Minimizing the Complexity of Fast Sphere Decoding of STBCs

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Abstract—Decoding of linear space-time block codes (STBCs) with sphere-decoding (SD) is well known. A fast-version of the SD known as fast sphere decoding (FSD) has been recently studied by Biglieri, Hong and Viterbo. Viewing a linear STBC as a vector space spanned by its defining weight matrices over the real number field, we define a quadratic form (QF), called as a vector space spanned by its defining weight matrices over the real number field, we define a quadratic form (QF), called a QF interpretation of the FSD complexity of a linear STBC. It is shown that the FSD complexity is only a function of the weight matrices defining the code and their ordering, and not of the channel realization (even though the equivalent channel when SD is used depends on the channel realization) or the number of receive antennas. It is also shown that the FSD complexity is completely captured into a single matrix obtained from the HRQF. Moreover, for a given set of weight matrices, an algorithm to obtain a best ordering of them leading to the least FSD complexity is presented. The well known classes of low FSD complexity codes (multi-group decodable codes, fast decodable codes and fast group decodable codes) are presented in the framework of HRQF.

I. INTRODUCTION & PRELIMINARIES

Consider a minimal-delay space-time coded Rayleigh quasi-static flat fading MIMO channel with full channel state information at the receiver (CSIR). The input output relation for such a system is given by

\[ Y = HX + N, \quad (1) \]

where \( H \in \mathbb{C}^{n_t \times n_t} \) is the channel matrix and \( N \in \mathbb{C}^{n_r \times n_t} \) is the additive noise. Both \( H \) and \( N \) have entries that are i.i.d. complex-Gaussian with zero mean and variance 1 and \( N_0 \) respectively. The transmitted codeword is \( X \in \mathbb{C}^{n_r \times n_t} \) and \( Y \in \mathbb{C}^{n_r \times n_t} \) is the received matrix. The ML decoding metric to minimize over all possible values of the codeword \( X \), is

\[ M(X) = \| Y - HX \|_F^2. \quad (2) \]

**Definition 1:** A linear STBC [1]: A linear STBC \( C \) over a real (1-dimensional) signal set \( S \), is a finite set of \( n_t \times n_t \) matrices, where any codeword matrix belonging to the code \( C \) is obtained from,

\[ X(x_1, x_2, \ldots, x_K) = \sum_{i=1}^{K} x_i A_i, \quad (3) \]

by letting the real variables \( x_1, x_2, \ldots, x_K \) take values from a real signal set \( S \), where \( A_i \) are fixed \( n_t \times n_t \) complex matrices defining the code, known as the weight matrices. The rate of this code is \( \frac{K}{2nm} \) complex symbols per channel use.

We are interested in linear STBCs, since they admit Sphere Decoding (SD) [2] which is a fast way of decoding for the variables. A further simplified version of the SD known as the fast sphere decoding (FSD) [3] (also known as condition ML decoding) was studied by Biglieri, Hong and Viterbo. The quadratic form (QF) approach has been used in the context of STBCs in [4] to determine whether Quaternion algebras or Biquaternion algebras are division algebras, an aspect dealing with the full diversity of the codes. This approach has not been fully exploited to study the other characteristics of STBCs. In this paper, we use this approach to study the fast sphere decoding (FSD) complexity of STBCs (a formal definition of this complexity is given in Subsection II-B).

Designing STBCs with low decoding complexity has been studied widely in the literature. Orthogonal designs with single symbol decodability were proposed in [5], [6], [7]. For STBCs with more than two transmit antennas, these came at a cost of reduced transmission rates. To increase the rate at the cost of higher decoding complexity, multi-group decodable STBCs were introduced in [8], [9], [10]. Fast decodable codes (codes that admit FSD) have reduced SD complexity owing to the fact that a few of the variables can be decoded as single symbols or in groups if we condition them with respect to the other variables. Fast decodable codes for asymmetric systems using division algebras have been recently reported [11]. Golden code and Silver code are also examples of fast decodable codes as shown in [12] and [13]. The properties of fast decodable codes and multi-group decodable codes were combined and a new class of codes called fast group decodable codes were studied in [14].

### A. Hurwitz-Radon Quadratic Form

In this subsection we define the Hurwitz Radon quadratic form (HRQF) on any STBC. We first recall some basics about quadratic forms. More details can be seen in [15].

**Definition 2:** Let \( F \) be a field with characteristic not 2, and \( V \) be a finite dimensional \( F \)-vector space. A quadratic form on \( V \) is defined as a map \( Q : V \rightarrow F \) such that it satisfies the following properties.

- \( Q(\alpha v) = \alpha^2 Q(v) \) for all \( v \in V \) and all \( \alpha \in F \).
- The map \( B(v, w) = \frac{1}{2} [Q(v + w) - Q(v) - Q(w)] \) for all \( v, w \in V \) is bilinear and symmetric.

If we consider \( V \) as an \( n \)-dimensional vector space over \( F \), then we can also consider the quadratic form as a homoge-
neous polynomial of degree two, i.e., for $1 \leq i, j \leq n$, we have scalars $m_{ij}$ such that

$$Q(v) = Q(v_1, v_2, ..., v_n) = \sum_{i,j=1}^{n} m_{ij} v_i v_j$$  \hspace{1cm} (4)

for all $v = [v_1, ..., v_n] \in V$. Hence, we can associate a matrix $M = (m_{ij})$ with the quadratic form such that $Q(v) = vMv^T$.

**Definition 3**: The Hurwitz-Radon quadratic form is a map from the STBC $C = \{X = \sum_{i=1}^{K} x_iA_i\}$ to the field of real numbers $\mathbb{R}$, i.e., $Q : C \rightarrow \mathbb{R}$ given by

$$Q(X) = \sum_{1 \leq i \leq j \leq K} x_i x_j d_{ij},$$  \hspace{1cm} (5)

where $X$ is an element of the STBC and

$$d_{ij} = \|A_iA_j^H + A_jA_i^H\|_F^2.$$

**Theorem 1**: The map defined by $Q$ is a quadratic form.

**Proof**: The map $Q$ needs to satisfy the conditions as defined in Definition 2. We have

$$Q(aX) = \sum_{i,j} a x_i a x_j d_{ij} = a^2 \sum_{i,j} x_i x_j d_{ij} = a^2 Q(X)$$

and

$$B(X, Y) = \frac{1}{2} [Q(X + Y) - Q(X) - Q(Y)]$$

should be bilinear and symmetric where $X = \sum_{i=1}^{K} x_iA_i$ and $Y = \sum_{i=1}^{K} (y_iA_i)$. Substituting and simplifying, we get

$$B(X, Y) = \frac{1}{2} \sum_{i,j} [x_i y_i d_{ii} + (x_i y_j + x_j y_i) d_{ij}].$$

It is clearly seen that this map is bilinear and symmetric. $\blacksquare$

We can associate a matrix with the HRQF. If we define the matrix $M = (m_{ij})$ where $i, j = 1, 2, ..., K$ such that $m_{ij} = d_{ij}$, then we can write the HRQF as $Q(x) = xMx^T$, where $x = [x_1 \ x_2 \ ... \ x_K]$. Notice that $M$ is a symmetric matrix and $m_{ij} = 0$ if and only if $A_iA_j^H + A_jA_i^H = 0$.

The following example shows that the FSD complexity depends on the ordering of the weight matrices or equivalently the ordering of the variables.

**Example 1**: Let us consider the Silver code given by:

$$X = X_a(s_1, s_2) + TX_b(z_1, z_2),$$  \hspace{1cm} (6)

where $X_a$ and $X_b$ take the Alamouti structure, and

$$X_a(s_1, s_2) = \begin{bmatrix} s_1 - s_2^* \\ s_2 \end{bmatrix}, \quad X_b(z_1, z_2) = \begin{bmatrix} z_1 - z_2^* \\ z_2 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $[z_1, z_2]^T = U [s_3, s_4]^T$, where $U$ is a unitary matrix chosen to maximize the minimum determinant and is given by $U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 + j & -1 + 2j \\ 1 + 2j & 1 - j \end{bmatrix}$.

Let all the variables take values from a signal set of cardinality $M$. If we order the variables (and hence the weight matrices) as $[s_{11}, s_{1Q}, s_{21}, s_{2Q}, s_{31}, s_{3Q}, s_{41}, s_{4Q}]$, then the $R$ matrix for SD has the following structure

$$R = \begin{bmatrix} t & 0 & 0 & t & t & t & t & t & t \\ 0 & t & 0 & t & t & t & t & t & t \\ 0 & 0 & t & t & t & t & t & t & t \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{bmatrix},$$

where $t$ denotes non-zero entries. We can clearly see that the Silver code admits fast decoding with this ordering with FSD complexity $M^5$. However, if we change the ordering to $[s_{11}, s_{1Q}, s_{41}, s_{2Q}, s_{31}, s_{3Q}, s_{21}, s_{4Q}]$, then the $R$ matrix for SD has the following structure

$$R = \begin{bmatrix} t & 0 & 0 & t & t & t & t & t & t \\ 0 & t & 0 & t & t & t & t & t & t \\ 0 & 0 & t & t & t & t & t & t & t \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{bmatrix},$$

where $t$ denotes non-zero entries. With this ordering, the FSD complexity increases to $M^7$.

The contributions of this paper are as follows: We give a formal definition of the FSD complexity of a linear STBC (Subsection II-B). With the help of HRQF, it is shown that the FSD complexity of the code depends only on the weight matrices of the code with their ordering, and not on the channel realization (even though the equivalent channel when SD is used depends on the channel realization) or the number of receive antennas.

A best ordering (not necessarily unique) of the weight matrices provides the least FSD complexity for the STBC. We provide an algorithm to be applied to the HRQF matrix which outputs a best ordering.

The remaining of the paper is organized as follows: In Section III the known classes of low ML decodable codes, the system model and the formal definition of the FSD complexity of a linear STBC are given. In Section IV we show that the FSD complexity depends completely on the HRQF and not on the channel realization or the number of receive antennas. In Section V we present an algorithm to modify the HRQF matrix in order to obtain a best ordering of the weight matrices to obtain the least FSD complexity. Concluding remarks constitute Section VI.

**Notations**: Throughout the paper, bold lower-case letters are used to denote vectors and bold upper-case letters to denote matrices. For a complex variable $x$, $x_T$ and $x_Q$ denote the real and imaginary part of $x$, respectively. The sets of all integers, all real and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. The operation of stacking the columns of $X$ one below the other is denoted by $\text{vec}(X)$. The Kronecker product is denoted by $\otimes$, $I_T$ and $D_T$ denote the $T \times T$ identity matrix and the null matrix, respectively. For a complex variable $x$, the $(\cdot)$ operator acting on $x$ is defined as follows

$$\hat{x} = \begin{bmatrix} x_T & -x_Q \\ x_Q & x_T \end{bmatrix}. $$
The \( (\cdot) \) operator can similarly be applied to any matrix \( X \in \mathbb{C}^{n \times m} \) by replacing each entry \( x_{ij} \) by \( \dot{x}_{ij} \), \( i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \), resulting in a matrix denoted by \( \dot{X} \in \mathbb{R}^{2n \times 2m} \). Given a complex vector \( \dot{x} = [x_1, x_2, \ldots, x_n]^T \), \( \dot{x} \) is defined as \( \dot{x} \triangleq [x_{11}, x_{1Q}, \ldots, x_{n1}, x_{nQ}]^T \).

### II. System Model and Definition of FSD Complexity

For any Linear STBC with variables \( x_1, x_2, \ldots, x_K \) given by (3), the generator matrix \( G \) is defined by \( \text{vec}(X) = \dot{G} \dot{x} \), where \( \dot{x} = [x_1, x_2, \ldots, x_K]^T \). In terms of the weight matrices, the generator matrix can be written as

\[
G = \left[ \text{vec}(A_1) \quad \text{vec}(A_2) \quad \cdots \quad \text{vec}(A_K) \right].
\]

Hence, for any STBC, (3) can be written as

\[
\text{vec}(Y) = \text{vec}(X) + \text{vec}(N),
\]

where \( \text{vec}(X) \) is an orthonormal matrix and

\[
\text{vec}(Y) = H_{eq} \dot{x} + \text{vec}(N),
\]

and \( \dot{x} = [x_1, x_2, \ldots, x_K]^T \), with each \( x_i \) drawn from a 1-dimensional (PAM) constellation. Using the above equivalent system model, the ML decoding metric (2) can be written as

\[
M(\dot{x}) = \| \text{vec}(Y) - H_{eq} \dot{x} \|^2_F.
\]

Using QR decomposition of \( H_{eq} \), we get \( H_{eq} = QR \) where \( Q \in \mathbb{R}^{2n_r \times n_r} \) is an orthonormal matrix and \( R \in \mathbb{R}^{n_r \times K} \) is an upper triangular matrix. Using this, the ML decoding metric now changes to

\[
M(\dot{x}) = \| Q^T \text{vec}(Y) - RX \|_F^2 \overset{2} = \| y' - RX \|_F^2.
\]

If we have \( H_{eq} = [h_1, h_2, \ldots, h_K] \), where \( h_i, i = 1, 2, \ldots, K \) are column vectors, then the \( Q \) and \( R \) matrices have the following form obtained by the Gram-Schmidt orthogonalization:

\[
Q = [q_1, q_2, \ldots, q_K], \quad (8)
\]

where \( q_i, i = 1, 2, \ldots, K \) are column vectors, and

\[
R = \begin{bmatrix}
\| r_1 \| & \langle q_1, h_2 \rangle & \langle q_1, h_3 \rangle & \cdots & \langle q_1, h_K \rangle \\
0 & \| r_2 \| & \langle q_2, h_3 \rangle & \cdots & \langle q_2, h_K \rangle \\
0 & 0 & \| r_3 \| & \cdots & \langle q_3, h_K \rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \| r_K \|
\end{bmatrix}, \quad (9)
\]

where \( r_1 = h_1 \), \( q_1 = \frac{r_1}{\| r_1 \|} \) and for \( i = 2, \ldots, K \),

\[
r_i = h_i - \sum_{j=1}^{i-1} \langle q_j, h_i \rangle q_j, \quad q_i = \frac{r_i}{\| r_i \|}.
\]

### A. Multi-group decodability, fast decodability and fast group decodability

In case of a multi-group decodable STBC, the variables can be partitioned into groups such that the ML decoding metric is decoupled into submetrics such that only the members of the same group need to be decoded jointly. It can be formally defined as [9], [16], [17]:

**Definition 4:** An STBC is said to be \( g \)-group decodable if there exists a partition of \( \{1, 2, \ldots, K\} \) into \( g \) non-empty subsets \( \Gamma_1, \Gamma_2, \ldots, \Gamma_g \) such that the following condition is satisfied:

\[
A_1 A_1^H + A_m A_m^H = 0,
\]

whenever \( l \in \Gamma_i \) and \( m \in \Gamma_j \) and \( i \neq j \).

If we group all the variables of the same group together in (7), then the \( R \) matrix for the SD [2], [18] in case of multi-group decodable codes will be of the following form:

\[
R = \begin{bmatrix}
\Delta_1 & 0 & \cdots & 0 \\
0 & \Delta_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Delta_g
\end{bmatrix}, \quad (10)
\]

where \( \Delta_i, i = 1, 2, \ldots, g \) is a square upper triangular matrix.

Now, consider the standard SD of an STBC. Suppose the \( R \) matrix as defined in (9) turns out to be such that when we fix values for a set of symbols, the rest of the symbols become group decodable, then the code is said to be fast decodable. Formally, it is defined as follows:

**Definition 5:** An STBC is said to be fast SD if there exists a partition of \( \{1, 2, \ldots, L\} \) where \( L \leq K \) into \( g \) non-empty subsets \( \Gamma_1, \Gamma_2, \ldots, \Gamma_g \) such that the following condition is satisfied

\[
(q_i, h_j) = 0 \quad (i < j),
\]

whenever \( i \in \Gamma_p \) and \( j \in \Gamma_q \) and \( p \neq q \). Where \( q_i \) and \( h_j \) are obtained from the QR decomposition of the equivalent channel matrix \( H_{eq} = [h_1, h_2, \ldots, h_K] = QR \) with \( h_i, i = 1, 2, \ldots, K \) as column vectors and \( Q = [q_1, q_2, \ldots, q_K] \) with \( q_i, i = 1, 2, \ldots, K \) as column vectors as defined in (8).

Hence, by conditioning \( K - L \) variables, the code becomes \( g \)-group decodable. As a special case, when no conditioning is needed, i.e., \( L = K \), then the code is \( g \)-group decodable. The \( R \) matrix for fast decodable codes will have the following form:

\[
R = \begin{bmatrix}
\Delta & B_1 \\
0 & B_2
\end{bmatrix},
\]

where \( \Delta \) is an \( L \times L \) block diagonal, upper triangular matrix, \( B_2 \) is a square upper triangular matrix and \( B_1 \) is a rectangular matrix.

Fast group decodable codes were introduced in [14]. These codes combine the properties of multi-group decodable codes and the fast decodable codes. These codes allow each of the groups in the multi-group decodable codes to be fast decoded. The \( R \) matrix for a fast group decodable code will have the
following form:

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{R}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}_g \end{bmatrix},$$

(13)

where each $\mathbf{R}_i, i = 1, 2, \ldots, g$ will have the following form:

$$\mathbf{R}_i = \begin{bmatrix} \Delta_i & \mathbf{B}_{i1} \\ 0 & \mathbf{B}_{i2} \end{bmatrix},$$

(14)

where $\Delta_i$ is an $L_i \times L_i$ block diagonal, upper triangular matrix, $\mathbf{B}_{i2}$ is a square upper triangular matrix and $\mathbf{B}_{i1}$ is a rectangular matrix. The structure of the $\mathbf{R}$ matrix for each of the codes defined above depends upon the ordering of the weight matrices. If we change the ordering of the weight matrices, the $\mathbf{R}$ matrix may lose its structure and no longer exhibit the desirable decoding properties. The Silver code of matrices, the $\mathbf{R}$ matrix be the matrix obtained by the $QR$ decomposition used to obtain the $\mathbf{R}$ matrix and by $\mathbf{B}_{i2}$ the number of variables that need to be conditioned, when we use FSD on the $\mathbf{R}$ th group, where $\mathbf{R}$ th group exhibit the desirable decoding properties. The Silver code of an STBC will have the following form:

$$R_{\lambda} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} & a_{16} & a_{17} & a_{18} \\ 0 & a_{22} & 0 & 0 & a_{25} & a_{26} & a_{27} & a_{28} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88} \end{bmatrix}$$

Definition 6: We define the FSD complexity of a single group decodable STBC for the given ordering to be $M^{t_1 + k_1}$.

In case of a multi-group decodable code with $g$ groups, the $\mathbf{R}$ matrix will be a block diagonal matrix. We then calculate the FSD complexity of each group independently as described above and choose the maximum among them as the FSD complexity of the STBC.

Definition 7: We define the FSD complexity of a multi-group decodable STBC with $g$ groups to be $\max_i (M^{t_1 + k_1})$, where $1 \leq i \leq g$.

We present a few examples to get a better understanding of FSD complexity.

Example 2: Let the $\mathbf{R}$ matrix be of the form:

$$\mathbf{R}_1 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{bmatrix}.$$ 

Now if we use FSD on this matrix, we will condition $l_1 = 4$ variables and obtain a $2$-group decodable code. We have

$$k_{1,1} = k_{1,2} = 0, \quad k_1 = \max (l_{1,1} + k_{1,1}, l_{1,2} + k_{1,2}) = 2.$$ 

The FSD complexity of this STBC for the given ordering is $M^{t_1 + k_1} = M^{l_{1,2} + k_{1,2}}$.

Example 3: Let the $\mathbf{R}$ matrix be of the form:

$$\mathbf{R} = \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} & 0 & 0 & 0 & 0 & t \\ 0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & t \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & a_{58} \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 & a_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & 0 & a_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{bmatrix}$$

Now if we use FSD on this matrix, we will condition $l_1 = 2$ variables and obtain a $2$-group decodable code. We have

$$\mathbf{R}_1 = \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$
\[
R_2 = \begin{bmatrix}
  a_{5,5} & 0 & 0 & a_{5,8} \\
  0 & a_{6,6} & 0 & a_{6,8} \\
  0 & 0 & a_{7,7} & a_{7,8} \\
  0 & 0 & 0 & a_{8,8}
\end{bmatrix}.
\]

Since the number of variables in each of the above matrices are 4, we have \( n_1 = n_2 = 4 \). Now if we use FSD on \( R_1 \), we can condition \( l_1, l_2 = 2 \) variables and obtain a 2-group decodable code. If we use FSD on \( R_2 \), we can condition \( l_1, l_2 = 1 \) variable and obtain a 3-group decodable code. We now have

\[
R_{1,1} = \begin{bmatrix} a_{1,1} \end{bmatrix}, \quad R_{1,2} = \begin{bmatrix} a_{2,2} \end{bmatrix}, \\
R_{2,1} = \begin{bmatrix} a_{5,5} \end{bmatrix}, \quad R_{2,2} = \begin{bmatrix} a_{6,6} \end{bmatrix}, \quad R_{2,3} = \begin{bmatrix} a_{7,7} \end{bmatrix},
\]

\( n_{1,1} = n_{1,2} = n_{2,1} = n_{2,2} = n_{2,3} = 1 \) and \( n_{1,3} = 0 \).

We cannot condition any more variables in any of these matrices. So the process stops here and we set

\[
l_{1,1,1} = l_{1,1,2} = l_{1,2,1} = l_{1,2,2} = l_{1,2,3} = 1.
\]

We have

\[
\begin{align*}
k_{1,1,1} &= k_{1,1,2} = k_{1,2,1} = k_{1,2,2} = k_{1,2,3} = 0, \\
k_{1,1} &= \max(l_{1,1,1} + k_{1,1,1}, l_{1,1,2} + k_{1,1,2}) = 1, \\
k_{1,2} &= \max(l_{1,2,1} + k_{1,2,1}, l_{1,2,2} + k_{1,2,2}, l_{1,2,3} + k_{1,2,3}) = 1, \\
k_1 &= \max(l_{1,1} + k_{1,1}, l_{1,2} + k_{1,2}) = 3.
\end{align*}
\]

The FSD complexity of this STBC for the given ordering is \( M^{l_1+k_1} = M^5 \).

### III. HRQF AND FSD COMPLEXITY

In this section we show that the HRQF matrix is enough to determine the FSD complexity of an STBC and hence the FSD complexity is independent of the channel matrix realization or the number of receive antennas. Towards this end, we prove that the zeros in the \( R \) matrix which determine the FSD complexity are also zeros in the HRQF matrix. First we define an ordered partition of a set.

**Definition 8:** We call a partition of \( \{a_1, a_2, ..., a_K\} \) into \( g \) non-empty subsets \( \Gamma_1, \Gamma_2, ..., \Gamma_g \) with cardinalities \( K_1, K_2, ..., K_g \) an ordered partition if \( \{a_1, ..., a_{K_1}\} \in \Gamma_1, \{a_{K_1+1}, ..., a_{K_1+K_2}\} \in \Gamma_2 \) and so on, til \( \{a_{\sum_{j=1}^{g-1} K_j+1}, ..., a_{\sum_{j=1}^{g} K_j}\} \in \Gamma_g \).

Now we address the class of multi-group decodable codes.

**Lemma 1:** Consider an STBC \( C = \sum_{i=1}^{K} x_i A_i \). Let \( M \) denote the HRQF matrix of this STBC. If there exists an ordered partition of \( \{1, 2, ..., K\} \) into \( g \) non-empty subsets \( \Gamma_1, \Gamma_2, ..., \Gamma_g \) such that \( m_{ij} = 0 \) whenever \( i \in \Gamma_p \) and \( j \in \Gamma_q \) and \( p \neq q \), then the code is \( g \)-group sphere decodable. In other words, the FSD complexity of the STBC is determined by the HRQF matrix.

**Proof:** Let \( R \) be the matrix obtained from the QR decomposition of \( H_{eq} \). For the code to be \( g \)-group sphere decodable, we need to prove that \( r_{ij} = 0 \), whenever \( i \in \Gamma_p \) and \( j \in \Gamma_q \) and \( p \neq q \). We know from [12] that if \( A_i A_j^H + A_j A_i^H = 0 \) is satisfied for some \( i, j \) then the corresponding columns in the \( H_{eq} \) matrix are orthogonal, i.e., \( \langle h_i, h_j \rangle = 0 \). We also know that \( m_{ij} = 0 \) if and only if \( A_i A_j^H + A_j A_i^H = 0 \). Let \( L_p = \sum_{q=1}^{g} \|\Gamma_q\| \) where \( p = 1, 2, ..., g \) and \( L_0 = 0 \).

For any group \( \Gamma_p \), we need to prove that \( r_{ij} = 0 \) for \( L_{p-1} + 1 \leq i \leq L_p \) and \( L_p + 1 \leq j \leq K \). Consider the first group \( \Gamma_1 \). We have \( m_{ij} = 0 \) for \( l \leq i \leq L_1 \) and \( L_1 + 1 \leq j \leq K \). We need to prove that the \( R \) matrix has zero entries at the same locations. The proof for this is by induction.

For \( i = 1 \) and for any \( j \geq L_1 + 1 \),

\[
\langle q_1, h_j \rangle = \frac{1}{\|r_i\|} (h_1, h_j) = 0
\]

since \( q_1 = \frac{1}{\|h_1\|} h_1 \). Now, let \( (q_i, h_j) = 0 \) for all \( l < i \) for any \( i \) such that \( 1 \leq i \leq L_1 \). We have,

\[
\langle q_i, h_j \rangle = \frac{1}{\|r_i\|} \left[ (h_i - \sum_{l=1}^{i-1} \langle q_i, h_l \rangle q_l, h_j) \right]
\]

\[
= \frac{1}{\|r_i\|} \left[ (h_i, h_j) - \sum_{l=1}^{i-1} \langle q_l, h_j \rangle q_l, h_j) \right] = 0,
\]

since \( (h_i, h_j) = 0 \) as \( m_{ij} = 0 \) and \( \langle q_i, h_j \rangle = 0 \) for \( l < i \) by induction hypothesis.

Now consider the \( p \)-th group \( \Gamma_p \). Let the induction hypothesis be true for all groups \( 1, 2, ..., p-1 \). Consider \( r_{ij} \) where \( L_{p-1} + 1 \leq i \leq L_p \) and \( L_p + 1 \leq j \leq K \). We have,

\[
r_{ij} = \langle q_i, h_j \rangle = \frac{1}{\|r_i\|} \left[ (h_i - \sum_{l=1}^{i-1} \langle q_i, h_l \rangle q_l, h_j) \right]
\]

\[
= \frac{1}{\|r_i\|} \left[ (h_i, h_j) - \sum_{l=1}^{i-1} \langle q_l, h_j \rangle q_l, h_j) \right] = 0,
\]

since \( (h_i, h_j) = 0 \) as \( m_{ij} = 0 \) and \( \langle q_i, h_j \rangle = 0 \) for \( l < i \) by induction hypothesis.

We now consider an example to illustrate the above lemma.

**Example 4:** Consider the \( 2 \times 2 \) ABBA code given by [19]:

\[
X = \begin{bmatrix} x_1 + jx_4 & -x_2 + jx_3 \\
-2x_2 + jx_3 & x_1 + jx_4 \end{bmatrix},
\]

where \( x_i \in \mathbb{R} \) for \( i = 1, 2, 3, 4 \). This is a two group decodable code with \( \{x_1, x_2\} \) belonging to one group and \( \{x_3, x_4\} \) belonging to the other. The structure of the HRQF matrix \( M \) and the \( R \) matrix are given below with \( \{x_1, x_2, x_3, x_4\} \) as the ordering of the variables and the weight matrices,

\[
M = \begin{bmatrix} t & t & 0 & 0 \\
t & t & 0 & 0 \\
0 & 0 & t & t \\
0 & 0 & t & t \end{bmatrix}, \quad R = \begin{bmatrix} t & t & 0 & 0 \\
t & t & 0 & 0 \\
0 & 0 & t & t \\
0 & 0 & t & t \end{bmatrix},
\]

where \( t \) denotes the non-zero entries. As it can be seen, the upper triangular portion of \( M \) matrix and the \( R \) matrix have the same structure.

Now we move on the class of fast decodable codes.

**Lemma 2:** Consider an STBC \( C = \sum_{i=1}^{K} x_i A_i \). Let \( M \) denote the HRQF matrix of this STBC. If there exists an ordered partition of \( \{1, 2, ..., K\} \) where \( L \leq K \) into \( g \) non-empty subsets \( \Gamma_1, \Gamma_2, ..., \Gamma_g \) such that \( m_{ij} = 0 \) whenever
$i \in \Gamma_p$ and $j \in \Gamma_q$ and $p \neq q$, then the code is fast decodable or conditionally $g$-group decodable.

**Proof.** The proof follows from the proof of Lemma 1 by replacing $K$ with $L$.

We now consider an example to illustrate the above lemma.

**Example 5:** Consider the Silver code as mentioned in Example 1 if we order the variables (and hence the weight matrices) in the following fashion $[s_{11}, s_{1Q}, s_{21}, s_{2Q}, s_{31}, s_{3Q}, s_{41}, s_{4Q}]$, then the HRQF matrix $M$ and the $R$ matrix will have the following structure:

$$
M = \begin{bmatrix}
t & 0 & 0 & 0 & t & t & t & t \\
0 & t & 0 & 0 & t & t & t & t \\
0 & 0 & t & t & 0 & t & t & t \\
0 & 0 & 0 & t & t & 0 & t & t \\
t & t & t & t & t & 0 & 0 & 0 \\
t & t & t & t & t & 0 & 0 & 0 \\
t & t & t & t & t & 0 & 0 & 0 \\
t & t & t & t & t & 0 & 0 & 0 \\
\end{bmatrix}
$$

$$
R = \begin{bmatrix}
t & 0 & 0 & 0 & t & t & t & t \\
0 & t & 0 & 0 & t & t & t & t \\
0 & 0 & t & t & 0 & t & t & t \\
0 & 0 & 0 & t & t & 0 & t & t \\
o & 0 & 0 & 0 & t & 0 & 0 & 0 \\
o & 0 & 0 & 0 & t & 0 & 0 & 0 \\
o & 0 & 0 & 0 & t & 0 & 0 & 0 \\
o & 0 & 0 & 0 & t & 0 & 0 & 0 \\
\end{bmatrix}
$$

where $t$ denotes the non-zero entries. As it can be seen, the upper triangular portion of the matrix $M$, has a structure that admits fast decodability which is conditionally 4-group decodable if considered as the $R$ matrix.

We now turn to the class of fast group decodable codes.

**Lemma 3:** Consider an STBC $C = \sum_{i=1}^{K} x_i A_i$. Let $M$ denote the HRQF matrix of this STBC. If there exists an ordered partition of $\{1, 2, ..., K\}$ into $g$ non-empty subsets $\Gamma_1, \Gamma_2, ..., \Gamma_g$ such that $m_{ij} = 0$ whenever $i \in \Gamma_p$ and $j \in \Gamma_q$ and $p \neq q$, and if any group $\Gamma_i$ admits fast decodability, i.e., there exists an ordered partition of $\{\sum_{l=1}^{t-1} K_l + 1, \sum_{l=1}^{t-1} K_l + 2, ..., \sum_{l=1}^{t-1} K_l + L_i\}$ where $L_i \leq K_i$, into $g_i$ non-empty subsets $\Upsilon_{l_1}, \Upsilon_{l_2}, ..., \Upsilon_{l_g}$ such that $m_{rs} = 0$ whenever $r \in \Upsilon_{l_i}$ and $s \in \Upsilon_{l_j}$ and $p \neq q$, $i = 1, 2, ..., g$, then the code is fast group decodable.

**Proof.** The proof follows from the proofs of lemmas 1 and 2.

We now consider an example to illustrate the above lemma.

**Example 6:** Consider the fast group decodable STBC [14] given in (15).

Let the ordering of the variables (and hence the weight matrices) be $[s_1, s_2, ..., s_{17}]$. This STBC is two group decodable with $s_1$ in one group and $\{s_2, s_3, ..., s_{17}\}$ in the other. The second group is conditionally five group decodable. The HRQF matrix $M$ and the $R$ matrix are given in (16) and (17) respectively, where $t$ denotes the non-zero entries.

As we have seen from Lemmas 1, 2, and 3, the FSD complexity of the STBC depends only upon the HRQF matrix $M$ and not on the $H_{eq}$ matrix, i.e., the FSD complexity is independent of the channel matrix and the number of receive antennas. It can be completely captured into a single matrix obtained from the set of weight matrices and their ordering.

IV. ALGORITHM FOR A BEST ORDERING OF THE WEIGHT MATRICES

As seen in Example 1 the ordering of weight matrices determines the FSD complexity of an STBC. We have also seen that the HRQF matrix completely determines the FSD complexity of an STBC. In this section, we present an algorithm that uses the HRQF matrix as an input and manipulates it in order to obtain a best possible ordering of weight matrices. We do so by using row and column permutations of the HRQF matrix. The rows and columns of the HRQF matrix are in one to one correspondence with the ordering of the weight matrices. Hence, if we change the ordering of the weight matrices, the HRQF matrix changes accordingly and vice versa. For example, any transposition in the ordering of the weight matrices will result in swapping the corresponding rows and columns (since HRQF matrix is symmetric) of the HRQF matrix.

**Remark 1:** Note that we cannot perform such a manipulation on the $R$ matrix since it depends not only on the order of weight matrices but on the channel matrix as well. Also, all the entries of the $R$ matrix do not depict the HR orthogonality of the weight matrices, i.e., the $(i, j)$-th entry of the matrix may not be zero even if the $i$-th and $j$-th weight matrices are HR orthogonal. Hence, the $R$ matrix needs to be calculated each
As these denote all the variables HR orthogonal with \( x_1 \). The current ordering of variables is \(- \{ x_1, \Lambda_1, \Lambda_3 \}\). For the working of the algorithm, the following set operations are synonymous with the following matrix operations on the HRQF matrix:

- Moving a variable \( x_j \) into \( \Lambda_1 \) - Suppose the variable \( x_j \) is the \( p \)-th element in the current ordering, we move the \( p \)-th column to the \((|\Lambda_1| + 1)\)-th column shifting the rest of the columns to the right and then we move the \( p \)-th row to the \((|\Lambda_1| + 1)\)-th row shifting the rest of the rows downwards. Update the ordering accordingly.
- Moving a variable \( x_j \) into \( \Lambda_3 \) - Suppose the variable \( x_j \) is the \( p \)-th element in the current ordering, we move the \( p \)-th column to the last column shifting the rest of the columns to the left and then we move the \( p \)-th row to the last row shifting the rest of the rows upwards. Update the ordering accordingly.
- Moving a variable \( x_j \) into \( \Lambda_2 \) - No change. Only the cardinalities of \( \Lambda_2 \) and \( \Lambda_1 \) will change accordingly.

The algorithm works in two stages.

- First, we find the largest \( L \leq K \) such that \(|\Lambda_1| + |\Lambda_2| = L\).
- In the second stage, since the variables in \( \Lambda_1 \) and \( \Lambda_2 \) will be decoded separately, we consider the submatrices representing them as HRQF matrices of some STBC and run the first step of the algorithm on them recursively.

Let \( x_k \) be the first variable in \( \Lambda_1 \). It is currently the second variable in the overall ordering. We now proceed to find all the variables that are HR orthogonal with \( x_k \) and not HR orthogonal with \( \Lambda_1 \), since these will need to be jointly decoded with the variables in \( \Lambda_1 \). This can be found as follows. If any variable is HR orthogonal with \( x_k \), it will have a zero entry in the column corresponding to \( x_k \). Hence we traverse down the column represented by \( x_k \) to find the next zero entry. Let it be found in the \( p \)-th row corresponding to the variable \( x_i \). We also need to ensure that this variable is not HR orthogonal with \( \Lambda_1 \). So, we traverse the \( p \)-th row from column 1 to column \( |\Lambda_1| \) and check for any non-zero entries. In case any of them are found, it means that this variable needs to be jointly decoded with \( \Lambda_1 \). We add \( x_i \) to \( \Lambda_1 \).

First, we fix the first variable in the given ordering. Let it be \( x_i \) for some \( 1 \leq i \leq K \). The algorithm starts with only \( x_1 \) in \( \Lambda_1 \), all variables which are HR orthogonal with \( x_1 \) in a temporary set \( \Lambda_1 \) and the rest of the variables in \( \Lambda_3 \). The variables HR orthogonal with \( x_i \) can be easily identified as they correspond to the zero entries in the first row. For ease of manipulation, we move all the variables in the set \( \Lambda_i \) adjacent \( x_i \) in the ordering. This is equivalent to grouping all the zeros in the first row and placing them adjacent to the \((1, 1)\) entry.
Input: The HRQF matrix - $M = (m_{ij})$, the size of the HRQF matrix - $K \times K$, the input ordering - input_ordering
Output: The best possible FSD complexity - best_decplxty and its corresponding ordering - best_ordering
- current_ordering = input_ordering
- best_decplxty = $K$; $i = 1$
repeat
  - Shift all the zero entries in the first row next to the first element
  - Let the number of zeros in the first row be num_zero_cols
  - grp_size = 1; cur_zero_col = 2. (Var under consideration)
if num_zero_cols = 0 then
  - decplxty = $K$
end
else
  repeat
    flag = 0
    - Let no. of consecutive zeros in cur_zero_col be n.
    if $n < grp_size$ then
      - Move the cur_zero_col-th variable to the end
        (Move the corresponding row and column to the end and moving the rest of the rows and columns upwards).
      - Update current_ordering.
      - num_zero_cols = num_zero_cols - 1
      end
    if $n > cur_zero_col - 1$ then
      - grp_size = cur_zero_col - 1
    end
    Marker:
    - Find the next zero along the cur_zero_col column from the cur_zero_col row.
    - Let the next zero be found in the j-th row.
    for $t = 1$ to grp_size do
      if $m_{ij} \neq 0$ then
        - flag = 1
      end
    end
    if flag = 1 then
      - Move up the j-th variable to the grp_size + 1-th position.
      - Update current_ordering.
      - grp_size = grp_size + 1
      - cur_zero_col = cur_zero_col + 1
      - jump back to Marker
    end
    if flag = 0 then
      - cur_zero_col = cur_zero_col + 1
    end
  until cur_zero_col <= grp_size + num_zero_cols
  - top_hrqf_matrix is upper left $grp_size \times grp_size$ matrix. (With ordering top_hrqf_ordering).
  - bot_hrqf_matrix is square matrix from row, column = $grp_size + 1$ to row, column = $grp_size + num_zero_cols$.
  (With ordering bot_hrqf_ordering).
  Run the current algorithm on the top and bottom matrices
  - [top_decplxty, best_top_ordering] = order_hrqf (top_hrqf_matrix, top_hrqf_size, top_ordering)
  - [bot_decplxty, best_bot_ordering] = order_hrqf (bot_hrqf_matrix, bot_hrqf_size, bot_ordering)
  - Update current_ordering with best_top-ordering and best_bot-ordering.
  - Number of variables conditioned:
    condvars = $K - grp_size - num_zero_cols$
    decplxty = condvars + max{(top_decplxty, bot_decplxty)}
  if best_decplxty > decplxty then
    - best_decplxty = decplxty
    - best_ordering = current_ordering
  end
  - Circularly shift the variables (rows and columns)
  - Update current_ordering
  - $i = i + 1$
until $i = K$
Algorithm 1: The algorithm to obtain a best ordering of weight matrices - order_hrqf

Remark 2: Since the algorithm recursively orders each set, it is capable of even ordering variables in scenarios where any group obtained from conditioning of variables admits admittable decoding.

We now illustrate the working of the algorithm with an example.

Example 7: Consider the Silver code presented in Example 1. If we order the variables as $\{s_{11}, s_{41}, s_{2Q}, s_{3Q}, s_{31}, s_{21}, s_{1Q}\}$, we get the following HRQF matrix and the $R$ matrix for this ordering:

$$
M = \begin{bmatrix}
t & t & t & 0 & t & 0 & 0 \\
t & t & 0 & t & 0 & 0 & t \\
t & 0 & t & t & 0 & t & t \\
t & 0 & 0 & t & 0 & t & t \\
t & 0 & 0 & 0 & t & 0 & t \\
t & 0 & 0 & t & t & 0 & t \\
t & 0 & t & t & 0 & t & 0 \\
t & 0 & t & t & 0 & t & 0 
\end{bmatrix}
$$
The FSD complexity for this ordering is $M^8$. And this ordering does not admit fast decoding as well.

When we run the algorithm on the given HRQF matrix, the two sets $\Lambda_1$ and $\Lambda_2$ are formed, which are HR orthogonal with each other. In this case, $\Lambda_2 = \{s_2Q, s_2I, s_1Q\}$ and $\Lambda_1 = \{s_1I\}$. The conditioned variables will be present in the set $\Lambda_3 = \{s_4I, s_4Q, s_3Q, s_3I\}$. The HRQF matrix at this stage is as given by (18). The ordering of the variables at the end of the stage is $\{s_1I, s_2Q, s_2I, s_1Q, s_4I, s_4Q, s_3Q, s_3I\}$. Now, the variables from both sets $\Lambda_1$ and $\Lambda_2$ are run through the algorithm again. So, the top left $1 \times 1$ matrix and the next block diagonal $3 \times 3$ matrix are both fed to the algorithm. Since this is already the best possible ordering of these sets, the matrix $M$ remains the same after this stage. And the final ordering obtained is $\{s_1I, s_2Q, s_2I, s_1Q, s_4I, s_4Q, s_3Q, s_3I\}$. The $R$ matrix for this ordering is given by (19).

The FSD complexity for this $R$ matrix is $M^{13}$ but the best possible FSD complexity is $M^{12}$. When we run the algorithm on the given HRQF matrix, the two sets $\Lambda_1$ and $\Lambda_2$ are formed, which are HR orthogonal with each other. In this case, $\Lambda_1 = \{s_2, s_3, s_4, s_5, s_6, s_9, s_{10}, s_{11}, s_{12}, s_7, s_8, s_{13}, s_{14}, s_{15}, s_{16}\}$ and $\Lambda_2 = \{s_1\}$. The set $\Lambda_3$ is empty as this provides a group decoding scenario. Now, the variables from both sets are run through the algorithm again. So, the top left $16 \times 16$ matrix and the bottom right $1 \times 1$ matrix are both fed to the algorithm. The top left matrix is ordered according to the fast decoding algorithm as presented in the previous example. Since the bottom right matrix is a $1 \times 1$ matrix, it is returned without change. The final ordering of variables obtained is $\{s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_{13}, s_{14}, s_{15}, s_{16}, s_9, s_{10}, s_{11}, s_{12}, s_1\}$. The $M$ and the $R$ matrix for this ordering is as shown below.

Example 8: Consider the fast group decodable code presented in Example 6. If we order the variables as $[s_2, s_3, ..., s_{10}, s_1, s_{11}, s_{12}, ..., s_{17}]$, we get the following HRQF matrix and the $R$ matrix:
This ordering gives us the FSD complexity of $M^{12}$.

V. CONCLUSION

In this paper we have analysed the FSD complexity of an STBC using quadratic forms. We have shown that the HRQF completely categorizes the FSD complexity of an STBC and hence it is independent of the channel and the number of receive antennas. We have provided an algorithm to obtain a best ordering of weight matrices to get the best decoding performance from the code.

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