CUSUM Average Run Length: Conditional or Unconditional?

Fred Lombard  
Department of Statistics  
University of Johannesburg  
fredl@uj.ac.za

Douglas M. Hawkins  
University of Minnesota  
Minneapolis, MN  
dhawkins@umn.edu

Abstract

The behavior of CUSUM charts depends strongly on how they are initialized. Recent work has suggested that self-starting CUSUM methods retain some dependence on their very first readings, and introduced the concept of “conditional average run length” (CARL) – the average run length conditioned on the first few process readings – as a result of which is it claimed that different practitioners using the same methodology could experience different ARLs because of the random differences in their earliest readings. We cast doubt on whether CARL is relevant to practitioners who use self-starting methods and argue that the unconditional ARL is the relevant measure there.

1 Introduction

Cumulative sum (CUSUM) charts are highly sensitive to even small shifts in the process parameters, and their behavior therefore depends strongly on how they are initialized. The traditional approach of plugging in estimates from a Phase I is well known to require inordinately large and unproductive Phase I samples (Jones, Champ and Rigdon, 2004, Jensen, et al., 2006), and this has led to the development of ”self-starting” approaches which circumvent the need for large Phase I samples but nevertheless control average run length behavior. In particular, the in-control average run length (ARL) of self-starting methods is exactly that obtained in the known-parameter setting. Recent work (Keefe, Woodall and Jones-Farmer, 2015) has highlighted the concept of ”conditional average run length” (CARL) – the average run length conditioned on the first few process readings – and practitioner to practitioner
variability, motivated by the idea that different practitioners using the same self-starting methodology could experience different CARLs because of the random differences in their earliest readings.

The exemplar cumulative sum control chart (CUSUM) is that defined by Page (1954) and explored in the monograph by Hawkins and Olwell (1998). It deals with a sequence of independent \( N(\mu, \sigma^2) \) readings \( X_1, X_2, \ldots X_m \). In its standardized form rescaling the data to \( N(0,1) \), the CUSUM defines

\[
U_n = (X_n - \mu)/\sigma \\
D_0 = 0
\]

and \( D_0 = 0 \),

\[
D_n = \max(0, D_{n-1} + U_n - \delta)
\]

for \( n \geq 1 \), where \( \delta \) denotes the reference value. This canonical form requires that \( \mu \) and \( \sigma \) be known. The traditional resolution to this was to conduct a separate preliminary Phase I study, and use its mean \( \hat{\mu} \) and standard deviation \( \hat{\sigma} \) in place of \( \mu \) and \( \sigma \), replacing \( U_m \) in the CUSUM defining equation by \( W_m = (X_m - \hat{\mu})/\hat{\sigma} \). However \( W_m \) does not follow the \( N(0,1) \) distribution, nor are the successive \( W_m \) independent thanks to their common dependence on \( \hat{\mu} \) and \( \hat{\sigma} \). Consequently, when the control limits applicable to the known parameter case are used, the run lengths of this plug-in CUSUM differ substantially from those of the known-parameter setting unless many hundreds of readings are included in the Phase I study.

Hawkins (1987) and Quesenberry (1991) - see also Zantek (2006) - proposed the self-starting methodology in which the estimates of \( \mu \) and \( \sigma \) are updated with each new observation. Write, for \( n \geq 2 \),

\[
\hat{\mu}_n = \frac{X_1 + \cdots + X_n}{n}, \quad \hat{\sigma}_n^2 = \frac{(X_1 - \hat{\mu}_n)^2 + \cdots + (X_m - \hat{\mu}_n)^2}{n-1}
\]

and for \( n \geq m \geq 3 \), \( W_n = (X_n - \hat{\mu}_{n-1})/\hat{\sigma}_{n-1} \). CUSUMs can be constructed starting from \( m \geq 3 \). These \( m \) initial observations constitute the “warmup” of a self-starting scheme, corresponding to the traditional Phase I study, from which it segues smoothly into the Phase II online monitoring by updating the estimates of \( \mu \) and \( \sigma \) upon the arrival of each new observation. The random variable \( W_n \) follows a scaled \( t \) distribution with \( n - 2 \) degrees of freedom and the successive \( W_n \), \( n \geq m \) are statistically independent. They may be transformed to independent exact \( N(0,1) \) quantities \( U_n \) by the double probability integral transform

\[
U_n = \Phi^{-1}(F_{n-2}(W_n \sqrt{(n-1)/n}))
\] (1)
where $\Phi^{-1}$ is the inverse normal CDF and $F_{n-2}$ is the cumulative distribution function of Student’s $t$ with $n-2$ degrees of freedom. The CUSUM recursion is $D_0 = 0$ and

$$D_k = \max(0, D_{k-1} + U_{m+k} - \delta)$$

for $k \geq 1$. The run length, $N$, is the first $k \geq 1$ at which $D_k$ exceeds the control limit. A crucial requirement underlying the self-starting method is that fresh warmup observations be taken upon every restart of the procedure. The effect of this requirement is that the successive $U_n$, $n \geq m$ are statistically independent with standard normal distributions and that the known-parameter CUSUM’s control limits can be used. The in-control ARL of the self-starting CUSUM will then equal that of the known-parameter CUSUM.

2 Practitioner-to-practitioner variability

Keefe, et al. (2015) argue that practitioners who use self-starting CUSUMS will experience different Phase II in-control ARLs simply because their warmup estimates of $\mu$ and $\sigma$ differ and (page 496) that this points to a possible defect in the self-starting CUSUM, to wit: "It is not the case, as stated by Hawkins and Olwell (1998, p. 162) that the self-starting approach removes the estimation issue from the problem completely."

The key issue in understanding the relevance of the CARL and practitioner-to-practitioner variation concepts is in how the CUSUM is re-initialized following a signal. We identify two scenarios:

Scenario 1: In each run gather $m$ fresh warmup readings and restart the CUSUM.

Scenario 2: Keep the original $m$ warmup readings and their resulting $\hat{\mu}_m$ and $\hat{\sigma}_m$ and restart every run from that baseline.

In what follows it is assumed that the underlying process is in control. The run length of the self-starting CUSUM is denoted by $N$ and the warmup data by $X_m$. The self-starting method demands a Scenario 1 initialization of every run, that is, $X_m$ varies from run to run. Then the run lengths $N_1$, $N_2$, \ldots observed in a long series of runs are i.i.d. copies of $N$ and their average will converge to the nominal in-control ARL $E[N]$.

Keefe, et Al. (2015), consider the conditional ARL $E[N|X_m]$ and find in a Monte Carlo study that it varies substantially with $X_m$, as would be expected
because \( E[N|X_m] \) is a random variable that depends upon \( X_m \). Different practitioners will therefore have different warmup sets \( X_m \) and the ARL observed by a practitioner whose warmup set is \( X_m = x_m \) is \( E[N|X_m = x_m] \). But this is the ARL of the self-starting CUSUM run under scenario 2 initialization. With this initialization the \( U_i \) are not statistically independent, nor do they have standard normal distributions. Thus, none of the CUSUMs being run by the various practitioners is the self-starting CUSUM defined by Hawkins (1987). It is therefore not at all clear how the behaviour of these CUSUMs can be construed as indicating a defect in the Hawkins (1987) self-starting CUSUM.

The link between the unconditional ARL of the self-starting CUSUM and its CARLs considered by Keefe, et Al. (2015) is provided by the well known formula,

\[
E[N] = E[E[N|X_m]],
\]

that is, the unconditional ARL \( E[N] \) is equal to the long run average of the CARLs \( E[N|X_w] \) over all warmup sets \( X_m \). The warmup readings \( X_m \) define the CARL of the CUSUM but, as seen from (2), this CARL is a notional rather than a real construct. The process owner who implements the self-starting CUSUM correctly, i.e. with Scenario 1 initializations, will see a single run length from any particular warmup set \( X_m \) after which that warmup set is discarded and a new one generated. In accordance with (2), the average of the process owner’s observed run lengths will therefore be the unconditional ARL, which equals the nominal value, no matter how the individual CARLs \( E[N|X_w] \) behave. The conclusion, referred to above, reached by Keefe, et Al. (2015) is therefore unwarranted.

In fact, the practitioner to practitioner concept can be used to cast doubt on the validity of many perfectly valid statistical procedures. As a case in point, consider the two-sample t-test for the equality of the means in two populations with a common but unknown variance \( \sigma^2 \). The test statistic is

\[
T = \frac{\bar{X} - \bar{Y}}{S}
\]

where \( \bar{X}_m \) and \( \bar{Y}_n \) denote the sample means and

\[
S^2 = \frac{\sum_{i=1}^{m}(X_i - \bar{X}_m)^2 + \sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2}{m + n - 2}
\]

denotes the pooled estimate of \( \sigma^2 \). The null hypothesis is rejected at the 100\(\alpha\)% level of significance whenever \( T \) exceeds the upper 100\((1-2\alpha)\)% point of the Student t distribution with \( m + n - 2 \) degrees of freedom. However, conditional upon \( \bar{Y} = y \), the level of significance is no longer 100\(\alpha\)% but
something quite different and it will vary with the $y$ values observed by different practitioners. Clearly, this fact cannot serve as justification for a statement that "It is not the case that the two sample t test removes the variance estimation issue from the problem completely". The t-test requires that fresh $X$ and $Y$ samples be used in every repetition of the test (scenario 1) while the conditional test restricts the mean of the $Y$ sample to a fixed value in every repetition of the test (scenario 2).

We conclude that the CARL and practitioner-to-practitioner variation concepts are not relevant to anyone using the self-starting CUSUM of Hawkins (1987) and that this CUSUM, based upon (1) does indeed solve the estimation problem completely.

A fully satisfactory parametric self-starting CUSUM is presently available only when the underlying distribution is normal. In other multi-parameter distributions the self-starting property applies only to a single parameter - the remaining parameters must be known. If these are estimated from Phase I data, the self-starting property is lost and we again have a plug-in CUSUM. Thus, it is perhaps appropriate to point out that there are available attractive distribution-free alternatives to some plugin CUSUMs. By distribution-free is meant that the in-control properties of the CUSUM do not depend upon a parametric specification of the underlying distribution. For instance, a straightforward approach to the problem is to use sequential rank CUSUMs (Lombard and Van Zyl, 2018 and Van Zyl and Lombard, 2018). Besides being distribution-free and not requiring any parameter estimates, hence no warmup data, methods are available for estimating, a priori, the out-of-control ARL of these CUSUMs. Furthermore, "once and for all situations" control limits are available. Since no parameter estimates or warmup data are required, the concept of practitioner-to-practitioner variability is vacuous in these CUSUMs. The Wilcoxon-type CUSUM of Hawkins and Deng (2010) does not require any parameter estimates and is completely distribution-free. Since no parameter estimates are required, the concept of practitioner-to-practitioner variability is vacuous in these CUSUMs.

Again in nonparametric settings, Chatterjee and Qiu (2009), Gandy and Kvaloy (2013) and Saleh, et al. (2016) show how the bootstrap may be used to obtain control limits that would yield ARLs close to the nominal value when a substantial amount of Phase I data are available. However, since the control limits depend on the (unknown) underlying distribution, ”once and for all situations” control limits do not exist. Furthermore, in any given application, new control limits must be generated whenever the in-control mean or variance has undergone a permanent shift. The normal self-starting and distribution-free CUSUMs (1) are not affected by such shifts and use "once and for all situations” control limits.
Some useful further work would entail comparisons between the out-of
control behaviors of the rank-based CUSUMs and the bootstrap-defined plug-
in CUSUMS.

References

[1] Chatterjee, S. and Qiu, P. (2009). Distribution-Free Cumulative Sum
Control Charts Using Bootstrap-based Control Limits. The Annals of
Applied Statistics, 3, 349-369.

[2] Gandy, A. and Kvaloy, J.T. (2013). Guaranteed Conditional Performance
of Control Charts via Bootstrap Methods. Scandinavian Journal
of Statistics, 40, 647-668.

[3] Hawkins, D.M. (1987). Self-Starting CUSUM Charts for Location and
Scale. Journal of the Royal Statistical Society. Series D, 36, 299-316.

[4] Jensen, W.A., Jones-Farmer, L.A., Champ, C.W. and Woodall, W.H.,
(2006). Effects of parameter estimation on control chart properties: A
literature review. Journal of Quality Technology, 38 (4), 349-364.

[5] Jones, A.L., Champ, C.W. and S.E. Rigdon, (2004). The Run Length
Distribution of the CUSUM with Estimated Parameters. Journal of
Quality Technology, 36 (1), 95-108.

[6] Keefe, M.J., Woodall, W.H. and Jones-Farmer, L.A. (2015). The Conditional In-Control Performance of Self-Starting Control Charts. Quality
Engineering, 27, 488-499.

[7] Hawkins, D.M. and Olwell, D.H. (1998). Cumulative Sum Charts and
Charting for Quality Improvement, Springer New York.

[8] Hawkins, D.M. and Deng, Q. (2010). A Nonparametric Change-Point
Control Chart. Journal of Quality Technology, 42, 165-173.

[9] Lombard, F. and Van Zyl, C. (2018). Signed Sequential Rank CUSUMs.
Computational Statistics and Data Analysis, 118, 30-39.

[10] Quesenberry, C.P., (1991). SPC Q charts for start-up processes and short
or long runs. Journal of Quality Technology, 23 (3), 213-224.

[11] Saleh, N.A., Zwetsloot, I.M., Mahmoud, A.M. and Woodall, W.H.
(2016). CUSUM charts with controlled conditional performance under
estimated parameters. Quality Engineering, 28, 402-425.
[12] Van Zyl, C. and Lombard, F., (2018). Sequential Rank CUSUMs for Location and Dispersion. *South African Statistical Journal*, **52**, 93-113.

[13] Zantek, F. (2006). Design of Cumulative Sum Schemes for Start-Up Processes and Short Runs. *Journal of Quality Technology*, **38** (4), 365-375.