placed on \( \beta \), and an inverse gamma prior is placed on \( \sigma^2 \) (Chib, 1992).

2.3.2 Type I TOBART Model

The Type I TOBART model replaces the linear combination \( x_i \beta \) with the sum-of-trees function \( f(x_i) \):

\[
Y_i^* = f(x_i) + \epsilon_i, \quad \epsilon_i \sim i.i.d. N(0, \sigma^2)
\]

\[
Y_i = \begin{cases} 
  a & \text{if } Y_i^* \leq a \\
  Y_i^* & \text{if } a < Y_i^* < b \\
  b & \text{if } b \leq Y_i^* 
\end{cases}
\]

where a BART prior is placed on \( f(x_i) \) and an inverse gamma prior is placed on \( \sigma^2 \).  

2.3.3 Type I TOBART Gibbs Sampler

Tobit can be implemented by MCMC with data augmentation (Chib, 1992). The realization, \( y_i^* \), of the variable \( Y_i^* \) is observed for uncensored outcomes, and is sampled from its full conditional for censored outcomes.

\[
y_i^* = y_i \text{ if } y_i \in (a, b), \quad \text{and}
\]

\[
y_i^* \sim \begin{cases} 
  \mathcal{TN}_{[-\infty, a]}(f(x_i), \sigma^2) & \text{if } y_i = a \\
  \mathcal{TN}_{[b, \infty]}(f(x_i), \sigma^2) & \text{if } y_i = b
\end{cases}
\]

where \( \mathcal{TN}_{[l, u]} \) denotes a normal distribution truncated to the interval \([l, u]\). The full conditionals for \( f(x_i) \) and \( \sigma^2 \) are standard full conditionals for BART with \( y_i^* \) as the dependent variable and \( x_i \) as the potential splitting variables. Appendix A contains a description of a sampler that produces draws \( f^{(1)}(x_i), \ldots, f^{(D)}(x_i) \) and \( \sigma^{(1)}, \ldots, \sigma^{(D)} \).

2.3.4 Predicting Outcomes with TOBART

The conditional mean of the latent variable is \( f(x_i) \). If censoring is also applied to the test data,
then the outcomes are predicted by averaging the standard Tobit expectation formula across MCMC iterations:

For all MCMC iterations \( d = 1, \ldots, D \) calculate

\[
E[Y_i|X_i = x_i, f^{(d)}, \sigma^{(d)}] = a \Phi \left( \frac{a - f^{(d)}(x_i)}{\sigma^{(d)}} \right) + \\
f^{(d)}(x_i) \left[ \Phi \left( \frac{b - f^{(d)}(x_i)}{\sigma^{(d)}} \right) - \Phi \left( \frac{a - f^{(d)}(x_i)}{\sigma^{(d)}} \right) \right] + \\
a^{(d)} \left[ \phi \left( \frac{a - f^{(d)}(x_i)}{\sigma^{(d)}} \right) - \phi \left( \frac{b - f^{(d)}(x_i)}{\sigma^{(d)}} \right) \right] + \\
b \left[ 1 - \Phi \left( \frac{b - f^{(d)}(x_i)}{\sigma^{(d)}} \right) \right]
\]

The predicted outcome is \( \frac{1}{D} \sum_{d=1}^{D} E[Y_i|X_i = x_i, f^{(d)}, \sigma^{(d)}]. \) The expectation conditional on the outcome not being in the censored range is:

\[
E[Y_i|a < Y_i < b, X_i = x_i, f^{(d)}, \sigma^{(d)}] = \\
f^{(d)}(x_i) + \sigma^{(d)} \left[ \phi \left( \frac{a - f^{(d)}(x_i)}{\sigma^{(d)}} \right) - \phi \left( \frac{b - f^{(d)}(x_i)}{\sigma^{(d)}} \right) \right] - \\
\Phi \left( \frac{b - f^{(d)}(x_i)}{\sigma^{(d)}} \right) - \Phi \left( \frac{a - f^{(d)}(x_i)}{\sigma^{(d)}} \right)
\]

for the error terms.

\[
y_i^* = f(x_i) + \varepsilon_i, \quad y_i = \begin{cases} 
  a & \text{if } y_i^* \leq a \\
  y_i^* & \text{if } a < y_i^* < b \\
  b & \text{if } b \leq y_i^*
\end{cases}
\]

\[
\varepsilon_i \sim \text{i.i.d. } N(0, \sigma^2_i), \quad \nu_i = (\gamma_i, \sigma_i) \sim G
\]

\[
G \sim \mathcal{DP}(G_0, \alpha)
\]

The distribution of the error term is specified similarly to George et al. (2019). The base distribution \( G_0 \) is defined as follows:

\[
p(\gamma, \sigma|\nu, \lambda_1, \gamma_0, k_0) = p(\sigma|\nu, \lambda)p(\gamma|\sigma, \gamma_0, k_0)
\]

\[
\sigma^2 \sim \frac{\nu \lambda}{\chi^2}, \quad \gamma|\sigma \sim N(\gamma_0, \frac{\sigma^2}{k_0})
\]

where, in contrast to the standard BART prior of Chipman et al. (2010), \( \nu \) is set to 10 instead of 3.\(^3\)

The parameter \( \lambda \) is set such that the \( q \)th quantile of the prior distribution of \( \sigma \) is the sample standard deviation of the outcome, or of the residuals from a linear model. For TOBART-NP, \( q = 0.9 \) instead of 0.95.\(^4\) The prior on \( \alpha \) is the \( \alpha \sim \Gamma(2, 2) \)

\(^3\)George et al. (2019) recommend \( \nu = 10 \) as the spread of the error increases when there are many components and the spread of a single component can be reduced by increasing \( \nu \). This gives better results than \( \nu = 3 \) for some DGPs in a simulation study in Appendix E.

\(^4\)This is complicated by the censoring of the outcome. Some options are: 1. Estimate the standard deviation assuming that censored outcome is normally distributed. 2. Estimate the standard deviation of a linear type I Tobit model (contains option 1 as a special case but not feasible when there are more regressors than observations). 3. Estimate the standard deviation of the censored outcome without accounting for censoring. We use option 2 for TOBART-NP.
prior introduced by Escobar and West (1995) and applied by Van Hasselt (2011).5

The outcome is scaled by subtracting the sample mean before applying the Gibbs sampler, therefore George et al. (2019) set \( \gamma_0 = 0.6 \). The parameter \( k_0 \) is scaled with the marginal distribution of \( \gamma (\gamma \sim \frac{\lambda^{E}}{\sqrt{k_0}}) \). Given \( k_s \) (set to 10 by default), \( k_0 \) is set such that \( \max_{i=1,...,n} \epsilon_i = k_s \frac{\lambda^{E}}{\sqrt{k_0}} \) where \( k_s = 10.0 \) and \( \epsilon_1,...,\epsilon_n \) are the residuals from a linear model.7 The Gibbs sampler for TOBART-NP is described in Appendix A.

For each MCMC iteration, \( d \), and observation \( i \), we obtain \( \theta_i(d) = (\gamma_i, \sigma_i) \). The conditional expectation, \( E[y_i|x_i, \theta_i(d), \gamma_i, \sigma_i] \), is calculated as outlined in section 2.3.4.

2.5 Treatment Effect Estimation for Censored Outcomes

Let a binary variable \( T_i \) equal 1 if unit \( i \) is assigned to treatment and 0 if \( i \) is assigned to the control group. The potential outcomes under treatment and control group allocation are denoted by \( Y_i(1) \) and \( Y_i(0) \) respectively. Similarly, the potential outcomes of the latent outcome are denoted by \( Y_i^*(1) \) and \( Y_i^*(0) \). Assume the data generating process is as follows:

\[
Y_i^* = \mu(x_i) + \tau(x_i)T_i + \epsilon_i, \epsilon_i \sim \mathcal{N}(0, \sigma^2)
\]

\[
Y_i = \begin{cases} 
  a & \text{if } Y_i^* \leq a \\
  Y_i^* & \text{if } a < Y_i^* < b \\
  b & \text{if } b \leq Y_i^*
\end{cases}
\]

where \( \mu(x_i) \) and \( \tau(x_i) \) are possibly nonlinear functions of covariates. Assume conditional unconfoundedness, i.e. \( Y_i^*(1), Y_i^*(0) \perp T_i | X_i \). The estimate is the conditional average treatment effect on \( Y_i^* \), i.e., \( E[Y_i^*(1) - Y_i^*(0)] | X_i = x_i = \tau(x_i) \). However, a model naively trained on only uncensored outcomes estimates the following effects

\[
E[Y_i(1)|a < y_i < b, X_i = x_i] - E[Y_i(0)|a < y_i < b, X_i = x_i] = \tau(x_i) + \\
\phi \left( \frac{a - \mu(x_i) + \tau(x_i)}{\sigma} \right) - \phi \left( \frac{b - \mu(x_i) + \tau(x_i)}{\sigma} \right) - \\
\phi \left( \frac{a - \mu(x_i)}{\sigma} \right) - \phi \left( \frac{b - \mu(x_i)}{\sigma} \right)
\]

A sufficiently flexible nonparametric method, without restrictive assumptions on the error term, will produce estimates that approximate the expression above. A model naively trained on the

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5The TobitBART package also includes an option for the prior described by Rossi (2014) and George et al. (2019), \( p(\alpha) \sim \left(1 - \frac{\alpha_{\min}}{\alpha_{\max}} \right)^{\alpha} \), where \( \alpha_{\min} \) and \( \alpha_{\max} \) are set so that the modal numbers of components are \( I_{\min} = 1 \) and \( I_{\max} = \lfloor 0.1n \rfloor \) respectively, and \( \psi = 0.5 \).

6However, the mean cannot be estimated for censored data without making further assumptions. Options include: 1. Estimate the mean (and variance) of a censored normal distribution. 2. Calculate the sample mean of the censored outcome without accounting for censoring. We use option 1.

7The residuals likely underestimate the true errors for censored observations.

8This bias occurs if all the uncensored observations are included in one regression and differences in predictions for \( T_i = 1 \) and \( T_i = 0 \) are obtained, i.e. an S-learner approach (Künzel et al., 2019), or if the two conditional expectations are obtained from separate regressions for treated and untreated uncensored observations, i.e. a T-learner approach. In both cases, the conditional expectations are not equal to the expectation of the latent outcome.
full data set with censoring similarly gives biased estimates (see Appendix B). By directly modelling \( Y^*_i \), censored outcome models avoid the bias described above. Similar biases occur if the error term is not normally distributed.

### 3 Simulation Studies

#### 3.1 Description of Prediction Simulations

We adapt the data generating processes (DGPs) introduced by Friedman (1991) to a censored regression setting. This DGP has often been applied in comparisons of semiparametric regression methods. We also make use of the censored outcome simulations described by Groot and Lucas (2012), Sigrist and Hirnschall (2019), and Jacobson and Zou (2022) for fair comparison against competing methods with existing synthetic censored data.

The covariates \( x_1, \ldots, x_p \) are independently sampled from the uniform distribution on the unit interval. The outcome before censoring is generating from one of the following functions:

- \( y^* = 10 \sin(\pi x_1 x_2) + 20(x_3 - 0.5)^2 + 10x_4 + 5x_5 + \varepsilon \), \( \varepsilon \sim \mathcal{N}(0, \sigma^2) \) with censoring from below at the 15\(^{th}\) percentile of the training data \( y^* \) values, and from above at the 85\(^{th}\) percentile of the training data \( y^* \) values (Friedman, 1991).

- \( y^* = 6(x_1 - 2)^2 \sin(2(6x_1 - 2)) + \varepsilon \), \( \varepsilon \sim \mathcal{N}(0, \sigma^2) \) with censoring from below at the 40\(^{th}\) percentile of the training data \( y^* \) values (Groot and Lucas, 2012).

- \( y^* = 6(\sum_{k=1}^{5} 0.3 \max(x_k, 0) + \sum_{k=1}^{3} \sum_{j=k+1}^{4} \max(x_k x_j, 0) + \varepsilon), \varepsilon \sim \mathcal{N}(0, \sigma^2) \) with censoring from above at the 95\(^{th}\) percentile of the training data \( y^* \) values (Sigrist and Hirnschall, 2019). For this simulation, \( x_1, \ldots, x_p \) are uniformly distributed on \([-1, 1]\) instead of \([0, 1]^{10}\).

- \( y^* = 3 + 5x_1 + x_2 + 2x_4 + \frac{x_5}{2} + \varepsilon \), \( \varepsilon \sim \mathcal{N}(0, \sigma^2) \) with censoring from below at the 25\(^{th}\) percentile of the training data \( y^* \) values (Jacobson and Zou, 2022).

The variance of the error, \( \sigma^2 \), is set to 1. See the Supplementary Appendix (Online Resource 1) for the results obtained from simulations with \( \sigma \in \{0.1, 2\} \). We also consider deviations from the assumption of normally distributed errors. In particular, we include results for simulations in which \( \varepsilon \) is generated from Skew-t, and Weibull\((1/2, 1/5)\).

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\(^{9}\)The original Friedman simulations did not involve censoring.

\(^{10}\)This simulation differs somewhat from the original simulation of Sigrist and Hirnschall (2019) for which the variable determining censoring was not perfectly correlated with the observed outcome before censoring.
The number of covariates, $p$, is set to 30. We generate 500 training and 500 test observations.

### 3.2 Prediction Simulation Results

We compare the performance of TOBART-1, TOBART-1-NP, Soft TOBART-1, and Soft TOBART-1-NP against Grabit (Sigrist and Hirnschall, 2019), linear Tobit (Tobin, 1958), BART (Chipman et al., 2010), Random Forests (RF) (Breiman, 2001), Gaussian Processes, and a Tobit Gaussian Process model (Groot and Lucas, 2012). The results for a Gaussian Process (GP) with only 5 variables (always including all informative variables) are included because GPs were observed to produce inaccurate predictions when applied to data with 30 variables. Censored outcome predictions are evaluated using Mean Squared Error (MSE), and predicted probabilities of censoring are evaluated using the Brier Score. All results are averaged over 5 repetitions.

The results for simulations with normally distributed errors are presented in Tables 1 and 2. The TOBART algorithms generally outperform competing methods across all DGPs, except unsurprisingly for the linear Jacobson and Zou (2022) simulations linear Tobit is outperformed only by Soft TOBART. TOBART-NP can slightly improve on TOBART in some cases, but generally the results are similar when errors are normally distributed. The differences in criteria across methods are small for the more linear DGPs from Sigrist and Hirnschall (2019) and Jacobson and Zou (2022), as linear Tobit is designed for a linear DGP, and the nonlinear methods BART and RF can model the relatively simple response surface well. It is worth noting that TOBART outperforms Grabit even though the true standard deviation, $\sigma = 1$, is included as one of five possible Grabit hyperparameter values in cross-validation. The same pattern of results can be observed for simulations with $\sigma = 0.1$ and $\sigma = 2$ in the Supplementary Appendix. The Supplementary Appendix contains comparisons of Area Under the Curve for all methods and DGPs, from which similar conclusions can be drawn.

The results for Skew-t and Weibull distributed errors are also presented in Tables 1 and 2. The TOBART models outperform all other methods for almost all DGPs and criteria. The results for the Weibull distribution generally favour TOBART-NP and Soft TOBART-NP, indicating

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11 Bradic and Guo (2016) considered Weibull(1/2, 1/5) errors in a simulation study.
12 Standard BART for continuous outcomes is trained on censored outcomes. Probit BART is trained on a binary variable indicating censorship. Similarly, Random Forests are separately trained on continuous censored outcomes and a binary censorship indicator.
13 The GP Matlab code was obtained from https://www.cs.ru.nl/~perry/software/tobit1.html.
14 See the Supplementary Appendix (Online Resource 1) for implementation details and parameter settings.
15 Latent outcome predictions similarly demonstrate that TOBART outperforms other methods, and these results are available on request. However, it is unsurprising that Tobit based latent outcome predictions outperform naive approaches due to the aforementioned censoring bias.
16 Computational times are included in Appendix D.
17 Results for t-distributed errors with $\nu = 3$ are in the Supplementary Appendix.
that there is some improvement from the Dirichlet Process model when the errors are sufficiently non-Gaussian.

The average coverage and length of 95% prediction intervals for the latent outcomes and the observed outcomes are given in the Supplementary Appendix (Online Resource 1). For most DGPs and error distributions, TOBART and Soft TOBART provides the closest to 95% coverage of prediction intervals for both latent and observed outcomes. For some DGPs with non-normal errors, the more conservative intervals produced by TOBART-NP and Soft TOBART-NP provide better coverage.

### 3.3 Description of Treatment Effect Simulations

A number of recent simulation studies have demonstrated that BART is among the most accurate treatment effect estimation methods (Wendling et al., 2018; McConnell and Lindner, 2019; Dorie et al., 2019; Hahn et al., 2019). However, in practice many data sets, including randomized trial data sets, contain censored outcomes. For example, antibody concentrations or environmental levels of chemicals can only be measured accurately within a certain range as a result of limitations of measuring equipment. Often economic data is censored due to privacy considerations, for example income might be censored above a certain threshold. TOBART provides a machine learning treatment effect estimation method with uncertainty quantification that can be applied to this data while still making use of the information provided by censored observations.

We demonstrate the effectiveness of TOBART by censoring the outcomes of DGPs from published studies of machine learning methods for treatment effect estimation. The chosen data generating processes contain linear and non-linear functions of covariates, constant and heterogeneous effects, and various degrees of confounding.

#### 3.3.1 Censored Caron et al. (2022) Simulations

\[ P = 10 \] covariates are generated from a multivariate Gaussian distribution, \( X_1, \ldots, X_{10} \sim MVN(0, \Sigma) \), with \( \Sigma_{jk} = 0.6^{|j-k|} + 0.1 \mathbb{1}(j \neq k) \). The binary treatment variable is Bernoulli distributed, \( Z_i \sim \text{Bern}(\pi(x_i)) \), where

\[ \pi(x_i) = \Phi(-0.4 + 0.3X_{i,1} + 0.2X_{i,2}) \]

and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution.

The prognostic score function, \( \mu(x_i) \), and CATE function, \( \tau(x_i) \), are defined as

\[ \mu(x_i) = 3 + X_{i,1} + 0.8 \sin(X_{i,2}) + 0.7X_{i,3}X_{i,4} - X_{i,5} \]
Table 1  Simulation Study, Mean Squared Error. Minimum values, excluding GP trained with only the 5 relevant variables, are in bold.

| Data                          | Tobit | BART | RF | Grabit | TOBART | TOBART NP |
|-------------------------------|-------|------|----|--------|--------|-----------|
| Friedman (1991)               | 4.764 | 1.765| 4.559 | 2.291 | 1.162 | 1.154     |
| Friedman (1991) 1 side        | 6.444 | 1.768| 6.004 | 2.743 | 1.457 | 1.509     |
| Groot and Lucas (2012)        | 12.886| 2.247| 4.612 | 0.702 | 0.631 | 0.617     |
| Jacobson and Zou (2022)       | 0.694 | 0.722| 0.855 | 0.739 | 0.718 | 0.720     |
| Sigrist and Hirnschall (2019)| 1.353 | 1.142| 1.170 | 1.146 | 1.072 | 1.074     |
| **Skew-t distribution, location = 1, scale = 1, \( \nu = 4 \)** |       |      |      |        |        |           |
| Friedman (1991)               | 5.075 | 2.172| 4.790 | 2.495 | 1.648 | 1.597     |
| Friedman (1991) 1 side        | 6.221 | 1.624| 5.962 | 2.650 | 1.344 | 1.154     |
| Groot and Lucas (2012)        | 11.297| 1.746| 4.071 | 0.760 | 0.787 | 0.648     |
| Jacobson and Zou (2022)       | 0.721 | 0.814| 0.873 | 0.783 | 0.960 | 0.696     |
| Sigrist and Hirnschall (2019)| 1.519 | 1.333| 1.337 | 1.340 | 1.279 | 1.267     |
| **Weibull distribution, shape = 0.5, scale = 0.2** |       |      |      |        |        |           |
| Friedman (1991)               | 4.599 | 1.502| 4.576 | 2.099 | 0.871 | 0.811     |
| Friedman (1991) 1 side        | 6.221 | 1.624| 5.962 | 2.650 | 1.344 | 1.154     |
| Groot and Lucas (2012)        | 11.297| 1.746| 4.071 | 0.760 | 0.787 | 0.648     |
| Jacobson and Zou (2022)       | 0.721 | 0.814| 0.873 | 0.783 | 0.960 | 0.696     |
| Sigrist and Hirnschall (2019)| 1.519 | 1.333| 1.337 | 1.340 | 1.279 | 1.267     |

\[
\tau(x_i) = 2 + 0.8X_{i,1} - 0.3X_{i,2}
\]

The outcome before censoring is generated as:

\[
Y_i^* = \mu(x_i) + \tau(x_i)Z_i + \varepsilon_i, \text{ where } \varepsilon_i \sim \mathcal{N}(0,1)
\]

The number of sampled observations is 200. The observed outcome \(Y_i\) is censored from below at the 15\(^{th}\) percentile of the generated \(Y_i^*\) values, and from above at the 85\(^{th}\) percentile.
### 3.3.2 Censored Friedberg et al. (2020)

**Simulations**

$P = 20$ covariates are generated from independent standard uniform distributions $X_1, \ldots, X_{20} \sim U(0, 1)$. There is no confounding as $\pi(x_i) = 0.5$ and $Z_i \sim \text{Bern}(\pi(x_i))$. The prognostic score function, $\mu(x_i)$, and CATE function, $\tau(x_i)$, are defined

\[
\begin{align*}
\text{normal distribution, } \sigma = 1 \\
\text{Skew-t distribution, location } = 1, \text{ scale } = 1, \nu = 4 \\
\text{Weibull distribution, shape } = 0.5, \text{ scale } = 0.2
\end{align*}
\]

| Data                  | Tobit | BART | RF | Grabit | TOBART | TOBART NP |
|-----------------------|-------|------|----|--------|--------|------------|
| Friedman (1991)       | 0.140 | 0.165| 0.195 | 0.135 | 0.069 | 0.070      |
| Friedman (1991) 1 side| 0.061 | 0.058| 0.077 | 0.067 | 0.032 | 0.033      |
| Groot and Lucas (2012)| 0.287 | 0.121| 0.171 | 0.158 | 0.116 | 0.115      |
| Jacobson and Zou (2022)| 0.099 | 0.104| 0.123 | 0.105 | 0.102 | 0.102      |
| Sigrist and Hirnschall (2019)| 0.052 | 0.052| 0.053 | 0.050 | 0.047 | 0.047      |

| Friedman (1991)       | 0.152 | 0.173| 0.197 | 0.143 | 0.083 | 0.081      |
| Friedman (1991) 1 side| 0.074 | 0.071| 0.094 | 0.070 | 0.047 | 0.044      |
| Groot and Lucas (2012)| 0.281 | 0.122| 0.172 | 0.177 | 0.119 | 0.119      |
| Jacobson and Zou (2022)| 0.098 | 0.103| 0.121 | 0.141 | 0.106 | 0.101      |
| Sigrist and Hirnschall (2019)| 0.049 | 0.049| 0.050 | 0.049 | 0.047 | 0.046      |

| Friedman (1991)       | 0.137 | 0.163| 0.197 | 0.170 | 0.058 | 0.053      |
| Friedman (1991) 1 side| 0.060 | 0.059| 0.081 | 0.061 | 0.028 | 0.024      |
| Groot and Lucas (2012)| 0.290 | 0.058| 0.144 | 0.116 | 0.077 | 0.073      |
| Jacobson and Zou (2022)| 0.056 | 0.061| 0.088 | 0.072 | 0.072 | 0.046      |
| Sigrist and Hirnschall (2019)| 0.046 | 0.045| 0.046 | 0.040 | 0.041 | 0.038      |

| Friedman (1991)       | 0.116 | 0.477| 0.120 | 0.477 | 0.055 | 0.055      |
| Friedman (1991) 1 side| 0.033 | 0.455| 0.055 | 0.456 | 0.025 | 0.025      |
| Groot and Lucas (2012)| 0.107 | 0.241| 0.233 | 0.241 | 0.106 | 0.107      |
| Jacobson and Zou (2022)| 0.100 | 0.440| 0.168 | 0.444 | 0.099 | 0.099      |
| Sigrist and Hirnschall (2019)| 0.050 | 0.054| 0.050 | 0.055 | 0.042 | 0.042      |

| Friedman (1991)       | 0.135 | 0.489| 0.144 | 0.489 | 0.064 | 0.062      |
| Friedman (1991) 1 side| 0.047 | 0.459| 0.078 | 0.459 | 0.034 | 0.033      |
| Groot and Lucas (2012)| 0.110 | 0.251| 0.258 | 0.251 | 0.111 | 0.111      |
| Jacobson and Zou (2022)| 0.098 | 0.444| 0.171 | 0.447 | 0.098 | 0.097      |
| Sigrist and Hirnschall (2019)| 0.048 | 0.049| 0.046 | 0.050 | 0.044 | 0.044      |

| Friedman (1991)       | 0.120 | 0.488| 0.120 | 0.488 | 0.037 | 0.032      |
| Friedman (1991) 1 side| 0.031 | 0.463| 0.060 | 0.463 | 0.016 | 0.012      |
| Groot and Lucas (2012)| 0.050 | 0.247| 0.206 | 0.247 | 0.067 | 0.065      |
| Jacobson and Zou (2022)| 0.049 | 0.440| 0.144 | 0.444 | 0.054 | 0.039      |
| Sigrist and Hirnschall (2019)| 0.044 | 0.045| 0.037 | 0.046 | 0.036 | 0.034      |

*Table 2* Simulation Study, Brier Score. Minimum values, excluding GP trained with only the 5 relevant variables, are in bold.
as $\mu(x_i) = 0$ and

$$
\tau(x_i) = \left(1 + \frac{1}{1 + \exp \left(\frac{-20(X_{i,1} - \frac{1}{3})}{\mu} \right)} \right) \times \left(1 + \frac{1}{1 + \exp \left(\frac{-20(X_{i,2} - \frac{1}{3})}{\mu} \right)} \right).
$$

The outcome before censoring is generated as:

$$
Y_i^* = \mu(x_i) + \tau(x_i)Z_i + \varepsilon_i, \text{ where } \varepsilon_i \sim \mathcal{N}(0,1)
$$

The number of sampled observations is 200. The observed outcome $Y_i$ is censored from below at the 15th percentile of the generated $Y_i^*$ values, and from above at the 85th percentile.

### 3.3.3 Censored Nie and Wager (2021) Simulations

The covariates are generated as follows across scenarios A to D. In simulation A, $X_1, ..., X_{12} \sim \mathcal{U}[0,1]$. In simulations B to D, $X_1, ..., X_{12} \sim \mathcal{N}(0,1)$.

$\pi(x_i)$ is defined as follows across scenarios A to D: (A) trim$_{0.1}$($\sin(\pi X_{i,1} X_{i,2})$), (B) constant equal to 0.5, (C) $1/\{1 + \exp(X_{i,2} + X_{i,3})\}$, (D) $1/\{1 + \exp(-X_{i,1}) + \exp(-X_{i,2})\}$.

$\mu(x_i)$ is defined as follows across scenarios A to D: (A) $\sin(\pi X_{i,1} X_{i,2}) + 2(X_{i,3} - 0.5)^2 + X_{i,4} + 0.5X_{i,5}$, (B) $\max\{X_{i,1} + X_{i,2} + X_{i,3}, 0\}$, (C) $2\log\{1 + \exp(X_{i,1} + X_{i,2} + X_{i,3})\}$, (D) $\frac{1}{2}\max\{X_{i,1} + X_{i,2} + X_{i,3}, 0\} + \max\{X_{i,4} + X_{i,5}, 0\}$.

$\tau(x_i)$ is defined as follows across scenarios A to D: (A) $(X_{i,1} + X_{i,2})/2$, (B) $X_{i,1} + \log\{1 + \exp(X_{i,2})\}$, (C) constant equal to 1, (D) $\max\{X_{i,1} + X_{i,2} + X_{i,3}, 0\} - \max\{X_{i,4} + X_{i,5}, 0\}$.

The outcome before censoring is generated as:

$$
Y_i^* = \mu(x_i) + \tau(x_i)(Z_i - 0.5) + \varepsilon_i, \text{ where } \varepsilon_i \sim \mathcal{N}(0,1)
$$

The number of sampled observations is 200. The observed outcome $Y_i$ is censored from below at the 15th percentile of the generated $Y_i^*$ values, and from above at the 85th percentile.

### 3.4 Treatment Effect Simulation Results

All methods are evaluated in terms of Precision in Estimation of Heterogeneous Effects (PEHE), which is defined as $\frac{1}{N} \sum_{i=1}^{N} (\hat{\tau}(x_i) - \tau(x_i))^2$. Confidence intervals are evaluated in terms of average coverage of 95% intervals and average length of intervals.

The results are presented in Table 3. For all DGPs, at least one TOBART method attains lower PEHE than all other methods, often by a large margin. Local Linear Forests (Friedberg et al., 2020) attain similar PEHE to TOBART and TOBART-NP for Nie and Wager (2021) DGP D, which involves partly linear prognostic and treatment effect functions, although soft TOBART is notably more accurate. The average coverages of
TOBART and soft TOBART credible intervals for $\tau(x_i)$ are generally much closer to 95% than the coverages of intervals produced by competing methods. TOBART-NP produces very wide credible intervals relative to TOBART. TOBART-NP produces better coverage than TOBART for four DGPs.

### 4 Data Application

For the data application, we consider the same methods as in section 3.2, excluding Gaussian Processes and adding a hurdle model combining linear regression and probit. For each data set, we average results over 10 training-test splits. Each split is defined by taking a random sample of floor$(0.7n)$ training observations stratified by censorship status. Categorical variables were encoded as sets of dummy variables. The numbers of observations, covariates, and proportions of censored observations are given in table 4. Appendix C contains descriptions of each data set with references to original sources.

#### 4.1 Data Application Results

The data application results are presented in Table 5. For most data sets the results are similar across methods, particularly when methods are evaluated in terms of Brier score for predicted probabilities of censoring. Similar results can be observed for the AUC in the Supplementary Appendix (Online Resource 1). TOBART can give notably lower MSE of outcome predictions relative to other methods for some data sets.

In contrast to the simulation studies above, there is not a clear winning method in Table 5. Although censored outcome models have been applied to these data sets in previous work, perhaps other models are more suitable for some data sets. This is evidenced by the fact that for many data sets the combination of probit and a linear model outperforms Tobit. Therefore for some data sets zero inflated, hurdle, or sample selection models might be more appropriate. For the data sets on which Tobit outperforms probit and a linear model in terms of MSE, namely Recon and Atrazine, the best method is Soft TOBART. The TOBART models also notably outperform other methods when applied to the BostonHousing and Missouri data sets.

A lesson from this study is that it is important to select the appropriate model for the data set. The TOBART and Grabit methods are designed for the same form of DGPs, therefore it is arguably fairer to compare these two methods. Soft TOBART produces lower MSE predictions than Grabit across almost all data sets.\(^ {18} \)

Nonetheless, the results are less impressive than

\(^ {18} \text{Potentially Grabit could produce better results with more hyperparameter tuning, although this would be computationally costly.} \)
Table 3: Treatment Effect Simulation Results. Caron et al. (2022), Friedberg et al. (2020), and Nie and Wager (2021) simulations with censoring from below at 15th percentile and from above at 85th percentile. PEHE = Precision in Estimation of Heterogeneous Effects, Cov = average coverage of 95% intervals, Len = average length of 95% intervals.

Table 4: Data Application: Number of observations (n), number of covariates (p), and proportions censored from below and above.

5 Conclusion

Type I TOBART produces accurate predictive probabilities of censoring, predictions of outcomes, and treatment effect estimates. TOBART-NP, gives better uncertainty quantification for some simulated DGPs. Advantages of TOBART over...
### Table 5 Data Application Results: Mean Squared Error of Outcome Predictions Relative to TOBART, and Brier Score for Predicted Probabilities of Censoring. Average over 10 random splits into 70% training data 30% test data. The numbers of observations and covariates are denoted by \( n \) and \( p \) respectively.

| Data Set     | \( n \) | \( p \) | TO BART | TO Soft BART | TO Soft Grabit | BART RF +LM | Probit Tobit |
|--------------|--------|--------|---------|--------------|---------------|-------------|--------------|
| antibody     | 330    | 3      | 0.98    | 0.97         | 0.96          | 1.00        | 0.99         | 0.99 |
| Recon        | 423    | 108    | 0.74    | **0.63**     | 0.77          | 0.66        | 0.74         | 0.68 | 0.85 | 0.81 |
| Atrazine     | 48     | 2      | 0.98    | **0.97**     | 0.98          | 0.99        | 1.01         | 1.01 | 1.02 | 1.00 |
| SedPb        | 42     | 2      | 0.97    | 0.94         | 0.95          | 1.03        | **0.93**     | 1.08 | 1.02 | 1.02 |
| Pollen,Thia  | 204    | 4      | **0.99**| 1.00        | 1.00          | **0.99**    | **0.99**     | **0.99** | 1.00 |
| Missouri     | 127    | 3      | 0.66    | **0.58**     | 0.73          | 0.88        | 0.76         | 0.77 | 0.93 | 0.93 |
| BostonHousing| 506    | 108    | **1.00**| 1.49        | 1.02          | 1.06        | 1.38         | 1.18 | 1.31 | 125.10 |

| Brier Score  |        |        |         |         |         |           |           |
|--------------|--------|--------|---------|---------|---------|-----------|-----------|
| antibody     | 330    | 3      | 0.22    | 0.20    | 0.22    | 0.20      | 0.19      | 0.19     | 0.19     | 0.22 |
| Recon        | 423    | 108    | 0.17    | 0.12    | 0.17    | 0.15      | 0.22      | 0.00     | 0.00     | 0.10 |
| Atrazine     | 48     | 2      | 0.11    | 0.12    | 0.19    | 0.17      | 0.22      | 0.01     | 0.02     | 0.00 |
| SedPb        | 42     | 2      | **0.04**| **0.04**| 0.05    | 0.07      | **0.04**  | 0.05     | 0.05     | 0.05 |
| Pollen,Thia  | 204    | 4      | 0.17    | 0.16    | 0.18    | 0.16      | 0.22      | **0.13** | **0.13** | **0.13** |
| Missouri     | 127    | 3      | 0.17    | 0.15    | 0.17    | 0.14      | 0.18      | **0.13** | **0.13** | **0.13** |
| BostonHousing| 506    | 108    | 0.02    | 0.03    | **0.01**| 0.02      | 0.02      | 0.03     | 0.13     | 0.03 |

competing methods include the fact that hyperparameter tuning is not required, and the straightforward combination of the method with other variations on BART to allow for smooth DGP and sparsity (Linero and Yang, 2018).

**Supplementary information.** The online supplementary appendix contains (A) additional simulation study results, (B) additional data application results, and (C) implementation details and parameter settings.

**Acknowledgments.** The author gratefully acknowledges helpful comments from Mikhail Zhelonkin, Chen Zhou, and participants at the Econometric Institute internal seminar.

### Declarations

**Conflict of interest:** The authors declare no competing interests.

### Appendix A  TOBART-1

#### Gibbs Sampler

**A.1 Gibbs Sampler Algorithms**

For completeness of exposition, we describe here the full conditional samples from $p((T_k, M_k) | \{(T_j, M_j) \}_{j \neq k}, \sigma, y^*)$, $k = 1, \ldots, m$ introduced by Chipman et al. (2010) in Algorithm 1. This sample is separated into a Metropolis-Hastings draw of $p(T_k | \{(T_j, M_j) \}_{j \neq k}, \sigma, y^*)$, $k = 1, \ldots, m$ following by a closed form (multivariate...
normal) draw from
\[ p(M_k|T_k, \{(T_j, M_j)\}_{j \neq k}, \sigma, y^*) \quad k = 1, \ldots, m). \]

The TOBART and TOBART-NP Gibbs samplers are described in algorithm 2.

\section*{A.2 TOBART-NP Out of sample distribution of the error}

For test data predictive intervals, we may sample TOBART-NP error term values for out of sample observations. Van Hasselt (2011) describes the sampling method as follows. Let \( \tilde{n} \) denote the index of an out of sample observation. At iteration \( t \in \{1, \ldots, T\} \) of the Markov chain, given \( \{\vartheta_{i,t}\}_{i=1}^n \), generate an out-of-sample value \( \vartheta_{\tilde{n},t} \) according to:

\[
\vartheta_{\tilde{n},t} \begin{cases} 
\vartheta_{i,t} \text{ with probability } \frac{1}{\alpha + n} \\
\sim G_0 \text{ with probability } \frac{\alpha}{\alpha + n}
\end{cases}
\]

An estimate of the posterior predictive distribution of the error is

\[
\hat{f}(u|y, s) = \frac{1}{T} \sum_{t=1}^T f(a|\mu_{\tilde{n},t}, \sigma_{\tilde{n},t}^2)
\]

Also, samples \( u^{(t)}_{\tilde{n}} \) can be made from \( \mathcal{N}(\mu_{\tilde{n},t}, \sigma_{\tilde{n},t}^2) \) for each iteration \( t \) of the MCMC sampler.
Algorithm 1 Full conditional sampler for Bayesian trees (Chipman et al., 2010)

**Input:** (Latent) outcome values $y^*$, covariates $X$, constant error variance for TOBART $\sigma^2$, trees and terminal node parameters $\{(T_k, M_k)\}_{k=1}^m$ or observation-specific error means and variances for TOBART-NP $(\gamma_i, \sigma_i^2)_{i=1}^n$.

for $k = 1, \ldots, m$ do

1. Create partial residuals. For TOBART $R_{k,i} = y^*_i - \sum_{j \neq k} g_k(x_i)$. For TOBART-NP $R_{k,i} = y^*_i - \sum_{j \neq k} g_k(x_i)$.

2. Draw from $T_k\{\{(T_j, M_j)\}_{j \neq k}, \sigma, y^* \}$ using a Metropolis-Hastings Sampler. Propose $T'_k$ using a PRUNE, CHANGE, or GROW proposal.

The PRUNE proposal removes (uniformly at random) a split that results in two terminal nodes from $T_k$. i.e. remove the children nodes from a random internal node without grandchildren.

The CHANGE proposal uniformly at random selects an internal node without grandchildren from tree $T_k$ and randomly samples a new splitting variable (uniformly) and splitting point (uniformly).

The GROW proposal uniformly at random selects a terminal node of tree $T_k$ (with some minimum number of observations) and uniformly at random samples a new splitting variable and splitting point to create tree $T'_k$.

3. The log-likelihoods of trees $T_k$ and $T'_k$, marginalizing out the terminal node parameters $M_k$ (or $M'_k$), are standard weighted linear regression log-likelihoods

$$
\sum_{i=1}^{b_k} \left[ -n_{kl} \log\left(\sqrt{2\pi} \sigma_1\right) - \log(\sigma_1) - \frac{1}{2} \log \left( 1 + \sum_{i \in n_{kl}} \frac{R_{ik}^2}{2\sigma_1^2} \right) - \sum_{i \in n_{kl}} \frac{R_{ik}^2}{2\sigma_1^2} + \frac{\left( \sum_{i \in n_{kl}} \frac{R_{ik}^2}{\sigma_1^2} \right)^2}{2(\frac{1}{\sigma_1^2} + \sum_{i \in n_{kl}} \frac{1}{\sigma_i^2})} \right]
$$

$b_k$ is the number of terminal nodes in $T_k$. There are $n_{kl}$ observations in the $l^{th}$ terminal node.

4. The Metropolis-Hastings step accepts the new tree proposal $T'_k$ with probability equal to

$$
\frac{p(T'_k \rightarrow T_k) p(R_k | T'_k, M_k, \sigma_1^2)}{p(T_k \rightarrow T'_k) p(R_k | T_k, M_k, \sigma_1^2)} \frac{p(T'_k)}{p(T_k)},
$$

where $p(T_k \rightarrow T'_k)$ is the probability of proposing tree $T'_k$ given the current tree is $T_k$.

The terminal node parameters $M_k = (\mu_1, \ldots, \mu_{b_k})'$ are drawn from the full conditional

$$
p(M_k | T_k, \{T_j, M_j\}_{j \neq k}, \gamma, y^*)
$$

$k = 1, \ldots, m$), which separates into independent univariate normal draws. For $\ell = 1, \ldots, b_k$ sample from

$$
p(\mu_\ell | \{R_{ik,i} \}_{i \in \ell}, \{\sigma_i^2\}_{i=1}^n, \sigma_\mu^2) = \mathcal{N}(\mu_\ell | \bar{\mu}_\ell, \bar{\sigma}_\mu^2), \quad \bar{\sigma}_\mu^2 = \frac{1}{\sum_{i \in \ell} \frac{1}{\sigma_i^2}}, \quad \bar{\mu}_\ell = \bar{\sigma}_\mu^2 \left( \sum_{i \in \ell} \frac{R_{i,k,i}}{\sigma_i^2} \right)
$$

end for
Algorithm 2 TOBART and TOBART-NP Gibbs Sampler

**Input:** Number of MCMC iterations, $B$. Outcome values $y$, covariates $X$, censoring limits $a$ and $b$, hyperparameter values $\alpha$, $\beta$, $\kappa$, $\nu$, $\lambda$ (or $q$). For TOBART-NP, hyperparameters $\gamma_0$ and $k_0$.

0. Set initial values of $f_y$, initialize all $m$ trees as stumps with no splits. For TOBART, initialize $\sigma$. For TOBART-NP, set initial values of $\gamma_i, \sigma_i$.

for $b = 1, \ldots B$ do

1. Draw latent variable from $y^*|\alpha, \sigma, f_y, y; y^*_\sim \left\{ \begin{array}{ll}
T \mathcal{N}(-\infty, a) (f(x_i) + \gamma_i, \sigma_i^2) & \text{if } y_i = a \\
T \mathcal{N}(b, \infty) (f(x_i) + \gamma_i, \sigma_i^2) & \text{if } y_i = b
\end{array} \right.$

2. Draw the sum of trees $f_y|y^*, \alpha, \sigma, i$, by applying Algorithm 1.

3. [TOBART] Draw $\sigma^2$ from an inverse gamma distribution $IG\left(\frac{a+1}{2}, \frac{\sum a_i(y_i - \bar{y})^2 + \nu \lambda}{2}\right)$.

4. [TOBART-NP] Sample from $(\gamma_i, \sigma_i)|y^* , \{(\gamma_k, \sigma_k), k \neq i\}, f_y$. This follows the procedure described in Escobar (1994); Escobar and West (1995, 1998). For a similar context see step 3 of algorithm 1 of Chib and Greenberg (2010). Define $\vartheta_i = (\gamma_i, \sigma_i)$. Let $\vartheta_{-i} = \{(\gamma_{-i}, \sigma_{-i})\} = \{(\gamma_k, \sigma_k), k \neq i\}$ be the set of pairs of parameters excluding $(\gamma_i, \sigma_i)$. Let $(\gamma^*_{-i, r}, \sigma^*_{-i, r})$ with $r = 1, ..., k_{-i}$ be the set of $k_{-i}$ unique pairs of parameters in the set $(\gamma_{-i, r})$.

(i) Calculate $q_{i,0} = \alpha t\nu_t(g_i^* f_y(x_i), \lambda(1 + \frac{1}{k_{-i}}))$ where $t\nu_t$ denotes the probability density function of a t distribution with $\nu$ degrees of freedom.

(ii) For $r = 1, ..., k_{-i}$ calculate $q_{i, r} = \frac{q_{i,0} \prod_{s \neq i} q_{j,0} \prod_{s \neq i} q_{j,0}}{q_{i,0} + \sum_{s \neq i} q_{j,0}}$, and $\tilde{q}_{i, r} = \frac{q_{i, r}}{q_{i,0} + \sum_{s \neq i} q_{j,0}}$ for $r = 1, ..., k_{-i}$.

(iii) Draw $r' \in \{0, 1, ..., k_{-i}\}$ from a categorical distribution with probabilities $\{q_{i,0}, \tilde{q}_{i, 1}, ..., \tilde{q}_{i, k_{-i}}\}$.

(iv) If $r' \neq 0$, set $(\gamma_i, \sigma_i) = (\mu_r, \sigma_r)$.

(v) If $r' = 0$, draw $\sigma_i^2 \sim IG\left(\frac{\nu + 1}{2}, \frac{\nu + \frac{1}{2}(\sum y_i^2 f(y_i) - f(x_i))^2}{2(1 + \frac{1}{k_{-i}})}\right)$ then $\gamma_i | \sigma_i^2 \sim \mathcal{N}\left(\frac{1}{k_{-i} + 1} (y_i - f(y_i)), \frac{\sigma_i^2}{k_{-i} + 1}\right)$

end for

4. [TOBART-NP] The following mixing step speeds up convergence of the Markov chain. Steps of this form were introduced by Bush and MacEachern (1996) and West et al. (1994).

Let $n_j$ denote the number of observations in cluster $j$, $N_j = \{i : q_i = j\}$ where the variable $q_i$ equals the index of the cluster to which observation $i$ belongs. Let $u_i = y_i - f_y(x_i)$ and let $\bar{u}^{(j)} = \frac{\sum_{i \in N_j} u_i}{n_j}$ be the mean of $n_j$ values for all observations in cluster $N_j$.

Note that $p(\gamma^*_{-i, r}, \sigma^*_{-i, r}|y^*, f_y) \propto \prod_{i \in N_j} N\left(y_i | f_y(x_i) + \gamma_j, \sigma_j^2\right) N\left(\gamma_j | 0, \frac{\sigma_j^2}{\nu_{-i}}\right) IG\left(\sigma_j^2 | \frac{\nu_{-i}}{2}, \frac{1}{2}\right)$. A standard conjugacy result implies that we can sample $\sigma_{-i, r}^2 \sim IG\left(\nu_{-i} + 1, \frac{\nu_{-i}}{2} + \frac{\sum_{i \in N_j} (u_i - \bar{u}^{(j)})^2}{\nu_{-i}}\right)$.

5. [TOBART-NP] Sample an auxiliary variable $\kappa \sim Beta(\alpha + 1, n)$ and sample $\alpha$ from the mixture distribution $\alpha|k \sim p_k Gamma(c_1 + k, c_2 - \log \kappa) + (1 - p_k) Gamma(c_1 + k - 1, c_2 - \log \kappa)$ where $k$ is the current number of mixture components, i.e. unique elements of $\{\theta_i\}_{i=1}^n = \{\gamma_i, \sigma_i\}_{i=1}^n$. $p_k$ is the mixing probability, set so that $\frac{p_k}{p_{k-1}} = \frac{\alpha_k}{\alpha_{k-1}} = \frac{1}{10^{c_2 - \log \kappa}}$. If the prior on $\alpha$ is the prior applied by George et al. (2019); McCulloch et al. (2021) and Conley et al. (2008), then samples are obtained from $\alpha|k$ by noting that $p(\alpha|k) \propto p(\kappa|\alpha) p(\alpha) \propto \kappa G(\alpha)_{(n + \alpha)} (1 - \frac{\alpha - \alpha_{\min}}{\alpha_{\max} - \alpha_{\min}})^{n}$. A sample can be obtained by discretizing the support and making a multinomial draw. McCulloch et al. (2021) use an equally spaced grid of 100 values from $\alpha_{\min}$ to $\alpha_{\max}$.
Appendix C  Description of Data Sets

- **antibody**: Measles vaccine response data set obtained from Moulton and Halsey (1995), originally from Job et al. (1991). The outcome is an antibody measurement censored from below at 0.1, $n = 330$ and $p = 3$.

- **Recon**: Atrazine concentrations in streams throughout the Midwestern United States. Data available in the R package NADA (Helsel et al., 2005) sourced from Mueller et al. (1997). The outcome is Atrazine concentration, censored from below at 0.05, $n = 423$ and $p = 108$.

- **Atrazine**: Atrazine concentrations in Nebraska ground water. Data available in the R package NADA (Helsel et al., 2005) sourced from Junk et al. (1980). The outcome is Atrazine concentration, censored from below at 0.01, $n = 48$ and $p = 2$.

- **SedPb**: Lead concentrations in stream sediments before and after wildfires. Data available in the R package NADA (Helsel et al., 2005). The outcome is Lead concentration, censored from below at 4, $n = 82$ and $p = 2$.

- **Pollen_Thia**: Thiamethoxam concentrations in pollen from the Ontario Pollen Monitoring Network. Data available in the R package NADA2 (Helsel et al., 2005) sourced from Junk et al. (1980). The outcome is Thiamethoxam concentration, censored from below at 0.05, $n = 204$ and $p = 4$.

- **Missouri**: TCDD concentrations used by Zirschky and Harris (1986) in a geostatistical analysis of Hazardous waste data in Missouri. Data available in the R package CensSpatial (Helsel et al., 2005). The outcome is censored from below at 0.1, $n = 127$ and $p = 3$.

- **BostonHousing**: Housing data for 506 census tracts of Boston from the 1970 census available in the R package mlbench (Leisch and Dimitriadou, 2010), sourced from Harrison Jr and Rubinfeld (1978); Pace and Gilley (1997). Outcome is median value of owner-occupied homes in USD 1000’s censored from above at 50. $n = 506$, $p = 108$.

Appendix D  Comparison of Simulation Study Computational Times

This appendix contains a comparison of average computational times, in minutes, across iterations for each DGP of the simulation study. The times for BART, RF, and Soft BART do not contain the time taken to train separate models for estimation of binary censoring probabilities. The Grabit time...
does not contain the considerable time required for parameter tuning by 5-fold cross-validation (the model was re-trained 135 times in each fold for different parameter settings).

The Gaussian Process MATLAB code written by Groot and Lucas (2012) was called in R via the R package R.matlab. All other functions were implemented in R. Therefore the Gaussian Process times are omitted for fair speed comparison. The Gaussian Process functions were fast, and ran for at most a few minutes per iteration.

**Appendix E  Simulation Study - TOBART prior settings**

This appendix contains a comparison of simulation study results for different prior parameter settings.

For standard TOBART, we present results for different $\lambda$ parameter settings. Recall that $\sigma^{-2} \sim Ga(\frac{v}{2}, \frac{v\lambda}{2})$ and $\lambda$ is set such that the $q^{th}$ quantile of the prior distribution of $\sigma$ is equal to some estimate $\hat{\sigma}$. For standard BART, $\hat{\sigma}$ is the sample standard deviation of the residuals from a linear model. However, a standard linear model does not account for censoring, and therefore may give poor prior calibration.

We consider the following options:

- naive sd: The sample standard deviation of the outcomes without accounting for censoring.
- Tobit sd: The maximum likelihood estimate of the standard deviation of the error term from a linear Tobit model (with covariates).
- cens sd: The maximum likelihood estimate of the standard deviation of the error term from an intercept-only linear Tobit model. This is an estimate of the standard deviation of $y^*$ that adjusts for censoring, assuming normality and no effects of covariates.
- lm sd: The default BART setting. The sample standard deviation of residuals from a linear model.

A limitation of the options that account for censoring is that the estimates rely on the assumption of normality. unsurprisingly, we observe that no setting provides the best results for all DGPs in table E4. The $\hat{\sigma}$ estimate from an intercept-only Tobit model gives good results. It is generally larger than the estimates from other options and results in a less informative prior. Therefore we apply this option for our main results.
| Data                              | Tobit | BART | RF   | Grabit | Tobit | Tobit | Tobit |
|----------------------------------|-------|------|------|--------|-------|-------|-------|
| Friedman (1991)                  | 0.028 | 6.800| 9.912| 0.675  | 3.142 | 27.762|
| Friedman (1991) 1 side           | 0.035 | 5.713| 10.269| 1.747  | 1.106 | 56.761|
| Groot and Lucas (2012)           | 0.167 | 6.046| 40.916| 1.153  | 33.681| 26.193|
| Jacobson and Zou (2022)          | 0.030 | 7.508| 13.267| 0.364  | 3.563 | 11.993|
| Sigrist and Hirnschall (2019)    | 0.030 | 6.262| 11.670| 0.265  | 13.910| 41.027|

Table D1  Simulation Study, normal distribution, $\sigma = 1$, Computational times, in minutes.

| Friedman (1991)                  | 22.214| 24.416| 109.241|
| Friedman (1991) 1 side           | 16.938| 34.236| 89.566 |
| Groot and Lucas (2012)           | 30.325| 89.028| 103.573|
| Jacobson and Zou (2022)          | 33.373| 9.905 | 150.921|
| Sigrist and Hirnschall (2019)    | 17.918| 44.855| 97.271 |

Table D2  Simulation Study, Skew-t Distribution, Computational times, in minutes.

| Friedman (1991)                  | 0.011 | 6.543| 9.505 | 0.461  | 2.912 | 48.585|
| Friedman (1991) 1 side           | 0.086 | 6.802| 28.386| 0.560  | 16.317| 29.928|
| Groot and Lucas (2012)           | 0.026 | 6.438| 11.567| 0.352  | 3.527 | 20.167|
| Jacobson and Zou (2022)          | 0.054 | 6.813| 19.126| 0.404  | 8.625 | 19.915|
| Sigrist and Hirnschall (2019)    | 0.068 | 7.536| 22.446| 0.353  | 10.291| 79.636|

Table D3  Simulation Study, Weibull Distribution, Computational times, in minutes.
| Data                        | Tobit BART | Tobit BART | Tobit BART | Tobit BART | Tobit BART NP, ν = 3, sd | Tobit BART NP, ν = 10, sd |
|-----------------------------|------------|------------|------------|------------|----------------------------|---------------------------|
| Friedman (1991)             | 1.196      | 1.269      | 1.162      | 1.180      | 1.176                      | 1.154                     |
| Friedman (1991) 1 side      | 1.478      | 1.510      | 1.457      | 1.462      | 1.518                      | 1.509                     |
| Groot and Lucas (2012)      | 0.638      | 0.634      | 0.631      | 0.644      | 0.625                      | 0.617                     |
| Jacobson and Zou (2022)     | 0.717      | 0.717      | 0.718      | 0.716      | 0.720                      | 0.720                     |
| Sigrist and Hirnschall (2019) | 1.072     | 1.072      | 1.072      | 1.075      | 1.075                      | 1.074                     |

Skew-t distribution, location = 1, scale = 1, ν = 4

| Friedman (1991)             | 1.715      | 1.725      | 1.648      | 1.696      | 1.677                      | 1.597                     |
| Friedman (1991) 1 side      | 2.220      | 2.307      | 2.163      | 2.200      | 2.079                      | 2.060                     |
| Groot and Lucas (2012)      | 1.170      | 1.189      | 1.149      | 1.170      | 1.068                      | 1.071                     |
| Jacobson and Zou (2022)     | 1.236      | 1.242      | 1.238      | 1.240      | 1.186                      | 1.184                     |
| Sigrist and Hirnschall (2019) | 1.281     | 1.280      | 1.279      | 1.283      | 1.268                      | 1.267                     |

Weibull distribution, shape = 0.5, scale = 0.2

| Friedman (1991)             | 0.933      | 0.910      | 0.871      | 0.928      | 0.807                      | 0.811                     |
| Friedman (1991) 1 side      | 1.447      | 1.396      | 1.344      | 1.371      | 1.146                      | 1.154                     |
| Groot and Lucas (2012)      | 0.771      | 0.782      | 0.787      | 0.846      | 0.649                      | 0.648                     |
| Jacobson and Zou (2022)     | 0.942      | 0.920      | 0.960      | 0.906      | 0.697                      | 0.696                     |
| Sigrist and Hirnschall (2019) | 0.417     | 0.424      | 0.426      | 0.424      | 0.965                      | 0.962                     |

| Friedman (1991)             | 2.406      | 2.498      | 2.459      | 2.420      | 2.339                      | 2.369                     |
| Friedman (1991) 1 side      | 3.761      | 3.786      | 3.667      | 3.679      | 3.629                      | 3.538                     |
| Groot and Lucas (2012)      | 2.073      | 2.113      | 2.028      | 2.105      | 1.963                      | 1.964                     |
| Jacobson and Zou (2022)     | 2.427      | 2.401      | 2.394      | 2.454      | 2.301                      | 2.306                     |
| Sigrist and Hirnschall (2019) | 2.635     | 2.603      | 2.615      | 2.599      | 2.493                      | 2.498                     |

Table E4 Simulation Study, Mean Squared Error. Different Prior calibration settings for error term distribution.
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