TRACE HARDY INEQUALITY FOR THE EUCLIDEAN SPACE WITH A CUT AND ITS APPLICATIONS

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ABSTRACT. We obtain a trace Hardy inequality for the Euclidean space with a bounded cut \( \Sigma \subset \mathbb{R}^d, d \geq 2 \). In this novel geometric setting, the Hardy-type inequality non-typically holds also for \( d = 2 \). The respective Hardy weight is given in terms of the geodesic distance to the boundary of \( \Sigma \). We provide its applications to the heat equation on \( \mathbb{R}^d \) with an insulating cut at \( \Sigma \) and to the Schrödinger operator with a \( \delta' \)-interaction supported on \( \Sigma \). We also obtain generalizations of this trace Hardy inequality for a class of unbounded cuts.

1. Introduction

The classical Hardy inequality in the Euclidean space \( \mathbb{R}^d, d \geq 3 \), can be stated as follows

\[ \int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \frac{(d - 2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx, \quad \forall u \in H^1(\mathbb{R}^d). \]  

(1.1)

This inequality and some of its generalizations can be found in e.g. [D95, §5.3]. They provide a technical tool in various studies of elliptic partial differential operators. Probably, the most famous of them is Kato’s proof of self-adjointness and semi-boundedness for many-body quantum Hamiltonians [K51]; cf. the discussion in the review [S18]. From this perspective, the physically observed stability of the hydrogen atom is just a simple consequence of the inequality (1.1). We also point out usefulness of (1.1) in the proof of self-adjointness and semi-boundedness for a class of Schrödinger operators with potentials having decoupled singularities [GMNT16]. Further applications of the Hardy inequality concern criticality properties of the Schrödinger operators [DFP14] and large time decay of the heat semigroups [VZ00], the latter being intimately connected with the notion of transiency of the Brownian motion in stochastic analysis; cf. [KK14] and the references therein.

Recently, considerable interest has been attracted by various trace versions of the Hardy inequality [FMT13, N11, T15, T16, vH16]. The progenitor of all the trace Hardy inequalities and also the most illustrative of them is frequently attributed to T. Kato and claims that

\[ \int_{\mathbb{R}^d} |\nabla u|^2 dx \geq 2 \left( \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-2}{2})} \right)^2 \int_{\mathbb{R}^{d-1}} \frac{|u|^2}{|x'|} dx', \quad \forall u \in H^1(\mathbb{R}^d). \]  

(1.2)
where \( \mathbb{R}^d_+ = \{ (x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1}, x_d > 0 \} \), \( d \geq 3 \), denotes the upper half-space in \( \mathbb{R}^d \) and \( \Gamma(\cdot) \) stands for the Euler \( \Gamma \)-function. The inequality (1.2) can be derived from the Hardy inequality for the relativistic Schrödinger operator [H77, K] and the Sobolev trace theorem for the half-space. It is applicable e.g. in the study of Schrödinger operators with surface potentials [ES88, FL08].

The main result of the present paper, formulated in Theorem 2.2, is a trace Hardy inequality in a novel geometric setting of the Euclidean space with a cut across an interface, the latter being a bounded, non-closed hypersurface \( \Sigma \subset \mathbb{R}^d \), \( d \geq 2 \). By saying that \( \Sigma \) is non-closed, we essentially mean that \( \mathbb{R}^d \setminus \Sigma \) is connected. In this setting, we replace the ordinary trace by its jump across \( \Sigma \). Under an additional Ahlfors-David-type regularity assumption, the weight on \( \Sigma \) involved in the inequality is estimated in terms of the geodesic distance to its boundary \( \partial \Sigma \). Trace Hardy inequalities for the jump hold also in the two-dimensional case, thus being substantially different from the classical trace Hardy inequalities.

The trace Hardy inequality for \( \mathbb{R}^d \setminus \Sigma \) implies an integral upper bound on the large time decay for the weighted \( L^2 \)-norm of the temperature-jump across \( \Sigma \) for the solutions of the heat equation with the (insulating) Neumann boundary condition on the cut \( \Sigma \) being imposed. The singularity of the respective weight on \( \partial \Sigma \) reflects the intuition that the temperature levels off quicker in a neighbourhood of \( \partial \Sigma \).

An analogous trace Hardy inequality is also proven in Theorem 6.1 for a class of unbounded cuts. Namely, we consider a cut across the hypersurface \( \Sigma \subset \Gamma := \mathbb{R}^{d-1} \times \{0\} \), \( d \geq 3 \), which is a (hyper)cone. Some generalizations for cuts with non-trivial topology such as Möbius strips are also discussed.

Considerations of the present paper are largely inspired by the spectral analysis of the Schrödinger operator with the \( \delta' \)-interaction supported on a non-closed curve carried out by two of us in [JL16]. The investigation of Schrödinger operators with \( \delta' \)-interactions supported on hypersurfaces became a topic of permanent interest – see, e.g., [BGLL15, BEL14, BLL13, EJ13, EKh15, EKh18, L18, MPS16a]. Such Hamiltonians appear, e.g., in the study of photonic crystals [FK96a, FK96b] and arise in the asymptotic limit for a class of structured thin Neumann obstacles [DFZ18, H70]. The obtained trace Hardy inequality for a bounded cut implies absence of the negative discrete spectrum for a weak attractive \( \delta' \)-interaction supported on a bounded non-closed surface \( \Sigma \). This is in shear contrast to attractive \( \delta' \)-interaction supported on a closed bounded hypersurface, which always induces negative discrete spectrum (see [BEL14, Thm. 4.4]).

**Organization of the paper.** In Section 2 we describe the setting in more detail and formulate our main results. Section 3 provides several statements on Sobolev spaces that are used in the proofs. Section 4 is devoted to the proof of the trace Hardy inequality for the jump of the trace across a cut. Finally, in Section 5 we apply the obtained inequality to the estimation of the large-time behaviour for the heat equation on \( \mathbb{R}^d \setminus \Sigma \) and to the Schrödinger operator with a \( \delta' \)-interaction.
supported on $\Sigma$. In Section 6 we consider extensions to a class of unbounded cuts and to topologically non-trivial cuts.

2. Setting of the problem and the main result

Throughout the paper, we consider a cut $\Sigma$ in $\mathbb{R}^d$ satisfying the following assumptions.

**Hypothesis 2.1.** We consider a relatively open subset $\Sigma \subset \Gamma$ of a bounded, orientable, and closed Lipschitz hypersurface$^1$ $\Gamma$. We implicitly assume that $\Gamma$ is the boundary of a bounded (Lipschitz) domain $\Omega_+ \subset \mathbb{R}^d$. We suppose moreover that $\mathbb{R}^d \setminus \Sigma$ is connected (thus $\Gamma \setminus \Sigma$ is not empty).

Let $\partial \Sigma = \overline{\Sigma} \setminus \Sigma$ denote the boundary of $\Sigma$ in $\Gamma$. For any $x \in \Sigma$, we abbreviate by $\rho_{\Sigma}(x)$ the geodesic distance between $x$ and $\partial \Sigma$, measured in the induced Riemannian metric of $\Sigma$ (cf. [CP91]) and by $\tilde{\rho}_\Sigma(x)$ we denote the respective Euclidean distance in $\mathbb{R}^d$.

We denote by $B_r(x) \subset \mathbb{R}^d$ the open ball of radius $r > 0$ centred at $x \in \mathbb{R}^d$ and by $S \subset \mathbb{R}^d$ the unit sphere. Recall also that $\mathcal{F} \subset \mathbb{R}^d$ is called a $(d-1)$-set if there exist $r_* > 0$ and $c_-, c_+ > 0$ such that for any $r \in (0, r_*)$ one has

$$c_- r^{d-1} \leq \Lambda_{d-1}(B_r(x) \cap \mathcal{F}) \leq c_+ r^{d-1}, \quad \forall x \in \mathcal{F},$$

where $\Lambda_{d-1}(\cdot)$ stands for the $(d-1)$-dimensional Hausdorff measure.

Sobolev spaces on domains with cuts are thoroughly investigated in the monographs [Gr85, D88]; see also [CHM17]. Roughly speaking, the Sobolev space $H^1(\mathbb{R}^d \setminus \Sigma)$ can be defined as a subspace of $H^1(\Omega_+) \oplus H^1(\mathbb{R}^d \setminus \overline{\Omega_+})$ in which the traces of functions agree on $\Gamma \setminus \Sigma$, cf. Section 3 for details.

For any $u \in H^1(\mathbb{R}^d \setminus \Sigma)$ the traces $u|_{\Sigma_\pm}$ onto two faces $\Sigma_\pm$ of $\Sigma$ are well-defined functions in $L^2(\Sigma)$. These functions need not be the same and, thus, the jump of the trace $|u|_{\Sigma} := u|_{\Sigma_+} - u|_{\Sigma_-}$ is a well-defined and non-trivial function in $L^2(\Sigma)$.

Our trace Hardy inequality holds for functions in the Sobolev space $H^1(\mathbb{R}^d \setminus \Sigma)$. If the complement $\Gamma \setminus \Sigma$ is a $(d-1)$-set, then the Hardy weight in this inequality can be controlled by the inverse of the geodesic distance $\rho_{\Sigma}(\cdot)$.

**Theorem 2.2.** Let $\Sigma \subset \mathbb{R}^d$ be a bounded non-closed Lipschitz hypersurface satisfying Hypothesis 2.1. Then there exists a constant $C = C(\Sigma) > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq C \int_{\Sigma} w(x) |u|_{\Sigma}^2 dx, \quad \forall u \in H^1(\mathbb{R}^d \setminus \Sigma),$$

where the weight $w : \Sigma \to \mathbb{R}_+$ is given by

$$w(x) := \int_{\Gamma \setminus \Sigma} \frac{d\sigma(y)}{|x-y|^d}$$

If, in addition, $\Gamma \setminus \Sigma$ is a $(d-1)$-set, then there exists $c > 0$ such that $w(x) \geq c \tilde{\rho}_\Sigma(x)^{-1} \geq c \rho_{\Sigma}(x)^{-1}$.

$^1$This means that $\Gamma$ is a Lipschitz manifold without boundary, i.e. $\Gamma$ can be covered by a finite collection of local maps in which $\Gamma$ coincides with the graph of a Lipschitz function.
The condition that $\Gamma \setminus \Sigma$ is a $(d - 1)$-set calls for the following remarks and examples:

**Remark 2.3.** (i) The upper bound $\Lambda_{d-1}(B_r(x) \cap \mathcal{F}) \leq c_r r^{d-1}$ is always satisfied for $\mathcal{F} = \Gamma \setminus \Sigma$ since $\Gamma$ itself is a $(d - 1)$-set.

(ii) If the uniform lower bound $c_r r^{d-1} \leq \Lambda_{d-1}(B_r(x) \cap \mathcal{F})$ is satisfied with $\mathcal{F} = \Gamma \setminus \Sigma$ for any $x \in \partial \Sigma$, this implies a similar bound for all $x \in \Gamma \setminus \Sigma$.

(iii) If $\Gamma'$ is another closed Lipschitz hypersurface containing $\Sigma$, and if $\Gamma \setminus \Sigma$ is a $(d - 1)$-set, then $\Gamma' \setminus \Sigma$ is a $(d - 1)$-set too.

**Example 2.4.** (i) If $\Sigma$ is (moreover) a Lipschitz hypersurface with Lipschitz boundary (its boundary $\partial \Sigma$ can be viewed as a $(d - 2)$-dimensional Lipschitz manifold) so $\Gamma \setminus \Sigma$ is a $(d - 1)$-set.

(ii) In the space dimension $d = 2$, $\Gamma \setminus \Sigma$ is a 1-set without any extra assumptions on $\Sigma$.

(iii) If $\Sigma$ has an inward cusp at a point $x_0$, then $\Gamma \setminus \Sigma$ has an outward cusp at the same point. The lower bound $c_r r^{d-1} \leq \Lambda_{d-1}(B_r(x) \cap \mathcal{F})$ is not satisfied with $\mathcal{F} = \Gamma \setminus \Sigma$ at this point $x = x_0$.

The existence of a weight singular on $\partial \Sigma$ is reminiscent of the Hardy inequality for the Dirichlet Laplacian on a bounded planar regular Euclidean domain [D95, Thms. 5.3.5 and 5.3.6]. We also remark that the inequality in Theorem 2.2 can be generalized to the setting of a cut in a Riemannian manifold. However, we do not pursue this goal here.

An inequality analogous to the trace Hardy inequality in Theorem 2.2 is proven in Theorem 6.1 for a class of unbounded cuts. These cuts are contained in the hyperplane $\Gamma := \mathbb{R}^{d-1} \times \{0\}$, $d \geq 3$ and a cut from this class is defined as

$$\Sigma = \left\{ x \in \mathbb{R}^d \setminus \{0\} : \frac{x}{|x|} \in \hat{\Sigma} \right\},$$

where $\hat{\Sigma}$ is an open set in $\hat{\Gamma} := \Gamma \cap \mathcal{S}$ such that $\hat{\Gamma} \setminus \hat{\Sigma}$ is a $(d - 2)$-set. The proof of Theorem 6.1 relies on splitting the Euclidean space $\mathbb{R}^d$ by dyadic spherical shells, inside which the method used in the proof of Theorem 2.2 is applied. Similar localization technique can be used to show generalizations of Theorem 2.2 for non-orientable cuts such as Möbius strips in $\mathbb{R}^3$; see Example 6.5.

### 3. The first order Sobolev space on a domain with a cut

In this section we recall several properties of and notions related to the Sobolev space on a domain with a cut. Let the hypersurface $\Sigma \subset \mathbb{R}^d$ satisfy Hypothesis 2.1. Recall that we implicitly assume existence of a bounded Lipschitz domain $\Omega_+ \subset \mathbb{R}^d$ with the boundary $\Gamma := \partial \Omega_+$ such that $\Sigma$ is a relatively open subset of $\Gamma$. From the topological point of view, $\Gamma$ and $\Sigma$ possess two faces $\Gamma_+ \subset \Sigma_+$ and $\Sigma_+ \subset \Gamma$, respectively. Additionally, we assume that $\Sigma$ is non-closed or in other terms the domain $\mathbb{R}^d \setminus \Sigma$ is assumed to be connected. Further, let $\Omega \subset \mathbb{R}^d$ be an open set such that $\overline{\Omega}_+ \subset \Omega$ and that $\Omega_- := \Omega \setminus \overline{\Omega}_+$ is a possibly unbounded Lipschitz domain of the type
described in [St, §VI.3]. The case $\Omega = \mathbb{R}^d$ and $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega_+}$ is for us the most typical.

For a function $u \in L^2(\Omega)$ we adopt the notation $u_{\pm} := u|_{\Omega_{\pm}}$. The Sobolev space $H^1(\Omega \setminus \Gamma) = H^1(\Omega_+) \oplus H^1(\Omega_-)$ is introduced in the conventional way. For a function $u = u_+ \oplus u_- \in H^1(\Omega \setminus \Gamma)$ we define $\nabla u := \nabla u_+ \oplus \nabla u_-$. The norm on $H^1(\Omega \setminus \Gamma)$ is given by

$$
\|u\|_1^2 = \|u\|^2_{H^1(\Omega \setminus \Gamma)} := \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega \setminus \Gamma)}.
$$

Next, we define the sub-space of $H^1(\Omega \setminus \Gamma)$

$$
C_c^\infty(\Omega \setminus \Sigma) := \{u = u_+ \oplus u_- \in C_c^\infty(\Omega_+) \oplus C_c^\infty(\Omega_-) : \text{supp } ([u]_\Gamma) \subset \Sigma\},
$$

where $[u]_\Gamma := u_+|_\Gamma - u_-|_\Gamma$.

**Definition 3.1.** The first-order $L^2$-based Sobolev space on the domain $\Omega \setminus \Sigma$ is defined as the closure of $C_c^\infty(\Omega \setminus \Sigma)$ in $H^1(\Omega \setminus \Gamma)$ with respect to its norm

$$
H^1(\Omega \setminus \Sigma) = C_c^\infty(\Omega \setminus \Sigma)^\perp_1.
$$

**Remark 3.2.** The above definition of the Sobolev space $H^1(\Omega \setminus \Sigma)$ is not intrinsic. The Sobolev space $H^1(\Omega \setminus \Sigma)$ can be alternatively intrinsically defined as the space of all square-integrable functions on $\Omega$ such that their distributional gradients on $\Omega \setminus \Sigma$ are also square-integrable. The intrinsic definition is for technical reasons less convenient. The equivalence between the two definitions can be shown via standard argument based on integration by parts.

Note that it is easy to see that the inclusions $H^1(\Omega) \subseteq H^1(\Omega \setminus \Sigma)$ and $H^1(\Omega \setminus \Sigma) \subseteq H^1(\Omega_+) \oplus H^1(\Omega_-)$ are proper.

Under the assumptions imposed, the trace maps

$$
H^1(\Omega \setminus \Sigma) \ni u \mapsto u|_{\Sigma_{\pm}} \in L^2(\Sigma)
$$

onto two different faces $\Sigma_{\pm}$ of $\Sigma$ are well-defined, continuous, linear operators. These traces need not coincide for a generic element of $H^1(\Omega \setminus \Sigma)$.

The jumps of the trace $[u]_\sigma := u_+|_\Gamma - u_-|_\Gamma$ extends by continuity to the Sobolev space $H^1(\Omega \setminus \Gamma)$. It is convenient to introduce the notation $[u]_\Sigma$ and $[u]_\Gamma$ for the restrictions of $[u]_\sigma$ onto $\Sigma$ and $\Gamma \setminus \Sigma$, respectively.

**Proposition 3.3.** For any $u \in H^1(\Omega \setminus \Sigma)$ one has $[u]_\Gamma = 0$.

**Proof.** Let $u \in H^1(\Omega \setminus \Sigma)$ be fixed and let $(u_n)_n$ be a sequence in $C_c^\infty(\Omega \setminus \Sigma)$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$. Then by continuity we get

$$
[u]_\Gamma = \lim_{n \to \infty} [u_n]_\Gamma = 0.
$$

The fractional Sobolev space $H^{1/2}(\Gamma)$ is defined as in [Gr85, §1.3.3] via local charts (see also [McL, Chap. 3]). There are several ways to define the norm on $H^{1/2}(\Gamma)$. The most convenient for us is the following Besov-type norm

$$
\|\psi\|_{1/2}^2 = \|\psi\|^2_{H^{1/2}(\Gamma)} := \|\psi\|^2_{L^2(\Gamma)} + \int_{\Gamma} \int_{\Gamma} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^d} d\sigma(x) d\sigma(y);
$$

where $d \in \mathbb{N}$ in $\mathbb{R}^d$.
According to [Gr85, Thm. 1.5.1.2] the trace maps are continuous linear operators 
\[ H^1(\Omega \setminus \Gamma) \ni u \mapsto u|_{\Gamma_\pm} \in H^{1/2}(\Gamma). \]

4. Trace Hardy inequality for a bounded cut

The aim of this section is to prove Theorem 2.2. To this end we need an auxiliary lemma.

Lemma 4.1. Let \( \Sigma \subset \mathbb{R}^d \) be a bounded Lipschitz hypersurface satisfying Hypothesis 2.1. Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain such that \( \Sigma \subset \Omega \). Then the following statements hold.

(i) For any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon = C_\varepsilon(\Sigma) > 0 \) such that
\[ \|[u]_\Sigma\|^2_{L^2(\Sigma)} \leq \varepsilon \|[\nabla u]\|^2_{L^2(\mathbb{R}^d \setminus \Sigma)} + C_\varepsilon \|u\|^2_{L^2(\mathbb{R}^d)} \]
holds for all \( u \in H^1(\mathbb{R}^d \setminus \Sigma) \).

(ii) There exists a constant \( \tilde{C} = \tilde{C}(\Omega, \Sigma) > 0 \) such that for any \( u \in H^1(\Omega \setminus \Sigma) \)
\[ \int_{\Sigma} w(x)[[u]_\Sigma(x)]^2 \, d\sigma(x) \leq \tilde{C} \|u\|^2_{H^1(\Omega, \Sigma)}, \quad \text{with } w(x) := \int_{\Gamma \setminus \Sigma} \frac{d\sigma(y)}{|x - y|^d}. \]

If, in addition, \( \Gamma \setminus \Sigma \) is a \((d - 1)\)-set in the sense of (2.1), then there exists a constant \( c > 0 \) such that \( w(x) \geq c\rho_{\Sigma}(x)^{-1} \) for all \( x \in \Sigma \).

Proof. (i) Let \( u \in H^1(\mathbb{R}^d \setminus \Sigma) \). Set \( \Omega_- := \mathbb{R}^d \setminus \overline{\Omega_+} \) and \( u_\pm := u|_{\Omega_\pm} \). By elementary computations we get
\[ \|[u]_\Sigma\|^2_{L^2(\Sigma)} \leq 2\|u_+|_r\|^2_{L^2(\Gamma)} + 2\|u_-|_r\|^2_{L^2(\Gamma)}. \]

Applying the trace theorem [M87] (see also [BEL14, Lem 2.6]), we get that for any \( \varepsilon > 0 \) there exist \( C_\varepsilon^\pm > 0 \) such that
\[ 2\|u_\pm|_r\|^2_{L^2(\Gamma)} \leq \varepsilon \|\nabla u_\pm\|^2_{L^2(\Omega_\pm ; \mathbb{R}^d)} + C_\varepsilon^\pm \|u_\pm\|^2_{L^2(\Omega_\pm)}. \]

Summing the above two inequalities for \( \Omega_+ \) and \( \Omega_- \), respectively, and combining with (4.2) we get the desired inequality with \( C_\varepsilon = \max\{C_\varepsilon^+, C_\varepsilon^-\} \).

(ii) Let \( u \in H^1(\Omega \setminus \Sigma) \). Set \( \Omega_- := \Omega \setminus \overline{\Omega_+} \) and \( u_\pm := u|_{\Omega_\pm} \). By the Sobolev trace theorem [Gr85, Thm. 1.5.1.2] we get that there exists a constant \( C = C(\Gamma) > 0 \) such that
\[ \|u|_r\|^2_{H^{1/2}(\Gamma)} \leq \|u_+|_r\|^2_{H^{1/2}(\Gamma)} + \|u_-|_r\|^2_{H^{1/2}(\Gamma)} \leq \tilde{C} \|u\|^2_{H^{1}(\Omega, \Sigma)}. \]

Using the expression (3.3) for the norm \( \| \cdot \|_{H^{1/2}} \) and Proposition 3.3 we get
\[ \|u|_r\|^2_{H^{1/2}(\Gamma)} \geq \int_{\Gamma} \int_{\Gamma} \frac{|[u](x) - [u](y)|^2}{|x - y|^d} \, d\sigma(x) \, d\sigma(y) \geq 2 \int_{\Sigma} w(x)[[u](x)]^2 \, d\sigma(x), \]
where the weight \( w: \Sigma \to \mathbb{R}_+ \) is as in (4.1). Suppose now that the closed set \( \Sigma^c := \Gamma \setminus \Sigma \) is a \((d-1)\)-set in the sense of (2.1). Let us fix \( x \in \Sigma \) and set \( r := 2 \text{dist}(x, \Sigma^c) = 2\rho_{\Sigma}(x) \) where the distance is measured in \( \mathbb{R}^d \). Let \( x_* \in \Sigma^c \) be such that \( |x - x_*| = \frac{r}{2} \). Then we get the following lower bound on \( w(x) \)

\[
w(x) \geq \int_{B_{r}(x) \cap \Sigma^c} \frac{d\sigma(y)}{|x - y|^d} \geq \frac{\Lambda_{d-1}(B_{r}(x) \cap \Sigma^c)}{r^d} \geq \frac{c_{-}}{2^{d-1}r} = \frac{c}{\rho_{\Sigma}(x)} \geq \frac{c}{\frac{r}{2}},
\]

where \( c := \frac{c_{-}}{2} \). In the course of these estimates we used that \( B_{\frac{r}{2}}(x_*) \subset B_r(x) \) and that \( \rho_{\Sigma}(x) \geq \frac{r}{2} \).

Now we have all the tools to prove the desired functional inequality. For the sake of convenience we introduce the notation

\[
\mathfrak{h}_{\Sigma}(u) := \int_{\Sigma} w(x)[|u|_{\Sigma}(x)]^2 d\sigma(x).
\]

**Proof of Theorem 2.2.** Recall that \( \Sigma \) is a bounded non-closed Lipschitz hypersurface as in Hypothesis 2.1. Let \( \Omega \subset \mathbb{R}^d \) be a bounded connected \( C^\infty \)-smooth domain such that the inclusion \( \Sigma \subset \Omega \) holds and that \( \Omega \setminus \Sigma \) is connected as well. For any \( u \in H^1(\mathbb{R}^d \setminus \Sigma) \) we clearly have \( u_\Omega := u|_\Omega \in H^1(\Omega \setminus \Sigma) \) and moreover

\[
\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \int_{\Omega} |\nabla u_\Omega|^2 dx.
\]

Since the domain \( \Omega \) has a finite measure, Cauchy-Schwarz inequality implies \( u_\Omega \in L^1(\Omega) \). Thus, the average

\[
(u_\Omega) = \frac{1}{|\Omega|} \int_{\Omega} u_\Omega(x) dx
\]

of \( u_\Omega \) is well defined. Rather elementary calculations give

\[
[u_\Omega - (u_\Omega)]_{\Sigma} = [u_\Omega]_{\Sigma} \quad \text{and} \quad \nabla (u_\Omega - (u_\Omega)) = \nabla u_\Omega.
\]

Note that the constant function on \( \Omega \) is the eigenfunction corresponding to the lowest eigenvalue \( \lambda^N_{\Sigma}(\Omega \setminus \Sigma) = 0 \) of the Neumann Laplacian on \( \Omega \setminus \Sigma \) and that \( u_\Omega - (u_\Omega) \) is orthogonal to it. Hence, by the min-max principle we can estimate the \( L^2 \)-norm of the difference \( u_\Omega - (u_\Omega) \) in the following way

\[
\lambda^N_{\Sigma}(\Omega \setminus \Sigma) \| u_\Omega - (u_\Omega) \|^2_{L^2(\Omega)} \leq \| \nabla (u_\Omega - (u_\Omega)) \|^2_{L^2(\Omega \setminus \Sigma)},
\]

where \( \lambda^N_{\Sigma}(\Omega \setminus \Sigma) \) denotes the second eigenvalue of the Neumann Laplacian on the domain \( \Omega \setminus \Sigma \). Observe that \( \lambda^N_{\Sigma}(\Omega \setminus \Sigma) > 0 \), because the domain \( \Omega \setminus \Sigma \) is connected and bounded.

Using the shorthand notation \( \kappa := (\lambda^N_{\Sigma}(\Omega \setminus \Sigma))^{-1} \), we rewrite the inequality in Lemma 4.1 (ii) for the function \( u_\Omega - (u_\Omega) \) as follows

\[
\mathfrak{h}_{\Sigma}[u] = \int_{\Sigma} w(x)[|u_\Omega|_{\Sigma}(x)]^2 d\sigma(x) = \int_{\Sigma} w(x)[|u_\Omega - (u_\Omega)|_{\Sigma}(x)]^2 d\sigma(x) \leq \tilde{C} \| u_\Omega - (u_\Omega) \|^2_{H^1(\Omega \setminus \Sigma)} \leq \tilde{C} (1 + \kappa) \| \nabla (u_\Omega - (u_\Omega)) \|^2_{L^2(\Omega \setminus \Sigma)} \leq \tilde{C} (1 + \kappa) \| \nabla u_\Omega \|^2_{L^2(\mathbb{R}^d \setminus \Sigma)}.
\]
Thus, we get the desired inequality with \( C^{-1} = \tilde{C}(1 + \kappa) \).

If, in addition, the \( \Gamma \setminus \Sigma \) is a \((d - 1)\)-set in the sense of \((2.1)\), we get the inequality

\[
\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq C \int_{\Sigma} \frac{|u|_\Sigma^2}{\rho_\Sigma(x)} \, dx \geq C \int_{\Sigma} \frac{|u|_\Sigma^2}{\tilde{\rho}_\Sigma(x)} \, dx. \quad \Box
\]

5. Applications

In this section we discuss applications of the obtained inequality. First, in Subsection 5.1 we apply our main result to the heat equation. Second, in Subsection 5.2 we apply our main result to the Schrödinger operator with a \( \delta' \)-interaction supported on a non-closed hypersurface.

5.1. Applications to the propagation of heat. The trace Hardy inequality in Theorem 2.2 finds a physically motivated application in the theory of the heat propagation.

Let \( \Sigma \subset \mathbb{R}^d \) be a bounded non-closed Lipschitz hypersurface as in Section 2. Consider the closed, non-negative, and densely defined sesquilinear form in the Hilbert space \( L^2(\mathbb{R}^d) \)

\[
(5.1) \quad a_\Sigma(u, v) := (\nabla u, \nabla v)_{L^2(\mathbb{R}^d, \mathbb{C}^d)}, \quad \text{dom} \, a_\Sigma := H^1(\mathbb{R}^d \setminus \Sigma).
\]

The above sesquilinear form defines by the first representation theorem a unique self-adjoint operator \( A_\Sigma \) in the Hilbert space \( L^2(\mathbb{R}^d) \). This operator corresponds to the (minus) Laplace operator with Neumann boundary conditions on the cut \( \Sigma \). In thermodynamics, the Neumann boundary condition describes the heat insulator; i.e. roughly speaking, a barrier through which the heat does not propagate. By [Ouh, Thm. 1.49], the operator \( A_\Sigma \) generates a strongly continuous contraction semigroup \( e^{-tA_\Sigma} \) on \( L^2(\mathbb{R}^d) \). This semigroup can be defined e.g. through the functional calculus for self-adjoint operators. Our main interest is the large-time behaviour of the function

\[
t \mapsto h_\Sigma \left( e^{-tA_\Sigma} u \right) = \int_{\Sigma} w(x) [e^{-tA_\Sigma} u]_\Sigma(x) \, d\sigma(s),
\]

where \( w \) is as in \((2.2)\). This function can be used to measure the jump of the temperature across the insulating hypersurface \( \Sigma \) at the time \( t > 0 \). Employing the Hardy inequality in Theorem 2.2 we get an integral estimate on \( h_\Sigma(u(\cdot, t)) \).

**Theorem 5.1.** Let \( \Sigma \subset \mathbb{R}^d \) be a bounded Lipschitz hypersurface as in Section 2. Let the self-adjoint operator \( A_\Sigma \) be associated with the form \((5.1)\). Then there exists a constant \( C = C(\Sigma) > 0 \) such that

\[
\int_0^\infty h_\Sigma(e^{-tA_\Sigma} u) \, dt \leq C \| u \|_{L^2(\mathbb{R}^d)}^2
\]

holds for all \( u \in H^1(\mathbb{R}^d \setminus \Sigma) \).
Proof. We denote by $V'$ the dual space to the Sobolev space $V := H^1(\mathbb{R}^d \setminus \Sigma)$ and by $\langle \cdot, \cdot \rangle$ the corresponding duality product, which is compatible with the inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$. Clearly, we have the chain of inclusions

$$ V \subset L^2(\mathbb{R}^d) \subset V'. $$

The form $a_\Sigma$ defines (see [Ouh, Section 1.4.2]) a unique linear operator $A_\Sigma$ in the Banach space $V'$ with $\text{dom } A_\Sigma = V$ such that

$$ a_\Sigma(u, v) = \langle A_\Sigma u, v \rangle, \quad \forall u, v \in \text{dom } a_\Sigma. \tag{5.2} $$

By [Ouh, Thm. 1.55] $A_\Sigma$ generates a strongly continuous semigroup $e^{-tA_\Sigma}$ on $V'$, which is compatible with the semigroup $e^{-t\Lambda_\Sigma}$ in the following sense

$$ e^{-tA_\Sigma}u = e^{-t\Lambda_\Sigma}u, \quad \forall u \in L^2(\mathbb{R}^d). \tag{5.3} $$

According to [Da, Lem. 6.1.11] we have $e^{-tA_\Sigma}u \in V$ for any $u \in V$. Moreover, by [Da, Lem. 6.1.13] the function $t \mapsto e^{-tA_\Sigma}u$ is norm continuously differentiable on $[0, \infty)$ and

$$ \frac{d}{dt} e^{-tA_\Sigma}u = -A_\Sigma e^{-tA_\Sigma}u. \tag{5.4} $$

For any $u \in V$ we get using (5.2), (5.3) and (5.4)

$$ \frac{d}{dt} \|e^{-tA_\Sigma}u\|^2_{L^2(\mathbb{R}^d)} = \frac{d}{dt} \langle e^{-2t\Lambda_\Sigma}u, u \rangle_{L^2(\mathbb{R}^d)} = \frac{d}{dt} \langle e^{-2tA_\Sigma}u, u \rangle = -2 \langle A_\Sigma e^{-2t\Lambda_\Sigma}u, u \rangle \\
= -2a_\Sigma(e^{-2tA_\Sigma}u, u). $$

Applying the second representation theorem [K, Thm. VI 2.23]

$$ \frac{d}{dt} \|e^{-t\Lambda_\Sigma}u\|^2_{L^2(\mathbb{R}^d)} = -2(\Lambda_\Sigma^{1/2} e^{-2t\Lambda_\Sigma}, \Lambda_\Sigma^{1/2} u)_{L^2(\mathbb{R}^d)} \\
= -2(\Lambda_\Sigma^{1/2} e^{-t\Lambda_\Sigma}u, \Lambda_\Sigma^{1/2} e^{-t\Lambda_\Sigma}u)_{L^2(\mathbb{R}^d)} = -2a_\Sigma(e^{-t\Lambda_\Sigma}u). $$

Integrating against $t$ the above equation on the interval $(0, \infty)$ and using the inequality in Theorem 2.2 and the fact that $e^{-t\Lambda_\Sigma}$ is a contraction semigroup we get that there exists a constant $C > 0$ such that

$$ C \int_0^\infty \langle e^{-t\Lambda_\Sigma}u \rangle dt \leq -\int_0^\infty \frac{d}{dt} \|e^{-t\Lambda_\Sigma}u\|^2_{L^2(\mathbb{R}^d)} dt \leq \|u\|^2_{L^2(\mathbb{R}^d)}, $$

by which the theorem is proved. \qed

5.2. Application to $\delta'$-interaction. Using Theorem 2.2 we can show the absence of negative discrete spectrum for Hamiltonians describing weak $\delta'$-interactions supported on non-closed hypersurfaces.

First, we briefly recall how these operators are defined. For a more complete discussion we refer reader to [BEL14, BLL13] for the case of closed hypersurfaces or to [ER16, JL16] for the non-closed case. Let $\Omega_+$ be a bounded Lipschitz domain
as in Section 2 such that \( \Sigma \) is a relatively open connected subset of its boundary \( \Gamma := \partial \Omega \). We introduce the quadratic form

\[
(5.5) \quad a_{\omega, \Gamma}(u) = \|\nabla u\|_{L^2(\mathbb{R}^d; C^d)}^2 - (\omega[u]_{\Sigma}, [u]_{\Sigma})_{L^2(\Sigma)}, \quad \text{dom } a_{\omega, \Gamma} := H^1(\mathbb{R}^d \setminus \Gamma),
\]

where \( \omega \in L^\infty(\Sigma; \mathbb{R}) \) denotes the coupling coefficient. The above form is closed, densely defined, symmetric and lower semi-bounded; cf. [BEL14, Prop. 3.1]. As the next step, we introduce a restriction of the form (5.5)

\[
(5.5) \quad a_{\omega, \Sigma}(u) = a_{\omega, \Gamma}(u), \quad \text{dom } a_{\omega, \Sigma} := \{ u \in \text{dom } a_{\omega, \Gamma} : [u]_{\Gamma \setminus \Sigma} = 0 \}.
\]

This form is obviously closed, densely defined, symmetric and lower semi-bounded. The operator \( A_{\omega, \Sigma} \) describing \( \delta' \)-interaction is defined as the unique self-adjoint operator representing the form \( a_{\omega, \Sigma} \) in the usual manner. Now we are ready to state the main result on \( \delta' \)-interactions.

**Proposition 5.2.** Let \( \Sigma \subset \mathbb{R}^d \) be a bounded non-closed Lipschitz hypersurface satisfying Hypothesis 2.1. Then there is a constant \( \omega_* = \omega_*(\Sigma) > 0 \) such that \( A_{\omega, \Sigma} \geq 0 \) holds if \( \omega(x) \leq \omega_* \) for all \( x \in \Sigma \).

**Proof.** Applying Theorem 2.2 we find that

\[
(5.6) \quad a_{\omega, \Sigma}(u) = \|\nabla u\|_{L^2(\mathbb{R}^d; C^d)}^2 - \int_{\Sigma} \omega|[u]_{\Sigma}|^2 d\sigma(x) \geq \int_{\Sigma} (w - \omega)[[u]_{\Sigma}]^2 d\sigma(x),
\]

where \( w \) is as in Theorem 2.2. We choose \( \omega_* := \inf_{x \in \Sigma} w(x) > 0 \). The right hand side of (5.6) is certainly non-negative provided \( \omega_* \geq \omega \) on \( \Sigma \), which completes the proof. \( \square \)

The above proposition yields the following elementary consequence.

**Corollary 5.3.** Let \( \Sigma \subset \mathbb{R}^d \) be a bounded non-closed Lipschitz hypersurface satisfying Hypothesis 2.1. Let the coupling coefficient be constant \( \omega \in \mathbb{R} \). Then there is a critical coupling coefficient \( \omega_c = \omega_c(\Sigma) > 0 \) such that \( A_{\omega, \Sigma} \geq 0 \) holds if and only if \( \omega \in (-\infty, \omega_c] \).

**Remark 5.4.** If we consider a family of nested Lipschitz hypersurfaces \( (\Sigma_\varepsilon)_{\varepsilon > 0} \) in \( \Gamma \) with \( \Sigma_{\varepsilon'} \supset \Sigma_\varepsilon \) for \( \varepsilon' < \varepsilon \) such that \( \Gamma \setminus \Sigma_\varepsilon \) shrinks to a point \( x_0 \) in \( \Gamma \), then we expect that the critical coupling coefficient \( \omega_c(\Sigma_\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). If the family of hypersurfaces \( (\Sigma_\varepsilon)_{\varepsilon > 0} \) enjoys some self-similarity properties around \( x_0 \), asymptotic expansion of \( \omega_c(\Sigma_\varepsilon) \) in the small parameter \( \varepsilon > 0 \) can be determined using the technique of [BFTT12, DTV10].

**6. Generalizations to unbounded and to non-orientable cuts**

So far, we have considered only the situation when the cut is bounded and orientable. In this section we will discuss a class of unbounded cuts. The same technique is applicable for cuts given by non-orientable hypersurfaces.
As a warm-up we will derive from (1.2) a simple trace Hardy inequality for a not necessarily bounded cut contained in a hyperplane. On the Euclidean space $\mathbb{R}^d$, $d \geq 3$ we define the coordinates $x = (x', x_d)'$ with $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. We also define the upper and lower half-spaces

$$\mathbb{R}^d_+ := \{(x', x_d) : x' \in \mathbb{R}^{d-1}, x_d > 0\}.$$ 

The common boundary of $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ is a hyperplane $\Gamma := \mathbb{R}^{d-1} \times \{0\}$. Let $\Sigma$ be a relatively open subset of $\Gamma$, which is not assumed to be bounded. The Sobolev space $H^1(\mathbb{R}^d \setminus \Sigma)$ can be defined in a similar way as in Subsection 3

$$H^1(\mathbb{R}^d \setminus \Sigma) = \{u = u_+ \oplus u_- \in H^1(\mathbb{R}^d_+) \oplus H^1(\mathbb{R}^d_-) : u_+|_{\Gamma \setminus \Sigma} = u_-|_{\Gamma \setminus \Sigma}\}.$$ 

For any $u \in H^1(\mathbb{R}^d \setminus \Sigma)$, we find using the inequality (1.2) that

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = \int_{\mathbb{R}^d_+} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d_-} |\nabla u|^2 \, dx \geq 2 \left( \frac{\Gamma(d)}{\Gamma(d-2)} \right) \int_{\Gamma} \left( \frac{|u_+|}{|x'|} + \frac{|u_-|}{|x'|} \right) \, dx'$$ 

$$\geq \left( \frac{\Gamma(d)}{\Gamma(d-2)} \right) \int_{\Sigma} \frac{|u_+|\Sigma - u_-|\Sigma|^2}{|x'|} \, dx'.$$

(6.1)

Modifying the technique of the proof of Theorem 2.2 we obtain for a class of unbounded cuts an improvement of the above trace Hardy inequality with the weight which is singular in the neighbourhood of $\partial \Sigma$ and behaves at infinity far from $\partial \Sigma$ as $|x'|^{-1}$. Recall that $\delta$ denotes the unit sphere of $\mathbb{R}^d$ and also that $\Gamma := \mathbb{R}^{d-1} \times \{0\}$, $d \geq 3$, and set $\hat{\Gamma} := \Gamma \cap S$. Let $\Sigma$ be an open set in $\hat{\Gamma}$ such that $\hat{\Gamma} \setminus \Sigma$ is a $(d - 2)$-set. Define the cone $\Sigma$ as

$$\Sigma = \left\{ x \in \mathbb{R}^d \setminus \{0\} : \frac{x}{|x|} \in \hat{\Sigma} \right\}.$$ 

Let $\rho_\Sigma$ be the geodesic distance in $\Sigma$ to $\partial \Sigma$. We also remark that it coincides with a counterpart Euclidean distance $\tilde{\rho}_\Sigma$.

**Theorem 6.1.** Let the cone $\Sigma \subset \mathbb{R}^d$ be as in (6.2). Then there exists a constant $C = C(\Sigma) > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx \geq C \int_{\Sigma} \rho_\Sigma(x')^{-1} \, [u]_{\Sigma}(x')^2 \, dx', \quad \forall u \in H^1(\mathbb{R}^d \setminus \Sigma).$$

**Remark 6.2.** Denote by $\tilde{\rho}_\Sigma$ the (Euclidean) distance to $\partial \Sigma$ and by $\tilde{\rho}$ the “angle” $\frac{x}{|x|}$ for any $x \in \mathbb{R}^d \setminus \{0\}$. There holds

$$\frac{1}{2} |x| \tilde{\rho}_\Sigma(x') \leq \rho_\Sigma(x') \leq |x| \tilde{\rho}_\Sigma(x'), \quad x' \in \Sigma \setminus \{0\}.$$ 

(6.3)

**Proof.** Consider the following “partition” of $\mathbb{R}^d$ by dyadic spherical shells: For any integer $j \in \mathbb{Z}$ let $A_j$ be the spherical shell

$$A_j = B_{2^{j+1}}(0) \setminus B_2(0) = \{ x \in \mathbb{R}^d : |x| \in (2^j, 2^{j+1}) \}.$$
Then for any function \( f \in L^2(\mathbb{R}^d) \) there holds
\[
\int_{\mathbb{R}^d} |f(x)|^2 dx = \sum_{j \in \mathbb{Z}} \int_{A_j} |f(x)|^2 dx.
\]
Set \( \Sigma_j = A_j \cap \Sigma \). Then the theorem will be proved if we show that
\[
(6.4) \quad \int_{A_j} |\nabla u(x)|^2 dx \geq C \int_{\Sigma_j} \rho_{\Sigma}(x')^{-1}||u|_{\Sigma_j}(x')|^2 dx', \quad \forall u \in H^1(A_j \setminus \Sigma_j),
\]
with a constant \( C \) independent of \( j \). Using the distance equivalence \((6.3)\), we see that \((6.4)\) follows from
\[
(6.5) \quad \int_{A_j} |\nabla u(x)|^2 dx \geq 2C \int_{\Sigma_j} |x'|^{-1}\tilde{\rho}_{\Sigma}(x')^{-1}||u|_{\Sigma_j}(x')|^2 dx', \quad \forall u \in H^1(A_j \setminus \Sigma_j).
\]
Now, the norms in both members of \((6.5)\) are homogeneous by dilatation and have the same homogeneity. As a consequence if we prove \((6.5)\) for \( j = 0 \) with a constant \( C = C_0 \), then we get \((6.5)\) for any \( j \in \mathbb{Z} \) with the same constant \( C = C_0 \) by virtue of the change of variables \( x \mapsto 2^j x \) from \( A_0 \) to \( A_j \). Using that \( \tilde{\rho}_{\Sigma_j}(x') \leq |x'|\tilde{\rho}_{\Sigma}(x') \), we see that the estimate \((6.5)\) for \( j = 0 \) follows from
\[
(6.6) \quad \int_{A_0} |\nabla u(x)|^2 dx \geq 2C_0 \int_{\Sigma_0} \tilde{\rho}_{\Sigma}(x')^{-1}||u|_{\Sigma}(x')|^2 dx', \quad \forall u \in H^1(A_0 \setminus \Sigma_0)
\]
for some positive constant \( C_0 \). Now we are (almost) back to what we have proved in Section 4. Let \( \Omega_{\pm} = A_0 \cap \mathbb{R}^d_{\pm} \). We are in a bounded configuration with the cut \( \Sigma_0 \) contained in the flat surface \( \Gamma_0 = \Gamma \cap A_0 \). Since \( \hat{\Gamma} \setminus \hat{\Sigma} \) is a \((d - 2)\)-set, we find that \( \Gamma_0 \setminus \Sigma_0 \) is a \((d - 1)\)-set and that \( \Omega := A_0 \setminus \overline{\Sigma_0} \) is a connected set. Both configurations (the present one and that in Section 4) differ by the fact that \( \Gamma_0 \) is not the full boundary of \( \Omega_0 \), any more. Nevertheless, we can check that all steps of the former proof are valid: characterization of traces from \( H^1(\Omega_{\pm}) \) to \( H^{1/2}(\Gamma_0) \) and positiveness of the second Neumann eigenvalue in \( \Omega \). This yields the proof of \((6.6)\), and hence the proof of the theorem.

Remark 6.3. Theorem 6.1 can be generalized in the following way: Let \( \hat{\Gamma} \) be a closed Lipschitz hypersurface in \( S \), such that \( S \setminus \hat{\Gamma} \) has two connected components \( \Omega_{\pm} \) that are Lipschitz subdomains of \( S \). Set \( \Gamma \) as the closed Lipschitz hypersurface of \( \mathbb{R}^d \) defined by
\[
\Gamma = \left\{ x \in \mathbb{R}^d \setminus \{0\}, \quad \frac{x}{|x|} \in \hat{\Gamma} \right\}.
\]
With these assumptions on \( \Gamma \) replacing that \( \Gamma \) is an hyperplane, Theorem 6.1 still holds.

The proof of Theorem 6.1 derives from Theorem 2.2 via a localization process. The localized estimates are not a direct consequence of Theorem 2.2, but are obtained by the same chain of arguments. The assumptions of Hypothesis 2.1 are indeed sufficient, but not necessary. Let us give two examples.
Example 6.4 (The penny shape hole). Let \( \Gamma \) be the hyperplane \( \mathbb{R}^{d-1} \times \{0\} \) and \( \Sigma = \Gamma \setminus \overline{B_1(0)} \). This configuration is smooth, but satisfies neither the assumptions of Theorem 2.2 nor those of Theorem 6.1. But if we localize to \( \Omega = B_2(0) \), we obtain, by the same process as before the local estimate (with \( \Sigma_0 = \Sigma \cap \Omega \), but \( \rho_{\Sigma} \) still the distance to \( \partial \Sigma \))

\[
\int_{\Sigma} |\nabla u(x)|^2 dx \geq C \int_{\Sigma_0} \rho_{\Sigma}(x')^{-1} |[u]_{\Sigma_0}(x')|^2 dx', \quad \forall u \in H^1(\Omega \setminus \Sigma_0).
\]

We note that outside \( \Omega \), the function \( x' \mapsto \rho_{\Sigma}(x') \) is equivalent to \( |x'| \). Hence, combining the latter inequality with the estimate (6.1), we obtain

\[
\int_{\Sigma} |\nabla u(x)|^2 dx \geq C \int_{\Sigma} \rho_{\Sigma}(x')^{-1} |[u]_{\Sigma}(x')|^2 dx', \quad \forall u \in H^1(\mathbb{R}^d \setminus \Sigma).
\]

Example 6.5 (The Möbius strip). Consider (in \( \mathbb{R}^{d} \)) a smooth Möbius strip \( \Sigma \) surrounded by a torus \( \Omega \). More specifically, suppose that \( \Sigma \) is generated by smoothly twisting a segment of length 1 along a circular curve \( \gamma \) of radius 2 (for any \( x \in \gamma \), the intersection of \( \Sigma \) with the plane \( \Pi_x \) orthogonal to \( \gamma \) at \( x \) is a segment \( I_x \) such that \( x \) is the middle of \( I_x \)) and \( \Omega \) is the torus of major radius 2 and minor radius 1. Choose as \( \Gamma \) the larger Möbius strip generated by the same curve \( \gamma \) and intervals \( J_x \supset I_x \) of length \( 2 \) and center \( x \). Signed jump of a function cannot be defined across such a \( \Sigma \) but the jump norm can. The torus \( \Omega \) can be cut into several slices so that the topology of each corresponding pieces of \( \Sigma \) and \( \Gamma \) is trivial. Adding the finitely many contributions one obtains

\[
\int_{\Omega} |\nabla u(x)|^2 dx \geq C \int_{\Sigma} \rho_{\Sigma}(x)^{-1} |[u]_{\Sigma}(x)|^2 d\sigma(x), \quad \forall u \in H^1(\Omega \setminus \Sigma).
\]

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