NONPERTURBATIVE RELATIONS IN
N=2 SUSY YANG-MILLS AND WDVV EQUATION

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Abstract

We find the nonperturbative relation between $\langle \text{tr} \phi^2 \rangle$, $\langle \text{tr} \phi^3 \rangle$ the prepotential $F$ and the vevs $\langle \phi_i \rangle$ in $N=2$ supersymmetric Yang-Mills theories with gauge group $SU(3)$. Nonlinear differential equations for $F$ including the Witten – Dijkgraaf – Verlinde – Verlinde equation are obtained. This indicates that $N=2$ SYM theories are essentially topological field theories and that should be seen as low-energy limit of some topological string theory. Furthermore, we construct relevant modular invariant quantities, derive canonical relations between the periods and investigate the structure of the beta function by giving its explicit form in the moduli coordinates. In doing this we discuss the uniformization problem for the quantum moduli space. The method we propose can be generalized to $N=2$ supersymmetric Yang-Mills theories with higher rank gauge groups.

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1. Seiberg-Witten exact results about $N = 2$ SUSY Yang-Mills concern the low-energy Wilsonian effective action with at most two derivatives and four fermions. These terms are completely described by the so-called prepotential $F$ whose most important property is holomorphy. Furthermore, it has been shown in that $F$ gets perturbative contributions only up to one-loop. Higher-order terms in the asymptotic expansion come as instanton contribution implicitly determined in

In [4], where a method to invert functions was proposed, it has been derived a nonperturbative equation which relates in a simple way the prepotential and the vevs of the scalar fields. In [3], proving a conjecture in [6], it has been shown that the above relation underlies the nonperturbative Renormalization Group Equation and the exact expression for the beta function in the $SU(2)$ case has been obtained. The problem of extending these results to the case of higher rank groups is a nontrivial task. An important step in this direction is the result in [6] where the nonperturbative relation in [4] has been generalized. However, it remains the problem of finding the nonperturbative relations between $\langle \text{tr} \phi^k \rangle$ for $k > 2$ and the prepotential. Also, one should find a set of equations for $F$ in a similar way to the $SU(2)$ case [4].

In this paper we will solve these problems for the $SU(3)$ case. In particular, we will find a complete set of non-linear differential equations completely characterizing the prepotential including the Witten – Dijkgraaf – Verlinde – Verlinde (WDVV) equation. This indicates that $N = 2$ SYM theories are essentially topological field theories and that should be seen as low-energy limit of some topological string theory.

Furthermore, we introduce a set of modular invariant quantities which will be useful to find the relation between $\langle \text{tr} \phi^k \rangle$ and $F$ and to formulate canonical relations between the periods. We also investigate the structure of the beta function and give its explicit form in the moduli coordinates.

2. The Seiberg-Witten curve for $SU(n)$, has been found in [3] for $n = 3$ and generalized to arbitrary $n$ in [10]. Let us denote by $a^i = \langle \phi^i \rangle$ and $a^D_i = \langle \phi^D_i \rangle = \partial F / \partial a^i$ the vevs of the scalar component of the chiral superfield and its dual. The effective couplings are given by $\tau_{ij} = \partial^2 F / \partial a^i \partial a^j$. We also set $u^2 \equiv u = \langle \text{tr} \phi^2 \rangle$, $u^3 \equiv v = \langle \text{tr} \phi^3 \rangle$ and $\partial_k \equiv \partial / \partial a^k$, $\partial_\alpha \equiv \partial / \partial u^\alpha$. Our starting point are the reduced Picard – Fuchs equations (RPFE’s) for
SU(3) introduced in \[\text{[11]}\]

\[
L_\beta \left( \frac{\alpha_i^D}{a^i} \right) = 0, \quad \beta = 2, 3, \tag{1}
\]

where

\[
L_2 = \frac{1}{u} P(u, v, \Lambda) \partial_u^2 + L, \quad L_3 = \frac{1}{3} P(u, v, \Lambda) \partial_v^2 + L,
\]

\[
P(u, v, \Lambda) = 27(v^2 - \Lambda^6) + 4u^3, \quad L = 12uv\partial_u\partial_v + 3v\partial_v + 1. \tag{2}
\]

Let us recall some transformation properties under the action of \(Sp(2n - 2, \mathbb{Z})\) which hold for the \(SU(n)\) case. We have

\[
\left( \begin{array}{c} a^D \\ a \end{array} \right) \rightarrow \left( \begin{array}{c} \tilde{a}^D \\ \tilde{a} \end{array} \right) = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \left( \begin{array}{c} a^D \\ a \end{array} \right).
\]

Integrating \(\tilde{a}^D = Aa^D + Ba = \partial F(\tilde{a})/\partial \tilde{a}\) yields \(\text{[12, 4, 6]}\)

\[
F(\tilde{a}) = F(a) + \frac{1}{2} a^D C^t Aa^D + \frac{1}{2} aB^t Da + aB^t Ca^D.
\]

We now re-write the equations \(\text{[1]}\) as non-linear differential equations with respect to the \(a^i\)-coordinates. To this end it is convenient to introduce the following notation

\[
U = u_2^2\partial_{11} - 2u_1u_2\partial_{12} + u_1^2\partial_{22}, \quad V = v_2^2\partial_{11} - 2v_1v_2\partial_{12} + v_1^2\partial_{22},
\]

\[
C = (u_1v_2 + v_1u_2)\partial_{12} - u_2v_2\partial_{11} - u_1v_1\partial_{22}, \quad D = u_1v_2 - u_2v_1,
\]

where \(\partial_{i_1...i_n} \equiv \partial^n/\partial a^{i_1}...\partial a^{i_n}\), \(u_i \equiv \partial_i u\) and \(v_i \equiv \partial_i v\). In computing the inversion of Eqs.\(\text{[1]}\) there are terms which simplify because the \(a^i\) are solutions of the RPFE’s themselves. We have

\[
\left[ 12uvC + \frac{1}{3} P(u, v, \Lambda)U + D^2(1 - a^i\partial_i) \right] F_l = 0 = \left[ 12uvC + \frac{1}{3} P(u, v, \Lambda)V + D^2(1 - a^i\partial_i) \right] F_l, \tag{3}
\]

where \(l = 1, 2\) and \(F_{i_1...i_n} \equiv \partial_{i_1...i_n} F\). Note that \(D\) is the Jacobian of \((u, v) \rightarrow (a^1, a^2)\) and therefore generally non-vanishing. Subtracting the LHS from the RHS of Eqs.\(\text{[3]}\), we obtain

\[
A_l \equiv x_{11}F_{22l} + x_{22}F_{11l} - 2x_{12}F_{12l} = 0, \tag{4}
\]

where \(l = 1, 2\) and

\[
x_{ij} = 3v_i v_j - uu_i u_j. \tag{5}
\]

We stress that Eqs.\(\text{[3]}\) have been obtained from the Eqs.\(\text{[1]}\). Therefore, since \((\tilde{a}^D, \tilde{a}^i)\) are still solutions of Eqs.\(\text{[1]}\), it follows that Eqs.\(\text{[3]}\) are modular invariant by construction.
3. On general grounds it seems that the Picard-Fuchs equations are related to the WDVV equation. The above construction allows us to show that this is actually true for the Picard-Fuchs equations arising in $N = 2$ supersymmetric Yang-Mills theory with gauge group $SU(3)$. Actually, a suitable linear combination of the equations $A_l = 0$, namely

$$A_1 (y_{22}F_{112} - 2y_{12}F_{122} + y_{11}F_{222}) - A_2 (-2y_{12}F_{112} + y_{11}F_{122} + y_{22}F_{111}) = 0,$$

where $y_{jk}$ are arbitrary parameters, can be written in the WDVV form

$$F_{ikl}\eta^{lm}F_{mnj} = F_{jkl}\eta^{lm}F_{mni}, \quad (6)$$

for $i, j, k, n = 1, 2$, where

$$\eta^{lm} = \begin{pmatrix}
2x_{22}y_{12} - 2x_{12}y_{22} & x_{11}y_{22} - x_{22}y_{11} \\
x_{11}y_{22} - x_{22}y_{11} & 2x_{12}y_{11} - 2x_{11}y_{12}
\end{pmatrix}. \quad (7)$$

We observe that for each choice of the metric, that is of the parameters $y_{jk}$, there is only one nontrivial equation in (6) which can be re-written as

$$\eta^{11}\Theta_{11} + 2\eta^{12}\Theta_{12} + \eta^{22}\Theta_{22} = 0, \quad (8)$$

where

$$\Theta_{ij} = \frac{1}{2} (F_{1i}F_{2j} + F_{1j}F_{2i}) - F_{12i}F_{12j},$$

which satisfy the identity

$$2F_{12l}\Theta_{12} = F_{22l}\Theta_{11} + F_{11l}\Theta_{22}, \quad l = 1, 2. \quad (9)$$

4. Let us introduce some modular invariant quantities which will be used later on. We set

$$I_\beta^\gamma = (\partial_k z)(\partial_\beta \tau)^{-1kl}\partial_l u^\gamma, \quad (10)$$

where $\beta, \gamma = 2, 3$ and $z$ is the modular invariant

$$z = a^i \partial_i F - 2F.$$

Another set of modular invariants which will be useful are

$$v_{(\beta)}^\alpha = I_\beta^\gamma(\partial_\gamma \partial_k u^\alpha)\partial_\beta a^k + a^k \partial_\beta u^\alpha, \quad \alpha, \beta = 2, 3. \quad (11)$$
There is an interesting structure underlying these invariants. Namely, introducing the brackets
\[
\{X, Y\}_{(\beta)} \equiv \partial_i X (\partial_\beta \tau)^{-1ij} \partial_j Y - \partial_i Y (\partial_\beta \tau)^{-1ij} \partial_j X,
\] (12)
the vector field components \(v^\alpha_{(\beta)}\) can be expressed in the form
\[
v^\alpha_{(\beta)} = \{u^\alpha, z\}_{(\beta)}.
\] (13)
Furthermore, the periods satisfy the following canonical relations
\[
\{\alpha^i, \alpha^j\}_{(\beta)} = 0, \quad \{\alpha^i, \alpha^D_j\}_{(\beta)} = 0, \quad \{\alpha^i, \alpha^D_j\}_{(\beta)} = \delta^i_j.
\] (14)

In order to extract the differential equations for \(F\), we re-write the operators in (2) in the following general form
\[
\mathcal{L}_\beta = \xi_{(\beta)} \partial_\beta + \eta_{(\beta)} + 1,
\] (15)
where \(\xi_{(\beta)} = \xi_{(\beta)}^\alpha \partial_\alpha = \xi_{(\beta)}^i \partial_i\) and \(\eta_{(\beta)} = \eta_{(\beta)}^\alpha \partial_\alpha = \eta_{(\beta)}^i \partial_i\) are vector fields. Considering the action of \(\mathcal{L}_\beta\) on \(fg\) with \(f\) and \(g\) arbitrary functions, we have
\[
\mathcal{L}_\beta f g = g \mathcal{L}_\beta f + f \mathcal{L}_\beta g - f g + \partial_\beta f \xi_{(\beta)} g + \xi_{(\beta)} f \partial_\beta g,
\]
and by Eqs. (1)
\[
\mathcal{L}_\beta (a^i a^D_i - 2F) = a^i a^D_i - 2F,
\] (16)
that is \(\mathcal{L}_\beta z = z\). Note that in (15), as in (2), for each value of \(\beta\) the second-order derivative terms contain always at least one \(\partial_\beta\) (note that Eq. (13) is independent from this peculiarity).

In order to find \(\xi_{(\beta)}\) and \(\eta_{(\beta)}\), we impose that the operators defined in (14) satisfy (1). From the lower components of (14) we obtain \(\eta_{(\beta)}^i = -a^i - \xi_{(\beta)} \partial_\beta a^i\), which substituted in the upper components of (14), yields \(\xi_{(\beta)}^i = (\partial_\beta z)(\partial_\beta \tau)^{-1ki}\). Therefore
\[
\mathcal{L}_\beta = I^\gamma_\beta \partial_\gamma \partial_\beta - v^\gamma_{(\beta)} \partial_\gamma + 1.
\] (17)

Comparing (17) with (2) we obtain a complete set of non-linear differential equations for the prepotential and its exact relation with the moduli coordinates, namely
\[
v^2_{(2)} = 0 = v^2_{(3)},
\] (18)
\[
v^3_{(2)} = -3v = v^3_{(3)},
\] (19)
\[
I^3_2 = 12uv = I^2_3,
\] (20)
\[ uI_2^2 = P(u, v, \Lambda) = 3I_3^3. \] (21)

The same procedure introduced above when applied to the SU(2) case gives \( u = Az + B \) (where the constant \( A \), which turns out to be \( \pi/2i \), is fixed by asymptotic analysis whereas the constant \( B \) turns out to be zero using the recursion relations which follow from the inversion of the reduced uniformizing equation) and

\[ I_2^2 = 4(u^2 - \Lambda^4), \] (22)

which is the equation for the prepotential obtained in [4, 13].

5. Let us define the modular invariant 1-form

\[ W = \left( a^i \partial_{\beta} a^D_i - a^D_i \partial_{\beta} a^i \right) du^\beta = dz, \] (23)

which, due to the existence of the prepotential, is closed, i.e. \( dW = 0 \). Substituting in (23) the expression of the periods in terms of Appell’s \( F_4 \) functions obtained in [11], we obtain \( W = \frac{2i}{\pi} du \). On the other hand, by (16) it follows that the components of \( W = W_\beta du^\beta \) satisfy the linear differential equations

\[ \xi(\beta) W_\beta = v(\beta) W_\alpha, \quad \beta = 2, 3, \] (24)

which are satisfied by \( W_2 = \frac{2i}{\pi}, \ W_3 = 0 \). Therefore, we have \( z = \frac{2i}{\pi} u \), that is

\[ u = \pi i \left( \mathcal{F} - \frac{a^i}{2} \partial_i \mathcal{F} \right), \] (25)

in agreement with [4, 7]. We stress that a possible non-vanishing additive constant in (25) cannot be excluded at this stage. Note that by (25), thanks to (13), Eq.(18) is identically satisfied.

We now use Eq.(4) to face the problem of finding the explicit relation between \( v \) and the prepotential. On general grounds it can be shown that the properties of special geometry imply that \( \Theta_{ii} \neq 0 \) [14]. By (3) the general solution of Eq.(4) is given by

\[ x_{ij} = \rho \Theta_{ij}, \] (26)

where \( \rho \) is determined by the compatibility condition \( (3v_1 v_2)^2 = (3v_1^2)(3v_2^2) \) applied to (3) and (26), that is

\[ \rho^2 \Delta = u\rho(\Theta_{11} u_2^2 + \Theta_{22} u_1^2 - 2u_1 u_2 \Theta_{12}), \] (27)
where $\Delta = \Theta_{12}^2 - \Theta_{11} \Theta_{22}$. Notice that $\rho \neq 0$ otherwise $3v_i v_j = uu_i u_j$ which implies $D = 0$. Since $\rho^2 \Delta = x_{12}^2 - x_{11} x_{22} = 3u D^2$, we have $\Delta \neq 0$, so that (27) can be solved as

$$\rho = \Delta^{-1} u(\Theta_{11} u_2^2 + \Theta_{22} u_1^2 - 2u_1 u_2 \Theta_{12}),$$

which implies

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \epsilon \sqrt{u} \frac{1}{3\Delta} \begin{pmatrix} \Theta_{12} & -\Theta_{11} \\ \Theta_{22} & -\Theta_{12} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

(28)

where $\epsilon = \pm 1$ and the relative sign between $v_1$ and $v_2$ has been fixed by $x_{12} = \rho \Theta_{12}$. Observe that we can set $\epsilon = 1$ by a suitable transformation on the moduli variables (we use the fact that the RPFE’s are invariant under the transformations $u \rightarrow e^{2i\pi k/3} u$ and $v \rightarrow -v$).

In order to find $v$ we first explicitly evaluate the $I_{\alpha}^{\gamma}$ invariants in terms of $u_i$, $v_i$ and $F_{ijk}$. Substituting (28) we obtain the relation between $v$ and $F$ and non-linear differential equations for the prepotential as well.

The essential point is that by (25) and (20) we have

$$v = \frac{I_2^2 I_3}{36 I_3^3}.$$  

(29)

On the other hand (10) can be written as

$$I_2^{\alpha} = \frac{2i}{\pi} D \left( u_2^2 \Theta_{11} + v_1^2 \Theta_{22} - 2v_1 v_2 \Theta_{12} \right)^{-1} (v_2 g_1^{\alpha} - v_1 g_2^{\alpha}),$$

and

$$I_3^{\alpha} = \frac{2i}{\pi} D \left( u_2^2 \Theta_{11} + u_1^2 \Theta_{22} - 2u_1 u_2 \Theta_{12} \right)^{-1} (u_1 g_2^{\alpha} - u_2 g_1^{\alpha}),$$

where

$$g_i^{\alpha} = u_2 u_i^{\alpha} F_{11i} + u_1 u_i^{\alpha} F_{22i} - (u_1 u_2^{\alpha} + u_2 u_1^{\alpha}) F_{12i}.$$

By (25) and (28) the $I_{\beta}^{\alpha}$’s are explicitly known in terms of $a^i$ and the prepotential. It follows that (29) solves the problem of finding the relation between $v$ and $F$.

By (13), we can re-write Eqs. (14) in the form

$$\{u, v\}(\beta) = \frac{6i}{\pi} v, \quad \beta = 2, 3,$$

(30)

which by (29) are two non-linear differential equations for $F$, that together with the two-parameters WDVV equations correspond to the four non-linear differential equations (3).
6. Let us now consider the modular properties of the prepotential and its homogeneity. The fact that $\tau_{ij}$ is dimensionless implies that

$$(\Lambda \partial_\Lambda + \Delta_{u,v}) \tau_{ij} = 0,$$  \hfill (31)

where $\Delta_{u,v} = 2u \partial_u + 3v \partial_v$ is the scaling invariant vector field. Let $\xi$ be an arbitrary modular invariant vector field. We have

$$\xi \tau \rightarrow \xi \tilde{\tau} = (\tau C^t + D^t)^{-1} \xi \tau (C \tau + D)^{-1},$$

which implies that (31) is a modular invariant equation. We also have

$$(\Lambda \partial_\Lambda + \Delta_{u,v}) a_i = a_i, \quad (\Lambda \partial_\Lambda + \Delta_{u,v}) a^D_i = a^D_i,$$

which are are compatible with a pseudo-homogeneity of degree 2 for the prepotential

$$(\Lambda \partial_\Lambda + \Delta_{u,v}) F = 2F + \Lambda^2 \cdot \text{const},$$

In our case the semiclassical analysis gives $\text{const}=0$.

Let us now discuss in the $SU(3)$ case the uniformization mechanism which generalizes the structure underlying the $SU(2)$ case. The structure of the covering of the quantum moduli space $M_{SU(3)}$ is encoded in the properties of the Appell’s functions. The Appell system $F_4$ is a two-dimensional generalization of the hypergeometric system also endowed with algebraic relations involving the functions and their derivatives.

It is known [11] that the period matrix $\tau_{ij}$ is a rational combination of Appell’s functions. By (31), the dependence on $u, v$ and $\Lambda$ is of the form

$$\tau = \tau(u/\Lambda^2, v/\Lambda^3).$$  \hfill (32)

Therefore the $\tau$-space is a subvariety $S$ of the genus 2 Siegel upper-half space of complex codimension 1 which covers the quantum moduli space. $S$ can be characterized as $s(\tau) = 0$, where the structure of $s$ is related to the equations satisfied by the prepotential.

Let $M_{SU(3)} \subset Sp(4, \mathbb{Z})$ be the monodromy group of $N = 2$ SYM with gauge group $SU(3)$ [11]. The above remarks imply that the Picard-Fuchs equations, from which Eqs.(1) are derived, are the uniformization equations for the quantum moduli space. Therefore

$$M_{SU(3)} \cong S/M_{SU(3)}.$$  \hfill (33)
The polymorphic matrix function \( \tau \) is the inverse covering with \( M_{SU(3)} \)-monodromy. Let
\[
  u/\Lambda^2 = u(\tau), \quad v/\Lambda^3 = v(\tau), \quad \tau \in \mathcal{S},
\]
be the covering map. From the above data we now derive the beta function of the theory.

Let us consider the following equations
\[
  0 = \Lambda \partial_\Lambda s(\tau) = \Sigma(\beta)s(\tau),
  
  0 = \Lambda \partial_\Lambda u = \Lambda^2 [\Sigma(\beta)u(\tau) + 2u(\tau)],
  
  0 = \Lambda \partial_\Lambda v = \Lambda^3 [\Sigma(\beta)v(\tau) + 3v(\tau)],
\]  
(34)

where \( \beta_{ij} = \Lambda \partial_\Lambda \tau_{ij} \) is the \( \beta \)-function and \( \Sigma(\beta) \) is the scaling operator
\[
  \Sigma(\beta) = \beta_{11} \frac{\partial}{\partial \tau_{11}} + \beta_{12} \frac{\partial}{\partial \tau_{12}} + \beta_{22} \frac{\partial}{\partial \tau_{22}}.
\]

Note that the solution of the system (34) completely determines the \( \beta \)-function of the theory.

We now derive the exact \( \beta \)-function projected on the natural moduli directions in terms of the modular invariants
\[
  J_{\alpha\beta\gamma} = \partial_\alpha a^i \partial_\beta \tau_{ij} \partial_\gamma a^j,
\]
which are completely symmetric in their indices. Actually, defining the projected \( \beta \) function
\[
  \beta_{\alpha\gamma} = \partial_\alpha a^i \beta_{ij} \partial_\gamma a^j,
\]
and using (31), we have
\[
  \beta_{\alpha\gamma} = -2u J_{\alpha 2\gamma} - 3v J_{\alpha 3\gamma},
\]  
(35)

The \( J_{\alpha\beta\gamma} \)'s are related to the \( I_{\beta} \gamma \)'s by
\[
  I_{\beta}^\gamma J_{\gamma 2} = \frac{\pi}{2i}, \quad I_{\beta}^\gamma J_{\gamma 3} = 0,
\]
that is
\[
  \frac{P(u, v, \Lambda)}{u} J_{223} + 12uv J_{233} = 0 = \frac{P(u, v, \Lambda)}{3} J_{333} + 12uv J_{233},
\]  
(36)

\[
  \frac{P(u, v, \Lambda)}{u} J_{222} + 12uv J_{223} = \frac{2i}{\pi} = \frac{P(u, v, \Lambda)}{3} J_{233} + 12uv J_{223},
\]  
(37)

Inserting the solution of this system in (34), we obtain
\[
  \beta_{22} = \frac{2Au}{3} [P(u, v, \Lambda) - 54v^2],
\]  
(38)
\[
\beta_{23} = \beta_{32} = \frac{3Av}{u} [P(u, v, \Lambda) - 8u^3], \quad (39)
\]
\[
\beta_{33} = 2A [P(u, v, \Lambda) - 54v^2], \quad (40)
\]
where \( A = \frac{24}{\pi} [(12uv)^2 - P^2(u, v, \Lambda)/3u]^{-1}. \)

7. Let us make some remarks. First of all we note that similar structures can be generalized to the case of gauge group \( SU(n), \ n \geq 3. \) Furthermore the condition \( s(\tau) = 0 \) and the WDVV equation suggest a relation with the condition on the lattices obtained in [14] (see Eq.(5.22) there). In this framework one should be able to connect the BPS mass formula with the area of degenerate metrics on a suitable Riemann surface. This surface should be related to the two-dimensional space which arises in compactifying \( N = 1 \) in \( D = 6 \) to obtain \( N = 2 \) in \( D = 4. \) In [13] a similar structure for the \( SU(2) \) case has been obtained.

We also observe that the way we use the Picard-Fuchs equations could be useful in investigating some algebraic-geometrical structure and some aspects concerning mirror symmetry (see [17] for related aspects). In this context we note that considering the branching points of the hyperelliptic Riemann surfaces as punctures on the Riemann sphere it should be possible to describe the Seiberg-Witten moduli space in terms of moduli space of Riemann spheres with punctures. Observe that already in the \( SU(2) \) case the moduli space is the Riemann sphere with three punctures which can be essentially seen as \( \overline{\mathcal{M}}_{0,4} \), the moduli space of Riemann spheres with four punctures. In this framework one can use relevant structures such as the Deligne-Mumford compactification \( \overline{\mathcal{M}}_{h,p} \), where “punctures never collide”, which allows us to consider natural embeddings (this problem is of interest also for softly supersymmetry breaking [16]). We also observe that the WDVV equation can be seen as an associativity condition for divisors on \( \overline{\mathcal{M}}_{0,p} \) [18]. These structures together with the restriction phenomenon of the Weil-Petersson metric, whose Kähler potential is the on-shell Liouville action, are at the basis of recursion relations arising in 2D quantum gravity.

In conclusion, we have found nonperturbative relations for \( N = 2 \) SYM with gauge groups \( SU(3) \) which generalize the results in [4] where the relation between \( u \) and the prepotential has been found in the \( SU(2) \) case. This relation has been recently verified in [19] up to two-instanton and at all orders in [20, 21]. The results of our investigation should be similarly verified for a more complete proof of the Seiberg-Witten theory.

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