On conjugacy of Smale homeomorphisms*

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Abstract

Given closed topological \( n \)-manifold \( M^n \), \( n \geq 2 \), one introduces the classes of Smale regular \( SRH(M^n) \) and Smale semi-regular \( SsRH(M^n) \) homeomorphisms of \( M^n \) with \( SRH(M^n) \subset SsRH(M^n) \). The class \( SRH(M^n) \) contains all Morse-Smale diffeomorphisms, while \( SsRH(M^n) \) contains A-diffeomorphisms with trivial and some nontrivial basic sets provided \( M^n \) admits a smooth structure. We select invariant sets that determine dynamics of Smale homeomorphisms. This allows us to get necessary and sufficient conditions of conjugacy for \( SRH(M^n) \) and \( SsRH(M^n) \). We deduce applications for some Morse-Smale diffeomorphisms and A-diffeomorphisms with codimension one expanding attractors.

Introduction

Let \( M^n \) be a topological closed \( n \)-manifold, \( n \geq 2 \). Recall that homeomorphisms \( f_1, f_2 : M^n \to M^n \) are called conjugate, if there is a homeomorphism \( h : M^n \to M^n \) such that \( h \circ f_1 = f_2 \circ h \). The homeomorphism \( h \) is a conjugacy from \( f_1 \) to \( f_2 \). One also says that \( f_1 \) and \( f_2 \) are (topologically) conjugate by \( h \). To check whether given \( f_1 \) and \( f_2 \) are conjugate one constructs usually an invariant of conjugacy which is some dynamical characteristic keeping under a conjugacy. Normally, such invariant is constructed in the frame of special class of dynamical systems. The famous invariant is Poincare’s rotation number in the class of transitive circle homeomorphisms [24]. This invariant is effective i.e. two transitive circle homeomorphisms are conjugate if and only if they have the same Poincare’s rotation number (see [20] and [4], ch. 7, concerning topological invariants of low dimensional dynamical systems).

Anosov [3] and Smale [27] were first who realize the fundamental role of hyperbolicity in a topological structure of dynamical systems. Numerous topological invariants were constructed for smooth dynamical systems satisfying Smale’s axiom A (non-wandering sets are hyperbolic and contain dense subsets of periodic orbits), including Morse-Smale systems (non-wandering sets consists of finitely many hyperbolic periodic orbits) and Anosov systems [9, 10, 11, 12, 21, 23]. Grines and Zhuzhoma [14] classify the structurally stable A-diffeomorphisms having orientable codimension one expanding attractors. Recently, one gets a great progress in the classification of 3-dimensional Morse-Smale diffeomorphisms by Bonatti, Grines, Medvedev, Pecou, and Pochinka [6, 7, 8].

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Taking in mind that there are manifolds that do not admit smooth structures \[22\], we consider homeomorphisms whose non-wandering sets have a hyperbolic type (see definitions below). Deep theory of topological dynamical systems was developed in \[11, 2\].

We introduce the classes of Smale regular \(SRH(M^n)\) and Smale semi-regular \(SsRH(M^n)\) homeomorphisms of closed topological manifold \(M^n\), \(SRH(M^n) \subset SsRH(M^n)\). The class \(SRH(M^n)\) contains all Morse-Smale diffeomorphisms, while \(SsRH(M^n)\) contains A-diffeomorphisms with trivial and some nontrivial basic sets provided \(M^n\) admits a smooth structure. We select invariant sets that determine dynamics of Smale homeomorphisms. This allows to get necessary and sufficient conditions of conjugacy for homeomorphisms of the classes \(SRH(M^n)\) and \(SsRH(M^n)\). In sense, we suggest a general approach for the topological classification of wide classes of regular and semi-regular dynamical systems. We illustrate our approaching for some concrete Morse-Smale diffeomorphisms and A-diffeomorphisms with codimension one expanding attractors.

Let us give previous definitions and formulate the main results. The generalization of the notion of conjugacy is a local conjugacy. To be precise, let \(\Omega_i\) be an invariant set of homeomorphism \(f_i : M \to M\), \(i = 1, 2\). One says that \(f_1\) and \(f_2\) are locally conjugate in neighborhoods \(U_1, U_2\) of \(\Omega_1\) and \(\Omega_2\) respectively if there are neighborhoods \(U_1, U_2\) of \(\Omega_1\) and \(\Omega_2\) respectively and a homeomorphism \(\varphi : U_1 \cup f_1(U_1) \to M\) such that

\[
\varphi(\Omega_1) = \Omega_2, \quad \varphi(U_1) = U_2, \quad \varphi \circ f_1|_{U_1} = f_2 \circ \varphi|_{U_1}.
\]

In short, the restrictions \(f_1|_{\Omega_1}, f_2|_{\Omega_2}\) are conjugate by \(\varphi\). To emphasize the main idea we begin for simplicity with the introducing Smale regular homeomorphisms.

Let \(F : L^n \to L^n\) be a \(C^1\)-diffeomorphism of smooth closed \(n\)-manifold \(L^n\), \(n \geq 2\), and \(z_0\) a periodic point of \(F\) with period \(p \in \mathbb{N}\). Then the differential \(DF^p(z_0) : T_{z_0}L^n \to T_{z_0}L^n\) is a linear isomorphism of the tangent space \(T_{z_0}L^n\) that is naturally isomorphic to \(\mathbb{R}^n\). The point \(z_0\) is called hyperbolic if non of the eigenvalues of \(DF^p(z_0)\) have modulus 1. Well-known \[18, 27\] that a hyperbolic \(z_0\) has the stable \(W^s(z_0)\) and unstable \(W^u(z_0)\) manifolds formed by points \(y \in L^n\) such that \(g_L(F^{pk}z_0, F^{pk}y) \to 0\) as \(k \to +\infty\) and \(k \to -\infty\) respectively, where \(g_L\) is a metric on \(L^n\). Moreover, \(W^s(z_0)\) and \(W^u(z_0)\) are homeomorphic (in the interior topology) to Euclidean spaces \(\mathbb{R}^{\dim W^s(z_0)}\), \(\mathbb{R}^{\dim W^u(z_0)}\) respectively. Note that \(\dim W^s(z_0) + \dim W^u(z_0) = n\).

Let \(x_0\) now be a periodic point of a homeomorphism \(f : M^n \to M^n\) of topological \(n\)-manifold \(M^n\), \(n \geq 2\). One says that the point \(x_0\) has a hyperbolic type or \(x_0\) is locally hyperbolic if there is a \(C^1\)-diffeomorphism \(F : L^n \to L^n\) with a hyperbolic periodic point \(z_0\) such that the restrictions \(f^p|_{z_0}, L^p|_{z_0}\) are conjugate where \(p\) is the period of \(x_0\) and \(z_0\). It follows immediately from this definition that there are stable \(W^s(x_0)\) and unstable \(W^u(x_0)\) manifolds with similar properties. A locally hyperbolic periodic point \(x_0\) is called sink point if \(\dim W^s(x_0) = \dim M^n\) (hence, \(\dim W^u(x_0) = 0\)). A locally hyperbolic periodic point \(x_0\) is called source point if \(\dim W^u(x_0) = \dim M^n\) (hence, \(\dim W^s(x_0) = 0\)). A locally hyperbolic periodic point \(x_0\) is called saddle point if \(1 \leq \dim W^s(x) \leq \dim M^n - 1\) (hence, \(1 \leq \dim W^u(x) \leq \dim M^n - 1\)).

A homeomorphism \(f : M^n \to M^n\) of topological \(n\)-manifold \(M^n\), \(n \geq 2\), is called a Smale regular homeomorphism if

- the non-wandering set \(NW(f)\) of \(f\) consists of a finitely many periodic points;
- every periodic point is locally hyperbolic.
- the non-wandering set \(NW(f)\) contains a non-empty set \(\alpha(f)\) of source periodic points and non-empty set \(\omega(f)\) of sink periodic points.
We denote by $SRH(M^n)$ the set of Smale regular homeomorphisms $M^n \to M^n$. Note that it is possible that $f \in SRH(M^n)$ has the empty set $\sigma(f)$ of saddle periodic points. In this case the set $\alpha(f)$ consists of a unique source and the set $\omega(f)$ consists of a unique sink, and $M^n = S^n$ is an $n$-sphere. Later on, we'll assume that $f \in SRH(M^n)$ has a non-empty set $\sigma(f)$ of saddle periodic points.

Let $F : M^n \to M^n$ be a diffeomorphism satisfying Smale axiom A (in short, A-diffeomorphism) \cite{27}. Then the non-wandering set $NW(F)$ is a finite union of pairwise disjoint $F$-invariant closed sets $\Omega_1, \ldots, \Omega_k$ such that every restriction $F|\Omega_i$ is topologically transitive. These $\Omega_i$ are called the basic sets of $F$. A basic set is nontrivial if it is not a periodic isolated orbit. By definition, each basic set $\Omega_i$ is hyperbolic and $\Omega_i \subset W^s(\Omega_i) \cap W^u(\Omega_i)$. One says that $\Omega_i$ is a sink basic set provided $W^u(\Omega_i) = \Omega_i$. A basic set $\Omega_i$ is a source basic set provided $W^s(\Omega_i) = \Omega_i$. A basic set $\Omega_i$ is a saddle basic set if it neither a sink nor a source basic set.

A homeomorphism $f : M^n \to M^n$ is called Smale A-homeomorphism if there is an A-diffeomorphism $F : M^n \to M^n$ such that the restrictions $f|_{NW(f)}$, $F|_{NW(F)}$ are conjugate. As a consequence, $NW(f)$ is a finite union of pairwise disjoint $f$-invariant closed sets $\Lambda_1, \ldots, \Lambda_k$ called basic sets of $f$ such that every restriction $f|_{\Lambda_i}$ is topologically transitive. Each basic set $\Lambda$ has the stable manifold $W^s(\Lambda)$, and the unstable manifold $W^u(\Lambda)$. Similarly to Smale homeomorphisms, one introduces the set $\omega(f)$ of sink basic sets, and the set $\alpha(f)$ of source basic sets, and the set $\sigma(f)$ of saddle basic sets which we assume to be non-empty.

A Smale A-homeomorphism $f$ is called Smale semi-regular homeomorphism if

- the non-wandering set $NW(f)$ contains a non-empty sets of source basic sets $\alpha(f)$, and sink basic sets $\omega(f)$, and saddle basic sets $\sigma(f)$;
- all source basic sets $\alpha(f)$ are trivial or all sink basic sets $\omega(f)$ are trivial.

Denote by $SsRH(M^n)$ the set of Smale semi-regular homeomorphisms $M^n \to M^n$. If all basic sets of Smale semi-regular homeomorphism $f$ are trivial, then $f$ is a Smale regular homeomorphism. Hence, $SRH(M^n) \subset SsRH(M^n)$.

Given any $f \in SsRH(M^n)$ or $f \in SRH(M^n)$, denote by $A(f)$ (resp., $R(f)$) the union of $\omega(f)$ (resp., $\alpha(f)$) and unstable (resp., stable) manifolds of saddle basic sets $\sigma(f)$ or saddle periodic orbits respectively:

$$A(f) = \omega(f) \bigcup_{\nu \in \sigma(f)} W^u(\nu), \quad R(f) = \alpha(f) \bigcup_{\nu \in \sigma(f)} W^s(\nu).$$

Let $f_1, f_2 : M^n \to M^n$ be homeomorphisms of closed topological $n$-manifold, $n \geq 2$, and $N_1, N_2$ invariant sets of $f_1, f_2$ respectively i.e. $f_i(N_i) = N_i$; $i = 1, 2$. We say that the sets $N_1, N_2$ have the same dynamical locally equivalent embedding if there are (open) neighborhoods $\delta_1, \delta_2$ of $\text{clo} N_1$, $\text{clo} N_2$ respectively and a homeomorphism $h_0 : \delta_1 \cup f_1(\delta_1) \to M^n$ such that

$$h_0(\delta_1) = \delta_2, \quad h_0(\text{clo} N_1) = \text{clo} N_2, \quad h_0 \circ f_1|_{\delta_1} = f_2 \circ h_0|_{\delta_1} \quad (1)$$

Here, $\text{clo} N$ means a topological closure of $N$. Actually, if $N_1, N_2$ are closed then the dynamical locally equivalent embedding coincides with the conjugation of the restrictions $f_1|_{N_1}, f_2|_{N_2}$. The main result of the paper are the following statements.

**Theorem 1** Let $M^n$ be a closed topological $n$-manifold $M^n$, $n \geq 2$. Homeomorphisms $f_1, f_2 \in SsRH(M^n)$, $n \geq 2$, are conjugate if and only if one of the following conditions holds:
• the basic sets $\alpha(f_1), \alpha(f_2)$ are trivial while the sets $A(f_1), A(f_2)$ have the same dynamical locally equivalent embedding;

• the basic sets $\omega(f_1), \omega(f_2)$ are trivial while the sets $R(f_1), R(f_2)$ have the same dynamical locally equivalent embedding.

As a consequence, one gets the following statement (recall that $SRH(M^n) \subset SsRH(M^n)$).

**Corollary 1** Let $M^n$ be a closed topological $n$-manifold $M^n$, $n \geq 2$. Homeomorphisms $f_1, f_2 \in SRH(M^n)$ are conjugate if and only if one of the following conditions holds:

• the sets $A(f_1), A(f_2)$ have the same dynamical locally equivalent embedding;

• the sets $R(f_1), R(f_2)$ have the same dynamical locally equivalent embedding.

The structure of the paper is the following. In Section 1 we give some previous results. In Section 2 we prove Theorem 1. At last, in Section 3 we discuss our approaching to the problem of classification comparing with the approaching by Bonatti, Grines, Medvedev, Pecou, and Pochinka [6, 7, 8]. We also give some applications of main results.

1 Properties of Smale homeomorphisms

We begin by recalling several definitions. Further details may be found in [4, 5, 27]. Denote by $\text{Orb}(x)$ the orbit of point $x \in M^n$ under a homeomorphism $f : M^n \to M^n$. The $\omega$-limit set $\omega(x)$ of the point $x$ consists of the points $y \in M^n$ such that $f^{k_i}(x) \to y$ for some sequence $k_i \to \infty$. Clearly that any points of $\text{Orb}(x)$ have the same $\omega$-limit. Replacing $f$ with $f^{-1}$, one gets an $\alpha$-limit set. Obviously, $\omega(x) \cup \alpha(x) \subseteq NW(f)$ for every $x \in M^n$.

Since $SRH(M^n) \subset SsRH(M^n)$, we formulate mainy properties for Smale semi-regular homeomorphisms. Given a family $C = \{c_1, \ldots, c_l\}$ of sets $c_i \subset M^n$, denote by $\bar{C}$ the union $c_1 \cup \ldots \cup c_l$. It follows immediately from definitions that

$$NW(f) = \bar{\alpha(f)} \cup \bar{\omega(f)} \cup \bar{\sigma(f)}, \quad f \in SsRH(M^n)$$

(2)

**Lemma 1** Let $f \in SsRH(M^n)$ and $x \in M^n$. Then

1. if $\omega(x) \subseteq \bar{\sigma(f)}$, then $x \in W^s(\sigma_s)$ for some saddle basic set $\sigma_s \subseteq \sigma(f)$.

2. if $\alpha(x) \subseteq \bar{\sigma(f)}$, then $x \in W^u(\sigma_s)$ for some saddle basic set $\sigma_s \subseteq \sigma(f)$.

**Proof.** Suppose that $\omega(x) \subseteq \bar{\sigma(f)}$. Since $\bar{\alpha(f)}$ and $\bar{\omega(f)}$ are invariant sets, $x \notin \bar{\alpha(f)} \cup \bar{\omega(f)}$. Therefore, there are exist a neighborhood $U(\alpha)$ of $\alpha(f)$ and neighborhood $U(\omega)$ of $\omega(f)$ such that the positive semi-orbit $\text{Orb}^+(x)$ belongs to the compact set $N = M^n \setminus (U(\omega) \cup U(\alpha))$. Let $V(\sigma_1), \ldots, V(\sigma_m)$ be pairwise disjoint neighborhoods of saddle basic sets $\sigma_1, \ldots, \sigma_m$ respectively such that $\bigcup_{i=1}^m V(\sigma_i) \subset N$. Since every $V(\sigma_i)$ does not intersect $\bigcup_{j \neq i} V(\sigma_j)$ and all saddle basic sets are invariant, one can take the neighborhoods $V(\sigma_1), \ldots, V(\sigma_m)$ so small that every $f(V(\sigma_i))$ does not intersect $\bigcup_{j \neq i} V(\sigma_j)$. Suppose the contrary, i.e. there is no a unique saddle basic set $\sigma_s \subseteq \sigma(f)$ with $x \in W^s(\sigma_s)$. Thus, there are at least two different saddle basic sets $\sigma_1, \sigma_2$ such that $x \in W^s(\sigma_1)$ and $x \in W^s(\sigma_2)$. Hence, $\omega(x)$ have to intersect $\sigma_1, \sigma_2$. It follows that the
compact set $N_0 = N \setminus (\bigcup_{i=1}^m V(\sigma_i))$ contains infinitely many points of the semi-orbit $\text{Orb}^+(x)$. This implies $\omega(x) \cap N_0 \neq \emptyset$ that contradicts (2). The second assertion is proved similarly. \hfill \Box

A set $U$ is a trapping region for $f$ if $f(\text{clos} U) \subset \text{int} U$. A set $A$ is an attracting set for $f$ if there exists a trapping set $U$ such that

$$A = \bigcap_{k \geq 0} f^k(U).$$

A set $A^*$ is a repelling set for $f$ if there exists a trapping region $U$ for $f$ such that

$$A^* = \bigcap_{k \leq 0} f^k(M^n \setminus U).$$

Another words, $A^*$ is an attracting set for $f^{-1}$ with the trapping region $M^n \setminus U$ for $f^{-1}$. When we wish to emphasize the dependence of an attracting set $A$ or a repelling set $A^*$ on the trapping region $U$ from which it arises, we denote it by $A_U$ or $A^*_U$ respectively.

Let $A$ be an attracting set for $f$. The basin $B(A)$ of $A$ is the union of all open trapping regions $U$ for $f$ such that $A_U = A$. One can similarly define the notion of basin for a repelling set.

Let $N$ be an attracting or repelling set and $B(N)$ the basin of $N$. A closed set $G(N) \subset B(N) \setminus N$ is called a generating set for the domain $B(N) \setminus N$ if

$$B(N) \setminus N = \bigcup_{k \in \mathbb{Z}} f^k(G(N)).$$

Moreover,

1) every orbit from $B(N) \setminus N$ intersects $G(N)$; 2) if an orbit from $B(N) \setminus N$ intersects the interior of $G(N)$, then this orbit intersects $G(N)$ at a unique point; 3) if an orbit from $B(N) \setminus N$ intersects the boundary of $G(N)$, then the intersection of this orbit with $G(N)$ consists of two points; 4) the boundary of $G(N)$ is the union of finitely many compact codimension one topological submanifolds.

**Lemma 2** Let $f \in SsRH(M^n)$.

1) Suppose that all basic sets $\alpha(f)$ are trivial. Then $\alpha(f)$ is a repelling set while $A(f)$ is an attracting set with

$$B\left(\alpha(f)\right) \setminus \alpha(f) = B\left(A(f)\right) \setminus A(f).$$

Moreover,

- there is a trapping region $T(\alpha)$ for $f^{-1}$ of the set $\alpha(f)$ consisting of pairwise disjoint open $n$-balls $b_1, \ldots, b_r$ such that each $b_i$ contains a unique periodic point from $\alpha(f)$;

- the regions $B(\alpha(f)) \setminus \alpha(f), B(A(f)) \setminus A(f)$ have the same generating set $G(\alpha)$ consisting of pairwise disjoint closed $n$-annuluses $a_1, \ldots, a_r$ such that $a_i = \text{clos} f^{p_i}(b_i) \setminus b_i$ where $p_i \in \mathbb{N}$ is a minimal period of a periodic point belonging to $b_i, i = 1, \ldots, r$:

$$G(\alpha) = \bigcup_{i=1}^r a_i = \bigcup_{i=1}^r \left(\text{clos} f^{p_i}(b_i) \setminus b_i\right);$$

- $B(A(f)) \setminus A(f) = \bigcup_{k \in \mathbb{Z}} f^k(G(\alpha))$. 

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2) Suppose that all basic sets \( \omega(f) \) are trivial. Then \( \omega(f) \) is an attracting set while \( R(f) \) is a repelling set with

\[ B(\omega(f)) \setminus \omega(f) = B(R(f)) \setminus R(f). \]

Moreover,

- there is a trapping region \( T(\omega) \) for \( f \) of the set \( \omega(f) \) consisting of pairwise disjoint open \( n \)-balls \( b_1, \ldots, b_l \) such that each \( b_i \) contains a unique periodic point from \( \omega(f) \);
- the regions \( B(\omega(f)) \setminus \omega(f) \), \( B(R(f)) \setminus R(f) \) have the same generating set \( G(\omega) \) consisting of pairwise disjoint closed \( n \)-annuluses \( a_1, \ldots, a_l \) such that \( a_i = b_i \setminus \text{int } f^{p_i}(b_i) \) where \( p_i \in \mathbb{N} \) is a minimal period of a periodic point belonging to \( b_i \), \( i = 1, \ldots, l \);

\[ G(\omega) = \bigcup_{i=1}^l a_i = \bigcup_{i=1}^l (b_i \setminus \text{int } f^{p_i}(b_i)); \]

- \( B(R(f)) \setminus R(f) = \bigcup_{k \in \mathbb{Z}} f^k(G(\omega)). \)

**Proof.** It is enough to prove the first statement only. Since all basic sets \( \alpha(f) \) are trivial and consists of locally hyperbolic source periodic points, there is a trapping region \( T(\alpha) \) for \( f^{-1} \) of the set \( \alpha(f) \) consisting of pairwise disjoint open \( n \)-balls \( b_1, \ldots, b_r \) such that each \( b_i \) contains a unique periodic point \( q_i \) from \( \alpha(f) \). Thus,

\[ T(\alpha) = \bigcup_{i=1}^r b_i, \quad \bigcap_{k \leq 0} f^{k}\alpha(f) = q_i, \quad i = 1, \ldots, r. \]

As a consequence, there is the generating set \( G(\alpha) = \bigcup_{i=1}^r (\text{clos } f^{p_i}(b_i) \setminus b_i) \) consisting of pairwise disjoint closed \( n \)-annuluses \( a_i = \text{clos } f^{p_i}(b_i) \setminus b_i, i = 1, \ldots, r. \)

Since the balls \( b_1, \ldots, b_r \) are pairwise disjoint and \( \text{clos } b_i \subset f^{p_i}(b_i) \), the balls \( f^{p_i}(b_i), \ldots, f^{p_r}(b_r) \) are pairwise disjoint also. For simplicity of exposition, we'll assume that \( \alpha(f) \) consists of fixed points (otherwise, \( \alpha(f) \) is divided into periodic orbits each considered like a point). Therefore,

\[ f(M^n \setminus \bigcup_{i=1}^r b_i) = M^n \setminus \bigcup_{i=1}^r f b_i \subset M^n \setminus \bigcup_{i=1}^r \text{clos } b_i \subset \text{int } (M^n \setminus \bigcup_{i=1}^r b_i). \]

Hence, \( M^n \setminus \bigcup_{i=1}^r b_i \) is a trapping region for \( f \). Clearly, \( A(f) \subset M^n \setminus \bigcup_{i=1}^r b_i \).

Take a point \( x \in M^n \setminus \bigcup_{i=1}^r b_i \). Obviously, \( \omega(x) \notin \alpha(f) \). It follows from (2) that \( \omega(x) = \omega(f) \cup \sigma(f) \). By Lemma \([1]\) \( \omega(x) \in A(f) \). Therefore, \( A(f) \) is an attracting set with the trapping region \( M^n \setminus \bigcup_{i=1}^r b_i \) for \( f : \)

\[ A(f) = A_{M^n \setminus \bigcup_{i=1}^r b_i}. \]

Moreover,

\[ M^n = \omega(f) \cup B(A(f)) \]

because of \( \bigcap_{k \leq 0} f^k = q_i, i = 1, \ldots, r \).

Let us prove the quality \( B(\omega(f)) \setminus \alpha(f) = B(A(f)) \setminus A(f) \). Take \( x \in B(\omega(f)) \setminus \alpha(f) \). Since \( x \notin \alpha(f) \) and \( M^n = \omega(f) \cup B(A(f)), x \in B(A(f)) \). Since \( x \in B(\omega(f)) \setminus \alpha(f) \), \( \alpha(x) \subset \alpha(f) \). Hence, \( x \notin \alpha(x) \) and \( x \in B(\alpha(f)) \setminus A(f) \). Now, set \( x \in B(\alpha(f)) \setminus A(f) \). Then \( x \notin \alpha(f) \). Since \( x \notin A(f), \alpha(x) \subset \sigma(f) \cup \alpha(f) \). If one assumes that \( \alpha(x) \subset \sigma(f) \), then according to Lemma \([2]\) \( x \in W^u(\nu) \) for
some saddle basic set \( \nu \). Thus, \( x \in A(f) \) which contradicts to \( x \notin A(f) \). Therefore, \( \alpha(x) \subset \tilde{\alpha}(f) \). Hence \( x \in B(\tilde{\alpha}(f)) \). As a consequence, \( x \in B(\tilde{\alpha}(f)) \setminus \alpha(f) \).

The last assertion of the first statement follows from the previous ones. This completes the proof. \( \square \)

In the next statement, we keep the notation of Lemma 2.

**Lemma 3** Let \( f \in SsRH(M^n) \).

1) Suppose that all basic sets \( \alpha(f) \) are trivial. Then given any neighborhood \( V_0(A) \) of \( A(f) \), there is \( n_0 \in \mathbb{N} \) such that

\[
\cup_{k \geq n_0} f^k(G(\alpha)) \subset V_0(A)
\]

where \( G(\alpha) \) is the generating set of the region \( B(\tilde{\alpha}(f)) \setminus \tilde{\alpha}(f) \).

2) Suppose that all basic sets \( \omega(f) \) are trivial. Then given any neighborhood \( V_0(R) \) of \( R(f) \), there is \( n_0 \in \mathbb{N} \) such that

\[
\cup_{k \leq -n_0} f^k(G(\omega)) \subset V_0(R)
\]

where \( G(\omega) \) is the generating set of the region \( B(\tilde{\omega}(f)) \setminus \tilde{\omega}(f) \).

**Proof.** It is enough to prove the first statement only. Take a closed tripping neighborhood \( U \) of \( A(f) \) for \( f \). Since \( \cap_{k \in \mathbb{N}} f^k(U) = A(f) \subset V_0(A) \), there is \( k_0 \in \mathbb{N} \) such that \( f^{k_0}(U) \subset V_0(A) \). Clearly, \( f^{k_0}(U) \) is a tripping region of \( A(f) \) for \( f \). Hence, \( f^{k_0+k}(U) \subset f^{k_0}(U) \subset V_0(A) \) for every \( k \in \mathbb{N} \).

Let \( G(\alpha) \) be a generating set of the region \( B(\tilde{\alpha}(f)) \setminus \tilde{\alpha}(f) \). By Lemma 2, \( G(\alpha) \) is the generating set of the region \( B(\tilde{\alpha}(f)) \setminus A(f) \) as well. Since \( G(\alpha) \) is a compact set, there is \( n_0 \in \mathbb{N} \) such that \( f^{n_0}(G(\alpha)) \subset f^{k_0}(U) \). It follows that \( f^{n_0+k}(G(\alpha)) \subset f^{k_0+k}(U) \subset f^{k_0}(U) \subset V_0(A) \) for every \( k \in \mathbb{N} \). As a consequence, \( \cup_{k \geq n_0} f^k(G(\alpha)) \subset V_0(A) \). \( \square \)

### 2 Proof of Theorem 1

Suppose that homeomorphisms \( f_1, f_2 \in SsRH(M^n) \) are conjugate. Since a conjugacy mapping \( M^n \to M^n \) is a homeomorphism, the sets \( A(f_1), A(f_2) \), as well as the sets \( R(f_1), R(f_2) \) have the same dynamical locally equivalent embedding.

To prove the inverse assertion, let us suppose for definiteness that the basic sets \( \alpha(f_1), \alpha(f_2) \) are trivial while the sets \( A(f_1), A(f_2) \) have the same dynamical locally equivalent embedding. Taking in mind that \( A(f_1) \) and \( A(f_2) \) are attracting sets, we see that there are neighborhoods \( \delta_1, \delta_2 \) of \( A(f_1), A(f_2) \) respectively, and a homeomorphism \( h_0 : \delta_1 \to \delta_2 \) such that

\[
h_0 \circ f_1|_{\delta_1} = f_2 \circ h_0|_{\delta_1}, \quad f_1(\delta_1) \subset \delta_1, \quad h_0(A(f_1)) = A(f_2). \quad (3)
\]

Without loss of generality, one can assume that \( \delta_1 \subset B(A(f_1)) \). Moreover, taking \( \delta_1 \) smaller if one needs, we can assume that \( \text{clos} \, \delta_1 \) is a trapping region for \( f_1 \) of the set \( A(f_1) \). By (3), one gets

\[
f_2(\text{clos} \, \delta_2) = f_2 \circ h_0(\text{clos} \, \delta_1) = h_0 \circ f_1(\text{clos} \, \delta_1) \subset h_0(\delta_1) = \delta_2.
\]

Thus, \( \text{clos} \, \delta_2 \) is a trapping region for \( f_2 \) of the set \( A(f_2) \). As a consequence, we get the following generalization of (3)

\[
h_0 \circ f_1^k|_{\delta_1} = f_2^k \circ h_0|_{\delta_1}, \quad k \in \mathbb{N}, \quad f_1(\text{clos} \, \delta_1) \subset \delta_1, \quad h_0(A(f_1)) = A(f_2). \quad (4)
\]
By Lemma 2 there is a trapping region $T(\alpha_1)$ for $f_1^{-1}$ of the set $\alpha(f_1)$ consisting of pairwise disjoint open $n$-balls $b_1, \ldots, b_r$ such that each $b_i$ contains a unique periodic point $q_i$ from $\alpha(f_1)$. In addition, the region $B(\alpha^{-1}(f_1) \setminus \alpha(f_1))$ has the generating set $G(\alpha_1)$ consisting of pairwise disjoint closed $n$-annuluses $a_1, \ldots, a_r$ such that $a_i = \text{clos} \ f_1^{p_i}(b_i) \setminus b_i$ where $p_i \in \mathbb{N}$ is a minimal period of the periodic point $q_i$.

Due to Lemma 3 one can assume without loss of generality that $G(\alpha_1) \overset{\text{def}}{=} G_1 \subset \delta_1$. Hence,

$$A(f_1) \cup \left( \cup_{k \geq 0} f^k(G_1) \right) = A(f_1) \cup N^+ \subset \delta_1, \quad N^+ = \cup_{k \geq 0} f^k(G_1).$$

According to Lemma 2 $G_1$ is a generating set of the region $B(A(f_1)) \setminus A(f_1)$. Let us show that $h_0(G_1) \overset{\text{def}}{=} G_2$ is a generating set for the region $B(A(f_2)) \setminus A(f_2)$. Take a point $z_2 \in G_2$. There is a unique point $z_1 \in G_1$ such that $h_0(z_1) = z_2$. Note that $z_2 \notin A(f_2)$ since $z_1 \notin A(f_1)$. Since $G_1 \subset (B(A(f_1)) \setminus A(f_1))$, $f^k(z_1) \to A(f_1)$ as $k \to \infty$. It follows from (4) that

$$f^k(z_2) = f^k \circ h_0(z_1) = h_0 \circ f^k_1(z_1) \to h_0(A(f_1)) = A(f_2) \quad \text{as} \quad k \to \infty.$$ 

Hence, $z_2 \in B(A(f_2))$ and $G_2 \in B(A(f_2)) \setminus A(f_2)$.

Take an orbit $\text{Orb}_{f_2} \subset B(A(f_2)) \setminus A(f_2)$. Since this orbit intersects a trapping region of $A(f_2)$, $\text{Orb}_{f_2} \cap \delta_2 \neq \emptyset$. Therefore there exists a point $x_2 \in \text{Orb}_{f_2} \cap \delta_2$. Since $h_0(A(f_1)) = A(f_2)$ and $x_2 \in B(A(f_2)) \setminus A(f_2)$, the orbit $\text{Orb}_{f_2}$ of the point $x_1 = h_0^{-1}(x_2) \subset \delta_1$ under $f_1$ belongs to $B(A(f_1)) \setminus A(f_1)$. Hence, $\text{Orb}_{f_1}$ intersects $G_1$ at some point $w_1 \in \delta_1$. Since $x_1, w_1 \in \text{Orb}_{f_2}$, there is $k \in \mathbb{N}$ such that either $x_1 = f_1^k(w_1)$ or $w_1 = f_1^k(x_1)$. Suppose for definiteness that $w_1 = f_1^k(x_1)$. Using (3), one gets

$$w_2 = h_0(w_1) = h_0 \circ f_1^k(x_1) = h_0 \circ f^k_1 \circ h_0^{-1}(x_2) = f^k_2(x_2) \in G_2 \cap \text{Orb}_{f_2}.$$

Similarly one can prove that if $\text{Orb}_{f_2}$ intersects the interior of $G_2$, then $\text{Orb}_{f_2}$ intersects $G_2$ at a unique point, and if $\text{Orb}_{f_2}$ intersects the boundary of $G_2$ then $\text{Orb}_{f_2}$ intersects $G_2$ at two points. Thus, $G_2$ is a generating set for the region $B(A(f_2)) \setminus A(f_2)$.

Set

$$\cup_{k \geq 0} f_i^{-k}(G_i) \overset{\text{def}}{=} O^-(G_i), \quad \cup_{k \geq 0} f_i^k(G_i) \overset{\text{def}}{=} O^+(G_i), \quad i = 1, 2.$$

We see that $O^-(G_1) \cup O^+(G_1)$ is invariant under $f_i, i = 1, 2$. Given any point $x \in O^-(G_1) \cup O^+(G_1)$, there is $m \in \mathbb{Z}$ such that $x \in f_i^{-m}(G_1)$. Let us define the mapping

$$h: O^-(G_1) \cup O^+(G_1) \to O^-(G_2) \cup O^+(G_2)$$

as follows

$$h(x) = f_2^{-m} \circ h_0 \circ f_1^m(x), \quad \text{where} \quad x \in f_1^{-m}(G_1).$$

Since $G_1$ and $G_2$ are generating sets, $h$ is correct. It is easy to check that

$$h \circ f_1|_{O^-(G_1) \cup O^+(G_1)} = f_2 \circ h|_{O^-(G_1) \cup O^+(G_1)}.$$

It follows from (3) that

$$h: A(f_1) \cup O^-(G_1) \cup O^+(G_1) \to A(f_2) \cup O^-(G_2) \cup O^+(G_2)$$

is the homeomorphic extension of $h_0$ putting $h|_{A(f_1)} = h_0|_{A(f_1)}$. Moreover,

$$h \circ f_1^k|_{A(f_1) \cup O^-(G_1) \cup O^+(G_1)} = f_2^k \circ h|_{A(f_1) \cup O^-(G_1) \cup O^+(G_1)}, \quad k \in \mathbb{Z}.$$
By Lemma\textsuperscript{2} $G_i$ is a generating set for the region $B \left( \alpha(f_1) \right) \setminus \alpha(f_1) = B \left( A(f_i) \right) \setminus A(f_i)$ and $B \left( A(f_i) \right) \setminus A(f_i) = \cup_{k \in \mathbb{Z}} f_k^{k}(G_i), i = 1, 2$. Thus, one gets the conjugacy $h : M^n \setminus \alpha(f_1) \rightarrow M^n \setminus \alpha(f_2)$ from $f_1|_{M^n \setminus \alpha(f_1)}$ to $f_2|_{M^n \setminus \alpha(f_2)}$:

$$h \circ f_1^k|_{M^n \setminus \alpha(f_1)} = f_2^k \circ h|_{M^n \setminus \alpha(f_1)}, \quad k \in \mathbb{Z}. \quad (5)$$

Recall that the sets $\alpha(f_1), \alpha(f_2)$ are periodic sources $\{\alpha_j(f_1)\}_{j=1}^{l_1}, \{\alpha_j(f_2)\}_{j=1}^{l_2}$ respectively. By Lemma\textsuperscript{2} the generating set $G_i$ consists of pairwise disjoint $n$-annuluses $a_j(f_i), i = 1, 2$. Take an annulus $a_r(f_1) = a_r \subset G_1$ surrounding a source periodic point $\alpha_r(f_1)$ of minimal period $p_r, 1 \leq r \leq l_1$. Then the set $\bigcup_{k \geq 0} f_1^{-k p_r}(a_r) \cup \{\alpha_r(f_1)\} = D_r^n$ is a closed $n$-ball. Since

$$M^n \setminus B(A(f_2)) = M^n \setminus (A(f_2) \cup_{k \in \mathbb{Z}} f_2^k(G_2))$$

consists of the source periodic points $\alpha(f_2)$, the annulus

$$\bigcup_{k \geq 0} f_2^{-k p_r} \circ h(a_r) = \bigcup_{k \geq 0} h \circ f_1^{-k p_r}(a_r) = D_r^*$$

surrounds a unique source periodic point $\alpha_{j(r)}(f_2)$ of the same minimal period $p_r$. Moreover, $D_r^* \cup \{\alpha_{j(r)}(f_2)\}$ is a closed $n$-ball. It implies the one-to-one correspondence $r \rightarrow j(r)$ inducing the one-to-one correspondence $j_0 : \alpha_r(f_1) \rightarrow \alpha_{j(r)}(f_2))$. Since $\alpha_r(f_1)$ and $\alpha_{j(r)}(f_2)$ have the same period, one gets

$$j_0 \left( f_1^k(\alpha_r(f_1)) \right) = f_2^k \left( j_0(\alpha_r(f_1)) \right) = f_2^k \left( \alpha_{j(r)}(f_2) \right), \quad 0 \leq k \leq p_r. \quad (6)$$

Put by definition, $h(\alpha_r(f_1)) = \alpha_{j(r)}(f_2))$. For sufficiently large $m \in \mathbb{N}$, the both $f_1^{-mp_r}(D_r^n)$ and $f_2^{-mp_r}(D_r^*)$ can be embedded in arbitrary small neighborhoods of $\alpha_r(f_1)$ and $\alpha_{j(r)}(f_2)$ respectively, because of $\alpha(f_1)$ and $\alpha(f_2)$ are repelling sets. Taking in mind (6), it follows that the constructed $h : M^n \rightarrow M^n$ is a conjugacy from $f_1$ to $f_2$. This completes the proof. $\square$

3 Discussions and applications

First, for references, we formulate the result which follows immediately from Corollary\textsuperscript{1}.

**Corollary 2** Let $f_1, f_2$ be Morse-Smale diffeomorphisms of closed smooth $n$-manifold $M^n, n \geq 2$. Then $f_1, f_2$ are conjugate if and only if one of the following conditions holds:

- the sets $A(f_1), A(f_2)$ have the same dynamical locally equivalent embedding;
- the sets $R(f_1), R(f_2)$ have the same dynamical locally equivalent embedding.

Now, we compare our approaching to the problem of classification with the approaching by Bonatti, Grines, Medvedev, Pecou, and Pochinka. The main idea of the last approaching consists of considering a space of orbits with corresponding traces of separatrices of periodic saddle points. To be precise, let us consider the starting article where one studies the class $MS(S^3, 4)$ of orientation preserving Morse-Smale diffeomorphisms $f : S^3 \rightarrow S^3$ of 3-sphere with the non-wandering set $NW(f)$ consisting of four periodic points: a saddle $\sigma$, two sources $\alpha_1$.
and $\alpha_2$, and a sink $\omega$. Let $S(f)$ be the space of orbits of $f$ and $p : S^3 \to S(f)$ the natural projection where $p(x) = p(y)$ iff the points $x$ and $y$ belong to the same orbit. Note that $S(f)$ is homeomorphic to $S^2 \times S^1$. The saddle $\sigma$ has one-dimensional separatrices, say $l_1$ and $l_2$. Then $p(l_1)$ and $p(l_2)$ are knots and one of them, say $p(l_2)$ is always trivial. Roughly speaking, Ch. Bonatti and V. Grines [6] proved that an embedding of the knot $p(l_1) \subset S^2 \times S^1$ denoted by $k(f)$ is a complete invariant of conjugacy in the class $MS(S^3, 4)$. The set $R(f) = \alpha_1 \cup \alpha_2 \cup l_1 \cup l_2$ is a repeller. Due to Corollary 2 if $f_1, f_2 \in MS(S^3, 4)$ are conjugate then $R(f_1), R(f_2)$ have the same dynamical locally equivalent embedding. It follows that $k(f_1)$ and $k(f_2)$ have the same embedding in $S(f_1)$ and $S(f_2)$ respectively. We see that the necessary condition of conjugacy in the frame of our approaching implies Bonatti-Grines’s invariant.

Consider now the class $MS(M^m, 3)$ of Morse-Smale diffeomorphisms $f : M^m \to M^m$ of closed $m$-manifolds $M^m$ with the non-wandering set $NW(f)$ consisting of three periodic points. It was proved in [19] that $MS(M^m, 3) \neq \emptyset$ iff $m \in \{2, 4, 8, 16\}$. Given any $f \in MS(M^m, 3)$, $NW(f)$ consists of a sink $\omega$, a source, and a saddle $\sigma$. Moreover, $\sigma$ has $\frac{m}{2}$-dimensional separatrices $W^u(\sigma)$, $W^s(\sigma)$. Let us restrict ourself for simplicity by orientation preserving diffeomorphisms embedded in flows. Then the knot $k(f) = p(W^s_f(\sigma))$ is homeomorphic to $S^{m-1} \times S^1$. This knot is trivially embedded in the space of orbit $S(f)$ that is homeomorphic to $S^{m-1} \times S^1$. For the dimensions $m = 8, 16$, there are non-homeomorphic manifolds supporting the Morse-Smale diffeomorphisms $MS(M^m, 3)$. Hence, there are non-conjugate diffeomorphisms $f \in MS(M^m, 3)$ having the knots $k(f) = p(W^s_f(\sigma))$ with the same embedding in the space of orbits. Therefore, $k(f)$ is not a complete invariant of conjugacy in the class $MS(M^m, 3)$. On the other hand, Corollary 2 implies the following application for $MS(M^m, 3)$.

**Corollary 3** Morse-Smale diffeomorphisms $f_1, f_2 \in MS(M^m, 3)$ are conjugate iff the unstable manifolds $W^u(\sigma_1), W^u(\sigma_2)$ or stable manifolds $W^s(\sigma_1), W^s(\sigma_2)$ have the same dynamical locally equivalent embedding where $\sigma_i$ is the saddle of $f_i$, $i = 1, 2$.

Let us give another applications of our approaching to the problem of classification beginning with simplest one. Again consider the class $MS(M^2, 3)$. The supporting manifold $M^2$ for any $f \in MS(M^2, 3)$ is the projective plane $M^2 = \mathbb{P}^2$ [19]. The attracting set $A(f)$ is a closed curve consisting of $\sigma$ and two one-dimensional unstable separatrices. A neighborhood $U$ of $A(f)$ is homeomorphic to Möbius band, Fig. 1. Since $U$ contains only two fixed points, $\sigma$ and the sink,

![Figure 1: Phase portrait for $f \in MS(M^2, 3)$: the diametrically opposite points are identified](image)

the dynamics of $f|_U$ depends completely on a local dynamics of $f$ at the saddle $\sigma$ which is defined by the four following types

$$T_1 = \{ \frac{\bar{x}}{y} = \frac{1}{2}x, \quad T_2 = \{ \frac{\bar{x}}{y} = -\frac{1}{2}x, \quad T_3 = \{ \frac{\bar{x}}{y} = \frac{1}{2}x, \quad T_4 = \{ \frac{\bar{x}}{y} = -\frac{1}{2}x.$$
As a consequence, one gets

**Corollary 4** The diffeomorphisms $f_1, f_2 \in MS(\mathbb{P}^2, 3)$ are conjugate if and only if the types of their saddles coincide. Given any type $T_i$, there is a diffeomorphism $f \in MS(\mathbb{P}^2, 3)$ with a saddle of the type $T_i$, $i = 1, 2, 3, 4$.

Thus, up to conjugacy, there are four classes of Morse-Smale diffeomorphisms $MS(\mathbb{P}^2, 3)$.

Let $AO(M^n, k + s)$ be the class of A-diffeomorphisms $M^n \to M^n$ of closed $n$-manifold $M^n$ with the non-wandering set consisting of an orientable codimension one expanding attractor, and $k \geq 1$ isolated periodic nodes, and $s \geq 0$ isolated periodic saddles. Denote by $\Lambda_f$ an orientable codimension one expanding attractor of $f \in AO(M^n, k + s)$. For $n \geq 3$, Plykin [23] proved that there are a codimension one Anosov automorphism $A(f) : \mathbb{T}^n \to \mathbb{T}^n$ with a finitely many periodic orbits $P(f) \subset \mathbb{T}^n$ of $A(f)$ and a continuous mapping $h : W^s(\Lambda_f) \to \mathbb{T}^n \setminus P(f)$ which is a semi-conjugacy from $f|W^s(\Lambda_f)$ to $A(f)|\mathbb{T}^n \setminus P(f)$. Moreover, $(A(f), P(f))$ is a complete invariant of conjugacy for $f$. To be precise, the pairs $(A_1, P_1)$ and $(A_2, P_2)$ are called commensurable if there is a homeomorphism $\psi : \mathbb{T}^n \to \mathbb{T}^n$ such that $\psi(P_1) = P_2$ and $\psi \circ A_1 = A_2 \circ \psi$. Plykin [23] proved that given any two diffeomorphisms $f_1 \in AO(M^n_1, k_1 + s_1)$ and $f_2 \in AO(M^n_2, k_2 + s_2)$, the restrictions $f_1|W^s(\Lambda_{f_1}) \to f_2|W^s(\Lambda_{f_2})$ are conjugate if and only if the pairs $(A(f_1), P(f_1))$, $(A(f_2), P(f_2))$ are commensurable. For $n = 2$ and $M^2 = \mathbb{T}^2$, the similar complete invariant $(A(f), P(f))$ was obtained by Grines [11,12]. For $n \geq 3$ and $M^n = \mathbb{T}^n$, the similar complete invariant $(A(f), P(f))$ was obtained in [13]. Our approaching to the problem of classification gives the following result.

**Corollary 5** Given any structurally stable $f \in AO(M^n, 2 + 1)$, $n \geq 3$, the supporting manifold $M^n$ is an $n$-torus $\mathbb{T}^n$. Moreover, any $f_1, f_2 \in AO(M^n, 2 + 1)$ are conjugate if and only if their Plykin’s pairs $(A_1, P_1)$ and $(A_2, P_2)$ are commensurable.

**Sketch of the proof.** It follows from a structural stability that $M^n = \mathbb{T}^n$ [14]. Again, the structural stability of $f$ implies that a unique saddle $\sigma_f \in NW(f)$ has $(n-1)$-dimensional unstable manifold $W^u(\sigma_f)$ that have to intersect $W^s(\Lambda_f)$. Now suppose that Plykin’s pairs $(A_1, P_1)$ and $(A_2, P_2)$ of $f_1, f_2 \in G(M^n, 2 + 1)$ respectively are commensurable. Hence, the codimension one expanding attractors $\Lambda_{f_1}, \Lambda_{f_2}$ have the same dynamical locally equivalent embedding. Since $W^u(\sigma_{f_1}) \cap W^s(\Lambda_{f_1}) \neq \emptyset$ and $W^u(\sigma_{f_2}) \cap W^s(\Lambda_{f_2}) \neq \emptyset$, the local conjugacy of $\Lambda_{f_1}, \Lambda_{f_2}$ can be easily extended to the unstable manifolds of the saddles $\sigma_{f_1}, \sigma_{f_2}$. Due to Theorem [1], $f_1$ and $f_2$ are conjugate. □

**Remark.** One can prove that every $f \in AO(M^n, 1 + 1)$, $n \geq 3$, is not structurally stable. Moreover, any $f_1, f_2 \in AO(M^n, 1 + 1)$ are conjugate if and only if the sets $W^u(\sigma_{f_1}) \cup \Lambda_{f_1}$, $W^u(\sigma_{f_2}) \cup \Lambda_{f_2}$ have the same dynamical locally equivalent embedding.

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