ON A POSSIBLE ALGEBRA MORPHISM
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OSCILLATOR ALGEBRA $W_q(N)$

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ABSTRACT

We formulate a conjecture, stating that the algebra of $n$ pairs of deformed Bose creation and annihilation operators is a factor-algebra of $U_q[osp(1/2n)]$, considered as a Hopf algebra, and prove it for $n = 2$ case. To this end we show that for any value of $q$, $U_q[osp(1/4)]$ can be viewed as a superalgebra, freely generated by two pairs $B_1^\pm, B_2^\pm$ of deformed para–Bose operators. We write down all Hopf algebra relations, an analogue of the Cartan–Weyl basis, the “commutation” relations between the generators and a basis in $U_q[osp(1/2n)]$ entirely in terms of $B_1^\pm, B_2^\pm$.

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I. Introduction

One way to describe completely a given simple Lie (super)algebra $A$ is in terms of its Chevalley generators. These generators are especially appropriate for a quantization of $A$, i.e., for a deformation of the universal enveloping algebra $U[A]$ of $A$ to a new associative algebra $U_q[A]$ in such a way that $U_q[A]$ remains a Hopf algebra.

Another possible way to define $A$ and $U[A]$ was outlined in Ref.1. It is based on the concept of creation and annihilation operators (CAO’s) of the simple Lie (super)algebra $A$ under consideration. Contrary to the Chevalley generators, the creation and annihilation operators of some algebras have direct physical significance. The CAO’s $F_1^\pm, F_2^\pm, \ldots, F_n^\pm$ of the orthogonal Lie algebra $B_n \equiv so(2n + 1)$ are known in quantum field theory as para-Fermi operators [2,3]; in a particular representation of $B_n$ they become usual Fermi operators. Similarly, the CAO’s $B_1^\pm, B_2^\pm, \ldots, B_n^\pm$ of the orthosymplectic Lie superalgebra $osp(1/2n)$ are the para-Bose operators [5,6]; in the representation, corresponding to an order of the statistics $p = 1$ they reduce to Bose creation and annihilation operators $b_1^+, b_2^+, \ldots, b_n^+$.

Clearly any deformation of $U[osp(1/2n)]$ will lead to a deformation of $B_1^\pm, B_2^\pm, \ldots, B_n^\pm$ and consequently to a deformation of the Bose operators $b_1^+, b_2^+, \ldots, b_n^+$. In this connection we wish to raise and discuss two questions.

1. Do the deformed CAO’s $B_1^\pm, B_2^\pm, \ldots, B_n^\pm$ of $osp(1/2n)$ define entirely $U_q[osp(1/2n)]$ (as this is the case with the deformed Chevalley generators)?

2. Is there any relation between the deformation of the Bose operators, obtained in this way, and the other known approaches to deform the Bose operators [7-11], which are unrelated to any Hopf algebra structure?

At present we do not know the answers neither to the first nor to the second question. There are good evidences, however, that the answer to both questions is positive and in particular that the Fock space representation of the $q-$deformed CAO’s $B_1^\pm, B_2^\pm, \ldots, B_n^\pm$ coincides with the deformed Bose operators as defined in Refs.8-10.

In order to formulate our conjecture more precisely let $W_q(n)$ be the deformed Weyl (or oscillator) algebra as defined by Hayashi [12]. The oscillator algebra $W_q(n)$ is an associative algebra with unity, free generators $k_i^+, k_i^-, i = 1, \ldots, n$ and the relations $(i, j = 1, \ldots, n)$

\[
\begin{align*}
    k_i^{-1} k_i &= k_i k_i^{-1} = 1, \\
    k_i k_i^+ &= q^{a_i} k_i^+ k_i, \\
    b_i^- b_i^+ - q^2 b_i^+ b_i^- &= k_i^{-2}, \\
    b_i^- b_i^+ - q^2 b_i^+ b_i^- &= k_i^2, \\
    a_i a_j &= a_j a_i, \quad i \neq j,
\end{align*}
\]  

(1.1)

where $a_i = b_i^+, k_i^\pm$.

To turn $W_q(n)$ into a superalgebra ( $\mathbb{Z}_2$-graded algebra) we set

\[
\begin{align*}
    \text{deg}(k_i^\pm) &= 1 \in \mathbb{Z}_2, \quad \text{deg}(k_i^\mp) = 0 \in \mathbb{Z}_2, \quad i = 1, \ldots, n.
\end{align*}
\]  

(1.2)
In the Fock representation of $W_q(n)$, namely when $k_i = q^{N_i}$, $b_i^\dagger$ are the deformed $q$-bosons [8-10] and $N_i$ is the $i$th boson number operator. With respect to the grading following from (1.2) $W_q(n)$ is an infinite-dimensional associative superalgebra. It is neither a Hopf algebra nor even a coalgebra. Our conjecture is the following one.

**CONJECTURE.** The deformed Weyl superalgebra $W_q(n)$ is a factor algebra of a deformed universal enveloping algebra $U_q[osp(1/2n)]$.

This conjecture holds in the nondeformed case [6]. In Sec.II we recall the idea of the proof. At $q \neq 1$ the conjecture has been proved so far for $n = 1$ [13]. Here we prove it for $n = 2$. To this end in Sec.III we deform $U[osp(1/4)]$ in terms of generators, which are in fact deformed para-Bose operators.

The case $n = 2$ was considered in Ref.14 in relation to the "supersingleton" Fock representation of $osp(1/4)$ and its "singleton" [15] structure. The authors have studied in details the quantum deformations of $sp(4)$ at $q$ being root of unity using deformed CAO's. We have been informed they have similar results to ours also for the deformed $osp(1/4)$ [16].

If the conjecture turns true, one can use the Hopf algebra structure in order to construct new representations of $U_q[osp(1/2n)]$ or of any of its subalgebras beginning with some known representations of it and in particular with the representation $\rho_F$ in the Fock space of the deformed Bose operators $F_q(n) \equiv \text{Fock}_q(n)$. To this end one can use the comultiplication $\Delta$. For instance the maps

$$\Delta^{(2)} = (\rho_F \otimes \rho_F) \circ \Delta : U_q[osp(1/2n)] \rightarrow \text{End}(F_q(n) \otimes F_q(n))$$

and

$$\Delta^{(3)} = (\rho_F \otimes \rho_F \otimes \rho_F) \circ [(id \otimes \Delta) \circ \Delta] : U_q[osp(1/2n)] \rightarrow \text{End}(F_q(n) \otimes F_q(n) \otimes F_q(n))$$

(1.3)

define representations of $U_q[osp(1/2n)]$ in $F_q(n) \otimes F_q(n)$ and $F_q(n) \otimes F_q(n) \otimes F_q(n)$, respectively .

Throughout we use the following abbreviations and notation:

**LS (LS's)— Lie superalgebra (Lie superalgebras),**

**lin.env.$\{X\}$ — the linear envelope of $X$,**

**$Z$ — all integers,**

**$Z_+$ — all nonnegative integers,**

**$Z_2 = \{0, 1\}$ — the ring of all integers modulo 2,**

**$[A, B] = AB - BA, \quad \{A, B\} = AB + BA,$**

**$< A, B > = AB - (-1)^{deg(A)deg(B)}BA,$**

**$[A, B]_q = AB - q^{nBA}, \quad \{A, B\}_q = AB + q^nBA.$**
II. The Nondeformed Case

Let $\text{Free}(n)$ be the associative superalgebra with unity, free generators $B^\pm_1, B^\pm_2, \ldots, B^\pm_n$, $\deg(B^\pm_i) = 1$ and the relations $(\xi, \eta, \epsilon = \pm)$ or $\pm 1, i, j, k = 1, 2, \ldots, n)$

$$[(B^\xi_i, B^\eta_j), B^\epsilon_k] = (\epsilon - \xi)\delta_{ik} B^\eta_j + (\epsilon - \eta)\delta_{jk} B^\xi_i,$$

(2.1)

Consider the subspace

$$B(0/n) = \text{lin.env.} \{ [B^\xi_i, B^\eta_j], B^\epsilon_k \mid \xi, \eta, \epsilon = \pm, i, j, k = 1, 2, \ldots, n \}$$

(2.2)

and define a supercommutator on it

$$\langle A, B \rangle = AB - (-1)^{\deg(A)\deg(B)} BA, \quad A, B \in B(0/n).$$

(2.3)

**PROPOSITION 1** [6]. $B(0/n)$ is a Lie superalgebra isomorphic to the orthosymplectic LS $\text{osp}(1/2n)$ with an even subalgebra

$$\text{sp}(2n) = \text{lin.env.} \{ [B^\xi_i, B^\eta_j] \mid \xi, \eta, = \pm, i, j = 1, 2, \ldots, n \}.$$  

(2.4)

**PROPOSITION 2.** The associative superalgebra $\text{Free}(n)$ is (isomorphic to) the universal enveloping algebra of $\text{osp}(1/2n)$,

$$\text{Free}(n) = U[\text{osp}(1/2n)].$$

(2.5)

From these propositions one concludes that $U[\text{osp}(1/2n)]$ is generated from $B^\pm_1, B^\pm_2, \ldots, B^\pm_n \in \text{osp}(1/2n)$. We point out that these $2n$ generators are very different from the Chevalley generators of the same algebra. The operators $B^\pm_1, B^\pm_2, \ldots, B^\pm_n$ were introduced by Green [2] as a possible generalization of the Bose statistics and are called para-Bose operators. Propositions 1 and 2 indicate that the representation theory of the para-Bose statistics is simply another name for the representation theory of the orthosymplectic Lie superalgebra.

Let $b^+_1, b^+_2, \ldots, b^+_n$ be Bose creation and annihilation operators and let $W(n)$ be the corresponding Weyl superalgebra, i.e., the set of all polynomials of $b^+_i$, considered as odd variables. It is straightforward to check that the Bose operators satisfy the para-Bose relations (2.1):

$$[[b^\xi_i, b^\eta_j], b^\epsilon_k] = (\epsilon - \xi)\delta_{ik} b^\eta_j + (\epsilon - \eta)\delta_{jk} b^\xi_i.$$  

(2.6)

This shows that the conjecture holds in the nondeformed case:

**PROPOSITION 3.** The Weyl superalgebra $W(n)$ is a factor-algebra of $U[\text{osp}(1/2n)]$.

Consequently any representation of $W(n)$ and in particular its Fock representation is a representation of $U[\text{osp}(1/2n)]$. Hence one has

**PROPOSITION 4.** The linear map $\rho$ defined by the replacement $B^\pm_i \rightarrow b^+_i, i = 1, \ldots, n$ is a morphism of $U[\text{osp}(1/2n)]$ onto $W(n)$.

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The above proposition is in the origin of the so called ladder (or oscillator) representations. From (2.4) and proposition 4 one obtains

\[ sp(2n) = \text{lin.env.}\{\{b_i^-, b_j^+\} \mid \xi, \eta, = \pm, i, j = 1, 2, \ldots, n\}. \] (2.7)

Similarly

\[ gl(n) = \text{lin.env.}\{\{b_i^-, b_j^+\} \mid i, j = 1, 2, \ldots, n\}. \] (2.8)

III. The Superalgebra \(U_q[osp(1/4)]\)

For a quantization of \(U_q[osp(1/4)]\) in terms of its Chevalley generators see Refs.17, 18. Here we proceed in a different way, which will make it easier to prove the conjecture for \(n = 2\) and is of independent interest.

Let \(\text{Free}_q(B^\pm_1, B^\pm_2, K^\pm_1, K^\pm_2)\) be the associative algebra with unity, free generators \(B^\pm_1, B^\pm_2, K^\pm_1, K^\pm_2\) and the relations \((\xi, \eta = \pm \text{ or } \pm 1)\)

\[
K_i K_j^{-1} = K_j^{-1} K_i = 1, \quad K_1 K_2 = K_2 K_1, \quad i = 1, 2, \quad (3.1)
\]

\[
K_i^\xi B^\eta_j = q^{\eta} B^\eta_j K^\xi_i, \quad K_i^\xi B^\eta_j = B_j^\eta K_i^\xi, \quad i \neq j = 1, 2, \quad (3.2)
\]

\[
\{B^\xi_i, B^\eta_j\} = \frac{q K^\xi_j - q^{-1} K^\eta_j}{q - q^{-1}}, \quad i = 1, 2, \quad (3.3)
\]

To turn \(\text{Free}_q(B^\pm_1, B^\pm_2, K^\pm_1, K^\pm_2)\) into an associative superalgebra we set

\[
\text{deg}(B^\pm_1) = 1 \in \mathbb{Z}_2, \quad \text{deg}(K^\pm_1) = 0 \in \mathbb{Z}_2, \quad i = 1, 2. \quad (3.6)
\]

PROPOSITION 5. \(\text{Free}_q(B^\pm_1, B^\pm_2, K^\pm_1, K^\pm_2)\) is a Hopf superalgebra with a comultiplication \(\Delta\), a counit \(\epsilon\) and an antipode \(S\) as follows:

\[
\Delta(B^\pm_1) = q^{-1/2} B^\pm_1 \otimes K_1 K_2^{-2} + q^{3/2} K_1^{-1} K_2^{-2} \otimes B^\pm_1 + (q - q^{-1}) q^{-1/2} B^\pm_2 K_1^{-1} K_2^{-1} \otimes \{B^\pm_1, B^\pm_2\} \otimes K_2^{-1},
\]

\[
\Delta(B^\pm_2) = q^{1/2} B^\pm_2 \otimes K_1 K_2^{-1} + q^{1/2} K_1^{-1} K_2^{-2} \otimes B^\pm_2 + (q^{-1} - q) q^{1/2} \{B^\pm_1, B^\pm_2\} \otimes K_2 \otimes B^\pm_2 K_1 K_2,
\]

\[
\Delta(B^\pm_3) = q^{1/2} B^\pm_3 \otimes K_2 + q^{-1/2} K_2^{-1} \otimes B^\pm_3, \quad \xi = \pm,
\]
\[ \Delta(K_i^\xi) = q^{\xi/2}K_i^\xi \otimes K_i^\xi, \ i = 1, 2, \ \xi = \pm 1, \ (3.7) \]

\[ \epsilon(B_i^\xi) = 0, \ \epsilon(K_i^\xi) = q^{-\xi/2}, \ i = 1, 2, \ \xi = \pm 1, \]

\[ S(K_i^\xi) = q^{-\xi}K_i^{-\xi}, \ i = 1, 2, \ \xi = \pm 1, \]

\[ S(B_i^+) = (q^7 - q^5)(B_i^0, B_i^+)_{q \rightarrow B_i^+}K_i^2 - q^7B_i^+K_i^2, \]

\[ S(B_i^-) = (q^7 - q^{-5})B_i^{-2}(B_i^1, B_i^+)_{q \rightarrow B_i^+}K_i^{-2} - q^{-7}B_i^-K_i^{-4}, \]

\[ S(B_i^0) = -q^4B_i^0, \ \xi = \pm 1. \]

**PROPOSITION 6.** The associative superalgebra \( \text{Free}_q(B_i^+, B_i^-, K_i^{\pm 1}) \) is a deformation of \( U[osp(1/4)] \).

For a proof set \( K_i = q^{H_i - 1/2} \). Then as \( q \rightarrow 1 \) the relations (3.1)–(3.5) reduce to the para-Bose relations (2.1). In particular \( H_i = \{B_i^-B_i^+ \} \).

In terms of the generators \( B_i^+, B_i^-, K_i^{\pm 1}, K_i^{\pm 1} \) one can construct a \( q \)-analog of the Cartan-Weyl basis.

For all values of \( q \) it is given with the following 14 generators:

\[ K_i, B_i^\pm, (B_i^\pm)^2, \{B_i^+, B_i^-\}_{q \rightarrow -1}, i = 1, 2, \ \xi, \eta = \pm or \pm 1. \ (3.8) \]

**PROPOSITION 7.** All ordered monomials \((n_i, m_i \in \mathbb{Z}_+, \ p_i \in \mathbb{Z})\)

\[ (B_i^+)^{n_1}(B_i^0)^{m_1}(B_i^-)^{n_2}(B_i^0)^{m_2} \]

constitute a basis in \( U_q[osp(1/4)] \).

This is a \( q \)-deformed version of the Poincare-Birkhoff-Witt theorem. The proof follows from eqs. (3.1)–(3.5) and the relations following from them, namely

\[ \{[B_i^-, B_i^+]_{q \rightarrow 2}, [B_i^+, B_i^+]_{q \rightarrow 2}\}_{q \rightarrow 2} = (1 + q^{-2})(B_i^+)^2K_i^2, \]

\[ \{[B_i^-, B_i^+]_{q \rightarrow 2}, [B_i^-, B_i^-]_{q \rightarrow 2}\}_{q \rightarrow 2} = -(1 + q^{-2})(1 + q^2)(B_i^-)^2K_i^2, \]

\[ \{[B_i^-, B_i^+]_{q \rightarrow 2}, [B_i^-, B_i^-]_{q \rightarrow 2}\}_{q \rightarrow 2} = \frac{q + q^{-1}}{q - q^{-1}}(K_i^2K_i^2 - K_i^2K_i^2), \]

\[ \{[B_i^-, B_i^-]_{q \rightarrow 2}, [B_i^-, B_i^+]_{q \rightarrow 2}\}_{q \rightarrow 2} = (1 + q^{-2})(1 + q^2)(B_i^-)^2K_i^{-2}. \ (3.10) \]
Observe the very interesting situation that appears at \( q = \pm i \) a case which is not considered in terms of the Chevalley basis [17, 18]. For these values of \( q \) the right hand sides of all equations (3.10) vanish. This particular case deserves further investigations.

**IV. Proof of the Conjecture for \( n=2 \)**

**PROPOSITION 8.** The Weyl superalgebra \( W_q(2) \) generated by the deformed Bose operators \( b_1^\pm, k_1^\pm \), a factor-algebra of \( U_q[osp(l/4)] \).

The proof is an immediate consequence of the observation that the deformed Bose operators \( b_1^\pm, k_1^\pm \) satisfy the defining relations for \( U_q[osp(l/4)] \). More precisely, eqs. (3-1)–(3.5) remain valid after the replacement \( b_1^+ \rightarrow b_1^+, K_1^\pm \rightarrow k_1^\pm \).

Consider the representation of \( W_q(2) \) in the Fock space \( Fock_q(2) \) [8-10]. Then using proposition 8 and eq.(1.4) we can write a representation of \( U_q[osp(l/4)] \) in \( Fock_q(2) \otimes Fock_q(2) \):

\[
\Delta^{(2)}(b_1^+) = b_1^+ \otimes q^{N_1-2N_2-1/2} + q^{-N_1-2N_2-3/2} \otimes b_1^+ + (q^{1/2} - q^{-7/2})b_2^+ q^{-N_1-N_2} \otimes b_2^- q^{-N_2},
\]

\[
\Delta^{(2)}(b_1^-) = b_1^- \otimes q^{N_1+2N_2+3/2} + q^{-N_1+2N_2+1/2} \otimes b_1^- + (q^{-1/2} - q^{-7/2})b_2^- q^{N_1+N_2} \otimes b_2^+ q^{N_2},
\]

\[
\Delta^{(2)}(b_2^\xi) = b_2^\xi \otimes q^{N_2+1/2} + q^{-N_2-1/2} \otimes b_2^\xi, \quad \xi = \pm,
\]

\[
\Delta^{(2)}(K_i^\xi) = q^{(N_i+1/2)} \otimes q^{\xi N_i}, \quad i = 1, 2, \quad \xi = \pm 1.
\]

The operator \( \Delta^{(2)} \) defines a morphism of \( U_q[osp(l/4)] \) into \( W_q(2) \otimes W_q(2) \). In all essential points (as far as the representations of \( U_q[osp(l/4)] \) or of any of its subalgebras are concerned) \( \Delta^{(2)} \) is a good substitute for a comultiplication in the Weyl algebra \( W_q(2) \). The operator \( \Delta^{(2)} \) however does not satisfy the requirements for a comultiplication in \( W_q(2) \). In fact it is impossible to define a comultiplication in the Weyl algebra even in the nondeformed case.

Equations (4.1) indicate how to construct new representations of the superalgebra \( U_q[osp(l/4)] \) using its oscillator representation, i.e., the Fock space representation of its factor-algebra \( W_q(2) \). If the conjecture turns true, then the same approach can be applied for any \( U_q[osp(1/2n)] \). Certainly, instead of the oscillator representation one can use any other representation. The point is however that other representations are at present unknown (contrary to the class of quantum superalgebras \( U_q[sl(n/1)] \) [19]). This statement holds even for the ordinary, the nondeformed case and even for \( osp(1/4) \). All finite-dimensional representations of the orthosymplectic LS's \( osp(2m+1/2n) \) are completely classified [4]. However (apart
from $osp(1/2)$ [20] and $osp(3/2)$ [21]) explicit expressions for the transformations of the finite-dimensional irreducible $osp(2m + 1/2n)$ modules are not available.

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