Two-Dimensional Divisor Problems Related to Symmetric L-Functions

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Abstract: In this paper, we study two-dimensional divisor problems of the Fourier coefficients of some automorphic product L-functions attached to the primitive holomorphic cusp form \( f(z) \) with weight \( k \) for the full modular group \( SL_2(\mathbb{Z}) \). Additionally, we establish the upper bound and the asymptotic formula for these divisor problems on average, respectively.

Keywords: Hecke L-function; symmetric L-function; divisor problem; Fourier coefficients

1. Introduction

As usual, let \( \mathcal{H}_k \) denote the set of primitive holomorphic cusp forms with even integral weight \( k \geq 2 \) for the full modular group \( SL_2(\mathbb{Z}) \); then \( \mathcal{H}_k \) is made up of the common eigenfunctions of all Hecke operators \( T_n \). Then the Fourier series expansion of Hecke eigenfunction \( f \) at the cusp \( \infty \) has the following form.

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(k-1)/2}e^{2\pi i nz} \quad (\text{Im} \, z > 0),
\]

where the coefficient \( \lambda_f(n) \) denotes the \( n \)-th normalized eigenvalue, which is the coefficient divided by \( n^{(k-1)/2} \) of the Hecke operator \( T_n \). Note that \( \lambda_f(n) \) is real valued and also multiplicative. Let \( n \) be an integer greater than one, Deligne [1] proved that

\[
|\lambda_f(n)| \leq \tau(n),
\]

where \( \tau(n) \) denotes the number of \( n \)'s positive divisors. For prime \( p \), we have

\[
\lambda_f(p) = \alpha_f(p) + \beta_f(p) \quad \text{and} \quad |\alpha_f(p)| = |\beta_f(p)| = 1.
\]

Studying the properties and average behaviors of various sums concerning \( \lambda_f(n) \) and \( \lambda_{f \times f}(n) \) is a meaningful and interesting problem. In number theory, classical problems are investigate mean value estimates of these Fourier coefficients and related problems with the corresponding automorphic L-functions (for examples, see [1–23], etc.). In particular, we give a brief introduction for the general divisor problem.

Let \( \omega \geq 1 \) is an integer, and

\[
\lambda_{\omega,f}(n) = \sum_{n=n_1 \cdots n_\omega} \lambda_f(n_1)\lambda_f(n_2)\cdots\lambda_f(n_\omega),
\]

\[
\lambda_{\omega,f \times f}(n) = \sum_{n=n_1 \cdots n_\omega} \lambda_{f \times f}(n_1)\lambda_{f \times f}(n_2)\cdots\lambda_{f \times f}(n_\omega).
\]

when \( \omega = 1 \), we actually have \( \lambda_{1,f}(n) = \lambda_f(n) \) and \( \lambda_{1,f \times f}(n) = \lambda_{f \times f}(n) \). Hecke [24] proved

\[
\sum_{n \leq x} \lambda_f(n) \ll x^{1/2}.
\]
Later, the above upper bound was improved by many authors (see [1,6,15]). Additionally, the best result up to now was due to Wu [18]:

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{3}{4}} \log^d x,$$

where

$$\rho = \frac{102 + 7\sqrt{21}}{210} \left( \frac{6 - \sqrt{21}}{5} \right)^{\frac{1}{2}} + \frac{102 - 7\sqrt{21}}{210} \left( \frac{6 + \sqrt{21}}{5} \right)^{\frac{1}{2}} - \frac{33}{35} = -0.118 \cdots.$$ 

Rankin [14] and Selberg [16] showed that

$$\sum_{n \leq x} \lambda_f(f(n)) = C_f x + O(x^{\frac{3}{5}}),$$

where $C_f$ is a positive constant depending on $f$. Kanemitsu, Sankaranarayanan and Tanigawa [25] considered a general divisor problem and established

$$\sum_{n \leq x} \lambda_{\omega, f}(n) \ll x^{1 - \frac{3}{2\omega} + \epsilon},$$

$$\sum_{n \leq x} \lambda_{\omega, f \times f}(n) = M_{\omega}(x) + O(x^{1 - \frac{1}{2\omega} + \epsilon}),$$

where $\omega \geq 2$ is an integer; $M_{\omega}(x)$ derives from a residue and has the form $xP_{\omega-1}(\log x)$; $P_{\omega-1}(t)$ represents a polynomial of $t$ with degree $\omega - 1$. Fomenko [26] also showed the same result for the sum $\sum_{n \leq x} \lambda_{\omega, f}(n)$. Later, Kanemitsu, Sankaranarayanan and Tanigawa’s result was improved by Lü [27], and in this direction, many scholars have obtained a series of results (see [28–30], etc.).

In this paper, we consider the two-dimensional divisor problems related to the Fourier coefficients $\lambda_{a,b}^f(n)$, $\lambda_{a,b}^{f \times f}(n)$. To state our results, we first introduce some notation. For any fixed integers $1 < a < b$, we write

$$\lambda_{a,b}^f(n) = \sum_{n=n_1 n_2^b} \lambda_f(n_1) \lambda_f(n_2)$$

and

$$\lambda_{a,b}^{f \times f}(n) = \sum_{n=n_1 n_2^b} \lambda_{f \times f}(n_1) \lambda_{f \times f}(n_2).$$

The two-dimensional divisor problems can be considered as the average behaviors of the coefficients $\lambda_{a,b}^f(n)$ and $\lambda_{a,b}^{f \times f}(n)$. We set

$$S_f(a, b; x) := \sum_{n \leq x} \lambda_{a,b}^f(n)$$

and

$$S_{f \times f}(a, b; x) := \sum_{n \leq x} \lambda_{a,b}^{f \times f}(n).$$

For these sums (4) and (5), we establish the upper bound and the asymptotic formula by considering the sizes of $a$ and $b$, respectively. We refer to Section 3 for our detailed results.

In the following Section 2, we first introduce some specific automorphic $L$-functions and quote some lemmas. We state our results in Section 3, and show their proofs in Sections 4 and 5. In Sections 6 and 7, we state an application of our results and give a conclusion, respectively.
2. Some Lemmas

In this section, to prove Theorems 1 and 2 we first introduce some specific automorphic L-functions, which is important for the proof of our results and also help us understand the Fourier coefficients in another way. For Re \( s > 1 \), we define the Hecke \( L \)-function \( L(s, f) \) attached to \( f \) as

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left( 1 - \frac{\alpha_f(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta_f(p)}{p^s} \right)^{-1}.
\]

Moreover, the Rankin–Selberg \( L \)-function attached to \( f \) could be defined as

\[
L(s, f \times f) = \prod_p \left( 1 - \frac{\alpha_f^2(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta_f^2(p)}{p^s} \right)^{-1}.
\]

Then \( L(s, f \times f) \) can be rewritten in the following form:

\[
L(s, f \times f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f^2(n)}{n^s} := \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}(n)}{n^s}.
\]

The \( j \)-th symmetric power \( L \)-function attached to \( f \) could be defined as follows.

\[
L(s, \text{sym}^j f) := \prod_p \prod_{m=0}^{j} \left( 1 - \frac{\alpha_f(p)^{j-m} \beta_f(p)^m}{p^s} \right), \quad \text{Re } s > 1. \tag{6}
\]

Additionally, the \( j \)-th symmetric power \( L \)-function attached to \( f \) could be expressed in the following Dirichlet series:

\[
L(s, \text{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^j f}(n)}{n^s} = \prod_p \left( 1 + \frac{\lambda_{\text{sym}^j f}(p)}{p^s} + \frac{\lambda_{\text{sym}^j f}(p^2)}{p^{2s}} + \frac{\lambda_{\text{sym}^j f}(p^3)}{p^{3s}} + \cdots \right), \quad \text{Re } s > 1. \tag{7}
\]

The \( j \)-th symmetric \( L \)-function \( L(s, \text{sym}^j f), j = 1, 2, 3, 4 \) could be analytic continued to an entire function over the whole complex plane \( \mathbb{C} \) and has a confirmed functional equation. We refer to papers of Hecke [31], Gelbert and Jacquet [32], Kim [33] and Kim and Shahidi [34,35] for these properties of \( L(s, \text{sym}^j f), j = 1, 2, 3, 4 \). Therefore, we can note that \( L(s, \text{sym}^j f), j = 1, 2, 3, 4 \) could be recognized as general \( L \)-functions in the sense of Perelli [36].

With the help of these automorphic \( L \)-functions, we then quote the following lemmas, which include the individual and averaged subconvexity bounds for Riemann zeta-function \( \zeta(s) \), symmetric square \( L \)-function \( L(s, \text{sym}^2 f) \) and corresponding Rankin–Selberg \( L \)-function \( L(s, f \times f) \). From the following Lemma 1 we know that the Rankin–Selberg \( L \)-function \( L(s, f \times f) \) could be decomposed into the product of Riemann zeta-function \( \zeta(s) \) and corresponding symmetric square \( L \)-function \( L(s, \text{sym}^2 f) \).

**Lemma 1.** For \( \text{Re } s > 1 \), one has

\[
L(s, f \times f) = \zeta(s)L(s, \text{sym}^2 f). \tag{8}
\]

**Proof.** By the comparison of Euler products of two sides of (8) and applying Deligne’s result (1), we could easily get this lemma. This lemma can also be found in [27,28]. \( \square \)
Lemma 2. For any \( \epsilon > 0 \), one has the mean value estimate
\[
\int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{12} dt \ll T^{2+\epsilon}
\] (9)
uniformly for \( T \geq 1 \) and the upper bounds
\[
\zeta(\sigma + it) \ll \begin{cases} 
(1 + |t|)^{\frac{11}{12}(1-\epsilon)+\epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\
1, & \text{if } \sigma > 1,
\end{cases}
\] (10)
where \( |t| \geq 1 \).

Proof. The mean value result (9) is due to Heath-Brown [37]. The upper bounds (10) are due to Bourgain [38]. □

Lemma 3. For any \( \epsilon > 0 \), one has the mean value estimate
\[
\int_1^T |L(\sigma + it, \text{sym}^2 f)|^2 dt \ll T^{3(1-\epsilon)+\epsilon}
\] (11)
uniformly for \( T \geq 1 \) and the upper bounds
\[
L(\sigma + it, \text{sym}^2 f) \ll \begin{cases} 
(1 + |t|)^{\frac{5}{6}(1-\epsilon)+\epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\
1, & \text{if } \sigma > 1,
\end{cases}
\] (12)
where \( |t| \geq 1 \).

Proof. The mean value result (11) follows from analytic properties of \( L(s, \text{sym}^2 f) \) and standard arguments in number theory. The upper bounds (12) are due to Nunes [13]. □

Lemma 4. For any \( \epsilon > 0 \), one has the mean value estimate
\[
\int_T^{2T} \left| L \left( \frac{1}{2} + it, f \right) \right|^2 dt \sim CT \log T
\] (13)
uniformly for \( T \geq 1 \) and the upper bounds
\[
L(\sigma + it, f) \ll \begin{cases} 
(1 + |t|)^{\frac{1}{2}(1-\epsilon)}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\
1, & \text{if } \sigma > 1,
\end{cases}
\] (14)
where \( |t| \geq 1 \).

Proof. The results in this lemma were established by Good [5]. □

3. Main Theorems

In this paper, we consider the two-dimensional divisor problems related to the Fourier coefficients \( \lambda_f(n) \), \( \lambda_{f \times f}(n) \) and establish the following two theorems. To establish these two theorems, we apply some classical methods and instruments, such as Perron’s formula, Cauchy’s residue theorem, decomposition of the Rankin–Selberg \( L \)-function, upper bounds and mean values of specific functions.

Theorem 1. Suppose that \( a \) and \( b \) are any fixed integers with \( 1 < a < b \). Then for any \( \epsilon > 0 \), one has
\[
S_f(a, b; x) = \sum_{n \leq x} \lambda_f^{ab}(n) \ll \begin{cases} 
x^{\frac{3(2a-1)}{2(b-a)}+\epsilon}, & \text{if } b \\n\frac{1}{2}+\epsilon, & \text{if } b > 2a.
\end{cases}
\]
Theorem 2. Suppose that $a$ and $b$ are any fixed integers with $1 < a < b$. Then for any $\epsilon > 0$, one has

$$S_{f,s}(a,b;x) = \sum_{n=1}^{\lambda_{a,b}(n)} L(1, \text{sym}^2 f) L(\frac{b}{T}, f) x^1 + O\left(x^{1 - \frac{8(2a-1)}{6(2a-1)\pi^2} + \epsilon}\right),$$

if $318a^2 - 131ab - 402a + 131b < 0, b \leq 2a$,

$$= \begin{cases} L(1, \text{sym}^2 f) L(\frac{b}{T}, f) x^1 + O\left(x^{\frac{2}{3} + \epsilon}\right), & \text{if } a = 2, b \geq 2a, \\ O\left(x^{1 - \frac{8(2a-1)}{6(2a-1)\pi^2} + \epsilon}\right), & \text{if } a \geq 3, b \geq 2a. \end{cases}$$

4. Proof of Theorem 1

In this section, we shall complete the proof of Theorem 1. Let $s = \sigma + it$ and $\eta = 1 + \epsilon$. We have

$$L(as, f) L(bs, f) = \sum_{n=1}^{\infty} \frac{\lambda_{a,b}(n)}{n^s}. \quad (15)$$

Then, by applying Perron’s formula (see the Proposition 5.54 in [39]), we can obtain

$$S_f(a,b;x) = \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} L(as, f) L(bs, f) \frac{x^s}{s} ds + O\left(\frac{x^{1+\epsilon}}{T}\right), \quad (16)$$

where $T$ is a parameter which will be decided later.

We shift the line of the integral of (16) to the line $\text{Re } s = \frac{1}{2\pi a}$. Then Cauchy’s residue theorem shows that

$$S_f(a,b;x) = G_1 + G_2 + G_3 + O\left(\frac{x^{1+\epsilon}}{T}\right), \quad (17)$$

where

$$\begin{aligned}
G_1 &= \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} L(as, f) L(bs, f) \frac{x^s}{s} ds, \\
G_2 &= \frac{1}{2\pi i} \int_{\eta + iT}^{\eta + iT} L(as, f) L(bs, f) \frac{x^s}{s} ds, \\
G_3 &= \frac{1}{2\pi i} \int_{\eta - iT}^{\eta - iT} L(as, f) L(bs, f) \frac{x^s}{s} ds.
\end{aligned}$$

The following work is to estimate these three terms, $G_1$, $G_2$, and $G_3$. The estimates of these integrals on the horizontal parts are analogous; thus, we always consider $G_2$ and $G_3$ firstly in the following parts. To get this goal, we consider two cases $b \leq 2a$ and $b > 2a$.

We first consider the case $b \leq 2a$. To estimate $G_2$ and $G_3$, we divide the integral interval into the following four short intervals $I_1, \ldots, I_4$ and apply Lemma 4.

Interval 1. $I_1 := \left\{s = \sigma + iT : \frac{a}{b} \leq \sigma \leq 1, \frac{a}{2b} \leq \sigma \leq 1 \right\} = \left\{s = \sigma + iT : \frac{1}{2a} \leq \sigma \leq \frac{1}{b} \right\}$

In this interval, we have

$$T^{-1} \times \int_{I_1} x^\sigma |L(a \sigma + iat, f)L(b \sigma + ibt, f)|d\sigma \leq \max_{\frac{a}{b} \leq \sigma \leq \frac{1}{b}} x^\sigma T^{\frac{1}{2} - \frac{a}{b}} T^{\frac{1}{2} - \frac{b}{b} + \epsilon} T^{-1} \leq \max_{\frac{a}{b} \leq \sigma \leq \frac{1}{b}} T^{\frac{1}{2} - \frac{a}{b} + \epsilon} \left(\frac{x}{T^{\frac{1}{2} - \frac{a}{b}}}ight)^{\sigma} \leq x^{\frac{1}{2b} T^{-\frac{1}{2} + \epsilon}} + x^{\frac{1}{2b} T^{-\frac{1}{2} - \frac{2a}{b} + \epsilon}}. \quad (18)$$
Interval 2. \( I_2 := \left\{ s = \sigma + iT : \frac{1}{2} \leq a \sigma \leq 1, \ 1 < b \sigma \leq b \eta \right\} = \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{a} \right\} \).

In this interval, we have

\[
T^{-1} \times \int_{I_2} x^\sigma |L(a \sigma + iat, f)L(b \sigma + ibt, f)|d\sigma \\
\ll \max_{\frac{1}{b} < \sigma \leq \frac{1}{a}} x^\sigma T^{\frac{1}{2}(1-a \sigma)+\epsilon} T^{-1} \\
\ll \max_{\frac{1}{b} < \sigma \leq \frac{1}{a}} T^{-\frac{1}{2}+\epsilon}\left(\frac{x}{T^{\frac{1}{2}}}\right)^\sigma \\
\ll x^{\frac{1}{2}} T^{-\frac{1}{2} - \frac{2a}{3b} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\] (19)

Interval 3. \( I_3 := \left\{ s = \sigma + iT : 1 < a \sigma \leq a \eta, \ \frac{b}{2a} \leq b \sigma \leq 1 \right\} \).

This interval is an empty set noting that \( \frac{1}{b} < \frac{1}{a} \).

Interval 4. \( I_4 := \left\{ s = \sigma + iT : 1 < a \sigma \leq a \eta, \ 1 < b \sigma \leq b \eta \right\} = \left\{ s = \sigma + iT : \frac{1}{a} < \sigma \leq \eta \right\} \).

In this interval, we have

\[
T^{-1} \times \int_{I_4} x^\sigma |L(a \sigma + iat, f)L(b \sigma + ibt, f)|d\sigma \\
\ll \max_{\frac{1}{a} < \sigma \leq \eta} x^\sigma T^{-1} \\
\ll x^{\frac{1}{2}} T^{-1+\epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\] (20)

Therefore, from (18)–(20) we have

\[
G_2 + G_3 \ll T^{-1} \int_{\frac{1}{2a}}^{\eta} x^\sigma |L(a \sigma + iat, f)L(b \sigma + ibt, f)|d\sigma \\
= T^{-1} \int_{I_1 \cup \ldots \cup I_4} x^\sigma |L(a \sigma + iat, f)L(b \sigma + ibt, f)|d\sigma \\
\ll x^{\frac{1}{2}} T^{-\frac{1}{2} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\] (21)

Now we turn to estimate \( G_1 \). We have

\[
G_1 \ll x^{\frac{1}{2}} \int_{1}^{T} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) \right| t^{-1} dt + x^{\frac{1}{2}} \\
\ll x^{\frac{1}{2}} \log T \max_{\frac{1}{2} \leq T} T^{-1} \int_{\frac{1}{2}}^{\tilde{T}_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) \right| dt + x^{\frac{1}{2}}.
\]

Then, by Lemma 4 and applying Cauchy’s inequality, we can deduce

\[
G_1 \ll x^{\frac{1}{2}} \log T \max_{\frac{1}{2} \leq T} T^{-1} \int_{\frac{1}{2}}^{\tilde{T}_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \left( \int_{\frac{1}{2}}^{\tilde{T}_1} 1 dt \right)^{\frac{1}{2}} + x^{\frac{1}{2}} \\
\ll x^{\frac{1}{2}} \log T \max_{\frac{1}{2} \leq T} T^{\frac{1}{2} - \frac{a}{b} + \epsilon} \\
\ll x^{\frac{1}{2}} T^{\frac{1}{2} - \frac{a}{b} + \epsilon}.
\] (22)

From (17), (21) and (22), we have

\[
S_f(a, b, x) \ll x^{\frac{1}{2}} T^{-\frac{1}{2} - \frac{2a}{3b} + \epsilon} + x^{\frac{1}{2}} T^{\frac{1}{2} - \frac{a}{b} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon},
\] (23)

Taking \( T = x^{\frac{3(2a-1)}{2b(3a-b)}} \) in (23), we can get

\[
S_f(a, b, x) \ll x^{1 - \frac{3(2a-1)}{2b(3a-b)} + \epsilon},
\]
which proves the first result of Theorem 1.

For the case \( b > 2a \), to estimate \( G_2 + G_3 \) we also divide the integral interval into four short intervals \( I_1^*, \cdots, I_4^* \), which are different from ones for the case \( b \leq 2a \). In fact, the corresponding short intervals \( I_1^* \) and \( I_2^* \) become empty sets at the current case. However, we still can estimate \( G_2 + G_3 \) by following a similar argument to the corresponding parts of the case \( b \leq 2a \) and get

\[
G_2 + G_3 \ll x^{\frac{1}{2}} T^{-\frac{3}{2}+\epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\]

The estimate of \( G_1 \) becomes the following at the current case by noting \( \frac{b}{2a} > 1 \).

\[
G_1 \ll x^{\frac{1}{2}} \int_1^T \left| L \left( \frac{1}{2} + iat, f \right) L \left( \frac{b}{2a} + ibt, f \right) \right| t^{-1} dt + x^{\frac{1}{2}}
\]

\[
\ll x^{\frac{1}{2}} \log T \max_{\frac{1}{2} \leq \sigma \leq 1} \int_{\frac{1}{T}}^{\frac{1}{T}} \left| L \left( \frac{1}{2} + iat, f \right) \right| L \left( \frac{b}{2a} + ibt, f \right) dt + x^{\frac{1}{2}}
\]

\[
\ll x^{\frac{1}{2}} \log T \max_{\frac{1}{2} \leq \sigma \leq 1} \left( \int_{\frac{1}{T}}^{\frac{1}{T}} \left| L \left( \frac{1}{2} + iat, f \right) \right|^2 dt \right)^{\frac{1}{2}} \left( \int_{\frac{1}{T}}^{\frac{1}{T}} dt \right)^{\frac{1}{2}} + x^{\frac{1}{2}}
\]

\[
\ll x^{\frac{1}{2}} \log T \max_{\frac{1}{2} \leq \sigma \leq 1} T^{-1+\frac{1}{2}+\epsilon} + x^{\frac{1}{2}}
\]

\[
\ll x^{\frac{1}{2}} T^\epsilon.
\]

Thus, we have, recalling (17),

\[
S_f(a, b; x) \ll x^{\frac{1}{4}} T^\epsilon + x^{1+\epsilon} T^{-1+\epsilon}.
\]  

(24)

Taking \( T = x^{1-\frac{1}{a}} \) in (24), we can obtain

\[
S_f(a, b; x) \ll x^{\frac{1}{2}+\epsilon},
\]

which proves the second result of Theorem 1.

5. Proof of Theorem 2

We shall prove Theorem 2, the process of which is more complicated than Theorem 1, in this section. Let also \( s = \sigma + it \) and \( \eta = 1 + \epsilon \). Note that

\[
L(as, f \times f) L(bs, f \times f) = \sum_{n=1}^{\infty} \lambda_{f \times f}^a(n) n^s.
\]  

(25)

Then, by applying Perron’s formula ( see the Proposition 5.54 in [11] ), we have

\[
S_{f \times f}(a, b; x) = \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} ds + O \left( \frac{x^{1+\epsilon}}{T} \right).
\]  

(26)

where \( T \) is a parameter which will be decided later. Then we shift the line of the integral of (26) to the line Re \( s = \frac{1}{2} \). In view of (8), we know that at the point \( s = 1 \), \( L(s, \text{sym}^2 f) \) is holomorphic, which was proved by Gelbart–Jacquet [32]. Thus, the points \( s = \frac{1}{2} \) and \( s = \frac{1}{b} \) are the only two possible simple poles of the integrand of (26) in the range \( R_T := \{ s = \sigma + it : \frac{1}{2} \leq \sigma \leq 1 + \epsilon, \ T \} \) depending on the size difference between \( b \) and \( 2a \). Thus, we consider two cases, \( b \leq 2a \) and \( b > 2a \).
We first consider the case \( b \leq 2a \). In this situation, the points \( s = \frac{1}{2} \) and \( s = \frac{1}{b} \) are all simple poles of the integrand of (26) in the range \( R_T \). Then, Cauchy's residue theorem gives

\[
S_{f \times f}(a, b; x) = \left\{ \text{Res}_{s = \frac{1}{2}} + \text{Res}_{s = \frac{1}{b}} \right\} L(as, f \times f)L(bs, f \times f) \frac{x^s}{s} + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\epsilon}}{T}\right)
\]

\[
= L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^{\frac{1}{2}} + L\left(\frac{a}{b}, f \times f\right)L(1, \text{sym}^2 f)x^{\frac{1}{2}} + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\epsilon}}{T}\right),
\]

(27)

where

\[
J_1 = \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} L(as, f \times f)L(bs, f \times f) \frac{x^s}{s} ds,
\]

\[
J_2 = \frac{1}{2\pi i} \int_{\frac{1}{b} - iT}^{\frac{1}{b} + iT} L(as, f \times f)L(bs, f \times f) \frac{x^s}{s} ds,
\]

\[
J_3 = \frac{1}{2\pi i} \int_{\frac{1}{b} - iT}^{\frac{1}{2} - iT} L(as, f \times f)L(bs, f \times f) \frac{x^s}{s} ds,
\]

and the main terms \( L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^{\frac{1}{2}} \) and \( L\left(\frac{a}{b}, f \times f\right)L(1, \text{sym}^2 f)x^{\frac{1}{2}} \) derive from the residues of \( L(as, f \times f)L(bs, f \times f) \frac{x^s}{s} \) at the simple poles \( s = \frac{1}{2} \) and \( s = \frac{1}{b} \), respectively.

Now the remaining work is to handle these three terms: \( J_1, J_2 \) and \( J_3 \). Additionally, the estimates of these integrals on the horizontal parts are analogous, and thus we deal with \( J_2 \) and \( J_3 \) firstly. To estimate \( J_2 \) and \( J_3 \), similarly to the method of estimating \( G_2 \) and \( G_3 \), we also divide the integral interval into the following four short intervals \( I'_1, \cdots, I'_4 \) and apply Lemmas 2 and 3.

**Interval 1.** \( I'_1 := \{ s = \sigma + iT : \frac{1}{2} \leq a \sigma \leq 1, \frac{1}{2b} \leq b \sigma \leq 1 \} = \{ s = \sigma + iT : \frac{1}{2b} \leq \sigma \leq \frac{1}{b} \} \).

In this interval, we have

\[
T^{-1} \times \int_{I'_1} x^\sigma | \zeta(\sigma a + i T) L(a \sigma + i a T, \text{sym}^2 f) \zeta(b \sigma + i b T)L(b \sigma + i b T, \text{sym}^2 f) | d \sigma
\]

\[
\ll \max_{\frac{1}{b} \leq \sigma \leq \frac{1}{2}} x^\sigma T^{1 + \frac{1}{2} + \frac{1}{4}}(1-a \sigma)^{\frac{1}{4}}(1-b \sigma)^{\frac{1}{4}} T^{1+\epsilon}
\]

\[
\ll \max_{\frac{1}{b} \leq \sigma \leq \frac{1}{2}} x^\sigma T^{\frac{1}{4} + \epsilon} \left(\frac{x}{T^{\frac{1}{4} + \frac{1}{4}}} \right)^{\sigma}
\]

\[
\ll x^{\frac{1}{4}} T^{\frac{1}{4} + \lambda + \frac{1}{4}} + x^{\frac{1}{4}} T^{\frac{1}{4} + \lambda + \frac{1}{4}} + x^{\frac{1}{4}} T^{\frac{1}{4} + \lambda + \frac{1}{4}} + \epsilon.
\]

(28)

**Interval 2.** \( I'_2 := \{ s = \sigma + iT : \frac{1}{b} \leq \sigma \leq 1, 1 < b \sigma \leq b \} \).

In this interval, we have

\[
T^{-1} \times \int_{I'_2} x^\sigma | \zeta(\sigma a + i T) L(a \sigma + i a T, \text{sym}^2 f) \zeta(b \sigma + i b T)L(b \sigma + i b T, \text{sym}^2 f) | d \sigma
\]

\[
\ll \max_{\frac{1}{b} \leq \sigma \leq \frac{1}{2}} x^\sigma T^{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}(1-a \sigma)^{\frac{1}{4}} T^{1+\epsilon}
\]

\[
\ll \max_{\frac{1}{b} \leq \sigma \leq \frac{1}{2}} x^\sigma T^{\frac{1}{4} + \epsilon} \left(\frac{x}{T^{\frac{1}{4} + \frac{1}{4}}} \right)^{\sigma}
\]

\[
\ll x^{\frac{1}{4}} T^{\frac{1}{4} + \epsilon} + x^{\frac{1}{4}} T^{\frac{1}{4} + \lambda + \frac{1}{4}} + \epsilon.
\]

(29)
Interval 3. \( I'_3 := \left\{ s = \sigma + iT : 1 < a\sigma \leq a\eta, \frac{b}{2a} \leq b\sigma \leq 1 \right\} \).

This interval is an empty set noting that \( \frac{1}{b} < \frac{1}{2} \).

Interval 4. \( I'_4 := \left\{ s = \sigma + iT : 1 < a\sigma \leq a\eta, 1 < b\sigma \leq b\eta \right\} = \left\{ s = \sigma + iT : \frac{1}{a} < \sigma \leq \eta \right\} \).

In this interval, we have

\[
T^{-1} \times \int_{I'_1} x^\sigma \left| \xi(\sigma + iat) L(\sigma + iat, \text{sym}^2 f) \zeta(\sigma + ibT) L(\sigma + ibT, \text{sym}^2 f) \right| \, d\sigma
\leq \max_{\frac{1}{a} < \sigma \leq \eta} x^\sigma T^{-1+\epsilon}
\leq x^{1+\epsilon} T^{-1+\epsilon}.
\]

(30)

From (28)–(30) we can obtain

\[
f_2 + f_3
\leq T^{-1} \int_{I'_1} x^\sigma \left| \xi(\sigma + iat) L(\sigma + iat, \text{sym}^2 f) \zeta(\sigma + ibT) L(\sigma + ibT, \text{sym}^2 f) \right| \, d\sigma
= T^{-1} \int_{I'_1 \cup I'_4} x^\sigma \left| \xi(\sigma + iat) L(\sigma + iat, \text{sym}^2 f) \zeta(\sigma + ibT) L(\sigma + ibT, \text{sym}^2 f) \right| \, d\sigma
\leq x^{1+\epsilon} T^{-1+\epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\]

(31)

For \( I_1 \), we have

\[
I_1
\leq x^{\frac{1}{2}} \int_{I'_1} \left| \left( \frac{1}{2} + iat \right) L \left( \frac{1}{2} + iat, \text{sym}^2 f \right) \zeta \left( \frac{b}{2a} + ibT \right) L \left( \frac{b}{2a} + ibT, \text{sym}^2 f \right) \right| \, dt
\leq x^{\frac{1}{2} + \epsilon}
\leq x^{\frac{1}{2} + \epsilon} + x^{\frac{1}{2}} \log T \max_{1 \leq T_1 \leq T} T_1^{-1}
\times \int_{\frac{T_1}{2}}^{T_1} \left| \left( \frac{1}{2} + iat \right) L \left( \frac{1}{2} + iat, \text{sym}^2 f \right) \zeta \left( \frac{b}{2a} + ibT \right) L \left( \frac{b}{2a} + ibT, \text{sym}^2 f \right) \right| \, dt.
\]

(32)

Then from Lemmas 2 and 3, and using Hölder’s inequality, we can obtain

\[
I_1
\leq x^{\frac{1}{2} + \epsilon} + x^{\frac{1}{2}} \log T \max_{1 \leq T_1 \leq T} T_1^{-1 + \frac{1}{2} + \frac{1}{2}} \leq x^{\frac{1}{2} + \epsilon}
\leq x^{\frac{1}{2} + \epsilon} + x^{\frac{1}{2}} \max_{1 \leq T_1 \leq T} T_1^{-1 + \frac{1}{2}} \leq x^{\frac{1}{2} + \epsilon} + x^{\frac{1}{2}} \log T \max_{1 \leq T_1 \leq T} T_1^{-1 + \frac{1}{2}} \leq x^{\frac{1}{2} + \epsilon}.
\]

(33)

Thus, by putting (27), (31) and (33) together, we have

\[
S_{f \times f}(a, b; x) = L(1, \text{sym}^2 f) L \left( \frac{b}{a}, f \times f \right) x^{\frac{1}{2}} + L \left( \frac{a}{b}, f \times f \right) L(1, \text{sym}^2 f) x^{\frac{1}{2}}
+ O \left( x^{\frac{1}{2}} T^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}} + x^{\frac{1}{2}} \right).
\]

(34)

Taking \( T = x^{\frac{84(2a-1)}{80(2a-1)}} \) in (34), we can obtain

\[
S_{f \times f}(a, b; x) = L(1, \text{sym}^2 f) L \left( \frac{b}{a}, f \times f \right) x^{\frac{1}{2}} + L \left( \frac{a}{b}, f \times f \right) L(1, \text{sym}^2 f) x^{\frac{1}{2}}
+ O \left( x^{\frac{1}{2} - \frac{84(2a-1)}{80(2a-1)}} \right).
\]

(35)
Note that \(1 - \frac{84(2a - 1)}{3864a - 131} > \frac{1}{2}\) always holds. Then, comparing the first term and the error term in (35), we have, recalling \(b \leq 2a\),

\[
S_{f \times f}(a,b;x) = \begin{cases} 
L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^\frac{1}{2} + O\left(x^{-\frac{3(2a-1)}{8a}}\right), & \text{if } 318a^2 - 131ab - 402a + 131b < 0, b \leq 2a, \\
O\left(x^{-\frac{3(2a-1)}{8a}}\right), & \text{if } 318a^2 - 131ab - 402a + 131b > 0, b \leq 2a,
\end{cases}
\]

which implies the first and second results of Theorem 2.

For the case \(b > 2a\), we use a similar argument to the corresponding case of Theorem 1. In this situation, the point \(s = \frac{1}{b}\) is the only simple pole of the integrand of (26) in the range \(R_T\) by noting \(\frac{1}{b} < \frac{1}{2a}\). Then Cauchy’s residue theorem shows

\[
S_{f \times f}(a,b;x) = \text{Res}_{s=\frac{1}{b}} L(\alpha, x \times f)L(\beta, x \times f)\frac{x^\frac{1}{2}}{s} + O\left(x^{1+\epsilon}\right) \\
+ \frac{1}{2\pi i} \int_{\gamma_T} L(\alpha, x \times f)L(\beta, x \times f)\frac{x^\frac{1}{2}}{s} ds
\]

where the main term \(L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^\frac{1}{2}\) derives from the residue of \(L(\alpha, x \times f)L(\beta, x \times f)\frac{x^\frac{1}{2}}{s}\) at the simple pole \(s = \frac{1}{b}\).

To estimate \(J_2' + J_3'\) we also divide the integral interval into four short intervals \(I_1', \ldots, I_4',\) which are different from ones for the case \(b \leq 2a\). In fact, the corresponding short intervals \(I_1, \ldots, I_4\) become empty sets in this situation. However, we still can estimate \(J_2' + J_3'\) by following a similar argument to the corresponding parts of the case \(b \leq 2a\) and get

\[
J_2' + J_3' \ll x^{1-\frac{1}{2}} T^{-\frac{3}{16}} + x^{1+\epsilon} T^{1-\epsilon}.
\]

The estimate of \(J_1'\) becomes the following at the current case by noting \(\frac{b}{2a} > 1\).

\[
J_1' \ll x^{1-\frac{1}{2}} \int_1^T \left|\frac{1}{2} + iat\right| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right)\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) dt \sim x^{1-\frac{1}{2}} + \ldots
\]

Thus, recalling (36) we have

\[
S_{f \times f}(a,b;x) = L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^\frac{1}{2} + O\left(x^{\frac{1}{2} T^{1+\epsilon} + x^{1+\epsilon} T^{-1+\epsilon}}\right). \tag{37}
\]

Taking \(T = x^{-\frac{3(2a-1)}{8a}}\) in (37), we can obtain

\[
S_{f \times f}(a,b;x) = L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^\frac{1}{2} + O\left(x^{-\frac{3(2a-1)}{8a} + \epsilon}\right). \tag{38}
\]

Note that when \(a \geq 3\), we have \(1 - \frac{3(2a-1)}{8a} > \frac{1}{2}\). Therefore, we have, recalling \(b > 2a\),

\[
S_{f \times f}(a,b;x) = \begin{cases} 
L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^\frac{1}{2} + O\left(x^{\frac{7}{8} + \epsilon}\right), & \text{if } a = 2, b > 2a, \\
O\left(x^{-\frac{3(2a-1)}{8a} + \epsilon}\right), & \text{if } a \geq 3, b > 2a,
\end{cases}
\]
6. Application

As an application of Theorems 1 and 2, we may consider detecting the sign changes of \( \lambda_{f}^{a,b}(n) \) and \( \lambda_{f \times f}^{a,b}(n) \), i.e., estimating the following two quantities:

\[
N_{f}(x) = \sum_{\substack{n \leq x \atop \lambda_{f}^{a,b}(n) \geq 0}} 1 \quad \text{and} \quad N_{f \times f}(x) = \sum_{\substack{n \leq x \atop \lambda_{f \times f}^{a,b}(n) \geq 0}} 1.
\]

To estimate these quantities \( N_{f}(x) \) and \( N_{f \times f}(x) \), we need to establish the lower and upper bounds to the sums

\[
\sum_{n \leq x} |\lambda_{f}^{a,b}(n)| \quad \text{and} \quad \sum_{n \leq x} |\lambda_{f \times f}^{a,b}(n)|,
\]

respectively. Then, comparing these bounds with Theorems 1 and 2, we can get the estimates of these quantities \( N_{f}(x) \) and \( N_{f \times f}(x) \).

7. Conclusions

In this paper, we studied the mean value estimates of two-dimensional divisor problems related to some automorphic \( L \)-functions. We focused on the average behaviors of \( \lambda_{f}^{a,b}(n) \) and \( \lambda_{f \times f}^{a,b}(n) \), respectively. In this research, we established the upper bounds for the sum \( \sum_{n \leq x} \lambda_{f}^{a,b}(n) \) and the asymptotic formulas for the sum \( \sum_{n \leq x} \lambda_{f \times f}^{a,b}(n) \). The conditions of the integers \( a, b \) satisfy \( 1 < a < b \), because of the complexities and difficulties. To overcome these complexities and difficulties, we need to estimate the integrals of the horizontal parts more carefully. Some classical methods and instruments, such as Perron’s formula and Cauchy’s residue theorem; the decomposition of the Rankin–Selberg \( L \)-function; upper bounds and mean values of the Riemann zeta-function; the Hecke \( L \)-function and the symmetric square \( L \)-function, are also indispensable. With the results of this paper, we can further understand the properties of the Fourier coefficients \( \lambda_{f}^{a,b}(n) \) and \( \lambda_{f \times f}^{a,b}(n) \).

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