Magnetic response from constant backgrounds

to Coulomb sources

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Abstract

Magnetically uncharged, magnetic linear response of the vacuum filled with arbitrarily combined constant electric and magnetic fields to an imposed static electric charge is found within general nonlinear electrodynamics. When the electric charge is point-like and external fields are parallel, the response found may be interpreted as a field of two point-like magnetic charges of opposite polarity in one point. Coefficients characterizing the magnetic response and induced currents are specialized to Quantum Electrodynamics, where the nonlinearity is taken as that determined by the Heisenberg-Euler effective Lagrangian.

1 Introduction

It is well understood that the vacuum filled with strong background field is, in Quantum Electrodynamics (QED), equivalent to a linear or nonlinear medium [1, 2, 3]. The simplest (one-loop) Feynman diagrams responsible for description of such a media are shown in Fig. 1.
The bold lines there stand for the electron propagators in the external field. These are known exactly for the constant background, for the background plane electromagnetic field, also for special combinations of the latter two. In the first case the equivalent medium is space- and time-homogeneous, otherwise it is not, and then the energy and momentum exchange between the field and the background occurs. Using the exact solutions to the Dirac equation in external field makes these diagrams belonging to the so-called Furry picture. The first diagram corresponds to linearization of the field equations above the background. It represents the (second-rank) polarization tensor $\Pi_{\mu\nu}(x,y)$, which contains in itself the linear polarizational properties of the equivalent medium, usually referred to as dielectric permeability and magnetic permittivity. It is responsible for the screening of charges and currents and transformations of their shapes due to the strong background, and for small-amplitude electromagnetic wave propagations in the background, especially for polarization of the eigen-modes and (different) modifications of the mass shell in each mode (the birefringence making a goal for observation [4] with the recent evidence for it obtained from the neutron star RX J1856.5-3764 [5]) by deviating the dispersion curves from the standard shape $k_0^2 = k^2$ known in the empty vacuum (the one with null background). The second diagram in Fig. 1, the third-rank polarization tensor $\Pi_{\mu\nu\rho}(x,y,z)$, takes into account the quadratic response of the background. When taken on the photon mass shell, it is responsible for the photon splitting and merging in an external field [6, 7]. Beyond the mass shell, it also describes the response of the medium to small perturbations with the quadratic accuracy relative to these perturbations. Analogously, the fourth-rank polarization tensor includes the cubic response, photon-by-photon scattering (the first experimental detection of this fundamental process, which is the source of nonlinearity of QED, was recently reported in [8]), photon splitting into three [7], and so on.

The polarization tensors may be defined as variational derivatives of the effective action $\Gamma$.
known as [9] the generating functional of the one-particle-irreducible vertex functions

\[ \Pi^{\alpha \rho}(x, y) = \frac{\delta^2 \Gamma}{\delta A_\alpha(x) \delta A_\rho(y)} \bigg|_{A=\bar{A}}, \quad \Pi^{\alpha \rho \beta}(x, y, z) = \frac{\delta^3 \Gamma}{\delta A_\alpha(x) \delta A_\rho(y) \delta A_\beta(z)} \bigg|_{A=\bar{A}}, \]

with respect to the potentials \( A_\alpha(x) \), taken at their background values \( A_\alpha(x) = \bar{A}_\alpha(x) \). This is equivalent to the Feynman diagram representation. Each differentiation adds an extra photon vertex to the diagram. The polarization tensors of all ranks participate in the nonlinear Maxwell equations

\[
\begin{align*}
[\eta_{\rho \alpha} \Box - \partial_\rho \partial_\alpha] a^\rho(x) + \int d^4 y \Pi^{\alpha \rho}(x, y) a^\rho(y) \\
+ \frac{1}{2} \int d^4 y d^4 z \Pi^{\alpha \rho \beta}(x, y, z) a^\rho(y) a^\beta(z) + O(a^3) = j_\alpha(x),
\end{align*}
\]

(1)

where \( a^\alpha(x) \) is the potential over the background, \( a_\alpha(x) = A_\alpha(x) - \bar{A}_\alpha(x) \), i.e., the response to the applied source (perturbation) \( j_\rho(x) \). Bearing in mind that each vertex in the diagrams carries a small quantity, the electron charge \( e \), we may approach this equation perturbatively. Then, at the classical level, the solution to (1) is

\[ a_{(0)}^\alpha(x) = \int D_{(0)}^{\alpha \alpha'}(x - x') j_{\alpha'}(x') d^4 x', \]

(2)

where \( D_{(0)}^{\nu \rho}(x - x') \) is the free photon propagator. The first correction to it (within the linearity of the equation) is

\[ a_{(1)}^\alpha(x) = \int D_{(0)}^{\alpha \alpha'}(x - x') \Pi_{\alpha' \beta'}(x', y') D_{(0)}^{\beta \rho}(y' - y) j_\rho(y) d^4 x' d^4 y d^4 y'. \]

(3)

The correction of the second power of the perturbation \( j_\rho(x) \) is

\[ a_{(2)}^\alpha(x) = \frac{1}{2} \int D_{(0)}^{\alpha \alpha'}(x - x') \Pi_{\alpha' \beta' \gamma'}(x', y', z') D_{(0)}^{\beta \gamma \rho}(y' - y) j_\rho(y) D_{(0)}^{\gamma \beta}(z, z) j_\beta(z) d^4 x' d^4 y' d^4 z' d^4 y d^4 z. \]

(4)

The terms (3) and (4) are shown graphically in Fig. 2, where the wiggly line stands for the free photon propagator \( D_{(0)}^{\nu \rho}(x - x') \).

An essential simplification of the calculations is achieved if one confines oneself to the approximation of the local effective action, where the functional \( \Gamma \) does not depend on the space-time derivatives of the fields, for instance, where the Euler-Heisenberg expression for it is taken in one-

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\(^1\)Greek indices span the 4-dimensional Minkowski space-time, e.g., \( \mu = (0, i) \), \( i = 1, 2, 3 \), \( \eta_{\mu \nu} = \text{diag}(+1, -1, -1, -1) \), and boldface letters denote three-dimensional Euclidean vectors (e.g., \( \mathbf{A}(x) = (A^i(x), i = 1, 2, 3) \)). The four-rank and three-rank Levi-Civita tensors are normalized as \( \varepsilon^{0123} = +1 \) and \( \varepsilon_{123} = 1 \), respectively.
loop [3] approximation in QED (also two-loop [10], [11] and even three-loop [12] results are being considered). This approximation is good as long as the fields slowly varying in time and space are dealt with. Contrary to (1), within this approximation the field equations are differential (not integral) ones, and they do not include higher derivatives, while (the Fourier transform of) the polarization tensor of \( n \)-th rank behaves as the \( n \)-th power of the momentum; in the language of optics this corresponds to disregard of the spatial and frequency dispersion\(^2\). We thoroughly traced the derivation of the Maxwell equations within the local approximation in [14], [15], [16], [17]. In [15] the self-interaction of magnetic and electric dipole moments, which modifies their values calculated within any version of the strong-interaction theory, was considered using the 4-th rank polarization tensor with no background. In [18] the self-interaction of a point-like charge was studied with the same tools leading to the result that its field-energy (beyond the perturbation approach) is finite, although the field in the center of the charge is not. Therefore, the point charge, with its field being a solution of a nonlinear equation, becomes a soliton at rest or in motion [19], [20]. Interaction between long-wave electromagnetic waves was considered taking into account, effectively, the polarization tensors up to 6-th rank [21], [22]. In [16], [23] we showed that the quadratic response of the vacuum with the background of a constant magnetic field to an applied electric field of a point-like or extended central-symmetric charge, governed by the 3-rd rank polarization tensor and corresponding to term (4) is purely magnetic, i.e. we face here the magneto-electric effect. Moreover, the magnetic response far from the charge is the field of a magnetic dipole with its dipole moment quadratically dependent upon the electric charge. The photon splitting on the basis of the same diagram was studied in [7]. In [24] and now we take an arbitrary combination of constant electric \( \vec{E} \) and magnetic \( \vec{B} \) fields as a background (see [25] beyond the local approximation), and we consider linear response to an applied electric charge following the information contained in the 2-nd rank polarization tensor (second term in Eq. (1)). This response may be both electric and magnetic. The electric response was studied in [24], resulting in descrip-

\(^2\)Various local Lagrangians that, like the Heisenberg-Euler approximation, do not contain space- and time-derivatives of fields, but are not associated with QED, are widely used together with the Einstein’s gravity, especially when studying magnetized black holes. See e.g. [13] and pertinent references therein.
tion of the induced charge density and modification of the Coulomb field far from the charge\(^3\). In the present paper we study the linear magnetic response of the constant background to an applied Coulomb source, complementary to our previous study [24], where only the linear electric response was found. Linearly induced currents and vector potentials are discussed in detail. Contrary to [26] and to our forthcoming work, only magnetic response with vanishing total magnetic charge is considered here. Correspondingly, the finally found magnetic field looks like a combination of two opposite point-like magnetic charges coexisting in one point. Applying the general results derived for any nonlinear theory to the special case where the nonlinearity is provided by QED at one-loop, we study relevant coefficients characterizing the results in terms of the Heisenberg-Euler effective action [3], in which proper-time representations and asymptotic regimes are discussed in detail.

The paper is organized as follows. In Sec. 2, after presenting the necessary Maxwell equations linearized near the background field and indicating the structure of the applied electric field, we obtain expressions for the current density induced in the “medium” inside and outside of the applied extended charge. In Subsecs. 2.1 and 3 we find the magnetic fields produced by this current and the vector potential corresponding to the magnetic response produced by a pointlike Coulomb source. All the results reported above are written in terms of the derivatives of the local effective Lagrangian over the field invariants taken at the background. Hence these may be used with every model Lagrangian, irrespective of its origin and of its connection to QED. On the contrary, in Sec. 4, we specialize the results to the one-loop Euler-Heisenberg Lagrangian of QED. Sec. 5 is devoted to the concluding remarks.

2 Linearly induced currents and magnetic responses in constant backgrounds

Let there be a background electromagnetic field, with its field tensor \(F_{\nu\mu}(x)\) equal to \(\mathbf{F}_{\nu\mu}(x)\), produced by the background current \(J_\mu\) via the (second set of) Maxwell equations

\[
\partial^\nu \mathbf{F}_{\nu\mu}(x) - \partial^\nu \left[ \frac{\delta \mathcal{L}(\mathbf{\Phi}, \mathbf{\Sigma})}{\delta \mathbf{\Phi}(x)} \right]_{F=\mathbf{F}} \mathbf{F}_{\nu\mu}(x) + \frac{\delta \mathcal{L}(\mathbf{\Phi}, \mathbf{\Sigma})}{\delta \mathbf{\Sigma}(x)} \left. \right|_{F=\mathbf{F}} = J_\mu(x), \tag{5}
\]

within a nonlinear local electrodynamics with the Lagrangian

\[
\mathcal{L}(x) = -\mathbf{\Phi}(x) + \mathcal{L}(x), \tag{6}
\]

where its nonlinear part \(\mathcal{L}(x)\) is taken as a function \(\mathcal{L}(x) = \mathcal{L}(\mathbf{\Phi}, \mathbf{\Sigma})\) of two field invariants \(\mathbf{\Phi} = (1/4) F^{\mu\nu} F_{\mu\nu} = (1/2) (\mathbf{B}^2 - \mathbf{E}^2), \mathbf{\Sigma} = (1/4) \tilde{F}^{\mu\nu} F_{\mu\nu} = -(\mathbf{E} \cdot \mathbf{B}), \tilde{F}_{\mu\nu}(x) = (1/2) \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}(x)\), and may be thought of, for instance, as the effective Lagrangian of Quantum Electrodynamics in

\(^3\)A brief review of our previous works may be found in [17].
the local (infra-red) approximation, i.e. the one where the dependence on the space- and time-derivatives of the fields is neglected\(^4\). Moreover, we shall be considering the constant background 
\[ F_{\nu\mu}(x) = \text{const.} \] 
here. This field does not require any current to be supported: it is seen that equation (5) is satisfied by \( J_\mu(x) = 0 \) in this case.

Let the constant background be disturbed by a small introduced current \( j_\mu(x) \). It causes the deviation \( f_{\nu\mu}(x) = F_{\nu\mu}(x) - F_{\nu\mu} \) of the field from the background. Expanding the Maxwell equations in powers of \( f_{\nu\mu}(x) \) we obtain in the first order the linear equation (see Refs. [14, 15, 16, 17, 18, 23, 24, 26] for equations for higher orders, which are nonlinear as containing the second and higher powers of \( f_{\nu\mu}(x) \)):

\[
\partial^\nu f_{\nu\mu}(x) = j_\mu^\text{lin}(x) + j_\mu(x),
\]

\[
j_\mu^\text{lin}(x) = \partial^\tau \left[ L_\delta f_{\tau\mu}(x) + \frac{1}{2} \left( L_\delta \tilde{F}_{\alpha\beta} + L_\delta \tilde{F}_{\alpha\beta} \right) F_{\tau\mu} f_{\alpha\beta}(x) \right] + \partial^\tau \left[ L_\phi \tilde{f}_{\tau\mu}(x) + \frac{1}{2} \left( L_\phi \tilde{F}_{\alpha\beta} + L_\phi \tilde{F}_{\alpha\beta} \right) \tilde{F}_{\tau\mu} f_{\alpha\beta}(x) \right],
\]

(7)

where the subscripts by \( L \) designate derivatives with respect to the indicated field invariants taken at their background value, for instance \( \frac{\partial^2 L}{\partial \delta \delta} \bigg|_{F=\tilde{F}} = L_\delta \). We have introduced here the notation for the linearly induced current \( j_\mu^\text{lin}(x) \) (nonlinearly induced currents were dealt with in [14, 15, 17, 18, 24, 16, 23, 26]). To avoid possible misunderstanding, we stress that nonlinearly induced currents are responsible for selfinteraction of the deviation fields \( f_{\nu\mu}(x) \), whereas the nonlinearity of the theory given by the Lagrangian (6) shows itself in the present framework as the interaction between the electromagnetic field \( f_{\nu\mu}(x) \) and the electromagnetic background \( F_{\alpha\beta} \).

In what follows we solve Eq. (7) perturbatively with respect to the above coefficients, whose connection with QED will be exploited in Section 4. To this end we represent the electromagnetic field-strength tensor as

\[
f_{\nu\mu}(x) = f_{\nu\mu}^{(0)}(x) + f_{\nu\mu}^{(1)}(x) + \ldots,
\]

(8)

where \( f_{\nu\mu}^{(0)}(x) \) is a solution of the classical field equation

\[
\partial^\nu f_{\nu\mu}^{(0)}(x) = j_\mu(x),
\]

while the linear response \( f_{\nu\mu}^{(1)}(x) \) is subject to the equation determined by the induced current

\(^4\)The special case with the Euler-Heisenberg Lagrangian accepted as the one-loop approximation of such local effective action will be treated for the purposes of the present article in Section 4 below.
taken on \( f^{(0)}_{\nu\mu}(x) \)

\[
\partial^\nu f^{(1)}_{\nu\mu}(x) = \partial^\nu \left[ \frac{1}{2} \left( \mathcal{L}_{\delta\delta} \mathcal{F}_{\alpha\beta} + \mathcal{L}_{\delta\theta} \tilde{\mathcal{F}}_{\alpha\beta} \right) \mathcal{F}_{\nu\mu} f^{(0)\alpha\beta}(x) + \mathcal{L}_{\delta} f^{(0)}_{\nu\mu}(x) \right] \\
+ \partial^\nu \left[ \frac{1}{2} \left( \mathcal{L}_{\delta\delta} \mathcal{F}_{\alpha\beta} + \mathcal{L}_{\delta\theta} \tilde{\mathcal{F}}_{\alpha\beta} \right) \tilde{\mathcal{F}}_{\nu\mu} f^{(0)\alpha\beta}(x) + \mathcal{L}_{\delta} f^{(0)}_{\nu\mu}(x) \right].
\]  

(9)

For the perturbation of the background we take the current corresponding to a static charge \( q \) homogeneously distributed over a sphere\(^5\) with the radius \( R \)

\[
j_\mu(x) = \delta_\mu \rho^{(0)}(r), \quad r = |\mathbf{r}|, \\
\rho^{(0)}(r) = \frac{3q}{4\pi R^3} \theta(R - r), \quad R = \text{const.},
\]

(10)

This charge density corresponds to a regularization of the pointlike static charge

\[
\rho^{(0)}(r) = q\delta^3(r), \quad \delta^3(r) = \delta(x^1)\delta(x^2)\delta(x^3),
\]

(11)

placed at origin \( \mathbf{r} = 0 \). It is a source of the regularized Coulomb field \( f^{(0)}_{\nu\mu}(r) = E^{(0)i}(r) \) and null magnetic field \( B^{(0)i}(r) = -(1/2)\varepsilon_{ijk}f^{(0)jk}(r) = 0 \)

\[
\partial^\nu f^{(0)}_{\nu\mu}(x) = j_\mu(x), \quad E^{(0)}(\mathbf{r}) = E^{(0)}_{\text{in}}(\mathbf{r}) \theta(R - r) + E^{(0)}_{\text{out}}(\mathbf{r}) \theta(r - R), \\
E^{(0)}_{\text{in}}(\mathbf{r}) = \frac{q \mathbf{r}}{4\pi R^3}, \quad E^{(0)}_{\text{out}}(\mathbf{r}) = \frac{q \mathbf{r}}{4\pi r^3}, \quad B^{(0)}(\mathbf{r}) = 0.
\]

(12)

Throughout the text, the indexes “in” and “out” classify electromagnetic quantities at points inside \((r < R)\) and outside \((r \geq R)\) of the spherical charge distribution, respectively.

In our previous work \cite{24}, we studied the electric response \( E^{(1)k}(\mathbf{r}) \neq 0 \), \( B^{(1)k}(\mathbf{r}) = -(1/2)\varepsilon_{ijk}f^{(1)jk}(\mathbf{r}) = 0 \) to equation (9) giving a correction to the Coulomb law (12); now we shall consider the magnetic solution. Substituting the zero-order solutions \( B^{(0)}(\mathbf{r}) = 0 \) and \( E^{(0)}(\mathbf{r}) \) (12) in Eq. (9), one finds that the first-order linear magnetic response \( B^{(1)}(\mathbf{r}) \) to the purely electric perturbation (10) is the solution of the differential equation

\[
\nabla \times [B^{(1)}(\mathbf{r}) - S^{(0)}(\mathbf{r})] = 0,
\]

(13)

\(^5\)Hereafter \( \theta(z) \) denotes the Heaviside step function defined as \( \theta(z) = 1 \) if \( z \geq 0 \) and zero otherwise.
where $\mathbf{f}^{(0)}(\mathbf{r})$ is the expression within the brackets in (7), taken in the zeroth order:

\[
\begin{align*}
\mathbf{f}^{(0)}(\mathbf{r}) &= -\mathbf{L}_\delta \mathbf{E}^{(0)}(\mathbf{r}) - \left[\mathbf{L}_\delta \mathbf{E} \cdot \mathbf{E}^{(0)}(\mathbf{r}) + \mathbf{L}_\delta \mathbf{B} \cdot \mathbf{E}^{(0)}(\mathbf{r})\right] \mathbf{B} \\
+ & \left[\mathbf{L}_\delta \mathbf{E} \cdot \mathbf{E}^{(0)}(\mathbf{r}) + \mathbf{L}_\delta \mathbf{B} \cdot \mathbf{E}^{(0)}(\mathbf{r})\right] \mathbf{E} \\
= & \mathbf{f}_{\text{in}}^{(0)}(\mathbf{r}) \theta(R - r) + \mathbf{f}_{\text{out}}^{(0)}(\mathbf{r}) \theta(r - R) ,
\end{align*}
\] (14)

in which

\[
\begin{align*}
\mathbf{f}_{\text{in}}^{(0)}(\mathbf{r}) &= \frac{q}{4\pi R^3} \left\{ -\mathbf{L}_\delta \mathbf{r} - \left[\mathbf{L}_\delta \mathbf{E} \cdot \mathbf{r} + \mathbf{L}_\delta \mathbf{B} \cdot \mathbf{r}\right] \mathbf{B} \\
+ & \left[\mathbf{L}_\delta \mathbf{E} \cdot \mathbf{r} + \mathbf{L}_\delta \mathbf{B} \cdot \mathbf{r}\right] \mathbf{E} \right\} , \quad r < R , \\
\mathbf{f}_{\text{out}}^{(0)}(\mathbf{r}) &= \frac{q}{4\pi R^3} \left\{ -\mathbf{L}_\delta \mathbf{r} - \left[\mathbf{L}_\delta \mathbf{E} \cdot \mathbf{r} + \mathbf{L}_\delta \mathbf{B} \cdot \mathbf{r}\right] \mathbf{B} \\
+ & \left[\mathbf{L}_\delta \mathbf{E} \cdot \mathbf{r} + \mathbf{L}_\delta \mathbf{B} \cdot \mathbf{r}\right] \mathbf{E} \right\} , \quad r \geq R .
\end{align*}
\] (15, 16)

Here the space- and time-independent electric and magnetic components of the background field are barred: $\mathbf{E}^i = \bar{F}_{0i}$, $\mathbf{B}^i = -(1/2) \varepsilon_{ijk} \bar{F}^{jk}$.

Consider the linearly induced current density (7) to the same first-order approximation. According to Eq. (13), it is

\[
\begin{align*}
\mathbf{j}^{(1)}(\mathbf{r}) &= \nabla \times \mathbf{f}^{(0)}(\mathbf{r}) = \theta(R - r) \left[\nabla \times \mathbf{f}_{\text{in}}^{(0)}(\mathbf{r})\right] + \theta(r - R) \left[\nabla \times \mathbf{f}_{\text{out}}^{(0)}(\mathbf{r})\right] .
\end{align*}
\] (17)

Note that the quantity (14) is continuous at the border of the charge $r = R$, because $\mathbf{E}^{(0)}(\mathbf{r})$ (12) is. For this reason the differentiation of the step functions has not contributed to the sum (17). Hence there does not appear any current at the surface of the charge.

Thus we may define the inner $\mathbf{B}_{\text{in}}^{(1)}(\mathbf{r})$ and outer $\mathbf{B}_{\text{out}}^{(1)}(\mathbf{r})$ magnetic responses as solutions to the equations

\[
\begin{align*}
\nabla \times \mathbf{B}_{\text{in}}^{(1)}(\mathbf{r}) &= \mathbf{j}_{\text{in}}^{(1)}(\mathbf{r}) , \quad \nabla \times \mathbf{B}_{\text{out}}^{(1)}(\mathbf{r}) = \mathbf{j}_{\text{out}}^{(1)}(\mathbf{r}) ,
\end{align*}
\] (18)

where the inner and outer parts of the first-order linearly induced current densities are

\[
\begin{align*}
\mathbf{j}_{\text{in}}^{(1)}(\mathbf{r}) &= \nabla \times \mathbf{f}_{\text{in}}^{(0)}(\mathbf{r}) = \frac{q}{4\pi R^3} (\mathbf{L}_\delta \mathbf{E} + \mathbf{L}_\delta \mathbf{B}) \left[\mathbf{B} \times \mathbf{E}\right] , \quad r < R ,
\end{align*}
\] (19)

\footnote{It should be noted that the relativistic covariance is deprived by selecting the reference frame in which the static charge is at rest.}
\[ j^{(1)}_{\text{out}}(r) = \nabla \times j^{(0)}_{\text{out}}(r) = \frac{q}{4\pi r^3} \left\{ \mathcal{L}_{\delta\delta} \left[ [\mathbf{B} \times \mathbf{E}] + \frac{3}{r^2} (\mathbf{E} \cdot \mathbf{r}) [\mathbf{r} \times \mathbf{B}] \right] \right. \\
+ \left. \mathcal{L}_{\phi\phi} \left[ [\mathbf{B} \times \mathbf{E}] - \frac{3}{r^2} (\mathbf{B} \cdot \mathbf{r}) [\mathbf{r} \times \mathbf{E}] \right] \right. \\
+ \left. \frac{3}{r^2} \mathcal{L}_{\delta\delta} \left[ (\mathbf{B} \cdot \mathbf{r}) [\mathbf{r} \times \mathbf{B}] - (\mathbf{E} \cdot \mathbf{r}) [\mathbf{r} \times \mathbf{E}] \right] \right\}, \quad r \geq R. \tag{20} \]

respectively. The induced current (17) is discontinuous at the edge of the sphere, the same as the charge density (10) is.

For the special case of parallel external backgrounds \( \mathbf{B} \parallel \mathbf{E} \), the induced current density inside the charge disappears, \( j^{(1)}_{\text{in}} = 0 \), while the current \( j^{(1)}_{\text{out}}(r) \) circles the coordinate axis parallel to their common direction. Letting \( \hat{z} \) be the unit vector (\( |\hat{z}| = 1 \)) along the common direction of the external fields, \( \hat{z} \parallel \mathbf{B} \parallel \mathbf{E} \), the current density \( j^{(1)}_{\text{out}}(r) \) acquires the form

\[ j^{(1)}_{\text{out}}(r) = \frac{3q}{4\pi r^5} \tilde{g} (\hat{z} \cdot \mathbf{r}) [\mathbf{r} \times \hat{z}], \tag{21} \]

where \( \tilde{g} \) is a combination of derivatives of the effective Lagrangian and field invariants,

\[ \tilde{g} = \mathcal{G} (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\delta\delta}) + 2\mathcal{G} \mathcal{L}_{\delta\delta}, \]

\[ \mathcal{G} = -\mathbf{B} \cdot \mathbf{E}, \quad \mathcal{G} = \frac{1}{2} \left( \mathbf{B}^2 - \mathbf{E}^2 \right). \tag{22} \]

The current flux (21) flows in opposite directions in the upper and lower hemispheres; see Fig. 3. Hence, the total current through the part of a fixed meridional plane \( \varphi = \varphi_0, \ 0 < \varphi_0 < 2\pi \), enclosed between any two coordinate spheres \( r_1 < r < r_2 \) is zero:

\[ \int j^{(1)}_{\text{out}}(r) \, ds = \frac{3q}{4\pi} \int^{r_2}_{r_1} \int^{r_2}_{r_1} \int_{0}^{\pi} d\theta \cos \theta \sin \theta = 0. \]

Here \( \cos \theta = (\hat{z} \cdot \mathbf{r}) / r \). Once any two mutually non-orthogonal constant fields may be reduced to parallelity by an appropriate Lorentz transformation to a special inertial frame, the current (20) differs from (21) by a contribution due to the motion of the charge in that frame.

**2.1 Magnetic response**

Besides the linearized Maxwell equations (13), the magnetic response \( \mathbf{B}^{(1)}(r) \) should obey also the equation

\[ \nabla \cdot \mathbf{B}^{(1)}(r) = 0, \tag{23} \]
Figure 3: (color online) Flux of the electric current \((21)\) linearly induced by static electric charge placed into parallel electric and magnetic background fields (outside the charge). The current revolves about the axis drawn along the common direction of the background fields. The brighter the arrow-head lines, the larger the current density.

that excludes an overall magnetic charge and makes the formulation of the theory in terms of potentials possible, in which case it corresponds to one of the Bianchi identities in electrodynamics. Equations (13) and (23) are satisfied by the magnetic response \(B^{(1)}(\mathbf{r})\)

\[
B^{(1)i}(\mathbf{r}) = \left( \delta^{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \mathbf{J}^{(0)j}(\mathbf{r}),
\]

identified as the transverse component of \(\mathbf{J}^{(0)}(\mathbf{r})\). The integral form of (24) reads

\[
B^{(1)i}(\mathbf{r}) = \mathbf{J}^{(0)i}(\mathbf{r}) + \frac{1}{4\pi} \partial_i \partial_j \int d\mathbf{y} \mathbf{J}^{(0)j}(\mathbf{y}) |\mathbf{r} - \mathbf{y}|.
\]

After computing these integrals (see Eqs. (60) - (64) in Appendix A) the inner part \(B_{in}^{(1)}(\mathbf{r})\) takes the form

\[
B_{in}^{(1)}(\mathbf{r}) = \frac{q}{4\pi} \left\{ \left[ \frac{3}{5R^3} \mathcal{L}_{\Phi} (\mathbf{E} \cdot \mathbf{r}) + \frac{1}{5R^3} (\mathcal{L}_{\Phi} + 4\mathcal{L}_{\Phi} + \mathbf{B} \cdot \mathbf{r}) \right] \mathbf{E} - \left[ -\frac{3}{5R^3} \mathcal{L}_{\Phi} (\mathbf{B} \cdot \mathbf{r}) - \frac{1}{5R^3} (4\mathcal{L}_{\Phi} + \mathcal{L}_{\Phi}) (\mathbf{E} \cdot \mathbf{r}) \right] \mathbf{B} + \frac{\tilde{g}}{5R^3} \mathbf{r} \right\},
\]

while the outer part \(B_{out}^{(1)}(\mathbf{r})\) can be conveniently written as

\[
B_{out}^{(1)}(\mathbf{r}) = B_{pl}^{(1)}(\mathbf{r}) + B_{out}^{(1)}(\mathbf{r}; R).
\]
Here \( B_{\text{pl}}^{(1)}(r) \) denotes an \( R \)-free part

\[
B_{\text{pl}}^{(1)}(r) = \frac{q}{4\pi} \left\{ \frac{g}{2r^3} - \frac{\mathcal{L} + \mathcal{L}_\varepsilon}{2r^3} \left[ (E \cdot r) B - (B \cdot r) E \right] \right. \\
+ \frac{3}{2} \left[ (\mathcal{L} - \mathcal{L}_\varepsilon) (E \cdot r) (B \cdot r) \right. \\
- \left. \left. \left[ -\frac{g}{r^2} (\mathcal{L} - \mathcal{L}_\varepsilon) (E \cdot r) (B \cdot r) \right] \right] \frac{r}{r^5} \right\},
\]

while \( B_{\text{out}}^{(1)}(r; R) \), the \( R \)-dependent part, reads

\[
B_{\text{out}}^{(1)}(r; R) = \frac{q}{8\pi} \left( \frac{3R^2}{5r^5} \right) \left\{ 2\mathcal{L}_\varepsilon \left[ (E \cdot r) B - (B \cdot r) E \right] \right. \\
- \left. \left( \mathcal{L} - \mathcal{L}_\varepsilon \right) \left[ (B \cdot r) E + (E \cdot r) B \right] \right. \\
+ \left[ -\frac{g}{r^2} (\mathcal{L} - \mathcal{L}_\varepsilon) (E \cdot r) (B \cdot r) \right] \\
+ \left. \left. \left[ \frac{5}{r^2} \mathcal{L}_\varepsilon \left( (B \cdot r)^2 - (E \cdot r)^2 \right) \right] \frac{r}{r^5} \right\}.
\]

The division in \( R \)-dependent and \( R \)-free terms, expressed in Eq. (27), is aimed to emphasize that Eq. (29) corresponds to a pure homogeneous solution \( \nabla \times B_{\text{out}}^{(1)}(r; R) = 0 \). This is a consequence of the fact that the outer induced current density, given by Eq. (20), does not depend on \( R \) or, in other words, there is no \( R \)-dependent source providing (29). Its real role is to provide continuity of the whole magnetic response \( B^{(1)}(r) = B_{\text{in}}^{(1)}(r) \theta (R - r) + B_{\text{out}}^{(1)}(r) \theta (r - R) \) at the border of the Coulomb source (10). A similar feature has been reported by us in [24], wherein \( R \)-dependent terms in the electric response come automatically from the projection operator with the same interpretation. These \( R \)-dependent solutions are a consequence of the Coulomb source being an extended charge distribution rather than a pointlike one. In contrast, the \( R \)-independent part \( B_{\text{pl}}^{(1)}(r) \) is the same as the first-order linear response of the pointlike Coulomb source (11), since it is the only survivor in the limit \( r \gg R \) (or \( R \to 0 \)), bearing in mind that for any nonspecial direction, \( B_{\text{pl}}^{(1)}(r) \) decreases as \( r^{-2} \), while \( B_{\text{out}}^{(1)}(r; R) \) decreases as \( r^{-4} \). Therefore \( B_{\text{pl}}^{(1)}(r) \) is identified as the first-order linear response to the pointlike Coulomb source (11). Moreover, according to Eq. (20), \( B_{\text{pl}}^{(1)}(r) \) is provided by the outer induced current \( j_{\text{out}}^{(1)}(r) \)

\[
\nabla \times B_{\text{pl}}^{(1)}(r) = \nabla \times j_{\text{out}}^{(0)}(r) = j_{\text{out}}^{(1)}(r).
\]

The first-order linear magnetic response calculated above does not carry any magnetic charge, in virtue of the triviality of the Gauss integral

\[
\int_S \left( B^{(1)}(r) \cdot \hat{n} \right) dS = 0,
\]

for an arbitrary closed surface \( S \) embracing the charge \( q \). This integral vanishes for each magnetic...
response \( \mathbf{B}_{\text{in}}^{(1)} (\mathbf{r}) \), \( \mathbf{B}_{\text{pl}}^{(1)} (\mathbf{r}) \) and \( \mathbf{B}_{\text{out}}^{(1)} (\mathbf{r}; R) \), independently. To begin with, taking \( S \) to be a sphere of radius \( R \), centered in the charge \( q \) (placed at the origin \( r = 0 \)), and choosing a reference frame in which \( \mathbf{B} \) is aligned along the \( z \)-axis and \( \mathbf{E} \) lies in the \( xz \)-plane (so that \( \mathbf{E} \cdot \mathbf{r} = ER \cos \gamma \), \( \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \varphi \) and \( \mathbf{E} \cdot \mathbf{B} = EB \cos \theta' \)), we find that only \( \mathbf{B}_{\text{in}}^{(1)} (\mathbf{r}) \) might contribute in Eq. (31) and that

\[
\oint_S (\mathbf{B}_{\text{in}}^{(1)} (\mathbf{r}) \cdot \hat{n}) \, dS = \frac{3q}{20\pi} \left( -\frac{\mathcal{J}(R)}{R^2} + \frac{4\pi \tilde{g}}{3} \right) = 0 ,
\]

(32)

where \( \mathcal{J}(R) \) denotes the integral over the surface \( S \)

\[
\mathcal{J}(R) = \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta \left\{ \left( \mathcal{L}_{\tilde{z}\tilde{n}} - \mathcal{L}_{\vartheta\varphi} \right) (\mathbf{E} \cdot \mathbf{r}) (\mathbf{B} \cdot \mathbf{r})
+ \mathcal{L}_{\varphi\theta} \left[ (\mathbf{B} \cdot \mathbf{r})^2 - (\mathbf{E} \cdot \mathbf{r})^2 \right] \right\}
= \frac{4\pi R^2}{3} \tilde{g}.
\]

(33)

Therefore Eq. (32) holds true. If one takes \( S \) to be a sphere of radius \( r > R \), then Eq. (31) takes the form

\[
\oint_S (\mathbf{B}_{\text{out}}^{(1)} (\mathbf{r}) \cdot \hat{n}) \, dS = \oint_S (\mathbf{B}_{\text{pl}}^{(1)} (\mathbf{r}) \cdot \hat{n}) \, dS + \oint_S (\mathbf{B}_{\text{out}}^{(1)} (\mathbf{r}; R) \cdot \hat{n}) \, dS ,
\]

in which it can be seen that both integrals vanish identically

\[
\oint_S (\mathbf{B}_{\text{out}}^{(1)} (\mathbf{r}; R) \cdot \hat{n}) \, dS = -\frac{3R^2}{5r^2} \oint_S (\mathbf{B}_{\text{pl}}^{(1)} (\mathbf{r}) \cdot \hat{n}) \, dS
= -\frac{3R^2}{5r^2} \frac{q}{8\pi} \left( 4\pi \tilde{g} - \frac{3\mathcal{J}(r)}{r^2} \right) = 0 .
\]

(34)

Here \( \mathcal{J}(r) \) is the same integral as (33), but with the \( R \) replaced by \( r \). One concludes that there is no magnetic charge attributed to the magnetic response \( \mathbf{B}^{(1)} (\mathbf{r}) \): the magnetic lines of force incomming to and ougoing from the charge \( q \), compensate each other, so that the corresponding magnetic flux be zero.

To visualize the structure of the magnetic lines of force, let us consider the particular case of parallel background fields, \( \mathbf{E} = \mathbf{E} \hat{z} \), \( \mathbf{B} = \mathbf{B} \hat{z} \) (\( |\mathbf{z}| = 1 \)), whose response acquires a simpler form

\[
\mathbf{B}_{\text{in}}^{(1)} (\mathbf{r}) = \frac{\tilde{g}}{4\pi R^3} \frac{1}{5} \left[ \mathbf{r} - 3 (\hat{z} \cdot \mathbf{r}) \hat{z} \right] ,
\]

(35)

and

\[
\mathbf{B}_{\text{out}}^{(1)} (\mathbf{r}) = \frac{\tilde{g}}{4\pi r^3} \left\{ \left[ 1 - \frac{3R^2}{5r^2} - 3 \left( 1 - \frac{R^2}{r^2} \right) \left( \frac{\hat{z} \cdot \mathbf{r}}{r} \right) \right]^2 \frac{\mathbf{r}}{2}
- \frac{3R^2}{5r^2} (\hat{z} \cdot \mathbf{r}) \hat{z} \right\} .
\]

(36)
Moreover, in the limit \( r/R \to \infty \), we are left with a single magnetic response, exclusively radial, although spherically nonsymmetric, corresponding to that of a pointlike Coulomb source (11)

\[
B_{pl}^{(1)}(r) = \lim_{r/R \to \infty} B^{(1)}(r) = \frac{q}{4\pi} \frac{\hat{g}}{2} \left[ 1 - 3 \left( \frac{\hat{z} \cdot r}{r} \right)^2 \right] \frac{r}{r^3}.
\]  

(37)

The magnetic lines of force are straight lines, vanishing at the angles \( \cos \theta = \zeta = \zeta_0 = 1/\sqrt{3} \). As no net magnetic charge exists for producing a nontrivial magnetic flux (34), there are inward magnetic lines (pointing to \( q \)) and outward magnetic lines (lines leaving \( q \)), in the same proportion (see Fig 4).

![Figure 4: (color online) Magnetic lines of force produced in a constant background by the static charge. The left pattern corresponds to the extended charge concentrated within the dashed circle, and the right pattern corresponds to the point charge (according to Eq. (37)). Magnetic field \( B_{in}^{(1)}(x) \) inside the charge is drawn following Eq. (35), and its outer part \( B_{out}^{(1)}(x) \) following Eq. (36). In the limit \( R \to 0 \), the red/blue arrows in the left pattern tend to straight lines, becoming “inward” and “outward” magnetic lines of force, as depicted in the right pattern. The inclined dashed lines indicate regions of zero magnetic field on the right, whereas, on the left, they only divide “inward” and “outward” magnetic lines of force in the limit \( R \to 0 \).]

The magnetic field found (37) may be understood as two pointlike magnetic poles of equal, but opposite, polarities superposed in one point, so to say, a pointlike magnetic dipole.
3 Vector potentials

In this section, we extend the consideration above to the level of electromagnetic potentials, restricting ourselves to the case of electromagnetic responses generated by a pointlike charge distribution (11) since, as discussed before, the role of the regularization (10) is simply to avoid divergent integrals in the calculation of magnetic responses (25) if a pointlike source is considered from the beginning. In this case, the vector potential is sought in the form

\[ A^{(1)}(r) = [\hat{z} \times r] \frac{A(\zeta, \xi)}{r^2} + [\hat{\nu} \times r] \frac{C(\zeta, \xi)}{r^2} + \frac{[\hat{z} \times \hat{\nu}]}{r} \mathcal{M}, \]

\[ \hat{z} = \frac{\mathbf{B}}{B}, \quad \hat{\nu} = \frac{\mathbf{E}}{E}, \quad \zeta = \frac{\hat{z} \cdot r}{r}, \quad \xi = \frac{\hat{\nu} \cdot r}{r}, \]

where \( \mathcal{M} \) is a constant and \( A(\zeta, \xi), C(\zeta, \xi) \) are functions of the cosines of the angles between directions of \( \mathbf{E}, \mathbf{B} \) and the radius vector \( r \). Although this form may be not the most general one, its use is sufficient for finding at least a certain class of the vector-potentials \( A^{(1)}(r) \), all the variety of other possible values for \( A^{(1)}(r) \) being gauge-equivalent to those found. The representation of the field in terms of the vector potential should be exploited I changed this phrase a little, Shabad:

\[ \mathbf{B}^{(1)}(r) = \nabla \times A^{(1)}(r) = \frac{r}{r^4} [2A(\hat{z} \cdot r) + 2C(\hat{\nu} \cdot r)] - \frac{r}{r^3} \left( \frac{\hat{z} \cdot \hat{\nu} - (\hat{z} \cdot r)(\hat{\nu} \cdot r)}{r^2} \right) \left( \partial_\zeta A + \partial_\xi C \right) \]

\[ - \frac{r}{r^3} \left\{ \left[ 1 - \left( \frac{\hat{z} \cdot r}{r} \right)^2 \right] \partial_\zeta A + \left[ 1 - \left( \frac{\hat{\nu} \cdot r}{r} \right)^2 \right] \partial_\xi C \right\} + \frac{\mathcal{M}}{r^3} [\hat{\nu} (\hat{z} \cdot r) - \hat{z} (\hat{\nu} \cdot r)]. \]  

(39)

Note that only the \( \mathcal{M} \)-term from (38) contributes the \( \hat{z} \) - and \( \hat{\nu} \) - components to (39). The two equations obtained by projecting the equation resulting from equating (28) and (39) onto these directions

\[ -\frac{q}{4\pi} \frac{G_{\phi \phi} + G_{\phi \phi}}{2r^3} \left[ (\mathbf{E} \cdot r) \mathbf{B} - (\mathbf{B} \cdot r) \mathbf{E} \right] = -\frac{\mathcal{M}}{r^2} [\hat{z} (\hat{\nu} \cdot r) - \hat{\nu} (\hat{z} \cdot r)], \]

are both satisfied with the unique choice

\[ \mathcal{M} = \frac{q}{8\pi} (G_{\phi \phi} + G_{\phi \phi}) B E. \]

(40)

---

\(^7\)For the sake of convenience, we remove the subscript “pl” on the magnetic response generated by the pointlike Coulomb distribution (11), given by Eq. (28). Thus \( \mathbf{B}_{pl}^{(1)}(r) \equiv \mathbf{B}^{(1)}(r) \) from now on.
This fact has an important consequence: after projecting (28), (39) onto the r-direction we are left with only one first-order partial differential equation for two functions \( A \) and \( C \)

\[
2 \left( \zeta A + \xi C \right) - \left[ (1 - \zeta^2) \partial_\zeta A + (1 - \xi^2) \partial_\xi C \right] - (\hat{z} \cdot \hat{v} - \zeta \xi) \left( \partial_\zeta A + \partial_\xi C \right) = \frac{q}{4\pi} \frac{g}{2} + \frac{3q}{8\pi} \left[ (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\phi}) B \mathcal{E} \zeta - \mathcal{L}_{\phi\phi} B^2 \zeta^2 + L_{\phi\phi} E^2 \xi^2 \right].
\]

(41)

Therefore, there is an arbitrary number of solutions to Eq. (41). Such arbitrariness corresponds to a certain part of the gauge freedom of vector potentials. It will be sufficient to seek for solutions of Eq. (41) in the following subclass of (38)

\[
A (\zeta, \xi) = \frac{q}{16\pi} (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\phi}) B \mathcal{E} \xi - \frac{q}{8\pi} \mathcal{L}_{\phi\phi} B^2 \zeta + Y (\zeta),
\]

\[
C (\zeta, \xi) = \frac{q}{16\pi} (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\phi}) B \mathcal{E} \zeta + \frac{q}{8\pi} \mathcal{L}_{\phi\phi} E^2 \xi + X (\xi),
\]

(42)

where the functions \( Y (\zeta) \) and \( X (\xi) \) satisfy one and the same differential equation,

\[
Z' (u) - \left( \frac{2u}{1 - u^2} \right) Z (u) = 0,
\]

(43)

in which \( Z = (Y, X) \) and \( u = (\zeta, \xi) \). The latter equation can be readily integrated,

\[
Z (u) = \frac{z^2 - 1}{u^2 - 1} Z (z),
\]

(44)

yielding the final form of the vector potential (38),

\[
A^{(1)} (r) = \left[ \frac{\mathbf{B} \times \mathbf{r}}{r^2} \right] \left[ \frac{q (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\phi}) B \mathcal{E} \cdot \mathbf{r}}{16\pi r} - \frac{q \mathcal{L}_{\phi\phi} B \cdot \mathbf{r}}{8\pi r} + \frac{z^2 - 1}{\zeta^2 - 1} Y (\zeta) \right]
\]

\[
+ \left[ \frac{\mathbf{E} \times \mathbf{r}}{r^2} \right] \left[ \frac{q (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\phi\phi}) E \cdot \mathbf{r}}{16\pi r} + \frac{q \mathcal{L}_{\phi\phi} E \cdot \mathbf{r}}{8\pi r} + \frac{\tilde{z}^2 - 1}{\xi^2 - 1} X (\tilde{z}) \right]
\]

\[
+ \frac{q}{8\pi} (\mathcal{L}_{\phi\phi} + \mathcal{L}_{\phi\phi}) \left[ \frac{\mathbf{B} \times \mathbf{E}}{r^2} \right],
\]

(45)

In Eqs. (44) and (45), \( z, \tilde{z} \in [-1, +1] \) are integration constants or boundary points, through which the boundary conditions \( Y (z) \), \( X (\tilde{z}) \) should be specified. The choice of the latter is a matter of gauge fixing. It should be noted that in the class of functions given by Eqs. (42), the boundary conditions cannot be fixed by imposing the Coulomb gauge, since

\[
\nabla \cdot A^{(1)} (r) = \frac{q (\mathcal{L}_{\phi\phi} + \mathcal{L}_{\phi\phi}) r \cdot [\mathbf{E} \times \mathbf{B}]}{8\pi r^3}.
\]

(46)
The arbitrariness due to different choices of the boundary conditions leaves us within this gauge. Nevertheless, there are two special choices for the integration constants, namely \( z = \tilde{z} = \pm 1 \), that do not require that these boundary conditions be specified. For any other choices of \( z \) and \( \tilde{z} \), the potential is singular along the entire \( z \)- and \( \nu \)-axes, corresponding to the direction of the external fields \( \overline{B}, \overline{E} \), respectively, provided \( Y(z) \) and \( X(\tilde{z}) \) are nontrivial.

Sticking to the choices \( z = \tilde{z} = \pm 1 \), the vector potential (45) acquires the final form

\[
\mathbf{A}^{(1)}(\mathbf{r}) = \frac{q}{16\pi r^2} \left[ \overline{B} \times \mathbf{r} \right] \left[ (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\overline{\phi}\overline{\phi}}) \left( \frac{\mathbf{E} \cdot \mathbf{r}}{r} \right) - 2\mathcal{L}_{\phi\overline{\phi}} \left( \frac{\mathbf{B} \cdot \mathbf{r}}{r} \right) \right] \\
+ \frac{q}{16\pi r^2} \left[ \overline{E} \times \mathbf{r} \right] \left[ (\mathcal{L}_{\phi\phi} - \mathcal{L}_{\overline{\phi}\overline{\phi}}) \left( \frac{\mathbf{B} \cdot \mathbf{r}}{r} \right) + 2\mathcal{L}_{\phi\overline{\phi}} \left( \frac{\mathbf{E} \cdot \mathbf{r}}{r} \right) \right] \\
+ \frac{q}{8\pi} \left( \mathcal{L}_{\overline{\phi}\overline{\phi}} + \mathcal{L}_{\phi\phi} \right) \frac{\left[ \mathbf{B} \times \mathbf{E} \right]}{r^2},
\]

(47)

which, in particular, is free of “string” singularities along directions of the external fields, specified by \( \xi, \zeta = \pm 1 \). For parallel backgrounds, the cosines \( \zeta, \xi \) coincide and so do the functions \( X(\xi), Y(\zeta) \). In this case, Eq. (47) reduces to

\[
\mathbf{A}^{(1)}(\mathbf{r}) = -\frac{q}{4\pi} \left[ \hat{z} \times \mathbf{r} \right] \frac{\hat{g}}{2} \zeta,
\]

(48)

Note that in this case all the potentials, to which the ansatz (39) reduces, obey the Coulomb gauge condition \( \nabla \cdot \mathbf{A}^{(1)}(\mathbf{r}) = 0 \).

4 Results in QED

To visualize how the magnetic responses and related effects, valid for any local nonlinear theory, depend on the constant background, we apply the former results to a specific theory, whose nonlinearity is provided by the local approximation of the effective Lagrangian of QED found within one-fermion-loop calculation by Heisenberg and Euler [3] (see e.g. [27])

\[
\mathcal{L} = \frac{M^4}{8\pi^2} \int_0^\infty dt \frac{e^{-t}}{t^3} \left\{ - (ta \cot ta) (tb \coth tb) + 1 - \frac{1}{3} (a^2 - b^2) t^2 \right\},
\]

(49)

where the integration contour is meant to circumvent the poles on the real axis of \( t \) supplied by \( \cot ta \) above the real axis. Here \( a \) and \( b \) are dimensionless combinations of the field invariants,

\[
a = \frac{e}{M^2} \sqrt{-\overline{\delta} + \sqrt{\overline{\delta}^2 + \overline{\Theta}^2}}, \\
b = \frac{e}{M^2} \sqrt{\overline{\delta} + \sqrt{\overline{\delta}^2 + \overline{\Theta}^2}},
\]

(50)
and have the meaning of the electric and magnetic field in the Lorentz frame where these are parallel, normalized to the characteristic field value \( M^2/e \), where \( M \) and \( e \) are the electron mass and charge, respectively. As it is well known, such a frame always exists when \( \mathbf{E} \neq 0 \).

We are primarily interested in strong magnetic-dominated backgrounds, in which vacuum polarization effects overcome vacuum instability ones. In such backgrounds, the electric contribution is sufficiently small in comparison to the magnetic part

\[
\frac{a}{b} \ll 1, \tag{51}
\]

irrespective of whether \( a \) and \( b \) are small or not as compared to the unity, implying the magnetic dominance \( \mathbf{B} \gg \mathbf{E} \) in any reference frame. Such condition is enough to probe vacuum nonlinear effects\(^8\), and should be applied in final expressions after all coefficients composing the magnetic responses (derivatives of the effective Lagrangian) have been calculated (with \( a \) and \( b \) arbitrary, i.e. not subjected to condition (51)). Thus, using general expressions for the derivatives of the effective Lagrangian, given by Eqs. (52) - (54) in Ref. [24], the coefficient \( \tilde{g} \) (22) takes the form

\[
\tilde{g} = \frac{M^4}{32\pi^2} \frac{b^2}{b^2 + \overline{\mathbf{E}}^2} \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau^3} \left\{ \mathcal{H}(\tau) \mathcal{Q}(\frac{a\tau}{b}) \left( -4\overline{\mathbf{E}} + 2\overline{\mathbf{E}}^2 \left( \frac{a}{b} - \frac{b}{a} \right) \right) \right.
\]
\[
+ \left. \tau \coth \tau \left[ \left( 2\overline{\mathbf{E}} \gamma_+ - 2\overline{\mathbf{E}}^2 \kappa - \frac{b}{a} \gamma_- \right) \mathcal{Q}(\frac{a\tau}{b}) \right] \right. 
\]
\[
+ \left. \left( \frac{a\tau}{b} \cot \frac{a\tau}{b} \right) \left[ 2 \left( \overline{\mathbf{E}} \gamma_- + \overline{\mathbf{E}}^2 \kappa \right) \mathcal{H}(\tau) \right] \right.
\]
\[
+ \left. \left( \frac{a\tau}{b} \cot \frac{a\tau}{b} \right) \left( 1 - \frac{a^2}{b^2} \right) - 2\overline{\mathbf{E}} \kappa \frac{a}{b} \right) \mathcal{H}(\tau) \right\}, \tag{52}
\]

where \( \mathcal{H}(\tau) \), \( Q(\tau) \), \( \tilde{\mathcal{H}}(\tau) \), \( \tilde{Q}(\tau) \) are auxiliary functions

\[
\mathcal{H}(\tau) = \tau \coth \tau - \frac{\tau^2}{\sinh^2 \tau}, \quad Q(\tau) = \tau \cot \tau - \frac{\tau^2}{\sin^2 \tau},
\]
\[
\tilde{\mathcal{H}}(\tau) = \frac{2\tau^2}{\sinh^2 \tau} (\tau \coth \tau - 1), \quad \tilde{Q}(\tau) = \frac{2\tau^2}{\sin^2 \tau} (\tau \cot \tau - 1); \tag{53}
\]

\( \kappa = \text{sgn} (\overline{\mathbf{E}}) \) and \( \gamma_\pm = 1 \pm \frac{2\overline{\mathbf{E}}}{\sqrt{\overline{\mathbf{E}}^2 + \overline{\mathbf{E}}^2}} \) are constants. Eq. (52) together with Eqs. (52) - (54) from Ref. [24], provide integral representations for all the necessary coefficients within the Euler-Heisenberg nonlinear electrodynamics to be substituted into Eqs. (26)-(29) for specializing the magnetic responses.

\(^8\)Note that the magnetic responses (26), (27) and the nonlinearly induced currents (19), (20), vanish identically in the pure magnetic background \( a/b = 0 \), since \( \mathbf{E} = 0 \).
Let us study the strong magnetic-dominated case, specified by the condition (51). To this end, we rewrite the coefficient (22) as \( \tilde{g} = -\mathcal{G}_- + 2\mathcal{F}_\mathcal{G}_- \), \( \mathcal{L}_- = \mathcal{L}_{\mathcal{G}} - \mathcal{L}_{\mathcal{G}_-} \), and expand trigonometric functions within \( \mathcal{L}_- \), \( \mathcal{L}_{\mathcal{G}} \) in power series of \( a/b \) (avoiding the poles at the real axis in Eq. (52), thereby). The first coefficient \( \mathcal{G}_- \) reads

\[
\mathcal{G}_- = \left( \frac{a}{b} \right) (\mathcal{G}_-)^{(1)} + O \left( (a/b)^3 \right),
\]

\[
(\mathcal{G}_-)^{(1)} = 2\kappa \left( \frac{\alpha}{2\pi} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[ \left( \frac{1}{\tau} - \frac{\tau}{3} \right) \coth \tau - \frac{\tau \coth \tau}{\sinh^2 \tau} \right],
\]

while the second \( \mathcal{F}_{\mathcal{G}_-} \) takes the form,

\[
\mathcal{F}_{\mathcal{G}_-} = \left( \frac{a}{b} \right) (\mathcal{F}_{\mathcal{G}_-})^{(1)} + O \left( (a/b)^3 \right),
\]

\[
(\mathcal{F}_{\mathcal{G}_-})^{(1)} = \kappa \left( \frac{\alpha}{2\pi} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[ \left( \frac{3}{\tau} - \frac{2\tau}{3} \right) \coth \tau - \left( 1 + \frac{2\tau^2}{3} \right) \frac{1}{\sinh^2 \tau} - 2\tau \coth \tau \sinh^2 \tau \right].
\]

As a result, the leading-order contribution for \( \tilde{g} \) is expressed as

\[
\tilde{g} = \left( \frac{a}{b} \right) \tilde{g}^{(1)} + O \left( (a/b)^3 \right),
\]

\[
\tilde{g}^{(1)} = 2\kappa \left( \frac{\alpha}{2\pi} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[ \left( \frac{2}{\tau} - \frac{\tau}{3} \right) \coth \tau - \left( 1 + \frac{2\tau^2}{3} \right) \frac{1}{\sinh^2 \tau} - \tau \coth \tau \sinh^2 \tau \right].
\]

To estimate the asymptotic behavior of \( \tilde{g} \) (56) in the infinite magnetic field limit \( b \to \infty \), it is convenient to express all integrals above in terms of the Hurwitz Zeta function \( \zeta (z, a) \), DiGamma function \( \psi (z) = \Gamma' (z)/\Gamma (z) \) and related functions [29]. Using Zeta-function regularization techniques (see e. g., [30, 31, 32]), the expansion coefficient is expressed as follows

\[
\tilde{g}^{(1)} = 2\kappa \left( \frac{\alpha}{2\pi} \right) \left\{ \frac{b}{3} + \frac{1}{6} - 12\zeta'\left( -1, \frac{1}{2b} \right) + \psi \left( \frac{1}{2b} \right) + \frac{1}{2b^2} \left[ \psi \left( \frac{1}{2b} \right) + \frac{1}{2} \right] \right. \\
\left. + \frac{1}{b} \left[ 2\log 2b - \log 2\pi + \frac{1}{2} + \frac{1}{3} \psi^{(1)} \left( \frac{1}{2b} \right) + 2\log \Gamma \left( \frac{1}{2b} \right) \right] \right\},
\]

where \( \zeta'(z, a) \) is the derivative of the Hurwitz Zeta function with respect to \( z \), \( \psi^{(j)} (z) \) is the \( j \)-th derivative of the DiGamma function and \( \gamma \approx 0.577 \) is the Euler constant [29]. Using asymptotic representations of special functions (74), (75), discussed in Appendix B, the above coefficient behaves as

\[
\tilde{g}^{(1)} \sim 2\kappa \left( \frac{\alpha}{2\pi} \right) \left( -\frac{b}{3} + K_g^{(1)} \right), \quad b \to \infty,
\]

\[
K_g^{(1)} = \frac{1}{6} - \gamma - 12\zeta'(-1),
\]
in the large-field limit. Therefore it is easily seen that the leading-order contribution to $\tilde{g}$

$$\tilde{g} \sim 2\kappa \left( \frac{\alpha}{2\pi} \left( \frac{a}{3} + \frac{a}{b}\kappa^{(1)} \right) \right), \quad b \to \infty.$$  \hspace{1cm} (59)

is proportional to the electric part $a$. The pseudoscalar quality to $\tilde{g}$ is imparted by the factor $\kappa = \text{sgn} \, \tilde{\mathcal{G}}$, because $\tilde{\mathcal{G}}$ changes its sign under spacial reflection.

5 Conclusions

Within a nonlinear local electrodynamics (6), we have obtained magnetic fields created by a static electric charge $q$ placed in a background of arbitrarily strong constant electric, $\mathbf{E}$, and magnetic, $\mathbf{B}$, fields by solving (the second pair of) the Maxwell equations (7) linearized near the background and treated in the approximation of small nonlinearity. All our formulas contain coefficients that are derivatives of the nonlinear part of the Lagrangian (6), where the background values of the fields are meant to be substituted after the derivatives have been calculated. These coefficients are related to dielectric permeability and magnetic permittivity of the equivalent “medium” formed by the background fields in the vacuum [28].

Before considering the necessary magnetic fields we establish the character of their source, which comprises of the currents induced in the equivalent “medium” by the static charge. The result for the current inside and outside of the charge is given by Eqs. (19), (20). The flow of this current (21) for the special case of parallel background fields is shown in Fig. 3. There is no induced current inside the charge in this special case.

The magnetic response $\mathbf{B}^{(1)}(\mathbf{r})$ to an introduced small extended electric charge (10) homogeneously distributed over a sphere of the radius $R$ is given by Eq. (26) inside the charged sphere and by Eqs. (28), (29) outside it. The case of the pointlike charge (11) is covered by the $R \to 0$ limit, Eq. (28). In the simplified case of parallel background fields $\mathbf{B} \parallel \mathbf{E}$, the magnetic response is given by Eqs. (35), (36) inside and outside the extended charge, respectively, and by Eq. (37) for the point charge. The pattern of magnetic lines of force is presented in Fig. 4. To complete the study, vector potentials associated with magnetic responses generated by a pointlike Coulomb source were calculated in Sec. 3.

Adjusting the present results with the realistic situation where the nonlinearity of the Maxwell equations is owing to nonlinearity stemming from the quantum interaction between electromagnetic fields inherent to QED, we give integral representations for all the nonlinearity coefficients, in terms of which our results for the fields are expressed, as they follow from appropriate differentiations with respect to the field invariants of the effective Lagrangian of QED in its local approximation taken as the Euler-Heisenberg (one-loop) effective Lagrangian. We consider the asymptotic regime when the magnetic background dominates over the electric one. We found that in that regime the
above integrals are conveniently expressed in terms of the Hurwitz Zeta function. The resulting formula for \( \tilde{g} \) (22), a common coefficient in magnetic responses, linearly induced current densities and vector potentials, is linear in the background electric field.

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A Projection operator

In this Appendix we present further details concerning the derivation of the first-order linear magnetic responses \( B^{(1)} (r) \) to the static charge in consideration, given by Eqs. (26) and (27). Referring to Eqs. (14) and (12), the integral in (25) can be expressed in terms of auxiliary integrals \( \mathcal{I}^j (r) \),

\[
\mathcal{I}^j (r) = w (r) x^j, \quad w (r) = \frac{1}{r^2} \int \frac{d y \Phi (y) (y \cdot r)}{|r - y|}, \\
\Phi (y) = \frac{\theta (R - y)}{R^3} + \frac{\theta (y - R)}{y^3}, \quad r = |r|, \quad y = |y|,
\]

in which the function \( \Phi (r) \), stemming from the regularized Coulomb field \( E^{(0)} (r) \) (12), provides magnetic responses over all points of the space. From the latter, the integral in the rhs. of Eq. (25) is expressed as

\[
\int \frac{d y \mathbf{S}^{(0)j} (y)}{|r - y|} = - \frac{q}{4\pi} \mathbf{\mathcal{I}}^j (r) - \frac{q}{4\pi} \mathcal{S}^{jk} \mathcal{I}^k (r), \\
\mathcal{S}^{jk} = \left( \mathbf{\mathcal{L}}_{\mathbf{\mathcal{S}}} \mathcal{B}^j - \mathbf{\mathcal{L}}_{\mathbf{\mathcal{S}}} \mathcal{E}^j \right) \mathcal{E}^k + \left( \mathbf{\mathcal{L}}_{\mathbf{\mathcal{S}}} \mathcal{B}^j - \mathbf{\mathcal{L}}_{\mathbf{\mathcal{S}}} \mathcal{E}^j \right) \mathcal{B}^k.
\]
Next, one can evaluate the scalar integral \( w(r) \) by selecting a system of reference in which \( r \) is aligned along the \( z \)-direction, such that \( y \cdot r = yr \cos \theta \) and it is reduced to a simple radial integral

\[
 w(r) = \frac{2\pi}{r} \int_0^\infty dy y^3 \Phi(y) \int_{-1}^1 d(cos \theta) \frac{\cos \theta}{\sqrt{r^2 + y^2 - 2ry \cos \theta}} = \frac{2\pi}{3r^3} \int_0^\infty dy \Phi(y) \left\{ (r^2 - ry + y^2) (r + y) - (r^2 + ry + y^2) |r - y| \right\} .
\]

Computing the remaining integrals above, the auxiliary integrals (60) acquires the final form

\[
\mathcal{I}^j(r) = 2\pi \varrho(r) x^j, \quad \varrho(r) = \varrho_{in}(r) \theta(R - r) + \varrho_{out}(r) \theta(r - R) ,
\]

\[
\varrho_{in}(r) = \frac{1}{R} \left( 1 - \frac{r^2}{5R^2} \right) , \quad \varrho_{out}(r) = \frac{1}{r} \left( 1 - \frac{R^2}{5r^2} \right) . \tag{62}
\]

From the definition of \( \mathcal{E}_B^{jk} \) (61), one may use the following set of identities

\[
\mathcal{E}_B^{jk}(x^i) = \mathcal{E}_B^{ij}(x^k) \delta^{jk} + \mathcal{E}_B^{ij}(x^k) \delta^{ik} + \mathcal{E}_B^{ik}(x^j) \delta^{jk} = \mathcal{E}_B^{ij}(x^k) \delta^{jk} , \quad \mathcal{E}_B^{jk}(x^i) = \mathcal{E}_B^{ij}(x^k) \delta^{jk} , \quad \mathcal{E}_B^{jk}(x^i) = \mathcal{E}_B^{ij}(x^k) \delta^{jk} ,
\]

\[
U_E = (\mathcal{E}_{\delta\delta} - \mathcal{L}_{\delta\delta})(\mathbf{E} \cdot r) - 2\mathcal{L}_{\delta\delta}(\mathbf{E} \cdot r) , \quad U_B = (\mathcal{E}_{\delta\delta} - \mathcal{L}_{\delta\delta})(\mathbf{E} \cdot r) + 2\mathcal{L}_{\delta\delta}(\mathbf{E} \cdot r) . \tag{63}
\]

to learn that the action of partial derivatives on Eq. (61) takes the final form

\[
\frac{\partial_i \partial_j}{4\pi} \int dy \frac{\tilde{f}^{(ij)}(y)}{|r - y|} = \left( \frac{\mathcal{L}_{\phi}}{3} - \frac{qU_E \rho'(r)}{8\pi r} \right) \mathcal{E}^i - \left( \frac{\mathcal{L}_{\tilde{\delta}}}{3} + \frac{qU_B \rho'(r)}{8\pi r} \right) \mathcal{B}^j - \frac{q}{8\pi} \left[ \tilde{g} \rho'(r) r + \mathcal{L}_{\phi} \left( \frac{4\rho'(r)}{r} + \rho''(r) \right) + \left( \rho''(r) - \frac{\rho'(r)}{r} \right) \frac{\mathcal{E}_B^{jk} x^k}{r^2} \right] \Phi \tag{64}
\]

It should be noted that the inner \( \varrho_{in}(r) \) and the outer \( \varrho_{out}(r) \) components of \( \varrho(r) \), defined in Eq. (62), are continuous at \( r = R \) (as well as its first and second derivatives). For these reasons, coefficients proportional to Dirac delta functions, stemming from the differentiation of Heaviside step functions, vanishes everywhere, including at \( r = R \). Accordingly, derivatives of \( \varrho(r) \) can be treated as \( \rho'(r) = \rho_{in}(r) \theta(R - r) + \rho_{out}(r) \theta(r - R) \) and \( \rho''(r) = \rho''_{in}(r) \theta(R - r) + \rho''_{out}(r) \theta(r - R) \).
B Expansion coefficients

In this Appendix we present exact expressions for the expansion coefficients of the derivatives of
the Heisenberg-Euler effective Lagrangian (49) in the strong magnetic-dominated case, discussed in
Sec. 4, in terms of the Hurwitz Zeta and related functions. Moreover, we supplement some of our
previous results concerning the electric response to the charge distribution (10) by the background
in consideration, namely, the nonlinearly induced total charge inside the distribution $Q$ and the
coefficient $\tilde{b}$, both given by Eqs. (32) and (33) in Ref. [24], respectively. Using formulae below, we
write the leading and next-to-leading expansion coefficients for these quantities.

Starting with parity-even coefficients $S_e = \{S_\delta, S_{\delta\delta}, S_{\delta\delta}\}$, which admits expansions of the form

$$ S_e = S_e^{(0)} + \left(\frac{a}{b}\right)^2 S_e^{(2)} + O\left(\left(\frac{a}{b}\right)^4\right), $$

(65)

the corresponding leading and next-to-leading contributions have the form:

$$ S_e^{(0)} = \frac{\alpha}{2\pi} \left\{ \frac{1}{2b^2} - \frac{1}{3} + \frac{2}{3} \log 2 + \frac{2}{3} \log b + \frac{1}{b} \log \left(\frac{\pi}{b}\right) \right. $$

$$ - \frac{2}{b} \log \Gamma \left(\frac{1}{2b}\right) + 8 \zeta' \left(-1, \frac{1}{2b}\right) \right\}, $$

(66)

$$ S_e^{(2)} = \frac{\alpha}{2\pi} \left\{ \frac{1}{3} + \frac{1}{2b^2} + \frac{1}{b} \log \left(\frac{b}{\pi}\right) \right. + \frac{2}{b} \log \Gamma \left(\frac{1}{2b}\right) + \frac{2}{3} \psi \left(\frac{1}{2b}\right) $$

$$ + \frac{1}{6b} \psi^{(1)} \left(\frac{1}{2b}\right) - 8 \zeta' \left(-1, \frac{1}{2b}\right) \right\}, $$
and

\[
(\bar{\mathcal{S}} + \sqrt{\bar{\mathcal{S}}^2 + \mathcal{G}^2}) \mathcal{L}^{(0)}_{\bar{\mathcal{S}\mathcal{G}}} = \frac{\alpha}{2\pi} \left\{ \frac{2}{3} \left[ 2 \frac{1}{b^2} + 1 \frac{1}{b} + 1 \log \frac{4b}{\pi} - \frac{2}{b} \log \Gamma \left( \frac{1}{2b} \right) + \frac{1}{b^2} \psi \left( \frac{1}{2b} \right) \right] \right. \\
+ \frac{1}{b} \left[ -2 - 7 \frac{\psi(1)}{b} \right] - 6 \log 2b + 2 \log 2\pi - 4 \log \Gamma \left( \frac{1}{2b} \right) \\
+ \frac{1}{b^2} \left[ \frac{2}{6} \zeta \left( 3, \frac{1}{2b} \right) - 2 \psi \left( \frac{1}{2b} \right) \right] \right\} \\
(\bar{\mathcal{S}} + \sqrt{\bar{\mathcal{S}}^2 + \mathcal{G}^2}) \mathcal{L}^{(2)}_{\bar{\mathcal{S}\mathcal{G}}} = \frac{\alpha}{2\pi} \left\{ \frac{2}{3} \left[ -2 - 7 \frac{\psi(1)}{b} \right] - 6 \log 2b + 2 \log 2\pi - 4 \log \Gamma \left( \frac{1}{2b} \right) \\
+ \frac{1}{b^2} \left[ \frac{2}{6} \zeta \left( 3, \frac{1}{2b} \right) - 2 \psi \left( \frac{1}{2b} \right) \right] \right\} \\
(\bar{\mathcal{S}} + \sqrt{\bar{\mathcal{S}}^2 + \mathcal{G}^2}) \mathcal{L}^{(2)}_{\mathcal{G}\mathcal{G}} = \frac{\alpha}{2\pi} \left\{ \frac{2}{3} \left[ -2 - 7 \frac{\psi(1)}{b} \right] - 6 \log 2b + 2 \log 2\pi - 4 \log \Gamma \left( \frac{1}{2b} \right) \\
+ \frac{1}{b^2} \left[ \frac{2}{6} \zeta \left( 3, \frac{1}{2b} \right) - 2 \psi \left( \frac{1}{2b} \right) \right] \right\} .
\]

The parity-even combination \(\bar{\mathcal{S}}\mathcal{L}_{\bar{\mathcal{S}\mathcal{G}}}\) has a trivial zero-order term

\[
\bar{\mathcal{S}}\mathcal{L}_{\bar{\mathcal{S}\mathcal{G}}} = \left( \frac{a}{b} \right)^2 \left( \bar{\mathcal{S}}\mathcal{L}_{\bar{\mathcal{S}\mathcal{G}}} \right)^{(2)} + O \left( \left( \frac{a}{b} \right)^4 \right),
\]

\[
(\bar{\mathcal{S}}\mathcal{L}_{\bar{\mathcal{S}\mathcal{G}}} )^{(2)} = \frac{\alpha}{2\pi} \left\{ \frac{2}{3} \left[ 2 \frac{1}{b^2} + 1 \frac{1}{b} + 1 \log \frac{4b}{\pi} - \frac{2}{b} \log \Gamma \left( \frac{1}{2b} \right) + \frac{1}{b^2} \psi \left( \frac{1}{2b} \right) \right] \right. \\
+ \frac{1}{b^2} \left[ 1 + \frac{1}{3} \psi(1) \left( \frac{1}{2b} \right) + 3 \log 2b - \log 2\pi + 2 \log \Gamma \left( \frac{1}{2b} \right) \right] \right\} .
\]

because it must vanish identically if the electric background is zero. The proper-time representations to all these coefficients are given by Eqs. (57), (58) in Ref. [24]. For the sake of completeness, we also include the parity-odd coefficient \(\mathcal{L}_{\mathcal{G}}\)

\[
\mathcal{L}_{\mathcal{G}} = -\frac{M^4}{16\pi^2} \frac{\kappa b^2}{\sqrt{\bar{\mathcal{S}}^2 + \mathcal{G}^2}} \left( \frac{a}{b} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau^3} \left\{ \left( \frac{b}{a} \right)^2 \tau \coth \tau \right\} Q \left( \frac{a \tau}{b} \right) \\
+ \left( \frac{a \tau}{b} \cot \frac{a \tau}{b} \right) \mathcal{H}(\tau) .
\]
which it is expanded as

\[ \mathcal{L}_\phi = \left( \frac{a}{b} \right) \mathcal{L}_\phi^{(1)} + O \left( \left( \frac{a}{b} \right)^3 \right) , \]

\[ \mathcal{L}_\phi^{(1)} = -\kappa \left( \frac{\alpha}{2\pi} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[ \left( \frac{1}{\tau} - \frac{2}{3\tau} \right) \coth \tau - \frac{1}{\sinh^2 \tau} \right] \]

\[ = -\kappa \left( \frac{\alpha}{2\pi} \right) \left\{ \frac{2b}{3} + \frac{1}{3} + \frac{2}{3} \psi \left( \frac{1}{2b} \right) - 8\zeta' \left( -1, \frac{1}{2b} \right) \right\} + \frac{1}{b} \left[ \log \frac{b}{\pi} + 2 \log \Gamma \left( \frac{1}{2b} \right) \right] \]

\[ + \frac{1}{b} \left[ \log \frac{b}{\pi} + 2 \log \Gamma \left( \frac{1}{2b} \right) \right] \]

(70)

With the help of these coefficients, the nonlinearly induced electric charge \( Q \) and the coefficient \( \tilde{b} \), corresponding to the electric response of the background to the charge (10),

\[ Q = q \left( \mathcal{L}_\delta + \frac{\tilde{b}}{3} \right) , \quad \tilde{b} = -\left( \mathcal{L}_\delta \mathcal{E}^2 + \mathcal{L}_\phi \mathcal{B}^2 - 2\mathcal{L}_\phi \mathcal{L}_\delta \right) , \]

are expanded as follows:

\[ \tilde{b} = \tilde{b}^{(0)} + \left( \frac{a}{b} \right)^2 \tilde{b}^{(2)} + O \left( \left( \frac{a}{b} \right)^4 \right) , \]

\[ \tilde{b}^{(0)} = \sqrt{\mathcal{L}_\phi^{(0)}} - \sqrt{\mathcal{L}_\phi^{(0)}} \mathcal{L}_\phi^{(0)} , \quad \tilde{b}^{(2)} = \sqrt{\mathcal{L}_\phi^{(2)}} - \sqrt{\mathcal{L}_\phi^{(2)}} \mathcal{L}_\phi^{(2)} + 2\mathcal{L}_\phi \mathcal{L}_\phi^{(2)} , \]

\[ \tilde{b}^{(0)} = \frac{\alpha}{2\pi} \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left[ \left( \frac{1}{\tau} - \frac{2\tau}{3} \right) \coth \tau - \frac{1}{\sinh^2 \tau} \right] \]

\[ = \frac{\alpha}{2\pi} \left\{ \frac{2b}{3} + \frac{1}{3} + \frac{2}{3} \psi \left( \frac{1}{2b} \right) - 8\zeta' \left( -1, \frac{1}{2b} \right) \right\} + \frac{1}{b} \left[ \log \frac{b}{\pi} + 2 \log \Gamma \left( \frac{1}{2b} \right) \right] \]

\[ + \frac{1}{b} \left[ \log \frac{b}{\pi} + 2 \log \Gamma \left( \frac{1}{2b} \right) \right] \]

(72)
where \( \mathfrak{L}_\pm = \mathfrak{L}_\mp \pm \mathfrak{L}_\circ \) and

\[
Q = Q^{(0)} + \left( \frac{a}{b} \right)^2 Q^{(2)} + O \left( \left( \frac{a}{b} \right)^4 \right), \quad Q^{(i)} = q \left( \mathfrak{L}^{(i)}_\delta + \bar{d}^{(i)}_\delta \right),
\]

\[
Q^{(0)} = q \left( \frac{\alpha}{3\pi} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left\{ 1 - \left( \frac{1}{\tau} + \frac{\tau}{3} \right) \coth \tau + \frac{1}{\sinh^2 \tau} \right\},
\]

\[
= q \left( \frac{\alpha}{3\pi} \right) \left\{ \frac{b}{3} + \log b - \frac{1}{3} + \log 2 + \frac{1}{3} \psi \left( \frac{1}{2b} \right) + 8 \zeta' \left( -1, \frac{1}{2b} \right) \right\},
\]

\[
+ \frac{1}{b} \left\{ \log \left( \frac{\pi}{b} \right) - 2 \log \Gamma \left( \frac{1}{2b} \right) - \frac{1}{2b^2} \right\},
\]

\[
Q^{(2)} = q \left( \frac{\alpha}{3\pi} \right) \int_0^\infty d\tau \frac{e^{-\tau/b}}{\tau} \left\{ \frac{1}{\tau} - \frac{\tau}{3} - \frac{2\tau^3}{15} \right\} \coth \tau - \left( 1 + \frac{\tau^2}{3} \right) \frac{1}{\sinh^2 \tau},
\]

\[
= q \left( \frac{\alpha}{3\pi} \right) \left\{ \frac{4b^3}{15} + \frac{b}{3} + \frac{2}{3} \psi \left( \frac{1}{2b} \right) - 8 \zeta' \left( -1, \frac{1}{2b} \right) - \frac{1}{15} \zeta \left( 3, \frac{1}{2b} \right) \right\},
\]

\[
+ \frac{1}{b} \left\{ \log \left( \frac{\pi}{b} \right) + \frac{1}{6} \psi^{(1)} \left( \frac{1}{2b} \right) + 2 \log \Gamma \left( \frac{1}{2b} \right) + \frac{1}{2b^2} \right\}. \tag{73}
\]

The representations above are useful to study the asymptotic regime for large \( b \). For example, using the expansions

\[
\log \Gamma \left( \frac{1}{2b} \right) = \log b - \frac{\gamma}{2b} + \log 2 + O \left( \left( \frac{1}{2b} \right)^2 \right),
\]

\[
\psi \left( \frac{1}{2b} \right) = -2b - \gamma + \frac{\pi^2}{12b} + O \left( \left( \frac{1}{2b} \right)^2 \right),
\]

\[
\psi^{(1)} \left( \frac{1}{2b} \right) = 4b^2 + \frac{\pi^2}{6} + \frac{\psi^{(2)} \left( 1 \right)}{2b} + O \left( \left( \frac{1}{2b} \right)^2 \right),
\]

\[
\zeta' \left( -1, \frac{1}{2b} \right) = \zeta' \left( -1 \right) - \frac{1}{4b} \log 2\pi - \frac{1}{4b} \left( 1 - \frac{1}{2b} \right) + \int_0^{1/2b} dx \log \Gamma \left( x \right),
\]

\[
= \zeta' \left( -1 \right) - \frac{1}{4b} \log 2\pi + \frac{1}{4b} + \frac{1}{2b} \log 2b + O \left( \left( \frac{1}{2b} \right)^2 \right). \tag{74}
\]
the asymptotic behaviour large $b$ limit of Eqs. (66) - (68) read

\begin{align}
\mathcal{L}^{(0)} &\sim \frac{\alpha}{2\pi} \left( \frac{2}{3} \log b + \mathcal{K}^{(0)}_\delta \right), \quad \mathcal{L}^{(2)} \sim \frac{\alpha}{2\pi} \left( -\frac{b}{3} + \mathcal{K}^{(2)}_\delta \right), \\
\mathcal{L}^{(1)}_\phi &\sim -\kappa \frac{\alpha}{2\pi} \left( -\frac{2}{3} b + \mathcal{K}^{(1)}_\phi \right), \quad \left( \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(0)} \sim \frac{\alpha}{3\pi}, \\
\left( \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(2)}_\phi &\sim \frac{\alpha}{2\pi} \left( b + \mathcal{K}^{(2)}_\phi \right), \\
\left( \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(0)}_\phi &\sim \alpha \frac{2b}{3} + \mathcal{K}^{(0)}_\phi, \quad \left( \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \right) \mathcal{L}^{(2)}_\phi \sim \frac{\alpha}{2\pi} \left( -\frac{2}{3} b + \mathcal{K}^{(2)}_\phi \right), \\
\left( \mathcal{F}^2 + \mathcal{G}^2 \right) \mathcal{L}^{(0)}_G &\sim \frac{\alpha}{2\pi} \left( \frac{8b^3}{15} - \frac{5b}{3} + \mathcal{K}^{(2)}_G \right), \quad (75)
\end{align}

as $b \to \infty$. The $\mathcal{K}$’s are constants

\begin{align*}
\mathcal{K}^{(0)}_\delta &= -\frac{1}{3} + \frac{2}{3} \log 2 + 8\zeta'(-1), \quad \mathcal{K}^{(2)}_\delta = -\frac{2}{3} \gamma - 8\zeta'(-1), \\
\mathcal{K}^{(1)}_\phi &= \frac{1}{3} \left( 1 - 2\gamma \right) - 8\zeta'(-1), \quad \mathcal{K}^{(2)}_\phi = -\frac{2}{3} + \frac{8}{3} \gamma + 8\zeta'(-1), \\
\mathcal{K}^{(0)}_\phi &= -\frac{1}{3} + \frac{2}{3} \gamma + 8\zeta'(-1), \quad \mathcal{K}^{(2)}_\phi = \frac{2}{3} - 40\zeta'(-1) - \frac{4}{15} \psi^{(2)}(1) - \frac{10}{3} \gamma, \\
\mathcal{K}^{(2)}_G &= \frac{2}{3} - \frac{4}{3} \gamma - 16\zeta'(-1),
\end{align*}

and $\gamma \simeq 0.577$ is the Euler constant.

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