Asymptotics of the counting function of $k$-th power-free elements in an arithmetic semigroup

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Abstract. For any $k \geq 2$, we find the asymptotics of the counting function of $k$-th power-free elements in an additive arithmetic semigroup with exponential growth of the abstract prime counting function. This paper continues the authors’ earlier research dealing with the case of $k = \infty$.

1. Introduction and statement of the problem

Let $G$ be an additive arithmetical semigroup \cite[pp. 11, 56]{1}; i.e.,
(i) $G$ is a commutative semigroup with identity element 1.
(ii) There exists a (uniquely determined) countable subset $P \subset G$ (whose elements are called the primes of $G$) such that every element $a \in G$, $a \neq 1$, has a factorization of the form
\begin{equation}
    a = p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s},
\end{equation}
with some positive integers $s, n_1, \ldots, n_s$ and elements $p_1, \ldots, p_s \in P$, and this factorization is unique up to the order of factors.
(iii) A mapping $\partial: G \to \mathbb{R}$ (called the degree mapping) is given such that $\partial(1) = 0$, $\partial(p) > 0$ for all $p \in P$, $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$, and the number $N_G^{\#}(x)$ of elements $a \in G$ such that $\partial(a) \leq x$ is finite for every $x > 0$.

For an integer $k \geq 2$, an element $a \in G$ is said to be $k$-th power-free if it has no divisors of the form $b^k$, where $1 \neq b \in G$. We denote the set

of $k$-th power-free elements in $G$ by $G_k \subset G$ and the number of $k$-th
power-free elements of degree $\leq x$ by $\mathcal{N}_{G,k}^\#(x)$. It is natural to extend
the definition to $k = \infty$ by setting $G_\infty = G$; then $\mathcal{N}_{G,\infty}^\#(x) = \mathcal{N}_{G}^\#(x)$.

Our aim is to find the asymptotics of $\mathcal{N}_{G,k}^\#(x)$ as $x \to \infty$ under the
assumption that the number $\pi_{G}^\#(x)$ of primes $p \in P$ such that $\partial(p) \leq x$ has the asymptotics

$$\pi_{G}^\#(x) = \rho x^\gamma e^x \left(1 + O(x^{-\delta})\right), \quad x \to \infty,$$

for some $\rho > 0$, $\gamma > -1$, and $\delta \in (0, 1]$. The limit case of $k = \infty$ was
considered in \[2–4\] (where one can also find more detailed bibliographi-
cal remarks). Here we essentially show that the results obtained there remain valid, mutatis mutandis, for the case of finite $k$.

Theorems deriving the asymptotic behavior of $\mathcal{N}_{G,k}^\#(x)$ as $x \to \infty$ from that of $\pi_{G}^\#(x)$ are known as (inverse) abstract prime number theo-
rems, and the corresponding theorems for $\mathcal{N}_{G,k}^\#(x)$ are a generalization
of these. Apart from the purely number-theoretic meaning, the func-
tion $\mathcal{N}_{G,k}^\#(x)$ has a natural interpretation in statistical mechanics. Let
us enumerate the elements of $P$ in some way, $P = \{p_1, p_2, \ldots\}$, and set $\lambda_j = \partial(p_j)$. Then $\mathcal{N}_{G,k}^\#(x)$ is the number of solutions of the inequality

$$\sum_{j=1}^{\infty} \lambda_j n_j \leq x$$

in integers $n_j$ such that

$$0 \leq n_j < k.$$

Inequality (3) describes the states with total energy $\leq x$ of a system
of noninteracting indistinguishable particles, $n_j$ being the number of particles at the energy level $j$ with energy $\lambda_j$. Inequalities (3) imply
that there are at most $k - 1$ particles at each energy level. In other
words, the particles obey the Gentile statistics (see \[5, 7, 8\, p. 258\]),
which becomes the well-known Bose–Einstein statistics (any number of particles at any level) and Fermi–Dirac statistics (at most one particle
at each level) in the limit cases of $k = \infty$ and $k = 2$, respectively.

The logarithm $\ln \mathcal{N}_{G,k}^\#(x)$ is the entropy of the system. Note, however,
that the counting function $\pi_{G}^\#(x)$ usually has a power-law asymptotics
rather than the exponential asymptotics \[2\] in statistical mechanics,
at least if the individual particles have finitely many degrees of free-
dom (e.g., see \[9\] and the survey \[10\], where further references can be found). Our interest in the asymptotics \[2\] is partly motivated by
problems arising when calculating the number of localized Gaussian
packets in the theory of dynamical systems on metric and decorated
crystals [11–13], where exponential growth is associated with positivity
of topological entropy of the manifolds in question (see \[14, 16\]).
2. Main results

Assume that condition (2) is satisfied. The Dirichlet series

\[ \zeta_{G,k}(s) = \sum_{a \in G_k} e^{-\partial(a)s}, \quad s = \sigma + it, \]

converges absolutely in the half-plane \( \sigma > 1 \), and one has the Euler identity

\[ \zeta_{G,k}(s) = \prod_{p \in P} \frac{1 - e^{-\partial(p)sk}}{1 - e^{-\partial(p)s}}. \]

The proof is the same as for the zeta function \( \zeta_G(s) \) of \( G \) (e.g., see [1] p. 36), which is the special case of (5) for \( k = \infty \).

Now we are in a position to state the main results of the paper.

**Theorem 1.** Under condition (2), the function \( N_{G,k}^\#(x) \) has the following asymptotics as \( x \to \infty \):

\[ N_{G,k}^\#(x) = \frac{e^{xs}\zeta_{G,k}(s)}{\sqrt{2\pi (\ln \zeta_{G,k}(s))'}} \bigg|_{s = \beta(x)} (1 + O(x^{-\kappa})), \]

where \( s = \beta(x) > 1 \) is the unique real solution of the equation

\[ x + (\ln \zeta_{G,k}(s))' = 0 \]

and \( \kappa > 0 \) is an arbitrary number such that

\[ \kappa < \frac{\delta}{2 + \gamma}, \quad \kappa \leq \frac{1 + \gamma}{2 + \gamma}. \]

Theorem 1 gives the asymptotics of \( N_{G,k}^\#(x) \) in terms of the function \( \zeta_{G,k}(s) \), which is itself given by the infinite product (6). The formulas for the logarithmic asymptotics are much simpler and depend only on the constants \( \rho, \gamma, \) and \( \delta \) occurring in the asymptotics (2) of the prime counting function. Namely, the following theorem holds.

**Theorem 2.** Under condition (2), the function \( \ln N_{G,k}^\#(x) \) has the following asymptotics as \( x \to \infty \):

\[ \ln N_{G,k}^\#(x) = x + 2(\rho \Gamma(\gamma + 2))^{\frac{1}{\gamma + 2}} x^{\frac{\gamma + 1}{\gamma + 2}} + R(x) \]

if \( \delta \leq \min\{1, 1 + \gamma\} \), where

\[ R(x) = O(x^{\frac{\gamma + 1}{\gamma + 2}}) \quad \text{if} \ \delta < 1 + \gamma \quad \text{and} \quad R(x) = O(\ln x) \quad \text{if} \ \delta = 1 + \gamma; \]

\[ \ln N_{G,k}^\#(x) = x + 2(\rho \Gamma(\gamma + 2))^{\frac{1}{\gamma + 2}} x^{\frac{\gamma + 1}{\gamma + 2}} - \frac{1}{2} \gamma + 3 \ln x + O(1) \]

if \( 1 \geq \delta > 1 + \gamma \).

**Remark 1.** In contrast to the asymptotics obtained in Theorem 1, the logarithmic asymptotics provided by Theorem 2 does not feel the difference between the cases of \( k = \infty \) and finite \( k \).
Remark 2. Similar results were obtained in [17] in a different setting. (The analysis in that paper only applies to the case in which the mapping $\partial$ is integer-valued.)

3. Proof of the theorems

The proof of both theorems is completely similar to that given in [4] for the case of $k = \infty$, and here we only give a brief outline of the reasoning. The argument relies on asymptotic formulas for the function $\ln \zeta_{G,k}(\sigma)$ and its derivatives as $\sigma \downarrow 1$. These formulas have the form

$$
\ln \zeta_{G,k}(\sigma) = \rho \Gamma(1 + \gamma)(\sigma - 1)^{-1-\gamma}(1 + o(1)),
$$

$$
(\ln \zeta_{G,k}(\sigma))' = -\rho \Gamma(\gamma + 2)(\sigma - 1)^{-\gamma - 2}(1 + o(1)),
$$

$$
(\ln \zeta_{G,k}(\sigma))'' = \rho \Gamma(\gamma + 3)(\sigma - 1)^{-\gamma - 3}(1 + o(1)) 
$$

(we only write out the leading terms of the asymptotics) and coincide modulo $O(1)$ with those obtained in [4] for $\ln \zeta_{G}(\sigma)$, because

$$
\ln \zeta_{G,k}(\sigma) - \ln \zeta_{G}(\sigma) = \sum_{p \in P} \ln(1 - e^{-\partial(p)\sigma k}) = O(1) \quad \text{as } \sigma \downarrow 1.
$$

(Recall that $k \geq 2$.)

**Outline of proof of Theorem 1.** We have

$$
\mathcal{N}_{G,k}(\sigma) = \sum_{a \in G_k} H\left(\frac{x - \partial(a)}{\varepsilon}\right),
$$

where $H(x)$ is the Heaviside step function and $\varepsilon > 0$ is arbitrary. Take smooth functions $\chi_{\pm}(x)$ such that

$$
\chi_-(x) \leq H(x) \leq \chi_+(x) \quad \text{for all } x, \quad \chi_{\pm}(x) = H(x), \quad |x| \geq 1;
$$

then

$$
\sum_{a \in G_k} \chi_-(\frac{x - \partial(a)}{\varepsilon}) \leq \mathcal{N}_{G,k}(\varepsilon) \leq \sum_{a \in G_k} \chi_+(\frac{x - \partial(a)}{\varepsilon}).
$$

Using the generalization in [4] Proposition 3 of the well-known Perron formula [18, p. 12, Theorem 13], we obtain

$$
I_-(x, \varepsilon) \leq \mathcal{N}_{G,k}(\varepsilon) \leq I_+(x, \varepsilon),
$$

where

$$
I_{\pm}(x, \varepsilon) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\sigma s} \zeta_{G,k}(s) \varepsilon \tilde{\chi}_{\pm}(\varepsilon s) ds,
$$

$\tilde{\chi}_{\pm}(s)$ is the two-sided Laplace transform of $\chi_{\pm}(x)$, and the integrals are independent of the choice of $\sigma > 1$.

Now we compute these integrals by the mountain pass method [19, Ch. 4, p. 170]. The phase function is

$$
S(x, s) = xs + \ln \zeta_{G,k}(s),
$$
and the amplitude is $\varepsilon \tilde{\chi}_\pm(\varepsilon s)$. The equation $\partial S(x, s)/\partial s = 0$ for the stationary points of the phase function $[19]$ coincides with $[8]$ and has a unique real solution $s = \beta(x) > 1$ for each $x > 0$. Further, $\beta(x) \to 1$ as $x \to \infty$, and it follows from $[13]$ that

$$
\beta(x) - 1 \sim C x^{-1/(\gamma+2)}, \quad C = (\rho \Gamma(\gamma + 2))^{1/2}.
$$

In $[18]$, we take the integration contour to be given by $\sigma = \beta(x)$; this is a mountain pass contour for this integral. We make a change of the integration variable $s$ by the formula $s = \beta(x) + i\xi$, so that the contour of integration with respect to the new variable $\xi$ coincides with the real line. Further, set

$$
x = x(\beta) \equiv -(\ln \zeta_{G,k}(\beta))';
$$

this is the inverse function of $\beta(x)$. These transforms give the integrals

$$
I_\pm(x(\beta), \varepsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{S(x(\beta), \beta + i\xi)} \varepsilon \tilde{\chi}_\pm(\varepsilon(\beta + i\xi)) d\xi,
$$

with the saddle point $\xi = 0$. By using the Mountain Pass Theorem $[19]$, Ch. 4, Theorem 1.3, p. 170 and the estimates in $[4]$ for the phase function $\Phi(\beta, \xi) = S(x(\beta), \beta + i\xi)$ and the amplitude $\varphi_\pm(\beta, \xi) = \varepsilon(\beta) \tilde{\chi}_\pm(\varepsilon(\beta)(\beta + i\xi))$ with $\varepsilon(\beta) = (\beta - 1)^{\kappa(2+\gamma)}$, $\kappa > 0$, we finally obtain

$$
I_\pm(x(\beta), \varepsilon(\beta)) = \frac{e^{x(\beta)\beta} \tilde{\zeta}_{G,k}(\beta)}{\sqrt{2\pi(\ln \zeta_{G,k}(\beta))'^2}} \times (1 + O(\varepsilon(\beta)) + O(\beta - 1) + O((\beta - 1)^{1+\gamma})),
$$

which, together with $[17]$, gives $[7]$.  □

OUTLINE OF PROOF OF THEOREM $2$. This theorem follows if one takes the logarithm of both sides of formula $[7]$ in Theorem $1$ and then uses the asymptotics $[20]$ of $\beta(x)$ together with the asymptotics of $\ln \zeta_{G,k}(\sigma)$ and $(\ln \zeta_{G,k}(\sigma))''$ whose leading terms are given by $[12]$ and $[14]$, respectively.  □

4. Simulation results

To illustrate the results, consider the arithmetical semigroup $G$ with primes $p_n \in P$, $n = 1, 2, \ldots$, and with $\partial(p_n) = \ln(n + \varepsilon)$. Then

$$
\pi^G_\#(x) = [p e^x - \rho] = p e^x + R(x), \quad -\rho \leq R(x) \leq -\rho + 1.
$$

We compare the exact values of $N^G_{\#}(x)$ and the asymptotic values given by Theorem $1$ for $\rho = 0.5, 1$, and $2$ at the points $x = 1, 2, \ldots, 7$. The results are presented in Fig. $1$ for $k = 2$ and $k = \infty$ (the Fermi and Bose cases). We also present the dependence of $N^G_{\#}$ on the parameter $k \in [2, 8]$ at the point $x = 7$. 
Figure 1. Left: the comparison of exact (points) and asymptotic (lines) values of $\mathcal{N}_{G,k}^\#(x)$ for $k = \infty$ (solid points and black lines) and $k = 2$ (empty points and gray lines). The parameter values are $\rho = 0.5, 1, 2$ (dotted lines and squares, dashed lines and triangles, and solid lines and circles, respectively). Right: the ratio of the asymptotic values to the exact values of $\mathcal{N}_{G,k}^\#$ for $k = 2, \infty$ and $\rho = 0.5, 1, 2$ with the same notation. Bottom: the dependence of $\mathcal{N}_{G,k}^\#(x)$ on the parameter $k \in [2, 8]$ at the point $x = 7$ for $\rho = 0.5, 1, 2$ (squares, triangles, and circles, respectively).

The figure illustrates the convergence of asymptotic formulas to the exact values (right). It also illustrates the fact, that the rate of growth $\mathcal{N}_{G,k}^\#(x)$ with $x$ is the same for all $k$ and asymptotics $\mathcal{N}_{G,k}^\#(x)$ differs by a factor (left). This fact follows from corollaries. We also can see from the bottom figure that this factor tends to 1 very rapidly with increasing parameter $k$ and the value $k = 10$ can already be treated as “infinity” from the viewpoint of convergence.

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References

[1] J. Knopfmacher, Abstract analytic number theory, North-Holland Mathematical Library, vol. 12, 1975.
[2] V. L. Chernyshev, D. S. Minenkov, and V. E. Nazaikinskii, On the Bose-Maslov statistics in the case of infinitely many degrees of freedom, Dokl. Math. 468 (6), 2016.
[3] V. E. Nazaikinskii, D. S. Minenkov, On the inverse theorem on distribution of abstract primes, Math. Notes, 2016, to appear.
[4] V. L. Chernyshev, D. S. Minenkov, V. E. Nazaikinskii, The asymptotic behavior of the number of elements in the additive arithmetic subgroup with an exponential counting function of the number of abstract primes. // Functional Analysis and Its Applications, 2016, to appear.
[5] G. Gentile, Osservazioni sopra le statistiche intermedie, Il Nuovo Cimento, 17 (10), pp 493-497, 1940.
[6] G. Gentile, Le Statistiche Intermedie e le Proprietà Dell’elio Liquido, Il Nuovo Cimento, 19 (4), pp 109-125, 1942.
[7] A. Khare, Fractional statistics and quantum theory, World Scientific Publishing Co. Pte. Ltd., 2005.
[8] I. A. Kvasnikov, Thermodynamics and Statistical Mechanics. Volume 2. The theory of equilibrium systems Statistical Physics. Editorial URSS, 2010.
[9] V. P. Maslov, Quasithermodynamics and a Correction to the Stefan–Boltzmann Law, Mat. Zametki, Volume 83, Issue 1, 77–85. 2008.
[10] V. P. Maslov, Undistinguishing statistics of objectively distinguishable objects: Thermodynamics and superfluidity of classical gas, Math. Notes, 94(5), 722–813. 2013.
[11] V. L. Chernyshev, A. I. Shafarevich, Statistics of gaussian packets on metric and decorated graphs. // Philosophical transactions of the Royal Society A. — Vol. 372. — Issue 2007. — Article number 20130145. — 2014.
[12] V. L. Chernyshev, A. A. Tolchennikov, Asymptotic estimate for the number of Gaussian packets on three decorated graphs, arXiv:math/1403.0263 [math-ph]. 2014.
[13] R. Schubert, R. O. Vallejos, F. Toscano, How do wave pakets spread, Time evolution on Ehrenfest time scales, Journal of Physics A: Mathematical and Theoretical 45:21, Article number 215307. 2012.
[14] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, // Publ. Math., Inst. Hautes Etud. Sci. — Vol. 51. — P. 137–173. — 1980.
[15] R. Mañé, On the topological entropy of geodesic flows, // Journal of Differential Geometry. — Vol. 45. — p. 7493. — 1997.
[16] M. Pollicott, A symbolic proof of a theorem of Margulis on geodesic arcs on negatively curved manifolds, Amer. J. Math., 117, 289-305. 1995.
[17] B. L. Granovsky, D. Stark, Developments in the Khintchine-Meinardus probabilistic method for asymptotic enumeration, arXiv:1311.2254 2013.
[18] G. H. Hardy, M. Riesz, The General Theory of Dirichlet’s Series, Cambridge, Cambridge University Press, 1915.
[19] M. V. Fedoryuk, The saddle-point method, Moscow, 1977.