Tight error bounds and facial residual functions for the $p$-cones and beyond

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Abstract

We prove tight Hölderian error bounds for all $p$-cones. Surprisingly, the exponents differ in several ways from those that have been previously conjectured; moreover, they illuminate $p$-cones as a curious example of a class of objects that possess properties in 3 dimensions that they do not in 4 or more. Using our error bounds, we analyse least squares problems with $p$-norm regularization, where our results enable us to compute the corresponding KL exponents for previously inaccessible values of $p$. Another application is a (relatively) simple proof that most $p$-cones are neither self-dual nor homogeneous. Our error bounds are obtained under the framework of facial residual functions and we expand it by establishing for general cones an optimality criterion under which the resulting error bound must be tight.

Keywords: error bounds, facial residual functions, Hölderian error bounds, amenable cones

1 Introduction

Consider the following conic feasibility problem:

\[
\text{find } x \in (\mathcal{L} + a) \cap \mathcal{K}, \quad (\text{Feas})
\]

where $\mathcal{K}$ is a closed convex cone contained in a finite dimensional Euclidean space $\mathcal{E}$, $\mathcal{L} \subseteq \mathcal{E}$ is a subspace and $a \in \mathcal{E}$. Here, we would like to tightly estimate the distance from $x$ to $(\mathcal{L} + a) \cap \mathcal{K}$ using the individual distances between $x$ and $\mathcal{L} + a$ and between $x$ and $\mathcal{K}$. This is an error bound question and is a classical topic in the optimization literature [12, 17, 26, 29, 38].

In this paper, we focus on the case where $\mathcal{K} = \mathcal{K}_p^{n+1}$, the $p$-cone for $p \in (1, \infty)$, which is defined as

\[
\mathcal{K}_p^{n+1} := \{ x = (x_0, \bar{x}) \in \mathbb{R}^{n+1} | x_0 \geq \|\bar{x}\|_p \},
\]

where $\|\bar{x}\|_p$ denotes the $p$-norm of $\bar{x}$. The cases $p = 1$ or $p = \infty$ are well-understood since they correspond to polyhedral cones and, therefore, Lipschitzian error bounds hold by Hoffman’s lemma [12].

The case $p = 2$ corresponds to the second-order cones and their error bounds are also well-understood: Luo and Sturm proved that if $\mathcal{K}$ is a direct product of second order cones then (Feas)
satisfies a Hölderian error bound with exponent \((1/2)\alpha\) and \(\alpha\) depends on the level of regularity of the problem [25, Theorems 7.4.1 and 7.4.2] which, in this case, is upper bounded by the number of cones. In particular, if \(\mathcal{K} = \mathcal{K}_2^{n+1}\) then the exponent is 1/2.

The case \(p \in (1, \infty), p \neq 2\) is quite peculiar. Although not as well-known as the 2-cones, it has applications in facility location problems, regularization of least squares problems and others [16, 32, 35, 38, 39]. The \(p\)-cone also appears in the recent push towards efficient algorithms and solvers for nonsymmetric cones [6,15,27,30,32].

There are significant differences between the cases \(p = 2\) and \(p \neq 2\), however. The former is a symmetric cone and, thus, enjoys a number of benefits that come with the Jordan algebraic structure that can be attached to it [9,10] such as closed form expressions for projections. The other \(p\)-cones are not symmetric and do not typically have closed form expressions for their projections. See [13,14] for a discussion on the extent to which they fail to be symmetric.

Despite the differences between general \(p\)-cones and 2-cones, they still have quite a few similarities, so one might be tempted to guess that if \(\mathcal{K} = \mathcal{K}_p^{n+1}\) then \((\text{Feas})\) satisfies a Hölderian error bound with exponent \(1/p\) as it was conjectured in [21, Section 5]. It turns out that this conjecture is wrong, and the true answer is far more interesting.

In this paper, our main contribution is to show for the first time that explicit Hölderian error bounds hold for all the \(p\)-cones and to determine the optimal exponents. As we will see in Theorem 4.7, for a fixed \(p\)-cone, there are situations where the exponent is 1/2 and others where the exponent becomes \(1/p\). It turns out that the correct exponent depends on the number of zeros that a certain vector exposing the feasible region of \((\text{Feas})\) has. Furthermore, there is one special case that only happens when \(p \in (1, 2)\). We also compute Hölderian error bounds for direct products of \(p\)-cones in Theorem 4.8. As an application of our results, we compute the KL exponent of the function associated to least squares problems with \(p\)-norm regularization; see section 5.1. Previously, an explicit exponent was only known when \(p \in [1, 2] \cup \{\infty\}\); see [38,39]. We also provide new “easy” proofs of some results about self-duality and homogeneity of \(p\)-cones; see section 5.2.

An important feature of our main \(p\)-cones error bound result is that the bound is optimal in a strong sense that implies, for example, that the exponents we found cannot be improved. It also precludes the existence of better error bounds beyond Hölderian ones; see Theorem 4.7 for the details.

Our results are obtained under the the facial residual function (FRF) framework developed in [19,21] which allows computation of error bounds for \((\text{Feas})\) without assuming constraint qualifications. Another major contribution in this work is that we expand the general framework in [19] to allow verification of the tightness of the error bound. In particular, when the facial residual function satisfies a certain optimality criterion and the problem can be regularized in a single facial reduction step, the obtained error bound must be optimal; see Theorem 3.6 and Corollary 3.7. We believe this expansion, and the new associated criterion will be useful for analysing other cones. For example, they easily verify the optimality of the FRFs constructed for the nontrivial exposed faces of the exponential cone in [19], while requiring no additional effort beyond what the authors used to merely show the FRFs were admissible; see Remark 3.8.

This paper is organized as follows. In section 2, we review preliminary materials including some essential tools developed in [19] for computing FRFs. In section 3, we build the tightness framework and establish the optimality criterion for certifying tight error bounds. In section 4, we derive explicit error bounds for \((\text{Feas})\) with \(\mathcal{K} = \mathcal{K}_p^{n+1}\) and certify their tightness. Finally, we discuss applications of our error bound results in section 5.
2 Preliminaries

In this paper, we will follow the notation and definitions used in [19], where we explained in more details some of the background behind the techniques we used. Here we will be terser and simply refer to the explanations contained in [19] as needed.

As a reminder, we assume throughout this paper that \( E \) is a finite dimensional Euclidean space \( \mathcal{E} \). The inner product of \( \mathcal{E} \) will be denoted by \( \langle \cdot, \cdot \rangle \) and the induced norm by \( \| \cdot \| \). With that, for \( x \in \mathcal{E} \) and a closed convex set \( C \subseteq \mathcal{E} \), we denote the distance between \( x \) and \( C \) by \( \text{dist}(x, C) := \inf_{y \in C} \| x - y \| \). We denote the projection of \( x \) onto \( C \) by \( P_C(x) \) so that \( P_C(x) = \arg \min_{y \in C} \| x - y \| \).

We will also write \( C^\perp \) for the orthogonal complement of \( C \). Next, let \( K \) be a closed convex cone contained in \( E \). We denote the relative interior, boundary, linear span, dual cone and dimension of \( K \) by \( \text{ri} K, \partial K, \text{span} K, K^* \) and \( \dim K \), respectively. We use \( \mathbb{R}_+ \) and \( \mathbb{R}_{++} \) to denote the set of nonnegative and positive real numbers, respectively. We also write \( B(\eta) := \{ x \mid \| x \| \leq \eta \} \) for any \( \eta \geq 0 \).

If \( F \subseteq K \) is a face of \( K \) we write \( F \subseteq K \). We say that a face \( F \) is proper if \( F \neq K \) and nontrivial if \( F \neq K \cap -K \) and \( F \neq K \). A finite collection of faces of \( K \) satisfying \( F_1 \subseteq \cdots \subseteq F_\ell \) is called a chain of faces and its length is defined to be \( \ell \). Then, the distance to polyhedrality of \( K \), denoted by \( \ell_{\text{poly}}(K) \), is the length minus one of the longest chain of faces of \( K \) such that only the final face \( F_\ell \) is polyhedral.

A cone \( K \) is said to be amenable [21] if for every \( F \subseteq K \) there exists \( \kappa > 0 \) such that \( \text{dist}(x, F) \leq \kappa \text{dist}(x, K) \) holds for all \( x \in \text{span} F \). By [21, Proposition 12], this is equivalent to the existence of \( \kappa > 0 \) such that

\[
\text{dist}(x, F) \leq \kappa \max\{\text{dist}(x, K), \text{dist}(x, \text{span} F)\}, \quad \forall x \in \mathcal{E}. \tag{2.1}
\]

Amenability implies facial exposedness [21, Proposition 13], i.e., every face \( F \subseteq K \) can be written as \( F = K \cap \{ z \}^\perp \) for some \( z \in K^* \). More information on amenability can be seen in [24]. A cone is said to be pointed if \( K \cap -K = \{ 0 \} \). Related to that, we note the following well-known fact for further reference

\[
z \in \text{ri} K^* \Rightarrow K \cap \{ z \}^\perp = K \cap -K, \tag{2.2}
\]

see, for example, [23, items (i) and (iv) of Lemma 2.2] applied to the dual cone.

Next, we will quickly review some ideas from facial reduction [5, 31, 34] and the FRA-poly algorithm of [22]; see also [19, Section 3]. In what follows we say that (Feas) satisfies the partial-polyhedral Slater (PPS) condition (see [22]) if one of the following three conditions hold: (i) \( K \) is polyhedral; (ii) \( (\mathcal{L} + a) \cap (\text{ri} K) \neq \emptyset \) (Slater’s condition holds); (iii) \( K \) can be written as \( K = K^1 \times K^2 \) where \( K^1 \) is polyhedral and \( (\mathcal{L} + a) \cap (K^1 \times (\text{ri} K^2)) \neq \emptyset \).

The next proposition follows from the correctness of the FRA-Poly algorithm [22, Proposition 8] and ensures that it is always possible to find a face of \( K \) that contains the feasible region of (Feas) and satisfies the PPS condition.

**Proposition 2.1** ([19, Proposition 3.2]). Let \( K = K^1 \times \cdots \times K^s \), where each \( K^i \) is a closed convex cone. Suppose (Feas) is feasible. Then there is a chain of faces

\[
F_\ell \subseteq \cdots \subseteq F_1 = K \tag{2.3}
\]

of length \( \ell \) and vectors \( \{ z_1, \ldots, z_{\ell-1} \} \) satisfying the following properties.

(i) \( \ell - 1 \leq \sum_{i=1}^s \ell_{\text{poly}}(K^i) \leq \dim K \).

(ii) For all \( i \in \{ 1, \ldots, \ell - 1 \} \), we have

\[
z_i \in F_i^* \cap \mathcal{L}^\perp \cap \{ a \}^\perp \quad \text{and} \quad F_{i+1} = F_i \cap \{ z_i \}^\perp.
\]
(iii) $F_e \cap (L + a) = K \cap (L + a)$ and $F_e, L + a$ satisfy the PPS condition.

If $(\text{Feas})$ is feasible, we define the distance to the PPS condition $d_{\text{PPS}}(K, L + a)$ as the length minus one of the smallest chain of faces satisfying Proposition 2.1. By [21, Proposition 24], we have

$$d_{\text{PPS}}(K, L + a) \leq \min \left\{ \sum_{i=1}^{s} \ell_{\text{poly}}(K^i), \dim(L^+ \cap \{a\}^+) \right\} \quad (2.4)$$

and there is a chain of faces as in Proposition 2.1 of length $d_{\text{PPS}}(K, L + a) + 1$.

### 2.1 Facial residual functions

First we recall the definition of facial residual functions [19, 21].

**Definition 2.2** (Facial residual functions (FRFs)). Let $K$ be a closed convex cone, $F \cong K$ be a face, and let $z \in F^*$. Suppose that $\psi_{F,z} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the following properties:

(i) $\psi_{F,z}$ is nonnegative, monotone nondecreasing in each argument and it holds that $\psi_{F,z}(0,t) = 0$ for every $t \in \mathbb{R}_+$.

(ii) The following implication holds for any $x \in \text{span } K$ and $\epsilon \geq 0$:

$$\text{dist}(x, K) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \text{dist}(x, \text{span } F) \leq \epsilon \quad \implies \text{dist}(x, F \cap \{z\}^\perp) \leq \psi_{F,z}(\epsilon, \|x\|).$$

Then, $\psi_{F,z}$ is said to be a facial residual function (FRF) for $F$ and $z$ with respect to $K$.

Next, we recall a few tools that allow us to compute facial residual functions. First we deal with a trivial case.

**Lemma 2.3** (FRFs for the zero face). Suppose $K$ is a pointed closed convex cone and let $z \in \text{ri } K^*$. Then, there exists $\kappa > 0$ such that $\psi_{K,z}$ defined by $\psi_{K,z}(\epsilon, t) := \kappa \epsilon$ is a facial residual function for $K$ and $z$ with respect to $K$.

**Proof.** Because $z \in \text{ri } K^*$ and $K$ is pointed, there exists $\kappa > 0$ such that $y \in K$ implies that $\|y\| \leq \kappa \langle y, z \rangle$; see for example [21, Lemma 26]. Next, suppose that $x \in \text{span } K$ satisfies $\text{dist}(x, K) \leq \epsilon$ and $\langle x, z \rangle \leq \epsilon$. Then, there exists $h$ satisfying $\|h\| \leq \epsilon$ such that $x + h \in K$. From (2.2), the pointedness of $K$ and the Cauchy-Schwarz inequality, we have

$$\text{dist}(x, K \cap \{z\}^\perp) = \|x\| \leq \|x + h\| + \|h\| \leq \kappa \langle x + h, z \rangle + \epsilon \leq \kappa \epsilon + \kappa \epsilon \|z\| + \epsilon.$$

Therefore, $\psi_{K,z}(\epsilon, t) := (\kappa + \kappa \|z\| + 1)\epsilon$ is an FRF for $K$ and $z$ with respect to $K$. $\square$

The next lemma shows how to go from error bounds to facial residual functions.

**Lemma 2.4** ([19, Lemma 3.11]). Suppose that $K$ is a closed convex cone and let $z \in K^*$ be such that $F = \{z\}^\perp \cap K$ is a proper face of $K$. Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be monotone nondecreasing with $g(0) = 0$, and let $\kappa_{z,s}$ be a finite monotone nondecreasing nonnegative function in $s \in \mathbb{R}_+$ such that

$$\text{dist}(x, F) \leq \kappa_{z,s} \|x\| g(\text{dist}(x, K)) \quad \text{whenever } x \in \{z\}^\perp. \quad (2.5)$$

Define the function $\psi_{K,z} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\psi_{K,z}(s, t) := \max \left\{ s, s/\|z\| \right\} + \kappa_{z,s} g(\max \left\{ s, s/\|z\| \right\}).$$

Then, $\psi_{K,z}$ is a facial residual function for $K$ and $z$ with respect to $K$. 


Next, we recall a result on how to compute error bounds suitable to be used in conjunction with Lemma 2.4.

**Theorem 2.5** ([19, Theorem 3.12]). Suppose that $K$ is a closed convex cone and let $z \in K^*$ be such that $\mathcal{F} = \{z\}^\perp \cap K$ is a nontrivial exposed face of $K$. Let $\eta > 0$, $\alpha \in (0, 1]$ and let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be monotone nondecreasing with $g(0) = 0$ and $g \geq |\cdot|^\alpha$. Define

$$
\gamma_{z,\eta} := \inf_v \left\{ \frac{g(|w - v|)}{\|w - u\|} \mid v \in \partial K \cap B(\eta) \backslash \mathcal{F}, \ w = P_{\{z\}^\perp} v, \ u = P_{\mathcal{F}} w, \ w \neq u \right\}. \tag{2.6}
$$

If $\gamma_{z,\eta} \in (0, \infty]$, then it holds that

$$
dist(x, \mathcal{F}) \leq \kappa_{z,\eta} g(dist(x, K)) \quad \text{whenever } x \in \{z\}^\perp \cap B(\eta),
$$

where $\kappa_{z,\eta} := \max\{2\eta^{1-\alpha}, 2\gamma_{z,\eta}^{-1}\} < \infty$.

Note that $\gamma_{z,0} = \infty$. For a given $K, z$ and $\mathcal{F}$ as in Theorem 2.5 and $\eta > 0$, we will typically prove that $\gamma_{z,\eta}$ in (2.6) is nonzero by contradiction. The next lemma will aid in this task.

**Lemma 2.6** ([19, Lemma 3.14]). Suppose that $K$ is a closed convex cone and let $z \in K^*$ be such that $\mathcal{F} = \{z\}^\perp \cap K$ is a nontrivial exposed face of $K$. Let $\eta > 0$, $\alpha \in (0, 1]$ and let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be monotone nondecreasing with $g(0) = 0$ and $g \geq |\cdot|^\alpha$. Let $\gamma_{z,\eta}$ be defined as in (2.6). If $\gamma_{z,\eta} = 0$, then there exist $\hat{v} \in \mathcal{F}$ and a sequence $\{v^k\} \subset \partial K \cap B(\eta) \backslash \mathcal{F}$ such that

$$
\lim_{k \to \infty} v^k = \lim_{k \to \infty} w^k = \hat{v} \quad \text{and} \quad \lim_{k \to \infty} g(\|w^k - v^k\|) = 0,
$$

where $w^k = P_{\{z\}^\perp} v^k$, $u^k = P_{\mathcal{F}} w^k$ and $w^k \neq u^k$.

The next result will be helpful in the analysis of one-dimensional faces.

**Lemma 2.7.** Let $K$ be a closed convex cone and let $z \in \partial K^* \setminus \{0\}$ be such that $\mathcal{F} := \{z\}^\perp \cap K$ is a one-dimensional proper face of $K$. Let $f \in K \setminus \{0\}$ be such that

$$
\mathcal{F} = \{tf : \ t \geq 0\}.
$$

Let $\eta > 0$ and $v \in \partial K \cap B(\eta) \backslash \mathcal{F}$, $w = P_{\{z\}^\perp} v$ and $u = P_{\mathcal{F}} w$ with $u \neq w$. Then it holds that $\langle f, z \rangle = 0$ and we have

$$
\|v - w\| = \frac{|z, v|}{\|z\|}, \quad \|u - w\| = \begin{cases} \|v - \frac{(z, v)}{\|z\|^2} z - \frac{(f, v)}{\|f\|^2} f\| & \text{if } \langle f, v \rangle \geq 0, \\ \|v - \frac{(z, v)}{\|z\|^2} z\| & \text{otherwise.} \end{cases}
$$

Moreover, when $\langle f, v \rangle \geq 0$ (or, equivalently, $\langle f, w \rangle \geq 0$), we have $u = P_{\text{span}, \mathcal{F}} w$. On the other hand, if $\langle f, v \rangle < 0$, we have $u = 0$.

**Proof.** The fact that $\langle f, z \rangle = 0$ follows from $f \in \mathcal{F} \subset \{z\}^\perp$. The formula for $\|v - w\|$ holds because the projection of $v$ onto $\{z\}^\perp$ is $w = v - \frac{(z, v)}{\|z\|^2} z$. Finally, notice that $u$ is obtained as $t^* f$, where

$$
t^* = \arg \min_{t \geq 0} \{\|w - tf\|\}.
$$

Then $t^*$ is given by $\frac{(w, f)}{\|f\|^2}$ if $\langle w, f \rangle \geq 0$ (in this case, we have $u = P_{\text{span}, \mathcal{F}} w$), and is zero otherwise. The desired formulas now follow immediately by noting that $\langle w, f \rangle = \langle v - \frac{(z, v)}{\|z\|^2} z, f \rangle = \langle v, f \rangle$. This completes the proof. \qed
2.2 Hölderian error bounds and error bounds based on facial residual functions

Here we recall some definitions and results related to error bounds.

**Definition 2.8** (Lipschitzian and Hölderian error bounds). Let $C_1, C_2 \subseteq \mathcal{E}$ be closed convex sets with $C_1 \cap C_2 \neq \emptyset$. We say that $C_1, C_2$ satisfy a uniform Hölderian error bound with exponent $\gamma \in (0, 1]$ if for every bounded set $B \subseteq \mathcal{E}$ there exists a constant $\kappa_B > 0$ such that

$$\text{dist}(x, C_1 \cap C_2) \leq \kappa_B \max\{\text{dist}(x, C_1), \text{dist}(x, C_2)\}^\gamma, \quad \forall x \in B.$$  

If $\gamma = 1$, then the error bound is said to be Lipschitzian.

From [2, Corollary 3] it follows that if (Feas) satisfies the PPS condition, then a Lipschitzian error bound holds. See [19, Section 2.2] for the details. We register this as proposition.

**Proposition 2.9** (Error bound under the PPS condition). If (Feas) satisfies the PPS condition, then a Lipschitzian error bound holds.

Next, we state the error bound based on facial residual functions proved in [19]. In what follows, for functions $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, we define the diamond composition $f \diamond g$ to be the function satisfying

$$(f \diamond g)(a, b) = f(a + g(a, b), b), \quad \forall a, b \in \mathbb{R}_+.$$  

(2.8)

Also if $\psi$ is a facial residual function, we say that $\tilde{\psi}$ is a positive rescaling of $\psi$ if there are positive constants $M_1, M_2, M_3$ such that

$$\tilde{\psi}(\epsilon, t) = M_3 \psi_{F, e}(M_1 \epsilon, M_2 t).$$  

(2.9)

**Theorem 2.10** ([19, Theorem 3.7]). Suppose (Feas) is feasible and let

$$F_\ell \subseteq \cdots \subseteq F_1 = \mathcal{K}$$

be a chain of faces of $\mathcal{K}$ together with $z_i \in F_i^* \cap \mathcal{L}^+ \cap \{a\}^+$ such that $F_i, \mathcal{L} + a$ satisfy the PPS condition and $F_{i+1} = F_i \cap \{z_i\}^+$ for every $i$. For $i = 1, \ldots, \ell - 1$, let $\psi_i$ be a facial residual function for $F_i$ and $z_i$ with respect to $\mathcal{K}$.

Then, there is a positive rescaling of the $\psi_i$ (still denoted as $\psi_i$ by an abuse of notation) such that for any bounded set $B$ there is a positive constant $\kappa_B$ (depending on $B, \mathcal{L}, a, F_\ell$) such that for any $\epsilon \geq 0$ and $x \in B \cap \text{span} \mathcal{K}$, the following implication holds

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon \quad \Rightarrow \quad \text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa_B(\epsilon + \varphi(\epsilon, \eta)),$$

where $\eta = \sup_{x \in B} \|x\|$, $\varphi = (\psi_{\ell-1} \hat{\diamond} \cdots \hat{\diamond} (\psi_2 \hat{\diamond} \psi_1))$, if $\ell \geq 2$. If $\ell = 1$, we let $\varphi$ be the function satisfying $\varphi(\epsilon, \eta) = \epsilon$.

To finish this section, we prove an auxiliary lemma that will be helpful to analyze Hölderian error bounds.

**Lemma 2.11.** let $\mathcal{K}, \mathcal{L}, a$ and $\psi_i$ be as in Theorem 2.10. Suppose that $\text{span} \mathcal{K} = \mathcal{E}$ and consider the following additional assumption on the $\psi_i$:

(i) there exists $a_i \in (0, 1]$ and nonnegative, monotone nondecreasing functions $\rho_i, \hat{\rho}_i$ such that $\psi_i(\epsilon, \eta) = \rho_i(\eta) \epsilon + \hat{\rho}_i(\eta) \epsilon^{a_i}$ for every $\epsilon \geq 0$ and $\eta \geq 0$. 

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Then, $K, \mathcal{L} + a$ satisfy a uniform Hölderian error bound with exponent $\prod_{i=1}^{\ell} \alpha_i$ if $\ell \geq 2$ or exponent 1 if $\ell = 1$.

**Proof.** If $\ell = 1$, then Theorem 2.10 implies that a Lipschitzian error bound holds, so the result is true. So suppose that $\ell \geq 2$, let $B \subseteq \mathcal{E}$ be an arbitrary bounded set and define $d$ as the function satisfying $d(x) := \max \{ \text{dist}(x, K), \text{dist}(x, \mathcal{L} + a) \}$. Theorem 2.10 implies that there exists $\kappa_B > 0$ such that

$$\text{dist} \left( x, (\mathcal{L} + a) \cap K \right) \leq \kappa_B (d(x) + \varphi(d(x), \eta)), \quad \forall x \in B,$$

where $\eta = \sup_{x \in B} ||x||$, $\varphi = (\psi_{\ell-1} \triangledown \cdots \triangledown (\psi_2 \triangledown \psi_1))$ and the $\psi_i$ might have been positively rescaled. Note however, that if $\psi_i$ is positively rescaled, then, adjusting $\rho_i$ and $\rho_1$ if necessary, $\psi_i(\epsilon, \eta) = \rho_i(\eta) \epsilon + \rho_i(\eta) \epsilon^{\alpha_i}$ still holds with the same $\alpha_i$.

In what follows we make extensive use of the following principle: if $\beta_1 \leq \beta_2$, then, for a fixed bounded set $B$, we can find a constant $\kappa$ such that $d(x)^{\beta_2} \leq \kappa d(x)^{\beta_1}$ for all $x \in B$. Indeed,

$$d(x)^{\beta_2} = d(x)^{\beta_2 - \beta_1} d(x)^{\beta_1} \leq \left( \sup_{y \in B} d(y)^{\beta_2 - \beta_1} \right) d(x)^{\beta_1}, \quad \forall x \in B,$$

where the sup is finite because $d(\cdot)$ is continuous, $B$ is bounded and $\beta_2 - \beta_1 \geq 0$.

First, we will prove by induction that there exists $\kappa_{\ell-1} > 0$ such that

$$\varphi(d(x), \eta) \leq \kappa_{\ell-1} d(x)^{\tilde{\alpha}_{\ell-1}}, \quad \forall x \in B,$$

where $\tilde{\alpha}_{\ell-1} := \prod_{i=1}^{\ell-1} \alpha_i$. If $\ell = 2$, then $\varphi = \psi_1$. From (2.11) and the assumption on the format of $\psi_1$ we can find a constant $\kappa > 0$ so that

$$\rho_1(\eta) d(x) + \hat{\rho}_1(\eta) d(x) = (\kappa \rho_1(\eta) + \hat{\rho}_1(\eta)) d(x) \leq \kappa \rho_1(\eta) + \hat{\rho}_1(\eta), \quad \forall x \in B.$$

This implies that (2.12) holds when $\ell = 2$. So suppose that (2.12) holds some $\hat{\ell} \geq 2$ and we will show that it holds for $\ell + 1$ (when a chain of faces of length $\hat{\ell} + 1$ exists). In this case, we have $\varphi = \psi_{\ell-1} \triangledown \varphi_{\hat{\ell} - 1}$, where $\varphi_{\hat{\ell} - 1} = \psi_{\hat{\ell} - 1} \triangledown (\cdots \triangledown (\psi_2 \triangledown \psi_1))$. Now, the induction hypothesis (applied to $\varphi_{\hat{\ell} - 1}$), the monotonicity of facial residual functions and the observation in (2.11) imply that for some constants $\kappa_{\hat{\ell} - 1}, \hat{\kappa}_{\hat{\ell} - 1}$ we have for all $x \in B$,

$$\varphi(d(x), \eta) = \psi_{\hat{\ell}}(d(x) + \varphi_{\hat{\ell} - 1}(d(x), \eta), \eta) \leq \psi_{\hat{\ell}}(d(x) + \kappa_{\hat{\ell} - 1} d(x)^{\tilde{\alpha}_{\hat{\ell} - 1}}, \eta)$$

$$\leq \psi_{\hat{\ell}}(\hat{\kappa}_{\hat{\ell} - 1} d(x)^{\tilde{\alpha}_{\hat{\ell} - 1}}, \eta).$$

Because of the assumption on the format of $\psi_{\hat{\ell}}$, we have that $\psi_{\hat{\ell}}(\hat{\kappa}_{\hat{\ell} - 1} d(x)^{\tilde{\alpha}_{\hat{\ell} - 1}}, \eta)$ can be written as $a_1 d(x)^{\tilde{\alpha}_{\hat{\ell} - 1}} + a_2 d(x)^{\alpha_{\hat{\ell} - 1}}$, for some constants $a_1, a_2$ which do not depend on $d(x)$. Since $\alpha_{\hat{\ell}} \in (0, 1]$, we have $\tilde{\alpha}_{\hat{\ell}} = \alpha_{\hat{\ell}} \tilde{\alpha}_{\hat{\ell} - 1} \leq \alpha_{\hat{\ell} - 1}$. Therefore, the observation in (2.11) together with (2.13) imply the existence of $\kappa_{\hat{\ell}} > 0$ such that

$$\varphi(d(x), \eta) \leq \psi_{\hat{\ell}}(\hat{\kappa}_{\hat{\ell} - 1} d(x)^{\tilde{\alpha}_{\hat{\ell} - 1}}, \eta) \leq \kappa_{\hat{\ell}} d(x)^{\tilde{\alpha}_{\hat{\ell}}}, \quad \forall x \in B,$$

which concludes the induction proof. We have thus established that (2.12) holds. Plugging (2.12) into (2.10) and applying the observation (2.11) yet again, we conclude that there exists a constant $\kappa > 0$ such that

$$\text{dist} \left( x, (\mathcal{L} + a) \cap K \right) \leq \kappa d(x)^{\tilde{\alpha}_{\ell-1}}, \quad \forall x \in B.$$

\[\square\]
3 Best error bounds

In this section, we will present some results that will help us identify when certain error bounds are the “best” possible. The error bounds discussed in this paper come from Theorem 2.10, which uses facial residual functions. The FRFs themselves are constructed from Lemma 2.4, and most of the properties are inherited from the function $g$ appearing in (2.5). We will show in this section that, under some assumptions, the criterion described in Definition 3.1 below will be enough to show optimality of the underlying error bound induced by $g$. In order to keep the notation compact, we will use $w_\epsilon$ instead of $w(\epsilon)$.

**Definition 3.1** (An optimality criterion for $g$). Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be monotone nondecreasing with $g(0) = 0$. Let $K$ be a closed convex cone and $z \in K^*$ be such that $F = \{z\}^* \cap K$ is a proper face of $K$. If there exist $\tau \in F \setminus \{0\}$ and a continuous function $w : (0, 1] \to \{z\}^* \cap \text{span}K \setminus F$ satisfying

$$
\lim_{\epsilon \downarrow 0} w_\epsilon = \tau \text{ and } \limsup_{\epsilon \downarrow 0} \frac{g(\text{dist}(w_\epsilon, K))}{\text{dist}(w_\epsilon, F)} =: L_g < \infty,
$$

then we say that $g$ satisfies the asymptotic optimality criterion for $K$ and $z$.

We will next show that any FRF built from a concave $g$ satisfying (G1) is optimal in the sense that, up to a constant, it must be better than any other possible FRF for the same sets. First, we need some preliminary lemmas.

**Lemma 3.2.** Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be concave with $g(0) = 0$. Then $g((1 + \lambda)s) \leq (1 + \lambda)g(s)$ for all positive numbers $\lambda$ and $s$.

**Proof.** Note that

$$
s = \frac{1}{1 + \lambda}(1 + \lambda)s + \frac{\lambda}{1 + \lambda}0.
$$

Hence, $g(s) \geq g((1 + \lambda)s)/(1 + \lambda) + (\lambda/(1 + \lambda))g(0) = g((1 + \lambda)s)/(1 + \lambda).

**Lemma 3.3.** Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be monotone nondecreasing with $g(0) = 0$. Let $K$ be a closed convex cone and $z \in K^*$ be such that $F = \{z\}^* \cap K$ is a proper face of $K$. Suppose that (G1) holds and let $\tau : (0, 1] \to \mathbb{R}_+$ satisfy $\lim_{\epsilon \downarrow 0} \tau_\epsilon = \tau \in \mathbb{R}_+$. Suppose further that the following partial sub-homogeneity holds:

$$
\exists S > 0 \text{ such that } s \in [0, S] \text{ and } \tau \geq 1 \text{ together imply } g(\tau s) \leq \tau g(s). \tag{SH}
$$

Then

$$
\limsup_{\epsilon \downarrow 0} \frac{g(\text{dist}(\tau_\epsilon w_\epsilon, K))}{\text{dist}(\tau_\epsilon w_\epsilon, F)} \leq \max \left\{ \frac{L_g}{\tau}, L_g \right\}. \tag{3.1}
$$

**Proof.** Since $w_\epsilon \to \tau \in F$ as $\epsilon$ goes to 0, for all sufficiently small $\epsilon$, we have $\text{dist}(w_\epsilon, K) \leq S$. For those $\epsilon$, if $\tau_\epsilon \geq 1$, we have

$$
\frac{g(\text{dist}(\tau_\epsilon w_\epsilon, K))}{\text{dist}(\tau_\epsilon w_\epsilon, F)} \leq \frac{\tau_\epsilon g(\text{dist}(w_\epsilon, K))}{\tau_\epsilon \text{dist}(w_\epsilon, F)} = \frac{g(\text{dist}(w_\epsilon, K))}{\text{dist}(w_\epsilon, F)},
$$

where (a) follows by (SH) and the fact that $\text{dist}(aw_\epsilon, K) = a \text{dist}(w_\epsilon, K)$ holds for every convex cone $K$ and nonnegative scalar $a$. If $\tau_\epsilon < 1$, we have

$$
\frac{g(\text{dist}(\tau_\epsilon w_\epsilon, K))}{\text{dist}(\tau_\epsilon w_\epsilon, F)} \leq \frac{g(\text{dist}(w_\epsilon, K))}{\tau_\epsilon \text{dist}(w_\epsilon, F)},
$$

1With an abuse of terminology, we will simply say “(G1) holds” when this happens.

2In view of Lemma 3.2, (SH) automatically holds for any $S > 0$ when $g$ is concave.
where (a) follows from the monotonicity of $g$. Overall, we conclude that for sufficiently small $\epsilon$, we have
\[
\frac{g(\text{dist}(\tau_1w_1,K))}{\text{dist}(\tau_1w_1,F)} \leq \max\left\{ \frac{g(\text{dist}(w_1,K))}{\text{dist}(w_1,F)} : \frac{\text{dist}(w_1,K)}{\text{dist}(w_1,F)} \right\}.
\]
The desired conclusion now follows immediately upon invoking (G1).

\[\square\]

**Theorem 3.4 (Optimality of FRFs satisfying (G1)).** Let $K$ be a closed convex cone, $z \in K^*$ with $\|z\| = 1$ and let $F := K \cap \{z\}^\perp$ be a nontrivial exposed face of $K$. Let $g$, $\gamma_{z,\eta}$ and $k_{z,\eta}$ be as in Theorem 2.5 such that $\gamma_{z,\eta} \in (0, \infty)$ for every $\eta > 0$. Let
\[
\psi_{K,z}(s,t) := s + k_{z,\eta}g(2s),
\]
so that $\psi_{K,z}(s,t)$ is an FRF for $K$ and $z$ with respect to $K$ (see Lemma 2.4). Suppose further that (SH) and (G1) hold.

Consider any $\bar{\eta} > 0$ and define $M : \mathbb{R}_+ \to \mathbb{R}_+$ as follows:
\[
M(t) := \frac{1}{2} \left[ 1 + k_{z,\eta} \cdot \left( \max\left\{ 1, \frac{\|\tau\|}{t} \right\} L\right)^{-1},
\]
where $\tau$ and $L\eta$ are as in (G1). Let $\psi_{K,z}$ be an arbitrary facial residual function for $K$ and $z$ with respect to $K$. Then, for any $\bar{\eta}_0 \in (0, \bar{\eta}]$, there exists $s_0 > 0$ such that
\[
M(\bar{\eta}_0)\psi_{K,z}(s,t) \leq \psi_{K,z}(s,\bar{\eta}_0), \quad \forall (s, b) \in [0, s_0] \times [0, \bar{\eta}].
\]

**Proof.** Let $w_1$ and $\tau$ be as in (G1). Let $\tau$ be the function defined as follows:
\[
\tau_\epsilon := \begin{cases} \bar{\eta}_0 & \text{if } \epsilon \in (0, 1], \\ \bar{\eta}_0 \|w_1\|^{\epsilon} & \text{if } \epsilon = 0. \end{cases}
\]
As a reminder, we are using $\tau_\epsilon$ as a shorthand for $\tau(\epsilon)$.

Note that $\text{dist}(\tau_\epsilon w_1,F) \neq 0$ because $w_1 \notin F$ by assumption. Using this and the definition of $\psi_{K,z}$ in (3.2), we have for all $\epsilon \in (0, 1]$ that
\[
\frac{\psi_{K,z}(\text{dist}(\tau_\epsilon w_1,K),\bar{\eta})}{\text{dist}(\tau_\epsilon w_1,F)} = \frac{\text{dist}(\tau_\epsilon w_1,K)}{\text{dist}(\tau_\epsilon w_1,F)} + \frac{k_{z,\eta}g(2\text{dist}(\tau_\epsilon w_1,K))}{\text{dist}(\tau_\epsilon w_1,F)}
\]
\[
\leq 1 + \frac{k_{z,\eta}g(2\text{dist}(\tau_\epsilon w_1,K))}{\text{dist}(\tau_\epsilon w_1,F)},
\]
where the inequality holds because $F \subseteq K$, which implies that $\text{dist}(\tau_\epsilon w_1,K) \leq \text{dist}(\tau_\epsilon w_1,F)$ for all $\epsilon \in (0, 1]$.

Next, notice that $\lim_{\epsilon \to 0} \text{dist}(\tau_\epsilon w_1,K) = 0$. Using this and (SH), we see that for all sufficiently small $\epsilon$,
\[
\frac{k_{z,\eta}g(2\text{dist}(\tau_\epsilon w_1,K))}{\text{dist}(\tau_\epsilon w_1,F)} \leq \frac{k_{z,\eta}g(2\text{dist}(\tau w_1,K))}{\text{dist}(\tau w_1,F)}. \tag{3.6}
\]
Combining (3.6) with Lemma 3.3 and recalling again that (G1) and (SH) hold, we deduce further that
\[
\limsup_{\epsilon \to 0} \frac{k_{z,\eta}g(2\text{dist}(\tau_\epsilon w_1,K))}{\text{dist}(\tau_\epsilon w_1,F)} \leq k_{z,\eta} \cdot 2 \left( \max\left\{ 1, \frac{1}{\tau_0} \right\} L\right) < \infty. \tag{3.7}
\]
Now, combining (3.5) and (3.7), we have that
\[
\limsup_{\epsilon \to 0} \frac{\psi_{K,z}(\text{dist}(\tau_\epsilon w_1,K),\bar{\eta})}{\text{dist}(\tau_\epsilon w_1,F)} \leq 1 + k_{z,\eta} \cdot 2 \left( \max\left\{ 1, \frac{1}{\tau_0} \right\} L\right) < \infty. \tag{3.8}
\]
Consequently, there must exist $\hat{\epsilon} \in (0, 1]$ such that
\[
\frac{\psi_{\mathcal{K}, z}(\text{dist}(\tau, w_\epsilon, \mathcal{K}), \bar{\eta})}{\text{dist}(\tau, w_\epsilon, \mathcal{F})} \leq 2 + \kappa_{z, \bar{\eta}} \cdot 2 \left( \max \left\{ 1, \frac{1}{\tau_0} \right\} L_\theta \right) = \frac{1}{M(\bar{\eta}_0)}, \quad \forall \epsilon \in (0, \hat{\epsilon}]. \tag{3.9}
\]
On the other hand, since $\psi_{\mathcal{K}, z}^*$ is an FRF and $w_\epsilon \in \text{span} \mathcal{K} \cap \{ z \}^\perp \setminus \mathcal{F}$, we have from [19, Remark 3.4] that for all $\epsilon \in (0, 1]$,
\[
1 \leq \frac{\psi_{\mathcal{K}, z}^*(\text{dist}(\tau, w_\epsilon, \mathcal{K}), \|\tau, w_\epsilon\|)}{\text{dist}(\tau, w_\epsilon, \mathcal{F})}. \tag{3.10}
\]
Combining (3.9) with (3.10), we deduce that for all $\epsilon \in (0, \hat{\epsilon}]$,
\[
M(\bar{\eta}_0) \psi_{\mathcal{K}, z}(\text{dist}(\tau, w_\epsilon, \mathcal{K}), \bar{\eta}) \leq \psi_{\mathcal{K}, z}^*(\text{dist}(\tau, w_\epsilon, \mathcal{K}), \|\tau, w_\epsilon\|). \tag{3.11}
\]
Since $\psi_{\mathcal{K}, z}$ is monotone in its second variable, we further obtain that for all $\epsilon \in (0, \hat{\epsilon}]$ and $b \in [0, \bar{\eta}]$,
\[
M(\bar{\eta}_0) \psi_{\mathcal{K}, z}(\text{dist}(\tau, w_\epsilon, \mathcal{K}), b) \leq M(\bar{\eta}_0) \psi_{\mathcal{K}, z}(\text{dist}(\tau, w_\epsilon, \mathcal{K}), \bar{\eta}) \leq \psi_{\mathcal{K}, z}^*(\text{dist}(\tau, w_\epsilon, \mathcal{K}), \|\tau, w_\epsilon\|) \tag{3.12}
\]
Since $\epsilon \mapsto \text{dist}(\tau, w_\epsilon, \mathcal{K})$ is continuous with $\lim_{\epsilon \downarrow 0} \text{dist}(\tau, w_\epsilon, \mathcal{K}) = 0$ but is positive on $(0, \hat{\epsilon}]$ (since $w_\epsilon \in \{ z \}^\perp \setminus \mathcal{F}$, hence $w_\epsilon \notin \mathcal{K}$), its image contains some interval of the form $(0, s_0]$, where $s_0 := \text{dist}(\tau, w_\epsilon, \mathcal{K})$. Therefore, (3.12) shows that for every $s \in (0, s_0]$ and for every $b \in [0, \bar{\eta}]$ we have
\[
M(\bar{\eta}_0) \psi_{\mathcal{K}, z}(s, b) \leq \psi_{\mathcal{K}, z}^*(s, \bar{\eta}_0).
\]
The proof is now complete upon noting that the above relation holds trivially when $s = 0$ because both sides of the inequality become zero when $s = 0$, according to the definition of facial residual function.

We now move on to an application of Theorem 3.4. The next result says that if $(\text{Feas})$ only requires a single facial reduction step and the FRF is as in Theorem 3.4, then the obtained error bound must be optimal. This will be discussed in the framework of consistent error bound functions developed in [20], which we now recall.

**Definition 3.5** (Consistent error bound functions). Let $C_1, \ldots, C_m \subseteq \mathcal{E}$ be closed convex sets with $C := \bigcap_{i=1}^m C_i \neq \emptyset$. A function $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be a *consistent error bound function* for $C_1, \ldots, C_m$ if:

(i) the following error bound condition is satisfied:
\[
\text{dist}(x, C) \leq \Phi \left( \max_{1 \leq i \leq m} \text{dist}(x, C_i), \|x\| \right), \quad \forall x \in \mathcal{E}; \tag{3.13}
\]

(ii) for any fixed $b \geq 0$, the function $\Phi(\cdot, b)$ is monotone nondecreasing on $\mathbb{R}^+$, right-continuous at 0 and satisfies $\Phi(0, 0) = 0$;

(iii) for any fixed $a \geq 0$, the function $\Phi(a, \cdot)$ is monotone nondecreasing on $\mathbb{R}^+$.

We say that (3.13) is the *consistent error bound* associated to $\Phi$.

Hölderian error bounds and the error bounds developed under the theory of amenable cones can all be expressed through the framework of consistent error bound functions; see [20]. With that, we can now state the main result of this section.
Theorem 3.6 (Best error bounds). Suppose (Feas) is feasible and consider the following three assumptions.

(i) There exist \( z \in \mathcal{K}^* \cap \mathcal{L}^\perp \cap \{a\}^\perp \) and \( \mathcal{F} := \mathcal{K} \cap \{z\}^\perp \) such that \( \mathcal{F} \) is a nontrivial exposed face and \( \mathcal{F}, \mathcal{L} + a \) satisfies the PPS condition.

(ii) The function \( \psi \) is a facial residual function for \( \mathcal{K} \) and \( z \) with respect to \( \mathcal{K} \) as in Lemma 2.4, for some \( g \) and \( \kappa_{z,\eta} \) as in Theorem 2.5 so that the \( \gamma_{z,\eta} \) in (2.6) satisfies \( \gamma_{z,\eta} \in (0, \infty) \) for every \( \eta > 0 \).

(iii) The function \( g \) from (ii) also satisfies (SH) and (G1) holds.

Then, the following hold.

(a) There is a positive rescaling of the \( \psi \) denoted by \( \hat{\psi} \) such that for any bounded set \( B \), there is a positive constant \( \kappa_B \) (depending on \( B, \mathcal{L}, a, \mathcal{F} \)) such that for every \( x \in B \cap \text{span} \mathcal{K} \) and \( \epsilon \geq 0 \) we have the following implication

\[
\text{dist}(x, \mathcal{K}) \leq \epsilon \quad \text{and} \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon \\
\implies \text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa_B(\epsilon + \hat{\psi}(\epsilon, \eta)),
\]

(3.14)

where \( \eta = \sup_{x \in B} \|x\| \).

(b) The error bound in (3.14) is optimal in the following sense: Suppose we also assume the following item holds:

\[
\|z\| = 1, \quad \mathcal{L} = \{z\}^\perp, \quad \text{and} \quad a = 0.
\]

Then for any consistent error bound function \( \Phi \) for \( \mathcal{K}, \mathcal{L} \) and any \( \hat{\eta} > 0 \), there are constants \( \hat{\kappa} > 0 \) and \( s_0 > 0 \) such that

\[
s + \hat{\psi}(s, \hat{\eta}) \leq \hat{\kappa}\Phi(\hat{\kappa}s, \hat{\eta}), \quad \forall s \in [0, s_0].
\]

Proof. Item (a) follows directly from Theorem 2.10 with \( \ell = 2 \).

We move on to item (b). Suppose that \( \Phi \) is a consistent error bound function for \( \hat{\mathcal{K}} = \{z\}^\perp \) and \( \mathcal{K} \). First, we will show that \( \Phi \) is a facial residual function for \( \mathcal{K} \) and \( z \) with respect to \( \mathcal{K} \). Let

\[
d(x) := \max\{\text{dist}(x, \mathcal{K}), \text{dist}(x, \{z\}^\perp)\},
\]

so that

\[
\text{dist}(x, \mathcal{K} \cap \{z\}^\perp) = \text{dist}(x, \mathcal{F}) \leq \Phi(d(x), \|x\|), \quad \forall x \in \mathcal{E}.
\]

(3.15)

Suppose that \( x \in \text{span} \mathcal{K} \) is such that \( \text{dist}(x, \mathcal{K}) \leq \epsilon, \langle x, z \rangle \leq \epsilon \). Then, there exists \( u \) with \( \|u\| \leq \epsilon \) such that \( x + u \in \mathcal{K} \). Therefore

\[
0 \leq \langle x + u, z \rangle \quad \text{and hence} \quad -\epsilon \leq \langle -u, z \rangle \leq \langle x + u - u, z \rangle = \langle x, z \rangle \leq \epsilon.
\]

That is \( |\langle x, z \rangle| = \text{dist}(x, \{z\}^\perp) \leq \epsilon \) and thus \( d(x) \leq \epsilon \). Then, (3.15) implies that \( \Phi \) is a facial residual function for \( \mathcal{K} \) and \( z \) with respect to \( \mathcal{K} \).

By assumption, the conditions of Theorem 3.4 are satisfied. Hence, for any \( \hat{\eta} > 0 \), there exists \( s_0 > 0 \) such that

\[
M(\hat{\eta})\psi(s, b) \leq \Phi(s, b), \quad \forall (s, b) \in [0, s_0] \times [0, \hat{\eta}],
\]

(3.16)

where \( M \) is as in (3.3). In addition, by the definition of positive rescaling, there are positive constants \( M_0, M_1, M_2 \) such that \( \psi(s, t) = M_0\psi(M_1s, M_2t) \). We have:

\[
s + \hat{\psi}(s, \hat{\eta}) = s + M_0\psi(M_1s, M_2\hat{\eta}) = s + M_0(M_1s + \kappa_{z,\eta}g(2M_1s))
\]
\[ s + \hat{\psi}(s, \hat{\eta}) \leq \kappa s + \kappa g(2\kappa s) \leq \max \left\{ 1, \frac{\kappa}{\kappa_{s, \hat{\eta}}} \right\} (\kappa s + \kappa_{s, \hat{\eta}}g(2\kappa s)) = \max \left\{ 1, \frac{\kappa}{\kappa_{s, \hat{\eta}}} \right\} \psi(\kappa s, \hat{\eta}). \]

Therefore, if \( s \in [0, s_0/\kappa] \), we have from (3.16) and the above display that

\[ s + \hat{\psi}(s, \hat{\eta}) \leq \kappa s + \kappa g(2\kappa s) = \kappa s + \kappa_{s, \hat{\eta}}g(2\kappa s) \]

where \( \kappa := \max \left\{ M_0M_1 + 1, M_0\kappa_{s, \hat{\eta}}, M_1 \right\} \) and the last inequality holds because of the monotonicity and nonnegativity of \( g \). Continuing, note that since \( \hat{\eta} > 0 \), we have \( \gamma_{z, \hat{\eta}} \in (0, \infty] \) and hence \( \kappa_{s, \hat{\eta}} = \max \{ 2\hat{\eta}^{1-\alpha}, 2\gamma_{z, \hat{\eta}} \} > 0 \). Thus,

\[ s + \hat{\psi}(s, \hat{\eta}) \leq \kappa s + \kappa g(2\kappa s) = \kappa s + \kappa_{s, \hat{\eta}}g(2\kappa s) \]

where \( \hat{\kappa} := \max \left\{ \kappa, \max \left\{ 1, \frac{\kappa_{s, \hat{\eta}}}{\kappa_{s, \hat{\eta}}} \right\} M(\hat{\eta})^{-1} \right\} \) and the second inequality follows from the monotonicity of \( \Phi \) in the first entry. This completes the proof.

\[ \square \]

**Corollary 3.7 (Best Hölderian bounds).** Suppose \((\text{Feas})\) is feasible and suppose items (i), (ii) and (iii) in Theorem 3.6 hold for some \( g = | \cdot |^{\alpha} \) with \( \alpha \in (0, 1) \). Then, the following items hold.

(a) \( K \) and \( L + a \) satisfy a uniform Hölderian error bound with exponent \( \alpha \).

(b) Suppose item (iv) in Theorem 3.6 also holds. Then for any consistent error bound function \( \Phi \) for \( K \), \( L \) and any \( \hat{\eta} > 0 \), there are constants \( \hat{\kappa} > 0 \) and \( s_0 > 0 \) such that

\[ s^\alpha \leq \hat{\kappa} \Phi(\hat{s} \hat{\eta}), \quad \forall s \in [0, s_0]. \]

In particular, \( K \), \( L \) do not satisfy a uniform Hölderian error bound with exponent \( \hat{\alpha} \) where \( \alpha < \hat{\alpha} \leq 1 \).

**Proof.** First, we prove item (a). Noting that \( z/\|z\| \in K^* \cap L^+ \cap \{ a \}^\perp \) and \( \{ z/\|z\| \}^\perp = \{ z \}^\perp \), we assume (without loss of generality) that \( \|z\| = 1 \). By assumption, we have \( g = | \cdot |^{\alpha} \), so that the facial residual function \( \psi \) from Lemma 2.4 satisfies

\[ \psi(s, t) = s + \kappa_{z,t}(2s)^\alpha, \quad (3.17) \]

where \( \kappa_{z,t} \) is as in Theorem 2.5 and is nonnegative monotone nondecreasing in \( t \). Then, applying Lemma 2.11 with \( \ell = 2 \), we conclude that a uniform Hölderian error bound with exponent \( \alpha \) holds.

Next, we move on to item (b). By item (b) of Theorem 3.6, for any \( \hat{\eta} > 0 \), there are constants \( \hat{\kappa} > 0 \) and \( s_0 > 0 \) such that for every \( s \in [0, s_0] \) we have

\[ s + \hat{\psi}(s, \hat{\eta}) \leq \hat{\kappa} \Phi(\hat{s} \hat{\eta}), \quad (3.18) \]

where \( \hat{\psi} \) is a positive rescaling of \( \psi \) in (3.17). The left-hand-side of (3.18), as a function of \( s \), now has the form \( a_1 s + a_2 s^\alpha \) for some constants \( a_1 > 0, a_2 > 0 \). Therefore, adjusting \( \hat{\kappa} \) if necessary, we have

\[ s^\alpha \leq \hat{\kappa} \Phi(\hat{s} \hat{\eta}), \quad \forall s \in [0, s_0]. \]

For the sake of obtaining a contradiction, suppose that a uniform Hölderian error bound holds for \( K \) and \( L \) with exponent \( \hat{\alpha} \) for some \( \hat{\alpha} \in (\alpha, 1] \). Then, there exists a monotone nondecreasing
function \( \hat{\rho} \) such that \( \Phi \) given by \( \Phi(s, t) = \hat{\rho}(t)s^\alpha \) is a consistent error bound function for \( \mathcal{K} \) and \( \mathcal{L} \); see [20] or this footnote\(^3\).

By what have been shown so far and in view of (3.19), there are constants \( \hat{\kappa} > 0 \) and \( s_0 > 0 \) such that for every \( s \in [0, s_0] \) we have
\[
s^\alpha \leq \hat{\kappa} \hat{\rho}(\tilde{s})(\hat{s})^\alpha.
\]
Dividing both sides by \( s^\alpha \) and letting \( s \downarrow 0 \), we get a contradiction: the left-hand-side blows up to infinity, while the right-hand-side is constant. □

**Remark 3.8 (On the exponential cone).** The exponential cone in \( \mathbb{R}^3 \) is
\[
\mathcal{K}_{\text{exp}} := \left\{ (x, y, z) \mid y > 0, z \geq ye^{x/y} \right\} \cup \mathcal{F}_{-\infty}, \quad \mathcal{F}_{-\infty} := \{(x, 0, z) \mid x \leq 0, z \geq 0 \}.
\]
Its nontrivial exposed faces are the 2-D face \( \mathcal{F}_{-\infty} \), infinitely many 1-D faces of form \( \mathcal{F}_\beta = \{(y - \beta y, ye^{-\beta y}) \mid y \geq 0 \} \) for \( \beta \in \mathbb{R} \), and the exceptional 1-D face \( \mathcal{F}_{\infty} := \{(x, 0, 0) \mid x \leq 0 \}; \) see [19, Section 4.1]. For \( t \) sufficiently near 0, the \( g \) corresponding to their (worst case) FRFs simplify to \( g_{-\infty}(t) = -t \ln(t), g_\beta(t) = \sqrt{t} \) and \( g_{\infty}(t) = -1/\ln(t) \), respectively; see Corollaries 4.4, 4.7 and 4.11 in [19]. Once admissibility of these \( g \) is established, the condition (G1) may be verified by letting \( w_\epsilon = (-1, \epsilon, 0) \) for \( \mathcal{F}_{\infty} \), \( w_\epsilon = (-\epsilon \ln(\epsilon), 0, 1) \) for \( \mathcal{F}_{-\infty} \), and \( w_\epsilon = P_{\{x\}}[(1 - \beta + \epsilon, 1, e^{1-\beta + \epsilon})] \) for \( \mathcal{F}_\beta \) when \( \beta \in \mathbb{R} \): Indeed, the arguments in [19, Remark 4.16] demonstrate that \( L_{g_\infty} = L_{g_{-\infty}} = 1 \) and \( L_{g_\beta} \in (0, \infty) \) (note that \( L_{g_\beta} \) is what the authors labeled as \( L_\beta \) in [19]). Thus, our framework can be used to show that the error bounds for the exponential cone are also tight in the sense of item (b) of Theorem 3.6.

## 4 Error bounds for \( p \)-cones

In this section, we will compute the facial residual functions for the \( p \)-cones, obtain error bounds and prove their optimality. First, we recall that for \( p \) satisfying \( 1 < p < \infty \), \( \bar{x} \in \mathbb{R}^n \), \( n \geq 2 \), the \( p \)-norm of \( \bar{x} \) and the \( p \)-cone are given by
\[
\|\bar{x}\|_p := \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}, \quad \mathcal{K}^{n+1}_p := \{x = (x_0, \bar{x}) \in \mathbb{R}^{n+1} \mid x_0 \geq \|\bar{x}\|_p\}; \quad (4.1)
\]
here, given a vector \( x \in \mathbb{R}^{n+1} \), we use \( x_0 \) to denote its first (0th) entry and \( \bar{x} \) to denote the subvector obtained from \( x \) by deleting \( x_0 \). Here, we fix \((\cdot, \cdot)\) the usual Euclidean inner product so that the dual cone of \( \mathcal{K}^{n+1}_p \) is the \( q \)-cone, where \( \frac{1}{p} + \frac{1}{q} = 1 \):
\[
\mathcal{K}^{n+1}_q := \{z = (z_0, \bar{z}) \in \mathbb{R}^{n+1} \mid z_0 \geq \|\bar{z}\|_q\}.
\]
We recall that \( \mathcal{K}^{n+1}_p \) is a pointed full-dimensional cone. In what follows, we will be chiefly concerned with the case \( p \in (1, \infty) \) and \( n \geq 2 \). We will also make extensive use of the following lemma.

**Lemma 4.1.** Let \( p, q \in (1, \infty) \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and let \( \zeta \in \mathbb{R}^n \) (\( n \geq 1 \)) satisfy \( \|\zeta\|_q = 1 \). Define
\[
\tilde{\zeta} := -\text{sgn}(\zeta) \circ |\zeta|^{q-1},
\]
where \( \circ \) is the Hadamard product, and \( \text{sgn} \), absolute value and the \( q - 1 \) power are taken componentwise. Then \( \|\tilde{\zeta}\|_p = 1 \). Moreover, there exist \( C > 0 \) and \( \epsilon > 0 \) so that
\[
1 + \langle \zeta, \omega \rangle \geq C \sum_{i \in I} |\omega_i - \tilde{\zeta}_i|^2 + \frac{1}{p} \sum_{i \in I} |\omega_i|^p \text{ whenever } \|\omega - \zeta\| \leq \epsilon \text{ and } \|\omega\|_p = 1, \quad (4.2)
\]
\(^3\)For any positive integer \( r \), there is a constant \( \kappa_r > 0 \) such that Definition 2.8 holds for \( B \) equal to the ball centered at the origin with radius \( r \). Adjusting the constants if necessary, we have \( \kappa_r \leq \kappa_{r'} \) if \( r \leq r' \), so we can let \( \hat{\rho}(b) \) be the \( \kappa_r \) such that \( r \) is the smallest integer larger than \( b \).
Thus, we must indeed have $\text{sgn}(\omega_i) = -1$. Furthermore, for any $\omega$ satisfying $||\omega||_p \leq 1$, it holds that $\langle \zeta, \omega \rangle \geq -1$, with the equality holding if and only if $\omega = \zeta$.

**Proof.** It is easy to check that $||\zeta||_p = 1$. Next, for each $i \in I$, by considering the Taylor series at $\zeta_i$ of the function $t \mapsto |t|^p$, we see that

$$
|\omega_i|^p = |\zeta_i|^p + p \text{sgn}(\zeta_i)|\zeta_i|^{p-1}(\omega_i - \zeta_i) + \frac{p(p-1)}{2}|\zeta_i|^{p-2}(\omega_i - \zeta_i)^2 + O(|\omega_i - \zeta_i|^3) \quad \text{as} \quad \omega_i \to \zeta_i.
$$

In particular, there exist $c_i > 0$ and $\epsilon_i > 0$ so that

$$
|\omega_i|^p \geq |\zeta_i|^p + p \text{sgn}(\zeta_i)|\zeta_i|^{p-1}(\omega_i - \zeta_i) + c_i(\omega_i - \zeta_i)^2 \quad \text{whenever} \quad |\omega_i - \zeta_i| \leq \epsilon_i.
$$

(4.3)

Let $\epsilon := \min_{i \in I} \epsilon_i$. Then we have for any $\omega \in \mathbb{R}^n$ satisfying $||\omega - \zeta|| \leq \epsilon$ and $||\omega||_p = 1$ that

$$
1 = ||\omega||_p^p = \sum_{i \in I} |\omega_i|^p + \sum_{i \in I} |\omega_i|^p \geq \sum_{i \in I} |\omega_i|^p + \sum_{i \in I} \left[|\zeta_i|^p + p \text{sgn}(\zeta_i)|\zeta_i|^{p-1}(\omega_i - \zeta_i) + c_i(\omega_i - \zeta_i)^2\right] = \sum_{i \in I} |\omega_i|^p + \sum_{i \in I} \left[|\zeta_i|^p + p \text{sgn}(\zeta_i)|\zeta_i|^{p-1}\omega_i - \omega_i(\omega_i - \zeta_i)^2\right] = \sum_{i \in I} |\omega_i|^p + \sum_{i \in I} \left[|\zeta_i|^p - p \text{sgn}(\zeta_i)|\zeta_i|^{p-1}\omega_i + c_i(\omega_i - \zeta_i)^2\right],
$$

where (a) follows from (4.3), (b) holds since $\text{sgn}(\zeta_i)|\zeta_i|^{p-1} = -\text{sgn}(\zeta_i)|\zeta_i|^{(q-1)(p-1)} = -\text{sgn}(\zeta_i)|\zeta_i| = -\zeta_i$, (c) holds because $\sum_{i \in I} |\zeta_i|^p = ||\zeta||_p^p = 1$. Rearranging terms in the above display, we see that (4.2) holds with $C = \frac{1}{p} \min_{i \in I} c_i$.

Finally, we see from the Hölder inequality and the fact $||\zeta||_q = 1$ that $\langle \zeta, \omega \rangle \geq -1$ whenever $||\omega||_p \leq 1$. We now discuss the equality case. It is clear that if $\omega = \zeta$, then $\langle \zeta, \omega \rangle = -1$. Conversely, suppose that $\langle \zeta, \omega \rangle = -1$ and $||\omega||_p \leq 1$. Then we have

$$
1 = ||\zeta||_q ||\omega||_p = ||\zeta||_q \leq 1.
$$

Thus, the Hölder’s inequality holds as an equality and we have $||\omega||_p = 1$. This means that there exists $c > 0$ so that $|\omega_i|^p = c|\zeta_i|^q$ for all $i$. Summing both sides of this equality for all $i$ and invoking $||\omega||_p = ||\zeta||_q = 1$, we see immediately that $c = 1$ and hence $|\omega_i|^p = |\zeta_i|^q$ for all $i$. Consequently,

$$
\omega_i = \text{sgn}(\omega_i)|\zeta_i|^\frac{q}{p} = \text{sgn}(w_i)|\zeta_i|^q - 1.
$$

(4.4)

Plugging the above relation into $\langle \zeta, \omega \rangle = -1$ yields

$$
-1 = \sum_{i=1}^n \zeta_i \omega_i = \sum_{i=1}^n \text{sgn}(\omega_i)\zeta_i |\zeta_i|^{q-1} \geq - \sum_{i=1}^n |\zeta_i|^q = -1.
$$

Thus, we must indeed have $\text{sgn}(\omega_i) = -\text{sgn}(\zeta_i)$ whenever $\zeta_i \neq 0$.\footnote{Since $|\zeta_i| = |\omega_i|$ for all $i$, $\zeta_i \neq 0$ is the same as $\omega_i \neq 0$.} Combining this with (4.4), we conclude that $\omega = \zeta$. This completes the proof.\qed
4.1 Facial structure of $K_{p}^{n+1}$ for $n \geq 2$ and $p \in (1, \infty)$

The $p$-cones $K_{p}^{n+1}$ for $p \in (1, \infty)$ are strictly convex, i.e., all faces are either $\{0\}$, $K_{p}^{n+1}$ or extreme rays (one-dimensional faces). All strictly convex cones are amenable [21, Proposition 9] so, in particular, $K_{p}^{n+1}$ is amenable and facially exposed. In this subsection, we characterize all the faces of $K_{p}^{n+1}$ in terms of the corresponding exposing hyperplanes.

Let $z \in K_{q}^{n+1}$, so $K_{p}^{n+1} \cap \{z\}^\perp$ is a face of $K_{p}^{n+1}$ and, because $K_{p}^{n+1}$ is facially exposed, all faces arise in this fashion. If $z \in \text{ri} K_{q}^{n+1}$ or $z = 0$ we have that $K_{p}^{n+1} \cap \{z\}^\perp = \{0\}$ (see (2.2)) or $K_{p}^{n+1}$, respectively.

Next, suppose that $z \in \partial K_{q}^{n+1}\backslash \{0\}$. Then $z = (z_0, \bar{z})$ with $z_0 = \|\bar{z}\|_q > 0$. Now, $x \in \{z\}^\perp$ if and only if

$$x_0 z_0 + \langle \bar{z}, \bar{x} \rangle = 0.$$ 

Suppose also that $x \in K_{p}^{n+1}\backslash \{0\}$. Then $x_0 > 0$ and the above display is equivalent to

$$1 + \langle z_0^{-1} \bar{z}, x_0^{-1} \bar{x} \rangle = 0.$$

Notice that $\|z_0^{-1} \bar{z}\|_q = 1$ and $\|x_0^{-1} \bar{x}\|_p \leq 1$. An application of Lemma 4.1 with $\zeta = z_0^{-1} \bar{z}$ then shows that

$$x_0^{-1} \bar{x} = -\text{sgn} (z_0^{-1} \bar{z}) \circ |z_0^{-1} \bar{z}|^{q-1} = -\text{sgn} (\bar{z}) \circ |z_0^{-1} \bar{z}|^{q-1},$$

where the second equality holds because $z_0 > 0$. Thus, it follows from the above displays that

$$F_z := \{z\}^\perp \cap K_{p}^{n+1} = \{tf \mid t \geq 0\}, \quad \text{where } f := \left[\frac{1}{-\text{sgn} (\bar{z}) \circ |z_0^{-1} \bar{z}|^{q-1}}\right].$$ (4.5)

4.2 Facial residual functions for $K_{p}^{n+1}$ when $n \geq 2$ and $p \in (1, \infty)$

In this subsection, we build the facial residual functions (FRFs) of $K_{p}^{n+1}$. Thus, let $F \leq K_{p}^{n+1}$ be a face and $z \in F^*$.

First, we take care of the cases where we have “Lipschitz” FRFs. If $F = \{0\}$, then it is clear from Definition 2.2 that $\psi_{F, z}(\epsilon, t) = \epsilon$ is an FRF for $(F, z)$ with respect to $K_{p}^{n+1}$. If $F$ is an extreme ray of $K_{p}^{n+1}$, then $F \cap \{z\}^\perp$ is either $F$ or $\{0\}$. In the former case, the amenability condition (2.1) implies that $\kappa \epsilon$ is an FRF for some $\kappa > 0$. In the latter case, since $F$ is a polyhedral cone and FRFs for polyhedral cones are of the form $\kappa \epsilon$ [21, Proposition 18], we have that $\kappa \epsilon$ is an FRF for $(F, z)$ with respect to $F$. Then, since $K_{p}^{n+1}$ is amenable, using [19, Proposition 3.15] we can lift this FRF to an FRF with respect to $K_{p}^{n+1}$ of the format $\sigma(t) \epsilon$, where $\sigma$ is a nonnegative monotone nondecreasing function. Next, if $F = K_{p}^{n+1}$ and $z \in \text{ri} K_{q}^{n+1}$, then we also have $\kappa \epsilon$ as an FRF, by Lemma 2.3. If $z = 0$, then $\epsilon$ is clearly an FRF.

The remainder of this section is focused on the truly nontrivial case where $F$ is exposed by some $z \in \partial K_{q}^{n+1}\backslash \{0\}$. With that in mind, we define

$$J_z := \{i \mid \bar{z}_i \neq 0\} \quad \text{and} \quad \alpha_z := \begin{cases} \frac{1}{q} & \text{if } |J_z| = n, \\ \frac{1}{p} & \text{if } |J_z| = 1 \text{ and } p < 2, \\ \min \left\{ \frac{1}{2}, \frac{1}{p} \right\} & \text{otherwise}, \end{cases}$$ (4.6)

where $|J_z|$ is the number of elements of $J_z$. Then we have the following result.

**Theorem 4.2.** Let $n \geq 2$ and $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $z \in \partial K_{q}^{n+1}\backslash \{0\}$ and let $F_z := \{z\}^\perp \cap K_{p}^{n+1}$. Let $\eta > 0$, $\alpha_z$ be as in (4.6), and define

$$\gamma_{z, \eta} := \inf_v \left\{ \frac{\|v - w\|^{\alpha_z}}{\|u - w\|} \mid v \in \partial K_{p}^{n+1} \cap B(\eta) \backslash F_z, \ w = P_{\{z\}^\perp} v, \ u = P_F w, \ u \neq w \right\}.$$ (4.7)
Then it holds that $\gamma_{z,\eta} \in (0, \infty)$ and that
\[
\text{dist}(x, \mathcal{F}_z) \leq \max\{2\eta^{-\alpha_z}, 2\gamma_{z,\eta}^{-1}\} \cdot \text{dist}(x, \mathcal{K}_p^{n+1})^{\alpha_z} \quad \text{whenever } x \in \{z\}^\perp \cap B(\eta).
\]

Proof. If $\gamma_{z,\eta} = 0$, in view of Lemma 2.6, there exist $\bar{v} \in \mathcal{F}_z$ and a sequence $\{v^k\} \subset \partial \mathcal{K}_p^{n+1} \cap B(\eta) \setminus \mathcal{F}_z$ such that
\[
\lim_{k \to \infty} v^k = \lim_{k \to \infty} w^k = \bar{v} \quad \text{and} \quad \lim_{k \to \infty} \frac{\|w^k - v^k\|_{\alpha_z}}{\|w^k - u^k\|} = 0,
\]
where $w^k = P_{\{z\}} v^k$, $u^k = P_{\mathcal{F}_z} w^k$ and $u^k \neq w^k$. Since $v^k \notin \mathcal{F}_z$ and $v^k \in \partial \mathcal{K}_p^{n+1}$, we must have $v^k_0 = \|v^k\|_p > 0$. By passing to a further subsequence if necessary, we may then assume that
\[
(v^k_0)^{-1} v^k \to \xi \quad \text{(4.9)}
\]
for some $\xi$ satisfying $\|\xi\|_p = 1$. Next, applying Lemma 4.1 with $\zeta := z_0^{-1} \bar{z}$, we have
\[
\bar{v} := -\text{sgn}(z_0^{-1} \bar{z}) \circ |z_0^{-1} \bar{z}|^{q-1} = -\text{sgn}(\bar{z}) \circ |z_0^{-1} \bar{z}|^{q-1}, \quad \|\bar{v}\|_p = 1,
\]
and there exist $C > 0$ and $\epsilon > 0$ so that (4.2) holds. Moreover, one can see that the $J_z$ in (4.6) equals the $I$ in Lemma 4.1. We consider two cases.

(I) $\xi = \bar{v}$;

(II) $\xi \neq \bar{v}$.

(II): Suppose that $\xi = \bar{v}$. Then $\lim_{k \to \infty} \langle (v^k_0)^{-1} v^k, \bar{v} \rangle = \|\bar{v}\|^2 > 0$. Thus, for all sufficiently large $k$, we have, upon using the definition of $I$ in (4.5), that
\[
\langle (v^k_0)^{-1} v^k, f \rangle = \left[ \frac{1}{(v^k_0)^{-1} v^k}, \frac{1}{(v^k_0)^{-1} v^k} \right] = 1 + \langle (v^k_0)^{-1} v^k, \bar{v} \rangle \geq 1 + \frac{\|\bar{v}\|^2}{2}.
\]
Consequently, we have $\langle v^k, f \rangle > 0$ for sufficiently large $k$. Thus, if we let
\[
Q := I - \frac{z z^T}{\|z\|^2} - \frac{f f^T}{\|f\|^2},
\]
then using Lemma 2.7 (with $f$ as in (4.5)), we deduce that for all sufficiently large $k$,
\[
\|u^k - w^k\| = \|Q v^k\| = v^k_0 \left[ \frac{1}{(v^k_0)^{-1} v^k} \right] = v^k_0 \left[ \frac{1}{(v^k_0)^{-1} v^k} \right] - Q \left[ \frac{1}{\bar{v}} \right] 
\]
\[
\leq v^k_0 \|Q (v^k_0)^{-1} v^k - \bar{v}\|, \quad \text{(4.10)}
\]
where (a) holds because $f = \left[ 1 \quad \bar{z}^T \right]^T$ and $Q f = 0$. Moreover, if it happens that $|J_z| = 1$, say, $J_z = \{i_0\}$, then $z_0 = |\bar{z}_{i_0}| \neq 0$ and $f_0 = |\bar{f}_{i_0}| \neq 0$, and we have $\bar{z}_i = \bar{f}_i = 0$ for all $i \neq i_0$ and $\bar{z}_{i_0} = -\bar{f}_{i_0}$. Then a direct computation shows that $Q$ is diagonal with $Q_{i_0 i_0} = Q_{i_0 i} = 0$, and $Q_{i_0 i} = 1$ otherwise. Thus, we have the following refined estimate on $\|u^k - w^k\|$ for all sufficiently large $k$ when $|J_z| = 1$:
\[
\|u^k - w^k\| = \|Q v^k\| = v^k_0 \left[ \frac{1}{(v^k_0)^{-1} v^k} \right] = v^k_0 \sqrt{\sum_{i \notin J_z} |(v^k_0)^{-1} \bar{v}_i|^2}. \quad \text{(4.11)}
\]

Next, in view of (4.2) and (4.9) and recalling that $\|v^k_0\| = 1$, we have for all sufficiently large $k$ that
\[
1 + \langle \zeta, (v^k_0)^{-1} \bar{v} \rangle \geq C \sum_{i \in I} |(v^k_0)^{-1} \bar{v}_i| + \frac{1}{p} \sum_{i \notin I} |(v^k_0)^{-1} \bar{v}_i|^p. \quad \text{(4.12)}
\]
If $|J_z| \neq 1$ or $p \geq 2$, then we see from (4.9) and (4.12) that for all sufficiently large $k$

$$1 + \langle \zeta, (v_0^k)^{-1}v^k \rangle \geq \min \left\{ C, \frac{1}{p} \right\} \left( \sum_{i \in I} |(v_0^k)^{-1}v^k_i|^2 - \zeta_i^{1/\alpha_z} + \sum_{i \notin I} |(v_0^k)^{-1}v^k_i|^{1/\alpha_z} \right)$$

(a) $\geq \min \left\{ C, \frac{1}{p} \right\} \| (v_0^k)^{-1}v^k - \zeta \|^{1/\alpha_z}$

(4.13)

$$\geq C_1 \| (v_0^k)^{-1}v^k - \zeta \|^{1/\alpha_z},$$

where (a) follows from the definition of $\alpha_z$ and the observation that $J_z = I$, and the last inequality follows from the equivalence of norms in finite-dimensional Euclidean spaces, and $C_1$ is a constant that depends only on $C$, $p$, $\alpha_z$ and the dimension $n$. Thus, combining (4.13) with Lemma 2.7 and recalling that $\zeta = z_0^{-1} \bar{z}$, we see that for all sufficiently large $k$,

$$\| v^k - w^k \| = \frac{|z_0 v_0^k + \zeta, (v_0^k)^{-1}v^k \rangle |}{\| z \|} = \frac{z_0 v_0^k + (z_0, (v_0^k)^{-1}v^k)}{\| z \|} \geq \frac{z_0 v_0^k}{\| z \|} \sum_{i \in I} |(v_0^k)^{-1}v^k_i|^p$$

(a) $\geq \frac{z_0 v_0^k}{\| z \|} \left( \sum_{i \notin I} |(v_0^k)^{-1}v^k_i|^2 \right)^{1/\alpha_z}$

(4.14)

$$\geq \frac{z_0 v_0^k}{\| z \|} \| (v_0^k)^{-1}v^k - \zeta \|^{1/\alpha_z},$$

where the last inequality follows from (4.10).

On the other hand, if $|J_z| = 1$ and $p < 2$, we have $\alpha_z = 1/p$. We can deduce from (4.12) and Lemma 2.7 that for all sufficiently large $k$,

$$\| v^k - w^k \| = \frac{z_0 v_0^k + \zeta, (v_0^k)^{-1}v^k \rangle |}{\| z \|} = \frac{z_0 v_0^k + (z_0, (v_0^k)^{-1}v^k)}{\| z \|} \geq \frac{z_0 v_0^k}{\| z \|} \sum_{i \in I} |(v_0^k)^{-1}v^k_i|^p$$

(a) $\geq \frac{z_0 v_0^k}{\| z \|} \left( \sum_{i \notin I} |(v_0^k)^{-1}v^k_i|^2 \right)^{1/\alpha_z}$

(4.15)

$$\geq \frac{z_0 v_0^k}{\| z \|} \| (v_0^k)^{-1}v^k - \zeta \|^{1/\alpha_z},$$

where (a) holds because $p < 2$ so that $p$-norm majorizes 2-norm, and we used (4.11) and the fact that $\alpha_z = 1/p$ for the last equality.

Combining (4.14) and (4.15), we see that there exists $C_2 > 0$ such that for all sufficiently large $k$,

$$\| v^k - w^k \| \leq \frac{C_2 \cdot (v_0^k)^{1-\alpha_z}}{\| v^k - w^k \|^{\alpha_z}} \leq \frac{C_2 \eta^{1-\alpha_z}}{\| v^k - w^k \|^{\alpha_z}},$$

where the last inequality holds because $v^k \in B(\eta)$. This contradicts (4.8) and hence case (I) cannot happen.

(II): In this case, $\xi \neq \zeta$. Since $\| \xi \|_p = 1$, we see from Lemma 4.1 that $1 + \langle \zeta, \xi \rangle > 0$. Thus, in view of (4.9), there exists $\ell > 0$ such that

$$1 + \langle \zeta, (v_0^k)^{-1}v^k \rangle \geq \ell > 0$$

for all large $k$.

Using this together with Lemma 2.7 and the fact that $\zeta = z_0^{-1} \bar{z}$, we deduce that for these $k$,

$$\| v^k - w^k \| = \frac{|z_0 v_0^k + \zeta, v^k \rangle |}{\| z \|} = \frac{z_0 v_0^k + (z_0, (v_0^k)^{-1}v^k)}{\| z \|} \geq \frac{z_0 v_0^k}{\| z \|} \| v^k \|_p$$

(a) $\geq \frac{z_0 v_0^k}{\| z \|} \| v^k \|_p \geq \frac{1}{2} \| v^k \|_p$ (b) $\geq \frac{C_3 \ell z_0}{2} \| v^k \|_p$ (c) $\geq \frac{C_3 \ell z_0}{2} \| u^k - w^k \|,$
where (a) follows from the triangle inequality and the fact that \( v^k_0 = \|v^k\|_p \), (b) holds for some constant \( C_3 > 0 \) that only depends on \( n \) and \( p \), (c) follows from the fact that \( \|u^k - w^k\| = \text{dist}(w^k, F_z) \leq \|w^k\| \leq \|v^k\| \) (which holds because of the properties of projections). The above display contradicts (4.8) and hence case (II) also cannot happen.

Thus, we must have \( \gamma_{z,q} \in (0, \infty) \) and the desired error bound follows from Theorem 2.5. \( \square \)

Using Theorem 4.2 together with Lemma 2.4 and recalling that an upper bound to an FRF is also an FRF, we obtain the following facial residual function for \( K_p^{n+1} \).

**Corollary 4.3.** Let \( n \geq 2 \) and \( p,q \in (1,\infty) \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( z \in \partial K_q^{n+1}\{0\} \) and let \( F_z := \{z\}^\perp \cap K_p^{n+1} \).

Let \( \alpha_z \) be as in (4.6) and let \( \gamma_{z,t} \) be as in (4.7).\(^5\) Let \( \kappa = \max\{1,1/\|z\|\} \).

Then the function \( \psi_{K,z} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) given by

\[
\psi_{K,z}(\epsilon, t) := \kappa \epsilon + \max\{2t^{1-\alpha_z}, 2\gamma_{z,t}^{-1}\}(\kappa + 1)^{\alpha_z} \epsilon^{\alpha_z}
\]

is a facial residual function for \( K_p^{n+1} \) and \( z \) with respect to \( K_p^{n+1} \).

Next, we will prove that the optimality criterion (G1) is satisfied for \( p \)-cones when \( g = |\cdot|^\alpha_z \), with \( \alpha_z \) as in (4.6). For that, we need two lemmas that assert the existence of functions \( \epsilon \mapsto w_\epsilon \) having certain desirable properties.

**Lemma 4.4 (A \( w_\epsilon \) of order \( \frac{1}{p} \) when \( |J_z| < n \)).** Let \( n \geq 2 \) and \( p,q \in (1,\infty) \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( z \in \partial K_q^{n+1}\{0\} \) be such that \( J_z \neq \{1,2,\ldots,n\} \), where \( J_z \) is as in (4.6). Let \( F_z := \{z\}^\perp \cap K_p^{n+1} \) and \( f \in F_z \setminus \{0\} \) be defined as in (4.5). Then there exists a continuous function \( w : (0,1) \to \{z\}^\perp \setminus F_z \) such that

\[
\lim_{\epsilon \downarrow 0} w_\epsilon = f \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \frac{\text{dist}(w_\epsilon,K_p^{n+1})^{\frac{1}{p}}}{\text{dist}(w_\epsilon,F_z)} < \infty.
\]

**Proof.** Fix any \( j \in \{1,2,\ldots,n\} \setminus J_z \) and define

\[
\bar{z} := -\text{sgn}(\bar{z}) \circ |z_0^{-1}\bar{z}|^{q-1}.
\]

Then \( \|\bar{z}\|_p = 1 \) and \( \bar{z}_i \neq 0 \) if and only if \( i \in J_z \). Moreover, we have \( f = \left[ 1 - \bar{z}^T \bar{z} \right]^T \). Define the (bounded) continuous function \( w : (0,1) \to \mathbb{R}^{n+1} \) by

\[
(w_\epsilon)_i = \begin{cases} f_i & \text{if } i \neq j \\ \epsilon & \text{if } i = j \end{cases}
\]

Then \( (z,w_\epsilon) = 0 \) for every \( \epsilon \in (0,1) \) and \( w_\epsilon \to f \in F_z \setminus \{0\} \). Now, observe from \( \|\bar{z}\|_p = 1 \) that the image of the function \( y : (0,1) \to \mathbb{R}^{n+1} \) defined by

\[
y_\epsilon = (1 + \epsilon^p)^{\frac{1}{p}}, \quad \bar{y}_\epsilon = \bar{w}_\epsilon.
\]

is entirely contained in \( K_p^{n+1} \). Hence, we have

\[
\text{dist}(w_\epsilon,K_p^{n+1}) \leq \|w_\epsilon - y_\epsilon\| = (1 + \epsilon^p)^\frac{1}{p} - 1 \leq \frac{1}{p} \epsilon^p,
\]

where the last inequality follows from the concavity of \( t \mapsto t^\frac{1}{p} \) and the supgradient inequality. In addition, we can also deduce from (4.5) and the definition of \( \bar{z} \) that dist\( (w_\epsilon,F_z) = \epsilon > 0 \). Thus

\[
\lim_{\epsilon \downarrow 0} \frac{\text{dist}(w_\epsilon,K_p^{n+1})^{\frac{1}{p}}}{\text{dist}(w_\epsilon,F_z)} \leq \left( \frac{1}{p} \right)^{\frac{1}{p}}.
\]

\( \square \)

\(^5\)We set \( \gamma_{z,t,0} = \infty \).
Lemma 4.5 (A $w_e$ of order $\frac{1}{2}$ when $|J_z| \geq 2$). Let $n \geq 2$ and $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $z \in \partial K_p^{n+1} \setminus \{0\}$ be such that $|J_z| \geq 2$, where $J_z$ is as in (4.6). Let $\mathcal{F}_z := \{ z \}^\perp \cap K_p^{n+1}$ and $f \in \mathcal{F}_z \setminus \{ 0 \}$ be defined as in (4.5). Then there exists a continuous function $w : (0, 1] \to \{ z \}^\perp \setminus \mathcal{F}_z$ such that

$$\lim_{\epsilon \to 0} w_\epsilon = f \quad \text{and} \quad \limsup_{\epsilon \to 0} \frac{\text{dist}(w_\epsilon, K_p^{n+1})^{\frac{1}{2}}}{\text{dist}(w_\epsilon, \mathcal{F}_z)} < \infty.$$  

**Proof.** Without loss of generality, we assume that $n \in J_z$ so that $\bar{z}_n \neq 0$; by symmetry, we will assume that $\bar{z}_n < 0$. Define

$$\zeta := -\text{sgn}(\bar{z}) \circ |z_0^{-1}\bar{z}|^{q-1}. $$

Then $\|\zeta\|_p = 1$, $\zeta_n > 0$ and $f = \begin{bmatrix} 1 & \zeta^T \end{bmatrix}^T$. Define the (bounded) continuous function $w : (0, \min\{|\bar{z}_n| \cdot \zeta_n, 1\}) \to \mathbb{R}^{n+1}$ by

$$(w_\epsilon)_i = \begin{cases} 1 - \epsilon z_0^{-1} & \text{if } i = 0 \\ \zeta_n + \epsilon z_n^{-1} & \text{if } i = n \\ \zeta_i & \text{otherwise.} \end{cases} $$

Then $(z, w_\epsilon) = 0$ and $w_\epsilon \to f \in \mathcal{F}_z \setminus \{ 0 \}$. Now, observe that the image of the function $y : (0, \min\{|\bar{z}_n| \cdot \zeta_n, 1\}) \to \mathbb{R}^{n+1}$ defined by

$$(y_\epsilon)_0 = \left[ \begin{array}{c} \sum_{i=1}^{n-1} \zeta_i^p \| \zeta_i^p + (\zeta_n + \epsilon z_n^{-1})^p \|_{\frac{1}{p}} \end{array} \right], \quad \bar{y}_\epsilon = \bar{w}_\epsilon,$$

is contained in $K_p^{n+1}$. Hence, we have

$$\text{dist}(w_\epsilon, K_p^{n+1}) \leq \| w_\epsilon - y_\epsilon \| = \left[ \sum_{i=1}^{n-1} \| \zeta_i^p + (\zeta_n + \epsilon z_n^{-1})^p \|_{\frac{1}{p}} \right] - 1 + \epsilon z_0^{-1} = \left[ \sum_{i=1}^{n-1} \| \zeta_i^p + \zeta_n^p + p \zeta_n^{p-1} \epsilon z_n^{-1} + O(\epsilon^2) \|_{\frac{1}{p}} \right] - 1 + \epsilon z_0^{-1} = \left[ 1 + p \zeta_n^{p-1} \epsilon z_n^{-1} + O(\epsilon^2) \right]^{\frac{1}{p}} - 1 + \epsilon z_0^{-1} = \left[ 1 + \epsilon \zeta_n^{p-1} z_n^{-1} - 1 + \epsilon z_0^{-1} + O(\epsilon^2) \right] = O(\epsilon^2) \quad \text{as } \epsilon \downarrow 0, \tag{4.16}$$

where (a) holds because $\|\zeta\|_p = 1$, and the last equality holds because

$$\zeta_n^{p-1} = |z_0^{-1} z_n^{q-1} (p-1)| = |z_0^{-1} z_n| = -z_0^{-1} z_n$$

as $\zeta_n > 0$ and $\bar{z}_n < 0$. We next estimate $\text{dist}(w_\epsilon, \mathcal{F}_z)$. Since $w_\epsilon \to f$, we must have $(w_\epsilon, f) > 0$ for all sufficiently small $\epsilon$. Thus, from the definition of $\mathcal{F}_z$ and Lemma 2.7, we see that for these $\epsilon$,

$$\text{dist}(w_\epsilon, \mathcal{F}_z)^2 = \left\| w_\epsilon - \frac{(w_\epsilon, f)}{\|f\|^2} f \right\|^2 = \|w_\epsilon\|^2 - \frac{(w_\epsilon, f)^2}{\|f\|^2}. $$

Now, a direct computation shows that

$$\|w_\epsilon\|^2 = (1 - \epsilon z_0^{-1})^2 + \sum_{i=1}^{n-1} \| \zeta_i \|^2 + (\zeta_n + \epsilon z_n^{-1})^2$$

$$= 1 + \sum_{i=1}^{n-1} \| \zeta_i \|^2 + \zeta_n + 2\epsilon (z_n^{-1} \zeta_n - z_0^{-1}) + \epsilon^2 z_n^{-2} + \epsilon^2 z_0^{-2}$$

$$= \|f\|^2 + 2\epsilon (z_n^{-1} \zeta_n - z_0^{-1}) + \epsilon^2 z_n^{-2} + \epsilon^2 z_0^{-2},$$

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where the last equality follows from the definition of \( f \) in (4.5). In addition, we also have

\[
(\langle w, f \rangle)^2 = \left( 1 - \epsilon z_0^{-1} + \sum_{i=1}^{n-1} |\zeta_i|^2 + \zeta_n + \epsilon \bar{z}_n^{-1} \zeta_n \right)^2
\]

\[
= \left[ ||f||^2 + \epsilon (\bar{z}_n^{-1} \zeta_n - z_0^{-1}) \right]^2
\]

\[
= ||f||^4 + 2\epsilon ||f||^2 (\bar{z}_n^{-1} \zeta_n - z_0^{-1}) + \epsilon^2 (\bar{z}_n^{-1} \zeta_n - z_0^{-1})^2.
\]

Thus, it holds that for all sufficiently small \( \epsilon \),

\[
\text{dist}(w, \mathcal{F}_z)^2 = \epsilon^2 \left( \bar{z}_n^{-2} + z_0^{-2} - \frac{(\bar{z}_n^{-1} \zeta_n - z_0^{-1})^2}{||f||^2} \right)
\]

\[
\geq \epsilon^2 \left( \bar{z}_n^{-2} + z_0^{-2} - \frac{(-\bar{z}_n^{-1} \zeta_n + z_0^{-1})^2}{1 + \zeta_n^2} \right).
\]

(4.17)

Since \(|J_z| \geq 2\) and \(||\zeta||_p = 1\), we must have \( \zeta_n < 1 \). Hence, we see from the Cauchy-Schwarz inequality that

\[
[\bar{z}_n^{-2} + z_0^{-2}] \cdot [\zeta_n^2 + 1] > (-\bar{z}_n^{-1} \zeta_n + z_0^{-1})^2,
\]

and the inequality is strict because

\[
\frac{\zeta_n}{-\bar{z}_n^{-1}} < \frac{|z_0^{-1} \bar{z}_n| q^{-1}}{|\zeta_n| q} = |z_0^{-1} \bar{z}_n|^q = |\zeta_n|^{\frac{q}{q-1}} < 1.
\]

Consequently, we have from (4.17) that there exists \( c > 0 \) such that \( \text{dist}(w, \mathcal{F}_z) \geq c \epsilon \) for all sufficiently small \( \epsilon \). Combining this with (4.16), we obtain

\[
\limsup_{\epsilon \downarrow 0} \frac{\text{dist}(w, \mathcal{K}_p^{n+1})^{\frac{1}{2}}}{\text{dist}(w, \mathcal{F}_z)} < \infty.
\]

To conclude, we observe that we can perform a change of variable \( \hat{\epsilon} = a \epsilon \) for some \( a > 0 \) so that \( w \) is defined for \( \hat{\epsilon} \in (0, 1] \) with \( w \not\in \mathcal{F}_z \) for all \( \hat{\epsilon} \in (0, 1] \). \( \square \)

We now have all the tools to prove the following theorem.

**Theorem 4.6** (The optimality criterion (G1) is satisfied for \( p \)-cones). Suppose that \( n \geq 2 \) and \( p, q \in (1, \infty) \) are such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( z \in \partial \mathcal{K}_p^{n+1} \backslash \{0\} \) and let \( \mathcal{F}_z := \{z\}^\perp \cap \mathcal{K}_p^{n+1} \). Let \( \alpha_z \) be as in (4.6). Then the function \( \mathfrak{g} = |\cdot|^q \) satisfies the asymptotic optimality criterion (G1) for \( \mathcal{K}_p^{n+1} \) and \( z \).

**Proof.** Let \( J_z \) be as in (4.6). We split the proof in a few cases:

(I) \( |J_z| = \{1, 2, \ldots, n\} \). In this case \( \alpha_z = 1/2 \). Since \( n \geq 2 \), we have \( |J_z| \geq 2 \) so we can invoke Lemma 4.5, which gives the required function \( \mathfrak{w} \) satisfying (G1) with \( \mathfrak{v} = f \) in (4.5).

(II) \( |J_z| = 1 \) and \( p < 2 \). In this case, \( \alpha_z = 1/p \). Since \( n \geq 2 \), we have \( J_z \neq \{1, \ldots, n\} \), so that we can invoke Lemma 4.4, which gives the required function \( \mathfrak{w} \) satisfying (G1) with \( \mathfrak{v} = f \) in (4.5).

(III) \( |J_z| = 1 \) and \( p \geq 2 \). In this case, \( \alpha_z = 1/p \). Similarly to the previous item, it follows from Lemma 4.4.

(IV) \( 2 \leq |J_z| < n \). In this case, \( \alpha_z = \min\{1/2, 1/p\} \). Similarly to the previous items, it follows from either Lemma 4.4 or Lemma 4.5.

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Theorem 4.6 shows an interesting property of $p$-cones: namely, in 3 dimensions, $n = 2$, so that the cases $|J_z| = n$ and $|J_z| \neq 1$ exactly coincide, thereby eliminating the minimum $\{\frac{1}{2}, \frac{1}{p}\}$ case in the definition of $\alpha_z$ when $p < 2$. Thus, $p$-cones of dimension 4 and higher exhibit a greater complexity in their optimal FRFs than those in $\mathbb{R}^3$.

4.3 Error bounds

In this subsection, we gather all the results we have proved so far and prove a tight error bound for problems involving a single $p$-cone.

**Theorem 4.7** (Error bounds for the $p$-cone and their optimality). Let $n \geq 2$ and $p \in (1, \infty)$. Let $\mathcal{L} \subseteq \mathbb{R}^{n+1}$ be a subspace and $a \in \mathbb{R}^{n+1}$ such that $(\mathcal{L} + a) \cap \mathcal{K}_p^{n+1} \neq \emptyset$. Then the following items hold.

(i) If $(\mathcal{L} + a) \cap \mathcal{K}_p^{n+1} = \{0\}$ or $(\mathcal{L} + a) \cap (\text{ri } \mathcal{K}_p^{n+1}) \neq \emptyset$, then $\mathcal{K}_p^{n+1}$ and $\mathcal{L} + a$ satisfy a Lipschitzian error bound.

(ii) Otherwise, $\mathcal{K}_p^{n+1}$ and $\mathcal{L} + a$ satisfy a uniform Hölderian error bound with exponent $\alpha_z$ satisfies Theorem 4.2. By Theorem 4.6, we have $K_p^{n+1}$ and $\mathcal{L} + a$ satisfy a Lipschitzian error bound by [21, Proposition 27]. This concludes the proof of item (i).

Next, we move on to item (ii). In this case we have $(\mathcal{L} + a) \cap (\text{ri } \mathcal{K}_p^{n+1}) \neq \emptyset$ then Proposition 2.9 implies that a Lipschitzian error bound holds.

If we have $\mathcal{K}_p^{n+1} \cap (\mathcal{L} + a) = \{0\}$, a Lipschitzian error bound holds by [21, Proposition 27]. This concludes the proof of item (i).

In particular, $\mathcal{K}_p^{n+1}$, $\{z\}^\perp$ do not satisfy a uniform Hölderian error bound with exponent $\alpha < \tilde{\alpha} \leq 1$.

Proof. If $(\mathcal{L} + a) \cap (\text{ri } \mathcal{K}_p^{n+1}) \neq \emptyset$ then Proposition 2.9 implies that a Lipschitzian error bound holds. If we have $\mathcal{K}_p^{n+1} \cap (\mathcal{L} + a) = \{0\}$, a Lipschitzian error bound holds by [21, Proposition 27]. This concludes the proof of item (i).

Next, we move on to item (ii). In this case we have $(\mathcal{L} + a) \cap (\text{ri } \mathcal{K}_p^{n+1}) = \emptyset$ and $\mathcal{K}_p^{n+1} \cap (\mathcal{L} + a) \neq \emptyset$. The $p$-cone for $p \in (1, \infty)$ only has faces of dimension 0, 1 or $n + 1$. As such, its distance to polyhedrality $\ell_{\text{poly}}(K_p^{n+1}) = 1$ (see Section 2). By (2.4), we have $d_{\text{PPS}}(K_p^{n+1}, \mathcal{L} + a) \leq 1$. Since $(\mathcal{L} + a) \cap (\text{ri } \mathcal{K}_p^{n+1}) = \emptyset$, we have $d_{\text{PPS}}(K_p^{n+1}, \mathcal{L} + a) = 1$. Therefore, there exists a chain of faces $\mathcal{F}_2 \subseteq \mathcal{K}_p^{n+1}$ satisfying items (ii) and (iii) of Proposition 2.1 together with $z \in \mathcal{K}_q^{n+1} \cap \mathcal{L}^\perp \cap \{a\}^\perp$ with $1/q + 1/p = 1$, such that $\mathcal{F}_2 = K_p^{n+1} \cap \{z\}^\perp$.

Since $(\mathcal{L} + a) \cap (\text{ri } \mathcal{K}_p^{n+1}) = \emptyset$, we have $z \neq 0$. Since $\mathcal{K}_p^{n+1} \cap (\mathcal{L} + a) \neq \{0\}$, we have $\mathcal{F}_2 \neq \{0\}$ (recall that $\mathcal{F}_2$ contains $\mathcal{K}_p^{n+1} \cap (\mathcal{L} + a)$ so that $z \notin \text{ri } \mathcal{K}_q^{n+1}$ by (2.2). We conclude that $z \in \partial \mathcal{K}_q^{n+1} \setminus \{0\}$ and

$$\mathcal{F}_2 = \mathcal{F}_z,$$

where $\mathcal{F}_z$ is as in (4.5). Let $\mathcal{g} = |\cdot|^{\alpha_z}$ and let $\psi$ be the facial residual function given by Corollary 4.3. By Theorem 4.2, we have $\gamma_{z, \eta} \in (0, \infty]$ for every $\eta > 0$, so that $\mathcal{g}$ satisfies Theorem 2.5. Furthermore, by Theorem 4.6, $\mathcal{g}$ satisfies the asymptotic optimality criterion (G1) for $K_p^{n+1}$ and $z$ besides satisfying (SH) (since $\mathcal{g}$ is concave, see Lemma 3.2).

In conclusion, all assumptions of Theorem 3.6 are satisfied. In particular, item (a) of Corollary 3.7 tells us that $\mathcal{K}$ and $\mathcal{L} + a$ satisfy a uniform Hölderian error bound with exponent $\alpha_z$ as in (4.6). By definition, $\alpha_z \geq \min\left\{\frac{1}{2}, \frac{1}{p}\right\}$ which concludes the proof of item (ii).

The optimality statement follows directly from item (b) of Corollary 3.7 by noting that $z$ can be scaled so that $\|z\| = 1$. □
The optimal error bound in Theorem 4.7 inherits its properties, in essence, from those of the optimal FRFs from Theorem 4.6. This highlights the importance of the framework we built in section 3. Worthy of additional note is that the optimal error bounds also differ dramatically from the hypothesized form in [21, Section 5].

We conclude this subsection with a result on the direct product of nonpolyhedral $p$-cones.

**Theorem 4.8 (Direct products of $p$-cones).** Let $K = K_{p_1}^{n_1+1} \times \cdots \times K_{p_s}^{n_s+1}$, where $n_i \geq 2$ and $p_i \in (1, \infty)$ for $i = 1, \ldots, s$.

Let $\mathcal{L}$ be a subspace and $a$ be a point such that $(\mathcal{L} + a) \cap K \neq \emptyset$. Then the following items hold.

(i) $d_{PPS}(K, \mathcal{L} + a) \leq 1$.

(ii) Let $\alpha = \min \{1, 1/|p_1|, 1/|p_2|, \ldots, 1/|p_s| \}$ and $d = d_{PPS}(K, \mathcal{L} + a)$. Then, $K$ and $\mathcal{L} + a$ satisfy a uniform H"{o}lderian error bound with exponent $\alpha^d$.

**Proof.** Each $K_{p_i}^{n_i+1}$ has only three types of faces: $\{0\}$, $K_{p_i}^{n_i+1}$ and extreme rays as in (4.5). Therefore, the distance to polyhedrality satisfies $\ell_{\text{poly}}(K_{p_i}^{n_i+1}) = 1$ for every $i$. With that, (i) follows from (2.4).

We move on to item (ii). We invoke Theorem 2.10 and let $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_s = K$ be a chain of faces of $K$ with $\ell = d_{PPS}(K, \mathcal{L} + a) + 1$. At least one such chain exist; see Proposition 2.1 and the subsequent discussion.

First, we need to determine the facial residual functions of $K$. Let $F \subseteq K$, then $F = F^1 \times \cdots \times F^s$ and each $F^i$ is a face of $K_{p_i}^{n_i+1}$. Also, let $z \in F^s$, so that $z = (z_1, \ldots, z_s)$ where $z_i \in (F^i)^s$ for every $i$. As discussed at the beginning of Section 4.1, each $K_{p_i}^{n_i+1}$ is an amenable cone. Therefore, by [21, Proposition 17] a facial residual function for $F, z$ with respect to $K$ can be obtained by positively rescaling the sum $\psi_{F_1, z_1} + \cdots + \psi_{F_s, z_s}$, where each $\psi_{F_i, z_i}$ is an FRF of $F^i, z_i$ with respect to $K_{p_i}^{n_i+1}$. By the discussion in Section 4.2, the FRFs for $K_{p_i}^{n_i+1}$ are of the form $\rho_i(t)\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i}$ where $\rho_i, \hat{\rho}_i$ are nonnegative functions and $\alpha_i$ is either $1$ (see the beginning of Section 4.2) or $\alpha_i$ as in Corollary 4.3. Since $\alpha \leq \alpha_i$, adjusting $\rho_i$ and $\hat{\rho}_i$ if necessary, $\rho_i(t)\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i}$ is also an FRF of $F^i, z_i$ with respect to $K_{p_i}^{n_i+1}$. Finally, summing $s$ functions of the form $\rho_i(t)\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i}$ and positively rescaling still leads to a function of the same form $\rho(t)\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i}$.

We conclude that all the FRFs for $K$ can be taken to be of the form $\rho(t)\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i}$. With that, the result follows from Lemma 2.11.

5 Applications

Using our framework for certifying optimal FRFs, we have built optimal error bounds for the $p$-cones. In this final section, we showcase two applications of these results.

5.1 Least squares with $p$-norm regularization

In this subsection we consider the following least squares problem with (sum of) $p$-norm regularization:

$$\theta = \min_{x \in \mathbb{R}^n} \ g(x) := \frac{1}{2}\|Ax - b\|_p^2 + \sum_{i=1}^s \lambda_i \|x_i\|_p$$

Specifically, if $\rho(t)\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i}$ is an FRF of $F^i, z_i$ with respect to $K_{p_i}^{n_i+1}$, since $\alpha \leq \alpha_i \leq 1$, we have

$$\rho_i(t)\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i} \leq \begin{cases} \rho_i(t)\epsilon + \hat{\rho}_i(t)\epsilon \quad & \text{if } \epsilon \in [0, 1], \\ \rho_i(t)\epsilon + \hat{\rho}_i(t)\epsilon \quad & \text{if } \epsilon > 1. \end{cases}$$

Then $(\rho_i(t) + \hat{\rho}_i(t))\epsilon + \hat{\rho}_i(t)\epsilon^{\alpha_i}$ is also an FRF.

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where \( A \) is an \( m \times n \) matrix, \( b \in \mathbb{R}^m \), \( \lambda_i > 0 \) for each \( i \), and \( x \) is partitioned in \( s \) blocks so that \( x = (x_1, \ldots, x_s) \) with \( x_i \in \mathbb{R}^{n_i} \) for some \( n_i \geq 2 \), \( i = 1, \ldots, s \). When \( p = 2 \), problem \((5.1)\) corresponds to the group LASSO model in statistics for inducing group sparsity \([37]\). The same model can also be used in compressed sensing when the original signal is known to belong to a union of subspaces; see \([8]\).

Instances of \((5.1)\) are usually presented in large scale and are solved via various first-order methods such as the proximal gradient algorithm. When \( p \in [1, 2] \) or \( p = \infty \), it has been shown in \([38, 39]\) that a certain first-order error bound condition holds for the \( g \) in \((5.1)\): as a consequence, when the proximal gradient algorithm is applied to \((5.1)\) with \( p \in [1, 2] \cup \{\infty\} \), the sequence generated converges locally linearly to a global minimizer. The first-order error bound condition for \( g \) is equivalent to the fact that \( g \) is a Kurdyka-Lojasiewicz (KL) function with exponent \( \frac{1}{2} \); see \([7, \text{Corollary 3.6}]\) and \([4, \text{Theorem 5}]\). Consequently, we know that the \( g \) in \((5.1)\) is a KL function with exponent \( \frac{1}{2} \) when \( p \in [1, 2] \cup \{\infty\} \). KL property (see \([1, \text{Definition 3.1}]\)) and the associated exponents (see \([18, \text{Definition 2.3}]\)) play an important role in the local convergence rate analysis of first-order methods; see, for example, \([1, 3, 18]\).

On the other hand, in view of \([39, \text{Example 4}]\), it is known that the \( g \) in \((5.1)\) is in general not a KL function with exponent \( \frac{1}{2} \) when \( p \in (2, \infty) \). While one can still show that \( g \) is a KL function by invoking \([3, \text{Theorem 3.3}]\) and noting that \( g \) is continuous subanalytic and level-bounded, its KL exponent is not explicitly known. Here, leveraging our error bound results in Section 4.3 on direct analysis of first-order methods; see, for example, \([1, 3, 18]\).

Let \( T : \mathbb{R}^{m+2} \to \mathbb{R}^{m+2} \) be the bijective linear map such that \( T(t, u, x) = (t + u, t - u, 2x) \). Then, \( TQ_r^{m+2} = K_{m+2}^2 \), i.e., \( Q_r^{m+2} \) and \( K_{m+2}^2 \) are linearly isomorphic cones. By \([21, \text{Proposition 17}]\), linearly isomorphic cones have the same facial residual functions up to positive rescaling. This implies that Theorem 4.8 is still valid if a cone \( K_{p_i}^{n_i+1} \) with \( p_i = 2 \) is replaced by \( Q_r^{n_i+1} \).

With that, we can write \((5.1)\) as

\[
\begin{aligned}
\min_{t,u,w,y,x} & \quad 0.5t + \sum_{i=1}^{s} \lambda_i y_i \\
\text{s.t.} & \quad Ax - w = b, \\
& \quad u = 1, \\
& \quad (t, u, w) \in Q_r^{m+2}, \\
& \quad (y_i, x_i) \in K_{p_i}^{n_i+1}, \quad i = 1, \ldots, s.
\end{aligned}
\]

The optimal values of \((5.1)\) and \((5.2)\) are the same (i.e., both are \( \theta \)) and an optimal solution to the former can be readily used to construct an optimal solution to the latter and vice-versa. For notational convenience, in what follows we write

\[
\mathbf{v} = (t, u, w, (y_1, x_1), \ldots, (y_s, x_s)).
\]

Then, the optimal set of \((5.2)\) can be written as the intersection of the affine space

\[
\mathcal{V} = \left\{ \mathbf{v} \left| 0.5t + \sum_{i=1}^{s} \lambda_i y_i = \theta, u = 1, Ax - w = b \right. \right\}
\]
with the cone

$$\mathcal{K} = Q_r^{m+2} \times K_p^{n+1} \times \cdots \times K_p^{n+1}.$$  \hspace{1cm} (5.5)

The feasible region of (5.2) will be denoted by $\mathcal{D}$, so that

$$\mathcal{D} = \{v \mid u = 1, Ax - w = b, v \in \mathcal{K}\}.$$  

We can then apply Theorem 4.8, since, as remarked previously, $Q_r^{m+2}$ and $K_p^{n+2}$ are linearly isomorphic. Therefore, there exists $\alpha \in (0, 1]$ such that for every bounded set $B$ there exists $\kappa_B > 0$ such that

$$\text{dist}(v, \mathcal{K} \cap \mathcal{V}) \leq \kappa_B \max(\text{dist}(v, \mathcal{K}), \text{dist}(v, \mathcal{V}))^\alpha, \quad \forall v \in B,$$  

and we will discuss the value of $\alpha$ later. Because $\mathcal{V}$ is an affine set, it follows from Hoffman’s lemma [12] that there exists a constant $\kappa_\mathcal{V} > 0$ such that if $v$ (as in (5.3)) satisfies $u = 1$ and $Ax - w = b$, we have

$$\text{dist}(v, \mathcal{V}) \leq \kappa_\mathcal{V} \left| 0.5t + \sum_{i=1}^s \lambda_i y_i - \theta \right|.$$  

Plugging this in (5.6), we obtain

$$\text{dist}(v, \mathcal{K} \cap \mathcal{V}) \leq \kappa \left| 0.5t + \sum_{i=1}^s \lambda_i y_i - \theta \right|^{\alpha}, \quad \forall v \in B \cap \mathcal{D},$$  

for some constant $\kappa > 0$. Next, denoting by $\delta_\mathcal{D}$ the indicator function of $\mathcal{D}$, we define $G$ by

$$G(v) = 0.5t + \sum_{i=1}^s \lambda_i y_i - \theta + \delta_\mathcal{D}(v),$$

so that $G$ is a proper convex lower semicontinuous function satisfying $\inf_v G(v) = 0$. Then (5.7) implies the following error bound condition: if $v^* \in \arg \min G$ and $B$ is any ball centered at $v^*$, then there exists $\kappa > 0$ such that

$$\text{dist}(v, \arg \min G) \leq \kappa G(v)^\alpha, \quad \forall v \in B \cap \mathcal{D}.$$  \hspace{1cm} (5.8)

By [4, Theorem 5], this means that $G$ satisfies the KL property at $v^*$ with exponent $1 - \alpha$. Now, recalling the definition of $v$ in (5.3) and writing $x = (x_1, \ldots, x_s)$ and $z = (t, u, w, y_1, \ldots, y_s)$, we can see that

$$g(x) = \inf_z G(v),$$

where $g$ is defined in (5.1). Let $Z(x) = \arg \min_z G(v)$. Then, $Z(x)$ is nonempty and if $z \in Z(x)$, it must be the case that $z = (t, u, w, y_1, \ldots, y_s)$ satisfies $t = \|Ax - b\|^2$, $u = 1$, $t \geq \|w\|^2$ and $y_i = \|x_i\|^p$. This shows that $Z(x)$ is compact.

We have thus fulfilled all the conditions necessary to invoke [36, Corollary 3.3] which says that the KL exponent of $G$ gets transferred to $g$. Therefore, if $x^*$ is an optimal solution to (5.1), then $g$ satisfies the KL property with exponent $1 - \alpha$ at $x^*$.

The final piece we need is a discussion on the value of $\alpha$. By Theorem 4.8, $\alpha$ can be chosen as

$$\min\{0.5, 1/p\}^d,$$

where $d = d_{PPS}(\mathcal{K}, \mathcal{V})$ and $d \leq s + 1$. So let us take a look at a situation under which we have $d \leq 1$. By selecting $b$, $c$ and $\mathcal{A}$ appropriately, we can write (5.1) and its dual as

$$\min_v \{\langle c, v \rangle \mid \mathcal{A}v = b, v \in \mathcal{K}\} \quad \text{and} \quad \max_y \{\langle b, y \rangle \mid c - \mathcal{A}^T y \in \mathcal{K}^*\}.$$  \hspace{1cm} (5.9)

Because of the format of (5.2), Slater’s condition is satisfied, since one can take $y_i$ and $t$ large enough so that $v$ is feasible and $v \in \text{ri} \mathcal{K}$. Therefore, the corresponding dual problem has an
optimal solution $\mathbf{v}^*$ attaining the same optimal value $\theta$. Let $\mathbf{s}^* = \mathbf{c} - A^T \mathbf{y}^*$, so that $\mathbf{s}^* \in \mathcal{K}^* \cap \mathcal{V}^\perp$ holds because $\mathbf{y}^*$ is dual optimal. Then $\langle \mathbf{v}^*, \mathbf{s}^* \rangle = \langle \mathbf{c}, \mathbf{v}^* \rangle - \langle A \mathbf{v}^*, \mathbf{y}^* \rangle = 0$ whenever $\mathbf{v}^*$ is a primal optimal solution. Thus, $\mathcal{F} := \mathcal{K} \cap \{\mathbf{s}^*\}^\perp$ defines a face of $\mathcal{K}$ containing the optimal set of (5.2). In particular, if the following strict complementarity-like condition holds for some optimal solution $\mathbf{v}^*$,

$$ \mathbf{v}^* \in \text{ri}(\mathcal{K} \cap \{\mathbf{s}^*\}^\perp), $$

(5.10)

then $\mathcal{V} \cap \mathcal{F} = \mathcal{V} \cap \mathcal{K}$ and $\mathcal{V} \cap (\text{ri} \mathcal{F}) \neq \emptyset$ holds, so that $d_{\text{PPS}}(\mathcal{K}, \mathcal{V}) \leq 1$. We have thus proved the following result.

**Theorem 5.1.** Let $x^*$ be an optimal solution to (5.1). Then $g$ satisfies the KL property at $x^*$ with exponent $1 - \alpha$, where $\alpha = \min\{0.5, 1/p\}^d$ and $d = d_{\text{PPS}}(\mathcal{K}, \mathcal{V})$, with $\mathcal{V}$ and $\mathcal{K}$ given in (5.4) and (5.5) respectively. Furthermore, $d \leq s + 1$, and if (5.2) satisfies strict complementarity (see (5.10)), then $d \leq 1$.

### 5.2 Self-duality and homogeneity of $p$-cones

The second-order cone $\mathcal{K}^{n+1}_2$ is quite special since it is symmetric, that is self-dual and homogeneous. Self-duality means that $(\mathcal{K}^{n+1}_2)^* = \mathcal{K}^{n+1}_2$ and homogeneity means that for every $x, y \in \text{ri} \mathcal{K}^{n+1}_2$, there exists a linear map $A$ such that $Ax = y$ and $AK^{n+1}_2 = K^{n+1}_2$. Symmetric cones have many nice properties coming from the theory of Jordan algebras [9, 10].

A basic question then is the following: are all $p$-cones symmetric? At first glance, the answer might seem obviously no; however, this is a subtle question, and the path to its solution is rife with tempting pitfalls. For example, a common source of confusion is as follows: in order to disprove that a cone is symmetric, it is not enough to show that $\mathcal{K}^* \neq \mathcal{K}$. The reason is that the self-duality requirement, in the Jordan algebra context, can be met by arbitrary inner products, and $\mathcal{K}^*$ changes if $\langle \cdot, \cdot \rangle$ varies. An interesting discussion on symmetrizing a cone by changing the inner product can be seen in [28]. In fact, the existence of an inner product making a cone $\mathcal{K}$ self-dual is equivalent to the existence of a positive definite matrix $Q$ such that $Q\mathcal{K} = \mathcal{K}^*$, where $\mathcal{K}^*$ is the dual cone obtained under the usual Euclidean inner product.

In what follows, we let $p \in (1, \infty)$, $p \neq 2$, $n \geq 2$ and $q$ be such that $1/p + 1/q = 1$. In order to prove that a $p$-cone is not a symmetric cone via the self-duality route, what is actually required is to show that $Q\mathcal{K}^{n+1}_p = \mathcal{K}^{n+1}_q$ never holds for any positive definite matrix $Q$. It might be fair to say that this is harder than merely showing that $\mathcal{K}^{n+1}_p \neq \mathcal{K}^{n+1}_q$. One might then try to focus on the homogeneity requirement instead, but this is also a nontrivial task. In fact, the homogeneity of general $p$-cones was one of the open questions mentioned by Gowda and Trott in [11].

These issues were later settled in [13, 14] using techniques such as T-algebras [33] and tools borrowed from differential geometry. In this final subsection, we show “easy” proofs for the questions above based on our error bound results. The only preliminary fact we need is that if $A$ is a matrix, then $AK^{n+1}_1 = \mathcal{K}^{n+1}_1$ if and only if

$$ A = \alpha \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, $$

(5.11)

for some $\alpha > 0$ and generalized permutation matrix $D$ (i.e., $\pm 1$ are allowed in the entries of $D$); see [11, Theorem 7]. Here, $\mathcal{K}^{n+1}_1 := \{(x_0, \bar{x}) \in \mathbb{R}^{n+1} \mid x_0 \geq \|\bar{x}\|_1\}$.

**Theorem 5.2** ([14, Theorem 11, Corollaries 13 and 14]). Suppose that $p \in (1, \infty)$ and $n \geq 2$. Then, the following items hold.

(i) If $\hat{p} > p$, then there is no matrix $A$ such that $AK^{n+1}_p = \mathcal{K}^{n+1}_\hat{p}$.
(ii) If \( p \neq 2 \) and \( AK_p^{n+1} = K_p^{n+1} \) holds for some matrix \( A \), then \( AK_1^{n+1} = K_1^{n+1} \); in particular, the matrix \( A \) must be as in (5.11).

(iii) If \( p \neq 2 \), then \( K_p^{n+1} \) is neither self-dual nor homogeneous.

Proof. First, we prove item (i). Suppose that \( \hat{p} > p \) and there is a matrix \( A \) such that \( AK_p^{n+1} = K_p^{n+1} \). Such an \( A \) must be invertible and if \( \psi \) is an FRF for a face of \( F_1 \subseteq K_p^{n+1} \), then \( A \) must map \( F_1 \) onto a face \( F_2 \subseteq K_p^{n+1} \) which admits an FRF that is a positive rescaling of \( \psi \); see [21, Proposition 17].

The FRFs we constructed for the extreme rays of the \( p \)-cones are built from functions that satisfy the optimality criterion (G1); see Theorem 4.6. This means the exponents appearing in Corollary 4.3 are the largest possible, which follows from Theorem 3.4 and an argument similar to the proof of item (b) of Corollary 3.7. Consequently, the only possibility of having \( AK_p^{n+1} = K_p^{n+1} \) is if \( K_p^{n+1} \) and \( K_p^{n+1} \) have FRFs with the same exponents.

If \( p = 2 \), then all the FRFs of \( K_2^{n+1} \) have exponent 1/2. Since this is not true for \( K_p^{n+1} \), this case cannot happen. The case \( p \in (1, 2) \) is also impossible because the largest exponent appearing in an FRF for \( K_p^{n+1} \) is \( \max\{1/2, 1/\hat{p}\} \) and \( 1/p > \max\{1/2, 1/\hat{p}\} \).

Finally, suppose that \( p > 2 \). Then, there is a face \( F_1 \subseteq K_p^{n+1} \) with (best) exponent \( 1/p \). If this face is mapped to a face of \( K_p^{n+1} \) with best exponent \( 1/\hat{p} \), then we have a contradiction since \( 1/p > 1/\hat{p} \). If it is mapped to a face with best exponent 1/2, a contradiction arises because \( A^{-1} \) can be used to construct an FRF for \( F_1 \) with exponent 1/2, contradicting optimality. This concludes the proof of item (i).

Next, we move on to item (ii). Suppose first that \( p \in (1, 2) \). Let \( \bar{e}_i \) denote the \( i \)-th unit vector in \( \mathbb{R}^n \). Then, the half-line along \( (1, \pm \bar{e}_i) \) is an extreme ray of \( K_p^{n+1} \) that has an FRF with exponent \( 1/p \), by Corollary 4.3. Observing (4.6), we see that those are the only extreme rays of \( K_p^{n+1} \) having exponent \( 1/p \), and there are 2n of them. Since \( AK_p^{n+1} = K_p^{n+1} \) and \( A \) must map a face to another face having an identical exponent, we conclude that \( A \) permutes this set of 2n extreme rays. However, they are also all the extreme rays of the 1-cone \( K_1^{n+1} \), so \( AK_1^{n+1} = K_1^{n+1} \).

If \( p \in (2, \infty) \), then taking duals we have \( A^{-T}K_1^{n+1} = K_q^{n+1} \), where \( 1/q + 1/p = 1 \), so that \( q \in (1, 2) \) and \( A^{-T}K_1^{n+1} = K_1^{n+1} \). Then, \( A^{-T} \) must be as in (5.11) so that \( A \) is another matrix having the same format and \( AK_1^{n+1} = K_1^{n+1} \). This completes the proof of item (ii).

Let \( p \neq 2 \). As discussed previously, self-duality implies the existence of a positive definite matrix \( Q \) such that \( QK_p^{n+1} = K_p^{n+1} \), which is impossible by item (i). Next, we disprove homogeneity. Notice that \( \text{ri} K_1^{n+1} \) is properly contained in \( \text{ri} K_p^{n+1} \), but they do not coincide since \( K_1^{n+1} \neq K_p^{n+1} \). So let \( x := (1, 0, \ldots, 0) \) and \( y \) be a point in \( \text{ri} K_p^{n+1} \). If \( AK_p^{n+1} = K_p^{n+1} \) then \( AK_1^{n+1} = K_1^{n+1} \) by item (ii). Since \( x \in \text{ri} K_1^{n+1} \), we have \( Ax \neq y \) for all \( A \) satisfying \( AK_p^{n+1} = K_p^{n+1} \). This shows that \( K_p^{n+1} \) is not homogeneous.

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