QUANTIZED VERSHIK–KEROV THEORY AND QUANTIZED CENTRAL MEASURES
ON BRANCHING GRAPHS

RYOSUKE SATO

ABSTRACT. We propose a natural quantized character theory for inductive systems of compact quantum groups based on KMS states on AF-algebras following Stratila–Voiculescu’s work \[26\] (or \[7\]), and give its serious investigation when the system consists of quantum unitary groups \(U_q(N)\) with \(q \in (0, 1)\). The key features of this work are: The “quantized trace” of a unitary representation of a compact quantum group can be understood as a quantized character associated with the unitary representation and its normalized one is captured as a KMS state with respect to a certain one-parameter automorphism group related to the so-called scaling group. In this paper we provide the approximation theorem for extremal quantized characters (called the ergodic method) and also compare our quantized character theory for the inductive system of \(U_q(N)\) with Gorin’s theory on \(q\)-Gelfand–Tsetlin graphs \[10\].

1. INTRODUCTION

Vershik and Kerov initiated the approximation theory of extremal characters on the infinite symmetric group \(S(\infty)\) and the infinite-dimensional unitary group \(U(\infty)\) in \[30\], \[31\]. The infinite dimensional unitary group \(U(\infty)\) is the inductive limit of unitary groups \(U(N)\), and its finite factor representations have been studied in a number of papers, see for instance \[33\], \[30\], \[2\], \[22\], \[23\], \[1\]. The tracial states associated with finite factor representations naturally correspond to extremal characters on \(U(\infty)\). Vershik–Kerov’s approximation method, which is called the ergodic method, allows us to compute extremal characters on \(U(\infty)\) as a limit of ordinary irreducible characters on \(U(N)\) and the method is useful to show the completeness of the parameterization of finite factor representations (or the extremal characters on \(U(\infty)\)) given by Voiculescu in \[26\].

Stratila and Voiculescu introduced a certain AF-algebra, each of whose factor representations naturally corresponds to the one of \(U(\infty)\) and also each tracial state on this AF-algebra corresponds to a character on \(U(\infty)\), see \[26\]. Furthermore, Borodin and Olshanski studied the decompositions of certain representations into the finite factor representations in \[23\], \[1\]. These works due to Vershik–Kerov, Stratila–Voiculescu and Borodin–Olshanski lead us to the probability theory on Bratteli diagrams or the Gelfand–Tsetlin graph. In particular, certain probability measures called central measures on the paths on the Gelfand–Tsetlin graph correspond to characters on \(U(\infty)\).

Furthermore, Gorin propounded a \(q\)-deformation of the Gelfand–Tsetlin graph in \[10\] and he gave the parameterization of extremal \(q\)-deformed central measures. His \(q\)-deformation was developed along the harmonic analysis on \(U(\infty)\) in \[11\]. However, his study has been done without representation theory of any appropriate mathematical objects like quantum groups.

In the present work we propose an appropriate object, which gives a natural representation-theoretic interpretation of Gorin’s work. This is plausibly obtained from quantum unitary groups \(U_q(N)\). Since compact groups are replaced with compact quantum groups, inductive systems of groups are replaced with projective systems of \(C^*\)-algebras of compact quantum groups, which are also called inductive systems of compact quantum groups. Similarly to Stratila–Voiculescu’s work, we will construct a certain AF-algebra for a given inductive system of compact quantum groups, that is, the projective system of their \(C^*\)-algebras, and investigate a certain one-parameter automorphism group related to the so-called scaling group. A certain class of KMS states for this action, whose members should be called quantized characters, naturally corresponds to a certain class of probability measures on the Brattelli diagram (or the Gelfand-Tsetlin graph in the case of quantum unitary groups \(U_q(N)\)). Moreover, we show that these probability measures are nothing less than Gorin’s \(q\)-central measures (by replacing \(q\) with \(q^2\)) and prove the approximation theorem of the extremal quantized characters. Also we will reprove the boundary theorem for the \(q\)-deformation of the Gelfand–Tsetlin graph by Gorin \[10\].

The paper is organized as follows. In Section 2 we review some basic definitions and some properties on compact quantum groups and fix some notations used throughout this paper. In Section 3 and 4 we introduce the dual objects, which are operator algebras, of compact quantum groups, and investigate certain properties on quantum subgroups and their duals, respectively. In particular, we investigate the dual actions of so-called
scaling actions on compact quantum groups. Due to the results in Section 3 and 4, we construct a certain AF-algebra and a certain flow on it of a given inductive system of compact quantum groups. Following Enomoto–Izumi [7], we call them the Stratila–Voiculescu AF-algebra and the Stratila–Voiculescu AF-flow.

In Section 6, we introduce a weighted branching graph and prove the approximation theorem for certain KMS states for the Stratila–Voiculescu AF-flow, which is called quantized characters. Due to the approximation theorem, in Section 9, we reprove the boundary theorem of the $q$-deformation of the Gelfand–Tsetlin graph by Gorin in [10].

As preliminaries, we review the representation theory of quantum unitary group $U_q(N)$ and some definitions and some notations of the symmetric polynomials in Section 7 and 8. In Section 10, we give the representation-theoretic interpretation of the generating functions of probability measures on the signatures introduced by Gorin in [10].

2. Compact quantum groups

We recall some basic definitions and some properties on compact quantum groups to fix notations used throughout this paper. A typical example is the quantum unitary group $U_q(N)$, which will be discussed later.

Let $G = (A, \delta)$ be a pair of a unital $C^*$-algebra and a unital $*$-homomorphism $\delta: A \to A \otimes A$ (called the comultiplication), where $\otimes$ denotes the minimal (or spatial) tensor product. The pair $G = (A, \delta)$ is called a $C^*$-algebraic compact quantum group (CQG) if it satisfies the following two conditions:

- (coassociativity) $(id \otimes \delta)\delta = (\delta \otimes id)\delta$ as homomorphisms from $A$ to $A \otimes A \otimes A$,
- (cancellation property) the spaces $\delta(A)(A \otimes 1_A)$ and $\delta(1_A \otimes A)$ are dense in $A \otimes A$.

For two given $f_1, f_2 \in A^*$ we define $f_1 \ast f_2 := (f_1 \otimes f_2)\delta = A^*$. It is well known (see e.g. [20, Theorem 1.2.1]) that any CQG $G = (A, \delta)$ has the so-called Haar state $h: A \to C$, which enjoys that $f \ast h = h \ast f = f(1_A)h$ for any linear functional $f \in A^*$, or equivalently $(id \otimes h)\delta(a) = (h \otimes id)\delta(a) = h(a)1_A$ for any $a \in A$.

Let $G = (A, \delta)$ be a CQG and $V$ be a finite-dimensional vector space. An invertible element $U \in B(V) \otimes A$ is called a representation of $G$ if it satisfies $(id \otimes \delta)(U) = U_{12}U_{13}$ in $B(V) \otimes A \otimes A$, where $U_{12}, U_{13}$, etc are leg numbering notations, see [20, Section 1.3]. Let $e_{ij}$ be a matrix unit system of $B(V)$. An invertible element $U = \sum_{i,j=1}^{\dim V} e_{ij} \otimes u_{ij} \in B(V) \otimes A$ is a representation of $G$ if and only if $\delta(u_{ij}) = \sum_{k=1}^{\dim V} u_{ik} \otimes u_{kj}$ holds for any $i, j, k = 1, \ldots, \dim V$. The element $u_{ij}$ is called the matrix coefficient of $U$ with respect to the matrix unit system $e_{ij}$ (or the basis of $V$). The dimension of $V$ is denoted by $\dim(U)$ (or $\dim(\pi)$ when $U = U_{\pi}$) and called the dimension of the representation $U$. If $V$ is a Hilbert space, that is, $V$ is equipped with an inner product and $U$ is a unitary, then $U$ is called a unitary representation.

For any two finite-dimensional representations $U, V$ on vector spaces $V_U, V_V$, respectively, a linear map $T: V_U \to V_V$ is an intertwiner from $U$ to $V$ if $(T \otimes id)U = V(T \otimes id)$ holds. We denote by $\text{Mor}(U, V)$ the intertwiners from $U$ to $V$. If there exists a bijective intertwiner from $U$ to $V$ (or from $V$ to $U$), then $U$ and $V$ are said to be equivalent, moreover if these representations are unitary and there exists a unitary intertwiner then they are said to be unitary equivalent, we write $U \sim V$ in the case. We denote by $\hat{G}$ all the equivalent classes of irreducible unitary representations, and call $\hat{G}$ the unitary dual of $G$.

Let $U$ be a finite-dimensional representation on a Hilbert space $\mathcal{H}$. Let $J: \mathcal{H} \to \mathcal{H}^*$ be a conjugate linear map sending a basis to its dual basis and let $J: B(\mathcal{H}) \ni a \mapsto Ja^*J^{-1} \in B(\mathcal{H}^*)$. Then a new representation $U^c := (j \otimes id)U^{-1} \in B(\mathcal{H}^*) \otimes A$ is called the contragredient representation of $U$. It is well known that, for any finite-dimensional unitary representation $U$, there exists a unique positive invertible intertwiner $F = F_U \in \text{Mor}(U, U^c)$ such that $\text{Tr}(F) = \text{Tr}(F^{-1})$ on $\text{End}(U) := \text{Mor}(U, U)$. The trace $\text{Tr}(F)$ is called the quantum dimension of the representation $U$, denoted by $\dim_q(U)$.

In what follows, we write $\mathcal{H}_\pi := \mathcal{H}_{U_{\pi}}, \dim(\pi) := \dim(\mathcal{H}_\pi), F_\pi := F_{U_{\pi}} \text{ and } \dim_q(\pi) := \dim_q(U_{\pi})$ when given representations $U_{\pi}$ have suffix $\pi$. In this paper, we call the matrix $F_{\pi}$ the density matrix of the representation $U_{\pi}$.

For a given unitary representation $U$, if $\text{End}(U)$ is 1-dimensional, then $U$ is said to be irreducible. It is well known that every finite dimensional representation of a CQG is a direct sum of irreducible ones, see [20, Theorem 1.3.7].

Let $\mathcal{A}$ be the subspace generated by matrix coefficients of finite-dimensional representations. By the definition of tensor product representations and contragredient representations, the subspace $\mathcal{A}$ becomes a $*$-subalgebra of $A$, see [20, section 1.6]. Conversely, we always assume that $A$ is the universal $C^*$-algebra generated by $\mathcal{A}$. 
For any $z \in \mathbb{C}$, we have a linear functional $f^G_z: \mathcal{A} \to \mathbb{C}$ determined by $(\text{id} \otimes f^G_z)(U) = F^G_U$ for any finite-dimensional unitary representation $U$. See [20] section 1.7. These functionals $\{f^G_z\}_z$ are called the Woronowicz characters of the CQG $G$. By definition, we have $F^G_U = [f^G_z(u_{ij})]_{ij=1}^{\dim(U)}$, where $u_{ij}$ is matrix coefficient of the representation $U$. For any $a \in \mathcal{A}$ and $f, g \in \mathcal{A}^*$, we define $f \ast a := (\text{id} \otimes f)(a)$, $a \ast f = (f \otimes \text{id})\delta(a)$ and $f \ast a \ast g := f \ast (a \ast g) = (f \ast a) \ast g$. The Woronowicz characters $\{f^G_z\}_z$ induce two actions $\sigma^G_z, \tau^G_z : \mathbb{C} \curvearrowright \mathcal{A}$ of the whole complex field $\mathbb{C}$ on $\mathcal{A}$ defined by $\sigma^G_z(a) := f^G_z(\sqrt{-1}z) \ast a \ast f^G_z$ and $\tau^G_z(a) := f^G_{\sqrt{-1}z} \ast a \ast f^G_{\sqrt{-1}z}$ for any $a \in \mathcal{A}$, and they are called the modular action (or modular group) and the scaling action (or scaling group), respectively. Note that

$$\langle \text{id} \otimes \sigma^G_z(U) \rangle = (F^G_{U \otimes 1_A} \otimes 1_A)(F^G_{U \otimes \sqrt{-1}} \otimes 1_A), \quad \langle \text{id} \otimes \tau^G_z(U) \rangle = (F^G_{U \otimes \sqrt{-1}} \otimes 1_A)(F^G_{U \otimes \sqrt{-1}} \otimes 1_A)$$

(2.1) for any finite-dimensional unitary representation $U$. We remark that the Haar state $h$ on $\mathcal{A}$ is $\sigma^G_z$-invariant for all $z \in \mathbb{C}$, and $h(ab) = h(\bar{a} \sqrt{-1}(a))$ for all $a, b \in \mathcal{A}$, see [20] Theorem 1.7.3]. This fact is the reason why $\sigma_z$ is called the modular action.

In closing of this section we recall the notion of quantum subgroups. Let $G = (A, \delta_G)$ and $H = (B, \delta_H)$ be CQGs. The CQG $H$ is a quantum subgroup of the CQG $G$ if there exists a surjective $*$-homomorphism $\theta: \mathcal{A} \to \mathcal{B}$ which satisfies $\delta_H \theta = (\theta \otimes \theta) \delta_G: \mathcal{A} \to \mathcal{B} \otimes \mathcal{B}$. Since $\theta$ is a $*$-homomorphism, we have the following: For any unitary representation $U$ on a Hilbert space $\mathcal{H}$ of $G$, it is easy to see that $(\text{id} \otimes \theta)(U)$ is a unitary representation of the quantum subgroup $H$, and this is called the restriction of $U$ to $H$. When the $U$ has a suffix $\pi$, that is, $U = U_{\pi}$, we denote the restriction of $U_{\pi}$ to $H$ by $\pi|_H$ and write $U_{\pi|_H} = (\text{id} \otimes \theta)(U_{\pi})$.

### 3. Duals of Compact Quantum Groups

Let $G = (A, \delta)$ be a CQG and $\mathcal{A}$ the $*$-subalgebra generated by the matrix coefficients of all finite-dimensional representations of $G$. It is known that $\mathcal{A}$ is a Hopf algebra with comultiplication $\delta |_A$, counit $\epsilon$ and antipode $S$ determined by $\epsilon(u_{ij}) = \delta_{ij}, S(u_{ij}) = u_{ji}^*$ for matrix coefficients $u_{ij}$ of every finite-dimensional representation $U$. See [20] Theorem 1.6.4.

The dual space $U(G) := \mathcal{A}^*$ as a linear space becomes a $*$-algebra with multiplication

$$U(G) \times U(G) \ni (f_1, f_2) \mapsto f_1 \ast f_2 := (f_1 \otimes f_2) \circ \delta \in U(G)$$

and involution $U(G) \ni f \mapsto f^* \in U(G)$ defined to be $f^*(a) := f(S(a^*))$, $a \in \mathcal{A}$. Let $\hat{G}$ be the unitary dual of $G$, and choose and fix a complete family, say $\{U_{\pi}\}_{\pi \in \hat{G}}$, of representatives of members of $\hat{G}$. It is known, see e.g. [20] Section 1.6], that for each $\pi \in \hat{G}$ the mapping $f \in U(G) \mapsto U_{\pi}(f) := (\text{id} \otimes f)(U_{\pi}) \in B(\mathcal{H}_\pi)$ defines a surjective $*$-homomorphism and moreover that

$$U_{\hat{G}}: U(G) \ni f \mapsto U_{\hat{G}}(f) := (U_{\pi}(f))_{\pi \in \hat{G}} \in \prod_{\pi \in \hat{G}} B(\mathcal{H}_\pi)$$

becomes a bijective $*$-homomorphism. The surjectivity and the bijectivity of the mappings $f \mapsto U_{\pi}(f)$ and $f \mapsto U_{\hat{G}}(f)$, respectively, are bit non-trivial and follow from the following consideration: For each $\pi \in \hat{G}$, we set

$$a_{ij}(\pi) := \dim(\pi) \sum_{p=1}^{\dim(\pi)} f_{-1}(u_{jp}(\pi)) u_{ip}(\pi)^*$$

with $U_{\pi} = \sum_{i,j=1}^{\dim(\pi)} e_{ij}(\pi) \otimes u_{ij}(\pi)$. Then we have

$$U_{\pi}(a_{ij}(\pi)) h = e_{ij}(\pi)$$

by the orthogonality relations for matrix coefficients, where $[a_{ij}(\pi)](\cdot) := h(a_{ij}(\pi))$. See [20] Theorem 1.4.3]. It follows that $f \mapsto U_{\pi}(f)$ and $f \mapsto U_{\hat{G}}(f)$ are surjective and moreover that

$$U_{\hat{G}} \left( \sum_{i,j=1}^{\dim(\pi)} a_{ij}(\pi) e_{ij}(\pi) \right) = \sum_{\pi \in \hat{G}} \sum_{i,j=1}^{\dim(\pi)} a_{ij}(\pi) a_{ij}(\pi) h,$$

(3.1)

whose right-hand side involves an infinite sum over $\pi \in \hat{G}$, but it is indeed a well-defined linear functional on $\mathcal{A}$, because

$$[a_{ij}(\pi)]h(u_{kl}(\rho)) = h(u_{kl}(\rho)a_{ij}(\pi)) = \delta_{\pi,\rho} \delta_{i,k} \delta_{j,l},$$

that is, the $a_{ij}(\pi)h$ form a dual basis of the $u_{ij}(\pi)$.

Here is a simple (probably well-known) lemma, which immediately follows from [21].
Lemma 3.1. For every $\pi \in \hat{G}$ we have

$$U_\pi(f \circ \tau^G_t) = F_\pi^{-\sqrt{-t}}U_\pi(f)F_\pi^{-\sqrt{-t}}, \quad f \in \mathcal{U}(G), \ t \in \mathbb{R}.$$  

Therefore, the dual scaling action $\hat{\tau}^G : \mathbb{R} \curvearrowright \mathcal{U}(G)$ defined by $\hat{\tau}^G_t(f) := f \circ \tau^G_t$ for every $f \in \mathcal{U}(G)$ and $t \in \mathbb{R}$ enjoys the formula

$$U_G(\hat{\tau}^G_t(f)) = \left(F_\pi^{-\sqrt{-t}}U_\pi(f)F_\pi^{-\sqrt{-t}} \right)_{\pi \in \hat{G}}, \quad f \in \mathcal{U}(G), \ t \in \mathbb{R},$$

and $\hat{\tau}^G_t(f) = f_{\sqrt{-t}} \ast f \ast f_{\sqrt{-t}}^\ast$ holds for every $f \in \mathcal{U}(G)$ and $t \in \mathbb{R}$.

There are three canonical *-subalgebras of $\mathcal{U}(G)$ or $\bigoplus_{\pi \in \hat{G}} B(H_\pi)$. The collection of all elements $(x_\pi)_{\pi \in \hat{G}}$ in $\bigoplus_{\pi \in \hat{G}} B(H_\pi)$ such that $\sup_{\pi \in \hat{G}} \|x_\pi\| < +\infty$ becomes a unital *-subalgebra and is denoted by $\mathcal{A}$ or $\mathcal{A}(\hat{G})$.

Remark that the notation $\mathcal{A}$ of the linear span of all matrix coefficients associated with $\pi$ in $\mathcal{U}(G)$ is a group algebra, which we call the $\pi$-group algebra $\mathcal{A}(\pi)$.

Finally, we have the *-norm closures of $\bigoplus_{\pi \in \hat{G}} B(H_\pi)$ and $\mathcal{C}[\mathcal{G}]$, in $W^*(G)$ by the mapping $f \mapsto U_\pi(f)$ is exactly the unital *-subalgebra of all linear combinations of the $a_{ij}(\pi)h$. This should be called the group algebra associated with $G$. (Remark that the notation $\mathcal{A}$ stands for $\mathcal{A}$ in [24] differently from here.)

Finally, we have the *-norm closures of $\bigoplus_{\pi \in \hat{G}} B(H_\pi)$ and $\mathcal{C}[\mathcal{G}]$, in $W^*(G)$ by the mapping $f \mapsto U_\pi(f)$ is exactly the non-unital *-subalgebra consisting of all linear combinations of the $a_{ij}(\pi)h$. This should be called the group algebra associated with $G$. (Remark that the notation $\mathcal{A}$ stands for $\mathcal{A}$ in [24] differently from here.)

4. QUANTUM SUBGROUPS AND THEIR DUALS

Let $G = (A, \delta_G)$, $H = (B, \delta_H)$ be CQGs and assume that the $H$ is a quantum subgroup of $G$ with surjective *-homomorphism $\theta : A \rightarrow B$. We fix complete families $\{U_\pi\}_{\pi \in \hat{G}}$ and $\{U_\rho\}_{\rho \in \hat{B}}$ of representatives.

We will investigate the relation between the dual actions $\tau^G$, $\hat{\tau}^H$ of the representations of $G$, $H$. Recall that for any $\pi \in \hat{G}$ the restriction $U_{\pi|_H} = (\text{id} \otimes \theta)(U_\pi)$ admits an irreducible decomposition, that is, there exist a finite subset $\mathcal{F}_\pi \subset \hat{H}$, natural numbers $m_\pi(\rho)$, $\rho \in \mathcal{F}_\pi$, and $S_{\rho,l,\pi} \in \text{Mor}(U_\rho, U_{\pi|_H})$, $1 \leq l \leq m_\pi(\rho)$, $\rho \in \mathcal{F}_\pi$, such that $S_{\rho,l,\pi}^*S_{\rho',l',\pi} = \delta_{\rho,\rho'}\delta_{l,l'}I_{H_\rho}$ for every $1 \leq l \leq m_\pi(\rho)$ and $\rho \in \mathcal{F}_\pi$, $\sum_{\rho \in \mathcal{F}_\pi} \sum_{l=1}^{m_\pi(\rho)} S_{\rho,l,\pi}^*S_{\rho,l,\pi} = I_{H_\pi}$ and

$$U_{\pi|_H} = \sum_{\rho \in \mathcal{F}_\pi} \sum_{l=1}^{m_\pi(\rho)} (S_{\rho,l,\pi} \otimes 1_B)U_\rho(U_{\rho,l,\pi} \otimes 1_B)^*.$$  

(4.1)

It is well known that the family $\mathcal{F}_\pi$ as well as the multiplicities $m_\pi(\rho)$ are uniquely determined and describe the branching rule. We write $\rho \prec \pi$ for $(\rho, \pi) \in \hat{H} \times \hat{G}$, if $\rho \in \mathcal{F}_\pi$. The $S_{\rho,l,\pi}$ are not unique, and thus we choose and fix them throughout this section.

We define the map $\Theta : \mathcal{U}(H) \rightarrow \mathcal{U}(G)$ by $\Theta(f) = f \circ \theta$ for any $f \in \mathcal{U}(H)$. It is easy to see that the map $\Theta$ becomes a *-injective unital homomorphism. Note that the injectivity follows from the fact that $\theta$ sends the linear span of all matrix coefficients with $G$ onto that associated with $H$, see [27, Lemma 2.8(1)]. We need the next simple proposition later.

Proposition 4.1. We have $\Theta(W^*(H)) \subset W^*(G)$.

Proof. For any $\pi \in \hat{G}$ and $f \in \mathcal{U}(H)$ we have

$$U_\pi(\Theta(f)) = \sum_{\rho \in \mathcal{F}_\pi} \sum_{l=1}^{m_\pi(\rho)} S_{\rho,l,\pi}U_\rho(f)S_{\rho,l,\pi}^*,$$  

(4.2)

thus, it follows that $\|U_\pi(\Theta(f))\| \leq \sup_{\rho \in \mathcal{F}_\pi} \|U_\rho(f)\|$. This immediately implies the desired assertion. □
We remark that both \( \Theta(\mathbb{C}[H]) \subset \Theta(C^*(H)) \) do not sit inside \( C^*(G) \) in general. This forces us to construct a canonical \( C^* \)-algebra associated with a given inductive system of CQGs in an indirect way; see the next section.

Formula (4.2) gives an explicit description of the embedding \( \Theta \). Indeed, we have

\[
U^G_\pi(\Theta(f)) = \left( \sum_{\rho \in H, \rho \prec \pi} \sum_{l=1}^{m_\rho(\pi)} S_{\rho,l,\pi} U_\rho(f) S_{\rho,l,\pi}^* \right)_{\pi \in \hat{G}}, \quad f \in \mathcal{U}(H).
\]

We then investigate how \( F_\pi = (\id \otimes f^H_\pi)(U_\pi) \), \( \pi \in \hat{G} \) are related to \( F_\rho = (\id \otimes f^H_\rho)(U_\rho), \rho \in \hat{H} \).

**Lemma 4.1.** Let \( \pi \in \hat{G} \) be arbitrarily given. Then

\[
F_{\pi|H} = (\id \otimes f^H_\pi)(U_{\pi|H}) = U_\pi(\Theta(f^H_\pi)) = \sum_{\rho \in H, \rho \prec \pi} \sum_{l=1}^{m_\rho(\pi)} S_{\rho,l,\pi} F_\rho S_{\rho,l,\pi}^* \subseteq U_\pi(\Theta(\mathcal{U}(H))),
\]

and there exits a unique positive invertible element \( W_\pi \in \End(U_{\pi|H}) = U_\pi(\Theta(\mathcal{U}(H)))' \) on \( \mathcal{H}_\pi \) such that

\[
F_\pi = W_\pi F_{\pi|H} = F_{\pi|H} W_\pi.
\]

**Proof.** The first formula is trivial.

Since \( U_{\pi|H}^{cc} = (\id \otimes \Theta)(U_{\pi|H}^{cc}) \), we observe that \( F_\pi \in \Mor(U_{\pi|H}, U_{\pi|H}^{cc}) \cong \Mor(U_{\pi|H}, U_{\pi|H})_\pi \). Hence \( W_\pi := F_{\pi|H}^{-1} F_\pi \) falls in \( \End(U_{\pi|H}) \). We observe that \( \End(U_{\pi|H}) = U_\pi(\Theta(\mathcal{U}(H)))' \) on \( \mathcal{H}_\pi \), and hence \( F_\pi = F_{\pi|H} W_\pi = W_\pi F_{\pi|H} = F_{\pi|H}^{1/2} W_\pi F_{\pi|H}^{1/2} \) and \( W_\pi = F_{\pi|H}^{-1/2} F_\pi F_{\pi|H}^{-1/2} \) is positive invertible. \( \square \)

**Remark 4.1.** Remark that the \( S_{\rho,l,\pi} S_{\rho,l',\pi}^* \), \( 1 \leq l, l' \leq m_\rho(\pi), \rho \prec \pi \), form a matrix unit system of \( \End(U_{\pi|H}) = U_\pi(\Theta(\mathcal{U}(H)))' \) on \( \mathcal{H}_\pi \). The above lemma shows, in particular, that one can re-choose, by perturbing them by a suitable unitary in \( U_{\pi}(\Theta(\mathcal{U}(H)))' \) to each \( \pi \), a complete family \( \{U_{\pi}\}_{\pi \in \hat{G}} \) of representatives and the intertwiners \( S_{\rho,l,\pi} \) with keeping \( (U_\rho)_{\rho \in \hat{H}} \) and \( \{U_{\pi|H}\}_{\pi \in \hat{G}} \) in such a way that all \( W_\pi \) are “diagonalized”, that is,

\[
W_\pi = \sum_{\rho \in H, \rho \prec \pi} \sum_{l=1}^{m_\rho(\pi)} w(\rho, l, \pi) S_{\rho,l,\pi} S_{\rho,l,\pi}^*
\]

with \( w(\rho, l, \pi) \) not decreasing (or not increasing) in \( l \). In this case, we have the following branching formula of the density matrix:

\[
F_\pi = \sum_{\rho \in H, \rho \prec \pi} \sum_{l=1}^{m_\rho(\pi)} w(\rho, l, \pi) S_{\rho,l,\pi} F_\rho S_{\rho,l,\pi}^*. \tag{4.3}
\]

This is an important observation to construct a canonical “weighted branching graph” for a given inductive system of CQGs later.

The next proposition seems fundamental in the study of quantum subgroups and might be well-known to experts. Indeed, it follows from e.g. [17] Proposition 3.15]. However, we include the proof for the convenience of the reader.

**Proposition 4.2.** We have \( \Theta \circ \tau^G_t = \tau^H_t \circ \Theta \), and hence \( \Theta \circ \tau^G_t = \tau^G_t \circ \Theta \) for every \( t \in \mathbb{R} \).

**Proof.** It suffices to confirm the first identity against any matrix coefficients. For every \( \pi \in \hat{G} \) and \( t \in \mathbb{R} \) we have, by formula (2.1) and Lemma 4.1

\[
(\id \otimes \Theta \circ \tau^G_t)(U_\pi) = (F_{\pi}^{\sqrt{-t}H} \otimes 1_B) U_{\pi|H} (F_{\pi}^{-\sqrt{-t}H} \otimes 1_B)
\]

\[
= \sum_{\rho \in H, \rho \prec \pi} \sum_{l=1}^{m_\rho(\pi)} (W_{\pi}^{\sqrt{-t}H} U_{\pi|H} S_{\rho,l,\pi} \otimes 1_B) U_\rho (S_{\rho,l,\pi}^{-\sqrt{-t}H} W_\pi^{-\sqrt{-t}H} \otimes 1_B)
\]

\[
= \sum_{\rho \in H, \rho \prec \pi} \sum_{l=1}^{m_\rho(\pi)} (W_{\pi}^{\sqrt{-t}H} S_{\rho,l,\pi} F_{\rho}^{-\sqrt{-t}H} \otimes 1_B) U_\rho (F_{\rho}^{-\sqrt{-t}H} S_{\rho,l,\pi} W_\pi^{-\sqrt{-t}H} \otimes 1_B)
\]

\[
= (W_{\pi}^{-\sqrt{-t}H} \otimes 1_B)(\id \otimes \tau^H_t)(U_{\pi|H})(W_{\pi}^{-\sqrt{-t}H} \otimes 1_B)
\]

\[
= (\id \otimes \tau^H_t)(U_{\pi|H}) = (\id \otimes \tau^H_t \circ \Theta)(U_\pi),
\]
Proposition 4.3. The family of intertwiners $S_{R,t,\pi}$ are uniquely determined up to left multiplication of unitary elements in $\text{End}(U_{\pi\mid H}) = U_{\pi}(\Theta(U(H))'$ on $H_{\pi}$ by $U_{\rho}$ and the $U_{\rho}$ with $\rho \in \hat{H}$, $\rho < \pi$.

5. Stratila–Voiculescu AF-flows and Quantized Characters

For a given inductive system of compact groups, Stratila and Voiculescu studied factor representations of its inductive limit group by using a certain AF-algebra, see e.g. [24]. Following Enomoto–Izumi [7] we call that AF-algebra the Stratila–Voiculescu AF-algebra. In this section, we introduce the same kind of AF-algebra with a certain one-parameter automorphism group for a given inductive system of CQGs, which we call the Stratila–Voiculescu AF-flow.

Let $G = (G_N, \theta_N)_{N=0}^{\infty}$ be an inductive system of CQGs, that is, each $G_N = (A_N, \delta_N)$ is a CQG and also a quantum subgroup of the next $G_{N+1}$ with the surjective *-homomorphism $\theta_N: A_{N+1} \to A_N$. For the convenience, we assume that $G_0 = (\mathbb{C}, \text{id}_\mathbb{C})$. As we saw in the previous section, we have an inductive system $\Theta_N: W^*(G_N) \to W^*(G_{N+1})$, $N = 0, 1, 2, \ldots$, where all $\Theta_N$ are injective unital *-homomorphisms. Hence we can take the $C^*$-inductive limit $\mathfrak{M}(G) := \varprojlim_{N} (W^*(G_N), \Theta_N)$. Then we can faithfully embed all $C^*(G_N) \subset W^*(G_N)$ into $\mathfrak{M}(G)$ and denote by $\mathfrak{A}(G)$ the unital $C^*$-subalgebra generated by those $C^*(G_N)$ inside $\mathfrak{M}(G)$. Here we remark that $C^*(G_0) = W^*(G_0) = \mathbb{C}$. We call this unital $C^*$-algebra $\mathfrak{A}(G)$ the Stratila–Voiculescu AF-algebra with the inductive system $G$. Indeed, it immediately follows from [31 Theorem 2.2] that $\mathfrak{A}(G)$ is an AF-algebra.

We remark that the inductive system $\Theta_N: W^*(G_N) \to W^*(G_{N+1})$ as well as the structure of $C^*(G_N) \subset W^*(G_N)$ are completely determined by the unitary duals $\hat{G}_N$ and their branching rule. Therefore, the AF-algebra $\mathfrak{A}(G)$ itself never remembers the effect of “$q$-deformation”. However, the dual actions $\hat{\tau}^G_N$ certainly remembers the $q$-deformation when one considers $q$-deformed classical groups like $U_q(G_N)$, and also they are known to be the modular actions associated with the dual Haar weights, see e.g. [25]. Fortunately, as we saw before, the dual actions $\hat{\tau}^G_N$ are compatible with the inductive system $\Theta_N$, and hence we obtain the action $\hat{\tau}^G: \mathbb{R} \cap \mathfrak{A}(G)$ as the restriction of $\varprojlim_{N} \hat{\tau}^G_N$, $t \in \mathbb{R}$ to $\mathfrak{A}(G)$. It is rather easy to see that $\hat{\tau}^G: \mathbb{R} \cap \mathfrak{A}(G)$ is norm continuous pointwisely. In what follows, we call this action $\hat{\tau}^G: \mathbb{R} \cap \mathfrak{A}(G)$ the Stratila–Voiculescu AF-flow associated with the inductive system $G$. We regard a certain class of KMS states with respect to $\hat{\tau}^G$ as the characters of the inductive system $G$ as follows.

Definition 5.1. A quantized character of the inductive system $G$ is a $\hat{\tau}^G$-KMS state $\chi$ on $\mathfrak{A}(G)$ of inverse temperature $-1$ such that the restriction of $\chi|_{C^*(G_N)}$ to $C^*(G_N)$ is of norm 1 for every $N = 1, 2, \ldots$.

If the inductive system $G$ comes from ordinary compact groups (or even compact Kac algebras), then the dual scaling actions $\hat{\tau}^G_N$ are all trivial and hence so is the Stratila–Voiculescu AF-flow $\hat{\tau}^G$, implying that any quantized characters are trivial in the case. It is quite natural to assume that the restriction of $\chi|_{C^*(G_N)}$ to $C^*(G_N)$ is of norm 1 for every $N = 1, 2, \ldots$ in the above. In fact, this assumption clearly holds true in the case of characters of ordinary compact groups. Conversely, if a tracial state satisfies this assumption, then the norm of the restriction of the state to the closed ideal $J_N$ generated by $\bigcup_{k \geq N+1} C^*(G_k)$ in $\mathfrak{A}(G)$ is equal to 1 for every $N = 0, 1, \ldots$, and such a trace comes from some character on the inductive limit group, see the proof of [7 Lemma 2.2(1)].

We denote the set of $\hat{\tau}^G$-KMS states of inverse temperature $-1$ by $\text{KMS}(\mathfrak{A}(G), \hat{\tau}^G)$ and the set of quantized characters by $\text{Ch}(\mathfrak{A}(G))$. The former is equipped with the topology of weak* convergence and the later is equipped with the relative topology. Since the set $\text{KMS}(\mathfrak{A}(G), \hat{\tau}^G)$ is a Choquet simplex, see e.g. [31 Theorem 5.3.30(2)], the next proposition follows. See [24] for the definition of Choquet simplexes and some basics of Choquet theory. In particular, see [24 Section 10] for Choquet–Meyer theorem.

Proposition 5.1. All of extremal points in $\text{Ch}(\mathfrak{A}(G))$ are also extremal in $\text{KMS}(\mathfrak{A}(G), \hat{\tau}^G)$, that is, $\text{ex}(\text{Ch}(\mathfrak{A}(G))) = \text{ex}(\text{KMS}(\mathfrak{A}(G), \hat{\tau}^G)) \cap \text{Ch}(\mathfrak{A}(G))$. 

Furthermore, for any quantized character \( \chi \in \text{Ch}(G) \) there exists a unique probability measure \( M \) on the set of extremal points \( \text{ex}(\text{Ch}(G)) \) such that
\[
\chi = \int_{\text{ex}(\text{Ch}(G))} \epsilon \, dM(\epsilon),
\]
that is,
\[
\chi(a) = \int_{\text{ex}(\text{Ch}(G))} \epsilon(a) \, dM(\epsilon)
\]
for any \( a \in \mathfrak{A}(G) \).

The first statement is proved similarly to [2] Lemma 2.2 (2) and the second statement is proved by Choquet–Mayer theorem.

The following lemma is necessary in the next section. For any \( N = 1,2,\ldots \) and any \( \pi \in \hat{G}_N \), by Lemma 5.1, we have \( \hat{\tau}^G_N |_{B(\mathcal{H}_\pi)} = \text{Ad}(F_N \pi^{-1}) \). Thus, the state \( \chi_\pi \) on \( W^*(G_N) \) defined to be \( \chi_\pi(x) := \text{Tr}(F_\pi p_\pi x)/\text{Tr}(F_\pi) \) is a KMS state on \( W^*(G_N) \) for the action \( \hat{\tau}^G_N : \mathbb{R} \curvearrowright W^*(G_N) \), where \( p_\pi \) is the projection onto \( \mathcal{H}_\pi \). See [4] Example 5.3.31.

**Lemma 5.1.** For any \( \chi \in \text{KMS}(\mathfrak{A}(G), \hat{\tau}^G) \) the restriction of \( \chi \) to \( C^*(G_N) \) is decomposed as
\[
\chi|_{C^*(G_N)} = \sum_{\pi \in \hat{G}_N} c_\pi \chi_\pi, \quad c_\pi = \chi(p_\pi).
\]

**Proof.** Firstly, we claim that the right-hand side is well defined. Let \( \chi_S \) be the linear functional on \( W^*(G_N) \) defined to be \( \sum_{\pi \in S} c_\pi \chi_\pi \) for any finite subset \( S \subset \hat{G}_N \). Then, we have
\[
\|\chi_S - \chi_{S'}\| = \left\| \sum_{\pi \in S} c_\pi \chi_\pi - \sum_{\pi \in S'} c_\pi \chi_\pi \right\| \leq \sum_{\pi \in S \setminus S' \cup S' \setminus S} c_\pi \to 0
\]
as \( S, S' \to \hat{G}_N \), that is, \( \{\chi_S\}_S \) is a Cauchy net in the predual \( W^*(G_N) \) of the von Neumann algebra \( W^*(G_N) \), and thus, it converges. The functional \( \sum_{\pi \in \hat{G}_N} c_\pi \chi_\pi \) is defined as the limit. Since a KMS state on a matrix algebra is unique, see [4] Example 5.3.31, we have \( \chi/c_\pi = \chi_\pi \) on \( B(\mathcal{H}_\pi) \). Thus, we have \( \chi = \sum_{\pi \in \hat{G}_N} c_\pi \chi_\pi \) on the algebraic direct sum \( \bigoplus_{\pi \in \hat{G}_N} B(\mathcal{H}_\pi) \) and hence on \( C^*(G_N) \).

By the above proof, the (KMS) state \( \chi \) is extended to a \( \sigma \)-weakly continuous functional on the von Neumann algebra \( W^*(G_N) \).

6. **Vershik-Kerov’s ergodic method**

**6.1. Branching graphs and weighted central measures, weighted coherent systems.** We recall the concept of branching graphs (or Bratteli diagrams) and introduce a certain class of probability measures on their path spaces.

**Definition 6.1.** Let \( G := (V,E,s,r) \) be a directed graph, that is, \( V \) is the vertex set, \( E \) is the edge set and \( s,r : E \to V \) are the source and the range maps. The graph \( G \) is called a branching graph if
\[
\begin{align*}
(1) & \ V_0 \text{ consists of only one element denoted by } *, \\
(2) & \ V_N \text{ is a countable set for every } N = 1, 2, \ldots , \\
(3) & \ |r^{-1}(\{v\})| < \infty \text{ for every } v \in V, \\
(4) & \ s(E_N) = V_{N-1} \text{ and } r(E_N) = V_N \text{ for every } N = 1, 2, \ldots .
\end{align*}
\]

Typical examples of branching graphs are Bratteli diagrams as well as the following example:

**Example 6.1** (The Gelfand–Tsetlin graph). Our main example of branching graph is the Gelfand-Tsetlin graph \( GT \). This graph is associated with the branching rule of (quantum) unitary groups, see [34], [21]. Let \( \text{Sign}_N \) be the set of signatures of size \( N \), that is,
\[
\text{Sign}_N := \{\nu = (\nu_1, \ldots , \nu_N) \in \mathbb{Z}^N : \nu_1 \geq \cdots \geq \nu_N\}.
\]

We set \( \text{Sign}_0 := \{*\} \). For each \( N \geq 1 \), two signatures \( \mu \in \text{Sign}_N \) and \( \nu \in \text{Sign}_{N+1} \) are joined by an edge if and only if
\[
\nu_1 \geq \mu_1 \geq \nu_2 \geq \cdots \geq \nu_N \geq \mu_N \geq \nu_{N+1}.
\]

We write \( \mu \prec \nu \) in this case. Moreover, we assume that \(* \in \text{Sign}_0\) is joined to each vertex in \( \text{Sign}_1 \) by only one edge. Define
\[
E_G^T_N := \{(\mu, \nu) \in \text{Sign}_{N-1} \times \text{Sign}_N : \mu \prec \nu\}
\]
for each $N \geq 1$ and
\[ G^\ast := \bigcap_{N \geq 0} \text{Sign}_N, \quad E^\ast := \bigcap_{N \geq 1} E_N. \]

The source and the range maps $s, r : E^\ast \rightarrow G^\ast$ are defined as the projections onto the first and the second components, respectively. In this way, we have obtained the branching graph $\mathcal{G} := (G^\ast, E^\ast, s, r)$, called the Gelfand-Tsetlin graph. Note that the number of the paths from $*$ to $\nu \in \text{Sign}_N$ is exactly the dimension of the irreducible representation of the unitary group $U(N)$ with label $\nu$.

**Definition 6.2** (The branching graph associated with an inductive system of CQGs) For each inductive system $\mathcal{G} = (G_N, \theta_N)_{N=0}^\infty$ with $G_0 = (\mathbb{C}, \text{id})$ we can construct the branching graph arising from the branching rule of the irreducible representations in the following way: The vertex set $V_N, N = 1, 2, \ldots$, is exactly the unitary dual $\hat{G}_N$. We set $V_0 = \{*, \}$ consisting only of the trivial representation. For any pair $(\rho, \pi) \in \hat{G}_{N-1} \times \hat{G}_N, \theta$, we define $m_{\pi}(\rho)$ to be the multiplicity of $\rho$ in $\pi|_{\hat{G}_{N-1}}$. Then the edge set $E_N, N = 1, 2, \ldots$, is defined to be $\{(\rho, l, \pi) : 1 \leq l \leq m_{\pi}(\rho), \rho \hookrightarrow \pi \in \hat{G}_N\}$. The source and the range maps $s, r : E := \bigcup_{N \geq 1} E_N \rightarrow V := \bigcup_{N \geq 0} V_N$ are the projections to the first and the third coordinates, respectively. Then, the quadlet $(V, E, s, r)$ is a branching graph. When all the multiplicities $m_{\pi}(\rho)$ are equal to 1, we simply denote each edge by $[\rho, \pi]$.

For any $K < N$ a sequence $(t_n)_{n=K}^N \in \prod_{n=K}^N E_n$ of edges is called a path on a branching graph $\mathcal{G}$ if $r(t_n) = s(t_{n+1})$ for every $n = K, \ldots, N - 1$. When $N < \infty$, the path is called a finite path; otherwise it is called an infinite path. For any $u \in V_K, v \in V_\infty$ with $K < N$ we denote by $\Omega(u, v)$ the set of all (finite) paths from $u$ to $v$, that is, the source of the first edge is $u$ and the range of the final edge is $v$. We define $\dim(u, v) := |\Omega(u, v)|$, the number of elements of $\Omega(u, v)$, called the relative dimension from $u$ to $v$. When $u = * \in V_0$, we call the dimension of $v$ and denote it by $\dim(v)$. When the branching graph $\mathcal{G}$ associated with an inductive system of CQGs, the dimension of each vertex is nothing but the dimension of the corresponding irreducible representation.

We introduce the notion of weighted branching graphs. Let $\mathcal{G} = (V, E, s, r)$ be a branching graph. A weight function is a function $w : E \rightarrow (0, \infty)$, and the branching graph $\mathcal{G}$ equipped with a weight function $w$ is called a weighted branching graph. For any finite path $t = (t_n)_{n=K}^N$, its weight $w(t)$ is defined to be $w(t_K)w(t_{K+1})\cdots w(t_N)$ and the weighted dimension of each vertex $v$ is defined to be $\sum_{t \in \Omega(v)} w(t)$, denoted by $w(\dim(v))$. Moreover, for any $u \in V_K, v \in V_N$ with $K < N$, the relative weighted dimensions from $u$ to $v$ is defined to be $\sum_{t \in \Omega(u, v)} w(t)$ and denoted by $w(\dim(v))$.

**Definition 6.3** (The weighted branching graph associated with an inductive system of CQGs) On the branching graph $\mathcal{G}$ associated with an inductive system $\mathcal{G}$ of CQGs, we can construct a canonical weight function as follows. By Remark 4.1 we can inductively select the sequence of families of representatives, say $(U_n)_{n=\infty}^N$, $N = 1, 2, \ldots$ with intertwiners $S_{\rho, l, \pi}$ with $(\rho, \pi) \in \hat{G}_{N-1} \times \hat{G}_N, \rho \hookrightarrow \pi, 1 \leq l \leq m_{\pi}(\rho)$, in such a way that
\[ W_\pi = \sum_{\rho \in \hat{G}_{N-1}, \rho \hookrightarrow \pi} \sum_{l=1}^{m_{\pi}(\rho)} w(\rho, l, \pi)S_{\rho, l, \pi}S_{\rho, l, \pi}^*, \quad \pi \in \hat{G}_N \]
with $w(\rho, l, \pi)$ not decreasing in $l$ for every $N = 1, 2, \ldots$. These $w(\rho, l, \pi)$ define a weight function $w$, i.e., $w(\rho, l, \pi) := w(\rho, l, \pi)$. In this way, we can define a canonical weight function $w$ on $\mathcal{G}$. Since the list $w(\rho, l, \pi)$ is nothing but the eigenvalues of $W_\pi$, this weight function is essentially independent of the choice of intertwiners $S_{\rho, l, \pi}$.

**Remark 6.1.** By Formula (1.3), we have
\[ w(\rho, l, \pi) = \frac{\text{Tr}(S_{\rho, l, \pi}^*F_\pi S_{\rho, l, \pi})}{\text{Tr}(F_\pi)} \]
for every edge $[\rho, l, \pi] \in E_N, N = 1, 2, \ldots$. Let $S_e := S_{\rho, l, \pi}$ and $S_t := S_{\rho, l, \pi}S_{\rho, l, \pi-1}\cdots S_t$ for every edge $e = [\rho, l, \pi]$ and every finite path $t = (t_n)_{n=1}^N$. By Formula (1.3) again we have
\[ F_\pi = \sum_{t \in \Omega(\pi, \pi)} w(t)S_tS_t^*. \]

The $S_tS_t^*, t, u \in \Omega(\pi, \pi)$ form a system of matrix units on $B(H_\pi)$.

Let $\Omega := \Omega(\mathcal{G})$ be the space of infinite paths starting at $*$ on a branching graph $\mathcal{G}$. For any finite path $t := (t_n)_{n=1}^N$ (starting at $*$), the cylinder set $C_t$ is defined by
\[ C_t := \{(\omega_n)_{n=1}^N \in \Omega : \omega_n = t_n, n = 1, \ldots, N\}. \]
Let $\mathcal{F}$ be the $\sigma$-algebra generated by the collection of cylinder sets. It is easy to see that the measurable space $(\Omega, \mathcal{F})$ is a unique standard Borel space. Following Gorin’s definition, see [10] Section 1.2, a probability measure $P$ on $(\Omega, \mathcal{F})$ is called $w$-central if the following holds: For any finite path $t$ starting at $*$ and terminating at $v \in V_N, N = 1, 2, \ldots$

$$\frac{P(C_t)}{w(t)} = \frac{P(X_N = v)}{w_\ast \dim(v)},$$

where the measurable function $X_N : (\Omega, \mathcal{F}) \to V_N$ is defined by $X_N(\omega) := r(\omega_N)$. The set of $w$-central probability measures is denoted by $\text{Cent}(\mathcal{G}, w)$ and it is clearly a convex set.

A sequence of probability measures $P_N$ on $V_N$ with $N = 1, 2, \ldots$ is called a $w$-coherent system if the following coherent relation holds:

$$P_N(v) = \sum_{v' \in V_{N+1}} \left( \sum_{e \in \Omega(v, v')} w(e) \right) \frac{w_\ast \dim(v)}{w_\ast \dim(v')} P_{N+1}(v')$$

(6.2)

for any $v \in V_N$ and $N = 1, 2, \ldots$. The set of $w$-coherent systems is denoted by $\text{Coh}(\mathcal{G}, w)$ and it is also a convex set. When a weighted branching graph $(\mathcal{G}, w)$ is associated with an inductive system $\mathcal{G}$, we denote the set $\text{Coh}(\mathcal{G}, w)$ by $\text{Coh}(\mathcal{G})$, also the set $\text{Coh}(\mathcal{G}, w)$ is denoted by $\text{Cent}(\mathcal{G})$.

There exists an affine bijection between the convex set of $w$-coherent systems and that of $w$-central measures. The bijection is simply given by $P_N(v) = P(X_N = v)$ for any $v \in V_N, N = 1, 2, \ldots$. We consider the topology of weak convergence on $\text{Cent}(\mathcal{G}, w)$ and the topology of component-wise weak convergence on $\text{Coh}(\mathcal{G}, w)$. The next proposition trivially holds true.

**Proposition 6.1.** The convex sets $\text{Cent}(\mathcal{G}, w)$ and $\text{Coh}(\mathcal{G}, w)$ are (affine-)homeomorphic by the correspondence $P \mapsto (P_N)_{N=1}^\infty$ with $P_N(v) = P(X_N = v)$ for any $v \in V_N, N = 1, 2, \ldots$.

Finally, we introduce a certain group of measurable transformations on $(\Omega, \mathcal{F})$. For any $v \in V_N, N = 1, 2, \ldots$, the group of all permutations of $\Omega(v, v)$ is denoted by $\mathfrak{S}_v^0$. It is naturally embedded into the measurable transformations on $(\Omega, \mathcal{F})$, that is, for any $\gamma \in \mathfrak{S}_v^0$ and any $\omega = (\omega_n)_n \in \Omega$, we define by

$$\gamma(\omega) := \begin{cases} (\gamma(\omega_1, \ldots, \omega_N), \omega_{N+1}, \ldots) & (r(\omega_N) = v) \\ \omega & (\text{otherwise}) \end{cases}.$$ 

Let $\mathfrak{S}_v(\Omega)$ be the embedding of $\mathfrak{S}_v^0$ and define $\mathfrak{S}_N(\Omega)$ to be the subgroup of all the measurable transformations on $\Omega$ generated by $\bigcup_{v \in V_N} \mathfrak{S}_v(\Omega)$ which is isomorphic to $\bigoplus_{v \in V_N} \mathfrak{S}_v(\Omega)$ as abstract groups. Trivially, the group $\mathfrak{S}_N(\Omega)$ is a subgroup of $\mathfrak{S}_{N+1}(\Omega)$ and hence we obtain the transformation group $\mathfrak{S}(\Omega) := \lim_{N \to \infty} \mathfrak{S}_N(\Omega) = (\bigcup_{N \geq 1} \mathfrak{S}_N(\Omega))$ on $\Omega$. For any $\gamma \in \mathfrak{S}(\Omega)$, any cylinder set $C$ and any $w$-central measure $P$ it follows that $P(C) = 0$ if and only if $P(\gamma(C)) = 0$ from the definition of $w$-central measures. Thus $w$-central measures are quasi-invariant under the transformation group $\mathfrak{S}(\Omega)$. In Subsection 6.3, we will show that when a weighted branching graph arising from an inductive system $\mathcal{G}$ of CQGs, a $w$-central measure $P$ is $\mathfrak{S}(\Omega)$-ergodic if and only if $P$ is extremal in $\text{Cent}(\mathcal{G})$ (see Theorem [6.2]).

### 6.2. $w$-central measures and quantized characters

For an inductive system $\mathcal{G} = (G_N, \theta_N)_{N=0}^\infty$ of CQGs with $G_0 = (\mathbb{C}, \text{id})$, we will investigate the relation between the set $\text{Ch}(\mathcal{G})$ of quantized characters and the set $\text{Cent}(\mathcal{G})$ of $w$-central measures (or the set $\text{Coh}(\mathcal{G})$ of $w$-coherent systems).

**Proposition 6.2.** There exists an affine homeomorphism between $\text{Ch}(\mathcal{G})$ and $\text{Coh}(\mathcal{G})$ such that the quantized character $\chi$ corresponding to a $w$-coherent system $(P_N)_{N}$ is decomposed as

$$\sum_{\pi \in \hat{G}_N} P_N(\pi) \chi_{\pi}$$

on the $C^*$-subalgebra $C^*(G_N)$. Furthermore, there exists an affine homeomorphism between $\text{Ch}(\mathcal{G})$ and $\text{Cnet}(\mathcal{G})$ such that the quantized character $\chi$ corresponding to a $w$-central measure $P$ is decomposed as

$$\sum_{\pi \in \hat{G}_N} P(X_N = \pi) \chi_{\pi}$$

on the $C^*$-subalgebra $C^*(G_N)$, where $X_N$ is the measurable function defined by $X_N(\omega) = r(\omega_N)$ for any $\omega = (\omega_n)_{n=1}^\infty \in \Omega$. 

Proof. For a given $w$-coherent system $(P_N)_N$, the KMS state $\chi_N$ on $\mathcal{W}^*(G_N)$ is defined to be

$$\chi_N := \sum_{\pi \in \hat{G}_N} P_N(\pi)\chi_\pi.$$ We define $p_\pi$ to be the projection onto $\mathcal{H}_\pi$. Since the net $a(S) := (a_\pi(S))_{\pi \in \hat{G}}$, where $a_\pi(S)$ is defined by

$$a_\pi(S) := \begin{cases} p_\pi & (\pi \in S) \\ 0 & (\pi \notin S) \end{cases}$$

and $S$ is a finite subset of $\hat{G}_N$, is an approximate unit of $\mathcal{C}^*(G_N)$, we have

$$\|\chi_N\| = \lim_{S \to \hat{G}_N} \chi_N((p_\pi)_{\pi \in S}) = \lim_{S \to \hat{G}_N} \sum_{\pi \in S} P_N(\pi) = 1.$$

See e.g. [5 Lemma I.9.5]. For every $x = (x_\rho)_{\rho \in \hat{G}_N} \in \mathcal{W}^*(G_N)$, by Formulas (12) and (13), we have

$$\chi_{N+1}(\Theta_N(x)) = \sum_{\pi \in \hat{G}_{N+1}} P_{N+1}(\pi)\chi_\pi \left( \sum_{\rho \in \hat{G}_N} \sum_{e \in \Omega(\rho, \pi)} S_e x_\rho S_e^* \right)$$

$$= \sum_{\pi \in \hat{G}_{N+1}} P_{N+1}(\pi) \sum_{\rho \in \hat{G}_N; \rho \prec \pi} \left( \sum_{e \in \Omega(\rho, \pi)} w(e) \frac{w(\dim(\rho))}{w(\dim(\pi))} \right) \chi_\rho(x_\rho)$$

$$= \sum_{\rho \in \hat{G}_N} \sum_{\pi \in \hat{G}_{N+1}; \rho \prec \pi} \frac{w(\dim(\rho))}{w(\dim(\pi))} \sum_{e \in \Omega(\rho, \pi)} P_{N+1}(\pi) \chi_\rho(x_\rho)$$

$$= \chi_N(x),$$

where we can freely change the order of sums since the above summations absolutely converge. Hence we can construct the state $\chi$ on $\mathfrak{A}(\mathfrak{G})$ by $\chi|_{\mathcal{C}^*(G_N)} = \chi_N$ on $\mathcal{C}^*(G_N)$. Recall $\bigcup_{N \geq 0} \mathcal{C}^*(G_N)$ is a norm-dense and $C^\mathfrak{G}$-invariant subset of $\mathfrak{A}(\mathfrak{G})$. By Lemma 5.1 Formulas (4.2) and (4.3), we have $\chi(w N \sqrt{\mathfrak{G}}(y)) = \chi(yx)$ for any $x, y \in \bigcup_{N \geq 0} \mathfrak{C}[G_N]$. Thus, by [5 Section 5.3.1], $\chi$ is a $\tau^\mathfrak{G}$-KMS state. Since the norm of the restriction $\chi$ to $C^*(G_N)$ is equal to 1 for every $N \geq 1$, the KMS state $\chi$ falls in $\mathfrak{C}(\mathfrak{G})$. Thus, we have the desired map from $\mathfrak{C}(\mathfrak{G})$ to $\mathfrak{C}(\mathfrak{G})$.

Next, we consider the inverse map. By Lemma 5.1 every quantized character $\chi \in \mathfrak{C}(\mathfrak{G})$ can be decomposed as

$$\chi|_{\mathcal{C}^*(G_N)} = \sum_{\pi \in \hat{G}_N} c_\pi \chi_\pi$$

with non-negative coefficients $c_\pi$ on $\mathcal{C}^*(G_N)$ for any $N = 1, 2, \ldots$. Since the restriction $\chi|_{\mathcal{C}^*(G_N)}$ to $\mathcal{C}^*(G_N)$ is of norm 1, we have

$$\sum_{\pi \in \hat{G}_N} c_\pi = \lim_{S \to \hat{G}_N} \chi(a(S)) = 1,$$

that is, the function $P_N$ defined to be $P_N(\pi) := c_\pi$ is a probability measure on $\hat{G}_N$. It suffices to show that the sequence of probability measures $(P_N)_N$ becomes a $w$-coherent system. Indeed, for any $\rho \in \hat{G}_N$, by Formula (13), we have

$$P_N(\rho) = \chi_N(p_\rho)$$

$$= \chi_{N+1}(\Theta_N(p_\rho))$$

$$= \sum_{\pi \in \hat{G}_{N+1}; \rho \prec \pi} P_{N+1}(\pi) \chi_\pi \left( \sum_{l=1}^{m_\rho(S)} S_{\rho, l, S_{\rho, l, \pi}}^* \right)$$

$$= \sum_{\pi \in \hat{G}_{N+1}; \rho \prec \pi} \left( \sum_{e \in \Omega(\rho, \pi)} w(e) \frac{w(\dim(\rho))}{w(\dim(\pi))} \right) P_{N+1}(\pi),$$

and thus the sequence $(P_N)_N$ satisfies the coherent relation (6.2).

From this correspondence and Proposition 5.1 the next unique integral representation theorem for $w$-central measures follows.
Theorem 6.1. For any \( w \)-central measure \( P \in \text{Cent}(\mathcal{G}) \), there exists a unique Borel probability measure \( m \) on \( \text{Cent}(\mathcal{G}) \) supported on the extremal points \( \text{ex}(\text{Cent}(\mathcal{G})) \) such that

\[
P = \int_{\text{ex}(\text{Cent}(\mathcal{G}))} Q \, dm(Q).
\]

Here we remark that the theorem was already proved by Gorin, see \cite{10} Proposition 5.18. However, we gave a new approach to the unique integral representation theorem. We would like to emphasize that the approach here is quite natural in view of the original work due to Vershik–Kerov \cite{14}.

6.3. Krieger constructions and GNS constructions. We assume that a weighted branching graph \((\mathcal{G}, w)\) is associated with an inductive system \( \mathcal{G} = (\mathcal{G}_N, \theta_N)_N \) of CQGs with \( \mathcal{G}_0 = (\mathbb{C}, \text{id}_\mathbb{C}) \). Let \( P \in \text{Cent}(\mathcal{G}) \) be a \( w \)-central measure and \( \chi \in \text{Ch}(\mathcal{G}) \) the corresponding quantized character. In the subsection, we will show the following theorem:

Theorem 6.2. A \( w \)-central probability measure \( P \) is \( \mathfrak{S}(\Omega) \)-ergodic if and only if the corresponding quantized character \( \chi \in \text{Ch}(\mathcal{G}) \) is factorial, namely, it is extremal. Therefore, \( P \) is \( \mathfrak{S}(\Omega) \)-ergodic if and only if \( P \) is extremal.

The second part of this theorem follows from Proposition \cite{62}. The proof of the first part is given by investigating two von Neumann algebras constructed by using the \( w \)-central measure \( P \) and the corresponding quantized character \( \chi \), respectively. The study of these von Neumann algebras is based on a kind of the crossed product construction in our context, which appears in the study of AF (or LS)-algebras by Vershik and Kerov, see \cite{32}, \cite{13}.

For the given \( w \)-central probability measure \( P \), we will construct the von Neumann algebra of the dynamical system \((\Omega, \mathcal{F}, P, \mathfrak{S}(\Omega))\). The equivalence relation \( \mathcal{R} \) is defined by

\[
\mathcal{R} := \{ (\omega, \omega') \in \Omega \times \Omega : \exists \gamma \in \mathfrak{S}(\Omega), \omega' = \gamma \omega \}
\]

and it is called the tail equivalence relation. We denote its equivalent classes by \([\cdot]\). The projection \( \text{pr} : \mathcal{R} \to \Omega \) is defined by \( \text{pr}(\omega, \omega') := \omega' \) for any \((\omega, \omega') \in \mathcal{R}\). It is known that the set function \( P_1 \) on the \( \sigma \)-algebra \( B := (\mathcal{F} \times \mathcal{F}) \cap \mathcal{R} \) defined by

\[
P_1(A) := \int_{\Omega} |A \cap \text{pr}^{-1}(\omega')| \, dP(\omega'), \quad A \in B
\]

becomes a measure on \((\mathcal{R}, \mathcal{F})\), see \cite{8} Theorem 2]. The measure \( P_1 \) is called the left measure of \( P \) on \( \mathcal{R} \). The following space of bounded “small support” functions

\[
K(\mathcal{R}) := \{ f \in L^\infty(\mathcal{R}, P_1) : \text{ess.sup}\{ |\omega' : f(\omega, \omega') \neq 0| \} < \infty, \ \text{ess.sup}\{ |\omega : f(\omega, \omega') \neq 0| \} < \infty \}
\]

becomes a \( \ast \)-algebra, whose multiplication and \( \ast \)-operation are defined by

\[
f(\omega, \omega') := \sum_{\omega'' \in [\omega]} f(\omega, \omega'')g(\omega'', \omega'),
\]

\[
f^\ast(\omega, \omega') := f(\omega', \omega)
\]

for any \( f, g \in K(\mathcal{R}) \) and \((\omega, \omega') \in \mathcal{R}\). The \( \ast \)-representation \( \varpi_1 \) of \( K(\mathcal{R}) \) on the Hilbert space \( L^2(\mathcal{R}, B, P_1) \) is defined by

\[
[\varpi_1(f)](\omega, \omega') := \sum_{\omega'' \in [\omega]} f(\omega, \omega'')\xi(\omega'', \omega')
\]

for any \( f \in K(\mathcal{R}) \) and \( \xi \in L^2(\mathcal{R}, B, P_1) \). The von Neumann algebra \( W^\ast(\mathcal{R}; P) \) is defined to be the double commutant \( \varpi_1(K(\mathcal{R}))'' \), that is, the closure with respect to the strong operator topology (see e.g. \cite{5} Theorem I.7.1]). This construction is called the Krieger construction. It is well known that

- the von Neumann algebra \( W^\ast(\mathcal{R}; P) \) is a factor if and only if \( P \) is \( \mathfrak{S}(\Omega) \)-ergodic, see \cite{8} Proposition 2.9].

Next, we construct another von Neumann algebra which also acts on \( L^2(\mathcal{R}, B, P_1) \). The equivalence relation \( \mathcal{R}_N \) is defined by

\[
\mathcal{R}_N := \{ (\omega, \omega') \in \Omega \times \Omega : \exists \gamma \in \mathfrak{S}_N(\Omega), \omega' = \gamma (\omega) \}
\]

and its equivalence classes are denoted by \([\cdot]_N\). For any \( \rho \in \widehat{\mathfrak{S}}_N \) and \( t, u \in \Omega(*) \rho \) we define \( f_{t,u} \) to be the characteristic function of \((C_t \times C_u) \cap \mathcal{R}_N\). Remark that \( f_{t,u} \in K(\mathcal{R}) \) by the definition. Every pair
$(\omega, \omega') \in (C_t \times C_u) \cap \mathcal{R}_N$ can be written as $(\omega, \omega') = ((t, \omega_{N+1}, \omega_{N+2}, \ldots), (u, \omega_{N+1}, \omega_{N+2}, \ldots))$. Using this, for any $t, u \in \Omega(\ast, \rho)$, $t', u' \in \Omega(\ast, \rho')$ and any $(\omega, \omega') \in \mathcal{R}$ we have

$$f_{t,u}f_{t',u'}(\omega, \omega') = \sum_{\omega'' \in [\omega]} f_{t,u}(\omega, \omega'')f_{t',u'}(\omega'', \omega') = \delta_{\rho,\rho'}\delta_{t,u'}1_{\mathcal{R}_N}(\omega, \omega')1_{C_t}(\omega)1_{C_u}(\omega') = \delta_{\rho,\rho'}\delta_{t,u'}f_{t,u}(\omega, \omega')$$

and

$$f_{t,u}^*(\omega, \omega') = f_{t,u}(\omega', \omega) = f_{t,u}(\omega, \omega').$$

Thus the functions $f_{t,u}$, $t, u \in \Omega(\ast, \rho)$, $\rho \in \hat{G}_N$ form a matrix unit system and we obtain the $*$-homomorphism $g_N : W^*(\mathcal{G}) \to W^*(\mathcal{R}; P)$ defined by $g_N(S_tS_u^*) := \varpi(f_{t,u}) \in \varpi(K(\mathcal{R}))$, where $S_tS_u^*$ is a matrix unit of $B(\ell^2(\Omega(\ast, \rho)))$, see Remark [6.1].

**Lemma 6.1.** We have $g_{N+1} \circ \Theta_N = g_N$ on $W^*(\mathcal{G})$ for all $N = 1, 2, \ldots$.  

**Proof.** It suffices to show that $g_{N+1}(\Theta_N(S_tS_u^*)) = g_N(S_tS_u^*) = \varpi(f_{t,u})$ for any $t, u \in \Omega(\ast, \rho)$ and any $\rho \in \hat{G}_N$, where $S_tS_u^*$ is a matrix unit, see Remark [6.1]. By Formula (4.2), we have

$$g_{N+1}(\Theta_N(S_tS_u^*)) = \sum_{\pi \in \hat{G} \cap \pi < \pi \in \hat{G}(\rho, \rho)} \rho_{N+1}(S_tS_uS_t^*S_u^*)$$

$$= \sum_{\pi \in \hat{G} \cap \pi < \pi \in \hat{G}(\rho, \rho)} \varpi(f_{t,u})(\omega, \omega')$$

$$= \varpi(f_{t,u}) = g_N(S_tS_u^*).$$

Hence we are done. \hfill \Box

By the lemma, we obtain the representation $\varrho$ of $\mathfrak{A}(\mathcal{G})$ on $L^2(\mathcal{R}, B, P)$ defined by $\varrho(C_t(\mathcal{G})) = g_N$ for any $N = 1, 2, \ldots$. By the construction, the double commutant $\varrho(\mathfrak{A}(\mathcal{G}))''$ is a von Neumann subalgebra of $W^*(\mathcal{R}; P)$. Furthermore, we have the following:

**Theorem 6.3.** The above two von Neumann algebras coincide, that is, $\varrho(\mathfrak{A}(\mathcal{G}))'' = W^*(\mathcal{R}; P)$.  

**Proof.** It suffices to show that $\varpi_1(K(\mathcal{R})) \subseteq g_N(\mathfrak{A}(\mathcal{G}))''$. For any function $f \in L^\infty(\Omega, \mathcal{F}, P)$ the function $\tilde{f}$ on $\mathcal{R}$ is defined to be $\tilde{f}(\omega, \omega') := \delta_{\omega, \omega}f(\omega)$. For any measurable bijection $\phi : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$ such that $\{(\omega, \phi(\omega)) : \omega \in \Omega\} \subseteq \mathcal{R}$, the function $F_\phi$ on $\mathcal{R}$ is defined to be

$$F_\phi(\omega, \omega') := \begin{cases} 1 & (\omega = \phi(\omega)) \\ 0 & (\text{otherwise}) \end{cases}$$

By Proposition 2.4, every function in $K(\mathcal{R})$ is a finite linear combination of functions of the form $\tilde{f}F_\phi$ with a function $f$ and a measurable bijection $\phi$. Thus, our goal is $\varpi_1(\tilde{f}), \varpi(F_\phi) \in g_N(\mathfrak{A}(\mathcal{G}))''$ for any $f \in L^\infty(\Omega, \mathcal{F}, P)$ and any measurable bijection $\phi : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$ such that $\{(\omega, \phi(\omega)) : \omega \in \Omega\} \subseteq \mathcal{R}$.

For any finite path $t$ we have $\varpi_1(1_{C_t}) = \varpi_1(f_{t,t}) = \varrho(S_tS_t^*) \in g_N(\mathfrak{A}(\mathcal{G})) \subseteq g_N(\mathfrak{A}(\mathcal{G}))''$. Note that the collection of all cylinder sets and the empty set is a $\pi$-system and the collection of the set $A$ such that $1_A \in g_N(\mathfrak{A}(\mathcal{G}))''$ is a $\lambda$-system. Thus, by the $\pi$-$\lambda$ theorem, we have $\varpi_1(1_A) \in g_N(\mathfrak{A}(\mathcal{G}))''$ for any $A \in \mathcal{F}$. For any $f \in L^\infty(\Omega, \mathcal{F}, P)$, there exists a sequence $(\psi_n)_{n \geq 1}$ of simple functions such that $\psi_n \to f$ as $n \to \infty$ $P$-a.s. with $0 \leq |\psi_1| \leq |\psi_2| \leq \cdots \leq |f|$. Then $\varpi_1(\tilde{f}F_\phi) \in g_N(\mathfrak{A}(\mathcal{G}))$ and $\|\varpi_1(\tilde{f}F_\phi)\| \leq \|\varpi_1(\tilde{f})\|$ for any $n \geq 1$. Moreover, for any $\eta \in L^2(\mathcal{R}, B, P)$, by the dominated convergence theorem,

$$\|\varpi_1(\tilde{f}F_\phi - \sum_{n=1}^N \tilde{1}_{g_n(D_n)}\phi_n)\eta\|^2 = \int_\mathcal{R} \|(f(\omega) - \psi_n(\omega))\eta(\omega, \omega')\|^2 dP(\omega, \omega') \to 0 \text{ as } n \to \infty,$$

that is, $\varpi_1(\tilde{f}F_\phi)$ converges to $\varpi_1(\tilde{f})$ in the strong operator topology. Thus, $\varpi_1(\tilde{f}) \in g_N(\mathfrak{A}(\mathcal{G}))''$.

For any measurable bijection $\phi : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$ such that $\{(\omega, \phi(\omega)) : \omega \in \Omega\} \subseteq \mathcal{R}$, Remark [3.3], there exist $g_n \in \mathcal{S}(\Omega)$ and $D_n \subseteq \mathcal{F}$ with $n = 1, 2, \ldots$ such that the collection $\{D_n\}_{n \geq 1}$ is a partition of $\mathcal{R}$ and $D_n \subseteq \{(\omega, \phi(\omega)) : \phi(\omega) = g_n(\omega)\}$. Then we have $F_g = \sum_{n \geq 1} \tilde{1}_{g_n(D_n)}F_{g_n}$. Remark that $\varpi_1(F_g) \in g_N(\mathfrak{A}(\mathcal{G})) \subseteq g_N(\mathfrak{A}(\mathcal{G}))''$ for any $g \in \mathcal{S}(\Omega)$. Thus, if the partition $\{D_n\}_{n \geq 1}$ is finite, then the operator $\varpi_1(F_g)$ belongs to $g_N(\mathfrak{A}(\mathcal{G}))''$. We suppose that the partition $\{D_n\}_{n \geq 1}$ is infinite. Since $0 \leq \sum_{n=1}^N \tilde{1}_{g_n(D_n)}F_{g_n} \to F_\phi$ as $N \to \infty$ $P$-a.s., we obtain

$$\|\varpi_1(F_\phi - \sum_{n=1}^N \tilde{1}_{g_n(D_n)}F_{g_n})\eta\| \to 0 \text{ as } N \to \infty.$$
for any $\eta \in L^2(\mathcal{R}, \mathcal{B}, \rho)$ by the dominated convergence theorem. Therefore, $\varpi_l(\sum_{n=1}^{N} 1_{g_n(D_n)}F_{g_n})$ converges to $\varpi_l(F_0)$ in the strong operator topology, that is, $\varpi_l(F_0)$ also belongs to $\rho(\mathfrak{A}(G))''$. Hence we are done. \hfill \Box

Recall that for any $\rho \in \hat{G}_N$ and any $t, u \in \Omega(\ast, \rho)$ every pair $(\omega, \omega') \in (C_t \times C_u) \cap \mathcal{R}_N$ can be written as $(\omega, \omega') = ((t, \omega_{N+1}, \omega_{N+2}, \ldots), (u, \omega_{N+1}, \omega_{N+2}, \ldots))$. Using this and Formula (6.1), we have

$$
\langle \varrho_N(S_tS_u^\ast)\eta_D, \eta_D \rangle = \int_{\Omega} \sum_{\omega \in \mathcal{E}} \langle \varrho_N(S_tS_u^\ast)\eta_D|((\omega, \omega')\eta_D)(\omega, \omega') \rangle dP(\omega')
$$

$$
= \int_{\Omega} f_{t,u}(\omega', \omega') dP(\omega')
$$

$$
= \delta_{t,u}P(C_t)
$$

$$
= \delta_{t,u}w(t)P(X_N = \rho) / w-\text{dim}(\rho) = \chi(S_tS_u^\ast).
$$

Therefore, we obtain $(L^2(\mathcal{R}, \mathcal{B}, \rho), \varrho, \eta_D)$ is the GNS-triple of $\mathfrak{A}(G)$ associated with $\chi$. By [4] Theorem 5.3.30(3)], the von Neumann algebra $\rho(\mathfrak{A}(G))'' = W^*(\mathcal{R}, \rho)$ is factor if and only if the KMS state $\chi$ is extremal in the convex set $\text{KMS}(\mathfrak{A}(G), \tau^{\mathcal{G}})$. Therefore, by Proposition [5.1] the von Neumann algebra $W^*(\mathcal{R}, \rho)$ is a factor if and only if the corresponding character $\chi$ is extremal in $\text{Ch}(G)$. Theorem [6.2] follows from this and the property of the Krieger construction mentioned in the first half of this subsection.

### 6.4. The Ergodic Method

Theorem 6.4 is a type of claim called the **ergodic method**. Let $\mathcal{G} = (V, E, s, r)$ be a branching graph with weight $w: E \to (0, \infty)$. We define $X_N: \Omega \to V$ by $X_N(\omega) = r(\omega_N)$ for any $\omega = (\omega_N)_N \in \Omega$.

**Theorem 6.4** (the ergodic method). If $P$ is $\mathfrak{S}(\Omega)$-ergodic and $w$-central, then there exists a path $\omega \in \Omega$ such that

$$
P(X_K = v) = \lim_{N \to \infty, K \leq N} w^{-\text{dim}}(v, X_N(\omega)),\quad \text{for any } v \in V_K.
$$

In particular, if a weighted branching graph associated with an inductive system of CQGs, then this holds true for any extremal $w$-central probability measures.

**Remark 6.2.** A general approximation theorem for weighed branching graphs without multiple edges was found by Okounkov and Olshanski, see [22] Theorem 6.1. Our method here allows a weighted branching graph to have multiple edges. The proof follows Vershik–Kerov’s original approach, see [13].

**Remark 6.3.** It is well known that Vershik–Kerov’s ergodic method follows from the backward martingale convergence theorem or the standard method of proving Birkhoff’s ergodic theorem, see [13] Section 1.1]. The proof here also uses the backward martingale convergence theorem. The final part of the theorem follows from Theorem [6.2].

**Proof.** We fix the vertex $v \in V_K$. For any $N \geq K+1$, the $\sigma$-algebra $\mathcal{E}_N$ is generated by $X_N, X_{N+1}, \ldots$ and the random variables $Z_N$ on the probability space $(\Omega, \mathcal{F}, P)$ are defined by

$$
Z_N(\omega) := \frac{w^{-\text{dim}}(v, X_N(\omega))}{w^{-\text{dim}}(X_N(\omega))}
$$

for $N = K+1, K+2, \ldots$. Note that $\mathcal{E}_{K+1} \supset \mathcal{E}_{K+2} \supset \ldots$ and $|Z_N| \leq 1/w^{-\text{dim}}(v)$ for any $N$. We claim that the stochastic process $(Z_N)_{N=K+1}$ is a $\{\mathcal{E}_N\}_N$-backward martingale. Clearly, $Z_N$ is $\mathcal{E}_N$-adapted and integrable. Thus it suffices to show that $E[Z_N|\mathcal{E}_{N+1}] = Z_{N+1}$ $P$-almost surely. For any $L \geq N+1$ and any $u_n \in V_n$ with $n = N, \ldots, L$ let $A := \cap_{n=N+1}^{L} (X_n = u_n) \in \mathcal{E}_{N+1}$ and assume $A \neq \emptyset$. It suffices to show that $E[Z_{N+1}|A] = E[Z_{N+1}|A]$ where $A$ denotes the characteristic function of $A$. Indeed, for each path $t$ from $u_N$ to $u_L$ along $u_{N+2}, \ldots, u_{L-1}$, we have

$$
E[Z_{N+1}A] = \frac{w^{-\text{dim}}(v, u_{N+1})w(t)}{w^{-\text{dim}}(u_L)}P(X_L = u_L) = E[Z_{N+1}A].
$$

Therefore, by the backward martingale theorem, see e.g. [3] Section 5.6], $Z_N$ converges to $E[Z_{K+1}|\mathcal{E}_N]$ $P$-almost surely as well as in $L^1$-norm, where $\mathcal{E}_\infty := \cap_{N=K+1}^{\infty} \mathcal{E}_N$. Since the collection of all $\mathfrak{S}(\Omega)$-invariant sets forms a $\sigma$-algebra, every measurable set in $\mathcal{E}_\infty$ is $\mathfrak{S}(\Omega)$-invariant. Since $P$ is $\mathfrak{S}(\Omega)$-ergodic, we have

$$
E[Z_{K+1}|\mathcal{E}_\infty] = E[Z_{K+1}] = \frac{P(X_K = v)}{w^{-\text{dim}}(v)}, \quad P\text{-a.s.}
$$
Therefore,

$$\frac{P(X_K = v)}{w \cdot \dim(v)} = \lim_{N \to \infty, K \leq N} \frac{w \cdot \dim(v, X_N(\omega))}{w \cdot \dim(X_N(\omega))}$$

for \(P\)-almost sure \(\omega \in \Omega\). Since the vertex set \(V\) is countable, we have done. The final part of the theorem follows from Theorem 6.2.

When the weighted branching graph \((G, w)\) associated with an inductive system \(G\) of CQGs, we have the following corollary.

**Corollary 6.1.** For any extremal quantized character \(\chi \in \text{ex}(\text{Ch}(G))\), there exists a sequence \(\pi(1) < \pi(2) < \cdots\) such that

$$\chi|_{C^*(G_K)}(\pi) = \lim_{N \to \infty, K \leq N} \chi(\pi(N)) \circ \Theta_{N,K}$$
on C^*(G_N), \text{where } \Theta_{N,K} := \Theta_{N-1} \circ \Theta_{N-2} \circ \cdots \circ \Theta_K.

**Proof.** Firstly, we give the following simple observation: Let \(P_K, P'_K, i = 1, 2, \ldots \) be probability measures on \(\hat{G}_K\). If \(P'_K\) converges to \(P_K\) weakly, then we have \(\sum_{\pi \in \hat{G}_K} |P_K(\pi) - P'_K(\pi)| \to 0 \text{ as } N \to \infty\). In order to show this, for any \(\epsilon > 0\) we take a finite subset \(A \subset \hat{G}_K\) such that \(P_K(A) > 1 - \epsilon/3\). (This can be done since \(G_K\) is at dos countable.) Since \(A\) is a finite set, there exists \(i_0\) such that for any \(i \geq i_0\) \(P_K(A) > 1 - \epsilon/3\) and \(|P_K(\pi) - P'_K(\pi)| < \epsilon/3|A|\) for any \(\pi \in A\). Then we have \(\sum_{\pi \in \hat{G}_K} |P_K(\pi) - P'_K(\pi)| \to 0 \text{ as } N \to \infty\).

We show the corollary. Let \((P_K)\) be the \(w\)-coherent system corresponding to the quantized character \(\chi\). By the ergodic method (Theorem 6.3), there exists a path \(\omega \in \Omega\) such that

$$\frac{P_K(\pi)}{w \cdot \dim(\pi)} = \lim_{N \to \infty, K \leq N} \frac{w \cdot \dim(\pi, X_N(\omega))}{w \cdot \dim(X_N(\omega))}$$

for any \(\pi \in \hat{G}_N\). By Proposition 6.2 Formula 13 and the observation given in the first paragraph, for any \(x = (x_\pi)_{\pi \in \hat{G}_K} \in C^*(G_K)\) we have

$$\chi(x) = \sum_{\pi \in \hat{G}_K} P_K(\pi) \chi_\pi(x_\pi)$$

$$= \lim_{N \to \infty, K \leq N} \sum_{\pi \in \hat{G}_K} \frac{w \cdot \dim(\pi, X_N(\omega))}{w \cdot \dim(X_N(\omega))} \chi_\pi(x_\pi)$$

$$= \lim_{N \to \infty, K \leq N} \sum_{\pi \in \hat{G}_K} \chi_{X_N(\omega)}(\Theta_{N,K}(x_\pi)).$$

Hence we are done.

### 7. Quantum Unitary Group

In this section, we review some basic concepts of the quantum universal enveloping algebra \(U_q(\mathfrak{gl}(N))\), the quantum unitary group \(U_q(\mathfrak{gl}(N))\) and their representation theories. We suppose that \(q\) belongs to the interval \((0, 1)\) throughout the rest of the paper.

The quantum universal enveloping algebra \(U_q(\mathfrak{gl}(N))\) is a unital algebra generated by the letters \(Q_i, Q_i^{-1}, E_j, F_j\) with \(i = 1, \ldots, N, j = 1, \ldots, N - 1\) which satisfy the relations:

$$Q_i Q_j = Q_j Q_i, \quad Q_i Q_j^{-1} = Q_j^{-1} Q_i = 1,$$

$$Q_i E_j Q_i^{-1} = q^{\delta_{i,j} / 2 - \delta_{i,j+1} / 2} E_j, \quad Q_i F_j Q_i^{-1} = q^{-\delta_{i,j} / 2 + \delta_{i,j+1} / 2} F_j,$$

$$E_i F_j - F_j E_i = \delta_{i,j} Q_i^2 Q_{i+1}^2 - Q_i^{-2} Q_{i+1}^2 = q - q^{-1},$$

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad |i - j| \geq 2,$$

$$E_i E_j + E_j E_i - (q + q^{-1})E_i E_j \geq 0, \quad F_i F_j + F_j F_i \geq 0,$$

$$E_i F_j + F_j E_i \leq 0.$$
It is known that every finite dimensional irreducible left $U_q(\mathfrak{g}(N))$-module $V$ has a highest weight vector and its weight becomes $(\omega_1 q^{\nu_1/2}, \ldots, \omega_N q^{\nu_N/2})$, where $\omega_i \in \{\pm 1, \pm \sqrt{-1}\}$ and $\nu_i \in \mathbb{Z}$ satisfy $\nu_1 \geq \cdots \geq \nu_N$, see [28] Proposition 1.2 and [14, Chapter 7]. Conversely, every weight of this form is the highest weight of a finite dimensional irreducible $U_q(\mathfrak{g}(N))$-module and every finite dimensional left $U_q(\mathfrak{g}(N))$-module with highest weight must be irreducible.

We review some concepts of the Gelfand–Tsetlin basis for finite dimensional $U_q(\mathfrak{g}(N))$-modules developed by Ueno–Takebayashi–Shibukawa. For $0 \leq j \leq i \leq N$, the lowering operators $D_{ij} \in U_q(\mathfrak{g}(N))$ are inductively defined by

$$D_{ii} := 1, \quad D_{i,i-1} := F_i,$$

$$D_{ij} := \frac{q^{i-j}Q_{i+1}^2Q_i^{-2} - q^{-i+j}Q_{i+1}^2Q_i^{-2}}{q - q^{-1}}F_iD_{i-1,j} - \frac{q^{-i-j-1}Q_{i+1}^{-2}Q_i^2 - q^{i-j+1}Q_{i+1}^{-2}Q_i^2}{q - q^{-1}}D_{i-1,j}F_i,$$

for any $\alpha = (\alpha_0, \ldots, \alpha_{i-1}) \in \mathbb{Z}_{\geq 0}^i$ with $1 \leq i \leq N$, $D^\alpha \in U_q(\mathfrak{g}(N))$ is defined to be

$$D^\alpha_t := D_{i_0,0}^{\alpha_0} \cdots D_{i_{i-1},i_{i-1}}^{\alpha_{i-1}}$$

and for any $\nu = (\nu_1 \geq \cdots \geq \nu_N)$ and any finite path $t = (t_n)_{n=1}^N \in \Omega(\nu)$, $D^\nu \in U_q(\mathfrak{g}(N))$ is defined to be

$$D^\nu_t := D_1^{(t_2) - r(t_1)}D_2^{(t_3) - r(t_2)} \cdots D_N^{(t_N) - r(t_{N-1})},$$

where $r(t_k) - r(t_{k-1}) := (r(t_k_1) - r(t_{k-1}_1), \ldots, r(t_k_{k-1}) - r(t_{k-1}_{k-1})$ and $r$ is the range map of the Gelfand–Tsetlin graph. Let $H_\nu$ be a finite-dimensional irreducible left $U_q(\mathfrak{g}(N))$-module with weight $q^{\nu/2} := (q^{\nu_1/2}, \ldots, q^{\nu_N/2})$ and $\nu = (\nu_1 \geq \cdots \geq \nu_N)$ and $\nu_0$ a highest weight vector of the weight $q^{\nu/2}$. Then the $D^\nu \nu_0$, $t \in \Omega(\nu)$ form a basis of $H_\nu$. Furthermore, $D^\nu \nu_0$ is a weight vector of the weight

$$(q^{r(t_1)/2}, q^{r(t_2) - r(t_1)/2}, \ldots, q^{r(t_N) - r(t_{N-1})}/2),$$

where $|\lambda| := \lambda_1 + \cdots + \lambda_K$ for any signature $\lambda = (\lambda_n)_{n=1}^K \in \text{Sign}_K$, $K = 1, 2, \ldots$. This basis is called the Gelfand–Tsetlin basis. See [28], [29], [14], Section 7.3 for more details.

Next, we review some concepts of the quantum unitary group $U_q(N)$ as a compact quantum group introduced by Woronowicz. The unital algebra $A_N$ is generated by the letters $\det_q^{-1}(N)$ and $u_{ij}(N)$ with $i, j = 1, \ldots, N$, with the next relations:

$$u_{ij}(N)u_{kj}(N) = qu_{kj}(N)u_{ij}(N), \quad i < k,$$

$$u_{ij}(N)u_{il}(N) = qu_{il}(N)u_{ij}(N), \quad j < l,$$

$$u_{ij}(N)u_{kl}(N) = u_{ik}(N)u_{lj}(N), \quad i < k, \quad j > l,$$

$$u_{ij}(N)u_{kl}(N) - qu_{il}(N)u_{kj}(N) = u_{ik}(N)u_{lj}(N) - q^{-1}u_{kj}(N)u_{il}(N), \quad i < k, \quad j < l,$$

$$x_{ij}(N)\det_q^{-1}(N) = \det_q^{-1}(N)x_{ij}(N), \quad \det_q(N)\det_q^{-1}(N) = \det_q^{-1}(N)\det_q(N) = 1,$$

where $\det_q(N)$ is defined by

$$\det_q(N) := \sum_{\sigma \in S(N)} (-q)^{\ell(\sigma)}u_{\sigma(1)1}(N) \cdots u_{\sigma(N)N}(N)$$

and $\ell(\sigma)$ is defined by the number of inversions appearing in the permutation $\sigma$, that is,

$$\ell(\sigma) := |\{(i, j) : i < j, \sigma(i) > \sigma(j)\}|.$$

The element $\det_q(N)$ is called the quantum determinant. For any two subsets $I := \{i_1 < \cdots < i_r\}$ and $J := \{j_1 < \cdots < j_r\}$ of $\{1, \ldots, N\}$, the quantum minor determinant is similarly defined to be

$$\det_I^{\sigma} := \sum_{\sigma \in S(r)} (-q)^{\ell(\sigma)}u_{\sigma(1)j_1}(N) \cdots u_{\sigma(r)j_r}(N),$$

where $\det_I^{\sigma}$ is simply written $\det_I$. It is well known that the algebra $A_N$ becomes a Hopf algebra whose comultiplication $\delta_N : A_N \to A_N \otimes A_N$ is defined by

$$\delta_N(u_{ij}(N)) := \sum_{k=1}^N u_{ik}(N) \otimes u_{kj}(N),$$

$$\delta_N(\det_q^{-1}(N)) := \det_q^{-1}(N) \otimes \det_q^{-1}(N).$$
See \cite{[14]} Section 7 for the definitions of the antipode $S$ and the counit $\epsilon$. Furthermore, the $*$-operation on $A_N$ is defined by 

$$ u_{ij}(N)^* := (-q)^{j-i} \hat{j}_1^i \hat{\det}_q^{-1}(N), $$

$$ \hat{\det}_q^{-1}(N)^* := \hat{\det}_q(N), $$

where $\hat{i} := \{1, \ldots, n\} \setminus \{i\}$ and $\hat{j}$ is similar. Thus $A_N$ also becomes $*$-algebra. If we regard the quadlet $(A_N, \delta_N, S_N, \epsilon)$ as a Hopf algebra, then it is called the quantum general linear group and denoted by $GL_q(N; \mathbb{C})$. On the other hand, if we regard it as a Hopf $*$-algebra, then it is called the (algebraic) quantum unitary group and denoted by $U_q(N)$. We denote the universal $C^*$-algebra of $A_N$ by $A_N$ and the continuous extension $\delta_N$ on $A_N$ by the same symbol. The pair $U_q(N) := (A_N, \delta_N)$ is called the quantum unitary group, where we denoted by $U_q(0)$ the trivial quantum group $(\mathbb{C}, \text{id}_\mathbb{C})$.

We note that the unitary quantum group $U_q(N)$ can be regarded as a quantum subgroup of $U_q(N+1)$ with the surjective unital $*$-homomorphism $\theta_N : A_{N+1} \to A_N$ defined by

$$ \theta_N(u_{ij}(N+1)) := \begin{cases} u_{ij}(N) & (1 \leq i, j \leq N) \\ \delta_{i,j}^1 & (\text{otherwise}), \end{cases} $$

$$ \theta_N(\hat{\det}_q^{-1}(N+1)) := \hat{\det}_q^{-1}(N). $$

We denote by $\mathbb{U}_q$ the inductive system of the unitary quantum groups $U_q(N)$ with $*$-homomorphisms $\theta_N$.

There exists a natural pairing of Hopf algebras between $U_q(\mathfrak{gl}(N))$ and $U_q(N)$. For given two Hopf algebras $(A, \delta_A, \epsilon_A, S_A)$ and $(U, \delta_U, \epsilon_U, S_U)$ over $\mathbb{C}$, a bilinear form $(\cdot, \cdot) : U \times A \to \mathbb{C}$ is called a pairing of Hopf algebras if it satisfies the relations:

$$ (a, \varphi \psi) = (\delta_U(a), \varphi \otimes \psi), \quad (a, 1_A) = \epsilon_U(a), $$

$$ (ab, \varphi) = (a \otimes b, \delta_A(\varphi)), \quad (1_U, \varphi) = \epsilon_A(\varphi), $$

$$ (S_U(a), \varphi) = (a, S_A(\varphi)) $$

for any $a, b \in U$ and $\varphi, \psi \in A$. The pairing of Hopf algebras between $U_q(\mathfrak{gl}(N))$ and $U_q(N)$

$$(\cdot, \cdot) : U_q(\mathfrak{gl}(N)) \times A_N \to \mathbb{C}$$

is defined by

$$ (Q_k, u_{ij}(N)) = \delta_{i,j}^k q^{k/2}, $$

$$ (E_k, u_{ij}(N)) = \delta_{i,k}^j \delta_{k+1,j}, \quad (F_k, u_{ij}(N)) = \delta_{k+1,i} \delta_{k,j}, $$

$$ (Q_k, \hat{\det}_q(N)^m) = \hat{\det}_q(N)^m, \quad m \in \mathbb{Z}, $$

$$ (E_k, \hat{\det}_q(N)^m) = (F_k, \hat{\det}_q(N)^m) = 0, \quad m \in \mathbb{Z}, $$

see \cite{[21]} Proposition 1.3]. The pairing naturally induces the representation of the algebra $U_q(\mathfrak{gl}(N))$ from a right coaction of $U_q(N)$. Indeed, for any right coaction $\pi : V \to V \otimes A_N$ we have the representation $\hat{\pi} : U_q(\mathfrak{gl}(N)) \to \mathcal{B}(V)$ defined by $\hat{\pi}(x) := (\text{id} \otimes f_x)\pi$, where the linear functional $f_x : A_N \to \mathbb{C}$ is defined by $f_x(\cdot) = (x, \cdot)$. Thus, (highest) weights and (highest) weight vectors of right $U_q(N)$-comodules make sense.

In the rest of this section, we review the representation theory of $U_q(N)$. The references here are \cite{[21]} and \cite{[14]}. The algebra $B^-_N$ is generated by the $z_{ij}$, $1 \leq j < i \leq N$ and the $z^{-1}_{ii}$, $i = 1, \ldots, N$ with the same relations as in Equations (7.1) for $j < i$ and

$$ z_{ii} z^{-1}_{ii} = z^{-1}_{ii} z_{ii} = 1, \quad i = 1, \ldots, N. $$

The comultiplication $\delta_- : B^-_N \to B^-_N \otimes B^-_N$ is defined by

$$ \delta_-(z_{ij}) := \sum_{k=j}^i z_{ik} \otimes z_{kj}, \quad \delta_-(z^{-1}_{ii}) := z^{-1}_{ii} \otimes z^{-1}_{ii}. $$

The pair $B^-_N(N) := (B^-_N, \delta_-)$ has a Hopf algebra structure and it is called the quantum Borel subgroups of $U_q(N)$. Indeed, we have a surjective Hopf algebra homomorphism $\pi_{B^-} : A_N \to B^-$ defined by

$$ \pi_{B^-}(u_{ij}(N)) := z_{ij}, \quad i \geq j, \quad \pi_{B^-}(u_{ij}(N)) := z^{-1}_{ii} \otimes z^{-1}_{NN}, \quad i < j, $$

$$ \pi_{B^-}(\hat{\det}_q^{-1}(N)) := z^{-1}_{ii} \cdots z^{-1}_{NN}. $$
The left coaction of $B_q^{-}(N)$ on $U_q(N)$ is defined by $(\pi_B^{-} \otimes \text{id})\delta_N: A_N \rightarrow B_q^{-} \otimes A_N$. For any signature $\nu = (\nu_1 \geq \cdots \geq \nu_N)$, the element $z^\nu \in B_q^{-}$ is defined by $z^\nu := z_{11}^{\nu_1} \cdots z_{NN}^{\nu_N}$ and the left relative invariant space with respect to $z^\nu$ is defined by

$$\mathcal{H}_\nu := \{ x \in A_N : (\pi_B^{-} \otimes \text{id})\delta_N(x) = z^\nu \otimes x \}.$$ 

Then $\pi_\nu: \mathcal{H}_\nu \rightarrow \mathcal{H}_\nu \otimes A_N$ defined by $\pi_\nu(x) := \delta_N(x)$ for any $x \in \mathcal{H}_\nu$ is well defined. It is known that there exists an $U_q(N)$-invariant inner product on $\mathcal{H}_\nu$, that is, it satisfies

$$\langle \pi_\nu(x), \pi_\nu(y) \rangle = \langle x, y \rangle_{A_N}$$

for any $x, y \in \mathcal{H}_\nu$, and has the orthonormal basis consisting of weight vectors, see [21] Proposition 3.3. For such a basis $x_1, \ldots, x_d$ with $d := \dim(\mathcal{H}_\nu)$, we define $u_{ij}(\nu) \in A_N$, $i, j = 1, \ldots, d$ by

$$\pi_\nu(x_{ij}) = \sum_{i=1}^{d} x_i \otimes u_{ij}(\nu).$$

Then, $U_\nu := \sum_{i,j=1}^{d} e_{ij}(\nu) \otimes u_{ij}(\nu)$ is an irreducible unitary representation of $U_q(N)$, where $e_{ij}(\nu)$ are matrix units with respect to the basis $x_1, \ldots, x_d$. It is known that every irreducible unitary representation is unitarily equivalent to $U_\nu$ for some signature $\nu$, see [21] Theorem 2.12, Theorem 3.7. Furthermore, the above irreducible right $U_q(N)$-comodule $\mathcal{H}_\nu$ is also irreducible left $U_q(\mathfrak{gl}(N))$-module with highest weight $q^{\nu/2} = (q^{\nu_1/2}, \ldots, q^{\nu_N/2})$, see [13] Proposition 11.50. Therefore, we can explicitly obtain these weights of the basis $x_1, \ldots, x_d$ by the Gelfand–Tsetlin basis. We rename the basis $x_1, \ldots, x_d$ to $x_t$ with $t \in \Omega(\ast, \nu)$ in the manner such that the vector $x_t$ and the vector of the Gelfand–Tsetlin basis corresponding to $t \in \Omega(\ast, \nu)$ have the same weight. Accordingly, we rename $u_{ij}(\nu), e_{ij}(\nu), i, j = 1, \ldots, d$ to $u_{tu}, e_{tu}$ with $t, u \in \Omega(\ast, \nu)$, that is,

$$U_\nu = \sum_{t, u \in \Omega(\ast, \nu)} e_{tu} \otimes u_{tu}.$$ 

Then, by [21] Theorem 3.7, we can describe the density matrix $F_\nu$ of the irreducible representation $U_\nu$ as

$$[F_\nu]_{tu} = \delta_{t, u} q^{-\frac{1}{2}(N-1)|\nu|+2\sum_{n=1}^{N-1} |r(t_n)|},$$

where $t = (t_n)_{n=1}^{N}, u \in \Omega(\ast, \nu)$. Thus, the weight function of the weighted branching graph associated with the inductive system $U_q$ of quantum unitary groups is given by

$$w(\nu) = q^{N|u|-(N-1)|\nu|},$$

(7.2)

where $\mu = s(\nu) \in \text{Sign}_{N-1}$ and $\nu = r(\nu) \in \text{Sign}_N$. Furthermore, in the case of the weighted branching graph associated with the inductive system $U_q$ of quantum unitary groups $U_q(N)$, our $w$-central measures coincide with Gorin’s $q$-central measures by replacing $q$ with $q^2$. Thus, we call $w$-central measures, $w$-coherent systems and weighted dimensions $q$-central measure, $q$-coherent systems and quantum dimensions, respectively in this case.

8. Symmetric Polynomials

Here we review some definitions and some notations on symmetric polynomials that we need for the explanation of the boundary theorem on the Gelfand–Tsetlin graph $\mathcal{G}_T$ with the weight function $w$ defined in the previous section. See e.g. Macdonald’s book [13] for the general theory of symmetric polynomials, and see [10] Chapter 3, [12] for the details of useful techniques for the boundary theorem on the Gelfand-Tsetlin graph $\mathcal{G}_T$. In the paper, the Schur polynomials, which defined below, play a primary role, but the double Schur polynomials investigated by Molev in [18], [19] are also useful.

For any signature $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_N) \in \text{Sign}_N$, a (rational) Schur polynomial $s_\nu(x_1, \ldots, x_N)$ is defined by

$$s_\nu(x_1, \ldots, x_N) := \frac{\det \left[ x_i^{\nu_j+N-j} \right]_{i,j=1}^{N}}{\prod_{i<j} (x_i - x_j)}.$$
When \( \nu_N \) is non-negative, the signature \( \nu \) is called a partition and \( s_\nu(x_1, \ldots, x_N) \) becomes a polynomial, which is called the Schur polynomial associated with \( \nu \). It is easy to see the branching rule for Schur polynomials:

\[
   s_\nu(x_1, \ldots, x_N) = \sum_{\mu \in \text{Sign}_{N-1}; \mu \prec \nu} x_1^{\nu(1)} x_2^{\nu(2)} \cdots x_N^{\nu(N)-\nu(N-1)},
\]

From this formula (8.1), it follows that

\[
   s_\nu(x_1, \ldots, x_N) = \sum_{\nu(1) < \cdots < \nu(N)} x_1^{\nu(1)} x_2^{\nu(2)} \cdots x_N^{\nu(N)-\nu(N-1)},
\]

where \( \nu(N) = \nu \). Furthermore, we have

\[
   s_{\nu'}(x_1, \ldots, x_N) = x_1^{\nu_1} \cdots x_N^{\nu_N} s_\nu(x_1, \ldots, x_N)
\]

for \( \nu' = (\nu_1 + l, \ldots, \nu_N + l) \) and any \( l \in \mathbb{Z} \).

The next formula is obtained from Formula (7.2), (8.1) and allows us to compute the quantum dimension \( \dim_q(\nu) \) of the irreducible representation of \( U_q(N) \) corresponding to \( \nu \) as follows.

\[
   \dim_q(\nu) = s_\nu(q^{N-1}, q^{N-3}, \ldots, q^{N+1}) = q^{-\nu(1)} \prod_{1 \leq i < j \leq N} \frac{q^{2(\nu_i - i)} - q^{2(\nu_j - j)}}{q^{2i} - q^{2j}}.
\]

By Formula (7.2), our \( w \)-central measures on the Gelfand–Tsetlin graph coincide with Gorin’s \( q \)-central measures by replacing \( q \) with \( q^2 \). Therefore, Gorin’s theory of generating functions of probability measures on each \( \text{Sign}_N \) is still useful. Let \( P_N \) be a probability measure on \( \text{Sign}_N \). Following Gorin’s definition, see [10] Section 4], its generating function \( S(x_1, \ldots, x_N; P_N) \) is defined by

\[
   S(x_1, \ldots, x_N; P_N) := \sum_{\nu \in \text{Sign}_N} P_N(\nu) \frac{s_\nu(x_1, \ldots, x_N)}{s_\nu(1, q^{-2}, \ldots, q^{-2(N-1)})}.
\]

It is known that the function \( S(x_1, \ldots, x_N; P_N) \) converges uniformly on

\[
   T_N := \{(x_1, \ldots, x_N) \in \mathbb{C}^N : |x_i| < q^{-2(i-1)}\}
\]

and also that the generating function \( S(x_1, \ldots, x_N; P_N) \) on \( T_N \), see [10] Proposition 4.10, Proposition 4.11].

9. THE BOUNDARY THEOREM

In this section we explain of the boundary theorem of the weighted graph associated with the inductive system \( U_q \) of quantum unitary groups \( U_q(N) \). Let \( \mathcal{N} := \{ \theta = (\theta_i)_{i=1}^\infty \in \prod_{i=1}^\infty \mathbb{Z} : \theta_1 \leq \theta_2 \leq \cdots \} \) endowed with the topology of component-wise convergence.

**Theorem 9.1.** There exists a homeomorphism from the set of extremal quantized characters \( \text{ex}(\text{Ch}(U_q)) \) of the inductive system \( U_q \) of quantum unitary groups to the set \( \mathcal{N} \).

For any \( \nu \in \text{Sign}_N \) and \( K < N \), the probability measure \( P_{\nu}^{K} \) on \( \text{Sign}_K \) is defined to be

\[
   P_{\nu}^{K}(\mu) := \dim_q(\mu) \frac{\dim_q(\mu, \nu)}{\dim_q(\nu)}
\]

and \( P_{\nu}^{N} \) on \( \text{Sign}_N \) as the Dirac measure \( \delta_\nu \). Clearly, the finite sequence \( (P_{\nu}^{K})_{K=0}^{N} \) is a “finite” \( q \)-coherent system. Thus, we have

\[
   S(x; P_1^{\nu}) = S(x, q^{-2}, \ldots, q^{-2(N-1)}; P_1^{\nu}) = \frac{s_\nu(x, q^{-2}, \ldots, q^{-2(N-1)})}{s_\nu(1, q^{-2}, \ldots, q^{-2(N-1)})}
\]

and also

\[
   P_1^{\nu}(\nu_N) = P_1^{\nu-N}(0) = S(0; P_1^{\nu-N}) = \frac{s_{\nu-N}(0, q^{-2}, \ldots, q^{-2(N-1)})}{s_{\nu-N}(1, q^{-2}, \ldots, q^{-2(N-1)})} \geq \prod_{i=1}^\infty (1 - q^{2i}), \quad (9.1)
\]
where $\nu - \nu_N := (\nu_1 - \nu_N, \ldots, \nu_{N-1} - \nu_N, 0)$. See [10] Proposition 5.3 for details. This estimation of the probability $P^{\nu}_i(\nu_N)$ will be a key of the proof of the next lemma, which is only a reproduction of the main part of Gorin’s work [10].

**Lemma 9.1.** Let $(\nu(N))_{N=0}^{\infty}$ be a path on signatures. The following conditions are equivalent.

1. The path $(\nu(N))_N$ is stable, that is, $\nu(N)_{N-i+1}$ converges as $N \to \infty$ for any $i = 1, 2, \ldots$.
2. For any $\mu \in \text{Sign}$, the limit
   \[
   \lim_{N \to \infty} \frac{\dim_q(\mu, \nu(N))}{\dim_q(\nu(N))}
   \]
   converges as $N \to \infty$.

**Proof.** Suppose that Condition (1) holds. Let $\theta_i := \lim_{N \to \infty} \nu(N)_{N-i+1}$ for any $i = 1, 2, \ldots$. By [10] Proposition 5.7, for any $K \geq 1$ the family of the functions
   \[
   S(x_1, \ldots, x_K; P^{-i}_K) = \frac{s_{\nu(N)}(x_1, \ldots, x_K, q^{-2K}, \ldots, q^{-2(N-1)})}{s_{\nu(N)}(1, q^{-2}, \ldots, q^{-2(N-1)})}
   \]
   is a relatively compact subset of the set of continuous functions on $T_K$ with respect to the topology of uniform convergence. By the assumption, there exists an integer $l \in \mathbb{Z}$ such that $\nu(N)_N \geq l$ for any $N$. By Formula [5.2], we have
   \[
   S(x_1, \ldots, x_K; P^{-i}_K) = \frac{x_1^N \cdots x_{K}^N}{q^{-2} \cdots q^{-2(N-1)}} S(x_1, \ldots, x_K; P^{-i}_K),
   \]
   where $\nu(N) - l = (\nu(N)_{N-1}, \ldots, \nu(N)_N - l)$. Therefore, by [10] Proposition 5.9, any possible limit functions of the subsequences are independent of the choice of subsequences. Thus, $S(x_1, \ldots, x_K; P^{-i}_K)$ converges, that is, we obtain Condition (2).

Suppose that Condition (2) holds. It suffices to show that $\nu(N)_N$ is bounded below because $\nu(N)_{N-i+1}$ is non-increasing and $\nu(N)_{N-i} \geq \nu(N)_N$ for all $i = 1, 2, \ldots$. Note that the sequence $(P^{-i}_K)_{N=K}^{\infty}$ of probability measures on $\text{Sign}_K$ converges weakly. Let $P_K$ be its limit probability measure on $\text{Sign}_K$ for each $K = 1, 2, \ldots$. We choose $K > 0$ in such a way that
   \[
   P_i(\{m \in \text{Sign}_1 : -k < m < k\}) > 1 - \frac{1}{2} \prod_{i=1}^{\infty} (1 - q^{2i}).
   \]
   Since $\{m \in \text{Sign}_1 : -k < m < k\}$ is a finite set, we can choose $N_0$ in such a way that
   \[
   P_i(\{m \in \text{Sign}_1 : -k < m < k\}) > 1 - \prod_{i=1}^{\infty} (1 - q^{2i})
   \]
   for all $N > N_0$. If $\nu(N)_N < -k$, then we have
   \[
   P_i(\nu(N)_N) < \prod_{i=1}^{\infty} (1 - q^{2i}).
   \]
   This contradicts to Inequality [9.1], that is, $\nu(N)_N \geq -k$ for all $N \geq N_0$. \hfill \Box

We give a proof of Theorem 9.1.

**Proof.** By Proposition [5.2] it suffices to show that the two sets $\text{ex}(\text{Cent}(\mathbb{G}_T, w))$ and $\mathcal{N}$ are homeomorphic.

The desired continuous map from $\text{ex}(\text{Cent}(\mathbb{G}_T, w))$ to $\mathcal{N}$ is given by the following way: For a given $P \in \text{ex}(\text{Cent}(\mathbb{G}_T, w))$, by the ergodic method (Theorem [6.3]), there exists a path $(\nu(N))_N$ on the Gelfand–Tsetlin graph such that
   \[
   \frac{P(X_K = \mu)}{\dim_q(\mu)} = \lim_{N \to \infty} \frac{\dim_q(\mu, \nu(N))}{\dim_q(\nu(N))}, \quad \mu \in \text{Sign}_K, \quad K = 0, 1, \ldots.
   \]
   By Lemma [9.1] we have $\theta = (\theta_i)_{i=1}^{\infty} \in \mathcal{N}$ which is the limit of the sequence $(\nu(N))_N$, that is, $\theta_i = \lim_{N \to \infty} \nu(N)_{N-i+1}$ for every $i = 1, 2, \ldots$. Note that a path $(\nu(N))_N$ is not necessarily unique. However, it follows that $\theta$ is unique from [10] Proposition 5.14. Therefore, we obtain the continuous map sending $P \in \text{ex}(\text{Cent}(\mathbb{G}_T, w))$ to $\theta \in \mathcal{N}$ and denote this map by $\Phi: \text{ex}(\text{Cent}(\mathbb{G}_T, w)) \to \mathcal{N}$. From [10] Proposition 5.16, Proposition 5.9, it follows that this map is continuous and injective.

The inverse map from $\mathcal{N}$ to $\text{ex}(\text{Cent}(\mathbb{G}_T, w))$ is given as follows. For any $\theta = (\theta_i)_{i=1}^{\infty} \in \mathcal{N}$, let $\nu(N) := (\theta_N, \theta_{N-1}, \ldots, \theta_1) \in \text{Sign}_N$. By Lemma [9.1] the $q$-central measure $P^\theta$ defined by
   \[
   P^\theta(X_K = \mu) = \dim_q(\mu) \lim_{N \to \infty} \frac{\dim_q(\mu, \nu(N))}{\dim_q(\nu(N))}, \quad \mu \in \text{Sign}_K, \quad K = 1, 2, \ldots
   \]
is well defined. By Theorem 6.1 there exists a unique probability measure \( m \) on \( \text{ex}(\text{Cent}(\mathbb{C}T, w)) \) such that
\[
P^\theta = \int_{\text{ex}(\text{Cent}(\mathbb{C}T, w))} Q \, dm(Q) = \int_{N} P^{\theta'} \, d\Phi^*_\theta m(\theta').
\]
From [10, Proposition 5.17], it follows that the push-forward measure \( \Phi^*_\theta m \) coincides with \( \delta_\theta \). Since the map \( \Phi \) is injective, \( m \) must be a Dirac measure. Therefore \( P^\theta \) is an extremal point and we obtain the inverse map. The continuity of the inverse map also follows from [10, Proposition 5.16].

We emphasize that the proof here uses the ergodic method, explicitly. More precisely, the proof of the boundary theorem consists of two parts. One is the approximation for an extremal \( q \)-central measure. Another is to construct the correspondence between the extremal points \( \text{ex}(\text{Cent}(\mathbb{C}, w)) \) and the parameters \( N \) by this approximation. The later was based on the Gorin’s work. The former was proved here by the ergodic method given in Section 6. We believe that the approach based on the ergodic method is quite natural \( q \)-deformation of the approximation theorem for ordinary characters of groups due to Vershik–Kerov in [30, 31].

10. Quantized Characters of the Inductive System \( U_q \) of Quantum Unitary Groups

In this final section of the paper, we regard quantized characters of the inductive system \( U_q \) of quantum unitary groups \( U_q(N) \) as functions on the inductive limit of \( N \)-dimensional torus \( \mathbb{T}^N \). Remark that the work here can also be regarded as a representation-theoretic interpretation of Gorin’s generating function \( S \) of probability measures on the set of signatures. See Section 9 for the definition of the generating functions. We write \( G_N := U_q(N) = (A_N, \delta_N) \) throughout this section.

We first introduce the quantum group \( T^N := (C(\mathbb{T}^N), \delta_{T^N}) \) of the compact group \( \mathbb{T}^N \), see [20, Example 1.1.2]. The continuous functions \( t_1, \ldots, t_N, t_1^{-1}, \ldots, t_N^{-1} : \mathbb{T}^N \to \mathbb{C} \) are defined to be
\[
t_i(z_1, \ldots, z_N) := z_i, \quad t_i^{-1}(z_1, \ldots, z_N) := \overline{z_i}
\]
for any \( i = 1, \ldots, N \). Then the functions \( t_1, \ldots, t_N, t_1^{-1}, \ldots, t_N^{-1} \) generate \( C(\mathbb{T}^N) \) and we have \( \delta_{T^N}(t_i) = t_i \otimes t_i \).

Remark that \( T^N \) is a quantum subgroup of the quantum unitary group \( G_N \) by the surjective \(*\)-homomorphism \( \pi_{T^N} : A_N \to C(\mathbb{T}^N) \) defined by
\[
\pi_{T^N}(u_{ij}(N)) = \delta_{i,j} t_i, \quad \pi_{T^N}(\det_q^{-1}(N)) = t_1^{-1} \cdots t_N^{-1}.
\]
On the other hand, \( T^N \) is a quantum subgroup of the quantum group \( T^{N+1} \) by the surjective \(*\)-homomorphism \( \theta_{T^N} : C(\mathbb{T}^{N+1}) \to C(\mathbb{T}^N) \) defined by
\[
[\theta_{T^N} f](z_1, \ldots, z_N) := f(z_1, 1, z_N)
\]
for any \( f \in C(\mathbb{T}^{N+1}) \). Remark that \( \theta_{T^N} \circ \pi_{T^{N+1}} = \pi_{T^N} \circ \theta_N \) for any \( N \geq 1 \).

Let
\[
V_N := (U_\nu)_{\nu \in \hat{G}_N} \subset \bigoplus_{\nu \in \hat{G}_N} B(\mathcal{H}_\nu) \otimes A_N \cong W^*(G_N) \otimes A_N.
\]
See Section 7 for the definition of \( U_\nu = \sum_{t,u \in \Omega(\nu)} e_{tu} \otimes u_{tu} \). Recall that the matrix \( F_\nu \) is diagonalized for any \( \nu \in \hat{G}_N \). For any state \( \chi \) on \( W^*(G_N) \) we define \( \Phi_\chi^{(N)} \in C(\mathbb{T}^N) \) by \( \Phi_\chi^{(N)} := (\chi \otimes \pi_{T^N}) V_N \). Note that for any \( \nu \in \hat{G}_N = \text{Sign}_N \) we have
\[
\Phi_\nu(z_1, \ldots, z_N) := \Phi_{\chi_\nu}^{(N)}(z_1, \ldots, z_N) = \frac{s_\nu(z_1, q^{-2} z_2, \ldots, q^{-2(N-1)} z_N)}{s_\nu(1, q^2, \ldots, q^{-2(N-1)})}, \quad (10.1)
\]
see Section 5 for the definition of \( \chi_\nu \) and [21, Section 3.2] for this computation. Let \( P_N \) be a probability measure on \( \hat{G}_N = \text{Sign}_N \) and \( \chi := \sum_{\nu \in \hat{G}_N} P_N(\nu) \chi_\nu \). Then we have
\[
\Phi_{\chi}^{(N)}(z_1, \ldots, z_N) = \sum_{\nu \in \text{Sign}_N} P_N(\nu) \Phi_\nu(z_1, \ldots, z_N) = S(z_1, q^{-2} z_2, \ldots, q^{-2(N-1)} z_N; P_N).
\]
Thus, we obtain that for any quantized character \( \chi, \chi_n \in \text{Ch}(U_q), n = 1, 2, \ldots \)
- \( \Phi_{\chi_1}^{(N+1)}(z_1, \ldots, z_N, 1) = \Phi_{\chi_1}^{(N)}(z_1, \ldots, z_N) \) for any \( N \geq 1 \),
- if \( \chi_n \to \chi \) as \( n \to \infty \) in the weak* topology, then the functions \( \Phi_{\chi_n}^{(N)} \) converge to \( \Phi_{\chi}^{(N)} \) uniformly on \( \mathbb{T}^N \) as \( n \to \infty \) for any \( N \geq 1 \),
by Proposition 6.2. See Section 8 for the properties of generating function $S$. By the former property, we obtain the function $\Phi_\chi$ on $T := \{(z_1, \ldots, z_N, 1, \ldots) : z_i \in \mathbb{T}, i = 1, \ldots, N, N \geq 1\} \cong \mathbb{T}^N$ defined by $\Phi_\chi(z_1, \ldots, z_N, 1, \ldots) := \Phi_\chi^{(N)}(z_1, \ldots, z_N)$. Furthermore, by these properties and the ergodic method (Corollary 6.1), we have the following proposition:

**Proposition 10.1.** For any extremal quantized character $\chi \in \text{ex}(\text{Ch}(U_q))$, there exists a sequence $\nu(1) \prec \nu(2) \prec \cdots$ of signatures such that

$$
\Phi_\chi(z_1, \ldots, z_N, 1, \ldots) = \lim_{L \to \infty, N \leq L} \Phi_\chi^{(L)}(z_1, \ldots, z_N, 1, \ldots) = \lim_{L \to \infty, N \leq L} \frac{s_{\nu(L)}(z_1, q^{-2}z_2, \ldots, q^{-2(N-1)}z_N, q^{-2N}, \ldots, q^{-2(L-1)})}{s_{\nu(L)}(1, q^{-2}, \ldots, q^{-2(L-1)})}
$$

for any $(z_1, \ldots, z_N, 1, \ldots) \in T$ and any $N \geq 1$.

Let $\theta = (\theta_i)_i \in N$ be the corresponding parameter. By [12] Theorem 6.5, we have the following formula.

**Corollary 10.1.** For any $(z_1, \ldots, z_N, 1, \ldots) \in T$

$$
\Phi_\chi(z_1, \ldots, z_N, 1, \ldots) = \frac{(-1)^{\binom{N}{2}} q^{-4}\binom{N}{2}}{\Delta(z_1, q^{-2}z_2, \ldots, q^{-2(N-1)}z_N) \prod_{i=1}^{N}(z_i q^{2(N-1)}; q^2)\infty}
$$

$$
\times \det \left[ D_{i,q^{-2}}^{j-1} \right]_{i,j=1}^{N} \prod_{i=1}^{N} F_\theta(q^{2(N-i)} z_i) (q^{2(N-i)} z_i; q^2)\infty,
$$

where

$$
\Delta(x_1, \ldots, x_N) := \det [x_i^{N-j} \prod_{i,j=1}^{N} (x_i - x_j),
$$

$$(a; q^2)\infty := \prod_{i \geq 0} (1 - a q^{2i}) \text{ for any } a \in \mathbb{C},
$$

$$
[D_{i,q^{-2}} F](x_1, \ldots, x_N) := \frac{F(x_1, \ldots, x_{i-1}, q^{-2}x_i, x_{i+1}, \ldots, x_N) - F(x_1, \ldots, x_N)}{q^{-2} - 1}
$$

for any function $F(x_1, \ldots, x_N)$ and

$$
F_\theta(x) := \sum_{i=0}^{\infty} \frac{1 - q^{2(i+1)}}{1 - q^{2(i+1)}x} \sum_{k=1}^{\infty} \frac{x^{\theta_k + k - 1}}{\prod_{j \neq k} (1 - q^{-2(\theta_k + k - \theta_j - j))}}.
$$

In this section we have considered, in a more direct fashion, the generating functions $S$ of probability measures on the set of signatures defined by Gorin in [10] in relation to quantized characters of the inductive system $U_q$ of quantum unitary groups. We hope that the study here is useful for the representation theory of the $\sigma$-$C^*$-quantum group $U_q(\infty)$ introduced in [16].

**Acknowledgment**

The author gratefully acknowledge the passionate guidance and continuous encouragement from his supervisor, Professor Yoshimichi Ueda. The author thanks Professor Yuki Arano for his useful comments on quantum subgroups and final section of the present paper. Particularly, he informed of the reference [17]. The author also thanks Professor Reiji Tomatsu and Professor Makoto Yamashita for their comments on this paper. In particular, Professor Reiji Tomatsu pointed out an inaccuracy in our previous proof of Lemma 9.1 and informed the author of the reference [27].

**References**

[1] A. Borodin, G. Olshanski, *Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes*, Ann. of Math. (2) 161 (2005), no. 3, 1319–1422.

[2] Robert P. Boyer, *Characters and factor representations of the infinite-dimensional classical groups*, J. Operator Theory 28 (1992), no. 2, 281–307.

[3] O. Bratteli, *Inductive limits of finite dimensional $C^*$-algebras*, Trans. Amer. Math. Soc. 171 (1972), 195–234.

[4] O. Bratteli, D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2. Equilibrium states. Models in quantum statistical mechanics. Second edition*, Texts and Monographs in Physics, Springer-Verlag, Berlin, Heidelberg, 1997.

[5] K. R. Davidson, *$C^*$-Algebras by Example*, Fields Institute Monographs, 6, Amer. Math. Soc., 1996.

[6] R. Durrett, *Probability: theory and examples*, Fourth edition, Cambridge Series in Statistical and Probabilistic Mathematics, 31. Cambridge University Press, 2010.
