Non-Perturbative Superpotentials in Landau-Ginzburg Compactification

Hitoshi Sato

Department of Physics, Faculty of Science, Osaka University
Toyonaka, Osaka 560, Japan
email address : sato@funpth.phys.sci.osaka-u.ac.jp

ABSTRACT

We study the Landau-Ginzburg models which correspond to Calabi-Yau fourfolds. We construct the index of the typical states which correspond to toric divisors. This index shows that whether a corresponding divisor can generate a non-perturbative superpotential. For an application, we consider the phase transition in terms of the orbifold construction. We obtain the simple method by which the divisor, which can not generate a superpotential in the original theory, can generate a superpotential after orbifoldization.
1 Introduction

Recently the Calabi-Yau four-folds have been paid attention to in the context of M-theory [1, 2] and F-theory [3] compactification. Witten found [4] the interesting property of the Calabi-Yau four-fold compactification, i.e. the special class of the divisors with wrapped 5-branes can generate the $N = 1$ superpotential. The condition whether or not a divisor $D$ can generate a superpotential is purely topological, i.e. the Euler character $\chi(D) = 1$ is necessary (not sufficient) and the holomorphic Hodge numbers $h^{1,0}(D) = h^{2,0}(D) = h^{3,0}(D) = 0$ are sufficient (not necessary) conditions.

In this note, we study the such topological properties of divisors in terms of Landau-Ginzburg models. It is well known that the Landau-Ginzburg models as well as their orbifolds with central charge $c = 3d$ can describe the Calabi-Yau manifolds of dimension $d$. For example, the Hodge numbers $h^{i,j}$ can be identified with the numbers of the $(c, c)$ or $(a, c)$ states with $U(1)$ charge $(d-i,j)$ or $(-i,j)$ respectively, where $c (a)$ denotes the (anti-)chiral ring of $N = 2$ conformal field theory [5]. These numbers can be calculated without any geometrical information and the simple formulas were obtained [6] for three-fold case. Since four-folds are of our interest, we will consider the Landau-Ginzburg models (and their orbifolds) with $c = 12$.

For geometrical description of the Calabi-Yau manifolds, toric geometry is most useful. In the work of Klemm, Lian, Roan and Yau [7], the topological properties for the divisors is described in terms of toric geometry. It is of our interest to find the corresponding result in terms of Landau-Ginzburg models. To do this, we will use the method obtained in [8, 9] where it was shown how these two description, the toric geometry and the Landau-Ginzburg models, can be identified. Once we obtain the Landau-Ginzburg description, we can find the simple conditions for the states which correspond to the divisors. These conditions, as expected, can be applied straightforwardly to the calculation without any geometrical information. Moreover, by the orbifold construction of [9], we can easily find the new model in which the divisor, which can not generate the superpotential in the original theory, can generate the superpotential in the orbifoldized theory. This implies that the phase transition in terms of our orbifoldization can be characterized by the property of the divisors which lead to the non-perturbatively generated superpotentials. Similar
transitions are discussed in [10].

This paper is organized as follows. In section 2, we briefly review the results of [7] and their Landau-Ginzburg interpretation is obtained in section 3. Moreover in section 3, we develop the simple method to decide whether or not a divisor with wrapped 5-brane can generate a superpotential. In the last section we consider the Landau-Ginzburg orbifolds, which correspond to the Calabi-Yau four-fold orbifolds, and their phase transitions. We can easily find the appropriate orbifoldization by the method obtained in [9], such that the divisor which cannot generate the superpotential in the original theory can generate the superpotential in the orbifoldized theory.

2 Toric description of Calabi-Yau four-folds and their divisors

The topological numbers for Calabi-Yau four-folds are studied in [11],[7],[14]. At first sight, the non-trivial Hodge numbers are \(h_{1,1}, h_{3,1}, h_{2,1}\) and \(h_{2,2}\) (up to dualities). However by the index theorem [11] it is shown that there is one relation

\[
h_{2,2} = 2(22 + 2h_{1,1} + 2h_{3,1} - h_{2,1}),
\]

so that only three Hodge numbers are independent, namely \(h_{1,1}, h_{3,1}\) and \(h_{2,1}\). It will be clear that this fact is important when one considers the relation between the Landau-Ginzburg description and the toric geometry. The Euler number can be simply written as

\[
\chi = 6(8 + h_{1,1} + h_{3,1} - h_{2,1}).
\]

There are formulas for these Hodge numbers \(h_{1,1}, h_{3,1}\) and \(h_{2,1}\) in terms of toric geometry. The formulas of the first two of these are obtained due to Batylev’s original work [15]:

\[
h_{1,1} = l(\Delta^*) - 6 - \sum_{\text{codimension } \Theta^* = 1} l'(\Theta^*) + \sum_{\Theta^* = 2, \Theta^* \in \Delta^*} l'(\Theta^*)l'(\Theta),
\]

\[
h_{3,1} = l(\Delta) - 6 - \sum_{\text{codimension } \Theta = 1} l'(\Theta) + \sum_{\Theta = 2, \Theta \in \Delta} l'(\Theta)l'(\Theta^*),
\]

\[
h_{2,1} = \text{Formula in terms of toric geometry}.
\]
where $l(\Delta) (l(\Delta^*))$ denotes the number of integral points in the Newton polyhedron \( \Delta \) (the dual polyhedron \( \Delta^* \)) and \( l'(\Theta) (l'(\Theta^*)) \) denotes the number of integral points interior of the face \( \Theta \) (the dual face \( \Theta^* \)). The formula for $h^{2,1}$ is obtained in \cite{7, 14} to be

\[
h^{2,1} = \sum_{\text{codimension } \Theta = 3, \Theta \in \Delta} l'(\Theta)l'(\Theta^*). \tag{2.5}
\]

The authors of ref. \cite{7} studied the topological numbers of the divisors in Calabi-Yau four-folds using toric geometry. They analyze the local structure of the divisors coming from the blowing up of the singularities on the hypersurface of the Calabi-Yau embedded in the weighted complex projective space. They have classified the divisors by their topological numbers. We will briefly review their results which we need.

**Case A** $d_{\Theta_k^*} = 3$, \( h^{0,0}(D_k) = l'(\Theta_k) + 1 \). This case explains the additional fourth term in (2.3).

**Case B** $d_{\Theta_k^*} = 2$, \( h^{0,0}(D_k) = 1, \ h^{1,0}(D_k) = l'(\Theta_k) \). In this case we get $l'(\Theta_k^*) \cdot l'(\Theta_k^*)$ (3, 2) forms, their dual (1, 2), (2, 3) and (2, 1) forms on a four-fold, where we have used the Poincaré and complex conjugation dualities.

**Case C** $d_{\Theta_k^*} = 1$, \( h^{0,0}(D_k) = 1, \ h^{2,0}(D_k) = l'(\Theta_k) \). On a four-fold, we get additional (3, 1) forms of the fourth term in (2.4).

and other $h^{i,j}(D_k) = 0$ in all the cases. As a result, they find the simple formula for the Euler number of the divisor $D_k$, i.e.

\[
\chi(D_k) = 1 + (-1)^{\dim\Theta_k + 1} \ l'(\Theta_k), \tag{2.6}
\]

where $l'(\Theta_k)$ is the number of integral points in the face $\Theta_k$ which is dual to the point in the dual polyhedron $\Delta^*$ corresponding to the divisor $D_k$ of our interest.

### 3 Landau-Ginzburg analysis

A Landau-Ginzburg model is characterized by a superpotential $W(X_i)$ where $X_i$ are $N=2$ chiral superfields. The Landau-Ginzburg orbifolds \cite{15} are obtained by
quotienting with an Abelian symmetry group $G$ of $W(X_i)$, whose element $g$ acts as an $N \times N$ diagonal matrix, $g : X_i \to e^{2\pi i \tilde{g}} X_i$, where $0 \leq \tilde{g}_i < 1$. Of course the $U(1)$ twist $j : X_i \to e^{2\pi i q_i} X_i$ generates the symmetry group of $W(X_i)$, where $q_i = \frac{w_i}{d}$. $W(\lambda^{w_i} X_i) = \lambda^d W(X_i)$ and $\lambda \in \mathbb{C}^*$. Using the results of Intriligator and Vafa [16], we can construct the $(c,c)$ and $(a,c)$ rings, where $c$ (a) denotes chiral (anti-ciral). Also we could have the left and right $U(1)$ charges of the ground state $|h\rangle_{(a,c)}$ in the $h$-twisted sector of the $(a,c)$ ring. In terms of spectral flow, $|h\rangle_{(a,c)}$ is mapped to the $(c,c)$ state $|h'\rangle_{(c,c)}$ with $h' = h j^{-1}$. Then the charges of the $(a,c)$ ground state of $h$-twisted sector $|h\rangle_{(a,c)}$ are obtained to be

$$
\left(
\begin{array}{c}
J_0 \\
J_0
\end{array}
\right)
|h\rangle_{(a,c)} = \left(
\begin{array}{c}
-\sum_{\tilde{g}_i > 0} (1 - q_i - \tilde{g}_i^{-h'}) + \sum_{\tilde{g}_i = 0} (2q_i - 1) \\
\sum_{\tilde{g}_i > 0} (1 - q_i - \tilde{g}_i^{-h'})
\end{array}
\right)
|h\rangle_{(a,c)},
\right.$$

(3.1)

Our purpose of this section is to consider the topological properties of the divisors in terms of the Landau-Ginzburg model. As we reviewed in the previous section, there are three classes of divisors of our interest which are called Case A, B and C. What is the corresponding classification in the Landau-Ginzburg description? To answer this question, we should recall the observation in ref.[4] where the Calabi-Yau three-folds are considered in terms of the Landau-Ginzburg model.

The observation of [4] is that if the $(-1,1)$ state $|h'\rangle_{(a,c)}$ exists in the $h'$-twisted sector then it is possible to exist the states written in the form $\prod_{\tilde{g}_i = 0} X_i^{h_i} |h\rangle_{(a,c)}$ in the $h$-twisted sector, where $h' = h j^{-1}$. The $U(1)$ charge of the state $\prod_{\tilde{g}_i = 0} X_i^{h_i} |h\rangle_{(a,c)}$ depends on the number of invariant fields under the $h'$ action. We denote by $I_{h'}$ that number of invariant fields. For three-folds, if $I_{h'} = 2$, 3, then the possible $U(1)$ charges are $(-1,1)$, $(-2,1)$, respectively.

Applying this observation to our four-fold case, we obtain the following results which correspond to the previous classification of divisors.

$I_{h'} = 2$ It is possible to exist $(-1,1)$ states $X_i^{h_i} X_j^{l_j} |h\rangle_{(a,c)}$. This case corresponds to the Case A.

$I_{h'} = 3$ It is possible to exist $(-2,1)$ states $X_i^{h_i} X_j^{l_j} X_k^{l_k} |h\rangle_{(a,c)}$. This case corresponds to the Case B.
$I_{h'} = 4$ It is possible to exist $(-3,1)$ states $X_i^{l_i}X_j^{l_j}X_k^{l_k}X_m^{l_m}|h\rangle_{(a,c)}$. This case corresponds to the Case C.

where at least one $l_i > 0$. Of course, we have assumed the existence of the $(-1,1)$ state $|h\rangle_{(a,c)}$. The well-known identification between $(i,j)$ forms and $(-i,j)$ states implies that we can classify by the number $I_{h'}$ the $(-1,1)$ states which correspond to the divisors. We will study each case more in depth and find the condition of the $(-1,1)$ states whether the corresponding divisors can generate the superpotential.

First, consider the case of $I_{h'} = 2$. In this case, all the divisors can generate the superpotentials. Since it is shown \[8, 9\] that a $(-1,1)$ state $|h\rangle_{(a,c)}$ corresponds to a integral point in the dual polyhedron $\Delta^*$, the number of $(-1,1)$ states $X_i^{l_i}X_j^{l_j}|h\rangle_{(a,c)}$ in the $h$-twisted sector is equal to $l'(\Theta)$.

Let us turn to the $(c,c)$ ring by spectral flow. For the four-fold case, the states which correspond to the $(1,1)$ forms have their $U(1)$ charge being $(3,1)$. Using the technique developed in \[6\], we obtain the index $\beta(h')$ which counts the number of $(3,1)$ states written in the form $X_i^{l_i}X_j^{l_j}|h\rangle_{(c,c)}$, i.e.

$$\beta(h') = \frac{1}{|G|} \sum_{\text{all } g \in G} \prod_{\tilde{\theta}_i^g = \tilde{\theta}_i^{h'} = 0} (1 - \frac{1}{q_i}). \quad (3.2)$$

(If the set of $i$'s satisfying $\tilde{\theta}_i^g = \tilde{\theta}_i^{h'} = 0$ is empty, then we define $\prod_{\tilde{\theta}_i^g = \tilde{\theta}_i^{h'} = 0} (1 - \frac{1}{q_i}) = 1$ ). Clearly, in this case we have $\beta(h') = l'(\Theta_k)$.

The above discussions can be extended to other $(-1,1)$ states $|h\rangle_{(a,c)}$ with 3 or 4 invariant fields under $h'$ action. Remember that if there are 3 (4) invariant fields under $h'$ action, it is possible to exist $(-2,1)$ $(-3,1)$ states $\prod_{\tilde{\theta}_i^{h'} = 0} X_i^{l_i}|h\rangle_{(a,c)}$. For the case with $I_{h'} = 4$, the index $\beta(h')$ is defined as in (3.2) over the $(c,c)$ ring in the $h'$-twisted sector. However in the case with $I_{h'} = 3$, the index $\beta(h')$ should be defined to be

$$\beta(h') = -\frac{1}{2} \frac{1}{|G|} \sum_{\text{all } g \in G} \prod_{\tilde{\theta}_i^g = \tilde{\theta}_i^{h'} = 0} (1 - \frac{1}{q_i}). \quad (3.3)$$

since in the same twisted sector there are the $(2,1)$ and $(1,2)$ states which corresponds to the $(2,1)$ and $(1,2)$ forms obeying the complex conjugation duality.

Finally we obtain

$$\beta(h') = l'(\Theta_k), \quad (3.4)$$

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in all the cases with \( I_{h'} = 2, 3, 4 \). Note that \( \dim\Theta_k + 1 = I_{h'} \). Comparison of (2.6) with (3.4) leads us to conclude that a non-perturbative superpotential can be generated if

\[
\beta(h') = 0. \tag{3.5}
\]

This condition is very useful because the calculation can be done straightforwardly without any knowledge of geometry. Moreover, our condition can be available for the Landau-Ginsburg orbifolds which correspond to the Calabi-Yau four-fold orbifolds. If one needs to know the toric data for the divisor which can generate the superpotential, one has only to use the techniques obtained in \([8, 9]\). We will consider the illustrative examples in the next section.

## 4 Applications

As a first example, we consider the Landau-Ginsburg model which corresponds to the hypersurface embedded in \( WCP_{1,1,1,1,4,4}[12] \)

\[
W_1 = X_1^{12} + X_2^{12} + X_3^{12} + X_4^{12} + X_5^3 + X_6^3. \tag{4.1}
\]

This model is already studied in \([4, 13]\). By our Landau-Ginsburg analysis, we first find two \((-1, 1)\) states \( |j^{-1}\rangle_{(a,c)} \) and \( |j^{-3}\rangle_{(a,c)} \). The state \( |j^{-1}\rangle_{(a,c)} \) corresponds to the canonical divisor. We should pay attention to the fact that only two fields \( X_5 \) and \( X_6 \) are invariant under \( j^{-3} \) action. So the state \( |j^{-3}\rangle_{(a,c)} \) corresponds to the Case A in the previous section and the index of (3.2) is obtained to be \( \beta(j^{-3}) = 2 \) (note that the index \( \beta \) is calculated over the \((c, c)\) ring). This implies there are more two \((-1, 1)\) states, namely \( X_5|j^{-2}\rangle_{(a,c)} \) and \( X_6|j^{-2}\rangle_{(a,c)} \). For divisors this implies \( h^{0,0}(\mathcal{D}) = 3 \), i.e. three independent divisors intersect the Calabi-Yau hypersurface in the same way \([13]\).

To resolve this unsatisfactory situation, we apply the orbifold construction developed in \([3]\). We orbifoldize this model by the \( \mathbb{Z}_3 \) twist \( g = \rho_5^2 \rho_6 \), where \( \rho_i X_j = e^{2\pi i q_i \delta_{ij}} X_j \). The orbifoldized potential is obtained to be

\[
W'_1 = X_1'^{12} + X_2'^{12} + X_3'^{12} + X_4'^{12} + X_5'^2X_6' + X_5'X_6'^2. \tag{4.2}
\]

The two \((-1, 1)\) states \( X_5|j^{-2}\rangle_{(a,c)} \) and \( X_6|j^{-2}\rangle_{(a,c)} \) in the original theory are mapped to in the orbifoldized theory \( |g^{-1}j^{-2}\rangle_{(a,c)} \) and \( |g^{-2}j^{-2}\rangle_{(a,c)} \), respectively. Applying
the method in [3], the toric data for the \((-1,1)\) state \(|j^{-3}\rangle_{(a,c)}\), \(|g^{-1}j^{-2}\rangle_{(a,c)}\) and \(|g^{-2}j^{-2}\rangle_{(a,c)}\) are obtained to be \((0,0,0,-1,-1)\), \((0,0,0,1,0)\) and \((0,0,0,0,1)\), respectively. Thus the three divisors, which can not have the toric data independently in the original theory, are represented independently in the orbifoldized theory. All of these divisors with wrapped 5-branes can generate superpotentials. These two theories of \(W_1\) and \(W'_1\) are connected by this orbifold transition as was shown in [3].

Our next example is the Landau-Ginzburg model whose potential is

\[
W_2 = X_1^{24} + X_2^{24} + X_3^{12} + X_4^6 + X_5^3 + X_6^3,
\]

which corresponds to the hypersurface embedded in \(\mathbb{WCP}_{(1,1,2,4,8,8)}\) [24]. This is the three-fold fibered Calabi-Yau four-fold [12] and its Hodge numbers are \(h^{1,1} = 6\), \(h^{3,1} = 803\), \(h^{2,1} = 1\) and \(h^{2,2} = 3, 278\).

We will concentrate on the pair of states \(|j^{-6}\rangle_{(a,c)}\) of \(U(1)\) charge \((-1,1)\) and \(X_4|j^{-5}\rangle_{(a,c)}\) of \((-2,1)\), since this pair corresponds to the Case B in the previous section. Thus the divisor corresponding to the \((-1,1)\) state \(|j^{-6}\rangle_{(a,c)}\) cannot generate a superpotential. The index of (3.3) is obtained to be \(\beta(j^{-6}) = 1\), as expected. The toric data of the \((-1,1)\) state \(|j^{-6}\rangle_{(a,c)}\) is obtained to be \((0,0,-1,-2,-2)\).

We will show that after taking an appropriate orbifoldization this divisor can generate a superpotential. If we apply the appropriate orbifoldization, then the \((-2,1)\) state \(X_4|j^{-5}\rangle_{(a,c)}\) is projected out, so that the divisor corresponds to the \((-1,1)\) state \(|j^{-6}\rangle_{(a,c)}\) can generate the superpotential. We can easily find such an appropriate orbifoldization by the method obtained in [3].

In this case, we should consider the \(\mathbb{Z}_3\) orbifoldization by the twist \(g = \rho_2^0\rho_5^0\rho_6^0\). In the orbifoldized theory, the \((-2,1)\) state \(X_4|j^{-5}\rangle_{(a,c)}\) is projected out and the index \(\beta(j^{-6})\) is calculated to be \(\beta(j^{-6}) = 0\) as expected. Thus the divisor corresponds to the \((-1,1)\) state \(|j^{-6}\rangle_{(a,c)}\) can generate a superpotential. As was shown in [3], the original and the orbifoldized theories are connected. This phase transition can be characterized by the property of the divisors which lead to the non-perturbatively generated superpotentials.

As usual, new twisted states will appear in the orbifolded theory. For our model, two new \((-2,1)\) states \(|g^{-1}j^{-23}\rangle_{(a,c)}\) and \(|g^{-2}j^{-23}\rangle_{(a,c)}\) arise. So the new divisors which correspond to the new \((-1,1)\) states \(|g^{-1}\rangle_{(a,c)}\) and \(|g^{-2}\rangle_{(a,c)}\) cannot generate the superpotentials.
It is interesting to note that the $(-2,1)$ state $X_4|j^{-5}\rangle_{(a,c)}$ in the original theory is mapped to the $(-1,1)$ state $|g^{-1}j^{-5}\rangle_{(a,c)}$ which can generate the superpotential. In general, by the $U(1)$ charge analysis of [9], we see that the $(-p,1)$ states in the original theory written in the form $\prod_{\bar{h}_i^{k'}=0} X_i^l |h\rangle_{(a,c)}$ for $p = 1, 2, 3$ can be mapped to the $(-1,1)$ states $|g^{-1}h\rangle_{(a,c)}$ in our orbifoldized theory. In other words, if there exists one divisor in the original four-fold which cannot generate the superpotential, we obtain several divisors in the orbifolded four-fold, at least one of which can generate a superpotential.

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