On the dynamics of mechanical failures in magnetized neutron-star crusts

Yuri Levin¹,² and Maxim Lyutikov³

¹ Monash Center for Astrophysics, Monash University, Clayton, VIC 3800, Australia
² Leiden Observatory, Leiden University, Niels Bohrweg 2, Leiden, the Netherlands
³ Department of Physics, Purdue University, 525 Northwestern Avenue, West Lafayette, IN, USA

yuri.levin@monash.edu.au, lyutikov@purdue.edu

ABSTRACT

We consider the dynamics of a mechanical failure induced by a shear stress in a strongly magnetized neutron-star crust. We show that even if the elastic properties of the crust allow the creation of a shear crack, the strongly sheared magnetic field around the crack leads to a back-reaction from the Lorentz force which does not allow large relative displacement of the crack surfaces. Instead, the global evolution of the crack proceeds on a slow resistive timescale, and is unable to release any substantial mechanical energy. Our calculations demonstrate that for some magnetic-field configurations, the magnetic forces cause, effectively, a plastic deformation of the crust when the resulting elastic shear stress exceeds the critical value for mechanical failure.

Subject headings: Neutron stars

1. Introduction

In two common astrophysical circumstances, a crust of a neutron star may be placed under considerable stress. Firstly, fast-spinning young pulsars, which are reducing their spin frequencies on a short (∼1000 years) timescale, change the shape of their rotational bulges which deforms their crusts. Secondly, in magnetars a ∼10¹⁵ G magnetic field exerts a strong twisting force on their crusts. In both cases the crust yields, and it is reasonable to assume that some of the yield proceeds via explosive ruptures which release the crustal strain and produce starquakes, in analogy with a similar process in the Earth’s crust.

Neutron-star quakes have attracted significant attention in the neutron-star astrophysics folklore. Firstly, it has been proposed that the starquakes in the spinning-down pulsar may help trigger the sudden spin-frequency increases known as glitches (see, e.g., Link & Epstein 1996). Secondly, it has been argued (Thompson & Duncan 1995, hereafter TD95, see also Blaes et al. 1989) that starquakes may be responsible for powering flares in Soft-Gamma Repeaters, a sub-class of magnetars. It is mainly this latter suggestion that motivates the present study. Briefly, the physical picture is as follows:

Magnetar emission (see Wood & Thompson 2004 for a review) is powered by dissipation of a non-potential (current-carrying) magnetic field (TD95, Thompson, Lyutikov, & Kulkarni 2002). The field exerts Lorentz force on the crust, which is balanced by induced elastic stress. For strong enough magnetic fields, Lorentz force may induce a stress that exceeds the critical stress of the lattice. This leads to breaking of the crust and a release of the seismic waves. The seismic waves are coupled to the magnetosphere via strong magnetic field and, after converting into the Alfven waves, deposit their energy in a fireball above the neutron-star surface.

TD95 did not specify how exactly the cracking would proceed; instead, they assumed that the mechanical failure would occur within some finite volume and that part of the magnetic/elastic energy contained within this volume would be converted into seismic waves (see discussion in their section 2.2). Molecular-dynamical simulations (Horowitz & Kadau 2009, hereafter HK09, Chugunov & Horowitz 2010, hereafter CH10) show that indeed the crust breaks down suddenly when it is shear-stressed above some critical level.

A viable alternative theory for the origin of the SGR flares has also been proposed (Lyutikov 2003, 2006), This theory does not invoke the starquakes as the flares’ origin. Instead, it proposes that slow untwisting of the in-
ternal magnetic field leads to gradual twisting of magnetospheric field lines, on time scales much longer than the flare duration. In this picture the magnetospheric field evolves slowly, perhaps due to the Hall drift (Goldreich & Reisenegger 1992, Pons & Geppert 2007) or plastic flow (Jones 2003). Eventually, the magnetosphere reaches a dynamical stability threshold due to increasing energy associated with current-carrying magnetic field. Then sudden relaxation of the twist outside the star and associated dissipation and magnetic topology change lead to flares, in analogy with Solar flares and Coronal Mass Ejections (CMEs). In this scenario, the flare itself is produced by a rapid reconfiguration of the magnetospheric field, which is assisted by a fast reconnection process, much like what happens in the better-studied case of a solar flare. Levin & van Hoven (2011) have shown that the external magnetic-field reconfiguration during a giant flare would excite the large-amplitude torsional oscillations of the NS crusts; thus in this picture a flare would precede the starquake and not the other way around.

It is therefore imperative to understand in detail the mechanics of a mechanical failure of the neutron-star crust under the strong shear stress. In terrestrial experience, the sudden mechanical failure in response to the shear stress is common and occurs via a macroscopic shear crack, i.e. the breakdown of mechanical elasticity within a thin layer of material. The crack’s rapid progress is easy to understand (e.g., Kostrov 1964): once it forms, the stress at the cracks edge is strongly enhanced which facilitates the crack’s further propagation along its plane. Often, the efficient crack formation requires the solid’s ability to form a small void at the crack’s location, in order for the slippage to be unhindered. Jones (2003) has previously argued that since in the neutron-star crust the pressure is greater than the shear modulus by two orders of magnitude, the conventional crack that relies on a formation of the void cannot occur. However, molecular-dynamical simulations of HK09 and CH10 demonstrate a sudden drop in the shear stress once the shear of the crust coulomb lattice exceeds a critical value of about 0.1. Therefore formation of the void is not necessary for the efficient slippage. The situation here is somewhat analogous to that of the deep-focus earthquakes. There it is known that the void-aided shear cracks do not form; instead it is thought that a localized shear-stress-induced phase transition reduces the stress and allows for the rapid slippage of tectonic plates (Bridgeman 1945).

Two remarks are in order. Firstly, molecular-dynamic simulations of HK09 and CH10 have not shown shear cracks. On the contrary, the mechanical failure propagated rapidly through the volume of the simulation domain. However, these simulations may not be capable of capturing macroscopic elasto-dynamical effects: the domain-size in these simulations is truly microscopic, of order of 100 inter-atomic spaces and is orders of magnitude smaller than e.g. a mean free path of a phonon. By contrast, the localization of the shear failure into a plane does not depend on the microscopic nature of the mechanical failure, but is instead a result of the redistribution of the elastic shear stresses once a localized failure is initiated.

Secondly, in the theory of mechanical failure it is conventional to distinguish between the brittle and ductile cracks (see, e.g., Ashby & Sammis 1990). The former is thought to be initiated at a single location and proceeds rapidly from the start. The latter begins with a series of unrelated micro-cracks and initially proceeds slowly. The runaway process is initiated when sufficient number of micro-cracks merge to form a propagating macro-crack. Some materials under shear stress display plastic flows which may become unstable and concentrate into so-called shear-bends (see, e.g., Molinari 1997). All of these processes have an important feature in common: the mechanical failure and the rapid shearing motion that follows is concentrated into a narrow layer. This feature plays the central role in our paper.

In this paper we show that the dynamics of a thin shear crack would be strongly affected by the magnetic tension inside the crust of a strongly magnetized neutron star. In particular we demonstrate that the energy release from a thin crack would be strongly suppressed, since the magnetic field provides mechanical connection between the two slipping sides of the crack and rapidly suppresses the slippage. The energy released from a thin crack falls short by several orders of magnitude to power SGR flares. The strong influence of the magnetic field on the rupture dynamics has no counterpart in the Earth crust, where magnetic field is relatively small and its coupling to the mechanical motion is weak due to the crust’s small conductivity. We show that for some magnetic-field configuration the crust response to the magnetically-induced shear stress results in what is effectively a plastic deformation rather than sudden rupture, and we construct an explicit model of such deformation.

The plan for this paper is as follows. In sections 2 we describe our dynamical solutions for the magnetized thin crack, and show how magnetic field suppresses seismic en-

1A useful terrestrial analogy is the brittle concrete reinforced with metal rods against fracture. Magnetic fields play a role of the metal rods.
energy release. In section 3 we discuss an explicit example with effectively plastic deformation under the action of the magnetar-strength magnetic field. In section 4 we conclude.

2. Dynamics of a thin crack

In order to estimate the maximal amount of energy released by a fracture, we assume that it appears instantly as a planar slab of infinitesimally small thickness and an infinite lateral extent, and that inside this slab the shear modulus and viscosity suddenly become zero. We build the simplest-possible model which faithfully represents the essential physical aspects of the system. The space is assumed to be filled with the homogeneous material of shear modulus \( \xi \), and the material is sheared in the \( x \) direction; the displacement depends on \( z \) only and coordinate \( y \) is redundant (i.e., the problem is 2-d; cf. e.g. Kostrov 1966). The crack appears at \( t = 0 \) along the plane \( z = 0 \), thus splitting elastic medium into lower and upper half-spaces. The vertical magnetic field component \( B_z \) is taken to be homogeneous, the horizontal magnetic field component \( B_x(z) \) is \( x \)-independent, and \( B_y = 0 \). The system is assumed to be initially in equilibrium, with

\[
\frac{\partial(T_{xx}^\text{el} + T_{xz}^\text{mag})}{\partial z} = 0
\]

where \( T_{xx}^\text{el} \) and \( T_{xz}^\text{mag} \) are the elastic and magnetic stress tensors, respectively. The above condition can be re-written as

\[
\mathbf{B} \times \mathbf{J} + T_{ik,k}^\text{el} = 0
\]

where

\[
T_{ik}^\text{el} = 2\mu\xi_{ik}, \quad \xi_{ik} = \frac{1}{2}(\xi_{i,k} + \xi_{k,i})
\]

The magnetic field, \( \mathbf{J} \) is the current density, \( \mu \) is the shear modulus (assumed spatially constant), \( \xi_{ik} \) is strain tensor, and \( \xi_k \) is the displacement vector. In the incompressible medium \( \xi_{kk} = 0 \), so that \( \sigma_{ik,k} = \mu\Delta\xi_k \). The static magnetic field can be expressed in terms of flux function \( P(x) \)

\[
B_x = B_z \partial_z P(z).
\]

The mechanical equilibrium in the \( x \) direction becomes

\[
\frac{B_z^2}{4\pi} P'' = \mu \xi''
\]

(primers denote differentiation with respect to \( z \)). Thus, the initial displacement is given by

\[
\xi_0 = \frac{B_z^2}{4\pi\mu} P + k_1 + k_2 \ast z,
\]

where \( k_1 \) and \( k_2 \) are constants.

The quantities \( T_{xz}^\text{el} \) and \( T_{xz}^\text{mag} \) represent the flow of the \( x \) component of momentum density in the \( z \) direction [see, e.g., Landau & Lifshitz 1956], and thus determine the dynamics of the system.

When the crack appears at \( t = 0 \) along the plane \( z = 0 \), thus splitting elastic medium into lower and upper half-spaces, it hinders the momentum flow since \( T_{xz}^\text{el} = 0 \) in the crack\(^2\). The momenta of opposite sign accumulate at the upper and lower sides of the crack, and they move in the opposite direction. In the unmagnetized case this motion is indefinite, and all of the elastic energy of the medium can be released. However, the magnetic tension in the \( z \) direction results in a rapid change of the magnetic \( xz \)-stress within the crack and, as we show below, a rapid re-establishment of the initial momentum flow through the crack. This results in the slipping motion being stopped and the associated energy release being suppressed.

2.1. Motion generated by the crack

Let \( \xi(z,t) \) be the \( x \)-displacement from the initial equilibrium, and let \( b_x(z,t) = \delta B_x/B_z \), where \( \delta B_x \) is the change in \( B_x \) compared to the initial value. The variables are chosen so that \( \xi(z,0) = \xi_z(0) = b_x(z,0) = 0 \). The force balance in the \( x \) direction and the induction equations become

\[
\ddot{\xi} = c_A^2 b_x' + c_{el}^2 \xi'' \quad \text{(7)}
\]

\[
b_x' = \xi' + \eta \xi'' \quad \text{(8)}
\]

Here \( \eta \) is the magnetic diffusivity, and \( c_A = B_z/\sqrt{4\pi\rho} \) and \( c_{el} = \sqrt{\mu/\rho} \) are the Alfvén and the elastic shear velocity, respectively, where \( \mu \) is the shear modulus. Our notation is \( \xi = \frac{\partial \xi}{\partial t}, \xi' = \frac{\partial \xi}{\partial z}, \) etc.

Eqs (7,8) describe the behavior of the elastic-resistive medium. Its normal modes are considered in Appendix A. There two types of modes: the slowly-damped mechanical shear waves of the elastic medium with magnetic field, and resistive diffusion-type modes. Next we demonstrate that a sudden crack in a magnetized elastic medium excites mostly the slow resistive modes, while the amplitude of the magnetized elastic modes is very small.

A convenient way to solve initial-value problem with moving boundaries is to use the Laplace transform. We use the following notation: the Laplace transform of a function

\[^2\text{This assumption is most favorable for the energy release. One could just as well assume that } T_{el} \text{ takes some lower than the initial but non-zero value}\]
function \( f(t) \) is given by
\[
\hat{f}(p) = \int_0^\infty f(t) \exp(-pt) dt.
\] (9)

The behavior of \( f(t) \) at late times \( t \gg 1 \) is determined by the behavior of \( \hat{f}(p) \) for small values of \( p \ll 1 \).

We now take Laplace transform of the system of homogeneous equations \([7]\) and \([8]\) and look for solutions of the form \( \exp[\lambda(p)z] \). We get the following general solution:
\[
\hat{\xi}(z, p) = \hat{A}(p) \exp(-\lambda_1 z) + \hat{B}(p) \exp(-\lambda_2 z),
\] (10)
where
\[
\lambda_1 = \frac{c_s^2}{c_{el}^2} \sqrt{\frac{p}{\eta}},
\] (11)
and
\[
\lambda_2 = \frac{p}{c_e t}.
\] (12)

The dispersion relation is obtained by using the smallness \( \eta p/c_{el}^2 \ll 1 \). This is an excellent approximation (see below) for the late-time behaviour of the system. The homogeneous solution with \( \lambda_2 \) corresponds to the familiar shear wave propagating under the combined action of the elastic and magnetic restoring forces, while that with \( \lambda_1 \) corresponds to the “diffusion wave”, i.e. the harmonic perturbation damped by magnetic diffusion in the \( z \)-direction.

We now match the general solution above to the appearance of a crack at \( z = 0 \), \( t = 0 \). In the upper half-plane \( z > 0 \) only the outgoing waves exist, so \( \hat{A} = \hat{B} = 0 \). The values of the coefficients \( \hat{A}(p) \) and \( \hat{B}(p) \) is obtained from two boundary conditions at the crack surface.

The first boundary condition is straightforward: the elastic stress at the boundary is zero, so the shear at the boundary has to be zero and therefore
\[
\xi'(0_+, t) = -\xi_0'(z = 0).
\] (13)

Here \( \xi_0(z) \) is displacement of the crust from the position of zero elastic stress, before the crack appears. In the case that the elastic stress balances out the magnetic one, we have
\[
\xi_0' = -\frac{B_x B_z}{4\pi \mu} = \frac{c_s^2 A B_x}{c_e^2 B_z},
\] (14)
where \( c_s = \sqrt{\mu/\rho} \) is the velocity of a purely elastic shear wave. Taking the Laplace transform of Eq. \([13]\), we obtain
\[
\hat{\xi}'(0_+, p) = -\frac{1}{p} \hat{\xi}_0'(z = 0).
\] (15)

From here onwards we shall write the shorthand of \( \xi_0 \) for \( \xi_0'(z = 0) \).

The second boundary condition is obtained by considering the singularity at the crack, \( \xi' = 2\xi(0_+)\delta(z) \). Integrating Eq. \([8]\) across the boundary, using the fact that \( b_z \) is finite, and the symmetry \( b_z'(z) = -b_z'(-z) \) we get
\[
\eta b_z'(0_+, t) + \hat{\xi}(0_+, t) = 0
\] (16)
Taking the Laplace transform of this equation, we obtain
\[
\eta \hat{b}_z'(0_+, p) + p \hat{\xi}(0_+, p) = 0.
\] (17)

From Eq. \([7]\) we get
\[
\hat{b}_z' = (1/c_A^2)[p^2 \xi - c_s^2 \xi''].
\] (18)

Substituting Eq. \([18]\) into Eq. \([17]\) we get at the boundary \( z = 0_+ \):
\[
p \left( \frac{p \eta}{c_A^2} + 1 \right) \hat{\xi} = \eta c_s^2 \xi''.
\] (19)

Substituting Eq. \([10]\) (with \( \hat{A} = \hat{B} = 0 \)) into Eq. \([19]\) we obtain (neglecting \( \eta p/c_{el}^2 \)):
\[
\hat{\xi} = \frac{c_s^2}{c_A^2} \hat{A}.
\] (20)

Substituting this and Eq. \([10]\) into the first boundary condition in Eq. \([13]\), we get
\[
\hat{B}(p) = \frac{c_s \xi_0'}{p^{3/2} (\sqrt{p} + \sqrt{p_0})},
\] (21)
where
\[
p_0 = \frac{c_s^2 c_A^2}{c_e^2 \eta}.
\] (22)

We consider two useful limiting cases of the above equation (the general solution will be obtained later in this section):

\textit{Case (i):} For unmagnetized crust, \( c_A = p_0 = 0 \), and one gets
\[
\hat{\xi}_0 = c_s \xi_0/p^2,
\]
\[
\hat{A} = 0.
\] (23)

The inverse Laplace transform of \( \hat{B}(p) \) is given by
\[
B(t) = c_s t \xi_0
\] (24)
and the corresponding displacement in the upper half-plane \( z > 0 \) is given by
\[
\xi(z, t) = c_s \xi_0 x \left( t - \frac{z}{c_s} \right) \Theta \left( t - \frac{z}{c_s} \right),
\] (25)
where $\Theta(t)$ is the Heaviside function. This solution represents the shear wave launched from the suddenly introduced unmagnetized crack. The velocity jumps discontinuously across the crack,
\[ \dot{\xi}(0+, t) - \dot{\xi}(0-, t) = 2c_{el}\dot{c}_0, \]
see Fig. 1.

![Fig. 1.— Examples of the displacement after the crack in a non-magnetized material (Left Panel) and magnetized material (Right Panel) for different times $t_0 < t_1 < t_2 < t_3$. Initial profile is $\xi_0 = \tanh z$, both shear and Alfvén velocities are unity, $\eta = 0.01$. At $t = t_0$ the crack appears at the $z = 0$ plane, launching two shear waves in the non-magnetized case and two shear and two resistive waves in the magnetized case. In the magnetized case the shear waves have very small amplitude, $\propto \sqrt{\eta}$ and do not carry large energy flux. The resistive waves are limited to a narrow region $\Delta z \sim l$.](image)

If magnetic field is present, however, this fast slippage is quickly stopped by the rapidly building magnetic tension, and all further slippage is controlled by magnetic-field diffusion. This leads us to consider another limiting case.

**Case (ii):** Of major interest for us is the solution at late times, $t \gg 1/p_0$. To these, only $p \ll p_0$ contribute substantially and in this limit we have
\[ \dot{B}(p) = \frac{c_0^2 c_{el}^2}{\sqrt{\pi} c_A c_l} p^{-3/2}, \]
and
\[ \dot{A}(p) = \frac{c_0^2 c_{el}}{c_A} \dot{B}(p) \]
Substituting this into Eq. (10), we obtain
\[ \dot{\xi}(z, p) = \dot{\xi}_B(z, p) + \xi_A(z, p), \]
where
\[ \dot{\xi}_B(z, p) = \frac{c_0^2 c_{el}}{\sqrt{\pi} c_A c_l} \frac{1}{p^{3/2}} \exp \left[ -\frac{zp}{c_l} \right], \]
and
\[ \dot{\xi}_A(z, p) = \frac{c_0^2 c_{el}}{c_A} p^{3/2} \exp \left[ -\frac{c_l}{c_A} \sqrt{\frac{p}{\eta}} \right]. \]

Taking the inverse Laplace transform, we obtain
\[ \xi_B(z, t) = \frac{c_0^2 c_{el}}{c_A} \frac{2}{\sqrt{\pi}} \eta \cdot (t - z/c_l) \Theta(t - z/c_l), \]
and
\[ \xi_A(z, t) = \frac{c_0^2 c_{el}}{c_A} \left[ \frac{\exp \left( -\frac{[z/l]^2}{\sqrt{\pi}} \right) - [z/l] \cdot \text{Erfc}(z/l)}{l} \right], \]
where
\[ l = \frac{2c_{el}}{c_l} \sqrt{\eta t} \]
\[ = \frac{0.02 c_{el}}{c_l} \left( \frac{\eta}{10^{-5}\text{cm}^2\text{s}^{-1}} \right)^{1/2} \left( \frac{t}{10^8} \right)^{1/2} \text{cm}, \]
(see Fig. 1).

The two contributions have distinct physical character. The $\xi_B(z, t)$ represents a shear wave which is launched by the slippage at $z = 0$ and propagates with the speed $c_l$. It is this contribution which is responsible for the release of seismic energy into the crust. On the other hand, the $\xi_A(z, t)$ represents the magnetic-diffusion-type process which operates locally within distance $\sim l$ given by the equation Eq. (35) from the crack. Within this thin layer the elastic stress of the crust is substantially reduced.

The surface $z = 0+$ moves according to
\[ \xi(z = 0+) = \xi_0 \frac{c_0^2 c_{el}}{c_A} \sqrt{2\eta t/\pi} \]
Thus, within a very short time,
\[ t \sim \frac{c_0^2 c_{el}^2 \eta}{2 \pi} \]
the slippage velocity falls below the elastic shear velocity: magnetic field effectively stops the relative motion of the crack’s surfaces.

For typical neutron-star crust parameters (Chamel & Haensel 2008), one evaluates
\[ p_0 = 10^{21}\text{s}^{-1} \left( \frac{c_l}{c_{el}} \right)^2 \left( \frac{c_0}{10^8\text{cm/s}} \right)^2 \beta^2, \]
where

$$\beta = c_A^2/c_D^2.$$  \hspace{1cm} (39)

For magnetar-strength fields, $\beta \sim 1$ and all times of interest satisfy $t \gg 1/p_0$. Therefore the approximations for the case considered here hold extremely well in a magnetar.

The energy flux going out in seismic waves is

$$\frac{d^2 E}{dxdydt} = \frac{1}{2} \rho \dot{\xi}_B(0, t)^2 c_t = \frac{1}{2} \frac{c^2_D}{c_t} \rho (\xi'_0)^2 \frac{\eta}{t},$$  \hspace{1cm} (40) \begin{align*}
= 3 \times 10^{14} \frac{\rho}{10^{14}} \frac{c^2_D}{c_t} \left( \frac{\xi'_0}{0.1} \right)^2 \frac{\eta}{10^{-5} t} \text{ erg cm}^2 \text{s}^{-1}
\end{align*}

where all the quantities in the expression above are expressed in the cgs units.

Taking an optimistically large crack area of $10^{12} \text{cm}^2$, we get the typical released energy of $\sim 10^{27} \text{erg}$ for the magnetar-strength field with $\beta \sim 1$. This is $\sim 12$ orders of magnitude short compared to typical weak SGR flares (Woods & Thompson 2006). Clearly, thin shear cracks cannot be responsible for the flares. We finish this section with 2 remarks.

Mathematics remark. One can evaluate the inverse Laplace transform of Eq. (21), thus finding the motion due to appearance of the crack without considering limiting cases. It is easiest to first evaluate "seismic velocity" $\dot{B}$, whose Laplace transform is $pB(p)$. The latter could be calculated by noticing that the inverse Laplace transform of

$$f(p) = \frac{1}{\sqrt{p(\sqrt{p} + 1)}}$$  \hspace{1cm} (41)

is given by

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-k^2t} dk = e^t \left[ 1 - \text{Erf} \left( \sqrt{t} \right) \right].$$  \hspace{1cm} (42)

This can be checked directly from the definition of Laplace transform (the integrals over first $t$ and then $k$ are elementary). The "seismic velocity" is then given by

$$\dot{B}(t) = c_t \xi'_0 f(p_0 t).$$  \hspace{1cm} (43)

It is straightforward to check that the limits $t \ll 1/p_0$ (free slip) and $t \gg 1/p_0$ (magnetic-diffusion-controlled slip) are recovered.\footnote{For $x \gg 1$, with high degree of precision $f(x) = \sqrt{1/(\pi x)}$.}

Physics remark. We have considered the configuration where there is a substantial magnetic field in the direction perpendicular to the crack. One could ask whether a crack direction could adjust itself in such a way that the crack surface would be tangential to the field lines. From Eq. (40), we see that the $B_z$ component would have to be smaller than the average $\overline{B}$ by a factor greater than $10^3$ in order for the crack to release sufficient energy for the flare (this is because $\beta \propto B_z^2$). It seems very unlikely to us that this type of fine-tuned alignment could be efficient on a large scale, unless there was an elasto-dynamical mechanism that would drive the crack into alignment with the field. We have so far failed to identify such a mechanism.

3. Plastic flows?

We now construct an explicit scenario where changing magnetic field configuration induces effectively a gradual plastic flow in the constant-density crust. We shall assume, as before, $B_z = \text{const}$ and consider $B_z$ that is initially increasing with time

$$B_z = C t [\cosh(z/L)]^{-2}$$  \hspace{1cm} (44)

due to some non-MHD process (e.g., Hall drift) the specifics of which are not important for our purpose. Here $C$ is some constant and $L$ is the vertical length scale. We assume that the $xz$-component of the magnetic stress is balanced precisely by the $xz$-component of the elastic stress, i.e.

$$-T_{x_z}^\text{mag}(z) = \frac{B_z B_x(z)}{4\pi} = T_{x_z}^\text{el}(z) = -\mu \xi'_0(z).$$  \hspace{1cm} (45)

Let $T_{x_z}^\text{crit}$ be the critical shear stress at which the lattice begins to slip. According to HL09, $T_{x_z}^\text{crit} \approx 0.1 \mu$. Once the external magnetic field exceeds the critical value at height $z$,

$$B_z^\text{crit} = T_{x_z}^\text{crit} / 4\pi B_z,$$  \hspace{1cm} (46)

the crust begins to break and forms a shear crack at that location. From Eq. (44), this occurs when

$$\cosh(z_{\text{crit}}/L) = \sqrt{\frac{C t}{B_z^\text{crit}}}.$$  \hspace{1cm} (47)

The vertical velocity with which the boundary of the critical domain moves is given by

$$v_{\text{crit}} = \frac{dz_{\text{crit}}}{dt} = \frac{1}{2} L \sqrt{\frac{C}{B_z^\text{crit}} [\cosh(z/L)]^{-1}}.$$  \hspace{1cm} (48)
Both the elastic stress and the $B_x$ are 0 at the crack’s location and are reduced within distance $\delta z \sim \sqrt{\eta \delta t}$, where $\delta t$ is the time interval that has passed from the crack’s formation; see Eq. (35). The time interval $\delta t$ between the formation of two sequential cracks is given by

$$\sqrt{\eta \delta t} \sim v_{\text{crit}} \delta t,$$

so that

$$\delta t \sim \frac{\eta}{v_{\text{crit}}^2},$$

and the separation between the cracks is

$$\delta l \simeq \frac{\eta}{v_{\text{crit}}}.$$

Some numerical estimates are in order. Assuming that the magnetar field evolves due to Hall drift on a timescale of $\sim 100\,\text{yr}$, one gets

$$v_{\text{crit}} \sim \frac{1\,\text{km}}{100\,\text{yr}} = 3 \times 10^{-5}\,\text{cm s}^{-1},$$

and therefore

$$\delta t \sim 0.3\,\text{cm}$$

and

$$\delta t \sim 10^3\,\text{s} \sim \text{day}.$$

It is on the latter timescale that the magnetic field $B_x$ and the elastic shear $\xi' - \xi'_0$ relax to zero within the stressed domain where multiple cracks have appeared.

4. Discussion

The neutron star crust cracks differently from that of the Earth. In the latter, the quakes are caused by a sudden elastic stress release along a 2-dimensional surface, the shear crack, which causes slippage and release of mechanical energy. The magnitude of the release is much greater than the amount of elastic energy stored within the volume of the crack. By contrast, in the neutron-star crust magnetic tension strongly suppresses the slippage and the energy release. Infinitely thin cracks release energy via magnetic-field diffusion, too slowly to be able to contribute to the energetics of magnetar flares. If SGR flares are associated with starquakes which are generated by a crystal failure of the stressed crust, these constraints place strict requirements on the geometry of the failures.

We have so far considered simple field geometries, and thus have not proved that plastic flow is the generic macroscopic response of the crust. One can imagine situations in which mechanical failure in some region will cause magnetic-field reconfiguration that will in turn cause supercritical stress in another part of the crust, thus leading to a runaway process. This possibility has to be investigated further using specific magneto-elastic calculations. For now we remark that it is an open question whether magnetar flares are associated with the crust failures; as was already explained in the introduction, viable alternative model is that the flares are caused by reconnection events in the magnetosphere, in a manner similar to that observed in the solar flares (Lyutikov 2003, 2006). Therefore, in our opinion the inferences about flare statistics based on phenomenological models for the crust crystal failure (e.g., Perna & Pons 2011) to have be made with caution.

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A. Normal modes in magnetised elastic medium with resistivity

In this Appendix we discuss the normal modes in the elastic-resistive medium. In Eqns (7-8), eliminating $B_z$ in favor of $\xi$, the equation for the normal modes becomes

$$\left( \partial_t - \eta \partial_z^2 \right) \left( \partial_t^2 - c_{el}^2 \partial_z^2 \right) \xi = v_A^2 \partial_t \partial_z \xi$$

(A1)

This is a linear equation with constant coefficients and it can generally be solved by separation of variable. Assuming $\xi = F(t)G(z)$, we find

$$\frac{F^{(3)}}{F} - \left( c_t^2 \frac{F'}{F} + \eta \frac{F''}{F} \right) \frac{G''}{G} - \eta c_{el}^2 \frac{G^{(4)}}{G} = 0$$

(A2)

For harmonic spatial oscillations $\propto e^{-\omega t+kz}$, the dispersion equation is

$$\omega^3 + ik^2 \eta \omega^2 - k^2 c_t^2 \omega - ic_{el}^2 \eta k^4 = 0$$

(A3)

In the limit $\eta_{res} \to 0$, the normal modes are are

$$\omega_{1,2}^2 = c_t^2 k^2 \left( 1 \pm \frac{i k c_t^2 \eta_{res}}{c_t^2} \right)$$

$$\omega_3 = -ic_{el}^2 k^2 \frac{2}{c_t^2 \eta}$$

(A4)

The modes $\{1,2\}$ are just the resistivity-modified normal modes of the elastic medium with magnetic field. They become over-damped (real part smaller than imaginary) for

$$k > \frac{c_t^3}{c_A^2 \eta_{res}}$$

(A5)

The mode 3 is the resistive mode.