Abstract. The set consisting of all rotations of the Euclidean plane is equipped with a quandle structure. We show that a knot is colorable by this quandle if and only if its Alexander polynomial has a root on the unit circle in $\mathbb{C}$. Further we enumerate all non-trivial colorings of a torus knot diagram by the quandle using PL trochoids. As an application of these results, we have the complete factorization of the Alexander polynomial of the torus knot.

1. Introduction

A quandle, introduced by Joyce [4] in 1982\(^1\), is an algebraic system which is characterized by certain three axioms. These three axioms are closely related to the three local moves of knot diagrams, known as the Reidemeister moves, respectively. Therefore a quandle is considered as a minimal algebra for knots [6]. Indeed, it is known by Joyce [4] and Matveev [5] independently that quandles derived from knots are isomorphic if and only if the knots are equivalent.

On the other hand, a special quandle called a kei (\(\pm\)) was studied by Takasaki [7] in 1943. The notion of a kei abstracts the behavior of symmetric transformations. Takasaki said in his paper that, to investigate geometric symmetry, using a kei is more suitable than using a group. A quandle also describes primitive symmetry well. For example, each subset of a group closed under conjugations is equipped with a quandle structure, while it is no longer not equipped with a group structure in general. This quandle still describes symmetry that the subset does as a part of the group.

How those two aspects of a quandle are related to each other? In this paper, we focus on the quandle $\text{RotE}^2$ consisting of all rotations of the Euclidean plane, and investigate colorability of knots by $\text{RotE}^2$. Of course, $\text{RotE}^2$ describes rotational symmetry of the Euclidean plane. Colorability of knots is related to the first aspect. We show in Section 3 that a knot is $\text{RotE}^2$-colorable if and only if its Alexander polynomial has a root on the unit circle in $\mathbb{C}$ (Theorem 3.4). Furthermore, in Section 4, we enumerate all $\text{RotE}^2$-colorings of a torus knot diagram using PL trochoids (Theorem 4.1). As an application of these results, we have the complete factorization of the Alexander polynomial of the torus knot (Corollary 4.2).

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\(^1\)The same notion was also introduced by Matveev [5] in 1982, under the name of a distributive groupoid.

\(^2\)It is a Chinese character which is invariant under reflection about the centerline and invariant under swapping top and bottom.
2. Preliminaries

In this section, we review the definitions of a quandle and colorability of a knot by a quandle briefly. We refer the reader to [2, 4, 6] for more details. We further define the quandle consisting of all rotations of the Euclidean plane in which we are mainly interested in this paper.

A quandle is a non-empty set $X$ equipped with a binary operation $\ast : X \times X \to X$ satisfying the following three axioms:

(Q1) For each $x \in X$, $x \ast x = x$.
(Q2) For each $x \in X$, a map $\ast x : X \to X$ ($w \mapsto w \ast x$) is bijective.
(Q3) For each $x, y, z \in X$, $(x \ast y) \ast z = (x \ast z) \ast (y \ast z)$.

A quandle is said to be a kei ($\ast$) if each bijection $\ast x$ is involutive. A typical example of a quandle is a subset $X$ of a group, which is closed under conjugations, with $x \ast y = y^{-1}xy$ for each $x, y \in X$. We call it a conjugation quandle.

Let $\text{Rot} \mathbb{E}^2$ be the set consisting of all rotations of the Euclidean plane $\mathbb{E}^2$. Since it is a subset of the isometry group of $\mathbb{E}^2$ and is closed under conjugations, $\text{Rot} \mathbb{E}^2$ is equipped with a conjugation quandle structure. This conjugation quandle $\text{Rot} \mathbb{E}^2$ is the quandle consisting of all rotations of the Euclidean plane. We note that $\text{Rot} \mathbb{E}^2$ is isomorphic to a quandle $\mathbb{C} \times \text{U}(1)$ whose binary operation $\ast$ is given by

$$(z, e^{\theta \sqrt{-1}}) \ast (w, e^{\eta \sqrt{-1}}) = ((z - w)e^{\eta \sqrt{-1}} + w, e^{\theta \sqrt{-1}}).$$

Under the identification of the complex plane $\mathbb{C}$ with $\mathbb{E}^2$, an element $(z, e^{\theta \sqrt{-1}}) \in \mathbb{C} \times \text{U}(1)$ corresponds to a rotation about $z$ by angle $\theta$. In the remaining of this paper, we do not distinguish $\text{Rot} \mathbb{E}^2$ from $\mathbb{C} \times \text{U}(1)$.

The axioms of a quandle are closely related to the Reidemeister moves of knot diagrams as follows. A coloring of a knot diagram $D$ by a quandle $X$ is a map \{arcs of $D$\} $\to X$ satisfying the condition depicted in Figure 1 at each crossing. We call an element of a quandle assigned to an arc by a coloring a color of the arc. Suppose $D'$ is a knot diagram obtained from $D$ by a Reidemeister move. Then, for each coloring $\mathcal{C}$ of $D$, we have a unique coloring of $D'$ whose restriction to the arcs unrelated to the deformation coincides with the restriction of $\mathcal{C}$. Indeed, the axioms (Q1), (Q2) and (Q3) of a quandle guarantee that we can perform the Reidemeister moves RI, RII and RIII fixing ends’ colors respectively. See Figure 2. Thus, for a fixed quandle, the number of all colorings gives rise to a knot invariant.

A constant map from the set consisting of the arcs of a knot diagram to a quandle always satisfies the coloring condition. We thus call this constant map a trivial coloring. For a quandle $X$, a knot $K$ is said to be $X$-colorable if there is a non-trivial coloring of a diagram of $K$ by $X$. We note that no non-trivial colorings are obtained from a trivial coloring by Reidemeister moves.

![Figure 1](image1.png)

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![Figure 2](image2.png)

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![Figure 3](image3.png)

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3. Necessary and sufficient condition

Suppose $\text{Rot}E^2$ is the quandle consisting of all rotations of the Euclidean plane, defined in the previous section. Which knots are $\text{Rot}E^2$-colorable?

Example 3.1 (trefoil). Consider a diagram of the right hand trefoil, depicted in the left-hand side of Figure 3, colored by $\text{Rot}E^2$. Here, $(z_1, e^{\theta_1\sqrt{-1}})$, $(z_2, e^{\theta_2\sqrt{-1}})$ and $(z_3, e^{\theta_3\sqrt{-1}})$ denote colors of the corresponding arcs. We then have the following equations

\[
(z_1, e^{\theta_1\sqrt{-1}}) = (z_3, e^{\theta_3\sqrt{-1}}) * (z_2, e^{\theta_2\sqrt{-1}}),
\]

\[
(z_2, e^{\theta_2\sqrt{-1}}) = (z_1, e^{\theta_1\sqrt{-1}}) * (z_3, e^{\theta_3\sqrt{-1}}),
\]

\[
(z_3, e^{\theta_3\sqrt{-1}}) = (z_2, e^{\theta_2\sqrt{-1}}) * (z_1, e^{\theta_1\sqrt{-1}})
\]

associated with the crossings. The first equation requires that $\theta_1 = \theta_3$ and $|z_1 - z_2| = |z_3 - z_2|$. The other equations require $\theta_2 = \theta_1$, $|z_2 - z_3| = |z_1 - z_3|$, $\theta_3 = \theta_2$ and $|z_3 - z_1| = |z_2 - z_1|$. These requests are satisfied if we take points $z_1, z_2, z_3$ so that the triangle $\triangle z_1 z_2 z_3$ is regular and set each $\theta_i$ to be $\pi/3$. See the right-hand side of Figure 3. Therefore the right hand trefoil is $\text{Rot}E^2$-colorable.
In the above example, it is required that all $\theta_i$ are the same. The same request is obviously imposed for any knot diagram, because one can go around all arcs of a knot diagram passing under over arcs:

**Lemma 3.2.** For any knot diagram colored by $\text{Rot} \mathbb{E}^2$, the second components (i.e., rotation angles) of colors of the arcs are the same.

**Example 3.3** (figure eight knot). We next consider a diagram of the figure eight knot, depicted in the left-hand side of Figure 4, colored by $\text{Rot} \mathbb{E}^2$. In this case, we have the following equations

\[
(z_2, e^{\theta \sqrt{-1}}) = (z_1, e^{\theta \sqrt{-1}}) \ast (z_4, e^{\theta \sqrt{-1}}), \quad (z_4, e^{\theta \sqrt{-1}}) = (z_3, e^{\theta \sqrt{-1}}) \ast (z_2, e^{\theta \sqrt{-1}}),
\]

\[
(z_2, e^{\theta \sqrt{-1}}) = (z_3, e^{\theta \sqrt{-1}}) \ast (z_4, e^{\theta \sqrt{-1}}), \quad (z_4, e^{\theta \sqrt{-1}}) = (z_1, e^{\theta \sqrt{-1}}) \ast (z_3, e^{\theta \sqrt{-1}})
\]

associated with the crossings. The first two equations require that $|z_1 - z_4| = |z_4 - z_2| = |z_2 - z_3|$ and $z_1 z_4 \parallel z_2 z_3$. Further the last two equations require that $|z_2 - z_1| = |z_1 - z_3| = |z_3 - z_4|$ and $z_2 z_1 \parallel z_3 z_4$. These requests are depicted in the right-hand side of Figure 4 respectively. There are no arrangements of points $z_1, z_2, z_3, z_4$ other than $z_1 = z_2 = z_3 = z_4$ which satisfy the requests simultaneously. Therefore the figure eight knot is not $\text{Rot} \mathbb{E}^2$-colorable.

![Figure 4](image_url)

We note that the Alexander polynomial of the trefoil ($\text{Rot} \mathbb{E}^2$-colorable) is

\[
\Delta_3(t) = t^2 - t + 1 = \left( t - \exp\left(\frac{\pi \sqrt{-1}}{3}\right) \right) \left( t - \exp\left(\frac{-\pi \sqrt{-1}}{3}\right) \right),
\]

and that of the figure eight knot (not $\text{Rot} \mathbb{E}^2$-colorable) is

\[
\Delta_4(t) = t^2 - 3t + 1 = \left( t - \frac{3 - \sqrt{5}}{2} \right) \left( t - \frac{3 + \sqrt{5}}{2} \right).
\]

**Theorem 3.4.** A knot $K$ is $\text{Rot} \mathbb{E}^2$-colorable if and only if its Alexander polynomial $\Delta_K(t)$ has a root on the unit circle in $\mathbb{C}$. More precisely, there is a non-trivial coloring of a diagram of $K$ by $\text{Rot} \mathbb{E}^2$, whose rotation angles are $\theta$, if and only if $\Delta_K(e^{\theta \sqrt{-1}}) = 0$.

**Proof.** Let $D$ be a diagram of $K$, $a_1, a_2, \ldots, a_n$ the arcs of $D$, $c_i$ the crossing of $D$ from which $a_i$ starts, and $\varepsilon_i$ the sign of $c_i$. Suppose $a_{\ell_i}$ and $a_{i}$ denote the under arc other than $a_i$ and the over arc related to $c_i$ respectively. See figure 5.

For a fixed $\theta \in \mathbb{R}$, consider an $n \times n$ matrix $X_\theta$ whose $(k_i, i)$ entry is $\exp(\varepsilon_i \theta \sqrt{-1})$, $(l_i, i)$ entry is $1 - \exp(\varepsilon_i \theta \sqrt{-1})$, $(i, i)$ entry is $-1$, and otherwise is $0$. Then a map...
a_i \mapsto (z_i, \theta) is a RotE^2-coloring of D if and only if \((z_1, z_2, \ldots, z_n)X_\theta\) is equal to \((0, 0, \ldots, 0)\). Since we always have trivial colorings, the rank of \(X_\theta\) is at most \(n - 1\). There is a non-trivial RotE^2-coloring, whose rotation angles are \(\theta\), if and only if the rank of \(X_\theta\) is less than \(n - 1\).

On the other hand, let \(A_D\) be the Alexander matrix for \(K\) derived from the Wirtinger presentation of the knot group related to \(D\) with Fox’s free differential calculus and the abelization of the knot group. See [1, Theorem 9.10], for example. Actually, \(A_D\) is an \(n \times n\) matrix whose \((k_i, i)\) entry is \(t^{\epsilon_i}\), \((l_i, i)\) entry is \(1 - t^{\epsilon_i}\), \((i, i)\) entry is \(-1\), and otherwise is 0. Since \(X_\theta = A_D|_{t = \exp(\theta \sqrt{-1})}\) and \(\Delta_K(t)\) is the greatest common divisor of all \((n - 1) \times (n - 1)\) minors of \(A_D\), the rank of \(X_\theta\) is less than \(n - 1\) if and only if \(e^{\theta \sqrt{-1}}\) is a root of \(\Delta_K(t)\).

\[\text{Remark 3.5.}\] For each \(i \ (0 \leq i \leq n - 2)\), the greatest common divisor of all \((n - i - 1) \times (n - i - 1)\) minors of \(A_D\) is called the \(i\)-th Alexander polynomial of \(K\) and is denoted by \(\Delta^{(i)}_K(t)\). We further let \(\Delta^{(n-1)}_K(t) = 1\). Since the elementary divisors of \(A_D\) are 0 and \(e^{(i)}_K(t) = \Delta^{(i)}_K(t)/\Delta^{(i+1)}_K(t)\) \((0 \leq i \leq n - 2)\), we have the following equation:

\[
\text{rank } X_\theta = n - 1 - |\{i \mid e^{(i)}_K(e^{\theta \sqrt{-1}}) = 0 \ (0 \leq i \leq n - 2)\}|.
\]

\[\text{Remark 3.6.}\] Assume that the rank of \(X_\theta\) is \(n - 2\). Let \(a_i \mapsto (z_i, \theta)\) be a non-trivial RotE^2-coloring of \(D\). Then, by the assumption, any non-trivial RotE^2-coloring, whose rotation angles are \(\theta\), is given by \(a_i \mapsto (\alpha z_i + \beta, \theta)\) with some \(\alpha \in \mathbb{C} \setminus \{0\}\) and \(\beta \in \mathbb{C}\). We note that corrections of points \(\{\alpha z_1 + \beta, \alpha z_2 + \beta, \ldots, \alpha z_n + \beta\}\) are related to each other by orientation preserving similarities. Therefore a non-trivial RotE^2-coloring, whose rotation angles are \(\theta\), is unique up to orientation preserving similarities in this case.

\[\text{Remark 3.7.}\] Let \(G\) be the group consisting of all orientation preserving similarities of \(\mathbb{C}\). Burde and Zieschang showed that there is a non-trivial representation of the knot group to \(G\) mapping a positive meridian to a rotation if and only if the Alexander polynomial of the knot has a root on the unit circle in \(\mathbb{C}\) (Proposition 14.5 in [1]). Since RotE^2 is also a subset of \(G\) closed under conjugations, in the light of Lemmas 3.3 and 3.5 in [3], there is a one-to-one correspondence between a RotE^2-coloring and a representation mapping a positive meridian to a rotation. Therefore Theorem 3.4 is essentially no different from Proposition 14.5 in [1].
4. Torus knot and PL trochoïd

The purpose of this section is to enumerate all non-trivial $\text{Rot}\mathbb{E}^2$-colorings of a torus knot diagram. Since the Alexander polynomial of the $(p, q)$-torus knot is
\[
\Delta_{T(p,q)}(t) = \frac{(tpq-1)(t-1)}{(tp-1)(tq-1)},
\]
in the light of Theorem 3.4, the $(p, q)$-torus knot is $\text{Rot}\mathbb{E}^2$-colorable. Indeed, each root of $\Delta_{T(p,q)}(t)$ is obviously a root of unity.

In order to achieve our goal, we need the following notations. Let $\Pi(m)$ be a convex regular $m$-gon in $\mathbb{C}$ ($m \geq 2$) and $v_0^m, v_1^m, \ldots, v_{m-1}^m$ the vertices of $\Pi(m)$ in counterclockwise order. For each $k$ and $i$ ($1 \leq k \leq m - 1$, $0 \leq i \leq m - 1$), we suppose that $v_{[ik]}^m$ denotes the vertex $v_{[r]}^m$, where $[r]$ denotes the integer satisfying $0 \leq [r] \leq m - 1$ and $[r] \equiv r \pmod{m}$ for each $r \in \mathbb{Z}$. Let $\Pi(m, k)$ be the regular polygon in $\mathbb{C}$ obtained by joining vertices $v_{[i]}^m$ and $v_{[i+1]}^m$ for each $i$. We note that some vertices and edges of $\Pi(m, k)$ are overlapped if $m$ and $k$ are not coprime. Further $\Pi(m, m - k)$ is the mirror image of $\Pi(m, k)$. Various $\Pi(5, \bullet)$ and $\Pi(6, \bullet)$ are depicted in Figure 6, for example. It is easy to see that
\[
\angle v_{[i]}^m v_{[i+1]}^m v_{[i+2]}^m = \left| \frac{(m - 2k)\pi}{m} \right|
\]
for each $i$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure6.png}
\caption{\textbf{Figure 6}}
\end{figure}

Suppose $m, n, k$ and $l$ are integers with $m, n \geq 2$, $1 \leq k \leq m-1$ and $1 \leq l \leq n-1$. We define a real number $\theta(m, k; n, l)$ by
\[
\theta(m, k; n, l) = \left( \frac{m - 2k}{m} - \frac{n - 2l}{n} \right) \pi.
\]
Consider two regular polygons $\Pi(m, k)$ and $\Pi(n, l)$ whose side lengths are the same. We arrange $\Pi(m, k)$ and $\Pi(n, l)$ so that $v_{[0]}^m = v_{[0]}^n$ and $v_{[1]}^m = v_{[1]}^n$. Fixing the position of $\Pi(n, l)$, we first rotate $\Pi(m, k)$ about $v_{[1]}^m = v_{[1]}^n$ by angle $\theta(m, k; n, l)$. 
Then $v_{2}^{m,k}$ meets with $v_{2}^{n,l}$. Thus we next rotate $\Pi(m,k)$ about $v_{2}^{m,k}$ instead of $v_{1}^{m,k} = v_{1}^{n,l}$ by angle $\theta(m,k; n,l)$, which produces that $v_{3}^{m,k} = v_{3}^{n,l}$. We continue this process in the same manner. The $i$-th step is the rotation of $\Pi(m,k)$ about $v_{i+1}^{m,k}$ by angle $\theta(m,k; n,l)$. We call this motion of $\Pi(m,k)$ related to $\Pi(n,l)$ the $(m,k; n,l)$-trochoid. Figure 7 illustrates the $(4, 1; 3, 1)$-trochoid, for example.

**Theorem 4.1.** Let $p$ and $q$ be non-zero coprime integers and $D(p,q)$ a diagram of the $(p,q)$-torus knot depicted in Figure 8. Then, for each $k$ and $l$ ($1 \leq k \leq |p| - 1$, $1 \leq l \leq |q| - 1$), there is a non-trivial Rot$E^2$-coloring of $D(p,q)$ derived from the $(|p|, k; |q|, l)$-trochoid. Rotation angles of the non-trivial coloring are $\theta(|p|, k; |q|, l)$.

**Proof.** First, we let $a_{ij}$ denote the arcs of $D(p,q)$ as depicted in Figure 8, although $a_{i0}$ and $a_{i+1,|p|-1}$ (resp. $a_{|q|i}$ and $a_{0j}$) mark the same arc.

For each $i$ and $j$ ($0 \leq i \leq |q|$, $0 \leq j \leq |p| - 1)$, let $z_{ij} \in \mathbb{C}$ be the coordinate of the vertex $v_{i+1}^{p,k}$ of $D(p,q)$ after the $i$-th step of the $(|p|, k; |q|, l)$-trochoid. Here, the $0$-th step means the arrangement of $\Pi(|p|, k)$ and $\Pi(|q|, l)$ into the initial position. These $z_{ij}$ for the $(4, 1; 3, 1)$-trochoid are depicted in Figure 9, for example.

By definition, $z_{i0}$ is equal to $z_{i+1,|p|-1}$ for each $i$ ($0 \leq i \leq |q| - 1$). Further the rotation about $z_{i0}$ by angle $\theta(|p|, k; |q|, l)$ sends $z_{ij}$ to $z_{i+1,j}$ for each $i$ and $j$ ($0 \leq i \leq |q| - 1$, $1 \leq j \leq |p| - 1$). Since $\Pi(|p|, k)$ returns to the initial position but is
if $pq > 0$

rotated by angle $(|p| - |q|) \cdot (2\pi/|p|)$ after the $|q|$-th step, we have that $v_{|p|}^{[p],[k]} = v_{|q|}^{[q],[l]}$ and $v_{|q|+1}^{[p],[k]} = v_{|q|}^{[q],[l]}$. Therefore $z_{|q|}$ is equal to $z_{q}$ for each $j$ ($0 \leq j \leq |p| - 1$). By the above argument, we have a non-trivial Rot$\mathbb{E}^2$-coloring $a_{ij} \mapsto (z_{ij}, \theta(|p|, k; |q|, l))$ of $D(p, q)$.

We note that $(|p| - 1)(|q| - 1)$ complex numbers $e^{\theta(|p|, k; |q|, l)} \sqrt{-1}$ ($1 \leq k \leq |p| - 1$, $1 \leq l \leq |q| - 1$) are mutually distinct if $p$ and $q$ are coprime. Further the span of the Alexander polynomial $\Delta_{T(p,q)}(t)$ of the $(p, q)$-torus knot is $(|p| - 1)(|q| - 1)$.

Therefore, in the light of Theorems 3.4 and 4.1, we have the following corollary:
**Corollary 4.2.** The Alexander polynomial of the \((p, q)\)-torus knot is factorized with a certain \(r \in \mathbb{Z}\) as follows:

\[
\Delta_{T(p,q)}(t) = t^r \prod_{k=1}^{\|p\| - 1} \prod_{l=1}^{\|q\| - 1} (t - e^{\theta([p], k; [q], l)} \sqrt{-1}).
\]

Suppose \(X_\theta\) is the \(n \times n\) matrix, defined in the proof of Theorem 3.4, derived from \(D(p, q)\) \((n = \|q\|([p] - 1))\). In the light of Corollary 4.2 and Remark 3.5, the rank of \(X_\theta\) is \(n - 2\) if \(\theta = \theta([p], k; [q], l)\) with some \(k\) and \(l\), and is \(n - 1\) otherwise. Therefore, in the light of Remark 3.6, all non-trivial \(\text{Rot}_2\)-colorings of the \((p, q)\)-torus knot diagram \(D(p, q)\) are essentially enumerated by Theorem 4.1.

**5. Related topics**

We close this paper with a discussion of some related topics.

Let \(\text{Rot}_n\) be the set consisting of all rotations of the \(n\)-dimensional Euclidean space \((n \geq 3)\). Obviously, \(\text{Rot}_n\) is equipped with a conjugation quandle structure. Although it is not \(\text{Rot}_2\)-colorable as we have seen in Example 3.3, the figure eight knot is \(\text{Rot}_3\)-colorable. Indeed, it is known that the figure eight knot is colorable by the *tetrahedral quandle* which is a subquandle of \(\text{Rot}_3\). See [2], for example. Thus it seems to be natural that we have the following questions:

**Question 5.1.** For a non-trivial knot \(K\) whose Alexander polynomial \(\Delta_K(t)\) has no roots on the unit circle in \(\mathbb{C}\), which \(n\) is the minimum number so that \(K\) is \(\text{Rot}_n\)-colorable?

**Question 5.2.** Is there a non-trivial knot which is not \(\text{Rot}_n\)-colorable for any \(n\)?

**Question 5.3.** For \(n \geq 3\), what is meaning of \(\text{Rot}_n\)-colorability? For example, are there relationships between \(\text{Rot}_n\)-colorability and other knot invariants?

The set consisting of all reflections of the Euclidean space is obviously equipped with a conjugation quandle structure. Further the sets consisting of all rotations or reflections of spherical or hyperbolic spaces are also equipped with conjugation quandle structures.

**Question 5.4.** Which knots are colorable by these “geometric” quandles?

**Question 5.5.** What is meaning of colorability by the quandles?
A coloring of a (multi-component) link diagram by a quandle is defined in a similar way for a knot diagram. A map from the arcs of a link diagram to $\operatorname{Rot} \mathbb{E}^2$ sending the arcs of the $i$-th component to a fixed $(w, e^{\theta_i \sqrt{-1}})$ satisfies the coloring condition, although $\theta_i \neq \theta_j$ for some $i \neq j$. Thus a link is said to be $\operatorname{Rot} \mathbb{E}^2$-colorable if there is a $\operatorname{Rot} \mathbb{E}^2$-coloring of its diagram other than all the above.

**Question 5.6.** Which links are $\operatorname{Rot} \mathbb{E}^2$-colorable? Further, which links are colorable by the other quandles?

**Question 5.7.** What is meaning of colorability of links by $\operatorname{Rot} \mathbb{E}^2$ and the others?

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