On variational principles in Euler’s and Lagrange’s descriptions

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Abstract. The correspondence between the variational principles of the equations of continuum mechanics in Euler’s and Lagrange’s descriptions is considered. It is shown that the introduction of Lagrange’s variables can be considered as the construction of a differential covering by introducing the potential into the mass conservation law. A correspondence between the variational principles of the $l$-normal equations of continuum mechanics in the Euler’s and Lagrange’s descriptions in terms of symplectic structures is proposed.

1. Introduction

It is known that a number of equations of continuum mechanics in Lagrange’s variables are Euler–Lagrange equations, that is, they admit variational principles. For example, the principle of least (stationary) action arises in gas dynamics in the Lagrange’s description of the motion of an ideal barotropic gas [1]. The question arises about the existence of corresponding variational principles in the Euler’s description. This requires a definition of what is meant by the correspondence between the variational principles in Lagrange’s and Euler’s descriptions.

The paper offers a description of the correspondence between the variational principles using the concept of the symplectic structure of a differential equation [2].

2. Lagrange’s description as a differential covering

The classical method of introducing Lagrange’s description in continuum mechanics is associated with the choice of the Cartesian coordinates of a particle of the medium at the initial moment of time as Lagrange’s variables. By introducing Lagrange’s variables, the continuity equation

$$
\rho_t + (u\rho)_x + (v\rho)_y + (w\rho)_z = 0
$$

reduces to the relation

$$
\rho = \rho_0(\xi^1, \xi^2, \xi^3) \det \left( \frac{\partial \xi}{\partial x} \right),
$$

where $\rho > 0$, $\rho_0 > 0$. In this case, the Lagrange’s variables $\xi^i(t, x, y, z)$ satisfy the conditions

$$
\frac{d\xi^i}{dt} = \xi^l_t + u\xi^i_x + v\xi^i_y + w\xi^i_z = 0, \quad i = 1, 2, 3.
$$
By direct calculation, it is established that the system of equations (1), (2) is equivalent to the following equality

\[ \rho \, dx \wedge dy \wedge dz - u \rho \, dt \wedge dy \wedge dz + v \rho \, dt \wedge dx \wedge dz - w \rho \, dt \wedge dx \wedge dy = \rho_0 \, d\xi^1 \wedge d\xi^2 \wedge d\xi^3, \]

in which, using a suitable replacement of Lagrange’s variables, one can always make \( \rho_0 = 1 \).

The continuity equation expresses the mass conservation law and is equivalent to the condition that the differential form

\[ \rho \, dx \wedge dy \wedge dz - u \rho \, dt \wedge dy \wedge dz + v \rho \, dt \wedge dx \wedge dz - w \rho \, dt \wedge dx \wedge dy \]

is closed. Thus, the introduction of Lagrange’s variables is equivalent to the introduction of the potential into the mass conservation law in the form

\[ \rho \, dx \wedge dy \wedge dz - u \rho \, dt \wedge dy \wedge dz + v \rho \, dt \wedge dx \wedge dz - w \rho \, dt \wedge dx \wedge dy = d\xi^1 \wedge d\xi^2 \wedge d\xi^3. \]

**Remark 1.** Transformations that preserve volume in variables \( \xi^1, \xi^2, \xi^3 \) form a group of symmetries for any equations written in Lagrange’s variables.

The introduction of the potential into the conservation law of a system of differential equations determines its differential covering [2]. Let the initial system of equations in Euler’s variables be of the form

\[ F_1 = 0, \]

\[ \ldots \]

\[ F_m = 0. \]

(3)

Then the covering with the potential for the mass conservation law is given by a system of equations

\[ F_1 = 0, \]

\[ \ldots \]

\[ F_m = 0, \]

\[ \rho = \det \left( \frac{\partial \xi}{\partial x} \right), \]

\[ \xi_i^1 + u \xi_i^1 + v \xi_i^2 + w \xi_i^3 = 0, \quad i = 1, 2, 3. \]

(4)

The final form of the system of equations (3) in Lagrange’s variables will be obtained if \( x, y, z \) are selected as new dependent variables in the system of equations (4).

**Remark 2.** The method of introducing Lagrange’s variables as non-local variables in the differential covering allows us to naturally transfer some geometric structures from Euler’s description to the Lagrange’s one.

The concept of \( \ell \)-normality plays an important role for determining of the one-to-one correspondence between variational principles and symplectic structures of systems of differential equations [2].
3. \( l \)-normal systems of differential equations

Consider a system of differential equations

\[
F_1 = 0, \\
\ldots \\
F_m = 0,
\]

where \( F_i \) are functions of independent variables \( x^1, \ldots, x^n \), dependent variables \( u^1, \ldots, u^m \) and their derivatives up to some finite order. Denote multi-index of the form \( \alpha_i x^i \) by \( \alpha \). Here and further we assume summation over repeated indices. Put

\[
D_\alpha = D_{\alpha_1 x^1} \circ \ldots \circ D_{\alpha_n x^n}, \quad u_\alpha = D_\alpha (u^i),
\]

where \( D_{x^i} \) are the operators of total derivatives.

We shall say that an infinitely prolonged system of differential equations is regular if

1) it has no nontrivial relations between dependent and independent variables;

2) for each \( k \) its projection to the space of independent variables, dependent variables and derivatives up to the order \( k \) can be written as a system of equations of the form

\[
\nabla_{ij}^n D_\alpha (F_i) = 0,
\]

where the Jacobi matrix of the left-hand side has maximal rank at \( (6) \).

Denote the infinite prolongation of the system of equations \( F = 0 \) by \( \mathcal{E} \)

\[
\mathcal{E}: F_1 = 0, \quad D_{x^i}(F_1) = 0, \quad D_{x^2}(F_1) = 0, \quad \ldots
\]

Consider the Frechet derivative of the left-hand side of the system \( (5) \). Its action on the vector-function \( \varphi = (\varphi^1, \ldots, \varphi^m) \) is given by the formula

\[
(l_F(\varphi))_i = l_{F_{ij}}(\varphi^j) = \frac{\partial F_i}{\partial u_\alpha} D_\alpha (\varphi^j).
\]

Denote by \( l_\mathcal{E} \) the restriction of the operator \( l_F \) to the system \( \mathcal{E} \).

Regular system of equations \( F = 0 \) is \( l \)-normal if the relation

\[
\Delta \circ l_\mathcal{E} = 0
\]

holds only for \( \Delta = 0 \). Here \( \Delta \) is an operator in total derivatives, which acts from the space of vector-functions of the form

\[
(f_1(x^i, u^i), \ldots, f_m(x^i, u^i))|_\mathcal{E}
\]

to scalar functions on \( \mathcal{E} \).

Further we consider \( l \)-normal systems only.

We shall say that the system of the form

\[
\begin{align*}
\varphi^1_{b_1 x^n} &= \varphi^1, \\
\ldots \\
\varphi^m_{b_m x^n} &= \varphi^m
\end{align*}
\]

is written in the extended Kovalevskaya form if all \( b_i \) are positive integers and the right-hand side \( (\varphi^1, \ldots, \varphi^m) \) is independent of the variables \( u^i_{b_i x^n} \) and their derivatives.
Remark 3. Systems of equations in the extended Kovalevskaya form are $l$-normal. Almost all systems of equations of continuum mechanics can be written in the extended Kovalevskaya form.

Example. In dimensionless variables the Navier-Stokes system of equations for incompressible fluid

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}, \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{align*}
\]

can be written in the extended Kovalevskaya form for $x^n = z$, $b = (2, 2, 1, 1)$. For this one we can eliminate $w_z$ from the fourth equation; eliminate $u_{zz}$ from the first equation; eliminate $v_{zz}$ from the second equation; eliminate $p_z$ from the third equation. The Euler system of equations can be written in the extended Kovalevskaya form in the same way.

4. Symplectic structures and variational principles

Let us consider the definition of a symplectic structure for a regular system of differential equations. Similar to the classical differential geometry, for $l$-normal systems of equations there are two equivalent descriptions of symplectic structures: as equivalence classes of differential forms and as equivalence classes of operators in total derivatives:

1. Symplectic structure for the system of equations $E$ is a closed variational 2-form on $E$, i.e. an element of the kernel of the variational differential

\[
\delta: E_1^{2,n-1}(E) \to E_1^{3,n-1}(E).
\]

Here $E_1^{p,n-1}(E)$ are groups of variational $p$-forms of the $C$-spectral sequence [2].

2. Variational 2-forms of an $l$-normal system of equations $E$ can be described as operators in total derivatives $\Delta$, for which holds the relation

\[
\Delta^* \circ l_E = l_E^* \circ \Delta,
\]

modulo operators of the form $\nabla \circ l_E$, where $\nabla = \nabla^*$. Here the operator $\Delta^*$ is formally adjoint to the operator $\Delta$.

Remark 4. The definition of a symplectic structure as a closed variational 2-form allows us to lift symplectic structures in coverings.

Let us some consequence of a system of equations $F = 0$ be variational, i.e. for some operator in total derivatives $A$ the corresponding system

\[
A(F) = 0
\]

is an Euler–Lagrange system of equations. Then the operator $\Delta = A^*|_E$ determines the symplectic structure for the initial system $F = 0$. Moreover, each symplectic structure of an $l$-normal system of equations can be obtained in this way. Beside this, for systems of equations in the extended Kovalevskaya form, there is a canonical way to derive a variational principle from a symplectic structure.

Therefore, if a system of equations is written in the extended Kovalevskaya form in both Euler’s and Lagrange’s descriptions, then the relation between it’s variational principles in
Euler’s and Lagrange’s descriptions can be described in terms of symplectic structures. Each variational principle in Euler’s description generates unique symplectic structure, which can be lifted to the Lagrange’s description. The last step is to derive the corresponding variational principle from this lift.

5. Conclusion

The relation between variational principles for equations of continuum mechanics in Euler’s and Lagrange’s descriptions is obtained in the paper. This relation is based on the fact that Lagrange’s variables can be considered as nonlocal variables in differential covering.

References

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