SLICE CONFORMALITY: RIEMANN MANIFOLDS AND LOGARITHM ON QUATERNIONS AND OCTONIONS

GRAZIANO GENTILI, JASNA PREZELJ, AND FABIO VLACCI

Abstract. We establish the quaternionic and octonionic analogs of the classical Riemann surfaces of the complex logarithm and \( n \)-th root function, and give a unifying definition of such functions in the quaternionic and octonionic settings. The construction of these manifolds has nice peculiarities and the scrutiny of Bernhard Riemann approach to Riemann surfaces, mainly based on conformality, leads to the definition of slice conformal or slice isothermal parameterization of a Riemann 4-manifold and 8-manifold. These new classes of manifolds include slice regular quaternionic and octonionic curves, graphs of slice regular functions, the 4 and 8 dimensional spheres, the helicoidal and catenoidal 4 and 8 dimensional manifolds.

1. Preface

The project originating this paper is that of giving a well structured and unifying definition of the logarithm and \( n \)-th root functions in the quaternionic and octonionic settings. The aim was, first of all, to construct the quaternionic and octonionic analogs of the well known Riemann surface of the complex logarithm, which in the complex setting allows a complete understanding of this function and of its branches. For the case of the principal branch of the logarithm, definitions were already given in the general setting of Clifford Algebras - see, e.g., [15, Definition 11.24, p. 231] - and also specialized to the case of quaternions - see, e.g., [9, Definition 3.4].

Indeed, a detailed contruction of manifolds which can be used as domains of definition for the quaternionic and octonionic logarithm and their different branches is one of the subjects that we present in this paper.

2020 Mathematics Subject Classification. 30G35; 30C35; 30F99.

Key words and phrases. Slice regular functions, Conformal mappings, Riemann surfaces.

The first and third authors were partly supported by INdAM, through: GNSAGA; INdAM project “Hypercomplex function theory and applications”. It was also partly supported by MIUR, through the projects: Finanziamento Premiale FOE 2014 “Splines for accurate NumRics: adaptive models for Simulation Environments”. The second author was partially supported by research program P1-0291 and by research projects J1-7256 and J1-9104 at Slovenian Research Agency. The third author was also partially supported by PRIN 2017 “Real and complex manifolds: topology, geometry and holomorphic dynamics”.

1
Let $\mathbb{K}$ denote either the division algebra of quaternions $\mathbb{H}$ or that of octonions $\mathbb{O}$, and let $S_\mathbb{K} \subset \mathbb{K}$ be the 2-sphere or, respectively, the 6-sphere of imaginary units, i.e. the sets of $I \in \mathbb{K}$ such that $I^2 = -1$. For the sake of simplicity, both in the case of quaternions and in the case of octonions we will simply write $S$ instead of $S_\mathbb{K}$ since no confusion can arise. The construction of the logarithm and its branches given in the complex case cannot be directly replicated in the quaternionic and octonionic environments. This is mainly due to the fact that the exponential function

$$\exp q = \sum_{n=1}^{\infty} \frac{q^n}{n!}$$

is an entire function (i.e., its domain of definition is $\mathbb{K}$), but cannot be used to define a covering of $\mathbb{K} \setminus \{0\}$. In fact, for all $0 > x \in \mathbb{R}$, the preimage of $x$ is not a discrete set but consists of infinitely many 2 or 6 dimensional spheres. Indeed, setting $S(2k+1)\pi = \{q(2k+1)\pi : q \in S\}$, we have

$$(\exp)^{-1}(x) = \{\log |x| + S(2k+1)\pi : k \in \mathbb{Z}\}$$

As a consequence, contrarily to what happens in the case of the complex logarithm, no continuous branch of the quaternionic or octonionic logarithm can be defined in any open neighborhood of any strictly negative $x \in \mathbb{R}$. A similar phenomenon happens for all strictly positive $x \in \mathbb{R}$. To overcome this difficulty, we turn our attention to the construction of a 4-dimensional, respectively 8-dimensional, manifold obtained by blowing-up $\mathbb{K}$ along the real axis, and “adapting” it to become a domain of definition for the quaternionic or octonionic logarithm. Our natural approach to perform this construction passes through the recent theory of slice regular functions - see, e.g., the monograph [7] and references therein - and leads to the helicoidal 4-manifold and 8-manifold (which are manifolds in the sense of [3]) inspired by the classical helicoidal surface of the space $\mathbb{R}^3$. These manifolds have new peculiar features and provide a natural environment for the definition for the quaternionic and octonionic logarithm, and for all their possible branches. Once done this for the logarithm, the construction of natural manifolds of definition for the 2-nd and $n$-th root quaternionic and octonionic functions is an easily doable step.

Some geometrical peculiarities of these manifolds, constructed with the purpose specified above, attracted the attention of the authors and encouraged them to go back to the scrutiny of Bernhard Riemann approach to holomorphic functions and Riemann surfaces, which was mainly based on conformality, as in [16]. All this led to a deeper appreciation of the work of Riemann, to a nice surprise and to Definition 3.2 of slice conformal or slice isothermal parameterization and of Riemann 4-manifold and Riemann 8-manifold. Let $\langle \ , \ \rangle$ denote the standard Euclidean scalar product in $\mathbb{K}$ and,
for any purely imaginary unit $I \in \mathbb{K}$, set
\[ C_I^\perp = \{ q \in \mathbb{K} : \langle q, x + Iy \rangle = 0, \forall (x + Iy) \in C_I \} \]
to be the orthogonal space to the slice $C_I = \mathbb{R} + I\mathbb{R}$. An at least $C^1$ injective $\mathbb{R}^N$-valued immersion $f$ defined on a suitable domain $\Omega$ of $\mathbb{K}$ is called slice conformal or slice isothermal if for any purely imaginary unit $I \in \mathbb{K}$ and any $x, y \in \mathbb{R}$ the differential $df(x + Iy)$ is such that both
\[ df(x + Iy)|_{C_I} \]
and
\[ df(x + Iy)|_{C_I^\perp} \]
are conformal. If this is the case, $f(\Omega)$ is called a Riemann manifold. The nice surprise was that the helicoidal 4-manifold and 8-manifold are Riemann manifolds as well as the manifolds constructed as domains for the $n$-th root functions on $\mathbb{K}$.

Slice conformality is indeed an extension of the definition of slice regularity, and reduces to it when the immersion $f$ is $\mathbb{K}$-valued. It is worthwhile noticing that to require that the real differential $df$ be conformal may be too restrictive: by a classical result due to Liouville, for $n > 2$ a conformal map from a domain of $\mathbb{R}^n$ to $\mathbb{R}^n$ is a Möbius transformation.

At this point the authors defined and used a four-curve apparatus and an eight-curve apparatus to study the real differential of at least $C^1$ injective $\mathbb{R}^N$-valued immersions $f$ defined on suitable domains $\Omega$ of $\mathbb{K}$. As a result, the paper can exhibit a collection of Riemann 4 and 8-manifolds, inspired by classical Riemann surfaces, which testify the interest of the approach.

The paper is organized as follows. After very few preliminaries, Section 3 is dedicated to the definition and construction of a collection of examples of Riemann 4-manifolds, studied with the use of the four-curve apparatus: quaternionic regular curves, graphs of quaternionic slice regular functions, the Riemann 4-sphere; this section includes the construction of the helicoidal 4-manifold, the catenoidal 4-manifold and the study of the relations between them. Section 4 contains the construction of the Riemann 4-manifold of the logarithm and its detailed study. The constructions of the Riemann 4-manifolds of the 2-nd and $n$-th root quaternionic functions appear in Section 5. At last, Section 6 easily extends all the results of the previous sections to the case of octonions, by means of the eight-curve apparatus. This last Section is mainly intended for the reader who has no familiarity at all with octonions.

2. Preliminaries

As we said, $\mathbb{K}$ denotes either $\mathbb{H}$ or $\mathbb{O}$, i.e., the algebras of quaternions or octonions, and $\mathbb{S} \subset \mathbb{K}$ denotes, respectively, the 2-sphere or 6-sphere of
imaginary units, i.e. the set of $I \in \mathbb{K}$ such that $I^2 = -1$. Given any non real $q \in \mathbb{K}$, there exist (and are uniquely determined) an imaginary unit of $\mathbb{K}$, and two real numbers $x$ and $y > 0$, such that $q = x + Iy$. With this notation, the conjugate of $q$ will be $\bar{q} := x - Iy$ and $|q|^2 = q\bar{q} = qq = x^2 + y^2$. In both cases, each imaginary unit $I$ generates (as a real algebra) a copy of the complex plane denoted by $\mathbb{C}_I$. We call such a complex plane a slice.

Let $\Omega$ be a slice domain of $\mathbb{K}$, i.e., an open and connected subset containing real points and such $\Omega_I = \Omega \cap \mathbb{C}_I$ is a domain of $\mathbb{C}_I$ for all purely imaginary units $I$ of $\mathbb{K}$. The set of slice-regular functions on $\Omega$ is defined using a family of Cauchy-Riemann operators (see e.g. [7, 8]).

**Definition 2.1.** Let $\Omega \subseteq \mathbb{K}$ be a slice domain and let $f : \Omega \to \mathbb{K}$ be a function. If, for an imaginary unit $I$ of $\mathbb{K}$, the restriction $f_I := f|_{\Omega_I}$ has continuous partial derivatives and

$$\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) \equiv 0,$$

then $f_I$ is called holomorphic. If $f_I$ is holomorphic for all imaginary units of $\mathbb{K}$, then the function $f$ is called slice regular.

The property of being holomorphic along the slices $\Omega_I$ for all imaginary units $I$ of $\mathbb{K}$, forces slice regular functions to be affine along entire regions of each sphere of type $x + S_I$. In fact, the local representation formula for quaternionic slice regular functions on slice domains (see, e.g., [5, 6]), states that, if $L, M, N \in S$, with $M \neq N$, are such that $x + Ly, x + My, x + Ny$ belong to a suitable open neighborhood of $x + Iy$ in the 2-sphere $x + S_I$, then

$$f(x + Ly) = (M - N)^{-1} [Mf(x + My) - Nf(x + Ny)] + L(M - N)^{-1} [f(x + My) - f(x + Ny)]$$

where

$$f'_s(x + Iy) := y^{-1}(M - N)^{-1} [f(x + My) - f(x + Ny)]$$

is the spherical derivative of $f$, constant in the same neighborhood of $x + Iy$ in $x + S_I$ (see, e.g., [6, Definition 3.1]). The analog of this representation formula holds for octonionic slice regular functions as well (see Subsection 6.1). A subclass of slice-regular functions particularly resemble the class of holomorphic functions of one complex variable; these functions are defined as follows: $f : \Omega \to \mathbb{K}$ is said to be slice-preserving if, and only if, for all imaginary units $I$ of $\mathbb{K}$, we have that $f(\Omega_I) \subseteq \mathbb{C}_I$, (see [8] for the case of octonions).

For most of the rest of the properties of slice regular functions that will be used in the sequel we will mainly refer the reader to [7, 8]. As for the
main applications and developments of this theory, the reader can consult [1, 2, 5, 6, 14], and e.g. [13] for generalizations.

3. PARAMETERIZED RIEMANN 4-MANIFOLDS

Following the case of classical parameterized surfaces and parameterized Riemann surfaces in \( \mathbb{R}^3 \), we will give the following definitions, useful in the quaternionic setting of slice regular functions. As customary, a differentiable map will be called an immersion if its differential is injective at all points of the domain of definition.

**Definition 3.1.** Let \( n, N \) be natural numbers with \( N \geq n \) and let \( \Omega \) be domain in \( \mathbb{R}^n \). An at least \( C^1 \) immersion

\[
f : \Omega \rightarrow \mathbb{R}^N
\]

will be called a conformal or isothermal map if the matrix of the differential of \( f \) is conformal, i.e., if it satisfies

\[
^t df(x)df(x) = k(x)I_n
\]

for a (never vanishing at least \( C^1 \)) function \( k : \Omega \rightarrow \mathbb{R} \).

If \( \langle \cdot, \cdot \rangle \) denotes the Euclidean scalar product in \( \mathbb{H} \cong \mathbb{R}^4 \), set \( C^I_4 = \{ q \in \mathbb{H} : \langle q, x + Iy \rangle = 0, \forall (x + Iy) \in C_I \} \) to be the orthogonal space to the slice \( C_I \).

**Definition 3.2.** Let \( \Omega \) be a slice domain in \( \mathbb{H} \cong \mathbb{R}^4 \) and let \( N \geq 4 \) be a natural number. Let

\[
f : \Omega \rightarrow \mathbb{R}^N
\]

be an at least \( C^1 \) immersion. If for any \( I \in \mathbb{S} \) and any \( x, y \in \mathbb{R} \) the differential \( df(x + Iy) \) is such that both

\[
df(x + Iy)|_{C_I}
\]

and

\[
df(x + Iy)|_{C^I_4}
\]

are conformal, then \( f \) will be called a slice conformal or slice isothermal immersion.

If, in addition, \( f \) is injective, then it will be called slice conformal or slice isothermal parameterization and the parameterized 4-manifold \( f(\Omega) \) in \( \mathbb{R}^N \) will be called a (parameterized) Riemann 4-manifold of \( \mathbb{R}^N \).

In case \( f : \Omega \rightarrow \mathbb{R}^N \) itself is a conformal parameterization, then the parameterized 4-manifold \( f(\Omega) \) in \( \mathbb{R}^N \) will be called a special (parameterized) Riemann 4-manifold of \( \mathbb{R}^N \).
The notion of parameterized Riemann 4-manifold turns out to be quite natural, as the significant examples that we will present show. To construct the examples we will need a direct and easy method to compute the differential of an at least $C^1$ immersion $f$ defined in a slice domain $\Omega$ of $\mathbb{H} \cong \mathbb{R}^4$ and with values in $\mathbb{R}^N$.

3.1. The four-curve apparatus and the case of the differential of a regular function. For $I \in \mathbb{S}$, let us consider a point $x + Iy \in \mathbb{C}_I \subset \mathbb{H}$ and choose $J \in \mathbb{S}$ orthogonal to $I$. Then if $K := IJ$, one has an orthonormal frame $\{1, I, J, K\}$ in $\mathbb{H}$ and can define the following set of four curves, called four-curve apparatus. When $y \neq 0$ we have

1. the curve $\alpha(t) = (x+t)+Iy$, such that $\alpha(0) = x+Iy$ and $\alpha'(0) = 1$;

2. the curve $\beta(t) = x+I(y+t)$, such that $\beta(0) = x+Iy$ and $\beta'(0) = I$;

3. the curve $\Gamma_J(t) = x + \gamma(t)y$, where $\gamma(t)$ is an arc of a maximum circle $C_\gamma$ of $\mathbb{S}$ such that $\gamma(0) = I$ and that $\gamma'(0) = \frac{J}{y}$; hence $\Gamma_J(0) = x + Iy$ and $\Gamma'_J(0) = J$;

4. the curve $\Gamma_K(t) = x + \delta(t)y$, where $\delta(t)$ is an arc of a maximum circle $C_\delta$ of $\mathbb{S}$ orthogonal to $C_\gamma$ such that $\delta(0) = I$ and that $\delta'(0) = \frac{K}{y}$; hence $\Gamma_K(0) = x + Iy$ and $\Gamma'_K(0) = K$.

Instead, when $y = 0$ and hence $x + Iy = x$, the first curve of the apparatus remains unchanged

1. $\alpha(t) = (x+t) + Iy$, such that $\alpha(0) = x + Iy$ and $\alpha'(0) = 1$;

and the three last curves coherently become:

2)-(4) $\beta_I(t) = x + It, \beta_J(t) = x + Jt, \beta_K(t) = x + Kt$, such that $\beta_I(0) = \beta_J(0) = \beta_K(0) = x$ and $\beta'_I(0) = I, \beta'_J(0) = J, \beta'_K = K$.

We desire now to test the four-curve apparatus and to point out some of its features. Indeed, when naturally used with a slice regular function $f : \Omega \to \mathbb{H}$, defined on a slice domain $\Omega$ of $\mathbb{H}$, this apparatus reveals an easy tool to compute and directly interpret the real differential $df(x + Iy) : \mathbb{R}^4 \to \mathbb{R}^4$ of the function $f$. But its full use will be seen in the sequel of this paper, in more general situations.

Fix $I \in \mathbb{S}$. Now, by the local representation formula for slice regular functions on slice domains (see Section 2), we have that, if $L, M, N \in \mathbb{S}$ with $M \neq N$ are such that $x + Ly, x + My, x + Ny$ belong to suitable open neighborhood of $x + Iy$ in the 2-sphere $x + \mathbb{S}_y$, then

3.2 $f(x + Ly) = (M - N)^{-1} [Mf(x + My) - Nf(x + Ny)] + L(M - N)^{-1} [f(x + My) - f(x + Ny)]$
where
\[ f_s'(x + Iy) := y^{-1}(M - N)^{-1} [f(x + My) - f(x + Ny)] \]
is the spherical derivative of \( f \).

For \( J, K \in \mathbb{S} \) with \( I \perp J \) and \( K =IJ \), the four units \( \{ 1, I, J, K \} \) are a standard basis of the division algebra of quaternions, and we can use the associated four-curve apparatus. A direct computation shows that
\[
\frac{df(x + Iy)1}{dt} = \frac{d}{dt} f(x + t + Iy) = \frac{df}{dx}(x + Iy)
\]
Analogously, and since \( f \) is slice regular,
\[
\frac{df(x + Iy)I}{dt} = \frac{d}{dt} f(x + I(y + t)) = \frac{df}{dy}(x + Iy)
\]
In particular we have that
\[
Idf(x + Iy)J = IJf_s'(x + Iy) = Kf_s'(x + Iy) = df(x + Iy)K
\]
is a conformal matrix. Let us now continue. The local representation formula \[3.2\] yields
\[
df(x + Iy)J = df(x + Iy)\Gamma_J'(0) = \frac{d}{dt} f(\Gamma_J(t))
\]
\[
= \frac{d}{dt} (\gamma(t)(M - N)^{-1} [f(x + My) - f(x + Ny)])
\]
\[
= Jy^{-1}(M - N)^{-1} [f(x + My) - f(x + Ny)]
\]
\[
= Jf_s'(x + Iy)
\]
\[
df(x + Iy)K = df(x + Iy)\Gamma_K'(0) = \frac{d}{dt} f(\Gamma_K(t))
\]
\[
= \frac{d}{dt} (\gamma(t)(M - N)^{-1} [f(x + My) - f(x + Ny)])
\]
\[
= Ky^{-1}(M - N)^{-1} [f(x + My) - f(x + Ny)]
\]
\[
= Kf_s'(x + Iy)
\]
Since
\[
Idf(x + Iy)J = IJf_s'(x + Iy) = Kf_s'(x + Iy) = df(x + Iy)K
\]
we have that
\[ df(x + Iy)_{|C^I} : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \]
is conformal. In conclusion, it turns out that, even if both \( df(x + Iy)_{|C^I} : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) and \( df(x + Iy)_{|C^I} : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) are conformal, the full differential
\[ df(x + Iy) : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \]
is not conformal in general.

3.2. Quaternionic slice regular curves. We will use the standard notion of curve in the quaternionic setting.

**Definition 3.3.** Let \( \Omega \subseteq \mathbb{H} \) be a slice domain, and let
\[ F : \Omega \rightarrow \mathbb{H}^2 \]
\[ F(q) = (f(q), g(q)) \]
be a map whose components \( f, g : \Omega \rightarrow \mathbb{H} \) are slice regular functions. If \( F \) is an immersion, then \( F \) will be called a slice regular curve (in \( \mathbb{H}^2 \)).

Let us now consider a slice regular curve \( F : \Omega \rightarrow \mathbb{H}^2 \), with slice regular components \( f, g : \Omega \rightarrow \mathbb{H} \), and choose any \( I \in \mathbb{S} \) and any \( J \in \mathbb{S} \) orthogonal to \( I \). Then, setting \( K = IJ \) and using, with the standard basis \( \{1, I, J, K\} \) of \( \mathbb{H} \), the four curves apparatus defined in Subsection 3.1, we get that the differential
\[ dF(x + Iy) : \mathbb{R}^4 \rightarrow \mathbb{H}^2 \]
assumes the form
\[ dF(x + Iy) = \begin{bmatrix}
\frac{\partial f}{\partial x}(x + Iy) & I \frac{\partial f}{\partial x}(x + Iy) & J f_s'(x + Iy) & K f_s'(x + Iy) \\
\frac{\partial g}{\partial x}(x + Iy) & I \frac{\partial g}{\partial x}(x + Iy) & J g_s'(x + Iy) & K g_s'(x + Iy)
\end{bmatrix} \]
which can be interpreted as a \( 8 \times 4 \) real matrix. The two first columns of this \( 8 \times 4 \) real matrix, and separately the last two columns of the same matrix, are orthogonal and with the same norm, and hence \( F \) is isothermal. In conclusion we have proved

**Proposition 3.4.** Let \( \Omega \subseteq \mathbb{H} \) be a slice domain, and let \( F : \Omega \rightarrow \mathbb{H}^2 \) be a slice regular curve. If \( F \) is injective, then \( F(\Omega) \) is a parameterized Riemann 4-manifold in \( \mathbb{H}^2 \), and the map \( F : \Omega \rightarrow F(\Omega) \) is a slice conformal parameterization.

As we already pointed out, in general \( F \) is (a slice conformal but) not a conformal parameterization. It is well known in fact that the slice regular functions \( f, g \) are in general not conformal at non real points of \( \Omega \) (see, e.g., [7]), and hence \( F \) cannot be a conformal parameterization in general.
We end this section with a natural question, on how the quaternionic parameter can be changed between slice regular quaternionic curves having the same image. Indeed, let us consider $\Omega, \Omega' \subseteq \mathbb{H}$ slice domains, and $F = (f_1, f_2) : \Omega \to \mathbb{H}^2$ and $G = (g_1, g_2) : \Omega' \to \mathbb{H}^2$ injective, slice regular curves with the same image $F(\Omega) = G(\Omega')$. In this situation, we may assume that locally $g_1$ is injective. Then the local equalities $f_1(q) = g_1(q')$ and $f_2(q) = g_2(q')$ imply

$$f_2 = g_2 \circ (g_1^{-1} \circ f_1)$$

and since $f_2, g_2 : \Omega' \to \mathbb{H}$ are slice regular functions, this functional equation is in general not valid. Nevertheless we know that it holds if, for instance, $g_1^{-1} \circ f_1 : \Omega \to \Omega'$ is a slice preserving regular function. Hence, we can make the following

**Remark 3.5.** Let $F$ and $G$ be injective, slice regular curves having the same image $\Gamma \subseteq \mathbb{H}^2$. If a change of quaternionic parameter between $F$ and $G$ is a slice preserving invertible function, then the two slice regular curves (both being slice conformal parameterizations) belong to a same atlas of the parameterized Riemann 4-manifold $\Gamma$.

What established in this section can be directly reformulated for the case of slice regular curves $F = (f_1, \ldots, f_n) : \Omega \to \mathbb{H}^n$ from a slice domain $\Omega \subseteq \mathbb{H}$ to $\mathbb{H}^n$. To conclude, we point out that remark 3.5 is valid in a more general setting. Indeed it easy to prove that

**Remark 3.6.** Let $F$ be a slice isothermal parameterization having the Riemann 4-manifold $\Gamma \subseteq \mathbb{R}^N$ as its image. Then, for every slice preserving invertible change of parameter $\phi$ between slice domains, the map $F \circ \phi$ is a slice isothermal parameterization which belongs to the same atlas of $F$ for the parameterized Riemann 4-manifold $\Gamma$.

### 3.3. Graphs of quaternionic slice regular functions

Let $\Omega \subseteq \mathbb{H}$ be a slice domain, and let $f : \Omega \to \mathbb{H}$ be a slice regular function. Let us consider the slice regular curve $F : \Omega \to \mathbb{H}^2 \cong \mathbb{R}^8$ defined as

$$F(q) = (q, f(q))$$

Since the curve $F$ is clearly injective, as a particular case of what established in Proposition 3.4 we obtain

**Proposition 3.7.** Let $\Omega \subseteq \mathbb{H}$ be a slice domain, and let $f : \Omega \to \mathbb{H}$ be a slice regular function. Then the graph of $f$

$$\Gamma(f) = \{(q, f(q)) : q \in \Omega\} \subseteq \mathbb{H} \times \mathbb{H}$$
is a parameterized Riemann 4-manifold, and the map $F : \Omega \to \Gamma(f)$ defined by $F(q) = (q, f(q))$ is a slice conformal parameterization.

The following remark should better explain the definition of slice conformal immersion that has been adopted.

**Remark 3.8.** Let $\{1, i, j, k\}$ be the standard basis of $\mathbb{H}$, and let $F : \mathbb{H} \to \mathbb{H}^2$ be the function

$$F(x + Iy) = (x + Iy, x + \psi(I)y)$$

where $\psi : S \to S$ is the odd $C^\infty$ function defined by

$$\psi(\alpha i + \beta j + \gamma k) = \frac{\alpha^3 i + \beta j + \gamma^3 k}{\sqrt{\alpha^6 + \beta^2 + \gamma^6}}$$

i.e., when $\langle \, , \, \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^4 \cong \mathbb{H}$, by

$$\psi(I) = \frac{\langle I, i \rangle^3 i + \langle I, j \rangle j + \langle I, k \rangle^3 k}{\sqrt{\langle I, i \rangle^6 + \langle I, j \rangle^2 + \langle I, k \rangle^6}}$$

For any $I \in S$, let us consider a point $x + Iy \in \mathbb{C}_I \subset \mathbb{H}$ and $J \in S$ orthogonal to $I$. Then if $K := IJ$, one has an orthonormal frame $\{1, I, J, K\}$ and can use the four-curve apparatus defined in Subsection 3.1 to compute the differential $dF : \mathbb{R}^4 \to \mathbb{R}^8$ at the point $x + Iy$:

$$dF(x + Iy)1 = dF(x + Iy)\alpha'(0) = \frac{d}{dt} \big|_0 F(\alpha(t)) = (1, 1)$$

$$dF(x + Iy)I = dF(x + Iy)\beta'(0) = \frac{d}{dt} \big|_0 F(\beta(t)) = (I, \psi(I))$$

$$dF(x + Iy)J = dF(x + Iy)\Gamma'_J(0) = \frac{d}{dt} \big|_0 F(\Gamma_J(t))$$

$$= \frac{d}{dt} \big|_0 (x + \Gamma_J(t)y, x + \psi(\Gamma_J(t))y)$$

$$= (\Gamma'_J(t)y, d\psi(\Gamma_J(t))\Gamma'_J(t)y)$$

$$= (J, d\psi(I)J)$$

$$dF(x + Iy)K = dF(x + Iy)\Gamma'_K(0) = \frac{d}{dt} \big|_0 F(\Gamma_K(t))$$

$$= \frac{d}{dt} \big|_0 (x + \Gamma_K(t)y, x + \psi(\Gamma_K(t))y)$$

$$= (\Gamma'_K(t)y, d\psi(\Gamma_K(t))\Gamma'_K(t)y)$$

$$= (K, d\psi(I)K)$$
While applying the four-curve apparatus take the point \( x + Iy = x + i y \) (i.e., \( I = i \)) with \( y \neq 0 \), choose \( J = j \) and use the curves

\[
\Gamma_j(t) = i\cos(t/y) + j\sin(t/y), \quad \Gamma_k(t) = i\cos(t/y) + k\sin(t/y)
\]

Direct computations show that

\[
dF(x + iy)j = \frac{d}{dt}|_{0} \left( x + (i\cos(t/y) + j\sin(t/y))y, x + \frac{i\cos^3(t/y) + j\sin(t/y)}{\sqrt{\cos^6(t/y) + \sin^2(t/y)}} y \right)
\]

\[
= (j, j)
\]

and

\[
dF(x + iy)k = \frac{d}{dt}|_{0} \left( x + (i\cos(t/y) + k\sin(t/y))y, x + \frac{i\cos^3(t/y) + j\sin^3(t/y)}{\sqrt{\cos^6(t/y) + \sin^6(t/y)}} y \right)
\]

\[
= (k, 0)
\]

Thus:

\[
dF(x + iy) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

and hence the last two columns have different norms. In conclusion such an \( F \) is not an isothermal parameterization.

3.4. The Riemann 4-sphere. This example generalizes to dimension 4 the case of the Riemann sphere in the complex setting. Let us consider the map

\[
f : \mathbb{R}^4 \cong \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5
\]

defined by

\[
f(x + Iy) = \begin{pmatrix} 2(x + Iy) \\ 1 + x^2 + y^2 \\ -1 + x^2 + y^2 \\ 1 + x^2 + y^2 \end{pmatrix}
\]

Notice that the two representations

\[
x + Iy = x + (-I)(-y)
\]
of the same quaternion are associated - via \( f \) - with the two representations of the same image pair of quaternions, namely

\[
\left( \frac{2(x + Iy)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) = \left( \frac{2(x + (-I)(-y))}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)
\]

We will now show that \( f \) is injective. Suppose indeed that for \( I, L \in S \) one has \( f(x + Iy) = f(u + Lv) \) i.e.,

\[
\left( \frac{2(x + Iy)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) = \left( \frac{2(u + Lv)}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right)
\]

It follows, first of all, that

\[
\frac{Lv}{1 + u^2 + v^2} = \frac{Iy}{1 + x^2 + y^2}
\]

which leads either to \( v = y = 0 \) or, when \( y \neq 0 \), to one of the following two possibilities

\[
\frac{v}{1 + u^2 + v^2} = \frac{y}{1 + x^2 + y^2} \quad \text{and} \quad L = I
\]

or

\[
\frac{v}{1 + u^2 + v^2} = \frac{-y}{1 + x^2 + y^2} \quad \text{and} \quad L = -I
\]

If \( v = y = 0 \) we get

\[
\left( \frac{2x}{1 + x^2}, \frac{-1 + x^2}{1 + x^2} \right) = \left( \frac{2u}{1 + u^2}, \frac{-1 + u^2}{1 + u^2} \right)
\]

The equality of the first components of the two vectors yields that \( x \) and \( u \) have the same sign, and since \( -\frac{1 + t}{1 + t} \) is injective, the equality of the last components implies \( x^2 = u^2 \) and so \( x = u \), whence \( x + Iy = u + Lv \).

In the second case, the equality of the last components implies that \( u^2 + v^2 = x^2 + y^2 \) and then the equality of the first components implies \( x = u \), whence \( y^2 = v^2 \) and therefore \( f \) is injective.

We will now prove that the real differential \( df \) of \( f \) has maximal rank at all points of \( \mathbb{H} \), and hence that \( f \) is an immersion.

To do this we will consider a point \( x + Iy \in C_I \subset \mathbb{H} \) with \( J \in S \) orthogonal to \( I \). Then if \( K := IJ \), one has an orthonormal frame \( \{1, I, J, K\} \) in \( \mathbb{H} \). By using the four-curve apparatus defined in Subsection 3.1 we can now compute:
\begin{align*}
\text{df}(x + Iy)1 &= \text{df}(x + Iy)\alpha'(0) = \frac{d}{dt|_0} f(\alpha(t)) \\
&= \frac{d}{dt|_0} \left( \frac{2((x + t) + Iy)}{1 + (x + t)^2 + y^2}, \frac{-1 + (x + t)^2 + y^2}{1 + (x + t)^2 + y^2} \right) \\
&= \frac{2}{(1 + x^2 + y^2)^2} (1 - x^2 + y^2 - 2Ixy, 2x) \\
\text{df}(x + Iy)I &= \text{df}(x + Iy)\beta'(0) = \frac{d}{dt|_0} f(\beta(t)) \\
&= \frac{d}{dt|_0} \left( \frac{2(x + I(y + t))}{1 + x^2 + (y + t)^2}, \frac{-1 + x^2 + (y + t)^2}{1 + x^2 + (y + t)^2} \right) \\
&= \frac{2}{(1 + x^2 + y^2)^2} (-2xy + I(1 + x^2 - y^2), 2y) \\
\text{df}(x + Iy)J &= \text{df}(x + Iy)\Gamma'_J(0) = \frac{d}{dt|_0} f(\Gamma_J(t)) \\
&= \frac{d}{dt|_0} \left( \frac{2(x + \gamma(t)y)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) \\
&= \frac{2}{(1 + x^2 + y^2)^2} (J(1 + x^2 + y^2), 0) \\
\text{df}(x + Iy)K &= \text{df}(x + Iy)\Gamma'_K(0) = \frac{d}{dt|_0} f(\Gamma_K(t)) \\
&= \frac{d}{dt|_0} \left( \frac{2(x + \delta(t)y)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) \\
&= \frac{2}{(1 + x^2 + y^2)^2} (K(1 + x^2 + y^2), 0) \\
\end{align*}

If we now set

\( \mathbb{H} \ni x_1 + x_2 I + x_3 J + x_4 K \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \)

and

\( \mathbb{H} \times \mathbb{R} \ni (x_1 + x_2 I + x_3 J + x_4 K, y_1) \cong (x_1, x_2, x_3, x_4, y_1) \in \mathbb{R}^5 \)
we have
\[
df(x + Iy) = \frac{2}{(1 + x^2 + y^2)^2} \begin{bmatrix}
1 - x^2 + y^2 & -2xy & 0 & 0 \\
-2xy & 1 + x^2 - y^2 & 0 & 0 \\
0 & 0 & 1 + x^2 + y^2 & 0 \\
2x & 2y & 0 & 0
\end{bmatrix}
\]

The four columns of \(df(x + Iy)\) are orthogonal with all norms equal to
\[
2 \frac{1}{(1 + x^2 + y^2)^2} \neq 0
\]
and so, \(f : \mathbb{R}^4 \cong \mathbb{H} \to \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5\) is conformal.

Summing up we can state:

**Proposition 3.9.** Let us consider the 4-sphere \(S^4 \subset \mathbb{R}^5 \cong \mathbb{H} \times \mathbb{R}\) and the inverse of the stereographic projection from the north pole \(N = (0, 0, 0, 1)\) of \(S^4\) onto the equatorial plane \(\mathbb{H} \cong \mathbb{R}^4\), namely
\[
f : \mathbb{R}^4 \cong \mathbb{H} \to S^4 \setminus \{N\} \subset \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5,
\]
defined by
\[
f(x + Iy) = \left( \frac{2(x + Iy)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)
\]
Then \(S^4 \setminus \{N\}\) is a special parameterized Riemann 4-manifold and the map \(f\) is a conformal parameterization.

In a completely analogous way, the following statement can be proved for the stereographic projection from the south pole

**Proposition 3.10.** Let us consider the 4-sphere \(S^4 \subset \mathbb{R}^5 \cong \mathbb{H} \times \mathbb{R}\) and the inverse of the stereographic projection from the south pole \(S = (0, 0, 0, -1)\) of \(S^4\) to the equatorial plane \(\mathbb{H} \cong \mathbb{R}^4\), namely
\[
g : \mathbb{R}^4 \cong \mathbb{H} \to S^4 \setminus \{S\} \subset \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5,
\]
defined by
\[
g(x + Iy) = \left( \frac{2(x + Iy)}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right)
\]
Then \(S^4 \setminus \{S\}\) is a special parameterized Riemann 4-manifold and the map \(g\) is a conformal parameterization.

The proof of the following consequence is straightforward

**Corollary 3.11.** Let us consider the 4-sphere \(S^4 \subset \mathbb{R}^5 \cong \mathbb{H} \times \mathbb{R}\) and the map
\[
h : \mathbb{R}^4 \cong \mathbb{H} \to S^4 \subset \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5
\]
defined by \( h(x + I y) = g(x - I y) \), i.e., by

\[
(3.10) \quad h(x + I y) = \left( \frac{2(x - I y)}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right)
\]

Then \( S^4 \setminus \{ S \} \) is a special parameterized Riemann 4-manifold, and the map \( h \) is a conformal parameterization.

We can now conclude by exhibiting the “Riemann” structure of 1-dimensional quaternionic manifold of the sphere \( S^4 \subset \mathbb{R}^5 \).

**Theorem 3.12.** Let \( f \) and \( h \) be the following maps

\[
f : \mathbb{R}^4 \cong \mathbb{H} \to S^4 \setminus \{ N \} \subset \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5
\]

\[
f(x + I y) = \left( \frac{2(x + I y)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)
\]

and

\[
h : \mathbb{R}^4 \cong \mathbb{H} \to S^4 \setminus \{ S \} \subset \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5
\]

\[
h(x + I y) = \left( \frac{2(x - I y)}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right)
\]

Then the differentiable conformal atlas \( \{(\mathbb{H}, f), (\mathbb{H}, h)\} \) endows \( S^4 \subset \mathbb{R}^5 \) with a structure of slice quaternionic manifold (see [3]).

**Proof.** A direct computation shows that the transition map

\[
h^{-1} \circ f = g^{-1} \circ f : \mathbb{H} \setminus \{ 0 \} \to \mathbb{H} \setminus \{ 0 \}
\]

has the form

\[
h^{-1} \circ f(q) = g^{-1} \circ f(q) = \frac{\bar{q}}{q^2} = \frac{1}{q}
\]

and hence is a slice regular map. \(\square\)

### 3.5. The helicoidal 4-manifold

This further example generalizes to dimension 4 the case of the helicoid in the complex setting, whose classical isothermal parameterization is given by

\[
(3.11) \quad g : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}
\]
defined as

\[
(3.12) \quad g(x + i y) = (\sinh x \cos y + i \sinh x \sin y, y)
\]

In the case of quaternions, let us consider the map

\[
(3.13) \quad f : \mathbb{H} \cong \mathbb{R}^4 \to \mathbb{R}^7 \cong \mathbb{H} \times \text{Im}(\mathbb{H})
\]
defined by

\[
(3.14) \quad f(x + I y) = (\sinh x \cos y + I \sinh x \sin y, I y)
\]
Notice first of all that the two representations
\[ x + Iy = x + (-I)(-y) \]
of the same quaternion via \( f \) are associated with the two representations of the same image pair of quaternions, namely
\[ (\sinh x \cos y + I \sinh x \sin y, Iy) = (\sinh x \cos(-y) + (-I) \sinh x \sin(-y), (-I)(-y)) \]
We will first prove that the real differential \( df \) of \( f \) has maximal rank at all points of \( \mathbb{H} \). To do this we will consider, as in Subsection 3.4, a point \( x + Iy \in \mathbb{C}_I \subset \mathbb{H} \) with \( y \neq 0 \) and \( J \in \mathbb{S} \) orthogonal to \( I \). Then if \( K := IJ \), one has an orthonormal frame \( \{1, I, J, K\} \) in \( \mathbb{H} \) and can use the same four-curve apparatus as in Subsection 3.1 to compute:

\[
\begin{align*}
\text{df}(x + Iy)1 &= df(x + Iy)\alpha'(0) = \frac{d}{dt}\big|_0 f(\alpha(t)) \\
&= \frac{d}{dt}\big|_0 (\sinh(x + t) \cos y + I \sinh(x + t) \sin y, Iy) \\
&= (\cosh x \cos y + I \cosh x \sin y, 0)
\end{align*}
\]

\[
\begin{align*}
\text{df}(x + Iy)I &= df(x + Iy)\beta'(0) = \frac{d}{dt}\big|_0 f(\beta(t)) \\
&= \frac{d}{dt}\big|_0 (\sinh x \cos(y + t) + I \sinh x \sin(y + t), I(y + t)) \\
&= (- \sinh x \sin y + I \sinh x \cos y, I)
\end{align*}
\]

\[
\begin{align*}
\text{df}(x + Iy)J &= df(x + Iy)\Gamma'_J(0) = \frac{d}{dt}\big|_0 f(\Gamma_J(t)) \\
&= \frac{d}{dt}\big|_0 (\sinh x \cos y + \gamma(t) \sinh x \sin y, \gamma(t)y) \\
&= (J \frac{\sinh x \sin y}{y}, J)
\end{align*}
\]

\[
\begin{align*}
\text{df}(x + Iy)K &= df(x + Iy)\Gamma'_K(0) = \frac{d}{dt}\big|_0 f(\Gamma_K(t)) \\
&= \frac{d}{dt}\big|_0 (\sinh x \cos y + \delta(t) \sinh x \sin y, \delta(t)y) \\
&= (K \frac{\sinh x \sin y}{y}, K)
\end{align*}
\]

If we then set
\[ \mathbb{H} \ni x_1 + x_2 I + x_3 J + x_4 K \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \]
and
\[
\mathbb{H} \times \text{Im}(\mathbb{H}) \ni (x_1 + x_2J + x_3J + x_4K, y_2I + y_3J + y_4K)
\]
we have
\[
df(x + Iy) = \begin{bmatrix}
\cosh x \cos y & -\sinh x \sin y & 0 & 0 \\
\cosh x \sin y & \sinh x \cos y & 0 & 0 \\
0 & 0 & \frac{\sinh x \sin y}{y} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Since the first column of the differential never vanishes, \(df(x + Iy)\) has maximum rank at all points of \(\mathbb{H} \setminus \mathbb{R}\). Moreover, for \(y = 0\), we have coherently with the use of the suitable four-curve apparatus:
\[
df(x) = \begin{bmatrix}
\cosh x & 0 & 0 & 0 \\
0 & \sinh x & 0 & 0 \\
0 & 0 & \sinh x & 0 \\
0 & 0 & 0 & \sinh x \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
In conclusion the differential \(df\) has maximal rank over the entire \(\mathbb{H}\), and hence \(f\) is an immersion. We will now show that \(f\) is injective. Suppose indeed, for \(I, L \in \mathbb{S}\) we have \(f(x + Iy) = f(u + Lv)\), i.e.
\[
(\sinh x \cos y + I \sinh x \sin y, Iy) = (\sinh u \cos v + L \sinh u \sin v, Lv).
\]
It follows, first of all, that
\[
Lv = Iy
\]
which leads either to
\[
v = y = 0
\]
or, when \(y \neq 0\), to one of the following two possibilities
\[
v = y \quad \text{and} \quad L = I
\]
or
\[
v = -y \quad \text{and} \quad L = -I
\]
In the first case we get
\[
(\sinh x, 0) = (\sinh u, 0)
\]
whence $u = x$ and $x + Iy = u + Lv$. In the second case
\[(\sinh x \cos y + I \sinh x \sin y, Iy) = (\sinh u \cos y + I \sinh u \sin y, Iy)\]
whence
\[
\sinh x \cos y + I \sinh x \sin y = \sinh x(\cos y + I \sin y) = \sinh u(\cos y + I \sin y)
\]
and injectivity of $\sinh$ implies $x = u$.

Therefore the map
\[(3.15) \quad f : \mathbb{H} \rightarrow \mathbb{H} \times \text{Im}(\mathbb{H}) \]
defined by
\[(3.16) \quad f(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy) \]
is an injective immersion. The performed computation of the differential $df(x + Iy)$, for any $I \in \mathbb{S}$, implies that the restrictions
\[
df(x + Iy)|_{C_I} : C_I \rightarrow \mathbb{H} \times \text{Im}(\mathbb{H}) \cong \mathbb{R}^7
\]
and
\[
df(x + Iy)|_{C_{I}^\perp} : C_{I}^\perp \rightarrow \mathbb{H} \times \text{Im}(\mathbb{H}) \cong \mathbb{R}^7
\]
are both conformal. We are now ready to state the following result

**Proposition 3.13.** Let the map
\[f : \mathbb{H} \rightarrow \mathbb{H} \times \text{Im}(\mathbb{H})\]
be defined by
\[f(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy)\]
for $I \in \mathbb{S}$, $x, y \in \mathbb{R}$. Then $f(\mathbb{H})$ is a parameterized Riemann 4-manifold (diffeomorphic to $\mathbb{H}$) and $f$ is a slice isothermal parameterization. This manifold will be called helicoidal 4-manifold and denoted by $E$.

### 3.6. The catenoidal 4-manifold.

The case of the catenoid in the complex setting
\[(3.17) \quad g : \mathbb{C} \cong \mathbb{R}^2 \rightarrow \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R} \]
defined as
\[(3.18) \quad g(x + iy) = (\cosh x \cos y + i \cosh x \sin y, x) \]
generalizes to dimension 4 as well. In the case of quaternions, let us consider the map
\[(3.19) \quad f : \mathbb{H} \cong \mathbb{R}^4 \rightarrow \mathbb{R}^5 \cong \mathbb{H} \times \mathbb{R} \]
defined by
\[(3.20) \quad f(x + Iy) = (\cosh x \cos y + I \cosh x \sin y, x) \]
Notice first of all that the two representations
\[ x + Iy = x + (-I)(-y) \]
of the same quaternion via \( f \) are associated with the two representations of
the same image pair of quaternions, namely
\[ (\cosh x \cos y + I \cosh x \sin y, y) \]
\[ = (\cosh x \cos(-y) + (-I) \cosh x \sin(-y), x) \]

We will first prove that the real differential \( df \) of \( f \) has maximal rank at
all points of \( \mathbb{H} \). To do this we will consider, as in Subsection 3.4, a point
\( x + Iy \in \mathbb{C}_I \subset \mathbb{H} \) with \( y \neq 0 \) and \( J \in \mathbb{S} \) orthogonal to \( I \). Then if
\( K := IJ \), one has an orthonormal frame \( \{1, I, J, K\} \) in \( \mathbb{H} \) and can use
the same four-curve apparatus as in as in Subsection 3.1 to compute the
differential \( df(x + Iy) \). Indeed if we set
\[ \mathbb{H} \ni x_1 + x_2 I + x_3 J + x_4 K \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \]
and
\[ \mathbb{H} \times \text{Im}(\mathbb{H}) \ni (x_1 + x_2 I + x_3 J + x_4 K, y_1) \cong (x_1, x_2, x_3, x_4, y_1) \in \mathbb{R}^5 \]
we have
\[
\begin{pmatrix}
\sinh x \cos y & -\cosh x \sin y & 0 & 0 \\
\sinh x \sin y & \cosh x \cos y & 0 & 0 \\
0 & 0 & \cosh x \sin y & 0 \\
1 & 0 & 0 & \cosh x \sin y \\
\end{pmatrix}
\]
Since the second column of the differential never vanishes, \( df(x + Iy) \) has
maximum rank at all points of \( \mathbb{H} \setminus \mathbb{R} \). Moreover, for \( y = 0 \), we coherently
obtain:
\[
\begin{pmatrix}
\sinh x & 0 & 0 & 0 \\
0 & \cosh x & 0 & 0 \\
0 & 0 & \cosh x & 0 \\
0 & 0 & 0 & \cosh x \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]
In conclusion the differential \( df \) has maximal rank over the entire \( \mathbb{H} \), and
hence \( f \) is an immersion. We will now show that \( f \) is injective. Suppose
indeed, for \( I, L \in \mathbb{S} \)
\[ f(x + Iy) = f(u + Lv) \]
i.e.,
\[ (\cosh x \cos y + I \cosh x \sin y, x) = (\cosh u \cos v + L \cosh u \sin v, u) \]
It follows, first of all, that
\[ u = x \]
which leads to
\[ \cos y + I \sin y = \cos v + L \sin v \]
Now, if \( I \neq -L \) we get
\[ v = y + 2k\pi \]
for some \( k \in \mathbb{Z} \). If instead \( I = -L \) we obtain
\[ v = -y + 2k\pi \]
for some \( k \in \mathbb{Z} \). In any case, if we set
\[ \mathbb{S}(-\pi, \pi) = \{ Iy : I \in \mathbb{S}, y \in (-\pi, \pi) \} \]
then the map
\[ f : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{H} \times \mathbb{R} \]
defined by
\[ f(x + Iy) = (\cosh x \cos y + I \cosh x \sin y, x) \]
is an injective immersion.

The performed computations show that, for any \( I \in \mathbb{S} \), both the restrictions of the differential,
\[ df(x + Iy)_{|C_I} : \mathbb{C}_I \to \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5 \]
and
\[ df(x + Iy)_{|C_I^\perp} : \mathbb{C}_I^\perp \to \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5 \]
are conformal for any \( I \) in \( \mathbb{S} \). Therefore

**Proposition 3.14.** Let the map
\[ f : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5 \]
be defined by
\[ f(x + Iy) = (\cosh x \cos y + I \cosh x \sin y, x) \]
Then \( f(\mathbb{R} \times \mathbb{S}(-\pi, \pi)) \) is a parameterized Riemann 4-manifold and \( f \) is a slice isothermal parameterization. This manifold will be called catenoidal 4-manifold.

As in the real case, once both naturally embedded in \( \mathbb{H}^2 \), the catenoidal 4-manifold can be transformed to a part of an embedded helicoidal 4-manifold through a family of parameterized Riemann 4-manifolds.

Let the part of the helicoidal manifold embedded in \( \mathbb{H}^2 \) be parameterized by
\[ h : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{H}^2, h(x + Iy) = (\sinh x (\cos y + I \sin y), Iy) \]
and the
embedded catenoidal manifold parameterized by \( c : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{H}^2 \),
\( c(x + Iy) = (\cosh x (\cos y + I \sin y), x) \). Then
\[
H_\theta := h \cos \theta + c \sin \theta, \theta \in [0, \pi/2]
\]
defines a family of slice–conformal injective immersions with \( H_0 = h \),
\( H_{\pi/2} = c \). Indeed, for any \( I \in \mathbb{S} \), the restriction of the differential
\[
dH_{\theta, I}(x + Iy)|_{C_I} : C_I \to \mathbb{H} \times \mathbb{H} \cong \mathbb{R}^8
\]
is given by
\[
dH_{\theta, I}(x + Iy)|_{C_I} = \begin{bmatrix}
A \cos y & -B \sin y \\
A \sin y & B \cos y \\
0 & 0 \\
0 & 0 \\
\sin \theta & 0 \\
0 & \cos \theta \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]
where \( A = (\cosh x \cos \theta + \sinh x \sin \theta) \) and \( B = (\cosh x \sin \theta + \sinh x \cos \theta) \). It is obvious that the columns are orthogonal and direct computation shows that using the equality \( \cosh^2 x = 1 + \sinh^2 x \) their norms are equal. Analogously we obtain that
\[
dH_{\theta, I}(x + Iy)|_{C_I^+} : C_I^+ \to \mathbb{H} \times \mathbb{H} \cong \mathbb{R}^8
\]
is conformal. The injectivity follows from the observation that the second quaternionic component of \( H_\theta(x + Iy) \) equals \( x \sin \theta + Iy \cos \theta \). If this equals \( u \sin \theta + Jv \cos \theta \), the second quaternionic component of \( H_\theta(u + Jv) \), then either \( \theta = 0 \) and then we have the parameterization of the helicoidal 4-manifold or \( x = u \). The imaginary component gives that either \( \theta = \pi/2 \) in which case we have the catenoidal surface or else \( Iy = Jv \in \mathbb{S}(-\pi, \pi) \) so \( x + Iy = u + Jv \).

4. THE QUATERNIONIC LOGARITHM

We will show here how the helicoidal 4-manifold defined in the previous section can be adapted to be the natural domain for the definition of a quaternionic logarithm.

**Proposition 4.1.** Let \( f : \mathbb{H} \to \mathbb{H} \times \text{Im}(\mathbb{H}) \) be the map defined by
\[
f(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy)
\]
for \( I \in \mathbb{S} \), \( x, y \in \mathbb{R} \). Let \( \mathbb{H}^+ = \{ q \in \mathbb{H} : \text{Re } q > 0 \} \), and set \( \mathcal{S}^+ = f(\mathbb{H}^+) \) to denote the semi-helicoidal 4-manifold.
The $\mathcal{E}^+\text{-exponential map}$

$$E : \mathbb{H} \to \mathcal{E}^+ \subset \mathbb{H} \times \text{Im}(\mathbb{H})$$

defined by:

$$E(x + Iy) = (\exp(x + Iy), Iy) = (\exp x \cos y + I \exp x \sin y, Iy)$$

is an immersion and a diffeomorphism between $\mathbb{H}$ and $\mathcal{E}^+$, which endows $\mathcal{E}^+$ with a structure of slice quaternionic manifold (see, e.g., [3]).

**Proof.** The proof replicates part of the one of Proposition 3.13. \( \square \)

**Remark 4.2.** If $\pi : \mathbb{H} \times \text{Im}(\mathbb{H}) \to \mathbb{H}$ denotes the projection on the first factor, then by definition the following equality holds

$$\pi \circ E(q) = \exp(q)$$

for all $q \in \mathbb{H}$.

We will now define the $\mathcal{E}^+\text{-logarithm}$ on the semi-helicoidal 4-manifold $\mathcal{E}^+$, and exhibit some of its properties.

**Definition 4.3.** Let $\mathcal{E}^+$ be the semi-helicoidal 4-manifold. The $\mathcal{E}^+\text{-logarithm}$

$$L : \mathcal{E}^+ \subset \mathbb{H} \times \text{Im}(\mathbb{H}) \to \mathbb{H}$$

is defined as follows, in terms of the real logarithm $\log$:

$$L(q, p) = \log |q| + p$$

Indeed, if $(q, p) \in \mathcal{E}^+$, then $q = r \exp p$ for $r = |q|$ and our definition can be rewritten as:

$$L(r \exp p, p) = \log r + p$$

The following result and definition explain why the semi-helicoidal 4-manifold is a natural domain of definition for the $\mathcal{E}^+\text{-logarithm}$. Indeed this 4-manifold plays the role of an “adapted” blow-up of $\mathbb{H}$ at points of the form $x + 2Ik\pi$, for $k \in \mathbb{Z}$ and $k \neq 0$.

**Proposition 4.4.** The map

$$L : \mathcal{E}^+ \to \mathbb{H}$$

is the inverse of the $\mathcal{E}^+\text{-exponential } E$, and a diffeomorphism from the semi-helicoidal 4-manifold $\mathcal{E}^+$ to $\mathbb{H}$.

**Proof.** Let us read the $\mathcal{E}^+\text{-logarithm}$ through the parameterization

$$E(x + Iy) = (\exp x \cos y + I \exp x \sin y, Iy)$$

of $\mathcal{E}^+$. By composition we get that $L \circ E$ becomes the identity map of $\mathbb{H}$

$$x + Iy \mapsto (\exp(x(\cos y + I \sin y)), Iy) \mapsto \log(\exp x) + Iy = x + Iy$$
Analogously, $E \circ L$ becomes the identity map of $\mathcal{E}^+$

$$(r \exp p, p) \mapsto \log r + p \mapsto (\exp(\log r) \exp p, p) \mapsto (r \exp p, p)$$

The assertion is now proved.

As a consequence, the map $L$ is a slice regular map from the semi-helicoidal 4-manifold $\mathcal{E}^+$ to $\mathbb{H}$, with respect to the structure of slice regular manifold induced by $E$ on $\mathcal{E}^+$ (see, e.g., [3]).

**Definition 4.5.** Let $\pi : \mathcal{E}^+ \subset \mathbb{H} \times \text{Im}(\mathbb{H}) \to \mathbb{H} \setminus \{0\}$ denote the natural projection $(q, p) \mapsto q$

and let $\Omega \subset \mathcal{E}^+$ be a path connected subset such that $\pi|_{\Omega}$ is injective. Then, the map

$$\log_{\mathbb{H}} : \pi(\Omega) \to \mathbb{H}$$

defined by

$$\log_{\mathbb{H}} q = L(\pi^{-1}_{|\Omega}(q))$$

is called a branch or a determination of the quaternionic logarithm on $\pi(\Omega)$.

Notice that, as expected, with the notations of Definition 4.5 we have that

$$\exp(\log_{\mathbb{H}} q) = \pi(E(L(\pi^{-1}_{|\Omega}(q)))) = \pi(\pi^{-1}_{|\Omega}(q)) = q$$

for all $q$ in $\pi(\Omega)$

**Remark 4.6.** It is important to notice that, with the approach used, one of the main differences between the quaternionic and the complex logarithm becomes quite clear. Indeed, while a branch of the complex logarithm can be defined in a small open neighborhood of a strictly negative real point, no branch of the quaternionic logarithm can be defined in any open set $A \subset \mathbb{H} \setminus \{0\}$ which contains a strictly negative real point $x_0$, and hence a small segment $(x_0 - \epsilon, x_0 + \epsilon) \subset \mathbb{R}^-$. In this setting, there exists no subset $\Omega \subset \mathcal{E}^+$ such $\pi(\Omega) = A$ and that $\pi|_{\Omega}$ is injective; indeed, for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ we have that, for any fixed $k \in \mathbb{Z}$,

$$\pi^{-1}_{|\Omega}(x) = (-x \exp(\mathbb{S}(2k + 1)\pi), \mathbb{S}(2k + 1)\pi) = (x, \mathbb{S}(2k + 1)\pi)$$

and hence no continuous coherent choice of $I \in \mathbb{S}$ can be made on the entire $\Omega$ to define

$$\log_{\mathbb{H}} x = L(\pi^{-1}_{|\Omega}(x)) = L(y, \mathbb{S}(2k + 1)\pi) = \log |x| + I(2k + 1)\pi$$

Nevertheless, this is not always the case if $A \subset \mathbb{H} \setminus \{0\}$ is not open. To see this, consider $I_0 \in \mathbb{S}$. Let $\alpha : [-a, a] \to \mathbb{C}_{I_0} \setminus \{0\} \subset \mathbb{H} \setminus \{0\}$ be a continuous curve having its image in a small disc $\Delta$ centered at $x < 0$ with $\Delta \subset \mathbb{C}_{I_0} \setminus \{0\}$, and such that $\alpha(0) = x$. In this case the function $\log_{\mathbb{H}} \alpha(t)$ will coincide with a local branch of the complex logarithm of the
leaf $\mathbb{C}_{I_0} \setminus \{0\}$ defined along the curve $\alpha$. We will address this issue in a forthcoming paper.

We conclude this section by pointing out a different possible definition of the Riemann 4-manifold on which to define the quaternionic logarithm.

**Remark 4.7.** The definition of quaternionic logarithm could be given, alternatively, using the graph of the exponential function

$$\Gamma(\exp) = \{(q, \exp q) : q \in \mathbb{H}\}$$

which has a natural structure of Riemann 4-manifold (see Subsection 3.3), with the function $F(q) = (q, \exp q)$ as a slice isothermal parameterization. Indeed the logarithm could be defined as the slice regular function from the “reversed” graph $\Lambda(\exp) = \{(\exp w, w) : w \in \mathbb{H}\}$ onto $\mathbb{H}$, coinciding with the projection onto the second factor. The advantage of the approach that we actually adopted in this paper stays in that it calls into scenery the helicoidal and semi-helicoidal 4 manifolds, which more closely follow the path of the complex setting.

## 5. THE QUATERNIONIC SQUARE ROOT AND n-TH ROOT

To give a proper definition of the square root function over the quaternions, we will first of all define a suitable Riemann 4-manifold, $\mathcal{Q}(2)$, which will be useful to find a possible domain for such a function. To this aim, let us set $\mathbb{S}(-2\pi, 2\pi) = \{Ir : I \in \mathbb{S}, r \in (-2\pi, 2\pi)\} \subset \mathbb{R}^3$, and consider the function

$$g : \mathbb{R} \times \mathbb{S}(-2\pi, 2\pi) \to \mathbb{H} \times \mathbb{H} \cong \mathbb{R}^8$$

defined by

$$g(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, 2 \exp(I\frac{y}{2}))$$

The function $g$ is $C^\infty$ and injective: indeed $g(x + Iy) = g(u + Jv)$ implies

$$\exp(I\frac{y}{2}) = \exp(J\frac{v}{2})$$

whence either $\frac{y}{2}, \frac{v}{2} \in \{-2\pi, 0, 2\pi\}$ yielding $y = v = 0$, or $Iy = Jv$. In both cases, from

$$\sinh x(\cos y + I \sin y) = \sinh u(\cos y + I \sin y)$$

we deduce $x = u$. The injectivity of $g$ is then proved.

We will now use the four-curve apparatus of Subsection 3.1 to compute the differential $dg(x + Iy)$. Since the first member of $g$ has already been
analyzed in Subsection 3.4, we will only need to compute the differential of the second member of the map \( g \), i.e. of \( g_2(x + Iy) = 2 \exp(I \frac{y}{2}) \).

\[
dg_2(x + Iy)1 = dg_2(x + Iy)\alpha'(0) = \left. \frac{d}{dt} \right|_0 g_2(\alpha(t))
= \left. \frac{d}{dt} \right|_0 (2 \exp(I \frac{y}{2}))
= 0
\]

\[
dg_2(x + Iy)I = dg_2(x + Iy)\beta'(0) = \left. \frac{d}{dt} \right|_0 g_2(\beta(t))
= \left. \frac{d}{dt} \right|_0 2 \exp(I \frac{y + t}{2})
= - \sin(\frac{y}{2}) + I \cos(\frac{y}{2})
\]

\[
dg_2(x + Iy)J = dg_2(x + Iy)\Gamma'_J(0) = \left. \frac{d}{dt} \right|_0 g_2(\Gamma_J(t))
= \left. \frac{d}{dt} \right|_0 \exp(\gamma(t) \frac{y}{2})
= J \frac{2 \sin(y/2)}{y}
\]

\[
dg_2(x + Iy)K = dg_2(x + Iy)\Gamma'_K(0) = \left. \frac{d}{dt} \right|_0 g_2(\Gamma_K(t))
= \left. \frac{d}{dt} \right|_0 \exp(\delta(t) \frac{y}{2})
= K \frac{2 \sin(y/2)}{y}
\]

If we then set

\[
\mathbb{H} \ni x_1 + x_2 I + x_3 J + x_4 K \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4
\]

and

\[
\mathbb{H} \times \mathbb{H} \ni (x_1 + x_2 I + x_3 J + x_4 K, y_1 + y_2 I + y_3 J + y_4 K) 
\cong (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{R}^8
\]
we have

\[
\begin{bmatrix}
\cosh x \cos y & -\sinh x \sin y & 0 & 0 \\
\cosh x \sin y & \sinh x \cos y & 0 & 0 \\
0 & 0 & \frac{2\sinh x \sin(y/2)}{y} & 0 \\
0 & 0 & 0 & \frac{2\sinh x \sin(y/2)}{y} \\
0 & -\sin(\frac{y}{2}) & 0 & 0 \\
0 & \cos(\frac{y}{2}) & 0 & 0 \\
0 & 0 & \frac{2\sin(y/2)}{y} & 0 \\
0 & 0 & 0 & \frac{2\sin(y/2)}{y}
\end{bmatrix}
\]

Since \( y \in (-2\pi, 2\pi) \) and hence \( \frac{y}{2} \in (-\pi, \pi) \), the differential \( dg(x + Iy) \) has maximum rank at all points of its domain, including at the real points, where, for \( y = 0 \), we coherently obtain:

\[
dg(x) = \begin{bmatrix}
\cosh x & 0 & 0 & 0 \\
0 & \sinh x & 0 & 0 \\
0 & 0 & \sinh x & 0 \\
0 & 0 & 0 & \sinh x \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Notice that, for any \( I \in \mathbb{S} \), the restrictions of the differential

\[
dg_I(x + Iy)_{\mathbb{C}_I^\perp} : \mathbb{C}_I^\perp \to \mathbb{H}^2 \cong \mathbb{R}^8
\]

and

\[
dg_I(x + Iy)_{\mathbb{C}_I^\perp} : \mathbb{C}_I^\perp \to \mathbb{H}^2 \cong \mathbb{R}^8
\]

are conformal. In conclusion we have proved

**Proposition 5.1.** Let the map

\[
g : \mathbb{R} \times \mathbb{S}(-2\pi, 2\pi) \to \mathbb{H} \times \mathbb{H} \cong \mathbb{R}^8
\]

be defined by

\[
g(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, 2 \exp(I \frac{y}{2}))
\]

for \( I \in \mathbb{S}, x, y \in \mathbb{R} \). Then \( g(\mathbb{R} \times \mathbb{S}(-2\pi, 2\pi)) \) is a parameterized Riemann 4-manifold (diffeomorphic to \( \mathbb{R} \times \mathbb{S}(-2\pi, 2\pi) \)) and \( g \) is a slice isothermal parameterization. This manifold will be denoted by \( \mathcal{Q}(2) \).

We will now see how to use \( \mathcal{Q}(2) \) to construct an appropriate domain for the quaternionic square root function.
Proposition 5.2. Let $\mathcal{Q}(2)^+ = g(\mathbb{R}^+ \times S(-2\pi, 2\pi))$ denote the square root 4-manifold. The map

$$G : \mathbb{R}^+ \times S(-2\pi, 2\pi) \to \mathcal{Q}(2)^+$$

defined by:
$$G(x + Iy) = (\exp(x + Iy), 2\exp(I\frac{y}{2}))$$

is an injective immersion and a diffeomorphism between $\mathbb{R}^+ \times S(-2\pi, 2\pi)$ and $\mathcal{Q}(2)^+$

Proof. The proof replicates the one used to establish Proposition 5.1. □

We will now define the quaternionic square root on the square root 4-manifold, and establish some of its properties.

Definition 5.3. Let $\mathcal{Q}(2)^+$ be the square root 4-manifold. The quaternionic square root

$$S : \mathcal{Q}(2)^+ \subset \mathbb{H} \times \mathbb{H} \to \mathbb{H}$$

is defined as follows, for all $r \in \mathbb{R}^+$ and $p \in S(-2\pi, 2\pi)$

$$S(r \exp p, 2 \exp(I\frac{p}{2})) = \sqrt{r} \exp(I\frac{p}{2})$$

or directly (and equivalently), for all $(q, s) \in \mathcal{Q}(2)^+$, by

$$S(q, s) = \sqrt{|q|} \cdot s$$

Indeed, this last formulation of the definition extends in a natural fashion, to $\mathcal{Q}(2)^+ = g((\mathbb{R}^+ \cup \{0\}) \times S[-2\pi, 2\pi])$ as

$$S(0, s) = 0$$

and

$$S(r, -2) = -\sqrt{r}$$

for all $s \in 2S^3 = \{2q : q \in S^3\}$ and all $r \geq 0$.

The case of the quaternionic $n$-th root is totally analogous to the one on the square root: proofs which replicate those presented in detail for the case of the square root allow the following statements and definitions.

Proposition 5.4. Let $n \in \mathbb{N}$ and let the map

$$h : \mathbb{R} \times S(-\pi n, \pi n) \to \mathbb{H} \times \mathbb{H} \cong \mathbb{R}^8$$

be defined by

$$h(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, n \exp(I\frac{y}{n}))$$
for $I \in \mathbb{S}$, $x y \in \mathbb{R}$. Then $h(\mathbb{R} \times \mathbb{S}(-\pi n, \pi n))$ is a parameterized Riemann 4-manifold (diffeomorphic to $\mathbb{R} \times \mathbb{S}(-\pi n, \pi n)$) and $h$ is a slice isothermal parameterization. This manifold will be denoted by $\mathcal{Q}(n)$.

In the new setting, we will use $\mathcal{Q}(n)$ to construct an appropriate domain for the quaternionic $n$-th root function.

**Proposition 5.5.** Let $\mathcal{Q}(n)^+ = h(\mathbb{R}^+ \times \mathbb{S}(-\pi n, \pi n))$ denote the $n$-th root 4-manifold. The map

$$H : \mathbb{R}^+ \times \mathbb{S}(-\pi n, \pi n) \to \mathcal{Q}(n)^+$$

defined by:

$$H(x + Iy) = (\exp(x + Iy), n\exp(Iy/n))$$

$$= (\exp x(\cos y + I\sin y), n(\cos(y/n) + I\sin(y/n)))$$

is an injective immersion and a diffeomorphism between $\mathbb{R}^+ \times \mathbb{S}(-\pi n, \pi n)$ and $\mathcal{Q}(n)^+$

The quaternionic $n$-th root is defined on the $n$-th root 4-manifold, as follows.

**Definition 5.6.** Let $\mathcal{Q}(n)^+$ be the $n$-th root 4-manifold. The quaternionic $n$-th root

$$R : \mathbb{H} \times \mathbb{H} \supset \mathcal{Q}(n)^+ \to \mathbb{H}$$

is defined as follows, for all $r \in \mathbb{R}^+$ and $p \in \mathbb{S}(-\pi n, \pi n)$

$$R(r \exp p, n\exp(p/n)) = \sqrt[r]{\exp(p/n)}$$

or directly (and equivalently), for all $(q, s) \in \mathcal{Q}(n)^+$, by

$$R(q, s) = \frac{\sqrt{|q|}}{n}s$$

Indeed, this last formulation of the definition extends in a natural fashion, to $\mathcal{Q}(n)^+ = h((\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}[-\pi n, \pi n])$ as

$$R(0, s) = 0$$

and

$$R(r, -n) = -\sqrt[r]{r}$$

for all $s \in nS^3$ and all $r \geq 0$. 

6. The case of octonions: Riemann 8-manifolds, logarithm and n-th root

The definition of parameterized Riemann 4-manifolds just studied in the case of quaternions naturally extends to the case of the division algebra \( \mathbb{O} \) of octonions. If \( \langle , \rangle \) denotes the Euclidean scalar product in \( \mathbb{R}^8 \cong \mathbb{O} \), set again \( \mathbb{C}_I^+ = \{ q \in \mathbb{O} : \langle q, x + Iy \rangle = 0, \forall (x + Iy) \in \mathbb{C}_I \} \) to be the orthogonal space to the slice \( \mathbb{C}_I \).

**Definition 6.1.** Let \( \Omega \) be a slice domain in \( \mathbb{O} \cong \mathbb{R}^8 \) and let \( N \geq 8 \) be a natural number. Let

\[ f : \Omega \to \mathbb{R}^N \]

be an at least \( C^1 \) immersion. If for any \( I \in \mathbb{S} \) and any \( x, y \in \mathbb{R} \) the differential \( df(x + Iy) \) is such that both

\[ df(x + Iy)|_{\mathbb{C}_I} \]

and

\[ df(x + Iy)|_{\mathbb{C}_I^+} \]

are conformal, then \( f \) will be called a slice conformal or slice isothermal immersion.

If in addition \( f \) is injective, then it will be called slice conformal or slice isothermal parameterization and the parameterized 8-manifold \( f(\Omega) \) in \( \mathbb{R}^N \) will be called a (parameterized) Riemann 8-manifold of \( \mathbb{R}^N \).

In case \( f : \Omega \to \mathbb{R}^N \) itself is a conformal parameterization, then the parameterized 8-manifold \( f(\Omega) \) in \( \mathbb{R}^N \) will be called a special (parameterized) Riemann 8-manifold of \( \mathbb{R}^N \).

Indeed, all the relevant examples presented in Section 3 in the setting of quaternions are also found in the case of octonions. Most of the proofs that are necessary to construct these examples exactly replicate those given in the case of quaternions, and will be skipped in the presentation.

6.1. The eight-curve apparatus in the case of octonions. Choose any \( I_1 \) in the 6-sphere \( \mathbb{S} \) of purely imaginary unit octonions, and any two \( I_2 \perp I_4 \in \mathbb{S} \), orthogonal to \( I_1 \) such that

\[ \{1, I_1, I_2, I_1I_2 = I_3, I_4, I_1I_4 = I_5, I_2I_4 = I_6, I_3I_4 = I_7 \} \]

form a standard basis of \( \mathbb{O} \). When \( y \neq 0 \), we can define the following set of eight curves, called eight-curve apparatus:

(1) the curve \( \alpha(t) = (x + t) + I_1y \), such that \( \alpha(0) = x + I_1y \) and \( \alpha'(0) = 1 \);
(2) the curve \( \beta(t) = x + I_1(y + t) \), such that \( \beta(0) = x + I_1 y \) and \( \beta'(0) = I_1 \); 

(3)-(8) for \( \ell = 2, \ldots, 7 \), the curve \( \Gamma_{I_\ell}(t) = x + \gamma_{I_\ell}(t) y \), where \( \gamma_{I_\ell}(t) \) is an arc of a maximum circle \( C_{\gamma_{I_\ell}} \) of \( S \), such that \( \gamma_{I_\ell}(0) = I_1 \) and that \( \gamma_{I_\ell}'(0) = I_\ell \); hence \( \Gamma_{I_\ell}(0) = x + I_1 y \) and \( \Gamma_{I_\ell}'(0) = I_\ell \).

When \( x + I y = x \), i.e., when \( y = 0 \), the eight-curve apparatus simplifies: the first curve remains the same

(1) \( \alpha(t) = x + t \), such that \( \alpha(0) = x \) and \( \alpha'(0) = 1 \),
while the other curves become

(2)-(8) for \( \ell = 2, \ldots, 8 \), the curve \( \beta_{I_\ell}(t) = x + I_\ell t \), such that \( \beta_{I_\ell}(0) = x \) and \( \beta_{I_\ell}'(0) = I_\ell \).

The eight-curve apparatus plays the role of the four-curve apparatus in the computation of the differential of slice regular octonionic functions as well as that of the immersions of domains \( \Omega \subseteq \mathbb{O} \) into \( \mathbb{R}^N \).

Let us now consider the case of a slice regular function \( f : \Omega \subseteq \mathbb{O} \to \mathbb{O} \). Given such a function, fix \( I \in S \). The representation formula used in the case of quaternions holds, in a totally similar form, for slice regular functions on (octonionic) slice domains (see, e.g., [10, Proposition 6], [12, Formula (5)]). Indeed, if \( L, M, N \in S \) with \( M \neq N \) are such that \( x + Ly, x + My, x + Ny \) belong to suitable open neighborhood of \( x + Iy \) in the 6-sphere \( x + S_y \), then

\[
\begin{align*}
(6.23) & \quad f(x + Ly) \\
& = (M - N)^{-1} [Mf(x + My) - Nf(x + Ny)] + \\
& + L \{(M - N)^{-1} [f(x + My) - f(x + Ny)]\}
\end{align*}
\]

where \( f'_L(x + I_1 y) := y^{-1} \{(M - N)^{-1} [f(x + My) - f(x + Ny)]\} \) is the spherical derivative of \( f \), constant in the same neighborhood of \( x + I_1 y \) in \( x + S_y \). Choose any \( I_1 \) in the 6-sphere \( S \) of purely imaginary unit octonions, and any two \( I_2 \perp I_4 \in S \) orthogonal to \( I_1 \) such that

\[
\{1, I_1, I_2, I_1I_2 = I_3, I_4, I_1I_4 = I_5, I_2I_4 = I_6, I_3I_4 = I_7\}
\]

form a standard basis of \( \mathbb{O} \). We can now use the associated eight-curve apparatus, and a direct computation shows that

\[
\begin{align*}
df(x + I_1 y) 1 & = df(x + I_1 y) \alpha'(0) = \frac{d}{dt}\bigg|_0 f(\alpha(t)) = \frac{d}{dt}\bigg|_0 f(x + t + I_1 y) \\
& = \frac{\partial f}{\partial x}(x + I_1 y)
\end{align*}
\]
Analogously, and since $f$ is slice regular,
\[
df(x + I_1y)I_1 = df(x + I_1y)\beta'(0) = \frac{d}{dt}_{|t=0} f(\beta(t)) = \frac{d}{dt}_{|t=0} f(x + I_1(y + t)) = \frac{\partial f}{\partial y}(x + I_1y) = I_1 \frac{\partial f}{\partial x}(x + I_1y)
\]
In particular we have that $I_1 df(x + I_1y)I_1 = I_1 \frac{\partial f}{\partial y}(x + I_1y) = df(x + I_1y)I_1$
and hence the differential
\[
df(x + I_1y)_{|C_{I_1}} : C_{I_1} \cong \mathbb{R}^2 \rightarrow \mathbb{O} \cong \mathbb{R}^8
\]
is conformal. Let us now continue. The local representation formula 6.23 yields, for all $\ell = 2, \ldots, 7$,
\[
\begin{align*}
df(x + I_1y)I_\ell &= df(x + I_1y)\Gamma'_\ell(0) = \frac{d}{dt}_{|t=0} f(\Gamma_\ell(t)) \\
&= \frac{d}{dt}_{|t=0} (\gamma_\ell(t)(M - N)^{-1} [f(x + My) - f(x + Ny)]) \\
&= I_\ell y^{-1}(M - N)^{-1} [f(x + My) - f(x + Ny)]) \\
&= I_\ell f'_s(x + I_1y)
\end{align*}
\]
Hence $df(x + I_1y)I_2 = I_2 f'_s(x + I_1y)$, ..., $df(x + I_1y)I_7 = I_7 f'_s(x + I_1y)$
are all mutually orthogonal columns with the same norm. As a consequence we have that
\[
df(x + I_1y)_{|C_{I_1}} : C_{I_1} \cong \mathbb{R}^6 \rightarrow \mathbb{O} \cong \mathbb{R}^8
\]
is conformal. Notwithstanding the conformality of the two examined components of the real differential of $f$ at $x + I_1y$, the entire differential $df(x + I_1y) : \mathbb{R}^8 \rightarrow \mathbb{R}^8$ is not conformal in general (as we already know from the quaternionic case.

6.2. **Octonion slice regular curves.** We will use the standard notion of curve in the octonionic setting.

**Definition 6.2.** Let $\Omega \subseteq \mathbb{O}$ be a slice domain, and let
\[
F : \Omega \rightarrow \mathbb{O}^2
\]
be a map whose components $f, g : \Omega \rightarrow \mathbb{O}$ are slice regular functions. If $F$
is a immersion, then $F$ will be called a slice regular curve (in $\mathbb{O}^2$).

Let us now consider a slice regular curve $F : \Omega \rightarrow \mathbb{O}^2$. Choose any $I_1$ in the 6-sphere $S$ of purely imaginary unit octonions, and any two $I_2 \perp I_4 \in S$
orthogonal to $I_1$ such that
\[
\{1, I_1, I_2, I_1I_2 = I_3, I_4, I_1I_4 = I_5, I_2I_4 = I_6, I_3I_4 = I_7\}
\]
form a standard basis of the division algebra \( \mathbb{O} \) (see, e.g., [8]). The eight-curve apparatus defined in Subsection 6.1 allows the computation of the differential

\[
dF(x + I_1 y) : \mathbb{R}^8 \to \mathbb{O}^2
\]

which assumes the form

\[
dF(x + I_1 y) = \begin{bmatrix}
\frac{\partial f}{\partial x} I_1 \frac{\partial f}{\partial x} I_2 f'_s & I_3 f'_s & I_4 f'_s & I_5 f'_s & I_6 f'_s & I_7 f'_s \\
\frac{\partial g}{\partial x} I_1 \frac{\partial g}{\partial x} I_2 g'_s & I_3 g'_s & I_4 g'_s & I_5 g'_s & I_6 g'_s & I_7 g'_s
\end{bmatrix}
\]

which can be interpreted as a \( 18 \times 8 \) real matrix. The two first columns of the \( 16 \times 8 \) real matrix, which represent \( dF(x + I_1 y) |_{C I_1} \), are orthogonal and with the same norm. In the same manner, the last six columns of the same matrix, which represent \( dF(x + I_1 y) |_{C I_1}^\perp \), are orthogonal and with the same norm. Hence \( F \) is isothermal. In conclusion we have proved.

**Proposition 6.3.** Let \( \Omega \subseteq \mathbb{O} \) be a slice domain, and let \( F : \Omega \to \mathbb{O}^2 \) be a slice regular curve. If \( F \) is injective, then \( F(\Omega) \) is a parameterized Riemann 8-manifold in \( \mathbb{O}^2 \), and the map \( F : \Omega \to F(\Omega) \) is a slice conformal parameterization.

As in the case of quaternions, and for the same reason, in general \( F \) is (a slice conformal but) not a conformal parameterization. Moreover, the remarks made in the quaternionic case on the “admitted” changes of parameter are also valid in the octonionic environment.

### 6.3. Graphs of octonionic slice regular functions.

Let \( \Omega \subseteq \mathbb{O} \) be a slice domain, and let \( f : \Omega \to \mathbb{O} \) be a slice regular function. Let us consider the slice regular curve

\[
F : \Omega \to \mathbb{O}^2 \cong \mathbb{R}^{16}
\]

defined as

\[
F(q) = (q, f(q))
\]

Since the curve \( F \) is clearly injective, as a particular case of what established in Proposition 6.3 we obtain

**Proposition 6.4.** Let \( \Omega \subseteq \mathbb{O} \) be a slice domain, and let \( f : \Omega \to \mathbb{O} \) be a slice regular function. Then the graph of \( f \)

\[
\Gamma(f) = \{(q, f(q)) : q \in \Omega \} \subseteq \mathbb{O}^2
\]

is a parameterized Riemann 8-manifold, and the map \( F : \Omega \to \Gamma(f) \) defined by \( F(q) = (q, f(q)) \) is a slice conformal parameterization.
6.4. **The Riemann 8-sphere.** This example generalizes to dimension 8 the case of the Riemann sphere in the complex setting and that of the quaternionic Riemann 4-sphere. Indeed the map

\[ f : \mathbb{R}^8 \cong \mathbb{O} \rightarrow \mathbb{O} \times \mathbb{R} \cong \mathbb{R}^9 \]

defined, for all \( x, y \in \mathbb{R} \) and all \( I \in \mathbb{S} \), by

\[ f(x + Iy) = \left( \frac{2(x + Iy)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) \]

is injective, and the real differential \( df : \mathbb{R}^8 \rightarrow \mathbb{R}^9 \) of \( f \) has maximal rank at all points of \( \mathbb{O} \cong \mathbb{R}^8 \). To prove this we will consider any \( I_1 \) in the 6-sphere \( S \) of purely imaginary unit octonions, and any two \( I_2, I_4 \in S \) orthogonal to \( I_1 \) such that

\[ \{1, I_1, I_2, I_1I_2 = I_3, I_4, I_1I_4 = I_5, I_2I_4 = I_6, I_3I_4 = I_7\} \]

form standard basis of \( \mathbb{O} \).

By using this eight-curve apparatus defined in Subsection 6.1, we can now find that \( df : \mathbb{R}^8 \rightarrow \mathbb{R}^9 \) has the following form, when \( a = 1 + x^2 + y^2 \)

\[
\begin{bmatrix}
2a^2 \\
1 - x^2 + y^2 & -2xy & 0 & 0 & 0 & 0 & 0 & 0 \\
-2xy & 1 + x^2 - y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\
2x & 2y & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The eight columns of \( df(x + I_1y) \) are mutually orthogonal with all norms equal to \( \frac{2}{1 + x^2 + y^2} \neq 0 \) and so, \( f : \mathbb{R}^8 \cong \mathbb{O} \rightarrow \mathbb{O} \times \mathbb{R} \cong \mathbb{R}^9 \) is conformal. In conclusion, we can state:

**Proposition 6.5.** Let us consider the 8-sphere \( S^8 \subset \mathbb{R}^9 \cong \mathbb{O} \times \mathbb{R} \) and the map

\[ f : \mathbb{R}^8 \cong \mathbb{O} \rightarrow S^8 \setminus \{N\} \subset \mathbb{O} \times \mathbb{R} \cong \mathbb{R}^9 \]

inverse of the stereographic projection from the north pole \( N = (0, 0, 0, 0, 0, 0, 1) \) of \( S^8 \) to the equatorial plane \( \mathbb{O} \cong \mathbb{R}^8 \), defined by

\[ f(x + Iy) = \left( \frac{2(x + Iy)}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) \]
Then $S^8 \setminus \{N\}$ is a special parameterized Riemann 8-manifold, and the map $f$ is a conformal parameterization.

The statement for the inverse of the stereographic projection from the south pole $S$ of $S^8$ is completely analogous to the last one, and will be skipped. As in the case of quaternions, it is now possible to exhibit the “Riemann” structure of 1-dimensional octonionic manifold of the sphere $S^8 \subset \mathbb{R}^9$. The proof is exactly as in the case of quaternions.

**Theorem 6.6.** Let $f$ and $h$ be the following maps

$$f : \mathbb{R}^8 \simeq \mathbb{O} \to S^8 \setminus \{N\} \subset \mathbb{O} \times \mathbb{R} \cong \mathbb{R}^9$$

and

$$h : \mathbb{R}^8 \simeq \mathbb{O} \to S^8 \setminus \{S\} \subset \mathbb{O} \times \mathbb{R} \cong \mathbb{R}^9$$

Then the differentiable conformal atlas $\{(\mathbb{H}, f), (\mathbb{H}, h)\}$ endows $S^8 \subset \mathbb{R}^9$ with a structure of slice octonionic manifold (see, e.g., [3]).

6.5. **The helicoidal and the catenoidal 8-manifolds.** This examples generalize to dimension 8 the case of the quaternionic helicoidal and catenoidal 4-manifolds. Let us begin by considering the map

$$(6.28) \quad f : \mathbb{O} \cong \mathbb{R}^8 \to \mathbb{R}^{15} \cong \mathbb{O} \times \text{Im}(\mathbb{O})$$

defined by

$$(6.29) \quad f(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy)$$

The $C^\infty$ map $f$ is injective, and the use of the eight-curve apparatus associated to a standard basis of $\mathbb{O}$ (see Subsection 6.4) easily allows the computation of

$$df(x + Iy) : \mathbb{R}^8 \to \mathbb{R}^{15}$$

which turns out to be of maximum rank at all points and such that the two restrictions

$$df_I(x + Iy)|_{C_I} : C_I \cong \mathbb{R}^2 \to \mathbb{R}^{15} \cong \mathbb{O} \times \text{Im}(\mathbb{O})$$

and

$$df_I(x + Iy)|_{C_I^\perp} : C_I^\perp \cong \mathbb{R}^6 \to \mathbb{R}^{15} \cong \mathbb{O} \times \text{Im}(\mathbb{O})$$

are conformal for any $I$ in $S$. We can then state:
Proposition 6.7. Let the map 
\[ f : \mathbb{O} \to \mathbb{O} \times \text{Im}(\mathbb{O}) \]
be defined by 
\[ f(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy) \]
for \( I \in \mathbb{S}, x, y \in \mathbb{R} \). Then \( f(\mathbb{O}) \) is a parameterized Riemann 8-manifold (diffeomorphic to \( \mathbb{O} \)) and \( f \) is a slice isothermal parameterization. This manifold will be called helicoidal 8-manifold and denoted by \( \mathbb{E} \).

Analogously, as done in Proposition 3.14 for the case of quaternions, we can construct the octonionic catenoidal 8-manifold.

Proposition 6.8. Let the map 
\[ f : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{O} \times \mathbb{R} \cong \mathbb{R}^9 \]
be defined by 
\[ f(x + Iy) = (\cosh x \cos y + I \cosh x \sin y, x) \]
Then \( f(\mathbb{R} \times \mathbb{S}(-\pi, \pi)) \) is a parameterized Riemann 8-manifold and \( f \) is a slice isothermal parameterization. This manifold will be called catenoidal 8-manifold.

As in the quaternionic case, once both embedded \( \mathbb{O} \times \mathbb{O} \), the catenoidal 8-manifold can be deformed through a family of octonionic 8-manifolds to a portion of a helicoidal 8-manifold.

6.6. The octonionic logarithm. As in the setting of quaternions studied in the previous section, with the same proof, we can state

Proposition 6.9. Let \( \mathbb{O}^+ = \{ q \in \mathbb{O} : \text{Re} \ q > 0 \} \), and set \( \mathbb{E}^+ = f(\mathbb{O}^+) \) to denote the semi-helicoidal 8-manifold. The \( \mathbb{E}^+ \)- exponential map
\[ E : \mathbb{O} \to \mathbb{E}^+ \subset \mathbb{O} \times \text{Im}(\mathbb{O}) \]
defined by:
\[ E(x + Iy) = (\exp(x + Iy), Iy) = (\exp x \cos y + I \exp x \sin y, Iy) \]
is an injective immersion and a diffeomorphism between \( \mathbb{O} \) and \( \mathbb{E}^+ \), which endows \( \mathbb{E}^+ \) with a structure of slice octonionic manifold (see, e.g., [3]).

The semi-helicoidal 8-manifold is used to define the octonionic logarithm and establish some of its properties, whose proofs closely replicate those given in the case of \( \mathbb{H} \).
Definition 6.10. Let $\mathbb{E}^+$ be the semi-helicoidal 8-manifold. The $\mathbb{E}^+$-logarithm

$$L : \mathbb{E}^+ \subset O \times \text{Im}(O) \to O$$

is defined as follows, in terms of the real logarithm $\log$:

$$L(q, p) = \log |q| + p$$

Indeed, if $(q, p) \in \mathbb{E}^+$, then $q = r \exp p$ for $r = |q|$ and our definition can be rewritten as:

$$L(r \exp p, p) = \log r + p$$

Proposition 6.11. The map

$$L : \mathbb{E}^+ \to O$$

is the inverse of the $\mathbb{E}^+$-exponential, and a diffeomorphism from the semi-helicoidal 8-manifold $\mathbb{E}^+$ to $O$.

As a consequence, $\mathbb{E}^+$-logarithm $L$ is a slice regular map from the semi-helicoidal 8-manifold $\mathbb{E}^+$ to $O$, with respect to the structure of slice regular manifold induced by $E$ on $\mathbb{E}^+$ (see, e.g., [3]).

These tools lead to the definition of the various branches of the octonionic logarithm. The semi-helicoidal 8-manifold is a natural domain of definition for it, and in turn plays the role of an “adapted” blow-up of $O$ at points of the form $x + 2Ik\pi$, for $I \in S, k \in \mathbb{Z}$ and $k \neq 0$.

Definition 6.12. Let $\pi : O \times \text{Im}(O) \supset \mathbb{E}^+ \to O \setminus \{0\}$ denote the natural projection

$$(q, p) \mapsto q$$

and let $\Omega \subset \mathbb{E}^+$ be an path connected subset such that $\pi_{\text{in}}$ is injective. Then, the map

$$\log_{\Omega} : \pi(\Omega) \to O$$

defined by

$$\log_{\Omega} q = L(\pi^{-1}_{\text{in}}(q))$$

is called a a branch or a determination of the octonionic logarithm on $\pi(\Omega)$.

The alternative approach to a definition of quaternionic logarithm described in Remark [4,7] could be used in the octonionic setting as well. The same reasons exposed in that Remark convinced us to maintain the approach that you have found in the paper.
6.7. The octonionic $n$-th root. The case of the octonionic $n$-th root can be studied as in the case of quaternions, with similar proofs.

**Proposition 6.13.** Let $n \in \mathbb{N}$ and let the map
\[
    h : \mathbb{R} \times S(\pi n, \pi n) \to \mathbb{O} \times \mathbb{O} \cong \mathbb{R}^{16}
\]
be defined by
\[
    h(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, n \exp(I \frac{y}{n}))
\]
for $I \in \mathbb{S}$, $x, y \in \mathbb{R}$. Then $h(\mathbb{R} \times S(\pi n, \pi n))$ is a parameterized Riemann 8-manifold (diffeomorphic to $\mathbb{R} \times S(\pi n, \pi n)$) and $h$ is a slice isothermal parameterization. This manifold will be denoted by $\mathcal{F}(n)$.

The 8-manifold $\mathcal{F}(n)$ is used to construct an appropriate domain for the octonionic $n$-th root function.

**Proposition 6.14.** Let $\mathcal{F}(n)^+ = h(\mathbb{R}^+ \times S(\pi n, \pi n))$ denote the $n$-th root 8-manifold. The map
\[
    H : \mathbb{R}^+ \times S(-\pi n, \pi n) \to \mathcal{F}(n)^+
\]
defined by:
\[
    H(x + Iy) = (\exp(x + Iy), n \exp(I \frac{y}{n}))
\]
\[
    = (\exp x (\cos y + I \sin y), n (\cos(x \frac{y}{n}) + I \sin(x \frac{y}{n})))
\]
is a regular parameterization which also represents a diffeomorphism between $\mathbb{R}^+ \times S(-\pi n, \pi n)$ and $\mathcal{F}(n)^+$.

We are now ready to define the octonionic $n$-th root function.

**Definition 6.15.** Let $\mathcal{F}(n)^+$ be the $n$-th root 8-manifold. The octonionic $n$-th root
\[
    R : \mathcal{F}(n)^+ \subset \mathbb{O} \times \mathbb{O} \to \mathbb{O}
\]
is defined as follows, for all $r \in \mathbb{R}^+$ and $p \in S(-\pi n, \pi n)$
\[
    R(r \exp p, n \exp(p \frac{y}{n})) = \sqrt[r]{r} \exp(p \frac{y}{n})
\]
or directly (and equivalently), for all $(q, s) \in \mathcal{F}(n)^+$, by
\[
    R(q, s) = \sqrt[n]{\sqrt|q|} s
\]
Indeed, this last formulation of the definition extends in a natural fashion, to $\mathcal{F}(n)^+ = h((\mathbb{R}^+ \cup \{0\}) \times S[-\pi n, \pi n])$ as
\[
    R(0, s) = 0
\]
and
\[ R(r, -n) = -\sqrt{r} \]
for all \( s \in nS^7 \) and all \( r \geq 0 \).

As it clearly appears, there is natural space and interest for the study of differential geometry of Riemann 4 and 8 manifolds, and in particular for the study of their curvature, of their mean curvature and minimality. This will be the subject of a forthcoming paper.

**References**

[1] D. Angella, C. Bisi, *Slice-Quaternionic Hopf Surfaces*, J. Geom. Anal. **29** (2019), 1837-1858. https://doi.org/10.1007/s12220-018-0064-9

[2] F. Colombo, I. Sabadini, and D. C. Struppa, *Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions*, vol. 289 of Progress in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2011.

[3] G. Gentili, A. Gori, G. Sarfatti, *A direct approach to quaternionic manifolds*, Math. Nachr., **290** (2017), 321-331. https://doi.org/10.1002/mana.201500489

[4] G. Gentili, A. Gori, G. Sarfatti, *On Compact Affine Quaternionic Curves and Surfaces*, J. Geom. Anal. **31** (2021), 1073-1092. https://doi.org/10.1007/s12220-019-00311-2

[5] G. Gentili, C. Stoppato, *A local representation formula for quaternionic slice regular functions*, to appear in: Proc. Amer. Math. Soc.

[6] G. Gentili, C. Stoppato, *Geometric function theory over quaternionic slice domains*, J. Math. Anal. Appl., **495** (2021), 1-38.

[7] G. Gentili, C. Stoppato, D. C. Struppa, *Regular Functions of a Quaternionic Variable*, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013.

[8] G. Gentili, D. C. Struppa, *Regular functions on the space of Cayley numbers*, Rocky Mountain J. Math., **40** (2010), 225-241.

[9] G. Gentili, I. Vignozzi, *The Weierstrass factorization theorem for slice regular functions over the quaternions*, Ann. Global Anal. Geom. **40** (2011), 435-466. https://doi.org/10.1007/s10455-011-9266-0

[10] R. Ghiloni, A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math., **226** (2011), 1662-1691. https://doi.org/10.1016/j.aim.2010.08.015.

[11] R. Ghiloni, A. Perotti, *Zeros of regular functions of quaternionic and octonionic variable: a division lemma and the camshaft effect*, Ann. Mat. Pura Appl. (4), **190** (2011), 539-551.

[12] R. Ghiloni, A. Perotti, C. Stoppato, *Singularities of slice regular functions over real alternative *-algebras*, Adv. Math., **305** (2017), 1085-1130. https://doi.org/10.1016/j.aim.2016.10.009.

[13] R. Ghiloni, A. Perotti, C. Stoppato. (2020). *Division algebras of slice functions*, Proc. Roy. Soc. Edinburgh Sect. A, **150** (2020), 2055-2082.

[14] A. Gori, F. Vlacci, *On a criterion of local invertibility and conformality for slice regular quaternionic functions*, Proc. Edimb. Math. Soc. **62** (2019), 97-105.

[15] K. Gürlebeck, K. Habetha, W. Sprössig, *Holomorphic Functions in the Plane and n-dimensional Space*, Birkhäuser Verlag, Basel (2008)

[16] B. Riemann, *Fondamenti di una teorica generale delle funzioni di una variabile complessa*, Ann. Mat. Pura Appl. **2**, (1859), 288-304.
DiMaI, Università di Firenze, Viale Morgagni 67/A, Firenze, Italy
Email address: graziano.gentili@unifi.it

Fakulteta za matematiko in fiziko Jadranska 21 1000 Ljubljana, Slovenija, UP FAMNIT, Glagoljaška 8, Koper Slovenija, IMFM Jadranska 19 1000 Ljubljana, Slovenija
Email address: jasna.prezelj@fmf.uni-lj.si

DiSpeS Università di Trieste Piazzale Europa 1, Trieste, Italy
Email address: fvlacci@units.it