On the asymptotics of solutions to the evolutionary form of the constraints

Károly Csukás *,1 and István Rácz †,1,2

1Wigner RCP, H-1121 Budapest, Konkoly Thege Miklós út 29-33, Hungary
2Faculty of Physics, University of Warsaw, Ludwika Pasteura 5, 02-093 Warsaw, Poland

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Abstract

A systematic investigation of the asymptotic behavior of near Schwarzschild vacuum initial data sets is carried out by making use of numerical solutions to the evolutionary form of the constraints. The decay rate of the monopole part of the trace of the tensorial projection of the extrinsic curvature is found to be of critical importance in controlling asymptotic flatness of initial data configurations.

1 Introduction

Near Schwarzschild vacuum initial data configurations are automatically regarded as being asymptotically flat. This preconception may stem from the fact that when the constraints are solved by using elliptic methods suitable fall off (or boundary) conditions at infinity can be imposed to guarantee asymptotic flatness. In such a case, at least if the trace of the extrinsic curvature is nearly vanishing, one can get implicit solutions to the corresponding boundary value problem. As opposed to this, while applying either of the evolutionary methods—two alternative approaches were introduced in [9, 10, 11]—to generate near Schwarzschild configurations, data for the

*E-mail address: csukas.karoly@wigner.mta.hu
†E-mail address: racz.istvan@wigner.mta.hu
constraint fields can only be freely specified at a 2-sphere at some finite location. This, in principle, may not allow one to acquire control on the asymptotic behavior of the yielded solutions. For this reason it is of obvious interest to know if asymptotic flatness can be guaranteed by choosing suitable initial data for the evolutionary form of the constraints [3,4]. In this paper the asymptotics of near Schwarzschild solutions are investigated by integrating numerically the parabolic-hyperbolic and algebraic-hyperbolic form of the constraints, respectively. Our most important finding is that once we can get a control on the monopole part of the trace of the extrinsic curvature's tensorial projection \( \hat{\gamma}^{kl} K_{kl} \)—see subsection 2.1 for definitions—even the strong asymptotic flatness of the solutions to the evolutionary form of the constraints can be guaranteed.

This paper is organized as follows. Section 2 is to introduce the analytic framework. In subsection 2.1 we start by specifying the basic variables and then, in subsections 2.1.1 and 2.1.2 the evolutionary form of the constraints are recalled. A short review of asymptotic flatness in terms of the applied new variables is given in subsection 2.2. The use of concept of near Schwarzschild configurations is explained in subsection 2.3. In subsection 2.4 purely spherically symmetric initial data configurations are investigated. Section 3 is to present all of our numerical findings. First the applied multipole expansion and the numerical setup is outlined in subsection 3.1. Then, in subsection 3.2 results based on the use of the full set of evolutionary forms are presented. These results, along with the ones reported in the succeeding section, are to convince the readers that a clear separation of the excitation modes from the Schwarzschild background are necessary to identify those modes which may get in the way of asymptotic flatness. For this reason a non-linear perturbative approach is introduced in subsection 3.3. A systematic investigation based on non-linear perturbative approach is carried out in section 4. In particular, all the relevant modes of the basic variables using either the parabolic-hyperbolic or the algebraic-hyperbolic systems are identified in subsections 4.1 and 4.2. An attempt to alter the asymptotic behavior of the critical variable, \( K_{kl} = \hat{\gamma}^{kl} K_{kl} \) is made in subsection 4.3 by relaxing slightly the conditions on near Schwarzschildian configurations. The corresponding findings may be considered as the first attempt towards a better controlling of the asymptotic behavior of initial data sets. The discussions are completed by our final remarks in section 5 whereas the paper is closed by an appendix providing the detailed form of the evolutionary forms of the constraints relevant for non-linear perturbations.

2 Preliminaries

This section is to introduce the applied analytic setup.
2.1 Evolutionary form of the equations

As the Schwarzschild spacetime is a vacuum solution to Einstein’s equations throughout this paper our considerations will be restricted to the vacuum constraints. Vacuum initial data configurations are represented by two symmetric tensor fields \( h_{ij} \) and \( K_{ij} \) defined on a 3-dimensional manifold \( \Sigma \) such that \( h_{ij} \) is a Riemannian metric there. These fields are not free as they are subject to the vacuum constraints

\[
\begin{align*}
\Box R + (K^j_j)^2 - K_{ij}K^{ij} &= 0 \quad (1) \\
D_j K^j_i - D_i K^j_j &= 0. \quad (2)
\end{align*}
\]

As the Schwarzschild spacetime is foliated by a 2-parameter family of metric spheres there exists a high variety of initial data surfaces that can be foliated by a one-parameter family of such surfaces. They are determined by the level surfaces of the area function \( r : \mathbb{R}^+ \rightarrow \Sigma \). As we are looking for near Schwarzschild initial data sets we shall assume that \( \Sigma \)—in the investigated case this will be a \( t_{KS} = \text{const} \) Kerr-Schild time slice—will also be assumed to be foliated by the \( r = \text{const} \) level surfaces, that are level surfaces of the area radius function \( r : \mathbb{R}^+ \rightarrow \Sigma \) with nowhere vanishing gradient. We shall also assume that a flow \( r^i \) interrelating the \( r = \text{const} \) level surfaces had been fixed. Then all the tensor fields on \( \Sigma \) can be decomposed using variables intrinsic to the \( r = \text{const} \) level surfaces and normal to them. In particular, the flow can be characterized by its lapse and shift, \( \hat{N} \) and \( \hat{N}^i \), and it can be given as \( r^i = \hat{N} \hat{n}^i + \hat{N}^i \), where \( \hat{n}^i \) denotes the unit normal to the leaves \( \mathcal{S}_r \). The lapse and shift, along with the 2-metric \( \hat{\gamma}_{ij} \), induced on the \( r = \text{const} \) level surfaces, give an algebraically equivalent representation of the content of \( h_{ij} \) \([9, 10, 11]\). Analogously, the geometric content of \( K_{ij} \) can be represented by its scalar, vector and tensorial projections \( \kappa, k_i, K_{ij} \) via the decomposition (for details see, e.g. \([9, 10, 11]\))

\[
K_{ij} = \kappa \hat{n}_i \hat{n}_j + [\hat{n}_i k_j + \hat{n}_j k_i] + K_{ij}. \quad (3)
\]

As pointed out in \([9, 10, 11]\) in getting the evolutionary form of the constraints it is also essential to split \( K_{ij} \) into its trace \( K^i_i = \hat{\gamma}^{ij} K_{ij} \) and trace-free part \( \hat{K}_{ij} = K_{ij} - \frac{1}{2} \hat{\gamma}_{ij} K^i_i \).

By construction the above set of variables \( \hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \kappa, k_i, K^i_i \) and \( \hat{K}_{ij} \) give an algebraically equivalent representation of the content of twelve components of the original pair \( h_{ij} \) and \( K_{ij} \). In particular, in coordinates \( (r, x^A) \), with \( A = 1, 2 \), adopted to the foliation \( \mathcal{S}_r \) and the flow \( r^i \), the 3-metric takes the form

\[
h_{ij} = \hat{N}^2 (dr)_i (dr)_j + \hat{\gamma}_{AB} [\hat{N}^A (dr)_i + (dx^A)_i] [\hat{N}^B (dr)_j + (dx^B)_j], \quad (4)
\]

i.e. the components of \( h_{ij} \) reads as

\[
h_{ij} = \begin{pmatrix}
\hat{N}^2 & \hat{\gamma}_{EF} \hat{N}^E \hat{N}^F & \hat{\gamma}_{AE} \hat{N}^E \\
\hat{\gamma}_{BF} \hat{N}^F & \hat{N}^E & \hat{\gamma}_{AB}
\end{pmatrix} . \quad (5)
\]
Analogously, in adopted coordinates \((r, x^A)\), the components of \(K_{ij}\) read as

\[
K_{ij} = K_{rr}(dr)_i(dr)_j + 2K_{rA}(dr)_i(dx^A)_j + K_{AB}(dx^A)_i(dx^B)_j,
\]

where, in virtue of (3), the relation

\[
K_{rr} = \kappa \hat{N}^2 + 2\hat{N}k_A\hat{N}^A + K_{AB}\hat{N}^A\hat{N}^B
\]

(7)

\[
K_{rA} = \hat{N}k_A + K_{AB}\hat{N}^B
\]

(8)

\[
K_{AB} = K_{AB},
\]

(9)

hold, i.e. in adopted coordinates the components of \(K_{ij}\) reads as

\[
K_{ij} = \begin{pmatrix}
\kappa \hat{N}^2 + 2\hat{N}k_E\hat{N}^E + K_{EF}\hat{N}^E\hat{N}^F
& \hat{N}k_A + K_{AB}\hat{N}^E \\
\hat{N}k_B + K_{BE}\hat{N}^E
& K_{AB}
\end{pmatrix}.
\]

(10)

Regardless of the choice of the representation—using \((h_{ij}, K_{ij})\) or \((\hat{N}, \hat{N}^A, \hat{\gamma}_{AB}, \kappa, k_A, K^{Ei}, \hat{K}_{AB})\)—we always need to select four of the twelve variables such that these four are subject to the constraints whereas the remaining eight are freely specifiable throughout \(\Sigma\).

Although the initial data surface \(\Sigma\) is foliated by \(r = \text{const}\) surfaces in the general case they need not to be metric spheres but merely topological two-spheres. Nevertheless, in either case we can represent all the tangential derivatives via the operators \(\partial_\theta\) and \(\partial_\phi\) (if needed supplemented by the use of \(\partial_\rho\)) and a very convenient set of variables can also be introduced. This latter is done by fixing a complex dyad \(\{q_i, \bar{q}_i\}\) on the unit sphere \(\mathbb{S}^2\) which, as proposed in [12, 13, 14], can be mapped first onto one of the leaves (say onto \(\mathcal{S}_0\)) and in the second step Lie propagate onto all the \(\mathcal{S}_r\) leaves along the flow \(r^i\). Having the complex dyad \(\{q_i, \bar{q}_i\}\) defined throughout \(\Sigma\) we will use instead of the variables \((\hat{N}, \hat{N}^A, \hat{\gamma}_{AB}, \kappa, k_A, K^{Ei}, \hat{K}_{AB})\) the ones yielded by contracting the involved tensorial expressions with the dyad vector \(q_i\) and/or its complex conjugate \(\bar{q}_i\) in their free indices. These new variables possess definite spin-weights whence they are also analogous to the basic variables applied in the Newman-Penrose formalism. They were defined in [12, 13, 14] but for convenience of the readers they are also recalled in Table I.

Before one attempts to apply the evolutionary form of the constraints it is important to decide which of the alternative methods are to be work with. One may use either the parabolic-hyperbolic or the algebraic–hyperbolic system. Using the parabolic-hyperbolic system one solves the constraints for \(\hat{N}, K\) and \(k\), whereas alternatively one solves the algebraic-hyperbolic system for \(\kappa, \bar{K}\) and \(\bar{k}\). The specific form of these equations are recalled in the following subsections.
| notation | definition | spin-weight |
|----------|------------|------------|
| a        | $\frac{1}{2} q^i \bar{q}^j \hat{\gamma}_{ij}$ | 0          |
| b        | $\frac{1}{2} q^i q^j \hat{\gamma}_{ij}$ | 2          |
| c        | $a^2 - b \bar{b}$ | 0          |
| A        | $q^a q^b C_{ab} g_c = d^{-1} \{ a [2 \bar{a} a - 2 \bar{b} \bar{b}] - \bar{b} \bar{a} b \}$ | 1          |
| B        | $q^a q^b C_{ab} g_c = d^{-1} \{ a \bar{b} \bar{b} - \bar{b} \bar{a} b \}$ | 1          |
| C        | $q^a q^b C_{ab} g_c = d^{-1} \{ a \bar{a} \bar{b} - \bar{b} \bar{a} b \}$ | 3          |
| H        | $\frac{1}{2} a^{-1} (2 R - \{ \bar{a} b - \bar{b} a + \frac{1}{2} [ C C - b \bar{b} b] \})$ | 0          |
| N        | $q^i \hat{N}_i = q^i \hat{\gamma}_{ij} \hat{N}_j$ | 1          |
| N        | $q_i \hat{N}^i = q_i \hat{\gamma}_{ij} \hat{N}_j = d^{-1}(a N - b \bar{b} N)$ | 1          |
| k        | $q^i k_i$ | 1          |
| K        | $K'_l = \hat{\gamma}_{kl} K_{kl}$ | 0          |
| K        | $q^k q^l \hat{K}_{kl}$ | 2          |
| K        | $(2 a)^{-1} [ b \bar{K}_{qq} + \bar{b} \hat{K}_{qq} ]$ | 0          |
| K        | $\hat{K}'_l = \hat{\gamma}_{ij} \hat{K}_{ij}$ | 0          |
| K        | $\frac{1}{2} \hat{N}^{-1} \{ 2 \partial_{\bar{a}} \bar{b} - 2 \bar{a} N + C N + A N \}$ | 2          |
| K        | $(2 a)^{-1} \{ d \cdot \hat{K} + \frac{1}{2} [ b \bar{K}_{qq} + \bar{b} \hat{K}_{qq} ] \}$ | 0          |
| K        | $\hat{K}'_l = \hat{\gamma}_{ij} \hat{K}_{ij}$ | 0          |
| K        | $\frac{1}{2} \{ 2 \partial_{\bar{a}} \bar{b} - 2 \bar{a} N + C N + A N \}$ | 2          |
| K        | $(2 a)^{-1} \{ d \cdot \hat{K} + \frac{1}{2} [ b \bar{K}_{qq} + \bar{b} \hat{K}_{qq} ] \}$ | 0          |

Table 1: The variables used in recasting the constraints in [12, 13, 14]. For detailed derivations for some of these and some other more involved expressions see references [12, 13, 14].
2.1.1 Parabolic–hyperbolic equations

The equations relevant for the parabolic-hyperbolic formulation are given as [14]

\[ \dot{K} \left[ \partial_r \hat{N} - \frac{1}{2} \hat{N} \hat{\sigma} \hat{N} - \frac{1}{2} \hat{\sigma} \hat{N} \right] - \frac{1}{2} \hat{d}^{-1} \hat{N}^2 \left[ a \left\{ \partial \hat{\sigma} \hat{N} - \hat{\sigma} \hat{N} \right\} - b \left\{ \sigma^2 \hat{N} - \frac{1}{2} \hat{K} \hat{\sigma} \hat{N} - \frac{1}{2} \hat{C} \hat{\sigma} \hat{N} \right\} + \text{cc.} \right] - A \hat{N} - B \hat{N}^3 = 0, \]  

(11)

\[ \partial_r k - \frac{1}{2} \hat{K} \hat{\sigma} k - \frac{1}{2} \hat{N} \hat{\sigma} k - \frac{1}{2} \hat{\sigma} \hat{K} + \hat{f} = 0, \]  

(12)

\[ \partial_r K - \frac{1}{2} \hat{N} \hat{\sigma} K - \frac{1}{2} \hat{\sigma} \hat{N} \hat{K} - \frac{1}{2} \hat{d}^{-1} \left\{ a(\partial K + \sigma k) - b \sigma K - \hat{b} \hat{\sigma} k \right\} + \hat{F} = 0, \]  

(13)

where the factors \( A, B \) and the source terms \( \hat{f}, \hat{F} \) read as

\[ A = \partial_r \hat{K} - \frac{1}{2} \hat{N} \hat{\sigma} \hat{K} - \frac{1}{2} \hat{\sigma} \hat{N} \hat{K} + \frac{1}{2} \left[ \hat{K}^2 + \hat{K}_{kl} \hat{K}^{kl} \right], \]  

(14)

\[ B = -\frac{1}{2} \left[ \hat{K} + 2 \kappa \hat{K} + \frac{1}{2} \hat{K}^2 - \hat{d}^{-1} \left\{ 2 a \kappa - b \hat{K} \hat{K} - \hat{b} \hat{\sigma} k \right\} - \hat{K}_{kl} \hat{K}^{kl} \right], \]  

(15)

\[ \hat{f} = -\frac{1}{2} \left[ k \hat{\sigma} \hat{N} + \hat{K} \hat{\sigma} \hat{N} \right] - \left[ \kappa - \frac{1}{2} \hat{K} \right] \partial \hat{N} + \dot{\hat{K}} \hat{\sigma} \hat{K} + \hat{N} \left[ - \partial \kappa + q^i \hat{\dot{K}}_{li} + q^i \hat{D}^l \hat{K}_{li} \right], \]  

(16)

\[ \hat{F} = \frac{1}{2} \hat{d}^{-1} \left\{ 2 a \hat{K} \hat{\sigma} k - b(\hat{\sigma} \hat{K} + \hat{K} \hat{\sigma}) + \text{cc.} \right\} - \hat{d}^{-1} \left\{ (a \hat{K} - b \hat{K}) \partial \hat{N} + \text{cc.} \right\} + \left[ \hat{K}_{ij} \hat{K}^{ij} + (\kappa - \frac{1}{2} \hat{K}) \hat{\dot{K}} \right], \]  

(17)

and where the explicit form of terms \( q^i \hat{\dot{K}}_{li}, q^i \hat{D}^l \hat{K}_{li}, \hat{K}_{ij} \hat{K}^{ij}, \hat{K}_{kl} \hat{K}^{kl}, \hat{K}_{kl} \hat{\dot{K}}^{kl} \) can be found e.g. in [14],

2.1.2 Algebraic-hyperbolic equations

Analogously, the equations relevant for the algebraic-hyperbolic case read as [14]

\[ \partial_r \hat{K} - \frac{1}{2} \hat{N} \hat{\sigma} \hat{K} - \frac{1}{2} \hat{\sigma} \hat{N} \hat{K} - \frac{1}{2} \hat{d}^{-1} \left\{ a(\partial \hat{K} + \sigma k) - b \sigma \hat{K} - \hat{b} \hat{\sigma} k \right\} + \hat{F} = 0, \]  

(18)

\[ \partial_r k - \frac{1}{2} \hat{N} \hat{\sigma} k - \frac{1}{2} \hat{\sigma} \hat{N} k + \hat{N} \hat{K}^{-1} \left\{ \kappa \hat{\sigma} \hat{K} - \hat{d}^{-1} [(a \hat{K} - b \hat{K}) \partial \hat{N} + (a \hat{K} - b \hat{K}) \partial \hat{K}] \right\} + \hat{f} = 0, \]  

(19)
\[ \kappa = \frac{1}{2} K^{-1} \left[ d^{-1} \left( 2a k k - b k^2 - \bar{b} k^2 \right) - \frac{1}{2} K^2 - \kappa_0 \right], \tag{20} \]

where
\[ \kappa_0 = \Omega R - \Omega_{k/l} K_{kl}, \tag{21} \]

and the pertinent source terms \( F, f \) read as
\[ F = \frac{1}{4} \hat{N} d^{-1} \left\{ 2a B k - b (C k + A k) + cc. \right\} - d^{-1} \left[ (a k - \bar{b} k) \partial \hat{N} + cc. \right] + \left[ \partial_{ij} K^{ij} - (\kappa - \frac{1}{2} K) \hat{K} \right], \tag{22} \]
\[ f = -\frac{1}{2} \left[ k \partial \hat{N} + \bar{k} \partial \hat{N} \right] + \frac{1}{2} \hat{N} \left( dK \right)^{-1} \left[ (a k - \bar{b} k) \left( B k + \bar{B} k \right) + (a k - \bar{b} k) \left( C k + A k \right) \right] - \left[ \kappa - \frac{1}{2} K \right] \partial \hat{N} + \hat{N} \left[ \frac{1}{2} K^{-1} \partial \kappa_0 + \hat{K} k - q^i \hat{n}^j \partial_{ij} K + q^i \hat{D}^j \partial_{ij} K \right]. \tag{23} \]

2.2 The strong and weak forms of asymptotic flatness

Before restricting considerations to specific classes of solutions it is rewarding to recall the notion of asymptotic flatness we shall apply.

As our aim is to study near Schwarzschild initial data configurations it suffices to consider initial data surfaces with a single asymptotically flat end that is diffeomorphic to a region complementing a ball \( B \) in \( \mathbb{R}^3 \). Hereafter an initial data set \( (\Sigma, h_{ij}, K_{ij}) \) is called asymptotically flat in the ’strong sense’, if the complement of a compact set in \( \Sigma \) can be mapped by an admissible Cartesian coordinate system \( \{ x_i \} \) diffeomorphically onto the complement of \( B \) in \( \mathbb{R}^3 \), and also there exist a positive constant \( C \), such that in these coordinates
\[ h_{ij} = \left( 1 + \frac{C}{r} \right) \delta_{ij} + O(r^{-2}), \tag{24} \]
\[ K_{ij} = O(r^{-2}) \tag{25} \]
hold as \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty \). Here the indices \( i, j, k... \) are also assumed to denote coordinates indices, taking the values 1, 2, 3, whereas \( \delta_{ij} \) denotes the flat metric with respect to the given admissible Cartesian coordinate system \( \{ x_i \} \). Notably the above conditions are known to guarantee that the mass, the momentum, and the angular momentum of the initial data set are well defined \([6]\).

Note also that there exist various weaker notions of asymptotic flatness \([1, 2]\). In what follows an initial data set \( (\Sigma, h_{ij}, K_{ij}) \) is called asymptotically flat in the “weak sense” if for some positive constant \( C \) and for some arbitrarily small positive \( \varepsilon \)
\[ h_{ij} = \left( 1 + \frac{C}{r} \right) \delta_{ij} + O(r^{-3/2-\varepsilon}), \tag{26} \]
\[ K_{ij} = O(r^{-3/2-\varepsilon}) \tag{27} \]
hold. It was shown in \([2]\) that if \((26)\) and \((27)\) holds, and also \( h_{ij} \) and \( K_{ij} \) satisfy the constraints, then the ADM mass and momentum are still well-defined as \( r \to \infty \).
2.2.1 The fall off properties of the new variables

Although all the arguments in this subsection will be outlined by assuming that the initial data set \((\Sigma, h_{ij}, K_{ij})\) are asymptotically flat in the strong sense by a straightforward modification the corresponding analysis can also be applied in case of weakly asymptotically flat initial data configurations.

To start off recall first that the coordinate basis fields \(\{(\partial_{x_i})^a\}\) of admissible Cartesian coordinates \(\{x_i\}\) can always be expressed as linear combinations of the coordinate basis fields \((\partial_r)^a, (\partial_\theta)^a, (\partial_\phi)^a\) (of the conventional spherical coordinates \((r, \theta, \phi)\)) with factors of order \(O(r^0), O(r^{-1})\) in \(r\), respectively.

In accordance with these observations, the relation \(h_{ij} = h_{ab}(\partial_{x_i})^a(\partial_{x_j})^b\) reads as

\[
h_{ij} = h_{rr} + 2r^{-1}h_{rA} + r^{-2}h_{AB},
\]

which, along with (5) and (24), implies that

\[
(\hat{N}^2 + \hat{N}_E\hat{N}^E) - 1 \sim O(r^{-1})
\]

\[
\hat{N}_A = \hat{\gamma}_{AB}\hat{N}^B \sim O(r^{-1})
\]

\[
\hat{\gamma}_{AB} - r^2\hat{\gamma}_{AB} \sim O(r^0),
\]

where \(\hat{\gamma}_{AB}\) denotes the standard unit sphere metric.

In virtue of (35) it follows immediately that

\[
\hat{\gamma}_{AB} \sim O(r^2) \quad \text{and} \quad \hat{\gamma}^{AB} \sim O(r^{-2}).
\]

This, along with (34), gives then \(\hat{N}^A\) should fall off as

\[
\hat{N}^A \sim O(r^{-3}).
\]

\[1\] To see this one may recall that in \(\mathbb{R}^3\) the Cartesian coordinates \((x_1, x_2, x_3)\) can be given in terms of the conventional spherical coordinates \((r, \theta, \phi)\) via the relations

\[
x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta,
\]

whereas the corresponding coordinate basis fields are related as

\[
(\partial_{x_1})^a = \sin \theta \cos \phi (\partial_r)^a + r^{-1} \left[ \cos \theta \cos \phi (\partial_\theta)^a - (\sin \theta)^{-1} \sin \phi (\partial_\phi)^a \right]
\]

\[
(\partial_{x_2})^a = \sin \theta \sin \phi (\partial_r)^a + r^{-1} \left[ \cos \theta \sin \phi (\partial_\theta)^a + (\sin \theta)^{-1} \cos \phi (\partial_\phi)^a \right]
\]

\[
(\partial_{x_3})^a = \cos \theta (\partial_r)^a - r^{-1} \sin \theta (\partial_\phi)^a.
\]
Finally, the last two relations, along with (33), gives that $\hat{N}^2 - 1 \sim O(r^{-1})$, i.e. $\hat{N}$ tends to 1 at infinity, which, by $\hat{N}^2 - 1 = (N + 1)(\hat{N} - 1)$ gives that $\hat{N} - 1$ should fall off as

$$\hat{N} - 1 \sim O(r^{-1}).$$  

(38)

Analogously, from

$$K_{ij} = K_{rr} + 2 r^{-1} K_{rA} + r^{-2} K_{AB}$$  

(39)

we get, along with (7), (8), (9) and (25) that

$$\kappa \hat{N}^2 + 2 \hat{N} k_A \hat{N}^A + K_{AB} \hat{N}^A \hat{N}^B \sim O(r^{-2})$$  

(40)

$$\hat{N} k_A + K_{AB} \hat{N}^B \sim O(r^{-1})$$  

(41)

$$K_{AB} \sim O(r^0).$$  

(42)

Correspondingly, (42), along with the relation $K_{AB} = \hat{K}_{AB} + \frac{1}{2} \hat{\gamma}_{AB} K_{EE} + O(r^0)$ and (36), gives then that $K_{AB}, \hat{K}_{AB}$ and $K_{EE} = \hat{\gamma}_{EF} K_{EF}$ should fall off as

$$K_{AB}, \hat{K}_{AB} \sim O(r^0) \quad \text{and} \quad K_{EE} \sim O(r^{-2}).$$  

(43)

Then, (41), along with (38), (37) and (43), implies

$$k_A \sim O(r^{-1}),$$  

(44)

and, finally, in virtue of (40), that

$$\kappa \sim O(r^{-2}).$$  

(45)

The fall off properties relevant for the constraint variables—these appear in either of the evolutionary forms of the constraint equations—are collected in Table 2.

| variable | strong form | weak form |
|----------|-------------|-----------|
| $\hat{N} - 1$ | $r^{-1}$ | $r^{-1}$ |
| $\kappa$ | $r^{-2}$ | $r^{-3/2 - \varepsilon}$ |
| $\hat{k}$ | $r^{-1}$ | $r^{-1/2 - \varepsilon}$ |
| $\hat{K}$ | $r^{-2}$ | $r^{-3/2 - \varepsilon}$ |

Table 2: The asymptotic behavior of some of the basic variables are collected.
2.3 Near Schwarzschild configurations

There are various ways one could consider an initial data configuration to be near Schwarzschildian. Throughout this paper, almost exclusively, we shall use the most obvious definition. In advance of spelling it out recall first that among the twelve scalar variables stored either in \((h_{ij}, K_{ij})\) or in \((\hat{N}, \hat{N}^i, \gamma_{ij}, \kappa, k_i, K'_i, \hat{K}_{ij})\), there are always eight which can be chosen on \(\Sigma\) as completely free, whereas the four constrained variables are also freely specifiable but this freedom applies only on one of the \(r = \text{const}\) level surfaces, say on \(\mathcal{I}_0\), in \(\Sigma\).

Hereafter, we shall refer to an initial data specification as near Schwarzschild one (in the strict sense) if each of the eight completely freely specifiable variables takes its Schwarzschildian form, whereas at least one of the four constrained variables is chosen to be slightly different from the corresponding Schwarzschildian form on \(\mathcal{I}_0\).

Note that this definition is somewhat rigid, i.e. there is a much higher variety of conditions which could be used to define near Schwarzschildian initial data configurations. One of these will be applied in subsection 4.3 where \(\hat{K}_{ij}\)—this is one of the freely specifiable variables—will also be chosen to differ from its Schwarzschildian form.

In the succeeding sections we study the asymptotic behavior of near Schwarzschild initial data configurations—yielded by numerical integration of the evolutionary form of the constraints—and investigate their decay rates. Before determining these solutions it is rewarding to recall the functional form of the basic variables, as deduced in [12], for the Schwarzschild solution on \(t_{KS} = \text{const}\) time slices. The non-identically vanishing elements (notably these are all of zero spin-weight) are listed in Table 3, whereas all the other variables \(b, A, B, C, N, \tilde{N}, \tilde{k}, \hat{K}_{qq}, \hat{K}_{qq}, \hat{K}_{q}, \hat{K}_{qq}, \hat{K}_{qq}, \hat{K}_{qq}\) vanish identically.

| variable | functional form | variable | functional form |
|----------|----------------|----------|----------------|
| \(a\)   | \(r^2\)        | \(\hat{K}\) | \(\frac{2}{r}\) |
| \(d\)   | \(r^4\)        | \(\hat{\bar{K}}\) | \(\frac{2}{r\sqrt{1+2M/r}}\) |
| \(\hat{N}\) | \(\sqrt{1+2M/r}\) | \(\hat{R}\) | \(\frac{2}{r^2}\) |
| \(\hat{K}\) | \(-\frac{4M}{r^2\sqrt{1+2M/r}}\) | \(\bar{R}\) | \(2\) |
| \(\kappa\) | \(\frac{2M(1+M/r)}{r^2(1+2M/r)^{1/2}}\) | \((3)\bar{R}\) | \(\frac{8M^2}{r^2(1+2M/r)^2}\) |

Table 3: The non-identically vanishing elements of the basic variables relevant for the Schwarzschildian initial data induced on a \(t_{KS} = \text{const}\) time slice.
2.4 The spherically symmetric solutions

Before proceeding investigating near Schwarzschild initial data configurations by integrating numerically the full set of evolutionary form of the constraints it is also rewarding to have a glance of the pure spherically symmetric solutions to these equations.

2.4.1 Spherical solutions to the parabolic-hyperbolic system

It is straightforward to see that the applied notion of near Schwarzschild initial data configurations does not allow to have spherically symmetric solutions unless the vectorial projection of the extrinsic curvature $k_A$, or equivalently the variable $k$, vanishes identically. If this happens the spherically symmetric form of (11) and (13) are

$$
\frac{d\hat{N}}{dr} = \frac{1}{2r} \hat{N} - \frac{4\kappa \hat{K} r^2 + \hat{K}^2 r^2 + 4}{8r} \hat{N}^3
$$  \hspace{1cm} (46)

$$
\frac{d\hat{K}}{dr} = \frac{2\kappa}{r} \frac{\hat{K}}{r} - \frac{8}{r^2} \hat{N}^3
$$  \hspace{1cm} (47)

Note first that the equations (46) and (47) decouple and the generic solutions to (47) can be seen to take the form

$$
\hat{K} = \hat{K}_{\text{Schw}} + C_{K} \hat{r}.
$$  \hspace{1cm} (48)

where $\hat{K}_{\text{Schw}} = -\frac{4M}{r^2 \sqrt{1 + \frac{2M}{r}}}$ is the Schwarzschildian form of $\hat{K}$ as given in Table 3, and $C_{K}$ is a constant of integration.

By substituting this solution into (46) and solving the yielded equation for $\hat{N}$ we get an analytic (but complicated) expression possessing the asymptotic form

$$
\hat{N}_{C_{K}} = \frac{2}{\sqrt{C_{K}^2 + 4}} + \frac{8 C_{K} M - 4 C_{\hat{N}}}{(C_{K}^2 + 4)^{3/2} r} + \mathcal{O} \left( r^{-2} \right) .
$$  \hspace{1cm} (49)

This solution cannot tend to one at infinity unless $C_{K} = 0$. Note than if $C_{K} \neq 0$ the solution for $\hat{K}$ in (48) does not fall sufficiently fast either. If, however, $C_{K} = 0$ the solution of (46) takes the form

$$
\hat{N} = \frac{\sqrt{r(2M+r)}}{\sqrt{2M \left( C_{\hat{N}} + r \right) + r \left( C_{\hat{N}} + r \right) + 4M^2}} ,
$$  \hspace{1cm} (50)

where $C_{\hat{N}}$ is another positive constant of integration which possesses the desired asymptotic decay

$$
\hat{N} = 1 - \frac{C_{\hat{N}}}{2r} + \frac{3C_{\hat{N}}^2}{8r^2} - \frac{2M^2}{r^2} + \mathcal{O} \left( r^{-9/4} \right) .
$$  \hspace{1cm} (51)
In summing up it worth to be emphasized here again that even in spherical symmetry near Schwarzschild solutions to (11) and (13) fail to be weakly asymptotically flat unless \( K \) takes exactly the Schwarzschildian functional form
\[
K_{\text{Schw}} = -\frac{4M}{r^2 \sqrt{1+\frac{2M}{r}}}
\]

2.4.2 Spherical solutions to the algebraic-hyperbolic system

Note that as in case of the parabolic-hyperbolic system the algebraic-hyperbolic equations (18)–(20) does not admit spherical symmetric solutions either unless \( k \) vanishes identically. If this happens (18) and (20) reduce to the form

\[
\frac{dK}{dr} = -\frac{3}{2r} K - \frac{(3)R}{K r}
\]
\[
k = -\frac{K}{4} - \frac{(3)R}{2K}
\]

By substituting
\[
(3)R = \frac{8M^2}{r^4 \left(1 + \frac{2M}{r}\right)^2}
\]
into (52) one gets the generic solution in the form (see also [3])
\[
K = -\sqrt{C_K \left(2M + r\right) + 16M^2} \frac{1}{r^2 \sqrt{1 + 2M/r}}
\]

where \( C_K \) is a positive constant of integration. This \( K \) and \( \kappa \), the latter determined by (53), have the asymptotic form

\[
K = -\sqrt{C_K} r^{-3/2} - \frac{8M^2}{\sqrt{C_K}} r^{-5/2} + O\left(r^{-7/2}\right)
\]
\[
\kappa = \frac{1}{4} \sqrt{C_K} r^{-3/2} + \frac{6M^2}{\sqrt{C_K}} r^{-5/2} + O\left(r^{-7/2}\right)
\]

If \( C_K = 0 \) both \( K \) and \( \kappa \) take their Schwarzschildian form
\[
K_{\text{Schw}} = -\frac{4M}{r^2 \sqrt{1+\frac{2M}{r}}}
\]

\[
\kappa_{\text{Schw}} = \frac{2M(1+M/r)}{r^2 (1+2M/r)^{3/2}}
\]

Notably these findings, likewise the ones at the end of the previous subsection, indicate that even the purely spherically symmetric near Schwarzschild solutions to the algebraic-hyperbolic equations (18)–(20) will not admit (neither strongly or weakly) asymptotically flat solution unless \( K \) takes exactly the Schwarzschildian functional form \( K_{\text{Schw}} = -\frac{4M}{r^2 \sqrt{1+\frac{2M}{r}}} \).
3 The numerical setup

This section is to introduce the applied numerical scheme and to give some results based on the use of the full set of the evolutionary form of the constraints. These later results indicate the need for a clear separation of the excitation modes from the Schwarzschild background. For this reason a non-linear perturbative approach is outlined in the last subsection.

3.1 Multipole expansion

The applied numerical method is designed such that all the basic variables are expanded by making use of spin-weighted spherical harmonics. As in equations (11)–(17) and (18)–(23) the angular derivatives can be expressed in terms of the $\bar{\sigma}$ and $\bar{\sigma}$ operators this allows us to evaluate all the angular derivatives analytically, whereas the evolution of the expansion coefficients, in the radial direction, is determined by applying a fourth order accurate finite differencing numerical integrator.

As outlined above, in solving the evolutionary form of the constraints all the basic variables will be expanded in terms of spin-weighted spherical harmonics. Accordingly, if $(s)P$ was such a spin-weight $s$ variable it had been replaced by the expansion

$$(s)P(r, \theta, \varphi) = \sum_{\ell = |s|}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} P_{\ell m}(r) \cdot s Y_{\ell m}(\theta, \varphi),$$

where $s Y_{\ell m}$ denote the spin-weight $s$ spherical harmonics. This way the evolutionary form of the constraints can be recast as a set of coupled ordinary differential equations (ODEs) for the expansion coefficients—these were $P_{\ell m}(r)$ for the variable $P^{(s)}$—, whereas all the angular derivatives are evaluated analytically by making use of the well-known action of the operators $\bar{\sigma}$ and $\bar{\sigma}$ on the spin-weight $s$ spherical harmonics $s Y_{\ell m}$ (detailed discussion of some of the related issues can be found e.g. in appendix B of [5]). As indicated in (58) the summation in $\ell$ goes from $\ell = |s|$ (instead of to infinity only) up to some $\ell = \ell_{\text{max}}$ value. In practice, $\ell_{\text{max}}$ was chosen to be $\ell_{\text{max}} = 5$ which was found to be satisfactory in keeping the truncation error tolerably small. The coupled set of ODEs were solved numerically by applying a 4th order accurate adaptive Runge–Kutta–Fehlberg integrator.

3.1.1 Convergence tests

This subsection is to demonstrate that regardless whether the full or non-linear perturbative form of the equations are used, and also regardless whether the algebraic hyperbolic or the parabolic-hyperbolic system are applied our numerical integrator always produces the expected fourth-order convergence rate. In particular, the figures—Fig. (1a) and (1b)—the convergence rate relevant for the evolution of the monopole
\((\ell = m = 0)\) part of \(K\) or \(\Delta K = K - K_{Schw}\) is shown for solutions yielded by the excitation \(K|_{\mathcal{I}_0} = K_{Schw}|_{\mathcal{I}_0} - \alpha \cdot 0 Y^0_1\), with \(\alpha = 0.1\) at \(r_0 = 1\). The convergence rates relevant for the evolution of the variables \(K\) or \(\Delta K\) are plotted for the full and non-linear perturbative forms of the algebraic-hyperbolic system on Fig. (1a), whereas for the full and non-linear perturbative forms of the parabolic-hyperbolic system on Fig. (1b), respectively.

\[ \text{Figure 1: The convergence rate relevant for the evolution of the monopole (} \ell = m = 0\text{) parts of } K \text{ and } \Delta K = K - K_{Schw} \text{ are plotted for solutions yielded by the excitation } K|_{\mathcal{I}_0} = K_{Schw}|_{\mathcal{I}_0} - \alpha \cdot 0 Y^0_1 \text{ or by } \Delta K|_{\mathcal{I}_0} = -\alpha \cdot 0 Y^0_1, \text{ with } \alpha = 0.1, \text{ and using the full or non-linear perturbative form of algebraic-hyperbolic system, respectively.} \]

3.2 Solutions to the full set of equations

This subsection is to present our numerical results relevant for the use of the full set of evolutionary form of the constraint equations. These results are to demonstrate that without a clear separation of the excitation modes from the Schwarzschild background it is really hard to identify those modes which may affect the asymptotic flatness of the yielded initial data configurations.

To see that this is indeed the case we start by inspecting the \(r\)-dependence of the absolute value of the monopole (\(\ell = m = 0\)) part of \(K\) is shown for solutions yielded by a non-spherical excitation. Fig. (2a) is relevant for integrating the full form of

\[ \text{Figure 2: The convergence rates of the evolution yielded by } K|_{\mathcal{I}_0} = K_{Schw}|_{\mathcal{I}_0} - \alpha \cdot 0 Y^0_1 \text{ or by } \Delta K|_{\mathcal{I}_0} = -\alpha \cdot 0 Y^0_1, \text{ with } \alpha = 0.1, \text{ and using the full or non-linear perturbative form of parabolic-hyperbolic system, respectively.} \]

As expected, in both cases solutions to the non-linear perturbative equations converge slightly better than they do for the full forms.
the algebraic-hyperbolic system while Figs. (2b) for the full form of the parabolic-hyperbolic system. The initial data specified at $S_{r_0}$, with $r_0 = 1$, was chosen to be the form $K|_{S_{r_0}} = K_{Schw}|_{S_{r_0}} - \alpha \cdot Y_1^0$, where $\alpha$ is a positive constant, in both cases (i.e. the excitation is for $K$ is of a pure axially symmetric, $m = 0$, spherical mode), while $k|_{S_{r_0}} = k_{Schw} = 0$ in both cases, and also $\hat{N}|_{S_{r_0}} = \hat{N}_{Schw}|_{S_{r_0}}$ for the parabolic-hyperbolic system. Three different solutions are plotted in each cases corresponding to the excitation amplitudes $\alpha = 1, 0.1, 0.01$. By inspecting the corresponding graphs it is immediately transparent that the smaller of the amplitude of the excitation the later the deviation from the desired fall off rate is showing up in the monopole ($\ell = m = 0$) part of $K$. In particular, in both the algebraic-hyperbolic and the parabolic-hyperbolic cases the effect of the excitation with amplitude $\alpha = 0.01$ remains almost completely hidden until the value $r \approx 10^7$ is reached.

Note also that hereafter in all of the figures in the present paper the absolute value of various multipole coefficients will be plotted against $r$. In addition, always log–log scale is applied which allows us to indicate the polynomial character of the pertinent fall off rates immediately.

(a) Evolution yielded by the excitation $K|_{S_{r_0}} = K_{Schw}|_{S_{r_0}} - \alpha \cdot Y_1^0$, with $\alpha = 1, 0.1, 0.01$, and $k|_{S_{r_0}} = k_{Schw} = 0$ by integrating the algebraic-hyperbolic system.

(b) Evolution yielded by the excitation $K|_{S_{r_0}} = K_{Schw}|_{S_{r_0}} - \alpha \cdot Y_1^0$, with $\alpha = 1, 0.1, 0.01$, $k|_{S_{r_0}} = k_{Schw} = 0$ and $\hat{N}|_{S_{r_0}} = \hat{N}_{Schw}|_{S_{r_0}}$ in case of the parabolic-hyperbolic system.

Figure 2: The $r$-dependence of the absolute value of the monopole ($\ell = m = 0$) part of $K$ is shown for solutions yielded by the excitation $K|_{S_{r_0}} = K_{Schw}|_{S_{r_0}} - \alpha \cdot Y_1^0$, with $\alpha = 1, 0.1, 0.01$ and evolved by the algebraic hyperbolic and the parabolic-hyperbolic system, respectively. The smaller the amplitude $\alpha$ of the excitation is the later its effect is showing up in the monopole part of $K$. In both the algebraic-hyperbolic and the parabolic-hyperbolic cases for $\alpha = 0.01$ the deviation from the desired fall off rate remain almost completely hidden until the values $r \approx 10^7$ is reached.

There are some remarks in order now. It is important to be mentioned that even though initially only one of the higher $\ell$-modes ($\ell \geq 1$) of $K$ is excited—likewise the
mode $K_1^0$ was excited in producing the solutions indicated on Figs. (2a) and (2b)—due to the non-linear couplings of various modes, for sufficiently large values of $r$, the monopole part $K_0^0$ gets always to be triggered. Note also that exactly the same type of argument applies when various modes of $k$ or (in the parabolic-hyperbolic case) that of $\hat{N}$ are excited. As the difference between the distinct choices of excitations affects only the specific $r$-span where the monopole part $K_0^0$ of $K$ gets excited the use of higher $\ell$-modes excitations of $K$ turned out to be optimal.

It is worth to be emphasized again that even if the decay rate appears to be optimal on convincingly large $r$-intervals—in virtue of the implications Figs. (2a) and (2b)—there is no guarantee that the observed favorable behavior will remain for arbitrarily large values of $r$. In practice, the above observations have another unfavorable consequence. As the smallest monopole perturbations of $K$ decay as $r^{-1}$ in certain situations even numerical errors may affect the integration of the constraint equations. This means that it may not possible to produce asymptotically flat solutions unless such a numerical noise can be suppressed effectively. Indeed, these two points motivated us to develop the non-linear perturbative scheme of the evolutionary forms of the constraint equations outlined in the next subsection.

### 3.3 Non-linear perturbative form of the constraint equations

As it is clearly indicated by Figs. (2a) and (2b) in many circumstances there may be an obvious need to investigate the evolution of mode excitations separately, meanwhile the background gets to be suppressed. In our present investigations this way we can get a much cleaner picture as the excitations will be separated from the Schwarzschild background and the smaller amplitude but slowly decaying modes can get to be immediately visible as they are not any more suppressed by the Schwarzschild background for considerably long radial intervals.

Note that the basic ideas outlined here had already been applied in using a parabolic-hyperbolic solver in generating Kerr-Schild type black hole initial data in [7]. Nevertheless, as the application and the explicit form of the non-linear perturbative approach is definitely new in case of the algebraic-parabolic system we decided to give here a systematic review of the basic ideas and in the appendix we provide the nonlinear perturbative form of the constraint equations for both the parabolic-hyperbolic and the algebraic-hyperbolic systems.

To start off note first that the non-linear perturbations can essentially be defined with respect to any fixed background. In the present paper this will be chosen to be the Schwarzschildian initial data. In proceeding note first that in both cases, in the parabolic-hyperbolic and the algebraic-hyperbolic ones, schematically the evolutionary form of the constraints can be put into the form

$$\partial_r \xi_{(i)} = \mathcal{R}_{(i)} \left( \partial_A \xi_{(j)}, \xi_{(j)} \right), \quad (59)$$
where \( f_i \) denote the dependent variables. Replacing these dependent variables by the sum \( f_i = (0)f_i + (\Delta)f_i \), where \((0)f_i\) stand for the unperturbed background variables whereas the variables \((\Delta)f_i\) denote the deviations from this background, i.e. 
\[
(\Delta)f_i = f_i - (0)f_i , \tag{60}
\]
we get
\[
\partial_r (\Delta)f_i = \mathcal{R}_i \left( \partial_A (0)f_j + (\Delta)f_j, (0)f_j + (\Delta)f_j \right) - \mathcal{R}_i \left( \partial_A (0)f_j, (0)f_j \right) , \tag{61}
\]
where in the last step it was assumed that the background fields are subject to some equations of the form 
\[
\partial_r (0)f_i = \mathcal{R}_i \left( \partial_A (0)f_j + (\Delta)f_j, (0)f_j + (\Delta)f_j \right) - \mathcal{R}_i \left( \partial_A (0)f_j, (0)f_j \right) \tag{59}.
\]
If the background fields \((0)f_i\) are also solutions to the constraint equations—i.e. to the schematic form (59)—then (the right hand sides) \(\mathcal{R}_i\) possess exactly the same functional form as \(\mathcal{R}_i\) do in (59).

In the particular case of the parabolic-hyperbolic system the above outlined splitting read as
\[
\hat{N} = \theta\hat{N} + (\Delta)\hat{N}, \quad K = \theta K + (\Delta)K, \quad k = \theta k + (\Delta)k \tag{62}
\]
whereas in case of the algebraic-hyperbolic system as
\[
\kappa = \theta\kappa + (\Delta)\kappa, \quad K = \theta K + (\Delta)K, \quad k = \theta k + (\Delta)k . \tag{63}
\]
The non-linear perturbative forms of the parabolic-hyperbolic and algebraic-hyperbolic systems relevant for these splittings are given explicitly in the Appendix.

## 4 Numerical results based on non-linear perturbations

In this section our numerical results relevant for the use of the non-linear perturbative equations deduced from the evolutionary forms of the constraints are presented. These results are to demonstrate that by separating the excitation modes from the Schwarzschild background one can get a much clearer picture and an effective framework that could also be used (if possible) to get control on the asymptotic flatness of the yielded initial data configurations.

### 4.1 Non-linear perturbation with parabolic-hyperbolic system

In this subsection we shall inspect the \(r\)-dependence of the absolute value of various modes of the constrained variables evolved by the non-linear perturbative form of
equations deduced from the parabolic-hyperbolic form of the constraints. The initial data specified at $S_{r_0}$, with $r_0 = 1$, was chosen to be of the form $(\Delta)K|_{S_{r_0}} = -\alpha \cdot 0Y_0^0$—with $\alpha = 0.001$ and $\ell = 1, 2, 3$—, $(\Delta)K|_{S_{r_0}} = 0$ and $(\Delta)\hat{N}|_{S_{r_0}} = 0$. Three different solutions corresponding to the choices $\ell = 1, 2, 3$ are plotted in case of each of the monitored modes. The pertinent set of decay rates for various $\ell$-modes of $(\Delta)K$, $(\Delta)k$ and $(\Delta)\hat{N}$ are indicated on figures 3, 4 and 5.

Start by a quick inspection of figures 3, 4 and 5. It gets immediately transparent that the individual modes pick up already at the very early phase, i.e. for relatively small values of the $r$-coordinate, their asymptotic decay rates. This, in particular, verifies that the separation of the excitation modes from the Schwarzschild background provide us a much clearer picture than the inspection of graphs yielded by the set of full evolution equations. Accordingly, it gets immediately transparent that some of the small amplitude modes may get in the way of getting asymptotically flat initial data configurations as they decay much slower than the higher amplitude background solution, thereby, they become dominant only for larger values of the $r$-coordinate.

There is a very important additional message conveyed by the panels of figures 3, 4 and 5. All the $\ell = 1, 2, 3$ modes of $(\Delta)K$ decay $r^{-2}$ so they would suit to asymptotically flat requirement even in the strong sense, likewise the $\ell = 1, 2, 3$ modes of $(\Delta)k$ and the $\ell = 0, 1, 2, 3$ modes of $(\Delta)\hat{N}$ decay as $r^{-1}$, thereby they all would suit to the strong asymptotically flat requirement. Nevertheless, the monopole part $(\Delta)K_0^0$ of $(\Delta)K$ decays only with the rate $r^{-1}$ which is far too slow to allow asymptotic flatness even in the weak sense.

Based on the behavior of spherically symmetric near Schwarzschild configurations—studied in subsection 2.4.1 in case of the parabolic-hyperbolic system—one expects that whenever $K$ does not have the Schwarzschildian form then $\hat{N}$ cannot tend to one at infinity. Nevertheless, this behavior did not show up in Fig.(5a). The panels in Fig.(6) are to demonstrate that the simple reason beyond this apparent discrepancy is that the monopole part of $\hat{N}$ gets to be triggered for much larger values of $r$ than it happens for $K_0^0$ as $C_K$—it is proportional to the extrapolated value of $(\Delta)K_0^0$ at $r_0 = 1$—is very small and, in virtue of (49), $\hat{N}$ has to tend to $1/\sqrt{1 + C_K/4}$. Notably, the larger the amplitude of the initial excitation in $K$ the earlier the anticipated deviation in the decay rate of $\hat{N}_0^0$ shows up. As the panels in Fig.(6) indicates, in both cases the initial decay rate of $(\Delta)\hat{N}_0^0$ is $r^{-1}$ but after a while the trend changes and $(\Delta)\hat{N}_0^0$ start to tend to some specific constant value, respectively.

### 4.2 Non-linear perturbation with algebraic-hyperbolic system

The $r$-dependence of the absolute value of various modes of the constrained variables evolved are plotted on figures 7 and 8. In the present case these modes are evolved by using the non-linear perturbative form of the algebraic-hyperbolic form of the
(a) The decay rate of the mode $\Delta_{K_00}$ of $\Delta_{K}$ is $r^{-1}$.

(b) The decay rate of the mode $\Delta_{K_{10}}$ of $\Delta_{K}$ is $r^{-2}$.

(c) The decay rate of the mode $\Delta_{K_{20}}$ of $\Delta_{K}$ is $r^{-2}$.

(d) The decay rate of the mode $\Delta_{K_{30}}$ of $\Delta_{K}$ is $r^{-2}$.

(e) The decay rate of the mode $\Delta_{K_{40}}$ of $\Delta_{K}$ is $r^{-2}$.

(f) The decay rate of the mode $\Delta_{K_{50}}$ of $\Delta_{K}$ is $r^{-2}$.

Figure 3: The decay rates of the modes $\Delta_{K_{\ell0}}$, with $\ell = 0, 1, 2, 3, 4, 5$, $m = 0$, of $\Delta_{K}$ are plotted. The evolution was yielded by the non-linear perturbative form of the parabolic-hyperbolic system, whereas the initial perturbation was of the form $\Delta_{K}|_{r=r_0} = -10^{-3} \cdot 0 Y_{\ell0}$ with $\ell = 1, 2, 3$. 
(a) The decay rate of the mode $\delta_{k 1}^0$ of $\delta_k$ is $r^{-1}$.

(b) The decay rate of the mode $\delta_{k 2}^0$ of $\delta_k$ is $r^{-1}$.

(c) The decay rate of the mode $\delta_{k 3}^0$ of $\delta_k$ is $r^{-1}$.

(d) The decay rate of the mode $\delta_{k 4}^0$ of $\delta_k$ is $r^{-1}$.

(e) The decay rate of the mode $\delta_{k 5}^0$ of $\delta_k$ is $r^{-1}$.

Figure 4: The decay rates of the modes $\delta_{k \ell}^0$ with $\ell = 1, 2, 3, 4, 5, m = 0$ of $\delta_k$ are plotted. The evolution was yielded by the non-linear perturbative form of the parabolic-hyperbolic system, whereas the initial perturbation was of the form $\left.\delta_k\right|_{r_0} = -10^{-3} \cdot Y_0^0$ with $\ell = 1, 2, 3$. 

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(a) The decay rate of the monopole part $^0\mathbf{\hat{S}}_N$ of $^0\mathbf{\hat{N}}$ is $r^{-1}$.

(b) The decay rate of the mode $^1\mathbf{\hat{N}}_1$ of $^0\mathbf{\hat{N}}$ is $r^{-1}$.

(c) The decay rate of the mode $^2\mathbf{\hat{N}}_2$ of $^0\mathbf{\hat{N}}$ is $r^{-1}$.

(d) The decay rate of the mode $^3\mathbf{\hat{N}}_3$ of $^0\mathbf{\hat{N}}$ is $r^{-1}$.

(e) The decay rate of the mode $^4\mathbf{\hat{N}}_4$ of $^0\mathbf{\hat{N}}$ is $r^{-1}$.

(f) The decay rate of the mode $^5\mathbf{\hat{N}}_5$ of $^0\mathbf{\hat{N}}$ is $r^{-1}$.

Figure 5: The decay rates of the modes $^\ell\mathbf{\hat{N}}_m^0$, with $\ell = 0, 1, 2, 3, 4, 5$, $m = 0$, of $^0\mathbf{\hat{N}}$ are plotted. The evolution was yielded by the non-linear perturbative form of the parabolic-hyperbolic system, whereas the initial perturbation was of the form $^\ell\mathbf{\hat{K}}|_{r=0} = -10^{-3} \cdot Y_0^0$ with $\ell = 1, 2, 3$. 

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The $r$-dependence of the monopole part $\hat{N}^0_{00}$ of $\hat{N}$ is plotted for the excitation $|\hat{K}|_{r_0} = -0.1 \cdot 0 Y^1_0$.

The $r$-dependence of the monopole part $\hat{N}^0_{00}$ of $\hat{N}$ is plotted for the excitation $|\hat{K}|_{r_0} = -1.0 \cdot 0 Y^1_0$.

Figure 6: The dependence on the amplitude of initial perturbation of $\hat{K}$ of the asymptotic behavior of the monopole part of $\hat{N}$ is indicated. The two panels clearly demonstrate that $\hat{N}^0_{00}$ does not tend to 0 as $r \to \infty$, and, in turn, $\hat{N}$ will not tend to 1, as it happens in the Schwarzschild case. Instead, after an initial $r^{-1}$ decay, once sufficiently small, $\hat{N}^0_{00}$ starts tending to some constant value.

As in the parabolic-hyperbolic case, figures 7 and 8 make it transparent that the individual modes get at almost immediately to their asymptotic decay rates. This clearly verifies that the separation of the excitation modes from the Schwarzschild background provide us a much clearer picture, as opposed to the case when the full evolution equations were used where the background fields dominated considerably large part of the yielded solutions. Here there is again a very important additional message conveyed by the individual panels of figures 7 and 8. Visibly, all the $\ell = 1, 2, 3, 4, 5$ modes of $|\hat{K}|$ decay at the rate $r^{-2}$ and all the $\ell = 1, 2, 3, 4, 5$ modes of $|\hat{k}|$ decay as $r^{-1}$, by which they all suit immediately to the strong asymptotic flatness requirement. Nevertheless, the monopole part $|\hat{K}|^0_0$ of $\hat{K}$ decays at the rate $r^{-1}$ which is again far too slow to allow asymptotic flatness even in the weak sense.

There is an interesting additional difference between the decay rates of the monopole part $|\hat{K}|^0_0$ of $\hat{K}$ and the decay rate of the generic spherically symmetric solution— the latter determined in subsection 2.4— depending whether the non-linear perturbative form is based on the parabolic-hyperbolic or on the algebraic hyperbolic form of
(a) The decay rate of the mode $\Delta K^0_0$ of $\Delta K$ is $r^{-1}$.

(b) The decay rate of the mode $\Delta K^0_1$ of $\Delta K$ is $r^{-2}$.

(c) The decay rate of the mode $\Delta K^0_2$ of $\Delta K$ is $r^{-2}$.

(d) The decay rate of the mode $\Delta K^0_3$ of $\Delta K$ is $r^{-2}$.

(e) The decay rate of the mode $\Delta K^0_4$ of $\Delta K$ is $r^{-2}$.

(f) The decay rate of the mode $\Delta K^0_5$ of $\Delta K$ is $r^{-2}$.

Figure 7: The decay rates of the modes $\Delta K^0_\ell$, with $\ell = 0, 1, 2, 3, 4, 5$, $m = 0$, of $\Delta K$ are plotted. The evolution was yielded by the non-linear perturbative form of the algebraic-hyperbolic system, whereas the initial perturbation was of the form $\Delta K|_{r_0} = -10^{-3} \cdot Y^0_{\ell}$ with $\ell = 1, 2, 3$. 

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(a) The decay rate of the mode $\Delta k_1^0$ of $\Delta K$ is $r^{-1}$.

(b) The decay rate of the mode $\Delta k_2^0$ of $\Delta K$ is $r^{-1}$.

(c) The decay rate of the mode $\Delta k_3^0$ of $\Delta K$ is $r^{-1}$.

(d) The decay rate of the mode $\Delta k_4^0$ of $\Delta K$ is $r^{-1}$.

(e) The decay rate of the mode $\Delta k_5^0$ of $\Delta K$ is $r^{-1}$.

Figure 8: The decay rates of the modes $\Delta k_\ell^0$, with $\ell = 1, 2, 3, 4, 5$, $m = 0$, of $\Delta K$ are plotted. The evolution was yielded by the non-linear perturbative form of the algebraic-hyperbolic system, whereas the initial perturbation was of the form $\Delta K |_{\gamma_0} = -10^{-3} \cdot Y_0^0$ with $\ell = 1, 2, 3$. 

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the constraints. In particular, in the parabolic-hyperbolic case the decay rate of the monopole parts \( (\Delta K)_{00} \) of \( ^{\omega}\kappa \) and of the generic spherically symmetric solution, the latter given by \( (48) \), are \( r^{-1} \). As opposed to this, in the algebraic-hyperbolic case the decay rate of the monopole part \( (\Delta K)_{00} \) of \( ^{\omega}\kappa \) is \( r^{-1} \) whereas, in virtue of \( (56) \), the decay rate of the generic spherically symmetric solutions is \( r^{-3/2} \).

### 4.3 Revisiting the near-Schwarzschild conditions

One of the most striking consequences of the results presented in the previous subsections is that the monopole part \( (\Delta K)_{00} \) of \( ^{\omega}\kappa \) appears to decay far too slow to fit to asymptotically flat initial data configurations. One might conclude that this is simply the consequence of the difference between the elliptic and the evolutionary methods as the latter appears to be unable to impose suitable asymptotic fall of conditions on the to be solutions. However, convincing such a claim may sound we should not forget about the fact that there is a one-to-one correspondence between the space of solutions produced by the elliptic and by either of the evolutionary methods. Accordingly, it seems to be more appropriate to question the suitableness of the conditions that have been applied in all of the investigations—in particular, in \( [3, 4] \) or in the former subsections of the present paper—in identifying near-Schwarzschild configurations. By recalling the determination of near-Schwarzschild configurations given in subsection 2.3 it gets clear immediately that there is an enormous freedom to explore in trying to get more appropriate selection rules. Nevertheless, here—instead of trying to be extremely ambitious—we simply aim to demonstrate that such a relaxation of the selection rules may be the proper direction for future investigations.

Our aim in applying this simple relaxation is to show that there may be really a large freedom in acquiring suitable control on the asymptotic behavior of the monopole part \( (\Delta K)_{00} \) of \( ^{\omega}\kappa \). By inspecting the basic relations relevant for the two alternative evolutionary methods it gets immediately obvious that the minimal alteration of the selection rule for near-Schwarzschild configurations can be achieved by replacing the freely specifiable variable \( ^{\omega}K_{ij} \) (or alternatively \( ^{\omega}K_{qq} \)), this is vanishing for “near-Schwarzschild configurations”, by applying a non-vanishing term \( ^{\omega}K_{ij}^{*} \) in (21), and, in turn, in (20), both used in the algebraic-hyperbolic form of the constraints.

Specifically, we applied the following concrete modifications. First, \( ^{\omega}K_{qq} \) was chosen to be the form \( ^{\omega}K_{qq} = 2Y_{20} \) (i.e. a pure spin-weighted spherical harmonics with \( s = 2, \ell = 2, m = 0 \)) whereas the system was excited by applying another single mode excitation \( ^{\omega}K_{ij} \) with exactly the same excitation. Note that both of these choices are compatible with the strong version of asymptotic flatness as, in virtue of \( (13) \), \( ^{\omega}K_{ij} \) was required only to be bounded. The corresponding asymptotic behaviors of the monopole part \( (\Delta K)_{00} \) of \( ^{\omega}\kappa \) are plotted on Figs. (9a) and (9b).
The freely specifiable variable $\tilde{\kappa}_{qq} = 2Y_0^0$ was chosen to be. The evolution was yielded by the non-linear perturbative form of the algebraic-hyperbolic system the decay rates of the monopole mode $|\Delta K|_0^0$ of $|\Delta K|$ are indicated for initial data with $|\Delta K|_{r=r_0} = -10^{-3} \cdot Y_0^0$.

Figure 9: Two alternative choices for the freely specifiable variable $\tilde{\kappa}_{qq}$ were made. The decay rates of the monopole mode $|\Delta K|_0^0$ of $|\Delta K|$ are indicated. The evolution in each cases was yielded by the non-linear perturbative form of the algebraic-hyperbolic system, whereas the initial perturbation, as in the previous two subsections, was chosen to be of the form $|\Delta K|_{r=r_0} = -10^{-3} \cdot Y_0^0$.

It gets promptly transparent on these figures that, likewise on figures (7a) and (8a), the evolution starts with a fall off nearly at the rate $r^{-1}$. Nevertheless, around $r \approx 10^3$ the tendency is changing on both Figs. (9a) and (9b). Independent of the alternative functional forms of $\tilde{\kappa}_{qq}$, for sufficiently large values of the $r$-coordinate, instead of $r^{-1}$, the more favorable asymptotic behavior with $r^{-3/2}$ fall off manifests itself. However, encouraging it is this result should merely be viewed as a first step in demonstrating that there is an obvious need to find a suitable relaxation of the conditions—applied so far in all the investigations—in identifying near-Schwarzschild configurations. If this can be done we supposedly will also have a much better control on the asymptotic behavior of solutions to the evolutionary forms of the constraints.

5 Final remarks

The two novel alternative evolutionary formulations—the parabolic-hyperbolic and the algebraic-hyperbolic forms—of the constraint equations were used to investigate the asymptotic behavior of “near Schwarzschild” vacuum initial data sets. In doing so a delicate combination of analytic and numerical investigations had to be applied. To generate near Schwarzschild configurations by making use of the evolutionary methods—after fixing eight of the completely freely specifiable variables—, as an input data only the constrained variables can be chosen freely on a two-sphere at...
some finite location. In principle, this should not allow to have any control on the asymptotic behavior of the pertinent solutions. Therefore, it is of fundamental interest to know if asymptotically flat initial data can be produced by the evolutionary forms of the constraints and if so how this can be done.

For this reason, in this paper, a systematic investigation of the asymptotic behavior of near Schwarzschild vacuum initial data sets is carried out by making use of numerical solutions to the evolutionary form of the constraints. Special attention was given to the asymptotic behavior of the yielded initial data configurations. To get a really adequate picture in addition to applying the full set of evolution equations the use of their non-linear perturbative forms—their explicit form was worked out in details and they are presented in appendix of the present paper—turned out to be of fundamental importance.

Our most important finding are that apart from the monopole part of the trace of the extrinsic curvature’s tensorial projection $\hat{K}^{ij} = \hat{\gamma}^{ij} K_{ij}$ all modes of the constrained variables—$K$ and $k$ in the algebraic-hyperbolic case and $\hat{N}$, $\hat{K}$ and $\hat{k}$ in the parabolic-hyperbolic case—decay sufficiently fast to support the strong asymptotic flatness of the solutions to the evolutionary form of the constraints. Nevertheless, the monopole part of $K^I_l$ was found to decay far too slow even to allow the initial data configurations to be weakly asymptotically flat. Therefore, getting control on the decay rate of $K^I_l$ is really of fundamental importance in producing asymptotic flatness of initial data configurations.

While inspecting what might explain the experienced lack of asymptotic flatness of the generated initial data one should not miss to have a closer look at the set of conditions used to determine near-Schwarzschild configurations. Indeed, the applied selection rule require—in all of the pertinent investigations, including [3, 4] and considerable part of the present one—each of the eight freely specifiable variables to take exactly its Schwarzschildian form. Our results indicate that by applying this selection rule the entire setup may get to be so rigid that it allows only the exact Schwarzschild initial data to be “asymptotically flat”. Based on these observations, in subsection 4.3 we used a simple relaxation of the previously applied selection rules. This was done by allowing the trace-free part $\tilde{K}_{ij}$ of tensorial projection $K_{ij}$ of the extrinsic curvature $K_{ij}$ to be non-vanishing. Note that on a $t_{KS} = const$ Kerr-Schild slice $\tilde{K}_{ij}$ vanishes in the exact Schwarzschild case. Notably, this slight change—this was done by respecting the fall off requirement imposed by strong asymptotic flatness on $\tilde{K}_{ij}$—by integrating the non-linear perturbative form of the algebraic-hyperbolic system for the monopole mode, $\omega K^0_0$, of $\omega K$ yields instead of the slow fall off rate $r^{-1}$ the more favorable $r^{-3/2}$ asymptotic behavior.

Obviously this result should merely be viewed as a first promising step in demonstrating that there is a real need to find suitable loosening of the conditions identify-
ing near-Schwarzschild configurations. Notably, in addition to $\hat{K}_{ij}$ (or $\hat{K}_{qq}$) the other freely specifiable variables $a, b, \widehat{N}, N$—they store six other real functions—could also be altered in an analogous way. The clear up whether appropriate modifications of these variables could be made such that a better control on the asymptotic behavior of solutions to the evolutionary forms of the constraints can be acquired remains for further investigations.

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**Appendix: The non-linear perturbative equations**

This appendix is to give the explicit non-linear perturbative forms of the parabolic-hyperbolic and algebraic-hyperbolic systems relevant for the splittings (62) and (63).
The non-linear perturbative form of the parabolic-hyperbolic system

\[ \dot{\mathbf{N}} \left[ \partial_r \left( \Theta^{\mathbf{N}} \right) - \frac{1}{2} \frac{\ddot{\mathbf{N}}}{2} \left( \partial_\perp \Theta^{\mathbf{N}} \right) - \frac{1}{2} \frac{\ddot{\mathbf{N}}}{2} \left( \partial_\parallel \Theta^{\mathbf{N}} \right) \right] \]

\[ - \frac{1}{2} \partial_\perp^{-1} \left\{ \left( \partial^{\mathbf{N}} \right)^2 + 2 \partial^{\mathbf{N}} \right\} \left[ a \left( \partial_\perp \Theta^{\mathbf{N}} - \partial_\parallel \Theta^{\mathbf{N}} \right) \right] \]

\[ - b \left( \partial^{2\mathbf{N}} - \frac{1}{2} \partial_\perp \left( \partial_\perp \Theta^{\mathbf{N}} \right) - \frac{1}{2} \partial_\parallel \left( \partial_\parallel \Theta^{\mathbf{N}} \right) \right) \]

\[ + \left( \partial^{\mathbf{N}} \right)^2 + 2 \theta^{\mathbf{N}} \left[ a \left( \partial_\perp \Theta^{\mathbf{N}} - \partial_\parallel \Theta^{\mathbf{N}} \right) \right] \]

\[ - b \left( \partial^{2\mathbf{N}} - \frac{1}{2} \partial_\perp \left( \partial_\perp \Theta^{\mathbf{N}} \right) - \frac{1}{2} \partial_\parallel \left( \partial_\parallel \Theta^{\mathbf{N}} \right) \right) \]

\[ - b \left( \partial^{2\mathbf{N}} \right) - \left( \partial^{\mathbf{N}} \right)^3 + 3 \theta^{\mathbf{N}} \left( \partial^{\mathbf{N}} \right)^2 + 3 \theta^{\mathbf{N}} \left( \partial^{\mathbf{N}} \right) \left( \partial^{\mathbf{N}} + \theta \mathbf{B} \right) = 0, \quad \text{(PH.1)} \]

where

\[ A = \theta \mathbf{A}, \quad \text{(PH.2)} \]

\[ \mathbf{B} = \mathbf{B} - \theta \mathbf{B} = - \kappa \mathbf{K} - \frac{1}{2} \left( \partial^{\mathbf{K}} + 2 \alpha \mathbf{K} \mathbf{K} \right) \]

\[ + \frac{1}{2} \partial_\perp^{-1} \left[ 2 \alpha \left( \partial_\perp \mathbf{K} \mathbf{K} + \alpha \mathbf{K} \mathbf{K} + \alpha \mathbf{K} \mathbf{K} \right) \right] \]

\[ - b \left( \partial^{2\mathbf{K}} + 2 \theta^{\mathbf{K}} \mathbf{K} \mathbf{K} \right) - \left( \partial^{\mathbf{K}} + 2 \theta^{\mathbf{K}} \mathbf{K} \mathbf{K} \right) \]. \quad \text{(PH.3)}

\[ \partial_r \left( \partial^{\mathbf{K}} \right) - \frac{1}{2} \frac{\ddot{\mathbf{N}}}{2} \left( \partial_\perp \Theta^{\mathbf{K}} \right) - \frac{1}{2} \frac{\ddot{\mathbf{N}}}{2} \left( \partial_\parallel \Theta^{\mathbf{K}} \right) \]

\[ - \frac{1}{2} \left[ \partial^{\mathbf{N}} \left( \partial_\perp \Theta^{\mathbf{K}} \right) + \partial^{\mathbf{N}} \left( \partial_\parallel \Theta^{\mathbf{K}} \right) + \partial^{\mathbf{N}} \left( \partial_\parallel \Theta^{\mathbf{K}} \right) \right] + \partial^{\mathbf{K}} = 0, \quad \text{(PH.4)} \]

where

\[ \partial_\perp \left( \partial^{\mathbf{N}} \right) = \left( \partial^{\mathbf{K}} \right) + \partial_\perp \left( \partial^{\mathbf{K}} \right) \]

\[ + \frac{1}{2} \left[ \partial^{\mathbf{K}} \left( \partial_\perp \Theta^{\mathbf{K}} \right) + \partial^{\mathbf{K}} \left( \partial_\parallel \Theta^{\mathbf{K}} \right) + \partial^{\mathbf{K}} \left( \partial_\parallel \Theta^{\mathbf{K}} \right) \right] - \kappa \left( \partial_\perp \Theta^{\mathbf{K}} \right) + \partial_\parallel \left( \partial^{\mathbf{K}} \right) \]

\[ + \frac{1}{2} \partial_\perp^{-1} \left[ \alpha \left( \partial^{\mathbf{K}} \right) \dot{\mathbf{K}}_{\perp \perp} + \alpha \left( \partial^{\mathbf{K}} \right) \dot{\mathbf{K}}_{\parallel \perp} - b \left( \partial^{\mathbf{K}} \right) \dot{\mathbf{K}}_{\perp \perp} - b \left( \partial^{\mathbf{K}} \right) \dot{\mathbf{K}}_{\parallel \perp} \right] \]

\[ + \left( q^2 \mathbf{D} \left( \partial^{\mathbf{K}} \right) \right) \left( \partial^{\mathbf{K}} \right). \quad \text{(PH.5)} \]
Finally,

\[
\partial_r^{(\alpha)K} - \frac{1}{2} \tilde{N}(\partial^{(\alpha)K}) - \frac{1}{2} \tilde{N}(\partial^{(\alpha)K}) \\
- \frac{1}{2} d^{-1}\left\{ \omega_{N} \left[ a \left( \partial^{(\alpha)K} + \bar{\partial}^{(\alpha)K} \right) - b \left( \partial^{(\alpha)K} - \bar{\partial}^{(\alpha)K} \right) \right] \right. \\
+ \omega_{N} \left[ a \left( \partial^{(\alpha)K} + \bar{\partial}^{(\alpha)K} \right) - b \left( \partial^{(\alpha)K} - \bar{\partial}^{(\alpha)K} \right) \right] \\
+ \omega_{N} \left[ a \left( \partial^{(\alpha)K} + \bar{\partial}^{(\alpha)K} \right) - b \left( \partial^{(\alpha)K} - \bar{\partial}^{(\alpha)K} \right) \right] \right\} + ^{(\alpha)F} = 0, \quad \text{(PH.6)}
\]

where

\[
^{(\alpha)F} = F - \theta^{(\alpha)F} = \frac{1}{4} d^{-1}\left\{ \omega_{N} [2 a B (\partial^{(\alpha)K}) - b (\bar{\partial}^{(\alpha)K}) + \bar{\partial}^{(\alpha)K}] \right. \\
+ 2 a B \omega_{K} - b (\bar{\partial}^{(\alpha)K} + \bar{\partial}^{(\alpha)K}) + cc.] \\
+ \omega_{N} [2 a B (\partial^{(\alpha)K}) - b (\bar{\partial}^{(\alpha)K} + \bar{\partial}^{(\alpha)K}) + cc.] \right\} \\
- d^{-1}\left[ \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \left( \partial^{(\alpha)N} + \partial^{(\alpha)N} \right) \right] \\
+ \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)N} + cc.] + \frac{1}{2} \tilde{N}^{(\alpha)K}. \quad \text{(PH.7)}
\]

The non-linear perturbative form of the algebraic-hyperbolic system

\[
\partial_r^{(\alpha)K} - \frac{1}{2} \tilde{N} \partial^{(\alpha)K} + \frac{1}{2} \tilde{N} \partial^{(\alpha)K} \\
- \frac{1}{2} d^{-1}\left[ a \left( \partial^{(\alpha)K} + \bar{\partial}^{(\alpha)K} \right) - b \left( \partial^{(\alpha)K} - \bar{\partial}^{(\alpha)K} \right) \right] + ^{(\alpha)F} = 0, \quad \text{(AH.1)}
\]

\[
\partial_r^{(\alpha)K} - \frac{1}{2} \tilde{N} \partial^{(\alpha)K} - \frac{1}{2} \tilde{N} \partial^{(\alpha)K} \\
+ \tilde{N}^{(\alpha)K} - d^{-1}\left[ \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)K} + \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)K} \right] \right. \\
+ \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)K} + \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)K} \right\} \\
- \tilde{N}^{(\alpha)K}(\partial^{(\alpha)K})^{-1}\left[ \partial^{(\alpha)K} - d^{-1}\left[ \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)K} + \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)K} \right] \right] + \partial^{(\alpha)F} = 0 \\
\quad \text{(AH.2)}
\]

where the lower order source terms are

\[
^{(\alpha)F} = \frac{1}{4} d^{-1}\left[ 2 a B (\partial^{(\alpha)K}) - b (\bar{\partial}^{(\alpha)K}) + cc.] \\
- d^{-1}\left[ \left( a^{(\alpha)K} - b^{(\alpha)K} \right) \partial^{(\alpha)N} + cc.] - \left( \partial^{(\alpha)K} - \frac{1}{2} \tilde{N} \right) \tilde{N} \right. \quad \text{(AH.3)}
\]
and
\[
\begin{align*}
&\omega_f = -\frac{1}{2} \left[ (\Delta_k \partial \tilde{N} + \tilde{\Delta}_k \partial \tilde{N}) 
+ \frac{1}{2} \hat{N} (d \theta^0 k)^{-1} \left[ (a (\Delta_k - b \tilde{B} k)(\tilde{B} \theta^0 k + B \theta^0 k) + (a k - b \tilde{K} k)(B k + B k) 
+ (a \tilde{\Delta}_k - \tilde{B} (\Delta k))(C \theta^0 k + A \theta^0 k) + (a k - b \tilde{K} k)(C k + A k) \right] 
- \left(\Delta_k - \frac{1}{2} \tilde{\Delta}_k k \right) \partial \tilde{N} - \frac{1}{2} \hat{N} (2 \theta^0 k)^{-1} \partial \theta^0 \right. 
\left. + \hat{K} (\Delta_k k) \right) 
\right) 
\end{align*}
\]

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