Online Robust Control of Linear Dynamical Systems with Prediction

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Abstract

We address the online robust control problem of a linear dynamical system with adversarial cost functions and adversarial disturbances. The goal is to find an online control policy that minimizes the disturbance gain, defined as the ratio of the cumulative cost and the cumulative energy in the disturbances. This problem is similar to the well-studied $\mathcal{H}_\infty$ problem in the robust control literature. However, unlike the standard $\mathcal{H}_\infty$ problem, where the cost functions are quadratic and fully known, we consider a more challenging online control setting where the cost functions are general and unknown a priori. We first propose an online robust control algorithm for the setting where the algorithm has access to an $N$-length preview of the future cost functions and future disturbances. We show that, under standard system assumptions, with $N$ greater than a threshold, the proposed algorithm can achieve a disturbance gain $(2 + \rho(N))\gamma^2$, where $\gamma$ is the best (minimum) possible disturbance gain for an oracle policy with full knowledge of the cost functions and disturbances, with $\rho(N) = O(1/N)$. We then propose an online robust control algorithm for a more challenging setting where only the preview of the cost functions is available. We show that under similar assumptions, with $N$ greater than the same threshold, the proposed algorithm achieves a disturbance gain of $6\gamma^2$ with respect to the maximum cumulative energy in the disturbances. Unlike the static regret, which is a relative metric, disturbance gain is a stronger metric that is directly indicative of the absolute performance.

1 Introduction

In this work we study the online robust control of the response of systems to disturbances, where the cost functions are adversarial. Recently there has been increased attention on online control algorithms for systems with uncertainties such as adversarial cost functions and disturbances. In online control, typically the objective is to minimize a performance objective such as the regret, which is defined as the difference between the cumulative cost of the online controller and the cumulative cost of the best policy from a comparator class over a finite duration of time. However, the absolute performance of the online controller designed to minimize this metric crucially depends on the comparator class. Most works that have studied this problem have considered the comparator class as the set of linear feedback policies (Dean et al., 2018; Cohen et al., 2019; Mania et al., 2019; Agarwal et al., 2019a,b; Simchowitz et al., 2020). While the class of linear feedback policies is the optimal for the classical Linear Quadratic Regulator (LQR) problem, this need not hold true for other cost functions such as the general convex functions that were considered in Agarwal et al. (2019a,b); Simchowitz et al. (2020). In this work our goal is to find an online robust control policy that minimizes the disturbance gain, defined as the ratio of the cumulative cost and the cumulative energy in the disturbances. Unlike the static regret, which is a relative metric, disturbance gain is a stronger metric that is directly indicative of the absolute performance.

Our problem is similar to the well studied $\mathcal{H}_\infty$ robust control problem. Here, the objective is to minimize the worst-case gain of the energy in the disturbance to the energy in the output; see Doyle et al. (1988). This problem has been well studied for linear dynamic systems and quadratic costs. Explicit characterization of the optimal state feedback controller has been given for this problem; see Doyle et al. (1988). Such characterization is harder when the cost functions are general and even harder in the online control settings where the cost functions need not be known apriori. In such cases, how do we develop online control algorithms to...
minimize the gain of the disturbances?

In this work we address the specific question of designing control algorithms for minimizing disturbance gain in the online setting. The system we consider is a known linear dynamic system with adversarial disturbances and adversarial and general cost functions. We study the setting where the controller has a limited preview of the future cost functions and disturbances. This is reasonable considering that in many practical problems such as robotics (Baca et al., 2018; Shi et al., 2019), energy systems (Vazquez et al., 2010), data-center management (Lazic et al., 2018), etc., a fixed finite-length preview of the future cost functions and/or disturbances are available to compute the current control input. Specifically we address the following question: how do we use the limited preview of future cost functions and disturbances to minimize the gain of the disturbances?

In the control theory literature, Receding Horizon Control (RHC), also known as Model Predictive Control (MPC), addresses the class problems where a preview of future cost functions are available to compute the current control input. The RHC is a well studied methodology in the control literature (Mayne et al., 2000; Angeli et al., 2011; Camacho and Alba, 2013; Grune and Stieler, 2014; Angeli et al., 2016; Borrelli et al., 2017; Rosolia and Borrelli, 2017; Baca et al., 2018; Grune and Pirkelmann, 2020). However, the control theory literature on RHC primarily focuses on asymptotic performance guarantees. Recently though, some works have studied the problem of online control with preview (Li et al., 2019; Goel and Hassibi, 2020; Yu et al., 2020). The metric that these works study is a stronger notion of regret called the dynamic regret, which is the regret where the comparator class is not restricted to any policy class. Thus, dynamic regret is a stronger metric than the static regret and is indicative of the absolute performance like the disturbance gain metric. While both Goel and Hassibi (2020) and Yu et al. (2020) assume full knowledge of the future cost functions and assume cost functions to be quadratic, we study the setting where the preview of the future cost functions is limited and the cost functions are sufficiently general than quadratic. Though Li et al. (2019) study systems with much general cost functions, they do not consider disturbances in the system dynamics as we do. The most important difference of our work from these works is that we present guarantees for the disturbance gain metric instead of the dynamic regret. In Section 2 we discuss in detail the comparison of our work with the dynamic regret online control literature.

In Section 2 we present our problem setting, elucidate its relation to prior work in robust control in both the full information and the online setting and then outline our contribution. In Section 3 we discuss the related works. In Section 4 we outline our algorithms, state the theorems for the performance of our algorithms and then interpret the results. In section 5 we present the detailed analysis. Finally, we conclude in Section 6. The detailed proofs are given in the Appendix.

1.1 Notations

For a sequence (of vectors/function) \((a_1, a_2, ..., a_T)\), we denote \(a_{1:t} = (a_1, a_2, ..., a_T)\). For a matrix \(M\), we denote its transpose by \(M^\top\). We denote the maximum eigenvalue of a matrix \(M\) by \(\lambda_{\max}(M)\). The two norm of a vector is denoted by \(\|\cdot\|_2\). When two matrices \(M_1\) and \(M_2\) are related by \(M_1 \geq M_2\) then it implies that \(M_1 - M_2\) is positive semi-definite. Similarly, when \(M_1 > M_2\) it implies that \(M_1 - M_2\) is positive definite. We denote \(\mathbb{R}^n\) as the \(n\)-dimensional euclidean space and \(\mathbb{R}_+\) as the non-negative part of the real line. We will introduce two system constants \(\alpha\) and \(\beta\) later. In terms of these constants, let \(\beta := \frac{\alpha}{\alpha + \beta}\). We compactly denote a time interval from \(t_1\) to \(t_2\) by \([t_1, t_2]\).

2 Problem Preliminaries

We consider the problem of online robust control of a linear dynamical system with cost functions and disturbance chosen by an adversary. The system evolution is represented by the equation

\[
x_{t+1} = Ax_t + Bu_t + w_t,
\]

where \(x_t \in \mathbb{R}^n\) is the system state, \(u_t \in \mathbb{R}^m\) is the control input, and \(w_t \in \mathbb{R}^n\) is the adversarial disturbance at time step \(t\). At each time step \(t\), the control algorithm selects a control input \(u_t\) and incurs a cost \(c_t(x_t, u_t)\). The cost function sequence \(\{c_t(\cdot, \cdot)\}\) is assumed to be adversarial and hence unknown to the algorithm. We assume that the system is fully observed, i.e., the algorithm can observe \(x_t\) at each time. We also assume that the system parameters \(A\) and \(B\) are known to the algorithm.

We consider the setting where a fixed horizon preview of the future cost functions and disturbances are available to the algorithm at each time step \(t\). In particular, the control input \(u_t\) is computed as \(u_t = \pi_t(x_{1:t}, u_{1:t-1}, w_{1:t+N_w-1}, c_{1:t+N_w-1})\), where \(N_w\) and \(N_c\) are fixed horizon lengths and \(\pi_t\) is the control policy at time \(t\). The goal of the agent is to select a control policy \(\pi = \pi_{1:T}\) in order to minimize the cumulative cost. This can be formulated as the following
optimization problem

\[
\min_{\pi} \ J(\pi; w_{1:T}, c_{1:T}) := \sum_{t=1}^{T} c_t(x_t, u_t),
\]

where, \( u_t = \pi_t(x_{1:t}, u_{1:t-1}, w_{1:t+N_w-1}, c_{1:t+N_c-1}) \).

(2)

Our goal is to characterize the performance of the optimal control policy \( \pi^* \) and the effect of prediction horizons \( N_w \) and \( N_c \). More specifically, our goal is to characterize the optimal cost in terms of the energy in the disturbance, i.e. to characterize

\[
\min_{\pi} \sup_{w_{1:T}} \frac{J(\pi; w_{1:T}, c_{1:T})}{\sum_{t=1}^{T} \|w_t\|^2}.
\]

(3)

We make the following assumptions on the adversarial disturbance and cost functions.

**Assumption 1** (Disturbance). The adversarial disturbance \( w_t \in W \) for all \( t \in [T] \), where \( W \) is compact and \( \|w\|_2 \leq w_c \) for all \( w \in W \).

We note that the boundedness of disturbance is a standard assumption in the online robust control literature; see [Agarwal et al. (2019a); Simchowitz et al. (2020)]. Let

\[
V_t(x_t, w_{t:t+N-1}) := \inf_{\tilde{x}_{t:t+N-1}} \sum_{j=t}^{t+N-1} c_j(\tilde{x}_j, \tilde{u}_j),
\]

\[
\tilde{x}_{j+1} = A\tilde{x}_j + B\tilde{u}_j + w_j, \quad \tilde{x}_t = x_t.
\]

**Assumption 2** (System). For all \( t \in [T] \),

(i) \( c_t(\cdot, \cdot) \) is continuous and non-negative

(ii) There exists a continuous non-negative function \( \sigma: \mathbb{R}^n \to \mathbb{R}_+, \alpha > 0 \) such that \( c_t(x, u_t) \geq \alpha \sigma(x) \)

(iii) There exist \( \sigma, \gamma > 0 \) such that \( V_t(x, w_{0:N-1}) \leq \sigma(x) + \gamma^2 \sum_{k=0}^{N-1} \|w_k\|_2^2 \)

The Assumption 2.(ii) states that the cost functions are lower bounded by a non-negative function. Such assumptions are used in control settings with prediction for general cost functions such as ours; see for e.g. [Grimm et al.] (2005). We show in the appendix that Assumption 2.(ii) is satisfied for the LQR problem where the cost \( c_t(x, u) = x^\top Q_t x + u^\top R_t u \), and \( Q_t > 0 \) and \( R_t > 0 \). The upper bound of \( V_t \) in Assumption 2.(iii) has two terms: the first term is the contribution of the initial state to the overall cost and the second term is the contribution of the disturbances over the period \( N \) to the overall cost. Here, the factor \( \gamma^2 \) is the minimum achievable disturbance gain by any sequence of control actions. The minimum achievable gain for the LQR cost function for linear systems satisfies this form of upper bound with \( \sigma(x) = \|x\|_2^2 \) (see Theorem 9.5.1, of [Hassibi et al. (1999)]. This assumption is a necessary assumption since the problem we address in this work has a solution only if a finite gain \( \gamma^2 \) is achievable.

Assumption 2 does not include the LQR case where \( Q_t \) is not positive definite and rather can be \( Q_t \geq 0 \). To include this case we require a more general form of Assumption 2(ii). We present the generalization of Assumption 2(ii) in the appendix. We then give a short proof in the appendix to show that the general time varying LQR cost function satisfies Assumption 2. We also give examples of non-quadratic and non-convex examples in appendix. Thus, the cost functions we consider are sufficiently general.

### 2.1 Connection to the classical \( H_\infty \)-Optimal Control problem

The robust control problem we study in this work has been addressed extensively in the control community using the framework of \( H_\infty \)-optimal control. Most of these works however focus on the linear systems with quadratic costs.

Consider the system \( T \) with \( c_t(x, u) = x^\top Q_t x + u^\top R_t u \), where \( Q_t \) is positive semi-definite and \( R_t \) is positive definite. Also, assume that \( N_c = T, N_w = 1 \). Then, the \( H_\infty \)-optimal control problem can be written as

\[
\min_{\pi} \sup_{w_{1:T}} \frac{J(\pi; w_{1:T}, c_{1:T})}{\sum_{t=1}^{T} \|w_t\|^2}.
\]

(4)

The goal, intuitively, is to minimize the worst-case gain from the energy in the disturbance \( w_{1:T} \) to the cost incurred by the controller. In general, \( \gamma \) is extremely hard to solve. A usual way to overcome the difficulty is to consider a relaxed problem with a given performance level \( \gamma \) as given below

\[
\min_{\pi} \sup_{w_{1:T}} \frac{J(\pi; w_{1:T}, c_{1:T})}{\sum_{t=1}^{T} \|w_t\|^2} < \gamma^2.
\]

(5)

Clearly, a solution may not exist for any arbitrary value of \( \gamma \). If the problem is feasible for a given \( \gamma \), then the optimal solution can be expressed in closed form, exploiting the quadratic nature of the cost function. The precise form of the \( H_\infty \)-optimal controller can be found in [Doyle et al. (1988)].

Compared to the \( H_\infty \) problem, the problem we study in this work is more challenging. Our setting is an online setting in which at any point of time the controller does not know the complete future of cost functions and disturbances, but only up to a horizon \( N_c < T \) and \( N_w \leq N_c \). We contrast this with the classical problem, where \( N_c = T, N_w = 1 \); and thus it needs complete
knowledge of future cost functions. Additionally, the cost functions we consider are general and need not be quadratic. Our methodology also does not need the knowledge of the optimal $\gamma$.

### 2.2 Connection to the Dynamic Regret Online Control Problem

In [Goel and Hassibi (2020)](Goel and Hassibi (2020)), the dynamic regret minimization problem is formulated as:

$$\min_{\pi} \sup_{w_{1:T}} J(\pi; w_{1:T}, c_{1:T}) - \min_u J(u; w_{1:T}, c_{1:T}),$$

$$\frac{\sum_{t=1}^{T} ||w_t||^2}{\sum_{t=1}^{T} ||w_t||^2}$$

(6)

where $u$ is the best sequence of actions (not restricted to a policy class) in hindsight. This way of posing the dynamic regret minimization problem has equivalence to the classical $H_\infty$ problem. In [Goel and Hassibi (2020)](Goel and Hassibi (2020)), the authors characterize the regret optimal controller in terms of the ratio defined above. As in $H_\infty$ they assume that $N_c = T$ and $N_w = 1$ and the cost functions to be quadratic. Thus, the comparisons we made of our setting and the $H_\infty$ framework apply here as well.

In [Yu et al. (2020)](Yu et al. (2020)), the authors present guarantees for dynamic regret for a fixed LQR cost function and the preview of disturbances that can be of any length. Thus, [Yu et al. (2020)](Yu et al. (2020)), similar to [Goel and Hassibi (2020)](Goel and Hassibi (2020)), assume $N_c = T$ and the cost functions to be quadratic. Though, their setting is similar to [Goel and Hassibi (2020)](Goel and Hassibi (2020)), they characterize dynamic regret in the standard form as $O(T^\alpha)$ and show how $\alpha$ decreases with the length of the preview of disturbances. Crucially, they require the preview to grow at least logarithmically with the control horizon to achieve sublinear dynamic regret. In contrast to [Yu et al. (2020)](Yu et al. (2020)), we present performance guarantees for the disturbance gain. We present these guarantees for the case where the preview of both the cost functions and the disturbances is a constant and need not grow with the control horizon. The cost functions we consider are also much more general than the fixed quadratic costs.

In [Li et al. (2013)](Li et al. (2013)), the authors characterize the effect of preview on the dynamic regret of any baseline policy. They present an algorithm that improves the dynamic regret of any baseline policy exponentially with the length of the preview. Their major shortcoming is that they leave the baseline policy unspecified. In contrast we present a complete algorithm and a complete characterization of an absolute performance metric. Secondly, it is assumed in [Li et al. (2013)](Li et al. (2013)) that the cost functions are strongly convex, Lipschitz and the system is controllable. These assumptions are restrictive. For example it does not include all cases of LQR such as the positive semi-definite $Q$s. Our cost functions need not satisfy these assumptions. Thirdly, [Li et al. (2013)](Li et al. (2013)) do not consider disturbances in their system model.

### 2.3 Main Contribution

We present online robust control algorithms for couple of cases: (i) where the controller has $N$-length preview of both the cost functions and disturbance and (ii) where the controller has $N$-length preview of only the cost functions. The online controller we propose for both the settings is a novel extension of the standard Receding Horizon Controller (RHC). To the best of our knowledge ours is the first work to characterize disturbance gain for online RHC for sufficiently general time varying cost functions.

We first present the case where the controller has preview of both the cost functions and the disturbances. Our algorithm for this setting trivially achieves the minimum achievable gain when the preview $N$ is $T$. Hence, we analyse in detail the case where $N$ is strictly less than $T$. We show that, when the initial state’s cost is zero, and $N > 4\epsilon^3$ and $\beta = \alpha/\pi \geq 1/\zeta$ for some $\zeta > 1$, for any $T$ the proposed RHC’s total cost is bounded by

$$J(\pi) \leq (2\zeta + \rho(N))T^2 \sum_{t=1}^{T} ||w_t||^2,$$

with $\rho(N) = O(1/N)$. Thus, we show that the disturbance gain achieved by the proposed RHC is given by $(2\zeta + \rho(N))T^2$. Thus, we show that the online RHC’s disturbance gain differs from $\gamma^2$ only by a constant factor and reduces with preview $N$ as $O(1/N)$.

We then present the second case, where only $N$-length preview of the cost functions are available. We show that, when the initial state’s cost is zero, and $N > 2(\zeta - 2)^2$ and $\beta \geq 1/(\zeta - 2)$ for some $\zeta > 3$, for any $T$ the proposed RHC’s total cost is bounded by

$$J(\pi) \leq 2\zeta T^2 \max_{w \in W} ||u||^2.$$

Thus, in this setting: (i) our guarantee is with respect to the maximum possible energy and (ii) our disturbance gain guarantee differs from the minimum achievable $\gamma^2$ only by the constant factor $\zeta$.

Our guarantees differ from the guarantees for disturbance gain in [Goel and Hassibi (2020)](Goel and Hassibi (2020)) in the following way: [Goel and Hassibi (2020)](Goel and Hassibi (2020)) characterize the suboptimal controller that achieves any given feasible disturbance ratio using full information of the cost functions. In contrast we prescribe an online controller that operates with limited preview of the future at any point of time and characterize its disturbance gain.
3 Other Related Works

**Robust Control**: The control of dynamical systems with uncertainties such as modeling errors, parametric uncertainty, and disturbances is a central challenge in control theory and so has been extensively studied. There is vast literature in the control literature on control synthesis for systems with such uncertainties. For instance, the robust control literature has studied the problem of feedback controller design for stability and performance guarantees with modeling uncertainty and disturbances (Skogestad and Postlethwaite, 2007). The adaptive control literature has studied the control of systems with parametric uncertainty (Sastry and Bodson, 2011; Åström and Wittenmark, 2013; Ioannou and Sun, 2012). However, classical control theory has primarily focused on asymptotic performance guarantees.

**Online Stochastic Control**: This area of online control studies the online control of stochastic systems. In Abbasi-Yadkori and Szepesvári (2011), the authors studied the online Linear Quadratic Regulator (LQR) problem for unknown systems and stochastic disturbances. The authors proposed an adaptive algorithm that achieves \(\sqrt{T}\) regret w.r.t. the best linear feedback controller policy. One drawback of their online controller was the inefficiency. In Dean et al. (2018), the authors proposed an algorithm for the same problem that was efficient and still achieved a sub-linear regret of \(O(T^{2/3})\). In Cohen et al. (2019) and Mania et al. (2019) this result was further improved by providing an efficient algorithm that was able to achieve \(O(\sqrt{T})\). In Mania et al. (2019), the authors studied the same setting but with partial observations of the state and established that \(O(\sqrt{T})\)-regret is achievable. Recently, Simchowitz and Foster (2020) showed that \(O(\sqrt{T})\) is the optimal regret for the online LQR problem. Cohen et al. (2013) studied a variant of the online LQR where the system is known and disturbance is stochastic but the controller cost function is an adversarially chosen quadratic function. They presented an algorithm that was able to achieve \(O(\sqrt{T})\) even when the cost functions are adversarial. In Agarwal et al. (2019b), the authors showed that the regret can be improved to poly logarithmic regret if the cost functions are strongly convex and the disturbances are well-conditioned stochastic processes.

**Online Robust Control**: This area of online control studies the online control of non-stochastic systems. In Agarwal et al. (2019a), the authors studied the control of a known linear dynamic system with adversarial disturbance and adversarial controller cost functions. They showed that an \(O(\sqrt{T})\)-regret is achievable with respect to the best linear feedback controller. In Hazan et al. (2020), the authors studied the same setting but for the unknown system and showed that a sub-linear regret of \(O(T^{2/3})\) is still achievable. Recently, Simchowitz et al. (2020) showed that similar results are achievable with partial observation of the state for both known and unknown systems.

**Receding Horizon Control**: Many receding horizon control based methods have been proposed for managing disturbances and uncertainties in the system dynamics. For example, some works handle disturbances or uncertainties by robust or chance constraints (Langson et al., 2004; Goulart et al., 2006; Limon et al., 2010; Tempo et al., 2012; Goulart et al., 2016). Adaptive RHC techniques that adapt online when the system model is unknown have also been proposed (Fukushima et al., 2007; Adetola et al., 2009; Aswani et al., 2013; Tanaskovic et al., 2019; Bujarbaruah et al., 2019). These methods primarily focus on constraint satisfaction, stability and in some cases performance improvement using the adapted models. In contrast to these works, we consider non-asymptotic performance of an online adaptive RHC. There are considerable amount of papers that provide performance analysis of RHC under both time-invariant costs (Angeli et al., 2011; Grune and Stieler, 2014; Grune and Panin, 2013) and time varying costs (Ferramosca et al., 2010; Angeli et al., 2016; Ferramosca et al., 2014; Grune and Pirkelmann, 2017). However, most of these studies focus on asymptotic performance.

4 Online Robust Control: Algorithms and Main Results

In this section, we present couple of online control algorithms for the robust control problem. We first present the algorithm for the case where \(N_c = N_u = N > 0\), i.e., when the preview of both the future disturbances and cost functions for period \(N\) are available. We then present our algorithm for the case \(N_c = N\) and \(N_u = 0\), i.e., where only the preview of the future cost functions is available. We present the performance guarantees for our algorithms in each setting and interpret the result.

4.1 Online Robust Control with Preview

In this case, we assume that the control algorithm has a fixed horizon preview of the future cost functions and disturbances. In particular, at each time \(t\), algorithm has access to \(c_{t:t+N-1}\) and \(u_{t:t+N-1}\), in addition to the history of observation until \(t\). The control policy we propose is a modification of the standard receding horizon controller. We denote this control policy by \(\pi_{\text{op}}\). Unlike the standard receding horizon control, which re-
computes the decision every time step, the policy $\pi_{op}$ recomputes only every $N - M + 1$th time step, where $M(< N)$ will be specified later. In the period between, where the control sequences are not recomputed, the policy $\pi_{op}$ selects the control input from the computed sequence for the current interval. The period $M$ overlap from one computation to the next determines how much of the preview is exploited and how frequently the control inputs are updated.

More formally, the horizon $T$ is split as the intervals $[t_1, t_2 - 1], [t_2, t_3 - 1], \ldots$ and so on till the end of the horizon. The intervals are indexed by $i$ and the beginning of this interval is given by $t_i = (i - 1)(N - M) + 1$. Thus, each interval is of duration $N - M$. The control policy solves the following optimization at the beginning of interval $i$:

$$\inf_{\tilde{u}_{t:t+N-1}} \sum_{j=t}^{t+N-1} c_j(\tilde{x}_j, \tilde{u}_j),$$

s.t. $\tilde{x}_{j+1} = A\tilde{x}_j + B\tilde{u}_j + w_j$, $\tilde{x}_t = x_t$. \hspace{1cm} (7)

The optimization computes the optimal control sequence for the cost-to-go for the duration $N$ from the beginning of every interval. We denote the output of the optimization by $\tilde{u}_{t:t+N-1}$. Given this computed control sequence, the policy $\pi_{op}$ sets the control input $u_t$ as $\tilde{u}_t$ for all $t \in [t_i, t_i + 1 - 1]$. The policy $\pi_{op}$ then repeats this process in every new interval. The complete algorithm for $\pi_{op}$ is described in Algorithm 1.

We note that policy $\pi_{op}$ trivially achieves the maximum possible attenuation when $N = T$. Hence, we do not discuss this case formally. In the next theorem we characterize the disturbance gain achieved by the $\pi_{op}$ when $N < T$.

Algorithm 1 Online Robust Control with Preview ($\pi_{op}$)

1: Input: $N, M$
2: Define: $t_i = (i - 1)(N - M) + 1$
3: Initialize: $i = 1$
4: for $t = 1, ..., T$ do
5:  if $t = t_i$ then
6:    Observe the current state $x_t$, and the preview $c_{t:t+N-1}, \tilde{u}_{t:t+N-1}$ to get $\tilde{u}_{t:t+N-1}$
7:    Solve the problem (7) to get $\tilde{u}_{t:t+N-1}$
8:    $i = i + 1$
9:  end if
10: Set the control input $u_t$ as $\tilde{u}_t$
11: end for

Theorem 1. Suppose Assumption 4 and 5 hold. Let $\pi = \pi_{op}$, $M = N/2$ and $\sigma(x_1) = 0$. Let $N > 4\zeta^3$ and $\beta \geq 1/\zeta$ for some $\zeta > 1$. Then, for any $T > N$

$$J(\pi_{op}) \leq \gamma^2_{op} \sum_{t=1}^{T} \|w_t\|_2^2,$$

where

$$\gamma^2_{op} = (2\zeta + \rho(N))\zeta^2, \quad \rho(N) \leq O(1/N),$$

i.e., $\pi_{op}$ achieves a disturbance gain $\gamma^2_{op}$.

We discuss the analysis in Section 4.

Remark 1. The complete characterization of the bound to the cost includes an additional term that is the cost of the initial condition, $\sigma(x_1)$, as in the $H_\infty$ characterization; see Theorem 9.5.1 Hassibi et al. (1999) for $H_\infty$ characterization. We present the complete characterization later. In the above theorem, we assume the initial state’s cost to be zero and only state the bound in terms of the disturbance gain.

Remark 2. It was shown in Li et al. (2019), although for the setting without disturbance, that exponential decrease of dynamic regret with preview is achievable for strongly convex and Lipschitz cost functions. In our case, the $\rho(N)$ term in the factor accompanying $\gamma^2$ implies that $1/N$ reduction is achievable for the general cost functions we consider.

Remark 3. An implication of our general result is the following: when $\zeta = (1 + o(1))$, Theorem 1 implies that, with preview of disturbances and cost functions $N > 4+o(1)$, the proposed RHC achieves a disturbance gain $\gamma^2_{op} = (2 + O(1/N))\zeta^2$.

4.2 Online Robust Control with Cost Preview Only

In this case, we assume that the control algorithm has the preview of only the future cost functions for $N$ time steps, i.e., $N_c = N$ and $N_w = 0$. Since the disturbance preview is not available, we propose a min-max receding horizon controller for the control policy. We denote this policy by $\pi_o$. Similar to Algorithm 1 the control horizon is split into intervals $[t_1, t_2 - 1], [t_2, t_3 - 1], \ldots$ and so on till the end of the control horizon. The difference being that the following min-max optimization is solved instead of Eq. (7) at the beginning of every interval:

$$\inf_{\tilde{u}_{t:t+N-1}} \sup_{\tilde{w}_{t:t+N-1}} \sum_{j=t}^{t+N-1} c_j(\tilde{x}_j, \tilde{u}_j),$$

s.t. $\tilde{x}_{j+1} = A\tilde{x}_j + B\tilde{u}_j + \tilde{w}_j$, $\tilde{w}_j \in W$, $\tilde{x}_t = x_t$. \hspace{1cm} (8)

Similar to the policy $\pi_{op}$, the policy $\pi_o$ sets the control input $u_t$ as $\tilde{u}_t$ for all $t \in [t_i, t_i + 1 - 1]$. It repeats this process to compute the control inputs for every new interval. The complete algorithm for $\pi_o$ is described in Algorithm 2.
Algorithm 2 Online Robust Control with Cost Preview Only ($\pi_o$)

1: **Input:** $N, M$
2: **Definition:** $t_i = (i - 1)(N - M) + 1$
3: **Initialize:** $i = 1$
4: for $t = 1,..., T$ do
5: if $t = t_i$ then
6: Observe the current state $x_t$, and the preview $c_{t:t+N-1}$
7: Solve the problem $\mathcal{P}$ to get $\bar{u}_{t:t+N-1}$
8: $i = i + 1$
9: end if
10: Set the control input $u_t$ as $\bar{u}_t$
11: end for

In the next theorem we characterize the disturbance gain achieved by the proposed controller for this setting.

**Theorem 2.** Suppose Assumption 1 and 2 hold. Let $\pi = \pi_o$, $M = N/2$ and $\sigma(x_1) = 0$. Let $N > 2\zeta(\zeta - 2)^2$ and $\beta \geq 1/(\zeta - 2)$ for some $\zeta > 3$. Then, for any $T > N$

$$J(\pi_o) \leq \gamma_o^2 T \max_{w \in W} \|w\|_2^2,$$

where

$$\gamma_o^2 = 2\zeta^2,$$

i.e., $\pi_o$ achieves a disturbance gain $\gamma_o^2$.

We discuss the analysis in Section 5.

**Remark 4.** The first difference here from Theorem 1 is that the disturbance gain is characterized with respect to the maximum possible cumulative energy. The second difference is that the disturbance gain does not have the $O(1/N)$ term in the factor accompanying $\gamma^2$. In the complete characterization we present later, we show that the disturbance gain decreases with preview $N$, although not as $O(1/N)$. The factor we presented in the above theorem is a bound that holds for all $N$.

**Remark 5.** An implication of our general result is the following: when $\zeta = (3 + o(1))$, Theorem 2 implies that, with preview of cost functions $N > 6 + o(1)$, the proposed RHC achieves a disturbance gain $\gamma_o^2 = (6 + o(1))\gamma^2$.

5 Online Robust Analysis

5.1 Online Robust Control with Preview

In the next theorem we give complete characterization of the performance of Algorithm 1. In appendix, we present the characterization for the generalization of Assumption 2.

**Theorem 3.** Suppose Assumption 1 and 2 hold. Let $\pi = \pi_o$, $N \geq 2M$ and $M > 1/\beta^2$. Then for any $T > N$

$$J(\pi_o) \leq O(\sigma(x_1)) + \omega_o \gamma^2 T \max_{w \in W} \|w\|_2^2,$$

where

$$\omega_o = \frac{2 - \beta + \kappa(M)}{\beta(1 - 1/(\beta^2 M))}, \quad \kappa(M) = \frac{3}{\beta M} + \frac{1}{(\beta M)^2} - 1.$$

Please see the appendix for the detailed proof. This characterization offers more insights on the behaviour of our Algorithm 2.

We observe that as $N$ increases, fixing $M$ to a constant does not change the $\omega_o$. This is because fixing $M$ as $N$ grows could limit the adaptability because of the less frequent updating of the control actions inspite of the increase in preview. This observation implies that $M$ cannot be very small. While the optimal value for $M$ depends on the specific realization of the disturbances, $M = N/2$ is a reasonable choice considering the guarantee in Theorem 3. In our case, when $M = N/2$, we observe that $\omega_o$ trivially decreases as atleast $1/N$ since $\kappa$ decreases as $1/N$.

The lower bound stated for $M$ is the minimum preview that is required for the response to be stable. This is because when $M$ is lower than this lower bound, the predictive controller, which in this case will only update its control actions less frequently, need not be stabilizing. Thus, we give guarantees for the problem of stabilization and disturbance gain minimization for the online robust control problem.

When $M = N/2$, since $\omega_o$ is decreasing with $N$, $\omega_o \geq (2 - \beta)1/\beta$. This implies that the minimum achievable disturbance gain when $N < T$ is dependent on the system constant $\beta$. This dependence arises from the fact that there has to be a period $M$ overlap between the two consecutive intervals to ensure stability. If period $M$ overlap is not required, then it is trivial to show that $\gamma^2$ can be achieved as the gain. In addition, since $\beta \leq 1$, $\omega_o \geq 1$, which is as expected.

5.2 Online Robust Control with only Cost Preview

In the next theorem, we give the complete characterization of Algorithm 2.

**Theorem 4.** Suppose Assumption 1 and 2 hold. Let $\pi = \pi_o$, $M \leq N/2$ and $M > 1/\beta^2$. Then for any $T > N$

$$J(\pi_o) \leq O(\sigma(x_1)) + \omega_o \gamma^2 T \max_{w \in W} \|w\|_2^2,$$

where

$$\omega_o = \frac{2(1 + \beta)}{\beta(1 - 1/(\beta^2 M))}.$$
Please see the appendix for the proof. As noted earlier, the characterization of the performance for this setting is given in terms of the maximum possible cumulative energy. Without the actual realization, the RHC approach can only optimize for the worst case and thus our guarantees are in terms of the maximum total energy. This is the same case as the algorithms of Li et al. (2019); Yu et al. (2020). Their regret guarantees are only valid when the controller has the preview of the disturbances. While Li et al. (2019); Yu et al. (2020) do not address the case without the disturbance preview, we present guarantees for this case too.

Here too, when $M$ is fixed and does not change with $T$, the disturbance gain does not change with the horizon $T$. Similarly, the interval $M$ has to be greater than $\beta^2$ for stability. Thus, the stability condition is dependent on the cost preview, which is evident from earlier RHC works (Grimm et al., 2003). By setting $M = N/2$, we observe that $\omega_o$ decreases with the preview $N$ as $\beta N(1 + \beta)/((\beta^2 N - 2)$, thus the minimum achievable gain is given by $(1 + \beta)/\beta$. Since $\beta \leq 1$, it follows as expected that $\omega_o \geq 1$.

5.3 Proofs of Main Theorems

Proof of Theorem 1

Proof. First, we note that $M = N/2$. Next, we note that $N > 4\zeta^3 > 2\zeta^2$. Then, given that $\zeta^2 \geq 1/\beta^2$, $N > 2/\beta^2$. That is $M > 1/\beta^2$. Thus, all the conditions in Theorem 3 are satisfied.

Then, given that $\sigma(x_1) = 0$, $\sum_{t=1}^T c_t(x_t, u_t) \leq \omega_o \sum_{t=1}^T \|u_t\|^2$. Substituting $M = N/2$, we get

$$\omega_o = \frac{2 - \beta + \kappa(N/2)}{\beta(1 - 2/(\beta^2 N))} \leq \frac{2 - \beta}{\beta(1 - 2/(\beta^2 N))} + \kappa(N/2) \beta(1 - 2/(\beta^2 N)).$$

Given that $N > 4\zeta^3$ and $\beta \geq 1/(\zeta)$, we get that

$$\frac{1}{2\zeta} > \frac{2\zeta^2}{N} \geq \frac{2}{\beta N^2}, \text{ i.e., } \frac{1}{\beta(1 - 2/(\beta^2 N))} < \frac{2\zeta^2}{2\zeta - 1}.$$

By definition $\beta \leq 1$. Hence,

$$\kappa(N/2) = \frac{6}{\beta N} + \frac{4}{(\beta N)^2} \frac{2}{N} = \frac{6 - 2\beta}{\beta N} + \frac{4}{(\beta N)^2} \leq \mathcal{O}(1/N).$$

Thus,

$$\frac{\kappa(N/2)}{\beta(1 - 2/(\beta^2 N))} \leq \mathcal{O}(1/N).$$

Then, given that $\beta \geq 1/\zeta$

$$\frac{2 - \beta}{\beta(1 - 2/(\beta^2 N))} \leq 2\zeta.$$

Thus,

$$\omega_o \leq 2\zeta + \rho(N), \text{ where } \rho(N) \leq \mathcal{O}(1/N).$$

From here $\gamma_o^2$ follows.

Proof of Theorem 2

Proof. First, we note that $M = N/2$. Next, we note that $N > 2\zeta(\zeta - 2)^2 > 2(\zeta - 2)^2$. Then, given that $(\zeta - 2)^2 \geq 1/\beta^2$, $N/2 > 1/\beta^2$. Thus all the conditions in Theorem 3 are satisfied.

Then, given that $\sigma(x_1) = 0$, $\sum_{t=1}^T c_t(x_t, u_t) \leq \omega_o \sigma^2 T \max_{w \in W} \|w\|^2$. Substituting $M = N/2$, we get

$$\omega_o = \frac{2(1 + \beta)}{\beta(1 - 2/(\beta^2 N))}.\n$$

Given that $N > 2\zeta(\zeta - 2)^2$ and $\beta \geq 1/(\zeta - 2)$,

$$\frac{1}{\zeta} > \frac{2(\zeta - 2)^2}{N} \geq \frac{2}{N\beta^2}.$$

Then, given that $\beta \geq 1/(\zeta - 2)$,

$$\frac{2(1 + \beta)}{\beta(1 - 2/(\beta^2 N))} \leq 2\zeta.$$

Thus,

$$\omega_o \leq 2\zeta.$$

From here $\gamma_o^2$ follows.

6 Conclusion

In this work we studied the online robust control problem for linear dynamic systems with adversarial cost functions and adversarial disturbances. The objective of the online controller is to minimize the disturbance gain, which is a well studied metric in the classical $H_{\infty}$ framework, a standard robust control framework for quadratic costs and full preview of costs. In contrast to the $H_{\infty}$ framework, we studied the setting where the controller only has a preview of the future disturbances and cost functions that is strictly less than the control horizon. The online robust controller we proposed is a novel variant of the standard Receding Horizon Controller (RHC). We showed that the proposed RHC can achieve a disturbance gain that differs from the minimum achievable by a constant factor only and decreases with the preview length $N$ as $\mathcal{O}(1/N)$. We gave explicit characterization of the gain and the required threshold for the preview for the disturbance gain guarantees to hold. In conclusion, our work contributes towards algorithms and mathematical techniques for the online robust control with limited preview.
References

Abbasi-Yadkori, Y. and Szepesvári, C. (2011). Regret bounds for the adaptive control of linear quadratic systems. In Proceedings of the 24th Annual Conference on Learning Theory, pages 1–26.

Adetola, V., DeHaan, D., and Guay, M. (2009). Adaptive model predictive control for constrained nonlinear systems. Systems & Control Letters, 58(5):320–326.

Agarwal, N., Bullins, B., Hazan, E., Kakade, S., and Singh, K. (2019b). Logarithmic regret for online control. In Advances in Neural Information Processing Systems, pages 10175–10184.

Angeli, D., Amrit, R., and Rawlings, J. B. (2011). On average performance and stability of economic model predictive control. IEEE transactions on automatic control, 57(7):1615–1626.

Angeli, D., Casavola, A., and Tedesco, F. (2016). Theoretical advances on economic model predictive control with time-varying costs. Annual Reviews in Control, 41:218–224.

Åström, K. J. and Wittenmark, B. (2013). Adaptive control. Courier Corporation.

Aswani, A., Gonzalez, H., Sastry, S. S., and Tomlin, C. (2013). Provably safe and robust learning-based model predictive control. Automatica, 49(5):1216–1226.

Baca, T., Hert, D., Loianno, G., Saska, M., and Kumar, V. (2018). Model predictive trajectory tracking and collision avoidance for reliable outdoor deployment of unmanned aerial vehicles. pages 6753–6760.

Borrelli, F., Bemporad, A., and Morari, M. (2017). Predictive control for linear and hybrid systems. Cambridge University Press.

Bujarbaruah, M., Zhang, X., Tanaskovic, M., and Borrelli, F. (2019). Adaptive mpc under time varying uncertainty: Robust and stochastic. arXiv preprint arXiv:1909.13473.

Camacho, E. F. and Alba, C. B. (2013). Model predictive control. Springer science & business media.

Cohen, A., Hassidim, A., Koren, T., Lazic, N., Mansour, Y., and Talwar, K. (2018). Online linear quadratic control. arXiv preprint arXiv:1806.07104.

Cohen, A., Koren, T., and Mansour, Y. (2019). Learning linear-quadratic regulators efficiently with only \( \sqrt{T} \) regret. In International Conference on Machine Learning, pages 1300–1309.

Daskalakis, C., Skoulakis, S., and Zampetakis, M. (2020). The complexity of constrained min-max optimization. arXiv preprint arXiv:2009.09623.

Dean, S., Mania, H., Matni, N., Recht, B., and Tu, S. (2018). Regret bounds for robust adaptive control of the linear quadratic regulator. In Advances in Neural Information Processing Systems, pages 4188–4197.

Doyle, J., Glover, K., Khargonekar, P., and Francis, B. (1988). State-space solutions to standard \( H_2 \) and \( H_{\infty} \) control problems. pages 1691–1696.

Facchinei, F. and Pang, J.-S. (2007). Finite-dimensional variational inequalities and complementarity problems. Springer Science & Business Media.

Ferramosca, A., Limon, D., and Camacho, E. F. (2014). Economic mpc for a changing economic criterion for linear systems. IEEE Transactions on Automatic Control, 59(10):2657–2667.

Ferramosca, A., Rawlings, J. B., Limón, D., and Camacho, E. F. (2010). Economic mpc for a changing economic criterion. In 49th IEEE Conference on Decision and Control (CDC), pages 6131–6136. IEEE.

Fukushima, H., Kim, T.-H., and Sugie, T. (2007). Adaptive model predictive control for a class of constrained linear systems based on the comparison model. Automatica, 43(2):301–308.

Goel, G. and Hassibi, B. (2020). Regret-optimal control in dynamic environments. arXiv preprint arXiv:2010.10473.

Goulart, P., Zhang, X., Kamgarpour, M., Georghiou, A., and Lygeros, J. (2016). Robust optimal control with adjustable uncertainty sets. Automatica, 75.

Goulart, P. J., Kerrigan, E. C., and Maciejowski, J. M. (2006). Optimization over state feedback policies for robust control with constraints. Automatica, 42(4):523–533.

Grimm, G., Messina, M. J., Tuna, S. E., and Teel, A. R. (2005). Model predictive control: for want of a local control lyapunov function, all is not lost. IEEE Transactions on Automatic Control, 50(5):546–558.

Grüne, L. and Panin, A. (2015). On non-averaged performance of economic mpc with terminal conditions. In 2015 54th IEEE Conference on Decision and Control (CDC), pages 4332–4337. IEEE.

Grüne, L. and Pirkelmann, S. (2017). Closed-loop performance analysis for economic model predictive control of time-varying systems. In 2017 IEEE 56th Annual Conference on Decision and Control (CDC), pages 5563–5569. IEEE.

Grüne, L. and Pirkelmann, S. (2020). Economic model predictive control for time-varying systems: Perfor-
Online Robust Control of Linear Dynamical Systems with Prediction

Grüne, L. and Stieler, M. (2014). Asymptotic stability and transient optimality of economic mpc without terminal conditions. *Journal of Process Control*, 24(8):1187–1196.

Hassibi, B., Sayed, A. H., and Kailath, T. (1999). *Indefinite-Quadratic estimation and control: a unified approach to $H_2$ and $H_{\infty}$ theories*. SIAM.

Hassani, M. (2019). Neural lander: Stable drone landing control using learned dynamics. pages 9784–9790.

Simchowitz, M. and Foster, D. (2020). Naive exploration is optimal for online lqr. In *International Conference on Machine Learning*, pages 8937–8948. PMLR.

Simchowitz, M., Singh, K., and Hazan, E. (2020). Improper learning for non-stochastic control. *arXiv preprint arXiv:2001.09254*.

Skogestad, S. and Postlethwaite, I. (2007). *Multivariable feedback control: analysis and design*, volume 2. Citeseer.

Vazquez, S., Rodriguez, J., Rivera, M., Franquelo, L. G., and Norambuena, M. (2016). Model predictive control for power converters and drives: Advances and trends. *IEEE Transactions on Industrial Electronics*, 64(2):935–947.

Yu, C., Shi, G., Chung, S.-J., Yue, Y., and Wierman, A. (2020). The power of predictions in online control. *arXiv preprint arXiv:2006.07569*.
A Discussion on Assumption 2

In the case, $Q_t > 0$ and $R_t > 0$, we observe that $c_t(x, u) = x^T Q_t x + u^T R_t u \geq x^T Q_t x \geq \lambda_{\min}(Q_t) \|x\|^2$. Hence, it follows that Assumption 2(ii) is satisfied with $\alpha = \min_{t \in [1, ..., T]} \lambda_{\min}(Q_t)$, and $\sigma(x) = \|x\|^2$. Assumption 2(iii) follows from [Hassibi et al., 1999, Theorem 9.5.1].

When $Q_0 > 0$, we note that the above set of parameter values will not satisfy Assumption 2(ii). We can set $\sigma(x) = x^T Q x$, where $Q$ is a positive semi-definite matrix such that $Q \leq Q_t$, $\forall \ t \in [1, ..., T]$. But this specification for $\sigma(x)$ will not satisfy Assumption 2(iii), since, in this upper bound, $\sigma(x)$ has to be a positive definite quadratic like $\|x\|^2$ (see [Hassibi et al., 1999, Theorem 9.5.1]). Next, we present the generalization of Assumption 2 that will include the $Q_t \geq 0$ case.

A.1 Generalization of Assumption 2

Assumption 2 can be generalized by generalizing Assumption 2(ii) as follows: There exits a continuous non-negative function $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $\sigma > 0, \sigma(x) \geq 0, \gamma \leq \gamma$ and a continuous function $\tilde{V} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $\tilde{V}(x_{t+1}) - \tilde{V}(x_t) \leq -\sigma(x_t) + c_t(x_t, u_t) + \gamma^2 \|w_t\|^2$, and $\tilde{V}(x_t) \leq \sigma(x_t) \sigma(x_t)$.

The difference in this generalization is the additional $\tilde{V}$ function. We note that this generalization trivially reduces to Assumption 2 (ii) when $\tilde{V} = 0$. Next, we show that the general time varying LQR case where $Q_t$s can be positive semi-definite satisfies this generalization. We give the proof for the control algorithm 1 for this generalization later in the appendix.

A.1.1 Proof for Time Varying LQR

First, we consider the time invariant LQR. When the cost function is LQR, i.e. $c(x, u) = x^T Q x + u^T R u$, where $Q \geq 0, R > 0, \exists \alpha > 0, \gamma \leq \gamma$ such that $\tilde{V}(x) = \sigma(x) P x$ for some $P > 0$ and $\sigma(x) = \|x\|^2$ satisfies the condition in Assumption 2. We use an argument similar to [Grimm et al., 2005] to show this. Suppose $(A, C)$ is detectable, where $(A, C)$ is detectable we can choose $L$ and $P = P^T > 0$ such that $A = A + LC$ satisfies $A^T P A - P = -2I$. Let $\tilde{V}(x) = \sigma(x) P x$. For simplicity, we remove the time index and denote $x_+$ as the next state. Then

$$\tilde{V}(x_+) - \tilde{V}(x) = \sigma(x_+) P x_+ - x^T P x = \sigma \left( (A + L C - L C)x + Bu + w \right)^T P \left( (A + L C - L C)x + Bu + w \right) - x^T P x.$$

Separating the terms we get

$$\tilde{V}(x_+) - \tilde{V}(x) = \sigma \left( x^T (A^T P A - P) x + x^T C^T L^T P L C x + u^T B^T P B u + w^T P w \right)$$

$$- 2x^T A^T P L C x + 2u^T B^T P A x - 2x^T C^T L^T P B u + 2w^T P A x - 2x^T C^T L^T P w + 2w^T P B u).$$

Now from Young's inequality, we have that $a^T b \leq \epsilon/2 a^T a + 2b^T b/\epsilon$ for any $\epsilon > 0$. Hence, we can choose $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6$ such that

$$\tilde{V}(x_+) - \tilde{V}(x) \leq \sigma \left( (1 + \epsilon_1 + \epsilon_2 + \epsilon_4) A^T P A - P \right) x + \sigma(1 + 4/\epsilon_1 + \epsilon_3 + \epsilon_5) x^T C^T L^T P L C x$$

$$+\sigma (1 + 4/\epsilon_2 + 4/\epsilon_3 + \epsilon_6) u^T B^T P B u + \sigma (1 + 4/\epsilon_4 + 4/\epsilon_5 + 4/\epsilon_6) w^T P w.$$ 

Then there exists $\tilde{\sigma}, \sigma > 0$, such that

$$\tilde{V}(x_+) - \tilde{V}(x) \leq -2\sigma \|x\|^2 + \sigma c(x, u) + \sigma w^T P w.$$

Then, we can choose $\sigma$ sufficiently small such that $\|x\|^2 \leq 1$ and $\sigma \lambda_{\max}(P) \leq \gamma^2$. Then, setting $\alpha = -2\sigma, \sigma_v = \sigma \lambda_{\max}(P)$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue, we see that the assumption is satisfied. This proof can be trivially extended to the time varying LQR cost by selecting $P$ as the solution of $A^T P A - P = -2I$ for a $Q \geq 0$ s.t. $Q \leq Q_t \forall t \in [1, T]$. Assumption 2(iii) follows from [Hassibi et al., 1999, Theorem 9.5.1].
A.2 Non-Quadratic, Non-Convex Cost Function Examples for Assumption 2

We can trivially construct many non-quadratic, non-convex cost functions that satisfy Assumption 2 for linear systems. Let the system \((A, B)\) be such that \(\rho(A) < 1\), and \(B = I\). Let the controller cost be

\[
c(x, u) = \|x\|^a + \|u\|^2,
\]

where \(a\) is any positive value. Let \(N_c = N_w = N > 1\). Given that \(N_w > 1\), the controller has access to \(w_t\) and so we can set \(u_t = -w_t\) to cancel out the disturbance. Then, given that \(\rho(A) < 1\), there exists a \(c_\rho \geq 1\), and \(\beta < 1\) when \(u_t = -w_t\) such that \(\|x_t\| \leq c_\rho \beta^{t-1} \|x_1\|\). Let \(\sigma(x) = \|x\|^a\). Then, when \(u_t = -w_t\),

\[
\sum_{t=1}^{N} c(x_t, u_t) \leq \left(\frac{c_\rho^a}{1 - \beta^a}\right) \sigma(x_1) + \sum_{t=1}^{N} \|w_t\|^2.
\]

Then

\[
V_t \leq \sum_{t=1}^{N} c(x_t, u_t) \leq \left(\frac{c_\rho^a}{1 - \beta^a}\right) \sigma(x_1) + \sum_{t=1}^{N} \|w_t\|^2.
\]

Then it is clear that \(\exists \gamma \leq 1\) such that \(\underline{\alpha} = 1, \overline{\alpha} = \left(\frac{c_\rho^a}{1 - \beta^a}\right)\), \(\sigma(x) = \|x\|^a\), and \(\gamma\) satisfy the assumptions. Thus, the cost functions can be non-convex or non-quadratic.

We give a second example. Consider the system

\[
x_{t+1} = \beta x_t + u_t + w_t, \quad 0 < \beta < 1, \text{ with } c(x, u) = \|x - b\|^a + \|u - b(1 - \beta)\|^2, \quad a > 0.
\]

Let \(u_t = b(1 - \beta) - w_t\). Let \(\sigma(x) = \|x - b\|^a\). Then, when \(u_t = b(1 - \beta) - w_t\),

\[
\sum_{t=1}^{N} c(x_t, u_t) \leq \frac{1}{1 - \beta^a} \|x_1 - b\|^a + \sum_{t=1}^{N} \|w_t\|^2 = \frac{1}{1 - \beta^a} \sigma(x_1) + \sum_{t=1}^{N} \|w_t\|^2.
\]

Thus, there exists \(\gamma \leq 1\), such that \(\underline{\alpha} = 1, \overline{\alpha} = \frac{1}{1 - \beta^a}\), \(\sigma(x) = \|x - b\|^a\), \(\gamma\) satisfy the assumptions. These examples illustrate that Assumption 2 includes non-convex and non-quadratic examples.

B Discussion on Complexity of Min-Max RHC

While RHC algorithms are complex because of the optimization, min-max optimization further exacerbate the difficulty because of the bi-level nature of the optimization. Min-max optimization of general non-convex and non-concave objective functions are known to be hard problems (Daskalakis et al., 2020). It was shown in Daskalakis et al. (2020) that finding an approximate local min-max point of large enough approximation is guaranteed to exist, but finding one such point is PPAD-complete. When the objective function \(L(x, y)\) is a convex-concave function, i.e., \(L\) is convex in \(x\) for all \(y\) and it is concave in \(y\) for all \(x\), the problem \(\min_{x, y} \max_{y} L(x, y)\) with constraints is guaranteed to have a solution, under compactness of the constraint set (Rosen, 1965), while computing a solution is amenable to convex programming. In fact, if \(L\) is \(L\)-smooth, the problem can be solved via first-order methods, and achieve an approximation error of \(\text{poly}(L, 1/T)\) in \(T\) iterations; see for e.g. Nemirovski (2004). When the function is strongly convex and strongly concave, the rate becomes geometric (Facchinei and Pang, 2007).
C Proof of Theorem 3

Proof. Let

$$\hat{u}_{t:t+N-1} := \arg \inf_{\hat{u}_{t:t+N-1}} \sum_{j=t}^{t+N-1} c_j(\hat{x}_j, \hat{u}_j), \quad \hat{x}_{j+1} = A\hat{x}_j + B\hat{u}_j + w_j, \quad \hat{x}_t = x_t.$$ 

Let $\phi_t(k, x_t, \hat{u}_{t:t+k-1}, w_{t:t+k-1})$ denote the state of the system at time $t + k$ following the control sequence $\hat{u}_{t:t+k-1}$ and disturbance $w_{t:t+k-1}$. For ease of illustration, we denote $V_t$, $x_t$, $\hat{u}_{t:t+k-1}$, $w_{t:t+k-1}$ and $\phi_t$ by $V_i$, $x_i$, $\hat{u}_{0:k-1}$, $w_{0:k-1}$ and $\phi_i$.

Let $n_m := N - M$, $x_{i+1} := \phi_{i+1}(j - 1, x_{i+1}, \hat{u}_{i+1}, w_{i+1})$ and $t_{i+1} = t_i + j - 1$. Now,

$$V_{i+1}(x_{i+1}, w_{0:N-1}^{i+1}) = \sum_{k=0}^{N-1} c_{i+1+k}(\phi_{i+1}(k, x_{i+1}, \hat{u}_{i+1}, w_{0:k-1}^{i+1}), \hat{u}_{k}^{i+1})$$

$$\leq \sum_{k=0}^{j-2} c_{i+1+k}(\phi_{i+1}(k, x_{i+1}, \hat{u}_{i+m}^{i}, w_{0:k-1}^{i+1}), \hat{u}_{k}^{i}) + \inf_{\hat{u}_0:N-j} \sum_{k=0}^{N-j} c_{i+1+k}(\phi_{i+1}(k, x_{i+1}, \hat{u}_{0:k-1}, w_{0:k-1}^{i+1}, \hat{u}_{j+1}^{i}), \hat{u}_{k})$$

$$\leq \sum_{k=0}^{j-2} c_{i+1+k}(\phi_{i+1}(k, x_{i+1}, \hat{u}_{i+m}^{i}, w_{0:k-1}^{i+1}), \hat{u}_{k}^{i}) + N \sigma(x_{i+1}^{j}) + \bar{\sigma}^2 \sum_{k=t_{i+1}}^{t_{i+1}+N} \|w_k\|^2.$$ 

Here, we get (a) by using the definition of $x_{i+1}$ and the fact that $\hat{u}_{0:N-1}^{i}$ is the optimal control sequence, and (b) by applying Assumption 2 (iii) to the second term.

Similarly,

$$V_i(x_i, w_{0:N-1}^{i}) = \sum_{k=0}^{N-1} c_{i+k}(\phi_i(k, x_i, \hat{u}_{0:k-1}^{i}, w_{0:k-1}^{i}), \hat{u}_{k}^{i})$$

$$\leq \sum_{k=0}^{n_m} c_{i+k}(\phi_i(k, x_i, \hat{u}_{i+m}^{i}, w_{0:k-1}^{i}), \hat{u}_{k}^{i}) + \sum_{k=0}^{M-n} c_{i+k}(\phi_i(k, x_i, \hat{u}_{n_m:n_m+k-1}^{i}, w_{n_m:n_m+k-1}^{i}), \hat{u}_{n_m+k}^{i})$$

$$\leq \sum_{k=0}^{n_m} c_{i+k}(\phi_i(k, x_i, \hat{u}_{i+m}^{i}, w_{0:k-1}^{i}), \hat{u}_{k}^{i}) + \sum_{k=0}^{j-2} c_{i+k}(\phi_i(k, x_i, \hat{u}_{n_m:n_m+k-1}^{i}, w_{n_m:n_m+k-1}^{i}), \hat{u}_{n_m+k}^{i})$$

$$+ \sum_{k=0}^{M-j} c_{i+j+k-1}(\phi_{i+1}(k, x_{i+1}, \hat{u}_{n_m+j-1:n_m+j+k-2}^{i}, w_{j-1:j+k-2}^{i}), \hat{u}_{n_m+j+k}^{i}).$$

Here, we get (c) by $\hat{u}_{0:n_m-1}^{i} = u_{t_i+n_m-1}$, $t_{i+1} = t_i + n_m$ and $\phi_i(n_m, x_i, \hat{u}_{0:n_m-1}^{i}, w_{0:n_m-1}^{i}) = x_{i+1}$, and (d) by splitting the second term further and using the definition of $x_{i+1}$.

Now, subtracting $V_i$ from $V_{i+1}$ and canceling the common terms, we get

$$V_{i+1}(x_{i+1}, w_{0:N-1}^{i+1}) - V_i(x_i, w_{0:N-1}^{i}) \leq \bar{\sigma}\sigma(x_{i+1}^{j}) + \bar{\sigma}^2 \sum_{k=t_{i+1}}^{t_{i+1}+N} \|w_k\|^2 - \sum_{k=0}^{n_m-1} c_{i+k}(\phi_i(k, x_i, u_{t_i:t_i+k-1}^{i}, w_{0:k-1}^{i}), u_{t_i+k}).$$

(10)

Let $\phi_i^j := \phi_i(t-t_i, x_i, \hat{u}_{0:t-t_i-1}^{i}, w_{0:t-t_i-1}^{i})$. Then, from Assumption 2 (ii), and recognizing that $x_{i+1}^j = \phi_{i+1}^{j+j-1}$, we get

$$\sum_{t=t_i}^{t_{i+1}-1} \sigma(\phi_i^j) + \sum_{j=1}^{M} \sigma(x_{i+1}^{j}) \leq \sum_{k=0}^{N-1} c_{i+k}(\phi_i(k, x_i, \hat{u}_{0:k-1}^{i}, w_{0:k-1}^{i}), \hat{u}_{k}^{i})$$

$$\Rightarrow \sum_{j=1}^{M} \sigma(x_{i+1}^{j}) \leq \sum_{k=0}^{N-1} c_{i+k}(\phi_i(k, x_i, \hat{u}_{0:k-1}^{i}, w_{0:k-1}^{i}), \hat{u}_{k}^{i}).$$
Then, by Assumption 2(iii), definition of \( \delta_{0,N-1} \) and recognizing that \( t_{i+1}^M = t_i^N \), we get

\[
\alpha \sum_{j=1}^{M} \sigma(x_{i+1}^j) \leq \sigma(x_i) + \gamma^2 \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2.
\]

Then, there exists a \( j^* \in [1, M] \) such that

\[
\sigma(x_{i+1}^{j^*}) \leq \frac{\gamma}{\alpha M} \sigma(x_i) + \frac{\gamma^2}{M} \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2.
\]

Then, setting \( j = j^* \) in Eq. (10), we get

\[
V_{i+1}(x_{i+1}, w_{0:N-1}^{i+1}) - V_i(x_i, w_{0:N-1}^i) \leq \frac{\gamma^2}{\alpha M} \sigma(x_i) + \frac{\gamma^2}{M} \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2 + \frac{\gamma^2}{M} \sum_{t=t_{i+1}^M+1}^{t_i+M-1} \|w_t\|^2
\]

\[
- \sum_{k=0}^{n_{-1}} c_{i+k}(\phi(k, x_i, u_{i:t_i+k-1}, w_{0:k-1}^i), u_{i+k})
\]

\[
= \frac{\gamma^2}{\alpha M} \sigma(x_i) + \frac{\gamma^2}{M} \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2 + \left( \frac{\gamma}{\alpha M} + 1 \right) \gamma^2 \sum_{t=t_{i+1}^M+1}^{t_{i+1}^M} \|w_t\|^2 + \frac{\gamma^2}{M} \sum_{t=t_{i+1}^M+1}^{t_i+M-1} \|w_t\|^2
\]

Then, applying Assumption 2(ii) to the last term we get

\[
V_{i+1}(x_{i+1}, w_{0:N-1}^{i+1}) - V_i(x_i, w_{0:N-1}^i) \leq \left( \frac{\gamma^2}{\alpha M} - 1 \right) \alpha \sigma(x_i) + \frac{\gamma^2}{M} \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2 + \left( \frac{\gamma}{\alpha M} + 1 \right) \gamma^2 \sum_{t=t_{i+1}^M+1}^{t_{i+1}^M} \|w_t\|^2
\]

Given that \( M > \frac{\gamma^2}{\alpha}, \frac{\alpha}{\alpha M} < 1 \). Then, using Assumption 2(iii), we get

\[
V_{i+1}(x_{i+1}, w_{0:N-1}^{i+1}) - V_i(x_i, w_{0:N-1}^i) \leq \frac{\alpha}{\alpha} \left( \frac{\gamma^2}{\alpha M} - 1 \right) \left( V_i(x_i, w_{0:N-1}^i) - \gamma \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2 \right) + \frac{\gamma^2}{M} \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2 + \left( \frac{\gamma}{\alpha M} + 1 \right) \gamma^2 \sum_{t=t_{i+1}^M+1}^{t_{i+1}^M} \|w_t\|^2
\]

Let \( a := 1 + \frac{\alpha}{\alpha} \left( \frac{\gamma^2}{\alpha M} - 1 \right) \). Then, it follows that \( 0 < a < 1 \). Then

\[
V_{i+1}(x_{i+1}, w_{0:N-1}^{i+1}) \leq a V_i(x_i, w_{0:N-1}^i) + \frac{\gamma^2}{\alpha} \sum_{t=t_i}^{t_{i+1}^M} \|w_t\|^2 + \left( \frac{\gamma}{\alpha} + 1 \right) \gamma^2 \sum_{t=t_{i+1}^M+1}^{t_{i+1}^M} \|w_t\|^2
\]

\[
\leq a V_i(x_i, w_{0:N-1}^i) + \frac{\gamma^2}{\alpha} \sum_{t=t_i}^{t_{i+N/2-1}} \|w_t\|^2 + \left( \frac{\gamma}{\alpha} + 1 \right) \gamma^2 \sum_{t=t_{i+N/2}^M}^{t_{i+1+N/2-1}} \|w_t\|^2 + \frac{\gamma^2}{M} \sum_{t=t_{i+N/2}^M}^{t_{i+1+N/2-1}} \|w_t\|^2.
\]
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Here, we get (e) by lowering the initial point of the sum in second term to $t_{i-1} + N/2$, increasing the final point of the sum in last term to $t_{i+2} + N/2 - 1$ and then absorbing terms from the first sum and the last sum in to the middle sum by using that $M \leq N/2$.

Then, repeatedly applying the previous inequality on the right of the inequality for each $i$, recognizing that $\sum_{t=1}^{t_{i+1}-1} c_t(x_t, u_t) \leq V_i(x_i, w_{i,N-1}^i)$, and summing over all $i$'s, we get

$$\sum_{t_1=1}^T c_t(x_t, u_t) \leq \frac{\pi}{1-a} \sigma(x_1) + \frac{(a^2 + a(\frac{\alpha}{\alpha} + 1) + \frac{\alpha}{\alpha})}{1-a} \sum_{t=1}^T \|w_t\|^2.$$

Let

$$\omega_{op} = \frac{(a^2 + a(\frac{\alpha}{\alpha} + 1) + \frac{\alpha}{\alpha})}{1-a}.$$

We first observe that $a = 1 + \beta(1/(\beta^2 M) - 1)$. Thus,

$$\frac{1}{1-a} = \frac{1}{\beta(1-1/(\beta^2 M))}.$$

Substituting for $a$ in the numerator of $\omega_{op}$, we get

$$a^2 + a(\frac{\alpha}{\alpha} + 1) + \frac{\alpha}{\alpha} = a^2 + a(\beta + 1) + \beta$$

$$= (1 - \beta)^2 + \frac{2}{\beta M} - \frac{2}{M} + \frac{2}{\beta^2 M^2} + (1 + \beta) \left(1 - \frac{1}{\beta M}\right) + \beta$$

$$= 1 - 2\beta + \beta^2 + \frac{2}{\beta M} - \frac{2}{M} + \frac{1}{\beta^2 M^2} + 1 - \beta^2 + \frac{1}{\beta M} + \frac{1}{M} + \beta$$

$$= 2 - \beta + \frac{3}{\beta M} + \frac{1}{\beta^2 M^2} + \frac{1}{M}.$$

Then, letting $\kappa(M) = \frac{3}{\beta M} + \frac{1}{\beta^2 M^2} - \frac{1}{M}$, we get

$$\omega_{op} = \frac{2 - \beta + \kappa(M)}{\beta(1-1/(\beta^2 M))}.$$
D Characterization of Algorithm 1 for Generalization of Assumption 2

First we state the main theorem and then present the proof.

**Lemma 1.** Suppose Assumption 1 and the generalization of Assumption 2 hold. Let \( N_c = N, N_w = N \). Let \( \pi = \pi_{op}, N \geq 2M \) and \( M > \frac{2c_T \gamma}{\alpha} \). Then for any \( T > N \),

\[
J(\pi_{op}) \leq \mathcal{O}(\sigma(x_1)) + (\omega_{op}^1 \mathbb{T}^2 + \omega_{op}^2 \gamma^2) \sum_{t=1}^{T} \|w_t\|_2^2, \quad \text{where}
\]

\[
\begin{align*}
\omega_{op}^1 &= \frac{2 - \beta_v + \kappa_1(M)}{\beta_v(1 - 1/(\beta_v M))}, \\
\omega_{op}^2 &= \frac{1 + \kappa_2(M)}{\beta_v(1 - 1/(\beta_v M))}, \\
\kappa_1(M) &= \frac{3}{\beta M} + \frac{1}{\beta^2 M^2} - \frac{\beta_v}{\beta M}, \\
\kappa_2(M) &= \frac{2}{\beta M} + \frac{1}{\beta^2 M^2} - \frac{\beta_v}{\beta M}.
\end{align*}
\]

**Proof.** The proof is exactly the same till Eq. (10) of the proof of Theorem 1. For the generalization of Assumption 2, the difference lies in the remaining part of the proof. To complete the proof, the argument below is to be applied after Eq. (10) instead of the argument in the proof of Theorem 1.

From the generalization of Assumption 2(iii) (see Appendix A.1), and recognizing that \( x_{i+1} = \phi_{i+1} \), we get

\[
\mathcal{A} \sum_{t=t_i}^{t_{i+1} - 1} \sigma(\phi_t^i) + \mathcal{A} \sum_{j=1}^{M} \sigma(x_{i+1}) \leq \sum_{t=t_i}^{N-1} c_{t+1+k}(\phi_t(k, x_i, u_{0:B-k-1}, w_{0:B-k-1}), \tilde{u}_k) + \tilde{V}(x_i) + \gamma^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|.
\]

This implies

\[
\mathcal{A} \sum_{j=1}^{M} \sigma(x_{i+1}) \leq \sum_{t=t_i}^{N-1} c_{t+1+k}(\phi_t(k, x_i, u_{0:B-k-1}, w_{0:B-k-1}), \tilde{u}_k) + \tilde{V}(x_i) + \gamma^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|.
\]

Then, by Assumption 2(iii), definition of \( u_{0:B-k-1} \) and recognizing that \( t_{i+1} = t_{i+1} \), we get that

\[
\mathcal{A} \sum_{j=1}^{M} \sigma(x_{i+1}^*) \leq \mathcal{A} \sigma(x_i) + \tilde{V}(x_i) + (\gamma^2 + \gamma^2) \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2.
\]

Then, there exists a \( j^* \in [1, M] \) such that

\[
\sigma(x_{i+1}^*) \leq \frac{\mathcal{A}}{\mathcal{A}M} \sigma(x_i) + \frac{\tilde{V}(x_i)}{\mathcal{A}M} + (\frac{\gamma^2 + \gamma^2}{\mathcal{A}M}) \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2.
\]

Then, using \( \tilde{V}(x_i) \leq \mathcal{A}, \sigma(x_i) \) we get that

\[
\sigma(x_{i+1}^*) \leq \frac{\mathcal{A} + \mathcal{A}}{\mathcal{A}M} \sigma(x_i) + (\frac{\gamma^2 + \gamma^2}{\mathcal{A}M}) \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2.
\]

Then, setting \( j = j^* \) in Eq. (10) and rearranging terms, we get

\[
\begin{align*}
V_{i+1}(x_{i+1}, w_{0:B-1}) - V_i(x_i, w_{0:B-1}) &\leq - \sum_{k=0}^{n_{i+1} - 1} c_{t+1+k}(\phi_t(k, x_i, u_{0:B-k-1}, w_{0:B-k-1}), \tilde{u}_k) + \mathcal{A} (\mathcal{A} + \mathcal{A}) \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2 + (\frac{\mathcal{A}}{\mathcal{A}M} + 1) \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2 + \frac{(\gamma^2 + \gamma^2)}{\mathcal{A}M} \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2 + \frac{\gamma^2}{\mathcal{A}M} \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2.
\end{align*}
\]
Let $Y_i := V_i + \bar{V}(x_i)$. Then, applying the generalization of Assumption 2(ii) to the first term and rearranging terms we get

$$Y_{i+1} - Y_i \leq \left(\frac{\alpha}{\alpha + \alpha_v} - 1\right) \omega \sigma(x_i) + \frac{\alpha}{\alpha + \alpha_v} \omega^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2 + \left(\frac{\alpha}{\alpha M} + 1\right) \gamma^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2$$

$$+ \left(\frac{\alpha}{\alpha + \alpha_v} + 1\right) \gamma^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2 + \left(\frac{\alpha}{\alpha M} + 1\right) \gamma^2 \sum_{t=t_{i+1}}^{t_{i+1} + M} \|w_t\|^2.$$

From Assumption 2(iii) we have $\sigma(x_i) \geq \frac{1}{\alpha + \alpha_v} \left(Y_i - \gamma^2 \sum_{t=t_i}^{t_i+N} \|w_t\|^2\right)$. Since $\left(\frac{\alpha}{\alpha + \alpha_v} - 1\right) < 0$, this implies

$$Y_{i+1} - Y_i \leq \left(\frac{\alpha}{\alpha + \alpha_v} - 1\right) \frac{\alpha}{\alpha + \alpha_v} \omega Y_i + \frac{\alpha}{\alpha + \alpha_v} \omega^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2$$

$$+ \left(\frac{\alpha}{\alpha + \alpha_v} + 1\right) \gamma^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2 \leq aY_i + \frac{\alpha}{\alpha + \alpha_v} \gamma^2 \sum_{t=t_i}^{t_{i+1} - N/2} \|w_t\|^2 + \frac{\alpha}{\alpha + \alpha_v} \gamma^2 \sum_{t=t_{i+1} + N/2}^{t_{i+1} + N/2 - 1} \|w_t\|^2$$

$$+ \left(\frac{\alpha}{\alpha M} + 1\right) \gamma^2 \sum_{t=t_i}^{t_{i+1} - 1} \|w_t\|^2.$$ 

Here, we get (a) by the same trick as in the proof of Theorem 1. Since $M > \frac{\alpha}{\alpha + \alpha_v}$, $a$ satisfies $0 < a < 1$. Then, repeatedly applying the previous inequality on the right of the inequality for each $i$, recognizing that $Y_i \geq \sum_{t=t_i}^{t_{i+1} - 1} c_t(x_i, u_t)$, and summing over all $i$, we get

$$\sum_{t=1}^{T} c_t(x_i, u_t) \leq \frac{\alpha}{1 - a} \omega \sigma(x_1) + \frac{a}{1 - a} \left(\frac{a}{\alpha + \alpha_v} + 1\right) \gamma^2 \sum_{t=1}^{T} \|w_t\|^2 + \frac{a}{1 - a} \frac{\alpha}{\alpha M} \left(\frac{\alpha}{\alpha M} + 1\right) \gamma^2 \sum_{t=1}^{T} \|w_t\|^2.$$ 

Let

$$\omega_1^1 = \frac{a^2 + a\left(\frac{\alpha}{\alpha + \alpha_v} + 1\right)}{1 - a}, \quad \omega_2^1 = \frac{a}{\alpha M} + \left(\frac{\alpha}{\alpha M} + 1\right) \gamma^2$$

We first observe that $a = 1 + \beta_v(1/(\beta_v M) - 1)$. Thus,

$$\frac{1}{1 - a} = \beta_v(1 - 1/(\beta_v M)).$$
Substituting for $a$ in the numerator of $\omega^1_{\text{op}}$,
\[
a^2 + a(\frac{\alpha}{\alpha + \alpha_v} + 1) + \frac{\alpha}{\alpha + \alpha_v} = a^2 + a(\beta_v + 1) + \beta_v = (1 - \beta_v)^2 + \frac{2}{\beta M} - \frac{2\beta_v}{\beta M} + \frac{1}{\beta^2 M^2} + 1 - \beta_v + \frac{1}{\beta M} + \frac{\beta_v}{\beta M} + \beta_v = 2 - \beta_v + \frac{3}{\beta M} + \frac{1}{\beta^2 M^2} - \frac{\beta_v}{\beta M}.
\]

Thus,
\[
\omega^1_{\text{op}} = \frac{2 - \beta_v + \kappa_1(M)}{\beta_v(1 - 1/(\beta \beta_v M))}, \quad \kappa_1(M) = \frac{3}{\beta M} + \frac{1}{\beta^2 M^2} - \frac{\beta_v}{\beta M}.
\]

Substituting for $a$ in the numerator of $\omega^2_{\text{op}}$, we get
\[
\omega^2_{\text{op}} = \frac{a \pi}{\alpha M} + \left( \frac{\pi}{\alpha M} + 1 \right) = (1 - \beta_v + \frac{1}{\beta M}) - \frac{1}{\beta M} + \frac{1}{\beta M} + 1 = 1 + \frac{2}{\beta M} + \frac{1}{\beta^2 M^2} - \frac{\beta_v}{\beta M}.
\]

Letting $\kappa_2(M) = \frac{2}{\beta M} + \frac{1}{\beta^2 M^2} - \frac{\beta_v}{\beta M}$, we get
\[
\omega^2_{\text{op}} = \frac{1 + \kappa_2(M)}{\beta_v(1 - 1/(\beta \beta_v M))}.
\]

We note that for the generalization of Assumption 2, $\pi$ need not be greater than $\alpha$. Rather it should be that $\pi + \pi_v \geq \alpha$. As in the specialized case, a reasonable value for $M$ is given by $M = N/2$. For this value of $M$, we observe that both $\omega^1_{\text{op}}$ and $\omega^2_{\text{op}}$ decrease as $1/N$ since $\kappa_1$ and $\kappa_2$ decrease as $1/N$. And when $N$ is large, it is easy to see that $\omega^1_{\text{op}} = (2 - \beta_v)/\beta_v, \omega^2_{\text{op}} = 1/\beta_v$. As in the specialized case, here too, we observe that the minimum achievable disturbance gain for our algorithm is dependent on the system constants $\pi, \pi_v, \alpha$, although of a slightly different form. Since $\beta_v \leq 1$, $\omega^1_{\text{op}} \geq 1$, which is as expected.

Next we use the above Lemma to prove the result for the generalization.

**Theorem 5.** Suppose Assumption 4 and generalization of Assumption 2 hold. Let $\pi = \pi_{\text{op}}, \sigma(x_1) = 0$ and $M = N/2$. Let $N > 4\zeta^3$ and $\beta_v \geq 1/\zeta$ for some $\zeta > 1$. Then, for any $T > N$
\[
J(\pi_{\text{op}}) \leq \gamma^2_{\text{op}} \sum_{t=1}^{T} \|w_t\|_2^2, \quad \text{where}
\]
\[
\gamma^2_{\text{op}} = (2\zeta + \rho_1(N))\kappa^2 + (2\zeta + \rho_2(N))\gamma^2, \quad \rho_1(N) \leq O(1/N), \quad \rho_2(N) \leq O(1/N),
\]

i.e., Algorithm 1 achieves a disturbance gain $\gamma^2_{\text{op}}$.

**Proof.** First, we note that $M = N/2$. Next, we note that $N > 4\zeta^3 > 2\zeta^2$. Then, given that $\zeta \geq 1/\beta_v \geq 1/\beta, N > 2/(\beta \beta_v)$. That is $M > 1/(\beta \beta_v)$. Thus, all the conditions in Lemma 1 are satisfied.

Then, given that $\sigma(x_1) = 0, J(\pi_{\text{op}}) \leq (\omega^1_{\text{op}} + \omega^2_{\text{op}}\gamma^2) \sum_{t=1}^{T} \|w_t\|_2^2$. Substituting $M = N/2$, we get
\[
\omega^1_{\text{op}} = \frac{2 - \beta + \kappa_1(N/2)}{\beta_v(1 - 1/(\beta \beta_v N))}, \quad \omega^2_{\text{op}} = \frac{1 + \kappa_2(N/2)}{\beta_v(1 - 1/(\beta \beta_v N))}.
\]

Given that $N > 4\zeta^3$ and $\beta \geq \beta_v \geq 1/(\zeta)$, we get that
\[
\frac{1}{2\zeta} \geq \frac{2\zeta^2}{N} \geq \frac{2}{N\beta \beta_v}, \quad \text{i.e.,} \quad \frac{1}{\beta_v(1 - 1/(\beta \beta_v N))} \leq \frac{2\zeta^2}{2\zeta - 1}.
\]
By definition $\beta_v \leq 1$. Hence 
\[ \kappa_1(N/2) = \frac{6}{\beta N} + \frac{4}{(\beta N)^2} - \frac{2\beta_v}{\beta N} = \frac{6 - 2\beta_v}{\beta N} + \frac{4}{(\beta N)^2} \leq \mathcal{O}\left(\frac{1}{N}\right). \]

Thus 
\[ \frac{\kappa_1(N/2)}{\beta_v(1 - 2/(\beta\beta_v N))} \leq \kappa_1(N/2) \frac{2\zeta^2}{2\zeta - 1} \leq \mathcal{O}\left(\frac{1}{N}\right). \]

Also, given that $\beta_v \geq 1/\zeta$, 
\[ \frac{2 - \beta_v}{\beta_v(1 - 2/(\beta\beta_v N))} \leq 2\zeta. \]

Thus, letting $\rho_1(N) = \kappa_1(N/2)/(\beta_v(1 - 2/(\beta\beta_v N)))$ 
\[ \omega_{op}^1 \leq 2\zeta + \rho_1(N), \quad \rho_1(N) \leq \mathcal{O}(1/N). \]

Similarly, since $\beta_v \leq 1$,
\[ \kappa_2(N/2) = \frac{4}{\beta N} + \frac{4}{\beta^2 N^2} - \frac{2\beta_v}{\beta N} = \frac{4 - 2\beta_v}{\beta N} + \frac{4}{\beta^2 N^2} \leq \mathcal{O}(1/N). \]

Then 
\[ \frac{\kappa_2(N/2)}{\beta_v(1 - 2/(\beta\beta_v N))} \leq \kappa_2(N/2) \frac{2\zeta^2}{2\zeta - 1} \leq \mathcal{O}(1/N). \]

Given that $N > 4\zeta^3$, $\beta \geq \beta_v \geq 1/(\zeta)$ and $\zeta > 1$, we also get that 
\[ \frac{1}{2} > \frac{1}{2\zeta} > \frac{2\zeta^2}{N} \geq \frac{2}{N\beta\beta_v}, \quad \text{i.e.,} \quad \frac{1}{\beta_v(1 - 2/(\beta\beta_v N))} \leq \frac{\zeta}{1 - 1/2} \leq 2\zeta. \]

Thus, letting $\rho_2(N) = \frac{\kappa_2(N/2)}{\beta_v(1 - 2/(\beta\beta_v N))}$, we get 
\[ \omega_{op}^2 \leq 2\zeta + \rho_2(N), \quad \rho_2(N) \leq \mathcal{O}(1/N). \]
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E Proof of Theorem 4

Proof. Let

\[ (\tilde{u}_{t:t+N-1}, \tilde{w}_{t:t+N-1}) := \arg \inf_{u_{i:t} \in \mathcal{W}_{t:N-1}} \sup_{\tilde{w}_{i:t} \in \mathcal{W}_{t:N-1}} \sum_{j=t}^{t+N-1} c_j(\tilde{x}_j, \tilde{u}_j), \quad \tilde{x}_{j+1} = A\tilde{x}_j + B\tilde{u}_j + \tilde{w}_j, \quad \tilde{x}_t = x_t. \]

Let \( \phi_i(k, x_t, \tilde{u}_{0:k-1}, \tilde{w}_{0:k-1}) \) denote the state of the system at time \( t+k \) under the control sequence \( \tilde{u}_{0:k-1} \) and disturbance sequence \( \tilde{w}_{0:k-1} \) while starting at \( x_t \). For ease of illustration, we denote \( V_t, x_t, \tilde{u}_{0:N-1}, \tilde{w}_{0:N-1} \) by \( V_i, x_i, \tilde{u}_{0:N-1}, \tilde{w}_{0:N-1}, \phi_i \).

Let \( n_m := N - M, x_{i+1}^j := \phi_{i+1}(j - 1, x_{i+1}, \tilde{u}_{n_m:n_m+j-2}, \tilde{w}_{0:j-2}) \) and \( t_i^{j+1} := t_i^j + 1 - j \). Now,

\[
V_{i+1}(x_{i+1}, \tilde{u}_{0:N-1}^i) = \sum_{k=0}^{N-1} c_{t_i+k}(\phi_{i+k}(k, x_{i+1}, \tilde{u}_{n_m:n_m+k-1}, \tilde{w}_{0:k-1}), \tilde{u}_{n_m+k}) + \inf_{\tilde{u}_{0:N-j}} \sum_{k=0}^{N-j} c_{t_i+k}(\phi_{i+k}(k, x_{i+1}, \tilde{u}_{n_m:n_m+k-1}, \tilde{w}_{0:k-1}), \tilde{u}_{n_m+k}) + \tilde{\sigma}(x_{i+1}^{j+1}) + \tilde{\sigma}(N-j) \max_{w \in \mathcal{W}} \|w\|^2.
\]

Here, we get (a) by using the definition of \( x_{i+1}^j \) and the fact that \( \tilde{u}_{0:N-1}^i \) is the optimal control sequence, and (b) by applying Assumption (iii) to the second term.

Similarly,

\[
V_i(x_i, \tilde{u}_{0:N-1}^i) = \sum_{k=0}^{N-i-1} c_{t_i+k}(\phi_{i+k}(k, x_i, \tilde{u}_{n_m:n_m+k-1}, \tilde{w}_{0:k-1}), \tilde{u}_{n_m+k}) + \sum_{j=0}^{M-j} c_{t_i+j}(\phi_{i+j}(k, x_i, \tilde{u}_{n_m:n_m+j+k-2}, \tilde{w}_{0:j+k-2}), \tilde{u}_{n_m+j+k-1})
\]

Here, we get (c) by recognizing that \( \tilde{u}_{0:n_m-1}^i = u_{t_i:t_i+n_m-1} \), and (d) by using the fact that the maximizing disturbance sequence under \( \tilde{u}_{0:N-1}^i \) is \( \tilde{w}_{0:N-1}^i \), then splitting the second term further and using the definition of \( x_{i+1} \).

Taking the difference between \( V_{i+1} \) and \( V_i \), and cancelling out the common terms, we get

\[
V_{i+1}(x_{i+1}, \tilde{w}_{0:N-1}^{i+1}) - V_i(x_i, \tilde{u}_{0:N-1}^i) \leq \tilde{\sigma}(x_{i+1}^{j+1}) + \tilde{\sigma}(N-j) \max_{w \in \mathcal{W}} \|w\|^2 - \sum_{k=0}^{n_m-1} c_{t_i+k}(\phi_{i+k}(k, x_i, u_{t_i:t_i+n_m-1}, \tilde{w}_{0:k-1}), u_{t_i+k}).
\]
Let $\phi^i_t := \phi_t(t-t_i, x_i, \tilde{u}_{0:M-1}^{i,t}, (w^i_{0:m-1}, \tilde{w}_{0:M-1}^{i,t})_{t=t_i-1})$. Then, by Assumption 2(ii), and recognizing that $x^j_{i+1} = \phi^{t_{i+1}}_i$, we get

$$
\sum_{t=i}^{t_{i+1}-1} \sigma(\phi^i_t) + \sum_{j=1}^{M} \sigma(x^j_{i+1}) \leq \sum_{k=0}^{N-1} c_{i+k}(\phi_k(k, x_i, \tilde{u}_{0,k-1}^i, (w^i_{0:m-1}, \tilde{w}_{0:M-1}^{i,t})_{t=k-1}, \tilde{u}_k^i))
$$

(c) $\leq \sum_{k=0}^{N-1} c_{i+k}(\phi_k(k, x_i, \tilde{u}_{0,k-1}^i, \tilde{w}_{0,k-1}^i), \tilde{u}_k^i) \leq \overline{\sigma}(x_i) + \gamma^2 N \max_{w \in W} \|w\|_2.$

Here, we get (e) by the fact that $\tilde{w}_{0:N-1}^i$ is the maximizing disturbance sequence under $\tilde{u}_{0:N-1}^i$ and (f) by Assumption 2(iii).

Hence, there exists a $j^* \in [1, M]$, such that

$$
\sigma(x^j_{i+1}) \leq \overline{\sigma} \alpha M + \gamma^2 \max_{w \in W} \|w\|_2.
$$

Then, setting $j = j^*$ in Eq. (11), we get

$$
V_{i+1}(x_{i+1}, \tilde{w}_{0:N-1}^{i+1}) - V_i(x_i, \tilde{w}_{0:N-1}^i) \leq \overline{\sigma} \alpha M + \gamma^2 \max_{w \in W} \|w\|_2
$$

and

$$
\leq \left( \frac{\overline{\sigma}}{\alpha M} - 1 \right) \alpha \sigma(x_i) + \left( \frac{\overline{\sigma}}{\alpha M} + 1 \right) \gamma^2 N \max_{w \in W} \|w\|_2.
$$

Here, we get (g) by applying Assumption 2(ii) to the last term in the previous line and rearranging terms.

Since $M > \overline{\sigma}^2/\alpha^2$, \left( \frac{\overline{\sigma}^2}{\alpha^2 M} - 1 \right) < 0$. Then, by applying Assumption 2(iii) to the first term on the right and rearranging terms we get

$$
V_{i+1}(x_{i+1}, \tilde{w}_{0:N-1}^{i+1}) \leq \left( 1 + \frac{\alpha}{\overline{\sigma}} \left( \frac{\overline{\sigma}^2}{\alpha^2 M} - 1 \right) \right) V_i(x_i, \tilde{w}_{0:N-1}^i) + \left( 1 + \frac{\alpha}{\overline{\sigma}} \right) \gamma^2 N \max_{w \in W} \|w\|_2.
$$

Let $a := \left( 1 + \frac{\alpha}{\overline{\sigma}} \left( \frac{\overline{\sigma}^2}{\alpha^2 M} - 1 \right) \right)$. Since $M > \overline{\sigma}^2/\alpha^2$, $a$ satisfies $0 < a < 1$. Hence, by repeatedly applying the previous inequality, we get

$$
V_{i+1}(x_{i+1}, \tilde{w}_{0:N-1}^{i+1}) \leq a^i \overline{\sigma}(x_1) + \left( 1 + \frac{\alpha}{\overline{\sigma}} \right) \gamma^2 N \max_{w \in W} \|w\|_2.
$$

By definition $V_{i+1}(x_{i+1}, \tilde{w}_{0:N-1}^{i+1}) \geq \sum_{t=i}^{t_{i+1}-1} c_t(x_t, u_t)$. Using this in the previous inequality and summing over all is we get

$$
\sum_{t=i}^{T} c_t(x_t, u_t) \leq \overline{\sigma} \alpha MT \max_{w \in W} \|w\|_2.
$$

Then, using the fact that $\beta = \alpha/\overline{\sigma}$, $\alpha \leq N/2$ and $1/(1-a) = \frac{1}{\beta(1-1/\beta^2 M)}$, we get the final result. $\square$