Algorithmic characterization results for the Kerr-NUT-(A)dS space-time. II. KIDs for the Kerr-(A)(de Sitter) family

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Abstract
We characterize Cauchy data sets leading to vacuum space-times with vanishing Mars-Simon tensor. This approach provides an algorithmic procedure to check whether a given initial data set $(\Sigma, h_{ij}, K_{ij})$ evolves into a space-time which is locally isometric to a member of the Kerr-(A)(dS) family.

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1 Introduction

In part I \cite{14} of this work we have provided an algorithm to check whether a given vacuum space-time with cosmological constant $\Lambda \in \mathbb{R}$ belongs to the Kerr-NUT-(A)dS family, or, more generally, admits a (possibly complex) Killing vector field such that the associated Mars-Simon tensor (MST) vanishes. An issue, which complements these kind of problems, is to derive analogous results from the point of view of an initial value problem. This will be the main object of this article.

It is generally agreed that a main source of understanding of dynamical black hole space-times will come from numerical simulations. These make often use of a $3+1$-decomposition of space-time, whence it becomes relevant to gain a better understanding of the $3+1$ features of the Kerr family (cf. \cite{6} and the references given therein). A $3+1$-decomposition hides the symmetries of a space-time unless it is adapted to them. Invariant characterization results are therefore of particular relevance for such kind of problems.

One would like to know whether a given Cauchy data set generates a development which is isometric to a portion of a Kerr or a Kerr-NUT-(A)dS-space-time. Such a result has been obtained in \cite{7} for the Kerr space-time based on a $3+1$-splitting of the MST and its space-time characterization as given in \cite{9,10}. However, in view of an algorithmic characterization, it suffers from a similar drawback as corresponding space-time characterization results obtained in \cite{9,10,12}, namely the need to solve PDEs before the results can be applied: Given vacuum Cauchy data $(\Sigma, h_{ij}, K_{ij})$, it is a non-trivial issue to check whether there exists a scalar function $\sigma$ and a vector field $Y^i$ complementing them to Killing initial data which generate a space-time with a Killing vector field.

As for the space-time approach we shall employ the restrictions, coming from the requirement that the emerging space-time admits a Killing vector field w.r.t. which the MST vanishes, to show that there is (up to rescaling) at most one candidate tuple $(\sigma, Y^i)$. That yields an algorithm to check whether a given set of Cauchy data $(\Sigma, h_{ij}, K_{ij})$, solution to the vacuum constraint equations, emerges into a space-time which admits a KVF w.r.t. which the MST vanishes. Moreover, it can be extended to provide an algorithm to check whether the triple $(\Sigma, h_{ij}, K_{ij})$ constitutes Kerr-NUT-(A)dS data, by which we mean that the Cauchy data evolve into a vacuum space-time which is locally isometric to a member of the Kerr-NUT-(A)dS family. The procedure will be algorithmic in the sense that, given $(\Sigma, h_{ij}, K_{ij})$, only differentiation and computation of roots is needed without any need to solve differential equations. So far such an algorithmic test has been given for Schwarzschild data \cite{6}, for Kerr-data \cite{4}, and for Petrov type D-data \cite{5}.

The paper is organized as follows: In Section 2 we will review definition and some properties of the MST, a space-time characterization result for the Kerr-
(A)dS family based on this tensor, as well as the notion of Killing initial data sets (KIDs). The definition of the MST comes along with a scalar function $Q$, which can be defined in several different ways. In Section 3 we analyze the vanishing of the MST on Cauchy surfaces for different choices of $Q$ and investigate the equivalence of these choices. In Section 4 we construct candidates for solving the KID equation and characterize conditions under which these candidates are in fact KIDs. This way we are led to an algorithmic characterization of Cauchy data which generate $\Lambda$-vacuum space-times which admit a Killing vector field whose associated MST vanishes, cf. Theorem 4.5. A somewhat shortened version of Theorem 4.5 reads

**Theorem 1.1** Consider Cauchy data $(\Sigma, h_{ij}, K_{ij})$ which solve the vacuum constraint equations and satisfy

$$\text{tr}(\mathcal{E} \cdot \mathcal{E}) \neq 0, \quad \text{tr}(\mathcal{E} \cdot \mathcal{E}) - \frac{2}{3} \Lambda^2 \neq 0, \quad \text{tr}(\mathcal{E} \cdot \mathcal{E}) - \frac{1}{6} \Lambda^2 \neq 0, \quad \text{tr}(\mathcal{E} \cdot \mathcal{E}) - \frac{8}{3} \Lambda^2 \neq 0,$$

where

$$\mathcal{E}_{ij} := \mathring{\mathcal{R}}_{ij} + KK_{ij} - K_{ik}K_{jk} - \frac{2}{3} \Lambda h_{ij} - i \epsilon_{ikl} \mathcal{D}_k K_{lj},$$

and where $\mathring{\mathcal{R}}_{ij}$ and $\mathcal{D}$ denote the Ricci tensor and the Levi-Civita covariant derivative of $h_{ij}$.

Then the emerging $\Lambda$-vacuum space-time admits a non-trivial (possibly complex) KVF such that the associated MST vanishes (at least in some neighborhood of $\Sigma$) if and only if two certain scalars ((4.62) and (4.76), cf. Section 4 for the details) which depend on $E$, $D E$, $h$ and $K$ vanish.

Finally, this result is combined in Section 5 with well-known space-time characterizations of Kerr-(NUT-)(A)dS to end up with an algorithmic characterization of these space-times in terms of their Cauchy data, Theorem 5.2.

## 2 Preliminaries

In this section we fix the notation and recall some results which will be relevant for the subsequent analysis. We will be rather brief here, for more details we refer the reader to part I [14].

### 2.1 Mars-Simon tensor and the function $Q$

Let $(\mathcal{M}, g)$ be a smooth 3 + 1-dimensional space-time which admits a Killing vector field (KVF) $X$. Let us denote by $C_{\mu\nu\sigma\rho}$ its conformal Weyl tensor, while $F_{\mu\nu} := \nabla_\mu X_\nu = \nabla_\nu X_\mu$ denotes the Killing form. We define the Mars-Simon tensor (MST) (compare [8] where somewhat different conventions are used) as

$$S_{\mu\nu\sigma\rho} := \mathcal{C}_{\mu\nu\sigma\rho} + Q Q_{\mu\nu\sigma\rho},$$

where

$$Q_{\mu\nu\sigma\rho} := - F_{\mu\nu} F_{\sigma\rho} + \frac{1}{3} F^2 I_{\mu\nu\sigma\rho},$$

$$I_{\mu\nu\sigma\rho} := \frac{1}{4} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma} + i \epsilon_{\mu\nu\sigma\rho}),$$

$$F^2 := F_{\mu\nu} F^{\mu\nu},$$

$$\mathcal{C}_{\mu\nu\sigma\rho} := 0.$$
and where
\[ C_{\mu\nu\rho} := C_{\mu\nu\rho} + iC^*_{\mu\nu\rho} , \quad (2.5) \]
\[ F_{\mu\nu} := F_{\mu\nu} + iF^*_{\mu\nu} , \quad (2.6) \]
denote the self-dual Weyl tensor and the self dual Killing form, respectively. At this stage \( Q : \mathcal{M} \to \mathbb{C} \) is an arbitrary function on \( \mathcal{M} \). The MST is a Weyl field, i.e. it has all the algebraic symmetries of the Weyl tensor.

Let us address the issue how the function \( Q \) is to be chosen. Denote by
\[ \chi_{\mu} := 2X^\alpha F_{\alpha\mu} \quad (2.7) \]
the Ernst 1-form. In a \( \Lambda \)-vacuum space-time it is well-known to be closed. Thus, at least locally, there exists a scalar field \( \chi \), the Ernst potential, such that \( \chi_{\mu} = \nabla_{\mu} \chi \). Note that \( \chi \) is only defined up to some additive complex “\( \chi \)-constant”. We further set
\[ C^2 := C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} . \quad (2.8) \]
Supposing that the corresponding denominators are non-zero we have the following natural\(^1\) definitions for the function \( Q \) \cite{11, 14}
\[ Q_0 := \frac{3}{2} F^{-4} F_{\mu\nu} F^{\rho\sigma} C_{\mu\nu\rho\sigma} , \quad (2.9) \]
\[ Q_{\text{ev}} := \frac{3F^2 + 4\Lambda \chi \pm 3\sqrt{F^2(F^2 + 4\Lambda \chi)}}{\chi F^2} , \quad (2.10) \]
\[ Q_{\text{F}} := \pm \sqrt{\frac{3}{2}} F^{-2} \sqrt{C^2} . \quad (2.11) \]
\[ Q_{\text{C}} := \kappa (C^2)^{5/6} \left( \pm \sqrt{C^2} - \sqrt{\frac{32}{3} \Lambda} \right)^{-2} , \quad \kappa \in \mathbb{C} \setminus \{0\} . \quad (2.12) \]

In our setting the square roots will be taken only of nowhere vanishing functions, so that one can prescribe the choice of square root at one point and extend it by continuity to the whole manifold, and since no branch point is ever met the root will be smooth everywhere. The definitions (2.10) and (2.12) both involve a complex constant which is arbitrary at this stage. The same is true for the choice of \( \pm \).

In \cite{14} we have established the following

**Proposition 2.1** Assume that the MST associated to some KVF \( X \) vanishes for some function \( Q \), and that the inequalities
\[ C^2 \neq 0 , \quad C^2 \neq \frac{32}{3} \Lambda^2 \neq 0 , \quad (2.13) \]
hold. Then (2.9), (2.11) and (2.12) are regular everywhere, and there exists a constant \( \kappa \in \mathbb{C} \setminus \{0\} \) and a choice of \( \pm \) such that
\[ Q = Q_0 = Q_{\text{F}} = Q_{\text{C}} . \quad (2.14) \]

\(^1\) “Natural” in the sense that each of these expressions is obtained by requiring a certain component of the MST to vanish, whence the function \( Q \) necessarily needs to coincide with each of these definitions whenever the MST vanishes, cf. Proposition 2.1 below.
Assume that, in addition,
\[ C^2 - \frac{8}{3} \Lambda^2 \neq 0, \quad C^2 - \frac{128}{3} \Lambda^2 \neq 0 \quad (2.15) \]
hold. Then (2.10) is regular, as well, and there exists an Ernst potential \( \chi \), i.e. a choice of the \( \chi \)-constant, such that
\[ Q = Q_0 = Q_{ev} = Q_\mathcal{F} = Q_C. \quad (2.16) \]

**Remark 2.2** The second condition in (2.13) and the conditions in (2.15) may be replaced by
\[ \pm \sqrt{C^2 - \frac{32}{3} \Lambda} \neq 0, \quad \pm \sqrt{C^2 + \frac{8}{3} \Lambda} \neq 0, \quad \pm \sqrt{C^2 + \frac{128}{3} \Lambda} \neq 0, \quad (2.17) \]
respectively, and merely need to hold for one sign, depending on the sign which one needs to take in (2.10)-(2.12) for the MST to vanish.

Proposition 2.1 shows that as long as one is interested in space-times with vanishing MST the definitions (2.9)-(2.12) of the functions \( Q \) are equally good in the sense that they are all necessary for a space-time to admit a KVF for which the associated MST vanishes.

In this article we want to analyze the vanishing of the MST in terms of an initial value problem. To derive sufficient conditions which ensure the existence of a KVF w.r.t. which the MST vanishes one needs evolution equations for the MST. More precisely, one would like to have homogeneous equations at hand which ensure that, given an appropriate set of zero initial data, the zero-solution is the only one. While it does not seem to be possible to derive such equations for \( Q_0, Q_\mathcal{F}, \) and \( Q_C \), it can be done \([8, 11]\) for \( Q = Q_{ev}: \)

**Proposition 2.3** Consider a smooth \( 3 + 1 \)-dimensional \( \Lambda \)-vacuum space-time which admits a KVF and which satisfies (cf. Remark 5.1)
\[ C^2 \neq 0, \quad C^2 - \frac{32}{3} \Lambda^2 \neq 0, \quad C^2 - \frac{8}{3} \Lambda^2 \neq 0, \quad C^2 - \frac{128}{3} \Lambda^2 \neq 0. \quad (2.18) \]
Then the MST with \( Q = Q_{ev} \) satisfies a regular linear homogeneous symmetric hyperbolic system of evolution equations,
\[ \nabla_\beta S_{\mu \alpha}^{(ev)\beta} = -Q_{ev} \left( F_{\alpha \beta} \delta^{\gamma} \delta_\delta - \frac{2}{3} F^{\gamma \delta} T_{\alpha \beta \mu \nu} \right) X^\lambda S_{\gamma \delta \lambda}^{(ev)\beta} \]
\[ -4\Lambda \frac{5}{Q_{ev}} Q_{ev} F^2 + 4\Lambda \frac{Q_{\mu \alpha \beta}}{Q_{ev}} F^{-4} F^{\gamma \delta} X^\lambda S_{\gamma \delta \lambda}^{(ev)\beta}. \quad (2.19) \]

Here and henceforth we denote the MST corresponding to the choice \( Q = Q_{ev} \) by \( S_{\alpha \beta \mu \nu}^{(ev)} \), the one corresponding to \( Q = Q_C \) by \( S_{\alpha \beta \mu \nu}^{(C)} \), etc. Because of Proposition 2.1 this distinction is less essential in the setting where the MST vanishes, whence we will occasionally omit the superscript.

### 2.2 Space-time characterization results

Our aim is to analyze the implications of the space-time characterization results for the Kerr-NUT-A(dS) metrics (in particular for vanishing NUT-charge)
obtained by Mars and Senovilla [12] on a Cauchy problem, and to derive an analog of [14, Theorem 5.3] for Cauchy data. In paper I [14] we have reviewed their results. Here, let us just recall a characterization result for the Kerr-A(dS) metric [14] which is a reformulation of a special case of [12, Theorem 1], and which is the most important one for our purposes.

To this end we define 4 real-valued functions $b_1$, $b_2$, $c$ and $k$ [12] (we assume that $Q F^2 - 4 \Lambda$ is nowhere vanishing),

\[ b_2 - i b_1 := - \frac{36Q(F^2)^{5/2}}{(Q F^2 - 4 \Lambda)^3}, \]  
\[ c := - |X|^2 - \text{Re} \left( \frac{6F^2(Q F^2 + 2 \Lambda)}{(Q F^2 - 4 \Lambda)^2} \right), \]  
\[ k := \left| \frac{36F^2}{(Q F^2 - 4 \Lambda)^2} \right| \nabla_\mu Z \nabla^\mu Z - b_2 Z + c Z^2 + \frac{\Lambda}{3} Z^4, \]

where

\[ Z := 6 \text{Re} \left( \sqrt{F^2 Q F^2 - 4 \Lambda} \right). \]

Moreover, we set (because of the assumptions (2.27)-(2.29) below the square roots will be real)

\[ \begin{align*}
\text{for } \Lambda = 0: & \quad \zeta_1 := \sqrt{\frac{k}{c}}, \\
\text{for } \Lambda > 0: & \quad \zeta_1 := \sqrt{- \frac{3c}{\Lambda} + \sqrt{\left( \frac{3c}{\Lambda} \right)^2 + \frac{3}{\Lambda} k}}, \\
\text{for } \Lambda < 0: & \quad \zeta_1 := \sqrt{- \frac{3c}{\Lambda} - \sqrt{\left( \frac{3c}{\Lambda} \right)^2 + \frac{3}{\Lambda} k}}.
\end{align*} \]

**Theorem 2.4** (cf. [12]) Let $(\mathcal{M}, g)$ be a smooth $3 + 1$-dimensional $\Lambda$-vacuum space-time which admits a KVF $X$ such that the associated MST vanishes for some function $Q$. Assume that $Q F^2$ and $Q F^2 - 4 \Lambda$ are not identically zero, and that $\text{Im} \left( \sqrt{\frac{F^2}{Q F^2 - 4 \Lambda}} \right)$ has non-zero gradient somewhere. Then the functions $b_1, b_2, c$, and $k$ are constant. Assume further that $b_2 = 0$ and that

\[ \begin{align*}
\text{for } \Lambda = 0: & \quad c > 0 \quad (\implies k \geq 0), \\
\text{for } \Lambda > 0: & \quad \begin{cases} c > 0 \quad (\implies k \geq 0) & \text{or} \\ c \leq 0 \text{ and } k > 0, & \end{cases} \\
\text{for } \Lambda < 0: & \quad c > 0 \text{ and } k < \frac{3}{|\Lambda|} \frac{c^2}{4} \quad (\implies k \geq 0). 
\end{align*} \]

Then $(\mathcal{M}, g)$ is locally isometric to a Kerr-(A)dS space-time with parameters $(\Lambda, m, a)$, where

\[ \begin{align*}
m = \frac{b_1}{2(\frac{\Lambda \zeta_1^2}{c} + c)^{3/2}}, & \quad a = \frac{\zeta_1}{\left( \frac{\Lambda \zeta_1^2}{c} + c \right)^{1/2}}.
\end{align*} \]

The Schwarzschild-(A)dS limit is obtained for $a = 0$, equivalently $k = 0$. 
2.3 Killing initial data sets (KIDs)

In order to define the MST the emerging space-time needs to admit a KVF. This is ensured by so-called Killing initial data sets (KIDs).

**Theorem 2.5 ([2, 3, 13], cf. [1])** Consider the tuple \((\Sigma, h_{ij}, K_{ij}, \sigma, Y^i)\), where \((\Sigma, h_{ij})\) is a Riemannian 3-manifold, and where \(K_{ij}\), \(\sigma\) and \(Y^i\) are a symmetric 2-tensor, a scalar function and a vector field on \(\Sigma\), respectively. Then there exists an (up to isometries) unique maximal globally hyperbolic space-time \((\mathcal{M}, g_{\mu\nu})\) such that

(i) \((\mathcal{M}, g_{\mu\nu})\) solves Einstein’s vacuum field equations \(R_{\mu\nu} = \Lambda g_{\mu\nu}\),

(ii) \((\mathcal{M}, g_{\mu\nu})\) contains \(\Sigma\) as a Cauchy surface with \(h_{ij}\) and \(K_{ij}\) being its first and second fundamental form, and

(iii) \((\mathcal{M}, g_{\mu\nu})\) admits a KVF \(X^\mu\) with \(\{X_t, X_i\}|_{\Sigma} = (\sigma, Y_i)\),

if and only if the vacuum constraint equations

\[
\begin{align*}
\hat{R} - |K|^2 + K^2 - 2\Lambda &= 0, \quad (2.31) \\
\mathcal{D}_j K^j_i - \mathcal{D}_i K &= 0, \quad (2.32)
\end{align*}
\]

and the KID equations

\[
\begin{align*}
\mathcal{D}(Y_j) + K_{ij}\sigma &= 0, \quad (2.33) \\
\mathcal{D}\mathcal{D}\sigma + \mathcal{L} Y K_{ij} - (\hat{R}_{ij} + KK_{ij} - 2K^k K_{jk} - \Lambda h_{ij})\sigma &= 0, \quad (2.34)
\end{align*}
\]

are fulfilled.

We use \(\hat{\cdot}\) to denote objects associated to the Riemannian metric \(h_{ij}\). \(\mathcal{D}\) denotes the covariant derivative associated to \(h_{ij}\).

3 A vanishing MST arising from a space-like Cauchy problem

Suppose we have been given KIDs, i.e. a tuple \((\Sigma, h_{ij}, K_{ij}, \sigma, Y^i)\) which solves \((2.31)-(2.34)\). We want to extract conditions which characterize the vanishing of the MST on the initial surface \(\Sigma\).

3.1 Weyl tensor and Killing form in adapted coordinates

To make computations as simple as possible let us impose certain gauge conditions: We assume that the initial surface \(\Sigma\) has (locally) been given in adapted coordinates \((t, x^i)\) with \(\Sigma = \{t = 0\}\), and we further assume a gauge where

\[
\begin{align*}
\bar{g}_{tt} = -1, \quad \bar{g}_{ti} = 0, \quad \partial_t \bar{g}_{\mu\nu} = 0, \quad (3.1)
\end{align*}
\]

where here and henceforth an overbar means restriction of the corresponding space-time object to the initial surface \(\Sigma\). In such a gauge the second fundamental form reads

\[
K_{ij} = \frac{1}{2} \partial_t g_{ij}, \quad (3.2)
\]
while the traces of the connection coefficients on the initial surface become
\[
\Gamma^t_{tt} = \Gamma^t_{ti} = \Gamma^i_{ti} = 0, \quad \Gamma^i_{ij} = K^i_j, \quad \Gamma^k_{ij} = \hat{\Gamma}^k_{ij}.
\] (3.3)
The electric and magnetic part of the conformal Weyl tensor are given by
\[
E_{ij} := \nabla_{ti} = \hat{R}_{ij} + KK_{ij} - K^k_i K^k_j - \frac{2}{3} \Lambda h_{ij},
\] (3.4)
\[
B_{ij} := \nabla^t_{ij} = -\epsilon^{kl}_{ij} K_k^l.
\] (3.5)
(Note that the vacuum constraints imply \(\epsilon^{[\ell}_{ij} K_{[kl]} = 0\).) It is useful to introduce the following fields on \(\Sigma\),
\[
\mathcal{P}_i := \nabla^t_{ti} = \partial_i \sigma + K^j_i Y^j + \frac{i}{2} \epsilon^{jk} \partial_j Y^k,
\] (3.6)
\[
\mathcal{E}_{ij} := \nabla^t_{ij} = E_{ij} + iB_{ij}.
\] (3.7)
Moreover, it is convenient to introduce a notation for the curls of the KVF \(Y\) and the co-vector field \(P\),
\[
Z' := \epsilon^{jk} \partial_j Y^k,
\] (3.8)
\[
Q_i := \epsilon^{jk} \partial_j P_k.
\] (3.9)
One straightforwardly checks that it follows from the vacuum constraint equations (2.31)-(2.32) that \(\mathcal{E}_{ij}\) satisfies the following relations,
\[
h_{ij} \mathcal{E}_{ij} = 0, \quad \partial_j \mathcal{E}^j = i\epsilon^{jk} K_j^l \mathcal{E}_{kl}.
\] (3.10)
For the sake of completeness let us collect some more useful formulas which are obtained by employing the self-duality of the objects under consideration (to compute the transverse derivative of the self-dual Weyl tensor the contracted second Bianchi identity has been used),
\[
\nabla^t_{ij} = -i\epsilon^{ijkl} P^k, \quad \nabla^t_{ij} = -i\epsilon^{ijkl} P^k, \quad \nabla^t_{ij} = -i\epsilon^{ijkl} P^k, \quad \nabla^t_{ij} = -i\epsilon^{ijkl} P^k,
\] (3.11)
\[
\nabla^t_{ijkl} = -i\epsilon^{ijkl} P^k, \quad \nabla^t_{ijkl} = -i\epsilon^{ijkl} P^k, \quad \nabla^t_{ijkl} = -i\epsilon^{ijkl} P^k, \quad \nabla^t_{ijkl} = -i\epsilon^{ijkl} P^k,
\] (3.12)
where \(.)\) denotes the trace-free part of the corresponding 2-tensor w.r.t. \(h_{ij}\).

### 3.2 Vanishing of the MST on \(\Sigma\)

Let us compute the trace of the MST on the initial hypersurface \(\Sigma\). First of all we observe that
\[
\mathcal{F}^2 |_{\Sigma} = 4F_{ti} \mathcal{F}^t_{ij} = -4P^2.
\] (3.15)
We further obtain
\[
\mathcal{Q}_{titj} |_{\Sigma} = -\nabla^t_{ti} \mathcal{F}^t_{ij} + \frac{1}{3} \mathcal{F}^t \nabla_{titj} = -(\mathcal{P}_i \mathcal{P}_j)^{\gamma}.
\] (3.16)
We conclude that
\[
\mathcal{S}_{titj} |_{\Sigma} = \mathcal{T}_{titj} + \mathcal{Q} \nabla_{titj} = \mathcal{E}_{ij} - q(\mathcal{P}_i \mathcal{P}_j)^{\gamma}.
\] (3.17)
where we have set
\[ q := Q|_{\Sigma}. \]  
(3.18)

Note that \( S_{ij}|_{\Sigma} \) encompasses all independent components of the MST. Consequently, the MST vanishes on \( \Sigma \) for some function \( q \) if and only if
\[ E_{ij} = q(P_i P_j'). \]  
(3.19)

It follows immediately from the definition of the function \( Q_0 \) [14] that whenever the MST vanishes (in space-time, on some hypersurface, or merely at one point) the function \( q \) needs to coincide with \( Q_0 \) there, supposing that \( F^2 \neq 0 \). In particular, \( q = Q_0|_{\Sigma} \) if the MST restricted to \( \Sigma \) vanishes for some function \( q \),
\[ q = Q_0|_{\Sigma} = \frac{3}{2} F^{-4} F^{\mu\nu} F^{\rho\sigma} C_{\mu\nu\rho\sigma} = 24 F^{-4} F^{ij} F^{kl} C_{ijkl} = \frac{3}{2} P^{-4} P^i P^j E_{ij}. \]  
(3.20)

We are thus led to the following result:

**Lemma 3.1** Suppose that \( P^2 \neq 0 \). Then a Killing initial data set \((\Sigma, h_{ij}, K_{ij}, \sigma, Y^i)\) yields a \( \Lambda \)-vacuum space-time with a KVF such that the associated MST vanishes on \( \Sigma \) for some function \( Q \) if and only if
\[ E_{ij} = \frac{3}{2} P^{-4} P^k P^l E_{kl}(P_i P_j), \]  
(3.21)

and in that case \( q = Q_0|_{\Sigma} \).

**Remark 3.2** Equation (3.21) is of the same form as the corresponding equation on \( \mathcal{I} \) derived in [11]. Note, however, that the tensors \( E_{ij} \) and \( P_i \) obey different equations as the corresponding ones in [11], whence \( P^k P^l E_{kl} \) behaves differently, cf. the considerations below.

### 3.3 Equivalence of the choices of \( Q \) on \( \Sigma \)

In the previous section we made the choice \( Q = Q_0 \). Here, analog to the proceeding in [14], our goal is to solve, on \( \Sigma \), the equation \( S_{\alpha\beta\mu\nu} = 0 \) for the KVF, and to do that it will be key to choose \( Q = Q_C \). Moreover, in order to employ the evolution equations for the MST the choice \( Q = Q_{\text{ev}} \) is essential.

Compared to [14], though, we consider a space-like hypersurface rather than the full space-time, whence Proposition 2.1 does not apply. A priori it might e.g. happen that \( S_{\alpha\beta\mu\nu}^{(\text{ev})} \) does not vanish on some space-like hypersurface \( \Sigma \) while \( S_{\alpha\beta\mu\nu}^{(0)} \) or \( S_{\alpha\beta\mu\nu}^{(C)} \) do. (On the other hand, if \( S_{\alpha\beta\mu\nu}^{(\text{ev})} \) vanishes on \( \Sigma \) it follows from the evolution equations that it also vanishes off \( \Sigma \), whence it follows from Proposition 2.1 that \( S_{\alpha\beta\mu\nu}^{(0)} \) and \( S_{\alpha\beta\mu\nu}^{(C)} \) needs to vanish, as well.) We therefore aim to make sure that whenever there exists a function \( Q \) for which the MST vanishes on a space-like hypersurface \( \Sigma \), then \( Q = Q_0 = Q_{\text{ev}} = Q_C \) holds there. Along the way we will obtain some relations which will be crucial later on.
First of all we employ the KID equations (2.33)-(2.34) to compute
\[ \mathcal{D}_i \mathcal{P}_j \equiv \mathcal{D}_i \mathcal{P}_j \sigma + Y^k \mathcal{D}_i K_{jk} + K_j^k \mathcal{D}_i Y_{kj} + K_k^j \mathcal{D}_i Y_{kl} + \frac{i}{2} \mathcal{D}_i Z_j \] (2.32)
\[
= \sigma \left( \mathcal{E}_{ij} - \frac{\Lambda}{3} h_{ij} \right) + i \varepsilon_{ijkl} B_{ij} k^l + \frac{1}{2} \varepsilon_{ijkl} K_i^k Z^l + i \varepsilon_{ijk} \tilde{K}_i^k \mathcal{D}_i \sigma \\
+ \frac{1}{2} i \tilde{K}_{ij} k^l Y^l - \frac{1}{2} (\tilde{K}_{ij} k^l Y^l) \] (2.33)
\[
= \sigma \left( \mathcal{E}_{ij} - \frac{\Lambda}{3} h_{ij} \right) + i \varepsilon_{ijkl} \left( \mathcal{E}_{jl} - \frac{\Lambda}{3} h_{jl} \right) Y^k + i \varepsilon_{ijkl} K_i^k \mathcal{P}_i \\
+ \frac{1}{2} i \left[ (|K|^2 - K^2) \tilde{K}_{ij} Y^k + i \varepsilon_{ijkl} (K K_j k Y_{ij} - K_j k K_k k Y_{ip}) \right] \\
+ i \varepsilon_{ijkl} (K K_j k Y_{ij} - K_j k K_k k Y_{ip}) \right) . \] (2.34)

In particular,
\[ \mathcal{D}_i \mathcal{P}_i = -\Lambda \sigma . \] (2.35)

From (2.34) we compute
\[ P^2 \mathcal{D}_i \mathcal{P} = 2 P^2 \mathcal{P} \mathcal{D}_i \mathcal{P} \] (2.36)
\[ = 2 \mathcal{P} \left[ \sigma \mathcal{P}^2 \mathcal{E}_{ij} - i \varepsilon_{ijkl} \mathcal{P}^k \mathcal{E}_j^k Y^l - \frac{\Lambda}{3} \mathcal{P}_i + i \varepsilon_{ijkl} \mathcal{P}^k \mathcal{P}^l Y^k \right] \] (2.37)
\[= \left( 2A - \frac{3}{3} \Lambda \sigma \right) \left[ \sigma \mathcal{P}_i - i \varepsilon_{ij} \mathcal{P}^j Y^k \right] , \] (2.38)
where we have set
\[ A := P^k \mathcal{P}_i \mathcal{E}_{kl} . \] (2.39)

We would like to derive an algebraic relation between \( A \) and \( P^2 \). We use (3.10), (2.34)-(2.35) and (2.38) to compute the divergence of (2.31) (for \( P^2 \neq 0 \),
\[ 0 = \mathcal{D}_i \left( 2 P^2 \mathcal{E}_{ij} - 3 A \left( \mathcal{P}_i \mathcal{P}_j \right) \right) \] (3.30)
\[ = 4 \mathcal{E}_i \mathcal{P}^2 \mathcal{D}_j \mathcal{P}^2 + 2 P^4 \mathcal{P}^2 \mathcal{E}_{ij} - 3 P^2, \mathcal{P}^2 \mathcal{D}_j A - 3 A P^2 \mathcal{P}_j \mathcal{P}_i \\
- 3 A P^2 \mathcal{P}_j \mathcal{P}_i + P^2 \mathcal{D}_i A + A \mathcal{D}_i P^2 \] (3.31)
\[= 9 \sigma A^2 \mathcal{P}^2 \mathcal{P}_i + \frac{3}{2} A^2 \mathcal{P}^2 \mathcal{P}_i + 3 i \varepsilon_{ijkl} \mathcal{P}^k \mathcal{P}^l Y^k + i \Lambda \mathcal{A} \varepsilon_{ij} \mathcal{P}^j Y^k \\
+ P^2 \mathcal{D}_i A - 3 P^2 \mathcal{P}_j \mathcal{D}_j A - A \mathcal{D}_i P^2 . \] (3.32)

Contraction with \( \mathcal{P}^i \) yields with (2.38)
\[ 2 P^2 \mathcal{P}^i \mathcal{D}_i A = 9 \sigma A^2 - A P^2 \mathcal{D}_i P^2 = 7 \sigma A^2 + \frac{2}{3} \Lambda \sigma A P^2 . \] (3.33)

We insert this into (3.32) to deduce that
\[ P^4 \mathcal{D}_i A - A P^2 \mathcal{D}_i P^2 = A \left( \frac{3}{2} A + \Lambda P^2 \right) \left( \sigma \mathcal{P}_i - i \varepsilon_{ij} \mathcal{P}^j Y^k \right) . \] (3.34)
Combining (3.34) and (3.28) we end up with the equation,

\[
(A - \frac{\Lambda}{3} P^2) P^2 \partial_t A = \left( \frac{7}{4} A + \frac{\Lambda}{6} P^2 \right) A \partial_t P^2 .
\]  

(3.35)

For

\[ P^2 \neq 0 \quad , \quad AP^{-2} \neq 0 , \quad AP^{-2} + \frac{2}{3} \Lambda \neq 0 \]  

(3.36)

this can be written as

\[
\partial_t \log P^2 = \frac{4 A P^{-2} - \frac{\Lambda}{3}}{3 A P^{-2} + \frac{2}{3} \Lambda} \partial_t \log (A P^{-2})
\]

(3.37)

\[
= \partial_t \log \left( \frac{A P^{-2} + \frac{2}{3} \Lambda}{(A P^{-2})^{2/3}} \right) .
\]

(3.38)

This PDE can be integrated straightforwardly to obtain a relation of the desired form,

\[
P^2 = \mu \left( A P^{-2} + \frac{4}{3} \Lambda \right)^{2} \left( A P^{-2} \right)^{2/3} , \quad \mu \in \mathbb{C} \setminus \{0\} .
\]

(3.39)

This equation holds whenever \( S_{\alpha \beta \mu \nu} |_{\Sigma} = 0. \)

Let us return to the equivalence issue concerning the various choices of \( Q. \)

It follows immediately from the computations in [14, Section 3.2] that

\[
Q_{F} |_{\Sigma} = Q_{0} |_{\Sigma}
\]

(3.40)

whenever \( S_{\alpha \beta \mu \nu} |_{\Sigma} = 0. \) Note for this that \( Q = Q_{F} \) can be derived algebraically from \( S_{\alpha \beta \mu \nu} = 0 \) without differentiation.

We employ (3.39) to express \( F^2 |_{\Sigma} \) in terms of \( Q_{F} F^2 |_{\Sigma} \), and finally in terms of \( C^2 |_{\Sigma} \) (we set \( x := \mp \frac{3}{2} \sqrt{\frac{3}{2} \mu^{-1}},\))

\[
F^2 |_{\Sigma} = \left( \frac{2}{3} \right)^{1/6} x^{-1} \left( \frac{Q_{0} F^2 - 4 A}{(Q_{0} F^2)^{2/3}} \right)
\]

(3.41)

\[
\equiv \pm \left( \frac{3}{2} x \right)^{1/3} \left( \pm \sqrt{C^2} - \sqrt{\frac{32}{3} \Lambda} \right)^{2} .
\]

(3.42)

Thus (recall that in our current setting where \( S_{\alpha \beta \mu \nu} |_{\Sigma} = 0, \) there is no freedom to choose \( \pm, \) cf. Proposition 2.1),

\[
Q_{F} |_{\Sigma} = x (C^2)^{5/6} \left( \pm \sqrt{C^2} - \sqrt{\frac{32}{3} \Lambda} \right)^{-2} = Q_{C} |_{\Sigma} .
\]

(3.43)

By (3.15), (3.20) and (3.29) we have

\[
AP^{-2} = -\frac{1}{6} \bar{Q}_{0} F^2 ,
\]

(3.44)

whence (3.36) is equivalent to

\[
Q_{F} F^2 |_{\Sigma} \neq 0 , \quad Q_{F} F^2 |_{\Sigma} - 4 \Lambda \neq 0
\]

\[
\iff C^2 |_{\Sigma} \neq 0 , \quad \pm \sqrt{C^2} |_{\Sigma} - \sqrt{\frac{32}{3} \Lambda} \neq 0 .
\]

(3.45)

(3.46)
and (3.41)-(3.43) are well-defined. Note that it depends on the sign in (3.43) for which the MST vanishes, for which sign (3.46) (and the corresponding conditions below) need to hold.

In view of \(Q\) let us compute the restriction of the Ernst potential to \(\Sigma\).

\[
\mathcal{P}_i \chi|\Sigma = \chi_i = 2X^\alpha F_{\alpha i} \quad (3.47)
\]

\[
= 2\sigma P_i - 2i\epsilon_{ijk} P^j Y^k \quad (3.48)
\]

\[
(3.28) = (\mathcal{A} P^{-2} - \frac{\Lambda}{3})^{-1} \mathcal{P}_i P^2, \quad (3.49)
\]

supposing that, in addition to (3.36),

\[
\mathcal{A} P^{-2} - \frac{\Lambda}{3} \neq 0. \quad (3.50)
\]

Because of (3.44) this is equivalent to

\[
Q F^2|\Sigma + 2\Lambda \neq 0 \iff \pm \sqrt{C^2|\Sigma} + \sqrt{\frac{8}{3}\Lambda} \neq 0. \quad (3.51)
\]

It follows from (3.39) and (3.49) that

\[
\mathcal{P}_i \chi|\Sigma = \mu \left(\mathcal{A} P^{-2} - \frac{\Lambda}{3}\right)^{-1} \mathcal{P}_i \left(\frac{(\mathcal{A} P^{-2} + \frac{3}{4}\Lambda)^2}{(\mathcal{A} P^{-2})^{2/3}}\right) \quad (3.52)
\]

\[
= \mathcal{P}_i \left(4\mu \frac{\mathcal{A} P^{-2} - \frac{\Lambda}{3}}{(\mathcal{A} P^{-2})^{2/3}}\right) \quad (3.53)
\]

\[
= \mathcal{P}_i \left(4\mu^2 \frac{\mathcal{A} P^{-2} - \frac{\Lambda}{3}}{(\mathcal{A} P^{-2} + \frac{3}{4}\Lambda)^2}\right), \quad (3.54)
\]

and thus, for an appropriate choice of the \(\chi\)-constant,

\[
\chi|\Sigma = 4\mu P^2 \frac{\mathcal{A} P^{-2} - \frac{\Lambda}{3}}{(\mathcal{A} P^{-2} + \frac{3}{4}\Lambda)^2} (3.44) \quad 6\epsilon^2 \frac{Q_0 F^2 + 2\Lambda}{(Q_0 F^2 - 4\Lambda)^2}. \quad (3.55)
\]

Given \(\chi|\Sigma\) and \(F^2|\Sigma\) this can be read as a quadratic equation for \(Q_0\), and it is precisely the same equation which is satisfied by \(Q_{ev}\) [14]. Consequently, one of its solutions satisfies

\[
Q_{ev}|\Sigma = Q_0|\Sigma \quad (3.56)
\]

for an appropriate choice of the \(\chi\)-constant. When working with \(Q_{ev}\) we also need to require \(Q F^2 + 8\Lambda \neq 0\) [14], or, equivalently,

\[
\pm \sqrt{C^2|\Sigma} + \sqrt{\frac{128}{3}\Lambda} \neq 0. \quad (3.57)
\]

We have

\[
C^2|\Sigma \equiv C^\alpha\beta\mu\nu C_{\alpha\beta\mu\nu}|\Sigma = 16\epsilon^2, \quad (3.58)
\]

whence, (3.46), and (3.51) & (3.57) are equivalent to

\[
\epsilon^2|\Sigma \neq 0, \quad \pm \sqrt{C^2|\Sigma} - \sqrt{\frac{2}{3}\Lambda} \neq 0, \quad (3.59)
\]
\[
\pm \sqrt{\mathcal{E}^2} |\Sigma| + \sqrt{\frac{1}{6}} \Lambda \neq 0, \quad \pm \sqrt{\mathcal{E}^2} |\Sigma| + \sqrt{\frac{8}{3}} \Lambda \neq 0. \tag{3.60}
\]

We have established the following lemma which is the Cauchy-surface-equivalent of Proposition 2.1:

**Lemma 3.3** Consider a Killing initial data set, i.e. a tuple \((\Sigma, h_{ij}, K_{ij}, \sigma, Y^i)\) which satisfies the vacuum constraints and the KID equations. Assume that the restriction to \(\Sigma\) of the MST associated to the KVF \(X\) generated by \((\sigma, Y^i)\) vanishes for some function \(Q\). Assume further that the conditions (3.59) hold.

(i) Then \(Q_0, Q_F\) and \(Q_C\) are regular near \(\Sigma\), and there exists a constant \(\kappa \in \mathbb{C} \setminus \{0\}\) and a choice of \(\pm\) such that \(Q|_{\Sigma} = Q_0|_{\Sigma} = Q_F|_{\Sigma} = Q_C|_{\Sigma}\).

(ii) Assume that, in addition, the inequalities (3.60) are valid. Then, the function \(Q_{ev}\) is regular near \(\Sigma\) as well, and there exists an Ernst potential \(\chi\) for \(Q_{ev}\) such that \(Q|_{\Sigma} = Q_0|_{\Sigma} = Q_{ev}|_{\Sigma} = Q_F|_{\Sigma} = Q_C|_{\Sigma}\).

Whenever (3.59)-(3.60) hold, the evolution equations (2.19) for \(S^{(ev)}_{\alpha\beta\mu\nu}\) are regular, at least sufficiently close to \(\Sigma\), and imply by standard result for symmetric hyperbolic systems that the MST vanishes in some neighborhood of \(\Sigma\):

**Corollary 3.4** Let \((\Sigma, h_{ij}, K_{ij}, \sigma, Y^i)\) be vacuum KIDs which satisfy (3.59)-(3.60). The MST of the emerging space-time \((\mathcal{M}, g_{\mu\nu})\) associated to the KVF generated by \((\sigma, Y^i)\) vanishes in some neighborhood of \(\Sigma\) for some function \(Q\) if and only if \(S^{(C)}_{\alpha\beta\mu\nu}|_{\Sigma} = 0\) (or \(S^{(0)}_{\alpha\beta\mu\nu}|_{\Sigma} = 0\) or \(S^{(F)}_{\alpha\beta\mu\nu}|_{\Sigma} = 0\) or \(S^{(ev)}_{\alpha\beta\mu\nu}|_{\Sigma} = 0\)).

The main advantage of the equation \(S^{(C)}_{\alpha\beta\mu\nu}|_{\Sigma} = 0\), as e.g. opposed to \(S^{(0)}_{\alpha\beta\mu\nu}|_{\Sigma} = 0\), cf. (3.21), is that it can be solved for \(P_i\) (supposing that a solution exists after all).

### 4 Construction of solutions to the KID equations

#### 4.1 Candidates for solving the KID equations

Similar to the proceeding in [14] we do not want to assume that Killing initial data \((\Sigma, h_{ij}, K_{ij}, \sigma, Y^i)\) have been given but only Cauchy data \((\Sigma, h_{ij}, K_{ij})\). Indeed, it is a grievance of Corollary 3.4 that, given \((\Sigma, h_{ij}, K_{ij})\), it is a non-trivial and non-algorithmic issue to check whether there exists \((\sigma, Y^i)\) complementing them to Killing initial data. Only then, it is straightforward to check whether \(S_{\mu\nu\rho}\|_{\Sigma} = 0\) holds. We therefore intend to derive conditions from the equation \(S_{\mu\nu\rho}\|_{\Sigma} = 0\) which impose restrictions on \((\sigma, Y^i)\). As in the space-time case [14] it turns out that up to rescaling only one candidate remains.
### 4.1.1 Candidate fields

Let us assume for the time being that we have been given a \( \Lambda \)-vacuum space-time which admits a KVF for which the associated MST vanishes, and which moreover satisfies \( F^2|_\Sigma \neq 0 \) and \( QF^2|_\Sigma + 2\Lambda \neq 0 \), or, equivalently,

\[
P^2 \neq 0 , \quad qP^2 - \frac{\Lambda}{2} \neq 0 . \tag{4.1}
\]

We will collect a number of necessary conditions, which need to be satisfied in any such space-time. Among other things, they will provide candidates for \( \sigma \) and \( Y^i \).

It has been shown in [14] that in vacuum space-times with vanishing MST and in which the space-time analog of (4.1) holds, the self-dual Killing form \( F_{\mu\nu} \) and the function \( Q \) necessarily satisfy the equation

\[
\nabla_{\mu}Q + \frac{1}{4} QF^2 + \frac{20\Lambda}{QF^2 + 2\Lambda} Q \nabla_{\mu} \log F^2 = 0 . \tag{4.2}
\]

For vanishing MST the different choices for \( Q \) are equivalent, whence we have written \( Q \) without any subscript. Let us consider its restriction to the initial surface \( \Sigma \), where it suffices to take the spatial components into account, and complement it by (3.19)

\[
\varepsilon_{ij} - q(P_iP_j) = 0 , \tag{4.3}
\]

\[
\mathcal{D}_i q + \frac{1}{4} qP^2 - \frac{5\Lambda}{4} qP^2 - \frac{\Lambda}{2} q\mathcal{D}_i \log P^2 = 0 . \tag{4.4}
\]

(Alternatively, (4.4) may be obtained by combining (3.20) and (3.37).) For the computations below, (4.4) will often be needed in the form

\[
\mathcal{D}_i(qP^2) = \frac{3}{4} \frac{qP^2 + \Lambda}{qP^2 - \frac{\Lambda}{2}} q\mathcal{D}_i P^2 . \tag{4.5}
\]

**Lemma 4.1** Assume that (4.1) and (4.3) hold. Then (4.4) holds if and only if \( q = Q_{\Sigma} \).

**Proof:** Equation (4.4) is equivalent to

\[
\mathcal{D}_i \log \left( (qP^2)^{-2/3}(qP^2 + \Lambda)^2 \right) = \mathcal{D}_i \log P^2 . \tag{4.6}
\]

It follows from (4.3) that \( qP^2 = \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{\mathcal{C}}^2 \), and, after integration, we deduce that (4.6) is equivalent to

\[
q = \kappa(\mathcal{C}^2)^{5/6} \left( \pm \sqrt{\mathcal{C}^2} - \sqrt{\frac{3\Lambda}{2}} \right)^{-2} = Q_{\Sigma} \tag{4.7}
\]

as claimed.  \( \square \)
Our aim is to derive candidate fields for $\sigma$ and $Y^i$ in terms of $P_i$ and $q$. Recall (3.24), whose derivation required the KID equations. With (4.3) it becomes

$$\mathcal{D}_i P_j = \sigma (q P_i P_j - \frac{1}{3} (q P^2 + \Lambda) h_{ij}) + \hat{\epsilon}_{ijk} \left( q P_j P_l - \frac{1}{3} (q P^2 + \Lambda) h_{jl} \right) Y^k + \hat{\epsilon}_{ijkl} K_{ik} K^l.$$  

(4.8)

Contraction with $P^j$ yields

$$\mathcal{D}_i P^2 = \frac{4}{3} \sigma \left( q P^2 - \frac{\Lambda}{2} \right) P_i + \frac{4}{3} \hat{\epsilon}_{ikl} \left( q P^2 - \frac{\Lambda}{2} \right) Y^k P_i.$$  

(4.9)

Contraction with $P^i$ provides an expression for $\sigma$,

$$\sigma = \frac{3}{4} \left( q P^2 - \frac{\Lambda}{2} \right)^{-1} P^i \mathcal{D}_i \log P^2.$$  

(4.10)

An application of $\hat{\epsilon}_{pq}$ to (4.9) and relabeling indices gives

$$Y_{[k} P_{l]} = \frac{i}{2} \sigma i_{kl} m P_m - \frac{3}{8} \left( q P^2 - \frac{\Lambda}{2} \right)^{-1} \epsilon_{klm} \mathcal{D}_m P^2.$$  

(4.11)

We insert (4.11) into (4.8) to obtain the useful relation

$$\mathcal{D}_i P_j = \frac{1}{3} \left( q P^2 + \Lambda \right) \left( \hat{\epsilon}_{ijk} Y^k - \sigma h_{ij} \right) + \frac{3}{4} \frac{q P^2}{q P^2 - \frac{\Lambda}{2}} P_j \mathcal{D}_i \log P^2 + i \hat{\epsilon}_{ijkl} K_{ik}^l P_l.$$  

(4.12)

Supposing that

$$q P^2 + \Lambda \neq 0,$$  

(4.13)

its anti-symmetric part yields an equation which can be solved for $Y$,

$$Y_i = -\frac{3}{2} \left( q P^2 + \Lambda \right)^{-1} \frac{3}{4} \hat{\epsilon}_{ikl} \left( q P^2 - \frac{\Lambda}{2} \right) P_k \mathcal{D}_i \log P^2 + i \hat{\epsilon}_{i kl} K_{ik}^l P_l - K P_i.$$  

(4.14)

Whenever a $\Lambda$-vacuum space-time with (4.1) and (4.13) admits a KVF such that the associated MST vanishes, the corresponding KIDs necessarily need to satisfy (4.10) and (4.14) where $(q, P_i)$ solve (4.3)-(4.4) (cf. Proposition 4.2 below).

We would like to gain some insight under which conditions the candidates (4.10) and (4.14) for $\sigma$ and $Y^i$ do provide a solution of the KID equations. For this purpose in turns out to be fruitful to derive a couple of relations between the co-vector field $P$ and the function $q$.

Of course, in general, there is no reason why $(\sigma, Y^i)$ should be real. As in [14] this does not cause any problems, and we can enlarge our space-times of interest to those which admit a complex KVF whose associated MST vanishes.

4.1.2 Necessary conditions on $P$

Let us compute the symmetric part of (4.12) which provides a useful relation satisfied by $P$ which does not involve $Y$,

$$\mathcal{D}_i (P_j) = -\frac{1}{3} \sigma (q P^2 + \Lambda) h_{ij} + \frac{3}{4} \frac{q P^2}{q P^2 - \frac{\Lambda}{2}} P_i \mathcal{D}_j \log P^2 + i \hat{\epsilon}_{ijkl} K_{ik}^l \mathcal{P}_l.$$  

(4.15)
Two special components will be of particular importance: Its contraction with \( P^j \)
\[
P^j \mathcal{D}(iP_j) = \frac{1}{6} \sigma (qP^2 - 2\Lambda)P_i + \frac{3}{8} qP^2 \mathcal{D}_iP^2 + \frac{i}{2} \epsilon_i^{jk} K^i_j P_k P_l ,
\]
and its trace
\[
\mathcal{D}_i P^i = -\Lambda \sigma .
\]

### 4.1.3 Vanishing of the transverse derivative of the MST on \( \Sigma \)

We would like to derive an expression, analog to (4.15), for the anti-symmetric part of the covariant derivative of \( P \). The anti-symmetric part of (4.12), though, was used to obtain an expression for the candidate field \( Y \), whence it does not seem to be usable for this.

Later on we will be interested in initial data \((\Sigma, h_{ij}, K_{ij})\) for which we do not know whether they admit a solution \((\sigma, Y_i)\) of the KID equations. Instead, we want to assume that we have been given a co-vector field \( P \) which solves (4.3). Then the “MST” \( S^\alpha{}_{\alpha\beta\mu\nu} \) vanishes on \( \Sigma \). The quotation marks are to emphasize that we do not know whether it is associated to a KVF: First of all, a solution of the KID equations does not need to exist. And secondly, even if a solution exists, there is a priori no reason why a solution \( P \) of (4.3) should arise from \((\sigma, Y_i)\) via (3.6). In both cases the “MST” is not the proper one, so a priori there is no reason to expect that the transverse derivative of the “MST” vanishes on \( \Sigma \) (which otherwise would follow from the fact that the MST satisfies the symmetric hyperbolic system (2.19)). For this reason, it seems promising to analyze the vanishing of the transverse derivative of the MST. Clearly, relations obtained this way necessarily need to be fulfilled by Cauchy data \((\Sigma, h_{ij}, K_{ij})\) which yield a \( \Lambda \)-vacuum space-time with vanishing MST.

For the computation of \( \nabla_t S^\alpha{}_{\alpha\beta\mu\nu}|_\Sigma \) we need to determine the transverse derivatives of \( Q \) and \( Q^\alpha{}_{\alpha\beta\mu\nu} \). We assume
\[
qP^2 \neq 0 , \quad qP^2 - \frac{\Lambda}{2} \neq 0 .
\]
In any vacuum space-time with vanishing MST the following relations hold [12],
\[
\nabla_\mu F^2 = \frac{4}{3} (QF^2 + 2\Lambda)X^\alpha f_{\alpha\mu} ,
\]
\[
\nabla_\mu f_{\alpha\beta} = QX^\kappa f_{\kappa\mu} f_{\alpha\beta} + \frac{1}{3} (QF^2 - 4\Lambda)X^\nu\mathcal{I}_{\alpha\beta\mu\nu} .
\]
We employ (4.2) and (4.19) to calculate
\[
\nabla_t Q|_\Sigma = -\frac{1}{3} (QF^2 + 20\Lambda)Q^\nu f_{\alpha\nu} f_{\alpha\mu} t^\mu .
\]
\[
= \frac{1}{3} (qP^2 - 5\Lambda)qP^{-2} Y^k P_k .
\]
Finally, using (3.13), (4.19) and (4.20) we determine the transverse derivative of the MST on \( \Sigma \),
\[
\nabla_t S^\alpha{}_{\alpha\beta\mu\nu}|_\Sigma = \nabla_t S^\alpha{}_{\alpha\beta\mu\nu}|_\Sigma - \nabla_t Q \left( \mathcal{P}^\alpha_{ti} f_{\alpha\mu} \mathcal{J}^j - \frac{1}{3} \mathcal{J}^j S^\alpha{}_{\alpha\beta\mu\nu} \right) - \left( \mathcal{P}^\alpha_{ti} \nabla_t f_{\alpha\mu} \mathcal{J}^j - \frac{1}{3} \mathcal{J}^j S^\alpha{}_{\alpha\beta\mu\nu} \right) + \frac{i}{3} qP^2 + \Lambda \left( 5P^{-2} Y^k P_k P_i P_j - 2P_{(i} Y_{j)} - Y^k P_k h_{ij} \right) .
\]
Assume now that the MST vanishes initially, i.e. $S_\alpha z_{\mu\nu}|\Sigma = 0$, or, equivalently, that (4.3) holds for a function $q = q(x^i)$ which satisfies (4.4). Then

\[
\nabla_t S_{itj}|\Sigma = \frac{i}{4} qP^2 - \frac{5\Lambda}{4} qP^2 - \frac{1}{2} q\epsilon_{(i}kP_{l)}P_k \log P^2 + iq\epsilon_{(i}kP_{l)}\partial_t \log P^2 + iqP_{(i}Q_{l)}
\]

\[
+ q\epsilon_{i}k\eta P_{kP_{l}}P_q + qK_{(i}kP_{l)}P_k - qK_{P_{l}P_{j}} + \frac{1}{2} q(qP^2 + \Lambda) \left( 5P^{-2}Y^kP_kP_{l} - 2P_{(i}P_{j)} - Y^kP_kh_{ij} \right).
\]

We plug in the expression (4.14) we derived for $Y$,

\[
\nabla_t S_{itj}|\Sigma = \frac{i}{4} qP^2 - \frac{5\Lambda}{4} qP^2 - \frac{1}{2} q\epsilon_{(i}kP_{l)}P_k \log P^2 + \frac{i}{2} q \left( \epsilon_{i}kP_{l}\partial_t (P_{j}) + \epsilon_{j}kP_{l}\partial_t (P_{i}) \right)
\]

\[
+ \frac{5}{2} iq \left( Q_{(i}P_{j)} - P^{-2}P^kQ_kP_{j} \right) + q\epsilon_{i}k\eta P_{kP_{l}}P_q + 2qK_{(i}kP_{l)}P_k
\]

\[
- \frac{1}{2} q \left( 5P^{-2}K_{kl}P_{k} - K \right)P_{j} + \frac{1}{2} q \left( K_{kl}P_{k}P_{l} - KP^2 \right)h_{ij}.
\]

Contracting this with $P^j$ yields

\[
\mathcal{P}^j \nabla_t S_{itj}|\Sigma = iqP^2(Q_i - P^{-2}P^kQ_kP_i) + \frac{i}{4} qP^2 - \frac{2\Lambda}{4} qP^2 - \frac{1}{2} qP^2K^kP_k - K^{kl}P_kP_lP_i.
\]

Vanishing of $\mathcal{P}^j \nabla_t S_{itj}|\Sigma = 0$ requires

\[
Q_i - P^{-2}P^kQ_kP_i = - \frac{1}{4} qP^2 - \frac{2\Lambda}{4} qP^2 - \frac{1}{2} qP^2K^kP_k - iK^kP_k - iP^{-2}K^kP_kP_i,
\]

which yields the desired relation for the anti-symmetric part of $\nabla_t P_j$. If one inserts (4.27) and (4.15) into (4.25), the right-hand side vanishes automatically, so no additional relation can be extracted from $\nabla_t S_{\alpha\beta\mu\nu}|\Sigma = 0$.

### 4.1.4 An intermediate result

It follows from (4.3) that

\[
qP^2 = \pm \sqrt{\frac{3}{2} \mathcal{E}^2},
\]

whence our assumptions on $qP^2$,

\[
qP^2 \neq 0 , \quad qP^2 + \Lambda \neq 0 , \quad qP^2 - \frac{\Lambda}{2} \neq 0 , \quad qP^2 - 2\Lambda \neq 0.
\]

can be expressed in terms of $\mathcal{E}^2$, (4.30) below, i.e. in terms of the Cauchy data.

Let us collect the equations we have found in the preceding sections. Taking also Corollary 3.4 into account we end up with the following
Proposition 4.2 Consider Cauchy data \((\Sigma, h_{ij}, K_{ij})\) which satisfy the vacuum constraint equations and\(^3\)

\[\mathcal{E}^2 \neq 0, \quad \mathcal{E}^2 - \frac{1}{6} \Lambda^2 \neq 0, \quad \mathcal{E}^2 - \frac{2}{3} \Lambda^2 \neq 0, \quad \mathcal{E}^2 - \frac{8}{3} \Lambda^2 \neq 0.\] (4.30)

A necessary condition for the emerging Cauchy development to admit a (possibly complex) KVF \(X\) such that the associated MST vanishes is:

(i) There exists a function \(q : \Sigma \rightarrow \mathbb{C}\) and a co-vector field \(\mathcal{P}\) such that (4.3), (4.4), (4.15), and (4.27) hold.

(ii) \(X^\mu|_\Sigma = (\sigma, Y^i)\) is given by (4.10) and (4.14).

If, in addition to (i)-(ii),

(iii) \((\sigma, Y^i)\) satisfies the KID equations, and

(iv) \((\sigma, Y^i)\) and \(\mathcal{P}_i\) are related via (3.6),

then the Cauchy development of \((\Sigma, h_{ij}, K_{ij})\) admits a KVF \(X\) with \(X^\mu|_\Sigma = (\sigma, Y^i)\) such that the associated MST vanishes in some neighborhood of \(\Sigma\).

Proof: The first part follows directly from the considerations above. (iii)-(iv) guarantee that the tensor \(S_{\alpha\beta\mu
u}\), whose vanishing on the Cauchy surface \(\Sigma\) is ensured by (i), is actually the MST associated to the KVF \(X\) generated by \((\sigma, Y^i)\). The result follows now from Lemma 4.1 and Corollary 3.4. \(\square\)

4.2 The KID equations

Let us analyze to what extent (iii)-(iv) follow from (i)-(ii). We consider Cauchy data \((\Sigma, h_{ij}, K_{ij})\) which satisfy the vacuum constraint equations, (4.1) and (4.13). Moreover, we assume that there exist a function \(q : \Sigma \rightarrow \mathbb{C}\) and a co-vector field \(\mathcal{P}\) such that (4.3), (4.4), (4.15), and (4.27) hold. Finally, we define a (possibly complex) function \(\sigma : \Sigma \rightarrow \mathbb{C}\) via (4.10), and a (possibly complex) vector field \(Y\) via (4.14).

Using (4.4) we find that

\[\mathcal{D}_k \sigma = \frac{3}{4} \left( q \mathcal{P}^2 - \frac{\Lambda}{2} \right)^{-1} \left( \mathcal{P}^i \mathcal{D}_i \mathcal{D}_1 \log \mathcal{P}^2 + \mathcal{D}_k \mathcal{P}^1 \mathcal{D}_1 \log \mathcal{P}^2 - \sigma \frac{q \mathcal{P}^2 + \Lambda}{q \mathcal{P}^2 - \frac{\Lambda}{2}} \mathcal{P}^2 \mathcal{D}_k \log \mathcal{P}^2 \right).\] (4.31)

Differentiating (4.15) yields with (4.4)

\[\mathcal{D}_k \mathcal{D}_i (\mathcal{P}_j) = \frac{1}{4} q \mathcal{P}^2 + \frac{\Lambda}{4} \left( \frac{3}{2} q \mathcal{P}^2 - \frac{1}{2} \right) \mathcal{D}_k \mathcal{P}^2 \mathcal{D}_i \mathcal{D}_j \log \mathcal{P}^2 - \mathcal{D}_k \mathcal{P}^1 \mathcal{D}_1 \log \mathcal{P}^2 - \mathcal{P}^1 \mathcal{D}_i \mathcal{D}_j \mathcal{D}_k \log \mathcal{P}^2 \right) h_{ij}

- \frac{9}{32} \left( q \mathcal{P}^2 + \frac{\Lambda}{2} \right) \mathcal{D}_k \mathcal{D}_i (\mathcal{P}_j) \log \mathcal{P}^2 \mathcal{D}_k \log \mathcal{P}^2 + \frac{3}{4} q \mathcal{P}^2 \mathcal{D}_k \mathcal{D}_i \mathcal{D}_j \mathcal{D}_k \log \mathcal{P}^2 \mathcal{D}_k \log \mathcal{P}^2

+ \frac{1}{4} q \mathcal{P}^2 \mathcal{D}_k \mathcal{D}_i (\mathcal{P}_j) \log \mathcal{P}^2 + i c (\mathcal{P}_j \mathcal{D}_k K_{ij} \mathcal{P}_q + K_{ij} \mathcal{P}_q \mathcal{D}_k \mathcal{P}_q).\] (4.32)

\(^3\) A similar comment as in Remark 5.1 applies: It is actually sufficient when the following conditions are satisfied for one sign \(\pm\), depending on the sign on which \(S_{\alpha\beta\mu\nu}\) is fulfilled,

\[\mathcal{E}^2 \neq 0, \quad \pm \sqrt{\mathcal{E}^2 + \frac{\sqrt{6} \Lambda}{\sqrt{3}} \neq 0}, \quad \pm \sqrt{\mathcal{E}^2 - \frac{\sqrt{2} \Lambda}{\sqrt{3}} \neq 0}, \quad \pm \sqrt{\mathcal{E}^2 + \frac{\sqrt{2} \Lambda}{\sqrt{3}} \neq 0}.\]
For the covariant derivative of $Y$ a somewhat lengthy calculation, which makes use of (4.32), (4.15), and the vacuum constraints, reveals that

\[
\mathcal{D}_i Y_j = \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} \mathcal{D}_i \log P^2 \left( K_j^k \mathcal{P}_k - K \mathcal{P}_j \right) \right. \\
+ i \epsilon_{ij}^{kl} \left( \mathcal{R}_{ik} \left( 2 R_{lm} P^m - R P_l \right) - 2 R_{il} P_k + \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} \mathcal{P}_k \mathcal{D}_i \log P^2 \right) \\
+ \frac{3}{4} \frac{1}{qP^2 - \frac{3}{2}} \left( 1 + \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} + \frac{3}{8} \frac{qP^2 + \Lambda}{(qP^2 - \frac{3}{2})^2} \right) qP^2 P_k \mathcal{D}_i \log P^2 \mathcal{D}_i \log P^2 \\
- \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} \mathcal{D}_i \mathcal{P}_k \mathcal{D}_i \log P^2 - \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} P^2 P_k \mathcal{D}_i \mathcal{P}_i - 2 P_k \mathcal{D}_i (P_l) \\
\left. - \mathcal{P}_k \mathcal{D}_i K_j^k + \mathcal{P}_j \mathcal{D}_i K - K_j^k \mathcal{D}_i \mathcal{P}_k + K \mathcal{D}_i \mathcal{P}_j \right] \\
= \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} \left( K_j^k \mathcal{P}_k - K \mathcal{P}_j \right) \mathcal{D}_i \log P^2 \right.
\]

\[
+ \frac{9}{16} \frac{(qP^2)^2}{(qP^2 - \frac{3}{2})^2} \epsilon_{ij}^{kl} \mathcal{P}_k \mathcal{D}_i \log P^2 \mathcal{D}_i \log P^2 - \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} \epsilon_{ij}^{kl} \mathcal{D}_i (\mathcal{P}_k) \mathcal{D}_i \log P^2 \\
+ \epsilon_{ij}^{kl} \epsilon_{pq} K_{lp} \mathcal{D}_k \mathcal{P}_q - 2 \epsilon_{ij}^{kl} \mathcal{R}_{ik} P_l - \Lambda K_i \sigma - K_j^k \mathcal{D}_j \mathcal{P}_k + K \mathcal{D}_i \mathcal{P}_j \\
+ 2 i \epsilon_{ij}^{kl} \mathcal{R}_{ik} P_l - i \epsilon_{ij}^{kl} \left( 2 \mathcal{R}_{kl} P^l - \mathcal{R} P_k \right) \\
- \frac{3}{4} \frac{1}{qP^2 - \frac{3}{2}} \epsilon_{ij}^{kl} \left( \frac{\Lambda}{2} \frac{qP^2}{qP^2 - \frac{3}{2}} \sigma qP^2 \delta_i^l - \mathcal{D}_i P^l - \mathcal{P}^l \mathcal{D}_k \right) \mathcal{D}_i \log P^2
\]

\[
= \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} \left( K_j^k \mathcal{P}_k + \mathcal{D}_j \mathcal{P}_k \log P^2 + (K_j^k \mathcal{P}_k - K \mathcal{P}_j) \mathcal{D}_i \log P^2 \right) \right.
\]

\[
+ \epsilon_{ij}^{kl} \epsilon_{pq} K_{lp} \left( \mathcal{D}_k \mathcal{P}_q - \frac{3}{4} \frac{qP^2}{qP^2 - \frac{3}{2}} \mathcal{P}_q \mathcal{D}_k \log P^2 \right) - 2 \epsilon_{ij}^{kl} \mathcal{R}_{ik} P_l - K_j^k \mathcal{D}_k \mathcal{P}_j \\
- K_j^k \mathcal{D}_k \mathcal{P}_k + K \mathcal{D}_i \mathcal{P}_j + 2 i \epsilon_{ij}^{kl} \mathcal{R}_{ik} P_l - i \epsilon_{ij}^{kl} \left( 2 \mathcal{R}_{kl} P^l - \mathcal{R} P_k \right) \left. - \frac{3}{2} K_i \sigma \right)
\]

\[
= \frac{3}{4} \frac{1}{qP^2 - \frac{3}{2}} \epsilon_{ij}^{kl} \left( \frac{qP^2 + \Lambda}{2} \frac{qP^2}{qP^2 - \frac{3}{2}} \sigma qP^2 \delta_i^l - \mathcal{D}_i P^l - \mathcal{P}^l \mathcal{D}_k \right) \mathcal{D}_i \log P^2 .
\]

Now we are ready to consider the KID equation (2.33). Taking the symmetric part of (4.35) another lengthy calculation shows (we use (4.32), (4.15),
(4.27), (4.3) and the vacuum constraints),

\[
\mathcal{P}_i(Y_j) = \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ \frac{3}{4} \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \left( K^{[i} k | P_k - K | P_{[i} \right) \mathcal{D}_j \log P^2 \\
-2i\tilde{\epsilon}_{(i} kl \tilde{R}_{j)l} P_k - \frac{3}{2} i \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \tilde{\epsilon}_{(i} kl \mathcal{D}_{(j)l} P_k \mathcal{D}_l \log P^2 \\
+ \frac{9}{16} \left( \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \right)^2 \tilde{\epsilon}_{(i} kl \mathcal{D}_{(j)l} \log P^2 \mathcal{D}_l \log P^2 - \Lambda \sigma K_{ij} - 2\tilde{\epsilon}_{(i} pq B_{j)p} P_q \\
+ \tilde{\epsilon}_{(i} kl \tilde{\epsilon}_{j)k} P_l \mathcal{D}_{(k)l} P_q - 2K_{[i} k \mathcal{D}_{(j)l} P_k + K \mathcal{D}_{(i} P_j) \right] \right) \quad (4.36)
\]

\[
= \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ \tilde{\epsilon}_{(i} kl \left( K_{[k} k_{l]} p + K_{[k} k \mathcal{D}_{l]} P_l - K K_{j}[k \mathcal{D}_{l]} P_l \right) \right]_{=0} \\
+ 2\tilde{\epsilon}_{(i} kl \tilde{\epsilon}_{j)k} \mathcal{P}_l \right] - \sigma K_{ij} \quad (4.37)
\]

\[
= -\sigma K_{ij} \quad (4.38)
\]

The first KID equation (2.33) is therefore automatically fulfilled in this setting.

Before we analyze the second KID equation, it is useful to focus attention to another equation first, namely (3.6),

\[
\mathcal{P}_i = \mathcal{D}_i \sigma + K_{ij} Y_j + \frac{i}{2} Z_i \quad (4.39)
\]

It ensures that \( \mathcal{P} \) and \( (\sigma, Y) \) are related in the right way, so that the "MST", given on \( \Sigma \) in terms of \( \mathcal{P} \), is actually the proper MST associated to a KVF, namely the one generated by \( (\sigma, Y) \). To check it, we determine the anti-symmetric part of (4.35). With (4.27), (4.31), (4.3) and the vacuum constraints we find

\[
\mathcal{D}_i(Y_j) = \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ \frac{3}{4} \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \left( K^{[i} k | P_j \mathcal{D}_k \log P^2 - (K^{[i} k | P_{[k} - K | P_{[i} \right) \mathcal{D}_j \log P^2 \\
- \tilde{\epsilon}_{[i} kl \tilde{\epsilon}_{j)k} P_l \mathcal{D}_{[k]l} \log P^2 + \frac{3}{2} \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} P_{[i} \mathcal{D}_{l]} P_l \right) + K^{[i} k | \mathcal{D}_k P_j - K^{[i} k \mathcal{D}_{k]l} P_l \\
+ K \mathcal{D}_i P_j - 2\tilde{\epsilon}_{[i} kl B_{j]k} P_l - 2\tilde{\epsilon}_{[i} kl \tilde{R}_{j]k} P_l - i\tilde{\epsilon}_{i} kl (2\tilde{R}_{k} P_{[l} - \mathcal{R}_{l]} P) + i\tilde{\epsilon}_{i} kl \mathcal{D}_k \sigma \right] \right) \\
= \frac{3}{4} \left( qP^2 - \frac{\Lambda}{2} \right) \left( K^{[i} k | P_j \mathcal{D}_k \log P^2 - \mathcal{P}_k K^{[i} k \mathcal{D}_{l]} P_l \log P^2 + K \mathcal{D}_i \mathcal{D}_j \log P^2 \right) \\
- \tilde{\epsilon}_{[i} kl \tilde{\epsilon}_{j)k} P_l \mathcal{D}_{[k]l} \log P^2 \right) \\
+ \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ \frac{1}{2} \left( \tilde{\epsilon}_{i} kl P_{[k} k_{l]} + 2K^{[i} k \tilde{\epsilon}_{j]k} P_l + K \tilde{\epsilon}_{ij} \right) \right]_{=\tilde{\epsilon}_{i} kl P_{[k} k_{l]}} \\
\times \left( \mathcal{P}^{-2} P_{[m} P_{n]} Q_m + iK_{i} \mathcal{P}_r - iP^{-2} K_{[m} P_{n]} P_{[r]} \right) \\
+ \frac{3}{2} \left( qP^2 + \Lambda \right)^{-1} \left[ -2i\tilde{\epsilon}_{[i} kl (KK)_{j]k} - K_{[i} k \tilde{\epsilon}_{j]k} P_l - \frac{2}{3} \Lambda \tilde{H}_{j]k} \mathcal{P}_l + 2i\tilde{\epsilon}_{[i} kl \tilde{\epsilon}_{j]k} P_l \\
- i\left( \tilde{\epsilon}_{i} kl (2\tilde{R}_{k} P_{[l} - \mathcal{R}_{l]} P) + 4\tilde{\epsilon}_{[i} kl \tilde{R}_{j]k} P_l \right) + i\tilde{\epsilon}_{i} kl \mathcal{D}_k \sigma \right]_{=\tilde{\epsilon}_{i} kl K \mathcal{D}_k P}
\]
\[ \frac{1}{4 q P^2} \left[ K_k \partial_k \log P^2 - \partial_k K_k \partial_k \log P^2 + K P_k \partial_k \log P^2 \right] \]

\[ - \frac{1}{4} \left( q P^2 + \Lambda \right)^{-1} \left( q P^2 - \frac{q P^2}{2} \partial_k \log P^2 \right) \]

\[ \times \left[ 2 \epsilon_{i kl} (K K_{jk} - K_{jk} m K_m n) P_l + \epsilon_{ij} p [(|K|^2 - K^2) P_p - K_p (K_k P_k - K P)] \right] = 0 \]

\[ + i \epsilon_{ij} k (\partial_k \sigma - \partial_k + K_k Y^k) . \]

Recall our definition (4.14) of \( Y \). With (4.27) it can be written as

\[ Y_i = - \frac{3}{2} \left( q P^2 + \Lambda \right)^{-1} \left( q P^2 - \frac{q P^2}{2} \partial_k \log P^2 \right) \]

\[ \times \left( q P^2 - \frac{q P^2}{2} \partial_k \log P^2 - \frac{q P^2}{2} \left( q P^2 + \Lambda \right)^{-1} (K_k P_k - K P) \right) . \]

We insert this into the expression we have derived for \( \partial_k Y_j \) to end up with

\[ \partial_k Y_j = i \epsilon_{ij} k (\partial_k \sigma - \partial_k + K_k Y^k) , \quad (4.40) \]

which is equivalent to (4.39), i.e. \( \partial \) and \( (\sigma, Y) \) are automatically related to each other in the desired way. Moreover, it follows immediately from (4.40) that for \( P^2 \neq 0 \) the emerging KV cannot be trivial on \( \Sigma \).

Finally, let us devote attention to the second KID equation (2.34). We differentiate (4.39). Using (4.15), (2.33), (4.3), (4.14), (4.27) yields

\[ \partial_i \partial_j \sigma = \partial_i (\partial_j p) - \frac{i}{2} \partial_i (\partial_j z) - Y^k \partial_i (K_{jk}) - K_{jk} \partial_j Y^k \]

\[ = \partial_i (\partial_j p) - \frac{i}{2} \epsilon_{ijl} \partial_k (\partial_j Y_l) + \epsilon_{ijl} \partial_k (\partial_j Y_l) + \frac{i}{2} \epsilon_{ijl} (\partial_l p) Y^k \]

\[ = - Y^k \partial_i (\partial_j p) - K_{jk} \partial_j Y^k \]

\[ = - \frac{1}{3} \sigma (q P^2 + \Lambda) h_{ij} + \frac{3}{4} \frac{q P^2}{P} \partial_i (\partial_j p) \log P^2 - i \sigma B_{ij} - Y^k \partial_i (K_{jk}) \]

\[ - K_k (\partial_j Y^k) + i \epsilon_{ijl} \left( K_{jk} p_l + K_{jjl} \partial_i \sigma + \tilde{R}_{ij} Y_l \right) \]

\[ = - \sigma (i B_{ij} + \frac{1}{3} q P^2 + \frac{q P^2}{3} h_{ij}) + \frac{3}{4} \frac{q P^2}{P} \partial_i (\partial_j p) \log P^2 - \partial_Y K_{ij} \]

\[ + K_{jk} (\partial_j Y_k) + i \epsilon_{ijl} (K_{jk} p_l Y^p + i \epsilon_{ijkl} (\tilde{R}_{ij} + i B_{ik}) Y_l) \]

\[ = \sigma \left( \tilde{R}_{ij} + K K_{ij} - 2 K_{ik} K_j^k - \Lambda h_{ij} - E_{ij} - \frac{1}{3} q P^2 \right) \]

\[ + \frac{3}{4} \frac{q P^2}{P} \partial_i (\partial_j p) \log P^2 + i \epsilon_{ijl} K_{jk} Y^p Y_l - \partial_Y K_{ij} \]

\[ + i \epsilon_{ijl} (K_{jk} K_l p Y_p + K_{jk} K_p Y_l - K K_{jk} Y_l) \]

\[ = \sigma \left( \tilde{R}_{ij} + K K_{ij} - 2 K_{ik} K_j^k - \Lambda h_{ij} \right) - \partial_Y K_{ij} , \quad (4.46) \]

i.e. the second KID equation holds automatically, as well.

We are thus led to the following improvement of Proposition 4.2.
Proposition 4.3 Consider Cauchy data \((\Sigma, h_{ij}, K_{ij})\) which satisfy the vacuum constraint equations and \((4.30)\) (cf. footnote 3). The emerging Cauchy development admits a (possibly complex) KVF \(X\) such that the associated MST vanishes if and only if there exists a function \(q : \Sigma \rightarrow \mathbb{C}\) and a co-vector field \(P\) such that \((4.3), (4.4), (4.15)\) and \((4.27)\) hold. In that case \(X^\nu|_{\Sigma} = (\sigma, Y^i)\), where \(\sigma\) and \(Y^i\) are given by \((4.10)\) and \((4.14)\), respectively, and \(X^\nu|_{\Sigma}\) is non-trivial.

4.3 The equations for \(P\) revisited

Proposition 4.3 requires the existence of a function \(q\) and a co-vector field \(P\) such that, for given Cauchy data \((\Sigma, h_{ij}, K_{ij})\) (recall the definition of \(E_{ij}\) \((3.7)\), \(\sigma\) \((4.10)\) and \(Q_i\) \((3.9)\)),

\[
\begin{align*}
E_{ij} & = q(P_j P_i) , \\
\mathcal{D}_i q & = -\frac{1}{4} qP^2 - 5\Lambda q \mathcal{D}_i \log P^2, \\
\mathcal{D}_i P_j & = \frac{3}{4} \frac{qP^2}{qP^2 - \frac{1}{2}} P_i \mathcal{D}_j \log P^2 - \frac{1}{2} \mathcal{D}_i (\mathcal{D}_j P) + \frac{2}{3} \Lambda h_{ij} \mathcal{D}_k P^2. \\
Q_i & = P^{-2} P^k Q_k P_i - \frac{1}{4} qP^2 - \frac{2}{4} qP^2 - \frac{1}{2} \mathcal{D}_i (\mathcal{D}_j P) + \frac{2}{12} \mathcal{D}_k P^2 + i K_{ij} P_k - i P^2 K_{ij} P_k P_i.
\end{align*}
\]

(4.47) (4.48)

In this section we want to analyze to what extent these equations are independent of each other, or rather if one of them is implied by the remaining ones. The most promising starting point is undoubtedly to differentiate \((4.47)\). With \((4.48)\) we obtain

\[
\mathcal{D}_k E_{ij} + \frac{1}{4} qP^2 - \frac{5}{4} qP_i P_j \mathcal{D}_k \log P^2 - 2 qP_i \mathcal{D}_j \log P^2 + \frac{1}{4} qP^2 + \Lambda q h_{ij} \mathcal{D}_k P^2 = 0.
\]

(4.51)

To extract one of the above equations, though, we need to eliminate the derivative of \(E_{ij}\). For this purpose, recall that the vacuum constraints impose restrictions \((3.10)\) on \(E_{ij}\). Taking also \((4.47)\) into account that yields an equation of a form we are looking for,

\[
\mathcal{D}_j E_{ij} = i q \mathcal{D}_k K_{ij} P_k P_i.
\]

(4.52)

On the other hand, applying \(h^{jk}\) to \((4.51)\) yields

\[
\mathcal{D}_j E_{ij} - 2 qP^j \mathcal{D}_i P_j - qP_i \mathcal{D}_j P^j + \frac{1}{4} qP^2 - \frac{5}{4} qP_i P_j \mathcal{D}_j \log P^2 + \frac{3}{4} qP^2 \mathcal{D}_j P^2 = 0.
\]

(4.53)

Combined we obtain (recall \((4.10)\) and note that \(q \neq 0\) in our current setting of Proposition 4.3)

\[
2 P^j \mathcal{D}_i P_j + P_i \mathcal{D}_j P^j - \frac{1}{3} \sigma (qP^2 - 5\Lambda) P_i - \frac{3}{4} qP^2 \mathcal{D}_j P^2 - i \mathcal{D}_j K_{ij} P_k P_i = 0.
\]

(4.54)

We contract this equation with \(P^i\) to recover \((4.17)\) as the trace of \((4.49)\),

\[
\mathcal{D}_j P^j = -\Lambda \sigma.
\]

(4.55)
We insert this into (4.54) to recover (4.16) as the contraction of (4.49) with $P^j$,

$$
P^j \mathcal{D}(\iota(P_j)) = \frac{1}{6} \sigma(qP^2 - 2\Lambda)P_i + \frac{3}{8} \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \mathcal{D}_i P^2 + \frac{i}{2} \mathcal{D}_i k^j P_k P_l . \quad (4.56)
$$

Let us rewrite (4.56). With (4.10) and the identity $P^j \mathcal{D}(\iota(P_j)) \equiv \frac{1}{2} \mathcal{D}_i P^2 - \frac{1}{2} \mathcal{D}_i j^k P_j Q_k$ we obtain, after contraction with $\mathcal{D}_i$,

$$
2P_{[i} Q_{j]} = \frac{1}{3} \sigma(qP^2 - 2\Lambda)\mathcal{D}_i k^k P_k + \frac{1}{4} \frac{qP^2 - \frac{\Lambda}{2}}{qP^2 - \frac{\Lambda}{2}} \mathcal{D}_i k^k P^2 - 2i(\mathcal{D}_i k^k P_j P_k) . \quad (4.57)
$$

If we contract this equation with $P^i$ we recover (4.50).

By way of summary, the vacuum constraints, (4.47)-(4.48) imply (4.50) and certain components of (4.49), namely its trace (4.55) and its contraction with $P^j$ (4.50).

Let us bring (4.49) in a form which takes care of the fact that some of its components are redundant. For this purpose, set

$$
\mathcal{A}_{ij} := P^2 \mathcal{D}(\iota(P_j)) , \quad (4.58)
$$

and note that (4.55) and (4.10) imply

$$
\text{tr}(\mathcal{A}) = -\Lambda \mathcal{A} P^2 = -\frac{3}{2} \Lambda \left(qP^2 - \frac{\Lambda}{2}\right)^{-1} p^k p^i \mathcal{A}_{ki} . \quad (4.59)
$$

On the other hand, we use (4.10) to write (4.56) as

$$
qP^i \mathcal{D}_i k^k P_k Q_i = \frac{2}{3} \sigma(qP^2 - 2\Lambda)\left(p^k A_{ik} - p^k p^i \mathcal{A}_{kl} P_l\right) - \frac{4}{3} i \left(qP^2 - \frac{\Lambda}{2}\right) p^2 \mathcal{D}_i k^l P_i P_k . \quad (4.60)
$$

Finally, we employ (4.59) and (4.60) to rewrite (4.49),

$$
\mathcal{A}_{ij} = \frac{3}{2} \frac{q}{qP^2 - \frac{\Lambda}{2}} p^i (p^k A_{jk}) - \frac{1}{2} \frac{qP^2 + \Lambda}{qP^2 - \frac{\Lambda}{2}} p^{-2} p^k p^i \mathcal{A}_{kl} h_{ij} + \frac{1}{4} \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \mathcal{D}_i k^k P_k Q_i + iP^2 \mathcal{D}_i k^k P_k Q_i - \frac{4}{3} \left(qP^2 - \frac{\Lambda}{2}\right) p^2 \mathcal{D}_i k^l P_i P_k + \frac{3}{2} \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \mathcal{D}_i k^k P_k Q_i + \frac{1}{2} \frac{qP^2 + \Lambda}{qP^2 - \frac{\Lambda}{2}} p^{-2} p^k p^l \mathcal{A}_{kl} \left(p^{-2} P_j P_j + h_{ij}\right) + \frac{4}{3} \left(qP^2 - \frac{\Lambda}{2}\right) p^2 \mathcal{D}_i k^l P_i P_k + \frac{3}{2} \frac{qP^2}{qP^2 - \frac{\Lambda}{2}} \mathcal{D}_i k^k P_k Q_i + \frac{1}{2} \frac{qP^2 + \Lambda}{qP^2 - \frac{\Lambda}{2}} p^{-2} p^k p^l \mathcal{A}_{kl} \left(p^{-2} P_j P_j + h_{ij}\right) + \frac{4}{3} \left(qP^2 - \frac{\Lambda}{2}\right) p^2 \mathcal{D}_i k^l P_i P_k . \quad (4.61)
$$

Conversely, this equation implies (4.49) when using (4.55) and (4.56) (which in turn follow from (4.47), (4.48), and the vacuum constraints). Its trace and its contraction with $P^j$ are automatically satisfied, which reflects the fact that the same is true for the corresponding components of (4.49). Equation (4.61) therefore has two non-trivial independent components which need to be fulfilled.

One may replace (4.61) by a scalar equation. Since $(\Sigma, h_{ij})$ is a Riemannian manifold, such an equation can immediately be obtained: We write all terms on one side and compute its norm. Proceeding this way we find that (4.61) is

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equivalent to (we set $B_i := A_{ij}P^j, C := A_{ij}P^iP^j, b_i := K_{ij}P^j, \epsilon := K_{ij}P^iP^j$),

$$
0 = \mathcal{R} := \mathcal{P}^4|A|^2 - 2\mathcal{P}^2|B|^2 + \frac{1}{2}|C|^2 + \mathcal{P}^2\text{tr}A\left(\mathcal{C} - \frac{1}{2}\mathcal{P}^2\text{tr}A\right) + \mathcal{P}^6\left(2\mathcal{P}^2|b|^2 - \mathcal{P}^4(|K|^2 + \frac{1}{2}K^2) - \frac{1}{2}|C|^2 - K\mathcal{P}^2\epsilon\right) + 2i\epsilon^{ijk}\mathcal{P}^4\mathcal{P}_k\left(\mathcal{B}_i\mathcal{b}_j - \mathcal{P}^2A_{il}K_{jl}\right). \tag{4.62}
$$

### 4.4 Cauchy data for vacuum space-times with vanishing MST

It follows from Lemma 3.1 that the conditions (4.47)-(4.48), needed to apply Proposition 4.3, can be replaced by the condition

$$
\mathcal{E}_{ij} - q_c(\mathcal{P}_i\mathcal{P}_j) = 0, \tag{4.63}
$$

where

$$
q_c := Q_c|_{\Sigma} = \pm \tilde{\kappa}^{-2}(\mathcal{E}^2)^{5/6}\left(\pm \sqrt{\mathcal{E}^2} - \sqrt{\frac{2}{3}\Lambda}\right)^{-2}, \quad \tilde{\kappa} \in \mathbb{C} \setminus \{0\}. \tag{4.64}
$$

One checks that $\tilde{\kappa} \mapsto \lambda \tilde{\kappa}, \lambda \in \mathbb{C}$ implies $(\sigma, Y^i) \mapsto (\lambda\sigma, \lambda Y^i)$. The constant $\tilde{\kappa}$ therefore provides a gauge freedom which reflects the freedom to choose a scale of the KVF. It may be set equal to 1.

In this section we discuss the solvability of (4.63). Assume that (3.59) holds. We deduce from (3.42),

$$
\mathcal{P}^2 = -\frac{1}{4}\mathcal{F}^2|_{\Sigma} = -\sqrt{\frac{3}{2}}\tilde{\kappa}^2(\mathcal{E}^2)^{-1/3}\left(\pm \sqrt{\mathcal{E}^2} - \sqrt{\frac{2}{3}\Lambda}\right)^2. \tag{4.65}
$$

Let us analyze the vanishing of the MST $S^{(C)}_{\alpha\beta\mu\nu}$ on $\Sigma$.

$$
S^{(C)}_{\alpha\beta\mu\nu}|_{\Sigma} = 0 \iff S^{(C)}_{ij}|_{\Sigma} = 0 \tag{4.66}
$$

$$
\iff (\mathcal{P}_i\mathcal{P}_j) = \mp \tilde{\kappa}^2\mathcal{E}_{ij} - \mathcal{E}_{ij} = \pm \sqrt{\mathcal{E}^2} - \sqrt{\frac{2}{3}\Lambda} \mathcal{E}_{ij} \tag{4.67}
$$

$$
\iff \mathcal{P}_i\mathcal{P}_j = \mp \tilde{\kappa}^2(\mathcal{E}^2)^{-5/6}\left(\pm \sqrt{\mathcal{E}^2} - \sqrt{\frac{2}{3}\Lambda}\right)^2 \mathcal{E}_{ij} = \mp \sqrt{\frac{2}{3}\Lambda} \mathcal{E}_{ij} = 0. \tag{4.68}
$$

Conditions which characterize solvability and uniqueness of equations of a form such as in (4.68), regarded as equations for $\mathcal{P}_i$, and different approaches to construct solutions thereof are discussed in paper I [14]. Let us summarize the results:

(i) A solution exists for at most one choice of $\pm$. It is then uniquely determined up to a sign.

(ii) A solution exists if and only if

$$
\mathcal{E}_{ik}\mathcal{E}^k_j = \sqrt{\frac{\mathcal{E}^2}{6}}\mathcal{E}_{ij} - \frac{\mathcal{E}^2}{3} h_{ij} = 0. \tag{4.69}
$$

Again, this happens at most for either $+$ or $-$.  

---

4 We note that (4.65) and (4.68) imply $\mathcal{P}^k\mathcal{P}_{li}\mathcal{E}^l_{ij} = \pm \sqrt{\mathcal{E}^2}$, and observe that (4.67) is equivalent (3.21), as one should expect from Lemma 3.3.
(iii) Let \( W^i \) be any vector with \(|(E_{ij} \pm \sqrt{\frac{e^2}{6}} h_{ij}) W^j|^2 = 1 \) (its existence is ensured by \( E^2 \neq 0 \)). Then

\[
P_i = \frac{3}{2} \, \hat{\sigma} (E^2)^{-1/6} \left( \pm \sqrt{E^2} - \sqrt{\frac{2}{3} \Lambda} \right) (E_{kl} W^k \pm \sqrt{\frac{e^2}{6}} W^k) \mathcal{D}_j \log E^2
\]

solves \((4.68)\), supposing that a solution exists, i.e. supposing that \((4.69)\) holds.

Because of \((4.69)-(4.70)\) we have \((4.65)\) and

\[
Q_i = \frac{3}{2} \, \hat{\sigma} (E^2)^{-1/6} \left[ \left( \pm \sqrt{E^2} - \sqrt{\frac{2}{3} \Lambda} \right) (E_{kl} W^k \pm \sqrt{\frac{e^2}{6}} W^k) \mathcal{D}_j \log E^2 + \left( \pm \sqrt{E^2} - \sqrt{\frac{2}{3} \Lambda} \right) \mathcal{D}_j \sigma \right].
\]

We employ \((4.28)\) and \((4.70)-(4.71)\) to express \( \sigma \) and \( Y^i \) (given by \((4.10)\) and \((4.14)\)) in terms of the Cauchy data and \( W^i \),

\[
\sigma = \frac{1}{\sqrt{6}} \left( \pm \sqrt{E^2} - \sqrt{\frac{2}{3} \Lambda} \right)^{-1} \mathcal{P}^k \mathcal{D}_k \log E^2
\]

and

\[
Y_i = \left( \frac{3}{2} \hat{\sigma} (E^2)^{-1/6} \left[ - \hat{\sigma} + \frac{1}{6} \hat{\sigma} \mathcal{D}_j \log E^2 + i K_k \delta^k \right] \right. \times \left( E_{kl} W^k \pm \sqrt{\frac{e^2}{6}} W^k \right),
\]

Remark 4.4: As for \((4.61)\), since \((\Sigma, h_{ij})\) is a Riemannian manifold, \((4.69)\) can be replaced by the equation which requires the vanishing of the norm of its left-hand side,

\[
\varphi := \text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}} \cdot \hat{\mathcal{E}}) \mp \sqrt{\frac{2}{3}} \sqrt{\text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}})} \text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}} \cdot \hat{\mathcal{E}}) - \frac{1}{6} (\text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}}))^2 = 0.
\]

We have proven the first main result, which provides an algorithmic characterization of Cauchy data which generate vacuum space-times with vanishing MST (cf. [14, Theorem 4.8] for the space-time pendant). It brings together all the results of the previous sections.

Theorem 4.5: Consider Cauchy data \((\Sigma, h_{ij}, K_{ij})\) which solve the vacuum constraint equations and satisfy (cf. footnote 3)

\[
\text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}}) \neq 0, \quad \text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}}) - \frac{2}{3} \Lambda^2 \neq 0, \quad \text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}}) - \frac{1}{6} \Lambda^2 \neq 0, \quad \text{tr}(\hat{\mathcal{E}} \cdot \hat{\mathcal{E}}) - \frac{8}{3} \Lambda^2 \neq 0,
\]

(4.77)
where
\[
\mathcal{E}_{ij} := \mathcal{R}_{ij} + KK_{ij} - K_{ik}K_j^k - \frac{2}{3}\lambda h_{ij} - i\epsilon_{ikl}h_{lj}K_{ij}.
\]

Moreover, let \(W^i\) be any vector with
\[
|\left(\mathcal{E}_{ij} \pm \sqrt{\mathcal{E}^2} h_{ij}\right)W^j|^2 = 1
\]
(which exists), and set
\[
\mathcal{P}_i := i\left(\frac{3}{2}\right)^{1/4} \bar{z} (\mathcal{E}^2)^{-1/6} \left(\pm \sqrt{\mathcal{E}^2} - \sqrt{\frac{2}{3}} \Lambda \right) \left(\mathcal{E}_{ij} \pm \sqrt{\mathcal{E}^2} h_{ij}\right)W^j
\]
(4.78)
(the “right” signs are determined by condition (i) below). Then the emerging \(\Lambda\)-vacuum space-time admits a non-trivial (possibly complex) KVF such that the associated MST vanishes (at least in some neighborhood of \(\Sigma\)) if and only if
\[
(4.62) \quad \text{and} \quad (4.76)
\]
hold, i.e.

(i) \(\delta_1(\mathcal{E}, h) \equiv \text{tr}(\mathcal{E} \cdot \mathcal{E} \cdot \mathcal{E} \cdot \mathcal{E}) \mp \frac{2}{\sqrt{3}} \text{tr}(\mathcal{E} \cdot \mathcal{E}) \text{tr}(\mathcal{E} \cdot \mathcal{E} \cdot \mathcal{E}) - \frac{1}{6} [\text{tr}(\mathcal{E} \cdot \mathcal{E})]^2 = 0\), and

(ii) \(\mathcal{R}(\mathcal{E}, \mathcal{D_E}, h, K) = 0\).

In that case the Cauchy data are complemented to Killing initial data via (4.73) and (4.75).

Remark 4.6 (i) and (ii) may be replaced by their tensor-equivalents (4.69) and (4.61). Alternatively, they may be combined into one single scalar equation,
\[
\mathcal{L} := \delta_1^2 + \mathcal{R}^2 = 0
\]
(4.79)
which depends only on \(h_{ij}, K_{ij}\) and derivatives thereof.

Corollary 4.7 The function \(\mathcal{L}\) provides, under the hypotheses (4.77), a measure for the deviation of \(\Lambda\)-vacuum initial data \((\Sigma, h_{ij}, K_{ij})\) to initial data which admit a (possibly complex) KVF whose associated MST vanishes.

Remark 4.8 The conditions (4.77) only make sure that the evolution equations for the MST are regular near \(\Sigma\), whence the vanishing of the MST can merely be concluded in a corresponding neighborhood. If the KIDs are real, though, the KVF will be real as well, and the KIDs will generate one of the vacuum space-times contained in the class of space-times described in [12]. All these space-times have the property that the MST vanishes everywhere, whence we conclude that the MST actually vanishes in the whole domain of dependence of \(\Sigma\), and not just in some neighborhood of \(\Sigma\). In fact, one should expect that the same is true for MSTs associated to complex KVF's.

5 Algorithmic characterization of Cauchy data for the Kerr-NUT-(A)dS family

5.1 Vanishing of the MST associated to real KIDs and the Kerr-NUT-(A)dS family

A necessary condition for Cauchy data \((\Sigma, h_{ij}, K_{ij})\) to generate a member of the Kerr-NUT-(A)dS family is that the MST vanishes w.r.t. a KVF which is real. This will be the case whenever there exists a choice \(\bar{z} \in \mathbb{C} \setminus \{0\}\) for which the
Killing initial data $\sigma$ and $Y^i$, as given by (4.73) and (4.75), are real. In that case there only remains the freedom to multiply $\tilde{z}$ with real constants $\lambda \in \mathbb{R} \setminus \{0\}$.

Once it is known that the initial data set $(\Sigma, h_{ij}, K_{ij})$ leads to a $\Lambda$-vacuum space-time which admits a real KVF w.r.t. which the MST vanishes, the characterization result in [12] (which we have recalled in paper I [14]) can be consulted to check whether the emerging space-time belongs to the Kerr-NUT-(A)dS family. Moreover, that result can be used to compute the Kerr-NUT-(A)dS parameters $m$, $a$ and $\ell$ from $h_{ij}$ and $K_{ij}$, so that one gains insight which member of the Kerr-NUT-(A)dS family is generated by $(\Sigma, h_{ij}, K_{ij})$. For this we need to determine the constants $b_1$, $b_2$, $c$ and $k$ (2.20)-(2.22).

Using (4.28), (4.64), and (4.65) we obtain (cf. [14], but note that $\tilde{z}$ differs from the one used there)

$$b_1 = 18 \left(\frac{2}{3}\right)^{1/4} \text{Im}(\tilde{z}^2),$$

$$b_2 = -18 \left(\frac{2}{3}\right)^{1/4} \text{Re}(\tilde{z}^3),$$

$$c = \sigma^2 - |Y|^2 + 6\text{Re}(\tilde{z}^2(\xi^2)^{1/6}) - \sqrt{6} \Lambda \text{Re}(\tilde{z}^2(\xi^2)^{-1/3}),$$

$$k = 9 \left(\frac{2}{3}\right)^{1/2} |\tilde{z}^2(\xi^2)^{-1/3}| \left(\mathcal{Q}_i \mathcal{Z}_j - (\nabla_0 \mathcal{Z})^2 \right) - b_2 Z + c Z^2 + \frac{\Lambda}{3} Z^4,$$

where

$$Z|_{\Sigma} = \left(\frac{2}{3}\right)^{1/4} \text{Re}(\tilde{z}(\xi^2)^{-1/6}),$$

$$\nabla_0 Z|_{\Sigma} \stackrel{(14)}{=} \left(\frac{2}{3}\right)^{1/4} \text{Re} \left(\frac{\sqrt{\xi^2} (Q^2 + 2\Lambda)}{2\xi^2} \right)|_{\Sigma}$$

and

$$\nabla_0 Z|_{\Sigma} \stackrel{(4.19)}{=} \left(\frac{2}{3}\right)^{1/4} \text{Re} \left(\tilde{z}^{-1} \frac{(\xi^2)^{1/6}}{\pm \sqrt{6} \Lambda - \sqrt{\frac{4\Lambda}{3}}} Y^i \mathcal{P}_i \right).$$

**Remark 5.1** The 6th root $(\xi^2)^{1/6}$ is determined by the requirement that the Killing initial data $(\sigma, Y)$ need to be real.

Since it is of particular physical interest and somewhat easier to analyze, we devote ourselves henceforth to Kerr-(A)dS family.

### 5.2 Kerr-(A)dS family

To end up with an algorithmic local characterization result for the Kerr-(A)dS metrics in terms of Cauchy data we will employ the space-time characterization Theorem 2.4. We assume that we have been given Cauchy data $(\Sigma, h_{ij}, K_{ij})$ which fulfill all hypotheses of Theorem 4.5. In particular (4.77) implies that $Q F^2$ and $Q F^2 - 4\Lambda$ are not identically zero, as required by Theorem 2.4. Then we supplement the data via (4.10) and (4.14) to Killing initial data $(\Sigma, h_{ij}, K_{ij}, \sigma, Y^i)$, where we assume that there exists a choice of $\tilde{z} \in \mathbb{C} \setminus \{0\}$ for which $\sigma$ and $Y^i$ are real (in other words we require $(\sigma, Y^i)$ to be real up to some multiplicative complex constant).

According to Theorem 2.4 a necessary condition for the Cauchy development of $(\Sigma, h_{ij}, K_{ij})$ to be locally isometric to a Kerr-(A)dS space-time is $b_2 = 0,
equivalently \( \text{Re}(\tilde{\kappa}^3) = 0 \). There remains a gauge freedom concerning the choice of the complex constant \( \tilde{\kappa} \), namely to prescribe its length. It arises from the freedom to rescale the KVF. One may therefore impose the gauge condition

\[
\tilde{\kappa} = i.
\]  
(5.8)

Let us analyze the validity of (4.77) in the KdS-case. In [14] it has been shown that the Kerr-(A)dS family satisfies

\[
C^2 = \frac{96m^2}{(r + ia \cos \theta)^6},
\]  
(5.9)

and as in [14] we define \( \sqrt{-} \) in such a way that

\[
\sqrt{C^2} = \frac{\sqrt{96m}}{(r + ia \cos \theta)^3}
\]  
(5.10)

then (4.76) holds with “−”). For \( m \neq 0 \) we thus have \( \mathcal{E}^2 \neq 0 \) on any Cauchy surface \( \Sigma \). In particular (4.77) holds everywhere in the \( \Lambda \neq 0 \)-case. Moreover, observe that that \( \text{grad}(\text{Re}(C^2^{-1/6})) \) is nowhere vanishing.

So let us consider the case \( \Lambda \neq 0 \) (and \( m \neq 0 \)). It has been shown in [14] that

\[
\sqrt{C^2} \neq \sqrt{\frac{2}{3}} \Lambda \text{ holds if and only if }
\]  

for \( a = 0 \): \( r = (3m\Lambda^{-1})^{1/3} \),

(5.11)

for \( a > 0 \): \( \theta = \pi/2 \) and \( r = (3m\Lambda^{-1})^{1/3} \)

or

\[
\cos \theta = \pm \left(\frac{9}{8} \sqrt{3} ma^{-3} \Lambda^{-1}\right)^{1/3} \text{ and } r = \mp \frac{a}{\sqrt{3}} \cos \theta.
\]  
(5.13)

Clearly, the solution (5.13) exists only for \( \frac{9}{8} \sqrt{3} ma^{-3} \Lambda^{-1} \leq 1 \). Moreover, \( \sqrt{C^2} \neq -\sqrt{\frac{2}{3}} \Lambda \) is equivalent to [14]

for \( a = 0 \): \( r = -(6m\Lambda^{-1})^{1/3} \),

(5.14)

for \( a > 0 \): \( \theta = \pi/2 \) and \( r = -(6m\Lambda^{-1})^{1/3} \)

or

\[
\cos \theta = \pm \left(\frac{9}{4} \sqrt{3} ma^{-3} \Lambda^{-1}\right)^{1/3} \text{ and } r = \pm \frac{a}{\sqrt{3}} \cos \theta ,
\]  
(5.15)

where (5.16) exists only for \( \frac{3}{2} \sqrt{3} ma^{-3} \Lambda^{-1} \leq 1 \). It remains to consider the last condition in (4.77):

\[
\sqrt{C^2} = -\sqrt{\frac{128}{3}} \Lambda \iff \frac{3}{2} m\Lambda^{-1} = -(r + ia \cos \theta)^3,
\]  
(5.17)

happens if and only if

for \( a = 0 \): \( r = -\left(\frac{3}{2} m\Lambda^{-1}\right)^{1/3} \),

(5.18)

for \( a > 0 \): \( \theta = \pi/2 \) and \( r = -\left(\frac{3}{2} m\Lambda^{-1}\right)^{1/3} \)

or

\[
\cos \theta = \pm \left(\frac{9}{16} \sqrt{3} ma^{-3} \Lambda^{-1}\right)^{1/3} \text{ and } r = \pm \frac{a}{\sqrt{3}} \cos \theta .
\]  
(5.20)
The solution (5.20) exists only for \( \frac{a}{2\pi} \sqrt{3} \ma^{-3} \Lambda^{-1} \leq 1 \).

To sum it up, for \( \Lambda = 0 \), the conditions (4.77) are satisfied everywhere by the Kerr family. For \( \Lambda \neq 0 \) and \( a \neq 0 \) some of the conditions in (4.77) are violated on certain \( \{r, \theta = \text{const.}\}-2\)-surfaces, in the Schwarzschild-(\( \Lambda \))de Sitter case on certain \( \{r = \text{const.}\}\)-hypersurfaces. For the latter ones,

\[ g(\partial_r, \partial_r) = \left(1 - \frac{2m}{r} - \frac{\Lambda}{r^2}\right)^{-1}, \tag{5.21} \]

so for appropriate choices of \( \Lambda \) and \( m \) these surfaces will be spacelike. Our results do not apply on these 3-surfaces.

Finally, we state our second main result:

**Theorem 5.2** Consider Cauchy data \((\Sigma, h_{ij}, K_{ij})\) which solve the vacuum constraint equations and satisfy (cf. footnote 3)

\[
\begin{align*}
\text{tr}(E \cdot E) &\neq 0, \quad \text{tr}(E \cdot E) - \frac{2}{3} \Lambda^2 \neq 0, \quad \text{tr}(E \cdot E) - \frac{1}{6} \Lambda^2 \neq 0, \quad \text{tr}(E \cdot E) - \frac{8}{3} \Lambda^2 \neq 0, \\
\end{align*}
\]

where

\[ E_{ij} := \mathring{R}_{ij} + KK_{ij} - \Lambda h_{ij} - i\mathring{e}_{ij} \mathcal{D} K_{ij}, \]

and for which \( \text{Im} \left( \sqrt{\frac{E^2}{Q^2 - 4\Lambda}} \right) \) has non-zero gradient somewhere.

Then the emerging \( \Lambda \)-vacuum space-time is locally isometric to a member of the Kerr-(\( \Lambda \))dS family if and only if (i)-(iv) hold:

(i) \( \text{tr}(E \cdot E \cdot E \cdot E) \mp \sqrt{\frac{2}{3} \text{tr}(E \cdot E) \text{tr}(E \cdot E \cdot E) - \frac{1}{6} [\text{tr}(E \cdot E)]^2} = 0. \)

Let \( W^i \) be any vector field with \( |(E_{ij} \pm \sqrt{\frac{E^2}{6}} h_{ij}) W^j|^2 = 1 \) (which exists because \( \text{tr}(E \cdot E) \neq 0 \)); then, set

\[ P_i = \mathcal{P}_i = -(\frac{3}{2})^{1/4} (E^2)^{-1/6} \left( \pm \sqrt{E^2 - \frac{2}{3} \Lambda} \right) (E_{ij} \pm \sqrt{\frac{E^2}{6}} h_{ij}) W^j, \]

(the signs are determined by (i)).

(ii) \( \mathfrak{R} = 0 \) (the scalar \( \mathfrak{R} \) has been defined in (4.62) in terms of the Cauchy data and \( \mathcal{P} \)).

(iii) The fields \( \sigma \) and \( Y \) are real, where

\[
\begin{align*}
\sigma &= \frac{1}{2} (\frac{2}{3})^{1/4} (E^2)^{-1/6} (E^k W_l \pm \sqrt{E^2/6} W_k) \mathcal{D}_k \log E^2, \\
Y^i &= i (\frac{3}{2})^{3/4} (E^2)^{-1/6} \left[ -\mathring{e}^{ijk} \mathcal{D}_j + \frac{1}{6} \mathring{e}^{ijk} \mathcal{D}_j \log E^2 + i K^{ik} - i K h^{ik} \right] \\
&\quad \times \left( E_k^l W_l \pm \sqrt{E^2/6} W_k \right). 
\end{align*}
\]

(iv) \( \text{grad} (\text{Re}[(E^2)^{-1/6}]) \) is not identically zero, and

(v) the constants \( c \) and \( k \), given by (5.3)-(5.4) (with (5.8)), satisfy, depending on the sign of the cosmological constant, (2.27)-(2.29), respectively.
If (i)-(iv) are fulfilled, the $K(A)dS$-space-time generated by $(\Sigma, h_{ij}, K_{ij})$ has parameters

$$m = 9 \left( \frac{2}{3} \right)^{1/4} \left( \frac{A}{3} \zeta_i^2 + c \right)^{-3/2}, \quad a = \zeta_i \left( \frac{A}{3} \zeta_i^2 + c \right)^{-1/2},$$  \hspace{1cm} (5.22)

where $\zeta_i$ is given by (2.24)-(2.26). The KVF whose associated MST vanishes (in Boyer-Lindquist-type coordinates this is a multiple of the $\partial_t$-KVF restricted to $\Sigma$) is then generated by $(\sigma, Y)$.

An issue of interest would be to do an analog analysis for the characteristic initial value problem.

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