No massless boson
in chiral symmetry breaking in NJL and Thirring models

Makoto HIRAMOTO and Takehisa FUJITA
Department of Physics, Faculty of Science and Technology
Nihon University, Tokyo, Japan

ABSTRACT

We show that the chiral symmetry breaking occurs in the vacuum of the massless
Nambu-Jona-Lasinio (NJL) and Thirring models without a Goldstone boson. The basic
reason of non-existence of the massless boson is due to the fact that the new vacuum
after the symmetry breaking acquires nonzero fermion mass which inevitably leads to
massive bosons. The new vacuum has a finite condensate of \( \langle \bar{\psi} \psi \rangle \) with the chiral current
conservation. Thus, it contradicts the Goldstone theorem, and we show that the proof of
the Goldstone theorem cannot be justified any more for fermion field theory models with
regularizations.

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1. Introduction

Symmetries play a most important role for the understanding of the basic behavior in quantum field theory. The breaking of symmetries is also important and interesting since the vacuum can violate the symmetry which is possessed in the Lagrangian.

In field theory, there is an interesting theorem for the spontaneous symmetry breaking. That is, the Goldstone theorem [1, 2], and it states that there appears a massless boson if the symmetry is spontaneously broken.

For the spontaneous symmetry breaking in fermion field theory, Nambu and Jona-Lasinio first pointed out that the current current interaction model (NJL model) presents a good example of the spontaneous symmetry breaking [3]. Indeed, they showed in their classic paper that the chiral symmetry is broken if they start from the massless fermion Lagrangian which possesses a chiral symmetry. Then, they construct the new vacuum with the Bogoliubov transformation, and proved that the new vacuum breaks the chiral symmetry, and besides they found that the originally massless fermion acquires an induced mass. To be more important, all of the physical observables like the boson mass is measured by the new fermion mass. Up to this point, it was indeed quite an interesting and convincing scenario that happened to the chiral symmetry breaking phenomena.

However, Nambu and Jona-Lasinio claimed that there appears a massless boson after the chiral symmetry breaking. This massless boson, if at all exists, must be constructed by the fermion and antifermion by its dynamics. But all of the calculations which have been carried out up to now show that the boson in the NJL model with the massive fermion is always massive, and the boson mass is proportional to the fermion mass. Therefore, a massless boson can be obtained only when one sets the fermion mass to zero [4]. But this is not allowed since the induced fermion mass can never become zero. A massless boson might be obtained when the interaction strength is at the strongest limit since all the fermion mass could be eaten up by the interaction. However, Nambu and Jona-Lasinio claim that there exists a massless boson regardless the strength of the coupling constant even if the strength of the coupling constant is quite small. But obviously, this is physically not acceptable.

In this paper, we show by explicit calculations that the NJL model has no massless boson, and the boson mass is always finite. But the symmetry breaking phenomenon is exactly the same as the one presented by Nambu and Jona-Lasinio. The new vacuum breaks the chiral symmetry since the energy of the new vacuum is lower
than the trivial one. In this case, the originally massless fermion can indeed acquire an induced mass which is measured by the cutoff momentum $\Lambda$. With this finite fermion mass, we calculate the boson mass, and obtain the mass which is indeed finite, and its magnitude certainly depends on the strength of the coupling constant. It is interesting to observe that there is no bound state of the bosonic state if the strength of the coupling constant is weaker than the critical value.

Here, we also carry out the calculation of the chiral symmetry breaking in the massless Thirring model which is a two dimensional field theory [5]. In fact, we show that the new vacuum breaks the chiral symmetry, and the massless fermion acquires an induced mass. Therefore, the massless Thirring model becomes just the massive Thirring model with the induced fermion mass, and it is clear that there exists always a massive boson even for the weak coupling region [6]. Therefore, even in two dimensional field theory models with the regularization, the continuous symmetry is broken, but there appears no massless boson. In this respect, the situation in the Thirring model is just the same as the NJL model. Since there is no massless boson in the symmetry breaking, this is not inconsistent with Coleman’s theorem [7]. But we should stress that the symmetry is, in fact, broken in the two dimensional field theory models. Further, a recent work [8] shows that a chiral symmetry in the two dimensional QCD with massless fermions is broken without the anomaly, but there appears no massless boson, and instead, there exists a massive boson.

This means that there must be some problems in the Goldstone theorem for the fermion field theory models with the regularization. Here, we show that the procedure of proving Goldstone theorem for the fermion field theory models cannot be justified any more when the fermion current is regularized with the point splitting. The defect of the Goldstone theorem for the fermion field theory with the regularization is essentially based on the fact that the boson is a complex object, and it must be constructed by the fermion and antifermion. But the fermion current must be regularized, and, in this case, the boson cannot be treated as a simple elementary particle. Therefore, the method of proving the Goldstone theorem which is successful for the boson field theory cannot be applied any more to the fermion field theory model with the regularization.

This paper is organized as follows. In the next section, we first treat the Bogoliubov transformation in the Nambu-Jona-Lasinio(NJL) model, and obtain a new vacuum. The mass of the boson is evaluated in the regularized NJL model. In section 3, we discuss the chiral symmetry breaking in the massless Thirring model and calculate
the boson mass with the Bogoliubov transformed vacuum. In section 4, we explain
the breaking down of the Goldstone theorem for the fermion field theory models with
the current regularization. Finally, section 5 summarizes what we have clarified in
this paper.

2. Nambu-Jona-Lasinio model

Here, we discuss the four dimensional current-current interaction model by Nambu
and Jona-Lasinio [3]. This paper is the original one that initiated the chiral symme-
try breaking phenomena. Here, we treat the NJL model just in the same manner as
the one given by Nambu and Jona-Lasinio as far as the symmetry breaking mech-
anism is concerned. However, the boson mass determination is different, and we
carry out the calculation in terms of the Fock space expansion. Also, we carry out
the RPA calculation, but it turns out that the RPA calculation gives the boson
spectrum which is similar to that calculated by the Fock space expansion. But
the boson mass is slightly lower than the result of the Fock space expansion as the
function of the coupling constant.

Now, we carry out the calculation which is based on the Bogoliubov transformation,
and show that the chiral symmetry is indeed broken. However, we also show that
there appears no massless boson in this regularized NJL model.

Here, we first quantize the fermion field in a box $L^3$

$$\psi(r) = \frac{1}{\sqrt{L^3}} \sum_{n,s} \left[ a(n, s) u(n, s) e^{i \frac{2\pi}{L} n \cdot r} + b^\dagger(n, s) v(n, s) e^{-i \frac{2\pi}{L} n \cdot r} \right] \quad (2.1)$$

where $s$ denotes the spin index, and $s = \pm 1$. Also, the spinors are defined as

$$u(n, s) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma \cdot \hat{n} \chi_{(s)} \\ \chi_{(s)} \end{pmatrix} ;$$

$$v(n, s) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_{(s)} \\ \sigma \cdot \hat{n} \chi_{(s)} \end{pmatrix} .$$

Now, we define new fermion operators by the Bogoliubov transformation,

$$c(n, s) = e^{-A} a(n, s) e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a(n, s) - s \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b^\dagger(-n, s) \quad (2.2a)$$

$$d^\dagger(-n, s) = e^{-A} b^\dagger(-n, s) e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b^\dagger(-n, s) + s \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a(n, s) \quad (2.2b)$$
where the generator of the Bogoliubov transformation is given by

$$\mathcal{A} = -\sum_{n,s} \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) \left( a^\dagger(n, s)b^\dagger(-n, s) - b(-n, s)a(n, s) \right). \quad (2.3)$$

$\theta_n$ denotes the Bogoliubov angle which can be determined by the condition that the vacuum energy is minimized. In this case, the new vacuum state is obtained as

$$|\Omega\rangle = e^{-\mathcal{A}}|0\rangle. \quad (2.4)$$

In what follows, we treat the NJL Hamiltonian with the Bogoliubov transformed vacuum state. In order to clearly see some important difference between the massive fermion and massless fermion cases, we treat the two cases separately.

(a) Massive fermion case

The Lagrangian density for the NJL model with the massive fermion can be written as

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m_0\bar{\psi}\psi + G \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right]. \quad (2.5)$$

Now, we can obtain the new Hamiltonian under the Bogoliubov transformation,

$$H = \sum_{n,s} \left\{ |p_n| \sin \theta_n + \left( m_0 + \frac{2G}{L^3} B \right) \cos \theta_n \right\} \left( c^\dagger(n, s)c(n, s) + d^\dagger(-n, s)d(-n, s) \right)$$

$$+ \sum_{n,s} \left\{ -|p_n| s \cos \theta_n + \left( m_0 + \frac{2G}{L^3} B \right) s \sin \theta_n \right\} \left( c^\dagger(n, s)d^\dagger(-n, s) + d(-n, s)c(n, s) \right)$$

$$+ H'_{int} \quad (2.6)$$

where $H'_{int}$ is the interaction term. Since the $H'_{int}$ is quite complicated, and besides its explicit expression is not needed in this context, we will not write it here. $B$ is defined as

$$B = \sum_{n,s} \cos \theta_n.$$

Now, we can define the renormalized fermion mass $m$

$$m = m_0 + \frac{2G}{L^3} B. \quad (2.7)$$

The Bogoliubov angle $\theta_n$ can be determined by imposing the condition that the $cd$ term in eq.(2.6) must vanish. Therefore, we obtain

$$\tan \theta_n = \frac{|p_n|}{m}. \quad (2.8)$$
This Bogoliubov angle $\theta_n$ does not change when the mass varies from $m_0$ to $m$. In this case, the vacuum is just the same as the trivial vacuum of the massive case, except that the fermion mass is replaced by the renormalized mass $m$. The rest of the theory becomes identical to the massive NJL model with the same interaction Hamiltonian $H'_{\text{int}}$. Therefore, there is no symmetry breaking, and this vacuum has no condensate.

(b) Massless fermion case

Here, we present the same procedure for the massless fermion case in order to understand why the fermion has to become massive.

We start from the Lagrangian density with no mass term in eq.(2.5). Under the Bogoliubov transformation, we obtain the new Hamiltonian

$$H = \sum_{n,s} \left\{ |p_n| \sin \theta_n + \frac{2G}{L^3} B \cos \theta_n \right\} \left( c_n^\dagger(n,s) c_n(n,s) + d_n^\dagger(-n,s) d(-n,s) \right)$$

$$+ \sum_{n,s} \left\{ -|p_n| s \cos \theta_n + \frac{2G}{L^3} B s \sin \theta_n \right\} \left( c_n^\dagger(n,s) d(-n,s) + d(-n,s) c(n,s) \right) + H'_{\text{int}}$$

(2.9)

where $H'_{\text{int}}$ is just the same as the one given in eq.(2.6). From this Hamiltonian, we get to know that the mass term is generated in the same way as the massive case. But we cannot make any renormalization since there is no mass term. Further, the new term is the only mass scale in this Hamiltonian since the coupling constant cannot serve as the mass scale. In fact, it is even worse than the dimensionless coupling constant case, since the coupling constant in the NJL model is proportional to the inverse square of the mass dimension. Thus, we define the new fermion mass $M_N$ by

$$M_N = \frac{2G}{L^3} B.$$  

(2.10)

The Bogoliubov angle $\theta_n$ can be determined from the following equation

$$\tan \theta_n = \frac{|p_n|}{M_N}.$$  

(2.11)

In this case, the vacuum changes drastically since the original vacuum is trivial. Further, the constraints of eqs.(2.10) and (2.11) give rise to the equation that determines the relation between the induced fermion mass $M_N$ and the cutoff momentum $\Lambda$.

$$M_N = \frac{4G}{(2\pi)^3} \int_0^\Lambda d^3p \frac{M_N}{\sqrt{M_N^2 + p^2}}.$$  

(2.12)
This equation has a nontrivial solution for $M_N$, and the vacuum energy becomes lower than the trivial vacuum ($M_N = 0$). Therefore, $M_N$ can be expressed in terms of $\Lambda$ as

$$M_N = \gamma \Lambda$$

where $\gamma$ is a simple numerical constant.

It should be noted that the treatment up to now is exactly the same as the one given by Nambu and Jona-Lasinio [3]. Further, we stress that the induced fermion mass $M_N$ can never be set to zero, and it is always finite.

(c) Boson mass

The boson state $|B\rangle$ can be expressed as

$$|B\rangle = \sum_{n,s} f_n c^\dagger(n, s)d^\dagger(-n, s)|\Omega\rangle,$$

(2.13)

where $f_n$ is a wave function in momentum space, and $|\Omega\rangle$ denotes the Bogoliubov vacuum state. The equation for the boson mass $\mathcal{M}$ for the NJL model is written in terms of the Fock space expansion at the large $L$ limit

$$\mathcal{M}f(p) = 2E_p f(p) - \frac{2G}{(2\pi)^3} \int_\Lambda d^3 q f(q) \left(1 + \frac{M^2}{E_p E_q} \right) + \frac{p \cdot q}{E_p E_q}$$

(2.14)

where $M$ should be taken to be $M = m$ for the massive case, and $M = M_N$ for the massless case. It is important to note that the fermion mass $M$ after the Bogoliubov transformation, therefore, cannot become zero.

Here, again, we note that the RPA calculation gives the similar boson spectrum to the Fock space expansion. But we do not know whether the RPA calculation is better than the Fock space expansion or not, since the derivation of the RPA equation in field theory is not based on the fundamental principle. In principle, the RPA calculation may take into account the effect of the deformation of the vacuum in the presence of the particle and antiparticle. However, this is extremely difficult to do it properly, and indeed the RPA eigenvalue equation is not hermite, and thus it is not clear whether the effect is taken into account in a better way or worse. The examination and the validity of the RPA equation will be given elsewhere.

The solution of eq.(2.14) can be easily obtained, and the boson mass spectrum for the NJL model is shown in Fig. 1. Note that the boson mass is measured in units of the cutoff momentum $\Lambda$. As can be seen from the figure, there is a massive boson for some regions of the values of the coupling constant.
Here, as we will see later, the NJL and the Thirring models are quite similar to each other. This is mainly because the current-current interaction is essentially a delta function potential in coordinate space. Indeed, as is well known, the delta function potential in one dimension can always bind the fermion and antifermion while the delta function potential in three dimension cannot normally bind them. Due to the finite cut off momentum, the delta function potential in three dimensions can make a weak bound state, depending on the strength of the coupling constant. This result of the delta function potential in quantum mechanics is almost the same as what is just shown in Fig. 1.

It should be noted that there is no serious difficulty of proving the non-existence of the massless boson. However, if it were to prove the existence of the massless boson, it would have been extremely difficult to do it. For the massless boson, there should be a continuum spectrum, and this continuum spectrum of the massless boson should be differentiated from the continuum spectrum arising from the many body nature of the system. This differentiation must have been an extremely difficult task. In fact, even if one finds a continuum spectrum which has, for example, the dispersion of $E = c_0 p^2$ as often discussed in solid state physics, one sees that the spectrum has nothing to do with the Goldstone boson.

3. Massless Thirring model

Now, we discuss the Thirring model which is a two dimensional field theory with the current-current interaction [5]. For the massive Thirring model, it is discussed in detail in [6], and therefore, we treat only the massless case in this paper.

Since the massless Thirring model does not contain any scale parameter, it is necessary to introduce a scale or a cutoff momentum if one wants to discuss physical quantities. In this respect, for the massless Thirring model, any calculations with the regularization is physically meaningful. For example, the equivalence between the massless Thirring model and the massless boson system is quite well known and interesting, but this is mathematically important. However, physically, this equivalence is somewhat more complicated than expected since there is no scale introduced in the systems, and thus one cannot measure physical observables.

The massless Thirring model is described by the following Lagrangian density

$$\mathcal{L} = i \bar{\psi} \gamma_\mu \partial^\mu \psi - \frac{1}{2} g j^\mu j_\mu$$

(3.1)
where the fermion current $j_\mu$ is given as

$$j_\mu = :\bar{\psi} \gamma_\mu \psi : \quad (3.2)$$

It is clear that this Lagrangian density has a chiral symmetry, and therefore, defining the chiral current by

$$j^5_\mu = :\bar{\psi} \gamma_5 \gamma_\mu \psi : \quad (3.3)$$

we see that the chiral current is conserved in the classical level

$$\partial^\mu j^5_\mu = 0. \quad (3.4)$$

In the recent paper, Faber and Ivanov [9] show that the true vacuum state has a chiral symmetry broken phase. They consider that the chiral symmetry breaking is spontaneous, and therefore they discuss the reason why the symmetry can be broken in the two dimensional field theory. Their discussions are concerned with the problem of Coleman’s theorem.

Here, we present the calculation which is based on the Bogoliubov transformation, and show that the chiral symmetry is indeed broken. However, we also show that there appears no massless boson in this regularized Thirring model.

Here, we first quantize the fermion field in a box $L$

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} e^{i \frac{2\pi nx}{L}}. \quad (3.5)$$

In this case, the Hamiltonian of the massless Thirring model can be written as

$$H = \sum_n \left[ p_n(a_n^\dagger a_n - b_n^\dagger b_n) + \frac{2g}{L} \tilde{j}_{1,n} j_{2,-n} \right]. \quad (3.6)$$

where the fermion currents in the momentum representation $\tilde{j}_{1,p_n}$ and $\tilde{j}_{2,p_n}$ are given by

$$\tilde{j}_{1,p_n} = \sum_l a_l^\dagger a_{l+n} \quad (3.7a)$$

$$\tilde{j}_{2,p_n} = \sum_l b_l^\dagger b_{l+n}. \quad (3.7b)$$

Now, we define new fermion operators by the Bogoliubov transformation,

$$c_n = e^{-A} a_n e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a_n - \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b_n \quad (3.8a)$$

$$d_{-n}^\dagger = e^{-A} b_n e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b_n + \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a_n \quad (3.8b)$$
where the generator of the Bogoliubov transformation is given by
\[ A = -\sum_n \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) (a_n^\dagger b_n - b_n^\dagger a_n). \] (3.9)

\( \theta_n \) denotes the Bogoliubov angle which can be determined by the condition that the vacuum energy is minimized. In this case, the new vacuum state is obtained as
\[ | \Omega \rangle = e^{-A} | 0 \rangle. \] (3.10)

Now, we can obtain the new Hamiltonian under the Bogoliubov transformation,
\[
H = \sum_n \left[ \left\{ p_n \sin \theta_n + \frac{g}{L} \mathcal{B} \cos \theta_n \right\} (c_n^\dagger c_n + d_n^\dagger d_n) \right] \\
+ \sum_n \left[ -p_n \cos \theta_n + \frac{g}{L} \mathcal{B} \sin \theta_n \right] (c_n^\dagger d_n^\dagger - d_n^\dagger c_n) + H'
\] (3.11)

where \( H' \) denotes the interaction terms of the Bogoliubov transformed state and is given in detail in [6]. Here, \( \mathcal{B} \) is defined as \( \mathcal{B} = \sum \cos \theta_n \), and the term \( \frac{g}{L} \mathcal{B} \) corresponds to an induced mass. Therefore, we define the induced mass \( M \) by
\[ M = \frac{g}{L} \mathcal{B}. \] (3.12)

The Bogoliubov angle \( \theta_n \) can be determined by imposing the condition that the vacuum energy must be minimized. Therefore, we obtain
\[ \tan \theta_n = \frac{p_n}{M}. \] (3.13)

In this case, we can express the self-consistency condition for \( M \), and obtain
\[ M = \frac{g}{\pi} M \ln \left( \frac{\Lambda}{M} + \sqrt{1 + \left( \frac{\Lambda}{M} \right)^2} \right) \] (3.14)

where \( \Lambda \) denotes the cutoff. Since the massless Thirring model has no scale, we should measure all of the observables in terms of \( \Lambda \). Therefore, we can express the induced mass \( M \) in terms of \( \Lambda \),
\[ M = \frac{\Lambda}{\sinh \left( \frac{\pi}{g} \right)}. \] (3.15)

Further, the vacuum energy \( E_{\text{vac}} \) as measured from the trivial vacuum is given
\[ E_{\text{vac}} = -\frac{L}{2\pi} \frac{\Lambda^2}{\sinh \left( \frac{\pi}{g} \right)} e^{-\frac{\pi}{g}}. \] (3.16)
From this value of the vacuum energy, we get to know that the new vacuum energy is indeed lower than the trivial one. Therefore, the chiral symmetry is broken in the new vacuum state since the fermion becomes massive.

Note that the present expression of the mass \( M \) is somewhat different from the one given in Faber and Ivanov \[9\]. This is related to the fact that we take the cutoff momentum in the normal momentum while they used the cutoff in the Lorentz invariant fashion.

In the same manner as \[6\], we carry out the calculations of the spectrum of the bosons in the Fock space expansion.

The boson state \(|B\rangle\) can be expressed as

\[
|B\rangle = \sum_n f_n c_n^d d_{-n}^\dagger |\Omega\rangle, \tag{3.17}
\]

where \( f_n \) is a wave function in momentum space, and \(|\Omega\rangle\) denotes the Bogoliubov vacuum state. The energy eigenvalue of the Hamiltonian for the large \( L \) limit can be written as

\[
\mathcal{M} f(p) = 2E_p f(p) - \frac{g}{2\pi} \int dq f(q) \left( 1 + \frac{M^2}{E_p E_q} + \frac{pq}{E_p E_q} \right) \tag{3.18}
\]

where \( \mathcal{M} \) denotes the boson mass. \( E_p \) is given as

\[
E_p = \sqrt{M^2 + p^2}. \tag{3.19}
\]

Eq.(3.18) can be solved exactly as shown in \[6\]. First, we define \( A \) and \( B \) by

\[
A = \int_{-\Lambda}^{\Lambda} dp f(p) \tag{3.20a}
\]
\[
B = \int_{-\Lambda}^{\Lambda} dp \frac{f(p)}{E_p}. \tag{3.20b}
\]

Using \( A \) and \( B \), we can solve Eq. (3.18) for \( f(p) \) and obtain

\[
f(p) = \frac{g/2\pi}{2E_p - \mathcal{M}} \left( A + \frac{m^2}{E_p} B \right). \tag{3.21}
\]

Putting this \( f(p) \) back into Eqs. (3.20), we obtain the matrix equations

\[
A = \frac{g}{2\pi} \int_0^\Lambda \frac{2dp}{2E_p - \mathcal{M}} \left( A + \frac{m^2}{E_p} B \right). \tag{3.22a}
\]
\[
B = \frac{g}{2\pi} \int_0^\Lambda \frac{2dp}{(2E_p - \mathcal{M})E_p} \left( A + \frac{m^2}{E_p} B \right). \tag{3.22b}
\]
Since the model is already regularized, we can easily calculate the boson spectrum which is given in Fig. 2 as the function of the coupling constant $g/\pi$. As can be seen from Fig. 2, there is no massless boson in this spectrum even though the boson mass for the very small coupling constant $g$ is exponentially small. Here, we also carry out the RPA calculation for the massive Thirring model. However, the spectrum predicted by the RPA calculation somewhat deviates from the other calculated results [11, 12, 13, 14] in the massive Thirring model. Since the derivation of the RPA equation in field theory is not based on the fundamental principle, we do not know to what extent the calculation of the RPA is reliable and the validity of the RPA equation will be examined elsewhere.

Now, we discuss the chiral current conservation. In order to examine it, we evaluate the following equation,

$$i\dot{Q}_5 = [H, Q_5]$$

(3.23)

where $Q_5$ is defined as

$$Q_5(t) = \int j_0^5(x, t)dx = \int \bar{\psi}\gamma_0\gamma_5\psi dx.$$  

(3.24)

Clearly, we can show that the right hand side of eq.(3.23) vanishes for the massless Thirring Hamiltonian of eq.(3.4). However, since the Bogoliubov transformation is unitary, $[H, Q_5]$ is invariant and thus it remains to be zero. Therefore, the chiral current is still conserved after the Bogoliubov transformation.

Here, we also calculate the fermion condensate $C$ for this vacuum state [10], and obtain

$$|C| = \langle \Omega | \frac{1}{L} \int \bar{\psi}\psi dx | \Omega \rangle = \frac{M}{g}$$

(3.25)

which is a finite value. Therefore, the massless Thirring model has a chiral symmetry broken vacuum, but, the chiral current is conserved. Therefore, this situation contradicts the Goldstone theorem since there appears no massless boson in this symmetry breaking. This point will be discussed in detail in the next section.

4. Goldstone Theorem

Here, we discuss the Goldstone theorem in connection with the massless NJL and the Thirring models. In the formal proof of the Goldstone theorem [1, 2], one assumes the existence of the vacuum expectation value of the following commutation relation,

$$\langle \Omega | [Q_5(t), \phi(0)] | \Omega \rangle \neq 0$$

(4.1)
where the boson field \( \phi(x) \) must be constructed by the fermion fields. Now, inserting intermediate boson states, one obtains

\[
\sum_n \delta(p_n) \left[ e^{iE_n t} \langle \Omega | \phi^\dagger_0(0) | n \rangle \langle n | \phi(0) | \Omega \rangle - e^{-iE_n t} \langle \phi(0) | n \rangle \langle n | \phi^\dagger_0(0) | \Omega \rangle \right] \neq 0
\]

(4.2)

The right hand side of eq.(4.2) is non-vanishing, and time-independent. Thus, eq.(4.2) can be satisfied only if there exists an intermediate state with \( E_n = 0 \) for \( p_n = 0 \). This is the proof of the Goldstone theorem, and it is indeed valid for boson field theory. One can also obtain the same result from rewriting the Lagrangian density at the new vacuum point for boson field theory models. This is reasonable since the Goldstone boson is understood as a kinematical effect.

On the other hand, the fermion field theory models are quite different, and one cannot obtain the Goldstone boson even if one rewrites the Lagrangian density at the new vacuum point. This is due to the fact that the Goldstone boson must be constructed by the fermion and anti-fermion fields, and therefore it should involve the dynamics. But it has been believed that the proof based on eqs.(4.1) and (4.2) holds for the fermion field theory models as well. However, one must be careful for the fermion field theory models with the regularization. In the above proof, the use of the translational property of a boson plays an essential role and it is assumed for the boson field \( \phi(x) \) and the chiral current \( j_5^\mu(x) \) even if they are constructed by fermions. However, the boson field \( \phi(x) \) together with the current \( j_5^\mu(x) \) must be regularized by the point splitting,

\[
j_5^\mu(x) = \bar{\psi}(x)\gamma_5\gamma^\mu\psi(x + \epsilon)
\]

(4.3a)

\[
\phi(x) = \bar{\psi}(x)\gamma_5\psi(x + \epsilon)
\]

(4.3b)

where we have to keep the \( \epsilon \) finite. In the NJL as well as the Thirring models, the \( \epsilon \) should be related to the cutoff momentum \( \Lambda \) in some way or the other. In this case, one cannot make use of the translation property of \( \phi(x) \)

\[
\phi(x) = e^{ipx} \phi(0)e^{-ipx}.
\]

Instead, \( \phi(x) \) is only written as

\[
\phi(x) = e^{ipx} \bar{\psi}(0)\gamma_5e^{ipx}\psi(0)e^{-ip(x+\epsilon)},
\]

(4.4)

and one cannot obtain the same equation as eq.(4.2) for fermion fields with the regularization. Therefore one cannot claim the existence of the massless boson any
more. In the massless NJL as well as the Thirring models, one needs to regularize
the fermion field, and therefore, one cannot claim that the Goldstone theorem should
hold true. In fact, as we saw in the previous sections, there appears no massless
boson, and this is just what is found here.

Further, we should note that no fermion field theory model except the NJL model
predicts a Goldstone boson. It is often discussed that pion may well be a Goldstone-
like boson. But for this, one should be careful. In four dimensional QCD, the
coupling constant has no dimension, and therefore, all the physical mass must be
measured by the quark mass if the quarks are massive. In this case, there is no chiral
symmetry, and there is no need to discuss the Goldstone boson. It is believed that
the pion mass is too light compared with other mesons. However, it is quite large
if one measures it in terms of the quark mass which should be around 10 MeV. In
this respect, the pion mass should be considered as an object which has nothing to
do with the chiral symmetry breaking. If the quarks were massless, then the chiral
symmetry should be broken in the new vacuum, and any physical observables like
meson masses should be measured by the cut off momentum \( \Lambda_{QCD} \). But this story
has nothing to do with nature in real QCD.

5. Conclusions

We have presented the chiral symmetry breaking of the massless NJL and the
Thirring models. The new vacuum has a finite chiral condensate. Also, it is shown
that the chiral symmetry breaking here does not accompany a massless boson. This
is not consistent with the Goldstone theorem since the chiral current conservation
still holds in the Bogoliubov transformed vacuum with the regularization.

This inconsistency is resolved in that the Goldstone theorem turns out to be invalid
for the fermion field theory models with the regularization. This is simply due to
the fact that the proof of the Goldstone theorem is essentially based on the use
of the translational property of the boson, which, however, cannot be valid for the
fermion and antifermion bound state due to the point splitting regularization.

To summarize the chiral symmetry breaking in fermion field theory models of the
NJL and the Thirring, we state that the new vacuum indeed violates the chiral
symmetry, and it has a finite condensate value. The fermion acquires the finite
mass, and the rest of the theory becomes just the massive fermion field theory with
the same interactions as the massive fermion case. This completes the symmetry
breaking business. There is no space and no freedom left for the Goldstone boson. The massive fermion field theory models of the NJL and the Thirring give a massive boson. The Thirring model has an exponentially small boson mass for the small coupling constant region while the boson mass of the NJL model strongly depends on the strength of the coupling constant. Indeed, below the critical value of the coupling constant, there is no bosonic bound state. However, the boson mass has little to do with the symmetry breaking phenomena, but it is determined by the coupling constant and the cutoff \( \Lambda \).

What is then the main difference between the boson field theory and the fermion field theory? In boson field theory models, one assumes that the potential term \( U(|\phi|) \) for the boson field has a nontrivial minimum, which has infinite degenerate states. This degeneracy of the "potential vacuum" is resolved when one considers the kinetic energy terms of the boson field. At this point, the symmetry is broken and one obtains the new vacuum state with a massless boson. Here, one notices that, if there were no nontrivial minimum in the potential vacuum, then there should exist no Goldstone boson. Further, if one finds a system whose total Hamiltonian has the vacuum with infinite degenerate states, then there is no way to resolve the degeneracy, and the vacuum should stay as it is. The fermion field theory models which we treat in this paper have no nontrivial minimum of the potential vacuum, and the vacuum is found only when one considers the total Hamiltonian of the system. Therefore, there is no place one finds any massless boson degree of freedom for the fermion field theory models.

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Fig. 1: The boson mass for the NJL model is plotted as the function of $G\Lambda^2$. It is measured by the cutoff $\Lambda$. 
Fig. 2: The boson mass for the massless Thirring model is plotted as the function of $g/\pi$. It is measured by the cutoff $\Lambda$. 
No massless boson
in chiral symmetry breaking in NJL and Thirring models

Makoto HIRAMOTO and Takehisa FUJITA
Department of Physics, Faculty of Science and Technology
Nihon University, Tokyo, Japan

ABSTRACT

We show that the chiral symmetry breaking occurs in the vacuum of the massless Nambu-Jona-Lasinio (NJL) and Thirring models without a Goldstone boson. The basic reason of non-existence of the massless boson is due to the fact that the new vacuum after the symmetry breaking acquires nonzero fermion mass which inevitably leads to massive bosons. The new vacuum has a finite condensate of $\langle \bar{\psi}\psi \rangle$ with the chiral current conservation. Thus, it contradicts the Goldstone theorem, and we show that the proof of the Goldstone theorem cannot be justified any more for fermion field theory models with regularizations.

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1. Introduction

Symmetries play a most important role for the understanding of the basic behavior in quantum field theory. The breaking of symmetries is also important and interesting since the vacuum can violate the symmetry which is possessed in the Lagrangian.

In field theory, there is an interesting theorem for the spontaneous symmetry breaking. That is, the Goldstone theorem [1, 2], and it states that there appears a massless boson if the symmetry is spontaneously broken.

For the spontaneous symmetry breaking in fermion field theory, Nambu and Jona-Lasinio first pointed out that the current current interaction model (NJL model) presents a good example of the spontaneous symmetry breaking [3]. Indeed, they showed in their classic paper that the chiral symmetry is broken if they start from the massless fermion Lagrangian which possesses a chiral symmetry. Then, they construct the new vacuum with the Bogoliubov transformation, and proved that the new vacuum breaks the chiral symmetry, and besides they found that the originally massless fermion acquires an induced mass. To be more important, all of the physical observables like the boson mass is measured by the new fermion mass. Up to this point, it was indeed quite an interesting and convincing scenario that happened to the chiral symmetry breaking phenomena.

However, Nambu and Jona-Lasinio claimed that there appears a massless boson after the chiral symmetry breaking. This massless boson, if at all exists, must be constructed by the fermion and antifermion by its dynamics. But all of the calculations which have been carried out up to now show that the boson in the NJL model with the massive fermion is always massive, and the boson mass is proportional to the fermion mass. Therefore, a massless boson can be obtained only when one sets the fermion mass to zero [4]. But this is not allowed since the induced fermion mass can never become zero. A massless boson might be obtained when the interaction strength is at the strongest limit since all the fermion mass could be eaten up by the interaction. However, Nambu and Jona-Lasinio claim that there exists a massless boson regardless the strength of the coupling constant even if the strength of the coupling constant is quite small. But obviously, this is physically not acceptable.

In this paper, we show by explicit calculations that the NJL model has no massless boson, and the boson mass is always finite. But the symmetry breaking phenomenon is exactly the same as the one presented by Nambu and Jona-Lasinio. The new vacuum breaks the chiral symmetry since the energy of the new vacuum is lower
than the trivial one. In this case, the *originally* massless fermion can indeed acquire an induced mass which is measured by the cutoff momentum $\Lambda$. With this finite fermion mass, we calculate the boson mass, and obtain the mass which is indeed finite, and its magnitude certainly depends on the strength of the coupling constant. It is interesting to observe that there is no bound state of the bosonic state if the strength of the coupling constant is weaker than the critical value.

Here, we also carry out the calculation of the chiral symmetry breaking in the massless Thirring model which is a two dimensional field theory [5]. In fact, we show that the new vacuum breaks the chiral symmetry, and the massless fermion acquires an induced mass. Therefore, the massless Thirring model becomes just the massive Thirring model with the induced fermion mass, and it is clear that there exists always a massive boson even for the weak coupling region [6]. Therefore, even in two dimensional field theory models with the regularization, the continuous symmetry is broken, but there appears no massless boson. In this respect, the situation in the Thirring model is just the same as the NJL model. Since there is no massless boson in the symmetry breaking, this is not inconsistent with Coleman’s theorem [7]. But we should stress that the symmetry is, in fact, broken in the two dimensional field theory models. Further, a recent work [8] shows that a chiral symmetry in the two dimensional QCD with massless fermions is broken without the anomaly, but there appears no massless boson, and instead, there exists a massive boson.

This means that there must be some problems in the Goldstone theorem for the fermion field theory models with the regularization. Here, we show that the procedure of proving Goldstone theorem for the fermion field theory models cannot be justified any more when the fermion current is regularized with the point splitting. The defect of the Goldstone theorem for the fermion field theory with the regularization is essentially based on the fact that the boson is a complex object, and it must be constructed by the fermion and antifermion. But the fermion current must be regularized, and, in this case, the boson cannot be treated as a simple elementary particle. Therefore, the method of proving the Goldstone theorem which is successful for the boson field theory cannot be applied any more to the fermion field theory model with the regularization.

This paper is organized as follows. In the next section, we first treat the Bogoliubov transformation in the Nambu-Jona-Lasinio(NJL) model, and obtain a new vacuum. The mass of the boson is evaluated in the regularized NJL model. In section 3, we discuss the chiral symmetry breaking in the massless Thirring model and calculate
the boson mass with the Bogoliubov transformed vacuum. In section 4, we explain
the breaking down of the Goldstone theorem for the fermion field theory models with
the current regularization. Finally, section 5 summarizes what we have clarified in
this paper.

2. Nambu-Jona-Lasinio model

Here, we discuss the four dimensional current-current interaction model by Nambu
and Jona-Lasinio [3]. This paper is the original one that initiated the chiral symme-
try breaking phenomena. Here, we treat the NJL model just in the same manner as
the one given by Nambu and Jona-Lasinio as far as the symmetry breaking mecha-
nism is concerned. However, the boson mass determination is different, and we
carry out the calculation in terms of the Fock space expansion. Also, we carry out
the RPA calculation, but it turns out that the RPA calculation gives the boson
spectrum which is similar to that calculated by the Fock space expansion. But
the boson mass is slightly lower than the result of the Fock space expansion as the
function of the coupling constant.

Now, we carry out the calculation which is based on the Bogoliubov transformation,
and show that the chiral symmetry is indeed broken. However, we also show that
there appears no massless boson in this regularized NJL model.

Here, we first quantize the fermion field in a box $L^3$

$$\psi(r) = \frac{1}{\sqrt{L^3}} \sum_{n,s} \left[ a(n, s) u(n, s) e^{i \hat{n} \cdot r} + b^\dagger(n, s) v(n, s) e^{-i \hat{n} \cdot r} \right]$$

(2.1)

where $s$ denotes the spin index, and $s = \pm 1$. Also, the spinors are defined as

$$u(n, s) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma \cdot \hat{n} \chi(s) \\ \chi(s) \end{pmatrix}$$

$$v(n, s) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi(s) \\ \sigma \cdot \hat{n} \chi(s) \end{pmatrix}$$

Now, we define new fermion operators by the Bogoliubov transformation,

$$c(n, s) = e^{-A} a(n, s) e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a(n, s) - s \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b^\dagger(-n, s)$$

(2.2a)

$$d^\dagger(-n, s) = e^{-A} b^\dagger(-n, s) e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b^\dagger(-n, s) + s \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a(n, s)$$

(2.2b)
where the generator of the Bogoliubov transformation is given by

\[ \mathcal{A} = - \sum_{n,s} \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) \left( a^\dagger(n, s) b^\dagger(-n, s) - b(-n, s) a(n, s) \right). \] (2.3)

\( \theta_n \) denotes the Bogoliubov angle which can be determined by the condition that the vacuum energy is minimized. In this case, the new vacuum state is obtained as

\[ |\Omega\rangle = e^{-\mathcal{A}} |0\rangle. \] (2.4)

In what follows, we treat the NJL Hamiltonian with the Bogoliubov transformed vacuum state. In order to clearly see some important difference between the massive fermion and massless fermion cases, we treat the two cases separately.

(a) Massive fermion case

The Lagrangian density for the NJL model with the massive fermion can be written as

\[ \mathcal{L} = i \bar{\psi} \gamma_\mu \partial^\mu \psi - m_0 \bar{\psi} \psi + G \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \right]. \] (2.5)

Now, we can obtain the new Hamiltonian under the Bogoliubov transformation,

\[ H = \sum_{n,s} \left\{ |p_n| \sin \theta_n + \left( m_0 + \frac{2G}{L^3} B \right) \cos \theta_n \right\} \left( c^\dagger(n, s) c(n, s) + d^\dagger(-n, s) d(-n, s) \right) \]

\[ + \sum_{n,s} \left\{ -|p_n| s \cos \theta_n + \left( m_0 + \frac{2G}{L^3} B \right) s \sin \theta_n \right\} \left( c^\dagger(n, s) d^\dagger(-n, s) + d(-n, s) c(n, s) \right) \]

\[ + H'_{int} \] (2.6)

where \( H'_{int} \) is the interaction term. Since the \( H'_{int} \) is quite complicated, and besides its explicit expression is not needed in this context, we will not write it here. \( B \) is defined as

\[ B = \sum_{n,s} \cos \theta_n. \]

Now, we can define the renormalized fermion mass \( m \)

\[ m = m_0 + \frac{2G}{L^3} B. \] (2.7)

The Bogoliubov angle \( \theta_n \) can be determined by imposing the condition that the \( cd \) term in eq.(2.6) must vanish. Therefore, we obtain

\[ \tan \theta_n = \frac{|p_n|}{m}. \] (2.8)
This Bogoliubov angle \( \theta_n \) does not change when the mass varies from \( m_0 \) to \( m \). In this case, the vacuum is just the same as the trivial vacuum of the massive case, except that the fermion mass is replaced by the renormalized mass \( m \). The rest of the theory becomes identical to the massive NJL model with the same interaction Hamiltonian \( H'_{int} \). Therefore, there is no symmetry breaking, and this vacuum has no condensate.

(b) Massless fermion case

Here, we present the same procedure for the massless fermion case in order to understand why the fermion has to become massive.

We start from the Lagrangian density with no mass term in eq.(2.5). Under the Bogoliubov transformation, we obtain the new Hamiltonian

\[
H = \sum_{n,s} \left\{ |p_n| \sin \theta_n + \frac{2G}{L^3} B \cos \theta_n \right\} \left( c^\dagger(n,s)c(n,s) + d^\dagger(-n,s)d(-n,s) \right) \\
+ \sum_{n,s} \left\{ -|p_n| s \cos \theta_n + \frac{2G}{L^3} Bs \sin \theta_n \right\} \left( c^\dagger(n,s)d^\dagger(-n,s) + d(-n,s)c(n,s) \right) \\
+ H'_{int}
\]

where \( H'_{int} \) is just the same as the one given in eq.(2.6). From this Hamiltonian, we get to know that the mass term is generated in the same way as the massive case. But we cannot make any renormalization since there is no mass term. Further, the new term is the only mass scale in this Hamiltonian since the coupling constant cannot serve as the mass scale. In fact, it is even worse than the dimensionless coupling constant case, since the coupling constant in the NJL model is proportional to the inverse square of the mass dimension. Thus, we define the new fermion mass \( M_N \) by

\[
M_N = \frac{2G}{L^3} B.
\]

The Bogoliubov angle \( \theta_n \) can be determined from the following equation

\[
\tan \theta_n = \frac{|p_n|}{M_N}.
\]

In this case, the vacuum changes drastically since the original vacuum is trivial. Further, the constraints of eqs.(2.10) and (2.11) give rise to the equation that determines the relation between the induced fermion mass \( M_N \) and the cutoff momentum \( \Lambda \).

\[
M_N = \frac{4G}{(2\pi)^3} \int_0^\Lambda d^3p \frac{M_N}{\sqrt{M_N^2 + p^2}}.
\]
This equation has a nontrivial solution for $M_N$, and the vacuum energy becomes lower than the trivial vacuum ($M_N = 0$). Therefore, $M_N$ can be expressed in terms of $\Lambda$ as

$$M_N = \gamma \Lambda$$

where $\gamma$ is a simple numerical constant.

It should be noted that the treatment up to now is exactly the same as the one given by Nambu and Jona-Lasinio [3]. Further, we stress that the induced fermion mass $M_N$ can never be set to zero, and it is always finite.

(c) Boson mass

The boson state $|B\rangle$ can be expressed as

$$|B\rangle = \sum_{n,s} f_n c^\dagger(n, s) d^\dagger(-n, s) |\Omega\rangle,$$ \hspace{1cm} (2.13)

where $f_n$ is a wave function in momentum space, and $|\Omega\rangle$ denotes the Bogoliubov vacuum state. The equation for the boson mass $\mathcal{M}$ for the NJL model is written in terms of the Fock space expansion at the large $L$ limit

$$\mathcal{M} f(p) = 2E_pf(p) - \frac{2G}{(2\pi)^3} \int^{\Lambda} d^3q f(q) \left( 1 + \frac{M^2}{E_p E_q} + \frac{p \cdot q}{E_p E_q} \right)$$ \hspace{1cm} (2.14)

where $M$ should be taken to be $M = m$ for the massive case, and $M = M_N$ for the massless case. It is important to note that the fermion mass $M$ after the Bogoliubov transformation, therefore, cannot become zero.

Here, again, we note that the RPA calculation gives the similar boson spectrum to the Fock space expansion. But we do not know whether the RPA calculation is better than the Fock space expansion or not, since the derivation of the RPA equation in field theory is not based on the fundamental principle. In principle, the RPA calculation may take into account the effect of the deformation of the vacuum in the presence of the particle and antiparticle. However, this is extremely difficult to do it properly, and indeed the RPA eigenvalue equation is not hermite, and thus it is not clear whether the effect is taken into account in a better way or worse. The examination and the validity of the RPA equation will be given elsewhere.

The solution of eq.(2.14) can be easily obtained, and the boson mass spectrum for the NJL model is shown in Fig. 1. Note that the boson mass is measured in units of the cutoff momentum $\Lambda$. As can be seen from the figure, there is a massive boson for some regions of the values of the coupling constant.
Here, as we will see later, the NJL and the Thirring models are quite similar to each other. This is mainly because the current-current interaction is essentially a delta function potential in coordinate space. Indeed, as is well known, the delta function potential in one dimension can always bind the fermion and anti-fermion while the delta function potential in three dimension cannot normally bind them. Due to the finite cut off momentum, the delta function potential in three dimensions can make a weak bound state, depending on the strength of the coupling constant. This result of the delta function potential in quantum mechanics is almost the same as what is just shown in Fig. 1.

It should be noted that there is no serious difficulty of proving the non-existence of the massless boson. However, if it were to prove the existence of the massless boson, it would have been extremely difficult to do it. For the massless boson, there should be a continuum spectrum, and this continuum spectrum of the massless boson should be differentiated from the continuum spectrum arising from the many body nature of the system. This differentiation must have been an extremely difficult task. In fact, even if one finds a continuum spectrum which has, for example, the dispersion of $E = c_0 p^2$ as often discussed in solid state physics, one sees that the spectrum has nothing to do with the Goldstone boson.

3. Massless Thirring model

Now, we discuss the Thirring model which is a two dimensional field theory with the current-current interaction [5]. For the massive Thirring model, it is discussed in detail in [6], and therefore, we treat only the massless case in this paper.

Since the massless Thirring model does not contain any scale parameter, it is necessary to introduce a scale or a cutoff momentum if one wants to discuss physical quantities. In this respect, for the massless Thirring model, any calculations with the regularization is physically meaningful. For example, the equivalence between the massless Thirring model and the massless boson system is quite well known and interesting, but this is mathematically important. However, physically, this equivalence is somewhat more complicated than expected since there is no scale introduced in the systems, and thus one cannot measure physical observables.

The massless Thirring model is described by the following Lagrangian density

$$\mathcal{L} = i \bar{\psi} \gamma_\mu \partial^\mu \psi - \frac{1}{2} g j^\mu j_\mu$$

(3.1)
where the fermion current $j_\mu$ is given as

$$j_\mu = : \bar{\psi} \gamma_\mu \psi : \quad (3.2)$$

It is clear that this Lagrangian density has a chiral symmetry, and therefore, defining the chiral current by

$$j^5_\mu = : \bar{\psi} \gamma_5 \gamma_\mu \psi : \quad (3.3)$$

we see that the chiral current is conserved in the classical level

$$\partial^\mu j^5_\mu = 0. \quad (3.4)$$

In the recent paper, Faber and Ivanov [9] show that the true vacuum state has a chiral symmetry broken phase. They consider that the chiral symmetry breaking is spontaneous, and therefore they discuss the reason why the symmetry can be broken in the two dimensional field theory. Their discussions are concerned with the problem of Coleman’s theorem.

Here, we present the calculation which is based on the Bogoliubov transformation, and show that the chiral symmetry is indeed broken. However, we also show that there appears no massless boson in this regularized Thirring model.

Here, we first quantize the fermion field in a box $L$

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} e^{i \frac{2\pi nx}{L}}. \quad (3.5)$$

In this case, the Hamiltonian of the massless Thirring model can be written as

$$H = \sum_n \left[ p_n (a_n^\dagger a_n - b_n^\dagger b_n) + \frac{2g}{L} \tilde{j}_{1,pn} \tilde{j}_{2,-pn} \right]. \quad (3.6)$$

where the fermion currents in the momentum representation $\tilde{j}_{1,pn}$ and $\tilde{j}_{2,pn}$ are given by

$$\tilde{j}_{1,pn} = \sum_l a_l^\dagger a_{l+n} \quad (3.7a)$$

$$\tilde{j}_{2,pn} = \sum_l b_l^\dagger b_{l+n}. \quad (3.7b)$$

Now, we define new fermion operators by the Bogoliubov transformation,

$$c_n = e^{-A} a_n e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a_n - \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b_n \quad (3.8a)$$

$$d_{-n}^\dagger = e^{-A} b_n e^A = \cos \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) b_n + \sin \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) a_n \quad (3.8b)$$
where the generator of the Bogoliubov transformation is given by

$$A = -\sum_n \left( \frac{\theta_n}{2} - \frac{\pi}{4} \right) (a_n^\dagger b_n - b_n^\dagger a_n).$$

(3.9)

$\theta_n$ denotes the Bogoliubov angle which can be determined by the condition that the vacuum energy is minimized. In this case, the new vacuum state is obtained as

$$|\Omega\rangle = e^{-A}|0\rangle.$$  

(3.10)

Now, we can obtain the new Hamiltonian under the Bogoliubov transformation,

$$H = \sum_n \left[ \left\{ p_n \sin \theta_n + \frac{g}{L} \mathcal{B} \cos \theta_n \right\} (c_n^\dagger c_n + d_n^\dagger d_n) \right]$$

$$+ \sum_n \left[ -p_n \cos \theta_n + \frac{g}{L} \mathcal{B} \sin \theta_n \right] (c_n^\dagger d_n^\dagger - d_n c_n) + H'$$

(3.11)

where $H'$ denotes the interaction terms of the Bogoliubov transformed state and is given in detail in [6]. Here, $\mathcal{B}$ is defined as $\mathcal{B} = \sum \cos \theta_n$, and the term $\frac{g}{L} \mathcal{B}$ corresponds to an induced mass. Therefore, we define the induced mass $M$ by

$$M = \frac{g}{L} \mathcal{B}.$$  

(3.12)

The Bogoliubov angle $\theta_n$ can be determined by imposing the condition that the vacuum energy must be minimized. Therefore, we obtain

$$\tan \theta_n = \frac{p_n}{M}.$$ 

(3.13)

In this case, we can express the self-consistency condition for $M$, and obtain

$$M = \frac{g}{\pi} M \ln \left( \frac{\Lambda}{M} + \sqrt{1 + \left( \frac{\Lambda}{M} \right)^2} \right)$$

(3.14)

where $\Lambda$ denotes the cutoff. Since the massless Thirring model has no scale, we should measure all of the observables in terms of $\Lambda$. Therefore, we can express the induced mass $M$ in terms of $\Lambda$,

$$M = \frac{\Lambda}{\sinh \left( \frac{\pi g}{M} \right)}.$$ 

(3.15)

Further, the vacuum energy $E_{\text{vac}}$ as measured from the trivial vacuum is given

$$E_{\text{vac}} = -\frac{L}{2\pi} \frac{\Lambda^2}{\sinh \left( \frac{\pi}{g} \right)} e^{-\frac{\pi}{g}}.$$  

(3.16)
From this value of the vacuum energy, we get to know that the new vacuum energy is indeed lower than the trivial one. Therefore, the chiral symmetry is broken in the new vacuum state since the fermion becomes massive.

Note that the present expression of the mass $M$ is somewhat different from the one given in Faber and Ivanov [9]. This is related to the fact that we take the cutoff momentum in the normal momentum while they used the cutoff in the Lorentz invariant fashion.

In the same manner as [6], we carry out the calculations of the spectrum of the bosons in the Fock space expansion.

The boson state $|B\rangle$ can be expressed as

$$|B\rangle = \sum_n f_n c_n^\dagger d_{-n}^\dagger |\Omega\rangle,$$

where $f_n$ is a wave function in momentum space, and $|\Omega\rangle$ denotes the Bogoliubov vacuum state. The energy eigenvalue of the Hamiltonian for the large $L$ limit can be written as

$$\mathcal{M} f(p) = 2E_p f(p) - \frac{g}{2\pi} \int dq f(q) \left( 1 + \frac{M^2}{E_p E_q} + \frac{pq}{E_p E_q} \right)$$

(3.18)

where $\mathcal{M}$ denotes the boson mass. $E_p$ is given as

$$E_p = \sqrt{M^2 + p^2}.$$  

(3.19)

Eq.(3.18) can be solved exactly as shown in [6]. First, we define $A$ and $B$ by

$$A = \int_{-\Lambda}^{\Lambda} dp f(p)$$

(3.20a)

$$B = \int_{-\Lambda}^{\Lambda} dp \frac{f(p)}{E_p}.$$  

(3.20b)

Using $A$ and $B$, we can solve Eq. (3.18) for $f(p)$ and obtain

$$f(p) = \frac{g/2\pi}{2E_p - \mathcal{M}} \left( A + \frac{m^2}{E_p} B \right).$$  

(3.21)

Putting this $f(p)$ back into Eqs. (3.20), we obtain the matrix equations

$$A = \frac{g}{2\pi} \int_0^{\Lambda} \frac{2dp}{2E_p - \mathcal{M}} \left( A + \frac{m^2}{E_p} B \right)$$

(3.22a)

$$B = \frac{g}{2\pi} \int_0^{\Lambda} \frac{2dp}{(2E_p - \mathcal{M})E_p} \left( A + \frac{m^2}{E_p} B \right).$$  

(3.22b)
Since the model is already regularized, we can easily calculate the boson spectrum which is given in Fig. 2 as the function of the coupling constant $g/\pi$. As can be seen from Fig. 2, there is no massless boson in this spectrum even though the boson mass for the very small coupling constant $g$ is exponentially small. Here, we also carry out the RPA calculation for the massive Thirring model. However, the spectrum predicted by the RPA calculation somewhat deviates from the other calculated results [11, 12, 13, 14] in the massive Thirring model. Since the derivation of the RPA equation in field theory is not based on the fundamental principle, we do not know to what extent the calculation of the RPA is reliable and the validity of the RPA equation will be examined elsewhere.

Now, we discuss the chiral current conservation. In order to examine it, we evaluate the following equation,

$$i\dot{Q}_5 = [H, Q_5] \tag{3.23}$$

where $Q_5$ is defined as

$$Q_5(t) = \int j_5^\phi(x, t) dx = \int \bar{\psi}\gamma_0\gamma_5\psi dx. \tag{3.24}$$

Clearly, we can show that the right hand side of eq.(3.23) vanishes for the massless Thirring Hamiltonian of eq.(3.4). However, since the Bogoliubov transformation is unitary, $[H, Q_5]$ is invariant and thus it remains to be zero. Therefore, the chiral current is still conserved after the Bogoliubov transformation.

Here, we also calculate the fermion condensate $C$ for this vacuum state [10], and obtain

$$|C| = \langle\Omega | \frac{1}{L} \int \bar{\psi}\psi dx |\Omega\rangle = \frac{M}{g} \tag{3.25}$$

which is a finite value. Therefore, the massless Thirring model has a chiral symmetry broken vacuum, but, the chiral current is conserved. Therefore, this situation contradicts the Goldstone theorem since there appears no massless boson in this symmetry breaking. This point will be discussed in detail in the next section.

4. Goldstone Theorem

Here, we discuss the Goldstone theorem in connection with the massless NJL and the Thirring models. In the formal proof of the Goldstone theorem [1, 2], one assumes the existence of the vacuum expectation value of the following commutation relation,

$$\langle\Omega | [Q_5(t), \phi(0)] |\Omega\rangle \neq 0 \tag{4.1}$$
where the boson field \( \phi(x) \) must be constructed by the fermion fields. Now, inserting intermediate boson states, one obtains

\[
\sum_n \delta(p_n) \left[ e^{iE_n t} \langle \Omega | j_0^5(0) | n \rangle \langle n | \phi(0) | \Omega \rangle - e^{-iE_n t} \langle \phi(0) | n \rangle \langle n | j_0^5(0) | \Omega \rangle \right] \neq 0 \tag{4.2}
\]

The right hand side of eq.(4.2) is non-vanishing, and time-independent. Thus, eq.(4.2) can be satisfied only if there exists an intermediate state with \( E_n = 0 \) for \( p_n = 0 \). This is the proof of the Goldstone theorem, and it is indeed valid for boson field theory. One can also obtain the same result from rewriting the Lagrangian density at the new vacuum point for boson field theory models. This is reasonable since the Goldstone boson is understood as a kinematical effect.

On the other hand, the fermion field theory models are quite different, and one cannot obtain the Goldstone boson even if one rewrites the Lagrangian density at the new vacuum point. This is due to the fact that the Goldstone boson must be constructed by the fermion and anti-fermion fields, and therefore it should involve the dynamics. But it has been believed that the proof based on eqs.(4.1) and (4.2) holds for the fermion field theory models as well. However, one must be careful for the fermion field theory models with the regularization. In the above proof, the use of the translational property of a boson plays an essential role and it is assumed for the boson field \( \phi(x) \) and the chiral current \( j_5^\mu(x) \) even if they are constructed by fermions. However, the boson field \( \phi(x) \) together with the current \( j_5^\mu(x) \) must be regularized by the point splitting,

\[
j_5^\mu(x) = \bar{\psi}(x)\gamma_5\gamma_\mu\psi(x + \epsilon) \tag{4.3a}
\]

\[
\phi(x) = \bar{\psi}(x)\gamma_5\psi(x + \epsilon) \tag{4.3b}
\]

where we have to keep the \( \epsilon \) finite. In the NJL as well as the Thirring models, the \( \epsilon \) should be related to the cutoff momentum \( \Lambda \) in some way or the other. In this case, one cannot make use of the translation property of \( \phi(x) \)

\[
\phi(x) = e^{ipx}\phi(0)e^{-ipx}.
\]

Instead, \( \phi(x) \) is only written as

\[
\phi(x) = e^{ipx}\bar{\psi}(0)\gamma_5e^{ipx}\psi(0)e^{-ip(x+\epsilon)}, \tag{4.4}
\]

and one cannot obtain the same equation as eq.(4.2) for fermion fields with the regularization. Therefore one cannot claim the existence of the massless boson any
more. In the massless NJL as well as the Thirring models, one needs to regularize
the fermion field, and therefore, one cannot claim that the Goldstone theorem should
hold true. In fact, as we saw in the previous sections, there appears no massless
boson, and this is just what is found here.

Further, we should note that no fermion field theory model except the NJL model
predicts a Goldstone boson. It is often discussed that pion may well be a Goldstone-
like boson. But for this, one should be careful. In four dimensional QCD, the
coupling constant has no dimension, and therefore, all the physical mass must be
measured by the quark mass if the quarks are massive. In this case, there is no chiral
symmetry, and there is no need to discuss the Goldstone boson. It is believed that
the pion mass is too light compared with other mesons. However, it is quite large
if one measures it in terms of the quark mass which should be around 10 MeV. In
this respect, the pion mass should be considered as an object which has nothing to
do with the chiral symmetry breaking. If the quarks were massless, then the chiral
symmetry should be broken in the new vacuum, and any physical observables like
meson masses should be measured by the cut off momentum $\Lambda_{QCD}$. But this story
has nothing to do with nature in real QCD.

5. Conclusions

We have presented the chiral symmetry breaking of the massless NJL and the
Thirring models. The new vacuum has a finite chiral condensate. Also, it is shown
that the chiral symmetry breaking here does not accompany a massless boson. This
is not consistent with the Goldstone theorem since the chiral current conservation
still holds in the Bogoliubov transformed vacuum with the regularization.

This inconsistency is resolved in that the Goldstone theorem turns out to be invalid
for the fermion field theory models with the regularization. This is simply due to
the fact that the proof of the Goldstone theorem is essentially based on the use
of the translational property of the boson, which, however, cannot be valid for the
fermion and antifermion bound state due to the point splitting regularization.

To summarize the chiral symmetry breaking in fermion field theory models of the
NJL and the Thirring, we state that the new vacuum indeed violates the chiral
symmetry, and it has a finite condensate value. The fermion acquires the finite
mass, and the rest of the theory becomes just the massive fermion field theory with
the same interactions as the massive fermion case. This completes the symmetry
breaking business. There is no space and no freedom left for the Goldstone boson. The massive fermion field theory models of the NJL and the Thirring give a massive boson. The Thirring model has an exponentially small boson mass for the small coupling constant region while the boson mass of the NJL model strongly depends on the strength of the coupling constant. Indeed, below the critical value of the coupling constant, there is no bosonic bound state. However, the boson mass has little to do with the symmetry breaking phenomena, but it is determined by the coupling constant and the cutoff $\Lambda$.

What is then the main difference between the boson field theory and the fermion field theory? In boson field theory models, one assumes that the potential term $U(|\phi|)$ for the boson field has a nontrivial minimum, which has infinite degenerate states. This degeneracy of the "potential vacuum" is resolved when one considers the kinetic energy terms of the boson field. At this point, the symmetry is broken and one obtains the new vacuum state with a massless boson. Here, one notices that, if there were no nontrivial minimum in the potential vacuum, then there should exist no Goldstone boson. Further, if one finds a system whose \textit{total Hamiltonian} has the vacuum with infinite degenerate states, then there is no way to resolve the degeneracy, and the vacuum should stay as it is. The fermion field theory models which we treat in this paper have no nontrivial minimum of the potential vacuum, and the vacuum is found only when one considers the total Hamiltonian of the system. Therefore, there is no place one finds any massless boson degree of freedom for the fermion field theory models.

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Fig. 1: The boson mass for the NJL model is plotted as the function of $GA^2$. It is measured by the cutoff $\Lambda$. 
Fig. 2: The boson mass for the massless Thirring model is plotted as the function of $g/\pi$. It is measured by the cutoff $\Lambda$. 