Higher-order differentials of the period map and higher Kodaira-Spencer classes

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In [K] we introduced two variants of higher-order differentials of the period map and showed how to compute them for a variation of Hodge structure that comes from a deformation of a compact Kähler manifold. More recently there appeared several works ([BG], [EV], [R]) defining higher tangent spaces to the moduli and the corresponding higher Kodaira-Spencer classes of a deformation. The \( n^{th} \) such class \( \kappa_n \) captures all essential information about the deformation up to \( n^{th} \) order.

A well-known result of Griffiths states that the (first) differential of the period map depends only on the (first) Kodaira-Spencer class of the deformation. In this paper we show that the second differential of the Archimedean period map associated to a deformation is determined by \( \kappa_2 \) taken modulo the image of \( \kappa_1 \), whereas the second differential of the usual period map, as well as the second fundamental form of the VHS, depend only on \( \kappa_1 \) (Theorems 2, 5, and 6 in Section 3).

Presumably, similar statements are valid in higher-order cases (see Section 4).

1 Constructing linear maps out of connections

We start by reviewing the definitions of higher-order differentials of the period map from [K], using a slightly different approach. Let \( S \) be a polydisc in \( \mathbb{C}^s \) centered at 0. Consider a free \( \mathcal{O}_S \)-module \( \mathcal{V} \) with a decreasing filtration by \( \mathcal{O}_S \)-submodules \( \ldots \subseteq \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \subseteq \ldots \) and an integrable connection

\[
\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_S
\]

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satisfying the Griffiths transversality condition $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_S$.

Lemma 1 (a) $\nabla$ induces an $O_S$-linear map

$$\Theta_S \rightarrow \text{Hom}_{O_S}(F^p, F^{p-1}/F^p)$$

$$\xi \rightarrow \nabla \xi \mod F^p$$

(b) Analogously, we also have an $O_S$-linear symmetric map

$$\Theta_S^\otimes 2 \rightarrow \text{Hom}_{O_S}(F^p, F^{p-2}/F^p + \text{span} \{\nabla_{\eta}(F^p) \mid \text{all } \eta \in \Gamma(S, \Theta_S)\})$$

$$\zeta \otimes \xi \rightarrow \nabla_{\zeta} \nabla_{\xi} \mod F^p + \text{span} \{\nabla_{\eta}(F^p)\} .$$

Proof. Both (a) and (b) are proved by straightforward computations; the fact that the map in (b) is symmetric follows from the integrability of $\nabla$:

$$\nabla_{\zeta} \nabla_{\xi} - \nabla_{\xi} \nabla_{\zeta} = [\zeta, \xi] ,$$

and so

$$\nabla_{\zeta} \nabla_{\xi} \equiv \nabla_{\xi} \nabla_{\zeta} \mod \text{span} \{\nabla_{\eta}(F^p) \mid \eta \in \Theta_S\} .$$

We will apply the Lemma to two connections arising from a deformation of a compact Kähler manifold $X$,

\begin{equation}
\begin{array}{ccc}
\mathcal{X} & \supset & X \\
\pi & \downarrow & \downarrow \\
S & \ni & 0
\end{array}
\end{equation}

1. The usual Gauss-Manin connection $\nabla$ on $\mathcal{H} = R^m \pi_* C_X$. In this case we denote the map in (a)

$$d\Phi : \Theta_S \rightarrow \text{Hom}_{O_S}(F^p, F^{p-1}/F^p) .$$

This is the (first) differential of the (usual) period map. The same notation and terminology will be applied to the induced map

$$\Theta_S \rightarrow \bigoplus_p \text{Hom}_{O_S}(F^p/F^{p+1}, F^{p-1}/F^p) .$$
The map given by part (b) of the Lemma is called the second differential of the (usual) period map and denoted
\[ d^2 \Phi : \Theta_S^2 \longrightarrow \text{Hom}_{\mathcal{O}_S}(F^p, F^{p-2}/F^p + \text{span} \{\nabla_\eta(F^p) \mid \eta \in \Theta_S\}) . \]

2. The Archimedean Gauss-Manin connection \( \nabla = \nabla_{ar} \) on
\[ \mathcal{H} \otimes B_{ar} = R^m \pi_* C_X[T, T^{-1}] \]
(see Appendix). The corresponding map from (a) is called the (first) differential of the Archimedean period map, denoted
\[ d\Psi : \Theta_S \longrightarrow \text{Hom}_{\mathcal{O}_S}(\mathcal{H}_{ar}, \mathcal{F}_{ar}/\mathcal{H}_{ar}) . \]
Again, we will abuse notation and write \( d\Psi \) to denote the induced map
\[ (2) \quad \Theta_S \rightarrow \text{Hom}_{\mathcal{O}_S}(Gr^0_{\mathcal{F}_{ar}}, Gr^{-1}_{\mathcal{F}_{ar}}) . \]

Finally, the map in part (b) of the Lemma, for \( \nabla = \nabla_{ar} \), is the second differential of the Archimedean period map and will be denoted
\[ d^2\Psi : \Theta_S^2 \longrightarrow \text{Hom}_{\mathcal{O}_S}(\mathcal{H}_{ar}, \mathcal{F}_{ar}/\mathcal{H}_{ar} + \text{span} \{\nabla_\eta(\mathcal{H}_{ar}) \mid \eta \in \Theta_S\}) . \]

We have an identification of \( \mathcal{O}_S \)-modules
\[ Gr^0_{\mathcal{F}_{ar}}(\mathcal{H} \otimes B_{ar}) \cong R^m \pi_*(Gr^0_F(\Omega^*_X/S \otimes B_{ar})) \cong R^m \pi_* \Omega^*_X/S = \mathcal{H} , \]
obtained from the obvious isomorphism of sheaf complexes
\[
\begin{array}{c}
\rightarrow \quad \Omega^p_{X/S},T^p \\
| \downarrow \cong | \\
\rightarrow \quad \Omega^p_{X/S},T^{p+1} \\
\end{array} 
\]
\[
\begin{array}{c}
d \quad \quad \rightarrow \\
\downarrow \cong \\
\rightarrow \quad \Omega^p_{X/S} \\
\end{array} 
\]
(“dropping the T’s”).

Similarly, \( Gr^{-1}_{\mathcal{F}_{ar}} \cong \mathcal{H} \). We use these identifications to obtain a version of (2) “without T’s,”
\[ \overline{d}\Psi : \Theta_S \longrightarrow \text{End}_{\mathcal{O}_S}(\mathcal{H}) . \]

Analogously, we also define
\[ \overline{d^2}\Psi : \Theta_S^{\otimes 2} \longrightarrow \text{End}_{\mathcal{O}_S}(\mathcal{H}) , \]
with \( \overline{d^2\Psi}(\zeta, \xi) \) being the composition

\[
\mathcal{H} \cong Gr_{\mathcal{F}_ar}^0 \xrightarrow{d^2\Psi(\zeta, \xi)} \mathcal{F}_{ar}^{-2} / \mathcal{H}_{ar} + \text{span} \{ \nabla_\eta(\mathcal{F}_{ar}^{-1}) \mid \eta \in \Theta_S \} \rightarrow Gr_{\mathcal{F}_ar}^{-2} \cong \mathcal{H}.
\]

**Remark.** Let \( t = (t_1, \ldots, t_s) \) be a coordinate system on \( S \) centered at 0. Then, in the notation of \([K]\], \( d\Psi(\partial/\partial t_i)|_{t=0} \) is \( \overline{L}_i \), \( d\Psi(\partial/\partial t_i \otimes \partial/\partial t_j)|_{t=0} \) is \( L_{ij} \), and \( d^2\Psi(\partial/\partial t_i \otimes \partial/\partial t_j)|_{t=0} \) is \( L^{ij} \). \( \mathcal{O}_S \)-linearity of \( d\Psi, d^2\Psi, \) etc. is essential for the ability to restrict to 0 in \( S \).

To formulate the next Lemma, we bring out the natural \( C^\infty_S \)-linear identification

\[
h : \mathcal{H} = \bigoplus_{p+q=m} \mathcal{H}^{p,q} \xrightarrow{\cong} \bigoplus_p Gr_p^\mathcal{H} =: Gr_p^\mathcal{F}\mathcal{H}.
\]

**Lemma 2** (a) For any \( \xi \in \Theta_S \) and all \( p \) we have

\[
\overline{d\Psi}(\xi)(\mathcal{F}^p\mathcal{H}) \subset \mathcal{F}^{p-1}\mathcal{H},
\]

the induced endomorphism of degree \(-1\) of \( Gr_p^\mathcal{F}\mathcal{H} \) coincides with \( d\Phi(\xi) \) and, in fact,

\[
\overline{d\Psi}(\xi) = h^{-1} \circ d\Phi(\xi) \circ h.
\]

(b) For any \( \zeta, \xi \in \Theta_S \) and all \( p \) we have

\[
\overline{d^2\Psi}(\zeta, \xi)(\mathcal{F}^p\mathcal{H}) \subset \mathcal{F}^{p-2}\mathcal{H},
\]

and

\[
d^2\Phi(\zeta, \xi) : \mathcal{F}^p \rightarrow \mathcal{F}^{p-2} / \mathcal{F}^p + \text{span} \{ \nabla_\eta(\mathcal{F}^p) \mid \eta \in \Theta_S \}
\]

factors through \( \overline{d^2\Psi}(\zeta, \xi) : \mathcal{F}^p \rightarrow \mathcal{F}^{p-2}\).

**Proof.** (a) For any \( \xi \in \Theta_S \)

\[
\nabla_\xi \mathcal{H}^{p,q} \subset \mathcal{H}^{p,q} \oplus \mathcal{H}^{p-1,q+1}
\]

and, correspondingly,

\[
\nabla^a_\xi \mathcal{H}^{p,q} T^p \subset \mathcal{H}^{p,q} T^p \oplus \mathcal{H}^{p-1,q+1} T^p.
\]

Therefore, \( d\Psi(\xi) \) maps \( \mathcal{H}^{p,q} T^p \) to its image under \( \nabla^a_\xi \) modulo \( \mathcal{H}_{ar} = \mathcal{F}_{ar}^0 \), i.e. into \( \mathcal{H}^{p-1,q+1} T^p \). Hence

\[
\overline{d\Psi}(\xi)(\mathcal{H}^{p,q}) \subset \mathcal{H}^{p-1,q+1}.
\]
which implies every statement in part (a) of the Lemma.

Part (b) is established by similar reasoning. □

The connection $\nabla$ on $\mathcal{H}$ naturally induces a connection $\nabla$ on $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{H})$ subject to the rule

$$\nabla(Ax) = (\nabla A)x + A\nabla x$$

for any $A \in \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{H})$ and $x \in \mathcal{H}$.

In accordance with Lemma 1, for any $\zeta \in \Theta_S$ the covariant derivative $\nabla_\zeta$ on $\mathcal{E}nd(\mathcal{H})$ determines an $\mathcal{O}_S$-linear map

$$\mathcal{E}_\zeta : \text{im}(\overline{d\Psi}) \rightarrow \mathcal{E}nd(\mathcal{H})/\text{im}(\overline{d\Psi})$$.

**Lemma 3** For any $\zeta, \xi \in \Theta_S$

(a) $\nabla_\zeta(\overline{d\Psi}(\xi))((\mathcal{F}^p) \subset \mathcal{F}^{p-1}$ for all $p$.

(b) $\mathcal{E}_\zeta(\overline{d\Psi}(\xi))$ determines an element of $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{Gr}_X^\cdot \mathcal{H})/\text{im}(d\Phi)$.

**Proof.** (a) $\nabla_\zeta(\overline{d\Psi}(\xi))\omega = \nabla_\zeta(\overline{d\Psi}(\xi)\omega) - \overline{d\Psi}(\xi)\nabla_\zeta\omega$ for any $\omega \in \mathcal{H}$. Assume $\omega \in \mathcal{H}^{p,q}$. We want to show that the $(p - 2, q + 2)$-component of the right-hand side is 0. But this component is

$$(\nabla_\zeta \nabla_\xi \omega)_{(p-2,q+2)} - (\nabla_\xi \nabla_\zeta \omega)_{(p-2,q+2)} = (\nabla_{[\zeta,\xi]} \omega)_{(p-2,q+2)} = 0 !$$

(b) follows from (a) and the relation between $\overline{d\Psi}(\xi)$ and $d\Phi(\xi)$ established in part (a) of the previous Lemma. □

**Definition.** The second fundamental form of the VHS

$$\Pi : \Theta_S^\otimes 2 \rightarrow \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{Gr}_X^\cdot) / \text{im}(d\Phi)$$

is defined by

$$\Pi(\zeta, \xi) := h \circ \nabla_\zeta(h^{-1}d\Phi(\xi)h) \circ h^{-1} \mod \text{im}(d\Phi) .$$

**Remark.** $\Pi|_{t=0}$ was denoted $d^2\Phi$ in [K].

We will omit the identification $h$ in what follows.
Proposition 1 ((2.4) in [K])

\[ II(ζ, ξ) \equiv \overline{d^2Ψ}(ζ, ξ) - \overline{dΨ}(ζ) \circ \overline{dΨ}(ξ) \mod \text{im}(dΦ). \]

Proof. \( \nabla_ζ(\overline{dΨ}(ξ)ω) = \nabla_ζ(\overline{dΨ}(ξ)ω) - \overline{dΨ}(ξ)∇_ζω \) for any \( ω \in H \). Now, let \( ω \) be the element of \( Gr^0_{F_{ar}}(H \otimes B_{ar}) \) corresponding to \( ω \) under the isomorphism \( H \cong Gr^0_{F_{ar}}(H \otimes B_{ar}) \). Using a similar identification of \( H \) with \( Gr^{-2}_{F_{ar}}(H \otimes B_{ar}) \), we have the following correspondences:

\[ \nabla_ζ(\overline{dΨ}(ξ)ω) \leftrightarrow (\nabla_ζ(\overline{dΨ}(ξ)ω) \mod F_{ar}^{-1}) = (d^2Ψ(ζ, ξ)ω \mod F_{ar}^{-1}) \in Gr^{-2}_{F_{ar}}, \]

and

\[ \overline{dΨ}(ξ)∇_ζω \leftrightarrow (dΨ(ζ) \circ dΨ(ξ)ω \mod F_{ar}^{-1}) \in Gr^{-2}_{F_{ar}}. \]

It remains to pass to \( H \) on the right-hand side, i.e. put bars on \( dΨ \) and \( d^2Ψ \).

\[ \square \]

2 The second Kodaira-Spencer class

First, let us recall the construction of the (first) Kodaira-Spencer map \( κ_1 = κ \) of the deformation (1): it is the connecting morphism in the higher-direct-image sequence

\[ \xymatrix{ \to \pi_*Ω_X \ar[r] & Θ_S \ar[r]^-κ & R^1\pi_*Ω_{X/S} \ar[r] & } \]

associated with the short exact sequence

\[ 0 \to Θ_{X/S} \to Θ_X \to π^*Θ_S \to 0. \]

Given \( ξ \in Θ_S \), the corresponding covariant derivative of the Gauss-Manin connection

\[ \nabla_ξ : R^mπ_*Ω^m_{X/S} \to R^mπ_*Ω^m_{X/S} \]

is computed as follows (see [Del], [KO], or [K]). Choose a Stein covering \( \mathcal{U} = \{U_i\} \) of \( X \). Then \( \{W_i = U_i \times S\} \) constitute a Stein covering \( W \) of \( X \). Consider a class in \( Γ(S, R^mπ_*Ω^m_{X/S}) \) represented by the Čech cocycle \( ω = \{ω_Q ∈ Γ(W_Q, Ω^p_{X/S})\} \), where \( Q = (i_1 < \ldots < i_q) \) and \( p + q + 1 = m \). Let the same symbol \( ω_Q \) denote a pull-back of \( ω_Q \in Γ(W_Q, Ω^p_{X/S}) \) to an
element of $\Gamma(W_Q, \Omega^p_X)$. Let $v = \{v_i\}$ denote liftings of $\xi \in \Theta_S$, or rather, in $\pi^*\Theta_S$, to $\Gamma(W_i, \Theta_X)$. Then
\[
\nabla \xi(\omega) = \left[ \tilde{\mathcal{L}}_v \omega \right]
\]
where the brackets denote cohomology classes in $R^m \pi_* \Omega^\cdot_X/S$ (more precisely, in $\Gamma(S, R^m \pi_* \Omega^\cdot_X/S)$), and $\tilde{\mathcal{L}}_v$ is the Lie derivative on $\check{C}^m(W, \Omega^\cdot_X)$ with respect to $v = \{v_i\} \in \check{C}^0(W, \Theta_X)$:
\[
\tilde{\mathcal{L}}_v \omega = \{ dv_{i_1} \omega_{i_1, \ldots i_q} + v_{i_1} \omega_{i_1, \ldots i_q} \}
\]
with $\check{D} = d \pm \delta$, $\delta = \check{\delta}$ being the Čech differential, as usual.

Now, when $\xi \in \Theta_S$ lifts to all of $\mathcal{X}$, i.e. $\xi$ lies in the image of $\pi_*\Theta_X \to \Theta_S$ ($= ker(\kappa)$!), the cochain $v \in \check{C}^0(W, \Theta_X)$ lifting $\xi$ is a cocycle, i.e. $\delta v = 0$. But then formula (5) reduces to
\[
\tilde{\mathcal{L}}_v \omega = \{ dv_{i_1} \omega_{i_1, \ldots i_q} + v_{i_1} \omega_{i_1, \ldots i_q} \} = \{ \mathcal{L}_{v_{i_1}} \omega_{i_1, \ldots i_q} \},
\]
where $\mathcal{L}_{v_{i_1}}$ now denotes the usual Lie derivative with respect to the vector field $v_{i_1}$. Evidently, in this case $\nabla \xi \mathcal{F}^p \subset \mathcal{F}^p$. Consequently, the (first) differential of the period map
\[
d\Phi : \Theta_S \longrightarrow \text{Hom}(\mathcal{F}^p, \mathcal{H}/\mathcal{F}^p)
\]
factors through $\Theta_S/\text{im} \{ \pi_*\Theta_X \to \Theta_S \} = \Theta_S/\ker(\kappa)$, and thus we arrive at

**Theorem 1 (Griffiths)** There is a commutative diagram
\[
\begin{array}{ccc}
\Theta_S & \xrightarrow{d\Phi} & \bigoplus_p \text{Hom}(\mathcal{F}^p, \mathcal{H}/\mathcal{F}^p) \\
\downarrow \kappa & & \uparrow \sim \\
T^1_{\mathcal{X}/S} & \xrightarrow{-1} & \text{End}(Gr_{\mathcal{F}}\mathcal{H})
\end{array}
\]
where $T^1_{\mathcal{X}/S} := R^1 \pi_* \Theta_X/S$ and the northeast arrow sends $x$ to the map $x \sim$ (the cup product with $x$).

Analogous results hold for $d\Psi$ and $\overline{d\Psi}$ (see [K]). We seek a similar statement for $d^2\Phi$ and $d^2\Psi$.

First we need to review the construction of the “second-order tangent space to the moduli.” Here we are following (a relativized version of) the
presentation in \[ R \]. We will make the simplifying assumption that \( X \) has no global holomorphic vector fields, i.e. \( \pi_* \Theta_{X/S} = 0 \).

Consider the diagram
\[
\begin{array}{ccc}
\mathcal{X} \times_S \mathcal{X} & \xrightarrow{g} & \mathcal{X}_2/S \\
f \searrow & & \nearrow p \\
& S &
\end{array}
\]

where \( \mathcal{X}_2/S \) denotes the symmetric product of \( \mathcal{X} \) with itself over \( S \) (fiber-wise). Let \( \mathcal{K}^- \) denote the complex of sheaves on \( \mathcal{X}_2/S \)
\[
-1 \to (g_* (\Theta_{\mathcal{X}/S}^\otimes 2))^- \xrightarrow{[\cdot]} \Theta_{X/S} \to 0
\]

where \( \otimes \) stands for the exterior tensor product on \( \mathcal{X} \times_S \mathcal{X} \), \( (\ )^- \) denotes the anti-invariants of the \( \mathbb{Z}/2\mathbb{Z} \)-action, and the differential is the restriction to the diagonal \( \Delta \subset \mathcal{X} \times_S \mathcal{X} \), followed by the Lie bracket of vector fields.

**Definition.** \( T^{(2)}_{X/S} := R^1 p_* \mathcal{K}^- \) is the sheaf on \( S \) whose fiber over each \( t \in S \) is the second-order (Zariski) tangent space to the base \( V_t \) of the miniversal deformation of \( X_t \), i.e.
\( T^{(2)}_{X_t} \cong (\mathfrak{m}_{V_t,0}/\mathfrak{m}_{V_t,0}^3)^* \).

This should not be confused with the sheaf \( T^2_{X/S} = R^2 \pi_* \Theta_{X/S} \) whose fiber over each \( t \in S \) is the obstruction space \( T^2_{X_t} \) for deformations of \( X_t \).

In fact, we have this exact sequence:
\[
0 \to T^1_{X/S} \to T^{(2)}_{X/S} \to \text{Sym}^2 T^1_{X/S} \xrightarrow{o} T^2_{X/S},
\]

where \( o \) is the first obstruction map, given by the \( \mathcal{O}_S \)-linear graded Lie bracket:
\[
\text{Sym}^2 R^1 \pi_* \Theta_{X/S} \xrightarrow{[\cdot]} R^2 \pi_* \Theta_{X/S}.
\]

We will find it easier to deal with an “unsymmetrized version” of \( T^{(2)}_{X/S} \).

**Definition.** \( \tilde{T}^{(2)}_{X/S} := R^1 f_* \mathcal{K}^- \), where \( \mathcal{K}^- \) is the complex on \( \mathcal{X} \times_S \mathcal{X} \),
\[
-1 \to \Theta_{\mathcal{X}_2/S}^\otimes 2 \xrightarrow{[\cdot]} \Theta_{X/S} \to 0
\]
\( \tilde{T}^{(2)}_{X/S} \) fits in the commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & T^1_{X/S} & \rightarrow & \tilde{T}^{(2)}_{X/S} & \rightarrow & (T^1_{X/S})^\otimes 2 & \rightarrow & T^2_{X/S} \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & T^1_{X/S} & \rightarrow & T^{(2)}_{X/S} & \rightarrow & Sym^2 T^1_{X/S} & \rightarrow & T^2_{X/S}
\end{array}
\]

**Definition.** \( T^{(2)}_S := D^{(2)}_S / O_S \) will denote the sheaf of *second-order tangent vectors* on \( S \).

As part of a more general construction in [EV], there is the *second Kodaira-Spencer map* associated to every deformation as in (1):

\[ \kappa_2 : T^{(2)}_S \longrightarrow T^{(2)}_{X/S} \]  

We will work with a natural lifting \( \tilde{\kappa}_2 \) of \( \kappa_2 \):

\[
\begin{array}{cccccc}
\Theta^1_S \oplus \Theta^2_S & \longrightarrow & \tilde{T}^{(2)}_{X/S} & \longrightarrow & T^{(2)}_{X/S} \\
\| & & \| & & \\
T^{(2)}_S & \longrightarrow & T^{(2)}_{X/S}
\end{array}
\]

(8)

or, rather, with the restriction of \( \tilde{\kappa}_2 \) to \( \Theta^2_S \).

It is easy to describe \( \tilde{\kappa}_2 \) explicitly. Let

\[ \kappa : \Theta_S \longrightarrow T^1_{X/S} = R^1 \pi_* \Theta_{X/S} \]

be the (relative, first) Kodaira-Spencer map of the family \( \pi \) as in (1). It is equivalent to the datum of a section of \( \Gamma(S, \Omega^1_S \otimes R^1 \pi_* \Theta_{X/S}) \). This section can be represented by a \( \hat{C}^1(\mathcal{U}, \Theta_X) \)-valued one-form on \( S \),

\[
\theta(t) dt := \sum_{\ell=1}^{s} \theta(t)_\ell dt_\ell = \sum_{\ell=1}^{s} \sum_{I \in Z_+} \theta^{(I)}_\ell t^I dt_\ell \quad (Z_+ := \{0\} \cup N) .
\]

Here each \( \theta^{(I)}_\ell = \{\theta^{(I)}_{ij,\ell}\}_{ij} \) is a cochain in \( \hat{C}^1(\mathcal{U}, \Theta_X) \) and each \( \theta(t)_\ell \) is a cocycle on \( X_t \) for every value of \( t \), but only the leading coefficients \( \theta^{(0)}_\ell \) (\( \ell = 1, \ldots, s \)) are Čech cocycles on \( X \) (\( t = 0 \)). The rest satisfy the “deformation equation”

\[
\delta \left( \frac{\partial \theta(t)_\ell}{\partial \theta_k} \right) = [\theta(t)_\ell, \theta(t)_k]
\]

(10)
When \( s = 1 \), this reduces to \( \delta \dot{\theta}(t) = [\theta(t), \theta(t)] \).
Thus, it is natural to make the following

**Definition.** \( \tilde{\kappa}_2 : \Theta_S^\otimes^2 \rightarrow \tilde{T}_{X/S}^{(2)} \) sends \( \frac{\partial}{\partial t_k} \otimes \frac{\partial}{\partial t_\ell} \) to the cohomology class of the cocycle

\[
(\theta(t)_k \times \theta(t)_\ell, \frac{\partial \theta(t)_\ell}{\partial t_k}) \in \hat{C}^1(W \times_S W, \kappa^-).
\]

(11)

For example, if

\[
\theta(t)dt = \sum_{\ell=1}^{s} (\theta^{(0)}_\ell + \sum_{k=1}^{s} \theta^{(k)}_\ell t_k) dt_\ell + O(t^2)
\]

is the expansion of \( \theta(t)dt \) at 0 up to order two, then \( \tilde{\kappa}_2 |_0 : \Theta_S^\otimes^2 |_0 \rightarrow \tilde{T}_{X}^{(2)} \) sends \( \frac{\partial}{\partial t_k} \otimes \frac{\partial}{\partial t_\ell} \) to the cohomology class of the cocycle

\[
(\theta^{(0)}_k \times \theta^{(0)}_\ell, \theta^{(k)}_\ell) \in \hat{C}^1(U \times U, \tilde{\kappa}^- |_0).
\]

(13)

Indeed, for the definition of \( \tilde{\kappa}_2 \) to make any sense, we must have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & \Theta_S & \rightarrow & \Theta_S \oplus \Theta_S^\otimes^2 & \rightarrow & \Theta_S^\otimes^2 & \rightarrow & 0 \\
\kappa_1 & \downarrow & \quad & \downarrow & \tilde{\kappa}_2 & \downarrow & \kappa_2^1 & & \\
0 & \rightarrow & T_{X/S}^1 & \rightarrow & \tilde{T}_{X/S}^{(2)} & \rightarrow & (T_{X/S}^1)^{\otimes^2} & \rightarrow & T_{X/S}^2.
\end{array}
\]

(14)

The square on the right induces a commutative triangle

\[
\begin{array}{ccc}
\Theta_S^\otimes^2 & \xrightarrow{\tilde{\kappa}_2} & \tilde{T}_{X/S}^{(2)} \\
\kappa_2^1 & \searrow & \\
\kappa_1^2 & \downarrow & (T_{X/S}^1)^{\otimes^2}.
\end{array}
\]

(15)

Therefore, \( \tilde{\kappa}_2(\frac{\partial}{\partial t_k} \otimes \frac{\partial}{\partial t_\ell}) \) must project onto

\[
\kappa_1(\frac{\partial}{\partial t_k}) \otimes \kappa_1(\frac{\partial}{\partial t_\ell}) = [\theta(t)_k] \otimes [\theta(t)_\ell].
\]

Since

\[
\hat{C}^1(W \times_S W, \kappa^-) = \hat{C}^2(W \times_S W, \kappa^-) + \hat{C}^1(\kappa^0),
\]

(10)
and
\[ \tilde{\mathcal{C}}^2(\mathcal{W} \times_S \mathcal{W}, \tilde{\mathcal{K}}^{-1}) \simeq \check{\mathcal{C}}^1(\mathcal{U}, \Theta_{\mathcal{X}/S}^{\otimes 2}), \]
this means that the \( \check{\mathcal{C}}^2(\tilde{\mathcal{K}}^{-1}) \)-component of a representative of \( \tilde{\kappa}_2(\partial_{\theta_k} \otimes \partial_{\theta_\ell}) \) in \( \check{\mathcal{C}}^1(\tilde{\mathcal{K}}^{-1}) \) may be taken to be
\[ \theta(t)_k \times \theta(t)_\ell. \]
And, in view of (10), the cochain (11) is indeed a cocycle in \( \check{\mathcal{C}}^1(\tilde{\mathcal{K}}^{-1}) \).

We still need to check that \( \tilde{\kappa}_2 \) is well-defined.

**Proposition 2** \( \tilde{\kappa}_2 : \Theta_S^{\otimes 2} \rightarrow \check{T}^{(2)}_{\mathcal{X}/S} \) can be presented as a connecting morphism in the higher-direct-image sequence of a short exact sequence.

**Proof.** The starting point is the sequence of \( \mathcal{O}_{\mathcal{X}} \)-modules (4), whose direct-image sequence (3) gives the first Kodaira-Spencer mapping \( \kappa_1 \) as a connecting morphism. Now, (4) contains an exact subsequence
\[ (16) \quad 0 \rightarrow \Theta_{\mathcal{X}/S} \xrightarrow{\alpha} \tilde{\Theta}_{\mathcal{X}} \xrightarrow{\beta} \pi^{-1}\Theta_S \rightarrow 0 \]
of \( \pi^{-1}\mathcal{O}_S \)-modules, whose direct-image sequence also has \( \kappa_1 \) as a connecting morphism. From now on we will work with (16) in place of (4).

We can splice two sequences produced from (16) by exterior tensor products with \( \Theta_{\mathcal{X}/S} \) and with \( \pi^{-1}\Theta_S \), respectively:
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Theta_{\mathcal{X}/S} & \otimes & \Theta_{\mathcal{X}/S} & \rightarrow & \tilde{\Theta}_{\mathcal{X}} & \otimes & \Theta_{\mathcal{X}/S} & \rightarrow & \pi^{-1}\Theta_{\mathcal{X}/S} & \otimes & \tilde{\Theta}_{\mathcal{X}} & \rightarrow & \pi^{-1}\Theta_S & \otimes & \Theta_{\mathcal{X}/S} & \rightarrow & (\pi^{-1}\Theta_S)^{\otimes 2} & \rightarrow & 0
\end{array}
\]

The resulting four-term exact sequence
\[ (17) \quad 0 \rightarrow \Theta_{\mathcal{X}/S}^{\otimes 2} \rightarrow \tilde{\Theta}_{\mathcal{X}} \otimes \Theta_{\mathcal{X}/S} \xrightarrow{\beta \otimes \alpha} \pi^{-1}\Theta_S \otimes \tilde{\Theta}_{\mathcal{X}} \rightarrow (\pi^{-1}\Theta_S)^{\otimes 2} \rightarrow 0 \]

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can be extended to a commutative diagram

\[
\begin{array}{ccc}
0 & \to & \Theta_{X/S}^2 \\
\downarrow & & \downarrow \\
\Theta_{X/S} & = & \Theta_{X/S}
\end{array}
\]

(18)

where the vertical maps are composed of the restriction to the diagonal \( \Delta \subset X \times S \) followed by Lie brackets.

**Remark.** Here we use the fact that the restriction of the Lie bracket

\[
\left[ \ , \right] : \Theta_{X/S}^2 \longrightarrow \Theta_X
\]

to \( \tilde{\Theta}_X \otimes \Theta_{X/S} \) takes values in \( \Theta_{X/S} \) (see [BS], and also [EV]).

We note that the first column of the diagram (18) constitutes the complex \( \tilde{K}^{-} \) computing \( \bar{T}^{(2)}_{X/S} \). Let \( L^{-} \) denote the complex

\[
\begin{array}{c}
\to \tilde{K}^{-} \\
\to L^{-} \\
\to (\pi^{-1} \Theta_{S} \otimes \tilde{\Theta}_X) \oplus \Theta_{X/S}
\end{array}
\]

with \( \ell = (\beta \otimes \alpha, [ \ , ] ) \). Then we can rewrite (18) as a short exact sequence of complexes on \( X \times_S X \):

\[
0 \longrightarrow \tilde{K}^{-} \longrightarrow L^{-} \longrightarrow (\pi^{-1} \Theta_{S} \otimes \tilde{\Theta}_X) \oplus \Theta_{X/S} \rightarrow 0
\]

(19)

The associated direct-image sequence yields

\[
\to R^0 f_* L^{-} \longrightarrow \Theta_{X/S}^2 \longrightarrow \bar{T}^{(2)}_{X/S} \rightarrow .
\]

(20)

Tracing out the definition of a connecting morphism (bearing in mind that if \( \zeta \in \Theta_X \) is a local lifting of \( \partial/\partial t_k \), and \( \theta(t) \) is any element of \( \Theta_{X/S} \), then \( [\zeta, \theta(t)] = \frac{\partial \theta(t)}{\partial t_k} \) ) shows that it is indeed the same as \( \bar{\kappa}_2 \) given by the explicit definition in coordinates given above. \( \square \)

**Remark.** The explicit construction above shows how the data, up to second order, of the (first) Kodaira-Spencer mapping

\[
\kappa : \Theta_{S,0}/m_{S,0}^2 \Theta_{S,0} \longrightarrow T^1_{X/S,0}/m_{S,0}^2 T^1_{X/S,0}
\]
determines the second Kodaira-Spencer class
\[ \tilde{\kappa}_2 : \Theta_S^{\otimes 2}|_0 = \Theta_{S,0}^{\otimes 2}/\mathfrak{m}_{S,0}\Theta_{S,0}^{\otimes 2} \rightarrow \tilde{T}_X^{(2)} \]
(see (12)). Conversely, if \( \zeta, \xi \in \Theta_S|_0 \), and \( \tilde{\kappa}_2(\zeta \otimes \xi) \) is represented by a cocycle \( (\zeta \times \xi, \theta) \in \check{C}^1(\kappa^-|_0) \), then we can choose coordinates \( t \) on \( S \) so that \( \zeta = \partial/\partial t_k, \xi = \partial/\partial t_\ell \), and the Kodaira-Spencer mapping of the deformation in question is represented in \( T_{X/S,0}^1 \otimes \Omega_{S,0}^1/\mathfrak{m}_{S,0}^2 T_{X/S,0}^1 \otimes \Omega_{S,0}^1 \) by the form
\[ \sum_{\mu=1}^s (\theta_0^{(\mu)} + \sum_{\nu=1}^s \theta_{\mu}^{(\nu)} t_k) dt_\ell \]
with \( \theta_k^{(0)} = \tilde{\zeta}, \theta_\ell^{(0)} = \tilde{\xi} \), and \( \theta_\ell^{(k)} = \theta \).

3 Main results

There is a natural composition map
\[ \Theta_S^{\otimes 2} \hookrightarrow \Theta_S \oplus \Theta_S^{\otimes 2} \twoheadrightarrow T_S^{(2)}. \]
However, this map is not \( \mathcal{O}_S \)-linear. For example, \( x \otimes y - y \otimes x \) is mapped to \([x, y]\), whereas for any \( f \in \mathcal{O}_S \)
\[ f.(x \otimes y - y \otimes x) = (f.x) \otimes y - y \otimes (f.x) \]
is sent to \([f, y] - y(f).x\) . Nevertheless, (21) induces an \( \mathcal{O}_S \)-linear map
\[ \Theta_S^{\otimes 2} \twoheadrightarrow T_S^{(2)}/\Theta_S \quad (\simeq \text{Sym}^2 \Theta_S). \]
The latter fits in a commutative square of \( \mathcal{O}_S \)-linear maps obtained from (8),
\[ \begin{array}{ccc}
\Theta_S^{\otimes 2} & \xrightarrow{\tilde{\kappa}_2} & \tilde{T}_{X/S}^{(2)}/\text{im}(\kappa_1) \\
\downarrow & & \downarrow \\
T_S^{(2)}/\Theta_S & \xrightarrow{\tilde{\kappa}_2} & \tilde{T}_{X/S}^{(2)}/\text{im}(\kappa_1) \\
\end{array} \]

**Theorem 2** The second differential of the Archimedean period map \( d^2\Psi \) factors through the diagonal of (22),
\[ \tilde{\kappa}_2 : \Theta_S^{\otimes 2} \rightarrow \tilde{T}_{X/S}^{(2)}/\text{im}(\kappa_1). \]
Proof. Since the statement deals with $O_S$-linear maps, it is enough to prove it pointwise, for each $t \in S$. It suffices to restrict to $0 \in S$. We need to show that $d^2\Psi(y) = 0$ for any $y \in \Theta_S^2|_0$ with $\overline{\kappa}_2(y) \in im(\kappa_1)$. The condition on $y$ implies that $\overline{\kappa}_2(y) \in T^{(2)}_X$ can be represented by a cocycle of the form

\[(0, \theta) \in \tilde{C}^2(\Theta_X^2[1]) \oplus \tilde{C}^1(\Theta_X) = \tilde{C}^1(\tilde{K}_|0),\]

where $\theta$ is a cocycle in $\tilde{C}^1(\Theta_X)$ representing $\kappa_1(\eta)$ for some $\eta \in \Theta_S|_0$.

At this point we “recall” two theorems from [K].

Theorem 3 ((5.4) in [K]) If $\kappa \in T^1_{X/S,0} \otimes \Omega^1_{S,0}/m^2_{S,0} T^1_{X/S,0} \otimes \Omega^1_{S,0}$ is represented by the form

\[
\sum_{\mu=1}^s (\theta^{(0)}_\mu + \sum_{\nu=1}^s \theta^{(\nu)}_\mu t_\kappa) dt_\ell
\]

with $\theta^{(0)}_k = \tilde{\zeta}, \theta^{(0)}_\ell = \tilde{\xi}$, and $\theta^{(k)}_\ell = \theta$, then the second differential of the Archimedean period map

\[d^2\Psi(\partial_{t_k} \otimes \partial_{t_\ell}) : H_{ar} \longrightarrow F_{ar}^{-2}/H_{ar} + \text{span} \{\nabla_\eta|_0(H_{ar}) \mid \eta \in \Theta_S\}
\]

is induced by the map

\[H_{ar} = H^m_{ar} \longrightarrow H^m_{ar}(\Omega_X \otimes B_{ar}/F_{ar}^0(\Omega_X \otimes B_{ar}) + \text{span} \{\tilde{\mathcal{L}}_\eta|_0 F^0_{ar} \mid \eta \in \Theta_S\})
\]

given on the cochain level by

\[
\omega_{1_i \ldots i_q} T^p = \omega Q T^p \mapsto
\]

\[
\tilde{\zeta}_{i-1i_0} \otimes \omega Q T^p - \tilde{\zeta}_{i_0i_1} \otimes \omega Q T^{p+1} + \theta_{i_0i_1} \omega Q T^p .
\]

Theorem 4 ((5.7) in [K]) $d^2\Psi$ on $\Theta_S^2|_0$ is determined by

\[\kappa \in T^1_{X/S,0} \otimes \Omega^1_{S,0}/m^2_{S,0} T^1_{X/S,0} \otimes \Omega^1_{S,0} .
\]
Reading the two theorems in light of the Remark at the end of Section 2, Theorem 4 shows that $d^2\Psi(y)$ is determined by $\tilde{\kappa}_2(y) = [(0, \theta)] \in \tilde{T}_X^{(2)}$, and Theorem 3 says that $d^2\Psi(y)$ is induced on the cochain level by the contraction with the cocycle $\theta$ representing $\kappa_1(\eta)$. This contraction is equivalent to $\nabla_\eta|_0$ modulo $H_{ar}$ (in fact, it is none other than $d\Psi(\eta)$), and so we have proved that $d^2\Psi(y) = 0$. □

**Theorem 5**  The graded version of the second differential of the Archimedean period map $d^2\Psi$, as well as the second differential of the usual period map $d^2\Phi$ and the second fundamental form of the VHS, $\Pi$, all factor through

$$\kappa_1^2 : \Theta_S^{\otimes 2} \rightarrow (T^{(1)}_{X/S})^{\otimes 2},$$

and thus depend on $\kappa_1$ only.

**Proof.** Again it suffices to restrict to $0 \in S$. Suppose

$$\tilde{\kappa}_2(y) = [(\hat{\zeta} \times \hat{\xi}, \theta)] \in \tilde{T}_X^{(2)}$$

for some $y = \zeta \otimes \xi \in \Theta_S^{\otimes 2}|_0$. Examining formula (24), we observe that the term involving $\theta$ lies in $F_{ar}^{-1}$. Therefore, $d^2\Psi(y)$ depends only on $\kappa_1^2(y) := \kappa_1(\zeta) \otimes \kappa_1(\xi)$. This proves the Theorem for $d^2\Psi$.

The statements for $d^2\Phi$ and $\Pi$ follow from this by Lemma 2 (b) and Proposition 1, respectively. □

Finally, all the maps in question are symmetric, and so we may pass from $\kappa_1^2$ to $\text{Sym}^2\kappa_1$ and from $\tilde{\kappa}_2$ to $\tilde{\pi}_2$ (see (8)). Referring to the following symmetrized version of (14),

$$
\begin{align*}
0 & \rightarrow \Theta_S \rightarrow T_S^{(2)} \rightarrow \text{Sym}^2\Theta_S \rightarrow 0 \\
0 & \rightarrow T^{(1)}_{X/S} \rightarrow T^{(2)}_{X/S} \rightarrow \text{Sym}^2T^{(1)}_{X/S} \rightarrow T^{2}_{X/S},
\end{align*}
$$

we conclude with

**Theorem 6**  $d^2\Psi$ factors through

$$\tilde{\pi}_2 : \text{Sym}^2\Theta_S \rightarrow T^{(2)}_{X/S}/\text{im} (\kappa_1),$$
whereas $d^2\Psi$, $d^2\Phi$ and $\Pi$ factor through

$$Sym^2\kappa_1 : Sym^2\Theta_S \rightarrow Sym^2T^1_{X/S}.$$ 

**Remark.** When the deformation is versal, i.e. $im(\kappa_1)$ is all of $T^1_{X/S}$, there is no difference between $\overline{\kappa}_2$ and $Sym^2\kappa_1$.

**4 The higher-order cases**

The definition of the second differential of the period map in Section 1 easily generalizes to higher-order cases (cf. [K]).

All three papers mentioned in the introduction define “tangent spaces to the moduli” $T^{(n)}_{X/S}$ of all orders $n$. However, these definitions seem more complicated than in the case $n = 2$.

Still, we should have a diagram analogous to (25),

$$
\begin{array}{ccccccccc}
0 & \rightarrow & T^{(n-1)}_S & \rightarrow & T^{(n)}_S & \rightarrow & Sym^n\Theta_S & \rightarrow & 0 \\
\kappa_{n-1} & \downarrow & \kappa_n & \downarrow & Sym^n\kappa_1 & \downarrow & o_n & \rightarrow & T^2_{X/S}, \\
0 & \rightarrow & T^{(n-1)}_{X/S} & \rightarrow & T^{(n)}_{X/S} & \rightarrow & Sym^nT^1_{X/S} & \rightarrow & T^2_{X/S}, \\
\end{array}
$$

where $o_n$ is the $n^{th}$ obstruction map, and we expect that the $n^{th}$ differential of the Archimedean period map $d^n\Psi$ factors through the $n^{th}$ Kodaira-Spencer map $\kappa_n$ modulo the image of $\kappa_{n-1}$, whereas the $n^{th}$ differential of the usual period map $d^n\Phi$ and the $n^{th}$ fundamental form of the VHS InI factor through $Sym^n\kappa_1$.

**5 Appendix: Archimedean cohomology**

In this section we summarize what we need about Archimedean cohomology. For more information on this subject we refer to [Den].

**Definition.** $B_{ar} = C[T, T^{-1}], \ L = C[T^{-1}]$. $B_{ar}$ is filtered by the $L$-submodules $F^p = T^{-p}L$.

Thus, if $X$ is a compact Kähler manifold, $H^m(X) \otimes_C B_{ar}$ receives the filtration $F_{ar}$ obtained as the tensor product of the Hodge filtration on
$H^m(X, C)$ and the filtration $F^r$ on $B_{ar}$. $F^r_{ar}$ is a decreasing filtration with infinitely many levels, and

$$G_{k}^{r}_{ar} \cong \bigoplus_{p+q=m} H^{p,q} T^{p-k}.$$ 

**Definition.** The Archimedean cohomology of $X$ is

$$H^m_{ar}(X) := F^0_{ar}(H^m(X, C) \otimes B_{ar}).$$

Consider the complex of sheaves $\Omega^\cdot_X \otimes_C B_{ar}$ with the differential

$$d(\omega, T^k) := d\omega, T^{k+1}.$$ 

This complex is also filtered by the tensor product of the stupid filtration on $\Omega^\cdot_X$ and $F^r$ on $B_{ar}$, and we have

$$H^m_{ar}(X) = F^0 H^m(X, \Omega^\cdot_X \otimes B_{ar}) \cong H^m(X, F^0(\Omega^\cdot_X \otimes B_{ar})) \cong H^m(X, \Omega^\cdot_X) \otimes L \cong H^m(X, C) \otimes L.$$ 

Note that $H^m(X, \Omega^\cdot_X \otimes B_{ar})$ is a complex infinite-dimensional Hodge structure (of weight $m$), and $(\Omega^\cdot_X \otimes B_{ar}, d)$ is a Hodge complex. Hence

$$G_{k}^{r}_{ar} H^m(X, \Omega^\cdot_X \otimes B_{ar}) \cong H^m(X, G_{k}^{r}_{ar} (\Omega^\cdot_X \otimes B_{ar})).$$

We will write boldface $\tilde{D}$ for the differential in the Čech cochain complex computing $H^m(X, \Omega^\cdot_X \otimes B_{ar})$, and

$$\tilde{\mathcal{L}}_v := \tilde{D} v + v \tilde{D}$$

for the corresponding Lie derivative with respect to a vector field $v$ on $X$.

These constructions extend without any difficulty to the relative situation. In particular, given a flat family $\pi : \mathcal{X} \rightarrow S$ of compact Kähler manifolds, the bundle

$$\mathcal{H} \otimes B_{ar} = \mathbb{R}^m \pi_*(\Omega^\cdot_{\mathcal{X}/S} \otimes B_{ar})$$

is filtered by $F_{ar}$, and the Gauss-Manin connection $\nabla$ extends to

$$\nabla_{ar} : \mathcal{H} \otimes B_{ar} \rightarrow \mathcal{H} \otimes B_{ar} \otimes \Omega^1_S,$$
with the usual Griffiths’ transversality property
\[
\nabla_{ar}(F_{ar}^r) \subset F_{ar}^{r-1} \otimes \Omega^1_S.
\]

Specifically, if \( x \) is a section of \( \mathcal{H} \), then
\[
\nabla_{ar}(x.T^p) = \nabla x.T^p .
\]

The real difference arises when one examines the definition of \( \nabla_{ar} \) on the cochain level, due to the fact that \( d \) increases the exponent at \( T \).

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