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CORRIGENDUM

Corrigendum: Heterogeneous behavioral adoption in multiplex networks (2018 New J. Phys. 20 125002)

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Heterogeneous behavioral adoption in multiplex networks

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Abstract

Heterogeneity is found widely in populations, e.g. different individuals have diverse personalities and a different willingness to accept novel ideas or behaviors. Whereas population heterogeneity is rarely considered in studying the social contagions on complex networks, especially on multiplex networks. To explore the effect of population heterogeneity on the dynamics of social contagions, a novel model based on double-layer multiplex networks is proposed, in which information diffuses synchronously on the two layers, and each layer is assigned with different adoption thresholds. Meanwhile, populations are classified into the activists and conservatives according to their willingness to adopt new behaviors. To qualitatively understand the effect of population heterogeneity on social contagions, a generalized edge-based compartmental theory is proposed. Through rigorous theoretical analysis and extensive simulations, we find the activists in the two layers promote the adoption of behavior. More interestingly, the crossover phenomena in phase transition are found in the growth of the final adopted size when increasing the information transmission rate. When the proportion of activists is relatively large, the phase transition exhibits continuous pattern. Reducing the proportion of activists induces a hybrid transition, in which the final adoption size first grows continuously at the first threshold, and then increases discontinuously at the second threshold. In addition, we find that the network heterogeneity will change the crossover phenomena in phase transition.

1. Introduction

Real complex systems ranging from economic network to the ecological network [1–3] can be described as multiplex networks [4–7], in which each subnetwork represents one distinct subsystem. Existing researches revealed that the multiplexity of networks have significant influence on the dynamics in multiplex networks [8–15]. For cascading failure in interdependent networks, the system exhibited a discontinuous phase transition unlike the continuous one on a single network [17]. For synchronizing process, when the specific microscopic correlation features between the natural frequencies of the oscillators and their effective coupling strengths vanish, Zhang et al. [18] found the far more general process can occur in adaptive and multiplex networks. Moreover, Boccaletti et al. discovered that instead of classic result of the second-order type, i.e. continuous and reversible, in complex networks' structure and dynamics, the synchronizing process rather reminds first-order (discontinuous and irreversible) transitions [19]. For evolutionary games in multiplex network, interactions between layers influence evolution of cooperation [20–22]. Researchers found spreading dynamics on multiplex network also behaves distinctively from the single-layer network [23–35]. A few interlayer links can induce the global outbreak of disease, although the epidemic cannot outbreak on a single-layer network [36]. Granell et al. revealed that metacritical point exists in the asymmetric coevolution of spreading dynamics in complex networks [37].
Recently, the social contagions in multiplex networks attract extensive studies and have been used to describe the dynamics of information diffusion, behavior and innovation adoption [38–40]. Different from the epidemic spreading [41–43], the social contagions display one inherent characteristic of social reinforcement effect [25], which means adopting a certain social behavior requires verification of its credibility and legitimacy. It was found that the multiplexity of networks promotes the spreading of Markovian social contagions [44–47]. Wang et al [48] considered that information transmits through multiple channels and switches among these channels. Based on the communication channel alteration (CCA) mechanism, a non-Markovian threshold model on multiplex networks was proposed. The authors found that the time delay induced by the CCA mechanism slows down the transmission rate of information, but it does not affect the final spreading size of the information nor change the type of phase transition. Chen et al [49] considered that the behavior adoption of a single person is affected by neighbors from different layers of a multiplex network simultaneously. They found that because of the synergy effect between different layers, the final spreading size is enhanced, and a few seeds can stimulate a global spreading of information. In addition, Wang et al [50] studied the effect of inter-layer correlation on the social contagions in multiplex networks, and a non-Markovian social contagion that considers the inter-layer correlation was proposed. They found that the correlation between the layers of multiplex networks promotes behavior adoption but will not alter the increment pattern of the final adoption size.

From the above literatures, previous researchers have ubiquitously investigated the heterogeneity in populations, such as different people have different attitudes towards new ideas or behaviors [51–56]. Nevertheless, in real systems, individuals can be in several social networks simultaneously, and they have different intimate relationships in different subnetworks, which can affect their willingness to adopt a new idea or behavior. For example, families or cronies can convince you more easily to adopt a new idea or behavior, but you need to verify an information many times before accepting it on a virtual social network. Naturally, we consider that the adoption thresholds in different layers of a multiplex network are different, and the adoption willingness is also different from person to person. Previous literatures mainly studied the effect of population heterogeneity on social contagions on a single network [57, 58]. However, the social contagions with heterogeneous populations on multiplex networks has not been systematically studied.

To study the effect of heterogeneous populations on the social contagions in multiplex network, a non-Markovian social contagion model based on two-layer multiplex network has been proposed in our paper. In this model, to reflect adoption heterogeneity of populations, we randomly select a fraction of nodes as activists, who have strong willingness to adopt the behavior, and the remaining nodes as the conservatives, who have weak willingness to adopt the behavior. In addition, to investigate the influence of different social circles on the spreading dynamics, different adoption thresholds are assigned to the two layers of the network. To theoretically analyze the dynamics of social contagions, a generalized edge-based compartmental theory is established. Through rigorous theoretical analysis and extensive simulations, we find that the population heterogeneity affects both the final adoption size and phase transition. Increasing the proportion of activists can promote the final adoption size. Most interestingly, the crossover phenomena [59] in phase transition of the final adoption size are found. The phase transition of the adoption size changes from the continuous to the hybrid pattern, which exhibits the characteristics of both continuous and discontinuous transitions, when the proportion of activists is adjusted from a large value to a small value. At last, we find that changing the degree heterogeneity in both layers of the network will alter the crossover phenomena in phase transition.

2. Model descriptions

To study the complex contagions, a double-layer multiplex network is adopted, in which the two layers A and B represent two different communication channels. Nodes correlate one to one in the two layers, and one pair of the inter-correlated nodes stand for the same person in two different social subnetworks. Edges in different layers represent different types of connections between individuals. To avoid the intra-degree–degree correlations, the uncorrelated configuration model [60] is used in our model following the two given independent degree distributions \( P_A(k_A) \) and \( P_B(k_B) \). Note that the self and multiple edges are avoided in building the networks. Degree of node \( i \) is denoted by \( k_A^i \) in layer A and \( k_B^i \) in layer B. Thus, the degree of node \( i \) can be denoted by \( k_i = (k_A^i, k_B^i) \). Assuming that there is no degree–degree correlations between two layers, we get a joint degree distribution \( P(k_i) = P_A(k_A^i)P_B(k_B^i) \). Each node \( i \) holds adoption thresholds \( T_A \) and \( T_B \) in layers A and B, respectively. The larger value of adoption threshold, the less willingness of behavior adoption. Besides, in reality individuals have different willingness to adopt the behavior when they receive the information. Usually, the individuals with greater willingness to adopt the behavior are named as activists, while, the individuals with weaker willingness are named as conservatives. To reflect the different willingness of individuals
in the network, a fraction of $q$ nodes are randomly selected as activists and the remaining $1-q$ nodes are selected as conservatives.

To describe the social contagion on the multiplex network, we adopt the generalized susceptible–adopted–recovered model, where each node in the two layers can be in any of the three states: susceptible (S), adopted (A) and recovered (R) state. Nodes in the susceptible state do not adopt the behavior. The nodes in the adopted state have adopted the behavior and are willing to transmit the behavior information to their neighbors. Finally, the nodes in the recovered state lose interest in the behavior and will not transmit the behavior information to their neighbors.

Initially, a fraction of $\rho_0$ nodes are selected randomly as the adopted nodes (seeds). We set $\rho_0 = 1/N$ in our paper, namely only one node in the network is selected as the seed. At each time step, each A-state node tries to diffuse the behavior information to its S-state neighbors in both layer $A$ and layer $B$ synchronously with rate $\lambda_A$ and $\lambda_B$, respectively. Note that once the information is transmitted successfully through an edge between the A–S node-pair, it will never be transmitted again. In other words, only non-redundant information transmission is allowed [61]. In addition, the A-state nodes can try many times to transmit the information to their susceptible neighbors until they enter into the recovered state. If a piece of information is successfully transmitted from an A-state node i to an S-state neighbor $j$ in layer $X \in \{A, B\}$, the accumulated pieces of information of the S-state $m^X_{ij}$ increases one, i.e. $m^X_{ij} = m^X_{ij} + 1$. Subsequently, node $j$ in layers $A$ and $B$ will compare its accumulated pieces of information with its adopt thresholds. The activists adopt the behavior when their accumulated pieces of information exceed any threshold in two layers, i.e. $m^A_{ij} > T_A$ or $m^B_{ij} > T_B$. In contrast, the conservatives adopt the behavior only if accumulated pieces of information in both layer $A$ and layer $B$ exceed the adopt thresholds simultaneously, i.e. $m^A_{ij} > T_A$ and $m^B_{ij} > T_B$. Obviously the adoption of behavior is determined by the accumulated pieces of information in both layers, so the non-Markovian effect is induced in the dynamics of behavior spreading. At each time step, the A-state nodes will lose interest in the behavior and enter into the recovered state with rate $\gamma$. When the nodes enter into the R-state from the A-state, they will not participate in the transmission of the behavior information, namely, they will neither transmit the information to nor receive information from A-state neighbors. Finally, the dynamics of the information spreading terminate when all A-state nodes enter into the R-state.

3. Theoretical analysis

To theoretically analyze the proposed model in section 2 and describe the strong dynamical correlations among the states of nodes in the process of information spreading, we establish an edge-based compartmental theory. In the edge-based compartmental theory, a node $i$ is assumed to be in the cavity state [62], which means that it can only receive information from its neighbors, but cannot transmit the information to its neighbors. In addition, we denote $\theta_A(t)(X \in \{A, B\})$ as the probability that the information about the behavior has not been transmitted through an randomly chosen edge to the susceptible neighbors in layer $X$ by time $t$. The probability that an randomly selected susceptible node $i$ with degree $\tilde{k}_i = (k^A_i, k^B_i)$, receives $m_A$ and $m_B$ pieces of information by time $t$ can be expressed as

$$\phi^A_{m_A}(k^A_i, t) = \binom{k^A_i}{m_A} \theta_A(t)^{k^A_i-m_A} [1-\theta_A(t)]^{m_A},$$

and

$$\phi^B_{m_B}(k^B_i, t) = \binom{k^B_i}{m_B} \theta_B(t)^{k^B_i-m_B} [1-\theta_B(t)]^{m_B},$$

respectively. According to the differences between the activists and conservatives in section 2, an activist in susceptible state indicates that the accumulated pieces of information he has received in layers $A$ or $B$ are less than the corresponding thresholds, namely $m_A < T_A$ and $m_B < T_B$. While a conservative in susceptible state means that the accumulated pieces of information in layer $A$ or $B$ is less than its threshold, namely $m_A < T_A$ or $m_B < T_B$. We define the probability that an activist with degree $\tilde{k} = (k^A, k^B)$ in the susceptible state as $s_a(\tilde{k}, t)$, and the probability that a conservative with degree $\tilde{k} = (k^A, k^B)$ in the susceptible state as $s_b(\tilde{k}, t)$. Thus, the probability that a randomly selected node with degree $\tilde{k} = (k^A, k^B)$ in susceptible at time $t$ can be expressed as

$$s(\tilde{k}, t) = q_a s_a(\tilde{k}, t) + (1-q) s_b(\tilde{k}, t),$$
where
\[
s_A(\vec{k}, t) = \sum_{m_x=0}^{T_x-1} \phi^{A}_{m_x}(k^A, t) \sum_{m_y=0}^{T_y-1} \phi^{B}_{m_y}(k^B, t),
\]
(4)

and
\[
s_B(\vec{k}, t) = 1 - \left( 1 - \sum_{m_x=0}^{T_x-1} \phi^{A}_{m_x}(k^A, t) \right) \times \left( 1 - \sum_{m_y=0}^{T_y-1} \phi^{B}_{m_y}(k^B, t) \right).
\]

The terms in the righthand of equation (4) stand for the probability that the accumulated pieces of information that an activist has received by time \(t\) in layer \(X\) are less than the corresponding thresholds \(\sum_{m_x=0}^{T_x-1} \phi^{X}_{m_x}(k^X, t), X \in \{A, B\}\). The second term in the righthand of equation (5) stands for the probability that both the accumulated pieces of information in layers \(A\) and \(B\) exceed the corresponding thresholds. Thus the probability that the number of the accumulated information of a randomly selected node in layer \(X(X \in \{A, B\})\) is less than the corresponding threshold at time \(t\) can be expressed as
\[
\eta_X(t) = \sum_{\vec{k}^X} P_X(k^X) \sum_{m_x=0}^{T_x-1} \phi^{X}_{m_x}(k^X, t).
\]
(5)

The probability that an activist stays susceptible is \(\eta_A \eta_B\), and a conservative keeps susceptible with probability \(\eta_A + \eta_B - \eta_A \eta_B\). Consequently, we can get the fraction of susceptible nodes at time \(t\) as
\[
S(t) = \sum_\vec{k} P(\vec{k}) s(\vec{k}, t) = q \eta_A \eta_B + (1 - q)(\eta_A + \eta_B - \eta_A \eta_B).
\]
(6)

The neighbor of a node in cavity state can be in any of the three states: susceptible, adopted and recovered, so \(\theta_A(t)\) is constituted by the following three parts
\[
\theta_A(t) = \xi^X_A(t) + \xi^X_B(t) + \xi^R(t),
\]
where \(\xi^X_A(t)\), \(\xi^X_B(t)\) and \(\xi^R(t)\), respectively, denote the probabilities that a neighbor of the node in the cavity state stays in susceptible, adopted and recovered state and has not transmitted the information to the node.

Next, we analyze the above three terms If node \(i\) is in the cavity state and node \(j\) with degree \(\vec{k}_j = (k^A_j, k^B_j)\) is a susceptible neighbor of node \(i\) in layer \(A(B)\), node \(j\) can only receive information from other \(k^A_j - 1\) \((k^B_j - 1)\) neighbors as node \(i\) is in cavity state in layer \(A(B)\). Thus the probability that node \(j\) has received \(n_A\) pieces of information in layers \(A\) is \(\phi^{A}_{n_A}(k^A_j - 1, t)\). If \(j\) is an activist, taking all possible values of \(n_A\) into consideration, we can get the probability of node \(j\) remaining in susceptible state as
\[
\Theta_A^A(\vec{k}, t) = \sum_{n_x=0}^{T_x-1} \phi^{A}_{n_x}(k^A_x - 1, t) \sum_{n_y=0}^{T_y-1} \phi^{B}_{n_y}(k^B_y, t),
\]
(8)

where the term \(\sum_{n_y=0}^{T_y-1} \phi^{B}_{n_y}(k^B_y, t)\) accounts for all possible pieces of information that node \(j\) receives in layer \(B\). When considering the case that node \(j\) is a susceptible neighbor of node \(i\) in layer \(B\), we can get the similar equation as
\[
\Theta_B^B(\vec{k}, t) = \sum_{n_x=0}^{T_x-1} \phi^{A}_{n_x}(k^A_x, t) \sum_{n_y=0}^{T_y-1} \phi^{B}_{n_y}(k^B_y - 1, t).
\]
(9)

In addition, if \(j\) is a conservative, the probability that it remains susceptible is
\[
\Theta_A^R(\vec{k}, t) = 1 - \left[ 1 - \sum_{n_x=0}^{T_x-1} \phi^{A}_{n_x}(k^A_x - 1, t) \right] \left[ 1 - \sum_{n_y=0}^{T_y-1} \phi^{B}_{n_y}(k^B_y, t) \right],
\]
(10)

and
\[
\Theta_B^R(\vec{k}, t) = 1 - \left[ 1 - \sum_{n_x=0}^{T_x-1} \phi^{A}_{n_x}(k^A_x, t) \right] \left[ 1 - \sum_{n_y=0}^{T_y-1} \phi^{B}_{n_y}(k^B_y - 1, t) \right].
\]
(11)

Accordingly, taking the characteristics of the susceptible neighbors into consideration, the probability that node \(i\) connects to a susceptible node is composed of two parts
\[
\Theta_A(\vec{k}, t) = q \Theta_A^A(\vec{k}, t) + (1 - q) \Theta_A^R(\vec{k}, t),
\]
(12)

and
\[
\Theta_B(\vec{k}, t) = q \Theta_B^A(\vec{k}, t) + (1 - q) \Theta_B^R(\vec{k}, t).
\]
(13)
Given the joint degree distribution $P(\tilde{k})$, in layer $X$, an edge connects to a susceptible neighbor with probability

$$
\xi^X_S(t) = \frac{\sum_{k} k^X_P(\tilde{k}) \Theta_X(k^X_j, t)}{\langle k^X \rangle},
$$

(14)

where expression $k^X_P(\tilde{k}) / \langle k^X \rangle$ stands for the probability that an edge connects to a neighbor with degree $k^X_j$ in layer $X$.

Next, we analyze the evolution of $\xi_A(t)$ and $\xi_B(t)$ in layers $A$ and $B$. Once the behavior information is transmitted successfully through an edge in layer $A(B)$, the value of $\theta_A(t)$ ($\theta_B(t)$) decreases with $\lambda_A \xi_A(t)$ ($\lambda_B \xi_B(t)$). Thus the evolution of $\theta_A(t)$ and $\theta_B(t)$ can be expressed as

$$
\frac{d\theta_A(t)}{dt} = -\lambda_A \xi_A^X(t),
$$

(15)

where $X \in \{A, B\}$. As for the evolution of $\xi_A^X(t)$, if the information is not transmitted through an edge and the adopted nodes enter into the recovered state at time $t$, then the value of $\xi_A^X(t)$ will increase. Thus we can get the dynamics of $\xi_A^X(t)$ as

$$
\frac{d\xi_A^X(t)}{dt} = \gamma(1 - \lambda_X) \xi_A^X(t).
$$

(16)

Initially we have $\theta_A(0) = 1$ and $\xi_A^X(0) = 0$. Based on equations (15) and (16), we can get the integration constant $\gamma(1 - \lambda_X) / \lambda_X$ and the function of $\xi_A^X(t)$ as

$$
\xi_A^X(t) = \frac{\gamma(1 - \lambda_X)[1 - \theta_A(t)]}{\lambda_X}.
$$

(17)

Inserting equations (14), (17) into (7), we can get

$$
\xi_A^X(t) = \theta_A(t) - \sum_{k} k^X_P(\tilde{k}) \Theta_X(k^X_j, t) \frac{\gamma(1 - \lambda_X)[1 - \theta_A(t)]}{\lambda_X}.
$$

(18)

Now, we can substitute equations (18) into (15) and get the time evolution of $\theta_A(t)$ in detail

$$
\frac{d\theta_A(t)}{dt} = -\lambda_X \left[ \theta_A(t) - \sum_{k} k^X_P(\tilde{k}) \Theta_X(k^X_j, t) \frac{\gamma(1 - \lambda_X)[1 - \theta_A(t)]}{\lambda_X} \right] - \gamma(1 - \lambda_X)[1 - \theta_A(t)].
$$

(19)

According to the evolution mechanism of node state, we can learn that the susceptible nodes move into the adopted state when they adopt the behavior. Meanwhile, the adopted nodes lose interest in the behavior and move into the recovered state. Thus time evolution of the fraction of adopted nodes and recovered nodes can be obtained easily by

$$
\frac{dA(t)}{dt} = -\frac{dS(t)}{dt} - \gamma A(t),
$$

(20)

and

$$
\frac{dR(t)}{dt} = \gamma A(t).
$$

(21)

Through equations (6), (20) and (21), the fraction of nodes in each state at arbitrary time step can be obtained by iteration. Moreover, the final adoption size $R_X(\infty)$ can also be obtained when $t \to \infty$.

To study contagion dynamics, we can analyze the fixed points of equation (19) at the steady state. By setting $d\theta_A(t) / dt = 0$ at $t \to \infty$ we obtain

$$
\theta_A(\infty) = \sum_{k} k^X_P(\tilde{k}) \Theta_X(k^X, \infty) \frac{\gamma(1 - \lambda_X)[1 - \theta_A(\infty)]}{\lambda_X} = f_X(\theta_A(\infty), \theta_B(\infty)).
$$

(22)

For the convenience to analyze, we denote $\theta_A(\infty) = f_A(\theta_A(\infty), \theta_B(\infty))$ as

$$
\theta_A = F_A(\theta_B),
$$

(23)

and $\theta_B(\infty) = f_B(\theta_A(\infty), \theta_B(\infty))$ as

$$
\theta_B = F_B(\theta_A).
$$

(24)

If equation (23) is tangent to equation (24) when $\theta_A < 1$ and $\theta_B < 1$, there exists an discontinuous first-order phase transition [63]. At critical point, the following condition satisfies
\[
\frac{\partial f_A(\theta_A(\infty), \theta_B(\infty))}{\partial \theta_B(\infty)} \frac{\partial f_B(\theta_A(\infty), \theta_B(\infty))}{\partial \theta_A(\infty)} = 1. \tag{25}
\]

As equations above are too complex to get theoretical solutions, especially when there are different adopted thresholds \(T_A\) and \(T_B\) in the two layers of the complex networks. Thus to intuitively analyze the dynamics of the complex contagions, we discuss the special case when \(T_A = T_B = 1\). In this case, equations (8) and (9) are simplified to

\[
\Theta_A^a(\vec{k}, t) = \theta_A(t)^{\gamma_A - 1} \theta_B(t)^{\gamma_B}, \tag{26}
\]

and

\[
\Theta_B^b(\vec{k}, t) = \theta_A(t)^{\gamma_A - 1} \theta_B(t)^{\gamma_B - 1} \tag{27}
\]
in addition, equations (10) and (11) can be, respectively, rewritten as

\[
\Theta_A^a(\vec{k}, t) = \theta_A(t)^{\gamma_A - 1} + \theta_B(t)^{\gamma_B} - \theta_A(t)^{\gamma_A - 1} \theta_B(t)^{\gamma_B}, \tag{28}
\]

and

\[
\Theta_B^b(\vec{k}, t) = \theta_A(t)^{\gamma_A - 1} + \theta_B(t)^{\gamma_B - 1} - \theta_A(t)^{\gamma_A - 1} \theta_B(t)^{\gamma_B - 1}. \tag{29}
\]

Equations (12) and (13) are, respectively, expressed as

\[
\Theta_A(\vec{k}, t) = q\Theta_A^a(\vec{k}, t) + (1 - q)\Theta_A^b(\vec{k}, t) = (1 - q)1\theta_A(t)^{\gamma_A - 1} + \theta_B(t)^{\gamma_B} + (2q - 1)\theta_A(t)^{\gamma_A - 1} \theta_B(t)^{\gamma_B}, \tag{30}
\]

and

\[
\Theta_B(\vec{k}, t) = q\Theta_B^b(\vec{k}, t) + (1 - q)\Theta_B^b(\vec{k}, t) = (1 - q)1\theta_A(t)^{\gamma_A - 1} + \theta_B(t)^{\gamma_B - 1} + (2q - 1)\theta_A(t)^{\gamma_A - 1} \theta_B(t)^{\gamma_B - 1}. \tag{31}
\]

We substitute equations (30) and (31) into (22), and obtain

\[
\theta_A(\infty) = \left[ G_A^0(\theta_A(\infty)) + G_B^0(\theta_B(\infty)) \right] = (2q - 1)G_A^{1,0}(\theta_A(\infty), \theta_B(\infty)) + \frac{\gamma[1 - \theta_A(\infty)](1 - \lambda_A)}{\lambda_A}, \tag{32}
\]

and

\[
\theta_B(\infty) = \left[ G_A^0(\theta_A(\infty)) + G_B^0(\theta_B(\infty)) \right] = (2q - 1)G_A^{1,0}(\theta_A(\infty), \theta_B(\infty)) + \frac{\gamma[1 - \theta_B(\infty)](1 - \lambda_B)}{\lambda_B}, \tag{33}
\]

where \(G_A^0(x) = \sum_{k_A} P_A(k_A) x^{k_A}\) is the generation function of degree distribution \(P_A(k_A), X \in \{ A, B \}\) and \(G_B^0(x) = \sum_{k_B} P_B(k_B) x^{k_B - 1}\), \(X \in \{ A, B \}\) is the generation function of excess degree distribution \(P_B(k_B)\). In addition, the generation function of degree distribution \(P(\vec{k})\) are as follows:

\[
G_{A\beta}^{0,1}(x, y) = \frac{1}{G_{B0}(1)} \frac{\partial G_{A\beta}^{0,0}(x, y)}{\partial x}, \tag{34}
\]

and

\[
G_{A\beta}^{1,1}(x, y) = \frac{1}{G_{B0}(1)} \frac{\partial G_{A\beta}^{0,0}(x, y)}{\partial y}, \tag{35}
\]

\[
G_{A\beta}^{0,0}(x, y) = \sum_{k_A, k_B} P(\vec{k}) x^{k_A} y^{k_B}, \tag{36}
\]

where \(\vec{k} = (k_A, k_B)\). The critical condition can be obtained by substituting equations (32) and (33) into (25).

### 4. Numerical verification and simulation results

We perform extensive numerical simulations on artificial two-layered multiplex networks based on Erdős-Rényi (ER) [64] and Scale-Free (SF) [60] models. The results are obtained by averaging over at least \(10^4\) independent dynamical realizations on 30 artificial networks. In our simulations, information transmission rate in the two layers are set at \(\lambda_A = \lambda_B = \lambda\), and the recovery probability is \(\gamma = 1.0\). In addition, there are \(N = 10^4\) nodes in each layer, and the mean degrees of the two layers are set at \(\langle k_A \rangle = \langle k_B \rangle = 10\). At the final state the adopted
nodes all move into the recovered state, so we can measure the propagation range using the fraction of the recovered nodes at the steady state \( R(\infty) \).

To determine the threshold \( \lambda_c \) from simulations, we adopt the relative variance \( \chi \), which has been used widely and successfully to determine the epidemic thresholds \([9, 65]\). The expression of \( \chi \) is as follows

\[
\chi = \frac{\langle (R(\infty) - \langle R(\infty) \rangle)^2 \rangle}{\langle R(\infty) \rangle^2},
\]

where \( \langle \ldots \rangle \) is the ensemble average, which exhibits a diverging peak at the critical point.

4.1. Homogeneous two-layered network

We first investigate the effect of heterogeneous behavioral adoption on the dynamics of the complex behavior contagions on ER–ER network with Poisson degree distribution \( P_A(k) = e^{-k_0} \frac{(k_0)^k}{k!} \) in layer A and \( P_B(k) = e^{-k_0} \frac{(k_0)^k}{k!} \) in layer B.

In figure 1, we explore the effect of activists on the dynamics of the complex contagion, where the adoption thresholds in the two layers are set at \( T_A = 1 \) and \( T_B = 3 \), respectively. Figure 1(a) exhibits the values of the final adopted nodes as a function of information transmission rate \( \lambda \) for three typical values of fraction of activists \( q \).

We find that the adoption behavior can be enhanced by the fraction of activists, and there exist crossover phenomena of phase transition by tuning the proportion of the activists and conservatives. Specifically, the final adoption fraction \( R(\infty) \) increases with the increment of \( q \). In addition, when there is a relatively large fraction of activists in the network, i.e. \( q = 0.8 \) (blue triangles) and \( q = 0.5 \) (green squares), it shows continuous phase transition of the final adopted nodes \( R(\infty) \) at the critical value of transmission rate \( \lambda_c \). When the proportion of activists reduces to \( q = 0.2 \) (red circles), the transition of \( R(\infty) \) exhibits a hybrid pattern. In the hybrid phase,

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Effect of the fraction of activists \( q \) on the social contagion on ER–ER network. The phase transition changes from hybrid to continuous when increasing the fraction of activists \( q \). (a) The value of the fraction of final adopted nodes \( R(\infty) \) as a function of information transmission rate \( \lambda \) for \( q = 0.2 \) (red circles), \( q = 0.5 \) (green squares), and \( q = 0.8 \) (blue triangles). Lines are theoretical results computed from equations (1)–(7), and (14)–(17). (b) The relative variance \( \chi \) as a function of transmission rate \( \lambda \) for \( q = 0.2 \) (red solid line), \( q = 0.5 \) (green dash line), and \( q = 0.8 \) (blue dashed–dotted line). Other parameters are \( \gamma = 1.0 \), average degree \( \langle k \rangle = 10 \), \( T_A = 1 \), and \( T_B = 3 \).}
\end{figure}
transition, the value of $R(\infty)$ increases continuously at the first threshold, then grows slowly with the increase of $\lambda$ until it reaches the second threshold, and exhibits an abrupt discontinuous increase of $R(\infty)$ at the second threshold. To distinguish the two critical values of $\lambda$ in hybrid transitions, we denote $\lambda^I$ as the threshold where discontinuous phase transition occurs, and $\lambda^{II}$ as the threshold where continuous phase transition occurs. Lines in figure 1(a) are theoretical results obtained from equations (1)–(7), and (14)–(17), which agree well with the numerical simulations. Peaks of $\chi$ in figure 1(b) show the critical points obtained from simulations. We can also observe that the threshold $\lambda_2$ decreases with the increment of $q$.

The continuous and discontinuous transitions are caused by the relative proportion of the activists and conservatives in the network. The activists will adopt the behavior once the accumulated pieces of information in any layer exceed the corresponding threshold. The threshold in layer $A$ is $T_{A}= 1$, which means only one piece of information in layer $A$ will induce the adoption of the behavior. Thus when there are large fraction of activists in the networks, the final adoption number of nodes increases quickly with the increment of $\lambda$. While when the proportion of conservatives in the network increases, the adoption of the behavior becomes harder as they will adopt the behavior only when the pieces of information in both layers exceed the thresholds. In this scenario, the activists will adopt the behavior first and then stimulate the conservatives to adopt the behavior. For the conservatives, there exists a subcritical state, in which these conservatives have not adopted the behavior, but only one piece of information will induce the adoption of these nodes. Similar to the so called ‘powder keg’ in explosive percolation [59], the discontinuous phase transition appears when the number of information pieces of those nodes in subcritical state exceed the threshold simultaneously.

Note that the two critical values $\lambda^I$ and $\lambda^{II}$ for $q = 0.2$ separate the parameter space into three regions. When $\lambda \leq \lambda^I$, the behavior is locally adopted, i.e. only finite small fraction of nodes adopt the behavior. When $\lambda^I < \lambda \leq \lambda^{II}$, more and more activists adopt the behavior with the increase of $\lambda$. Thus there is a continuous increase of $R(\infty)$. At last when $\lambda > \lambda^{II}$, conservatives adopt the behavior simultaneously, leading to a discontinuous increase of $R(\infty)$. The relative variance $\chi$ is used to numerically locate the critical points, as shown in figure 1(b).

From the analysis above, it can be obtained that both the final adoption size $R(\infty)$ and the phase transitions are affected by parameters $q$ and $\lambda$. Thus the dependence of $R(\infty)$ on $q$ and $\lambda$ is studied in figure 2. Colors in figures 2(a) and (b) represent the values of $R(\infty)$. We find that the parameter plane $(\lambda, q)$ is divided into three regions by two critical values $q_1$ (white dotted line), $q_2$ (white line). In region I, where $q < q_1$, the proportion of activists is extremely small, blocking the propagation of information since there is not enough activists to propagate the information in the initial stage. Consequently, in this region, the global adoption of behavior cannot be stimulated and the final fraction of adopted nodes keeps an extremely small value no matter what value of $\lambda$ is. In region II, where $q_1 \leq q < q_{II}$, the proportion of activists increases to a relatively large value. In this region, the transition of $R(\infty)$ is continuous at $\lambda^{II}$ (see green squares in figure 2(a)), and then $R(\infty)$ increases slowly as more and more activists adopt the behavior. When $\lambda$ increases to $\lambda^I$, a considerable number of conservatives move into the subcritical state. Consequently when $\lambda > \lambda^I$, $R(\infty)$ increases abruptly to a large value, and a discontinuous phase transition occurs at $\lambda = \lambda^{II}$ (see red triangles in figure 2(a)). In region III, where $q > q_{II}$, the activists dominate the contagion dynamics. With the increase of $\lambda$, more and more activists adopt the behavior. At the same time conservatives will adopt the behavior gradually as a result of being stimulated by the activists. Consequently, $R(\infty)$ increases continuously along with $\lambda$. In addition, with the increment of the fraction of activists $q$, the adoption threshold $\lambda_2$ decreases. Our theoretical predictions of $\lambda^I$, $\lambda^{II}$ and $\lambda^I$ agree well with the numerical predictions, as shown in figure 2(b).

Moreover, we find that the average degree does not qualitatively alter the phenomena presented in figure 3. Specifically, the small fraction of activists $q$ induces hybrid phase transition and the large $q$ arouses a continuous phase transition. In addition, increasing average degree $\langle k \rangle$ will enlarge the spreading scale and decrease the outbreak threshold. Our theoretical method agrees with the above phenomena very well.

4.2. Heterogeneous two-layered network
We next study the effect of network structure on the phase transition of the social contagions. First of all, we focus on the ER–SF networks with average degree $\langle k_A \rangle = \langle k_B \rangle = 10$. For the ER–SF networks, layer $A$ is an ER network with Poisson degree distribution $P_k(k_A) = e^{-\langle k_A \rangle} \frac{(\langle k_A \rangle)^k}{k_A!}$, and layer $B$ is a SF network with scale-free degree distribution $P_k(k_B) = \zeta_B k_B^{-\nu_B}$, where $\zeta_B = 1/\sum_k k_B^{-\nu_B}$ and parameter $\nu_B$ represents the degree exponent of layer $B$. In the SF subnetwork, the minimum and maximum degree are set at $k_{min} = 4$ and $k_{max} = 100$, respectively.

To explore the effect of degree heterogeneity of layer $B$ on the dynamics of social contagions, we focus on three typical values of $\nu_B$, i.e. $\nu_B = 2.1$, 3 and 4. Figure 4 exhibits $R(\infty)$ as a function of $\lambda$ with different values of $q$. In figure 4, the adoption thresholds are $T_A = 1$ and $T_B = 3$. We find that similar to the case of ER–ER network, there exist crossover phenomena of phase transition on the networks with $\nu_B = 2.1$, 3 and 4.
Specifically, the transition type changes from continuous pattern when there is a large proportion of activists, e.g. $q = 0.5$ and $0.8$, to hybrid pattern with a small proportion, e.g. $q = 0.12$ for $v_B^B = 2.1$, $3$ (see red circles in figures 4(a1) and (b1)) and $q = 0.2$ for $v_B^B = 4$ (see red circles in figure 4(c1)). Lines correspond to the theoretical results. Figures 4(a2)–(c2) show the relative variance $\chi$ as a function of $\lambda$ when $v_B^B = 2.1$ (a2), $3$ (b2) and $4$ (c2).
Multiplex networks. We considered that individuals on social networks have different willingness to accept new behaviors. Thus, the fraction of individuals in the subcritical state decreases, and the growth of susceptible neighbors gradually adopt the behavior. As a result, in SF networks, there are few hubs, which induces the individuals to adopt the behavior simultaneouly. Individuals in the subcritical state means that his accumulative received information equals to $R_B = 1$. For SF networks, there are few hubs, which induces the individuals to adopt the behavior gradually. Thus, the fraction of individuals in the subcritical state decreases, and the discontinuous disappears.

At last, to further explore the impact of network structure on the dynamics of social contagions, we study the process of social contagions on SF networks. The degree distributions of layers $A$ and $B$ are $P_k(\lambda) = \zeta_k \delta^{\nu k}$ and $P_k(\nu) = \zeta_k \delta^{\nu B}$, where $\zeta_k = 1/\sum_{k} k^{\nu A}$, $\zeta_B = 1/\sum_{k} k^{\nu B}$, $\nu A$ and $\nu B$ indicate the degree exponents of layer $A$ and $B$. Figure 5 exhibits the numerical simulations (see the symbols) and theoretical solutions (see the lines), under $\nu_A = \nu_B = 2.1$ (a1), $3$ (b1), and $4$ (c1), respectively. In figure 5, the adoption thresholds are set as $T_A = 1$ and $T_B = 3$. Interestingly, we find that when the degree heterogeneity is strong enough, i.e. $\nu_A = \nu_B = 2.1$, the hybrid phase transition disappears (see figure 5(a1)), and there is only one continuous phase transition for all values of $q$ (see figure 5(a1)). When the power exponent $\nu_A = \nu_B$ increases, the hybrid phase transition appears when there is at a relatively small value of activists, e.g., $q = 0.1$ for $\nu_A = \nu_B = 3$ (see figures 5(b1) and (b2)) and $q = 0.2$ for $\nu_A = \nu_B = 4$ (see figures 5(c1) and (c2)). The simultaneous adoption of the behavior by a large fraction of individuals in the subcritical state raises the discontinuous growth of $R(\infty)$. With fixed average degree, the more heterogeneous the degree distribution, the more the nodes, respectively, with big degree and small degree. Since behavior propagates on complex networks in a hierarchical way [53], a small number of nodes with large degree make the susceptible neighbors gradually adopt the behavior. As a result, in SF network there hardly exist a fraction of nodes simultaneously in the subcritical state. Thus, the discontinuous growth of $R(\infty)$ disappears.

5. Discussions

In summary, we studied the effect of heterogeneity populations on the dynamics of social contagions on multiplex networks. We considered that individuals on social networks have different willingness to accept new ideas or behaviors, and then heterogeneity in populations appears. To represent the heterogeneity of the populations, we randomly selected a fraction of $q$ nodes in the network as activists. The remaining $1 - q$ nodes...
of phenomena of phase transition appear when tuning the value of thresholds. We assumed that the populations in different subnetworks have different adoption thresholds. We assigned two information are needed to convince a person on virtual social networks to adopt the behavior. Consequently, we different credibility, e.g. information transmitted among families or friends will be more credible than the

pattern at the an ER heterogeneity has signiﬁcantly effects on the dynamics of social contagions on multiplex networks. For the behavior spreading on multiplex networks, the

small value of \(\lambda\) for exponent parameter \(v^A = v^B = 2.1\) on both SF networks at subgraph (a1), for \(v^A = v^B = 3\) at (b1), and for \(v^A = v^B = 4\) at (c1). The simulated results are plotted with active node fraction \(q = 0.1 (0.2) \) (red circles), \(q = 0.5 \) (green squares), and \(q = 0.8 \) (blue triangles). The lines are theoretical values of \(R(\infty)\) computed from equations (1)–(7), and (14)–(17). Other parameters are \(\gamma = 1.0\), average degree \((k) = 10\), \(TA = 1\), and \(TB = 3\). The theoretical solutions agree well with numerical simulations. The relative variance \(\chi\) as a function of \(\lambda\) on both SF networks at subgraph (a2), for \(v^A = v^B = 3\) at (b2), and for \(v^A = v^B = 4\) at (c2).

are defined as conservatives. We also considered that information spreadsings in different networks have different credibility, e.g. information transmitted among families or friends will be more credible than the information transmitted on virtual social networks, such as Facebook or Twitter. Thus more pieces of information are needed to convince a person on virtual social networks to adopt the behavior. Consequently, we assumed that the populations in different subnetworks have different adoption thresholds. We assigned two thresholds \(TA\) and \(TB\) to the two layers of the network, respectively. Information is transmitted synchronously in the two layers of the network. The activists will adopt the behavior once the accumulated pieces of information they received in any layer exceed the corresponding threshold. While the conservatives will adopt the behavior only if the accumulated pieces of information in both layers of the network exceed the corresponding thresholds simultaneously.

To theoretically analyze the model, a generalized edge-based compartmental theory was established in our paper. Through theoretical analysis and simulation veriﬁcation, we found that the property of population heterogeneity has signiﬁcant effects on the dynamics of social contagions on multiplex networks. First of all, on an ER–ER multiplex network, the adoption behavior can be enhanced by the fraction of of activists. The final adoption size grows with the increment of \(q\), and the threshold decreases with \(q\). More importantly, crossover phenomena of phase transition appear when tuning the value of \(q\). When \(q\) is relatively large, the phase transition of ﬁnal adoption size \(R(\infty)\) increases continuously with transmission rate \(\lambda\). While when there is a relatively small value of \(q\), there is a hybrid transition of \(R(\infty)\). The value of \(R(\infty)\) versus \(\lambda\) ﬁrst exhibits a continuous pattern at the ﬁrst critical value \(\lambda^1\), and then follows a discontinuous pattern at the second critical value \(\lambda^2\). At last, we studied the effect of degree heterogeneity of the social contagions. On an ER–SF network, we ﬁnd that there are still crossover phenomena in phase transition when tuning the fraction of activists. However, on an SF–SF network, the crossover phenomenon in phase transition disappears when degree heterogeneity is strong enough.

Population heterogeneity is a crucial in studying the social contagion that is often overlooked in previous studies. By considering simultaneously the network multiplicity and the population heterogeneity in studying social contagion, we can better reveal the underlying mechanism of social contagions on complex networks. The main contribution of our study lies in providing a qualitative and quantitative view on the impact of heterogeneous populations and heterogeneous adoption threshold on the dynamics of social contagions. Our work enriches the studies about phase transition phenomenon, and our theory developed in this paper can offer new inspirations to the researches of other spreading dynamics, such as epidemic spreading, innovation spreading, marketing, and diffusion of computer virus. For the behavior spreading on multiplex networks, the number of subnetworks and the total number of the adoption neighbors in all subnetworks are extremely important, and deserve being investigated in the future researches.
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