On the Special Role of Symmetric Periodic Orbits in a Chaotic System

L. Benet\textsuperscript{a}, C. Jung\textsuperscript{b}, T. Papenbrock\textsuperscript{c}, T. H. Seligman\textsuperscript{b}

\textsuperscript{a}Inst. de Física, National University of Mexico (U.N.A.M.), Cuernavaca, México
\textsuperscript{b}Inst. de Matemáticas, National University of Mexico (U.N.A.M.), Cuernavaca, México
\textsuperscript{c}Inst. for Nuclear Theory, University of Washington, Seattle, USA

Abstract

After early work of Hénon it has become folk knowledge that symmetric periodic orbits are of particular importance. We reinforce this belief by additional studies and we further find that invariant closed symplectic submanifolds caused by discrete symmetries prove to be an important complement to the long known role of the orbits. The latter have particular importance in semi-classics. Based on the structural stability of hyperbolic horseshoes we give an argument that opens an avenue to the understanding of these facts.

1 Introduction

In 1965 [1] Hénon charts the simple symmetric periodic orbits of the so-called Copenhagen problem, i.e. the planar reduced circular three-body problem with equal masses for the two heavy bodies or, in other words, the planar orbits of a small planet or comet in a circular double star system. He states that the importance of these orbits resides in the fact that in a subsequent paper [2], where he analyzes the general periodic orbits of the same system, he can discuss the latter essentially in terms of the former.

More recently we find a discussion of symmetric short periodic orbits in simpler systems [3], and in some sense it seems to be folk knowledge that such orbits are not only easy to find but usually also particularly relevant. In this context it is important to state what we mean by relevance. This can not be a strict mathematical concept, but we may at least give one family of physical and one of mathematical examples, where this relevance is obvious.
In semi-classical considerations of chaotic systems trace-formulae of the Selberg-Gutzwiller type [4] play a central role. These formulae are sums over periodic orbits and from their structure it is clear that the leading terms are associated with short periodic orbits that are not too unstable. It is precisely such periodic orbits that according to [3] may be readily found among the symmetric ones and specific calculations have often confirmed this as may be exemplified by the collinear states of Helium [5]. Particular relevant work in this context was presented by Cvitanovic and Eckhardt on the symmetry decomposition of the cycle expansion for the zeta functions [6].

On the other hand the discussion of the chaotic saddle of scattering problems in two dimensions implies a Smale horseshoe construction starting from a few fundamental periodic orbits which overshadow the entire saddle [7]. Again experience shows that in presence of symmetry, the symmetric orbits seem to be favored as fundamental periodic orbits.

Recently we came across the importance of symmetric orbits in quite different contexts and the relevance is marked to an extent that can no longer be assumed to be casual. The purpose of this presentation is not only to show the essential role symmetric orbits play in certain systems. We shall show that their role subsists under significant deformations of the systems where the symmetry is destroyed. This should provide a clue as to the reason of their importance.

We shall start with an analysis of the chaotic saddle of the circular reduced three-body problem that shows that in this complicated case the infinite order horseshoe is well described by symmetric basic orbits though the parabolic ones (orbits that are asymptotically parabolae, not marginally stable orbits) have to be included [8,9], thus extending the findings of Hénon to the full scattering problem. Next we shall consider results obtained for systems of many identical particles. We find short periodic orbits that are not only invariant under some subgroup of the symmetry group, but are mapped pointwise on themselves. Such periodic orbits and indeed the entire invariant submanifolds in which they lie are shown to be fundamental. Finally we shall return to very simple systems such as the three-disc problem where we shall see that due to structural stability the relevance of such symmetric orbits persists under deformations, where the symmetry is largely destroyed. This gives us a first hint as to the reason for the recurrent importance of these orbits.

2 Scattering of a comet from a double star

Consider the coplanar scattering motion of a small body or comet off a binary star system on a circular orbit. Due to the huge ratio between the mass of
the stars and that of the comet we can consider this to be a reduced three-body problem where we neglect the influence of the comet on the motion of the stars. The Hamiltonian of this problem in rotating coordinates does not depend explicitly on time and therefore it is a constant of the motion, the Jacobi integral, which is given by:

\[ J = \frac{1}{2}(p_x^2 + p_y^2) - \omega(xp_y - yp_x) + V_g(x, y), \]  

where the gravitational potential has the form

\[ V_g(x, y) = -\frac{Gm_1}{\sqrt{(x - x_1)^2 + y^2}} - \frac{Gm_2}{\sqrt{(x - x_2)^2 + y^2}}. \]

Here, \( m_1 \) and \( m_2 \) are the masses of the stars, \( D \) their separation and \( x_1 = Dm_2/M \) and \( x_2 = -Dm_1/M \) their positions, with their center of mass at the origin. \( \omega \) is the angular velocity of their circular motion. One of the best explored systems is the so-called Copenhagen problem, where the two stars have equal masses.

For the general case (different masses of the stars) it follows from Eqs. (1-2) that the system has a symmetry resulting from simultaneous reflection with respect to the x-axis and time reversal. For the Copenhagen problem Hénon [1] explored the periodic orbits symmetric under this operation, focusing on the ones which crossed the x-axis only once with \( \dot{y} > 0 \). These he called the simple symmetric periodic orbits. He soon discovered that some related families of double and higher order symmetric periodic orbits also entered the picture in a relevant way, and included them on the chart. These new families of orbits have all many crossings with the x-axis (\( \dot{y} > 0 \)), but only one or at most two perpendicular ones respectively for orbits with an odd or even number of crossings. Figure 1 shows Hénon's chart of simple symmetric periodic orbits for the Copenhagen problem.

This chart shows several regions with very distinct behaviour as we vary the Jacobi integral [8]. In particular, for the Jacobi integral interval defined by \( J_1 \) and \( J_2 \) in Fig. 1, we find the families \( h_n, c_n \) and \( f_n \) of simple symmetric periodic orbits (the \( c_n \) and \( f_n \) families are symmetry related by the reflection on the y-axis and time reversal). The subscript \( n \) denotes the number of x-axis crossings with \( \dot{y} > 0 \) before the periodic orbit is closed. Increasing values of \( n \) imply longer periodic orbits and farther position of the turning points. In the Jacobi interval where we have the families \( h_\infty \) and \( c_\infty \), it is not enough to include all the finite symmetric periodic orbits, but we have to include also their accumulation points in our considerations. The latter are related to symmetric parabolic orbits that reach out to infinity; this implies that the
chaotic saddle also reaches out to infinity which is no surprise because of the long-range nature of the Coulomb potential.

We shall demonstrate this statement for the value $J = 0.5$ of the Jacobi integral which belongs to the above mentioned interval in Hénons chart. The other regions are discussed in detail in [8]. In Fig. 2a we present a Poincaré section constructed only with the shortest symmetric periodic orbits; Fig. 2b is an enlargement of the region between the stars. The surface of section is defined by the position on the x-axis where the comet crosses this axis ($\dot{y} > 0$), and by the angle $\gamma = \tan^{-1}(v_y/v_x)$ that the velocity vector forms in the synodic frame with the x-axis.

Notice first that the manifolds of the shortest periodic orbits lie deep inside the corresponding horseshoe construction, and that they appear within some external manifolds. It is known that the most important manifolds in the Smale horseshoe construction are precisely the outer ones, which build the so-called fundamental (curvilinear) rectangle [7]. By combining the arrangement of symmetric periodic orbits of Fig. 1 with the structure of the surface of section of Fig. 2b one is led to conclude that the outermost manifolds of the present chaotic saddle correspond precisely to those of the accumulation points, i.e. to the limiting parabolic orbits. Thus, to have a full understanding of the structure of the chaotic saddle for this case, beside the symmetric periodic orbits, one has to consider in the description the limiting parabolic orbits.
Fig. 2. Surface of section of the Copenhagen problem for $J = 0.5$. Only the manifolds of the hyperbolic and inverse hyperbolic fixed points associated with the periodic orbits $c_{11}$, $h_{6}$ and $f_{7}$ have been plotted. (a) Global structure of the stable and unstable manifolds. (b) Enlargement of the region between the stars. The outer manifolds belong to parabolic orbits ($c_{\infty}$, $h_{\infty}$ and $f_{\infty}$).

as part of the set of primitive periodic orbits. Moreover, since the number of independent symmetric periodic orbits needed for the whole construction of this saddle is infinite, we are in the case of an infinite order horseshoe.

A similar argument can be made in the case of small mass ratio between the less massive star and the total mass of the binary system. One starts by analyzing the symmetric collision orbits one obtains assuming zero mass for the smaller star but considering its collisions with the comet. In this case no chaotic scattering occurs since no homoclinic or heteroclinic connections between such orbits exist. Yet it turns out that the symmetric periodic collision orbits dominate the situation for small but non-zero mass ratios where homoclinic and heteroclinic connections between such orbits do occur. Details of this more complicated situations are forthcoming in [9]. In all the cases discussed above and in refs. [1,8,9] the dominant role of the symmetric periodic orbits is manifest.
To conclude this section, we shall briefly mention another interesting case in celestial mechanics where the symmetry of the periodic orbits is crucial, the central configurations. A central configurations is defined in the center of mass frame as one where the net force acting on all particles at a given time is radial and proportional to their distance from the origin [10]. Note that the constant of proportionality may be time dependent. In this case, the interest is focused on constraining the initial conditions in such a way that the particles stay for all times within the submanifold spanned by the central configurations. These situations are important in the analysis of collision orbits and of expanding gravitational systems, among others [10]. Notice that this case is different from the situation discussed above, where the complete periodic orbits maps onto itself by the symmetry operation, whereas now each point of such an orbit maps onto itself. Such configurations are a special case of the situation discussed in the next section.

3 Closed invariant submanifolds in interacting few–body systems

The central configurations discussed at the end of the previous section constitute a particular case of a larger family of closed invariant subspaces resulting from symmetries of the system. In our discussion we wish to exclude the well-known symmetry reduction of the dimensionality of a system in presence of a continuous symmetry that gives rise to additional constants of motion that are in involution. Rather we are interested in the following situation: The discrete symmetry group of a system or a discrete subgroup of a symmetry group may act trivially on a submanifold of phase space. Such a submanifold is invariant under the Hamiltonian flow of the system. The simplest example thereof is a reflection symmetry on a line or plane in configuration space. In this case the two or four dimensional phase space generated by points on this line or plane and momenta along this line or in this plane constitute the submanifold. Any periodic orbit that exists in the submanifold is obviously symmetric and indeed trivially so as the symmetry maps the orbit pointwise onto itself. The Hamiltonian may be restricted to such submanifolds and we can find periodic orbits in a comparatively low-dimensional space. This is a technical advantage, but there seems to be more to these invariant submanifolds. Prosen recently showed that such symmetry planes may carry scars of the wave-functions [11].

In rotationally invariant systems of identical particles such invariant submanifolds occur systematically [12]. Indeed they occur whenever we have a direct product of two symmetry groups that have a common subgroup on a submanifold. In this case we can use the elements of one group to annihilate the action of those of the other group. The most trivial example is obtained by arranging the $n$ particles on a ring at equal distances and with momenta perpendicular to the ring. Not only are the $n$-cycles of the permutation group $S_n$
and the rotations $C_n$ isomorphic, but they act in the same way on such a configuration. We can therefore consider a direct product $C_n \times C_n$ where the first represents the permutational $n$-cycles and the second the corresponding inverse rotation about an axis perpendicular to the ring in its center. While this group acts non-trivially on the entire phase space it clearly acts trivially on the phase space generated by such ring-configurations. Note that the original phase space was 4-n dimensional while the submanifold is only 2-dimensional. More complicated configurations leading to smaller invariance groups and thus larger dimensional manifolds can be constructed [12].

An interesting question here will be to analyze the stability of such periodic orbits in the manifold and perpendicular to it. A number of examples have been studied [12,13] and a wide variety of behaviour has been found in systems from three to nine particles in central harmonic and anharmonic fields interacting through Coulomb and quartic interactions. The small number of degrees of freedom does not imply that only a few particles are moving; rather it implies some form of collective movement of all particles. We shall show that this collectivity can have a physical meaning if the instability perpendicular to the submanifold is fairly small.

The obvious application derives from semi-classical considerations in quantum mechanics. Semi-classical periodic orbit theory, as developed by Balian, Bloch, Gutzwiller and Berry [4,14], has been used successfully to analyze chaotic systems with few degrees of freedom [5,15]. Within the semi-classical approximation, the level density is given in terms of all periodic orbits of the classical system

$$\rho(E) = \sum_p A_p(E) \exp \frac{i}{\hbar} S_p(E).$$

Here the amplitude $A$ depends mainly on the stability of the orbit labeled by $p$, and $S_p$ is its action. Including only the shortest orbits in the sum (3) yields a coarse grained spectrum, and therefore the long range correlations in the density of states are determined by the shortest and most stable orbits. Generically, individual periodic orbits are not associated with individual energy levels or wave functions. However, short and fairly stable periodic orbits may scar individual wave functions, i.e. a wave function displays higher than average intensities in the vicinity of a particular periodic orbit [16].

In chaotic systems of two degrees of freedom, periodic orbits are the only generic closed invariant sets in classical phase space besides the energy shell. Symmetries can introduce other such sets as we have seen above. In systems with more than two degrees of freedom a rich structure of such subsets may develop. Especially in the case of rotationally invariant systems of identical particles [17], the interplay between the discrete subgroups of the rotation
Fig. 3. Schematic plots of configurations that restrict the classical motion to invariant submanifolds in the case of a 4-particle system. Positions and momenta of the particles are indicated by points and arrows, respectively.

On these submanifolds the classical motion of the particles is highly correlated and depends on a small number of "collective" degrees of freedom only. In practice the summation over the short periodic orbits of submanifolds depending on one or two collective degrees of freedom is possible. The search for these orbits as well as the computation of their period and action may be performed in the appropriate low dimensional submanifold. The computation of their stability exponents and their Maslov index however has to take place in full phase space and incorporates the many-body effects.

The importance of any invariant submanifold depends crucially on its stability
properties. In what follows we show (i) that periodic orbits inside such submanifolds may be fairly stable and short [12] and thus important for the long range correlations in the density of states and (ii) that entire submanifolds may display a classical motion that is weakly unstable in transverse directions while being very unstable inside the submanifold. We show that such a behavior leads to a notable enhancement of the revival in the autocorrelation function and thus indicates some degree of localization around an invariant submanifold which has collective character and is associated with scars [11,13].

3.1 Short periodic orbits inside invariant submanifolds

We consider in two dimensions a system of \( N \) electrons bound by an harmonic oscillator. In polar coordinates the Hamiltonian reads

\[
H = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{p_i^2}{r_i^2} + \frac{l_i^2}{r_i^2} \right) + \frac{1}{2} \sum_{i<j} \frac{1}{\sqrt{r_i^2 + r_j^2 - 2 r_i r_j \cos(\psi_i - \psi_j)}}. \tag{4}
\]

The simplest periodic orbit may be constructed from the requirement that initial conditions should be connected by the group \( C_{Nv} \), i.e. we enter the equations of motion with the ansatz

\[
r_i = r(t), \quad p_i = p(t), \quad \psi_i = 2\pi \frac{i}{N}, \quad l_i = 0, \quad i = 1...N. \tag{5}
\]

Then the time evolution of the functions \( r(t) \) and \( p(t) \) is governed by the Hamiltonian

\[
\tilde{H} = \frac{1}{2} p^2 + \frac{1}{2} r^2 + \frac{c_N}{r}, \quad c_N = \frac{1}{N} \sum_{i<j} \frac{1}{\sqrt{2 - 2 \cos(2\pi \frac{i-j}{N})}}. \tag{6}
\]

At zero total angular momentum and for numbers of particles in the range \( 3 \leq N \leq 9 \) we computed a total of 40 periodic orbits. The configurations considered depend at most on two degrees of freedom and evolve from small oscillations near equilibrium configurations. They are shown schematically in Fig. 3 for \( N = 4 \). We traced each orbit for a range of energy and computed its period \( T \) and its full phase space monodromy matrix \( M \). Periods are approximately in the range \( 1.5 < T < 4 \). By comparison with periods of true many–body orbits they are found to be among the short ones. The considered periodic orbits enter into the periodic–orbit sum of eq. (3) with a factor proportional to \( |\det(1 - M)|^{-1/2} \), where the eigenvalues corresponding to zero Lyapunov exponent due to the continuous symmetries of the problem have
been eliminated. Most periodic orbits are very unstable but for any $N$ we also found a few periodic orbits that are stable or only slightly unstable. In Fig. 4 a logarithmic plot of $|\text{det}(1 - M)|^{-1/2}$ versus energy is shown for $N = 4$.

![Fig. 4. Stability factor $|\text{det}(1 - M)|^{-1/2}$ as function of energy $E/N$ for $N = 4$. (Orbit 1: circles, orbit 2: triangles, orbit 3: diamonds, orbit 4: filled diamonds, orbit 5: squares, orbit 6: filled circles, orbit 7: filled triangles. The orbit–number refers to the corresponding configuration shown in Fig. 3)](image)

### 3.2 Scars of closed invariant submanifolds

We consider a system of four particles in two dimensions with Hamiltonian

$$H = \sum_{i=1}^{4} \left( \frac{1}{2m} p_i^2 + 16\alpha |r_i|^4 \right) - \alpha \sum_{1 \leq i < j \leq 4} |r_i - r_j|^4,$$  \hspace{0.5cm} (7)$$

where $p_i = (p_{ix}, p_{iy})$ and $r_i = (x_i, y_i)$ with $i = 1, \ldots, 4$ are the two–dimensional momentum and position vectors of the $i^{th}$ particle, respectively. We use units where $m = \alpha = 1, \hbar = 0.01$; then coordinates and momenta are given in units of $\hbar^{1/3} \alpha^{-1/6} m^{-1/6}$ and $\hbar^{2/3} \alpha^{1/6} m^{1/6}$, respectively. The corresponding classical Hamiltonian has the scaling relation $H(\gamma^{1/2} p, \gamma^{1/2} r) = \gamma H(p, r)$. This shows that the structure of classical phase space is independent of energy. Moreover, energy and total angular momentum are the only integrals of motion, and the system is non–integrable.

We choose the initial conditions in such a way that positions and momenta exhibit the symmetry of a rectangle. Such a configuration is shown in Fig. 3, configuration 3. The appropriate manifold is spanned by the two–dimensional vectors $p = (p_x, p_y)$ and $r = (x, y)$ giving the momentum and position of particle 1. The associated Hamiltonian $\tilde{H}(p, r) = \frac{1}{2} p^2 + 16x^2 y^2$ has been studied extensively in the literature, both classical and the quantum mechanics [18]. The classical system is essentially chaotic. One very small island of stability is
The stability exponents of periodic orbits in the central region of the manifold are quite large. However, the picture changes when one considers the stability in the directions transverse to the submanifold. The sums of the transverse stability exponents of the central orbits are found to be considerably smaller than the stability exponents inside the manifold. This shows that a trajectory may be trapped quite a time in the vicinity of the submanifold before leaving it. One might therefore expect that this entire manifold rather than a specific periodic orbit may scar wave functions of the corresponding quantum system.

To demonstrate this conjecture we consider the time evolution of a Gaussian wave packet

$$\Psi(r, t) = c \exp \left[ -\frac{1}{2}(r - r_0)^TA(r - r_0) + \frac{i}{\hbar}p_0^T(r - r_0) \right]$$

(8)

where $<p> = p_0$ and $<r> = r_0$ define a point on the submanifold. We have used the shorthand notation $r = (r_1, r_2, r_3, r_4)$ and $p = (p_1, p_2, p_3, p_4)$ for configuration and momentum space vectors, respectively. The autocorrelation function $C(t) = <\Psi(t = 0)|\Psi(t)>$ is computed in the semi-classical approximation. Within the manifold, we used Heller’s cellular dynamics [20,21] which takes into account the nonlinearity of the classical motion. In the transverse direction we used linearized dynamics. This approximation is justified in the time scales considered, since the classical return probability to the manifold of transversely escaping trajectories is negligible.

Fig. 5. Autocorrelation function $C(t)$ of a symmetrized wave packet launched on a periodic orbit with period $T$ inside a weakly unstable manifold. In addition to the linear revival around $t = T/2$, strong nonlinear revival is seen for larger times.

We launch wave packets along periodic or aperiodic orbits within the submanifold and consider their revival as measured by the autocorrelation function. To achieve shorter recurrence times the initial packet was symmetrized with respect to the reflection symmetry of the system within the submanifold. For not too unstable periodic orbits we expect a fairly strong revival after one period, known as the linear revival [21]. As an example, we show in Fig. 5...
the real part of the autocorrelation function starting on the shortest periodic orbit with period $T$ inside the manifold. We indeed find strong linear revival. However, at larger times we find scattered strong revivals, the revival corresponding to the second period not being dominant. This implies that a significant fraction of the original amplitude remains within the submanifold, and that this fact is not related to the periodic orbit we started on. Revivals calculated for packets started on aperiodic orbits show similar features except for the obvious absence of the linear revival.

Numerical studies show that the configurations (6) and (7) of Fig. 3 belong to two dimensional submanifolds that are very unstable in the transverse directions. Launching wave packets inside this manifold does not lead to any significant linear or nonlinear revival.

In summary, we have seen that invariant submanifolds of rotationally invariant systems of identical particles constitute a remarkable structure in classical phase space. On such manifolds short periodic orbits may be easily found. Furthermore a chaotic submanifold with small transverse stability exponents may trap classical trajectories quite a time in their vicinity and lead to scaring of wave functions. As such submanifolds involve motion of many particles in a low-dimensional space we may hope that they constitute an approach to collective motion in chaotic systems.

4 Beyond symmetric Hamiltonians

At last let us turn to an interesting observation that extends the usefulness of symmetric periodic orbits even to non-symmetric systems. Consider the usual symmetric three-disc scattering system with sufficient distance between the discs to obtain the usual binary dynamics [22]. In this case we have a $C_{3v}$ symmetry and five fundamental orbits, three bouncing between two discs and two cyclic ones with opposite directions. Cyclic symmetry makes the three bouncing orbits equivalent and reflection symmetry the two cyclic ones.

It is easy to see that this fits with our expectations of the importance of symmetric orbits. A periodic orbit with the full $C_{3v}$ symmetry does not exist. So we have to consider the subgroups. What is more surprising [23] is that the structural stability of a complete horseshoe guarantees that these same orbits will be the fundamental ones even if we distort the system a little e.g. by choosing different sizes for the discs and/or distort the equilateral triangle on which the centers of the discs are placed. Inspection of the specific system shows that if the discs are sufficiently far apart as compared to their radii the deformations can become fairly large because the binary dynamics will only break down if one of the discs will interfere with an orbit connecting the other
Fig. 6. Complete ternary horseshoe formed by the invariant manifolds of the two outer unstable fixed points. The Poincaré map is taken in the surface $y = 0$ and $E = 0.5$. The parameter values are $a = 0.7$ and $b = 1.05$.

Conversely we can ask the question whether we can construct fundamental orbits that do not have the corresponding symmetries. For this purpose we shall look at the development of a two hill system into a four hill system.

We consider the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \exp[-(x - a)^2 - (y - A)^2] + \exp[-(x + a)^2 - (y - A)^2] + \exp[-(x - b)^2 - (y + A)^2] + \exp[-(x + b)^2 - (y + A)^2],$$

where $A$ is the fixed separation of the original two hills. For $a = b$ this system has a symmetry group of order 4 generated by two reflections, one along the axis uniting the two hills ($y$-axis) and one perpendicular to this axis at the center between the two hills ($x$-axis). Now we restrict our considerations to the single energy value $E = 0.5$ and start the development scenario at $a = b = 0.7$ where the equipotential line $V = 0.5$ still consists of two convex components. Then the whole invariant set consists of the single unstable periodic trajectory which oscillates along the $y$-axis between the two potential mountains. Despite of the fact that this system is integrable the invariant set is hyperbolic and by consequence structurally stable against small deformations.

Keeping $a = 0.7$ fixed and varying $b$ the symmetry of the system is reduced to a simple reflection symmetry in the $y$-axis. The original hyperbolic periodic orbit undergoes a pitchfork bifurcation (at $b \approx 0.73$) to one elliptic orbit along the symmetry line ($y$-axis) and a pair of non symmetric hyperbolic orbits that are reflection images of each other. As we increase $b$ further the elliptic orbit undergoes a period doubling bifurcation and becomes inverse hyperbolic. The horseshoe develops to a complete ternary one for $b = 1.05$ shown in Fig. 6.
Fig. 7. Incomplete horseshoe with five fixed points in the same surface of section as in previous figure. We show the invariant manifolds of the two outer fixed points, several KAM islands in the region of stability around the three inner fixed points and one small chaotic strip.

If we keep $b$ fixed at $b = 1.0$ and increase $a$ starting from 0.7 the horseshoe of order three reduces its degree of development but at some point the central orbit undergoes a further pitchfork bifurcation for fixed $b = 1.0$ when $a \approx 0.97$. Note that this value of $a$ is quite near to the value of $b$. If we further increase $a$ such that $a = b = 1.0$ we have a symmetric horseshoe of order five as shown in Fig. 7. In this case we are back to the typical situation where all fundamental periodic orbits are symmetric under a subgroup of the symmetry group which is again generated by two reflections taking pairs of hills into each other. This shows that the five orbits we found after the bifurcation are again simple deformations of symmetric ones.

If instead of increasing $a$ we further increase $b$ the central orbit will undergo a further pitchfork bifurcation generating two orbits with opposite rotational orientation in order finally to become a homoclinic connection of a new periodic orbit that is formed on the saddle perpendicular to the former central orbit. At this point we have a three hill problem which upon sufficient separation of the hills will have a horseshoe of the type discussed at the beginning of this section for the three disc system.

It thus seems that we have found that the horseshoe of order three really has orbits that are not deformations of symmetric ones. This appearance is fallacious because we can take a different route of deformation. Choosing the value of $b = 0.9$, i.e. sufficiently small we can vary $a$ up to $a = b$ without causing the pitchfork bifurcation of the central orbit as we can see in Fig. 8. The fact that the picture is not hiding a very small splitting of the central orbit is readily checked by verifying that it is still inverse hyperbolic.

The question whether the fundamental periodic orbits of a chaotic problem can always be viewed as deformations of those orbits symmetric under a non-trivial subgroup of the symmetry group of a more symmetric problem remains
Fig. 8. Incomplete ternary horseshoe in the same surface of section as in previous figures. We show the invariant manifolds of the two outer fixed points, several KAM islands in the region of stability around the inner fixed point and a secondary structure of order two.

an open question. That it will be often the case is quite evident. As we know from refs. [1,3], there are particularly efficient methods to find symmetric periodic orbits. Therefore it might in certain cases be well worth considering such a deformation to determine the structure of a system without or with low symmetry.

5 Summary and conclusions

A modern analysis of the chaotic saddle of the Copenhagen problem proves a thirty year old statement of Hénon to be right from a novel point of view. The simple (according to Hénon) periodic orbits are sufficient to explain the horseshoe structure of the entire chaotic saddle in terms of their stable and unstable manifolds. The only addition that is necessary are the symmetric parabolic orbits which are the limits of various families of symmetric periodic orbits and which generate the edges of what is known as the fundamental rectangle.

The concept of symmetry generated invariant submanifolds presents a particular challenge in this context: The periodic (and incidentally also the aperiodic) orbits that lie within these manifolds are all symmetric in a very special way. The symmetry transforms such orbits pointwise into each other. Such manifolds provide very easily some of the periodic orbits we may need to describe our system. We furthermore note that in the context of semi-classics these manifolds may be of fundamental importance. Actually there are strong signs that such manifolds may carry collective states with fairly long life times.

Finally we use the structural stability to argue that symmetric horseshoes may be readily deformed into non-symmetric ones and vice versa. This implies
that the topological structure of the fundamental periodic orbits does not change over some finite interval for the parameter that measures the symmetry breaking deformation of our system. Thus the symmetry even governs systems in which it is broken.

The remarkable and seemingly entirely uncorrelated roles of symmetric orbits in the systems discussed in sections 2 and 3 as well as others given in the literature [3–6] is the central question of our considerations. The survival of basic symmetric orbits in the horseshoe construction for deformed systems gives an indication of the origin of their importance.

Finally we wish to emphasize that the entire paper raises more questions than it answers, and indeed it is meant to be a stimulus for researchers in the field not to take the common knowledge for granted, but rather evaluate the situation in their specific fields of interest.

References

[1] M. Hénon, Ann. Astrophys. 28 (1965) 499.
[2] M. Hénon, Bull. Astron. (3) 1 fasc 1 (1966) 57; fasc 2 (1966) 49.
[3] P.H. Richter, H.J. Scholz and A. Wittek, Nonlinearity 3 (1990) 45.
[4] M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics, (Springer, Berlin) 1990; J. Math. Phys. 12 (1971) 343.
[5] D. Wintgen, K. Richter and G. Tanner, Chaos 2 (1992) 32.
[6] P. Cvitanović and B. Eckhardt, Nonlinearity 6 (1993) 277.
[7] B. Rückerl, C. Jung, J. Phys. A bf 27 (1994) 55.
[8] L. Benet, et al., Celest. Mech. & Dyn. Astr. 66 (1997) 203.
[9] L. Benet, et al., Celest. Mech. & Dyn. Astr., submitted.
[10] A. Wintner, The Analytical Foundations of Celestial Mechanics (Princeton University Press, 1941); D.G. Saari, Celest. Mech. & Dyn. Astr. 21 (1980) 9.
[11] T. Prosen, Phys. Lett. A 233 (1997) 332.
[12] T. Papenbrock and T.H. Seligman, Phys. Lett. A 218 (1996) 229.
[13] T. Papenbrock, T.H. Seligman, and H.A. Weidenmüller Phys. Rev. Lett. 80 (1998) 3057
[14] R. Balian and C. Bloch, Ann. Phys. (N.Y.) 69 (1972) 76; M.V. Berry, in Chaotic Behavior of Deterministic Systems, edited by G. Iooss et al. (North–Holland, Amsterdam, 1983).
[15] D. Delande and J.C. Gay, Phys. Rev. Lett. 57 (1986) 2006; M.C. Du and J.B. Delos, Phys. Rev. Lett. 58 (1987) 1731; See Chaos and Quantum Physics, eds. M.-J. Giannoni et al. (North–Holland, Amsterdam, 1991) and references therein.

[16] E.J. Heller, Phys. Rev. Lett. 53 (1984) 1515; E.B. Bogomolny, Physica D 31 (1988) 169; M.V. Berry, Proc. Roy. Soc. Lond. A 423 (1989) 219; O. Agam and S. Fishman, J. Phys. A 26 (1993) 2113, Corrigendum 6595; Phys. Rev. Lett. 73 (1994) 806.

[17] M.H. Sommermann and H.A. Weidenmüller, Europhys. Lett. 23 (1993) 79; H.A. Weidenmüller, Phys. Rev. A48 (1993) 1819; T.H. Seligman and H.A. Weidenmüller, J. Phys. A 27 (1994) 7915.

[18] B. Simon, Ann. Phys. 146 (1983) 209; C.C. Martens, R.L. Waterland and W.P. Reinhardt, J. Chem. Phys. 90 (1989) 2328; S. Tomsovic, J. Phys. A 24 (1991) L733; P. Dahlqvist and G. Russberg, J. Phys. A 24 (1991) 4763; O. Bohigas, S. Tomsovic, D. Ullmo, Phys. Rep. (1993) 43; B. Eckhardt, G. Hose, E. Pollak, Phys. Rev. A 39 (1989) 3776.

[19] P. Dahlqvist and G. Russberg, Phys. Rev. Lett. 65 (1990) 2837.

[20] E.J. Heller, J. Chem. Phys. 94 (1991) 2723.

[21] M.A. Sepulveda, E.J. Heller, J. Chem. Phys. 101 (1994) 8004.

[22] B. Eckhardt, J. Phys. A 20 (1987) 5971; P. Gaspard and S. Rice, J. Chem. Phys. 90 (1989) 2225.

[23] C. Jung, C. Lipp and T.H. Seligman, Ann. Phys. (N.Y.) submitted.