Continuous spin particles from a string theory

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Abstract

It has been shown that the massless irreducible representations of the Poincaré group with continuous spin can be obtained from a classical point particle action which admits a generalization to a conformally invariant string action. The continuous spin string action is quantized in the BRST formalism. We show that the vacuum carries a continuous spin representation of the Poincaré group and that the spectrum is ghost-free.

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1 Introduction

In this article we study the string action

\[ S[X^\mu, h_{mn}] = \mu \int d^2 \sigma \sqrt{-h} \sqrt{\Box X^\mu \Box X^\nu \eta_{\mu\nu}}. \] (1)

It shares with the classical Polyakov action the two-dimensional local reparametrization and Weyl symmetries as well as the global target space Poincaré invariance. It also introduces a mass scale \( \mu \) similar to the string tension. It differs however in two important aspects: it is non-polynomial and it is a higher derivative action. The action (1) was first considered by Savvidy in [7, 8, 9] where it was conjectured to play a role in the tensionless limit of the Nambu-Goto-Polyakov action [13, 14, 15], it can be also motivated as a generalization of the point particle action [18] describing a massless particle belonging to the continuous spin representations of the Poincaré group [1, 2, 3, 4, 5, 6]. The quantization of the action (1), due to its non-polynomiality and higher derivative nature, is not straightforward. These two difficulties can be overcome if one introduces an auxiliary field, called \( \Xi^\mu \) in the following and a Lagrange multiplier. The action becomes second order and polynomial. More explicitly the classically equivalent action is

\[ S[X^\mu, \Xi^\mu, h_{mn}, \lambda] = -\mu \int d^2 \sigma \sqrt{-h} \left[ h^{mn} \partial_m X^\mu \partial_n \Xi^\nu \eta_{\mu\nu} + \lambda (\Xi^2 - 1) \right]. \] (2)

In turn, this action presents some difficulties: its kinetic term has signature \((D, D)\) signalling potential ghosts and the constraints that arise from this action are bilocal constraints. It was shown in [18] that one can transform these bilocal constraints to local one at the expense of breaking manifest Lorentz covariance. The quantization can then be performed with the standard BRST techniques [10, 11, 12]. It was initiated in [18] where it was found that the critical dimension of the theory is 28. It this article, we pursue this quantization in more details and find that:

- The ground state carries a continuous spin representation of the Poincaré group.
- The BRST-closed higher level states are all of zero norm.

This implies that the only physical state is the ground state. In this respect it has some similarities with the N=2 superstring [17] with a finite number of physical states. It differs however in that the signature of spacetime is \((1, D - 1)\). Although the manifest Lorentz covariance was broken to transform the constraints, the resulting spectrum is relativistic. The absence of ghosts in the spectrum, in spite of the \((D, D)\) signature of the sigma model (2), is due to the additional constraints contained in (2) and proves the consistency of the free theory.

This theory may prove to be useful to explore some properties of string theory. The states being all massless, it may be relevant to some very high energy regime of string theory. This remains to be explored. It is also of crucial importance to study interactions and to elucidate the issue of unitarity in the resulting amplitudes.
In this respect let us note that consistent interactions of particles belonging to the massless continuous spin representations are not known. An interesting possibility would be that string theory is needed in order to get consistent interactions. The theory may also be considered as a simple toy model for the quantization of systems with higher derivatives or bilocal constraints.

The plan of this paper is the following. In Section 2 we briefly recall the derivation of the constraints. Section 3 is devoted to the BRST quantization where we calculate the critical dimension and the normal ordering constant. The latter is crucial for the ground state to belong to the continuous spin representation. In Section 4 we show that the higher level BRST-closed states are all of zero norm. The proof that the spectrum is ghost-free relies heavily on the requirement that the zero modes of the antighosts annihilate the physical states. Finally, Section 5 contains the conclusions.

2 String action and associated constraints

The classical point particle action [18] (for earlier treatments see [5], see also [19]) for the dynamical variables \(x^\mu\), the space-time position of the particle, and \(\xi^\mu\), the “internal” degree of freedom,

\[
S[x^\mu, \xi, e, \lambda] = \int d\tau \frac{\dot{x} \cdot \dot{\xi}}{e} + e\tilde{\mu} + \lambda(\xi^2 - 1),
\]

(3)
gives rise to the constraints

\[
p \cdot q - \tilde{\mu} = 0, \quad \xi^2 - 1 = 0, \quad \xi \cdot p = 0, \quad p^2 = 0.
\]

(4)

Here \(p\) and \(q\) are the conjugate momenta to \(x\) and \(\xi\) respectively. These constraints are precisely those used by Wigner [2] to describe in a manifestly covariant way the massless continuous spin representation \(^4\). The generalization of (3) to a two-dimensional world-sheet is given by

\[
S[X^\mu, \Xi^\mu, h_{mn}, \lambda] = -\mu \int d^2\sigma \sqrt{-h} \left[ h^{mn} \partial_m X^\mu \partial_n \Xi^\nu \eta_{\mu\nu} + \lambda(\Xi^2 - 1) \right],
\]

(5)

where \(h_{mn}\) is the two-dimensional metric and the two target space coordinates \(X\) and \(\Xi\) depend on the two world-sheet coordinates \(\sigma^0\) and \(\sigma^1\) with \(\sigma^1 = \sigma^1 + 2\pi\). The action is classically invariant under reparametrizations and Weyl rescaling of the metric. The equations of motion are

\[
\Box \Xi^\mu = 0, \quad \Box X^\mu = 2\lambda \Xi^\mu,
\]

(6)

\(^4\)Recall that these representations have \(p^2 = 0\) and \(M_{\mu\nu} p^\nu M^{\mu\rho} p_\rho = \tilde{\mu}^2\), where \(p^\mu\) and \(M_{\mu\nu}\) are the generators of translations and Lorentz transformations and \(\tilde{\mu}\) is the nonvanishing continuous spin parameter. Notice that \(\tilde{\mu}\) is not necessarily positive.
where $\Box = \frac{1}{\sqrt{-h}} \partial_m \sqrt{-h} h^{mn} \partial_n$. The primary constraints obtained by varying with respect to $\lambda$ and $h_{mn}$ are
\begin{align*}
(\Xi^2 - 1) &= 0, \\
\partial_m X^\mu \partial_n \Xi_\mu + \partial_n X^\mu \partial_m \Xi_\mu - h_{mn} \partial_i X^\mu \partial^i \Xi_\mu &= 0.
\end{align*}

Using (6) and (7) we get for $\lambda$
\begin{equation}
(2\lambda)^2 = (\Box X)^2,
\end{equation}

The second equation in (6) together with (9) can be used to determine $\Xi$ in terms of $X$ as
\begin{equation}
\Xi = \frac{\Box X}{\sqrt{(\Box X)^2}}.
\end{equation}

We can eliminate $\Xi$ from the action to get the higher derivative action
\begin{equation}
S = -\mu \int d^2 \sigma \sqrt{-h} h^{mn} \partial_m X^\mu \partial_n \frac{\Box X_\mu}{\sqrt{(\Box X)^2}}
\end{equation}
\begin{equation} = \mu \int d^2 \sigma \sqrt{-h} \sqrt{\Box X^\mu \Box X_\mu},
\end{equation}

which is the form proposed in [7].

It was shown in [18] that, in the conformal gauge, in addition to the primary constraints
\begin{align*}
\mathcal{H}_0 &= P.Q + \mu^2 \partial_i X \partial_i \Xi = 0 \\
\mathcal{H}_1 &= P.\partial_i X + Q.\partial_i \Xi = 0, \\
\phi_1 &= \Xi^2 - 1 = 0,
\end{align*}

the string action gives rise to the secondary constraint
\begin{equation}
P.\Xi(\sigma),
\end{equation}
and the bilocal constraints
\begin{equation}
(P(\sigma) - \mu \partial_i \Xi(\sigma))(P(\sigma') + \mu \partial_i \Xi(\sigma')) = 0.
\end{equation}
Here $P$ and $Q$ are respectively the conjugate momenta to $X$ and $\Xi$. In the following we shall use the definitions $P_R(\sigma) = P + \mu \partial_i \Xi$; $P_L(\sigma) = P - \mu \partial_i \Xi$ of the right and left momenta and similarly for $Q_R$ and $Q_L$ so that the first two constraints in (12) become $P_R.Q_R = P_L.Q_L = 0$.

We start by analyzing the bilocal constraint (14) rewritten as $P_R(\sigma).P_L(\sigma') = 0$. Let $V_R$ be the vector space spanned by $P_R(\sigma)$ when $\sigma$ varies from 0 to $2\pi$ and
similarly for $V_L$, then $V_R$ and $V_L$ are orthogonal. Let $p$ be the common zero mode of $P_R$ and $P_L$. By taking the integral on both $\sigma$ and $\sigma'$ of the constraint (14) we get that $p^2 = 0$. All the string modes are thus massless. Furthermore $p$ is contained in both $V_R$ and $V_L$. Suppose that the only nonvanishing component of $p$ is $p^+ \, ^5$, and split the spacelike and transverse indices $i = 1, \ldots D - 2$ into $a$ which belong to $V_R$ and $a'$ which belong to $V_L$. We thus have

$$P^a_L(\sigma) = 0, \quad a = 1, \ldots N, \quad P^a_R(\sigma) = 0, \quad a' = N + 1, \ldots D - 2,$$

(15)

and since $p^+$ is in both $V_R$ and $V_L$ we also have

$$P_L(\sigma) = 0 = P_R(\sigma).$$

(16)

The latter two constraints are equivalent to $\Xi^- = \xi^-$ and $P^- = 0$, with $\xi^-$ a constant zero mode. We have thus transformed the bilocal constraints into local ones at the expense of breaking the manifest Lorentz covariance.

The remaining constraints are

$$P.\Xi = 0, \quad \Xi^2 - 1 = 0.$$

(17)

We wish to transform them, using (14), into constraints involving only left movers or right movers. We shall be able to do so except for a zero mode which involves the sum of right moving and left moving variables. We have

$$P = \frac{P_R + P_L}{2}, \quad \Xi' = \frac{P_R - P_L}{2\mu}.$$

(18)

Let $\hat{P}_R$ and $\hat{P}_L$ be the nonzero mode parts of $P_R$ and $P_L$ and define $\hat{P}_R$ and $\hat{P}_L$ by $\hat{P}_R' = \hat{P}_R$ the integration constant being chosen so that $\hat{P}_R$ has no zero mode. Integrate the second equation in (18) as

$$\Xi(\sigma) = \frac{\hat{P}_R - \hat{P}_L}{2\mu} + \xi,$$

(19)

Here $\xi$ is a constant $D$-vector. Notice that we have $\hat{P}_R.\hat{P}_L = 0$ and $P_R.\hat{P}_L = 0$.

Using the derivative of $\phi_1$, $\Xi'.\Xi = 0$, we get

$$P_R.\Xi = 0, \quad P_L.\Xi = 0,$$

(20)

which yield

$$G = \xi.P_R + \frac{1}{2\mu} \hat{P}_R.P_R(\sigma) = 0,$$

(21)

\footnote{We are using here the light-cone coordinates $V^\pm = (V^0 \pm V^{D-1})/\sqrt{2}$.}
and
\[ \tilde{G} = \xi P_L - \frac{1}{2\mu} \hat{P}_L P_L(\sigma) = 0, \]  
(22)

Notice that \( G \) and \( \tilde{G} \) have the same zero mode
\[ G_0 = \frac{\xi p}{2\pi} = -\frac{p^+ \xi^-}{2\pi}. \]  
(23)

We have thus obtained one left moving and one right moving constrain ts. It remains to take into account the zero mode constraint contained in \( \phi_1 \). This is accomplished by
\[ g = \xi^2 + \frac{1}{4\mu^2} \int d\sigma \frac{d\sigma}{2\pi} (\hat{P}_R^2 + \hat{P}_L^2) - 1 = 0. \]  
(24)

Notice that it is the sum of left moving variables and right moving ones.

### 3 BRST quantization

Recall that for a system with first class constraints \( G_i \) which form a Lie algebra \( [G_i, G_j] = C_{ij}^k G_k \) the first step in the BRST quantization is the enlargement of the Hilbert space by the introduction of the ghosts \( c^i \) and antighosts \( b_j \) which verify \( \{ b_i, c^j \} = \delta^j_i, \{ c^i, c^j \} = \{ b_i, b_j \} = 0 \). The BRST charge is defined by \( Q = c^i G_i - \frac{1}{2} C_{ij}^k c^i c^j b_k \), and verifies \( Q^2 = 0 \). The physical states are in the kernel of \( Q \) and two states are equivalent if they differ by an exact state, i.e. of the form \( Q|\phi> \) for some \( |\phi> \).

In string theory, the classical constraints form a closed Lie algebra. When one turns the dynamical variables into operators, an anomaly appears in the commutators of the energy-momentum tensor. This anomaly has two contributions: the first is dependent on the normal ordering constant in the energy-momentum tensor and the second is the central charge. More precisely, if one defines the normal ordered matter energy-momentum tensor by \( T^{(m)} = \sum \partial^2 R : T_{R} : \) with \( a^m \) a constant, then it obeys the commutation relations
\[ [T^{(m)}(\sigma), T^{(m)}(\sigma')] = 2\pi i(T^{(m)}(\sigma) + T^{(m)}(\sigma')) \delta'(\sigma - \sigma') \]
\[ - 4\pi a^m \delta'(\sigma - \sigma') + 2\pi i \frac{c^m}{12} \delta''''(\sigma - \sigma'), \]  
(25)

where \( c^m \) is the central charge of the matter sector given by \( 2D \).

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\(^6\)We are using the canonical commutation relations \([X^\mu(\sigma), P^\nu(\sigma')] = i\eta^{\mu\nu} \delta(\sigma - \sigma') \) and so on.
The other constraints, for the right moving part, are given by

\[ P_a'(\sigma) = 0, \quad P_R^- (\sigma) = 0, \]

\[ G = \xi P_R + \frac{1}{2\mu} \dot{P}_R . P_R (\sigma) = 0. \]  

One has also to add the zero mode constraints \( g = 0 \). The constraints (26-28) commute among each other and with \( g \) and they have conformal weights equal to one, that is if we denote generically one of the constraints (26-27) by \( K \) then we have

\[ \left[ T^{(m)} (\sigma'), K (\sigma) \right] = -2\pi i K' (\sigma) \delta (\sigma - \sigma') - 2\pi i K (\sigma) \delta' (\sigma - \sigma'). \]  

We also have

\[ [T (\sigma), g] = -\frac{i}{\mu} G (\sigma), \]

and

\[ \left[ T^{(m)} (\sigma'), G (\sigma) \right] = -2\pi i G' (\sigma) \delta (\sigma - \sigma') - 2\pi i G (\sigma) \delta' (\sigma - \sigma') + i\delta (\sigma - \sigma') \frac{p_+}{2\mu} P_R^- (\sigma). \]  

Equation (31) means that \( G \) is weakly a field of conformal weight 1. Let the ghosts fields associated to the constraints (26-28) and \( T^{(m)} \) be denoted respectively by \( c_{a'}, c_-, d \) and \( c \) and the corresponding antighosts by \( b^{a'}, b^-, e \) and \( b \). All the ghosts except \( c \) have conformal weights 0 and \( c \) has the conformal weight \(-1\). Let the ghost and antighost associated to \( g \) be denoted by \( \gamma \) and \( \omega \). These are not fields, they have only zero modes. The naive BRST charge \( Q \) which results when ignoring the anomalous terms in the commutation relations of the energy-momentum tensor reads

\[
Q = \int \frac{d\sigma}{2\pi} \left[ c(\sigma) T^{(m)} + c_{a'} P_R^{a'} + c_- P_R^- + dG \right] + \gamma g \\
+ \int \frac{d\sigma}{2\pi} : c(\sigma) \left[ \frac{1}{2} T^{(c)} + T^{(c_{a'})} + T^{(c_-)} + T^{(d)} \right] - \frac{i}{2\pi\mu} \gamma e - \frac{ip^+}{4\pi\mu} db^- + a \right]:, \quad (32)
\]

\( T^{(c)} \) is the energy-momentum tensor of the ghost system \( c \) and \( b \) and so on for the other terms in (32). The energy-momentum tensor of the weight \( h \) ghost \( \gamma \) system is given by

\[ T_h = -i : [h \partial (c_h b_h) - c_h \partial b_h] : -\frac{1}{12} + a^h, \]

\( \gamma \)Here \( h \) is the conformal weight of the antighosts \( b_h \) which is the same as that of the corresponding constraint; we use the conventions \( \{ c_h (\sigma), b_h (\sigma') \} = 2\pi \delta (\sigma - \sigma') \).
which satisfies commutation relations analogous to (25) with $a^m$ replaced by $a^h$ and a central charge given by $[16, 17] c_h = 1 - 3(2h - 1)^2$. In (32), we also allowed for a normal ordering constant $a$. The BRST charge depends on the normal ordering constants only through the combination $a^m + \sum a^h + a$. Without loss of generality it is thus possible to choose $a^m = a^h = 0$.

The left moving BRST charge is similarly defined with its corresponding left ghosts and antighosts except for $\gamma$ and $\omega$ and the zero modes of $d$ and $e$ which are the same. This is due to the structure of $g$ as a sum of left movers and right movers and the fact that the zero modes of $G$ and $\tilde{G}$ coincide.

The crucial property of $Q$ is its nilpotency. The calculation of $Q^2$, using the full commutation relations gives

\[ Q^2 = -\frac{c_T}{24} \int \frac{d\sigma}{2\pi} (\partial^3 c(\sigma)) c(\sigma) - ia \int \frac{d\sigma}{2\pi} c(\sigma) \partial c(\sigma), \]

(34)

where $c_T = c^m + \sum c_h$ is the total central charge of the matter and ghost system which is given by $2D - 26 - 2(D - N)$. Thus the nilpotency of $Q$ requires that the total central charge $c_T$ and the normal ordering constant $a$ vanish. A similar conclusion is of course valid for the left moving sector whose total central charge is given by $2D - 26 - 2(N + 2)$. The vanishing of both central charges gives $D = 28$ and $N = 13$.

### 4 Spectrum

The physical states are equivalence classes of states annihilated by $Q$, two states being equivalent if their difference is $Q$-exact. One has also to add some supplementary conditions originating from the ghosts zero modes. Notice that among the ghosts $c, d$ and of course $\gamma$ together with their antighosts have zero modes. This results in a degeneracy of the ground state. By analogy with the usual string theory [17] we impose that the zero modes of the antighosts annihilate the physical states

$$ b_0 |\Psi\rangle = \tilde{b}_0 |\Psi\rangle = e_0 |\Psi\rangle = w |\Psi\rangle = 0. $$

(35)

This allows to impose correctly the zero mode constraints on the physical states.

It will be convenient to use the following Fourier expansions

\[ P_R(\sigma) = \frac{p}{2\pi} + 2\mu \sum_{n\neq 0} \beta_n e^{in\sigma}, \quad Q_R(\sigma) = \frac{q}{2\pi} + \frac{1}{2\pi} \sum_{n\neq 0} \alpha_n e^{in\sigma}, \]

(36)

\[ c(\sigma) = \sum_{n=-\infty}^{+\infty} c_n e^{in\sigma}, \quad b(\sigma) = \sum_{n=-\infty}^{+\infty} b_n e^{in\sigma} \]

(37)

\[ T(\sigma) = \sum_{n=-\infty}^{+\infty} L_n e^{in\sigma}, \quad G(\sigma) = \sum_n G_n e^{in\sigma} \]

(38)
\[ \hat{P}_R(\sigma) = -2i\mu \sum_{n \neq 0} \frac{\beta_n}{n} e^{in\sigma}. \] (39)

The Fourier coefficients of the fields satisfy the commutation relations
\[ [\beta^\mu_n, \alpha^\nu_m] = m\eta^{\mu\nu} \delta_{n+m,0}, \quad \{c_n, b_m\} = \delta_{n+m,0}, \quad [q^\nu, \xi^\mu] = -i\eta^{\mu\nu}, \] (40)
the others being zero. We have
\[ L^{(m)}_0 = \frac{p,q}{4\pi\mu} + \sum_{n \neq 0} :\beta_n,\alpha_{-n}:+ \left( \frac{7}{3} \right), \quad L^{(m)}_m = \frac{p,\alpha_m}{4\pi\mu} + q,\beta_m + \sum_{n \neq 0, m} \beta_n,\alpha_{m-n}, \]
\[ L^{(ch)}_0 = \sum_n n :c_n b_{-n}: - \frac{1}{12}, \quad L^{(ch)}_m = \sum_n [(h-1)m+n] c_n b_{m-n}, \]
\[ G_n = 2\mu (\xi,\beta_n - i \sum_{m \neq 0} \frac{\beta_m}{m} \beta_{n-m}). \] (41)

In equation (41) we used \( \beta_0 = p/(4\pi\mu) \).

Notice that the total normal ordering constant coming from the ghost sector is \( (D + 4)/24 = -2/3 \) which when added to that of the matter sector gives 1.

If one defines similarly the left moving Fourier modes and denotes them with a tilde then the total BRST charge, including left and right contributions, reads
\[ Q = \sum_n c_n (L_{tot}^{(m)} - \tilde{L}_{tot}^{(m)}) + 2\mu \left( \sum_{n \neq 0} c_n \beta_{-n}^\prime + \sum_{n \neq 0} c_{-n} \beta_{-n}^\prime \right) + \sum_n d_n G_{-n} \]
\[ + \sum_n \tilde{c}_n (\tilde{L}_{tot}^{(m)} - \tilde{L}_{tot}^{(m)}) + 2\mu \left( \sum_{n \neq 0} \tilde{c}_n \tilde{\beta}_{-n}^\prime + \sum_{n \neq 0} \tilde{c}_{-n} \tilde{\beta}_{-n}^\prime \right) + \sum_{n \neq 0} \tilde{d}_n \tilde{G}_{-n} \]
\[ - \frac{i p^+}{4\pi\mu} \left[ \sum_{n,m \neq 0} c_n d_m b_{-(n+m)}^- + \tilde{c}_n \tilde{d}_m \tilde{b}_{-(n+m)}^- \right] \]
\[ + \gamma \left[ g + \frac{i}{2\pi\mu} \left( \sum_{n \neq 0} (c_{-n} e_n + \tilde{c}_{-n} \tilde{e}_n) + (c_0 + \tilde{c}_0) e_0 \right) \right] \]
\[ + d_0 \left[ G_0 + \frac{i p^+}{4\pi\mu} \sum_{n \neq 0} c_n b_{-n}^- + \tilde{c}_n \tilde{b}_{-n}^- \right], \] (42)

where
\[ g = \xi^2 - 1 + \sum_{n \neq 0} \left( \frac{\beta_n \beta_{-n}}{n^2} + \frac{\tilde{\beta}_n \tilde{\beta}_{-n}}{n^2} \right). \] (43)

Notice that as anticipated \( \gamma \) and \( d_0 \) multiply sums of right movers and left movers. The vacuum state is defined by
\[ \alpha_n^\mu |0 >= \beta_n^\mu |0 >= c_n^{(h)} |0 >= b_n^{(h)} |0 >= 0, \quad \forall n > 0 \] (44)
and for all the ghosts labelled by $h$ here. The BRST operator acting on a vacuum state gives

$$Q|0> = \left[ c_0 \left( \frac{p.q}{4\pi\mu} + 1 \right) + d_0 G_0 + \gamma \left( g + \frac{i}{2\pi\mu} c_0 e_0 \right) \right] |0>.$$  \hspace{1cm} (45)

If we now use the supplementary conditions (35) then we get the following constraints on the zero-mode part of the state

$$(p.q + 4\pi\mu) |0> = 0, \quad \xi.p |0> = 0, \quad (\xi^2 - 1) |0> = 0.$$  \hspace{1cm} (46)

These are precisely the conditions (4) defining a continuous spin state [2].

The additional zero mode constraints (35), when anticommutated with the BRST charge $Q$ give the compatibility conditions:

$$L_{0}^{tot} |\Psi> = 0, \quad \tilde{L}_{0}^{tot} |\Psi> = 0$$  \hspace{1cm} (47)

$$\left[ G_0 + i \frac{p^+}{4\pi\mu} \sum_{n \neq 0} c_n \bar{b}_{-n} + \bar{c}_n \bar{b}_{-n} \right] |\Psi> = 0, \quad (48)$$

$$\left[ g + i \frac{1}{2\pi\mu} \sum_{n \neq 0} (c_{-n} e_n + c_{-n} \bar{e}_n) \right] |\Psi> = 0. \quad \hspace{1cm} (49)$$

Two states are equivalent if they differ by a $Q$ exact state of the form $Q|\Phi>$. The constraints (35) imply that $|\Phi>$ is not arbitrary but has to verify

$$b_0|\Phi> = 0, \quad \tilde{b}_0|\Phi> = 0, \quad L_{0}^{tot} |\Phi> = 0, \quad \tilde{L}_{0}^{tot} |\Phi> = 0$$  \hspace{1cm} (50)

$$w|\Phi> = 0, \quad [g + i \frac{1}{2\pi\mu} \sum_{n \neq 0} (c_{-n} e_n + c_{-n} \bar{e}_n)]|\Phi> = 0$$  \hspace{1cm} (51)

$$e_0|\Phi> = 0, \quad [G_0 + i \frac{p^+}{4\pi\mu} \sum_{n \neq 0} c_n \bar{b}_{-n} + \bar{c}_n \bar{b}_{-n}]|\Phi> = 0.$$  \hspace{1cm} (52)

Let us first determine the physical states with only matter excitations, the ghosts being in the ground state. The condition $Q|\psi> \otimes |0> = 0$ gives for $|\psi>$, the state in the matter sector,

$$\left( p.q + 4\pi\mu(\sum_{n \neq 0} : \beta_n \cdot \alpha_{-n} : + 1) \right) |\psi> = 0, \quad L_n^{(m)} |\psi> = 0, \quad n > 0$$  \hspace{1cm} (53)

$$\beta_n^o |\psi> = 0, \quad n > 0, \quad \beta_n^- |\psi> = 0, \quad n > 0$$  \hspace{1cm} (54)

$$G_n |\psi> = 0, \quad n \geq 0, \quad g |\psi> = 0.$$  \hspace{1cm} (55)

There are of course similar conditions for the left moving sector. The first condition determines the continuous spin parameter as $p.q = 4\pi\mu(N - 1)$, where $N$ is the
level of the state. Notice that for \( N = 1 \), we have a reducible representation of the Poincaré group containing an infinite number of fixed helicity states.

Consider in more details the first level states

\[
|\psi> = (A_\mu \alpha_{-1}^\mu + B_\mu \beta_{-1}^\mu)|0>, \tag{56}
\]

The \( L_0 \) condition gives \( p.q = 0 \), this state belongs to standard helicity representations. The \( L_1 \) condition yields

\[
\frac{p.B}{4\pi\mu} + q.A = 0. \tag{57}
\]

The \( G_0 \) and \( G_1 \) conditions give

\[
\xi.p = 0, \quad (\xi - \frac{i}{4\pi\mu}p)^\mu A_\mu = 0. \tag{58}
\]

The \( \beta_1 \) conditions implies

\[
A_+ = 0 = A_{\nu'}. \tag{59}
\]

It remains to consider the \( g \) condition which gives

\[
(\xi^2 - 1)A_\mu = 0, \quad (\xi^2 - 1)B_\mu = A_\mu. \tag{60}
\]

The nonzero solution of equation (60) is

\[
(\xi^2 - 1) = 0, \quad A_\mu = 0. \tag{61}
\]

The first level physical state is thus

\[
|\psi> = B_\mu \beta_{-1}^\mu|0>, \tag{62}
\]

with \( p.B = 0 \). The norm of the state is proportional to \( A_\mu B^\mu \) which is zero. The standard helicity states are thus of zero norm.

Notice that the most constraining condition was \( g|\psi> = 0 \) which implied alone the vanishing of \( A_\mu \). We shall prove that this is also true for the higher level states, that is the states annihilated simultaneously by \( L_0 \) and \( g \) do not contain \( \alpha \) oscillators and are thus of zero norm. Define \( N_\alpha \) by \( -\sum_{m=1}^\infty \alpha_{-m}.\beta_m/m \) and \( N_\beta = -\sum_{m=1}^\infty \beta_{-m}.\alpha_m/m \). \( N_\alpha \) counts the number of \( \alpha \) oscillators acting on the ground state and \( N_\beta \) counts the number of \( \beta \) oscillators, for example if

\[
|\phi> = \alpha_{-n_1}^\mu \ldots \alpha_{-n_n}^\mu |0>, \tag{63}
\]

with all the \( n_i \) strictly positive then

\[
N_\alpha |\phi> = n|\phi>. \tag{64}
\]
Let $A$ be the part of $g$ depending on the right oscillators $A = \sum_{m \neq 0} \beta_m, \beta_{-m}/m^2$. Then $A$ decreases the number of $\alpha$ oscillators by one and increases the number of $\beta$ oscillators by one:

$$[N_\alpha, A] = -A, \quad [N_\beta, A] = A. \tag{65}$$

We want to show that an eigenstate of $A$ with a finite number of oscillators (an eigenstate of $L_0$) has necessarily eigenvalue zero and does not contain $\alpha$ oscillators. For this decompose the eigenstate as

$$|\psi > = \sum_{n_\alpha = 0}^N |\psi_{n_\alpha} >, \tag{66}$$

with $|\psi_{n_\alpha} >$ a state containing $n_\alpha$ $\alpha$ oscillators acting on the vacuum, and $A|\psi > = a|\psi >$, $a$ being the eigenvalue. Since $A$ decreases the number of $\alpha$ oscillators by one, we have

$$A|\psi > = \sum_{n_\alpha = 1}^N |\chi_{n_\alpha - 1} >, \tag{67}$$

where $|\chi_{n_\alpha - 1} > = A|\psi_{n_\alpha} >$. Equation (67) implies that $|\chi_{n_\alpha} > = a|\psi_{n_\alpha} >$ for $n_\alpha = 0, \ldots N - 1$ and $a|\psi_N > = 0$. Suppose that $a$ is not zero then $|\psi_N > = 0$ and $a|\psi_{N - 1} > = |\chi_{N - 1} > = A|\psi_N > = 0$ and so $|\psi_{N - 1} >$ vanishes and all the other $|\psi_n >$ are zero. So $a$ is necessarily 0 and $|\psi >$ does not contain any $\alpha$ oscillator. We conclude from the $g$ condition that physical states are linear combinations of states of the form

$$|\beta_{-n_1}^\mu \ldots \beta_{-n_N}^\mu |0 >, \tag{68}$$

and these are all of zero norm.

Consider now the general state in the matter and ghosts Hilbert space. The condition $g|\psi > = 0$ is now replaced by (49) which implies again that the state does not contain $\alpha$ oscillators and also that the state does not contain $b$ nor $d$ oscillators. The absence of $\alpha$ oscillators implies that a physical state of non-zero norm is necessarily of the form $|0 > \otimes |\chi >$, where $|\chi >$ depends only on ghosts excitations and $|0 >$ is the ground state of the matter sector. The physical state condition implies, in addition, that $|\chi >$ does not contain $b, b^+, b^-$ or $e$ fermionic oscillators. In turn, this implies that $|\chi >$ has a positive norm. This completes the proof that there are no physical states with a strictly negative norm.

## 5 Conclusion

Our main result is the absence of ghosts in the spectrum and the presence of a physical state carrying the continuous spin representation. These results do not seem to be dependent on the particular way we used to handle the bilocal constraint,
that is on its replacement by a number of equivalent local ones in a given frame. Notice in this respect that the physical spectrum we obtained is Lorentz covariant. It would be interesting to treat the bilocal constraint in a manifestly covariant way. The results are however strongly dependent on the requirement that physical states are annihilated by the zero modes of the antighosts which seems to be the key point in this BRST quantization. Our results prove the consistency of the free theory. This is but the first step towards a consistent theory with interactions where the role of the critical dimension and the zero norm states should be very important.

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References

[1] E. P. Wigner, Annals Math. 40 (1939) 149 [Nucl. Phys. Proc. Suppl. 6 (1989) 9].

[2] E.P. Wigner, Z. Physik 124 (1947) 665; V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. 34, 211 (1948).

[3] J. Yngvason, Commun. Math. Phys. 18 (1970) 195; A. Chakrabarty, J. Math. Phys. 12, 1813 (1971); G.J. Iverson and G. Mack, Annals of Physics 64 (1971) 211.

[4] L. F. Abbott, Phys. Rev. D 13, 2291 (1976); K. Hirata, Prog. Theor. Phys. 58 (1977) 652; J. Mund, B. Schroer and J. Yngvason, Phys. Lett. B 596 (2004) 156 [arXiv:math-ph/0402043].

[5] D. Zoller, Class. Quant. Grav. 11, 1423 (1994).

[6] L. Brink, A. M. Khan, P. Ramond and X. z. Xiong, J. Math. Phys. 43 (2002) 6279 [arXiv:hep-th/0205145].

[7] G. K. Savvidy, Phys. Lett. B 552 (2003) 72.

[8] G. K. Savvidy, Int. J. Mod. Phys. A 19 (2004) 3171 [arXiv:hep-th/0310085]. I. Antoniadis and G. Savvidy, arXiv:hep-th/0402077.

[9] G. Savvidy, arXiv:hep-th/0409047; G. Savvidy, arXiv:hep-th/0502114.
[10] C. Becchi, A. Rouet and R. Stora, Annals Phys. 98 (1976) 287; I.V. Tyutin, Lebedev Institute preprint N39 (1975). E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. B 55 (1975) 224. I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B 69 (1977) 309.

[11] M. Kato and K. Ogawa, Nucl. Phys. B 212 (1983) 443.

[12] M. Henneaux, Phys. Rept. 126 (1985) 1.

[13] A. Schild, Phys. Rev. D 16 (1977) 1722.

[14] A. Karlhede and U. Lindstrom, Class. Quant. Grav. 3, L73 (1986); F. Lizzi, B. Rai, G. Sparano and A. Srivastava, Phys. Lett. B 182 (1986) 326; J. Isberg, U. Lindstrom and B. Sundborg, Phys. Lett. B 293, 321 (1992) [arXiv:hep-th/9207005]; U. Lindstrom, B. Sundborg and G. Theodoridis, Phys. Lett. B 253, 319 (1991); S. Hassani, U. Lindstrom and R. von Unge, Class. Quant. Grav. 11, L79 (1994); JHEP 0201, 034 (2002) [arXiv:hep-th/0112206].

[15] B. Sundborg, Nucl. Phys. Proc. Suppl. 102 (2001) 113 [arXiv:hep-th/0103247]; U. Lindstrom and M. Zabzine, Phys. Lett. B 584 (2004) 178 [arXiv:hep-th/0305098]; G. Bonelli, Nucl. Phys. B 669 (2003) 159 [arXiv:hep-th/0305155]; A. Sagnotti and M. Tsulaia, Nucl. Phys. B 682 (2004) 83 [arXiv:hep-th/0311257].

[16] D. Friedan, E. J. Martinec and S. H. Shenker, Nucl. Phys. B 271 (1986) 93.

[17] M. Green, J. Schwarz and E. Witten, Superstring theory, Vol. 1, Cambridge university press (1987), chap. 3; J. Polchinski, String theory, Vol. 1, Cambridge university press (1998).

[18] J. Mourad, “Continuous spin and tensionless strings,” arXiv:hep-th/0410009.

[19] L. Edgren, R. Marnelius and P. Salomonson, arXiv:hep-th/0503136.