SUPPORT OF EXTREMAL DOUBLY STOCHASTIC ARRAYS

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Abstract. An $n \times m$ array with nonnegative entries is called doubly stochastic if the sum of its entries at each column is $n$ and at each row is $m$. The set of all $n \times m$ doubly stochastic arrays is a convex polytope with finitely many extremal points. The main result of this paper characterizes the possible sizes of the supports of all extremal $n \times m$ doubly stochastic arrays. In particular we prove that the minimal size of the support of an $n \times m$ doubly stochastic array is $n + m - \gcd(n, m)$. Moreover, for $m = kn + 1$ we also characterize the structure of the support of the extremal arrays.

1. Introduction

1.1. An $n \times m$ matrix $A = (a_{ij})$ is called a doubly stochastic array (with uniform marginals) if its entries are nonnegative, $a_{ij} \geq 0$, and
\[
\sum_{i=1}^{n} a_{ij} = n, \quad 1 \leq j \leq m, \tag{1.1}
\]
\[
\sum_{j=1}^{m} a_{ij} = m, \quad 1 \leq i \leq n, \tag{1.2}
\]
that is, the sum of the entries at each column is $n$ and at each row is $m$.

For example (see [Car96, p. 65]), the two matrices
\[
\begin{bmatrix}
3 & 0 & 0 & 1 \\
0 & 3 & 0 & 1 \\
0 & 0 & 3 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 3 \\
2 & 2 & 0 & 0 \\
0 & 1 & 3 & 0
\end{bmatrix}
\tag{1.3}
\]
are $3 \times 4$ doubly stochastic arrays.

We will use $S_{n,m}$ to denote the set of all $n \times m$ doubly stochastic arrays. This set is a convex polytope (of dimension $(n-1)(m-1)$) in the linear space of all $n \times m$ matrices. Hence $S_{n,m}$ coincides with the convex hull of its finitely many extremal points. We recall that an element $A \in S_{n,m}$ is extremal if it cannot be represented as a convex combination of two other elements of $S_{n,m}$ that are both different from $A$.

For square $n \times n$ matrices, a classical theorem due to Birkhoff [Bir46] characterizes the extremal doubly stochastic arrays as follows: an array $A \in S_{n,n}$ is extremal if and only if $\frac{1}{n}A$ is a permutation matrix.

Let us say that two $n \times m$ matrices are equivalent if one can be transformed into the other by a permutation of rows and columns. In this language, Birkhoff’s theorem...
states that up to equivalence there is only one extremal \( n \times n \) doubly stochastic array, namely, the identity matrix times the scalar \( n \).

\textit{Nonsquare} \( n \times m \) extremal doubly stochastic arrays have been studied by different authors, see [Car96], [Li96] and the references therein. This case is more complicated and in general there is more than one equivalence class of extremal arrays. For example, the two matrices in (1.3) are both extremal but not equivalent.

1.2. In [KP21, Question 7] the authors posed the following question, which arose in connection with a tiling problem in finite abelian groups: what is the smallest possible size of the support of a doubly stochastic \( n \times m \) array?

We recall that the \textit{support} of an \( n \times m \) matrix \( A = (a_{ij}) \) is the set
\[
\text{supp } A = \{(i, j) : a_{ij} \neq 0\}.
\]

In this paper we obtain a complete answer to the question above:

\textbf{Theorem 1.1.} For all \( n, m \) the minimal size of the support of an array \( A \in S_{n,m} \) is equal to \( n + m - \gcd(n, m) \).

This result was also obtained by M. Loukaki in [Lou22] and we acknowledge her priority. In [KP21, Section 4] the authors established that the result holds in the special case where either \( m = kn \) or \( m = kn + 1 \).

1.3. It is known, see [Li96, Corollary 2], that an array \( A \in S_{n,m} \) is extremal if and only if its support is minimal with respect to inclusion, that is, there is no \( B \in S_{n,m} \) such that \( \text{supp } B \) is a proper subset of \( \text{supp } A \). This implies that any array which minimizes the size of the support must also be an extremal array.

For example, each one of the two \( 3 \times 4 \) matrices in (1.3) has \( 6 = 3 + 4 - 1 \) nonzero entries, which is the minimal size of the support of a doubly stochastic \( 3 \times 4 \) array. It follows that these two matrices are extremal.

One can thus pose a more general version of the question above, namely, what are the possible sizes of the support of an extremal \( n \times m \) doubly stochastic array?

In the special case where one of \( n, m \) divides the other, say, if \( m = kn \), it has been known that there is only one equivalence class of extremal \( n \times kn \) doubly stochastic arrays, and the support size of every extremal array is \( kn \), see [Car96, Proposition 4].

Our next result provides a complete answer to the question in the more complicated case where neither one of \( n, m \) divides the other. That is, the result characterizes the possible sizes of the supports of all extremal \( n \times m \) arrays.

\textbf{Theorem 1.2.} Assume that neither one of \( n, m \) divides the other. Then any extremal array \( A \in S_{n,m} \) has support size of the form
\[
|\text{supp } A| = n + m - s \tag{1.4}
\]

where \( 1 \leq s \leq \gcd(n, m) \). Conversely, for any integer \( s, 1 \leq s \leq \gcd(n, m) \), there exists an extremal array \( A \in S_{n,m} \) whose support size is given by (1.4).

For certain special values of \( n, m \) and \( s \) this result was obtained in [Lou22, Theorem 3].
It follows from Theorem 1.2 that if \( n, m \) are coprime, then all the extremal \( n \times m \) arrays are of the same support size, namely, \( n + m - 1 \). Hence an array is extremal if and only if it minimizes the support size among all the \( n \times m \) doubly stochastic arrays. So in this case one can characterize extremality by the size of the support:

**Theorem 1.3.** Let \( n, m \) be coprime. Then an array \( A \in S_{n,m} \) is extremal if and only if 
\[
|\text{supp } A| = n + m - 1.
\]

This result was obtained also in [Lou22] using a different approach. In the special case where \( m = kn + 1 \) the result was proved in [Car96, Proposition 6].

If one of \( n, m \) divides the other, say, \( m = kn \), then there is only one equivalence class of extremal arrays and the support size of every extremal array is equal to \( kn \). Hence in this case an array \( A \in S_{n,kn} \) is extremal if and only if 
\[
|\text{supp } A| = kn.
\]

However, for all the remaining values of \( n, m \), that is, if \( n, m \) are neither coprime nor any one of them divides the other, there is no characterization of extremality by the size of the support. One can show that the only sufficient condition for the extremality of \( A \in S_{n,m} \) is 
\[
|\text{supp } A| = n + m - \gcd(n, m),
\]
but this condition is not necessary.

A more general problem is to determine not only the possible size, but also a structural characterization, of the support of an extremal \( n \times m \) doubly stochastic array. That is, to obtain an effective necessary and sufficient condition for a subset \( \Omega \subset \{1, \ldots, n\} \times \{1, \ldots, m\} \) to be the support of some extremal \( n \times m \) array.

We note that if an extremal \( n \times m \) array with a given support does exist, then it is unique, see [Li96, Theorem 1].

In this paper we solve this problem in the special case where \( m = kn + 1 \).

**Theorem 1.4.** Let \( m = kn + 1 \). Then a subset \( \Omega \subset \{1, \ldots, n\} \times \{1, \ldots, m\} \) is the support of an extremal array \( A \in S_{n,m} \) if and only if the following two conditions hold:

(i) For each \( 1 \leq i \leq n \), the set \( \{j : (i, j) \in \Omega\} \) contains exactly \( k + 1 \) elements;

(ii) \( \Omega \) does not contain any “cycle”, that is, any sequence of the form
\[
(i_1, j_1), (i_2, j_1), (i_2, j_2), (i_3, j_2), \ldots, (i_s, j_s), (i_1, j_s) \quad (s \geq 2)
\]
where \( i_1, \ldots, i_s \) are distinct row indices and \( j_1, \ldots, j_s \) are distinct column indices.

The necessity of the conditions (i), (ii) is known, see [Car96, Propositions 2, 6]. The novel part of the result is that these two conditions are also sufficient.

For example, it is easy to check that the subset
\[
\Omega = \begin{bmatrix}
\bullet & 0 & 0 & \bullet \\
\bullet & \bullet & 0 & 0 \\
0 & \bullet & \bullet & 0
\end{bmatrix}
\]

of the \( 3 \times 4 \) array, whose elements are indicated by ‘\( \bullet \)’, satisfies the conditions (i) and (ii) in Theorem 1.4 (with \( n = 3 \) and \( k = 1 \)). Hence \( \Omega \) must be the support of some extremal array \( A \in S_{3,4} \). Indeed, \( A \) is the matrix on the right hand side of (1.3).

As an application of Theorem 1.4 we find the total number of extremal \( n \times (n + 1) \) doubly stochastic arrays:

**Theorem 1.5.** The total number of extremal arrays in \( S_{n,n+1} \) is exactly \( n!(n+1)^{n-1} \).
1.6. Loukaki posed in [Lon22, Remark 2] the following question: Do there exist two extremal $n \times m$ doubly stochastic arrays $A, B$ whose supports are of the least possible size $n + m - \gcd(n, m)$, and such that $A, B$ have the same set of entries (counted with multiplicities) but $A$ and $B$ are not equivalent?

We will prove that for certain values of $n, m$ the answer is affirmative:

**Theorem 1.6.** For any $n \geq 6$ there exist two extremal arrays $A, B \in S_{n, n+1}$ that have the same set of entries counted with multiplicities, but $A$ and $B$ are not equivalent.

In fact the two arrays $A, B$ in this result are not only extremal but moreover have supports of the least possible size, since $n$ and $n + 1$ are coprime (Theorem 1.3).

We note that the result in Theorem 1.6 does not hold for $n \leq 5$.

1.7. The rest of the paper is organized as follows. In Section 2 we give some preliminary background. In Section 3 we prove that the minimal size of the support of an $n \times m$ doubly stochastic array is $n + m - \gcd(n, m)$ (Theorem 1.1). In particular we present methods for constructing doubly stochastic arrays of minimal support size.

In Section 4 we characterize the possible sizes of the supports of all the extremal $n \times m$ doubly stochastic arrays (Theorem 1.2). In Section 5 we characterize the support structure of the extremal $n \times (kn + 1)$ arrays (Theorem 1.4). We then use this result in order to determine the total number of extremal $n \times (n + 1)$ arrays (Theorem 1.5).

In Section 6 we construct examples of non-equivalent extremal arrays with the same set of entries counted with multiplicities (Theorem 1.6).

In the last Section 7 we give some motivational remarks, relating the notion of doubly stochastic arrays to some other mathematical topics.

## 2. Preliminaries

In this section we provide some preliminary background on doubly stochastic arrays. First we discuss the notion of the graph associated to an $n \times m$ matrix, and then state several basic results on doubly stochastic arrays that will be used later on.

### 2.1. The associated graph of a matrix.

To every $n \times m$ matrix $A = (a_{ij})$ one can associate a weighted (undirected) bipartite graph $G(A) = (U, V, E, w)$ defined as follows. The set of vertices is the union of two disjoint sets $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ (corresponding to rows and columns of $A$ respectively) which form a bipartition of the graph. The set of edges $E$ includes the edge $\{u_i, v_j\}$ joining $u_i$ and $v_j$ if and only if the matrix entry $a_{ij}$ is nonzero. Finally, the graph is endowed with a weight function $w : E \to \mathbb{R}$ which assigns the (nonzero) weight $a_{ij}$ to the edge $\{u_i, v_j\}$.

For example, Figure 2.1 shows the graphs associated to the two matrices in (1.3).

If $x$ is any one of the vertices of the graph $G(A)$, then the sum of the weights of all the edges incident to $x$ will be called the weight of the vertex $x$.

We observe that an $n \times m$ matrix $A$ is doubly stochastic if and only if the graph $G(A)$ has the following three properties: (i) the edges of the graph have all positive weights; (ii) every vertex from $V$ has weight $n$; and (iii) every vertex from $U$ has weight $m$. 
2.2. Basic results on extremal arrays. We collect some basic results on extremal $n \times m$ doubly stochastic arrays that will be used in the paper.

**Proposition 2.1** (see [Li96, Theorem 1]). An array $A \in S_{n,m}$ is extremal if and only if the two conditions $B \in S_{n,m}$, $\text{supp } B = \text{supp } A$ imply that $B = A$.

**Proposition 2.2** (see [Li96, Corollary 2]). An array $A \in S_{n,m}$ is extremal if and only if there is no array $B \in S_{n,m}$ such that $\text{supp } B$ is a proper subset of $\text{supp } A$.

**Proposition 2.3** (see [Car96, Proposition 2]). An array $A \in S_{n,m}$ is extremal if and only if the graph $G(A)$ contains no cycles.

**Proposition 2.4.** Let $A \in S_{n,m}$ be an extremal array. Then $A$ is an integer matrix. Moreover, all the entries of $A$ are integral multiples of $\gcd(n, m)$.

**Proof.** Let $d = \gcd(n, m)$. Since $A$ is doubly stochastic, then for every vertex $x$ of the graph $G(A)$ the sum of the weights of all the edges incident to $x$ is in $\mathbb{Z}d$ (since this sum is either $n$ or $m$). Hence if a vertex $x$ is incident to some edge $e$ of weight not in $\mathbb{Z}d$, then $x$ must be incident to at least one other edge $e'$ also of weight not in $\mathbb{Z}d$. This implies that if $H$ is the subgraph of $G(A)$ induced by all the edges with weights not in $\mathbb{Z}d$, then every vertex of $H$ must have degree at least two. It follows that if $H$ is nonempty, then $H$ must contain a cycle, which is not possible due to the extremality of the matrix $A$ (Proposition 2.3). Thus $H$ has to be empty, that is, the weight of every edge of $G(A)$ is in $\mathbb{Z}d$, and so all the entries of $A$ are integral multiple of $d$. □

**Proposition 2.5** (see [Car96, Proposition 4]). If $m = kn$, then any extremal $n \times m$ doubly stochastic array is equivalent to the block diagonal matrix

$$
\begin{bmatrix}
  n & n & \cdots & n \\
  n & n & \cdots & n \\
  \vdots & \vdots & & \vdots \\
  n & n & \cdots & n 
\end{bmatrix}
$$

where in each row there are exactly $k$ nonzero entries.

The following proof is different from the one in [Car96].
Proof of Proposition 2.5. By Proposition 2.4, all the entries of $A$ are integral multiples of $n$. Since the entries are nonnegative and the sum at each column is $n$, there can be no entry greater than $n$. Hence every nonzero entry must be equal to $n$. Since the sum of the entries at each row is $kn$, then in each row there are exactly $k$ nonzero entries. But in each column there can be only one nonzero entry, since the sum at each column is $n$. Thus $A$ must be equivalent to the matrix in (2.1). □

Remark. Notice that Birkhoff’s theorem is the case $k = 1$ in Proposition 2.5.

3. Arrays with minimal support size

In this section our main goal is to prove Theorem 1.1, which states that the minimal size of the support of an $n \times m$ doubly stochastic array is $n + m - \gcd(n, m)$.

To prove this result we will first show that the support size of any $n \times m$ doubly stochastic array is bounded from below by the value $n + m - \gcd(n, m)$.

Then we will turn to construct examples of arrays that attain the minimal support size. First, we will show that the problem can be reduced to the special case when $n, m$ are coprime. Then, we will present two different methods for constructing $n \times m$ doubly stochastic arrays of minimal support size.

3.1. Lower bound for the support size of a doubly stochastic array. We begin by showing that an $n \times m$ doubly stochastic array cannot have support of size smaller than $n + m - \gcd(n, m)$, which establishes the lower bound in Theorem 1.1. In fact we will prove the following more general version of the result, where the entries of the matrix are not required to be nonnegative.

Theorem 3.1. Let $A = (a_{ij})$ be an $n \times m$ complex-valued matrix satisfying (1.1) and (1.2), that is, the sum of the entries at each column is $n$ and at each row is $m$. Then

$$|\text{supp}A| \geq n + m - \gcd(n, m). \quad (3.1)$$

Proof. Let $G(A) = (U, V, E, w)$ be the graph of the matrix $A$, and let $r$ be the number of connected components of $G(A)$. Since the graph has $n + m$ vertices, the number of edges in the graph must be at least $n + m - r$ (for adding an edge to a graph can reduce the number of connected components by at most one). So we obtain

$$|\text{supp}A| = |E| \geq n + m - r. \quad (3.2)$$

Let $H_1, \ldots, H_r$ denote the connected components of $G(A)$. For each $1 \leq k \leq r$, assume that $H_k$ has $n_k$ vertices in $U$, and $m_k$ vertices in $V$. Since the sum of the weights of the edges incident to a vertex from $U$ is $m$, and the sum of the weights of the edges incident to a vertex from $V$ is $n$, then the sum of the weights of all the edges of the subgraph $H_k$ is equal to $mn_k$ on one hand, and to $nm_k$ on the other hand. Hence

$$\frac{n_k}{m_k} = \frac{n}{m} \quad (1 \leq k \leq r). \quad (3.3)$$

Let us now write $n = dp$, $m = dq$, where $d = \gcd(n, m)$ and $p, q$ are coprime. It follows from (3.3) that each one of $n_1, \ldots, n_r$ can be no smaller than $p$. But $n_1 + \cdots + n_r = n$, so we must have $rp \leq n$, or equivalently, $r \leq d$. Together with (3.2) this implies (3.1). □
3.2. Block structure of arrays with minimal support size. Now that we have established the lower bound (3.1), our next goal is to look into the structure of the \(n \times m\) doubly stochastic arrays for which an equality in (3.1) is attained. We keep using the notation and terminology from the proof of Theorem 3.1. In particular we denote

\[n = dp, \quad m = dq, \quad d = \gcd(n, m), \quad p, q \text{ are coprime.} \tag{3.4}\]

Suppose that \(A\) is an \(n \times m\) doubly stochastic array whose support is of size exactly \(n + m - d\). Then all the inequalities in the proof above become equalities. So the number of connected components of the graph \(G(A)\) is exactly \(d\), and each connected component has exactly \(p\) vertices in \(U\) and \(q\) vertices in \(V\). This implies that by a permutation of rows and columns we may assume that \(A\) has the structure of a block diagonal matrix consisting of \(d\) blocks of size \(p \times q\) each. Moreover, for each block the sum of the entries at each column is \(n\) and at each row is \(m\), so each block is equal to the scalar \(d\) times a certain \(p \times q\) doubly stochastic array. That is to say,

\[A = d \times \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ & & & A_d \end{bmatrix} \tag{3.5}\]

where each one of \(A_1, A_2, \ldots, A_d\) is a \(p \times q\) doubly stochastic array.

We also observe that the support size of each \(A_k\) can be no smaller than \(p + q - 1\), again by Theorem 3.1. So we have

\[|\text{supp } A| = \sum_{k=1}^{d} |\text{supp } A_k| \geq d(p + q - 1) = n + m - d. \tag{3.6}\]

Hence \(|\text{supp } A| = n + m - d\) implies that \(|\text{supp } A_k| = p + q - 1\) for each \(1 \leq k \leq d\).

It is obvious that these conditions, necessary for an \(n \times m\) doubly stochastic array to have support of size \(n + m - d\), are also sufficient. That is, suppose that we are given \(d\) matrices \(A_1, A_2, \ldots, A_d\) such that each \(A_k\) is a \(p \times q\) doubly stochastic array with support of size \(p + q - 1\). Then the \(n \times m\) matrix \(A\) given by (3.5) is an \(n \times m\) doubly stochastic array with support of size \(n + m - d\). We have thus proved the following:

**Theorem 3.2.** For any \(n, m\), let the numbers \(d, p, q\) be defined by (3.4). Then an \(n \times m\) doubly stochastic array \(A\) has support of size \(n + m - d\) if and only if, possibly after a permutation of rows and columns, \(A\) has the form (3.5) where each one of \(A_1, \ldots, A_d\) is a \(p \times q\) doubly stochastic array with support of size \(p + q - 1\).

The main point of this result is that it reduces the analysis of \(n \times m\) doubly stochastic arrays with minimal support size to the special case when \(n, m\) are coprime.

3.3. Constructing arrays with minimal support size. Our next goal is to prove that the lower bound (3.1) is sharp, that is, to establish the existence of \(n \times m\) doubly stochastic arrays for which an equality in (3.1) is attained.

**Theorem 3.3.** For any \(n, m\) there exists an \(n \times m\) doubly stochastic array \(A\) such that

\[|\text{supp } A| = n + m - \gcd(n, m). \tag{3.7}\]
In fact we only have to prove this for coprime $n, m$, thanks to Theorem 3.2.

Below we present two different methods for constructing $n \times m$ doubly stochastic arrays of minimal support size. We call the first one the “discrete trapezoid method” and the second one the “Euclidean algorithm method”.

3.3.1. The discrete trapezoid method. Let $\mathbb{Z}_{nm}$ be the additive group of residue classes modulo $nm$, and let $\chi_k$ denote the indicator function of the subset $\{0, 1, \ldots, k - 1\}$ of $\mathbb{Z}_n$. We consider a function $\tau : \mathbb{Z}_{nm} \to \mathbb{R}$ defined as the convolution

$$
\tau(t) = (\chi_n * \chi_m)(t) = \sum_{s \in \mathbb{Z}_{nm}} \chi_n(t - s)\chi_m(s), \quad t \in \mathbb{Z}_{nm}.
$$

(3.8)

By a straightforward calculation one can verify that

$$
\tau(t) = \min \{t + 1, n + m - t - 1, n, m\}, \quad 0 \leq t \leq n + m - 2,
$$

(3.9)

and

$$
\tau(t) = 0, \quad n + m - 1 \leq t \leq nm - 1.
$$

(3.10)

We call $\tau$ the discrete trapezoid function on the group $\mathbb{Z}_{nm}$.

For example, in Figure 3.1 the discrete trapezoid function is shown for $n = 5, m = 7$.

![Figure 3.1. The discrete trapezoid function for $n = 5, m = 7$.](image)

Now assume that $n, m$ are coprime. We denote by $\mathbb{Z}_n$ and $\mathbb{Z}_m$ the additive groups of residue classes modulo $n$ and $m$ respectively. By the Chinese remainder theorem there is a group isomorphism $\varphi : \mathbb{Z}_{nm} \to \mathbb{Z}_n \times \mathbb{Z}_m$ given by $\varphi(t) = (t \mod n, t \mod m)$. This isomorphism allows us to lift the discrete trapezoid function $\tau$ to a new function

$$
A : \mathbb{Z}_n \times \mathbb{Z}_m \to \mathbb{R}
$$

(3.11)

defined by $A(\varphi(t)) = \tau(t)$, $t \in \mathbb{Z}_{nm}$. We use (3.11) as an alternative way to represent a real $n \times m$ matrix $A$, in which the rows of the matrix are indexed by residue classes modulo $n$, while the columns indexed by residue classes modulo $m$.

We claim that the matrix $A$ thus constructed is doubly stochastic. To see this, let $H_k$ denote the subgroup of $\mathbb{Z}_{nm}$ generated by the element $k$. We then observe that the discrete trapezoid function $\tau$ tiles the group $\mathbb{Z}_{nm}$ by translations along each one of the two subgroups $H_n$ and $H_m$. That is to say,

$$
\sum_{s \in H_n} \tau(t - s) = m, \quad \sum_{s \in H_m} \tau(t - s) = n, \quad t \in \mathbb{Z}_{nm}.
$$

(3.12)

This can be verified directly using the definition (3.8) of the function $\tau$. 

We next observe that the isomorphism \( \varphi \) maps the subgroup \( H_n \) of \( \mathbb{Z}_{nm} \) onto the subgroup \( \{0\} \times \mathbb{Z}_m \) of \( \mathbb{Z}_n \times \mathbb{Z}_m \). Hence for each \( i \in \mathbb{Z}_n \), the set \( \{(i, j) : j \in \mathbb{Z}_m\} \) is the image under \( \varphi \) of a certain coset of \( H_n \) in \( \mathbb{Z}_{nm} \), say, the coset \( a_i - H_n \). It follows that
\[
\sum_{j \in \mathbb{Z}_m} A(i, j) = \sum_{s \in H_n} \tau(a_i - s) = m,
\]
where in the last equality we used (3.12). In a similar way, \( \varphi \) maps the subgroup \( H_m \) onto \( \mathbb{Z}_n \times \{0\} \), so for each \( j \in \mathbb{Z}_m \) the set \( \{(i, j) : i \in \mathbb{Z}_n\} \) is the image under \( \varphi \) of a coset \( b_j - H_m \), and we obtain
\[
\sum_{i \in \mathbb{Z}_n} A(i, j) = \sum_{s \in H_m} \tau(b_j - s) = n,
\]
again using (3.12). Since the values of \( A \) are clearly nonnegative, we conclude from (3.13) and (3.14) that \( A \) is a doubly stochastic array.

We note that the correspondence between the \( n \times m \) doubly stochastic arrays and the nonnegative functions which tile the group \( \mathbb{Z}_{nm} \) by translations along each one of the two subgroups \( H_n \) and \( H_m \) (where \( n, m \) are coprime) was pointed out in an earlier version of [KP21].

Finally we notice, using (3.9) and (3.10), that
\[
|\text{supp } A| = |\text{supp } \tau| = n + m - 1,
\]
so that the support of \( A \) is of the smallest possible size according to Theorem 3.1.

For example, in the case \( n = 5, m = 7 \) we obtain the matrix
\[
\begin{bmatrix}
1 & 1 & 5 \\
2 & 3 & 5 \\
4 & 3 & 5 \\
3 & 4 & 5 \\
2 & 5 & 0
\end{bmatrix}
\]
which is a \( 5 \times 7 \) doubly stochastic array that has \( 11 = 5 + 7 - 1 \) nonzero entries.

One other example is the matrix on the right hand side of (1.3), which is obtained by the discrete trapezoid method for \( n = 3, m = 4 \).

3.3.2. The Euclidean algorithm method. We now turn to describe our second method for constructing \( n \times m \) doubly stochastic arrays of minimal support size. The same method was proposed also in [Lou22].

Assume that \( n \leq m \), and write
\[
m = kn + r, \quad k \geq 1, \quad 0 \leq r \leq n - 1.
\]

If \( r = 0 \) then \( m = kn \), and in this case we already know from Proposition 2.5 how to construct an \( n \times m \) doubly stochastic array of the minimal support size \( kn \).

In the remaining case \( 1 \leq r \leq n - 1 \) we proceed by induction as follows. We first use the inductive hypothesis to find an \( r \times n \) doubly stochastic array \( B \) of support size \( r + n - \gcd(r, n) \). Next, let \( D_n \) denote the \( n \times n \) identity matrix times the scalar \( n \), that is, \( D_n \) is an \( n \times n \) diagonal matrix whose main diagonal entries are all equal to \( n \). We then construct an \( n \times m \) matrix \( A \) given by
\[
A = [D_n \quad D_n \quad \cdots \quad D_n \quad B^\top],
\]
(3.18)
that is, \( A \) is obtained by horizontally concatenating \( k \) copies of \( D_n \) and the transpose of the matrix \( B \). Then \( A \) has nonnegative entries and the sum of the entries at each column is \( n \) and at each row is \( m \). Hence \( A \) is an \( n \times m \) doubly stochastic array.

It remains to find the size of the support of \( A \). We have

\[
|\text{supp} \ A| = k|\text{supp} \ D_n| + |\text{supp} \ B| = kn + r + n - \gcd(r, n).
\]  

(3.19)

But \( kn + r = m \) and \( \gcd(r, n) = \gcd(n, m) \), so we conclude from (3.19) that

\[
|\text{supp} \ A| = m + n - \gcd(n, m),
\]  

(3.20)

which is indeed the smallest possible support size.

For example, applying this algorithm for \( n = 5, m = 7 \) yields the matrix

\[
\begin{bmatrix}
5 & 2 \\
5 & 2 \\
5 & 2 \\
5 & 1 \\
1 & 1
\end{bmatrix}
\]  

(3.21)

which is a \( 5 \times 7 \) doubly stochastic array with support of size \( 11 = 5 + 7 - 1 \).

One more example is the matrix on the left hand side of (1.3), which is obtained by the Euclidean algorithm method for \( n = 3, m = 4 \).

3.3.3. Remarks. In general, the discrete trapezoid method and the Euclidean algorithm method may yield different, and in fact non-equivalent, doubly stochastic arrays. For example, the \( 5 \times 7 \) matrices in (3.16) and (3.21), obtained by the two different methods, are obviously not equivalent to each other.

We also point out that when performing the Euclidean algorithm method for \( n, m \) coprime, we may choose at any stage of the induction to use the discrete trapezoid method rather than continue further with the inductive process. In this way one can obtain more examples of \( n \times m \) doubly stochastic arrays of minimal support size.

4. Extremal arrays with non-minimal support size

In this section we prove Theorem 1.2 that characterizes the possible sizes of the supports of all extremal \( n \times m \) arrays. Recall that the theorem asserts that if neither one of \( n, m \) divides the other, then any extremal \( A \in S_{n,m} \) has support size of the form

\[
|\text{supp} \ A| = n + m - s,
\]  

(4.1)

where

\[
1 \leq s \leq \gcd(n, m).
\]  

(4.2)

Moreover, this condition is sharp, that is, for any integer \( s \) satisfying (4.2) there exists an extremal array \( A \in S_{n,m} \) whose support size is given by (4.1).

(The reader is reminded that this is no longer true if one of \( n, m \) does divide the other. In fact, if \( m = kn \) then every extremal array in \( S_{n,km} \) has support of size \( kn \), by Proposition 2.5.)
4.1. Estimating the support size of an extremal array. We first show that the conditions (1.1) and (4.2) hold for any extremal \( n \times m \) doubly stochastic array.

**Theorem 4.1.** For all \( n,m \) the support size of any extremal array in \( S_{n,m} \) is of the form \( n + m - s \), where \( 1 \leq s \leq \gcd(n,m) \).

**Proof.** By Theorem 3.1 we know that every \( A \in S_{n,m} \) has support of size no less than \( n + m - \gcd(n,m) \). Now let \( A \) be extremal, then by Proposition 2.3 the graph \( G(A) \) contains no cycles. Since the graph has \( n + m \) vertices, this implies that the number of edges in the graph must be \( n + m - r \), where \( r \) is the number of connected components of the graph. Hence \( \text{supp} \ A \) is of size \( n + m - r \), which can be no greater than \( n + m - 1 \). \( \square \)

4.2. Constructing extremal arrays of given support size. Next we turn to prove that if neither one of \( n, m \) divides the other, then the estimate given in Theorem 4.1 is sharp. That is, we will construct examples of extremal \( n \times m \) doubly stochastic arrays whose support sizes attain all values of the form \( n + m - s \), where \( 1 \leq s \leq \gcd(n,m) \).

**Theorem 4.2.** Assume that neither one of \( n, m \) divides the other. Then for any \( s \) such that \( 1 \leq s \leq \gcd(n,m) \), there exists an extremal array \( A \in S_{n,m} \) whose support is of size \( n + m - s \).

Our proof consists of two parts. In the first part, which is the key one in the proof, we establish the result in the special case where \( \gcd(n,m) = m - n \). Then in the second part we use the Euclidean algorithm method to extend the result to the general case.

4.2.1. The key part of our construction is performed in the following lemma:

**Lemma 4.3.** Let \( n = dp \), \( m = d(p+1) \), where \( d > 1 \), \( p > 1 \). Then for any \( 1 \leq s \leq d-1 \), there is an extremal array \( A \in S_{n,m} \) whose support is of size \( n + m - s \).

**Proof.** It will be more convenient to construct the matrix \( B = \begin{smallmatrix} 1 \end{smallmatrix} \times A \) (recall that by Proposition 2.3 all the entries of \( A \) must be integral multiples of \( d \), so that \( B \) would be an integer matrix). We will obtain the matrix \( B \) by constructing its associated graph \( G(B) = (U, V, E, w) \). The graph \( G(B) \) will be decomposed from a system of \( d \) weighted bipartite subgraphs \( G_j = (U_j, V_j, E_j, w_j), 1 \leq j \leq d \), together with additional edges that join vertices between these subgraphs.

The subgraphs \( G_1, \ldots, G_d \) will be constructed from four basic graph prototypes that will be now introduced. Each prototype is a weighted bipartite graph containing no cycles and such that the edges have positive weights. Furthermore, the weight of each \( U \)-vertex is \( p + 1 \) and the weight of each \( V \)-vertex is \( p \), with possibly the exception of certain “special” vertices whose weight has deficiency one. That is, a special \( U \)-vertex has weight \( p \) and a special \( V \)-vertex has weight \( p - 1 \).

The four graph prototypes are illustrated in Figures 4.1 and 4.2. The first prototype, type I, has \( p - 1 \) vertices in \( U \) and \( p \) vertices in \( V \), and has one special \( V \)-vertex. The second, type II, has \( p + 1 \) vertices in \( U \) and \( p + 2 \) vertices in \( V \), and has one special \( U \)-vertex. Types III and IV have \( p \) vertices in \( U \) and \( p + 1 \) vertices in \( V \), but type III has one special \( U \)-vertex and one special \( V \)-vertex, while type IV has no special vertices.

We now use these four graph prototypes to construct the subgraphs \( G_1, \ldots, G_d \) as follows. We first let \( G_1 \) be a graph of type I. Next, for each \( 2 \leq j \leq d - s \) we let \( G_j \) be a graph of type III (in the case \( s = d - 1 \) there will be no subgraphs of type III). Then,
we let $G_{d-s+1}$ be a graph of type II. Finally, for each $d - s + 2 \leq j \leq d$ we let $G_j$ be a graph of type IV (if $s = 1$, there will be no subgraphs of type IV).

Next, we connect the first $d - s + 1$ subgraphs $G_1, \ldots, G_{d-s+1}$ by adding edges between them as follows. For each $1 \leq j \leq d - s$, we add an edge of weight one joining the special $V$-vertex of $G_j$ to the special $U$-vertex of $G_{j+1}$. In this way, the subgraphs $G_1, \ldots, G_{d-s+1}$ together with the new edges become a single connected component. Note that we do not add any edges to the subgraphs $G_j$ for $d - s + 2 \leq j \leq d$.

This construction yields a weighted bipartite graph $G = (U, V, E, w)$ with edges of positive weights, that has $dp = n$ vertices in $U$ and $d(p + 1) = m$ vertices in $V$. Moreover, the weight of each $U$-vertex is $p + 1$ and the weight of each $V$-vertex is $p$, with no exceptions (since by adding the new edges we have increased the weight of every special vertex by exactly one, so that in the final graph the special vertices have no weight deficiency). We therefore have $G = G(B)$ for a certain $n \times m$ matrix $B$, such that $A := d \times B$ is an $n \times m$ doubly stochastic array.
Finally, we note that by our construction the graph $G(B)$, and hence also the graph $G(A)$, contains no cycles. This implies that $A$ is extremal by Proposition 2.3. Since the graph $G(A)$ has $n + m$ vertices, this also implies that the number of edges must be $n + m - r$, where $r$ is the number of connected components of the graph. But $G(A)$ has exactly $s$ connected components, so that $|\text{supp } A| = |E| = n + m - s$ as required. □

4.2.2. As an example, if $d = 3$, $p = 2$, $s = 1$, then the proof of Lemma 4.3 yields the $6 \times 9$ doubly stochastic array

\[
3 \times \begin{bmatrix}
2 & 1 & & & & & & & \\
1 & 2 & 1 & & & & & & \\
& 1 & 1 & 1 & & & & & \\
& & 2 & 1 & & & & & \\
& & 1 & 1 & 1 & & & & \\
& & & 1 & 2 & & & & \\
\end{bmatrix}
\]  

(4.3)

which is extremal and has support of the maximal size $6 + 9 - 1 = 14$, while we know that the minimal support size of a $6 \times 9$ doubly stochastic array is $6 + 9 - 3 = 12$. 

Figure 4.2. Graph prototypes of type III (left) and type IV (right).
Proof of Theorem 4.2. Let \( d = \gcd(n, m) \). If \( s = d \) then the result is a consequence of Theorem 3.3, so we may suppose that \( 1 \leq s \leq d - 1 \). In particular this means that \( d > 1 \), that is, \( n, m \) are not coprime. The construction is done by induction.

Assume that \( n \leq m \), and write

\[
m = kn + r, \quad k \geq 1, \quad 0 \leq r \leq n - 1.
\]  

(4.4)

The remainder \( r \) is an integral multiple of \( d \). We observe that \( r \) cannot be zero, since \( n \) does not divide \( m \) by assumption. Hence we must have \( d \leq r \leq n - 1 \).

The induction base case is when \( k = 1 \) and \( r = d \). In this case we have \( n = dp \), \( m = d(p + 1) \) for some integer \( p \) which must be at least two (since \( n \) does not divide \( m \)), and so the result follows from Lemma 4.3.

If we have either \( k \geq 2 \), or \( k = 1 \) and \( d + 1 \leq r \leq n - 1 \), then we proceed by induction as follows. Suppose that we can find an extremal \( B \in S_{n,m-n} \) of support size \( m - s \). Let \( D_n \) be the \( n \times n \) identity matrix times the scalar \( n \), and let \( A = [D_n \ B] \) be the \( n \times m \) matrix obtained by horizontally concatenating \( D_n \) and \( B \). Then \( A \in S_{n,m} \) and

\[
|\text{supp } A| = |\text{supp } D_n| + |\text{supp } B| = n + m - s.
\]  

(4.5)

We claim that \( A \) is extremal. To see this, it would suffice by Proposition 2.3 to show that the graph \( G(A) \) contains no cycles. Indeed, \( G(A) \) is obtained from \( G(B) \) by adding \( n \) new vertices (corresponding to the columns of \( D_n \)) where each new vertex is joined by a single new edge to some vertex of \( G(B) \) (since each column of \( D_n \) has exactly one nonzero entry). Then all the new vertices have degree one, i.e. they are leaves. Since \( G(B) \) contains no cycles (\( B \) being extremal) and since adding leaves creates no cycle in the graph, we see that also \( G(A) \) contains no cycles and \( A \) is extremal.

It remains to show that indeed there exists an extremal \( B \in S_{n,m-n} \) of support size \( m - s \). This would follow from the inductive hypothesis if we can verify that (i) neither one of \( n, m - n \) divides the other; and (ii) \( \gcd(n, m - n) = d \). The property (ii) is obvious, so we turn to verify (i). But property (i) is equivalent to the assertion that \( \min\{n, m - n\} \) does not coincide with \( d = \gcd(n, m - n) \). And indeed, if \( k \geq 2 \) then we have \( d < n < m - n \). If, on the other hand, \( k = 1 \) and \( d + 1 \leq r \leq n - 1 \), then \( m - n \) is equal to \( r \), and we have \( d < r < n \). Thus in any case both (i) and (ii) are established and the existence of \( B \) follows from the inductive hypothesis.

\[\Box\]

4.3. Remark. Suppose that neither one of \( n, m \) divides the other, and let \( A \) be an \( n \times m \) doubly stochastic array with support of size \( n + m - s \). If \( s = \gcd(n, m) \) then this implies that \( A \) is extremal, by Theorem 1.1 and Proposition 2.2. On the other hand, one can show that for any \( 1 \leq s \leq \gcd(n, m) - 1 \), the condition \( |\text{supp } A| = n + m - s \) does not imply extremality.

5. Support structure for extremal \( n \times (kn + 1) \) arrays

In this section we obtain a structural characterization of the support of an extremal \( n \times m \) doubly stochastic array in the special case where \( m = kn + 1 \). That is, for
these values of $n, m$ we characterize the subsets $\Omega \subset \{1, \ldots, n\} \times \{1, \ldots, m\}$ such that $\Omega = \text{supp} A$ for some extremal $n \times m$ array (Theorem 1.4).

As an application, we will construct a one-to-one correspondence between the extremal $n \times (n+1)$ doubly stochastic arrays and the trees on $n+1$ labeled vertices having their edges also labeled. This will allow us to determine the total number of extremal arrays in $S_{n,n+1}$ (Theorem 1.5), as well as the number of equivalence classes of these extremal arrays.

5.1. Proof of Theorem 1.4. Our goal now is to prove that if $m = kn + 1$, then a subset $\Omega \subset \{1, \ldots, n\} \times \{1, \ldots, m\}$ is the support of an extremal array $A \in S_{n,m}$ if and only if $\Omega$ satisfies the following two conditions:

(i) For each $1 \leq i \leq n$, the set $\{j : (i, j) \in \Omega\}$ contains exactly $k + 1$ elements;
(ii) $\Omega$ does not contain any cycle, that is, any sequence of the form

$$(i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots, (i_s, j_s), (i_1, j_1) \quad (s \geq 2)$$

where $i_1, \ldots, i_s$ are distinct row indices and $j_1, \ldots, j_s$ are distinct column indices.

We remind the reader that if an extremal $n \times m$ array with a given support does exist, then it is unique, due to Proposition 2.1.

5.1.1. Proof of the necessity of conditions (i) and (ii). Suppose that $\Omega = \text{supp} A$ where $A \in S_{n,m}$ is extremal. We first show that condition (i) must hold. In fact this follows from [Car96, Proposition 6] but we give a different proof, based on Theorem 1.3.

Indeed, the entries of $A$ are nonnegative and the sum of the entries at each column is $n$, so there can be no entry greater than $n$. Since the sum of the entries at each row is $kn + 1$, then in each row there must be at least $k + 1$ nonzero entries. Now if there was a row with more than $k + 1$ nonzero entries, then the size of the support of $A$ would be greater than $n(k + 1) = n + m - 1$. But this is not possible due to Theorem 1.3, since $n$ and $m$ are coprime. Hence each row of $A$ contains exactly $k + 1$ nonzero entries, and since $\Omega = \text{supp} A$ this establishes that condition (i) holds.

To see that condition (ii) must hold as well, we observe that if $\Omega = \text{supp} A$ then this condition means that the graph $G(A)$ contains no cycles. Hence condition (ii) follows from Proposition 2.3.

5.1.2. Proof of the sufficiency of conditions (i) and (ii). We now turn to prove the converse part of the result. That is, we assume that $\Omega$ satisfies the two conditions (i) and (ii), and we will show that there is an extremal $A \in S_{n,m}$ with $\text{supp} A = \Omega$.

We will obtain the matrix $A$ by constructing its associated graph $G(A) = (U, V, E, w)$. Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ be two disjoint vertex sets constituting the bipartition of the graph. We define the set of edges $E$ to include the edge $\{u_i, v_j\}$ if and only if $(i, j) \in \Omega$. Our goal now is to construct a weight function $w$ on $E$, which assigns a positive weight to each edge, in such a way that each vertex from $U$ has total weight $m$, while each vertex from $V$ has total weight $n$.

Consider the unweighted graph $G = (U, V, E)$. By condition (ii) this graph contains no cycles. Since the graph has $n + m$ vertices, this implies that the number of edges in the graph must be $n + m - r$, where $r$ is the number of connected components of the graph. On the other hand, it follows from condition (i) that the size of $\Omega$, and hence
also the number of edges in the graph, is equal to \( n(k + 1) = n + m - 1 \). The number of connected components is therefore one, which means that the graph \( G \) is connected. So \( G \) is both connected and contains no cycles, that is, the graph \( G \) is a tree.

For each edge \( e = \{u, v\} \) of the graph \( G \) \((u \in U, v \in V)\) we let \( G_e = (U, V, E \setminus \{e\}) \) be the subgraph of \( G \) obtained by the removal of the edge \( e \). Then the graph \( G_e \) has exactly two connected components. We let \( H_e \) denote the connected component of \( G_e \) that contains the vertex \( u \). The set of vertices of \( H_e \) is then the union of two disjoint sets \( U_e \) and \( V_e \), where \( U_e \subset U \) and \( V_e \subset V \). We now make the following claim:

**Claim.** For every edge \( e = \{u, v\} \in E \) \((u \in U, v \in V)\) we have

\[
|V_e| = k|U_e|.
\]  

To see this, observe that by condition [1] the degree in \( G \) of each vertex from \( U \) is exactly \( k + 1 \). Hence the degree in \( H_e \) of each vertex from \( U_e \) is also \( k + 1 \), with the exception of the vertex \( u \) that has degree only \( k \) (since in the graph \( G_e \) the edge \( e \) has been removed). Hence the total number of edges in \( H_e \) is equal to

\[
\sum_{u' \in U_e} \deg_{H_e}(u') = (|U_e| - 1)(k + 1) + k = (k + 1)|U_e| - 1,
\]

where \( \deg_{H_e}(u') \) denotes the degree of the vertex \( u' \) in the graph \( H_e \). On the other hand, observe that the graph \( H_e \) does not contain any cycles (since it is a subgraph of \( G \)), and \( H_e \) is connected by its definition. So the number of edges in \( H_e \) is one less than the number of vertices, that is, \( H_e \) must have exactly

\[
|U_e| + |V_e| - 1
\]

edges. Comparing (5.3) and (5.4) we conclude that (5.2) holds, which proves the claim.

We now endow the graph \( G \) with a weight function \( w \) given by

\[
w(e) := |U_e|, \quad e \in E.
\]

It is obvious that this weight function is positive. Indeed, if \( e = \{u, v\} \) \((u \in U, v \in V)\) is an edge in \( E \), then the set \( U_e \) contains at least the vertex \( u \), hence \( U_e \) is nonempty and \( w(e) \) is a positive integer.

Let us show that each vertex \( v \in V \) has total weight \( n \). Indeed, since the graph \( G \) is a tree (that is, \( G \) is connected and contains no cycles) then the subgraphs \( H_e \), as \( e \) goes through the edges incident to \( v \), are vertex disjoint and their union contains all the vertices of \( G \) except \( v \) itself. Hence the vertex sets \( U_e \), as \( e \) goes through the same edges, are disjoint and their union is all of \( U \). The total weight of the vertex \( v \) is therefore

\[
\sum_e w(e) = \sum_e |U_e| = |U| = n,
\]

where in the sum \( e \) goes through the edges incident to \( v \).

Next we show that each vertex \( u \in U \) has total weight \( m \). To see this, we fix the vertex \( u \) and recall that \( u \) has degree \( k + 1 \) in the graph \( G \). Since \( G \) is a tree, it follows that a given vertex \( v \in V \) belongs to \( H_e \) for every edge \( e \) incident to \( u \), with the exception of the edge \( e \) that lies on the unique path connecting \( u \) and \( v \). Hence each vertex \( v \in V \) belongs to exactly \( k \) sets \( V_e \) as \( e \) goes through the edges incident to \( u \), which yields

\[
\sum_e |V_e| = k|V| = km.
\]
But together with (5.2) this implies that the total weight of the vertex \( u \) is

\[
\sum_e w(e) = \sum_e |U_e| = \frac{1}{k} \sum_e |V_e| = m,
\]

where in the sum \( e \) goes through the edges incident to \( u \), in both (5.7) and (5.8).

We conclude that the weighted graph \( G = (U, V, E, w) \) thus constructed is the graph associated to an \( n \times m \) doubly stochastic array \( A = (a_{ij}) \). Moreover, \( A \) is extremal, as the graph contains no cycles (Proposition 2.3). Lastly, we notice that \( \text{supp} \ A = \Omega \), since the matrix entry \( a_{ij} \) is nonzero if and only if \( \{u_i, v_j\} \) is an edge in \( E \), which is the case if and only if \( (i, j) \in \Omega \). This completes the proof of Theorem 1.4.

\[\square\]

5.2. An application to extremal \( n \times (n+1) \) arrays. We can now use Theorem 1.4 in order to construct a one-to-one correspondence between the extremal \( n \times (n+1) \) doubly stochastic arrays and the trees on \( n+1 \) labeled vertices having their edges also labeled. This allows us to find the exact total number of extremal arrays in \( S_{n,n+1} \) (Theorem 1.5), and also to determine the number of equivalence classes of these extremal arrays.

5.2.1. We begin by introducing the type of trees that will be of interest to us:

**Definition 5.1.** Let \( T \) be a tree (that is, a graph which is both connected and contains no cycles) on \( n+1 \) vertices. We say that \( T \in T_n \) if the vertices of \( T \) are labeled as \( v_1, \ldots, v_{n+1} \), and the edges of \( T \) are also labeled (in some order) as \( e_1, \ldots, e_n \).

We think of the edges \( e_1, \ldots, e_n \) of the tree \( T \in T_n \) as corresponding to the rows of an \( n \times (n+1) \) matrix \( A \), and of the vertices \( v_1, \ldots, v_{n+1} \) of the tree as corresponding to the columns of \( A \). We then have the following result:

**Theorem 5.2.** To each tree \( T \in T_n \) there corresponds a unique extremal array \( A \in S_{n,n+1} \) with the following property: \( e_i \) is incident to \( v_j \) in \( T \) if and only if \( (i, j) \in \text{supp} \ A \), for all \( i, j \). Moreover, this correspondence constitutes a bijection from the set of trees in \( T_n \) to the set of extremal arrays in \( S_{n,n+1} \).

**Proof.** Given a tree \( T \in T_n \), let \( B = (b_{ij}) \) be its \( n \times (n+1) \) incidence matrix, defined by \( b_{ij} = 1 \) if \( e_i \) is incident to \( v_j \), and \( b_{ij} = 0 \) otherwise. We then claim that \( \Omega := \text{supp} \ B \) satisfies the conditions (ii) and (iii) (with \( k = 1 \)) in Theorem 1.4. Indeed, each edge of \( T \) is incident with exactly two vertices, hence each row of \( B \) contains exactly two nonzero entries, and so (i) is satisfied. To see that (ii) holds as well, we observe that a cycle in \( \Omega \) corresponds to a cycle in the graph \( T \). But since \( T \) is a tree, it contains no cycles, and (ii) follows. We can therefore use Theorem 1.4 to conclude that \( \Omega \) is the support of some extremal array \( A \in S_{n,n+1} \), having the property that \( e_i \) is incident to \( v_j \) in \( T \) if and only if \( (i, j) \in \text{supp} \ A \), for all \( i, j \). Moreover, such an \( A \) is unique, since an extremal array is uniquely determined by its support (Proposition 2.1).

Conversely, given any extremal array \( A \in S_{n,n+1} \), we construct an \( n \times (n+1) \) matrix \( B = (b_{ij}) \) defined by \( b_{ij} = 1 \) if \( (i, j) \in \text{supp} \ A \), and \( b_{ij} = 0 \) otherwise. The conditions (i) and (ii) in Theorem 1.4 ensure that \( B \) is the incidence matrix of a cycle-free graph \( T \) with \( n+1 \) vertices labeled as \( v_1, \ldots, v_{n+1} \) and with \( n \) edges labeled as \( e_1, \ldots, e_n \). This means that \( T \) is a tree in \( T_n \) with the property that \( e_i \) is incident to \( v_j \) in \( T \) if and only if \( (i, j) \in \text{supp} \ A \), for all \( i, j \). Moreover, the incidence matrix \( B \) determines the tree \( T \) uniquely, which shows that the correspondence from the set of trees in \( T_n \) to the set of extremal arrays in \( S_{n,n+1} \) is indeed a bijection. \[\square\]
5.2.2. We can now use the latter result in order to prove Theorem 1.5, which asserts that the total number of extremal arrays in $S_{n,n+1}$ is exactly $n!(n+1)^n-1$.

**Proof of Theorem 1.5.** We invoke the classical Cayley’s formula, which states that the number of trees with $n+1$ labeled vertices (but with unlabeled edges) is $(n+1)^n-1$. Notice that there are $n!$ ways to label also the edges of such a tree. Hence the number of trees in $T_n$ is exactly $n!(n+1)^n-1$. But the set of trees in $T_n$ has the same cardinality as the set of extremal arrays in $S_{n,n+1}$, due to Theorem 5.2, and the result follows. □

Theorem 5.2 also allows us to determine the number of extremal $n \times (n+1)$ doubly stochastic arrays up to equivalence. Recall that two arrays are said to be equivalent if one can be transformed into the other by a permutation of rows and columns. Then two equivalent extremal arrays in $S_{n,n+1}$ correspond through the bijection of Theorem 5.2 to two isomorphic labeled trees, where two labeled trees are said to be isomorphic if one can be transformed into the other by a permutation of vertex labels and edge labels. Hence we arrive at the following conclusion:

**Theorem 5.3.** The number of equivalence classes of the extremal arrays in $S_{n,n+1}$ is equal to the number of unlabeled trees on $n+1$ vertices.

Thus, for instance, any extremal $3 \times 4$ doubly stochastic array is equivalent to one of the two matrices in (1.3), since there are only two unlabeled trees on 4 vertices.

Unfortunately, no closed-form expression is known for the number of unlabeled trees on $n+1$ vertices as a function of $n$.

6. Non-equivalent extremal arrays with the same entries

The following question was posed by Loukaki in [Lou22, Remark 2]: Do there exist two extremal arrays $A, B \in S_{n,m}$, whose supports are both of the least possible size $n + m - \gcd(n, m)$, and such that $A, B$ have the same set of entries counted with multiplicities but $A$ and $B$ are not equivalent?

In this section we prove that the question admits an affirmative answer in $S_{n,n+1}$ for each $n \geq 6$, by constructing an example of two extremal arrays $A, B$ with the properties above (Theorem 6.1). We also observe that the same does not hold for $n \leq 5$.

Our strategy is based on the correspondence in Theorem 5.2 between the extremal arrays in $S_{n,n+1}$ and the trees on $n+1$ labeled vertices having their edges also labeled. Recall that two equivalent extremal arrays in $S_{n,n+1}$ correspond to two isomorphic trees. Hence our goal is to construct two non-isomorphic trees which induce the same multiset of entries in the corresponding extremal arrays.

6.1. Let $T$ be a tree in $T_n$, that is, $T$ is a tree on $n+1$ vertices labeled as $v_1, \ldots, v_{n+1}$, and the edges of $T$ are also labeled as $e_1, \ldots, e_n$ (recall Definition 5.1). Suppose that $i, j$ are such that the edge $e_i$ is incident to the vertex $v_j$ in the tree $T$. If we remove the edge $e_i$ from the tree then we obtain a subgraph $G_i(T)$ with exactly two connected components. We denote by $H_{ij}(T)$ the connected component of $G_i(T)$ that does not contain the vertex $v_j$.

**Theorem 6.1.** Let $T$ be a tree in $T_n$, and let $A = (a_{ij})$ be the extremal array in $S_{n,n+1}$ which corresponds to $T$ through the bijection of Theorem 5.2. For each $(i, j) \in \text{supp} \ A$, the entry $a_{ij}$ is equal to the number of vertices in the connected component $H_{ij}(T)$. 
This result complements Theorem 5.2 by specifying the values of the nonzero entries of the extremal \( n \times (n+1) \) array \( A \) that corresponds to a given tree \( T \in \mathcal{T}_n \).

**Proof of Theorem 6.1.** Let \( G(A) = (U, V, E, w) \) be the graph associated to the extremal \( n \times (n+1) \) array \( A \), where \( U = \{u_1, \ldots, u_n\} \) and \( V = \{v_1, \ldots, v_{n+1}\} \). By Theorem 5.2, \( (i,j) \in \text{supp} \ A \) if and only if \( e_i \) is incident to \( v_j \) in the tree \( T \). Hence each vertex \( u_i \) in the graph \( G(A) \) is connected to exactly two \( V \)-vertices, that constitute the endpoints of the edge \( e_i \) in \( T \). This means that the (unweighted) graph \( G(A) \) is obtained from the tree \( T \) by inserting a new vertex \( u_i \) in the middle of each edge \( e_i \) of \( T \).

Now the matrix entry \( a_{ij} \) is the weight of the edge \( \{u_i, v_j\} \) in \( G(A) \). We recall from the proof of Theorem 1.4 (with \( k = 1 \)) that the weight of the edge \( \{u_i, v_j\} \) is equal to the number of \( U \)-vertices in the connected component containing the vertex \( u_i \) in the subgraph of \( G(A) \) obtained by the removal of the edge \( \{u_i, v_j\} \). But each \( U \)-vertex corresponds to the edge of the tree \( T \) in the middle of which it was inserted, so the weight of the edge \( \{u_i, v_j\} \) is also equal to the number of edges in the connected component \( H_{ij}(T) \) together with the edge \( e_i \) itself (which corresponds to the vertex \( u_i \)). But this number is the same as the number of vertices in \( H_{ij}(T) \), which proves our claim. \( \square \)

6.2 Consider now the two trees shown in Figure 6.1. Each tree has 7 vertices, but the trees are not isomorphic, since one of them has a vertex of degree 4 while the other does not. We will show that these two trees correspond to two non-equivalent extremal \( 6 \times 7 \) doubly stochastic arrays \( A, B \) that have the same multiset of entries.

Let \( A \) be the extremal array corresponding to the left tree in Figure 6.1. Then by Theorem 1.4, each row of \( A \) contains exactly two nonzero entries, and by Theorem 6.1 the values of the two nonzero entries at the \( i \)’th row of \( A \) are equal to the number of
vertices in each one of the two connected components created by the removal of the edge $e_i$ in the left tree.

For example, removing the edge $e_1$ in the left tree creates two connected components, one containing 6 vertices and the other consisting of a single vertex. Hence the two nonzero values in the first row of $A$ are 6 and 1.

Now consider the extremal array $B$ corresponding to the right tree in Figure 6.1. Notice that we have labeled the edges of this tree as $\phi(e_1), \ldots, \phi(e_6)$, namely, there is a bijection $\phi$ between the edges of the left tree and the edges of the right tree. It is now straightforward to verify that for each $1 \leq i \leq 6$, the removal of the edge $e_i$ in the left tree, or the removal of the edge $\phi(e_i)$ in the right tree, creates two connected components whose respective number of vertices is the same for the two trees. This implies that the $i$'th row of $A$ contains the same two nonzero values as the $i$'th row of $B$, although these two nonzero values may appear on different columns of $A$ and $B$.

For example, removing the edge $\phi(e_1)$ in the right tree creates a connected component with 6 vertices and another component with a single vertex. Hence the two nonzero values in the first row of $B$ are 6 and 1, the same two values as in the first row of $A$.

The two extremal $6 \times 7$ doubly stochastic arrays obtained from this construction are

\[
\begin{bmatrix}
6 & 1 & 6 \\
1 & 6 & 4 \\
3 & 2 & 5 \\
1 & 6 & 1 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
6 & 1 & 6 \\
1 & 6 & 3 \\
4 & 5 & 2 \\
1 & 6 & 1 \\
\end{bmatrix},
\]

which indeed have the same two nonzero values in each row, and as a consequence, have the same multiset of entries. On the other hand these two arrays are not equivalent, as one of them has a column with 4 nonzero entries, while the other does not (which, in turn, is due to one of the trees having a vertex of degree 4 while the other tree not having such a vertex).

We have thus proved the case $n = 6$ in Theorem 1.6.

6.3. We now continue with the proof of Theorem 1.6 for all $n \geq 6$. It would suffice for us to find two non-isomorphic trees $T = (V, E)$ and $T' = (V', E')$ with $n + 1$ vertices each, and an edge bijection $\phi : E \to E'$, such that the removal of the edge $e_i$ in the tree $T$, or the removal of the edge $\phi(e_i)$ in the tree $T'$, creates two connected components whose respective number of vertices is the same for the two trees.

The construction is done by induction as illustrated in Figure 6.2. Assume that the two trees on $n$ vertices have already been constructed. We then insert a new edge $e_n$ into the left tree, between the edge $e_{n-1}$ and the vertex common to $e_3$ and $e_4$. Similarly, we insert a new edge $\phi(e_n)$ into the right tree, between the edge $\phi(e_{n-1})$ and the vertex common to $\phi(e_2)$, $\phi(e_3)$ and $\phi(e_5)$. Thus we obtain two trees on $n + 1$ vertices and also extend the edge bijection $\phi$ from the previous step to these new trees.

We must show that the new bijection $\phi$ has the required property. Indeed, observe that for each $1 \leq i \leq n$, the removal of the edge $e_i$ in the left tree, as well as the removal of the edge $\phi(e_i)$ in the right tree, creates one connected component whose number of
vertices is a certain constant $c_i$ which does not depend on $n$, and a second connected component which contains the remaining $n + 1 - c_i$ vertices of the tree.

We conclude that the two trees thus constructed correspond to two extremal $n \times (n+1)$ doubly stochastic arrays that have the same two nonzero values in each row, and hence also the same multiset of entries. It remains only to notice that the two trees are not isomorphic (since, as before, one of the trees has a vertex of degree 4 while the other does not) and therefore the two corresponding extremal arrays are not equivalent. The theorem is therefore proved.

\[\Box\]

6.4. **Remark.** We note that the result in Theorem 1.6 does not hold for $n \leq 5$. Indeed, there are only 13 unlabeled trees on $n + 1$ vertices with $1 \leq n \leq 5$ in total (see, for instance, [Har69, Appendix III]), and one can check that no two of them correspond to two extremal doubly stochastic arrays that have the same multiset of entries.

7. **Remarks**

We conclude the paper with some motivational remarks, relating the notion of doubly stochastic arrays to some other mathematical topics.

7.1. The notion of an $n \times m$ doubly stochastic array admits an obvious probabilistic interpretation. If $A \in S_{n,m}$ then the matrix $\frac{1}{nm}A$ represents a joint distribution of two random variables $X, Y$ such that $X$ is uniformly distributed in $\{1, 2, \ldots, n\}$ and $Y$ uniformly distributed in $\{1, 2, \ldots, m\}$. Thus minimizing the support size of an array $A \in S_{n,m}$ is the same as minimizing the support size of a joint distribution over all couplings of the two random variables $X, Y$.

7.2. It is also possible to give a combinatorial interpretation of the $n \times m$ doubly stochastic array as *transportation plans* as follows. Suppose that we are given $n$ blue bins and $m$ green bins, such that each blue bin contains $m$ balls while the green bins
are empty from balls. The problem is to move the balls from the blue bins to the green ones, so that each green bin would contain exactly \(n\) balls. Then a transportation plan corresponds to an integer-valued array \(A = (a_{ij})\) in \(S_{n,m}\) where \(a_{ij}\) is the number of balls to be moved from the \(i\)'th blue bin to the \(j\)'th green bin. If we seek to minimize the number of operations of the form “move \(a_{ij}\) balls from the \(i\)'th blue bin to the \(j\)'th green bin”, then the minimal number of operations needed to move the balls from the blue bins to the green ones as required is \(n + m - \gcd(n, m)\). This follows from Theorem 1.1 and the fact that an array in \(S_{n,m}\) which minimizes the support size is automatically integer-valued (Propositions 2.2 and 2.4).

7.3. Let \(G\) be a finite abelian group, and suppose that \(G\) is the direct sum of two subgroups \(H_1\) and \(H_2\) of sizes \(n\) and \(m\) respectively. A function \(f\) on the group \(G\) is said to tile by translations along each one of the two subgroups \(H_1\) and \(H_2\) if we have
\[
\sum_{s \in H_1} f(t - s) = n, \quad \sum_{s \in H_2} f(t - s) = m, \quad t \in G. \tag{7.1}
\]

In [KP21, Section 4] the authors pointed out a correspondence between the \(n \times m\) doubly stochastic arrays and the nonnegative functions \(f\) which tile by translations along each one of the two subgroups \(H_1\) and \(H_2\). They posed the question as to what is the smallest possible size of the support of such a function \(f\). It follows from Theorem 1.1 that the answer is \(n + m - \gcd(n, m)\). In fact, Theorem 3.1 implies that the answer remains the same even for complex-valued functions \(f\) (that is, even if we do not require \(f\) to be nonnegative).

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