The Weyl Curvature Conjecture

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Abstract

In this paper we review Penrose’s Weyl curvature conjecture which states that the concept of gravitational entropy and the Weyl tensor is somehow linked, at least in a cosmological setting. We give a description of a certain entity constructed from the Weyl tensor, from the very early history of our universe until the present day. Inflation is an important mechanism in our early universe for homogenisation and isotropisation, and thus it must cause large effects upon the evolution of the gravitational entropy. Therefore the effects from inflationary fluids and a cosmological constant are studied in detail.

1 The arrow of time and gravitational entropy in the context of a cosmology

There is a strange omission in the traditional version of the second law of thermodynamics (SLT). It does not take gravity into account. However the existence of the arrow of time is usually explained with reference to SLT. And most discussions of the origin of the arrow of time appeal ultimately to the initial condition and the evolution of the universe. Gravity plays an essential role for this evolution.

As pointed out by P.C.W. Davies \cite{1, 2} there seems to be a paradox that the material contents of the universe began in a condition of thermodynamic equilibrium, whereas the universe today is far from equilibrium. Hence the thermodynamic entropy has been reduced in conflict with SLT which says that

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the entropy of the universe is an increasing function of cosmic time. This is due
to the tendency of self-gravitating systems irreversibly to grow inhomogeneous.

In order to include this gravitational effect into a generalized version of SLT,
one has to define a gravitational entropy. Several tentative definitions have been
given based on Penrose’s Weyl curvature hypothesis [3, 4, 5].

The Weyl curvature tensor vanishes identically in the homogeneous and
isotropic Friedmann-Robertson-Walker universe models. It was suggested by
Penrose [3, 4, 5] to use this tensor as a measure of inhomogeneities of the uni-
verse models. The Weyl curvature scalar is expressed by scalars constructed
from the Riemann curvature tensor as follows

\[ C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{3}R^2 \]  \hspace{1cm} (1)

Penrose has given several formulations of the conjecture. A version that
emphasizes the entropy aspect was given in ref. [3] p.178. It seems that in
some way the Weyl tensor gives a measure of the entropy in the space-
time geometry. The initial curvature singularity would then be one with large Ricci
tensor and vanishing Weyl tensor (zero entropy in the geometry); the final
curvature singularity would have Weyl tensor much larger than Ricci tensor
(large entropy in the geometry).

In order to quantify Penrose’s conjecture Wainwright and collaborators [6, 7]
have suggested that the quantity

\[ P^2 = \frac{C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}}{R_{\mu\nu} R^{\mu\nu}} \]  \hspace{1cm} (2)

may represent a “gravitational entropy”, at least in a cosmological context with
a non-vanishing Ricci-tensor.

We could of course ask ourself whether the entropic behavior of the Weyl
curvature invariant (or an invariant composed thereof) results because the Weyl
tensor is directly related to the gravitational entropy, or whether it is only a side-
effect because the universe evolves towards a state of maximal entropy. Hence,
even though the results may show that the Weyl curvature invariant has an
entropic behaviour, it is by no means a proof that the Weyl tensor should be
identified as the “gravitational entropy”.

2 The behaviour of the Weyl tensor

In an earlier article [8] we discussed the Weyl curvature tensor in a homogeneous
but anisotropic model as well as in an inhomogeneous model. Our survey will
be summarized in this and the next sections, as well as bringing new arguments
into the discussion.

As a basis for our study, we used two different models which induce Weyl
curvature effects in two conceptually different ways. One was the anisotropic
Bianchi type I model, the other was the inhomogeneous Lemaitre-Tolman model.
The Weyl curvature conjecture has been investigated in the Szekeres cosmo-
logical model, that generalize the Lemaitre-Tolman model, by W.B. Bonnor [9].
Inhomogeneous modes are in general local modes for general relativity while
anisotropic modes are global modes. The global topology and geometry of our
universe has significant consequences for the possibilities for anisotropic modes
The role of inhomogeneous modes can be considered as more local in origin, but the effect on the global geometry and topology are not known in detail. Hence, it is important to study both an inhomogeneous model as well as a homogeneous model to get a more complete picture of the behaviour of the Weyl tensor in generic cosmological models.

2.1 The Bianchi type I model

The Bianchi type I model is the simplest anisotropic generalisation of the FRW models. It has flat spatial sections, and hence, can be a good candidate for the universe we live in. The metric for this model can be written as

$$ds^2 = -dt^2 + e^{2\alpha} \left[ e^{2\beta} \right]_{ij} dx^i dx^j$$

(3)

where $\beta = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$. In [14] we solved the Einstein field equations for dust and a cosmological constant $\Lambda$. By introducing the volume element $v = e^{3\alpha}$, the Einstein field equations gives the following equation for $v$:

$$\dot{v}^2 = 3\Lambda v^2 + 3Mv + A^2$$

(4)

where $M$ is the total mass of the dust, and $A$ is an anisotropy parameter. The equations for $\beta_\pm$ are in terms of the volume element

$$\dot{\beta}_\pm = \frac{a_\pm}{3v}$$

(5)

where the constants $a_\pm$ are related to $A$ via $a_+^2 + a_-^2 = A^2$. It is useful to define an angular variable $\gamma$ by $a_+ = A\sin(\gamma - \pi/6)$ and $a_- = A\cos(\gamma - \pi/6)$. An important special case of the Bianchi type I is the Kasner vacuum solutions. The Kasner solutions are characterised only by the angular variable $\gamma$. These solutions have a Weyl scalar

$$\left( C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} \right)_{I} = \frac{16}{27} \frac{A^4}{v^4} \left( 1 - 2z \cos 3\gamma + z^2 \right)$$

(6)

where $z = \dot{v}/A$ and $z = 1$ for the Kasner solutions which have vanishing Ricci tensor. So in a sense, the Kasner solutions are the anisotropic counterpart to the inhomogeneous Schwarzschild solution.

The Weyl tensor will decay as the volume expands, even at late times. One should expect that this entity would increase if it represents gravitational entropy. But it decreases monotonically, and hence, it is doubtful that it is the correct measure in these models.

Inserting dust and a cosmological constant will not have a significant effect on this decreasing behaviour. If $M$ is the mass of the dust inside the volume $v$, then eq. (6) still holds but with

$$z = \sqrt{1 + \frac{3Mv}{A^2} + \frac{3\Lambda v^2}{A^2}}.$$ 

(7)
Hence, except for the special case $\gamma = 0$, the Weyl tensor diverges as $v^{-4}$ as $v \to 0$.

For the Bianchi type I universe models the entity $P^2$ defined in eq. (2), turns out to be

$$(P^2)_I = \frac{4}{27} A^4 \frac{1 + z^2 - 2z \cos 3\gamma}{v^2 M^2 + 2\Lambda M v + 4\Lambda^2 v^2}. \quad (8)$$

Also this measure for the Weyl entropy diverges as $v \to 0$.

### 2.2 The Lemaître-Tolman models

Let us now consider the inhomogeneous Lemaître-Tolman (LT) models. The line element for the LT models can be written as

$$ds^2 = -dt^2 + Q^2 dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (9)$$

where $Q = Q(r, t)$ and $R = R(r, t)$. In $[15]$ we used this model and investigated the solutions of the Einstein field equations where the spacetime contains a cosmological constant $\Lambda$ and dust. The equations then turn into

$$R' = FQ$$

where $F = F(r)$ is an arbitrary function, and

$$\frac{1}{2} R \dot{R}^2 + \frac{1}{2} (1 - F^2) R - \frac{\Lambda}{6} R^3 = m. \quad (10)$$

The function $m = m(r)$ is given by the integral

$$m(r) = \int_0^r 4\pi \rho R^2 R' dr, \quad (11)$$

where prime denotes derivative with respect to $r$ and $\rho$ is the dust density. Hence, $m(r)$ can be interpreted as the total mass of the dust inside the spherical shell of coordinate radius $r$. Inverting the equation for $m(r)$, the dust density can be written in terms of $R$ and $m(r)$:

$$4\pi \rho = \frac{m'}{R^2 R'}. \quad (12)$$

It is also useful to define the mean dust density function $\bar{\rho}(r)$ by the relation

$$m(r) = \frac{4}{3} \pi \bar{\rho} R^3. \quad (13)$$

Interestingly, the Weyl curvature scalar can now be written quite elegantly as

$$\left( C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \right)_{LT} = \frac{16}{3} \pi^2 (\bar{\rho} - \rho)^2. \quad (14)$$

\footnote{The special case of $\gamma = 0$ is somewhat interesting. For $\gamma = 0$ it can be shown that the Kasner solution is just a special part of Minkowski spacetime, which has no singularities. It seems a bit odd that for any $\gamma \neq 0$ we will have an initial singularity, while for $\gamma = 0$ we do not have one. However, if we compactify the spatial sections this oddity disappears. The model has a singularity at $v = 0$ even for the case $\gamma = 0$.}
This relation provides us with a physical interpretation of the Weyl scalar in the LT models. It is just the difference between the mean dust density and the actual dust density. Also, the Weyl tensor is everywhere zero, if and only if \( \bar{\rho} = \rho \). In this case the LT models turn into the FRW models with homogeneous spatial sections. The FRW models are conformally flat, hence they have a zero Weyl tensor.

Let us return to the equations of motion for the LT model. The mass equation (10) can be written as an “energy” equation:

\[
\frac{1}{2} \ddot{R}^2 + V(r, R) = E(r) \tag{15}
\]

where the “potential" \( V \) and the “energy" \( E \) are given by

\[
V = -\frac{m}{R} - \frac{\Lambda}{6} R^2 \\
E = -\frac{1}{2}(1 - F^2) \tag{16}
\]

This energy equation may be integrated and solved exactly. A summary of the results and a qualitative description of the physical meaning of the solutions is given in [16]. The solutions may be written in terms of the Weierstrass’ elliptic functions [17]. The actual expressions are not very informative unless the reader has massive knowledge of these elliptic functions. However a lot of qualitative information can be extracted from simple classical considerations.

The classical solutions will move on level curves of the energy function \( E(R, \dot{R}) = \frac{1}{2} \dot{R}^2 - \frac{m}{R} - \frac{\Lambda}{6} R^2 \), since the total energy \( E \) is independent of \( t \).

In figure 1 the level curves of a typical energy function are drawn. If \( \Lambda > 0 \) there will exist a saddle-point of the energy function. This saddle point will be at \( R = \left(\frac{2m}{\Lambda}\right)^{\frac{3}{2}}, \dot{R} = 0 \) where the energy function will have the value \( E_s = -\frac{1}{2}(9m^2\Lambda)^{\frac{1}{2}} \). The saddle point solution is static, and is as a matter of fact the Einstein static universe\(^2\). If \( E < E_s \), the solutions fall into two distinct classes:

\(^2\)This is easily seen if we define the new radial variable to be \( R \). This can be done since we have to assume that \( m'(r) > 0 \) on physical grounds. In the case \( m'(r) = 0 \), the metric become degenerate.
1. Schwarzschild-like solutions: These solutions expand, but they do not possess enough energy to escape the gravitational collapse, so they end as black holes. If $m$ is constant these solutions are those of a Schwarzschild black hole in Lemaître coordinates.

2. de Sitter-like solutions: solutions where the universe evolves approximately as that of de Sitter solutions with positively curved hypersurfaces. If $E > E_s$ the (test) matter has enough energy to escape the gravitational collapse (expanding solutions) or enough energy to prevent the gravitational repulsion from the cosmological constant (contracting solutions).

Near the initial singularity, both $\bar{\rho}$ and $\rho$ diverge. Unless they are identically equal, the Weyl tensor will diverge near the initial singularity. Explicitly we have near the initial singularity

$$4\pi(\bar{\rho} - \rho) = \frac{3m}{R^3} \frac{2t'_0}{2t'_0 - \frac{m'}{m}(t - t_0)}$$

where we have used that near the initial singularity, we can approximate the solutions with

$$R \approx \left(\frac{9}{2m}\right)^\frac{1}{3} (t - t_0(r))^\frac{2}{3}.$$ \hspace{1cm} (18)

The free function $t_0(r)$ is the big bang time. To avoid intersecting world-lines we have to assume $t'_0 < 0$, thus $(\bar{\rho} - \rho) > 0$. The Weyl scalar will diverge as $R \to 0$ unless $t'_0 = 0$. Also the entity $P$ will diverge in general near the initial singularity. $P$ is given by

$$\left(P^2\right)_{LT} = \frac{4}{3} \frac{\left(\frac{\bar{\rho}}{\rho} - 1\right)^2}{\left(\frac{\Lambda}{4\pi\rho}\right)^2 + \left(\frac{\Lambda}{4\pi\rho}\right) + 1}$$

which near the initial singularity can be approximated by

$$\left(P^2\right)_{LT} \propto \left(\frac{2mt'_0}{m'(t - t_0)}\right)^2.$$ \hspace{1cm} (20)

Hence, $P$ diverges as $t \to t_0$ unless $t'_0 = 0$, i.e. unless the big bang is homogeneous.

This does not prove to be a very promising behaviour for the WCC. The curvature scalars diverge near the initial singularity, and hence, come in conflict with the WCC. But let us analyse the situation in the LT a bit more carefully. As the universe expands, both $\bar{\rho}$ and $\rho$ will decrease. They both decrease from an infinite value at the initial singularity. As the universe expands the value of $\bar{\rho}$ decrease as $t^{-2}$ while $\rho$ decreases as $t^{-1}$ close to the initial singularity. Hence, the value of $\bar{\rho}$ approaches the value of $\rho$ by a factor of $t^{-1}$, but since they both diverge, the Weyl tensor and $P$ diverge as well.

However, even though R. Penrose and S. Hawking showed that according to the general theory of relativity the big bang must have started in a singularity, it has later been emphasized that the physical universe must obey not only the
general relativistic laws of nature, but also the quantum mechanical laws. Hence, the initial singularity is a fiction which not corresponds to physical reality. The classical laws can be applied only after the Planck time. We should therefore study the behaviour of $P^2$ not only in the limit $t \to 0$, but rather at a very small cosmic time. Choosing the origin of cosmic time $T$ at $t = t_0$ we introduce $T = t - t_0(r)$. Hence close to $T = 0$

$$|P| = 2 \frac{m |t'_0|}{m'} T$$

(21)

which implies that $[\partial |P|/\partial T]_{T \to 0} < 0$. Bonnor [20] considered the model with $1 - F^2 > 0$ and a vanishing cosmological constant, and found the opposite result. However he restricted his investigation to models with homogeneous initial singularity, i.e. $t'_0 = 0$. As we will show this is an exceptional case. The behaviour of more general models with $t'_0 \neq 0$ is different. To see what is actually happening if we push $t'_0$ towards zero, we utilize that close to $T = 0$

$$R^3 = \frac{9}{2} m T^2 \left( 1 - \epsilon(r) T^{\frac{2}{3}} \right)$$

(22)

where $\epsilon(r)$ is assumed to be a small function of $r$. From eqs. (12), (13) and (14) with $\Lambda = 0$ we get

$$|P| = \frac{2}{\sqrt{3}} \left( \frac{3 m R'}{m' R} - 1 \right).$$

(23)

Using (22) we obtain

$$|P| = \frac{1}{m'} \left[ 2 m |t'_0| \left( T^{-1} - \frac{1}{3} \epsilon T^{-\frac{2}{3}} \right) - \epsilon' T^{\frac{2}{3}} \right].$$

(24)

This function is plotted in fig. 2.

Note that this function diverges for $T \to 0$, obtains a minimum for some small $T$ and increases thereafter. As $t'_0$ is driven towards zero this minimum goes towards $T = 0$.
3 The Gravitational entropy revisited

Let us consider a finite 3-volume $V$ comoving with the cosmic gas in our space

time. According to the first law of thermodynamics the matter entropy $S_M$ and

the internal energy $U$ will evolve as:

$$TdS_M = dU + p dV. \quad (25)$$

If the matter content is dust then $p = 0$. There are a couple of things to

note. Firstly, in a dense dust cloud, we expect the internal energy of the dust
to be large, thus we expect the entropy to be large. Secondly, the entropy is

an increasing function of the volume. Let us therefore consider a co-moving

volume $V$ in our spacetime. What could the expression for a gravitational

entropy be? From the previous sections we noticed that an LT model with

$\bar{\rho} = \rho$ is homogeneous. The quantity $\frac{\bar{\rho} - \rho}{\rho}$ is an inhomogeneity measure in the

LT models. In the absence of a cosmological constant we notice that

$$P = \frac{2}{\sqrt{3}} \left( \frac{\bar{\rho} - \rho}{\rho} \right). \quad (26)$$

The sign of $P$ is here chosen so that the configuration $\bar{\rho} < \rho$ is associated with

$P < 0$ while the more realistic configuration $\bar{\rho} > \rho$ has $P > 0$ corresponding to

eq. (26) positive. Let us consider the entity defined by:

$$S = \int_V P dV. \quad (27)$$

Introducing co-moving coordinates $x^i$ we write $dV = \sqrt{h} d^3x$ where $\sqrt{h}$ is the

3-volume element. The integration range is now constant as a function of time

and if we integrate over a unit coordinate volume which is so small that the

integrand is approximately constant, we may write

$$S = \int_V P dV \approx P \sqrt{h} \quad (28)$$

This formulation of the gravitational entropy is an intuitive one, but it is not

a covariant formulation. To find a covariant formulation we define the entropy

current vector by [21]

$$\Psi = s u + \varphi \quad (29)$$

where $u$ and $\varphi$ are orthogonal, $s$ is the entropy density, $u$ is the material flow

vector ($u \cdot u = -1$) and $\varphi$ is the entropy flux. The second law of thermodynamics

can now be expressed as

$$\Psi_{\mu, \mu} \geq 0. \quad (30)$$

Writing this in a local coordinate system, the divergence is

$$\Psi_{\mu, \mu} = \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} \Psi^\mu \right). \quad (31)$$

In our case we take the entropy density proportional to $P$ and assume a vanishing

entropy flux. Hence,

$$\Psi \propto P u. \quad (32)$$
In comoving coordinates \( u = e_t \) so that

\[
\Psi_{\mu} = \frac{1}{\sqrt{h}} \partial_{\mu}(\sqrt{h}P) = \frac{1}{\sqrt{h}} \dot{S}.
\]  

(33)

For the “second law to hold”, we have to check whether the entity \( S = \sqrt{h}P \) is increasing. If we take the corresponding one-form of the entropy vector, and take the dual of this form (Hodge dual), we obtain the entropy three-form which is given by the contraction of the space-time volume form \( \epsilon \) with the entropy current vector

\[
\omega = i \Psi \epsilon.
\]  

(34)

In our case, this three-form is simply

\[
\omega = \sqrt{h}P dx \wedge dy \wedge dz
\]  

(35)

Hence, if the coordinates are comoving, then it is the component of this three-form that is increasing as long as the matter and fields obey the SEC. This explains why the entropy should scale as the volume. It is usually this entropy three-form we are thinking of. It has a more intuitive behaviour than the entropy current vector.

We will therefore use \( S \) in the further to study the WCC. The square of the Weyl tensor and the entity \( P \) do not, as we have seen, capture the entropic behaviour properly. They both diverge at the initial singularity and thus, cannot be a proper measure of the gravitational entropy. The entity \( S \) on the other hand, seems to be a more promising candidate for the Weyl entropy.

4 The behaviour of \( S \)

4.1 The Lemaître-Tolman model

Let us start with the LT model. The motivation for studying the entity \( S \) given in (28) came from the LT models. Therefore it is natural to start with this model. In all the LT models, we can approximate the behaviour near the initial singularity, \( R \to 0 \) with

\[
S_{LT} = 2\sqrt{3} \frac{m|t'_0|}{Fm'} (m'(t - t_0) - 2t'_0).
\]  

(36)

For physically realistic spacetimes, \( t'_0 < 0 \), so as \( t \to t'_0 \), \( S \) is positive and finite. Note also that \( S \) is increasing, as suggested by the WCC. In the absence of a cosmological constant and if \( F = 1 \), this will be the exact expression for \( S \). If a cosmological constant is present, the universe will eventually go into a de Sitter phase if the universe is allowed to expand for ever.

For the sake of illustration, it is useful to investigate a specific case. Let us choose \( F^2 = 1 \). The energy equation eq. (15), can now be solved to yield

\[
R = \left( \frac{6m}{\Lambda} \right)^{\frac{2}{3}} \sinh^{\frac{2}{3}} \left[ \frac{3}{2} H(t - t_0(r)) \right]
\]  

(37)

where \( H = \sqrt{\Lambda/3} \). In general, the late time behaviour of the LT model (if the universe has one) is that of a de Sitter universe. This exact solution provides
Figure 3: The evolution of $S$ in the LT models is dependent on the value of $\Lambda$. The upper graph has a small value of $\Lambda$ which makes $S$ increase to a large value. The lowest graph has a large value of $\Lambda$.

us with a solution which connects the initial singularity with the late time de Sitter era. The initial behaviour of $R$ is the same as in eq. (18). The late time behaviour of $R$ is

$$R = \left( \frac{3m}{2\Lambda} \right)^{\frac{1}{2}} e^{H(t-t_0(r))}. \quad (38)$$

Hence, at late times the entity $S$ is approximately constant:

$$S = \frac{2m|t'_0|}{\Lambda^{\frac{1}{2}}}. \quad (39)$$

At late times, $S$ approaches a constant value which is inversely proportional to $H$. The larger the value of the cosmological constant is, the smaller the value of the final value of $S$. In fig. 3, we have plotted, using the exact solution (37), the evolution of $S$ for three different values of $\Lambda$.

The volume under consideration expands at late times exponentially as $V \propto V_0(r) \exp (3Ht)$, where $H$ is the Hubble parameter. Hence, the entropy per unit volume will decrease exponentially:

$$\frac{S}{V} \propto e^{-3Ht}. \quad (40)$$

The entropy in a unit volume will therefore decrease rapidly during a de Sitter stage. This is a very important consequence of inflation. At the exit of the inflationary period, the entropy in a unit volume is very low, there are very little inhomogeneities left of the primordial ones.

Let us summarize the evolution of the entity $S$ in the LT models:
• **Large $\Lambda$ and ever expanding:** In the initial epoch the dust dominates and $S$ is increasing linearly in $t$. The universe is becoming more and more inhomogeneous. After the universe has grown considerably, the cosmological constant becomes dominant, $S$ stops growing and if the cosmological constant is large enough, it evolves asymptotically towards a constant value. The universe is smoothened out. This scenario corresponds approximately to the lowest two graphs in Fig. 3.

• **Small $\Lambda$ and ever expanding:** Again the dust dominates initially. The cosmological constant is too small to make $S$ decreasing. The entity $S$ is ever increasing but is bounded from above by a relatively large constant value.

• **Zero $\Lambda$ and ever expanding:** The $S$ will again be ever increasing and will asymptotically approach a function $f(t) = c + bt^p$ where $c$ and $b$ are constants and $p = 3$ iff $F^2 > 1$ and $p = 1$ iff $F^2 = 1$.

• **Recollapsing universe:** Due to the dust term, the final singularity will not be similar to the initial singularity. Hence this entity is asymmetric in time for a recollapsing universe.

In the LT models we see that $S$ behaves in agreement with the WCC.

Let us investigate more carefully the Schwarzschild spacetime, where the whole motivation of gravitational entropy comes from [22, 23]. The Schwarzschild spacetime is a special case of the general LT models; it has $m(r) = constant$ and a vanishing Ricci tensor. If we look at the entity $S$ in the region outside the Schwarzschild singularity, $S$ will diverge. This is in some sense the maximal possible value of $S$, the Weyl tensor is as large as possible and the Ricci tensor is the smallest as possible. Thus at this classical level it seems that the Schwarzschild spacetime has the largest possible $S$ which is a good thing if one wants to connect $S$ with the entropy of the gravitational field. Outside a real black hole, $R_{\mu\nu} = 0$ is probably impossible. Even though the classical vacuum has vanishing Ricci tensor, the quantum vacuum will probably not have $R_{\mu\nu} = 0$. The quantum fields will fluctuate and cause the expectation value of the square of the Ricci tensor to be non-zero: $\langle R_{\mu\nu} R^{\mu\nu} \rangle \neq 0$. Hence, there will probably be an upper bound of how large $S$ can be, even in a vacuum.

### 4.2 The Bianchi type I model

In the Bianchi type I with dust, the explicit expression for the entity $S$ is

$$S_I = \sqrt{h} P = \frac{2 A^2}{3 \sqrt{3}} \left( \frac{1 + z^2 - 2 z \cos 3 \gamma}{M^2 + 2 \Lambda M v + 4 \Lambda^2 v^2} \right)^{\frac{3}{2}} \tag{41}$$

where $\sqrt{h} = v$. In a neighbourhood of the singularity we can make a Taylor expansion to first order in $v$:

$$S_I^2 \approx \frac{8}{27} \frac{A^4}{M^2} (1 - \cos 3 \gamma) \left( 1 + \left( \frac{3M}{2A^2} - \frac{2 \Lambda}{M} \right) v \right).$$

We interpret $m = M/A$ as the dust density in coordinate space [4]. Hence, as we go towards the initial singularity $S$ will be finite. For the vacuum case,
m = 0 and Λ = 0, and the solutions are the Kasner solutions. The Kasner solutions have a vanishing Ricci tensor, but a non-zero Weyl tensor. Hence, as in the Schwarzschild case, the entity \( S \) will diverge.

From the Taylor expansion we see that, if \( 3m^2 > 4\Lambda \) then \( S \) will increase immediately after the initial singularity. If \( 3m^2 < 4\Lambda \) then \( S \) will decrease. This can be understood as follows. If the cosmological constant is too large, the universe will increase too rapidly initially to allow the dust to contribute significantly to the anisotropy. Initially the universe will become more and more isotropic. In the presence of a cosmological constant, the universe will eventually enter a de Sitter phase. Again \( S \) will evolve towards a constant value given by

\[
\lim_{t \to \infty} S_I = \frac{A}{3\Lambda^{\frac{1}{2}}}. 
\]

If the cosmological constant vanishes \( S \) is monotonically increasing:

\[
S_I = \frac{2A}{3\sqrt{3m}} \left( (2 + 3mt)(1 - \cos 3\gamma) + \frac{9}{4} m^2 t^2 \right)^{\frac{1}{2}}
\]

Note that for large \( t \) we have \( S_I = \frac{A}{\sqrt{3}m} t \), which is according to the same power law as the \( F^2 = 1 \) and \( \Lambda = 0 \) case of the LT model (compare with eq. (36)).

To summarize our investigation of \( S \) in the Bianchi type I model we can say that the entity \( S \) at the initial singularity is a constant determined by the inverse of the dust density. For \( 3m^2 > 4\Lambda \) it will increase immediately after the initial singularity. In most cosmological considerations it is assumed that the cosmological constant is small. The exception is in the inflationary era in which the vacuum energy dominates over all other matter degrees of freedom. The inflationary era will smooth out anisotropies as well as inhomogeneities, and the behavior of \( S \) in this case is therefore expected. In the de Sitter limit \( S \) will asymptotically move towards a constant. Large \( \Lambda \) means small value, while small \( \Lambda \) corresponds to a large value. This is in full agreement with the LT models. It is also interesting that the entity \( S \) is very sensitive to different matter configurations. This makes it a lot easier to check whether the \( S \) has the right entropic behaviour.

The question now arises: How generic is this behaviour? Does the entity behave in the correct way for all physically realistic models?

We know that our universe today is close to homogeneous on a scale larger than a billion light years. As mentioned earlier, the FRW models are conformally flat and hence, they have a zero Weyl tensor. So a question would be, at late times when the universe is close to isotropic and homogeneous, does the entity \( S \) still behave in the correct manner? Does it still increase? We saw that at late times, both the LT model and the Bianchi type I model, did behave correctly for \( \Lambda = 0 \) even though they both isotropise and evolve towards homogeneity.

Let us choose a more general model, a flat model that allows for inhomogeneities and anisotropies. We will assume that the model at late times asymptotically evolve towards a FRW model. In [24, 25], Barrow and Maartens investigated the general equations of motion for such models with \( \Lambda = 0 \). In the velocity dependent regime they derived some approximate solutions to the field equation close to a FRW model, having both inhomogeneities and anisotropy. Also an anisotropic stress of the form \( \pi_{\mu\nu} = \lambda_{\mu\nu} \rho \) where \( \lambda_{\mu\nu} \) is a constant
matrix, was included. Their result was that the late time behaviour of such a model is consistent with a perfect fluid where the equation of state parameter $\gamma$ for the perfect fluid obeys $\gamma \leq 4/3$. For $\gamma > 4/3$ the FRW model is unstable at late times. In the latter case one can get for instance different stable anisotropic magnetic solutions $[24, 27]$.

Let us therefore assume that $2/3 \leq \gamma \leq 4/3$ (so that the matter obeys the SEC). The ratio of the Weyl tensor squared and the Ricci tensor squared is in this case

$$P^2 = \frac{C_{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}}{R_{\mu\nu}R_{\mu\nu}} \propto \mathcal{O}(t^{-2n}) + \mathcal{O}(t^{-3n}) + \mathcal{O}(t^{-4n})$$  \hspace{1cm} (42)

where $n = \frac{2(4-3\gamma)}{\gamma}$ for $n \neq 4/3$. In the radiation case, $\gamma = 4/3$, we get a logarithmic decay of $P^2$:

$$P^2_{\gamma=4/3} \propto \frac{1}{(\ln t)^2}.$$  \hspace{1cm} (43)

Thus in the range of validity of the assumption $\gamma \leq \frac{4}{3}$, this entity will decrease in the future. However, the entity $S$ shows more promising behaviour

$$S = P\sqrt{h} = a^{3}P \propto t^{\frac{2}{3}-n}$$  \hspace{1cm} (44)

for $2/3 \leq \gamma < 4/3$, while the $\gamma = 4/3$ case yields

$$S \propto (\ln t)^{-1}t^{\frac{2}{3}}.$$  \hspace{1cm} (45)

Hence, $S$ increases as long as

$$\frac{2}{3} \leq \gamma \leq \frac{4}{3}.$$  \hspace{1cm} (46)

This entity increases in the future (as any entity describing entropy should do) and tells us that even if the universe itself asymptotically goes towards isotropy, the entropy of the gravitational field actually increases (if we should believe the WCC).

Note that this result is slightly different than the case where no anisotropic stress is present. If the anisotropic stress is not present, we get $S \propto t$ at late times for all $\gamma$.

We should also mention a work done by Hervik $[28]$ which investigates the evolution of the Weyl curvature invariant for generic solutions of spatially homogeneous models containing a $\gamma$-law perfect fluid with $\gamma \geq 2/3$. The conclusion was that all spatially homogeneous models, except for sets of measure zero, had an increasing $S$ at late times.

4.3 Inflation

As long as the matter obeys the strong energy condition (SEC), matter will behave more or less attractive. Hence, when the SEC is fulfilled, we should expect the gravitational entropy to increase.

During inflation the SEC is violated and gravity is not necessarily attractive. We have already seen how the inclusion of a cosmological constant could alter the
behaviour of $\mathcal{S}$. The late time behaviour of $\mathcal{S}$ in the presence of a cosmological constant is that $\mathcal{S}$ evolves approximately as a constant. The constant itself is a decreasing function of $\Lambda$ (see eq. (39)).

What happens for a more general inflationary fluid? Consider a perfect fluid with equation of state

$$p = (\gamma - 1)\rho$$

(47)

for $0 \leq \gamma < \frac{2}{3}$. This perfect fluid will violate the strong energy condition and will in general cause a power law inflation. Note that the case $\gamma = 0$ can be considered the same as including a cosmological constant. In the FRW cases, all the models will now have a late time behaviour similar to the flat case. The FRW models will become inflationary and will be dominated by this fluid at some stage in the future (provided that the universe is ever-expanding). To simplify, we will therefore consider a flat universe, and perturb the FRW flat universe model with $\gamma < \frac{2}{3}$. Following [24, 25], the anisotropic stresses will not dominate the shear modes at late times. The shear will under these assumptions decrease as

$$\sigma_{\mu\nu} \propto t^{-\frac{2}{3}}$$

(48)

while the Hubble parameter is the same as in the FRW case (to lowest order)

$$H = \frac{2}{3\gamma}t^{-1}. \quad (49)$$

The late time behaviour of $P^2$ is now

$$P^2 = \mathcal{O}\left(\gamma^2 t^{-\frac{2(2-\gamma)}{\gamma}}\right)$$

(50)

which is decreasing for all $0 < \gamma < 2/3$. The late time behaviour of $\mathcal{S}$ is

$$\mathcal{S} = \sqrt{\pi P} = \mathcal{O}(\gamma t)$$

(51)

which is increasing linearly in $t$ at late times. For $\gamma = 0$ we recover the cosmological constant case where $\mathcal{S}$ is approximately a constant at late times. Note also that this result coincide with eq. (44) for $\gamma = 2/3$.

Hence, during the inflationary period, $\mathcal{S}$ increases much slower than for ordinary matter ($\mathcal{S} \propto t^{\frac{4}{3}}$ for dust and $\mathcal{S} \propto (\ln t)^{-1}t^{\frac{4}{3}}$ for radiation).

5 Using Quantum Cosmology to determine the initial state

Maybe the closest to a quantum theory of gravitation that have been obtained, is what we call quantum cosmology (QC). We will in this section show how one might be able to determine the likelihood of a certain initial state to occur. QC is perhaps best described as a “theory of initial conditions”. We will again use the results of the previous paper to try determine which of the initial states are more probable.

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In QC, the wave function of the universe satisfies the *Wheeler-DeWitt (WD) equation*:

\[
\left( -G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + V(h_{ij}) \right) \Psi = 0 \tag{52}
\]

where \( G_{ijkl} \) is called DeWitt’s supermetric, and \( h_{ij} \) is the metric on the 3-dimensional spatial hypersurfaces. The potential term \( V(h_{ij}) \) consists of the Ricci scalar of the three-dimensional hypersurfaces and possibly matter potentials and a cosmological constant.

We want now to calculate the expectation value for the Weyl scalar, in these models. All of the curvature invariants have to go over to their respective curvature operators. Especially, the entity \( S \) goes over to the curvature operator \( \hat{S} \). The question is now, what is more likely, creation of a universe with large expectation value of \( \hat{S} \) or a universe with a low value?

### 5.1 The LT model

In the paper [15] we considered semi-classical tunneling wave functions which are solutions of the WD equation. The universe was tunneling from a matter-dominated universe classically confined to a finite size, into a de Sitter like universe. After tunneling across the classically forbidden region the universe became \( \Lambda \)-dominated, similarly to an inflationary model of the universe. Thus we have a “flow” from matter-dominated universes towards de Sitter like universes.

The actual expectation values of entities like \( C^{\alpha \beta \gamma \delta} C_{\alpha \beta \gamma \delta} \), \( P^2 \) and \( S \) are not explicitly obtained for these models, because the actual calculations suffer from “endless” expressions and highly time-consuming quantities. We will therefore give a more general description of the evolution of the Weyl tensor for the LT models.

It would be useful first to recapitulate some of the discussion done in [15, 8]. First of all we discussed tunneling wave functions in the WKB approximation. In the WKB approximation we assume that the wave function has the form \( \Psi_{WKB} = \exp(\pm iS) \), where \( S \) will to the lowest order satisfy the Hamilton-Jacobi equation:

\[
\left( \frac{\delta S}{\delta R} \right)^2 - \frac{F'^2}{F^4} \left[ 2mR - R^2(1 - F^2) + \frac{\Lambda}{3} R^4 \right] = 0 \tag{53}
\]

Here \( \delta / \delta R \) denotes the functional derivative with respect to the function \( R \). If we assume that \( S = \int \sigma(r) dr \), the resulting equation will be the Hamilton-Jacobi equation for a point particle with action \( \sigma(r) \) (\( r \) is only a parameter and the functional derivatives turn into ordinary partial derivatives). In the Hamilton-Jacobi equation the functional \( S \) turns out to be the action at the classical level. Since the classical action can be written as an integral over \( r \) the assumption \( S = \int \sigma(r) dr \) is therefore reasonable at the lowest order WKB level. We can interpret the action \( \sigma \) as the action of a point particle moving in a potential \( V(R) = \frac{c^2}{2} \left[ -mR + \frac{1}{2}(1 - F^2)R^2 - \frac{\Lambda}{6} R^4 \right] \) with zero energy. The WKB wave function \( \psi_{WKB} \) for the point particle can then be written \( \psi_{WKB} = \exp(\pm i\sigma) \). The two WKB wave functions can therefore be related by \( \Psi_{WKB} = \)
exp(\int dr \ln \psi_{WKB}). Finding first the wave function \psi we can then relate its WKB approximation to \Psi_{WKB} through \Psi_{WKB} = \exp(\int dr \ln \psi_{WKB}).

Let us now ask the question: Is it more likely for a universe with small Weyl tensor to tunnel through the classical barrier than a universe with a large Weyl tensor? The question is difficult to answer in general but we shall make some simple considerations in order to shed some light upon it.

We assume that the dust density near the origin of the coordinates is larger than further out. We define the homogeneous mass function for a closed universe \(k = 1\) as \(m_h(r) = \frac{4}{3} \pi \rho_h r^3\) where \(\rho_h\) is a constant. The constant \(\rho_h\) is determined by demanding \(m_h(r_{\text{max}}) = m(r_{\text{max}})\). If the dust density is larger near the origin of the \(r\)-coordinate than for larger values of \(r\) then \(m(r) \geq m_h(r)\). This will not in general change the size of the universe so we can look at the effects from \(m(r)\) alone. Since \(m(r)\) is greater in general for an inhomogeneous universe than for a homogeneous universe, we see that the potential barrier will be smaller for an inhomogeneous universe than for a homogeneous universe. Thus an inhomogeneous universe will tunnel more easily through the classical barrier than the homogeneous universe. Since an inhomogeneous universe will have a larger Weyl tensor than an almost homogeneous one, we see that universes with large Weyl tensor tunnel more easily than those with a small Weyl tensor.

If we look at the tunneling amplitude concerning effects from the \(\Lambda\) term, it is evident that larger \(\Lambda\) will yield a larger tunneling probability. In the initial era inhomogeneities will increase the value of \(S\). We saw that an inhomogeneous state will tunnel more easily through the potential barrier than a homogeneous state. The largest tunneling probability amplitude thus occurs for universes with a large cosmological constant and large local inhomogeneities. From a classical point of view the value of \(S\) initially was large (but increasing thereafter), but as the universe entered the inflationary era the cosmological constant had a large value, hence the value of \(S\) at the end of the inflationary era was relatively small.

In the initial epoch the universe is not believed to be dust dominated. The dust does not exert any pressure and dust particles do therefore not interact with each other. A more probable matter content is matter which has internal pressure. Even though gravitation tends to make the space inhomogeneous, internal pressure from the matter will try to homogenise the space. Since dust is the only matter source in our model, the model only indicates the tendency for gravitation itself to create inhomogeneities.

Since the universes tunnel into a de Sitter-like state, the cosmological constant will rapidly dominate the evolution. The larger the cosmological constant the lower will the entity \(S\) be after the inflationary era ends.

### 5.2 The Bianchi type I model

We can write the general solution of the WD-equation for the Bianchi type I models as:

\[
\Psi(v, \beta) = \int d^2 k \left[ C(\vec{k}) \psi_E(v) \rho(\vec{k}) e^{i \beta \vec{k}} \right]
\]

where \(\psi_E(v) = v^{-\frac{3}{2}} W_{L, \mu}(2Hv)\) is a particular solution of the WD equation (with dust), \(C(\vec{k})\) is a “normalizing constant”, and \(\rho(\vec{k})\) is a distribution function in
momentum space. This distribution function satisfies the equation:

$$\int d^2 k |\rho(\vec{k})|^2 = 1$$

The function $W_{L,\mu}(x)$ is the Whittaker function.

There is still a factoring problem in turning the classical entities to operators. Let us first investigate the expectation value of $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$. In the expression for the Weyl tensor squared, there is a term

$$\frac{1}{v^4} (p_+^2 + p_\perp^2)^2$$

(55)

where $p_\perp$ is the conjugated momenta to $\beta_\perp$. Upon quantization, the above term is replaced with the operator:

$$\frac{1}{v^4} (p_+^2 + p_\perp^2)^2 \rightarrow \frac{1}{v^4} \left( \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta^2} \right)^2$$

(56)

In the expectation value of the square of the Weyl tensor, the above term will contribute with

$$\int d^2 \beta \Psi^* \frac{1}{v^4} \left( \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_+^2} \right)^2 \Psi = \frac{1}{v^4} \int d^2 \beta \rho(\vec{k})^2 |\psi(\vec{k}(v))|^2 |\psi(\vec{k})|^2 |\vec{k}|^4$$

Unless we have a delta-function distribution at $\vec{k} = 0$; $\rho(\vec{k}) = \delta^2(\vec{k})$, the contribution from this term to the Weyl curvature invariant will diverge as $v^{-4}$ for small $v$. Thus, we have to conclude that, in the small $v$ limit the expectation value of $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ goes as:

$$\langle C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} \rangle \propto \frac{1}{v^4}$$

(57)

just as in the classical case.

Investigating the invariant $R^{\mu\nu}R_{\mu\nu}$, we notice that things are not so easy. The Ricci square also has a term which presumably would contribute with a $k^4$ term. However, looking at the classical expression we see that the Ricci square is independent of the anisotropy parameter $A$. This indicates that at the classical level all terms involving the anisotropy parameters, have to cancel exactly. This is not the case quantum mechanically. In the quantum case operators do not necessarily commute. Hence there may be contributions from terms which classically would cancel each other. In other words, the fact that the classical vacuum has $R_{\mu\nu} = 0$, does not mean that the quantum vacuum has $\hat{R}_{\mu\nu} = 0$.

We assume that $(\xi_j)$ is a set of factor-ordering parameters which represents the “true” quantum mechanical system in such a way that $\xi_j = 0$ represents the classical system. With this parameterization of the factor ordering we would expect the Ricci square expectation value to be:

$$\langle R^{\mu\nu}R_{\mu\nu} \rangle = 4\Lambda^2 + 2\Lambda \frac{M}{v^2} + \frac{M^2}{v^4} + \frac{f_j(v)}{v^4} \xi_j + O(\xi_j^2)$$
where $f_i$ is some function of $v$ which has the property: $v \approx 0, f_j(v) \approx \text{constant}$. Thus for small $v$ and $\xi_j$ the expectation value would behave as

$$\langle R^{\mu\nu}R_{\mu\nu} \rangle \propto \frac{f_j(0)}{v^4} \xi_j$$

and

$$\frac{\langle C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} \rangle}{\langle R^{\mu\nu}R_{\mu\nu} \rangle} \propto \frac{1}{f_j(0)} \xi_j \cdot \text{constant} \quad (58)$$

The Weyl square divided by the Ricci square is in general finite as $v \to 0$ for a quantum system. In some sense, the quantum mechanical effects renormalizes the infinity that the classical system possesses at $v = 0$. The expectation value at $v = 0$ is, however, strongly dependent on the factor-ordering. As the factor ordering parameters approach zero, the value will diverge. Quantum effects in the early epoch are essential for the behaviour of this entity near the initial singularity. As $v \to 0$ we expect the quantum effect to be considerable, thus expecting the factor-ordering parameters $\xi_j$ to be large.

As indicated in the above discussion, the quantum mechanical expectation value of $\hat{P}$ will be lower and presumably finite at the initial singularity. Therefore the expectation value of $\hat{S}$ is also presumed to be considerably lower in the initial stages than its classical counterpart.

Comparing different tunnelling amplitudes in the Bianchi type I model is difficult and more speculative because the Bianchi type I universe has no classically forbidden region for $\Lambda \geq 0$. This causes the lowest order WKB approximation to be purely oscillatory. The lowest order WKB wave function will therefore be approximately constant. In the paper [14] we did however construct under some assumptions a wave function which clearly peaked at small values of the anisotropy parameter. Thus these wave functions predicts universes that have a relatively low value of $S$.

6 Cosmic evolution of the Weyl Curvature

Let us now recapitulate how the evolution in the context of the WCC might have been.

The universe was created in a rather arbitrary state. As the time ticked past the Planck time $10^{-43}$ s a rather inhomogeneous universe appeared. QC suggests that this state was rather inhomogeneous, but it is more or less a guess how inhomogeneous the universe was at that state. Nevertheless, as the universe grew larger, the expectation value of the Weyl entropy increased initially. At what rate the Weyl entropy increased is highly uncertain, it depends very much on the true nature of our universe. It depends on what matter fields that were present, the topology of the universe, whether it was anisotropic or not, quantum effects etc.

Nevertheless, at some time very short after the Big Bang, an enormous effective cosmological constant appeared. The universe was driven unconditionally into an inflationary period. The Weyl entropy stopped increasing and began instead to evolve asymptotically towards a constant. Whether or not the Weyl entropy had earlier a higher value than at the exit of the inflationary regime,
is difficult to say. During the inflationary period, the universe was more or less in an adiabatic expanding state, the Weyl entropy was more or less constant. Since the universe was expanding exponentially during this period, the entropy per unit volume dropped exponentially. If the scale factor increased by a factor of 60 $e$-foldings during the inflationary epoch, then the entropy per unit volume of space would have decreased by a factor of

$$\frac{s}{s_0} \sim e^{-3 \cdot 60} \approx 10^{-78}. \quad (59)$$

This is quite a drastic decrease, and shows how powerful inflation is when it comes to smoothing out the inhomogeneities and anisotropies of our universe.

At the exit of the inflationary regime, the Weyl entropy had dropped by an enormous factor compared to what it would have been if no inflation had occurred. The Weyl entropy was very small, compared to the maximal possible value. The universe was more or less uniform and homogeneous. This homogeneity can be seen in the CMB radiation today. However, small fluctuations in the spectrum can also be seen, reflecting the state of the universe 300 000 years after the Big Bang. After the inflation, small seeds of inhomogeneities from the quantum fluctuations of quantum fields were the only thing left of the initial inhomogeneities. Nevertheless, these seeds were large enough to gradually clump together and form galaxies and stars. The Weyl entropy began to increase again after the inflationary regime was over.

The radiation-, and later the matter-dominated universe caused the Weyl entropy to grow steadily and firmly for almost 15 billion years. During the radiation era, there were only small inhomogeneities and anisotropies left of the primordial fluctuations. Using the calculations from the earlier section, the Weyl entropy increases as $S \propto (\ln t)^{-1} t^{3/2}$ in the radiation era.

At about $t = 10 000$ years the radiation become sub-dominant. The universe evolved into a matter dominated era and the Weyl entropy increased as $S \propto t^{4/3}$. Today it is still growing. Recent observations suggest that the universe
has entered a new era with accelerated expansion. The surprising fact that the universe appears to be in an accelerating state today, can be explained with the presence of a vacuum energy. If this is true, the Weyl entropy will increase steadily and asymptotically towards a constant. However, the late time behaviour of our universe is still quite speculative, and whether or not this vacuum dominated period will persist, is very uncertain. It might happen that the period ends like inflation did, and perhaps the vacuum period is followed by a curvature dominated period. If so, the Weyl entropy might again rise to new heights, increasing towards a value where all the matter is collected in black holes.

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