STANLEY’S CONJECTURE FOR CRITICAL IDEALS
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Abstract. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables over a field $K$. Stanley’s conjecture holds for the modules $I$ and $S/I$, when $I \subset S$ is a critical monomial ideal. We calculate the Stanley depth of $S/I$ when $I$ is a canonical critical monomial ideal. For non-critical monomial ideals we show the existence of a Stanley ideal with the same depth and Hilbert function.

1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring with standard grading over a field $K$. Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. Any decomposition of the module $M$ as a finite direct sum of $\mathbb{Z}^n$-graded $K$-subspaces of the form $uK[Z]$, where each $uK[Z]$ is a free $K[Z]$-module, is called a Stanley decomposition of the module $M$. In other words, a Stanley decomposition of $M$ has the form $D: M = \bigoplus_{i=1}^{r} u_i K[Z_i]$, where the $u_i \in M$ are homogenous elements and each $Z_i$ is a subset of $\{x_1, \ldots, x_n\}$. The number $sdepth(D) = \min\{|Z_i|, i = 1, 2, \ldots, r\}$ is called the Stanley depth of the decomposition $D$. The Stanley depth of the module $M$ is defined to be $sdepth(M) = \max\{sdepth(D) : D \text{ is Stanley decomposition}\}$.

In [8] Stanley conjectured that $sdepth(M) \geq \text{depth}(M)$. We call $I$ a Stanley ideal, if Stanley’s conjecture holds for $S/I$. There are not many known classes of Stanley ideals [6].

Let $I \subset S$ be a monomial ideal. We denote by $G(I)$ the unique minimal monomial system of generators of $I$ and $H_{S/I}$ the Hilbert function of the quotient algebra $S/I$. Consider the lexicographic order $<_{\text{lex}}$ on $S$ induced by the ordering $x_1 > x_2 > \ldots > x_n$ of the variables. A lexsegment ideal is a monomial ideal $I$ such that for a monomial $u \in I$ and for a monomial $v \in S$ with $\deg u = \deg v$ and $v >_{\text{lex}} u$, one has $v \in I$. A lexsegment ideal $I$ is called a universal lexsegment ideal if $I$ is a lexsegment ideal in $K[x_1, \ldots, x_{n+m}]$ for any natural number $m \geq 0$.

Recall that for any graded ideal $I \subset S$, there exists a unique lexsegment ideal, denoted by $I^{\text{lex}}$, such that $S/I$ and $S/I^{\text{lex}}$ have the same Hilbert function. Hibi and Murai [4] call a monomial ideal $I$ critical if $I^{\text{lex}}$ is universal lexsegment.

Let $m_1, m_2, \ldots, m_t \in \text{Mon}(S)$ for $1 \leq t \leq n$ where $m_i \in K[x_1, x_{i+1}, \ldots, x_n]$ and $\deg m_t > 0$. Then we define the ideal

(1) $I_{(m_1, m_2, \ldots, m_t)} = (x_1m_1, x_2m_1m_2, \ldots, x_{t-1}m_1m_2 \cdots m_{t-1}, m_1m_2 \cdots m_t)$

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In analogy to the definition of Hibi and Murai [5] we call a monomial ideal canonical critical, if it is of the form $I_{(m_1,m_2,...,m_t)}$, up to the permutation of variables. By [5, Theorem 1.1] canonical critical ideals are critical. In Lemma 2.1 we show that Stanley’s conjecture holds for $I$ and $S/I$. For a canonical critical monomial ideal $I$ we calculate the Stanley depth of $S/I$ (Theorem 2.2), and obtain a Stanley decomposition (Theorem 2.3) which exactly gives the Stanley depth of $S/I$. We also show that for a canonical critical monomial ideal one has $sdepth(I) \geq 1 + sdepth(S/I)$, thereby giving in this special case an affirmative answer to a question raised by Rauf in [7]. In Proposition 2.6 we show that for each non critical monomial ideal $I$ there exists a Stanley ideal which has the same depth and Hilbert function as the ideal $I$.

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2. STANLEY DEPTH AND CRITICAL MONOMIAL IDEALS

First we show that the Stanley conjecture holds for modules $I$ and $S/I$, when $I$ is critical monomial ideal.

**Lemma 2.1.** Let $I \subset S$ be a critical monomial ideal. Then

(i) $sdepth(S/I) \geq \depth(S/I)$;

(ii) $sdepth(I) \geq \depth(I)$.

**Proof.** (i) If $I \subset S$ is a critical monomial ideal, then $\depth(S/I) = n - |G(I)|$ (see [4, Theorem 1.6]) and for any monomial ideal $I$, $sdepth(S/I) \geq n - |G(I)|$ (see [2, Proposition 1.3]).

(ii) follows from the fact $\depth(I) = 1 + \depth(S/I)$ and $sdepth(I) \geq \max\{1, n - |G(I)| + 1\}$ (see [3, Proposition 3.4]).

We show that the equality holds in (i) of Lemma 2.1 for canonical critical monomial ideals.

**Theorem 2.2.** Let $I \subset S$ be a canonical critical monomial ideal. Then $sdepth(S/I) = \depth(S/I) = n - |G(I)|$.

**Proof.** If $I$ be a canonical critical monomial ideal, then

$$I = (x_1m_1, x_2m_1m_2, \ldots, x_{t-1}m_1m_2 \cdots m_{t-1}, m_1m_2 \cdots m_t),$$

where for $1 \leq t \leq n$, $m_i$ is a monomial belonging to $K[x_i, x_{i+1}, \ldots, x_n]$ and $\deg m_t > 0$.

We set $S_i = K[x_i, x_{i+1}, \ldots, x_n]$ and for $1 \leq i \leq t - 1$ we define the ideals

$$I_i = (x_im_i, x_{i+1}m_{i+1}, \ldots, x_{t-1}m_{i+1}m_{i+2} \cdots m_{t-1}, m_{i+1}m_{i+2} \cdots m_t),$$

and

$$I'_i = (x_1, x_{i+1}m_{i+1}, \ldots, x_{t-1}m_{i+1}m_{i+2} \cdots m_{t-1}, m_{i+1}m_{i+2} \cdots m_t).$$

Moreover, we set $I'_t = (m_t)$.

Then $S = S_1$, $I = I_1$, $I_i = m_iI'_i$ and $I'_i = (x_i, I_{i+1})$. The ideals $I_i$, $I'_i$ and $I'_t$ are critical monomial ideals in $S_j = [x_j, x_{j+1}, \ldots, x_n]$ for each $1 \leq j \leq i$.

By [2, Proposition 1.3] and the fact that $\gcd\{u | u \in G(I)\} = m_1$, we have $sdepth(S/I) = sdepth(S/I'_1)$, and further $sdepth(S/I'_1) = sdepth(S/2/2)$, since $S/I'_1 \cong S_2/I_2$. Hence we
conclude that \( \text{sdepth}(S/I) = \text{sdepth}(S_2/I_2) \). Continuing in this way we get \( \text{sdepth}(S/I) = \text{sdepth}(S_t/I'_t) = n - |G(I)| \). The last equation follows since \( I'_t = (m_t) \) is a principal monomial ideal, and since \( t = |G(I)| \) (see [5, Corollary 2.3]).

\[ \square \]

**Corollary 2.3.** Let \( I \) be a canonical critical monomial ideal. Then

\[ \text{sdepth}(I) \geq 1 + \text{sdepth}(S/I). \]

**Proof.** If \( I \) is a canonical critical monomial ideal, then by using [3, Proposition 3.4] we have \( \text{sdepth}(I) \geq n - |G(I)| + 1 \), and the assertion follows from Theorem 2.2. \[ \square \]

For a canonical critical monomial ideal \( I \), the following decomposition is obtained from [5, Lemma 2.2].

**Lemma 2.4.** As a vector space over \( K \) the canonical critical monomial ideal \( I \) is the direct sum \( I = \bigoplus_{j=1}^{l-1} x_j (\prod_{k=1}^{j} m_k) K[x_j, x_{j+1}, \ldots, x_n] \bigoplus \prod_{k=1}^{l} m_k K[x_t, \ldots, x_n] \).

Using the decomposition in Lemma 2.4 we will give an explicit Stanley decomposition of \( S/I \) which in fact gives the Stanley depth.

Let \( S_i = K[x_1, \ldots, x_n] \) and \( m_i \in S_i \) with \( \deg m_i = d_i \) for \( 1 \leq i \leq t \leq n \). We define for any monomial \( m \in S \) a positive number \( v(m) = \min\{k \mid x_k \text{ divides } m\} \) and monomials \( w_i(d+1) = \frac{w_i d}{x_v(w_i d)} \), where \( w_i = m_i^{-1}, 1 \leq d \leq d_i - 1 \) and \( 1 \leq i \leq t \). For monomial \( m_i \) we have \( m_i = \prod_{j=1}^{d_i} x_v(w_{ij}) \). Finally, by using the monomials \( w_{id} \), we define \( u_{ij} = \prod_{k=1}^{j-1} x_v(w_{ik}) \) with \( u_{i1} = 1 \) and \( Z_{ij} = \{x_i, x_{i+1}, \ldots, x_n\} \setminus \{x_v(w_{ij})\} \).

With the notation introduced we have,

**Theorem 2.5.** Let \( I = I_{(m_1, m_2, \ldots, m_t)} \) be a critical monomial ideal. We set \( n_1 = 1 \) and \( n_i = m_1 m_2 \cdots m_{i-1} \) for \( i = 2, \ldots, t \). Then for \( S/I \) we have the following Stanley decomposition

\[ \mathcal{D} : S/I = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{d_i} u_{ij} n_i K[Z_{ij}] \quad \text{with } d_i = \deg(m_i). \]

Moreover we have \( \text{sdepth}(\mathcal{D}) = \text{sdepth}(S/I) \).

**Proof.** We decompose \( S_i \) for the variable \( x_v(w_{i1}) = u_{i2} \),

\[ S_i = K[Z_{i1}] \oplus x_v(w_{i1}) S_i. \]

Again for \( x_v(w_{i2}) \) we decompose \( S_i \) in the above equation,

\[ S_i = K[Z_{i1}] \oplus x_v(w_{i1}) K[Z_{i2}] \oplus x_v(w_{i1}) x_v(w_{i2}) S_i = u_{i1} K[Z_{i1}] \oplus u_{i2} K[Z_{i2}] \oplus u_{i3} S_i. \]

We know that \( m_i = \prod_{j=1}^{d_i} x_v(w_{ij}) \), so continuing in this way we obtain

\[ S_i = \bigoplus_{j=1}^{d_i} u_{ij} k[Z_{ij}] \oplus m_i S_i. \]

(2)
As $S_i = S_{i+1} \oplus x_iS_i$ for $1 \leq i \leq t - 1$, it follows from (2) that

$$S_i = \bigoplus_{j=1}^{d_i} u_{ij}k[Z_{ij}] \oplus m_iS_{i+1} \oplus x_im_iS_i. \quad (3)$$

By using the above recursive relation for $1 \leq i \leq t - 1$ we get

$$S_1 = \bigoplus_{i=1}^{t-1} \bigoplus_{j=1}^{d_i} u_{ij}n_iK[Z_{ij}] \oplus \bigoplus_{j=1}^{t-1} x_jn_{j+1}S_j \oplus n_tS_t. \quad (4)$$

Now for $S_t$ we substitute the decomposition obtained in (2)

$$S_1 = \bigoplus_{i=1}^{t-1} \bigoplus_{j=1}^{d_i} u_{ij}n_iK[Z_{ij}] \oplus \bigoplus_{j=1}^{t-1} x_jn_{j+1}S_j \oplus n_tS_t,$$

and obtain

$$S_1 = \bigoplus_{i=1}^{t-1} \bigoplus_{j=1}^{d_i} u_{ij}n_iK[Z_{ij}] \oplus \bigoplus_{j=1}^{t-1} x_jn_{j+1}S_j \oplus n_tS_t. \quad (5)$$

Since $I$ is a critical monomial ideal, Lemma 2.3 implies that $I = \bigoplus_{j=1}^{t-1} x_jn_{j+1}S_j \oplus n_tS_t$ and that $S_1 = S$. Thus (5) yields

$$S/I = \bigoplus_{i=1}^{t-1} \bigoplus_{j=1}^{d_i} u_{ij}n_iK[Z_{ij}], \quad (6)$$

where for each $i$ and $j$ we have $|Z_{ij}| = n - i \geq n - t$. This implies $|Z_{ij}| \geq n - |G(I)|$ as $t = |G(I)|$. Hence the desired conclusion follows from Theorem 2.2.

Recall that a numerical function $H: \mathbb{N} \to \mathbb{N}$ is the Hilbert function of $S/I$ for some graded ideal $I \subset S = K[x_1, \ldots, x_n]$, if and only if $H(0) = 1$, $H(1) \leq n$ and $H(d + 1) \leq H(d)^{<d>}$ for all $d \geq 1$ (See [1] Theorem 4.2.10).

If $I$ is a non critical monomial ideal we prove the following.

**Proposition 2.6.** Let $K$ be an infinite field. Then for any non critical monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ there exists a Stanley ideal $L \subset S$ such that $S/I$ and $S/L$ have the same depths and the same Hilbert function.

**Proof.** Let $I \subset S$ be a non critical monomial ideal and $I^{\text{lex}} \subset S$ is the corresponding lexsegment ideal then by [1] Corollary 1.3 we have $|G(I^{\text{lex}})| > n$. If depth$(S/I) = b$, then there exists a regular sequence $(\theta_1, \theta_2, \ldots, \theta_b)$ of $S/I$ with each deg$(\theta_i) = 1$. It then follows that there exists a homogeneous ideal $J$ of $S'$ such that $J \subset K[x_1, \ldots, x_{n-b}]$ such that the ideal $JS$ of $S$ satisfies $H_{S/JS} = H_{S/I}$.

We now claim that the lexsegment ideal $J^{\text{lex}} \subset S'$ of $J$ cannot be universal lexsegment. In fact, if $J^{\text{lex}}$ is universal lexsegment, then $J^{\text{lex}}$ remains being lexsegment in the polynomial ring $K[x_1, \ldots, x_m]$ for each $m \geq n - b$. In particular, the ideal $J^{\text{lex}}S$ of $S$ is universal lexsegment. Since $H_{S/JS} = H_{S/J^{\text{lex}}S} = H_{S/I}$, it follows that $I^{\text{lex}} = J^{\text{lex}}S$, because we
have unique lexsegment ideal corresponding to every ideal \( I \subset S \). Thus \( I \) has a universal lexsegment ideal \( I^{\text{lex}} \), which is a contradiction.

Since the lexsegment ideal \( J^{\text{lex}} \) of \( J \) cannot be universal lexsegment, it follows from [4, Corollary 1.3,1.4] that \( \text{depth}(S'/J^{\text{lex}}) = 0 \). Thus \( \text{depth}(S/J^{\text{lex}}S) = b \leq \text{sdepth}_S(S'/J^{\text{lex}}) + b = \text{sdepth}_S(S/J^{\text{lex}}S) \), by ([3, Lemma 3.6]). \( \square \)

**References**

[1] W. Bruns and J. Herzog, Cohen-Macaulay rings, Revised Edition, Cambridge University Press, 1998.
[2] M. Cimpoeas, Stanley depth of monomial ideals in three variables, arXiv: math.AC/0807.2166v3 (2008).
[3] J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal, to appear in J. Alg.
[4] S. Murai, T. Hibi, The depth of an ideal with a given Hilbert function, Proc. Am. Math. Soc. **136**, 1533-1538 (2008).
[5] S. Murai, T. Hibi, Gotzmann ideals of the polynomial ring, Math. Z. **260**, 629-646 (2008).
[6] D. Popescu, Stanley depth of multigraded modules, arXiv: math.AC/0801.2632v1 (2008).
[7] A. Rauf, Depth and Stanley depth of multigraded modules, to appear in Communications in Algebra.
[8] R. P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. **68**, 175-193 (1982).

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