The Loewner Energy via Moving Frames and Surfaces of Finite Renormalised Area Bounding Weil-Petersson Curves

Alexis Michelat∗ Yilin Wang†

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Abstract

We obtain a new formula for the Loewner energy of simple curves of the sphere as the renormalised energy of moving frames of the two domains of the sphere delimited by the given curve. We also construct examples of surfaces of finite Willmore energy (or renormalised area) in $\mathbb{H}^3$ bounding (a sub-class of) curves of finite Loewner energy using equipotential lines.

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1 Introduction

1.1 Background on the Weil-Petersson Class

In [53, 43], the second author and Steffen Rohde introduced the Möbius-invariant Loewner energy to measure the roundness of simple closed curves on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ using the Loewner transform [34]. In this article, we will also view simple close curves as curves on $S^2 \subset \mathbb{R}^3$ and give new characterisations of the Loewner energy in terms of the moving frames on $S^2$. Note that in this article,

∗Mathematical Institute, University of Oxford (UK) michelat@maths.ox.ac.uk/ alexis.michelat@normalesup.org
†Massachusetts Institute of Technology. yilwang@mit.edu
S² refers to the sphere of radius 1 centered at the origin in \( \mathbb{R}^3 \) equipped with the induced round metric \( g_0 \) from its embedding into \( \mathbb{R}^3 \). Therefore, \( S^2 \) is isometric to \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) endowed with the metric
\[
g_{\hat{\mathbb{C}}} = \frac{4|dz|^2}{(1 + |z|^2)^2}
\]
by the stereographic projection. To distinguish the two setups, we will let \( \gamma \) denote a closed curve in \( \mathbb{C} \cup \{ \infty \} \) and let \( \Gamma \) denote a closed curve in \( S^2 \).

The motivation of introducing Loewner energy comes from the probabilistic theory of Schramm-Loewner evolutions. However, it was proved in [54] that the class of curves of finite energy corresponds exactly to the Weil-Petersson class of quasi-circles which has already been studied extensively by both physicists and mathematicians since the eighties, see, e.g., [9, 55, 37, 36, 44, 14, 20, 45, 22, 46, 23, 50, 51, 30], and is still a very active research area. See the introduction of [5] (see also [4] and [8] for earlier versions of Bishop’s work, and the companion papers [6] and [7] for more on this topic) for a summary and a list of currently more than twenty equivalent definitions of very different nature. Let us give more details on this topic and list first a few equivalent definitions which are more classical and relevant to this work.

**Theorem 1.1** (Cui, [14], Takhtajan-Teo, Theorem 1.12 [20], Shen, [46]). Let \( \gamma \) be a simple closed curve, \( \Omega \) be the bounded connected component of \( \mathbb{C} \setminus \gamma \), and let \( f : \mathbb{D} \to \Omega \) and \( g : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \Omega \) be biholomorphic maps such that \( g(\infty) = \infty \). The following conditions are equivalent:

1. There exists a quasiconformal extension of \( g \) to \( \hat{\mathbb{C}} \) such that the Beltrami coefficient \( \mu = \frac{\partial g}{\partial z} : \mathbb{D} \to \mathbb{D} \) of \( g|_\mathbb{D} \) satisfies

\[
\int_{\mathbb{D}} |\mu(z)|^2 \frac{|dz|^2}{(1 - |z|^2)^2} < \infty.
\]

2. \( \int_{\mathbb{D}} |\nabla \log |f'(z)||^2|dz|^2 = \int_{\mathbb{D}} \frac{|f''(z)|^2}{f'(z)} |dz|^2 < \infty. \)

3. \( \int_{\mathbb{D}} |S(f)(z)|^2(1 - |z|^2)^2|dz|^2 < \infty. \)

4. \( \int_{\mathbb{D} \setminus \hat{\mathbb{D}}} \frac{|g''(z)|^2}{g'(z)} |dz|^2 < \infty. \)

5. \( \int_{\mathbb{D} \setminus \hat{\mathbb{D}}} |S(g)|^2(1 - |z|^2)^2|dz|^2 < \infty; \)

6. the (conformal) welding function \( \varphi = g^{-1} \circ f|_\mathbb{D} \), satisfies \( \log |\varphi'| \) belongs to the Sobolev space \( H^{\frac{1}{2}}(S^1) \).

If \( \gamma \) satisfies any of those conditions, \( \gamma \) is called a Weil-Petersson quasi-circle.

In (3), \( S(f) \) is the Schwarzian derivative
\[
S(f)(z) = \frac{f'''(z)}{f'(z)} - 3 \left( \frac{f''(z)}{f'(z)} \right)^2, \quad z \in \mathbb{D}.
\]

The equivalences (1), (2) and (3) are due to Cui, and independently of Takhtajan and Teo who proved the equivalences (1), (2), (3), (4), (5). In (6), the continuous extension of \( f, g \) to \( S^1 \) is well-defined by a classical theorem of Carathéodory [12]. The equivalence between (6) and other conditions is proved by Shen. The second condition is perhaps the simplest one since it corresponds to the condition \( \log |\varphi'| \in W^{1,2}(\mathbb{D}) \), the Sobolev space of functions with squared-integrable weak derivatives.

After appropriate normalisation, the space of Weil-Petersson quasi-circles is identified with the homogeneous space \( T_0(1) = \text{Möb}(S^1) \setminus \text{WP}(S^1) \) via the conformal welding as in (6), where \( \text{WP}(S^1) \) denotes the group of welding functions of Weil-Petersson quasi-circles and \( \text{Möb}(S^1) \simeq \text{PSU}(1,1) \simeq \text{PSL}(2, \mathbb{R}) \) is the subgroup of Möbius map preserving \( S^1 \). The space \( T_0(1) \) is contained in the universal Teichmüller space \( T(1) = \text{Möb}(S^1) \setminus \text{QS}(S^1) \), where \( \text{QS}(S^1) \) is the group of quasisymmetric homeomorphisms that are exactly the welding functions of quasi-circles. Takhtajan and Teo showed that \( T_0(1) \) has a remarkable structure of an infinite-dimensional homogeneous Kähler-Einstein manifold. Furthermore, the Kähler metric is unique up to a constant factor and reminiscent to the Weil-Petersson metric on Teichmüller
spaces of hyperbolic surfaces. Therefore, this Kähler metric is also called Weil-Petersson metric and \( T_0(1) \) is called Weil-Petersson Teichmüller space. Furthermore, they introduced the universal Liouville action \( S_1 \) and showed that it is a Kähler potential on \( T_0(1) \) of critical importance for the Kähler geometry. We take an analytic instead of a Teichmüller theoretic viewpoint, so we will consider \( S_1 \) as defined for Weil-Petersson quasi-circles instead of their welding functions. Explicitly, for every Weil-Petersson quasi-circle \( \gamma \),

\[
S_1(\gamma) = \int_{\Gamma} \left| f'(z) \right|^2 |dz|^2 + \int_{\Gamma} \left| g'(z) \right|^2 |dz|^2 + 4\pi \log |f'(0)| - 4\pi \log |g'(\infty)|. \tag{1.1}
\]

**Theorem 1.1.** This expression is crucial in the proof of our main theorem and also of particular interest \( \gamma \). We take an analytic instead of a Teichmüller theoretic viewpoint, so we will consider action \( S \) spaces of hyperbolic surfaces. Therefore, this Kähler metric is also called Weil-Petersson quasi-circle, or simply “Weil-Petersson curve”. As we did not define explicitly the Loewner transform in [43]. Provided that \( \gamma \) separates 0 from \( \infty \), we may choose the biholomorphic functions \( f \) and \( g \) as in Theorem 1.1 and assume further that \( f(0) = 0 \). Applying the invariance of the Loewner energy under \( \iota \), we get

\[
I^L(\gamma) = I^L(\iota(\gamma)) = \frac{1}{\pi} S_2(\gamma) \tag{1.2}
\]

\[
= \frac{1}{\pi} \int_D \left| f''(z) \right|^2 |dz|^2 + \frac{1}{\pi} \int_{\Delta \subset \mathbb{D}} \left| g''(z) \right|^2 |dz|^2 + 4 \log |f'(0)| - 4 \log |g'(0)|. \tag{1.3}
\]

Let us also mention another result expressing the Loewner energy of smooth curves using zeta-regularised determinants (see [39] for example; the notations will be explained below when we will use this theorem). Notice that smooth curves are Weil-Petersson by, e.g., conditions (2) and (6) in Theorem 1.1. This expression is crucial in the proof of our main theorem and also of particular interest to understand the motivations behind this work.

**Theorem 1.2 (Y. Wang, [54]).** A simple closed curve \( \gamma \) has finite Loewner energy \( I^L(\gamma) \) if and only if \( \gamma \) is a Weil-Petersson quasi-circle. Furthermore, we have

\[
I^L(\gamma) = \frac{1}{\pi} S_1(\gamma). \tag{1.2}
\]

We will therefore use interchangeably the terms “simple closed curve of finite Loewner energy”, “Weil-Petersson quasi-circle”, or simply “Weil-Petersson curve”. As we did not define explicitly the Loewner energy \( I^L(\gamma) \), readers may consider (1.2) as its definition. It is not obvious from the expression of \( S_1 \) that the Loewner energy is invariant under Möbius transformations, such as the inversion \( \iota : z \mapsto 1/z \). However, it would be very clear from the original definition using Loewner transform in [43]. Provided that \( \gamma \) separates 0 from \( \infty \), we may choose the biholomorphic functions \( f \) and \( g \) as in Theorem 1.1 and assume further that \( f(0) = 0 \). Applying the invariance of the Loewner energy under \( \iota \), we get

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**Theorem 1.3 (Y. Wang [54]).** Let \( \alpha \in C^\infty(S^2, \mathbb{R}) \), \( g = e^{2\alpha}g_0 \) be a metric conformally equivalent to the spherical metric \( g_0 \) of \( S^2 \), and \( \Gamma \subset S^2 \) be a simple smooth curve. Let \( \Omega_1, \Omega_2 \subset S^2 \) be the two disjoint open connected components of \( S^2 \setminus \Gamma \). Then we have

\[
I^L(\Gamma) = 12 \log \det \xi(-\Delta_{\Omega_2}, g) \det \xi(-\Delta_{\Omega_1}, g) \tag{1.4}
\]

where \( S^2_\pm \) (resp. \( S^2_+ \)) is the southern hemisphere (resp. the northern hemisphere).

In this article, basing ourselves on a new result of Laurain-Petrides ([32]), we show that the Loewner energy can be computed with respect to the (renormalised) energy of moving frames. An indication on why such an expression could be relevant is found in previous work of T. Rivière and Rivière-Mondino. In [35], the latter authors introduce a frame energy that is closely linked to the Willmore energy in the case of tori and they proved a Willmore-type conjecture for such an energy. In a subsequent work ([42]), T. Rivière introduced a more general frame energy for surfaces of arbitrary genus. A main component of this energy, named the Onofri energy, is nothing else than—up to lower-order terms—the Polyakov-Alvarez anomaly formula for the zeta-regularised determinants:

\[
\mathcal{O}(\alpha) = \frac{1}{8\pi} \int_{S^2} |d\alpha|^2_{g_0} d\text{vol}_{g_0} + \int_{S^2} \alpha d\text{vol}_{g_0} - 2\pi \log \left( \frac{1}{4\pi} \int_{S^2} e^{2\alpha} d\text{vol}_{g_0} \right). \nonumber
\]

Although in [42], the energy is only defined for conformal metrics associated to an immersion, it can be defined for arbitrary Weyl scalings. Below, we will rather use an Onofri energy for surfaces with boundary, but the analogy seems striking to us.
Another link to the Willmore energy is given by Christopher Bishop in a recent preprint, which showed several remarkable links to minimal surfaces in the hyperbolic 3-space $\mathbb{H}^3$ along with other geometric characterisations of Weil-Petersson curves. Our model of $\mathbb{H}^3$ in this article is $\mathbb{C} \times \mathbb{R}_+$ endowed with the metric
\[ g_{\text{hyp}} = \frac{|dz|^2 + dt^2}{t^2}. \]

Let us list a few relevant characterisations below.

**Theorem 1.4** (Bishop, [5]). Let $\gamma \subset \mathbb{C}$ be a simple closed curve. The following conditions are equivalent:

1. $\gamma \subset \mathbb{C}$ is a Weil-Petersson curve.
2. The curve $\gamma$ is chord-arc and the unit tangent $\tau : \gamma \to S^1$ belongs to $H^{\frac{3}{2}}(\gamma)$.
3. There exists an embedding $\Phi : \mathbb{D} \to \mathbb{H}^3$ with asymptotic boundary $\gamma$ (i.e. such that $\partial_{\infty}\Phi(\mathbb{D}) = \gamma$), whose second fundamental form $A$ satisfies $|A(z)|_{|z|=1} \to 0$ and with finite total curvature
   \[ \int_{\mathbb{D}} |A|^2 \text{dvol}_g < \infty, \]
   where $g = \Phi^* g_{\text{hyp}}$ is the induced metric on $\mathbb{D}$.
4. Every minimal surface $\Sigma \subset \mathbb{H}^3$ with asymptotic boundary $\gamma$ has finite Euler characteristic and finite total curvature i.e.
   \[ \int_{\Sigma} |\hat{A}|^2 \text{dvol}_\Sigma < \infty. \]
5. Every minimal surface $\Sigma \subset \mathbb{H}^3$ with asymptotic boundary $\gamma$ has finite renormalised area, i.e.,
   \[ \mathcal{R}\mathcal{A}(\Sigma) = \lim_{t \to 0} (\text{Area}(\Sigma_t) - \text{Length}(\partial\Sigma_t)) = -2\pi \chi(\Sigma) - \int_{\Sigma} |\hat{A}|^2 \text{dvol}_\Sigma > -\infty, \quad (1.5) \]
   where for all $t > 0$, $\Sigma_t = \Sigma \cap \{(x, y, z) : z > t\}$ and $\partial\Sigma_t = \Sigma \cap \{(x, y, z) : z = t\}$.

We recall that a simple closed curve is chord-arc if there exists $K < \infty$ such that for all $x, y \in \gamma$, we have $\ell(x, y) \leq K|x - y|$, where $\ell(x, y)$ is the length of the shortest arc joining $x$ to $y$. We mention that Weil-Petersson curves are not only chord-arc but even asymptotically smooth, namely, the ratio $\ell(x, y)/|x - y|$ tends to 1 as $x$ tends to $y$. However, these curves are not necessarily $C^1$ for they allow infinite spirals. See Section 6.2 for an explicit construction of such spirals. Recall that for any simple closed chord-arc curve $\gamma$, a function $u : \gamma \to \mathbb{C}$ belongs to the Sobolev space $H^{\frac{3}{2}}(\gamma)$ if and only if
\[ \int_\gamma \int_{\gamma} \left| \frac{u(z) - u(w)}{z - w} \right|^2 |dz|dw < \infty, \quad (1.6) \]
where $|dz|$ is the arc-length measure.

The first equivalence was proven independently by Y. Shen and L. Wu ([49]; see also [28, 46, 47, 48]). The last characterisation in Theorem 1.4 using the notion of renormalised area was first investigated for Willmore surfaces by Alexakis and Mazzeo ([1], [2]) which has strong motivations arising from string theory [27]. The integral of the squared trace-free second fundamental form in (4) is the Willmore energy of $\Sigma$ which is of particular interest for being conformally invariant. We note that results in [5] are qualitative and Bishop did not aim at giving quantitative relations between the Loewner energy and the renormalised area or total curvature. We will provide an explicit construction of surfaces bounding a class of Weil-Petersson curves with quantitative bounds on the Willmore energy. Amongst the important previous contribution that inspired this work, we should mention Epstein’s work ([18], [19]).
1.2 Moving Frames and the Ginzburg-Landau Equations

In [32], Paul Laurain and Romain Petrides suggested a new approach to express the Loewner energy with respect to the renormalised energy of moving frames using the Ginzburg-Landau energy in a minimal regularity setting (which is of independent interest). Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and $\gamma = \partial \Omega$. Then the Ginzburg-Landau functional is defined for all $\varepsilon > 0$ and $u \in W^{1,2}(\Omega, \mathbb{C})$ by

$$E_{\varepsilon}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 |dz|^2 + \frac{1}{4\varepsilon} \int_{\Omega} (1 - |u|^2)^2 |dz|^2.$$  

It appears in models of super-conductivity, and here, we will be rather interested in its role to construct harmonic maps with values into $S^1$ with singularities. Indeed, provided that $h_0 \in H^{1/2}(\gamma, S^1)$ has non-zero degree (the degree is well-defined for $H^{1/2}$ maps by classical work of Boutet de Monvel and Gabber [15]), then the Sobolev space

$$W^{1,2}_{h_{0}}(\Omega, S^1) = W^{1,2}(\Omega, S^1) \cap \{ u : u|\partial \Omega = h_0 \}$$

is empty. Therefore, the energy of the minimiser $u_{\varepsilon}$ (that exists by classical arguments) of $E_{\varepsilon}$ with fixed boundary data $g_0$ needs to blow-up as $\varepsilon \to 0$.

More precisely, they showed the following result.

**Theorem 1.5** (Laurain-Petrides, [32], Theorem 0.1). Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain such that $\gamma = \partial \Omega$ is chord-arc, and let $h_0 \in H^{1/2}(\gamma, S^1)$ have degree 1. Then, there exists $p \in \Omega$ and a sequence $\{ \varepsilon_k \}_{k \in \mathbb{N}} \subset (0, \infty)$ such that $\varepsilon_k \to 0$, and a harmonic map $u_0 \in C^\infty(\Omega \setminus \{ p \}, S^1)$ such that $u_0 = h_0$ on $\partial \Omega$ and $u_{\varepsilon_k} \to u_0$ in $W^{1,2}_{loc}(\Omega \setminus \{ p \})$ as $k \to \infty$.

In order to state the next theorem, we first introduce a technical result on the existence of the Green’s function.

**Theorem 1.6** (Jones [31], Wallin [52]). Let $\Omega \subset \mathbb{C}$ (resp. $\Omega \subset S^2$) be a connected bounded domain, $g_0$ be the flat metric on $\Omega$ (resp. $g_0$ be the round metric on $S^2$) and assume that $\partial \Omega$ is chord-arc. Then for all $p \in \Omega$, there exists a unique Green’s function $G_{\Omega, p} \in C^\infty(\Omega \setminus \{ p \}, \mathbb{R})$ such that

$$\begin{cases} 
\Delta_{g_0} G_{\Omega, p} = 2\pi \delta_p & \text{in } \Omega \\
G_{\Omega, p} = 0 & \text{on } \partial \Omega.
\end{cases}$$

Furthermore, for all $h \in H^{1/2}(\partial \Omega, \mathbb{R})$, there exists a unique function $u \in W^{1,2}_{h}(\Omega, \mathbb{R})$ such that

$$\begin{cases} 
\Delta_{g_0} u = 0 & \text{in } \Omega \\
u = h & \text{on } \partial \Omega.
\end{cases}$$

Whenever it is clear from context, we will write $G_{\Omega}$ for $G_{\Omega, p}$.

**Remark 1.7.** The existence of a Green’s function follows by classical work of P. Jones ([31]) and H. Wallin [52]. In fact, the Green’s function exists in any Jordan domain thanks to the conformal invariance of the Dirichlet energy. Indeed, if $\Omega$ is a Jordan domain, and $f : D \to \Omega$ is a biholomorphic map such that $f(0) = p$, and $G_{D,0} = \log |z|$, then $G_{\Omega, p} = G_{D,0} \circ f^{-1}$. We assumed $\partial \Omega$ is chord-arc so that the trace theorems apply. The passage from $\mathbb{C}$ to $S^2$ is easy using a stereographic projection and the conformal invariance of Green’s function.

**Theorem 1.8** (Laurain-Petrides, [32], Theorem 0.2, Theorem 0.3). Under the assumptions of Theorem 1.5, let $\bar{e} : \Omega \setminus \{ p \} \to S^1$ be the harmonic map $w_0$ of Theorem 1.5 with boundary data $\tau : \Gamma \to S^1$ which is the unit tangent vector of $\partial \Omega = \Gamma$, $f = -i\bar{e}$, and $\omega = (\bar{e}, df)$. Then there exists a harmonic function $\mu : \Omega \to \mathbb{R}$ such that $\omega = * d (G_{\Omega} + \mu)$, and a conformal map $f : D \to \Omega$ such that

$$\begin{cases} 
\frac{1}{r} \partial_\theta f = e^{\mu f} \bar{e} \circ f. \\
\partial_r f = e^{\mu f} f \circ f
\end{cases}$$

(1.7)
Furthermore, we have

\[ \int_{\Omega} |\omega - * dG_{\Omega}|^2 dx = \int_{\Omega} |\nabla \mu|^2 dx = \int_{\Omega} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2, \quad (1.8) \]

and \( f \) maximises \( |f'(0)| \) amongst all uniformisation maps \( f : \mathbb{D} \to \Omega \).

**Remark 1.9.** If \( f : \mathbb{D} \to \Omega \) is a biholomorphic map such that \( f(0) = p \), and define \( \tilde{e} \) and \( \tilde{f} \) from \( \Omega \setminus \{p\} \) into \( S^1 \) such that

\[
\begin{cases} 
\tilde{e} = \frac{\partial f}{|\partial f|} \circ f^{-1} \\
\tilde{f} = \frac{\partial \tilde{f}}{|\partial \tilde{f}|} \circ f^{-1},
\end{cases}
\]

then \( \mu \) is given by \( \mu = \log |\partial f| \circ f^{-1} = \log |\frac{1}{2} \partial_{\bar{z}}f| \circ f^{-1} = \log |f'| \circ f^{-1} \). The last identities follow from the conformality of \( f \).

To obtain the second half of the Loewner energy involving

\[ \int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2, \]

we cannot easily use the Ginzburg-Landau equation to construct the moving frames since that would force us to work on the non-compact domain \( \mathbb{C} \setminus \overline{\mathbb{D}} \). Using the inversion \( \iota \) will not suffice either. If we choose the biholomorphic map \( \tilde{g} : \mathbb{D} \to \iota(\mathbb{C} \setminus \overline{\mathbb{D}}) \) so that \( \tilde{g} = \iota \circ g \circ \iota \), we have

\[ \int_{\mathbb{D}} \left| \frac{\tilde{g}''(z)}{\tilde{g}'(z)} \right|^2 |dz|^2 = \int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g''(z)}{g'(z)} - 2 \frac{g'(z)}{g(z)} \right|^2 |dz|^2 \]

which is in general different from \( \int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g'(z)}{g(z)} \right|^2 |dz|^2 \). To overcome this difficulty, we work directly on \( S^2 \) to obtain a formula of the Loewner energy in terms of moving frames.

### 1.3 Main Results

**Theorem A.** Let \( \Gamma \subset S^2 \) be a Weil-Petersson curve, \( \Omega_1, \Omega_2 \subset S^2 \) be the two disjoint open connected components of \( S^2 \setminus \Gamma \). Fix some \( j = 1, 2 \). Then, for all \( p_j \in \Omega_j \), there exists harmonic moving frames \( (\tilde{e}_j, \tilde{f}_j) : \Omega_j \setminus \{p_j\} \to U \Omega_j \times U \Omega_j \) such that the Cartan form \( \omega_j = (\tilde{e}_j, d\tilde{f}_j) \) admits the decomposition

\[ \omega_j = * d(G_{\Omega_j} + \mu_j), \quad (1.9) \]

where \( G_{\Omega_j} : \Omega_j \setminus \{p_j\} \to \mathbb{R} \) is the Green’s function of the Laplacian \( \Delta_{g_0} \) on \( \Omega_j \) with Dirichlet boundary condition, and \( \mu_j \in C^\infty(\Omega_j) \) satisfies

\[ \begin{cases} -\Delta_{g_0} \mu_j = 1 & \text{in } \Omega_j \\
\partial_{\bar{z}} \mu_j = k_{g_0} - \partial_{\bar{z}} G_{\Omega_j} & \text{on } \partial \Omega_j, \quad (1.10) \end{cases} \]

where \( k_{g_0} \) is the geodesic curvature on \( \Gamma = \partial \Omega_j \). Define the functional (that we call the renormalised energy associated to the frames \( (\tilde{e}_1, \tilde{f}_1) \) and \( (\tilde{e}_2, \tilde{f}_2) ) \) \( \delta \) by

\[ \delta(\Gamma) = \int_{\Omega_1} |d\mu_1|_{g_0}^2 d\text{vol}_{g_0} + \int_{\Omega_2} |d\mu_2|_{g_0}^2 d\text{vol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_2} G_{\Omega_2} K_{g_0} d\text{vol}_{g_0} + 4\pi. \quad (1.11) \]

Then there exists conformal maps \( f_1 : \mathbb{D} \to \Omega_1 \) and \( f_2 : \mathbb{D} \to \Omega_2 \) such that \( f_1(0) = p_1 \), \( f_2(0) = p_2 \) and

\[ I^L(\Gamma) = \frac{1}{2\pi} \delta(\Gamma) + 4 \log |\nabla f_1(0)| + 4 \log |\nabla f_2(0)| - 12 \log(2) = \frac{1}{\pi} \delta_0(\Gamma). \quad (1.12) \]
Remark 1.10.  (1) In the theorem above, we wrote $U \Omega_j$ ($j=1,2$) for the unit tangent bundle. Here, the function $\mu_j$ explicitly given by
\[
\mu_j = \frac{1}{2} \log \left( \frac{\left| \nabla f_j \right|^2}{2} \right) = \log |\nabla f_j| - \frac{1}{2} \log(2) \tag{1.13}
\]
extactly correspond to the conformal parameter of the conformal maps $f_1, f_2 : \mathbb{D} \to S^2 \subset \mathbb{R}^3$.

(2) The factor $4\pi$ in the definition of $\mathcal{E}$ is arranged so that $\mathcal{E}(S^1) = 0$ (see Remark 2.5). Furthermore, the name renormalised energy is justified by the following identity
\[
\mathcal{E}(\Gamma) = \int_{\Omega_1} \left( |d\tilde{c}_1|_{g_0}^2 + |d\tilde{c}_1|_{g_0}^2 - 2|dG_{\Omega_1}|_{g_0}^2d\text{vol}_{g_0} + \int_{\Omega_2} \left( |d\tilde{c}_2|_{g_0}^2 + |d\tilde{c}_2|_{g_0}^2 - 2|dG_{\Omega_2}|_{g_0}^2d\text{vol}_{g_0} \right),
\]
where no constant term is involved.

(3) The solution to the Dirichlet problem is unique, and so is the moving frame once the singularities $(p_1, p_2) \in \Omega_1 \times \Omega_2$ are fixed. See Theorem 4.2 and 4.5.

This theorem corresponds to Theorem 2.3 in the article in the smooth case and to Theorem 4.2 in the general case. The general case follows essentially from the following result which can also be viewed as a restatement of Theorem A without any mention of moving frames.

**Theorem B.** Let $\Gamma \subset S^2$ be a closed simple curve of finite Loewner energy. If $\Omega_1, \Omega_2 \subset S^2 \setminus \Gamma$, for all conformal maps $f_1 : \mathbb{D} \to \Omega_1$ and $f_2 : \mathbb{D} \to \Omega_2$, we have
\[
I^L(\Gamma) = \frac{1}{\pi} \left\{ \int_{\mathbb{D}} |\nabla \log |\nabla f_1|^2|dz|^2 + \int_{\mathbb{D}} |\nabla \log |\nabla f_2|^2|dz|^2 + \int_{\mathbb{D}} \log |z||\nabla f_1|^2|dz|^2 + \int_{\mathbb{D}} \log |z||\nabla f_2|^2|dz|^2 + 4\pi + 4\pi \log |\nabla f_1(0)| + 4\pi \log |\nabla f_2(0)| - 12\pi \log(2) \right\}. \tag{1.14}
\]

This theorem corresponds to Theorem 3.5 below.

**Theorem C.** Let $\gamma \subset \mathbb{C}$ be a simple curve of finite Loewner energy, let $\Omega$ be the bounded open connected component of $\mathbb{C} \setminus \gamma$ and let $f : \mathbb{D} \to \Omega$ be a uniformisation map. Let $0 < \alpha < \frac{1}{2}$ and assume that
\[
\int_{\mathbb{D}} \frac{|f''(z)|^2}{|f'(z)|^\alpha} < \infty. \tag{1.15}
\]
Then the embedding $\tilde{\Phi} : \mathbb{D} \to \mathbb{H}^3 = \left( \mathbb{C} \times \mathbb{R}^+_+, g_{\text{hyp}} = \frac{|dz|^2 + dt^2}{t^2} \right)$ defined for all $z \in \mathbb{D}$ by
\[
\tilde{\Phi}(z) = (\text{Re}(f(z)), \text{Im}(f(z)), (1 - |z|^2)^\alpha)
\]
is orthogonal to $\partial_{\infty} \mathbb{H}^3 = \mathbb{R}^2 \times \{0\}$ and has finite total curvature, i.e.
\[
\int_{\mathbb{D}} |\tilde{A}|^2d\text{vol}_{g_{\text{hyp}}} < \infty,
\]
where \( \hat{A} \) is the trace-less fundamental form and \( g = \bar{\Phi}^*g_{hyp} \) is the induced metric on the unit disk \( \mathbb{D} \). Furthermore, there exists a universal constant \( C = C_\alpha < \infty \) independent of \( f \) such that

\[
\int_{\mathbb{D}} |\hat{A}|^2 d\text{vol}_g \leq C \left( 1 + \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)(1-|z|^2)^{1-\alpha}|} \exp \left( C \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right) \right).
\]

(1.16)

![Figure 2: Surface in \( \mathbb{H}^3 \) constructed from lifting the equipotentials.](image)

This theorem corresponds to Theorem 5.3.

The proof in fact proceeds from the proof of the first theorem in the smooth case to the one of the second, where we directly construct the moving frame from the biholomorphic maps. Finally, we give of construction a surfaces orthogonal to a planar curve satisfying a stronger condition than the finiteness of the Loewner energy. The construction is free of branch points, and works for more general curves (given those boundary conditions) as the ones given in [33] since in this work, the authors require that the boundary curve is \( C^{1,1} \) and its normal \( C^{1,1} \), while for all \( \varepsilon > 0 \), this condition is satisfies for curves \( C^{1,1+\varepsilon} \) (see Proposition 5.7).

We also remark (see Theorem 5.5) that this condition produces a surface of \( L^1 \) norm of Gauss curvature (which is sometimes called Lipschitz–Killing curvature or total curvature contrary to the more common terminology in the theory of immersion, see [17]) for any curve of finite Loewner energy provided that \( \alpha < \frac{1}{2} \).

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2 Moving Frame Energy via Zeta-Regularised Determinants for Smooth Curves

We will now use a formula giving the Loewner energy with respect to zeta-regularised determinants to link it with the renormalised energy of moving frame on domains of \( S^2 \). First, let \( g_0 = g_{S^2} \) be the standard round metric on \( S^2 \). Let \( \Gamma \subset S^2 \) be a simple smooth curve, and let \( \Omega_1, \Omega_2 \subset S^2 \) the two disjoint open connected components of \( S^2 \setminus \Gamma \). Since we are working on a curved manifold, we cannot directly use the result of [32] to construct moving frames with the Ginzburg-Landau method. However, its existence follows from directly in Section 4. Therefore, let assume that \((\bar{e}_1, \bar{f}_1) : \Omega_1 \setminus \{p_1\} \to US^2 \times US^2 \) are harmonic vector fields such that \( \bar{e}_1 = \tau \) on \( \partial \Omega_1 = \Gamma \) (where \( \tau \) is the unit tangent on \( \Gamma \)), and the 1-form \( \omega = \langle \bar{e}, d\bar{f} \rangle \) satisfies

\[
\omega = \ast d \left( G_{\Omega_1} + \mu_1 \right) \quad \text{in} \quad \mathcal{D}'(\Omega_1)
\]

(2.1)

*It is necessary to assume that the curve is more regular than the curves of finite Loewner energy that can happen not to be \( C^1 \) for one will need to recurse to the Froebenius theorem below. Furthermore, the formula for the Loewner energy using the zeta-regularised determinants ([54]) only works for smooth (or at least \( C^2 \)) curves.
where $G_{\Omega_1} : \Omega_1 \setminus \{p_1\} \to \mathbb{R}$ is the Green’s function for the Laplacian on $\Omega_1 \setminus \{p_1\}$, which satisfies (using Theorem 4.5).

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\Delta_{g_0} G_{\Omega_1} = 2\pi\delta_p & \text{in } \mathcal{D}'(\Omega_1) \\
G_{\Omega_1} = 0 & \text{on } \partial\Omega_1,
\end{array} \right.
\end{aligned}
\]  

(2.2)

and $\mu_1 : \Omega_1 \to \mathbb{R}$ is a smooth function satisfying

\[
\begin{aligned}
\left\{ \begin{array}{ll}
-\Delta_{g_0} \mu_1 = 1 & \text{in } \Omega_1 \\
\partial_r \mu_1 = k_{g_0} - \partial_r G_{\Omega_1} & \text{on } \partial\Omega_1,
\end{array} \right.
\end{aligned}
\]  

(2.3)

where $k_{g_0}$ is the geodesic curvature with respect to the round metric $g_0$, and the normal derivative is taken with respect to the $g_0$. Now, using Proposition 5.1. of [32], we see that the proof using the Frobenius also works for domains of the sphere, and we get a conformal diffeomorphism $\psi : (-\infty, 0) \times S^1(0, \rho) \to \Omega_1 \setminus \{p_1\}$ such that

\[
\begin{aligned}
\partial_s \psi(s, \theta) &= e^{G_{\Omega_1} \circ \psi + \mu_1 \circ \psi} \vec{e}_1 \circ \psi \\
\partial_\theta \psi(s, \theta) &= e^{G_{\Omega_1} \circ \psi + \mu_1 \circ \psi} \vec{e}_1 \circ \psi.
\end{aligned}
\]

Notice that the Proposition 5.1 of [32] gives a privileged $p_1 \in \Omega_1$, but we will show in Theorem 4.2 that $p_1$ can be taken arbitrarily (see also Theorem 3.5). However, the proof works for an arbitrary harmonic moving frame whose Cartan form admits an expansion as in (2.1) where $\mu_1$ solves (2.3). Since $\mu_1$ is defined up to a constant, we can assume that $\rho = 1$ in the following. Now, defining $f_1 : \mathbb{D} \to \Omega_1$ by

\[
f_1(r, \theta) = \psi(\log(r), \theta),
\]

we can continuously extend $f_1$ at $z = 0$ such that $f_1(0) = p_1$, and since $f$ is conformal, the function $G = G_{\Omega_1} \circ f : \mathbb{D} \setminus \{0\} \to \mathbb{R}$ is harmonic on $\mathbb{D} \setminus \{0\}$, satisfies $G = 0$ on $\partial\mathbb{D}$, so by (2.2), we deduce that

\[
G = G_{\mathbb{D}} = \log |z|.
\]

Therefore, we have

\[
\begin{aligned}
\partial_r f_1 &= \frac{1}{r} \partial_r \psi(\log(r), \theta) = \frac{1}{r} e^{\log(r) + \mu_1 \circ f_1} \vec{e}_1 \circ f_1 = e^{\mu_1 \circ f_1} \vec{f} \circ f_1 \\
\frac{1}{r} \partial_\theta f_1 &= \frac{1}{r} \partial_\theta \psi(\log(r), \theta) = e^{\mu_1 \circ f_1} \vec{e}_1 \circ f_1.
\end{aligned}
\]  

(2.4)

Since $|\vec{e}| = |\vec{f}| = 0$, and $\langle \vec{e}, \vec{f} \rangle = 0$, we deduce that

\[
|\partial_r f_1|^2 = \frac{1}{r^2} |\partial_\theta f_1|^2 = e^{2\mu_1 \circ f_1},
\]

\[
\langle \partial_r f_1, \partial_\theta f_1 \rangle = 0,
\]

which shows that the conformal parameter of $f$ is

\[
\frac{1}{2} |\nabla f_1|^2 = e^{2\lambda} = e^{2\mu_1 \circ f_1},
\]

which implies that

\[
\mu_1 = \log |\nabla f_1| \circ f_1^{-1} - \frac{1}{2} \log(2).
\]

In particular, we have

\[
\mu_1(p_1) = \log |\nabla f_1(0)| - \frac{1}{2} \log(2),
\]  

(2.5)
where \( p_1 \in \Omega_1 \) is the singularity of the moving frame \((\vec{e}_1, \vec{f}_1) : \Omega_1 \setminus \{p_1\} \to US^2 \times US^2\). If \( \iota : \Omega_1 \subset S^2 \hookrightarrow \mathbb{R}^3 \) is the inclusion map, we have \( g_{S^2|\Omega_1} = \iota^* g_{S^2} \). As \( f \) is conformal, we have

\[
\begin{align*}
    f_1^* g_{S^2|\Omega_1} &= f_1^* \iota^* g_{S^2} = (\iota \circ f_1)^* g_{S^2} = |\nabla f_1(z)|^2 |dz|^2 = e^{2 \mu_1 \circ f_1(z)} |dz|^2 = e^{2 \mu_1 \circ f_1(z) - 2 \psi(z)} \frac{4|dz|^2}{(1 + |z|^2)^2} \\
    &= e^{2 \mu_1 \circ f_1 - 2 \psi((\pi^{-1})^* g_{S^2})}|z|,
\end{align*}
\]

where

\[
\psi(z) = \log \left( \frac{2}{1 + |z|^2} \right),
\]

and \( \pi^{-1} : \mathbb{C} \to S^2 \setminus \{N\} \) is the inverse stereographic projection. Writing for simplicity

\[
g_0 = \frac{4|dz|^2}{(1 + |z|^2)^2} = e^{2\psi(z)} |dz|^2,
\]

we deduce by (2.6) that

\[
g_{S^2|\Omega_1} = (f_1 \circ f_1^{-1})^* (g_{S^2}|\Omega_1) = (f_1^{-1})^* f_1^* g_{S^2|\Omega_1} = e^{2 \mu_1 - 2 \psi f_1^{-1} (f_1)^* (g_0)},
\]

so that (by an abuse of notation for the last identity)

\[
g_1 = (f_1^{-1})^* (g_0) = e^{-2 \mu_1 + 2 \psi f_1^{-1}} g_{S^2|\Omega_1} = e^{2 \alpha_1} g_{S^2|\Omega_1} = e^{2 \alpha_1} g_0 \tag{2.7}
\]

where

\[
\alpha_1(z) = - \mu(z) + \psi(f_1^{-1}(z)).
\]

**Remark 2.1.** Notice that thanks to the constructions in Section 4, we do not need to use the approach of [32] in the discussion above.

**Definition 2.2.** Define the open subsets \( S^2_+, S^2_- \subset S^2 \) by

\[
S^2_+ = S^2 \cap \{(x, y, z) : z > 0\} \\
S^2_- = S^2 \cap \{(x, y, z) : z < 0\}.
\]

**Theorem 2.3.** Let \( \Gamma \subset S^2 \) be a simple smooth curve, and let \( \Omega_1, \Omega_2 \subset S^2 \) the two disjoint open connected components of \( S^2 \setminus \Gamma \). Fix some \( j = 1, 2 \). Then, for all \( p_j \in \Omega_j \) and for all harmonic moving frames \((\vec{e}_j, \vec{f}_j) : \Omega_j \setminus \{p_j\} \to U\Omega_j \times U\Omega_j \) such that the Cartan form \( \omega_j = (\vec{e}_j, d\vec{f}_j) \) admits the decomposition

\[
\omega_j = * d (G_{\Omega_j} + \mu_j),
\]

where \( G_{\Omega_j} : \Omega_j \setminus \{p_j\} \to \mathbb{R} \) is the Green’s function of the Laplacian \( \Delta_{g_0} \) on \( \Omega_j \) with Dirichlet boundary condition, and \( \mu_j \in C^\infty(\Omega_j) \) satisfies

\[
\begin{cases}
- \Delta_{g_0} \mu_j = 1 & \text{in } \Omega_j \\
\partial_r \mu_j = k_{g_0} - \partial_r G_{\Omega_j} & \text{on } \partial \Omega_j,
\end{cases} \tag{2.8}
\]

where \( k_{g_0} \) is the geodesic curvature on \( \Gamma = \partial \Omega_j \). Define the functional (that we call the renormalised energy associated to the frames \((\vec{e}_1, \vec{f}_1) \) and \((\vec{e}_2, \vec{f}_2) \)) \( \mathcal{E} \) by

\[
\mathcal{E}(\Gamma) = \int_{\Omega_1} |d\mu_1|^2_{g_0} \dvol_{g_0} + \int_{\Omega_2} |d\mu_2|^2_{g_0} \dvol_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} \dvol_{g_0} + 2 \int_{\Omega_2} G_{\Omega_2} K_{g_0} \dvol_{g_0} + 4 \pi.
\]

Then there exists conformal maps \( f_1 : \mathbb{D} \to \Omega_1 \) and \( f_2 : \mathbb{D} \to \Omega_2 \) such that \( f_1(0) = p_1, f_2(0) = p_2 \) and

\[
I^L(\Gamma) = \frac{1}{4 \pi} \mathcal{E}(\Gamma) + 4 \log |\nabla f_1(0)| + 4 \log |\nabla f_2(0)| - 12 \log(2) = \frac{1}{4 \pi} \mathcal{E}_0(\Gamma).
\]
Remark 2.4. By the previous discussion, we know that there exists at least one such harmonic moving frame.

Proof. If $\pi^{-1}: \mathbb{C} \to S^2 \setminus \{N\}$ is the inverse stereographic projection,

$$g_0 = \frac{4|dz|^2}{(1 + |z|^2)^2} = e^{2\psi(z)}|dz|^2,$$

and $S^2$ is the southern hemisphere, we deduce that

$$\det_{\zeta}(-\Delta_{S^2,g_0}) = \det_{\zeta}(-\Delta_{\mathbb{P}}) = \det_{\zeta}(-\Delta_{\mathbb{P},e^*g_0}) = \det_{\zeta}(-\Delta_{\Omega_1,g_0}),$$

and by the Alvarez-Polyakov formula and (2.7), we have

$$\int_{\Omega_2} |d(-\mu_1 + \theta_1)|^2_{g_0} d\nu_{g_0} + 2 \int_{\Omega_1} K_{g_0}(-\mu_1 + \theta_1) d\nu_{g_0} + 2 \int_{\partial\Omega_1} k_{g_0}(-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0} + 3 \int_{\partial\Omega_1} \partial_\nu(-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0},$$

if we choose $\Gamma$ to be given with the same orientation of $\partial\Omega_1$, and

$$\theta_1 = \psi \circ f^{-1}_{\Omega_1}.$$

Therefore, using subscripts with evident notations, we deduce by Theorem 1.3 with $g = g_0$ that

$$-\pi I^L(\Gamma) = -12\pi \log \frac{\det_{\zeta}(-\Delta_{S^2,g_0}) \det_{\zeta}(-\Delta_{\mathbb{P}})}{\det_{\zeta}(-\Delta_{\mathbb{P},e^*g_0})} = -12\pi \log \frac{\det_{\zeta}(-\Delta_{\Omega_1,g_0}) \det_{\zeta}(-\Delta_{\mathbb{P}})}{\det_{\zeta}(-\Delta_{\Omega_1,\mathbb{P}})}$$

$$= \int_{\Omega_1} |d(-\mu_1 + \theta_1)|^2_{g_0} d\nu_{g_0} + 2 \int_{\Omega_1} K_{g_0}(-\mu_1 + \theta_1) d\nu_{g_0} + 2 \int_{\partial\Omega_1} k_{g_0}(-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0} + 3 \int_{\partial\Omega_1} \partial_\nu(-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0}$$

$$+ \int_{\Omega_2} |d(-\mu_2 + \theta_2)|^2_{g_0} d\nu_{g_0} + 2 \int_{\Omega_2} K_{g_0}(-\mu_2 + \theta_2) d\nu_{g_0} + 2 \int_{\partial\Omega_2} k_{g_0}(-\mu_2 + \theta_2) d\mathcal{H}^1_{g_0} + 3 \int_{\partial\Omega_2} \partial_\nu(-\mu_2 + \theta_2) d\mathcal{H}^1_{g_0},$$

(2.9)

Notice that provided that $\Gamma$ be given with the same orientation of $\partial\Omega_1$, we have

$$2 \int_{\partial\Omega_1} k_{g_0}(-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0} + 2 \int_{\partial\Omega_1} k_{g_0}(-\mu_2 + \theta_2) d\mathcal{H}^1_{g_0} = 2 \int_{\Gamma} k_{g_0}(-\mu_1 + \theta_1 + \mu_2 - \theta_2) d\mathcal{H}^1_{g_0},$$

(2.10)

Since $K_{g_0} = 1 = -\Delta_{g_0}\mu_1$ on $\Omega_1$, we have

$$\int_{\Omega_1} K_{g_0}(-\mu_1) d\nu_{g_0} = \int_{\Omega_1} \mu_1 \Delta_{g_0}\mu_1 d\nu_{g_0} = - \int_{\Omega_1} |d\mu_1|^2_{g_0} d\nu_{g_0} + \int_{\Gamma} \mu_1 \partial_\nu \mu_1 d\mathcal{H}^1_{g_0}$$

$$= - \int_{\Omega_1} |d\mu_1|^2_{g_0} d\nu_{g_0} + \int_{\Gamma} (k_{g_0} - \partial_\nu G_{\Omega_1}) \mu_1 d\mathcal{H}^1_{g_0}$$

(2.11)

by (2.3). Therefore, we deduce that

$$2 \int_{\Omega_1} K_{g_0}(-\mu_1 + \theta_1) d\nu_{g_0} + 2 \int_{\partial\Omega_1} k_{g_0}(-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0},$$

11
\[\begin{align*}
&= -2 \int_{\Omega_1} |d\mu_1|^2_{g_0} \text{dvol}_{g_0} - 2 \int_{\Gamma} \partial_{\nu} G_{\Omega_1} \mu_1 d\mathcal{H}^1_{g_0} + 2 \int_{\Omega_1} \theta_1 \text{dvol}_{g_0} + 2 \int_{\Gamma} k_{g_0} \theta_1 d\mathcal{H}^1_{g_0}. \quad (2.12)
\end{align*}\]

Now, we have
\[\begin{align*}
\int_{\Omega_1} |d(-\mu_1 + \theta_1)|^2_{g_0} \text{dvol}_{g_0} &= \int_{\Omega_1} |d(-\mu_1 + \psi \circ f^{-1})|^2_{g_0} \text{dvol}_{g_0} \\
&= \int_{\Omega_1} |d\mu_1|^2_{g_0} \text{dvol}_{g_0} - 2 \int_{\Omega_1} \langle d\mu_1, d(\psi \circ f^{-1}) \rangle_{g_0} \text{dvol}_{g_0} + \int_{\Omega_1} |d(\psi \circ f^{-1})|^2_{g_0} \text{dvol}_{g_0}. \quad (2.13)
\end{align*}\]

Since \(-\Delta_{g_0} \mu_1 = 1\), we deduce that
\[\begin{align*}
-2 \int_{\Omega_1} \langle d\mu_1, d(\psi \circ f^{-1}) \rangle_{g_0} \text{dvol}_{g_0} &= -2 \int_{\Omega_1} \langle d\mu_1, d\theta_1 \rangle_{g_0} \text{dvol}_{g_0} = 2 \int_{\Omega_1} \theta_1 \Delta_{g_0} \mu_1 \text{dvol}_{g_0} - 2 \int_{\Omega_1} \theta_1 \partial_{\nu} \mu_1 d\mathcal{H}^1_{g_0} \\
&= -2 \int_{\Omega_1} \theta_1 \text{dvol}_{g_0} - 2 \int_{\Omega_1} k_{g_0} \theta_1 d\mathcal{H}^1_{g_0} + 2 \int_{\Omega_1} \partial_{\nu} G_{\Omega_1} \theta_1 d\mathcal{H}^1_{g_0}. \quad (2.14)
\end{align*}\]

Now, by conformal invariance of the Dirichlet energy, we have
\[\begin{align*}
\int_{\Omega_1} |d(\psi \circ f^{-1})|^2_{g_0} \text{dvol}_{g_0} &= \int_{\mathbb{D}} |\nabla \psi|^2 |dz|^2 \\
\end{align*}\]

Since \(\psi(z) = \log(2) - \log(1 + |z|^2)\) and \(\psi\) is real, we have
\[\begin{align*}
\int_{\mathbb{D}} |\nabla \psi|^2 |dz|^2 &= 4 \int_{\mathbb{D}} |\partial_z \psi|^2 |dz|^2 = 4 \int_{\mathbb{D}} \frac{|z|^2 |dz|^2}{(1 + |z|^2)^2} = 4\pi \log(2) - 2\pi. \quad (2.15)
\end{align*}\]

Therefore, we get (2.12), (2.13), (2.14) and (2.15)
\[\begin{align*}
\int_{\Omega_1} |d(-\mu_1 + \theta_1)|^2_{g_0} \text{dvol}_{g_0} &= \int_{\Omega_1} |d\mu_1|^2_{g_0} \text{dvol}_{g_0} - 2 \int_{\Omega_1} \langle d\mu_1, d\theta_1 \rangle_{g_0} \text{dvol}_{g_0} + \int_{\Omega_1} |d\theta_1|^2_{g_0} \text{dvol}_{g_0} \\
&= \int_{\Omega_1} |d\mu_1|^2_{g_0} \text{dvol}_{g_0} - 2 \int_{\Omega_1} \langle d\mu_1, d\theta_1 \rangle_{g_0} \text{dvol}_{g_0} + 2 \int_{\Omega_1} \theta_1 \partial_{\nu} \mu_1 d\mathcal{H}^1_{g_0} \\
&= \int_{\Omega_1} |d\mu_1|^2_{g_0} \text{dvol}_{g_0} - 2 \int_{\Omega_1} \theta_1 \text{dvol}_{g_0} - 2 \int_{\Omega_1} k_{g_0} \theta_1 d\mathcal{H}^1_{g_0} + 2 \int_{\Omega_1} \partial_{\nu} G_{\Omega_1} \theta_1 d\mathcal{H}^1_{g_0} \\
&= - \int_{\Omega_1} |d\mu_1|^2_{g_0} \text{dvol}_{g_0} + 2 \int_{\Omega_1} \theta_1 \partial_{\nu} \mu_1 d\mathcal{H}^1_{g_0} + 4\pi \log(2) - 2\pi. \quad (2.16)
\end{align*}\]

Now, since \(\theta_1 = \psi \circ f^{-1}\), and \(\psi(z) = 0\) for all \(z \in S^1\), we have \(\theta_1 = 0\) on \(\Gamma\). Therefore, we have
\[\begin{align*}
2 \int_{\Omega_1} \partial_{\nu} G_{\Omega_1} (-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0} &= -2 \int_{\Gamma} \partial_{\nu} G_{\Omega_1} \mu_1 d\mathcal{H}^1_{g_0}. \quad (2.17)
\end{align*}\]

Now, since \(-\Delta_{g_0} \mu_1 = 1\) and \(K_{g_0} = 1\), we have
\[\begin{align*}
\int_{\Omega_1} G_{\Omega_1} K_{g_0} \text{dvol}_{g_0} &= - \int_{\Omega_1} G_{\Omega_1} \Delta_{g_0} \mu_1 \text{dvol}_{g_0} \\
&= - \int_{\Omega_1} \mu_1 \Delta_{g_0} G_{\Omega_1} \text{dvol}_{g_0} - \int_{\Omega_1} (G_{\Omega_1} \partial_{\nu} \mu_1 - \mu_1 \partial_{\nu} G_{\Omega_1}) d\mathcal{H}^1_{g_0} \\
&= -2\pi \mu_1(p_1) + \int_{\Omega_1} \mu_1 \partial_{\nu} G_{\Omega_1} d\mathcal{H}^1_{g_0}. \quad (2.18)
\end{align*}\]
where we used the Dirichlet condition \( G_{\Omega_1} = 0 \) on \( \partial \Omega_1 = \Gamma \). Therefore, (2.17) and (2.18) imply that
\[
2 \int_{\Gamma} \partial_{\nu} G_{\Omega_1} (-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0} = -2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} - 4\pi \mu_1 (p_1). \tag{2.19}
\]
Gathering (2.16) and (2.19) yields
\[
\int_{\Omega_1} |d(-\mu_1 + \theta_1)|^2_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} K_{g_0} (-\mu_1 + \theta_1) d\text{vol}_{g_0} + 2 \int_{\partial \Omega_1} k_{g_0} (-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0}
= - \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} - 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} - 4\pi \mu_1 (p_1) + 4\pi \log(2) - 2\pi \tag{2.20}
\]
We also have
\[
\int_{\partial \Omega_2} \partial_{\nu} (-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0} + \int_{\partial \Omega_2} \partial_{\nu} (-\mu_2 + \theta_2) d\mathcal{H}^1_{g_0} = 0. \tag{2.21}
\]
Indeed, we have by the boundary conditions (2.8)
\[
\int_{\partial \Omega_1} \partial_{\nu} \mu_1 d\mathcal{H}^1_{g_0} = \int_{\Gamma} k_{g_0} d\mathcal{H}^1_{g_0} - \int_{\partial \Omega_2} \partial_{\nu} G_{\Omega_1} d\mathcal{H}^1_{g_0} = \int_{\Gamma} k_{g_0} d\mathcal{H}^1_{g_0} - \int_{\Omega_1} \Delta_{g_0} G_{\Omega_1} d\text{vol}_{g_0}
= \int_{\Gamma} k_{g_0} d\text{vol}_{g_0} - 2\pi \tag{2.22}
\]
We also have by the conformal invariance of the Dirichlet energy
\[
\int_{\partial \Omega_1} \partial_{\nu} \theta_1 d\mathcal{H}^1_{g_0} = \int_{\Omega_1} \Delta_{g_0} \theta_1 d\text{vol}_{g_0} = \int_{\Omega_1} \Delta \psi |dz|^2 = \int_{S^1} \partial_{\nu} \psi d\mathcal{H}^1 = -2\pi. \tag{2.23}
\]
Therefore, we finally get by (2.22) and (2.23)
\[
\int_{\partial \Omega_1} \partial_{\nu} (-\mu_1 + \theta_1) d\mathcal{H}^1_{g_0} + \int_{\partial \Omega_2} \partial_{\nu} (-\mu_2 + \theta_2) d\mathcal{H}^1_{g_0}
= - \left( \int_{\Gamma} k_{g_0} d\text{vol}_{g_0} - 2\pi \right) - 2\pi - \left( - \int_{\Gamma} k_{g_0} d\text{vol}_{g_0} - 2\pi \right) - 2\pi = 0
\]
which proves (2.21).

Finally, we deduce by (2.9), (2.20) and (2.21) that
\[
- \pi I^L (\Gamma) = - \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} - 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} - 4\pi \mu_1 (p_1) + 4\pi \log(2) - 2\pi
- \int_{\Omega_2} |d\mu_2|^2_{g_0} d\text{vol}_{g_0} - 2 \int_{\Omega_2} G_{\Omega_2} K_{g_0} d\text{vol}_{g_0} - 4\pi \mu_2 (p_2) + 4\pi \log(2) - 2\pi
= - \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} - \int_{\Omega_2} |d\mu_2|^2_{g_0} d\text{vol}_{g_0} - 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} - 2 \int_{\Omega_1} G_{\Omega_2} K_{g_0} d\text{vol}_{g_0}
- 4\pi \mu_1 (p_1) - 4\pi \mu_2 (p_2) + 8\pi \log(2) - 4\pi \tag{2.24}
\]
Recalling the identity (2.5), we finally deduce that
\[
\pi I^L (\Gamma) = \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} + \int_{\Omega_2} |d\mu_2|^2_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_2} K_{g_0} d\text{vol}_{g_0} + 4\pi
+ 4\pi \log |\nabla f_1 (0)| + 4\pi \log |\nabla f_2 (0)| - 12\pi \log(2) \tag{2.25}
\]
Now introduce the functional
\[
\mathcal{E} (\Gamma) = \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} + \int_{\Omega_2} |d\mu_2|^2_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_2} K_{g_0} d\text{vol}_{g_0} + 4\pi. \tag{2.26}
\]
If \( W_j (j = 1, 2) \) is the renormalised energy associated to the moving frame \((\tilde{e}_j, \tilde{f}_j)\) (see [3] and [32]), we have
\[
W_1 + W_2 = \mathcal{E}(\Gamma) + 2\pi \log |\nabla f_1(0)| + 2\pi \log |\nabla f_2(0)|,
\]

since \( \omega_1 - * dG_{\Omega_1} = * d\mu_1 \). Thanks to (2.2) and (2.26), we deduce that
\[
I^L(\Gamma) = \frac{1}{\pi} \mathcal{E}(\Gamma) + 4 \log |\nabla f_1(0)| + 4 \log |\nabla f_2(0)| - 12 \log(2). \tag{2.27}
\]

This concludes the proof of the theorem. \(\square\)

**Remark 2.5.** Now, we need to check that equality holds for the hemisphere in (2.27) since the Loewner energy of the circle vanishes (using the formula). First, let us check that
\[
\mathcal{E}(S^1) = 0.
\]

This identity justifies the term \(4\pi\) in the definition of \(\mathcal{E}\) as we remarked earlier. Since \(K_{\Omega_0} = 1\) on \(S^2_+\), after making a stereographic projection, we find
\[
\int_{S^2_+} G_{S^2_+} K_{\Omega_0} d\text{vol}_{\Omega_0} = \int_{\mathbb{D}} G_{\mathbb{D}}(z) \frac{4|dz|^2}{(1 + |z|^2)^2} = \int_{\mathbb{D}} \frac{4 \log |z| |dz|^2}{(1 + |z|^2)^2} = 8\pi \int_0^1 \frac{r \log r}{(1 + r^2)^2} dr
\]
\[
= 8\pi \left[ -\frac{1}{2} \log(r) + \frac{1}{4} \log(1 + r^2) \right]_0^1 = -2\pi \log(2). \tag{2.28}
\]

Remember the decomposition
\[
\omega = * dG_{S^2_+} + * d\mu
\]
of Theorem 6.7. Furthermore, since \(\omega = 0\) on \(\partial S^2_+ = S^1\), we deduce that \(\partial_\nu \mu = -1\) on \(S^1\). Now, since
\[
-\Delta_{\gamma_0} \mu = 1 \quad \text{in} \quad S^2_+,
\]
we can rewrite this equation as
\[
-\Delta\mu = \frac{4}{(1 + |z|^2)^2} \quad \text{in} \quad \mathbb{D}.
\]

Since \(\partial_\nu \mu = -1\) by (2.3), we deduce by a direct verification have \(\mu(z) = -\log(1 + |z|^2)\) (this is easy to guess this result since, as we will see below in Section 4, \(\mu\) corresponds to the conformal factor of the conformal map from the disk into the domain \(\Omega_1 = S^2_+\); in this case, the map is simply the restriction to the disk of the inverse stereographic projection). Therefore, we have by the conformal invariance of the Dirichlet energy
\[
\int_{S^2_+} |\omega - * dG_{S^2_+}\omega_0|^2 d\text{vol}_{\omega_0} = \int_{S^2_+} |d\mu_0|^2 d\text{vol}_{\omega_0} = \int_{\mathbb{D}} |\nabla \mu(x)|^2 dx = \int_{\mathbb{D}} \frac{2x}{1 + |x|^2} dx = 8\pi \int_0^1 \frac{r^3}{(1 + r^2)^2} dr
\]
\[
= 8\pi \left[ \frac{r}{1 + r^2} - \frac{r}{(1 + r^2)^2} \right]_0^1 = 8\pi \left[ \frac{1}{2} \log(1 + r^2) - \frac{1}{2} \right]_0^1 = 4\pi \log(2) - 2\pi. \tag{2.29}
\]

Finally, by (2.28) and (2.29), we have
\[
\int_{S^2_+} |\omega - * dG_{S^2_+}\omega_0|^2 d\text{vol}_{\omega_0} + 2 \int_{S^2_+} G_{S^2_+} K_{\Omega_0} d\text{vol}_{\Omega_0} + 2\pi = (4\pi \log(2) - 2\pi) - 4\pi \log(2) + 2\pi = 0.
\]

Now we need to show that
\[
4 \log |\nabla f_1(0)| + 4 \log |\nabla f_2(0)| - 12 \log(2) = 0,
\]

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where \( f_1 : \mathbb{D} \to S^2 \) and \( f_2 : \mathbb{D} \to S^2 \) are conformal maps such that \( f_1(0) = p_1 \) and \( f_2(0) = p_2 \) (indeed, by a symmetry argument, we have \( p_1 = S = (0, 0, -1) \) and \( p_2 = N = (0, 0, 1) \)). Therefore, we have \( f_2 = -f_1 \), and

\[
4 \log |\nabla f_1(0)| + 4 \log |\nabla f_2(0)| - 12 \log(2) = 4(2 \log |\nabla f_1(0)| - 3 \log(2)). \tag{2.30}
\]

Furthermore, we know (by [32], also see also Lemma 6.12 below) that \( |\nabla f_1(0)| \) maximises \( |\nabla f(0)| \) among conformal maps \( f : \mathbb{D} \to S^2 \). Up to a translation in the domain (which are isometries), such functions \( f \) are given by

\[
f = \pi \circ \varphi_\theta,
\]

where \( \theta \in [0, 2\pi) \), \( w \in \mathbb{D} \), the function \( \psi_{\theta, w} : \mathbb{D} \to \mathbb{D} \) is given by

\[
\psi_{\theta, w}(z) = e^{i\theta} \frac{z - w}{1 - \overline{w}z},
\]

where \( \pi^{-1} : \mathbb{D} \to S^2 \) is by abuse of notation the restriction to the disk of the inverse stereographic projection which is given by

\[
\pi^{-1}(z) = \left( \frac{2 \Re(z)}{1 + |z|^2}, \frac{2 \Im(z)}{1 + |z|^2}, \frac{-1 + |z|^2}{1 + |z|^2} \right).
\]

Now, an easy computation shows that (remember that the conformal factor \( e^{2\lambda} \) satisfies \( |\nabla(\pi \circ \varphi_{\theta, w})|^2 = 2e^{2\lambda} \))

\[
|\nabla(\pi^{-1} \circ \varphi_{\theta, w})(0)|^2 = \frac{8(1 - |w|^2)^2}{(1 + |w|^2)^2} \leq 8,
\]

with equality if and only if \( w = 0 \), which shows that \( f_1 = \pi \), and that

\[
|\nabla f_1(0)| = 2\sqrt{2}.
\]

In particular, (2.30) vanishes and we are done.

## 3 Extension of the Results to the Non-Smooth Curves

In order to extend our results to the non-smooth setting, we will obtain another formula for \( E_0 \) in terms of conformal maps and that holds true for any closed simple curve of finite Loewner energy. Using this additional formula, the convergence result will be easily obtained.

Under the preceding notations, if \( \Gamma \subset S^2 \) is a simple curve of finite Loewner energy, then there exists moving frames \((\vec{e}_1, \vec{f}_1)\) and \((\vec{e}_2, \vec{f}_2)\) on \( \Omega_1 \) and \( \Omega_2 \) respectively such that

\[
E(\Gamma) = \int_{\Omega_1} |d\mu_1|^2 d\sigma_{\Omega_1} + \int_{\Omega_2} |d\mu_2|^2 d\sigma_{\Omega_2} + 2 \int_{\Omega_1} G_{\Omega_1} K_{\Omega_0} d\sigma_{\Omega_0} + 2 \int_{\Omega_2} G_{\Omega_2} K_{\Omega_0} d\sigma_{\Omega_0} + 4\pi.
\]

where \( \omega_j = (\vec{e}_j, d\vec{f}_j) = * d(G_{\Omega_j} + \mu_j) \) in \( \mathcal{S}'(\Omega_j) \) for \( j = 1, 2 \). Now, if \( \pi : S^2 \setminus \{p_2\} \to \mathbb{C} \) is the standard stereographic projection, since \( f_j : \mathbb{D} \to \Omega_j \) is conformal and \( \pi \) is also conformal, we deduce that \( \pi \circ f_j : \mathbb{D} \to \pi(\Omega_j) \subset \mathbb{R} \) is also conformal. Therefore, these maps are holomorphic or anti-holomorphic, so up to a complex conjugate (which is an isometry), we can assume that there are holomorphic. Notice that \( \Omega = \pi(\Omega_1) \) is bounded, while \( \pi(\Omega_2) = \mathbb{C} \setminus \overline{\Omega} \) is unbounded. Therefore, if \( \iota : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\} \) is the inversion, and \( g = \pi \circ f_j \circ \iota : \mathbb{C} \setminus \overline{\Omega} \to \mathbb{C} \setminus \overline{\Omega} \), while \( f : \mathbb{D} \to \Omega \), by conformal invariance of the Loewner energy, if \( \gamma = \pi(\Gamma) \), we have

\[
I^L(\Gamma) = I^L(\gamma) = \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_{\mathbb{D} \setminus \overline{\mathbb{D}}} \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2 + 4\pi \log |f'(0)| - 4\pi \log |g'(\infty)|. \tag{3.1}
\]
Indeed, since $f_2(0) = p_2$, we have $\pi \circ f_2(0) = \infty$ and $g(\infty) = \infty$, so that the functions $f$, $g$ satisfy the needed conditions for (3.1) to hold. Furthermore, up to a translation of $\Omega$ we can assume that $0 \in \Omega$ and up to a conformal transformation of $f$, that $f(0) = 0$. Therefore, to our maps $f_1: \mathbb{D} \to \Omega_1$, and $f_2: \mathbb{D} \to \Omega_2$, we canonically associate biholomorphic maps $f: \mathbb{D} \to \Omega$ and $g: \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\Omega}$ such that $f(0) = 0$ and $g(\infty) = \infty$. Furthermore, since $|\nabla f_1(0)|$ and $|\nabla f_2(0)|$ are maximal among conformal maps $f_1: \mathbb{D} \to \Omega_1$ such that $f_1(0) = p_1$ and $f_2: \mathbb{D} \to \Omega_2$ such that $f_2(0) = p_2$ respectively, we have $|f'(0)| > 0$ and $|g'(\infty)| > 0$.

Now, with the previous notions, define the functional

$\mathcal{E}_0(\gamma) = \mathcal{E}(\gamma) + 4\pi \log |\nabla f_1(0)| + 4\pi \log |\nabla f_2(0)| - 12\pi \log(2)$.

In order to show the general equality, we will need to use another formula for the Loewner energy. We will now be able to prove the following general result and we need a definition for that

**Definition 3.1.** Let $\gamma$ be a Jordan curve with finite Loewner energy. Let $f: \mathbb{D} \to \Omega$, $g: \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\Omega}$ be biholomorphic maps such that $g(\infty) = \infty$, we define the third universal Liouville action $S_3$ by

$$S_3(\gamma) = \int_{\mathbb{D}} \frac{|f''(z)|^2}{f'(z)^2} + 2 \int_{\mathbb{C} \setminus \overline{\mathbb{D}}} \frac{|g''(z)|^2}{g'(z)^2} + 2 \pi \int_{\mathbb{D}} \log |z| + 4\pi \log |f'(0)| - 4\pi \log |g'(\infty)| - 4\pi \log(1 + |f(0)|^2)$$

We will now prove the last two equalities in

$$S_1 = S_2 = S_3 = \mathcal{E}_0 = \frac{1}{\pi} \mathcal{L}.$$
Furthermore, we have $u \in W^{1,2}(B(0,2) \setminus \overline{B}(0,1))$ and there exists a universal constant $C_0 > 0$ such that

$$\|\nabla u\|_{L^2(B(0,2) \setminus \overline{B}(0,1))} \leq C_1 \|\log |f'|\|_{H^2(S^1)} = C_1 \left( \int \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}} < \infty, \quad (3.5)$$

provided that the $H^2$ norm here is the infimum of the Dirichlet energy of functions of the disk, i.e. for all $h : S^1 \to \mathbb{R}$,

$$\|h\|_{H^2(S^1)} = \inf \left\{ \int_D |\nabla u|^2 dx : u = h \text{ on } \partial D = S^1 \right\},$$

and noticing that $\log |f'|$ is a harmonic map. This is an equivalent norm to the definition given above in (1.6) by the Douglas formula ([16]). Furthermore, since $u = 0$ on $\partial B(0,2)$, its extension by 0 denoted by $\tilde{u} : \mathbb{C} \setminus D \to \mathbb{R}$ satisfies $\tilde{u} \in W^{1,2}(\mathbb{C} \setminus \overline{D})$ and

$$\|\nabla \tilde{u}\|_{L^2(\mathbb{C} \setminus \overline{D})} = \|\nabla u\|_{L^2(B(0,2) \setminus \overline{B}(0,1))}.$$

Finally, if $\varphi : \mathbb{C} \to \mathbb{R}$ is defined by

$$\varphi(z) = \begin{cases} \log |f'(z)| & \text{if } z \in D \\ \tilde{u}(z) & \text{if } z \in \mathbb{C} \setminus D, \end{cases}$$

we have $\varphi \in W^{1,2}(\mathbb{C})$ and by (3.3) and (3.4), we get for all $-\infty < p < \infty$

$$\log \int_D |f'(z)|^p \frac{|dz|^2}{\pi(1 + |z|^2)^2} = \log \left( \int_D e^{p \log |f'(z)|} \frac{|dz|^2}{\pi(1 + |z|^2)^2} \right) \leq \log \int_D e^{p \varphi(z)} \frac{|dz|^2}{\pi(1 + |z|^2)^2} \leq \frac{p^2}{16\pi} \int_D \frac{f''(z)}{f'(z)}^2 |dz|^2 + \frac{p^2}{16\pi} \int_{B(0,2) \setminus \overline{B}(0,1)} |\nabla u|^2 |dz|^2 + p \int_D \varphi(z) |dz|^2 \frac{|dz|^2}{\pi(1 + |z|^2)^2} \leq 1 + \frac{C_1}{16\pi} p^2 \int_D \frac{f''(z)}{f'(z)}^2 |dz|^2 + p \int_D \varphi(z) |dz|^2 \frac{|dz|^2}{\pi(1 + |z|^2)^2}$$

Now, notice that by the Sobolev inequality and (3.5)

$$\left| \int_D \varphi(z) \frac{|dz|^2}{\pi(1 + |z|^2)^2} \right| \leq \|\varphi\|_{L^1(\mathbb{C})} \leq C_S \|\nabla \varphi\|_{L^2(\mathbb{C})} \leq C_S(1 + C_1) \left( \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}}. \quad (3.7)$$

Finally, by (3.6) and (3.7), we deduce that

$$\log \int_D |f'(z)|^p \frac{|dz|^2}{\pi(1 + |z|^2)^2} \leq 1 + \frac{C_1}{16\pi} p^2 \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + C_S(1 + C_1)p \left( \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}},$$

which finally shows since

$$\frac{1}{\pi(1 + |z|^2)^2} \geq \frac{1}{4\pi} \quad \text{for all } z \in D$$

that provided that $C_0 = \max \left\{ \frac{1 + C_1}{16\pi}, C_S(1 + C_1) \right\}$, we have

$$\int_D |f'(z)|^p |dz|^2 \leq \frac{4}{\pi} \exp \left( C_0 p^2 \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + C_0 p \left( \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}} \right).$$

Likewise, for all $-\infty < p < \infty$, an analogous construction shows that

$$\int_{\mathbb{C} \setminus \overline{D}} |g'(z)|^p \frac{|dz|^2}{(1 + |z|^2)^2} < \infty$$

which concludes the proof of the Lemma. \(\square\)
\textbf{Theorem 3.3.} Let $\Gamma \subset S^2$ be a simple curve of finite Loewner energy. Then we have
\[ \mathcal{E}_{0}(\Gamma) = S_{3}(\Gamma) \]

\textit{Proof.} If $\Gamma \subset S^2$ is a curve of finite Loewner energy $\Omega_1$ and $\Omega_2$ the two connected components of $S^2 \setminus \Gamma$, and $f_1: \mathbb{D} \to \Omega_1$, $f_2: \mathbb{D} \to \Omega_2$ are the canonical conformal maps associated to $\Gamma$ in the definition of $\mathcal{E}$. Now, recall that
\[ |\nabla f_1|^2 = 2e^{2\mu_1 \circ f_1}, \]
which implies that
\[ \log |\nabla f_1| = \frac{1}{2} \log (2) + \mu_1 \circ f_1. \]

Therefore, we have by conformal invariance of the Dirichlet energy
\[ \int_{\mathbb{D}} |\nabla \log |\nabla f_1||^2 |dz|^2 = \int_{\mathbb{D}} |\nabla (\mu_1 \circ f_1)|^2 |dz|^2 = \int_{\Omega_1} |d\mu_1|^2 \cdot \text{dvol}_{g_0}. \quad (3.8) \]

Since $f_1$ is conformal, we deduce that $G = G_{\Omega_1} \circ f_1$ satisfies
\[ \begin{cases} \Delta G = 2\pi \delta_0 & \text{in } \mathbb{D} \\ G = 0 & \text{on } S^1. \end{cases} \]

Therefore, we deduce that $G = G_2 = \log |z|$, and by a change of variable since $f_1$ is conformal with conformal factor
\[ e^{2\lambda} = \frac{1}{2} |\nabla f_1|^2, \]
we get
\[ 2 \int_{\Omega_1} G_{\Omega_1} \cdot \text{dvol}_{g_0} = \int_{\mathbb{D}} \log |z| |\nabla f_1|^2 |dz|^2. \quad (3.9) \]

Finally, we deduce by (3.8) and (3.9) that
\[ \begin{align*}
\mathcal{E}(\gamma) &= \int_{\Omega_1} |d\mu_1|^2 \cdot \text{dvol}_{g_0} + \int_{\Omega_2} |d\mu_2|^2 \cdot \text{dvol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} \cdot \text{dvol}_{g_0} + 2 \int_{\Omega_2} G_{\Omega_2} K_{g_0} \cdot \text{dvol}_{g_0} + 4\pi \\
&\quad + 4\pi \log |\nabla f_1(0)| + 4\pi \log |\nabla f_2(0)| - 12\pi \log (2) \\
&= \int_{\mathbb{D}} |\nabla \log |\nabla f_1||^2 |dz|^2 + \int_{\mathbb{D}} |\nabla \log |\nabla f_2||^2 |dz|^2 + \int_{\mathbb{D}} \log |z| |\nabla f_1|^2 |dz|^2 + \int_{\mathbb{D}} \log |z| |\nabla f_2|^2 |dz|^2 + 4\pi \\
&\quad + 4\pi \log |\nabla f_1(0)| + 4\pi \log |\nabla f_2(0)| - 12\pi \log (2). \quad (3.10) \end{align*} \]

Up to a rotation of $S^2$, we can assume that $p_2 = N$ and if $\pi: S^2 \setminus \{N\} \to \mathbb{C}$ is the standard stereographic projection, let
\[ f = \pi \circ f_1: \mathbb{D} \to \Omega, \]
\[ \tilde{g} = \pi \circ f_2: \mathbb{D} \to \mathbb{C} \setminus \overline{\Omega}. \]

Recall that $f$ and $\tilde{g}$ are univalent conformal maps, and up to a complex conjugation, we can assume without loss of generality that those maps are holomorphic. Now, since
\[ f_1(z) = \pi^{-1}(f(z)) = \left( \frac{2 \text{Re}(f(z))}{1 + |f(z)|^2}, \frac{2 \text{Im}(f(z))}{1 + |f(z)|^2}, -1 + |f(z)|^2 \right), \]
and immediate computation shows that
\[ \partial_z f_1 = f' \left( \frac{(1 - \overline{f}^2)}{(1 + |f|^2)^2}, \frac{-i(1 + \overline{f}^2)}{(1 + |f|^2)^2}, \frac{2\overline{f}}{(1 + |f|^2)^2} \right), \]
which implies that
\[
|\partial_{z} f_{1}|^2 = \frac{|f'|^2}{(1 + |f|^2)^2} \left( |1 - f'^2|^2 + |1 + f^2|^2 + 4|f|^2 \right) = \frac{2|f'|^2}{(1 + |f|^2)^2}.
\]

Since \( f_1 \) is \( \mathbb{R}^3 \)-valued, we deduce that
\[
|\nabla f_1|^2 = 4|\partial_{z} f_1|^2 = \frac{8|f'|^2}{(1 + |f|^2)^2}
\]

Therefore, we have

\[
\log |\nabla f_1| = \log |f'| - \log(1 + |f|^2) + \frac{3}{2} \log(2),
\]

so that

\[
\log |\nabla f_1(0)| = \log |f'(0)| - \log(1 + |f(0)|^2) + \frac{3}{2} \log(2).
\]

In particular, we have
\[
4\pi \log |\nabla f_1(0)| = 4\pi \log |f'(0)| - 4\pi \log(1 + |f(0)|^2) + 6\pi \log(2) \tag{3.12}
\]

Since \( \Gamma \) is a Weil-Petersson curve, we have \( \log |f'| \in L^2(\mathcal{D}) \), which implies that \( f' \in L^p(\mathcal{D}) \) for all \( p < \infty \) by Lemma 3.2. In particular, we have
\[
\int_{\mathcal{D}} |f'(z)|^2 \frac{|f(z)|^2|dz|^2}{(1 + |f(z)|^2)^2} \leq \frac{1}{4} \|f'\|^2_{L^2(\mathcal{D})}. \tag{3.13}
\]

Therefore, (3.13) implies that \( \log |\nabla f_1| \in W^{1,2}(\mathcal{D}) \) and
\[
\int_{\mathcal{D}} |\nabla \log |\nabla f_1|^2|dz|^2 = \int_{\mathcal{D}} \left| f''(z) - \frac{2f'(z)}{f(z)} |f(z)|^2 \right|^2 |dz|^2 < \infty, \tag{3.14}
\]

while (3.11) implies that
\[
\int_{\mathcal{D}} \log |z||\nabla f_1|^2|dz|^2 = \int_{\mathcal{D}} \log |z| \frac{8|f'(z)|^2|dz|^2}{(1 + |f(z)|^2)^2} \tag{3.15}
\]

Since the function \( \tilde{g} : \mathcal{D} \to \mathbb{C} \setminus \overline{\Omega} \) is unbounded at 0, we do not see trivially that
\[
\int_{\mathcal{D}} \left| \tilde{g''}(z) - \frac{2\tilde{g'}(z)|\tilde{g}(z)|}{1 + |\tilde{g}(z)|^2} \right|^2 |dz|^2 = \int_{\mathcal{D}} \left| \tilde{g''}(z) - \frac{2\tilde{g'}(z)|\tilde{g}(z)|}{g(z) + |\tilde{g}(z)|^2} \right|^2 |dz|^2 < \infty.
\]

As \( \tilde{g} \) is univalent and \( \tilde{g}(0) = \infty \), we deduce that \( \tilde{g} \) admits the following meromorphic expansion at \( z = 0 \) for some \( a \in \mathbb{C} \setminus \{0\} \) and \( a_0, a_1 \in \mathbb{C} \)
\[
\tilde{g}(z) = \frac{a}{z} + a_0 + a_1 z + O(|z|^2).
\]

Therefore, we have by a direct computation
\[
\frac{\tilde{g''}(z)}{\tilde{g'}(z)} = -\frac{2}{z} - \frac{2a_1}{a} z + O(|z|^2)
\]

\[
\frac{\tilde{g'}(z)}{g(z)} \frac{|\tilde{g}(z)|^2}{1 + |\tilde{g}(z)|^2} = -\frac{1}{z} + \frac{a_0}{a} + \left( \frac{2a_1}{a^2} - \frac{a_0}{a^2} \right) z + \frac{z}{|a|^2} + O(|z|^2).
\]

Finally, we get
\[
\frac{\tilde{g''}(z)}{\tilde{g'}(z)} - \frac{2\tilde{g'}(z)}{g(z) + |\tilde{g}(z)|^2} = -\frac{a_0}{a} + \left( \frac{a_0}{a^2} - \frac{4a_1}{a^2} \right) z - \frac{z}{|a|^2} + O(|z|^2) \in L^\infty_{\text{loc}}(\mathcal{D}). \tag{3.16}
\]
Since $\Gamma$ is a Weil-Petersson curve, we deduce that $\log |\tilde{g}'| \in W^{1,2}(\overline{\mathbb{D}} \setminus \overline{\mathbb{V}(0, \varepsilon)})$ for all $\varepsilon > 0$ and we finally deduce that
\[
\int_{\mathbb{D}} \left| \frac{\tilde{g}''(z)}{\tilde{g}'(z)} - 2 \frac{\tilde{g}'(z)}{\tilde{g}(z)} \frac{|\tilde{g}(z)|^2}{1 + |\tilde{g}(z)|^2} \right|^2 |dz|^2 < \infty.
\]
We notice that the integrand is never holomorphic since
\[
\int_{\mathbb{D}} \left| \frac{\tilde{g}''(z)}{\tilde{g}'(z)} - 2 \frac{\tilde{g}'(z)}{\tilde{g}(z)} \frac{|\tilde{g}(z)|^2}{1 + |\tilde{g}(z)|^2} \right|^2 |dz|^2 = - \frac{1}{|a|^2} \neq 0.
\]
Now, if $g = \tilde{g} \circ \iota : \mathbb{C} \setminus \overline{\mathbb{V}} \to \mathbb{C} \setminus \overline{\Omega}$, we compute
\[
\tilde{g}''(z) = - \frac{1}{z^2} g' \left( \frac{1}{z} \right), \quad \quad \quad \tilde{g}''(z) = \frac{1}{z^4} g'' \left( \frac{1}{z} \right) + \frac{2}{z^3} g' \left( \frac{1}{z} \right)
\]
and
\[
\frac{\tilde{g}''(z)}{\tilde{g}'(z)} - 2 \frac{\tilde{g}'(z)}{\tilde{g}(z)} \frac{|\tilde{g}(z)|^2}{1 + |\tilde{g}(z)|^2} = - \frac{1}{z^2} g'' \left( \frac{1}{z} \right) \frac{1}{z} - \frac{2}{z} + \frac{2}{z^2} g' \left( \frac{1}{z} \right) \frac{g(1/z)^2}{1 + |g(1/z)|^2}
\]
\[
= - \frac{1}{z^2} \left( g'' \left( \frac{1}{z} \right) - \frac{2}{g' \left( \frac{1}{z} \right)} \frac{g(1/z)^2}{1 + |g(1/z)|^2} + 2z \right).
\]
A change of variable shows that
\[
\int_{\mathbb{D}} \left| \frac{\tilde{g}''(z)}{\tilde{g}'(z)} - 2 \frac{\tilde{g}'(z)}{\tilde{g}(z)} \frac{|\tilde{g}(z)|^2}{1 + |\tilde{g}(z)|^2} \right|^2 |dz|^2 = \int_{\mathbb{C} \setminus \overline{\Omega}} \left| \frac{g'' \left( \frac{1}{z} \right)}{g' \left( \frac{1}{z} \right)} - \frac{2}{z} \frac{g(1/z)^2}{1 + |g(1/z)|^2} + \frac{2}{z} \right|^2 |dz|^2 \tag{3.17}
\]
Furthermore, we directly get
\[
\int_{\mathbb{D}} \log |z| |\tilde{g}'(z)|^2 |dz|^2 = \int_{\mathbb{D}} \log |z| \left| \frac{g'(1/z)^2}{(1 + |g(1/z)|^2)^2} \right|^2 |dz|^2 = \int_{\mathbb{C} \setminus \overline{\Omega}} \log \left( \frac{1}{|z|} \right) \left| \frac{g'(z)^2}{(1 + |g(z)|^2)^2} \right|^2 |dz|^2
\]
\[
= - \int_{\mathbb{C} \setminus \overline{\Omega}} \log |z| \left| \frac{g'(z)^2}{(1 + |g(z)|^2)^2} \right|^2 |dz|^2 \tag{3.18}
\]
Now, notice that
\[
|\nabla f_2|^2 = \frac{8 |g'(z)^2}{(1 + |g(z)|^2)^2} = \frac{8}{|a|^2} + O(|z|),
\]
which implies that
\[
\log |\nabla f_2| = \frac{3}{2} \log(2) - \log |a|,
\]
where $a \in \mathbb{C} \setminus \{0\}$ is defined by
\[
a = \lim_{z \to 0} (z \tilde{g}(z)).
\]
Furthermore, the previous expansion shows that as $|z| \to \infty$, we have
\[
g(z) = az + O(1),
\]
so that $|a| = |g'(\infty)|$, and
\[
4\pi \log |\nabla f_2(0)| = -4\pi \log |g'(\infty)| + 6\pi \log(2) \tag{3.19}
\]
Finally, we deduce by (3.12) and (3.19) that
\[
4\pi \log |\nabla f_1(0)| + 4\pi \log |\nabla f_2(0)| - 12\pi \log(2) = 4\pi \log |f'(0)| - 4\pi \log g'(\infty) - 4\pi \log(1 + |f(0)|^2),
\]
(3.20)

Gathering (3.10), (3.14), (3.15), (3.17), (3.18), and (3.20), we finally deduce that
\[
\mathcal{E}(\Gamma) = \int_\mathbb{D} |\nabla \log |\nabla f_1|^2|dz|^2 + \int_\mathbb{D} |\nabla \log |\nabla f_2|^2|dz|^2 + \int_\mathbb{D} \log |z| |\nabla f_1|^2|dz|^2 + \int_\mathbb{D} |z| |\nabla f_2|^2|dz|^2 + 4\pi
\]
\[
+ 4\pi \log |\nabla f_1(0)| + 4\pi \log |\nabla f_2(0)| - 12\pi \log(2)
\]
\[
= \int_\mathbb{D} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_\mathbb{D} \left| \frac{f'(z)}{f(z)} \right|^2 \left| \frac{|f(z)|^2}{1 + |f(z)|^2} \right|^2 |dz|^2 + \int_\partial \mathbb{D} \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2
\]
\[
+ 2 \int_\mathbb{D} \log |z| \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_\partial \mathbb{D} \log |z| \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2 + 4\pi
\]
\[
+ 4\pi \log |f'(0)| - 4\pi \log g'(\infty) - 4\pi \log(1 + |f(0)|^2)
\]
\[
= S_3(\Gamma)
\]
which concludes the proof of the theorem. 

Remark 3.4. If \( \Gamma = S^1 \), then we can take \( f = \text{Id}_\mathbb{D} \) and \( g = \text{Id}_{\mathbb{D} \setminus \overline{\mathbb{D}}} \), and we compute
\[
S_3(\Gamma) = \int_\mathbb{D} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_\mathbb{D} \left| \frac{f'(z)}{f(z)} \right|^2 \left| \frac{|f(z)|^2}{1 + |f(z)|^2} \right|^2 |dz|^2
\]
\[
+ 2 \int_\mathbb{D} \log |z| \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_\partial \mathbb{D} \log |z| \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2 + 4\pi
\]
\[
+ 4\pi \log |f'(0)| - 4\pi \log g'(\infty) - 4\pi \log(1 + |f(0)|^2)
\]
\[
= 8 \int_\mathbb{D} \left| \frac{|z|^2}{1 + |z|^2} \right|^2 |dz|^2 + 16 \int_\mathbb{D} \log |z| \left| \frac{|z|^2}{1 + |z|^2} \right|^2 |dz|^2 + 4\pi
\]
\[
= 16\pi \left( 2 \log(2) - 1 \right) + 32\pi \left( -\frac{1}{4} \log(2) \right) + 4\pi
\]
\[
= 0
\]
as expected.

In the next theorem, we finally prove the afore-mentioned equality and obtain as a consequence a direct formula for the Loewner energy of simple curves in \( S^2 \) as the Dirichlet energy of conformal maps into domains of the sphere (up to some additional lower-order terms).

Theorem 3.5. Let \( \Gamma \subset S^2 \) be a closed simple curve of finite Loewner energy. Then we have
\[
I^L(\Gamma) = \frac{1}{\pi} \mathcal{E}_0(\Gamma),
\]
where \( \mathcal{E}_0 \) is defined in Theorem A. Furthermore, if \( \Omega_1, \Omega_2 \subset S^2 \setminus \Gamma \) are the two connected components of \( S^2 \setminus \Gamma \), for all conformal maps \( f_1 : \mathbb{D} \to \Omega_1 \) and \( f_2 : \mathbb{D} \to \Omega_2 \), we have
\[
I^L(\Gamma) = \frac{1}{\pi} \left\{ \int_\mathbb{D} |\nabla \log |\nabla f_1|^2|dz|^2 + \int_\mathbb{D} |\nabla \log |\nabla f_2|^2|dz|^2 + \int_\mathbb{D} \log |z| |\nabla f_1|^2|dz|^2 + \int_\mathbb{D} \log |z| |\nabla f_2|^2|dz|^2 + 4\pi
\]
\[
+ 4\pi \log |\nabla f_1(0)| + 4\pi \log |\nabla f_2(0)| - 12\pi \log(2) \right\}.
\]
Now, by Corollary A.4, we have the identity $I^L(\Gamma) = \frac{1}{2} S_0(\Gamma)$ for all smooth $\Gamma$, and by the preceding Theorem 3.3, we have $\delta_0(\Gamma) = S_2(\Gamma)$ for any simple closed curve $\Gamma$ of finite Loewner energy. Therefore, we will prove that $I^L = \frac{1}{2} S_4$ which will imply our result.

We now let $\Omega_1, \Omega_2 \subset S^2 \setminus \Gamma$ be the two connected components of $S^2 \setminus \Gamma$, and $f_1 : \mathbb{D} \rightarrow \Omega_1$, $f_2 : \mathbb{D} \rightarrow \Omega_2$ be the two canonical conformal maps associated to $\Omega_1$ and $\Omega_2$ in the definition of $\delta_0$, and let as previously $p_1 = f_1(0)$ and $p_2 = f_2(0)$. Up to a rotation on $S^2$ (which does not change any of the energies considered), we can assume that $p_2 = N$. If $\pi : S^2 \setminus \{N\} \rightarrow \mathbb{C}$ is the standard stereographic projection, let $\gamma = \pi(\Gamma)$, and $\Omega$ the bounded component of $\mathbb{C} \setminus \gamma$ and define $f = \pi \circ f_1 : \mathbb{D} \rightarrow \pi(\Omega_1) = \Omega$ and $g = \pi \circ f_2 \circ \iota : \mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{C} \setminus \Omega$ such that (using Theorem 3.3)

$$
\delta_0(\Gamma) = S_3(\gamma) = \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f(z)} \frac{|f(z)|^2}{1 + |f(z)|^2} \right|^2 |dz|^2 + \int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g'(z)}{g(z)} - \frac{g'(z)}{g(z)} \frac{|g(z)|^2}{1 + |g(z)|^2} \right|^2 |dz|^2
$$

$$
+ 2 \int_{\mathbb{D}} \log |z| \left[ 4 \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} - 2 \int_{\mathbb{C} \setminus \mathbb{D}} \log |z| \frac{|g'(z)|^2}{(1 + |g(z)|^2)^2} + 4\pi \right]
$$

$$
+ 4\pi \log |f'(0)| - 4\pi \log |g'(\infty)| - 4\pi \log (1 + |f(0)|^2).
$$

(3.21)

Now, by Corollary A.4 of [20] and Theorem 8.1 [54], if $\{\gamma_n\}_{n \in \mathbb{N}}$ is a sequence of smooth curves converging uniformly to a simple curve $\gamma$ and such that for a sequence of maps $f_n : \mathbb{D} \rightarrow \mathbb{C}$ such that $f_n(0) = 0$, $f'_n(0) = 1$ and $f_n(\mathbb{D}) = \Omega_n$, where $\Omega_n$ is the bounded component of $\mathbb{C} \setminus \gamma_n$, and satisfies

$$
\lim_{n \rightarrow \infty} \int_{\mathbb{D}} \left| \frac{f''_n(z)}{f'_n(z)} - \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 = 0,
$$

where $f : \mathbb{D} \rightarrow \Omega$ is a univalent function such that $f(0) = 0$ and $f'(0) = 1$, we have

$$
I^L(\Gamma_n) \longrightarrow I^L(\Gamma).
$$

(3.22)

In particular, for any sequence of holomorphic maps $g_n : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{\Omega_n}$ such that $g_n(\infty) = \infty$, since

$$
S_1(\Gamma_n) = \pi I^L(\Gamma_n) = \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g''_n(z)}{g'_n(z)} \right|^2 |dz|^2 + 4\pi \log |f'(0)| - 4\pi \log |g'_n(\infty)|
$$

$$
= \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g''_n(z)}{g'_n(z)} \right|^2 |dz|^2 - 4\pi \log |g'(\infty)|
$$

we deduce that

$$
\int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g''_n(z)}{g'_n(z)} \right|^2 |dz|^2 - 4\pi \log |g'_n(\infty)| \longrightarrow n \rightarrow \infty \int_{\mathbb{C} \setminus \mathbb{D}} \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2 - 4\pi \log |g'(\infty)|
$$

(3.23)
for all univalent function $g : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \setminus \overline{D}$ such that $g(\infty) = \infty$. Now, if $\gamma = \pi(\Gamma) \subset \mathbb{C}$, let $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ such that $\varepsilon_n \to 0$, and define
\[ f_n : D \to \mathbb{C} \]
\[ z \mapsto f((1 - \varepsilon_n)z). \]
Then $\gamma_n = f_n(S^1)$ is smooth and converges uniformly to $\gamma$. Furthermore, we have immediately
\[
\int_D \left| \frac{f''_n(z)}{f'_n(z)} \right|^2 |dz|^2 \to 0 \quad (n \to \infty). \tag{3.24}
\]
which implies that
\[
I^L(\gamma_n) = \frac{1}{\pi} S_1(\gamma_n) \to \frac{1}{\pi} S_1(\gamma) = I^L(\gamma).
\]
Furthermore, by Sobolev embedding theorem, $\{f_n\}_{n \in \mathbb{N}}$ is compact in $L^q(D)$ for all $q < \infty$ which implies in particular that
\[
\frac{f'_n}{f_n} \to \frac{f'}{f} \quad \text{in } L^2(D) \quad \text{and} \quad \frac{f''_n}{f'_n} \to \frac{f''}{f'} \quad \text{in } L^q(D) \quad \text{for all } q < \infty
\]
so that
\[
\int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \to 0 \quad (n \to \infty)
\]
\[
\int_D \log |z| \frac{4 |f'(z)|^2 |dz|^2}{(1 + |f(z)|^2)^2} \to 0 \quad (n \to \infty).
\]
Finally, we also have $f_n(0) = f(0)$ and
\[
4 \pi \log |f'_n(0)| = 4 \pi \log |f'(0)| + 4 \pi \log (1 - \varepsilon_n) \to 4 \pi \log |f'(0)|. \tag{3.26}
\]
Therefore, if $\Omega_n = f_n(D)$, and $g_n : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \setminus \overline{D}$ is any univalent map such that $g_n(\infty) = \infty$, since $\gamma_n \to \gamma$ uniformly, we can assume without loss of generality that $g'_n(\infty) = g'(\infty)$. Furthermore, by Corollary A.4 of [20], we also get
\[
\lim_{n \to \infty} \int_{\mathbb{C} \setminus \overline{D}} \left| \frac{g''_n(z)}{g'_n(z)} \right| = 0 \tag{3.27}
\]
\[
\lim_{n \to \infty} \int_{\mathbb{C} \setminus \overline{D}} \left( \frac{g'(z)}{g(z)} - 2 \frac{g'(z)}{g(z)} + \frac{2}{z} \right)^2 |dz|^2 = 0. \tag{3.28}
\]
As previously, we have
\[
\frac{g'_n}{g_n} \to \frac{g'}{g} \quad \text{in } L^2(\mathbb{C} \setminus \overline{D})
\]
\[
\frac{g'_n}{1 + |g_n|^2} \to \frac{g'}{1 + |g|^2} \quad \text{in } L^q(\mathbb{C} \setminus \overline{D}) \quad \text{for all } 2 \leq q < \infty. \tag{3.29}
\]
Therefore, (3.27) and (3.29) imply that
\[
\int_{\mathbb{C} \setminus \overline{D}} \left| \frac{g''_n(z)}{g'_n(z)} \right|^2 |dz|^2 \to 0 \quad (n \to \infty)
\]
\[
\int_{\mathbb{C} \setminus \overline{D}} \log |z| \frac{4 |g''_n(z)|^2 |dz|^2}{(1 + |g_n(z)|^2)^2} \to 0 \quad (n \to \infty).
\]
Finally, we deduce by (3.21), (3.25), (3.26) and (3.30) that
\[ S_0(\gamma_n) \xrightarrow{n \to \infty} S_2(\gamma) \]
which concludes the proof of the theorem by (3.22), Theorem 2.3 and Theorem 3.3.

\[\square\]

**Remark 3.6.** This formula suggests a way to define a Loewner energy for separating curves on Riemann surfaces of higher genus by taking (up to lower-order terms) the Dirichlet energy of uniformisation maps of the simply connected sub-domains.

**Remark 3.7.** To construct the moving frames on \( \Omega_1 \) and \( \Omega_2 \), we must assume that the unit tangent \( \tau : \Gamma \to S^1 \) satisfies \( \tau \in H^{1/2}(\Gamma) \), which is equivalent to \( I^L(\gamma) < \infty \) by a theorem of Bishop ([5]).

## 4 The Loewner Energy as a Renormalised Frame Energy

Before stating the main theorem of this section, recall an easy lemma on harmonic vector fields.

**Lemma 4.1.** Let \( \Sigma \subset \mathbb{R}^3 \) be a smooth surface, \( \bar{n} : \Sigma \to S^2 \) its unit normal, and \( g = \ast_{\mathbb{R}^3} \) be the induced metric. Assume that \( \bar{u} : \Sigma \to S^2 \) is a smooth critical point of the Dirichlet energy amongst \( S^2 \)-valued maps such that \( \langle \bar{u}, \bar{n} \rangle = 0 \). Then \( \bar{u} \) satisfies the following Euler-Lagrange equation:
\[
-\Delta_g \bar{u} = |d\bar{u}|^2_g \bar{u} + (\langle d\bar{u}, d\bar{n} \rangle_g + \langle \bar{u}, \Delta_g \bar{n} \rangle) \bar{n}. \tag{4.1}
\]

**Proof.** We proceed as in [29] in Lemma (1.4.10), taking variations that also satisfy \( \langle \bar{X}, \bar{n} \rangle = 0 \). \[\square\]

**Theorem 4.2.** Let \( \Gamma \subset S^2 \) a simple curve of finite Loewner energy and let \( \Omega_1, \Omega_2 \) the two open connected components of \( S^2 \setminus \Gamma \). Fix some \( j=1,2 \). Then, for all \( p_j \in \Omega_j \), there exists harmonic moving frames \((\bar{e}_j, \bar{f}_j) : \Omega_j \setminus \{p_j\} \to U\Omega_j \times U\Omega_j \) such that the Cartan form \( \omega_j = (\bar{e}_j, d\bar{f}_j) \) admits the decomposition
\[
\omega_j = \ast d(G_{\Omega_j} + \mu_j), \tag{4.2}
\]
where \( G_{\Omega_j} = G_{\Omega_j, p_j} : \Omega_j \setminus \{p_j\} \to \mathbb{R} \) is the Green’s function of the Laplacian \( \Delta_{g_0} \) on \( \Omega_j \) with Dirichlet boundary condition and singularity \( p_j \in \Omega_j \), and \( \mu_j \in C^\infty(\Omega_j) \) satisfies
\[
\begin{aligned}
-\Delta_{g_0} \mu_j &= 1 \quad \text{in } \Omega_j \\
\partial_{\nu} \mu_j &= k_{g_0} - \partial_{\nu} G_{\Omega_j} \quad \text{on } \partial \Omega_j, \tag{4.3}
\end{aligned}
\]
where \( k_{g_0} \) is the geodesic curvature on \( \Gamma = \partial \Omega_j \), and there exists conformal maps \( f_1 : \mathbb{D} \to \Omega_1 \) and \( f_2 : \mathbb{D} \to \Omega_2 \) such that \( f_1(0) = p_1 \) and \( f_2(0) = p_2 \) and
\[
I^L(\Gamma) = \frac{1}{\pi} \left\{ \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} + \int_{\Omega_2} |d\mu_2|^2_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_2} G_{\Omega_2} K_{g_0} d\text{vol}_{g_0} + 4 \right\} \\
+ 4 \log |\nabla f_1(0)| + 4 \log |\nabla f_2(0)| - 12 \log(2) \tag{4.4}
\]

**Remark 4.3.** To see another derivation of the equation satisfied by \( \mu \) in a more general setting, see Theorem 6.7 in the Appendix 6.3.

**Proof.** To overcome the issue of regularity to get the functions \( f_j : \mathbb{D} \to \Omega_j \) in the Frobenius theorem used in [32] and treat this non-flat case (notice that in [32], one need not use the Frobenius theorem in general by using the moving frame given by the Ginzburg-Landau minimisation, but we do not have such a frame on the sphere in a trivial way), we will construct them directly with the associated moving frame by using the Uniformisation theorem that does not suppose any regularity of the simply connected domain.

**Step 1.** Definition of \((\bar{e}_j, \bar{f}_j)\) and \( \mu_j \).
Let \( \pi : S^2 \setminus \{ N \} \to \mathbb{C} \) be the standard stereographic projection and assume without loss of generality that \( N \in \Omega_2 \). If \( \Omega = \pi(\Omega_1) \subset \mathbb{C} \) and \( \gamma = \pi(\Gamma) \subset \mathbb{C} \), by the Uniformisation Theorem, there exists univalent holomorphic maps \( f : \mathbb{D} \to \Omega \) and \( g : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \Gamma \) such that

\[
I^L(\Gamma) = I^L(\gamma) = \frac{1}{\pi} \left\{ \int_{\partial \mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_{\Gamma \setminus \mathbb{D}} \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2 + 4\pi \log |f'(0)| - 4\pi \log |g'(\infty)| \right\}.
\]

Now, let \( f_1 = \pi^{-1} \circ f : \mathbb{D} \to \Omega_1 \) and \( f_2 = \pi^{-1} \circ g \circ \iota : \mathbb{D} \to \Omega_2 \) be the induced conformal maps on \( S^2 \), where \( \iota : \mathbb{C} \setminus \{ 0 \} \to \mathbb{C} \setminus \{ 0 \} \) is the standard inversion. Then a computation in the proof of Theorem 3.3 shows that

\[
\partial_z f_1 = f' \left( \frac{(1 + \overline{f'})^2}{(1 + |f(z)|^2)^2} \right).
\]

Now, by analogy with the previous construction in Section 2 (see also [32], Proposition 5.1), define \( \mu_1 : \Omega_1 \to \mathbb{R} \) and \( \iota_1 : \Omega_1 \to US^2 \) and \( \bar{f}_1 : \Omega_1 \to US^2 \) by

\[
\begin{aligned}
&\partial_z f_1 = e^{\mu_1 \circ f_1} \bar{f}_1 \circ f_1 \\
&\frac{1}{r} \partial_\theta f_1 = e^{\mu_1 \circ f_1} \iota_1 \circ f_1.
\end{aligned}
\]

Then we have

\[
e^{2\mu_1 \circ f_1} = |\partial_z f_1|^2 = \frac{1}{r^2} |\partial_\theta f_1|^2 = 2|\partial_z f_1|^2 = \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2}.
\]

Therefore, we deduce if \( \mu = \mu_1 \circ f_1 \) that

\[
\mu(z) = \log |f'(z)| - \log(1 + |f(z)|^2) - \frac{1}{2} \log(2), \tag{4.5}
\]

Since \( \partial_z = \frac{1}{2} (\partial_x - i \partial_y) \), we have

\[
\begin{aligned}
&\partial_z f_1 = \cos(\theta) \partial_x f_1 + \sin(\theta) \partial_y f_1 = \operatorname{Re} \left( \frac{z}{|z|^2} \right) \operatorname{Re} (\partial_z f_1) - \operatorname{Im} \left( \frac{z}{|z|^2} \right) \operatorname{Im} (\partial_z f_1) \\
&\frac{1}{r} \partial_\theta f_1 = -\sin(\theta) \partial_x f_1 + \cos(\theta) \partial_y f_1 = -\operatorname{Im} \left( \frac{z}{|z|^2} \right) \operatorname{Re} (\partial_z f_1) - \operatorname{Re} \left( \frac{z}{|z|^2} \right) \operatorname{Im} (\partial_z f_1).
\end{aligned}
\]

By the elementary identities for all \( a, b \in \mathbb{C} \)

\[
\begin{aligned}
&\operatorname{Re} (a) \operatorname{Re} (b) + \operatorname{Im} (a) \operatorname{Im} (b) = \operatorname{Re} (ab) \\
&\operatorname{Re} (a) \operatorname{Im} (b) + \operatorname{Im} (a) \operatorname{Re} (b) = \operatorname{Im} (ab),
\end{aligned}
\]

we deduce that

\[
\begin{aligned}
&\partial_z f_1 = \operatorname{Re} \left( \frac{z}{|z|^2} \partial_x f_1 \right) = \operatorname{Re} \left( \frac{z}{|z|^2} \partial_y f_1 \right) \\
&= \operatorname{Re} \left( \frac{z}{|z|^2} \overline{f'(z)} \left( \frac{1 - f(z)^2}{(1 + |f(z)|^2)^2} \right) \right) = \operatorname{Re} \left( \frac{z}{|z|^2} \overline{f'(z)} \left( \frac{1 + f(z)^2}{(1 + |f(z)|^2)^2} \right) \right) \\
&\frac{1}{r} \partial_\theta f_1 = -\operatorname{Im} \left( \frac{z}{|z|^2} \partial_x f_1 \right) = \operatorname{Im} \left( \frac{z}{|z|^2} \partial_y f_1 \right) \\
&= \operatorname{Im} \left( \frac{z}{|z|^2} \overline{f'(z)} \left( \frac{1 - f(z)^2}{(1 + |f(z)|^2)^2} \right) \right) = \operatorname{Im} \left( \frac{z}{|z|^2} \overline{f'(z)} \left( \frac{1 + f(z)^2}{(1 + |f(z)|^2)^2} \right) \right).
\end{aligned}
\]

More generally, if \( \varphi : \mathbb{C} \to \mathbb{C} \) is a smooth complex function, we have

\[
\partial_\theta \varphi = -\operatorname{Im} (z) (\partial_x + \partial_y) \varphi + \operatorname{Re} (z) i (\partial_z - \partial_x) \varphi = i (\partial_z \varphi - \partial_x \varphi). \tag{4.6}
\]
Since
\[ |\partial_z f_1| = \frac{1}{r} |\partial_y f_1| = \frac{2|f'(z)|}{1 + |f(z)|^2}, \]
we deduce that
\[
\begin{align*}
\tilde{f}_1 \circ f_1 &= \text{Re} \left( \frac{zf'(z)}{|z f'(z)|} \left( \frac{(1 - f(z)^2)}{(1 + |f(z)|^2)^2} + \frac{2f(z)}{(1 + |f(z)|^2)^2} \right) \right) \\
\tilde{e}_1 \circ f_1 &= \text{Im} \left( \frac{zf'(z)}{|z f'(z)|} \left( \frac{(1 - f(z)^2)}{(1 + |f(z)|^2)^2} + \frac{2f(z)}{(1 + |f(z)|^2)^2} \right) \right).
\end{align*}
\tag{4.7}
\]
Notice that
\[ F(z) = ((1 - f(z)^2), i(1 + f(z)^2), 2f(z)) \]
is a holomorphic null vector, i.e. \( \langle F(z), F(z) \rangle = 0 \), so we see directly since \( |\tilde{e}_1| = |\tilde{f}_1| = 1 \) that
\[ \langle \tilde{e}_1, \tilde{f}_1 \rangle = 0. \]

**Step 2.** Verification of the system (4.3).
First, recalling the definition of \( \mu \) in (4.5), we check that
\[ -\Delta \mu = e^{2\nu} \tag{4.8} \]
Indeed, since \( \log |f'| \) is harmonic, we have
\[ \Delta \mu = 4 \partial_z \left( \frac{f'(z)\bar{f}'(z)}{1 + |f(z)|^2} \right) = 4 \left( \frac{|f'(z)|^2}{1 + |f(z)|^2} - \frac{|f'(z)|^2 f(z)^2}{(1 + |f(z)|^2)^2} \right) = \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2} = e^{2\nu}. \]
Recalling that
\[ g_0 = \frac{4|dz|^2}{(1 + |z|^2)^2} = (\pi^{-1})^* g_{S^2}, \]
we deduce that
\[ \frac{4|f'(z)|^2|dz|^2}{(1 + |f(z)|^2)^2} = f^* g_0 = f^* (\pi^{-1})^* g_{S^2} = (\pi^{-1} \circ f)^* g_{S^2} = f_1^* g_{S^2}. \]
Therefore, (4.8) can be rewritten as
\[ -\Delta f_1^* g_{S^2} (\mu_1 \circ f_1) = 1 \]
or by conformal invariance of the Dirichlet energy
\[ -\Delta g_{S^2} \mu_1 = 1. \tag{4.9} \]

Now, let us check that \( \mu \) satisfies the adequate boundary conditions. If \( \varphi : \mathbb{C} \to \mathbb{R} \) is a smooth function, we have
\[
\partial_{\nu} \varphi = \frac{x}{\sqrt{x^2 + y^2}} \partial_x \varphi + \frac{y}{\sqrt{x^2 + y^2}} \partial_y \varphi = \frac{\text{Re}(z)}{|z|} (\partial_z + \partial_{\bar{z}}) \varphi + \frac{\text{Im}(z)}{|z|} i (\partial_z - \partial_{\bar{z}}) \varphi = 2 \text{Re} \left( \frac{z}{|z|} \partial_z \varphi \right).
\]
This implies since \( \mu(z) = \log |f'(z)| - \log(1 + |f(z)|^2) + \log(2) \) by (4.5) that
\[ \partial_z \mu(z) = \frac{1}{2} \frac{f''(z)}{f'(z)} - \frac{f'(z)^2}{f(z)} \frac{|f(z)|^2}{1 + |f(z)|^2}, \]
\[ |\partial_{\nu} \varphi| = \frac{1}{r} |\partial_y \varphi| = \frac{2|f''(z)|}{1 + |f(z)|^2}, \]
\[ \partial_{\nu} \varphi = \frac{1}{2} \frac{f''(z)}{f'(z)} - \frac{f'(z)^2}{f(z)} \frac{|f(z)|^2}{1 + |f(z)|^2}. \]

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and
\[ \partial_\nu \mu = \text{Re} \left( \frac{zf''(z)}{f'(z)} - 2zf'(z) \frac{|f(z)|^2}{1 + |f(z)|^2} \right). \]

Recall that the geodesic curvature on \( \Gamma \) is given (see [13]) by
\[ k_{\gamma u} = \langle \bar{e}_1, \partial_\nu \bar{f} \rangle. \]

Now, let
\[ \varphi(z) = \frac{zf'(z)}{|z|f'(z)} \left( \frac{(1 - f(z)^2) i(1 + f(z)^2)}{(1 + |f(z)|^2)}, \frac{2f(z)}{(1 + |f(z)|^2)} \right) = \chi(z) \psi(z), \]
where
\[
\begin{align*}
\chi(z) &= \frac{zf'(z)}{|z|f'(z)} = \exp \left( \frac{1}{2} \log \left( \frac{zf'(z)}{|z|f'(z)} \right) - \frac{1}{2} \log (zf'(z)) \right) \\
\psi(z) &= \left( \frac{1 - f(z)^2}{1 + |f(z)|^2}, \frac{i(1 + f(z)^2)}{1 + |f(z)|^2}, \frac{2f(z)}{1 + |f(z)|^2} \right). \tag{4.10}
\end{align*}
\]

Therefore, we deduce that
\[
\begin{align*}
\partial_2 \chi &= \frac{1}{2} \left( \frac{zf''(z)}{f'(z)} + \frac{1}{z} \right) \chi \\
\partial_2 \chi &= \frac{1}{2} \left( \frac{zf''(z)}{f'(z)} + \frac{1}{z} \right) \chi. \tag{4.11}
\end{align*}
\]

We also get
\[
\begin{align*}
\partial_2 \psi &= -\frac{f'(z)zf(z)}{1 + |f(z)|^2} \psi + \frac{2f'(z)}{1 + |f(z)|^2} (-f(z), i f(z), 1) \\
\partial_2 \psi &= -\frac{f'(z)f(z)}{1 + |f(z)|^2} \psi. \tag{4.12}
\end{align*}
\]

Since \( \langle \psi, \psi \rangle = 0 \) and \( |\psi|^2 = 2 \), we have \( \langle \partial_2 \psi, \psi \rangle = \langle \partial_2 \psi, \psi \rangle = 0 \). In particular, we have
\[ \langle (f(z), i f(z), 1), \psi \rangle = \frac{1}{1 + |f(z)|^2} \langle ((-f(z), i f(z), 1), (1 - f(z)^2, i(1 + f(z)^2), 2f(z))) = 0, \tag{4.13} \]
while
\[
\begin{align*}
\langle (-f(z), i f(z), 1), &\left( \frac{1 - f(z)^2}{1 + |f(z)|^2}, -i \frac{1 + f(z)^2}{1 + |f(z)|^2}, 2f(z) \right) \rangle \\
&= -f(z) + \frac{f(z)|f(z)|^2 + f(z) + |f(z)|f(z)|^2 + 2f(z)}{2|f(z)|1 + |f(z)|^2}.
\end{align*}
\]
so that
\[ \langle (-f(z), i f(z), 1), \psi \rangle = 2f(z). \tag{4.14} \]

Therefore, we deduce by (4.12), (4.13) and (4.14) that
\[
\begin{align*}
\langle \varphi, \varphi \rangle &= \langle \partial_2 \varphi, \varphi \rangle = \langle \partial_2 \varphi, \varphi \rangle = \langle \psi, \psi \rangle = \langle \partial_2 \psi, \psi \rangle = \langle \partial_2 \psi, \psi \rangle = 0 \\
\langle \partial_2 \psi, \psi \rangle &= \frac{2f'(z)f(z)}{1 + |f(z)|^2} \\
\langle \partial_2 \psi, \psi \rangle &= -\frac{2f'(z)f(z)}{1 + |f(z)|^2}. \tag{4.15}
\end{align*}
\]

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The identities (4.11) and (4.15) imply that
\[ z\partial_z \varphi - \overline{\varphi} \partial_{\overline{z}} \varphi = - \left( \Re \left( z \frac{f''(z)}{f'(z)} \right) + 1 \right) \varphi + \chi(z) (z\partial_z \psi - \overline{\varphi} \partial_{\overline{z}} \psi), \]
and since \( |\chi|^2 = 1 \) and \( |\psi|^2 = 2 \), we have
\[ \langle z\partial_z \varphi - \overline{\varphi} \partial_{\overline{z}} \varphi, \varphi + \overline{\varphi} \rangle = -2 \left( \Re \left( z \frac{f''(z)}{f'(z)} \right) + 1 \right) \]
\[ + \chi(z) \left( \chi(z)(\partial_z \varphi, \varphi) + \chi(z)(\partial_{\overline{z}} \varphi, \overline{\varphi}) - \overline{\chi(z)}(\partial_z \psi, \varphi) \right) \]
\[ = -2 \left( \Re \left( z \frac{f''(z)}{f'(z)} \right) + 1 \right) + 2z \frac{f'(z)}{f(z)} \frac{|f'(z)|^2}{f(z) + |f(z)|^2} + 2\pi f'(z) \frac{|f(z)|^2}{f(z) + |f(z)|^2} \]
\[ = -2 \Re \left( z \frac{f''(z)}{f'(z)} - 2z \frac{f'(z)}{f(z)} \frac{|f(z)|^2}{f(z) + |f(z)|^2} \right), \]
so that
\[ k_{y_0} = \langle \vec{e}, \partial_y \vec{f} \rangle = - \langle \partial_y \vec{e}, \vec{f} \rangle = - \langle \partial_y \Re(\varphi), \Re(\varphi) \rangle = - \Re \langle \partial_y \varphi, \Re(\varphi) \rangle \]
\[ = -\frac{1}{2} \Im \langle \langle i(z\partial_z \varphi - \overline{\varphi} \partial_{\overline{z}} \varphi, \varphi + \overline{\varphi}) \rangle \rangle = -\frac{1}{2} \Re \langle z\partial_z \varphi - \overline{\varphi} \partial_{\overline{z}} \varphi, \varphi + \overline{\varphi} \rangle \]
\[ = \Re \left( z \frac{f''(z)}{f'(z)} - 2z \frac{f'(z)}{f(z)} \frac{|f(z)|^2}{f(z) + |f(z)|^2} \right) + 1 = \partial_{\nu} \mu + 1 = \partial_{\nu} \mu + \partial_{\nu} G_{D}, \]
which concludes the proof of the claim by the conformal invariance of the Dirichlet energy (we denoted for simplicity \( G_{D} = G_{D,0} = \log |\cdot| \)).

**Step 3.** Verification that \((\vec{e}_1, \vec{f}_1)\) is a harmonic moving frame.

Now, thanks to Lemma 4.1 and (4.1), the maps \( \vec{e}_1 \) and \( \vec{f}_1 \) are unit harmonic moving frames if and only if they satisfy in the distributional sense (see Theorem 4.5) the system (writing \( \vec{e}_1 \) for \( \vec{e}_1 \circ f_1 \) and \( \vec{f}_1 \) for \( \vec{f}_1 \circ f_1 \) for simplicity)
\[
\begin{cases}
-\Delta \vec{e}_1 = |\nabla \vec{e}_1|^2 \vec{e}_1 + (2 \langle \nabla \vec{e}_1, \nabla \vec{n} \rangle + \langle \vec{e}_1, \Delta \vec{n} \rangle) \vec{n}
- \Delta \vec{f}_1 = |\nabla \vec{f}_1|^2 \vec{f}_1 + (2 \langle \nabla \vec{f}_1, \nabla \vec{n} \rangle + \langle \vec{f}_1, \Delta \vec{n} \rangle) \vec{n}.
\end{cases}
\]
where \( \vec{n} : D \to S^2 \) is the composition by \( f \) of the identity map \( \text{Id} : S^2 \to S^2 \). It is given by
\[
\vec{n}(z) = \left( \frac{2 \Re(f(z))}{1 + |f(z)|^2}, \frac{2 \Im(f(z))}{1 + |f(z)|^2}, -1 + |f(z)|^2 \right).
\]
Notice that \( \vec{n} \) is nothing else than the Gauss map associated to the branched minimal immersion of the disk from \( D \) into \( \mathbb{R}^3 \) with Weierstrass data \( (f, dz) \). For example, taking \( f(z) = z \) if \( \Gamma = S^1 \), the associated minimal immersion is branched at 0 of order 2 and is a part of the Enneper surface, given by
\[
\vec{\Phi} : D \to \mathbb{R}^3
\]
\[ z \mapsto \Re \left( -\frac{1}{3}z^3 + z, i \left( \frac{1}{3}z^3 + z \right), z^2 \right). \]
The Gauss map satisfies by an immediate computation
\[
|\nabla \vec{n}(z)|^2 = \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2} - \Delta \vec{n} = |\nabla \vec{n}|^2 \vec{n}.
\]
Furthermore, we immediately check that
\[
\langle F(z), \vec{n}(z) \rangle = \left( 1 - f(z)^2, i(1 + f(z)^2), 2f(z), \left( \frac{2 \Re(f(z))}{1 + |f(z)|^2}, \frac{2 \Im(f(z))}{1 + |f(z)|^2}, -1 + |f(z)|^2 \right) \right) = 0,
\]
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so that $(\vec{e}_1, \vec{n}) = (\vec{f}_1, \vec{n}) = 0$ as expected. In particular, the previous equation (4.17) must reduce to

$$\begin{align*}
-\Delta \vec{e}_1 &= |\nabla \vec{e}_1|^2 \vec{e}_1 + 2(\nabla \vec{e}_1, \nabla \vec{n}) \vec{n} \\
-\Delta \vec{f}_1 &= |\nabla \vec{f}_1|^2 \vec{f}_1 + 2(\nabla \vec{f}_1, \nabla \vec{n}) \vec{n}.
\end{align*}$$

(4.18)

However, since $(\vec{e}_1, \Delta \vec{n}) = 0$, we immediately deduce that $-\langle \Delta \vec{e}_1, \vec{n} \rangle = 2(\nabla \vec{e}_1, \nabla \vec{n})$, and since $|\nabla \vec{e}_1|^2 = 1$, we also get $-\langle \Delta \vec{e}_1, \vec{e}_1 \rangle = |\nabla \vec{e}_1|^2 \vec{e}_1$. Therefore, we need only check that

$$\langle \Delta \vec{e}_1, \vec{f}_1 \rangle = \langle \Delta \vec{f}_1, \vec{e}_1 \rangle = 0$$

(4.19)

to show that $\vec{e}_1$ and $\vec{f}_1$ satisfy the equations (4.18). Since $\vec{e}_1 = \text{Re} (\varphi)$ and $\vec{f}_1 = \text{Im} (\varphi)$, we deduce that

$$\Delta \vec{e}_1 = \text{Re} (\Delta \varphi), \quad \Delta \vec{f}_2 = \text{Im} (\Delta \varphi),$$

and we have

$$\begin{align*}
&\begin{cases}
\langle \text{Re} (\Delta \varphi), \text{Im} (\varphi) \rangle = \frac{1}{2} \text{Im} (\langle \Delta \varphi, \varphi \rangle) - \frac{1}{2} \text{Im} (\langle \Delta \varphi, \overline{\varphi} \rangle) \\
\langle \text{Im} (\Delta \varphi), \text{Re} (\varphi) \rangle = \frac{1}{2} \text{Im} (\langle \Delta \varphi, \varphi \rangle) + \frac{1}{2} \text{Im} (\langle \Delta \varphi, \overline{\varphi} \rangle).
\end{cases}
\end{align*}$$

(4.20)

Therefore, the equations (4.19) are equivalent to

$$\text{Im} (\langle \Delta \varphi, \varphi \rangle) = \text{Im} (\langle \Delta \varphi, \overline{\varphi} \rangle) = 0.$$  

(4.21)

Using (4.12), (4.13) and (4.15), we get

$$\langle \Delta \varphi, \varphi \rangle = -4(\partial_z \varphi, \partial_{\overline{\varphi}})$$

(4.22)

$$= -4 \left\langle -\frac{1}{2} \left( f''(z) + \frac{1}{z} \right) \varphi - \frac{f'(z)f(z)}{1 + |f(z)|^2} (-f(z), f(z), 1), \left( \frac{1}{2} \left( \frac{f''(z)}{f'(z)} + \frac{1}{z} \right) - \frac{f'(z)f(z)}{1 + |f(z)|^2} \right) \varphi \right\rangle = 0,$$

which implies in particular that $\text{Im} (\langle \Delta \varphi, \varphi \rangle) = 0$. Then, we compute

$$\langle \partial_{\overline{\varphi}} \varphi \rangle = \left\langle \left( \frac{1}{2} \left( \frac{f''(z)}{f'(z)} + \frac{1}{z} \right) - \frac{f'(z)f(z)}{1 + |f(z)|^2} \right) \varphi, \varphi \right\rangle = \left( \frac{1}{2} \left( \frac{f''(z)}{f'(z)} + \frac{1}{z} \right) - 2 \frac{f'(z)f(z)}{1 + |f(z)|^2} \right).$$

(4.23)

Therefore, we have

$$\frac{1}{4} \langle \Delta \varphi, \overline{\varphi} \rangle = \partial_z \langle \partial_{\overline{\varphi}} \varphi \rangle - \langle \partial_{\overline{\varphi}} \varphi, \partial_z \varphi \rangle = \partial_z \langle \partial_{\overline{\varphi}} \varphi, \varphi \rangle - \langle \partial_{\overline{\varphi}} \varphi, \partial_z \varphi \rangle = \partial_z \langle \partial_{\overline{\varphi}} \varphi \rangle.$$  

where we used $\partial_z \overline{\varphi} = \partial_{\overline{\varphi}} \varphi$. By (4.23), we deduce that

$$\partial_z \langle \partial_{\overline{\varphi}} \varphi \rangle = -2 \frac{|f'(z)|^2}{1 + |f(z)|^2} + 2 \frac{|f'(z)|^2 |f(z)|^2}{(1 + |f(z)|^2)^2} = -\frac{2 |f'(z)|^2}{(1 + |f(z)|^2)^2},$$

so that

$$\langle \Delta \varphi, \overline{\varphi} \rangle = -\frac{8 |f'(z)|^2}{(1 + |f(z)|^2)^2} - 4 |\partial_{\overline{\varphi}} \varphi|^2 \in \mathbb{R},$$

which implies that

$$\text{Im} (\langle \Delta \varphi, \overline{\varphi} \rangle) = 0.$$  

(4.24)

Therefore, we deduce that (4.21) holds, which implies that $\vec{e}_1$ and $\vec{f}_1$ solve the equations (4.18).

**Step 4.** Obtention of the decomposition $\omega_1 = * d (G_{R_1} + \mu_1)$.

Define

$$\omega = \langle \vec{e}, \vec{d} \vec{f} \rangle = \langle \vec{e}, \partial_{\overline{\varphi}} \vec{f} \rangle + \langle \vec{e}, \partial_{\overline{\varphi}} \vec{f} \rangle.$$
Recall that since \( *dx = dy \) and \( *dy = -dx \), we have
\[
*dz = *(dx + i
dy) = dy - i
dx = -i(dx + i
dy) = -i
dz
\]
\[
*dz^2 = i
dz^2.
\]
Therefore, \( \omega = *d(\mu + G) \) (where we write for simplicity \( G = G_0 = \log |\cdot| \)) if and only if
\[
\langle \vec{e}, \partial \vec{f} \rangle + \langle \vec{e}, \vec{f} \rangle = * (\partial (\mu + G) + \vec{f} (\mu + G)) = -i \partial (\mu + G) + i \vec{f} (\mu + G),
\]
which is equivalent to the identity
\[
\langle \vec{e}, \partial \vec{f} \rangle = -i \partial (\mu + G) .
\] (4.25)

We have by (4.11) and (4.12)
\[
\partial_z \vec{f} = \partial_z \text{Re} (\varphi) = \frac{1}{2} (\partial_z \varphi + \partial \varphi)
\]
\[
= \frac{1}{2} \left( -\frac{1}{2} \left( f''(z) + \frac{1}{z} \right) \varphi - \frac{f'(z)f(z)}{1 + |f(z)|^2} \varphi + \frac{2f'(z)}{1 + |f(z)|^2} \chi(z) - f'(z), i f(z), 1 \right)
\]
\[
+ \frac{1}{2} \left( \frac{f''(z)}{f(z)} + \frac{1}{z} \right) \varphi - \frac{f'(z)f(z)}{1 + |f(z)|^2} \varphi
\]
\[
= -\frac{i}{2} \left( f''(z) + \frac{1}{z} \right) \text{Im} (\varphi) - \frac{f'(z)f(z)}{1 + |f(z)|^2} \text{Re} (\varphi) + \frac{f'(z)}{1 + |f(z)|^2} \chi(z) - f'(z), i f(z), 1
\]
\[
= -\frac{i}{2} \left( f''(z) - \frac{2f'(z)f(z)}{1 + |f(z)|^2} + \frac{1}{z} \right).
\]
Therefore, using (4.13), (4.14), (4.15), and \( \langle \vec{e}, \vec{f} \rangle = 0 \), we deduce that
\[
\langle \vec{e}, \partial_z \vec{f} \rangle = \langle \text{Im} (\varphi), \partial_z \text{Re} (\varphi) \rangle = -\frac{i}{2} \left( f''(z) + \frac{1}{z} \right) - \frac{f'(z)f(z)}{1 + |f(z)|^2} \langle (-f'(z), i f(z), 1), \varphi \rangle
\]
\[
= -\frac{i}{2} \left( f''(z) - \frac{2f'(z)f(z)}{1 + |f(z)|^2} + \frac{1}{z} \right),
\]
and this concludes the proof of (4.25) since by (4.5)
\[
\partial_z (\mu(z) + \log |z|) = \partial_z \left( \log |f'(z)| - \log(1 + |f(z)|^2) - \frac{1}{2} \log(2) + \log |z| \right)
\]
\[
= \frac{1}{2} \left( f''(z) - \frac{2f'(z)f(z)}{1 + |f(z)|^2} + \frac{1}{z} \right).
\]
Finally, the identity (4.4) follows by Theorem 3.5, and this concludes the proof of the theorem. 

Remarks 4.4. (1) Using the next Theorem 4.5, it is easy to check that \( (\vec{e}_j, \vec{f}_j) \ (j = 1, 2) \) are harmonic vector fields since by (4.10) and (4.11), we have
\[
-\Delta \chi = \left| f''(z) + \frac{1}{z} \right| \chi = |\nabla \chi|^2 \chi,
\]
\it i.e. \( \chi : \mathbb{D} \rightarrow S^1 \) is a harmonic map with values into \( S^1 \).

(2) To define the geodesic curvature on \( \Gamma \), we see by the formula (4.16) that we need to make sense of the trace of the holomorphic function \( h = \frac{f'}{f} \) on \( \partial \mathbb{D} \). Since we only have \( h \in L^2(\mathbb{D}) \), it is \it a priori \ unclear whether the trace of \( h \) exists on \( \partial \mathbb{D} = S^1 \) or not. However, since \( k_{\varphi_0} = (\vec{e}, \partial_0 \vec{f}) \) (with the notation of the proof of Theorem 4.2), and \( \vec{e}, \vec{f} \in W^{1,2}(\mathbb{D} \setminus \bar{\mathbb{D}}(0, \varepsilon)) \) for all \( \varepsilon > 0 \), we deduce by trace theory that
\( \vec{e}, \vec{f} \in H^{\frac{1}{2}}(S^1) \). In particular, the product \( k_{g_0} = \langle \vec{e}, \partial_\theta \vec{f} \rangle \) is well-defined by the \( H^{\frac{1}{2}}/H^{-\frac{1}{2}} \) duality in the distributional sense. In fact, \( k_{g_0} \in H^{-\frac{1}{2}}(S^1) \) (see Lemma 6.3). Conversely, it would be interesting to characterise the simple closed curves \( \Gamma \subset S^2 \) (or equivalently \( \gamma \subset \mathbb{C} \)) such that their geodesic curvature \( k_{g_0} \) satisfies \( k_{g_0} \in \mathcal{D}(\Gamma) \) or \( k_{g_0} \in H^{-\frac{1}{2}}(\Gamma) \).

Notice that by C. Bishop’s result ([5]), a closed simple curve \( \Gamma \) is in the Weil-Petersson class if and only if and only if \( \Gamma \) is chord-arc and its arc-length parametrisation is in \( H^{\frac{1}{2}} \) parametrisation. Due to \( \Gamma \) being chord-arc, we can as above define the Sobolev space \( H^{\frac{1}{2}}(\Gamma) \), and the unit tangent \( \tau \) satisfies \( \tau \in H^{\frac{1}{2}}(\Gamma) \). Therefore, we deduce directly if \( \nu \) is the unit normal of \( \Gamma \) that \( k_{g_0} = \langle \tau, \partial_\nu \nu \rangle \) is a tempered distribution.

The problem of characterising the curves whose geodesic curvature is a distribution or lies in \( L^1 \) is reminiscent of the one of characterising the functions with values into \( S^1 \) that admit a degree (this problem was mostly considered by H. Brezis, L. Nirenberg, P. Mironescu, J. Bourgain, and collaborators, see [10], [11]). Recall that for all function \( u : S^1 \to S^1 \subset \mathbb{R}^2 \), we have

\[
\text{deg}(u) = \frac{1}{2\pi} \int_{S^1} u \wedge \partial_\tau u \, d\mathcal{H}^1. \tag{4.26}
\]

Equivalently, seeing \( u \) as a complex-valued function, we have

\[
\text{deg}(u) = \frac{1}{2\pi} \int_{S^1} \overline{u} \partial_\theta u \, d\theta.
\]

As above, we see that this quantity is well-defined provided \( u \in H^{\frac{1}{2}}(S^1) \). Furthermore, writing \( u(z) = \sum_{n \in \mathbb{Z}} a_n z^n \), we notice that

\[
\int_{S^1} \overline{u} \partial_\theta u \, d\theta = \sum_{n,m \in \mathbb{Z}} \int_{S^1} n \overline{a_m} a_n e^{i(n-m)\theta} \, d\theta = 2\pi \sum_{n \in \mathbb{Z}} \lvert a_n \rvert^2,
\]

so that

\[
\text{deg}(u) = \sum_{n \in \mathbb{Z}} \lvert a_n \rvert^2
\]

while \( u \in H^{\frac{1}{2}}(S^1) \) if and only if

\[
\sum_{n \in \mathbb{Z}} \lvert n \rvert \lvert a_n \rvert^2 < \infty,
\]

which is obviously a stronger property. In [10] and [11], Brezis and Nirenberg define a notion of degree that satisfies to the natural axiomatic properties of the degree for the space \( \text{VMO}(S^1) \) (the space of functions that have a vanishing mean oscillation) which is—by definition—the closure of smooth functions in the space \( BMO(S^1) \) (bounded mean oscillation). By the Sobolev embedding \( H^{\frac{1}{2}}(S^1) \hookrightarrow \text{VMO}(S^1) \), this permits to extend the notion of degree beyond the space \( H^{\frac{1}{2}}(S^1) \). Coming back to our problem, we can ask the following natural question: if \( \tau : \Gamma \to S^1 \) is the unit tangent vector to a simple closed curve \( \Gamma \), does \( \tau \in \text{VMO}(\Gamma) \) imply that \( k_{g_0} \in \mathcal{D}(\Gamma)? \) Characterising curves such that \( k_{g_0} \in L^1(\Gamma) \) (or \( k_{g_0} \) is a measure) seems to be an interesting question too.

(3) Notice that we can also directly define the Loewner energy using moving frames. First, we trivially have

\[
I^L(\Gamma) = \frac{1}{\pi} \left\{ \int_{\Omega_1} \lvert \omega_1 - * dG_{\Omega_1} \rvert_{g_0}^2 \, d\text{vol}_{g_0} + \int_{\Omega_2} \lvert \omega_2 - * dG_{\Omega_2} \rvert_{g_0}^2 \, d\text{vol}_{g_0} + 4\pi \right\}
+ 4 \log \lvert \nabla f_1(0) \rvert + 4 \log \lvert \nabla f_2(0) \rvert - 12 \log(2).
\]

Alternatively, we have

\[
I^L(\Gamma) = \frac{1}{2} \int_{\Omega_1} \left( |\vec{e}_1|_{g_0}^2 + |\vec{f}_1|_{g_0}^2 - 2|dG_{\Omega_1}|_{g_0}^2 \right) \, d\text{vol}_{g_0} + \frac{1}{2} \int_{\Omega_2} \left( |\vec{e}_2|_{g_0}^2 + |\vec{f}_2|_{g_0}^2 - 2|dG_{\Omega_2}|_{g_0}^2 \right) \, d\text{vol}_{g_0}
\]

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Indeed, since $\tilde{c}_1$, $\tilde{f}_1$ and $n$ are unitary, we have
\[
|d\tilde{c}_1|^2_{g_0} = |d\tilde{c}_1, f_1|^2_{g_0} + |d\tilde{c}_1, n|^2_{g_0} = |\omega_1|^2_{g_0} + |d\tilde{n}, \tilde{c}_1|^2_{g_0}
\]
\[
|d\tilde{f}_1|^2_{g_0} = |\omega_1|^2_{g_0} + |(d\tilde{n}, f_1)|^2_{g_0}
\]
\[
|d\tilde{c}_1|^2_{g_0} + |d\tilde{f}_1|^2_{g_0} = 2|\omega_1|^2_{g_0} + |d\tilde{n}|^2_{g_0} = 2|\omega_1|^2_{g_0} + 2.
\]
Then integrating by parts and using that $G_{\Omega_1} = 0$ on $\partial\Omega_1$, we deduce by Stokes theorem—and the equation (that follows from (4.9))
\[
d (\omega_1 - * dG_{\Omega_1}) = -K_{g_0} d\text{vol}_{g_0},
\]
where $K_{g_0} = 1$ is the Gauss curvature of the sphere—that
\[
\frac{1}{2} \int_{\Omega_1} \left( |d\tilde{c}_1|^2_{g_0} + |d\tilde{f}_1|^2_{g_0} - 2|dG_{\Omega_1}|^2_{g_0} \right) d\text{vol}_{g_0} = \int_{\Omega_1} (|\omega_1|^2_{g_0} - |dG_{\Omega_1}|^2_{g_0}) d\text{vol}_{g_0} + \text{Vol}_{g_0}(\Omega_1)
\]
\[
= \int_{\Omega_1} (\omega_1 - * dG_{\Omega_1}, \omega_1 + * dG_{\Omega_1}) d\text{vol}_{g_0} + \text{Vol}_{g_0}(\Omega_1)
\]
\[
= \int_{\Omega_1} |\omega_1 - * dG_{\Omega_1}|^2_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} (\omega_1 - * dG_{\Omega_1}) \wedge dG_{\Omega_1} + \text{Vol}_{g_0}(\Omega_1)
\]
\[
= \int_{\Omega_1} |\omega_1 - * dG_{\Omega_1}|^2_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} (\omega_1 - * dG_{\Omega_1}) \wedge dG_{\Omega_1} + \text{Vol}_{g_0}(\Omega_1)
\]
\[
= \int_{\Omega_1} |\omega_1 - * dG_{\Omega_1}|^2_{g_0} - 2 \int_{\Omega_1} G_{\Omega_1} d(\omega_1 - * dG_{\Omega_1}) + \text{Vol}_{g_0}(\Omega_1)
\]
\[
= \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} + \text{Vol}_{g_0}(\Omega_1)
\]
which implies since $\text{Vol}_{g_0}(S^2) = 4\pi$ that
\[
\frac{1}{2} \int_{\Omega_1} \left( |d\tilde{c}_1|^2_{g_0} + |d\tilde{f}_1|^2_{g_0} - 2|dG_{\Omega_1}|^2_{g_0} \right) d\text{vol}_{g_0} + \frac{1}{2} \int_{\Omega_1} \left( |d\tilde{c}_2|^2_{g_0} + |d\tilde{f}_2|^2_{g_0} - 2|dG_{\Omega_2}|^2_{g_0} \right) d\text{vol}_{g_0}
\]
\[
= \int_{\Omega_1} |d\mu_1|^2_{g_0} d\text{vol}_{g_0} + \int_{\Omega_2} |d\mu_2|^2_{g_0} + 2 \int_{\Omega_1} G_{\Omega_1} K_{g_0} d\text{vol}_{g_0} + 2 \int_{\Omega_2} G_{\Omega_2} K_{g_0} d\text{vol}_{g_0} + 4\pi.
\]
Notice that it gives another explanation for the factor $4\pi$ in the definition of $\delta'$.

Finally, we will establish the uniqueness of distributional solutions of the system 4.2 with appropriate boundary conditions. This is the exact analogous of Remark I.1 of [3]. First, we need to define explicit maps that yield trivialisations of vector fields on simply connected domains of the sphere. Let $\Omega_1 \subset S^2$ be as Theorem 4.2.

Seeing $S^2$ as $\mathbb{C} \cup \{\infty\}$, we have one holomorphic chart $z$ on $S^2 \setminus \{N\} = \mathbb{C}$, and for all domain $\Omega_1 \subset S^2 \setminus \{N\}$, it yields the following trivialisation of $T(S^2 \setminus \{N\})$:
\[
\Psi : T\Omega_1 \rightarrow \Omega \times \mathbb{C}
\]
\[
(z, v) \mapsto (z, v).
\]
Although this map is the identity map, since we consider vector fields as $\mathbb{R}^3$-valued maps, we will describe the trivialisation more explicitly. Therefore, let $X : \Omega \rightarrow \mathbb{R}^3$ such that $\langle X, n \rangle = 0$, where $n = \text{Id}_\Omega : \Omega \rightarrow S^2$ is the unit normal, using the chart given by the stereographic projection $\pi : S^2 \setminus \{N\} \rightarrow \mathbb{C}$, and letting $\Omega = \pi(\Omega_1) \subset \mathbb{C}$, we get a global chart $\pi^{-1} : \Omega \rightarrow T\Omega_1$, where we recall that
\[
\pi^{-1}(z) = \bar{n}(z) = \begin{pmatrix} 2 \text{Re}(z) \\ 2 \text{Im}(z) \\ -1 + |z|^2 \end{pmatrix} \left( \frac{1}{1 + |z|^2}, \frac{1}{1 + |z|^2}, \frac{1}{1 + |z|^2} \right).
\]
Now, we introduce the function $\psi : \mathbb{C} \to \mathbb{C}^3$, given by

$$\psi(z) = \left( \frac{1 - z^2}{1 + |z|^2}, \frac{i(1 + z^2)}{1 + |z|^2}, \frac{2z}{1 + |z|^2} \right),$$

and we easily check that

$$\langle \psi, \psi \rangle = 0 \quad |\psi|^2 = \langle \psi, \overline{\psi} \rangle = 2.$$  \hspace{1cm} (4.27)

Therefore, we deduce that $(\vec{e}_1, \vec{e}_2)$ defined as follows is a tangent unit moving frame (orthogonal to $\vec{n}$ )

$$\vec{e}_1(z) = \text{Re} (\psi(z)), \quad \vec{e}_2(z) = \text{Im} (\psi(z)).$$

The trivialisation map on $\Omega_1 \subset S^2 \setminus \{N\}$ is then given by

$$T\Omega \to \Omega \times \mathbb{C}$$

$$(z, v) \mapsto (z, (v, \vec{e}_1(z)) + i (v, \vec{e}_2(z))),$$  \hspace{1cm} (4.28)

while the trivialisation map of sections is given by

$$\Psi_{\Omega_1} : \Gamma(T\Omega_1) \to C^\infty(\Omega_1, \mathbb{C})$$

$$X \mapsto \langle X, \vec{e}_1 \rangle + i \langle X, \vec{e}_2 \rangle.$$  \hspace{1cm} (4.29)

Since $\langle X, \vec{n} \rangle = 0$, there exists real functions $\lambda_1, \lambda_2 : \Omega_1 \to \mathbb{R}$ such that

$$X = \lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2.$$

**Theorem 4.5.** Under the conditions of Theorem 4.2, let $\Omega_1 = \Omega_1 \setminus \{p_1\}$ and $\vec{u} \in \Gamma_{loc}^1(T\Omega_1) \cap \Gamma^{1,1}(T\Omega_1)$ be a unit vector field in $\Omega_1$, and let $\vec{u}_0 = \Psi_{\Omega_1}(\vec{u}) : \Omega_1 \to S^1$. Then $\vec{u}$ is a harmonic vector field on $\Omega_1$, i.e.

$$-\Delta_g \vec{u} = |d\vec{u}|^2_{g_0} \vec{u}_0 + 2(d\vec{u}, d\vec{n})_{g_0} + \langle \vec{u}, \Delta_g \vec{n} \rangle \vec{n}$$

if and only if $\vec{u}_0$ is a harmonic map with values into $S^1$, i.e.

$$-\Delta_g \vec{u}_0 = |d\vec{u}_0|^2_{g_0} \vec{u}_0.$$

In particular, for all degree 1 boundary data $h : \partial \Omega_1 \to S^1$ and $p \in \Omega_1$, there exists a unique unit vector-field $\vec{u} \in \Gamma_{loc}^{1,2}(T(\Omega_1 \setminus \{p_1\})) \cap \Gamma^{1,1}(T\Omega_1)$ such that $\vec{u} = \Psi_{\Omega_1}^{-1}(g)$, and such that $\vec{u}_0 = \Psi_{\Omega_1}(\vec{u})$ satisfies in the distributional sense

$$\text{div} (\vec{u}_0 \times \nabla \vec{u}_0) = 0 \quad \text{in } \mathcal{D}'(\Omega_1).$$

**Remark 4.6.** If $\vec{u}_0 : \mathbb{D} \to S^1$, writing locally $\vec{u}_0 = e^{i\varphi}$ for some real-valued function $\varphi$, we deduce that $\vec{u}_0$ is harmonic if and only if

$$-\Delta \vec{u}_0 = (|\nabla \varphi|^2 - i (\Delta \varphi)) \vec{u}_0 = |\nabla \vec{u}_0|^2 \vec{u}_0.$$

Therefore, $\vec{u}_0$ is harmonic as a map with values into $S^1$ if and only if $\varphi$ is harmonic, i.e. $\Delta \varphi = 0$.

**Proof.** By making a stereographic projection, thanks to the conformal invariance of the harmonic equation, we deduce that for all unit vector-field $\vec{u} \in \Gamma(T\Omega_1)$ is given in $\Omega = \pi_N(\Omega_1)$ as

$$\vec{u} = \lambda_1 \text{Re} (\psi) + \lambda_2 \text{Im} (\psi),$$  \hspace{1cm} (4.30)

where

$$\psi(z) = \left( \frac{1 - z^2}{1 + |z|^2}, \frac{i(1 + z^2)}{1 + |z|^2}, \frac{2z}{1 + |z|^2} \right).$$
Furthermore, we have $\lambda_1^2 + \lambda_2^2 = 1$, which implies that there exists a measurable function $\phi$ such that $\lambda_1 + i\lambda_2 = e^{-i\varphi}$. In particular, we can rewrite (4.30) as
\[
\bar{u} = \cos(\varphi) \text{Re} (\psi) - \sin(\varphi) \text{Im} (\psi) = \text{Re} (e^{-i\varphi}) \text{Re} (\psi) + \text{Im} (e^{-i\varphi}) \text{Im} (\psi) = (e^{i\varphi}) \psi,
\]
where we used the identity
\[
\text{Re} (a) \text{Re} (b) + \text{Im} (a) \text{Im} (b) = \text{Re} (\bar{a} b)
\]
valid for all $a, b \in \mathbb{C}$. If
\[
\bar{v} = \sin(\varphi) \text{Re} (\psi) + \cos(\varphi) \text{Im} (\psi) = \text{Im} (e^{i\varphi}) \psi,
\]
we immediately have $\langle \bar{u}, \bar{v} \rangle = 0$, and since $|\bar{u}|^2 = |\bar{v}|^2$, while $-\Delta \bar{u} = |\nabla \bar{u}|^2 \bar{n}$, we get
\[
\begin{align*}
\langle \Delta \bar{u}, \bar{v} \rangle &= -|\nabla \bar{u}|^2 \bar{u} \\
\langle \Delta \bar{u}, \bar{v} \rangle &= 2(\nabla \bar{u}, \nabla \bar{n}) - \langle \bar{u}, \Delta \bar{v} \rangle = -2(\nabla \bar{u}, \nabla \bar{n}),
\end{align*}
\]
and similar formulae for $\bar{u}$. Therefore, we deduce that $(\bar{u}, \bar{v})$ solves the system
\[
\begin{align*}
-\Delta \bar{u} &= |\nabla \bar{u}|^2 \bar{u} + 2(\nabla \bar{u}, \nabla \bar{n}) \bar{n} \quad \text{in } \Omega \\
-\Delta \bar{v} &= |\nabla \bar{v}|^2 \bar{v} + 2(\nabla \bar{v}, \nabla \bar{n}) \bar{n} \quad \text{in } \Omega,
\end{align*}
\]
if and only if
\[
\langle \Delta \bar{u}, \bar{v} \rangle = \langle \Delta \bar{v}, \bar{u} \rangle = 0.
\]

Now, we compute
\[
\begin{align*}
\Delta \bar{u} &= \text{Re} ((i \Delta \varphi - |\nabla \varphi|^2) e^{i\varphi} \psi) + 2 \text{Re} (i e^{i\varphi} \nabla \varphi \cdot \nabla \psi) + \text{Re} (e^{i\varphi} \Delta \psi) \\
&= -(\Delta \varphi) \text{Im} (e^{i\varphi} \psi) - |\nabla \varphi|^2 \text{Re} (e^{i\varphi} \psi) + \text{Re} (e^{i\varphi} \Delta \psi) \\
&= -(\Delta \varphi) \psi - |\nabla \varphi|^2 \psi + 2 \text{Re} (i e^{i\varphi} \nabla \varphi \cdot \nabla \psi) + \text{Re} (e^{i\varphi} \Delta \psi) \\
\Delta \bar{u} &= (\Delta \varphi) \bar{u} - |\nabla \varphi|^2 \bar{u} + 2 \text{Im} (i e^{i\varphi} \nabla \varphi \cdot \nabla \psi) + \text{Im} (e^{i\varphi} \Delta \psi).
\end{align*}
\]
We have since $\langle \nabla \varphi, \varphi \rangle = 0$ the identity
\[
\begin{align*}
\langle \text{Re} (i e^{i\varphi} \nabla \varphi \cdot \nabla \psi), \bar{v} \rangle &= \text{Re} \left( i e^{i\varphi} \nabla \varphi \cdot \nabla \psi, \frac{e^{i\varphi} \psi - e^{-i\varphi} \bar{\psi}}{2i} \right) = -\frac{1}{2} \text{Re} \left( \nabla \varphi, \langle \nabla \psi, \bar{\psi} \rangle \right) \\
\langle \text{Im} (i e^{i\varphi} \nabla \varphi \cdot \nabla \psi), \bar{u} \rangle &= \text{Im} \left( i e^{i\varphi} \nabla \varphi \cdot \nabla \psi, \frac{e^{i\varphi} \psi + e^{-i\varphi} \bar{\psi}}{2} \right) = \frac{1}{2} \text{Re} \left( \nabla \varphi, \langle \nabla \psi, \bar{\psi} \rangle \right) \\
\langle \text{Re} (e^{i\varphi} \Delta \psi), \bar{v} \rangle &= \text{Re} \left( e^{i\varphi} \Delta \psi, \frac{e^{i\varphi} \psi - e^{-i\varphi} \bar{\psi}}{2i} \right) = \frac{1}{2} \text{Im} \left( e^{2i\varphi} \langle \Delta \psi, \psi \rangle \right) - \frac{1}{2} \text{Im} \langle \Delta \psi, \bar{\psi} \rangle \\
\langle \text{Im} (e^{i\varphi} \Delta \psi), \bar{u} \rangle &= \frac{1}{2} \text{Im} \left( e^{2i\varphi} \langle \Delta \psi, \psi \rangle \right) + \frac{1}{2} \text{Im} \langle \Delta \psi, \bar{\psi} \rangle
\end{align*}
\]
In particular, we have
\[
\begin{align*}
\langle \Delta \bar{u}, \bar{v} \rangle &= -(\Delta \varphi) - \text{Re} \left( \nabla \varphi, \langle \nabla \psi, \bar{\psi} \rangle \right) + \frac{1}{2} \text{Im} \left( e^{2i\varphi} \langle \Delta \psi, \psi \rangle \right) - \frac{1}{2} \text{Im} \langle \Delta \psi, \bar{\psi} \rangle \\
\langle \Delta \bar{u}, \bar{v} \rangle &= (\Delta \varphi) + \text{Re} \left( \nabla \varphi, \langle \nabla \psi, \bar{\psi} \rangle \right) + \frac{1}{2} \text{Im} \left( e^{2i\varphi} \langle \Delta \psi, \psi \rangle \right) + \frac{1}{2} \text{Im} \langle \Delta \psi, \bar{\psi} \rangle
\end{align*}
\]
Summing those equations and substracting the first one to the second one yields the system
\[
\begin{align*}
\text{Im} \left( e^{2i\varphi} \langle \Delta \psi, \psi \rangle \right) &= 0 \\
2(\Delta \varphi) + 2 \text{Re} \left( \nabla \varphi, \langle \nabla \psi, \bar{\psi} \rangle \right) + \text{Im} \langle \Delta \psi, \bar{\psi} \rangle &= 0.
\end{align*}
\]
We will show that for all smooth real-valued function $\varphi : \Omega \to \mathbb{R}$
\[
\text{Re} (\nabla \varphi \cdot (\nabla \psi, \overline{\psi})) = \langle \Delta \psi, \psi \rangle = \text{Im} \langle \Delta \psi, \overline{\psi} \rangle = 0,
\]
which will imply that $(\bar{u}, \bar{v})$ solves the system (4.31) if and only if $\Delta \varphi = 0$, or $\varphi$ is harmonic. Now, we compute
\[
\partial_z \psi = -\frac{\overline{\varphi}}{1 + |z|^2} \psi + \frac{2}{1 + |z|^2} (-z, iz, 1)
\]
\[
\partial_{\overline{z}} \psi = -\frac{\varphi}{1 + |z|^2} \psi\]

We have
\[
\nabla \varphi \cdot \nabla \psi = 2 \partial_z \varphi \cdot \partial_z \psi + 2 \partial_{\overline{z}} \varphi \cdot \partial_{\overline{z}} \psi
\]
\[
= -2 \frac{\overline{\varphi}}{1 + |z|^2} \psi + \frac{4}{1 + |z|^2} (-z \partial_z \varphi, iz \partial_z \varphi, \partial_z \varphi) - 2 \frac{z \partial_{\overline{z}} \varphi}{1 + |z|^2} \psi
\]
\[
= -4 \text{Re} \left( \frac{\overline{\varphi}}{1 + |z|^2} \right) \psi + \frac{4}{1 + |z|^2} (-z \partial_z \varphi, iz \partial_z \varphi, \partial_z \varphi).
\]

Then we have
\[
\frac{1}{4} \Delta \psi = \partial^2 \psi = -\frac{1 + |z|^2}{1 + |z|^2} \psi - \frac{2}{(1 + |z|^2)^2} (-z^2, iz^2, z)
\]

(4.33)

Now, notice that
\[
\langle (-z^2, iz^2, z), \psi \rangle = \frac{1}{1 + |z|^2} ((-z^2, iz^2, z), (1 - z^2, i(1 + z^2), 2z))
\]
\[
= \frac{1}{1 + |z|^2} (-z^2(1 - \bar{z}^2) + z^2(1 + \bar{z}^2) + 2z^2) = 0,
\]

which implies as $\langle \psi, \psi \rangle = 0$ and by (4.33) that
\[
\langle \Delta \psi, \psi \rangle = 0.
\]

(4.34)

Now, we have
\[
\langle (-z^2, iz^2, z), \overline{\psi} \rangle = \frac{1}{1 + |z|^2} (-z^2(1 - \bar{z}^2) + z^2(1 + \bar{z}^2) + 2|z|^2) = 2|z|^2.
\]

Since $|\psi|^2 = \langle \psi, \overline{\psi} \rangle = 2$, we deduce that
\[
\langle \Delta \psi, \overline{\psi} \rangle = 4 \left( \frac{2(-1 + |z|^2)}{(1 + |z|^2)^2} - \frac{4|z|^2}{(1 + |z|^2)^2} \right) = \frac{8}{1 + |z|^2} \in \mathbb{R}.
\]

(4.35)

We now compute
\[
\langle (-z, iz, 1), \overline{\psi} \rangle = \frac{1}{1 + |z|^2} (-z(1 - \bar{z}^2) + z(1 + \bar{z}^2) + 2z) = 2\pi
\]

which shows as $|\psi|^2 = 2$ that
\[
\langle \partial_z \psi, \psi \rangle = -\frac{2\pi}{1 + |z|^2} + \frac{4\pi}{1 + |z|^2} = \frac{2\pi}{1 + |z|^2}
\]
\[
\langle \partial_{\overline{z}} \psi, \psi \rangle = -\frac{2z}{1 + |z|^2}.
\]

Therefore, we have
\[
\nabla \varphi \cdot (\nabla \psi, \overline{\psi}) = 2 \partial_{\overline{z}} \varphi \cdot (\partial_z \psi, \psi) + 2 \partial_z \varphi \cdot (\partial_{\overline{z}} \psi, \psi) = \frac{4\pi \partial_z \varphi}{1 + |z|^2} - \frac{4\pi \partial_{\overline{z}} \varphi}{1 + |z|^2} = 8i \text{Im} \left( \frac{\overline{\varphi}}{1 + |z|^2} \right) \in i\mathbb{R},
\]

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and this immediately implies that
\[
\text{Re} \left( \nabla \varphi \cdot \langle \nabla \psi, \psi \rangle \right) = 0.
\]
(4.36)

Finally, we deduce by (4.34), (4.35) and (4.36) that (4.32) holds and that the system (4.31) holds if and only if \( \Delta \varphi = 0 \). If \( \vec{u} = g_{\Omega_1} = \Psi_{\Omega_1}(g) \) for some \( g : \partial \Omega_1 = \Gamma \to S^1 \), then we have
\[
\lambda_1 + i \lambda_2 = g,
\]
or
\[
e^{-i\varphi} = g \quad \text{on } \Gamma.
\]
In particular, the function \( \vec{u}_0 = e^{-i\varphi} : \Omega_1 \setminus \{p_1\} \to S^1 \) is a harmonic map on \( \Omega_1 \setminus \{p_1\} \) satisfying \( \vec{u}_0 = h \) on \( \Gamma \). Now, notice that provided \( \vec{u} \in W^{1,1}(\Omega_1) \), one can rewrite the equation distributionally as
\[
\text{div} \left( \vec{u} \times \nabla \vec{u} \right) = 2 \langle \nabla \vec{u}, \nabla \vec{n} \rangle \vec{n} \times \vec{u}.
\]

In particular, we deduce as \( \vec{u}_0 \) is harmonic that
\[
\text{div} \left( \vec{u}_0 \times \nabla \vec{u}_0 \right) = \frac{\partial}{\partial x_1} \left( \vec{u}_0 \times \frac{\partial \vec{u}_0}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \vec{u}_0 \times \frac{\partial \vec{u}_0}{\partial x_2} \right) = 0.
\]

By Theorem I.5 and Remark I.1 of [3], we deduce that \( \vec{u}_0 \) is the unique harmonic function with a singularity at \( p_1 \) such that \( \vec{u}_0 = h \) on \( \partial \Omega_1 \). This concludes the proof of the theorem.

5 Construction of Surfaces of Finite Willmore Energy Bounding Loewner Curves

First, recall an elementary lemma from [20].

**Lemma 5.1.** Let \( \varphi : \mathbb{D} \to \mathbb{C} \) be a holomorphic function such that \( \varphi \in L^2(\mathbb{D}) \). Then we have
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|\varphi(z)| \leq \frac{1}{\sqrt{\pi}} \| \varphi \|_{L^2(\mathbb{D})}.
\]

**Proof.** We recall the straightforward proof for the sake of completeness. Write
\[
\varphi(z) = \sum_{n \in \mathbb{N}} a_n z^n.
\]
Then we have
\[
\int_{\mathbb{D}} |\varphi(z)|^2 |dz|^2 = 2\pi \sum_{n \in \mathbb{N}} \int_0^1 |a_n|^2 r^{2n+1} dr = \pi \sum_{n \in \mathbb{N}} \frac{|a_n|^2}{n+1}.
\]
Therefore, we have by Cauchy-Schwarz inequality
\[
|\varphi(z)| \leq \sum_{n \geq 0} |a_n| |z|^n \leq \left( \sum_{n \in \mathbb{N}} \frac{|a_n|^2}{n+1} \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{N}} (n+1) |z|^{2n} \right)^{\frac{1}{2}} = \frac{1}{(1-|z|^2)^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}} \| \varphi \|_{L^2(\mathbb{D})},
\]
which concludes the proof of the lemma.

**Proposition 5.2.** Let \( \Omega \subset \mathbb{C} \) be a simply connected domain and \( f : \mathbb{D} \to \Omega \) be a holomorphic univalent map, and assume that \( \gamma = \partial \Omega \) is a Weil-Petersson curve. For all \( 0 < \varepsilon \leq \infty \), define \( \Gamma_{\varepsilon} \in \mathbb{R}_+^{*} \) by
\[
\Gamma_{\varepsilon} = 2 \exp \left( 2C_0 \left( 1 + \frac{1}{\varepsilon} \right) \frac{1}{2} \int_{\mathbb{D}} \frac{|f''(z)|^2}{|f'(z)|^2} |dz|^2 + C_0 \left( \frac{1}{2} \int_{\mathbb{D}} \frac{|f''(z)|^2}{|f'(z)|^2} |dz|^2 \right)^{\frac{1}{2}} \right),
\]
(5.1)
where $C_0$ is the constant appearing in Lemma 3.2. Then, the following inequality holds true for all $0 < \varepsilon < 1$

$$\Gamma_{n-1}^{-1}(1 - |z|^2)\varepsilon \leq |f'(z)| \leq \frac{\Gamma_{n}}{(1 - |z|^2)\varepsilon}$$

for all $z \in \mathbb{D}$. \hfill (5.2)

**Proof.** Using Lemma 3.2, we have $f' \in L^p(\mathbb{D})$ for all $p < \infty$. Let $n \in \mathbb{N}^+$ such that $\frac{1}{n+1} < \varepsilon \leq \frac{1}{n}$. By (3.5) from Lemma 3.2, we have

$$\int_{\mathbb{D}} |f'(z)|^{2n+2} |dz|^2 \leq 4\pi \exp \left(4C_0(n + 1)^2 \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 + 2C_0(n + 1) \left( \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 \right)^{\frac{1}{2}} \right).$$

Furthermore, $(f')^{n+1}$ is a holomorphic function and $(f')^{n+1} \in L^2$ so we deduce by Lemma 5.1 and (5.3) that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1+\varepsilon} |f'(z)|^{n+1} \leq \frac{1}{\sqrt{2\pi}} \left\| (f')^{n+1} \right\|_{L^2(\mathbb{D})}$$

$$\leq 2 \exp \left( 2C_0(n + 1)^2 \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 + C_0(n + 1) \left( \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 \right)^{\frac{1}{2}} \right).$$

Therefore, we deduce that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1+\varepsilon} |f'(z)| \leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1+\varepsilon} |f'(z)|$$

$$\leq 2 \exp \left( 2C_0(n + 1) \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 + C_0(n + 1) \left( \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 \right)^{\frac{1}{2}} \right),$$

which concludes the proof of the first estimate using the inequality $n + 1 \leq 1 + \frac{1}{\varepsilon}$. Finally, applying the theorem to $(f')^{-(n+1)}$ yields the other estimate and this concludes the proof of the proposition. \hfill \Box

**Theorem 5.3.** Let $\gamma \subset \mathcal{C}$ be a simple curve of finite Loewner energy, $\Omega$ be the bounded open connected component of $\mathcal{C} \setminus \gamma$ and let $f : \mathbb{D} \to \Omega$ be a uniformisation map. Let $0 < \alpha < \frac{1}{2}$ and assume that

$$\int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} < \infty.$$ \hfill (5.4)

Then the embedding $\Phi : \mathbb{D} \to \mathbb{H}^3 = \left( \mathbb{C} \times \mathbb{R}_+^+, g_{hyp} = \frac{|dz|^2 + dt^2}{t^2} \right)$ defined for all $z \in \mathbb{D}$ by

$$\Phi(z) = (\text{Re}(f(z)), \text{Im}(f(z)), (1 - |z|^2)^{\alpha})$$

is orthogonal to $\partial_\infty \mathbb{H}^3 = \mathbb{R}^2 \times \{0\}$ and has finite total curvature, i.e.

$$\int_{\mathbb{D}} |\tilde{A}|^2 dvol_\mathcal{D} < \infty,$$

where $\tilde{A}$ is the trace-less fundamental form and $g = \Phi^*g_{hyp}$ is the induced metric on the unit disk $\mathbb{D}$. Furthermore, there exists a universal constant $C = C_0 \in \mathbb{R}$ independent of $f$ such that

$$\int_{\mathbb{D}} |\tilde{A}|^2 dvol_\mathcal{D} \leq C \left( 1 + \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 + \int_{\mathbb{D}} \left| f'''(z) \right|^2 |dz|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} \right) \exp \left( C \int_{\mathbb{D}} \left| f''(z) \right|^2 |dz|^2 \right).$$ \hfill (5.5)

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Remark 5.4. By (5.19), we can replace (5.5) by the improved estimate
\[ \int_D |\tilde{A}|^2 d\text{vol}_g \leq C \left( 1 + \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \int_D \left| \text{Re} \left( \frac{z f''(z)}{f'(z)} \right) \right|^2 \frac{|dz|^2}{|f'(z)|^2(1 - |z|^2)^{1-\alpha}} \right) \exp \left( C \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right) \]
which only requires that
\[ \int_D \left| \text{Re} \left( \frac{z f''(z)}{f'(z)} \right) \right|^2 \frac{|dz|^2}{|f'(z)|^2(1 - |z|^2)^{1-\alpha}} < \infty \]
compared to (5.4).

Proof. Define the immersion \( \tilde{\Phi} : \mathbb{D} \to \mathbb{R}^3 \) by
\[ \tilde{\Phi}(z) = (\text{Re}(f(z)), \text{Im}(f(z)), g_3(z)), \]
where \( g_3 \in C^0(\mathbb{D}, \mathbb{R}_+) \), \( g_3(z) \to 0 \) when \( z \to \partial \mathbb{D} \) and \( g_3(z) > 0 \) for all \( z \in \mathbb{D} \). Here, we see \( \tilde{\Phi} \) as an immersion of the Euclidean space, and we denote \( g = \varphi^* g_{\mathbb{R}^3} \) the induced metric on \( \mathbb{D} \). Therefore, we have
\[
\partial_x \tilde{\Phi} = \left( \text{Re}(f'(z)), \text{Im}(f'(z)), \partial_x g_3(z) \right) \\
\partial_y \tilde{\Phi} = \left( -\text{Im}(f'(z)), \text{Re}(f'(z)), \partial_y g_3(z) \right) \]
We deduce that
\[
\partial_x \tilde{\Phi} \times \partial_y \tilde{\Phi} = 2i \partial_x \tilde{\Phi} \times \partial_x \tilde{\Phi} = 2i \begin{pmatrix} \frac{1}{2} f'(z) \\ \frac{i}{2} f'(z) \\ \partial_z g_3(z) \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} f'(z) \\ \frac{-i}{2} f'(z) \\ \partial_z g_3(z) \end{pmatrix} = \begin{pmatrix} -2 \text{Re}(\partial_z g_3(z) f'(z)) \\ -2 \text{Im}(\partial_z g_3(z) f'(z)) \end{pmatrix}.
\]
Then we get
\[ e^{4\lambda} = |\partial_x \tilde{\Phi} \times \partial_y \tilde{\Phi}|^2 = 4|\partial_x g_3(z)|^2 |f'(z)|^2 + |f'(z)|^4 = |\nabla g_3(z)|^2 |f'(z)|^2 + |f'(z)|^4, \quad (5.6) \]
and
\[
\begin{cases}
|g_{1,1}| = |\partial_x \tilde{\Phi}|^2 = |f'(z)|^2 + |\partial_x g_3(z)|^2 \\
|g_{2,2}| = |\partial_y \tilde{\Phi}|^2 = |f'(z)|^2 + |\partial_y g_3(z)|^2 \\
|g_{1,2}| = |\partial_z g_3(z) \partial_y g_3(z) |
\end{cases}
\]
Assume furthermore that \( g_3 \) is radial and that \( g_3(z) = h(|z|^2) \) for some decreasing smooth function \( h : [0, 1) \to \mathbb{R}_+ \) such that \( h \in C^0([0, 1]) \), we deduce that
\[
\partial_x g_3(z) = \partial_x g (x^2 + y^2) = 2x h'(|z|^2) = 2 \text{Re} (z) h'(|z|^2) \\
\partial_y g_3(z) = 2 \text{Im} (z) h'(|z|^2).
\]
and
\[
\begin{cases}
\partial_x \tilde{\Phi} = (\text{Re}(f'(z)), \text{Im}(f'(z)), 2 \text{Re}(z) h'(|z|^2)) \\
\partial_y \tilde{\Phi} = (-\text{Im}(f'(z)), \text{Re}(f'(z)), 2 \text{Im} (z) h'(|z|^2)) \end{cases}, \quad (5.7)
\]
Therefore, the coefficients of the metric \( g = g_{i,j} dx_i \otimes dx_j \) (see [21] 1.9 for notations) are given by
\[
\begin{cases}
g_{1,1} = |f'(z)|^2 + 2 \text{Re}(z)^2 h'(|z|^2)^2 \\
g_{2,2} = |f'(z)|^2 + 2 \text{Im}(z)^2 h'(|z|^2)^2 \\
g_{1,2} = 2 \text{Re}(z) \text{Im}(z) h'(|z|^2)^2 \end{cases}, \quad (5.8)
\]
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We check that
\[ e^{4\lambda} = g_{1,1}g_{2,2} - g_{1,2}^2 = |f'(z)|^4 + 4 \Re (z^2) |h'(z)|^2 + 4 \Im (z^2) |h'(z)|^2 + 4 \Re (z^2) \Im (z^2) |h'(z)|^4 - 4 \Re (z^2) \Im (z^2) |h'(z)|^2 = |f'(z)|^4 + 4 |z|^2 |h'(z)|^2 |f'(z)|^2 \]
\[ |h'(z)|^2 |f'(z)|^2 \left( 4|z|^2 + \frac{|f'(z)|^2}{|h'(z)|^2} \right) = |h'(z)|^2 |f'(z)|^2 e^{4\omega(z)}, \] (5.9)

where
\[ e^{4\omega(z)} = 4|z|^2 + \frac{|f'(z)|^2}{|h'(z)|^2}, \]
which checks with (5.6). Then, we obtain
\[ \partial_z \bar{\Phi} \times \partial_y \bar{\Phi} = (-2 h'(|z|^2) \Re (z f'(z)), -2 h'(|z|^2) \Im (z f'(z)), |f'(z)|^2) \]
and
\[ \bar{n} = e^{-2\lambda} \partial_x \bar{\Phi} \times \partial_y \bar{\Phi} - e^{-2\omega(z)} \left( 2 \Re \left( \frac{z f'(z)}{|f'(z)|} \right), 2 \Im \left( \frac{z f'(z)}{|f'(z)|} \right), \frac{|f'(z)|}{h'(z)^2} \right), \]

since we assumed that \( h \) is a decreasing function. We assume that \( \bar{n}_3(z) \rightarrow 0 \) as \( z \to \partial \Omega \), which implies in particular that \( \omega \in L^\infty(\Omega) \). Now, we compute as previously
\[ \partial_x f'(z) \right| f'(z) \right| = \frac{f''(z)}{|f'(z)|} - \frac{f'(z)}{2} \frac{f''(z)}{|f'(z)|^3} = \frac{1}{2} \frac{f''(z)}{|f'(z)|^2} \]
\[ \partial_x \frac{f'(z)}{|f'(z)|} = -\frac{1}{2} \frac{f'(z) f''(z)}{|f'(z)|^3}. \]

Therefore, we have
\[ \partial_x \Re \left( \frac{z f'(z)}{|f'(z)|} \right) = \frac{1}{2} \frac{f'(z)}{|f'(z)|} + i \frac{f''(z)}{f'(z)} \Im \left( \frac{z f'(z)}{|f'(z)|} \right) \]
\[ \partial_x \Im \left( \frac{z f'(z)}{|f'(z)|} \right) = -i \frac{f'(z)}{2 |f'(z)|} - i \frac{f''(z)}{f'(z)} \Re \left( \frac{z f'(z)}{|f'(z)|} \right) \]
\[ \partial_x \left( \frac{f'(z)}{|h'(z)|^2} \right) = \frac{1}{2} \frac{f''(z)}{|h'(z)|^2} - \frac{f'(z)}{|f'(z)|} \frac{2 h''(|z|^2) \bar{h}'(z)}{|h'(z)|^2}. \]

This implies that
\[ \partial_x \bar{n} = 2 (\partial_x \omega) \bar{n} - e^{-2\omega(z)} \left( \begin{array}{c} \frac{1}{2} \frac{f'(z)}{|f'(z)|} + i \frac{f''(z)}{f'(z)} \Im \left( \frac{z f'(z)}{|f'(z)|} \right) \\ -i \frac{f'(z)}{2 |f'(z)|} - i \frac{f''(z)}{f'(z)} \Re \left( \frac{z f'(z)}{|f'(z)|} \right) \\ \frac{1}{2} \frac{f''(z)}{|h'(z)|^2} - \frac{f'(z)}{|f'(z)|} \frac{2 h''(|z|^2) \bar{h}'(z)}{|h'(z)|^2} \end{array} \right) \]
\[ = \frac{1}{2} h_x \bar{n} + \frac{1}{2} \bar{H}_z = \frac{1}{2} (h_1 - ih_2) \bar{n} + \frac{1}{2} \left( \bar{H}_1 - i \bar{H}_2 \right). \]

Now, we compute
\[ e^{4\omega} |\bar{H}_z|^2 = 1 + 4|z|^2 \left| \frac{f''(z)}{f'(z)} \right|^2 + 4 \Re \left( \frac{z f''(z)}{f'(z)} \left( \frac{f'(z)}{|f'(z)|} \right)^2 \right) \]
\[ + \frac{1}{|h'(z)|^2} \left| \frac{f''(z)}{f'(z)} \right|^2 + 4|z|^2 |f'(z)|^2 |h''(|z|^2)|^2 |h'(z)|^2 - 4 \frac{h''(|z|^2)}{|h'(z)|^2} \Re (z f''(z)), \]

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where we used
\[ \text{Re} (a) \text{Re} (b) - \text{Im} (a) \text{Im} (b) = \frac{1}{4} ((a + \overline{a})(b + \overline{b}) - (a - \overline{a})(b - \overline{b})) = \text{Re} (a\overline{b}) . \]

We have
\[ \partial_z \left( |z|^2 + \frac{|f'(z)|^2}{|h'(z)|^2} \right) = \tau + \frac{f''(z)f'(z)}{|h'(z)|^2} - \frac{2\pi h''(|z|^2)|f'(z)|^2}{h'(z)|z|^2} , \]
so that
\[ h_z = 4(\partial_z \omega) = e^{-4\omega} \partial_z (e^{4\omega}) = e^{-4\omega} \left( \tau + \frac{f''(z)f'(z)}{|h'(z)|^2} - \frac{2\pi h''(|z|^2)|f'(z)|^2}{h'(z)|z|^2} \right) . \]

In particular, we have
\[ \begin{align*}
|h_z|^2 &= 4|\nabla \omega|^2 = 16|\partial_z \omega|^2 = e^{-8\omega} \left( |z|^2 + \frac{|f''(z)|^2|f'(z)|^2}{|h'(z)|^2} + 4|z|^2 \frac{h''(|z|^2)}{|h'(z)|^2} |f'(z)|^4 \right) \\
+ 2 &Re \left( \frac{z f''(z)f'(z)}{|h'(z)|^2} - 4|z|^2 |f'(z)|^2 \frac{h''(|z|^2)}{|h'(z)|^2} - 4 \frac{h''(|z|^2)}{|h'(z)|^2} \right) \right) . \end{align*} \]

Therefore, we have
\[ \begin{align*}
|h_z|^2 - |h| z^2 &= e^{-4\omega} \left( 1 + 4|z|^2 \frac{f''(z)}{f'(z)} - 4 \frac{h''(|z|^2)}{|h'(z)|^2} \right) \\
+ &\frac{1}{|h'(z)|^2} \left( f''(z) \right)^2 - 4 \frac{h''(|z|^2)}{|h'(z)|^2} \frac{f''(|z|^2)}{|f'(z)|^2} - 4 \frac{h''(|z|^2)}{|h'(z)|^2} \right) \right) \right) . \end{align*} \]

Then, since $|\overline{\nu}|^2 = 0$, we have
\[ \langle \overline{\nu} (z), \overline{\nu} \rangle = -h_1(z) \quad \text{and} \quad \langle \overline{\nu} (z), \overline{\nu} \rangle = -h_2(z) \quad \text{for} \quad \text{(5.10)} \]

Therefore, we have by (5.11)
\[ \langle \partial_z \overline{\nu}, \partial_z \nu \rangle = h_1(z) h_2(z) + h_1(z) \langle \overline{\nu} (z), \overline{\nu} \rangle + h_2(z) \langle \nu (z), \nu \rangle + \langle \overline{\nu} (z), \overline{\nu} \rangle = -h_1(z) h_2(z) + \langle \overline{\nu} (z), \overline{\nu} \rangle . \]

Likewise, we have by (5.11)
\[ \langle \partial_z \nu, \partial_z \overline{\nu} \rangle = h_1(z) h_2(z) + h_1(z) \langle \overline{\nu} (z), \overline{\nu} \rangle + h_2(z) \langle \nu (z), \nu \rangle + \langle \overline{\nu} (z), \overline{\nu} \rangle = -h_1(z) h_2(z) + \langle \overline{\nu} (z), \overline{\nu} \rangle . \]

Lastly, (5.8) implies that
\[ \begin{align*}
g_{11} &= e^{-4\lambda} g_{11} = e^{-4\lambda} \left( |f'(z)|^2 + 2 \text{Im} (z)|h'(|z|^2)|^2 \right) \\
g_{22} &= e^{-4\lambda} g_{22} = e^{-4\lambda} \left( |f'(z)|^2 + 2 \text{Re} (z)|h'(|z|^2)|^2 \right) \\
g_{12} &= e^{-4\lambda} g_{12} = -2 e^{-4\lambda} \text{Re} (z)|h'(|z|^2)|^2 . \end{align*} \]
Therefore, we deduce that
\[
|\tilde{d}^2|_g = e^{-2\lambda} |f'(z)|^2 \left( |h_1^2| - |h_2^2| \right) + e^{-2\lambda} \frac{|f'(z)|^2}{|h'(|z|^2)|^2} \left( 1 + 4|z|^2 \right) \frac{f''(z)}{|f'(z)|^2} \]
\[
+ \frac{1}{|h'(|z|^2)|^2} \left( 1 + 4|z|^2 \right) \frac{f''(z)}{|f'(z)|^2} - 4 \frac{h''(|z|^2)}{|f'(z)|^2} \left( z f''(z) f'(z) - 2e^{-4\lambda} \frac{z f''(z) f'(z)}{|f'(z)|^2} \right)
\]
\[
+ 4 e^{-4\lambda} \frac{h''(|z|^2)}{|f'(z)|^2} + 4 e^{-4\lambda} \frac{h''(|z|^2)}{|f'(z)|^2} \left( z f''(z) f'(z) - 2e^{-4\lambda} \frac{z f''(z) f'(z)}{|f'(z)|^2} \right)
\]
\[
\frac{1}{|h'(|z|^2)|^2} \left( 1 + 4|z|^2 \right) \frac{f''(z)}{|f'(z)|^2} - 4 \frac{h''(|z|^2)}{|f'(z)|^2} \left( z f''(z) f'(z) - 2e^{-4\lambda} \frac{z f''(z) f'(z)}{|f'(z)|^2} \right)
\]

We first have
\[
e^{-2\lambda} \frac{|f'(z)|}{|h'(|z|^2)|^2} \left( |h_1^2| - |h_2^2| \right) = e^{-6\lambda} \frac{|f'(z)|}{|h'(|z|^2)|^2} \left( 1 + 4|z|^2 \right) \frac{f''(z)}{|f'(z)|^2} + \frac{1}{|h'(|z|^2)|^2} \frac{f''(z)}{|f'(z)|^2}
\]
\[
+ |z|^2 \frac{|f'(z)|^2}{|h'(|z|^2)|^2} \frac{h''(|z|^2)}{|f'(z)|^2} + 4 \frac{h''(|z|^2)}{|h'(|z|^2)|^2} \frac{f''(z)}{|f'(z)|^2} - 4 \frac{h''(|z|^2)}{|f'(z)|^2} \left( z f''(z) f'(z) - 2e^{-4\lambda} \frac{z f''(z) f'(z)}{|f'(z)|^2} \right)
\]
\[
+ 4 e^{-4\lambda} \frac{h''(|z|^2)}{|f'(z)|^2} + 4 e^{-4\lambda} \frac{h''(|z|^2)}{|f'(z)|^2} \left( z f''(z) f'(z) - 2e^{-4\lambda} \frac{z f''(z) f'(z)}{|f'(z)|^2} \right)
\]
\[
Now, we fix some 0 < \alpha < 1 and define
\[
h(r) = (1 - r)^\alpha.
\]

Then we have
\[
h'(r) = -\alpha (1 - r)^{\alpha - 1}
\]
\[
h''(r) = \alpha (1 - \alpha)(1 - r)^{\alpha - 2}.
\]
We have
\[
\frac{(h''(r))^2}{|h'(r)|^4} = \frac{\alpha^2(1-\alpha)^2(1-r)^{2\alpha-4}}{\alpha^3(1-r)^{3\alpha-3}} = \frac{(1-\alpha)^2}{\alpha} \cdot \frac{1}{(1-r)^{1+\alpha}}
\]
\[
\frac{h''(r)}{|h'(r)|^2} = \frac{\alpha(1-\alpha)(1-r)^{\alpha-2}}{(-\alpha(1-r)^{\alpha-1})^2} = \frac{1-\alpha}{\alpha} \cdot \frac{1}{(1-r)^{\alpha}}
\]
\[
\frac{h''(r)}{|h'(r)|^4} = \frac{\alpha(1-\alpha)(1-r)^{\alpha-2}}{\alpha^4(1-r)^4(\alpha-4)} = \frac{(1-\alpha)}{\alpha^3} \cdot (1-r)^{2-3\alpha}
\]
and
\[
\frac{1}{|h'(r)|} \left| \frac{h''(z)}{|h'(r)|^2} \right|^2 = \frac{(1-\alpha)^2}{\alpha^2} \cdot (1-r)^{1-3\alpha}.
\]
By Proposition 5.2, for all \(\varepsilon > 0\), there exists \(\Gamma_{\varepsilon}\) such that
\[
\Gamma_{\varepsilon}^{-1}(1-|z|^2) \leq |f'(z)| \leq \frac{\Gamma_{\varepsilon}}{(1-|z|^2)^{\varepsilon}} \quad \text{for all} \quad z \in \mathbb{D}.
\]
Explicitly, we have
\[
\Gamma_{\varepsilon} = 2 \exp \left( 2C_0 \left( 1 + \frac{1}{\varepsilon} \right) \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + C_0 \left( \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}} \right) \quad (5.13)
\]
Recalling that
\[
e \quad (1-|z|^2)^{1-\alpha} \left| f'(z) \right|^2 = 4|z|^2 + 2(1-|z|^2)^{2(1-\alpha)} |f'(z)|^2,
\]
we deduce that
\[
\exp \left( 2 \varepsilon \right) \leq 2|z| + \alpha(1-|z|^2)^{1-\alpha} |f'(z)| \leq 2 + \alpha \Gamma_{1-\alpha}
\]
\[
= 2 + 2\alpha \exp \left( 2C_0 \left( 1 + \frac{1}{1-\alpha} \right) \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + C_0 \left( \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}} \right) \quad (5.14)
\]
We estimate directly
\[
\int_{\mathbb{D}} e^{-2\omega \frac{|f''(z)|}{|h'(z)|^2}} \left( |\bar{H}|^2 - |h_z|^2 \right) |dz|^2 \leq \int_{\mathbb{D}} e^{-2\omega \frac{|f''(z)|}{|h'(z)|^2}} |\bar{H}|^2 |dz|^2
\]
\[
\leq e^{6\|\omega\|_{L^\infty}} \left( \frac{1}{\alpha} \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{1-\alpha} |dz|^2 + \frac{4}{\alpha} \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |z|^2 |f'(z)|(1-|z|^2)^{1-\alpha} |dz|^2
\]
\[
+ \frac{4}{\alpha} \int_{\mathbb{D}} |z||f''(z)|(1-|z|^2)^{1-\alpha} |dz|^2 + \frac{1}{\alpha^2} \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |f'(z)|(1-|z|^2)^{5(1-\alpha)} |dz|^2
\]
\[
+ \frac{1}{\alpha - 1} \int_{\mathbb{D}} |f'(z)|^3(1-|z|^2)^{1-3\alpha} |dz|^2 + 4 \frac{(1-\alpha)}{\alpha^3} \int_{\mathbb{D}} \text{Re} \left( \frac{f''(z)}{f'(z)} \right) |f'(z)|(1-|z|^2)^{2-3\alpha} |dz|^2 \right) \quad (5.15)
\]
We successively estimate by Lemma 3.3, (5.12) and Cauchy-Schwarz inequality
\[
\int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{1-\alpha} |dz|^2 \leq 4\pi \exp \left( C_0 \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + C_0 \left( \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}} \right)
\]
\[ \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} |z|^{\alpha} |f'(z)|(1 - |z|^{2})^{-\alpha} \left| dz \right|^{2} \leq \Gamma_{1-\alpha} \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \]

\[ \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} |z|^{\alpha} \left| f'(z) \right|(1 - |z|^{2})^{-\alpha} \left| dz \right|^{2} \leq \Gamma_{1-\alpha} \left( \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \right)^{\frac{1}{2}} \left( \int_{D} \left| f'(z) \right|(1 - |z|^{2})^{-\alpha} \left| dz \right|^{2} \right)^{\frac{1}{2}} \]

\[ \int_{D} \left| f'(z) \right|^{2} (1 - |z|^{2})^{-3\alpha} \left| dz \right|^{2} \leq \Gamma_{z} \int_{D} (1 - |z|^{2})^{-3\alpha - \varepsilon} \left| dz \right|^{2} = \frac{\pi}{2 - 3\alpha - \varepsilon} \]

\[ \int_{D} \left| f'(z) \right|^{2} (1 - |z|^{2})^{2(2 - 3\alpha) - \varepsilon} \left| dz \right|^{2} \leq C_{z} \int_{D} (1 - |z|^{2})^{2(2 - 3\alpha) - \varepsilon} \left| dz \right|^{2} = \frac{\pi}{5 - 6\alpha + \varepsilon} C_{z} < \infty. \]

Therefore, by (5.15) and (5.16), we deduce that

\[ \int_{D} e^{-2\omega} \left| \frac{f'(z)}{|h'|^{\alpha}|z|^{\alpha}} \right|^{2} |h_{z}|^{2} \left| dz \right|^{2} \leq (8 + 8\alpha^{3} \Gamma_{1-\alpha}) \]

\[ \times \left( \frac{4\pi}{\alpha} \exp \left( C_{0} \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} + C_{0} \left( \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \right)^{\frac{1}{2}} \right) + \frac{4}{\alpha} \Gamma_{1-\alpha} \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \]

\[+ 4\sqrt{\pi} \frac{(1 - \alpha)}{\alpha^{3}} \Gamma_{1-\alpha} \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \frac{1}{2} + \Gamma_{1-\alpha} \int_{D} \left| f''(z) \right|^{2} \left| dz \right|^{2} \frac{2}{7} \left( \frac{1}{\alpha} - 1 \right)^{2} \Gamma_{\frac{7}{3}} \]

\[+ 8\sqrt{\pi} \frac{(1 - \alpha)}{\alpha^{3}} \exp \left( 2C_{0} \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} + C_{0} \left( \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \right)^{\frac{1}{2}} \right) \left( \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \right)^{\frac{1}{2}} \]

\[ \leq (8 + 8\alpha^{3} \Gamma_{1-\alpha}) \left( \frac{2\pi}{\alpha} \Gamma_{\infty} + \frac{4}{\alpha^{3}} \Gamma_{1-\alpha} \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \right) \]

\[+ \frac{4\sqrt{\pi}}{\alpha} \left( \Gamma_{1-\alpha} + \frac{2(1 - \alpha)}{\alpha^{2}} \Gamma_{1} \right) \left( \int_{D} \left| \frac{f''(z)}{f'(z)} \right|^{2} \left| dz \right|^{2} \right)^{\frac{1}{2}} + \frac{2\pi}{7} \left( \frac{1}{\alpha} - 1 \right)^{2} \Gamma_{\frac{7}{3}} \]

(5.17)

Now, we compute the second part of \(|\Delta|_{g}^{2} d\text{vol}_{g} \). We first have

\[ \text{Im} (zh_{z}) = \text{Im} \left( e^{-2\omega} \left( \varepsilon + \frac{f''(z)h'(z)}{|h'|^{2}} \frac{2\pi |f'(z)|^{2}}{|h'(z)|^{2}} \right) \right) \]

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\[= e^{-4\omega} \text{Im} \left( |z|^2 + \frac{f''(z) f'(z)}{|h'||z|^2|} - \frac{2|z|^2 h''(|z|^2)|f'(z)|^2}{h''(|z|^2)|} \right) = e^{-4\omega} \text{Im} \left( \frac{f''(z) f'(z)}{|h'||z|^2|} \right),\]

while
\[\text{Im} \left( z\tilde{H}_z \right) = -e^{-2\omega} \left( \text{Im} \left( \frac{f'(z)}{f'(z)} \right) + 2 \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \right) - 2 \text{Re} \left( \frac{f''(z)}{f'(z)} \right).\]

Therefore, we find that
\[
\left| \text{Im} \left( z\tilde{H}_z \right) \right|^2 = e^{-4\omega} \left( |z|^2 + 4|z|^2 \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \right) + \frac{1}{|h'||z|^2|} \left| \text{Im} \left( \frac{f''(z)}{f'(z)} \right) \right|^2
+ 4|z|^2 \left| \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \right|^2
+ \frac{1}{|f'(z)|^2} \left( 1 - 4e^{-4\omega} \left| \frac{f'(z)}{f'(z)} \right|^4 \right) \left| \text{Im} \left( \frac{f''(z)}{f'(z)} \right) \right|^2
+ 4|z|^2 \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \left| \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \right|^2.
\]

Then, we have
\[e^{-2\omega} \frac{|h'||z|^2|}{|f'(z)|} \left( \left| \text{Im} \left( z\tilde{H}_z \right) \right|^2 - \left| \text{Im} \left( zh_z \right) \right|^2 \right)
= e^{-6\omega} \left( |z|^2 \frac{|h'||z|^2|}{|f'(z)|} + 4|z|^2 \frac{|h'|(|z|^2)|}{|f'(z)|} \right) \left| \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \right|^2
+ \frac{1}{|f'(z)|^2} \left( 1 - 4e^{-4\omega} \left| \frac{f'(z)}{f'(z)} \right|^4 \right) \left| \text{Im} \left( \frac{f''(z)}{f'(z)} \right) \right|^2
+ 4|z|^2 \frac{|h'|(|z|^2)|}{|f'(z)|} \left| \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \right|^2.
\]

Notice that simply using the Golusin’s inequality ([25], [26]), we would only obtain the inequality
\[|f'(z)| \geq 1 - \frac{1}{|z|^2},\]

which does not imply the boundedness of
\[\int_{D} |z|^2 \frac{|h'|(|z|^2)|}{|f'(z)|} |dz|^2 = \alpha \int_{D} |z|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}}.\]

However, for all \(\alpha < 1\), this quantity is finite by virtue of inequality (5.2). Then, we estimate
\[
\int_{D} e^{-2\omega} \frac{|h'|(|z|^2)|}{|f'(z)|} \left( \left| \text{Im} \left( z\tilde{H}_z \right) \right|^2 - \left| \text{Im} \left( zh_z \right) \right|^2 \right) |dz|^2
\leq e^{\delta \|h\|_{L^\infty(D)}} \left( \alpha \int_{D} |z|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} + 4 \alpha \int_{D} |z|^2 \left| \text{Re} \left( \frac{f''(z)}{f'(z)} \right) \right|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} \right),
\]

which is finite.
\(\int |z|^2 \left| \frac{\text{Im} \left( z f''(z) \right)}{f'(z)} \right|^2 \frac{(1 - |z|^2)^{1-\alpha}|dz|^2}{|f'(z)|} + 4 \alpha \int |z|^2 \text{Re} \left( \frac{z f''(z)}{f'(z)} \right) \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}}. \) (5.18)

We successively estimate

\[
\int_D |z|^2 \frac{|dz|^2}{|f'(z)|} \leq C_D \int_D \frac{|dz|^2}{(1 - |z|^2)^{1-\alpha}} = 2\pi C_D \int_0^1 r(1 - r^2)^{\frac{\alpha}{2} - 1} dr = \frac{2\pi}{\alpha} C_D
\]

\[
\int_D |z|^2 \left| \frac{\text{Re} \left( z f''(z) \right)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|} \leq \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} \leq \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|} \leq \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} \leq \left( \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|} \right)^{\frac{1}{2}} \sqrt{\frac{2\pi}{\alpha} C_D}. \quad (5.19)
\]

where we used the Cauchy-Schwarz inequality with the measure \(\frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}}\) in the last inequality.

Therefore, we finally deduce by (5.18) and (5.19) that

\[
\int_D e^{-2s |f'(z)|} \left( |z H(z)|^2 - |\text{Im} (z H(z))|^2 \right) |dz|^2 \leq (8 + 8\Gamma_1^{3-\alpha}) \left( 2\pi \Gamma_2^{3-\alpha} + 4\alpha \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} + \frac{\Gamma_1^{1-\alpha}}{\alpha} \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)
\]

\[
+ 4\sqrt{2\pi \Gamma_2^{3-\alpha}} \left( \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} \right)^{\frac{1}{2}} \leq (8 + 8\Gamma_1^{3-\alpha}) \left( 4\pi \Gamma_2^{3-\alpha} + 8\alpha \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} + \frac{\Gamma_1^{1-\alpha}}{\alpha} \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right). \quad (5.21)
\]

Putting together (5.17) and (5.20), we deduce that there exists \(C'_\alpha < \infty\) such that

\[
\int_D |A|^2 \text{vol}_g \leq (8 + 8\Gamma_1^{3-\alpha}) \left( \frac{2\pi}{\alpha} \Gamma_1^{1-\alpha} + 4\pi \Gamma_2^{3-\alpha} + \frac{2\pi}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \right)^2 \Gamma_2^{3-\alpha} + \frac{5}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \Gamma_2^{3-\alpha} + \frac{4\pi}{\alpha} \left( \Gamma_1^{1-\alpha} + \frac{2(1 - \alpha)}{\alpha^2} \right) \left( \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right)^{\frac{1}{2}}
\]

\[
+ 8\alpha \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{|f'(z)|(1 - |z|^2)^{1-\alpha}} \leq (8 + 8\Gamma_1^{3-\alpha}) \left( \frac{2\pi}{\alpha} \Gamma_1^{1-\alpha} + 4\pi \Gamma_2^{3-\alpha} + \frac{2\pi}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \right)^2 \Gamma_2^{3-\alpha} + \frac{4\pi}{\alpha} \left( \Gamma_1^{1-\alpha} + \frac{2(1 - \alpha)}{\alpha^2} \right) \Gamma_1^{1-\alpha} + \frac{2(1 - \alpha)}{\alpha^2} \Gamma_1^{1-\alpha} + \frac{2(1 - \alpha)}{\alpha^2} \Gamma_1^{1-\alpha} \right).
\]

Using the conformal invariance of the Willmore energy, we get the same inequality for the \(L^2\) norm of the trace-less second fundamental form for \(\Phi\) as an immersion of \(\mathbb{H}^3\). This concludes the end of the proof of the theorem.

\textbf{Theorem 5.5.} Let \(\gamma \subset \mathbb{C}\) be a simple curve of finite Loewner energy, let \(\Omega\) be the bounded open connected component of \(\mathbb{C} \setminus \gamma\) and let \(f : D \to \mathbb{C}\) be any univalent holomorphic map. Assume that for some \(0 < \alpha < 1\) we have

\[
\int_D \left| f''(z) \right| \frac{|dz|^2}{(1 - |z|^2)^{\alpha}} < \infty, \quad (5.22)
\]

\(45\)
and fix some $\varepsilon > 0$. Then the embedding $\bar{\Phi} : \mathbb{D} \to \mathbb{R}^4_+$ defined for all $z \in \mathbb{D}$ by

$$\bar{\Phi}(z) = (\text{Re}(f(z)), \text{Im}(f(z)), (1 - |z|^2)^\alpha)$$

is orthogonal to $\partial \mathbb{R}^4_+ = \mathbb{R}^2 \times \{0\}$ and has finite Lipschitz–Killing curvature, i.e.

$$\int_{\mathbb{D}} |K_g| \text{dvol}_g < \infty,$$

where $K_g$ is the Gauss curvature and $g = \bar{\Phi}^* g_{\mathbb{R}^4_+}$ is the induced metric on the disk $\mathbb{D}$. Furthermore, there exists a universal constant $C'_\alpha > 0$ such that

$$\int_{\mathbb{D}} |K_g| \text{dvol}_g \leq C'_\alpha \left( 1 + \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{(1 - |z|^2)^\alpha} + \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{(1 - |z|^2)^{\alpha + 2\alpha}} \right) \exp \left( C'_\alpha \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 \right). \tag{5.23}$$

**Remark 5.6.** If $\alpha < \frac{1}{2}$, then the condition in (5.22) is weaker than the finiteness of Loewner energy since by (5.12), we have

$$\int_{\mathbb{D}} |f''(z)| \frac{|dz|^2}{(1 - |z|^2)^\alpha} \leq \left( \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 \frac{|dz|^2}{(1 - |z|^2)^\alpha} \right)^{\frac{1}{2}} \left( \int_{\mathbb{D}} \frac{|dz|^2}{(1 - |z|^2)^{\alpha + 2\alpha}} \right)^{\frac{1}{2}} \leq \sqrt{\frac{2\pi \Gamma(1 - \alpha)}{1 - 2\alpha}} \left( \int_{\mathbb{D}} \frac{|dz|^2}{f'(z)} \right)^{\frac{1}{2}} < \infty.$$

**Proof.** By (5.7), we compute

$$\partial_x \bar{\Phi} = \left( \frac{1}{2} f'(z), -\frac{i}{2} f''(z), \tau h'(|z|^2) \right),$$

$$\partial_y \bar{\Phi} = \left( \frac{1}{2} f''(z), -\frac{i}{2} f''(z), \tau^2 h''(|z|^2) \right),$$

$$\partial^2_{xy} \bar{\Phi} = \left( 0, 0, h'(|z|^2) + |z|^2 h''(|z|^2) \right).$$

Therefore, we obtain

$$\partial_x^2 \bar{\Phi} = 2 \text{Re} \left( \partial_x^2 \bar{\Phi} \right) + 2 \partial^2_{xy} \bar{\Phi} = \left( \text{Re} \left( f''(z) \right), \text{Im} \left( f''(z) \right), 2 h'(|z|^2) + 2 \left( \text{Re} \left( f'(z) \right) + |z|^2 \right) h''(|z|^2) \right)$$

$$\partial_y^2 \bar{\Phi} = -2 \text{Re} \left( \partial_y^2 \bar{\Phi} \right) + 2 \partial^2_{xy} \bar{\Phi} = \left( -\text{Re} \left( f''(z) \right), -\text{Im} \left( f''(z) \right), 2 h''(|z|^2) + 2 \left( -\text{Re} \left( f'(z) \right) + |z|^2 \right) h''(|z|^2) \right)$$

$$\partial^2_{xy} \bar{\Phi} = -2 \text{Im} \left( \partial^2_{xy} \bar{\Phi} \right) = \left( -\text{Im} \left( f''(z) \right), \text{Re} \left( f''(z) \right), 2 \text{Im} \left( z^2 \right) h''(|z|^2) \right)$$

Recalling that

$$\bar{n} = -e^{-2\omega(z)} \left( 2 \text{Re} \left( z \frac{f'(z)}{f'(z)} \right), 2 \text{Im} \left( z \frac{f'(z)}{f'(z)} \right), \frac{|f'(z)|}{h'(|z|^2)} \right),$$

we deduce that

$$\begin{align*}
I_{1,1} &= \langle \partial^2_{xy} \bar{\Phi}, \bar{n} \rangle = -e^{-2\omega} \left( 2 \text{Re} \left( \frac{\tau f''(z)}{|f'(z)|} + 2 f'(z) + 2 \left( \text{Re} \left( f'(z) \right) + |z|^2 \right) \frac{h''(|z|^2)}{h'(|z|^2)} \frac{|f'(z)|}{h'(|z|^2)} \right) \\
&= -e^{-2\omega} |f'(z)| \left( 2 \text{Re} \left( \frac{\tau f''(z)}{|f'(z)|} \right) + 2 \left( \text{Re} \left( f'(z) \right) + |z|^2 \right) \frac{h''(|z|^2)}{h'(|z|^2)} \right) \\
I_{1,2} &= -e^{-2\omega} |f'(z)| \left( -\text{Re} \left( \frac{\tau f''(z)}{|f'(z)|} \right) + 2 \left( -\text{Re} \left( f'(z) \right) + |z|^2 \right) \frac{h''(|z|^2)}{h'(|z|^2)} \right) \\
I_{1,2} &= -e^{-2\omega} |f'(z)| \left( -\text{Im} \left( \frac{\tau f''(z)}{|f'(z)|} \right) + 2 \text{Im} \left( z^2 \right) \frac{h''(|z|^2)}{h'(|z|^2)} \right) \\
\end{align*}$$

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where we used
\[
\text{Re}(a)\text{Re}(b) + \text{Im}(a)\text{Im}(b) = \frac{1}{4} ((a + \pi)(b + \overline{b}) - (a - \pi)(b - \overline{b})) = \text{Re}(a\overline{b})
\]
\[
-\text{Im}(a)\text{Re}(a) + \text{Re}(a)\text{Im}(b) = \frac{1}{4i} ((a - \pi)(b + \overline{b}) + (a + \pi)(b - \overline{b})) = -\text{Im}(a\overline{b})
\]
Then, we have
\[
K_\gamma = \det \left\{ \begin{pmatrix} g_1^{1,1} & g_1^{1,2} \\ g_2^{1,1} & g_2^{1,2} \end{pmatrix}, \begin{pmatrix} I_{1,1} \\ I_{1,2} \end{pmatrix}, \begin{pmatrix} I_{2,1} \\ I_{2,2} \end{pmatrix} \right\} = \frac{I_{1,1}I_{2,2} - I_{1,2}^2}{\text{deg}(g)}
\]
so that
\[
K_\gamma d\text{vol}_g = \frac{I_{1,1}I_{2,2} - I_{1,2}^2}{\sqrt{\text{deg}(g)}} |dz|^2 = G(z)^{-1} (I_{1,1}I_{2,2} - I_{1,2}^2) e^{-2\omega(z)}|dz|^2.
\]
(5.24)
We easily compute
\[
I_{1,1}I_{2,2} - I_{1,2}^2 = e^{-4\omega} |f'(z)|^2 \left( -|\text{Re} \left( z f''(z) \right) |^2 + 4 + 4 |\text{Re}(z^2)|^2 + |z|^2 \right) \left| \frac{h''(|z|^2)}{h'(|z|^2)} \right|^2 - 4 \text{Re} \left( z f''(z) \right) \left| \frac{h''(|z|^2)}{h'(|z|^2)} \right|^2 + 4 \text{Im} \left( z f''(z) \right) \left| \frac{h''(|z|^2)}{h'(|z|^2)} \right|^2
\]
and finally
\[
\int_D |K_\gamma|d\text{vol}_g \leq e^{6|\omega|_{L^\infty}} \left( \frac{1}{\alpha} \int_D |z|^2 \left| \frac{f''(z)}{f'(z)} \right|^2 |f'(z)|^2 (1 - |z|^2)^{1-\alpha} |dz|^2
\]
\[
+ 4 \left( \frac{1}{\alpha} - 1 \right) \int_D |z|^2 |f'(z)| \left| \frac{|dz|^2}{(1 - |z|^2)^\alpha} \right|^2 + \frac{4}{\alpha} \int_D |f'(z)| (1 - |z|^2)^{1-\alpha} |dz|^2
\]
\[
+ 4 \left( \frac{1}{\alpha} - 1 \right) \int_D |z|^2 |\text{Re} \left( z f''(z) \right) | \left| f'(z) \right| \left| \frac{|dz|^2}{(1 - |z|^2)^\alpha} \right|
\]
\[
\leq (8 + 8\alpha^3 \Gamma_{1-\alpha} \left( \frac{1}{\alpha} \right) \left( \frac{1}{\alpha} - 1 \right) \Gamma_{1-\alpha} + \frac{16\pi}{\alpha} \Gamma_{1-\alpha})
\]
where we used
\[
\text{Re}(a)\text{Re}(b) - \text{Im}(a)\text{Im}(b) = \text{Re}(ab)
\]
and
\[
\int_D \frac{|dz|^2}{(1 - |z|^2)^{\frac{1}{2} + \frac{\alpha}{2}}} = 2\pi \int_0^1 r(1 - r^2)^{-\frac{1}{2} - \frac{\alpha}{2}} dr = \frac{2\pi}{1 - \alpha}
\]
which concludes the proof of the Proposition.

Now, we will show which Hölder condition implies the previous conditions.

**Proposition 5.7.** Let $\gamma$ be a $C^{1,\beta}$ curve. If $\beta > 0$, then condition (5.22) is satisfied for some $0 < \alpha < 1$. If $\beta > 2/3$, then condition (5.4) is satisfied for some $0 < \alpha < 1$.
The main tool is Kellogg’s theorem which states that the boundary extension of a conformal map \( f \) of the disk \( \mathbb{D} \) onto the interior of the curve has the same regularity as the arc-length parametrisation of the curve. See for instance [40] or [24].

**Theorem 5.8** (Kellogg, Theorem 4.3, [24]). Let \( k \geq 1 \) and \( 0 < \beta < 1 \). Then the following conditions are equivalent:

1. \( g \) is of class \( C^{k,\beta} \).
2. \( \arg f' \in C^{k-1,\beta}(\partial \mathbb{D}) \).
3. \( f \in C^{k,\beta}(\mathbb{D}) \) and \( f' \neq 0 \) on \( \overline{\mathbb{D}} \).

**Proof of Proposition 5.7.** Since \( g \) is \( C^{1,\beta} \), Kellogg’s theorem shows that \( f' \in C^{0,\beta}(\overline{\mathbb{D}}) \). In particular, there exists \( C > 0 \) such that \( |f'(z)| \geq C^{-1} \) and \( |f'(z) - f'(w)| \leq C|z - w|^\beta \), for all \( z, w \in \overline{\mathbb{D}} \). Let \( z \in \mathbb{D} \) and \( \delta = \frac{1 - |z|}{2} \), since \( f' \) is holomorphic, Cauchy’s integral formula gives

\[
|f''(z)| = \frac{1}{2\pi} \int_0^{2\pi} \frac{f'(z + \delta e^{i\theta}) - f'(z)}{\delta e^{i\theta}} d\theta \leq C\delta^{\beta-1} \leq C'(1 - |z|^2)^{\beta-1},
\]

for some constant \( C' > 0 \). Hence

\[
\int_{\mathbb{D}} |f''(z)| \left| \frac{dz^2}{(1 - |z|^2)^\alpha} \right| \leq 2\pi C' \int_0^1 (1 - r^2)^{\beta-1-\alpha} r dr
\]

is finite when \( \beta > \alpha \), and since \( \alpha > 0 \) is arbitrary, the condition is equivalent to \( \beta > 0 \) which is the condition (5.22). Similarly,

\[
\int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 \left| \frac{dz^2}{(1 - |z|^2)^{1-\alpha}} \right| \leq C^3(C')^2 2\pi \int_0^1 (1 - r^2)^{2\beta-3+\alpha} r dr,
\]

which is finite when \( 2\beta - 2 - (1 - \alpha) > -1 \), i.e. \( \beta > 1 - \frac{\alpha}{2} \). \( \square \)

## 6 Appendix

In this appendix, we will prove some additional properties related to moving frames and the Loewner energy and explore similar formulae and Grunsky-type identities.

### 6.1 Cartesian Frames for Weil-Petersson Curves

Here, we show that if one chooses the Cartesian frame and not the polar frame, this permits to give another characterisation of the Loewner energy with respect to the Dirichlet energy of moving frames.

Let \( \Gamma \subset S^2 \) be a simple curve, and let \( \Omega_1, \Omega_2 \subset S^2 \) be the two connected components of \( S^2 \setminus \Gamma \). Assume without loss of generality that \( N \notin \Gamma \), and letting \( \pi: S^2 \setminus \{N\} \to \mathbb{C} \) be the stereographic projection, an easy computation (see Section 4 for similar computations) shows that the Coulomb frame \((\vec{e}_1, \vec{f}_1)\) associated to a uniformisation \( f_1 = \pi^{-1} \circ f: \mathbb{D} \to \Omega_1 \subset S^2 \) is given by

\[
\vec{e}_1 \circ f_1 = \frac{\partial_x f_1}{|\partial_x f_1|} = e^{-\psi} \text{Re} (\partial_z f_1(z))
\]

\[
= \text{Re} \left( \frac{f''(z)}{|f'(z)|} \left( \frac{1 - f(z)^2}{(1 + |f(z)|^2)^2} \right) \left( 1 + \frac{f(z)}{1 + |f(z)|^2} \right) \frac{2f(z)}{1 + |f(z)|^2} \right) = \text{Re} \left( \frac{f'(z)}{|f'(z)|^3} \psi(z) \right)
\]

\[
\vec{f}_1 \circ f_1 = \frac{\partial_y f_1}{|\partial_y f_1|} - e^{-\psi} \text{Im} (\partial_z f_1(z))
\]

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\[= \text{Im} \left( \frac{f'(z)}{|f'(z)|} \left( \frac{(1 - f(z)^2)}{(1 + |f(z)|^2)}, \frac{i(1 + f(z)^2)}{(1 + |f(z)|^2)}, \frac{2f(z)}{(1 + |f(z)|^2)} \right) \right) = \text{Im} \left( \frac{f'(z)}{|f'(z)|} \psi(z) \right),\]

where

\[\psi(z) = \left( \frac{(1 - f(z)^2)}{(1 + |f(z)|^2)}, \frac{i(1 + f(z)^2)}{(1 + |f(z)|^2)}, \frac{2f(z)}{(1 + |f(z)|^2)} \right)\]

Now define

\[\chi(z) = \frac{f'(z)}{|f'(z)|}.\]

First, we easily have

\[|\nabla \chi|^2 = \left| \frac{f''(z)}{f'(z)} \right|^2.\]

Furthermore, a computation shows that

\[\partial z \chi(z) = -\frac{1}{2} \frac{f''(z)}{f'(z)} \chi(z),\]

\[\partial \bar{z} \chi(z) = \frac{1}{2} \frac{f''(z)}{f'(z)} \chi(z),\]

\[\partial z \psi(z) = -\frac{f'(z)^2 f(z)}{1 + |f(z)|^2} \psi(z) + \frac{2f'(z)}{1 + |f(z)|^2} (-f(z), if(z), 1),\]

\[\partial \bar{z} \psi(z) = -\frac{f'(z)f(z)}{1 + |f(z)|^2} \psi(z).\]

Therefore, we have in particular

\[|\partial z \psi(z)|^2 = \frac{2|f'(z)|^2|f(z)|^2}{(1 + |f(z)|^2)^2} + 4\left| \frac{f'(z)^2(1 + 2|f(z)|^2)}{(1 + |f(z)|^2)^2} \right| - 4 \text{Re} \left( \frac{|f(z)|^2 f'(z)}{(1 + |f(z)|^2)^2} \langle \psi(z), (-f(z), -if(z), 1) \rangle \right) = \frac{2|f'(z)|^2|f(z)|^2}{(1 + |f(z)|^2)^2} + \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2} \]

since

\[\langle \psi(z), (-f(z), -if(z), 1) \rangle = 2f(z).\]

Notice also that \(\langle \psi, \psi \rangle = 0\) implies that \(\langle \partial z \psi, \psi \rangle = 0\) and both identities show that

\[\langle \psi(z), (-f(z), if(z), 1) \rangle = 0,\]

\[\langle \partial z \psi, \partial \bar{z} \psi \rangle = 0.\]

Therefore, the identity

\[|\partial z \psi(z)|^2 = \frac{2|f'(z)|^2|f(z)|^2}{(1 + |f(z)|^2)^2}\]

shows that

\[|\nabla \psi(z)|^2 = 2|\partial z \psi(z)|^2 + 2|\partial \bar{z} \psi(z)|^2 = \frac{8|f'(z)|^2}{(1 + |f(z)|^2)}.\]

Finally, we get

\[\partial_z \text{Re}(\chi(z)\psi(z)) = \frac{1}{2} \left( \partial_z \chi(z)\psi(z) + \chi(z)\partial_z \psi(z) + \partial \bar{z} \chi(z)\psi(z) + \chi(z) \partial \bar{z} \psi(z) \right).\]
Let us deduce the following result. Since $\langle \psi, \psi \rangle = 0$ and $|\psi|^2 = \langle \psi, \overline{\psi} \rangle = 2$, we have

$$|\nabla \tilde{e}_1|^2 = 4|\partial_z \tilde{e}_1|^2 = 2|\partial_x \chi(z)|^2 + 2|\partial_y \chi(z)|^2 + |\partial_z \psi(z)|^2 + |\partial_y \psi(z)|^2 + 2 \text{Re} \left( \chi(z) \overline{\partial_x \chi(z)} \langle \overline{\psi}, \partial_y \psi(z) \rangle \right)$$

$$+ 2 \text{Re} \left( \chi(z) \partial_x \chi(z) \langle \overline{\psi}, \partial_y \psi(z) \rangle \right)$$

$$= |\nabla \chi(z)|^2 + \frac{1}{2} |\nabla \psi(z)|^2 - 4 \text{Re} \left( \frac{f''(z)}{f'(z)} \frac{f'(z)}{f''(z)} \right)$$

$$= \left| \frac{f''(z)}{f'(z)} \right|^2 + \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2} - 4 \text{Re} \left( \frac{f''(z)}{f'(z)} \frac{f'(z)}{f''(z)} \right)$$

$$= \left( \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f''(z)} \right)^2 + \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2} = \left| \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f''(z)} \right|^2 + \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2},$$

where we used

$$\chi(z) \partial_x \chi(z) = \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)$$

$$\chi(z) \partial_y \chi(z) = - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)$$

$$\langle \overline{\psi}(z), \partial_y \psi(z) \rangle = \frac{2f'(z)\overline{f(z)}}{(1 + |f(z)|^2)^2}$$

$$\langle \overline{\psi}(z), \partial_y \psi(z) \rangle = - \frac{2f'(z)\overline{f(z)}}{(1 + |f(z)|^2)^2}.$$

Notice that this identity implies as

$$\frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f''(z)}$$

cannot vanish uniformly that

$$\int_{\Omega_1} |d\tilde{e}_1|^2_{g_0} d\text{vol}_{g_0} > \text{Vol}_{g_0}(\Omega_1).$$

Likewise, we have

$$|\nabla \tilde{f}_1|^2 = \left| \frac{f''(z)}{f'(z)} \right|^2 + \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2} - 4 \text{Re} \left( \frac{f''(z)}{f'(z)} \frac{f'(z)}{f''(z)} \right).$$

(6.1)

Since

$$\int_{\mathbb{D}} 4|f'(z)|^2 |dz|^2 \leq (1 + \|f\|_{L^\infty(\mathbb{D})}^2) \int_{\mathbb{D}} \frac{4|f'(z)|^2 |dz|^2}{(1 + |f(z)|^2)^2} = (1 + \|f\|_{L^\infty(\mathbb{D})}^2) \text{Vol}_{g_0}(\Omega_1) < \infty,$$

we deduce the following result.

**Theorem 6.1.** Let $g_0$ be the standard round metric on $S^2$, $\Gamma \subset S^2$ be a simple closed curve, and let $\Omega_1, \Omega_2 \subset S^2 \setminus \Gamma$ be the two connected components of $S^2 \setminus \Gamma$. Then the following conditions are equivalent:

1. $\Gamma$ is a curve of finite Loewner energy.
2. There exists a uniformisation $F_1 : \mathbb{D} \to \Omega_1$ such that the unit vector field $\tilde{e}_1 : \Omega_1 \to U \Omega_1 \subset T \Omega_1$ defined by $\tilde{e}_1 = \frac{\partial_z F_1}{|\partial_z F_1|} \circ F_1^{-1}$ satisfies $\tilde{e}_1 \in W^{1,2}(\Omega_1, d\text{vol}_{g_0})$.
3. There exists a uniformisation $F_1 : \mathbb{D} \to \Omega_1$ such that the unit vector field $\tilde{f}_1 : \Omega_1 \to U \Omega_1 \subset T \Omega_1$ defined by $\tilde{f}_1 = \frac{\partial_y F_1}{|\partial_y F_1|} \circ F_1^{-1}$ satisfies $\tilde{f}_1 \in W^{1,2}(\Omega_1)$.  

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Therefore, we deduce that the geodesic curvature is given by

\[ \frac{\partial_x F_2}{|x_2 F_2|} \circ F_2^{-1} \] satisfies \( \vec{e}_2 \in W^{1,2}(\Omega_2) \).

There exists a uniformisation \( F_2 : \mathbb{D} \rightarrow \Omega_2 \) such that the vector field \( \vec{f}_2 : \Omega_2 \rightarrow U \Omega_2 \subset T \Omega_2 \) defined by \( \vec{f}_2 = \frac{\partial_y F_2}{|y_2 F_2|} \circ F_2^{-1} \) satisfies \( \vec{f}_2 \in W^{1,2}(\Omega_2) \).

(7) There exists a uniformisation \( F_2 : \mathbb{D} \rightarrow \Omega_2 \) such that the Cartan form \( \omega = (\vec{e}_2, d\vec{f}_2) \) associated to the Coulomb moving frame \( (\vec{e}_2, \vec{f}_2) \) defined in 2. and 3. satisfies \( \omega_2 \in L^2(\Omega_2) \).

We will also need of the formula of the geodesic curvature for immersions of the Poincaré half-plane.

**Lemma 6.2.** Let \( \mathbb{H} = \mathbb{C} \cap \{ z : \text{Im} (z) > 0 \} \) be the Poincaré half-plane, and \( f : \mathbb{H} \rightarrow \mathbb{C} \) a univalent holomorphic map, \( \Omega = f(\mathbb{H}) \), and assume that \( \gamma = \partial \Omega \) is a simple curve of finite Loewner energy. Then the geodesic curvature \( k_{\gamma_0} \) of \( \gamma \) is given in the distributional sense by

\[ k_{\gamma_0} = \text{Im} \left( \frac{f''(z)}{f'(z)} \right) \quad \text{for all } z \in \partial_\infty \mathbb{H} = \mathbb{R}. \tag{6.2} \]

**Proof.** The geodesic curvature is given by

\[ k_{\gamma_0} = \langle \partial_x \vec{e}, \vec{f} \rangle, \]

if \( (\vec{e}, \vec{f}) \) is the Cartesian frame given by (in the following formulae, \( f \) is seen as a \( \mathbb{R}^2 \)-valued function)

\[
\begin{align*}
\vec{e} &= \frac{\partial_x f}{|\partial_x f|} = (\text{Re} \left( \frac{f'(z)}{|f'(z)|} \right), \text{Im} \left( \frac{f'(z)}{|f'(z)|} \right)) = \frac{f'(z)}{|f'(z)|}, \\
\vec{f} &= \frac{\partial_y f}{|\partial_y f|} = (-\text{Im} \left( \frac{f'(z)}{|f'(z)|} \right), \text{Re} \left( \frac{f'(z)}{|f'(z)|} \right)) = i \frac{f'(z)}{|f'(z)|}.
\end{align*}
\]

Define \( \vec{e}_z = \frac{f'(z)}{|f'(z)|} \). Then we have

\[
\begin{align*}
\partial_x \vec{e}_z &= \partial_x \left( \frac{f'(z)}{|f'(z)|} \right) = \frac{f''(z)}{|f'(z)|} - \frac{1}{2} \frac{f'(z)^2}{|f'(z)|^3}, \\
\partial_{\vec{e}} \vec{e}_z &= \partial_{\vec{e}} \left( \frac{f'(z)}{|f'(z)|} \right) = -\frac{1}{2} \frac{f'(z)^2 f''(z)}{|f'(z)|^3} - \frac{1}{2} \frac{f'(z) f''(z)}{|f'(z)|^3}.
\end{align*}
\]

Therefore, we deduce that

\[
\begin{align*}
\partial_x \text{Re} \left( \frac{f'(z)}{|f'(z)|} \right) &= \frac{1}{2} \left( \partial_x \left( \frac{f'(z)}{|f'(z)|} \right) + \partial_{\vec{e}} \left( \frac{f'(z)}{|f'(z)|} \right) \right) = \frac{1}{2} \frac{f''(z)}{|f'(z)|} \vec{e}_z, \\
\partial_{\vec{e}} \text{Re} \left( \frac{f'(z)}{|f'(z)|} \right) &= -\frac{i}{2} \left( \frac{1}{2} \frac{f''(z)}{|f'(z)|} + \frac{1}{2} \frac{f''(z)}{|f'(z)|} \right) = -\frac{i}{2} \frac{f''(z)}{|f'(z)|} \vec{e}_z.
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
\partial_x \text{Re} \left( \frac{f'(z)}{|f'(z)|} \right) &= 2 \text{Re} \left( \partial_x \text{Re} \left( \frac{f'(z)}{|f'(z)|} \right) \right) = -\text{Im} \left( \frac{f''(z)}{|f'(z)|} \right) \text{Im} \left( \frac{f'(z)}{|f'(z)|} \right),
\end{align*}
\]

\[
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\]
\[ \partial_x \text{Im} \left( \frac{f'(z)}{|f'(z)|} \right) = \text{Im} \left( \frac{f''(z)}{f'(z)} \right) \text{Re} \left( \frac{f'(z)}{f''(z)} \right), \]

so that

\[ \partial_x \epsilon = \text{Im} \left( \frac{f''(z)}{f'(z)} \right) \bar{f}, \]

and

\[ k_{\gamma_0} = \langle \partial_x \epsilon, \bar{f} \rangle = \text{Im} \left( \frac{f''(z)}{f'(z)} \right), \]

which concludes the proof of the lemma.

\[ \square \]

6.2 Properties of the Geodesic Curvature for Weil-Petersson Curves

Lemma 6.3. Let \( \gamma \subset \mathbb{C} \) be a Weil-Petersson curve. Then the geodesic curvature \( k_{\gamma_0} : S^1 \to \mathbb{R} \) is a tempered distribution of order at most 2. More precisely, we have \( k_{\gamma_0} \in H^{-\frac{1}{2}}(S^1) \).

Proof. Either using the Poincaré half-plane \( \mathbb{H} \) and the formula (6.2) or (4.16), we get

\[ k_{\gamma_0} = \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) + 1. \tag{6.3} \]

Now, if \( 0 < \epsilon < 1 \) and \( f_\epsilon : \mathbb{D} \to \Omega \) is defined by

\[ f_\epsilon(z) = \frac{1}{1-\epsilon} f((1-\epsilon)z), \quad z \in \mathbb{D}, \]

we have (see [54], Lemma 8.2)

\[ \lim_{\epsilon \to 0} \int_\mathbb{D} \frac{|f''(z)|}{|f'(z)|} \left| \frac{zf''(z)}{f'(z)} \right|^2 |dz|^2 = 0, \]

which is equivalent by trace theory to

\[ \lim_{\epsilon \to 0} \| \log |f_\epsilon'| - \log |f'| \|_{H^\frac{1}{2}(S^1)} = 0, \]
\[ \lim_{\epsilon \to 0} \| \arg(f_\epsilon') - \arg(f') \|_{H^\frac{1}{2}(S^1)} = 0, \]

and using the equivalent norm for \( H^s \) spaces given by

\[ \|u\|_{H^s(S^1)} = \sum_{n \in \mathbb{Z}} |a_n|^{2s} |a_n|^2 \quad \text{if} \quad u(z) = \sum_{n \in \mathbb{Z}} a_n z^n, \]

we deduce that

\[ \lim_{\epsilon \to 0} \| \partial_\theta \log(f_\epsilon') - \partial_\theta \log(f') \|_{H^{-\frac{1}{2}}(S^1)} = 0. \]

Since

\[ \frac{f''(z)}{f'(z)} = \frac{1}{ie^{i\theta}} \partial_\theta \log(f'(z)), \]

we deduce that \( f''/f' \in H^{-\frac{1}{2}}(S^1) \) and that concludes the proof of the lemma by (6.3).

\[ \square \]

Remark 6.4. For other considerations related to trace spaces, see the Definition 5 of [8] and [6].
**Proposition 6.5.** For all $\varepsilon > 0$, let $D_+(0,\varepsilon) = \mathbb{H} \cap \{z : |z| < \varepsilon\}$, where $\mathbb{H} = \mathbb{C} \cap \{z : \text{Im}(z) > 0\}$ is the Poincaré half-plane. For $\varepsilon > 0$ small enough, the map $f : D_+(0,\varepsilon)$ defined by

$$f(z) = z e^{i \log \log(z)},$$

where $\log(z)$ is the principal value of the logarithm on $\mathbb{H}$, is an immersion and $\log|f'| \in W^{1,2}(\mathbb{D})$. In particular, the curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ such that $\gamma(t) = t e^{i \log \log(t)}$ for all $t \in (-\varepsilon, \varepsilon)$ is a part of a Weil-Petersson curve. Furthermore, its geodesic curvature $k_{\gamma_0}$ is given by

$$k_{\gamma_0} = -\frac{1}{t(1 + \log^2(t))} + \text{p.v.} \int_{-\varepsilon}^{\varepsilon} \frac{dt}{t \log(t)}.$$

**Proof.** We compute

$$f'(z) = e^{i \log \log(z)} + \frac{i}{\log(z)} e^{i \log \log(z)} = \left(1 + \frac{i}{\log(z)}\right) e^{i \log \log(z)}$$

$$f''(z) = -\frac{i}{z \log^2(z)} e^{i \log \log(z)} + \frac{i}{z \log(z)} \left(1 + \frac{i}{\log(z)}\right) e^{i \log \log(z)}$$

Therefore, we have

$$\frac{f''(z)}{f'(z)} = -\frac{i}{z \log(z)(i + \log(z))} + \frac{i}{z \log(z)} i = -1 + i + \log(z).$$

Notice that we have

$$|i + \log(z)|^2 = |\log(z) + i(1 + \arg(z))|^2 \log^2 |z|.$$ 

Therefore, we have

$$\left|\frac{f''(z)}{f'(z)}\right|^2 \leq \frac{2}{|z|^2 \log^4 |z|} + \frac{2}{|z|^2 \log^2 |z|},$$

and

$$\int_{D(0,\varepsilon)} \left|\frac{f''(z)}{f'(z)}\right|^2 |dz|^2 \leq 4\pi \int_0^1 \frac{dr}{r \log^4(r)} + 4\pi \int_0^1 \frac{dr}{r \log^2(r)} = \frac{4\pi}{3 \log^2(2)} < \infty,$$

which shows that $\gamma$ is a Weil-Petersson curve. Then, we have by Lemma 6.2 for $z \in \mathbb{R}$

$$k_{\gamma_0} = \text{Im} \left(\frac{f''(z)}{f'(z)}\right) = \frac{1 - \log |z| + \log^2 |z|}{|z| \log |z|(1 + \log^2 |z|)} = \frac{1}{|z|(1 + \log^2 |z|)} + \frac{1}{|z| \log |z|},$$

which concludes the proof of the proposition. \hfill \Box

**Remark 6.6.** In particular, we see that there exists curves whose geodesic curvature is a distribution of order 1.

This curve is an example of spiral mentioned earlier in the introduction (Section 1).

### 6.3 Properties of Harmonic Moving Frames on Surfaces

**Theorem 6.7.** Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface with boundary, and let $g = \iota^* g_{\mathbb{R}^3}$. Let $\bar{u} : \Sigma \setminus \{p\} \to U \Sigma$ be a unit harmonic vector-field such that $\bar{u} \in W^{1,1}(\Sigma)$, and $\bar{v} = \bar{u} \times \bar{u}$, where $\bar{u} : \Sigma \to S^2$ is the unit normal, so that $(\bar{u}, \bar{v})$ is a moving frame. Let $\omega = \langle \bar{u}, d\bar{v}\rangle$ be the Cartan form. Then there exists $c_0, c_1 \in \mathbb{R}$ such that $\omega$ satisfies in the distributional sense in $\Sigma$

$$
\begin{cases}
  d^* \omega = 2 \pi c_0 \delta_p - 2 I_{1,2} H & \text{in } \mathcal{D}'(\Sigma) \\
  d \omega = 2 \pi c_0 \delta_p - K_g & \text{in } \mathcal{D}'(\Sigma),
\end{cases}
$$

*Beware that the log function here is defined as the trace of our continuous determination of the logarithm on the upper-half plane and is not the standard log function on $(0, \infty)$.  

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and there exists smooth functions $\mu, \nu : \Sigma \to \mathbb{R}$ satisfying respectively
\begin{align}
-\Delta_\Sigma \mu &= K_g & \text{in } \Sigma \\
-\Delta_\Sigma \nu &= 2 \mathbb{I}_{1,2} H & \text{in } \Sigma
\end{align}
(6.4)
such that
\[ \omega = \ast d (c_0 G_\Sigma + \mu) + d (c_1 G_\Sigma + \nu) \]
(6.5)
where $G_\Sigma = G_{\Sigma,p}$.

**Remark 6.8.**
(1) An similar theorem in the flat case is proved in [32] (in Proposition 4.1, see also the Proposition 4.3) and more precise properties are obtained there if some assumptions are made on the boundary data of $\vec{u}$.

(2) Notice that $(\vec{u}, \vec{v})$ is not a Coulomb moving frame in general. Recalling that the Weingarten tensor can be written as
\[ h_0 = ((\mathbb{I}_{1,1} - \mathbb{I}_{2,2}) - 2 i \mathbb{I}_{1,2}) \, dx^2, \]
and since $H$ vanishes at isolated point only (since $\Sigma$ cannot be minimal), we deduce that $(\vec{u}, \vec{v})$ is a Coulomb moving frame if and only if
\[ \text{Im} (h_0) = 0, \]
which means that $\Sigma$ is a isothermic surface (see [41]). This implies that in the construction, one could not obtain an expansion $\omega = \ast d (G_\Sigma + \mu)$ for a surface different from the sphere.

**Proof.** Recall that $\vec{u} : \Sigma \to U\Sigma$ satisfies on $\Sigma \setminus \{p\}$ the equation
\[ -\Delta \vec{u} = |d\vec{u}|^2 \vec{u} + (2\langle du, d\vec{u} \rangle + \langle u, \Delta \vec{u} \rangle) \vec{n}. \]
We now define the Cartan form as $\omega = \langle \vec{u}, \nabla \vec{v} \rangle = -\langle \vec{v}, \nabla \vec{u} \rangle$. Since $\langle \vec{u}, \vec{v} \rangle = 0$, we deduce that
\[ \langle \vec{v}, \Delta \vec{u} \rangle = -\langle \vec{u}, |d\vec{u}|^2 \vec{u} - (2\langle du, d\vec{u} \rangle + \langle u, \Delta \vec{u} \rangle) \vec{n} \rangle = 0. \]
Therefore, we have on $\Sigma \setminus \{p\}$
\[ \text{div} (\langle \vec{v}, \nabla \vec{u} \rangle) = \langle \nabla \vec{u}, \nabla \vec{v} \rangle = \langle \vec{v}, \Delta \vec{u} \rangle = \langle \nabla \vec{u}, \nabla \vec{v} \rangle. \]
Recall that by the properties of the vector product, for all $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in \mathbb{R}^3$, we have
\[ \langle \vec{w}_1 \times \vec{w}_2, \vec{w}_3 \rangle = \langle \vec{w}_1, \vec{w}_2 \times \vec{w}_3 \rangle \]
Since $\vec{v} = \vec{n} \times \vec{u}$, we get
\[ \langle \nabla \vec{v}, \nabla \vec{u} \rangle = \sum_{j=1}^{2} (\partial_{x_j} \vec{v}, \partial_{x_j} \vec{u}) = \sum_{j=1}^{2} (\partial_{x_j} \vec{n} \times \vec{u} + \vec{n} \times \partial_{x_j} \vec{u}, \partial_{x_j} \vec{u}) \]
\[ = \sum_{j=1}^{2} (\partial_{x_j} \vec{n} \times \vec{u}, \partial_{x_j} \vec{u}) + \sum_{j=1}^{2} (\vec{n}, \partial_{x_j} \vec{u} \times \partial_{x_j} \vec{u}) = \sum_{j=1}^{2} (\partial_{x_j} \vec{n}, \vec{u} \times \partial_{x_j} \vec{u}). \]
(6.6)
Since $|\vec{n}|^2 = 1$, and we have
\[ \partial_{x_j} \vec{u} = \langle \partial_{x_j} \vec{u}, \vec{v} \rangle \vec{v} + \langle \partial_{x_j} \vec{u}, \vec{n} \rangle \vec{n} \]
Since $\vec{u} \times \vec{v} = \vec{n}$ and $\vec{v} = \vec{n} \times \vec{u}$, while $|\vec{n}|^2 = 1$ we deduce that
\[ \langle \partial_{x_j} \vec{n}, \vec{u} \times \partial_{x_j} \vec{u} \rangle = \langle \partial_{x_j} \vec{n}, \partial_{x_j} \vec{u}, \vec{v} \rangle \vec{v} + \langle \partial_{x_j} \vec{n}, \vec{n} \rangle \vec{n} = \langle \partial_{x_j} \vec{n}, \vec{v} \rangle \vec{v} - \langle \partial_{x_j} \vec{n}, \vec{n} \rangle \vec{v} \]
\[ = -\langle \partial_{x_j} \vec{n}, \vec{v} \rangle \vec{v} + \langle \partial_{x_j} \vec{n}, \vec{n} \rangle = \langle \partial_{x_j} \vec{n}, \vec{v} \rangle \vec{v} \vec{n} + \langle \partial_{x_j} \vec{n}, \vec{n} \rangle = \mathbb{I}_{1,2,j}, \]
(6.7)
where we used $\langle \partial_{x_j} \tilde{u}, \tilde{n} \rangle + \langle \tilde{u}, \partial_{x_j} \tilde{n} \rangle = 0$ in the last line. Thanks to (6.6) and (6.7), we get
\[
\langle \nabla \tilde{u}, \nabla \tilde{v} \rangle = I_{1,1}^2 I_{2,1} + I_{1,2}^2 I_{2,2} = 2 I_{1,2} H.
\]
Since $\omega \in W^{1,1}(\Sigma)$, there exists $c_0 \in \mathbb{R}$ such that
\[
d (\ast \omega) = 2 \pi c_0 \delta_p - \langle \nabla \tilde{u}, \nabla \tilde{v} \rangle = 2 \pi c \delta_p - 2 I_{1,2} H \quad \text{in } \mathcal{D}'(\Sigma).
\]
Furthermore, if $\nu \in W^{1,2} \cap C^\infty(\Sigma)$ is a smooth solution to
\[
- \Delta_g \nu = 2 I_{1,2} H \quad \text{in } \Sigma,
\]
we deduce that
\[
d (\ast (\omega - d\nu - c_0 dG_\Sigma)) = 0 \quad \text{in } \mathcal{D}'(\Sigma).
\]
Therefore, there exists by the Poincaré lemma a function $\eta : \Sigma \to \mathbb{R}$ such that $\eta \in W^{1,1}(\Sigma)$ and
\[
\omega = d\nu + c_0 dG_\Sigma + \ast d\eta. \quad (6.8)
\]
On the other hand, we have by the Cartan equation on $\Sigma \setminus \{p\}$ the identity
\[
d \omega = -K_g \quad \text{in } \Sigma \setminus \{p\}.
\]
To check for the sign, observe that for all $j, k \in \{1, 2\}$, we have
\[
\langle \partial_{x_j} \tilde{u}, \partial_{x_k} \tilde{v} \rangle = \langle \partial_{x_j} \tilde{u}, \partial_{x_j} \tilde{n} \times \tilde{u} + \tilde{n} \times \partial_{x_k} \tilde{u} \rangle = \langle \partial_{x_j} \tilde{n}, \tilde{u} \times \partial_{x_k} \tilde{u} \rangle + \langle \tilde{n}, \partial_{x_k} \tilde{u} \times \partial_{x_j} \tilde{u} \rangle.
\]
Since
\[
\partial_{x_j} \tilde{u} = (\partial_{x_j} \tilde{u}, \tilde{v}) \tilde{v} + (\partial_{x_j} \tilde{u}, \tilde{n}) \tilde{n},
\]
and $\tilde{v} \times \tilde{n} = \tilde{u}$, we deduce that
\[
\langle \tilde{n}, \partial_{x_j} \tilde{u} \times \partial_{x_j} \tilde{u} \rangle = 0.
\]
Now, we have
\[
\tilde{u} \times \partial_{x_k} \tilde{u} = (\partial_{x_j} \tilde{u}, \tilde{v}) (\tilde{u} \times \tilde{v}) + (\partial_{x_j} \tilde{u}, \tilde{n}) (\tilde{u} \times \tilde{n}) = (\partial_{x_j} \tilde{u}, \tilde{v}) \tilde{n} - (\partial_{x_j} \tilde{u}, \tilde{n}) \tilde{v}.
\]
Therefore, we have since $|\tilde{n}|^2 = 1$
\[
\langle \partial_{x_j} \tilde{n}, \tilde{u} \times \partial_{x_k} \tilde{u} \rangle = - \langle \partial_{x_j} \tilde{n}, \tilde{v} \rangle \langle \partial_{x_k} \tilde{u}, \tilde{n} \rangle = \langle \partial_{x_j} \tilde{n}, \tilde{v} \rangle \langle \partial_{x_k} \tilde{u}, \tilde{n} \rangle.
\]
Therefore, we have
\[
\langle \nabla^\perp \tilde{u}, \nabla \tilde{v} \rangle = \langle \partial_{x_j} \tilde{n}, \tilde{v} \rangle (\partial_{x_k} \tilde{n}, \tilde{u}) - \langle \partial_{x_j} \tilde{n}, \tilde{v} \rangle (\partial_{x_k} \tilde{u}, \tilde{n}) = I_{1,1}^2 I_{2,2} - I_{1,2}^2 = K_g.
\]
Letting $c_1 \in \mathbb{R}$ such that
\[
d \omega = 2 \pi c_1 \delta_p - K_g \quad \text{in } \mathcal{D}'(\Sigma),
\]
Therefore, we have
\[
d (\omega - c_1 dG_\Sigma) = -K_g \quad \text{in } \mathcal{D}'(\Sigma).
\]
Therefore, if $\mu$ is a solution to the equation
\[
- \Delta_g \mu = K_g \quad \text{in } \Sigma
\]
Notice that by Wente lemma, we can find a solution such that $\mu \in W^{1,2} \cap C^\infty(\Sigma)$. Therefore, we get
\[
d (\omega - c_1 dG_\Sigma - d\mu) = 0 \quad \text{in } \mathcal{D}'(\Sigma).
\]
We deduce by the Poincaré lemma that there exist $\kappa : \Sigma \to \mathbb{R}$ such that $\kappa \in W^{1,1}(\Sigma)$ and
\[
\omega = \ast c_1 \, dG_\Sigma + \ast d\mu + d\kappa. \tag{6.9}
\]
Comparing (6.8) to (6.9), we see that
\[
\ast c_1 \, dG_\Sigma + \ast d\mu + d\kappa = \ast c_1 \, dG_\Sigma + \ast d\eta. \tag{6.10}
\]
Applying successively the operators $d$ to this equation, we deduce that
\[
\omega = \ast c_1 \, dG_\Sigma + c_0 \, dG_\Sigma + \ast d\mu + d\kappa = \ast c_1 \, dG_\Sigma + \ast d\eta + d\nu
\]
where $\mu, \nu, \eta, \kappa$ are smooth and $\nu - \pi$ and $\mu - \eta$ are conjugate harmonic function, meaning that $\Delta (\nu - \pi) = \Delta (\mu - \eta) = 0$, and
\[
\ast d (\mu - \eta) = d (\nu - \pi).
\]
This concludes the proof of the theorem. \hfill \[ \square \]

### 6.4 Cylindrical Expression

We assume that $\gamma$ is a Weil-Petersson curve surrounding 0. Recall that the Loewner energy of $\gamma$ is defined as
\[
I^L(\gamma) = \frac{1}{\pi} S_1 (\gamma) = \frac{1}{\pi} \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 |dz|^2 + \frac{1}{\pi} \int_{\mathbb{C} \setminus \overline{D}} \left| \frac{g''(z)}{g'(z)} \right|^2 |dz|^2 + 4 \log |f'(0)| - 4 \log |g'(\infty)|, \tag{6.11}
\]
where $f : \mathbb{D} \to \Omega$ (resp. $g : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \setminus \overline{\Omega}$) is the conformal map from the unit disk $\mathbb{D}$ (resp. the exterior disk $\mathbb{C} \setminus \overline{D}$) to inside $\Omega$ (inside $\mathbb{C} \setminus \overline{D}$) of $\gamma$, such that $f(0) = 0$ (resp. $g(\infty) = \infty$). Let $\iota : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ be the inversion map $z \mapsto 1/z$. The Loewner energy is conformally invariant. In particular, we have
\[
I^L(\gamma) = I^L(\iota(\gamma)). \tag{6.12}
\]
We show another identity of the Loewner energy whose expression is well-behaved under inversion. For this, consider the conformal maps $\tilde{f} = \iota \circ g \circ \iota$ and $\tilde{g} = \iota \circ f \circ \iota$ associated to $\iota(\gamma)$. We have $\tilde{f}'(0) = g'(\infty)^{-1}$ and
\[
\tilde{f}''(z) = 1 \left( \frac{g''(1/z)}{g'(1/z)} - \frac{2g'(1/z)}{g(1/z)} + 2 \right). \tag{6.13}
\]
Similarly for $\tilde{g}$.

**Lemma 6.9.** We have the identity
\[
\text{Re} \left[ \int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \left| \frac{1}{z} \right| |dz|^2 + \int_{\mathbb{C} \setminus \overline{D}} \left| \frac{g''(z)}{g'(z)} \right|^2 \left| \frac{1}{z} \right| |dz|^2 \right]
\]
\[
= \int_D \left| \frac{f'(z)}{f(z)} \right|^2 \left| \frac{1}{z} \right| |dz|^2 + \int_{\mathbb{C} \setminus \overline{D}} \left| \frac{g'(z)}{g(z)} \right|^2 \left| \frac{1}{z} \right| |dz|^2 \tag{6.14}
\]
\[
= 2 \pi \left( \log |g'(\infty)| - \log |f'(0)| \right). \tag{6.15}
\]

**Proof.** From (6.13), we have
\[
I^L(\iota(\gamma)) = \frac{1}{\pi} \int_{\mathbb{C} \setminus \overline{D}} \left| \frac{g''(z)}{g'(z)} - \frac{2g'(1/z)}{g(1/z)} + \frac{2}{z} \right|^2 |dz|^2 + \frac{1}{\pi} \int_D \left| \frac{f''(z)}{f'(z)} - \frac{2f'(1/z)}{f(1/z)} + \frac{2}{z} \right|^2 |dz|^2 + 4 \log \left| \frac{f'(0)}{g'(\infty)} \right|
\]
\[
= I^L(\gamma) - 4 \int_{\mathbb{C} \setminus \overline{D}} \text{Re} \left[ \frac{g''(z)}{g'(z)} \left( \frac{1}{z} \right) \right] |dz|^2 - 4 \int_D \text{Re} \left[ \frac{f''(z)}{f'(z)} \left( \frac{1}{z} \right) \right] |dz|^2
\]
\[
= I^L(\gamma) - 4 \int_{\mathbb{C} \setminus \overline{D}} \text{Re} \left[ \frac{g''(z)}{g'(z)} \left( \frac{1}{z} \right) \right] |dz|^2 - 4 \int_D \text{Re} \left[ \frac{f''(z)}{f'(z)} \left( \frac{1}{z} \right) \right] |dz|^2
\]
\[
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Lemma 6.10 (see also [51, Theorem 8.4]). We have the identity

\[ I^L(\gamma) = \frac{1}{\pi} \int_{D} |\nabla \arg \left( \frac{f'(z)z}{f(z)} \right)|^2 \, dz^2 + \frac{1}{\pi} \int_{C \setminus \overline{D}} |\nabla \arg \left( \frac{g'(z)z}{g(z)} \right)|^2 \, dz^2 + 2 \log \left| \frac{f'(0)}{f'(\infty)} \right|. \]

Proof. We expand the first two terms:

\[ \frac{1}{\pi} \int_{D} \left| \nabla \arg \left( \frac{f'(z)z}{f(z)} \right) \right|^2 \, dz^2 + \frac{1}{\pi} \int_{C \setminus \overline{D}} \left| \nabla \arg \left( \frac{g'(z)z}{g(z)} \right) \right|^2 \, dz^2 = \frac{1}{\pi} \int_{D} \left| f''(z) - \frac{f'(z)}{f(z)} \right| 2 \, dz^2 + \frac{1}{\pi} \int_{C \setminus \overline{D}} \left| g''(z) - \frac{g'(z)}{g(z)} \right| 2 \, dz^2 \]

\[ = \frac{1}{\pi} \int_{D} \left| f''(z) \right|^2 |dz|^2 + \frac{1}{\pi} \int_{C \setminus \overline{D}} \left| g''(z) \right|^2 |dz|^2 - \frac{1}{\pi} \int_{D} \left| f'(z) \right|^2 |dz|^2 + \frac{1}{\pi} \int_{C \setminus \overline{D}} \left| g'(z) \right|^2 |dz|^2 + \frac{1}{\pi} \int_{C \setminus \overline{D}} \left| f'(z) \right|^2 |dz|^2 + \frac{1}{\pi} \int_{C \setminus \overline{D}} \left| g'(z) \right|^2 |dz|^2 \]

which gives the desired equality. The last two equalities follow from Lemma 6.9.

Let \( \tilde{f} = \iota \circ g \circ \iota : \mathbb{D} \to \iota(C \setminus \overline{D}) \). We have from direct computation that

\[ \int_{C \setminus \overline{D}} \left| \nabla \arg \left( \frac{g'(z)z}{g(z)} \right) \right|^2 \, dz^2 = \int_{D} \left| \nabla \arg \left( \frac{\tilde{f}'(z)z}{\tilde{f}(z)} \right) \right|^2 \, dz^2. \]

Alternatively, one can see this relation by the geometric meaning of \( \arg(g'(z)z/g(z)) \): it is the angle at \( g(z) \) between \( g(|z|S^1) \) (the image of the circle of radius \( |z| \) and centre 0 under \( g \)) and the circle \( |g(z)|S^1 \) centred at 0 and passing through \( g(z) \). This angle is preserved by the inversion \( \iota \) which is a conformal map.

Corollary 6.11 (Cylindrical expression). We have

\[ I^L(\gamma) = \frac{1}{\pi} \int_{D} \left| \nabla \arg \left( \frac{f'(z)z}{f(z)} \right) \right|^2 \, dz^2 + \frac{1}{\pi} \int_{D} \left| \nabla \arg \left( \frac{\tilde{f}'(z)z}{\tilde{f}(z)} \right) \right|^2 \, dz^2 + 2 \log |f'(0)| + 2 \log |\tilde{f}'(0)|. \]

The reason of calling this identity the cylindrical expression of the Loewner energy is the following. Consider the projection \( \Pi_{cyl} \) from 0 of \( S^2 \) to the cylinder \( S^1 \times \mathbb{R} \), tangent at the equator \( S^1 \times \{0\} \). The projection is conformal. We denote the stereographic projection \( S^2 \to \mathbb{D} \) from the south pole \( S \). Then \( \Pi = \Pi_{cyl} \circ \Pi_S^2 : C \to S^1 \times \mathbb{R} \) maps the concentric circles centred at 0 to horizontal circles of the form \( S^1 \times \{t\} \). The function \( F = \Pi \circ f \circ \Pi^{-1} \) maps conformally \( S^1 \times \mathbb{R} \) onto \( \Pi(\Omega) \subset S^1 \times \mathbb{R} \). We have

- \( \arg \left( \frac{\tilde{f}'(z)z}{\tilde{f}(z)} \right) \) is “the angle change by \( F \), or the argument of \( F' \) at \( \Pi(z) \);”
- \( \log |f'(0)| = \lim_{t \to -\infty} (t - p_2 \circ F(\theta, t)) \) for any \( \theta \in S^1 \) and \( p_2 : S^1 \times \mathbb{R} \to \mathbb{R} \) is the projection into the second (vertical) coordinate of the cylinder. Namely, \( \log |f'(0)| \) is the vertical translation of the positive end of the cylinder under \( F \). Similarly for \( g \).

Is that more natural consider the frame energy in the cylinder?
One may also interpret the identity in Corollary 6.11 from the conformal factor. The trivial observation is that the Dirichlet energy of a harmonic function coincides with the Dirichlet energy of its harmonic conjugate. Therefore,

$$\int_D |\nabla \arg \left( \frac{f'(z)z}{f(z)} \right)|^2 |dz|^2 = \int_D |\nabla \sigma_f|^2 |dz|^2,$$

where $\sigma_f = \log \left| \frac{f'(z)z}{f(z)} \right|$ can be interpreted as the conformal factor with respect to the flat metric on the cylinder $g_{cyl} = |z|^{-2} |dz|^2$. Namely, the pull-back of $g_{cyl}$ by $f$ is

$$f^*(g_{cyl}) = e^{2\sigma_f} g_{cyl}.$$

Thus the Loewner energy is expressed as the Dirichlet energy of this conformal factor.

### 6.5 Another Generalised Grunsky-type Identity

The new identity of $\pi I^L = S_3$ from Theorem 3.3 and Theorem 3.5 provides a new identity about holomorphic univalent maps of the plane which is reminiscent of the Grunsky identity (6.15). We use the same notations as in Section 6.4.

**Lemma 6.12.** Let $\gamma \subset \mathbb{C}$ be a curve with finite Loewner energy. We have

$$\begin{align*}
4 \Re \int_D \left( \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f(z)} + \frac{2}{z} \right) \left( \frac{f'(z)}{f(z)} - \frac{1}{1 + |f(z)|^2} - \frac{2}{z} \right) |dz|^2 + 4 \int_D \left( \frac{f'(z)}{f(z)} - \frac{1}{1 + |f(z)|^2} - \frac{1}{z} \right)^2 |dz|^2 \\
+ 4 \Re \int_{\mathbb{C} \setminus \mathbb{C}} \left( \frac{g''(z)}{g'(z)} - 2 \frac{g'(z)}{g(z)} + \frac{2}{z} \right) \left( \frac{g'(z)}{g(z)} - \frac{1}{1 + |g(z)|^2} \right) |dz|^2 + 4 \int_{\mathbb{C} \setminus \mathbb{C}} \left( \frac{g'(z)}{g(z)} - \frac{1}{1 + |g(z)|^2} \right)^2 |dz|^2 \\
+ 2 \int_D \log |z| \left( \frac{4|f'(z)|^2 |dz|^2}{(1 + |f(z)|^2)^2} - \frac{2}{1 + |g(z)|^2} \right)^2 + 4\pi = 0
\end{align*}$$

**Proof.** Recall the definition of $S_3(\gamma)$ in Definition 3.1:

$$S_3(\gamma) = \int_D \left| \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f(z)} + \frac{2}{z} \right| \frac{|f'(z)|^2}{1 + |f(z)|^2} |dz|^2 + \int_{\mathbb{C} \setminus \mathbb{C}} \left| \frac{g''(z)}{g'(z)} - 2 \frac{g'(z)}{g(z)} + \frac{2}{z} \right| \frac{|g'(z)|^2}{1 + |g(z)|^2} |dz|^2 + 2 \int_D \log |z| \frac{|f'(z)|^2 |dz|^2}{(1 + |f(z)|^2)^2} + 4\pi \log \frac{|f'(0)|}{g'(\infty)}$$

and that $\pi I^L(\gamma) = S_1(\gamma) = S_3(\gamma)$ from Theorem 3.5. Using again the identities (6.12) and (6.13), we obtain

$$S_3(\gamma) = S_4(\gamma)$$

$$= \int_D \left( \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f(z)} + \frac{2}{z} \right) \frac{|f'(z)|^2}{1 + |f(z)|^2} |dz|^2 + \int_{\mathbb{C} \setminus \mathbb{C}} \left( \frac{g''(z)}{g'(z)} - 2 \frac{g'(z)}{g(z)} + \frac{2}{z} \right) \frac{|g'(z)|^2}{1 + |g(z)|^2} |dz|^2 + 2 \int_D \log |z| \frac{|f'(z)|^2 |dz|^2}{(1 + |f(z)|^2)^2} + 4\pi \log \frac{|f'(0)|}{g'(\infty)}$$

$$= \int_D \left( \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f(z)} + \frac{2}{z} \right) \frac{1}{1 + |f(z)|^2} |dz|^2 + \int_{\mathbb{C} \setminus \mathbb{C}} \left( \frac{g''(z)}{g'(z)} - 2 \frac{g'(z)}{g(z)} + \frac{2}{z} \right) \frac{1}{1 + |g(z)|^2} |dz|^2 + 4\pi \log \frac{|f'(0)|}{g'(\infty)}$$

$$+ 4 \Re \int_D \left( \frac{f''(z)}{f'(z)} - 2 \frac{f'(z)}{f(z)} + \frac{2}{z} \right) \frac{1}{1 + |f(z)|^2} |dz|^2$$



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\[ + 4 \text{Re} \int_{\mathbb{C}\setminus \mathbb{D}} \left( \frac{g''(z)}{g'(z)} - 2 \frac{g'(z)}{g(z)} + \frac{2}{z} \right) \left( \frac{g'(z)}{g(z)} - \frac{1}{1 + |z|^2} \right) |dz|^2 \]
\[ + 4 \int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} \frac{1}{1 + |f(z)|^2} - \frac{1}{z} \right|^2 |dz|^2 + 4 \int_{\mathbb{C}\setminus \mathbb{D}} \left| \frac{g'(z)}{g(z)} - \frac{1}{1 + |z|^2} \right|^2 |dz|^2 \]
\[ + 2 \int_{\mathbb{D}} \log |z| \left| \frac{4g'(z) |dz|^2}{(1 + |f(z)|^2)^2} \right|^2 |dz|^2 - 2 \int_{\mathbb{C}\setminus \mathbb{D}} \log |z| \left( \frac{4g'(z) |dz|^2}{(1 + |z|^2)^2} \right)^2 + 4\pi. \]

Comparing (6.16) and (6.17), we get the claimed identity. \(\square\)

Let us check the formula in the case \(\gamma = S^1\). In the case, we have \(f(z) = z\), and \(g(z) = z\), and the sum in Lemma 6.12 simplifies to

\[ 4 \int_{\mathbb{D}} \left| \frac{1}{z} \right|^2 |dz|^2 + 4 \int_{\mathbb{C}\setminus \mathbb{D}} \left| \frac{1}{z} \right|^2 |dz|^2 + 2 \int_{\mathbb{D}} 4 \log |z| |dz|^2 + 2 \int_{\mathbb{C}\setminus \mathbb{D}} \frac{4 \log |z| |dz|^2}{(1 + |z|^2)^2} + 4\pi. \]

(6.18)

First, we have

\[ \int_{\mathbb{D}} \left| \frac{1}{z} \right|^2 |dz|^2 = \int_{\mathbb{D}} \left| \frac{\pi}{1 + |z|^2} \right|^2 |dz|^2 = \int_{\mathbb{D}} \frac{|z|^2 |dz|^2}{(1 + |z|^2)^2}, \]

and an immediate change of variable \(z \mapsto \frac{1}{z}\) shows that

\[ \begin{cases} \int_{\mathbb{C}\setminus \mathbb{D}} \left| \frac{1}{z} \right| |dz|^2 \left| \frac{1}{z} + |z|^2 \right|^2 |dz|^2 = \int_{\mathbb{D}} \frac{|z|^2 |dz|^2}{(1 + |z|^2)^2} \\ \int_{\mathbb{C}\setminus \mathbb{D}} \frac{\log |z| |dz|^2}{(1 + |z|^2)^2} = - \int_{\mathbb{D}} \frac{\log |z| |dz|^2}{(1 + |z|^2)^2}. \end{cases} \]

(6.20)

By previous computations (Remark 2.5), we have

\[ \begin{cases} \int_{\mathbb{D}} \frac{4|z|^2 |dz|^2}{(1 + |z|^2)^2} = 4\pi \log(2) - 2\pi \\ \int_{\mathbb{D}} \frac{4 \log |z| |dz|^2}{(1 + |z|^2)^2} = -2\pi \log(2), \end{cases} \]

(6.21)

which shows by (6.19), (6.20) and (6.21) that the sum (6.18) equals to

\[ 2 \int_{\mathbb{D}} \frac{4|z|^2 |dz|^2}{(1 + |z|^2)^2} + 4 \int_{\mathbb{D}} \frac{4 \log |z| |dz|^2}{(1 + |z|^2)^2} + 4\pi = 2 (4\pi \log(2) - 2\pi) + 4 (-2\pi \log(2)) + 4\pi = 0, \]

as expected.

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