Low Rank Vectorized Hankel Lift for Matrix Recovery via Fast Iterative Hard Thresholding

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Abstract—We propose a VHL-FIHT algorithm for matrix recovery in blind super-resolution problem in a non-convex schema. Under certain assumptions, we show that VHL-FIHT converges to the ground truth with linear convergence rate. Numerical experiments are conducted to validate the linear convergence and effectiveness of the proposed approach.

Index Terms—Fast Iterative Hard Thresholding, Vectorized Hankel Lift, Blind Super-resolution.

I. INTRODUCTION

MATRICES recovery considers the problem of reconstructing a data matrix from a small number of measurements. This problem has many applications, such as recommendation system [12], X-ray crystallography [13], quantum tomography [14] and blind deconvolution [15], etc.

Motivated by blind super-resolution of point sources, we study a different low rank structured matrix recovery problem, where the target data matrix can be modeled as $X^k = \sum_{k=1}^{s} d_k h_k a_k^T \in \mathbb{C}^{s \times n}$. Here $h_k$ is normalized vector such that $\|h_k\|_2 = 1$ and $a_r$ is a vector defined as

$$a_r := \begin{bmatrix} 1 & e^{-2\pi ir} & \cdots & e^{-2\pi ir(n-1)} \end{bmatrix}^T.$$  

The linear measurements of $X^k$ is given by

$$y = A(X^k),$$

where $A : \mathbb{C}^{s \times n} \rightarrow \mathbb{C}^n$ is a linear operator that performs the linear observation $y[j] = \langle b_j e_j^T, X^k \rangle$ and $b_j \in \mathbb{C}^s$ is a random vector. Let $\mathcal{H}$ be the vectorized Hankel lift operator $\mathcal{H}$ which maps a matrix $\mathbb{C}^{s \times n}$ into an $sn_1 \times n_2$ vectorized Hankel matrix,

$$\mathcal{H}(X) = \begin{bmatrix} x_1 & \cdots & x_{n_2} \\ \vdots & \ddots & \vdots \\ x_{n_1} & \cdots & x_{n_1+n_2-1} \end{bmatrix}$$

where $x_i \in \mathbb{C}^s$ is the $i$-th column of $X$ and $n_1 + n_2 = n + 1$. We will investigate matrix recovery problem based on the low rank structure of vectorized Hankel matrix $\mathcal{H}(X^k)$. Specifically, assuming

$$\operatorname{rank}(\mathcal{H}(X^k)) = r.$$  

We are particularly interested in under what condition can the data matrix be recovered from a small number of linear measurements, given the following non-convex recovery problem

$$\min_{X \in \mathbb{C}^{s \times n}} \frac{1}{2} \|y - A(X)\|_2^2 \text{ subject to } \operatorname{rank}(\mathcal{H}(X)) = r,$$  

We tend to solve it in a fast and provable way and this leads us to the Fast Iterative Hard Thresholding algorithm.

A. Related work

The blind super-resolution problem is explored in many literatures [6, 9, 15, 18]. More specifically, an atomic norm minimization method has been proposed in [15, 18] to solve the blind super-resolution problem. Recently, Chen et al. [6] proposed a nuclear minimization method called Vectorized Hankel Lift to recover $X^k$. Indeed, all of these recovery methods are for convex optimization problems which can be solved by many sophisticated algorithms such as interior point methods, gradient descent etc. However, one common drawback of the convex optimization approach is that the high computational complexity of solving the equivalent semi-definite programming.
B. Our contribution

In this paper, we consider a non-convex recovery approach for blind super-resolution problem [13]. Here, the objective function is a standard least squares function which minimizes the distance from the observed point sources to the recovered ones while the constraint function enforces low rank structure in the transform domain. We proposed a non-convex algorithm called Vectorized Hankel Lifted Fast Iterative Hard Thresholding (VHL-FIHT for short) to solve this low rank structured matrix recovery procedure [13]. The algorithm is presented in Algorithm 1. We also establish the linear convergence rate of this method provided that number of measurements is of the order $O(s^2r^2\log^2(sn))$ and given a suitable initialization.

C. Notations and preliminaries

The vectorized Hankel matrix $\mathcal{H}(X^t)$ admits the following low rank decomposition [6]

$$\mathcal{H}(X^t) = (E_L \odot H) \text{diag} (d_1, \ldots, d_r) E_R^T,$$  

where $\odot$ is the Khatri-Rao product of matrix and the matrices $E_L \in \mathbb{C}^{n \times r}$ as well as $E_R \in \mathbb{C}^{n \times r}$ are full column rank matrices.

The adjoint of $\mathcal{H}$, denoted $\mathcal{H}^*$, is a linear mapping from $sn_1 \times n_2$ matrices to matrices of size $s \times n$. In particular, for any matrix $Z \in \mathbb{C}^{sn_1 \times n_2}$, the $i$-th column of $\mathcal{H}^*(Z)$ is given by

$$\mathcal{H}^*(Z)e_i = \sum_{(j,k) \in W_i} z_{j,k},$$

where $z_{j,k} = Z[js : (j+1)s-1, k]$ and $W_i$ is the set

$$\{(j, k) \mid j + k = i, 0 \leq j \leq n_1 - 1, 0 \leq k \leq n_2 - 1\}$$

Let $D : \mathbb{C}^{s \times n} \rightarrow \mathbb{C}^{s \times n}$ be an operator such that

$$D(X) = X \text{diag} (\sqrt{w_0}, \ldots, \sqrt{w_n-1})$$

for any $X$, where the scalar $w_i$ is defined as $w_i = \#W_i$ for $i = 0, \ldots, n-1$. The Moore-Penrose pseudoinverse of $\mathcal{H}$ is given by $\mathcal{H}^! = D^{-2} \mathcal{H}^*$ which satisfies $\mathcal{H}^! \mathcal{H} = I$. The adjoint of the operator $A(\cdot)$, denoted $A^*(\cdot)$, is defined as $A^*(y) = \sum_{j=0}^{n-1} y[j]b^*_j$. Denote $Z = \mathcal{H}(X)$ and $\mathcal{G} = \mathcal{H} D^{-1}$. The adjoint of $\mathcal{G}$, denoted $\mathcal{G}^*$, is given by $\mathcal{G}^* = D^{-1} \mathcal{H}^*$.

D. Organization

The remainder of this letter is organized as follows. In Section 2, we will introduce VHL-FIHT algorithm. In Section 3, we will introduce two assumptions and establish our main results. The performance of VHL-FIHT is evaluated from numerical examples in Section 4. In Section 5, we give the detailed proofs of main result. We close with a conclusion in Section 6.

II. VECTORIZED HANKEL LIFTED FAST ITERATIVE HARD THRESHOLDING

Let $Z^t = U^t \Sigma^t V^t H^t \in \mathbb{C}^{sn_1 \times n_2}$ be the compact singular value decomposition of a rank-$r$ matrix, where $U^t \in \mathbb{C}^{sn_1 \times r}$, $V^t \in \mathbb{C}^{n_2 \times r}$ and $\Sigma^t \in \mathbb{R}^{r \times r}$. It is known that the tangent space of the fixed rank-$r$ matrix manifold at $Z$ is given by

$$T = \{UN^H + MV^H : M \in \mathbb{C}^{sn_1 \times r}, N \in \mathbb{C}^{n_2 \times r}\}.$$ 

Given any matrix $W \in \mathbb{C}^{sn_1 \times n_2}$, the projection of $W$ onto $T$ can be computed using the formula

$$P_T(W) = UU^H W + WVV^H - UU^H WWV^H.$$ 

The VHL-FIHT method is shown in Algorithm 1 when used to solve [13]. An initial guess was obtained with spectrum method. In each iteration of VHL-FIHT, the current estimate $X^t$ is first updated along the gradient descent direction of the objective in (I.3). Then, the Hankel matrix corresponding to the update is formed via the application of the vectorized Hankel lift operator $\mathcal{H}$, followed by a projection operator $P_T$ onto the $T$ space. Finally, it imposes a hard thresholding operator $T_\mu$ to $W^t$ by truncated SVD process and then apply $\mathcal{H}$ on the low rank matrix $Z^{t+1}$.

Algorithm 1 VHL-FIHT

Input: Initialization $X^0 = \mathcal{H}^! T \mathcal{H} \mathcal{A}^*(y)$. 
Output: $X^T$

1: for $t = 0, 1, \ldots, T - 1$ do
2: $\tilde{X}^t = X^t - \mathcal{A}^*(\mathcal{A}(X^t) - y)$
3: $W^t = P_T(\mathcal{H}(\tilde{X}^t))$
4: $Z^{t+1} = T_\mu(W^t)$
5: $X^{t+1} = \mathcal{H}^!(Z^{t+1})$

III. MAIN RESULT

In this section, we establish our main results. To this end, we make two assumptions.

Assumption III.1. The column vectors $\{b_j\}_{j=0}^{n-1} \subset \mathbb{C}^s$ of the subspace matrix $B^H$ are i.i.d random vectors which obey

$$\mathbb{E}[b_j b^H_j] = I_s \text{ and } \max_{0 \leq \ell \leq s-1} |b_j[\ell]|^2 \leq \mu_0,$$
The coefficient vectors $N$ are generated via $U$ and the measurements are obtained by $(\text{??})$ in future work. On $s$ target matrix performance of FIHT. In the numerical experiments, the $\{U\}$ distribution (i.e., $U$) is commonly used in spectrally sparse signal recovery [1], [2], [8] and blind super-resolution [6]. As-

Assumption III.1 is introduced in [5] and has been used in blind super-resolution [6], [9], [15], [18]. Assumption III.2 is commonly used in our proof.

**Theorem III.1.** Under Assumption III.1 and III.2, with probability at least $1 - c_1 n^{-c_2}$, the iterations generated by FIHT (Alg. 1) with the initial guess $X^0 = H^1 T H(A^*(y))$ satisfies

$$
\|X^t - X^0\|_F \leq \left(\frac{1}{2}\right)^t \|X^0 - X^1\|_F \quad (\text{III.1})
$$

provided

$$n \geq C \kappa^2 \mu_0^2 \mu_1 s^2 r^2 \log^2 (sn) / \varepsilon^2,
$$

where $c_1, c_2$ and $C$ are absolute constants and $\kappa = \sigma_{\text{max}}(H(X^0)) / \sigma_{\text{min}}(H(X^0))$.

**Remark III.1.** The sample complexity established in [6] for the Vectorized Hankel Lift is $n \geq c_0 \mu_0 \mu_1 s^r r \log^4 (sn)$. While the sample complexity is sub-optimal dependence on $s$ and $r$, our recovery method requires low per iteration computational complexity. Similar to [3], the per iteration computational cost of FIHT is about $O(sn + r n \log (sn) + r)$. To improve the dependence on $s$, we will investigate other initialization procedures in future work.

**IV. Numerical Results**

Numerical experiments are conducted to evaluate the performance of FIHT. In the numerical experiments, the target matrix $X^0$ is generated by $X^0 = \sum_{l=1}^r d_k h_k a_{l,k}^T$ and the measurements are obtained by $(\text{??})$, where the location $\{t_k\}_{k=1}^r$ are generated from a standard uniform distribution (i.e., $U(0,1)$), and the amplitudes $\{d_k\}_{k=1}^r$ are generated via $d_k = (1 + 10^c \varepsilon - e^{-10^c \varepsilon})$ where $\varepsilon$ follows $U(0, 2\pi)$ and $c_k$ follows $U(0, 1)$. Each entry of subspace matrix $B$ is independently and identically generated from a standard normal distribution (i.e., $N(0,1)$). The coefficient vectors $\{h_k\}_{k=1}^r$ are generated from a standardized multivariate Gaussian distribution (i.e., $MVN(0, I_{s \times s})$, where $I_{s \times s}$ is the identity matrix). We set the dimension of signal $n = 256$ and the number of point sources $r = 5$. Fig. 1 presents the logarithmic recovery error $\log_{10} \|X_t - X^0\|_F / \|X^0\|_F$ with respect to the number of iteration. Numerical experiment shows that FIHT converges linearly.

**V. Proof of Main Result**

We first introduce three auxiliary lemmas that will be used in our proof.

**Lemma V.1 ([6] Corollary III.9).** Suppose $n \geq C \varepsilon^{-2} \mu_0 \mu_1 s^r r \log^4 (sn)$. The event

$$
\|P_T (GG^* - GA^* AG^*) P_T\| \leq \varepsilon \quad (\text{V.1})
$$

occurs with probability exceeding $1 - c_1 n^{-c_2}$.

**Lemma V.2 ([16] Lemma 5.2).** Suppose that $n \geq C \varepsilon^{-2} \kappa^2 \mu_0^2 \mu_1 s^2 r^2 \log^2 (sn)$. Then with probability at least $1 - c_1 n^{-c_2}$, the initialization $Z_0 = H(X^0)$ obeys

$$
\|Z^0 - Z^0\| \leq \frac{\sigma_r (Z^0) \varepsilon}{16 \sqrt{1 + \varepsilon} \mu_0 s}.
$$

**Lemma V.3.** Suppose

$$
\|Z^t - Z^0\| \leq \frac{\sigma_r (Z^t) \varepsilon}{16 \sqrt{1 + \varepsilon} \cdot \mu_0 s}. \quad (\text{V.2})
$$

**Conditioned on** (V.1), one has

$$
\|P_T G^* P_T\| \leq 3 \varepsilon \sqrt{1 + \varepsilon}, \quad (\text{V.3})
$$

$$
\|P_T G (I - A^* A) G^* P_T\| \leq 2 \varepsilon. \quad (\text{V.4})
$$

We can rewrite the iteration as

$$
Z^{t+1} = T_r (Z^t - G A^* G^* (Z^t - Z^0)). \quad (\text{V.5})
$$

Fig. 1. $\log_{10} \|X_t - X^0\|_F / \|X^0\|_F$ with respect to the number of iteration.
Notice that \( \|X^t - X^{\ast}\|_F = \|H^t(Z^t - Z^{\ast})\|_F \leq \|Z^t - Z^{\ast}\|_F \). Our proof follows the line of (1). We first assume that in the \( t \)-th iteration \( Z^t \) obeys
\[
\|Z^t - Z^{\ast}\|_F \leq \frac{\sigma_r(Z^t)\varepsilon}{16\sqrt{(1 + \varepsilon)}\mu_0 s}. \tag{V.6}
\]
Denote \( W^t = P_{T_1}(Z^t - G^{A^{\ast}}G^\ast Z^t) \). We have that \( Z^{t+1} = I_t(W^t) \) and \( \|Z^{t+1} - Z^{\ast}\|_F \) can be bounded as follows:
\[
\begin{align*}
\|Z^{t+1} - Z^{\ast}\|_F &\leq \|Z^{t+1} - W^t\|_F + \|W^t - Z^{\ast}\|_F \\
&\leq 2\|P_{T_1}(Z^t - G^{A^{\ast}}G^\ast Z^t - \langle AG^\ast P_T(Z)\rangle)\|_F \\
&\quad + 2\|I - P_{T_1}\|_F (\|Z^t - Z^{\ast}\|_F) \\
&\leq 2\|I - P_{T_1}\|_F (\|Z^t - Z^{\ast}\|_F) \\
&\quad + 2\|P_{T_1}(G^{A^{\ast}}G^\ast Z^t) - P_{T_1}(Z^t - Z^{\ast})\|_F \\
&\quad + 2\|P_{T_1}(G^{A^{\ast}}G^\ast Z^t) - P_{T_1}\|_F (\|Z^t - Z^{\ast}\|_F) \\
&\quad + 2\|P_{T_1}(G^{A^{\ast}}G^\ast Z^t) - I - P_{T_1}\|_F (\|Z^t - Z^{\ast}\|_F) \\
&\quad + 2\|P_{T_1}(G^{A^{\ast}}G^\ast Z^t) - I - P_{T_1}\|_F (\|Z^t - Z^{\ast}\|_F) \\
&\triangleq I_1 + I_2 + I_3 + I_4.
\end{align*}
\]
Applying Lemma (17) Lemma 4.1 yields that
\[
I_1 + I_3 \leq 4\|Z^t - Z^{\ast}\|_F^2 / \sigma_r(Z^t).
\]
A simple computation obtains that \( I_2 \leq 2\varepsilon\|Z^t - Z^{\ast}\|_F \). Finally, Lemma (V.3) implies that
\[
I_5 \leq 3\sqrt{1 + \varepsilon} \cdot \|Z^t - Z^{\ast}\|_F^2 / \sigma_r(Z^t).
\]
Combining these terms together, we have
\[
\|Z^{t+1} - Z^{\ast}\|_F \leq \left(2\varepsilon + \frac{4 + 3\sqrt{1 + \varepsilon}}{\sigma_r(Z^t)}\right)\|Z^t - Z^{\ast}\|_F \leq \left(2\varepsilon + \frac{4 + 3\sqrt{1 + \varepsilon}}{16\sqrt{(1 + \varepsilon)}\mu_0 s}\right)\varepsilon \cdot \|Z^t - Z^{\ast}\|_F \tag{V.7}
\]
\[
\leq 3\varepsilon\|Z^t - Z^{\ast}\|_F \leq \frac{1}{2}\|Z^t - Z^{\ast}\|_F, \tag{V.8}
\]
where (V.7) has used (V.2) and (V.8) is due to \( \varepsilon \leq 1/6 \). Finally, it remains to verify (V.6). By Lemma (V.2) the inequality (V.6) is valid for \( t = 0 \). Since \( \|Z^t - Z^{\ast}\|_F \) is a contractive sequence following from (V.8), the inequality (V.6) holds for all \( t \geq 0 \) by induction.

A. Proof of Lemma (V.2)

Proof. For any \( Z \in \mathbb{C}^{s n_1 \times n_2} \), we have
\[
\|AG^\ast P_T(Z)\|^2 = (AG^\ast P_T(Z), AG^\ast P_T(Z)) = (Z, P_T G (A^{\ast} A - I) G^\ast P_T(Z)) + (Z, P_T G G^\ast P_T(Z)) \leq (1 + \varepsilon) \|Z\|_F^2,
\]
where the last inequality is due to Lemma III in (6). So it follows that \( \|AG^\ast P_T\| \leq \sqrt{1 + \varepsilon} \) and
\[
\|AG^\ast P_T\| = \|AG^\ast (P_{T_1} - P_{T_1})\| \leq \sqrt{1 + \varepsilon} + \|P_{T_1} - P_{T_1}\| \leq \sqrt{1 + \varepsilon} + \frac{2\sqrt{\mu_0 s} \|Z^t - Z^{\ast}\|_F}{\sigma_m(Z^t)} \leq 3\sqrt{1 + \varepsilon}.
\]
Finally, a straightforward computation yields that
\[
\|P_{T_1} G (I - A^{\ast} A) G^\ast P_T\| \leq \|P_{T_1} - P_T\| G G^\ast (P_{T_1} - P_T) G^\ast P_T + \|P_{T_1} - P_T\| \|G A^\ast G^\ast (P_{T_1} - P_T)\| + \|P_{T_1} G A^\ast G^\ast (P_{T_1} - P_T)\| + \|P_{T_1} G (I - A^{\ast} A) G^\ast P_T\| \leq 4\|Z^t - Z^{\ast}\|_F \frac{2\|Z^t - Z^{\ast}\|_F}{\sigma_r(Z^t)} + \frac{\varepsilon \sqrt{\mu_0 s}}{\sigma_r(Z^t)} \cdot \|Z^t - Z^{\ast}\|_F + \|P_{T_1} G A^\ast\| + \|P_{T_1} G A^\ast\| + \varepsilon \tag{V.9}
\]
\[
\leq \frac{4\varepsilon}{16\sqrt{(1 + \varepsilon)}\mu_0 s} + \frac{8\varepsilon}{16\sqrt{(1 + \varepsilon)}\mu_0 s} + \varepsilon \tag{V.10}
\]
\[\leq 2\varepsilon, \]
where (V.9) is due to Lemma (17) Lemma 4.1 and the fact that \( \|A^{\ast}\| = \|A\| \leq \sqrt{\mu_0 s} \) and \( \|P_{T_1} G A^{\ast}\| = \|A G^\ast P_{T_1}\| \), step (V.10) follows (V.2).

VI. Conclusion

We propose a VHL-FIHT method to solve the blind super-resolution problem in a non-convex schema. The convergence analysis has been established for VHL-FIHT, showing that the algorithm will linearly converge to the target matrix given suitable initialization and provided the number of samples is large enough. The numerical experiments validate our theoretical results.

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REFERENCES

[1] Jian-Feng Cai, Suhui Liu, and Weiyu Xu. A fast algorithm for reconstruction of spectrally sparse signals in super-resolution. In Wavelets and Sparsity XVI, volume 9597, page 95970A. International Society for Optics and Photonics, 2015.

[2] Jian-Feng Cai, Tianming Wang, and Ke Wei. Spectral compressed sensing via projected gradient descent. SIAM Journal on Optimization, 28(3):2625–2653, 2018.

[3] Jian-Feng Cai, Tianming Wang, and Ke Wei. Fast and provable algorithms for spectrally sparse signal reconstruction via low-rank hankel matrix completion. Applied and Computational Harmonic Analysis, 46(1):94–121, 2019.

[4] Jian-Feng Cai and Ke Wei. Exploiting the structure effectively and efficiently in low rank matrix recovery. Handbook of Numerical Analysis, abs/1809.03652, 2018.

[5] Emmanuel J Candes and Yaniv Plan. A probabilistic and ripless theory of compressed sensing. IEEE transactions on information theory, 57(11):7235–7254, 2011.

[6] Jinch Chen, Weiguo Gao, Sihan Mao, and Ke Wei. Vectorized hankel lift: A convex approach for blind super-resolution of point sources, 2020.

[7] Yudong Chen and Yuejie Chi. Harnessing structures in big data via guaranteed low-rank matrix estimation: Recent theory and fast algorithms via convex and nonconvex optimization. IEEE Signal Processing Magazine, 35(4):14–31, 2018.

[8] Yuxin Chen and Yuejie Chi. Robust spectral compressed sensing via structured matrix completion. IEEE Transactions on Information Theory, 60(10):6576–6601, Oct 2014.

[9] Yuejie Chi. Guaranteed blind sparse spikes deconvolution via lifting and convex optimization. IEEE Journal of Selected Topics in Signal Processing, 10(4):782–794, 2016.

[10] Yuejie Chi, Yue M Lu, and Yuxin Chen. Nonconvex optimization meets low-rank matrix factorization: An overview. IEEE Journal of Selected Topics in Signal Processing, 10(4):608–622, 2016.

[11] Mark A Davenport and Justin Romberg. An overview of low-rank matrix recovery from incomplete observations. IEEE Journal of Selected Topics in Signal Processing, 10(4):608–622, 2016.

[12] David Goldberg, David Nichols, Brian M. Oki, and Douglas Terry. Using collaborative filtering to weave an information tapestry. Communications of the ACM, 35(6):1–70, 1992.

[13] Robert W. Harrison. Phase problem in crystallography. J. Opt. Soc. Am. A, 10(5):1046–1055, May 1993.

[14] Martin Kliesch, Richard Kueng, Jens Eisert, and David Gross. Guaranteed recovery of quantum processes from few measurements. Quantum, 3, 01 2017.

[15] Shuang Li, Michael B Wakin, and Gongguo Tang. Atomic norm denoising for complex exponentials with unknown waveform modulations. IEEE Transactions on Information Theory, 66(6):3893–3913, 2019.

[16] Sihan Mao and Jinch Chen. Fast blind super-resolution of point sources via projected gradient descent. In preparation, 2021.

[17] Ke Wei, Jian-Feng Cai, Tony F Chan, and Shingyu Leung. Guarantees of riemannian optimization for low rank matrix recovery. SIAM Journal on Matrix Analysis and Applications, 37(3):1198–1222, 2016.

[18] Dehui Yang, Gongguo Tang, and Michael B Wakin. Super-resolution of complex exponentials from modulations with unknown waveforms. IEEE Transactions on Information Theory, 62(10):5809–5830, 2016.

[19] Shuai Zhang, Yingshuai Hao, Meng Wang, and Joe H Chow. Multichannel Hankel matrix completion through nonconvex optimization. IEEE Journal of Selected Topics in Signal Processing, 12(4):617–632, 2018.