SCALING LIMITS FOR CRUMP-MODE-JAGERS PROCESSES WITH IMMIGRATION VIA STOCHASTIC VOLterra EQUATIONS

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In this paper, we firstly give a reconstruction for Crump-Mode-Jagers processes with immigration as solutions to a class of stochastic Volterra integral equations, which offers us a new insight for the evolution dynamics of age-dependent population. Based on this new representation, we prove the weak convergence of rescaled Crump-Mode-Jagers processes with immigration to a class of continuous-state branching processes with immigration. Moreover, the limits reveal that the individual law mainly changes the branching mechanism and immigration mechanism proportionally. This covers the results obtained by Lambert et al. [35] for subcritical binary Crump-Mode-Jagers processes.

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1. Introduction. Crump-Mode-Jagers processes (CMJ-processes), as the general continuous-time and discrete-state branching process models with age-dependent reproduction mechanism, were introduced by [7, 8, 26] in the study of biological populations. A CMJ-process is usually described as follows. It starts with a single individual at time 0 and this individual gives birth during its lifetime to a random number of offsprings at a sequence of random times. Every child that is born evolves in the same way. The aim of this work is to explore the contribution of branching law and individual law to the evolution dynamic of population.

In the past few decades, CMJ-processes have been widely applied in many fields. For instance, they have been closely connected to professor-sharing queue; see [10, 15, 42]. Moreover, they also

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have been widely studied in mathematics by many authors and a large number of interesting results have been obtained; see Chapter 6 in [27] and [16]. Since CMJ-processes are neither Markov processes (unless the lifetime distribution is exponential) nor semimartingales, methods and tools developed for Galton-Watson processes (GW-processes) usually cannot be applied to study CMJ-processes. In order to overcome these difficulties, authors have tried to relate CMJ-processes to other classic Markov processes. For instance, He et al. [20] considered a measure-valued Markov process whose total mass evolves according to a CMJ-process and its support represents the residual life times of those existing particles. Schertzer and Simatos [43] studied the height and contour processes of the random forest defined from a CMJ-process. These two connections offer effective ways to study the related random trees and measure-valued processes via their related CMJ-processes. However, it does not work very well to study a CMJ-process via the corresponding random tree or measure-valued process.

As a milestone, Lambert [32] established a connection between spectrally positive Lévy processes and homogeneous, binary CMJ-processes (constant birth rate and one birth at every successive time). For any homogeneous, binary CMJ-process with life-length distribution having finite variance, he showed that the contour process of the splitting tree defined from this CMJ-process is a spectrally positive Lévy process with finite variation and negative drift killed when it hits 0. Conversely, given such a Lévy process, one can construct a CMJ process as the local time process of Lévy process. Based on this connection and the abundant properties of Lévy processes, they also have explored some interesting properties of homogeneous, binary CMJ-processes. Lambert [33] gave an exact representation for their one-dimensional marginal distributions with the scale function of Lévy processes. Via excursion theoretic arguments, Lambert et al. [35] showed that conditioned by their total offspring, the renormalized homogeneous, binary CMJ processes starting from one individual would converge to a reflected Brownian motion with drift.

Unfortunately, limited by the fact that only one jump occurs each time in Lévy processes, similar connection in [32] can not be established between Lévy processes and CMJ-processes with general branching mechanism. Thus some new descriptions for CMJ-processes are necessary. This also motivates us to explore the relationship between CMJ-processes and self-exciting point processes. Hawkes processes, as a special kind of random point processes with self-exciting property, was firstly introduced in [17, 18]. Hawkes and Oakes [19] represented them with a class of general branching processes. The marked Hawkes processes with general immigrants were firstly introduced in [5] and Boumezoud [4] considered them as a multi-type population with ages, including immigration and births with mutations. As an infinite-dimensional extension of Hawkes processes, Hawkes random measures were firstly introduced by Horst and Xu in the study of limit order books with cross-dependencies existing in the order flows; see [21]. Their cluster representation with a special kind of branching particle systems with nonlocal branching mechanism was given in [22]. Conversely, in this work we represent the CMJ-processes with immigration into the form of Hawkes random measures; see Section 3. Based on the properties of Hawkes random measures, we reconstruct CMJ-processes with immigration as solutions to a special class of stochastic Volterra integral equations. This new representation provides us not only a new insight for the evolution dynamic of CMJ-processes, but also a new way to explore their properties via the related stochastic Volterra integral equations. For instance, applying the Burkholder-Davis-Gundy inequality, we can get the fractional moment estimations for CMJ-processes much easily; see Section 6.2.

As an application of this new representation, in this work we mainly study the weak convergence of rescaled CMJ-processes with immigration. As one of the most important topics, the scaling limits of discrete stochastic dynamic systems not only reveal the fascinating connections between microscopic stochastic systems and the macroscopic phenomena, they also contribute to the better and deeper understanding of the asymptotic features of some phenomena. There has been a wealth
of relative results for GW-processes appeared since Feller [13] firstly considered the convergence to a class of diffusion processes. Lamperti [36] showed in detail that such scaling limits form a class of Markov processes called \textit{continuous-state branching processes} (CB-processes), which were firstly introduced in [29]. Kawazu and Watanabe [30] studied the convergence of GW-processes with immigration in the sense of finite-dimensional distributions and characterized the limit processes as \textit{continuous-state branching processes with immigration} (CBI-processes), which was proved by Li [37] in the space of càdlàg functions $D([0,\infty],\mathbb{R}_+)$. Compared with a large number of results about scaling limits for GW-processes, we only find several papers related to the convergence of (non-Markovian) CMJ-processes. Sagitov [40] considered a sequence of non-Markovian CMJ-processes with regularly varying generating function, which converged to some CB-processes with stable branching mechanism in the sense of finite-dimensional distributions. Benefiting greatly from the wealth of literature about the convergence of the local time processes associated to random walks, Lambert et al. [35] proved the weak convergence of rescaled subcritical, homogeneous, binary CMJ-processes, whose life-length distribution had finite variance, to the Feller diffusion. For the infinite variance case, the convergence in the sense of finite-dimensional distributions was proved in [34]. Except for the non-Markovian property, they did not give any other properties for the limits. For any supercritical CMJ-process, since the contour process of its splitting tree will tend to infinity with positive probability, methods developed in [35] will not work. To the best of our knowledge, there is no literature that considers the weak convergence of rescaled CMJ-processes with immigration.

Effectively by the fact that CMJ-processes usually are not Markov (except the exponential life-length case), we can neither prove the tightness in the standard ways such as the generator methods; see Corollary 8.9 in [11], nor characterize the limits following the standard argument of martingale problems. Meanwhile, since CMJ-processes are not semimartingales, it is difficult to get the moment estimation for their maximum from Doob’s martingale inequality. The methods developed in [24, 46] to establish the maximal inequality established for stochastic Volterra equations driven by Brownian motion strongly depend on Kolmogorov’s continuity criterion, which will not work for stochastic Volterra equations with càdlàg solutions. Hence it also difficult to obtain the tightness from the Aldous criterion; see Theorem 1 in [1]. Fortunately, our stochastic Volterra representation for CMJ-processes with immigration offers a new way to keep us away from these problems. Enlightened by the observation that the contribution of every particle to the future population decreases at an exponential rate, we can approximate the evolution dynamic of CMJ-process by a stochastic Volterra integral equation with exponential kernel. The integral representation for exponential kernel allows us to rewrite the stochastic Volterra integral equation into Itô’s type stochastic integral equations driven by semimartingales. Applying results in [31] about weak convergence of stochastic integrals and differential equations driven by infinite-dimensional semimartingales, we prove the weak convergence of rescaled CMJ-processes with immigration and the limit is a strong Markov process. At the same time, we also show that the limit process is the unique solution to the stochastic differential equation with jumps studied in [9, Theorem 3.1], which induces that the limit process is a CBI-process.

The difficulties we encounter in the proofs mainly derive from two factors: collapse at time 0 and approximation errors. Indeed, if ancestors has the same life-length distribution as their offsprings, the rescaled CMJ-processes will only converge in the sense of finite-dimensional distributions to some process which is not right-continuous at time 0. This phenomenon also has been observed in [35]. Moreover, in the progress of approximation, we try to transfer the bad characteristics of CMJ-processes with immigration into the error processes including non-Markov property and non-semimartingale, which means that the internal structure of error processes is very confusing. To overcome these problems, we firstly assume that the life-length of ancestors is distributed according to some weighted size-biased distribution. This assumption will keep ancestors surviving for a long
time such that collapse at time 0 will not occur. To identify that error processes converge to 0 uniformly, we consider their linear interpolations with the mesh dense enough such that the differences can be uniformly bounded. Instead of the errors processes, we just need to prove that the sequence of linear interpolations converges weakly to 0 in the space of continuous functions $C(R_+, R)$.

The connection established in [32] between homogeneous, binary CMJ-processes and Lévy processes makes it possible to apply the abundant results of Lévy processes into the study of CMJ-processes. Our new representation also offers another way to study the general CMJ-processes with immigration via the theory of stochastic Volterra integrals. However, compared to the method developed in [35] which only works for the subcritical case, our method is more effective and works for all cases. In order to simplify the statements and proofs, in this work we mainly consider the homogenous CMJ-processes with general branching mechanism and homogenous immigration. For the inhomogeneous case with predictive immigration, similar results in this work also can be gotten in the same way.

The remainder of this paper is organized as follows. Firstly, we recall Hawkes random measures introduced in [21] and give another representation for their density processes in Section 2. In Section 3, for any given CMJ-process with immigration, we represent it into the form of Hawkes random measure and then reconstruct it as the unique solution of stochastic Volterra integral equation. In Section 4, we show that the rescaled CMJ-processes with immigration converge weakly in $D(R_+, R_+)$ and the limits are CBI-processes. Before showing the proof for the main theorem, we make some preparations about the asymptotic results for the resolvent kernels in Section 5, which play an important role in approximating CMJ-processes with immigration by stochastic Volterra integral equations with exponential kernel. We carry out the proof for the main result in Section 6. In details, in Subsection 6.1, we show the main ideas of proof with some technical estimations are postponed into the following subsections. In Subsection 6.2, we show several uniform upper bound estimations for the fractional moments of CMJ-processes with immigration. In Subsection 6.3, we prove that the error processes converge weakly to 0. In Subsection 6.4, we prove the weak convergence of semimartingales which drive the stochastic Volterra integral equations.

**Notation:** Let $Z_+ = \{1, 2, \cdots \}$, $R_+ = [0, \infty)$ and $R_+^Z := \cup_{d=1}^\infty R_+^d$. We write $(k, y) \in Z_+ \times R_+^Z$, which means that $k \in Z_+$ and $y := (y_1, \cdots, y_k, 0, \cdots)$. Let $S(R_+)$ be the space of $\sigma$-finite Borel signed measures on $R_+$ endowed with the $\sigma$-algebra generated by the mappings $\nu \mapsto \nu(A)$ for any $A \in B(R_+)$. We denote Dirac measure by $\delta$. For any probability measure $\Lambda(\cdot)$ on $R$, denote its tail probability by $\hat{\Lambda}(x) := \Lambda((x, \infty))$ for any $x \in R$. For any measurable space $(E, \mathcal{E})$, let $D(R_+, E)$ be the space of $E$-valued functions on $R_+$ that are right-continuous and have left-hand limits. Denote by $C(R_+, E)$ the subspace of $D(R_+, E)$ contains all continuous functions. For any function $F, G$ on $R$, denote by $F * G$ the convolution of $F$ and $G$, and $F^{(n)}$ the $n$-th convolution of $F$. We make the convention that for any $t_1 \leq t_2 \in R$

$$\int_{t_1}^{t_2} = \int_{(t_1, t_2]} \quad \text{and} \quad \int_{t_1}^{\infty} = \int_{(t_1, \infty]}.$$  

2. **Hawkes random measures.** We begin this section by recalling a special class of random point measures—Hawkes random measures, which are firstly introduced by Horst and Xu [21]. From the theory of Volterra-Fredholm integral equations, we give another representation for the density processes of Hawkes random measures.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with filtration $\{\mathcal{F}_t : t \geq 0\}$ that satisfies the usual hypotheses. Let $(U, \mathcal{U})$ be a measurable space endowed with a basis measure $\mathfrak{m}(du)$. 
We say a real-valued two-parameter stochastic process \( \{h(t, u) : t \geq 0, u \in U\} \) is \((\mathcal{F}_t)\)-progressive if for any \( t \geq 0 \) the mapping \((\omega, s, u) \mapsto h(\omega, s, u)\) restricted to \( \Omega \times [0, t] \times U \) is measurable to \( \mathcal{F}_t \times \mathcal{B}([0, t]) \times \mathcal{U} \). Let \( \{p_t : t \geq 0\} \) be a \((\mathcal{F}_t)\)-progressive random point process on \( U \) and \( N(dt, du) \) be its random point measure defined by

\[
N(I, A) = \# \{t \in I : p_t \in A\}, \quad I \in \mathcal{B}(\mathbb{R}_+), \ A \in \mathcal{U}.
\]

We say a \((\mathcal{F}_t)\)-progressive process \( \lambda(t, u) \) is the density process of \( N(dt, du) \) with respect to the base measure \( m(du) \) if for any nonnegative \((\mathcal{F}_t)\)-predictable process \( H(t, u) \) have

\[
E\left[ \int_0^t \int_U H(s, u)N(ds, du) \right] = E\left[ \int_0^t ds \int_U H(s, u)\lambda(s, u)m(du) \right].
\]

Now we give the definition for a special kind of Hawkes random measures; more general definition can be found in [21].

**Definition 2.1.** We say \( N(dt, du) \) is a Hawkes random measure on \( \mathbb{R}_+ \times U \) if its density process \( \lambda(t, u) \) can be written as

\[
\lambda(t, u) = \mu(t, u) + \int_0^t \int_U K(t - s, u, v)N(ds, dv), \quad t \geq 0
\]

where \( \mu(t, u) : \mathbb{R}_+ \times U \mapsto \mathbb{R}_+ \) and \( K(t, u, v) : \mathbb{R}_+ \times U \times U \mapsto \mathbb{R}_+ \).

Here \( \mu(t, u) \) and \( K(t, u, v) \) is called the **exogenous intensity** and **kernel** of Hawkes random measure \( N(dt, du) \) respectively. For any \( \mu(t, u) \) and \( K(t, u, v) \) satisfying that

\[
\sup_{t \in [0, T], u \in U} \left\{ \int_U \mu(t, u)m(du) + \int_U K(t, u, v)m(du) \right\} < \infty,
\]

we can always find a Hawkes random measure with density process \( \lambda(t, u) \) defined by (2.1); see Theorem 2.3 in [21]. Like the argument in [23, p.93], on an extension of the original probability space we can always find a Poisson random measure \( N_0(dt, du, dz) \) on \( \mathbb{R}_+ \times U \times \mathbb{R}_+ \) with intensity \( dt m(du) dz \) such that

\[
N(I, A) = \int_I \int_J \int_0^\lambda t(u) N_0(dt, du, dz), \quad I \in \mathcal{B}(\mathbb{R}_+), \ A \in \mathcal{U}.
\]

Suppose \( \nu(du) \) is a \( \sigma \)-finite measure on \( U \) and \( N_1(dt, du) \) is a Poisson random measure on \( \mathbb{R}_+ \times U \) with intensity \( dt \nu(du) \), which is independent of \( N_0(dt, du, dz) \). We say \( N(dt, du) \) is a Hawkes random measure with immigration if the exogenous intensity \( \mu(t, u) \) is defined by

\[
\mu(t, u) := \mu_0(t, u) + \int_0^t \int_U K(t - s, u, v)N_1(ds, dv),
\]

where \( \mu_0(t, u) \) is \( \mathcal{F}_0 \)-measurable for any \( t \geq 0 \).

**Theorem 2.2.** Assume that (2.2) holds almost surely, then the density process \( \lambda(t, u) \) of Hawkes random measure with immigration \( N(dt, du) \) satisfies the following equation

\[
\lambda(t, u) = \mu_0(t, u) + \int_0^t \int_U R(t - s, u, v)\mu_0(s, v)m(du) + \int_0^t \int_U R(t - s, u, v)N_1(ds, dv)
\]
\begin{align}
(2.5) & \quad + \int_0^t \int_U \int_0^{\lambda(s-v)} R(t-s,u,v)\tilde{N}_0(ds,dv,dz),
\end{align}

where \( \tilde{N}_0(ds,dv,dz) := N_0(ds,dv,dz) - ds\mathbf{m}(dv)dz \) and \( R(t,u,v) \) is the resolvent kernel associated with the kernel \( K(t,u,v) \), which solves the following Volterra-Fredholm integral equation

\begin{align}
R(t,u,v) &= K(t,u,v) + \int_0^t \int_U K(t-s,u,w)R(s,w,v)ds\mathbf{m}(dw) \\
&= K(t,u,v) + \int_0^t \int_U R(t-s,u,w)K(s,w,v)ds\mathbf{m}(dw).
\end{align}

\textbf{Proof.} Like the proof of Theorem 2.1.13 in [6, p.76], we can prove that the solution to (2.6) exists uniquely. From (2.3) and (2.4), we can see that

\begin{align}
\lambda(t,u) &= \mu_0(t,u) + \int_0^t \int_U K(t-s,u,v)N_1(ds,dv) + \int_0^t \int_U \int_0^{\lambda(s-v)} K(t-s,u,v)N_0(ds,dv,dz) \\
&= \mu_0(t,u) + \int_0^t \int_U K(t-s,u,v)N_1(ds,dv) + \int_0^t \int_U \int_0^{\lambda(s-v)} K(t-s,u,v)\tilde{N}_0(ds,dv,dz) \\
&\quad + \int_0^t \int_U K(t-s,u,v)\lambda(s,v)ds\mathbf{m}(dv).
\end{align}

Applying Theorem 2.1.13 in [6, p.76], we have

\begin{align}
\lambda(t,u) &= \mu_0(t,u) + \int_0^t \int_U R(t-s,u,v)\mu_0(s,v)ds\mathbf{m}(dv) + \int_0^t \int_U K(t-s,u,v)N_1(ds,dv) \\
&\quad + \int_0^t \int_U R(t-s,u,w)\int_0^s \int_U K(s-r,w,v)N_1(dr,dv)ds\mathbf{m}(dw) \\
&\quad + \int_0^t \int_U \int_0^{\lambda(s-v)} K(t-s,u,v)\tilde{N}_0(ds,dv,dz) \\
&\quad + \int_0^t \int_U R(t-s,u,w)\int_0^s \int_U \int_0^{\lambda(r-v)} K(s-r,w,v)\tilde{N}_0(dr,dv,dz)ds\mathbf{m}(dw) \\
&= \mu_0(t,u) + \int_0^t \int_U R(t-s,u,v)\mu_0(s,v)ds\mathbf{m}(dv) + \int_0^t \int_U K(t-s,u,v)N_1(ds,dv) \\
&\quad + \int_0^t \int_U \int_0^t \int_U R(t-r,u,w)K(r-s,w,v)dr\mathbf{m}(dw)N_1(ds,dv) \\
&\quad + \int_0^t \int_U \int_0^{\lambda(s-v)} K(t-s,u,v)\tilde{N}_0(ds,dv,dz) \\
&\quad + \int_0^t \int_U \int_0^{\lambda(s-v)} \int_0^t \int_U R(t-r,u,w)K(r-s,w,v)dr\mathbf{m}(dw)\tilde{N}_0(ds,dv,dz) \\
&= \mu_0(t,u) + \int_0^t \int_U R(t-s,u,v)\mu_0(s,v)ds\mathbf{m}(dv) + \int_0^t \int_U K(t-s,u,v)N_1(ds,dv) \\
&\quad + \int_0^t \int_U \int_0^{t-s} \int_U R(t-s-r,u,w)K(r,w,v)dr\mathbf{m}(dw)N_1(ds,dv) \\
&\quad + \int_0^t \int_U \int_0^{\lambda(s-v)} K(t-s,u,v)\tilde{N}_0(ds,dv,dz)
\[
+ \int_0^t \int_U \int_0^{t-s} \int_0^r R(t-s-r,u,w)K(r,w,v)drm(dw)N_0(ds,dv,dz)
\]
\[
= \mu_0(t) + \int_0^t \int_U R(t-s,u,v)\mu_0(s,v)ds \nu_1(ds,dv) + \int_0^t \int_U R(t-s,u,v)\tilde{N}_0(ds,dv,dz).
\]

Here we have finished the proof. \qed

3. Stochastic Volterra representations for CMJ-processes with immigration. Based on the preparation in the last section, in this section we reconstruct CMJ-processes with immigration as solutions to a special kind of stochastic Volterra integral equations. This new representation allows us to apply results and methods from stochastic analysis into the study of CMJ-processes with immigration.

Let us firstly give a brief description of CMJ-processes. A fuller and more rigorous description can be found in [26, 27, 28]. Recall the well defined filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). We consider a sequence of general homogeneous CMJ-processes with immigration \(\{Z^{(n)}(t) : t \geq 0\}_{n \geq 1}\) on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), which is defined by the following properties:

(P1) (Initial state) There are \(Z^{(n)}(0)\) ancestors at time 0.

(P2) (Lifespan) The common distribution of life-length is \(\Lambda^{(n)}(dy)\), where \(\Lambda^{(n)}(dy)\) is a probability measure on \(\mathbb{R}_+\) with finite first and second moment

\[
\eta^{(n)} := \int_0^\infty y\Lambda^{(n)}(dy) \quad \text{and} \quad \sigma^{(n)} := \frac{1}{2} \int_0^\infty y^2\Lambda^{(n)}(dy).
\]

(P3) (Branching rate) Conditioned on the life-length \(y\), the successive ages \(0 < t_1 < t_2 < \cdots < y\) at which the particle gives birth is a Poisson point process on \((0, y)\) with intensity \(\lambda^{(n)}\).

(P4) (Branching mechanism) At each successive age, the particle gives birth to a random number of offsprings according to the probability law \(p^{(n)}\), which is given by the following generating function

\[
g^{(n)}(z) := \sum_{k=1}^{\infty} p_k^{(n)} z^k, \quad z \in [0, 1].
\]

(P5) (Immigration rate) The entry times of new immigrating particles are governed by a Poisson point process with intensity \(\zeta^{(n)}\).

(P6) (Immigration mechanism) The number of immigrating particles at any entry time is distributed as the probability distribution \(q^{(n)}\) with generating function

\[
h^{(n)}(z) := \sum_{k=1}^{\infty} q_k^{(n)} z^k, \quad z \in [0, 1].
\]

Define two useful quantities as

\[
m^{(n)} := \sum_{k=1}^{\infty} kp_k^{(n)} \quad \text{and} \quad a^{(n)} := \sum_{k=1}^{\infty} kq_k^{(n)}.
\]
We claim that the CMJ-process with immigration \( Z^{(n)} \) is subcritical, critical or supercritical if \( \lambda^{(n)} \eta^{(n)} m^{(n)} < 1, = 1 > 1; \) see Chapter 6 in [27].

Notice that each ancestor gives birth to its descendants independently according to a Poisson point process and a random number of offsprings are born at every successive age. All offsprings will give birth to their own descendants independently in the same way. Thus each individual \( j \) in the system can be labelled with the pair \((\tau_j, e_j)\), where \( \tau_j \) and \( e_j \) represent its birth time and life-length respectively. Denote by \( I_0 \) the collection of all particles in the family derived by the ancestors; see the red pots in Figure 1. The population alive at time \( t \), denote by \( X_0^{(n)}(t) \), can be written as

\[
X_0^{(n)}(t) = \sum_{j \in I_0} 1\{\tau_j \leq t < \tau_j + e_j\}.
\]

We record the entry time of new immigrating particles as \( 0 < \varsigma_1 < \varsigma_2 < \cdots \), which are governed by a Poisson point processes. For any \( k \geq 1 \), let \( I_k \) be the collection of all particles in the family derived by the new particles which entry into the population in the \( k \)-th immigration; see the blue and black pots in Figure 1. Then the whole population \( Z^{(n)}(t) \) at time \( t \) can be written as

\[
Z^{(n)}(t) = \sum_{k=0}^{\infty} \sum_{j \in I_k} 1\{\tau_j \leq t < \tau_j + e_j\}.
\]

Now we start to construct the new representation for CMJ-processes with immigration. Instead of horizontal moving, particles in Figure 1 move decreasingly to the time axis at the rate 1, e.g. the particle \( j \) born at time \( \tau_j \) with life-length \( e_j \) will move uniformly from \((\tau_j, e_j)\) to \((\tau_j + e_j, 0)\) at the rate 1. Particles will die when they arrive at the time axis; see Figure 2. Then (3.1) can be rewritten into

\[
Z^{(n)}(t) = \sum_{k=0}^{\infty} \sum_{j \in I_k} 1\{\tau_j \leq t, e_j > t - \tau_j\}.
\]

Precisely, the population consists of the following four kinds of particles: ancestors and their descendants, immigrations and their descendants. Denote by \( \{e^{(n)}_i : i = 1, \cdots, Z^{(n)}(0)\} \) the life-length of ancestors. Let \( N_{I_k} \) be the number of new particles in the \( k \)-th immigration and \( \{e^{(n)}_{k,i} : i = 1, \cdots, N_{I_k}\} \) be their life-length. Let \( I_k^0 \) be the collection of all descendants of ancestors and \( I_k \) be the collection of all descendants of the particles in the \( k \)-th immigration. Then (3.2) can be rewritten into

\[
Z^{(n)}(t) = \sum_{j=1}^{Z^{(n)}(0)} 1\{e^{(n)}_j > t\} + \sum_{k=1}^{\infty} \sum_{j=1}^{N_{I_k}} 1\{\varsigma_k \leq t, e^{(n)}_{k,i} > t - \varsigma_k\} + \sum_{k=0}^{\infty} \sum_{j \in I_k^0} 1\{e_j > t - \tau_j\}.
\]

We introduce two random point processes \( p_0^{(n)}(t) \) and \( p_1^{(n)}(t) \) on \( \mathbb{Z}_+ \times \mathbb{R}_+^k \), which record branching and immigration respectively. For instance, \( p_0^{(n)}(t) = (k, y) \in \mathbb{Z}_+ \times \mathbb{R}_+^k \) means that there exists a particle alive at time \( t \) that gives birth to \( k \) offsprings with life-length \( y \). Similarly, when \( p_1^{(n)}(t) = (k, y) \in \mathbb{Z}_+ \times \mathbb{R}_+^k \), then there are \( k \) new particles with life-length \( y \) migrate into the population at time \( t \). Let \( N^{(n)}(dt, dk, dy) \) and \( N_1^{(n)}(dt, dk, dy) \) be two random point measures on \( \mathbb{R}_+ \times \mathbb{Z}_+ \times \mathbb{R}_+^k \) associated with the point processes \( p_0^{(n)}(t) \) and \( p_1^{(n)}(t) \) respectively. From properties (P4) and (P5),
Fig 1. Part of a sample path of a CMJ-process with immigration. Every line represents a particle. The abscissa and ordinate of the left point on every line are the birth time and life-length of the particle respectively. The red lines belong to the family derived by ancestors. The black and blue lines are families derived by the first and second immigrations.

Fig 2. Redraw the sample path of a CMJ-process with immigration in Figure 1. The particle born at time $\tau_i$ with life-length $e_i$, moves from $(\tau_i, e_i)$ to the horizontal axis at the rate 1 and dies when it arrives at the horizontal axis.
it is easy to see that $N^{(n)}_1(dt, dk, dy)$ is a Poisson random measure with intensity $\zeta^{(n)} dt \nu^{(n)}_1(dk, dy)$, where

\begin{equation}
\nu^{(n)}_1(dk, dy) = \sum_{i=1}^{\infty} q_i^{(n)} 1_{\{i\}}(dk) \prod_{j=1}^{i} \Lambda^{(n)}(dy_j).
\end{equation}

From the properties (P2)-(P3) and the branching property, we also have that $N^{(n)}_1(dt, dk, dy)$ is a random point measure with intensity $\lambda^{(n)} Z^{(n)}(t-) dt \nu^{(n)}_0(dk, dy)$, where

\begin{equation}
\nu^{(n)}_0(dk, dy) = \sum_{i=1}^{\infty} p_i^{(n)} 1_{\{i\}}(dk) \prod_{j=1}^{i} \Lambda^{(n)}(dy_j).
\end{equation}

It is easy to check that $\nu^{(n)}_0(dk, dy)$ and $\nu^{(n)}_1(dk, dy)$ are two probabilities on $\mathbb{Z}_+ \times \mathbb{R}^{Z^+_+}$. Based on the argument above, we can see (3.3) can be rewritten into

\begin{equation}
Z^{(n)}(t) = Z^{(n)}_\beta(t) + \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}^{Z^+_+}} 1_{\{y_j > t-s\}} N^{(n)}_1(ds, dk, dy)
\end{equation}

\begin{equation}
+ \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}^{Z^+_+}} 1_{\{y_j > t-s\}} N^{(n)}(ds, dk, dy),
\end{equation}

where

\begin{equation}
Z^{(n)}_\beta(t) := \sum_{j=1}^{Z^{(n)}(0)} 1_{\{e_j^{(n)} > t\}}.
\end{equation}

Denote by $\bar{N}^{(n)}(dt, dk, dy)$ and $\bar{N}^{(n)}_1(dt, dk, dy)$ the compensated measures of $N^{(n)}(dt, dk, dy)$ and $N^{(n)}_1(dt, dk, dy)$ defined as follows:

\begin{equation}
\bar{N}^{(n)}(dt, dk, dy) := N^{(n)}(dt, dk, dy) - \lambda^{(n)} Z^{(n)}(t-) dt \nu^{(n)}_0(dk, dy),
\end{equation}

\begin{equation}
\bar{N}^{(n)}_1(dt, dk, dy) := N^{(n)}_1(dt, dk, dy) - \zeta^{(n)} dt \nu^{(n)}_1(dk, dy).
\end{equation}

From the independence between immigration and branching, we can see that for any function $H_1(k, y), H_2(k, y)$ defined on $\mathbb{Z}_+ \times \mathbb{R}^{Z^+_+}$,

\begin{equation}
\left\langle \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}^{Z^+_+}} H_1(k, y) \bar{N}^{(n)}(ds, dk, dy), \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}^{Z^+_+}} H_2(k, y) \bar{N}^{(n)}_1(ds, dk, dy) \right\rangle_t = 0, \hspace{1em} a.s.
\end{equation}

Like the argument in [23, p.93], on an extension of the original probability space we can always find a Poisson random measure $N^{(n)}_0(dt, dk, dx, du)$ on $\mathbb{R}_+ \times \mathbb{Z}_+ \times \mathbb{R}^{Z^+_+} \times \mathbb{R}_+$ with intensity $\lambda^{(n)} dt \nu^{(n)}_0(dk, dy) du$ such that

\begin{equation}
N^{(n)}(dt, dk, dy) = \int_0^{Z^{(n)}(t-)} N^{(n)}_0(dt, dk, dy, du).
\end{equation}

Thus we can rewrite (3.6) into

\begin{equation}
Z^{(n)}(t) = Z^{(n)}_\beta(t) + \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}^{Z^+_+}} 1_{\{y_j \geq t-s\}} N^{(n)}_1(ds, dk, dy)
\end{equation}
where unique solution to the following Volterra-Fredholm integral equation:

\[
(3.7) \quad + \int_0^t \int_{Z_+} \int_{R_+} \int_{Z_+} \int_{R_+} Z^{(n)}(s-) \sum_{j=1}^k 1_{\{y_j \geq t-s\}} N^{(n)}_0(dt, dk, dy, du).
\]

We introduce several function-valued processes derived from \( Z^{(n)} \), which will play an important role in the following reconstruction for CMJ-processes with immigration. For any \( t \geq 0, k, l \in Z_+ \) and \( x, y \in R_{Z+} \), define

\[
(3.8) \quad Z^{(n)}(t, k, x) = Z^{(n)}(t)1_{\{k \geq 1, x > 0\}}, \quad Z^{(n)}_{\beta}(t, k, x) = Z^{(n)}_{\beta}(t)1_{\{k \geq 1, x > 0\}}
\]

and

\[
(3.9) \quad K(t, k, x, l, y) = \sum_{i=1}^l 1_{\{k \geq 1, x > 0, y_i > t\}}.
\]

Based on these notations and (3.7), it is easy to see that \( Z^{(n)}(t, k, x) \) satisfies the following equation:

\[
(3.10) \quad Z^{(n)}(t, k, x) = Z^{(n)}_{\beta}(t, k, x) + \int_0^t \int_{Z_+} \int_{R_+} K(t - s, k, x, l, y) N^{(n)}_1(ds, dl, dy)
\]

\[ \quad + \int_0^t \int_{Z_+} \int_{R_+} Z^{(n)}(s-, l, y) K(t - s, k, x, l, y) N^{(n)}_0(ds, dl, dy, du). \]

From Definition 2.1, (2.5) and (3.10), we can see that \( N^{(n)}(dt, dk, dy) \) is a Hawkes random measure on \( R_+ \times Z_+ \times R_{Z+} \) with density process \( \lambda^{(n)} Z^{(n)}(t, k, y) \). From Theorem 2.2, we can get the following result directly.

**Proposition 3.1.** The function-valued process \( Z^{(n)}(t, k, x) \) satisfies the following stochastic Volterra-Fredholm integral equation:

\[
Z^{(n)}(t, k, x) = Z^{(n)}_{\beta}(t, k, x) + \int_0^t \int_{Z_+} \int_{R_+} R^{(n)}(t - s, k, x, l, y) Z^{(n)}_{\beta}(s, l, y) d\nu^{(n)}_0(dl, dy)
\]

\[ \quad + \int_0^t \int_{Z_+} \int_{R_+} R^{(n)}(t - s, k, x, l, y) \nu^{(n)}_1(ds, dl, dy)
\]

\[ \quad + \int_0^t \int_{Z_+} \int_{R_+} Z^{(n)}(s-, l, y) R^{(n)}(t - s, k, x, l, y) \tilde{N}^{(n)}_0(ds, dl, dy, du), \]

where \( \tilde{N}^{(n)}_0(ds, dl, dy, du) := N^{(n)}_0(ds, dl, dy, du) - \lambda^{(n)} d\nu^{(n)}_0(dl, dy)du \) and \( R^{(n)}(t, k, x, l, y) \) is the unique solution to the following Volterra-Fredholm integral equation:

\[
R^{(n)}(t, k, x, l, y)
\]

\[ = \lambda^{(n)} K(t, k, x, l, y) + \lambda^{(n)} \int_0^t ds \int_{Z_+} \int_{R_+} R^{(n)}(t - s, k, x, \theta, z) K(s, \theta, z, l, y) \nu^{(n)}_0(d\theta, dz)
\]

\[ = \lambda^{(n)} K(t, k, x, l, y) + \lambda^{(n)} \int_0^t ds \int_{Z_+} \int_{R_+} K(t - s, k, x, \theta, z) R^{(n)}(s, \theta, z, l, y) \nu^{(n)}_0(d\theta, dz). \]
We introduce two relative integrals of the resolvent kernel \( R^{(n)}(t, k, x, l, y) \). Let

\[
R^{(n)}(t) := \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} R^{(n)}(t, k, x, l, y) \nu_0^{(n)}(dl, dy) \lambda_0^{(n)}(dk, dx)
\]

\[
= \lambda^{(n)} m^{(n)} (t) + \lambda^{(n)} m^{(n)} \int_0^t R^{(n)}(t - s) \bar{\lambda}^{(n)}(s) ds
\]

(3.12)

\[
= \lambda^{(n)} m^{(n)} \bar{\lambda}^{(n)} (t) + \lambda^{(n)} m^{(n)} \int_0^t \bar{\lambda}^{(n)}(t - s) R^{(n)}(s) ds
\]

and

\[
R^{(n)}(t, l, y) := \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} R^{(n)}(t, k, x, l, y) \nu_0^{(n)}(dk, dx)
\]

(3.13)

\[
= \lambda^{(n)} \sum_{j=1}^l 1_{\{y_j > t\}} + \lambda^{(n)} \int_0^t R^{(n)}(t - s) \sum_{j=1}^l 1_{\{y_j > s\}} ds = \sum_{j=1}^l R^{(n)}(t, 1, y_j).
\]

Since \( \nu_0^{(n)}(dk, dy) \) is a probability measure on \( \mathbb{Z}_+ \times \mathbb{R}_+^2 \), we have

\[
Z^{(n)}(t) = \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} Z^{(n)}(t, k, y) \nu_0^{(n)}(dk, dy).
\]

Now we show the stochastic Volterra representation for CMJ-process with immigration \( \{Z^{(n)}(t) : t \geq 0\} \) in the following theorem, which can be proved directly by integrating both sides of (3.11).

**Theorem 3.2.** The CMJ-process with immigration \( \{Z^{(n)}(t) : t \geq 0\} \) is a solution to the following stochastic Volterra integral equation

\[
Z^{(n)}(t) = Z^{(n)}_\beta (t) + \int_0^t R^{(n)}(t - s) Z^{(n)}_{\beta l} (s) ds + \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} R^{(n)}(t - s, l, y) N^{(n)}_1 (ds, dl, dy)
\]

\[
+ \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} \int_0^{Z^{(n)}(s-l)} R^{(n)}(t - s, l, y) \tilde{N}^{(n)}_0 (ds, dl, dy, du).
\]

(3.14)

Since \( Z^{(n)}_\beta (t) \) is \( \mathcal{F}_0 \)-measurable for any \( t \geq 0 \), from (3.14) we can see that \( \mathbb{E}[Z^{(n)}(t)|\mathcal{F}_0] \) is finite almost surely and satisfies the following equation

\[
\mathbb{E}[Z^{(n)}(t)|\mathcal{F}_0] = Z^{(n)}_\beta (t) + \int_0^t R^{(n)}(t - s) Z^{(n)}_{\beta l} (s) ds
\]

\[
+ \zeta^{(n)} \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} R^{(n)}(t - s, l, y) ds \nu_1^{(n)}(dl, dy)
\]

(3.15)

and

\[
Z^{(n)}(t) = \mathbb{E}[Z^{(n)}(t)|\mathcal{F}_0] + \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} R^{(n)}(t - s, l, y) \tilde{N}^{(n)}_1 (ds, dl, dy)
\]

\[
+ \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^2} \int_0^{Z^{(n)}(s-l)} R^{(n)}(t - s, l, y) \tilde{N}^{(n)}_0 (ds, dl, dy, du).
\]

(3.16)

Now we have reconstructed CMJ-processes with immigration as solutions to a special class of stochastic Volterra integral equations. Based on these two representations (3.14) and (3.16), the
results and methods from the theory of stochastic Volterra integrals can be applied to study CMJ-processes with immigration. For example, applying the Burkholder-Davis-Gundy inequality, we can easily give some fractional moment estimations for CMJ-processes with immigration, which are usually difficult to be obtained from their generating functions (or Laplace transforms); see Section 6.

4. Scaling limits for CMJ-processes with immigrations. In this section, we show the main results about weak convergence of rescaled CMJ-processes with immigration in $D(\mathbb{R}_+, \mathbb{R}_+)$ and the limits are CBI-processes. We start with introducing assumptions about the convergence of the individual law, the branching law and immigration law.

Condition 4.1. Let $\{\gamma_n\}$ be a sequence of positive numbers satisfying that $\gamma_n \to \infty$ and $\gamma_n \sim \gamma_s n$ for some $\gamma_s \geq 0$. For any $n \geq 1$ and $z \in [0, n]$, define

$$
(4.1) \quad \phi^{(n)}(z) := n\gamma_n \left[ g^{(n)} \left( 1 - \frac{z}{n} \right) - \left( 1 - \frac{z}{n} \right) \right] \quad \text{and} \quad \psi^{(n)}(z) := \gamma_n \left[ 1 - h^{(n)} \left( 1 - \frac{z}{n} \right) \right].
$$

The following statements hold:

1. There exist constants $\lambda, \eta, \sigma > 0$ and $b \in \mathbb{R}$ such that $\lambda^{(n)} \to \lambda$, $\eta^{(n)} \to \eta$, $\sigma^{(n)} \to \sigma$ and

$$
(4.2) \quad \gamma_n (1 - \lambda^{(n)} \eta^{(n)}) \to b.
$$

2. There exists a function $\psi(z)$ on $[0, \infty)$ such that $\psi^{(n)}(z) \to \psi(z)$ uniformly on $[0, z_0]$ for any $z_0 \geq 0$ as $n \to \infty$.

3. The sequence $\{\phi^{(n)}\}$ is uniformly Lipschitz continuous on any bounded interval. Moreover, there exists a continuous function $\phi(z)$ on $[0, \infty)$ such that $\phi^{(n)}(z) \to \phi(z)$ uniformly on $[0, z_0]$ for any $z_0 \geq 0$ as $n \to \infty$.

Condition 4.1(2)-(3) have been widely used to study the convergence of rescaled GW-processes with immigration; see [14, Theorem 3.1] and [2, 37]. From Lemma 2.1(i) in [37], the limiting functions $\phi(z)$ and $\psi(z)$ have the following representation:

$$
(4.3) \quad \phi(z) = mz + cz^2 + \int_0^\infty (e^{-zu} - 1 + zu)\nu_0(du)
$$

and

$$
(4.4) \quad \psi(z) = az + \int_0^\infty (1 - e^{-zu})\nu_1(du),
$$

where $m \leq 0$, $a, c \geq 0$, $(u \wedge u^2)\nu_0(du)$ and $(1 \wedge u)\nu_1(du)$ are two finite measures on $\mathbb{R}_+$.

Remark 4.2. From Condition 4.1, it is easy to see that $\lambda \eta = 1$, $\gamma_n (1 - m^{(n)}) \to m$ and

$$
(4.5) \quad \gamma_n (1 - \lambda^{(n)} \eta^{(n)} m^{(n)}) \to b + m.
$$

As we have mentioned before, since CMJ-processes usually are neither Markov processes nor semimartingales, we cannot prove the weak convergence in the standard way. Thanks to the stochastic Volterra representation, we will prove the weak convergence of rescaled CMJ-processes with immigration based on the results in [31]. Here we need the following moment condition.

Condition 4.3. For some $\alpha \in (1, 2)$, the following statements hold:
(1) There exist constants $C, k_0 > 0$ such that for any $n \geq 1$

$$n \gamma_n \sum_{k=k_0}^{\infty} \left| \frac{k}{n} p_k^{(n)} + \sum_{k=1}^{\infty} k^\alpha q_k^{(n)} \right| \leq C.$$ 

Moreover,

$$\lim_{k_1 \to \infty} \limsup_{n \geq 1} \gamma_n \sum_{k=k_1}^{\infty} kp_k^{(n)} = 0.$$ 

(2) There exist a constant $C_0 > 0$ and a probability measure $\Lambda^*(dt)$ on $\mathbb{R}_+$ satisfying that for any $n \geq 1$ and $t \geq 0$,

$$\int_0^\infty t^{2\alpha} \Lambda^*(dt) < \infty \quad \text{and} \quad \bar{\Lambda}^{(n)}(t) \leq C_0 \bar{\Lambda}^*(t).$$

Notice that this condition is in fact not really restrictive. Indeed, from Remark 4.2, we have

$$n \gamma_n \sum_{k=2}^{\infty} \frac{k}{n} p_k^{(n)} \sim m \quad \text{as} \quad n \to \infty.$$ 

Thus the first statement in Condition 4.3(1) holds if the support of $p^{(n)}$ is $\{1, \cdots Kn\}$ for some $K \in \mathbb{Z}_+$. Moreover, without loss of generality we can always assume the constant $C_0 = 1$, i.e. under Condition 4.3(2), we may always find a constant $t_0$ such that $C_0 \bar{\Lambda}^*(t_0) < 1$. Define a new probability measure

$$\Lambda^{**}(dt) := (1 - C_0 \bar{\Lambda}^*(t_0)) \delta_{t_0}(dt) + C_0 1_{\{t > t_0\}} \Lambda^*(dt).$$

It is easy to see that $\bar{\Lambda}^{(n)}(t) \leq \Lambda^{**}(t)$ for any $t \geq 0$.

For any $\beta \geq 0$, let $S^{(n)}(dt)$ be the $\beta$-weighted size-biased distribution of $\Lambda^{(n)}(dt)$ defined by

$$S^{(n)}(dt) := \frac{1}{\eta^{(n)}_\beta} \int_t^\infty e^{-\frac{\beta}{m} (s-t)} \Lambda^{(n)}(ds) dt,$$

where

$$\eta^{(n)}_\beta = \int_0^\infty dt \int_t^\infty e^{-\frac{\beta}{m} (s-t)} \Lambda^{(n)}(ds) = \int_0^\infty e^{-\frac{\beta}{m} \bar{\Lambda}^{(n)}(t)} dt.$$ 

Specially, when $\beta = 0$, $S^{(n)}(dt)$ is the size-biased distribution (or forward recurrence time) of $\Lambda^{(n)}(dt)$. From Condition 4.1(1) and Condition 4.3(2), there exists a constant $C > 0$ such that

$$\eta^{(n)}_\beta = \int_0^\infty \gamma^{(n)}_\beta \left( 1 - e^{-\frac{\beta t}{m}} \right) t \Lambda^{(n)}(dt) \leq C \int_0^\infty t \Lambda^{(n)}(dt) < C$$

and

$$\liminf_{n \to \infty} \eta^{(n)}_\beta = \liminf_{n \to \infty} \int_0^\infty \gamma^{(n)}_\beta \left( 1 - e^{-\frac{\beta t}{m}} \right) t \Lambda^{(n)}(dt) \geq C \liminf_{n \to \infty} \int_0^\infty t \Lambda^{(n)}(dt) > 0.$$ 

**Condition 4.4**. For some $\beta \in [0, \infty) \cap (-\frac{b+m}{\sigma \Lambda}, \infty)$, the life-length of ancestors in the $n$-model is distributed according to the $\beta$-weighted size-biased distribution $S^{(n)}_\beta$. 

Actually, this condition is not unconventional. In the study of the convergence of subcritical homogeneous, binary CMJ-processes, Lambert et al. \[35\] and Lambert and Simatos \[34\] also assumed that the life-length of the ancestors is distributed according to the size-biased distribution of \[\Lambda^{(n)}\]; see Theorem 5.4 in \[35\] and Theorem 6.2 in \[34\]. Now we give the main result in this work.

**Theorem 4.5.** Suppose Condition 4.1-4.4 hold. If \(Z^{(n)}(0)/n\) converges to \(Z(0)\) in distribution, then the sequence \(\{Z^{(n)}(\gamma_{n}t)/n : t \geq 0\}_{n \geq 1}\) converges to \(\{Z(t) : t \geq 0\}\) weakly in \(D(\mathbb{R}_{+}, \mathbb{R}_{+})\). Moreover, on an extension of the original probability space, there exist a white noise \(W(dt, du)\) on \(\mathbb{R}_{+}^{2}\) with intensity \(\lambda dt du\), two independent Poisson random measures \(N_{0}(dt, dz, du)\) and \(N_{1}(dt, dz)\) on \(\mathbb{R}_{+}^{3}\) and \(\mathbb{R}_{+}^{2}\) with intensity \(\lambda dt \nu(z)du\) and \(\zeta dt \nu_{1}(dz)\) respectively, such that the limit process \(\{Z(t) : t \geq 0\}\) solves the following stochastic differential equation:

\[
Z(t) = Z(0) + \int_{0}^{t} \left[ \frac{\eta}{\sigma} \alpha \zeta - \frac{\eta}{\sigma} (b + m) Z(s) \right] ds + \int_{0}^{t} \int_{0}^{\infty} \frac{\eta}{\sigma} z N_{1}(ds, dz) \\
+ \int_{0}^{t} \int_{0}^{Z(s)} \frac{\eta}{\sigma} \sqrt{2c + 2\gamma_{s} \sigma \lambda} W(ds, du) + \int_{0}^{t} \int_{0}^{\infty} \frac{Z(s-)}{\sigma} z \tilde{N}_{0}(dt, dz, du),
\]

where \(\tilde{N}_{0}(dt, dz, du) = N_{0}(dt, dz, du) - \lambda dt \nu_{0}(dz)du\).

We will prove this theorem in Section 6. For any \(z \geq 0\), let

\[
\varphi(z) = \frac{\eta}{\sigma} bz + \frac{\gamma_{s}}{\sigma} z^{2}
\]

and \(\phi_{\lambda}(z)\) be a modification of \(\varphi(z)\) defined by

\[
\phi_{\lambda}(z) := \frac{m}{\lambda} z + cz^{2} + \int_{0}^{\infty} \left( e^{-zu} - 1 + zu \right) \nu_{0}(du).
\]

From Theorem 2.1 in \[38\] and Theorem 2.5 in \[9\], we can easily get the following proposition.

**Proposition 4.6.** There is a unique nonnegative strong solution to (4.8). Moreover, the solution \(\{Z(t) : t \geq 0\}\) is a strong Markov process with Feller transition semigroup \((Q_{t})_{t \geq 0}\) on \(\mathbb{R}_{+}\) defined by

\[
\int_{0}^{\infty} e^{-zy} Q_{t}(x, dy) = \exp \left\{ -xV_{t}(z) - \zeta \int_{0}^{t} \psi \left( \frac{y}{\sigma} V_{s}(z) \right) ds \right\},
\]

where \(\{V_{t}(z) : t \geq 0, z \geq 0\}\) is the unique solution to the follow ordinary differential equation

\[
V_{t}(z) = z - \lambda \int_{0}^{t} \phi_{\lambda} \left( \frac{y}{\sigma} V_{s}(z) \right) ds - \int_{0}^{t} \varphi \left( V_{s}(z) \right) ds.
\]

From Theorem 1.1 and 1.2 in \[30\], the solution \(\{Z(t) : t \geq 0\}\) to (4.6) is a conservative CBI-process with infinitesimal generator \(A\) defined by: for any \(f \in C^{2}(\mathbb{R}_{+})\)

\[
Af(x) = \lambda \left\{ - \frac{m}{\lambda} x f'(x) + e \frac{n^{2}}{\sigma^{2}} x f''(x) + \int_{0}^{\infty} \left[ f \left( x + \frac{\eta}{\sigma} z \right) - f(x) - \frac{\eta}{\sigma} zf'(z) \right] \nu_{0}(dz) \right\} \\
+ \zeta \left( \frac{\eta}{\sigma} x f'(x) + \int_{0}^{\infty} \left[ f \left( x + \frac{\eta}{\sigma} z \right) - f(x) \right] \nu_{1}(dz) \right) - \frac{\eta}{\sigma} bxf(x) + \gamma_{s} \frac{\lambda}{\sigma} xf''(x).
\]

Comparing this to (1.14) in \[30\], we can see that the individual law mainly affects the evolution of the population in the following three ways:
(1) The branching rate and immigration rate are changed proportionally with proportionality constants \( \lambda \) and \( \zeta \), respectively.
(2) The jump sizes derived from branching and immigration are changed proportionally with proportionality constant \( \eta/\sigma \).
(3) The randomness of individual’s life-length increases the volatility of population by \( \gamma \lambda/\sigma \).

In order to emphasize the impact factors, we say \( \{Z(t) : t \geq 0\} \) is a CBI-process with branching law \((\lambda, \phi)\), immigration law \((\zeta, \psi)\) and individual parameter \((b, \eta, \sigma)\).

**Example 4.1.** (Homogeneous, binary CMJ-processes) Suppose \( p_1^{(n)} = 1, \zeta^{(n)} = 0 \) and \( \gamma_n = n \) for any \( n \geq 1 \). Under conditions in Theorem 4.5, the sequence \( \{Z^{(n)}/n\} \) converges weakly to \( Z \), which is a Feller diffusion

\[
Z(t) = Z(0) - \int_0^t \frac{b}{\sigma \lambda} Z(s) ds + \int_0^t \int_0^s \sqrt{\frac{2\lambda}{\sigma}} W(ds, du).
\]

This weak convergence also have been considered in [35, Theorem 5.4].

**5. Asymptotic properties of resolvent kernels.** From Theorem 3.2, we can see that the resolvent kernels \( \{R^{(n)}(t) : t \geq 0\} \) and \( \{R^{(n)}(t, k, y) : t \geq 0, (k, y) \in \mathbb{Z}_+ \times \mathbb{R}_+^2\} \) play an important role in the study of \( Z^{(n)} \). In this section, we mainly study their asymptotic behaviors, which will be widely used in the proof of Theorem 4.5. Here we always assume that Condition 4.1 and 4.3 hold. Recall the parameter \( \beta \in (0, \infty) \cap (\frac{b+m}{\lambda \sigma \gamma}, \infty) \). For any \( t \geq 0 \) and \( (k, y) \in \mathbb{Z}_+ \times \mathbb{R}_+^2 \), define

\[
\bar{\Lambda}^{(n)}_{\beta}(t) := e^{-\frac{\beta}{\gamma n} t} \bar{\Lambda}^{(n)}(t), \quad \bar{R}^{(n)}_{\beta}(t) := e^{-\frac{\beta}{\gamma n} t} \bar{R}^{(n)}(t) \quad \text{and} \quad \bar{R}^{(n)}_{\beta}(t, k, y) := e^{-\frac{\beta}{\gamma n} t} \bar{R}^{(n)}(t, k, y).
\]

From (3.12) and (3.13), we can see that \( \bar{R}^{(n)}_{\beta}(t) \) and \( \bar{R}^{(n)}_{\beta}(t, k, y) \) satisfy

\[
\bar{R}^{(n)}_{\beta}(t) = \lambda^{(n)} m^{(n)} \bar{\Lambda}^{(n)}_{\beta}(t) + \lambda^{(n)} m^{(n)} \int_0^t \bar{R}^{(n)}_{\beta}(t-s) \bar{\Lambda}^{(n)}_{\beta}(s) ds
\]

(5.1)

\[
= \lambda^{(n)} m^{(n)} \bar{\Lambda}^{(n)}_{\beta}(t) + \lambda^{(n)} m^{(n)} \int_0^t \bar{\Lambda}^{(n)}_{\beta}(t-s) \bar{R}^{(n)}_{\beta}(s) ds
\]

and

\[
\bar{R}^{(n)}_{\beta}(t, k, y) = \lambda^{(n)} e^{-\frac{\beta}{\gamma n} t} \sum_{j=1}^k 1_{(y_j > t)} + \lambda^{(n)} \int_0^t \bar{R}^{(n)}_{\beta}(t-s) e^{-\frac{\beta}{\gamma n} s} \sum_{j=1}^k 1_{(y_j > s)} ds
\]

(5.2)

\[
= \sum_{j=1}^k \bar{R}^{(n)}_{\beta}(t, 1, y_j).
\]

**Lemma 5.1.** Recall \( \eta^{(n)}_{\beta} \) defined by (4.7). We have

\[
\lim_{n \to \infty} \gamma_n (1 - \lambda^{(n)} m^{(n)} \eta^{(n)}_{\beta}) = \lim_{n \to \infty} \gamma_n \left(1 - \lambda^{(n)} m^{(n)} \int_0^\infty \bar{\Lambda}^{(n)}_{\beta}(t) dt\right) = b + m + \beta \sigma \lambda > 0.
\]

**Proof.** From Condition 4.1(1) and the definition of \( \eta^{(n)}_{\beta} \),

\[
\lim_{n \to \infty} \gamma_n \left(1 - \lambda^{(n)} m^{(n)} \int_0^\infty e^{-\frac{\beta}{\gamma n} t} \bar{\Lambda}^{(n)}(t) dt\right)
\]
\[
\begin{align*}
&= \lim_{n \to \infty} \gamma_n \left( 1 - \lambda^{(n)} \eta^{(n)} m^{(n)} \right) + \lim_{n \to \infty} \lambda^{(n)} m^{(n)} \gamma_n \left( \eta^{(n)} - \int_{0}^{\infty} e^{-\frac{\beta}{\gamma_n} t} \Lambda^{(n)}(t) dt \right) \\
&= b + m + \beta \lambda \lim_{n \to \infty} \int_{0}^{\infty} t \Lambda^{(n)}(t) dt + \lambda \lim_{n \to \infty} \int_{0}^{\infty} \gamma_n \left[ 1 - \frac{\beta}{\gamma_n} t - e^{-\frac{\beta}{\gamma_n} t} \right] \Lambda^{(n)}(t) dt \\
&= b + m + \beta \sigma \lambda + \lambda \lim_{n \to \infty} \int_{0}^{\infty} \gamma_n \left[ 1 - \frac{\beta}{\gamma_n} t - e^{-\frac{\beta}{\gamma_n} t} \right] \Lambda^{(n)}(t) dt.
\end{align*}
\]

By Condition 4.3(2) and the dominated convergence theorem,
\[
\lim_{n \to \infty} \int_{0}^{\infty} \gamma_n \left| 1 - \frac{\beta}{\gamma_n} t - e^{-\frac{\beta}{\gamma_n} t} \right| \Lambda^{(n)}(t) dt \leq \lim_{n \to \infty} \int_{0}^{\infty} \gamma_n \left| 1 - \frac{\beta}{\gamma_n} t - e^{-\frac{\beta}{\gamma_n} t} \right| \Lambda^*(t) dt \\
= \lim_{n \to \infty} \int_{0}^{\infty} \left[ \beta t \wedge \frac{\beta^2 t^2}{\gamma_n} \right] \Lambda^*(t) dt \\
= \int_{0}^{\infty} \lim_{n \to \infty} \left[ \beta t \wedge \frac{\beta^2 t^2}{\gamma_n} \right] \Lambda^*(t) dt = 0.
\]

Taking this back into (5.4), we will get the desired result directly. \(\Box\)

**Lemma 5.2.** There exist constants \(n_0 \geq 1\) and \(C > 0\) such that for any \(n > n_0\),

\[
\sup_{t \geq 0} R^{(n)}_{\beta}(t) \leq C.
\]

**Proof.** From Lemma 5.1, there exists \(n_0 \geq 1\) such that for any \(n \geq n_0\),

\[
1 - \lambda^{(n)} \eta^{(n)}_\beta m^{(n)} > 0.
\]

Without loss of generality, we always assume that this inequality holds for all \(n \geq 1\). Let

\[
f^{(n)}_{\beta}(t) := |\eta^{(n)}_\beta|^{-1} \Lambda^{(n)}_{\beta}(t) \mathbf{1}_{\{t > 0\}},
\]

which is a probability density on \(\mathbb{R}\). From (5.1), we have

\[
R^{(n)}_{\beta}(t) = \lambda^{(n)} \eta^{(n)}_\beta m^{(n)} f^{(n)}_{\beta}(t) + \lambda^{(n)} \eta^{(n)}_\beta m^{(n)} \int_{0}^{t} R^{(n)}_{\beta}(t-s) f^{(n)}_{\beta}(s) ds.
\]

It is easy to check that the resolvent kernel \(R^{(n)}_{\beta}(t)\) can be rewritten as

\[
R^{(n)}_{\beta}(t) = \sum_{k=1}^{\infty} \left[ \lambda^{(n)} \eta^{(n)}_\beta m^{(n)} f^{(n)}_{\beta} \right]^{(sk)}(t).
\]

Let \(\{X^{(n)}_i : i = 1, 2, \cdots\}\) be a sequence of i.i.d. random variables with probability density \(f^{(n)}_{\beta}(t)\) and \(N^{(n)}_G\) be a geometric random variable on \(\mathbb{Z}_+\) with parameter \(1 - \lambda^{(n)} \eta^{(n)}_\beta m^{(n)}\). Let \(g^{(n)}_N(t)\) be the density of the geometric summation \(\frac{1}{n} \sum_{i=1}^{N^{(n)}_G} X^{(n)}_i\). From Lemma A.3 in Appendix A, there exists a constant \(C\) independent of \(n\) such that \(\sup_{t \in \mathbb{R}} g^{(n)}_N(t) < C\). Moreover, we also have

\[
g^{(n)}_N(t) = \gamma_n (1 - \lambda^{(n)} \eta^{(n)}_\beta m^{(n)}) \sum_{k=1}^{\infty} \left[ \lambda^{(n)} \eta^{(n)}_\beta m^{(n)} f^{(n)}(t) \right]^{(sk)}(\gamma_n t) = \gamma_n (1 - \lambda^{(n)} \eta^{(n)}_\beta m^{(n)}) \frac{R^{(n)}_{\beta}(\gamma_n t)}{(\lambda^{(n)} \eta^{(n)}_\beta m^{(n)})}.
\]
and
\[ R^{(n)}_\beta(\gamma nt) \leq C \frac{\lambda^{(n)}\eta^{(n)}_\beta m^{(n)}}{\gamma_n(1 - \lambda^{(n)}\eta^{(n)}_\beta m^{(n)})}. \]

The desired result follows directly from this and Lemma 5.1.

**Proposition 5.3.** We have
\[ \lim_{n \to \infty} \int_0^\infty R^{(n)}_\beta(\gamma nt)dt \leq \frac{1}{b + m + \beta\sigma\lambda} < \infty. \]  

Moreover, there exist constants \( C > 0 \) and \( n_0 \geq 1 \) such that for any \( n > n_0 \) and \( \kappa \geq 1 \),
\[ \int_0^\infty |R^{(n)}_\beta(\gamma nt)|^\kappa dt \leq C^\kappa. \]

**Proof.** From (5.1), we can see that
\[ R^{(n)}_\beta(\gamma nt) = \lambda^{(n)}m^{(n)}\bar{\Lambda}^{(n)}_\beta(\gamma nt) + \lambda^{(n)}m^{(n)} \int_0^{\gamma nt} R^{(n)}_\beta(\gamma nt - s)\bar{\Lambda}^{(n)}_\beta(s)ds. \]

Integrating both sides of this equation, we have
\[ \int_0^\infty R^{(n)}_\beta(\gamma nt)dt = \int_0^\infty \bar{\Lambda}^{(n)}_\beta(\gamma nt)dt + \int_0^\infty dt \int_0^{\gamma nt} R^{(n)}_\beta(\gamma nt - s)\bar{\Lambda}^{(n)}_\beta(s)ds. \]

Applying variable substitution, we have
\[ \int_0^\infty R^{(n)}_\beta(\gamma nt)dt = \frac{1}{\gamma_n} \int_0^\infty R^{(n)}_\beta(t)dt, \quad \int_0^\infty \bar{\Lambda}^{(n)}_\beta(\gamma nt)dt = \frac{1}{\gamma_n} \int_0^\infty \bar{\Lambda}^{(n)}_\beta(t)dt \]
and
\[ \int_0^\infty dt \int_0^{\gamma nt} R^{(n)}_\beta(\gamma nt - s)\bar{\Lambda}^{(n)}_\beta(s)ds = \frac{1}{\gamma_n} \int_0^\infty R^{(n)}_\beta(t)dt \int_0^\infty \bar{\Lambda}^{(n)}_\beta(t)dt. \]

Putting all results above together, we have
\[ \int_0^\infty R^{(n)}_\beta(t)dt = \frac{\lambda^{(n)}m^{(n)} \int_0^\infty \bar{\Lambda}^{(n)}_\beta(t)dt}{1 - \lambda^{(n)}m^{(n)} \int_0^\infty \bar{\Lambda}^{(n)}_\beta(t)dt} = \frac{\lambda^{(n)}m^{(n)}\eta^{(n)}_\beta}{1 - \lambda^{(n)}m^{(n)}\eta^{(n)}_\beta} \]
and
\[ \int_0^\infty R^{(n)}_\beta(\gamma nt)dt = \frac{\lambda^{(n)}m^{(n)}\eta^{(n)}_\beta}{\gamma_n(1 - \lambda^{(n)}m^{(n)}\eta^{(n)}_\beta)}. \]

From this and (5.3), we can get (5.6) directly. Moreover, (5.7) follows directly from (5.5) and (5.6).

Here we have finished the proof.

Now we start to consider the convergence of the sequence \( \{R^{(n)}(nt) : t \geq 0\}_{n \geq 1} \). From Proposition 5.3, their Fourier transforms can be well defined. In the following lemma, we will show that their Fourier transforms converge to the characteristic function of an exponential function.
Lemma 5.4. For any $u \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \int_0^\infty e^{iut} R_\beta^{(n)}(\gamma_n t) dt = \int_0^\infty e^{i ut} \frac{1}{\sigma \lambda} \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} dt. \quad (5.9)$$

**Proof.** Applying Fourier transform on the both sides of (5.1), we have

$$\int_0^\infty e^{iut} R_\beta^{(n)}(\gamma_n t) dt = \lambda^{(n)} m^{(n)} \int_0^\infty e^{i ut} \tilde{\Lambda}_\beta^{(n)}(\gamma_n t) dt$$

$$+ \lambda^{(n)} m^{(n)} \int_0^\infty e^{i ut} \int_0^{\gamma_n t} R_\beta^{(n)}(\gamma_n t - s) \tilde{\Lambda}_\beta^{(n)}(s) ds. \quad (5.10)$$

Applying variable substitution, we have

$$\int_0^\infty e^{iut} R_\beta^{(n)}(\gamma_n t) dt = \frac{1}{\gamma_n} \int_0^\infty e^{i \frac{bt}{\gamma_n}} R_\beta^{(n)}(t) dt, \quad \int_0^\infty e^{iut} \tilde{\Lambda}_\beta^{(n)}(\gamma_n t) dt = \frac{1}{\gamma_n} \int_0^\infty e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(t) dt$$

and

$$\int_0^\infty e^{iut} \int_0^{\gamma_n t} R_\beta^{(n)}(\gamma_n t - s) \tilde{\Lambda}_\beta^{(n)}(s) ds = \frac{1}{\gamma_n} \int_0^\infty e^{i \frac{ut}{\gamma_n}} R_\beta^{(n)}(t) dt \int_0^\infty e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(s) ds.$$ 

Taking them back to (5.10), we have

$$\int_0^\infty e^{i \frac{bt}{\gamma_n} t} R_\beta^{(n)}(t) dt = \frac{\lambda^{(n)} m^{(n)}}{1 - \lambda^{(n)} m^{(n)}} \int_0^\infty e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(t) dt \quad (5.11)$$

and

$$\int_0^\infty e^{iut} R_\beta^{(n)}(\gamma_n t) dt = \frac{\lambda^{(n)} m^{(n)} \int_0^\infty e^{i \frac{bt}{\gamma_n} t} \tilde{\Lambda}_\beta^{(n)}(t) dt}{\gamma_n \left[ 1 - \lambda^{(n)} m^{(n)} \int_0^\infty e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(t) dt \right]}$$

$$= \frac{\lambda^{(n)} m^{(n)} \int_0^\infty e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(t) dt}{\gamma_n \left[ 1 - \lambda^{(n)} \eta^{(n)} \right] m^{(n)} + \lambda^{(n)} m^{(n)} \gamma_n \left[ \eta^{(n)} - \int_0^\infty e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(t) dt \right]} \quad (5.12)$$

From Condition 4.3(2) and the dominated convergence theorem,

$$\lim_{n \to \infty} \int_0^\infty \left| 1 - e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(t) dt \right| = 0.$$ 

From this and (4.5),

$$\lim_{n \to \infty} \lambda^{(n)} m^{(n)} \int_0^\infty e^{i \frac{ut}{\gamma_n}} \tilde{\Lambda}_\beta^{(n)}(t) dt = \lim_{n \to \infty} \lambda^{(n)} m^{(n)} \eta^{(n)} - \lim_{n \to \infty} \int_0^\infty \left[ 1 - e^{i \frac{ut}{\gamma_n}} \right] \tilde{\Lambda}_\beta^{(n)}(t) dt = 1.$$

Moreover, since $|e^{iy} - 1 - iy| \leq |y| \wedge |y|^2$ for any $y \in \mathbb{R}$, applying the dominated convergence theorem again, we have

$$\lim_{n \to \infty} \int_0^\infty \gamma_n \left| e^{i \frac{ut}{\gamma_n}} t - 1 - i \frac{ut}{\gamma_n} \right| \tilde{\Lambda}_\beta^{(n)}(t) dt \leq \lim_{n \to \infty} \int_0^\infty \left( \frac{ut}{\gamma_n} \wedge \frac{|ut|^2}{\gamma_n} \right) \tilde{\Lambda}_\beta^{(n)}(t) dt$$

$$= \int_0^\infty \lim_{n \to \infty} \left( \frac{ut}{\gamma_n} \wedge \frac{|ut|^2}{\gamma_n} \right) \tilde{\Lambda}_\beta^{(n)}(t) dt = 0.$$
∫_0^∞ e^{it λ} R_β^n(γ_n t) dt = \frac{1}{b + m + βσλ - iuσλ} 
= \frac{1}{σλ(b + m + βσλ) + β - iu} 
= \int_0^∞ e^{iu t} \frac{1}{σλ} \exp \left\{ - \left( \frac{b + m + β}{σλ} + β \right) t \right\} dt.

Here we have finished the proof.  

**Lemma 5.5.** There exist constants C > 0 and n_0 ≥ 1 such that for any (k, y) ∈ \mathbb{Z}_+ × \mathbb{R}_+ and n ≥ n_0,

\[ \sup_{t ≥ 0} R_β^n(t, k, y) ≤ C \sum_{j=1}^k (1 + y_j). \]

**Proof.** From (5.2) and (5.5), we have

\[ R_β^n(t, k, y) = λ^n \sum_{j=1}^k \left[ e^{-\frac{β}{m}} 1_{\{y_j > t\}} + \int_0^{t \land y_j} R_β^n(t - s) e^{-\frac{β}{m} s} ds \right] ≤ λ^n \sum_{j=1}^k (1 + Cy_j). \]

The desired result follows directly from this inequality.  

**Proposition 5.6.** For any (k, y) ∈ \mathbb{Z}_+ × \mathbb{R}_+^Z, we have

\[ \lim_{n \to \infty} \int_0^∞ R_β^n(γ_n t, k, y) dt = \frac{λ \sum_{j=1}^k y_j}{b + m + βσλ} < ∞. \]

Moreover, there exist constants C > 0 and n_0 ≥ 1 such that for any n > n_0 and κ ≥ 1,

\[ \int_0^∞ \left| R_β^n(γ_n t, k, y) \right|^κ dt ≤ C^κ \left[ \sum_{j=1}^k (1 + y_j) \right]^κ. \]

**Proof.** Integrating both sides of (5.2), we have

\[ \int_0^∞ R_β^n(γ_n t, k, y) dt = λ^n \sum_{j=1}^k \left[ \int_0^∞ e^{-βt} 1_{\{y_j > γ_n t\}} dt + \int_0^∞ dt \int_0^{γ_n t} R_β^n(γ_n t - s) e^{-\frac{β}{m} s} 1_{\{y_j > s\}} ds \right]. \]

Applying variable substitution, we have

\[ \int_0^∞ R_β^n(γ_n t, k, y) dt = \frac{1}{γ_n} \int_0^∞ R_β^n(t, k, y) dt \quad \int_0^∞ e^{-βt} 1_{\{y_j ≥ γ_n t\}} dt = \frac{1}{γ_n} \int_{y_j} e^{-\frac{β}{m} t} dt. \]
and
\[ \int_0^\infty dt \int_0^{\gamma_n t} e^{-\frac{\beta}{\gamma_n}(\gamma_n t - s)} R_\gamma^{(n)}(\gamma_n t - s) e^{-\frac{\beta}{\gamma_n}s} 1_{\{y_j > s\}}ds = \frac{1}{\gamma_n} \int_0^\infty R_\gamma^{(n)}(t) dt \int_0^{y_j} e^{-\frac{\beta}{\gamma_n}t} dt. \]

Based on all results above, we have
\[ \int_0^\infty R_\gamma^{(n)}(t, k, y) dt = \lambda^{(n)} \sum_{j=1}^k \int_0^{y_j} e^{-\frac{\beta}{\gamma_n}t} dt \left[ 1 + \int_0^\infty R_\gamma^{(n)}(t) dt \right] \]
and
\[ \int_0^\infty R_\gamma^{(n)}(\gamma_n t, k, y) dt = \lambda^{(n)} \sum_{j=1}^k \int_0^{y_j} e^{-\frac{\beta}{\gamma_n}t} dt \times \frac{1}{\gamma_n} \left[ 1 + \int_0^\infty R_\gamma^{(n)}(t) dt \right]. \]

From (5.3) and (5.8),
\[ \lim_{n \to \infty} \frac{1}{\gamma_n} \left[ 1 + \int_0^\infty R_\gamma^{(n)}(t) dt \right] = \lim_{n \to \infty} \frac{1}{\gamma_n (1 - \lambda^{(n)} m^{(n)})} = \frac{1}{b + m + \beta \sigma \lambda}. \]

By the dominated convergence theorem,
\[ \lim_{n \to \infty} \sum_{j=1}^k \int_0^{y_j} e^{-\frac{\beta}{\gamma_n}t} dt = \frac{\lambda \sum_{j=1}^k y_j}{b + m + \beta \sigma \lambda} \]
Here we have gotten the first result. The second result follows directly from (5.13) and (5.14). \[ \square \]

**Lemma 5.7.** For any \((k, y) \in \mathbb{Z}_+ \times \mathbb{R}_+^2\) and \(u \in \mathbb{R}\), we have
\[ \lim_{n \to \infty} \int_0^\infty e^{int} R_\gamma^{(n)}(\gamma_n t, k, y) dt = \left( \sum_{j=1}^k y_j \right) \int_0^\infty e^{int} \frac{1}{\sigma} \exp \left\{ -\left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} dt. \]

**Proof.** Applying Fourier transform on the both sides of (5.2), we have
\[ \int_0^\infty e^{int} R_\gamma^{(n)}(\gamma_n t, k, y) dt = \sum_{j=1}^k \int_0^\infty e^{int} R_\gamma^{(n)}(\gamma_n t, 1, y_j) dt \]
and
\[ \int_0^\infty e^{int} R_\gamma^{(n)}(\gamma_n t, 1, y_j) dt = \lambda^{(n)} \int_0^\infty e^{int} e^{-\beta t} 1_{\{y_j > \gamma_n t\}} dt + \lambda^{(n)} \int_0^\infty e^{int} \int_0^{y_j} \gamma_n t \ R_\gamma^{(n)}(\gamma_n t - s) e^{-\frac{\beta}{\gamma_n}s} 1_{\{y_j > s\}} ds. \]

Applying variable substitution, we have
\[ \int_0^\infty e^{int} e^{-\beta t} 1_{\{y_j > \gamma_n t\}} dt = \frac{1}{\gamma_n} \int_0^{y_j} e^{-\frac{\beta}{\gamma_n}t} e^{-\beta t} dt, \]
\[
\int_{0}^{\infty} e^{i\omega t} R_{\beta}^{(n)}(\gamma n t, 1, y_j) dt = \frac{1}{\gamma_n} \int_{0}^{\infty} e^{i\frac{\omega}{\gamma_n} t} R_{\beta}^{(n)}(t, 1, y_j) dt
\]

and
\[
\int_{0}^{\infty} e^{i\omega t} dt \int_{0}^{\gamma n t} R_{\beta}^{(n)}(\gamma n t - s) e^{-\frac{\beta}{\gamma_n} s} 1_{\{y_j > s\}} ds = \frac{1}{\gamma_n} \int_{0}^{\infty} e^{i\frac{\omega}{\gamma_n} t} R_{\beta}^{(n)}(t) dt \int_{0}^{y_j} e^{i\frac{\omega}{\gamma_n} t} e^{-\frac{\beta}{\gamma_n} t} dt.
\]

Putting them together, we have
\[
\int_{0}^{\infty} e^{i\frac{\omega}{\gamma_n} t} R_{\beta}^{(n)}(t, 1, y_j) dt = \frac{\lambda(n) \int_{0}^{y_j} e^{i\frac{\omega}{\gamma_n} t} e^{-\frac{\beta}{\gamma_n} t} dt}{1 - \lambda(n) m(n) \int_{0}^{\infty} e^{i\frac{\omega}{\gamma_n} t} \Lambda_{\beta}^{(n)}(t) dt}
\]

and
\[
(5.18) \quad \int_{0}^{\infty} e^{i\omega t} R_{\beta}^{(n)}(\gamma n t, 1, y_j) dt = \frac{\lambda(n) \int_{0}^{y_j} e^{i\frac{\omega}{\gamma_n} t} e^{-\frac{\beta}{\gamma_n} t} dt}{\gamma_n [1 - \lambda(n) m(n) \int_{0}^{\infty} e^{i\frac{\omega}{\gamma_n} t} \Lambda_{\beta}^{(n)}(t) dt]}
\]

Like the argument in the proof of Lemma 5.4, we have
\[
\lim_{n \to \infty} \int_{0}^{\infty} e^{i\omega t} R_{\beta}^{(n)}(\gamma n t, 1, y_j) dt = y_j \int_{0}^{\infty} e^{i\omega t} \frac{1}{\sigma} \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} dt.
\]

The desired result follows directly by taking this back to (5.17). \qed

In Lemma 5.4 and 5.7, we have proved the convergence of Fourier transforms of the sequences \( \{R_{\beta}^{(n)}(nt) : t \geq 0\}_{n \geq 1} \) and \( \{R_{\beta}^{(n)}(nt, k, y) : t \geq 0\}_{n \geq 1} \). From the Fourier isometry, we are going to prove the \( \mathcal{L}^2 \)-convergence in the follows. Firstly, we need to give some uniform bound estimations for their Fourier transforms.

**Proposition 5.8.** There exist constants \( T_0 > 0 \) and \( n_0 \geq 1 \) such that for any \( n \geq n_0 \) and \( |u| < 1/T_0 \),
\[
(5.19) \quad \left| \int_{0}^{\infty} \sin(ut) \Lambda_{\beta}^{(n)}(t) dt \right| \geq \frac{3}{16} \sigma |u|.
\]

**Proof.** From Condition 4.1(1), there exist constants \( n_0 \geq 1 \) and \( T_0 > 0 \) such that for any \( n \geq n_0 \),
\[
\int_{0}^{T_0} \frac{t \Lambda_{\beta}^{(n)}(t) dt}{n} = \int_{0}^{T_0} t e^{-\frac{\beta}{\gamma_n} t} \Lambda_{\beta}^{(n)}(t) dt \geq \frac{3}{4} \sigma \quad \text{and} \quad \int_{0}^{T_0} t \Lambda_{\beta}^{(n)}(t) dt \geq \frac{5}{8} \sigma.
\]

Moreover, from Condition 4.3(2), we can always find some constant \( T_0 > 0 \) large enough such that
\[
\int_{T_0}^{\infty} \frac{t \Lambda_{\beta}^{(n)}(t) dt}{n} \leq \int_{T_0}^{\infty} t e^{-\frac{\beta}{\gamma_n} t} \Lambda_{\beta}^{(n)}(t) dt \leq \frac{1}{8} \sigma.
\]

Since \( \cos(x) \geq 1/2 \) for any \( |x| \leq 1 \), we have for any \( |u| < 1/T_0 \),
\[
\frac{\partial}{\partial u} \int_{0}^{\infty} \sin(ut) \Lambda_{\beta}^{(n)}(t) dt = \int_{0}^{\infty} \cos(ut) t \Lambda_{\beta}^{(n)}(t) dt
\]
From the mean value theorem, we have for any $|u| < 1/T_0$,

$$| \int_0^\infty \sin(ut) \tilde{\Lambda}^{(n)}_\beta(t) dt | \geq \frac{3}{16}\sigma |u|.$$  

Here is the end of the proof. \hfill \Box

**Proposition 5.9.** There exists a constant $C > 0$ such that for any $u \in \mathbb{R}$,

$$\left| \int_0^\infty e^{int} \tilde{\Lambda}^{(n)}_\beta(t) dt \right| \leq C \left( \frac{1}{|u|} \wedge 1 \right).$$  

**Proof.** From Condition 4.3(2), we have

$$\left| \int_0^\infty e^{int} \tilde{\Lambda}^{(n)}_\beta(t) dt \right| \leq \int_0^\infty \tilde{\Lambda}^{(n)}_\beta(t) dt \leq C.$$

For any $\epsilon > 0$ and $n \geq 1$, we can find a non-increasing function $g^{(n)}_\epsilon(t)$ defined on $\mathbb{R}_+$ satisfying that for any $t \geq 0$,

$$g^{(n)}_\epsilon(t) \leq 2\tilde{\Lambda}^*(t) \quad \text{and} \quad \int_0^\infty |\tilde{\Lambda}^{(n)}(t) - g^{(n)}_\epsilon(t)| dt \leq \epsilon.$$

Thus

$$\left| \int_0^\infty e^{int} \tilde{\Lambda}^{(n)}_\beta(t) dt \right| \leq \left| \int_0^\infty e^{int} e^{-\frac{\beta}{m}t} (\tilde{\Lambda}^{(n)}(t) - g^{(n)}_\epsilon(t)) dt \right| + \left| \int_0^\infty e^{int} e^{-\frac{\beta}{m}t} g^{(n)}_\epsilon(t) dt \right|$$

$$\leq \epsilon + \frac{g^{(n)}_\epsilon(0)}{|u|} + \int_0^\infty \frac{1}{iu} \frac{\partial}{\partial t} e^{int} e^{-\frac{\beta}{m}t} \frac{\partial}{\partial t} e^{-\frac{\beta}{m}t} g^{(n)}_\epsilon(t) dt$$

$$\leq \epsilon + \frac{g^{(n)}_\epsilon(0)}{|u|} + \frac{1}{|u|} \int_0^\infty \left| \frac{\partial}{\partial t} e^{-\frac{\beta}{m}t} g^{(n)}_\epsilon(t) \right| dt = \epsilon + \frac{2g^{(n)}_\epsilon(0)}{|u|} \leq \epsilon + \frac{4}{|u|}.$$

The desired result follows directly from the arbitrariness of $\epsilon$. \hfill \Box

**Proposition 5.10.** For any $\vartheta > 0$, there exist constants $n_\vartheta \geq 1$ and $\epsilon > 0$ such that for any $n \geq n_\vartheta$ and $|u| \geq \vartheta$,

$$\left| 1 - \lambda^{(n)} m^{(n)} \int_0^\infty e^{int} \tilde{\Lambda}^{(n)}_\beta(t) dt \right| \geq \epsilon.$$

**Proof.** From Proposition 5.9, there exists a constant $u_0 > 0$ such that for any $n \geq 1$ and $|u| \geq u_0$,

$$\left| \int_0^\infty e^{int} \tilde{\Lambda}^{(n)}_\beta(t) dt \right| \leq \frac{1}{2}.$$

If $\vartheta \geq u_0$, the desired result follows directly from this and (4.2). Here we assume $\vartheta < u_0$. From Lemma 5.1, without loss of generality we may assume that

$$\lambda^{(n)} \eta^{(n)}_{\beta} m^{(n)} \leq 1.$$
Moreover, since \( \cos(\theta) \) is periodic and \( |\eta_\beta^{(n)}|^{-1}\tilde{\Lambda}_{\beta}^{(n)}(t) \) is a non-increasing probability density, thus \( u_n \) is well defined and for any \( r \geq \frac{\pi}{2u_n} \),

\[
\int_{\frac{\pi}{2u_n}}^{r} \cos(u_n t)|\eta_\beta^{(n)}|^{-1}\tilde{\Lambda}_{\beta}^{(n)}(t)dt \leq 0.
\]

Moreover, since \( \cos(u_n t) \) is decreasing on \([0, \frac{\pi}{2u_n}]\), we have

\[
\int_{0}^{\frac{\pi}{2u_n}} \cos(u_n t)|\eta_\beta^{(n)}|^{-1}\tilde{\Lambda}_{\beta}^{(n)}(t)dt \leq |\eta_\beta^{(n)}|^{-1}\int_{0}^{\frac{\pi}{2u_n}} \cos(u_n t)dt \leq \frac{\sin(u_n \eta_\beta^{(n)})}{u_n \eta_\beta^{(n)}}.
\]

From Condition 4.1(1), we have \( \eta_\beta^{(n)} \to \eta \) and \( u_n \eta_\beta^{(n)} \geq \vartheta \eta / 2 \) for \( n \) large enough. If \( \vartheta \eta \leq \pi \), we have

\[
\frac{\sin(u_n \eta_\beta^{(n)})}{u_n \eta_\beta^{(n)}} \leq \frac{\sin(\vartheta \eta / 2)}{\vartheta \eta / 2} < 1 < \frac{2}{\pi}.
\]

If \( \vartheta \eta \geq \pi \), we also have

\[
\frac{\sin(u_n \eta_\beta^{(n)})}{u_n \eta_\beta^{(n)}} \leq \frac{2}{\pi}.
\]

Putting all results above together, we can find a constant \( \epsilon \in (0, 1) \) such that

\[
\sup_{u \geq \eta} \max_{1 \leq n \leq [\vartheta, u_0]} \left\{ \int_{0}^{\infty} \cos(ut)|\eta_\beta^{(n)}|^{-1}\tilde{\Lambda}_{\beta}^{(n)}(t)dt \right\} < 1 - \epsilon
\]

and

\[
\sup_{u \geq \eta} \left| 1 - \lambda^{(n)} m^{(n)} \int_{0}^{\infty} e^{int}\tilde{\Lambda}_{\beta}^{(n)}(t)dt \right| \geq \frac{1}{2} \land \epsilon.
\]

Here we have finished the proof. \( \square \)

**Proposition 5.11.** There exists a constant \( C > 0 \) such that

\[
(5.22) \quad \left| \int_{0}^{\infty} e^{int} R_{\beta}^{(n)}(\gamma_n t)dt \right| \leq C \left( 1 + \frac{1}{|u|} \right).
\]

Moreover, for any \( (k, y) \in \mathbb{Z}_+ \times \mathbb{R}_+ \) and \( u \in \mathbb{R} \), we also have

\[
(5.23) \quad \left| \int_{0}^{\infty} e^{int} R_{\beta}^{(n)}(\gamma_n t, k, y)dt \right| \leq C \sum_{j=1}^{k} (1 + y_j) \left( 1 + \frac{1}{|u|} \right).
\]
PROOF. It is easy to see that (5.22) follows directly from (5.23), i.e.
\[
\left| \int_0^\infty e^{iut} R_\beta^{(n)} (\gamma_n t) dt \right| = \left| \int_{Z_+} \int_{R_+} \int_0^\infty e^{iut} R_\beta^{(n)} (\gamma_n t, k, y) dt \nu_0^{(n)} (dk, dy) \right|
\leq \int_{Z_+} \int_{R_+} \int_0^\infty e^{iut} R_\beta^{(n)} (\gamma_n t, k, y) dt \nu_0^{(n)} (dk, dy)
\leq \int_{Z_+} \int_{R_+} C \sum_{j=1}^k (1 + y_j) \nu_0^{(n)} (dk, dy) \left( 1 \wedge \frac{1}{|u|} \right) \leq C \left( 1 \wedge \frac{1}{|u|} \right).
\]

Now we start to prove (5.23). Firstly, from (5.15),
\[
\left| \int_0^\infty e^{iut} R_\beta^{(n)} (\gamma_n t, k, y) dt \right| \leq \int_0^\infty \left| e^{iut} \right| R_\beta^{(n)} (\gamma_n t, k, y) dt
\leq \int_0^\infty R_\beta^{(n)} (\gamma_n t, k, y) dt \leq C \sum_{j=1}^k (1 + y_j).
\]

Moreover, from (5.16) and (5.17),
\[
(5.24) \quad \left| \int_0^\infty e^{iut} R_\beta^{(n)} (\gamma_n t, k, y) dt \right| \leq \sum_{j=1}^k \frac{|\lambda^{(n)} \int_0^y e^{i\frac{ut}{\gamma_n}} t - \frac{\beta}{\gamma_n} t dt|}{\gamma_n |1 - \lambda^{(n)} m^{(n)} \int_0^\infty e^{i\frac{ut}{\gamma_n}} \Lambda^{(n)} _\beta (t) dt|}.
\]

From Proposition 5.8, for any $|u|/n \leq 1/T_0$,
\[
\gamma_n \left| 1 - \lambda^{(n)} m^{(n)} \int_0^\infty e^{i\frac{ut}{\gamma_n}} \Lambda^{(n)} _\beta (t) dt \right|
= \gamma_n \left| 1 - \lambda^{(n)} m^{(n)} \int_0^\infty \left[ \cos \left( \frac{ut}{\gamma_n} \right) + i \sin \left( \frac{ut}{\gamma_n} \right) \right] \Lambda^{(n)} _\beta (t) dt \right|
\geq \gamma_n \lambda^{(n)} m^{(n)} \int_0^\infty \sin \left( \frac{ut}{\gamma_n} \right) \Lambda^{(n)} _\beta (t) dt \geq \gamma_n \lambda^{(n)} m^{(n)} \frac{3 |u|}{16 \gamma_n} = \frac{3}{16} \lambda^{(n)} m^{(n)} \sigma|u|
\]
and
\[
\frac{|\lambda^{(n)} \int_0^y e^{i\frac{ut}{\gamma_n}} t - \frac{\beta}{\gamma_n} t dt|}{\gamma_n |1 - \lambda^{(n)} m^{(n)} \int_0^\infty e^{i\frac{ut}{\gamma_n}} \Lambda^{(n)} _\beta (t) dt|} \leq C \frac{y_j}{|u|}.
\]

From Proposition 5.9 and 5.10, for any $|u|/\gamma_n > 1/T_0$,
\[
\frac{|\lambda^{(n)} \int_0^y e^{i\frac{ut}{\gamma_n}} t - \frac{\beta}{\gamma_n} t dt|}{\gamma_n |1 - \lambda^{(n)} m^{(n)} \int_0^\infty e^{i\frac{ut}{\gamma_n}} \Lambda^{(n)} _\beta (t) dt|} \leq \frac{C \gamma_n}{\gamma_n |u|} \leq \frac{C}{|u|}.
\]

Taking these two estimation back to (5.24), we will get the desired result. \square

**Theorem 5.12.** For any $\kappa \geq 1$, we have
\[
(5.25) \quad \lim_{n \to \infty} \int_0^\infty \left| R_\beta^{(n)} (\gamma_n t) - \frac{1}{\sigma \lambda} \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} \right|^{2\kappa} dt = 0.
\]

Moreover, for any $(k, y) \in Z_+ \times R_+^k$,
\[
(5.26) \quad \lim_{n \to \infty} \int_0^\infty \left| R_\beta^{(n)} (\gamma_n t, k, y) - \frac{1}{\sigma} \left( \sum_{j=1}^k y_j \right) \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} \right|^{2\kappa} dt = 0.
\]
PROOF. Here we just prove the second result and the first one can be proved similarly. From Lemma 5.5,

\[
\int_0^\infty |R^{(n)}_{\beta}(\gamma_n t, k, y) - \frac{1}{\sigma} \left( \sum_{j=1}^k y_j \right) \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} |^{2\alpha} dt \\
\leq C \left( \sum_{j=1}^k (1 + y_j) \right)^{2\alpha - 2} \int_0^\infty \left| R^{(n)}_{\beta}(\gamma_n t, k, y) - \frac{1}{\sigma} \left( \sum_{j=1}^k y_j \right) \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} \right|^2 dt.
\]

From the Fourier isometry, Proposition 5.13 and the dominated convergence theorem, we have

\[
\lim_{n \to \infty} \int_0^\infty \left| R^{(n)}_{\beta}(\gamma_n t, k, y) - \frac{1}{\sigma} \left( \sum_{j=1}^k y_j \right) \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} \right|^2 dt = \lim_{n \to \infty} \int_0^\infty e^{iut} R^{(n)}_{\beta}(\gamma_n t, k, y) dt - \left( \sum_{j=1}^k y_j \right) \int_0^\infty e^{iut} \frac{1}{\sigma} \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} dt = 0.
\]

Here the last equality follows from Lemma 5.7. We have finished the proof. \(\square\)

**Corollary 5.13.** For any \(\delta \geq 0\), there exists a constant \(C > 0\) such that for any \(\kappa \geq 1\) and \((k, y) \in \mathbb{Z}_+ \times \mathbb{R}^{\mathbb{Z}_+} \),

\[
(5.27) \quad \int_0^\infty \left| R^{(n)}_{\beta}(\gamma_n(t + \delta), k, y) - R^{(n)}_{\beta}(\gamma_n t, k, y) \right|^{2\kappa} dt \leq C \left( \sum_{j=1}^k (1 + y_j) \right)^{2\kappa} (\delta + \delta^2).
\]

**Proof.** We firstly prove this result with \(\kappa = 1\). Let

\[
(5.28) \quad \Delta_{\gamma_n \delta} R^{(n)}_{\beta}(\gamma_n t, k, y) := R^{(n)}_{\beta}(\gamma_n(t + \delta), k, y) - R^{(n)}_{\beta}(\gamma_n t, k, y).
\]

Applying variable substitution, we have

\[
\int_\mathbb{R} e^{iut} \Delta_{\gamma_n \delta} R^{(n)}_{\beta}(\gamma_n t, k, y) dt = (e^{-i\delta} - 1) \int_\mathbb{R} e^{iut} R^{(n)}_{\beta}(\gamma_n t, k, y) dt.
\]

It is easy to see that \(|e^{-iut} - 1| \leq |ut| \wedge 2\). From Proposition 5.11,

\[
\int_\mathbb{R} \left| \Delta_{\gamma_n \delta} R^{(n)}_{\beta}(\gamma_n t, k, y) \right|^2 dt = \int_\mathbb{R} \int_\mathbb{R} e^{ius} \Delta_{\gamma_n \delta} R^{(n)}_{\beta}(\gamma_n t, k, y) dt \right|^2 du \\
\leq C \left( \sum_{j=1}^k (1 + y_j) \right)^2 \int_\mathbb{R} \left( |u\delta|^2 \wedge 4 \right) \times \left( 1 \wedge \frac{1}{|u|^2} \right) du.
\]

By some simple calculation, we have

\[
\int_\mathbb{R} \left( |u\delta|^2 \wedge 4 \right) \left( 1 \wedge \frac{1}{|u|^2} \right) du \leq \int_{|u| \leq 1} |u\delta|^2 du + \int_{1 < |u| \leq \frac{2}{\delta}} \frac{|u\delta|^2}{|u|^2} du + \int_{|u| > \frac{2}{\delta}} \frac{4}{|u|^2} du \leq C(\delta + \delta^2).
\]
When \( \kappa > 1 \), from Lemma 5.5,
\[
\int_{\mathbb{R}} \left| R^{(n)}_\beta (\gamma_n (t + \delta), k, y) - R^{(n)}_\beta (\gamma_n t, k, y) \right|^{2\kappa} dt
\]
\[
\leq C \left( \sum_{j=1}^{k} (1 + y_j) \right)^{2\kappa-2} \int_{\mathbb{R}} \left| R^{(n)}_\beta (\gamma_n (t + \delta), k, y) - R^{(n)}_\beta (\gamma_n t, k, y) \right|^{2} dt
\]
\[
\leq C \left( \sum_{j=1}^{k} (1 + y_j) \right)^{2\kappa} (\delta + \delta^2).
\]
Here we have finished the proof. \( \square \)

At the end of this section, we give a representation for the local integral of the kernel \( R^{(n)}_\beta (nt) \), which will be used in the proof of Theorem 4.5.

**Corollary 5.14.** For any \( T \geq 0 \), we have
\[
\int_0^T R^{(n)}_\beta (t) dt = \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \int_0^T R^{(n)}_\beta (T - t) \int_0^t \bar{\Lambda}^{(n)}_\beta (s) ds dt
\]
(5.29)
\[
= \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t) \int_0^{T-t} R^{(n)}_\beta (s) ds dt
\]
and
\[
\frac{1 - \lambda^{(n)} \eta^{(n)}_\beta m^{(n)}}{\lambda^{(n)} m^{(n)}} \int_T^\infty R^{(n)}_\beta (t) dt = \int_T^\infty \bar{\Lambda}^{(n)}_\beta (t) dt + \int_0^T \bar{\Lambda}^{(n)}_\beta (T - t) \int_t^\infty R^{(n)}_\beta (s) ds dt
\]
(5.30)
\[
= \int_T^\infty \bar{\Lambda}^{(n)}_\beta (t) dt + \int_0^T \bar{\Lambda}^{(n)}_\beta (T - t) \int_0^{T-t} R^{(n)}_\beta (s) ds dt.
\]

**Proof.** Integrating both sides of (5.1) on the interval \( (0, T] \), we have
\[
\int_0^T R^{(n)}_\beta (t) dt = \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t-s) R^{(n)}_\beta (s) ds dt
\]
\[
= \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \int_0^T R^{(n)}_\beta (t) \int_0^t \bar{\Lambda}^{(n)}_\beta (s-t) ds dt
\]
\[
= \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \int_0^T R^{(n)}_\beta (t) \int_0^{T-t} \bar{\Lambda}^{(n)}_\beta (s) ds dt
\]
\[
= \lambda^{(n)} m^{(n)} \int_0^T \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \int_0^T R^{(n)}_\beta (T-t) \int_0^t \bar{\Lambda}^{(n)}_\beta (s) ds dt.
\]
Here we have gotten the first equality in (5.29) and the second one can be proved similarly. Now we start to prove (5.30). Integrating both sides of (5.1) on \( (0, \infty) \) again, we obtain
\[
\int_0^\infty R^{(n)}_\beta (t) dt = \lambda^{(n)} m^{(n)} \int_0^\infty \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \int_0^\infty \bar{\Lambda}^{(n)}_\beta (t) dt \int_0^\infty R^{(n)}_\beta (s) ds.
\]
Splitting the integral intervals, we have
\[
\int_0^T R^{(n)}_\beta (t) dt + \int_T^\infty R^{(n)}_\beta (t) dt = \lambda^{(n)} m^{(n)} \left( \int_0^T \bar{\Lambda}^{(n)}_\beta (t) dt + \int_T^T R^{(n)}_\beta (T-t) \int_0^t \bar{\Lambda}^{(n)}_\beta (s) ds dt \right)
\]
\[ + \lambda^{(n)} m^{(n)} \int_0^T R^{(n)}_\beta (T-t) \int_t^\infty \bar{\Lambda}^{(n)}_\beta (s) ds dt \\
+ \lambda^{(n)} m^{(n)} \int_T^\infty \bar{\Lambda}^{(n)}_\beta (t) dt + \lambda^{(n)} m^{(n)} \eta^{(n)}_\beta \int_T^\infty R^{(n)}_\beta (t) dt. \]

From this and (5.29), we have
\[ [1 - \lambda^{(n)} m^{(n)} \eta^{(n)}_\beta] \int_T^\infty R^{(n)}_\beta (t) dt = \lambda^{(n)} m^{(n)} \left[ \int_T^\infty \bar{\Lambda}^{(n)}_\beta (t) dt + \int_T^\infty R^{(n)}_\beta (T-t) \int_t^\infty \bar{\Lambda}^{(n)}_\beta (s) ds dt \right]. \]

Here we have proved the first equality in (5.30) and the second one can be proved similarly. \[\Box\]

6. Proof for the main theorem. In this section, we give the proof for Theorem 4.5. In order to make the whole proof much clearer and easier to be understood, in the subsection 6.1 we just show the main idea of the proof. Some technical estimations for the error processes will be given in Subsection 6.3. The weak convergence of semimartingales driving the stochastic Volterra integral equations will be proved in Subsection 6.4.

6.1. Proof for Theorem 4.5. From Theorem 3.2, we can rewrite \(e^{-\beta t} Z^{(n)}(\gamma_n t)/n\) as
\[
\frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} = \frac{1}{n} e^{-\beta t} Z^{(n)}_\beta (\gamma_n t) + \frac{1}{n} \int_0^{\gamma_n t} e^{-\beta t} R^{(n)}(\gamma_n t - s) Z^{(n)}_\beta (s) ds \\
+ \frac{1}{n} \int_0^{\gamma_n t} \int_{Z^+} \int_{R^+} \int_0^{Z^{(n)}(s)} e^{-\beta t} R^{(n)}(\gamma_n t - s, k, y) N^{(n)}_1 (ds, dk, dy) \\
+ \frac{1}{n} \int_0^{\gamma_n t} \int_{Z^+} \int_{R^+} \int_0^{Z^{(n)}(s)} e^{-\beta t} R^{(n)}(\gamma_n t - s, k, y) \tilde{N}^{(n)}_0 (ds, dk, dy, du) \\
= \frac{1}{n} e^{-\beta \gamma_n t} Z^{(n)}_\beta (\gamma_n t) + \frac{1}{n} \int_0^{\gamma_n t} R^{(n)}_\beta (\gamma_n t - s) e^{-\frac{\beta}{\gamma_n} s} Z^{(n)}_\beta (s) ds \\
+ \frac{1}{n} \int_0^{t} \int_{Z^+} \int_{R^+} \int_0^{Z^{(n)}(s)} R^{(n)}_\beta (\gamma_n (t-s), k, y) e^{-\beta s} \tilde{N}^{(n)}_0 (ds, dk, dy, du) \\
+ \frac{1}{n} \int_0^{t} \int_{Z^+} \int_{R^+} R^{(n)}_\beta (\gamma_n (t-s), k, y) e^{-\beta s} N^{(n)}_1 (ds, dk, dy, du).
\]

(6.1)

From Theorem 5.12, we can see that the resolvent kernels \(R^{(n)}_\beta (\gamma_n t)\) and \(R^{(n)}_\beta (\gamma_n t, k, y)\) can be approximated respectively by
\[
\frac{1}{\sigma \lambda} \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} \text{ and } \frac{1}{\sigma} \exp \left\{ - \left( \frac{b + m}{\sigma \lambda} + \beta \right) t \right\} \left( \sum_{j=1}^{k} y_j \right).
\]

Thus we may rewrite (6.1) into
\[
\frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} = \sum_{k=1}^{4} e^{(n)}_k (t) + \frac{Z^{(n)}(0)}{n} \left[ e^{-\frac{\beta}{\gamma_n} \gamma_n t} \tilde{S}^{(n)}_\beta (\gamma_n t) + \int_0^{\gamma_n t} R^{(n)}_\beta (\gamma_n t - s) e^{-\frac{\beta}{\gamma_n} s} \tilde{S}^{(n)}_\beta (s) ds \right] \\
+ \int_0^{t} \int_{Z^+} \int_{R^+} \int_0^{Z^{(n)}(s)} \frac{1}{n} e^{-\left( \frac{b + m}{\sigma \lambda} + \beta \right) (t-s)} \left( \frac{1}{n} \sum_{j=1}^{k} y_j \right) e^{-\beta s} \tilde{N}^{(n)}_0 (ds, dk, dy, du).
\]
(6.2) \[ + \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{1}{\sigma} e^{-\left(\frac{b+\iota m}{\sigma^2} + \beta\right)(t-s)} \left(\frac{1}{n} \sum_{j=1}^k y_j\right) e^{-\beta s} N_1^{(n)}(d\gamma n s, dk, dy), \]

where \( \{\varepsilon_k^{(n)} : k = 1, \ldots, 4\} \) are error processes defined by

(6.3) \[ \varepsilon_1^{(n)}(t) := \frac{1}{n} \sum_{i=1}^n \left[ e^{-\beta t} \mathbf{1}_{\{\varepsilon_i^{(n)} > \gamma n t\}} - e^{-\beta t} \tilde{S}_\beta^{(n)}(\gamma n t) \right], \]

(6.4) \[ \varepsilon_2^{(n)}(t) := \frac{1}{n} \int_0^{\gamma n t} R_\beta^{(n)}(\gamma n t - s) \sum_{i=1}^n \left[ e^{-\beta s} \mathbf{1}_{\{\varepsilon_i^{(n)} > s\}} - e^{-\beta s} \tilde{S}_\beta^{(n)}(s) \right] ds, \]

(6.5) \[ \varepsilon_3^{(n)}(t) := \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_0^{Z(n)(t-s) - 1} \frac{1}{n} \left[ R_\beta^{(n)}(\gamma n (t-s), k, y) \right. \]

\[ \left. - \frac{1}{\sigma} e^{-\left(\frac{b+\iota m}{\sigma^2} + \beta\right)(t-s)} \left(\sum_{j=1}^k y_j\right) e^{-\beta s} \tilde{N}_0^{(n)}(d\gamma n s, dk, dy, dnu) \right], \]

(6.6) \[ \varepsilon_4^{(n)}(t) := \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_0^{Z(n)(t-s) - 1} \frac{1}{n} \left[ R_\beta^{(n)}(\gamma n (t-s), k, y) \right. \]

\[ \left. - \frac{1}{\sigma} e^{-\left(\frac{b+\iota m}{\sigma^2} + \beta\right)(t-s)} \left(\sum_{j=1}^k y_j\right) e^{-\beta s} N_1(d\gamma n s, dk, dy) \right]. \]

In order to simplify the following statement, we introduce a sequence of Lévy processes \( \{L^{(n)}\} \) and a sequence of martingale measures \( \{M^{(n)}\} \) on \( \mathbb{R}_+^2 \), which are defined by

(6.7) \[ L^{(n)}(t) := \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left(\frac{1}{n} \sum_{j=1}^k y_j\right) N_1^{(n)}(d\gamma n s, dk, dy) \]

and

(6.8) \[ M^{(n)}(dt, du) := \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left(\frac{1}{n} \sum_{j=1}^k y_j\right) \tilde{N}_0^{(n)}(d\gamma n t, dk, dy, dnu). \]

From (4.6) and (5.30), we have

\[ e^{-\frac{b+\iota m}{\sigma^2} \gamma n t} S_\beta^{(n)}(\gamma n t) + \int_0^{\gamma n t} R_\beta^{(n)}(\gamma n t - s) e^{-\frac{b+\iota m}{\sigma^2} s} \tilde{S}_\beta^{(n)}(s) ds \]

\[ = \frac{1}{\lambda(n)} \int_{\gamma n t}^{\infty} \tilde{\Lambda}_\beta^{(n)}(r) dr + \int_0^{\gamma n t} R_\beta^{(n)}(\gamma n t - s) \int_s^{\infty} \tilde{\Lambda}_\beta^{(n)}(r) dr ds \]

\[ = \frac{1}{\lambda(n)} \int_{\gamma n t}^{\infty} R_\beta^{(n)}(s) ds = \frac{\gamma n (1 - \lambda(n)) \eta_\beta^{(n)} m^{(n)}}{\lambda(n) \eta_\beta^{(n)} m^{(n)}} \int_{\gamma n t}^{\infty} R_\beta^{(n)}(s) ds. \]

From Lemma 5.4, we can see that the last term in the above equation can be approximated by \( e^{-\left(\frac{b+\iota m}{\sigma^2} + \beta\right)t} \). Based on results and notation above, we can rewrite (6.2) as

\[ \frac{e^{-\beta t} Z^{(n)}(\gamma n t)}{n} = \sum_{k=1}^5 \varepsilon_k^{(n)}(t) + \int_0^t \int_{\mathbb{R}^+} \frac{Z^{(n)}(\gamma n s)}{n} \frac{1}{\sigma} e^{-\left(\frac{b+\iota m}{\sigma^2} + \beta\right)(t-s)} e^{-\beta s} M^{(n)}(ds, du) \]
Switching the order of integrals, we have

\[ Z^{(n)}(0) = \frac{\gamma_n (1 + m \eta_{\beta}^{(n)})}{\lambda^{(n)} \eta_{\beta}^{(n)} m^{(n)}} \int_{t}^{\infty} R_{\beta}^{(n)}(\gamma_{n} s) ds - e^{-\left(\frac{b + m + \beta}{\sigma \lambda} + \eta_{\beta}^{(n)} m^{(n)}\right)^{r}}. \]

where

\[ \varepsilon_{5}^{(n)}(t) := \int_{t}^{\infty} R_{\beta}^{(n)}(\gamma_{n} s) ds - e^{-\left(\frac{b + m + \beta}{\sigma \lambda} + \eta_{\beta}^{(n)} m^{(n)}\right)^{t}}. \]

It is easy to see that

\[ e^{-\left(\frac{b + m + \beta}{\sigma \lambda} + \eta_{\beta}^{(n)} m^{(n)}\right)^{t}} = 1 - \left(\frac{b + m + \beta}{\sigma \lambda} + \eta_{\beta}^{(n)} m^{(n)}\right)^{t}. \]

Switching the order of integrals, we have

\[
\int_{0}^{t} \frac{1}{\sigma} e^{-\beta s} dL^{(n)}(s) \int_{s}^{t} e^{-\left(\frac{b + m}{\sigma \lambda} + \beta\right)^{r}} dr
\]

\[ = \int_{0}^{t} ds \int_{0}^{s} \frac{1}{\sigma} e^{-\left(\frac{b + m}{\sigma \lambda} + \beta\right)^{(s-r)}} \left(\frac{1}{n} \sum_{j=1}^{k} y_{j}\right) e^{-\beta r} dL^{(n)}(r) \]

and

\[
\int_{0}^{t} \int_{0}^{\infty} e^{-\beta s} M^{(n)}(ds, du) \frac{1}{\sigma} \int_{s}^{t} e^{-\left(\frac{b + m}{\sigma \lambda} + \beta\right)^{r}} dr
\]

\[ = \int_{0}^{t} ds \int_{0}^{s} \frac{1}{\sigma} e^{-\left(\frac{b + m}{\sigma \lambda} + \beta\right)^{(s-r)}} e^{-\beta r} M^{(n)}(dr, du). \]

From these two equalities and (6.9), we have

\[
\int_{0}^{t} \frac{1}{\sigma} e^{-\beta s} dL^{(n)}(s) \int_{s}^{t} e^{-\left(\frac{b + m}{\sigma \lambda} + \beta\right)^{r}} dr
\]

\[ + \int_{0}^{t} \int_{0}^{\infty} e^{-\beta s} M^{(n)}(ds, du) \frac{1}{\sigma} \int_{s}^{t} e^{-\left(\frac{b + m}{\sigma \lambda} + \beta\right)^{r}} dr
\]

\[ = \int_{0}^{t} \left[ e^{-\beta s} Z^{(n)}(\gamma_{n} s) - \sum_{k=1}^{5} \varepsilon_{k}^{(n)}(s) - \frac{Z^{(n)}(0)}{n} e^{-\left(\frac{b + m + \beta}{\sigma \lambda} + \eta_{\beta}^{(n)} m^{(n)}\right)^{s}} \right] ds. \]

Taking there back to (6.9), we have

\[
\frac{e^{-\beta t} Z^{(n)}(\gamma_{n} t)}{n} = \sum_{k=1}^{5} \varepsilon_{k}^{(n)}(t) + \int_{0}^{t} \frac{1}{\sigma} e^{-\left(\frac{b + m + \beta}{\sigma \lambda} + \eta_{\beta}^{(n)} m^{(n)}\right)^{s}} ds
\]

\[ + \left(\frac{b + m}{\sigma \lambda} + \beta\right) \sum_{k=1}^{5} \int_{0}^{t} \varepsilon_{k}^{(n)}(s) ds - \left(\frac{b + m}{\sigma \lambda} + \beta\right) \int_{0}^{t} e^{-\beta s} Z^{(n)}(\gamma_{n} s) \frac{1}{n} e^{-\beta s} M^{(n)}(ds, du) \]
\[
\sum_{k=1}^{5} \varepsilon_k^{(n)}(t) + \left( \frac{b+m}{\sigma \lambda} + \beta \right) \int_{t}^{0} \varepsilon_k^{(n)}(s) ds - \left( \frac{b+m}{\sigma \lambda} + \beta \right) \int_{0}^{t} e^{-\beta s} \frac{Z(n)(\gamma_n s)}{n} ds \\
+ \frac{Z(n)(0)}{n} + \int_{0}^{t} e^{-\beta s} dL(n)(s) + \int_{0}^{t} \int_{0}^{\infty} \frac{1}{\sigma} e^{-\beta s} M(n)(ds, du).
\]

From Lemma 6.5-6.8 in Subsection 6.3, as \( n \to \infty \) the sequence \( \{\sum_{k=1}^{5} \varepsilon_k^{(n)}\} \) converges to 0 weakly and hence uniformly on any finite interval \([0, T] \); see [3, p.124]. Thus the sequence \( \{\sum_{k=1}^{5} \varepsilon_k^{(n)}\} \) also converges to 0 weakly. Moreover, from Theorem 6.16 in Subsection 6.4, the sequence \( \{(L^{(n)}, M^{(n)})\} \) is uniformly tight. Moreover, there is a \( \mathbb{R}_+ \times \mathcal{S}(\mathbb{R}_+) \)-valued process \( \{(L(t), M([0,t], \cdot)) : t \geq 0\} \) such that \( (L^{(n)}, M^{(n)}) \to (L, M) \) weakly in \( \mathbf{D}(\mathbb{R}_+, \mathbb{R}_+ \times \mathcal{S}(\mathbb{R}_+)) \). From Theorem 6.18 in Subsection 6.4, on an extension of the probability space, there exist two independent Poisson random measures \( N_0(ds, dz, du) \) and \( N_1(ds, dz) \) defined on \( \mathbb{R}_+^2 \) and \( \mathbb{R}_+^2 \) with intensity \( \lambda ds \nu_0(dz, du) \) and \( \zeta ds \nu_1(dz) \) respectively, such that

\[
L(t) = a\zeta \eta t + \int_{0}^{t} \int_{0}^{\infty} \eta z N_1(ds, dz)
\]

and for any function \( f(u) \) on \( \mathbb{R}_+ \),

\[
\int_{0}^{t} \int_{0}^{\infty} f(u) M(ds, du) = \int_{0}^{t} \int_{\mathbb{R}_+} \sqrt{2c|\eta|^2 + 2\gamma \sigma f(u)} W(ds, du) \equiv \int_{0}^{t} \int_{0}^{\infty} f(u) \eta z N_0(dt, dz, du).
\]

From these results and Corollary 3.33 in [25, p.353], we have as \( n \to \infty \),

\[
\left( \sum_{k=1}^{5} \varepsilon_k^{(n)}, \sum_{k=1}^{5} \varepsilon_k^{(n)}, L^{(n)}, M^{(n)} \right) \to (0, 0, L, M)
\]

weakly in \( \mathbf{D}(\mathbb{R}_+, \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathcal{S}(\mathbb{R}_+)) \). Thus conditions of Theorem 7.5 in [31] are satisfied and

\[
\left( \frac{e^{-\beta s} Z(n)(\gamma_n s)}{n}, \sum_{k=1}^{5} \varepsilon_k^{(n)}, \sum_{k=1}^{5} \varepsilon_k^{(n)}, L^{(n)}, M^{(n)} \right) \to (\hat{Z}(0), 0, L, M)
\]

weakly in \( \mathbf{D}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathcal{S}(\mathbb{R}_+)) \), where \{\( \hat{Z}(t) : t \geq 0 \)\} solves

\[
\hat{Z}(t) = Z(0) - \int_{0}^{t} \left( \frac{\eta}{\sigma} (b + m) + \beta \right) \hat{Z}(s) ds + \int_{0}^{t} \frac{\eta}{\sigma} a\zeta e^{-\beta s} ds + \int_{0}^{t} \int_{0}^{\infty} \frac{\eta}{\sigma} z e^{-\beta s} N_1(ds, dz) \\
+ \int_{0}^{t} \int_{0}^{\infty} e^{\beta s} \hat{Z}(s) \frac{\eta}{\sigma} \sqrt{2c + 2\gamma \sigma \lambda^2} e^{-\beta s} W(ds, du) + \int_{0}^{t} \int_{0}^{\infty} e^{\beta s} \hat{Z}(s) \frac{\eta}{\sigma} z e^{-\beta s} N_0(dt, dz, du).
\]

Thus the sequence \( \{Z^{(n)}(\gamma_n t)/n : t \geq 0\}_{n \geq 1} \) converges weakly to \( \{Z(t) := e^{\beta t} \hat{Z}(t) : t \geq 0\} \) in \( \mathbf{D}(\mathbb{R}_+, \mathbb{R}_+) \). Applying Itô’s formula to \( Z(t) = e^{\beta t} \hat{Z}(t) \), we have

\[
Z(t) = Z(0) + \int_{0}^{t} \left[ \frac{\eta}{\sigma} a\zeta - \frac{\eta}{\sigma} (b + m) Z(s) \right] ds + \int_{0}^{t} \int_{0}^{\infty} \frac{\eta}{\sigma} z N_1(ds, dz) \\
+ \int_{0}^{t} \int_{0}^{\infty} \frac{\eta}{\sigma} \sqrt{2c + 2\gamma \sigma \lambda^2} W(ds, du) + \int_{0}^{t} \int_{0}^{\infty} \frac{Z(s)}{\sigma} z N_0(dt, dz, du).
\]

Here we have finished the proof. □
6.2. Moment estimations. In this section, we give several moment estimations for $e^{-\beta t}Z^{(n)}(\gamma nt)/n$, which will be used to prove the weak convergence of the error processes. It is usually very difficult to get them from the generating functions or Laplace transforms of CMJ-processes. Recall which will be used to prove the weak convergence of the error processes. It is usually very difficult to get them from the generating functions or Laplace transforms of CMJ-processes. Recall that for any $n, k \geq 1$,

$$E[|e_k^{(n)}|^\alpha] \leq C.$$  

**Lemma 6.1.** There exists a constant $C > 0$ such that for any $n \geq 1$,

$$\sup_{t \geq 0} E\left[\frac{e^{-\beta t}Z^{(n)}(\gamma nt)}{n}\right] \leq C \left(1 + \frac{1}{n} \sum_{k=1}^{Z^{(n)}(0)} e_k^{(n)}\right)$$

and

$$\sup_{t \geq 0} E\left[\frac{e^{-\beta t}Z^{(n)}(\gamma nt)}{n}\right] \leq C.$$

**Proof.** Obviously, (6.13) follows directly from (6.11) and (6.12). For the first equality, from (6.1), we have

$$E\left[\frac{e^{-\beta t}Z^{(n)}(\gamma nt)}{n}\right] = \frac{1}{n} e^{-\beta t} Z^{(n)}(\gamma nt) + \frac{1}{n} \int_0^{\gamma nt} e^{-\beta t} R^{(n)}(\gamma nt - s) Z^{(n)}(s) ds$$

$$+ \frac{c^{(n)}}{n} \sum_{k=1}^{Z^{(n)}(0)} q_k^{(n)} \int_0^{\gamma nt} ds \sum_{j=1}^k R^{(n)}(\gamma nt - s, 1, y_j) \prod_{j=1}^k \Lambda^{(n)}(dy_j)$$

$$\leq \frac{Z^{(n)}(0)}{n} + \frac{1}{n} \int_0^{\gamma nt} R^{(n)}(\gamma nt - s) Z^{(n)}(s) ds$$

$$+ \frac{c^{(n)}}{n} \sum_{k=1}^{Z^{(n)}(0)} k q_k^{(n)} \int_0^{\gamma nt} ds \int_0^s R^{(n)}(\gamma nt - s, 1, y) \Lambda^{(n)}(dy)$$

$$= \frac{Z^{(n)}(0)}{n} + \frac{1}{n} \int_0^{\gamma nt} R^{(n)}(\gamma nt - s) Z^{(n)}(s) ds$$

$$+ c^{(n)} \gamma n \sum_{k=1}^{Z^{(n)}(0)} k q_k^{(n)} \int_0^{\gamma nt} ds \int_0^s R^{(n)}(\gamma n(t - s), 1, y) \Lambda^{(n)}(dy).$$

From Lemma 5.2, we can see that the second term on the right side of the last equality can be bounded by

$$\frac{1}{n} \int_0^{\gamma nt} R^{(n)}(\gamma nt - s) Z^{(n)}(s) ds \leq C \frac{Z^{(n)}(0)}{n} \sum_{k=1}^{Z^{(n)}(0)} \int_0^{\gamma nt} 1_{\{e_k^{(n)} > s\}} ds \leq \frac{1}{n} \sum_{k=1}^{Z^{(n)}(0)} e_k^{(n)}.$$

Moreover, from (5.2), for any $r \geq 0$,

$$\int_0^{\infty} R^{(n)}(\gamma nt - r, 1, y) \Lambda^{(n)}(dy) = \lambda^{(n)} R^{(n)}(\gamma nr) + \lambda^{(n)} \int_0^{\gamma nr} R^{(n)}(\gamma nr - s) \Lambda^{(n)}(s) ds = \frac{R^{(n)}(\gamma nr)}{m^{(n)}}.$$

From this and Proposition 5.3,

$$\int_0^t ds \int_0^{\infty} R^{(n)}(\gamma n(t - s), 1, y) \Lambda^{(n)}(dy) \leq \frac{1}{m^{(n)}} \int_0^{\infty} R^{(n)}(\gamma ns) ds \leq C.$$
Combining all results above together, we will get the desired result.

\[ \sup_{t \geq 0} E \left[ \left| \frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} \right|^\alpha \right] \leq C. \]

**Proof.** From (3.16) and (6.1), we have

\[ \frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} = E\left[ \frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} \bigg| \mathcal{F}_0 \right] \]

\[ + \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t-s), k, y) e^{-\beta s} \tilde{N}^{(n)}_1 (d\gamma_n s, dk, dy) \]

\[ + \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t-s), k, y) e^{-\beta s} \tilde{N}^{(n)}_0 (d\gamma_n s, dk, dy, dnu). \]

From the Cauchy-Schwarz inequality,

\[ E\left[ \left| \frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} \right|^\alpha \right] = CE \left[ E\left[ \frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} \bigg| \mathcal{F}_0 \right] \right]^\alpha \]

\[ + CE \left[ \left| \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t-s), k, y) e^{-\beta s} \tilde{N}^{(n)}_1 (d\gamma_n s, dk, dy) \right|^\alpha \right] \]

\[ + CE \left[ \left| \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t-s), k, y) e^{-\beta s} \tilde{N}^{(n)}_0 (d\gamma_n s, dk, dy, dnu) \right|^\alpha \right]. \]

From (6.11) and (6.12),

\[ E\left[ \left| \frac{e^{-\beta t} Z^{(n)}(\gamma_n t)}{n} \bigg| \mathcal{F}_0 \right| \right]^{\alpha} \leq CE \left[ 1 + \frac{1}{n} \sum_{k=1}^{Z^{(n)}(0)} e_k^{(n)} \right] \leq C + CE \left[ e_k^{(n)} \right] \leq C. \]

Applying the Burkholder-Davis-Gundy inequality, we have

\[ E \left[ \left| \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t-s), k, y) e^{-\beta s} \tilde{N}^{(n)}_1 (d\gamma_n s, dk, dy) \right|^\alpha \right] \]

\[ \leq E \left[ \left| \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \frac{1}{n^2} |R^{(n)}_\beta (\gamma_n(t-s), k, y)|^2 e^{-2\beta s} N_1^{(n)} (d\gamma_n s, dk, dy) \right|^\alpha \right] \]

\[ \leq \left| \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} R^{(n)}_\beta (\gamma_n(t-s), k, y) \right|^{\alpha} e^{-\alpha \beta s} N_1^{(n)} (d\gamma_n s, dk, dy) \]

\[ \leq CE \left[ \int_0^t \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \frac{1}{n^\alpha} \sum_{j=1}^k (1 + y_j) \right]^{\alpha} e^{-\alpha \beta s} N_1^{(n)} (d\gamma_n s, dk, dy) \]

\[ \leq C \gamma_n \int_0^t e^{-\alpha \beta s} ds \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^n} \left| \sum_{j=1}^k (1 + y_j) \right|^{\alpha} \nu_1^{(n)} (dk, dy) \]

\[ \leq C \gamma_n \int_0^t \int_{\mathbb{R}_+^n} 1^{\alpha-1} (1 + y_j)^\alpha \nu_1^{(n)} (dk, dy) \]

\[ \leq C \gamma_n \int_0^t \int_{\mathbb{R}_+^n} k^{\alpha-1} (1 + y_j)^\alpha \nu_1^{(n)} (dk, dy) \]
Here the second inequality follows from the fact that \((x + y)^{\alpha/2} \leq x^{\alpha/2} + y^{\alpha/2}\) for any \(x, y \geq 0\), the third inequality follows from Lemma 5.5, the fifth inequality follows from Hölder’s inequality and the last inequality follows from Condition 4.3. Similarly, we also have

\[
E \left[ \left\| \int_0^t \int_{Z_+} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t,s), k, y) e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, dk, dy, dnu) \right\|^{\alpha} \right] \\
\leq C E \left[ \left\| \int_0^t \int_{Z_+ \setminus \{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t,s), k, y) e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, dk, dy, dnu) \right\|^{\alpha/2} \right] \\
+ C E \left[ \left\| \int_0^t \int_{Z_+ \setminus \{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t,s), k, y) e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, dk, dy, dnu) \right\|^{\alpha} \right].
\]

Like the deduction in (6.14), from Condition 4.3 and Lemma 6.1 we also have

\[
E \left[ \left\| \int_0^t \int_{Z_+ \setminus \{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t,s), k, y) e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, dk, dy, dnu) \right\|^{\alpha} \right] \\
\leq E \left[ \left\| \int_0^t \int_{Z_+ \setminus \{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n^2} R^{(n)}_\beta (\gamma_n(t,s), k, y) e^{-2\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, dk, dy, dnu) \right\|^{\alpha/2} \right] \\
\leq E \left[ \left\| \int_0^t \int_{Z_+ \setminus \{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n^2} R^{(n)}_\beta (\gamma_n(t,s), k, y) e^{-\alpha \beta s} \tilde{N}_0^{(n)} (d\gamma_n s, dk, dy, dnu) \right\|^{\alpha} \right] \\
\leq C n^{2-\alpha} \int_0^t E \left[ \frac{e^{-\beta s} Z^{(n)}(\gamma_{n,s})}{n} \right] e^{-(\alpha-1)\beta s} ds \int_{Z_+ \setminus \{1\}} \int_{\mathbb{R}_+^d} \left| \sum_{j=1}^k (1 + y_j) \right|^\alpha \nu_0^{(n)} (dk, dy) \\
\leq C n^{2-\alpha} \sum_{k=2}^{\infty} k^{\alpha} p_k \int_0^\infty (1 + y)^\alpha \Lambda^{(n)} (dy) \leq C.
\]

From the Burkhoder-Davis-Gundy inequality,

\[
E \left[ \left\| \int_0^t \int_{\{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n} R^{(n)}_\beta (\gamma_n(t,s), k, y) e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, dk, dy, dnu) \right\|^{\alpha} \right] \\
\leq C E \left[ \left\| \int_0^t \int_0^\infty \int_{\{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n^2} R^{(n)}_\beta (\gamma_n(t,s), 1, y) e^{-2\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, \{1\}, dy, dnu) \right\|^{\alpha/2} \right] \\
\leq C \left\| \int_0^t \int_0^\infty \int_{\{1\}} \int_{\mathbb{R}_+^d} \frac{Z^{(n)}(\gamma_{n,s})}{n} \frac{1}{n^2} R^{(n)}_\beta (\gamma_n(t,s), 1, y) e^{-2\beta s} \tilde{N}_0^{(n)} (d\gamma_n s, \{1\}, dy, dnu) \right\|^{\alpha/2} \\
\leq C \left\| \int_0^t e^{-\beta s} E \left[ \frac{e^{-\beta s} Z^{(n)}(\gamma_{n,s})}{n} \right] ds \int_0^\infty (1 + y)^2 \Lambda^{(n)} (dy) \right\|^{\alpha/2} \leq C \left\| \int_0^t e^{-\beta s} ds \right\|^{\alpha/2} \leq C.
\]

Here the second inequality follows from Jensen’s inequality and the forth inequality follows from Condition 4.3(2) and Lemma 6.1. Putting all results above together, we will get the desired result directly.

\[\square\]
Proposition 6.3. For any \( k \geq 1 \), there exists a constant \( C > 0 \) independent of \( n \) such that for any \( 0 \leq t_1 \leq t_2 \),
\[
\mathbb{E} \left[ \left| \int_{t_1}^{t_2} \int_{\mathbb{R}_+} \int_0^n e^{-\beta s} \left( \sum_{i=1}^k y_i \right) \tilde{N}_0^{(n)} (d\gamma_{n} s, \{ k \}, dy, dnu) \right|^{2\alpha} \right] \leq C \gamma_n |t_2 - t_1| e^{-C_n \gamma_n |t_2 - t_1|}.
\]

Proof. Here we just prove this result with \( k = 1 \) and other cases can be proved similarly. From the Burkholder-Davis-Gundy inequality,
\[
\mathbb{E} \left[ \left( \int_{t_1}^{t_2} \int_0 \int_0 e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu) \right)^{2\alpha} \right] \leq \mathbb{E} \left[ \left( \int_{t_1}^{t_2} \int_0 \int_0 e^{-\beta s} N_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu) \right)^{2\alpha} \right] =: \mathbb{E} \left[ \left| J_2^{(n)} (t_1, t_2) \right|^\alpha \right].
\]

For any \( t \geq t_1 \), applying Itô’s formula to \( J_2^{(n)} (t_1, t) \), we have
\[
\left| J_2^{(n)} (t_1, t) \right|^\alpha = \int_{t_1}^{t} \int_0 \int_0 e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu) \left[ \left| J_2^{(n)} (t_1, s) + y e^{-\beta s} \right|^\alpha - \left| J_2^{(n)} (t_1, 1) \right|^\alpha \right] N_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu)
\]
\[
= \alpha \int_{t_1}^{t} \int_0 \int_0 e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu) \times y e^{-\beta s} N_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu).
\]

Here the second equality follows from the mean value theorem with \( \vartheta_s \in [0, 1] \). Since \( \alpha \in (1, 2) \) and \( (x + y)^{\alpha - 1} \leq x^{\alpha - 1} + y^{\alpha - 1} \) for any \( x, y \geq 0 \), we have
\[
\left| J_2^{(n)} (t_1, t) \right|^\alpha \leq \alpha \int_{t_1}^{t} \int_0 \int_0 e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu) \left| J_2^{(n)} (t_1, s) \right|^{\alpha - 1} x e^{-\beta s} N_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu)
\]
\[
+ \alpha \int_{t_1}^{t} \int_0 \int_0 e^{-\beta s} \tilde{N}_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu) y^{2\alpha} e^{-2\beta s} N_0^{(n)} (d\gamma_{n} s, \{ 1 \}, dy, dnu).
\]

From Young’s inequality and Corollary 6.2,
\[
\mathbb{E} \left[ \left| J_2^{(n)} (t_1, t) \right|^\alpha \right] \leq C \gamma_n \int_{t_1}^{t} \mathbb{E} \left[ \frac{e^{-\beta s} Z^{(n)} (\gamma_{n} s)}{n} \left| J_2^{(n)} (t_1, s) \right|^{\alpha - 1} \right] ds \int_0^\infty y^{2\alpha} \Lambda^{(n)} (dy)
\]
\[
+ C \gamma_n \int_{t_1}^{t} \mathbb{E} \left[ \frac{e^{-\beta s} Z^{(n)} (\gamma_{n} s)}{n} \right] ds \int_0^\infty y^{2\alpha} \Lambda^{(n)} (dy)
\]
\[
\leq C \gamma_n \int_{t_1}^{t} \mathbb{E} \left[ \left| J_2^{(n)} (t_1, s) \right|^\alpha \right] ds + C \gamma_n \int_{t_1}^{t} \mathbb{E} \left[ \frac{e^{-\beta s} Z^{(n)} (\gamma_{n} s)}{n} \right] ds
\]
\[
+ C \gamma_n \int_{t_1}^{t} \mathbb{E} \left[ \frac{e^{-\beta s} Z^{(n)} (\gamma_{n} s)}{n} \right] ds
\]
\[
\leq C \gamma_n \int_{t_1}^{t} \mathbb{E} \left[ \left| J_2^{(n)} (t_1, s) \right|^\alpha \right] ds + C \gamma_n |t - t_1|.
\]

Applying Grönwall’s inequality, we will get the desired result. \( \square \)
Proposition 6.4. For any $T > 0$ and $k \geq j \geq 1$, there exists a constant $C > 0$ independent of $n$ such that for any $0 \leq t_1 \leq t_2$,

\begin{equation}
\mathbb{E} \left[ \left\| \int_0^T ds \int_{\gamma_n(t_1-s)}^{\gamma_n(t_2-s)} e^{-\beta s} Z(n)(\gamma_n s) \Lambda(n)(dy) \right\|^\alpha \right] \leq C |t_2 - t_1|^\alpha
\end{equation}

and

\begin{equation}
\mathbb{E} \left[ \left\| \int_0^T \int_{\mathbb{R}_+^k} \int_0^1 e^{-\beta s} 1_{\{y_j \in [\gamma_n(t_1-s), \gamma_n(t_2-s)]\}} \tilde{N}_0(n)(d\gamma_n s, \{k\}, dy, dnu) \right\|^{2\alpha} \right] 
\leq C n |\gamma_n| |t_2 - t_1| + C n |\gamma_n|^\alpha |t_2 - t_1|^\alpha.
\end{equation}

Proof. For the first statement, from Hölder’s inequality and Corollary 6.2,

\begin{align*}
\mathbb{E} \left[ \left\| \int_0^T ds \int_{\gamma_n(t_1-s)}^{\gamma_n(t_2-s)} e^{-\beta s} Z(n)(\gamma_n s) \Lambda(n)(dy) \right\|^\alpha \right] 
\leq \left\| \int_0^T ds \int_{\gamma_n(t_1-s)}^{\gamma_n(t_2-s)} \Lambda(n)(dy) \right\|^{\alpha-1} \int_0^T ds \int_{\gamma_n(t_1-s)}^{\gamma_n(t_2-s)} \mathbb{E} \left[ \left\| \frac{e^{-\beta s} Z(n)(\gamma_n s)}{n} \right\|^\alpha \right] \Lambda(n)(dy) 
\leq C \left\| \int_0^T \tilde{\Lambda}(n)(\gamma_n(t_1-s)) ds - \int_0^T \tilde{\Lambda}(n)(\gamma_n(t_2-s)) ds \right\|^{\alpha} 
\leq C |\gamma_n|^\alpha \left\| \int_{\gamma_n(t_1-T)}^{\gamma_n(t_1)} \tilde{\Lambda}(n)(s) ds - \int_{\gamma_n(t_2-T)}^{\gamma_n(t_2)} \tilde{\Lambda}(n)(s) ds \right\|^{\alpha}
\leq C |\gamma_n|^\alpha \left\| \int_{\gamma_n(t_1)}^{\gamma_n(t_1-t_2)} \tilde{\Lambda}(n)(s) ds - \int_{\gamma_n(t_1-t_2)}^{\gamma_n(t_1-T)} \tilde{\Lambda}(n)(s) ds \right\|^{\alpha} 
\leq C |\gamma_n|^\alpha |t_2 - t_1|^\alpha.
\end{align*}

Here we have gotten (6.15). For (6.16), from the Burkholder-Davis-Gundy inequality,

\begin{align*}
\mathbb{E} \left[ \left\| \int_0^T ds \int_{\mathbb{R}_+^k} \int_0^1 e^{-\beta s} 1_{\{y_j \in [\gamma_n(t_1-s), \gamma_n(t_2-s)]\}} \tilde{N}_0(n)(d\gamma_n s, \{k\}, dy, dnu) \right\|^{2\alpha} \right] 
\leq \mathbb{E} \left[ \left\| \int_0^T ds \int_{\mathbb{R}_+^k} \int_0^1 e^{-\beta s} 1_{\{y_j \in [\gamma_n(t_1-s), \gamma_n(t_2-s)]\}} N_0(n)(d\gamma_n s, \{k\}, dy, dnu) \right\|^\alpha \right] 
\leq C |\gamma_n|^\alpha \mathbb{E} \left[ \left\| \int_0^T ds \int_{\gamma_n(t_1-s)}^{\gamma_n(t_2-s)} e^{-2\beta s} Z(n)(\gamma_n s) \Lambda(n)(dy) \right\|^\alpha \right] 
\leq C n |\gamma_n|^\alpha |t_2 - t_1|^\alpha
\end{align*}

Since $\alpha \in (1, 2)$, we have

\begin{align*}
\mathbb{E} \left[ \left\| \int_0^T ds \int_{\mathbb{R}_+^k} \int_0^1 e^{-4\beta s} 1_{\{y_j \in [\gamma_n(t_1-s), \gamma_n(t_2-s)]\}} N_0(n)(d\gamma_n s, \{k\}, dy, dnu) \right\|^{\alpha/2} \right]
\end{align*}
\[ \leq C n^{\gamma_n} \int_0^T ds \int_0^\infty \mathbb{E} \left[ \frac{e^{-\beta s} Z^{(n)}(\gamma_n s)}{n} \right] 1_{\{y \in [\gamma_n(t_1-s), \gamma_n(t_2-s)]\}} \Lambda^{(n)}(dy) \]

Here we have finished the proof. \( \square \)

6.3. Weak convergence of error processes. In this subsection we mainly prove that for any \( k = 1, \cdots, 5 \), the error sequence \( \{\varepsilon_k^{(n)}\} \) converges to 0 weakly in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}) \) and hence it converges uniformly on any finite time interval \([0, T]\); see [3, p.124]. Thus the sequence \( \{\int \varepsilon_k^{(n)}\} \) also converges to 0 uniformly on any interval \([0, T]\).

6.3.1. Weak convergence of \( \{(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, \varepsilon_5^{(n)})\} \). In the following two lemmas, we firstly prove the uniform convergence of the sequences \( \{\varepsilon_1^{(n)}\} \) and \( \{\varepsilon_2^{(n)}\} \).

**Lemma 6.5.** The sequence \( \{\varepsilon_1^{(n)}\} \) converges to 0 uniformly, i.e.
\[ \lim_{n \to \infty} \sup_{t \geq 0} |\varepsilon_1^{(n)}(t)| = 0, \ a.s. \]

**Proof.** From (6.3), we have
\[ \sup_{t \geq 0} |\varepsilon_1^{(n)}(t)| = \sup_{t \geq 0} \frac{Z^{(n)}(0)}{n} e^{-\beta t} \left| \frac{1}{Z^{(n)}(0)} \sum_{k=1}^{Z^{(n)}(0)} 1_{\{\varepsilon_k^{(n)} > \gamma_n t\}} \right| S^{(n)}_\beta(\gamma_n t) \]
\[ \leq C \sup_{t \geq 0} \left| \frac{1}{Z^{(n)}(0)} \sum_{k=1}^{Z^{(n)}(0)} 1_{\{\varepsilon_k^{(n)} > t\}} \right| S^{(n)}_\beta(t) \]

It is easy to see that the term on the right side of the last inequality is a sequence of empirical processes of a triangular array of row-independent random variables. From the Glivenko-Cantelli-type theorem (or the main theorem of statistics); see Theorem 2.1 in [44],

\[ \sup_{t \geq 0} \left| \frac{1}{Z^{(n)}(0)} \sum_{k=1}^{Z^{(n)}(0)} 1_{\{\varepsilon_k^{(n)} > t\}} \right| S^{(n)}_\beta(t) \to 0, \ a.s. \]

Here we have finished the proof. \( \square \)

**Lemma 6.6.** The sequence \( \{\varepsilon_2^{(n)}\} \) converges to 0 uniformly in probability, i.e.
\[ \lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \geq 0} |\varepsilon_2^{(n)}(t)| \right] = 0. \]

**Proof.** From Lemma 5.2 and Fubini’s theorem, we have
\[ \mathbb{E} \left[ \sup_{t \geq 0} |\varepsilon_2^{(n)}(t)| \right] \leq C \mathbb{E} \left[ \sup_{t \geq 0} \int_0^{\gamma_n t} R^{(n)}_\beta(\gamma_n t - s) \left| \frac{1}{Z^{(n)}(0)} \sum_{k=1}^{Z^{(n)}(0)} 1_{\{\varepsilon_k^{(n)} > s\}} - S^{(n)}_\beta(s) \right| ds \right] \]
\begin{align*}
\leq C & \mathbb{E} \left[ \int_0^\infty \frac{1}{Z^{(n)}(0)} Z^{(n)}(0) \sum_{k=1}^{n} 1_{\{\epsilon_k^{(n)} > \tau\}} - \bar{S}_\beta^{(n)}(s) \right] ds \\
= C & \int_0^\infty \mathbb{E} \left[ \frac{1}{Z^{(n)}(0)} \sum_{k=1}^{n} 1_{\{\epsilon_k^{(n)} > \tau\}} - \bar{S}_\beta^{(n)}(s) \right] ds.
\end{align*}

From Condition 4.3(2), we have
\begin{align*}
\mathbb{E} \left[ \frac{1}{Z^{(n)}(0)} \sum_{k=1}^{n} 1_{\{\epsilon_k^{(n)} > \tau\}} - \bar{S}_\beta^{(n)}(t) \right] & \leq \mathbb{E} \left[ \frac{1}{Z^{(n)}(0)} \sum_{k=1}^{n} 1_{\{\epsilon_k^{(n)} > \tau\}} \right] + \bar{S}_\beta^{(n)}(t) \\
& = 2\bar{S}_\beta^{(n)}(t) \leq \frac{2}{\eta^{(n)}_\beta} \int_t^\infty e^{-\frac{\theta}{n} \bar{\Lambda}}(s) ds \\
& \leq C \int_t^\infty \bar{\Lambda}(s) ds
\end{align*}

and
\begin{align*}
\int_0^\infty \int_t^\infty \bar{\Lambda}(s) ds dt = \int_0^\infty \bar{\Lambda}(t) dt < \infty.
\end{align*}

From the dominated convergence theorem and (6.17),
\begin{align*}
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \geq 0} |\epsilon_2^{(n)}(t)| \right] \leq C \int_0^\infty \lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{Z^{(n)}(0)} \sum_{j=1}^{n} 1_{\{\epsilon_j^{(n)} > \tau\}} - \bar{S}_\beta^{(n)}(s) \right] ds = 0.
\end{align*}

Here we have finished the proof. \hfill \square

**Lemma 6.7.** The sequence \{\epsilon_5^{(n)}\} converges to 0 uniformly on any finite time interval \([0, T]\), i.e.
\begin{align*}
\lim_{n \to \infty} \sup_{t \in [0, T]} |\epsilon_5^{(n)}(t)| = 0.
\end{align*}

**Proof.** From (5.8) and (6.10), we have
\begin{align*}
|\epsilon_5^{(n)}(t)| &= \frac{Z^{(n)}(0)}{n} \gamma_n(1 - \lambda^{(n)} \eta^{(n)}_\beta m^{(n)}) \int_0^t R^{(n)}_\beta(\gamma_n s) ds - \int_0^t \left( \frac{b + m}{\sigma \lambda} + \beta \right) e^{-\left( \frac{b + m}{\sigma \lambda} + \beta \right) s} ds \\
& \leq C \gamma_n(1 - \lambda^{(n)} \eta^{(n)}_\beta m^{(n)}) \int_0^t R^{(n)}_\beta(\gamma_n s) ds \\
& \quad + C(b + m + \beta \sigma \lambda) \int_0^t |R^{(n)}_\beta(\gamma_n s) - \frac{1}{\sigma \lambda} e^{-\left( \frac{b + m}{\sigma \lambda} + \beta \right) s} | ds.
\end{align*}

From (5.6) and (5.8), we can see that the first term on the right side of the last inequality vanishes uniformly as \(n \to \infty\). For the second term, by Hölder’s inequality,
\begin{align*}
\sup_{t \in [0, T]} \int_0^t |R^{(n)}_\beta(\gamma_n s) - \frac{1}{\sigma \lambda} e^{-\left( \frac{b + m}{\sigma \lambda} + \beta \right) s} | ds \leq CT \int_0^\infty |R^{(n)}_\beta(\gamma_n s) - \frac{1}{\sigma \lambda} e^{-\left( \frac{b + m}{\sigma \lambda} + \beta \right) s} |^2 ds,
\end{align*}

which tends to 0 as \(n \to \infty\); see (5.25) in Theorem 5.12. Here we have finished the proof. \hfill \square
6.3.2. Weak convergence of \( \{(\varepsilon_3^{(n)}, \varepsilon_4^{(n)})\} \). In this part, we mainly prove weak convergence of the sequences \( \{\varepsilon_3^{(n)}\} \) and \( \{\varepsilon_4^{(n)}\} \) to 0, which is the most difficult part in whole proof.

**Lemma 6.8.** The two sequences \( \{\varepsilon_3^{(n)}\} \) and \( \{\varepsilon_4^{(n)}\} \) converge to 0 weakly in \( D(\mathbb{R}_+, \mathbb{R}) \).

Here we mainly give the precise proof for the weak convergence of the sequence \( \{\varepsilon_3^{(n)}\} \). Actually, it is easy to see that the structure of \( \varepsilon_4^{(n)} \) is similar but much more simpler than \( \varepsilon_3^{(n)} \). Thus the weak convergence of the sequence \( \{\varepsilon_4^{(n)}\} \) can proved in the same way. From (6.5) and (5.2), we have

\[
\varepsilon_3^{(n)}(t) = \sum_{k=1}^{\infty} \frac{1}{n} \int_0^t \int_0^1 \frac{z^{(n)}(\gamma n s - k, t)}{n} \left[R^{(n)}_\beta(\gamma n(t - s), k, y)\right] \frac{-1}{\sigma} e^{-\left(\frac{b_m + \beta}{\sigma} + \beta\right)(t-s)} \left(\sum_{j=1}^{k} y_j\right) e^{-\beta s} \tilde{N}_0^{(n)}(d\gamma n s, \{k\}, dy, dnu) \\
= \sum_{k=1}^{\infty} \sum_{j=1}^{k} \int_0^t \int_0^1 \frac{z^{(n)}(\gamma n s - k, t)}{n} \frac{1}{n} \left[R^{(n)}_\beta(\gamma n(t - s), 1, y_j)\right] \frac{-1}{\sigma} e^{-\left(\frac{b_m + \beta}{\sigma} + \beta\right)(t-s) y_j} e^{-\beta s} \tilde{N}_0^{(n)}(d\gamma n s, \{k\}, dy, dnu) =: \sum_{k=1}^{\infty} \sum_{j=1}^{k} \varepsilon_{3,k,j}^{(n)}(t).
\]

For any \( k, j \geq 1 \), we firstly prove the weak convergence of the sequence \( \{\varepsilon_{3,k,j}^{(n)}\} \) in the following four steps:

1. The sequence \( \{\varepsilon_{3,k,j}^{(n)}\} \) converges to 0 in the sense of finite-dimensional distributions; see Proposition 6.9.
2. For some constant \( \theta > \frac{\alpha}{\sigma+1} \), the sequence of continuous stochastic processes \( \{\varepsilon_{3,k,j}^{(n)}\} \) defined by

\[
(6.18) \quad \varepsilon_{3,k,j}^{(n)}(t) := \varepsilon_{3,k,j}^{(n)}(\left\lfloor n^{\theta} t \right\rfloor) + \left( n^{\theta} t - \left\lfloor n^{\theta} t \right\rfloor \right) \left[\varepsilon_{3,k,j}^{(n)}\left(\frac{\left\lfloor n^{\theta} t \right\rfloor}{n^{\theta}}\right) - \varepsilon_{3,k,j}^{(n)}\left(\frac{n^{\theta} t}{n^{\theta}}\right)\right],
\]

is tight in \( C(\mathbb{R}_+, \mathbb{R}) \); see Proposition 6.11.
3. In Proposition 6.12, we show that \( \{\varepsilon_{3,k,j}^{(n)}\} \) can be approximated by \( \{\varepsilon_{3,k,j}^{(n)}\} \) uniformly on any finite time interval, i.e.

\[
\sup_{t \in [0,T]} \left| \varepsilon_{3,k,j}^{(n)}(t) - \varepsilon_{3,k,j}^{(n)}(t) \right| \to 0, \quad \text{in probability.}
\]

4. Based on the results gotten in the previous steps, we prove that the sequence \( \{\varepsilon_{3,k,j}^{(n)}\} \) converges to 0 weakly in \( D(\mathbb{R}_+, \mathbb{R}) \); see Proposition 6.13.

**Proposition 6.9.** The sequence \( \{\varepsilon_{3,k,j}^{(n)}\} \) converges to 0 in the sense of finite-dimensional distributions, i.e. for any \( t \geq 0 \),

\[
\mathbb{E}[\varepsilon_{3,k,j}^{(n)}(t)]^{2\alpha} \to 0.
\]

**Proof.** Here we just prove this result with \( k = j = 1 \) and other cases can be proved similarly. From the Burkholder-Davis-Gundy inequality and the Cauchy-Schwarz inequality,

\[
\mathbb{E}[\varepsilon_{3,1,1}^{(n)}(t)]^{2\alpha} \leq I_1^{(n)}(t) + I_2^{(n)}(t),
\]
Here we have gotten the desired result. Other cases can be proved similarly. For any $0 < \alpha < 1$, there exists a constant $C > 0$ independent of $n$ such that for any $t_1, t_2 \geq 0$,

$$\mathbb{E}\left[\left|\varepsilon_{3,k,j}^{(n)}(t_2) - \varepsilon_{3,k,j}^{(n)}(t_1)\right|^{2\alpha}\right] \leq C \left(\frac{\gamma_n}{n^{2\alpha-1}}|t_1 - t_2| + |t_1 - t_2|^{1+\frac{\alpha-1}{\alpha}}\right).$$  

**Proposition 6.10.** For any $k \geq j \geq 1$, there exists a constant $C > 0$ independent of $n$ such that for any $t_1, t_2 \geq 0$,

$$
\mathbb{E}\left[\left|\varepsilon_{3,k,j}^{(n)}(t_2) - \varepsilon_{3,k,j}^{(n)}(t_1)\right|^{2\alpha}\right] \leq C \left(\frac{\gamma_n}{n^{2\alpha-1}}|t_1 - t_2| + |t_1 - t_2|^{1+\frac{\alpha-1}{\alpha}}\right).
$$

**Proof.** In order to simplify the following argument, here we just prove this result with $k = j = 1$. Other cases can be proved similarly. For any $0 \leq t_1 \leq t_2$, we have

$$\varepsilon_{3,1,1}^{(n)}(t_2) - \varepsilon_{3,1,1}^{(n)}(t_1) = \varepsilon_{3,1,1}^{(n)}(t_1, t_2) - \varepsilon_{3,1,1}^{(n)}(t_1, t_2) + \varepsilon_{3,1,1}^{(n)}(t_1, t_2),$$

where

$$I_{1}^{(n)}(t) := C \left[\int_0^t \int_0^\infty \int_0^\infty Z^{(n)}(\gamma_n s) \left|R_{\beta}^{(n)}(\gamma_n(t-s), 1, y)\right| \frac{1}{\varepsilon \sigma(n)} e^{-\left(\frac{b+m}{\varepsilon \sigma(n)}+\beta\right)(t-s)y} ds\Lambda^{(n)}(dy)\right].$$

By Hölder’s inequality and Corollary 6.2,

$$I_{1}^{(n)}(t) \leq C \int_0^t \int_0^\infty \mathbb{E}\left[\left|\frac{e^{-\beta s} Z^{(n)}(\gamma_n s)}{\varepsilon \sigma(n)}\right|^\alpha\right] \left|R_{\beta}^{(n)}(\gamma_n(t-s), 1, y)\right| \frac{1}{\varepsilon \sigma(n)} e^{-\left(\frac{b+m}{\varepsilon \sigma(n)}+\beta\right)(t-s)y} ds\Lambda^{(n)}(dy)$$

$$\leq C \int_0^t \int_0^\infty \left|R_{\beta}^{(n)}(\gamma_n(t-s), 1, y)\right| \frac{1}{\varepsilon \sigma(n)} e^{-\left(\frac{b+m}{\varepsilon \sigma(n)}+\beta\right)(t-s)y} ds\Lambda^{(n)}(dy).$$

From Lemma 5.5, Condition 4.3(2) and the dominated convergence theorem,

$$\lim_{n \to \infty} I_{1}^{(n)}(t) = 0.$$

Applying the Burkholder-Davis-Gundy inequality to $I_{2}^{(n)}(t)$, we have

$$I_{2}^{(n)}(t) \leq C \int_0^t \int_0^\infty \int_0^\infty \mathbb{E}\left[\left|\frac{e^{-\beta s} Z^{(n)}(\gamma_n s)}{\varepsilon \sigma(n)}\right|^\alpha\right] \left|R_{\beta}^{(n)}(\gamma_n(t-s), 1, y)\right| \frac{1}{\varepsilon \sigma(n)} e^{-\left(\frac{b+m}{\varepsilon \sigma(n)}+\beta\right)(t-s)y} ds\Lambda^{(n)}(dy)$$

$$\leq C \int_0^t \int_0^\infty \left|R_{\beta}^{(n)}(\gamma_n(t-s), 1, y)\right| \frac{1}{\varepsilon \sigma(n)} e^{-\left(\frac{b+m}{\varepsilon \sigma(n)}+\beta\right)(t-s)y} ds\Lambda^{(n)}(dy) \to 0.$$
where

\[
(6.20) \quad \varepsilon_{3,1,1}(t_1, t_2) := \int_0^{t_2} \int_0^\infty \int_0^{\frac{\varepsilon(n+1)}{n}} \frac{1}{n} \left[ R_{\beta}^{(n)}(\gamma_n(t_2 - s), 1, y) - R_{\beta}^{(n)}(\gamma_n(t_1 - s), 1, y) \right] e^{-\beta s \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu)},
\]

\[
(6.21) \quad \varepsilon_{3,1,2}(t_1, t_2) := \int_0^{t_1} \int_0^\infty \int_0^{\frac{\varepsilon(n+1)}{n}} \frac{1}{n} \sigma e^{-\left(\frac{b+m}{n}+\beta\right)(t_2-s)} e^{-\beta s \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu)},
\]

\[
(6.22) \quad \varepsilon_{3,1,3}(t_1, t_2) := \int_0^{t_1} \int_0^\infty \int_0^{\frac{\varepsilon(n+1)}{n}} \left( e^{-\left(\frac{b+m}{n}+\beta\right)(t_2-t_1)} - 1 \right) e^{-\beta s \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu)},
\]

It suffices to prove that

\[
\mathbb{E}\left[ \varepsilon_{3,1,1}(t_1, t_2)^{2\alpha} \right] \leq C \left( \frac{\gamma_n}{n^{2\alpha-1}} |t_1 - t_2| + |t_1 - t_2|^{1+\frac{2-\alpha}{\alpha}} \right), \quad i = 1, 2, 3.
\]

Here we just prove this inequality with \(i = 1\) and the other two cases can be proved similarly. From the Burkholder-Davis-Gundy inequality,

\[
\mathbb{E}\left[ \varepsilon_{3,1,1}(t_1, t_2)^{2\alpha} \right] \leq C \mathbb{E}\left[ \int_0^{t_2} \int_0^\infty \int_0^{\frac{\varepsilon(n+1)}{n}} \frac{1}{n^2} \left| R_{\beta}^{(n)}(\gamma_n(t_2 - s), 1, y) - R_{\beta}^{(n)}(\gamma_n(t_1 - s), 1, y) \right|^2 e^{-2\beta s \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu)} \right]^{\alpha}.
\]

For any \(t \geq 0\), define

\[
J_1^{(n)}(t_1, t_2, t) := \int_0^t \int_0^\infty \int_0^{\frac{\varepsilon(n+1)}{n}} e^{-2\beta s} \left| R_{\beta}^{(n)}(\gamma_n(t_2 - s), 1, y) - R_{\beta}^{(n)}(\gamma_n(t_1 - s), 1, y) \right|^{2\alpha} \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu).
\]

Applying Itô’s formula to \(|J_1^{(n)}(t_1, t_2, t)|^\alpha\), we have

\[
|J_1^{(n)}(t_1, t_2, t)|^\alpha = \int_0^t \int_0^\infty \int_0^{\frac{\varepsilon(n+1)}{n}} \left[ \left| J_1^{(n)}(t_1, t_2, s) + \frac{e^{-2\beta s}}{n^2} R_{\beta}^{(n)}(\gamma_n(t_2 - s), 1, y) - R_{\beta}^{(n)}(\gamma_n(t_1 - s), 1, y) \right|^2 - |J_1^{(n)}(t_1, t_2, s)|^\alpha \right] \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu).
\]

From the mean value theorem, there exists a sequence of constants \(\{\tilde{\vartheta}_s^{(n)} \in [0, 1] : s \geq 0\} \) such that

\[
|J_1^{(n)}(t_1, t_2, t)|^\alpha = \alpha \int_0^t \int_0^\infty \int_0^{\frac{\varepsilon(n+1)}{n}} \left| J_1^{(n)}(t_1, t_2, s) + \tilde{\vartheta}_s^{(n)} e^{-2\beta s} \frac{n^2}{n^2} R_{\beta}^{(n)}(\gamma_n(t_2 - s), 1, y) - R_{\beta}^{(n)}(\gamma_n(t_1 - s), 1, y) \right|^2 \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu).
\]

\[
- R_{\beta}^{(n)}(\gamma_n(t_1 - s), 1, y) \right|^2 \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu).
\]

\[
- R_{\beta}^{(n)}(\gamma_n(t_1 - s), 1, y) \right|^2 \tilde{N}_0^{(n)}(d\gamma_{ns}, \{1\}, dy, dnu).
\]
Since $\alpha \in (1, 2)$ and $(x + y)^{\alpha - 1} \leq x^{\alpha - 1} + y^{\alpha - 1}$ for any $x, y \geq 0$, we have

$$|J_1^{(n)}(t_1, t_2, t)|^\alpha \leq J_{1,1}^{(n)}(t_1, t_2, t) + J_{1,2}^{(n)}(t_1, t_2, t),$$

where

$$J_{1,1}^{(n)}(t_1, t_2, t) := \alpha \int_0^t \int_0^\infty \int_0^\infty \frac{Z(n)(\gamma_{n,s})}{n} \left|J_1^{(n)}(t_1, t_2, s)\right|^{\alpha - 1} \times \frac{e^{-2\beta s}}{n^2} |R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y) - R_\beta^{(n)}(\gamma_n(t_1 - s), 1, y)|^2 N_0^{(n)}(d\gamma_n s, \{1\}, dy, dnu),$$

$$J_{1,2}^{(n)}(t_1, t_2, t) := \alpha \int_0^t \int_0^\infty \int_0^\infty \frac{Z(n)(\gamma_{n,s})}{n} \left|\frac{e^{-2\alpha \beta s}}{n^{2\alpha}} |R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y) - R_\beta^{(n)}(\gamma_n(t_1 - s), 1, y)|^2 N_0^{(n)}(d\gamma_n s, \{1\}, dy, dnu).$$

Taking the expectation on the both sides of (6.24), from (6.13) and (5.7),

$$\mathbb{E}\left[J_{1,2}^{(n)}(t_1, t_2, t)\right] \leq C\frac{\gamma_n}{n^{2\alpha - 1}} \int_0^t \int_0^\infty |R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y) - R_\beta^{(n)}(\gamma_n(t_1 - s), 1, y)|^{2\alpha} ds \Lambda^{(n)}(dy)$$

$$\leq C\frac{\gamma_n}{n^{2\alpha - 1}} \int_0^t \int_0^\infty |R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y) - R_\beta^{(n)}(\gamma_n(t_1 - s), 1, y)|^{2\alpha} ds \Lambda^{(n)}(dy)$$

Before considering (6.23), we need to make some preparation. Since $\alpha \in (1, 2)$, it is easy to see that

$$\mathbb{E}\left[|J_1^{(n)}(t_1, t_2, t)|^{\alpha - 1} \frac{e^{-\beta t Z(n)(\gamma_n t)}}{n}\right]$$

$$\leq \mathbb{E}\left[\frac{e^{-\beta t Z(n)(\gamma_n t)}}{n} \int_0^t \int_0^\infty \int_0^\infty \frac{Z(n)(\gamma_{n,s})}{n} \frac{e^{-2\beta s}}{n^2} |R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y) - R_\beta^{(n)}(\gamma_n(t_1 - s), 1, y)|^{-1} \right]$$

$$+ \mathbb{E}\left[\frac{e^{-\beta t Z(n)(\gamma_n t)}}{n} \int_0^t ds \int_0^\infty \frac{e^{-2\beta \beta s Z(n)(\gamma_{n,s})}}{n} |R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y)|^{\alpha - 1} \right].$$

By Hölder’s inequality and Corollary 6.2,

$$(6.26) \leq \left\{ \mathbb{E}\left[\int_0^t ds \int_0^\infty \frac{e^{-2\beta \beta s Z(n)(\gamma_{n,s})}}{n} |R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y)|^{\alpha - 1} \right] \right\}^{1/\alpha} \{ \mathbb{E}\left[\frac{e^{-\beta t Z(n)(\gamma_n t)}}{n} \right]^{\alpha - 1} \}$$

$$\leq C \left\{ \int_0^t ds \int_0^\infty \left| R_\beta^{(n)}(\gamma_n(t_2 - s), 1, y) - R_\beta^{(n)}(\gamma_n(t_1 - s), 1, y) \right|^{2\alpha} \right\}.$$
Here the last inequality comes from (5.27) and Condition 4.3(2). Similarly, since \( \alpha/2 < 1 \), we also have

\[
(6.25) \leq \left\{ \mathbb{E} \left[ \frac{e^{-\beta t} Z^{(n)}(\gamma t)}{n} \right] \right\}^{1/\alpha} \left\{ \mathbb{E} \left[ \int_0^t \int_0^\infty \int_0^\infty \frac{\mathcal{Z}^{(\alpha-\beta,\gamma-\gamma)}}{n^2} R^{(n)}(\gamma(n_t-2-s),1,y) \right] \right\}^{\alpha-1/\alpha}
\]

Thus

\[
\mathbb{E} \left[ |J_1^{(n)}(t_1,t_2,t)|^{\alpha-1} \frac{e^{-\beta t} Z^{(n)}(\gamma t)}{n} \right] \leq C |t_1 - t_2|^{\alpha-1/\alpha}.
\]

Now we start to consider (6.23). Taking the expectation on the both sides of (6.23), from this result and (5.27),

\[
\mathbb{E} \left[ J_1^{(n)}(t_1,t_2,t) \right] \leq C \int_0^t \int_0^\infty \mathbb{E} \left[ |J_1^{(n)}(t_1,t_2,s)|^{\alpha-1} \frac{e^{-\beta s} Z^{(n)}(\gamma n s)}{n} \right]
\]

\[
\times |R^{(n)}(\gamma(n_t-2-s),1,y) - R^{(n)}(\gamma(n_t-1-s),1,y)|^{2\alpha} \Lambda^{(n)}(dy)
\]

\[
\leq C |t_1 - t_2|^{\alpha-1/\alpha} \int_0^\infty \int_0^\infty |R^{(n)}(\gamma(n_t-2-s),1,y)|^{2\alpha} \Lambda^{(n)}(dy)
\]
Putting all results above together, we will get the desired result. \qed

**Proposition 6.11.** For any \( k \geq j \geq 1 \), there exist two constants \( C > 0 \) and \( \varphi \in (0, \frac{2\alpha - 2}{\alpha}) \) such that for any \( t_1, t_2 \geq 0 \),

\[
|E[|\hat{\xi}_{3,k,j}(t_2) - \hat{\xi}_{3,k,j}(t_1)|^{2\alpha}]| \leq C(|t_2 - t_1|^{1+\varphi} + |t_2 - t_1|^{1+\frac{\alpha-1}{\alpha}}).
\]

Hence the sequence \( \{\hat{\xi}_{3,k,j}\} \) is tight in \( C(\mathbb{R}_+, \mathbb{R}) \).

**Proof.** From Proposition 10.3 in [11, p.149] or Theorem 13.5 in [3, p.142], the tightness of the sequence \( \{\hat{\xi}_{3,k,j}\} \) follows directly from (6.27). We start to prove the first statement with \( k = j = 1 \) and other cases can be proved similarly. If there exists \( i \geq 0 \) such that \( t_1, t_2 \in [in^{-\theta}, (i+1)n^{-\theta}] \), from Proposition 6.10,

\[
E[|\hat{\xi}_{3,1,1}(t_2) - \hat{\xi}_{3,1,1}(t_1)|^{2\alpha}] = n^{2\alpha}|t_2 - t_1|^{2\alpha}E[|\hat{\xi}_{3,1,1}(in^{-\theta}) - \hat{\xi}_{3,1,1}((i+1)n^{-\theta})|^{2\alpha}]
\]

\[
\leq n^{2\alpha}|t_2 - t_1|^{2\alpha}n^{-\theta-2\alpha+2}
\]

\[
\leq n^{2\alpha-\theta-2\alpha+2}|t_2 - t_1|^{2\alpha-1-\theta}|t_2 - t_1|^{1+\varphi}
\]

\[
\leq n^{2\alpha-\theta-2\alpha+2}n^{-\theta(2\alpha-1-\theta)}|t_2 - t_1|^{1+\varphi} \leq |t_2 - t_1|^{1+\varphi}.
\]

If there exists \( i \geq 0 \) such that \( in^{-\theta} \leq t_1 \leq (i+1)n^{-\theta} \leq t_2 \leq (i+2)n^{-\theta} \), we have

\[
E[|\hat{\xi}_{3,1,1}(t_2) - \hat{\xi}_{3,1,1}(t_1)|^{2\alpha}] \leq C E[|\hat{\xi}_{3,1,1}(t_2) - \hat{\xi}_{3,1,1}(i+1)n^{-\theta})|^{2\alpha}]
\]

\[
+ C E[|\hat{\xi}_{3,1,1}((i+1)n^{-\theta}) - \hat{\xi}_{3,1,1}(t_1)|^{2\alpha}]
\]

\[
\leq C|t_2 - (i+1)n^{-\theta}|^{1+\varphi} + C|(i+1)n^{-\theta} - t_1|^{1+\varphi}
\]

\[
\leq 2C|t_2 - t_1|^{1+\varphi}.
\]

If \( |t_2 - t_1| > n^{-\theta} \), from these two results above and Proposition 6.10,

\[
E[|\hat{\xi}_{3,1,1}(t_2) - \hat{\xi}_{3,1,1}(t_1)|^{2\alpha}] \leq C E[|\hat{\xi}_{3,k,j}(t_2) - \hat{\xi}_{3,k,j}([n^\theta t_2]n^{-\theta})|^{2\alpha}]
\]

\[
+ C E[|\hat{\xi}_{3,k,j}([n^\theta t_2]n^{-\theta}) - \hat{\xi}_{3,k,j}([n^\theta t_1] + 1)n^{-\theta})|^{2\alpha}]
\]

\[
+ C E[|\hat{\xi}_{3,k,j}(([n^\theta t_1] + 1)n^{-\theta}) - \hat{\xi}_{3,k,j}(t_1)|^{2\alpha}]
\]

\[
\leq C|t_2 - [n^\theta t_2]n^{-\theta}|^{1+\varphi} + C|[n^\theta t_1] + 1)n^{-\theta} - t_1|^{1+\varphi}
\]

\[
+ C \left[ \frac{\gamma n}{n^{2\alpha-1}}|t_1 - t_2| + |t_1 - t_2|^{1+\frac{\alpha-1}{\alpha}} \right]
\]
\[
\leq C \left( |t_2 - t_1|^{1+\theta} + |t_2 - t_1|^{1+\frac{\alpha-1}{\alpha}} \right).
\]

Here we have finished the proof. \(\square\)

**Proposition 6.12.** For any \(k \geq j \geq 1, T > 0\) and \(\epsilon > 0\), we have as \(n \to \infty\),

\begin{equation}
(6.28) \quad \sup_{t_1, t_2 \in [0,T], |t_2 - t_1| < n^{-\theta}} |\varepsilon_{5,k,j}^{(n)}(t_2) - \varepsilon_{5,k,j}^{(n)}(t_1)| \to 0, \quad \text{in probability.}
\end{equation}

Moreover, we also have

\[
\sup_{t \in [0,T]} |\varepsilon_{3,k,j}^{(n)}(t) - \varepsilon_{3,k,j}^{(n)}(t)| \to 0, \quad \text{in probability.}
\]

**Proof.** It is easy to see that the second result follows directly from the first one, i.e. from the definition of \(\varepsilon_{3,k,j}^{(n)}\), we have as \(n \to \infty\),

\[
\sup_{t \in [0,T]} |\varepsilon_{3,k,j}^{(n)}(t) - \varepsilon_{3,k,j}^{(n)}(t)| \leq \sup_{t \in [0,T]} |\varepsilon_{3,k,j}^{(n)}(t) - \varepsilon_{3,k,j}^{(n)}(|n^\theta t| n^{-\theta})|
\]

\[
+ \sup_{t \in [0,T]} |\varepsilon_{3,k,j}^{(n)}(t) - \varepsilon_{3,k,j}^{(n)}((|n^\theta t| + 1)n^{-\theta})|
\]

\[
\leq 2 \sup_{t_1, t_2 \in [0,T], |t_1 - t_2| < n^{-\theta}} |\varepsilon_{3,k,j}^{(n)}(t_2) - \varepsilon_{3,k,j}^{(n)}(t_1)| \to 0.
\]

Now we start to prove (6.28). In order to simplify the following statement, here we just prove it with \(k = j = 1\) and \(T = 1\). Other cases can be proved similarly. For any \(0 \leq t_1 \leq t_2 \leq 1\), recall \(\{\varepsilon_{3,1,1,i}^{(n)}(t_1, t_2) : i = 1, 2, 3\}\) defined by (6.20)-(6.22). It suffices to prove that for any \(\epsilon > 0\),

\begin{equation}
(6.29) \quad \lim_{n \to \infty} \mathbb{P}\left\{ \sup_{t_1, t_2 \in [0,1], |t_2 - t_1| < n^{-\theta}} |\varepsilon_{3,1,1,i}^{(n)}(t_1, t_2)| \geq \epsilon \right\} = 0, \quad i = 1, 2, 3.
\end{equation}

**Step 1.** From (5.2), we can split \(\varepsilon_{3,1,1,1}^{(n)}(t_1, t_2)\) into the following four parts:

\[
\varepsilon_{3,1,1,1}^{(n)}(t_1, t_2) = I_{1,1}^{(n)}(t_1, t_2) + I_{1,2}^{(n)}(t_1, t_2) + I_{1,3}^{(n)}(t_1, t_2) + I_{1,4}^{(n)}(t_1, t_2),
\]

where

\[
I_{1,1}^{(n)}(t_1, t_2) := \int_0^{t_2} \int_0^{\gamma_n(t_2-s)} \frac{e^{-\beta t_2} - e^{-\beta t_1}}{n} \int_{\gamma_n(t_2-s)-y} R^{(n)}(\xi) d\xi \tilde{N}_0^{(n)}(d\gamma_s, \{1\}, dy, dnu),
\]

\[
I_{1,2}^{(n)}(t_1, t_2) := \int_0^{t_2} \int_0^{\gamma_n(t_2-s)} \frac{1}{n} \left[ e^{-\beta(t_2-s)} R^{(n)}(\xi) d\xi \right] e^{-\beta s} \tilde{N}_0^{(n)}(d\gamma_s, \{1\}, dy, dnu),
\]

\[
I_{1,3}^{(n)}(t_1, t_2) := (e^{-\beta t_2} - e^{-\beta t_1}) \int_0^{t_2} \int_0^{\gamma_n(t_2-s)} \frac{1}{n} \tilde{N}_0^{(n)}(d\gamma_s, \{1\}, dy, dnu),
\]

\[
I_{1,4}^{(n)}(t_1, t_2) := \int_0^{t_2} \int_{\gamma_n(t_2-s)} \frac{e^{-\beta t_1}}{n} \tilde{N}_0^{(n)}(d\gamma_s, \{1\}, dy, dnu).
\]
From Lemma 6.1,

\[
|I_{1,1}^{(n)}(t_1, t_2)| \leq \frac{C|t_2 - t_1|}{n} \int_0^{t_2} \int_0^{t_1} \left( \frac{\gamma_{n,s}}{n} \right)^{-2} ye^{-\beta s} N_{0}^{(n)}(d\gamma_{n,s}, \{1\}, dy, dnu) \]

\[
+ C\gamma_{n}|t_2 - t_1| \int_0^{t_2} \frac{e^{-\beta s} Z^{(n)}(\gamma_{n,s})}{n} ds
\]

and

\[
E\left[ \sup_{t_1,t_2 \in [0,1],|t_2-t_1|<n^{-\theta}} |I_{1,1}^{(n)}(t_1, t_2)| \right] \leq \frac{C\gamma_{n}}{n^{\theta}}.
\]

By Chebyshev’s inequality,

\[
\lim_{n \to \infty} P\left\{ \sup_{t_1,t_2 \in [0,1],|t_2-t_1|<n^{-\theta}} |I_{1,1}^{(n)}(t_1, t_2)| \geq \epsilon \right\} = 0.
\]

Similarly, we can also get the same result for $I_{1,2}^{(n)}$ and $I_{1,3}^{(n)}$. Now we start to consider $I_{1,4}^{(n)}$. Obviously, for any $|t_2 - t_1| < n^{-\theta}$,

\[
I_{1,4}^{(n)}(t_1, t_2) = \frac{1}{n} e^{-\beta t_1} \left( \int_0^{t_2} \int_{\gamma_{n,(t_1-s)}}^{\gamma_{n,(t_1-s)}} \frac{\gamma_{n,s}}{n} N_{0}^{(n)}(d\gamma_{n,s}, \{1\}, dy, dnu) \right)
\]

\[
\leq \frac{C}{n} \int_0^{t_1} \left( \int_{\gamma_{n,(t_1-s)}}^{\gamma_{n,(t_1-s)}} \frac{\gamma_{n,s}}{n} e^{-\beta s} N_{0}^{(n)}(d\gamma_{n,s}, \{1\}, dy, dnu) \right)
\]

\[
+ C\gamma_{n} \int_0^{1} ds \left( \int_{\gamma_{n,(t_1-s)}}^{\gamma_{n,(t_1-s)}} e^{-\beta s} Z^{(n)}(\gamma_{n,s}) \Lambda^{(n)}(dy) \right).
\]

Thus

\[
\sup_{t_1,t_2 \in [0,1],|t_2-t_1|<n^{-\theta}} |I_{1,4}^{(n)}(t_1, t_2)|
\]

\[
\leq \sup_{t_1 \in [0,1]} \left\{ \frac{C}{n} \int_0^{t_1} \left( \int_{\gamma_{n,(t_1-s)}}^{\gamma_{n,(t_1-s)}} \frac{\gamma_{n,s}}{n} e^{-\beta s} N_{0}^{(n)}(d\gamma_{n,s}, \{1\}, dy, dnu) \right)
\]

\[
+ C\gamma_{n} \int_0^{1} ds \left( \int_{\gamma_{n,(t_1-s)}}^{\gamma_{n,(t_1-s)}} e^{-\beta s} Z^{(n)}(\gamma_{n,s}) \Lambda^{(n)}(dy) \right) \right\}
\]

\[
\leq \max_{j=0,\ldots,[n\theta]} \left\{ \frac{C}{n} \int_0^{1} \left( \int_{\gamma_{n,(j+2)n^{-\theta}-s}}^{\gamma_{n,(j+2)n^{-\theta}-s}} \frac{\gamma_{n,s}}{n} e^{-\beta s} N_{0}^{(n)}(d\gamma_{n,s}, \{1\}, dy, dnu) \right)
\]

\[
+ C\gamma_{n} \int_0^{1} ds \left( \int_{\gamma_{n,(j+2)n^{-\theta}-s}}^{\gamma_{n,(j+2)n^{-\theta}-s}} e^{-\beta s} Z^{(n)}(\gamma_{n,s}) \Lambda^{(n)}(dy) \right) \right\}
\]

\[
\leq \max_{j=0,\ldots,[n\theta]} \frac{C}{n} \int_0^{1} \left( \int_{\gamma_{n,(j+2)n^{-\theta}-s}}^{\gamma_{n,(j+2)n^{-\theta}-s}} \frac{\gamma_{n,s}}{n} e^{-\beta s} N_{0}^{(n)}(d\gamma_{n,s}, \{1\}, dy, dnu) \right)
\]

\[
+ \max_{i=0,\ldots,[n\theta]} C\gamma_{n} \int_0^{1} ds \left( \int_{\gamma_{n,(j+2)n^{-\theta}-s}}^{\gamma_{n,(j+2)n^{-\theta}-s}} e^{-\beta s} Z^{(n)}(\gamma_{n,s}) \Lambda^{(n)}(dy) \right).
\]

From Chebyshev’s inequality and Proposition 6.4, we have

\[
P\left\{ \max_{j=0,\ldots,[n\theta]} C\gamma_{n} \int_0^{1} ds \left( \int_{\gamma_{n,(j+2)n^{-\theta}-s}}^{\gamma_{n,(j+2)n^{-\theta}-s}} e^{-\beta s} Z^{(n)}(\gamma_{n,s}) \Lambda^{(n)}(dy) \right) \geq \epsilon \right\} = 0.
\]
Step 2. Now we start to prove (6.29) with \( i = 2 \). From (6.21),

\[
\left| \varepsilon_{3,1,1,2}^{(n)}(t_1, t_2) \right| \leq \frac{C}{n} \int_{t_1}^{t_2} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{z^{(n)}(\gamma_s n)}{n} \gamma_s^{n-\theta} \gamma_0(n)(dy, s, \{1\}, dy, d\nu u) \geq \varepsilon \]

\[
+ C \gamma_n \int_{t_1}^{t_2} e^{-\beta s} \gamma_0(n)(\gamma_s n) dy ds.
\]

Similar to the argument in Step 1, we also have

\[
\sup_{t_1, t_2 \in [0, 1], |t_2 - t_1| < n^{-\theta}} \left| \varepsilon_{3,1,1,2}^{(n)}(t_1, t_2) \right| \leq \max_{j=0, \ldots, [n^\theta]} C \gamma_n \int_{jn^{-\theta}}^{(j+2)n^{-\theta}} e^{-\beta s} \gamma_0(n)(\gamma_s n) ds + \max_{j=0, \ldots, [n^\theta]} \left| \frac{C}{n} \int_{jn^{-\theta}}^{(j+2)n^{-\theta}} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{z^{(n)}(\gamma_s n)}{n} \gamma_s^{n-\theta} \gamma_0(n)(dy, s, \{1\}, dy, d\nu u) \right|.
\]

From Chebyshev’s inequality and Hölder’s inequality,

\[
\mathbb{P} \left\{ \max_{j=0, \ldots, [n^\theta]} 2C \gamma_n \int_{jn^{-\theta}}^{(j+2)n^{-\theta}} e^{-\beta s} \gamma_0(n)(\gamma_s n) ds \geq \epsilon \right\} \leq \sum_{j=0}^{[n^\theta]} \mathbb{P} \left\{ 2C \gamma_n \int_{jn^{-\theta}}^{(j+2)n^{-\theta}} e^{-\beta s} \gamma_0(n)(\gamma_s n) ds \geq \epsilon \right\}
\]
Finally, we prove (6.29) with $\alpha = 3$. It is easy to see that
\[
\left| \varepsilon_{3,1,1}^{(n)}(t_1, t_2) \right| \leq C\gamma_n |t_2 - t_1| \int_0^1 ds \int_0^\infty \frac{e^{-\beta s} Z(n) (\gamma_n s)}{n} ds
\]
\[
+ C|t_2 - t_1| \int_0^1 \int_0^\infty \int_0^\infty \frac{\gamma s e^{\beta s} N_0(n) (d\gamma_n s, \{1\}, dy, dnu)}{n}
\]
and
\[
\mathbb{E} \left[ \sup_{t_1, t_2 \in [0,1], |t_2 - t_1| < n^{-\theta}} \left| \varepsilon_{3,1,1,3}^{(n)}(t_1, t_2) \right| \right] \leq \frac{C\gamma_n}{n^\theta} \int_0^1 \mathbb{E} \left[ \frac{e^{-\beta s} Z(n) (\gamma_n s)}{n} \right] ds \leq \frac{C\gamma_n}{n^\theta}.
\]
Applying Chebyshev's inequality again, we will get the desired result. Here we have finished the whole proof.

Since $\varepsilon_{3,k,j}^{(n)} = \varepsilon_{3,k,j}^{(n)} + \varepsilon_{3,k,j}^{(n)} - \varepsilon_{3,k,j}^{(n)}$ from Proposition 6.11-6.12 and Theorem 3.1 in [3, p.27], we can get the weak convergence of the sequence $\{\varepsilon_{3,k,j}^{(n)}\}$ directly; see the following proposition.

**Proposition 6.13.** For any $k \geq j \geq 1$, the sequence $\{\varepsilon_{3,k,j}^{(n)}\}$ converges to 0 weakly in $D(\mathbb{R}_+, \mathbb{R})$.

**Proposition 6.14.** For any $T > 0$ and $\epsilon > 0$, we have
\[
\lim_{k_0 \to \infty} \limsup_{n \to \infty} \mathbb{P} \left\{ \sup_{t \in [0,T]} \left| \sum_{k = k_0}^k \sum_{j = 1}^k \varepsilon_{3,k,j}^{(n)}(t) \right| \geq \epsilon \right\} = 0.
\]
Proof. For any $k \geq 1$, it is easy to see that \( \{\varepsilon_{3,k,j}^{(n)} : j = 1, \ldots, k\} \) is identically distributed. From Lemma 6.1,

\[
|\varepsilon_{3,k,1}^{(n)}(t)| \leq \int_0^t \int_0^\infty \int_0^\infty \frac{1}{n} (1 + y) e^{-\beta s} \Lambda_0^{(n)}(d\gamma_n, \{k\}, dy, dnu) e^{-\beta s} Z_0^{(n)}(\gamma_n, s) \int_0^\infty (1 + y) \Lambda^{(n)}(dy)
\]

and

\[
E \left[ \sup_{t \in [0,T]} |\varepsilon_{3,k,1}^{(n)}(t)| \right] \leq C \gamma_n \Lambda^{(n)} p_k^{(n)} \int_0^T E \left[ \frac{e^{-\beta s} Z_0^{(n)}(\gamma_n, s)}{n} \right] ds \leq CT p_k^{(n)} \gamma_n.
\]

From Condition 4.3(1),

\[
\lim_{k_0 \to \infty} \limsup_{n \to \infty} E \left\{ \sup_{t \in [0,T]} \sum_{k=k_0}^\infty \sum_{j=1}^k |\varepsilon_{3,k,j}^{(n)}(t)| \right\} \leq \lim_{k_0 \to \infty} \limsup_{n \to \infty} \sum_{k=k_0}^\infty \sum_{j=1}^k E \left\{ \sup_{t \in [0,T]} |\varepsilon_{3,k,j}^{(n)}(t)| \right\} \leq CT \lim_{k_0 \to \infty} \limsup_{n \to \infty} \gamma_n \sum_{k=k_0}^\infty k p_k^{(n)} = 0.
\]

Here we have finished the proof.

Based on all results above, we start to prove Lemma 6.8.

Proof for Lemma 6.8. From Proposition 6.13 and the argument in [3, p.124], for any $k_0 \geq 1$

\[
\limsup_{n \to \infty} \mathbb{P} \left\{ \sup_{t \in [0,T]} \sum_{k=1}^{k_0} |\varepsilon_{3,k,j}^{(n)}(t)| \geq \varepsilon \right\} \leq \sum_{k=1}^{k_0} \limsup_{n \to \infty} \mathbb{P} \left\{ \sup_{t \in [0,T]} |\varepsilon_{3,k,j}^{(n)}(t)| \geq \frac{\varepsilon}{k_0} \right\} = 0.
\]

From this and Proposition 6.14,

\[
\limsup_{n \to \infty} \mathbb{P} \left\{ \sup_{t \in [0,T]} |\varepsilon_{3}^{(n)}(t)| \geq \varepsilon \right\} \leq \lim_{k_0 \to \infty} \limsup_{n \to \infty} \mathbb{P} \left\{ \sup_{t \in [0,T]} \sum_{k=1}^{k_0} \sum_{j=1}^k |\varepsilon_{3,k,j}^{(n)}(t)| \geq \frac{\varepsilon}{2} \right\} + \lim_{k_0 \to \infty} \limsup_{n \to \infty} \mathbb{P} \left\{ \sum_{k=k_0+1}^\infty \sum_{j=1}^k |\varepsilon_{3,k,j}^{(n)}(t)| \geq \frac{\varepsilon}{2} \right\} = 0.
\]

Here we have finished the proof.

6.4. Weak convergence of semimartingales. In this section, we mainly prove the weak convergence of the sequence \( \{(L^{(n)}, M^{(n)})\} \) defined by (6.7) and (6.8) in \( \mathbf{D}(\mathbb{R}_+, \mathbb{R}_+ \times S(\mathbb{R}_+)) \). It is easy to see that \( \{M^{(n)}(dt, du)\} \) is a sequence of orthogonal martingale random measure on \( \mathbb{R}_+^2 \). From Mitoma’s theorem; see Theorem 6.13 in [45], it suffices to prove the weak convergence of the sequence...
\{ (L^{(n)}, M^{(n)}_f) \} in D(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}) for any f \in C_K(\mathbb{R}_+), where C_K(\mathbb{R}_+) is the space of continuous functions on \mathbb{R}_+ with compact support and

\[ M^{(n)}_f(t) := \int_0^t \int_0^\infty f(u)M^{(n)}(ds, du) = \int_0^t \int_{\mathbb{R}_+^2} \int_0^\infty \frac{f(u)}{n} \left( \sum_{j=1}^k y_j \right) \tilde{N}^{(n)}_0(d\gamma^n_s, dk, dy, du). \]

It is easy to check that \( (L^{(n)}, M^{(n)}_f) \) is a two-dimensional strong Markov process with generator \( \mathcal{L}^{(n)} \) defined by: for any \( F(x_1, x_2) \in C^2(\mathbb{R}^2), \)

\[
\mathcal{L}^{(n)} F(x_1, x_2) := \gamma_n \zeta^{(n)} \int_{\mathbb{R}_+^2} \left[ F \left( x_1 + \frac{1}{n} \left( \sum_{j=1}^k y_j \right), x_2 \right) - F(x_1, x_2) \right] \nu_1^{(n)}(dk, dy)
\]

\[ + n \gamma_n \alpha^{(n)} \int_{\mathbb{R}_+^2} \left[ F \left( x_1, x_2 + \frac{f(u)}{n} \left( \sum_{j=1}^k y_j \right) \right) - F(x_1, x_2) \right] \frac{\partial F}{\partial x_2} \nu_0^{(n)}(dk, dy) du. \]

**Lemma 6.15.** For any \( T > 0 \) and \( f \in C_K(\mathbb{R}_+) \), there exists a constant \( C > 0 \) such that

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \sup_{n \geq 0} |L^{(n)}(t)| \right] + \sup_{t \in [0, T]} \mathbb{E} \left[ \sup_{n \geq 0} |M^{(n)}_f(t)| \right] \leq CT.
\]

**Proof.** From (6.7) and Condition 4.1-4.3, for any \( n \geq 1, \)

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |L^{(n)}(t)| \right] = \mathbb{E} \left[ L^{(n)}(T) \right] = \int_0^T \int_{\mathbb{R}_+^2} \int_0^\infty \frac{1}{n} \left( \sum_{j=1}^k y_j \right) \gamma_n \zeta^{(n)} ds \nu_1^{(n)}(dk, dy)
\]

\[ = \frac{\gamma_n}{n} \zeta^{(n)} T \sum_{k=1}^\infty k d\zeta^{(n)} \int_0^\infty y \Lambda^{(n)}(dy) \leq CT. \]

Moreover, from (6.8),

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |M^{(n)}_f(t)| \right] = \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t \int_{\mathbb{R}_+^2} \int_0^\infty \frac{f(u)}{n} \left( \sum_{j=1}^k y_j \right) \tilde{N}^{(n)}_0 (d\gamma^n_s, dk, dy, du) \right]
\]

\[ \leq \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t \int_{\{1\}} \int_{\mathbb{R}_+^2} \int_0^\infty \frac{f(u)}{n} \left( \sum_{j=1}^k y_j \right) \tilde{N}^{(n)}_0 (d\gamma^n_s, dk, dy, du) \right]
\]

\[ + \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t \int_{\mathbb{R}_+^2 \setminus \{1\}} \int_{\mathbb{R}_+^2} \int_0^\infty \frac{f(u)}{n} \left( \sum_{j=1}^k y_j \right) \tilde{N}^{(n)}_0 (d\gamma^n_s, dk, dy, du) \right]. \]

From Jensen’s inequality and the Burkholder-Davis-Gundy inequality,

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t \int_{\{1\}} \int_{\mathbb{R}_+^2} \int_0^\infty \frac{f(u)}{n} \left( \sum_{j=1}^k y_j \right) \tilde{N}^{(n)}_0 (d\gamma^n_s, dk, dy, du) \right]
\]

\[ \leq \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t \int_{\{1\}} \int_0^\infty \frac{f(u)}{n} y \tilde{N}^{(n)}_0 (d\gamma^n_s, \{1\}, dy, du) \right] \right\}^{1/2} \]
Moreover, we also have

\[
\left\{ \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \int_0^\infty \left| \frac{f(u)}{n^2} y^2 N_0^{(n)}(d\gamma_n s, \{1\}, dy, dnu) \right|^2 \right] \right\}^{1/2} \leq C \sum_{k=2}^{\infty} k p_k^{(n)} \leq CT.
\]

Here the last inequality follows from Condition 4.3. Putting all results above together, we will get the desired result.

Like the standard argument, the tightness of the sequence \( \{(L^{(n)}, M_f^{(n)})\} \) can be proved from the Aldous criterion and Lemma 6.15; see the following theorem (the proof will be omitted).

**Theorem 6.16.** The sequence \( \{(L^{(n)}, M_f^{(n)})\} \) is uniformly tight in \( D(\mathbb{R}_+, \mathbb{R}^2) \) and hence the sequence \( \{(L^{(n)}, M^{(n)})\} \) is uniformly tight in \( D(\mathbb{R}_+, \mathbb{R}_+ \times \mathcal{S}(\mathbb{R}_+)) \).

For any \( F \in C^2(\mathbb{R}^2) \) and \( f \in C_K(\mathbb{R}_+) \), define a map \( \mathcal{L} F : \mathbb{R}^2 \rightarrow \mathbb{R} \) by:

\[
\mathcal{L} F(x_1, x_2) = a \zeta_\eta \frac{\partial^2 F(x_1, x_2)}{\partial x_1^2} + \frac{1}{2} \left( 2c|\eta|^2 + 2\gamma_\sigma \right) \int_0^\infty |f(u)|^2 \lambda du \frac{\partial^2 F(x_1, x_2)}{\partial x_2^2} \\
+ \int_0^\infty \left( F(x_1 + z, x_2) - F(x_1, x_2) \right) \zeta \nu_1 \left( \frac{dz}{\eta} \right) \\
+ \int_{\mathbb{R}_+} \int_0^\infty \left( F(x_1, x_2 + f(u)z) - F(x_1, x_2) - f(u)z \frac{\partial F(x_1, x_2)}{\partial x_2} \right) \lambda_0 \left( \frac{dz}{\eta} \right) du.
\]

**Lemma 6.17.** Suppose \( (L, M_f) \) is a cluster point of the sequence \( \{(L^{(n)}, M_f^{(n)})\} \). Then for any \( (z_1, z_2) \in \mathbb{R}^2 \),

\[
\mathcal{M}_f(t) := \exp \left\{ i z_1 L(t) + i z_2 M_f(t) \right\} - 1 - \int_0^t \mathcal{L} \exp \left\{ i z_1 L(s) + i z_2 M_f(s) \right\} ds
\]

is a complex-valued local martingale.
Putting all results above together, we have

Now we prove that

\[\mathcal{M}_f^{(n)}(t) := \exp \left\{ i z_1 L^{(n)}(t) + i z_2 M_f^{(n)}(t) \right\} - 1 - \int_0^t \mathcal{L}^{(n)} \exp \left\{ i z_1 L^{(n)}(s) + i z_2 M_f^{(n)}(s) \right\} ds\]

is a complex-valued local martingale. Here

\[\mathcal{L}^{(n)} \exp \left\{ i z_1 L^{(n)}(s) + i z_2 M_f^{(n)}(s) \right\}\]

\[= \exp \left\{ i z_1 L^{(n)}(s) + i z_2 M_f^{(n)}(s) \right\} \times \left\{ \int_{Z^+} \int_{R^2_+} \zeta(n) \gamma_n \left[ \exp \left\{ i z_1 n \left( \sum_{j=1}^k y_j \right) \right\} - 1 \right] \nu_1^{(n)}(dk, dy)\right.\]

\[+ \int_{Z^+} \int_{R^2_+} \lambda(n) n \gamma_n \left[ \exp \left\{ i z_2 f(u) n \left( \sum_{j=1}^k y_j \right) \right\} - 1 - i z_2 \frac{f(u)}{n} (\sum_{j=1}^k y_j) \right] \nu_0^{(n)}(dk, dy) du \left. \right\}.

From Condition 4.1(2) and the mean value theorem,

\[\lim_{n \to \infty} \int_{Z^+} \int_{R^2_+} \zeta(n) \gamma_n \left[ \exp \left\{ i z_1 n \left( \sum_{j=1}^k y_j \right) \right\} - 1 \right] \nu_1^{(n)}(dk, dy)\]

\[= \lim_{n \to \infty} c^{(n)} \gamma_n \left[ \int_{Z^+} \int_{R^2_+} \exp \left\{ i z_1 n \left( \sum_{j=1}^k y_j \right) \right\} \nu_1^{(n)}(dk, dy) - 1 \right]\]

\[= \lim_{n \to \infty} c^{(n)} \gamma_n \left[ h^{(n)} \left( \int_0^\infty \exp \left\{ i z_1 y \right\} \Lambda^{(n)}(dy) - 1 \right) \right]\]

\[= \lim_{n \to \infty} c^{(n)} \gamma_n \left[ h^{(n)} \left( \int_0^\infty \exp \left\{ i z_1 y \right\} \Lambda^{(n)}(dy) \right) - h^{(n)} \left( 1 + i z_1 \eta^{(n)} \right) - 1 \right]\]

\[= -\zeta \psi(-iz_1 \eta) = iz_1 a \eta \zeta + \int_0^\infty (e^{iz_1 \eta} - 1) \zeta \nu_1(du).

Similarly, from Condition 4.1(3), we also have

\[\lim_{n \to \infty} \lambda^{(n)} \int_{Z^+} \int_{R^2_+} \int_0^\infty n \gamma_n \left[ \exp \left\{ i z_2 f(u) n \left( \sum_{j=1}^k y_j \right) \right\} - 1 - i z_2 \frac{f(u)}{n} (\sum_{j=1}^k y_j) \right] \nu_0^{(n)}(dk, dy) du \]

\[= \int_0^\infty \left[ \phi(-iz_2 \eta f(u)) - |z_2|^2 \gamma_\sigma |f(u)|^2 + iz_2 m \eta f(u) \right] \lambda du\]

\[= -|z_2|^2 (c \eta^2 + \gamma_\sigma) \int_0^\infty |f(u)|^2 \lambda du + \int_0^\infty \int_0^\infty \left[ e^{iz_2 \eta f(u) \xi} - 1 - iz_2 \eta f(u) \xi \right] \nu_0(\xi) d\xi du.

Putting all results above together, we have

\[\mathcal{M}_f^{(n)}(\cdot) \to \mathcal{M}_f(\cdot), \quad \text{a.s. in } D(R^+, \mathbb{C}).\]

Now we prove that \(\mathcal{M}_f^{(n)}\) is a complex-valued local martingale. Since \(|e^{ix} - 1 - ix| \leq |x| \wedge |x|^2\), we have

\[n \gamma_n \int_{Z^+} \int_{R^2_+} \int_0^\infty \left[ \exp \left\{ i z_2 f(u) n \left( \sum_{j=1}^k y_j \right) \right\} - 1 - i z_2 \frac{f(u)}{n} (\sum_{j=1}^k y_j) \right] \nu_0^{(n)}(dk, dy) du \]
\[ \leq n \gamma_n \int_0^\infty \int_0^\infty |\exp\left\{ \frac{iz_2 f(u)}{n} y \right\} - 1 - \frac{iz_2 f(u)}{n} y \nu_0^{(n)}(\{1\}, dy) du \]
\[ + n \gamma_n \int_{Z_+ \setminus \{1\}} \int_{R_+^2} |\exp\left\{ \frac{iz_2 f(u)}{n} \left( \sum_{j=1}^k y_j \right) \right\} - 1 - \frac{iz_2 f(u)}{n} \left( \sum_{j=1}^k y_j \right) \nu_0^{(n)}(dk, dy) du \]
\[ \leq C |z_2|^2 \int_0^\infty |f(u)|^2 du \int_0^\infty |y|^2 \Lambda^{(n)}(dy) + |z_2| \gamma_n \int_0^\infty |f(u)| du \int_{Z_+ \setminus \{1\}} \int_{R_+^2} \left( \sum_{j=1}^k y_j \right) \nu_0^{(n)}(dk, dy) \]
\[ \leq C |z_2|^2 + C |z_2| \gamma_n \sum_{k=2}^\infty k \rho_k^{(n)} \leq C (|z_2|^2 + |z_2|) . \]

Similarly, since \(|e^{ix} - 1| < |x|\), we also have
\[ \gamma_n \int_{Z_+} \int_{R_+^2} \left[ \exp\left\{ \frac{i z_1}{n} \left( \sum_{j=1}^k y_j \right) \right\} - 1 \right] \nu_1^{(n)}(dk, dy) \leq |z_1| \int_{Z_+} \int_{R_+^2} \left( \sum_{j=1}^k y_j \right) \nu_1^{(n)}(dk, dy) \leq C |z_1| . \]

Here all the bounds above are independent of \(n\), which induces that \(\{M_f^{(n)}(t)\}\) is uniformly integrable. Since the sequence \(\{M_f^{(n)}(t)\}\) converges almost surely, hence it also converges in \(L^1(\mathbb{P})\). By the standard stopping time argument, we can show that \(M_f\) is a local martingale. Here we have finished the proof. \(\square\)

**Theorem 6.18.** Suppose \((L, M)\) is a cluster point of the sequence \(\{(L^{(n)}, M^{(n)})\}\). On an extension of probability space, there exist a white noise \(W(ds, du)\) on \(R_+^2\) with intensity \(\lambda ds du\) and two independent Poisson random measures \(N_0(ds, dz, du)\) and \(N_1(ds, dz)\) defined on \(R_+^3\) and \(R_+^2\) with intensity \(\lambda ds \nu_0(dz) du\) and \(\zeta ds \nu_1(dz)\) respectively, such that
\[ L(t) = a \eta \zeta t + \int_0^t \int_0^\infty \eta z \nu_1(ds, dz) \]
and
\[ M_f(t) = \int_0^t \int_0^\infty \sqrt{2c|\eta|^2 + 2 \gamma_\sigma f(u)} W(ds, du) + \int_0^t \int_0^\infty \int_0^\infty f(u) \eta z N_0(dt, dz, du), \]
where \(\tilde{N}_0(dt, dz, du) := N_0(dt, dz, du) - \lambda ds \nu_0(dz) du\).

**Proof.** From Lemma 6.17 and Theorem 2.42 in [25, p.86], we can see that \((L, M_f)\) is a semi-martingale with the following canonical representation:
\[ L(t) := \zeta \eta \left( a + \int_0^\infty z \nu_1(dz) \right) t + M^f_L(t) + M^{\mathcal{L}}(t), \]
\[ M_f(t) := M^f_L(t) + M^f_f(t), \]
where \(M^f_L(t)\) and \(M^f_f(t)\) are independent continuous martingales with quadratic variation processes
\[ \langle M^f_L \rangle_t = 0, \quad \langle M^f_f \rangle_t = t \left( 2c |\eta|^2 + 2 \gamma_\sigma \right) \int_0^\infty |f(u)|^2 \lambda du, \]
and $\mathcal{M}^d_1(t)$ and $\mathcal{M}^d_2(t)$) are two independent purely discontinuous martingale. By Theorem III-6 in [12], on some extension of the probability space we can find a white noise $W(ds, du)$ on $\mathbb{R}^2_+$ with intensity $\lambda ds du$ such that

$$\mathcal{M}^d_j(t) = \int_0^t \int_0^\infty \sqrt{2c|\eta|^2 + 2\gamma_1f(u)} W(ds, du).$$

Moreover, there is an optional random measure $N_1(ds, dz)$ on $\mathbb{R}^2_+$ with compensator $\tilde{N}_1(ds, dz) = \zeta ds \nu_1(dz)$ such that

$$\mathcal{M}^d_L(t) = \int_0^t \int_0^\infty \eta \tilde{N}_1(ds, dz),$$

where $\tilde{N}_1(ds, dz) = N_1(ds, dz) - N_1(ds, dz)$. Similarly, there is an optional random measure $N_2(ds, d\nu)$ on $\mathbb{R}_+ \times \mathcal{S}(\mathbb{R}_+)$ with compensator

$$\tilde{N}_2(ds, d\nu) = \lambda dt \int_0^\infty \int_0^\infty z\delta_u(d\nu) d\nu_0\left(\frac{d\tilde{z}}{\eta}\right),$$

such that

$$\mathcal{M}^d_j(t) = \int_0^t \int_{\mathbb{S}(\mathbb{R}_+)} \nu(f) \tilde{N}_2(ds, d\nu).$$

For any bounded function $\varphi(z, u)$ on $\mathbb{R}^2_+$, we can define a purely discontinuous martingale $M^d_j(\varphi)$ by

$$M^d_j(\varphi) := \int_0^t \int_{\mathbb{S}(\mathbb{R}_+)} \int_0^\infty \varphi(z, x) \nu(dx) \tilde{N}_0(ds, d\nu).$$

Then $\{M^d_j : t \geq 0\}$ determines a martingale measure $M^d(dt, dz, du)$ on $\mathbb{R}^3_+$ with compensator $\hat{M^d}(dt, dz, du)$ satisfying

$$\int_0^t \int_0^\infty \int_0^\infty \varphi(z, u) \hat{M^d}(dt, dz, du) := \int_0^t \int_{\mathbb{S}(\mathbb{R}_+)} \int_0^\infty \varphi(z, x) \nu(dx) \lambda ds \int_0^\infty \int_0^\infty z\delta_u(d\nu) d\nu_0\left(\frac{d\tilde{z}}{\eta}\right)$$

$$= \int_0^t \lambda ds \int_0^\infty \int_0^\infty zd\nu_0\left(\frac{d\tilde{z}}{\eta}\right) \int_{\mathbb{S}(\mathbb{R}_+)} \int_0^\infty \varphi(z, x) \nu(dx) \delta_u(d\nu)$$

$$= \int_0^t \int_0^\infty \int_0^\infty \varphi(z, u) z \lambda ds \nu_0\left(\frac{d\tilde{z}}{\eta}\right) du.$$

By the argument in [23, p.93], on an extension of the probability space, there is a Poisson random measure $\tilde{N}_0(ds, d\tilde{z}, du)$ on $\mathbb{R}^3_+$ with intensity $\lambda ds \nu_0(d\tilde{z}, du)$ so that

$$\int_0^t \int_0^\infty \int_0^\infty \varphi(z, u) \hat{M^d}(dt, dz, du) = \int_0^t \int_0^\infty \int_0^\infty \varphi(z, u) z \tilde{N}_0(dt, d\tilde{z}, du)$$

$$= \int_0^t \int_0^\infty \int_0^\infty \varphi(z, u) \eta z \tilde{N}_0(dt, dz, du).$$

Specially, the purely discontinuous martingale $\mathcal{M}^d_j(t)$ can be presented into

$$\mathcal{M}^d_j(t) = \int_0^t \int_0^\infty \int_0^\infty f(u) \eta z \tilde{N}_0(dt, dz, du).$$

Here we have finished the proof. □
APPENDIX A: APPENDIX

**Lemma A.1** (Rogozin (1965)). Suppose \( \{X_i\}_{i \geq 1} \) is a sequence of i.i.d random variables with unimodal density \( f(x) \) satisfying that

\[
\sup_{x \in \mathbb{R}} f(x) \leq C_0,
\]

where \( C_0 \) is a positive constant. Let \( \{U_i\}_{i \geq 1} \) be a sequence of i.i.d uniformly distributed random variables on \([-1/2C_0, 1/2C_0]\). For any \( n \geq 1 \), denote by \( f_n(x) \) and \( h_n(x) \) the density of \( \sum_{i=1}^n X_i \) and \( \sum_{i=1}^n U_i \) respectively. Then for any \( n \geq 1 \)

\[
\sup_{x \in \mathbb{R}} |f_n(x)| \leq \sup_{x \in \mathbb{R}} |h_n(x)| = h_n(0).
\]

**Lemma A.2** (Shakhaidarova (1966)). Let \( \{Y_i\}_{i \geq 1} \) be a sequence of i.i.d random variables with probability density \( g(x) \) satisfying that

\[
\mathbb{E}[|Y_i|^3] < \infty, \quad \mu := \mathbb{E}[Y_i], \quad \sigma^2 := \mathbb{E}[|Y_i|^2], \quad \sup_{x \in \mathbb{R}} g(x) < \infty.
\]

Let \( g_n(x) \) be the density of \( \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \). Then there exists a constant \( C > 0 \) such that

\[
\sup_{x \in \mathbb{R}} \left| g_n(x) - \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} \right| \leq \frac{C}{\sqrt{n}}.
\]

**Lemma A.3.** Recall \( \{X_i\}_{i \geq 1} \) and \( f_n(x) \) defined in Lemma A.1. Then there exists a constant \( C > 0 \) such that

\[
\sup_{n \geq 1} \sup_{x \in \mathbb{R}} f_n(x) \leq C.
\]

Moreover, let \( N \) be an \( \mathbb{Z}_+ \)-valued random variable independent of \( \{X_i\}_{i \geq 1} \). Denote by \( f_N(x) \) the density of the random summation \( \sum_{i=1}^N X_i \), we have

\[
\sup_{x \in \mathbb{R}} f_N(x) \leq C.
\]

**Proof.** Recall \( \{U_i\}_{i \geq 1} \) and \( h_n(x) \) defined in Lemma A.1. From Lemma A.1, there exists a constant \( C > 0 \) such that for any \( n \geq 1 \),

\[
\sup_{x \in \mathbb{R}} f_n(x) \leq \sup_{x \in \mathbb{R}} h_n(x) = h_n(0).
\]

It is easy to check that \( \{U_i\}_{i \geq 1} \) satisfies conditions in Lemma A.2, thus

\[
\sup_{x \in \mathbb{R}} \left| h_n(x) - \sqrt{\frac{6}{\pi}} C_0 e^{-6C_0^2 \pi^2} \right| \leq \frac{C}{\sqrt{n}}
\]

and

\[
\sup_{x \in \mathbb{R}} h_n(x) = h_n(0) \leq \sqrt{\frac{6}{\pi}} C_0 + \frac{C}{\sqrt{n}}.
\]

Here we have gotten the first result. For the second one, from the law of total probability, we have

\[
\sup_{x \in \mathbb{R}} f_N(x) = \sup_{x \in \mathbb{R}} \sum_{n=1}^\infty \mathbb{P}(N = n) f_n(x) \leq \sum_{n=1}^\infty \mathbb{P}(N = n) \sup_{x \in \mathbb{R}} f_n(x) \leq \sqrt{\frac{6}{\pi}} C_0 + C.
\]

Here we gotten the second result. \( \square \)
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REFERENCES

[1] Aldous, D. (1978). Stopping times and tightness. Ann. Probab., 6(2), 335-340.
[2] Aliev S. A. (1985). A limit theorem for the Galton-Watson branching processes with immigration. Ukrainian Math. J., 37, 535-438.
[3] Billingsley, P. (1999). Convergence of Probability Measures. John Wiley & Sons.
[4] Brémaud, P. and Massoulié, L. (2002). Power spectra of general shot noise processes and Hawkes point processes with a random excitation. Adv. in Appl. Probab., 34(1), 205-222.
[5] Brunner, H. (2004). Collocation Methods for Volterra Integral and Related Functional Differential Equations (Vol. 15). Cambridge University Press.
[6] Crump, K. S. and Mode C. J. (1968). A general age-dependent branching process I. J. Math. Anal. Appl., 24(3), 494-508.
[7] Dawson, D. A. and Li, Z. (2012). Stochastic equations, flows and measure-valued processes. Ann. Probab., 40(2), 813-857.
[8] De Meyer, A. and Teugels, J. L. (1980). On the asymptotic behaviour of the distributions of the busy period and service time in M/G/1. J. Appl. Probab., 17(3), 802-813.
[9] Ethier, S. N. and Kurtz, T. G. (1986). Markov Processes: Characterization and Convergence. Wiley, New York.
[10] Ethier, S. N. and Kurtz, T. G. (1986). Markov Processes: Characterization and Convergence. Wiley, New York.
[11] Freund, R. J. (1971). Introduction to Statistics. Prentice-Hall Inc., Englewood Cliffs, N.J.
[12] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[13] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[14] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[15] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[16] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[17] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[18] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[19] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[20] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[21] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[22] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[23] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[24] Feller, W. (1957). An introduction to probability theory and its applications. John Wiley & Sons, New York.
[25] Ito, K. (1951). Stochastic integral. Mem. Coll. Sci. Univ. Kyushu Ser. A. Math., 3(1), 19-41.
[26] Jacod, J., Shiryaev, A. N. (2003). Limit Theorems for Stochastic Processes. Springer, Berlin.
[27] Jagers, P. (1975). Branching Processes with Biological Applications. John Wiley & Sons, London and New York.
[28] Jagers, P. and Nerman, O. (1984). The growth and composition of branching populations. Adv. in Appl. Probab., 16(2), 221-259.
[29] Jiřína, M. (1958). Stochastic branching processes with continuous state space. *Czechoslovak Math. J.*, 8(2), 292-313.

[30] Kawazu, K. and Watanabe, S. (1971). Branching processes with immigration and related limit theorems. *Theory Probab. Appl.*, 16(1), 36-54.

[31] Kurtz, T. G. and Protter, P. E. (1996). Weak convergence of stochastic integrals and differential equations II: Infinite dimensional case. In *Probabilistic models for nonlinear partial differential equations* (p.197-285). Springer, Berlin, Heidelberg.

[32] Lambert, A. (2010). The contour of splitting trees is a Lévy process. *Ann. Probab.*, 38(1), 348-395.

[33] Lambert, A. (2011). Species abundance distributions in neutral models with immigration or mutation and general lifetimes. *J. Math. Biol.*, 63(1), 57-72.

[34] Lambert, A. and Simatos, F. (2015). Asymptotic behavior of local times of compound Poisson processes with drift in the infinite variance case. *J. Theoret. Probab.*, 28(1), 41-91.

[35] Lamperti, J. (1967). The limit of a sequence of branching processes. *Z. Wahrsch. Verw. Gebiete*, 7(4), 271-288.

[36] Zhang, X. (2010). Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J. Functional Analysis*, 258, 1361-1425.