ANALYSIS OF THE FRACTIONAL DESCRIPTOR DISCRETE-TIME LINEAR SYSTEMS BY THE USE OF THE SHUFFLE ALGORITHM

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Abstract. The shuffle algorithm is applied to analysis of the fractional descriptor discrete-time linear systems. Using the shuffle algorithm the singularity of the fractional descriptor linear system is eliminated and the system is decomposed into dynamic and static parts. Procedures for computation of the solution and dynamic and static parts of the system are proposed. Sufficient conditions for the positivity of the fractional descriptor discrete-time linear systems are established.

1. Introduction. Fractional calculus is the branch of mathematics in which integrals and derivatives of non integer order are considered. Mathematical fundamentals of the fractional calculus are given in the monographs, e.g. [19], [20], [22].

In the recent years, fractional calculus has been used in many fields of science and engineering for modeling physical phenomena. Fractional order models have become more accurate than the corresponding classical integer order models. Numerous applications have been found in physics, mechanics, electricity, chemistry, signal processing, etc. [10], [13], [14], [21], [23], [27]. Some recent developments have been collected in the paper [25].

The descriptor system, which is also known as singular system, is a mathematical representation for modeling many real process found in electrical, mechanical and chemical engineering [4], [6], [7], [15], [16], [17], [18]. The positivity of descriptor linear systems has been analyzed in [1], [2], [26]. The dynamical system preserves the positivity property if the trajectory remains in nonnegative space for nonnegative inputs and initial conditions. The positive systems theory is given in monographs [8], [11]. The stability of positive fractional discrete-time linear systems has been investigated in [3], where the notion of practical stability for a class of the positive fractional discrete-time systems has been introduced. The problem of stabilization of positive descriptor fractional discrete-time linear systems has been presented in [24], where the method for finding the decentralized controller has been proposed.

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The shuffle algorithm has been introduced in the paper [17]. It is based on row reduction and shuffles of blocks in a matrix. The use of this algorithm for deriving the Jordan blocks for a square matrix with known eigenvalues has been presented in [5]. An extension of the classic shuffle algorithm has been used to analyse descriptor models of electrical circuits in [9].

In this paper the fractional descriptor discrete-time linear system will be considered. The shuffle algorithm will be applied to eliminate the singularity and to decomposed the system into dynamic and static parts. The positivity of the system will be also analysed. The paper is organized as follows. In Section 2 the shuffle algorithm is applied to eliminate the singularity of the fractional descriptor linear system and in Section 3 to decompose the system into dynamic and static parts. In Section 4 sufficient condition for the positivity of the fractional descriptor discrete-time systems are established. Concluding remarks are given in Section 5.

The following notations will be used:

\( \mathbb{R} \) - the set of real numbers,

\( \mathbb{R}^{n \times m} \) - the set of \( n \times m \) real matrices,

\( \mathbb{R}^{n \times m}_{+} \) - the set of \( n \times m \) real matrices with nonnegative entries and \( \mathbb{R}^{n}_{+} = \mathbb{R}^{n \times 1}_{+} \),

\( I_{n} \) - the \( n \times n \) identity matrix.

2. Shuffle algorithm for fractional descriptor discrete-time linear systems. Consider the fractional descriptor discrete-time linear system

\[
E \Delta^{\alpha} x_{i+1} = Ax_{i} + Bu_{i}, \quad i \in Z_{+} = 0, 1, ..., (1)
\]

\[
y_{i} = Cx_{i}, \quad (2)
\]

where \( x_{i} \in \mathbb{R}^{n}, u_{i} \in \mathbb{R}^{m}, y_{i} \in \mathbb{R}^{p} \) are the state, input and output vectors and \( E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \) and

\[
\Delta^{\alpha} x_{i} = \sum_{k=0}^{i} (-1)^{k} \binom{\alpha}{k} x_{i-k}, \quad i \in Z_{+} (3)
\]

is the \( \alpha \)-order difference of \( x_{i} \),

\[
\binom{\alpha}{k} = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for } k = 1, 2, ... \end{cases} \quad (4)
\]

Substituting (3) and (4) into (1) we obtain

\[
Ex_{i+1} = A_{\alpha}x_{i} + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} Ex_{i-k+1} + Bu_{i}, \quad i \in Z_{+}, \quad (5)
\]

where

\[
A_{\alpha} = A + \alpha E. \quad (6)
\]

It is assumed that \( \text{rank} \ E < n \) and

\[
\det[Ez - A] \neq 0 \quad \text{for some } \ z \in \mathbb{C}, \quad (7)
\]

where \( \mathbb{C} \) is the field of complex numbers.

In the shuffle algorithm the following elementary operations on real matrices will be used [13]:

1. Multiplication of any \( i \)-th row (column) by the number \( a \). This operation will be denoted by \( L[i \times a] \) for row operation and by \( R[i \times a] \) for column operation.

2. Addition to any \( i \)-th row (column) of the \( j \)-th row (column) multiplied by any number \( b \). This operation will be denoted by \( L[i + j \times b] \) for row operation and by \( R[i + j \times b] \) for column operation.
3. The interchange of the $i$-th and $j$-th rows (columns). This operation will be denoted by $L[i, j]$ for interchange of rows and $R[i, j]$ for interchange of columns.

Performing elementary row operations on the array

\[
\begin{bmatrix}
E & A & B \\
\end{bmatrix}
\]

or equivalently on (1) we obtain

\[
\begin{bmatrix}
E_1 & A_{\alpha 1} & B_1 \\
0 & A_{\alpha 2} & B_2 \\
\end{bmatrix}
\]

and

\[
E_1 x_{i+1} = A_{\alpha 1} x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} E_1 x_{i-k+1} + B_1 u_i,
\]

\[
i \in Z_+,
\]

\[
0 = A_{\alpha 2} x_i + B_2 u_i,
\]

where $E_1$ has full row rank equal to $r < n$. From (11) we have

\[
A_{\alpha 2} x_{i+1} = -B_2 u_{i+1},
\]

\[
i \in Z_+.
\]

The equations (10) and (12) can be written in the form

\[
\begin{bmatrix}
E_1 & A_{\alpha 2} \\
A_{\alpha 1} & 0 \\
B_1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_{i+1} \\
x_i \\
u_i \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{\alpha 1} & E_1 & B_1 \\
A_{\alpha 2} & 0 & -B_2 \\
0 & 0 & -B_2 \\
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_{i+1} \\
u_i \\
\end{bmatrix},
\]

\[
i \in Z_+.
\]

The array

\[
\begin{bmatrix}
E_1 & A_{\alpha 1} & B_1 & 0 \\
0 & A_{\alpha 2} & 0 & -B_2 \\
\end{bmatrix}
\]

can be obtained from (9) by performing a shuffle.

If the matrix

\[
\begin{bmatrix}
E_1 & A_{\alpha 2} \\
A_{\alpha 1} & 0 \\
B_1 & 0 \\
\end{bmatrix}
\]

is nonsingular then solving (13) we obtain

\[
\begin{bmatrix}
x_{i+1} \\
x_i \\
u_i \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
E_1 & A_{\alpha 2} \\
A_{\alpha 1} & 0 \\
B_1 & 0 \\
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{\alpha 1} & E_1 & B_1 & 0 \\
A_{\alpha 2} & 0 & -B_2 & 0 \\
0 & 0 & -B_2 \\
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_{i+1} \\
u_i \\
\end{bmatrix},
\]

\[
i \in Z_+.
\]

If the matrix (15) is singular then performing elementary row operations on (14) (or equivalently on (13)) we obtain

\[
\begin{bmatrix}
E_2 & A_{\alpha 3} & B_3 & C_1 \\
0 & A_{\alpha 4} & B_4 & C_2 \\
\end{bmatrix}
\]

and

\[
E_2 x_{i+1} = A_{\alpha 3} x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} E_2 x_{i-k+1} + B_3 u_i + C_1 u_{i+1},
\]

\[
0 = A_{\alpha 4} x_i + B_4 u_i + C_2 u_{i+1},
\]

where $E_2$ has full row rank and $\text{rank } E_2 \leq \text{rank } E_1$.

From (19) we have

\[
A_{\alpha 4} x_{i+1} = -B_4 u_{i+1} - C_2 u_{i+2}.
\]
The equations (18) and (20) can be written in the form

\[
\begin{bmatrix}
E_2 \\
A_{\alpha}
\end{bmatrix} x_{i+1} = \begin{bmatrix}
A_{\alpha3} \\
0
\end{bmatrix} x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} \begin{bmatrix}
E_2 \\
0
\end{bmatrix} x_{i-k+1}
+ \begin{bmatrix}
B_3 \\
0
\end{bmatrix} u_i + \begin{bmatrix}
C_1 \\
-B_4
\end{bmatrix} u_{i+1} + \sum_{k=2}^{i+1} \begin{bmatrix}
0 \\
-C_2
\end{bmatrix} u_{i+k}, \quad i \in \mathbb{Z}^+.
\] (21)

The array

\[
\begin{bmatrix}
E_2 & A_{\alpha3} & B_3 & C_1 & 0 \\
A_{\alpha4} & 0 & -B_4 & -C_2 & 0
\end{bmatrix}
\] (22)

can be obtained from (17) by performing a shuffle.

If the matrix

\[
\begin{bmatrix}
E_2 \\
A_{\alpha}
\end{bmatrix}
\] (23)

is nonsingular we can solve the equation (21) with respect to \( x_{i+1} \).

If the matrix is singular we repeat the procedure for (22).

In a similar way as for standard systems [11] it can be shown that if the condition (7) is satisfied then after \( q \) steps we obtain a nonsingular matrix

\[
\begin{bmatrix}
E_q \\
A_{\alpha q+2}
\end{bmatrix}
\] (24)

and

\[
x_{i+1} = \begin{bmatrix}
E_q \\
A_{\alpha q+2}
\end{bmatrix}^{-1} \begin{bmatrix}
A_{\alpha q+1} \\
0
\end{bmatrix} x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} \begin{bmatrix}
E_q \\
0
\end{bmatrix} x_{i-k+1}
+ \begin{bmatrix}
B_{q+1} \\
0
\end{bmatrix} u_i + \begin{bmatrix}
0 \\
-B_{q+2}
\end{bmatrix} u_{i+1} + \cdots + \begin{bmatrix}
0 \\
H_{q+q}
\end{bmatrix} u_{i+q}, \quad i \in \mathbb{Z}^+.
\] (25)

To analysis of (25) well-known methods [12], [13] of standard linear systems can be applied. From the above considerations we have the following procedure for eliminating the singularity of the equation (1).

**Procedure 2.1.**

**Step 1.** Performing elementary row operations on (8) find (9), where \( E_1 \) has full row rank.

**Step 2.** Shuffle the array (9) to (14). If the matrix (15) is nonsingular, find the desired solution (16). If the matrix is singular, performing elementary row operations on (14) find (17).

**Step 3.** If the condition (7) is satisfied the by repeating steps 1 and 2 we finally obtain the regular system (25).

From the above considerations we have the following theorem.

**Theorem 2.1.** The fractional order equation (1) satisfying the condition (7) of fractional descriptor discrete-time linear system by the use of the shuffle algorithm (Procedure 2.1) can be transformed to the standard one (25).

**2.1. Example 1.** Consider the equation (1) with the matrices

\[
E = \begin{bmatrix}
0 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{bmatrix}, \quad A_{\alpha} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 2
\end{bmatrix}.
\] (26)
Applying to (26) Procedure 2.1 we obtain the following

**Step 1.** Performing elementary row operations \( L[2 + 1 \times 1], L[2, 3] \) on the array

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & 2
\end{bmatrix}
\]

we obtain

\[
\begin{bmatrix}
E_1 & A_{\alpha 1} & B_1 \\
0 & A_{\alpha 2} & B_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

**Step 2.** A shuffle of (28) yields

\[
\begin{bmatrix}
E_1 & A_{\alpha 1} & B_1 \\
A_{\alpha 2} & 0 & -B_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\]

The matrix

\[
\begin{bmatrix}
E_1 \\
A_{\alpha 2}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0 & 2 \\
1 & 1 & 0
\end{bmatrix}
\]

is nonsingular and using (16) we obtain

\[
x_{i+1} = \left[ \begin{bmatrix} E_1 \\ A_{\alpha 2} \end{bmatrix} \right]^{-1} \left[ \begin{bmatrix} A_{\alpha 1} \\ 0 \end{bmatrix} \right] x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \left( \begin{array}{c}
\alpha \\
k
\end{array} \right) \left[ \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \right] e_{i-k+1}
\]

\[
+ \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_{i+1}
\]

\[
= \begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \left( \begin{array}{c}
\alpha \\
k
\end{array} \right) \begin{bmatrix} 0 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} e_{i-k+1}
\]

\[
+ \begin{bmatrix} -1 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix} u_i + \begin{bmatrix} -1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} u_{i+1}, \quad i \in \mathbb{Z}^+.
\]

3. **Decomposition of fractional descriptor discrete-time linear systems into dynamic and static parts.** In this section the shuffle algorithm will be applied to decompose the fractional descriptor discrete-time linear systems (1) into dynamic and static parts.

Firstly the essence of the method will be explained by the following example, which is a continuation of Example 1.

3.1. **Example 2.** Find the dynamic and static parts of the fractional descriptor discrete-time linear system (1) with the matrices (26).

After first shuffle of (28) we obtain (29) with nonsingular matrix (30). In this case we choose \( x_{1,i} = [ x_{2,i} \quad x_{3,i} ]^T, \bar{x}_{2,i} = x_{1,i} \) and from (29) we obtain

\[
E_{11} \bar{x}_{1,i+1} + E_{12} \bar{x}_{2,i+1} = A_{\alpha 11} \bar{x}_{1,i} + A_{\alpha 12} \bar{x}_{2,i} + \sum_{k=2}^{i+1} (-1)^{k+1} \left( \begin{array}{c}
\alpha \\
k
\end{array} \right) E_{11} \bar{x}_{1,i-k+1}
\]

\[
+ \sum_{k=2}^{i+1} (-1)^{k+1} \left( \begin{array}{c}
\alpha \\
k
\end{array} \right) E_{12} \bar{x}_{2,i-k+1} + B_1 u_i, \quad i \in \mathbb{Z}^+.
\]

\[
0 = A_{\alpha 21} \bar{x}_{1,i} + A_{\alpha 22} \bar{x}_{2,i} + B_2 u_i,
\]

\[
E_{11} \bar{x}_{1,i+1} + E_{12} \bar{x}_{2,i+1} = A_{\alpha 11} \bar{x}_{1,i} + A_{\alpha 12} \bar{x}_{2,i}
\]

\[
+ \sum_{k=2}^{i+1} (-1)^{k+1} \left( \begin{array}{c}
\alpha \\
k
\end{array} \right) E_{11} \bar{x}_{1,i-k+1}
\]

\[
+ \sum_{k=2}^{i+1} (-1)^{k+1} \left( \begin{array}{c}
\alpha \\
k
\end{array} \right) E_{12} \bar{x}_{2,i-k+1} + B_1 u_i, \quad i \in \mathbb{Z}^+.
\]

\[
0 = A_{\alpha 21} \bar{x}_{1,i} + A_{\alpha 22} \bar{x}_{2,i} + B_2 u_i,
\]
where
\[ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{\alpha 11} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \quad A_{\alpha 12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \]
\[ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_{\alpha 21} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_{\alpha 22} = [1]. \quad (34) \]

From (33) we have
\[ \ddot{x}_{2,i} = -A_{\alpha 22}^{-1} A_{\alpha 21} \ddot{x}_{1,i} - A_{\alpha 22}^{-1} B_2 u_i, \quad i \in Z_+. \quad (35) \]
\[ \ddot{x}_{2,i+1} = -A_{\alpha 22}^{-1} A_{\alpha 21} \ddot{x}_{1,i+1} - A_{\alpha 22}^{-1} B_2 u_{i+1}. \quad (36) \]
Substituting (35) into (32) we obtain the equation of the dynamic part
\[ \ddot{x}_{1,i+1} = \left( E_{11} - E_{12} A_{\alpha 22} A_{\alpha 21} \right)^{-1} \left( (A_{\alpha 11} - A_{\alpha 12} A_{\alpha 22} A_{\alpha 21}) \ddot{x}_{1,i} 
+ \sum_{k=2}^{i+1} (-1)^{k+1} \begin{pmatrix} \alpha \\ k \end{pmatrix} E_{11} \ddot{x}_{1,i-k+1}
+ \sum_{k=2}^{i+1} (-1)^{k+1} \begin{pmatrix} \alpha \\ k \end{pmatrix} E_{12} \ddot{x}_{2,i-k+1}
+ (B_1 - A_{\alpha 12} A_{\alpha 22} B_2) u_i + E_{12} A_{\alpha 22}^{-1} B_2 u_{i+1} \right) 
= \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \ddot{x}_{1,i} + \sum_{k=2}^{i+1} (-1)^{k+1} \begin{pmatrix} \alpha \\ k \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{x}_{1,i-k+1} 
+ \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_i. \quad (37) \]
The equation (36) of the static part has the form
\[ \ddot{x}_{2,i} = -A_{\alpha 22}^{-1} A_{\alpha 21} \ddot{x}_{1,i} - A_{\alpha 22}^{-1} B_2 u_i = \begin{bmatrix} 1 & 0 \end{bmatrix} (\ddot{x}_{1,i} + u_i), \quad i \in Z_. \quad (38) \]

3.2. General case. In general case performing elementary row operations on (8) after \( h \) shuffles we obtain
\[ E_h A_{\alpha 21} B_{11} ... B_{1h} \\
0 A_{\alpha 21} B_{21} ... B_{2h} \]
with
\[ \det \begin{bmatrix} E_h \\ A_{\alpha 21} \end{bmatrix} \neq 0 \]
and \( \text{rank } E_h = n_h < n. \)
If the condition (40) is satisfied then it is possible to choose the new state vectors \( \ddot{x}_{1,i} \in \mathbb{R}^{n_1} \) and \( \ddot{x}_{2,i} \in \mathbb{R}^{n_2}, n_1 + n_2 = n \) such that
\[ E_{h1} \ddot{x}_{1,i+1} + E_{h2} \ddot{x}_{2,i+1} = \dddot{A}_{\alpha 11} \ddot{x}_{1,i} + \dddot{A}_{\alpha 12} \ddot{x}_{2,i} 
+ \sum_{k=2}^{i+1} (-1)^{k+1} \begin{pmatrix} \alpha \\ k \end{pmatrix} E_{h1} \dddot{x}_{1,i-k+1} 
+ \sum_{k=2}^{i+1} (-1)^{k+1} \begin{pmatrix} \alpha \\ k \end{pmatrix} E_{h2} \ddot{x}_{2,i-k+1} + B_{11} u_i + ... + B_{1h} u_{i+h}, \quad i \in Z_+. \quad (41) \]
and \( \det E_{h1} \neq 0, \quad \det \dddot{A}_{\alpha 22} \neq 0. \)
From (42) we have
\[ \ddot{x}_{2,i} = -\ddot{A}_{\alpha 22}^{-1} (\dddot{A}_{\alpha 21} \ddot{x}_{1,i} + B_{21} u_i + ... + B_{2h} u_{i+h}), \quad i \in Z_+, \quad (43) \]
and
\[ \ddot{x}_{2,i+1} = -\ddot{A}_{\alpha 22}^{-1} (\dddot{A}_{\alpha 21} \ddot{x}_{1,i+1} + B_{21} u_{i+1} + ... + B_{2h} u_{i+h+1}). \quad (44) \]
Substitution of (43) and (44) into (41) yields

\[
E_{h1}\bar{x}_{1,i+1} = E_{h2}\tilde{A}_{\alpha_{22}}^{-1} (\tilde{A}_{\alpha_{21}} \bar{x}_{1,i+1} + B_{21} u_{i+1} + ... + B_{2h} u_{i+h+1}) \\
= A_{\alpha_{11}} \bar{x}_{1,i} - A_{\alpha_{12}} A_{\alpha_{22}}^{-1} (A_{\alpha_{21}} \bar{x}_{1,i} + B_{21} u_{i} + ... + B_{2h} u_{i+h}) \\
+ \sum_{k=2}^{i+1} (-1)^{k+1} \left( \frac{\alpha}{k} \right) E_{h1} \bar{x}_{1,i-k+1} \\
- \sum_{k=2}^{i+1} (-1)^{k+1} \left( \frac{\alpha}{k} \right) E_{h2} A_{\alpha_{22}}^{-1} (A_{\alpha_{21}} \bar{x}_{1,i-k+1} + ... + B_{2h} u_{i-k+1+h}) \\
+ B_{21} u_{i-k+1} + ... + B_{2h} u_{i-k+1+h}) + B_{11} u_{i} + ... + B_{1h} u_{i+h}, \quad i \in Z_+
\]

and the equation of the dynamic part

\[
\bar{x}_{1,i+1} = \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right]^{-1} \left[ A_{\alpha_{11}} - A_{\alpha_{12}} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right] \bar{x}_{1,i} \\
+ \sum_{k=2}^{i+1} (-1)^{k+1} \left( \frac{\alpha}{k} \right) (E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}}) \bar{x}_{1,i-k+1} \\
- \sum_{k=2}^{i+1} (-1)^{k+1} \left( \frac{\alpha}{k} \right) E_{h2} A_{\alpha_{22}}^{-1} (B_{21} u_{i-k+1} + ... + B_{2h} u_{i-k+1+h}) \\
+ \left[ B_{11} - A_{\alpha_{12}} A_{\alpha_{22}}^{-1} B_{21} \right] u_{i} \\
+ \left[ B_{12} - A_{\alpha_{12}} A_{\alpha_{22}}^{-1} B_{22} + E_{h2} A_{\alpha_{22}}^{-1} B_{21} \right] u_{i+1} \\
+ ... + \left[ B_{1h} - A_{\alpha_{12}} A_{\alpha_{22}}^{-1} B_{2h} + E_{h2} A_{\alpha_{22}}^{-1} B_{21} \right] u_{i+h} \\
+ E_{h2} A_{\alpha_{22}}^{-1} B_{2h} u_{i+h+1}) \\
= \hat{A}_{1} \bar{x}_{1,i} + \sum_{k=2}^{i+1} (-1)^{k+1} \left( \frac{\alpha}{k} \right) \hat{E}_{1} \bar{x}_{1,i-k+1} \\
+ \sum_{k=2}^{i+1} (-1)^{k+1} \left( \frac{\alpha}{k} \right) \hat{B}_{21} u_{i-k+1} + ... \\
+ \sum_{k=2}^{i+1} (-1)^{k+1} \left( \frac{\alpha}{k} \right) \hat{B}_{h} u_{i+h}, \quad i \in Z_+
\]

where

\[
\hat{A}_{1} = \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right]^{-1} \left[ A_{\alpha_{11}} - A_{\alpha_{12}} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right], \\
\hat{E}_{1} = \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right]^{-1} \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right], \\
\hat{B}_{21} = \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right]^{-1} E_{h2} A_{\alpha_{22}}^{-1} B_{21}, \\
\hat{B}_{2h} = \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right]^{-1} E_{h2} A_{\alpha_{22}}^{-1} B_{2h}, \\
\hat{B}_{h} = \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right]^{-1} E_{h2} A_{\alpha_{22}}^{-1} B_{2h}, \\
\hat{B}_{h+1} = \left[ E_{h1} - E_{h2} A_{\alpha_{22}}^{-1} A_{\alpha_{21}} \right]^{-1} E_{h2} A_{\alpha_{22}}^{-1} B_{2h}.
\]

The static part is described by the equation

\[
\bar{x}_{2,i} = \tilde{A}_{\alpha_{21}} \bar{x}_{1,i} + \tilde{B}_{21} u_{i} + ... + \tilde{B}_{2h} u_{i+h}, \quad i \in Z_+, \quad (47)
\]

where

\[
\tilde{A}_{\alpha_{21}} = -A_{\alpha_{22}}^{-1} A_{\alpha_{21}}, \quad \tilde{B}_{21} = -A_{\alpha_{22}}^{-1} B_{21}, \quad ..., \quad \tilde{B}_{2h} = -A_{\alpha_{22}}^{-1} B_{2h}.
\]
From the above considerations the following procedure for the decomposition of the fractional descriptor system (1) into dynamic and static parts follows.

**Procedure 3.1.** **Step 1.** Applying the shuffle algorithm to (1) find (39), (41) and (42).

**Step 2.** Substituting (43) and (44) into (41) find the equation (45) of the dynamic part of the system.

**Step 3.** Using (43) find the equation of the static part of the system.

Therefore, the following theorem of the decomposition of descriptor fractional linear systems has been proved.

**Theorem 3.1.** The fractional descriptor discrete-time linear system (1) satisfying the condition (7) can be decomposed by the use of Procedure 3.1 into the dynamic and static parts of the system.

4. **Positive systems.** In this section sufficient conditions for the positivity of the fractional descriptor linear systems (1), (2) will be established.

**Definition 4.1.** The fractional descriptor linear system (1), (2) is called (internally) positive if

\[ x_i \in \mathbb{R}^n, \quad y_i \in \mathbb{R}^p, \quad i = 0, 1, \ldots \]

for any consistent initial conditions \( x_0 \in \mathbb{R}^n \) and all admissible inputs \( u_i \in \mathbb{R}^m \) for \( i = 0, 1, \ldots \).

Using the shuffle algorithm and the method presented in Section 3 the system (1) can be reduced to the one described by (45).

**Theorem 4.2.** The fractional descriptor linear system (45) (the system (1), (2) is positive if

\[
\hat{A}_1 \in \mathbb{R}^{n_1 \times n_1}, \quad \hat{E}_1 \in \mathbb{R}^{n_1 \times n_1}, \quad \hat{B}_2l \in \mathbb{R}^{n_1 \times m}, \quad \hat{B}_2l \in \mathbb{R}^{n_2 \times m},
\]

\[
\hat{B}_k \in \mathbb{R}^{n_1 \times m}, \quad \hat{A}_n2l \in \mathbb{R}^{n_2 \times n_2}, \quad C \in \mathbb{R}^p \times n, \quad l = 1, 2, \ldots, h, \quad k = 0, 1, \ldots, h + 1. \tag{49}
\]

**Proof.** It is well known [12], [13] that the system (1) is positive if \( \bar{x}_1 \in \mathbb{R}^n, \quad \bar{x}_2 \in \mathbb{R}^n, \quad n_1 + n_2 = n \). In this case \( x \in \mathbb{R}^n \) since the vectors have the same state variables. It is well-known that if \( 0 < \alpha < 1 \) then the coefficients \((-1)^{k+1} \binom{\alpha}{k}\) are positive. Therefore, the fractional descriptor linear system (45) is positive if the conditions (49) are satisfied.

The fractional descriptor system (45) is asymptotically stable if and only if the matrix \( \hat{A}_1 \) is Schur matrix (asymptotically stable).

4.1. **Example 3.** Consider the fractional descriptor linear system (1) with the matrices

\[
E = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 0 & 0.2 \\
0 & 0.3 & 0 \\
1 & -1 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 0
\end{bmatrix}, \quad \tag{50}
\]

\[0 < \alpha < 1.\]

The shuffle of

\[
\begin{bmatrix}
E_1 & A_1 & B_1 \\
0 & A_2 & B_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0.2 & 0 & 1 \\
0 & 0 & 1 & 0 & 0.3 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0
\end{bmatrix} \tag{51}
\]
yields
\[
E_1 \quad A_1 \quad B_1 \quad 0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.3 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (52)

The matrix
\[
\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}
\] (53)

is nonsingular since
\[
\det \left[ \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \right] = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 1. 
\] (54)

From (6) and (50) we have
\[
A_\alpha = \begin{bmatrix} A_{\alpha 1} \\ A_{\alpha 2} \end{bmatrix} = A + \alpha E = \begin{bmatrix} 0 & \alpha & 0.2 \\ 0 & 0.3 & \alpha \\ 1 & -1 & -1 \end{bmatrix}
\] (55)

and
\[
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_{i+1} = \begin{bmatrix} 0 & \alpha & 0.2 \\ 0 & 0.3 & \alpha \end{bmatrix} x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \begin{bmatrix} \alpha \\ k \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_{i-k+1} + \begin{bmatrix} 0 & 1 \end{bmatrix} u_i,
\] (56)

\[
0 = A_{\alpha 2} x_i + B_2 u_i = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} x_i \quad \text{or} \quad x_{1,i} = x_{2,i} + x_{3,i}. 
\] (57)

Substituting (57) and
\[
x_{1, i+1} = x_{2, i+1} + x_{3, i+1}
\] (58)

into (56) we obtain the dynamical part
\[
\begin{bmatrix} x_{2, i+1} \\ x_{3, i+1} \end{bmatrix} = \begin{bmatrix} \alpha & 0.2 \\ 0.3 & \alpha \end{bmatrix} \begin{bmatrix} x_{2,i} \\ x_{3,i} \end{bmatrix} + \sum_{k=2}^{i+1} (-1)^{k+1} \begin{bmatrix} \alpha \\ k \end{bmatrix} \begin{bmatrix} x_{2,i-k+1} \\ x_{3,i-k+1} \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} u_i.
\] (59)

From (59) it follows that the dynamical part is positive and asymptotically stable for \( \alpha \) satisfying the condition \( \alpha + 0.3 < 1 \). The static part of the system is given by (57).

5. Concluding remarks. In this paper the shuffle algorithm has been applied to analysis of the fractional descriptor discrete-time linear systems. Using the shuffle algorithm the singularity of the fractional descriptor linear system has been eliminated (Theorem 2.1) and the system has been decomposed into dynamic and static parts (Theorem 3.1). Procedures for computation of the solution and dynamic and static parts of the system have been proposed. The numerical efficiency of this approach has been demonstrated on numerical examples.

Many different physical phenomena may be expressed by the presented model. Therefore, the results obtained are useful to analyse properties such as stability, controllability, observability for a class of fractional descriptor discrete-time systems. Sufficient conditions for the positivity of the considered system have been established (Theorem 4.2). Moreover, using presented approach, after elimination
of the singularity the well-known methods can be applied to solve the standard models of the fractional linear systems.

The considerations can be extended to descriptor discrete-time linear systems of different fractional orders.

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