Cosmological Analysis of Modified Holographic Ricci Dark Energy in Chern-Simons Modified Gravity

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Abstract
In this paper, we study the cosmological analysis of the modified holographic Ricci dark energy model and reconstruct different scalar field models in the context of Chern-Simons modified gravity. We investigate the deceleration parameter, which shows that the universe is in the accelerating expansion phase. The equation of state parameter in this case also favors the fact that dark energy is the dominant component of universe which is responsible for the accelerated expansion. A number of scalar fields, such as: quintessence, tachyon, K-essence and dilaton models are reconstructed using modified holographic Ricci dark energy model in the context of dynamical CS modified gravity. The quintessence and K-essence models represent exponentially increasing behaviors while Tachyon model shows decreasing behavior. Unfortunately, the dilaton model has no numerical solution for modified holographic Ricci dark energy model in the framework of dynamical Chern-Simon modified gravity.

PACS: 04.20.-q

Keywords: Dynamical Chern-Simon Modified Gravity, Dark Energy, Scalar Fields.

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1 Introduction

A large number of evidences have been provided in favor of the accelerated expansion of the universe by Type Ia supernovae [1], Cosmic Microwave Background (CMB) [2], weak lensing [3], Large Scale Structures [4] and integrated Sachs-Wolfe effect [5]. It is postulated that there exist a component in the universe which has negative pressure is responsible for the accelerated expansion of the universe, called Dark Energy (DE). The most familiar candidate of DE model is cosmological constant $\Lambda$ which satisfy the cosmological observations [6] but fails to resolve the fine tuning problem and cosmic coincidence [7]. In literature, there are many DE models such as an evolving canonical scalar field [8] quintessence, the phantom energy [9], quintom energy [10] and so forth.

In recent studies to understand the nature of the universe, a new DE model has been constructed in the context of quantum gravity on holographic principle named as holographic dark energy (HDE) model [11]. This principle is extensively used to study the quantum behavior of black holes. The energy density of HDE is defined as $\rho = \frac{3c^2M_{pl}^2}{L^2}$, where $c$ is constant, $M_{pl}$ plank mass and $L$ is supposed to be size of the universe. For the Hubble radius $H^{-1}$, this HDE model density is very similar to the observational results. Gao et al. [12] motivated by the holographic principle, introduced a new DE model which is inversely proportional to Ricci scalar curvature called Ricci Dark Energy (RDE). Their investigation shows that this RDE model solve the causality problem and the evolution of density perturbations of matter power spectra and CMB anisotropy is not much affected by such modification. Granda and Oliveros [13] introduced a new infrared cut-off for the HDE model and reconstruct the potentials and fields for different DE models such as the quintessence, tachyon, K-essence and dilaton for FRW universe. Karami and Fehri [14] using Granda and Oliveros cut-off studied the non-flat FRW universe to find the DE density, the deceleration parameter and the equation of state (EoS).

Jackiw and Pi [15] introduced Chern-Simons (CS) modified gravity in which the Einstein-Hilbert action is modified as the sum of parity-violating CS term and scalar field. Silva and Santos [16] analysis the RDE of FRW universe and found it similar to GCG in the context of CS modified gravity. Jamil and Sarfraz [17] work the same for amended FRW universe and present their results graphically. Jawad and Sohail [18] considering modified QCD ghost dark energy model investigate the dynamics of scalar field and
potentials of various scalar field models in the framework of dynamical CS modified gravity. Jamil and Sarfraz [19] considering HDE model found the accelerated expansion behavior of the universe under the certain restrictions on the parameter $\alpha$. We study the correspondence between the quintessence, K-essence, tachyon and dilaton field models and holographic dark energy model. Pasqua et al. [20] investigated the HDE, modified holographic Ricci dark energy (MHRDE) and another model which is a combination of higher order derivatives of the Hubble parameter in the framework of CS modified gravity.

In this paper, working on same lines using the MHRDE model, we explore the energy density, deceleration parameter, EoS parameter and correspondence between different models. The paper organized in following order. The basic formulism of CS modified gravity is discussed in section 2. The section 3, is devoted for the investigation of energy density, deceleration parameter and EoS parameter. The correspondence between scalar field models such that quintessence, tachyon, K-essence and dilaton model is given in section 4. Summery and concluding remarks are in the last section.

2 ABC of Chern-Simon Modified Gravity

Jackiw and Pi modify the 4-dimensional GR theory introducing a Chern-Simon term in Einstein-Hilbert action, is given by

$$S = \int d^4x \sqrt{-g}[\chi R + \frac{\zeta}{4} \Theta^* RR - \frac{\eta}{2}(g^{\mu\nu} \nabla_\mu \Theta \nabla_\nu \Theta + 2V[\Theta])] + S_{\text{mat}},$$

(1)

the terms used in this relation are defined as $\chi = (16\pi G)^{-1}$, $R$ is the usual Ricci scalar, the term $^*RR$ is defined as $^*RR = ^*R^a_{\ b\ cd} R^b_{acd}$, is topological invariant, called Pontryagin term where $^*R^a_{\ b\ cd}$ is the dual of Reimann tensor $R^b_{acd}$ can be expressed as $^*R^a_{\ b\ cd} = \frac{1}{2} \varepsilon^{cdef} R^a_{bdef}$, the $\nabla_\mu$ is titled as covariant derivative, the dimensionless parameters $\zeta$ and $\eta$ are treated here as coupling constants and $V[\Theta]$ is potential and in the context of string theory it is assumes that $V[\Theta] = 0$. The most important term is $\Theta$, called CS coupling field works as a deformation function of the spacetime which is always other than constant (If $\Theta$ is constant function then the CS theory reduces to GR).

The variation of Einstein-Hilbert action $S$ corresponding to metric tensor $g_{\mu\nu}$ and scalar field $\Theta$ resulted into a set of field equations of CS modified
gravity given by

\[ G_{\mu\nu} + lC_{\mu\nu} = \chi T_{\mu\nu}, \]  
\[ g^{\mu\nu} \nabla_\mu \nabla_\nu \Theta = -\frac{\zeta}{4} * RR, \]  

where \( G_{\mu\nu}, C_{\mu\nu}, l, \) and \( T_{\mu\nu} \) are called Einstein tensor, Cotton tensor (C-tensor), coupling constant and energy-momentum tensor respectively. The energy-momentum tensor is a combination of matter part \( T^m_{\mu\nu} \) and the external field part \( T^\Theta_{\mu\nu} \). The mathematical expressions for these terms are followed as

\[ C_{\mu\nu} = -\frac{1}{2\sqrt{-g}} [v_\sigma \epsilon^{\sigma\mu\nu\eta} \nabla_\zeta R^\eta_{\zeta\eta} + \frac{1}{2} v_\sigma \epsilon^{\sigma\nu\zeta\eta} R^\eta_{\zeta\mu}] + (\mu \leftrightarrow \nu), \]  
\[ T^m_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu}, \]  
\[ T^\Theta_{\mu\nu} = \zeta (\partial_\mu \Theta)(\partial_\nu \Theta) - \frac{\zeta}{2} g_{\mu\nu} (\partial^\lambda \Theta)(\partial_\lambda \Theta). \]  

Here \( \rho \) is energy density, \( p \) is pressure and \( u_\mu = (1, 0, 0, 0) \) denote standard time-like 4-velocity respectively. The terms \( v_\sigma \equiv \nabla_\sigma \Theta, v_{\sigma\tau} \equiv \nabla_\sigma \nabla_\tau \Theta \). On the basis of choice of \((\zeta \neq 0 \text{ and } \chi \neq 0)\) and \( \chi \neq 0 \text{ and } \zeta = 0 \), this theory is divided into two distinct theories named as dynamical and non-dynamical Chern-Simon Modified gravity respectively.

### 3 Modified Holographic Ricci Dark Energy Model

In this paper we study the FRW universe in the framework of dynamical CS modified gravity. The 00-component of field equation of FRW universe using Eq.(2), we get,

\[ H^2 = \frac{1}{3} \rho_D + \frac{1}{6} \dot{\Theta}^2. \]  

Here \( H = \frac{\dot{a}}{a} \) is called Hubble parameter and \( \dot{a} \) is time derivative of scale factor \( a(t) \). The Pontryagin term \( *RR \) vanishes for FRW metric identically, so, Eq.(3) takes the form

\[ g^{\mu\nu} \nabla_\mu \nabla_\nu \Theta = g^{\mu\nu} [\partial_\mu \partial_\nu \Theta - \Gamma^\rho_{\mu\nu} \partial_\rho \Theta] = 0. \]
Keeping in view, the $\Theta$ is a function of spacetime, we consider, $\Theta = \Theta(t)$, Eq.(8) turns out to be.

$$\dot{\Theta} = Ca^{-3}. \quad (9)$$

$C$ is constant of integration other than zero (as if this $C$ is zero then the function $\Theta$ become constant which reduces the CS theory to GR).

Using holographic principle, Hooft [22] proposed very simple and convenient model to investigate the issues raised in DE, named as HDE model. This model is used in different scenarios such that Hubble radius and cosmological conformal time of particle horizon [23, 27]. An interesting holographic RDE model defined as $L = |R|^{-\frac{1}{2}}$, where $R$ is Ricci curvature scalar, was purposed by Gao et al [28].

In this paper, we use HDE model suggested by Granda and Oliveros in [29] defined as

$$\rho_{MHRDE} = \frac{2}{\alpha - \beta} (\dot{H} + \frac{3\alpha}{2} H^2), \quad (10)$$

here $\alpha$ and $\beta$ are constants and in limiting case if $(\alpha = \frac{1}{3}, \beta = 1)$, the Granda and Oliveros IR cut-off reduces to Gao et.al. IR cut-off. Using Eq.(9) and Eq.(10) in Eq.(7) we arrive at

$$H^2 = \frac{2}{3(\alpha - \beta)} (\dot{H} + \frac{3\alpha}{2} H^2) + \frac{1}{6} C^2 a^{-6} \quad (11)$$

and solving the differential equation, the scale factor $a(t)$ is explored as

$$a(t) = \left[ \frac{12C}{C(\alpha - 4)} - \frac{3C(\alpha - 4)}{4} t^2 \right]^\frac{1}{3}. \quad (12)$$

To avoid the singular solution, it is provided that $\alpha \neq 4$. The Hubble parameter $H(t) = \frac{\dot{a}}{a}$ can be evaluated as

$$H(t) = -\frac{Ct(\alpha - 4)}{4 \left( \frac{12C}{C(\alpha - 4)} - \frac{3Ct^2(\alpha - 4)}{4} \right)}. \quad (13)$$

Since, the scale factor has been explored by assuming $\beta = 4$ in Eq.(10) so it takes the form

$$\rho_{MHRDE} = \frac{2}{\alpha - 4} (\dot{H} + \frac{3\alpha}{2} H^2), \quad (14)$$
Using corresponding values of $H(t)$ and $\dot{H}(t)$, the expression for MHRDE density turned out to be

$$\rho_{MHRDE} = \frac{(\alpha - 4)\left(C^4t^2(\alpha - 4)^2(\alpha - 2) - 32C^2C_1\right)}{3\left(C^2t^2(\alpha - 4)^2 - 16C_1\right)^2}. \quad (15)$$

In term of redshift parameter the density is given as

$$\rho(z) = \frac{1}{4}(1 + z)^6[-C(\alpha - 2) + 12C_1(1 + z)^6]. \quad (16)$$

The energy density of this model is increasing for all values of $C$, $\alpha < 2$ and $C_1 > 0$. We plot a graph for different values of these parameters.

![Figure 1: Density vs Redshift Parameter](image)

The graphical behavior of the density is exponentially increasing after $z > 0$ in the context of CS gravity using this MHRDE model.

### 3.1 Deceleration Parameter

The rate of expansion of the universe remained unchanged at constant values of $\dot{a}(t)$ and deceleration term $q$ along with condition imposed on scale factor $a(t)$ that is $a(t) \propto t$, where $t$ is cosmic time. The Hubble parameter $H$ remain constant and deceleration term $q = -1$, when de Sitter and steady-state universes are under consideration. Furthermore, the deceleration parameter
varies with time for some universes available in literature. Using the vari-
ational values of $H$ and $q$, we can classify all the defined universe models
whether, they are in expansion or contraction mode, acceleration or deceler-
ation mode:
1. $H > 0, \ q > 0$, expanding and decelerating
2. $H > 0, \ q = 0$, expanding, zero deceleration
3. $H < 0, \ q = 0$, contracting, zero deceleration
4. $H < 0, \ q > 0$, contracting and decelerating
5. $H < 0, \ q < 0$, contracting and accelerating
6. $H = 0, \ q = 0$, static.

The deceleration parameter $q$ in term of Hubble parameter $H$ is defined as

$$q(t) = -1 - \frac{\dot{H}}{H^2}, \quad (17)$$

where $\dot{H}$ represent the derivative of $H$ with respect to $t$, termed as

$$\dot{H}(t) = -\frac{C^2(\alpha - 4)^2(C^2t^2(\alpha - 4)^2 + 16C_1)}{3(C^2t^2(\alpha - 4)^2 - 16C_1)^2}. \quad (18)$$

Substituting these values in Eq.(17), we find deceleration parameter $q$,

$$q(t) = 2 + \frac{48C_1}{(\alpha - 4)^2C^2t^2}. \quad (19)$$

Representing in the form of redshift, it becomes

$$q(z) = \frac{2[C(\alpha - 4) - 30C_1(1 + z)^6]}{C(\alpha - 4) - 12C_1(1 + z)^6} \quad (20)$$

It’s obvious that the deceleration parameter depends on $C, C_1, \alpha$ and redshift parameter $z$. At present epoch $z = 0$, the deceleration parameter $q < 0$ for each of case given below

1. $\alpha < 4, \ C_1 < 0$ and $\frac{12C_1}{\alpha - 4} < C < \frac{30C_1}{\alpha - 4}$,
2. $\alpha > 4, \ C_1 < 0$ and $\frac{30C_1}{\alpha - 4} < C < \frac{12C_1}{\alpha - 4}$,
3. $\alpha < 4, \ C_1 > 0$ and $\frac{30C_1}{\alpha - 4} < C < \frac{12C_1}{\alpha - 4}$,
4. $\alpha > 4, \ C_1 > 0$ and $\frac{12C_1}{\alpha - 4} < C < \frac{30C_1}{\alpha - 4}$. \quad (21)
For all these conditions, the modal under consideration in CS gravity advocates that the universe is in accelerated expansion phase.

### 3.2 Equation of State Parameter

The nature of component which is dominating universe can be study with the EoS parameter $\omega$. In fact, it illustrate the era of dominance of universe by certain component. For example, $\omega = 0, \frac{1}{3}$ and 1 predict that the universe is under dust, radiation and stiff fluid influence respectively. While $\omega = -\frac{1}{3}, -1$ and $\omega < -1$ stand for quintessence DE, $\Lambda$CDM and Phantom era respectively. Now, differentiating Eq.(15) with respect to time $t$

$$\dot{\rho}(t) = -\frac{2C^4 t(\alpha - 4)^3(C^2 t^2(\alpha - 4)^2(\alpha - 2) + 16C_1(\alpha - 6))}{3(C^2 t^2(\alpha - 4)^2 - 16C_1)^3}. \quad (22)$$

The conservation equation in CS modified gravity in given by [30]

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (23)$$

The expression for the EoS parameter $\omega$ can be explore using Eq.(23) such that

$$\omega(t) = -1 - \frac{\dot{\rho}(t)}{3H(t)\rho(t)}. \quad (24)$$

Making use of Eq.(13), Eq.(18) and Eq.(22) in Eq.(24), we found analytic solution of EoS parameter as given below

$$\omega(t) = -1 - \frac{-2C^2t^2(\alpha - 4)^2(\alpha - 2) - 32C_1(\alpha - 6)}{C^2 t^2(\alpha - 4)^2(\alpha - 2) - 32C_1}. \quad (25)$$

The EoS parameter $\omega$ in term of redshift parameter $z$ look like

$$\omega(z) = \frac{C(\alpha - 2) - 36 C_1(1 + z)^6}{C(\alpha - 2) - 12 C_1(1 + z)^6}. \quad (26)$$

Obviously, EoS parameter $\omega$ is a function of variable redshift parameter $z$ depends on $C$, $C_1$ and $\alpha$. At the present epoch $z = 0$ the EoS is $\omega < -1$ for
each of the case described below

1. $C < 0, \ C_1 < 0$ and $\frac{12C_1 + 2C}{C} < \alpha < \frac{24C_1 + 2C}{C}$,

2. $C < 0, \ C_1 > 0$ and $\frac{24C_1 + 2C}{C} < \alpha < \frac{12C_1 + 2C}{C}$,

3. $C > 0, \ C_1 < 0$ and $\frac{24C_1 + 2C}{C} < \alpha < \frac{12C_1 + 2C}{C}$,

4. $C > 0, \ C_1 > 0$ and $\frac{12C_1 + 2C}{C} < \alpha < \frac{24C_1 + 2C}{C}$.

For different values of these parameters, the EoS $\omega < -1$ favor the fact that universe is dominated by DE.

4 Study of MHRDE Model Using Scaler Field Models

In this section, we discuss different scalar field models like quintessence, tachyon, K-essence and dilaton models in the framework of CS modified gravity. To study, the behavior of quantum gravity, we explore the potential and scalar field.

4.1 Quintessence Model

A DE model is developed to explain the late-time cosmic acceleration called quintessence, is a simplest scalar field which have no theoretical problem like ghosts and Laplacian instabilities appearance [32]. This model is useful to settle down the issue of fine tuning in cosmology considering the time dependent EoS. Using this model, we can explain the cosmic acceleration having negative pressure when potential energy dominates the kinetic energy. The energy and pressure densities are defined as

$$\rho_Q = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_Q = \frac{1}{2} \dot{\phi}^2 - V(\phi),$$

where the scalar field $\phi$ is differentiated with respect to $t$. The EoS parameter for the quintessence is

$$\omega_\phi = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}. \quad (29)$$
The comparison of EoS $\omega_\phi$ formulated for quintessence model given in Eq.(29) and EoS $\omega$ calculated for MHRDE modal given in Eq.(25) turned as

$$\frac{1}{2}\dot{\phi}^2 - V(\phi) = \frac{C^2 t^2 (\alpha - 4)^2 (\alpha - 2) + 32C_1(\alpha - 5)}{C^2 t^2 (\alpha - 4)^2 (\alpha - 2) - 32C_1}. \quad (30)$$

Now, equating density of quintessence model given in Eq.(28) and density evaluated from MHRDE model represented in Eq.(15) expressed as

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{(\alpha - 4)
\left(C^4 t^2 (\alpha - 4)^2 (\alpha - 2) - 32C_2C_1\right)}{3\left(C^2 t^2 (\alpha - 4)^2 - 16C_1\right)^2}. \quad (31)$$

Using Eq.(30) and (31), we arrive at

$$\dot{\phi}^2 = \frac{2C^2(\alpha - 4)
\left(C^2 t^2 (\alpha - 4)^2 (\alpha - 2) + 16C_1(\alpha - 6)\right)}{3\left(C^2 t^2 (\alpha - 4)^2 - 16C_1\right)^2} \quad (32)$$

and integrating the last equation with respect to $t$

$$\phi(t) = \sqrt{\frac{2}{3}} \left[-\sqrt{2} \tanh^{-1} \left(\frac{\sqrt{2} Ct(\alpha - 4)^{\frac{3}{2}}}{\sqrt{C^2 t^2 (\alpha - 4)^2 (\alpha - 2) + 16C_1(\alpha - 6)}}\right)\right]$$

$$+ \sqrt{\frac{\alpha - 2}{\alpha - 4}} \ln \left[C\left(C t(\alpha^2 - 2\alpha + 8) + \sqrt{\alpha - 2}\right.)\right.$$ 

$$\times \sqrt{C^2 t^2 (\alpha - 4)^2 (\alpha - 2) + 16C_1(\alpha - 6)}} \right]. \quad (33)$$

In terms of redshift parameter $z$, it turns out to be

$$\phi(z) = -\frac{2}{3(\alpha - 4)} \tanh^{-1} \left[\frac{(\alpha - 4)^{\frac{3}{2}}}{\sqrt{8 - 2\alpha}} \sqrt{\frac{\left(\alpha - 2\right)^3}{\left(\alpha - 4\right)^3}} \right]$$

$$+ \frac{\sqrt{(8 - 6\alpha + \alpha^2)}}{2\sqrt{4 - \alpha}} \log \left[\frac{2C}{\sqrt{\alpha - 2}} \sqrt{\frac{(\alpha - 4)(-c(\alpha - 2) + 24(1 + z)^6C_1)}{(1 + z)^6}}\right] $$

$$+ \frac{\sqrt{\left(8 - 6\alpha + \alpha^2\right)}}{2\sqrt{4 - \alpha}} \sqrt{\frac{\left(\alpha - 4\right)^3}{(1 + z)^6}} + \frac{4C_1}{c(4 - \alpha)} \right]. \quad (34)$$

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Again using Eq.(30) and Eq.(31), the potential for quintessence model can be explored as

$$V(t) = -\frac{16C^2C_1(\alpha - 4)^2}{3(C^2t^2(\alpha - 4)^2 - 16C_1)^2},$$  \hspace{1cm} (35)$$

and making it convenient to discuss we change into redshift parameter

$$V(z) = -3C_1(1 + z)^{12}.$$  \hspace{1cm} (36)

It is obvious that the potential of quintessence model depends only on the values of constant of integration $C_1$. It shows the increasing behavior for all $C_1 < 0$ and decreasing for $C_1 > 0$. It is interesting that the parameters $C$ and $\alpha$ are not appeared in the final expression of potential in the framework of CS Modified gravity.

### 4.2 Tachyon Model

Much attention have been given to tachyon field models in last few decades in string theory and cosmology [33]-[39]. In fact, isotropic cosmological models whose radius depends on time and their potential can be constructed using minimally coupled scalar field model [40]. Since the same procedure for the correspondence between minimally coupled scalar field models and tachyon can be utilize to study the similar cosmological evolution [39]. The energy and pressure densities for the tachyon field model are expressed as

$$\rho = \frac{V(\phi)}{\sqrt{1 - \dot{\phi}^2}}, \hspace{1cm} p = -V(\phi)\sqrt{1 - \dot{\phi}^2}. \hspace{1cm} (37)$$

Since $p = \rho \omega$, using the above expressions, the EoS parameter can be evaluated as

$$\omega = \dot{\phi}^2 - 1.$$ \hspace{1cm} (38)

The comparison of Eq.(25) with Eq.(38) gives kinetic energy $\phi(t)$ such that

$$\phi(t) = \frac{C_1(6 - \alpha)}{\sqrt{C^2(\alpha - 4)^2(\alpha - 2)}} E[\arcsin(t\sqrt{\frac{C^2(\alpha - 4)^2(\alpha - 2)}{32C_1}}); \frac{2}{\alpha - 6}]. \hspace{1cm} (39)$$

The tachyon potential in this case is

$$V(t) = \frac{C^2(\alpha - 4)}{3(C^2t^2(\alpha - 4)^2 - 16C_1)^2} \times \sqrt{[C^2t^2(\alpha - 4)^2(\alpha - 2) - 32C_1][C^2t^2(\alpha - 4)^2(\alpha - 2) + 32C_1(\alpha - 5)]}. \hspace{1cm} (40)$$
Now, the kinetic and potential energies of tachyon model in term of redshift parameter $z$ respectively,

$$\phi(z) = \frac{4\sqrt{2C_1(\alpha - 6)}}{\sqrt{C^2(\alpha - 2)(\alpha - 4)^2}} EllipticE \left\{ \sin^{-1} \left[ \frac{\sqrt{\frac{C^2(\alpha - 2)(\alpha - 4)^2}{C_1^2}} \left( \frac{4}{(1+z)^6} + \frac{48C_1}{C(4-\alpha)} \right)}{4\sqrt{6C(4-\alpha)}} \right] - \frac{2}{\alpha - 6} \right\} \quad (41)$$

and

$$V(z) = -\frac{1}{4}(1+z)^6 \left( C(\alpha - 2) - 12(1+z)^6C_1 \right) \sqrt{1 - \frac{1}{2}(1+z)^6 \left( -c(\alpha - 2) + 24(1+z)^6C_2 \right)} \quad (42)$$

To investigate the behavior of potential, we plot a graph of $V(z)$ vs $z$

![Figure 2: Potential vs Redshift Parameter](image)

The graph plotted for the potential $V(z)$ of tachyon model against redshift $z$ parameter shows the decreasing behavior irrespective of the values of parameters $\alpha$, $C$ and $C_1$. This graph is plotted by taking particular values of these parameters to elaborate the result.
4.3 K-essence

Armendariz et al. [21] introduced the dynamical concept of k-essence to explain the fact of accelerated expansion of universe. This model solves the fine tuning problem of parameters. Actually, k-essence is developed on the principle of dynamical attractor solution that's why it works as cosmological constant at the onset of matter domination. The energy and pressure densities of this model are given as

\[ \rho = V(\phi)(-X + 3X^2), \quad p = V(\phi)(-X + X^2). \]  

(43)

Where \( X = \frac{\dot{\phi}^2}{2} \), the EoS parameter for this model is

\[ \omega = \frac{1 - X}{1 - 3X}. \]  

(44)

Equating Eq.(25) and Eq.(44), we obtain

\[ X(t) = \frac{16C_1(\alpha - 4)}{C^2t^2(\alpha - 4)^2(\alpha - 2) + 16C_1(3\alpha - 14)}. \]  

(45)

Since \( X(t) = \frac{\dot{\phi}^2}{2} \), the integration of the above equation provides \( \phi(t) \)

\[ \phi(t) = \frac{4C_1}{\sqrt{(\alpha - 4)(\alpha - 2)}} \ln[C(t^2 - 6\alpha + 8) + \sqrt{\alpha - 2}\sqrt{C^2t^2(\alpha - 4)^2(\alpha - 2) + 16C_1(3\alpha - 14)}]. \]  

(46)

The kinetic energy \( \phi \) in term of redshift parameter \( z \) is turned as

\[ \phi(z) = \frac{4C_1}{c\sqrt{\alpha - 2}\sqrt{(\alpha - 4)C_1}} \times \log \left[ \frac{2C}{\sqrt{3}} \left( \frac{\alpha - 4}{(1 + z)^6} \right)^{\frac{1}{6}} - \frac{48C_1}{C(4 - \alpha)} \sqrt{\frac{8 - 6\alpha + \alpha^2}{} \sqrt{4} + \frac{48C_1}{C(4 - \alpha)}} \right]. \]  

(47)

The k-essence potential is calculated using Eq.(43),(44) and Eq.(25) as

\[ V(t) = -\frac{C^2(C^2t^2(\alpha - 4)^2(\alpha - 2) + 16C_1(3\alpha - 14))^2}{48C_1(C^2t^2(\alpha - 4)^2 - 16C_1)^2}. \]  

(48)
Now, we convert this function in term of redshift parameter $z$ and investigate its behavior.

$$V(z) = -\frac{\left(c(\alpha - 2) - 48(1 + z)^6C_1\right)^2}{48C_2}.$$  
(49)

The potential $V(z)$ is increasing function for all values of $C$ and $\alpha$ at value of constant of integration $C_1 = -1$. We plot a graph for particular values of these parameters just for example. After present epoch $z = 0$, it increasing exponentially.

![Figure 3: Potential vs redshift parameter](image)

4.4 Dilaton Model

The negative kinetic energy of the phantom field creates the problem of quantum instability. To resolve this puzzle of instability, dilaton model is proposed and further it used to study the nature of DE. The dilaton model is defined as 4-dimensional effective low-energy model in the context of string theory. The pressure and energy densities are presented as

$$\rho = -X + c_1 e^{\lambda \phi} X^2, \quad p = -X + 3c_1 e^{\lambda \phi} X^2.$$  
(50)
where $X = \frac{\dot{\phi}^2}{2}$, $c_1$ and $\lambda$ are positive constant. The EoS parameter $\omega = \frac{\rho}{p}$ for these densities is calculated as

$$\omega = \frac{1 - c_1 e^{\lambda \phi}}{1 - 3c_1 e^{\lambda \phi}}. \quad (51)$$

The comparison of Eq.(25) with Eq.(51) yields as

$$c_1 e^{\lambda \phi} \dot{\phi}^2 = \frac{16C_1(\alpha - 4)}{C^2t^2(\alpha - 4)^2(\alpha - 2) + 16C_1(3\alpha - 14)}. \quad (52)$$

Solving for $\phi(t)$, we reached at

$$\phi(t) = \frac{2}{\lambda} \ln \left[ \sqrt{16C_1} \sinh^{-1} \left( t \sqrt{\frac{C^2(\alpha - 4)^2(\alpha - 2)}{16C_1(3\alpha - 14)}} \right) \right]. \quad (53)$$

In term of redshift parameter $z$

$$\phi(z) = \frac{2}{\lambda} \log \left[ \sqrt{\frac{16C_1}{C^2(\alpha - 2)}} \sinh^{-1} \left( \sqrt{\frac{C^2(\alpha - 4)^2(\alpha - 2)}{C_1(3\alpha - 14)}} \sqrt{\frac{4}{(1+z)^\alpha} + \frac{48C_1}{C(4-\alpha)}} \right) \right]. \quad (54)$$

Analytically, it is found that there is no combination of parameters $\alpha, C, C_1$ for which this function is defined.

5 Summary and Discussion

This work is devoted to study the cosmological analysis of MHRDE model in the context of CS modified gravity. The energy density for this model is calculated and observed in Fig.1. From the graph, it is obvious that density of the universe for MHRDE model is increasing for all values of CS modified gravity constant $C$, $\alpha < 2$ and $C_1 > 0$. The deceleration parameter $q < 0$ for the different combination of $C$, $C_1$ and $\alpha$ which advocates the accelerating expansion. The EoS parameter $\omega < -1$ is found which favors that DE is dominant at present epoch in case of MHRDE model in the context of CS modified gravity.

Futhermore, we reconstruct different scalar filed models using MHRDE in the context of dynamical CS modified gravity and found interesting results plotting them graphically. It is obvious that the potential of quintessence
model depends only on the value of constant of integration $C_1$. It shows the increasing behavior for all $C_1 < 0$ and decreasing for $C_1 > 0$. It is interesting that the potential in Eq.(36) is independent of CS parameter $C$ and MHRDE parameter $\alpha$ identically, although, we are working in the frame work of CS Modified gravity using MHRDE model. The graph plotted in Fig.2 for the potential of tachyon model shows the exponentially decreasing behavior irrespective of the values of parameters $\alpha$, $C$ and $C_1$. In case of k-essence, the potential is increasing function for all values of $C$ and $\alpha$ at particular value of $C_1 = -1$. After present epoch $z = 0$, the graph increasing exponentially given by Fig.3. Analytically, it is found that there is no combination of parameters $\alpha$, $C$, $C_1$ for which $\phi(z)$ is defined in case of Dilaton model.

**Acknowledgment** We acknowledge the remarkable assistance of the Higher Education Commission Islamabad, Pakistan. I would like to thanks Department of Physics and Astronomy University of British Columbia Canada for giving me space in their Astro Lab for six months as Visiting International Research Scholar. I really thankful to Dr. Douglus Scott for his supervision and valuable suggestions on this work.

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