ENUMERATION OF STRONG DICHOTOMY PATTERNS

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Abstract. We apply the version of Pólya-Redfield theory obtained by White to count patterns with a given automorphism group to the enumeration of strong dichotomy patterns, that is, bicolor self-complementary patterns of $\mathbb{Z}_{2^k}$ with respect to the action of $\text{Aff} (\mathbb{Z}_{2^k})$ with trivial isotropy group.

1. Introduction

In a short and beautiful paper [10], White proved an analogue of Cauchy-Frobenius-Burnside lemma tailored for the purpose of counting patterns with a fixed group of automorphisms. Before stating it, we warn the reader that we will use the Iverson bracket\(^1\) as defined by Graham, Knuth and Patashnik [4, p. 24]: if $P$ is a property, then

$$[P] = \begin{cases} 1, & P \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 1** (D. E. White, 1975). Let $S$ be a finite set, $G$ a finite group acting on $S$ and $\Delta$ a system of orbit representatives for $G$ acting on $S$. Suppose \{\( G_1, \ldots, G_N \)\} is a traversal of the orbits of the subgroups of $G$ under the conjugation action, such that $|G_1| \geq \cdots \geq |G_N|$.

Given a weight function $w : S \to T$ such that $w(\sigma s) = w(s)$ for all $\sigma \in G$ and all $s \in S$, we have

$$\sum_{s \in \Delta} w(s) [G_s \sim G_i] = \sum_{j=1}^{N} b_{i,j} \sum_{s \in S} w(s) [G_j s = s],$$

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\(^1\)White uses the similar notation $\chi(P)$ in his articles, which was introduced first by Adriano Garsia in a paper from 1979 according to Knuth [6], although White’s paper predates it by four years.
where \( B = (b_{i,j}) \) is the inverse of the table of marks matrix

\[
M_{i,j} = \frac{1}{|G_j|} \sum_{\sigma \in G} [\sigma G_i \sigma^{-1} \subseteq G_j].
\]

Note that the table of marks matrix is invertible because it is triangular and no element of the diagonal is 0.

Let \( S = R^D \), where both \( R \) and \( D \) are finite sets. If \( G \) acts on \( R \) (the set of colors), it is well known that the action can be extended to \( S \) defining \( \sigma \cdot f = f \circ \sigma^{-1} \), where \( \sigma \) is reinterpreted as a member of the permutation group of \( D \). We know that the action of \( G_i \) on \( D \) defines a set of disjoint orbits

\[
O_{G_i;D} := \{G_i x_1, \ldots, G_i x_\ell\}
\]

which is a partition of \( D \), so we can define

\[
q_{G_i}(d) = \sum_{i=1}^\ell |G x_i| = d.
\]

This allows us to define the orbit index monomial as

\[
P_i(z_1, z_2, \ldots, z_{|D|}) := \prod_{d \in D} z_d^{q_{G_i}(d)},
\]

which can be used in a straightforward manner to obtain a pattern inventory polynomial.

**Theorem 2** (D. E. White, 1975). The pattern inventory polynomial for patterns fixed by the subgroup \( G_i \) is

\[
Q_i = \sum_{j=1}^N b_{i,j} P_j(z_1, z_2, \ldots, z_{|D|}),
\]

where the substitution \( y_i = \sum_{r \in R} x_i^r \) is made.

Bicolor patterns under certain types of groups are of particular interest for mathematical musicology because (for instance) they represent rhythmic patterns if they are interpreted as onsets in a measure. See [2] and [5] and the references therein for more information. These patterns can also be seen as abstractions of the concepts of consonance and dissonance in Renaissance counterpoint. In particular, **self-complementary** (that is, those whose complement belongs to its orbit) and **rigid** (which means that they are invariant only under the identity) patterns, hereafter called **strong**, are known to be used in both Western and Eastern music [7, Part VII], and that their combinatorial structure lead to significant musicological results [7, Chapter 31]. Note that self-complementarity forces the patterns to be subsets...
of cardinality $k$ of sets of even cardinality $2k$. In general, dichotomy patterns are those of cardinality $k$ within a set of cardinality $2k$.

2. TWO EASY EXAMPLES

Suppose we color black or white the vertices of a rectangle that is not a square. The group of symmetries acting on the colorings of the vertices is the Klein four-group

$$V = \langle a, b | a^2 = b^2 = (ab)^2 = e \rangle.$$

We will find the patterns that are invariant under $G_1 = V$, $G_2 = \langle a \rangle$, $G_3 = \langle b \rangle$, $G_4 = \langle ab \rangle$ and $G_5 = \langle e \rangle$ using White’s formulas. Since all the proper subgroups of $V$ are normal, we easily calculate the table of marks matrix

$$M_V = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 \\
1 & 2 & 2 & 2 & 4
\end{pmatrix},$$

whose inverse is

$$B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}
\end{pmatrix}.$$

It is illustrative to make explicit the formula of Theorem 1 for this simple example. Let us code the colorings of the vertices with the strings $u_1u_2u_3u_4$ over the alphabet $\{n, b\}$, using the clockwise order and beginning from the upper left corner. Then

$$\Delta = \{nnnn, nnbb, nnbb, nbbn, nbmb, nbb, nbbb, nnnn\}.$$

For $G_1$ the formula trivially asserts that the only patterns invariant under the full group are the monochromatic ones. For $G_2$ we have

$$-\frac{1}{2}(w(nnnn) + w(bbbb)) + \frac{1}{2}(w(nnnn) + w(nnbb) + w(bbmn) + w(bbbb)) = \frac{1}{2}(w(bbmn) + w(nnbb)) = w(nnbb)$$

because $w(bbmn) = w(nnbb)$, by hypothesis. The colorings $bbmn$ and $nnbb$ are precisely those who represent the only pattern which is invariant under the reflection with vertical axis. The cases of $G_3$ and $G_4$ are
analogous. Finally, the case of the trivial subgroup is more interesting:

\[
\frac{1}{2}(w(nnnn) + w(bbbb))
- \frac{1}{4}(w(nnnn) + w(nnbb) + w(bbn) + w(bbbb))
- \frac{1}{4}(w(nnnn) + w(nbbn) + w(bnn) + w(bbbb))
- \frac{1}{4}(w(nnnn) + w(nbmb) + w(bnbn) + w(bbbb))
- \frac{1}{4}(w(nnnn) + w(nbbn) + w(bnm) + w(bbbb))
= w(nnnb) + w(nbmb),
\]

and it informs us of the two patterns that are invariant under the action of the trivial subgroup only; they are precisely those with only one black or only one white vertex.

Let us confirm the former using the orbit index polynomials for each subgroup. For \( G_1 \), we have only one orbit of four elements, thus

\[ P_1 = z_4. \]

The orbits defined by \( G_2, G_3 \) and \( G_4 \) are all of cardinality two, thus

\[ P_2 = P_3 = P_4 = z_2^2. \]

Finally, there are four orbits of cardinality one for the trivial subgroup, hence

\[ P_5 = z_1^4. \]

Using these polynomials, we can calculate all the pattern inventories at once:

\[
\begin{pmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4 \\
Q_5
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
z_4 \\
z_2^2 \\
z_2^2 \\
z_2^2 \\
z_1^4
\end{pmatrix}
=
\begin{pmatrix}
z_4 \\
\frac{1}{2}z_2^2 - \frac{1}{2}z_4 \\
\frac{1}{2}z_2^2 - \frac{1}{2}z_4 \\
\frac{1}{2}z_2^2 - \frac{1}{2}z_4 \\
\frac{1}{2}z_4 - \frac{3}{4}z_2^2 + \frac{1}{4}z_1^4
\end{pmatrix}.
Upon the substitution $z_i = 1 + x^i$, that allows us to count the number of patterns according to the number of black elements (say), we find

$$
\begin{pmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4 \\
Q_5
\end{pmatrix}
= 
\begin{pmatrix}
1 + x^4 \\
x^2 \\
x^2 \\
x^2 \\
x + x^3
\end{pmatrix}.
$$

We proceed now with a more complicated example that serves as an introduction to our main computation. Define

$$\text{Aff}(\mathbb{Z}_{2k}) = \mathbb{Z}/2k\mathbb{Z} \rtimes \mathbb{Z}/2k\mathbb{Z}^\times.$$

Denote an element $(u, v) \in \text{Aff}(\mathbb{Z}_{2k})$ by $e^u.v$. The action of $\text{Aff}(\mathbb{Z}_{2k})$ on $\mathbb{Z}/2k\mathbb{Z}$ is given by

$$e^u.v(x) = vx + u.$$

Let us compute the number of patterns of the action of $\text{Aff}(\mathbb{Z}_6)$. We have the following sequence of normal subgroups,

$$G_1 = \langle e^1.1, e^{0.5} \rangle, G_2 = \langle e^4.1, e^{5.5} \rangle,$$

$$G_3 = \langle e^1.1, e^{0.5} \rangle, G_4 = \langle e^{1.1} \rangle,$$

$$G_5 = \langle e^3.1, e^{0.5} \rangle, G_6 = \langle e^{2.1} \rangle,$$

$$G_7 = \langle e^{5.5} \rangle, G_8 = \langle e^{0.5} \rangle,$$

$$G_9 = \langle e^{3.1} \rangle, G_{10} = \{e^{0.1}\},$$

The computation of the table of marks matrix is not as direct as before, in part because the subgroups $G_5, G_7$ and $G_9$ are not normal. But using GAP \cite{GAP} we readily find

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 2 & 0 & 4 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 6 & 0 \\
1 & 2 & 2 & 2 & 3 & 4 & 6 & 6 & 6 & 12
\end{pmatrix}.$$
whose inverse is

\[
B = \begin{pmatrix}
\frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{12} & 0 & 0 & 0 & 1 \frac{1}{4} & 0 & 0 & 0 \\
-\frac{1}{12} & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\
-\frac{1}{12} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 \\
-\frac{1}{12} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & 1
\end{pmatrix}
\]

The orbit index polynomials are

\[P_1 = z_6, P_2 = z_6, P_3 = z_3^2, P_4 = z_6, P_5 = z_2 z_4,\]
\[P_6 = z_3^2, P_7 = z_2, P_8 = z_3^2 z_2^2, P_9 = z_2^3, P_{10} = z_1^6\]

whence

\[
Q(z_1, \ldots, z_6) = BP = \begin{pmatrix}
z_6 \\
0 \\
\frac{1}{2} z_6^2 - \frac{1}{2} z_6^2 \\
z_2 z_4 - z_6 \\
0 \\
-\frac{1}{2} z_2^3 - \frac{1}{4} z_2 z_4 \\
\frac{1}{2} z_6 - \frac{1}{2} z_2 z_4 - \frac{1}{2} z_3^2 + \frac{1}{4} z_2^2 z_2 \\
\frac{1}{3} z_6 - \frac{1}{3} z_2 z_4 + \frac{1}{2} z_3^2 \\
-\frac{1}{6} z_6 + \frac{1}{2} z_2 z_4 + \frac{1}{6} z_3^2 - \frac{1}{3} z_2^3 - \frac{1}{4} z_2^2 z_2 + \frac{1}{12} z_1^6
\end{pmatrix}
\]

thus

\[
Q(1 + x, \ldots, 1 + x^6) = \begin{pmatrix}
x^6 + 1 \\
x^3 \\
0 \\
x^4 + x^2 \\
x^4 + x^2 \\
x^5 + x^4 + x^3 + x^2 + x \\
0 \\
x^3
\end{pmatrix}
\]

It is interesting to learn that there are no patterns that are exclusively invariant under the subgroups generated, respectively, by the translations \(e^1.1, e^2.1, e^3.1\). In other words: arithmetic progressions
3. Calculations

Denoting by $\mathcal{D}$ the set of dichotomies, and by $\mathcal{S}$ and $\mathcal{R}$ the subsets of the self-complementary and rigid dichotomies (respectively), we know by the principle of inclusion and exclusion (PIE) that

$$|\mathcal{D}| \geq |\mathcal{S} \cup \mathcal{R}| = |\mathcal{S}| + |\mathcal{R}| - |\mathcal{S} \cap \mathcal{R}|$$

where $|\mathcal{S} \cap \mathcal{R}|$ is precisely the number of strong dichotomies. Hence

$$|\mathcal{S} \cap \mathcal{R}| \geq |\mathcal{S}| + |\mathcal{R}| - |\mathcal{D}|.$$ 

We can calculate $|\mathcal{D}|$ and $|\mathcal{S}|$ with the classical Pólya-Redfield theory, and $|\mathcal{R}|$ with White’s formulas, so we may expect this inequality to provide a reasonably good bound on the number of strong dichotomies. But, unfortunately, in general it does not, as we can readily see in Table 1.

However, not everything is lost. After examining the cases when the PIE yields a nontrivial bound, we discover that this happens when $k$ is a power of a prime. On the other hand, it is known that the classical pattern inventory polynomials of the Pólya-Redfield theory exhibit a form of the cyclic sieving phenomenon [8, Corollary 6.2], which means that if $p(x)$ is the generating function of the number of patterns according to its number of black elements, then $p(-1)$ yields the number of self-complementary patterns.

Since the polynomials for White’s formulas do not count cycles but orbits, in general they fail to cyclically sieve patterns, safe for the cases when $\mathbb{Z}_{2k}^\times$ is cyclic. Indeed, if the group of units is generated by a single element, it is plausible to think that all the orbits are cycles of $e^{1.1}$ and a generator of $\mathbb{Z}_{2k}^\times$. Furthermore, it is a well-known fact that the group of units of $\mathbb{Z}_n$ is cyclic precisely when $n = 1, 2, p^k$, where $p$ is a prime number. This discussion, however, is not a full proof, so we formalize it as a conjecture. Hopefully, it will be proved soon.

**Conjecture 1.** Let $G_N = \text{Aff}(\mathbb{Z}_{2k})$ and $\{G_i\}$ be a set of representatives of the orbits of the conjugation action such that $|G_N| \geq \cdots \geq |G_1|$ and let $B = (b_{i,j})$ be the inverse of its table of marks. If $k$ is equal to 1, 2 or a power of an odd prime number, then the pattern inventory polynomial for bicolor patterns fixed by the subgroup $G_i$

$$Q_i = \sum_{j=1}^{N} b_{i,j} P_j(1 + x, 1 + x^2, \ldots, 1 + x^{|D|}),$$
Table 1. Summary of the information that can be obtained via classical Pólya-Redfield theory and White’s extension.

| $2k$ | $|D|$ | $|S|$ | $|R|$ | PIE bound | $|Q_1(-1)|$ |
|------|------|------|------|----------|----------|
| 2    | 1    | 1    | 1    | 1        | 1        |
| 4    | 2    | 2    | 0    | 0        | 0        |
| 6    | 3    | 3    | 0    | 0        | 1        |
| 8    | 6    | 4    | 1    | −1       | 1        |
| 10   | 9    | 7    | 5    | 3        | 3        |
| 12   | 34   | 18   | 10   | −6       | 4        |
| 14   | 47   | 15   | 37   | 5        | 9        |
| 16   | 129  | 21   | 83   | −25      | 1        |
| 18   | 471  | 55   | 436  | 20       | 40       |
| 20   | 1280 | 134  | 1052 | −94      | 66       |
| 22   | 3235 | 115  | 3181 | 61       | 105      |
| 24   | 15008| 440  | 13331| −1237    | 33       |
| 26   | 33429| 385  | 33253| 209      | 355      |
| 28   | 121466|1194 | 117422|−2850| 886     |
| 30   | 648819|3365 | 643901|−1153| 3007   |
| 32   | 1182781|2189 | 1165498|−15094|1432    |
| 34   | 4290533|4375 | 4288913|2755| 4305    |
| 36   | 21082620|18404| 20933318|−130898|15518    |
| 38   | 51677171|15347| 51671611|9787| 15267   |
| 40   | 215804540|49684| 214972319|−782537|25659    |
| 42   | 1068159497|133285|1067785287|−240925|130839  |
| 44   | 2392981542|171662|2389064994|−3744886|155346  |
| 46   | 8135833183|198943|8135769049|134809|198753  |
| 48   | 42007923187|786707|41970277573|−36858907|643019  |
| 50   | 126410742103|872893|126410471144|601934|871992  |

is such that $|Q_1(-1)|$ counts the number of self-complementary dichotomies with automorphism group $G_i$. In particular, $|Q_1(-1)|$ counts the number of strong dichotomies.

For the general case, we can use another formula of White [9]. Now we need to consider the swapping action on the colors of the patterns simultaneously with that of the affine group, so we see $G = \text{Aff}(\mathbb{Z}_{2k}) \times \mathbb{Z}_2$ as acting both in $R$ and $D$, according to

$$(\sigma, \tau) \cdot r = \tau \cdot r \quad \text{and} \quad (\sigma, \tau) \cdot d = \sigma \cdot d;$$
hence $G$ acts doubly on $R^D$ in the following manner

$$g \cdot f = g \circ f \circ g^{-1}.$$

We provide a quick sketch of White’s reasoning to obtain the counting formula, in part because our problem’s conditions lead to a simpler statement and in part because his original paper has some minor (but misleading) typographical errors.

In order to apply Theorem 1, we must characterize first the patterns $f$ such that $H' \subseteq G_f$. If a $g \in H'$ leaves a pattern invariant, it means that it sends an element of $O_{H'D}$ to an element of $O_{H'R}$.

Thus, let $B \in O_{H':D}$ and $f(B) = C \in O_{H':R}$. Taking arbitrary elements $b \in B$ and $c \in C$, we deduce that $f$ must be defined by $f(\gamma_1 b) = \gamma_1 c$. This relation, however, might not be functional, for it may happen that $f(\gamma_1 b) \neq f(\gamma_2 b)$ when $\gamma_1 b = \gamma_2 b$, unless $\gamma_1 c = \gamma_2 c$, or $\gamma_1^{-1} \gamma_2 \in H_c$. In other words, if the function is well defined then

$$\gamma_1^{-1} \gamma_2 \in H_b \implies \gamma_1^{-1} \gamma_2 \in H_c$$

or, equivalently,

$$H_b \subseteq H_c$$

(note that these isotropy groups are relative to $H$). To check that (1) is also sufficient is direct, like the fact that the election of $b$ is irrelevant.

We have

$$\sum_{f \in S} w(s)[G_j f = f] = \sum_{f \in O_{H':D}} \prod_{j=0}^{\lfloor |f(B)| \rfloor - 1} \sum_{H_b \subseteq H_{\tau_j c}} [H_b \subseteq H_{\tau_j c}] w(f)$$

and, reorganizing the terms (in what White calls sum-product interchange), we get

$$\sum_{f \in S} w(f)[G_j f = f] = \prod_{B \in O_{H':D}} \sum_{C \in O_{H':R}} \sum_{j=0}^{\lfloor |C| \rfloor - 1} [H_b \subseteq H_{\tau_j c}] w(f)$$

where

$$w(f) = \prod_{i=0}^{\lfloor |B| \rfloor - 1} x_{\tau_j c}.$$

For bicolor patterns we have $C = \{0, 1\}$, therefore the invariance under the action of the whole group reduces the weights $w(f)$ to the following choices:

$$w(f) = \begin{cases} x_0^{\lfloor |B|/2 \rfloor} x_1^{\lfloor |B|/2 \rfloor}, & H \text{ swaps colors}, \\ x_0^{|B|} = x_1^{|B|}, & \text{otherwise.} \end{cases}$$
Thus we can reuse the previous algorithm that involves the inverse of the table of marks matrix, but with the larger group \( \text{Aff}(\mathbb{Z}_{2k}) \times \mathbb{Z}_2 \) and calculating the corresponding vector of polynomials with (2) and (3). The only remaining detail is that no longer we may read the total number of strong dichotomy patterns in a single entry of the output vector, for such patterns have automorphism groups of cardinality two; namely, the identity \((e^{0.1}, 0)\) and the color swap \((e^{0.1}, 1)\) composed with a unique symmetry of \( \text{Aff}(\mathbb{Z}_{2k}) \), which is called the polarity of the pattern. Hence, we gain a feature and not an inconvenience, for now we can know the number of strong dichotomy patterns for each polarity.

The first case that is not covered by Theorem 1 is \( n = 8 \), but the table of marks matrix is of size 148 \( \times \) 148, so we will not display it here. Let us simply state that there is only one strong dichotomy, whose polarity is \( e^5 \). In Table 2 we summarize the information that can be calculated with this algorithm up to \( n = 48 \).

### 4. Concluding remarks

The enumerations of strong dichotomies done here coincide with the explicit ones performed in [1] and subsequent verifications done by the
author, with a variation of the original algorithm presented in [1]. It is interesting to note that Conjecture 1 is of practical interest, since it significantly simplifies the computation of the table of marks: consider that the volume of calculations is exacerbated when we have to calculate with the product \( \text{Aff}(\mathbb{Z}_{2k}) \times \mathbb{Z}_2 \); its table of marks can be much bigger than the one of its largest factor.

Harald Fripertinger noted in a personal communication with the author that the number of self-complementary patterns \(|S|\) seems to approach asymptotically to the number of the strong ones (or, equivalently, that the vast majority of dichotomies is rigid). In particular, \(|S|\) provides a direct and fast way (it does not require to compute the table of marks) to determine a very good upper bound for the number of strong patterns, a useful fact in order to partially validate the exact (but lengthy) calculations.

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