On Sharpness of Error Bounds for Multivariate Neural Network Approximation

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Abstract

Sharpness of error bounds for best non-linear multivariate approximation by sums of logistic activation functions and piecewise polynomials is investigated. The error bounds are given in terms of moduli of smoothness. They describe approximation properties of single hidden layer feedforward neural networks with multiple input nodes. Sharpness with respect to Lipschitz classes is established by constructing counterexamples with a non-linear, quantitative extension of the uniform boundedness principle.

Keywords: Neural Networks, Rates of Convergence, Sharpness of Error Bounds, Counter Examples, Uniform Boundedness Principle

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1 Introduction

A feedforward neural network with an activation function $\sigma$, $m$ input nodes, one output node, and one hidden layer of $n$ neurons implements a multivariate real-valued function $g : \mathbb{R}^m \rightarrow \mathbb{R}$ of type

$$g(x) \in \mathcal{M}_n := \mathcal{M}_{n,\sigma} := \left\{ \sum_{k=1}^{n} a_k \sigma(w_k \cdot x + c_k) : a_k, c_k \in \mathbb{R}, w_k \in \mathbb{R}^m \right\}.$$  (1.1)

For vectors $w_k = (w_{k,1}, \ldots, w_{k,m})$ and $x = (x_1, \ldots, x_m)$,

$$w_k \cdot x = \sum_{j=1}^{m} w_{k,j} x_j$$

is the standard inner product of $w_k, x \in \mathbb{R}^m$. Summands $\sigma(w_k \cdot x + c_k)$ are ridge functions. They are constant on hyperplanes $w_k \cdot x = c, c \in \mathbb{R}$.

We discuss error bounds for best approximation by functions of $\mathcal{M}_n$ in terms of moduli of smoothness for an arbitrary number $m$ of input nodes in Section 2.

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Many papers deal with the univariate case \( m = 1 \). In [6], Chen proved an estimate against a first order modulus for general sigmoid activation functions. An overview of other estimates against first order moduli is given in doctoral thesis [7], cf. [8]. Under additional assumptions on activation functions, estimates against higher order moduli are possible. For example, one can easily extend the first order estimate of Ritter for approximation with “nearly exponential” activation functions in [24] to higher moduli, see [12]. Similar results can be obtained for activation functions that are arbitrarily often differentiable on some open interval such that they are not a polynomial on that interval, see [23, Theorem 6.8, p. 176] in combination with [12].

With respect to the general multivariate case, Barron applied Fourier methods in [4] to establish a convergence rate for a certain class of smooth functions in the \( L^2 \)-norm. Approximation errors for multi-dimensional bell shaped activation functions were estimated by first order moduli of smoothness or related Lipschitz classes by Anastassiou (e.g. [2]) and Costarelli and Spiegler (see e.g. [9] including a literature overview). However, discussed neural network spaces differ from (1.1). They do not consist of linear combinations of ridge functions. A special network with four layers is introduced in [19] to obtain a Jackson estimate in terms of a first order modulus of smoothness.

Maiorov and Ratsby establish an upper bound for functions in Sobolev spaces based on pseudo-dimension in [20, Theorem 2]. Pseudo-dimension is an upper bound of the Vapnik-Chervonenkis dimension (VC dimension) that will also be used in this paper to obtain lower bounds.

With respect to the standard situation (1.1), we apply results of Pinkus [23], Maiorov and Meir [22] to obtain error bounds for a large class of activation functions either based on K-functional techniques or on known estimates for best approximation with multivariate polynomials in Section 2. Both \( L^p \)- and sup-norms are considered.

In Section 3 we prove for the logistic activation function that counterexamples \( f_\alpha \) exist for all \( \alpha > 0 \) such that sup-norms as well as \( L^p \)-norm bounds are in \( O(n^{-\alpha}) \) but the error of best approximation is not in \( O(n^{-\beta}) \) for \( \beta > \alpha \). This result is a multivariate extension of univariate counterexamples (\( m = 1 \), one single input node, sup-norm) in [13]. A similar result is shown for piecewise polynomial activation functions with respect to an \( L^2 \)-norm bound.

In fact, the non-linear variant of a quantitative uniform boundedness principle in [12] can be applied to construct univariate and multivariate counterexamples. This principle is based on theorems of Dickmeis, Nessel and van Wickeren, cf. [11], that can be used to analyze error bounds of linear approximation processes. Its application, both in a linear and in the given non-linear context, requires the construction of a resonance sequence. To this end, a known result [5] on the VC dimension of networks with logistic activation is used. Theorem 3.2 in Section 3 is formulated as a general means to derive discussed counterexamples from VC dimension estimates. Also, [22] already provides sequences of counter examples that can be condensed to a single counter example with the uniform boundedness principle.

There are some published attempts to show sharpness of error bounds for neural network approximation in terms of moduli of smoothness based on inverse theorems. Inverse and equivalence theorems estimate the values of moduli of smoothness by approximation rates. For example, they determine membership to certain Lipschitz classes from known approximation errors. However, the letter [13] proves that the inverse theorem for neural network approximation in [26] as well as the inverse theorems in some related papers are wrong. Smoothness is one feature that favors high approximation rates. But in this non-linear situation, other features (e.g. the “nearly
2 Notation and Direct Estimates

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$ with non-negative integer components, let

$$|\alpha| := \sum_{j=1}^{m} \alpha_j$$

be its order. We write

$$x^\alpha := \prod_{j=1}^{m} x_j^{\alpha_j}.$$ 

With $P_k$ we denote the set of polynomials with degree at most $k$, i.e., each polynomial in $P_k$ is a linear combination of homogeneous polynomials of degree $j \in \{0, \ldots, k\}$. To this end, let

$$H_j := \left\{ f : \mathbb{R}^m \to \mathbb{R} : f(x) = \sum_{\alpha \in \mathbb{N}_0^m, |\alpha| = j} c_\alpha x^\alpha \right\}$$

be the space of homogeneous polynomials of degree $j$.

The set of all univariate polynomials with degree at most $k$ is denoted by $\Pi_k$, i.e., $\Pi_k = P_k$ for $m = 1$. Let

$$s := \dim H_k = \binom{m+k-1}{k} \leq (k+1)^{m-1}.$$ 

To obtain the upper estimate, we choose exponents independently from $\{0, \ldots, k\}$ for first $m-1$ variables. If the sum of these exponents does not exceed $k$ then the last exponent is $k$ minus sum of other exponents. Otherwise, we have counted a polynomial with degree greater than $k$. Thus, the estimate only is a coarse upper bound.

Multivariate polynomials can be represented by univariate polynomials, cf. [23, p. 164]: For a given degree $k \in \mathbb{N}$ there exist $s \leq (k+1)^{m-1}$ vectors $w_1, \ldots, w_s \in \mathbb{R}^m$ such that

$$P_k = \left\{ \sum_{j=1}^{s} p_j(w_j \cdot x) : p_j \in \Pi_k \right\}.$$ 

(2.1)

We use the result [23, p. 176], cf. [18]: Let $\sigma : \mathbb{R} \to \mathbb{R}$ be arbitrarily often differentiable on some open interval in $\mathbb{R}$ and let $\sigma$ be no polynomial on that interval. Then univariate polynomials of degree at most $k$ can be uniformly approximated arbitrarily well by choosing parameters $a_j, b_j, c_j \in \mathbb{R}$ in

$$\sum_{j=1}^{k+1} a_j \sigma(b_j x + c_j).$$ 

Thus due to (2.1), also multivariate polynomials of degree at most $k$ can be approximated by functions of $\mathcal{M}_{(k+1)}$ arbitrarily well, i.e., in the sup-norm

$$P_k \subset \mathcal{M}_{(k+1)}.$$ 

(2.2)
Theorem 3.1 in [15] describes a more general class of multivariate functions that can be approximated arbitrarily well like polynomials.

There holds following lemma from [16, Proposition 4] that extends (2.2) to simultaneous approximation.

**Lemma 2.1.** Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be arbitrarily often differentiable on an open interval around the origin with \( \sigma^{(i)}(0) \neq 0, i \in \mathbb{N}_0 \). Then for any polynomial \( \pi \in P_k \) of degree at most \( k \), any compact set \( I \subset \mathbb{R}^m \), and each \( \varepsilon > 0 \) there exists a sufficiently often differentiable function \( g \in M_{s(k+1)} \) such that simultaneously for all \( \alpha \in \mathbb{N}_m^0, |\alpha| \leq k \),

\[
\left\| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} (\pi(x) - g(x)) \right\|_{C(I)} < \varepsilon.
\]

The requirement that derivatives at zero must not be zero can be replaced by the requirement that \( \sigma \) is no polynomial on the open interval, see [16].

With \( n \) summands of the activation function, polynomials of degree \( k \) and their derivatives can be simultaneously approximated arbitrarily well for such values of \( k \) that fulfill

\[ n \geq (k + 1)^m, \text{ i.e. } k \leq \sqrt{\frac{n}{s}} - 1, \]

because

\[ (k + 1)^m = (k + 1)^{m-1}(k + 1) \geq s(k + 1). \]

Especially, polynomials of degree at most

\[ k := \lfloor \sqrt{\frac{n}{s}} \rfloor - 1 \tag{2.3} \]

can be approximated arbitrarily well.

Let \( \Omega \subset \mathbb{R}^m \) be an open set. By \( X^p(\Omega) := L^p(\Omega) \) with norm

\[ \|f\|_{L^p(\Omega)} := \left( \int_{\Omega} |f(x)|^p dx \right)^{1/p} \]

for \( 1 \leq p < \infty \) and \( X^\infty(\Omega) := C(\Omega) \) with norm

\[ \|f\|_{C(\Omega)} := \sup\{|f(x)| : x \in \Omega\} \]

we denote the usual Banach spaces. Let \( f \in X^p(\Omega), \nu \in \mathbb{R}^m, r \in \mathbb{N}_0 \) und \( t \in \mathbb{R} \). The \( r \)th radial difference (with direction \( \nu \)) is given via

\[
\Delta^r_\nu f(x) := \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f(x + j\nu)
\]

if defined. Thus, \( \Delta^r_\nu f = \Delta^{r-1}_\nu \Delta^1_\nu f = \Delta^1_\nu \Delta^{r-1}_\nu f \). Let

\[ \Omega(\nu) := \{x \in \Omega : x + t\nu \in \Omega, 0 \leq t \leq 1\}. \]

Then the \( r \)th radial modulus of smoothness of a function \( f \in L^p(\Omega), 1 \leq p < \infty, \) or \( f \in C(\Omega) \) is defined via

\[
\omega_r(f, \delta)_{p,\Omega} := \sup\{\|\Delta^r_\nu f\|_{X^p(\Omega(\nu))} : \nu \in \mathbb{R}^m, |\nu| \leq \delta\}.
\]

Our aim is to discuss errors \( E \) of best approximation. For \( S \subset X^p(\Omega) \) and \( f \in X^p(\Omega) \) let

\[ E(S, f)_{p,\Omega} := \inf\{\|f - g\|_{X^p(\Omega)} : g \in S\}. \]
Thus, $E(S, f)_{p, \Omega}$ is the distance between $f$ and $S$.

As an application of a multivariate equivalence theorem between K-functional and moduli of smoothness, an estimate for best polynomial approximation is proved on Lipschitz graph domains (LG-domains) in [17, Corollary 4, p. 139]. For the definition of not necessarily bounded LG-domains, see [1, p. 66]. For bounded domains, the LG property is equivalent to a Lipschitz boundary. Especially, later discussed bounded $m$-dimensional open intervals like $(0, 1)^m$ and the unit ball $\{x \in \mathbb{R}^m : |x| < 1\}$ are examples for LG-domains.

Let $\Omega$ be a bounded LG-domain in $\mathbb{R}^n$ and $1 \leq p \leq \infty$, then

$$E(P_k, f)_{p, \Omega} \leq C_r \omega_r \left( f, \frac{1}{k} \right)_{p, \Omega} \quad (2.4)$$

with a constant $C_r$ that is independent of $f$ and $k$, see [17].

**Theorem 2.1 (Arbitrarily Often Differentiable Functions).** Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be arbitrarily often differentiable on some open interval in $\mathbb{R}$, and let $\sigma$ be no polynomial on that interval, $f \in X^p(\Omega)$ for an LG-domain $\Omega \in \mathbb{R}^m$, $1 \leq p \leq \infty$, and $r \in \mathbb{N}$. For $n \geq 4^m$ there exists a constant $C$ that is independent of $f$ and $k$ such that

$$E(M_n, f)_{p, \Omega} \leq C \omega_r \left( f, \frac{1}{\sqrt[n]{n}} \right)_{p, \Omega}.$$

**Proof.** We combine (2.2) and (2.3) with (2.4) to get

$$E(M_n, f)_{p, \Omega} \leq C_r \omega_r \left( f, \frac{1}{\sqrt[n]{n}} - 1 \right)_{p, \Omega} \leq C_r \omega_r \left( f, \frac{1}{\sqrt[n]{n} - 2} \right)_{p, \Omega}$$

$$\leq C_r \omega_r \left( f, \frac{1}{\sqrt[n]{n} - 1} \right)_{p, \Omega} = C_r \omega_r \left( f, \frac{2}{\sqrt[n]{n}} - \frac{\sqrt[n]{n}}{2} \right)_{p, \Omega} \leq C_r \frac{2^r}{\omega_r^r} \omega_r \left( f, \frac{1}{\sqrt[n]{n}} \right)_{p, \Omega}. \quad \square$$

By using an error bound for best polynomial approximation we are not able to consider advantages of non-linear approximation. However, we will see in the next section that non-linear neural network approximation does not really perform better than polynomial approximation in the worst case.

Most activation functions, that are not piecewise polynomials, fulfill the requirements of Theorem 2.1. For example, it provides an error bound for approximation with the sigmoid activation function based on inverse tangent

$$\sigma(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x),$$

the logistic function

$$\sigma(x) = \frac{1}{1 + e^{-x}} = \frac{1}{2} \left( 1 + \tanh \left( \frac{x}{2} \right) \right),$$

and "Exponential Linear Unit" (ELU) activation function

$$\sigma(x) = \begin{cases} \alpha(e^x - 1), & x < 0 \\ x, & x \geq 0 \end{cases}$$

for $\alpha \neq 0$.
A direct bound for simultaneous approximation of a function and its partial derivatives in the sup-norm can be obtained similarly based on a corresponding estimate for simultaneous approximation by polynomials using a Jackson estimate from 9:

**Lemma 2.2.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function with compact support such that all partial derivatives up to order \( k \in \mathbb{N}_0 \) are continuous. Let \( \overline{\Omega} \subset \mathbb{R}^n \) be a compact set that contains the support of \( f \). Then there exists a constant \( C \in \mathbb{R} \) (independent of \( n \) and \( f \)) such that for each \( n \in \mathbb{N} \) a polynomial \( \pi \in \mathcal{P}_n \) can be found such that for all \( \alpha \in \mathbb{N}_0^m \) with \( |\alpha| \leq \min\{k, n\} \)

\[
\left\| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} (f(x) - \pi(x)) \right\|_{C(\overline{\Omega})} \leq C \max_{\beta \in \mathbb{N}_0^m, |\beta| = k} \omega_1 \left( \frac{\partial^k f}{\partial x_1^{\beta_1} \cdots \partial x_m^{\beta_m}} \right)_{\infty, \Omega},
\]

for \( \alpha = (\alpha_1, \ldots, \alpha_m) \) with \( \alpha_j \geq 0 \) and each component \( 0 \leq j \leq m \).

Similar to the proof of Theorem 2.1, we combine this cited result with Lemma 2.1 to obtain (cf. [27])

**Theorem 2.2 (Synchronous Sup-Norm Approximation).** Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be arbitrarily often differentiable without being a polynomial. For each function \( f : \mathbb{R}^n \to \mathbb{R} \) with compact support and continuous partial derivatives up to order \( k \in \mathbb{N}_0 \) and each compact set \( \overline{\Omega} \subset \mathbb{R}^n \) containing the support of \( f \) following estimate holds true: For each \( n \in \mathbb{N} \) and each \( m \in \mathbb{N} \), there exists a constant \( C \in \mathbb{R} \) (independent of \( n \) and \( f \)) such that for all \( \alpha \in \mathbb{N}_0^m \) with \( |\alpha| \leq \min\{k, n\} \)

\[
\inf \left\{ \left\| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} (f(x) - g(x)) \right\|_{C(\overline{\Omega})} : g \in \mathcal{M}_n \right\}
\leq C \frac{1}{n} \max_{\beta \in \mathbb{N}_0^m, |\beta| = k} \omega_1 \left( \frac{\partial^k f}{\partial x_1^{\beta_1} \cdots \partial x_m^{\beta_m}} \right)_{\infty, \Omega}.
\]

(2.5)

Requirements of Theorems 2.1 and 2.2 are not fulfilled for activation functions that are of type

\[
\sigma(x) = \begin{cases} 
0, & x < 0 \\
k, & x \geq 0
\end{cases}
\]

for \( k \in \mathbb{N} \). The often used ReLU function is obtained for \( k = 1 \). Corollary 6.11 in [23, p. 178] is an \( L^2 \)-norm Jackson estimate for this class of functions. To work with this estimate, we need to introduce Sobolev spaces.

Let \( W^r_p(\Omega), 1 \leq p < \infty \), be the \( L^p \)-Sobolev space of \( r \)-times partially differentiable functions (in the weak sense) on \( \Omega \subset \mathbb{R}^n \) with semi-norms

\[
|f|_{W^r_p(\Omega)} = \sum_{\alpha \in \mathbb{N}_0^m, |\alpha| = r} \left\| \frac{\partial^r f}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} \right\|_{L^p(\Omega)}
\]

and norm

\[
\|f\|_{W^r_p(\Omega)} = \sum_{k=0}^r |f|_{W^r_p(\Omega)}.
\]

For \( r \)-times continuously differentiable functions \( f \) (case \( p = \infty \)) or functions \( f \in W^r_p(\Omega) \) on LG-domains \( \Omega, 1 \leq p < \infty \), the estimate

\[
\omega_r(f, \delta)_{p, \Omega} \leq C_r \delta^r \sum_{\alpha \in \mathbb{N}_0^m, |\alpha| = r} \left\| \frac{\partial^r f}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} \right\|_{L^p(\Omega)}
\]

(2.7)
Inequality: Let constant $D$ and (2.9) and (2.12). The functionals should also fulfill not only (2.11) but a stability condition. Let Lemma 2.3 conditions.

This estimate can be extended to moduli of smoothness using K-functional techniques. To this end, we need some definitions that will also be needed in the next section for discussing sharpness. A functional $T$ on a normed space $X$, i.e., $T$ maps $X$ into $\mathbb{R}$, is non-negative-valued, sub-linear, and bounded, iff for all $f, g \in X$, $c \in \mathbb{R}$

\[ T(f) \geq 0, \]
\[ T(f + g) \leq T(f) + T(g), \]
\[ T(cf) = |c|T(f). \]

\[ \|T\|_{X^\infty} := \sup\{T(f) : f \in X, \|f\|_X \leq 1\} < \infty. \]

The set $X^\sim$ consists of all non-negative-valued, sub-linear, bounded functionals $T$ on $X$.

Since we deal with non-linear approximation, error functionals will not be sub-linear. Instead we discuss remainders $(E_n)_{n=1}^\infty, E_n : X \rightarrow [0, \infty)$ that fulfill following conditions for $m \in \mathbb{N}, f, f_1, f_2, \ldots, f_m \in X$, and constants $c \in \mathbb{R}$:

\[ E_m(f) := \left( \sum_{k=1}^m E_k(f_k) \right) \leq m \max_{k=1}^m E_k(f_k), \]

(2.9)

\[ E_m(cf) = |c|E_m(f), \]

(2.10)

\[ E_n(f) \leq D_n\|f\|_X, \]

(2.11)

\[ E_n(f) \geq E_{n+1}(f). \]

(2.12)

Constant $D_n$ is independent of $f$. Functional $E_n(f) := E(\mathcal{M}_n, f)_{p, \Omega}$ fulfills these conditions.

**Lemma 2.3 (K-functional).** Let functionals $(E_n)_{n=1}^\infty, E_n : C[0, 1] \rightarrow [0, \infty)$ fulfill (2.9) and (2.12). The functionals should also fulfill not only (2.11) but a stability inequality: Let constant $D_n$ in (2.11) be independent of $n$, i.e.,

\[ E_n(f) \leq D_0\|f\|_{[0,1]} \]

(2.13)

for a constant $D_0 > 0$ and all $n \in \mathbb{N}$. Also, a Jackson-type inequality $0 < \phi(n) \leq 1$)

\[ E_n(g) \leq D_1\phi(n)[\|g\|_X + |g|_V], \]

(2.14)
Let \( n \geq 2 \) and a constant \( D_2 > 0 \), the sequence \((\varphi(n))_{n=1}^{\infty}\) has to fulfill
\[
\varphi \left( \left\lceil \frac{n}{2} \right\rceil \right) \leq D_2 \varphi(n). \tag{2.15}
\]

Via the Peetre \( K \)-functional
\[
K(\delta,f,X,U) := \inf \{ \| f - g \|_X + \delta |g| : g \in U \}
\]
one can estimate
\[
E_n(f) \leq C [K(\varphi(n), f, X, U) + \varphi(n) \| f \|_X]
\]
for \( n \geq 2 \) with a constant \( C \) that is independent of \( f \) and \( n \).

Proof. Let \( g \in U \). Then
\[
E_{2n}(f) = E_{2n}(f - g + g) \leq E_n(f - g) + E_n(g) \\
\leq D_0 \| f - g \|_X + D_1 \varphi(n) \| g \|_X + |g| \|_U \\
\leq D_0 \| f - g \|_X + D_1 \varphi(n) \| f \|_X + |g - f| \|_X + |g| \|_U \\
\leq (D_0 + D_1) \| f - g \|_X + D_1 \varphi(n) |g| \|_U + D_1 \varphi(n) \| f \|_X,
\]

thus for \( n \geq 3 \):
\[
E_n(f) \leq E_{2\lceil \frac{n}{2} \rceil}(f) \\
\leq (D_0 + D_1) \left[ \inf \left\{ \| f - g \|_X + \varphi \left( \left\lceil \frac{n}{2} \right\rceil \right) |g| : g \in U \right\} + \varphi \left( \left\lceil \frac{n}{2} \right\rceil \right) \| f \|_X \right] \\
\leq (D_0 + D_1) \left[ \inf \left\{ \| f - g \|_X + D_2 \varphi(n) |g| : g \in U \right\} + D_2 \varphi(n) \| f \|_X \right] \\
\leq (D_0 + D_1) \max \{1, D_2\} [K(\varphi(n), f, X, U) + \varphi(n) \| f \|_X].
\]

We apply the lemma to \( \varphi(n) \) with \( X = L^2(\Omega), U = W^2_2(\Omega), \varphi(n) = n^{-\frac{\kappa}{m}} \). Error functional \( E(M_n,f,\omega) \) fulfills all prerequisites. In connection with the equivalence between \( K \)-functionals and moduli of smoothness \cite{[17], p. 120} we get

\textbf{Theorem 2.3 (Piecewise Polynomial Functions).} Let \( m \geq 2, \Omega \subset \mathbb{R}^m \) be the \( m \)-dimensional unit ball and \( \sigma \) a piecewise polynomial activation function of type \( \varphi(n) \).

Constants \( C_1, C_2 \in \mathbb{R} \) exist such that for each \( f \in L^2(\Omega), n \geq 2, r < k + 1 + \frac{m}{\sigma} \):
\[
E(M_n,f,\omega) \leq C_1 \left[ K \left( \frac{1}{n^m}, f, L^2(\Omega), W^2_2(\Omega) \right) + \frac{1}{n^m} \| f \|_{L^2(\Omega)} \right] \\
\leq C_2 \left[ \left( f, \frac{1}{\sqrt[4]{m}} \right)_{2,\Omega} + \frac{1}{n^m} \| f \|_{L^2(\Omega)} \right]. \tag{2.16}
\]

The saturation order of the modulus is \( n^{-\frac{m}{2}} \), so the term \( n^{-\frac{m}{2}} \| f \|_{L^2(\Omega)} \) is only technical. The estimate also holds for ReLU (\( k = 1 \)) with only one (\( m = 1 \)) input node for \( r = 2 \), see \cite{[12]}. It can be extended to the cut activation function because cut can be written as a difference of ReLU and translated ReLU.
3 Sharpness due to Counterexamples

A coarse lower estimate can be obtained for all integrable activation functions in the $L^2$-norm based on an estimate for ridge functions in \cite{21}. However, the general setting leads to an exponent $\frac{m-1}{m}$ instead of $\frac{m}{m-1}$.

The space of all measurable, real-valued functions that are integrable on every compact subset of $\mathbb{R}$ is denoted by $L(\mathbb{R})$.

**Lemma 3.1.** Let $\sigma$ be an arbitrary activation function in $L(\mathbb{R})$ and $r \in \mathbb{N}$, $m \geq 2$.

Let $\Omega$ be the $m$-dimensional unit ball.

Then there exists a sequence $(f_n)_{n=1}^{\infty}$, $f_n \in W_2^r(\Omega)$, with $\|f_n\|_{W_2^r(\Omega)} \leq C_0$, and a constant $c > 0$ such that (\cf\ Theorem 2.7)

$$\omega_r \left( f_n, \frac{1}{\sqrt{n}} \right)_{2,\Omega} = O \left( \frac{1}{n^{\frac{m}{2}}} \right)$$

and

$$E(\mathcal{M}_n, f_n)_{2,\Omega} \geq \frac{c}{n^{\frac{m}{2}}}.$$  

**Proof.** This is a direct corollary of Theorem 1 in \cite{21}: For $A \subset \mathbb{R}^m$ with cardinality $|A|$ let $R(A)$ be the linear space that is spanned by all functions $h(w \cdot x)$, $h \in L(\mathbb{R})$, $w \in A$. Thus in contrast to one activation function, different nearly arbitrary functions $h$ are allowed to be used with different vectors $w$ in linear combinations. Let $\mathcal{R}_n := \bigcup_{A \subset \mathbb{R}^m, |A| \leq n} R(A)$ be the space of functions that can be represented as $\sum_{k=1}^{n} a_k h_k(w_k \cdot x)$, $a_k \in \mathbb{R}$, $h_k \in L(\mathbb{R})$, $w_k \in \mathbb{R}^m$. Then for all activation functions $\sigma \in L(\mathbb{R})$ one has $h_k(x) := \sigma(x + c_k) \in L(\mathbb{R})$ for $c_k \in \mathbb{R}$, i.e. $\mathcal{M}_n \subset \mathcal{R}_n$. According to \cite{21}, for $m \geq 2$ there exist constants $0 < c \leq C$ independently of $n$ such that

$$\frac{c}{n^{\frac{m}{2}}} \leq \sup_{f \in W_2^r(\Omega), \|f\|_{W_2^r(\Omega)} \leq C_0} \inf_{h \in \mathcal{R}_n} \|f - h\|_{L^2(\Omega)} \leq \frac{C}{n^{\frac{m}{2}}}.$$  

From this condition, we obtain functions $f_n \in W_2^r(\Omega)$, $\|f_n\|_{W_2^r(\Omega)} \leq C_0$, such that

$$\frac{1}{2} \frac{c}{n^{\frac{m}{2}}} \leq \inf_{h \in \mathcal{R}_n} \|f_n - h\|_{L^2(\Omega)} \leq \inf_{h \in \mathcal{M}_n} \|f_n - h\|_{L^2(\Omega)} = E(\mathcal{M}_n, f_n)_{2,\Omega},$$

and (\see \cite{21})

$$\omega_r \left( f_n, \frac{1}{\sqrt{n}} \right)_{2,\Omega} \leq C_1 \frac{1}{n^{\frac{m}{2}}} \sum_{a \in \mathcal{R}_m, |a| = r} \left\| \frac{\partial^a f_n}{\partial x_1^{a_1} \cdots \partial x_m^{a_m}} \right\|_{L^2(\Omega)} \leq C_2(r, m) \frac{1}{n^{\frac{m}{2}}}.$$  

\hfill $\Box$

By considering properties of the activation function, better lower estimates are possible. For the logistic activation function and activation functions that are splines of fixed polynomial degree with finite number of knots like \cite{20}, Maiorov and Meir show that there exists a sequence $(f_n)_{n=2}^{\infty}$, $f_n \in W_2^r(\Omega)$, $r \in \mathbb{N}$, with $\|f_n\|_{W_2^r(\Omega)}$ uniformly bounded, and constant $c > 0$ (independent of $n \geq 2$) such that (\see \cite{22} Theorem 4 and Theorem 5, p. 99, Corollary 2, p. 100)

$$E(\mathcal{M}_n, f_n)_{p,\Omega} \geq \frac{c}{(n \log_2(n))^{\frac{m}{2}}}$$  \hspace{1cm} (3.1)
for $1 \leq p < \infty$ (and $L^\infty(\Omega)$, but we consider $C(\Omega)$ due to the definition of moduli of smoothness). Without explicitly saying so, the proof is based on a VC dimension argument similar to the proof of Theorem 3.2 that follows in this section. It uses [22, Lemma 7, p. 99]. The formula in line 4 on page 98 shows that (by choosing parameter $m$ as in the proof of [22, Theorem 4]) one additionally has

$$\|f_n\|_{L^p(\Omega)} \leq \frac{C}{(n \log_2(n))^{\frac{m}{p}}} \left[ \frac{1 + \log_2(n)}{\log_2(n)} \right]^{\frac{m}{p}} \leq \frac{2C}{(n(1 + \log_2(n)))^{\frac{m}{p}}}.$$ (3.2)

This result was proved for $\Omega$ being the unit ball. But similar to Theorem 3.2 below, a grid is used that can also be adjusted to $\Omega = (0, 1)^m$. We now apply a resonance principle from [12] that is a straightforward extension of a general theorem by Dickmeis, Nessel and van Wickern, see [11]. With this principle, we condense sequences $(f_n)_{n=1}^\infty$ like the one in (3.1) to single counterexamples.

To measure convergence rates, abstract moduli of smoothness $\omega$ are often used, see [25, p. 96ff]. To this end, let $\omega$ be a continuous, increasing function on $[0, \infty)$ such that for $0 < \delta_1, \delta_2, 0 = \omega(0) < \omega(\delta_1) \leq \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2).$ (3.3)

Typically, Lipschitz classes are defined via $\omega(\delta) := \delta^\alpha, 0 < \alpha \leq 1.

**Theorem 3.1** (Adapted Uniform Boundedness Principle, see [12]). Let $(E_n)_{n=1}^\infty$ be a sequence of remainders that map elements of a real Banach space $X$ to non-negative numbers, i.e.,

$$E_n : X \to [0, \infty).$$

The sequence has to fulfill conditions (2.9)–(2.12). Also, a family of sub-linear bounded functionals $S_\delta \in X^\sim$ for all $\delta > 0$ is given. These functionals will represent moduli of smoothness.

To express convergence rates, let $\mu : (0, \infty) \to (0, \infty)$ and $\varphi : [1, \infty) \to (0, \infty)$ be strictly decreasing with $\lim_{x \to \infty} \varphi(x) = 0$. Since remainder functionals $E_n$ are not required to be sub-linear, $\varphi$ also has to fulfill following condition. For each $0 < \lambda < 1$ there has to be a point $X_0 = X_0(\lambda) \geq \lambda^{-1}$ and constant $C_\lambda > 0$ such that for all $x > X_0$ there holds

$$\varphi(\lambda x) \leq C_\lambda \varphi(x).$$ (3.4)

If test elements $h_n \in X$ and a number $n_0 \in \mathbb{N}$ exist such that for all $n \in \mathbb{N}$ with $n \geq n_0$ and for all $\delta > 0$

$$\|h_n\|_X \leq C_1,$$ (3.5)

$$S_\delta(h_n) \leq C_2 \min \left\{ 1, \frac{\mu(\delta)}{\varphi(n)} \right\},$$ (3.6)

$$E_n(h_n) \geq c_3 > 0,$$ (3.7)

then for each abstract modulus of smoothness $\omega$ satisfying (3.3) and

$$\lim_{\delta \to 0^+} \frac{\omega(\delta)}{\delta} = \infty$$ (3.8)
a counterexample \( f_\omega \in X \) exists such that

\[ S_\delta(f_\omega) = O(\omega(\mu(\delta))) \text{ for } \delta \to 0+ \]

and

\[ E_n(f_\omega) \neq o(\omega(\varphi(n))) \text{ for } n \to \infty, \text{ i.e., } \limsup_{n \to \infty} \frac{E_n(f_\omega)}{\omega(\varphi(n))} > 0. \]

When dealing with the sup-norm, one can generally apply the resonance theorem in connection with known VC dimensions of indicator functions. The general definition of VC dimension based on sets is as follows.

Let \( X \) be a finite set and \( A \subset P(X) \) a family of subsets of \( X \). Set \( S \subset X \) is said to be shattered by \( A \) iff each subset \( B \subset S \) can be represented as \( B = S \cap A \) for a family member \( A \in A \). Thus, the set \( \{ S \cap A : A \in A \} \) has \( 2^{|S|} \) elements, \( |S| \) denoting the number of elements of \( S \).

\[ \text{VC-dim}(A) := \sup \{ k \in \mathbb{N} : \exists S \subset X \text{ with cardinality } |S| = k \text{ such that } S \text{ is shattered by } A \} \]

is called the VC dimension of \( A \).

For our purpose, we discuss a (non-linear) set \( V \) of functions \( g : X \to \mathbb{R} \) on a set \( X \subset \mathbb{R}^m \). Using Heaviside-function \( H : \mathbb{R} \to \{0, 1\} \),

\[ H(x) := \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \]

let

\[ A := \{ A \subset X : \exists g \in V : (\forall x \in A : H(g(x)) = 1) \land (\forall x \in X \setminus A : H(g(x)) = 0) \}. \]

Then we define \( \text{VC-dim}(V) := \text{VC-dim}(A) \). Thus, \( k := \text{VC-dim}(V) \) is the largest cardinality of a subset \( S = \{ x_1, \ldots, x_k \} \subset X \) such that for each sign sequence \( s_1, \ldots, s_k \in \{-1, 1\} \) a function \( g \in V \) can be found that fulfills (cf. [3])

\[ H(g(x_i)) = H(s_i), \quad 1 \leq i \leq k. \]

**Theorem 3.2 (Sharpness due to VC Dimension).** Let \( (V_n)_{n=1}^\infty \) be a sequence of (non-linear) function spaces \( V_n \) of bounded real-valued functions on \([0, 1]^m\) such that

\[ E_n(f) := \inf \{ \| f - g \|_{C([0,1]^m)} : g \in V_n \} \tag{3.9} \]

fulfills conditions (2.9)–(2.12). An equidistant grid \( X_n \subset [0,1]^m \) with a step size \( \frac{1}{\tau(n)} \),

\[ \tau : \mathbb{N} \to \mathbb{N}, \]

is given via

\[ X_n := \left\{ j \frac{1}{\tau(n)} : j \in \{0,1,\ldots,\tau(n)\} \right\}^m. \]

Let

\[ V_{n,\tau(n)} := \{ h : X_n \to \mathbb{R} : \text{ a function } g \in V_n \text{ exists with } h(x) = g(x) \text{ for all } x \in X_n \} \]

be the set of functions that are generated by restricting functions of \( V_n \) to this grid. As in Theorem 3.1, convergence rates are expressed via a function \( \varphi(x) \) that fulfills the
requirements of Theorem 3.1 including condition (3.4). Let VC dimension of $V_{n,\tau(n)}$ and function values of $\tau$ and $\varphi$ be coupled via inequalities

$$\text{VC-dim}(V_{n,\tau(n)}) < [\tau(n)]^m,$$

(3.10)

$$\tau(4n) \leq \frac{C}{\varphi(n)},$$

(3.11)

for all $n \geq n_0 \in \mathbb{N}$ with a constant $C > 0$ that is independent of $n$.

Then, for $r \in \mathbb{N}$ and each abstract modulus of smoothness $\omega$ satisfying (3.9) and (3.10), there exists a counterexample $f_{\omega} \in C([0,1]^m)$ such that for $\delta \to 0+$ and $n \to \infty$

$$\omega_r(f_{\omega},\delta)_{\infty,(0,1)^m} = O(\omega(\delta^r)) \text{ and } E_n(f_{\omega}) = o(\omega([\varphi(n)]^r)).$$

Proof. Condition (3.10) implies for $4n \geq n_0$ that a sequence of signs $s_{\mathbf{z}} \in \{-1,1\}$ for points $\mathbf{z} \in X_{4n}$ exists such that no function in $V_{4n}$ can reproduce the sign of the sequence in each point of $X_{4n}$, i.e., for each $g \in V_{4n}$ there exists a point $\mathbf{z}_0 \in X_{4n}$ such that

$$H(g(\mathbf{z}_0)) \neq H(s_{\mathbf{z}_0}).$$

Based on this sign sequence, we construct an arbitrarily often partially differentiable resonance function $h_n$ such that its function values equal the signs on the grid $X_{4^n}$. To this end, we use the arbitrarily often differentiable function

$$h(x) := \begin{cases} 
\exp \left( 1 - \frac{1}{1 - x^2} \right) & \text{for } |x| < 1, \\
0 & \text{for } |x| \geq 1,
\end{cases}$$

with properties $h(0) = 1$ and $\|h\|_{B(\mathbb{R})} = 1$. Based on $h$, we define $h_n$:

$$h_n(x) := \sum_{\mathbf{z} \in X_{4n}} s_{\mathbf{z}} \cdot \prod_{k=1}^{m} h(2\tau(4n)(x_k - z_k)).$$

Scaling factors $2\tau(4n)$ are chosen such that supports of summands only intersect at their borders. Therefore, $\|h_n\|_{C([0,1]^m)} \leq 1$ and $h_n(z) = s_{\mathbf{z}}$ for all $\mathbf{z} \in X_{4^n}$.

All partial derivatives of order up to $r$ are in $O([\varphi(n)]^{-r})$ because of (3.11). Additionally to $h_n$, we choose parameters in Theorem 3.1 as follows:

$$X = C([0,1]^m), \quad S_\delta(f) := \omega_r(f,\delta)_{\infty,(0,1)^m}, \quad \mu(\delta) := \delta^r,$$

and $E_n(f)$ as in (3.9). We do not directly use $\varphi(x)$ with Theorem 3.1. Instead, function $[\varphi(x)]^r$ fulfills the requirements of the function also called $\varphi(x)$ in Theorem 3.1.

Requirements (3.5) and (3.6) can be easily shown due to the sup-norms of $h_n$ and its partial derivatives, cf. [27].

Resonance condition (3.7) is fulfilled due to the definition of $h_n$: For each $g \in V_{4n}$ there exists at least one point $\mathbf{z}_0 \in X_{4n}$ such that

$$H(g(\mathbf{z}_0)) \neq H(s_{\mathbf{z}_0}) = H(h_n(\mathbf{z}_0)).$$

Function $h_n$ is defined to fulfill $|h_n(\mathbf{z}_0)| = |s_{\mathbf{z}_0}| = 1$. Thus, $h_n - g \in C([0,1]^m) \geq |h_n(\mathbf{z}_0) - g(\mathbf{z}_0)| \geq 1$,

and $E_{4n}h_n \geq 1$.

All preliminaries of Theorem 3.1 are fulfilled such that counterexamples exist as stated.
Theorem 3.3 (Sharpness for Logistic Function Approximation in Sup-Norm). Let $\sigma$ be the logistic function and $r \in \mathbb{N}$. For each abstract modulus of smoothness $\omega$ satisfying (3.3) and (3.8), a counterexample $f_\omega \in C([0,1]^m)$ exists such that for $\delta \to 0^+$
\[
\omega_r(f_\omega, \delta)_{\infty, (0,1)^m} = O(\omega(\delta))
\]
and for $n \to \infty$
\[
E(M_n, f_\omega)_{\infty, (0,1)^m} \neq o\left(\omega\left(\frac{1}{m(1 + \log_2(n))}\right)\right).
\]

For univariate approximation, i.e., $m = 1$, the theorem is proved in [12]. This proof can be generalized as follows.

Proof. Let $D \in \mathbb{N}$. In [5], an upper bound for the VC dimension of function spaces
\[
\Delta_n := \{g : \{-D, -D + 1, \ldots, D\}^m \to \mathbb{R} : g(x) = a_0 + \sum_{k=1}^n a_k \sigma(w_k \cdot x + c_k), a_0, a_k, c_k \in \mathbb{R}, w_k \in \mathbb{R}^m\}
\]
is derived. Functions are defined on a discrete set with $(2D + 1)^m$ points. Please note that the constant function $a_0$ is not consistent with the definition of $M_n$. It provides an additional degree of freedom.

We apply Theorem 2 in [5]: There exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ the VC dimension of $\Delta_n$ is upper bounded by
\[
2 \cdot (nm + 2n + 1) \cdot \log_2(24e(nm + 2n + 1)D),
\]
i.e., there exists an $n_0 \geq \max\{2, n^*\}$, $n_0 \in \mathbb{N}$, and a constant $C_m > 0$, dependent on $m$, such that for all $n \geq n_0$
\[
\text{VC-dim}(\Delta_n) \leq C_m n[\log_2(n) + \log_2(D)].
\]

Let constant $E > 1$ be chosen such that
\[
\frac{1 + \log_2(E)}{E} < \frac{1}{4C_m}, \text{ i.e. } 4C_m[1 + \log_2(E)] < E. \quad (3.12)
\]
This is possible because
\[
\lim_{E \to \infty} \frac{1 + \log_2(E)}{E} = 0.
\]
Now we choose a suitable value of $D = D(n)$ such that the VC dimension of $\Delta_n$ is less than $[D(n)]^m$. To this end, let
\[
D = D(n) := \left\lceil \sqrt[4]{En(1 + \log_2(n))} \right\rceil.
\]
Then we get for $n \geq n_0$ with (3.12):
\[
\text{VC-dim}(\Delta_n) \leq C_m n[\log_2(n) + \log_2(\sqrt[4]{En(1 + \log_2(n))})] = C_m n[\log_2(n) + m^{-1} \log_2(En(1 + \log_2(n)))] \leq C_m n[2 \log_2(n) + \log_2(E) + \log_2(2 \log_2(n))] \leq C_m n[3 \log_2(n) + \log_2(E) + 1] \leq 4C_m n \log_2(n)[1 + \log_2(E)]
\]
3 Sharpness due to Counterexamples

\[ < En \log_2(n) \leq \left[ \sqrt[2^m]{En(1 + \log_2(n))} \right]^m = [D(n)]^m. \]

One can map interval \([-D, D]\) to \([0, 1]^m\) with an affine transform. By also omitting constant \(a_0\), we estimate the VC dimension of \(V_{n, \tau(n)}\) with parameters \(V_n := \mathcal{M}_n\) and \(\tau(n) := 2D(n)\):

\[ \text{VC-dim}(V_{n, \tau(n)}) < [D(n)]^m < [2D(n)]^m = [\tau(n)]^m. \]

Thus, (3.10) is fulfilled. Conditions (2.9)–(2.12) are chosen such that they fit with error functionals

\[ E_n := E(\mathcal{M}_n, \cdot)_{\infty,(0,1)^m}. \]

For strictly decreasing function

\[ \varphi(x) := \frac{1}{\sqrt{x[1 + \log_2(x)]}} \]

conditions \(\lim_{x \to \infty} \varphi(x) = 0\) and (3.4) hold. Latter can be shown for \(x > X_0(\lambda) := \lambda^{-2}\) because \(\log_2(\lambda) > -\log_2(x)/2\) and

\[
\begin{align*}
\varphi(\lambda x) &= \frac{1}{\sqrt[2^m]{\lambda^2(1 + \log_2(\lambda) + \log_2(z))}} \\
&\leq \frac{1}{\sqrt[2^m]{\lambda^2(1 + \frac{1}{2}\log_2(x))}} < m \sqrt[2^m]{\lambda^2(1 + \frac{1}{2}\log_2(x))} < m \sqrt[2^m]{\lambda^2(1 + \frac{1}{2}\log_2(x))}.
\end{align*}
\]

Finally, (3.11) follows from

\[ \tau(4n) = 2D(4n) \leq 2 \frac{\sqrt[2^m]{4E(1 + \log_2(4n))}}{\sqrt[2^m]{E(1 + \log_2(4n))}} < \frac{2 \sqrt[2^m]{4E(1 + \log_2(4))}}{\sqrt[2^m]{E(n)}} = 2 \frac{\sqrt[2^m]{12E}}{\sqrt[2^m]{E(n)}}. \]

Thus, Theorem 3.2 can be applied to obtain the counterexample.

The theorem can also be proved based on the sequence \((f_n)_{n=1}^\infty\) from [22] with properties (3.1) and (3.2). We use this sequence to obtain the sharpness in \(L^p\) norms for approximation with piecewise polynomial activation functions as well as with the logistic function.

Theorem 1 in [20] provides a general means to obtain such bounded sequences in Sobolev spaces for which approximation by functions in \(\mathcal{M}_n\) is lower bounded with respect to pseudo-dimension.

We condense sequence \((f_n)_{n=1}^\infty\) to a single counterexample with the next theorem.

**Theorem 3.4 (Sharpness with \(L^p\)-Norms).** Let \(\sigma\) be either the logistic function or a piecewise polynomial activation function (2.6) and \(r \in \mathbb{N}\). Let \(\Omega\) be the \(m\)-dimensional unit ball, \(m \in \mathbb{N}, 1 \leq p < \infty\). For each abstract modulus of smoothness \(\omega\) satisfying (3.6) and (3.8), a counterexample \(f_\omega \in L^p(\Omega)\) exists such that for \(\delta \to 0^+\)

\[ \omega_\delta(f_\omega, \delta)_{p,\Omega} = O(\omega(\delta') \delta^r) \]

and for \(n \to \infty\)

\[ E(\mathcal{M}_n, f_\omega)_{p,\Omega} \neq o\left(\omega\left(\frac{1}{(n[1 + \log_2(n)])^m}\right)\right). \]
Proof. We apply Theorem 3.1 with following parameters for \( n \geq 2 \):

\[
E_n(f) := E(M_{n, f, p, \Omega}, X = L^p(\Omega), S_\delta(f) = \omega_r(f, \delta, p, \Omega),
\]

\[
\varphi(x) = \frac{1}{|x(1 + \log_2(x))|^\delta}, \quad \mu(\delta) = \delta.
\]

Function \( \varphi(x) \) satisfies the prerequisites of Theorem 3.1 similarly to the proof of Theorem 3.3. Also, conditions (2.9) – (2.12) hold true for \( E_n \). For \( r \in \mathbb{N} \), we use the sequence \( (f_n)_{n=2}^\infty \) of (3.1) to define resonance elements

\[
h_n := \frac{1}{\varphi(4n)} \cdot f_4n
\]

such that functions \( h_n \) are uniformly bounded in \( L^p(\Omega) \) due to (3.2) in L^p(\Omega). Thus, (3.3) is fulfilled. From (3.1) we obtain resonance condition (3.7)

\[
E(M_{4n, h_n, p, \Omega}) \geq [4n(1 + \log_2(4n))]^\frac{r}{m} \cdot \frac{c}{(4n(\log_2(4n))} \geq c > 0.
\]

Since \( \|f_n\|_{W_2^{\sigma}(\Omega)} \) is bounded, estimate (2.7) yields (3.6):

\[
\omega_r(h_n, \delta, p, \Omega) = \frac{1}{\varphi(4n)} \cdot \omega_r(f_4n, \delta, p, \Omega) \leq C \frac{\delta^r}{\varphi(4n)} \leq 12 \frac{\delta^r}{\varphi(4n)},
\]

Thus, all prerequisites of Theorem 3.1 are fulfilled such that counterexamples exist as stated.

With respect to error bound (2.9) for synchronous approximation, counterexamples can be obtained due to the following observation for \( \alpha \in \mathbb{N}_0^m \), \( |\alpha| = k \):

\[
\inf \left\{ \left\| \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}} (f(x) - g(x)) \right\|_{X_p(\Omega)} : g \in M_{n, \sigma}, \right\} \geq E \left( M_{n, \sigma}^{(k)}, \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}} \right)_{p, \Omega}.
\]

In the univariate case \( m = 1 \), a counterexample for approximation with \( \sigma^{(k)} \) can be integrated to become a counterexample that shows sharpness of (2.5). For example, \( \sigma(x) = \frac{1}{2} + \frac{1}{2} \arctan(x) \) is discussed in [12, Corollary 4.2]. The given proof shows that for each abstract modulus of smoothness \( \omega \) satisfying (3.8) and (3.9), a continuous counterexample \( f_\omega \) exists such that \( \omega_1(f_\omega, \delta)_{\infty, \Omega} = O(\omega(\delta)) \) and \( E(M_{n, \sigma, f_\omega})_{\infty, \Omega} \neq O(\omega(\delta)) \). Thus, one can choose \( f_\omega(x) := \int_0^x f_\omega(t) \, dt \). In the multivariate case however, integration with respect to one variable does not lead to sufficient smoothness with regard to other variables.

4 Conclusions

By setting \( \omega(\delta) := \delta^{m \alpha} \), we have shown the following for the logistic function. For each \( 0 < \alpha < \frac{1}{m} \) condition (3.9) is fulfilled, and according to Theorem 3.3 there exists a counterexample \( f_\omega \in C([0, 1]^m) \) with

\[
\omega_r(f_\omega, \delta)_{\infty, (0, 1)^m} = O(\delta^{m \alpha}) \quad \text{and} \quad E(M_n, f_\omega)_{\infty, (0, 1)^m} = O\left(\frac{1}{n^m}\right)
\]

such that for all \( \beta > \alpha \)

\[
E(M_n, f_\omega)_{\infty, (0, 1)^m} \neq O\left(\frac{1}{n^\beta}\right) \quad \text{because} \quad \frac{1}{n^\beta} = o\left(\frac{1}{(n[1 + \log_2(n)])^\alpha}\right).
\]
With Theorem 3.4, similar $L^p$ estimates for the logistic function and $L^2$ estimates for piecewise polynomial activation functions hold true, see direct $L^2$-norm estimate (2.14). With one input node ($m = 1$), a lower estimate for piecewise polynomial activation functions without the log-factor can be proved easily, see [12]. Thus, the bound in Theorem 3.4 might be improvable.

Future work can deal with sharpness of error bound (2.5) for synchronous approximation in the multivariate case. By extending quantitative uniform boundedness principles with multiple error functionals (cf. [10], [14], [15]) to non-linear approximation (cf. proof of Theorem 3.1 in [12]), one might be able to show simultaneous sharpness in different (semi-) norms.

References

[1] Adams, R.A.: Sobolev Spaces. Academic Press, New York, NY (1975)
[2] Anastassiou, G.: Rate of convergence of some multivariate neural network operators to the unit. Comput. Math. Appl. 40, 1–19 (2000)
[3] Bagby, T., Bos, L., Levenberg, N.: Multivariate simultaneous approximation. Constructive Approximation 18, 569–577 (2002)
[4] Barron, A.R.: Universal approximation bounds for superpositions of a sigmoidal function. IEEE Transactions on Information Theory 39(3), 930–945 (1993)
[5] Bartlett, P.L., Williamson, R.C.: The VC dimension and pseudodimension of two-layer neural networks with discrete inputs. Neural Computation 8(3), 625–628 (1996)
[6] Chen, D.: Degree of approximation by superpositions of a sigmoidal function. Approximation Theory and its Applications 9(3), 17–28 (1993)
[7] Costarelli, D.: Sigmoidal Functions Approximation and Applications. Doctoral thesis, Roma Tre University, Rome (2014)
[8] Costarelli, D., Spigler, R.: Approximation results for neural network operators activated by sigmoidal functions. Neural Networks 44, 101–106 (2013)
[9] Costarelli, D., Spigler, R.: Multivariate neural network operators with sigmoidal activation functions. Neural Networks 48C, 72–77 (2013)
[10] Dickmeis, W.: On quantitative condensation of singularities on sets of full measure. Approx. Theory Appl. 1, 71–84 (1985)
[11] Dickmeis, W., Nessel, R.J., van Wickeren, E.: Quantitative extensions of the uniform boundedness principle. Jahresber. Deutsch. Math.-Verein. 89, 105–134 (1987)
[12] Goebbels, S.: On sharpness of error bounds for approximation by single hidden layer feedforward neural networks. arXiv (2019). URL http://arxiv.org/abs/1811.05199
[13] Goebbels, S.: A counterexample regarding “New study on neural networks: the essential order of approximation”. Neural Networks 123, 234–235 (2020)
[14] Imhof, L., Nessel, R.J.: The sharpness of a pointwise error bound for the Fejér-Hermite interpolation process on sets of positive measure. Appl. Math. Lett. 7, 57–62 (1994)
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[15] Imhof, L., Nessel, R.J.: A resonance principle with rates in connection with
pointwise estimates for the approximation by interpolation processes. Numer.
Funct. Anal. Optim. 16, 139–152 (1995)

[16] Ito, Y.: Extension of approximation capability of three layered neural networks to
derivatives. In: M. Marinaro, R. Tagliaferri (eds.) IEEE International Conference
on Neural Networks, pp. 377–381 vol. 1 (1993)

[17] Johnen, H., Scherer, K.: On the equivalence of the K-functional and moduli of
continuity and some applications. In: W. Schempp, K. Zeller (eds.) Constructive
Theory of Functions of Several Variables. Proc. Conf. Oberwolfach 1976, pp. 119–
140 (1976)

[18] Kůrková, V.: Rates of approximation of multivariable functions by one-hidden-
layer neural networks. In: M. Marinaro, R. Tagliaferri (eds.) Neural Nets WIRN
VIETRI-97, Perspectives in Neural Computing, pp. 147–152. Springer, London
(1998)

[19] Lin, S., Rong, Y., Xu, Z.: Multivariate jackson-type inequality for a new type
neural network approximation. Applied Mathematical Modelling 38(24), 6031 –
6037 (2014)

[20] Maiorov, V., Ratsaby, J.: On the degree of approximation by manifolds of finite
pseudo-dimension. Constructive Approximation 15, 291–300 (1999)

[21] Maiorov, V.E.: On best approximation by ridge functions. J. Approx. Theory
90, 66–94 (1999)

[22] Maiorov, V.E., Meir, R.: On the near optimality of the stochastic approximation
of smooth functions by neural networks. Advances in Computational Mathematics
13, 79–103 (2000)

[23] Pinkus, A.: Approximation theory of the MLP model in neural networks. Acta
Numerica 8, 143–195 (1999)

[24] Ritter, G.: Efficient estimation of neural weights by polynomial approximation.
IEEE Transactions on Information Theory 45(5), 1541–1550 (1999)

[25] Timan, A.: Theory of Approximation of Functions of a Real Variable. Pergamon
Press, New York, NY (1963)

[26] Wang, J., Xu, Z.: New study on neural networks: the essential order of approxi-
mation. Neural Networks 23(5), 618 – 624 (2010)

[27] Xie, T., Cao, F.: The errors of simultaneous approximation of multivariate func-
tions by neural networks. Computers & Mathematics with Applications 61(10),
3146 – 3152 (2011)