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In and out of equilibrium quantum metrology with mean-field quantum criticality

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We study the influence that collective transition phenomena have on quantum metrological protocols. The single spherical quantum spin (SQS) serves as stereotypical toy model that allows analytical insights on a mean-field level. First, we focus on equilibrium quantum criticality in the SQS and obtain the quantum Fisher information analytically, which is associated with the minimum lower bound for the precision of estimation of the parameter driving the phase transition. We compare it with the Fisher information for a specific experimental scenario where photoconouting-like measurements are employed. We find how quantum criticality and squeezing are useful resources in the metrological scenario. Second, we obtain the quantum Fisher information for the out of equilibrium transition in the dissipative non-equilibrium steady state (NESS), and investigate how the presence of dissipation affects the parameter estimation. In this scenario, it is known that the critical point is shifted by an amount which depends on the dissipation rate. This is used here to design high precision protocols for a whole range of the transition-driving parameter in the ordered phase. In fact, for certain values of the parameter being estimated, dissipation may be used to obtain higher precision when compared to the equilibrium scenario.

I. INTRODUCTION

Bridging the gap between fields such as quantum optics, quantum information and statistical mechanics is a challenging but fruitful research direction. Interdisciplinary knowledge transfer allows these fields to mutually benefit from one another and, it is often the case that certain techniques or concepts initially tailored for a certain problem, say in statistical mechanics, yield a distinct perspective on and a deeper understanding of challenges in quantum information or optics. Perhaps the most prominent example is the study of phase transitions from the perspective of quantum information. Phase transitions and bipartite entanglement, for instance, are known to be closely related at arbitrary temperatures [1]. Scaling of entanglement near a quantum critical point has been studied in different systems [2, 3]. Other quantum information tools such as distance measures and state fidelities [4–10], state coherence [11], and general quantum correlations [12, 13] have also been extensively used in the study of quantum phase transitions.

In general, physical parameters such as coupling constants cannot be directly measured, and therefore need to be estimated in an indirect way through observables or generalized measurements. Quantum metrology is concerned with the estimation of such quantities by exploiting quantum resources in order to improve the precision of the estimation protocol. For example, quantum features such as entanglement [14], squeezing [15], quantum temporal correlations [16, 17], and quantum invasiveness [18] can be associated with a significant improvement of sensitivity in metrological protocols. If the global state of a system under inspection is highly sensitive to small variations of a certain parameter, this may be used to increase the precision with which this parameter can be estimated [19–21]. This provides a route to investigate critical systems from the point of view of quantum metrology, as we are going to pursue in this work [20, 22–25].

Here, we focus on a quantum mean-field version of the celebrated spherical model [26–29] which is a useful tool in statistical mechanics for the study of strongly interacting degrees of freedom. Based on the idea of introducing transition phenomena through adequate constraints or appropriate limits of physical parameters in few body problems [30, 31], we study here the case of a single SQS [32]. From the point of view of statistical mechanics, the SQS may be interpreted as a mean-field version of the spherical model, while from the point of view of quantum optics and quantum thermodynamics, it may be seen as a highly controlled quantum system subjected to work protocols, as explained in [33]. Originally introduced to overcome the lack of an analytical solution of the three dimensional Ising model [26], the spherical model has quickly proven itself as an excellent starting ground to study various transition phenomena in and out of equilibrium and routinely obtain results that go beyond mean-field statements [29, 34–36].

The SQS may be viewed as one of the simplest quantum systems that still allows for transition phenomena to take place. In thermal equilibrium, the SQS has a quantum critical point separating a paramagnetic (disordered) from a ferromagnetic (ordered) phase. Therefore, states close to this critical value do show macroscopically different properties due to large fluctuation effects at quantum criticality. Furthermore, the transition phenomenon is stable against thermal perturbations, meaning that the SQS shows a continuous critical line for $T > 0$. We shall therefore propose a metrological protocol for the SQS. Typically, a metrological protocol is divided into four different stages [37, 38]: (i) preparation: a certain system state is initialized, (ii) sensing: a parameter is imprinted in the system’s state via a certain dynamics, (iii) readout:
the system is measured, \( \text{iv) estimation:} \) the parameter is estimated from the measurement outcomes.

In our case, however, the sensing stage is to be seen as part of the preparation of a steady state which depends on the parameter driving the phase transition. We intend to estimate this parameter in two distinct steady state scenarios: First, in a \( T = 0 \) thermal equilibrium state and second in a specific NESS. The use of NESS transitions in quantum metrology [39–41] is rather unexplored in the literature compared to its equilibrium counterpart [20, 42]. Previous studies have shown examples illustrating the usefulness of driven and dissipative dynamics [43] and NESS transitions [23] for metrology. Here, we provide an extension to the non-equilibrium scenario by exploiting the metrological potential of NESS transitions in the SQS. This study helps to highlight the potential of NESS compared to equilibrium states as both cases can be worked out and compared explicitly in the SQS.

This paper is organized as follows. In section II we analyze the equilibrium quantum metrology. Therefore we first briefly introduce the Fisher information and the quantum Fisher information as central metrological quantities. We then discuss the thermal equilibrium of the SQS in the language of quantum optics. Next, we discuss how the zero temperature quantum phase transition may be used for a precision gain in parameter estimation protocols. In section III we introduce the driven SQS and review its NESS transition, which we shall then exploit as a quantum metrological resource. Finally we discuss how such NESS transitions may be beneficial compared to equilibrium quantum metrology.

II. SQS EQUILIBRIUM QUANTUM METROLOGY

In the present work, we shall put forth a metrological study of the SQS [32] by evaluating the quantum Fisher information and the Fisher information associated with a specific measurement scenario. Before moving on, we briefly review the meaning of these quantities in metrological protocols.

The Fisher information is a figure of merit for a metrological protocol concerned with the estimation of a certain parameter \( g \) encoded in the physical state of the system. It gives the sensitivity of the estimation as it is connected to the lower bound of the standard deviation \( \Delta g \).

\[

\Delta g \geq \frac{1}{\sqrt{\nu F(g)}}, \quad (1)

\]

where \( \nu \) is the number of realizations of the experiment, for unbiased measurements [44, 45]. Therefore, the Fisher information depends on the specific measurement that one performs on the system. If we term the probabilities associated with each possible result \( l \) of the measurement as \( p_l(g) \), satisfying \( \sum_l p_l(g) = 1 \), then the Fisher information may be written as

\[

F(g) = \sum_l p_l(g) \left[ \frac{\partial \ln p_l(g)}{\partial g} \right]^2. \quad (2)

\]

By expressing the probabilities as \( p_l(g) = \text{Tr}(\rho(g) E_l) \) where \( \rho(g) \) depends on \( g \) and \( \{E_l\} \) is a positive operator valued measure (POVM), the Fisher information is generalized to quantum mechanics. The upper bound for \( F(g) \) is called the quantum Fisher information \( \mathcal{F} \) [46–49], and corresponds to the maximization of \( F(g) \) over all quantum measurements \( \{E_l\} \).

\[

\mathcal{F} = \max_{\{E_l\}} F(\rho, \{E_l\}). \quad (3)

\]

Therefore, the quantum Fisher information is the Fisher information associated with the optimal measurement, i.e. the one associated with the maximal precision for the estimation of \( g \).

A. The SQS in Thermal Equilibrium

The SQS is a simple toy model that was designed to study the effects of collaborative transition phenomena in a rather easy quantum optics language. As such, it is described by a single and coherently driven quantum harmonic oscillator with the Hamiltonian

\[

H = \frac{g}{2} p^2 + \frac{\omega^2}{2g} x^2 - h_0 x, \quad (4a)

\]

that is subject to the following two distinct constraints [32, 33]

\[

\langle x^2 \rangle = 1, \quad h_0 = \langle x \rangle. \quad (4b)

\]

Here, \( x \) and \( p \) are canonically conjugate variables describing the position and momentum of the oscillator and satisfying the standard bosonic commutation relation \([x, p] = i, \ \omega \) is the frequency of the oscillator, \( h_0 \) parametrizes the coherent driving and \( g \) is the (inverse) mass of the quantum oscillator. The constraints that define the SQS are formulated on statistical averages \( \langle O \rangle := \text{tr}(\rho O) \) with \( \rho \) being the density matrix of the system.

The SQS can be derived as a molecular-field approximation of the full \( N \) body quantum spherical model [28, 29] and is thus a mean-field version of the latter [32]. In this context the parameter \( g \) is usually referred to as quantum parameter since the case \( g = 0 \) corresponds to the classical spherical model [26]. We shall adopt this terminology and therefore refer to the mass as quantum parameter. The first constraint in Eq. (4b) is the one-body version of the \( N \)-body spherical constraint [26] and confines fluctuations in the position degree of freedom. The second constraint in Eq. (4b) can be seen as the aforementioned molecular field approximation. Here, a complicated many-body interaction is replaced by a self
interaction of a single degree of freedom, as it is routinely done for the Weiss theory of magnetism [50], for example. Together, these constraints render the system effectively non-linear and therefore allow for different physical phases to exist. We shall review the order-disorder transition of this system in the following.

We first introduce bosonic ladder operators \((a, a^\dagger)\) obeying the bosonic commutation relation \([a, a^\dagger] = 1\),

\[
x = \sqrt{\frac{g}{2\omega}} (a + a^\dagger), \quad p = i \sqrt{\frac{\omega}{2g}} (a^\dagger - a).
\]

In terms of these creation and annihilation operators, the Hamiltonian reads

\[
H = \omega \left( a^\dagger a + \frac{1}{2} \right) - \hbar \left( a^\dagger a \right),
\]

with the rescaled driving given by \(\hbar = h_0 \sqrt{g/2\omega}\). In order to diagonalize the Hamiltonian, we need to introduce displaced operators

\[
b = a - \alpha = D(\alpha) a D^\dagger(\alpha),
\]

where \(D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)\) is the displacement operator. The displaced vacuum is \(|0_\alpha\rangle = D(\alpha) |0_a\rangle\), which corresponds to the ground state of the transformed Hamiltonian

\[
\tilde{H} = \omega \left( b^\dagger b + \frac{1}{2} \right) - \frac{\hbar^2}{\omega},
\]

obtained by choosing \(\alpha = \hbar/\omega\). In order to study the equilibrium properties of the system we first imagine it being weakly coupled to a thermal heat bath that eventually thermalizes the system. Such a process may be described by the following standard relaxational Lindblad master equation for the reduced density matrix \(\rho\)

\[
\partial_t \rho = -i \left[ \tilde{H}, \rho \right] + \gamma (n + 1) \left( b^\dagger b \rho - \frac{1}{2} \{ b^\dagger b, \rho \} \right) + \gamma n \left( b^\dagger \rho b - \frac{1}{2} \{ b^\dagger, \rho \} \right),
\]

where \(n = \frac{1}{2} \coth (\omega/2T) - \frac{1}{2}\) is the Bose-Einstein distribution. In the steady state \(\rho_{ss}\), which corresponds to the equilibrium state, the ladder operators obey the following set of expectation values

\[
\langle b \rangle = 0, \quad \langle bb \rangle = 0, \quad \langle b^\dagger b \rangle = \frac{1}{2} \coth (\omega/2T) - \frac{1}{2}.
\]

In the non-displaced picture these expectation values read

\[
\langle a \rangle = \frac{\hbar}{\omega}, \quad \langle aa \rangle = \frac{\hbar^2}{\omega^2}, \quad \langle a^\dagger a \rangle = \frac{\coth (\omega/2T) - 1}{2} + \frac{\hbar^2}{\omega^2},
\]

and allow us to determine straightforwardly the self-consistent parameters \(\omega\) and \(\hbar\) from the external con-

---

**FIG. 1.** Equilibrium phase diagram of the SQS. Two distinct phases are present, namely a disordered phase with an average magnetization equal to zero and an ordered phase with a non-zero average magnetization. The phases are separated by a thermal critical line that ends in a zero temperature quantum phase transition.

---

straints. These allow for two distinct solutions viz \([33]\)

\[
\omega = \frac{g}{2} \coth \left( \frac{\omega}{2T} \right), \quad h = 0.
\]

\[
\omega = \sqrt{\gamma}, \quad h = \left[ \frac{\sqrt{\gamma}}{2}, \frac{g}{4} \coth \left( \frac{\sqrt{\gamma}}{2T} \right) \right]^{1/2}.
\]

Eq. (12) is called the *disorder solution* and Eq. (13) is called the *ordered solution*. This nomenclature comes from the fact that the self-consistent magnetic field is proportional to the parameter \(\alpha := \langle a \rangle\). This parameter quantifies the displacement of the oscillator in its equilibrium position, and therefore is an *order parameter* for the system. It is interesting to observe that the shift in Eq. (7) is only needed to diagonalize the Hamiltonian in the ordered phase.

The critical \(g-T\) line that separates the ordered from the disordered phase is found from comparing both solutions and, therefore, is described by the functional relation

\[
\frac{2}{\sqrt{\gamma_c}} = \coth \left( \frac{\sqrt{\gamma_c}}{2T_c} \right).
\]

The equilibrium phase diagram is depicted in Fig. 1.

---

**B. Equilibrium Parameter Estimation**

As mentioned before, we shall be concerned with the estimation of the quantum parameter \(g\), which is the parameter driving the quantum phase transition. From here on, we shall work at zero temperature \(T = 0\) since we want to explore the effects of true quantum criticality rather than thermal criticality. This is motivated by the
fact that we want to understand the impact of a genuine quantum transition for metrology and that the $T > 0$ transition is effectually classical.

First, we calculate the quantum Fisher information. At $T = 0$, the system’s equilibrium state, in which the quantum parameter $g$ is imprinted, corresponds to the ground state of the SQS, $\rho = |0_{\text{SQS}}(g)\rangle \langle 0_{\text{SQS}}(g)|$. As this is a pure state, the quantum Fisher information can therefore be expressed as

$$\mathcal{F} = 4 \left( \frac{d (|0_{\text{SQS}}\rangle \langle 0_{\text{SQS}}|)}{dg} - \frac{d (|0_{\text{SQS}}\rangle \langle 0_{\text{SQS}}|)}{dg} \right)^2. \quad (15)$$

According to Eq. (15), we need to evaluate the derivative of the SQS ground-state with respect to the imprinted quantum parameter $g$. Several ways of how this is achieved can be found in appendix A. The quantum Fisher information then reads

$$\mathcal{F} = 2 \left( \frac{d\alpha}{dg} + \frac{\alpha \omega - g \frac{dg}{d\omega}}{\omega} \right)^2 + \frac{1}{2} \left( \frac{1}{g} - \frac{d \log(\omega)}{dg} \right)^2. \quad (16)$$

In order to investigate whether the upper bound set by the quantum Fisher information can be saturated, we evaluate the Fisher information considering photon-counting-like measurements. Such measurements correspond to simple projective measurements $\{|m\Omega\rangle \langle m\Omega|\}_{m=0,1,...}$, where $\{|m\Omega\rangle\}_{m=0,1,...}$ is a Fock basis described by a characteristic frequency $\Omega$.

It is clear from the previous section that the SQS can always be described by a $g$-dependent displacement of an underlying harmonic oscillator as

$$|0_{\text{SQS}}(g)\rangle = \mathcal{D}(\alpha)|0_g\rangle. \quad (17)$$

In order to calculate the Fisher information we need to evaluate the probabilities of counting $m$ photons,

$$p_m(g, \Omega) = \left| \langle m\Omega | 0_{\text{SQS}} \rangle \right|^2 = |\langle m\Omega | \mathcal{D}(\alpha) |0_g\rangle |^2. \quad (18)$$

To proceed, it is useful to express the underlying harmonic oscillator ground-state $|0_g\rangle$ in the $\{|m\Omega\rangle\}_{m=0,1,...}$ basis. Since both of these Fock basis describe harmonic oscillators, they may be connected by a Bogoliubov transformation. Such Bogoliubov transformations can be expressed through a squeezing operation which is associated with a unitary transformation $[51, 52]$. In this way, we write

$$a_\Omega = S^\dagger(\zeta) a_g S(\zeta), \quad (19)$$

with the squeezing operator

$$S(\zeta) = \exp \left( \frac{\zeta}{2} a_g^\dagger - \frac{\zeta}{2} (a_g)^2 \right) \quad (20)$$

and the complex squeezing parameter $\zeta(g, \Omega) = r \exp(2i\phi)$. The single mode squeezed state is then the vacuum of the transformed operators $|\Omega\rangle = S(\zeta)|0\rangle$.

Introducing the squeezed states

$$|\alpha, \zeta\rangle = \mathcal{D}(\alpha) S(\zeta)|0\rangle \quad (21)$$

we may write the probabilities as overlap probabilities between the Fock state and the squeezed state. By means of [51, Eq. (3.7.5)] and [53, Eq. (B.5)] we obtain

$$p_m(g, \Omega) = \frac{\alpha^{2m} e^{-|\alpha|^2} \text{sech}(r)}{\Gamma(m+1) \pi^\frac{1}{2}} \left| \sum_{n=0}^{\infty} \sqrt{\Gamma(n + \frac{1}{2})} \left( -\frac{e^{2\phi}}{\alpha} \tanh(r) \right)^n L_n^{(m-n)}(\alpha^2) \right|^2. \quad (22)$$

with the relevant squeezing parameter characterized by

$$\tanh(r) = \frac{\omega - g\Omega}{\omega + g\Omega}, \quad \phi = -\frac{1}{2} \arg (\omega - g\Omega). \quad (26)$$

In Fig. 2, we focus on the ordered phase and plot the Fisher information as a function of the quantum parameter $g$. In line with previous results for equilibrium states [23, 42], we see that the quantum Fisher information diverges at the critical point ($g_c = 4$), which means that arbitrarily large precision associated with the estimation of the quantum parameter $g$ could in principle be attained. Furthermore, the quantum Fisher information diverges as $g \to 0$ as well, which implies that at the quantum to classical limit $g \to 0$, arbitrarily large precision can be attained. This means that the ground-state is
very sensitive to quantum fluctuations around the classical fix point $g = 0$, which is harvested as a resource for the quantum Fisher information.

In the right panel of Fig. 2, we show the normalized Fisher information $F/F$ associated with the projections $\{m_\Omega\}_{m=0,1,\ldots}$ on the Fock basis, as a function of $g$ and for different values of the reference frequency $\Omega$. First, we see that in the interval $0 \leq g \leq 0.3$, different projective measurements associated with different Fock basis (i.e. for different values of $\Omega$) give the same $F/F$. In particular, as $g \to 0$, $F/F \to 1$. It is also noticeable that projections associated with decreasing values of $\Omega$ can provide a lower standard deviation $\Delta g$ in the range $3 \leq g \leq 3.8$, as $F/F$ increases in this interval as $\Omega$ decreases. However, in the range $3.8 \leq g < 4$, different $\Omega$ tend to be associated with the same normalized Fisher information, which goes to 1 as $g$ tends to 4.

We verified that criticality is a resource for the estimation of $g$, as $F/F$ is larger in the region close to $g = 4$. This is only valid when the system is in the ordered phase, since the quantum Fisher information is zero in the disordered phase ($g > 4$). In other words, for any choice of POVM in (2), the Fisher information will be zero in the disordered phase, meaning that it is not possible to establish a lower bound for precision in the estimation of $g$, i.e. there is no lower bound for the standard deviation $\Delta g$.

Moreover, we shed light on the region $g \ll 1$ where quantum fluctuations around the classical fix point yield a strong $g$ dependence of the ground-state properties. This delicate dependence may then be exploited as a metrological resource.

To conclude this analysis, we want to point out that squeezing helps the parameter estimation. Therefore, in Fig. 3, we show the Fisher information as a function of the magnitude of the squeezing parameter $\zeta$ for fixed values of $g$. We see that the Fisher information increases monotonically as the magnitude of $\zeta$ increases, which shows that the quality of the estimation of $g$ in the metrological protocol can be improved by increasing the magnitude of the squeezing parameter in quantum mean field systems. This is in line with previous results which shows that precision benefit from squeezing in metrological protocols [15].

**FIG. 2.** Left panel: The Fisher information $F$ for the SQS system as a function of $g$ for different measurements parametrized by the parameter $\Omega$, and the quantum Fisher information as a function of $g$ as well, which establishes the upper bound for the Fisher information. Right panel: Normalized Fisher information $F/F$ as a function of $g$. We can see that photon count measurements tend to be optimal for the estimation of the quantum parameter $g$ for small values of $g$ and as $g$ approaches the critical value $g_c = 4$. For measurements associated with smaller values of $\Omega$, however, we see that $F/F$ can attain larger values in the region close to $g = 4$.

**FIG. 3.** The Fisher information $F$ for a SQS system parametrized by the fluctuation parameter $g$ as a function of the associated magnitude of the squeezing parameter. We can see that the Fisher information increases monotonically as the magnitude of the squeezing parameter increases.
III. SQS OUT OF EQUILIBRIUM QUANTUM METROLOGY

Having studied the effects of equilibrium criticality on quantum metrology, we now address the question of whether non-equilibrium state transitions may be beneficial for the estimation of the parameter \( g \). In this section, we first shall review the type of NESS transition in the SQS that we will study as a probe in the metrological protocol. [32]

A. The NESS transition in the SQS

In this section we shall review the non-equilibrium steady state of the SQS that emerges from the following Lindblad master equation

\[
\partial_t \rho = -i[H, \rho] + \gamma (n + 1) \left( a g a^\dagger - \frac{1}{2} \{ a^\dagger a, \rho \} \right) + \gamma n \left( a^\dagger g a - \frac{1}{2} \{ a a^\dagger, \rho \} \right). \tag{27}
\]

Instead of thermalizing the eigenmode of the coherently driven oscillator, this dynamics thermalizes the eigenmodes of the non-driven oscillator which results in a competition between the dissipative dynamics and the coherent drive. It is known that this system shows a dissipative NESS transition [32]. The time evolution governed by Eq. (27) can be engineered by adding controlled degrees of freedom [55] and by using external lasers with appropriate intensity and phase fluctuations [56]. It is well-established that this dynamics relaxes towards a coherent state with an amplitude dependent on the driving strength and dissipation rate.

In the NESS, i.e. \( \partial_t \rho = 0 \), one finds [32]

\[
\langle a \rangle = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{2\omega}{g} \left( i - \frac{\gamma}{2\omega} \right)} , \tag{28}
\]

\[
\langle aa \rangle = \frac{g - 2\omega}{4g\omega} \left[ 1 - \frac{2\gamma^2 - 2g}{2\omega} - i\gamma \right] . \tag{29}
\]

These expectation values allow us again to impose the external constraints, as it was done in the equilibrium case. It turns out that there are again two distinct solutions, and while the disordered equilibrium solution persists, the ordered solution is altered. The protocols to satisfy the external constraints in the ordered phase are given by [32]

\[
\omega = \frac{1}{\sqrt{2}} \sqrt{4g - \gamma^2} , \quad h = -\frac{g\gamma}{4g - \gamma^2} \sqrt{\frac{1}{g^2} - \frac{4g - \gamma^2}{g^2}} . \tag{30}
\]

As in the equilibrium case, these self-consistent protocols determine the displacement parameter \( \alpha = h/\omega \). In the disordered phase, the displacement is \( \alpha = 0 \) or equivalently \( h = 0 \). This means that the external force term in Hamiltonian Eq. (4a) is null. Therefore, the master equation in Eq. (27) is effectively that of an harmonic oscillator in contact with a thermal bath. Thus, it is clear that the disordered phase is indeed an equilibrium phase while the ordered phase is a non-equilibrium phase. The critical \( g-\gamma \) line separating these phases is again found from the condition \( h = 0 \) and it is depicted in Fig. 4.

Note that for certain values of \( \gamma \) two distinct phase transitions are observed. The first, on the right-hand side of Fig. 4 is “similar” to the equilibrium case, as increasing quantum fluctuations parametrized by \( g \) destroy macroscopic order. Nevertheless, this transition is a true non-equilibrium transition. In turn, the second transition on the left-hand side of Fig. 4 is a striking non-equilibrium effect, as increasing quantum fluctuations induce macroscopic order [32]. For the following metrological study, it is important to note that the values of \( g \) corresponding to transition points are dependent on the choice of the dissipation rate \( \gamma \).

B. NESS Parameter estimation

We are now interested in how the non-equilibrium dissipative phase transition may affect the metrological protocol. Therefore, we study the influence of this transition on the quantum Fisher information. As we have already mentioned in Sec. III A, the NESS is still a coherent state whose displacement and squeezing parameters are altered with respect to the equilibrium scenario. Therefore, the general formula for the quantum Fisher information, cf. Eq. (16), remains valid in the NESS upon the correct replacement of the self-consistent parameters \( \omega \) and \( h \) from Eq. (30).\(^1\)

In Fig. 5, the quantum Fisher information \( \mathcal{F} \) is plotted for two different values of the dissipation rate \( \gamma \). As dis-

\(^1\) Recall that the displacement parameter may be obtained from the self-consistent parameters viz. \( \alpha = h/\omega \).
FIG. 5. The quantum Fisher information $F$ as a function of $g$ for the case with dissipation (out of equilibrium NESS case) in the ordered phase. We see that the values for the $F$ are lower than in equilibrium, see Fig. 2. Nonetheless, the critical point, which is associated with arbitrarily large $F$ is shifted to the left depending on $\gamma$. This may allow one to obtain larger precisions for other values of $g$ by using out of equilibrium scenarios.

cussed before, there are two phase transitions now, and therefore two critical points, as shown in Fig. 4. The first, on the right-hand side in Fig. 5, is a transition where increasing quantum fluctuations destroy order. In turn, the second, on the left-hand side in Fig. 5, is a true non-equilibrium transition where increasing quantum fluctuations induce order. At these two critical points, the quantum Fisher information diverges, and therefore arbitrarily large precision may be obtained. For the critical value of $g$ for the transition on the right-hand side in Fig. 5, as $\gamma$ increases, its value decreases as predicted in Fig. 4. In this way, the critical point can be shifted to the left by increasing $\gamma$. By doing this, one is able to obtain larger precisions to estimate $g$ in this region, when compared to the equilibrium case $\gamma = 0$. To be more specific, one fixes a value of $g$ in that region, and evaluate the quantum Fisher information $F$ with $\gamma = 0$. One finds a finite value for $F$, as depicted in the left panel of Fig. 2. Then, by introducing dissipation, one can make the chosen value of $g$ arbitrarily close to a critical point, what makes the value of $F$ increases as much as one wants. This shows that dissipation may help in the estimation of the quantum parameter $g$ of the SQS.

IV. SUMMARY

We investigate how collective quantum mean field systems may be useful for metrological protocols by using equilibrium and out of equilibrium states. The SQS, a simple system which possesses a quantum phase transition between an ordered and a disordered phase, is used as a toy-model. We focus on the estimation of the parameter driving the phase transition, the quantum parameter $g$. As a figure of merit for the quantum metrological protocol, we use the Fisher information and quantum Fisher information, the latter being an upper bound for the first. As a result, the quantum Fisher information is associated with the largest possible precision in the metrological protocol with quantum systems.

For equilibrium states, we find that the quantum Fisher information diverges at the critical point $(g_c = 4)$, meaning that one can obtain arbitrarily large precision as $g \rightarrow g_c = 4$. This also happens in the classical limit $g \rightarrow 0$, and therefore the estimation of small values of $g$ can be done with increasingly large precision as $g$ decreases. We also evaluate the Fisher information for different photon count measurements, corresponding to different values of $\Omega$. We found some dependence on $\Omega$ for a certain interval of $g$. Interestingly enough, different sets of projectors (different values of $\Omega$) lead to the same precision in the estimation of $g$ as it gets arbitrarily close to the critical point $(g_c = 4)$.

We then studied the impact that a dissipative driving may have on the parameter estimation. Therefore, we coupled the SQS to a heat bath at zero temperature that does not thermalize the eigenmode of the SQS. In this manner, the competition between the coherent and the dissipative drive create an ordered and an disordered phase out of equilibrium with two distinct phase transitions due to a re-entrance phenomena in the NESS phase diagram of out of equilibrium states. Arbitrarily large precisions are now observed at both these phase transition points. Since the two critical points where the Fisher information diverges depend explicitly on the dissipation rate $\gamma$, dissipation may help the estimation of values of the quantum parameter $g$ in the ordered phase when compared to the case of thermal equilibrium at $T = 0$. This is because, while at equilibrium the system may not be critical, we can move it to its NESS transition by dissipatively driving it, therefore obtaining a significant improvement in the parameter estimation. We also note that, since criticality is a resource for quantum metrology [20], the thermal regime may also be useful for parameter estimation in a certain range of temperature, as the only critical point is shifted to the left as the temperature increases, see Fig. 1. As a result, it would be interesting to study this in detail elsewhere. This effect is quite counter-intuitive, since one would expect a thermal bath to interfere with any measurement in a negative way. We thus conclude that it may be rather beneficial to measure systems far away from equilibrium.

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Appendix A: Quantum Fisher information

1. Infinitesimal Squeezing

We shall evaluate this derivative using an infinitesimal squeezing transformation since this approach highlights the different contributions arising from displacement and squeezing operations. Therefore, we proceed by formally writing the derivative of the SQS ground-state with respect to the parameter \( g \) as difference quotient as

\[
\frac{d}{dg} \langle 0_{\text{sq}s}(g) \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \langle 0_{\text{sq}s}(g + \epsilon) \rangle - \langle 0_{\text{sq}s}(g) \rangle \right) \quad (A1)
\]

Fig. 6 illustrates how the distinct ground-states are connected via a sequence of squeezing and displacement operations. In this manner the expression

\[
\frac{d}{dg} \langle 0_{\text{sq}s}(g) \rangle = \left[ \frac{dD(\alpha)}{dg} S(\zeta) + D(\alpha) \frac{dS(\zeta)}{dg} \right] \langle 0_{g} \rangle \quad (A2)
\]

is readily deduced. The derivative of the displacement operator for a real-valued displacement \( \alpha \in \mathbb{R} \) may be generally written as

\[
\frac{dD(\alpha)}{dg} = f(g) \left( a^\dagger - a \right) D(\alpha), \quad \text{with} \quad f(g) = \frac{d\alpha}{dg} + \frac{\alpha(g) \omega(g) - g \omega'(g)}{2g} \quad (A3)
\]

The derivative of the squeezing operator for real-valued squeezing parameters \( \zeta \in \mathbb{R} \) is given by

\[
\frac{dS(-\zeta)}{dg} = -\frac{1}{2} \frac{d\zeta}{dg} \left( a^2 - (a^\dagger)^2 \right) S(\zeta) \quad (A5)
\]

\[
\langle 0_{\text{sq}s}(g + \epsilon) \rangle \quad \langle 0_{\text{sq}s}(g) \rangle
\]

\[
D(\gamma) \quad D^{\dagger}(\alpha)
\]

\[
\langle 0_{g + \epsilon} \rangle \quad S(-\zeta) \quad \langle 0_{g} \rangle
\]

The first term is identical to the previous calculation, compare Eq. (A3). For the second term we may introduce a position representation viz

\[
\frac{d}{dg} \langle 0_{\text{SQS}}(g) \rangle = f(g) \langle 1_{\text{SQS}}(g) \rangle + \frac{1}{2g} |2_{\text{SQS}}(g)\rangle
\]

\[
\int_{-\infty}^{\infty} dx \frac{d}{dg} \langle 0_{g} \rangle = \int_{-\infty}^{\infty} dx \left[ \frac{1}{4} \left( \frac{\omega}{\pi g} \right)^{\frac{1}{2}} \frac{\omega' g - \omega}{\pi g^2} - \left( \frac{\omega}{\pi g} \right)^{\frac{1}{2}} x^2 \frac{\omega' g - \omega}{2g^2} e^{-\frac{\pi g}{\omega} x^2} \right] \langle 0_{g} \rangle \quad (A10)
\]

\[
= f(g) \langle 1_{\text{SQS}}(g) \rangle + \frac{1}{2g} |2_{\text{SQS}}(g)\rangle
\]

\[
\int_{-\infty}^{\infty} dx \frac{d}{dg} \langle 0_{g} \rangle = \int_{-\infty}^{\infty} dx \left[ \frac{1}{4} \left( \frac{\omega}{\pi g} \right)^{\frac{1}{2}} \frac{\omega' g - \omega}{\pi g^2} - \left( \frac{\omega}{\pi g} \right)^{\frac{1}{2}} x^2 \frac{\omega' g - \omega}{2g^2} e^{-\frac{\pi g}{\omega} x^2} \right] \langle 0_{g} \rangle \quad (A10)
\]

\[
= f(g) \langle 1_{\text{SQS}}(g) \rangle + \frac{1}{2g} |2_{\text{SQS}}(g)\rangle
\]
[1] B.-B. Wei, Phys. Rev. A 97, 042115 (2018).
[2] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature 416, 680 (2002).
[3] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[4] H.-Q. Zhou and J. P. Barjaktarević, Journal of Physics A: Mathematical and Theoretical 41, 412001 (2008).
[5] P. Zanardi and N. Paunković, Phys. Rev. E 74, 031123 (2006).
[6] P. Zanardi, M. Cozzini, and P. Giorda, Journal of Statistical Mechanics: Theory and Experiment 2007, L02002 (2007).
[7] M. Cozzini, P. Giorda, and P. Zanardi, Phys. Rev. B 75, 014439 (2007).
[8] M. Cozzini, R. Ionicioiu, and P. Zanardi, Phys. Rev. B 76, 104420 (2007).
[9] M. Bina, A. Mandarino, S. Olives, and M. G. A. Paris, Phys. Rev. A 89, 012305 (2014).
[10] Y.-C. Tzeng and M.-F. Yang, Phys. Rev. A 77, 012311 (2008).
[11] Y.-C. Li and H.-Q. Lin, Sci. Rep. 6, 26365 (2016).
[12] M. S. Sarandy, T. R. de Oliveira, and L. Amico, International Journal of Modern Physics B 27, 1345030 (2013).
[13] T. Werlang, G. A. P. Ribeiro, and G. Rigolin, International Journal of Modern Physics B 27, 1345032 (2013).
[14] G. Tóth, Phys. Rev. A 85, 022322 (2012).
[15] L. Maccone and A. Riccardi, (2019), arXiv:1901.07482.
[16] S. V. Moreira, A. Keller, T. Coudreau, and P. Milman, Phys. Rev. A 92, 062132 (2015).
[17] S. V. Moreira, G. Adesso, L. A. Correa, T. Coudreau, A. Keller, and P. Milman, Physical Review A 96, 012110 (2017).
[18] S. V. Moreira and M. T. Cunha, Physical Review A 99, 022124 (2019).
[19] H. Yuan and C.-H. F. Fung, npj Quantum Information 3, 14 (2017).
[20] P. Zanardi, M. G. A. Paris, and L. Campos Venuti, Phys. Rev. A 78, 042105 (2008).
[21] V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).
[22] I. Frérot and T. Roscilde, Phys. Rev. Lett. 121, 020402 (2018).
[23] L. Garbe, M. Bina, A. Keller, M. G. A. Paris, and S. Felicetti, (2019), arXiv:1910.00604.
[24] A. Carollo, B. Spagnolo, A. A. Dubkov, and D. Valenti, Journal of Statistical Mechanics: Theory and Experiment 2019, 094010 (2019).
[25] D. Braun, G. W. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, Rev. Mod. Phys. 90, 035006 (2018).
[26] T. H. Berlin and M. Kac, Physical Review 86, 821 (1952).
[27] H. W. Lewis and G. H. Wannier, Physical Review 88, 682 (1952).
[28] M. Henkel and C. Hoeger, Zeitschrift für Physik B Condensed Matter 55, 67 (1984).
[29] T. Vojta, Physical Review B 53, 710 (1996).
[30] S. Ashhab, Phys. Rev. A 87, 013826 (2013).
[31] M.-J. Hwang and M. B. Plenio, Phys. Rev. Lett. 115, 180404 (2015).
[32] S. Wald and M. Henkel, Journal of Physics A: Mathematical and Theoretical 49, 125001 (2016).
[33] A. M. Timpanaro, S. Wald, F. Semiao, and G. T. Landi, Physical Review A 100, 012117 (2019).
[34] S. Shukla and S. Singh, Phys. Rev. B 23, 4661 (1981).
[35] S. Wald and M. Henkel, Journal of Statistical Mechanics: Theory and Experiment 07006, 34 (2015), arXiv:1503.06713.
[36] S. Wald, G. T. Landi, and M. Henkel, Journal of Statistical Mechanics: Theory and Experiment 2018, 013103 (2018), arXiv:1707.06273.
[37] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nature Physics 7, 406 (2011).
[38] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Brazilian Journal of Physics 41, 229 (2011).
[39] L. Banchi, P. Giorda, and P. Zanardi, Phys. Rev. E 89, 022102 (2014).
[40] A. Carollo, B. Spagnolo, and D. Valenti, Scientific Reports 8, 9852 (2018).
[41] U. Marzolino and T. c. v. Prosen, Phys. Rev. B 96, 104402 (2017).
[42] M. Bina, I. Amelio, and M. G. A. Paris, Phys. Rev. E 93, 052118 (2016).
[43] M. Zuppardo, J. P. Santos, G. De Chiara, M. Paternostro, F. L. Semiao, and G. M. Palma, Phys. Rev. A 91, 033631 (2015).
[44] R. A. Fisher, Messenger of Mathematics 41 (1912).
[45] M. G. A. Paris, International Journal of Quantum Information 07, 125 (2009).
[46] C. W. Helstrom, Quantum detection and estimation theory (Academic Press, New York, 1976).
[47] A. S. Holevo, Probabilistic and statistical aspects of quantum theory (North-Holand, Amsterdam, 1982).
[48] S. L. Braunstein, C. M. Caves, and G. Milburn, Annals of Physics 247, 135 (1996).
[49] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. 72, 3499 (1994).
[50] J. Yeomans, Statistical Mechanics of Phase Transitions (Clarendon Press, 1992).
[51] S. Barnett and P. Radmore, Methods in Theoretical Quantum Optics (Oxford University Press, 2002).
[52] D. Walls and G. Milburn, Quantum Optics (Springer Berlin Heidelberg, 2008).
[53] K. E. Cahill and R. J. Glauber, Physical Review 177, 1857 (1969).
[54] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, ninth dover printing, tenth gpo printing ed. (Dover, New York, 1964).
[55] J. A. Dirac, A. S. Parkinson, and P. Zoller, Phys. Rev. Lett. 70, 556 (1993).
[56] S. Schneider and G. J. Milburn, Phys. Rev. A 57, 3748 (1998).