Noncommutative Cosmology

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February 7, 2008

Abstract

We show that the large number coincidence can be interpreted as giving the filling factor in a Landau problem. The analogy with the Landau problem leads to noncommutativity between the gravitational and matter degrees of freedom. We present a toy model that supports this view.

We present, also, some of the physical consequences this noncommutativity implies like a different insight into the semiclassical approximation of quantum gravity and a different tackling of the cosmological constant problem.

The large number coincidence [1] fascinated [2], [3] and still fascinates [4] many people. The aim of focusing upon it was to find the reasons of the link between micro and macro cosmos the coincidence seems to reveal.

Our aim here is to show that the large number coincidence hints to a commutation relation. Unlike the noncommutativity in spacetime assumed in many recent papers [5], the noncommutativity suggested by the large number coincidence is in superspace, between the gravitational and matter degrees of freedom.

The large number coincidence, expressed in its most usual form

\[ \left( \frac{\hbar^2 H}{G \cdot c} \right)^{1/3} \sim m \]  

connects the Hubble parameter \( H = \frac{a}{a} \) with the fundamental constants \( h, c, G \) and a typical hadron mass \( m \). For power law cosmologies, the scale factor varies like \( a(t) \propto t^\alpha \) and one can replace the Hubble constant with the horizon distance \( R_H \)

\[ R_H = c \cdot a(t) \int_0^t \frac{dt'}{d(t')} = \frac{ct}{1-\alpha} \quad ; \quad H = \frac{\alpha}{t} \quad ; \quad H = \frac{\alpha c}{(1-\alpha)R_H} \]
Replacing also for mass in (1), the associated Compton wavelength \( \lambda \), we get

\[
\frac{\hbar^2 \cdot \alpha}{(1 - \alpha) R_H \cdot G} = \frac{h^3 \cdot \alpha}{\lambda \cdot c_3}
\]

or, after a little rearrangement

\[
\left( \frac{R_H}{\lambda} \right)^3 = O(...) \frac{R_H \cdot \lambda}{2\pi l_p^2}
\]

where \( l_p \) is the Planck length and \( O(...) \) is a numerical factor, something between 0.01 and 1, depending on the type of cosmology we consider (\( \alpha \)), the definition of the Planck length, the hadron mass we choose. Using a reasoning dating back to Eddington, we can say the LHS of (4) represents, roughly, the number of particles in a universe with horizon \( R_H \). If a single particle was present in the universe, its extension would be of same order as the extension of that universe \( \lambda/R_H = \delta R_H / R_H \propto 1 \). When \( N \) particles are present, the statistical argument gives \( \lambda/R_H = \delta R_H / R_H \propto N^{-1/2} \), wherefrom \( N \propto (R_H/\lambda)^2 \).

The RHS is more interesting; it resembles the degeneracy of an energy level in the Landau problem. We remind that, in the Landau problem of a charged particle placed a strong magnetic field and confined to a two dimensional plane, say \( X, Y \) the degeneracy of an energy level and the filling factor are, respectively

\[
D = \frac{L_x L_y}{2\pi l_B^2} ; \quad \nu = \frac{N}{D}
\]

where \( l_B = (\hbar c/eB)^{1/2} \) is the magnetic length and \( L_x, L_y \) are the sizes of the bidimensional sample. In the analogy above, the sizes are \( R_H \) and \( \lambda \), and the role of the magnetic length is played by the Planck length. Note that, in this case, the analogs of the coordinates \( X, Y \) must be two degrees of freedom, one associated to a gravitational degree of freedom, say the scale factor \( a \) or a function of it, and the other one, to a matter degree of freedom, say a scalar field. Relation (4) also points to a filling factor for Universe \( \nu \leq 1 \), i.e. the Universe accommodates (almost) the maximum allowed number of particles.

It is a well known fact [6] that the dynamics of a charged particle in 2D subject to a strong constant magnetic field \( B \) is equivalent to the dynamics of the same particle, with no magnetic field present but confined to a noncommutative plane with the algebra of coordinates

\[
[x, y] = i \frac{\hbar c}{eB}
\]

The equivalence holds true in the limit of very strong magnetic field when the magnetic length \( l_B \) (and here the Planck length) is much smaller than any other length scale occurring in the problem and when the Hilbert space is truncated to the lowest Landau level (\( n = 1 \)); it is probably, also true for higher Landau levels, when the Hilbert space is truncated to the first \( n \) Landau levels and the RHS of relation (6) is multiplied by a factor \( n \) [7].
When we draw the analogy between the degeneracy in the Landau problem and the RHS of (4) we do not suppose the existence of a mysterious magnetic field; rather, we think relation (4) hints to a nonvanishing commutator between two degrees of freedom, yet to be specified, one associated to the gravitation, the other, to matter.

We might expect that the degrees of freedom alluded above, the analogs of the noncommuting coordinates, shall be the variables in a minisuperspace model; described by Wheeler-DeWitt equation. Consequently, Wheeler-DeWitt equation must bear some resemblances to Schrodinger equation in Landau problem. This does not happen, at first sight. First, the very appearance of Schrodinger equation in Landau problem depends on the chosen gauge (but the expression of degeneracy is gauge invariant). Then, despite its name of Schrodinger equation for quantum gravity, Wheeler-DeWitt equation is hyperbolic, the signature in the general case is -+++++ unlike the stationary Schrodinger equation. That is the main obstruction to any comparison.

Let us consider, however, the following simple minisuperspace model of a spatially homogenous and isotropic universe with metric

$$ds^2 = \sigma^2 (dt^2 - a^2 d\Omega_3^2)$$

(7)

where $d\Omega_3^2$ is the metric on a 3-sphere of unit radius. The only gravitational degree of freedom, $a$, which for later convenience we denote by $X_2 \equiv a$, is made dimensionless by choosing $\sigma = (2/3\pi)^{1/2}l_P$. The matter degree of freedom is represented by a conformally invariant scalar field $\phi$. The Wheeler-DeWitt equation is then [8]

$$\frac{1}{2} \left[ \frac{\partial^2}{\partial X_1^2} + X_1^2 + \frac{\partial^2}{\partial X_2^2} - X_2^2 \right] \Psi(X_1, X_2) = 0$$

(8)

where $X_1 = \pi^{3/2}3^{1/2}\phi/l_P$. The Hamiltonian is of the form

$$H = H_1 - H_2 \quad; \quad H_i = \frac{P_i^2 + X_i^2}{2}, \quad i = 1, 2$$

(9)

and it constituted object of special interest for ’t Hooft in a recent series of papers [9]. Eventually, we can transform the above Hamiltonian into

$$H = yp_x - xp_y$$

(10)

using the transformations

$$P_1 = \frac{1}{\sqrt{2}} (p_x + y) \quad; \quad P_2 = \frac{1}{\sqrt{2}} (x + p_y)$$

$$X_1 = \frac{1}{\sqrt{2}} (x - p_y) \quad; \quad X_2 = \frac{1}{\sqrt{2}} y - p_x$$

(11)

Neither the Hamiltonian (9) nor the Hamiltonian (10) is bounded from below. We invoke now the procedure advocated by ’t Hooft for Hamiltonians (9)
or (10). The lack of lower bound for the Hamiltonian (10) is cured changing to a positive definite function $\rho^2$ that commute with (9)

$$[\rho^2, H] = 0$$  \hspace{1cm} (12)

so that

$$H_{1,2} = \frac{1}{4\rho^2} (\rho^2 \pm H)^2 ; \quad H = H_1 - H_2$$  \hspace{1cm} (13)

To get a lower bounded Hamiltonian, one imposes as a constraint, motivated by information loss

$$H_2 |\Psi\rangle = 0$$  \hspace{1cm} (14)

whence

$$H \rightarrow H_1 \rightarrow \rho^2 \geq 0$$  \hspace{1cm} (15)

A positive function that fulfills the above conditions is

$$\rho^2 = \frac{1}{2} (x^2 + y^2)$$  \hspace{1cm} (16)

A minimal requirement for ’t Hooft prescription to work is that the Hamiltonian (16) shall generate the same equations of motion as (10). It is easy to check that this is possible iff the following bracket holds

$$\{x, y\} = 1$$  \hspace{1cm} (17)

To see more clearly the connection with the Landau problem we turn the bracket into the commutator

$$[x, y] = i$$  \hspace{1cm} (18)

The commutation relation can be implemented with the star Moyal product [10] in momentum space and the Hamiltonian becomes

$$\rho^2 * \Psi = \frac{1}{2} (x^2 + y^2) * \Psi = \frac{1}{2} \left( \frac{P_x^2}{4} + \frac{P_y^2}{4} + x^2 + y^2 + p_x y - p_y x \right) \Psi$$  \hspace{1cm} (19)

It represents exactly the Hamiltonian (in symmetric gauge) for a charged particle of mass $\mu = 2$ in constant magnetic field ($eB = 4$). Using commutator (18) and going through transformations (11) we get

$$[\phi, a] = iC$$  \hspace{1cm} (20)

where, for this toy model, $C = \frac{2l_p}{(3\pi^2 a^2)^{1/2}}$. The relation between ’t Hooft procedure and Landau problem has also been proved by Banerjee [11], in a slightly different manner and in a completely different context.
Can we go beyond this simple model equipped with a conformal scalar field? A realistic superspace model involves, in general, interaction terms between $a$ and $\Phi$, and the two Hamiltonians do not decouple so nicely.

We can consider, however, the more realistic model of a homogenous isotropic universe with a (nonconformal) scalar field $\phi$ with mass $\mu$. The Wheeler-DeWitt equation is

$$\frac{1}{2} \left[ \frac{\partial^2}{\partial a^2} - a^2 - \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} + a^4 m^2 \phi^2 \right] \Psi(a, \phi) = 0 \quad (21)$$

where $\phi = \sigma \phi$ and $m = \sigma \mu$. As long as the mass of the field is much smaller than the Planck mass $m \ll 1$, the last term in (21) is negligible small. Then, with the change of variables

$$x = a \sinh \Phi ; \quad x = a \cosh \Phi \quad (22)$$

equation (21) is brought to equation (8) to which the above analysis can be applied. The commutation relation (20) will survive but with a different $C$.

Let us follow now the implications of a nonvanishing commutator between the gravitational and matter degrees of freedom. An obvious consequence is the fact that the problem of the singularity $a \to 0$ is alleviated and less worrisome. Due to the uncertainty relation

$$\Delta a \Delta \phi \geq \frac{l_p}{(3\pi^3)^{1/2} \langle a \rangle} \quad (23)$$

the geometry becomes, as the singularity is approached, more and more fuzzy.

Another consequence concerns the so called semiclassical approximation in quantum gravity. We recall that the semiclassical approximation consists in treating classically the gravitational field while matter fields are treated quantum mechanically. Relation (20) shows this procedure can be consistent; one can not quantize both the gravitational and matter fields at a time simply because they are not compatible observables.

The commutation relation sheds new light on the cosmological constant problem. The cosmological constant in Einstein equations can be thought of as a purely geometrical term $\Lambda$, proportional to the scalar curvature; let us call it the geometrical cosmological constant $\Lambda_g$. A cosmological constant can also occur due to matter from a stress tensor with the special equation of state $p = -\rho$; let us call it the matter cosmological constant. In terms of a scalar field $\phi$ the previous equation entails zero kinetic energy and, since the field is constant, so is any arbitrary function of it, in particular, the potential energy density $V(\phi)$. The cosmological term reads:

$$\Lambda_m g_{\mu \nu} = -8\pi G T g_{\mu \nu} \quad (24)$$
where $T = V(\phi)/2$. The main point is that the two cosmological constants are completely equivalent. There is no operational way to distinguish between geometrical and matter cosmological constant at the classical level.

$$\Lambda_g = \Lambda_m$$  \hspace{1cm} (25)

On the other hand, when we compute the commutators by means of (20), we have

$$[\Lambda_g g_{\mu \nu}, a] = 0 \ ; \ [\Lambda_m g_{\mu \nu}, a] = -8\pi G T g_{\mu \nu} \frac{dT}{d\phi} [\phi, a] \neq 0$$  \hspace{1cm} (26)

Relations (25) and (26) are in clear contradiction. In physical terms, the conflict is between the necessity of a strictly constant scalar field for the cosmological constant and the forever fluctuating scalar field in (20). Put differently, the tension is between the equivalence (25) and the broken equivalence introduced by (20). A way out of these contradictions is to set to zero in Einstein equations any term proportional to the metric. May be this requirement does not solve completely the cosmological constant problem but it forbids, for instance, the constant term occurring in a phase transition governed by Higgs mechanism. The requirement above does not preclude the existence of a scalar field with a small but nonvanishing kinetic energy; the newly discovered accelerated expansion [13] could be driven by such a field with negative pressure if $|p| < \rho$.

Both the empirical evidence, (the large number coincidence) and the model above point to a nonnull commutator between matter and gravitational degrees of freedom. At first sight the idea of a nonnull commutator between the matter and gravitational degrees of freedom might seem crazy. We think it is crazy enough to be true.

References

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[3] R. H. Dicke- Nature 192, 440 (1961)

[4] G.A. Mena Marugan and S. Carneiro-Phys. Rev. D65 087303 (2002) and the references therein

[5] L. Susskind-hep-th/0101029, R. Jackiw-hep-th/0110057, V. P. Nair-hep-th/0112114, A. P. Balachandran-hep-th/0203259. For a review see I. Hinchliffe and N. Karsting-hep-th/0205040

[6] see, for instance, L. Mezincescu -hep-th/0007046 and the references therein
Appendix
We skipped in text some calculations. For complemtitude we give them here.

1. The Hamiltonian (16) with the bracket (17) engenders the same equations of motion as the Hamiltonian (10).

\[ H = yp_x - xp_y \]

a) \[ \frac{\partial H}{\partial p_x} = \dot{x} = y ; \quad \frac{\partial H}{\partial p_y} = \dot{y} = x \]

b) \[ \{x, y\} ; \quad \rho = \frac{1}{2} (x^2 + y^2) \]

\[ \dot{y} = \{y, \rho\} = \{y, \frac{1}{2} (x^2 + y^2)\} = \{y, x\} x = -x \]

\[ \dot{x} = \{x, \rho\} = \{x, \frac{1}{2} (x^2 + y^2)\} = \{x, y\} y = y \]

2. A realization of commutation relation (18) is made by the star (\(\ast\)) product. The star (\(\ast\)) product leads to (19).

We define
\[
\ast = \exp \left( \frac{i}{2} \left( \overrightarrow{\partial}_x \overrightarrow{\partial}_y - \overrightarrow{\partial}_y \overrightarrow{\partial}_x \right) \right)
\]

where \( \overrightarrow{\partial}_x = \frac{\partial}{\partial x} \) acts at left etc.

\[
x \ast y = x \left( \exp \left( \frac{i}{2} \left( \overrightarrow{\partial}_x \overrightarrow{\partial}_y - \overrightarrow{\partial}_y \overrightarrow{\partial}_x \right) \right) \right) y = x \left( 1 + \frac{i}{2} \left( \overrightarrow{\partial}_x \overrightarrow{\partial}_y - \overrightarrow{\partial}_y \overrightarrow{\partial}_x \right) + \ldots \right) y = xy + \frac{i}{2}
\]

\[
y \ast x = y \left( 1 + \frac{i}{2} \left( \overrightarrow{\partial}_x \overrightarrow{\partial}_y - \overrightarrow{\partial}_y \overrightarrow{\partial}_x \right) + \ldots \right) x = xy - \frac{i}{2}
\]

\[
[x, y] = x \ast y - y \ast x = i
\]

\[
x^2 \ast \Psi(x, y) = x^2 \left( 1 + \frac{i}{2} \left( \overrightarrow{\partial}_x \overrightarrow{\partial}_y - \overrightarrow{\partial}_y \overrightarrow{\partial}_x \right) + 1 \frac{i^2}{8} \left( \overrightarrow{\partial}_x \overrightarrow{\partial}_y - \overrightarrow{\partial}_y \overrightarrow{\partial}_x \right)^2 + \ldots \right) \Psi =
\]

\[
= x^2 \Psi + \frac{i}{2} 2x \overrightarrow{\partial}_y \Psi + \frac{i^2}{8} x \overrightarrow{\partial}_x \overrightarrow{\partial}_y \overrightarrow{\partial}_x \overrightarrow{\partial}_y \Psi =
\]

\[
= x^2 \Psi - xp_y + \frac{i^2}{4} \overrightarrow{\partial}_y \overrightarrow{\partial}_y \Psi = x^2 \Psi - xp_y + \frac{1}{4} p_y^2
\]

where we took into account \( i \overrightarrow{\partial}_y = -p_y \).

\[
\frac{1}{2} (x^2 + y^2) \ast \Psi(x, y) = x^2 + y^2 - xp_y + yp_x + \frac{1}{4} p_y^2 + \frac{1}{4} p_x^2.
\]

\[
\frac{\partial^2 \Psi}{\partial a^2} - \frac{1}{a^2 \Phi^2} - a^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} - y^2 \Psi + x^2 \Psi
\]

\[
\frac{\partial^2 \Psi}{\partial a^2} - \frac{1}{a^2 \Phi^2} - a^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} - y^2 \Psi + x^2 \Psi
\]

3. Equation (21) with the change of variables (22) is of the same form as (8).

\[
\left\{ \begin{array}{c}
x = a \sinh \Phi \\
y = a \cosh \Phi
\end{array} \right.
\]

\[
\frac{\partial \Psi}{\partial a} = \frac{x}{a} \frac{\partial \Psi}{\partial x} + \frac{y}{a} \frac{\partial \Psi}{\partial y} ; \quad \frac{\partial^2 \Psi}{\partial a^2} = \frac{x}{a^2} \frac{\partial \Psi}{\partial x} + \frac{y}{a^2} \frac{\partial \Psi}{\partial y} + \left( \frac{x}{a} \right)^2 \frac{\partial^2 \Psi}{\partial x^2} + \left( \frac{y}{a} \right)^2 \frac{\partial^2 \Psi}{\partial y^2}
\]

\[
\frac{\partial \Psi}{\partial \Phi} = a \left( \cosh \Phi \frac{\partial \Psi}{\partial x} + \sinh \Phi \frac{\partial \Psi}{\partial y} \right) ; \quad \frac{\partial^2 \Psi}{\partial \Phi^2} = \frac{x}{a} \frac{\partial \Psi}{\partial x} + \frac{y}{a} \frac{\partial \Psi}{\partial y} + y^2 \frac{\partial \Psi}{\partial x^2} + x^2 \frac{\partial \Psi}{\partial y^2}
\]

\[
\frac{\partial^2 \Psi}{\partial a^2} - \frac{1}{a^2 \Phi^2} - a^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} - y^2 \Psi + x^2 \Psi
\]

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