SCHUR ALGORITHM FOR STIELTJES INDEFINITE MOMENT PROBLEM

VLADIMIR DERKACH AND IVAN KOVALYOV

Abstract. Nondegenerate truncated indefinite Stieltjes moment problem in the class $N^k_\kappa$ of generalized Stieltjes functions is considered. To describe the set of solutions of this problem we apply the Schur step-by-step algorithm, which leads to the expansion of these solutions in generalized Stieltjes continuous fractions studied recently in [16]. Explicit formula for the resolvent matrix in terms of generalized Stieltjes polynomials is found.

1. Introduction

Classical Stieltjes moment problem consists in the following: given a sequence of real numbers $s_j$ ($j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$) find a positive measures $\sigma$ with a support on $\mathbb{R}_+$, such that

$$\int_{\mathbb{R}_+} t^j d\sigma(t) = s_j \quad (j \in \mathbb{Z}_+). \quad (1.1)$$

In [15] T. Stieltjes described piecewise solutions $\sigma$ of this problem in connection with small vibration problem for a massless thread with a countable set of point masses. Full description of all positive measures $\sigma$, which satisfy (1.1), was given by M.G. Krein in [21]. The problem (1.1), when $\sigma$ is recovered from a finite sequence $\{s_j\}_{j=0}^{2n}$ is called the truncated Stieltjes moment problem and was studied in [20].

By the Hamburger–Nevanlinna theorem [2] the truncated Stieltjes moment problem can be reformulated in terms of the Stieltjes transform

$$f(z) = \frac{d\sigma(t)}{t-z}, \quad z \in \mathbb{C}\setminus\mathbb{R}_+, \quad (1.2)$$

of $\sigma$ as the following interpolation problem at $\infty$

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \quad z \to \infty. \quad (1.3)$$

The notation $z \to \infty$ means that $z \to \infty$ nontangentially, that is inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$. It follows easily from (1.1) that the inequalities

$$S_{n+1}^+ := \left(s_{i+j}\right)_{i,j=0}^n \geq 0, \quad S_n^+ := \left(s_{i+j+1}\right)_{i,j=0}^{n-1} \geq 0 \quad (1.4)$$

are necessary for solvability of the moment problem (1.3). If the matrices $S_{n+1}$ and $S_n^+$ are nondegenerate, then the inequalities $S_{n+1} > 0$, $S_n^+ > 0$ are also sufficient.

1991 Mathematics Subject Classification. Primary 30E05; Secondary 15B57, 46C20, 47A57.

Key words and phrases. Stieltjes moment problem, Continued fraction, Generalized Stieltjes fraction, Schur algorithm, Solution matrix.

This work was supported by grants of Ministry of Education and Science of Ukraine (project numbers: 0115U000136, 0115U000556); V.D. is indebted to the German Science Foundation (DFG) for support under Grant TR 903/16-1.
for solvability of the moment problem \cite{13}, see \cite{20}. The degenerate case is more subtle and was studied in \cite{4}.

The function \( f \) in \cite{12} belongs to the class \( N \) of functions holomorphic on \( \mathbb{C}\setminus \mathbb{R} \) with nonnegative imaginary part in \( \mathbb{C}_+ \) and such that \( f(\overline{z}) = \overline{f(z)} \) for \( z \in \mathbb{C}_+ \).

Moreover, \( f \) belongs to the Stieltjes class \( S \) of functions \( f \in \mathbb{N} \), which admit holomorphic and nonnegative continuation to \( \mathbb{R}^- \). By M.G. Krein criterion, see \cite{22}

\begin{equation}
(1.5)
\quad f \in S \iff f \in N \quad \text{and} \quad zf \in N.
\end{equation}

Indefinite version of the class \( N \) was introduced in \cite{22}.

**Definition 1.1.** \cite{22} A function \( f \) meromorphic on \( \mathbb{C}\setminus \mathbb{R} \) with the set of holomorphy \( \mathbb{R}_f \) is said to be in the generalized Nevanlinna class \( N_\kappa \) \( (\kappa \in \mathbb{N}) \), if for every set \( z_j \in \mathbb{C}_+ \cap \mathbb{R}_f \) \( (j = 1, \ldots, n) \) the form

\[
\sum_{i,j=1}^{n} \frac{f(z_i) - f(z_j)}{z_i - z_j} \bar{z}_i \bar{z}_j
\]

has at most \( \kappa \) and for some choice of \( z_j \) \( (j = 1, \ldots, n) \) exactly \( \kappa \) negative squares.

The generalized Stieltjes class \( N_\kappa^+ \) was defined in \cite{23} as the class of functions \( f \in N_\kappa \), such that \( zf \in N \). Similarly, in \cite{8, 9} the class \( N_\kappa^k \) \( (\kappa, k \in \mathbb{N}) \) was introduced as the set of functions \( f \in N_\kappa \), such that \( zf \in N_k \).

In \cite{24} the moment problem in the class \( N_\kappa \) \( (N_\kappa^+ \) was considered in the following setting: Given a real sequence \( \{s_j\}_{j=0}^{\infty} \), find \( f \in N_\kappa \) \( (N_\kappa^+ \) such that \( 1.3 \) holds for every \( n \in \mathbb{N} \). In particular, it was shown in \cite{24} that the problem \( 1.3 \) is solvable in \( N_\kappa^+ \) if the number \( \nu_-(S_n) \) of negative eigenvalues of \( S_n \) does not exceed \( \kappa \) for all \( n \) big enough and \( S_n^+ > 0 \) for all \( n \in \mathbb{N} \). The indefinite moment problem in generalized Stieltjes class \( N_\kappa^k \) was studied in \cite{10}.

In the present paper we consider the following truncated indefinite moment problem.

**Problem** \( MP^k_\kappa(s, \ell) \). Given \( \ell, \kappa, k \in \mathbb{Z}_+ \), and a sequence \( s = \{s_j\}_{j=0}^{\ell} \) of real numbers, describe the set \( M^k_\kappa(s) \) of functions \( f \in N_\kappa^k \), which satisfy the asymptotic expansion

\begin{equation}
(1.6)
\quad f(z) = \frac{s_0}{z} - \cdots - \frac{s_{\ell}}{z^{\ell+1}} + o\left( \frac{1}{z^{\ell+1}} \right) \quad (z = iy, \; y \uparrow \infty).
\end{equation}

Such a moment problem is called even or odd regarding to the oddness of the number \( \ell + 1 \) of moments. To study this problem we use the Schur algorithm, which was elaborated in \cite{5, 6} and \cite{11} for the class \( N_\kappa \). Let us explain it for the even case, i.e. when \( s = \{s_i\}_{i=0}^{2n-1} \). Recall, that a number \( s_j \in \mathbb{N} \) is called a normal index of the sequence \( s \), if \( \det S_{s_j} \neq 0 \). The ordered set of normal indices

\[
n_1 < n_2 < \cdots < n_N
\]

of the sequence \( s \) is denoted by \( N(s) \). For every \( n_j \in N(s) \) polynomials of the first and the second kind \( P_{n_j}(z) \) and \( Q_{n_j}(z) \) can be defined by standard formulas, see \cite{5, 11}. A sequence \( s \) is called regular (see \cite{10}), if

\begin{equation}
(1.7)
\quad P_{n_j}(0) \neq 0 \quad \text{for} \quad (1 \leq j \leq N).
\end{equation}

The latter condition is equivalent to the condition \( \det S_{n_j}^+ \neq 0 \) for all \( j = 1, \ldots, N \).
If the set \( \mathcal{N}(s) \) consists of \( N \) indices \( \mathcal{N}(s) = \{n_j\}_{j=1}^N \) and \( n = n_N \), a function \( f \in \mathcal{M}^k_N(s) \) can be expanded into a \( P \)-fraction

\[
(1.8) \quad f = \frac{-b_0}{a_0(z) - \frac{b_1}{a_1(z) - \cdots - \frac{b_{N-1}}{a_{N-1}(z) + \tau(z)}}},
\]

where \( b_j \neq 0 \) are real numbers and \( a_j \) are monic polynomials of degree \( k_j = n_{j+1} - n_j \), by using \( N \) steps of the Schur algorithm, see [6]. \( P \)-fractions were introduced and studied in [26], see also [27]. In the present paper we show that for \( f \in \mathcal{M}^k_N(s) \) with regular \( s \) one step of the Schur algorithm can be splitted into two substeps, which lead to the following representation of \( f \)

\[
(1.9) \quad f(z) = \frac{1}{-zm_1(z) + \frac{1}{l_1 + f_1(z)}},
\]

where \( m_1(z) \) is a polynomial, \( l_1 \in \mathbb{R}\setminus\{0\}, f_1 \in \mathbb{N}^{k-k_1}_{\kappa-k_1}, \) and \( \kappa_1 = \nu_\lambda(S_{n_1}), k_1 = \nu_{\lambda}(S_{n_1}^\nu) \). By iterating this process, we show that for \( s = \{s_i\}_{i=0}^{2N-1} \) and \( \mathcal{N}(s) = \{n_{j}\}_{j=1}^N \) the problem \( MP^k_N(s) \) is solvable, if and only if

\[
(1.10) \quad \kappa_N \coloneqq \nu_\lambda(S_N) \leq \kappa, \quad k_N \coloneqq \nu_{\lambda}(S_N^\nu) \leq k,
\]

and every solution \( f \in \mathcal{M}^k_N(s) \) admits the representation as the continued fraction

\[
(1.11) \quad f(z) = \frac{1}{-zm_N(z) + \frac{1}{l_N + \tau(z)}},
\]

where \( m_j \) are polynomials, \( l_j \in \mathbb{R}\setminus\{0\} \) and \( \tau \) is a parameter function from some generalized Stieltjes class \( \mathbb{N}^{k-k_N}_{\kappa-k_N} \), such that \( \tau(z) = o(1) \) as \( z \to \infty \). Such continued fractions were studied in [16].

Associated with the continued fraction (1.11) is a system of difference equations

\[
(1.12) \quad \begin{cases} y_{2j} - y_{2j} - 2 = l_j y_{2j} - 1, \\ y_{2j+1} - y_{2j} - 1 = -zm_{j+1}(z) y_{2j} \end{cases}
\]

see [28] Section 1]. Following [15] (see also [25] Section 5.3], [12] we introduce Stieltjes polynomials \( P_j^+ \) and \( Q_j^+ \) in such a way, that \( u_j = Q_j^+ \) and \( v_j = P_j^+ \) are solutions of the system (1.12) subject to the initial conditions

\[
(1.13) \quad u_{-1} \equiv -1, \quad u_0 \equiv 0; \quad v_{-1} \equiv 0, \quad u_0 \equiv 1.
\]

This implies that the convergents \( \frac{u_j}{v_j} \) of the continued fraction (1.11) take the form

\[
(1.14) \quad \frac{u_j}{v_j} = \frac{Q_j^+}{P_j^+} \quad (j = 1, \ldots, 2N).
\]

In view of (1.14) the representation (1.11) can be rewritten as

\[
(1.15) \quad f(z) = \frac{Q_{2N-1}^+(z) \tau(z) + Q_{2N}^+(z)}{P_{2N-1}^+(z) \tau(z) + P_{2N}^+(z)},
\]
Moreover, the solution matrix, i.e. the $2 \times 2$ matrix $W_{2N}(z)$ of coefficients of the linear-fractional transform (1.15) admits the factorization

$$W_{2N}(z) = \begin{pmatrix} Q^+_1(z) & Q^+_2(z) \\ P^+_1(z) & P^+_2(z) \end{pmatrix} = M_1(z)L_1 \ldots M_N(z)L_N,$$

where the matrices $M_j$ and $L_j$ are defined by

$$M_j(z) = \begin{pmatrix} 1 & 0 \\ -zm_j(z) & 1 \end{pmatrix}, \quad L_j = \begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix} \quad j = 1, N.$$

In the case when the sequence $s$ satisfies the conditions

$$S_N > 0, \quad S^+_N > 0,$$

$s$ is automatically regular in the sense of (1.7) and $m_j$, $l_j$ are positive numbers.

In this case the system (1.12) describes small vibrations of a massless thread with masses $m_j$ and distances $l_j$ between them, see [2, Appendix]. The case, when $S^+_N > 0$ and $\mu_+(S_N) > 0$ was studied by M.G Krein and H. Langer [24]. In this case it may happen that $m_j$ is either a negative real or even a polynomial of degree 1, and the system (1.12) was interpreted in [24] as a generalized Stieltjes string with negative masses and dipoles. In the general case, when $s$ is a regular sequence and all $l_j$ are positive, one can treat system (1.12) as a generalized Stieltjes string with multipoles, cf. [10].

Continued fractions of the form (1.11) with negative masses $m_j$ were studied by Beals, Sattinger and Szmigielski [3] in connection with the theory of multi-peakon solutions of the Camassa-Holm equation. In particular, they noticed that in the indefinite case, the inverse problem is not always solvable in the class of such continued fractions. In [17] it was shown that the inverse spectral problem for multi-peakon solutions of the Camassa-Holm equation is solvable in the class of continued fractions of the form (1.11) with polynomials $m_j(z) = d_jz + m_j$ of formal degree 1 ($d_j \geq 0, m_j \in \mathbb{R}$). These result is in the full correspondence with the description of solutions of the Stieltjes indefinite moment problem given in [24].

A description of the set of solutions of odd Stieltjes moment problem, corresponding to a sequence $s = \{s_j\}_{j=0}^{2n-2}$, is also found in a form similar to (1.15). If $N(s) = \{n_j\}_{j=1}^N$ and $n = \sum_{j=1}^N n_j$, then the factorization formula for the corresponding solution matrix $W_{2N-1}$ takes the form

$$W_{2N-1}(z) = M_1(z)L_1 \ldots L_{N-1}M_N(z).$$

In the case of a non-regular sequence $s$ every solution $f \in \mathcal{M}_k(s)$ admits an expansion in a continued fraction of type (1.11), where $l_j$ are polynomials. The corresponding results will be published elsewhere. Notations in the present paper are quite tricky: all the objects which appear on the $j$-th step are endowed with the index $j$, regardless to the substep. To make difference between substeps, the moments which appear on the 1-st substep are denoted by Fraktur script, while moments which appear on the 2-nd substep are denoted by Latin script. The only exception is made for the solution matrix - the solution matrix, corresponding to an odd Stieltjes moment problem is denoted by $W_{2N-1}$, while solution matrix, corresponding to an even Stieltjes moment problem is denoted by $W_{2N}$.

Now, briefly describe the content of the paper. Section 2 contains some preliminary statements concerning the class $\mathcal{N}^k_\mathcal{R}$ of generalized Stieltjes functions, class
2. Preliminaries

2.1. Generalized Nevanlinna and Stieltjes classes. The class $N_\kappa$, introduced in Definition 1.1, is called the generalized Nevanlinna class. For $f \in N_\kappa$ let us write $\kappa-(f) = \kappa$. In particular, if $\kappa = 0$ then the class $N_0$ coincides with the class $N$ of Nevanlinna functions (see [25]).

Every real polynomial $P(t) = p_0 t^\nu + p_{\nu-1} t^{\nu-1} + \ldots + p_1 t + p_0$ of degree $\nu$ belongs to a class $N_\kappa$, where the index $\kappa = \kappa-(P)$ can be evaluated by (see [25, Lemma 3.5])

\[
\kappa-(P) = \begin{cases} 
\left\lfloor \frac{\nu+1}{2} \right\rfloor, & \text{if } p_\nu < 0; \text{ and } \nu \text{ is odd}; \\
\left\lfloor \frac{\nu}{2} \right\rfloor, & \text{otherwise}.
\end{cases}
\]

Denote by $\nu_-(S)$ ($\nu_+(S)$) the number of negative (positive, resp.) eigenvalues of the matrix $S$. Let $H$ be the set of finite real sequences $s = \{s_j\}_{j=0}^\ell$ and let $H_{\kappa,\ell}$ be the set of sequences $s = \{s_j\}_{j=0}^\ell \in H$, such that

\[
\nu_-(s_n) = \kappa \quad (n = \left\lfloor \ell/2 \right\rfloor + 1)
\]

where $S_n$ is defined by (1.4). The index $\nu_-(s_n)$ for a Hankel matrix $S_n$ can be calculated by the Frobenius rule (see [18, Theorem X.24]). In particular, if all the determinants $D_j := \det S_j$ ($j \in \mathbb{Z}_+$) do not vanish, then $\nu_-(s_n)$ coincides with the number of sign alterations in the sequence

\[
D_0 := 1, \quad D_1, \quad D_2, \ldots, \quad D_n.
\]

Let us remind some statements concerning the classes $N_\kappa$ and $H_{\kappa,\ell}$ from [23,24].

Proposition 2.1. (23) Let $f \in N_\kappa$, $f_1 \in N_{\kappa_1}$, $f_2 \in N_{\kappa_2}$. Then

1. $-f^{-1} \in N_{\kappa}$;
2. $f_1 + f_2 \in N_{\kappa'}$, where $\kappa' \leq \kappa_1 + \kappa_2$;
3. If, in addition, $f_1(iy) = o(y)$ as $y \to \infty$ and $f_2$ is a polynomial, then
\[
f_1 + f_2 \in N_{\kappa_1 + \kappa_2}.
\]
4. If a function $f \in N_\kappa$ has an asymptotic expansion (1.6), then there exists $\kappa' \leq \kappa$, such that $\{s_j\}^\ell_{j=0} \in H_{\kappa',\ell}$.

Recall, that a Nevanlinna function $f$ is said to be from the Stieltjes class $S^+$ ($S^-$), if it admits a holomorphic and nonnegative (nonpositive, resp.) continuation to the negative half-line. By the M.G. Krein criterion (21)

\[
f \in S^\pm \iff f \in N \text{ and } z^{\pm 1} f(z) \in N.
\]

The following generalization of the class $S^+$ was introduced in [8,10]. A function $f \in N_\kappa$ is said to be from the generalized Stieltjes class $N_{\kappa}^{\pm k}$, if $z^{\pm 1} f(z) \in N_k$. 

$U_\kappa(J)$ of generalized $J$-unitary matrix functions, normal indices of finite real sequences and some inversion formulas for asymptotic expansions. Solutions to odd and even basic moment problems will be described in Section 3. Section 4 presents a general Schur recursion algorithm, which allows to parametrize solutions of odd and even Stieltjes indefinite moment problems $MP_\kappa(s,2n_N-2)$ and $MP_\kappa(s,2n_N-1)$, respectively. Factorization formulas for solution matrices $W_{2N-1}$ and $W_{2N}$ for odd and even Stieltjes indefinite moment problems based on the Schur algorithm are found. In Section 5 we introduce Stieltjes polynomials and find explicit formulas for solution matrices $W_{2N-1}$ and $W_{2N}$ in terms of Stieltjes polynomials.
In the case $\kappa = k = 0$ the class $N^\pm_{00}$ coincides with the class $S^\pm$. The classes $N^\pm_{\kappa} := N^\pm_{0\kappa}$ and $S^\pm k := N^\pm_{0k}$ were studied in $[24]$ and $[11, 13]$, respectively.

Denote by $H^k_{\kappa, \ell}$ the set of real sequences $s = \{s_j\}_{j=0}^\ell \in H^k_{\kappa, \ell}$, such that $\{s_{j+1}\}_{j=0}^{\ell-1} \in H^k_{\kappa, \ell-1}$, i.e.,

\[(2.4) \quad \nu_-(S_{(\ell+1)/2}) = k.\]

**Proposition 2.2.** (23) The following equivalences hold:

1. $f \in N^k_{\kappa} \iff \frac{1}{f} \in N^{-k}_{\kappa};$
2. $f \in N^k_{\kappa} \iff zf(z) \in N^{-k}_{\kappa}$, in particular, $f \in N^k_{\kappa} \iff zf(z) \in S^{-\kappa};$
3. If a function $f \in N^k_{\kappa}$ has an asymptotic expansion (1.6) then

\[(2.5) \quad \{s_j\}_{j=0}^\ell \in H^\kappa_{\kappa', \ell} \quad \text{with} \quad \kappa' \leq \kappa, \quad k' \leq k.\]

2.2. **Normal indices.** Let $\mathcal{N}(s) = \{n_j\}_{j=1}^N$ be the set of normal indices of the sequence $s = \{s_j\}_{j=0}^\ell$, defined by the properties

\[(2.6) \quad \det S_{n_j} \neq 0 \quad (j \in \{1, 2, \ldots, N\}).\]

and enumerated in the increasing order. It follows from the Sylvester identity (see [16, Proposition 3.1]), that $\mathcal{N}(s)$ is the union of two not necessarily disjoint subsets

\[(2.7) \quad \mathcal{N}(s) = \nu_j \{1, 2, \ldots, N\} \cup \mu_j \{1, 2, \ldots, N\},\]

which are selected by

\[(2.8) \quad \det S_{\nu_j} \neq 0 \quad \text{and} \quad \det S^+_{\nu_j-1} \neq 0, \quad \text{for all} \quad j = 1, N_1\]

and

\[(2.9) \quad \det S_{\mu_j} \neq 0 \quad \text{and} \quad \det S^+_{\mu_j-1} \neq 0, \quad \text{for all} \quad j = 1, N_2.\]

Moreover, the normal indices $\nu_j$ and $\mu_j$ satisfy the following inequalities

\[(2.10) \quad 0 < \nu_1 \leq \mu_1 < \nu_2 \leq \mu_2 < \ldots\]

**Corollary 2.3.** If a function $f \in N^k_{\kappa}$ has an asymptotic expansion (1.6) with $\ell = 2\mu_j - 1$ and $\mu_j$ satisfy (2.9), then

\[(2.11) \quad \nu_-(S_{\mu_j}) \leq \kappa, \quad \nu_-(S^+_{\mu_j}) \leq k.\]

If a function $f \in N^k_{\kappa}$ has an asymptotic expansion (1.6) with $\ell = 2\nu_j - 2$ and $\nu_j$ satisfy (2.8), then

\[(2.12) \quad \nu_-(S_{\nu_j}) \leq \kappa, \quad \nu_-(S^+_{\nu_j-1}) \leq k.\]

Notice, that the number $\nu_1$ can be found from the conditions

\[(2.13) \quad s_0 = \ldots = s_{\nu_1-2} = 0, \quad s_{\nu_1-1} \neq 0,\]

since for such $\nu_1$ one has $\det S_i = 0$ for $i \leq \nu_1$ and

\[(2.14) \quad \det S_{\nu_1} \neq 0 \quad \text{and} \quad \det S^+_{\nu_1-1} \neq 0.\]

Therefore, the first normal index of $s$ coincides with $\nu_1$, i.e. $n_1 = \nu_1$.  

2.3. Toeplitz matrices and asymptotic expansions. A sequence \((c_0, \ldots, c_n)\) of real numbers determines an upper triangular Toeplitz matrix \(T(c_0, \ldots, c_n)\) of order \((n + 1) \times (n + 1)\) with entries \(t_{i,j} = c_{j-i}\) for \(i \leq j\) and \(t_{i,j} = 0\) for \(i > j\):

\[
T(c_0, \ldots, c_n) = \begin{pmatrix}
    c_0 & \cdots & c_n \\
    \vdots & \ddots & \vdots \\
    c_0 & & 
\end{pmatrix},
\]

Some of the calculations of the present paper will be based on the following

**Lemma 2.4.** Let the functions \(c\) and \(d\) (meromorphic on \(\mathbb{C} \setminus \mathbb{R}\)) have the asymptotic expansions

\[
c(z) = c_0 + \frac{c_1}{z} + \cdots + \frac{c_n}{z^n} + o\left(\frac{1}{z^n}\right), \quad z \to \infty;
\]

\[
d(z) = d_0 + \frac{d_1}{z} + \cdots + \frac{d_n}{z^n} + o\left(\frac{1}{z^n}\right), \quad z \to \infty.
\]

and let \(c(z)d(z) = 1\). Then the Toeplitz matrices \(T(c_0, \ldots, c_n)\) and \(T(d_0, \ldots, d_n)\) are connected by

\[
T(c_0, \ldots, c_n)T(d_0, \ldots, d_n) = I_{n+1}.
\]

Assume that a sequence \(s = \{s_j\}_{j=0}^\ell\) satisfies the conditions \((2.13)\) with \(\nu_1\) replaced by \(\nu\), i.e.

\[
s_0 = \ldots = s_{\nu-2} = 0, \quad s_{\nu-1} \neq 0.
\]

If \(\ell \geq 2\nu - 1\) then one can define a polynomial \(a\) and a constant \(b\) by

\[
a(z) = \frac{1}{D_{\nu}} \begin{vmatrix}
    s_0 & \cdots & s_{\nu-1} & s_{\nu} \\
    \vdots & \ddots & \vdots & \vdots \\
    s_{\nu-1} & \cdots & s_{2\nu-2} & s_{2\nu-1} \\
    1 & z & \cdots & z^{\nu}
\end{vmatrix}, \quad b = s_{\nu-1}.
\]

In the case when \(\ell = 2\nu - 2\) let us set \(s_{2\nu-1}\) to be an arbitrary real number. This number impacts only the last coefficient \(a_0\) of the polynomial

\[
a(z) = a_0z^{\nu} + \cdots + a_1z + a_0.
\]

The following lemma is a direct corollary of Lemma 2.4. It collects some statements concerning asymptotic expansions of the reciprocal function from [5] Lemma 2.1 and [13] Lemma 2.13, Lemma A3].

**Lemma 2.5.** Assume that a sequence \(s = \{s_j\}_{j=0}^\ell\) satisfies the conditions \((2.18)\) with \(\ell \geq 2\nu - 1\), let \(N(s) = \{n_j\}_{j=1}^N\), \(n = \lfloor \ell/2 \rfloor\) and let \(b\) and the polynomial \(a(z) = \sum_{j=0}^{\nu} a_j z^j\) be defined by \((2.19)\). Then a function \(f\) (meromorphic on \(\mathbb{C} \setminus \mathbb{R}\)) admits the asymptotic expansion

\[
f(z) = -\frac{s_{\nu-1}}{z^{\nu}} - \cdots - \frac{s_{\ell}}{z^{\ell+1}} + o\left(\frac{1}{z^{\ell+1}}\right), \quad z \to \infty,
\]

if and only if the function \(-1/f(z)\) admits the asymptotic expansion

\[
-\frac{1}{f(z)} = \frac{a(z)}{b} + g(z), \quad z \to \infty,
\]

where \(g(z)\) satisfies one of the following conditions:
Due to (2.19),

\[ (2.29) \]

\[ n \]

\[ (2.25) \]

\[ (2.27) \]

\[ \nu \]

\[ (2.30) \]

\[ \nu \]

\[ (2.28) \]

\[ n_j = n_{j+1} - \nu \quad (j = 1, \ldots, N - 1). \]

Let us define the following polynomial \( m \) by

\[ (2.30) \]

\[ m(z) = \frac{a(z) - a(0)}{b_0} \quad (\deg(m) = \nu - 1). \]

Due to (2.19), \( m(z) \) takes the form

\[ (2.31) \]

\[ m(z) = \frac{(-1)^{\nu + 1}}{D_\nu} \begin{vmatrix} 0 & \cdots & 0 & s_{\nu-1} & s_{\nu} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{\nu-1} & \cdots & \cdots & \cdots & s_{2\nu-2} \\ 1 & z & \cdots & z^{\nu-2} & z^{\nu-1} \end{vmatrix} \quad (D_\nu := \det S_\nu). \]

and the leading coefficient of \( m \) is calculated by

\[ (2.32) \]

\[ (-1)^{\nu + 1} \frac{D_{\nu - 1}^+}{D_\nu} = \frac{1}{s_{\nu-1}}. \]

Let us reformulate Lemma 2.6 in terms of the polynomial \( m \).

**Lemma 2.6.** Let a real sequence \( s = \{s_j\}_{j=0}^\ell \) satisfy the conditions (2.13) \((\ell \geq 2\nu - 1)\), let \( N(s) = \{n_j\}_{j=1}^N \) and let the polynomial \( m(z) = \sum_{j=0}^{\nu - 1} m_j z^j \) be defined by (2.31). Then a function \( f \) (meromorphic on \( \mathbb{C} \setminus \mathbb{R} \)) admits the asymptotic expansion (2.21) if and only if the function \(-1/f(z)\) admits the asymptotic expansion

\[ (2.33) \]

\[ -1/f(z) = zm(z) + g(z), \quad z \rightarrow \infty, \]

where \( g(z) \) satisfies one of the following conditions:
Remark 2.7. It follows from the equality \((2.24)\) and \([5, \text{Proposition 2.1}]\) that the sequence \(\{s_i\}_{i=0}^{\ell-2\nu_1}\) can be found by the equalities

\[
\begin{align*}
\hat{s}_{-1} &= \frac{(-1)^{\nu_1+1}}{s_{\nu_1-1}} D D^+ \nu_1, \\
\hat{s}_i &= \frac{(-1)^{i+\nu_1}}{s_{\nu_1-1}^{i+2}} \begin{pmatrix}
\nu_1 & s_{\nu_1-1} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots \\
\nu_1 & & & s_{\nu_1-1} \\
\end{pmatrix}, \quad i = 0, \ell - 2\nu_1.
\end{align*}
\]

Next statement is an analog of Lemma 2.4 which is applicable for expansions containing constants.

Lemma 2.8. Let \(s = \{s_j\}_{j=-1}^{\ell} \neq 0\). Let \(N(s) = \{n_j\}_{j=1}^{\infty}, n = [\ell/2]\) and let \(l = 1/s_{-1}\). Then a function \(g\) (meromorphic on \(\mathbb{C} \setminus \mathbb{R}\)) admits the asymptotic expansion \((2.34)\) if and only if the function \(-1/g(z)\) admits the representation

\[
-1/g(z) = l + f(z), \quad z \rightarrow \infty,
\]

where \(f(z)\) satisfies one of the following conditions:

(i) if \(l = -1\) then \(f(z) = o(1), z \rightarrow \infty;\)

(ii) if \(l \geq 0\) then \(f(z)\) has the asymptotic expansion

\[
f(z) = -\frac{s_0}{z} - \ldots - \frac{s_{\ell}}{z^{\ell+1}} + o\left(\frac{1}{z^{\ell+1}}\right), \quad z \rightarrow \infty,
\]

where the sequence \(\{s_i\}_{i=0}^{\ell-2\nu_1}\) is determined by the matrix equation

\[
T(s_{-1}, \ldots, s_{\ell}) T(l, -s_0, \ldots, -s_{\ell}) = I_{\ell+2}.
\]

The indices \(\nu_{\pm}(S_p), \nu_0(S_p)\) are given by

\[
\nu_0(S_p) = \nu_0(S_p), \quad \nu_{\pm}(S_p) = \nu_{\pm}(S_p) \quad (p = 0, \ldots, n + 1).
\]

Proof. If \(l = -1\), then \((2.22)\) takes the form

\[
g(z) = -s_{-1} + o(1), \quad z \rightarrow \infty,
\]

and hence (i) is clear.

Assume that \(l \geq 0\). Then by Lemma 2.5 one obtains the representation \((2.22)\), \((2.23)\) for \(-1/g\) with coefficients \(s_j (j = 0, \ldots, \ell)\), satisfying \((2.40)\). Multiplying
Applying Lemma 2.6 to \(zf\) yields
\[
A_\nu = \begin{pmatrix}
0_{(n+1)\times(n+1)} & A_{12} \\
A_{12} & A_{22}
\end{pmatrix}
\begin{pmatrix}
B_{11} & B_{12} \\
B_{12} & 0_{(n+1)\times(n+1)}
\end{pmatrix} = I_{2n+2},
\]
where
\[
A_{12} = \begin{pmatrix}
0 & \cdots & s_{n-1} \\
\vdots & \ddots & \vdots \\
s_{-1} & \cdots & s_{n-1}
\end{pmatrix} \in \mathbb{C}^{(n+1)\times(n+1)},
\]
and
\[
A_{22} = S_{n+1} \in \mathbb{C}^{(n+1)\times(n+1)}, \quad B_{11} = -J_{n+1}S_{n+1}J_{n+1} \in \mathbb{C}^{(n+1)\times(n+1)}
\]
and \(B_{12}, B_{12}^*\) are some matrices from \(\mathbb{C}^{(n+1)\times(n+1)}\). Notice that the matrix \(A\) is invertible. If in addition, the matrix \(A_{22}\) is invertible then its Schur complement
\[
B_{11}^{-1} = -A_{12}A_{22}^{-1}A_{12}^*
\]
and hence the matrix \(B_{11} = (A_{11})^{-1}\) is also invertible. In view of (2.43) this implies that the matrix \(S_{n+1}\) is invertible. The converse is also true by similar arguments. This proves the equalities (2.41). \(\Box\)

For a sequence \(s = \{s_i\}_{i=1}^{2n-1}\) let us set
\[
S_n^- = \begin{pmatrix}
s_{-1} & \cdots & s_{n-2} \\
\vdots & \ddots & \vdots \\
s_{n-2} & \cdots & s_{2n-3}
\end{pmatrix} \quad (n \in \mathbb{N}).
\]

**Corollary 2.9.** Under the assumptions of Lemma 2.7 the indices \(\nu_0(S_p^-)\) and \(\nu_\pm(S_p^-)\) for matrices \(S_p^- = (s_{i+j-1})_{i,j=0}^{p-1}\) are evaluated by the equalities
\[
\nu_0(S_p^-) = \nu_0(S_{p+p-1}^+) \quad (p = 1, \ldots, n - \nu + 1, n = \lceil \ell/2 \rceil);
\]
\[
\nu_\pm(S_p^-) = \nu_\pm(S_{p+p-1}^+) - \nu_\pm(S_{p-1}^+) \quad \text{if } s_0 = 0 \quad (p = 1, \ldots, n - \nu + 1);
\]
\[
\nu_\pm(S_p^-) = \nu_\pm(S_{p+p}^+) \quad \text{if } s_0 \neq 0 \quad (p = 1, \ldots, n).
\]

**Proof.** Assume that \(s_0 = 0\). Then it follows from (2.22), (2.23) that
\[
zf(z) = -\frac{s_{\nu-1}}{z^{\nu-1}} - \cdots - \frac{s_{2i-1}}{z^{2i-1}} + o\left(\frac{1}{z^{2i-2}}\right), \quad z \to \infty,
\]
and
\[
-\frac{1}{zf(z)} = m(z) - \frac{s_{-1}}{z} - \cdots - \frac{s_{2i-1-2\nu}}{z^{2i-2\nu}} + o\left(\frac{1}{z^{2i-2\nu}}\right), \quad z \to \infty,
\]
Applying Lemma 2.6 to \(zf(z)\) and using the expansions (2.48) and (2.49) one obtains
\[
\nu_0(S_{p-p-1}^-) = \nu_0(S_p^+) \quad (p = \nu, \ldots, \lceil \ell/2 \rceil);
\]
\[
\nu_\pm(S_{p-p-1}^-) = \nu_\pm(S_p^+) - \nu_\pm(S_{p-1}^+) \quad (p = \nu, \ldots, \lceil \ell/2 \rceil).
\]
If \(s_0 \neq 0\), then \(\nu = 1\) and the expansions (2.48) and (2.49) take the form
\[
zf(z) = -s_0 - \frac{s_{1}}{z} - \cdots - \frac{s_{2i-1}}{z^{2i-1}} + o\left(\frac{1}{z^{2i-2}}\right), \quad z \to \infty,
\]
and
\[
-\frac{1}{zf(z)} = m - \frac{s_{-1}}{z} - \cdots - \frac{s_{2i-3}}{z^{2i-2}} + o\left(\frac{1}{z^{2i-2}}\right), \quad z \to \infty,
\]
2.4. Class $U(J)$ and linear fractional transformations. Let $\kappa_1 \in \mathbb{N}$ and let $J$ be a $2 \times 2$ signature matrix

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$ 

A $2 \times 2$ matrix valued function $W(z) = (w_{i,j}(z))_{i,j=1}^{2}$ that is meromorphic in $\mathbb{C}_+$ belongs to the class $U_\kappa(J)$ of generalized $J$-inner matrix valued functions if:

(i) the kernel

$$K_\omega^W(z) = \frac{J - W(z)JW(\omega)^*}{-i(z - \omega)}$$

has $\kappa$ negative squares in $S_\kappa^+ \times S_\kappa^+$ and

(ii) $J - W(\mu)JW(\mu)^* = 0$ for a.e. $\mu \in \mathbb{R}$,

where $S_\kappa^+$ denotes the domain of holomorphy of $W$ in $\mathbb{C}_+$.

Consider the linear fractional transformation

$$T_W[\tau] = (w_{11}\tau(z) + w_{12})(w_{21}\tau(z) + w_{22})^{-1}$$

associated with the matrix valued function $W(z)$. The linear fractional transformation associated with the product $W_1W_2$ of two matrix valued function $W_1(z)$ and $W_2(z)$, coincides with the composition $T_{W_1} \circ T_{W_2}$.

As is known, if $W \in U_{\kappa_1}(J)$ and $\tau \in \mathbb{N}_{k_2}$ then $T_W[\tau] \in \mathbb{N}_{k'}$, where $k' \leq \kappa_1 + \kappa_2$. In the present paper two partial cases, in which the preceding inequality becomes equality, will be needed.
**Lemma 2.11.** Let \( m(z) \) be a real polynomial such that \( \kappa_-(zm) = \kappa_1, \kappa_-(m) = k_1 \), let \( M \) be a \( 2 \times 2 \) matrix valued function
\[
(2.56) \quad M(z) = \begin{pmatrix} 1 & 0 \\ -zm(z) & 1 \end{pmatrix}
\]
and let \( \tau \) be a meromorphic function, such that \( \tau(z)^{-1} = o(z) \) as \( z \to \infty \). Then the following equivalences hold:
\[
(2.57) \quad \tau \in \mathcal{N}_{\kappa_2} \iff T_M[\tau] \in \mathcal{N}_{\kappa_1 + \kappa_2}, \\
(2.58) \quad \tau \in \mathcal{N}_{k_2} \kappa_2 \iff T_M[\tau] \in \mathcal{N}_{k_1 + k_2}.
\]

**Proof.** Let us set \( f = T_M[\tau] \). Then
\[
(2.59) \quad -\frac{1}{f(z)} = zm(z) - \frac{1}{\tau(z)}.
\]
It follows from (2.59) and Proposition 2.1 (3) that \(-\frac{1}{f} \in \mathcal{N}_{\kappa_1 + \kappa_2}\). In view of Proposition 2.1 (1) this implies (2.57).

Dividing (2.59) by \( z \) one obtains
\[
(2.60) \quad -\frac{1}{zf(z)} = m(z) - \frac{1}{z\tau(z)}.
\]
Since \( z\tau(z) = o(1) \) as \( z \to \infty \), then by Proposition 2.1 (3) \(-\frac{1}{zf} \in \mathcal{N}_{k_1 + k_2}\) and hence \( zf \in \mathcal{N}_{k_1 + k_2} \). This proves (2.58). \( \Box \)

**Lemma 2.12.** Let \( l(z) \) be a real polynomial such that \( \kappa_-(l) = \kappa_1, \kappa_-(zl(z)) = k_1 \), let \( L(z) \) be a \( 2 \times 2 \) matrix valued function
\[
(2.61) \quad L(z) = \begin{pmatrix} 1 & l(z) \\ 0 & 1 \end{pmatrix}
\]
and let \( \tau \) be a meromorphic function, such that \( \tau(z)^{-1} = o(1) \) as \( z \to \infty \). Then the following equivalences hold:
\[
(2.62) \quad \tau \in \mathcal{N}_{\kappa_2} \iff T_L[\tau] \in \mathcal{N}_{\kappa_1 + \kappa_2}, \\
(2.63) \quad \tau \in \mathcal{N}_{k_2} \kappa_2 \iff T_L[\tau] \in \mathcal{N}_{k_1 + k_2}.
\]

**Proof.** Let us set \( f = T_L[\tau] \). Then (2.62) is implied by the equality
\[
(2.64) \quad f(z) = l(z) + \tau(z).
\]
and Proposition 2.1 (3). Multiplying (2.64) by \( z \) one obtains
\[
(2.65) \quad zf(z) = zl(z) + z\tau(z).
\]
Since \( z\tau(z) = o(1) \) as \( z \to \infty \), then by Proposition 2.1 (3) \( zf \in \mathcal{N}_{k_1 + k_2} \). This proves (2.63). \( \Box \)

3. Basic moment problem in \( \mathcal{N}_{\kappa}^k \)

In this section we consider a basic moment problem in Nevanlinna class \( \mathcal{N}_{\kappa}^k \) and describe its solutions. Odd and even moment problems will be treated separately. In both cases one step of the Schur algorithm will be considered.
3.1. Basic odd moment problem $MP^k_n(s, 2\nu_1 - 2)$. An odd moment problem $MP^k_n(s, 2n - 2)$ is called nondegenerate if

$$D_n \neq 0 \quad \text{and} \quad D_{n-1}^+ \neq 0.$$  

By definition (2.8) this means that $n \in \mathcal{N}(s)$. A nondegenerate odd moment problem $MP^k_n(s, 2n - 2)$ will be called basic, if $n$ is the only normal index of $s$, i.e. $n = \nu_1$ and $\mathcal{N}(s) = \{ \nu_1 \}$. This case can be characterized by the conditions (2.13).

The basic moment problem $MP^k_n(s, 2\nu_1 - 2)$ can be reformulated as follows:

Given a sequence $s = \{ s_j \}_{j=0}^{2\nu_1-2}$ with $\mathcal{N}(s) = \{ \nu_1 \}$, find all functions $f \in N^k_n$ such that

$$f(z) = -\frac{s_{\nu_1-1}}{z^{\nu_1}} - \cdots - \frac{s_{2\nu_1-2}}{z^{2\nu_1-1}} + o \left( \frac{1}{z^{2\nu_1-1}} \right), \quad z \to \infty.$$  

Let $s = \{ s_j \}_{j=0}^{2\nu_1-2}$ be a sequence of real numbers from $\mathcal{H}$ and let (2.13) holds. Then $s \in H^{k_1, 2\nu_1-2}_{\kappa_1}$, where $\kappa_1$ and $k_1$ are defined by

$$\kappa_1 = \nu_-(S_{\nu_1}) = \left\{ \frac{\nu_1 + 1}{2} \right\}, \quad \text{if} \ \nu_1 \ \text{is odd and} \ s_{\nu_1-1} < 0;$$

$$k_1 = \nu_-(S_{\nu_1}^+ - 1) = \left\{ \frac{\nu_1 + 1}{2} \right\}, \quad \text{if} \ \nu_1 \ \text{is even and} \ s_{\nu_1-1} < 0;$$

It follows from (3.3) and (3.4), that

$$\nu_+(S_{\nu_1}^+) = \begin{cases} \kappa_1 - 1, & \text{if} \ \nu_1 \ \text{is odd and} \ s_{\nu_1-1} < 0; \\ \kappa_1 - 1, & \text{if} \ \nu_1 \ \text{is even and} \ s_{\nu_1-1} > 0; \\ \kappa_1, & \text{otherwise}. \end{cases}$$

Let $m_1(z)$ be the polynomial defined by (2.31) with $\nu = \nu_1$. Then it follows from (2.1) and (3.3), (3.4), that

$$\kappa_1 = \kappa_-(z m_1), \quad k_1 = \kappa_-(m_1).$$

**Lemma 3.1.** Let $\nu_1$ be the first normal index of the sequence $s = \{ s_j \}_{j=0}^{2\nu_1-2}$, let polynomial $m_1$ be defined by (2.31) and let $f \in N^k_n$ have the asymptotic expansion (3.2). Then $f$ admits the following representation

$$f(z) = -\frac{1}{zm_1(z) + g(z)},$$

where

$$g \in N_{\kappa - \kappa_1} \quad \text{and} \quad g(z) = o(z), \quad z \to \infty.$$  

**Conversely, if $g$ satisfies (3.3) and $f$ is defined by (3.7), then $f \in N^k_n$.**

**Proof.** By Lemma 2.6 $f$ admits the representation (3.2), where $g(z) = o(z)$ as $z \to \infty$. Next, since $f \in N^k_n$ then also $-1/f \in N^k_n$ and then it follows from the equality

$$-1/f(z) = zm_1(z) + g(z)$$

and Proposition 2.1 (3) that $g \in N_{\kappa - \kappa_-(z m_1)}$. Since by (3.6) $\kappa_-(z m_1) = \kappa_1$ one gets $g \in N_{\kappa - \kappa_1}$.

Conversely, if $g$ satisfies (3.8) then by Lemma 2.6 $f$ has the asymptotic expansion (3.2) and by (3.9) and Proposition 2.1 (3) $f \in N_{\kappa_1 + (\kappa - \kappa_1)} = N^k_n$. □
Remark 3.2. Replacing \( g \) by \(-1/g \) in (3.7), we can rewrite it as follows

\[
\tag{3.10}
f(z) = T_{M_1}[g_1] = \frac{g_1(z)}{-zm_1(z)g_1(z) + 1},
\]
where the polynomial \( m_1(z) \) is defined by (2.31), and the matrix valued function

\[
\tag{3.11}
M_1(z) = \begin{pmatrix} 1 & 0 \\ -zm_1(z) & 1 \end{pmatrix}
\]
belongs to the class \( U_{\kappa_1}(J) \). The statement of Lemma 3.1 can be reformulated as follows

\[
\tag{3.12}
T_{M_1}[g_1] \in \mathbf{N}_\kappa \iff g_1 \in \mathbf{N}_{\kappa - \kappa_1} \quad \& \quad \frac{1}{g_1(z)} = o(z), \quad z \to \infty.
\]
Moreover, it follows from Lemma 2.11 that

\[
\tag{3.13}
T_{M_1}[g_1] \in \mathbf{N}_k^k \iff g_1 \in \mathbf{N}_{k - k_1}^k \quad \& \quad \frac{1}{g_1(z)} = o(z), \quad z \to \infty.
\]

In fact, the reason for switching to reciprocal function \( g_1 \) is motivated by (3.13), it helps to keep \( g_1 \) staying in a generalized Stieltjes class \( \mathbf{N}_{k - k_1}^k \).

Combining Lemma 3.1 and Remark 3.2 with calculations in (3.6) one obtains

**Theorem 3.3.** Let \( \nu_1 \) be the first normal index of the sequence \( s = \{ s_i \}_{i=0}^{2\nu_1 - 2} \), let \( m_1, \kappa_1 \) and \( k_1 \) be defined by (2.31), (3.3) and by (3.4), respectively, and let \( \ell \geq 2\nu_1 - 2 \). Then:

1. The problem \( MP^k_\kappa(s, \ell) \) is solvable if and only if

\[
\tag{3.14}
\kappa_1 \leq \kappa \quad \text{and} \quad k_1 \leq k,
\]

2. \( f \in \mathcal{M}^k_\kappa(s, 2\nu_1 - 2) \) if and only if \( f \) admits the representation

\[
\tag{3.15}
f = T_{M_1}[\tau],
\]
where \( \tau \) satisfies the conditions

\[
\tag{3.16}
\tau \in \mathbf{N}^{k-k_1}_{\kappa - \kappa_1} \quad \text{and} \quad \frac{1}{\tau(z)} = o(z), \quad z \to \infty.
\]

3. If \( \ell > 2\nu_1 - 2 \), then \( f \in \mathcal{M}^k_\kappa(s, \ell) \) if and only if \( f \) admits the representation

\[
\tag{3.17}
f = T_{M_1}[g_1], \quad \text{where} \quad g_1 \in \mathbf{N}^{k-k_1}_{\kappa - \kappa_1} \quad \text{and} \quad \frac{1}{g_1(z)} \text{ has the following asymptotic expansion}
\]

\[
\frac{1}{g_1(z)} = -s_{-1} - \frac{s_0}{z} - \cdots - \frac{s_{n-2\nu_1}}{z^{n-2\nu_1+1}} + o\left(\frac{1}{z^{n-2\nu_1+1}}\right), \quad z \to \infty,
\]

and the sequence \( \{ s_i \}_{i=-1}^{n-2\nu_1} \) is determined by the matrix equation

\[
\tag{3.18}
T(m^{(1)}_{\nu_1-1}, \ldots, m^{(1)}_0, -s^{(1)}_{-1}, \ldots, -s^{(1)}_{\ell-2\nu_1}) T(s_{\nu_1-1}, \ldots, s_{\ell}) = I_{\ell-\nu_1+2}.
\]

**Proof.** (1) Assume that \( f \in \mathcal{M}^k_\kappa(s, \ell) \). The inequality \( \kappa_1 \leq \kappa \) is implied by Proposition 2.1(4). Next, since \( zf \in \mathbf{N}_k \) and

\[
\tag{3.19}
zf(z) + s_0 = -\frac{s_1}{z} - \frac{s_2}{z^2} - \cdots - \frac{s_{\ell}}{z^{\ell}} + o\left(\frac{1}{z^{\ell}}\right), \quad z \to \infty,
\]
then necessarily, by Corollary 2.3(4) \( k_1 = \nu_-(S^+_{\nu_1-1}) \leq k \).
(2) Assume $f$ belongs to $N_k^s$ and has the asymptotic expansion (3.2). Then by Lemma 3.1 and Remark 3.2, the function $f \in M_{k}^s(s, 2\nu_1 - 2)$ has the representation (3.10) if and only if (3.10) holds.

(3) Suppose $f$ belongs to $M_{k}^s(s, \ell)$. By Lemma 2.6 and Remark 3.2, the function $f$ admits the representation $f = T_{M_1}(g_1)$, where $g_1$ satisfies (3.17) and the sequence $\{g_i^{(1)}\}_{i=-1}^{n-2\nu_1}$ is determined by (3.18). Moreover, $g_1 \in N_{k-1}^{s}$ by Lemma 2.11.

The converse also follows from Lemma 2.6 and Lemma 2.11 □

Remark 3.4. It follows from the equality (2.21) and [3] Proposition 2.1 that the sequence $\{g_i^{(1)}\}_{i=-1}^{\ell-2\nu_1}$ can be found by the equalities

\[
\begin{align*}
\text{(3.20)} & \quad g_i^{(1)} = \frac{(-1)^{\nu_1+1} D^+_{\nu_1}}{s_{\nu_1-1}} \\
\text{(3.21)} & \quad g_i^{(1)} = \frac{(-1)^{l+\nu_1}}{s_{\nu_1-1}^{l+\nu_1+2}} \\
\end{align*}
\]

3.2. Basic even moment problem $MP_{k}^s(s, 2\mu_1 - 1)$. An even moment problem $MP_{k}^s(s, 2n - 1)$ is called nondegenerate, if

\[
D_n \neq 0 \quad \text{and} \quad D_n^+ \neq 0.
\]

By classification (2.8), (2.9) this means that $n \in N(s)$ and $n = \mu_j$ for some $j$. A nondegenerate even moment problem $MP_{k}^s(s, 2n - 2)$ will be called basic, if $n$ is the smallest index such that (3.22) holds. Therefore, the basic even moment problem coincides with the problem $MP_{k}^s(s, 2\mu_1 - 1)$. Regarding to the conditions $\nu_1 = \mu_1$ or $\nu_1 < \mu_1$ the set of normal indices consists either of one element $\nu_1$ or of two elements $\nu_1$ and $\mu_1$.

The basic even moment problem $MP_{k}^s(s, 2\mu_1 - 1)$ can be reformulated as follows: Given a sequence $s = \{s_{j}\}_{j=0}^{2\mu_1-1} \in \mathcal{H}$, where $\mu_1$ is the smallest index $n$ such that (3.22) holds, find all functions $f \in N_k^s$, such that

\[
(3.23) \quad f(z) = -\frac{s_{\nu_1-1}}{z^{\nu_1}} - \cdots - \frac{s_{2\mu_1-1}}{z^{2\mu_1}} + o\left(\frac{1}{z^{2\mu_1}}\right), \quad z \to \infty.
\]

Solution of the basic even moment problem will be splitted into two steps. On the first step one applies Lemma 2.6 to construct a sequence $\{g_i^{(1)}\}_{i=1}^{2(\mu_1 - \nu_1) - 1}$. If $f \in M_{k}^s(s, 2\mu_1 - 1)$ then by Theorem 3.3, $f(z)$ admits the representation (3.10) which can be rewritten as

\[
(3.24) \quad -\frac{1}{f(z)} = zm_1(z) - \frac{1}{g_1(z)},
\]

and where $-g_1^{-1}$ has the following asymptotic expansion

\[
(3.25) \quad -\frac{1}{g_1(z)} = -g_{-1}^{(1)} - \frac{g_0^{(1)}}{z} - \cdots - \frac{g_{2(\mu_1 - \nu_1) - 1}^{(1)}}{z^{2(\mu_1 - \nu_1)}} + o\left(\frac{1}{z^{2(\mu_1 - \nu_1)}}\right), \quad z \to \infty.
\]
with \( s_i^{(1)} \) defined by (3.18). Moreover, \( f \in N^k \) if and only if \( g_1 \in N^{k-k_-(\mu_1)} \). Now two cases may occur.

1. If \( \nu_1 = \mu_1 \), then \( s_{-1}^{(1)} \neq 0 \) and by Lemma 2.12 \( g_1 \) admits the representation

\[
(3.26)
\]

where \( l_1 \) is a constant

\[
(3.27)
\]

\( L_1 \) is defined by (2.61) and \( f_1(z) = o(1) \) as \( z \to \infty \). Moreover, by Lemma 2.12

\( g_1 \in N_k^\kappa \) if and only if \( f_1 \in N_{k-k_-(z\ell_1)}^\kappa \).

2. If \( \nu_1 < \mu_1 \), then \( s_1^{(1)} = 0 \) and by Lemma 2.5 \( g_1 \) admits the representation (3.26), where \( l_1 = l_1(z) \) is a polynomial

\[
(3.28)
\]

\( L_1 \) is defined by (2.61) and \( f_1(z) = o(1) \) as \( z \to \infty \). Moreover, by Lemma 2.12

\( g_1 \in N_k^\kappa \) if and only if \( f_1 \in N_{k-k_-(\ell_1)}^\kappa \).

Combining the formulas (3.24) and (3.26) and summarising the above reasonings one obtains the first two statements of the following

Theorem 3.5. Let \( s = \{s_j\} \) \( j=0 \) be a sequence from \( \mathcal{H} \), such that \( N(s) = \{\nu_1, \mu_1\} \) \( \nu_1 \leq \mu_1 \), and let \( m_1, l_1 \) be defined by (2.31) and (3.26), respectively. Then:

1. The problem \( MP^s_\kappa(s, 2\mu_1 - 1) \) is solvable if and only if

\[
(3.29)
\]

where

\[
(3.30)
\]

2. \( f \in \mathcal{M}_\kappa^s(s, 2\mu_1 - 1) \) if and only if \( f \) admits the representation

\[
(3.31)
\]

where

\[
(3.32)
\]

3. If \( \ell > 2\mu_1 - 1 \), then \( f \in \mathcal{M}_\kappa^s(s, \ell) \), if and only if \( f \) admits the representation (3.30), where

\[
(3.33)
\]

\( \kappa_1 \) and \( k_1^+ \) are determined by (3.29) and the sequence \( \{s_i^{(1)}\}_{i=1}^{s_1^{(1)}} \) is determined by the matrix equation

\[
(3.34)
\]

if \( \mu_1 = \nu_1 \), and if \( \nu_1 < \mu_1 \) by the following equation

\[
(3.35)
\]
Proof. The items (1) and (2) are proved above.
Let us prove (3). Assume that ℓ > 2µ₁ - 1 and f \in \mathcal{M}_k^d(s, \ell). Then by Theorem 3.3 f(z) admits the representation (3.24) where
\[ -g_1^{-1} = \frac{-s_1^{(1)}}{z} - \cdots - \frac{s_{\ell-2\nu_1}^{(1)}}{z^{\ell-2\nu_1}} + o\left(\frac{1}{z^{\ell-2\nu_1+1}}\right), \quad z \to \infty, \]
and s_j^{(1)} are defined by (3.18). Moreover, f \in N^k if and only if g_1 \in N^{k-\kappa-(\nu_1)}.
Consider two cases:
(1) If ν₁ = µ₁, then s^{(1)} \neq 0 and by Lemma 2.8 g_1 admits the representation (3.26), (3.27), where L₁ is defined by (2.61) and f_1(z) has the following asymptotic
\[ f_1(z) = -\frac{s_0^{(1)}}{z} - \cdots - \frac{s_{\ell-2\mu_1}^{(1)}}{z^{\ell-2\mu_1+1}} + o\left(\frac{1}{z^{\ell-2\mu_1+1}}\right), \quad z \to \infty \]
with s_j^{(1)} defined by the matrix equation (3.34). By Lemma 2.12
\[ g_1 \in N^k \iff f_1 \in N^{k-\kappa-(\nu_1)}. \]
This proves that f_1 \in \mathcal{M}_k^{k-\ell}(s^{(1)}, \ell - 2\mu_1), since \kappa(l_1) = 0 and \kappa = \kappa-(\nu_1) in this case.
(2) If ν₁ < µ₁, then s^{(1)} = 0 and by Lemma 2.8 g_1 admits the representation (3.26), where L₁ = l₁(z) is a polynomial given by (3.28), L₁ is defined by (2.61) and f_1(z) has the asymptotic (3.37) as z \to \infty. By Lemma 2.12
\[ g_1 \in N^k \iff f_1 \in N^{k-\kappa-(\nu_1)}. \]
This proves that f_1 \in \mathcal{M}_k^{k-\ell}(s^{(1)}, \ell - 2\mu_1) also in the case ν₁ < µ₁.
The proof of the converse statement is similar and is based on Lemmas 2.6, 2.8, 2.11, 2.12.

Remark 3.6. It follows from the equality (3.24) and Proposition 2.1 that the sequence \{s_i^{(1)}\}_{i=0}^{\ell-2\mu_1} can be found by the equalities
\[ s_i^{(1)} = \frac{(-1)^i + \mu_1 - \nu_1}{(s_{\mu_1-\nu_1+2}^{(1)})} \]
where i = 0, \ell - 2\mu_1.

Remark 3.7. The solution matrix of the basic even moment problem \mathcal{M}_k^d(s, 2\mu_1 - 1)
\[ W_2(z) = \begin{pmatrix} 1 & l_1(z) \\ -zm_1(z) & -zm_1(z)l_1(z) + 1 \end{pmatrix} \]
admits the following factorization
\[ W_2(z) = M(z)L(z), \]
where the matrices $M_1(z)$ and $L_1(z)$ are defined by \((2.50), (2.61)\) and the corresponding linear fractional transform is defined by
\[
T_{W_2}[f_1] = \frac{f_1(z) + l_1(z)}{-zm_1(z)f_1(z) - zm_1(z)l_1 + 1}.
\]

4. Schur algorithm.

4.1. Regular sequences. A general nondegenerate indefinite truncated moment problem in the class $N_k^\kappa$ can be studied by the step-by-step algorithm based on the elementary steps, introduced in the previous section. In this section we will demonstrate this algorithm in the case when the sequence $s$ belongs to the class $H_{\kappa, \ell}$ of so-called regular sequences. This class $H_{\kappa, \ell}$ was introduced in \([14]\).

**Definition 4.1.** \([16]\) Let $s = \{s_j\}_{j=0}^\ell \in H_{\kappa, \ell}$ and let $N(s) = \{n_j\}_{j=1}^N$. A sequence $s$ is related to the class $H_{\kappa, \ell}$ and is said to be regular, if one of the following equivalent conditions holds:
\begin{enumerate}
  \item $P_{n_j}(0) \neq 0$ for every $j \leq N$;
  \item $D_{n_j - 1}^+ \neq 0$ for every $j \leq N$;
  \item $D_{n_j}^- \neq 0$ for every $j \leq N$;
  \item $\nu_j = \mu_j$ for all $j$, such that $\nu_j, \mu_j \in N(s)$.
\end{enumerate}

The equivalence of the conditions \((1) - (4)\) was proved in \([16, \text{Lemma 3.1}]\). The class of regular $H_{\kappa, \ell}$-sequences is defined by $H_{\kappa, \ell}^{reg} := H_{\kappa, \ell} \cap H_{\kappa, \ell}$.

For a regular sequence $s \in H_{\kappa, \ell}^{reg}$ the normal indices $n_j$ \((1 \leq j \leq N)\) of $s$ satisfy
\[
n_j = \nu_j = \mu_j \quad (1 \leq j \leq N),
\]
where $\nu_j$ and $\mu_j$ are introduced in \((2.50)\) and \((2.49)\). As was shown in \([16]\) for every sequence $s \in H_{\kappa, \ell}^{reg}$ there are polynomials $m_j$ of degree $\nu_j - n_{j-1} - 1$ and real numbers $l_j$ such that the $2j$-th convergent $\frac{m_j}{l_j}$ of the generalized $S$-fraction
\[
(1) \quad \frac{1}{zm_1(z) + \frac{1}{l_1 + \ldots + \frac{1}{zm_j(z) + \frac{1}{l_j + \ldots}}}}
\]
has the following asymptotic expansion
\[
(2) \quad f(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \ldots - \frac{s_{2n_j - 1}}{z^{2n_j}} + O\left(\frac{1}{z^{2n_j + 1}}\right), \quad z \to \infty.
\]

We will show that the Schur process leads to the same continued fraction and gives descriptions of solutions of odd and even problems $MP_{\kappa}^k(s, 2n_j - 2)$ and $MP_{\kappa}^k(s, 2n_j - 1)$ in terms of these continued fractions.

4.2. Odd moment problem. Let $MP_{\kappa}^k(s, 2n_N - 2)$ be a nondegenerate odd moment problem, i.e.
\[
(3) \quad D_{n_N} \neq 0 \quad \text{and} \quad D_{n_N - 1}^+ \neq 0.
\]
Assume that $f \in M_{\kappa}^k(s, 2n_N - 2)$ \((N > 1)\), i.e. $f \in N_k^\kappa$ and
\[
f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \ldots - \frac{s_{2n_N - 2}}{z^{2n_N - 1}} + o\left(\frac{1}{z^{2n_N - 1}}\right), \quad z \to \infty.
\]
Then by Theorem 3.3 the function $f$ can be represented as

$$f(z) = \frac{1}{-zm_1(z) + \frac{1}{l_1 + f_1(z)}},$$

where the polynomial $m_1$ and number $l_1$ are defined by (2.31) and (3.27), respectively. Here the function $f_1$ has the asymptotic expansion (3.37) with the sequence $s^{(1)} = \{s_i^{(1)}\}_{i=1}^{2(nN - n_1) - 2}$ determined consequently by (3.18) and (3.35). The set of normal indices of the sequence $s^{(1)}$ is $N(s^{(1)}) = \{n_j - n_{l-1}\}_{j=2}^{N}$. Continuing this process and applying Theorem 3.3 $N - 1$ times one obtains on each step some function $f_j \in N^{k-k_j}$ ($j = 1, \ldots, N - 1$) with an induced asymptotic expansion

$$f_j(z) = -\frac{s^{(j)}_0}{z} - \frac{s^{(j)}_1}{z^2} - \cdots - \frac{s^{(j)}_{2(nN - n_j) - 2}}{z^{2(nN - n_j) - 1}} + o\left(\frac{1}{z^{2(nN - n_j) - 1}}\right), \quad z \to \infty,$$

such that $f_{j-1}$ has the following representation in terms of $f_j$:

$$f_{j-1}(z) = \frac{1}{-zm_j(z) + \frac{1}{l_j + f_j(z)}} \quad (i = 1, \ldots, j),$$

Here the sequence $s^{(j)} = \{s^{(j)}_i\}_{i=1}^{2(nN - n_j) - 2}$ is determined recursively by (3.18) and (3.35) and $m_j$ and $l_j$ are defined by the formulas

$$m_j(z) = \frac{(-1)^{\nu + 1}}{D^{(j-1)}_{\nu}} \begin{vmatrix} 0 & \cdots & 0 & s^{(j-1)}_{\nu-1} & s^{(j-1)}_{\nu} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & z & \cdots & z^{\nu - 2} & s^{(j-1)}_{\nu-2} \\ \end{vmatrix},$$

where $D^{(j)}_{\nu} := \det s^{(j)}_i$, $\nu = n_j - n_{j-1}$ and

$$l_j = (-1)^{\nu + 1} \frac{D^{(j-1)}_{\nu}}{D^{(j-1)}_{\nu}} \quad (j = 1, \ldots, N - 1).$$

Let the matrix functions $M_j(z)$ and $L_j(z)$ be defined by

$$M_j(z) = \begin{pmatrix} 1 & 0 \\ -zm_j(z) & 1 \end{pmatrix} \quad \text{and} \quad L_j(z) = \begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix} \quad (j = 1, \ldots, N - 1).$$

Then it follows from (4.3) that

$$f_{j-1}(z) = T_{M_j(z)L_j(z)}[f_j(z)] \quad (j = 1, \ldots, N - 1).$$

On the last step we get the function $f_{N-1}(z)$, which is a solution of the basic moment problem $MP^{k}$ $(s^{(N-1)}, 2(nN - n_{N-1}) - 2)$. By Theorem 3.3 the function $f_{N-1}(z)$ can be represented as

$$f_{N-1}(z) = \frac{1}{-zm_N(z) + \frac{1}{f_N(z)}},$$

where the polynomial $m_N(z)$ is defined by (4.3) and $f_N(z)$ is a function from $N^{k-k_N}$ such that $f_N(z)^{-1} = o(z)$ as $z \to \infty$.

Combining the statements (4.3) and (4.7) and replacing $f_N(z)$ by $\tau(z)$, one obtains the following
Theorem 4.2. Let \( s = \{ s_i \}_{i=0}^{2n_N-2} \in \mathcal{H}^{k,reg}_{\kappa, 2n_N-2} \), let \( \mathcal{N}(s) = \{ n_j \}_{j=1}^N \), and let \( m_j(z) \) and \( l_j(z) \) are defined by (4.5) and (4.6), respectively. Then:

1. A nondegenerate odd moment problem \( MP^k_\kappa(s, 2n_N - 2) \) is solvable, if and only if

\[
\kappa_N := \nu_-(S_{n_N}) \leq \kappa \quad \text{and} \quad k_N := \nu_-(S_{n_N-1}^+) \leq k.
\]

2. \( f \in \mathcal{M}^k_\kappa(s, 2n_N - 2) \) if and only if \( f \) admits the representation

\[
f = T_{W_{2N-1}}[\tau],
\]

where

\[
W_{2N-1}(z) := M_1(z)L_1 \ldots (z)L_{N-1}M_N(z)
\]

and \( \tau(z) \) satisfies the conditions

\[
\tau \in \mathcal{N}^{k-k_N}_{\kappa-k_N} \quad \text{and} \quad \frac{1}{\tau(z)} = o(z) \quad as \quad z \to \infty.
\]

3. The representation (4.9) can be rewritten as a continued fraction expansion

\[
f(z) = \frac{1}{-zm_1(z) + \frac{1}{l_1 + \frac{1}{-zm_2(z) + \ldots + \frac{1}{-zm_N(z) + \tau(z)}}}}
\]

4. The indices \( \kappa_N \) and \( k_N \) are related to \( m_j \) and \( l_j \) by

\[
\kappa_N = \sum_{j=1}^N \kappa_-(z m_j), \quad k_N = \sum_{j=1}^N \kappa_-(m_j) + \sum_{j=1}^{N-1} \kappa_-(z l_j).
\]

4.3. Even moment problem. Let \( s = \{ s_i \}_{i=0}^{2n_N-1} \in \mathcal{H}^{k,reg}_{\kappa, 2n_N-1} \), let \( \mathcal{N}(s) = \{ n_N \}_{j=1}^N \) and let \( MP^k_\kappa(s, 2n_N - 1) \) be a nondegenerate even moment problem, i.e.

\[
D_{n_N} \neq 0 \quad \text{and} \quad D_{n_N}^* \neq 0.
\]

Applying Theorem 3.5 \( N - 1 \) times in the same way as in the odd case one obtains the equalities (4.14) and a sequence of functions \( f_j \in \mathcal{M}^{k-k_j}_{\kappa-k_j}(s^{(j)}, 2(n_N - n_j) - 1) \). On the last step we obtain the function \( f_{N-1}(z) \), which is a solution of the basic even moment problem \( MP^{k-k_N-1}_{\kappa-k_N-1}(s^{(N-1)}, 2(n_N - n_{N-1}) - 1) \). By Theorem 3.5, the function \( f_{N-1} \) can be represented as follows:

\[
f_{N-1}(z) = \frac{1}{-m_N(z) + \frac{1}{l_N + f_N(z)}}
\]

where \( m_N(z) \) and \( l_N \) are defined by (4.5) and (4.6), and \( f_N(z) \) is a function from \( \mathcal{N}^{k-N-1}_{\kappa-N-1}(z m_N) \), such that \( f_N(z) = o(1) \) as \( z \to \infty \).

Combining the statements (4.4) and (4.13) one obtains the following

Theorem 4.3. Let \( s = \{ s_i \}_{i=0}^{2n_N-1} \in \mathcal{H}^{k,reg}_{\kappa, 2n_N-1} \) and let \( \mathcal{N}(s) = \{ n_j \}_{j=1}^N \).

1. A nondegenerate odd moment problem \( MP^k_\kappa(s, 2n_N - 1) \) is solvable, if and only if

\[
\kappa_N := \nu_-(S_{n_N}) \leq \kappa \quad \text{and} \quad k_N^+ := \nu_+(S_{n_N}^+) \leq k.
\]
(2) \( f \in \mathcal{M}^k(s, 2n_N - 1) \) if and only if \( f \) admits the representation
\[
f = T_{W_{2N}^+}[\tau],
\]
where
\[
W_{2N}(z) := W_{2N-1}(z)L_N = M_1(z)L_1 \ldots M_N(z)L_N
\]
and \( \tau(z) \) satisfies the conditions
\[
\tau \in \mathbb{N}^{k-k^+_N} \quad \text{and} \quad \frac{1}{\tau(z)} = o(1), \quad z \to \infty.
\]
(3) The representation (4.9) can be rewritten as a continued fraction expansion
\[
f(z) = \frac{1}{-zm_1(z) + \frac{1}{l_1 + \frac{1}{ \ldots + \frac{1}{-zm_N(z) + \frac{1}{l_N + \tau(z)}}}}},
\]
where \( m_j(z) \) and \( l_j \) are defined by (2.31) and (3.27), respectively.
(4) The indices \( \kappa_N \) and \( k^+_N \) can be found by
\[
\kappa_N = \sum_{j=1}^N \kappa_-(zm_j), \quad k^+_N = \sum_{j=1}^N k_-(m_j) + \sum_{j=1}^N \kappa_-(z \ell_j).
\]

5. Solution matrices

In the case of a regular sequence \( s \), the solution matrices \( W_{2N-1}(z) \) and \( W_{2N}(z) \) defined by (4.10) and (4.11) can be represented explicitly in terms of polynomials of the first and the second kind.

5.1. Polynomials of the first and the second kind. Let \( s = \{s_i\}_{i=0}^\ell \in \mathcal{H}_{n, \ell} \) and let the sequence \( \mathcal{N}(s) = \{n_j\}_{j=1}^N \) be extended by \( n_{-1} := -1, \ n_0 := 0 \). Recall (see [2, 6]) that polynomials of the first and the second kind are defined by
\[
P_{n_j}(\lambda) = \frac{1}{D_{n_j}} \det \begin{pmatrix}
    s_0 & s_1 & \cdots & s_{n_j} \\
    \cdots & \cdots & \cdots & \cdots \\
    s_{n_j-1} & s_{n_j} & \cdots & s_{2n_j-1} \\
    1 & \lambda & \cdots & \lambda^{n_j}
\end{pmatrix},
\]
\[
Q_{n_j}(\lambda) = \mathcal{S}_t \left( \frac{P_{n_j}(\lambda) - P_{n_j}(t)}{\lambda - t} \right) \quad (j = 1, \ldots, N),
\]
where \( \mathcal{S}_t \) is a functional defined on span \( \{1, t, \ldots, t^\ell\} \) by \( \mathcal{S}_t(t^i) = s_i \ (i = 0, 1, \ldots, \ell) \). As is known ([7] see also [9, 27]), there are real numbers \( b_0 = s_{n_1-1}, \ b_1, \) and monic polynomials \( a_j \) of degree \( n_{j+1} - n_j \ (0 \leq j \leq N - 1) \), such that the \( j \)-th convergent of the continued fraction (1.8) has the asymptotic expansion (4.2) for \( j = 1, \ldots, N \). The polynomials \( P_{n_j}(\lambda) \) and \( Q_{n_j}(\lambda) \) are solutions of the following difference equations
\[
b_{j+1}y_{n_{j+1}}(\lambda) - a_j(\lambda)y_{n_j}(\lambda) + y_{n_{j+1}}(\lambda) = 0 \quad (j = 1, \ldots, N - 1)
\]
subject to the initial conditions
\[
P_{-1}(\lambda) \equiv 0, \ P_0(\lambda) \equiv 1, \ Q_{-1}(\lambda) \equiv -1, \ Q_0(\lambda) \equiv 0.
\]
It follows from (5.2) that \( P_n(\lambda) \) and \( Q_n(\lambda) \) are monic polynomials of degree \( n_j \) and \( n_j - n_1 \), respectively. Moreover, the \( j \)-th convergent of the continued fraction (1.8) takes the form
\[
f^{[j]}(z) = -\frac{Q_n(z)}{P_n(z)} \quad (1 \leq j \leq N - 1).
\]

5.2. System of difference equations and Stieltjes polynomials. Let us consider a system of difference equations associated with the continued fraction (4.1)
\[
\begin{align*}
y_{2j} - y_{2j-2} &= l_j(z) y_{2j-1}, \\
y_{2j+1} - y_{2j-1} &= -z m_{j+1}(z) y_{2j}
\end{align*}
\]
(5.4)
If the \( j \)-th convergent of this continued fraction is denoted by \( \frac{u_j}{v_j} \), then \( u_j, v_j \) can be found as solutions of the system (see [28, Section 1]) subject to the following initial conditions
\[
u_{-1} = 1, \quad u_0 = 0; \quad v_{-1} = 0, \quad u_0 = 1.
\]
The first two convergents of the continued fraction (4.1) take the form
\[
\frac{u_1}{v_1} = \frac{1}{1 - z m_1(z)} = T_{M_1}[\infty], \quad \frac{u_2}{v_2} = \frac{l_1(z)}{-z l_1(z) m_1(z) + 1} = T_{M_1 L_1}[0].
\]
Similarly, the \((2j - 1)\)-th and \((2j)\)-th convergents
\[
\frac{u_{2j-1}}{v_{2j-1}} = T_{W_{2j-1}}[\infty], \quad \frac{u_{2j-2}}{v_{2j}} = T_{W_{2j}}[0].
\]

Theorem 5.1. Let \( s \in \mathcal{H}_{\kappa, \ell}^{k, \text{reg}} \). Then the \( 2j \)-th convergent \( \frac{u_{2j-1}}{v_{2j}} \) of the generalized S-fraction (4.1) coincides with the \( j \)-th convergent of the P-fraction (1.8) corresponding to the sequence \( s \). The parameters \( l_j \) and \( m_j(z) \) \((j \in \mathbb{Z}_+)\) of the generalized S-fraction (4.1) are connected with the parameters \( b_j \) and \( a_j(z) \) \((j \in \mathbb{N})\) of the P-fraction (1.8) by the equalities
\[
b_0 = \frac{1}{d_1}, \quad a_0(z) = \frac{1}{d_1} \left( z m_1(z) - 1 \right),
\]
(5.6)
\[
b_j = \frac{1}{d_j d_{j+1}}, \quad a_j(z) = \frac{1}{d_{j+1}} \left( z m_{j+1}(z) - \left( 1 + \frac{1}{l_{j+1}} \right) \right),
\]
(5.7)
where \( d_j \) is the leading coefficient of \( m_j(z) \) \((j = 1, \ldots, N - 1)\).

In particular, it follows from (5.7) that
\[
b_0 \ldots b_j = \frac{1}{d_{j+1}} \prod_{i=1}^{j} \left( \frac{1}{d_i l_i} \right)^2 \quad (j = 1, \ldots, N - 1).
\]

Definition 5.2. Let \( s \in \mathcal{H}_{\kappa, \ell}^{k, \text{reg}} \). Define polynomials \( P^+_j(z), Q^+_j(z) \) by
\[
P^+_{-1}(z) = 0, \quad P^+_0(z) = 1, \quad Q^+_{-1}(z) = 1, \quad Q^+_0(z) = 0,
\]
(5.9)
\[
P^+_{2i-1}(z) = \frac{-1}{b_0 \ldots b_{i-1}} \begin{vmatrix} P_n(z) & P_{n-1}(z) \\ P_n(0) & P_{n-1}(0) \end{vmatrix}, \quad P^+_{2i}(z) = \frac{P_n(z)}{P_{n-1}(0)},
\]
\[
Q^+_{2i-1}(z) = \frac{1}{b_0 \ldots b_{i-1}} \begin{vmatrix} Q_n(z) & Q_{n-1}(z) \\ Q_n(0) & Q_{n-1}(0) \end{vmatrix}, \quad Q^+_{2i}(z) = -\frac{Q_n(z)}{P_{n-1}(0)}.
\]
The polynomials \( P^+_j(z), Q^+_j(z) \) will be called the Stieltjes polynomials corresponding to the sequence \( s \).
Lemma 5.3. ([16], [19]) Let $P_n(\lambda)$ be the polynomials of the first kind, then

\begin{equation}
(5.10) \quad P_n(0) = (-1)^j \frac{1}{d_i l_i} (j = 1, \ldots, N-1),
\end{equation}

\begin{equation}
(5.11) \quad P_n(0)^2 = d_{j+1} \prod_{i=0}^{j} b_i \quad (j = 1, \ldots, N-1)
\end{equation}

\begin{equation}
(5.12) \quad P_{n-1}(0) P_n(0) = -\frac{1}{l_j} \prod_{i=0}^{j-1} b_i \quad (j = 1, \ldots, N-1).
\end{equation}

**Proof.** The first statement was proved in [19] (see also [16, Corollary 4.1]). The second statement follows from (5.10) and (5.8)

\[ P_n(0)^2 = \prod_{i=1}^{j} \frac{1}{(d_i l_i)^2} = d_{j+1} \prod_{i=0}^{j} b_i \quad (j = 1, \ldots, N-1). \]

The third statement is implied by (5.10), (5.8) and the following calculations:

\[ P_{n-1}(0) P_n(0) = -\prod_{i=1}^{j} \frac{1}{d_i l_i} \prod_{i=1}^{j-1} \frac{1}{d_i l_i} = -\frac{1}{l_j} \prod_{i=1}^{j-1} \left( \frac{1}{d_i l_i} \right)^2 = -\frac{1}{l_j} \prod_{i=0}^{j-1} b_i. \]

\[ \blacksquare \]

**Proposition 5.4.** Let $s \in \mathcal{H}_{k, reg}^\kappa$ and let $P_j^+(z)$ and $Q_j^+(z)$ be the Stieltjes polynomials defined by (5.2). Then solutions $\{u_j\}_{j=0}^{N}$ and $\{v_j\}_{j=0}^{N}$ of the system (5.3), (5.4) take the form

\begin{equation}
(5.13) \quad u_j = Q_j^+(z), \quad v_j = P_j^+(z) \quad (j = -1, 0, \ldots, N).
\end{equation}

**Proof.** Since by Definition 5.2

\[ P_{1j}^+(z) \equiv 0, \quad P_{0j}^+(z) \equiv 1, \quad Q_{1j}^+(z) \equiv 1, \quad Q_{0j}^+(z) \equiv 0, \]

it is necessary to prove the formulas

\begin{equation}
(5.14) \quad P_{2j-1}^+(z) = -zm_1(z)P_{2j-2}^+(z) + P_{2j-3}^+(z),
\end{equation}

\begin{equation}
(5.15) \quad Q_{2j-1}^+(z) = -zm_1(z)Q_{2j-2}^+(z) + Q_{2j-3}^+(z),
\end{equation}

First, we prove the formula (5.14). Calculating $P_1^+(z)$ and $P_2^+(z)$, and using (5.2), (5.3) and (5.6) we get

\[ P_1^+(z) = -b_0^{-1} \begin{vmatrix} P_{n1}(z) & P_{n0}(z) \\ P_{n1}(0) & P_{n0}(0) \end{vmatrix} = -d_1 \begin{vmatrix} \frac{zm_1(z)}{d_1} - \frac{1}{d_1 l_1} & 1 \\ -\frac{1}{d_1 l_1} & 1 \end{vmatrix} = -zm_1(z) = -zm_1(z)P_0^+(z) + P_1^+(z),
\]

\[ P_2^+(z) = \frac{P_{n1}(z)}{P_{n1}(0)} = \frac{zm_1(z)}{d_1} - \frac{1}{d_1 l_1} = -l_1 zm_1(z) + 1 = l_1 P_1^+(z) + P_0^+(z).
\]
Next, by \((5.2), (5.3)\) and \((5.6)\) one gets for \(i = 1, \ldots, N\)
\[
P_{2i-1}^+(z) = -\frac{1}{b_0 \ldots b_{i-1}} \begin{vmatrix} P_{n_i}(z) & P_{n_i-1}(z) \\ P_{n_i}(0) & P_{n_i-1}(0) \end{vmatrix}
= -\frac{1}{b_0 \ldots b_{i-1}} \left( \frac{z m_i(z) P_{n_i-1}(0)}{d_i} + a_{i-1}(0) P_{n_i-1}(z) - b_{i-1} P_{n_i-2}(z) \right) \frac{P_{n_i-1}(z)}{P_{n_i-1}(0)}
= -z m_i(z) \frac{P_{n_i-1}(0)}{d_i b_0 \ldots b_{i-1}}
- \frac{P_{n_i-1}(z)(a_{i-1}(0) P_{n_i-2}(0) - P_{n_i}(0)) - b_{i-1} P_{n_i-2}(z) P_{n_i-1}(0)}{b_0 \ldots b_{i-1}}
\]

Using \((5.11)\) and \((5.2)\) one obtains
\[
\frac{P_{n_i-1}(0)}{d_i b_0 \ldots b_{i-1}} = \frac{1}{P_{n_i-1}(0)}, \quad a_{i-1}(0) P_{n_i-2}(0) - P_{n_i}(0) = b_{i-1} P_{n_i-2}(0)
\]
and hence by \((5.9)\)
\[
P_{2i-1}^+(z) = -z m_i(z) P_{2i-2}^+(z) + P_{2i-3}^+(z).
\]
This proves the first equality in \((5.14)\). The second equality in \((5.14)\) is immediate from Definition \((5.4)\) and \((5.12)\). Indeed,
\[
P_{2i}^+(z) - P_{2i-2}^+(z) = \frac{P_{n_i}(z)}{P_{n_i-1}(0)} - \frac{P_{n_i-1}(z)}{P_{n_i-1}(0)} = \frac{1}{P_{n_i}(0) P_{n_i-1}(0)} \begin{vmatrix} P_{n_i}(z) & P_{n_i-1}(z) \\ P_{n_i}(0) & P_{n_i-1}(0) \end{vmatrix}
= -\frac{l_i}{b_0 \ldots b_{i-1}} \begin{vmatrix} P_{n_i}(z) & P_{n_i-1}(z) \\ P_{n_i}(0) & P_{n_i-1}(0) \end{vmatrix} = l_i P_{2i-1}^+(z).
\]
Similarly, one proves the recurrence formula \((5.15)\). □

**Remark 5.5.** Notice, that the formula \((5.12)\) corresponds to the formula \((4.20)\) in [16, Corollary 4.1], which contains a misprint. The formulas \((5.10)\) with formally different coefficients \(\gamma_i = (l_i P_{n_i}(0) P_{n_i-1}(0))^{-1}\) were found in [16, Corollary 4.1]. However, these formulas coincide, since by \((5.12)\)
\[
l_i P_{n_i}(0) P_{n_i-1}(0) = -b_0 \ldots b_{i-1}.
\]

5.3. **Odd case.** Let \(s = \{s_i\}_{i=0}^{2N-2}\) and let \(N(s) = \{n_j\}_{j=1}^{2N}\) be the set of normal indices of the sequence \(s\). Then polynomials \(m_j\) \((1 \leq j \leq N)\) and numbers \(l_j\) \((1 \leq j \leq N - 1)\) are well defined by the formulas \((5.3)\), and thus the polynomials \(P_j^+\) and \(Q_j^+\) \((1 \leq j \leq 2N - 1)\) can be computed by the formulas \((5.14)\) and \((5.15)\). Notice, that the polynomials \(P_N\) and \(Q_N\) cannot be calculated by the formulas \((5.1)\) unless the sequence \(s = \{s_i\}_{i=0}^{2N-2}\) is completed by one more number \(s_{2N-1}\). As follows from Proposition \((5.4)\), the choice of \(s_{2N-1}\) does not impact the coefficients of \(P_{2N-1}^+\) and \(Q_{2N-1}^+\) in \((5.9)\), but does impact \(P_{2N}^+\) and \(Q_{2N}^+\).

**Lemma 5.6.** Let \(s = \{s_i\}_{i=0}^{2N-2} \in \mathcal{H}^k_{\kappa,2N-2}, N(s) = \{n_j\}_{j=1}^{2N}\), \(\kappa, k \in \mathbb{Z}_+, N \in \mathbb{N}\), let polynomials \(m_j(z)\) \((1 \leq j \leq N)\) and numbers \(l_j\) \((1 \leq j \leq N - 1)\) be defined by \((5.3)\), and let the matrices \(M_j(z), L_j\) and \(W_{2j-1}(z)\) be defined by
\[
M_j(z) = \begin{pmatrix} 1 & 0 \\ -z m_j(z) & 1 \end{pmatrix}, \quad L_j = \begin{pmatrix} 1 \ 0 \\ 0 \ 1 \end{pmatrix}, \quad j = 1, N,
\]
\[
W_{2j-1}(z) = M_1(z) L_1 \ldots L_{j-1} M_j(z),
\]
Then the matrix $W_{2j-1}(z)$ admits the following representation

\begin{equation}
W_{2j-1}(z) = \begin{pmatrix}
Q_{2j-1}^+(z) & Q_{2j-2}^+(z) \\
P_{2j-1}^+(z) & P_{2j-2}^+(z)
\end{pmatrix} \quad (j = 1, \ldots, N),
\end{equation}

where the polynomials $P_j^+(z)$ and $Q_j^+(z)$ ($0 \leq j \leq 2N-1$) are defined either by \((5.14)\) and \((5.15)\), or by \((5.9)\).

**Proof.** Let us prove \((5.18)\) by induction. If $j = 1$, then $W_1(z) = M_1(z)$ and hence

$$W_1(z) = \begin{pmatrix} 1 & 0 \\ -zm_j(z) & 1 \end{pmatrix} = \begin{pmatrix} Q_1^+(z) & Q_0^+(z) \\ P_1^+(z) & P_0^+(z) \end{pmatrix}.$$ 

Assume that \((5.18)\) holds for some $j < N$ and let us prove \((5.18)\) for $j := j + 1$. By \((5.14)\) and \((5.15)\)

$$W_{2j+1}(z) = M_1(z)L_1 \ldots L_j M_j+1(z) = W_{2j-1}(z)L_j M_j+1(z) =$$

\begin{align*}
&= \begin{pmatrix} Q_{2j-1}^+(z) & Q_{2j-2}^+(z) \\ P_{2j-1}^+(z) & P_{2j-2}^+(z) \end{pmatrix} \begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -zm_{j+1}(z) & 1 \end{pmatrix} = \\
&= \begin{pmatrix} Q_{2j-1}^+(z) & Q_{2j}^+(z) \\ P_{2j-1}^+(z) & P_{2j}^+(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -zm_{j+1}(z) & 1 \end{pmatrix} = \\
&= \begin{pmatrix} Q_{2j}^+(z) \\ P_{2j}^+(z) \end{pmatrix}.
\end{align*}

This completes the proof. □

Combining Theorem 4.2 and Lemma 5.6 one obtains the following

**Theorem 5.7.** Let $s = \{s_i\}_{i=0}^{2N-2} \in \mathcal{H}_{k;2N-2}^{k,reg}$, $N(s) = \{n_j\}_{j=1}^N$ ($k, \ell \in \mathbb{Z}_+$, $N \in \mathbb{N}$) and let the polynomials $P_j^+(z)$ and $Q_j^+(z)$ ($0 \leq j \leq 2N-1$) be defined by \((5.14)\) and \((5.15)\). Then:

1. A nondegenerate odd moment problem $MP_k^+(s, 2N-2)$ is solvable, if and only if

\begin{equation}
k_N := \nu_-(S_{n,N}^+) \leq \kappa \quad \text{and} \quad k_N^+ := \nu_-(S_{n,N-1}^+) \leq k.
\end{equation}

2. $f \in \mathcal{M}_k^+(s, 2N-2)$ if and only if $f$ admits the representation

\begin{equation}
f(z) = \frac{Q_{2N-1}^+(z)}{P_{2N-1}^+(z)} \tau(z) + \frac{Q_{2N-2}^+(z)}{P_{2N-2}^+(z)}
\end{equation}

where

\begin{equation}
\tau \in N_{k-k_N^+} \quad \text{and} \quad \frac{1}{\tau(z)} = o(z), \quad z \to \infty.
\end{equation}

**5.4. Even case.** Consider now the case when $s = \{s_i\}_{i=0}^N \in \mathcal{H}_k^{k,\ell}$ and $\ell$ is odd, i.e. $\ell = 2N - 1$ and $n_N$ is the largest normal index of $s$, $N(s) = \{n_j\}_{j=1}^N$. Then polynomials $m_j$ and numbers $l_j$ ($1 \leq j \leq N - 1$) are well defined by the formulas \((4.10)\) for all $j = 1, N$, and thus the polynomials $P_j^+$ and $Q_j^+$ ($1 \leq j \leq 2N$) can be computed by \((5.14)\) and \((5.15)\). In the even case the polynomials $P_{2j}$ and $Q_{2j}$ are also well defined by the formulas \((5.11)\) for all $j = 1, 2N$ and, thus, $P_j^+$ and $Q_j^+$ for $j = 1, N$ can be computed by the formulas \((5.9)\) as well.
Lemma 5.8. Let $s = \{s_i\}_{i=0}^{2nN-1} \in H^k_{\kappa,2nN-1}$, $N(s) = \{n_j\}_{j=1}^N (\kappa, k \in \mathbb{Z}_+, N \in \mathbb{N})$, let $n_j(z), l_j, M_j(z)$ and $L_j (1 \leq j \leq N-1)$ be defined by (5.15), (5.16) and let the matrix $W_{2j}(z)$ be defined by

\[(5.22) \quad W_0(z) = I, \quad W_{2j}(z) = M_1(z)L_1 \ldots M_j(z)L_j \quad (j = 0, \ldots, N). \]

Then the matrix $W_{2j}(z)$ admits the following representation

\[(5.23) \quad W_{2j}(z) = \begin{pmatrix} Q_{2j-1}(z) & Q_{2j}^+(z) \\ P_{2j-1}(z) & P_{2j}^+(z) \end{pmatrix} \quad (j = 0, \ldots, N), \]

where the polynomials $P_j^+(z)$ and $Q_j^+(z) (-1 \leq j \leq 2N)$ are defined by (5.9).

**Proof.** Let us prove (5.22) by induction. If $j = 0$, then $W_0 = I$ and hence by (5.9)

\[W_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Q_1^+(z) & Q_0^+(z) \\ P_0^+(z) & P_0^+(z) \end{pmatrix}. \]

Assume that (5.22) holds for some $j < N$. Then by (5.14) and (5.15)

\[W_{2j}(z) = M_1(z)L_1 \ldots M_j(z)L_j = W_{2j-2}(z)M_j(z)L_j = \]

\[= \begin{pmatrix} Q_{2j-3}(z) & Q_{2j-2}(z) \\ P_{2j-3}(z) & P_{2j-2}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -zm_j(z) & 1 \end{pmatrix} = \begin{pmatrix} Q_{2j-1}(z) & Q_{2j-2}(z) \\ P_{2j-1}(z) & P_{2j-2}(z) \end{pmatrix}. \]

This proves (5.22). \(\square\)

Combining Theorem 4.3 and Lemma 5.8 one obtains the following

**Theorem 5.9.** Let $s = \{s_i\}_{i=0}^{2nN-1} \in H^k_{\kappa,2nN-1}$, $N(s) = \{n_j\}_{j=1}^N (\kappa, k \in \mathbb{Z}_+, N \in \mathbb{N})$ and let the polynomials $P_j^+(z)$ and $Q_j^+(z) (0 \leq j \leq 2N)$ be defined by (5.14) and (5.15). Then:

1. The even moment problem $MP^k_\kappa(s, 2nN - 1)$ is solvable, if and only if

\[(5.24) \quad \kappa_N := n_-(S_{nN}) \leq \kappa \quad \text{and} \quad k_N^+ := n_-(S^+_{nN}) \leq k. \]

2. $f \in M^k_\kappa(s, 2nN - 1)$ if and only if $f$ admits the representation

\[(5.25) \quad f(z) = \frac{Q_{2N}^+(z) + Q_{2N-1}^+(z)\tau(z)}{P_{2N}^+(z) + P_{2N-1}^+(z)\tau(z)}, \]

where

\[(5.26) \quad \tau \in N^{k-k_N^+}_{\kappa-\kappa_N} \quad \text{and} \quad \tau(z) = o(1), \quad z \to \infty. \]

**References**

[1] D. Alpay, A. Dijksma, H. Langer. Factorization of $J$-unitary matrix polynomials on the line and a Schur algorithm for generalized Nevanlinna functions. *Linear Algebra Appl.*, 387:313–342, 2004.

[2] N.I. Akhiezer. *The classical moment problem*. Oliver and Boyd, Edinburgh, 1965.

[3] R. Beals, D.H. Sattinger, J. Szmigielski. Multipeakons and the classical moment problem. *Adv. Math.*, 154(2):229–257, 2000.

[4] R.E. Curto, L.A. Fialkow. Recursiveness, positivity, and truncated moment problems. *Houston J. Math.*, 17:603–635, 1991.
[5] M. Derevyagin. On the Schur algorithm for indefinite moment problem. Methods Functional Anal. Topol., 9(2):133–145, 2003.
[6] M. Derevyagin, V.Derkach. Spectral problems for generalized Jacobi matrices. Linear Algebra Appl., 382:1–24, 2004.
[7] M. Derevyagin, V.Derkach. Darboux transformations of Jacobi matrices and Padé approximation. Linear Algebra and Its Applications, 435(12):3056–3084, 2012.
[8] V. Derkach. Generalized resolvents of a class of Hermitian operators in a Krein space. Dokl. Akad. Nauk SSSR, 317(4):807–812, 1991.
[9] V. Derkach. On Weyl function and generalized resolvents of a Hermitian operator in a Krein space. Integral Equations Operator Theory, 23:387–415, 1995.
[10] V. Derkach. On indefinite moment problem and resolvent matrices of Hermitian operators in Krein spaces. Math. Nachr., 184:135–166, 1997.
[11] V. Derkach, M. Malamud. On Weyl function and Hermitian operators with gaps. Doklady Akad. Nauk SSSR, 293(5):1041–1046, 1987.
[12] V. Derkach, M. Malamud. The extension theory of Hermitian operators and the moment problem. J.of Math.Sci., 73(2):141–242, 1995.
[13] V. Derkach, M. Malamud. On some classes of Holomorphic Operator Functions with Nonnegative Imaginary Part. In 16th OT Conference Proceedings, Operator theory, operator algebras and related topics, pages 113–147, Timisoara, 1997. Theta Found.Bucharest.
[14] T.J. Stieltjes. Recherches sur les fractions continues. Ann. Fac. Sci. de Toulouse, 8:1–122, 1894.
[15] F.R. Gantmacher. The theory of matrices. Chelsey Publishing Company, New York, 1964.
[16] I. Kovalyov. Darboux transformation of generalized Jacobi matrices. Methods of Funct. Anal. and Topology, 20(4):301–320, 2014.