THE LYAPUNOV SPECTRUM FOR
CONDITIONED RANDOM DYNAMICAL SYSTEMS

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ABSTRACT. We establish the existence of a full spectrum of Lyapunov exponents for memoryless random dynamical systems with absorption. To this end, we crucially embed the process conditioned to never being absorbed, the $Q$-process, into the framework of random dynamical systems, allowing us to study multiplicative ergodic properties. We show that the finite-time Lyapunov exponents converge in conditioned probability and apply our results to iterated function systems and stochastic differential equations.

1. Introduction

A central part of modern mathematical theory and modelling is the description of evolving systems subject to uncertainty. A classical object of study are Markov processes on some state space $E$ which are given by a tuple $(\Omega, (\mathcal{G}_t), (X_t), (\mathcal{P}_t), (\mathbb{P}_x))$. Here, the law of the $\mathcal{G}_t$-adapted stochastic process $X_t$ under the probability measure $\mathbb{P}_x$ describes the evolution of the modelled system, initialised at $x \in E$, giving rise to a semigroup structure $\mathcal{P}_t$. In that sense, this formalism only describes the statistics of the one-point motion of trajectories and joint probability distributions for different initial conditions are not defined. Hence, classical questions from dynamical systems, in particular concerning the sensitivity on initial conditions associated to chaos, cannot be addressed.

The correct framework for studying such questions is given by the theory of random dynamical systems (RDS) [1] which model the stochastic system as a (deterministic) skew-product $(\theta, \varphi)$ where $\varphi$ evolves as a cocycle over the underlying noise dynamics given by $\theta$. In fact, since every RDS induces a Markov process in a canonical way, it contains, in principle, more information. Specifically, the framework of RDS allows for the definition and analysis of Lyapunov exponents which describe the asymptotics of the sensitivity to initial conditions. However, for systems with a unique ergodic component, such as for those driven by unbounded noise, the classical theory of Lyapunov exponents only captures global dynamical properties, for instance, the contraction of bounded sets to a single random fixed point [19, 24, 25]. This is one of the reasons why a stochastic extension of the local bifurcation theory for deterministic dynamical systems, describing changes of stability in dynamical behaviour, has been only developed along single examples and phenomena [2, 3, 6, 20, 22].

One first step towards a more convenient setting for the description of local stability properties in globally noisy systems has been undertaken by Engel, Lamb and Rasmussen [23] in the context of stochastic differential equations (SDEs) with additive noise. They have transferred the notion of a dominant Lyapunov exponent to the setting of conditioning a stochastic system to remain within...
Birkhoff averages satisfy this process. Sometimes this measure is also called quasi-ergodic [10] since it turns out that the Markov process and that the measure given by

\[ \nu \]

is the translation of a notion of stationarity for asymptotical survival processes to an appropriate invariant measure for the conditioned RDS.

In more detail, consider a RDS \((\theta, \phi)\) with filtered probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})\) on a state space \(E\) which is decomposable as \(E = M \cup \{\partial\}\); here \(M\) is a manifold and \(\partial\) is a cemetery (or absorbing) state for \(\phi\), i.e. \(\phi_s \in \{\partial\}\) implies that \(\phi_t \in \{\partial\}\) for all \(t \geq s\). Accordingly, we introduce the stopping time

\[ \tau(\omega, x) := \inf_{t \geq 0} \{\phi_t(\omega, x) = \partial\}, \quad (\omega, x) \in \Omega \times M. \]

The two classes of RDSs we consider are those given by solutions of SDEs (in continuous time) and iterations of random maps (in discrete time). As indicated above, \((\theta, \phi)\) induces a Markov process

\[ \phi := (\Omega \times E, (\mathcal{F}_t \otimes \mathcal{B}(E))_{t \geq 0}, (\phi_t)_{t \geq 0}, (\mathbb{P}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E}), \]

where \(\mathbb{P}_x := \mathbb{P} \otimes \delta_x\) and the usage of \(\phi\) as cocycle or Markov process becomes clear from the context. Conditioning a stochastic system to never reach the cemetery state is a well-studied problem for Markov processes [16, 33, 34], going back to the pioneering work of Yaglom [39], with recent advances [12, 14, 15, 17] on the statistical properties of the conditioned process, in particular on its ergodic properties. In these works one can find general, readily verifiable assumptions that guarantee the following hypothesis which can be found in more detailed form as Hypothesis (H) in Section 2 and describes exponential convergence to a unique quasi-stationary distribution:

**Hypothesis (H').** The Markov process as introduced above admits a unique quasi-stationary distribution (QSD) \(\mu\), i.e. \(\mu(M) = 1\) and

\[ \mu(A) = \int_M \mathbb{P}_x(\phi_t \in A \mid \tau > t) =: \mathbb{P}_\mu(\phi_t \in A \mid \tau > t), \quad \text{for all } A \in \mathcal{B}(M), \]

for which there exist \(C(x)\) and \(\alpha > 0\) such that for every \(x \in M\)

\[ \|\mathbb{P}_x(\phi_t \in \cdot \mid \tau > t) - \mu\|_{TV} \leq C(x)e^{-\alpha t}. \]

Furthermore, there is a positive bounded function \(\eta\) on \(M\) and \(\beta > 0\), such that

\[ \lim_{t \to \infty} \sup_{x \in M} \left| e^{\beta t} \mathbb{P}_x(\tau > t) - \eta(x) \right| = 0. \]

A key ingredient for the following is the notion of the \(Q\)-process, which describes the process \(\phi\) conditioned on asymptotic survival. It is given by the \(Q\)-measures

\[ Q_x(A) := \lim_{t \to \infty} \mathbb{P}_x(A \mid \tau > t) \quad \text{for all } A \in \mathcal{F}_s \otimes \mathcal{B}(E), \text{ for any fixed } s > 0. \]

In the setting of Hypothesis (H'), it can be shown that this limit exists, these measures define a Markov process and that the measure given by \(\nu(dx) = \eta(x)\mu(dx)\) is a stationary distribution of this process. Sometimes this measure is also called quasi-ergodic [10] since it turns out that the Birkhoff averages satisfy

\[ \lim_{t \to \infty} E_x\left[ \frac{1}{t} \int_0^t f(\phi_t) \, dt \mid \tau > t \right] = \int_M f \, d\nu \quad \text{for all } f \in L^1(\nu). \]
In [23], this property was exploited to obtain the notion of a dominant conditioned Lyapunov exponent via a modified Furstenberg–Khasminskii formula. Specifically, it was shown that for additive noise SDEs with linearisation $D\varphi_t$, the following limit exists

$$\Lambda_1 = \lim_{t \to \infty} E_x \left[ \frac{1}{t} \log \|D\varphi_t\| \bigm| \tau > t \right].$$

Consequently, it was conjectured in [23, Conjecture 3.5] that additional exponents $\{\Lambda_i\}_{i=1}^d$ can be found as limits

$$\Lambda_i = \lim_{t \to \infty} E_x \left[ \frac{1}{t} \log \delta_i(D\varphi_t) \bigm| \tau > t \right] \quad i \in \{1, \ldots, d\},$$

where $\delta_i(D\varphi_t)$ denotes the $i^{th}$ singular value of $D\varphi_t$. To show this conjectured existence of a spectrum of conditioned Lyapunov exponents, we now find an appropriate invariant, ergodic measure for the random dynamical system corresponding to the quasi-ergodic distribution.

**Theorem A** (Ergodic measure for conditioned RDS). Let $\Theta := (\theta, \varphi)$ be a random dynamical system on $M$ with absorption at $\{\partial\}$ satisfying Hypothesis (H') with quasi-ergodic distribution $\nu$. Then $\Theta$ has an invariant, ergodic (even strongly mixing) probability measure given by

$$Q_\nu(\cdot) := \int_M Q_x(\cdot) \nu(dx).$$

This new crucial insight allows for the application of the multiplicative ergodic theorem to obtain the following theorem as a corollary:

**Theorem B** (Lyapunov spectrum for the $Q$-process). Assume that the linear cocycle $\Phi := (D\varphi_t)_{t \geq 0}$, as the linearisation over the $C^1$ random dynamical system $\Theta := (\theta, \varphi)$ as in Theorem A, is invertible and fulfills the integrability condition

$$E_x^Q \left[ \sup_{0 \leq t \leq 1} \log^+ \|\Phi_t^{-1}\| \right] < \infty.$$

Then there exists a full spectrum of constant Lyapunov exponents $\Lambda_1 \geq \cdots \geq \Lambda_d > -\infty$ such that for all $i \leq d$

$$\lim_{t \to \infty} E_x^Q \left[ \Lambda_i - \frac{1}{t} \log \delta_i(\Phi_t) \right] = 0.$$

Here, the expression $E_x^Q$ denotes expectation with respect to the measure $Q_\nu$. In more detail, we also obtain Oseledeits flags, i.e. dynamically invariant subspaces that constitute a filtration of the tangent space, which are associated with the distinct Lyapunov exponents (cf. Theorem 2.8). Finally, we use results for the $Q$-process to show convergence to Lyapunov exponents in conditional probability, and under stronger assumptions, that are satisfied for SDEs with additive noise, even convergence in conditional expectation. In particular, this confirms [23, Conjecture 3.5].

**Theorem C** (Convergence of finite-time Lyapunov exponents). Let us assume the same hypotheses as in Theorem B such that there exist conditioned Lyapunov exponents $\{\Lambda_i\}_{i=1}^d$.

1. Then for all $\varepsilon > 0$, for $\nu$-almost every $x \in M$,

$$\lim_{t \to \infty} P_x \left[ \left| \Lambda_i - \frac{1}{t} \log \delta_i(\Phi_t) \right| > \varepsilon \bigm| \tau > t \right] = 0.$$
(2) If, additionally, for some $p \in (1, \infty]$, we have
\[
\sup_{t \geq 0} \left\| \frac{1}{t} \log^+ \| \Phi_t^\pm \| \right\|_{L^p(\Omega \times M, \mathbb{P}_x(\cdot | \tau>t))} < \infty,
\]
then for $\nu$-almost every $x \in M$
\[
\lim_{t \to \infty} \mathbb{E}_x \left[ \Lambda_i - \frac{1}{t} \log \delta_i(\Phi_t) \middle| \tau > t \right] = 0.
\]

The remainder of the paper is structured as follows. In this paper Theorem A is contained in the statement of Propositions 2.5 and 2.6; Theorem B is contained in the statement of Proposition 2.6 and Theorems 2.7 and 2.8; and Theorem C is contained in the statement of Theorems 2.11 and 2.13. In Section 2, we introduce the setting of this paper and state our main results in more detail. In Section 3, we provide results that make the theory of $Q$-process applicable to random dynamical systems with absorption, proving Theorem A and Theorem B. In Section 4, we link this framework back to finite-time conditioned dynamics proving Theorem C. In Section 5, we show how this can be applied to the study of the conditioned dynamics of a large class of stochastic differential equations, significantly generalising the results of [23].

2. General setting and main results

Let $M$ be a $d$-dimensional Riemannian manifold (possibly with boundary) embedded in $\mathbb{R}^n$. We aim to study random dynamical systems originating inside the domain $M$ and being killed when exiting this region. We denote by $\{\partial\}$ the cemetery state where the flow is absorbed after escape. Moreover, we let $E_M := M \cup \{\partial\}$ be the topological space generated by the topological basis $\mathcal{T} = \{U; U$ is open in $M\} \cup \{\partial\}$, where $\bigcup$ denotes disjoint union.

Throughout this paper, the time $\mathbb{T}$ can be taken to be either the semi-group $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ or $\mathbb{R}_+ := [0, \infty)$. Let $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$ be a memoryless noise space (see Appendix A) where $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{F})$ fulfils the usual measurability conditions (see [36, Definition II.67.1]). In this paper, we focus on two different noise spaces, given by

\begin{align}
(2.1) \quad (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{F}) &= \left( \Omega, \{\mathcal{F}(\pi_s)_{0 \leq s \leq t}\}_{t \geq 0}, \mathcal{F}(\pi_s)_{s \geq 0} \right) & \text{if } \mathbb{T} = \mathbb{R}_+,
\end{align}

where $\Omega \in \{\mathcal{D}(\mathbb{R}_+, \mathbb{R}^n), \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^n)\}$ or

\begin{align}
(2.2) \quad (\Omega, (\mathcal{F}_n)_{n \in \mathbb{N}_0}, \mathbb{F}) &= \left( X^{\mathbb{N}_0}, \{\mathcal{F}(\pi_m)_{0 \leq m \leq n}\}_{n \in \mathbb{N}_0}, \mathcal{F}(\pi_m)_{m \in \mathbb{N}_0} \right) & \text{if } \mathbb{T} = \mathbb{N}_0,
\end{align}

where $X$ is a Polish space and $\pi_s$ is the canonical processes (see Section 3 for details). These noise spaces are natural for applications to iterated function systems and stochastic differential equations, which we discuss at the end of Section 2.3 and in Section 2.4.

Throughout this paper we consider $(\theta, \varphi)$ as a $C^1$-random dynamical system on the state space $(E_M, \mathcal{B}(E_M))$ and with absorption at $\partial$. We further assume that the cocycle $\varphi$ is perfect in the sense of Definition A.3 (see Appendix A for details about random dynamical systems).

2.1. Absorbed Markov processes. Under the assumption of $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{F}, (\theta_t)_{t \in \mathbb{T}})$ being a memoryless noise space, $(\theta, \varphi)$ induces a time-homogeneous Markov process

\[
\varphi = (\Omega \times E_M, (\mathcal{G}_t := \mathcal{F}_t \otimes \mathcal{B}(E_M))_{t \in \mathbb{T}}, (\varphi_t)_{t \in \mathbb{T}}, (\mathcal{P}_t)_{t \in \mathbb{T}}, (\mathcal{P}_x := \mathbb{P} \otimes \delta_x)_{x \in E_M})
\]

in the sense of [36, Definition III.1.1] where $\mathcal{P}_t(x, dy) := \mathbb{E}_x(\varphi_t \in dy)$ for every $x \in E_M$, i.e.

(i) $(\Omega \times E_M, \mathcal{G}_t)$ is a filtered space;

(ii) $\varphi_t$ is an $\mathcal{G}_t$-adapted process with state space $E_M$;
(iii) $\mathcal{P}^t$ a time-homogeneous transition probability function of the process $\varphi_t$ satisfying the usual measurability assumptions and the Chapman-Kolmogorov equation;

(iv) $(P_x)_{x \in E_M}$ is a family of probability function satisfying $P_x[\varphi_0 = x] = 1$ for every $x \in E_M$; and

(v) for all $t, s \in \mathbb{T}$ and every bounded measurable function $f$ on $M$

$$E_x[f \circ \varphi_{t+s} | G_t] = (\mathcal{P}^s f)(\varphi_t) \quad \mathbb{P}_x\text{-almost surely.}$$

For a proof, refer to [32, Chapter 2.5].

Since $\varphi_t$ is absorbed at $\partial$, we can define the stopping time

$$\tau(\cdot, x) = \inf \{ t \geq 0 : \varphi_t(\cdot, x) = \partial \}.$$

Below, we introduce some notation used throughout the present paper.

**Notation 2.1.** Given a measure $\mu$ on $M$, we denote

$$P_\mu(\cdot) := \int_M P_x(\cdot) \mu(dx).$$

We consider the set $\mathcal{F}_b(M)$ as the set of bounded Borel measurable functions on $M$. Given $f \in \mathcal{F}_b(M)$, by abuse of notation we write

$$\mathcal{P}^t(f)(x) := \mathcal{P}^t(1_M f)(x) = \int_M f(y) \mathcal{P}^t(x,dy),$$

$$E_x[f] := E_x[1_M f], \quad \text{for all } x \in M,$$

and

$$f \circ \varphi_t := (1_M \circ \varphi_t) \cdot (f \circ \varphi_t)$$

We denote by $C^0(M)$ the space of continuous functions $f : M \to M$, and by $\mathcal{M}(M)$ the set of Borel signed-measures on $M$.

An essential tool to the study of random dynamical systems are invariant (so-called Markov) measures which correspond with stationary measures of the associated Markov process. However in the context of absorbed dynamics, such measures do not exist due to the exponential loss of mass of $\varphi$ on $M$. These measures are instead replaced by so-called quasi-stationary measures $\mu$ (QSM).

**Definition 2.2 (Quasi-stationary measure).** A probability measure $\mu$ on $(M, \mathcal{B}(M))$ is said to be a quasi-stationary measure for the random dynamical system $(\theta, \varphi)$ if for all $A \in \mathcal{B}(M)$

$$\mathbb{P}_\mu(\varphi_t \in A | \tau > t) := \mu(A) \quad \text{for all } t \in \mathbb{T}.$$ 

Note that, since $\varphi_t$ is absorbed at $\partial$, we have

$$\mathbb{P}_\mu(\varphi_t \in A | \tau > t) = \frac{\mathbb{P}_\mu(\varphi_t \in A)}{\mathbb{P}_\mu(\tau > t)} = \frac{\int_M \mathcal{P}^t(x, A) \mu(dx)}{\int_M \mathcal{P}^t(x, M) \mu(dx)}, \quad \text{for all } t \in \mathbb{T}.$$ 

Furthermore, if the absorbed dynamics evolve under the statistics of a unique quasi-stationary measure, in contrast, the history of the surviving trajectories at time $T > 0$ do not in general follow the quasi-stationary statistics. Instead the asymptotic distribution of the history of surviving trajectories is given by the so-called quasi-ergodic measure $\nu$ (QEM).

**Definition 2.3 (Quasi-ergodic measure).** A probability measure $\nu$ on $(M, \mathcal{B}(M))$ is said to be a quasi-ergodic measure for the random dynamical system $(\theta, \varphi)$ if it satisfies Birkhoff’s ergodic
theorem, i.e. for all \( f : X \to \mathbb{R} \) bounded and \( \mathcal{B}(X) \)-measurable

\[
\begin{align*}
\lim_{t \to \infty} \mathbb{E}_x \left[ \frac{1}{t} \int_0^t f(\varphi_s) \, ds \right] & = \int_M f \, d\nu \quad \text{for all } x \in M, \quad \text{if } T = \mathbb{R}_+ , \\
\lim_{n \to \infty} \mathbb{E}_x \left[ \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi_i) \right] & = \int_M f \, d\nu \quad \text{for all } x \in M, \quad \text{if } T = \mathbb{N}_0 .
\end{align*}
\]

We impose a suitable setting that ensures the existence and uniqueness of a QSM and QEM. Namely, we require the RDS with absorption \( \varphi \) to have pointwise exponential convergence towards the QSM in the total variation norm. Notably, our setting ensures the existence of the Q-process \([14]\) in the strongest possible sense, a key element to the proof of the multiplicative ergodic theorem in the conditioned setting.

**Hypothesis (H).** Let \( (\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}}) \) be a memoryless noise space of the form (2.1) or (2.2), and \( (\theta, \varphi) \) an absorbed random dynamical system on \( E_M \) absorbed at \( \partial \).

(H1) The Markov process \( (\Omega \times E_M, (\mathcal{G}_t)_{t \in \mathbb{T}}, (\mathbb{P}_t)_{t \in \mathbb{T}}, (\mathbb{P}_t x \in E_M) \) admits a unique quasi-stationary measure \( \mu \) and a unique quasi-ergodic measure \( \nu \) on \( M \).

(H2) For every \( x \in M \), there exist \( C(x) \) and \( \alpha > 0 \) such that

\[
\| \mathbb{P}_x(\varphi_t \in \cdot | \tau > t) - \mu \|_{TV} \leq C(x)e^{-\alpha t} .
\]

(H3) There exists a positive bounded function \( \eta \) on \( M \) and \( \beta > 0 \), such that

\[
\lim_{t \to \infty} \sup_{x \in M} e^{\beta t} \mathbb{P}_x(\tau > t) - \eta(x) = 0 .
\]

The literature on absorbed Markov processes generally assumes the conditions given by \([14]\) which imply exponential convergence of (2.3) uniformly on \( x \). While these conditions are well suited for the study of stochastic differential equations with escape, the uniform convergence on \( x \) of (2.3) turns out to be too restrictive for discrete-time systems with escape, specifically with bounded noise (see \([12]\)).

For criteria ensuring this hypothesis, see \([7, 12, 13, 14]\). Some properties of the conditioned process induced by these conditions are given by Proposition 3.1.

Under Hypothesis (H), we may further assume the existence of the Q-process shown by \([14, 15]\), the process \( (\varphi_t)_{t \in \mathbb{T}} \) conditioned on asymptotic survival.

**Definition 2.4 (Q-process).** A family of probability measures \( (Q_x)_{x \in M} \) on \( (\Omega \times M, \mathcal{G}) \) is called Q-process, if

1. for each \( x \in M \) and every \( s \geq 0 \) and \( A \in \mathcal{G}_s \) we have

\[
Q_x(A) := \lim_{t \to \infty} \mathbb{P}_x(A | \tau > t) ,
\]

2. the tupel

\[
(\Omega \times M, (\mathcal{G}_t)_{t \in \mathbb{T}}, (\varphi_t)_{t \in \mathbb{T}}, (Q^t)_{t \in \mathbb{T}}, (Q_x)_{x \in M})
\]

is a Markov process, where we define

\[
Q^t(x, A) := Q_x(\varphi_t \in A) \quad \text{for all } t \in \mathbb{T}, \ x \in M \text{ and } A \in \mathcal{B}(M).
\]
Note that, by the definition, a $Q$-process is unique.
In previous works such as \cite{14, 15} the probability measures $(Q_x)_{x \in M}$ were only defined on $\bigcup_{t \geq 0} G_t$. However, for our application to random dynamical systems, it is important that the measures $(Q_x)_{x \in M}$ can be extended to $\mathcal{G}$.

To understand why one cannot expect the limit (2.4) to hold for all $A \in G$, consider the set 
\[ \{ \tau = \infty \} \in G. \]
Under Hypothesis (H), we have $P_x(\tau = \infty) = 0$ for all $x \in M$ and thus also $P_x(\tau = \infty \mid \tau > t) = 0$, for all $x \in M$ and $t > 0$. On the other hand,
\[ Q_x(\{ \tau > s \}) = \lim_{t \to \infty} P_x(\tau > s \mid \tau > t) = 1, \]
for all $x \in M$ and $s > 0$.

and thus
\[ Q_x(\tau = \infty) = 1 \neq 0 = \lim_{t \to \infty} P_x(\tau = \infty \mid \tau > t). \]

**Proposition 2.5** (Existence of the $Q$-process). Under Hypothesis (H), there exists a $Q$-process $(Q_x)_{x \in M}$ with transition kernels given by
\[ Q^t(x, dy) = e^{\alpha t} \frac{\eta(y)}{\eta(x)} \mathcal{P}^t(x, dy). \]

Furthermore, the measure $\nu$ is the unique stationary measure of the Markov process
\[ \left( \Omega \times M, (\mathcal{G}_t)_{t \in T}, (\varphi_t)_{t \in T}, (Q^t)_{t \in T}, (Q_x)_{x \in M} \right) \]
and we have
\[ \lim_{t \to \infty} \|Q_x(\varphi_t \in \cdot) - \nu\|_{TV} = 0. \]

Proposition 2.5 is proved in Section 3.
Recall that the skew product $(\Theta_t)_{t \in T}$ of $(\theta, \varphi)$ defined as
\[ \Theta_t : \Omega \times E_M \to \Omega \times E_M \]
\[ (\omega, x) \mapsto \Theta_t(\omega, x) := (\theta_t(\omega, \varphi(t, \omega, x)) \]
is a family of measurable mappings generating a semi-flow, i.e. a measurable dynamical system.

The proof of the existence of conditioned Lyapunov exponents relies on finding a suitable ergodic probability measure for $\Theta_t$ giving full measure to paths never to be absorbed, i.e. to
\[ (2.5) \quad \Xi := \{ (\omega, x) \mid \tau(\omega, x) = \infty \} = \bigcap_{t \in T} \{ (\omega, x) \in \Omega \times M \mid \tau(\omega, x) > t \}. \]

The most appropriate choice for such a measure turned out to be
\[ Q_\nu(\cdot) := \int_M Q_x(\cdot) \nu(dx). \]
Observe that the measure $Q_\nu$ on $(\Omega \times M, \mathcal{F} \otimes \mathcal{B}(M))$ satisfies
\[ Q_\nu(\Xi) = 1 \]
and $P_x(\Xi) = 0$, for every $x \in M$. Given a function $f \in L^1(M \times \Omega, Q_\nu)$, we denote
\[ E_\nu^Q[f] = \int_{\Omega \times M} f(\omega, x) \ Q_\nu(d\omega, dx). \]

Note here that we impose the assumption of a perfect cocycle as for all $x \in M$. However, this can be loosened: if the cocycle is not perfect, even under the $Q$-measures, the cocycle property holds almost surely (see Remark 3.4 below).
The next theorem and a crucial insight of this paper states that the measure \( \mathbb{Q}_\nu \) fulfils an essential condition for the proof of the existence of conditioned Lyapunov exponents.

**Proposition 2.6.** Let \((\theta, \varphi)\) be a random dynamical system fulfilling Hypothesis (H), and consider the skew product \((\Theta_t)_{t \in \mathbb{T}}\) of \((\theta, \varphi)\). Then the measure \( \mathbb{Q}_\nu \) is invariant and mixing, and thus ergodic, with respect to the measurable dynamical system \((\Theta_t)_{t \in \mathbb{T}}\).

**2.2. Conditioned Lyapunov exponents.** Since \( M \) is a manifold embedded in \( \mathbb{R}^n \), we can consider

\[
TM = \{(x, v) \in M \times \mathbb{R}^n, x \in M \text{ and } v \in T_x M\},
\]

as the tangent bundle of \( M \), where \( T_x M \) denotes the tangent space of \( M \) at \( x \) (for a complete description of the tangent bundle and its properties see [30, page 65]). From Proposition 2.6 and equation (2.6) we have that \( \mathbb{Q}_\nu \) is an ergodic measure to the dynamical system \((\Theta_t)_{t \in \mathbb{T}}\) and

\[
\mathbb{Q}_\nu(\Xi) = \mathbb{Q}_\nu[\tau = \infty] = 1.
\]

Observe that for \( \mathbb{Q}_\nu \)-a.e. \((x, \omega) \in \Omega \times M\) the linear map

\[
\Phi_t(\omega, x) : T_x M \to T_{\varphi_t(\omega, x)} M
\]

\[v \mapsto D\varphi_t(\omega, x)v\]

is well-defined where \( D \) denotes the space derivative of variable \( x \). Throughout this paper, \( \Phi_t \) might both refer to the map above or be considered as acting on the state space \( TM \).

Moreover, since \( \varphi_{t+s}(\omega, x) = \varphi_t(\theta_s \omega, \varphi_s(\omega, x)) \), for every \( t, s \in \mathbb{T} \), by differentiation in \( x \) and using the chain rule we obtain

\[
\Phi_{t+s}(\omega, x) = \Phi_t(\Theta_s(\omega, x)) \circ \Phi_s(t, \omega), \text{ for } \mathbb{Q}_\nu\text{-almost every } (\omega, x) \in \Omega \times M,
\]
i.e. \( \Phi_t \) forms a cocycle over the dynamical system \((\Theta_t)_{t \in \mathbb{T}}\).

For \( \mathbb{Q}_\nu \)-almost every \((\omega, x) \in \Omega \times X\), we wish to show the convergence of the following limits

\[
\Lambda_i(\omega, x) = \lim_{t \to \infty} \frac{1}{t} \log \delta_i(\Phi_t(\omega, x)) \quad \text{for all } i \in \{1, \ldots, d\},
\]

where \( \delta_i(\Phi_t) \) denotes the \( i \)th singular value of \( \Phi_t \), i.e. the square root of the \( i \)th eigenvalue of \( \Phi_t^* \Phi_t \), when \( \Phi_t \) is seen as an \( \mathbb{R}^d \)-endomorphism. For \((\omega, x, v) \in \Omega \times TM\), one may define the finite-time Lyapunov exponents \( \lambda_v(t, \omega, x) \)

\[
\lambda_v(t, \omega, x) = \frac{1}{t} \log \frac{\|\Phi_t(\omega, x)v\|}{\|v\|}
\]

where \( \| \cdot \| \) is induced by the Riemannian metric on \( M \) and their limit superiors \( \lambda_v(\omega, x) \), the *characteristic Lyapunov exponents*

\[
\lambda_v(\omega, x) = \limsup_{t \to \infty} \frac{1}{t} \log \frac{\|\Phi_t(\omega, x)v\|}{\|v\|}.
\]

We observe that, in our setting, \( \lambda_v(\omega, x) \) exists as an actual limit, and take one of the values of \( \Lambda_i(\omega, x) \). In fact, it is directly related to the ergodicity of \((\Theta_t)_{t \in \mathbb{T}}\) with respect to \( \mathbb{Q}_\nu \) and an application of the Furstenberg–Kesten theorem, that the RDS associated with the \( Q \)-process has a spectrum of Lyapunov exponents, which do not depend on initial conditions \((\omega, x)\).
Theorem 2.7 (Spectrum of Lyapunov exponents). Let \((\Phi_t)_{t \in \Theta}\) be as above, i.e. an RDS over the dynamical system \((\Omega \times M, \mathcal{F} \otimes \mathcal{B}(M), (\Theta_t)_{t \in \Theta})\) with ergodic invariant measure \(\nu\). Assume further that

\[
E^\nu \left[ \sup_{0 \leq t \leq 1} \log^+ \|\Phi_t\| \right] < \infty.
\]

(1) Then there exists a \(\Theta\)-forward-invariant set \(\Delta \in \mathcal{F} \otimes \mathcal{B}(M)\) of \(\nu\)-full measure such \(\Delta \subset \Xi\) (see (2.5)) and constant Lyapunov exponents \(\Lambda_1 \geq \ldots \geq \Lambda_d \geq -\infty\) such that for all \((\omega, x) \in \Delta\)

\[
\Lambda_i = \lim_{t \to \infty} \frac{1}{t} \log \delta_t(\Phi_t(\omega, x)) \quad \text{for all } i \in \{1, \ldots, d\}
\]

(2) If in addition, \(\Phi_t(\omega, x)\) is invertible for all \((t, \omega, x) \in \mathbb{T} \times \Omega \times M\) and

\[
E^\nu \left[ \sup_{0 \leq t \leq 1} \log^+ \|\Phi_t^{-1}\| \right] < \infty,
\]

then the Lyapunov exponents are finite and the convergence (2.8) holds in \(L^1(\nu)\).

(3) Let \(\lambda_1, d_1\) respectively denote the distinct Lyapunov exponents and their multiplicities; let \(p\) be the number of distinct Lyapunov exponents. Then we define the Lyapunov spectrum

\[
S(\theta, \varphi) = \{ (\lambda_i, d_i) : i = 1, \ldots, p \}.
\]

Furthermore, the setting of the \(Q\)-process also yields the existence of Oseledets flags.

Theorem 2.8 (Multiplicative ergodic theorem). Let \((\Theta, \Phi), \Delta, S(\theta, \varphi)\) be as above. Then for all \((\omega, x) \in \Delta\), the following statements hold:

(1) the random matrix limit \(\Psi(\omega, x) := \lim_{t \to \infty} (\Phi_t(\omega, x)^* \Phi_t(\omega, x))^{1/2t}\) exists and has distinct eigenvalues \(e^{\lambda_1} > \ldots > e^{\lambda_p}\).

(2) Let \(E_1(\omega, x), \ldots, E_p(\omega, x) \leq T_x M\) denote the corresponding random eigenspaces of \(\Psi(\omega, x)\)

\[
\text{such that } \dim E_i = d_i \text{ and defining}
\]

\[
U_i(x, \omega) = \bigoplus_{k=1}^p E_k(\omega, x).
\]

So that \(\Phi_t(\omega, x) U_i(\omega, x) = U_i(\Theta_t(\omega, x))\) and the \(U_i\)'s form a random filtration of \(T_x M\):

\[
\{0\} \subset U_p(\omega, x) \subset U_{p-1}(\omega, x) \subset \ldots \subset U_2(\omega, x) \subset U_1(\omega, x) = T_x M.
\]

(3) Furthermore for all \(v \in T_x M\), the finite-time Lyapunov exponents converge and

\[
\lambda_v(\omega, x) = \lim_{t \to \infty} \frac{1}{t} \log \|\Phi_t(\omega, x)v\| = \lambda_i \iff v \in U_i(\omega, x) \setminus U_{i+1}(\omega, x).
\]

These two theorems are obtained directly from Proposition 2.6 in combination with the classical theory of random dynamical systems (see section 3.2 for more details). This shows that the \(Q\)-process setting is well-suited to the study of conditioned dynamics. However, we wish to ensure in Section 4 that in the particular context of absorbed diffusion processes, this corresponds exactly to the framework introduced by Engel et al. [23].

In [23], the existence of the top Lyapunov exponent is proved by introducing an extended process on the unit tangent bundle. This can be generalised as in [5] for the full spectrum of Lyapunov exponents by introducing a process on the Grassmannian bundle \(\text{Gr}_k(M)\), whose fibers \(\text{Gr}(T_x M)\).
are the manifolds consisting of subspaces of the tangent spaces $T_x M$ (see Section 3.2). Defining the space of the alternating $k$-multivectors

$$\bigwedge^k T_x M = \{ v_1 \wedge \cdots \wedge v_k \mid v_1, \ldots, v_k \in T_x M \}$$

generating the vector space $\bigwedge^k T_x M$. One can equivalently identify $\text{Gr}_k(T_x M)$ as the set $\mathbf{P}(\bigwedge^k T_x M)$, which is a $(d-k)$-dimensional submanifold of the projective space $\mathbf{P}(\bigwedge^k T_x M)$. Furthermore, let us define the vector space homomorphism

$$\bigwedge^k \Phi_t(\omega, x) : \bigwedge^k T_x M \to \bigwedge^k T_{\tilde{\varphi}_t(\omega, x)} M$$

defined on $\bigwedge^k T_x M$ by

$$\bigwedge^k \Phi_t (v_1 \wedge \cdots \wedge v_k) := \Phi_t(\omega, x)v_1 \wedge \cdots \wedge \Phi_t(\omega, x)v_k.$$

See for instance [18] for a concise introduction of exterior powers in the context of Lyapunov exponents. This allows us to state the following proposition.

**Proposition 2.9.** Assume that $(\Theta, \Phi)$ satisfies integrability conditions (2.7) and (2.9) and for $k \leq d$, let $\rho^k$ be the Borel measure on $\text{Gr}_k(M)$ defined as

$$\rho^k(dx \times dv) = \sigma_x^k (dv \cap \text{Gr}_k(T_x M)) \nu(dx),$$

where $\sigma_x^k$ is the Borel measure on the $\text{Gr}_k(T_x M)$ defined in (3.11). Then there exists a set $\bar{G} \subset \text{Gr}_k(M)$, such that $\rho^k(\bar{G}) = 1$ and

$$\lim_{t \to \infty} \mathbb{E}_\nu^\omega \left[ \frac{1}{t} \log \left\| \bigwedge^k \Phi_t v \right\| \right] = 0 \quad \text{for all } (x, v) \in \bar{G},$$

where $\lambda^{(k)} = \Lambda_1 + \cdots + \Lambda_k$ and $\{\Lambda_i\}_{i=1}^d$ are given by Theorem 2.7.

Note that the integrability conditions (2.7) and (2.9) of the multiplicative ergodic theorem can be formulated in terms of the conditioned process. Indeed from Proposition 3.2, with the QSM $\mu$, we have

$$\mathbb{E}_\nu^\omega \left[ \sup_{0 \leq t \leq 1} \log^+ \|\Phi_t^{(1)}\| \right] = \int_M \mathbb{E}_x^\omega \left[ \sup_{0 \leq t \leq 1} \log^+ \|\Phi_t^{(1)}\| \right] \nu(dx)$$

$$= e^\beta \int_M \mathbb{E}_x \left[ \eta(\varphi_1) \sup_{0 \leq t \leq 1} \log^+ \|\Phi_t^{(1)}\| \right] \mu(dx)$$

$$\leq e^\beta \|\nu\| \mathbb{E}_\nu \left[ \sup_{0 \leq t \leq 1} \log^+ \|\Phi_t^{(1)}\| \right].$$

### 2.3. Convergence in conditional probability

Under the same setting and mild conditions, we prove the convergence of the finite-time Lyapunov exponents towards the $Q$-process Lyapunov exponents in conditional probability. To this end, we exhibit the following result in the more general settings of Markov processes with $Q$-processes: Under suitable conditions, we show that any convergence $\Gamma_t \to \Gamma^*$ in mean or in probability under $Q_x$ also holds respectively in conditional mean or probability under $P_\nu$. For Theorem 2.10 below, although we keep the notation $(\varphi_t)_{t \in \mathbb{T}}$, it does not necessarily denote a cocycle but any absorbed Markov process $(\tilde{\Omega}, (\tilde{G}_t)_{t \in \mathbb{T}}, (\varphi_t)_{t \in \mathbb{T}}, (\mathcal{P}^t)_{t \in \mathbb{T}}, (\tilde{\mathbb{P}}_x)_{x \in M \cup \{\beta\}})$ for which there is a corresponding $Q$-process $(\tilde{\Omega}, (\tilde{G}_t)_{t \in \mathbb{T}}, (\varphi_t)_{t \in \mathbb{T}}, (\mathcal{Q}^t)_{t \in \mathbb{T}}, (Q_x)_{x \in M})$ under Hypothesis (H).
**Theorem 2.10.** Let \((\varphi_t)_{t \in T}\) be a Markov process satisfying Hypothesis (H), such that the existence of the corresponding \(Q\)-process is guaranteed. Let \(x \in M\) and \((\Gamma_t)_{t \in T}\) be a collection of \(\mathcal{G}_t\)-measurable random variables.

(i) Suppose that \((\Gamma_t)_{t \in T}\) convergences in probability to some \(\Gamma^* \in \mathbb{R}\) under \(Q_x\), i.e. for all \(\varepsilon > 0\),

\[
\lim_{t \to \infty} Q_x \left[ |\Gamma_t - \Gamma^*| > \varepsilon \right] = 0.
\]

Then this convergence holds in \(P_x\)-conditional probability, i.e. for all \(\varepsilon > 0\),

\[
\lim_{t \to \infty} P_x \left[ |\Gamma_t - \Gamma^*| > \varepsilon \mid \tau > t \right] = 0.
\]

(ii) If in addition, there exists \(p \in (1, \infty]\) such that

\[
\lim_{t \to \infty} E_x^Q[|\Gamma_t - \Gamma^*|] = 0 \quad \text{and} \quad \sup_{t \geq 0} \|\Gamma_t\|_{L^p(M \times \Omega, P_x(\cdot | \tau > t))} < \infty
\]

then

\[
\lim_{t \to \infty} E_x[|\Gamma_t - \Gamma^*| \mid \tau > t] = 0.
\]

Using these insights, we obtain the following convergence theorems for finite-time Lyapunov exponents under conditional probabilities.

**Theorem 2.11.** Assume that \((\Theta, \Phi)\) fulfils the integrability condition (2.7) so that the multiplicative ergodic theorem holds. Let \(k \leq d\) be such that

\[
\lambda^{(k)} = \Lambda_1 + \cdots + \Lambda_k > -\infty
\]

where \(\Lambda_1, \ldots, \Lambda_k\) are the \(k\) first Lyapunov exponents given by Theorem 2.7. Then for every \(\varepsilon > 0\) and \(\rho^k\)-almost every \((x,v) \in \text{Gr}_k(M)\),

\[
\lim_{t \to \infty} P_x \left[ \left\{ \left| \lambda^{(k)} - \frac{1}{t} \log \lambda^k \Phi v \right| > \varepsilon \right\} \mid \tau > t \right] = 0.
\]

Similarly, for all \(\varepsilon > 0\) and \(\nu\)-almost every \(x \in M\)

\[
\lim_{t \to \infty} P_x \left[ \left\{ \left| \lambda^{(k)} - \frac{1}{t} \log \lambda^k \Phi t \right| > \varepsilon \right\} \mid \tau > t \right] = 0.
\]

**Remark 2.12.** Observe that a slightly different version of Theorem 2.11 is the following: For all \(v \in S^{d-1}\) such that \(\lambda_v := \lambda(\cdot, \cdot, v) : (\omega, x) \to \lambda(\omega, x, v)\) is constant \(Q_\nu\)-almost surely, there exists \(k \leq p \leq d\) such that \(\lambda(\cdot, v) = \lambda_k\) holds \(Q_\nu\)-almost surely. Furthermore,

\[
\lim_{t \to \infty} P_x \left[ \left\{ \left| \lambda_k - \frac{1}{t} \log \lambda^k \Phi t \right| > \varepsilon \right\} \mid \tau > t \right] = 0.
\]

This is useful in cases where some of the Oseledets’ flag are not random or degenerate (see for instance Example 2.16). As a matter of fact, it is believed that the Oseledets’ spaces are either constant or that their distribution is non-degenerate at least for a large class of stochastic differential equations. This would immediately imply that the conditioned characteristic Lyapunov exponents \(\lambda_v\) are constant \(Q_\nu\)-almost surely for all \(v \in S^{d-1}\). However, we were not able to find such known general result that would most likely rely on the use of Malliavin calculus in the spirit of [28].

Under stronger assumptions, the convergences (2.11) and (2.12) can be strengthened in conditional expectation.
Theorem 2.13. Assume now that \((\Theta, \Phi)\) is an invertible linear cocycle fulfilling the integrability conditions (2.7) and (2.9) and let \(\rho^k\) be as above. Assume further that for some \(p \in (1, \infty)\)

\[
\alpha^\pm = \sup_{t \geq 0} \left\| \frac{1}{t} \log^n \| \Phi_t \| \right\|_{L^p(\Omega \times M, \mathbb{P}_\nu(\cdot \mid \tau > t))} < \infty.
\]

Then \(\lambda^{(d)} > -\infty\) and for all \(k \leq d\), for \(\rho^k\)-almost every \((x, v) \in \text{Gr}_k(M)\),

\[
\lim_{t \to \infty} \mathbb{E}_x \left[ \lambda^{(k)} - \frac{1}{t} \log \left\| \Lambda^k \Phi_t v \right\| \mid \tau > t \right] = 0
\]

and for \(\nu\)-almost every \(x \in M\)

\[
\lim_{t \to \infty} \mathbb{E}_x \left[ \lambda^{(k)} - \frac{1}{t} \log \left\| \Lambda^k \Phi_t v \right\| \mid \tau > t \right] = 0.
\]

Theorems 2.10, 2.11 and 2.13 are proved in Section 4.

Example 2.14 (Iterated function systems). Let \(X\) be a compact metric space and \(\Pi\) a Borel probability measure on \(X\); then they generate a memoryless noise space 

\((X^{N_0}, B(X) \otimes N_0, (\mathcal{F}_n)_{n \in N_0}, (\theta_n)_{n \in N_0}, \Pi \otimes N_0)\)

of the form (3.2), where

\[
\theta := \theta_1 : X^{N_0} \to X^{N_0}
\]

\[
\omega = \omega_0 \omega_1 \omega_2 \cdots \mapsto \theta \omega := \omega_1 \omega_2 \omega_3 \cdots
\]

Furthermore, let \(E\) be a \(d\)-dimensional manifold and \(M\) the compact closure of a \(d\)-dimensional submanifold of \(\mathbb{R}^n\) and denote \(\{\partial\} = E \setminus M\). Suppose

\[
f : X \times E \to E
\]

\[
(\omega, x) \mapsto f_{\omega}(x)
\]

is continuously differentiable and \(\partial_x f\) is invertible on \(X \times M\). Now define recursively the cocycle \((\theta, \varphi)\) as

\[
\varphi_{n+1}(\omega, x) = \begin{cases} 
  f_{\omega_n} \circ \varphi_n(\omega, x), & \varphi_n(\omega, x) \in M, \\
  \partial, & \varphi_n(\omega, x) = \partial.
\end{cases}
\]

Thus \((\theta, \varphi)\) forms a \(C^1\)-random dynamical system with absorption at \(\{\partial\}\) with linearised flow \(\Phi\) generated by \(\Phi_1 = \partial_x f\). Assume finally that the absorbed Markov process \((\varphi_n)_{n \in N_0}\) fulfils [12, Hypothesis (H) & Theorem 3-(M1)] so that Hypothesis (H) holds. Now by compactness of \(X \times M\), \((\Theta, \Phi)\) fulfils the integrability condition (2.14) and thus Theorems 2.7, 2.8 and 2.13 apply and there exist conditionally Lyapunov exponents \(L_1, \cdots, L_d\) such that for \(\rho^k\)-almost every \((x, v) \in \text{Gr}_k(M)\), \(k \leq d\),

\[
\lim_{n \to \infty} \mathbb{E}_x \left[ \lambda^{(k)} - \frac{1}{n} \log \left\| \Lambda^k \Phi_n v \right\| \mid \tau > n \right] = 0.
\]
2.4. Application to stochastic differential equations. In this section we aim to apply the results of section 2.2 to stochastic differential equations with escape. Let $M \subset \mathbb{R}^d,$

$$(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P}) = \left( C_0(\mathbb{R}_+, \mathbb{R}^m), (\sigma(\pi_s, 0 \leq s \leq t))_{t \geq 0}, \sigma(\pi_s, s \geq 0), \mathbb{P} \right),$$

where $\mathbb{P}$ is the Wiener measure, and consider the stochastic differential equation

$$(2.17) \quad dX_t = V_0(X_t)dt + \sum_{i=1}^{m} V_i(X_t) \circ dW^i_t, \quad X_0 \in M,$$

on $M$, where $(W^1_t, \ldots, W^m_t)$ denotes an $m$-dimensional standard Brownian motion, and $V_i : M \to \mathbb{R}^d$ are vector fields on $M$.

The assumption below ensures that the cocycle $\varphi$, generated by the SDE (2.17), converges exponentially to the quasi-stationary measure, i.e. it satisfies Hypothesis (H).

**Hypothesis (H) SDE.** We say that the stochastic differential equation (2.17) fulfils Hypothesis ($H_{SDE}$) if

- (H1) $M$ is an open connected and bounded subset of $\mathbb{R}^d$ with $C^2$-boundary.
- (H2) The vector fields $\{V_i : M \to \mathbb{R}^d\}_{i=0}^{m}$ admit vector field extensions $\{\tilde{V}_i : \mathbb{R} \to \mathbb{R}^d\}$ such that
  - (i) $\tilde{V}_i|_M = V_i$, for every $i \in \{0, 1, \ldots, m\}$; and
  - (ii) $\tilde{V}_0$ is a $C^1$-vector field and $\tilde{V}_i$ is a $C^2$-vector field for every $i \in \{1, \ldots, m\}$. Note that since $\overline{M}$ is compact, this ensures the boundness of the derivatives of $V_i$ on $M$.
- (H3) The generator of (2.17) is uniformly elliptic, i.e. there exist $c > 0$ such that for all $x \in M$ and for all $\xi \in \mathbb{R}^m \setminus \{0\}$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{d} \xi_i V^k_i V^j_i \xi_j > c \|\xi\|^2.$$

The results below can be applied identically (without any further modification of the proofs) to a setting where $M$ is a compact connect manifold on which $(\theta, \varphi)$ satisfies Hypothesis (H) and assuming some regularity of the vectors fields $\{V_i\}_{i=1}^{m}$ (see for instance [13, 7]).

The following theorem yields that the Lyapunov exponents given by the $Q$-process equate to the conditioned Lyapunov exponents conjectured in [23].

**Theorem 2.15.** Let $M$ be an open connected and bounded subset of $\mathbb{R}^d$ with $C^2$-boundary, and suppose that the stochastic differential equation

$$(2.18) \quad dX_t = V_0(X_t)dt + \sum_{i=1}^{m} V_i(X_t) \circ dW^i_t, \quad X_0 \in M,$$

satisfies Hypothesis ($H_{SDE}$). Then the random dynamical system $(\theta, \varphi)$ associated to (2.18) satisfies Hypothesis (H). Furthermore, the linearised flow $\Phi_t$ fulfils condition (2.7) so that the multiplicative ergodic theorem holds.

Let $\nu$ denote the unique quasi-ergodic measure of $(\theta, \varphi)$ on $M$ and $\sigma^k$ the Borel measure on the $k$th Grassmannian of $\mathbb{R}^d$, $\text{Gr}_k(T_x M) = \text{Gr}_k(\mathbb{R}^d)$, defined in (3.11) and Remark 3.6. Then for $(\nu \times \sigma^k)$-almost every $(x, v) \in \text{Gr}_k(M) \simeq M \times \text{Gr}_k(\mathbb{R}^d)$, we have that for every $\varepsilon > 0$,

$$(2.19) \quad \lim_{t \to \infty} \mathbb{P}_x \left[ \left| \lambda^{(k)} - \frac{1}{t} \log \left\| \Lambda_t \Phi_t v \right\| \right| > \varepsilon \left|\tau > t \right. \right] = 0,$$
where

\[\lambda^{(k)} = \Lambda_1 + \cdots + \Lambda_k\]

and \(\Lambda_1, \ldots, \Lambda_k\) are the \(k\) first Lyapunov exponents given by Theorem 2.7.

Theorem 2.15 is proved in Section 5. This result generalises the theorem of convergence of finite-time Lyapunov exponents towards the average conditioned first Lyapunov exponent in conditional probability [23, Theorem 3.9]. Indeed, now recall that the Lyapunov exponents \(\Lambda_i\) are not defined with respect to a quasi-ergodic measure of an extended process anymore. As a matter of fact, the existence and uniqueness of such measure need not to be assumed and the \(Q\)-process setting covers cases where this does not hold (see Example 2.16). However, in non-degenerate examples, one can in general compute the full spectrum of Lyapunov exponents with formulae we derive in Section 5.

**Example 2.16** (Uncoupled stochastic differential equation). Consider the simple uncoupled two-dimensional SDE

\[
\begin{aligned}
\frac{dX_t}{dt} &= (X_t - X_t^3)dt + \sigma_1 dW_t^1 \\
\frac{dY_t}{dt} &= (Y_t - Y_t^3)dt + \sigma_2 dW_t^2
\end{aligned}
\]

with absorption at the boundary of the square domain \([-1.5, 1.5] \times [-1.5, 1.5]\). Denote \(\varphi = (\varphi^1, \varphi^2)\) the stochastic flow given by \(\nu_1\) and \(\nu_2\) the unique quasi-ergodic measures on \([-1.5, 1.5]\) of \((\varphi_t^1)_{t \geq 0}\) and \((\varphi_t^2)_{t \geq 0}\) respectively. Hence, one can define

\[
\Lambda_i = \int_{-1.5}^{1.5} (1 - 3z^2) \nu_i(dz).
\]

the conditioned average Lyapunov exponent achieved by \(\varphi^i\). In this case, the conditioned process \((\varphi_t)_{t \geq 0}\) converges exponentially to quasi-stationarity and has quasi-ergodic measure \(\nu_1 \times \nu_2\). However, the top conditioned Lyapunov in the sense of Engel et al. [23] cannot be defined. Indeed the process \((\varphi_t, s_t)_{t \geq 0}\), where

\[s_t(\omega, x, v) := \frac{D\varphi_t(\omega, x)v}{\|D\varphi_t(\omega, x)v\|}\]

for some \(v \in S^1\), does not have unique quasi-stationary and quasi-ergodic measures. However, one can deduce that

**Case 1**: \(\sigma_1 > \sigma_2 \implies \Lambda_1 > \Lambda_2\): The conditioned process displays two (quasi)-ergodic components \(\nu_1 \times \nu_2 \times \delta_{(\pm 1, 0)}\) and \(\nu_1 \times \nu_2 \times \delta_{(0, \pm 1)}\) achieving \(\Lambda_1\) and \(\Lambda_2\) respectively. Thus, this system yields a Lyapunov spectrum \(\{(\Lambda_1, 1), (\Lambda_2, 1)\}\). In addition, with respect to the Lebesgue measure, almost every \(v \in S^1\) achieves \(\Lambda_1\). One can be even more precise: all \(v \in S^1 \setminus \{(0, \pm 1)\}\) achieve \(\Lambda_1\), and \((0, \pm 1)\) achieves \(\Lambda_2\). In other words, this exhibits the structure of an Oseledets flag \(\{(0, 0)\} \subset \text{Span}\{(0, 1)\} \subset \mathbb{R}^2\).

**Case 2**: \(\sigma_1 = \sigma_2 \implies \Lambda_1 = \Lambda_2\): In this case, we obtain that every \(v \in S^1\) achieves \(\Lambda_1 = \Lambda_2\). Thus the Lyapunov spectrum is \(\{(\Lambda_1, 2)\}\), i.e. \(\Lambda_1\) has multiplicity 2. Here, the Oseledets flag is given by \(\{0\} \subset \mathbb{R}^2\).

We emphasise that Theorem 2.15 covers such degenerate examples.
3. The $Q$-process for random dynamical systems

In this section we prove the existence of the $Q$-process process for a random dynamical system fulfilling Hypothesis (H) and the ergodicity of $\mathbb{Q}_\nu$ under $\Theta$. In this paper, we restrict to the two following noise spaces:

(i) In the case $\mathbb{T} = \mathbb{R}_+$, we always consider

$$\mathcal{Q} = (\Omega, (\mathcal{E}_t)_{t \geq 0}, \mathcal{F}) = \left( \Omega, (\sigma(\pi_s, 0 \leq s \leq t))_{t \geq 0}, \sigma(\pi_s, s \geq 0) \right),$$

where $\Omega \in \{D(\mathbb{R}_+, \mathbb{R}^m), C_0(\mathbb{R}_+, \mathbb{R}^m)\}$ and

- $C_0(\mathbb{R}_+, \mathbb{R}^m) = \{\omega : \mathbb{R}_+ \to \mathbb{R}^m; \omega \text{ is a continuous and } \omega(0) = 0\}$,
- $D(\mathbb{R}_+, \mathbb{R}^m) = \{\omega : \mathbb{R}_+ \to \mathbb{R}^m; \omega \text{ is càdlàg}\}$,

and

$$\pi_t : \omega \in \Omega \to \omega(t) \in \mathbb{R}^m.$$

(ii) In the case that $\mathbb{T} = \mathbb{N}_0$,

$$\mathcal{Q} = (\Omega, (\mathcal{E}_n)_{n \in \mathbb{N}_0}, \mathcal{F}) = (X^{\mathbb{N}_0}, (\sigma(\pi_m, 0 \leq m \leq n))_{n \in \mathbb{N}_0}, \sigma(\pi_m, m \geq 0)),

where $X$ is a Polish space,

$$X^{\mathbb{N}_0} = \{f : \mathbb{N}_0 \to X\},$$

and

$$\pi_n : \omega \in X^{\mathbb{N}_0} \to \omega(n) \in X.$$

Let us also recall some properties of the conditioned process under quasi-stationarity.

Proposition 3.1. If Hypothesis (H) is fulfilled, then

(i) $\nu(dx) = \eta(x)\mu(dx)$;

(ii) $\int_M \mathcal{P}^t(x, \cdot) d\mu = e^{-\beta t} \mu(\cdot)$ for every $t \in \mathbb{T}$;

(iii) $\int_M \mathcal{P}^t(\eta)(x) \mu(dx) = e^{-\beta t}$ for every $t \in \mathbb{T}$.

Proposition 3.1 is proved in Appendix B.

In the following we prove Proposition 2.5. The proof relies on the following proposition.

Proposition 3.2. Let $(\theta, \varphi)$ be an absorbed random dynamical system fulfilling Hypothesis (H). Then, for every $x \in M$, there exists a unique probability measure $\mathbb{Q}_x$ on $(\Omega \times M, \mathcal{G} = \mathcal{F} \otimes \mathcal{B}(M))$, such that for every $s \geq 0$ and $A \in \mathcal{G}_s$,

$$\mathbb{Q}_x[A] = \frac{e^{\beta s}}{\eta(x)} \mathbb{E}_x[1_A \eta \circ \varphi_s] = \lim_{t \to \infty} \mathbb{P}_x[A | \tau > t].$$

Proof. Let us fix $x \in M$. We divide the proof in four steps.

Step 1. We show that for every $x \in M$, $s \in \mathbb{T}$ and $A \in \mathcal{G}_s$,

$$\lim_{t \to \infty} \mathbb{P}[A | \tau(\cdot, x) > t] = \frac{e^{\beta s}}{\eta(x)} \mathbb{E}[1_A \eta(\varphi_s(\cdot, x))].$$
By a direct computation,

\[
\lim_{t \to \infty} \mathbb{P}_x[A \mid \tau > t] = \lim_{t \to \infty} \frac{\mathbb{E}_x[\mathbb{1}_{A, \{\tau > t\}}]}{\mathcal{P}(x, M)}
\]

\[
= \lim_{t \to \infty} \frac{\mathbb{E}_x[\mathbb{1}_{A, \{\tau > t\}}]}{\mathcal{P}(x, M)}
\]

\[
= \lim_{t \to \infty} \frac{\mathbb{E}_x[\mathbb{1}_{A} \mathcal{P}^{t-s}(\varphi_s, M)]}{\mathcal{P}(x, M)}
\]

\[
= \lim_{t \to \infty} \frac{e^{\beta_s} \mathbb{E}_x[\mathbb{1}_{A} \mathcal{P}^{t-s}(\varphi_s, M)]}{e^{\beta t} \mathcal{P}(x, M)} = \frac{e^{\beta_s}}{\eta(x)} \mathbb{E}_x[\mathbb{1}_{A} \mathbb{1}_{\varphi_s}].
\]

This proves Step 1.

**Step 2.** We show that there exists a measure \( \bar{Q}_x \) on \((\Omega, \mathcal{F})\), such that

\[
\bar{Q}_x \big|_{\mathcal{F}_s} [d\omega] = \frac{e^{\beta t}}{\eta(x)} \mathbb{E}_x[\varphi_t(\omega, x)] d\mathbb{P}[d\omega].
\]

In the case \( T = N_0 \) this follows immediately from Kolmogorov’s extension theorem (see [36, Theorem II 3.26.1]). In the case \( T = \mathbb{R}_+ \) and \( \Omega = \mathcal{D}(\mathbb{R}_+, \mathbb{R}^m) \) consider the measure space isomorphism

\[
\mathcal{E} : (\mathcal{D}(\mathbb{R}_+, \mathbb{R}^m), \sigma((\pi_s)_{s \geq 0})) \to \left( \bar{\Omega}, \bar{\mathcal{F}} \right)
\]

\[
\omega \mapsto (\omega |_{[0,1]}, \omega |_{[1,2]} - \omega(1), \omega |_{[2,3]} - \omega(2), \ldots),
\]

where

\[
(\bar{\Omega}, \bar{\mathcal{F}}) := \left( \mathcal{D}([0,1], \mathbb{R}^m), \sigma \left( (\pi_s)_{0 \leq s \leq 1} \right) \right) \otimes (\mathcal{D}_0([0,1], \mathbb{R}^m), \sigma((\pi_s)_{0 \leq s \leq 1})^{\otimes N_0}
\]

and \( \mathcal{D}_0([0,1], \mathbb{R}^m) = \{ \omega \in \mathcal{D}([0,1], \mathbb{R}^m) : \omega(0) = 0 \} \).

Consider the filtration \( (\bar{\mathcal{F}}_n)_{n \in \mathbb{N}} \) on \((\bar{\Omega}, \bar{\mathcal{F}})\) by \( \bar{\mathcal{F}}_n := \mathcal{E}(\mathcal{F}_n) \). Note that for each \( n \in \mathbb{N} \) the \( \sigma \)-algebra \( \bar{\mathcal{F}}_n \) is precisely the one induced by projection on the first \( n \) components of \( \mathcal{D}([0,1], \mathbb{R}^m) \otimes \mathcal{D}_0([0,1], \mathbb{R}^m)^{\otimes N_0} \). Since \( \mathcal{D}([0,1], \mathbb{R}^m) \) and \( \mathcal{D}_0([0,1], \mathbb{R}^m) \) are a Polish spaces, we can apply Kolmogorov’s extension theorem to the measures

\[
\bar{Q}_x^n := \mathcal{E}_n \left( \frac{e^{\beta t}}{\eta(x)} \mathbb{E}_x[\varphi_t(\omega, x)] d\mathbb{P}[d\omega] \right) \quad \text{on} \quad \mathcal{D}([0,1], \mathbb{R}^m) \times \mathcal{D}_0([0,1], \mathbb{R}^m)^{\otimes N_0}, \bar{\mathcal{F}}_n
\]

for every \( n \in N_0 \), to get a measure \( \bar{Q}_x \) on \((\bar{\Omega}, \bar{\mathcal{F}})\) with \( \bar{Q}_x^n = \bar{Q}_x |_{\bar{\mathcal{F}}_n} \). Now we can set \( \bar{Q}_x := \mathcal{E}^{-1}_n \bar{Q}_x \) to get the desired measure.

If \( \Omega = C_0(\mathbb{R}_+, \mathbb{R}^m) \) the exact same argument can be applied changing \( \mathcal{D} \) to \( C_0 \) in the above proof.

**Step 3.** We complete the proof.

Consider \( Q_x := \bar{Q}_x \times \delta_x \). From Steps 1-2 it is clear that \( Q_x \) is a Borel measure on \( \mathcal{G} = \mathcal{F} \otimes \mathcal{B}(M) \) and for every \( x \in M \), \( s \in \mathbb{T} \) and \( A \in \mathcal{G}_s \),

\[
Q_x[A] = \frac{e^{\beta s}}{\eta(x)} \mathbb{E}_x[\mathbb{1}_A \mathbb{1}_{\varphi_s}] = \lim_{t \to \infty} \mathbb{P}_x[A \mid \tau > t].
\]
The uniqueness of $Q_x$ follows directly from the monotone class theorem and
\[
\sigma \left( \bigcup_{s \geq 0} G_s \right) = \mathcal{G}.
\]
This finishes the proof.

In the following, we prove a useful lemma.

**Lemma 3.3.** Let $A_1 \in \mathcal{G}_t$ and $A_2 \in \mathcal{G}_s$. Then we have for all $x \in \mathcal{M}$
\[
Q_x(A_1 \cap \Theta_t^{-1}(A_2)) = \int_{\mathcal{M}} Q_y(A_2) \ Q_x(A_1 \cap \{ \varphi_t \in dy \}).
\]
As an easy consequence, we also have
\[
Q_x(A_1 \cap \Theta_t^{-1}(A_2)) = \int_{\mathcal{M}} Q_y(A_2) \ Q_x(A_1 \cap \{ \varphi_t \in dy \}).
\]

**Proof.** Clearly, the second equality follows from the first by integration with respect to $\nu$. Thus, we only show the first equality. Observe that
\[
Q_x(A_1 \cap \Theta_t^{-1}(A_2)) = E_x^Q [1_{A_1} \cdot 1_{A_2} \circ \Theta_t]
\]
\[
= \frac{e^{\beta t + s}}{\eta(x)} E_x [1_{A_1} \cdot 1_{A_2} \circ \Theta_t \cdot \eta \circ \varphi_s + t]
\]
(3.3)
\[
= \frac{e^{\beta t + s}}{\eta(x)} E_x [1_{A_1} \cdot 1_{A_2} \circ \Theta_t \cdot \eta \circ \varphi_s \circ \Theta_t].
\]
Since the $\sigma$-algebras $\mathcal{F}_t$ and $\Theta_t^{-1}\mathcal{F}_s$ are $\mathbb{P}$-independent, we obtain
\[
E_x [1_{A_1} \cdot 1_{A_2} \circ \Theta_t \cdot \eta \circ \varphi_s \circ \Theta_t] = E_x [1_{A_1} \cdot E [1_{A_2} \circ (\theta_t, \varphi_t(\cdot, x)) \cdot \eta \circ \varphi_s \circ (\theta_t, \varphi_t(\cdot, x)) \mid \mathcal{F}_t]]
\]
\[
= E_x [1_{A_1} \cdot E_{\varphi_t(\cdot, x)} [1_{A_2} \circ (\theta_t, \cdot) \cdot \eta \circ \varphi_s \circ (\theta_t, \cdot) \mid \mathcal{G}_t]]
\]
(3.4)
\[
= E_x [1_{A_1} \cdot E_{\varphi_t(\cdot, x)} [1_{A_2} \cdot \eta \circ \varphi_s]].
\]
Combining (3.3) and (3.4), we achieve
\[
Q_x(A_1 \cap \Theta_t^{-1}(A_2)) = \frac{e^{\beta t}}{\eta(x)} E_x \left[ 1_{A_1} \cdot \eta \circ \varphi_t \cdot \frac{e^{\beta s}}{\eta \circ \varphi_t} E_{\varphi_t(\cdot, x)} [1_{A_2} \cdot \eta \circ \varphi_s] \right]
\]
\[
= \frac{e^{\beta t}}{\eta(x)} E_x \left[ 1_{A_1} \cdot \eta \circ \varphi_t \cdot Q_{\varphi_t(\cdot, x)} [A_2] \right]
\]
\[
= E_x^Q [1_{A_1} \cdot Q_{\varphi_t(\cdot, x)} [A_2]]
\]
\[
= \int_{\mathcal{M}} Q_y(A_2) \ Q_x(A_1 \cap \{ \varphi_t \in dy \}),
\]
which yields the statement.

Now we can prove Proposition 2.5.

**Proof of Proposition 2.5.** Let $\{Q_x\}_{x \in \mathcal{M}}$ be the family of measures given by Proposition 3.2. We divide the proof in three steps.

**Step 1.** We show that $\{Q^t\}_{t \in \mathbb{T}}$ fulfils the Chapman-Kolmogorov equation.
Given \( t, s \in T \) and \( B \in \mathcal{B}(M) \), we have that
\[
Q^{t+s}(x, B) = \frac{e^{\beta(t+s)}}{\eta(x)} P^{t+s}(\mathbb{1}_B \eta)(x) = \frac{e^{\beta t}}{\eta(x)} P^t \left( \frac{e^{\beta s}}{\eta(y)} P^s(\mathbb{1}_B \eta)(y) \right)(x)
\]
\[
= \frac{e^{\beta t}}{\eta(x)} P^t (Q^s(\mathbb{1}_B)(y))(x)
\]
\[
= \int_M Q^s(y, B) Q^t(x, dy).
\]

**Step 2.** We show that
\[
(3.5) \quad \left( \Omega \times M, (\mathcal{G}_t)_{t \in T}, (\varphi_t)_{t \in T}, (Q^t)_{t \in T}, (Q_x)_{x \in M} \right)
\]
is a Markov Process.

To conclude that (3.5) is a Markov process, the only non-trivial property that must be verified is that for all \( t, s \in T \) and every bounded measurable function \( f \) on \( M \)
\[
E^\mathbb{Q}_x[f \circ \varphi_{t+s} | \mathcal{G}_s] = (Q^t f)(\varphi_s) \quad \text{\( \mathbb{Q}_x \)-a.s.}
\]
To verify this let \( A \in \mathcal{G}_s \). We can compute
\[
E^\mathbb{Q}_x[1_A \cdot f \circ \varphi_{t+s}] = E^\mathbb{Q}_x[1_A \cdot f \circ \varphi_t \circ \Theta_s]
\]
\[
\text{Lem. 3.3} \quad \int_M E_{Q_\nu}[f \circ \varphi_t] Q_x(A \cap \{ \varphi_s \in dy \})
\]
\[
= \int_M (Q_t f)(y) Q_x(A \cap \{ \varphi_s \in dy \})
\]
\[
= E^\mathbb{Q}_x[1_A \cdot (Q_t f)(\varphi_s)].
\]

**Step 3.** We show that (3.5) admits the quasi-ergodic measure \( \nu \) as its unique ergodic invariant measure and for every \( x \in M \)
\[
\lim_{t \to \infty} \|Q_x(\varphi_t(\cdot)) - \nu(\cdot)\|_{TV} = 0.
\]

Let \( B \in \mathcal{B}(M) \), then we have that for every \( t \in T \),
\[
|Q_x(\varphi_t \in B) - \nu(B)| = \left| \frac{e^{\beta t}}{\eta(x)} E_x[\mathbb{1}_B \circ \varphi_t \cdot \eta \circ \varphi_t] - \nu(B) \right|
\]
\[
= \frac{e^{\beta t} \mathbb{P}_x[\tau > t]}{\eta(x)} \mathbb{P}_x[\mathbb{1}_B \circ \varphi_t \cdot \eta \circ \varphi_t | \tau > t] - \int_B \eta d\mu
\]
\[
\leq \frac{e^{\beta t} \mathbb{P}_x[\tau > t]}{\eta(x)} - 1 + \mathbb{P}_x[\mathbb{1}_B \circ \varphi_t \cdot \eta \circ \varphi_t | \tau > t] - \int_B \eta d\mu.
\]
(3.6)

From (H2)-(H3) we have that
\[
|\mathbb{P}_x[\mathbb{1}_B \circ \varphi_t \cdot \eta \circ \varphi_t | \tau > t] - \int_B \eta d\mu| \leq \|\eta\|_{\infty} C(x)e^{-\alpha t}
\]
and
\[
(3.7) \quad \lim_{t \to \infty} \left| \frac{e^{\beta t} \mathbb{P}_x[\tau > t]}{\eta(x)} - 1 \right| = 0.
\]
From equations (3.6), (3.7) and (3.8), we obtain
\[ \lim_{t \to \infty} \| Q_t x [\varphi_t(\cdot)] - \nu \|_{TV} = 0. \]

The above equation implies that \( \nu \) is the unique stationary measure of the Markov process \((\Omega \times M, (G_t)_{t \in T}, (\varphi_t)_{t \in T}, (Q^t)_{t \in T}, (Q_x)_{x \in M})\) and therefore ergodic.

\[ \square \]

**Remark 3.4.** Note that the initial probabilities \((Q_x)_{x \in M}\) associated to the \(Q\)-process of the stochastic differential equation (2.18) do not depend on chosen modifications of \( \varphi \) (see [36, Definition II. 36.2]). Indeed, let \( \tilde{\varphi} \) be a modification of \( \varphi \) and consider the stopping time
\[ \tilde{\tau}(\omega, x) = \inf\{ t \geq 0 : \tilde{\varphi}(t, \omega, x) \notin M \} . \]
We have that for every \( x \in M \)
\[ \tau = \tilde{\tau} \quad \mathbb{P}_x \text{-almost surely}. \]
This implies that for every \( t \geq 0 \),
\[ \mathbb{P}_x [\cdot | \tau > t] = \mathbb{P}_x [\cdot | \tau > t], \]
and therefore \( \tilde{\varphi} \) and \( \varphi \) generate the same family of probabilities \((Q_x)_{x \in M}\). Furthermore, properties such as continuity, càdlàg paths and the cocycle property of \( \varphi \) under \( \mathbb{P}_x \) are preserved under \( Q_x \).

### 3.1. The \( Q\)-process dynamical framework.

Let \((\theta, \varphi)\) be a random dynamical system fulfilling Hypothesis \((H)\), \((Q_x)_{x \in M}\) the family of measures given by Proposition 3.2 and \( \nu \) the unique quasi-ergodic measure of \((\theta, \varphi)\) on \( M \) given by Hypothesis \((H1)\).

In this section we prove that the measure
\[ Q_x := \int_M Q_x \nu(dx), \]
on \( \Omega \times M \) is an ergodic measure for the skew product \((\Theta_t)_{t \in T}\) of the random dynamical system \((\theta, \varphi)\).

**Lemma 3.5.** Let \((\theta, \varphi)\) be a random dynamical system fulfilling Hypothesis \((H)\), and let \((\Theta_t)_{t \in T}\) be its skew-product. Then
\[ (\mathbb{P} \times \mu) \left( \Theta_t^{-1}(C) \right) = e^{-\beta t} (\mathbb{P} \times \mu)(C), \forall t \geq 0 \text{ and } C \in G, \]
where
\[ \Theta_t(\omega, x) = (\theta_t(\omega), \varphi_t(\omega, x)). \]

**Proof.** Fix \( s \geq 0 \). Consider \( A \times B \in \mathcal{F}_s \otimes \mathcal{B} \), then
\[ (\mathbb{P} \times \mu) \left( \Theta_t^{-1}(A \times B) \right) = \int_{\Theta_t^{-1}(A \times B)} \mathbb{1}_{A \times B}(\Theta_t(\omega, x))(\mathbb{P} \times \mu)(d\omega \times dx) \]
\[ = \int_{\Theta_t^{-1}(A \times B)} \mathbb{1}_{A \times B}(\theta_t(\omega), \varphi_t(\omega, x))(\mathbb{P} \times \mu)(d\omega \times dx) \]
\[ = \int_M \mathbb{E}[\mathbb{1}_A \circ \theta_t(\cdot) \mathbb{1}_B \circ \varphi_t(\cdot, x)] \mu(dx) \]
\[ = \int_M \mathbb{P}[A] \mathbb{E}[\mathbb{1}_B \circ \varphi_t(\cdot, x)] \mu(dx) \]
\[ = \mathbb{P}[A] \int_M P^t(x, A) \mu(dx) = e^{-\beta t} (\mathbb{P} \times \mu)(A \times B). \]
By the monotone class theorem, this implies \( \sigma \left( \left\{ A \times B; \ A \in \bigcup_{s \geq 0} \mathcal{F}_s, \ B \in \mathcal{B}(M) \right\} \right) = \mathcal{G} \),

we obtain from the monotone class theorem that

\[
(\mathbb{P} \times \mu) \left( \Theta_t^{-1}(C) \right) = e^{-\beta t} (\mathbb{P} \times \mu) (C), \quad \forall \ t \geq 0 \text{ and } C \in \mathcal{G},
\]

which finishes the proof.

In the following we prove Proposition 2.6.

**Proof of Proposition 2.6.** We show that the measure \( Q_\nu \) is strongly mixing under \( \Theta \), i.e. that we have

\[
(3.9) \quad \lim_{t \to \infty} Q_\nu(A_1 \cap \Theta_t^{-1}(A_2)) = Q_\nu(A_1)Q_\nu(A_2),
\]

for all \( A_1, A_2 \in \mathcal{G} \). In particular, this implies both ergodicity and invariance.

**Step 1.** We prove that (3.9) holds in the case where \( A_1 \in \mathcal{G}_s \) and \( A_2 \in \mathcal{G}_r \).

Under these additional assumptions we have

\[
\lim_{t \to \infty} Q_\nu(A_1 \cap \Theta_t^{-1}(A_2)) = \lim_{t \to \infty} Q_\nu(A_1 \cap \Theta_s^{-1} \Theta_{t-s}^{-1}(A_2))
\]

\[
\overset{\text{Lem. 3.3}}{=} \lim_{t \to \infty} \int_M Q_x(\Theta_{(t-s)}^{-1}(A_2)) \ Q_\nu(A_1 \cap \{ \varphi_s \in dx \})
\]

\[
\overset{\text{Dom. Conv. Thm.}}{=} \int_M \left( \lim_{t \to \infty} Q_x(\Theta_{(t-s)}^{-1}(A_2)) \right) Q_\nu(A_1 \cap \{ \varphi_s \in dx \})
\]

\[
\overset{\text{Lem. 3.3}}{=} \int_M \left( \lim_{t \to \infty} \int_M Q_y(A_2) \ Q_x(\varphi_{t-s}(.x) \in dy) \right) Q_\nu(A_1 \cap \{ \varphi_s \in dx \})
\]

\[
\overset{\text{Prop. 2.5}}{=} \int_M \left( \int_M Q_y(A_2) \nu(dy) \right) Q_\nu(A_1 \cap \{ \varphi_s \in dx \})
\]

\[
= Q_\nu(A_2) \int_M Q_\nu(A_1 \cap \{ \varphi_s \in dx \})
\]

\[
= Q_\nu(A_1)Q_\nu(A_2).
\]

**Step 2.** We prove that (3.9) holds for every \( A_1, A_2 \in \mathcal{G} \).

For each \( t \geq 0 \) and \( A_1 \in \mathcal{G}_s \) we define a subfamily \( \Sigma(A_1) \subset \mathcal{G} \) by

\[
\Sigma(A_1) := \left\{ A_2 \in \mathcal{G} : \lim_{t \to \infty} Q_\nu(A_1 \cap \Theta_t^{-1}(A_2)) = Q_\nu(A_1)Q_\nu(A_2) \right\}.
\]

It is easy to verify that \( \Sigma(A_1) \) is a monotone class. Also, Step 1 implies that, whenever there exist an \( s \geq 0 \) such that \( A_1 \in \mathcal{G}_s \), we have

\[
\bigcup_{r \geq 0} \mathcal{G}_r \subset \Sigma(A_1).
\]

By the monotone class theorem, this implies \( \Sigma(A_1) = \mathcal{G} \). Now we can define another subfamily \( \Sigma \subset \mathcal{G} \) by

\[
\Sigma = \left\{ A_1 \in \mathcal{G} : \Sigma(A_1) = \mathcal{G} \right\}.
\]

By the same procedure, we can show \( \Sigma = \mathcal{G} \), which concludes the proof.
3.2. The multiplicative ergodic theorem for the Q-process.

Proof of Theorems 2.7 and 2.8. We adapt the proof of [1, Theorem 4.2.6] to one-sided time. For more details, we refer to [1, Chapters 3 & 4]. Let \( \langle \cdot, \cdot \rangle_x \) denote the Riemannian structure of \( M \). Then there exists a global trivialisation \( \psi : TM \to M \times \mathbb{R}^d \) (here in the sense of a bimeasurable bijection) such that for all \( x \in M \)

\[
\psi_x = \psi_{|T_xM} : (T_xM, \langle \cdot, \cdot \rangle_x ) \to (\mathbb{R}^d, \langle \cdot, \cdot \rangle )
\]

is an isometry (see [1, Lemma 4.2.4]). Furthermore, let \( \Phi \) is an isometry (see [1, Lemma 4.2.4]). Furthermore, let

\[
\Phi_t((\omega, x), v) : T \times (\Omega \times M) \times \mathbb{R}^d \to \mathbb{R}^d
\]

\[
(t, (\omega, x), v) \mapsto \Phi_t((\omega, x), v) := \psi_{\varphi_t(\omega, x)} \circ \Phi_t(\omega, x) \circ \psi_x^{-1}(v).
\]

Then \( \Phi \) forms a linear cocycle over the ergodic DS \( (\Omega \times X, \mathcal{F} \otimes \mathcal{B}(M), \mathbb{Q}_\nu, (\Theta_t)_{t \in T}) \). Hence, we can apply the one-sided time versions of Furstenberg-Kesten theorem [1, Theorem 3.3.3] and the multiplicative ergodic theorem [1, Theorem 3.4.1] for linear cocycles on \( \Phi \). Since \( \Phi \) is Lyapunov cohomologous to \( \Phi \), this yields the desired result.

Observe that we can choose \( \Delta \) as subset of \( \Xi \), since \( \Xi \) is a full measure \( \Theta \)-forward-invariant set. □

Another characterisation of the Lyapunov exponent is obtained via the growth rates of \( k \)-volume forms along the trajectories of \( \varphi \). In more detail, for \( k \leq d \), \( (\omega, x) \in \Delta \), and \( v_1, \ldots, v_k \in T_x M \) linearly independent,

\[
\lambda(k) := \Lambda_1 + \cdots + \Lambda_k = \lim_{t \to \infty} \frac{1}{t} \log \mathrm{Vol}(\Phi_t(\omega, x)v_1, \ldots, \Phi_t(\omega, x)v_k)
\]

where \( \mathrm{Vol}(u_1, \ldots, u_k) \) denotes the volume of the parallelepiped spanned by \( u_1, \ldots, u_k \). For \( v = v_1 \wedge \cdots \wedge v_k \in \wedge^k(T_x M) \), one can compute this volume as

\[
\left\| \wedge^k \Phi_t(\omega, x)v \right\| = \left\| \Phi_t(\omega, x)v_1 \wedge \cdots \wedge \Phi_t(\omega, x)v_k \right\| = \mathrm{Vol}(\Phi_t(\omega, x)v_1, \ldots, \Phi_t(\omega, x)v_k)
\]

where \( \| \cdot \| \) is induced by the inner product on \( \wedge^k(T_x M) \) defined on \( \wedge^k_0(T_x M) \) as

\[
\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle := \det(\langle u_i, v_j \rangle)_{k \times k}.
\]

This motivates the introduction of

\[
(3.10) \quad r^k_t(\omega, x, v) := \left\| \wedge^k \Phi_t(\omega, x)v \right\| \quad \text{and} \quad s^k_t(\omega, x, v) = \frac{\wedge^k \Phi_t(\omega, x)v}{\left\| \wedge^k \Phi_t(\omega, x)v \right\|} \in \mathrm{Gr}_k(T_x M),
\]

where \( \mathrm{Gr}_k(T_x M) \) denotes the Grassmannian manifold \( \mathbb{P}(\wedge^k_0 T_x M) \), which is a \( (d-k) \)-dimensional submanifold of the projective space \( \mathbb{P}(\wedge^k T_x M) \) (Plücker embedding [26, Page 209]). \( \mathrm{Gr}_k(T_x M) \) can also be thought as the space of \( k \)-dimensional subspaces of \( T_x M \). Note that, here we have implicitly identified antipodal points as they achieve the same Lyapunov exponents.

One cannot, in general, equip \( \mathrm{Gr}_k(T_x M) \) with a Lebesgue measure. Hence we endow it with a measure \( \sigma^k \) in the most meaningful way. This is done as follows: from [4, p. 68, Example (5)], one can identify \( \mathrm{Gr}_k(T_x M) \) to the homogeneous space

\[
O_m(T_x M) / (O_k(T_x M) \times O_{m-k}(T_x M)),
\]

where

\[
O_m(T_x M) := \{ A \in \mathrm{End}(T_x M) : \langle Av, Aw \rangle_x = \langle v, w \rangle_x \text{ for all } v, w \in T_x M \},
\]

This motivates the introduction of

\[
(3.10) \quad r^k_t(\omega, x, v) := \left\| \wedge^k \Phi_t(\omega, x)v \right\| \quad \text{and} \quad s^k_t(\omega, x, v) = \frac{\wedge^k \Phi_t(\omega, x)v}{\left\| \wedge^k \Phi_t(\omega, x)v \right\|} \in \mathrm{Gr}_k(T_x M),
\]

where \( \mathrm{Gr}_k(T_x M) \) denotes the Grassmannian manifold \( \mathbb{P}(\wedge^k_0 T_x M) \), which is a \( (d-k) \)-dimensional submanifold of the projective space \( \mathbb{P}(\wedge^k T_x M) \) (Plücker embedding [26, Page 209]). \( \mathrm{Gr}_k(T_x M) \) can also be thought as the space of \( k \)-dimensional subspaces of \( T_x M \). Note that, here we have implicitly identified antipodal points as they achieve the same Lyapunov exponents.

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\[
O_m(T_x M) / (O_k(T_x M) \times O_{m-k}(T_x M)),
\]

where

\[
O_m(T_x M) := \{ A \in \mathrm{End}(T_x M) : \langle Av, Aw \rangle_x = \langle v, w \rangle_x \text{ for all } v, w \in T_x M \},
\]
Finally, we can define the natural Borel measure \( \sigma \)

\[
\sigma = \left\{ A \in O_m(T_xM) \mid A(\text{span}\{e_1, \ldots, e_k\}) = \text{span}\{e_1, \ldots, e_k\} \text{ and } A(\text{span}\{e_k, \ldots, e_m\}) = \text{span}\{e_k, \ldots, e_m\} \right\},
\]

where \( \langle \cdot, \cdot \rangle_x \) is again the inner product on \( T_xM \) induced by the Riemannian structure of \( M \).

We have that for every \( x \in M \) the manifold \( O_m(T_xM) \) is a Lie Group ([4, p. 10, Example (8)]). Therefore, there exists a unique Haar probability measure \( \sigma_x^k \) on \( O_m(T_xM) \), i.e., the unique Borel probability measure on \( O_m(T_xM) \) that is invariant under all rotations of \( T_xM \) (see [4, Theorem 2.4]). Finally, we can define the natural Borel measure \( \sigma^k \) on \( Gr_k(T_xM) \) by

\[
\sigma^k(\cdot) = \sigma_x^k(\pi^{-1}(\cdot)),
\]

where \( \pi : (O_m(T_xM) \rightarrow O_m(T_xM)/O_k(T_xM)) \times O_{m-k}(T_xM) \), is the canonical projection.

Denote the \( k\)th Grassmannian bundle \( Gr_k(M) \) as the fiber bundle whose fiber at \( x \in M \) is \( Gr_k(T_xM) \), i.e.

\[
Gr_k(M) := \bigcup_{x \in M} Gr_k(T_xM).
\]

**Remark 3.6.** In the case that \( M \) is an open subset of \( \mathbb{R}^m \), we have that \( Gr_k(M) \cong M \times Gr_k(\mathbb{R}^d) \) as fiber bundle, which implies that every \( \{\sigma^k_x, x \in M\} \) can be canonically identified with a single measure \( \sigma^k \).

This gives us the following corollary as a consequence of the multiplicative ergodic theorem.

**Corollary 3.7.** For \( k \leq d \), let \( \Lambda_1, \ldots, \Lambda_k \) be as above and let \( \sigma^k_x \) be the canonical measure on \( Gr_k(T_xM) \) above. Then for \( Q_\rho \)-almost every \( (\omega, x) \in \Omega \times M \), \( \sigma^k_x \)-almost every \( v \in Gr_k(T_xM) \)

\[
\lim_{t \to \infty} \frac{1}{t} \log r^k_t(\omega, x, v) = \lim_{t \to \infty} \frac{1}{t} \log \|A^k \Phi_t(\omega, x)v\| = \Lambda_1 + \cdots + \Lambda_k.
\]

**Proof.** The proof follows in the exact same way as the one of [5, Corollary 2.1], replacing the measure \( \rho \otimes \mathbb{P} \) defined above [5, Theorem 2.1] by the measure \( Q_\rho \).

Equivalentlty, introducing the Borel measure \( \mathcal{V}^k \) on \( \Omega \times Gr_k(M) \) such that

\[
\mathcal{V}^k(A \times B) = \int_{A \times M} \sigma^k_x(B \cap Gr_k(T_xM)) Q_\rho(d\omega, dx), \quad \forall A \times B \in \mathcal{F} \otimes \mathcal{B}(Gr_k(M))
\]

we have

\[
\lim_{t \to \infty} \frac{1}{t} \log r^k_t(\omega, x, v) = \Lambda_1 + \cdots + \Lambda_k \quad \mathcal{V}^k\text{-almost surely}.
\]

This finishes the proof. \( \square \)

**Remark 3.8.** Observe that \( r^k_t(\omega, x, v) \) was originally defined on (3.10) just for values of \( v \in \bigwedge^k_0(T_xM) \). However it is clear that if \( v_1, v_2 \in \bigwedge^k_0(T_xM) \) are linearly dependent then

\[
\lim_{t \to \infty} \frac{1}{t} \log r^k_t(\omega, x, v_1) = \lim_{t \to \infty} \frac{1}{t} \log r^k_t(\omega, x, v_2),
\]

implying that the limit (3.12) is well defined for \( v \in \bigwedge^k(T_xM) \).

We can now prove Proposition 2.9.
Proof of Proposition 2.9. We use estimates from the proof of [1, Theorem 3.3.3]. Observe that for every \((x, \omega) \in \Omega \times M\) and \(v \in \text{Gr}_k(\mathbb{R}^d)\) we obtain
\[
(3.13) \quad |\log \| \Lambda^k \Phi_t v \| | \leq \max \left\{ |\log \| \Lambda^k \Phi_t \| |, |\log \| \Lambda^k \Phi_{t-1}^{-1} \| | \right\}.
\]
But by [1, Theorem 3.3.3, Proof of Part (B)(b)], subbaditivity and \(\Theta_t\)-invariance of \(Q_{\nu}\), we observe
\[
\frac{1}{t} \mathbb{E}_{\nu}^Q \left[ |\log \| \Lambda^k \Phi_{t}^{\pm 1} \| | \right] \leq \frac{1}{t} \sum_{n=0}^{[t]} \mathbb{E}_{\nu}^Q \left[ \sup_{0 \leq \ell \leq 1} |\log \| \Lambda^k \Phi_{\ell}^{\pm 1} \circ \Theta_n \| | \right]
\leq 2 \mathbb{E}_{\nu}^Q \left[ \sup_{0 \leq \ell \leq 1} |\log \| \Lambda^k \Phi_{\ell}^{\pm 1} \| | \right]
\leq 2k \left( \mathbb{E}_{\nu}^Q \left[ \sup_{0 \leq \ell \leq 1} \log^+ \| \Phi_{\ell} \| + \sup_{0 \leq \ell \leq 1} \log^+ \| \Phi_{\ell}^{-1} \| \right] \right) < \infty,
\]
where we have used the integrability conditions (2.7) and (2.9). Now, combining the above equation (3.13), Proposition 2.6, Corollary 3.7 and [37, Part Three-Theorem 4.18], we obtain the existence of a set \(\tilde{G} \subset \text{Gr}_k(M)\) such that \(\rho^k(\tilde{G}) = 1\) and
\[
\lim_{t \to \infty} \mathbb{E}_{\nu}^Q \left[ \lambda^{(k)}(\cdot) - \frac{1}{t} \log \| \Lambda^k \Phi_t v \| \right] = 0 \quad \text{for all } (x, v) \in \tilde{G}.
\]
This finishes the proof of this proposition. \(\square\)
4. Convergence of finite-time Lyapunov exponents

We start this section with the proof of Theorem 2.10.

Proof of Theorem 2.10. We divide the proof in five steps.

Step 1. We show that if there exists \( p \in (1, \infty] \) such that
\[
\sup_{t \geq 0} E_x [ |\Gamma_t|^p \mid \tau > t] < \infty,
\]
then
\[
\lim_{a \to \infty} \sup_{t \geq 0} E_x [ \mathbb{1}_{|\Gamma_t| > a} |\Gamma_t| \mid \tau > t] = 0.
\]

It should be noted that the above step can be viewed as a conditioned version of the de la Vallée Poussin principle [29].

By a direct computation, we obtain
\[
|\Gamma_t|^p \geq a^{-1} \mathbb{1}_{|\Gamma_t| > a} |\Gamma_t|
\]
and, thus,
\[
\lim_{a \to \infty} \sup_{t \geq 0} E_x [ \mathbb{1}_{|\Gamma_t| > a} |\Gamma_t| \mid \tau > t] \leq \lim_{a \to \infty} a^{1-p} \sup_{t \geq 0} E_x [ |\Gamma_t|^p \mid \tau > t] = 0.
\]

Step 2. We show that if \( \{\Gamma_t\}_{t \geq 0} \) is a family of random variables fulfilling the assumptions in \((ii)\) and
\[
\sup_{t \geq 0} \| \mathbb{1}_{\{\tau > t\}} \Gamma_t \|_{L^\infty(\Omega \times M, P_x)} < \infty,
\]
then
\[
\lim_{t \to \infty} E_x [ |\Gamma_t - \Gamma^*| \mid \tau > t] = 0.
\]

Since \( |\Gamma_t - \Gamma^*| \) is a \( G_t \)-random variable, Proposition 3.2 implies
\[
\lim_{t \to \infty} \frac{e^{\beta t}}{\eta(x)} E_x [ |\Gamma_t - \Gamma^*| \cdot \eta \circ \varphi_t] = \lim_{t \to \infty} Q_x [ |\Gamma_t - \Gamma^*|] = 0.
\]

Since for every \( x \in M \)
\[
\lim_{t \to \infty} \frac{e^{\beta t}}{\eta(x)} P_x [\tau > t] = 1,
\]
we obtain
\[
E_x [ |\Gamma_t - \Gamma^*| \cdot \eta \circ \varphi_t \mid \tau > t] = 0.
\]

Moreover, given \( \delta > 0 \) and \( t \in T \),
\[
\delta \cdot \mathbb{1}_{\{\eta > \delta\}} \circ \varphi_t \cdot \mathbb{1}_{\{\tau > t\}} \leq \mathbb{1}_{\{\tau > t\}} = \mathbb{1}_{\{\tau > t\}}.
\]

Combining equations (4.2)-(4.3) yields
\[
\lim_{t \to \infty} E_x [ |\Gamma_t - \Gamma^*| \cdot \mathbb{1}_{\{\eta > \delta\}} \circ \varphi_t \mid \tau > t] \leq \frac{1}{\delta} \lim_{t \to \infty} E_x [ |\Gamma_t - \Gamma^*| \cdot \eta \circ \varphi_t \mid \tau > t] = 0.
\]

Observe that, defining
\[
K = |\Gamma^*| + \sup_{t \geq 0} \| \mathbb{1}_{\{\tau > t\}} \Gamma_t \|_{L^\infty(\Omega \times M, P_x)} < \infty,
\]
we obtain
\[ E_x \left[ |\Gamma_t - \Gamma^*| \cdot 1_{\{\eta \leq \delta\}} \circ \varphi_t \mid \tau > t \right] \leq K E_x \left[ 1_{\{\eta \leq \delta\}} \circ \varphi_t \mid \tau > t \right] . \]

The above equation implies that
\[ \lim_{t \to \infty} E_x \left[ |\Gamma_t - \Gamma^*| \cdot 1_{\{\eta \leq \delta\}} \circ \varphi_t \mid \tau > t \right] = K \mu(\{ \eta \leq \delta \}) . \]

This implies that for every \( \delta > 0 \),
\[ 0 \leq \limsup_{t \to \infty} E_x \left[ |\Gamma_t - \Gamma^*| \mid \tau > t \right] = \limsup_{t \to \infty} \left\{ E_x \left[ |\Gamma_t - \Gamma^*| \cdot 1_{\{\eta > \delta\}} \circ \varphi_t \mid \tau > t \right] + E_x \left[ |\Gamma_t - \Gamma^*| \cdot 1_{\{\eta \leq \delta\}} \circ \varphi_t \mid \tau > t \right] \right\} \]
\[ \leq K \mu(\{ \eta \leq \delta \}) . \]

Since \( \delta \) is arbitrary small and
\[ \mu(\{ \eta = 0 \}) = 0 , \]
we obtain
\[ \lim_{t \to \infty} E_x \left[ |\Gamma_t - \Gamma^*| \mid \tau > t \right] = 0 . \]

**Step 3.** Let \( \{ \Gamma_t \}_{t \geq 0} \) be a sequence of random variables fulfilling the assumptions of (ii). Define \( \Gamma^{(a)}_t := 1_{|\Gamma_t| \leq a} \Gamma_t \). We show that for each \( a \geq 2 |\Gamma^*| \) we have
\begin{equation}
(4.4) \lim_{t \to \infty} E_x \left[ |\Gamma_t^{(a)} - \Gamma^*| \mid \tau > t \right] = 0 .
\end{equation}

First observe that the restriction \( a \geq 2 |\Gamma^*| \) implies \( |\Gamma_t^{(a)} - \Gamma^*| \leq |\Gamma_t - \Gamma^*| \) and thus by (2.10) we also have
\[ \lim_{t \to \infty} E_x \left[ |\Gamma_t^{(a)} - \Gamma^*| \right] = 0 . \]

Furthermore, since \( |\Gamma_t^{(a)} - \Gamma^*| \leq a + |\Gamma^*| \), equation 4.1 is fulfilled and from Step 2 we obtain
\[ \lim_{t \to \infty} E_x \left[ |\Gamma_t^{(a)} - \Gamma^*| \mid \tau > t \right] = 0 . \]

**Step 4.** We show (ii).

Observe that for each \( a \geq 2 |\Gamma^*| \) we have
\[ \limsup_{t \to \infty} E_x \left[ |\Gamma_t - \Gamma^*| \mid \tau > t \right] \leq \limsup_{t \to \infty} \left( E_x \left[ |\Gamma_t^{(a)} - \Gamma^*| \mid \tau > t \right] + E_x \left[ |\Gamma_t - \Gamma_t^{(a)}| \mid \tau > t \right] \right) \]
\[ = \limsup_{t \to \infty} E_x \left[ 1_{|\Gamma_t| > a} |\Gamma_t| \mid \tau > t \right] \]
\[ \leq \sup_{t \geq 0} E_x \left[ 1_{|\Gamma_t| > a} |\Gamma_t| \mid \tau > t \right] . \]

Considering \( a \to \infty \) yields
\[ \lim_{t \to \infty} E_x \left[ |\Gamma_t - \Gamma^*| \mid \tau > t \right] = 0 , \]
completing the proof.

**Step 5.** We show (i).
Given \( \varepsilon > 0 \), consider the family of random variables 
\[
\{ \tilde{\Gamma}_t = 1(\{ |\Gamma_t - \Gamma^*| > \varepsilon \}) \}_{t \geq 0},
\]
and define \( \tilde{\Gamma}^* = 0 \). From Step 4, we obtain 
\[
\lim_{n \to \infty} \mathbb{P}_x \left[ |\Gamma_t - \tilde{\Gamma}^*| > \varepsilon \mid \tau > t \right] = \lim_{n \to \infty} \mathbb{E}_x \left[ |\tilde{\Gamma}_t - \tilde{\Gamma}^*| \mid \tau > t \right] = 0,
\]
proving the theorem. \( \square \)

In the following we use Theorem 2.10 to prove Theorems 2.11 and 2.13.

**Proof of Theorem 2.11.** Let \( \varepsilon > 0 \). From Corollary 3.7, we obtain that for \( \rho^k \)-almost every \((x, v) \in \text{Gr}_k(M)\),
\[
\lim_{t \to \infty} \mathbb{Q}_x \left\{ \left( \lambda^{(k)} - \frac{1}{t} \log \left\| \Lambda^k \Phi_t v \right\| \right) > \varepsilon \right\} = 0.
\]

Let us define \( \Gamma^* = \lambda^{(k)} \) and for every \( t \geq 0 \) the \( \mathcal{G}_t \)-measurable random variable
\[
\Gamma_t = \frac{1}{t} \log \left\| \Lambda^k \Phi_t v \right\|.
\]

Applying Theorem 2.10 to the family \( \{ \Gamma_t \}_{t \geq 0} \), we immediately obtain the desired result and similarly to show the convergence (2.12). \( \square \)

**Proof of Theorem 2.13.** From Proposition 2.9, we immediately get that there exists a subset \( \widetilde{G} \subset \text{Gr}_k(M) \) of full \( \rho^k \)-measure such that for all \((x, v) \in \widetilde{G}\),
\[
\lim_{t \to \infty} \mathbb{E}_x^{\widetilde{G}} \left( \left( \lambda^{(k)} - \frac{1}{t} \log \left\| \frac{1}{t} \Phi_t v \right\| \right) > \varepsilon \right\} = 0.
\]

Therefore, taking \( \Gamma^* = \lambda^{(k)} \) and the \( \mathcal{G}_t \)-measurable random variable
\[
\Gamma_t = \frac{1}{t} \log \left\| \Lambda^k \Phi_t (\cdot, x)v \right\|,
\]
in combination with [1, Theorem 3.3.3, Proof of Part (B)(b)] and (2.14), yields
\[
\sup_{t \geq 0} \|\Gamma_t\|_{L^p(\Omega \times M, \mathbb{P}_x(\cdot | \tau > t))} < \infty, \text{ for } \nu\text{-almost every } x \in M.
\]

Applying Theorem 2.10 immediately yields the desired result and similarly for (2.16). This completes the proof of the theorem. \( \square \)
5. Application to Stochastic flows

Let us recall that we work on the probability space

\[(Ω, (F_t)_{t≥0}, F, P) = \left(C_0(\mathbb{R}_+, \mathbb{R}^m), (σ(π_s, 0 ≤ s ≤ t))_{t≥0}, σ(π_s, s ≥ 0), P\right),\]

as in Eq. (3.1) where P is the Wiener measure.

Let us assume that equation (2.17) satisfies Hypothesis (H\text{SDE}). By [27, Theorem 1.2.9.], then for each initial condition \(x ∈ M\), there exists a unique solution \(ϕ_t(·, x) : \mathbb{R}_+ × Ω → E_M = M ∪ \{∂\}\) of the stochastic differential equation

\[dX_t = V_0(X_t)dt + \sum_{i=1}^{m} V_i(X_t) ◦ dW^i_t, \quad X_0 = x,\]

on \(M\) up until explosion, i.e. defining the stopping time

\[τ(ω, x) = \min\{t ≥ 0 : ϕ_t(ω, x) \not∈ M\}.\]

The stochastic process \(ϕ_t(·, x)\) fulfills the following conditions

(i) \(ϕ_0(·, x) = x\);
(ii) \(ϕ_t(·, x) = \int_0^t V_0(ϕ_s(·, x))ds + \sum_{i=1}^{m} \int_0^t V_i(ϕ(·, x))ds, \quad ∀ 0 ≤ t ≤ τ(·, x);\)
(iii) for every \(t ≥ 0\), \(ϕ_t(·, x)\) is \(F_t\)-measurable;
(iv) \(ϕ_t(·, x) = ∂, \quad \text{for every } τ(·, x) ≤ t;\)
(v) for every \(ω ∈ Ω\), the path \(t ↦ ϕ_t(ω, x)\) is continuous on \(E_M\);
(vi) if \(Y_t : Ω → E_M\) is another stochastic process satisfying conditions (i)–(v) then

\[P[Y_t = ϕ_t(·, x), \quad ∀ t ≥ 0] = 1.\]

Thus, this defines a stochastic flow which we denote by the same symbol \(ϕ\)

\[ϕ : \mathbb{R}_+ × Ω × E_M → E_M\]

\[(t, ω, x) ↦ ϕ_t(ω, x).\]

From [1, Chapter 2.3] and [31, Proposition 2.5], we may assume without loss of generality that \(ϕ\) forms a perfect cocycle (by taking a modification, see Remark 3.4), i.e.

\[ϕ_{t+s}(ω, x) = ϕ_t(θ_sω, ϕ_s(ω, x)), \quad \text{for every } s, t ≥ 0,\]

where

\[θ_t : Ω → Ω\]

\[ω ↦ ω(t + ·) − ω(t).\]

With the above notation, we say that \((θ, ϕ)\) is the random dynamical system induced by the stochastic differential equation (2.17).

From [1, Theorem 2.3.32], we obtain that the linearised flow \(Φ_t(ω, x) := Dϕ_t(ω, x)\) solves the stochastic differential equation

\[\begin{align*}
    dΦ_t &= DV_0(ϕ_t)Φ_t dt + \sum_{i=1}^{m} DV_i(ϕ_t)Φ_t ◦ dW^i_t, \quad ∀ 0 ≤ t ≤ τ, \\
    Φ_0 &= \text{Id}.
\end{align*}\]
Fixing $k \leq d$ and $v = v_1 \wedge \cdots \wedge v_k \in \text{Gr}_k(T_xM) \simeq \text{Gr}_k(\mathbb{R}^d)$, recall that we denote
\[
\rho^k_t(\omega, x, v) := \left\| \bigwedge^k \Phi_t(\omega, x)v \right\| \quad \text{and} \quad \sigma^k_t(\omega, x, v) = \frac{\bigwedge^k \Phi_t(\omega, x)v}{\left\| \bigwedge^k \Phi_t(\omega, x)v \right\|} \in \text{Gr}_k(T_xM)
\]

From [5, Theorem 3.1], there exist continuous (and hence bounded) functions $\psi^k : \text{Gr}_k(M) \to \mathbb{R}$ and $\phi^k_t : \text{Gr}_k(M) \to \mathbb{R}$ for $t \in \{1, \ldots, m\}$ such that
\[
(5.2) \quad d \left( \log \rho^k_t \right) = \psi^k_t(\varphi_t, \sigma^k_t) dt + \sum_{i=1}^{m} \phi^k_t(\varphi_t, \sigma^k_t) dW^i_t, \quad \forall \ 0 \leq t \leq \tau(\cdot, x).
\]
The Formulae for these functions were derived in [5] and are given by
\[
\psi^k_t(x, s) := \text{tr}(V_0'(x)P_s) + \sum_{i=1}^{m} \left\{ \text{tr}(V_i'(x)V_i(x)P_s) - \text{tr}(V_i'(x)(I - P_s)V_i'(x)P_s) \right\}
\]
and
\[
\phi^k_t(x, s) := \text{tr}(V_i'(x)P_s)
\]
where $P_s$ denotes the projection onto the subspace $s \in \text{Gr}_k(\mathbb{R}^d)$. We can now prove Theorem 2.15.

**Proof of Theorem 2.15.** From [13, Chapter 3.2] (see also [21, Theorem 6.1.7]) it is clear that the random dynamical system $(\theta, \varphi)$ associated to (2.18) satisfies Hypothesis (H). In the remainder of this proof, in the interest of readability, we fix $k \leq d$ and drop this superscript.

It suffices for us to show that the integrability conditions (2.7) and (2.9) are fulfilled allowing us to apply Theorem 2.7. Moreover, for all $T > 0$ and for every $k \in \{1, \ldots, d\}$,
\[
(5.3) \quad \sup_{0 \leq t \leq T} \log \left\| \bigwedge^k \Phi_t \right\|, \quad \sup_{0 \leq t \leq T} \log \left\| \bigwedge^k \Phi_t^{-1} \right\| \in L^1(\Omega \times M, \mathcal{F} \otimes \mathcal{B}(M), \mathbb{Q}_t),
\]

We follow ideas of [1, Remark 6.2.12]. First note that for $A \in GL(d, \mathbb{R})$
\[
\frac{\left\| \left( \bigwedge^k A \right)(e_{i_1} \wedge \cdots \wedge e_{i_k}) \right\|}{\left\| e_{i_1} \wedge \cdots \wedge e_{i_k} \right\|} \leq \left\| \bigwedge^k A \right\| \leq \max_{1 \leq i_1 \leq \cdots i_{k} \leq d} \frac{\left\| \left( \bigwedge^k A \right)(e_{i_1} \wedge \cdots \wedge e_{i_k}) \right\|}{\left\| e_{i_1} \wedge \cdots \wedge e_{i_k} \right\|}.
\]

So, for the first integrability condition, it suffices for us to prove that for $v \in \text{Gr}_k(\mathbb{R}^d)$,
\[
(5.4) \quad \mathbb{E}^\nu_0 \left[ \sup_{0 \leq t \leq T} \log \rho_t(\cdot, \cdot, v) \right] < \infty
\]
But for $t \leq T$,
\[
(5.5) \quad \log \rho_t = \int_0^t \psi(\varphi_t, s_t) d\ell + \sum_{i=1}^{d} \int_0^t \phi_i(\varphi_t, s_t) dW^i_t \leq t \|\psi\|_\infty + \sum_{i=1}^{d} \int_0^t \phi_i(\varphi_t, s_t) dW^i_t
\]
Now for $i \in \{1, \ldots, d\}$
\[
\mathbb{E}^\nu_0 \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \phi_i(\varphi_t, s_t) dW^i_t \right| \right] = \mathbb{E}_M \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \phi_i(\varphi_t, s_t) dW^i_t \right| \right] \nu(dx)
\]
\[
\leq \int_M \mathbb{E}_x \left[ \eta(x) \sup_{0 \leq t \leq T} \left| \int_0^t \phi_i(\varphi_t, s_t) dW^i_t \right| 1_{\{\tau > T\}} \right] \eta(x) \nu(dx)
\]

more, achieved by following ideas of Baxendale [5] below. A remarka
table aspect of these formulae is to compute the top Lyapunov exponent and provide an equivalent for lower exponents. This is, once  

\begin{equation}
(5.6)
\end{equation}

where we use Hölder’s inequality in the last step. Now observe that \( \int_{0}^{t_{\tau}} \phi_{i}(\varphi_{t}, s_{t}) dW_{t} \) is a stopped martingale. Therefore, by the Burkholder-Davis-Gundy inequality [35, Chapter IV, Corollary 4.2],

\[
\left( \mathbb{E}_{x} \left[ \sup_{0 \leq t \leq T} \int_{0}^{t_{\tau}} \phi_{i}(\varphi_{t}, s_{t}) dW_{t} \right] \right)^{1/2} \leq 2T^{1/2} \| \phi_{i} \|_{\infty}
\]

by (5.4) holds uniformly over all \( v \in \mathcal{G}_{\ell}(\mathbb{R}^{d}) \). This proves the first integrability condition of equation (5.3).

Now, observe that \( \Phi_{i}^{-1} \) solves the stochastic differential equation

\[
\begin{cases}
\frac{d\Phi_{i}^{-1}}{dt} = -D^{*}V_{0}(\varphi_{t})\Phi_{i}^{-1} dt - \sum_{i=1}^{m} D^{*}V_{i}(\varphi_{t})\Phi_{i}^{-1} \circ dW_{t}, & \forall 0 \leq t \leq \tau,
\Phi_{0}^{-1} = \text{Id}.
\end{cases}
\]

Since \( \| \Phi_{i}^{-1} \| = \| \Phi_{i}^{*^{-1}} \| \), applying the same reasoning as above proves the second integrability condition of equation (5.3).

These immediately imply integrability conditions (2.7) and (2.9). Thus, Theorems 2.7, 2.11 and 2.8 hold and yield the existence of conditioned Lyapunov exponents \( \Lambda_{1} \geq \cdots \geq \Lambda_{d} > -\infty \).

\[
\lim_{t \to \infty} \mathbb{E}_{\nu}^{Q} \left[ \lambda^{(k)} - \frac{1}{t} \log \| \Lambda^{k} \Phi_{t} \| \right] = 0
\]

Furthermore, by an application of the subadditive ergodic theorem (similarly to [38, Chapter 3]), we have

\[
\lim_{t \to \infty} \mathbb{E}_{\nu}^{Q} \left[ (\Lambda_{d-k+1} + \cdots + \Lambda_{d}) + \frac{1}{t} \log \| \Lambda^{k} \Phi_{t}^{-1} \| \right] = 0
\]

Theorem 2.11 proves the desired convergence in conditional probability: for every \( \varepsilon > 0 \) and \( \rho^{k} \)-almost every \( (x, v) \in \mathcal{G}_{\ell}(M) \),

\[
\lim_{t \to \infty} \mathbb{P}_{x} \left[ \left\{ \lambda^{(k)} - \frac{1}{t} \log \| \Lambda^{k} \Phi_{t} v \| > \varepsilon \right\} \tau > t \right] = 0,
\]

which finishes the proof.

Furthermore, we wish to generalise the Furstenberg–Khasminskii formula given by [23] to compute the top Lyapunov exponent and provide an equivalent for lower exponents. This is, once more, achieved by following ideas of Baxendale [5] below. A remarkable aspect of these formulae is
the apparent impossibility to derive the multiplicative noise case without the use of the $Q$-process. This additionally demonstrates the usefulness of the $Q$-process to study conditioned finite-time dynamics.

**Proposition 5.1.** For $k \leq d$, if there exist unique quasi-stationary and unique quasi-ergodic measures $\mu^k$ and $\nu^k$ on $\text{Gr}_k(M)$ for the process $(\varphi_t, s^k_t)$, then

\begin{equation}
\lambda^{(k)} = \Lambda_1 + \cdots + \Lambda_k = \int_{\text{Gr}_k(M)} \psi^k d\nu^k + \sum_{i=1}^{m} \sum_{j=1}^{d} \int_{\text{Gr}_i(M)} \phi_i^k V^2_i \partial_j \eta d\mu^k
\end{equation}

and similarly for the last $k$ Lyapunov exponents. In particular,

\begin{equation}
\lambda = \Lambda_1 + \cdots + \Lambda_k = \int_{\text{Gr}_k(M)} \psi^k d\nu^k + \sum_{i=1}^{m} \sum_{j=1}^{d} \int_{\text{Gr}_i(M)} \phi_i^k V^2_i \partial_j \eta d\mu^k
\end{equation}

where $\lambda^+$, $\lambda^-$ denote the extremal Lyapunov exponents $\Lambda_1$ and $\Lambda_d$ respectively and $\psi^\pm_1$, $\phi^\pm_1$ are given by (5.3) and its analogue for the inverse linearised flow and similarly for $\nu^\pm_1$ and $\mu^\pm_1$.

**Proof.** Recall that for $\mathcal{V}$-almost every $(\omega, x, v) \in \Omega \times \text{Gr}_k(M)$,

\begin{equation}
\lambda^{(k)} = \Lambda_1 + \cdots + \Lambda_k = \lim_{t \to \infty} \frac{1}{t} \log r^k_t(\omega, x, v)
\end{equation}

Also recall, formula (5.2) below (where we again drop the superscript $k$)

\begin{equation}
d(\log r^k_t) = \psi^k(\varphi_t, s_t) dt + \sum_{i=1}^{m} \phi^k_i(\varphi_t, s_t) dW^i_t, \quad 0 \leq t \leq \tau(\cdot, x).
\end{equation}

The time-average of the first term converges by Birkhoff’s ergodic theorem

\begin{equation}
\lim_{t \to \infty} \mathbb{E}^\mathcal{Q}_x \left[ \frac{1}{t} \int_0^t \psi(\varphi_\ell, s_\ell) d\ell \right] = \int_{\text{Gr}_x(M)} \psi d\nu^k.
\end{equation}

Now for the second term, by Girsanov’s Theorem one can show the existence of a $h$-transform

\begin{equation}
dW^i_t = \sum_{j} V^j_i(\varphi_t(\cdot, x)) \partial_j (\log \eta(\varphi_t(\cdot, x))) dt + dB^i_t
\end{equation}

where $(B^i_t)_{t \geq 0}$ is a $Q_x$-standard Brownian motion. Now, on the one hand by Hölder’s inequality and Itô isometry we obtain

\begin{equation}
\left| \frac{1}{t} \mathbb{E}^\mathcal{Q}_x \left[ \int_0^t \phi_i(\varphi_\ell, s_\ell) dB^i_\ell \right] \right| \leq \frac{1}{t} \left( \mathbb{E}^\mathcal{Q}_x \left[ \int_0^t \phi_i^2(\varphi_\ell, s_\ell) d\ell \right] \right)^{1/2}
\end{equation}

\begin{equation}
\leq \frac{1}{t} \left( \mathbb{E}^\mathcal{Q}_x \left[ \int_0^t \phi_i^2(\varphi_\ell, s_\ell) d\ell \right] \right)^{1/2}
\end{equation}

\begin{equation}
\leq \frac{\|\phi_i\|_\infty}{\sqrt{t}} \to 0 \quad \text{as} \quad t \to \infty
\end{equation}

On the other hand,
\[
\mathbb{E}_\nu^Q \left[ \frac{1}{t} \int_0^t \phi_t(\varphi_t, s_t) V_i^j(\varphi_t) \partial_j (\log \eta(\varphi_t)) \, d\ell \right] \xrightarrow{\text{Birkhoff}} \lim_{t \to \infty} \int_{\text{Gr}_k(M)} \phi_t V_i^j \partial_j (\log \eta) \, d\nu^k \\
= \int_{\text{Gr}_k(M)} \phi_t V_i^j \eta \, d\mu^k.
\]

Yielding the desired result (5.7). Note that, although it explodes near the boundary \( \partial M \), the integrand \( \phi_t V_i^j \partial_j (\log \eta) \in L^1(\nu^k) \) since \( d\nu^k = \eta \, d\mu^k \).

This proposition generalises the definition of conditioned Lyapunov exponents given in [23] which treats the additive case with \( k = 1 \). The Lyapunov exponents can then be computed recursively: \( \Lambda_k = \lambda^{(k)} - \lambda^{(k-1)} \). This is particularly useful for numerical estimations of the conditioned Lyapunov exponents. Note that the process \( (\varphi_t, s_t) \) is degenerate, making the uniqueness of its quasi-stationary and quasi-ergodic measures unclear in general. Some criteria for the exponential convergence of this process to quasi-stationarity such as the Hörmander condition are discussed in [7].

A particular case of this proposition is the Liouville’s formula below.

**Corollary 5.2** (Liouville’s formula). Let \( \mu \) and \( \nu \) be the quasi-stationary and quasi-ergodic measures of \( \varphi \) on \( M \), then

\[
\lambda^{(d)} = \lim_{t \to \infty} \mathbb{E}_\nu^Q \left[ \frac{1}{t} \log \det(\Phi_t) \right] = \lim_{t \to \infty} \mathbb{E}_\nu^Q \left[ \frac{1}{t} \log \|A^d \Phi_t\| \right] = \int_M \psi^d \, d\nu + \sum_{i=1}^m \sum_{j=1}^d \int_M \phi_i^d V_i^j \partial_j \eta \, d\mu.
\]

Finally we give the corollary below as an application of Theorem 2.13 for absorbed diffusions with additive noise.

**Corollary 5.3** (Additive noise case). Let \( (\theta, \varphi) \) be as in Theorem 2.15 and assume further that the vector fields \( \{V_i\}_{i=1}^m \) are constants, i.e. \( (\theta, \varphi) \) is generated by a stochastic differential equation with additive noise

\[
(5.8) \quad dX_t = V_0(X_t) \, dt + \sum_{i=1}^m V_i \circ dW_t^i, \quad X_0 = x,
\]
on \( M \) up until explosion. Then the convergence of the finite-time Lyapunov exponents occurs in conditional expectation in the sense that for all \( k \leq d \), for \( \rho^k \)-almost every \( (x, v) \in \text{Gr}_k(M) \),

\[
(5.9) \quad \lim_{t \to \infty} \mathbb{E}_x \left[ \lambda^{(k)} - \frac{1}{t} \log \|\wedge^k \Phi_t v\| \bigg| \tau > t \right] = 0.
\]

and

\[
\lambda^{(k)} = \int_{\text{Gr}_k(M)} \langle s, \hat{A}^k \rangle \nu^k (dx, ds)
\]

where for \( A \in \text{End}(\mathbb{R}^d) \), \( \hat{A}^k \in \text{End}(\wedge^k \mathbb{R}^d) \) is defined on \( \wedge^k \mathbb{R}^d \) as

\[
\hat{A}^k(v_1 \wedge \cdots \wedge v_k) := \sum_{i=1}^k (v_1 \wedge \cdots \wedge A v_i \wedge \cdots \wedge v_k).
\]
Proof. Observe that for all $k \leq d$ and for all $i \in \{1, \ldots, m\}$, $\phi^k_i = 0$ and $\psi^k : (x,s) \mapsto \langle s, \hat{D}V_0^k(x)s \rangle$. Therefore, (5.5) directly implies that the integrability condition (2.14) is fulfilled and Theorem 2.13 yields the desired result.

\[\square\]

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Appendix A. Random Dynamical Systems

In this appendix, we recall the definition a random dynamical system. Let $T$ be equal $\mathbb{N}_0$ or $\mathbb{R}_+$. In the interest of clarity, our notations correspond to the ones of continuous time, e.g. sums over discrete time are denoted as integrals.

**Definition A.1** (Metric Dynamical System). $(\Omega, (\mathcal{F}_t)_{t \in T}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. A family of mappings $\theta = \{\theta_t : (\Omega, (\mathcal{F}_t)_{t \in T}, \mathcal{F}, \mathbb{P}) \to (\Omega, (\mathcal{F}_t)_{t \in T}, \mathcal{F}, \mathbb{P})\}_{t \in T}$ is said to be a metric dynamical system (or measure preserving DS) if it satisfies the following:

1. $(\omega, t) \mapsto \theta_t \omega$ is $(\mathcal{F} \otimes \mathcal{B}(\mathbb{T}) - \mathcal{F})$-measurable;
2. $\theta_0 = \text{id}_\Omega$;
3. Semiflow property: $\theta_{s+t} = \theta_s \circ \theta_t$ for all $s,t \in T$;
4. $\mathbb{P}$ is $\theta_t$-invariant for all $t \in T$, i.e. $(\theta_t)_* \mathbb{P} = \mathbb{P}$ where $(\theta_t)_* \mathbb{P}(A) = \mathbb{P}(\theta_t^{-1}(A))$ for all $A \in \mathcal{F}$;
5. $\theta$ is said to be a filtered DS if $\theta_t^{-1} \mathcal{F}_s \subseteq \mathcal{F}_{s+t}$ for all $s,t \in T$;
6. Furthermore, $\theta$ is said to be ergodic if for all $t \in T$, $\theta_t$-invariant sets have measure 0 or 1, i.e. for all $A \in \mathcal{F}$, $\theta_t^{-1} A = A$ implies $\mathbb{P}(A) \in \{0,1\}$.

When the context is clear, the quadruplet $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T})$ denotes a metric dynamical $\theta$. If $\theta$ is a filtered DS, $(\Omega, (\mathcal{F}_t)_{t \in T}, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T})$ might be referred to as the noise space.

We may impose the following additional condition on our noise space.

**Definition A.2** (Memoryless Noise Space). A noise space $(\Omega, (\mathcal{F}_t)_{t \in T}, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T})$ is said to be memoryless if for any $s,t \in T$, $\theta_s^{-1} \mathcal{F}_t$ are independent under $\mathbb{P}$.

We also recall the definition of a random dynamical system.

**Definition A.3.** A random dynamical system on a measurable state space $(X, \mathcal{B})$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T})$ is a mapping

$$\varphi : \mathbb{T} \times \Omega \times X \to X$$

$$(t, \omega, x) \mapsto \varphi_t(\omega, x)$$

which satisfies the following properties

1. Measurability: $\varphi$ is a $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B} - \mathcal{B})$-measurable mapping; and
2. Cocycle property: $\varphi$ forms a perfect cocycle over $\theta$, i.e. $\omega \in \Omega$
   (a) $\varphi_0(\cdot, \omega) = \text{id}_X$
   (b) $\varphi_{t+s}(\omega, x) = \varphi_t(\theta_s \omega, \varphi_s(\omega, x))$ for all $s, t \in \mathbb{T}$ and for all $x \in X$.

When the context is clear, we denote the random dynamical system $\varphi$ over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T})$ simply by $(\theta, \varphi)$. When $(\Omega, (\mathcal{F}_t)_{t \in T}, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T})$ is a memoryless noise space and $\varphi_t$ is $\mathcal{F}_t \otimes \mathcal{B}$ measurable for every $t \in T$, we say that $(\theta, \varphi)$ is a memoryless random dynamical system.

**Definition A.4** (RDS with absorption). Let $X$ be a topological state space that can be decomposed as $X = M \sqcup \{\partial\}$ where $M \subset X$ and $\{\partial\}$ denotes a so-called “cemetery” or “coffin” state. A measurable RDS $(\theta, \varphi)$ over a metric dynamical system $(\Omega, (\mathcal{F}_t)_{t \in T}, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T})$ is said to form a random dynamical system with absorption $(\theta, \varphi)$ on $X = M \sqcup \{\partial\}$ if for all $\omega \in \Omega$, $x \in X$, $\varphi_s(\omega, x) = \partial$ implies $\varphi_t(\omega, x) = \partial$ for all $t \geq s$. This justifies the definition of the following stopping time for each $x \in M$

$$\tau(\cdot, x) = \inf \{ t \geq 0 : \varphi_t(\cdot, x) = \partial \}$$

In this context, a measurable RDS $(\theta, \varphi)$ is said to be
• continuous if for all $\omega \in \Omega$, the mappings
  \[ \varphi_t(\omega, \cdot) : \{(t, x) \in \mathbb{T} \times X \mid \tau(\omega, x) > t\} \to M \]
  \[ (t, x) \mapsto \varphi_t(\omega, x) \]
  are continuous.
• If $X$ is furthermore endowed with a smooth structure, i.e. if it is a manifold, then $(\theta, \varphi)$ is said to be of class $C^k$ ($1 \leq k \leq \infty$), if for all $t \in \mathbb{T}$ and $\omega \in \Omega$, the mappings
  \[ \varphi_t(\omega) = \varphi_t(\omega, \cdot) : \{\tau(\omega, \cdot) > t\} \subset M \to M \]
  \[ x \mapsto \varphi_t(\omega, x) \]
  are $k$-times differentiable (in the sense of [30, Page 645]) where $\{\tau(\omega, \cdot) > t\} = \{x \mid \tau(\omega, \cdot) > t\}$.

Appendix B. Proof of Proposition 3.1

In this section, we give a proof of proposition 3.1. The proof below is based on the techniques developed in [15], where similar results where proven assuming that the function $C(x)$ (in (H2) of Hypothesis (H)) is constant.

Proof of Proposition 3.1. The proof is done assuming $\mathbb{T} = \mathbb{R}_+$. If $\mathbb{T} = \mathbb{N}_0$, the same proof holds with minor adaptations. We divide the proof in four steps.

Step 1. We show that for every non-negative measurable bounded function $g : M \to \mathbb{R}_+$,
\[
\lim_{t \to \infty} e^{\beta t} \mathcal{P}^t(g)(x) = \eta(x) \int_M g \, d\mu, \text{ for every } x \in M.
\]
Since $g$ is non-negative and bounded, from (H2) we obtain that for every $t \geq 0$,
\[
\left| \frac{\mathcal{P}^t(g)(x)}{\mathcal{P}^t(x, M)} - \int_M g \, d\mu \right| \leq \|g\|_\infty C(x) e^{-\alpha t},
\]
where $\|g\|_\infty := \sup_{x \in M} |g(x)|$. Therefore, for every $t \geq 0$,
\[
\left| e^{\beta t} \mathcal{P}^t(g)(x) - e^{\beta t} \mathcal{P}^t(x, M) \int_M g \, d\mu \right| \leq \|g\|_\infty C(x) e^{-\alpha t} e^{\beta t} \mathcal{P}^t(x, M).
\]
Since
\[
\lim_{t \to \infty} e^{\beta t} \mathcal{P}^t(x, M) = \eta(x),
\]
We obtain that
\[
\lim_{t \to \infty} e^{\beta t} \mathcal{P}^t(g)(x) = \eta(x) \int_M g \, d\mu.
\]
This proves Step 1.

Step 2. We prove (i).

Let $f : M \to \mathbb{R}$ be a non-negative measure function. Let us consider the function
\[
h_u(x) = \min \left\{ \eta(x), \inf_{r \geq u} \{e^{\beta r} \mathcal{P}^r(x, M)\} \right\},
\]
observe that $h_u$ is a bounded function and $h_u \uparrow \eta$, for every $x \in M$.
\[
\mathbb{E}_x \left[ \frac{1}{t} \int_0^t f \circ \varphi_s \, ds \Big| \tau > t \right] = \frac{1}{t} \int_0^t \mathbb{E}_x \left[ f \circ \varphi_s \cdot 1_{\tau > t} \right] \, ds \frac{1}{\mathcal{P}^t(x, M)}
\]
\[\frac{1}{t} \int_0^t P_s (f \cdot P^{t-s}(1)) (x) ds \]
\[\frac{1}{t} \int_0^t e^{s\beta} P_s (f \cdot e^{(t-s)\beta} P^{t-s}(1)) (x) ds \]
\[\geq \frac{\eta(x)}{e^{\beta t} P_t(x, M)} \int_0^{t-u} e^{s\beta} P_s (f h_u)(x) ds.\]

From Step 1, we get
\[\liminf_{t \to \infty} E_x \left[ \frac{1}{t} \int_0^t f \circ \varphi_s ds \bigg\vert \tau > t \right] \geq \frac{\eta(x) \int_M f h_u d\mu}{\eta(x)} = \int_M f h_u d\mu, \text{ for every } u \geq 0.\]

Since \(h_u(x) \uparrow \eta(x)\) as \(u \to \infty\), we conclude that for every \(x \in M\),
\[\liminf_{t \to \infty} E_x \left[ \frac{1}{t} \int_0^t f \circ \varphi_s ds \bigg\vert \tau > t \right] \geq \int_M f \eta d\nu.\]

Repeating the same argument to \(\|f\|_{\infty} - f\), we obtain that for every \(x \in M\),
\[\limsup_{t \to \infty} E_x \left[ \frac{1}{t} \int_0^t f \circ \varphi_s ds \bigg\vert \tau > t \right] \leq \int_M f \eta d\nu.\]

This implies that \(\eta(x) \nu(dx)\) is a quasi-ergodic measure for \((\theta, \varphi)\) on \(M\). From (H1) we obtain that
\[\nu(dx) = \eta(x) \mu(dx).\]

**Step 3.** We prove (ii).

Let \(x \in M\) and \(A \in \mathcal{B}(M)\). From Step 1 we obtain that for every \(t \geq 0\),
\[\mu(A) = \lim_{s \to \infty} \frac{P^{t+s}(x, A)}{P^{t+s}(x, M)} = \lim_{s \to \infty} \frac{e^{\beta s} P^s \left(e^{\beta t} P^t(\cdot, A)\right)(x)}{e^{\beta(t+s)} P^{t+s}(x, M)} = \frac{\eta(x) \int_M e^{\beta t} P^t(x, A) \mu(dx)}{\eta(x)} = e^{\beta t} \int_M P^t(x, A) \mu(dx).\]

This proves step 3.

**Step 4.** We prove (iii).

From (H3) and Lebesgue dominated convergence,
\[P^t(\eta)(x) = e^{-\beta t} \lim_{s \to \infty} \int_M e^{\beta(s+t)} P^s(y, M) P^t(x, dy) = e^{-\beta t} \lim_{s \to \infty} e^{\beta(t+s)} P^{t+s}(x, M) = e^{-\beta t} \eta(x).\]

The proof follows directly from Steps 2-4. \(\square\)