Diquark condensation effects on hot quark star configurations

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Abstract

The equation of state for quark matter is derived for a nonlocal, chiral quark model within the mean field approximation. We investigate the effects of a variation of the formfactors of the interaction on the phase diagram of quark matter. Special emphasis is on the occurrence of a diquark condensate which signals a phase transition to color superconductivity and its effects on the equation of state under the condition of β-equilibrium and charge neutrality. We calculate the quark star configurations by solving the Tolman- Oppenheimer- Volkoff equations and obtain for the transition from a hot, normal quark matter core of a protoneutron star to a cool diquark condensed one a release of binding energy of the order of $\Delta M c^2 \sim 10^{53}$ erg. We find that this energy could not serve as an engine for explosive phenomena since the phase transition is not first order. Contrary to naive expectations the mass defect increases when for a given temperature we neglect the possibility of diquark condensation.

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1 Introduction

Color superconductivity (CS) in quark matter [1] is one interesting aspect of
the physics of compact star interiors [2]. Since calculations of the energy gap
of quark pairing predict a value $\Delta \sim 100$ MeV and corresponding critical tem-
peratures for the phase transition to the superconducting state are expected
to follow the BCS relation $T_c = 0.57 \Delta$, the question arises whether diquark
condensation can lead to remarkable effects on the structure and evolution
of compact objects. If positively answered, CS of quark matter could provide
signatures for the detection of a deconfined phase in the interior of compact
objects (pulsars, low-mass X-ray binaries) along the lines of previously sug-
gested strategies [3,4,5,6,7].

Diquarks gained popularity in astrophysics about a decade ago, when they
were suggested to influence the supernova collapse and “bounce-off” [8,9,10,11,12],
and to enhance the neutrino cooling of quark-stars. The latter effect is now
subject to much research within improved scenarios Refs. [13,14,15]. Possible
consequences for the magnetic field and glitch-type phenomena are discussed
in [16,17,18].

Hong, Hsu and Sannino have conjectured [19] that the release of binding energy
due to Cooper pairing of quarks in the course of protoneutron star evolution
could provide an explanation for the unknown source of energy in supernovae,
hypernovae or gamma-ray bursts, see also [20]. Their estimate of energy release
was based on the simple formula

$$\Delta E \sim \left( \frac{\Delta}{\mu} \right)^2 M c^2 \sim 10^{53} \text{ erg},$$

which did not take into account the change in the gravitational binding energy
due to the change in the structure of the stars quark core. In the present work
we will reinvestigate the question of a possible binding energy release due to
a CS transition by taking into account changes in the equation of state (EoS)
and the configuration of the quark star selfconsistently.

As a first step in this direction we will discuss here the two flavor color super-
conducting (2SC) quark matter phase which occurs at lower baryon densities
than the color-flavor-locking (CFL) one, see [21,22]. Studies of three-flavor
quark models have revealed a very rich phase structure (see [22] and refer-
ences therein). However, for applications to compact stars the omission of the
strange quark flavor within the class of nonlocal chiral quark models consid-
ered here may be justified by the fact that central chemical potentials in stable
star configurations do barely reach the threshold value at which the mass gap
for strange quarks breaks down and they appear in the system [23]. Therefore
we will not discuss here first applications to calculate compact star configurations with color superconducting quark matter phases that have employed non-dynamical quark models [24,25]. Their results depend on the somewhat arbitrary choice of the strange quark mass $m_s$, the diquark gap $\Delta$, and the bag constant. Dynamical calculations like the one we are going to present here fix their model parameters with the constraint to describe observable hadron properties. They include a selfconsistent solution of the chiral and diquark gap equations which shows that the bulk of a compact star interior does not contain strange quark matter [23,26].

We will investigate the influence of the formfactor of the interaction on the phase diagram and the EoS of dense quark matter under the conditions of charge neutrality and isospin asymmetry due to $\beta$-equilibrium relevant for compact stars.

Finally we consider the question whether the effect of diquark condensation which occurs in the earlier stages of the compact star evolution ($t \simeq 100$ s) [15,27,28] at temperatures $T \sim T_c \sim 20 - 50$ MeV can be considered as an engine for explosive astrophysical phenomena like supernova explosions due to the release of a binding energy of about $10^{52} \div 10^{53}$ erg, as has been suggested before [19,20].

2 Thermodynamic potential for asymmetric 2SC quark matter

We consider a nonlocal chiral quark model described by the effective action

$$S[\bar{\psi},\psi] = \sum_p \bar{\psi}(\not{p} - \hat{m})\psi + S_{qq}[\bar{\psi},\psi] + S_{q\bar{q}}[\bar{\psi},\psi], \quad (2)$$

$$S_{qq}[\bar{\psi},\psi] = G_1 \sum_{p,p'} \left[ (\bar{\psi}g(p)\psi)(\bar{\psi}g(p')\psi) + (\bar{\psi}i\gamma_5 \vec{r}g(p)\psi)(\bar{\psi}i\gamma_5 \vec{r}g(p')\psi) \right], \quad (3)$$

$$S_{q\bar{q}}[\bar{\psi},\psi] = G_2 \sum_{p,p'} (\bar{\psi}i\gamma_5 \tau_2 \lambda_2 C g(p)\not{\psi}) (\psi^T C i\gamma_5 \tau_2 \lambda_2 g(p')\psi), \quad (4)$$

which generalizes the approach of Ref. [29] by including the scalar diquark pairing interaction channel with a coupling strength $G_2$. Here, the matrix $C = i\gamma_0 \gamma_2$ is the charge conjugation matrix operator for fermions, the matrices $\tau_2$ and $\lambda_2$ are Pauli and Gell-Mann matrices acting in, respectively, the SU(2) flavor and SU(3) color spaces; $\hat{m}$ is the diagonal current quark mass matrix $\hat{m} = \text{diag}(m_u, m_d)$. The function $g(p)$ is the formfactor of the separable four-fermion interaction. In the present work it is assumed to be instantaneous, so that it depends only on the modulus of the three-momentum. After bosonization and integrating out the quark fields, the mean field approximation introduces the following order parameters: the diquark gap $\Delta$, which can
be seen as the gain in energy due to diquark condensation, and the mass gaps \( \phi_u, \phi_d \), which indicate chiral symmetry breaking. The grand canonical thermodynamic potential per volume can be obtained as

\[
\Omega_q(\phi_u, \phi_d, \Delta; \mu_u, \mu_d, T) = -T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \text{Tr} \ln \left( \frac{1}{T} \tilde{S}^{-1}(i\omega_n, \vec{p}) \right) + \frac{\phi_u^2 + \phi_d^2}{8 \, G_1} + \frac{|\Delta|^2}{4 \, G_2},
\]

(5)

where \( \omega_n = (2n+1)\pi T \) are the Matsubara frequencies for fermions. The chemical potential matrix \( \hat{\mu} = \text{diag}(\mu_u, \mu_d) \) as well as the dynamical quark mass matrix \( \hat{M} = \text{diag}(m_u + \phi_u g(p), m_d + \phi_d g(p)) \) are included into the definition of the inverse quark propagator

\[
\tilde{S}^{-1}(p_0, \vec{p}) = \begin{pmatrix}
\hat{\mu} - \hat{\hat{M}} - \hat{\mu} \gamma_0 & \Delta \gamma_5 \tau_2 \lambda_2 g(p) \\
-\Delta^* \gamma_5 \tau_2 \lambda_2 g(p) & \hat{\mu} - \hat{\hat{M}} + \hat{\mu} \gamma_0
\end{pmatrix}.
\]

(6)

One can combine the chemical potentials \( \mu_u, \mu_d \) of \( u \) and \( d \) quarks by introducing \( \mu_q = (\mu_u + \mu_d) / 2 \) and \( \mu_I = (\mu_u - \mu_d) / 2 \) as the Lagrange multipliers related to, respectively, the quark number density \( n_q \) and the isospin asymmetry \( n_I \). One can easily check the relations

\[
n_q = n_u + n_d = -\partial \Omega / \partial \mu_q, \quad n_I = n_u - n_d = -\partial \Omega / \partial \mu_I.
\]

Following Ref. [30] we use in the following that the chiral gaps of both flavors are degenerate \( \phi_u = \phi_d = \phi \) (and therefore also the masses \( m_u = m_d = m \)). The trace operations in the bispinor, flavor and color spaces as well as the Matsubara summations can be performed with the following result for the thermodynamic potential of asymmetric quark matter [31,32,33]

\[
\Omega_q(\phi, \Delta; \mu_q, \mu_I, T) = \frac{\phi^2}{4 \, G_1} + \frac{\Delta^2}{4 \, G_2} - 2 \int_0^{\infty} \frac{q^2 \, dq}{2\pi^2} \left\{ 2 \, E_\phi \ight. \\
+ T \ln \left[ 1 + \exp \left( -\frac{E_\phi - \mu_q - \mu_I}{T} \right) \right] + T \ln \left[ 1 + \exp \left( -\frac{E_\phi - \mu_q + \mu_I}{T} \right) \right] \\
+ T \ln \left[ 1 + \exp \left( -\frac{E_\phi + \mu_q - \mu_I}{T} \right) \right] + T \ln \left[ 1 + \exp \left( -\frac{E_\phi + \mu_q + \mu_I}{T} \right) \right] \\
- 4 \int_0^{\infty} \frac{q^2 \, dq}{2\pi^2} \left\{ E_+ + E_- \right. \\
+ T \ln \left[ 1 + \exp \left( -\frac{E_- - \mu_I}{T} \right) \right] + T \ln \left[ 1 + \exp \left( -\frac{E_- + \mu_I}{T} \right) \right] \\
+ T \ln \left[ 1 + \exp \left( -\frac{E_+ - \mu_I}{T} \right) \right] + T \ln \left[ 1 + \exp \left( -\frac{E_+ + \mu_I}{T} \right) \right] \right\} \]

(7)

\[- \Omega_q^{\text{vac}}.\]
The number of color degrees of freedom in our model is fixed to be $N_c = 3$. Only the coupling constant $G_1$ is considered to be a free parameter of the model, while $G_2$ could be chosen according to Ref. [34] as $G_2 = G_1 N_c/(2N_c - 2)$. We discuss later the special choices $G_2 = 0.75 G_1$ and $G_2 = G_1$.

The dispersion relation $E_\phi = \sqrt{q^2 + (m + \phi g(q))^2}$ holds for unpaired quarks also in color-symmetry broken 2SC phase. The quarks with paired colors have a modified dispersion relation

$$E_{\pm} = (E_\phi \pm \mu_q) \sqrt{1 + \frac{\Delta^2 g^2(q)}{(E_\phi \pm \mu_q)^2}}.$$ 

For nonvanishing $\Delta$ in Eq. (7) the color symmetry is broken. Two of the three quark color degrees of freedom are coupled to bosonic Cooper pairs in the color antitriplet state which can form a Bose condensate. The dispersion relation for quarks of these colors contains the diquark gap $\Delta$ and becomes

$$E_{f\pm}^2 = (E_{\phi f} \pm \mu_f)^2 + \Delta^2 g^2(q).$$ (8)

For quarks of the remaining third color there is no pairing gap in their dispersion relation which reads

$$E_{\phi f} = \sqrt{q^2 + (m_f + \phi_f g(q))^2}.$$ (9)

In thermal equilibrium characterized by a fixed set of thermodynamic variables $\{\mu_u, \mu_d, T\}$, the thermodynamic potential $\Omega_q$ shall attain a global minimum with respect to all its order parameters. In the corresponding state the gap equations for the order parameters are fulfilled. However, the opposite is not always true. Satisfying the gap equations is a necessary, but not a sufficient condition for the thermodynamic equilibrium state.

In Eq. (7), we subtract the term $\Omega_q^{\text{vac}}$ in order to define the thermodynamic potential such that the vacuum pressure vanishes,

$$P_q(0, 0, 0) = 0.$$ (10)

On the mean-field level the present approach is analogous to a flavor-asymmetric generalization of the instanton-motivated model of Ref. [34] for two-flavor color superconductivity (2SC).

From BCS theory it is known, that in order to form Cooper pairs at $T = 0$ in a dense Fermi system, the difference in the chemical potentials of the Fermions to be paired should not exceed the size of the gap. As previous calculations
within this type of models have shown [34], there is a critical chemical potential for the occurrence of quark matter \( \mu_f \geq 300 \text{ MeV} \) and values of the gap in the region \( \Delta \leq 150 \text{ MeV} \) have been found. Therefore it is natural to consider the problem of the color superconducting (2SC) phase with the assumption, that quark matter is symmetric or very close to being symmetric \( (\mu_u \simeq \mu_d) \).

The number densities derived from thermodynamic potential can be expressed in an integral form as

\[
n_q(\mu_q, \mu_I, T) = 2 \int_0^\infty \frac{q^2 \, dq}{2\pi^2} \left\{ f(E_\phi - \mu_q - \mu_I, T) + f(E_\phi - \mu_q + \mu_I, T) \ight. \\
- f(E_\phi + \mu_q - \mu_I, T) - f(E_\phi + \mu_q + \mu_I, T) \\
- 2 \partial E_+/\partial \mu_q [f(E_+ - \mu_I, T) + f(E_+ + \mu_I, T) - 1] \\
- 2 \partial E_-/\partial \mu_q [f(E_- - \mu_I, T) + f(E_- + \mu_I, T) - 1]\right\} \tag{11}
\]

and

\[
n_I(\mu_q, \mu_I, T) = 2 \int_0^\infty \frac{q^2 \, dq}{2\pi^2} \left\{ f(E_\phi - \mu_q - \mu_I, T) - f(E_\phi - \mu_q + \mu_I, T) \\
+ f(E_\phi + \mu_q - \mu_I, T) - f(E_\phi + \mu_q + \mu_I, T) \\
+ 2 [f(E_+ - \mu_I, T) - f(E_+ + \mu_I, T)] \\
+ f(E_- - \mu_I, T) - f(E_- + \mu_I, T)]\right\} \tag{12}
\]

Here, \( f(E, T) = [1 + \exp(E/T)]^{-1} \) is the Fermi distribution function and the derivatives are given by \( \partial E_\pm/\partial \mu_q = (\mu_q \pm E_\phi)/E_\pm \).

Using the entropy density \( s_q = -\partial \Omega_q/\partial T \), the pressure \( P_q = -\Omega_q \) and the thermodynamic identity \( P_q + \varepsilon_q = s_q T + \mu_q n_q + \mu_I n_I \), the EoS for the energy density as a function of temperature and chemical potentials becomes

\[
\varepsilon(\mu_q, \mu_I, T) = 2 \int_0^\infty \frac{q^2 \, dq}{2\pi^2} \left\{ E_\phi [f(E_\phi - \mu_q - \mu_I, T) + f(E_\phi - \mu_q + \mu_I, T) \\
+ f(E_\phi + \mu_q - \mu_I, T) + f(E_\phi + \mu_q + \mu_I, T) - 2] \\
+ 2 (E_+ - \mu_q \partial E_+/\partial \mu_q) [f(E_+ - \mu_I, T) + f(E_+ + \mu_I, T) - 1] \\
+ 2 (E_- - \mu_q \partial E_-/\partial \mu_q) [f(E_- - \mu_I, T) + f(E_- + \mu_I, T) - 1] \\
+ \phi^2/4 \, G_1 + \Delta^2/4 \, G_2 - \Omega_{\text{vac}} \right\} \tag{13}
\]

In the next sections we will apply this model for the construction of the EoS for stellar matter in \( \beta \)-equilibrium.
3 EoS for 2SC quark matter in $\beta$-equilibrium

We consider stellar matter in the quark core of compact stars consisting of electrons in chemical equilibrium with $u$ and $d$ quarks. Hence $\mu_d = \mu_u + \mu_e$, where $\mu_e$ is the electron chemical potential. The thermodynamic potential of such matter is

$$\Omega(\phi, \Delta; \mu_q, \mu_I, \mu_e, T) = \Omega_q(\phi, \Delta; \mu_q, \mu_I, T) + \Omega_e(\mu_e, T), \quad (14)$$

with

$$\Omega_e(\mu_e, T) = -\frac{1}{12\pi^2} \mu_e^4 - \frac{1}{6} \mu_e^2 T^2 - \frac{7}{180} \pi^2 T^4$$ \quad (15)

for ultrarelativistic electrons. The number density of electrons is then $n_e = -\partial \Omega / \partial \mu_e$. The condition of local charge neutrality

$$\frac{2}{3} n_u - \frac{1}{3} n_d - n_e = 0, \quad (16)$$

in the new notation reads

$$n_q + 3n_I - 6n_e = 0. \quad (17)$$

In our model we assume the neutrino chemical potential to be zero, which corresponds to a scenario where the neutrinos (and/or antineutrinos) leave the star as soon as they are created. The limits of such an approximation could be based on the calculations for the case of trapping of neutrinos in the hot stars interior [21]. For possible consequences of antineutrino trapping on hot quark star evolution, see [35].

In [36,32] it is shown that at given $\mu_q$ the diquark gap is independent of the isospin chemical potential for $|\mu_I(\mu_q)| < \mu_{Ic}(\mu_q)$, otherwise vanishes. Increase of isospin asymmetry forces the system to undergo a first order phase transition by tunneling through a barrier in the thermodynamic potential (14). In order to describe this transition we have to choose the appropriate set of Lagrange multipliers. Under the constraints of $\beta-$equilibrium and charge neutrality, the free enthalpy density can be written in the form

$$G = \mu_u n_u + \mu_d n_d + \mu_e n_e = \mu_B n_B, \quad (18)$$

where $\mu_B = \mu_u + 2\mu_d$ is the chemical potential conjugate to the baryon number density $n_B = (n_u + n_d)/3$ and remains as an independent thermodynamical
variable under the given constraints. Note that $\mu_B \neq \mu_q/3$ because of the isospin violation in weak interactions, which determine the equilibrium conditions for compact stars. This is in contrast to equilibrium conditions for dense matter in heavy-ion collisions. The corresponding set of equations of state is obtained from the thermodynamic potential (14) in the form

\begin{align}
P(T, \mu_B) &= -\Omega(T, \mu_B), \\
n_B(T, \mu_B) &= -\frac{\partial \Omega(T, \mu_B)}{\partial \mu_B}, \\
s(T, \mu_B) &= \frac{\partial \Omega(T, \mu_B)}{\partial T}, \\
\varepsilon(T, \mu_B) &= T s(T, \mu_B) - P(T, \mu_B) + \mu_B n_B(T, \mu_B),
\end{align}

where due to the above constraints the dependence on the set of thermodynamic potentials ($\mu_q, \mu_I, \mu_e$) is reduced to the dependence on $\mu_B$ only.

In concluding this section we would like to comment on the order of the CS phase transition within above approach. The phase transition to the 2SC phase coming from the hot normal state of matter by cooling the system occurs as a second order phase transition without a jump of concentrations, whereas the phase transition from an isospin asymmetric state to a superconducting one by decreasing the asymmetry is of first order and has a jump in the densities (in particular in the electron fraction).

4 Results of model calculations

4.1 EoS of quark matter in 2SC phase for finite temperature

The gaps have been calculated by minimizing the thermodynamic potential $\Omega$, Eq. (14), in the space of order parameters ($\phi, \Delta$) and the results are shown in Figs. 1 and 2.

In order to compare the effects of the quark interaction formfactors on the EoS, we study the Gaussian (G), Lorentzian (L) and cutoff (NJL) type formfactors defined as

\begin{align}
g_L(q) &= [1 + (q/\Lambda_L)^2\alpha]^{\alpha - 1}, \quad \alpha > 1, \\
g_G(q) &= \exp(-q^2/\Lambda_G^2), \\
g_{NJL}(q) &= \theta(1 - q/\Lambda_{NJL}).
\end{align}

The Lorentzian (L) interpolates between a soft (Gaussian-type, $\alpha \sim 2$) and a hard cutoff (NJL, $\alpha > 30$) depending on the value of the parameter $\alpha$. 

8
We will employ parametrisations of the nonlocal quark model which reproduce pion properties: mass $m_{\pi} = 140$ MeV, decay constant $f_{\pi} = 93$ MeV and which have the same quark mass gap $\phi(T = 0, \mu = 0) = 330$ MeV in the vacuum. The results for the parameterization are taken from [29], see Tab. 1. $G_2$ remains a free parameter of the approach. For $G_2 = G_1$, the 2SC phase with nonvanishing diquark condensate is obtained whereas for $G_2 = 0.75 G_1$ no 2SC phase is possible under neutron star conditions.

Figure 1 displays the $T = 0$ solutions of the chiral gap $\phi$ and diquark gap $\Delta$ for different formfactors in $\beta$–equilibrium for $G_2 = G_1$. For the densities relevant for stable star configurations, $\mu_q \leq 450$ MeV, the critical chemical potential $\mu_{q_c}$ for the chiral transition and for the onset of diquark condensation does depend on the type of the formfactor. The value of the diquark gap reaches $\Delta \sim 200$ MeV in the range of densities relevant for compact stars. It is rather insensitive to changes of the formfactor of the quark interaction.

Figure 2 shows the solutions for the chiral gap $\phi$ and the diquark gap $\Delta$ for the Gaussian model formfactor at different temperatures $T = 0, 40, 60$ MeV. We compare results in the chiral limit ($m_0 = 0$) with those for finite current quark mass $m_0 = 2.41$ MeV and observe that the diquark gap is not sensitive to the presence of the current quark mass, which holds for all formfactors in Eqs. (23)- (25). However, the choice of the formfactor influences the critical values of the phase transition as displayed in the quark matter phase diagram ($\mu_q - T$ plane) of Fig. 3, see also Fig. 1. A softer formfactor in momentum space gives lower critical values for $T_{c}$ and $\mu_{c}$ at the borders of chiral symmetry restoration and diquark condensation. The numerical results for the critical chemical potentials at $T = 0$ are summarised in Tab. 1 for the different formfactors.

The inset of Fig. 3 shows that the generalisation of the BCS relation $T_{c} \simeq 0.57 \Delta(T = 0, \mu_q) g(\mu_q)$, between the critical temperature $T_{c}$ of the superconducting phase transition and the pairing gap $\Delta$ at $T = 0$ is satisfactorily fulfilled in the domain of the phase diagram relevant for compact stars.

Figure 4 shows the resulting equation of state (EoS) for quark matter, as pressure $P$ versus baryochemical potential $\mu_B$ for the isotherms $T = 0, 40, 60$ MeV and in $\beta$-equilibrium. The Gaussian formfactor has been used. The Figure displays the defect of nonconfining quark models: at nonzero temperature quasifree quarks contribute to the pressure at low chemical potentials $\mu_B$ (corresponding to energy densities $\varepsilon \leq 300$ MeV/fm$^3$). The dot-dashed curve in Fig. 4 shows that the presence of a diquark condensate stabilizes the quark matter system and entails a hardening of the EoS.

In Figure 5 we show the matter distribution inside the quark star and the accumulated mass for quark stars with the same central baryon density $n_B(0) =$
5 \, n_0 \text{ for different temperatures. We have cut the density distribution at finite temperatures for } n_B < 0.5 \, n_0 \text{ due to the following reason. At nonzero temperatures the mass gap decreases as a function of the chemical potential already in the phase with broken chiral symmetry. Hence the model here gives unphysical low-density excitations of quasi-free quarks. A systematic improvement of this situation should be obtained by including the phase transition construction to hadronic matter. However, in the present work we circumvent the confinement problem by considering the quark matter phase only for densities } n_B > 0.5 \, n_0, \text{ where } n_0 \text{ is nuclear saturation density.}

4.2 Configurations of hot quark stars

Here we compare configurations with and without CS, in order to investigate the effect of diquark condensation on the total energy of a quark star and to decide whether the corresponding phase transition could serve as an engine for explosive phenomena like supernovæ and gamma ray bursts. The CS transition can occur during the cooling evolution of a hot protoneutron star [15,27]. We suppose that the approximation of an isothermal temperature distribution [40] in the star interior is sufficient for the estimate of the mass defect \(^2\). In the present paper we consider selfbound configurations of pure quark matter as models for quark cores in hybrid stars which should be the general case to be discussed in a separate paper, see e.g. Ref. [41] for \(T = 0\) configurations. The extrapolation from pure quark star configurations to hybrid ones, however, is strongly model dependent and cannot be done without a calculation. Quark matter effects are expected to be still valid in hybrid stars when rescaled with a monotonously rising function of the ratio of the quark core volume to the quark star volume.

The spherically symmetric, static star configurations are defined by the well-known Tolman-Oppenheimer-Volkoff equations [38] for the mechanical equilibrium of self-gravitating matter (see also [37,39])

\[
\frac{dP(r)}{dr} = - \frac{[\varepsilon(r) + P(r)][m(r) + 4\pi r^3 P(r)]}{r[r - 2m(r)]}.
\] (26)

Here \(\varepsilon(r)\) is the energy density and \(P(r)\) the pressure at the distance \(r\) from the center of the star. The mass enclosed in a sphere with radius \(r\) is defined by

\[
m(r) = 4\pi \int_0^r \varepsilon(r')r'^2dr'.
\] (27)

\(^2\) For a discussion of isentropic hot quark star configurations, see [37].
These equations are solved with the boundary condition of a given central baryon number density \( n_B(0) \) defining a configuration. In Fig. 6 we show the results for the dependence of the masses on the central energy density as well as on the radius of the configurations for different temperatures and two formfactor models: a Gaussian and a Lorentzian one. Stable configurations correspond to the rising branch on the mass-central density plane. In Tab. 2 we give for fixed temperatures \( T = 0, 40, 60 \) MeV the values of maximal possible mass, corresponding radius, central baryon number density and energy density for a Gaussian and a Lorentzian formfactor, respectively. For comparison, we have included \( T = 0 \) configurations for a small coupling constant \( G_2 = 0.75 G_1 \), for which the diquark condensate does not occur. The comparison shows that the maximum mass decreases with increasing temperature due to a softening of the EoS. Also the disappearance of the diquark condensate (in our case due to lowering of the coupling \( G_2 \)) softens the EoS and lowers the maximum mass. The Gaussian formfactor has the lowest critical baryochemical potential when compared with the Lorentzian and the NJL model. Therefore the Gaussian quark star configuration can extend to larger radii where the density is lower and can thus carry more mass than the Lorentzian model with the same central baryon density, see Fig.6.

Since in our calculations we do not consider the existence of a possible hadronic shell of the star where the transition to the phase with broken chiral symmetry and confinement occurs, we introduce a phenomenological cutoff in the baryon number density at the value \( n_B = 0.5 \) \( n_0 \), see Fig. 5. With this condition we define the radius of the quark core \( R \). The total mass \( M \) of the object is defined by the sphere with that radius \( M = m(R) \). It is easy to see from that Figure that temperature plays no role for the mechanical equilibrium of the quark core close to the center of the configuration due to its high density. It has an influence, however, on the peripheral region and changes the total mass for a given central density, see Figs. 6. We have checked numerically, that a change of the somewhat arbitrary cutoff density does not change the qualitative result for the energy release reported below.

For the discussion of evolutionary variations of the internal structure during the cooling it is important to take into account the conservation of the total baryon number of the star. Therefore, one has to compare configurations with different central densities, but with the same baryon number \( N_B \), see Figs. 7, 8.

Here we will discuss two scenarios for the protoneutron star cooling which we denote by \( A \) and \( B \), where \( A \) stands for cooling of a star configuration with SC whereas \( B \) is a scenario without SC. The initial states for both scenarios are chosen to have the same mass \( M_i(A) = M_i(B) \) for a given initial temperature of \( T = 40 \) MeV. The final states at \( T = 0 \), however, have different masses \( M_f(A) \neq M_f(B) \) while the total baryon number is conserved in the cooling
evolution. The resulting mass differences are $\Delta M(A) = 0.06 \, M_\odot$, $\Delta M(B) = 0.09 \, M_\odot$ and $\Delta M(A) = 0.05 \, M_\odot$, $\Delta M(B) = 0.07 \, M_\odot$ for the Gaussian and Lorentzian models, respectively.

We make the observation that, contrary to naive expectations of the mass defect as a result of the binding energy from Cooper pairing [19], $\Delta M \, c^2 \sim (\Delta/\mu)^2 M_\odot \, c^2 \propto 10^{53} \text{ erg}$, the mass defect in the cooling evolution with diquark condensation (A) is about 40\% smaller than in the scenario without SC (B), regardless of the choice of formfactor. The difference amounts to $\simeq 0.02 M_\odot$ for initial configurations with masses of about 1.4 – 1.5 $M_\odot$. Therefore, we conclude that general relativity effects play an important role in the estimate of the mass defect since the effects of diquark condensation on the mass of configurations via the change in the equation of state overcompensate effects due to the occurrence of the diquark gap in the one-particle energies.

5 Conclusion

We have investigated the influence of diquark condensation on the thermodynamics of quark matter under the conditions of $\beta$-equilibrium and charge neutrality relevant for the discussion of compact stars. The EoS has been derived for a nonlocal chiral quark model in the mean field approximation, and the influence of different formfactors of the nonlocal, separable interaction (Gaussian, Lorentzian, NJL) has been studied. The model parameters are chosen such that the same set of hadronic vacuum observables is described. We have shown that the critical temperatures and chemical potentials for the onset of the chiral and the superconducting phase transition are the lower the smoother the momentum dependence of the interaction formfactor is.

The phase transition to color superconducting quark matter from the lower density regions at small temperatures ($T < 5 \div 10$ MeV) is of first order, while the melting of the diquark condensate and the corresponding transition to normal quark matter at high temperatures is of second order. The flavor asymmetry in neutral quark matter increases when the electron fraction drops.

When the diquark coupling constant is large enough ($G_2 \gtrsim G_1$) then the asymmetry which is present in the system due to the $\beta$-equilibrium condition does not destroy the diquark condensation. The electron fraction under these conditions does not exceed $n_e/n_{\text{total}} \simeq 0.01$. The masses of the quark core configurations could be up to 1.8 $M_\odot$ for the Gaussian and up to 1.6 $M_\odot$ for the Lorentzian formfactor models. The quark core radii with Lorentzian formfactors are up to 9 km, while for Gaussian models could extend up to 11 km.
Diquark condensation makes the EoS harder, which leads to an increase in the maximum mass of the quark star configuration when compared to the case without diquark condensation. For finite temperatures the masses are smaller than at $T = 0$. For asymptotically high temperatures and densities the EoS behaves like a relativistic ideal gas, where the relation pressure versus energy density is temperature independent. In contrast to the bag model where this behavior occurs immediately after the deconfinement transition, our model EoS has a temperature dependent $P(\varepsilon)$ relation also beyond this point.

It has been shown for a hybrid star model which uses the quark matter EoS presented in this work that the possibility to obtain a stable star configuration with 2SC quark matter core depends on the formfactor of the quark interaction [42]. The Gaussian and Lorentzian formfactor models do allow a quark matter core, whereas the NJL formfactor model does not. Other NJL model calculations [26] confirm this conclusion.

A mass defect of about $0.1 \, M_\odot$ occurs in the cooling of a hot configuration to a cold one already without diquark condensation. It is mainly determined by the change of the configuration due to changes in the EoS. Cooling the star with diquark condensation results even in a lowering of the mass defect contrary to naive expectations. For a more realistic calculation of the cooling evolution one has to start from isentropic initial temperature profiles, which may change the size of the mass defect but not our main conclusions.

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Table 1
Parameters for the different formfactors models discussed in the text. Critical temperatures $T_c$ for the chiral phase transition at $\mu_q = 0$ and critical chemical potentials $\mu^c_q(T = 0)$ for the onset of diquark condensation at $T = 0$.

| Formfactor   | $\Lambda$ [GeV] | $G_1 \Lambda^2$ | $m$ [MeV] | $T_c(\mu = 0)$ [MeV] | $\mu^c_q(T = 0)$ [MeV] |
|--------------|-----------------|-----------------|-----------|----------------------|------------------------|
| Gaussian     | 1.0250          | 3.761           | 2.41      | 175                  | 314                    |
| Lorentzian $\alpha = 2$ | 0.8937          | 2.436           | 2.34      | 188                  | 330                    |
| NJL          | 0.9000          | 1.944           | 5.10      | 213                  | 356                    |

Table 2
Characteristic quark star properties for Gaussian (left) and Lorentzian (right) formfactor models: maximal possible masses $M_{max}$, radii $R$, central baryon densities $n_B(0)$ and central energy densities $\varepsilon(0)$ for different temperatures.

| $T$ [MeV] | $M_{max}$ [$M_\odot$] | $R$ [km] | $n_B(0)$ [$n_0$] | $\varepsilon(0)$ [GeV/fm$^3$] | $M_{max}$ [$M_\odot$] | $R$ [km] | $n_B(0)$ [$n_0$] | $\varepsilon(0)$ [GeV/fm$^3$] |
|-----------|----------------------|---------|-----------------|-----------------------------|----------------------|---------|-----------------|-----------------------------|
| 0$^*$     | 1.80                 | 10.10   | 6.97            | 1.196                       | 1.54                 | 8.72    | 10.34           | 2.115                       |
| 0         | 1.85                 | 10.00   | 7.54            | 1.213                       | 1.63                 | 8.63    | 9.24            | 1.813                       |
| 40        | 1.72                 | 9.67    | 7.70            | 1.553                       | 1.53                 | 8.44    | 9.81            | 1.893                       |
| 60        | 1.61                 | 9.46    | 7.95            | 1.653                       | 1.45                 | 8.55    | 9.97            | 1.925                       |

* )Diquark gap is set to $\Delta = 0$. 
Fig. 1. Solutions for the chiral gap $\phi$ and the diquark gap $\Delta$ for different formfactors at $T = 0$ in $\beta$–equilibrium quark matter, $G_2 = G_1$. 
Fig. 2. Solutions of the chiral gap $\phi$ and the diquark gap $\Delta$ for the Gaussian model formfactor in the chiral limit (left panel) and for finite current quark mass $m_0 = 2.41$ MeV (right panel) at different temperatures $T = 0, 40, 60$ MeV for $\beta$–equilibrium quark matter, $G_2 = G_1$. 
Fig. 3. Phase diagrams for different formfactor models: Gaussian (solid lines), Lorentzian $\alpha = 2$ (dashed lines) and NJL (dash-dotted) under conditions of $\beta$-equilibrium and charge neutrality. The color-superconducting phase which exists for $G_2 = G_1$ is absent for $G_2 = 0.75 \, G_1$. In the inset we show for the Gaussian model the comparison of the numerical result with the modified BCS formula $T_c^\Delta = 0.57 \, \Delta(T = 0, \mu_q) \, g(\mu_q)$ for the critical temperature of the superconducting phase transition.
Fig. 4. Dependence of the pressure on the baryochemical $\mu_B$ of quark matter for the Gaussian formfactor model for different temperatures $T = 0, 40, 60$ MeV. To demonstrate the effect of the stiffening of the EoS due to diquark condensation, we show the $T = 0$ result for both $\Delta \neq 0$ (solid line, $G_2 = G_1$) and $\Delta = 0$ (dash-dotted line, $G_2 = 0.75 \times G_1$).
Fig. 5. Matter distribution inside the quark star (lower panel) and accumulated mass (upper panel) for the quark stars with the same central baryon density $n_B(0) = 5 n_0$ and cut on $n_B(R) = 0.5 n_0$, for different temperatures. The case $\Delta = 0$ is calculated for $G_2 = 0.75 G_1$, the others for $G_2 = G_1$. 
Fig. 6. Stable configurations of quark stars for different temperatures $T = 0, 40, 60$ MeV. Mass as a function of central baryon density (left panels) and of the radius (right panels) for the Gaussian model (upper graph) and for the Lorentzian model (lower graph). The case $\Delta = 0$ is calculated for $G_2 = 0.75 \, G_1$, the others for $G_2 = G_1$. 

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Fig. 7. Hot ($T = 40$ MeV) versus cold ($T = 0$) quark star configurations for the Gaussian model; case A: with diquark condensation (dashed versus full lines) and case B: without diquark condensation (dash-dash-dotted versus dash-dotted lines). When a quark star with initial mass $M_i$ cools down from $T = 40$ MeV to $T = 0$ at fixed baryon number $N_B$ the mass defect $\Delta M$ occurs. The case $\Delta = 0$ is calculated for $G_2 = 0.75 \ G_1$, the others for $G_2 = G_1$. 

$\Delta M(A) = 0.06 \ M_{\text{sun}} \quad \Delta M(B) = 0.09 \ M_{\text{sun}}$

$M_i(A) = 1.51 \ M_{\text{sun}} \quad M_i(B) = 1.51 \ M_{\text{sun}}$

$M_f(B) = 1.44 \ M_{\text{sun}} \quad M_f(B) = 1.42 \ M_{\text{sun}}$

Conserved $N_B(A) = 1.68 \ N_{\text{sun}}\quad N_B(B) = 1.57 \ N_{\text{sun}}$

- $T = 0$
- $T = 0, \Delta = 0$
- $T = 40$ MeV, $\Delta = 0$
- $T = 40$ MeV
\[ \Delta M(A) = 0.05 \, M_{\odot} \]
\[ \Delta M(B) = 0.07 \, M_{\odot} \]

\[ M_i(A) = 1.45 \, M_{\odot} \]
\[ M_i(B) = 1.45 \, M_{\odot} \]

\[ M_f(A) = 1.40 \, M_{\odot} \]
\[ M_f(B) = 1.38 \, M_{\odot} \]

\[ N_{B}(A) = 1.58 \, N_{\odot} \]
\[ N_{B}(B) = 1.43 \, N_{\odot} \]

Fig. 8. Same as Fig. 7 for the Lorentzian model.