The Feedback Effect of Hedging in Portfolio Optimization

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In this short note, we will show how to optimize the portfolio of a large trader whose hedging strategy affects the price of his assets.

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1. Introduction

Since the famous papers of Black-Scholes on option pricing \(^1\) and of Markowitz on portfolio optimization \(^2\) (for a review \(^3\)), some progress has been made in order to extend these results to a more realistic arbitrage-free market model. As a reminder, the Black-Scholes theory and the Markowitz’s theory consist in following a (hedging) strategy to decrease the risk of loss given a fixed amount of return. Both theories are based on three important hypothesis which are satisfied only to a certain extent:

- The traders can revise their decisions continuously in time. This first hypothesis is not realistic for obvious reasons. A major improvement was recently introduced by Bouchaud-Sornette \(^4\) in their time-discrete model. They introduce an elementary time \(\tau\) after which a trader is able to revise his decisions again. The optimal strategy is fixed by the minimization of the risk defined by the variance of the portfolio. The resulting risk is no longer zero and in the continuous-time limit where \(\tau\) goes to zero, one recovers the classical result of Black-Scholes: the risk vanishes (one can show by using the Ito’s formalism that this is true for any definition of the risk in the context of continuous-time log-normal price processes).

- The price fluctuations obey a log-normal law. This second hypothesis doesn’t truthfully reflect the market. A data analysis would rather suggest that the (uncorrelated) price increments \(\delta x_n\) at time \(t = n\tau\) (with \(\tau\) about 15-30 minutes) are well fitted with a truncated Levy law or a Student distribution \(^5\). Both distributions exhibit large jumps for each increment and are distributed with fat tails, thus far from being a log-normal law. Moreover, the \(\delta x_n\) are no longer uncorrelated below the 15-30 minutes interval (incomplete markets). Thus the Bouchaud-Sornette model goes beyond the classical financial theory and allows to
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incorporate non log-normal price processes such as a truncated Levy law. If one tries to reproduce the price of an option given by the Bouchaud-Sornette theory by using the Black-Scholes model, one must introduce an implied volatility which will depend on the kurtosis of the non log-normal law. This theoretical implied volatility then reproduces the experimental "smile law".

- Markets are assumed to be completely elastic. This third hypothesis means that small traders don’t modify the prices of the market by selling or buying large amounts of assets. The limitation of this last assumption lies with the fact it is only justified when the market is liquid.

While so far research has mainly focused on relaxing the first and/or the second hypothesis described above, in this paper we shall relax only the third one. In relation to this, we note that some recent articles have appeared in the case of the hedging of derivatives in markets which are not perfectly liquid. In this context, the classical Black-Scholes equation is replaced by a non-linear partial differential equation (PDE) which we give (in the case of our analysis) in the appendix.

The aim of this paper is to analyze the feedback effect of hedging in portfolio optimization. In the first section, we model the market as composed of small traders and a large one whose demand is given by a hedging strategy. We derive from this toy-model the parameters of the stochastic process of the price with the influence of the large trader. Taking into account the influence on the volatility and the return, we compute in the second section the hedging strategy of the large trader in order to optimize dynamically a portfolio composed of a risky asset and a bond. This leads to a well-defined stochastic optimization problem.

2. The feedback effect of hedging on price

By definition, a market is liquid when the elasticity parameter is small. The elasticity parameter $\epsilon$ is given by the ratio of relative change in price $S_t$ to change in the net demand $D$:

$$\frac{dS_t}{S_t} = \epsilon dD_t$$

(2.1)

We observe experimentally that when the demand increases (resp. decreases), the price rises (decreases). The parameter $\epsilon$ is therefore positive and we will assume that it is also constant. We will then reproduce the log-normal process when the feedback hedging effect is negligible. Another interesting (because more realistic) assumption would be to define $\epsilon$ as a stochastic variable.

In a crash situation, the small traders who tend to apply the same hedging strategy can be considered as a large trader and the hedging feedback effects become very important. This can speed up the crash. In the following, we will use the simple relation (2.1) to analyze the influence of dynamical trading strategies on the prices in financial markets. The dynamical trading $\phi(S_t, t)$ which represents the number of shares of a given stock that a large trader holds will be determined in order
to optimize his portfolio. An analytic solution will not be possible and we will resolve the non-linear equations by expanding the solution as a formal series in the parameter $\epsilon$. The first order correction will be obtained.

Let’s call $D(t, W, S_t)$ the demand of all the traders in the market which depends on time $t$, a Brownian variable $W$ and the price $S_t$. $S_t$ is assumed to satisfy the stochastic equation

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW$$

(2.2)

with $\mu(S_t, t)$ the return and $\sigma(S_t, t)$ the volatility. The stochastic process $W$ models the information the traders have on the demand. If we take the derivative of $D(t, S_t, W)$, i.e $dD$, we obtain (using the Ito’s symbolic “rules” $dW^2 = dt$, $dWdS = \sigma S dt$ and $dS^2 = \sigma^2 S^2 dt$)

$$dD = \frac{1}{\epsilon} \left[ (\partial_t D + \frac{1}{2} \sigma^2 S^2 \partial_S^2 D + \frac{1}{2} \sigma \partial_W D + \sigma S \partial_{SW} D)dt + \partial_S DDt + \partial_W DDW\right]$$

(2.3)

By identifying the coefficients for $dt$ and $dW$ in the above stochastic equation, one obtains $\mu$ and $\sigma$ as a function of the derivatives of $D$:

$$\sigma(t, S_t) = \frac{\epsilon \partial_W D}{(1 - \epsilon S \partial_S D)}$$

(2.5)

$$\mu(t, S_t) = \frac{\epsilon [\partial_t D + \frac{1}{2} \sigma^2 S^2 \partial_S^2 D + \sigma S \partial_{SW} D + \frac{1}{2} \sigma \partial_W D]}{(1 - \epsilon S \partial_S D)}$$

(2.6)

We will assume that the market is composed of a group of small traders who don’t modify the prices of the market by selling or buying large amounts of assets on the one hand and a large trader one the other hand. One can consider a large trader as an aggregate of small traders following the same strategy given by the hedging position $\phi$.

We will choose the demand $D_{\text{small}}$ of small traders in order to reproduce the classic log-normal random walk motion (2.2) with a constant return $\mu = \mu_0$ and a constant volatility $\sigma = \sigma_0$. The simple solution is given by

$$D_{\text{small}} = \frac{1}{\epsilon} (\mu_0 t + \sigma_0 W)$$

(2.7)

Now, we include the effect of a large trader whose demand $D_{\text{large}}$ is generated by the trading strategy $D_{\text{large}} = \phi(t, S_t)$. It is implicitly assumed that $\phi$ depends only on $S$ and $t$ and not on the history of the Brownian motion. This hypothesis is proven in the appendix in the case of option pricing by using a Black-Scholes analysis. Moreover, we will see in the following section that this hypothesis is not strictly valid for our portfolio trading strategy. Indeed, $\phi$ will depend on $S$, $t$ and also the value of our portfolio $\Pi$. This last value depends implicitly on the history of the
Brownian motion. To simplify the discussion, we will take the mean value of the portfolio $\Pi$ in order to obtain a trading strategy which will depend only on $S$ and $t$.

The hedging position $\phi$ is then added to the demand of the small traders $D_{\text{small}}$ and the total net demand is given by

$$D = D_{\text{small}} + \phi(t, S_t) \quad (2.8)$$

By inserting (2.8) in (2.5)-(2.6), one finds the new volatility and return as a function of $\phi$:

$$\sigma = \frac{\sigma_0}{(1 - \epsilon S \partial_S \phi)} \quad (2.9)$$

$$\mu = \mu_0 + \epsilon(\partial_t \phi + \frac{1}{2} \sigma_0^2 S^2 S \partial^2_S \phi) \quad (2.10)$$

This relation describes the feedback effect of dynamical hedging on volatility and return. The hypothesis that the hedging function depends only on $S$ and $t$ allows to obtain a volatility and a return independent of historic effects. For derivatives and portfolio hedging strategies, the feedback effect on the volatility will not be the same.

Let’s take the case of derivatives hedging first. If the price rises, then $\partial_S \phi(S, t) > 0$ meaning that the trader buys additional shares. For completeness, it is shown in the appendix that the Black-Scholes relation $\phi = \partial_S C$ (with $C$ the value of an option) is still valid and the Gamma derivative $\Gamma = \partial_S \phi(S, t) = \partial^2_S C$ is positive. The volatility $\sigma$ is then greater than $\sigma_0$ and the price decreases. A destabilizing effect is then obtained.

On the other hand, in dynamical portfolio optimization, we have the expression $\partial_S \phi(S, t) < 0$ (as we will see in the next section) according to which a trader should sell stocks when his price increases and buy more stocks when his price decreases. As a result $\sigma < \sigma_0$ and the price increases. In this situation, a trader could buy large amounts of shares at a price $S$ and the price would then move to a higher price $S'$. By selling his shares, it would make a free-risk profit $S' - S$ per share.

The key feature that allows this manipulation is that the price reacts with a delay that allows the trader to buy at low price and sell at a higher price before the price goes down. It is shown in $11$ that to prevent manipulation strategies, the price must not react with delays.

In $6$, a slightly different approach is used to model the influence of the hedging on the volatility and the return. We will show that this approach can lead to the same result as ours for the volatility. One considers the difference $\chi$ between the demand $D(t, S_t, W)$ and the supply $S(t, S_t, W)$ which is the total number of shares available in the market:

$$\chi(t, S_t, W) = D - S \quad (2.11)$$

$^a$The zero-order solution of (2.2) (in the parameter $\epsilon$) is $S_t = S_0 e^{(\mu_0 - \frac{1}{2} \sigma_0^2) t + \sigma_0 W}$ and $S_t$ decreases as $\sigma_0$ increases.
It is clear that if $\chi$ increases (resp. decreases), the price will increase (resp. decrease) and the equilibrium price is reached when $\chi = 0$. By taking the derivative of $\chi$ (d$\chi = 0$, see (2.3)), one can then express $\mu$ and $\sigma$ appearing in (2.2) as a function of $\chi$:

$$\sigma = -\frac{\partial W}{S \partial S}$$  \hspace{1cm} (2.12)$$

$$\mu = -\frac{(\partial_t \chi + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 S}{\partial S^2} \chi + \frac{1}{2} \sigma^2 W \chi + \sigma S \partial SW D)}{S \partial S D}$$  \hspace{1cm} (2.13)$$

The $\chi$ for the small traders is then chosen in order to find a constant return and volatility. One should note that the solution is not unique. One can take (this is not the choice of 6)

$$\chi_{\text{small}}(S, W, t) = \frac{1}{\epsilon}(\epsilon u_0 - \frac{1}{2} \sigma^2_0 t + \sigma_0 W - \ln(S))$$  \hspace{1cm} (2.14)$$

If we include a large trader with a hedging position $\phi(t, S)$, the equilibrium equation is easily modified as

$$\chi_{\text{small}}(t, S, W) + \phi(t, S) = 0$$  \hspace{1cm} (2.15)$$

The modified volatility $\sigma$ then reproduces precisely the formula (2.9) and the return differs slightly from (2.10) and is given by:

$$\mu = \frac{\mu_0 + \frac{1}{2} \sigma^2 + \epsilon(\partial_t \phi + \frac{1}{2} \sigma^2 S^2 \partial^2 S \phi)}{1 - \epsilon S \partial S \phi}$$  \hspace{1cm} (2.16)$$

By using a different $\chi_{\text{small}}$, 6 found expressions for the volatility and the return which differ from (2.9)-(2.10)-(2.16).

The parameters $\mu$, $\sigma$ can be written as a formal series in $\epsilon$ and we will see in the following section that this is also the case for the hedging function $\phi = \sum_{i=0}^{\infty} \phi_i(S, t) \epsilon^i$.

We give the first order expansion of $\sigma$ (2.9) and $\mu$ (2.10):

$$\sigma = \sigma_0(1 + \epsilon S \partial S \phi) + o(\epsilon^2)$$  \hspace{1cm} (2.17)$$

$$\mu = \mu_0 + \epsilon(\partial_t \phi + \frac{1}{2} \epsilon \sigma^2 S^2 \partial^2 S \phi + \mu_0 S \partial S \phi) + o(\epsilon^2)$$  \hspace{1cm} (2.18)$$

The equation above will be used throughout the next section.

3. Optimized portfolio of a large trader

Let’s assume that a large trader holds at time $t$ a certain number of shares $\phi(S, t)$ of which the price at time $t$ is $S_t$. The price $S_t$ satisfies the stochastic differential equation (2.2) which depends implicitly on the hedging position $\phi(S, t)$ (see the volatility (2.9) and the return (2.10)). The trader holds also a number $b_t$ of bonds the price of which satisfies the following deterministic equation $dB_t = r(t) B_t dt$ with $r(t)$ the interest rate of the bond. The change in his portfolio $\Pi = \phi(S, t) S + b_t B_t$ during an elementary time $dt$ is then
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\[ d\Pi = \phi(t, S_t) dS_t + r(t) b_t dB_t dt \] (3.19)

\[ = r(t) \Pi dt + \phi(t, S_t) (\mu - r(t)) dt + \sigma dW \] (3.20)

One should note that \( \phi \) can be negative (which is equivalent to a short position).

At a maturity date \( T \), the value of the portfolio \( \Pi_T \) is

\[ \Pi_T = \Pi_0 + \int_0^T d\Pi \] (3.21)

The basic strategy of the trader is to find the optimal strategy \( \phi^*(S, t) \) so that the risk \( R \) is minimized for a given fixed value of the profit \( E[\Pi_T] = \mathcal{G} \) with \( E[\cdot] \) the mean operator. By definition, the risk is given by the variance of the portfolio \( \Pi_T \):

\[ R = E[\Pi_T^2] - E[\Pi_T]^2 = E[\Pi_T^2 - \mathcal{G}^2] \] (3.22)

It is clear that the result of this paper depends on the above mentioned definition for the risk, its main advantage being that it gives simple computations. The mean-variance portfolio selection problem is then formulated as the following optimization problem parameterized by \( \mathcal{G} \):

\[ \min E[\Pi_T^2 - \mathcal{G}^2] \] (3.23)

subject to

\[ E[\Pi_T] = \mathcal{G} , , (S(.), \Pi(.)) \text{ satisfying (2.2) - (3.20)} \] (3.24)

This problem is equivalent to the following one in which we have introduced a Lagrange multiplier \( \zeta \):

\[ \min E[\Pi_T^2 - \mathcal{G}^2 + \zeta (\Pi_T - \mathcal{G})] \] (3.25)

subject to

\[ (S(.), \Pi(.)) \text{ satisfying (2.2) - (3.20)} \] (3.26)

To simplify, we will assume in the following that the interest rates are negligible (ie \( r(t) = 0 \)) although this hypothesis can be easily relaxed. To derive the optimality equations, the above problem can be restated in a dynamic programming form so that the Bellman principle of optimality can be applied \(^{12,13}\). To do this, let’s define

\[ J(t, S, \Pi) = \min E_t[\Pi_T^2 - \mathcal{G}^2 + \zeta (\Pi_T - \mathcal{G})] \] (3.27)

where \( E_t \) is the conditional expectation operator at time \( t \) with \( S_t = S \) and \( \Pi_t = \Pi \). Then, \( J \) satisfies the famous Hamilton-Jacobi-Bellman equation:

\[ \min_{\phi, \zeta} \{ \partial_t J + \mu S \partial_S J + \frac{1}{2} \sigma^2 S^2 \partial^2_S J + \mu S \phi \partial_{\Pi} J + \frac{1}{2} \phi^2 \sigma^2 S^2 \partial^2_{\Pi} J + \phi \sigma^2 S^2 \partial_S \partial_{\Pi} J = 0 \} \] (3.28)
subject to

\[ J(T, S, \Pi) = \Pi_T^2 - G^2 + \zeta (\Pi_T - G) \]  
\[ = (\Pi_T + \frac{\zeta}{2})^2 - G^2 - \zeta G - \frac{\zeta^2}{4} \]  

The equation (3.28) is quite complicated due to the explicit dependence of the volatility and the return in the control \( \phi \). A simple method to solve the H-J-B equation (3.28) perturbatively in the liquidity parameter \( \epsilon \) is to use a mean-field approximation well known in statistical physics: one computes the optimal control \( \phi^*(S, t) \) up to a given order \( \epsilon^i \), say \( \phi^*_i(S, t) \). This gives a mean volatility \( \sigma_i(S, t) = \sigma(\phi^*_i(S, t), S, t) \) and mean return \( \mu_i(S, t) = \mu(\phi^*_i(S, t), S, t) \) up to a given order \( \epsilon^{i+1} \). The optimal control, at the order \( \epsilon^{i+1} \), is then easily found by taking the functional derivative of the expression (3.28) according to \( \phi \):

\[ \phi^*_{i+1} = - \frac{(\mu_i \partial_n J + \sigma_i^2 S \partial_S J)}{\sigma_i^2 S \partial_n J} \]  

By inserting this expression in the H-J-B equation (3.28), one obtains:

\[ \partial_t J + \mu_i S \partial_S J + \frac{1}{2} \sigma_i^2 S^2 \partial_S^2 J - \frac{(\mu_i \partial_n J + \sigma_i^2 S \partial_S J)^2}{2 \sigma_i^2 \partial_n J^2} = 0 \]  

The resolution of the above equation gives \( \phi(S, t)_{i+1} \) and the procedure can be restarted. We will now apply this method up to order one. The form of (3.32) and the boundary conditions (3.30) suggest than \( J \) takes the following form:

\[ J(t, S, \Pi) = A(t, S)(\Pi + \frac{\zeta}{2})^2 - G^2 - \zeta G - \frac{\zeta^2}{4} \]  

with \( \zeta \) such as \( \partial_\zeta J = 0 \) and with the boundary condition \( A(T, S) = 1 \).

By inserting this expression in (3.32) and (3.31), one then obtains the equations:

\[ \phi^*(u, t) = -S^{-1}(\frac{\mu_i}{\sigma_i^2} + \partial_u X)(\Pi + \frac{\zeta}{2}) \]  
\[ \partial_t X - (\mu_i + \frac{1}{2} \sigma_i^2) \partial_u X - \frac{1}{2} \sigma_i^2 (\partial_u X + (\partial_u X)^2) = \frac{\mu_i^2}{\sigma_i^2} \]  

with \( u = \ln(S) \) and \( A(S, t) = e^{X(S, t)} \). The Lagrange condition \( \partial_\zeta J = 0 \) gives

\[ \Pi + \frac{\zeta}{2} = \frac{G - \Pi}{e^X - 1} \]  

Let's define the parameter \( \lambda \equiv \frac{\mu_i^2}{\sigma_i^2} \). In the zero-order approximation, the volatility and return are constant and in this case, \( A \) will be a function of \( t \) only given by

\[ A(t, S) = e^{\lambda(t-T)} \]
By inserting this expression in (3.34) and using (3.36), one finds the optimal control (at zero-order):

\[ \phi^*(S, t, \Pi) = \frac{\lambda}{\mu_0} S^{-1} \frac{\Pi - G}{e^{\lambda(t-T)} - 1} \] (3.38)

This expression allows to derive a stochastic equation for the portfolio \(\Pi\) subject to the condition \(\Pi(0) = \Pi_0\):

\[ \frac{d\Pi}{(\Pi - G)} = \frac{1}{(e^{\lambda(t-T)} - 1)} (\lambda dt + \sqrt{\lambda} dW) \] (3.39)

The mean of the portfolio \(E[\Pi]\) is then given:

\[ \frac{dE[\Pi]}{dt} = \lambda E[\Pi] - G \] (3.40)

The solution is

\[ E[\Pi] - G = (G - \Pi_0) \frac{1 - e^{-\lambda(t-T)}}{(e^{\lambda T} - 1)} \] (3.41)

One can then verify that the condition \(E[\Pi_T] = G\) is well satisfied. Finally, the optimal control \(\phi_0^*\) at order zero satisfies the following stochastic equation:

\[ \frac{d[\phi_0^*S]}{\phi_0^*S} = -\lambda t + \frac{\sqrt{\lambda}}{(e^{\lambda(t-T)} - 1)} dW_t \] (3.42)

and the solution is

\[ \phi_0^* = \frac{\lambda}{\mu_0} \frac{S^{-1} (G - \Pi_0)}{(1 - e^{-\lambda T})^2} \left(1 - e^{-\lambda(t-T)}\right)^{1/2} e^{-\lambda t} \frac{e^{-\lambda t/2}}{e^{\lambda(t-T)/2}} + \frac{1}{e^{\lambda t/2}} \int_0^t e^{\lambda(t-T)/2} dW_t \] (3.43)

As explained in the first section, the hedging strategy depends explicitly on the history of the Brownian motion. To obtain a return and a volatility that depend only on \(S\) and \(t\), we will take \(\bar{\phi}_0^*(S, t) = \phi^*(S, t, E[\Pi])\) as the demand of the large trader:

\[ \bar{\phi}_0^*(S, t) = \frac{\lambda}{\mu_0} S^{-1} (G - \Pi_0) \frac{e^{-\lambda t}}{(1 - e^{-\lambda T})} \] (3.44)

We should note that the only models considered by Merton are models of dynamical portfolio optimization with no age effects 13.

Let’s derive now the one-order correction to this hedging function. First, we give from (2.17)-(2.18) \(\frac{\mu^2}{\sigma^2}\) up to the first order in \(\epsilon\) (using (3.44)):

\[ \frac{\mu^2}{\sigma^2} = \frac{1}{\sigma_0^2} \left( \mu_0^2 + 2\epsilon \partial_t \phi^*_0 + \frac{1}{2} \sigma_0^2 S^2 \partial^2 S^2 \phi^*_0 \right) \] (3.45)

\[ = \frac{1}{\sigma_0^2} \left( \mu_0^2 + 2\epsilon \phi^*_0 (-\lambda + \sigma_0^2) \right) \] (3.46)
Then the equation (3.35) at order one is

\[ \partial_t X_1 - (\mu_0 + \frac{1}{2} \sigma_0^2) \partial_u X_1 - \frac{1}{2} \sigma_0^2 \partial_u^2 X_1 = \frac{2 \tilde{\phi}_0^*}{\sigma_0^2} (-\lambda + \sigma_0^2) \]  

(3.47)

with boundary condition \( X_1(0, u) = 0 \). The solution is then given by

\[ X_1(S, t) = 2S^{-1} \frac{(G - \Pi_0)}{(e^{\lambda T} - 1)} \frac{\lambda}{\mu_0} \frac{\sigma_0^2 - \lambda}{\mu_0 - \lambda} \frac{X_1}{e^{\lambda (T-t)} - e^{\mu_0 (T-t)}} \]  

(3.48)

and after a sightly long computation we find the correction to \( \phi_0^* \) which depends on \( S, t \) and \( \Pi \):

\[ \phi^*(S, t, \Pi) = \frac{(\Pi - G) S^{-1} \lambda}{\mu_0 (e^{\lambda (T-t)} - 1)} \frac{\sigma_0^2 - \lambda}{\mu_0 - \lambda} \frac{X_1}{1 - e^{-\lambda (T-t)}} \]  

(3.49)

To obtain the next corrections for the implied volatility and return, on should take \( \phi^*(S, t, E[\Pi]) \) as the demand of the large trader. We have found that the large trader strategy induces an implied volatility which is given at the first-order by:

\[ \sigma_{\text{implied}} = \sigma_0 (1 - \epsilon \alpha e^{-\lambda T} S^{-1}) + o(\epsilon^2) \]  

(3.50)

with \( \alpha \) a constant depending on the characteristic of the portfolio \( (\Pi_0, G, T, \mu_0, \sigma_0) \).

4. Conclusion

In this note, we have explained how to incorporate easily the effect of the hedging strategy in the market prices. The derivative hedging gives a negative effect and the portfolio optimization hedging a positive one. The main difficulty with our optimization scheme was that the hedging function depends on age effects, which was simply solved by taking the mean over the portfolio value for \( S \) fixed. As explained the portfolio optimization and the option pricing lead to relatively simple non-linear partial differential equations (3.32)-(A.4) that can be numerically solved.

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Appendix: Modified Black-Scholes PDE

The Black-Scholes analysis can be trivially modified to incorporate the hedging feedback effect. Let’s \( \Pi \) be the portfolio and \( C(t, S_t) \) the value of the option at time \( t \):

\[ \Pi = -C(t, S_t) + \phi(t, S_t) S_t \]  

(A.1)
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As a result of Ito’s lemma, $d\Pi$ is given by

$$d\Pi = -dC + \phi dS_t$$

$$= -(\partial_t C + \frac{1}{2} \sigma^2 S^2 \partial^2_{S^2} C) dt + (\phi - \partial_S C) dS_t$$

(A.2)

(A.3)

A free-arbitrage condition gives $d\Pi = r(t)\Pi(t)$ with $r$ the interest rate. The risk is therefore zero iff $\phi = \partial_S C$ which is the usual hedging position. The Black-Scholes PDE (modulo adequate boundary conditions) is then

$$\partial_t C + \frac{1}{2} \sigma^2 S^2 \partial^2_{S^2} C + r(-C + S \partial_S C) = 0$$

(A.4)

This non-linear PDE can be solved by expanding $C$ (and so $\phi$) as a formal sum in $\epsilon$. A numerical computation can also be quite interesting and leads to the determination of an implied volatility.

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