A Simplified Novel Technique for Solving Linear Programming Problems with Triangular Fuzzy Variables via Interval Linear Programming Problems

Ladji Kané (Corresponding Author)
Department of Applied Mathematics Faculté des Sciences Economiques et de Gestion, Bamako, Mali
Email: fsegmath@gmail.com

Lassina Diabaté
Department of Applied Mathematics Faculté des Sciences Economiques et de Gestion, Bamako, Mali

Daouda Diawara
Department of Applied Mathematics Faculté des Sciences Economiques et de Gestion, Bamako, Mali

Moussa Konaté
Department of Applied Mathematics Faculté des Sciences Economiques et de Gestion, Bamako, Mali

Souleymane Kané
Department of Applied Mathematics Faculté des Sciences Economiques et de Gestion, Bamako, Mali

Abstract
This study proposes a novel technique for solving Linear Programming Problems with triangular fuzzy variables. A modified version of the well-known simplex method and Existing Method for Solving Interval Linear Programming problems are used for solving linear programming problems with triangular fuzzy variables. Furthermore, for illustration, some numerical examples and one real problem are used to demonstrate the correctness and usefulness of the proposed method. The proposed algorithm is flexible, easy and reasonable.

Keywords: Interval numbers; Triangular fuzzy variables; Linear programming.

1. Introduction

In the real-world environment, there are many problems which are concern to the linear programming models and sometimes it is necessary to formulate these models with parameters of uncertainty. Many numbers from these problems are linear programming problems with fuzzy variables. Interval analysis is an efficient and reliable tool that allows us to handle such problems effectively.

Linear programming problems with interval coefficients have been studied by several authors, such as Atanu and Tapan [1], Atanu, et al. [2], Bitran [3], Chanas and Kuchta [4], Nakahara, et al. [5], Steuer [6] and Tong Shaocheng [7]. Numerous methods for comparison of interval numbers can be found as in Atanu and Tapan [1], Atanu, et al. [2], Ganesan and Veeramani [8] Ganesan [9], etc. By taking maximum value range and minimum value range inequalities as constraint conditions, Tong Shaocheng [7] reduced the interval linear programming problem in to two classical linear programming problems and obtained an optimal interval solution to it. Ramesh and Ganesan [10], proposed a method for solving interval number linear programming problems without converting them to classical linear programming problems.

Bellman and Zadeh [11], introduced for the first time the concept of a fuzzy decision process as an intersection of the fuzzy objective function and resource constraints. From this idea, the authors of Mahdavi-Amiri and Nasseri [12], Seyed [13] introduced linear programming problems with fuzzy variables and semi-fully fuzzy linear programming problems. Some authors considered these problems and have developed various methods for solving these problems. Recently, some authors [12, 13] considered linear programming problems with trapezoidal fuzzy data and/or variables and semi-fully fuzzy linear programming problems and stated a fuzzy simplex algorithm to solve these problems. Moreover, they developed the duality results in fuzzy environment and presented a dual simplex algorithm for solving linear programming problems with trapezoidal fuzzy variables and semi-fully fuzzy linear programming problems. Furthermore, Seyed [13] showed that the presented dual simplex algorithm directly using the primal simplex tableau algorithm tending the capability for sensitivity (or post optimality) analysis using primal simplex tableaus. In this paper, we attempt to develop the solving for Linear Programming Problems with triangular fuzzy variables via Interval Linear Programming problems without converting them to classical linear programming problems.

The rest of this paper is organized as follows: In section 2, we recall the definitions of interval numbers, some related results of interval arithmetic on them and the existing method for solving linear programming problem involving interval numbers. In section 3, we define introduced linear programming problems with fuzzy variables and semi-fully fuzzy linear programming problems. In section 4, a numerical example is provided to illustrate the theory developed in this paper.
2. Material and Method

In this section, some basic definitions, arithmetic operations for closed interval numbers and of linear programming problems involving interval numbers are presented [10, 14, 15].

2.1. Arithmetic Operations for Closed Interval Numbers

In this section, some arithmetic operations for two intervals are presented [14, 16, 17].

Let us denote by \( \mathbb{I} \) the class of all closed intervals in \( \mathbb{R} \). If \( A \) is closed interval, we also adopt the notation \( \bar{a} = [a^2, a^3] \), where \( a^2 \) and \( a^3 \) means the lower and upper bounds of \( \bar{a} \) respectively and \( a^2 \leq a^3 \) with \( a^2, a^3 \in \mathbb{R} \).

If \( \bar{a} = a^1 = a^3 \), then \( \bar{a} = [a, a] = a \). For any two intervals \( \bar{a} = [a^1, a^3] \) and \( \bar{b} = [b^1, b^3] \) and the arithmetic operations on \( \bar{a} \) and \( \bar{b} \) are defined as:

- **Addition**: \( \bar{a} + \bar{b} = [a^1 + b^1, a^3 + b^3] \)
- **Subtraction**: \( \bar{a} - \bar{b} = [a^1 - b^3, a^3 - b^1] \)
- **Multiplication**: \( \bar{a} \times \bar{b} = [\text{Min}(a^1 \times b^1, a^1 \times b^3, a^3 \times b^1, a^3 \times b^3), \text{Max}(a^1 \times b^1, a^1 \times b^3, a^3 \times b^1, a^3 \times b^3)] \)
- **Division**: \( \bar{a} \div \bar{b} = \left[ \text{Min}\left\{ \frac{a^1}{b^3}, \frac{a^1}{b^1}, \frac{a^3}{b^1}, \frac{a^3}{b^3} \right\}, \text{Max}\left\{ \frac{a^1}{b^3}, \frac{a^1}{b^1}, \frac{a^3}{b^1}, \frac{a^3}{b^3} \right\} \right] \) with \( 0 \notin \bar{b} \).

2.2. A New Interval Arithmetic

In this section, some arithmetic operations for two intervals are presented [10, 15].

Let \( \mathbb{R} = (\bar{a} = [a^1, a^3]; a^1 \leq a^3 \text{ with } a^3 > a^2 \text{ with } a^1, a^3 \in \mathbb{R} ) \) be the set of all proper intervals and \( \mathbb{R} \) = \( (\bar{a} = [a^1, a^3]; a^3 > a^2 \text{ with } a^1, a^3 \in \mathbb{R} ) \) be the set of all improper intervals on the real line \( \mathbb{R} \). We shall use the terms "interval" and "interval number" interchangeably. The midpoint and width (or half-width) of an interval number are defined as:

- **Midpoint**: \( \mu(\bar{a}) = \frac{a^1 + a^3}{2} \)
- **Width**: \( w(\bar{a}) = \frac{a^3 - a^1}{2} \).

The interval number \( \bar{a} \) can also be expressed in terms of its midpoint and width as:

\( \bar{a} = [\mu(\bar{a}), w(\bar{a})] = \left( \frac{a^3 + a^1}{2}, \frac{a^3 - a^1}{2} \right) \).

For any two intervals \( \bar{a} = [a^1, a^3] \) and \( \bar{b} = [b^1, b^3] \), the arithmetic operations on \( \bar{a} \) and \( \bar{b} \) are defined as:

- **Addition**: \( \bar{a} + \bar{b} = [a^1 + b^1, a^3 + b^3] \)
- **Subtraction**: \( \bar{a} - \bar{b} = [a^1 - b^3, a^3 - b^1] \)
- **Multiplication**: \( \bar{a} \times \bar{b} = [\text{Min}(a^1 \times b^1, a^1 \times b^3, a^3 \times b^1, a^3 \times b^3), \text{Max}(a^1 \times b^1, a^1 \times b^3, a^3 \times b^1, a^3 \times b^3)] \)
- **Division**: \( \bar{a} \div \bar{b} = \left[ \text{Min}\left\{ \frac{a^1}{b^3}, \frac{a^1}{b^1}, \frac{a^3}{b^1}, \frac{a^3}{b^3} \right\}, \text{Max}\left\{ \frac{a^1}{b^3}, \frac{a^1}{b^1}, \frac{a^3}{b^1}, \frac{a^3}{b^3} \right\} \right] \) with \( 0 \notin \bar{b} \).

2.3. Ramesh and Ganesan’s Method for Solving Linear Programming Problem Involving Interval Numbers

In this section, to overcome all the limitations of the method, presented in Ladji, et al. [14], Ramesh and Ganesan [10] is presented to find the exact optimal solution of linear programming problems involving interval numbers in which all the parameters are represented by intervals numbers.

2.3.1. Formulation of a Linear Programming Problem Involving Interval Numbers (INLP)

Now we are in a position to prove interval analogue of some important relationships between the primal and dual linear programming problems. We consider the primal and dual linear programming problems involving interval numbers as follows [10, 14, 15]:

For all the rest of this paper, we will consider the following primal linear programming problem involving interval numbers (INLP):

\[
\begin{align*}
\text{Max } & \quad \sum_{j=1}^{n} \tilde{c}_j \times \tilde{x}_j \\
\text{Subject to the constraints } & \quad \sum_{j=1}^{n} a_{ij} \times \tilde{x}_j \leq \tilde{b}_i \\
\end{align*}
\]

For all the rest of this paper, we will consider the following dual linear programming problem involving interval numbers (INLP):

\[
\begin{align*}
\text{Min } & \quad \sum_{i=1}^{m} \tilde{b}_i \times \tilde{y}_i \\
\text{Subject to the constraints } & \quad \sum_{i=1}^{m} a_{ij} \times \tilde{y}_i \geq \tilde{c}_j \\
\end{align*}
\]
2.3.2. Table $\overline{T}^*$ Optimal for Linear Programming Problem Involving Interval Numbers (INLP2)

In this section, we will describe how to determine the values of the primal (P) and dual (D) variables.

Calculation of the values of the primal (P) variables $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_{n+m}^*)$:

Assume that $\overline{T}^*$ is optimal, then the current basis is $\tilde{x}_b^* = (\tilde{x}_j^1, \tilde{x}_j^2, ..., \tilde{x}_j^m)$ and the corresponding solution is $\tilde{x}_b = (\tilde{x}_j^1, \tilde{x}_j^2, ..., \tilde{x}_j^m)$.

Moreover, the current nonbasic variables is $\tilde{x}_N = \left\{ \tilde{x}_d^1, \tilde{x}_d^2 \in \tilde{x}_b \right\}$ and the corresponding solution is $\tilde{x}_d = (\tilde{x}_d^1, \tilde{x}_d^2 \in \tilde{x}_b)$. Hence the optimal solution to the problem can be written as $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_{n+m}^*)$ with the associated value of the objective function $\tilde{Z}(\tilde{x}^*) \approx \tilde{c}\tilde{x} = [\tilde{Z}^1, \tilde{Z}^2]$.

Calculation of the values of the dual (D) variables $\tilde{y}^* = (\tilde{y}_1^*, \tilde{y}_2^*, ..., \tilde{y}_{n+m}^*)$ in $\overline{T}^*$:

Assume that $\overline{T}^*$ is optimal, then the current basis is $\tilde{x}_b = (\tilde{x}_j^1, \tilde{x}_j^2, ..., \tilde{x}_j^m)$ and the corresponding coefficient is matrix $\tilde{C}_b = (\tilde{c}_j^1, \tilde{c}_j^2, ..., \tilde{c}_j^m)$. So $\tilde{y}_b^{n+j} = [\tilde{C}_bA_j - \tilde{c}_j I]$ and $\tilde{y}_b^{l} = [\tilde{C}_bA_{n+j} - \tilde{c}_n + 1]$ with the associated value of the objective function Min $\tilde{W}(\tilde{y}^*) \approx \tilde{y}^T\tilde{b} = [\tilde{W}^1, \tilde{W}^2]$.

3. Main Results

In this section, we will describe our method of solving.

3.1. A New Interval Arithmetic for Triangular Fuzzy Numbers via Intervals Numbers

The aim of this section is to present some notations, notions and results which are of useful in our further consideration.

A number $\tilde{a} = (a^1, a^2, a^3)$ (where $a^1 \leq a^2 \leq a^3$) is said to be a triangular fuzzy number if its membership function is given by Iwona, et al. [16], Jagdeep and Amit [17], Seyed [13]:

$$\mu_{\tilde{a}}(x) = \begin{cases} a^1 & a^1 \leq x \leq a^2, \\ a^2 - a^1 & a^2 \leq x \leq a^3, \\ \end{cases}$$

Assume that $\tilde{a} = (a^1, a^2, a^3) = (a^2[\tilde{a}]) = (a^2[(a^1, a^3)] = (\frac{a^2 + a^1}{2}, \frac{a^2 - a^1}{2})$ and $\tilde{b} = (b^1, b^2, b^3) = (b^2[\tilde{b}]) = (b^2[(b^1, b^3)] = (\frac{b^2 + b^1}{2}, \frac{b^2 - b^1}{2})$ are two triangular fuzzy numbers. For any two triangular fuzzy numbers $\tilde{a} = (a^2[\tilde{a}])$ and $\tilde{b} = (b^2[\tilde{b}])$, the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as:

Addition: $\tilde{a} + \tilde{b} = (a^2 + b^2[\tilde{a} + \tilde{b}]) = (a^2 + b^2[(a^1, a^3) + (b^1, b^3)])$

Subtraction: $\tilde{a} - \tilde{b} = (a^2 - b^2[\tilde{a} - \tilde{b}]) = (a^2 - b^2[(a^1, a^3) - (b^1, b^3)])$

Multiplication : $\tilde{a} \times \tilde{b} = (a^2 \times b^2[\tilde{a} \times \tilde{b}]) = (a^2 \times b^2[(a^1, a^3) \times (b^1, b^3)])$

For all the rest of this paper, we will consider the following notations:

Assume that $\tilde{c}_j = (c^1_j, c^2_j, c^3_j)$, $\tilde{s}_j = (s^1_j, s^2_j, s^3_j)$ and $\tilde{b}_j = (b^1_j, b^2_j, b^3_j)$ are triangular fuzzy numbers

$$\tilde{x} = \begin{bmatrix} x^1_j \\ x^2_j \\ x^3_j \end{bmatrix} \quad (\text{for } j = 1, ..., n)$$

$$\tilde{y} = \begin{bmatrix} y^1_j \\ y^2_j \\ y^3_j \end{bmatrix} \quad (\text{for } j = 1, ..., n)$$

$$\tilde{c}_j = \begin{bmatrix} c^1_j \\ c^2_j \\ c^3_j \end{bmatrix} \quad (\text{for } j = 1, ..., n)$$

Then

3.2. Formulation of a Linear Programming with Triangular Fuzzy Variables (FVLP) Problems

In this section, we introduce a kind of linear programming problems where the right-hand-side vector and the decision variables are a type of triangular fuzzy numbers, simultaneity. We name such problems as Linear Programming with Triangular Fuzzy Variables (FVLP) problems.
We consider the primal (P) and dual (D) Linear Programming with Triangular Fuzzy Variables (FVLP) problems as follows \cite{13,18,19}:

\[
\begin{align*}
(P) & \quad \text{Max } \bar{Z}(\bar{x}) \approx \sum_{j=1}^{n} c_j \times \bar{x}_j \\
& \text{Subject to the constraints } \sum_{j=1}^{n} a_{ij} \times \bar{x}_j \leq \bar{b}_i \\
(D) & \quad \text{Min } \bar{W}(\bar{y}) \approx \sum_{i=1}^{m} \bar{b}_i \times y_i \\
& \text{Subject to the constraints } \sum_{i=1}^{m} a_{ij} \times y_i \geq c_j
\end{align*}
\]

For all the rest of this paper, we will consider the following primal Linear Programming with Triangular Fuzzy Variables (FVLP) problem:

\[
\begin{align*}
\text{(FVLP1)} & \quad \text{Min/Max } \bar{Z}(\bar{x}_1, ..., \bar{x}_n) \approx \sum_{j=1}^{n} c_j \bar{x}_j \\
& \text{Subject to the constraints } \sum_{j=1}^{n} a_{ij} \times \bar{x}_j \leq \bar{b}_i \\
& \quad 1 \leq j \leq n \text{ and } 1 \leq i \leq m
\end{align*}
\]

\[
\begin{align*}
\text{(FVLP2)} & \quad \text{Min/Max } \bar{W}(\bar{y}) \approx \sum_{i=1}^{m} \bar{b}_i \times \bar{y}_i \\
& \text{Subject to the constraints } \sum_{i=1}^{m} a_{ij} \times \bar{y}_i \geq c_j \\
& \quad 1 \leq j \leq n \text{ and } 1 \leq i \leq m
\end{align*}
\]

3.3. Formulation of a Semi-fully Fuzzy Linear Programming Problem (SFFLP)

In this section, we introduce a kind of linear programming problems where the coefficients in objective function, the right-hand-side vector and the decision variables are a type of triangular fuzzy numbers, simultaneity. We name such problems as Semi-Fully Fuzzy Linear Programming (SFFLP) problems.

We consider the primal (P) and dual (D) Semi-fully Fuzzy Linear Programming Problem (SFFLP) as follows \cite{13,18,19}:

\[
\begin{align*}
(P) & \quad \text{Max } \bar{Z}(\bar{x}_1, ..., \bar{x}_n) \approx \sum_{j=1}^{n} c_j \bar{x}_j \\
& \text{Subject to the constraints } \sum_{j=1}^{n} a_{ij} \times \bar{x}_j \leq \bar{b}_i \\
& \quad 1 \leq j \leq n \text{ and } 1 \leq i \leq m
\end{align*}
\]

\[
\begin{align*}
(D) & \quad \text{Min } \bar{W}(\bar{y}) \approx \sum_{i=1}^{m} \bar{b}_i \times \bar{y}_i \\
& \text{Subject to the constraints } \sum_{i=1}^{m} a_{ij} \times \bar{y}_i \geq c_j \\
& \quad 1 \leq j \leq n \text{ and } 1 \leq i \leq m
\end{align*}
\]

3.4. Our Method for Solving Linear Programming Problems With Triangular Fuzzy Variables (FVLP) and Semi-Fully Fuzzy Linear Programming Problems (SFFLP)

In this section, a method to find a fuzzy optimal solution of Linear Programming problems with Triangular Fuzzy Variables (FVLP) and Semi-Fully Fuzzy Linear Programming problems (SFFLP) is presented.

3.4.1. Formulation of a Linear Programming (LP) Problem

For all the rest of this paper, we assume that \(x^{2} = (x_1^{2}, x_2^{2}, ..., x_n^{2})\) is unique optimal solution of (LP) with the associated value of the objective function \(Z^{2}(x^{2}) = \sum_{j=1}^{n} c_j^{2} \times x_j^{2}\)

\[
\begin{align*}
\text{(LP)} & \quad \text{Max } Z^{2}(x^{2}) = \sum_{j=1}^{n} c_j^{2} \times x_j^{2} \\
& \text{Subject to the constraints } \sum_{j=1}^{n} a_{ij} \times x_j^{2} + x_{n}^{2} = b_i^{2}, 1 \leq i \leq m \\
& \quad 1 \leq j \leq n + m \\
& \quad x_j^{2} \text{ are real numbers } (\mathbb{R})
\end{align*}
\]

Note 1. For any two triangular fuzzy numbers

\[
\begin{align*}
\bar{a} = (a^1, a^2, a^3) = (a^2, [a^1, a^3]) = (a^3, [a^1, a^2]) \quad \text{and} \\
\bar{b} = (b^1, b^2, b^3) = (b^2, [b^1, b^3]) = (b^3, [b^1, b^2]) \quad \text{then:}
\end{align*}
\]

\[
\begin{align*}
(i) & \quad \bar{a} = \bar{b} \text{ if and only if } a^1 = b^1, a^2 = b^2 \text{ and } a^3 = b^3.
\end{align*}
\]

\[
\begin{align*}
(ii) & \quad \bar{a} = \bar{b} \text{ if and only if } a^2 = b^2 \text{ and } a^1 = b^1.
\end{align*}
\]
(iii) \( a = b \) if and only if \( a^2 = b^2 \) and \([a^2, a^3] = [b^1, b^3] \).

(iv) \( a = b \) if and only if \( a^2 = b^2 \) and \( m(a) = m(b) \) and \( w(a) = w(b) \).

Now we can write the following propositions using Ramesh and Ganesan [10] and Ladji, et al. [14]:

**Proposition 1.** \( x^*_b \approx B^{-1}_{n+1}b \) is a optimal solution to the (INLP2) if and only if
\[ x^*_b \approx B^{-1}_{n+1}b \] is an optimal solution to the (INLP1).

Proof. Just assume that \( a = [a^1, a^3] = (m(a), w(a)) \).

We can write the following propositions:

**Proposition 2.** \( x^* = (x^*_1 \; x^*_2 \; ... \; x^*_{n+m})^t \) is an optimal solution to the (SFFLP2) if and only if
\[ x^* = (x^*_1 \; x^*_2 \; ... \; x^*_{n+m})^t \] is an optimal solution to the (SFFLP1).

Proof. Assume that \( x^*_j = (x^*_j^1, x^*_j^2, x^*_j^3) \) and \( x^*_j^1 = (x^*_j^1, x^*_j^2, x^*_j^3) \) for \( 1 \leq j \leq n \).

If \( x^*_j = (m(x^*_j^1), w(x^*_j^1)) \) is an optimal solution to the (SFFLP2), then
\[ x^*_j = (m(x^*_j^1), w(x^*_j^1)) \] is an optimal solution to the (SFFLP2) and \( x^*_j^1 = (x^*_j^1, x^*_j^2, x^*_j^3) \) is an optimal solution to the (LP) and vice versa and see Note 1.

**Proposition 3.** \( x^* = (x^*_1' \; x^*_2' \; ... \; x^*_{n+m}')^t \) is an optimal solution to the (SFFLP2) if and only if
\[ x^* = (x^*_1' \; x^*_2' \; ... \; x^*_{n+m}')^t \] is an optimal solution to the (SFFLP1).

Proof. Assume that \( x^*_j^1 = (x^*_j^1, x^*_j^2, x^*_j^3) \) and \( x^*_j^1 = (x^*_j^1, x^*_j^2, x^*_j^3) \) for \( 1 \leq j \leq n \).

If \( x^*_j^1 = (m(x^*_j^1), w(x^*_j^1)) \) is an optimal solution to the (SFFLP2), then \( x^*_j^1 = (x^*_j^1, x^*_j^2, x^*_j^3) \) is an optimal solution to the (SFFLP2) and vice versa and see Note 1.

**Corollary 1.** If \( \tilde{x}^*_j = [x_j^*, x_j^*] \) is an optimal solution to the (INLP1) and
\[ \tilde{x}^*_j = [x_j^*, x_j^*] \] is an optimal solution to the (INLP1) and
\[ x^*_j^2 = (x^*_j^2, x^*_j^2, ... , x^*_j^2) \] is an optimal solution to the (LP) then
\[ x^*_j^2 = (x^*_j^2, x^*_j^2, ... , x^*_j^2) \] is an optimal solution to the (LP) and vice versa.

3.4.2. The Steps of Our Method

The steps of our method for solving Linear Programming problems with Triangular Fuzzy Variables (FVLP) and Semi-Fully Fuzzy Linear Programming problems (SFFLP) as follows:

**Step 1** Consider a Linear Programming problem with Triangular Fuzzy Variables (FVLP) or Semi-Fully Fuzzy Linear Programming problem (SFFLP):

Identify the Linear Programming problem involving Interval Numbers (INLP1) and the Linear Programming problem (LP)

**Step 2** Solving the primal Linear Programming problem involving Interval Numbers (INLP1) by Ramesh and Ganesan [10] and Ladji, et al. [14] to determine the optimal solution:

If \( m(\tilde{x}^*_b) > w(\tilde{b}_1) \), then \( \tilde{x}^*_j^1 = [x^*_j^1, x^*_j^3] \) with \( \tilde{x}^*_b \approx B^{-1}_{n+1}b \) and \( \tilde{b}_1 = [b^1, b^3] \).

If \( m(\tilde{x}^*_b) < w(\tilde{b}_1) \), then \( \tilde{x}^*_j^1 = (m(x^*_j^1), w(x^*_j^1)) \) with \( \tilde{x}^*_b \approx B^{-1}_{n+1}b \) and \( \tilde{b}_1 = (m(b_1), w(b_1)) \). In addition, the values of \( \tilde{Z}^* = (m(Z^*), w(Z^*)) = [Z^1, Z^2] \).

**Step 3** To determine the optimal solution of the dual (D) of Linear Programming problem involving Interval Numbers (INLP1):

If \( m(\tilde{x}^*_b) > w(\tilde{b}_1) \), then \( \tilde{y}^*_j^1 = [y^*_j^1, y^*_j^3] \) with \( \tilde{y}^*_j^1 = [C^*_j^1 A^*_j^1 - c_1^j] \) and
\[ \tilde{y}^*_j^2 = [C^*_j^2 A^*_j^2 + c_2^j] \] and \( \tilde{y}^*_j^3 = [C^*_j^3 A^*_j^3 + c_3^j] \) and \( c_j = (c_1^j, c_2^j) \).

If \( m(\tilde{x}^*_b) < w(\tilde{b}_1) \), then \( \tilde{y}^*_j^1 = (m(\tilde{y}^*_j^1), w(\tilde{y}^*_j^1)) \) with \( \tilde{y}^*_j^1 = [C^*_j^1 A^*_j^1 - c_1^j] \) and
\[ \tilde{y}^*_j^2 = [C^*_j^2 A^*_j^2 - c_2^j] \] and \( \tilde{y}^*_j^3 = (m(c_j), w(c_j)) \).

In addition, the values of \( \tilde{W}^* = (m(W^*), w(W^*)) = [W^1, W^2] \).

**Step 4** Formulation of a classical primal (P) and dual (D) Linear Programming (LP) problem to determine the values of \( x^*_j^1, j = 1, ... , n + m \) and \( y^*_j^1 = [\Delta^*_j^1, j = 1, ... , n \). In addition, the values of \( Z^1, Z^2, W^1, W^2, W^3 \).

**Step 5** Fuzzy optimal solution of Linear Programming problems with Triangular Fuzzy Variables (FVLP) or Semi-Fully Fuzzy Linear Programming problems (SFFLP).

(i) Primal optimal solution:
\[ x^*_j^1 = [x^*_j^1, x^*_j^3] = (\tilde{x}^*_j^1, x^*_j^2, x^*_j^3) \] (i, j = 1, ... , n + m) with the associated value of the objective function
\[ \text{Max } \tilde{Z}^* = (Z^1, Z^2, Z^3) = (Z^1, \tilde{Z}^*) \] (j).

(ii) Then corresponding Dual optimal solution is given by
\[ y^*_j^1 = (y^*_j^1, y^*_j^3) = (\tilde{y}^*_j^1, y^*_j^2, y^*_j^3) \] (i, j = 1, ... , n + m) with the associated value of the objective function
\[ \text{Min } \tilde{W}^* = (W^1, W^2, W^3) = (W^1, \tilde{W}^*) \] (i).

(iii) Comparisons using Ranking function [17]:
\[ \mathcal{R}(	ext{Max } \tilde{Z}^*) = \frac{1}{2} (Z^1 + Z^2 + Z^3) \] and \[ \mathcal{R}(	ext{Min } \tilde{W}^*) = \frac{1}{4} (W^1 + 2W^2 + W^3) \].

NB: Let \( \tilde{a} = (a^1, a_2, a^3) \) and \( \tilde{b} = (b^1, b^2, b^3) \) then: \( \tilde{a} < \tilde{b} \) iff \( \mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b}) \), \( \tilde{a} \approx \tilde{b} \) iff \( \mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b}) \).
## 4. Numerical Examples

Numerical examples are provided to illustrate the theory developed in this paper.

### Example 4.1. Step 1

Consider the following Semi-Fully Fuzzy Primal Linear Programming (SFFLP) problem: (P)

\[
\begin{align*}
\text{Max} & \quad (29,30,31)\tilde{x}_1 + (22,23,24)\tilde{x}_2 + (28,29,30)\tilde{x}_3 \\
\text{Subject to the constraints} & \quad 6\tilde{x}_1 + 5\tilde{x}_2 + 3\tilde{x}_3 \leq (25,26,27) \\
& \quad 4\tilde{x}_1 + 2\tilde{x}_2 + 5\tilde{x}_3 \leq (6,7,8)
\end{align*}
\]

where \(\tilde{x}_1, \tilde{x}_2\) and \(\tilde{x}_3\) are triangular fuzzy numbers.

Then the corresponding dual problem is given by: (D)

\[
\begin{align*}
\text{Min} & \quad (25,26,27)\tilde{y}_1 + (6,7,8)\tilde{y}_2 + (7,8,9)\tilde{y}_3 \\
\text{Subject to the constraints} & \quad 6\tilde{y}_1 + 4\tilde{y}_2 \geq (29,30,31) \\
& \quad 5\tilde{y}_1 + 2\tilde{y}_2 \geq (22,23,24) \\
& \quad 3\tilde{y}_1 + 5\tilde{y}_2 \geq (28,29,30)
\end{align*}
\]

where \(\tilde{y}_1, \tilde{y}_2\) and \(\tilde{y}_3\) are triangular fuzzy numbers.

### Step 2

We convert the primal and dual linear programming problem involving interval numbers (INLP1) to its canonical form by adding slack variables \(\tilde{x}_4\) and \(\tilde{x}_5\) as follows: Max \(\tilde{Z}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (30,1)\tilde{x}_1 + (23,1)\tilde{x}_2 + (29,1)\tilde{x}_3\) Subject to the constraint \(6\tilde{x}_1 + 5\tilde{x}_2 + 3\tilde{x}_3 \leq (26,1)\) \(4\tilde{x}_1 + 2\tilde{x}_2 + 5\tilde{x}_3 \leq (6,1)\) \(\tilde{x}_j \in \mathbb{R}\)

### Step 3

We get the matrix \(B^{-1}\) \(\approx \begin{pmatrix} 1 & -9/2 \\ 0 & 5/2 \end{pmatrix}\) with \(\tilde{b} = (0,22,24)\) and \(\tilde{b}_i = (0,22,24)\):

\[
\begin{align*}
\tilde{x}_1 & \approx B^{-1}\tilde{b} \approx \begin{pmatrix} 1 & -9/2 \\ 0 & 5/2 \end{pmatrix} \begin{pmatrix} 25/27 \\ 6/8 \end{pmatrix} \approx \begin{pmatrix} 5/12 \\ 3/4 \end{pmatrix}
\end{align*}
\]

\(\tilde{x}_1 \approx 0, \tilde{x}_2 \approx 5, \tilde{x}_3 \approx 0\) with Max \(\tilde{Z}(\tilde{x}^*) \approx 66,96\).

### Step 4

We convert the dual linear programming problem involving interval numbers (INLP2) to its canonical form by adding slack variables \(\tilde{y}_4\) and \(\tilde{y}_5\) as follows: Min \(\tilde{W}(\tilde{y}_1, \tilde{y}_2) = (26,1)\tilde{y}_1 + (7,1)\tilde{y}_2\) Subject to the constraints

\[
\begin{align*}
6\tilde{y}_1 + 4\tilde{y}_2 \geq (29,31), \\
5\tilde{y}_1 + 2\tilde{y}_2 \geq (22,24), \\
3\tilde{y}_1 + 5\tilde{y}_2 \geq (28,30)
\end{align*}
\]

We have \(m(\tilde{x}_j) > w(\tilde{b}_i)\), then \(\tilde{x}_j^* = [x_j^1, x_j^2]\) with \(\tilde{x}_j \approx B_n^{-1}\tilde{b}\) and \(\tilde{b}_i = [b_i^1, b_i^2]\):

\[
\begin{align*}
\tilde{x}_j & \approx B_n^{-1}\tilde{b} = \begin{pmatrix} 1 & -9/2 \\ 0 & 5/2 \end{pmatrix} \begin{pmatrix} 25/27 \\ 6/8 \end{pmatrix} \approx \begin{pmatrix} 5/12 \\ 3/4 \end{pmatrix}
\end{align*}
\]

\(\tilde{x}_1 \approx 0, \tilde{x}_2 \approx 5, \tilde{x}_3 \approx 0\) with Max \(\tilde{Z}(\tilde{x}^*) \approx 66,96\).
\[
\begin{align*}
\text{Max } Z^2 &= 30x_1^2 + 23x_2^2 + 29x_3^2 \\
\text{Subject to the constraints} & \\
6x_1^2 + 5x_2^2 + 3x_3^2 + x_4^2 &= 26 \\
4x_1^2 + 2x_2^2 + 5x_3^2 + x_5^2 &= 7 \\
3 & \leq x_1^2 \leq 4 \\
5 & \leq x_4^2 \leq 12 \\
y_3^{2^2} &= 16, y_4^{2^2} = 0, \text{ and } y_5^{2^2} = \frac{57}{2} \text{ with} \\
\text{Max } Z^{2^2} &= \text{Min } W^{2^2} = \frac{161}{2}.
\end{align*}
\]

**Step 5** Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:
\[
\begin{align*}
\tilde{\alpha}_1 &= (x_1^1, x_1^2, x_1^3) = (x_1^2|\tilde{x}_1) = (0|0.0) = (0,0,0), \\
\tilde{\alpha}_2 &= (x_2^1, x_2^2, x_2^3) = (x_2^2|\tilde{x}_2) = \left(\frac{7}{2}, 3.7, 4\right), \\
\tilde{\alpha}_3 &= (x_3^1, x_3^2, x_3^3) = (x_3^3|\tilde{x}_3) = (0|0.0) = (0,0,0), \\
\tilde{\alpha}_4 &= (x_4^1, x_4^2, x_4^3) = (x_4^2|\tilde{x}_4) = \left(\frac{12}{2}, 5.12, 12\right), \\
\tilde{\alpha}_5 &= (x_5^1, x_5^2, x_5^3) = (x_5^2|\tilde{x}_5) = (0|0.0) = (0,0,0), \\
\text{Max } \tilde{Z} &= (Z^1, Z^2, Z^3) = \left(Z^{2^2}|\tilde{Z}\right) = \left(\frac{161}{2}, 66, 96\right) = \left(66, \frac{161}{2}, 96\right).
\end{align*}
\]

(ii) Dual optimal solution:
\[
\begin{align*}
\tilde{\gamma}_1 &= (y_1^1, y_1^2, y_1^3) = (y_1^2|\tilde{y}_1) = (0|0.0) = (0,0,0), \\
\tilde{\gamma}_2 &= (y_2^1, y_2^2, y_2^3) = (y_2^2|\tilde{y}_2) = \left(\frac{23}{2}, 11, 12\right) = \left(11, \frac{23}{2}, 12\right), \\
\tilde{\gamma}_3 &= (y_3^1, y_3^2, y_3^3) = (y_3^2|\tilde{y}_3) = (16|13,19) = \left(13, 16, 19\right), \\
\tilde{\gamma}_4 &= (y_4^1, y_4^2, y_4^3) = (y_4^2|\tilde{y}_4) = (0|0.0) = (0,0,0), \\
\tilde{\gamma}_5 &= (y_5^1, y_5^2, y_5^3) = (y_5^2|\tilde{y}_5) = \left(\frac{57}{2}, 25, 32\right) = \left(25, \frac{57}{2}, 32\right), \\
\text{Min } \tilde{W} &= (W^{1^2}, W^{2^2}, W^{3^2}) = \left(W^{2^2}|\tilde{W}\right) = \left(\frac{161}{2}, 66, 96\right) = \left(66, \frac{161}{2}, 96\right).
\end{align*}
\]

(iii) Comparisons using Ranking function [17]:
\[
\text{Max } \tilde{Z} = \text{Min } \tilde{W} = \left(66, \frac{161}{2}, 96\right) \text{ and } \mathcal{R}(\text{Max } \tilde{Z}) = \mathcal{R}(\text{Min } \tilde{W}) = \frac{323}{4}.
\]

**Example 4.2. Step 1**

Consider the following Semi-Fully Fuzzy Linear Programming (SFFLP) problem: (P)

\[
\begin{align*}
\text{Max } \tilde{Z} &= (5,6, 8)\tilde{x}_1 + (4, 4, 4)\tilde{x}_2 \\
\text{Subject to the constraints} & \\
3\tilde{x}_1 + 2\tilde{x}_2 & \leq (140, 150, 150) \\
4\tilde{x}_1 + 3\tilde{x}_2 & \leq (155, 160, 165) \\
\text{where } \tilde{x}_1 \text{ and } \tilde{x}_2 & \text{ are triangular fuzzy numbers.}
\end{align*}
\]

Then the corresponding dual problem is given by: (D)

\[
\begin{align*}
\text{Min } \tilde{W} &= (140, 150, 150)\tilde{y}_1 + (155, 160, 165)\tilde{y}_2 \\
\text{Subject to the constraints} & \\
3\tilde{y}_1 + 4\tilde{y}_2 & \geq (5, 6, 8) \\
2\tilde{y}_1 + 3\tilde{y}_2 & \geq (4, 4, 4) \\
\text{where } \tilde{y}_1 \text{ and } \tilde{y}_2 & \text{ are triangular fuzzy numbers.}
\end{align*}
\]

**Step 2**

\[
\begin{align*}
\text{Max } \tilde{Z}(\tilde{x}_1, \tilde{x}_2) & \approx [5,8]\tilde{x}_1 + [4,4]\tilde{x}_2 \\
\text{Subject to the constraints} & \\
3\tilde{x}_1 + 2\tilde{x}_2 & \leq [140, 150] \\
4\tilde{x}_1 + 3\tilde{x}_2 & \leq [155, 165] \\
\tilde{x}_1, \tilde{x}_2 & \in \mathbb{R} \\
\end{align*}
\]

(INLP1)

We convert the primal and dual linear programming problem involving interval numbers (INLP2) to its canonical form by adding slack variables \(\tilde{x}_3\) and \(\tilde{x}_4\) as follows: Max \(\tilde{Z}(\tilde{x}_1, \tilde{x}_2) \approx (\frac{13}{2}, \frac{3}{2})\tilde{x}_1 + (4, 0)\tilde{x}_2\) Subject to the constraints
\[
\begin{align*}
3\tilde{x}_1 + 2\tilde{x}_2 + \tilde{x}_3 & \approx (145, 5), 4\tilde{x}_1 + 3\tilde{x}_2 + \tilde{x}_4 & \approx (160, 5) \\
\text{Max } \tilde{Z}(\tilde{x}_1, \tilde{x}_2) & \approx (\frac{13}{2}, \frac{3}{2})\tilde{x}_1 + (4, 0)\tilde{x}_2 \text{ Subject to the constraints} \\
3\tilde{x}_1 + 2\tilde{x}_2 + \tilde{x}_3 & \approx (145, 5), 4\tilde{x}_1 + 3\tilde{x}_2 + \tilde{x}_4 & \approx (60, 5)
\end{align*}
\]
Basic variables $\bar{x}_B$  
Coefficients $\bar{c}_B$  
$[5,8] = 13/2$  
$[4,4] = (4,0)$  
$0$  
$0$  
Current values $\bar{X}_B$  
$\langle m(\bar{X}_B), w(\bar{b}_1) \rangle$

\begin{align*}
\bar{x}_3 &= 0 & A_1 = & 0 & -1/4 & 1 & -3/4 & \langle 25, 3/4 \rangle \\
\bar{x}_1 &= 13/2 & A_2 = & 1 & 3/4 & 0 & 1/4 & \langle 40, 3/4 \rangle \\
\bar{c}_j &= \bar{c}_B A_j - \bar{c}_1 & & 0 & \langle 7/8, 2 \rangle & 0 & 13/8 & \langle 260, 3/8 \rangle \\
\end{align*}

$m(\bar{X}_B) > w(\bar{b}_1)$, then $\bar{x}_j^* = [x_j^1, x_j^2]$ with $\bar{x}_B \approx B^{-1}_{n1} \bar{b}$ and $\bar{b}_1 = [b_1, b_1^2]$. We get the matrix $B_{n1}^{-1} = \begin{bmatrix} 140, 150 \\ 155, 165 \end{bmatrix}$ and $\bar{c}_B = (0, [5,8])$. $\langle \bar{x}_1^1, \bar{x}_1^2 \rangle \approx B^{-1}_{n1} \bar{b}$ avec $\bar{x}_1 \approx [155, 165]$, $\bar{x}^1 \approx 0$, $\bar{x}^2 \approx [65, 135]$, and $\bar{c}_4 \approx 0$ with $\max Z(\bar{x}) \approx \langle 775, 330 \rangle$.

**Step 3** Then the corresponding dual problem (D) is given by:

Min $\bar{W}(\bar{y}_1, \bar{y}_2) \approx [140, 150] \bar{y}_1 + [155, 165] \bar{y}_2$ Subject to the constraints $3 \bar{y}_1 + 4 \bar{y}_2 \geq [5,8]$, $2 \bar{y}_1 + 3 \bar{y}_2 \geq [4,4]$ and $\bar{y}_1, \bar{y}_2 \in \mathbb{R}, i = 1,2$.

We convert the dual linear programming problem involving interval numbers to its canonical form by adding slack variables $\bar{y}_3$ and $\bar{y}_4$ as follows:

Min $\bar{W}(\bar{y}_1, \bar{y}_2) \approx (145, 5) \bar{y}_1 + (160, 5) \bar{y}_2$ Subject to the constraints $3 \bar{y}_1 + 4 \bar{y}_2 - \bar{y}_3 \approx [13/2, 2]$. $2 \bar{y}_1 + 3 \bar{y}_2 - \bar{y}_4 \approx (4,0)$. We have $m(\bar{X}_B^*) > w(\bar{b}_1)$, then $\bar{y}_j^* = [y_j^1, y_j^2]$ with $\bar{y}_m\bar{y}_j = \bar{c}_B A_j - \bar{c}_1$ and $\bar{y}_j^* = \bar{c}_B A_j + c_{n+1}$ and $\bar{c}_j = [c_j^1, c_j^2]$. We get $\bar{y}_j^* \approx 0$, $\bar{y}_j^* \approx [5/4, 2]$, $\bar{y}_3 \approx 0$ and $\bar{y}_4 \approx [1/4, 2]$ with $\min \bar{W}(\bar{y}) \approx \langle 775, 330 \rangle$.

**Step 4** Formulation of a classical primal (P) Linear Programming (LP) problem

Max $Z^2 = 6x_1^2 + 4x_2^2$ Subject to the constraints $3x_1^2 + 2x_2^2 \leq 150$, $4x_1^2 + 3x_2^2 \leq 160$. We get $x_1^2 = 40$, $x_2^2 = 0$, $x_3^2 = 30$, $x_4^2 = 0$.

\begin{align*}
3\bar{x}_1^2 & \leq 150 \\
4\bar{x}_1^2 & \leq 160 \\
65/4 \leq x_3^2 & \leq 135/4 \\
x_1^2 \geq 0 \:
\end{align*}

$y_1^2 = 0$, $y_2^2 = 3/2$, $y_3^2 = 0$, and $y_4^2 = 1/2$ with Max $Z^2 = \min W^2 = 240$.

**Step 5** Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:

Max $\bar{Z} = (Z^1, Z^2, Z^3) = (Z^1, Z^2)$

$\bar{x}_1^* = (x_1^1, x_1^2, x_1^3) = (x_1^2) \bar{x}_1^* = \langle 40, 155/4, 165/4 \rangle = \langle 40, 165/4 \rangle$.

$\bar{x}_2^* = (x_2^1, x_2^2, x_2^3) = (x_2^2) \bar{x}_2^* = \langle 0, 0, 0 \rangle$.

$\bar{x}_3^* = (x_3^1, x_3^2, x_3^3) = (x_3^2) \bar{x}_3^* = \langle 30, 65/4, 135/4 \rangle = \langle 30, 135/4 \rangle$.

$\bar{x}_4^* = (x_4^1, x_4^2, x_4^3) = (x_4^2) \bar{x}_4^* = \langle 0, 0, 0 \rangle$.

Max $\bar{Z} = (Z^1, Z^2, Z^3) = (Z^1, Z^2) = \langle 240, 775/4, 330/4 \rangle = \langle 775/4, 240, 330 \rangle$.

(ii) Dual optimal solution:

$\bar{y}_1^* = (y_1^1, y_1^2, y_1^3) = (y_1^2) \bar{y}_1^* = \langle 0, 0, 0 \rangle$.

$\bar{y}_2^* = (y_2^1, y_2^2, y_2^3) = (y_2^2) \bar{y}_2^* = \langle 1/4, 2 \rangle = \langle 1/4, 2 \rangle$.

$\bar{y}_3^* = (y_3^1, y_3^2, y_3^3) = (y_3^2) \bar{y}_3^* = \langle 0, 0, 0 \rangle$.

$\bar{y}_4^* = (y_4^1, y_4^2, y_4^3) = (y_4^2) \bar{y}_4^* = \langle 1/4, 2 \rangle = \langle 1/4, 2 \rangle$.

Min $\bar{W} = (W^1, W^2, W^3) = (W^1) \bar{W}^* = \langle 240, 775/4, 330/4 \rangle = \langle 775/4, 240, 330 \rangle$.

(iii) Comparisons using Ranking function [17]:

Max $\bar{Z} = \min \bar{W} = \langle 775/4, 240, 330 \rangle$ and $\mathcal{R}(\max \bar{Z}) = \mathcal{R}(\min \bar{W}) = \langle 4015/16 \rangle$. 

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**Example 4.3. Step 1**

Consider the following Triangular Fuzzy Variables Linear Programming (TFVLP) problem:

\[
\text{Max } \tilde{Z} = 40\tilde{x}_1 + 30\tilde{x}_2 \\
\text{Subject to the constraints} \quad \tilde{x}_1 + \tilde{x}_2 \leq (395, 400, 405) \\
\tilde{x}_1 + \tilde{x}_2 \leq (493,500,507) \\
\text{where } \tilde{x}_1 \text{ and } \tilde{x}_2 \text{ are triangular fuzzy numbers.}
\]

Then the corresponding dual problem is given by:

\[
\text{(D)} \\
\text{Min } \tilde{W} = (395,400,405)y_1 + (493,500,507)y_2 \\
\text{Subject to the constraints} \quad y_1 + 2y_2 \geq 40 \\
y_1 + y_2 \geq 30 \\
\text{where } y_1 \text{ and } y_2 \text{ are real numbers.}
\]

**Step 2**

\[
\begin{align*}
\text{Max } \tilde{Z} &= 40\tilde{x}_1 + 30\tilde{x}_2 \\
\text{Subject to the constraints} \\
\tilde{x}_1 + \tilde{x}_2 &\leq (395, 400, 405) \\
2\tilde{x}_1 + \tilde{x}_2 &\leq (493,500,507) \\
\tilde{x}_1, \tilde{x}_2 &\in \mathbb{R} \\
\text{(INLP1)}
\end{align*}
\]

We convert the primal and dual linear programming problem involving interval numbers (INLP2) to its canonical form by adding slack variables \(\tilde{x}_3\) and \(\tilde{x}_4\) as follows: \(\text{Max } \tilde{Z} = 40\tilde{x}_1 + 30\tilde{x}_2\) Subject to the constraints \(\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \approx (400,5)\)

\(2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_4 \approx (500,7)\).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Basic variables} & \text{Coefficients} & \text{40} & \text{30} & \text{0} & \text{0} & \text{Current values} \\
\tilde{x}_1 & \tilde{C}_1 & 30 & 1 & 2 & -1 & \tilde{X}_1 (300,7) \\
\tilde{x}_2 & \tilde{C}_2 & 40 & 1 & -1 & 1 & \tilde{X}_2 (100,7) \\
\tilde{A}_1^* - \tilde{c}_j & & 0 & 0 & 20 & 10 & \tilde{X}_1 (13000,7) \\
\tilde{b}_1 & & & & & & \tilde{X}_2 \tilde{b}_1 \approx 13000,7 \\
\hline
\end{array}
\]

**Step 3**

We convert the primal and dual linear programming problem involving interval numbers to its canonical form by adding slack variables \(y_3\) and \(y_4\) as follows: \(\tilde{W} = [395,400,405]y_1 + [493,500,507]y_2\)

\(\text{Min } \tilde{W} \approx [12010,13990]\).

We convert the dual linear programming problem involving interval numbers to its canonical form by adding slack variables \(y_3\) and \(y_4\) as follows:

\(\text{Min } \tilde{W} = (400,5)y_1 + (500,7)y_2\) Subject to the constraints \(y_1 + 2y_2 \geq 40\), \(y_1 + y_2 \geq 30\) and \(y_i \in \mathbb{R}, i = 1,2,\ldots,A\) We have \(\tilde{m}(\tilde{b}_1) > \tilde{W}(\tilde{b}_1)\), then 

\[
\tilde{y}_i^* = [y_{i1}^*, y_{i2}^*, y_{i3}^*] \text{ with } \tilde{y}_m^* = \tilde{C}_n^* A_n - \tilde{c}_n^* \text{ and } \tilde{y}_i^* = [\tilde{C}_n^* A_{n+i} - \tilde{c}_n^*] \text{ and } \tilde{c}_n^* = [c_{i1}^*, c_{i2}^*] \\
\tilde{y}_i^* \approx 20, y_2^* \approx 10, y_3^* \approx 0 \text{ and } y_4^* \approx 0 \text{ with } \tilde{W}(\tilde{y}) \approx [12830,13170].
\]

**Step 4**

Formulation of a classical primal (P) Linear Programming (LP) problem

\[
\text{Max } Z^* = 40x_1^* + 30x_2^* \\
\text{Subject to the constraints} \\
x_1^2 + x_2^2 + x_3^2 \leq 400 \\
2x_1^2 + x_2^2 + x_3^2 \leq 500 \\
\text{we get } x_1^2 \leq 100, x_2^2 \leq 300, \\
88 \leq x_1^2 \leq 112 \\
283 \leq x_2^2 \leq 317 \\
x_1^2 \geq 0 \\
x_1^2 = 0, x_2^2 = 0, y_1^2 = 20, y_2^2 = 10, y_3^2 = 0, \text{ and } y_4^2 = 0 \text{ with } \\
\text{Max } Z^* = \text{Min } W^* = 13000.
\]

**Step 5**

Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:

\[
\tilde{x}_1^* = (x_1^1, x_1^2, x_1^3) = (x_1^2|\tilde{x}_1^*|) = (100|[88,112]) = (88,100,112), \\
\tilde{x}_2^* = (x_2^1, x_2^2, x_2^3) = (x_2^2|\tilde{x}_2^*|) = (300|[283,317]) = (283,300,317), \\
\tilde{x}_3^* = (x_3^1, x_3^2, x_3^3) = (x_3^2|\tilde{x}_3^*|) = (0|[0,0]) = (0,0,0), \\
\tilde{x}_4^* = (x_4^1, x_4^2, x_4^3) = (x_4^2|\tilde{x}_4^*|) = (0|[0,0]) = (0,0,0).
\]

\[
\text{Max } \tilde{Z} = (Z^1, Z^2, Z^3) = (Z^2|\tilde{Z}^*) = (13000|[12010,13990]) = (12010,13000,13990).
\]

(ii) Dual optimal solution:
\( y_1^* = 20, y_2^* = 10, y_3^* = 0, y_4^* = 0 \)

Min \( \bar{W} = (W^{x_1}, W^{x_2}, W^{x_3}) = (W^{x_1} \| W^{x_2} \| W^{x_3}) = (13000[12830, 13170]) = (12830, 13000, 13170) \).

(iii) Comparisons using Ranking function [17]:

\[
\mathcal{R}(\text{Max} \bar{Z}) = \frac{1}{4}(12010 + 26000 + 13990) = 13000 \quad \text{and} \quad \mathcal{R}(\text{Min} \bar{W}) = \frac{1}{4}(12830 + 26000 + 13170) = 13000
\]

Example 4.4. Step 1

Consider the following Triangular Fuzzy Variables Linear Programming (FVLP) problem:

\[
\begin{align*}
\text{Max} & \quad \tilde{z} = 4x_1 + 3x_2 \\
\text{Subject to the constraints} & \quad \tilde{x}_1 + 2\tilde{x}_2 \leq (4, 8, 12) \\
& \quad 2\tilde{x}_1 + \tilde{x}_2 \leq (6, 9, 12)
\end{align*}
\]

where \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are triangular fuzzy numbers.

Then the corresponding dual problem is given by:

\[
\begin{align*}
\text{Min} & \quad \bar{w} = (4, 8, 12)y_1 + (6, 9, 12)y_2 \\
\text{Subject to the constraints} & \quad y_1 + 2y_2 \geq 4 \\
& \quad 2y_1 + y_2 \geq 3
\end{align*}
\]

where \( y_1 \) and \( y_2 \) are real numbers.

Step 2

\[
\begin{align*}
\text{Max} \bar{Z}(\tilde{x}) & = 4x_1 + 3x_2 \\
\text{Subject to the constraints} & \quad \tilde{x}_1 + 2\tilde{x}_2 \leq (4, 12) \\
& \quad 2\tilde{x}_1 + \tilde{x}_2 \leq (6, 12) \\
\end{align*}
\]

We convert the primal and dual linear programming problem involving interval numbers (INLP2) to its canonical form by adding slack variables \( \tilde{x}_3 \) and \( \tilde{x}_4 \) as follows: Max \( \bar{Z} = 4\tilde{x}_1 + 3\tilde{x}_2 \) Subject to the constraints \( \tilde{x}_1 + 2\tilde{x}_2 + \tilde{x}_3 \leq (8, 4) \)

\[
2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_4 \leq (9, 3) \).
\]

Step 3 Then the corresponding dual problem (D) is given by:

\[
\begin{align*}
\text{Min} & \quad \bar{W} = (8, 9, 3)y_1 + (9, 3, 2)y_2 \\
\text{Subject to the constraints} & \quad y_1 + 2y_2 - y_3 \geq 4 \\
& \quad 2y_1 + y_2 - y_4 \geq 3
\end{align*}
\]

If \( m(\tilde{x}_b) < w(\tilde{b}_i) \), then \( \tilde{x}_i^* = (m(\tilde{x})^*, w(\tilde{x})^*) \) with \( \tilde{x}_i^* \approx B(\tilde{x}_i)^* \) and

\[
\tilde{b}_i = (m(\tilde{b})_i, w(\tilde{b})_i): \text{We get the matrix } \tilde{B} = \frac{1}{2} \begin{pmatrix}
-1 & -1 \\
2 & 2
\end{pmatrix}.
\]

Step 4 Formulation of a classical primal (P) Linear Programming (LP) problem

\[
\begin{align*}
\text{Max} Z^2 & = 4x_1^2 + 3x_2^2 \\
\text{Subject to the constraints} & \quad x_1^2 + 2x_2^2 + x_3^2 \leq 8 \\
& \quad 2x_1^2 + x_2^2 + x_4^2 \leq 9
\end{align*}
\]

we have \( x_1^2 \approx \frac{10}{3}, x_2^2 \approx \frac{7}{3}, x_3^2 = 0, x_4^2 = 0 \).

\[
y_1^* = 2, y_2^* = \frac{5}{3}, y_3^* = 0, \text{ and } y_4^* = 0 \text{ with } \text{Max} Z^2 = \text{Min} W^2 = \frac{61}{3}.
\]
Step 5 Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:

\[ \bar{x}_1^* = (x_1^1, x_1^2, x_1^3) = (x_1^1|\bar{x}_1^1) = \left(\frac{10}{3}, \frac{10}{3}, \frac{22}{3}\right) \]

\[ \bar{x}_2^* = (x_2^1, x_2^2, x_2^3) = (x_2^1|\bar{x}_2^1) = \left(\frac{7}{3}, \frac{10}{3}, \frac{19}{3}\right) \]

\[ \bar{x}_3^* = (x_3^1, x_3^2, x_3^3) = (x_3^1|\bar{x}_3^1) = (0, 0, 0) \]

\[ \bar{x}_4^* = (x_4^1, x_4^2, x_4^3) = (x_4^1|\bar{x}_4^1) = (0, 0, 0) \]

\[ \text{Max} \bar{Z} = (Z^1, Z^2, Z^3) = \left(\frac{49}{3}, \frac{61}{3} \right) \]

(ii) Dual optimal solution:

\[ y_1^* = \frac{2}{3}, y_2^* = \frac{5}{3}, y_3^* = 0, y_4^* = 0 \]

\[ \text{Min} \bar{W} = (W^{-1}, W^{-2}, W^{-3}) = \left(\frac{61}{3}, \frac{49}{3}, \frac{73}{3}\right) \]

(iii) Comparisons using Ranking function [17]:

\[ R(\text{Max} \bar{Z}) = R(\text{Min} \bar{W}) = \frac{1}{4} \left(\frac{49}{3} + \frac{122}{3} + \frac{73}{3}\right) = \frac{61}{3} \]

5. Conclusion

We introduced the notation of primal Linear Programming problems with triangular Fuzzy Variables and primal Semi-Fully Fuzzy Linear Programming problems as the way of primal linear programming problems involving interval numbers. We discuss the solution concepts of primal Linear Programming problems with interval numbers and fuzzy numbers, we have found the optimal solution of dual linear programming problems with triangular fuzzy variables and semi-fuzzy linear programming problems.

These results will be useful for fully fuzzy linear programming problems. A numerical example is provided to show that both primal and dual problems have optimal solutions and the two Ranking values are equal.

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