We present the full three-loop $\beta$-functions for the MSSM generalised to include additional matter multiplets in 5, 10 representations of SU(5). We analyse the effect of three-loop running on the sparticle spectrum for the MSSM Snowmass Benchmark Points. We also consider the effect on these spectra of additional matter multiplets (the semi-perturbative unification scenario).
1. Introduction

The LHC will soon resolve the question as to whether low energy supersymmetry is the solution to the hierarchy problem; and if it is, moreover, the LHC and a future $e^+e^-$ linear collider (LC) will lead to very precise measurements of the sparticle spectrum and couplings. The success of gauge unification in the MSSM suggests a Desert, the existence of which would mean that extrapolation of the MSSM couplings and masses to high scales will lead to immediate information about the underlying theory; for example regarding the commonly assumed universality of soft scalar masses, gaugino masses and cubic scalar interactions.

One component of this analysis is the running of masses and couplings between the weak and gauge unification scales, which is governed by the renormalisation group $\beta$-functions. In this paper we compare the results for this process using one, two and three-loop $\beta$-functions. In each case we generally use the same one-loop corrections for the relationship between running and pole masses for the various particles, with some use of two-loop results such as for the top quark mass. We anticipate that by the time sparticles are discovered complete two-loop threshold corrections will be available; the effect of these we would expect to be of the same order of magnitude as the effect of using the three-loop (as opposed to two-loop) $\beta$-functions, which, as we shall see, is surprisingly large for squarks.

The plan of this paper is as follows. In section 2 we review the exact results that relate the $\beta$-functions for the soft masses and interactions\cite{1}–\cite{3} to the $\beta$-functions of the dimensionless gauge and Yukawa couplings \cite{4}–\cite{6}, which we then give through three loops for the MSSM generalised to incorporate $n_5$ and $n_{10}$ sets of $SU_5 5(5)$ and $10(10)$ representations respectively. (A motive for grouping additional matter in this way is that complete $SU_5$ representations do not (at one loop) change the prediction of $\sin^2 \theta_W$ (or alternatively of $g_3^2(M_Z)$) that follows from imposing $g_{1,2,3}$ gauge unification. Also unchanged at one loop is the gauge unification scale, $M_X$; but at higher loops this scale increases and can approach the string scale.) We also give a simplified example of a three-loop soft $\beta$-function; general results for all the $\beta$-functions are available at Ref. \cite{7}.

In section 3 we present and discuss our results for the sparticle spectrum for a set of Snowmass Benchmark Points\cite{8}, all corresponding to the standard universal boundary conditions at unification, except for one case with non-universal gaugino masses. We compare our results with the useful website Ref. \cite{9} (see also Refs. \cite{10}, \cite{11}).
In section 4 we consider the effect of additional matter fields in $SU_5$ representations, as discussed in Refs. [12], [13] (for earlier work see for example Refs. [14]) and by ourselves in a previous paper [15]. We give some further examples of the effect on the sparticle spectrum of such matter. Finally section 5 contains our conclusions.

2. The Soft Beta functions

For a general $N = 1$ supersymmetric gauge theory with superpotential

$$W(\phi) = \frac{1}{2} \mu^{ij} \phi_i \phi_j + \frac{1}{6} Y^{ijk} \phi_i \phi_j \phi_k,$$

the standard soft supersymmetry-breaking scalar terms are as follows

$$V_{\text{soft}} = \left( \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \text{c.c.} \right) + (m^2)^{ij} \phi_i \phi_j,$$

where we denote $\phi^i \equiv \phi^*_i$ etc. (For the generalisation to the case when $V_{\text{soft}}$ includes a term linear in $\phi$ see [16].)

The complete exact results for the soft $\beta$-functions are given by:

$$\beta_M = 2 \mathcal{O} \left[ \frac{\beta_g}{g} \right],$$

$$\beta_h^{ij} = h^{l(kj)\gamma^i} - 2 Y^{l(jk)\gamma^i},$$

$$\beta_b^{ij} = b^{l(i\gamma^j)} - 2 \mu^{(i\gamma^j)},$$

$$(\beta_m^2)^{ij} = \Delta \gamma^{ij},$$

where $\gamma$ is the matter multiplet anomalous dimension, and

$$\mathcal{O} = M g^2 \frac{\partial}{\partial g^2} - h^{lmn} \frac{\partial}{\partial Y^{lmn}},$$

$$(\gamma_1)^{ij} = \mathcal{O} \gamma^{ij},$$

$$\Delta = 2 \mathcal{O} \mathcal{O}^* + 2 M M^* g^2 \frac{\partial}{\partial g^2} + \left[ \tilde{Y}^{lmn} \frac{\partial}{\partial Y^{lmn}} + \text{c.c.} \right] + X \frac{\partial}{\partial g}.$$ (2.4c)

Here $M$ is the gaugino mass and $\tilde{Y}^{ijk} = (m^2)^i_l Y^{jkl} + (m^2)^j_l Y^{ikl} + (m^2)^k_l Y^{ijl}$. Eq. (2.3) holds in a class of renormalisation schemes that includes the DRED’-one [17], which we will use throughout.

Finally the $X$ function above is given (in the NSVZ scheme [18]) by

$$X_{\text{NSVZ}} = -2 \frac{g^3}{16 \pi^2} \frac{S}{1 - 2g^2 C(G)(16\pi^2)^{-1}}$$

(2.5)
where

\[ S = r^{-1} \text{tr}[m^2 C(R)] - MM^* C(G), \]  

(2.6)

\( C(R), C(G) \) being the quadratic Casimirs for the matter and adjoint representations respectively. There is no corresponding exact form for \( X \) in the DRED’ scheme\[17\]; we will require the leading and sub-leading contributions, which are given by\[19\]:

\[ 16\pi^2 X^{\text{DRED}'}(1) = -2g^3 S \]  

(2.7)

and

\[(16\pi^2)^2 X^{\text{DRED}'}(2) = (2r)^{-1} g^3 \text{tr}[WC(R)] - 4g^5 C(G)S - 2g^5 C(G)QMM^*, \]  

(2.8)

where

\[ W^j_i = \frac{1}{2} Y_{ipq} Y^{pqn}(m^2)^j n + \frac{1}{2} Y_{ipq} Y^{pqn}(m^2)^i n + 2Y_{ipq} Y^{jpr}(m^2)^q r \]

+ \( h_{ipq} h^{jpr} - 8g^2 MM^* C(R)^j_i. \)  

(2.9)

and \( Q = T(R) - 3C(G) \), and \( r T(R) = \text{tr}[C(R)] \), \( r \) being the number of group generators.

We now present the results for the gauge \( \beta \)-functions and anomalous dimensions. These results are valid in the DRED’ scheme\[17\] (or indeed the DRED one\[20\], which differs from DRED’ only when we come to the soft \( \beta \)-functions). The MSSM superpotential is:

\[ W = H_2 QY_t \tau^c + H_1 QY_b \tau^c + H_1 L Y_\tau \tau^c + \mu H_1 H_2 \]  

(2.10)

where \( Y_t, Y_b, Y_\tau \) are \( n_g \times n_g \) Yukawa matrices\[3\], and we define

\[ T = Y_t Y_t^\dagger, B = Y_b Y_b^\dagger, E = Y_\tau Y_\tau^\dagger, \tilde{T} = Y_t^\dagger Y_t, \tilde{B} = Y_b^\dagger Y_b, \tilde{E} = Y_\tau^\dagger Y_\tau \].  

(2.11)

The \( SU_3 \otimes SU_2 \otimes U_1 \) gauge \( \beta \)-functions are as follows:

\[ \beta_{g_i} = (16\pi^2)^{-1} b_1 g_i^3 + (16\pi^2)^{-2} g_i^3 \left( \sum_j b_{ij} g_j^2 - a_i \right) + (16\pi^2)^{-3} \beta_{g_i}^{(3)} + \cdots \]  

(2.12)

where

\[ b_1 = \frac{1}{2} n_5 + \frac{3}{2} n_{10} + \frac{33}{2}, \quad b_2 = \frac{1}{2} n_5 + \frac{3}{2} n_{10} + 1, \quad b_3 = \frac{1}{2} n_5 + \frac{3}{2} n_{10} - 3 \]

\[ a_1 = \frac{26}{5} \text{tr}T + \frac{14}{5} \text{tr}B + \frac{18}{5} \text{tr}E, \quad a_2 = 6\text{tr}T + 6\text{tr}B + 2\text{tr}E, \quad a_3 = 4\text{tr}T + 4\text{tr}B \]  

(2.13)

\[^2 \text{Y}_{t,b,\tau} \text{ here are the transposes of the Yukawa matrices used in Ref.}\[3\].\]
\[
\begin{pmatrix}
\frac{199}{25} + \frac{7}{30} n_5 + \frac{23}{10} n_{10} & \frac{27}{5} + \frac{9}{10} n_5 + \frac{3}{10} n_{10} & \frac{88}{5} + \frac{16}{5} n_5 + \frac{24}{5} n_{10} \\
\frac{9}{5} + \frac{3}{10} n_5 + \frac{1}{10} n_{10} & 25 + \frac{7}{2} n_5 + \frac{21}{2} n_{10} & 24 + 8 n_{10} \\
\frac{11}{5} + \frac{2}{15} n_5 + \frac{3}{5} n_{10} & 9 + 3 n_{10} & 14 + \frac{17}{3} n_5 + 17 n_{10}
\end{pmatrix}.
\]

(2.14)

For the anomalous dimensions of the chiral superfields we have at one loop:

\[
\begin{align*}
16\pi^2\gamma_t^{(1)} &= 2\tilde{T} - \frac{8}{3} g_2^2 - \frac{8}{15} g_1^2, \\
16\pi^2\gamma_b^{(1)} &= 2\tilde{B} - \frac{8}{3} g_3^2 - \frac{2}{15} g_1^2, \\
16\pi^2\gamma_Q^{(1)} &= B + T - \frac{8}{3} g_3^2 - \frac{3}{2} g_2^2 - \frac{1}{30} g_1^2, \\
16\pi^2\gamma_H^{(1)} &= 3\text{tr}T - \frac{3}{2} g_2^2 - \frac{3}{10} g_1^2, \\
16\pi^2\gamma_H^{(1)} &= \text{tr}E + 3\text{tr}B - \frac{3}{2} g_2^2 - \frac{3}{10} g_1^2, \\
16\pi^2\gamma_L^{(1)} &= E - \frac{3}{2} g_2^2 - \frac{3}{10} g_1^2, \\
16\pi^2\gamma_{r}^{(1)} &= 2\tilde{E} - \frac{6}{5} g_1^2,
\end{align*}
\]

and at two loops[21]:

\[
\begin{align*}
(16\pi^2)^2\gamma_t^{(2)} &= -2\tilde{T}^2 - 6(\text{tr}T)\tilde{T} - 2\tilde{Y}_t^\dagger BY_t + (6g_2^2 - \frac{2}{3} g_1^2)\tilde{T} \\
&+ \left(\frac{8}{15} b_1 + \frac{64}{225}\right) g_1^4 + \frac{128}{45} g_2^2 g_3^2 + \left(\frac{8}{3} b_3 + \frac{64}{9}\right) g_3^4, \\
(16\pi^2)^2\gamma_b^{(2)} &= -2\tilde{B}^2 - 6(\text{tr}B)\tilde{B} - 2\tilde{Y}_b^\dagger TY_b - 2(\text{tr}E)\tilde{B} + (6g_2^2 + \frac{2}{3} g_1^2)\tilde{B} \\
&+ \left(\frac{2}{15} b_1 + \frac{4}{225}\right) g_1^4 + \frac{32}{15} g_2^2 g_3^2 + \left(\frac{8}{3} b_3 + \frac{64}{9}\right) g_3^4, \\
(16\pi^2)^2\gamma_Q^{(2)} &= -2T^2 - 3(\text{tr}T)T - 2B^2 - 3(\text{tr}B)B \\
&- (\text{tr}E)B + g_2^2 \left(\frac{4}{7} T + \frac{2}{7} B\right) + \frac{1}{10} g_1^2 g_2^2 + 8g_3^2 g_2^2 + \frac{8}{15} g_1 g_3 g_2^2 \\
&+ \left(\frac{8}{3} b_3 + \frac{64}{9}\right) g_3^4 + \left(\frac{3}{2} b_2 + \frac{9}{4}\right) g_2^4 + \left(\frac{1}{30} b_1 + \frac{1}{900}\right) g_1^4, \\
(16\pi^2)^2\gamma_{H_2}^{(2)} &= -9\text{tr}T^2 - 3\text{tr}BT + (16g_3^2 + \frac{4}{3} g_1^2) \text{tr}T + (\frac{3}{2} b_2 + \frac{9}{4}) g_2^4 \\
&+ \left(\frac{9}{10} g_1^2 g_2^2 + \left(\frac{3}{2} b_2 + \frac{9}{4}\right) g_2^4 + \left(\frac{1}{10} b_1 + \frac{9}{100}\right) g_1^4, \\
(16\pi^2)^2\gamma_{H_4}^{(2)} &= -9\text{tr}B^2 - 3\text{tr}BT - 3(\text{tr}E^2) + (16g_3^2 - \frac{2}{5} g_1^2) \text{tr}B + \frac{6}{5} g_1^2 \text{tr}E \\
&+ \left(\frac{3}{2} b_2 + \frac{9}{4}\right) g_1^4 + \frac{1}{10} g_1^2 g_2^2 + \left(\frac{3}{10} b_1 + \frac{9}{100}\right) g_1^4, \\
(16\pi^2)^2\gamma_L^{(2)} &= -2E^2 - E(3\text{tr}B + \text{tr}E - \frac{6}{5} g_1^2) + \left(\frac{3}{2} b_2 + \frac{9}{4}\right) g_2^4 \\
&+ \left(\frac{9}{10} g_1^2 g_2^2 + \left(\frac{3}{2} b_2 + \frac{9}{4}\right) g_2^4 + \left(\frac{1}{10} b_1 + \frac{9}{100}\right) g_1^4, \\
(16\pi^2)^2\gamma_{r}^{(2)} &= -2\tilde{E}^2 - \tilde{E}(6\text{tr}B + 2\text{tr}E - 6g_2^2 + \frac{6}{5} g_1^2) + \left(\frac{6}{5} b_1 + \frac{36}{25}\right) g_1^4.
\end{align*}
\]
The three-loop gauge $\beta$-function terms are (henceforth we suppress $(16\pi^2)^{-L}$ factors).

\[
\beta_{g_1}^{(3)} = g_1^3 \left[ \frac{84}{5} \text{tr}T^2 + 18(\text{tr}T)^2 + \frac{54}{5} \text{tr}B^2 + \frac{36}{5} (\text{tr}B)^2 + \frac{58}{5} \text{tr}TB + \frac{54}{5} \text{tr}E^2 + \frac{24}{5} (\text{tr}E)^2 
\right.
\]
\[
+ \frac{84}{5} \text{tr}E \text{tr}B - \left( \frac{169}{75} g_1^2 + \frac{87}{75} g_2^2 + \frac{352}{75} g_3^2 \right) \text{tr}T - \left( \frac{49}{75} g_1^2 + \frac{33}{75} g_2^2 + \frac{256}{75} g_3^2 \right) \text{tr}B
\]
\[
- \left( \frac{81}{25} g_1^2 + \frac{63}{25} g_2^2 \right) \text{tr}E + \left( \frac{484}{15} - \frac{4}{5} n_5^2 - \frac{506}{45} + 6n_{10} \right) n_5 - \frac{154}{5} n_{10} - \frac{54}{5} n_{10}^2 \right] \text{tr}E
\]
\[
- \left[ \left( \frac{24}{5} + \frac{8}{5} n_{10} \right) g_2^2 + \left( \frac{64}{225} n_5 + \frac{344}{75} n_{10} + \frac{1096}{75} \right) g_1^2 g_3^2
\]
\[
- \left( \frac{27}{40} n_5^2 + \left( \frac{27}{4} + \frac{9}{4} n_{10} \right) n_5 + \frac{81}{5} + \frac{261}{40} n_{10} + \frac{27}{40} n_{10}^2 \right) g_4^2 - \left( \frac{1}{50} n_{10} + \frac{27}{50} n_{10} + \frac{69}{25} \right) g_1^2 g_2
\]
\[
- \left( \frac{7}{40} n_5^2 + \left( \frac{7507}{900} + \frac{9}{4} n_{10} \right) n_5 + \frac{12859}{300} n_{10} + \frac{207}{40} n_{10}^2 + \frac{32117}{375} \right) g_1^4 \right].
\]

\[
(2.17a)
\]

\[
\beta_{g_2}^{(3)} = g_2^3 \left[ 24(\text{tr}T^2 + \text{tr}B^2) + 18[(\text{tr}T)^2 + (\text{tr}B)^2] + 12 \text{tr}BT + 12 \text{tr}B \text{tr}E + 8 \text{tr}E^2
\right.
\]
\[
+ 2(\text{tr}E)^2 - (32g_3^2 + 33g_2^2)(\text{tr}T + \text{tr}B) - g_1^2 \left( \frac{29}{5} \text{tr}T + \frac{11}{5} \text{tr}B \right) - (11g_2^2 + \frac{21}{5} g_1^2) \text{tr}E
\]
\[
- \left[ (6n_{10} + 18) n_5 + \frac{118}{3} n_{10} + 18n_{10}^2 - 44 \right] g_3^4 + \left[ \left( 8n_{10} + 24 \right) g_2^2 - \left( \frac{8}{15} n_{10} + \frac{4}{5} \right) g_1^2 \right] g_3^2
\]
\[
- \left( \frac{13}{8} n_5^2 + \left( \frac{33}{4} + \frac{39}{4} n_{10} \right) n_5 + \frac{99}{4} n_{10} + \frac{117}{8} n_{10}^2 - 35 \right) g_2^4 + \left( \frac{1}{10} n_{10} + \frac{9}{5} + \frac{3}{10} n_{10} \right) g_1^2 g_2
\]
\[
- \left( \frac{7}{40} n_5^2 + \left( \frac{3}{4} n_{10} + \frac{441}{100} \right) n_5 + \frac{9}{40} n_{10}^2 + \frac{457}{25} + \frac{1513}{300} n_{10} \right) g_1^4 \right],
\]

\[
(2.17b)
\]

\[
\beta_{g_3}^{(3)} = g_3^3 \left[ 18(\text{tr}T)^2 + 12 \text{tr}T^2 + 8 \text{tr}BT + 18(\text{tr}B)^2 + 12 \text{tr}B^2 + 6 \text{tr}E \text{tr}B
\right.
\]
\[
- \left( \frac{104}{3} g_3^2 + 12g_2^2 \right) (\text{tr}T + \text{tr}B) - g_1^2 \left( \frac{44}{15} \text{tr}T + \frac{32}{15} \text{tr}B \right)
\]
\[
+ \left( \frac{347}{3} + \frac{215}{3} n_{10} - \frac{11}{3} n_5^2 + \left( \frac{215}{9} - \frac{33}{2} n_{10} \right) n_5 - \frac{99}{4} n_{10}^2 \right) g_3^4
\]
\[
+ \left[ (2n_{10} + 6) g_2^2 + \left( \frac{3}{5} n_{10} + \frac{4}{15} n_5 + \frac{22}{15} \right) g_1^2 \right] g_3^2
\]
\[
- \left[ \left( \frac{27}{4} + \frac{9}{4} n_{10} \right) n_5 + \frac{27}{4} n_{10}^2 + \frac{117}{4} n_{10} + 27 \right] g_2^4 - \left( \frac{1}{5} n_{10} + \frac{3}{5} \right) g_1^2 g_2
\]
\[
- \left( \frac{1}{10} n_5^2 + \left( \frac{2689}{900} + \frac{3}{4} n_{10} \right) n_5 + \frac{27}{20} n_{10}^2 + \frac{1702}{75} + \frac{3353}{300} n_{10} \right) g_1^4 \right].
\]

\[
(2.17c)
\]

The three-loop results for the anomalous dimensions are as follows:

6
\( \gamma^{(3)}_Q = k(T^3 + B^3) + 4BTD + 4TB^2 + 6T^2trT + B^2(6trB + 2trE) \\
+ B[6tr(TB) - 9(trB)^2 - 6trBtrE + 18tr(B^2) - (trE)^2 + 6tr(E^2)] \\
- 9T(trT)^2 + 18Ttr(T^2) + 6Ttr(TB) + g_1^2[T^2(\frac{11}{3} - k) + B^2(\frac{7}{15} - \frac{1}{5}k)] \\
+ [(3k - 3)g_2^2 + \frac{54}{3}g_3^2](T^2 + B^2) + [(2 - \frac{4}{5}k)g_1^2 + 18g_2^2 + (8k - 8)g_3^2]TtrT \\
+ g_1^2B[(\frac{16}{5} - \frac{4}{5}k)trB + (\frac{3}{5}k - \frac{8}{5})trE] + g_2^2B(18trB + 6trE) \\
+ 8g_3^2B[(k - 1)trB + trE] + g_1^4T(\frac{143}{900}k - \frac{3767}{300} - \frac{51}{20}n_10 - \frac{17}{20}n_5) \\
+ g_1^4B(\frac{7}{180}k - \frac{633}{100} - \frac{27}{20}n_10 - \frac{9}{20}n_5) - (\frac{13}{30}trT + \frac{7}{30}trB + \frac{3}{10}trE)g_1^4 \\
+ (\frac{3}{2}k - \frac{54}{10}g_1^2g_2^2T + (\frac{3}{10}k - \frac{41}{10})g_1^2g_2^2B + g_1^2g_2^2T(\frac{64}{15}k - \frac{68}{10}) + g_1^2g_3^2B(\frac{64}{15}k - \frac{76}{15}) \\
- g_2^4[(T + B)(\frac{45}{10} + \frac{24}{5}k + \frac{27}{5}n_10 + \frac{9}{4}n_5) + \frac{45}{2}(trT + trB) + \frac{15}{2}trE] \\
- 4g_2^4g_3^3(T + B) + g_4^4[(T + B)(\frac{8}{5} - \frac{23}{10}k - 12n_10 - 4n_5) - \frac{80}{5}(trT + trB)] \\
+ (\frac{25}{4} - \frac{3}{40}kn_10 - \frac{9}{40}kn_5 - \frac{27}{20}n_10 + \frac{3}{10}n_5 + \frac{11}{10}n_5)g_1^2g_2^4 - \frac{8}{5}g_1^2g_2^2g_3^2 \\
+ g_4^6(\frac{28457}{14100} - k(\frac{23}{600}n_10 + \frac{7}{1800}n_5 + \frac{199}{1500}) + (\frac{1}{20}n_5 + \frac{17}{20} + \frac{3}{10}n_10)n_10 + \frac{43}{180}n_5 + \frac{1}{120}n_5^2) \\
+ g_1^4g_2^2(\frac{11}{100} - \frac{1}{200}kn_10 - \frac{3}{10}kn_5 - \frac{9}{100}k - \frac{1}{10}n_10 + \frac{1}{20}n_5) \\
+ g_1^4g_3^2(\frac{134}{225} - \frac{2}{15}kn_10 - \frac{4}{225}kn_5 - \frac{22}{75}k + \frac{4}{15}n_10 + \frac{2}{15}n_5) \\
+ g_1^4g_3^2(\frac{608}{45} - \frac{4}{5}kn_10 - \frac{8}{45}kn_5 - \frac{44}{15}k + \frac{58}{15}n_10 + \frac{38}{45}n_5) \\
+ g_2^4g_3^2(50 - 6kn_10 - 18k + 24n_10 - 2n_5) + g_2^4g_3^2(8 - 4kn_10 - 12k + 14n_10 - 2n_5) \\
+ g_2^6(\frac{345}{4} + \frac{45}{8}kn_10 + \frac{15}{16}kn_5 + \frac{105}{4}k + \frac{9}{4}n_10n_5 + \frac{81}{4}n_10 + \frac{27}{2}n_10 + \frac{27}{2}n_5 + \frac{3}{8}n_5^2) \\
+ g_3^6(\frac{2720}{27} + k(\frac{40}{3}n_10 + \frac{40}{9}n_5 + \frac{160}{3}) + n_10(4n_5 + \frac{236}{3} + 6n_10) + \frac{236}{9}n_5 + \frac{2}{7}n_5^2) \\
(2.18) \\

\gamma^{(3)}_L = kE^3 + E^2(6trB + 2trE) \\
+ E[6tr(TB) - 9(trB)^2 - 6trBtrE + 18tr(B^2) - (trE)^2 + 6tr(E^2)] \\
+ g_1^2E^2(9 - \frac{9}{5}k) + (8 - 2k)g_1^2EtrB + (3k - 3)g_2^2E^2 + g_2^2E(18trB + 6trE) \\
+ (8k - 32)g_3^2EtrB + g_1^4E(-\frac{549}{20} + \frac{27}{100}k - \frac{99}{20}n_10 - \frac{33}{20}n_5) \\
+ g_1^4E(-\frac{39}{10}trT - \frac{21}{10}trB - \frac{27}{10}trE) \\
+ g_2^2g_3^2E(-\frac{81}{10} + \frac{27}{10}k) + g_2^4E(-\frac{45}{4} - \frac{21}{4}k - \frac{27}{4}n_10 - \frac{9}{4}n_5) \\
+ g_4^4(-\frac{45}{2}trT - \frac{45}{2}trB - \frac{15}{2}trE) + \Xi \\
(2.19)
\[
\gamma^{(3)}_t = (6 + 2k)\bar{T}^3 + 6\bar{T}^2trT - 2Y_t^\dagger BY_t - 2Y_t^\dagger TBY_t + 6Y_t^\dagger B^2Y_t \\
+ \bar{T}(36tr(T^2) + 12tr(TB) - 18(trT)^2) + Y_t^\dagger BY_t(-6trT + 12trB + 4trE) \\
+ g_t^2\left[\bar{T}^2(-\frac{1}{3} + k) + 7(1 + \frac{k}{5})\bar{T}tr(T) + Y_t^\dagger BY_t(\frac{10}{15} + \frac{2}{5}k)\right] \\
+ g_t^2\left[(9 - 3k)\bar{T}^2 + (27 - 9k)\bar{T}trT + (9 - 3k)Y_t^\dagger BY_t\right] \\
+ g_t^2\left[16(k - 1)\bar{T}trT + \frac{64}{3}(\bar{T}^2 + Y_t^\dagger BY_t)\right] \\
+ g_t^2\left[\bar{T}(\frac{-799}{36} - \frac{247}{456}k - \frac{18}{5}n_{10} - \frac{6}{5}n_5) - \frac{104}{15}trT - \frac{56}{15}trB - \frac{24}{5}trE\right] \\
+ g_t^2g_t^2\bar{T}(\frac{-8}{15} - \frac{112}{45}k) + g_t^2g_t^2\bar{T}(\frac{-67}{5} + \frac{13}{5}k) \\
+ g_t^2\bar{T}(\frac{-87}{2} - \frac{3}{k} - 18n_{10} - 6n_5) + g_t^2g_t^2\bar{T}(-88 + 16k) \\
+ g_t^3\left[\bar{T}(\frac{16}{3} - \frac{272}{9}k - 24n_{10} - 8n_5) - \frac{80}{3}trT - \frac{80}{3}trB\right] \\
+ g_t^2g_t^2\left(-\frac{172}{15} - \frac{4}{5}kn_{10} - \frac{8}{25}kn_5 - \frac{44}{15}k + \frac{28}{15}n_{10} + \frac{8}{5}n_5\right) \\
+ g_t^2g_t^2\left(\frac{36}{5} - \frac{2}{25}kn_{10} - \frac{6}{25}kn_5 - \frac{36}{25}k + \frac{8}{5}n_{10} + \frac{6}{5}n_5\right) \\
+ g_t^2g_t^2\left(\frac{2144}{225} - \frac{32}{25}kn_{10} - \frac{64}{225}kn_5 - \frac{352}{75}k + \frac{64}{15}n_{10} + \frac{32}{15}n_5\right) \\
+ g_t^2\left(\frac{272}{27} + \frac{40}{9}kn_{10} + \frac{40}{9}kn_5 + \frac{160}{3}k + 4n_{10}n_5 + \frac{236}{3}n_{10} + 6n_1^2 + \frac{236}{9}n_5 + \frac{2}{3}n_5^2\right) \\
+ g_t^3\left(-\frac{106868}{3375} - k(\frac{46}{7}n_{10} + \frac{14}{225}n_5 + \frac{796}{375}) + n_{10}(\frac{4}{5}n_5 + \frac{66}{5} + \frac{6}{5}n_10) + \frac{166}{45}n_5 + \frac{2}{15}n_5^2\right) \\
+ g_t^2g_t^2\left(60 - 4kn_{10} - 12k + 20n_{10}\right)
\]

(2.20)

\[
\gamma^{(3)}_{H_1} = (k + 1)\left[3tr(B^3) + tr(E^3)\right] + 9tr(T^2B) + 18trTr(TB) \\
+ 6(trE + 3trB)\left[tr(E^2) + 3tr(B^2)\right] + (24 - 8k)g_t^2[tr(TB) + 3tr(B^2)] \\
+ g_t^2[18tr(TB) + (9k + 9)tr(B^2) + (3k + 3)tr(E^2)] \\
+ g_t^2[-\frac{12}{5}tr(TB) + \frac{7}{5}tr(B) + 3tr(B^2) + \frac{9}{5}ktr(B^2) + 9tr(E^2) - \frac{7}{9}tr(E^2)] \\
+ g_t^2[-\frac{30}{10}trT - trB(\frac{77}{50}k + \frac{77}{300}k + \frac{57}{10}n_{10} + \frac{19}{20}n_5) + trE(\frac{27}{100}k - \frac{603}{20} - \frac{99}{20}n_{10} - \frac{33}{20}n_5)] + g_t^2[\frac{2}{3}trB - \frac{3}{5}ktrB - \frac{8}{15}trE + \frac{27}{10}ktrE] \\
+ trB[g_t^2g_t^2(\frac{284}{15} + \frac{56}{15}k) + g_t^2g_t^2(-132 + 24k)] + g_t^2[-\frac{45}{2}trT \\
- trB(\frac{225}{4} + \frac{63}{4}k + \frac{81}{4}n_{10} + \frac{27}{4}n_5) - trE(\frac{75}{4} + \frac{21}{4}k + \frac{27}{4}n_{10} + \frac{9}{4}n_5)] \\
- g_t^4trB(\frac{160}{3} + \frac{8}{3}k + 48n_{10} + 16n_5) + \Xi
\]

(2.21)
\[\gamma_b^{(3)} = (2k + 6)\bar{B}^3 + 6Y_b^\dagger T^2 Y_b - 2Y_b^\dagger B T Y_b + Y_b^\dagger T Y_b (12\text{tr} T - 6\text{tr} B - 2\text{tr} E) - 2Y_b^\dagger T B Y_b + \bar{B}^2 (6\text{tr} B + 2\text{tr} E) + \bar{B}[12\text{tr} (T B) - 18(\text{tr} B)^2 - 12\text{tr} B \text{tr} E + 36\text{tr} (B^2) - 2(\text{tr} E)^2 + 12(\text{tr} E^2)] + g_1^2 Y_b^\dagger T Y_b (-\frac{29}{15} + \frac{3}{5} k) + g_1^2 \bar{B}^2 (-\frac{1}{3} + \frac{1}{5} k) + g_1^2 \bar{B}[(7 - k)\text{tr} B + (k - 3)\text{tr} E] + g_2^2 Y_b^\dagger T Y_b (9 - 3k) + g_2^2 \bar{B}^2 (9 - 3k) + g_2^2 \bar{B}[(27 - 9k)\text{tr} B + (9 - 3k)\text{tr} E] + \frac{64}{3} g_3^2 Y_b^\dagger T Y_b + \frac{64}{3} g_3^2 \bar{B}^2 + g_3^2 \bar{B}[(16k - 16)\text{tr} B + 16\text{tr} E] + g_1^4 \bar{B}(-\frac{87}{2} - \frac{3}{2} k - 18n_{10} - 6n_5) + g_2^4 \bar{B}(-\frac{87}{2} - \frac{3}{2} k - 18n_{10} - 6n_5) + g_3^4 \bar{B}(\frac{15}{6} - \frac{27}{9} k - 24n_{10} - 8n_5) - \frac{80}{3} \text{tr} T - \frac{80}{3} \text{tr} B] + g_3^6 (\frac{2720}{27} + \frac{40}{9} k n_{10} + \frac{40}{9} k n_5 + \frac{160}{9} k + 4n_{10} n_5 + \frac{236}{9} n_{10} + 6n_{10} n_5 + \frac{7}{5} n_5^2) + g_1^6 (\frac{5629}{675} - k (\frac{23}{150} n_{10} + \frac{7}{150} n_5 + \frac{199}{375}) + n_{10} (\frac{1}{5} n_5 + \frac{160}{50} + \frac{3}{10} n_5) + \frac{427}{450} n_5 + \frac{1}{30} n_5^2) + g_4^6 (\frac{9}{5} - \frac{1}{5} k n_{10} - \frac{3}{50} k n_5 - \frac{9}{25} k + \frac{1}{10} n_{10} + \frac{3}{10} n_5) + g_1^6 (\frac{728}{225} - \frac{8}{25} k n_{10} - \frac{16}{225} k n_5 - \frac{88}{75} k + \frac{16}{15} n_{10} + \frac{8}{45} n_5) + g_1^6 (\frac{452}{45} - \frac{4}{5} k n_{10} - \frac{16}{45} k n_5 - \frac{44}{15} k + \frac{52}{15} n_{10} + \frac{32}{45} n_5) + g_2^6 (60 - 4k n_{10} - 12k + 20n_{10}) \]
\[ \gamma^{(3)}_{\tau} = (6 + 2k)\tilde{E}^3 + (6trB + 2trE)\tilde{E}^2 \]
\[ + \tilde{E}(12tr(TB) - 18(trB)^2 - 12trBtrE + 36tr(B^2) - 2(trE)^2 + 12tr(E^2)) \]
\[ + g_2^2\tilde{E}^2(\frac{9}{5} + \frac{9}{5}k) + g_2^2\tilde{E}^2(9 - 3k) + g_2^2\tilde{E}(\frac{107}{10} - \frac{7}{5}k)trB + \frac{2}{5}k(trE) \]
\[ + g_2^2\tilde{E}trB(16k - 64) + g_2^2\tilde{E}(trB(27 - 9k) + trE(9 - 3k)) \]
\[ + g_1^4\tilde{E}(-\frac{1503}{90} - 27\frac{1}{k} - \frac{36}{5}n_{10} - \frac{12}{5}n_5) + g_1^2g_2^2\tilde{E}(-27 + \frac{27}{5}k) \]
\[ + g_1^4(-\frac{78}{5}trT - \frac{42}{5}trB + \frac{54}{5}trE) + g_1^4\tilde{E}(-\frac{87}{2} - \frac{3}{2}k - 18n_{10} - 6n_5) \]
\[ + g_1^6(\frac{7899}{125} - \frac{69}{50}kn_{10} - \frac{7}{50}kn_5 - \frac{597}{125}k + \frac{9}{5}n_{10}n_5 + \frac{27}{2}n_{10}^2 + \frac{77}{10}n_5 + \frac{3}{10}n_5^2) \]
\[ + g_1^4g_2^2(\frac{81}{5} - \frac{9}{50}kn_{10} - \frac{27}{50}kn_5 - \frac{81}{25}k + \frac{9}{10}n_{10} + \frac{27}{10}n_5) \]
\[ + g_1^4g_3(\frac{264}{5} - \frac{72}{25}kn_{10} - \frac{16}{25}kn_5 - \frac{264}{25}k + \frac{72}{5}n_{10} + \frac{16}{5}n_5) \]
\[ + \Xi = g_2^6(\frac{345}{4} + k(\frac{45}{8}n_{10} + \frac{15}{8}n_5 + \frac{105}{4}) + n_{10}(\frac{9}{4}n_5 + \frac{81}{2} + \frac{27}{8}n_{10} + \frac{27}{2}n_5 + \frac{3}{8}n_5^2) \]
\[ + g_1^6(\frac{1839}{100} - k(\frac{69}{200}kn_{10} + \frac{7}{200}kn_5 + \frac{597}{500}) + n_{10}(\frac{9}{20}n_5 + \frac{753}{100} + \frac{27}{40}n_{10} + \frac{211}{100}n_5 + \frac{3}{40}n_5^2) \]
\[ + g_1^4g_2^2(\frac{27}{100} - \frac{9}{200}kn_{10} - \frac{27}{200}kn_5 - \frac{81}{100}k - \frac{9}{20}n_{10} + \frac{9}{20}n_5) \]
\[ + g_1^4g_3(\frac{66}{5} - \frac{18}{25}kn_{10} - \frac{4}{25}kn_5 - \frac{66}{25}k + \frac{18}{5}n_{10} + \frac{4}{5}n_5) \]
\[ + g_2^2g_4^2(\frac{9}{4} - \frac{3}{40}kn_{10} - \frac{27}{40}kn_5 - \frac{27}{30}k - \frac{3}{10}n_{10} + \frac{9}{10}n_5) \]
\[ + g_2^4g_3^2(90 - 6kn_{10} - 18k + 30n_{10}) \]  
\[ (2.24) \]

where \( k = 6\zeta(3) \), and

\[ \Xi = g_2^6(\frac{345}{4} + k(\frac{45}{8}n_{10} + \frac{15}{8}n_5 + \frac{105}{4}) + n_{10}(\frac{9}{4}n_5 + \frac{81}{2} + \frac{27}{8}n_{10} + \frac{27}{2}n_5 + \frac{3}{8}n_5^2) \]
\[ + g_1^6(\frac{1839}{100} - k(\frac{69}{200}kn_{10} + \frac{7}{200}kn_5 + \frac{597}{500}) + n_{10}(\frac{9}{20}n_5 + \frac{753}{100} + \frac{27}{40}n_{10} + \frac{211}{100}n_5 + \frac{3}{40}n_5^2) \]
\[ + g_1^4g_2^2(\frac{27}{100} - \frac{9}{200}kn_{10} - \frac{27}{200}kn_5 - \frac{81}{100}k - \frac{9}{20}n_{10} + \frac{9}{20}n_5) \]
\[ + g_1^4g_3(\frac{66}{5} - \frac{18}{25}kn_{10} - \frac{4}{25}kn_5 - \frac{66}{25}k + \frac{18}{5}n_{10} + \frac{4}{5}n_5) \]
\[ + g_2^2g_4^2(\frac{9}{4} - \frac{3}{40}kn_{10} - \frac{27}{40}kn_5 - \frac{27}{30}k - \frac{3}{10}n_{10} + \frac{9}{10}n_5) \]
\[ + g_2^4g_3^2(90 - 6kn_{10} - 18k + 30n_{10}) \]  
\[ (2.25) \]

In terms of the anomalous dimensions, the Yukawa \( \beta \)-functions are:

\[ \beta_{Y_1} = \gamma_QY_1 + Y_1(\gamma_t + \gamma_{H_2}), \quad \beta_{Y_6} = \gamma_QY_6 + Y_6(\gamma_b + \gamma_{H_1}), \quad \beta_{Y_7} = \gamma_LY_7 + Y_7(\gamma_{\tau} + \gamma_{H_1}), \]
\[ (2.26) \]

and the \( \beta \)-function for the Higgs \( \mu \)-term is

\[ \beta_{\mu} = \mu(\gamma_{H_2} + \gamma_{H_1}). \]
\[ (2.27) \]

We will also require the anomalous dimensions of the constituents of the extra 5 and 10 representations, which are easily obtained by setting \( T = B = E = 0 \), except retaining terms that contain \( T, B, E \) only inside traces; such terms occur for the first time at three loops.
From the above expressions for $\beta_{g_i}$ and $\gamma$ we have calculated the three-loop soft $\beta$-functions using Eq. (2.3) and FORM. The resulting expressions are very unwieldy; as an example we give the one, two and three-loop results for $\beta_{m_Q^2}$, in the approximation that we retain only $g_3^2$ and the top quark Yukawa coupling $\lambda_t$ (in what follows we denote the third generation squarks as $Q_t, t^c, b^c$, and the first or second generation squarks as $Q_u, u^c, d^c$):

$$\beta_{m_Q^2}^{(1)} = 2\lambda_t^2 (\Sigma_t + A_t^2) - 8\left(\frac{1}{10}g_1^2M_1^2 + \frac{2}{3}g_2^2M_2^2 + \frac{4}{3}g_3^2M_3^2\right)$$

$$\beta_{m_Q^2}^{(2)} = -20\lambda_t^4 (\Sigma_t + 2A_t^2) + 16g_3^4M_3^2(n_5 + 3n_{10} - \frac{5}{3})$$
$$+ \frac{16}{3}g_3^4(2m_{Q_u}^2 + m_{t^c}^2 + m_{b^c}^2 + (n_{10} + 2)(m_{u^c}^2 + 2m_{Q_u}^2) + (n_5 + 2)m_{d^c}^2)$$

$$\beta_{m_Q^2}^{(3)} = [(1280k + \frac{20512}{9}) + 16n_5^2 + \left(\frac{6224}{9} + \frac{330}{3}k\right)(n_5 + 3n_{10})$$
$$+ 96n_{10}n_5 + 144n_{10}^2)M_3^2 + \left(\frac{330}{3} - \frac{16}{3}(n_5 + 3n_{10})\right)(m_{t^c}^2 + m_{b^c}^2 + 2m_{Q_u}^2)$$
$$+ (2m_{Q_u}^2 + m_{u^c}^2)\left(\frac{640}{9} - \frac{32}{3}n_5 + \frac{32}{9}n_{10} - \frac{16}{3}n_5n_{10} - 16n_{10}^2\right)$$
$$+ n_{10}^2\left(\frac{640}{9} + \frac{224}{9}n_5 - 32n_{10} - 16n_5n_{10} - \frac{16}{3}n_{10}^2\right)M_3^2$$
$$- \left[(288 + \frac{544}{3}k + 48(n_5 + 3n_{10}))M_3^2 - (192 + \frac{1088}{9}k + 32(n_5 + 3n_{10}))A_tM_3\right.$$
3. The Snowmass Benchmark Points

In this section we examine the effect of the three-loop corrections on the standard running analysis, that is for $n_5 = n_{10} = 0$. We will focus on the standard treatment with universal boundary conditions at gauge unification, often termed CMSSM or MSUGRA. Thus we assume that at $M_X$ we have universal soft scalar masses ($m_0$), gaugino masses ($m_1$) and $A$-parameters ($A$), and work in the third-generation-only Yukawa coupling approximation. This is for ease of comparison with existing results rather than because we find the scenario particularly compelling. We will present results for the set of MSUGRA Snowmass Benchmark Points shown in Table 1:

| Point | $\tan \beta$ | $m_1$  | $m_0$   | $A$     | sign $\mu$ |
|-------|--------------|--------|---------|---------|------------|
| SPS1a | 10           | 250GeV | 100GeV  | $-100$GeV | +          |
| SPS1b | 30           | 400GeV | 200GeV  | 0       | +          |
| SPS2  | 10           | 300GeV | 1450GeV | 0       | +          |
| SPS3  | 10           | 400GeV | 90GeV   | 0       | +          |
| SPS4  | 50           | 300GeV | 400GeV  | 0       | +          |
| SPS5  | 5            | 300GeV | 150GeV  | $-1$TeV | +          |
| SPS6  | 10           | see footnote$^3$ | 150GeV | 0 | +          |

Table 1: Input parameters for the SPS Benchmark Points

Other input parameters are shown in Table 2:

| $m_t^{\text{pole}}$ | $m_b^{\text{pole}}$ | $m_{\tau}^{\text{pole}}$ | $\alpha_3(M_Z)$ | $\alpha_2(M_Z)$ | $\alpha_1(M_Z)$ |
|---------------------|----------------------|--------------------------|-----------------|-----------------|-----------------|
| 178GeV              | 4.9GeV               | 1.777GeV                 | 0.1172          | 0.033823        | 0.016943        |

Table 2: Input parameters for the running analysis

In Table 2 the input couplings $\alpha_{1..3}$ correspond to the Standard Model $\overline{MS}$ results; we calculate the appropriate dimensionless coupling input values for the running analysis by an iterative procedure involving the sparticle spectrum. We define gauge unification

\textsuperscript{3} except for the SPS6 point. The SPS6 point corresponds to non-unified gaugino masses, $M_1 = 480$GeV, $M_2 = M_3 = 300$GeV.
to be the scale where $\alpha_2$ and $\alpha_1$ meet; we speed up the determination of this by (at each iteration) adjusting the unification scale using the solution of the one-loop $\beta$-functions for the gauge couplings from the previous value of the scale. We employ one-loop radiative corrections as detailed in Ref. \[24]\; thus we run up from $M_Z$ using the full supersymmetric $\beta$-functions. For most particles we evaluate the pole mass at a renormalisation scale equal to the pole mass itself, and determine this value by iteration; the exception being the light CP-even Higgs, where we use a scale equal to the average squark mass.

### 3.1. Benchmark point SPS 1a

This point is a “typical” point in MSUGRA parameter space. In Table 3 we compare our results for a selection of sparticle masses (at $n_5 = n_{10} = 0$) with the spread of results taken from Ref. \[9]\, denoted AKP (note our convention that the predominantly right-handed top squark is $\tilde{t}_2$). 

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\[4\] In the first line of Eq. 37 of Ref. \[24]\, the first term in the square bracket should read $-(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2)B_0(m_{\tilde{t}_2}, m_{\tilde{t}_1}, 0)$: i.e. it should have a minus sign. The corresponding exact result in Eq. D49 is correct, however.
| Sparticle | 1loop | 2loops | 3loops | AKP       |
|----------|-------|--------|--------|-----------|
| $\tilde{g}$ | 628   | 613    | 611    | 604 − 612 |
| $\tilde{t}_1$ | 594   | 590    | 583    | 577 − 588 |
| $\tilde{t}_2$ | 400   | 399    | 391    | 396 − 401 |
| $\tilde{u}_L$ | 573   | 565    | 557    | 565 − 569 |
| $\tilde{u}_R$ | 552   | 548    | 539    | 547 − 549 |
| $\tilde{b}_1$ | 520   | 514    | 507    | 514 − 518 |
| $\tilde{b}_2$ | 551   | 548    | 540    | 539 − 548 |
| $\tilde{d}_L$ | 579   | 571    | 563    | 571 − 574 |
| $\tilde{d}_R$ | 551   | 548    | 539    | 546 − 548 |
| $\tilde{\tau}_1$ | 212   | 207    | 206    | 208 − 211 |
| $\tilde{\tau}_2$ | 139   | 135    | 135    | 134 − 136 |
| $\tilde{e}_L$ | 209   | 202    | 202    | 204 − 207 |
| $\tilde{e}_R$ | 147   | 144    | 144    | 143 − 146 |
| $\tilde{\nu}_e$ | 192   | 186    | 185    | 186 − 191 |
| $\tilde{\nu}_\tau$ | 191   | 185    | 184    | 185 − 191 |
| $\chi_1$ | 104   | 97     | 97     | 95.6 − 97.4 |
| $\chi_2$ | 193   | 180    | 179    | 181 − 182 |
| $\chi_3$ | 351   | 369    | 364    | 362 − 371 |
| $\chi_4$ | 376   | 388    | 384    | 381 − 390 |
| $\chi_1^\pm$ | 193   | 179    | 178    | 180 − 182 |
| $\chi_2^\pm$ | 376   | 388    | 384    | 380 − 390 |
| $h$ | 114   | 114    | 114    | 112 − 115 |
| $H$ | 392   | 403    | 399    | 403 − 407 |
| $A$ | 391   | 403    | 399    | 400 − 406 |
| $H^\pm$ | 400   | 412    | 408    | 410 − 415 |

Table 3: Sparticle masses (in GeV) for the SPS1a point

We would expect our two-loop results to correspond most closely to AKP and we see that they are indeed broadly consistent, typically being within the range defined by the other programs or within a GeV of it. The effect of inclusion of three-loop running is never greater than 2%; note, however, that the shift caused by three-loop running effects is comparable for $\tilde{u}_L$ and larger for $\tilde{t}_2, \tilde{u}_R$ than that produced by two-loop running effects.
3.2. Benchmark point SPS 1b

This is another “typical” point but with a higher value of tan $\beta$. Our results are given in Table 4.

| Mass | 1 loop | 2 loops | 3 loops | AKP  |
|------|--------|---------|---------|------|
| $\tilde{g}$ | 967 | 946 | 943 | $933 - 943$ |
| $\tilde{t}_1$ | 848 | 841 | 832 | $836 - 839$ |
| $\tilde{t}_2$ | 657 | 656 | 646 | $652 - 661$ |
| $\tilde{u}_L$ | 891 | 878 | 868 | $878 - 882$ |
| $\tilde{u}_R$ | 854 | 849 | 837 | $848 - 850$ |
| $\tilde{b}_1$ | 781 | 773 | 763 | $773 - 778$ |
| $\tilde{b}_2$ | 831 | 827 | 816 | $819 - 828$ |
| $\tilde{d}_L$ | 895 | 882 | 872 | $882 - 885$ |
| $\tilde{d}_R$ | 851 | 847 | 835 | $844 - 848$ |
| $\tilde{\tau}_1$ | 353 | 347 | 346 | $347 - 349$ |
| $\tilde{\tau}_2$ | 208 | 199 | 200 | $196 - 202$ |
| $\tilde{e}_L$ | 348 | 339 | 338 | $341 - 342$ |
| $\tilde{e}_R$ | 258 | 254 | 254 | $253 - 256$ |
| $\tilde{\nu}_e$ | 338 | 329 | 328 | $329 - 332$ |
| $\tilde{\nu}_\tau$ | 328 | 318 | 318 | $319 - 322$ |
| $\chi_1$ | 173 | 162 | 162 | $159 - 163$ |
| $\chi_2$ | 327 | 305 | 304 | $308 - 308$ |
| $\chi_3$ | 507 | 532 | 526 | $521 - 534$ |
| $\chi_4$ | 526 | 546 | 541 | $534 - 546$ |
| $\chi_1^\pm$ | 327 | 305 | 304 | $307 - 308$ |
| $\chi_2^\pm$ | 526 | 547 | 541 | $535 - 547$ |
| $h$ | 118 | 118 | 118 | $117 - 119$ |
| $H$ | 528 | 544 | 539 | $540 - 544$ |
| $A$ | 529 | 545 | 540 | $538 - 544$ |
| $H^\pm$ | 535 | 551 | 547 | $547 - 551$ |

*Table 4:* Sparticle masses (in GeV) for the SPS1b point
3.3. Benchmark point SPS 2

This is a “focus point region” point [23], characterised by the large value of $m_0$. Our results are given in Table 5.

| mass    | 1loop | 2loops | 3loops | AKP    |
|---------|-------|--------|--------|--------|
| $\tilde{g}$ | 835   | 816    | 814    | 778 – 805 |
| $\tilde{t}_1$  | 1322  | 1292   | 1287   | 1291 – 1318 |
| $\tilde{t}_2$  | 942   | 921    | 913    | 913 – 942 |
| $\tilde{u}_L$  | 1597  | 1562   | 1558   | 1566 – 1591 |
| $\tilde{u}_R$  | 1584  | 1556   | 1552   | 1556 – 1581 |
| $\tilde{b}_1$  | 1303  | 1273   | 1268   | 1280 – 1309 |
| $\tilde{b}_2$  | 1571  | 1544   | 1540   | 1527 – 1568 |
| $\tilde{d}_L$  | 1600  | 1564   | 1560   | 1567 – 1593 |
| $\tilde{d}_R$  | 1584  | 1556   | 1553   | 1555 – 1580 |
| $\tilde{\tau}_1$ | 1463 | 1454   | 1454   | 1455 – 1460 |
| $\tilde{\tau}_2$ | 1444 | 1440   | 1441   | 1439 – 1443 |
| $\tilde{e}_L$  | 1468  | 1459   | 1459   | 1460 – 1465 |
| $\tilde{e}_R$  | 1457  | 1453   | 1453   | 1453 – 1455 |
| $\tilde{\nu}_e$ | 1465 | 1456   | 1456   | 1457 – 1463 |
| $\tilde{\nu}_\tau$ | 1459 | 1450   | 1450   | 1451 – 1457 |
| $\chi_1$   | 132   | 123    | 123    | 121 – 124 |
| $\chi_2$   | 257   | 237    | 237    | 240 – 241 |
| $\chi_3$   | 562   | 579    | 582    | 528 – 596 |
| $\chi_4$   | 574   | 589    | 592    | 539 – 605 |
| $\chi_1^\pm$ | 257  | 237    | 237    | 240 – 241 |
| $\chi_2^\pm$ | 574  | 590    | 592    | 539 – 605 |
| $h$        | 119   | 119    | 119    | 117 – 117 |
| $H$        | 1548  | 1545   | 1546   | 1542 – 1555 |
| $A$        | 1548  | 1545   | 1546   | 1532 – 1555 |
| $H^\pm$   | 1550  | 1547   | 1548   | 1544 – 1557 |

Table 5: Sparticle masses (in GeV) for the SPS2 point
3.4. Benchmark point SPS 3

This is a “co-annihilation region” point, its distinctive feature being a light stau not much heavier than the neutralino LSP. Our results are given in Table 6.

| mass   | 1loop | 2loops | 3loops | AKP    |
|--------|-------|--------|--------|--------|
| ˜g     | 964   | 943    | 940    | 930 – 940 |
| ˜t_1   | 851   | 845    | 835    | 836 – 843 |
| ˜t_2   | 645   | 644    | 634    | 640 – 650 |
| ˜u_L   | 872   | 860    | 849    | 861 – 863 |
| ˜u_R   | 835   | 830    | 818    | 828 – 831 |
| ˜b_1   | 794   | 787    | 776    | 786 – 793 |
| ˜b_2   | 830   | 826    | 814    | 816 – 825 |
| ˜d_L   | 876   | 864    | 853    | 864 – 867 |
| ˜d_R   | 831   | 828    | 816    | 825 – 829 |
| ˜τ_1   | 300   | 291    | 290    | 293 – 294 |
| ˜τ_2   | 180   | 173    | 173    | 172 – 176 |
| ˜e_L   | 299   | 288    | 288    | 291 – 293 |
| ˜e_R   | 186   | 181    | 181    | 179 – 183 |
| ˜ν_e   | 287   | 277    | 276    | 277 – 281 |
| ˜ν_τ   | 286   | 276    | 275    | 276 – 280 |
| χ_1    | 172   | 161    | 161    | 158 – 162 |
| χ_2    | 325   | 302    | 301    | 305 – 306 |
| χ_3    | 512   | 538    | 531    | 528 – 540 |
| χ_4    | 533   | 554    | 548    | 543 – 555 |
| χ_1^±  | 324   | 302    | 301    | 304 – 306 |
| χ_2^±  | 533   | 554    | 548    | 542 – 555 |
| h      | 117   | 118    | 117    | 116 – 118 |
| H      | 579   | 597    | 591    | 593 – 600 |
| A      | 579   | 597    | 591    | 589 – 600 |
| H^±    | 585   | 603    | 597    | 598 – 605 |

Table 6: Sparticle masses (in GeV) for the SPS3 point
This is a point with large tan $\beta$. Our results are given in Table 7.

| mass | 1loop | 2loops | 3loops | AKP   |
|------|-------|--------|--------|-------|
| $\tilde{g}$ | 759   | 743    | 741    | 729 - 738 |
| $\tilde{t}_1$ | 705   | 700    | 693    | 693 - 697 |
| $\tilde{t}_2$ | 544   | 541    | 533    | 540 - 544 |
| $\tilde{u}_L$ | 777   | 764    | 757    | 766 - 772 |
| $\tilde{u}_R$ | 755   | 747    | 739    | 747 - 751 |
| $\tilde{b}_1$ | 624   | 619    | 611    | 614 - 619 |
| $\tilde{b}_2$ | 693   | 690    | 683    | 679 - 692 |
| $\tilde{d}_L$ | 782   | 769    | 761    | 770 - 776 |
| $\tilde{d}_R$ | 753   | 746    | 738    | 746 - 749 |
| $\tilde{\tau}_1$ | 423   | 420    | 420    | 414 - 421 |
| $\tilde{\tau}_2$ | 272   | 268    | 268    | 253 - 269 |
| $\tilde{e}_L$ | 455   | 450    | 449    | 451 - 452 |
| $\tilde{e}_R$ | 419   | 417    | 418    | 417 - 419 |
| $\tilde{\nu}_e$ | 447   | 441    | 441    | 442 - 445 |
| $\tilde{\nu}_\tau$ | 395   | 390    | 390    | 387 - 393 |
| $\chi_1$ | 128   | 120    | 120    | 119 - 121 |
| $\chi_2$ | 242   | 226    | 225    | 228 - 228 |
| $\chi_3$ | 400   | 419    | 415    | 406 - 420 |
| $\chi_4$ | 420   | 435    | 431    | 422 - 436 |
| $\chi_{1}^{\pm}$ | 242   | 226    | 225    | 227 - 228 |
| $\chi_{2}^{\pm}$ | 421   | 436    | 432    | 422 - 436 |
| $h$ | 116   | 116    | 116    | 114 - 116 |
| $H$ | 370   | 386    | 385    | 355 - 367 |
| $A$ | 371   | 388    | 387    | 355 - 367 |
| $H^\pm$ | 381   | 397    | 396    | 366 - 379 |

*Table 7: Sparticle masses (in GeV) for the SPS4 point*
3.6. Benchmark point SPS 5

| mass     | 1loop | 2loops | 3loops | AKP           |
|----------|-------|--------|--------|---------------|
| \( \tilde{g} \) | 743   | 729    | 727    | 719 – 729    |
| \( \tilde{t}_1 \) | 653   | 654    | 646    | 629 – 651    |
| \( \tilde{t}_2 \) | 265   | 278    | 263    | 258 – 280    |
| \( \tilde{u}_L \) | 684   | 677    | 668    | 676 – 685    |
| \( \tilde{u}_R \) | 658   | 656    | 646    | 655 – 660    |
| \( \tilde{b}_1 \) | 563   | 563    | 554    | 554 – 567    |
| \( \tilde{b}_2 \) | 654   | 653    | 643    | 630 – 656    |
| \( \tilde{d}_L \) | 688   | 681    | 673    | 681 – 689    |
| \( \tilde{d}_R \) | 656   | 655    | 645    | 653 – 658    |
| \( \tilde{\tau}_1 \) | 264   | 259    | 258    | 259 – 262    |
| \( \tilde{\tau}_2 \) | 186   | 182    | 183    | 182 – 184    |
| \( \tilde{e}_L \) | 263   | 257    | 257    | 258 – 261    |
| \( \tilde{e}_R \) | 195   | 192    | 193    | 192 – 194    |
| \( \tilde{\nu}_e \) | 251   | 245    | 245    | 246 – 249    |
| \( \tilde{\nu}_\tau \) | 249   | 243    | 243    | 244 – 247    |
| \( \chi_1 \) | 128   | 120    | 120    | 119 – 120    |
| \( \chi_2 \) | 247   | 229    | 228    | 230 – 236    |
| \( \chi_3 \) | 608   | 626    | 621    | 626 – 631    |
| \( \chi_4 \) | 621   | 637    | 632    | 637 – 641    |
| \( \chi_1^\pm \) | 247   | 229    | 228    | 230 – 236    |
| \( \chi_2^\pm \) | 620   | 637    | 632    | 636 – 641    |
| \( h \) | 117   | 118    | 118    | 116 – 122    |
| \( H \) | 667   | 682    | 676    | 681 – 694    |
| \( A \) | 667   | 682    | 677    | 682 – 690    |
| \( H^\pm \) | 672   | 687    | 681    | 687 – 698    |

Table 8: Sparticle masses (in GeV) for the SPS5 point with \( m_t = 178 \)GeV

This point differs from the previous ones in having a large value of the \( A \)-parameter. The contributions of \( \mu, A \) to the off-diagonal term in the stop mass matrix have the same sign, and the magnitude of \( A \) is large, resulting in a light stop. For this point we have
calculated both using in Table 8 $m_t = 178\text{GeV}$ (as for the previous tables) and for comparison in Table 9 with $m_t = 174.3\text{GeV}$. This illustrates the sensitivity to the input $m_t$, with the light stop changing over 20GeV due to this small change in $m_t$.

| mass     | 1loop | 2loops | 3loops | AKP     |
|----------|-------|--------|--------|---------|
| $\tilde{g}$   | 743   | 729    | 727    | 718−728 |
| $\tilde{t}_1$ | 652   | 653    | 645    | 628−649 |
| $\tilde{t}_2$ | 243   | 257    | 240    | 232−258 |
| $\tilde{u}_L$ | 684   | 677    | 668    | 676−684 |
| $\tilde{u}_R$ | 658   | 656    | 646    | 653−660 |
| $\tilde{b}_1$ | 561   | 560    | 551    | 551−564 |
| $\tilde{b}_2$ | 654   | 653    | 643    | 629−655 |
| $\tilde{d}_L$ | 689   | 681    | 673    | 680−689 |
| $\tilde{d}_R$ | 656   | 655    | 645    | 651−658 |
| $\tilde{\tau}_1$ | 264   | 259    | 258    | 258−262 |
| $\tilde{\tau}_2$ | 186   | 182    | 182    | 182−184 |
| $\tilde{e}_L$ | 263   | 257    | 257    | 258−260 |
| $\tilde{e}_R$ | 195   | 192    | 192    | 192−194 |
| $\tilde{\nu}_e$ | 251   | 245    | 245    | 246−249 |
| $\tilde{\nu}_\tau$ | 249   | 243    | 243    | 244−246 |
| $\chi_1$ | 128   | 120    | 120    | 119−121 |
| $\chi_2$ | 247   | 229    | 228    | 230−236 |
| $\chi_3$ | 615   | 632    | 628    | 632−637 |
| $\chi_4$ | 628   | 644    | 639    | 643−646 |
| $\chi_{1}^{\pm}$ | 247   | 229    | 228    | 230−236 |
| $\chi_{2}^{\pm}$ | 627   | 643    | 639    | 643−646 |
| $h$     | 115   | 115    | 115    | 112−119 |
| $H$     | 674   | 688    | 683    | 687−693 |
| $A$     | 674   | 688    | 683    | 689−693 |
| $H^{\pm}$ | 679   | 692    | 687    | 694−702 |

*Table 9*: Sparticle masses (in GeV) for the SPS5 point with $m_t = 174.3\text{GeV}
3.7. Benchmark point SPS6

This is a point with un-unified gaugino masses so we are unable to compare with Ref. [9]. We instead use the paper by Ghodbane and Martyn (GM), Ref. [11], which also compares the results for various programs (Isajet, Susygen and Pythia). The results for these three programs are reasonably consistent with each other; this is due to some extent, however, to the fact that the Isajet gauge unification outputs are used as inputs for the other two programs; in our table we show only the Isajet predictions. Agreement with our results is less impressive; however we should notice that Ref. [11] uses an earlier version of Isajet (7.58) than Ref. [9]. Thus if we return to SPS1a and compare the Isajet 7.58 prediction for the gluino mass (595GeV) with the Isajet 7.69 one of 612GeV obtained from Ref. [9], we can anticipate that for SPS6 the more recent Isajet would give results more consistent with our (two-loop) ones, making the reasonable assumption that the newer version will give, for example, a higher gluino mass prediction for SPS6 as well. Our results for SPS6 are given in Table 10.
A clear feature of the results is that the corrections due to two and three-loop running can be quite large for squarks, but are typically smaller for weakly-interacting particles. In particular the light CP-even Higgs mass is very stable. The large three-loop $\alpha_3$ corrections stem mainly from the $M_3^2$ contributions to the three-loop $m^2$ $\beta$-functions; note that for

| mass     | 1loop | 2loops | 3loops | GM  |
|----------|-------|--------|--------|-----|
| $\tilde{g}$ | 744   | 726    | 724    | 708 |
| $\tilde{t}_1$ | 686   | 681    | 673    | 661 |
| $\tilde{t}_2$ | 498   | 496    | 488    | 476 |
| $\tilde{u}_L$ | 684   | 674    | 665    | 639 |
| $\tilde{u}_R$ | 665   | 659    | 650    | 628 |
| $\tilde{b}_1$ | 620   | 613    | 605    | 589 |
| $\tilde{b}_2$ | 657   | 652    | 643    | 624 |
| $\tilde{d}_L$ | 689   | 679    | 670    | 644 |
| $\tilde{d}_R$ | 658   | 653    | 644    | 622 |
| $\tilde{\tau}_1$ | 278   | 271    | 271    | 270 |
| $\tilde{\tau}_2$ | 235   | 229    | 229    | 228 |
| $\tilde{\nu}_L$ | 274   | 266    | 266    | 265 |
| $\tilde{\nu}_R$ | 243   | 238    | 238    | 237 |
| $\nu_e$ | 261   | 253    | 253    | 252 |
| $\nu_\tau$ | 260   | 252    | 252    | 252 |
| $\chi_1$ | 201   | 190    | 190    | 189 |
| $\chi_2$ | 239   | 222    | 221    | 218 |
| $\chi_3$ | 399   | 419    | 414    | 399 |
| $\chi_4$ | 425   | 439    | 435    | 420 |
| $\chi_1^\pm$ | 237   | 220    | 219    | 215 |
| $\chi_2^\pm$ | 423   | 438    | 434    | 419 |
| $h$ | 115   | 115    | 115    | 115 |
| $H$ | 469   | 481    | 477    | 464 |
| $A$ | 469   | 481    | 477    | 463 |
| $H^\pm$ | 477   | 489    | 484    | 470 |

Table 10: Sparticle masses (in GeV) for the SPS6 point

3.8. Discussion

A clear feature of the results is that the corrections due to two and three-loop running can be quite large for squarks, but are typically smaller for weakly-interacting particles. In particular the light CP-even Higgs mass is very stable. The large three-loop $\alpha_3$ corrections stem mainly from the $M_3^2$ contributions to the three-loop $m^2$ $\beta$-functions; note that for
the only MSUGRA point such that $m_0 > m_{\frac{1}{2}}$, i.e. SPS2, the three-loop correction to the squark masses is *smaller* than the two-loop one.

Generally speaking we would anticipate that for regions of parameter space where the three-loop corrections are comparable to or exceed the two-loop ones, the four-loop ones will be at least as large. This suggests that we are already at three loops approaching the asymptotic region for the $\beta$-functions. So it appears that squark mass predictions with an accuracy greater than a few per cent will not be possible using perturbation theory.

Overall our results agree reasonably well with those of existing programs [9]. One place where we have a significant difference is for the $H, A, H^\pm$ results for SPS4. This is a large tan $\beta$ point; however our results for the $b$-squark and $d$-squark masses (which one would expect to be sensitive to large tan $\beta$) agree quite well, so for the moment we have no explanation for this discrepancy.

4. The Semi-perturbative Region

The addition of additional matter representations in complete $SU_5$ multiplets does not affect gauge unification (and the unification scale) at one loop. Beyond one loop this is no longer the case, and increasing the amount of matter relevant to the running analysis requires the presumption of larger threshold corrections at the unification scale in order to restore gauge unification; one is thus forced to argue that the success of gauge unification in the MSSM is coincidental [3].

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Historically gauge unification was implemented by using $\alpha_3(M_Z)$ as an input and computing $\sin^2 \theta_W$, although the latter was more accurately measured, because $\sin^2 \theta_W$ varies very slowly with $\alpha_3(M_Z)$, and conversely (of course) $\alpha_3(M_Z)$ varies rapidly as a function of $\sin^2 \theta_W$. The current experimental results for $\alpha_3(M_Z)$ already require us to suppose the existence of *some* high scale radiative corrections in the MSSM; but the fact remains that things get worse as we add more matter [13].
In Fig. 1 we show the evolution of the gauge couplings $\alpha_i = g_i^2/(4\pi)$ for $n_{10} = 1.7$, using three-loop $\beta$-functions for all couplings. (As remarked in Ref. [12], the mass scale of these additional multiplets being unknown it makes sense to parametrise their effects by taking $n_5, n_{10}$ to be continuous variables.) The couplings are plotted against $\tau = \frac{1}{2\pi} \ln(Q/M_Z)$; evidently we are still in the perturbative regime. The input parameters at $M_Z$ correspond to a typical supersymmetric mass spectrum; specifically, the Benchmark point SPS1a. One sees clearly the need for large corrections to restore gauge unification.

We gave a number of examples of the effect of additional matter on the sparticle spectrum predictions in a previous paper[13]; here we contrast the effect on the first and third generation squark masses. Thus in Fig 2 we plot, for the SPS5 point, the ratio of the $\tilde{u}_L$ and gluino masses against $n_{10}$ for $n_5 = 0$; as already noted in Ref. [12], the mass increases with $n_{10}$. It is interesting that the effect of the three-loop correction to this ratio almost precisely cancels the two-loop correction, for all $n_{10}$. We contrast this with Fig 3
where we show the behaviour of the light stop mass for the same SPS point; in this case the ratio decreases smoothly, and the three-loop correction only cancels the two-loop one at $n_{10} = 0$. For the SPS5 point the electroweak vacuum fails around $n_{10} = 0.48$. (The change in this value and in Fig. 3 from our previous paper [15] is due to the change in the input top pole mass, and to an improved treatment of the Higgs potential minimisation.)

In Fig. 4 we plot the light CP-even Higgs mass for SPS1a as a function of $n_{10}$ (for $n_5 = 0$). We see that it is fairly stable both with respect to loop corrections and the addition of extra matter. In the case of SPS1a the electroweak vacuum fails at around $n_{10} = 1.8$.

![Graph of u_L/gluino mass ratio vs n_{10} for SPS5. Solid, dashed and dotted lines correspond to one, two and three-loop running respectively.](image)

*Fig. 2: Plot of the $u_L$/gluino mass ratio against $n_{10}$ for SPS5. Solid, dashed and dotted lines correspond to one, two and three-loop running respectively.*
Fig.3: Plot of the light stop/gluino mass ratio against $n_{10}$ for SPS5. Solid, dashed and dotted lines correspond to one, two and three-loop running respectively.

Fig.4: Plot of the light CP-even Higgs mass against $n_{10}$ for SPS1a. Solid, dashed and dotted lines correspond to one, two and three-loop running respectively.
5. Conclusions

We have extended typical detailed running coupling analyses for the MSUGRA MSSM SPS benchmark points to incorporate three-loop $\beta$-function corrections for the running masses and couplings. We compare our results to those obtained by existing programs using two-loop running. The spread in the results from these programs is probably due to a mixture of program errors and genuine theoretical uncertainties such as the choice of scale appropriate for the evaluation of the pole mass. Presumably over time the results used by these programs will converge; we would argue that a more reliable estimate of the ultimate theoretical error in these spectrum calculations is currently provided by the difference between our two and three-loop calculations, as opposed to the spread in the various available two-loop results.

Generally speaking the effect of the three-loop running corrections is small for weakly-interacting particles but larger for the squark masses. For the light stop mass at the SPS5 point, we see an 8% effect, but more typically the effect is between 1% and 2%. This appears to us to represent a fundamental limit on the theoretical precision of squark mass theoretical predictions.

Finally we show how additional matter in $SU_5$ multiplets can affect the sparticle spectrum; more dramatically as the “semi-perturbative unification” regime [12] is approached.

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