Normalized ground states of nonlinear biharmonic Schrödinger equations with Sobolev critical growth and combined nonlinearities

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Abstract

This paper is devoted to studying the following nonlinear biharmonic Schrödinger equation with combined power-type nonlinearities

\[ \Delta^2 u - \lambda u = \mu |u|^q u + |u|^{4^*-2}u \quad \text{in } \mathbb{R}^N, \]

where \( N \geq 5, \mu > 0, 2 < q < 2 + \frac{8}{N}, \ 4^* = \frac{2N}{N-4} \) is the \( H^2 \)-critical Sobolev exponent, and \( \lambda \) appears as a Lagrange multiplier. By analyzing the behavior of the ground state energy with respect to the prescribed mass, we establish the existence of normalized ground state solutions. Furthermore, all ground states are proved to be local minima of the associated energy functional.

Keywords: Biharmonic Schrödinger equations, Ground state, Normalized solutions, Critical growth, Combined nonlinearities

1. Introduction and main results

In this paper, we are concerned with the following nonlinear biharmonic Schrödinger equation

\[ \Delta^2 u - \lambda u = \mu |u|^q u + |u|^{4^*-2}u \quad \text{in } \mathbb{R}^N \]  

(1.1)

under the constraint

\[ \int_{\mathbb{R}^N} |u|^2 \, dx = c, \]  

(1.2)

where \( c > 0, N \geq 5, \mu > 0, 2 < q < 2 + \frac{8}{N}, \ 4^* = \frac{2N}{N-4} \) and \( \lambda \in \mathbb{R} \) is a Lagrange multiplier.

The interest in studying (1.1)-(1.2) comes from seeking the standing wave with form \( \psi(t, x) = e^{-i\lambda t} u(x) \) of the following time-dependent biharmonic Schrödinger equation

\[ i\partial_t \psi - \Delta^2 \psi + f(|\psi|)\psi = 0. \]  

(1.3)

This equation was considered in [1, 2] to study the stability of solitons in magnetic materials once the effective quasi particle mass becomes infinite. For some recent studies on (1.3), one can see [3, 4, 5, 6] etc. The biharmonic operator \( \Delta^2 \) was also used in [7, 8] to reveal the effects of higher-order dispersion terms in the mixed-dispersion fourth-order Schrödinger equations.

In recent years, much attention has been paid to the study of normalized solutions of nonlinear Schrödinger equations, see for example [9, 10, 11, 12] and the references therein. For some related studies on the mixed-dispersion fourth-order Schrödinger equations, we refer the reader to [13, 14, 15]. As far as we know, there are only very few references to the study of (1.1)-(1.2) in the literature.

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Let $f(|u|) = |u|^{p-2}$. Then (1.3) is usually called $L^2$-subcritical if $p \in (2, 2 + \frac{4}{N})$, $L^2$-critical if $p = 2 + \frac{4}{N}$, and $L^2$-supercritical if $p > 2 + \frac{4}{N}$. Recently, Phan [16] obtained the existence of normalized ground state solutions of the biharmonic Schrödinger equation with a coercive potential and the $L^2$-critical nonlinearity.

In this paper, inspired by [16], we are interested in the existence of normalized ground state solutions of (1.1)-(1.2), where the nonlinearity consists of a $L^2$-subcritical term $\mu|u|^{q-2}u$ and the $H^2$-critical term $|u|^{4-2}u$. We consider the associated energy functional

$$J(u) := \frac{1}{2} \|\Delta u\|_2^2 - \frac{\mu}{q} \|u\|_q^q - \frac{1}{4} \|u\|_4^4, \forall u \in H^2 := H^2(\mathbb{R}^N).$$

By standard arguments as in [17], $J$ is of $C^1$ and any critical point of $J$ restricted to the set $S(c) := \{u \in H^2 : \|u\|_2^2 = c\}$ corresponds to a normalized solution of (1.1).

**Definition 1.1.** We say that a function $u_c \in S(c)$ is a normalized ground state solution to (1.1) if it satisfies $J(u_c) = \inf \{J(u), u \in S(c), (J|_{S(c)}) (u) = 0\}$.

For $\rho > 0$, set

$$\Lambda_\rho(c) := \{u \in S(c) : \|\Delta u\|_2^2 < \rho\}, \quad \partial \Lambda_\rho(c) := \{u \in S(c) : \|\Delta u\|_2^2 = \rho\}.$$

Then we can state our main result:

**Theorem 1.2.** For any $\mu > 0$, there exist $c_0 := c_0(\mu, q, N) > 0$ and $\rho_0 := \rho(c_0) > 0$ such that, for any $c \in (0, c_0)$, $J|_{\partial \Lambda_0(c)}$ has a ground state, which is a local minimizer of $J$ in the set $\Lambda_{\rho_0}(c)$. Furthermore, any ground state of $J|_{\partial \Lambda_{\rho_0}(c)}$ is a local minimizer of $J$ in $\Lambda_{\rho_0}(c)$.

The paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.2.

### 2. Preliminaries

In this section, we establish some useful preliminary results. We recall that $\|u\|_{H^2} := (\int_{\mathbb{R}^N} |\Delta u|^2 + |u|^2 dx)^{\frac{1}{2}}$ is a norm on $H^2$ which is equivalent to the standard one $\|u\|_{H^2} := (\int_{\mathbb{R}^N} |D^2 u|^2 + |\nabla u|^2 + |u|^2 dx)^{\frac{1}{2}}$. We denote by $\| \cdot \|_q$ the standard norm in $L^q(\mathbb{R}^N)$.

Assume that $N \geq 5$. For any $u \in H^2$, we have the following Gagliardo-Nirenberg inequality:

$$\|u\|_r \leq C_{N,r} \|\Delta u\|_2^\frac{\alpha}{2} \|u\|_2^{1-\frac{\beta}{2}}, \quad \forall r \in (2, 4^*), \beta := \frac{N}{2} \left(\frac{1}{2} - \frac{1}{r}\right). \tag{2.1}$$

By similar arguments as in [17], we have the following Lions’ type lemma in $H^2$.

**Lemma 2.1.** If $\{u_n\}$ is bounded in $H^2$ such that $\sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} |u_n|^2 dx \to 0$ as $n \to \infty$, then $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 4^*$.

Define

$$f(c, \rho) := \frac{1}{2} \frac{\mu}{q} C_{N,q} \rho^{\alpha_0} e^{\alpha_1} - \frac{S^4}{4^*} \rho^{\alpha_2}, \forall c > 0, \rho > 0,$$

where

$$\alpha_0 = \frac{(q - 2)N}{8} - 1 \in (-1, 0), \quad \alpha_1 = \frac{2N - q(N - 4)}{8} \in \left(0, \frac{4}{N}\right), \quad \alpha_2 = \frac{4}{N - 4} \in (0, 4],$$

and $S$ is the optimal constant such that

$$\|u\|_{4^*} \leq S \|\Delta u\|_2. \tag{2.2}$$

(See [18]). Using (2.1)-(2.2), we get

$$J(u) \geq \|\Delta u\|_2^2 f(c, \|\Delta u\|_2^2). \tag{2.3}$$
By direct computations, for any \( c > 0 \), the function \( h_c(\rho) := f(c, \rho) \) has a unique global maximum point
\[
\rho_c := \left[ \frac{\alpha_0 \mu C_{q, q}^q}{\alpha_2} \right]^{\frac{1}{q-2}} \frac{\alpha_0}{\alpha_2} c \frac{\mu \tilde{\alpha}_q}{q^{\frac{q-2}{q}}}.
\]
And the maximum value is
\[
\max_{\rho > 0} h_c(\rho) = h_c(\rho_c) = \frac{1}{2} - M c^\frac{q}{q-2},
\]
where
\[
M := \mu C_{q, q}^q \left[ \frac{\alpha_0 \mu C_{q, q}^q}{\alpha_2} \right]^{\frac{1}{q-2}} \frac{\alpha_0}{\alpha_2} + \frac{S^q}{4^q} \left[ \frac{\alpha_0 \mu C_{q, q}^q}{\alpha_2} \right]^{\frac{1}{q-2}} > 0.
\]
Specially, \( f(c_0, \rho_0) = \max_{\rho > 0} h_c(\rho) = 0 \) for \( c_0 := (\frac{1}{M})^\frac{q}{q-2} > 0 \).

**Lemma 2.2.** Let \( c_1 > 0, \rho_1 > 0 \). Then, for any \( c_2 \in (0, c_1) \), we have
\[
f(c_2, \rho) \geq f(c_1, \rho_1), \quad \forall \rho \in \left( 0, \rho_1, \rho_0 \right].
\]

**Proof.** Since \( f(c, \rho) \) is decreasing with respect to \( c > 0 \), we have \( f(c_2, \rho_1) \geq f(c_1, \rho_1) \). In addition, by the facts that \( \alpha_0 + \alpha_1 = \frac{4-2}{2^q}, c_2 \leq c_1, c_2 \in (0, 4] \) and \( 2 < q < \frac{4}{q} + 2 \), we obtain
\[
f(c_2, \frac{c_2}{c_1} \rho_1) - f(c_1, \rho_1) = \mu C_{q, q}^q \rho_1 \left( 1 - \frac{c_2}{c_1} \right)^{\frac{q-2}{q}} \left( 1 - \left( \frac{c_2}{c_1} \right)^{\frac{q-2}{q}} \right) \geq 0.
\]
Since the function \( h_{c_2}(\rho) \) has a unique global maximum, we get (2.5) holds. 

Let \( u \in S(c) \) be arbitrary but fixed. Set \( u_s(x) := \frac{s^2}{4} u(\frac{s^2}{4} x), \forall s > 0 \). Then \( u_s \in S(c) \) for any \( s > 0 \). Set \( \rho_0 := \rho_c > 0 \). Define \( B_{\rho_0} := \{ u \in H^2 : \| \Delta u \|^2 < \rho_0 \} \). Clearly, \( \Lambda_{\rho_0}(c) := S(c) \cap B_{\rho_0} \).

**Lemma 2.3.** For any \( c \in (0, c_0) \), we have
\[
m(c) := \inf_{u \in \Lambda_{\rho_0}(c)} J(u) < 0 < \inf_{u \in \partial \Lambda_{\rho_0}(c)} J(u).
\]

**Proof.** For the first part of (2.6), we define
\[
\psi_s(s) := J(u_s) = \frac{s^2}{2} \| \Delta u_s \|^2 + \frac{\mu}{q} s^{4(q-2) / q} \| u_s \|_q - \frac{S^q}{4^q} \| u_s \|_{4q}^q.
\]
Since \( \frac{N(q-2)}{4} \in (0, 2) \), we deduce that \( \psi_s(s) \to -\infty \) as \( s \to 0^+ \). Then, there exists \( s_0 \) small enough such that \( \| \Delta u_{s_0} \|^2 = s_0^2 \| \Delta u \|^2 < \rho_0 \) and \( J(u_{s_0}) = \psi_{s_0}(s_0) < 0 \). Hence \( m(c) < 0 \).

For any \( u \in \partial \Lambda_{\rho_0}(c) \), we have \( \| \Delta u \|^2 = \rho_0 \). By the choice of \( \rho_0 \), we get \( f(c_0, \rho_0) = 0 \) and \( f(c, \rho_0) > 0 \) for all \( c \in (0, c_0) \). Then, by (2.3) it follows that
\[
J(u) \geq \| \Delta u \|^2 f(\| u \|^2, \| \Delta u \|^2) = \rho_0 f(c, \rho_0) > 0.
\]

**Lemma 2.4.** The function \( c \mapsto m(c) \) is decreasing in \((0, c_0)\).

**Proof.** Assume that \( c_1, c_2 \in (0, c_0) \) and \( c_2 > c_1 \). For any \( \epsilon > 0 \) small enough, there exists \( u \in \Lambda_{\rho_0}(c_1) \) such that
\[
J(u) \leq m(c_1) + \epsilon \quad \text{and} \quad J(u) < 0.
\]
Define \( v(x) := \left( \frac{c_2}{c_1} \right)^{\frac{q-2}{q}} u((\frac{c_2}{c_1})^2 x), \forall x \in \mathbb{R}^N \). Clearly, \( v \in \Lambda_{\rho_0}(c_2) \). Then
\[
m(c_2) \leq J(v) = \frac{1}{2} \| \Delta u \|^2 - \frac{\mu}{q} \left( \frac{c_2}{c_1} \right)^{\frac{q(N-q)}{2}} \| u \|_q - \frac{S^q}{4^q} \| u \|_{4q}^q < J(u).
\]
Hence, together with (2.7) we get \( m(c_2) < m(c_1) + \epsilon \). Since \( \epsilon \) is arbitrary, the conclusion follows. 

Lemma 2.5. (i) The function \( c \in (0, c_0) \mapsto m(c) \) is continuous.

(ii) Let \( c \in (0, c_0) \). Then \( m(c) \leq m(\alpha) + m(c - \alpha) \) for all \( \alpha \in (0, c) \). The inequality is strict if \( m(\alpha) \) or \( m(c - \alpha) \) is reached.

Proof. (i) For any \( c \in (0, c_0) \), we take \( \{c_n\} \subset (0, c_0) \) be such that \( c_n \to c \). We first assume that \( c_n > c \). By Lemma 2.4, we get \( m(c_n) \leq m(c) \). On the other hand, by Lemma 2.3, for any \( \epsilon > 0 \) small enough, there exists \( u_n \in \Lambda_{\rho_0}(c_n) \) such that

\[
J(u_n) \leq m(c_n) + \epsilon \quad \text{and} \quad J(u_n) < 0.
\]

Take \( z_n = \sqrt{c_n} u_n \). Clearly, \( z_n \in \Lambda_{\rho_0}(c) \). Then, noting that \( J(z_n) - J(u_n) \to 0 \), it follows that

\[
m(c) \leq J(z_n) = J(u_n) + (J(z_n) - J(u_n)) = J(u_n) + o_n(1).
\]

By (2.8), we get \( m(c) \leq m(c_n) + \epsilon + o_n(1) \). Hence \( m(c_n) \to m(c) \) as \( c_n \to c^+ \).

In what follows, we consider the case \( c_n < c \). By Lemma 2.4 again we get \( m(c) \leq m(c_n) \). For the opposite side, for any \( \epsilon > 0 \) small, we let \( u \in \Lambda_{\rho_0}(c) \) be such that

\[
J(u) \leq m(c) + \epsilon \quad \text{and} \quad J(u) < 0.
\]

Let \( u_n := \sqrt{c_n} u \). Clearly, \( u_n \in \Lambda_{\rho_0}(c_n) \) for \( n \) large enough, and \( J(u_n) \to J(u) \). Hence

\[
m(c_n) \leq J(u_n) = J(u) + (J(u) - J(u_n)) \leq m(c) + \epsilon + o_n(1).
\]

Since \( \epsilon > 0 \) is arbitrary, we obtain \( m(c_n) \leq m(c) + o_n(1) \). Thus \( m(c_n) \to m(c) \) as \( c_n \to c^- \).

(ii) For any \( \alpha \in (0, c) \), we shall prove that \( m(\theta \alpha) \leq \theta m(\alpha) \), \( \forall \theta \in (1, \frac{1}{\alpha}) \). In fact, by Lemma 2.3, for any \( \epsilon > 0 \) small, there exists \( u \in \Lambda_{\rho_0}(\alpha) \) such that

\[
J(u) \leq m(\alpha) + \epsilon \quad \text{and} \quad J(u) < 0.
\]

Using Lemma 2.2, we have \( f(\alpha, \rho) \geq f(c_0, \rho_0) = 0 \) for any \( \rho \in [\frac{\alpha}{c_0} \rho_0, \rho_0] \). Then, by (2.23) and (2.24) it follows that \( \|\Delta u\|_\rho^2 < \frac{\alpha}{c_0} \rho_0 \). Take \( v = \sqrt{\theta} u \) with \( \theta \in (1, \frac{1}{\alpha}) \). Clearly, \( v \in \Lambda_{\rho_0}(\theta \alpha) \). Then, in view of (2.20),

\[
m(\theta \alpha) \leq J(v) = \frac{1}{2} \theta \|\Delta u\|_\rho^2 - \frac{\mu}{q} \theta \|u\|_\rho^2 < \frac{\alpha}{4} \theta \|u\|_\rho^2 < \theta J(u) \leq \theta (m(\alpha) + \epsilon).
\]

Since \( \epsilon \) is arbitrary, we get \( m(\theta \alpha) \leq \theta m(\alpha) \), which implies that

\[
m(c) = \frac{c - \alpha}{c} m(c) + \frac{\alpha}{c} m(c) = \frac{c - \alpha}{c} m(c - \alpha) + \frac{\alpha}{c} m(\frac{c}{\alpha}) \leq m(c - \alpha) + m(\alpha).
\]

If \( m(\alpha) \) is reached, we can choose \( \epsilon = 0 \). Then the strict inequality follows.

3. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2. Set \( \mathcal{K}_c := \{ u \in \Lambda_{\rho_0}(c) : J(u) = m(c) \} \).

Theorem 3.1. For any \( c \in (0, c_0) \), \( \mathcal{K}_c \neq \emptyset \). Furthermore, the set \( \mathcal{K}_c \) is compact in \( H^2 \), up to translation.

Proof. Fix \( c \in (0, c_0) \). Let \( \{u_n\} \subset B_{\rho_0} \) be such that \( \|u_n\|_2^2 \to c \) and \( J(u_n) \to m(c) \). Then \( J(u_n) = m(c) + o_n(1) \) and \( J(u_n) < 0 \) for \( n \) large enough. Clearly, \( \{u_n\} \) is bounded in \( H^2 \). We claim that \( \{u_n\} \) is non-vanishing. In fact, if not, we may apply Lemma 2.1 to get \( \|u_n\|_q \to 0 \). Then, by (2.21), (2.22) and \( f(c_0, \rho_0) = 0 \) we get, for \( n \) large enough,

\[
0 \geq J(u_n) \geq \|\Delta u_n\|_2^2 \left( \frac{1}{2} - \frac{\|\Delta u_n\|_q^2}{\|\Delta u_n\|_2} \right) + \|u_n\|_q^2 = \frac{\mu}{q} C_{\theta_0}^q \|u_n\|_q^2 + o_n(1) > 0,
\]
Due to (3.3), we have
\[ u \] follows that this local minimizer is a ground state, and any ground state of
\[ J \] follows. Then, by the splitting properties of Brezis-Lieb and the translation invariance, we obtain
\[ J(u_n) = J(u_n(x - y_n)) = J(u_n) + o_n(1). \] (3.4)
Now, we claim that \( \|w_n\|_2^2 \to 0 \). Denote \( \mathcal{T} = \{ \|w_n\|_2^2 > 0 \} \). We assume by contradiction that \( \mathcal{T} < c \). In view of (3.2) and (3.3), for \( n \) large enough, we have \( w_n \in \Lambda_{\rho_0}(\|w_n\|_2^2) \). Then, using (3.4), we get
\[ m(c) = J(w_n) + J(u_c) + o_n(1) \geq m(\|w_n\|_2^2) + J(u_c) + o_n(1). \]
Due to (3.3), we have \( u_c \in \Lambda_{\rho_0}(\mathcal{T}) \). Thus, using Lemma 2.5 (i) and (3.2), we deduce that
\[ m(c) \geq m(c - \mathcal{T}) + J(u_c) \geq m(c - \mathcal{T}) + m(\mathcal{T}). \] (3.5)
If \( J(u_c) > m(\mathcal{T}) \), then it follows from Lemma 2.5 (ii) and (3.5) that
\[ m(c) > m(c - \mathcal{T}) + m(\mathcal{T}) \geq m(c - \mathcal{T} + \mathcal{T}) = m(c), \]
which produces a contradiction. Hence, \( J(u_c) = m(\mathcal{T}) \). Since \( m(\mathcal{T}) \) is reached, by (3.3) and the strict inequality in Lemma 2.5 (ii), we also obtain a contradiction. Thus, the claim holds and then \( \|w_n\|_2^2 \to c \).

In the following, we prove that \( \|\Delta w_n\|_2^2 \to 0 \). Using (3.5) we have \( u_c \in \Lambda_{\rho_0}(c) \), which implies that \( J(u_c) \geq m(c) \). Then (3.4) and \( J(u_n) \to m(c) \) imply that \( J(w_n) \leq o_n(1) \). However, by (3.3) again it follows that \( \{w_n\} \subset B_{\rho_0} \). Then, by \( \|w_n\|_2^2 \to 0 \) we get \( \|\Delta w_n\|_2^2 \to 0 \). Therefore, using (3.4) we obtain \( \|\Delta w_n\|_2^2 \to 0 \).

This together with \( \|\Delta w_n\|_2^2 \to 0 \) shows that \( w_n(x) \to 0 \) in \( H^2 \) and completes the proof.

Define the Pohozaev set by
\[ Q_c := \{ u \in S(c) : Q(u) = 0 \}, \]
where \( Q(u) := \|\Delta u\|_2^2 - \frac{\mu N(q - 2)}{4q} \|u\|_q^q - \|u\|_4^4 \).

**Theorem 3.2.** For any \( c \in (0, c_0) \), if \( m(c) \) is reached, then any ground state is contained in \( \Lambda_{\rho_0}(c) \).

**Proof.** For any \( v \in S(c) \) and \( s > 0 \), by direct calculations we get
\[ \psi_{v,s}'(s) = \frac{1}{s} Q(v_s). \] (3.6)
It is well known that any critical points of \( J \) stay in the set \( Q_c \). Hence, by (3.6) if \( w \in S(c) \) is a ground state, then there exists a \( v \in S(c) \) and a \( s_0 > 0 \) such that \( w = v_{s_0} \). \( J(w) = \psi_{v,s}(s_0) \) and \( \psi_{v,s}'(s_0) = 0 \).

We claim that the function \( \psi_{v,s}(s) \) has at most two zeros. It suffices to show that the function \( \xi(s) := \frac{\psi_{v,s}'(s)}{s} \) has at most two zeros. By direct computations,
\[ \xi'(s) = -2\alpha_0 \frac{\mu N(q - 2)}{4q} s^{2\alpha_0 - 1} \|u\|_q^q - 2\alpha_2 s^{2\alpha_2 - 1} \|u\|_4^4. \]
In view of \( \alpha_0 < 0 \) and \( \alpha_2 > 0 \), the function \( \xi(s) \) has a unique zero, which implies that the claim holds.

Since \( \psi_{v,s}(s) \to 0 \) as \( s \to 0 \), \( \psi_{v,s}'(s) \to -\infty \) as \( s \to \infty \), and \( \psi_{v,s}(s) = J(v_s) > 0 \) when \( v_s \in \partial \Lambda_{\rho_0}(c) = \{ u \in S(c) : \|\Delta u\|_2^2 = \rho_0 \} \), necessarily \( \psi_{v,s}' \) has a first zero \( s_1 > 0 \) corresponding to a local minimum, while \( \psi_{v,s}' \) has a second zero \( s_2 > 0 \) corresponding to a local maximum. In particular, \( v_{s_1} \in \Lambda_{\rho_0}(c) \) and \( J(v_{s_1}) = \psi_{v,s}(s_1) < 0 \). Hence, if \( m(c) \) is reached, then it is the ground state level and the theorem comes true.

**Proof of Theorem 1.2** By Theorem 3.1 (3.2) \( J \) admits a local minimizer in \( \Lambda_{\rho_0}(c) \). Then, by Theorem 3.2 it follows that this local minimizer is a ground state, and any ground state of \( J \) on \( S(c) \) corresponds to a local minimizer in \( \Lambda_{\rho_0}(c) \).
Acknowledgements

This research was supported by NSFC(11971095). This work was done when X. J. Chang visited the Laboratoire de Mathématiques, Université de Bourgogne Franche-Comté during the period from 2021 to 2022 under the support of China Scholarship Council (202006625034), and he would like to thank the Laboratoire for their support and kind hospitality.

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