Electromagnetic wall torques from magnetically confined plasmas

I H Hutchinson

Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Received 16 September 2000, in final form 5 December 2000

Abstract

A direct method to measure the localized magnetic shear stress and hence electromagnetic torque or force exerted on a wall by a fluctuating magnetic field is presented. A model of the wall is needed when the magnetic measurements do not include the normal component. Explicit formulae for thin and thick walls are given. Estimates of the force in illustrative Alcator C-Mod tokamak plasmas provide evidence against the theory that resistive magnetic drag determines mode locking.

1. Introduction

Propagating magnetic fluctuations interact with fixed conducting structures in such a way as to transfer momentum to them. As a result, for example, magnetic perturbations generated by MHD instabilities experience a drag in the wall, which they generally pass on to the plasma in which they are embedded. These processes are extremely important in understanding the possible stabilization of MHD instabilities by the effects of the walls [1], and most previous discussions of this topic focus on the way in which the magnetic perturbations are affected by the presence of the wall and transfer their influence back to the plasma, see for example [2–4]. In these calculations an extremely simplified model of the conducting wall has almost universally been adopted. The present paper focuses instead on the interaction with the wall to show, first, that it is possible, and indeed in many cases theoretically essential, to adopt more realistic models for the wall effect; second, that regardless of the actual wall structure it is in principle possible to measure the momentum flux into the wall; and, third, to illustrate actual cases in which magnetic measurements bear on the importance of magnetic fluctuation momentum transfer to the plasma.

2. Magnetic stress calculation

Consider the situation shown schematically in figure 1. The plasma is considered to be enclosed by a surface $S$ which contains inside it no fixed conducting components capable of affecting the momentum. If necessary, the volume enclosed by this surface will be considered to be periodic in a particular direction so that the contributions from the ends of the volume cancel and can
be ignored. We are thus considering a toroidal plasma to be represented approximately by a periodic linear approximation. In fact, the analysis can be carried out for a torus, in terms of torque and angular momentum rather than force and linear momentum, *mutatis mutandis*. No significant new physics is introduced thereby, so we will analyse the linear case for simplicity. We consider the force/momentum in a particular direction specified by the unit vector \( e_h \). This direction need not be the direction of periodicity.

Changes of the total momentum of the plasma within this surface can occur only if momentum is transferred across the surface \( S \). That transfer can be mediated either by particles, as represented by the pressure tensor (including viscous stress) in a fluid description, or by electric or magnetic fields. The rate at which electromagnetic fields transfer momentum across the surface is given, perfectly generally, by the integral over the surface,

\[
\frac{dM_h}{dt} = \int e_h \cdot T \cdot dS
\]  

(1)

of the Maxwell stress tensor,

\[
T \equiv \epsilon_0 \left[ EE - \frac{1}{2} E^2 I \right] + \frac{1}{\mu_0} \left[ BB - \frac{1}{2} B^2 I \right].
\]  

(2)

We focus on the magnetic components and write, ignoring \( E \),

\[
\frac{dM_h}{dt} = \int e_h \cdot \frac{1}{\mu_0} \left[ BB - \frac{1}{2} B^2 I \right] \cdot dS.
\]  

(3)

This equation is true for the steady-state components as well as any fluctuations. Thus, if there were a steady momentum transfer from the plasma to the magnetic field inside this volume, which might arise if the volume enclosed one end of a magnetic mirror for example, this momentum transfer is, in principle, measurable by performing this integral. In practice, however, such an approach would face severe difficulties for a low-\( \beta \) plasma because there would be large values of the stress tensor components whose sign changed from place to place in the integral cancelling each other out. That cancellation is exact, of course, when there is no net magnetic force on the plasma. In practice, taking the sensitive difference of large quantities in order to construct the integral would require accuracy far beyond what is normally possible in routine magnetic measurements.

If, however, we are concerned only with the magnetic fluctuations we shall see that it is plausible that we might be able to measure the fields accurately enough to construct this integral because the momentum transfer associated with those fluctuations does not necessarily involve large cancellations over the extent of the integral.

So, let us suppose that the magnetic field is composed of a steady part and a fluctuating part

\[
B = \langle B \rangle + \tilde{B}
\]  

(4)
where the angular brackets indicate an average over time much longer than the fluctuation period. When this decomposition is substituted into equation (3), and then averaged, the cross terms between \( \langle B \rangle \) and \( \tilde{B} \) will average to zero, and we are left with a term arising purely from the fluctuations:

\[
\frac{d\langle M_h \rangle}{dt} = \int e_h \cdot \frac{1}{\mu_0} \left( \tilde{B} \tilde{B} - \frac{1}{2} \tilde{B}^2 I \right) \cdot dS.
\]

(5)

The extent to which this quantity can meaningfully be measured experimentally will depend largely on whether or not there are major cancellations. The \( \tilde{B}^2 \) term has a sign that is that of \( e_h \cdot dS \) and will generally therefore experience significant cancellations if it is non-zero, because its sign changes from place to place on the surface. In the circumstance that the surface normal is perpendicular to the direction of momentum, \( e_h \cdot dS = 0 \), this term vanishes. This will be the case if we take an axisymmetric surface in a tokamak, for example (equivalent to a translationally symmetric surface, independent of \( h \), here.) In a stellarator, or other non-symmetric systems, it appears likely that this fluctuating magnetic pressure term will render impractical the direct measurement of momentum transfer through this integral, unless it can be constructed over a surface that is perpendicular to \( e_h \), utilizing the periodicity of the system.

Let us therefore consider only the case \( e_h \cdot dS = 0 \) (and omit the end surfaces by our periodic assumption). The only remaining magnetic stress tensor component is \( e_h \cdot (\tilde{B} \tilde{B}) \cdot \hat{n} = (\tilde{B}_h \tilde{B}_n) \), where \( \hat{n} \) is the surface normal direction. This component is measurable in practice by simultaneous measurements of the tangential (\( e_h \)) and normal components of the fluctuating field and forming the average.

Generally, of course, we do not have such measurements all over the surface in question. Therefore it will normally be necessary either to perform this measurement locally in a part of the surface that is representative of the whole, or else to construct a model that enables us to derive an estimate of the full integral from partial measurements.

The important point is that this measurement is completely independent of the complicated details of the mode behaviour inside the plasma. The momentum transfer to the plasma region can be obtained at the measurement surface directly from measured fields. To the best of the author’s knowledge, a direct reconstruction of the fluctuating magnetic stress tensor average has not been measured on tokamaks, although that would be an interesting experiment. However there are models that can be used to derive an estimate of the term from the routine field measurements for simplified approximations of experimental situations.

3. Wall penetration models

As a simple model, widely used since the earliest fusion research [5], we approximate the configuration as having circular cylindrical symmetry but suffering from a magnetic perturbation that is a function only of the helical coordinate \( h \equiv m\theta + kz \) and the radius \( r \). Magnetic measurements are generally made in close proximity to the wall, and it is usually only the poloidal component \( \tilde{B}_\theta \) that is measured. However, in vacuum, that is in the measurement region, the field can be represented as the gradient of a scalar potential, \( \phi \). Then the component of \( \tilde{B} \) that is perpendicular to \( r \) points in the gradient direction:

\[
e_h \equiv \frac{\nabla h}{|\nabla h|} = \frac{1}{g}(e_\theta + (kr/m)e_z)
\]

where \( g \equiv \sqrt{1 + (kr/m)^2} \). This fact allows us to immediately derive the required \( h \) component of \( \tilde{B} \) from \( \tilde{B}_\theta \) measurements alone, knowing that \( \tilde{B}_z = (kr/m)\tilde{B}_\theta \).
If the perturbation is harmonic in the helical coordinate, proportional to $\exp i h$, and we adopt a complex representation, dropping the tilde on the complex terms for brevity, then the vacuum field equation for a scalar magnetic potential, satisfying Laplace’s equation $\nabla^2 \phi = 0$, consists of constants $p$ and $q$ multiplied by modified Bessel functions:

$$\phi(r) = p I_m(kr) + q K_m(kr).$$  (7)

The application of boundary conditions is more convenient, however, if we use a helical flux function, $\psi$, to represent the field as

$$B = \frac{1}{g} (e_r \wedge e_h) \wedge \nabla \psi = \frac{1}{g} \frac{\partial \psi}{\partial r} e_h - \frac{m}{r} \frac{\partial \psi}{\partial h} e_r.$$  (8)

In terms of $\psi$, the vacuum solutions are then

$$\psi(r) = \frac{kr}{m} \left[ p I_m'(kr) + q K_m'(kr) \right].$$  (9)

The model that has most widely been used for the magnetic transfer of momentum to the wall is to approximate the wall as a concentric thin resistive sheet, and either ignore any helical current outside the sheet [2, 3] or sometimes impose a perfectly conducting boundary further outside it [4]. Extremely rare examples of thick-wall calculations include [6, 7]. The burden of our present argument is that no matter what model is adopted, the entire effect of the wall is contained in the boundary condition at the inner surface of the wall relating $\psi$ and $(\partial \psi / \partial r)$, or in other words, the relationship (phase and amplitude) between $B_r$ and $B_h$. This statement holds true even for ‘active’ walls (see, e.g., [8]) involving powered feedback, provided the feedback algorithm is linear.

Two (passive wall) cases represent the idealized extremes of wall behaviour, the thin wall referred to above and the thick wall, in which the perturbed field does not escape through the outer face of the wall. In either case, the solution of the problem follows from elementary matching of solutions at the inner radius, $b$, of the wall.

### 3.1. Thin wall

For the thin wall, the matching condition is

$$\left[ \frac{\partial \psi}{\partial r} \right]_{b-}^{b+} = \tau_b \frac{\partial \psi}{\partial t}$$  (10)

where $\tau_b \equiv \mu_\sigma b \delta$ is the time-constant of the wall whose conductivity is $\sigma$ and thickness $\delta \ll b$ and the notation $b-$ and $b+$ denote the radii just inside or just outside the wall. Taking $p = 0$ in the outer region ($r = b+$), to make the solution analytic at $r \to \infty$ the outer solution has a logarithmic derivative:

$$\frac{1}{L} \equiv \left. \frac{\partial \psi}{\partial r} / \psi \right|_{b+} = \frac{1}{b} + \frac{k K_m''(kb)}{K_m'(kb)}. $$  (11)

If small argument approximations of the Bessel function are warranted, i.e. when $(kr)^2 \ll 4m$, this becomes $1/L = -m/b$.

Considering harmonic time variation, $\partial / \partial t \to -i \omega$, we can then substitute into the previous equation to find

$$\left. \frac{\partial \psi}{\partial r} \right|_{b-} = \left( \frac{1}{L} + \frac{i \omega \tau_b}{b} \right) \psi_b.$$  (12)
This is an equation relating the radial field to the tangential field at \( r = b^- \):

\[
B_r = -\frac{m}{r} \frac{\partial \psi}{\partial h} = -\frac{m}{r} \frac{\partial \psi}{\partial h} \bigg|_{b^-} = -i \frac{m}{b} \frac{g}{(1/L) + (i \omega \tau \delta / b)} B_h
\]

which naturally has the property that \( B_r \to 0 \) as \( \omega \tau \delta \to \infty \) in which limit the wall becomes perfectly conducting and no field can penetrate normal to it.

The time-averaged component of the magnetic shear stress is

\[
\left( \frac{1}{\mu_0} \langle \tilde{B}_h \tilde{B}_r \rangle \right) = \frac{1}{2 \mu_0} \Re \left( B_h B_r^* \right) = \frac{1}{2 \mu_0} \Re \left( \frac{m}{b} \frac{g}{(1/L) - (i \omega \tau \delta / b)} \right) \left| B_h \right|^2
\]

where \( \Re(x) \) denotes the real part of \( x \) and \( B_h \) is the amplitude of the oscillating component; so \( \left| B_h \right|^2 \) is twice the mean-square value of \( B_h \). This component gives the magnetic momentum flux rate across the surface \( r = b \) at any point, i.e. the force per unit area exerted by the wall on the plasma. Within the helically symmetric model it is uniform over the wall surface. However it is noticeable that \( \tau \delta / b = \mu_0 \sigma \delta / k_h \) is dependent only on the wall properties and not the mode; so in the limit \( kb \ll 2 \sqrt{m} \) when \( 1/L = -m/b \), the only dependence of the complex factor on the mode is through the ratio \( m/b \) which is the local value of the total wavenumber of the perturbation, which we will denote as \( k_h \). Thus, in the thin-wall limit we can regard the magnetic stress at the wall as given by

\[
\langle T_{hr} \rangle = f(k_h, \omega) \frac{1}{2 \mu_0} \left| B_h \right|^2
\]

where the function \( f \) is given by

\[
f = \Re \left( \frac{1}{i - \omega \tau h} \right) = \frac{-\omega \tau h}{1 + (\omega \tau h)^2}
\]

and the time constant is \( \tau_h = \mu_0 \sigma \delta / k_h \). It is clear that this form is not dependent on geometry. We could equally well have performed the calculation in a slab. Moreover, in so far as \( k_h \) can be regarded as a well defined quantity, that is provided the variation of the magnetic perturbation along the wall can be represented as proportional to \( \exp(ik_h L) \) where \( L \) is distance parallel to \( e_h \), the expression for \( f \) can be used for any geometry, even one without a convenient symmetry.

3.2. Thick wall

The opposite limit, rarely considered in the tokamak literature but probably often more important, is when the wall is so thick that we can suppose that no field fluctuation penetrates to its outer surface. In other words, the solution domain may be divided into two, the vacuum region (for \( r < b \)) and a conducting wall region for all \( r > b \). For this to be a reasonable approximation in practice the solution must fall off very rapidly within the wall. The equation governing the flux in the wall is effectively

\[
\nabla^2 \psi = -i \omega \mu_0 \sigma \psi.
\]

We can therefore ignore the value of \( k_h^2 \approx m^2 / r^2 \) provided it is much less than \( \omega \mu_0 \sigma \). The solution in the wall is then

\[
\psi \propto \exp \left( \pm r \sqrt{-i \omega \mu_0 \sigma} \right) = \exp \left[ \pm r (i - 1) \sqrt{\omega \mu_0 \sigma / 2} \right]
\]

which shows that we are consistent in ignoring the outer wall interface provided the wall is much thicker than the standard skin depth, \( \sqrt{2 / \omega \mu_0 \sigma} \) and, by the way, that the thin-wall approximation is invalid if the wall is this thick, that is, if \( \omega \mu_0 \sigma \delta^2 > 1 \). Only the decaying
solution (plus sign) is allowed for the thick-wall case, and then the boundary condition required for the vacuum solution is

$$\frac{\partial \psi}{\partial r} \bigg|_b = (i - 1)\sqrt{\mu_0 \sigma/2} \psi_b.$$  \hspace{1cm} (19)

No discontinuity occurs at \( r = b \) in this case.

The calculation proceeds just as before, arriving at

$$\langle T_{tr} \rangle = f(k_h, \omega) \frac{1}{2\mu_0} |B_h| \quad \hspace{1cm} (20)$$

but now the function \( f \) is given by

$$f = -\frac{i}{(i + 1)\sqrt{\omega \tau_t}} = -\frac{1}{2\sqrt{\omega \tau_t}}$$  \hspace{1cm} (21)

in which the thick-wall time constant is \( \tau_t = \mu_0 \sigma/2k_h^2 \).

Again we have what is really a local expression for the shear stress, to a large degree independent of the geometry.

4. Example from Alcator C-Mod

We illustrate the models discussed in the previous section by application to the Alcator C-Mod tokamak. A typical equilibrium is shown in figure 2. Notable features of this tokamak are the thick conducting stainless-steel wall, whose shape is not at all conformal to the plasma surface, conducting (molybdenum) tiles lining the wall, and a large gap on the outboard side in which there are toroidally localized antennae and limiters. Clearly such a wall is far from the idealized cylindrical case, nevertheless the localized formulae, equations (16) and (21) can be used with reasonable confidence to estimate the contribution of particular parts of the wall to toroidal torques.

First we point out that the fluctuations observed are practically never in the thin-wall limit. Even if we take account only of the stainless-steel wall, 1.3 cm thick, ignoring the molybdenum tiles, which are small and have gaps between them, one obtains \( \mu_0 \sigma \delta^2 = 4.2 \times 10^{-4} \) s. So any
fluctuations faster than this timescale, i.e. with a frequency greater than $1/(2\pi \times 4.2 \times 10^{-4}) = 375$ Hz experience ‘thick’ walls. Practically all observed magnetic fluctuations associated with instabilities are faster than this unless they are already ‘locked’ (i.e. stationary).

Moreover, the presence of the tiles, which have electrical conductivity ($\sigma \approx 2 \times 10^7 \Omega^{-1} m^{-1}$) ten times higher than the stainless-steel wall, would make the time constants even longer, if the tile currents are effective in shielding the fluctuating fields. (Although they usually appear not to be, presumably because of the gaps between them.)

In figure 3 we show examples of poloidal (vertical) field time-derivative ($\dot{B}_v$) measurements, made at the inboard midplane between the wall and the tiles, of large MHD instabilities: one that locks and one that does not. The former is a tearing mode at or near the $q = 2$ surface, caused by excessive gas influx and leading to a disruption, and the latter arises from a large ‘snake’ at the $q = 1$ surface, caused by pellet injection. Table 1 shows a summary of their characteristics. The effective vertical wavenumber of these modes at the inner wall ($k_v$) is determined from the relative phase of the mode at coils ranging up the wall. They are both $n = 1$ modes with a toroidal wavenumber therefore $k_\phi = 2\pi / R = 15$ m$^{-1}$. The frequencies and amplitudes of the modes are quoted at their slowest rotation (prior to locking). Field perturbations at the outboard limiter (not wall) position are similar in magnitude, which confirms that the inboard measurements are not substantially shielded by the tiles.

| Shot reference | 1000824025 | 980220030 |
|----------------|------------|------------|
| Mode character | Tearing mode | Snake |
| Frequency $\omega/2\pi$ (kHz) | 4.5 | 0.8 |
| $\dot{B}_v$ amplitude (T s$^{-1}$) | 19 | 10 |
| Field amplitude (mT) | 0.67 | 1.99 |
| $k_v$ (m$^{-1}$) | 16 | 19 |
| Wall time constant $\tau_t$ (ms) | 2.6 | 2.2 |
| Stress ($T$) (N m$^{-2}$) | 0.017 | 0.41 |
| Mode slowing timescale (ms) | 40 | 0.4 |

Figure 3. Examples of MHD $B$ oscillations at the inner wall.
The shear drag stress at the inner wall is calculated using the thick-wall formula as

\[ \langle T \rangle = \frac{1}{2} \sqrt{\frac{\omega}{\mu_0}} \frac{|\dot{B}_t|^2}{\omega} \left( 1 + \frac{k^2}{k^2_s} \right). \]  

(22)

The total force on the plasma is then obtained by integrating over the inner wall. The measured profile up the inner wall of the field fluctuation provides an effective area of about 1.5 m² over which the quoted stress acts, giving the force as 1.5\(\langle T \rangle\). We are here assuming that the fluctuation magnitude is not significantly non-axisymmetric, which is confirmed by measurements at different toroidal angles. We ignore the contribution of conducting structures such as antennae and limiters on the outboard side. The fluctuating fields are similar in magnitude at them, but their toroidal extent is substantially smaller. The fluctuating magnetic stress at the vessel wall at the top, bottom and outside is much smaller.

These plasmas have quite similar shapes and densities (\(\bar{n}_e = 3.6 \times 10^{20} \text{ m}^{-3}\)) leading to a total deuterium plasma mass of approximately 1.1 \times 10^{-6} kg. The velocity of the mode is \(\omega/k_s\) and then the ratio of the mass multiplied by velocity to the total magnetic drag force gives a timescale for slowing down, indicated in the last row of the table.

The remarkable conclusion of these estimates is that the tearing mode, which locks in about 2 ms has too little a magnetic drag by a factor of 20 to slow the whole plasma down in this short a time, whereas the snake, which does not lock, and in fact speeds up from time to time, apparently has more than enough drag to slow the whole plasma down.

The resolution of this apparent contradiction must surely be that the rotation of the plasma in the case of the snake is in fact not determined by the magnetic drag of the mode on the wall, but rather by some balance of forces intrinsically larger than this drag force. However, if this is true for the snake it must presumably also be true a fortiori for the tearing mode in which the force as estimated from magnetic measurements is far smaller, mostly because the field perturbation is smaller.

We conclude that the experimental estimate of the magnetic drag forces from direct measurements does not support the widespread view that in mode locking, or at least in the slowing of mode rotation prior to locking, a predominant effect is wall drag.

In seeking an explanation for the experimental observations, one should note that in so far as the wall model is correct, the snake case, which does not slow down, nevertheless does experience the estimated stress at the inner wall. Moreover, the wall conductivity enters only as a square-root factor in the stress estimate. Thus, even if the complicated effects of the tiles somehow give rise to a different effective conductivity, it is hardly likely to reduce the stress by the factor of at least 20 required to explain the lack of slowing down on the basis of inertia. Nor can the slowness of the mode cause a significant theoretical reduction in the stress. Finally, our estimate of the total force is necessarily a lower bound since only the stress at the inner wall is accounted for. What is therefore required to explain the absence of slowing is a mechanism to counterbalance the measured momentum outflow via magnetic fluctuations. Viscous stress, that is momentum carried by particles, is one possibility. There undoubtedly is such a viscous stress near the plasma edge because of the interaction of the substantial edge velocity with neutrals and the wall. However, no direct measurement of its magnitude or sign is available so one can only speculate on its role.

Other mechanisms for momentum transport may include the direct effects of non-axisymmetric perturbations. These are presumed to be the final cause of locking; but they are also capable of directly influencing even rotating modes through nonlinear coupling. The momentum flux associated with such an interaction is not measured by the Maxwell stress technique discussed here if the nonlinear interaction is such as to transform the relevant magnetic component to effectively zero frequency, because our \(B\) measurement excludes...
such a frequency. It may also be the case that the $m = 2$ perturbations characteristic of the edge do not entrain the whole plasma and thus do not require the whole plasma to slow down. Arguments like this have been adduced to explain such quantitative discrepancies as are present in the tearing mode case; but they do not explain the snake observations.

5. Summary

The Maxwell stress tensor provides a natural direct approach to measuring the local momentum flux due to magnetic fluctuations. This approach can give direct measurements of wall torques and forces in tokamaks. Linear wall models are completely specified when the complex ratio of the helical flux and its derivative is specified at the wall. These boundary conditions, when known, are sufficient, even with existing measurements, to provide quantitative drag stress estimates that are independent of the behaviour of the perturbation in the plasma. However, the commonly adopted thin-wall approximation is not appropriate for Alcator C-Mod. A thick-wall approximation is more appropriate. Applying the estimates to illustrative cases shows that although the drag forces may be large enough to be significant for large-amplitude MHD modes, modes that lock actually have much smaller drag force than other modes that do not lock. Therefore it would appear that magnetic fluctuation wall drag cannot be the predominant factor in determining the mode frequency.

Acknowledgments

I am grateful to R S Granetz and J A Snipes for their efforts on the magnetic measurements on Alcator C-Mod. This work was supported by DOE grant DE-FC02-99ER54512.

References

[1] Strait E J et al 1999 Nucl. Fusion 39 1977
[2] Nave M F F and Wesson J A 1990 Nucl. Fusion 30 2575
[3] Fitzpatrick R 1993 Nucl. Fusion 33 1049
[4] Finn J M and Sovinec C R 1998 Phys. Plasmas 5 461
[5] Kruskal M D, Johnson J L, Gottlieb M B and Goldman L M 1958 Phys. Fluids 1 421
[6] Nalesso G F and Costa S 1980 Nucl. Fusion 20 443
[7] Gimblett C G 1986 Nucl. Fusion 26 617
[8] Fitzpatrick R and Jensen T H 1996 Phys. Plasmas 3 2641