Abstract

We study the spontaneous magnetization and the magnetic hysteresis using the gauge/gravity duality. We first propose a novel and general formula to compute the magnetization in a large class of holographic models. By using this formula, we compute the spontaneous magnetization in a model like a holographic superconductor. Furthermore, we turn on the external magnetic field and build the hysteresis curve of magnetization and charge density. To our knowledge, this is the first holographic model realizing the hysteresis accompanied with spontaneous symmetry breaking. By considering the Landau-Ginzburg type effective potential in the symmetry broken phase, we obtain the mass of the magnon from the bulk geometry data.

Keywords: gauge/gravity correspondence, magnetization, hysteresis curve
1 Introduction

The magnetism is one of the most intriguing topics in various fields of physics. One of the reasons is that interesting subjects of the magnetism are closely related to a fundamental quantum degrees of freedom that distinguish kinds of particles, the particle’s statistics and phases of the system. One of the significant phenomena of magnetic systems is the ferromagnetism given by the spontaneous symmetry breaking. In addition, the ferromagnetic system usually shows hysteresis curves, which are representative phenomena of the ferromagnetism. In this work, we will consider a simple model to build the ferromagnetism and the hysteresis curve using gauge/gravity duality [1,2]. In earlier studies, magnetized systems have already attracted much attention and have been studied many times in the holographic approach, e.g., [3–17].

The gauge/gravity duality is a very powerful tool to study strongly coupled field theory systems. The main advantage of the duality is that the generating functional of the strongly coupled field theory can be obtained by the on-shell action of the corresponding gravity theory [18,19]. This correspondence has been applied to various area of physics. One of the active applications is about the condensed matter theories, e.g., [4,20–25].

More specifically, the gauge/gravity duality is a relation between a \(d\)-dimensional field theory living on the boundary and the corresponding \((d+1)\)-dimensional bulk gravitational system with an appropriate boundary condition. Such a boundary condition tells us the prescription for the external sources and the non-normalizable modes of the field theory and the gravitational system, respectively. In this subject, we mainly consider a system with the sources given by the external magnetic field and chemical potential. These sources induce the magnetization, charge density and vacuum expectation value (VEV) of a scalar operator. We take into account \((2+1)\)-dimensional systems in the present work.

The \((2+1)\)-dimensional spacetime we considered has a very interesting magnetic nature because the direction of magnetic field is always orthogonal to the 2-dimensional space when the spacetime is embedded in higher dimensions. The magnetization is described as a real value, just as the magnetic field is. Therefore, the spontaneous magnetization is displayed by breaking \(Z_2\) symmetry. To realize this spontaneous symmetry breaking, we use the well-known hairy black brane solution with a real scalar field \(\phi\). Such a real scalar field is dual to a scalar operator in the boundary theory. Since our final goal is to construct spontaneous magnetization and hysteresis curves, we introduce an interaction among the scalar operator, the magnetic field and the charge density operator. In bulk point of view, this interaction is achieved by adding an axionic term, \(\int d^4x W(\phi)\epsilon^{MNPQ} F_{MN} F_{PQ}\), to the action. And we
consider a large class of model which can describe the spontaneous magnetization. We propose a general formula for the magnetization by using a 'scaling symmetry' trick [26–28]. This formula will be confirmed in a specific model.

The proposed large class of models enable us to describe the spontaneous magnetization by a Landau-Ginzburg type effective theory. We will show later how to associate gravity calculation to the parameters in the potential of the effective theory. Furthermore we find the effective mass of elementary excitation due to the quantization of magnetic fluctuations in the broken phase. In addition to this, we also give a comment on various local and global excitation in the effective theory.

As we mentioned, one of the characteristics of the ferromagnetic material is the hysteresis curve of magnetization. In order to construct the hysteresis curves, the interaction potential $W(\phi)$ plays an important role. This is because the interaction potential $W(\phi)$ should be an odd function to break the $\mathbb{Z}_2$ symmetry changing the sign of the scalar field $\phi$. We will consider an odd function for a specific model. Also, in fact, the hysteresis is obtained by the slowly varying magnetic field. In the real physics, we must consider a time-dependent background. Since, however, we assume a very slowly varying magnetic field, we ignore the effect from the time dependent dynamics. See [29] for an explicit time-dependent consideration for another holographic model.

This paper is organized as follows: In section 2, we propose the general form of the magnetization using the ‘scaling symmetry’ trick and introduce a simple model. Then we check that our general formula works in the specific model. In addition, we find the expression of the on-shell action dual to the free energy in the boundary field theory. In section 3, we investigate the spontaneous magnetization and the hysteresis curve. The Landau-Ginzburg type potential is also constructed. In section 4, we discuss our result and introduce possible future directions. Several formulas related to the holographic renormalization are summarized in Appendix.

## 2 Basic Formulation

In this section, we consider a large class of holographic models describing spontaneous magnetization. The magnetization is given by derivative of free energy with respect to external magnetic field $B$ as follows:

$$\mathcal{M} = -\frac{\delta\Omega}{\delta B},$$  \hspace{1cm} (1)

\footnote{See [4] for details.}
where $\Omega$ is the free energy of $d$-dimensional system. This is the most essential quantity to understand physics in various magnetic materials. The main goal of the present work is to study this quantity through holographic approach.

The fundamental relation of thermodynamics is the first law describing a constraint among variations of thermodynamic quantities. The gauge/gravity duality enables us to interpret this relation in the bulk geometry. $(d+1)$-dimensional black holes also have relations among their parameters by on-shell variation. These two relations have a similarity, which leads to the holographic principle and the gauge/gravity correspondence. Especially, we are interested in the magnetic variation. It consists of the external magnetic field variation and the magnetization. The form of 4-dimensional magnetic black brane solutions is simpler than those in the higher dimensions. Thus we will focus on the asymptotically $AdS_4$ black branes in this paper.

Furthermore, this first law of thermodynamics can be integrated to a relation among finite thermodynamic parameters. The resulting relation is consistent with the definition of free energy density by construction. One may write down the relation as follows:

$$\omega \equiv \Omega/V = \epsilon - sT - \mu \tilde{Q},$$

where $\omega$ is the free energy density of the system. In addition, spatial volume, energy density, entropy density, temperature, chemical potential and charge density are denoted by $V$, $\epsilon$, $s$, $T$, $\mu$, $\tilde{Q}$, respectively. Also, it is well known that the pressure $P$ is same with $-\omega$ for homogeneous systems.

When the system has the magnetization $\mathcal{M}$ under the magnetic field $B$, the pressure consists of the magnetic part and the other contribution as follows:

$$P = P_{\text{others}} + \mathcal{M}B.$$  \hspace{1cm} (3)

From this decomposition one can easily read off the magnetization $\mathcal{M}$. In the bulk geometry, the above relation (2) corresponds to the so-called Smarr relation. It is the constraint among black hole parameters given by vanishing time-time component of the metric at the horizon. We will show this explicitly.

Recently the general form of the Smarr relation has been studied and was derived by applying a modified Noether theorem to a reduced action of gravitational systems [26–28, 30–32]. This is very useful even in a hairy black hole background. We will extend the method for hairy magnetic black branes.
2.1 Einstein-Maxwell-Dilaton Model

Let us start with a large class of holographic models to describe magnetized systems.

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ R + \frac{6}{L^2} - \frac{Z(\phi)}{4} F^2 - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right\} \]
\[ + \frac{\theta}{16\pi G} \int d^4x W(\phi) \epsilon^{MNPQ} F_{MN} F_{PQ} , \]  

(4)

where the last term plays an important role for the spontaneous magnetization and the hysteresis curve. In 3+1 dimension, this term is linear in the magnetic field. So one may notice that there can exist a non-vanishing magnetization even in the absence of magnetic field. See (1) for this speculation. Furthermore, turning on the magnetic field with even functions of \( Z(\phi) \) and \( V(\phi) \), an odd function of \( W(\phi) \) breaks a \( \mathbb{Z}_2 \) symmetry by changing the sign of the scalar field \( \phi \rightarrow -\phi \). In this case, however, the Lagrangian is invariant under another transformation that simultaneously changes the sign of the magnetic field \( (\mathcal{B} \rightarrow -\mathcal{B}) \). Since the latter symmetry is essential to build the hysteresis curve, we will take into account the odd function \( W(\phi) \) with even \( Z(\phi) \) and \( V(\phi) \) in a specific calculation. It will be clear in section 3.

Now, we will derive the Smarr relation of black branes in the gravity model using a ‘scaling symmetry’ method in \([26, 28, 31, 32]\). In order to apply the method, we take the following metric ansatz:

\[ ds^2 = -U(r)e^{-\mathcal{W}(r)} dt^2 + \frac{r^2}{L^2} (dx^2 + dy^2) + \frac{dr^2}{U(r)} , \]  

(5)

\[ \phi = \phi(r) , \quad A = A_t(r) dt + \frac{H}{2} (xdy - ydx) \]  

(6)

We will use a convenient time coordinate \( \tau = e^{-\mathcal{W}(\infty)/2} t \) to keep the boundary metric \( \eta_{\mu\nu} \). As a result, the coordinate system of dual field theory becomes \((\tau, x, y)\) whose temporal component of the metric and gauge field is given by

\[ g_{\tau\tau}(r) = -U(r)e^{-(\mathcal{W}(r) - \mathcal{W}(\infty))} , \quad A_\tau(r) = A_t(r) e^{\mathcal{W}(\infty)/2} . \]  

(7)

For simplicity, we define \( \bar{\mathcal{W}}(r) \equiv \mathcal{W}(r) - \mathcal{W}(\infty) \). Then the Hawking temperature and entropy density are given by the following forms:

\[ T_H = \frac{U'(r_h)}{4\pi} e^{-\bar{\mathcal{W}}(r_h)/2} , \quad s = \frac{r_h^2}{4GL^2} , \]  

(8)

where \( r_h \) is the location of the horizon defined by \( U(r_h) = 0 \).
Plugging our ansatz into the gravitational action, we obtained the following reduced action:

\[ S = \frac{1}{16\pi G} \int d\tau dx dy \int dr L_{red} , \]  

where

\[ L_{red} = -\frac{\tilde{W}(r)}{L^2} \left( \frac{6r^2}{L^2} - \frac{H^2 L^4 Z(\phi)}{2r^2} - \frac{r^2 U'(r)}{2} - 2r V(\phi) - 2V'(r) - 2U(r) \right) \]

\[ + \frac{r^2 Z(\phi)}{2L^2} + \frac{e^{-\tilde{W}(r)/2} A'(r)^2}{2L^2} - 8\theta H W(\phi) A'_{\tau}(r) . \]  

Here we dropped out total derivative terms. When \( H = 0 \), the reduced action has a scaling symmetry. One can show that this reduced action without magnetic field is invariant under the following ‘scaling transformation’ up to total derivatives. We defined the scaling transformation as follows:

\[ \delta (e^{-\tilde{W}/2}) = -\lambda \left( 3e^{-\tilde{W}/2} + r (e^{-\tilde{W}/2})' \right) , \]  

\[ \delta U = -\lambda (-2 U + r U') , \]  

\[ \delta A_{\tau} = -\lambda (2 A_{\tau} + r A'_{\tau}) , \]  

\[ \delta \phi = -\lambda r \phi' , \]  

where \( \lambda \) is a small parameter. On the other hand, this symmetry can be broken by turning on the magnetic field \( H \). So we considered the invariant part of the Lagrangian. And one can show that

\[ \delta (L_{red} - L_H) = \lambda (-r (L_{red} - L_H))' , \]  

where \( L_H \) is the non-invariant part of the reduced Lagrangian. It is defined as follows:

\[ L_H = -\left( 8\theta H W(\phi) A'_{\tau} + e^{-\tilde{W}/2} \frac{H^2 L^2 Z(\phi)}{2r^2} \right) . \]  

Using the equations of motion, \( \delta L_{red} \) is given as follows:

\[ \delta L_{red} = \left( \sum_i \delta \Psi_i \frac{\partial L_{red}}{\partial \Psi_i'} \right)' , \]  

where \( \Psi_i \) denotes a collective expression of the fields such as \( \{ e^{-\tilde{W}/2}, U, A_{\tau}, \phi \} \).  

\[ \lambda \frac{d}{dr} C = \frac{1}{32\pi G} (\delta L_H + \lambda (r L_H')) = -\frac{1}{32\pi G} \delta H L_H , \]  

\[ C \]
where $C$ is defined by

$$C \equiv \frac{1}{32\pi G \lambda} \left( \sum_i \delta \Psi_i \frac{\partial L_{\text{red}}}{\partial \dot{\Psi}_i} + \lambda r L_{\text{red}} \right). \quad (19)$$

and $\delta_H$ is the complementary transformation acting on $H$ to make the $L_{\text{red}}$ invariant under a fake ‘scaling transformation’:

$$\delta_H H = 2 \lambda H. \quad (20)$$

The origin of the PCC equation [18] is nothing but the Hamiltonian constraint along the radial direction with other equations of motion. Even though PCC does not introduce a new symmetry, it is very useful to describe thermodynamics of black holes.

Now, we can easily find the form of PCC as follows:

$$C = -A_T Q + \frac{1}{32\pi G L^2} r e^{-\tilde{\mathcal{W}}/2} \left( 2r U' + r^2 U \phi'^2 - 4U \right), \quad (21)$$

where $Q$ is the charge density of the black brane, which is given by

$$Q = \frac{1}{16\pi G L^2} \left( r^2 e^{\tilde{\mathcal{W}}/2} Z(\phi) A_T'(r) - 8 \Theta W(\phi) H L^2 \right). \quad (22)$$

We may take an integration of PCC over the outer horizon region.

$$C(r) - C(r_h) = -\frac{1}{32\pi G \lambda} \int_{r_h}^r dr' \delta_H L_H(r') \quad (23)$$

where PCC at the horizon is given by $C(r_h) = sT_H$ with adopting a regularity condition $A_T(r_h) = U(r_h) = 0$.

Since the PCC is finite at the boundary of AdS space, it is legitimate to consider the PCC at $r = \infty$. Therefore, the integration over the outer region gives us the following relation:

$$sT_H = \left( C(\infty) + \frac{1}{32\pi G \lambda} \int_{r_h}^\infty dr' \delta_H L_H(r') \right). \quad (24)$$

We will identify the temperature in the dual field theory $T$ with the Hawking temperature. In addition the asymptotic value of the PCC is given by

$$C(\infty) = -\mu \tilde{Q} + \lim_{r \to \infty} \frac{1}{32\pi G L^2} r e^{-\tilde{\mathcal{W}}/2} \left( 2r U' + r^2 \phi'^2 U - 4U \right), \quad (25)$$

where $\mu \equiv \frac{1}{\sqrt{16\pi G}} A_T(\infty)$ and $\tilde{Q}$ are the chemical potential and the charge density in the dual field theory and we also identified $Q$ with $\frac{1}{\sqrt{16\pi G}} \tilde{Q}$. 
Since we are taking into account a homogeneous system, we may expect that the following relation using the thermodynamic potential density \( \omega = -P \) in (2):

\[
\epsilon + \mathcal{P} = \mu \tilde{Q} + s T_H
\]

\[
= \lim_{r \to \infty} \frac{1}{32 \pi G L^2} r e^{-\tilde{W}/2} \left( 2r U' + r^2 \phi^2 U - 4U \right) + \frac{1}{32 \pi G \lambda} \int_{r_H}^\infty dr' \delta_H L_H(r'),
\]

where \( \epsilon \) is the internal energy given by one component of the holographic tensor \( T_{\tau \tau} \). Furthermore the pressure \( \mathcal{P} \) can be decomposed into the two parts: \( \mathcal{P}_{\text{others}} + MB \), where we have used the magnetic field \( B \equiv \sqrt{L H/\sqrt{16 \pi G}} \) as a field theory quantity. Since we don’t consider sources except for the chemical potential and the magnetic field, \( \mathcal{P}_{\text{others}} \) is simply given by the holographic pressure, \( T_{xx} \). The holographic energy-momentum tensor is usually obtained from asymptotic values of fields. Therefore we may claim that the magnetization density is given by

\[
\mathcal{M} = \frac{1}{32 \pi G \lambda B} \int_{r_h}^\infty dr' \delta_H L_H(r')
\]

\[
= -L^2 \int_{r_h}^\infty dr e^{-\tilde{W}/2} \left( \frac{BZ(\phi)}{Lr^2} + \frac{64B\theta^2 W^2(\phi)}{Lr^2 Z(\phi)} + \frac{8 \theta \tilde{Q} W(\phi)}{\sqrt{Lr^2 Z(\phi)}} \right),
\]

where the charge density \( \tilde{Q} = \sqrt{16 \pi G Q} \) in the dual field theory. By using an appropriate coordinate transformation, it turns out that this expression is same with formula for the magnetization in [11, 13, 17]. In particular, the authors discussed \( \theta(\phi) F \wedge F \) term in [17], where \( \phi \) is a dilaton field which is not condensed. The authors of Ref. [17] were focusing on the DC transport and did not say anything about spontaneous magnetization because they didn’t consider scalar condensation. Nevertheless, formulas in the paper are very useful to study DC transport. We will mention a future direction related to this work in discussion section.

As we mentioned about the magnetization, one can notice that something interesting happens. Even though \( B = 0 \), the magnetization still exist if scalar field has finite expectation value. This means that \( \mathbb{Z}_2 \) symmetry of the magnetization is spontaneously broken with the help of the hairy configuration of the scalar field. The resultant spontaneous magnetization is given as follows:

\[
\mathcal{M}_f \equiv \mathcal{M}_{B=0} = -L^{3/2} \int_{r_h}^\infty dr e^{-\tilde{W}/2} \left( \frac{8 \theta \tilde{Q} W(\phi)}{r^2 Z(\phi)} \right).
\]

One can see that the sign of \( \theta \) determines that of the magnetization. Under proper conditions, the spontaneous magnetization is accompanied by a hysteresis curve which is a
trajectory in \((B, M)\)-configuration space. In order to obtain such a curve, we have to consider black brane solutions with a real scalar hair in a numerical method. In the following sections, we will discuss the spontaneous magnetization and the corresponding hysteresis curves in a particular model. Also, we will check whether our expression (27) is valid or not in the model.

2.2 A Simple Model

In this subsection we consider a specific model which is relevant to a spontaneous magnetization and a hysteresis curve. We start with adopting \(Z(\phi) = 1, V(\phi) = -\frac{2}{L^2} \phi^2\) and \(W(\phi) = -\phi^n\). As we discussed, we will consider only \(n = 1\) as a representative of odd \(W(\phi)\) cases. Together with this choice, the real scalar field is supposed to be dual to a dimension 2 operator \((\Delta = 2)\) for simplicity. Then the total action consists of the bulk part and the boundary part as follows:

\[
S_{\text{total}} = S_B + S_b ,
\]

\[
S_B = \frac{1}{16\pi G} \int_M d^4 x \sqrt{-g} L_{\text{bulk}}
\]

\[
= \frac{1}{16\pi G} \int_M d^4 x \sqrt{-g} \left\{ R + \frac{6}{L^2} - \frac{1}{4} F^2 - \frac{1}{2} (\partial \phi)^2 + \frac{1}{L^2} \phi^2 \right\}
\]

\[
- \frac{\theta}{16\pi G} \int_M d^4 x \phi^n \epsilon^{MNPQ} F_{MN} F_{PQ} ,
\]

\[
S_b = - \frac{1}{16\pi G} \int_{\partial M} d^3 x \sqrt{-\gamma} \left( 2K + \frac{4}{L} + \frac{\phi^2}{2L} \right) ,
\]

where \(K\) is the Gibbons-Hawking term which is the tace of extrinsic curvature. The other terms in the boundary action \(S_b\) are counter terms for holographic renormalization. The equations of motion for matter fields are given by

\[
\nabla_M \left( F^{MN} + 4 \theta \phi^n \frac{1}{\sqrt{-g}} \epsilon^{MNPQ} F_{PQ} \right) = 0
\]

\[
\left( \nabla^2 + \frac{2}{L^2} \right) \phi - n \theta \phi^{n-1} \frac{1}{\sqrt{-g}} \epsilon^{MNPQ} F_{MN} F_{PQ} = 0 .
\]

And the Einstein equation is

\[
R_{MN} - \frac{1}{2} g_{MN} \left( R + \frac{6}{L^2} - \frac{1}{4} F^2 - \frac{1}{2} (\partial \phi)^2 + \frac{1}{L^2} \phi^2 \right) - \frac{1}{2} F_{MP} F_N^P - \frac{1}{2} \partial_M \phi \partial_N \phi = 0 .
\]

In addition, the magnetization formula (27) becomes

\[
\mathcal{M} = -L \int_{r_h}^{\infty} dr e^{-\hat{W}/2} \left( \frac{B}{r^2} + \frac{64 \theta^2 B \phi^{2n}}{r^2} - \frac{8 \sqrt{L} \theta \hat{Q} \phi^n}{r^2} \right) ,
\]

\[\text{where } x^\mu = (x^\tau, r) = (\tau, x^i, r) = (\tau, x, y, r)\]
2.3 Magnetization with Holographic Renormalization

Now, we will show that our proposal for magnetization (27) is correct in the explicit model (29) using holographic renormalization. In order to show the validity, we first need to find the Euclidean on-shell action. We extend similar calculations for the on-shell action in [23,33]. The trace of the Einstein Equation (34) is given by

\[ R = 2 \left( \mathcal{L}_{\text{bulk}} + \frac{\theta}{\sqrt{-g}} \phi^n F_{MN} F_{PQ} \epsilon^{MNPQ} \right) + \frac{1}{2} F^2 + \frac{1}{2} (\partial \phi)^2 \]  \hspace{1cm} (36)

or

\[ 2g^{xx} G_{xx} + G^t_t + G^r_r + 2R \]
\[ = 2 \left( \mathcal{L}_{\text{bulk}} + \frac{\theta}{\sqrt{-g}} \phi^n F_{MN} F_{PQ} \epsilon^{MNPQ} \right) + \frac{1}{2} F^2 + \frac{1}{2} (\partial \phi)^2 , \]

where \( G_{MN} \) is the Einstein tensor. And one can find \( G_{xx} \),

\[ G_{xx} = \frac{r^2}{2L^2} \left( \mathcal{L}_{\text{bulk}} - R + \frac{\theta}{\sqrt{-g}} \phi^n F_{MN} F_{PQ} \epsilon^{MNPQ} \right) + \frac{L^2}{2r^2} H^2 \]  \hspace{1cm} (38)

from \( xx \)-component of the Einstein equation. Using the above three equations, the \( \mathcal{L}_{\text{bulk}} \) is given by

\[ \mathcal{L}_{\text{bulk}} = -G^t_t - G^r_r - \frac{L^4}{r^4} H^2 - \frac{\theta}{\sqrt{-g}} \phi^n F_{MN} F_{PQ} \epsilon^{MNPQ} . \]  \hspace{1cm} (39)

To get the Euclidean bulk action, we should take the wick rotation of \( \tau \). The bulk action is given by the following integration\(^3\)

\[ S_B^E = - \int_0^{1/T_H} d\tau_E \int d^2 x \int_{r_h}^\infty dr \]
\[ \left( -\frac{2r U e^{-\tilde{\gamma} / 2}}{16\pi G L^2} - BL^2 e^{-\tilde{\gamma} / 2} \left( \frac{B}{r^2} - \frac{8\theta \tilde{Q} \phi^n}{r^2} + \frac{64\theta^2 B \phi^{2n}}{r^2} \right) \right) , \]

where \( \tau_E \) is the Euclidean time and one can notice that the proposed magnetization expression (35) appears in the bulk on-shell action.

To finish the calculation of the total on-shell action, let us move on to the boundary part. The Euclidean boundary action is given by

\[ S_b^E = \lim_{r \to \infty} \frac{1}{16\pi G} \int_0^{1/T_H} d\tau_E \int d^2 x \sqrt{-\gamma} \left( 2K + \frac{4}{L} + \frac{\phi^2}{2L} \right) \]
\[ \hspace{1cm} (41) \]

\(^3\)Here \( \tau = -i\tau_E \) and \( S_B^E + S_b^E = -i(S_B + S_b) \).
\[ \frac{1}{16\pi GL^2} \frac{1}{T_H} \int d^2 x \lim_{r \to \infty} r e^{-\tilde{W}/2} \left( r\sqrt{U} \left( \frac{4}{L} + \frac{\phi^2}{2L} \right) - r U' - U(4 - r\tilde{W}') \right) . \]  

(42)

By plugging the asymptotic behavior of the fields in Appendix, finally we obtained the on-shell action as follows:

\[ S^E_B + S^E_b = \frac{\nu}{T_H} \omega = -\frac{\nu}{T_H} \left( \frac{\epsilon}{2} + M B \right) , \]

(43)

where \( M \) is same with (35). Also, one can notice that \( \epsilon/2 \) is equivalent to the internal pressure \( \epsilon/2 = T_{xx} = 1/2 T_{\tau\tau} \) which is derived in Appendix. This relation supports the fact that the proposed form of magnetization (27) is correct.

### 3 Spontaneous Magnetization and Hysteresis Curve

In this section we analyze the spontaneous magnetization and hysteresis curve in the explicit model (29) by numerical methods. For numerical calculations, we will take \( 16\pi G = 1, L = 1 \) and \( r_h = 1. \) In contrast to usual discussions in holographic models, solutions with larger free energy are also important in our discussion. We will explain the role of such unstable solutions in more details in the following subsections.

#### 3.1 Spontaneous Magnetization

We concentrate on the case with vanishing magnetic field in this subsection. Then the equations of motion are reduced to those without the magnetic field. In high temperature, the AdS Reisner-Nördstrom(RN) Black brane is the preferable solution dual to normal phase. On the contrary, the RN black brane becomes unstable and changes to a hairy black hole dual to an ordered phase in low temperature. The hairy scalar describes the VEV of a dimension 2 operator \( \mathcal{O} \). As we discussed, this hairy scalar contributes to the magnetization (35) even in the absence of the magnetic field. Therefore this holographic model can describe a phase transition from a diamagnetism phase to a ferromagnetism phase. Here the sign of the magnetization is determined by the sign of \( \theta \) and we have to fix it with a suitable value. For an odd \( n \), both signs of magnetization are possible with a given \( \theta \). On the other hand the sign of magnetization is fixed for an even \( n \). In the latter case the direction of magnetization is always same for both positive and negative scalar condensations. This is the reason why we concentrate on the \( n = 1 \) case because we need the both signs of magnetization to construct magnetic hysteresis.

Also, if we allow only solutions without source of scalar condensation, i.e, \( J_\mathcal{O} = 0 \) in (55), an instability of the RN black brane happens when \( T_H = T_c \). Figure 1(a) shows a set
Figure 1: (a) The set of black brane solutions satisfying zero source condition (b) The difference of free energy of RN black branes(red) and hair black branes(blue). (c) Temperature dependence of the absolute value of the condensation. (d) Temperature dependence of the absolute value of the magnetization, where we took $\theta = -0.2$ for numerical calculation.

of black brane solutions satisfying the sourceless condition. Above this critical temperature, the only solution is that of zero condensation which is nothing but the RN black brane. On the other hand, two more kinds of hairy black brane solutions show up below $T_c$(solid curve connecting C and D). They describe positive and negative values of $\langle O \rangle$ naturally coming from the $\mathbb{Z}_2 (\phi \leftrightarrow -\phi)$ symmetry in the action (29).

Figure 1 (b) presents free energy difference between RN black brane solutions and the hairy ones. Red solid line denotes the free energy of the RN black branes as a reference and blue line corresponds to the free energy of the hairy black branes. Below $T_c$, the free energy of the hairy solutions is lower than RN solutions. Therefore there is a second order phase transition from RN to hairy black brane solution at $T = T_c$.

We also plot the condensation diagram in Figure 1 (c). Since the integrand of magnetization density is proportional to $\phi$ near critical temperature, the magnetization shows similar behavior as shown in Figure 1 (d). In addition, the charge density is essential to produce the magnetization. Thus it turns out that the charge carrier also carries magnetic
moment and there exists an interaction between the charge operator $\tilde{Q}$ and the dimension 2 operator $\mathcal{O}$ in the dual field theory.

The scalar condensation comes from the spontaneous symmetry breaking. In the dual gravity picture, this corresponds to advent of normalizable hairy black brane solutions. The holographic superconductor model admits a complex scalar hair which is associated with spontaneous breaking of continuous $U(1)$ symmetry [22]. For a real scalar condensation and a real scalar hairy black brane, the associated symmetry is now $\mathbb{Z}_2$ symmetry. This phase transition is usually explained by a Landau-Ginzburg type theory whose effective field is a real scalar field. We denote this field by $\Phi$. Therefore, the potential of the Landau-Ginzburg model governs this phase transition. The typical shape of potential for the broken phase should also admit $\mathbb{Z}_2$ symmetry so it has two global minima. At these minima, the field has values, $\Phi = \pm \Phi_0$ and free energy of the system is given by the on-shell action of hairy black branes.

Now, we construct the Landau-Ginzburg type effective potentials using the data in Figure 1. One can naturally deduce that the free energy density of the RN black brane solution, such as B in Figure 1, corresponds to the unstable extremum due to $\mathbb{Z}_2$ symmetry and vanishing condensation. This is quite reasonable because the RN black branes are solutions of the dual gravity system and the extrema of the potentials are also solutions of the corresponding effective theory. In addition to this observation, we may assume that the condensation $\langle \mathcal{O} \rangle$ and the magnetization $\mathcal{M}$ are very small near the critical temperature. Then the effective potential of free energy can be approximated by a polynomial of $\mathcal{O}$ or $\mathcal{M}$. The suggested form of the potential is given as follows

$$V(\Phi) = \omega_{RN} + \frac{\omega_{RN} - \omega_{HB}}{\Phi_0^4} \Phi^2 (\Phi^2 - 2\Phi_0^2) + \mathcal{O} (\Phi^6), \quad (44)$$

where $\Phi$ is the effective field which in our case denotes the magnetization $\mathcal{M}$ or $\mathcal{O}$ and $\Phi_0$ denotes $\mathcal{M}$ or $\langle \mathcal{O} \rangle$ of the hairy black branes, respectively. In addition, $\omega_{RN(HB)}$ is the free energy density of the RN black brane (Hairy black brane). Here, we interestingly point out that the parameters of the Landau-Ginzburg potential can be obtained by the dual geometry data. We plot the effective potentials in Figure 2. The figure shows how the potential changes as the temperature decreases.

In addition, we may consider the fluctuation of $\Phi$ near the local minima. This fluctuation corresponds to an effective quanta in the broken phase. In this phase, the mass of quasi-particles governs major characteristic of the effective theory. We can easily obtain the mass of the quasi particle as follows:

$$m_{eff}^2 = 8 \frac{\omega_{RN} - \omega_{HB}}{\Phi_0^2}. \quad (45)$$
Then, the low energy effective potential for the broken phase up to quartic order is as follows:

\[ V_{BP} \sim \frac{1}{2} \omega_{HB} + \frac{1}{2} m_{\text{eff}}^2 \delta \Phi^2 + \frac{\lambda_3}{3!} \delta \Phi^3 + \frac{\lambda_4}{4!} \delta \Phi^4 , \]  \hspace{1cm} (46)

where \( \delta \Phi \) is the fluctuation near \( \Phi = \Phi_0 \) and

\[ \lambda_3 = 4! \frac{\omega_{RN} - \omega_{HB}}{\Phi_0^3} \quad \text{and} \quad \lambda_4 = 4! \frac{\omega_{RN} - \omega_{HB}}{\Phi_0^4}. \]  \hspace{1cm} (47)

These couplings describe the cubic and quartic interactions among the quasi particles. Of course, all the mass and couplings are also determined from data of the bulk geometry.

When \( \Phi = M \), \( \Phi \) describes the fluctuation of the magnetization. This quasi-particle can be interpreted as the magnon whose mass and self-couplings are given by

\[ m^2_{\text{magnon}} = 8 \frac{\omega_{RN} - \omega_{HB}}{(\langle M \rangle)^2}, \lambda_3 = 4! \frac{\omega_{RN} - \omega_{HB}}{(\langle M \rangle)^3}, \lambda_4 = 4! \frac{\omega_{RN} - \omega_{HB}}{(\langle M \rangle)^4}, \]  \hspace{1cm} (48)

where \( \langle M \rangle \) is obtained by the black brane corresponding D in Figure 2. For more clear understanding, we provide cartoons in Figure 3. In the figure, the length and direction of arrows depict the magnitude and the sign of the magnetization. Figure 3 (a) shows a fluctuation in the broken phase D in Figure 1 (a). This excitation plays a qualitatively same role of a magnon in condensed matter physics. In the figure, the convex part corresponds to a magnon and the concave part describes an anti-magnon.

One more interesting feature of these potentials is that the free energy potential in Figure 2 implies existence of a kink or anti-kink solution connecting two vacua D and C, which is depicted in (b) of Figure 3. In more general cases, the magnetization can have locally opposite sign in 2-dimensional space. See Figure 3 (c) for such a configuration which describe the fragmentation of the magnetization. This plays an important role in the hysteresis curve. To realize these complicated backgrounds, we have to solve partial
differential equations, which is left as a future study. Later, we will give a comment on the role of these configurations in hysteresis curves.

Figure 3: **Cartoons for various excitations in broken phase**: (a) shows a magnon (convex) and an anti-magnon (concave) and (b) depicts a magnetization kink configuration. In addition, (c) is an intersection of a locally opposite sign configuration. This kind of configurations can describe a fragmentation of magnetization.

### 3.2 Hysteresis Curve

In the previous subsections, we showed that a holographic model can describe the spontaneous magnetization and the hairy black brane solution is dual to a ferromagnetic phase. As we mentioned in the introduction, one of important properties of the ferromagnetism is the hysteresis curve. We now show how to construct the hysteresis curves. For this purpose, we first find the hairy black hole solutions with the non-vanishing magnetic field using a numerical method. We impose the boundary condition $J_0 = 0$ for the scalar field $\phi$ in (55) to simplify the problem. Using the solutions below the critical temperature, we display the relation of external magnetic field and magnetization in Figure 4: there exist three solutions for small magnetic field. The solutions on (D-O-E) line have higher free energy than those on (B-C_1-D) and (E-C_2-F) lines at the same magnetic field. So the former represent unstable extrema of the free energy potential, while the other two describe the stable and metastable extrema of the potential. If the black brane solution changes following the local minimum of the potential under slowly varying magnetic field, such process can give rise to magnetic hysteresis. See Figure 5.

Let us describe how the effective potential changes in more detail. We start with A in Figure 4 and increase the external magnetic field slowly. Until the magnetic field reaches B, there is a single minimum and it should be located at the global minimum of the effective potential, see Figure 5(a). When the magnetic field becomes zero, there are three solutions. Two of them are at the minima and the other one is at the local maximum (RN black brane), see Figure 5(b). Further variation of the magnetic field gives rise to an asymmetric potential. If the thermal fluctuation and the changing rate of the magnetic field are very small with
Figure 4: External magnetic field dependence of the magnetization.

Figure 5: Magnetization dependence of the effective potential for given magnetic field corresponding to Figure 4. The potentials (a) (b) (c) and (d) are given up to $\mathcal{M}^6$-order.

in infinitely large volume, then the system can be placed on a metastable point in the potential. See Figure 5 (c). When the magnetic field reaches D in Figure 4, the local minimum where
the solution is located and the local maximum coincides each other, Figure 5(d). Then, the solution cannot stay there and falls to the local minimum F which stands sudden jump of the magnetization and will stay there as the magnetic field increases further. If we start from large magnetic field and decrease it, then the process will be reversed. Our result describing magnetic hysteresis is reminiscent of a bistable model accompanied by the Barkhausen effect in condensed matter physics. See [34] for a text book.

While generating the hysteresis curve, we need to pay attention to several things. First, the time-variation of magnetic field should be slow to prevent a phase transition from the local minimum to the global minimum. Otherwise, the time variation $\dot{B}$ increases $\dot{M}$ and the effective kinetic energy of $M$. If the kinetic energy is large enough, then the system jumps over the potential barrier. To prevent this situation we assume that the changing rate of magnetic field is very small. Second, we need to care the thermal fluctuation of the system. In high temperature the thermal fluctuation becomes large, however we are investigating a broken phase from the real scalar hair. It is well known that real scalar hairs appear below very low temperature and so is our case.$^4$ In fact, such an effect changes hysteresis curves in Figure 6 slightly, but it is not significant, so we ignore that effect in the present work. Finally, there exist many nontrivial configurations such as (b) and (c) in Figure 3. These configurations contribute to the fragmentation of magnetization. The actually measured hysteresis curves have shapes that change less dramatically than (a) in Figure 6. The fragmentation must be considered to produce milder shapes. All the above issues are very interesting and important, but they are not the main topic of the present work. So we will investigate these issues in a future study.$^{38}$

$^4$In our numerical calculation, the black brane solutions have $T_c = 3.97 \times 10^{-4}$ in the unit of $L = 1$ and $16\pi G = 1$.

Figure 6: (a) Hysteresis curve of the magnetization. (b) Hysteresis curve of the charge density.
Therefore, the resulting process of the magnetization hysteresis is drawn in Figure 6 (a). Figure 6 (b) shows hysteresis behavior of the charge density for (a). The charge carrier density is decreasing for both direction and suddenly increase when the magnetization changes. In this paper, we do not consider the time dependence of the falling process, so the magnetization is suddenly dropped at the transition point. In fact, this dynamics should be considered with time dependent background and it is actually governed by a non-equilibrium physics.\footnote{An example using holography was discussed for the chiral phase transition in \cite{29}. See Figure 2 in that paper.}

In Figure 6, non-zero values of the magnetization cause the Hall effect and it gives nontrivial Hall conductivity ($\sigma_{xy}$) or Hall resistance ($\rho_{xy}$). We expect that there also be hysteresis behavior to the Hall resistance. On the other hand, the charge carrier density $Q$ would be proportional to the longitudinal conductivity $\sigma_{xx}$. The results are qualitatively same to the transport on the surface state of the topological insulator. See Figure 3 and 4 in \cite{35}. In addition, Figure 7 shows temperature dependence of the hysteresis curve. As temperature decreases, the size of hysteresis curve increases. It is natural that the expectation value of the scalar is bigger at lower temperature and so is the magnetization.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{Temperature dependence of the hysteresis curve with $T/T_C = 0.98$ (red), 0.96 (blue), 0.93 (black).}
\end{figure}

\section{Discussion}

In this paper, we have studied the magnetic properties of the strongly interacting system using gauge/gravity duality. To do this, we first defined the magnetization of the boundary theory in terms of the bulk data using the ‘scaling symmetry’ trick. Then, we applied it to a specific model and confirmed that the formula agreed with the magnetization obtained from the holographic renormalization and satisfied thermodynamics laws.

In our model, the magnetization comes from the condensation of the scalar field due to the spontaneous symmetry (discrete $\mathbb{Z}_2$) breaking. By introducing a coupling between
magnetic field and the real scalar, we showed that a spontaneous magnetization occurs in our model. Using these gravity backgrounds admitted in our model, we constructed the Landau-Ginzburg type potential in terms of bulk data just below the critical temperature. In the case without a magnetic field, we provided the closed form of the potential (44) in terms of the on-shell actions and the expectation values which can be easily obtained by a simple numerical calculation. This formula gives us a systematic way to relate the black hole physics to the Landau-Ginzburg type effective theory. As a by-product, We obtained magnon mass and its self-couplings from this effective potential.

Also, we analyzed the field theory systems with the magnetic field dual to the magnetic hairy black branes. Using the black brane solutions, we found the corresponding Landau-Ginzburg potentials under varying the magnetic field. This fact tells us that the gravity theory seems to be a kind of Landau-Ginzburg model. Since it is difficult to quantize Landau-Ginzberg Models in a generic situation, one can see the reason why quantization of the gravity theory is so hard without any dual description. Our result provide another direct explanation for difficulty of quantum gravity.

The effective potential implies that the magnetization hysteresis can be generated by slowly varying magnetic field. In this hysteresis process, the time scale of the variation and the thermal fluctuations are very important. So we assume very slow change of the system and we argued that thermal fluctuations are very small due to the low temperature. More detailed discussion would be interesting as a future direction. In addition to the issues, our resulting hysteresis has rapidly varying shapes and the fragmentation can make the shapes more smoothly. However, we did not consider the fragmentation of magnetization because the corresponding black brane solution is not easy to obtain. Thus we leave it as another future study.

As a final comment, it is expected that the magnetization and the charge carrier density are proportional to the Hall resistivity and longitudinal conductivity, respectively. Therefore, there exists hysteresis behavior of these quantities [35–37]. A study on this subject will be reported soon [38].

Appendix

A. Holographic Renormalization

In this section we provide expression of boundary tensors such as energy-momentum tensor, current and condensation of scalar operator dual to bulk scalar field $\phi$ in the subsection 2.2. In order to get these formulas using holographic renormalization, we introduce ADM
decomposition without the shift vector. Thus the metric (5) can be decomposed as follows:

\[ ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu + N^2 dr^2, \]  

(49)

where \( \gamma_{\mu\nu} = \text{diag} \left( -U(r) e^{\tilde{W}(r)}, \frac{r^2}{L^2}, \frac{r^2}{L^2} \right) \) and \( N = 1/\sqrt{U(r)} \). Then the extrinsic curvature tensor and the Gibbons-Hawking term \( K \) are given by \( K_{\mu\nu} = -\frac{1}{2N} \gamma'_{\mu\nu} \) and \( \gamma_{\mu\nu} K_{\mu\nu} \), respectively. A boundary tensor is usually given by variations of the total action \( S_B + S_b \) with respect to corresponding sources. In this model, they are given by

\[
\langle T_{\mu\nu} \rangle = \frac{1}{16\pi G} \lim_{r \to \infty} 2 \frac{r}{L} \left[ K_{\mu\nu} - K \gamma_{\mu\nu} - \left( \frac{\phi^2}{4L} + 2 \right) \gamma_{\mu\nu} \right] \]  

(50)

\[
\langle J^\mu \rangle = \frac{-1}{\sqrt{16\pi G}} \lim_{r \to \infty} \sqrt{\gamma} F^{r\mu} \]  

(51)

\[
\langle O \rangle = \frac{L}{\sqrt{16\pi G}} \lim_{r \to \infty} \frac{1}{r} \left( -\sqrt{\gamma} \nabla^r \phi - \sqrt{-\gamma} \phi/L \right) \]  

(52)

On the other hand, the asymptotic behavior of the bulk geometry is given by

\[
\tilde{W}(r) \sim 4\pi G \left( J_O^2 \left( \frac{L}{r} \right)^2 + \frac{8L J_O \tilde{O}}{3} \left( \frac{L}{r} \right)^3 + \ldots \right) \]  

(53)

\[
U(r) \sim \left( \frac{r}{L} \right)^2 + 4\pi G \left( J_O^2 - 2L \epsilon \left( \frac{L}{r} \right) + \ldots \right) \]  

(54)

\[
\phi(r) \sim \sqrt{16\pi G} \left( J_O \left( \frac{L}{r} \right) + L \tilde{O} \left( \frac{L}{r} \right)^2 + \ldots \right), \]  

(55)

\[
A_r(r) \sim \sqrt{16\pi G} \left( \mu - L \tilde{Q} \left( \frac{L}{r} \right) + \ldots \right), \]  

(56)

where we are considering \( J_O = 0 \) case since we don’t want to introduce another source except for the chemical potential and the magnetic field\(^6\). Then, we can obtain the boundary tensors as follows:

\[
\langle T_{\mu\nu} \rangle = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon/2 & 0 \\ 0 & 0 & \epsilon/2 \end{pmatrix}, \quad \langle J^\mu \rangle = (\tilde{Q}, 0, 0), \quad \langle O \rangle = \tilde{O}. \]  

(57)

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\(^6\) One can easily extend the case with a non-vanishing \( J_O \).
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