The Moyal bracket and the dispersionless
limit of the KP hierarchy

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Abstract

A new Lax equation is introduced for the KP hierarchy which avoids the use of pseudo-differential operators, as used in the Sato approach. This Lax equation is closer to that used in the study of the dispersionless KP hierarchy, and is obtained by replacing the Poisson bracket with the Moyal bracket. The dispersionless limit, under which the Moyal bracket collapses to the Poisson bracket, is particularly simple.

1. Introduction

One of the simplest nonlinear equations that can be completely be solved, albeit implicitly, is

\[ 4U_T - 12UU_X = 0, \]

the solution to which can be obtained using the method of characteristics. This equation can be described in two ways, either as the dispersionless KdV equation (i.e. the KdV equation without the dispersion \( U_{XXX} \) term) or as the simplest example of an equation of hydrodynamic type. This connection between dispersionless and hydrodynamic equations persists into the theory of \((2 + 1)\)-dimensional systems. The dispersionless KP equation (hereafter referred to as the dKP equation)

\[ (4U_t - 12UU_X)_X = 3U_{YY} \]

may be obtained from the KP equation itself

\[ (4u_t - 12u u_x - u_{xxx})_x = 3u_{yy} \]

via the scaling transformation

\[ \begin{align*}
X &= \epsilon x, \\
Y &= \epsilon y, \\
T &= \epsilon t, \\
U(X,Y,T) &= u(x,y,t)
\end{align*} \quad (1) \]

in the limit as \( \epsilon \to 0 \). The dKP may be reduced to the hydrodynamic-type equation

\[ \frac{\partial w}{\partial T} = A(w) \frac{\partial w}{\partial X}, \]

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where $A$ is an $N \times N$ matrix and $w$ is an $N$-component column vector ($N$, which characterises this reduction, is an arbitrary integer). Such a reduction enables one to construct solutions, and study the properties of, the dKP equation by using existing results on the theory of equation of hydrodynamic type [1].

The dKP and KP equations are important examples of $(2 + 1)$-dimensional integrable systems, both having associated Lax equation. For the KP equation (or more generally, for the KP hierarchy) the Lax equation is [2]

$$\frac{\partial L}{\partial t_n} = \left[(L^n)_+, L\right], \quad (2)$$

where $\partial = \frac{\partial}{\partial x}$,

$$L = \partial + \sum_{n=2}^{\infty} v_n(x, t_2, t_3, \ldots) \partial^{-n+1}$$

and $\Lambda_+$ denotes the projection onto the differential operator part of the pseudo-differential operator $\Lambda$. The bracket $[A, B]$ is just the commutator of the differential operators, i.e. $[A, B] = AB - BA$.

The Lax equation for the dKP hierarchy is somewhat different, as it does not involve the use of pseudo-differential operators. The Lax equation is [1]

$$\frac{\partial L}{\partial t_n} = \left\{(L^n)_+, L\right\}, \quad (3)$$

where

$$L = \lambda + \sum_{n=2}^{\infty} u_n(x, t_2, t_3, \ldots) \lambda^{-n+1}$$

and $\Omega_+$ denotes the projection onto positive (and zero) powers of $\lambda$ in the Laurent series $\Omega$. The bracket $\{f, g\}$ is just the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \lambda} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial \lambda}, \quad (4)$$

One interesting point to notice is that, although the scaling transformation takes one from the KP equation to the dKP equation, if one applies it to the Lax equation for the KP equation one does not obtain the Lax equation for the dKP equation, at least not in any naive way. This may be summarised as the failure of the following diagram to commute:
The aim of this letter is to introduce a new Lax equation for the KP hierarchy so that the above diagram does commute. In fact one will obtain the KP equation in the form

\[(4u_t - 12uu_x - 4\kappa^2 u_{xxx})_x = 3u_{yy}\]

and so the dispersionless limit corresponds to \(\kappa \to 0\). This then avoids the scaling transformation. This will be achieved by replacing the Poisson bracket in (3) by the Moyal bracket, and the dispersionless limit is the limit in which the Moyal bracket collapses to the Poisson bracket.

2. The Moyal Bracket

The Moyal bracket [3] may be thought of as a deformation of the Poisson bracket by the introduction of higher order derivative terms. It turns out that the Jacobi identity is highly restrictive as to the nature of these terms, and one is lead uniquely [4] to the Moyal bracket:

\[
\{f, g\}_\kappa = \sum_{s=0}^{\infty} \frac{\kappa^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} \left( \partial_x^j \partial_\lambda^{2s+1-j} f \partial_x^j \partial_\lambda g \right).
\]

It has all the standard properties one would expect of such a bracket:

(a) \(\{f, g\}_\kappa = -\{g, f\}_\kappa\), antisymmetry,
(b) \(a\{f, g\}_\kappa = a\{f, h\}_\kappa + b\{g, h\}_\kappa\), linearity,
(c) \(\{f, \{g, h\}_\kappa\} + \{g, \{h, f\}_\kappa\} + \{h, \{f, g\}_\kappa\} = 0\), Jacobi identity

(where \(a, b\) are independent of \(x\) and \(\lambda\)). Moreover it has the important property that

\[
\lim_{\kappa \to 0} \{f, g\}_\kappa = \{f, g\},
\]

i.e. in the limit as \(\kappa \to 0\) the bracket collapses to the Poisson bracket (4). It also has many other interesting properties [5], amongst which is the fact that it may be written in terms of an associative \(\star\)-product defined by

\[
f \star g = \sum_{s=0}^{\infty} \frac{\kappa^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} \left( \partial_x^j \partial_\lambda^{s-j} f \partial_x^j \partial_\lambda g \right).
\]

This has the property that

\[
\lim_{\kappa \to 0} f \star g = fg,
\]

and with this the Moyal bracket takes the form:

\[
\{f, g\}_\kappa = \frac{f \star g - g \star f}{2\kappa}.
\]

The hierarchy to be considered here is obtained by replacing the Poisson bracket in (3) by the Moyal bracket:
\[
\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}_\kappa , \tag{8}
\]

where

\[
\mathcal{B}_n = (\mathcal{L} \star \ldots \star \mathcal{L})_+^{n-\text{terms}}
\]

and \(\mathcal{L}\) remains unchanged. This is equivalent to the zero-curvature relations

\[
\frac{\partial \mathcal{B}_n}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial t_n} + \{\mathcal{B}_n, \mathcal{B}_m\}_\kappa = 0 . \tag{9}
\]

Since the \(\mathcal{B}_i\) are all polynomial in \(\lambda\) the Moyal bracket will automatically truncate after a finite number of terms, and so one obtains a well defined set of evolution equation for the independent variables. These equations differ from those of the dKP hierarchy by a finite number of \(\kappa\)-dependent terms. From (8) and (9) one may prove a number of general properties of the hierarchy. For example, using equations (6) and (7), as \(\kappa \to 0\) the hierarchy reduces to the dKP hierarchy.

**Example**

With \(n = 2\) and \(n = 3\) equation (8) yields

\[
\begin{align*}
\mathcal{B}_2 &= \lambda^2 + 2u_2 , \\
\mathcal{B}_3 &= \lambda^3 + 3\lambda u_2 + 3u_3
\end{align*}
\]

(\(\kappa\)-dependent terms only appear in \(\mathcal{B}_n\) for \(n > 3\)), and one obtains from (9):

\[
\begin{align*}
-3u_{2,t_2} + 6u_{3,x} &= 0 , \\
2u_{2,t_3} - 6u_{2,x} - 2\kappa^2 u_{2,xxx} - 3u_{3,t_2} &= 0 .
\end{align*}
\]

On eliminating \(u_3\) one obtains a single equation for \(u_2\):

\[
(4u_{2,t_3} - 12u_{2,x} - 4\kappa^2 u_{2,xxx})_x = 3u_{2,t_2} ,
\]

the KP equation itself. For this to agree with the KP equation obtained from (2) one has to set \(\kappa^2 = \frac{1}{4}\). Further, as \(\kappa \to 0\) one obtains the dKP directly without the need of the scaling transformation (1).

It therefore seems plausible that (8) is the KP hierarchy, and this has been proved for the first four members of the hierarchy by direct calculation. However, a direct proof that (8) is equivalent to the hierarchy given by the Sato approach is lacking at present, though it does seem highly unlikely that in addition to having the same dispersionless limit as the KP hierarchy, it should agree with the KP hierarchy for the first four terms and then, after that, diverge. An additional problem is that the functions \(v_n\) and \(u_n\) appearing in \(L\) and \(\mathcal{L}\) are not identical, but are related by simple relations, the first few being (with \(\kappa^2 = \frac{1}{4}\))
\begin{align*}
u_2 &= v_2, \\
u_3 &= v_3 + \frac{1}{2} v_{2,x}, \\
u_4 &= v_4 + v_{3,x} + \frac{1}{4} v_{2,xx}.
\end{align*}

It is therefore conjectured that this Moyal-KP hierarchy (with $\kappa^2 = \frac{1}{4}$) is the same as the KP hierarchy given by equation (2). Until this is proved, equation (8) will be called the Moyal-KP hierarchy. A similar approach was studied in [7], however the formalism used there is slightly different and the $\kappa \to 0$ limit does not yield the dKP equation directly, not without a scaling transformation, the avoidance of which was one of the motivations of this section.

3. The reduction to the KdV hierarchy

The KdV hierarchy may be obtained by imposing the constraint $\mathcal{L} \star \mathcal{L} = B_2$, and the evolution of $u_2$ is given by

\[ \frac{\partial B_2}{\partial t_{2n+1}} = \{B_{2n+1}, B_2\}_\kappa, \]

all functions being independent of the even time variables. The first couple of equations are given below, to show how the terms depend on the parameter $\kappa$:

\begin{align*}
u_{2,t_3} &= \kappa^2 v_{2,xxx} + 3 v_{2,x}, \\
u_{2,t_5} &= 10 \kappa^4 v_{2,xxxxx} + 5 \kappa^2 v_{2,xxx} + 10 \kappa^2 v_{2,x} v_{2,xx} + \frac{15}{2} \kappa^2 v_{2,xx}.
\end{align*}

In the limit as $\kappa \to 0$ one obtains the dispersionless KdV hierarchy. Other $(1 + 1)$-dimensional hierarchies may be obtain by imposing the appropriate constraints, as in the standard Sato theory. Once again, a direct proof that this is equivalent to the KdV hierarchy obtains using differential operators, as in the Sato approach, is lacking.

4. The geometry of the Moyal-KP hierarchy

A more geometrical way to describe the dKP hierarchy, equivalent to the Lax equation (3), is to introduce a 2-form\footnote{In this section it will be notationally convenient to set $x = t_1$.}

\[ \omega(\lambda) = \sum_{n=1}^{\infty} dB_n \wedge dt_n, \]

where $B_n = (\mathcal{L}^n)^+_\kappa$, i.e. the $\kappa \to 0$ limit of the $B_n$. The dKP hierarchy then becomes the following conditions on the 2-form $\omega$:

\[ \omega(\lambda) \wedge \omega(\lambda) = 0, \]
\[ d\omega(\lambda) = 0. \]
These equations imply, by Frobenious's theorem, the local existence of functions \( P(\lambda) \) and \( Q(\lambda) \) such that \( \omega = dP \wedge dQ \). In fact, one such pair of functions is given by

\[
P(\lambda) = \mathcal{L}, \\
Q(\lambda) = \sum_{n=1}^{\infty} nt_n \mathcal{L}^{n-1}, \\
def \mathcal{M}(\lambda),
\]

and hence:

**Proposition [7]**

The dispersionless KP hierarchy is governed by the exterior differential equation

\[
\omega = d\mathcal{L} \wedge d\mathcal{M},
\]

with

\[
\{\mathcal{L}, \mathcal{M}\} = 1.
\]

To discuss the geometry of the KP hierarchy it is first convenient to redefine the Moyal bracket

\[
\{f, g\}_\kappa = f \star g - g \star f.
\]

This amounts to rescaling the time variables, so now the limit \( \kappa \to 0 \) is singular. The basic definitions of \( \mathcal{L} \) and \( B_n \) remains unchanged. In Sato theory the pseudo-differential operator

\[
W = 1 + \sum_{n=1}^{\infty} w_i \partial^{-n}
\]

plays a more fundamental role than the Lax operator \( L \), and the evolution of \( W \) is governed by

\[
\frac{\partial W}{\partial t_n} = B_n W - W \partial^n,
\]

where \( L = W \partial W^{-1} \) and \( B_n = (L^n)_+ \). From these equations it is straightforward to derive the Lax equation and the zero curvature relations. For the Moyal version of the KP hierarchy one may similarly define a function

\[
\mathcal{W} = 1 + \sum_{n=1}^{\infty} w_n \lambda^n
\]

governed by

\[
\frac{\partial \mathcal{W}}{\partial t_n} = \mathcal{B}_n \star \mathcal{W} - \mathcal{W} \star \lambda^n.
\]
The Lax functions is then $L = W \star \lambda \star W^{-1}$ (where $W^{-1}$ is defined uniquely by the relations $W \star W^{-1} = W^{-1} \star W = 1$), and this satisfies the Lax equation (8).

Recently, another pseudo-differential operator was introduced in [8],

\[ M = W \left( \sum_{n=1}^{\infty} nt_n \partial^{-1} \right) W^{-1}. \]

this satisfying the equations

\[
\frac{\partial M}{\partial t_n} = [B_n, M], \\
[L, M] = 1.
\]

With this one may study the symmetries and other properties of the KP hierarchy in terms of the infinite dimensional Grassmannian manifold. Similarly, there is a Moyal version of this operator:

\[ \mathcal{M} = W \star \left( \sum_{n=1}^{\infty} nt_n \lambda^{n-1} \right) \star W^{-1}. \]

This satisfies the relations

\[
\frac{\partial \mathcal{M}}{\partial t_n} = \{B_n, \mathcal{M}\}_{\kappa}, \\
\{L, \mathcal{M}\}_{\kappa} = 1.
\]

This suggests that one may characterise solutions of the Moyal KP hierarchy in terms of a Riemann-Hilbert problem in the Moyal loop group.

Another multidimensional integrable system that admits such a description is the anti-self-dual Einstein equation [9]. These describe a complex 4-metric with vanishing Ricci and anti-self-dual Weyl tensors. The metric may be written in terms of a single function $\Omega$, the Kähler potential, which is governed by the equation

\[ \Omega_{,xx} \Omega_{,yy} - \Omega_{,xy} \Omega_{,yx} = 1, \]

or, using the Poisson bracket (4) (with respect to $\tilde{x}$ and $\tilde{y}$ variables):

\[ \{\Omega_{,x}, \Omega_{,y}\} = 1. \]  

(11)

This equation (Plebanski’s first heavenly equation [10]) may also be written in the form (10), with

\[ \omega(\lambda) = dx \wedge dy + \lambda(\Omega_{,xx} dx \wedge d\tilde{x} + \Omega_{,xy} dx \wedge d\tilde{y} + \Omega_{,yy} dy \wedge d\tilde{x} + \Omega_{,yx} dy \wedge d\tilde{y}) + \lambda^2 d\tilde{x} \wedge d\tilde{y} \]

and the additional constraint $d\lambda = 0$. Once again one may show the existence of function $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ with $\omega = d\mathcal{P} \wedge d\mathcal{Q}$ and $\{\mathcal{P}(\lambda), \mathcal{Q}(\lambda)\} = 1$, connected by Riemann-Hilbert problems. A Moyal algebraic deformation of (11), obtained by replacing the Poisson bracket by the Moyal bracket was introduced in [11], and has been studied further by Takasaki [12] and Castro [13], the former showing that it may be described in terms of a Riemann-Hilbert problem in the corresponding Moyal loop group.
5. Comments

The Moyal bracket was first introduced in an attempt to reformulate quantum mechanics in terms of a distribution $f$ on phase space. From the equation

$$\frac{\partial f}{\partial t} = \{f, H\}_\kappa$$

(together with an auxiliary equations for $f$) one can derive a wavefunction satisfying the Schrödinger equation. Note that the use of commutator relations is avoided. The theory outlined in this paper is somewhat analogous; the use of differential operator and commutator relations is replaced by the use of the Moyal bracket. Perhaps these ideas may be useful in a proof of the conjecture that the hierarchy (8) is the KP hierarchy.

It therefore seems likely that the entire theory of the KP hierarchy may be reformulated in terms of the Moyal bracket and $\star$-products, thus totally avoiding the use of pseudo-differential operators. One attraction of this approach is that it is closer in spirit to the formulation of other multidimensional integrable systems such as the anti-self-dual vacuum equations. For this equation the Riemann-Hilbert problem may be used to define an associated 3-dimensional complex manifold known as twistor space. Such a twistorial description of the KP equation has been sort for many years, after the conjecture of Ward [14] that all classical integrable systems should admit such a description. The failure to find this has lead to the suggestion [15] that a more general version to twistor theory is needed to encompass systems such as the KP equation. The use of the Moyal bracket in the study of the KP hierarchy, as outlined in this paper, suggests that what might be required is some sort of $\kappa$-deformed twistor space which would make use of the Moyal bracket, rather than the Poisson bracket as used in conventional twistor theory. This, however, remains pure speculation.

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