Scattering wave functions for Aharonov-Bohm-Coulomb field: Path integral treatment.

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Abstract

Exact Green’s functions related to Dirac particle submitted to the combination of Aharonov-Bohm and Coulomb fields in \((2 + 1)\) coordinate space are analytically calculated via path integral formalism in both global and local representations. The scattering normalized wavefunctions as well as the corresponding continuous energy eigenvalues are extracted following this approach. The interesting properties of the spinors are thus deduced after symmetrization. According to the symmetric form for the Green’s function, it is shown that the equivalence with Dirac equation is undertaken with much ease.

Some particular cases are also considered.

1 Introduction

The \((2 + 1)\)-spinor field theory has given rise to a tremendous interest in recent years due to the existence of quantum systems that can be effectively described by the \((2 + 1)\) Dirac equation. Consequently, we can see that two-dimensional relativistic models of charged fermions, related to the uniform as well as non-uniform magnetic fields can be used to study such quantum electrodynamic effects.

The main purpose of this paper is to study the relativistic quantum Aharonov-Bohm effect in the presence of the physical Coulomb field in \((2 + 1)\) coordinates space by showing how to extract the continuous energy eigenvalues and its corresponding scattering wave functions in terms of the Green’s function.

We should note that this work is a continuation of the previous paper \cite{1} which treated the discrete case of the Aharonov-Bohm-Coulomb field via \((2 + 1)\)path integral formalism.
To our knowledge, research on the combination of Aharonov-Bohm and Coulomb potentials is limited except for some important works, among these latter we quote the work done by [2] who gave an analytical solution via algebraic approach. However, an exact solution to the first order Dirac equation for the Aharonov-Bohm-Coulomb field via (2 + 1) path integral formalism related to scattering case, is still non-existent.

Is it well-known that the path integral formalism is becoming a working tool of modern physics, being the easiest and most elegant means of unification. It serves, for example, to unify the dynamics of continuous quantities such as the position and the impulsion with that of discrete quantities such as the spin. This path integral formalism gave a new breath to the research of the analytical and exact expressions of the relativistic spinning propagators in the presence of external fields. Generally, the advantage of this alternative path integral formalism to the Dirac equation lies in the extraction of existing symmetries thanks to the properties of the exact obtained Green’s function, thus the use of these symmetries allowed us to extract normalized eigenspinors.

This paper is composed of three sections and a conclusion. Section 2 finds exact Green’s functions in (2 + 1) dimensions for a combination of the Aharonov-Bohm and Coulomb potentials in both local and global representations. Section 3 derives normalized scattering wave functions and continuous energy eigenvalues based on different physical parameters and quantum numbers and shows that the solutions provided by this research have interesting properties and verify the first-order Dirac equation by exchanging quantum numbers. To be more effective, it would be interesting to get the exact and explicit results in particular cases. Therefore, section 4 shows that the nonrelativistic limit is undertaken with much ease according to the symmetric form for the Green’s function. The conclusion summarizes the whole research and presents its findings.

The configuration of the field is characterized by the following features:

\[
A_0 = -\frac{\alpha}{\epsilon \sqrt{x^2+y^2}}, \quad A_x = -\frac{By}{x^2+y^2}, \quad A_y = \frac{Bx}{x^2+y^2},
\]

where \( \alpha > 0 \) and \( B \) is the flux of the magnetic field.

The propagator related to a Dirac particle in an external electromagnetic field is the causal Green’s function \( S_c(x_b, x_a) \) \[1,3\]

\[
(\gamma^\mu \pi_\mu - m) \, S_c(x_b, x_a) = -\delta^3(x_b - x_a),
\]

where \( \pi_\mu = (i\partial_\mu - eA_\mu) \); the subscript \( \varsigma = \pm 1 \); \( \eta^{\mu\nu} = \text{diag} \, (1, -1, -1) \); \( x^T = (x, y, t) \), \( \gamma^\varsigma = \frac{i\varsigma}{2} \epsilon^{\mu\nu\lambda} \gamma_\varsigma \gamma_\nu \gamma_\lambda \), \( [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu} \); \( \mu, \nu = 0, 2 \)

The matrices \( \gamma^\mu \) generating the Clifford algebra are equivalently

\[
\gamma^0 = i\varsigma_1 \gamma^2 = \sigma_3, \quad \gamma^1 = i\varsigma_2 \gamma^0 = i\sigma_2, \quad \gamma^2 = -i\varsigma_0 \gamma^1 = -i\sigma_1.
\]

Formally, \( S_c(x_b, x_a) \) is the matrix element \( \langle x_b | S_c | x_a \rangle \) in the coordinate space of the inverse Dirac operator, namely

\[
S_c = (\gamma^\mu \pi_\mu - m)^{-1},
\]

\( S_c(x_b, x_a) \) is fulfilling the following equation

\[
S_c(x_b, x_a) = (\gamma^\mu \pi_\mu + m) \, S_g(x_b, x_a),
\]
\[ S_g^c(x_b, x_a) \] is the global Green’s function which verifies the quadratic Dirac equation with the respect of the boundary conditions for the \( x \)-space path integral

\[
x(0) = x_a, \quad x(1) = x_b.
\]

It is well known that the so-called global approach is related to the square of the Dirac equation, where the superfluous, or non-physical states, are eliminated thereafter by activating an operator. Consequently, it finds out that the so-called local approach is related to the first-order Dirac equation formulation, where there is no operator and the states are all physical without any being superfluous.

The object of the following section is to see how from the path integral formalism, we can obtain the Green’s functions calculations by determining exactly the corresponding Green’s functions for both global and local representations.

2 Green’s functions calculations

It is remarkable to see that the first steps of the computation of the global description \( S_g^c \) related to discrete treatment given in the previous work [1] are similar to those of the continuous one. Consequently, the Green function \( S_g^c \) has the following form [1]

\[
S_g^c = -\frac{i}{2} \frac{1}{\sqrt{|b'|}} \sum_{s=\pm 1}^{\pm \infty} \sum_{l=-\infty}^{\infty} e^{i \left( \nu_l - \nu_s \right)} \frac{1}{2\pi} \int \frac{dp_0}{2\pi} e^{-i \frac{s+1}{2} \nu_l e^{+\kappa e^{\nu_3 s} + e^{-\kappa e^{3+1} \nu_l}}} c_a \times e^{-i p_0 t_a - t_a} (r_b | r_a)_{E_c, l_c(s)} ,
\]

where

\[
(r_b | r_a)_{E_c, l_c(s)} = -i \frac{1}{\sqrt{-2E_c}} \frac{\Gamma(\nu+|l_c(s)|+\frac{1}{2})}{\Gamma(2|l_c(s)|+1)} W_{\nu,|l_c(s)|} \left( 2\sqrt{-2E_c} r_b \right) M_{\nu,|l_c(s)|} \left( 2\sqrt{-2E_c} r_a \right) ,
\]

with

\[
l_c(s) = \sqrt{\left( (l - eB) \zeta - 1/2 \right)^2 - a^2 - \frac{\kappa}{2}}, \quad \kappa = sign \left( (l - eB) \zeta - 1/2 \right). \]

\[
E_c = \frac{1}{2} \left( p_0^2 - m^2 \right), \quad \nu = \frac{\alpha p_0}{\sqrt{-2E_c}}\]

knowing that the spin operator \( \hat{S} = \sigma_3, \sigma_3 \chi_s = s \chi_s, \sum_{s=\pm 1} \chi_s \chi_s^\dagger = I \) and \( M_{\nu,|l_c(s)|} \), \( W_{\nu,|l_c(s)|} \) are the Whittaker functions of the first and second kind.

We can see that the sign of \( \sqrt{-2E_c} \) involves two different cases: the first, is related to the bound states and the corresponding discrete energy spectrum when \( E_c < 0 \) [1], and the second one, is related to the scattering states and the corresponding continuous energy spectra when \( E_c \geq 0 \), in what follows we are interested to expose in details the different steps of the computation for such case.

To evaluate the contribution of \( \sqrt{-2E_c} \) to the Green’s function, let us express [5] as

\[
(r_b | r_a)_{E_c, l_c(s)} = -i \frac{1}{\Gamma(2|l_c|+1)} \int_C \frac{dz}{E_c - \frac{z}{2}} \frac{\Gamma(\nu+|l_c|+\frac{1}{2})}{\sqrt{-2E_c}}
\]

\[
\times W_{\nu,|l_c|} \left( 2\sqrt{-2E_c} r_b \right) M_{\nu,|l_c|} \left( 2\sqrt{-2E_c} r_a \right) ,
\]

\[ (11) \]
we find it convenient to consider the following relations for the closed contour $C$, for more details see for example [4]

$$ C : \left\{ \begin{array}{ll}
  z = k, & k \in [-R, +R] \\
  z = R e^{i\phi}, & \phi \in [\pi, 2\pi]
\end{array} \right. $$

and to take the limit $R \to \infty$.

With the aid of the asymptotic behavior of the Whittaker functions the integral over the semicircle vanishes and the equation is solved as follows:

$$ (r_b | r_a)_{E_c \pm i0, l_C(s)} = -i \frac{1}{\Gamma(2 |l_C| + 1)} \int_{-\infty}^{+\infty} \frac{dk}{E_c \pm i0 - \frac{E}{2}} \Gamma\left(-\frac{\alpha \rho}{\pi} + |l_C| + \frac{1}{2}\right) $$

$$ \times W_{\alpha, \frac{\rho}{\pi} |l_C|} \left( \mp 2i |k| r_b \right) M_{\alpha, \frac{\rho}{\pi} |l_C|} \left( \mp 2i |k| r_a \right). $$

We can extract the scattering wave functions from the discontinuity of the Green’s function.

The discontinuity is given by [5]

$$ \text{disc} (r_b | r_a)_{E_c, l_C(s)} = (r_b | r_a)_{E_c \pm i0, l_C(s)} - (r_b | r_a)_{E_c \mp i0, l_C(s)} $$

$$ = \frac{2}{\Gamma(2 |l_C| + 1)} \int_{0}^{\infty} \frac{dk}{k} \frac{\Gamma\left(-\frac{\alpha \rho}{\pi} + |l_C| + \frac{1}{2}\right)}{E_c - \frac{E}{2}} $$

$$ \times e^{\mp i\alpha \frac{\rho}{\pi}} M_{\alpha, \frac{\rho}{\pi} |l_C(s)|} \left( -2i k r_b \right) M_{\alpha, \frac{\rho}{\pi} |l_C(s)|} \left( 2i k r_a \right), $$

where we have used the following relations [6]

$$ M_{a,b}(z) = e^{-i\pi(b+\frac{1}{2})} M_{-a,b}(-z), 2b \neq -1, -2, -3, ... $$

$$ M_{a,b}(z) = \Gamma(2b + 1) e^{i\pi a} \left\{ \frac{W_{a,b}(e^{i\pi z})}{\Gamma(b-a+\frac{1}{2})} + e^{-i\pi(b+\frac{1}{2})} \frac{W_{a,b}(z)}{\Gamma(b+a+\frac{1}{2})} \right\}, -\frac{3\pi}{2} < \text{arg} z < \frac{\pi}{2} $$

therefore, by substituting the previous results we will have

$$ S_c^{we}(x_b, x_a) = -i \frac{1}{2^{\alpha+\frac{1}{2}}} \sum_{s=\pm 1}^{+\infty} \left\{ \frac{e^{i\theta} (\varphi_e - \varphi_a)}{2\pi} \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} e^{-ip_0 \frac{\sigma_1 + \varphi_e + \kappa \sigma_2}{2} \chi_{\sigma_2} e^{-\kappa \sigma_2 e^{ikr_0}} \varphi_a} \right\} $$

$$ \times e^{-ip_0(t_b - t_a)} \frac{4}{\Gamma^2(2 |l_C| + 1)} \int_{0}^{\infty} \frac{dk}{k} \frac{\Gamma\left(-\frac{\alpha \rho}{\pi} + |l_C| + \frac{1}{2}\right)}{p_0^2 - (m^2 + k^2)} $$

$$ \times e^{\mp i\alpha \frac{\rho}{\pi}} M_{\alpha, \frac{\rho}{\pi} |l_C(s)|} \left( -2i k r_b \right) M_{\alpha, \frac{\rho}{\pi} |l_C(s)|} \left( 2i k r_a \right), $$

with the continuous energy spectra

$$ E_k = \sqrt{m^2 + k^2}. $$

It is easy to verify that the poles for continuous spectra are

$$ p_0 = \varepsilon E_k = \varepsilon \sqrt{m^2 + k^2}, \varepsilon = \pm 1 $$

and the property of symmetry is

$$ l_C(-s) = \sqrt{\left( l - eB \right) \zeta - \frac{1}{2}} - \alpha^2 + \frac{\kappa s}{2} = l_C(s) + \kappa s. $$
Let us as well use the residue theorem which gives
\[ \int_{-\infty}^{+\infty} f(p_0) \frac{dp_0}{2\pi i} e^{-ip_0(t_b-t_a)} = -i \sum_{\varepsilon=\pm} \int_{\varepsilon E_k} f(\varepsilon E_k) \frac{e^{-i\varepsilon E_k(t_b-t_a)}}{2E_k} \Theta(\varepsilon (t_b-t_a)), \]
where $\Theta(x)$ is the Heaviside function.

After a set of calculations, we obtain
\[ S_g^c(x_b, x_a) = -\frac{1}{\sqrt{\varepsilon_s r_a}} \sum_{\varepsilon=\pm} \int_{\varepsilon E_k} f(\varepsilon E_k) \frac{e^{-i\varepsilon E_k(t_b-t_a)}}{2E_k} \Theta(\varepsilon (t_b-t_a)) \times \frac{2}{\Gamma(2\kappa' t_C(s)+1)} \int_0^{\infty} \frac{dk}{k} e^{\pm \alpha s E_k} \frac{\Gamma\left(-i\kappa' E_k^2 \kappa' t_C(s)+\frac{1}{2}\right)}{\kappa'} \Theta(\varepsilon (t_b-t_a)) \times e^{+\pi \alpha E_k s E_k} \left\{ M_{10 \kappa' t_C(s)} (2ikr_b) M_{10 -\kappa' t_C(s)} (2ikr_a) \right\}, \kappa' = \text{sign}(t_C(s)). \]
It is easy to check the identity
\[ \sum_{s=\pm} f_s \sum_{s=\pm} g_s = \sum_{s=\pm} f_s (g_s + g_{-s}), \]
and considering the mapping $s \rightarrow -s$ only for terms containing $\Theta(-s(t_b-t_a))$, the global Green’s function related to Dirac particle is finally expressed as follows
\[ S_g^c(x_b, x_a) = -4 \sum_{s=\pm} \int_{\varepsilon E_k} f(\varepsilon E_k) \frac{e^{-i\varepsilon E_k(t_b-t_a)}}{2E_k} \Theta(\varepsilon (t_b-t_a)) \times \left\{ e^{-i\kappa' t_C(s)+\frac{1}{2}} \chi_s \chi_s^+ e^{\kappa' \sigma_2} e^{i\kappa' t_C(s)+\frac{1}{2}} \varphi_a \times \frac{\Gamma\left(-i\kappa' E_k^2 \kappa' t_C(s)+\frac{1}{2}\right)}{\kappa'} \frac{M_{10 \kappa' t_C(s)+\eta} (\eta)}{\sqrt{\kappa'}} \frac{M_{10 -\kappa' t_C(s)+\eta} (\eta)}{\sqrt{\kappa'}} \right\}, \eta = \kappa' \kappa s = \pm 1. \]

It is very well knowing that the dynamic of the system is totally determined by the causal Green’s function $S_g^c(x_b, x_a)$ which verifies the first order of Dirac equation given by (2) and obtained according to (4).

The projection operator in polar coordinates is given by
\[ (\gamma^\mu \pi_\mu + m)(b) = \sigma_3 \left( \frac{i}{\tau_6} \partial_\varphi \varphi_b + \frac{i}{\tau_5} \right) - i \frac{\gamma_5 \beta_1 e^{i\kappa \sigma_3 \varphi_b}}{\gamma_5 \beta_1 e^{i\kappa \sigma_3 \varphi_b}} - \sigma_2 e^{i\kappa \sigma_3 \varphi_b} \frac{\partial}{\partial \varphi_b} + \frac{1}{r_1} \sigma_1 e^{i\kappa \sigma_3 \varphi_b} \frac{\partial}{\partial \varphi_b} + m, \]
we can easily check the relations
\[ \left\{ \begin{array}{l}
\sigma_1 \chi_s = \chi_{-s} \\
\sigma_2 \chi_s = i \chi_{-s}
\end{array} \right., \text{and their conjugates} \left\{ \begin{array}{l}
\chi^+_s \sigma_1 = \chi^-_{-s} \\
\chi^+_s \sigma_2 = -i \chi^-_s
\end{array} \right., \]
for the vectors
\[ \chi^{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]
Then, we map \( \mathcal{A} \) and \( \mathcal{B} \) into \( \mathcal{C} \) to end up with

\[
S^c (x_b, x_a) = -4 \sum_{s=\pm 1} \sum_{l=\pm +} e^{i \phi b - \phi a} \int_0^\infty dk \frac{e^{-i \omega_b (t_b - t_a)}}{2 \pi} e^{+i \pi \alpha E_k} \Theta (s (t_b - t_a)) \\
\times e^{-i \frac{\sigma s + 1}{2} \phi c} e^{+i \alpha d} \left\{ \left( (\cosh (2 \alpha \eta) E_k + m) \chi_s \chi_s^+ - i s \pi \right) (z_b) \chi_s \chi_s^+ \right\} \\
\times \left| \Gamma \left( \frac{\alpha s + 1}{2} \right) \chi_s \chi_s^+ \right\| \frac{M_{a,b}^s (s + 1)}{\sqrt{z_b}} \frac{M_{a,b}^s (s + 1)}{\sqrt{z_b}} \right\} e^{-i \sigma s} e^{i \frac{\sigma s + 1}{2} \eta}, \eta = \kappa' \kappa s = \pm 1,
\]

(28)

where

\[
\pi_s (z) = -2i k \frac{\partial}{\partial z} + 2i k s \sqrt{\frac{c}{z}} + \kappa' \kappa \frac{E_k}{\sqrt{z}}.
\]

(29)

In what follows we derive normalized scattering wave functions and the corresponding continuous energy eigenvalues.

### 3 Continuous energy spectra and scattering states

In what follows, we deduce the scattering wave functions and the corresponding positive and negative continuous spectra.

Thanks to the use of the following derivative form of the confluent hypergeometric functions [7]

\[
\frac{dF}{dz} (\alpha, \gamma, z) = \frac{\alpha}{\gamma} F (\alpha + 1, \gamma + 1, z),
\]

(30)

and the two recurrence relations [7]

\[
\alpha F (\alpha + 1, \gamma + 1, z) = (\alpha - \gamma) F (\alpha, \gamma + 1, z) + \gamma F (\alpha, \gamma, z),
\]

(31)

\[
z F (\alpha + 1, \gamma + 1, z) = \gamma \left( F (\alpha + 1, \gamma, z) \right) - \gamma F (\alpha, \gamma, z),
\]

(32)

we easily get

\[
\frac{dF (\alpha, \gamma, z)}{dz} = \frac{\alpha}{\gamma} F (\alpha + 1, \gamma + 1, z) = \frac{\alpha}{\gamma} F (\alpha, \gamma, z) - \frac{\alpha - \gamma}{\gamma} \frac{z}{\gamma + 1} F (\alpha + 1, \gamma + 2, z),
\]

(33)

\[
\frac{dF (\alpha, \gamma, z)}{dz} = \frac{1 - \gamma}{z} \left[ F (\alpha, \gamma, z) + \frac{\alpha - \gamma + 1}{\gamma - 2} \frac{z}{\gamma - 2} + 1 \right] F (\alpha, \gamma, z) - F (\alpha - 1, \gamma - 2, z). \]

(34)

One can, immediately, verify that

\[
\frac{d}{dz} \frac{M_{a,b}(z)}{z^2} = \frac{a}{2b + 1} \frac{M_{a,b}(z)}{z^2} + \frac{b}{z} M_{a,b} (z) + \frac{b}{z} \frac{M_{a,b-1}(z)}{z^2}
\]

(35)

\[
\frac{d}{dz} \frac{M_{a,b}(z)}{z^2} = \frac{a}{2b + 1} \frac{M_{a,b}(z)}{z^2} + \frac{b}{z} M_{a,b} (z) - \frac{b - a + 1}{1 + 2b} \frac{1}{1 + 2b} \\
+ \frac{1}{1 + 2b} \frac{M_{a,b}(z)}{z^2}.
\]

(36)
Consequently, we check the two interesting properties for the two values \( \eta = \pm 1 \)

\[
\pi_s (r) \left( \frac{M \cdot s_{iE_k}^{\kappa'I_{LC}(s)} (z)}{\sqrt{2}} \right) = -2ik \left( 2 \kappa'I_{LC}(s) \frac{1+\eta}{2} + \frac{1-\eta}{2} \frac{(\alpha^{E_k})^2 + (\kappa'I_{LC}(s)+\frac{1}{2})^2}{(\kappa'I_{LC}(s)+\eta)(\kappa'I_{LC}(s)+1)^2} \right) \left( \frac{M \cdot s_{iE_k}^{\kappa'I_{LC}(s)+\eta} (z)}{\sqrt{2}} \right),
\]

\[
\pi_{-s} (r) \left( \frac{M \cdot s_{iE_k}^{\kappa'I_{LC}(s)+\eta} (z)}{\sqrt{2}} \right) = -2ik \left( 2 \kappa'I_{LC}(s) + \eta \right) \frac{1+\eta}{2} + \frac{1-\eta}{2} \frac{(\alpha^{E_k})^2 + (\kappa'I_{LC}(s)+\eta+\frac{1}{2})^2}{(\kappa'I_{LC}(s)+\eta+1)(\kappa'I_{LC}(s)+2\eta+1)^2} \right) \left( \frac{M \cdot s_{iE_k}^{\kappa'I_{LC}(s)} (z)}{\sqrt{2}} \right).
\]

We can make use of the well known formula

\[
\Gamma(z+1) = z \Gamma(z), \quad \Gamma(z+2) = z(z+1) \Gamma(z),
\]

\[
\Gamma(z-1) = \frac{1}{z-1} \Gamma(z), \quad \Gamma(z-2) = \frac{1}{z-2} \frac{1}{z-1} \Gamma(z),
\]

or else, in more convenient forms

\[
\Gamma(z+\theta) = (z+\theta) \frac{1+\theta}{2} + \frac{1-\theta}{2} \frac{1}{(z+1)^{\theta}} \Gamma(z), \quad \theta = \pm 1
\]

\[
\Gamma(z+2\theta) = (z+1) \frac{1+\theta}{2} + \frac{1-\theta}{2} \frac{1}{(z+1)^{2\theta}} \Gamma(z), \quad \theta = \pm 1
\]

by taking respectively \( z = -i \frac{\alpha^{E_k}}{k} + \kappa'I_{LC}(s) + \frac{1}{2} \), \( z = -i \frac{\alpha^{E_k}}{k} + \kappa'I_{LC}(s) + \frac{1}{2} \) with \( \theta = \eta \), we find the following interesting properties

\[
\Gamma \left( -i \frac{\alpha^{E_k}}{k} + \kappa'I_{LC}(s) + \eta + \frac{1}{2} \right) = \left( \left( -i \frac{\alpha^{E_k}}{k} + \kappa'I_{LC}(s) + \frac{1}{2} \right) \frac{1+\eta}{2} + \frac{1-\eta}{2} \left( -i \frac{\alpha^{E_k}}{k} + \kappa'I_{LC}(s) - \frac{1}{2} \right) \right) \Gamma \left( -i \frac{\alpha^{E_k}}{k} + \kappa'I_{LC}(s) + \frac{1}{2} \right),
\]

\[
\Gamma \left( 2 \kappa'I_{LC}(s) + 1 + 2\eta \right) = \left( 2 \left( 2 \kappa'I_{LC}(s) + 1 \right) \left( \kappa'I_{LC}(s) + 1 \right) \frac{1+\eta}{2} + \frac{1-\eta}{2} \frac{1}{2 \kappa'I_{LC}(s) - 1} \frac{1}{2 \kappa'I_{LC}(s) + 1} \right) \Gamma \left( 2 \kappa'I_{LC}(s) + 1 \right),
\]

thus, we get

\[
\pi_s (z) G_{\kappa'I_{LC}(s)} (z) = -ie^{i\eta \arctan \left( \frac{\alpha^{E_k}}{\kappa'I_{LC}(s) + \frac{1}{2}} \right)} \sqrt{\cosh (2\kappa\theta) E_k - m} \sqrt{\cosh (2\kappa\theta) E_k + mG},
\]

\[
\pi_{-s} (z) G_{\kappa'I_{LC}(s)+\eta} (z) = -ie^{-i\eta \arctan \left( \frac{\alpha^{E_k}}{\kappa'I_{LC}(s) + \frac{1}{2}} \right)} \sqrt{\cosh (2\kappa\theta) E_k + m} \sqrt{\cosh (2\kappa\theta) E_k - mG},
\]

which can be further simplified to end up with

\[
S^c(x_b, x_a) = \sum_{s=\pm 1} \int_{0}^{+\infty} dk \Psi^c_{k,l,s} (r_b, \varphi_b; t_b) \left( \Psi^c_{k,l,s} (r_a, \varphi_a; t_a) \right)^{\pm} \sigma_3 s \Theta (s (t_b - t_a)),
\]
where the scattering normalized wavefunctions are finally given by
\[
\Psi_{k,l,s}(r, \varphi; t) = \frac{2e^{il\varphi} e^{-i\alpha E_k t}}{\sqrt{2\pi} \sqrt{2E_k}} e^{i\pi \frac{sE_k}{2}} e^{-i\frac{\pi}{2}\frac{r^2}{2}} e^{i\theta \sigma_2} \\
\times \left[ \sqrt{\cosh (2\kappa \theta)} E_k + mG_{\alpha E_k, \kappa \ell C(s)}(z) \chi_s \right.
\]
\[
- se^{-i\eta \text{arctan} \left( \frac{\kappa E_k}{\kappa' \ell C(s) + \frac{1}{2}} \right)} \sqrt{\cosh (2\kappa \theta)} E_k - mG_{\alpha E_k, \kappa' \ell C(s)}(z) \chi_{-s} \right], \quad z = -2ikr, \quad (48)
\]
with
\[
G_{\alpha E_k, \kappa \ell C(s)}(z) = \frac{\Gamma \left( -i\frac{\alpha E_k}{k} + \kappa' \ell C(s) + \frac{1}{2} \right)}{\Gamma \left( 2\kappa' \ell C(s) + 1 \right)} M_{\alpha E_k, \kappa' \ell C(s)}(z) \sqrt{z}, \quad z = -2ikr, \quad (49)
\]
\[
G_{\alpha E_k, \kappa' \ell C(s)}(z) = \frac{\Gamma \left( -i\frac{\alpha E_k}{k} + \kappa' \ell C(s) + \eta + \frac{1}{2} \right)}{\Gamma \left( 2\kappa' \ell C(s) + 2\eta + 1 \right)} M_{\alpha E_k, \kappa' \ell C(s) + \eta}(z) \sqrt{z}, \quad z = -2ikr, \quad (50)
\]
and the corresponding positive and negative continuous energy spectra are
\[
\mathcal{E}_k^s = s \sqrt{m^2 + k^2}. \quad (51)
\]
Finally, in order to determine the phase shift, let us study the asymptotic form of the wavefunction. The asymptotic development of confluent hypergeometric functions is found for example in the ref [7].

\[
F(\lambda, \gamma; z) \xrightarrow{|z| \to +\infty} \frac{\Gamma(\gamma)}{\Gamma(\lambda)} e^z |z|^\lambda - \gamma e^{-i\frac{\pi}{2}(\lambda-\gamma)} + \frac{\Gamma(\gamma)}{\Gamma(\gamma - \lambda)} |z|^{-\lambda} e^{-i\frac{\pi}{2}(\lambda-\gamma)}, \quad z = |z| e^{-i\frac{\pi}{2}}, \quad (52)
\]
from which we have
\[
M_{a,b}(z) \xrightarrow{|z| \to +\infty} \frac{\Gamma(1+2b)}{\Gamma(b-a+\frac{1}{2})} e^{z^2} |z|^{a} e^{i\frac{\pi}{4}a} + \frac{\Gamma(1+2b)}{\Gamma(b-a+\frac{1}{2})} |z|^{a} e^{-\frac{\pi}{4}a} e^{-i\frac{\pi}{2}(2b-a+1)}, \quad (53)
\]
if \(z = -2ikr = 2kre^{-i\frac{\pi}{2}}\) the function \(G_{\alpha E_k, \kappa \ell C(s)}(z)\) behave as
\[
G_{\alpha E_k, \kappa \ell C(s)}(z) \xrightarrow{r \to +\infty} e^{-ikr(2\kappa \eta)} M_{\alpha E_k, \kappa \ell C(s)}(z) e^{-\frac{\pi}{2} \kappa' \ell C(s)} \left( \frac{E_k}{2} \ln(2kr) + \frac{\pi}{2} \kappa' \ell C(s) \right) e^{-i\kappa' \ell C(s)}(z) e^{i\frac{\pi}{2}(2\kappa' \ell C(s) + 2\eta - \kappa' \ell C(s) + \frac{1}{2})}. \quad (54)
\]
Consequently, the scattering eigenspinors \(\Psi_{k,l,s}(r, \varphi; t)\) behave as
\[
\Psi^c_{k,l,s}(r, \varphi; t) \xrightarrow{r \to +\infty} \frac{2e^{il\varphi} e^{-i\alpha E_k t}}{\sqrt{2\pi} \sqrt{2E_k}} e^{-i\pi \frac{S_{c1}+1}{2}} e^{i\alpha \sigma_2} \\
\times \left[ \sqrt{\cosh (2\kappa \theta)} E_k + mG_{\alpha E_k, \kappa \ell C(s)}(z) \chi_s \right.
\]
\[
- se^{-i\eta \text{arctan} \left( \frac{\alpha E_k}{\kappa' \ell C(s) + \frac{1}{2}} \right)} \sqrt{\cosh (2\kappa \theta)} E_k - mG_{\alpha E_k, \kappa' \ell C(s)}(z) \chi_{-s} \right], \quad z = -2ikr, \quad (55)
\]
we can make sure that
\[
\Gamma\left(\kappa' l_{C}(s) + \eta + \frac{1}{2} + i\alpha \frac{sE_{k}}{k}\right) = \left[\left(i\alpha E_{k} + \kappa' l_{C}(s) + \frac{1}{2}\right)^{1+n} + \frac{1}{\kappa l_{C}(s) + \frac{1}{2}}\right] e^{i\delta_{l}}.
\]  
(56)

thus, the scattering wavefunctions are finally expressed as follows
\[
\Psi_{k,l,s}^{\pm}(r, \varphi; t) \xrightarrow{r \rightarrow \infty} \frac{2 e^{iE_{k} t}}{\sqrt{2\pi}} e^{i\delta_{l}} e^{-i\delta_{AB} e^{i\frac{\alpha}{2} l_{C}(s) + \frac{1}{2}} \sigma_{2}}
\]
\[
\times \left[ \sqrt{\cosh (2\kappa \vartheta)} E_{k} + m \cos \left( \delta_{AB} + \delta_{l} - \alpha \frac{E_{k}}{k} \ln(2kr) - \frac{\pi l}{4} \right) \chi s
\]
\[
- is \sqrt{\cosh (2\kappa \vartheta)} E_{k} - m \sin \left( \delta_{l} + \delta_{AB} + \eta \arctan \left( \alpha \frac{E_{k}}{k} \ln(2kr) - \frac{\pi l}{4} \right) \right) \chi_{-s},
\]
\]  
(57)

where the two-phases shifts are given by
\[
\delta_{AB} = \frac{\pi}{2} \kappa' l_{C}(s) + \frac{\pi}{4} + \frac{\pi}{2}, \quad (58)
\]
\[
\delta_{l}^{\alpha} = \arctan \left( \Gamma \left( \kappa' l_{C}(s) + \frac{1}{2} + i\alpha \frac{sE_{k}}{k}\right) \right). \quad (59)
\]

In the following subsection we can make sure that the obtained scattering eigenspinors are fulfilling the Dirac wave equation.

3.1 Equivalence with Dirac equation

The first-order Dirac equation is given by
\[
(\gamma_{\mu}^{\alpha} \pi_{\mu} - m) \Psi_{k,l,s}^{\pm}(r, \varphi; t) = 0,
\]  
(60)

where
\[
(\gamma_{\mu}^{\alpha} \pi_{\mu} - m) = \sigma_{3} \left( i \frac{\partial}{\partial t} + \frac{\partial}{\partial \rho} \right) - i\sigma_{2} \left( -i \frac{\partial}{\partial \varphi} + \frac{eB_{y}}{x^{2} + y^{2}} \right) + i\varsigma \sigma_{1} \left( -i \frac{\partial}{\partial \rho} - \frac{eB_{x}}{x^{2} + y^{2}} \right) - m,
\]  
(61)

we can make sure that the operator \((\gamma_{\mu}^{\alpha} \pi_{\mu} - m)\) in polar coordinates is
\[
(\gamma_{\mu}^{\alpha} \pi_{\mu} - m) = \sigma_{3} \left( i \frac{\partial}{\partial t} + \frac{\partial}{\rho} \right) - i\frac{eB_{y}}{r} \sigma_{1} e^{i\varsigma \varphi} \sigma_{2} e^{i\varsigma \varphi} \frac{\partial}{\sigma_{2}} + \varsigma \frac{1}{r} \sigma_{1} e^{i\varsigma \varphi} \frac{\partial}{\sigma_{2}} - m.
\]  
(62)

Let us suppose that
\[
(\gamma_{\mu}^{\alpha} \pi_{\mu} - m) \Psi_{k,l,s}^{\pm}(r, \varphi; t) = \frac{2 e^{iE_{k} t}}{\sqrt{2\pi}} e^{i\alpha \frac{E_{k}}{k} \ln(2kr) - \frac{\pi l}{4}} e^{-i\delta_{l}^{\alpha} + \eta} \left(A \chi_{s} + B \chi_{-s} \right).
\]  
(63)

Through a straightforward calculus, we obtain
\[
A = \sqrt{\cosh (2\kappa \vartheta)} E_{k} + m \left( \cosh (2\kappa \vartheta) E_{k} - m \right) G \left( \frac{sE_{k}}{k} \kappa' l_{C}(s) + \frac{1}{2}, \frac{1}{2} \right) e^{-i\frac{\alpha}{2} l_{C}(s) + \frac{1}{2}}
\]
\[
- se \arctan \left( \frac{\alpha E_{k}}{k' l_{C}(s) + \frac{1}{2}} \right) \sqrt{\cosh (2\kappa \vartheta)} E_{k} - m \left( \frac{\partial}{\rho} + s \kappa' l_{C}(s) + \frac{1}{2} \right) + \kappa \alpha \frac{E_{k}}{l_{C}(s) - s} \right) \left( \frac{sE_{k}}{k' l_{C}(s) + \frac{1}{2}} \right)
\]  
(64)
\[ B = -is\sqrt{\cosh (2\kappa \vartheta)} E_k + m \left( \frac{\partial}{\partial r} - s \kappa \frac{l_c(s)}{r} + \kappa \alpha \frac{E_k}{l_c(s) + \frac{|s|}{2}} \right) G \left( z \right) \]
\[ -se^{i \varphi} \arctan \left( \frac{\alpha E_k}{k l_c(s) + \frac{|s|}{2}} \right) \sqrt{\cosh (2\kappa \vartheta)} E_k - m \left( -\cosh (2\kappa \vartheta) E_k - m \right) G \left( z \right). \]  
(65)

By introducing the moments \( \pi_s(z) \) and \( \pi_{-s}(z) \) given in (29), A and B take the following forms
\[ A = \sqrt{\cosh (2\kappa \vartheta)} E_k + m \pi_s(z) G \left( z \right) \]
\[ -se^{i \varphi} \arctan \left( \frac{\alpha E_k}{k l_c(s) + \frac{|s|}{2}} \right) \sqrt{\cosh (2\kappa \vartheta)} E_k - m \pi_{-s}(z) G \left( z \right). \]  
(66)

\[ B = -is\sqrt{\cosh (2\kappa \vartheta)} E_k + m \pi_s(z) G \left( z \right) \]
\[ -se^{i \varphi} \arctan \left( \frac{\alpha E_k}{k l_c(s) + \frac{|s|}{2}} \right) \sqrt{\cosh (2\kappa \vartheta)} E_k - m \pi_{-s}(z) G \left( z \right). \]  
(67)

Then, using (65) and (66), we get \( A = 0 \) and \( B = 0 \). This confirms that the scattering eigenspinors \( \Psi_{n,l,s}^\pm(r, \varphi; t) \) satisfy the first order Dirac equation
\[ (\gamma^\mu \pi_\mu - m) \Psi_{n,l,s}^\pm(r, \varphi; t) = 0. \]  
(68)

### 4 particular cases

#### 4.1 Aharonov-Bohm potential

At first we consider the case of the Aharonov-Bohm field [8], for this we suppress the Coulomb field by setting in (45) \( \alpha \rightarrow 0 \) then \( \vartheta \rightarrow 0 \) and \( l_c(s) \rightarrow \kappa \left( (l - eB) \varsigma - \frac{|s|}{2} \right) \), we get the scattering eigenspinors
\[ \Psi_{k,l,s}^\pm(r, \varphi; t) = \frac{2e^{il\varphi} e^{-isEt}}{\sqrt{2\pi}} e^{-ik \rho + \frac{|s|}{2} \rho} \times \left( \sqrt{E_k + mG_0(l-eB)z - \frac{1-z}{2}} (-2ikr) \chi_s - s \sqrt{E_k - mG_0(l-eB)z - \frac{1-z}{2}} (-2ikr) \chi_s \right). \]  
(69)

following [6]
\[ M_{0,\nu}(z) = 2^\nu \Gamma(\nu + 1) \sqrt{\pi} I_\nu(z), \]  
(70)

with the use of the relation between the first and second kind Bessel functions,
\[ I_\nu(z) = e^{-\frac{1}{2}z \pi i} J_\nu \left( z e^{\frac{1}{2} \pi i} \right), -\pi < \arg z \leq \frac{1}{2} \pi, \]  
(71)

and the well-known property of gamma function
\[ \Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma \left( x + \tfrac{1}{2} \right), \]  
(72)
we obtain the following results

\[
G_{0,|l-eB|\zeta-\frac{1}{2}}(-2ikr) = \sqrt{\pi}e^{-\frac{1}{2}|l-eB|\zeta-\frac{1}{2}}|\psi|J_{|l-eB|\zeta-\frac{1}{2}}(kr),
\]

(73)

\[
G_{0,|l-eB|\zeta-\frac{3}{2}}(-2ikr) = \sqrt{\pi}e^{-\frac{1}{2}|l-eB|\zeta-\frac{3}{2}}|\psi|J_{|l-eB|\zeta-\frac{3}{2}}(kr).
\]

(74)

thus, the two scattering eigenspinors are finally given by the following expressions

\[
\Psi_{k,l,s=1}^{\zeta}(r,\varphi; t) = \frac{2e^{il\varphi}}{\sqrt{2\pi}}e^{-iE_k t}\sqrt{\pi}
\left(\frac{\sqrt{E_k + me^{-i\varphi}e^{-\frac{1}{2}|l-eB|\zeta-1}|\psi|J_{|l-eB|\zeta-1}}(kr)}{\sqrt{E_k - me^{-i\varphi}e^{-\frac{1}{2}|l-eB|\zeta+1}|\psi|J_{|l-eB|\zeta+1}}(kr)}\right),
\]

(75)

\[
\Psi_{k,l,s=-1}^{\zeta}(r,\varphi; t) = \frac{2e^{il\varphi}}{\sqrt{2\pi}}e^{iE_k t}\sqrt{\pi}
\left(\frac{\sqrt{E_k - me^{-i\varphi}e^{-\frac{1}{2}|l-eB|\zeta+1}|\psi|J_{|l-eB|\zeta+1}}(kr)}{\sqrt{E_k + me^{-i\varphi}e^{-\frac{1}{2}|l-eB|\zeta-1}|\psi|J_{|l-eB|\zeta-1}}(kr)}\right).
\]

(76)

The obtained results (73) and (76) are more general than those of the ref [2] because they are containing the phases shifts \(\frac{1}{2}|l-eB|\zeta - \frac{1}{2}\pi i\) and \(\frac{1}{2}|l-eB|\zeta |\pi i\) in AB-effect.

4.2 Nonrelativistic limit

We require the limit \(\alpha \ll |l_c(s)| + \frac{1}{2}\) to recover the nonrelativistic limit for Dirac particle [2].

We can see that, the obtained solutions of the continuous energy spectrum are behave as

\[\vartheta (l) \to 0,\]

(77)

\[l_c (s) \to \kappa \left( l - eB \right) \zeta - \frac{1}{2} - \frac{\kappa s}{2} = \kappa \left( l - eB \right) - \frac{s + 1}{2},\]

(78)

\[E_k = m \left( 1 + \frac{k^2}{m^2} \right)^{\frac{1}{2}} \sim m + \frac{k^2}{2m} \sim m\]

(79)

and then the spinor in (48) behaves as

\[
\Psi_{k,l,s}^{\zeta}(r,\varphi; t) \to 2e^{il\varphi}e^{-is\left(\frac{k^2}{2m}\right)t}e^{\pi\alpha \frac{m}{m^2}}e^{-i\zeta \frac{s+1}{2}r}G_{\zeta \frac{s+1}{2}|l-eB|}(z) \chi_s.
\]

(80)

Now taking \(l \to l + \zeta \frac{s+1}{2}\), to end up with usual eigenfunctions for the nonrelativistic problem

\[
\Psi_{k,l,s=1}^{\zeta}(r,\varphi; t) \to e^{-imt}\begin{pmatrix}
\Psi_{k,l}^{NR}(r,\varphi; t) \\
0
\end{pmatrix},
\]

(81)

or

\[
\Psi_{k,l,s=-1}^{\zeta}(r,\varphi; t) \to e^{imt}\begin{pmatrix}
0 \\
\Psi_{k,l}^{NR}(r,\varphi; t)
\end{pmatrix},
\]

(82)

where \(\Psi_{k,l}^{NR}(r,\varphi; t)\) are the nonrelativistic functions defined by

\[
\Psi_{k,l}^{NR}(r,\varphi; t) = \frac{2e^{il\varphi}}{\sqrt{2\pi}}e^{-iE_k^{NR} t}e^{\pi\alpha \frac{m}{m^2}}\Gamma\left(\frac{-\alpha m}{2m}+1+\frac{1}{2}\right)M_{\alpha \frac{m}{m^2} |l-eB|}(kr)\frac{1}{\sqrt{-2ikr}},
\]

with

\[
\frac{M_{\alpha \frac{m}{m^2} |l-eB|}(kr)}{\sqrt{-2ikr}} = e^{ikr}\frac{1}{\sqrt{-2ikr}}F\left(\frac{|l-eB|+\alpha m}{2m}+\frac{1}{2}+\frac{1}{2}|l-eB|,-2ikr\right),
\]

(83)
and the corresponding nonrelativistic energy is

\[ E^{NR} = \frac{E_k^2 - m^2}{2m} = \frac{k^2}{2m}. \]  

(85)

These results agree exactly with those given in ref [8].

5 Conclusion

In this paper, we have proposed for (2 + 1)-dimensional Dirac wave equation related to a relativistic half spin particle in interaction with Aharonov-Bohm-Coulomb field, an analytical and exact solution by means of (2+1)path integral technique. We have shown following this approach that the continuous energy eigenvalues, as well as the corresponding scattering normalized eigenspinors are extracted from the symmetric form of the causal Green’s function.

We should note, however, that we have considered some special cases and we have shown that the results obtained in this paper are in agreement with the literature.

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