Energetic Pseudospectrum Stability of Near Extremal Spacetimes

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It has been suggested that the pseudospectrum of quasinormal modes of rotating black holes is unstable, as the linear perturbation analysis with additional potential terms is not available without well-defined self-adjoint inner product for the mode wave function space. We point out that a self-adjoint construction has already been realized previously and an associated perturbation analysis was applied to study the quasinormal modes of weakly charged Kerr-Newman black holes. The proposed pseudospectrum instability should be restated as an “energetic” instability specifically assuming the type of inner products defined by Jaramillo et al., which preserve the physical meaning of energy. We argue that it is necessary to address the energetic stability of all previous mode analysis results to reveal the their susceptibility to perturbations with infinitesimal energies. In particular, for near extremal Kerr spacetime, we show that the pseudospectrum of zero-damping modes, which have slow decay rates, is only energetically unstable (with order unity fractional change in decay rates) with fined-tuned modification of wave potential. The decay rates are however always positive with energetically infinitesimal perturbations. If finite potential modifications are allowed near the black hole, it is possible to find superradiantly unstable modes, i.e., “black hole bomb” without an explicit outer shell. For the zero-damping modes in near extremal Reissner-Nordström-de Sitter black holes, which are relevant for the breakdown of Strong Cosmic Censorship, we find that the corresponding pseudospectrum is energetically stable.

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Introduction – Modal analysis of black hole spacetimes plays an important role in gravitational-wave astronomy as the quasinormal mode (QNM) excitation and ringdown contribution is a vital part of generic black hole perturbations, especially for the post-merger black holes. Black hole spectroscopy, i.e., measuring the frequencies and damping rates of QNMs, can be used to infer the black hole mass and spin, and to test General Relativity. So far the fundamental mode ($\ell = 2, m = 2$) has been convincingly detected in some of the LIGO events [1–3], including GW150914 [4]. There have been claims of detecting high-overtone modes as well [5], albeit concerns from independent analyses [6]. The detection of QNMs with higher $\ell$ generally requires higher event signal-to-noise ratio (SNR), which may benefit from coherently stacking multiple events [7].

It has been recently (re)-claimed that distant and/or short wavelength perturbation of the wave potential of QNMs may significantly change the mode frequencies, despite the perturbation amplitudes are infinitesimal. This observation was initially pointed out in Refs. [8, 9] and recently used to demonstrate the instability of QNM pseudospectrum of Kerr black holes [10]. We point out that according to the mathematical definition, the pseudospectrum of Kerr black hole is actually stable with previously constructed self-adjoint inner product for the mode wave functions. Instead, the observation in Ref. [10] should be re-interpreted as a type of energetic instability, which is defined for inner products preserving the physical meaning of energy.

It is crucial to examine the energetic pseudospectrum stability (EPS) for various mode analysis results. If the mode frequencies are significantly modified with perturbations in the potential of infinitesimal energy costs, associated claims need be treated with extra caution due to the susceptibility to external perturbers, or even internal variations due to nonlinearities. A robust claim should have converging measures with respect to small perturbations. In this work, we focus on the EPS of near extremal black holes, which general host a class of zero-damping modes (ZDMs) with slow damping rates. For different kind of background near extremal spacetimes, these modes have been used to demonstrate parametric nonlinear instability in connection with turbulence [11], possible breakdown of Strong Cosmic Censorship [12], and near-horizon critical behaviour that leads to the instability of extremal black holes [13, 14], as possibly connected to gravitational critical collapse [15]. Using the example of near extremal Kerr, we will discuss whether these modes will be unstable against small perturbations, and under what condition the mode pseudospectrum become unstable. We also study the ZDMs for near extremal Reissner-Nordström-de Sitter (RNdS) black holes, as an example for non-asymptotically flat spacetimes, and comment on whether the EPS affects the divergence on the Cauchy horizon. Throughout the analysis we adopt the natural unit that $G = c = 1$.

Pseudospectrum stability – It is argued in Ref. [10] that an eigenvalue perturbation analysis is not possible as the inner products for the QNM wave functions is not self-adjoint. This claim is used to explain the significant variations of high-overtone QNMs by small-amplitude yet high-frequency perturbations of the wave potential. Mathematically, for an eigenvalue problem with $\hat{H}(\omega_0)\psi_0 = 0$ and a perturbation in the potential: $\hat{H}(\omega) \rightarrow \hat{H}(\omega) + \epsilon \hat{V}$, we may formally expand the eigenfrequency and wave function as $\omega = \omega_0 + \epsilon \omega_1 + O(\epsilon^2)$ and $\psi = \psi_0 + \epsilon \psi_1 + O(\epsilon^2)$. With a self-adjoint inner product: $\langle \chi | \hat{H} | \eta \rangle = \langle \hat{H} \chi | \eta \rangle$, it can be shown that
\[ \omega_1 = -\frac{\langle \psi_0 | i \delta V | \psi_0 \rangle}{\langle \psi_0 | \partial_x \tilde{H} | \psi_0 \rangle}. \] (1)

Such construction is explicitly given in Ref. [16], with the inner product defined along the contour in the complex \( r \) plane. The corresponding eigenvalue analysis is applied for computing the QNM frequency of weakly charged Kerr-Newman black holes [17]. The high-frequency and the distant Gaussian-bump perturbations considered in Refs. [10, 18] can all be consistently treated within this formalism, although the normalization of these operators are dramatically different from those in Ref. [16], according to the different inner products defined. In fact, similar perturbation exercise has also been performed in Ref. [19] for a scalar field in Schwarzschild spacetime, where for a bump centred around \( x_0 \), the mode frequency perturbation is shown to scale as \( e^{2i\omega_0/x_0^2} \). Since \( \text{Im}(\omega_0) < 0 \), the importance of the potential perturbation is exponentially amplified by its distance \( x_0 \). Nevertheless the pseudospectrum of Kerr is stable following these definitions.

Despite not being self-adjoint, and hence not suitable for perturbative analysis, the inner product defined in Ref. [10] closely fits the intuitive expectation of the magnitude (or normalization) of a potential. The phenomena observed are physically relevant. To reconcile with the above discussion, we shall refer the analysis in Ref. [10] as “the energetic pseudospectrum” of QNMs to emphasize the special choice of inner product. For a generic mode analysis, it is necessary to address the EPS, as it characterizes the robustness of the mode spectrum and the associated implications, such as the mode stability and the Strong Cosmic Censorship.

**EPS of near extremal Kerr** – It is of particular interest and importance to study the EPS of near extremal black holes because of the existence of ZDMs, whose decay rates approaching zero in the extremal limit. For Kerr black holes, it has been shown that ZDMs start to emerge when the dimensionless spin of black hole \( a \) is greater than certain critical value [20]. We shall use Kerr spacetime as an example of asymptotically flat spacetimes to address the EPS.

Perturbations of a bosonic field in Kerr spacetime satisfy separable wave equations in the frequency domain, i.e., the Teukolsky equations [21]. Writing the Teukolsky master variable \( \psi = R(r)S(\theta)e^{im\phi-i\omega t} \), the radial Teukolsky equation, \( \mathcal{L}_\omega R = 0 \), reduces to

\[ R''(x) + \frac{2(s+1)}{x} R'(x) + \frac{\omega^2 (x+2) + 2i\omega \Omega - \lambda}{x^2} R(x) = 0 \] (2)

in the exterior regime where \( x \gg \sqrt{\epsilon_\omega} \). Here we have defined \( x \equiv r/M-1 \) with \( M \) being the black hole mass, and considered the near-extremal case such that \( \epsilon_\omega \equiv 1 - a/c < 1 \), while \( s \) is the spin weight of the bosonic field, and \( \lambda \equiv s A_{\text{ext}} + \omega^2 - 2i\omega \Omega \) with \( A_{\text{ext}} \) being the eigenvalue of the angular Teukolsky equation. The homogeneous solutions of Eq. (2) may be expressed as

\[ R(x \gg \sqrt{\epsilon_\omega}) = AF_+(x) + BF_-(x) \] (3)

with

\[ F_\pm(x) = e^{-i\omega x} x^{-1/2-\pm i\eta} _1F_1 \left( \frac{1}{2} + s \pm i\eta + 2i\omega, 1 \pm 2i\eta, 2i\omega x \right), \]

where \( _1F_1(z) \) is the confluent hypergeometric function and \( \eta^2 \equiv 7m^2/4 - (s + 1/2)^2 - s A_{\text{ext}} \). Imposing the outgoing boundary condition at infinity leads to the physical solution

\[ R_o \left( x \gg \sqrt{\epsilon_\omega} \right) = A_o F_+(x) + B_o F_-(x) \] (4)

with

\[ \frac{A_o}{B_o} = e^{\pi^2 + 2i\omega \log 2}\frac{\Gamma(-2i\omega)(1/2 - s + i\omega - 2i\omega)}{\Gamma(2i\omega)(1/2 - s - i\omega - 2i\omega)}. \] (5)

Now we shall consider the perturbed radial Teukolsky equation

\[ \mathcal{L}_\omega R + \epsilon \delta V(x) R = 0 \] (6)

where the support of \( \delta V(x) \) is compact in the exterior regime. The presence of perturbations modifies the radial function. To the first order in \( \epsilon \), we have \( R = \tilde{R} + O(\epsilon^2) \) with

\[ \mathcal{L}_\omega \tilde{R} = -i\delta V(x)R_0. \] (7)

To evaluate the modified radial function \( \tilde{R} \), let us consider a Green function \( g(x, x') \) which satisfies

\[ \mathcal{L}_\omega g(x, x') = \delta(x - x') \] (8)

and out-going boundary condition at infinity. For \( x \neq x' \), the Green function \( g(x, x') \) in the exterior regime can be described by the homogenous solution (3). For \( x > x' \) the coefficients \( A \) and \( B \) are \( A_0 \) and \( B_0 \) satisfying Eq. (5) given the out-going boundary condition, while for \( x < x' \) the coefficients, say \( A_m \) and \( B_m \), can be obtained by matching \( g(x, x') \) at \( x = x' \). In particular, according to Eq. (8), the value of \( g \) is continuous at \( x = x' \), leading to

\[ A_m F_+(x') + B_m F_-(x') = A_o F_+(x') + B_o F_-(x') \] (9)

and the derivative of \( g' \) satisfies

\[ A_in F'_+(x') + B_in F'_-(x') + 1 = A_o F'_+(x') + B_o F'_-(x'). \] (10)

Hence, we have

\[ A_m = A_o - \frac{1}{W(x')} \quad \text{and} \quad B_m = B_o + \frac{1}{W(x')}, \] (11)

where \( W(x) \equiv F_+(x')F'_-(x) - F_-(x')F'_+(x) \) is the Wronskian of the two homogeneous solutions. The modified radial function, using the Green’s function, is

\[ \tilde{R} = R_o(x) - \epsilon \int \, dx' g(x, x') \delta V(x')R_0(x'). \] (12)

For \( x \) on the right side of the support of \( \delta V, \tilde{R} \) is

\[ \tilde{R}_o \equiv [1 - \epsilon a] \times R_o(x), \] (13)
and for $x$ on the left side of the support of $δV$, $\tilde{R}$ is
\[ \tilde{R}_c \equiv R_\circ + \epsilon β [F_+(x) - F_-(x)] \]  
where
\[ α \equiv \int dx' δV(x') R_\circ(x') \quad β \equiv \int dx' δV(x') \frac{R_\circ(x')}{W(x')} . \]  

The radial Teukolsky equation in the interior regime where $x ≪ 1$ can also be simplified, and the solutions can be expressed by hypergeometric functions. A detailed derivation can be found in Ref. [22]. In the intermediate regime where
\[ x \ll 1 \text{ but } x \gg \sqrt{e_o}, \]  
both the interior and exterior solutions can be expressed as linear combinations of $x^{-1/2 + iδ}$ and $x^{-1/2 - iδ}$. For example, the modified radial function behaves like
\[ \tilde{R}_c \left( \sqrt{e_o} \ll x \ll 1 \right) \rightarrow (A_0 - εcA_α + εβ) x^{-1/2 + s + iδ} + (1 - εc - εβ) x^{-1/2 - s - iδ} , \]
where we have set $R_0 = 1$ without loss of generality. The interior and exterior solutions can be matched in the intermediate regime by identifying the ratio of the coefficients of $x^{-1/2 + s + iδ}$ and $x^{-1/2 - s - iδ}$, leading to the equation for the QNM frequency,
\[ e^{-\delta - 2iδ \ln(m - iδ \ln(8e_e^e)} \frac{\Gamma^2(2iδ)Γ(1/2 + s - im - iδ)Γ(1/2 - s - im + iδ)Γ[1/2 + im + δ - \sqrt{2ω}]}{\Gamma^2(-2iδ)Γ(1/2 + s + im + iδ)Γ(1/2 - s - im + iδ)Γ[1/2 + im + δ + \sqrt{2ω}]} = Π , \]
where $Π$ is defined by the ratio from the exterior solution
\[ A_0 Π ≡ A_0 \frac{1 - εcα + εβ}{1 - εcα - εβ} \]  
and $ω ≡ (ω - mΩ_H)/\sqrt{e_o}$ with $Ω_H$ being the horizon frequency.

In the case of unperturbed Kerr spacetime, we have $Π = 1$, and the frequency of the ZDNMs is obtained by noticing that $Γ[1/2 + i(m - δ - \sqrt{2ω})]$ is near its pole [22], as the rest of terms multiplied together is rather small. It is therefore instructive to look at the magnitude of $Π$ in more detail. Let us consider a narrow gaussian centered around $x_ν$ with $\int dx δV = 1$. Such type of perturbation is capable of significantly shifting the fundamental mode of Kerr for generic spins. Correspondingly, we have
\[ α ≃ R_\circ(x_ν) \quad \text{and} \quad β ≃ R_\circ(x_ν)/W(x_ν) . \]  

The dominant amplitude growth in $R_\circ$ comes from the $e^{imx}$ factor. In the case that $\text{Re}(imx) ≪ O(\log 1/ε)$, $x ≪ O(1/\sqrt{e_o} \log 1/ε)$, the $eR_\circ(x)$ term is small, so that $Π$ is close to one. In order to have significant impact on $ω$, we need $Π ≃ 0$. This can only be achieved by rather fine-tuned $x_ν$ and $ε$. We will now focus on this regime, and in particular, investigate whether it possibly allows unstable modes. The requirement becomes
\[ R_\circ(x_ν) \left[ 1 - \frac{1}{W(x_ν)A_0} \right] \approx \frac{1}{ε} . \]  

If the decay rate is positive, for any infinitesimal $ε$, we can also find $x_ν$ such that the solution exists (as illustrated in Fig. 1), although the values are rather fine tuned as $R_\circ$ is oscillatory. If the decay rate is negative or zero, $R_\circ(x_ν)$ (notice that $R_\circ \sim e^{imx}/\sqrt{x^{1/2 + iδ} - 2imω}$ for $x \gg 1$) and the LHS of the above equation are bounded for all $x_ν$, so unstable modes are not allowed for infinitesimal $ε$. In fact, for more general $δV$, if the mode decay rate is positive, the corresponding $α, β$ are all bounded as $R_\circ$ is bounded, so that $Π ≃ 0$ can not be realized by infinitesimal $ε$, i.e., ZDNMs are always stable by energetically infinitesimal perturbations.

However, for finite (and large enough) $ε$ it seems we still can find $x_ν$ to have the above equation satisfied, so that the unstable mode is allowed. For example, for scalar modes with $(ℓ, m) = (2, 2)$ and a $δ$-type potential $εδ(x - x_ν)$, we find $(x_ν/M, ε_ν) ≈ (7.26, -0.0178)$ gives rise to $ω = mΩ_H$. For modes with modified frequencies and decay rates, the corresponding fined-tuned position and amplitude are shown in Fig. 2. Part of the modes there are unstable. This kind of realization is different from putting a reflective boundary on finite radius that has been proposed, to make a black hole bomb [23, 24]. It will be interesting to investigate whether the unstable mode similar to the one shown in Fig. 2 is still allowed with the black hole spin away from unity, for small but not infinitesimal perturbations.

**EPS of near extremal RNdS** – A RNdS black hole possesses three horizons: the Cauchy horizon, event horizon and cosmological horizon, from inside to outside, locating at $r_-$, $r_+$ and $r_\circ$ respectively. Considering a minimally coupled scalar field on a RNdS background, Ref. [12] shows that the mode spectrum of the scalar field allows universal “fast enough” decay rates, so that the initial data is sufficiently regular on the Cauchy horizon to violate the Strong Cosmic Censorship. Note that the non-asymptotic-flatness is crucial for the argument here, as the spacetime is free of the power-law tail that introduces the mass-inflation effect [25]. In the near-extremal regime ($τ ≡ (r_+ - r_-)/r_+ \rightarrow 0$), the mode with slowest decay rate, namely the dominating mode, belongs to a set of purely decaying modes. The frequencies of such modes are approximately
\[ ω_{NE} \rightarrow -(l + n + 1)κ_ν , \]
independent of $r_\circ$ [12]. Here $κ_ν$ is the surface gravity at the event horizon. As the magnitude of $ω_{NE}$ decides the regularity of the Cauchy horizon, it is important to show whether the values are susceptible to small perturbations of the spacetime.
from horizons, so that potential perturbations that possibly affect the EPS can only appear near the cosmological horizon, which we focus on below.

In the region near the cosmological horizon, the radial function of the dominating mode can be described by

$$R(x \to 1/c) = A_\alpha F_\alpha(x) + B_\alpha F_-(-x)$$

with

$$F_\alpha(x) = \Gamma(1 \pm 2iw)J_{\pm2iw}\left(\frac{2}{\sqrt{1-c^2x^2}}\right).$$

Here $J_{\pm2iw}$ is the Bessel function of the first kind, and we define $x \equiv r/r_+$. For near-extremal RN black holes, this is in agreement with Ref. [26].

$$j^2 \equiv \left(\frac{(1 + 4c + 6c^2)}{2 + 4c}\right)\ell(\ell + 1) \quad \text{and} \quad w \equiv \frac{\omega}{2\kappa_c}$$

with $\kappa_c$ being the surface gravity at $r_c$. The outgoing boundary condition indicates $A_\alpha/B_\alpha = 0$ in the absence of perturbations.

In the presence of a potential perturbation $\delta V = \epsilon \delta(x - x_f)$, the corrections on $A_\alpha/B_\alpha$ can be obtained in a similar way as in the Kerr case, c.f. Eq. (18),

$$\frac{\hat{A}_\alpha}{\hat{B}_\alpha} = \frac{(1 - \epsilon\alpha)A_\alpha - \epsilon\beta}{(1 + \epsilon\alpha)B_\alpha + \epsilon\beta} \approx A_\alpha/B_\alpha + \epsilon\frac{A_\alpha + B_\alpha R_\ast(x_f)}{B_\alpha} \frac{\omega}{W(x_f)}.$$  

In particular, we find that the correction on $A_\alpha/B_\alpha$ is of the order of $\epsilon(1 - c x_f)^{1+iw/2c}$ (see App. A for more details). Considering $\omega \sim \tau$, we conclude that the corrections on $A_\alpha/B_\alpha$ due to infinitesimal perturbations in the wave potential is bounded.

To further understand the effects of perturbations on the QNM frequencies, we need to address how the frequency changes with $A_\alpha/B_\alpha$. In practice, we numerically solve the radial wave equation of different $\omega$ with in-going boundary condition at the event horizon, and then extract $A_\alpha/B_\alpha$ by matching the numerical solution with Eq. (22) near the cosmological horizon. Sample numerical results are shown in Fig. 3, where we introduce $\gamma = (\omega - \omega_{NE})|\omega_{NE}|$ to describe the relative deviation from $\omega_{NE}$, and the contours show the absolute value of $A_\alpha/B_\alpha$. As $A_\alpha/B_\alpha = 0$ in the absence of perturbation, it should correspond to $\gamma = 0$ up to numerical errors. As shown in Fig. 3, the frequency of ZDMs is stable against infinitesimal perturbations. This is especially case for relatively larger $\Lambda$. For small $\Lambda$, it is possible to fine tune the perturbation so that it can modify the frequency at a relatively small cost. Namely, by fine tuning the perturbations, the frequency could drift along the blue “valley” in the left plot in Fig. 3. The cost becomes smaller as $\Lambda$ decreases, and eventually we expect the frequency becomes unstable in the limit of $\Lambda \to 0$. This pattern is generic. Actually, the position of the valley can be numerically fitted by $Im\gamma = p(Re\gamma)^2 + q$, where $q$ is tiny and affected only by numerical errors. As shown in Fig. 4, $p$ increases as $\Lambda$ decreases, and we expect it approaches $-\infty$ as $\Lambda$ goes to zero.

In fact, the QNM frequencies in the limit of $\Lambda \to 0$, i.e. the near-extremal RN black hole, have been studied in Ref. [26]. Following a similar matched expansion analysis as in the Kerr case and in Ref. [26], we find that the corrections on $A_\alpha/B_\alpha$ is proportional to $x_V^{-2i\omega \tau}e^{i\omega \tau} x_V$, which is unbound as $x_V$ goes to infinity. Therefore, even with an infinitesimal perturbation in potential (maybe fine tuned), the frequencies of the ZDMs can change significantly.
Regarding the cosmic censorship, our results also show that infinitesimal perturbations only infinitesimally modify the frequencies. We expect the decay rate of the scalar perturbations on the extremal RNdS spacetime is still sufficiently fast for the violation of the strong cosmic censorship [12] even in the presence of perturbations. Fig. 4 actually indicates that the decay rates tend to be larger with potential perturbations.

Discussion—The analysis for perturbations in near-extremal Kerr and RNdS background spacetimes indicates that the pseudospectrum is energetically stable for near-extremal spacetimes with nonzero cosmological constants, and unstable with respect to fine-tuned small perturbations if the cosmological constant is zero. It is not explicitly shown but nevertheless reasonable to expect that this applies for general Kerr-Newmann-type black holes. On the other hand, it seems the infinitesimal perturbations on the potential does not change the stability of modes, nor does them influence the robustness of the claims regarding the Strong Cosmic Censorship. Of course these claims ought to be explicitly checked in the relevant spacetimes.

It will also be interesting to investigate the shift of time domain signals in response to an small modification in the potential. The zero-damping modes of extremal spacetimes are known to excite collectively by external sources [22], giving rise to (transient) power-law signals. In the presence of potential perturbations, the time domain signal may exhibit other intriguing behavior beyond the understanding for individual modes.

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Appendix A: QNMs of a massless scalar field in the near-extremal RN(dS) spacetime

In this Appendix, we discuss the QNMs of a massless scalar field in the near-extremal RN(dS) spacetime. The metric of the RNdS spacetime can be written as

\[ \text{d}x^2 = -f(r) \text{d}t^2 + \frac{1}{f(r)} \text{d}r^2 + r^2 \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right) \quad (A1) \]

where

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} \quad (A2) \]

with \( M, Q \) and \( \Lambda \) are the BH mass, BH charge and the cosmological constant, while the metric RN spacetime is given by Eq. (A1) with \( \Lambda = 0 \). A RNdS black hole possesses three horizons. From inside to outside, they are the Cauchy horizon, event horizon and cosmological horizon, and are located at \( r_-, r_+ \) and \( r_c \) respectively. We introduce \( \kappa_{a,c} = |f'(r_{a,c})/2| \) to denote the surface gravity of each horizon.

The equation of the massless scalar field is

\[ \Box \Phi = 0 \quad (A3) \]

where \( \Box \) is the d’Alembert operator on the RNdS spacetime. Substituting

\[ \Phi = \int d\omega \sum_{lm} e^{-im\phi} R_{a lm}(r) Y_{lm}(\theta) e^{im\phi} \quad (A4) \]

into Eq. (A3) leads to the equation of radial function

\[ f(r) \frac{d}{dr} \left[ r^2 f'(r) \frac{dR}{dr} \right] + U(r) R = 0. \quad (A5) \]

where

\[ U(r) = \omega^2 r^2 - \ell(\ell + 1) f(r). \quad (A6) \]

1. RN spacetime

We shall start with the near-extremal RN spacetime by closely following and summarizing some results of Ref. [26], where the QNMs of a scalar field on the near-extremal RN spacetime has been studied. Introducing the dimensionless parameters

\[ x \equiv \frac{r - r_+}{r_+}, \quad \tau \equiv \frac{r_+ - r_-}{r_+}, \quad k \equiv 2\omega r_+, \quad \sigma \equiv \frac{k}{\tau}, \quad (A7) \]

and

\[ \beta^2 = (\ell + 1/2)^2 - k^2. \quad (A8) \]

Eq. (A5) can be written as

\[ x(x + \tau) \frac{d^2 R}{dx^2} + (2x + \tau) \frac{dR}{dx} + U(x) R = 0 \quad (A9) \]

where

\[ U(x) = \left( \frac{\omega r_+ x^2 + kx + \sigma \tau/2}{x(x + \tau)} \right)^2 - \ell(\ell + 1). \quad (A10) \]

We assume \( 0 < \beta \in \mathbb{R} \).

In the near region, i.e. \( x \ll 1 \), we have \( U \rightarrow (kx + \sigma \tau/2)^2 / [x(x + \tau)] - \ell(\ell + 1) \), and the physical solution, i.e. the one satisfies the in-going boundary condition, is

\[ R(x) = x^{-i\frac{\beta}{\tau}} \left( \frac{x}{\tau} + 1 \right)^{i\frac{\beta}{\tau} - ik} 2F_1 \left( \frac{1}{2} - \beta - i\sigma; \frac{1}{2} + \beta - ik; 1 - i\tau; \frac{x}{\tau} \right), \quad (A11) \]

where \( 2F_1(a; b; c; z) \) is the hypergeometric function. In the intermediate region, \( \tau \times \text{max}(1, \sigma) \ll x \ll 1 \), we have Eq. (A11) reducing to

\[ R(x) = \frac{\Gamma(1 - i\sigma)\Gamma(-2\beta\tau^{1/2} - i\sigma/2)}{\Gamma(1/2 - \beta - i\sigma)\Gamma(1/2 - \beta + i\sigma) x^{-\beta}} + (\beta \rightarrow -\beta). \quad (A12) \]

In the far region, i.e. \( x \ll \tau \times \text{max}(1, \sigma) \), Eq. (A9) reduces to

\[ x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + \left[ (\omega r_+ + k)^2 - \ell(\ell + 1) \right] R = 0, \quad (A13) \]

the solution of which can be written as

\[ R(x) = N_1 \times (2\nu)^{\beta/2} x^{-\beta - 2} e^{-2\nu x} 1F_1 \left( \frac{1}{2} - \beta - ik; 1 - 2\beta, 2\nu x \right) + N_2 \times (\beta \rightarrow -\beta), \quad (A14) \]
where $\nu \equiv i \omega r_+$. In the intermediate region, the far region solution Eq. (A14) reduces to
\[ R(x) = N_1 \times (2\nu)^{1+\beta} x^{-\beta} + N_2 \times (\beta \rightarrow -\beta). \quad (A15) \]
Matching Eq. (A15) with Eq. (A12) in the overlap region, we find that
\[ N_1(\beta) = \frac{\Gamma(1-i\sigma) \Gamma(-2\beta)}{\Gamma(1/2 - \beta - ik) \Gamma(1/2 - \beta + ik - i\sigma)} (2\nu)^{1+\beta} \tau^{1+\beta-
u}, \]
\[ N_2(\beta) = N_1(-\beta). \quad (A16) \]
Therefore, the solution at spatial infinity approaches
\[ R(x \rightarrow \infty) \rightarrow \left[ N_1(\beta) \times (2\nu)^{-ik} \Gamma(1-2\beta) \frac{\Gamma(1/2 - \beta - ik)}{\Gamma(1/2 - \beta + ik)} \right] x^{-1-ik} e^{\nu x} \]
\[ + \left[ N_1(\beta) \times (2\nu)^{ik} \Gamma(1-2\beta) \frac{\Gamma(1/2 - \beta + ik)}{\Gamma(1/2 - \beta - ik)} \right] x^{1+ik} e^{-\nu x}. \quad (A17) \]
The outgoing boundary condition requires $R(x \rightarrow \infty) \rightarrow e^{\nu x}$, thus
\[ N_1(\beta) \times (2\nu)^{ik} \Gamma(1-2\beta) \frac{\Gamma(1/2 - \beta + ik)}{\Gamma(1/2 - \beta - ik)} (-1)^{-1-\beta-ik} + (\beta \rightarrow -\beta) = 0, \quad (A18) \]
which determined the frequencies of the QNMs.
As we explained in the main text, a small perturbations in the potential effectively modified the boundary condition. In particular, we showed that for a potential perturbation $\delta V = \epsilon \delta(x - x_V)$, the boundary condition to the leading order in $\epsilon$ is given by
\[ N_1(\beta) \times (2\nu)^{ik} \Gamma(1-2\beta) \frac{\Gamma(1/2 - \beta + ik)}{\Gamma(1/2 - \beta - ik)} (-1)^{-1-\beta+ik} + (\beta \rightarrow -\beta) = \epsilon \Delta(x_V), \quad (A19) \]
where
\[ \Delta(x_V) = \frac{F_-(x_V)}{F_+(x_V) F'_+(x_V) - F_-(x_V) F'_-(x_V)}. \quad (A20) \]
As $x \rightarrow \infty$, we have $F_+(x) \rightarrow x^{1+ik} e^{\nu x}$ and hence
\[ \Delta(x \rightarrow \infty) \propto x^{1-2i\omega r_+} e^{i\omega r_+ x}. \quad (A21) \]
Eq. (A19) can be rewritten as
\[ \left[ \frac{\Gamma(2\beta)}{\Gamma(-2\beta)} \right] \frac{\Gamma(\frac{1}{2} - \beta - ik)}{\Gamma(\frac{1}{2} + \beta - ik)} \frac{\Gamma(\frac{1}{2} - \beta + ik - i\sigma)}{\Gamma(\frac{1}{2} + \beta + ik + i\sigma)} \]
\[ = (2\nu)^{2\beta} \left[ 1 + \frac{\epsilon \Delta(x_V)}{N_1(\beta) \times (2\nu)^{ik} \Gamma(1/2 + \beta + ik)} \frac{\Gamma(1/2 + \beta - ik)}{\Gamma(1/2 - \beta + ik)} \right]. \quad (A22) \]
In the absence of the perturbation, we can set $\epsilon = 0$ in Eq. (A22). Since $\tau \ll 1$ in the near-extremal case, we find the frequency is near the poles of $\Gamma\left(\frac{1}{2} + \beta + ik - i\sigma\right)$, namely
\[ \omega = -i(\ell + n + 1)\kappa_+^c, \quad (A23) \]
where $n = 0, 1, 2, \ldots$. While in the presence of the potential perturbation, we find that $|\Delta(x)|$ is unbounded as $x$ approaches to infinity.

2. RNDS spacetime

For RNDS spacetime, we further define
\[ c = \frac{r_+}{r_c - r_+}, \quad (A24) \]
and Eq. (A5) can be written as
\[ C_2 R''(x) + C_1 R'(x) + C_0 R(x) = 0 \quad (A25) \]
where $\nu$ denotes the derivative with respect to $x$, and
\[ C_2 = x(x + \tau)(1 - cx)[1 + c(x + 4 + \tau)] \]
\[ C_1 = -4c^2 x^3 - 12c^2 x^2 + 2\left[1 + c(4 - \tau) - c^2(4 - \tau)\right] x \]
\[ + \tau + c(4 - \tau) \tau \]
\[ C_0 = \left[1 + c(4 - \tau) + c^2(6 - 4\tau + \tau^2)\right] \left\{ (\ell + 1) + \omega^2 r_+^2 (1 + x)^2 \left[1 + c(4 - \tau) + c^2(6 - 4\tau + \tau^2)\right] \right\}. \quad (A26) \]
In the extremal limit, we have $\tau \rightarrow 0$. In the regime where $1 - cx \ll 1$, Eq. (A26) becomes
\[ (1 - cx) R''(x) - c R'(x) + \left( c^2 \beta^2 + \frac{c^2 w^2}{1 - cx} \right) R(x) = 0, \quad (A27) \]
where
\[ \beta^2 = \frac{(1 + 4c + 6c^2)}{2 + 4c}, \quad w = \frac{\omega}{2\kappa_c}, \quad (A28) \]
\[ \kappa_c = \frac{c(1 + 2c)}{(1 + c)^2(1 + 4c + 6c^2)r_+}. \]
The general solution in this region can be written as
\[ R(x \rightarrow 1/c) = A_0 \Gamma(1-2iw)J_{-2iw}(2\sqrt{1/c - cx}) + B_0 \Gamma(1+2iw)J_{2iw}(2\sqrt{1/c - cx}) \quad (A29) \]
where $J_\nu(z)$ is the Bessel function of the first kind. The outgoing boundary condition indicates $A_0 = 0$. In the presence of the infinitesimal potential perturbation, the correction on $A_0/B_0$ is proportional to $\epsilon(1-cx)\nu^{1+iw}$. In the near-extremal limit, we have $\omega \propto \tau/c$, while $\tau \rightarrow 0$. Hence, unless $c$ goes to zero faster than $\tau$, e.g., in the case of RN black hole, we find that the correction on $A_0/B_0$ is bounded as $x_V \rightarrow 1/c$. 
