ON THE TRANSITION DENSITY FUNCTION OF THE DIFFUSION PROCESS GENERATED BY THE GRUSHIN OPERATOR

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Abstract

The short-time asymptotic behavior of the transition density function of the diffusion process generated by the Grushin operator with parameter $\gamma > 0$ will be investigated, by using its explicit expression in terms of expectation. Further the dependence on $\gamma$ of the asymptotics will be seen.

1 Introduction

Let $d, d' \in \mathbb{N}$ and $\gamma > 0$. Define the vector fields on $\mathbb{R}^{d+d'}$ by

\[ V_i = \frac{\partial}{\partial x^i} \quad \text{and} \quad W_j = |x|^\gamma \frac{\partial}{\partial y^j} \quad \text{for} \ 1 \leq i \leq d \ \text{and} \ 1 \leq j \leq d', \]

where $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^{d'})$ are the standard coordinate systems of $\mathbb{R}^d$ and $\mathbb{R}^{d'}$, respectively. The Grushin operator with parameter $\gamma$ is the differential operator on $\mathbb{R}^{d+d'}$ given by

\[ \Delta_{(\gamma)} = \sum_{i=1}^{d} V_i^2 + \sum_{j=1}^{d'} W_j^2. \]

The operator is also represented as

\[ \Delta_{(\gamma)} = \Delta_x + |x|^{2\gamma} \Delta_y, \]

where $\Delta_x$ and $\Delta_y$ are the Laplacians in the variables $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d'}$, respectively. The studies of the Grushin operator go back to those by Baouendi in 1967 ([1]) and by Grushin in the beginning of the 1970’s ([6, 7]). After them, many researches corresponding to the operator have been made. In particular, in the case when $\gamma$ is an even integer, the associated heat kernel, i.e. the transition density function, is studied in details (cf. [2, 3, 4]).

The main aim of this paper is to investigate the short-time asymptotics of the transition density function of the diffusion process generated by $\frac{1}{2} \Delta_{(\gamma)}$ for general $\gamma$s in a

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the existence of the transition density function
stochastically analytic method. More precisely, let \{\{Z^{(x,y)}(t)\}_{t \in [0,\infty)}, (x, y) \in \mathbb{R}^d \times \mathbb{R}^{d'}\} be the diffusion process on \(\mathbb{R}^{d+d'}\) generated by \(\frac{1}{2}\Delta_{(\gamma)}\). For our purposes, we shall first show the existence of the transition density function \(p_T((x, y), (\xi, \eta))\) of this diffusion process:

\[
E[f(Z^{(x,y)}(T))] = \int_{\mathbb{R}^d \times \mathbb{R}^{d'}} f(\xi, \eta)p_T((x, y), (\xi, \eta))d\xi d\eta
\]

for any \(T > 0\), \((x, y) \in \mathbb{R}^d \times \mathbb{R}^{d'}\), and bounded \(f \in C(\mathbb{R}^d \times \mathbb{R}^{d'})\), where \(E\) stands for the expectation with respect to the underlying probability measure. At the same time, we shall establish an explicit expression of \(p_T((x, y), (\xi, \eta))\) in terms of expectation, by which we will conclude the continuity of \(p_T\). See Theorem 3.1. Further, with the help of this explicit expression, the short-time asymptotics of \(p_T((x, y), (\xi, \eta))\) will be investigated in the on-diagonal case (Theorem 3.2) and in the off-diagonal case (Theorems 4.1 and 4.2). In all cases, we shall see that the parameter \(\gamma\), i.e., the degeneracy of \(\Delta_{(\gamma)}\) on the plane \(\{\gamma = 0\}\), affects the asymptotics. For example, in Theorem 4.1 we shall show the convergence

\[
\lim_{|\eta - y| \to 0, T \to 0} T \log p_T((x, y), (\xi, \eta)) = C(\gamma),
\]

where \(C(\gamma)\) is a constant depending only on \(\gamma\). The explicit expression of \(C(\gamma)\) will be given in the theorem.

2 Preliminaries

2.1 Density function

Let \(T > 0\). Denote by \(W_T\) (resp. \(\hat{W}_T\)) the space of continuous functions \(w\) from \([0, T]\) to \(\mathbb{R}^d\) (resp. \(\mathbb{R}^{d'}\)) with \(w(0) = 0\), and by \(\mu_T\) (resp. \(\hat{\mu}_T\)) the Wiener measure on \(W_T\) (resp. \(\hat{W}_T\)). The product space \(W_T \times \hat{W}_T\) equipped with the product measure \(\mu_T \times \hat{\mu}_T\) is the \((d + d')\)-dimensional Wiener space. Let \(b = \{b(t) = (b^1(t), \ldots, b^{d'}(t))\}_{t \in [0,T]}\) (resp. \(\hat{b} = \{\hat{b}(t) = (\hat{b}^1(t), \ldots, \hat{b}^{d'}(t))\}_{t \in [0,T]}\)) be the coordinate process on \(W_T\) (resp. \(\hat{W}_T\)); \(b: W_T \ni w \mapsto b(t)(w) = w(t) \in \mathbb{R}^d\) and \(\hat{b}: \hat{W}_T \ni \hat{w} \mapsto \hat{b}(t)(\hat{w}) = \hat{w}(t) \in \mathbb{R}^{d'}\). Denote by \(\mathcal{F}_t\) the \(\sigma\)-field on \(W_T\) generated by \(\{b(s)^{-1}(A) \mid s \leq t, A \in \mathcal{B}(\mathbb{R}^d)\}\), \(\mathcal{B}(\mathbb{R}^{d'})\) being the Borel field of \(\mathbb{R}^{d'}\). In the standard manner, \(\mathcal{F}_t\) can be thought of as a \(\sigma\)-field on \(W_T \times \hat{W}_T\).

Let \(N \in \mathbb{N}\), and \(\phi = \{\phi(t) = (\phi_j(t))_{1 \leq i \leq N, 1 \leq j \leq d'}\}_{t \in [0,T]}\) be an \(\mathbb{R}^{N \times d'}\)-valued \((\mathcal{F}_t)\)-progressively measurable process on \(W_T\) satisfying that

\[
\int_0^T \|\phi(t)\|^2 dt \in L^\infty(\mu_T) \equiv \bigcap_{p \in (1, \infty)} L^p(\mu_T),
\]

(1)

where \(\mathbb{R}^{N \times d'}\) is the space of real \(N \times d'\)-matrices and \(\| \cdot \|\) denotes the Hilbert-Schmidt norm on it. Let \(\mathcal{G}_t\) be the \(\sigma\)-field on \(W_T \times \hat{W}_T\) generated by \(\{(b(s), \hat{b}(s))^{-1}(A) \mid s \leq t, A \in \mathcal{B}(\mathbb{R}^{d+d'})\}\). Thinking of \(\phi\) as a \((\mathcal{G}_t)\)-progressively measurable stochastic process on \(W_T \times \hat{W}_T\), we define the \(\mathbb{R}^N\)-valued random variable \(F\) by the stochastic integral

\[
F = \int_0^T \phi(t)\hat{d}b(t),
\]

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that is, the $i$th component $F^i$ of $F$ is given by

$$F^i = \sum_{j=1}^d \int_0^T \phi_j^i(t) \hat{d}^j(t) \text{ for } 1 \leq i \leq N.$$ 

Set

$$V_\phi = \int_0^T \phi(t)\phi(t)^\dagger dt,$$

where $A^\dagger$ stands for the transposed matrix of $A$.

**Lemma 2.1.** Suppose that

$$\frac{1}{\det V_\phi} \in L^{\infty-}(\mu_T).$$

Define $q_F : \mathbb{R}^N \to \mathbb{R}$ by

$$q_F(\eta) = \int_{W_T} \frac{1}{\sqrt{2\pi}^N \sqrt{\det V_\phi}} \exp \left( -\frac{\langle V_\phi^{-1}\eta, \eta \rangle_{\mathbb{R}^N}}{2} \right) d\mu_T \text{ for } \eta \in \mathbb{R}^N,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$ stands for the inner product of $\mathbb{R}^N$. Then $q_F$ is the $C^\infty$-distribution density function of $F$ with respect to the Lebesgue measure on $\mathbb{R}^N$.

**Proof.** Let $\tilde{V}_\phi$ be the cofactor matrix of $V_\phi$. Then, $V_\phi^{-1} = (\det V_\phi)^{-1}\tilde{V}_\phi$. By the assumptions (1) and (2), $\|V_\phi^{-1}\| \in L^{\infty-}(\mu_T \times \hat{\mu}_T)$. Applying the dominated convergence theorem, we see that $q_F \in C^\infty(\mathbb{R}^N)$.

For a Borel measurable $A : [0, T] \to \mathbb{R}^{N \times d'}$, set

$$I_A = \int_0^T A(t) \hat{d}b(t) \quad \text{and} \quad v_A = \int_0^T A(t)A(t)^\dagger dt.$$

It holds that

$$E[f(F)|\mathcal{F}_T] = E[f(I_A)] \bigg|_{A=\phi} \text{ for any bounded } f \in C(\mathbb{R}^N),$$

where $E[\cdot | \mathcal{F}_T]$ denotes the conditional expectation with respect to $\mu_T \times \hat{\mu}_T$ given $\mathcal{F}_T$ and $E[\cdot]$ does the expectation with respect to $\mu_T \times \hat{\mu}_T$. Moreover, if $\det v_A \neq 0$, then $I_A$ obeys the $N$-dimensional normal distribution with mean 0 and covariance $v_A$. Thus, it holds

$$\int_{W_T \times \tilde{W}_T} f(F) d(\mu_T \times \hat{\mu}_T) = \int_{W_T \times \tilde{W}_T} E[f(F)|\mathcal{F}_T] d(\mu_T \times \hat{\mu}_T)$$

$$= \int_{W_T \times \tilde{W}_T} \left( \int_{\mathbb{R}^N} \frac{f(\eta)}{\sqrt{2\pi}^N \sqrt{\det V_\phi}} \exp \left( -\frac{\langle V_\phi^{-1}\eta, \eta \rangle_{\mathbb{R}^N}}{2} \right) d\eta \right) d(\mu_T \times \hat{\mu}_T)$$

for any bounded $f \in C(\mathbb{R}^N)$. This implies

$$\int_{W_T \times \tilde{W}_T} f(F) d(\mu_T \times \hat{\mu}_T) = \int_{\mathbb{R}^N} f(\eta) q_F(\eta) d\eta,$$

which means $q_F$ is the distribution density function of $F$ with respect to the Lebesgue measure. \qed
2.2 Exponential integrability

Let $W_0^T = \{ w \in W_T \mid w(T) = 0 \}$, $\mu_0^T$ be the pinned Wiener measure on it, and $\beta = \{ \beta(t) = (\beta^1(t), \ldots, \beta^d(t)) \}_{t \in [0, T]}$ be the coordinate process on $W_0^T$, i.e., $\beta(t) = b(t)|_{W_0^T}$.

$W_0^T$ is a real separable Banach space with the uniform convergence norm $\| \cdot \|_{W_0^T}$ inherited from $W_T$;

$$\| w \|_{W_0^T} = \sup_{t \in [0, T]} |w(t)| \quad \text{for} \quad w \in W_0^T.$$ 

Given $\theta \in (0, \frac{1}{2})$ and $p \in (1, \infty)$ with $p\theta > 1$, define

$$\| \psi \|_{T,p,\theta} = \left( \int_{(0,T)^2} \frac{|\psi(u) - \psi(v)|^p}{|u - v|^{1+p\theta}} |dudv| \right)^{\frac{1}{p}} \quad \text{for} \quad \psi \in C([0, T]; \mathbb{R}^d).$$

The aim of this subsection is to show

**Proposition 2.1.** There exists a $\delta \in (0, \infty)$ such that $\exp(\delta \| \cdot \|_{T,p,\theta}^2) \in L^1(\mu_0^T)$.

For $x, \xi \in \mathbb{R}^d$, define $\ell^{T,x,\xi} \in C([0, T]; \mathbb{R}^d)$ by

$$\ell^{T,x,\xi}(t) = x + t \frac{\xi - x}{T} \quad \text{for} \quad t \in [0, T].$$

This proposition yields the following estimation.

**Corollary 2.1.** For the same $\delta$ as described in Proposition 2.1, it holds

$$\sup_{|\xi - x| \leq R} \int_{W_0^T} \exp \left( \frac{\delta}{2} \| \beta + \ell^{T,x,\xi} \|_{T,p,\theta}^2 \right) d\mu_0^T < \infty \quad \text{for any} \quad R \in [0, \infty).$$

**Proof.** By a straightforward computation, we have

$$\| \ell^{T,x,\xi} \|_{T,p,\theta} = \left( \frac{2}{T^{p\theta - 1}(1-\theta)p\{(1-\theta)p + 1\}} \right)^{\frac{1}{p}} |\xi - x|.$$ 

Since $\| \beta + \ell^{T,x,\xi} \|_{T,p,\theta} \leq \| \beta \|_{T,p,\theta} + \| \ell^{T,x,\xi} \|_{T,p,\theta}$, the desired estimation follows from Proposition 2.1. \qed

In the remaining of this subsection, we prove Proposition 2.1. Let $H_0^T$ be the Cameron-Martin subspace of $W_0^T$; it is the space of all $h \in W_0^T$ which is absolutely continuous and possesses a square integrable derivative $h'$ on $[0, T]$. $H_0^T$ is a real separable Hilbert space with the norm $\| \cdot \|_{H_0^T}$ corresponding to the inner product

$$\langle h, g \rangle_{H_0^T} = \int_0^T \langle h'(t), g'(t) \rangle_{\mathbb{R}^d} dt \quad \text{for} \quad h, g \in H_0^T.$$ 

Observe

$$|h(t) - h(s)| \leq \| h \|_{H_0^T} |t - s|^\frac{1}{2} \quad \text{for} \quad h \in H_0^T \text{ and } s, t \in [0, T].$$ (3)
Hence
\[ \|h\|_{T,p,\theta} \leq \left( \frac{2T(\frac{1}{2} - \theta)p + 1}{\frac{1}{2} - \theta} \right)^{\frac{1}{p}} \|h\|_{\mathcal{H}_T^0} \quad \text{for } h \in \mathcal{H}_T^0. \] (4)

Let \( \mathcal{W}_{T,p,\theta}^0 \) be the completion of \( \mathcal{H}_T^0 \) with respect to \( \| \cdot \|_{T,p,\theta} \).

The Garsia-Rodemich-Ruey lemma (cf. [11] Lemma 3.1) asserts
\[ |\psi(t) - \psi(s)| \leq 2^{3+\frac{p}{\theta}} \frac{T}{p\theta - 1} \|\psi\|_{T,p,\theta}|t-s|^{\theta - \frac{1}{p}} \] (5)
for any \( \alpha > 2, r > 0, s,t \in [0,T], \) and \( \psi \in C([0,T];\mathbb{R}^d) \) with \( \int_{(0,T)^2} \frac{|\psi(u)-\psi(v)|^r}{|u-v|^\alpha} dudv < \infty. \)
Setting \( r = p \) and \( \alpha = 1 + p\theta, \) we obtain
\[ |\psi(t) - \psi(s)| \leq 2^{3+\frac{p}{\theta}} \frac{p}{p\theta - 1} \|\psi\|_{T,p,\theta}|t-s|^{\theta - \frac{1}{p}} \] (6)
for \( s, t \in [0,T] \) and \( \psi \in C([0,T];\mathbb{R}^d) \) with \( \|\psi\|_{T,p,\theta} < \infty. \) This inequality with \( s = 0 \) yields
\[ \sup_{t \in [0,T]} |\psi(t)| \leq 2^{3+\frac{p}{\theta}} \frac{p}{p\theta - 1} T^{\theta - \frac{1}{p}} \|\psi\|_{T,p,\theta} \] (7)
for \( \psi \in \mathcal{W}_T^0 \) with \( \|\psi\|_{T,p,\theta} < \infty. \)

Let \( \{h_n\}_{n=1}^\infty \subset \mathcal{H}_T \) be a Cauchy sequence with respect to \( \| \cdot \|_{T,p,\theta}. \) By (6), it holds
\[ \|h_n - h_m\|_{\mathcal{W}_T^0} \leq 2^{3+\frac{p}{\theta}} \frac{p}{p\theta - 1} T^{\theta - \frac{1}{p}} \|h_n - h_m\|_{T,p,\theta} \quad \text{for } n, m \in \mathbb{N}. \]

Hence \( \{h_n\}_{n=1}^\infty \) converges in \( \mathcal{W}_T^0 \) to some point in \( \mathcal{W}_T^0. \) Moreover, if two Cauchy sequences \( \{h_n\}_{n=1}^\infty, \{h_n\}_{n=1}^\infty \subset \mathcal{H}_T^0 \) with respect to \( \| \cdot \|_{T,p,\theta} \) are equivalent, i.e., \( \lim_{n \to \infty} \|h_n - \hat{h}_n\|_{T,p,\theta} = 0, \) then, by (6) again, \( \lim_{n \to \infty} \|h_n - \hat{h}_n\|_{\mathcal{W}_T^0} = 0. \) Thus, each equivalent class of Cauchy sequences with respect to \( \| \cdot \|_{T,p,\theta} \) is identified with the limit point in \( \mathcal{W}_T^0. \) In this manner, we obtain the inclusion
\[ \mathcal{W}_{T,p,\theta}^0 \subset \mathcal{W}_T^0. \] (7)
Further, (6) also yields the continuity of this inclusion;
\[ \|w\|_{\mathcal{W}_T^0} \leq 2^{3+\frac{p}{\theta}} \frac{p}{p\theta - 1} T^{\theta - \frac{1}{p}} \|w\|_{T,p,\theta} \quad \text{for } w \in \mathcal{W}_T^0. \] (8)

Denoting by \( \mathcal{W}_T^{0*} \) and \( \mathcal{W}_{T,p,\theta}^{0*} \) the duals spaces of \( \mathcal{W}_T^0 \) and \( \mathcal{W}_{T,p,\theta}^0, \) respectively, we then have
\[ \mathcal{W}_T^{0*} \subset \mathcal{W}_{T,p,\theta}^{0*}. \]

We moreover prepare a functional analytical lemma.

**Lemma 2.2.** (i) Let \( \alpha \in (0, \frac{1}{2}) \) and denote by \( C_\alpha^0 \subset \mathcal{W}_T^0 \) be the space of all \( \alpha \)-Hölder continuous \( w \in \mathcal{W}_T^0. \) Then \( C_\alpha^0 \subset \mathcal{W}_{T,p,\theta}^0. \)

(ii) \( \mathcal{W}_T^{0*} \) is dense in \( \mathcal{W}_{T,p,\theta}^{0*}. \)
Proof. (i) Let \( f \in C^0_T \). Extend \( f \) to \( \overline{f} \in C(\mathbb{R}) \) by setting \( \overline{f} = 0 \) outside of \([0, T]\). Defining
\[
C_f = \sup_{s,t \in [0,T], s \neq t} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty,
\]
we have
\[
|\overline{f}(t) - \overline{f}(s)| \leq C_f |t - s|^\alpha \quad \text{for any } s, t \in \mathbb{R}. \tag{9}
\]

Take a non-negative \( \varphi \in C^\infty_0(\mathbb{R}) \) such that \( \varphi = 0 \) on \((-\infty, 0)\), and \( \int_{\mathbb{R}} \varphi(u)du = 1 \). Set \( \varphi_n(y) = n\varphi(ny) \) for \( y \in \mathbb{R} \). Put
\[
f_n(t) = \int_{\mathbb{R}} \overline{f}(t-u)\varphi_n(u)du - \frac{t}{T} \int_{\mathbb{R}} \overline{f}(T-u)\varphi_n(u)du \quad \text{for } t \in [0, T].
\]
Then \( f_n(0) = f_n(T) = 0 \) and \( f_n \in C^\infty([0, T]; \mathbb{R}^d) \). In particular, \( f_n \in \mathcal{H}^0_T \).

Since \( \int_{\mathbb{R}} \varphi_n(u)du = 1 \) and \( \overline{f}(T) = 0 \), by (9),
\[
|f_n(t) - f(t)| = \left| \int_{\mathbb{R}} (\overline{f}(t-u) - \overline{f}(t))\varphi_n(u)du - \frac{t}{T} \int_{\mathbb{R}} (\overline{f}(T-u) - \overline{f}(T))\varphi_n(u)du \right|
\leq 2C_f n^{-\alpha} \int_{\mathbb{R}} |v|^\alpha \varphi(v)dv \quad \text{for } t \in [0, T].
\]
Thus
\[
\|f_n - f\|_{\mathcal{W}^\alpha_T} \to 0 \quad \text{as } n \to \infty. \tag{10}
\]

By (9), \( \sup_{t \in \mathbb{R}} |\overline{f}(t)| \leq C_f T^\alpha \). Due to this and (9) again, we have
\[
|f_n(t) - f_n(s)| \leq \int_{\mathbb{R}} |\overline{f}(t-u) - \overline{f}(s-u)|\varphi_n(u)du + \frac{|t-s|}{T} \int_{\mathbb{R}} |\overline{f}(T-u)\varphi_n(u)du|
\leq 2C_f |t - s|^\alpha \quad \text{for } t, s \in [0, T].
\]
This yields the domination
\[
\frac{|(f_n(t) - f_m(t)) - (f_n(s) - f_m(s))|^p}{|t - s|^{1+\theta(p-1)}} \leq (4C_f)^p |t - s|^{(\alpha-\theta)p-1} \quad \text{for } t, s \in [0, T].
\]
By (10) and the dominated convergence theorem, we obtain
\[
\lim_{n,m \to \infty} \|f_n - f_m\|_{T,p,\theta} = 0.
\]
Thus \( \{f_n\}_{n=1}^\infty \subset \mathcal{H}^0_T \) is a Cauchy sequence with respect to \( \| \cdot \|_{T,p,\theta} \). Due to (10) and the inclusion (7), we have \( f \in \mathcal{W}^0_{T,p,\theta} \).

(ii) For \(-\infty \leq a < b \leq \infty\), let \( L^p(a, b) \) be the \( L^p \)-space with respect to the Lebesgue measure on \((a,b)\). Put
\[
\mathcal{W}^p_{\theta}(a, b) = \{ f \in L^p(a, b) \mid \|f\|_{\mathcal{W}^\theta(a, b)} < \infty \},
\]
where
\[
\|f\|_{\mathcal{W}^p_{\theta}(a, b)} = \left( \int_{(a,b]} |f(u)|^pdu \right)^{\frac{1}{p}} + \left( \int_{(a,b]^2} \frac{|f(u) - f(v)|^p}{|u - v|^{1+\theta p}}dudv \right)^{\frac{1}{p}}.
\]
Let \(-\infty < a < b < \infty\) and \(R_{a,b} : L^p(-\infty, \infty) \to L^p(a, b)\) be the restriction mapping onto \((a, b)\). It is known ([12]) that there exists a bounded linear operator \(S_{a,b} : W^p_0(a, b) \to W^p_0(-\infty, \infty)\) such that the composition \(R_{a,b} \circ S_{a,b}\) is the identity mapping of \(W^p_0(a, b)\). Thus \(W^p_0(a, b)\) can be thought of as a closed subspace of \(W^p_0(-\infty, \infty)\). Since \(W^p_0(-\infty, \infty)\) is reflexive ([12]), \(W^p_0(a, b)\) is also reflexive.

By ([5]), \(\| \cdot \|_{W^0_T(0, T)}\) and \(\| \cdot \|_{T, p, \theta}\) are equivalent on \(W^0_{T,p,\theta}\). Hence \(W^0_{T,p,\theta}\) is a closed subspace of \(W^0_0(0, T)\), and hence is reflexive.

Since \(W^0_{T,p,\theta}\) is reflexive and imbedded continuously in \(W^0_{T,p,\theta}\) by ([5]), it is an elementary exercise of functional analysis to show \(W^0_{0,T}\) is dense in \(W^0_{T,p,\theta}\).

As is well known, \(\mu^0_T(C^\infty_T) = 1\). Hence, by the above lemma, we obtain the probability measure \(\mu^0_{T,p,\theta}\) on \(W^0_{T,p,\theta}\) by restricting \(\mu^0_T\) to \(W^0_{T,p,\theta}\).

**Proof of Proposition 2.1.** The proof completes once we have shown \((W^0_{T,p,\theta}, H^0_T, \mu^0_{T,p,\theta})\) is an abstract Wiener space; (i) \(H^0_T\) is imbedded in \(W^0_{T,p,\theta}\) densely and continuously, and (ii) it holds

\[
\int_{W^0_{T,p,\theta}} e^{\sqrt{-1} \ell} d\mu^0_{T,p,\theta} = \exp\left(-\frac{\|\ell\|_{H^0_T}^2}{2}\right) \text{ for any } \ell \in W^0_{0,T}.
\]

where we have identified \(H^0_T\) with its dual space and thought of \(W^0_{0,T}\) as a subspace of \(H^0_T\). In fact, the Fernique theorem ([5]) applied to this abstract Wiener space yields the existence of \(\delta \in (0, \infty)\) such that

\[
\int_{W^0_{T,p,\theta}} \exp(\delta \| \cdot \|_{T,p,\theta}^2) d\mu^0_{T,p,\theta} < \infty.
\]

By the definition of \(\mu^0_{T,p,\theta}\), this means \(\exp(\delta \| \cdot \|_{T,p,\theta}^2) \in L^1(\mu^0_T)\).

The denseness and the continuity of the imbedding of \(H^0_T\) into \(W^0_{T,p,\theta}\) follow immediately from the definition of \(W^0_{T,p,\theta}\) and ([3]). To complete the proof, it remains to show ([3]). To do this, given \(\ell \in W^0_{T,p,\theta}\), apply Lemma 2.2(ii) to take a sequence \(\{\ell_n\}_{n=1}^\infty \subset W^0_{0,T}\) converging to \(\ell\) in \(W^0_{T,p,\theta}\). Since \((W^0_T, H^0_T, \mu^0_T)\) is an abstract Wiener space, we have

\[
\int_{W^0_{T,p,\theta}} e^{\sqrt{-1} \ell_n} d\mu^0_{T,p,\theta} = \int_{W^0_T} e^{\sqrt{-1} \ell_n} d\mu^0_T = \exp\left(-\frac{\|\ell_n\|_{H^0_T}^2}{2}\right) \text{ for } n \in \mathbb{N}.
\]

By ([1]), \(W^0_{T,p,\theta}\) is imbedded in \(H^0_T\) continuously. Then, letting \(n \to \infty\) in the above identity, we arrive at ([3]).

### 3 Transition density function and on-diagonal asymptotics

We continue to use the same notation introduced in the preceding sections. Our first aim of this section is to give an explicit expression of the transition density function \(p_T((x, y), (\xi, \eta))\) as follows.
**Theorem 3.1.** Let $x, \xi \in \mathbb{R}^d$. Define the random variable $v_{T,x,\xi}$ on $\mathcal{W}_T^0$ by

$$v_{T,x,\xi} = \int_0^T |(\beta + \ell_{T,x,\xi})(t)|^{2\gamma} dt.$$ 

(i) $v_{T,x,\xi}^{-1} \in L^{\infty} (\mu_T^0)$, and it holds

$$p_T((x,y),(\xi,\eta)) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{T^d}} \exp\left(-\frac{|\xi - x|^2}{2T}\right) \int_{\mathcal{W}_T^0} v_{T,x,\xi}^{-\frac{d}{2}} \exp\left(-\frac{|\eta - y|^2}{2v_{T,x,\xi}}\right) d\mu_T^0. \quad (12)$$

(ii) $p_T$ is continuous on $(\mathbb{R}^d \times \mathbb{R}^d)^2$.

To prove the theorem, we prepare several lemmas.

**Lemma 3.1.** For $\zeta \in \mathbb{R}^d$, put $M_{T,\zeta} = \max_{t \in [0,T]} |(\beta + \ell_{T,0,\zeta})(t)|$. Then it holds

$$\mu_T^0(M_{T,\zeta} \leq a) \leq \exp\left(\frac{|\zeta|^2}{2T}\right) \frac{\exp\left(-\frac{\pi^2 T}{8a^2}\right)}{a^d} \left(1 - \exp\left(-\frac{\pi^2 T}{8a^2}\right)\right)^d \quad \text{for any } a > 0. \quad (13)$$

In particular,

$$\sup_{|\zeta| \leq R} \|M_{T,\zeta}^{-1}\|_{L^p(\mu_T^0)} < \infty \quad \text{for any } R > 0 \text{ and } p \in (1, \infty). \quad (14)$$

**Proof.** Let $(\beta + \ell_{T,0,\zeta})^i(t)$ be the $i$th component of $(\beta + \ell_{T,0,\zeta})(t)$. It was shown in [11, p.429] that

$$\mu_T^0\left(\max_{t \in [0,T]} |(\beta + \ell_{T,0,\zeta})^i(t)| \leq a\right) \leq \exp\left(\frac{(\xi^i)^2}{2T}\right) \frac{\exp\left(-\frac{\pi^2 T}{8a^2}\right)}{a^d} \left(1 - \exp\left(-\frac{\pi^2 T}{8a^2}\right)\right)^d, \quad (15)$$

where $\zeta = (\zeta^1, \ldots, \zeta^d)$. The independence of components of $(\beta + \ell_{T,0,\zeta})(t)$ implies

$$\mu_T^0(M_{T,\zeta} \leq a) \leq \prod_{i=1}^d \mu_T^0\left(\max_{t \in [0,T]} |(\beta + \ell_{T,0,\zeta})^i(t)| \leq a\right).$$

Plugging (15) into this, we obtain (13). \hfill \Box

For $x, \xi \in \mathbb{R}^d$, define the random variable $B_{T,x,\xi}$ on $\mathcal{W}_T^0$ by

$$B_{T,x,\xi} = \|\beta + \ell_{T,x,\xi}\|_{T,12, \frac{1}{4}}.$$ 

It holds $B_{T,x,\xi} = B_{T,0,\xi-x}$. By Corollary 2.1, there exists $\delta \in (0, \infty)$ such that

$$\sup_{|\xi-x| \leq R} \int_{\mathcal{W}_T^0} \left\{ \exp(\delta B_{T,0,0}^2) + \exp\left(\frac{\delta^2}{2} B_{T,x,\xi}^2\right) \right\} d\mu_T^0 < \infty \quad \text{for any } R > 0. \quad (16)$$

In particular, $B_{T,x,\xi} < \infty \mu_T^0$-a.s. Moreover, by virtue of (13),

$$|(\beta + \ell_{T,x,\xi})(t) - (\beta + \ell_{T,x,\xi})(s)| \leq 96B_{T,0,\xi-x}|t-s|^{\frac{\gamma}{2}} \quad \text{for } t, s \in [0,T], \mu_T^0 \text{-a.s.} \quad (17)$$
Lemma 3.2. It holds
\[ \sup_{|x|,|ξ| \leq R} \left\| v_{T,x,ξ}^{-1} \right\|_{L^p(µ_T^0)} < \infty \text{ for any } R > 0 \text{ and } p \in (1, \infty). \] (18)

In particular, \( v_{T,x,ξ}^{-1} \in L^\infty(µ_T^0) \) for any \( x, ξ \in \mathbb{R}^d \).

Proof. Set \( A_{T,x,ξ} = \{ M_{T,ξ,x} ≥ 4|x| \} \) and take a random variable \( σ \in [0, T] \) such that \( M_{T,ξ,x} = |(β + \ell_{T,0,ξ,x})(σ)| \).

On \( A_{T,x,ξ} \), by (17) with \( s = σ \), if \( t \in [0, T] \) satisfies \( 96B_{T,0,ξ,x} |σ - t| \leq \frac{M_{T,ξ,x}}{2} \), then
\[ |(β + \ell_{T,x,ξ})(t)| \geq \frac{M_{T,ξ,x}}{2} - |x| \geq \frac{M_{T,ξ,x}}{4}. \]

This implies
\[ v_{T,x,ξ} \geq \left( \frac{M_{T,ξ,x}}{4} \right)^2 \left( \frac{M_{T,ξ,x}^6}{192B^6_{T,0,ξ,x}} \wedge \frac{T}{2} \right) \text{ on } A_{T,x,ξ}. \] (19)

Hence we have
\[ v_{T,x,ξ}^{-1} 1_{A_{T,x,ξ}} \leq 4^{2γ} M_{T,ξ,x}^{-2γ} \left( \frac{192B^6_{T,0,ξ,x}}{M_{T,ξ,x}^6} + \frac{2}{T} \right) 1_{A_{T,x,ξ}}. \]

By (14) and (16), we obtain
\[ \sup_{|ξ-x| \leq R} \int_{A_{T,x,ξ}} v_{T,x,ξ}^{-p} dµ_T^0 < \infty \text{ for any } R > 0 \text{ and } p \in (1, \infty). \] (20)

Let \( A_{T,x,ξ}^c \) be the complement set of \( A_{T,x,ξ} \). If \( x = 0 \), then \( A_{T,x,ξ}^c = \emptyset \), and hence
\[ \int_{A_{T,x,ξ}^c} v_{T,x,ξ}^{-p} dµ_T^0 = 0. \]

Suppose \( x \neq 0 \). By (17) with \( s = 0 \), if \( t \in [0, T] \) satisfies \( 96B_{T,0,ξ,x} t^{\frac{1}{p}} \leq \frac{|x|}{2} \), then
\[ |(β + \ell_{T,x,ξ})(t)| \geq \frac{|x|}{2}. \]

Thus
\[ v_{T,x,ξ} \geq \left( \frac{|x|}{2} \right)^{2γ} \left( \frac{|x|^6}{192B^6_{T,0,ξ,x}} \wedge T \right). \] (21)

Hence, by (13) with \( a = 4|x| \), we have
\[ \int_{A_{T,x,ξ}^c} v_{T,x,ξ}^{-p} dµ_T^0 \leq \frac{2^{2γp}}{|x|^{2γp+6p}} \left( 192^6B^6_{T,0,ξ,x} + \frac{|x|^6}{T} \right)^p \times \exp \left( \frac{|ξ - x|^2}{4T} \right) \frac{\sqrt{2πT^2}}{2^d |x|^d} \frac{\exp \left( -\frac{dπ^2T}{256|x|^2} \right)}{2^d |x|^d} \left( 1 - \exp \left( -\frac{π^2}{128|x|^2} \right) \right)^{\frac{d}{2}}. \]
We now have
\[
\sup_{|x|, |ξ| \leq R} \int_{A_T} v_T^{-p} dμ_T^0 < \infty \quad \text{for any } R > 0 \text{ and } p \in (1, \infty).
\]
In conjunction with (20), this implies (18).

**Lemma 3.3.** Let \( F_{t,x} = \int_0^T |x + b(t)|^\gamma \hat{b}(t) \) and define the function \( q_{t,x,ξ} \) by
\[
q_{t,x,ξ}(η) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2πv_{t,x,ξ}}} \exp\left(-\frac{1}{2v_{t,x,ξ}} |η|^2\right) dμ_T^0 \quad \text{for } η \in \mathbb{R}^d.
\]
Then it holds
\[
E[f(F_{t,x}) \mid x + b(T) = ξ] = \int_{\mathbb{R}^d} f(η)q_{t,x,ξ}(η) dη \quad \text{for any bounded } f \in C(\mathbb{R}^d),
\]
where \( E[\cdot \mid x + b(T) = ξ] \) stands for the conditional expectation given \( x + b(T) = ξ \) with respect to \( μ_T \times \hat{μ}_T \).

**Proof.** Realize the \( d \)-dimensional Brownian bridge \( \{ρ(t)\}_{t \in [0,T]} \) with \( ρ(0) = ρ(T) = 0 \) by the stochastic differential equation
\[
dρ(t) = \hat{b}(t) dt - \frac{ρ(t)}{T-t} dt.
\]
Setting \( φ(t) = |(ρ + ξ^T,T)(t)|^\gamma I_{d^2} \), we have
\[
E[f(F_{t,x}) \mid b(T) = ξ] = \int_{\mathbb{R}^d} f(φ) d(μ_T \times \hat{μ}_T).
\]
Since the distribution of \( \int_0^T φ(t) dt \) under \( μ_T \) coincides with that of \( v_{t,x,ξ} I_{d^2} \) under \( μ_T^0 \), by Lemmas 2.1 and 3.2 we obtain the assertion. □

**Proof of Theorem 3.1.** (i) The integrability of \( v_{t,x,ξ}^{-1} \) was already seen in Lemma 3.2.

To show (12), realize the the diffusion process \( \{Z^{(x,y)}(t)\}_{t \in [0,T]} \) generated by \( \frac{1}{2} \Delta(γ) \) starting at \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^{d^2} \) by the the Itô type stochastic differential equation
\[
dZ^{(x,y)}(t) = \sum_{i=1}^d V_i(Z^{(x,y)}(t))d\hat{b}_i(t) + \sum_{j=1}^{d^2} W_j(Z^{(x,y)}(t))d\hat{b}_j(t) \quad \text{with } Z^{(x,y)}(0) = (x, y).
\]
Let \( X^{(x,y)}(t) \) (resp. \( Y^{(x,y)}(t) \)) be the \( \mathbb{R}^d \)-part (resp. \( \mathbb{R}^{d^2} \)-part) of \( Z^{(x,y)}(t) \). Then
\[
X^{(x,y)}(t) = x + b(t) \quad \text{and} \quad Y^{(x,y)}(t) = y + \int_0^t |x + b(s)|^\gamma d\hat{b}(s).
\]
For bounded \( f \in C(\mathbb{R}^d \times \mathbb{R}^{d^2}) \), we have
\[
\int_{\mathbb{R}^d} f(Z^{(x,y)}(T)) d(μ_T \times \hat{μ}_T)
\]
\[
= \int_{\mathbb{R}^d} \mathbb{E}[f(ξ, Y^{(x,y)}(T)) \mid x + b(T) = ξ] \frac{1}{\sqrt{2πT}} \exp\left(-\frac{|ξ - x|^2}{2T}\right) dξ.
\]
Due to \( (23) \), using the function \( q_{T,x,\xi}(t) \) defined in Lemma 3.3, we have
\[
\mu_T \times \hat{\mu}_T(Z^{(x,y)}(t) \in d\xi d\eta) = \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{|\xi - x|^2}{2T}\right) q_{T,x,\xi}(\eta - y) d\xi d\eta.
\]
This completes the proof of \( (12) \).

(ii) Let \( R > 0 \). By Lemma 3.2, the family
\[
\left\{ \frac{\alpha}{\sqrt{2\pi T}} \exp\left(-\frac{|\eta - y|^2}{2\alpha}\right) ; |x|, |\xi| \leq R, y, \eta \in \mathbb{R}^d \right\}
\]
is uniformly integrable. Hence the mapping
\[
(B(R) \times \mathbb{R}^d)^2 \ni ((x, y), (\xi, \eta)) \mapsto \int_{W_1^0} \frac{\alpha}{\sqrt{2\pi T}} \exp\left(-\frac{|\eta - y|^2}{2\alpha}\right) d\mu_T^0,
\]
where \( B(R) = \{ x \in \mathbb{R}^d \mid |x| \leq R \} \), is continuous. By virtue of the expression \( (12) \), we arrive at the desired continuity of \( p_T \). \( \Box \)

In the remaining of this section, we apply Theorem 3.1 to the short-time on-diagonal asymptotics of \( p_T \). Our goal will be

**Theorem 3.2.** Let \( y \in \mathbb{R}^d \).

(i) For \( x \neq 0 \), it holds
\[
\sqrt{T^{d+d'}} p_T((x, y), (x, y)) \to \frac{1}{\sqrt{2\pi^d |x|^{d'}}} \text{ as } T \to 0.
\] (24)

(ii) It holds
\[
\sqrt{T^{d+(1+\gamma)d'}} p_T((0, y), (0, y)) = \frac{1}{\sqrt{2\pi^d t^{d'}}} \int_{W_1^0} \left( \int_0^1 |\beta(t)|^{2\gamma} dt \right)^{\frac{d'}{2}} d\mu_1^0 \text{ for } T \in (0, 1]. \] (25)

Since \( \int_0^1 |\beta(t)|^{2\gamma} dt = v_{1.0.0} \), by Lemma 3.2 the integral in \( (25) \) is finite. To show the theorem, we prepare several lemmas.

**Lemma 3.4.** Define the random variable \( \hat{v}_{T,x,\xi} \) on \( W_1^0 \) by
\[
\hat{v}_{T,x,\xi} = \int_0^1 |(\sqrt{T}\beta(t) + \ell^{1,x,\xi})(t)|^{2\gamma} dt.
\]
Then it holds
\[
p_T((x, y), (\xi, \eta)) = \frac{1}{\sqrt{2\pi T^{d+d'}}} \exp\left(-\frac{|\xi - x|^2}{2T}\right) \int_{W_1^0} \hat{v}_{T,x,\xi}^{\frac{d'}{2}} \exp\left(-\frac{|\eta - y|^2}{2T\hat{v}_{T,x,\xi}}\right) d\mu_1^0. \] (26)

**Proof.** Under \( \mu_T^0 \), \( \{ \beta(t) \}_{t \in [0,T]} \) is a continuous Gaussian process with mean 0 and covariance \( \{ t \wedge s - \frac{t}{2} \alpha \} I_d \). Since so is \( \{ \sqrt{T}\beta(t) \}_{t \in [0,T]} \) under \( \mu_1^0 \), they have the same law. Hence \( v_{T,x,\xi} \) under \( \mu_T^0 \) and \( T\hat{v}_{T,x,\xi} \) under \( \mu_1^0 \) have the same law. Plugging this into \( (12) \), we obtain \( (26) \). \( \Box \)
Lemma 3.5. Let \( y \in \mathbb{R}^d \). If \((x, \xi) \neq (0, 0) \in (\mathbb{R}^d)^2\), then
\[
\sqrt{2\pi T}^{d+d'} \exp\left(\frac{|\xi - x|^2}{2T}\right) p_T((x, y), (\xi, y)) \to \left(\int_0^1 |\ell^{1,x,\xi}(t)|^{2\gamma} dt\right)^{-\frac{d'}{2}}. \tag{27}
\]

Proof. Suppose \((x, \xi) \neq (0, 0)\). Define
\[
B'_{T,x,\xi} = \|\sqrt{T}\beta + \ell^{1,x,\xi}\|_{1,12}.4.
\]
Since \(B'_{T,x,\xi} \leq \sqrt{T}B_{1,0,0} + \|\ell^{1,x,\xi}\|_{1,12.4}\), by (16), we obtain
\[
\sup_{T \in [0,1]} \|B'_{T,x,\xi}\|_{L^p(\mu^0)} < \infty \quad \text{for any } p \in (1, \infty). \tag{28}
\]

If \(x \neq 0\), then, in repetition of the argument used to show (21), we have
\[
\hat{v}_{T,x,\xi} \geq \left(\frac{|x|}{2}\right)^{2\gamma} \left(\frac{|x|^6}{1926(B'_{T,x,\xi})^6} \wedge 1\right).
\]

If \(\xi \neq 0\), then by (17) with \(s = 1\), it holds
\[
|(\sqrt{T}\beta + \ell^{1,x,\xi})(t)| \geq \frac{|\xi|}{2} \quad \text{for } t \in [0,1] \text{ satisfying } 96B'_{T,x,\xi}|1-t|^\frac{1}{6} \leq \frac{|\xi|}{2}.
\]
Hence, for \(\xi \neq 0\), we have
\[
\hat{v}_{T,x,\xi} \geq \left(\frac{|\xi|}{2}\right)^{2\gamma} \left(\frac{|\xi|^6}{1926(B'_{T,x,\xi})^6} \wedge 1\right).
\]

Thus we obtain
\[
\hat{v}_{T,x,\xi} \geq \left(\frac{|x| \vee |\xi|}{2}\right)^{2\gamma} \left(\frac{(|x| \vee |\xi|)^6}{1926(B'_{T,x,\xi})^6} \wedge 1\right).
\]

In conjunction with (28), this yields
\[
\sup_{T \in [0,1]} \|\hat{v}^{-1}_{T,x,\xi}\|_{L^p(\mu^0)} < \infty \quad \text{for any } p \in (1, \infty). \tag{29}
\]

By Lemma 3.4 it holds
\[
\sqrt{2\pi T}^{d+d'} \exp\left(\frac{|\xi - x|^2}{2T}\right) p_T((x, y), (\xi, y)) = \int_{W_p} \hat{v}^{-d'}_{T,x,\xi} d\mu^0_{1}. \tag{30}
\]
Since the family \(\{\hat{v}^{-d'}_{T,x,\xi} : T \in (0,1)\}\) is uniformly integrable by (29), and \(\hat{v}_{T,x,\xi}\) converges to \(\int_0^1 |\ell^{1,x,\xi}(t)|^{2\gamma} dt\) point-wise as \(T \to 0\), we see
\[
\int_{W_p} \hat{v}^{-d'}_{T,x,\xi} d\mu^0_{1} \to \left(\int_0^1 |\ell^{1,x,\xi}(t)|^{2\gamma} dt\right)^{-\frac{d'}{2}}.
\]
Plugging this into (30), we obtain (27). \(\square\)

Proof of Theorem 3.2. The convergence (24) was already seen in Lemma 3.5 for \(\ell^{1,x,x} \equiv x\). Since \(\hat{v}_{T,0,0} = T^{\gamma} \int_0^1 |\beta(t)|^{2\gamma} dt\), (25) follows from (26). \(\square\)
4 Off-diagonal asymptotics

4.1 When \((x, \xi) \neq (0, 0)\)

The aim of this subsection is to obtain the off-diagonal asymptotics of \(p_T((x, y), (\xi, \eta))\) as \(T \to 0\) when \((x, \xi) \neq (0, 0)\). To do so, we will apply the theory of large deviations.

We first introduce quantities used in the expression of the asymptotics. For \(x, \xi \in \mathbb{R}^d, w \in W_0^1, h \in H_0^1,\) and \(a \geq 0,\) put

\[
\nu_{x, \xi}(w) = \int_0^1 |w(t) + \ell_{1, x, \xi}(t)|^{2\gamma} dt, \quad \Phi_{x, \xi, a}(h) = \frac{a^2}{\nu_{x, \xi}(h)} + \|h\|_{H_0^1}^2. \tag{31}
\]

If \(w + \ell_{1, x, \xi} \neq 0,\) then \(\nu_{x, \xi}(w) > 0.\) Hence \(\nu_{x, \xi}(w) > 0\) for any \(w \in W_0^1\) when \((x, \xi) \neq (0, 0),\) and \(\nu_{0, 0}(w) > 0\) for \(w \neq 0.\)

Define

\[
m(x, \xi, a) = \inf_{h \in H_0^1} \Phi_{x, \xi, a}(h).
\]

We first see the properties of \(m(x, \xi, a);\)

**Lemma 4.1.** (i) \(m(x, \xi, 0) = 0,\) the function \([0, \infty) \ni a \mapsto m(x, \xi, a)\) is non-decreasing, and it holds

\[
(a \wedge 1)^2 m(x, \xi, 1) \leq m(x, \xi, a) \leq \frac{a^2}{\nu_{x, \xi}(0)} \quad \text{for any } a \geq 0.
\]

(ii) It holds

\[
\lim_{a \to \infty} a^{-\frac{2}{1+\gamma}} m(x, \xi, a) = c^-_{\gamma} (1 + \gamma)^{-\frac{2\gamma}{1+\gamma}}, \quad \text{where}
\]

\[
c^-_{\gamma} = \sup_{h \in H_0^1} \frac{\|h\|_{H_0^1}^{2\gamma}}{\|h\|_{H_0^1}^{\gamma}} \quad \text{with } \|h\|_{2\gamma} = \left( \int_0^1 |h(t)|^{2\gamma} dt \right)^{\frac{1}{2\gamma}}. \tag{32}
\]

(iii) For each \(a > 0,\) there exists \(h_a \in H_0^1\) such that \(m(x, \xi, a) = \Phi_{x, \xi, a}(h_a).\)

(iv) \(m(x, \xi, a) > 0\) for \(a > 0.\)

**Proof.** (i) The assertion is an immediate consequence of the definition of \(m(x, \xi, a).\)

(ii) For \(p, q > 0\) and \(r \geq 0,\) set

\[
\Psi_{p, q, r}(\lambda) = \begin{cases} 
\frac{p}{\lambda^{2\gamma} + r} + q\lambda^2 & \text{when } \gamma < \frac{1}{2}, \\
\frac{p}{(\lambda + r)^{2\gamma}} + q\lambda^2 & \text{when } \gamma \geq \frac{1}{2}, 
\end{cases} \quad \text{for } \lambda \in (0, \infty).
\]

It is easily seen that

\[
\inf_{\lambda \in (0, \infty)} \Psi_{p, q, r}(\lambda) = q \left( 1 + \frac{1}{\gamma} \right) \lambda^2_{p, q, r} + \frac{qr}{\gamma} \lambda^{1+(1-2\gamma)v_0}_{p, q, r}, \tag{33}
\]
where \( \lambda_{p,q,r} > 0 \) is determined by

\[
q = \begin{cases} 
\frac{\gamma p}{\lambda_{p,q,r}^{2-2\gamma} (\lambda_{p,q,r}^2 + r)^2} & \text{when } \gamma < \frac{1}{2}, \\
\frac{\gamma p}{(\lambda_{p,q,r} + r)^{2\gamma+1}} & \text{when } \gamma \geq \frac{1}{2}.
\end{cases}
\]

The definition of \( \lambda_{p,q,r} \) yields

\[
\lim_{r \to 0} \lambda_{p,q,r} = \lambda_{p,q,0} = \left( \frac{\gamma p}{q} \right) \frac{1}{\gamma+1}.
\]

Plugging this into (33), we obtain

\[
\lim_{r \to 0} \inf_{\lambda \in (0,\infty)} \Psi_{p,q,r}(\lambda) = \inf_{\lambda \in (0,\infty)} \Psi_{p,q,0}(\lambda) = (1 + \gamma) \gamma^{-\frac{\gamma}{1+\gamma}} (pq)^{\frac{1}{1+\gamma}}.
\]

Set \( \alpha = (1 + \gamma)^{-1} \) and observe

\[
\Phi_{x,\xi,a}(a^\alpha h) = a^{2\alpha} \Phi_{a^{-\alpha}x,a^{-\alpha}\xi,1}(h) \quad \text{for any } h \in \mathcal{H}_1^0.
\]

Thus, to show (32), it suffices to prove

\[
\lim_{a \to \infty} \inf m(a^{-\alpha} x, a^{-\alpha} \xi, 1) \geq c_{\gamma}^{-\frac{2\gamma}{1+\gamma}} (1 + \gamma) \gamma^{-\frac{\gamma}{1+\gamma}},
\]

\[
\lim_{a \to \infty} \sup m(a^{-\alpha} x, a^{-\alpha} \xi, 1) \leq c_{\gamma}^{-\frac{2\gamma}{1+\gamma}} (1 + \gamma) \gamma^{-\frac{\gamma}{1+\gamma}}.
\]

We first show (35). Applying the inequality \((a + b)^{2\gamma} \leq a^{2\gamma} + b^{2\gamma}\) when \( \gamma < \frac{1}{2} \) and the Minkowski inequality when \( \gamma \geq \frac{1}{2} \), we have

\[
\int_0^1 |h(t) + \ell^{1,a^{-\alpha}x,a^{-\alpha}\xi}(t)|^{2\gamma} dt \leq \begin{cases} 
\frac{c_{\gamma}^2 \|h\|_{\mathcal{H}_1^0}^{2\gamma} + r(a)^{2\gamma}}{\gamma} & \text{when } \gamma < \frac{1}{2}, \\
(\gamma \|h\|_{\mathcal{H}_1^0} + r(a))^{2\gamma} & \text{when } \gamma \geq \frac{1}{2}
\end{cases}
\]

for any \( h \in \mathcal{H}_1^0 \), where \( r(a) = \|\ell^{1,a^{-\alpha}x,a^{-\alpha}\xi}\|_{2\gamma} \). Hence

\[
\Phi_{a^{-\alpha}x,a^{-\alpha}\xi,1}(h) \geq \begin{cases} 
\inf_{\lambda \in (0,\infty)} \Psi_{c_{\gamma}^{-\gamma},1,r(a)^{2\gamma}}(\lambda) & \text{when } \gamma < \frac{1}{2}, \\
\inf_{\lambda \in (0,\infty)} \Psi_{c_{\gamma}^{-\gamma},1,r(a)}(\lambda) & \text{when } \gamma \geq \frac{1}{2}
\end{cases}
\]

for any \( h \in \mathcal{H}_1^0 \).

Since \( \lim_{a \to \infty} r(a) = 0 \), in conjunction with (34), this implies (35).

We next show (36). It holds

\[
m(a^{-\alpha} x, a^{-\alpha} \xi, 1) \leq \Phi_{a^{-\alpha}x,a^{-\alpha}\xi,1}(h) \xrightarrow{a \to \infty} \Phi_{0,0,1}(h) \quad \text{for any } h \in \mathcal{H}_1^0.
\]

Hence we have

\[
\lim_{a \to \infty} \sup m(a^{-\alpha} x, a^{-\alpha} \xi, 1) \leq \inf_{\lambda \in (0,\infty)} \Phi_{0,0,1}(\lambda h) = \inf_{\lambda \in (0,\infty)} \Psi_{p(h)q(h),0}(\lambda) \quad \text{for any } h \in \mathcal{H}_1^0.
\]
where \( p(h) = \| h \|_{2\gamma}^2 \) and \( q(h) = \| h \|_{\mathcal{H}_1^0}. \) By (34),
\[
\limsup_{a \to \infty} m(a^{-\alpha} x, a^{-\alpha} \xi, 1) \leq (1 + \gamma) \gamma^{-\frac{1}{2+\gamma}} \left( p(h) q(h)^2 \right)^{\frac{1}{2+\gamma}}.
\]
Taking the infimum over \( h \) and noticing
\[
\inf_{h \in \mathcal{H}_1^0} p(h) q(h) = c_\gamma^{-2\gamma},
\]
we arrive at (36).

(ii) Since \( v(h_n) \to m(x, \xi, a) \) as \( n \to \infty. \) Since \( \| h \|^2_{\mathcal{H}_1^0} \leq \Phi_{x,\xi,a}(h) \) for \( h \in \mathcal{H}_1^0, \)
\[
\sup_{n \in \mathbb{N}} \| h_n \|_{\mathcal{H}_1^0} < \infty.
\] (37)

Hence taking a subsequence if necessary, we may assume \( h_n \) converges weakly to some \( h_a \in \mathcal{H}_1^0. \)

By (3), (37), and the Ascoli-Arzelà theorem, \( \{h_n\}_{n=1}^\infty \) have a subsequence \( \{h_{n_j}\}_{j=1}^\infty \) which converges in \( \mathcal{W}_1^0. \) Using the inclusion \( \mathcal{W}_1^{0*} \subset \mathcal{H}_1^0, \) we observe that
\[
\ell(h_{n_j}) = \langle \ell, h_{n_j} \rangle_{\mathcal{H}_1^0} \xrightarrow{j \to \infty} \langle \ell, h_a \rangle_{\mathcal{H}_1^0} = \ell(h_a)
\]
for any \( \ell \in \mathcal{W}_1^{0*}, \) which means \( h_{n_j} \) converges to \( h_a \) weakly in \( \mathcal{W}_1^0. \) Hence \( \| h_{n_j} - h_a \|_{\mathcal{W}_1^0} \to 0 \) as \( j \to \infty. \)

In particular, \( v_{x,\xi}(h_{n_j}) \to v_{x,\xi}(h_a) \) as \( j \to \infty. \)

Since \( \| h_a \|_{\mathcal{H}_1^0} \leq \liminf_{j \to \infty} \| h_{n_j} \|_{\mathcal{H}_1^0}, \) we obtain
\[
\Phi_{x,\xi,a}(h_a) \leq \liminf_{j \to \infty} \Phi_{x,\xi,a}(h_{n_j}) = m(x, \xi, a).
\]

This implies the identity \( \Phi_{x,\xi,a}(h_a) = m(x, \xi, a). \)

(iii) Take a sequence \( \{h_n\}_{n=1}^\infty \) such that \( \Phi_{x,\xi,a}(h_n) \to m(x, \xi, a) \) as \( n \to \infty. \) Since \( \| h \|^2_{\mathcal{H}_1^0} \leq \Phi_{x,\xi,a}(h) \) for \( h \in \mathcal{H}_1^0, \)
\[
\sup_{n \in \mathbb{N}} \| h_n \|_{\mathcal{H}_1^0} < \infty.
\] (37)

Remark 4.1. Let \( h \in \mathcal{H}_1^0 \) and \( t \in [0, 1]. \) Since \( h(t) = \int_0^1 (1_{[0,t]}(s) - t) h(s) ds, \) we have \( |h(t)| \leq (t - t^2) \frac{1}{2} \| h \|_{\mathcal{H}_1^0}. \) Hence
\[
c_\gamma \leq B(1 + \gamma, 1 + \gamma)^\frac{1}{2},
\]
where \( B(\cdot, \cdot) \) is the Beta function.

The goal of this subsection is

**Theorem 4.1.** Assume \( (x, \xi) \neq (0, 0). \) Then it holds
\[
\lim_{T \to 0} T \log p_T((x, y), (\xi, \eta)) = -\frac{1}{2} \{ |\xi - x|^2 + m(x, \xi, |\eta - y|) \} \quad \text{for any } y, \eta \in \mathbb{R}^d. \quad (38)
\]

Moreover,
\[
\lim_{|\eta - y| \to \infty} |\eta - y|^\frac{2}{1+\gamma} \lim_{T \to 0} T \log p_T((x, y), (\xi, \eta)) = c_\gamma \frac{2^\gamma}{1+\gamma} (1 + \gamma)^{\gamma} \quad (39)
\]
where \( c_\gamma \) is the constant given in (32).
To show the theorem, we recall the results in the theory of large deviations. Define \( \mathcal{W}_1^0 \rightarrow \mathcal{W}_1^0 \) by \( z_T(w) = \sqrt{T}w \) for \( w \in \mathcal{W}_1^0 \), and set \( \nu_T = \mu_0 \circ z_T^{-1} \). Then the Schilder theorem and the Varadhan integral lemma hold on \( \mathcal{W}_1^0 \) as follows.

**Lemma 4.2.** (i) \( \{\nu_T\} \) satisfies the large deviation principle with rate function

\[
I(w) = \begin{cases} 
\frac{1}{2} \|w\|_{\mathcal{H}_0}^2 & \text{if } w \in \mathcal{H}_0, \\
\infty & \text{if } w \in \mathcal{W}_1^0 \setminus \mathcal{H}_0.
\end{cases}
\]

(ii) If \( F \in C(\mathcal{W}_1^0) \) is bounded from above, then

\[
\lim_{T \to 0} T \log \int_{\mathcal{W}_1^0} \exp \left( \frac{1}{T} F \right) d\nu_T = \sup_{w \in \mathcal{W}_1^0} \{ F(w) - I(w) \}.
\]

**Proof.** See [10, Chapter 8] for (i) and [5, Theorem 4.3.1] for (ii). \( \square \)

**Proof of Theorem 4.1.** Notice that \( v_{x,\xi}(x, \xi, \eta, \eta) = \hat{v}_{T,x,\xi} \). By (26), we obtain

\[
p_T((x, y), (\xi, \eta)) = \frac{1}{\sqrt{2\pi T}} \exp \left( -\frac{|\xi - x|^2}{2T} \right) \mathcal{I}_{T,x,\xi},
\]

where

\[
\mathcal{I}_{T,x,\xi} = \int_{\mathcal{W}_1^0} v_{x,\xi}^T \exp \left( -\frac{|\eta - y|^2}{2T v_{x,\xi}} \right) d\nu_T.
\]

Thus, to see (38), it suffices to show

\[
\lim_{T \to 0} T \log \mathcal{I}_{T,x,\xi} = -\frac{1}{2} m(x, \xi, |\eta - y|).
\]

We prove this by showing the upper and lower estimations

\[
\limsup_{T \to 0} T \log \mathcal{I}_{T,x,\xi} \leq -\frac{1}{2} m(x, \xi, |\eta - y|),
\]

\[
\liminf_{T \to 0} T \log \mathcal{I}_{T,x,\xi} \geq -\frac{1}{2} m(x, \xi, |\eta - y|).
\]

Since \( v_{x,\xi} > 0, v_{x,\xi}^{-1} \in C(\mathcal{W}_1^0) \). Moreover, it holds

\[
\sup_{w \in \mathcal{W}_1^0} \left\{ -\frac{a^2}{2v_{x,\xi}(w)} - I(w) \right\} = -\frac{1}{2} m(x, \xi, a).
\]

By Lemma 4.2(ii), we then have

\[
\lim_{T \to 0} T \log \int_{\mathcal{W}_1^0} \exp \left( -\frac{a^2}{2T v_{x,\xi}} \right) d\nu_T = -\frac{1}{2} m(x, \xi, a) \quad \text{for } a \geq 0.
\]

For \( p \in (1, \infty) \), by the Hölder inequality, (29), (44), and Lemma 4.1 we obtain

\[
\limsup_{T \to 0} T \log \mathcal{I}_{T,x,\xi} \leq -\frac{1}{2p} m(x, \xi, \sqrt{p}|\eta - y|) \leq -\frac{1}{2p} m(x, \xi, |\eta - y|).
\]
Letting $p \downarrow 1$, we obtain (42).

Let $R > \sqrt{m(x, \xi, |\eta - y|)}$ and $B_R = \{w \in \mathcal{W}_1^0 \mid \|w\|_{\mathcal{W}_1^0} < R\}$. Noting that $\sup_{w \in B_R} \nu_{x, \xi}(w) \leq (R + |x| + |\xi|)^{2\gamma}$, we have

$$I_{T, x, \xi} \geq (R + |x| + |\xi|)^{-\gamma \delta} \int_{B_R} \exp\left(-\frac{|\eta - y|^2}{2T \nu_{x, \xi}}\right) d\nu_T \geq \int_{\mathcal{W}_1^0} \exp\left(-\frac{|\eta - y|^2}{2T \nu_{x, \xi}}\right) d\nu_T - \nu_T(B_R^c).$$

(45)

Since $\|h\|_{\mathcal{W}_1^0} \leq \|h\|_{H_1^0}$ for $h \in H_1^0$, by Lemma 4.2(i), we obtain

$$\limsup_{T \to 0} T \log \nu_T(B_R^c) \leq -\inf_{w \in B_R} I_T(w) \leq -\frac{R^2}{2}.$$  

In conjunction with this and (44), the inequality (45) yields (42).

The equation (39) follows from Lemma 4.1 and (38).

4.2 When $(x, \xi) = (0, 0)$

In this subsection, we investigate the off-diagonal asymptotics of $p_T((0, y), (0, \eta))$ as $T \to 0$. Set

$$\delta_0 = \sup\{\delta > 0 \mid \exp(\delta B_{1,0,0}^2) \in L^1(\mu_1^0)\}.$$  

By (16) with $T = 1$, $\delta_0 > 0$.

**Theorem 4.2.** For $\delta > 0$ define $\varepsilon(\delta) = 0$ by

$$\varepsilon(\delta)^2 = \frac{\delta}{8(192)^2}.$$  

Then it holds

$$-2^{\frac{3\gamma}{2\gamma + d}} (1 + \gamma)^{-\frac{\gamma}{2\gamma + d}} \left((2\varepsilon(\delta_0))^6 \land 1\right)^{-\frac{\gamma}{2\gamma + d}} |\eta - y|^{\frac{2\gamma}{2\gamma + d}}$$

$$\leq \liminf_{T \to 0} T \log p_T((0, y), (0, \eta))$$

$$\leq \limsup_{T \to 0} T \log p_T((0, y), (0, \eta)) \leq -2^{\frac{3\gamma}{2\gamma + d}} d^{-\frac{\gamma}{2\gamma + d}} |\eta - y|^{\frac{2\gamma}{2\gamma + d}}$$

(46) for any $y, \eta \in \mathbb{R}^d$.

The dependence on the parameter $\gamma$ can be seen in these estimations, especially in the growth order in $|\eta - y|$ as it tends to infinity, which is similar as in the case when $(x, \xi) \neq (0, 0)$ (cf. Theorem 1.1). The proof is divided into several steps, each step being a lemma. In what follows, let $M_{1,0}$ be the random variable given in Lemma 3.1 $M_{1,0}(w) = \|w\|_{\mathcal{W}_1^0}$ for $w \in \mathcal{W}_1^0$.

**Lemma 4.3.** It holds

$$\mu_1^0(M_{1,0} \geq a) \leq 2d \exp\left(-\frac{2a^2}{d}\right),$$

$$\mu_1^0(a \leq M_{1,0} \leq 2\sqrt{a}d) \geq 2 \exp(-2a^2) \{1 - (d + 1) \exp(-6a^2)\} \quad \text{for } a > 0.$$  

(48)
Proof. It is known ([9, p.39 (12)]) that

\[ \mu_1^0 \left( \max_{t \in [0,1]} |\beta^i(t)| \geq a \right) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp(-2k^2a^2) \] for \( 1 \leq i \leq d \) and \( a > 0 \),

where \( \beta^i(t) \) is the \( i \)th component of \( \beta(t) \). Since \( k \mapsto \exp(-k^2a^2) \) is non-increasing, we have

\[ 2 \exp(-2a^2) \{1 - \exp(-6a^2)\} \leq \mu_1^0 \left( \max_{t \in [0,1]} |\beta^i(t)| \geq a \right) \leq 2 \exp(-2a^2). \]

Combining this with the inclusion

\[ \left\{ \max_{t \in [0,1]} |\beta^1(t)| \geq a \right\} \subset \{M_{1,0} \geq a\} \subset \bigcup_{i=1}^{d} \left\{ \max_{t \in [0,1]} |\beta^i(t)| \geq \frac{a}{\sqrt{d}} \right\}, \]

we obtain [48] and

\[ 2 \exp(-2a^2) \{1 - \exp(-6a^2)\} \leq \mu_1^0(M_{1,0} \geq a). \]

This and [48] with \( 2\sqrt{d}a \) for \( \eta \) yield [49].

Lemma 4.4. It holds

\[ \limsup_{T \to 0} T \log p_T((0, y), (0, \eta)) \leq -2^{-\frac{d+1}{1+\gamma}} d^{-\frac{1}{1+\gamma}} \eta - y|^{\frac{2}{1+\gamma}}. \] (50)

In particular, the upper estimate [17] in Theorem 4.2 holds.

Proof. Due to Theorem 3.2, [50] holds if \( y = \eta \). Thus we assume \( y \neq \eta \).

By [11] and the very definition of \( \nu_T \), we obtain

\[ p_T((0, y), (0, \eta)) = \frac{T^{-d/2}}{\sqrt{2\pi}} \int_{W_{0,0}} e^{-\frac{2}{T} \nu_{0,0} \cdot \nu} \exp\left(-\frac{|\eta - y|^2}{2T^{1+\gamma} \nu_{0,0}}\right) d\nu_0. \] (51)

Let \( p > 1 \). By Hölder’s inequality, we have

\[ \limsup_{T \to 0} T \log p_T((0, y), (0, \eta)) \leq \frac{1}{p} \limsup_{T \to 0} T \log \left( \int_{W_{0,0}} \exp\left(-\frac{p|\eta - y|^2}{2T^{1+\gamma} \nu_{0,0}}\right) d\nu_0 \right). \] (52)

Since \( \nu_{0,0} \leq M_{1,0}^{2\gamma} \), it holds

\[ \int_{W_{0,0}} \exp\left(-\frac{p|\eta - y|^2}{2T^{1+\gamma} \nu_{0,0}}\right) d\nu_0 \leq \int_{W_{0,0}} \exp\left(-\frac{p|\eta - y|^2}{2T^{1+\gamma} M_{1,0}^{2\gamma}}\right) d\nu_0. \]

For \( \lambda > 0 \), using the decomposition \( W_{0,0}^0 = \{M_{1,0} < \lambda T^{-\frac{1}{2}}\} \cup \{M_{1,0} \geq \lambda T^{-\frac{1}{2}}\} \) and Lemma 4.3, we obtain

\[ \int_{W_{0,0}} \exp\left(-\frac{p|\eta - y|^2}{2T^{1+\gamma} \nu_{0,0}}\right) d\nu_0 \leq \exp\left(-\frac{p|\eta - y|^2}{2T^{1+\gamma} M_{1,0}^{2\gamma}}\right) + 2d \exp\left(-\frac{2\lambda^2}{dT}\right). \]

Substituting this into [52], we obtain

\[ \limsup_{T \to 0} T \log p_T((0, y), (0, \eta)) \leq \frac{1}{p} \sup_{\lambda > 0} \left( \frac{p|\eta - y|^2}{2\lambda^{2\gamma}} + \frac{2\lambda^2}{d} \right) \]

\[ = -p^{-1} \lambda \lambda^{2\gamma} d^{-\frac{1}{1+\gamma}} \eta - y|^{\frac{2}{1+\gamma}}. \]

Letting \( p \searrow 1 \), we obtain [50].

\[ \square \]
Lemma 4.5. It holds
\[
\liminf_{T \to 0} T \log p_T((0, y), (0, \eta)) \geq -2^{\frac{\gamma}{\gamma + 1}} \left(1 + \gamma\right) \gamma^{-\frac{2}{\gamma + 1}} \left((2\varepsilon(\delta_0))^6 \wedge 1 \right)^{-\frac{1}{\gamma + 1}} |\eta - y|^{\frac{2}{\gamma + 1}}. \tag{53}
\]
In particular, the lower estimate (46) in Theorem 4.2 holds.

Proof. Due to Theorem 3.2, (53) holds if \( y = \eta \). Thus we assume \( y \neq \eta \).

Let \( \delta \in (0, \delta_0) \). For \( \lambda, T > 0 \), put
\[
A_{\lambda, T} = \left\{ B_{1,0,0} \leq \frac{\lambda T^{-\frac{1}{\gamma}}}{192 \varepsilon(\delta)} \right\}.
\]
It holds
\[
\mu_1^0(A_{\lambda, T}^c) \leq C_\delta \exp \left(-\frac{8\lambda^2}{T}\right), \tag{54}
\]
where
\[
C_\delta = \int_{\mathcal{W}_1^0} \exp(\delta B_{1,0,0}^2) d\mu_1^0 < \infty.
\]

In repetition of the argument used to show (19), this time with \( x = \xi = 0 \), we can show
\[
\nu_{0,0} \geq \left(\frac{M_{1,0}}{2}\right)^{2\gamma} \left(\frac{M_{1,0}^6}{192^6 B_{1,0,0}^6} \wedge \frac{1}{2}\right).
\]
Thus
\[
\nu_{0,0} \mathbf{1}_{A_{\lambda, T}} \geq \left(\frac{M_{1,0}}{2}\right)^{2\gamma} \left(\left(M_{1,0} T\frac{1}{2} \lambda^{-1} \varepsilon(\delta)\right)^6 \wedge \frac{1}{2}\right) \mathbf{1}_{A_{\lambda, T}}.
\]

By this, setting
\[
B_{\lambda, T} = \{ \lambda T^{-\frac{1}{2}} \leq M_{1,0} \leq 2\sqrt{d} \lambda T^{-\frac{1}{4}} \}
\]
and remembering \( \nu_{0,0} \leq M_{1,0}^2 \), we obtain
\[
\int_{\mathcal{W}_1^0} \nu_{0,0}^{-\frac{\delta}{2}} \exp \left(-\frac{|\eta - y|^2}{2T^{1+\gamma} \nu_{0,0}}\right) d\mu_1^0 \geq \int_{A_{\lambda, T} \cap B_{\lambda, T}} M_{1,0}^{\gamma d} \exp \left(-\frac{2^{\gamma} |\eta - y|^2}{T^{1+\gamma} M_{1,0}^2 (\{2(M_{1,0} T^{-\frac{1}{2}} \lambda^{-1} \varepsilon(\delta))^6 \wedge 1\})}\right) d\mu_1^0 \geq (2\sqrt{d} \lambda)^{-\gamma d \frac{T^{1+\gamma}}{2}} \exp \left(-\frac{2^{\gamma} |\eta - y|^2}{T \lambda^{2\gamma} ((2\varepsilon(\delta)^6) \wedge 1)}\right) \mu_1^0(A_{\lambda, T} \cap B_{\lambda, T}). \tag{55}
\]

By Lemma 4.3 and (54), we have
\[
\mu_1^0(A_{\lambda, T} \cap B_{\lambda, T}) \geq \mu_1^0(B_{\lambda, T}) - \mu_1^0(A_{\lambda, T}^c) \geq \exp \left(-\frac{2\lambda^2}{T}\right) \left\{ 2 - (2 + 2d + C_\delta) \exp \left(-\frac{6\lambda^2}{T}\right) \right\}.
\]

Plugging this into (55), we obtain
\[
\liminf_{T \to 0} T \log \left(\int_{\mathcal{W}_1^0} \nu_{0,0}^{-\frac{\delta}{2}} \exp \left(-\frac{|\eta - y|^2}{2T^{1+\gamma} \nu_{0,0}}\right) d\mu_1^0\right) \geq - \inf_{\lambda \in (0, \infty)} \left\{ \frac{2^{\gamma} |\eta - y|^2}{\lambda^{2\gamma} ((2\varepsilon(\delta)^6) \wedge 1)} + 2\lambda^2 \right\}.
\]
Due to the observation on $\Psi_{p,q,r}$ in the proof of Lemma 4.1, we see
\[
\inf_{\lambda \in (0, \infty)} \left\{ \frac{2^{2\gamma} |\eta - y|^2}{\lambda^{2\gamma} \{(2\varepsilon(\delta)^6) \wedge 1\}} + 2\lambda^2 \right\} = 2^{\frac{3\gamma}{1+r}} (1 + \gamma)^{\frac{7}{1+r}} \left((2\varepsilon(\delta))^6 \wedge 1\right)^{\frac{1}{1+r}} |\eta - y|^{\frac{2}{1+r}}.
\]
Thus, by (51), letting $\delta \nearrow \delta_0$, we obtain (53).

**Remark 4.2.** Dominating $v_{0,0}$ by $M_{1,0}^{2\gamma}$ is best in the sense that
\[
\sup_{w \in \mathcal{W}_1^0} \frac{v_{0,0}(w)}{M_{1,0}(w)^{2\gamma}} = 1.
\]
This identity is seen as follows. It is obvious that the supremum is bounded by 1 from above. For $n \in \mathbb{N}$, take $w_n = (w_n^1, \ldots, w_n^d) \in \mathcal{W}_1^0$ given by
\[
w_n^1(t) = \begin{cases} nt & \text{for } t \in [0, \frac{1}{n}), \\ 1 & \text{for } t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right), \\ 1 + n \left(t - \left(1 - \frac{1}{n}\right)\right) & \text{for } t \in \left[1 - \frac{1}{n}, 1\right], \end{cases}
w_n^2 = \cdots = w_n^d = 0.
\]
Then
\[
\frac{v_{0,0}(w_n)}{M_{1,0}(w_n)^{2\gamma}} \geq 1 - \frac{2}{n} \quad \text{for } n \in \mathbb{N}.
\]
Letting $n \to \infty$, we obtain that the supremum is bounded by 1 from below.

**5 Remark**

For $\gamma \geq 1$, we can sharpen the assertion of Lemma 3.5. Define
\[
f_{x,\xi}(a) = \int \mathcal{W}_1^0 \left( \int_0^1 |a \beta(t) + \ell_{1,x,\xi}(t)|^{2\gamma} dt \right)^{-\frac{p}{2}} d\mu_1^0 \quad \text{for } a \in \mathbb{R}.
\]
By (26), it holds
\[
p_T((x, y), (\xi, y)) = \frac{1}{\sqrt{2\pi T^{1+d}}} \exp\left(-\frac{|\xi - x|^2}{2T}\right) f_{x,\xi}(\sqrt{T}) \quad \text{for } T > 0 \text{ and } y \in \mathbb{R}^d.
\]
Modifying the proof of Lemma 3.2 slightly, we have
\[
\sup_{|a| \leq R} \int \mathcal{W}_1^0 \left( \int_0^1 |a \beta(t) + \ell_{1,x,\xi}(t)|^{2\gamma} dt \right)^{-p} d\mu_1^0 < \infty \quad \text{for any } p \in (1, \infty) \text{ and } R > 0.
\]
This implies that $f_{x,\xi} \in C^{m(\gamma)}(\mathbb{R})$, where
\[
m(\gamma) = \begin{cases} \infty & \text{if } \gamma \in \mathbb{N}, \\ [\gamma] & \text{if } \gamma \notin \mathbb{N}. \end{cases}
\]
By the Taylor expansion, we have
\[ f_{x,\xi}(a) = f_{x,\xi}(0) + \sum_{k=1}^{m} \frac{f_{x,\xi}^{(k)}(0)}{k!} a^k + O(a^{m+1}) \quad \text{for any } m < m(\gamma) \text{ as } a \to 0. \]

Since \( \{\beta(t)\}_{t \in [0,1]} \stackrel{\text{law}}{\sim} \{-\beta(t)\}_{t \in [0,1]} \), it holds
\[ f_{x,\xi}(-a) = f_{x,\xi}(a) \quad \text{for } a \in \mathbb{R}. \]

This yields \((-1)^k f_{x,\xi}^{(k)}(-a) = f_{x,\xi}^{(k)}(a)\) for \( k \in \mathbb{N} \) and \( a \in \mathbb{R} \). In particular,
\[ f_{x,\xi}^{(k)}(0) = 0 \quad \text{for } k \notin 2\mathbb{N}. \]

Hence the Taylor expansion comes down to
\[ f_{x,\xi}(a) = f_{x,\xi}(0) + \sum_{k \in \mathbb{N}, 2k \leq m} \frac{f_{x,\xi}^{(2k)}(0)}{(2k)!} a^{2k} + O(a^{m+1}) \quad \text{for any } m < m(\gamma) \text{ as } a \to 0. \]

Thus we obtain
\[ p_T((x,y), (\xi, y)) = \frac{1}{\sqrt{2\pi}T^{d+\sigma}} \exp\left(-\frac{|\xi - x|^2}{2T}\right) \]
\[ \times \left\{ \left( \int_0^1 |\ell^{1,x,\xi}(t)|^{2\gamma} dt \right)^{-\frac{d'}{2}} + \sum_{k \in \mathbb{N}, 2k \leq m} \frac{f_{x,\xi}^{(2k)}(0)}{(2k)!} T^k + O\left(T^{m+1}\right) \right\} \]

as \( T \to 0 \) for any \( m < m(\gamma) \).

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