We discuss the transition from a quantum to a classical domain for a model where a separation into environment and system is explicitly not given. Utilizing the coarse graining procedure for free quantum fields we also apply the projection method and the Hamiltonian principle to study possible cases of emergent classicality. General conditions for classical dynamics are given. Eventually, they lead to the equations of motion for a perfect classical fluid.

I. INTRODUCTION

The transition from a quantum mechanical description to a classical one for systems with many degrees of freedom is still an active field of research. An interesting issue among it is the emergence of classical hydrodynamical equations out of a many-particle quantum system. It is commonly believed that one has to accomplish two main tasks in order to warrant this emergence. One is to show that entering the classical domain implies that a small set of variables (local densities of particle number, momentum and energy) experience decoherence, where this notion qualitatively means that quantum superpositions that give rise to non-classical phenomena (e.g., non-locality) disappear. The other is that those variables turn out to obey classical, hydrodynamic equations. Among different approaches to this problem, Gell-Mann and Hartle present the idea of how local densities may become classical when integrated over small volumes of space (method of coarse graining). As a consequence, the coarse grained variables are accompanied by dissipation and fluctuations. Furthermore, Gell-Mann and Hartle describe how decoherence, dissipation and noise may interact in such a manner that classicality becomes present. Physically, this approach should be appealing in systems where it is practically impossible to distinguish between a system and its environment, like in gases or in fluids. However, until now there were hardly any physically interesting models on which one could explicitly show this way to emergent classicality when a clear system-environment-separation is missing.

In this work we discuss the possibility of emergent classicality exactly for these kind of systems. First of all, we have to point out what is actually meant by the notion ‘classical’. The underlying assumption is that the classical realm is characterized by two general properties. This is determinism and locality in space as well as in time. Thus we want to find out, under which conditions these properties become valid. We start from a non-relativistic and non-interacting quantum field theory. Using Zwanzig’s projection method, we give an evolution equation for the expectation values of the coarse grained fields. The expectation values themselves are represented in a fixed \( N \)-particle Hilbertspace. Then we apply Hamilton’s principle to express this evolution through equations of motion for local densities. It turns out that under certain conditions determinism and locality in time are consistent with the results from the Hamiltonian principle. In fact, this condition is the existence of certain values of the averaging length, \( l_{av} \), which parametrizes the roughness of coarse graining.

The other classical property, locality in space, needs further discussion. In this context we discuss the problem of nodal regions. Their existence leads inevitably to additional quantization rules. These rules have no classical meaning and hence we want them to become trivial in the classical regime. This turns out to happen when the coarse-grained density is positive everywhere. As next we show that non-local quantum effects become unimportant when the quantum potential in the equations of motion fulfills an inequality. This inequality gives an upper bound for the contribution of quantum forces and relates them to microscopic fluctuations and dissipation. Again this relation sets a condition for \( l_{av} \).

If all these conditions become consistent we find the classical hydrodynamical equations for a perfect fluid, namely the mass continuity equation and the Euler equation. Finally, we give heuristic arguments of how \( l_{av} \) can be interpreted as the temperature of the classical macrosystem.

II. EVOLUTION EQUATIONS FOR COARSE-GRAINED FIELDS

We briefly summarize the formal approach commonly known as the projection method which goes back to the original work of Zwanzig (see [3], and the references therein). It provides us with an evolution equation for the relevant
part of the quantum system’s statistical operator $\rho$. The subspace containing the relevant states is represented by a projector $\mathcal{P}$. Of course, the realization of $\mathcal{P}$ is motivated by certain physical considerations but for the projection method itself the choice of $\mathcal{P}$ is arbitrary. Let $\mathcal{H}$ be the Hamiltonian of the system and $A$ an operator. Then the evolution equation for the expectation value of the relevant part of $A$, that is $(A) (t) = \text{tr} [\rho(t) \mathcal{P} A]$, reads as
\[
i \frac{\partial}{\partial t} (A)(t) - \langle \mathcal{P} L A \rangle (t) + i \int_0^t d\tau \langle \mathcal{P} L \mathcal{Q} \exp (i L \tau) \mathcal{Q} \mathcal{P} A \rangle (t - \tau) = \zeta_A (t),
\]
where $\mathcal{Q} = 1 - \mathcal{P}$ and $L \equiv [\mathcal{H}, \cdot]$ denotes the Liouvillian (We set $\hbar = 1$). The term on the right hand side represents the influence of the irrelevant states at initial time, that are states orthogonal to $\mathcal{P} \rho (0)$. It is
\[
\zeta_A (t) = - \text{tr} [\rho (0) \mathcal{Q} \exp (i L t) \mathcal{Q} \mathcal{P} A] .
\]
We apply this result for non-interacting quantum theory of fields in the non-relativistic limit ($c \to \infty$). Thus $A$ is now considered to be a field operator in the Heisenberg picture, $\hat{a}(x)$, such that $\hat{a}(x; t) = \exp (i H t) \hat{a}(x) \exp (-i H t)$ satisfies the Schrödinger equation. The aim is to then represent equation (1) in the $N$-particle Hilbert space $\mathcal{H}^N$ and to use it later for the construction of effective hydrodynamic equations. However, these equations will not be classical a priori. Only if we find certain conditions for a classical description then hydrodynamic equations will possibly emerge.

Let $a(1, \ldots, N; t) := a(x_1, \ldots, x_N; t) = \text{tr} \rho_N [\rho (t) \mathcal{P} \hat{a}(x; 0)]$ be the relevant part of the $N$-particle wave function represented in position space. The evolution equation for $a(1, \ldots, N; t)$ can then be written as
\[
i \frac{\partial}{\partial t} a(1, \ldots, N; t) + \int d1' \ldots dN' H^P (1', \ldots, N'; t) a(1', \ldots, N'; t) =
\]
\[
\zeta_a (1, \ldots, N; t) - i \int d1' \ldots dN' \int_0^t d\tau G^P (1, \ldots, N, 1', \ldots, N'; \tau) a(1', \ldots, N', t - \tau).
\]
Herein $H^P$ and $G^P$ are the integral kernels representing the operators $H^P a = \mathcal{P} H a$ and of
\[
G^P a = \int_0^t d\tau \mathcal{P} \mathcal{H} \mathcal{Q} \exp (i H \tau) \mathcal{Q} \mathcal{H} a (t - \tau). 
\]
The fluctuation term $\zeta_a (1, \ldots, N; t)$ is also written as an integral operator, viz.
\[
\zeta_a (1, \ldots, N; t) = \int d1' \ldots dN' F (1', \ldots, N', 1, \ldots, N; t) a(1', \ldots, N', 0),
\]
where $a(1, \ldots, N; 0)$ is the original, i.e. the ‘unsmear’, wavefunction at initial time. A similar equation for $a^\dagger (1, \ldots, N; t) = \text{tr} \rho_N^* [\rho (t) \mathcal{P} \hat{a}^\dagger (x; 0)]$ is obtained by taking the complex conjugate of equation (3). The actual realization of the operator kernels depends on the choice of the projector $\mathcal{P}$. A natural choice of $\mathcal{P}$ is that of covering the whole space with small cubes of volume $l_{\text{av}}^3$, taking the average on each of these cubes such that the set of all average values defines the coarse grained function. Therefore the spatial components of the (coarse grained) wave functions $a(1, \ldots, N; t)$ are defined on a cubic lattice $\Sigma_{l_{\text{av}}}$ with grid size $l_{\text{av}}$. In that case the integral kernel $P(x, x')$ of $\mathcal{P}$ is the characteristic function of the set $[x - l_{\text{av}}/2, x + l_{\text{av}}/2]^3 \subset \mathbb{R}^3$ for a given $x \in \Sigma_{l_{\text{av}}}$. For other choices of $\mathcal{P}$ we require that for any $x \in \Sigma_{l_{\text{av}}}$ the support of $P(x, \cdot)$ is basically a set of volume $l_{\text{av}}^3$ such that for all $x'$ with $|x - x'| > l_{\text{av}}$ the corresponding value of $P(x, x')$ is suppressed at least exponentially. This requirement reflects the physical assumption that all wavelengths shorter than $l_{\text{av}}$ should not contribute to the physical description on scales much larger than $l_{\text{av}}$. Independently of this particular choice we can state that if $l_{\text{av}} \ll l_{\text{obs}}$, where $l_{\text{obs}}$ is the minimum length scale on which macroscopic functions significantly vary, then the grid $\Sigma_{l_{\text{av}}}$ is taken to be the continuum $\mathbb{R}^3$ again. Until now, we have not given any physical motivation or mathematical indication that confirms the inequality $l_{\text{av}} \ll l_{\text{obs}}$. A possible confirmation of this assumption will be given later, when the emergence of classicality will be

\footnote{In general, the operator $\mathcal{P}$ satisfying the above condition will not be a projector anymore, i.e. $\mathcal{P}^2 \neq \mathcal{P}$, but on the macroscopic level the deviation $|\mathcal{P}^2 - \mathcal{P}|$ will be only of the order $(l_{\text{av}}/l_{\text{obs}})^4$.}
discussed.
As an obvious realization of $\mathcal{P}$ we take Gaussian distribution for $P(x, x')$,
\[ P(x, x') = \int dk \exp[-ik(x - x')] \exp[-k^2 \ell_{av}^2/2], \tag{6} \]
where the short form $dk$ stands for $(2\pi)^{-d/2} dk_1 \ldots dk_d$. This expression is generalized for the $N$-particle state via
\[ P(x_1, \ldots, x_N, x'_1, \ldots, x'_N) = \int dk_1 \ldots dk_N \exp[-i \sum_{j=1}^{N} k_j (x_j - x'_j)] \exp[-\sum_{j=1}^{N} k_j^2 l_{av}^2/2]. \tag{7} \]
Using $P(k_1, \ldots, k_N) = \exp[-1/4 \sum_{j=1}^{N} k_j^2 \ell_{av}^2]$ we can write down the expressions for $G^P$ and $H^P$, which are
\[ F^P(x_1, \ldots, x_N, x'_1, \ldots, x'_N; t) = \int dk_1 \ldots dk_N \exp[-i \sum_{j=1}^{N} k_j (x_j - x'_j) - i \omega_{k_1, \ldots, k_N} t] \tag{8} \]
\[ G^P(x_1, \ldots, x_N, x'_1, \ldots, x'_N; t) = \int dk_1 \ldots dk_N \exp[-i \sum_{j=1}^{N} k_j (x_j - x'_j) - i \omega_{k_1, \ldots, k_N} t] \tag{9} \]
\[ H^P(x_1, \ldots, x_N, x'_1, \ldots, x'_N; t) = \int dk_1 \ldots dk_N \exp[-i \sum_{j=1}^{N} k_j (x_j - x'_j)]. \tag{10} \]
Herein, $\omega_{k_1, \ldots, k_N}$ denotes the eigenvalues of $\mathcal{H}$ in Fourier-space. These expressions seem quite complicated, but can be simplified significantly by expanding $P(k_1, \ldots, k_N)$ to the lowest order in $\ell_{av}$, when it is $k^2 \ell_{av}^2 \ll 1$ which is equivalent to $\ell_{av} \ll \ell_{obs}$. This gives for the Gaussian kernel of $\mathcal{P}$
\[ H^P(x_1, \ldots, x_N, x'_1, \ldots, x'_N; t) = \int dk_1 \ldots dk_N \exp[-i \sum_{j=1}^{N} k_j (x_j - x'_j)]. \tag{11} \]
\[ G^P(x_1, \ldots, x_N, x'_1, \ldots, x'_N; t) = \frac{\ell_{av}}{64} \int dk_1 \ldots dk_N \exp[-i \sum_{j=1}^{N} k_j (x_j - x'_j) - i \omega_{k_1, \ldots, k_N} t], \tag{12} \]
and for the kernel acting on the 'unsmearred' states
\[ F^P(x_1, \ldots, x_N, x'_1, \ldots, x'_N; t) = \frac{\ell_{av}}{16} \int dk_1 \ldots dk_N \exp[-i \sum_{j=1}^{N} k_j (x_j - x'_j) - i \omega_{k_1, \ldots, k_N} t]. \tag{13} \]
Under this approximation, the averaging-length $\ell_{av}$ appears as a coupling constant determining the strength of the dissipation and the fluctuation term.

In appearance, equation (13) shows uncommon features in the classical regime. First there is the non-local behavior in time. Second, the fluctuation term makes it questionable whether eqn. (13) really describes a deterministic evolution in time. Therefore, in order to assign classicality to the evolution equation (13) we have to show how these 'uncommon features' become unimportant. This is the issue of the following section and one of the main issues of this work.
III. THE CLASSICAL EQUATIONS

A. The formal equations

For a hydrodynamical description of equation (3), we wish to find a representation by a small set of variables – in general those will be local densities of particle number, momentum and energy – such that these variables follow dynamic equations similar to the fluid mechanical equations. This representation should be independent of the number of particles in the system, if only this number is large enough and does not strongly vary in time. Then the resulting equations should be equivalent to the evolution equation (3). Consequently, we expect that the hydrodynamical modes are subject to dissipation and noise.

In the remarkable work of Holzwarth and Schütte [4] a fluid-dynamical description of a multiple particle quantum system is given. Using a variational scheme, it generalizes earlier attempts by considering also two-particle correlations. This generalization leads to a velocity field with non-vanishing vorticity being a direct consequence of two-particle correlations.

We adopt this idea for the relevant wave function

$$a(1,...,N;t) = \phi(1,...,N;t) \exp[i m S(1,...,N;t)] ,$$

with the real phase $S$ including a two-particle term of the form

$$S(1,...,N;t) = \sum_{i=1}^{N} \varphi(i;t) + \frac{1}{2} \sum_{i\neq j} \mu(i;t)\mu(j;t) .$$

Using this representation of $a$ we construct a Lagrangian functional $L$ out of equation (3)

$$L = \langle a | (i \partial_t |a⟩ - \mathcal{H}^P |a⟩ - \mathcal{G}^P |a⟩ + \zeta_a) .$$

Denoting $(\langle a|\mathcal{G}^P|a⟩ - \langle a|\zeta_a)$ by $E_P$ and making use of norm conservation, we have

$$L = -m \int d1...dN \phi^2(1,...,N;t) \dot{S}(1,...,N;t) - \frac{m}{2} \int d1...dN \phi^2(1,...,N;t) \sum_{i=1}^{N} \nabla_i S \cdot \nabla_i S - E_{qm} - E_P$$

with

$$E_{qm} = \frac{m}{2} \int d1...dN \phi^2(1,...,N;t) \sum_{i=1}^{N} \Delta_i \phi(1,...,N;t) .$$

Introducing the one-, two- and three-particle moments $\rho^{(1)}, \rho^{(2)}$ and $\rho^{(3)}$ by the formula

$$\rho^{(i)} = \frac{N!}{(N-i)!} \int d(i+1)...dN \phi^2(1,...,N;t) ,$$

where we denote $\rho^{(1)}$ by $\rho$ and the functions $\lambda(1;t), \kappa^2(1;t)$ by

$$\rho(1;t) \lambda(1;t) = N \int d2...dN \phi^2(1,...,N;t) \sum_{j>1} \mu(j;t) = \int \rho^{(2)}(1,2) \mu(2) d2 ,$$

$$\rho(1;t)\kappa^2(1;t) = N \int d2...dN \phi^2(1,...,N;t) \left[ \sum_{j>1} \mu(j;t) - \lambda(1;t) \right]^2$$

$$= \int \rho^{(3)}(1,2,3) \mu(2) \mu(3) d2d3 + \int \rho^{(2)}(1,2) \mu^2(2) d2 - \rho(1)\lambda^2(1) ,$$

we are able to write the Lagrangian (16) in the following form
\[ L = -m \int d1 \rho(1; t) \left[ \dot{\varphi}(1; t) + \lambda(1; t) \dot{\mu}(1; t) \right] \]
\[ - \frac{m}{2} \int d1 \rho(1; t) \left[ (\nabla \varphi(1; t) + \lambda(1; t) \nabla \mu(1; t))^2 + \kappa^2(1; t)(\nabla \mu(1; t))^2 \right] - E_{qm} - E_P. \] (22)

Therein, the function \( E_{qm} \) only depends on the local density \( \rho(1; t) \), whereas the function \( E_P \) in general depends on \( \rho, \varphi, \mu \), and \( \kappa \). If we now identify the expression \( \nabla \varphi + \lambda \nabla \mu \) with a general velocity field \( u \), then \( L \) already has the form familiar from fluid dynamics \[ \Box \]. It is remarkable that \( u \) has a non-vanishing vorticity \( \nabla \lambda \times \nabla \mu \). The corresponding equations of motion follow from the Langragian derivatives

\[ [L]_\sigma = \frac{\delta L}{\delta \sigma} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\sigma}} \equiv 0, \quad \sigma \in I = \{ \rho, \varphi, \lambda, \mu, \kappa \}. \] (23)

They are

\[ [L]_\varphi = \dot{\rho} + \nabla \cdot (\rho u) + \frac{1}{m} \frac{\delta}{\delta \varphi} E_P = 0, \] (24)
\[ [L]_\lambda = \rho(\dot{\mu} + u \cdot \nabla \mu) + \frac{1}{m} \frac{\delta}{\delta \lambda} E_P = 0, \] (25)
\[ [L]_\mu = (\rho \lambda) + \nabla \cdot (\rho \lambda u + \rho \kappa^2 \nabla \mu) + \frac{1}{m} \frac{\delta}{\delta \mu} E_P = 0, \] (26)
\[ [L]_\rho = \dot{\varphi} + \lambda \mu + \frac{1}{2} u^2 + \frac{1}{2} \kappa^2(\nabla \mu)^2 + \frac{1}{m} \frac{\delta}{\delta \rho} (E_{qm} + E_P) = 0, \] (27)
\[ [L]_\kappa = \rho \kappa(\nabla \mu)^2 + \frac{1}{m} \frac{\delta}{\delta \kappa} E_P = 0. \] (28)

The gradient of equation (26) together with the other equations of motion leads to

\[ (\rho u) + \nabla \cdot (\rho u \otimes u + \rho \kappa^2 \nabla \mu \otimes \nabla \mu) = -\frac{1}{m} \left( \rho \nabla \frac{\kappa}{2 \rho} \frac{\delta}{\delta \kappa} + \nabla \rho \frac{\delta}{\delta \rho} - \sum_{\sigma \in I \setminus \{\kappa\}} (\nabla \sigma) \frac{\delta}{\delta \sigma} \right) (E_{qm} + E_P). \] (29)

This equation looks very similar to the Euler equation except for the right hand side, where additional terms appear. Those are the forces originating from the coarse grained description and from the gradient of the quantum potential \( \delta_{P} E_{qm} \). It is important to note that the value of the 'thermal pressure' \( P = \rho \kappa^2 \nabla \mu \otimes \nabla \mu \) is always positive. This holds independently of the parity (odd or even) of \( S \), see \[ \Box \].

B. Constraints for classicality

To establish classical behavior of eqn. \[ \Box \] it is necessary to show its deterministic and local-in-time behavior as well as its independence of any quantum effects. We utilize these guiding principles in order formulate sufficient conditions for the emergence of macroscopic classicality. Eventually, these conditions will specify our choice of the averaging length, \( l_{av} \).

1. Determinism and locality in time

At first, we discuss the important question of how to 'choose' a value of \( l_{av} \) such that equation \[ \Box \] loses the non-local evolution in time and the non-deterministic nature due to the terms in \( E_P \). We approach this problem by introducing a generalization of the variational scheme presented in the previous section. Until now, \( l_{av} \) was regarded as a free parameter in our equations. It appears in the Lagrange functional \[ \Box \] and therefore we change the value of \( L \) whenever we change \( l_{av} \). Thus it is obvious to vary \( L \) also with respect to \( t := l_{av} \) and add the resulting equation to the sample of equations \( \Box \) - \( \Box \). In doing so, the functional derivative in \( [L]_t \) becomes an ordinary one and by virtue of equation \( \Box \) we obtain

\[ \frac{\partial}{\partial l} \int d1 \rho \frac{\delta}{\delta \rho} (E_{qm} + E_P) = \frac{\partial}{\partial l} (E_{qm} + E_P). \] (30)
The most trivial solution to this equation is the case when \((E_{qm} + E_P)\) can be represented as
\[
(E_{qm} + E_P) = \int d1 \rho^{1/2} \epsilon(\lambda, \varphi, \kappa, \mu) \rho^{1/2}.
\] (31)
with \(\epsilon\) being independent of \(\rho\). As we shall see, this representation is only possible when \(P = 1\), i.e. when there is no coarse graining at all, which means that \(l_{av}\) vanishes. This is because eqn. (31) implies that the term \(\langle a|\hat{\zeta} a\rangle\) in \(E_P\) must look like
\[
\langle a|\hat{\zeta} a\rangle = \int d1 \rho^{1/2} \epsilon'(\lambda, \varphi, \kappa, \mu, \hat{a}(x; 0)) \rho^{1/2}.
\] (32)
But \(\epsilon'\) is not independent of \(\rho\), because it is a function of the irrelevant part of the field operator at initial time and in general, a change of the initial state will change the values of the relevant variables. Only in the case of \(P = 1\), i.e. \(l_{av} = 0\), one can overcome this dependence for an arbitrary initial state. In order to seek for possible other solutions of (30), we rewrite this equation in the following way
\[
\int d1 \left( \frac{\partial \rho}{\partial l} \right) \frac{\delta}{\delta \rho} (E_{qm} + E_P) + \int d1 \rho \frac{\partial}{\partial l} \frac{\delta}{\delta \rho} (E_{qm} + E_P) = \frac{\partial}{\partial l} (E_{qm} + E_P),
\]
and in this form, it tells us already how \(l_{av}\) has to be in order to satisfy equation (30) resp. (33). Namely, if there is an \(l_{av}\) such that
\[
\left[ \frac{\partial}{\partial l} \rho \right]_{l=l_{av}} = 0, \quad (33)
\]
then the above equation becomes valid, when at the same time \(E_{qm} + E_P\) satisfies
\[
\left[ \frac{\partial}{\partial l} (E_{qm} + E_P) \right]_{l=l_{av}} = 0 \quad (34)
\]
and since \(E_{qm}\) does only depend on \(\rho\) it follows
\[
\left[ \frac{\partial}{\partial l} E_P \right]_{l=l_{av}} = 0 \quad (35)
\]
Furthermore, because of the consistency with the 'continuity' equation (24), the validity of (33) and (34) implies that it must even be
\[
\left[ \frac{\partial \sigma}{\partial l} \right]_{l=l_{av}} = 0, \quad \sigma = \{ \varphi, \lambda, \kappa, \mu \}. \quad (36)
\]
Condition (33) states that if we increase our averaging length scale being in the neighborhood of \(l_{av}\), then the values of \(\lambda, \rho, \varphi, \mu\) and \(\kappa\) do not change at all, or in other words, the smallest length-scale on which the functions \(\lambda, \rho, \varphi, \mu\) and \(\kappa\) significantly vary, i.e. \(l_{obs}\), is much bigger than the regarded scale \(l_{av}\). This is a remarkable result, because it shows explicitly that it must be \(l_{av} \ll l_{obs}\). Since \(E_P\) involves the coupling factor \(l_{av}^4/16\), c.f. equation (12) and (13), it follows that microscopic dissipation and fluctuations are coupled very weakly to the macroscopic evolution. However, we should stress out, that the actual existence of a critical value \(l_{av}\) has not been proven, rather we showed that its existence would be consistent with the results obtained from our variational approach.

2. Nodal regions

It is well known, that \(\delta_{\rho} E_{qm}\) becomes singular whenever \(a(1, ..., N; t)\) is on a nodal region, i.e. a region where the wave function vanishes. In this situation, equation (3) is not consistent with equation (2) and additional 'quantization
rules’ are necessary to re-establish consistency \cite{footnote}. In the classical regime, of course, we do not want to fulfill any additional constraints which are of pure quantum nature. To do so, we want the amplitude $\phi(1, ..., N; t)$ to be strictly positive, that means, $a(1, ..., N; t)$ must not have any nodal regions. This leads us to the following assumption: For a fixed $t$ and a given $l_{\text{av}}$ let $\Omega_t^P \subset \mathbb{R}^3$ be a set defined by the relation

$$\mathcal{P} \hat{a}(x; t) = 0 \quad \text{iff} \quad x \in \Omega_t^P,$$

(37)

then there is a $l_{\text{av}}' > l_{\text{av}}$ such that $\Omega_t^P = \emptyset$ for all $t$. In case of its existence, we take the smallest value $l_{\text{av}}'$ to be the actual averaging length scale. Obviously, this choice guaranties that the relevant part of $\hat{a}(x; t)$, that is all $\hat{a}$ satisfying $\mathcal{P}\hat{a} = \hat{a}$, never vanishes. Consequently, the amplitude of the relevant $N$-particle wavefunction, $\phi(1, ..., N; t)$, does not vanish, either. This ‘prohibition’ of singularities in $\delta_p E_{\text{qm}}$ sets a lower bound for $l_{\text{av}}$. It can be viewed as a condition for the system being not too far from equilibrium, thus the locally averaged particle number density, $\rho$, is positive everywhere.

3. Quantum potential

Because the quantum potential $\delta_p E_{\text{qm}}$ is defined in configuration space, it reveals an inherent non-local character in position space: variation of the density of one particle can instantly change the motion of another particle over an arbitrary spatial distance. This term is a product of quantum coherence and describes a completely non-classical phenomenon. Thus for the emergence of classicality, we want to give a natural bound for the quantum potential $\delta_p E_{\text{qm}}$ such that its influence remains minor at the macroscopic level of description. Recall that for a given set $O(x, l') = [x - l_{\text{av}}/2, x + l_{\text{av}}/2]^3 \subset \mathbb{R}^3$ and an arbitrary $x$ it is

$$\int_{O(x,l_{\text{av}})} \mathrm{d}x' \rho \nabla \delta \frac{\delta}{\delta \rho} E_{\text{qm}} \to 0, \quad \text{as} \quad l_{\text{av}} \to \infty,$$

(38)

which means that by increasing $l_{\text{av}}$, the value of $\mathcal{P} \rho \nabla \delta \frac{\delta}{\delta \rho} E_{\text{qm}}$ decreases, see \cite{footnote}. Therefore, we can give a natural upper bound for the quantum force by relating it to the dissipative and stochastic term via

$$\left| \left[ \rho \nabla \delta \frac{\delta}{\delta \rho} E_{\text{qm}} \right]_{t=l_{\text{av}}} \right| \leq \left| \left[ \left( \rho \nabla \frac{\kappa}{2\rho} \delta + \nabla \rho \frac{\delta}{\delta \rho} - \sum_{\sigma \in I \setminus \{\kappa\}} \left( \nabla \sigma \cdot \frac{\delta}{\delta \sigma} \right) E_P \right) \right]_{t=l_{\text{av}}} \right|.$$

(39)

This inequality demands for a lower bound of $l_{\text{av}}$. Thus if $l_{\text{av}}$ turns out to be consistent with that bound, then relation (39) tells us that ‘quantum forces’ never exceed those force terms coming from microscopic dissipation and fluctuation. The latter in turn are only weakly coupled to the macroscopic system, because $E_P$ involves the factor $l_{\text{av}}^4/16$. In particular, the above condition for a proper choice of the averaging length scale reveals a rather delicate interaction between the suppression of quantum coherence and classical determinism of the equation of motion. This is because to accomplish the former one has to increase $l_{\text{av}}$ in order to diminish the influence of the quantum potential. But on the other hand, the same mechanism increases the strength of the fluctuation term, which possibly leads to a highly stochastic, i.e. non-deterministic, dynamical evolution.

Hence, for the equations of motion (24) - (28), the classical realm is entered if $l_{\text{av}}$ is large enough such that the local density is positive and if its value additionally implies that equation (24) is consistent with (39). But at the same time it must also comply with $l_{\text{av}} \ll l_{\text{obs}}$, which is in agreement with equation (33) and (34). Then – on macroscopical scales – these equations turn out to be of the same form as the hydrodynamical equations. They are conservation laws representing mass continuity, vortex dynamics and momentum flow.

C. Temperature

Apart from mass and momentum density, classical macroscopic systems at statistical equilibrium are additionally characterized by their temperature. Thus any model that asserts to describe classical systems adequately must also give a description of this particular physical term. In the previous section we showed that averaging length could be vanishing or infinite. However, if a $l_{\text{av}}$ exists inbetween those bounds such that it fulfills the constraints (33) - (36) then the emergent system is deterministic and local in time. In this situation the system is in a local equilibrium, because the average over a volume of size $\sim l_{\text{obs}}$ of an arbitrary macroscopic function is the same when averaged over
a smaller part of this volume. This holds obviously because any macrkosscopic function becomes constant in space on spatial scales smaller than \( l_{\text{obs}} \). In this case it should be possible to assign a certain temperature to the system. This assignment is quite clear for the extreme cases, when \( l_{\text{av}} \) is either infinite or zero: For the latter case there is no coarse grained system that gives rise to an extremum of the action \( S = \int L \, dt \). In this situation the system is described by the original theory and from the classical point of view the corresponding temperature is infinite. On the other hand, when \( l_{\text{av}} = \infty \) is consistent with the equations of motion and the constraints coming from the Hamiltonian principle then the system is described completely by constant functions in space and time. This case corresponds to a zero temperature scenario where there are no fluctuations and inhomogeneities. Of course, these two cases represent the trivial critical points of infinite and zero temperature. In any other situation, this is where \( l_{\text{av}} \) turns out to be positive and finite, the system’s temperature \( T \) should be a function of \( l_{\text{av}} \) and possibly of all other relevant parameters (such as the particle mass \( m \)) of the original theory. If these ‘other’ parameters remain fixed then \( l_{\text{av}} \) can actually be identified with the temperature \( T \). For non-interacting and non-relativistic fields this relation should read as

\[
\frac{\hbar^2}{2 m l_{\text{av}}^2} = k_B T ,
\]

where \( k_B \) is Boltzmann’s constant.

Thus temperature can only be assigned to the macroscopic system, when the above mentioned conditions hold and from which it follows that the system is in a local equilibrium. In general however, the averaging length \( l_{\text{av}} \) will not have anything to do with a physical temperature. This situation is very similar to the role of temperature in non-equilibrium statistical mechanics, where the parameter \( T \) appearing in the distribution function turns out to be the temperature only if the system described by this distribution function is in (local) equilibrium.

## IV. CONCLUSIONS AND REMARKS

In this work, we show that it is possible to study the emergence of classical dynamics of systems with many degrees of freedom directly by applying the coarse graining principle to the quantum equations of motion. This lead us to a evolution equation for the relevant part of the N-particle wave function. Out of this we are able to construct a Lagrangian functional \( L \). This functional is expressed by means of local densities that are defined through the kinetic part of the Lagrangian. Variation of \( L \) then leads us to a sample of dynamic equations which explicitly contain terms arising from dissipation, fluctuation and quantum coherence. This approach is somewhat similar to the deduction of the Madelung equations (or quantum fluid dynamic equations, henceforth QFD equations) from the Schrödinger equation. However, there are formal and conceptual distinctions between the latter and our approach. Equation (3) is equal to the N-particle Schrödinger equation only in case of \( l_{\text{av}} = 0 \). For \( l_{\text{av}} > 0 \) it is non-linear in the relevant part of the wavefunction, because the noise term \( \zeta \) breaks the original linearity. Moreover, we obtain non-local behavior in time due to the dissipative term \( G^2 a \). All these features are not present in ordinary derivations of the QFD equations. The most prominent difference is perhaps that QFD equations do not indicate how a classical limit could ever be achieved. Only by carefully applying the coarse graining principle we were able to describe this limit and the transition towards it.

The classical limit itself is – as we believe – characterized by the following properties: locality in space, locality in time and determinism. To accomplish the demand for determinism and locality in time, we vary the Lagrangian \( L \) also with respect to \( l_{\text{av}} \), which is a generalization of the variational approach presented in section III. A. Then we show that the resulting constraints becomes valid when a critical value of \( l_{\text{av}} \) exists being consistent with the conditions for classicality stated above.

To avoid non-local quantum behavior in space we additionally need to show that the influence of the quantum potential \( \delta\psi_E_{\text{av}} \) becomes negligible. This happens when the quantum potential does not exceed a bound given by the dissipative and fluctuation force terms, c.f. condition (3). Additionally, we have to demand that there are no nodal regions in the coarse grained wavefunctions. Then, provided such a value of \( l_{\text{av}} \) exists, the equations of motion become classical. In our case, we recover the Euler equation together with the equations for continuity and for the motion of vortex lines.

We think that our approach to the problem of emergent classicality is quite general, because it requires only basic principles. Starting from a quantum field theory one additionally introduces a minimum level of spatial resolution. Then the immediate question arises how the original theory looks like when is represented on an arbitrary level of coarse graining. Finally, locality in space/time and determinism – as principles of the classical realm – are questioned and eventually the coarse grained equations of motion tell us whether they are consistent with these principles or not.

This approach should also work for interacting fields as long as not too high energy densities are considered. In general, interacting fields do not posses a representation in a fixed N-particle Hilbertspace anymore. This is because
of the creation and annihilation of particles. Thus if \( N \) varies strongly in time the Lagrangian functional approach presented here breaks down completely and our description of emergent classicality becomes inadmissible. In this situation we expect the original (quantum field) theory to be an appropriate model of physical description.

As a final remark we want to point out that there is another parameter in the Lagrangian functional \( L \) which has not been considered explicitely, yet. This is the number of spatial dimensions \( d \). When analytically continued along the real line, it could be used as an extra variable in the variational scheme presented in this work. The outcome of a corresponding constraint could then give some further insight to questions like: Does the existence of a classical domain require a certain number of spatial dimensions? The hope is that this number is '3' in the low-energy limit of the known interactions.

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