ON SIGNED $p$-KOSTKA NUMBERS AND THE INDECOMPOSABLE SIGNED YOUNG PERMUTATION MODULES

EUGENIO GIANNELLI, KAY JIN LIM, AND MARK WILDON

ABSTRACT. We study the modular structure of signed Young permutation modules for the symmetric groups. In particular, we prove new reduction theorems for the signed $p$-Kostka numbers, defined to be the multiplicities of indecomposable signed Young modules as direct summands of signed Young permutation modules. In the second part of the article we classify the indecomposable signed Young permutation modules and determine their endomorphism algebras.

1. Introduction

Let $F$ be a field of prime characteristic $p$ and let $\mathfrak{S}_n$ denote the symmetric group of degree $n$. In this article we investigate the modular structure of the $p$-permutation $F\mathfrak{S}_n$-modules obtained by inducing a linear representation of a Young subgroup $\mathfrak{S}_\lambda \leq \mathfrak{S}_n$ to $\mathfrak{S}_n$, where $\lambda$ is a partition of $n$.

The module obtained by inducing the trivial representation of $\mathfrak{S}_\lambda \leq \mathfrak{S}_n$ to $\mathfrak{S}_n$ is the Young permutation module $M^\lambda$. The family of Young permutation modules has been studied extensively by James [15], Klyachko [17] and Grabmeier [9]. In particular, each $M^\lambda$ has a distinguished summand $Y^\lambda$. The Young modules $Y^\lambda$ for $\lambda$ a partition parametrize the isomorphism classes of the indecomposable summands of Young permutation modules. Young modules were originally defined as the images of the projective covers of certain simple modules of Schur algebras under the Schur functor. More recently Erdmann in [6], later corrected in [5], proved the basic properties of Young modules and determined their vertices and Green correspondents using only the representation theory of the symmetric group.

A natural question is to determine the multiplicity of a given Young module $Y^\mu$ in the direct sum decomposition of a given Young permutation module $M^\lambda$. The multiplicities are called $p$-Kostka numbers and are denoted by $[M^\lambda : Y^\mu]$. At the time of writing, a complete understanding of $p$-Kostka numbers seems to be out of reach, but in recent years many partial results and significant advances have been obtained: see for example [7], [8] and [12].
Signed Young permutation modules and indecomposable signed Young modules were introduced by Donkin \[4\]. These modules are defined as follows. Let \( p \geq 3 \), so the sign and trivial representations of \( S_n \) are distinct. Let \( \mathcal{P}^2(n) \) be the set consisting of all the pairs of partitions \((\alpha|\beta)\) such that \( |\alpha| + |\beta| = n \). For \((\alpha|\beta) \in \mathcal{P}^2(n)\), the signed Young permutation module \( M(\alpha|\beta) \) is defined by

\[
M(\alpha|\beta) = \text{Ind}_{S_{\alpha} \times S_{\beta}}^S (F(S_\alpha) \boxtimes \text{sgn}(S_\beta)).
\]

A linear source \( F S_n \)-module having as a Green vertex a Sylow \( p \)-subgroup of a Young subgroup of \( S_n \) is called a signed Young \( F S_n \)-module. Donkin showed in \[4\] that the indecomposable signed Young modules are precisely the indecomposable direct summands of signed Young permutation modules, and that the isomorphism classes of indecomposable signed Young modules for \( F S_n \) are parametrized by pairs of partitions \((\lambda|\mu)\) such that \( |\lambda| + |\mu| = n \) (see \[4, 2.3(6)\]). Let \( Y(\lambda|\mu) \) denote the signed Young module corresponding to \((\lambda|\mu) \in \mathcal{P}^2(n)\). In \[4, 2.3(8)\] it is proved that \( M(\alpha|\beta) \) is isomorphic to a direct sum of indecomposable signed Young modules \( Y(\lambda|\mu) \) for \((\lambda|\mu) \in \mathcal{P}^2(n)\) such that \((\lambda|\mu) \geq (\alpha|\beta)\). (The dominance order on partition pairs is defined in Section 2.2 below.) Moreover, \([M(\lambda|\mu) : Y(\lambda|\mu)] = 1\).

A strong connection between simple Specht modules and indecomposable signed Young modules has been established by Hemmer \[11\] and by Danz and the second author \[3\]. More precisely, Hemmer showed that every simple Specht module is isomorphic to an indecomposable signed Young module, and Danz and the second author established their labels when \( p \geq 5 \).

The aim of this paper is to study signed \( p \)-Kostka numbers, defined to be the multiplicities of indecomposable signed Young modules as direct summands of signed Young permutation modules. In particular we will generalize some of the results obtained by Gill in \[8\] on the \( p \)-Kostka numbers \([M^\lambda : Y^\mu]\).

Our first main theorem is a relation between signed \( p \)-Kostka numbers. We refer the reader to Notation \[4,8\] for the definitions of the composition \( \delta_0 \) and the set \( \Lambda((\alpha|\beta), \rho) \).

**Theorem 1.1.** Let \((\alpha|\beta), (\lambda|\mu) \in \mathcal{P}^2(n)\). Then

\[
[M(\rho \alpha|\rho \beta) : Y(p \lambda|p^2 \mu)] \leq [M(\alpha|\beta) : Y(\lambda|\mu)].
\]

Furthermore, if \( \delta_0 = \emptyset \) for any \((\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)\) then equality holds.

Example \[5,4\] shows that strict inequality may hold in Theorem 1.1. However, in Corollary \[5,3\], we obtain the following asymptotic stability of the signed \( p \)-Kostka numbers

\[
[M(\alpha|\beta) : Y(\lambda|\mu)] \geq [M(\rho \alpha|\rho \beta) : Y(p \lambda|p^2 \mu)] = [M(p^2 \alpha|p^2 \beta) : Y(p^2 \lambda|p^3 \mu)] = \cdots .
\]

When \( \beta = \emptyset \) we have \( \delta_0 = \emptyset \) for any \((\gamma|\delta) \in \Lambda((\alpha|\emptyset), \rho)\). Theorem 1.1 is therefore a generalization of Gill’s result \[8, Theorem 1\].

Our second main theorem describes the relation between signed \( p \)-Kostka numbers for partitions differing by a \( p \)-power of a partition. Let \( \mathcal{P}^2(m) \) be the set consisting of all pairs of compositions \((\alpha|\beta)\) such that \( |\alpha| + |\beta| = m \). We refer the reader to Equation \[3,2\] for the definition of \( \ell_\rho(\lambda|\mu) \).
Theorem 1.2. Let $m$, $n$ and $k$ be natural numbers. Let $(π|\bar{π}) \in \mathcal{C}^2(m)$, $(λ|pμ) \in \mathcal{P}^2(m)$, $(φ|\bar{φ}) \in \mathcal{C}^2(n)$ and $(α|pβ) \in \mathcal{P}^2(n)$. If $k > ℓ(λ|pμ)$, then

$$[M(π + p^k φ|\bar{π} + p^k \bar{φ}) : Y(λ + p^k α|p(μ + p^k β))]$$

$$\geq [M(π|\bar{π}) : Y(λ|pμ)] \cdot [M(φ|p\bar{φ}) : Y(pα|p^2 β)].$$

Moreover, if $p^k > \max\{π_1, \bar{π}_1\}$, then equality holds.

In particular, taking $φ = α = (r)$ and $\bar{φ} = β = ∅$, we see that $[M(π + p^k(r)|\bar{π}) : Y(λ + p^k(r)|pμ)] \geq M[π|\bar{π}] : Y(λ|pμ)]$ with equality whenever $p^k > \max(π_1, \bar{π}_1)$.

The proofs of Theorem 1.1 and Theorem 1.2 rely heavily on a careful examination of the Brauer quotients of the signed Young permutation modules and the Broué correspondents of the indecomposable signed Young modules. We recall the results we need on Broué correspondents in Section 2 below.

Our third main theorem classifies the indecomposable signed Young permutation modules.

Theorem 1.3. Let $(α|β) \in \mathcal{P}^2(n)$. A signed Young permutation module $M(α|β)$ is indecomposable if and only if one of the following conditions holds.

(i) $(α|β) = ((m)|(n))$ for some non-negative integers $m, n$ such that either
   (a) $m = 0$,
   (b) $n = 0$, or
   (c) $m + n$ is divisible by $p$.

(ii) $(α|β)$ is either $((kp − 1, 1)|∅)$ or $∅((kp − 1, 1))$ for some $k \in \mathbb{N}$.

In cases (i)(a) and (i)(b), we have $\text{End}_{F\mathfrak{S}_n}(M(α|β)) \cong F$. If neither case applies then $\text{End}_{F\mathfrak{S}_n}(M(α|β)) \cong F[x]/x^2$.

Since $M((n − 1)|(1)) \cong M((n − 1, 1)|∅)$ and $M(∅|(n − 1)) \cong M(∅(n − 1, 1), 1)$, Case (i) includes all indecomposable signed Young permutation modules up to isomorphism. If $M(α|β)$ is indecomposable then there exist unique partitions $λ$ and $μ$ such that $M(α|β) \cong Y(λ|pμ)$. These partitions are determined in Proposition 6.1.

There is a close connection with polynomial representations of general linear groups. Fix $n, e \in \mathbb{N}$ with $e \geq n$ and let $ρ : \text{GL}_e(F) → \text{GL}_d(F)$ be a representation of $\text{GL}_e(F)$ of dimension $d$. We say that $ρ$ is a polynomial representation of degree $n$ if the matrix coefficients $ρ(X)_{ij}$ for each $i, j \in \{1, \ldots, d\}$ are polynomials of degree $n$ in the coefficients of the matrix $X$. Given a polynomial representation $ρ : \text{GL}_e(F) → \text{GL}(V)$ of degree $n$, the image of $V$ under the Schur functor $f$ is the subspace of $V$ on which the diagonal matrices $\text{diag}(a_1, \ldots, a_e) \in \text{GL}_e(F)$ acts as $a_1 \ldots a_e$. It is easily seen that $f(V)$ is preserved by the permutation matrices in $\text{GL}_e(F)$ that fix the final $e − n$ vectors in the standard basis of $F^e$. Thus $f(V)$ is a module for $F\mathfrak{S}_n$. By [10] §6.1, $f$ is an exact functor from the category of polynomial representations of $\text{GL}_e(F)$ of degree $n$ to the category of $F\mathfrak{S}_n$-modules.

Let $E$ denote the natural $\text{GL}_e(F)$-module. The mixed powers $\text{Sym}^k E \otimes \wedge^k E$ for $(λ|μ) \in \mathcal{P}^2(n)$ generate the category of $\text{GL}_e(F)$-modules of degree $n$. In [4], Donkin defines a listing module to be an indecomposable direct summand of a mixed
power. (As the name suggests, listing modules generalize tilting modules.) By [4, Proposition 3.1c] for each \((\alpha|\beta)\in \mathcal{P}^2(n)\) there exists a unique listing module \(\text{List}(\alpha|p\beta)\) such that \(f(\text{List}(\alpha|p\beta)) \cong Y(\alpha|p\beta)\). By [4, Proposition 3.1a], we have \(f(\text{Sym}^\lambda E \otimes \wedge^\mu E) \cong M(\lambda|\mu)\). Moreover, by [4, Proposition 3.1b], the Schur functor induces an isomorphism

\[
\text{End}_{\text{GL}_e(F)}(\text{Sym}^\lambda E \otimes \wedge^\mu E) \cong \text{End}_{S_n}(M(\lambda|\mu)).
\]

Thus each of our three main theorems has an immediate translation to a result on multiplicities of listing modules in certain mixed powers. For example Theorem 1.3 classifies the indecomposable \(\text{GL}_e(F)\)-mixed powers and shows that each has an endomorphism algebra, as a \(\text{GL}_e(F)\)-module, of dimension at most 2.

Outline. In Section 2 we recall the main ideas concerning the Brauer construction for \(p\)-permutation modules. We also fix the standard notation concerning indecomposable signed Young modules. We describe the Broué correspondents of indecomposable signed Young modules and Brauer quotients of signed Young permutation modules in Sections 3 and 4, respectively. Finally, in Sections 5 and 6 we prove Theorems 1.1, 1.2 and 1.3.

2. Preliminaries

We work with left modules throughout. For background on modular representation theory we refer the reader to [1]. For an account of the representation theory of the symmetric group we refer the reader to [14] or [16].

Let \(G\) be a finite group. Let \(M\) and \(N\) be \(FG\)-modules. We write \(N \mid M\) if \(N\) is isomorphic to a direct summand of \(M\). If \(N\) is indecomposable then, as already seen, \([M : N]\) denotes the number of summands in a direct sum decomposition of \(M\) that are isomorphic to \(N\).

**Lemma 2.1.** Let \(M\) and \(N\) be \(FG\)-modules, and let \(N\) be indecomposable. Suppose that \(H\) is a normal subgroup of \(G\) acting trivially on both the modules \(M\) and \(N\). Let \(\overline{M}\) and \(\overline{N}\) be the corresponding \(F[G/H]\)-modules. Then \([M : N] = [\overline{M} : \overline{N}]\).

**Proof.** Clearly, any indecomposable summand \(L\) of \(M\) gives an indecomposable \(F[G/H]\)-module \(\overline{L}\). Conversely, any indecomposable \(F[G/H]\)-module is an indecomposable \(FG\)-module through inflation. Hence \([M : N] = [\overline{M} : \overline{N}]\) by the Krull–Schmidt Theorem. \(\square\)

2.1. **Broué correspondence.** An \(FG\)-module \(V\) is a \(p\)-permutation module if for every Sylow \(p\)-subgroup \(P\) of \(G\) there exists a linear basis of \(V\) that is permuted by \(P\). A useful characterization of \(p\)-permutation modules is given by the following theorem (see [2, (0.4)]).

**Theorem 2.2.** An indecomposable \(FG\)-module \(V\) is a \(p\)-permutation module if and only if there exists a \(p\)-subgroup \(P\) of \(G\) such that \(V \mid \text{Ind}_P^G F\); equivalently, \(V\) has trivial Green source.
It easily follows that the class of $p$-permutation modules is closed under restriction and induction and under taking direct sums, direct summands, tensor products.

We now recall the definition and the basic properties of Brauer quotients. Given an $FG$-module $V$ and $Q$ a $p$-subgroup of $G$, the set of fixed points of $Q$ on $V$ is denoted
\[ V^Q = \{ v \in V : gv = v \text{ for all } g \in Q \}. \]
It is easy to see that $V^Q$ is an $FN_G(Q)$-module on which $Q$ acts trivially. For $P$ a proper subgroup of $Q$, the relative trace map $\text{Tr}_P^Q : V^P \rightarrow V^Q$ is the linear map defined by
\[ \text{Tr}_P^Q(v) = \sum_{g \in Q/P} gv, \]
where the sum is over a complete set of left coset representatives for $P$ in $Q$. The definition of this map does not depend on the choice of the set of representatives. We observe that
\[ \text{Tr}_P^Q(V) = \sum_{P < Q} \text{Tr}_P^Q(V^P) \]
is an $FN_G(Q)$-module on which $Q$ acts trivially. We define the Brauer quotient of $V$ with respect to $Q$ to be the $F[N_G(Q)/Q]$-module
\[ V(Q) = V^Q/\text{Tr}_P^Q(V). \]
If $V$ is an indecomposable $FG$-module and $Q$ is a $p$-subgroup of $G$ such that $V(Q) \neq 0$, then $Q$ is contained in a Green vertex of $V$. Broué proved in [2] that the converse holds for $p$-permutation modules.

**Theorem 2.3** ([2], Theorem 3.2). Let $V$ be an indecomposable $p$-permutation module and $Q$ be a Green vertex of $V$. Let $P$ be a $p$-subgroup of $G$. Then $V(P) \neq 0$ if and only if $P \leq gQ$ for some $g \in G$.

Here $gQ$ denotes the conjugate $gQg^{-1}$ of $Q$. If $V$ is an $FG$-module with $p$-permutation basis $B$ with respect to a Sylow $p$-subgroup $P$ of $G$ and $R \leq P$, then, taking for each orbit of $R$ on $B$ the sum of the vectors in that orbit, we obtain a basis for $V^R$. Each sum over an orbit of size $p$ or more is a relative trace from a proper subgroup of $R$. Hence $V(R)$ is isomorphic to the $F$-span of
\[ B(R) = \{ v \in B : gv = v \text{ for all } g \in R \}. \]
Thus Theorem 2.3 has the following corollary,

**Corollary 2.4.** Let $V$ be a $p$-permutation $FG$-module with $p$-permutation basis $B$ with respect to a Sylow $p$-subgroup $P$ of $G$, and let $R \leq P$. The Brauer quotient $V(R)$ has $B(R)$ as a basis. Moreover, $V$ has an indecomposable summand with a Green vertex containing $R$ if and only if $B(R) \neq \emptyset$.

The next result explains what is now known as the Broué correspondence.

**Theorem 2.5** ([2], Theorems 3.2 and 3.4). An indecomposable $p$-permutation module $V$ has Green vertex $Q$ if and only if $V(Q)$ is a projective $F[N_G(Q)/Q]$-module. Furthermore,
Lemma 2.7. Let $P_{g,g'}x$. By the previous paragraph, the coefficient $P_{g}$ since $g$ is a $1$-permutation $g$, and $a \in A$. Then, for any $p$-subgroup $P$ of $G$, there exist coefficients $a_1, \ldots, a_r \in A$ such that $B = \{a_1m_1, \ldots, a_rm_r\}$ is a $p$-permutation basis of $M$ with respect to $P$.

Proof. Let $\Omega = \{m_1, \ldots, m_r\}$, and let $\{i_1, \ldots, i_s\}$ be a subset of $\{1, \ldots, r\}$ such that $\Omega$ is the disjoint union of $\Omega_{i_1}, \ldots, \Omega_{i_s}$ where, for each $1 \leq j \leq s$, $\Omega_j = \{m_k : gm_{i_j} = a_gm_k \text{ for some } g \in P \text{ and } a_g \in A\}$.

Suppose that $gm_{i_j} = am_k$ and $g'm_{i_j} = a'm_k$ for some $g, g' \in P$ and $a, a' \in A$. Then $g^{-1}g'm_{i_j} = a'a^{-1}m_{i_j}$ and, consequently, $Fm_{i_j}$ is a one dimensional $F\langle g^{-1}g' \rangle$-module.

Since $P$ is a $p$-subgroup, $Fm_{i_j}$ is the trivial $F\langle g^{-1}g' \rangle$-module. Hence $a = a'$. Thus the coefficient $a_g$ is independent of the choice of $g$, and depends only on $m_{i_j}$ and $m_k$.

For each $1 \leq j \leq s$, let $A_j = \{a_km_k : gm_{i_j} = a_km_k \text{ for some } g \in P \text{ and } a_k \in A\}$.

By the previous paragraph, $\bigcup_{j=1}^s A_j$ is a basis of $M$. It is sufficient to prove that each $A_j$ is permuted by $P$. Let $x \in P$, and let $a_km_k, a_km_{k'} \in A_j$. Suppose that $x(a_km_k) = b(a_km_k)$ for some $b \in F$. We have $gm_{i_j} = a_km_k$ and $g'm_{i_j} = a_km_{k'}$ for some $g, g' \in P$. Thus $g^{-1}xg'm_{i_j} = bm_{i_j}$. Repeating the argument above we see that $Fm_{i_j}$ is the trivial $F\langle g^{-1}g' \rangle$ module and so $b = 1$. □

Lemma 2.6. Let $A$ be a subset of $F^\times$. Let $M$ be an FG-module with a linear basis $\{m_1, \ldots, m_r\}$ such that, if $g \in G$ and $1 \leq i \leq r$, then $gm_i = am_j$ for some $1 \leq j \leq r$ and $a \in A$. Then, for any $p$-subgroup $P$ of $G$, there exist coefficients $a_1, \ldots, a_r \in A$ such that $B = \{a_1m_1, \ldots, a_rm_r\}$ is a $p$-permutation basis of $M$ with respect to $P$.

Proof. Let $\Omega = \{m_1, \ldots, m_r\}$, and let $\{i_1, \ldots, i_s\}$ be a subset of $\{1, \ldots, r\}$ such that $\Omega$ is the disjoint union of $\Omega_{i_1}, \ldots, \Omega_{i_s}$ where, for each $1 \leq j \leq s$, $\Omega_j = \{m_k : gm_{i_j} = a_gm_k \text{ for some } g \in P \text{ and } a_g \in A\}$.

Suppose that $gm_{i_j} = am_k$ and $g'm_{i_j} = a'm_k$ for some $g, g' \in P$ and $a, a' \in A$. Then $g^{-1}g'm_{i_j} = a'a^{-1}m_{i_j}$ and, consequently, $Fm_{i_j}$ is a one dimensional $F\langle g^{-1}g' \rangle$-module.

Since $P$ is a $p$-subgroup, $Fm_{i_j}$ is the trivial $F\langle g^{-1}g' \rangle$-module. Hence $a = a'$. Thus the coefficient $a_g$ is independent of the choice of $g$, and depends only on $m_{i_j}$ and $m_k$.

For each $1 \leq j \leq s$, let $A_j = \{a_km_k : gm_{i_j} = a_km_k \text{ for some } g \in P \text{ and } a_k \in A\}$.

By the previous paragraph, $\bigcup_{j=1}^s A_j$ is a basis of $M$. It is sufficient to prove that each $A_j$ is permuted by $P$. Let $x \in P$, and let $a_km_k, a_km_{k'} \in A_j$. Suppose that $x(a_km_k) = b(a_km_k)$ for some $b \in F$. We have $gm_{i_j} = a_km_k$ and $g'm_{i_j} = a_km_{k'}$ for some $g, g' \in P$. Thus $g^{-1}xg'm_{i_j} = bm_{i_j}$. Repeating the argument above we see that $Fm_{i_j}$ is the trivial $F\langle g^{-1}g' \rangle$ module and so $b = 1$. □

Lemma 2.7. Let $M_1, M_2$ be $p$-permutation FG$_1$- and FG$_2$-modules, and let $P_1$ and $P_2$ be $p$-subgroups of $G_1, G_2$, respectively. Then

$$(M_1 \boxtimes M_2)(P_1 \times P_2) \cong M_1(P_1) \boxtimes M_2(P_2)$$

as a representation of $N_{G_1 \times G_2}(P_1 \times P_2)/(P_1 \times P_2) \cong (N_{G_1}(P_1)/P_1) \times (N_{G_2}(P_2)/P_2)$.

Proof. Let $B_1, B_2$ be permutation bases for $M_1, M_2$ with respect to the $p$-subgroups $P_1, P_2$ respectively. Then $M_1 \boxtimes M_2$ has permutation basis $B_1 \otimes B_2 = \{m_1 \otimes m_2 : m_1 \in B_1, m_2 \in B_2\}$ with respect to the subgroup $P_1 \times P_2$. So $(M_1 \boxtimes M_2)(P_1 \times P_2)$ has a basis $(B_1 \otimes B_2)(P_1 \times P_2) = \{m_1 \otimes m_2 : m_1 \in B_1(P_1), m_2 \in B_2(P_2)\}$. The lemma now follows from Corollary 2.4. □
2.2. Partitions and compositions. Let \( n \in \mathbb{N}_0 \). A composition of \( n \) is a sequence of non-negative integers \( \alpha = (\alpha_1, \ldots, \alpha_r) \) such that \( \alpha_r \neq 0 \) and \( \alpha_1 + \cdots + \alpha_r = n \). In this case, we write \( \ell(\alpha) = r \) and \( |\alpha| = n \). The unique composition of 0 is denoted by \( \emptyset \); we have \( \ell(\emptyset) = 0 \). The Young subgroup \( \mathfrak{S}_\alpha \) is the subgroup

\[
\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_r}
\]

of \( \mathfrak{S}_n \) where the \( i \)th factor \( \mathfrak{S}_{\alpha_i} \) acts on the set \( \{ \alpha_1 + \cdots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i \} \). Let \( \alpha = (\alpha_1, \ldots, \alpha_r) \) and \( \beta = (\beta_1, \ldots, \beta_s) \) be compositions and let \( q \in \mathbb{N} \). We denote by \( qa \) and \( \alpha \bullet \beta \) the compositions of \( q|\alpha| \) and \( |\alpha| + |\beta| \) defined by

\[
qa = (qa_1, \ldots, qa_r), \\
\alpha \bullet \beta = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s),
\]

respectively. We set \( 0\alpha = \emptyset \). We denoted by \( \alpha + \beta \) the composition of \( |\alpha| + |\beta| \) defined by

\[
\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_r + \beta_r, \beta_{r+1}, \ldots, \beta_s)
\]

where we have assume, without loss of generality, that \( s \geq r \).

The composition \( \alpha \) is a partition if it is non-increasing. The partition \( \alpha \) is \( p \)-restricted if \( \alpha_i - \alpha_{i+1} < p \) for all \( i \geq 1 \). We denote the set of compositions, partitions and \( p \)-restricted partitions of \( n \) by \( \mathcal{C}(n) \), \( \mathcal{P}(n) \) and \( \mathcal{RP}(n) \), respectively. The partition \( \alpha \) is \( p \)-regular if its conjugate \( \alpha' \), defined by \( \alpha'_j = |\{ i : \alpha_i \geq j \}| \), is \( p \)-restricted. It is well known that if \( \lambda \) is a partition then there exist unique \( p \)-restricted partitions \( \lambda(i) \) for \( i \in \mathbb{N}_0 \) such that

\[
\lambda = \sum_{i \geq 0} p^i \lambda(i).
\]  

We call this expression the \( p \)-adic expansion of \( \lambda \).

Let \( \mathcal{P}^2(n) \), \( \mathcal{C}^2(n) \) and \( \mathcal{RP}^2(n) \) be the sets consisting of all pairs \( (\lambda|\zeta) \) of partitions, compositions and \( p \)-restricted partitions, respectively, such that \( |\lambda| + |\zeta| = n \). Here \( \lambda \) or \( \zeta \) may be the empty composition \( \emptyset \). For \( (\lambda|\zeta), (\alpha|\beta) \in \mathcal{P}^2(n) \), we say that \( (\lambda|\zeta) \) dominates \( (\alpha|\beta) \), and write \( (\lambda|\zeta) \succeq (\alpha|\beta) \), if, for all \( k \geq 1 \), we have

\[
\begin{align*}
(a) & \quad \sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \alpha_i, \\
(b) & \quad |\lambda| + \sum_{i=1}^k \zeta_i \geq |\alpha| + \sum_{i=1}^k \beta_i.
\end{align*}
\]

(If \( i > \ell(\lambda) \) then take \( \lambda_i = 0 \), and similarly for \( \zeta, \alpha \) and \( \beta \).) This defines a partial order on the set \( \mathcal{P}^2(n) \) called the dominance order. This order becomes the usual dominance order on partitions when restricted to the subsets \( \{ (\lambda|\emptyset) \in \mathcal{P}^2(n) \} \) or \( \{ (\emptyset|\zeta) \in \mathcal{P}^2(n) \} \) of \( \mathcal{P}^2(n) \).

2.3. Modules for symmetric groups. Let \( n \in \mathbb{N}_0 \). Given a subgroup \( H \) of \( \mathfrak{S}_n \), we denote the trivial representation of \( H \) by \( F(H) \), and the restriction of the sign representation of \( \mathfrak{S}_n \) to \( H \) by \( \text{sgn}(H) \). In the case when \( H = \mathfrak{S}_n \) for some composition \( \gamma \) of \( n \) we write \( F(\gamma) \) and \( \text{sgn}(\gamma) \) for \( F(H) \) and \( \text{sgn}(H) \), respectively. If \( \gamma = (n) \) we simplify notation slightly by writing \( F(n) \) and \( \text{sgn}(n) \), respectively.

For \( \lambda \) a \( p \)-regular partition of \( n \), let \( D^\lambda \) be the \( F\mathfrak{S}_n \)-module defined by

\[
D^\lambda = S^\lambda / \text{rad}(S^\lambda),
\]
where \(S^\lambda\) is the Specht module labelled by \(\lambda\) (see \cite[Chapter 4]{FG}). By \cite[Theorem 11.5]{FG} each \(D^\lambda\) is simple, and each simple \(F\mathfrak{S}_n\)-modules is isomorphic to a unique \(D^\lambda\). The simple \(F\mathfrak{S}_n\)-modules can also be labelled by \(p\)-restricted partitions. For \(\lambda \in \mathcal{RP}(n)\) we set \(D_\lambda = \text{Soc}(S^\lambda)\). The connection between the two labelings is given by

\[
D_\lambda \cong D^\lambda \otimes \text{sgn}(n).
\]

For \(\lambda \in \mathcal{RP}(n)\), let \(P(\lambda)\) denote the projective cover of the simple \(F\mathfrak{S}_n\)-module \(D_\lambda\).

### 2.4. Sylow \(p\)-subgroups of \(\mathfrak{S}_n\)

Let \(P_p\) be the cyclic group \(\langle 1, 2, \ldots, p \rangle \leq \mathfrak{S}_p\) of order \(p\). Let \(P_1 = \{1\}\) and, for \(d \geq 1\), set

\[
P_{p^{d+1}} = P_{p^d} \uparrow P_p = \{(\sigma_1, \ldots, \sigma_p; \pi) : \sigma_1, \ldots, \sigma_p \in P_{p^d}, \pi \in P_p\}.
\]

(Our notation for wreath products is taken from \cite[Section 4.1]{FG}.) By \cite[4.1.22, 4.1.24]{FG}, \(P_{p^d}\) is a Sylow \(p\)-subgroup of \(\mathfrak{S}_{p^d}\).

Let \(n \in \mathbb{N}\). Let \(n = \sum_{i=0}^{r} n_ip^i\), where \(0 \leq n_i < p\) for \(i \in \{0, \ldots, r\}\), and \(n_r \neq 0\). By \cite[4.1.22, 4.1.24]{FG}, the Sylow \(p\)-subgroups of \(\mathfrak{S}_n\) are each conjugate to the direct product \(\prod_{i=0}^{r} (P_{p^i})^{n_i}\). Hence if we define \(P_n\) to be a Sylow \(p\)-subgroup of the Young subgroup \(\prod_{i=0}^{r} (\mathfrak{S}_{p^i})^{n_i}\) then \(P_n\) is a Sylow \(p\)-subgroup of \(\mathfrak{S}_n\). The normaliser \(N\mathfrak{S}_n(P_n)\) of \(P_n\) in \(\mathfrak{S}_n\) is denoted by \(N_n\).

Whenever \(\rho = (\rho_1, \ldots, \rho_k) \in \mathcal{C}(n)\), we denote by \(P_{\rho}\), a Sylow \(p\)-subgroup of \(\mathfrak{S}_{\rho}\), defined so that \(P_{\rho} = \prod_{i=1}^{k} P_{\rho_i}\). In the special case when

\[
\rho = (1^{m_0}, p^{m_1}, \ldots, p^{m_r}) = (1^{m_0}, 1, p, p, \ldots, p^{m_r}),
\]

where \(m_i \in \mathbb{N}_0\) for each \(i\), we have \(P_{\rho} = \prod_{i=0}^{r} (P_{p^i})^{m_i}\); in particular, the group \(P_{\rho}\) has precisely \(m_i\) orbits of size \(p^i\) on the set \(\{1, 2, \ldots, n\}\) for each \(i\).

### 3. Broué correspondents of indecomposable signed Young modules

#### 3.1. Modules for wreath products

Let \(m \in \mathbb{N}\) and let \(G\) be a finite group. Recall that the multiplication in the group \(G \wr \mathfrak{S}_m\) is given by

\[
(g_1, \ldots, g_m; \sigma)(g'_1, \ldots, g'_m; \sigma') = (g_1g'_{\sigma^{-1}(1)}, \ldots, g_mg'_{\sigma^{-1}(m)}; \sigma\sigma'),
\]

for \((g_1, \ldots, g_m; \sigma), (g'_1, \ldots, g'_m; \sigma') \in G \wr \mathfrak{S}_m\). Let \(M\) be a \(FG\)-module. The \(m\)-fold tensor product of \(M\) becomes an \(F[G \wr \mathfrak{S}_m]\)-module with the action given by

\[
(g_1, \ldots, g_m; \sigma) \cdot (v_1 \otimes \cdots \otimes v_m) = \text{sgn}(\sigma)g_1v_{\sigma^{-1}(1)} \otimes \cdots \otimes g_mv_{\sigma^{-1}(m)}
\]

for \((g_1, \ldots, g_m; \sigma) \in G \wr \mathfrak{S}_m\) and \(v_1, \ldots, v_m \in M\). We denote this module by \(\hat{M}^{\otimes m}\).

Note that we have twisted the action of the top group \(\mathfrak{S}_m\) by the sign representation. Thus, in the notation of \cite[4.3.14]{FG}, we have

\[
\hat{M}^{\otimes m} = (\# M)^{-m} \otimes \text{Inf}_{\mathfrak{S}_m}^G(\text{sgn}(m)).
\]

We note that if \(k\) is odd then

\[
\text{sgn}(k) = \text{Res}_{\mathfrak{S}_k}^{\mathfrak{S}_m} \text{sgn}(km)
\]
3.2. Building blocks for Broué correspondents. To describe the Broué correspondents of indecomposable signed Young modules, we need the modules $R_k(\alpha|\beta)$, $Q_k(\alpha|\beta)$ and $\overline{Q}_k(\alpha|\beta)$ defined below.

Recall that the projective covers $P(\lambda)$ of simple $F\mathfrak{S}_n$-modules were defined in §2.3.

**Definition 3.1.** Let $k \in \mathbb{N}$ and let $(\alpha|\beta) \in \mathcal{R}\mathcal{P}^2(m)$ where $m \in \mathbb{N}_0$. Let $m_1 = |\alpha|$ and let $m_2 = |\beta|$. The $F[\mathfrak{S}_k \wr \mathfrak{S}_m]$-module $R_k(\alpha|\beta)$ is defined by

$$R_k(\alpha|\beta) = \text{Ind}_{\mathfrak{S}_k \wr (\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2})}^{\mathfrak{S}_k \wr \mathfrak{S}_m} \left( \text{Inf}_{\mathfrak{S}_{m_1}}^{\mathfrak{S}_k \wr \mathfrak{S}_m} (P(\alpha)) \otimes \text{Inf}_{\mathfrak{S}_{m_2}}^{\mathfrak{S}_k \wr \mathfrak{S}_m} (P(\beta) \otimes \text{sgn}(k)^{m_1} \otimes \text{sgn}(k)^{m_2}) \right).$$

By convention, $R_k(\emptyset|\beta) = \text{Inf}_{\mathfrak{S}_m}^{\mathfrak{S}_k \wr \mathfrak{S}_m} (P(\beta) \otimes \text{sgn}(k)^{m_2})$, and similarly for $R_k(\alpha|\emptyset)$. Furthermore if $m = 0$, then $R_k(\alpha|\emptyset)$ is the trivial $F\mathfrak{S}_0$-module. If $k = 1$ then we identify $\mathfrak{S}_k \wr \mathfrak{S}_m$ with $\mathfrak{S}_m$ and get

$$R_1(\alpha|\beta) = \text{Ind}_{\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}}^{\mathfrak{S}_m} (P(\alpha) \otimes (P(\beta) \otimes \text{sgn}(m_2))).$$

Recall that $P_k$ is a fixed Sylow $p$-subgroup of $\mathfrak{S}_k$ and that $N_k = \mathfrak{N}_k(P_k)$.

**Definition 3.2.** Let $k \in \mathbb{N}$ and let $(\alpha|\beta) \in \mathcal{R}\mathcal{P}^2(m)$ where $m \in \mathbb{N}_0$. Let $Q_k(\alpha|\beta)$ be the $F[(N_k/P_k) \wr \mathfrak{S}_m]$-module defined by

$$Q_k(\alpha|\beta) = \text{Res}_{N_k \wr \mathfrak{S}_m}^{\mathfrak{S}_k \wr \mathfrak{S}_m} R_k(\alpha|\beta)$$

via the canonical surjection $(N_k \wr \mathfrak{S}_m)/(P_k)^m \cong (N_k/P_k) \wr \mathfrak{S}_m$.

It is clear that

$$Q_k(\alpha|\beta) = \text{Ind}_{N_k \wr (\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2})}^{N_k \wr \mathfrak{S}_m} (\text{Inf}_{\mathfrak{S}_{m_1}}^{N_k \wr \mathfrak{S}_m} (P(\alpha)) \otimes \text{Inf}_{\mathfrak{S}_{m_2}}^{N_k \wr \mathfrak{S}_m} (P(\beta) \otimes \text{sgn}(N_k)^{m_2})), $$

again regarded as an $F[(N_k/P_k) \wr \mathfrak{S}_m]$-module by this canonical surjection. When $k = 1$ we identify $(N_1/P_1) \wr \mathfrak{S}_m$ with $\mathfrak{S}_m$ and get

$$Q_1(\alpha|\beta) = \text{Ind}_{\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}}^{\mathfrak{S}_m} (P(\alpha) \otimes (P(\beta) \otimes \text{sgn}(m_2))) = R_1(\alpha|\beta).$$

Let $\mathfrak{A}_n$ be the alternating group of degree $n$. Since $(\mathfrak{A}_k)^m$ acts trivially on $R_k(\alpha|\beta)$ we see that $(N_k(P_k)/P_k)^m$ acts trivially on $Q_k(\alpha|\beta)$.

**Definition 3.3.** Let $k \in \mathbb{N}$ and let $(\alpha|\beta) \in \mathcal{R}\mathcal{P}^2(m)$. If $k \geq 2$ let $\overline{Q}_k(\alpha|\beta)$ be the $F[C_2 \wr \mathfrak{S}_m]$-module obtained from $Q(\alpha|\beta)$ via the canonical surjection

$$((N_k/P_k) \wr \mathfrak{S}_m)/(N_k(P_k)/P_k)^m \cong C_2 \wr \mathfrak{S}_m.$$

We define the $F[C_2 \wr \mathfrak{S}_m]$-module $\overline{Q}_1(\alpha|\beta)$ by

$$\overline{Q}_1(\alpha|\beta) = \text{Inf}_{\mathfrak{S}_m}^{C_2 \wr \mathfrak{S}_m} Q_1(\alpha|\beta).$$
We note that when \( k \geq 2 \) the \( F[C_2 \wr \mathfrak{S}_m] \)-module \( \overline{Q}_k(\alpha|\beta) \) can also be obtained directly from \( R_k(\alpha|\beta) \) via the canonical surjection \((\mathfrak{S}_k \wr \mathfrak{S}_m)/\mathfrak{A}_k^n \cong (\mathfrak{S}_k/\mathfrak{A}_k) \wr \mathfrak{S}_m \cong C_2 \wr \mathfrak{S}_m \). Provided \( k \geq 2 \), the modules \( R_k(\alpha|\beta) \) and \( Q_k(\alpha|\beta) \) are indecomposable by \cite[Proposition 5.1]{E}. Thus \( \overline{Q}_k(\alpha|\beta) \) is also indecomposable.

**Lemma 3.4.** For any \( k \geq 2 \) and \( (\alpha|\beta) \in \mathcal{P}^2(\mathcal{P}) \) for some \( m \in \mathbb{N} \), we have
\[
\overline{Q}_k(\alpha|\beta) \cong R_2(\alpha|\beta)
\]
as \( F[C_2 \wr \mathfrak{S}_m] \)-modules.

**Proof.** It suffices to show that \( \overline{Q}_k(\alpha|\beta) \cong R_2(\alpha|\beta) \) as \( F[C_2 \wr \mathfrak{S}_m] \)-modules, where \( \text{sgn}(k) \) is regarded as \( F\mathfrak{S}_2 \)-module via the canonical surjection \( \mathfrak{S}_k/\mathfrak{A}_k \cong \mathfrak{S}_2 \). This is clear since \( \text{sgn}(k) \cong \text{sgn}(2) \) as \( F\mathfrak{S}_2 \)-modules in this regard.

We pause to give a small example of these modules, showing the exceptional nature of the case \( k = 1 \).

**Example 3.5.** Let \( p = 3 \), and let \( k \geq 2 \). Let \( U = \text{Inf}_{\mathfrak{S}_k \wr \mathfrak{S}_3}(\text{sgn}(3)) \). There are four 1-dimensional simple \( F[\mathfrak{S}_k \wr \mathfrak{S}_3] \)-modules, namely
\[
\overline{F}(k)^{\otimes 3}, \text{sgn}(k)^{\otimes 3}, \overline{F}(k)^{\otimes 3} \otimes U, \text{sgn}(k)^{\otimes 3} \otimes U,
\]
where the trivial module appears as \( \overline{F}(\mathfrak{S}_k)^{\otimes 3} \otimes U \). The projective covers of these modules are \( R_k((1,1,1)|\varnothing), R_k(\varnothing|(2,1)), R_k((2,1)|\varnothing), R_k(\varnothing|(1,1,1)) \), respectively. The corresponding modules \( \overline{Q}_k(\alpha|\beta) \) for \( F(\mathfrak{S}_2 \wr \mathfrak{S}_3) \) are precisely the projective covers of the four one-dimensional simple modules for \( F(\mathfrak{S}_2 \wr \mathfrak{S}_3) \). There are four remaining simple modules for \( F[\mathfrak{S}_2 \wr \mathfrak{S}_3] \), each projective; the Clifford theory in \cite[Section 4.3]{E} shows that they are isomorphic to the modules \( \overline{Q}_2(\alpha|\beta) \) where both \( \alpha \) and \( \beta \) are non-empty. By contrast, when \( k = 1 \), identifying \( \mathfrak{S}_1 \wr \mathfrak{S}_3 \) with \( \mathfrak{S}_3 \) as described after Definition \ref{def:3.1}, we have \( \overline{Q}_1((1,1,1)|\varnothing) \cong \overline{Q}_1(\varnothing|(2,1)) \cong P(1,1,1) \cong M^{(2,1)} \otimes \text{sgn} \) and \( \overline{Q}_1((2,1)|\varnothing) \cong \overline{Q}_1(\varnothing|(1,1,1)) \cong P(2,1) \cong M^{(2,1)} \).

### 3.3. Broué correspondents

Suppose that \( (\lambda|\mu) \in \mathcal{P}^2(n) \). Let \( \lambda = \sum_{i \geq 0} p^i \lambda(i) \) and \( \mu = \sum_{i \geq 0} p^i \mu(i) \) be the \( p \)-adic expansions of the partitions \( \lambda \) and \( \mu \) respectively, as defined in \cite{21}. Let \( n_0 = |\lambda(0)| \), and let \( n_i = |\lambda(i)| + |\mu(i - 1)| \) for all \( i \geq 1 \). Let \( \rho = (p^{n_0}, p^{n_1}, \ldots, (p^r)^{n_r}) \) where \( r \) is maximal such that \( n_r \neq 0 \). In this case, we write
\[
\ell_p(\lambda|\mu) = r.
\]
Let \( N_\rho = N_{\mathfrak{S}_n}(P_\rho) \) where \( P_\rho \) is the Sylow \( p \)-subgroup of \( \mathfrak{S}_\rho \) defined in Section 2.4 above. We have
\[
P_\rho = (P_1)^{n_0} \times (P_p)^{n_1} \times \cdots \times (P_{p^r})^{n_r},
N_\rho = \mathfrak{S}_{n_0} \times (N_\rho \wr \mathfrak{S}_{n_1}) \times \cdots \times (N_\rho \wr \mathfrak{S}_{n_r}),
N_\rho / P_\rho = (N_\rho / P_\rho) \wr (\mathfrak{S}_{n_0} \times (N_\rho \wr (\mathfrak{S}_{n_1}) \times \cdots \times (N_\rho / P_\rho) \wr (\mathfrak{S}_{n_r}) \equiv (3.3)) \).
By the remark before Definition 3.3, for each $1 \leq i \leq r$, the subgroup $(P_{\rho})^{n_i}$ of $P_{\rho}$ acts trivially on $R_{\rho}((\lambda(i)||\mu(i-1))$. Hence we obtain the $F[(N_{\rho}/P_{\rho}) \wr S_{n_i}]$-module $Q_{\rho}(\lambda(i)||\mu(i-1))$, and the $F[N_{\rho}/P_{\rho}]$-module

$$Q(\lambda|\mu) = Q_1(\lambda(0)|\varnothing) \otimes Q_{\rho}(\lambda(1)||\mu(0)) \otimes \cdots \otimes Q_{\rho}((\lambda(r)||\mu(r-1)).$$

Similarly, since $(\mathcal{A}_{\rho})^{n_i}$ acts trivially, we obtain the $F[(C_2 \wr S_{n_0}) \times (C_2 \wr S_{n_1}) \times \cdots \times (C_2 \wr S_{n_r})]$-module

$$\overline{Q}(\lambda|\mu) = \overline{Q}_1(\lambda(0)|\varnothing) \otimes \overline{Q}_{\rho}(\lambda(1)||\mu(0)) \otimes \cdots \otimes \overline{Q}_{\rho}((\lambda(r)||\mu(r-1)).$$

The next theorem, describing the Broué correspondents of indecomposable signed Young modules, is fundamental to the proofs of our main theorems. The result was first proved by Donkin in [4, 5.2(2)]. A short proof can be found in [3, Appendix A].

**Theorem 3.6.** The indecomposable signed Young module $Y(\lambda|\mu)$ has Green vertex $P_{\rho}$ and Broué correspondent

$$Y(\lambda|\mu)(P_{\rho}) \cong Q(\lambda|\mu).$$

3.4. **Applications of Theorem 3.6.** We end this section with two applications of Theorem 3.6. The proof of the following lemma is very easy and is left to the reader.

**Lemma 3.7.** Let

$$\rho = (1^{m_0}, p^{m_1}, \ldots, (p^r)^{m_r}),$$

$$\gamma = (1^{m_0}, p^{n_1}, \ldots, (p^s)^{n_s}),$$

be partitions of $m$ and $n$ respectively, and let $k > r$. Then

$$P_{\rho \times p^k \gamma} = P_{\rho} \times P_{p^k \gamma},$$

$$N_{S_{m+p^k n}}(P_{\rho} \times P_{p^k \gamma}) = N_{S_{m}}(P_{\rho}) \times N_{S_{p^k n}}(P_{p^k \gamma}),$$

$$N_{\rho \times p^k \gamma}/P_{\rho \times p^k \gamma} = (N_{\rho}/P_{\rho}) \times (N_{p^k \gamma}/P_{p^k \gamma}).$$

Using Lemma 3.7 we get the following corollary of Theorem 3.6.

**Lemma 3.8.** Let $(\lambda|\mu) \in \mathcal{A}^2(n)$ and let $P_{\rho}$ be the Green vertex of the indecomposable signed Young module $Y(\lambda|\mu)$.

(i) The indecomposable signed Young module $Y(p\lambda|p^2 \mu)$ has Green vertex $P_{pp}$.

(ii) Suppose that $k > l_{\rho}(\lambda|\mu)$ and let $(\alpha|\beta) \in \mathcal{A}^2(m)$ for some $m \in \mathbb{N}$. Then $Y(\lambda+p^k \alpha | p(\mu+p^k \beta))$ has Green vertex $P_{\rho} \times P_{p^k \gamma}$, where $P_{\gamma}$ is the Green vertex of $Y(\alpha|p \beta)$. Moreover,

$$Y(\lambda|\mu)(P_{\rho}) \otimes Y(p^k \alpha|p^{k+1} \beta)(P_{p^k \gamma})$$

is isomorphic to the Broué correspondent $Y(\lambda+p^k \alpha | p(\mu+p^k \beta))(P_{\rho} \times P_{p^k \gamma}).$

**Proof.** For part (i), suppose that $\lambda$ and $\mu$ have $p$-adic expansions $\sum_{i \geq 0} p^i \lambda(i)$ and $\sum_{i \geq 0} p^i \mu(i)$ respectively. It is clear that the partitions $p\lambda$ and $p\mu$ have $p$-adic expansions $p\lambda = \sum_{i \geq 1} p^i \lambda(i-1)$ and $p\mu = \sum_{i \geq 1} p^i \mu(i-1)$, respectively. So $|(p\lambda)(0)| = 0$, and $|(p\lambda)(i)| + |(p\mu)(i-1)| = |\lambda(i-1)| + |\mu(i-2)|$ for all $i \geq 1$, where we set $|\mu(-1)| = 0$. By Theorem 3.6 $Y(p\lambda|p^2 \mu)$ has Green vertex $P_{pp}$.
Let \( r = \ell_p(\lambda | p \mu) \). For part (ii), since \( k > r \), the \( p \)-adic expansions of \( \lambda + p^k \alpha \) and \( \mu + p^k \beta \) are

\[
\lambda + p^k \alpha = \sum_{0 \leq i \leq r} p^i \lambda(i) + \sum_{i \geq k} p^i \alpha(i - k),
\]

\[
\mu + p^k \beta = \sum_{0 \leq i \leq r} p^i \mu(i) + \sum_{i \geq k} p^i \beta(i - k),
\]

respectively. Theorem 3.6 shows that \( Y(\lambda + p^k \alpha | p(\mu + p^k \beta)) \) has Green vertex \( P_\eta \) where

\[
\eta = (1^{\lfloor \lambda(0) \rfloor}, p^{\lfloor \lambda(1) + |\mu(0)| \rfloor}, \ldots, (p^r)^{\lfloor \lambda(r) + |\mu(r-1)| \rfloor}, (p^k)^{\lfloor \alpha(0) \rfloor}, (p^{k+1})^{\lfloor \alpha(1) + |\beta(0)| \rfloor}, \ldots) = \rho \bullet p^k \gamma.
\]

Thus \( P_\eta = P_\rho \rtimes p^k \gamma \). By Theorem 3.6 and Lemma 3.7, the Broué correspondent of \( Y(\lambda + p^k \alpha | p(\mu + p^k \beta)) \) with respect to \( P_\rho \rtimes P_{p^k \gamma} = P_{p^{k+1}} \) is

\[
Q(\lambda + p^k \alpha | p(\mu + p^k \beta)) = Q_1(\lambda(0)|\emptyset) \boxtimes Q_p(\lambda(1)|\mu(0)) \boxtimes \cdots \boxtimes Q_{p^r}(\lambda(r)|\mu(r-1)) \boxtimes \cdots
\]

\[
= Q(\lambda|\rho \mu) \boxtimes Q(\rho^k | \mu) \boxtimes \cdots \boxtimes Q_{p^{k+1}}(\beta(0)|\beta(1)) \boxtimes \cdots
\]

\[
\cong Y(\lambda|\rho \mu)(P_\rho) \boxtimes Y(\rho^k | \mu)(P_{p^{k+1}}),
\]

as required. \( \square \)

The following result is a special case of [8, Theorem 3.18]. The proof is included as an example of Theorem 3.6 and to illustrate a technique used again in the proof of Proposition 6.1.

**Lemma 3.9.** Let \( n \in \mathbb{N} \). If \( n = mp + r \) where \( m \in \mathbb{N}_0 \) and \( r < p \) then \( \text{sgn}(n) \cong Y((r)|(mp)) \).

**Proof.** Let \( n = \sum_{i=0}^r p^i n_i \) where \( n_i < p \) for each \( i \), and let \( \rho = (1^{n_0}, p^{n_1}, \ldots, (p^r)^{n_r}) \). By Theorem 3.6, the indecomposable signed Young module \( Y((r)|(mp)) \) has \( P_\rho \) as a vertex and

\[
Y((r)|(mp))(P_\rho) \cong Q_1((r)|\emptyset) \boxtimes Q_p(\emptyset|(n_1)) \boxtimes \cdots \boxtimes Q_{p^r}(\emptyset|(n_r))
\]

as a module for \( F[(N_\rho/P_\rho)]] \). Since \( n_i < p \) we have

\[
Q(\emptyset|(n_i)) \cong \text{Inf}_{\mathfrak{S}_{n_i}}(F(n_i)) \boxtimes \text{sgn}(k) \cong \text{Res}_{N_\rho}^{S_{n_i}}(\text{sgn}(n_i))
\]

where the second isomorphism follows from (3.1), regarding the right-hand side as a representation of \( N_\rho/P_\rho \). Hence there is an isomorphism of \( FN_\rho \)-modules, \( Y((r)|(mp))(P_\rho) \cong \text{Res}_{N_\rho}^{S_{n_i}}(\text{sgn}(n_i)) \). On the other hand, since \( P_\rho \) is a Sylow \( p \)-subgroup of \( \mathfrak{S}_n \), it is a vertex of \( \text{sgn}(n) \), and clearly \( \text{sgn}(n)(P_\rho) \cong \text{Res}_{N_\rho}^{S_{n_i}}(\text{sgn}(n_i)) \) as an \( FN_\rho \)-module. The Broué correspondence is bijective (see Theorem 2.5), so we have \( Y((r)|(mp)) \cong \text{sgn}(n) \). \( \square \)
4. The Brauer quotients of signed Young permutation modules

In this section, we determine the Brauer quotients of signed Young permutation modules with respect to Sylow subgroups of Young subgroups. Our main result is Proposition 4.12. The description of the Brauer quotients is combinatorial, using the \((\alpha|\beta)\)-tableaux defined below.

Fix \( n \in \mathbb{N} \) and \((\alpha|\beta) \in \mathcal{S}(n)\). Suppose that \( \alpha = (\alpha_1, \ldots, \alpha_r) \) and \( \beta = (\beta_1, \ldots, \beta_s) \), where \( r = \ell(\alpha) \) and \( s = \ell(\beta) \). The diagram \([\alpha] \bullet [\beta]\) is the set consisting of the boxes \((i, j) \in \mathbb{N}^2\) for \( i \) and \( j \) such that \( \text{either } 1 \leq i \leq r \) and \( 1 \leq j \leq \alpha_i \) or \( r + 1 \leq i \leq r + s \) and \( 1 \leq j \leq \beta_{i-r} \). A box \((i, j)\) is said to be in row \( i \). The subset of \([\alpha] \bullet [\beta]\) consisting of the boxes belonging to the first \( r \) rows (respectively, the last \( s \) rows) is denoted by \([\alpha] \bullet \emptyset \) (respectively, \( \emptyset \bullet [\beta]\)).

**Definition 4.1.** An \((\alpha|\beta)\)-tableau is a bijective function \([\alpha] \bullet [\beta] \rightarrow \{1, \ldots, n\}\). For \((i, j) \in [\alpha] \bullet [\beta]\), we define the \((i, j)\)-entry of an \((\alpha|\beta)\)-tableau \( T \) to be \( T(i, j) \).

We represent an \((\alpha|\beta)\)-tableau \( T \) by putting the \((i, j)\)-entry of \( T \) in the box \((i, j)\) of the diagram \([\alpha] \bullet [\beta]\). Considering \([\alpha] \bullet \emptyset \) as the Young diagram \([\alpha]\), we denote the \( \alpha \)-tableau \( T([\alpha] \bullet \emptyset) \) by \( T_+ \). Similarly, we denote the \( \beta \)-tableau \( T(\emptyset \bullet [\beta]) \) by \( T_- \). It will sometimes be useful to write \( T = (T_+|T_-) \).

The \((\alpha|\beta)\)-tableau \( T \) is row standard if the entries in each row of \( T \) are increasing from left to right, \( i.e. \) both \( T_+ \) and \( T_- \) are row standard in the usual sense. We denote by \( T^{a|b} \) the unique row standard \((\alpha|\beta)\)-tableau such that for all \( i, j \in \{1, \ldots, n\} \), if \( i \) is in row \( a \) of \( T \) and \( j \) is in row \( b \) of \( T \) and \( i \leq j \) then \( a \leq b \). For example,

\[
T^{(2,1)|(3)} = \begin{bmatrix}
1 & 2 \\
3 \\
4 & 5 & 6
\end{bmatrix}
\]

Let \( \mathcal{T}(\alpha|\beta) \) be the set of all \((\alpha|\beta)\)-tableaux. If \( T \in \mathcal{T}(\alpha|\beta) \) and \( g \in \mathfrak{S}_n \) then we define \( g \cdot T \) to be the \((\alpha|\beta)\)-tableau obtained by applying \( g \) to each entry of \( T \), \( i.e. \) \((g \cdot T)(i, j) = g(T(i, j))\). This defines an action of \( \mathfrak{S}_n \) on the set of \((\alpha|\beta)\)-tableaux. The vector space \( F\mathcal{T}(\alpha|\beta) \) over \( F \) with basis \( \mathcal{T}(\alpha|\beta) \) is therefore a permutation \( FG_n \)-module.

For each \( T \in \mathcal{T}(\alpha|\beta) \), let \( R(T) \leq \mathfrak{S}_n \) be the row stabilizer of \( T \) in \( \mathfrak{S}_n \), consisting of those \( g \in \mathfrak{S}_n \) such that the rows of \( T \) and \( g \cdot T \) coincide as sets. Then \( R(T) = R(T_+) \times R(T_-) \), where \( R(T_+) \) and \( R(T_-) \) are the row stabilizers of \( T_+ \) and \( T_- \) respectively, in the usual sense. Denote by \( U(\alpha|\beta) \) the subspace of \( F\mathcal{T}(\alpha|\beta) \) spanned by

\[
\{ T - \text{sgn}(g_2)g_1g_2 \cdot T : T \in \mathcal{T}(\alpha|\beta), (g_1, g_2) \in R(T_+) \times R(T_-) \}.
\]

In fact \( U(\alpha|\beta) \) is an \( FG_n \)-submodule of \( F\mathcal{T}(\alpha|\beta) \), since for all \( h \in \mathfrak{S}_n \) and for any \( T - \text{sgn}(g_2)g \cdot T \in U(\alpha|\beta) \), where \( T \in \mathcal{T}(\alpha|\beta) \), \((g_1, g_2) \in R(T_+) \times R(T_-) \) and \( g = g_1g_2 \), we have

\[
h \cdot (T - \text{sgn}(g_2)g \cdot T) = (h \cdot T) - \text{sgn}(^hg_2)^h g \cdot (h \cdot T) \in U(\alpha|\beta),
\]

since \(^hg \in ^hR(T) = R(h \cdot T) \) and \(^hg_2 \in R((h \cdot T)_-).\)
Definition 4.2. For each $T \in \mathcal{T}(\alpha|\beta)$, we write

$$\{T\} = \{(T_+|T_-)\}$$

for the element $T + U(\alpha|\beta) \in F\mathcal{T}(\alpha|\beta)/U(\alpha|\beta)$ and call it an $(\alpha|\beta)$-tabloid.

Note that $g(T) = \{g \cdot T\}$ for all $g \in \mathfrak{S}_n$ and $T \in \mathcal{T}(\alpha|\beta)$. If $T, T' \in \mathcal{T}(\alpha|\beta)$ are such that $T_- = T'_-$ and $T'_+$ is obtained by swapping two entries in the same row of $T_+$ then $\{T\} = \{T'\}$. On the other hand, if $T_+ = T'_+$ and $T'_-$ is obtained by swapping two entries in the same row of $T_-$ then $\{T'\} = -\{T\}$.

Let

$$\Omega(\alpha|\beta) = \{\{T\} : T \text{ is a row standard (} \alpha|\beta \text{)-tableau}\} \subseteq F\mathcal{T}(\alpha|\beta)/U(\alpha|\beta).$$

It is clear that $\Omega(\alpha|\beta)$ is an $F$-basis of $F\mathcal{T}(\alpha|\beta)/U(\alpha|\beta)$. We write $F\Omega(\alpha|\beta)$ for the $F\mathfrak{S}_n$-module $F\mathcal{T}(\alpha|\beta)/U(\alpha|\beta)$.

Lemma 4.3. Let $(\alpha|\beta) \in \mathcal{C}^2(n)$.

(i) The $F\mathfrak{S}_n$-module $F\Omega(\alpha|\beta)$ is isomorphic to the signed Young permutation $M(\alpha|\beta)$.

(ii) For any $p$-subgroup $P$ of $\mathfrak{S}_n$, there exist coefficients $a_{\{T\}} \in \{\pm 1\},$ one for each $\{T\} \in \Omega(\alpha|\beta)$, such that

$$\{a_{\{T\}}\{T\} : \{T\} \in \Omega(\alpha|\beta)\}$$

is a $p$-permutation basis for $F\Omega(\alpha|\beta) \cong M(\alpha|\beta)$ with respect to $P$.

Proof. By the remarks after Definition 4.2 there is an isomorphism $F(\alpha)\boxtimes \text{sgn}(\beta) \cong F(1^{a|\beta})$ of $F[\mathfrak{S}_n \times \mathfrak{S}_\beta]$-modules. Since $|\Omega(\alpha|\beta)| = \dim F M(\alpha|\beta)$, part (i) follows from the characterization of induced modules in [1, §8, Corollary 3, Lemma 4]. Part (ii) follows from Lemma 2.6, since, for all $\{T\} \in \Omega(\alpha|\beta)$ and $\sigma \in \mathfrak{S}_n$, we have $\sigma\{T\} = \pm\{T'\}$ for some $\{T'\} \in \Omega(\alpha|\beta)$. \hfill \Box

In view of Lemma 4.3(i), we shall identify $M(\alpha|\beta)$ with $F\Omega(\alpha|\beta)$, so that $M(\alpha|\beta)$ has the set of $(\alpha|\beta)$-tabloids as a basis.

The following corollary follows from Lemma 4.3 and Corollary 2.4.

Corollary 4.4. Let $(\alpha|\beta) \in \mathcal{C}^2(n)$ and let $P$ be a $p$-subgroup of $\mathfrak{S}_n$.

(i) The Brauer quotient $F[N_{\mathfrak{S}_n}(P)/P]$-module $M(\alpha|\beta)(P)$ has a linear basis consisting of all the $(\alpha|\beta)$-tabloids $\{T\}$ that are fixed by $P$.

(ii) Let $\rho = (1^{n_0}, p^{n_1}, \ldots, (p')^{n_r})$ be a partition of $n$. The group

$$N_\rho/P_\rho = \mathfrak{S}_{n_0} \times ((N_{p'}/P_{p'}) \wr \mathfrak{S}_{n_1}) \times \cdots \times ((N_{p'}/P_{p'}) \wr \mathfrak{S}_{n_r})$$

acts on the set of $P_\rho$-fixed $(\alpha|\beta)$-tabloids by transitively permuting the $P_\rho$-orbits of size $p^i$ according to $\mathfrak{S}_{n_i}$ and, within each $P_\rho$-orbit of size $p^i$, permuting its entries according to $N_{p^i}/P_{p^i}$, for all $i \in \{0, 1, \ldots, r\}$.

More explicitly, the basis in (i) consists of all $(\alpha|\beta)$-tabloids $\{T\}$ such that $T$ is row standard and each row of $T$ is a union of orbits of the $P$-action on $\{1, \ldots, n\}$. This can be seen in the following example.
Example 4.5. Let $p = 3$. Consider the 3-subgroups $Q_1 = \langle(4,5,6),(7,8,9)\rangle$ and $Q_2 = \langle(1,2,3),(4,5,6),(7,8,9)\rangle$ of $\mathcal{G}_6$. By Corollary 4.4 $M((2,1)|(6))(Q_1)$ has a basis consisting of the $\langle(2,1)|(6)\rangle$-tableaux

\[
\begin{align*}
\begin{array}{ccccccc}
1 & 2 & \ & \ & \ & 3 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array},
\begin{array}{ccccccc}
1 & 3 & \ & \ & \ & 2 \\
2 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array},
\begin{array}{ccccccc}
2 & 3 & \ & \ & \ & 1 \\
1 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\end{align*}
\]

where the horizontal line separates each $T_+$ from $T_-$. Since there are no $\langle(2,1)|(6)\rangle$-tableaux satisfying the condition in Corollary 4.4 when $Q_1$ is replaced with $Q_2$, we have $M((2,1)|(6))(Q_2) = 0$. By (3.3) taking $\rho = (3,3,1,1,1)$ we have

\[N_{\mathcal{G}_6}(Q_1) = \mathcal{G}_3 \times N_{\mathcal{G}_3}(P_3) \setminus \mathcal{G}_2 = \mathcal{G}_3 \times \mathcal{G}_3 \setminus \mathcal{G}_2.\]

The first factor $\mathcal{G}_3$ permutes the entries 1, 2, 3 of each tableau without sign, and the second factor $\mathcal{G}_3 \setminus \mathcal{G}_2$ permutes the entries 4, 5, 6, 7, 8, 9 with sign. The subgroup $Q_1$ acts trivially. Thus if $\{U\}$ and $\{V\}$ are the first two $\langle(2,1)|(3)\rangle$-tableaux above then

\[\text{Res}_{N_{\mathcal{G}_6}(Q_1)}^F\{U\} \cong \text{sgn}(3)\Rightarrow \]

and $(23)(45)\{V\} = -\{U\}$. Note that the first isomorphism above requires the sign twist in the definition of $\tilde{M}$ that we commented on in §3.1.

The modules $V_k(\gamma|\delta)$, $W_k(\gamma|\delta)$ and $W_k(\gamma|\delta)$ in Definition 4.6 are the analogues of the modules $R_k(\alpha|\beta)$, $Q_k(\alpha|\beta)$ and $Q_k(\alpha|\beta)$ defined in Definitions 3.1, 3.2 and 3.3.

Definition 4.6. Let $m \in \mathbb{N}_0$. Let $(\gamma|\delta) \in \mathcal{G}^2(m)$ be such that $|\gamma| = m_1$ and $|\delta| = m_2$. Let $k \in \mathbb{N}$.

(i) We define $V_k(\gamma|\delta)$ to be the $F[\mathcal{G}_k \wr \mathcal{G}_m]$-module

\[V_k(\gamma|\delta) = \text{Ind}_{\mathcal{G}_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m} \left( \text{Inf}_{\mathcal{G}_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m}(M^\gamma) \boxtimes \left( \text{Inf}_{\mathcal{G}_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m}(M^\delta) \otimes \text{sgn}(k) \Rightarrow \right) \right).\]

(ii) We define $W_k(\gamma|\delta)$ to be the $F[(N_k/P_k) \wr \mathcal{G}_m]$-module obtained from

\[\text{Res}_{N_k \wr \mathcal{G}_m}^F V_k(\gamma|\delta) \cong \text{Ind}_{N_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m} \left( \text{Inf}_{\mathcal{G}_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m}(M^\gamma) \boxtimes \left( \text{Inf}_{\mathcal{G}_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m}(M^\delta) \otimes \text{sgn}(N_k) \Rightarrow \right) \right)\]

via the canonical surjection $(N_k/P_k) \wr \mathcal{G}_m/(P_k)^m \cong (N_k/P_k) \wr \mathcal{G}_m$.

(iii) For $k \geq 2$, let $W_k(\gamma|\delta)$ be $F[C_2 \wr \mathcal{G}_m]$-module obtained from $W_k(\gamma|\delta)$ via the canonical surjection $((N_k/P_k) \wr \mathcal{G}_m)/(N_{\mathcal{G}_4}(P_k)/P_k)^m \cong C_2 \wr \mathcal{G}_m$. We define

\[W_1(\gamma|\delta) = \text{Inf}_{\mathcal{G}_m}^{C_2 \wr \mathcal{G}_m} W_1(\gamma|\delta).\]

We note that $W_k(\gamma|\delta)$ may equivalently be defined to be the $F[(N_k/P_k) \wr \mathcal{G}_m]$-module obtained from

\[\text{Ind}_{N_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m} \left( F(N_k \wr \mathcal{G}_m) \boxtimes \text{sgn}(N_k) \Rightarrow \right) \]

via the canonical surjection in (ii). (The $F[N_k \wr \mathcal{G}_m]$-module $\text{sgn}(N_k) \Rightarrow$ was defined in (3.1).) We have $W_1(\gamma|\delta) = V_1(\gamma|\delta)$ as $F\mathcal{G}_m$-modules. When $k \geq 2$, the $F[C_2 \wr \mathcal{G}_m$-modules

\[\text{Ind}_{N_k \wr \mathcal{G}_m}^{\mathcal{G}_k \wr \mathcal{G}_m} \left( F(N_k \wr \mathcal{G}_m) \boxtimes \text{sgn}(N_k) \Rightarrow \right) \]

via the canonical surjection in (ii). (The $F[N_k \wr \mathcal{G}_m]$-module $\text{sgn}(N_k) \Rightarrow$ was defined in (3.1).) We have $W_1(\gamma|\delta) = V_1(\gamma|\delta)$ as $F\mathcal{G}_m$-modules. When $k \geq 2$, the $F[C_2 \wr \mathcal{G}_m$-modules
standard \(|\alpha|\beta\rangle\) is isomorphic to both the \(F[C_2 \wr S_m]\)-modules \(V_k(\alpha|\beta)\) and \(\text{Res}_{S_k}^{S_n} V_k(\alpha|\beta)\) via the canonical surjections \((S_k \wr S_m)/2^m \cong (S_k/2^k) \wr S_m \cong C_2 \wr S_m\) and \((N_k \wr S_m)/(N_k (P_k))/2^m \cong C_2 \wr S_m\), respectively.

The proof of the following lemma is very similar to the proof of Lemma 3.4. For this reason it is left to the reader.

**Lemma 4.7.** For all \(k \geq 2\) and all \((\gamma|\delta) \in \mathcal{C}^2(m)\), we have
\[
\overline{W}_k(\gamma|\delta) \cong V_2(\gamma|\delta),
\]
as \(F[C_2 \wr S_m]\)-modules.

**Notation 4.8.** Let \((\alpha|\beta) \in \mathcal{C}^2(n)\) and let \(\rho = (p_1^{r_1}, p_2^{r_2}, \ldots, p_r^{r_r})\) be a partition of \(n\). We write \(\Lambda((\alpha|\beta), \rho)\) for the set consisting of all pairs of compositions \((\gamma|\delta) = (\gamma_0, \gamma_1, \ldots, \gamma_r|\delta_0, \delta_1, \ldots, \delta_r)\) such that:

1. \(\alpha = \sum_{i=0}^{r} p_i^\gamma_i, \beta = \sum_{i=0}^{r} p_i^\delta_i\), and
2. \(|\gamma_i| + |\delta_i| = n_i\) for \(i \in \{0, \ldots, r\}\).

Let \((\alpha|\beta) \in \mathcal{C}^2(n)\). Recall that \(\Omega((\alpha|\beta))\) is the basis of \(M((\alpha|\beta))\) consisting of all standard \((\alpha|\beta)\)-tabloids. As remarked after Corollary 4.4 an \(F\)-basis of \(M((\alpha|\beta))(P_\rho)\) is obtained by taking those \((\alpha|\beta)\)-tabloids \(\{(T_+|T_-)\} \in \Omega((\alpha|\beta))\) such that the rows of \(T_+\) and \(T_-\) are unions of the orbits of \(P_\rho\). Given such a basis element \(\{(T_+|T_-)\}\) and \(i \in \{0, \ldots, r\}\), let \((\gamma_i)\) and \((\delta_i)\) be the numbers of \(P_\rho\)-orbits of length \(p_i\) in rows \(j\) and \(k\) of \(T_+\) and \(T_-\), respectively. For each \(i \in \{0, \ldots, r\}\), let
\[
\gamma_i := ((\gamma_i)_1, (\gamma_i)_2, \ldots),
\delta_i := ((\delta_i)_1, (\delta_i)_2, \ldots).
\]
Note that \(|\gamma_i| + |\delta_i| = n_i\) for each \(i\), and so \((\gamma_0, \ldots, \gamma_r, \delta_0, \ldots, \delta_r) \in \Lambda((\alpha|\beta), \rho)\). We say that the \((\alpha|\beta)\)-tableau \(\{(T_+|T_-)\}\) is of \(\rho\)-type \((\gamma|\delta)\). We denote the set of all \((\alpha|\beta)\)-tabloids of \(\rho\)-type \((\gamma|\delta)\) by \(\Omega((\alpha|\beta), \rho)_{(\gamma|\delta)}\). Then the disjoint union
\[
\Omega((\alpha|\beta), \rho) := \bigcup_{(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)} \Omega((\alpha|\beta), \rho)_{(\gamma|\delta)} \tag{4.2}
\]
is an \(F\)-basis of \(M((\alpha|\beta))(P_\rho)\). Thus, as \(F\)-vector spaces, we have
\[
M((\alpha|\beta))(P_\rho) = F\Omega((\alpha|\beta), \rho) = \bigoplus_{(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)} F\Omega((\alpha|\beta), \rho)_{(\gamma|\delta)}. \tag{4.3}
\]
It is clear that (4.3) is in fact a decomposition of \(FN_\rho\)-modules, since \(N_\rho\) permutes orbits of \(P_\rho\) of the same size as blocks for its action, and therefore preserves the \(\rho\)-type in its action on \((\alpha|\beta)\)-tabloids. Furthermore, \(P_\rho\) fixes all \((\alpha|\beta)\)-tabloids of \(\rho\)-type. Therefore we obtain the following lemma.

**Lemma 4.9.** Let \((\alpha|\beta) \in \mathcal{C}^2(n)\) and \(\rho = (p_1^{r_1}, p_2^{r_2}, \ldots, p_r^{r_r})\) be a partition of \(n\). The Brauer quotient of \(M((\alpha|\beta))\) with respect to the subgroup \(P_\rho\) has the following direct sum decomposition into \(F[N_\rho/P_\rho]\)-modules:
\[
M((\alpha|\beta))(P_\rho) = \bigoplus_{(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)} F\Omega((\alpha|\beta), \rho)_{(\gamma|\delta)}.
\]
By Lemma \[4.9\], to understand the Brauer quotient \(M(\alpha|\beta)(P_\rho)\) of the signed Young module \(M(\alpha|\beta)\), it suffices to understand each of the \(F[N_\rho/P_\rho]\)-modules \(F\Omega((\alpha|\beta), \rho)(\gamma|\delta)\).

**Definition 4.10.** Suppose that \((\alpha|\beta) \in \mathcal{C}^2(n)\) and that \(\rho = (1^{n_0}, p^{n_1}, \ldots, (p^r)^{n_r})\) is a partition of \(n\). Let the orbits of \(P_\rho\) of size \(p^i\) be \(O_{i_1}, \ldots, O_{i_{n_i}}\). Let

\[\Theta : \Omega((\alpha|\beta), \rho) \rightarrow \bigcup_{(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)} \prod_{i=0}^r \Omega(\gamma_i|\delta_i)\]

be the bijective function defined as follows. Suppose that \(\{T\} \in \Omega((\alpha|\beta))\) is of \(\rho\)-type \((\gamma|\delta)\). For each \(0 \leq i \leq r\), let \(\{T_i\}\) be the \((\gamma_i|\delta_i)\)-tabloid such that \(T_i\) is row standard, and row \(k\) of \((T_i)_+\) (respectively, \((T_i)_-\)) contains \(j\) if and only if row \(k\) of \(T_+\) (respectively, \(T_-\)) contains the orbit \(O_{i,j}\). Define \(\Theta(\{T\}) = \{(T_i)\}_{i=0,1,\ldots,r}\).

We note that, by definition of \(P_\rho\),

\[O_{i,j} = \left\{ (j-1)p^i + 1 + \sum_{\ell=0}^{i-1} n_\ell p^\ell, \ldots, j p^i + \sum_{\ell=0}^{i-1} n_\ell p^\ell \right\},\]

for \(i = 0, \ldots, r\) and \(j = 1, \ldots, n_i\).

Clearly, the bijection \(\Theta\) in Definition \[4.10\] restricts to a bijection, also denoted \(\Theta\),

\[\Theta : \Omega((\alpha|\beta), \rho)(\gamma|\delta) \rightarrow \prod_{i=0}^r \Omega(\gamma_i|\delta_i)\]

Since \(\vert \Omega(\gamma_i|\delta_i) \vert = \dim_F M(\gamma_i|\delta_i) = [\mathfrak{S}_{n_i} : (\mathfrak{S}_{\gamma_i} \times \mathfrak{S}_{\delta_i})]\), we obtain the following lemma.

**Lemma 4.11.** Let \((\alpha|\beta) \in \mathcal{C}^2(n), \rho = (1^{n_0}, p^{n_1}, \ldots, (p^r)^{n_r})\) such that \(\vert \rho \vert = n\), and let \((\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)\). Set

\[H := \prod_{i=0}^r N_{p^i} \cdot (\mathfrak{S}_{\gamma_i} \times \mathfrak{S}_{\delta_i}) = \prod_{i=0}^r (N_{p^i} \cdot \mathfrak{S}_{\gamma_i}) \times (N_{p^i} \cdot \mathfrak{S}_{\delta_i}) \leq N_\rho.\]

Then \(\vert \Omega((\alpha|\beta), \rho)(\gamma|\delta) \vert = [N_\rho : H]\).

The next proposition is a generalization of \[6\] Proposition 1.

**Proposition 4.12.** Suppose that \((\alpha|\beta) \in \mathcal{C}^2(n)\) and that \(\rho = (1^{n_0}, p^{n_1}, \ldots, (p^r)^{n_r})\) is a partition of \(n\). Regarded as an \(F[N_\rho/P_\rho]\)-module, the Brauer quotient of \(M(\alpha|\beta)(P_\rho)\) of the signed Young permutation module \(M(\alpha|\beta)\) with respect to \(P_\rho\) satisfies

\[M(\alpha|\beta)(P_\rho) \cong \bigoplus_{(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)} W_1(\gamma_0|\delta_0) \otimes W_p(\gamma_1|\delta_1) \otimes \cdots \otimes W_p(\gamma_r|\delta_r).\]

**Proof.** Recall that for each \((\lambda|\mu) \in \mathcal{C}^2(n)\), we have a row-standard \((\lambda|\mu)\)-tableau \(T^{(\lambda|\mu)}\), as defined immediately after Definition \[4.1\]. Fix \((\gamma|\delta) \in \Lambda((\alpha|\beta)|\rho)\) and let
\[ Z = F \Omega((\alpha|\beta), \rho)(\gamma|\delta). \] By Lemma 4.9, it suffices to show that \( Z \cong \bigoplus_{i=0}^{r} W_{\rho}(\gamma_i|\delta_i) \) as \( FN_\rho \) modules with \( P_\rho \) acting trivially, or equivalently, by (4.1), that
\[
Z \cong \bigoplus_{i=0}^{r} \text{Ind}^{N_\rho}_{N_\rho'}(\mathcal{S}_{\gamma_i} \times \mathcal{S}_{\delta_i}) \left( F(N_\rho \lhd \mathcal{S}_{\gamma_i}) \boxtimes \text{sgn}(N_\rho')^{\otimes \delta_i} \right). \tag{4.4}
\]

Let \( \{S\} \in \Omega((\alpha|\beta), \rho)(\gamma|\delta) \) be the unique \((\alpha|\beta)\)-tabloid such that
\[
\Theta(\{S\}) = (T^{\gamma_0|\delta_0}, T^{\gamma_1|\delta_1}, \ldots, T^{\gamma_r|\delta_r}) \in \prod_{i=0}^{r} \Omega(\gamma_i|\delta_i).
\]

Using the \( N_\rho \)-action on \( Z \), we observe that \( Z \) is a cyclic \( FN_\rho \)-module generated by \( \{S\} \). Let \( X \) be the subspace of \( Z \) linearly spanned by \( \{S\} \). By the definition of \( \{S\} \), the subspace \( X \) is an \( FH \)-module where
\[
H = \prod_{i=0}^{r} N_\rho \lhd \mathcal{S}_{\gamma_i} \times \mathcal{S}_{\delta_i} = \prod_{i=0}^{r} (N_\rho \lhd \mathcal{S}_{\gamma_i}) \times (N_\rho \lhd \mathcal{S}_{\delta_i}) \leq N_\rho,
\]
and there is an isomorphism
\[
X \cong \left( F(N_\rho \lhd \mathcal{S}_{\gamma_0}) \boxtimes \text{sgn}(N_\rho')^{\otimes \delta_0} \right) \boxtimes \cdots \boxtimes \left( F(N_\rho \lhd \mathcal{S}_{\gamma_r}) \boxtimes \text{sgn}(N_\rho')^{\otimes \delta_r} \right)
\]
of \( FH \)-modules. Since \( \dim_F Z = [N_\rho : H] \dim_F X \) by Lemma 4.11 we have \( Z \cong \text{Ind}_H^N X \) by the characterization of induced modules in [1, §8, Corollary 3, Lemma 4]. Hence we obtain the isomorphism (4.4) as desired. \( \square \)

5. Signed \( p \)-Kostka Numbers

In this section we prove Theorem 1.1 and Theorem 1.2. The main ingredients are the Brûe correspondents of indecomposable signed Young modules and Brauer quotients of signed Young permutation modules, as described in Sections 3 and 4. We keep the notation used in these sections.

We begin with a key lemma for the proof of Theorem 1.1

**Lemma 5.1.** Let \( n \in \mathbb{N} \). For any \((\gamma|\delta) \in \mathcal{C}^2(n)\) and \((\lambda|\mu) \in \mathcal{R} \mathcal{P}^2(n)\) we have

(i) \([W_{\rho}^{i+1}(\gamma|\delta) : Q_{\rho}^{i+1}(\lambda|\mu)] = [W_{\rho}(\gamma|\delta) : Q_{\rho}(\lambda|\mu)]\) for all \( i \geq 1 \),

(ii) \([W_{\rho}(\gamma|\emptyset) : Q_{\rho}(\lambda|\emptyset)] = [W_{1}(\gamma|\emptyset) : Q_{1}(\lambda|\emptyset)]\),

(iii) \([W_{\rho}(\gamma|\delta) : Q_{\rho}(\lambda|\emptyset)] = 0\) if \( \delta \neq \emptyset \).

**Proof.** By Lemma 3.4 and Lemma 4.7 we have \( \overline{Q}_{\rho}^{i}(\lambda|\mu) \cong R_{2}(\lambda|\mu) \) and \( \overline{W}_{\rho}^{i}(\gamma|\delta) \cong V_{2}(\gamma|\delta) \) for all \( j \geq 1 \). Part (i) now follows by applying Lemma 2.1.

For part (ii), if \( \delta = \emptyset = \emptyset \) then
\[
\overline{W}_{\rho}(\gamma|\emptyset) \cong V_{2}(\gamma|\emptyset) = \text{Inf}_{\mathcal{S}_{\gamma}}\mathcal{S}_{\gamma} \left( M^\gamma = \overline{W}_{1}(\gamma|\emptyset), \right)
\]
\[
\overline{Q}_{\rho}(\lambda|\emptyset) \cong R_{2}(\lambda|\emptyset) = \text{Inf}_{\mathcal{S}_{\gamma}}\mathcal{S}_{\gamma} \left( P(\lambda) = \overline{Q}_{1}(\lambda|\emptyset). \right)
\]
So \([\overline{W}_{\rho}(\gamma|\emptyset) : \overline{Q}_{\rho}(\lambda|\emptyset)] = [\overline{W}_{1}(\gamma|\emptyset) : \overline{Q}_{1}(\lambda|\emptyset)]\). Now apply Lemma 2.1.

For part (iii), let \( m_1 = |\gamma| \), let \( m_2 = |\delta| \) and let \( m = m_1 + m_2 \). Suppose that \([W_{\rho}(\gamma|\delta) : Q_{\rho}(\lambda|\emptyset)] \neq 0\) so that \( Q = \overline{Q}_{\rho}^{i}(\lambda|\emptyset) \) is isomorphic to a direct summand of
\[ W = \overline{W}_p(\gamma|\delta) \] as \( F[C_2 \wr S_m]\)-modules. Denote by \( B \) the base group of the wreath product \( C_2 \wr S_m \), namely
\[
B = C_2 \times \cdots \times C_2. 
\]

Let \( \Gamma \) be a complete set of representatives for the left cosets of \( S_{m_1} \times S_{m_2} \) in \( S_m \).
Then \( \Gamma \) is also a complete set of representatives for the double cosets of \( (B,C_2 \wr (S_{m_1} \times S_{m_2})) \) in \( C_2 \wr S_m \). Since \( B \) is normal, we have \( g \in \Gamma \) for any \( g \in \Gamma \). Hence, by the Mackey decomposition formula,
\[
F \oplus \cdots \oplus F \cong \text{Res}^C_{B} \otimes_{\text{sgn}} W \cong \bigoplus_{g \in \Gamma} \bigoplus_{i=1}^{k} g(F(C_2)^{m_1} \otimes \text{sgn}((C_2)^{m_2})),
\]
where \( k = (\dim F M^\gamma)(\dim F M^\delta) \) and \( B \) acts on \( g(F(C_2)^{m_1} \otimes \text{sgn}((C_2)^{m_2})) \) via the conjugated action. By the Krull–Schmidt Theorem, \( F \cong \text{Res}^C_{B} \otimes_{\text{sgn}} W \) as \( FB \)-modules for some \( g \in \Gamma \). Since \( m_2 \neq 0 \), we arrive at a contradiction.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( P_\rho \) be a Green vertex of \( Y(\lambda|p\mu) \). By Theorem 3.6 we have
\[
\rho = (1^{n_0}, p^{n_1}, (p^2)^{n_2}, \ldots, (p^r)^{n_r}),
\]
where \( n_0 = |\lambda(0)| \) and \( n_i = |\lambda(i)| + |\mu(i - 1)| \) for all \( i \in \{1, \ldots, r\} \). By the Broué correspondence (see Theorem 2.5) and the description of the Broué correspondents of indecomposable signed Young modules in Lemma 3.8, to prove Theorem 1.1 it is equivalent to show that
\[
[M(\rho|p\beta)(P_\rho) : Y(p\lambda|p^2\mu)(P_\rho)] \leq [M(\alpha|\beta)(P_\rho) : Y(\lambda|p\mu)(P_\rho)].
\]

Let \( \Lambda = \Lambda((\alpha|\beta), \rho) \) and \( \Lambda' = \Lambda((\rho|p\beta), pp) \) be as defined in Notation 4.8. Observe that \( \Lambda' \) consists of all compositions
\[
(\emptyset, \gamma_0, \gamma_1, \ldots, \gamma_r) | \emptyset, \delta_0, \delta_1, \ldots, \delta_r)
\]
where \( (\gamma_0, \gamma_1, \ldots, \gamma_r, \delta_0, \delta_1, \ldots, \delta_r) \in \Lambda \). By Lemma 4.12 applied to the module \( M(\rho|p\beta)(P_\rho) \), we have
\[
M(\rho|p\beta)(P_\rho) \cong \bigoplus_{(\gamma|\delta) \in \Lambda'} W_1(\gamma'|\delta_0) \otimes W_p(\gamma'|\delta_1') \otimes \cdots \otimes W_{p^{r+1}}(\gamma'_{r+1}|\delta'_{r+1})
\]
\[
= \bigoplus_{(\gamma|\delta) \in \Lambda} W_1(\emptyset|\emptyset) \otimes W_p(\gamma_0|\delta_0) \otimes \cdots \otimes W_{p^{r+1}}(\gamma_r|\delta_r).
\]

By Theorem 3.6 and Lemma 3.8(i), we obtain both
\[
[M(\rho|p\beta)(P_\rho) : Y(p\lambda|p^2\mu)(P_\rho)] = \sum_{(\gamma|\delta) \in \Lambda} \prod_{i=0}^{r} [W_{p^{i+1}}(\gamma_i|\delta_i) : Q_{p^{i+1}}(\lambda(i)|\mu(i - 1))],
\]
\[
[M(\alpha|\beta)(P_\rho) : Y(\lambda|p\mu)(P_\rho)] = \sum_{(\gamma|\delta) \in \Lambda} \prod_{i=0}^{r} [W_{p^{i}}(\gamma_i|\delta_i) : Q_{p^{i}}(\lambda(i)|\mu(i - 1))],
\]

where, by convention, $\mu(-1) = \emptyset$. By Lemma 5.1, we have

$$[W_{p,i+1}(\gamma|\delta) : Q_{p,i+1}(\lambda(\mu(i)|\mu(i-1))] = [W_{p,i}(\gamma|\delta) : Q_{p,i}(\lambda(i)|\mu(i-1))]$$

for all $i \geq 1$, and for $i = 0$ whenever $\delta_0 = \emptyset$. Otherwise, when $i = 0$ and $\delta_0 \neq \emptyset$, we have

$$0 = [W_p(\gamma_0|\delta_0) : Q_p(\lambda(0)|\emptyset)] \leq [W_1(\gamma_0|\delta_0) : Q_1(\lambda(0)|\emptyset)].$$

This completes the proof.

**Corollary 5.2.** Let $(\alpha|\beta), (\lambda|\mu) \in \mathcal{B}_2(n)$. Suppose that $\lambda(0) = \emptyset$. Then

$$[M(p\alpha|p\beta) : Y(p\lambda|p^2\mu)] = [M(\alpha|\beta) : Y(\lambda|p\mu)].$$

**Proof.** Let $\rho$ be the partition of $n$ defined by

$$\rho = (1^{\lambda(0)}, p^{\lambda(1)+|\mu(0)|}, \ldots, (p^r)^{\lambda(r)+|\mu(r-1)|}).$$

The Green vertex $P_\rho$ of $Y(\lambda|p\mu)$ has no fixed points in $\{1, 2, \ldots, n\}$. Hence $\delta_0 = \emptyset$ for any $(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)$. The result now follows from Theorem 1.1.

It is now very easy to deduce the asymptotic stability of signed $p$-Kostka numbers mentioned in the introduction.

**Corollary 5.3.** Let $(\alpha|\beta), (\lambda|\mu) \in \mathcal{B}_2(n)$. Then, for every natural number $k \geq 2$, we have

$$[M(p^k\alpha|p^k\beta) : Y(p^k\lambda|p^{k+1}\mu)] = [M(p\alpha|p\beta) : Y(p\lambda|p^2\mu)] \leq [M(\alpha|\beta) : Y(\lambda|p\mu)].$$

**Proof.** This follows immediately from Corollary 5.2 and Theorem 1.1.

**Example 5.4.** We present an example when the inequality in Theorem 1.1 is strict. Let $0 < r < p$ and let $m \in \mathbb{N}$. Since $\mathfrak{S}_{mp} \times \mathfrak{S}_r$ has index coprime to $p$ in $\mathfrak{S}_{mp+r}$, the trivial module $Y((mp+r)|\emptyset)$ is a direct summand of $M((mp,r)|\emptyset)$; the multiplicity is 1 since $M((mp,r)|\emptyset)$ corresponds to a transitive action of $\mathfrak{S}_{mp+r}$. By Lemma 3.9 we have $\text{sgn}(n) \cong Y((1^r)|(mp))$. Thus

$$[M(\emptyset|(mp,r)) : Y((1^r)|(mp))]$$

$$= [M(\emptyset|(mp,r)) \otimes \text{sgn}(mp+r) : Y((1^r)|(mp)) \otimes \operatorname{sgn}(mp+r)]$$

$$= [M((mp,r)|\emptyset) : Y((mp+r)|\emptyset)]$$

$$= 1.$$

On the other hand, $[M((mp^2, rp)|\emptyset) : Y((mp^2)|(p1^r))] = 0$ because, by [1, 2.3(6)], the indecomposable signed Young modules are pairwise non-isomorphic and so the indecomposable signed Young module $Y((mp^2)|(p1^r)))$ is not isomorphic to a Young module. Thus we have

$$[M(\emptyset|(mp^2, rp)) : Y((p1^r)|(mp^2))]$$

$$= [M(\emptyset|(mp^2, rp)) \otimes \operatorname{sgn}(mp^2+rp) : Y((p1^r)|(mp^2)) \otimes \operatorname{sgn}(mp^2+rp)]$$

$$= [M((mp^2, rp)|\emptyset) : Y((mp^2)|(p1^r))]$$

$$= 0.$$
where the penultimate equation is obtained using [3, Theorem 3.18]. This shows that \( [M(\emptyset | (mp^2, rp)) : Y((p(1^r))|(mp^2))] < [M(\emptyset | (mp, r)) : Y((1^r)|(mp))] \).

We now turn to the proof of Theorem 1.2. We need a further result on the Brauer quotients of signed Young permutation modules.

**Proposition 5.5.** Let \( m, n \in \mathbb{N} \) and let \( (\pi | \bar{\pi}) \in \mathcal{S}^2(m) \) and \( (\phi | \bar{\phi}) \in \mathcal{S}^2(n) \). Let \( \rho \) and \( \gamma \) be the partitions of \( m \) and \( n \), respectively, defined by

\[
\rho = (1^{m_0}, p^{m_1}, \ldots, (p^r)^{m_r}), \\
\gamma = (1^{n_0}, p^{n_1}, \ldots, (p^s)^{n_s}),
\]

where \( m_i, n_j \in \mathbb{N} \) for all admissible \( i, j \). For all \( k \in \mathbb{N} \) such that \( k > r \), we have that \( M(\pi | \bar{\pi})(P_\rho) \bigotimes M(p^k \phi | p^k \bar{\phi})(P_{\rho^k \gamma}) \) is isomorphic to a direct summand of \( M(\pi + p^k \phi | \bar{\pi} + p^k \bar{\phi})(P_{\rho \bullet p^k \gamma}) \). Furthermore, if \( p^k > \max\{\pi_1, \bar{\pi}_1\} \), then

\[
M((\pi | \bar{\pi}))(P_\rho) \bigotimes M(p^k \phi | p^k \bar{\phi})(P_{\rho^k \gamma}) \cong M(\pi + p^k \phi | \bar{\pi} + p^k \bar{\phi})(P_{\rho \bullet p^k \gamma}),
\]

as \( F[N_{\mathbb{E}_{m+p^k n}}(P_{\rho \bullet p^k \gamma})]/P_{\rho \bullet p^k \gamma} \)-modules.

**Proof.** Since \( k > r \), by Lemma 3.7 we have

\[
P_{\rho \bullet p^k \gamma} = P_\rho \times P_{p^k \gamma}.
\]

To ease the notation, we denote by \( M \), \( M_1 \) and \( M_2 \) the modules \( M(\pi + p^k \phi | \bar{\pi} + p^k \bar{\phi}), M(\pi | \bar{\pi}) \) and \( M(p^k \phi | p^k \bar{\phi}) \), respectively. Furthermore, let \( P = P_{\rho \bullet p^k \gamma} \). By Corollary 4.4, \( M(P) \) has as a basis the subset \( \mathcal{B} \) of \( \Omega(\pi + p^k \phi | \bar{\pi} + p^k \bar{\phi}) \) consisting of all \( \{R\} \) such that \( R \) is a row standard \( (\pi + p^k \phi | \bar{\pi} + p^k \bar{\phi}) \)-tableau whose rows are unions of \( P \)-orbits. Similarly, we define bases \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) of \( \Omega(\pi | \bar{\pi}) \) and \( \Omega(p^k \phi | p^k \bar{\phi}) \) for \( M_1(P_{\rho}) \) and \( M_2(P_{p^k \gamma}) \), respectively; here each \( (p^k \phi | p^k \bar{\phi}) \)-tableau \( S \) of \( \mathcal{B}_2 \) is filled with the numbers \( m + 1, m + 2, \ldots, m + p^k n \).

For \( \{T\} \in \mathcal{B}_1 \) and \( \{S\} \in \mathcal{B}_2 \), let

\[
\psi : \mathcal{B}_1 \times \mathcal{B}_2 \longrightarrow \mathcal{B}
\]

be the map defined by

\[
\psi(\{T\}, \{S\}) = \{(R_+ | R_-)\},
\]

where \( R_+ \) is the row standard \( (\pi + p^k \phi) \)-tableau such that row \( i \) of \( R_+ \) is the union of row \( i \) of \( T_+ \) and row \( i \) of \( S_+ \), and \( R_- \) is the row standard \( (\bar{\pi} + p^k \bar{\phi}) \)-tableau such that row \( i \) of \( R_- \) is the union of row \( i \) of \( T_- \) and row \( i \) of \( S_- \); here we have used the convention row \( i \) of \( T_+ \) is empty if \( i > \ell(\pi) \), and so on. The map \( \psi \) is well defined since the rows of \( R = (R_+ | R_-) \) are union of orbits of \( P = P_\rho \times P_{p^k \gamma} \) in its action on \( \{1, 2, \ldots, m + p^k n\} \).

Clearly \( \psi \) is injective and it induces an injection of vector spaces

\[
\theta : M_1(P_{\rho}) \bigotimes M_2(P_{p^k \gamma}) \longrightarrow M(P)
\]
defined by \( \theta(\{T\} \otimes \{S\}) = \psi(\{T\} \setminus \{S\}) \). By Lemma 3.7, we may regard the domain and codomain of \( \theta \) as \( FN_{\mathfrak{S}_{m+p^n}}(P) \)-modules with the trivial \( P \)-action. It is not difficult to check that
\[
\theta(g(\{T\} \otimes \{S\})) = g\theta(\{T\} \otimes \{S\}),
\]
for all \( g \in N_{\mathfrak{S}_{m+p^n}}(P) \), \( \{T\} \in B_1 \) and \( \{S\} \in B_2 \). Therefore, \( \theta \) is an injective homomorphism of \( FN_{\mathfrak{S}_{m+p^n}}(P) \)-modules, and hence an injective homomorphism of \( F[N_{\mathfrak{S}_{m+p^n}}(P)/P] \)-modules. Since both \( M_1(P_\rho) \) and \( M_2(P_{\rho\gamma}) \) are projective and hence injective, their outer tensor product is also injective. Therefore, the map \( \theta \) splits and we obtain that \( M_1(P_\rho) \boxtimes M_2(P_{\rho\gamma}) \) is a direct summand of \( M(P) \).

The second assertion follows easily by observing that, if \( p^k > \max\{\pi_1, \pi_1\} \), then the map \( \psi \) defined above is a bijection.

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \rho \) and \( \gamma \) be the partitions of \( m \) and \( n \) defined by
\[
\rho = (1^{\lambda(0)}, p^{\lambda(1)+|\mu(0)|}, \ldots, (p^r)^{\lambda(r)+|\mu(r-1)|}),
\]
\[
\gamma = (1^{\alpha(0)}, p^{\alpha(1)+|\beta(0)|}, \ldots, (p^s)^{\alpha(s)+|\beta(s-1)|}),
\]
where \( r = \ell_p(\lambda|\mu) \) and \( s = \ell_p(\alpha|\beta) \), respectively. As stated in Theorem 3.6, \( P_\rho \) is a Green vertex of \( Y(\lambda|\mu) \) and \( P_\gamma \) is a Green vertex of \( Y(\alpha|\beta) \). Since \( k > r \), by Lemma 3.7, we have \( P_{\rho\gamma} = P_\rho \times P_{\rho\gamma} \) and
\[
N_{\mathfrak{S}_{m+p^n}}(P_\rho \times P_{\rho\gamma}) = N_{\mathfrak{S}_{m}}(P_\rho) \times N_{\mathfrak{S}_{p^n}}(P_{\rho\gamma}).
\]
By Lemma 3.8, \( Y(p^k\alpha|p^{k+1}\beta) \) has Green vertex \( P_{\rho\gamma} \) and \( Y(\lambda+p^k\alpha|p(\mu+p^k\beta)) \) has Green vertex \( P_{\rho\gamma} \). Moreover, the Broué correspondent of \( Y(\lambda+p^k\alpha|p(\mu+p^k\beta)) \) is
\[
Y(\lambda|\mu)(P_\rho) \boxtimes Y(p^k\alpha|p^{k+1}\beta)(P_{\rho\gamma}).
\]
By Proposition 5.5, we have
\[
M(\pi|\pi')(P_\rho) \boxtimes M(p^k\phi|p^{k+1}\tilde{\phi})(P_{\rho\gamma}) = M(\pi+p^k\phi|\pi+p^{k+1}\tilde{\phi}) (P_{\rho\gamma}).
\]
Therefore, using Theorem 2.5(ii), we deduce that
\[
[M(\pi+p^k\phi|\pi+p^{k+1}\tilde{\phi}) : Y(\lambda+p^k\alpha|p(\mu+p^k\beta))]
\]
\[
= [M(\pi+p^k\phi|\pi+p^{k+1}\tilde{\phi})(P_{\rho\gamma}) : Y(\lambda+p^k\alpha|p(\mu+p^k\beta))(P_{\rho\gamma})]
\]
\[
\geq [M(\pi|\pi)(P_\rho) \boxtimes M(p^k\phi|p^{k+1}\tilde{\phi})(P_{\rho\gamma}) : Y(\lambda|\mu)(P_\rho) \boxtimes Y(p^k\alpha|p^{k+1}\beta)(P_{\rho\gamma})]
\]
\[
= [M(\pi|\pi)(P_\rho) : Y(\lambda|\mu)(P_\rho)] \cdot [M(p^k\phi|p^{k+1}\tilde{\phi})(P_{\rho\gamma}) : Y(p^k\alpha|p^{k+1}\beta)(P_{\rho\gamma})]
\]
\[
= [M(\pi|\pi) : Y(\lambda|\mu)] \cdot [M(p^k\phi|p^{k+1}\tilde{\phi}) : Y(p^k\alpha|p^{k+1}\beta)],
\]
where the final equality follows from Corollary 5.3.

If \( p^k > \max\{\pi_1, \pi_1\} \), then Proposition 5.5 implies that we have equalities throughout. \( \square \)
6. Indecomposable signed Young permutation modules

In this section, in the spirit of Gill’s result [8, Theorem 2], we classify all indecomposable signed Young permutation modules over the field $F$ and determine their endomorphism algebras and their labels as indecomposable signed Young modules. By [8], any indecomposable Young permutation module is of the form $M^{(m)}$ or $M^{(kp-1,1)}$. It is immediate from the definition of signed Young permutation modules in (1.1) that

$$M(\alpha|\beta) \cong \text{Ind}^S_{\mathcal{S}_n} (M^\alpha \boxtimes (M^\beta \otimes \text{sgn}(|\beta|))).$$

As such, by Gill’s result, any indecomposable signed Young permutation module is of one of the forms $M((m)|(n))$, $M((m)|(kp-1,1))$, $M((kp-1,1)|(m))$ or $M((kp-1,1,1)|(\ell p-1,1,1))$. Since $M((m)|(kp-1,1)) \otimes \text{sgn}(m + kp) \cong M((kp-1,1)|(m))$, there are three essentially different forms to consider.

*Proof of Theorem 1.3.* Let $M_1 = M((m)|(n))$. If $m = 0$ then $M_1$ is the sign representation, and if $n = 0$ then $M_1$ is the trivial representation. In these cases, $M_1$ is simple with 1-dimensional endomorphism ring. Suppose that both $m, n$ are non-zero. By the Littlewood–Richardson rule the module $M_1$ has a Specht series with top Specht factor $S^{(m+1,1^{n-1})}$ and bottom Specht factor $S^{(m,1^n)}$. If $m + n$ is not divisible by $p$ then the $p$-cores of $(m + 1, 1^{n-1})$ and $(m, 1^n)$ are non-empty and distinct and so $S^{(m+1,1^{n-1})}$ and $S^{(m,1^n)}$ lie in different blocks. Consequently, $M_1$ is decomposable. Now suppose that $m + n$ is divisible by $p$. In this case, by Peel’s result [18],

$$S^{(m+1,1^{n-1})} = \begin{cases} F & n = 1, \\ D^\gamma & n \geq 2, \end{cases} \quad S^{(m,1^n)} = \begin{cases} \text{sgn}(m+n) & m = 1, \\ D^\mu & m \geq 2, \end{cases}$$

where $\mu, \lambda, \gamma$ are the $p$-regularization of the partitions $(m, 1^n)$, $(m + 1, 1^{n-1})$ and $(m + 2, 1^{n-2})$ respectively. If $m = 1$ then $M((1)|(n)) \cong M(\varnothing|(n, 1))$ is indecomposable. Similarly, if $n = 1$ then $M((m)|(1)) \cong M((m, 1)|\varnothing)$ is indecomposable. Moreover, since $M((1, 1)|\varnothing)$ has a Loewy series with factors $F, D^{(m-1,1)}, F$, we see that $\text{End}_{F\mathcal{S}_{m+1}} M((m, 1)|\varnothing)$ is 2-dimensional. Tensoring by the sign representation gives the same result for $\text{End}_{F\mathcal{S}_{n+1}} (\varnothing|(n, 1))$.

We now study the case when $m, n \geq 2$. In this case, both the head and socle of $M_1$ contain the simple module $D^\lambda$. Also, as a signed Young permutation module, $M_1$ is self-dual. Suppose that $D^\gamma$ is not isomorphic to a composition factor of any direct summand of $M_1$ containing $D^\lambda$ in its head (and hence in its socle). Then $D^\gamma$ is necessarily isomorphic to a direct summand of $M_1$. From the Specht series, there is a surjection $\psi$ from $M_1$ onto the Specht module $S = S^{(m+1,1^{n-1})}$. Since $S$ has composition factors $D^\gamma$ and $D^\lambda$, we have $\psi(D^\gamma) \neq 0$ and so $\psi(D^\gamma) \cong D^\gamma$. Let $Y$ be an indecomposable direct summand of $M_1$ such that $\psi(Y)$ contains a composition factor $D^\lambda$. This shows that $\psi(Y) \cong D^\lambda$ and hence

$$S = \psi(D^\gamma \oplus Y) \cong D^\gamma \oplus Y/(Y \cap \ker \psi) \cong D^\gamma \oplus D^\lambda.$$

This is absurd since $S$ is indecomposable. Hence there exists an indecomposable direct summand of $M_1$ containing $D^\lambda$ in its head and that does not contain $D^\gamma$ in
By the Littlewood–Richardson rule, $M$ and if

and a 2-dimensional endomorphism ring.

Let $M_2 = M((kp-1, 1)|(m))$. By Gill’s result, if $m = 0$ then $M_2$ is indecomposable and if $m = 1$ then $M_2 \cong M((kp-1, 1^2)|\varnothing)$ is decomposable. Suppose that $m \geq 2$.

By the Littlewood–Richardson rule, $M_2$ has a Specht series with Specht factors

$$
S_1 = S^{(kp+1, 1^{m-1})}, \quad S_2 = S^{(kp, 2, 1^{m-2})}, \quad S_3 = S^{(kp, 1^m)},
$$

$$
S_4 = S^{(kp-1, 2, 1^{m-1})}, \quad S_5 = S^{(kp-1, 1^{m+1})},
$$

with $S_3$ occurring twice. If $m \not\equiv 0 \mod p$, then $S_1$ and $S_3$ lie in different blocks. If $m \equiv 0 \mod p$ then $S_3$ and $S_4$ lie in different blocks. Thus we conclude that $M_2$ is decomposable whenever $m \geq 2$.

Let $M_3 = M((kp-1, 1)|(lp-1, 1))$. Then $M_3 \cong M((kp-1, 1^2)|(lp-1))$. By Gill’s result, since $M((kp-1, 1^2)$ is decomposable, we have that

$$
M((kp-1, 1^2)|(lp-1)) = \text{Ind}_{\mathfrak{S}_{kp+1} \times \mathfrak{S}_{lp-1}}^{\mathfrak{S}_{kp} \times \mathfrak{S}_{lp}} (M((kp-1, 1^2) \otimes (M((lp-1) \otimes \text{sgn}(lp-1)))
$$

is decomposable.

We end by determining the labels of the indecomposable signed Young permutation modules. By the remark immediately following the statement of Theorem 1.3, it suffices to consider the modules $M((m)|(n))$ where either $m = 0$, $n = 0$ or $m + n$ is divisible by $p$.

**Proposition 6.1.** Let $m, n \in \mathbb{N}$. Let $n = \nu_0 + pn'$ where $\nu_0 < p$. There are isomorphisms $M((m)|\varnothing) \cong Y((m)|\varnothing)$, $M(\varnothing|(n)) \cong Y((1^{\nu_0})|(pn'))$ and, provided $m + n$ is divisible by $p$, $M((m)|(n)) \cong Y((m, 1^{\nu_0})|(pn'))$.

**Proof.** Clearly $M((n)|\varnothing) \cong Y((n)|\varnothing) \cong F(n)$. The second isomorphism follows from Lemma 3.9. In the remaining case $m, n > 0$ and $m + n$ is divisible by $p$. Let $m = \sum_{i \geq 0} m_i p^i$ and let $n = \sum_{i \geq 0} n_i p^i$. Let $r$ be greatest such that $m_r + n_r \neq 0$. Let $P$ be a Sylow $p$-subgroup of $\mathfrak{S}_m \times \mathfrak{S}_n$. By Proposition 4.12 we have an isomorphism of $F \mathfrak{N}_{\mathfrak{e}_{m+n}}(P)/P$-modules

$$
M((a)|(b))(P) \cong W_1((m_0)|(n_0)) \boxtimes W_2((m_1)|(n_1)) \boxtimes \cdots \boxtimes W_{p'}((m_{r'})|(n_{r'})).
$$

By Theorem 3.3, the indecomposable signed Young module $Y(\lambda|\rho \mu)$ satisfies

$$
Y(\lambda|\rho \mu)(P) = Y((\nu_0, 1^{\nu_0})|\varnothing) \boxtimes Q^1_p((m_1)|(n_1)) \boxtimes \cdots \boxtimes Q^p_{p'}((m_r)|(n_r)),
$$

where $Q^p_r((m_r)|(n_r))$ is the $F[(N_k/P_k) \times \mathfrak{S}_m]$-module defined in Definition 3.2. The Broué correspondence is bijective (see Theorem 2.5), so it suffices to prove that the tensor factors in these two modules agree.
Observe that \(m_0 + n_0\) is a multiple of \(p\) and \(m_0 + n_0 < 2p\). If \(m_0 = n_0 = 0\) there is nothing left to prove, so we may assume that \(m_0 + n_0 = p\). The \(F\mathfrak{S}_p\)-module \(W_1((m_0)|(n_0)) \cong M((m_0)|(n_0))\) is indecomposable by Theorem 1.3. The only indecomposable signed Young module for \(F\mathfrak{S}_p\) that is not a Young module is the sign representation. Since \(n_0 < p\) we see that \(M((m_0)|(n_0))\) is a Young module. The proof of Theorem 1.3 show that it has a Specht filtration with \(S^{(m_0,1^{n_0})}\) at the bottom and \(S^{(m_0+1,1^{n_0-p})}\) at the top. Therefore \(M((m_0)|(n_0)) \cong Y((m_0,1^{n_0})|\emptyset)\), as required.

Finally suppose that \(i \geq 1\). By Definition 4.6(ii) \(W_p^i((m_i)|(n_i))\) is the \(F[[N_p/P_p]]\mathfrak{S}_{m_i+n_i}\)-module obtained from

\[
\text{Ind}_{N_p/P_p}^{F[N_p/P_p]}(F(m_i) \boxtimes ((\text{Ind}_{N_p/P_p}^{F[N_p/P_p]}(F(n_i)) \otimes \text{sgn}(N_p) \otimes_{m_2}^{m_1})).
\]

by the canonical surjection \((N_p/P_p)(\mathfrak{S}_{m_i+n_i})/(P_p)^{m_i+n_i} \cong (N_p/P_p)\mathfrak{S}_{m_i+n_i}\). Since \(m_i, n_i < p\) the projective covers \(P((m_i))\) and \(P((n_i))\) are the trivial \(F\mathfrak{S}_{m_i}\) and \(F\mathfrak{S}_{n_i}\)-modules, respectively. Therefore, by the equation following Definition 3.2 we have \(W_p^i((m_i)|(n_i)) \cong Q_p^i((m_i)|(n_i))\), again as required.

\[\square\]

REFERENCES

[1] Alperin, J. L. Local representation theory, vol 11 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1986.
[2] Broué, M. On Scott modules and \(p\)-permutation modules: an approach through the Brauer homomorphism. Proc. Amer. Math. Soc., 93 no. 3 (1985), 401–408.
[3] Danz, S. and Lim, K. J. Signed Young modules and simple Specht modules, arXiv:1504.02823v2.
[4] Donkin, S. Symmetric and exterior powers, linear source modules and representations of Schur superalgebras. Proc. London Math. Soc. (3), 83 (2001), 647–670.
[5] Erdmann, K. and Schroll, S. On Young modules of general linear groups. J. Algebra, 310 no. 1, (2007), 434–451.
[6] Erdmann, K. Young modules for symmetric groups. J. Aust. Math. Soc., 71 no. 2 (2001), 201–210.
[7] Fang, M., Henke, A. and Koenig, S. Comparing \(GL(n)\)-representations by characteristic-free isomorphisms between generalized Schur algebras. Forum Math., 20 no. 1 (2008), 45–79. With an appendix by Stephen Donkin.
[8] Gill, C. Young module multiplicities, decomposition numbers and the indecomposable Young permutation modules. J. Algebra Appl., 13, no. 5 (2014) 1350147, 23 pp.
[9] Grabmeier, J. Unzerlegbare Moduln mit trivialer Younquelle und Darstellungstheorie der Schuralgebra, Bayreuth. Math. Schr., 20 (1985), 9–152.
[10] Green, J. A. Polynomial representations of \(GL_n\), vol. 830 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1980.
[11] Hemmer, D. J. Irreducible Specht modules are signed Young modules. J. Algebra, 305 (2006) 433–441.
[12] Henke, A. and Koenig, S. Relating polynomials \(GL(n)\)-representations in different degrees. J. Reine Angew. Math., 551 (2002) 219–235.
[13] Külshammer, B. Some indecomposable modules and their vertices. J. Pure Appl. Algebra, 86 (1993), 65–73.
[14] James, G. D. The representation theory of the symmetric groups, vol. 682 of Lecture Notes in Mathematics. Springer-Verlag, 1978.
[15] James, G. D. Trivial source modules for symmetric groups. *Arch. Math. (Basel)*, 41 no. 4 (1983), 294–300.
[16] James, G. D., and Kerber, A. *The representation theory of the symmetric group*. Addison-Wesley, 1981.
[17] Klyachko, A. A. Direct summands of permutation modules. *Selecta Math. Soviet.* (I) 3 (1983–1984), 45–55.
[18] Peel, M. H. Hook representations of the symmetric groups. *Glasgow Math. J.* 12 (1971), 136–149.

(E. Giannelli) Department of Mathematics, University of Kaiserslautern, P.O. Box 3049, 67655 Kaiserslautern, Germany
E-mail address: gianelli@mathematik.uni-kl.de

(K. J. Lim) Division of Mathematical Sciences, Nanyang Technological University, SPMS-MAS-03-01, 21 Nanyang Link, Singapore 637371.
E-mail address: limkj@ntu.edu.sg

(M. Wildon) Department of Mathematics, Royal Holloway, University of London, United Kingdom.
E-mail address: mark.wildon@rhul.ac.uk