Unveiling A Hidden Classical-Quantum Link

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Abstract

The conceptual divide between classical physics and quantum mechanics has not been satisfactorily bridged as yet. The purpose of this paper is to show that such a bridge exists naturally in the Green-Wolf complex scalar representation of electromagnetic fields and its extension to massive fields.

Keywords: classical electrodynamics, quantum mechanics, hidden link

Part I: Electrodynamics

1 Complex Scalar Representation of Classical Electrodynamics

Green and Wolf showed that classical electromagnetic fields in vacuum can be rigorously derived from a single complex scalar potential [1, 2, 3]. The Lagrangian density is

\[ \mathcal{L}_\gamma = \partial_\mu \psi^* \partial^\mu \psi - \frac{1}{c^2} \dot{\psi}^* \dot{\psi} - \nabla \psi^* \nabla \psi. \]  

(1)

By comparing with the conventional Lagrangian density of free electromagnetic fields in terms of the vector potential \( \mathbf{A} \) which satisfies the subsidiary condition \( \nabla . \mathbf{A} = 0,0 \)

\[ \mathcal{L}_{em} = \frac{1}{8\pi} \left[ \frac{\epsilon_0}{c^2} \dot{\mathbf{A}}^* \dot{\mathbf{A}} - \frac{1}{\mu_0} (\nabla \times \mathbf{A})^* . (\nabla \times \mathbf{A}) \right], \]  

(2)

we find the correspondence

\[ \sqrt{\frac{8\pi}{\epsilon_0 c^2}} \dot{\psi} \hat{i} \equiv \mathbf{E}, \]

(3)

\[ \sqrt{8\pi \mu_0} (\nabla \psi) \hat{j} \equiv \mathbf{B}, \]

(4)

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where $\hat{i}$ is a unit vector in the direction of $E$ and $\hat{j}$ a unit vector in the direction of $B$.

The variational equation that follows from (1) is

$$
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(x, t) = 0
$$

which is the classical wave equation of a complex massless potential ($x := x$).

For stationary monochromatic $\psi$'s of the form $\psi = \psi(x) \exp(-i\omega t)$ [4], application of the time derivative $\partial/\partial t$ once on $\psi$ results in the equation

$$
i \dot{\psi} = -\frac{1}{\omega/c^2} \nabla^2 \psi.
$$

This equation might look nonrelativistic at first sight because of the appearance of a single time derivative and two space derivatives, but a careful look reveals that $\omega \dot{\psi}$ transforms like a second time derivative. Evaluating a single time derivative of a Lorentz invariant equation cannot obviously destroy its invariance. (See the Appendix for further justification.)

Now, writing the nonrelativistic Schrödinger equation in the form

$$
i \dot{\psi} = -\frac{\hbar}{2m} \nabla^2 \psi
$$

and comparing with eqn (6), one finds a surprising ‘correspondence’ between the two: the left hand sides (including the coefficient $i$) are identical, and the right hand sides differ only in the coefficient of $\nabla^2$. However, note that

$$
\frac{1}{\omega/c^2} = \frac{\hbar}{\hbar \omega/c^2} := \frac{\hbar}{2m^*}, \quad m^* = \frac{\hbar \omega}{2c^2}
$$

and so eqn (6) has the form

$$
i \dot{\psi} = -\frac{\hbar}{2m^*} \nabla^2 \psi
$$

with $m^*$ as an ‘effective mass’ which transforms like $\omega$ under Lorentz transformations. This is therefore the relativistic Schrödinger equation for a massless particle with an ‘effective mass’ $m^*$.

What has transpired is that the use of a stationary wave solution has enabled the setting of a scale $\omega$ in parameter space which converts to the mass/energy scale $m^*$ in the presence of a fundamental unit of action $\hbar$. We will return to this point later to probe its further implications.

Since this must hold for every frequency $\omega$ and eqn (6) is linear, it must hold for an arbitrary superposition of all frequencies. The relativistic invariance of the phase of the wave function $k^\mu x_\mu = \omega t - \vec{k} \cdot \vec{x}$ then implies that it must also hold for arbitrary superpositions of the wave-vector $\vec{k}$. Hence, the restriction to monochromaticity is not a serious limitation.

We will see in Appendix 2 that the restriction to stationary wave functions is also not a serious limitation. In fact, free states are stationary in the Interaction Picture.

Applying the time derivative in (6) on $\psi$, one obtains the classical Helmholtz equation

$$
(\nabla^2 + k^2) \psi = 0, \quad k^2 = \frac{\omega^2}{c^2}
$$

(10)
where \( k \) is the wave number in vacuum (refractive index \( n = 1 \)). Most interestingly, this classical equation (10) is derivable from the classical wave equation (5) via the intermediate equation (6) which has the mathematical structure of the Schrödinger equation!

To see the crucial difference between equations (6) and (10), consider a general stationary solution of eqn (6) in the polar form

\[
\psi(x, t) = \sqrt{\rho(x)} \exp(i\phi(x, t)), \tag{11}
\]

\[
\phi(x, t) = k \cdot x - \omega t \tag{12}
\]

where \( \rho(x) \) and \( \phi(x, t) \) are real functions. Substituting this in eqn (6) and separating the real and imaginary parts, one obtains the coupled equations

\[
\frac{\omega}{c^2} \frac{\partial \phi}{\partial t} + (\nabla \phi)^2 = \frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho(x)}}, \tag{13}
\]

\[
\nabla \cdot (\nabla \phi \rho(x)) = 0. \tag{14}
\]

Since, according to eqn (12), \( \partial \phi / \partial t = -\omega \) and \( \nabla \phi = k \), one gets

\[
k^2 = \frac{\omega^2}{c^2} + \frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho(x)}}. \tag{15}
\]

However, substitution of the same solutions (11, 12) for \( \psi \) in the classical Helmholtz equation (10) and separation of the real and imaginary parts result in the constraint

\[
\frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho(x)}} = 0 \tag{16}
\]

from the real part. This shows that the additional \( x \)-dependent term in eqn (15), which causes dispersion, vanishes in the classical case, ensuring that classical wave packets are non-dispersive.

But there is no such constraint on a wave function that satisfies eqn (6). This opens up the possibility of a non-classical wave mechanics based on eqn (6) in which wave packets are dispersive because of the relation (15). Since eqn (6) is the same as eqn (9), let us see what the implications are of incorporating a fundamental unit of action \( \hbar \) into it. Since the action is \( S = \int \mathcal{L}_\gamma \, dx \), scaling the wave function \( \psi \) by an arbitrary parameter \( \lambda \) implies that \( S \) scales by the factor \( \lambda^2 \). However, if the scale of the action is set by a fundamental constant \( \hbar \), then it is no longer permissible to scale the wave function \( \psi \) arbitrarily, which means it must be normalized. That in turn implies that \( \psi^* \psi \) can be interpreted as a probability density. That is, indeed, Born’s rule.

Writing \( \phi = S / \hbar \), one can rewrite the solution (11) in the form

\[
\psi(x, t) = \sqrt{\rho(x)} \exp(iS(x, t)/\hbar), \tag{17}
\]

where both \( \rho \) and \( S \) are real functions. Let us consider the stationary cases for which \( S(x, t) = W(x) - \omega t \). Separating the real and imaginary parts and substituting in eqn (9), one obtains the coupled equations

\[
\frac{\partial S(x, t)}{\partial t} + \frac{(\nabla S(x, t))^2}{2m^*} + Q = 0 \tag{18}
\]
with

\[ H = \frac{(\nabla S(x,t))^2}{2m^*} + Q = \frac{(\nabla W(x))^2}{2m^*} + Q, \]  

\[ Q = -\frac{\hbar^2}{2m^*} \frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho(x)}}, \]

and

\[ \nabla (\rho \nabla W) = 0. \]

Eqn. (18) is the Hamilton-Jacobi equation in stationary electrodynamics and Eqn. (21) is a conservation law (essentially the Poynting theorem for stationary radiation). \( Q \) is known in the literature as the ‘quantum potential’. Eqn (18) shows that the evolution of the phase \( \phi(x,t) = S(x,t)/\hbar \) is dependent on the real part of the wave function \( \sqrt{\rho(x)} \). This is a special feature of quantum mechanics absent in classical wave theory in which condition (16) holds, making \( Q \) vanish even though \( \hbar \neq 0 \).

Using the relation \( S/\hbar = (W - Et)/\hbar = \phi = kx - \omega t \) for an eigenstate of energy and momentum, one gets the familiar quantum mechanical results

\[ E = \hbar \omega, \nabla W = p = h k. \]

It now follows from eqns (18) and (19) that

\[ E = \frac{\hbar^2 k^2}{2m^*} + Q = pc + Q. \]

This is a scaled version of eqn (15) and is a very significant result which shows that \( Q \), the quantum potential, is the purely quantum mechanical energy which vanishes by condition (16) in classical wave theory, independent of \( \hbar \).

To give a concrete example of \( Q \), one can consider the case of a photon in a 1D box of length \( a \). The well known solution is \( \psi(x) = \sqrt{\rho(x)} \sin kx, \ k = n\pi/a, \ n = 1, 2, \cdots \). Hence,

\[ Q_n = \frac{\hbar^2 n^2 \pi^2}{2m^* a^2} = \frac{n\hbar c\pi}{a} \]

The lowest energy level corresponds to \( n = 1 \), and \( Q_1 = \hbar\pi c/a \) is the zero-point energy. Instead of a box one can consider other time independent potentials also. It is straightforward to see that for a harmonic oscillator potential \( \frac{1}{2} \beta x^2 \), for example, the zero-point energy is \( \frac{1}{2} \hbar \omega_0 \), \( \omega_0 = \sqrt{\beta/m^*} = \sqrt{2\beta c^2/\hbar \omega} \). Thus, the zero-point energy depends on the shape of the confining potential. This has important implications for physics and astrophysics, which we will explore in the next section.

Before passing on to the next topic, it would be worthwhile noting that eqn (23) can be written as

\[ H = E = pc + Q \]

from which follows the Hamiltonian equations

\[ \dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial Q}{\partial x_i}, \]

\[ \dot{x} = \frac{\partial H}{\partial p} = c. \]
The first of these equations has the form of Bohm’s equation for a nonrelativistic massive particle and may be interpreted as a relativistic generalization of it [5].

**Commutation Relations**

Now consider the general operator equations

\[ [D_i, x_j] = -i\delta_{ij} \] (28)

where \( D_i = -i\partial_i \) is the displacement operator, which must hold in classical field theories. If one defines the momentum operators by \( p_i = \hbar D_i \), this commutator can be written in the standard quantum mechanical form

\[ [p_i, x_j] = -i\hbar\delta_{ij}. \] (29)

It is usually argued that this commutator vanishes in the limit \( \hbar \to 0 \), the classical limit. The mathematics, however, shows that in the limit \( \hbar \to 0 \) what one actually gets is 0 = 0. The underlying non-commutative structure (28) is independent of \( \hbar \). In classical theory the translation operator \( D_i = -i\partial_i \) and \( x_i \) do not commute. In quantum mechanics the operator \( \hbar D_i \), interpreted as the momentum operator \( p_i \), does not commute with the position operator \( x_i \). Although the physical interpretations are different, the underlying mathematical structure is the same. The change in physics comes through the Planck constant \( h \) which sets a new scale for action missing in the classical theory.

**Classical and Quantum Waves Functions**

Finally, let us write \( \psi(x, t) = \langle x, t | \psi \rangle \). Let \( |\psi\rangle = \sum_i c_i |\psi_i\rangle \) where \( |\psi_i\rangle \) form a complete basis in a Hilbert space. If one defines the operator \( P_i = |\psi_i\rangle \langle \psi_i| \) and scale \( |\psi\rangle \) by \( \lambda \), \( P_i^2 = \lambda^2 P_i \) and it cannot be idempotent, i.e. it cannot be a projection operator. However, if \( |\psi\rangle \) is normalized, \( \sum_i |c_i|^2 = 1 \) and \( P_i \) is idempotent. Let us consider the pure state \( \rho = |\psi\rangle \langle \psi| \). It satisfies the conditions \( \rho^2 = \rho \), \( \text{Tr} \rho = 1 \). If an observable \( \hat{O} = \sum_i a_i P_i \) with discrete and nondegenerate eigenvalues \( a_i \) and projectors \( P_i = |i\rangle \langle i| \) is measured on the system, then according to the Lüders rule [6] the state is updated to

\[ \rho \to \rho'_k = \frac{P_k \rho P_k}{\text{Tr} P_k \rho} \] (30)

on the condition that the result \( a_k \) was obtained. However, if one considers the total state without selection or reading of individual results, the state transforms to

\[ \hat{\rho} = \sum_k p_k \rho'_k = \sum_k P_k \rho P_k \] (31)

where \( p_k = \text{Tr} P_k \rho \) is the probability weight of the state \( \rho'_k \) in the full ensemble [7]. This is the von Neumann rule. The Lüders rule clarifies its meaning and applicability.

It is clear from these discussions that the fundamental differences between classical and quantum wave functions arise from two features. First, classical wave functions satisfy condition (16) but quantum wave functions do not. Second, the Planck constant \( h \) sets a new
scale which requires the quantum wave function to be normalized, giving rise to discreteness in energy and momentum, projection operators and the special nature of projective measurements. The classical wave function can be arbitrarily scaled and has no such features. This scaling enables the amplitude and hence the intensity of classical waves to be varied arbitrarily. That freedom is not available to quantum wave functions which are normalized. Interestingly, \textit{wave functions that satisfy the quantum mechanical equation (9) are readily derivable from the classical wave equation (5)}.

In support of this one can cite the well known experiments of Aspect and his group [8], [9] who have shown very clearly that classical light pulses remain classical no matter how weak (low intensity) they are made by inserting neutral density filters—they always produce classically expected coincident counts on a beam splitter. The idea that a sufficiently low intensity light pulse cannot contain more than one photon and hence must be quantum mechanical, is contradicted by experiments. To observe the particle or quantum nature of light, one has to produce single photon light pulses (or squeezed states) which, when of sufficiently low flux, produce ‘anti-coincidence on a beam splitter’, the unambiguous signature of particle-like behaviour. Hence, a state of light is either classical or quantum depending on how it is prepared or produced—\textit{there is no transition from one to the other}. A coherent state of quantum light is the nearest one can get to classical light, but it is essentially quantum in nature. There is thus a \textit{contextuality and complementarity between classical and quantum light: the full nature of light can only be comprehended by taking into account the mutually exclusive methods of preparing these two forms of light}. And this is readily understood in terms of the theory outlined above.

An important feature of time-independent quantum mechanical wave functions is their single-valuedness. In nonrelativistic quantum mechanics this follows from the ellipticity of the Schrödinger equation and the fact that Euclidean space is simply-connected [10]. Since the Helmholtz equation is also elliptic, classical wave functions that are solutions of this equation must also be single-valued in simply connected spaces.

\textit{Helicity}

Let us next see how the helicity of electromagnetic radiation is described in the Green-Wolf scalar theory. Following Wolf [2] we write

\[ \psi(x, t) = \psi_+(x, t) + \psi_-(x, t) \] (32)

with

\[
\psi_+(x, t) = \int_{-\infty}^{0} \Psi(x, \omega) \exp(-i\omega t) \, d\omega = \int_{0}^{\infty} \Psi(x, -\omega) \exp(i\omega t),
\] (33)

\[
\psi_-(x, t) = \int_{0}^{\infty} \Psi(x, \omega) \exp(-i\omega t)
\] (34)

satisfying the relativistic Schrödinger-like equation, but \( \psi \) being complex, they are not complex conjugates of each other in general. They have been termed \textit{partial waves} by Wolf who has shown that on time averaging, \( \psi_+ \) and \( \psi_- \) are incoherent and represent two independent circularly polarized components of light with helicity \( \pm 1 \), i.e.

\[
\int_V \psi_+^* \hat{\lambda} \psi_\mp = \pm 1
\] (35)
where $\hat{\lambda} = \frac{\sigma_{\lambda}}{|\sigma| \sigma}$. It follows from this that
\[
\int_V \psi^* \hat{\lambda} \psi = 0 \tag{36}
\]
which shows that the convection current
\[
j = -\frac{i\hbar}{2m} [\psi^* \nabla \psi - \nabla \psi^* \psi] \tag{37}
\]
does not have any helicity. However, the currents
\[
\begin{align*}
j_+ &= -\frac{i\hbar}{2m} [\psi^+_+ \nabla \psi_+ - \nabla \psi^+_+ \psi_+] \tag{38} \\
j_- &= -\frac{i\hbar}{2m} [\psi^-_+ \nabla \psi_- - \nabla \psi^-_+ \psi_-] \tag{39}
\end{align*}
\]
carry $\pm 1$ helicities.

Finally, the current $j$ satisfies the continuity equation
\[
\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \tag{40}
\]
where $j_0 = \rho = |\psi|^2 > 0$.

Notice that the time component of the conserved current $j^\mu = \psi^* \partial^\mu \psi - \partial^\mu \psi^* \psi$ associate with the classical wave equation (3) is not always positive definite due to the second time derivative in the equation requiring the choice of two initial conditions. Consequently, $\rho = cj^0 = \dot{\psi}^* \psi - \dot{\psi}^* \psi$ cannot be interpreted as a probability density. This was the historical reason for the Pauli-Weisskopf second quantization of the Klein-Gordon equation [11]. By contrast, the time component of the conserved Schrödinger-like current $j^\mu = (\rho, j)$ is positive definite and can be interpreted as a probability density. Hence, the Pauli-Weisskopf second quantization is not mandated.

**Entanglement in Classical Optics**

Finally, let us consider entanglement. It is by now well known that entanglement occurs in classical optics [12, 13, 14, 15]. The reason is now clear—the classical and quantum mechanical wave functions are mathematically related.

Suppose there is a bipartite classical state $|\psi\rangle_{AB} \in H_A \otimes H_B$ where $H_A$ and $H_B$ are two Hilbert spaces. Then, according to the Schmidt decomposition theorem (which dates back to 1907 [16] and is pre-quantum) it is always possible to express this state as
\[
|\pi\rangle_{AB} = \sum_{i=0}^{d-1} \lambda_i |i\rangle_A |i\rangle_B \tag{41}
\]
where $\lambda_i$ are real and strictly positive, $\sum_i \lambda_i^2 = 1$, and $\{|i\rangle_A\}, \{|i\rangle_B\}$ are orthonormal bases in $H_1, H_2$ respectively. The Schmidt rank $d$ of a bipartite state is equal to the number of Schmidt coefficients $\lambda_i$ in its Schmidt decomposition and satisfies
\[
d \leq \min\{\dim(H_A), \dim(H_B)\} \tag{42}
\]
If the Schmidt rank $d > 1$, the state is entangled, i.e. it cannot be written as a product state.

In classical optics it is always possible to consider light of unit intensity without implying normalization in the quantum mechanical sense. Hence, the mathematical result is equally applicable to classical and quantum mechanical optics.

The two Hilbert spaces $H_A, H_B$ in the Schmidt decomposition have disjoint bases: $(\{i\}_A \cap \{i\}_B) = \emptyset$. There is, obviously, nothing in the mathematical theorem that tells us how the disjointness is to be physically realized. For intra-system bipartite entanglement (i.e. entanglement between two different degrees of freedom of a single system), one can have, for example, path-polarization entanglement in classical optics and path-spin entanglement in quantum mechanics where the choice of paths (strictly speaking, disjoint spatial modes) is restricted to two. For inter-system entanglement (entanglement between two different systems) one can have polarization-polarization entanglement in both classical and quantum optics, the dimension of the Hilbert spaces being 2 in both cases. In this case the spatial wave functions of the two systems remain in product form. So far only intra-system entanglement has been experimentally studied in classical optics, but extension to inter-system entanglement is possible in principle.

It should be clear therefore that the mathematical structure of entanglement is fundamentally the same for classical and quantum mechanical radiation, though because of the normalization of the quantum states forced by the Planck constant, projective measurements play a role in quantum radiation that has no counterpart in classical radiation.

**Interaction with Matter**

Before passing on to the implications, let us briefly consider the interaction of radiation with Dirac particles. The Lagrangian density is

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu \partial_\mu - mc/\hbar) \Psi + \partial_\mu \psi^* \partial^\mu \psi + e\bar{\Psi} \gamma^\mu \Psi \partial_\mu \psi$$

which is invariant (to within total four-divergence terms) under the local gauge transformations $\Psi' = e^{i\theta} \Psi, \psi' = \psi + \theta$ ($\theta$ real) with the restriction $\Box \theta = 0$.

**Part II: Massive Electrodynamics**

The Green-Wolf complex scalar representation of electromagnetic fields in vacuum turns out to be crucial in formulating a satisfactory theory of radiation encompassing both its classical and quantum aspects. We will now show that it can be extended to massive fields.

Let us consider the Lagrangian density of a massive complex scalar field in vacuum,

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - \mu^2 \psi^* \psi = \frac{1}{c^2} \dot{\psi}^* \dot{\psi} - \nabla \psi^* \cdot \nabla \psi - \mu^2 \psi^* \psi$$

where $\mu$ is an arbitrary constant with the dimension of inverse length. The classical wave equation that follows from it is

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \mu^2 \right) \psi(x,t) = 0$$
For stationary monochromatic \( \psi \)'s of the form \( \psi = \psi(x) \exp(-i\omega t) \), application of the operator \( \partial/\partial t \) once on \( \psi \) results in the equation

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \left[ -\frac{e^2}{\omega} \nabla^2 + \frac{\mu^2 c^2}{\omega} \right] \psi
\]

(46)

which, as we have seen in Part I, has the mathematical structure of the Schrödinger equation with a \( \omega \) dependent potential, and can indeed be written in the form

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar}{2m^*} \nabla^2 + V_0 \right] \psi
\]

(47)

where

\[
m^* = \frac{\hbar \omega}{2c^2}, \quad \hbar \omega = +\sqrt{m_0^2 c^4 + p^2 c^2},
\]

\[
\mu = \frac{m_0 c}{\hbar}, \quad V_0 = \frac{m_0^2 c^4}{\hbar^2 \omega}.
\]

(48)

The principal difference from the non-relativistic Schrödinger equation is the occurrence of the effective mass \( m^* \) which transforms like \( \omega \) under Lorentz transformations. Writing \( m^* = \gamma m_0, \gamma = (1 - v^2/c^2)^{-1/2} \) and ignoring terms of \( \mathcal{O}(v^2/c^2) \), one obtains

\[
\frac{i\hbar}{\hbar} \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m_0} \nabla^2 + V'_0 \right] \psi, \quad V'_0 = m_0 c^2,
\]

(49)

which is the nonrelativistic Schrödinger equation with a constant potential. This shows that eqn. (47) is the correct relativistic generalization of the Schrödinger equation.

Applying the time derivative on \( \psi \) in eqn (46), one obtains the Helmholtz equation

\[
(\nabla^2 + k^2) \psi = 0,
\]

(50)

\[
k^2 = \omega^2 c^2 - \mu^2
\]

(51)

where \( k \) is the wave number in vacuum (refractive index \( n = 1 \)). Most interestingly, this classical equation (50) is derivable from the classical wave equation (45) via the intermediate equation (46) which has the mathematical structure of the Schrödinger equation!

Eqn (47) ensures that the convection current

\[
\mathbf{j} = -\frac{i\hbar}{2m^*} [\psi^* \nabla \psi - \nabla \psi^* \psi]
\]

(52)

is conserved,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad \rho = \psi^* \psi > 0
\]

(53)

and its time component \( \rho > 0 \). It is therefore possible to interpret \( \psi^* \psi \) as the position probability density and \( \mathbf{j} \) as the probability current density when \( \psi \) is normalized. Hence, eqn (47) can be given a quantum mechanical particle interpretation.
The classical wave equation (45) does not have this property because the time component of its conserved current \( j^\mu = \psi^* \partial^\mu \psi - \partial^\mu \psi^* \psi \) is not always positive definite due to the second time derivative in the equation requiring the choice of two initial conditions. Consequently, \( \rho = cj^0 = \psi^* \psi - \psi^* \dot{\psi} \) cannot be interpreted as a probability density. This was the historical reason for the Pauli-Weisskopf second quantization of the Klein-Gordon equation which is no longer mandated.

Now consider a general stationary solution of eqn (46) in the polar form
\[
\psi(x, t) = \sqrt{\rho(x)} \exp(i\phi(x, t)), \quad (54)
\]
\[
\phi(x, t) = k \cdot x - \omega t \quad (55)
\]
where \( \rho(x) \) and \( \phi(x, t) \) are real functions. Substituting this in eqn (46) and separating the real and imaginary parts, one obtains the coupled equations
\[
\frac{\omega}{c^2} \frac{\partial \phi}{\partial t} + (\nabla \phi)^2 + \mu^2 = \frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho(x)}},
\]
\[
\nabla \cdot (\nabla \phi \rho(x)) = 0. \quad (57)
\]
Since, according to eqn (54), \( \partial \phi / \partial t = -\omega \) and \( \nabla \phi = k \), one gets
\[
k^2 = \frac{\omega^2}{c^2} - \mu^2 + \frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho(x)}},
\]
Substitution of the same solution (eqns (54) and (55)) in the classical Helmholtz equation (50) and separation of the real and imaginary parts result in the condition
\[
\frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho(x)}} = 0 \quad (59)
\]
from the real part. This shows that the additional \( x \)-dependent term in eqn (58), which causes dispersion, vanishes in classical theory, ensuring that classical wave packets are non-dispersive.

But, as in the massless case studied in Part I, there is no such restriction on a wave function satisfying eqn (46), which therefore forms the basis of a non-classical wave mechanics with dispersive wave packets.

Since eqn (47) with \( m^* = \hbar \omega / 2c^2 \) is the same as eqn (46), as pointed out in Part I, the scale of the action \( S \) is set by \( \hbar \) and it is no longer permissible to scale the wave function \( \psi \) arbitrarily, which means it must be normalized and can be interpreted as a probability density. Other important consequences of normalization of the wave function have been discussed in Part I.

Since \( \psi \) describes a massive field, there is a longitudinal component of the polarization vector in this case in addition to two transverse components.
2 Quantum and Classical Particles

Let $\eta \neq \hbar$ be an arbitrary unit of action. Multiplying eqns (56) and (57) by $\eta$ and writing $\eta \phi = S$, we get

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V_0 + Q = 0,$$

and

$$Q = -\frac{\eta^2 \nabla^2 \sqrt{\rho(x)}}{2m} \sqrt{\rho(x)}, \quad V_0 = \frac{\eta \mu^2 c^2}{\omega}, \quad 2m = \eta \omega/c^2.$$

Eqn (60) is the Hamilton-Jacobi equation for a massive particle in a potential $V_0 + Q$. For stationary eigenstates of energy and momentum one can set $S = W - Et$, $p(= \gamma m v) = \nabla S = \nabla W$. Then,

$$H = \frac{p^2}{2m} + V_0 + Q$$

and hence

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial Q}{\partial x_i}$$

and eqn (61) becomes

$$\nabla \cdot (p \rho(x)) = 0.$$

Eqn (64) would the relativistic version of Bohm’s equation for a massive particle in a quantum potential $Q$ if one were to identify $\eta$ with $\hbar$. It is the quantum potential that gives rise to interference of quantum particles [17].

Notice that condition (59) prevents dispersion and at the same time causes $Q$, the term responsible for quantum mechanical coherence, to vanish. It is therefore a sufficient condition for Newton’s equation to hold. Eqn (60) then takes the form

$$\frac{\partial S_{cl}}{\partial t} + H = 0,$$

and

$$H = \frac{(\nabla W_{cl})^2}{2m} + V_0 = \frac{p^2}{2m} + V_0 = 0,$$

It follows from this that

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} = 0.$$

The absence of interference indicates that there is no fixed phase relationship between different points of the wave amplitude. This follows from eqn (66) which shows that the phase
\[ \phi = S_{cl}/\eta \] is independent of \( \sqrt{\rho(x)} \). Hence, one cannot write a coherent superposition \( \sum_i c_i \psi_i \) of wave functions describing a classical particle. However, one can still write a density matrix:

\[
\hat{\rho} = \sum_i |c_i|^2 |\psi_i\rangle \langle \psi_i|.
\]  

(70)

A similar situation obtains in the Koopman-von Neumann wave theory of nonrelativistic classical mechanics in which the particle wave function satisfies the Liouville equation \[18, 19\].

All this shows that eqn (46) for a wave function \( \psi \) that satisfies condition (59) is equivalent to Newton’s equation of motion for a free massive particle. Hence, \textit{the same equation, namely eqn (46), holds for both quantum and classical mechanics of relativistic particles depending on whether or not the wave function satisfies a certain condition.} It describes quantum particles if the wave function is normalized and does not satisfy condition (59), and classical particles if it is not normalized and satisfies condition (59).

Eqsns (64) and (69) are second order differential equations in time and their solutions require two initial conditions specifying the position and velocity which can be varied independently. In quantum mechanics this is not permissible and solutions of eqn (64) require special care. There is no such restriction on eqn (69) which is classical. Further, in a theory in which the classical and quantum aspects of a system are intrinsically linked, they have the same ontology, and hence the de Broglie-Bohm type of interpretation \[5\] is a natural choice.

### 3 Measurements

Quantum mechanics presumes classical measuring apparatus with which quantum systems interact. This has been a fundamental problem since the inception of quantum mechanics because the two systems appeared so disparate, the quantum system being described by a ray in a Hilbert space and the classical system by a point in phase space. The option of treating the measuring apparatus also as a quantum system gave rise to the measurement problem which refuses to go away. A new option is now available, namely the use of a wave function in a Hilbert space for the classical measuring apparatus.

Let us consider the case of an observation designed to measure some observable \( \hat{P} \) of a stationary quantum system \( S \) with wave function \( \psi_S(x, t) \). Let the stationary classical wave function of the apparatus \( A \) be \( \psi_A(y, t) \) where \( y \) is the coordinate of the ‘pointer’. The initial state is a product state

\[
\Psi^{SA}(x_0, y_0, 0) = \psi^S(x_0, 0) \otimes \psi^A(y_0, 0) = \psi^A(y_0, 0) \sum_p c_p \psi^S_p(x_0, 0)
\]  

(71)

where \( \hat{P} \psi^S_p(x) = p \psi^S_p(x) \). This is a hybrid wave function. This kind of wave function was first introduced by Sudarshan \[20\].

Following von Neumann, let us assume that the measurement interaction is impulsive, and that during this impulsive interaction the free evolutions of the quantum particle and the classical apparatus can be ignored because the mass of the particle is very large and the mass of the apparatus (the massive particle) can always be chosen to be sufficiently large.
If one sets \( \hbar = 1 \) for convenience, the evolution operator of the system takes the form
\[
U = \exp(-i \hat{\Omega} t),
\]
where \( g \) is a suitable coupling strength and \( \hat{D}_y = -i \partial/\partial y \) is the classical displacement operator corresponding to the coordinate \( y \) of the apparatus. The form (73) of the measurement interaction has been chosen to be of the von Neumann type. Then,
\[
U \psi^A(y, t) = e^{-igp \hat{D}_y t} \psi^A(y_0, 0) = e^{-yp \hat{D}_y} \psi^A(y_0, 0) = \psi^A(y_0 - yp), \quad yp = gpt
\]
for every \( p \) and for \( t \leq \tau \), the measurement time which is assumed to be extremely short. For \( t > \tau \) there is no further displacement of the pointer. Hence, in accordance with (70), the final stationary state is of the form
\[
\hat{\rho}^{SA} = \sum_p |c_p|^2 |p\rangle^S \langle p| \hat{\rho}^{A} \langle p|.
\]
Each pointer position is correlated with a particular outcome \( p \) with probability \( |c_p|^2 \), the correlation being exact in the limits of both \( g \) and the number of trials tending to infinity.

The mixed state \( \hat{\rho}^S \) of the quantum system \( S \) alone after the measurement can be obtained by tracing \( \hat{\rho}^{SA} \) over the apparatus states:
\[
\hat{\rho}^S = \text{Tr}_A \hat{\rho}^{SA} = \sum_p |c_p|^2 |p\rangle^S \langle p|.
\]
which is formally the same as the standard von Neumann mixed density matrix but does not imply a process of collapse.

Thus, we have a unified theory of classical and quantum systems (intrinsically relativistic) in which measurement does not occupy any special significance, and the two systems naturally share the same ontology.

4 Concluding Remarks

That the Green-Wolf complex scalar representation of electromagnetic fields would reveal the much needed mathematical and conceptual link between the ‘quantum mechanics’ of radiation and the classical field theory of radiation comes as a surprise. It lays the mathematical and conceptual foundation of wave-particle duality originally discovered by Einstein in the energy fluctuations of Planck radiation. The fundamental role of the Planck constant in forcing the normalization of the wave function and hence the Born rule, becomes transparent. The classical time independent Helmholtz eqn (10) is derivable from the time dependent classical wave equation (5) through the intermediate equation (6) which has an essentially Schrödinger-like structure.

The function \( \nabla^2 \sqrt{\rho} / \sqrt{\rho} \) plays a fundamental role in the theory. Its presence or absence determines whether the theory is quantum mechanical or classical. The Helmholtz equation forces
this term to vanish, ensuring dispersion free classical waves in vacuum. Its presence allows quantum mechanical waves to disperse in vacuum. It also determines the functional form of the quantum potential $Q$ responsible for all quantum mechanical features like quantum coherence and quantized energy levels. It is noteworthy that the quantum potential, a typical feature of nonrelativistic de Broglie-Bohm theory \[5\], emerges naturally in a relativistic theory.

The generalization to massive electrodynamics is straightforward and leads to a theory of classical massive particles, and hence of classical measuring devices, obeying the Schrödinger equation with the supplementary condition $\nabla^2 \sqrt{\rho} \sqrt{\rho} = 0$ on the wave amplitude, and hence a satisfactory theory of measurement that does not require a collapse postulate.

The generalization of the Green-Wolf complex scalar representation to massless classical Yang-Mills fields and their quantum mechanical theory is under investigation. That will be of great importance for the standard model of particle physics as well as for Einstein’s gravitational equations which have close relationships with Yang-Mills equations \[21\] \[22\].

5 Acknowledgement

I am grateful to A. K. Rajagopal for some critical and helpful comments.

Appendix: Manifest Lorentz Invariance

The choice of stationary states with a clear separation of the spatial and time components might appear to violate Lorentz invariance, particularly because of the occurrence of the Schrödinger-like equation (9). That this is not the case can be shown by foliating the Minkowski manifold by a continuous set $\{\sigma\}$ of space-like hypersurfaces whose normal at every point is time-like. The whole sequence of these space-like slices is generated by time evolution. Each point $p \in \sigma$ has its own local time $t_p$ so that there is no preferred time frame that defines absolute simultaneity. This is therefore a Lorentz invariant procedure. Lorentz transformations will transform one set of surfaces $\{\sigma\}$ to another set $\{\sigma'\}$ with local times $t_{p'}, p' \in \sigma'$, and one is free to choose any of these foliations. Thus, space-time separation can be done fully relativistically, and a scalar field can be defined as a functional $\psi[\sigma]$. This foliation of the Minkowski manifold was first used by Tomonaga \[23\] and Schwinger \[24\] \[25\] to formulate quantum electrodynamics in a manifestly Lorentz covariant form.

In the interaction or Dirac representation the standard Schrödinger equation takes the form

$$i\hbar \partial_t \psi_I(t) = \hat{V}_I(t)\psi_I(t) \tag{77}$$

where the wave function in the interaction representation is

$$\psi_I(t) = \exp(i\hat{H}_{0,S}t/\hbar)\psi_S(t),$$

$\psi_S(t)$ being the wave function in the Schrödinger representation, $\hat{H}_S = \hat{H}_{0,S} + \hat{V}_S$ the Hamiltonian in the Schrödinger representation, and $\hat{V}_I(t)$ the interaction Hamiltonian in the interaction representation related to $\hat{V}_S$ by

$$\hat{V}_I(t) = e^{i\hat{H}_{0,S}t/\hbar}\hat{V}_Se^{-i\hat{H}_{0,S}t/\hbar}. \tag{78}$$
This can be written in the manifestly Lorentz covariant form

\[
\left[ -i\hbar \frac{\delta}{\delta \sigma_p} + \hat{V}_{Ip} \right] \psi_I[\sigma] = 0
\]  

(79)

provided \( \hat{V}_{Ip} \) is Lorentz invariant, where \( \frac{\delta}{\delta \sigma_p} \) is a functional derivative. In interaction free cases this implies that \( \psi_I \) is \( \sigma \)-independent, i.e. the same on the entire set of space-like hypersurfaces generated by a future-directed time-like vector field. Since this is generally true of all \( \psi_I \) in the interaction representation, and since all three representations are equivalent, this shows that free wave functions are essentially stationary. Hence, the restriction to stationary waves adopted in this paper to derive the main result is not a serious limitation.

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