The chemical potential in the transfer matrix and in the path integral formulation of QCD on a lattice

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November 10, 2021

Abstract

We define the chemical potential as the Lagrange multiplier of the baryon charge operator in the transfer matrix formalism of QCD on a lattice. Transforming the partition function into an euclidean path integral we get the Hasenfratz-Karsh action both for Wilson and Kogut-Susskind fermions. In the latter case the chemical potential in the spin-diagonal basis is half that in the flavour basis. Some open problems in the spin-diagonal basis are pointed out.

*This work has been partially supported by EEC under TMR contract ERB FMRX-CT96-0045
1 Introduction

Hasenfratz and Karsh [1] noticed that if in the euclidean path integral of relativistic field theories regularized on a lattice the chemical potential is coupled to the Fermi fields as in classical systems, specific counterterms are needed which would be extremely unconvenient in numerical simulations. Guided by the analogy between chemical potential and gauge fields, they exponentiated the chemical potential in the same way as the gauge fields are exponentiated in the link variables of Wilson regularization. They also showed that this definition does not require any counterterms (in the free theory), which makes it a good candidate for numerical simulations.

From the point of view of renormalization the way the chemical potential is introduced is obviously not unique [2], so that one is free to look for the form which is most convenient for a specific purpose. But Gavai [3] gave an argument whereby in any case the chemical potential cannot couple linearly to the fermion fields in the path integral.

There are a few points which in our view should be reexamined for a fully satisfactory settlement of the issue, the more so in view of the long standing difficulties met in numerical simulations [4].

For instance, the heuristic argument does not apply with a noncompact regularization [5], where the gauge fields still couple linearly to the Fermi fields. Should one in such a case choose a different coupling of the chemical potential? Moreover in statistical mechanics the chemical potential is the Lagrange multiplier of a conserved operator, a feature which is lost if it is exponentiated.

A last point to be clarified concerns the existence of the transfer matrix for the different fermion regularizations. This is of course a prerequisite for the lattice formulation to be correct, but its role becomes apparent in the present approach. We start from the observation that in classical statistical mechanics the partition function is the trace of the exponential of (minus) the hamiltonian. Now in the corresponding formulation of QCD the partition function is the trace of the transfer matrix, whose logarithm defines the hamiltonian. It is in this formulation that the chemical potential should behave as a Lagrange multiplier. We adopt this definition and show how the manipulations necessary to transform the trace into a path integral change the coupling of the chemical potential to the Fermi fields from linear to exponential. Following this prescription we find in fact exactly the result of Hasenfratz and Karsh both for Wilson and Kogut-Susskind fermions, with
the qualification discussed below. In the latter case the value of the chemical potential in the spin-diagonal basis is half that in the flavour basis, namely their ratio is the inverse of that of the cutoffs.

In our derivation we assume a given form for the transfer matrix. Now for Wilson fermions the transfer matrix was constructed by Lüscher \cite{6} in terms of quark-antiquark creation-annihilation operators, whereby the hamiltonian and the baryonic charge find the proper definition. The proof of reflection positivity was given in the gauge $U_0 = 1$. This gauge fixing is too strong, but Menotti and Pelissetto \cite{7} have extended the proof to the gauge $U_0 \sim 1$, where $U_0 = 1$ with the exception of a single time slice.

For the case of Kogut-Susskind we have not found in the literature a transfer matrix with the same desirable features. In the only paper on the subject \cite{8} we are aware of, the transfer matrix is constructed in the spin-diagonal basis, where it is found necessary to introduce auxiliary fields whose relation to quark-antiquarks is not transparent. Another feature, possibly related to the above, might be a difficulty in the way of the application we are interested in: the transfer matrix is linear in the creation-annihilation operators, rather than exponential, so that the hamiltonian contains the logarithm of these operators.

We have therefore tackled \cite{9} the problem in the flavour basis, and we have found that the formalism developed by Lüscher \cite{6} for Wilson fermions can easily be extended to the case of Kogut-Susskind, provided the elementary gauge variables are attached to the links of the blocks rather than of the lattice. This excludes the possibility of getting the action in the spin-diagonal basis from that in the flavour basis by a simple change of variables. If instead the elementary gauge variables are associated to the links of the lattice, we can transform to the spin-diagonal basis, but we meet with a formal difficulty and a practical complication. In fact the proof by Menotti and Pelissetto cannot be adapted straightforwardly, because their gauge fixing has in such a case a nontrivial Faddeev-Popov determinant (as explained in detail below) which must be taken into account to establish the reflection positivity of the transfer matrix. This is the formal difficulty. The practical complication is the following. We have constructed the transfer matrix in the flavour basis with the ”minimal” gauge coupling, the one where the link variables join the quark fields along the shortest path. Now it is well known that a gauge coupling simple in one basis becomes very complicated in the other \cite{10}, and in fact we find an awkward result in the spin-diagonal basis.

Both shortcomings can hopefully be overcome, but the construction of
the transfer matrix with Kogut-Susskind fermions in the spin-diagonal basis requires further study.

Two final observations. The first is that no claim of uniqueness is made about the coupling of the chemical potential in the path integral formulation for a given transfer matrix. The question of uniqueness will be discussed in separate paper.

The second is that the procedure just outlined appears also appropriate in the study of a system in the canonical ensemble, where since one has to integrate over the (imaginary) chemical potential \( \mu \), the exact way this is introduced can be of practical importance.

The paper is organized in the following way. We first show how the present prescription reproduces the Hasenfratz-Karsh result for Wilson fermions, by using Lüscher’s \( [6] \) construction of the transfer matrix. We then apply the same procedure to Kogut-Susskind fermions, adopting the notations of Montvay and Münster \( [10] \), and using the transfer matrix especially devised for the present application \( [9] \).

In this paper the pure gauge part of the transfer matrix and the partition function will be omitted because it does not play any explicit role (but with the Kogut-Susskind fermions in the spin-diagonal basis) and the lattice spacing is set equal to 1.

2 Transfer matrix and chemical potential

We start from the definition of the grand canonical partition function at finite baryon density according to the ordered product

\[
Z = Tr \left\{ \exp \left( \frac{\mu}{T} \hat{Q}_B \right) \prod_{n_0} \hat{T}_q(n_0) \right\},
\]

where the temperature \( T \) is the inverse of the number \( N_0 \) of links in the temporal direction, \( \hat{Q}_B \) is the baryon charge operator and \( \hat{T}_q \) is the quark transfer matrix. The chemical potential \( \mu \) is determined by the condition that the expectation value of the baryon number be

\[
q_B = Z^{-1} Tr \left\{ \hat{Q}_B \exp(\mu N_0 \hat{Q}_B) \prod_{n_0} \hat{T}_q(n_0) \right\}.
\]
The transfer matrix $\hat{T}_q$ can be written in terms of an auxiliary operator $\hat{T}_q(n_0)$

$$\hat{T}_q(n_0) = J^{-1} \hat{T}_q^\dagger(n_0) \hat{T}_q(n_0 + s),$$

where $J$ is a function of the gauge fields which will be defined later and $s = \pm 1$ for Wilson/Kogut-Susskind fermions respectively. This difference in sign does not reflect any intrinsic difference, but is only due to the different conventions adopted by Lüscher and Montvay-Münster, which we maintain for easy reference. As a consequence of these conventions the quarks propagate backwards in time with the Wilson definition and forwards with the Kogut-Susskind one.

$\hat{T}_q$ is defined in terms of quark-antiquark creation-annihilation operators $\hat{x}^\dagger, \hat{y}^\dagger, \hat{x}, \hat{y}$ acting in a Fock space. It depends on the time coordinate $n_0$ only through the dependence on it of the gauge fields. In fact the creation and annihilation operators do not depend on $n_0$. They depend on the spatial coordinates $n$ of the sites or, in the case of Kogut-Susskind in the flavour basis, of the blocks, and on Dirac, flavour and color indices, $\alpha, f, c$ sometimes comprehensively represented by $I$.

In the transfer matrix formalism most often one has to do with quantities at a given (euclidean) time $n_0$. For this reason we adopt a summation convention over spatial coordinates and intrinsic indices at fixed time. So for instance we will write

$$\overline{q}(m_0)Q(m_0, n_0)q(n_0) = \sum_{m, n, I, J} \overline{q}_{m, I}(m_0)Q_{m, I, n, J}(m_0, n_0)q_{n, J}(n_0),$$

where $Q$ is the quark matrix and $q$ the quark field. $Q$ is a function of time dependent link operators $U_\mu(m_0)$ which have the standard Wilson variables $U_\mu(m_0, m)$ as spatial matrix elements

$$(U_\mu(m_0))_{m, n} = \delta_{m, n}U_\mu(m_0, m).$$

In this notation the quark action $S_q$, the baryonic charge $\hat{Q}_B$ and the auxiliary operator $\hat{T}_q(n_0)$ can be written

$$S_q = \sum_{m_0, n_0} \overline{q}(m_0)Q(m_0, n_0)q(n_0)$$

$$\hat{Q}_B = \hat{x}^\dagger \hat{x} - \hat{y}^\dagger \hat{y}$$

$$\hat{T}_q(n_0) = \exp \left[ -\hat{x}^\dagger M(n_0) \hat{x} - \hat{y}^\dagger M(n_0) \hat{y} \right] \exp \left[ \hat{y} N(n_0) \hat{x} \right].$$
Their form is the same for Wilson and Kogut-Susskind fermions but the matrices $Q$, $M$ and $N$ are different in the two cases and will be specified later. The expression of $\hat{T}_q(n_0)$ is valid in the gauge $U_0 = \mathbb{1}$ which is not admissible. In the construction of the path integral formulation of QCD at finite baryon density we do not need to fix the gauge, but to lighten the formalism we will nevertheless put $U_0 = \mathbb{1}$ and we will reinstate $U_0$ in the final result. The reader can check that keeping $U_0$ in the intermediate steps one arrives at the same result, provided some care is exercised: for instance when $U_0 \neq \mathbb{1}$ the expression $\hat{x}^\dagger M(n_0) \hat{x} + \hat{y}^\dagger M(n_0) \hat{y}$ appearing in Eq. (8) changes and does not commute with $Q_B$ any longer. We anticipate that $N$ is hermitean and also $M$ is hermitean in the gauge $U_0 = \mathbb{1}$.

Now we transform the trace into a Berezin integral. To this end we introduce between the factors in Eq. (1) the identity

$$\mathbb{1} = \int [dx^+ dx dy^+ dy] \exp(-x^+ x - y^+ y)|xy><xy|,$$  

where the basis vectors

$$|xy> = |\exp(-x \hat{x}^\dagger - y \hat{y}^\dagger)>$$

are coherent states and the $x^+, x, y^+, y$ are Grassmann variables. They are labeled by the time slice where the unit operator is introduced. For the other indices they are subject to the same convention as the creation and annihilation operators.

We then use the following equations [6]. First

$$<x| \exp(\hat{x}^\dagger M \hat{x}) |x'> = \exp(x^+ e^M x'),$$  

In particular

$$<x, y| \exp(\mu Q_B)|x', y'> = \exp\left(e^\mu x^+ x' + e^{-\mu} y^+ y'\right),$$  

which shows how the chemical potential gets exponentiated going from the trace of the transfer matrix to the euclidean path integral.

Second, for arbitrary matrices $M$ and $N$

$$<x| \exp(\hat{x}^\dagger M \hat{x}) \exp(\hat{x}^\dagger N \hat{x}) |x'> = \exp\left(x^+ e^M e^N x'\right).$$  

Finally, if $B = B(\hat{x}^\dagger)$ and $C = C(\hat{x})$ are operators which depend on $\hat{x}^\dagger, \hat{x}$ only

$$<x|BM C|x'> = B(x^+) <x|M|x' > C(x').$$  

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Using these equations we can evaluate the kernel of the transfer matrix
\[
< x(n_0), y(n_0) | \mathcal{T}_\tau(n_0) | x(n_0 + s), y(n_0 + s) > = \mathcal{J}^{-1} \exp \left( x^+(n_0) N(n_0) y^+(n_0) \right)
\cdot \exp \left( x^+(n_0) \exp (-M(n_0)) \exp (-M(n_0 + s)) \exp(\mu) x(n_0 + s) \right)
\cdot \exp \left( y^+(n_0) \exp (-M(n_0)) \exp (-M(n_0 + s)) \exp(-\mu) y(n_0 + s) \right)
\cdot \exp \left( y(n_0 + s) N(n_0 + s) x(n_0 + s) \right).
\] (15)

Collecting all the pieces we arrive at the euclidean path integral form of the partition function
\[
Z = \mathcal{J}^{-1} \int [dx^+ dx dy^+ dy] \exp S'
\] (16)

with the action
\[
S' = \exp(N_0 \mu) x^+(1) x(1 + s) + \exp(-N_0 \mu) y^+(1) y(1 + s)
+ \sum_{n_0} \left\{ (1 - \delta_{n_0,1}) x^+(n_0) \exp(-M(n_0)) \exp(-M(n_0 + s)) x(n_0 + s)
+ (1 - \delta_{n_0,1}) y^+(n_0) \exp(-M(n_0)) \exp(-M(n_0 + s)) y(n_0 + s)
+ x^+(n_0) N(n_0) y^+(n_0) + y(n_0 + s) N(n_0 + s) x(n_0 + s)
- x^+(n_0) x(n_0) - y^+(n_0) y(n_0) \right\}.
\] (17)

The chemical potential appears only on the first site. We can arrive at a uniform distribution by the change of variables
\[
x(n_0) \rightarrow \exp(s\alpha(n_0)) x(n_0), \quad x^+(n_0) \rightarrow \exp(-s\alpha(n_0)) x^+(n_0),
\]
\[
y(n_0) \rightarrow \exp(-s\alpha(n_0)) y(n_0), \quad y^+(n_0) \rightarrow \exp(s\alpha(n_0)) y^+(n_0)
\] (18)

where
\[
\alpha(n_0) = \alpha(1) + (n_0 - 1) \mu - \frac{1}{2} (1 + s) N_0 \mu, \quad n_0 > 1,
\] (19)

with \(\alpha(1)\) arbitrary and \(\alpha(n_0) = \alpha(n_0 + N_0)\) to respect the antiperiodicity in time of the quark field. We thus get
\[
S' = \rightarrow \sum_{n_0} \exp(\mu) x^+(n_0) \exp(-M(n_0)) \exp(-M(n_0 + s)) x(n_0 + s)
+ \exp(-\mu) y^+(n_0) \exp(-M(n_0)) \exp(-M(n_0 + s)) y(n_0 + s)
+ x^+(n_0) N(n_0) y^+(n_0) + y(n_0 + s) N(n_0 + s) x(n_0 + s)
- x^+(n_0) x(n_0) - y^+(n_0) y(n_0).
\] (20)
3 Wilson fermions

We set $s = 1$ in Eq. (20) and assume

\[ M(n_0) = -\ln \left( (2K)^{\frac{1}{2}} B^{-\frac{1}{2}}(n_0) \right), \]
\[ N(n_0) = 2KB(n_0)^{-\frac{1}{2}}c(n_0)B(n_0)^{-\frac{1}{2}}, \]

(21)

where K is the hopping parameter and

\[ B(n_0) = 1 - K \sum_{j=1}^{3} \left( U_j(n_0)T_j^{(+)} + T_j^{(-)}U_j^+(n_0) \right), \]
\[ c(n_0) = \frac{1}{2} \sum_{j=1}^{3} i \sigma_j \left( U_j(n_0)T_j^{(+)} - T_j^{(-)}U_j^+(n_0) \right). \]

(22)

We have introduced the translation operators $T_j^{(\pm)}$ with matrix elements

\[ \left( T_j^{(\pm)} \right)_{n_1,n_2} = \delta_{n_2,n_1 \pm e_j}, \]

(23)

where the unit vectors $e_\mu$ have matrix elements

\[ (e_\mu)_\nu = \delta_{\mu,\nu}. \]

(24)

Next we define the Dirac spinors $q$ by the transformation

\[ x = B^{\frac{1}{2}}P_0^{(+)}q, \quad y = B^{\frac{1}{2}}P_0^{(-)}q, \]
\[ \overline{q} = q^\dagger \gamma_0 \]

(25)

where

\[ P_0^{(\pm)} = \frac{1}{2} \left( 1 \pm \gamma_0 \right). \]

(26)

The jacobian of this transformation is the function $J$ introduced in Eq. (3). The charge conjugation transformation is

\[ q = C^{-1} \overline{q}', \quad \overline{q} = -C^T q' \]

(27)

where, with Lüscher’s conventions

\[ C = \gamma_0 \gamma_2. \]

(28)
It exchanges $x$ and $y$ according to
\begin{align*}
x &= \gamma_2 y' \\
y &= \gamma_2 x',
\end{align*}
justifying our definition (7) of the baryon charge. Now we reinstate $U_0$. The partition function takes the form
\[ Z = \int [d\eta dq] \exp S_q, \]
with the action
\[ S_q = \sum_{n_0} \left\{ \exp(\mu)2K\eta(n_0)P_0^{(+)}U_0(n_0)q(n_0 + 1) \right. \\
&\left. + \exp(-\mu)2K\eta(n_0 + 1)P_0^{(-)}U_0^{(+)}(n_0)q(n_0) \\
+ \eta(n_0) \left[ 2KC(n_0) - B(n_0) \right] q(n_0) \right\} \]
where
\[ C(n_0) = \frac{1}{2} \sum_{j=1}^3 \gamma_j \left( U_j(n_0)T_j^{(+)} - T_j^{(-)}U_j^{+}(n_0) \right). \]
This is the Hasenfratz-Karsch action for Wilson fermions with Wilson parameter $r = 1$. Notice the "plus" sign in the exponential of the action, to comply with Lüscher's convention.

4 Kogut-Susskind fermions in the flavour basis

The Kogut-Susskind fermions can be defined in the flavour as well as in the spin-diagonal basis. If we want to be able to transform from one basis to the other, the gauge fields must be defined on the links of the lattice. But if we want to stay in the flavour basis, as we will do in this Section, we can associate the gauge fields to the links of the blocks and forget the lattice.

We set $s = -1$ in Eq. (20) and assume
\[ M(n_0) = 0, \]
\[ N(n_0) = \sum_{j=1}^3 \left[ \gamma_5 \otimes t_5t_j + \gamma_j \left( P_j^{(-)}U_j(n_0)T_j^{(+)} \\
- P_j^{(+)}T_j^{(-)}U_j^{+}(n_0) \right) \right] + \frac{m}{K} \mathbb{1} \otimes \mathbb{1} + \gamma_5 \otimes t_5t_0, \]
where \( m \) is the quark mass parameter and \( K \) the hopping parameter.

In the tensor product the \( \gamma \)-matrices act on Dirac indices, while the \( t \)-matrices

\[
t_{\mu} = \gamma_{\mu}^{T}
\]

act on flavor indices. The projection operators \( P_{\mu}^{(\pm)} \) are given by

\[
P_{\mu}^{(\pm)} = \frac{1}{2} [\mathbb{1} \otimes \mathbb{1} \pm \gamma_{\mu} \gamma_{5} \otimes t_{5} t_{\mu}] .
\]

The Dirac spinors \( q \) are obtained by the transformation

\[
x = 4\sqrt{K} P_{0}^{(+)} q, \quad y^{+} = 4\sqrt{K} P_{0}^{(-)} q
\]
\[
\bar{q} = q^\dagger \gamma_{0}
\]

whose jacobian is the function \( J \) introduced in Eq. (3). The charge conjugation transformation

\[
q = C^{-1} \bar{q}, \quad \bar{q} = -C^{T} q^\dagger
\]

where

\[
C = \gamma_{0} \gamma_{2} \otimes t_{0} t_{2},
\]

exchanges \( x \) and \( y \) according to

\[
x = \gamma_{2} \otimes t_{0} t_{2} y',
\]
\[
y = -\gamma_{2} \otimes t_{0} t_{2} x',
\]

justifying our definition (7) of the baryon charge. Now we reinstat \( E_{0} \). The partition function takes the form

\[
Z = \int [d\bar{q} dq] \exp (-16 S_{q}),
\]

with the action

\[
S_{q} = K \sum_{n_{0}} \left\{ \exp(-\mu)\bar{q}(n_{0}) \gamma_{0} P_{0}^{(-)} U_{0}(n_{0}) q(n_{0} + 1)
\right.
\]
\[
- \exp(\mu)\bar{q}(n_{0} + 1) \gamma_{0} P_{0}^{(+)} U_{0}^{+}(n_{0}) q(n_{0}) + \bar{q}(n_{0}) N(n_{0}) q(n_{0}) \right\} .
\]

The factor 16 in front of the action accounts for the fact that the volume element with Kogut-Susskind fermions is 16 times larger than in the Wilson case. The chemical potential appears in the same way as with Wilson fermions apart from the sign. This is due to the fact that with the conventions adopted the quarks propagate backwards in time with the Wilson definition and forwards with the Kogut-Susskind one.
The spin-diagonal basis

We want to explore the possibility of using the results of the previous Section to construct QCD at finite density in the spin-diagonal basis. To do this the gauge fields must be defined on the links so that the block link operators $W_\mu(n_0)$ have matrix elements

\[
(W_\mu(n_0))_{m,n} = \delta_{m,n} W_\mu(n_0, n)
\]

\[
W_\mu(n_0, n) = U_\mu(2n_0, 2n)U_\mu(2n_0 + e_\mu, 2n + e_\mu).
\] (42)

The relation of the coordinates of the blocks $n_\mu$ to the coordinates of the sites $x_\mu$ (not to be confused with the Grassmann variables of the previous Sections) is

\[
x_\mu = 2n_\mu + \eta_\mu, \quad \eta_\mu = 0, 1.
\] (43)

Then the action in the flavor basis becomes

\[
S_q = K \sum_{n_0} \left\{ \exp(-\mu)q(n_0)\gamma_0 P_0^(-)W_0(n_0)q(n_0 + 1)
\right.
\]

\[
- \exp(\mu)q(n_0 + 1)\gamma_0 P_0^+(n_0)W_0^+(n_0)q(n_0) + \bar{q}(n_0)N(n_0)q(n_0) \right\}
\] (44)

with

\[
N(n_0) = \sum_{j=1}^{3} [\gamma_5 \otimes t_5 t_j + \gamma_j \left( P_j(n_0) T_j^(+)
\right.
\]

\[
- P_j^+(n_0) T_j^{-}(n_0) \right) + \frac{m}{K} \mathbb{1} \otimes \mathbb{1} + \gamma_5 \otimes t_5 t_0.
\] (45)

But the gauge fixing $W_0 \sim \mathbb{1}$ has a nontrivial Fadeev-Popov determinant and we cannot say that the transfer matrix is positive definite before its effect is taken into proper account. We then proceed in the following way. We consider first the free case and then the interacting case to show the nature of the practical complication if it can be shown that transfer matrix remains positive definite in the gauge $W_0 \sim \mathbb{1}$.

In this Section we do not find convenient our summation convention and therefore we will abandon it.

5.1 The free case

The spin-diagonal and the flavour bases are related by

\[
q_{\mu;\alpha,f}(n_0)^c = \frac{1}{8} \sum_\eta \Gamma_{\alpha,f}^{(n)}(2n + \eta)^c, \quad \eta_\mu = 0, 1,
\] (46)
where
\[ \Gamma^{(\eta)}_{\alpha,f} = (\gamma_1^{m} \gamma_2^{\eta_2} \gamma_3^{\eta_3} \gamma_0^{\eta_0})_{\alpha,f}. \] (47)

Performing this transformation in Eq. (44) with \( W_\mu = 1 \) we find the action in the spin-diagonal basis
\[ S_\psi = \sum_x \left\{ m\bar{\psi}(x)\psi(x) + K \sum_\mu \alpha_\mu(x) \left[ \bar{\psi}(x)\psi(x + e_\mu) - \bar{\psi}(x + e_\mu)\psi(x) \right] + \delta S_\psi \right\}, \] (48)
where
\[ \alpha_\mu(x) = (-1)^{x_1 + x_2 + \ldots x_{\mu-1}} \] (49)
and \( \delta S_\psi \) is the contribution of the chemical potential. It is obtained by the transformation of the corresponding contribution in the flavor basis
\[ \delta S_q = 16K \sum_{n_0} \left\{ -\left(1 - \exp(-\mu)\right) \bar{q}(n_0 - 1)\gamma_0 P_0^{(-)} q(n_0) \right. \] \[ \left. + \left(1 - \exp(\mu)\right) \bar{q}(n_0)\gamma_0 P_0^{(+)} q(n_0 - 1) \right\} \] (50)
and is
\[ \delta S_\psi = K \sum_{x_0 \text{ even}, x} \left\{ -(1 - e^{-\mu})\alpha_0(x)\bar{\psi}(x - e_0)\psi(x) \right. \] \[ \left. + (1 - e^{\mu})\alpha_0(x)\bar{\psi}(x)\psi(x - e_0) \right\}. \] (51)

Note that the sum extends only over the sites with even temporal coordinate. Finally the total action in the spin-diagonal basis is
\[ S_\psi = \sum_{x_0 \text{ even}, x} \alpha_0(x) \left\{ \bar{\psi}(x)\psi(x + e_0) - \bar{\psi}(x + e_0)\psi(x) \right. \] \[ \left. + e^{-\mu}\bar{\psi}(x - e_0)\psi(x) - e^{\mu}\bar{\psi}(x)\psi(x - e_0) \right\} \] \[ + \sum_x \left\{ K \sum_j \alpha_j(x) \left[ \bar{\psi}(x)\psi(x + e_j) - \bar{\psi}(x + e_j)\psi(x) \right] \right. \] \[ \left. + m\bar{\psi}(x)\psi(x) \right\}. \] (52)

The chemical potential appears only on half the sites. We get a uniform coupling by the following change of variables in the sites with even temporal coordinate
\[ \psi(n) \rightarrow \psi(n) \exp(\mu/2), \quad \bar{\psi}(n) \rightarrow \bar{\psi}(n) \exp(-\mu/2), \quad n_0 \text{ even}. \] (53)
The quark action assumes then the standard form
\[
S_\psi = \sum_x \left\{ K \alpha_0(x) \left[ \exp(-\mu/2)\bar{\psi}(x)\psi(x + e_0) - \exp(\mu/2)\bar{\psi}(x + e_0)\psi(x) \right] \\
+ K \sum_j \alpha_j(x) \left[ \bar{\psi}(x)\psi(x + e_j) - \bar{\psi}(x + e_j)\psi(x_0) \right] + m\bar{\psi}(x)\psi(x) \right\}. \tag{54}
\]

The chemical potential now appears in all the sites in the Hasenfrat-Karsh form, but its value is halved.

5.2 The interacting case

When the gauge fields are present we must change the transformation (46) to make it gauge covariant. This can be done by introducing a string of link variables \( V_\eta(n) \) connecting the site \( 2n \) to the site \( 2n + \eta \)
\[
q^c_{\eta,\alpha,f} = \frac{1}{8} \sum_\eta \Gamma^{(\eta)}_{\alpha,f} (V_\eta(n)\psi(2n + \eta))^c, \quad \eta_\mu = 0, 1. \tag{55}
\]

A discussion of different ways to construct \( V_\eta(n) \) can be found in [11]. For instance we can assume
\[
V_\eta(n) = [U_1(2n)]^{\eta_0} [U_2(2n + \delta_{1,\eta_1}e_1)]^{\eta_2} [U_3(2n + \delta_{1,\eta_1}e_1 + \delta_{1,\eta_2}e_2)]^{\eta_3} [U_0(2n + \delta_{1,\eta_1}e_1 + \delta_{1,\eta_2}e_2 + \delta_{1,\eta_3}e_3)]^{\eta_0}. \tag{56}
\]

The contribution of the chemical potential to the action in the presence of the gauge fields can be read off from Eq. (14)
\[
\delta S_q = K \sum_{n_0} \left\{ -(1 - \exp(-\mu)) \bar{\psi}(n_0 - 1)\gamma_0 W_0(n_0 - 1) P^{-}_{0}(n_0) q(n_0) \\
+ (1 - \exp(\mu)) \bar{\psi}(n_0)\gamma_0 P^{+}_{0}(n_0) W^{+}_{0}(n_0 - 1) q(n_0 - 1) \right\}. \tag{57}
\]

After the transformation (55) this becomes
\[
\delta S_\psi = K \sum_{n,\eta} \alpha_0(\eta) \left\{ (1 - \exp(\mu)) \delta_{\eta_0,0} \\
\cdot \bar{\psi}(2n + \eta) V^+_\eta(n) W^+_0(n - e_0) V_\eta(n + e_0) \psi(2n + \eta - e_0) \\
- (1 - \exp(-\mu)) \delta_{\eta_0,1} \\
\cdot \bar{\psi}(2n + \eta) V^+_\eta(n) W_0(n) V_{\eta - e_0}(n + e_0) \psi(2n + \eta + e_0) \right\}. \tag{58}
\]
Therefore the total action in the spin-diagonal basis is

\[
S_\psi = \sum_{n,\eta} \left\{ K \sum_\mu \alpha_\mu(\eta) \bar{\psi}(2n + \eta) V_\eta^+(n) \left[ A_\mu(n, \eta) \psi(2n + \eta + e_\mu) - B_\mu(n, \eta) \psi(2n + \eta - e_\mu) \right] + m \bar{\psi}(2n + \eta) \psi(2n + \eta) \right\}
\]

where

\[
A_0(n, \eta) = \delta_{\eta,0} \exp(-\mu) W_0(n) V_{\eta-e_0}(n + e_0) + \delta_{\eta,0} V_{\eta+e_0}(n),
\]

\[
B_0(n, \eta) = \delta_{\eta,0} \exp(\mu) W_0^+(n - e_0) V_{\eta+e_0}(n - e_0) + \delta_{\eta,0} V_{\eta-e_0}(n),
\]

\[
A_j(n, \eta) = \delta_{\eta_j,0} W_j(n) V_{\eta-e_j}(n + e_j) + \delta_{\eta_j,0} V_{\eta+e_j}(n),
\]

\[
B_j(n, \eta) = \delta_{\eta_j,0} W_j^+(n - e_j) V_{\eta+e_j}(n - e_j) + \delta_{\eta_j,0} V_{\eta-e_j}(n).
\]

Again the chemical potential appears only on half the sites and we get a uniform coupling by the change of variables (53) which changes \(A_0, B_0 \rightarrow A'_0, B'_0\) according to

\[
A_0(n, \eta)' = e^{-\mu/2} \left[ \delta_{\eta,0} W_0(n) V_{\eta-e_0}(n + e_0) + \delta_{\eta,0} V_{\eta+e_0}(n) \right]
\]

\[
B_0(n, \eta)' = e^{\mu/2} \left[ \delta_{\eta,0} W_0^+(n - e_0) V_{\eta+e_0}(n - e_0) + \delta_{\eta,0} V_{\eta-e_0}(n) \right].
\]

The chemical potential now appears in all the sites in the Hasenfratz-Karsh form, but its value is halved.

6 Summary

We have seen that we can couple the chemical potential linearly to the Fermi fields, conserving its meaning of a Lagrange multiplier in the transfer matrix formalism, which is equivalent to the hamiltonian formalism in statistical mechanics. Starting from this definition we have derived the Hasenfratz-Karsh action showing that the exponentiation of the chemical potential is a property of the Grassmannian kernel of fermionic operators, and is not related to the exponential dependence of the gauge fields.

In the case of Kogut-Susskind, the value of the chemical potential depends on the basis, and this can be important when the chemical potential has a physical interpretation, which in the present scheme is only possible in the flavour basis.

The formulation of QCD at finite baryon density in the spin-diagonal basis cannot be considered settled. Within our approach one should prove
the reflection positivity of the transfer matrix in the gauge $W_0 \sim \mathbb{1}$. Then it might also be possible to justify the minimal coupling in the spin-diagonal basis, by appropriately changing the gauge coupling in the flavour basis. Alternatively one should use a transfer matrix constructed directly in the spin-diagonal basis.

We postpone to separate paper the discussion of the uniqueness of the path integral for a given transfer matrix.

**Acknowledgments**

It is a pleasure to thank dr. G. DiCarlo for many conversations about QCD at finite baryonic density.
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