On the number of eigenvalues of a model operator in fermionic Fock space

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Abstract. We consider a model describing a truncated operator $H$ (truncated with respect to the number of particles) acting in the direct sum of zero-, one-, and two-particle subspaces of a fermionic Fock space $F_{a}(L^{2}(T^{3}))$ over $L^{2}(T^{3})$. We admit a general form for the “kinetic” part of the hamiltonian $H$, which contains a parameter $\gamma$ to distinguish the two identical particles from the third one.

In this note:
(i) We find a critical value $\gamma^{*}$ for the parameter $\gamma$ that allows or forbids the Efimov effect (infinite number of bound states if the associated generalized Friedrichs model has a threshold resonance) and we prove that only for $\gamma < \gamma^{*}$ the Efimov effect is absent, while this effect exists for any $\gamma > \gamma^{*}$.
(ii) In the case $\gamma > \gamma^{*}$ we also establish the following asymptotics for the number $N(z)$ of eigenvalues $z$ below $E_{\text{min}}$, the lower limit of the essential spectrum of $H$:
\[
\lim_{z \to E_{\text{min}}} \frac{N(z)}{\log |E_{\text{min}} - z|} = U_{0}(\gamma) \quad (U_{0}(\gamma) > 0), \quad \forall \gamma > \gamma^{*}.
\]

1. Introduction
The spectral theory of continuous and lattice three particle Schrödinger operators in $\mathbb{R}^{3}$ shows the remarkable phenomenon known as Efimov effect: if all hamiltonians of all the two-body subsystems are positive and if at least two of them have a zero-energy resonance, then the three-body system has an infinite number of negative eigenvalues accumulating at zero.

This remarkable spectral property was discovered by V. Efimov [11] and has since become the subject of many papers [2, 3, 7, 9, 12, 20, 22, 23, 24, 26]. The first mathematical proof of the existence of this effect was given by Yafaev ([26]) and A. Sobolev established [22] the asymptotics of the number of eigenvalues near the threshold of the essential spectrum.

Recently, Wang [25] has proved the existence of the Efimov effect in the system with $N \geq 4$ particles in $\mathbb{R}^{3}$ but in this case the properties of the spectrum have not been fully comprehended yet.

In statistical physics Minlos and Shpon [18], Malishev and Minlos [16], solid-state physics Mattis [17], Mogilner [19] and the theory of quantum fields Friedrichs [13], Buhler et.al. [8] some important problems arise where the number of quasi-particles is bounded, but not fixed. In Sigal, Soffer [21] has developed geometric and commutator techniques to find the location of the spectrum and to prove absence of singular continuous spectrum for Hamiltonians without conservation of the particle number.
Notice that the study of systems describing \( n \) particles in interaction, without conservation of the number of particles is reduced to the investigation of the spectral properties of self-adjoint operators acting in the \( \text{cut subspace} \ \mathcal{H}^{(n)} \) of the Fock space, consisting of \( r \leq n \) particles \([8, 13, 18, 19, 21, 27]\).

The model operator \( H \), associated with a system describing two bosons and one particle another nature in interaction, without conservation of the number of particles, was considered in Albeverio et. al. \([5],[6]\) and there existence of the Efimov effect was proved. This model described a truncated operator (truncated with respect to the number of particles) that acts in the direct sum of zero-, one-, and two-particle subspaces of the \textit{bosonic Fock space}.

In the present paper we consider a model describing a truncated operator \( H \) (truncated with respect to the number of particles) acting in the direct sum of zero-, one-, and two-particle subspaces of a \textit{fermionic Fock space} \( \mathcal{F}_a(L^2(\mathbb{T}^3)) \) over \( L^2(\mathbb{T}^3) \). The model operator \( H \) is a lattice analogue of the Hamiltonians of the spin-fermion models for diluted magnetic semiconductors \([8]\).

The main aim of the present paper is to give a thorough mathematical treatment of the spectral properties for a model operator \( H \) with emphasis on the asymptotics for the number of infinitely many eigenvalues (Efimov’s effect case).

We admit a general form for the “kinetic” part of the hamiltonian \( H \), which contains a parameter \( \gamma \) to distinguish the two identical particles from the third one. Under some smoothness assumptions we obtain the following results:

(i) We find a critical value \( \gamma^* \) for the parameter \( \gamma \) that allows or forbids the Efimov effect and we prove that only for \( \gamma < \gamma^* \) the Efimov effect is absent, while this effect exists for any \( \gamma > \gamma^* \).

(ii) In the case \( \gamma > \gamma^* \) we also establish the following asymptotics for the number \( N(z) \) of eigenvalues \( z \) below \( E_{\text{min}} \), the lower limit of the essential spectrum of \( H \):

\[
\lim_{z \to E_{\text{min}}^-} \frac{N(z)}{\log |E_{\text{min}} - z|} = u_0(\gamma) \quad (u_0(\gamma) > 0), \quad \forall \gamma > \gamma^*.
\]

We notice that the assertion (i) is surprising and similar assertions have not yet been proved for the models in the \textit{bosonic Fock space} \([5][6]\).

We remark that for the three-particle discrete Schrödinger operator \( H \) the authors Dell’Antonio et.al. \([10]\) found an explicit value of the parameter (which is nicely characterized as the only positive solution of a transcendental equation in one of the appendices) \( \gamma \), say \( \gamma^* \), such that only for values of \( \gamma \) below this number, the Efimov effect is absent for the sector of the Hilbert space which contains functions which are antisymmetric with respect to the two identical particles, while it is present for all values of the parameter \( \gamma \) on the symmetric sector. Moreover, in this work, striking similarities between the Efimov and Thomas effects, aside the occurrence of infinitely many bound states are discussed.

The organization of the present paper is as follows. Section 1 is an introduction to the whole work. In Section 2, the model operator is described as a bounded self-adjoint operator \( H \) in \( \mathcal{H}^{(3)} \) and the main results of the present paper are formulated. Some spectral properties of the corresponding Friedrichs models \( h(p) \), \( p \in \mathbb{T}^3 \), are recalled and the location and structure of the essential spectrum of \( H \) are given. Section 3 deals with the review the Birman-Schwinger principle for the operator \( H \). In Section 4, we represent the sketch of the proof of the main results. We follow closely \([10]\) to derive the proof of our main results (Theorem 2.10).

2. The model operator and statement of the main results

Let \( \mathbb{T}^3 \) be the three – dimensional torus, the cube \((-\pi, \pi)^3\) with appropriately identified sides. We remark that the torus \( \mathbb{T}^3 \) will always be considered as an abelian group with respect to the addition and multiplication by the real numbers regarded as operations on \( \mathbb{R}^3 \) modulo \((2\pi \mathbb{Z})^3\). Denote by \( L^2(\mathbb{T}^3) \) the subspace of antisymmetric functions of the Hilbert space \( L^2((\mathbb{T}^3)^2) \).

Set

\[
\mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_1 = L^2(\mathbb{T}^3), \quad \mathcal{H}_2 = L^2_{\text{as}}((\mathbb{T}^3)^2).
\]
The Hilbert space $\mathcal{H}^{(3)} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ is called the direct sum of zero-, one-, and two-particle subspaces of a fermionic Fock space $\mathcal{F}_\omega (L^2 (\mathbb{T}^3))$ over $L^2 (\mathbb{T}^3)$.

Let $H_{ij}$ be annihilation (creation) operators [13] defined in the Fock space for $i < j (i > j)$. We note that in physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

In this paper, we consider the case, where the number of annihilations and creations of the particles of the considering system equal to 1. It means that $H_{ij} \equiv 0$ for all $|i - j| > 1$. So, a model describing a truncated operator $H$ acts in the Hilbert space $\mathcal{H}^{(3)}$ as a matrix operator:

$$H = \begin{pmatrix}
H_{00} & H_{01} & 0 \\
H_{10} & H_{11} & H_{12} \\
0 & H_{21} & H_{22}
\end{pmatrix},$$

where the operators $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0, 1, 2$ are defined by the forms:

$$(H_{00} f_0) = u_0 f_0, \quad (H_{01} f_1) = \int_{\mathbb{T}^3} b(q') f_1(q') dq', \quad (H_{10} f_0) = b(p) f_0,$$

$$(H_{11} f_1) = u(p) f_1(p), \quad (H_{12} f_2) = \int_{\mathbb{T}^3} b(q') f_2(p, q') dq',$$

$$(H_{21} f_1) = \frac{1}{2} (b(q) f_1(p) - b(p) f_1(q)), \quad (H_{22} f_2) = E_\gamma (p, q) f_2(p, q).$$

Here $f_i \in \mathcal{H}_i$, $i = 0, 1, 2$, $u_0$ - fixed real number, $b(\cdot), u(\cdot)$ - real-valued analytic functions on $\mathbb{T}^3$, $E_\gamma (\cdot, \cdot) : (\mathbb{T}^3)^2 \to \mathbb{R}$, $\gamma > 0$, is given in the form

$$E_\gamma (p, q) = \gamma \varepsilon (p + q) + \varepsilon (p) + \varepsilon (q), \quad p, q \in \mathbb{T}^3,$$

where the function $\varepsilon (\cdot)$ is real valued conditionally negative definite three times differentiable function on $\mathbb{T}^3$ with a unique non-degenerate minimum at the origin.

Recall that a complex-valued bounded function $\varepsilon : \mathbb{T}^m \to \mathbb{C}$ is called conditionally negative definite if $\varepsilon (p) = \varepsilon (-p)$ and

$$\sum_{i,j=1}^n \varepsilon (p_i - p_j) z_i \overline{z}_j \leq 0$$

for all $p_1, p_2, \ldots, p_n \in \mathbb{T}^m$ and all $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ satisfying $\sum_{i=1}^n z_i = 0$.

2.1. The Friedrichs model operator

To formulate the main results of the paper we introduce a family of Friedrichs models $h(p)$, $p \in \mathbb{T}^3$ which act in $\mathcal{H}^{(2)} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with the entries

$$(h_{00}(p) f_0) = u(p) f_0, \quad h_{01} = \frac{1}{\sqrt{2}} H_{01},$$

$$h_{10} = h_{01}^*, \quad (h_{11}(p) f_1(q)) = w_p(q) f_1(q),$$

where $w_p(q) = E_\gamma (p, q)$.

Let the operator $h_0(p)$, $p \in \mathbb{T}^3$, act in $\mathcal{H}^{(2)}$ as

$$h_0(p) \begin{pmatrix}
f_0 \\
f_1(q)
\end{pmatrix} = \begin{pmatrix}
0 \\
w_p(q) f_1(q)
\end{pmatrix}.$$
The perturbation $h(p) - h_0(p)$ of the operator $h_0(p)$ is a self-adjoint operator of rank 2. Therefore in accordance with the invariance of the essential spectrum under finite rank perturbations, the essential spectrum $\sigma_{ess}(h(p))$ of $h(p)$ fills the following interval on the real axis:

$$\sigma_{ess}(h(p)) = [m(p), M(p)],$$

where

$$m(p) = \min_{q \in \mathbb{T}^3} w_p(q), \quad M(p) = \max_{q \in \mathbb{T}^3} w_p(q).$$

**Definition 2.1** Let $u(0) \neq E_{\text{min}}$. The operator $h(0)$ is said to have a threshold resonance if the number 1 is an eigenvalue of the operator

$$(G\psi)(q) = \frac{b(q)}{2(u(0) - E_{\text{min}})} \int_{\mathbb{T}^3} \frac{b(t)\psi(t)dt}{w_0(t) - E_{\text{min}}}, \quad \psi \in C(\mathbb{T}^3),$$

and the associated eigenfunction $\psi$ (up to a constant factor) satisfies the condition $\psi(0) \neq 0$.

**Remark 2.2** The spectrum and resonances of this Friedrichs model are studied in [5, 6].

For any $p \in \mathbb{T}^3$ we define an analytic function $\Delta(p, z)(\text{the Fredholm determinant associated with the operator } h(p))$ in $\mathbb{C} \setminus [m(p), M(p)]$ by

$$\Delta(p, z) = u(p) - z - \frac{1}{2} \int_{\mathbb{T}^3} \frac{b^2(t)dt}{E_\gamma(p, t) - z}.$$  

Since $\Delta(0, \cdot)$ is continuous in $z \leq E_{\text{min}}$ the following limit exists

$$\Delta(0, E_{\text{min}}) = \lim_{z \to E_{\text{min}}} \Delta(0, E_{\text{min}}).$$

The lemmas below play crucial role in the proof of the main results when the operator $h(0)$ has a threshold resonance, and the proof of the lemmas were given in [5] and [6].

**Lemma 2.3** (i) If $u(0) \leq E_{\text{min}}$, then the operator $h(0)$ has no a threshold resonance.

(ii) If $u(0) > E_{\text{min}}$, then the operator $h(0)$ has a threshold resonance if $\Delta(0, E_{\text{min}}) = 0$ and $b(0) \neq 0$.

(iii) Assume that $u(0) > E_{\text{min}}$ and $\Delta(0, E_{\text{min}}) = 0$.

a) If $b(0) \neq 0$, then the operator $h(0)$ has a threshold resonance and the vector $f = (f_0, f_1)$, where

$$f_0 = \text{const} \neq 0, \quad f_1(q) = -\frac{b(q)f_0}{\sqrt{2(u(0) - E_{\text{min}})}} \in L^1(\mathbb{T}^3) \setminus L^2(\mathbb{T}^3),$$

obeys the equation $h(0)f = E_{\text{min}}f$.

b) If $b(0) = 0$, then the number $z = E_{\text{min}}$ is an eigenvalue of the operator $h(0)$ and the vector $f = (f_0, f_1)$, where $f_0 \in \mathbb{C}^1$ and $f_1 \in L^2(\mathbb{T}^3)$ defined by (1), is the corresponding eigenvalue.

Let $W$ be the $3 \times 3$-matrix of the second order partial derivatives of function $\varepsilon(\cdot)$ at the point $p = 0$.

**Lemma 2.4** Let the operator $h(0)$ has a threshold resonance. Then for any $p \in U_\delta(0)$, $\delta > 0$ sufficiently small, and $z \leq E_{\text{min}}$ the following decomposition

$$\Delta(p, z) = \frac{4\sqrt{2}\pi^2 b^2(0)}{(1 + \gamma)^3 \det(W)^{\frac{3}{2}}} \sqrt{m(p) - z + \Delta^{(02)}(m(p) - z) + \Delta^{(20)}(p, z)}$$

holds, where $\Delta^{(02)}(m(p) - z)$ (resp. $\Delta^{(20)}(p, z)$) is a function behaving like $O((m(p) - z)^{\frac{1+\delta}{2}})$ (resp. $O(|p|^2)$) as $|m(p) - z| \to 0$ (resp. $p \to 0$) uniformly in $z \leq E_{\text{min}}$.

**Lemma 2.5** Let the operator $h(0)$ has a threshold resonance. Then there exist positive numbers $c, C$ and $\delta$ such that

$$c|p| \leq \Delta(p, E_{\text{min}}) \leq C|p| \quad \text{for any } \quad p \in U_\delta(0),$$

and

$$\Delta(p, E_{\text{min}}) \geq c \quad \text{for any } \quad p \in \mathbb{T}^3 \setminus U_\delta(0).$$
2.2. The essential spectrum of the operator $H$

The following theorem describes the essential spectrum of the operator $H$.

**Theorem 2.6** For the essential spectrum $\sigma_{\text{ess}}(H)$ of the operator $H$ the equality

$$\sigma_{\text{ess}}(H) = \bigcup_{p \in \mathbb{T}^3} \sigma_d(h(p)) \cup [E_{\text{min}}, E_{\text{max}}]$$

holds, where $\sigma_d(h(p))$ is the discrete spectrum of the operator $h(p), p \in \mathbb{T}^3$.

**Assumption 2.7** Let $u(\cdot), b(\cdot)$ be even functions and $u(p) > u(0), \quad 0 \neq p \in \mathbb{T}^3$.

The following lemma describes the location of the essential spectrum of the operator $H$.

**Lemma 2.8** Let Assumption 2.7 be fulfilled and $h(0)$ has a threshold resonance at $E_{\text{min}}$. Then

$$\sigma_{\text{ess}}(H) = [E_{\text{min}}, E_{\text{max}}].$$

2.3. Statement of the main results

Henceforth we assume that $h(0)$ has a threshold resonance.

By $N(z)$ we denote the number of eigenvalues and counted according their multiplicities of $H$ lying below $z \leq E_{\text{min}}$.

Let $\gamma^*$ be a solution of the equation

$$\frac{2(1 + \gamma)^2}{\pi \gamma \sqrt{1 + 2\gamma}} - \frac{2(1 + \gamma)^2}{\pi \gamma^2} \arcsin \frac{\gamma}{1 + \gamma} = 1, \quad \gamma > 0.$$

**Remark 2.9** This equation has a unique positive solution (see [10, Appendix A]).

In the following main theorem, we precisely describe the dependence of the number of eigenvalues of $H$ on the parameters $\gamma > 0$.

**Theorem 2.10** Let Assumption 2.7 be fulfilled and $h(0)$ has a threshold resonance at $E_{\text{min}}$. Then

(i) For any $0 < \gamma < \gamma^*$ the operator $H$ has a finite number of eigenvalues lying below the bottom $E_{\text{min}}$ of the essential spectrum.

(ii) For any $\gamma > \gamma^*$ the operator $H$ has infinitely many eigenvalues lying below $E_{\text{min}}$. The function $N(z)$ obeys the relation

$$\lim_{z \to E_{\text{min}}^-} \frac{N(z)}{\log |E_{\text{min}} - z|} = U_0(\gamma) \quad (U_0(\gamma) > 0).$$

**Remark 2.11** In [10], a result analogue to Theorem 2.10 has been proven for the three-particle Schrödinger operators on the lattice $\mathbb{Z}^3$.

3. The Birman-Schwinger principle

For a bounded self-adjoint operator $B$, we define $n(\lambda, B)$ by

$$n(\lambda, B) = \sup \{ \dim F : (Bu, u) > \lambda, \ u \in F, \ ||u|| = 1 \}.$$  

$n(\lambda, B)$ is equal to infinity if $\lambda$ is in the essential spectrum of $B$ and if $n(\lambda, B)$ is finite, it is equal to the number of the eigenvalues of $B$ larger than $\lambda$. By the definition of $N(z)$ we have

$$N(z) = n(-z, -H), \ -z > -E_{\text{min}}.$$
In our analysis of the spectrum of $H$ the crucial role is played by the self-adjoint compact Faddeev-Newton type integral operator $T(z), z < E_{\text{min}}$, in the space $\mathcal{H}^{(2)}$ with the entries

$$(T_{00}(z)f_0)_0 = (1 - u_0 - z)f_0, \quad (T_{01}(z)f_1)_0 = -\int_{T^3} \frac{b(q')f(q')dq'}{\sqrt{\Delta(q', z)}},$$

$$T_{10}(z) = T_{01}^*(z), \quad (T_{11}(z)f_1)_1 = \frac{b(p)}{2\sqrt{\Delta(p, z)}} \int_{T^3} \frac{b(q')f(q')dq'}{\sqrt{\Delta(q', z)}(E_{\gamma}(p, q') - z)}.$$

The following lemma follows from the well known Birman-Schwinger principle for the operator $H$ (see [4, 22, 24]). In a system describing three particles in interaction, without conservation of the number of particles, case the Birman-Schwinger principle was obtained in [5], [6]. We refer to these paper for the proof.

**Lemma 3.1** The operator $T(z)$ is compact and continuous in $z < E_{\text{min}}$ and

$$N(z) = n(1, T(z)).$$

### 3.1. The Birman-Schwinger principle at the threshold

It should be noted that the operator $T(z)$ can be defined as a bounded operator even for the point $z = E_{\text{min}}$ by

$$(T_{00}(E_{\text{min}})f_0)_0 = (1 - u_0 - E_{\text{min}})f_0, \quad (T_{01}(E_{\text{min}})f_1)_0 = -\int_{T^3} \frac{b(q')f(q')dq'}{\sqrt{\Delta(q', z)}},$$

$$T_{10}(E_{\text{min}}) = T_{01}^*(E_{\text{min}}), \quad (T_{11}(E_{\text{min}})f_1)_1 = \frac{b(p)}{2\sqrt{\Delta(p, E_{\text{min}})}} \int_{T^3} \frac{b(q')f(q')dq'}{\sqrt{\Delta(q', E_{\text{min}})(E_{\gamma}(p, q') - E_{\text{min}})}}.$$

**Remark 3.2** The operator $T(z)$ converges strongly (but not uniformly) as $z \to E_{\text{min}} - 0$ to $T(E_{\text{min}})$. Here we do not give the proof of this convergence. The convergence of the these type of operators was shown in [26], [14].

Next lemma is an analogue of Lemma 3.1, and we omit its proof.

**Lemma 3.3** For any $z, z \leq E_{\text{min}}$, the inequality

$$N(z) \leq n(1, T(E_{\text{min}}))$$

occurs.

### 4. The sketch of the proof of the main results

By the definition of $z(\cdot)$ we get

$$E_{\gamma}(p, q) = E_{\text{min}} + \frac{1}{2} \left((1 + \gamma)(Wp, p) + 2\gamma(Wp, q) + (1 + \gamma)(Wq, q)\right) + O(|p|^{3+\theta} + |q|^{3+\theta}),$$

as $p, q \to 0$ and

$$m(k) = E_{\text{min}} + \frac{1 + 2\gamma}{1 + \gamma} (Wk, k) + O(|k|^{3+\theta}) \quad \text{as} \quad k \to 0.$$

Applying the asymptotics for $m(p)$ and using Lemma 2.4 we have

$$\Delta(p, z) = \frac{4\pi^2 b^2(0)}{(1 + \gamma)^{3/2} \det(W)^{1/2}} \left[n(Wp, p) + 2(E_{\text{min}} - z)^{3/2} + O(\frac{|p|^2 + |E_{\text{min}} - z|^{1+\theta}}{1+\gamma})\right] \quad \text{as} \quad p, |E_{\text{min}} - z| \to 0,$$

where

$$n = \frac{1 + 2\gamma}{1 + \gamma}.$$
4.1. The infiniteness of the discrete spectrum of $H$

In this subsection we shall derive the asymptotics (2) for the number of eigenvalues of $H$.

We recall that in the subsection we closely follow A. Sobolev’s method (see [22]) to derive the asymptotics for the number of eigenvalues of the operator $H$ (see Theorem 2.10).

Let $T(\delta; |z - E_{\text{min}}|)$ be the operator in $\mathcal{H}^{(2)}$ defined by

$$T(\delta; |z - E_{\text{min}}|) = \begin{pmatrix} 0 & 0 \\ 0 & T_{11}(\delta; |z - E_{\text{min}}|) \end{pmatrix},$$

where the $T_{11}(\delta; |E_{\text{min}} - z|)$ is the integral operator in $\mathcal{H}_1$ with the kernel

$$T(\delta, |E_{\text{min}} - z|; p, q) =$$

$$-d_0 \hat{\chi}_\delta(p)\hat{\chi}_\delta(q)(n(Wp, p) + 2|E_{\text{min}} - z|)^{-\frac{1}{4}}(n(Wq, q) + 2|E_{\text{min}} - z|)^{-\frac{1}{4}}(1 + \gamma)(Wp, p) + 2\gamma(Wp, q) + (1 + \gamma)(Wq, q) + 2|E_{\text{min}} - z|^{-\frac{1}{4}},$$

where $\hat{\chi}_\delta(\cdot)$ is the characteristic function of the region $\hat{U}_\delta(0) = \{ p \in T^3 : |W^{\frac{1}{2}}p| < \delta \}$ and

$$d_0 = \frac{\det W^{\frac{1}{2}}}{2\pi^2}(1 + \gamma)^{\frac{3}{2}}.$$

**Lemma 4.1** Let the conditions of Theorem 2.10 be fulfilled. The operator $T(z) - T(\delta; |E_{\text{min}} - z|)$ belongs to the Hilbert-Schmidt class and is continuous in $z \leq E_{\text{min}}$.

**Proof.** Applying asymptotics (3), (4) one can estimate the kernel of the operator $T(z) - T(\delta; |E_{\text{min}} - z|)$, $z \leq E_{\text{min}}$, by the square-integrable function

$$C\left(\frac{|p|^{1+\theta} + |q|^{1+\theta}}{|p|^{\frac{1}{2}}(p^2 + q^2)|q|^{\frac{1}{2}}} + \frac{|E_{\text{min}} - z|^\frac{1}{2}(p^2 + q^2)^{-1}}{(|p|^2 + |E_{\text{min}} - z|)^{\frac{1}{2}}(|q|^2 + |E_{\text{min}} - z|)^{\frac{1}{2}}} + 1\right).$$

Hence the operator $T(z) - T(\delta; |E_{\text{min}} - z|)$ belongs to the Hilbert-Schmidt class for all $z \leq E_{\text{min}}$. In combination with the continuity of the kernel of the operator in $z < E_{\text{min}}$ this gives the continuity of $T(z) - T(\delta; |E_{\text{min}} - z|)$ in $z \leq E_{\text{min}}$. \hfill $\Box$

Let $\hat{T}_0(\delta; |E_{\text{min}} - z|)$ be the restriction of the integral operator $T(\delta; |E_{\text{min}} - z|)$ to the subspace $L^2(\hat{U}_\delta(0))$. One verifies that the operator $\hat{T}_0(\delta; |E_{\text{min}} - z|)$ is unitarily equivalent to the integral operator $T_1(r)$ acting in $L^2(U_r(0))$, where $r = |E_{\text{min}} - z|^{-\frac{1}{2}}$ and $U_r(0) = \{ p \in \mathbb{R}^3 : |p| < r \}$, with the kernel

$$T_1(r; p, q) = -d_1 \frac{(np^2 + 2)^{-1/4}(nq^2 + 2)^{-1/4}}{(1 + \gamma)p^2 + 2\gamma(p, q) + (1 + \gamma)q^2 + 2},$$

where

$$d_1 = \frac{(1 + \gamma)^{3/2}}{2\pi^2}.$$

The equivalence is given by the unitary dilatation

$$B : L^2(\hat{U}_\delta(0)) \to L^2(U_r(0)), \quad (B_rf)(p) = (\frac{r}{\delta})^{-3/2}f(\frac{\delta}{r}W^{\frac{1}{2}}p).$$

Further, we may replace

$$(np^2 + 2)^{-1/4}, (nq^2 + 2)^{-1/4} \quad \text{and} \quad (1 + \gamma)p^2 + 2\gamma(p, q) + (1 + \gamma)q^2 + 2$$
by

\[(np^2)^{-1/4}(1 - \chi_1(p)), \quad (nq^2)^{-1/4}(1 - \chi_1(q))\]

and \((1 + \gamma)p^2 + 2\gamma(p, q) + (1 + \gamma)q^2\), respectively, since the error will be a Hilbert-Schmidt operator continuous up to \(z = E_{\text{min}}\).

We have denoted by \(\chi_1(\cdot)\) the characteristic function of the ball \(U_1(0)\). By the the replacement we obtain the integral operator \(T_2(r)\) in \(L^2(U_r(0) \setminus U_1(0))\) with a kernel

\[T_2(r; p, q) = -\frac{d_1}{n^2} \frac{|p|^{-1/2}|q|^{-1/2}}{(1 + \gamma)p^2 + 2\gamma(p, q) + (1 + \gamma)q^2}.\]

By the dilation

\[M : L^2(U_r(0) \setminus U_1(0)) \to L^2((0, r), S^2), \quad r = 1/2 \log |E_{\text{min}} - z|,\]

where \(S^2\) is the unit sphere in \(\mathbb{R}^3\), \((M f)(x, w) = e^{i\pi/2} f(e^x w), \quad x \in (0, r), \quad w \in S^2\), one sees that the operator \(T_2(r)\) is unitarily equivalent to the integral operator \(S_r\) with the kernel \(S_r(x - x'; < \xi, \eta >), \quad \xi, \eta \in S^2, \quad x, x' \in \mathbb{R}^+, \quad \text{where}\)

\[S_r(x; t) = -(2\pi)^{-2} \frac{u}{\cosh x + st}, \quad u = \frac{1 + \gamma}{\sqrt{1 + 2\gamma}}, \quad s = \frac{\gamma}{1 + \gamma}, \quad t = < \xi, \eta >.\]

Recall the lemma in [22].

**Lemma 4.2** Let \(A(z) = A_0(z) + A_1(z)\), where \(A_0\) (\(A_1\)) is compact and continuous in \(z < 0 \quad (z \leq 0)\). Assume that for some function \(f(\cdot), \quad f(z) \to 0, \quad z \to -0\) the limit

\[\lim_{z \to -0} f(z)n(\lambda, A_0(z)) = l(\lambda),\]

exists and is continuous in \(\lambda > 0\). Then the same limit exists for \(A(z)\) and

\[\lim_{z \to -0} f(z)n(\lambda, A(z)) = l(\lambda).\]

The following theorem is important for the proof of the asymptotics (2) and can be proved in similar way as Theorem 6.4 in [10].

**Theorem 4.3** Let the conditions of the part (ii) of Theorem 2.10 are fulfilled. The following equalities

\[\lim_{|E_{\text{min}} - z| \to 0} \frac{n(1, T(z))}{|\log |E_{\text{min}} - z||} = \lim_{r \to \infty} r^{-1} n(1, S_r) = \U_0(\gamma), \quad \U_0(\gamma) > 0,
\]

hold.

This theorem, together with Lemmas 3.1 and 4.2 completes the proof of part (ii) of Theorem 2.10.

4.2. The finiteness of the discrete spectrum of \(H\)

The following lemma and Lemma 3.3 completes the proof of part (i) of Theorem 2.10.

**Lemma 4.4** Let \(\gamma < \gamma^*\) and the hypothesis of part (i) of Theorem 2.10 be fulfilled. Then there exists the number \(\epsilon = \epsilon_{\gamma}\) depending on \(\gamma\) so that

\[\sup \sigma_{\text{ess}}(T(E_{\text{min}})) < 1 - \epsilon_{\gamma}.\]

**Proof.** See [10, appendix B].
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