Analysis of the maximal posterior partition in the Dirichlet Process Gaussian Mixture Model

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Abstract
Mixture models are a natural choice in many applications, but it can be difficult to place an a priori upper bound on the number of components. To circumvent this, investigators are turning increasingly to Dirichlet process mixture models (DPMMs). It is therefore important to develop an understanding of the strengths and weaknesses of this approach. This work considers the MAP (maximum a posteriori) clustering for the Gaussian DPMM (where the cluster means have Gaussian distribution and, for each cluster, the observations within the cluster have Gaussian distribution). Some desirable properties of the MAP partition are proved: ‘almost disjointness’ of the convex hulls of clusters (they may have at most one point in common) and (with natural assumptions) the comparability of sizes of those clusters that intersect any fixed ball with the number of observations (as the latter goes to infinity). Consequently, the number of such clusters remains bounded. Furthermore, if the data arises from independent identically distributed sampling from a given distribution with bounded support then the asymptotic MAP partition of the observation space maximises a function which has a straightforward expression, which depends only on the within-group covariance parameter. As the operator norm of this covariance parameter decreases, the number of clusters in the MAP partition becomes arbitrarily large, which may lead to the overestimation of the number of mixture components.

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1 Introduction

1.1 Motivation and new contributions
Clustering is a central task in statistical data analysis. A Bayesian approach is to model data as coming from a random mixture of distributions and de-
rive the posterior distribution on the space of possible divisions into clusters. When there is not a natural a priori upper bound on the number of clusters, an increasingly popular technique to use is Dirichlet Process Mixture Models (DP-MMs). It is therefore important to develop an understanding of the strengths and weaknesses of this approach.

Miller and Harrison (2014) restrict attention to the number of clusters produced by such a procedure and are somewhat critical of the method. Their main result implies that in a very general setting, the Bayesian posterior estimate of the number of clusters is not consistent, in the sense that for any \( t \in \{1, 2, \ldots\} \) almost surely

\[
\limsup_{n \to \infty} \mathbb{P}(T_n = t | X_1, \ldots, X_n) < 1,
\]

where \( X_1, X_2, \ldots \) is an i.i.d. sample from a mixture with \( t \) components and \( T_n \) denotes the number of clusters to which the data are assigned. Here \( \mathbb{P} \) is the probability in the probability space on which \( X_1, X_2, \ldots \) are defined.

The Miller and Harrison inconsistency result relates to estimation of the number of clusters, not the classification itself. While they do not pursue this, they do provide examples of the structure estimators, such as the MAP (maximal a posteriori) partition, which maximises the posterior probability and the mean partition, introduced in Huelsenbeck and Andolfatto (2007), which minimises the sum of the squared distance between the mean partition and all partitions sampled by the MCMC algorithm which they run, where the distance is the minimum number of individuals that have to be deleted from both partitions to make them the same.

This article presents developments that concern the properties of the MAP estimator in a Gaussian mixture model, where the cluster centres are generated according to a Gaussian distribution and, conditioned on the cluster centre, the observations within a cluster are generated by Gaussian distribution. The clusters are generated according to a Dirichlet Process. Analysing the MAP partition seems to be a natural choice. It is listed, for example, in Fritsch et al. (2009) as an established method. Of course, the set of possible candidates for the maximiser has to be restricted, since the space of partitions is too large for an exhaustive search. For example, Dahl (2006) suggests choosing the MAP estimator from a sample from the posterior. He notes, however, a potential problem of this approach; there may be only a small difference in the posterior probability between two significantly different partitions. This may indicate that the classifier is giving the wrong answer as a consequence of mis-specification of the within-cluster covariance parameter. We investigate such instability in our examples.

The conclusions of our analysis may be summarised as follows:

1. The convex hulls of the clusters are pairwise ‘almost disjoint’ (they may have at most one point in common, which must be a data point).
2. The clusters are of reasonable size; if \( \left( \frac{1}{n} \sum_{j=1}^{n} \| x_j \|^2 \right)_{n=1}^{\infty} \) (the sequence of means of squared Euclidean norms) is bounded, then \( \liminf_{n \to \infty} \frac{m_{[r]}}{n} > 0 \)
for any $r > 0$, where $m^{[r]}_n$ denotes the number of observations in the smallest cluster (in the MAP partition of the first $n$ observations) with non-empty intersection with $B(0, r)$ (the ball of radius $r$, centred at the origin).

3. This implies that for any $r > 0$ the number of clusters in the $n$-th MAP partition required to cover observations inside $B(0, r)$ remains bounded as $n \to \infty$.

4. When the data sequence comes from an i.i.d. sampling with bounded support there is an elegant formula to describe the limit of the MAP clustering; it is the partition of the observation space that maximises the function $\Delta$ given by Equation (2.3). In general, though it is a hard problem to find the global maximiser for this expression. Furthermore, the only parameter that this function depends on is the within-group covariance parameter.

5. The negative finding of the paper is that the clustering is very sensitive to the specification of the within-cluster variance and model mis-specification can lead to very misleading clustering. For example, if the data is i.i.d. from an input distribution which is uniform over a ball of radius $r$ in $\mathbb{R}^2$ and the within-cluster variance parameter is $\sigma^2 I$, then for small $\sigma$, the classifier partitions the ball into several, seemingly arbitrary, convex sets. This classifier therefore has to be treated with caution.

1.2 Organisation of the article

We now present a brief overview of the structure of the paper. In Section 2 we give key definitions and provide complete mathematical statements of the main results together with intuitive explanations. Section 3 presents examples which illustrate the results obtained in the article. These examples show the MAP clustering obtained in various situations where the data comes from i.i.d. sampling. They indicate that this procedure may fail to produce reasonable output. The examples are supported by numerical simulations, which are described in Supplement B. Section 4 contains a detailed presentation of the asymptotic proposition together with some related developments. In Section 5 we state the open problems and plans for future work.

2 Main results

2.1 The Model

This section presents definitions of fundamental notions of our considerations together with some of their basic properties and relevant formulas. We show how they can be used to construct a statistical model in which we expect the data to be generated from different sources of randomness, without an a priori upper
bound on the number of these sources a priori. We start with the definition of
the Dirichlet Process, formally introduced in Ferguson (1973).

**Definition.** Let $\Omega$ be a space and $\mathcal{F}$ a $\sigma$-field of its subsets. Let $\alpha > 0$ and $G_0$ be a probability measure on $(\Omega, \mathcal{F})$. The **Dirichlet Process** on $\Omega$ with parameters $\alpha$ and $G_0$ is a stochastic process $(G(A))_{A \in \mathcal{F}}$ such that for every finite partition $\{A_1, \ldots, A_p\} \subseteq \mathcal{F}$ of $\Omega$ the random vector $(G(A_1), \ldots, G(A_p))$ has Dirichlet distribution with parameters $\alpha G_0(A_1), \ldots, \alpha G_0(A_p)$. In this case we write $G \sim \text{DP}(\alpha, G_0)$.

As considered in Antoniak (1974), the Dirichlet Process can be used to construct a mixture model in which the number of clusters is not known a priori. The details are given in the following definition.

**Definition.** Let $(\Theta, \mathcal{F})$ be the parameter space and $(X, \mathcal{B})$ the observation space. Let $\alpha > 0$ and $G_0$ be a probability measure on $(X, \mathcal{F})$. Let $\{F_\theta\}_{\theta \in \Theta}$ be a family of probability distributions on $(X, \mathcal{B})$. The **Dirichlet Process mixture model** is defined by the following scheme for generating a random sample from the space $(X, \mathcal{F})$

\[
G \sim \text{DP}(\alpha, G_0)
\]

\[
\theta = (\theta_1, \ldots, \theta_n) \mid G \overset{iid}{\sim} G
\]

\[
x_i \mid \theta, G \sim F_{\theta_i} \text{ independently for } i \leq n.
\]

In Blackwell and MacQueen (1973) it is shown that the first two stages of (2.1) may be replaced by the following recursive procedure:

\[
\theta_1 \sim G_0, \quad \theta_i \mid \theta_1, \ldots, \theta_{i-1} \sim \frac{\alpha}{\alpha + i - 1} G_0 + \sum_{j=1}^{i-1} \frac{1}{\alpha + i - 1} \delta_{\theta_j},
\]

where $\delta_\theta$ is the probability measure that assigns probability 1 to the singleton $\{\theta\}$. Of course, this procedure can be used to generate sequences of arbitrary length: the distribution of the resulting infinite sequence $(\theta_i)_{i=1}^{\infty}$ produced in this way is called the **Hoppe urn scheme**. Note that a realisation of this scheme defines a partition of $\mathbb{N}$ by a natural equivalence relation $(i \sim j) \equiv (\theta_i = \theta_j)$. Restriction of this partition to sets $[n]$ for $n \in \mathbb{N}$ is called the **Chinese Restaurant Process**.

**Definition.** The **Chinese Restaurant Process** with parameter $\alpha$ is a sequence of random partitions $(J_n)_{n \in \mathbb{N}}$, where $J_n$ is a partition of $[n] = \{1, 2, \ldots, n\}$, that satisfies

\[
J_{n+1} \mid J_n = \{J_1, \ldots, J_k\} \sim \left\{
\begin{array}{ll}
\{J_1, \ldots, J_i \cup \{n+1\}, \ldots, J_k\} & \text{with probability } \frac{|J_i|}{\alpha + \alpha} \\
\{J_1, \ldots, J_k, \{n+1\}\} & \text{with probability } \frac{1}{\alpha + \alpha}
\end{array}
\right.
\]

We write $J_n \sim \text{CRP}(\alpha)_n$. 

4
The Dirichlet Process mixture model for \( n \) observations is therefore equivalent to

\[
\mathcal{J} \sim \text{CRP}(\alpha)_n \\
\theta = (\theta_J)_{J \in \mathcal{J}} | J \sim \text{iid } G_0 \\
x_J = (x_{jJ})_{J \in \mathcal{J}} | J \sim \text{iid } F_{\theta} \quad \text{for } J \in \mathcal{J}.
\]

We will refer to this formulation as the \textit{CRP-based model}. In this paper we focus our attention on the Gaussian case, in which \( \Theta = \mathbb{R}^d \), \( X = \mathbb{R}^d \), \( F \) and \( B \) are \( \sigma \)-fields of Borel sets, \( G_0 = N(\mu, T) \) and \( F_{\theta} = N(\theta, \Sigma) \) for \( \theta \in \Theta \), where \( \mu \in \mathbb{R}^d \) and \( T, \Sigma \in \mathbb{R}^{d,d} \) are the parameters of the model. This will be called the \textit{CRP-based Gaussian model}. We also limit ourselves to the case where \( \mu = 0 \), however it may be easily seen that this is not a real restriction; the sampling from the zero-mean Gaussian model and transposing the output by the vector \( \mu \) is equivalent to sampling from the Gaussian model with mean \( \mu \). Therefore all the clustering properties of the model can be investigated with the assumption that \( \mu = 0 \).

\textbf{Remark 2.1.} The conditional probability of partition \( \mathcal{J} \) in the zero-mean Gaussian model, given the observation vector \( x = (x_j)_{j=1}^n \), is proportional to

\[
C^{\mathcal{J}} \prod_{J \in \mathcal{J}} \frac{|J|!}{|J|(d+2)/2 \det R_{|J|}} \cdot \exp \left\{ \frac{1}{2} \sum_{J \in \mathcal{J}} |J| \cdot \| R_{|J|}^{-1} R_{|J|} x_J \|^2 \right\} =: Q_x(\mathcal{J}) \tag{2.2}
\]

where \( C = \alpha / \sqrt{\det T} \), \( R = \Sigma^{-1} \), \( R_m = (\Sigma^{-1} + T^{-1}/m)^{1/2} \) for \( m \in \mathbb{N} \), \( \| \cdot \| \) is the standard Euclidean norm in \( \mathbb{R}^d \) and \( x_J = \frac{1}{|J|} \sum_{j \in J} x_j \).

\textbf{Proof.} See Supplement A.

Having established the model we are now able to use it for inference about the data structure. A natural choice is to choose the partition that maximises the posterior probability given by (2.2). This leads to the notion of the MAP partition.

\textbf{Definition.} The \textit{maximal a posteriori} (MAP) partition of \([n]\) with observed \( x = (x_j)_{j=1}^n \) is any partition of \([n]\) that maximises \( Q_x(\cdot) \) (equivalently, the posterior probability). We denote a maximiser by \( \hat{\mathcal{J}}(x) \) (note: a priori this may not be unique).

\section{2.2 Results}

The first result is \textbf{Proposition 1} which states that the MAP partition divides the data into clusters whose convex hulls are disjoint, with the possible exception of one datum.

\textbf{Proposition 1.} For every \( n \in \mathbb{N} \) if \( J_1, J_2 \in \hat{\mathcal{J}}(x_1, \ldots, x_n) \), \( J_1 \neq J_2 \) and \( A_k \) is the convex hull of the set \( \{x_i : i \in J_k\} \) for \( k = 1, 2 \) then \( A_1 \cap A_2 \) is an empty set or a singleton \( \{x_i\} \) for some \( i \leq n \).
Proof. See Supplement A.

Figure 1: Illustration of the convexity property of a partition of the data. Clusters are indicated by the shape and colour of the points.

We say that a partition satisfying the property described by Proposition 1 is a convex partition. As Figure 1 indicates, this is a rather desirable feature of a clustering mechanism.

The next development give information about the size and number of the clusters. Proposition 2 states that when the sequence of sample ‘second moments’ is bounded then the size of the smallest cluster in the MAP partition among those that intersect a ball of given radius is comparable with the sample size.

Proposition 2. If \( \sup_n \frac{1}{n} \sum_{i=1}^{n} \|x_n\|^2 < \infty \) then

\[
\liminf_{n \to \infty} \min \{|J| : J \in \hat{J}(x_1, \ldots, x_n), \exists j \in J \|x_j\| < r \}/n > 0
\]

for every \( r > 0 \).

Proof. See Supplement A.
The assumption $\sup_n \frac{1}{n} \sum_{i=1}^n \|x_n\|^2 < \infty$ allows the data sequence to be unbounded but it does ensure that it does not grow too quickly. It is easy to see that an assumption of this kind is necessary, otherwise it would be possible for each new observation to be large enough to create a new singleton cluster.

A simple consequence of Proposition 2 is that under these assumptions the number of components in the MAP partition that intersect a given ball is almost surely bounded.

**Corollary 2.2.** If $\left( \frac{1}{n} \sum_{i=1}^n \|x_i\|^2 \right)_{n=1}^\infty$ is bounded then for every $r > 0$ the number of clusters that intersect $B(0, r)$ is bounded, i.e.

$$\limsup_{n \to \infty} \{|J \in J(x_1, \ldots, x_n): \exists j \in J \|x_j\| < r\} < \infty.$$  

**Proof.** The proof follows easily from the fact that the size of the smallest cluster that intersects $B(0, r)$ is bounded from above by the number of observations divided by the number of clusters intersecting the ball. □

In order to formulate the central result of the paper we need to introduce several notions. Let $P$ be a probability distribution on $\mathbb{R}^d$ and $X$ a random variable with distribution $P$. Let $\Delta$ be the function on the space of finite families of measurable sets defined by the following formula

$$\Delta(\mathcal{G}) = \frac{1}{2} \sum_{G \in \mathcal{G}} P(G) \|R\mathbb{E}(X | X \in G)\|^2 + \sum_{G \in \mathcal{G}} P(G) \ln P(G). \quad (2.3)$$

where $R^2$ is the inverse of the within-cluster covariance matrix $\Sigma$ and $\mathbb{E}(X | X \in G)$ is the expected value of $X$ conditioned on $X \in G$.

We consider the **symmetric distance metric** over $P$-measurable sets, which is defined by $d_P(A, B) = P((A \setminus B) \cup (B \setminus A))$. This can be easily extended to a metric $d_P$ over finite families of measurable subsets of $\mathbb{R}^d$ (details are given in Section 4.3). Also we say that a family of measurable sets $\mathcal{A}$ is a $P$-partition if $P(\bigcup_{A \in \mathcal{A}} A) = 1$ and $P(A \cap B) = 0$ for all $A, B \in \mathcal{A}$, $A \neq B$. Let $M_\Delta$ denote the set of finite $P$-partitions that maximise the function $\Delta$. 

Figure 2: Illustration of Proposition 2 and Corollary 2.2. The red circle is arbitrarily fixed and the clusters it intersects are coloured. The number of observations in each coloured cluster is proportional to $n$ and the number of these clusters remains bounded as $n \to \infty$. 
Consider $X_1, X_2, \ldots \sim \text{iid } P$ and let $\hat{A}_n$ be the family of the convex hulls of clusters of observations in $\hat{J}(X_1, \ldots, X_n)$.

**Proposition 3.** Assume that $P$ has bounded support and is continuous with respect to Lebesgue measure. Then $M_\Delta \neq \emptyset$ and almost surely $\inf_{M \in M_\Delta} d_P(\hat{A}_n, M) \to 0$.

**Proof.** The proof follows from Theorem 4.12. See Supplement A for details.

The function $\Delta$ does not depend on the concentration parameter $\alpha$ or the between-groups covariance parameter. It therefore follows, somewhat surprisingly, that in the limit the shape of the MAP partition does not depend on these two parameters.

It can be shown that as the norm of the within group covariance matrix tends to 0, the variance of the conditional expected value gains larger importance in maximising the function $\Delta$ in formula (2.3) and this variance increases as the number of clusters increases. Therefore by manipulating the within group covariance parameter, when the input distribution is bounded it is possible to obtain an arbitrarily large (but fixed) number of clusters in the MAP partition as $n \to \infty$, as Proposition 4 states. This is also an indication of the inconsistency of the procedure used since it implies that when the input comes from a finite mixture of distributions with bounded support, then setting the $\Sigma$ parameter too small leads to an overestimation of the number of clusters.

**Proposition 4.** Assume that $P$ has bounded support and is continuous with respect to Lebesgue measure. Then for every $K \in \mathbb{N}$ there exists an $\varepsilon > 0$ such that if $\|\Sigma\| < \varepsilon$ then $|\hat{J}_n| > K$ for sufficiently large $n$.

**Proof.** See Supplement A.

It is worth pointing out that Proposition 1 and Proposition 2 hold also for finite Gaussian mixture models with Dirichlet prior on the probabilities of belonging to a given cluster. Proposition 3 also remains true with $M_\Delta$ replaced by $M_K^\Delta$ – the set of $P$-partitions with at most $K$ clusters that maximise the function $\Delta$, where $K$ is the number of clusters assumed by the model. The details are left for Supplement A.

### 3 Examples

This section presents some examples which illustrate the main propositions of the article. In Section 3.1 we compute the convex partition that maximises $\Delta$ when $P$ is a uniform distribution on the interval $[-1, 1]$. Section 3.2 gives an example of a distribution with well-defined moments, for which the maximiser of $\Delta$ necessarily has infinitely many clusters, although for any $r < \infty$, the number of clusters that intersect a ball of radius $r$ is finite. This example illustrates the content of Theorem 4.4, where it is shown that with appropriate choice of model parameters, if the input distribution is exponential then the
number of clusters in the sequence of MAP partitions becomes arbitrarily large. Section 3.3 investigates Gaussian mixture models; the MAP partition does not properly identify the two clusters when the mixture distribution is bi-modal. Finally, in Section 3.4 we consider the uniform distribution on the unit disc in \( \mathbb{R}^2 \). The partition maximising the function \( \Delta \) cannot be obtained by analytical methods, but it may be approximated. The results approximate the optimal partition of the unit disc and illustrate the convexity of Proposition 1. All examples are substantiated with computer simulations, presented in the main text or in Supplement B.

3.1 Uniform distribution on an interval

We find the convex partition that maximises \( \Delta \) if \( P \) is a uniform distribution on \([-1, 1]\). Firstly we find an optimal partition with fixed number of clusters \( K \). Since it is convex, it is defined by the lengths of \( K \) consecutive subintervals of \([-1, 1]\). Let those be \( 2p_1, \ldots, 2p_n \). Computations in Supplement A show that with \( K \) fixed the optimal division is \( p_1 = p_2 = \ldots = p_K = 1/K \). Using this, it is computed that the optimal number of clusters is

\[
K = \left\lfloor \frac{R}{\sqrt{3}} \right\rfloor \quad \text{or} \quad K = \left\lceil \frac{R}{\sqrt{3}} \right\rceil,
\]

where \( \left\lfloor x \right\rfloor \) and \( \left\lceil x \right\rceil \) are the largest integer not greater than \( x \) and the smallest integer not less than \( x \), respectively. It is worth noting that the variance of the data within a segment of length \( 2R/\sqrt{3} \) is equal to \( R \), so in this case the MAP clustering splits the data in a way that adjusts the empirical within-group covariance to the model assumptions.

It should be underlined that in this example, if \( \Sigma \) is small, the MAP partition has more than one cluster. The clustering is therefore misleading, since in this case there is exactly one population (which is uniform \([-1, 1]\)). The number of clusters in the MAP partition becomes arbitrarily large as \( \Sigma \) goes to 0, as Proposition 4 states.

This would suggest that, in general, a sensible choice of \( \Sigma \) should be made a priori. The sample variance would give an upper bound on \( \Sigma \) (since the data variance is the sum of between-group and within-group variances), but there is no natural lower bound for this parameter. In this example the partitioning mechanism itself is clearly far from satisfactory when it produces more than two clusters; the divisions seem very arbitrary.

3.2 Exponential distribution

When the input distribution is exponential with parameter 1, then for a relevant choice of model parameters (e.g. \( \alpha = T = 1, \Sigma = 4 \)) there is no finite partition that maximises \( \Delta \); the value of the function \( \Delta \) for a given convex partition can be increased by taking any interval of length larger than 3 and dividing it into two equally probable parts. See Supplement A for the proof.

Since the exponential distribution does not have bounded support, our considerations regarding the relation between the function \( \Delta \) and the MAP clustering cannot be applied directly. However, by using similar methods we can establish that for exponential input the MAP procedure creates an arbitrarily
large number of clusters. This is stated in Theorem 4.4, whose proof is presented in Supplement A.

3.3 Mixture of two normals

Let the input distribution be a mixture of two normals \( P = \frac{1}{2}(\nu_{-1.01} + \nu_{1.01}) \), where \( \nu_m \) is the normal distribution with mean \( m \) and variance 1. It can be proved that this distribution is bi-modal (however slightly; see Supplement A). Choose the model parameters consistent with the input distribution, i.e. \( d = \alpha = \Sigma = T = 1 \). It can be computed numerically that \( \Delta(\{(-\infty, 0], (0, \infty)\}) \approx -0.0046 < 0 = \Delta(\{\mathbb{R}\}) \). An intuitive partition of the data into positive and negative is induced by the partition \( \{(-\infty, 0], (0, \infty)\} \) and hence, by Corollary 4.6, for sufficiently large data input the posterior score for the two clusters partition is smaller than the posterior score for a single cluster. This may be taken as an indication of inconsistency of the MAP estimator in this setting.

3.4 Uniform distribution on a disc

This gives an example of non-uniqueness of the optimal partition, since the family of optimal partitions is clearly invariant under rotation around \((0, 0)\). Let \( P \) be uniform distribution on \( B(0, 1) \). It can be easily seen that \( \Delta(B(0, 1)) = 0 \). Let \( R \) be the identity matrix and let \( B^+_r \) \( (B^-_r) \) be a subset of \( B(0, 1) \) with non-negative (negative) first coordinate. Then \( \Delta(\{B^+_r, B^-_r\}) = 2r^2/9 - \ln 2 \). Therefore, for sufficiently large \( r \), a partition of \( B(0, 1) \) into halves is better than a single cluster, hence the optimal convex partition \( E \) is not a single cluster. Since a single cluster is the only convex partition of \( B(0, 1) \) that is rotationally invariant about the origin, it follows that the optimal partition is not unique.

The simulation in this case also give a nice illustration of the convexity of the MAP partition, proved in Proposition 1 and show the arbitrary nature of the partitioning when \( r \) is large.
Figure 3: Clustering in the MAP partition of the first \( k = 100, 500, 1000, 1500, 2000 \) observations (in columns) in the i.i.d. sample from the uniform distribution of disc \( B(0, 1) \). Different clusters are denoted by different colours.

3.5 The MAP clustering properties

This short simulation study presents the performance of the MAP estimator when the input distribution is a mixture of uniform distributions on three pairwise disjoint ellipses. The output is shown on Figure 4. It shows that the MAP clustering detects the mixture components or at least the clusters it creates are the sub-groups of the true mixture components (all depending on the within-group covariance parameter \( \Sigma \)). It also provides a nice illustration for two properties of the MAP partition: firstly the convexity property (Proposition 1) and secondly – the fact that when the within-group covariance parameter is decreasing, the number of cluster in the MAP partition grows, as stated in Proposition 4.

4 Detailed presentation of Proposition 3

4.1 Classification of Randomly Generated Data

Let \( P \) be a probability distribution on \((\mathbb{R}^d, \mathcal{B})\) and let \((X_n)_{n=1}^\infty\) be a sequence of independent copies of a random variable \( X \) with distribution \( P \). Then \( \mathcal{J}_n = \mathcal{J}(X_1, \ldots, X_n) \) goes a random partition of \([n]\). Note that if \( \mathbb{E}\|X\|^4 < \infty \) (here and subsequently, \( \mathbb{E} \) denotes the expected value) then by the strong law of large numbers almost surely \( \frac{1}{n} \sum_{i=1}^n \|X_i\|^2 \to \mathbb{E}\|X\|^2 < \infty \) and therefore the
Figure 4: Clustering in the MAP partition of the first $k = 50, 100, 200, 500$ observations (in columns) in the i.i.d. sample from the mixture of three uniform distributions on a disjoint ellipses. The MAP clustering was constructed for $\alpha = 1$, $T = I$ and $\Sigma = \sigma^2 I$ where $\sigma^2 \in \{1, 0.1, 0.01, 0.0025\}$ (in rows). Different clusters are denoted by different colours, the convex hulls of the clusters are also marked. It is clear that some of the partitions presented are not convex, particularly for large $\sigma^2$. This is due to the fact that the method is less than perfect. As $\sigma^2$ increases, the likelihood component of the formula for the posterior is less significant and hence partitions with the same prior (where clusters are of the same size) have similar posterior score. Therefore, with high probability, sampling from the posterior will not choose the MAP partition, or even a partition that reasonably resembles the MAP clustering. We mentioned this instability in Section 1.1.
assumptions of Proposition 2 are satisfied almost surely. Useful corollaries of this observation are listed below.

**Corollary 4.1.** If $E \|X\|^4 < \infty$ then for every $r > 0$

(a) $\liminf_{n \to \infty} \min \{|J| : J \in \mathcal{J}_n, \exists j \in J \|X_j\| < r\}/n > 0$ almost surely.

(b) the number of clusters in $\mathcal{J}_n$ that intersect $B(0, r)$ is bounded.

An easy consequence of Corollary 4.1 is

**Corollary 4.2.** If the support of $P$ is bounded then

(a) $\liminf_{n \to \infty} \min \{|J| : J \in \mathcal{J}_n\}/n > 0$ almost surely.

(b) $|\mathcal{J}_n|$ is almost surely bounded.

**Proof.** If the support of $P$ is bounded then $E \|X\|^4 < \infty$. Therefore we can use Corollary 4.1 where we take $r$ sufficiently large so that $B(0, r)$ contains the support of $P$. \qed

The assumptions of Corollary 4.2 cannot be relaxed to those of Corollary 4.1. It turns out that there exists a probability distribution $P$ with a countable number of atoms sufficiently far apart, whose probabilities are chosen so that $E \|X\|^4 < \infty$ and almost surely the most recent observation creates a singleton in the sequence of MAP partitions infinitely often, i.e. there exists a sequence $(n_k)_{k=1}^{\infty}$ such that $\{x_{n_k}\} \in \mathcal{J}_{n_k}$. This violates part (a) of Corollary 4.2. On the other hand, for appropriate parameter choice, sampling from the exponential distribution leads to the number of clusters in the MAP partition tending to infinity, which contradicts part (b) of Corollary 4.2. Proofs of these facts are left for Supplement A. These facts are now formally stated in the following two theorems:

**Theorem 4.3.** If $d = 1$ and $\alpha = T = \Sigma = 1$ then for $P = \sum_{m=0}^{\infty} q(1-q)^m \delta_{18m},$ where $q = (2 \cdot 18)^{-1}$, almost surely $\liminf_{n \to \infty} m(\mathcal{J}_n) = 1.$

**Theorem 4.4.** If $P = \text{Exp}(1)$ and the CRP model parameters are $\alpha = T = 1$, $\Sigma < (32 \ln 2)^{-1}$ then the number of clusters in the sequence of MAP partitions almost surely goes to infinity.

### 4.2 The Induced Partition

Instead of searching for the MAP clustering, one may choose a simpler (and more arbitrary) way to partition the data. The idea is to choose a partition of the observation space in advance and then divide the sample assigning each datum to the element of this partition which contains it. We call this decision rule an **induced partition**. In this section we give a formal definition and investigate how it behaves when the input is identically distributed and how it relates to the formula for the posterior probability given by (2.2).

**Definition.** Let $\mathcal{A}$ be a fixed partition of $\mathbb{R}^d$. For $n \in \mathbb{N}$ and $A \in \mathcal{A}$ let $J_{n}^{A} = \{i \leq n : X_i \in A \}$ and define a random partition of $[n]$ by $\mathcal{J}^{A}_n = \{J_n^{A} \neq \emptyset : A \in \mathcal{A}\}$. We say that this partition of $[n]$ is induced by $\mathcal{A}$. 

13
In the following part of the text, for two sequences \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\) of nonzero real numbers, we use the notation \(a_n \approx b_n\) to denote \(\lim_{n \to \infty} a_n/b_n = 1\).

**Lemma 4.5.** Let \(A\) be a finite \(P\)-partition of \(\mathbb{R}^d\) consisting of Borel sets with positive \(P\) measure. Then almost surely \(\sqrt{Q_{X_1:n}(J^A)} \approx \frac{2}{\pi} \exp\{\Delta(A)\}\), where \(\Delta\) is the function defined by (2.3).

*Proof.* See Supplement A. \(\Box\)

**Corollary 4.6.** If \(A, B\) are two finite \(P\)-partitions of \(\mathbb{R}\) such that \(\Delta(A) > \Delta(B)\) then almost surely \(Q_{X_1:n}(J^A_n) > Q_{X_1:n}(J^B_n)\) for sufficiently large \(n\).

*Proof.* The proof is straightforward and therefore omitted. \(\Box\)

**Corollary 4.6** implies that if we look for the optimal, finite induced partition, it will be a partition of the data induced by the finite partition of the observation space that maximises the function \(\Delta\). This formulation suggests a strong relationship between the MAP partition and the finite maximisers of \(\Delta\), which will be investigated further in Section 4.3, in the case where \(P\) has bounded support. The case where \(P\) does not have bounded support is beyond the scope of this work, for reasons presented in Section 5. This is a goal for future research.

At the end of this Section, let us provide an interpretation of the function \(\Delta\). Let \(A\) be a finite partition and \(Z_A = \mathbb{E}(X|1_A(X): A \in A)\) be the conditional expected value of \(X\) given the indicators \(1_A(X)\) for \(A \in A\). Then \(Z_A\) is a discrete random variable which is equal to \(\mathbb{E}(X | X \in A)\) with probability \(P(A)\). This implies that \(\Delta(A) = \frac{1}{2} \mathbb{E} \|RZ_A\|^2 - H(Z_A)\), where the function \(H\) assigns to a random variable its entropy. Moreover

\[
\mathbb{E} \|RZ_A\|^2 = \operatorname{tr}(V(RZ_A)) + \|E RZ_A\|^2 = \operatorname{tr}(RV(Z_A)R^t) + \|RE Z_A\|^2
\]

in which \(\operatorname{tr}(\cdot)\) is the trace function and \(V(\cdot)\) is the covariance matrix of a given random vector. Since \(E Z_A = E X\) we obtain that

\[
\Delta(A) = \frac{1}{2} \operatorname{tr}(RV(Z_A)R^t) - H(Z_A) + \frac{1}{2} \|RE X\|^2. \tag{4.1}
\]

Equation (4.1) justifies the following description of the function \(\Delta\): up to a constant, it may be treated as a difference between the variance and the entropy of the conditional expected value of a linearly transformed, \(P\)-distributed random variable given its affiliation to one of the sets in the partition.

### 4.3 Convergence of the MAP partitions

**Corollary 4.6** gives us a convenient characterisation of the partitions of \(\mathbb{R}^d\) that in the limit induce the best possible partitions of sets \([n]\). At this stage however we do not know yet if the best induced partitions relate to overall best partitions, namely the MAP partitions. A natural question is if the behaviour of the
MAP partition resembles the induced classification introduced in Section 4.2, as the sample size goes to infinity, and under what conditions. This section presents partial answers in this regard; it should be stressed however that all the developments presented here are limited to the case when the input distribution has bounded support. The reasons for such limitation are briefly described in Section 5.

As we already know that clusters in the MAP partition create disjoint convex sets, the analysis of the approximate behaviour of these partitions would be easier if a form of ‘uniform law of large numbers’ with respect to the family of convex sets were true. More precisely if we let

\[ P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \]

we need the following to hold:

\[ \lim_{n \to \infty} \sup_{C \text{ convex}} |P_n(C) - P(C)| = 0 \quad \text{almost surely.} \quad (\ast) \]

In other words we require that the class of convex sets is a Glivenko-Cantelli class with respect to \( P \). A convenient condition for this to hold is given in Elker et al. (1979), Example 14:

**Lemma 4.7.** If for each convex set \( C \) the boundary \( \partial C \) can be covered by countably many hyperplanes plus a set of \( P \)-measure zero, then \( \ast \) holds for \( P \).

In particular, it can easily be seen that the assumptions of Lemma 4.7 are satisfied if \( P \) has a density with respect to Lebesgue measure \( \lambda_d \) on \( \mathbb{R}^d \) (since in this case the Lebesgue measure \( \lambda_d \) of the boundary of any convex set is 0, and hence is also \( P \) measure 0).

We can now formulate a functional relation between the posterior probability of the MAP partition and the value of the function \( \Delta \) on the family of convex hulls of the sets in the MAP partition.

**Lemma 4.8.** Assume that \( P \) has bounded support and satisfies \( \ast \). Let \( \hat{A}_n \) be the family of the convex hulls of the clusters in the MAP partition, i.e. \( \hat{A}_n = \{ \text{conv}\{X_j : j \in J\} : J \in \hat{J} \} \). Then almost surely

\[ \sqrt{n} Q_{X_1} \sim \frac{n}{e} \exp\{\Delta(\hat{A}_n)\}. \]

**Proof.** See Supplement A.

Now we investigate the convergence of the sequence \( \hat{A}_n \) defined in Lemma 4.8. In order to do so we need a topology on relevant subspaces of \( \mathbb{R}^d \). We begin by recalling two standard metrics used in this context.

**Definition.** Let \( \mathcal{D} \) be a class of closed subsets of \( \mathbb{R}^d \). Then the function \( \varrho_H : \mathcal{D}^2 \to \mathbb{R} \) defined by

\[ \varrho_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq (B)_\varepsilon, B \subseteq (A)_\varepsilon\}, \]

where \( (X)_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, X) < \varepsilon\} \), is a metric on \( \mathcal{D} \). It is called the Hausdorff distance. The fact that it is a metric follows from 1.2.1 in Moszyńska (2005).
Definition. Let $\mathcal{M}$ be a $\sigma$-field on $\mathbb{R}^d$ and $\mu$ be a measure on $(\mathbb{R}^d, \mathcal{M})$. Then the function $d_\mu : \mathcal{M}^2 \rightarrow \mathbb{R}$ defined by $d_\mu(A, B) = \mu((A \setminus B) \cup (B \setminus A))$ is a pseudometric on $\mathcal{M}$, which by definition means that it is symmetric, nonnegative and satisfies the triangle inequality. It is called the symmetric difference metric. The fact that it is a pseudometric is explained in the beginning of Section 13, Chapter III of Doob (1994). Note that since $d_\mu(A, B) = 0$ does not imply $A = B$, formally $d_\mu$ is not a metric on $\mathcal{M}$. Although for our consideration the difference of measure 0 is of no importance, we keep on using the proper pseudometric term in this context.

The two following theorems are crucial for establishing the limits of maximisers. Theorem 4.9 is Theorem 3.2.14 in Moszyńska (2005); it ensures the existence of $d_H$-converging subsequence in every bounded sequence of convex sets. Theorem 4.10 is a straightforward consequence of Theorem 12.7 in Valentine (1964) (in the latter $P$ is taken to be the Lebesgue measure). It states that when $P$ has a density with respect to the Lebesgue measure then the Hausdorff metric restricted to $K$ is stronger than the symmetric difference metric.

Theorem 4.9. The space $(K, \varrho_H)$ is finitely compact (i.e. every bounded sequence has a convergent subsequence).

Theorem 4.10. If $P$ is continuous with respect to the Lebesgue measure then convergence in $\varrho_H$ implies convergence in $d_P$ in the space $K$.

Note that the Hausdorff and symmetric difference metrics are defined on sets. However we are interested in MAP partitions, which are families of sets. Therefore it is convenient to extend the definitions of these metrics to families of sets, as presented below. Remark 4.11 ensures that the desirable properties of compactness are preserved by such extension.

Definition. Let $d$ be a pseudometric on the family of sets $\mathcal{F}$. For $K \in \mathbb{N}$ we define $F_K(\mathcal{F})$ to be the space of finite subfamilies of $\mathcal{F}$ that have at most $K$ elements. Moreover $A = \{A^{(1)}, \ldots, A^{(k)}\} \in F_K(\mathcal{F})$ and $B = \{B^{(1)}, \ldots, B^{(l)}\} \in F_K(\mathcal{F})$ we define

$$\bar{d}(A, B) = \min_{\sigma \in \Sigma_K} \max_{i \leq K} d(A^{(i)}, B^{(\sigma(i)))},$$

where $\Sigma_K$ is the set of all permutations of $[K]$ and we assume $A^{(i)} = \emptyset$ and $B^{(j)} = \emptyset$ for $i > k$ or $j > l$ respectively.

Remark 4.11. If $(\mathcal{F}, d)$ is a pseudometric space then $(F_K(\mathcal{F}), \bar{d})$ is also a pseudometric space. Moreover, if $(\mathcal{F}, d)$ is finitely compact then $(F_K(\mathcal{F}), \bar{d})$ is also finitely compact.

Proof. The proof is straightforward. See Supplement A for details.

Now assume that $P$ has bounded support. Then by Theorem 4.9 and Remark 4.11 it follows that $(\mathcal{A}_n)_{n=1}^\infty$ has convergent subsequences which have a limit under $d_H$ (note that as the support of $P$ is bounded, sets $A$ are also bounded in the $d_H$ metric). Let us denote the (random) set of their limits by
Note that by the properties of $d_H$ distance each family in $E$ consists of convex, closed sets. If we assume that $P$ is continuous with respect to the Lebesgue measure then it follows from Lemma 4.8 together with Theorem 4.10 that $E$ consists of finite $P$-partitions that maximise the function $\Delta$.

**Theorem 4.12.** Assume that $P$ has bounded support and is continuous with respect to Lebesgue measure. Then every partition in $E$ is a finite $P$-partition that maximises $\Delta$.

*Proof.* See Supplement A. \qed

Now Proposition 3 is a straightforward, topological consequence of Theorem 4.12. This is shown in Supplement A.

## 5 Discussion

It should be clearly stated that the scope of the paper is limited in two ways. Firstly, only the Gaussian model is considered. It is natural to ask if the methods used here can be applied for other combinations of base measure and component distributions. The author is sceptical in this regard. The proofs of the key Proposition 1 and Proposition 2 rely strongly on the formula (2.2). It is difficult to find a computationally feasible choice of the base and component measures so that the resulting formula for the posterior probabilities has similar properties.

Secondly, the limiting results contained in Section 4.3 are proved in the case where the support of the input distribution is bounded. In this case the model is clearly misspecified. A significant effort was put in order to extend the results from Section 4.3 at least to the case where $P$ is Gaussian. Unfortunately, there are some technical hurdles which the author was not able to overcome, which we now outline. The first result in which the boundedness of the input distribution is used is Lemma 4.8 – here we use both parts of Corollary 4.2 which, as shown by Theorem 4.3 and Theorem 4.4, cannot be easily generalised. A natural approach is to fix large $r > 0$ and use Corollary 4.1 – then the product of those factors in (2.2) which come from the clusters that intersect $B(0, r)$ may be well approximated using Lemma 4.7, since by Corollary 4.1 there are finitely many clusters intersecting $B(0, r)$ and the number of observations in the cluster is comparable with $n$ for each cluster. Unfortunately in this way there is no control over the impact of the clusters outside $B(0, r)$ as there are no lower bounds on their size and upper bounds on their number. However the author believes that these obstacles are possible to overcome and this remains subject for the future work.

It should be also underlined the setting of our analysis was not the usual one for the consistency analysis. Indeed, in our formulation of the CRP model our parameter space is the space of partitions of $[n]$, which is changing with $n$. To perform a classical consistency analysis we need the parameter space to be fixed regardless of the number of observations. On the other hand, if we consider the DPMM formulation, in which the parameter space is the space of all possible
realisations of the Dirichlet Process (i.e. the space of discrete measures on $\mathbb{R}^d$ with infinitely many atoms) then again our input should come from an infinite mixture of normals, which was not the case in our examples.

However some of our results from Section 4 can be applied when the input sequence is a realisation of the DPMM. Indeed, the convexity result of Proposition 1 does not have any assumptions on the data sequence. As for Proposition 2, it requires the sequence of mean squared norms to be bounded. It is easy to prove (see Supplement A) that for a realisation of the DPMM this assumption holds almost surely and hence for every $r > 0$ the clusters intersecting $B(0, r)$ in the sequence of the MAP partitions constructed on the sample from DPMM are of size comparable with the number of observation and their number is bounded. However, some fundamental questions remain unanswered in this case (e.g. does the number of clusters in the MAP partition tend to infinity in this case?) and they are open for further investigation.

Note that the machinery presented can be used for a different task. The $P$-partitions that maximise the function $\Delta$ seem to be interesting objects in their own right. Note that for dimension greater than 1 it seems to be extremely difficult to derive the maximisers simply by analytical means. Remark 4.11 and Proposition 3 give us a convenient tool to examine those maximisers as they may be approximated by performing sampling from the posterior. This cannot be done faithfully as the normalizing constant in the formula (2.2) cannot be computed explicitly, however there are standard MCMC techniques that can be applied there (e.g. Neal (2000)). Further examination of the maximisers of the function $\Delta$ is left for future research.

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Supplement A
Proofs

Proof of Remark 2.1

It is easy to see that in Equation (2.1) when the $F_\theta$'s have densities $f_\theta$ with respect to some $\sigma$-finite measure $\nu$, sampling of $\theta$ may be omitted by taking the marginal distribution of $x_J$ under $J$. Then the model takes the form

$$ J \sim \text{CRP}(\alpha)_n $$
$$ x_J | J \sim F_{j|J}^{G_0} \quad \text{for } J \in J $$

where for $F_{j|J}^{G_0}$ is a probability distribution on $x_j$ with the density

$$ f_{j|J}^{G_0}(x) = \int_{\Theta} \prod_{j \in J} f_\theta(x_j) dG_0(\theta) $$

with respect to product measure $\nu_{J\theta}$. We now compute the exact formula for $f_{j|J}^{G_0}$ when $G_0 = \mathcal{N}(0, T)$ and $F_\theta = \mathcal{N}(\theta, \Sigma)$.

In order to simplify computations it is convenient to use the notation $[A] = A^tA$, where $A$ is a matrix of any dimensionality. This is ambiguous as it is the same as the notation introduced in the main text, $[n] = \{1, 2, \ldots, n\}$, where $n$ is a natural number. However we don't use the latter in the following proof.

Let $U = T^{-1/2}$. Since the densities of $\theta$ and $x_J | \theta$ are given by

$$ \theta \sim \frac{\det U}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left[ U \theta \right] \right\} \quad \text{and} \quad x_J | \theta \sim \left( \frac{\det R}{\sqrt{2\pi}} \right)^{|J|} \exp \left\{ -\frac{1}{2} \sum_{j \in J} [R(x_j - \theta)] \right\} $$

we obtain that

$$ f_{j|J}^{G_0}(x) = \frac{\det U}{\sqrt{2\pi}} \left( \frac{\det R}{\sqrt{2\pi}} \right)^{|J|} \int_{\Theta} \exp \left\{ -\frac{1}{2} \left[ U \theta \right] - \frac{1}{2} \sum_{j \in J} [R(x_j - \theta)] \right\} \ d\theta. \quad (A.1) $$

Let $H_J$ be a positive definite symmetric matrix such that $[H_J] = |J| \cdot [R] + [U]$, then

$$ [U \theta] + \sum_{j \in J} [R(x_j - \theta)] = [U \theta] + \left( \sum_{j \in J} [R x_j] - 2\theta^t [R] \sum_{j \in J} x_j + |J| [R \theta] \right) = \left[ H_J \theta \right] - 2\theta^t [R] \sum_{j \in J} x_j + \sum_{j \in J} [R x_j] = \left[ H_J (\theta - [H_J]^{-1} [R] \sum_{j \in J} x_j) \right] - \left[ H_J^{-1} [R] \sum_{j \in J} x_j \right] + \sum_{j \in J} [R x_j] $$

and hence

$$ \int_{\Theta} \exp \left\{ -\frac{1}{2} \left[ U \theta \right] - \frac{1}{2} \sum_{j \in J} [R(x_j - \theta)] \right\} \ d\theta = \frac{\sqrt{2\pi^d}}{\det H_J} \exp \left\{ \frac{1}{2} \left( [H_J^{-1} [R] \sum_{j \in J} x_j] - \sum_{j \in J} [R x_j] \right) \right\}. \quad (A.2) $$
By joining equalities (A.1) with (A.2) and substituting \( H_j = \sqrt{|J|} \cdot R_j \) we obtain that
\[
 f_{j}^{\text{Gi}}(x_{J}) = \left( \frac{\det R}{\sqrt{2\pi}} \right)^{|J|} \frac{\det U}{\sqrt{|J|} \det R_{|J|}} \exp \left\{ \frac{1}{2} \left( |J| \cdot \left\| \frac{1}{|J|} R_{|J|}^{-1} R_{|J|}^{2} x_{J} \right\|^{2} - \sum_{j \in J} \| R_{x_{j}} \|^2 \right) \right\}.
\]
Therefore
\[
x_{J} \sim \prod_{j \in J} f_{j}^{\text{Gi}}(x_{J}) = \left( \frac{\det R}{\sqrt{2\pi}} \right)^n \exp \left\{ - \sum_{j \leq n} \| R_{x_{j}} \|^2 \right\} \left( \frac{\det U}{\sqrt{|J|} \det R_{|J|}} \right)^{|J|} \prod_{j \in J} \frac{1}{\sqrt{|J|} \det R_{|J|}} \exp \left\{ \frac{1}{2} \left( |J| \cdot \left\| \frac{1}{|J|} R_{|J|}^{-1} R_{|J|}^{2} x_{J} \right\|^{2} \right) \right\} \propto \exp \left\{ \frac{1}{2} \left( |J| \cdot \left\| \frac{1}{|J|} R_{|J|}^{-1} R_{|J|}^{2} x_{J} \right\|^{2} \right) \right\}.
\]
(A.3)

It is easy to see that the probability weights in CRP(\(\alpha\)) are given by the formula
\[
\mathbb{P}(\mathcal{J}_n = \mathcal{J}) = \frac{\alpha^{n_{\mathcal{J}}}}{\alpha^{n_{\mathcal{J}}} \prod_{j \in J} (|J| - 1)!} \propto \alpha^{n_{\mathcal{J}}} \prod_{j \in J} (|J| - 1)!, \tag{A.4}
\]
where \(\alpha^{(k)} = \alpha^{(a + 1)} \ldots (a + k - 1)\) and \(|J|\) is the number of sets in the partition \(\mathcal{J}\).

The proof of Remark 2.1 follows from (A.3), (A.4) and the Bayes formula.

**Proof of Proposition 1**

Take any \(I, J \in \mathcal{J}_n\). Consider all partitions of \([n]\) that are obtained by replacing sets \(I, J\) with \(\hat{I}\) and \(\hat{J}\) that satisfy \(|\hat{I}| = |I|, |\hat{J}| = |J|\) and \(I \cup J = I \cup J\). Note that by such operation we do not alter either the number of clusters or the size of the clusters and therefore the posterior probability of such partitions is an increasing function of
\[
|\hat{I}| \cdot \left\| \frac{1}{|\hat{I}|} R_{|\hat{I}|}^{-1} R_{|\hat{I}|}^{2} x_{\hat{I}} \right\|^{2} + |\hat{J}| \cdot \left\| \frac{1}{|\hat{J}|} R_{|\hat{J}|}^{-1} R_{|\hat{J}|}^{2} x_{\hat{J}} \right\|^{2} = \frac{1}{|\hat{I}|} \cdot \left\| \frac{1}{|\hat{I}|} R_{|\hat{I}|}^{-1} R_{|\hat{I}|}^{2} \sum_{i \in \hat{I}} x_{i} \right\|^{2} + \frac{1}{|\hat{J}|} \cdot \left\| \frac{1}{|\hat{J}|} R_{|\hat{J}|}^{-1} R_{|\hat{J}|}^{2} \left( S - \sum_{i \in \hat{J}} x_{i} \right) \right\|^{2} \tag{A.5}
\]
where \(S = \sum_{k \in \hat{I} \cup \hat{J}} x_{k} = \sum_{k \in \hat{I} \cup \hat{J}} x_{k}\). It may be seen quite easily that (A.5) defines a strictly convex quadratic function with respect to \(\sum_{i \in \hat{I}} x_{i}\). We investigate its value over a finite number of possible replacements. Therefore it achieves its maximal value at the vertices of convex hull of all possible values of \(\sum_{i \in \hat{I}} x_{i}\). Since \(\mathcal{J}_n\) is the MAP partition it follows that \(\sum_{i \in \hat{I}} x_{i}\) maximises (A.5).

Suppose that \(\text{conv}\{x_{i} : i \in I\}\) and \(\text{conv}\{x_{j} : j \in J\}\) have a point in common, which is not \(x_{i}\) for any \(i \leq n\). Then there exist two equal convex combinations of points in \(\{x_{i} : i \in I\}\) and \(\{x_{j} : j \in J\}\), at least one of which is non-trivial, i.e.
\[
\sum_{i \in I} \lambda_{i}^I x_{i} = \sum_{j \in J} \lambda_{j}^J x_{j}, \quad \sum_{i \in I} \lambda_{i}^I = \sum_{j \in J} \lambda_{j}^J = 1, \quad \lambda_{i}^I, \lambda_{j}^J \in [0, 1).
\]
(a convex combination is non-trivial if at least two of ‘lambdas’ are non-zero). From
this we can deduce that
\[
\sum_{i \in I} x_i = \sum_{i' \in I} \left( \lambda_i^f \sum_{i \in I} x_i \right) = \sum_{i \in I} \lambda_i^f x_i + \sum_{(i,i') \in I^2 \setminus \{(i,i') \}} \lambda_i^f x_i = \sum_{(i,i') \in I^2 \setminus \{(i,i') \}} \lambda_i^f x_i + \sum_{i \in J} \lambda_i^f x_i = \sum_{j \in J} \lambda_j^f \left( \sum_{i' \in I} \lambda_i^{f,j} x_i \right)
\]
\[
= \sum_{j \in J} \lambda_j^f \left( \sum_{i' \in I} \lambda_i^{j} x_i \right) + \sum_{(i',j) \in I \times J} \lambda_i^f \sum_{j \in J} \lambda_j^f \left( \sum_{i' \in I \setminus \{i'\}} \sum_{j \in J \setminus \{j\}} x_i \right).
\]
\[
(A.6)
\]
Moreover \(\lambda_i^f \lambda_j^f \in [0, 1]\) for \(i' \in I, j \in J\) and \(\sum_{(i',j) \in I \times J} \lambda_i^f \lambda_j^f = \sum_{i' \in I} \lambda_i^f \lambda_j^f = 1\), so (A.6) gives a representation of \(\sum_{i \in I} x_i\) as a non-trivial (since at least one of the two combinations was non-trivial) convex combination of \(\sum_{i \in I} x_i\). This is a contradiction and the proof follows.

Proof of Proposition 2

For the reader’s convenience the proof is split into three parts. In Subsection Preliminary lemmas we list some facts important for further analysis. Important properties of the MAP partition presents lemmas regarding the MAP, which are further used in Subsection Proof of Proposition 2, where the proof of one of the main results of the paper is presented.

Preliminary lemmas

**Remark A.1.** Symmetric, positive definite matrices have the following properties
(a) the sum of symmetric positive definite matrices is symmetric positive definite.
(b) the inverse of symmetric positive definite matrix is symmetric positive definite.
(c) for each symmetric positive matrix \(A\) there exist an uniquely defined symmetric positive matrix \(B\) such that \(A = B^2\). We use the notation \(B = A^{1/2}\).
(d) if \(A, B\) are symmetric positive definite matrices and also \(A - B\) is symmetric positive definite then \(B^{-1} - A^{-1}\) is symmetric positive definite.
(e) if \(A, B\) are positive definite then \(\det(A + B) \geq \det A\).

**Proof.** Let \(A, B \in \mathbb{R}^{d \times d}\).

(a) If \(A, B\) are symmetric then \(A + B\) is also symmetric. If \(A, B\) are positive definite then for every \(x \in \mathbb{R}^d \setminus \{0\}\) we have \(x'(A + B)x = x'Ax + x'Bx > 0\) and hence \(A + B\) is also positive definite.

(b) If \(A\) is symmetric then \(A^{-1}\) is also symmetric. If \(A\) is positive definite then by Theorem 7.1 from Zhang (2011) it may be expressed as \(U^* \text{diag}(\lambda_1, \ldots, \lambda_d) U\) where \(U\) is unitary matrix and \(U^*\) its conjugate transpose and \(\lambda_1, \ldots, \lambda_d > 0\). Therefore \(A^{-1} = U^* \text{diag}(\lambda_1^{-1}, \ldots, \lambda_d^{-1}) U\) and again by using Theorem 7.1 we obtain that \(A^{-1}\) is positive definite.

(c) Since if \(A\) is a symmetric matrix then \(A^T A = A^2\) and this point is an easy consequence of Theorem 7.4 in Zhang (2011).

(d) Let \(P\) be a symmetric matrix that satisfy \(P^2 = B\). Positive definiteness of \(A - B\) is equivalent to \(x'Ax > x'Bx\) for all \(x \in \mathbb{R}^d\). By substituting \(y = Px\) this is equivalent to \(y'P^{-1}AP^{-1}y > y'y\) for all \(y \in \mathbb{R}^d\). Note that \(P^{-1}AP^{-1}\) is positive
Lemma A.3. We get that may be ordered so that it is term-wise not less than finishing the proof of the lemma.

Clearly the existence of such ordering establishes the lemma. For

Assume that

Proof. The proof is straightforward and therefore omitted.

Remark A.2. Let $R_m$ be defined as in the statement of Remark 2.1, then

(a) $\det R_m \to \det R$ (b) if $y_m \to y$ then $R_m y_m \to Ry$

Proof. We prove by induction on $n_k$ that the sequence

may be ordered so that it is term-wise not less than $c = (1, \ldots, n_1, 1, \ldots, n_2, \ldots, 1, \ldots, n_k)$. Clearly the existence of such ordering establishes the lemma. For $n_k = 1$ this is self-evident. For $n_k > 1$ we apply 'greedy' approach – put all $a+1$ (or $a$ in case $n_k|n$) $n_k$'s in the places of $n_k, n_k-1, \ldots, n_k-a+1$. The fact that $n_k \geq n_{k-1} \geq \ldots \geq n_1$ ensures that it is possible and all of $n_k-1, n_k-1-1, \ldots, n_k-a+1-1, n_k-a, \ldots, n_1$ are less or equal to $n_k-1$. Therefore we may apply inductive assumptions to these numbers thus finishing the proof of the lemma.

Lemma A.4. For every $\varepsilon > 0$ there exist $K \in \mathbb{N}$ such that if $n_1, \ldots, n_k \leq n/K$, where $n = \sum_{i=1}^k n_i$, then $\sqrt[n]{\prod_{i=1}^k a_i/n!} \leq \varepsilon$.

Proof. Assume that $n_1 \leq \ldots \leq n_k \leq n/K$ and let $n = an_k + r$, where $0 \leq r < n_k$. By Lemma A.3 we get that

$$\frac{\prod_{i=1}^k a_i}{n!} \leq \frac{(n_k!)^a(n_k-r+1)\ldots n_k}{n!} \leq \frac{1}{1^{n_k}} \cdot \frac{1}{2^{n_k}} \cdot \ldots \cdot \frac{1}{a^{n_k}} \cdot \frac{1}{(a+1)^r} \leq \frac{1}{a^{n_k}}.$$

Therefore

$$\sqrt[n]{\prod_{i=1}^k a_i/n!} \leq \frac{1}{\sqrt[n]{a^{n_k}}} = \frac{1}{\sqrt[k]{a^{n_k}}}.$$

For $K$ large enough this might be arbitrarily small, so the proof follows.
Important properties of the MAP partition

Let us fix a sequence \((x_n)_{n=1}^\infty\) in \(\mathbb{R}^d\) and let \(\mathcal{J}_n = \mathcal{J}(x_1, \ldots, x_n)\). In order to facilitate the analysis, we introduce the following notation: let \(m_n = \min_{J \in \mathcal{J}_n} |J|\) and \(M_n = \max_{J \in \mathcal{J}_n} |J|\) be the minimum and the maximum cluster size in the partition \(\mathcal{J}_n\). Moreover for \(r > 0\) let
\[
m_n^{(r)} = \min\{|J|: J \in \mathcal{J}_n, \|\mathbf{x}_J\| < r\}, \quad M_n^{(r)} = \max\{|J|: J \in \mathcal{J}_n, \|\mathbf{x}_J\| < r\}
\]
be the minimal and the maximal cluster size in the partition \(\mathcal{J}_n\) among the clusters whose center of mass lies in \(B(0, r)\). Finally let
\[
m_n^{[r]} = \min\{|J|: J \in \mathcal{J}_n, \exists j \in J \|x_j\| < r\}, \quad M_n^{[r]} = \max\{|J|: J \in \mathcal{J}_n, \exists j \in J \|x_j\| < r\}
\]
be the minimal and the maximal cluster size in the partition \(\mathcal{J}_n\) among the clusters that intersect the ball \(B(0, r)\).

Let \(J_n^m, J_n^M \in \mathcal{J}_n\) satisfy \(|J_n^m| = m_n\) and \(|J_n^M| = M_n\). We define \(J_n^{m,(r)}, J_n^{M,(r)}, J_n^{m,[r]}\) and \(J_n^{M,[r]}\) similarly (e.g. \(J_n^{m,(r)} \in \mathcal{J}_n\) satisfies \(\|\mathbf{x}_{J_n^{m,(r)}}\| < r\) and \(|J_n^{m,(r)}| = m_n^{(r)}\)).

**Proposition A.5.** If \(\frac{1}{n} \sum_{i=1}^n \|x_i\|^2 \xrightarrow{n \to \infty} \infty\) is bounded then \(\liminf_{n \to \infty} M_n/n > 0\).

**Proof.** Suppose that \(\lim \inf M_n/n = 0\). Then there exists an increasing sequence \((n_k)_{k \in \mathbb{N}}\) such that \(M_{n_k}/n_k < 1/k\) for every \(k \in \mathbb{N}\). We now prove that
\[
\lim_{k \to \infty} \frac{n_k}{\sqrt[n_k]{\mathcal{Q}_k(\mathcal{J}_{n_k})/\mathcal{Q}_k([n_k])}} = 0,
\]
hence obtaining a contradiction with the definition of the MAP partition. By (2.2)
\[
\frac{n}{\sqrt[n]{\mathcal{Q}_k(\mathcal{J}_{n_k})/\mathcal{Q}_k([n_k])}} = \frac{n}{\sqrt[n]{\mathcal{C}(|J_{n,k}|)/C}} \cdot \prod_{J \in \mathcal{J}_{n_k}} \frac{|J|/n_k!}{\sqrt[n]{n_k^{(d+2)/2} \det R_{n_k}}} \cdot \exp\left\{ \frac{1}{2n_k} \left( \sum_{J \in \mathcal{J}_{n_k}} |J| \left\| R_{|J|}^{-1} R_{n_k}^2 \mathbf{x}_J \right\|^2 - n_k \left\| R_{n_k}^{-1} R_{n_k}^2 \mathbf{x}_{\mathcal{J}_{n_k}} \right\|^2 \right) \right\}
\]
(A.7)

Firstly note that
\[
\lim_{k \to \infty} \frac{n}{\sqrt[n]{\mathcal{C}(|J_{n,k}|)/C}} = \lim_{k \to \infty} \mathcal{C}(|J_{n,k}|^{-1}/n_k) \leq \max\{1, C\}.
\]
(A.8)

By **Lemma A.4**, it follows that, under the assumptions,
\[
\lim_{k \to \infty} \frac{n}{\sqrt[n]{\prod_{J \in \mathcal{J}_{n_k}} |J|/n_k!}} = 0.
\]
(A.9)

From **Remark A.2**
\[
\limsup_{k \to \infty} \frac{n_k^{(d+2)/2} \det R_{n_k}}{\prod_{J \in \mathcal{J}_{n_k}} |J|^{(d+2)/2} \det R_{|J|}} \leq \limsup_{k \to \infty} \frac{n_k^{(d+2)/2} \det R_{n_k}}{\liminf_{k \to \infty} n_k^{(d+2)/2} \det R_{|J|}} \leq \frac{1}{\min\{1, \det R\}}.
\]
(A.10)
Recall the inequality between linear and quadratic means which states that for every sequence \( y_1, \ldots, y_l \) of real numbers we have
\[
\left| \sum_{i=1}^l y_i \right| \leq \sqrt{\sum_{i=1}^l y_i^2} \iff l \cdot \left( \frac{\sum_{i=1}^l y_i}{l} \right)^2 \leq \sum_{i=1}^l y_i^2.
\]  
(A.11)

If we apply (A.11) to every coordinate of vectors \( y_1, \ldots, y_d \in \mathbb{R}^d \) and sum up obtained inequalities we obtain that
\[
l \cdot \left\| \sum_{i=1}^l y_i \right\|^2 \leq \sum_{i=1}^l \|y_i\|^2.
\]

Therefore, by linearity of multiplication by matrix
\[
\sum_{J \in \mathcal{J}_n} |J| \left\| R^{-1}_{J} R^2 x_J \right\|^2 \leq \sum_{J \in \mathcal{J}_n} \sum_{J \subseteq J} \left\| R^{-1}_{J} R^2 x_J \right\|^2 \leq \sum_{J \in \mathcal{J}_n} \sum_{J \subseteq J} \left\| R^{-1}_{J} R^2 x_J \right\|^2 \leq \|R\|^2 \sum_{i \in [n]} \|x_i\|^2,
\]
and hence, using Remark A.1, we have
\[
\sum_{J \in \mathcal{J}_n} |J| \left\| R^{-1}_{J} R^2 x_J \right\|^2 \leq \sum_{J \in \mathcal{J}_n} \sum_{J \subseteq J} \left\| R^{-1}_{J} R^2 x_J \right\|^2 \leq \sum_{J \in \mathcal{J}_n} \sum_{J \subseteq J} \left\| R^{-1}_{J} R^2 x_J \right\|^2 \leq \|R\|^2 \sum_{i \in [n]} \|x_i\|^2,
\]
where \( \| \cdot \|_2 \) is a matrix norm induced by \( \| \cdot \| \) (i.e. \( \|A\|_2 = \sup\|x\|=1 \|Ax\| \)). From this and assumptions of the Proposition we can easily deduce that
\[
\frac{1}{n_k} \left( \sum_{J \in \mathcal{J}_n} |J| \left\| R^{-1}_{J} R^2 x_J \right\|^2 - n_k \left\| R^{-1}_{n_k} R^2 x_{[n_k]} \right\|^2 \right)
\]
is bounded from above.  
(A.12)

Gathering (A.7), (A.8), (A.9), (A.10) and (A.12) together, we obtain that
\[
\limsup_n \frac{\sqrt{Q_x(\mathcal{J}_n)} / Q_x([n])}{Q_x([n_k])} = 0.
\]

Hence there exists a sufficiently large \( n \) that satisfies \( P(\mathcal{J}_n \mid x) < P([n] \mid x) \). This is a contradiction.

**Corollary A.6.** If \( \frac{1}{n} \sum_{i=1}^n \|x_i\|^2 \) is bounded then there exist \( r_0 > 0 \) such that \( \|\pi_{J^2}\| \leq r_0 \) for all \( n > 0 \).

**Proof.** By Proposition A.5 we know that \( \gamma := \liminf_{n \to \infty} M_n / n > 0 \), so there exists \( N > 0 \) such that \( M_n / n > \gamma / 2 \) for \( n > N \). Suppose that there exists a sequence \( (n_k)_{k=1}^\infty \) such that \( \|\pi_{J^2}\| \geq k \) for all \( k \in \mathbb{N} \). Note that for \( n_k > N \)
\[
\frac{1}{n_k} \sum_{i=1}^{n_k} \|x_i\| \geq \frac{1}{n_k} \sum_{i \in J^2_{n_k}} \|x_i\| \geq \frac{1}{n_k} \sum_{i \in J^2_{n_k}} \|x_i\| \geq \frac{M_{n_k}}{n_k} \|\pi_{J^2}\| \geq \gamma / 2 \cdot k,
\]
which, together with the inequality between the arithmetic and quadratic mean, contradicts the assumption that the sequence \( \frac{1}{n} \sum_{i=1}^n \|x_i\|^2 \) is bounded. The proof of the Lemma follows from the contradiction.

**Proposition A.7.** If \( \frac{1}{n} \sum_{i=1}^n \|x_i\|^2 \) is bounded then \( \liminf_{n \to \infty} m_n^{(r)} / n > 0 \) for every \( r > 0 \).
Proof. Firstly note that it is enough to prove the statement of Proposition A.7 for all $r > r_0$ for some given $r_0 > 0$ – indeed, $m_r$ is decreasing with $r$. We take $r_0$ from the statement of Corollary A.6.

Fix $r > r_0$. Note that $J_n^{m,M} = J_n^M$ and hence $\lim \inf_{n \to \infty} M_n^{M}/n > 0$. Now we prove that $\lim \inf_{n \to \infty} m_n^{M}/n > 0$. Suppose the contrary. We show that for sufficiently large $n$, the posterior probability of $\hat{J}_n$ increases if we create one cluster out of $J_n^{m,M}$ and $J_n^{M,M}$. Let $\hat{J}_n$ be a partition of $[n]$ obtained from $\hat{J}_n$ by joining $J_n^{m,M}$ with $J_n^{M,M}$, i.e.,

$$\hat{J}_n = \hat{J}_n \setminus \{J_n^{m,M}, J_n^{M,M}\} \cup \{J_n^{m,M}, J_n^{M,M}\}.$$  

In order to simplify the notation, we write $m, M$ instead of $m_n^{M}, M_n^{M}$ respectively and remember that they are both functions of $n$. Similarly let us write $\mathbf{x}, \mathbf{y}$ instead of $\mathbf{x}_n^{m,M}, \mathbf{y}_n^{M,M}$ and $\mathbf{x}_n^{m,M}\cup\mathbf{y}_n^{M,M}$. When taking a quotient $P(\hat{J}_n | x)/P(\hat{J}_n | x)$ most factors in (2.2) cancel out, giving

$$\frac{P(\hat{J}_n | x)}{P(\hat{J}_n | x)} = C \frac{m! M!}{(m + M)!} \left( \frac{m + M}{m M} \right)^{(d+2)/2} \frac{\det R_{m+M}}{\det R_m \cdot \det R_M} \cdot \exp \{ D_n \}^{1/2},$$

where

$$D_n = m \| R_{m+1}^{-1} R_2 \mathbf{x}_m \|^2 + M \| R_{m+1}^{-1} R_2 \mathbf{y}_M \|^2 - (m + M) \| R_{m+1}^{-1} R_2 \mathbf{x}_{m,M} \|^2.$$

It is now straightforward to verify that

$$m \| R_2 \mathbf{x}_m \|^2 + M \| R_2 \mathbf{y}_M \|^2 - (m + M) \| R_2 \mathbf{x}_{m,M} \|^2 = \frac{m M}{m + M} \| R(\mathbf{x}_m - \mathbf{y}_M) \|^2 \leq m \| R(\mathbf{x}_m - \mathbf{y}_M) \|^2 \leq m \| R \|^2 \cdot 4 \cdot r^2.$$  

Moreover

$$(m + M) I - (m + M) R (R_{m+M}^{-1})^2 R = (m + M) (I - (R_{m+M}^{-1}) R_{m+M} R) R =$$

$$= (m + M) R (I - (I + (UR^{-1})^2 UR^{-1} / (m + M) )^{-1} R =$$

$$= R (UR^{-1})^2 UR^{-1} (I + (UR^{-1})^2 UR^{-1} / (m + M) )^{-1} R$$

and therefore

$$\limsup_{n \to \infty} \left( (m + M) \| R_2 \mathbf{x}_{m,M} \|^2 - (m + M) \| R_{m+1}^{-1} R_2 \mathbf{x}_{m,M} \|^2 \right) \leq \| U \|^2 r^2.$$  

By Remark A.1, together with (A.14) and (A.15),

$$\limsup_{n \to \infty} D_n \leq m \| R \|^2 \cdot 4 \cdot r^2 + \| U \|^2 r^2.$$  

Stirling formula, which is valid for every $n \in \mathbb{N}$ (cf. Feller (1968)), states that

$$\sqrt{2\pi n} (n/e)^n e^{\pi n} \epsilon < n! \leq \sqrt{2\pi n} (n/e)^n e^{\pi n}.$$  

This gives:

$$\frac{m! M!}{(m + M)!} \leq \sqrt{2\pi} \left( \frac{m M}{m + M} \right)^{1/2} \frac{m! M!}{(m + M)^{(m+M)} e} \leq \sqrt{2\pi} e \left( \frac{m M}{m + M} \right)^{1/2} \left( \frac{m M}{m + M} \right).$$
Now by applying (A.16) and (A.18) to (A.13) we obtain that for some constants $C’$ and $C”$

$$\liminf_{n \to \infty} \frac{\mathbb{P}(\tilde{J}_n | x)}{\mathbb{P}(\tilde{J}_n | x)} \leq \liminf_{n \to \infty} C’ \left(\frac{m + M}{mM}\right)^{(d+1)/2} \left(\frac{mC”}{M}\right)^m = 0, \quad (A.19)$$

as $\liminf_{n \to \infty} m/M \to 0$. Hence there exist $n$ such that the posterior probability of $\tilde{J}_n$ is smaller than the posterior probability of $\tilde{J}_n$. This contradicts the definition of $\tilde{J}_n$ and finishes the proof of the Lemma. \hfill \Box

**Proof of Proposition 2**

Assume that $(\frac{1}{2} \sum_{i=1}^{n} \|x_i\|^2)_{n=1}^{\infty}$ is bounded. We want to prove that $\liminf_{n \to \infty} m^{(r)}_{n}/n > 0$ for every $r > 0$.

Take $r_0$ from the statement of Corollary A.6. Note that, as in proof of Proposition A.7 it is enough to prove the statement of Proposition 2 for $r > r_0$.

Fix $r > r_0$. Suppose that $\liminf_{n \to \infty} m^{(r)}_{n}/n = 0$ and let $(n_k)_{k=1}^{\infty}$ be a sequence such that $\lim_{k \to \infty} m^{(r)}_{n_k}/n_k = 0$. By Proposition A.7 we obtain that $\lim_{k \to \infty} \|\overline{x}_{m^{(r)}_{n_k}}\| = \infty$ (otherwise we would obtain a contradiction). Let

$$I_a^n = \{ j \in J^{(r)}_n : |x_j| \leq r \}, \quad I_b^n = \{ j \in J^{(r)}_n : |x_j| > r \}.$$

Consider a partition $\tilde{J}_n$ obtained from $\tilde{J}_n$ by taking $I_a^n$ from $J^{(r)}_n$ and adding it to $J^M_n$, i.e.

$$\tilde{J}_n = \tilde{J}_n \setminus \{ J^{(r)}_n \setminus I^M_n \} \cup \{ J^{(r)}_n \setminus I^a_n, J^M_n \cup I^b_n \}.$$

When taking a quotient $\mathbb{P}(\tilde{J}_n | x)/\mathbb{P}(\tilde{J}_n | x)$ most factors in (2.2) cancel out, giving

$$\frac{\mathbb{P}(\tilde{J}_n | x)}{\mathbb{P}(\tilde{J}_n | x)} = (a + b)!M! \left(\frac{b(a + M)}{(a + b)M}\right)^{(d+2)/2} \frac{\det R_a \cdot \det R_{a+b} - \det R_{a+b} \cdot \det R_m}{\det R_a \cdot \det R_{a+b} \cdot \det R_m} \exp \{ \hat{D}_n \}^{1/2}, \quad (A.20)$$

where $M = |J^M_n|$, $a = |I^a_n|$, $b = |I^b_n|$ (in order to simplify the notation we skip the index $n_k$) and

$$\hat{D}_{nk} = (a + b) \left\| R_{a+b}^{-1} R^2 \mathbf{x}_{a+b} \right\|^2 + M \left\| R^{-1}_{a+b} R^2 \mathbf{x}_{a+b} \right\|^2 - b \left\| R_{a+b}^{-1} R^2 \mathbf{x}_b \right\|^2 - (a + M) \left\| R_{a+b}^{-1} R^2 \mathbf{x}_{a+b} \right\|^2.$$

in which $\mathbf{x}_{a+b} = \mathbf{x}_{I_{n_k}^a \cup I_{n_k}^b}$ and we define $\mathbf{x}_a, \mathbf{x}_M, \mathbf{x}_{a+b}$ similarly. Note that

$$\frac{(a + b)!M!}{(a + b)M!} = \frac{(b + 1)^{(a)}}{(a + b)^{(a)}} = \frac{b + 1}{M + 1} \xrightarrow{k \to \infty} 0, \quad (A.21)$$

since $\lim_{k \to \infty} (a + b)/n_k = \lim_{k \to \infty} m^{(r)}_{n_k}/n_k = 0$ and $\liminf_{n} M/n > 0$. For the similar reason

$$\frac{b(a + M)}{(a + b)M} \xrightarrow{k \to \infty} 1. \quad (A.22)$$

Now let us investigate $\hat{D}_n$. The notation is easier after a linear substitution $y_i = R^2 x_i$ (so that $\mathbf{y}^T = R^2 \mathbf{x}$), hence obtaining

$$\hat{D}_{nk} = (a + b) \left\| R_{a+b}^{-1} \mathbf{y}_{a+b} \right\|^2 + M \left\| R^{-1}_{a+b} \mathbf{y}_{a+b} \right\|^2 - b \left\| R_{a+b}^{-1} \mathbf{y}_b \right\|^2 - (a + M) \left\| R_{a+b}^{-1} \mathbf{y}_{a+b} \right\|^2.$$
Note that
\[ (a + b)\| R_{a+b}^{-1} y_{a+b} \|^2 - b \| R_b^{-1} y_b \|^2 = (a + b) \left\| R_{a+b}^{-1} \left( \frac{a}{a + b} y_a + \frac{b}{a + b} y_b \right) \right\|^2 - b \left\| R_b^{-1} y_b \right\|^2 = \]
\[ = \frac{1}{a + b} \left( a^2 \left\| R_{a+b}^{-1} y_a \right\|^2 + 2ab \left\| R_{a+b}^{-1} y_a \right\| \left\| R_{a+b}^{-1} y_b \right\| + b^2 \left\| R_{a+b}^{-1} y_b \right\|^2 - \right) - b(a + b)\| R_b^{-1} y_b \|^2 \]
\[ = \frac{1}{a + b} \left( a^2 \left\| R_{a+b}^{-1} y_a \right\|^2 + 2ab \left\| R_{a+b}^{-1} y_a \right\| \left\| R_{a+b}^{-1} y_b \right\| + b^2 \left\| R_{a+b}^{-1} y_b \right\|^2 - \right) \]
\[ T_1 = b^2 R_{a+b}^{-1} - b(a + b)R_a^{-1}. \] For two positive/negative matrices $M_1, M_2$ we write $M_1 \geq M_2$ when $M_1 - M_2$ is positive definite. Then
\[ T_1 = b^2 \left( R^2 + U^2 / (a + b) \right)^{-1} - b(a + b) \left( R^2 + U^2 / b \right)^{-1} = \]
\[ = b^2 (a + b) \left( (a + b) R^2 + U^2 \right)^{-1} - \left( b R^2 + U^2 \right)^{-1} = \]
\[ = b^2 (a + b) \left( (a + b) R^2 + U^2 \right)^{-1} - \left( b R^2 + U^2 \right)^{-1} = \]
\[ = -ab^2 (a + b) \left( (a + b) R^2 + U^2 \right)^{-1} R^2 \left( b R^2 + U^2 \right)^{-1} = \]
\[ = -ab \left( R^2 + U^2 / (a + b) \right)^{-1} R^2 \left( R^2 + U^2 / b \right)^{-1} \leq \]
\[ = -ab \left( R^2 + U^2 / (a + b) \right)^{-1} R^2 \left( R^2 + U^2 / b \right)^{-1} = -ab T_2 \]

Using (A.23) and (A.24) we have that
\[ (a + b)\| R_{a+b}^{-1} y_{a+b} \|^2 - b \| R_b^{-1} y_b \|^2 \leq \frac{1}{a + b} \left( a^2 \left\| R_{a+b}^{-1} y_a \right\|^2 + 2ab \left\| R_{a+b}^{-1} y_a \right\| \left\| R_{a+b}^{-1} y_b \right\| - ab \left\| T_2 y_b \right\|^2 \right) = \]
\[ = a \left( \frac{a}{a + b} \left\| R_{a+b}^{-1} y_a \right\|^2 + \frac{b}{a + b} \left( \left\| R_{a+b}^{-1} y_a \right\| \left\| R_{a+b}^{-1} y_b \right\| - \left\| T_2 y_b \right\|^2 \right) = \right) \]
\[ \leq \left( \frac{a}{a + b} \left\| R_{a+b}^{-1} y_a \right\|^2 + \frac{b}{a + b} \left( \left\| R_{a+b}^{-1} y_a \right\| \left\| R_{a+b}^{-1} y_b \right\| - \left\| T_2 y_b \right\|^2 \right) \right) \]
\[ = \left( \frac{a}{a + b} \right) \left\| y_a \right\|^2 + \frac{b}{a + b} \left\| y_b \right\|^2 - \left\| y_b \right\|^2 \right) \]

(A.25) where by $\nu_A$ we denote the minimal eigenvalue of the square matrix $A$. Similarly we note that
\[ M \left\| R_{M}^{-1} y_M \right\|^2 - (a + M) \left\| R_{a+M}^{-1} y_{a+M} \right\|^2 \leq M \left\| R_{M}^{-1} y_M \right\|^2 - (a + M) \left\| R_{a+M}^{-1} y_{a+M} \right\|^2 = \]
\[ = M \left\| R_M^{-1} y_M \right\|^2 - \frac{1}{a + M} \left( a^2 \left\| R_{a+M}^{-1} y_a \right\|^2 + 2aMR_{a+M}^{-1} y_a \cdot R_{a+M}^{-1} y_M + M^2 \left\| R_{a+M}^{-1} y_M \right\|^2 \right) = \]
\[ = \frac{1}{a + M} \left( (a + M) \left\| R_M^{-1} y_M \right\|^2 - a^2 \left\| R_{a+M}^{-1} y_a \right\|^2 + 2aMR_{a+M}^{-1} y_a \cdot R_{a+M}^{-1} y_M + M^2 \left\| R_{a+M}^{-1} y_M \right\|^2 \right) \leq \]
\[ \leq \frac{1}{a + M} \left( M \left\| R_M^{-1} y_M \right\|^2 - M \left\| R_{a+M}^{-1} y_M \right\|^2 \right) \leq 2aMR_{a+M}^{-1} y_a \cdot R_{a+M}^{-1} y_M \right) \]

(A.26)
We can write
\[(a + M)(R^2 + U^2/\mathcal{M})^{-1} - M(R^2 + U^2/(a + M))^{-1} =
\]
\[= M(a + M)(MR^2 + U^2)^{-1} - M(a + M)((a + M)R^2 + U^2)^{-1} =
\]
\[= aM(a + M)(MR^2 + U^2)^{-1}R^2((a + M)R^2 + U^2)^{-1} =
\]
\[= a(R^2 + U^2/M)^{-1}R^2(R^2 + U^2/(a + M))^{-1} \leq
\]
\[\leq a(R^2)^{-1}R^2(R^2)^{-1} = aR^2.
\]
\[\text{(A.27)}
\]
and hence, by (A.26) and (A.27)
\[M\|\text{R}_M\frac{y}{\sqrt{M}}\|^2 - (a + M)\|\text{R}^{-1}_{a+M}\frac{y}{\sqrt{a+M}}\|^2 \leq \frac{1}{a + M} \left( aM\|\text{R}\frac{y}{\sqrt{M}}\|^2 - 2aMR_{a+M}\frac{y}{\sqrt{a+M}} \cdot R^{-1}_{a+M}\frac{y}{\sqrt{a+M}} \right) =
\]
\[= a \frac{M}{a + M} \left( \|\text{R}\frac{y}{\sqrt{M}}\|^2 - 2R_{a+M}\frac{y}{\sqrt{a+M}} \cdot R^{-1}_{a+M}\frac{y}{\sqrt{a+M}} \right) \leq
\]
\[\leq a \frac{M}{a + M} \left( \|\text{R}\frac{y}{\sqrt{M}}\|^2 + 2\|\text{R}^{-1}\frac{y}{\sqrt{a+M}}\| \cdot \|\text{R}^{-1}\frac{y}{\sqrt{M}}\| \right).
\]
\[\text{(A.28)}
\]
Joining (A.25) and (A.28) we get that
\[\hat{D}_{\text{tk}} \leq a \left( \frac{a}{a + b} \|\frac{y}{\sqrt{a+M}}\|^2 + 2 \frac{b}{a + b} \|\frac{y}{\sqrt{a+M}}\| \left( \frac{2}{a+M} \|\frac{y}{\sqrt{a+M}}\| - \frac{L_2^2}{a+M} \|\frac{y}{\sqrt{a+M}}\| \right) +
\]
\[+ \frac{M}{a + M} \left( \|\text{R}\frac{y}{\sqrt{M}}\|^2 + 2\|\text{R}^{-1}\frac{y}{\sqrt{a+M}}\| \cdot \|\text{R}^{-1}\frac{y}{\sqrt{M}}\| \right) \right).
\]
\[\text{(A.29)}
\]
Note that since \(\|\frac{y}{\sqrt{a+M}}\| \xrightarrow{k \to \infty} \infty\) and \(\|\frac{y}{\sqrt{a+M}}\| \leq \sqrt{R^2} \cdot \|\frac{x}{\sqrt{a+M}}\| \leq \sqrt{R^2} \sqrt{a+M} \leq \sqrt{R^2} r \) (\(\sqrt{a} \) is the largest eigenvalue of the square matrix \(A\)) we have that
\[\frac{b}{a + b} \|\frac{y}{\sqrt{a+M}}\| \geq \|\frac{y}{\sqrt{a+M}}\| \xrightarrow{k \to \infty} \infty.
\]
\[\text{(A.30)}
\]
Moreover \(\|\frac{y}{\sqrt{M}}\| \leq \sqrt{R^2} \cdot \|\frac{x}{\sqrt{a+M}}\| \leq \sqrt{R^2} \sqrt{a+M} \leq \sqrt{R^2} r \) and therefore by (A.29) and (A.30) we have
\[\lim_{k \to \infty} D_{\text{tk}} = -\infty
\]
\[\text{(A.31)}
\]
By taking (A.20) and using (A.21), (A.22) and (A.31) we obtain that \(\lim_{k \to \infty} \frac{\sqrt{P(D_{\text{tk}} | x)}}{\sqrt{\text{E}_{\text{tk}} | x}} = 0\); from this contradiction the proof follows.

**Proof of Proposition 3**

Let \(K_r\) be the space of all closed and convex subsets of \(B(0, r)\). Note that \(M_\Delta\) is closed in \((F_\Delta(K_r), \bar{d}_P)\) as an intersection of the set of maximisers of \(M_\Delta\) in \((F_\Delta(K_r), \bar{d}_P)\) and the subspace of \(P\)-partitions, both of them being closed subspaces of \((F_\Delta(K_r), \bar{d}_P)\).

By Theorem 4.12 we know that \(E \subseteq M_\Delta\). Now the proof of Proposition 3 follows from the following Lemma A.8.
Lemma A.8. Let \((X, d)\) be a finitely compact metric space, \(D \subseteq X\) a closed set and 
\((a_n)_{n=1}^\infty\) a bounded sequence in \(X\). If every converging subsequence of 
\((a_n)_{n=1}^\infty\) has a limit in \(D\) then \(\text{dist}(a_n, D) \to 0\), where \(\text{dist}(\cdot, \cdot)\) is the distance function, i.e.

\[
\text{dist}(x, D) = \inf_{y \in D} d(x, y).
\]

**Proof.** Suppose that \(\lim \sup \text{dist}(a_n, D) > 0\). Then there exist a subsequence \((a_{n_k})_{k=1}^\infty\)
and \(\varepsilon > 0\) such that \(\text{dist}(a_{n_k}, D) > \varepsilon > 0\). This contradicts the fact that 
\((a_{n_k})_{k=1}^\infty\) as a bounded sequence in \(X\) has a converging subsequence whose limit must belong to 
the closed set \(D\). \(\Box\)

**Proof of Proposition 4**

**Lemma A.9.** If \(P\) is a measure on \((\mathbb{R}^d, \mathcal{B})\) with bounded support and absolutely continuous
with respect to the Lebesgue measure then for every \(\alpha > 0\)

\[
\Psi(\alpha) := \inf_{A \in \mathcal{K}_{r}} \sup_{\alpha_{1}, \alpha_{2} \in B \atop \alpha_{1} \cup \alpha_{2} = A} P(A_{1}) \cdot P(A_{2}) \cdot \|E(A_{1}) - E(A_{2})\|^2 > 0
\]

where \(E(B) = \int_{B} xdP(x)/P(B)\) for \(B \in \mathcal{B}\).

**Proof.** Fix \(\alpha > 0\). As an easy consequence of Theorem 4.10 we obtain that \(P(\cdot)\) is a continuous function
in \((\mathcal{K}_{r}, \mathcal{G}_{H})\). Therefore \(\mathcal{K}_{r}^\alpha := \{A \in \mathcal{K}_{r} : P(A) \geq \alpha\}\) is a closed
subspace of compact (by Theorem 4.9) topological space, therefore it is compact itself.

Assume that the support of \(P\) is contained in the ball \(B(0, r)\) and let \(r > 1\).
Consider the function

\[
\varphi(A) = \sup_{A_{1}, A_{2} \in B \atop A_{1} \cup A_{2} = A} P(A_{1}) \cdot P(A_{2}) \cdot \|E(A_{1}) - E(A_{2})\|^2 \geq 0
\]

in the compact topological space \((\mathcal{K}_{r}, \mathcal{G}_{H})\). We prove that this function is continuous.

Firstly note that since we operate in a bounded space then if \(P(B) \to 0\) then
\(\int_{B} \psi \to 0\). From this it can be easily seen that for every \(\varepsilon > 0\) there exist \(\delta > 0\) such that if \(d_{P}(A, B) < \delta\) then \(\|E(A) - E(B)\| < \varepsilon\) for \(A, B \in \mathcal{K}_{r}^\alpha\).

Fix \(0 < \varepsilon < 1\). There exist \(\delta_{1} < \varepsilon\) such that if \(d_{P}(A, A') < \delta_{1}\) then \(\|E(A) - E(A')\| < \varepsilon/2\). There exist \(\delta_{2}\) such that if \(g_{H}(A, A') < \delta_{2}\) then \(d_{P}(A, A') < \delta_{1}\) (this is because of Theorem 4.10 and the fact that \((\mathcal{K}_{r}, \mathcal{G}_{H})\) is compact and therefore the continuity implies the uniform continuity). Let us take \(A, A' \in \mathcal{K}_{r}\) such that \(g_{H}(A, A') < \delta_{2}\). Let \(A_{1}, A_{2} \in \mathcal{K}_{r}\) be such that \(A_{1} \cap A_{2} = \emptyset, A_{1} \cup A_{2} = A\) and

\[
\varphi(A) - \varepsilon \leq P(A_{1}) \cdot P(A_{2}) \cdot \|E(A_{1}) - E(A_{2})\|^2
\]

Consider \(A'_{1} = A_{1} \cup (A' \setminus A) \setminus (A \setminus A')\) and \(A'_{2} = A_{2} \setminus (A \setminus A')\). Then \(A_{1} \cap A'_{2} = \emptyset, A_{1} \cup A'_{2} = A'\) and

\[
d_{P}(A_{1}, A'_{1}), d_{P}(A_{2}, A'_{2}) \leq d_{P}(A, A') \leq \delta_{1}.
\]

Therefore \(|P(A_{i}) - P(A'_{i})| < \delta_{1} < \varepsilon, \|E(A_{i}) - E(A'_{i})\| < \varepsilon/2\) for \(i = 1, 2\). Since
\(|P(A_{i})| \leq 1\) and \(\|E(A_{i})\| \leq r\) for \(i = 1, 2\) we get

\[
|P(A_{1}) \cdot P(A_{2}) \cdot \|E(A_{1}) - E(A_{2})\|^2 - P(A'_{1}) \cdot P(A'_{2}) \cdot \|E(A'_{1}) - E(A'_{2})\|^2| < 50r^{2}\varepsilon.
\]
By (A.32) and (A.33) we obtain
\[ \varphi(A) - \varepsilon - 50r^2\varepsilon \leq P(A') \cdot P(A_2') \cdot \|E(A_1) - E(A_2')\|^2 \leq \varphi(A'). \]
By symmetry we get \( \varphi(A') - \varepsilon - 50r^2\varepsilon \leq \varphi(A) \) which means that \( |\varphi(A) - \varphi(A')| < (1 + 50r^2)\varepsilon \) for \( g_H(A, A') < \delta_2 \) which proofs the continuity of \( \varphi \) in the topological space \( (K_r, g_H) \). Therefore by Weierstrass Theorem we get that
\[ \inf_{\Delta(A) \geq \alpha} \varphi(A) = \varphi(A_0) \]
for some \( A_0 \in K_r \) such that \( P(A_0) \geq \alpha \). It is easy to see that \( \varphi(A_0) > 0 \) (it is enough to divide \( A_0 \) into two sets of positive measure by a hyperplane so that the center of masses of two parts do not coincide) and the Lemma follows.

**Proof of Proposition 4:** Fix \( K > 0 \). Let \( \Psi \) be defined as in the statement of Lemma A.9. We first prove that for \( \varepsilon = \frac{1}{8}e\Psi(K^{-1}) \) if \( \|\Sigma\| < \varepsilon \) then every maximiser of the \( \Delta \) function is of size larger than \( K \). Take any finite partition \( G \) of \( \mathbb{R}^d \) that consists of at most \( K \) convex sets with positive \( P \) measure. Let \( A \in G \) be the set of the largest probability in \( G \); note that \( P(A) \geq K^{-1} \). By definition of \( \Psi \) we can divide \( A \) into two sets \( A_1, A_2 \) \( (A_1 \cup A_2 = A, A_1 \cap A_2 = \emptyset) \) such that
\[ P(A_1) \cdot P(A_2) \cdot \|E(A_1) - E(A_2)\|^2 > \Psi(K^{-1})/2. \]
Let \( G' = G \cup \{A_1, A_2\} \setminus \{A\} \). Then
\[ \Delta(G') - \Delta(G) = \frac{1}{2} (P(A_1)\|R \cdot E(A_1)\|^2 + P(A_2)\|R \cdot E(A_2)\|^2 - P(A)\|R \cdot E(A)\|^2) - \]
\[ - P(A_1)\ln \frac{1}{P(A_1)} - P(A_2)\ln \frac{1}{P(A_2)} + P(A)\ln \frac{1}{P(A)}. \]
(A.34)
It is straightforward to verify that \( p\ln p^{-1} \in [0, \frac{1}{e}] \) for \( p \in [0, 1] \) and, since \( P(A_1)E(A_1) + P(A_2)E(A_2) = P(A)E(A) \) we have
\[ P(A_1)\|R \cdot E(A_1)\|^2 + P(A_2)\|R \cdot E(A_2)\|^2 - P(A)\|R \cdot E(A)\|^2 = \frac{P(A_1)P(A_2)}{P(A)}\|R \cdot (E(A_1) - E(A_2))\|^2. \]
Therefore by (A.34) and Lemma A.9 we get
\[ \Delta(G') - \Delta(G) \geq \frac{P(A_2)P(A_2)}{P(A)}\|R \cdot (E(A_1) - E(A_2))\|^2 - 2e^{-1} \geq \]
\[ \geq \frac{P(A_1)P(A_2)}{P(A)}\|R \cdot E(A_1)\|^2 - 2e^{-1} = \]
\[ = \frac{P(A_1)P(A_2)}{P(A)}\|E(A_1) - E(A_2)\|^2 - 2e^{-1} \geq \]
\[ \geq \varepsilon^{-1}P(A_1)P(A_2)\|E(A_1) - E(A_2)\|^2 - 2e^{-1} \geq \]
\[ \geq \varepsilon^{-1}\Psi(K^{-1})/2 - 2e^{-1} > 2e^{-1} > 0. \]
Hence \( G \) is not a maximiser of \( \Delta \) function.

Now let \( X_1, X_2, \ldots, X_n \rightharpoonup P \) and \( \mathcal{A}_n \) be the family of convex hulls of groups of observations defined by the sequence of the MAP partitions based on \( X_1, \ldots, X_n \) (where the MAP partitions were computed in the model with the within group covariance.
matrix of the norm less than $\varepsilon$). Suppose that there exists a subsequence $(n_i)_{i=1}^\infty$ such that $|\mathcal{A}_{n_i}| \leq K$ for $i \in \mathbb{N}$. By the compactness of the space $(\mathcal{F}_K(\mathcal{K}), \overline{\rho_n})$ (cf. Remark 4.11) we get that there is a subsequence $(\mathcal{A}_{n_j})$ that is convergent in this space to a $P$-partition $\mathcal{E}$ of $\mathbb{R}^d$ which is a maximiser of $\Delta$ (cf. Theorem 4.12). By our previous analysis, $|\mathcal{E}| > K$. On the other hand the probabilities of sets in $\mathcal{A}_n$ are separated from 0 (this is a consequence of Corollary 4.1) and this yields a contradiction.

**Proofs for Section 3 (Examples)**

**Uniform distribution on an interval**

We now find the convex partition that maximises $\Delta$ if $P$ is a uniform distribution on $[-1, 1]$. Since it is convex it is defined by the lengths of consecutive subsegments of $[-1, 1]$; let those be $2p_1, \ldots, 2p_n$. Let $s_k = \sum_{i=1}^n p_i$ for $k \geq 1$ and $\mathcal{A}_k = [s_{k-1}, s_k]$, where $s_0 = 0$. Then it follows that the optimal partition maximises

$$F(p_1, \ldots, p_n) = \rho \sum_{i=1}^n p_i(2s_{i-1} - 1 + p_i)^2 + \sum_{i=1}^n p_i \ln p_i,$$

where $\rho = R^2/2$, with the constraint $\sum_{i=1}^n p_i = 1$. This problem can be solved using Lagrange multipliers. We are looking for the local maximum of a function

$$F_\lambda(p_1, \ldots, p_n) = F(p_1, \ldots, p_n) - \lambda \sum_{i=1}^n p_i.$$

We now compute its partial derivatives

$$\frac{\partial}{\partial p_k} \sum_{i=1}^n p_i(2s_{i-1} - 1 + p_i)^2 = \frac{\partial}{\partial p_k} p_k(2s_{k-1} - 1 + p_k)^2 + \frac{\partial}{\partial p_k} \sum_{i=k+1}^n p_i(2s_{i-1} - 1 + p_i)^2 =$$

$$= (2s_{k-1} - 1 + p_k)^2 + 4(2s_{k-1} - 1 + p_k) + 4\sum_{i=k+1}^n p_i(2s_{i-1} - 1 + p_i) =$$

$$= (2s_{k-1} - 1 + p_k)^2 - 2p_k \cdot (2s_{k-1} - 1 + p_k) + 4\sum_{i=k}^n p_i(2s_{i-1} - 1 + p_i) =$$

$$= (2s_{n-1} - 1)^2 - p_k^2 + 4\sum_{i=k}^n p_i(2s_{i-1} - 1 + p_i) =$$

$$= (2s_{n-1} - 1)^2 - p_k^2 = 1 - p_k^2$$

from which

$$\frac{\partial}{\partial p_k} F_\lambda(p_1, \ldots, p_k) = \ln p_k - \rho p_k^2 + 1 + \rho - \lambda.$$

If all partial derivatives are zero then $\ln p_k - \rho p_k^2 = C$ for all $1 \leq k \leq n$ and some $C \in \mathbb{R}$. Therefore we may restrict the search of maximum of $F$ on the set of probability weights to the subset where $\ln p_k - \rho p_k^2 = C$. On that set function $\tilde{F}$ is equal to

$$\tilde{F}(p_1, \ldots, p_n) = \rho \sum_{i=1}^n p_i(2s_{i-1} - 1 + p_i)^2 + \sum_{i=1}^n p_i (C + \rho p_i^2)$$

and the derivative of this function is equal to

$$\frac{\partial}{\partial p_k} \tilde{F}(p_1, \ldots, p_n) = \rho(1 - p_k^2) + 3\rho p_k^2 + C.$$

If we apply Lagrange multipliers to the function $\tilde{F}$ then we obtain a condition of the form $p_k^2 + C = 0$ for all $k \leq n$ and some $C \in \mathbb{R}$. Since $p_k \in [0, 1]$ and $\sum_{i=1}^n p_k = 1$ we
get that \( p_1 = p_2 = \ldots = p_n = 1/n \). Here we also have the maximum of \( F \) on the set of probability weights.

For \( p_1 = p_2 = \ldots = p_n = 1/n \) we have

\[
E(\mathbb{E}(X \mid A))^2 = \frac{1}{\pi} \sum_{i=1}^{n} \frac{(2i-1)}{(2i)^2 - 1 + \frac{1}{\pi}} = \frac{1}{\pi} \sum_{i=1}^{n} \frac{1}{(2i)^2 - 1 + \frac{1}{\pi}} = \frac{1}{\pi} \left( \frac{1}{n^2} \sum_{i=1}^{n} \frac{1}{(2i)^2 - 1 + \frac{1}{\pi}} \right) = \frac{1}{\pi} \left( \frac{1}{n^2} \sum_{i=1}^{n} \frac{1}{(2i)^2} \right) = \frac{1}{\pi} \left( \frac{1}{n^2} \sum_{i=1}^{n} \frac{1}{2i} \right) = \frac{1}{\pi} \left( \sum_{i=1}^{n} \frac{1}{2i} \right) = \frac{1}{\pi} \left( \sum_{i=1}^{n} \frac{1}{2i} \right)
\]

and hence

\[
f(n) := F(1/n, \ldots, 1/n) = \frac{\rho}{3} \left( 1 - \frac{1}{n^2} \right) - \ln n = \frac{\rho}{3} \left( 1 - \frac{1}{n^2} \right) - \ln n.
\]

The derivative of the function \( \frac{\rho}{3} \ln x - \frac{2\rho}{3\pi} + \frac{1}{2} \) so it is increasing for \( x < \frac{\sqrt{2\rho}}{3} \) and decreasing for \( x > \frac{\sqrt{2\rho}}{3} \) and therefore \( f(n) \) achieves its maximum for \( [1/\sqrt{2\rho}] \) or \( [1/\sqrt{2\rho}] \), where \( \lfloor x \rfloor \) and \( \lfloor x \rfloor \) are the largest integer not greater than \( x \) and the smallest integer not less than \( x \) respectively. Hence setting \( \Sigma \leq \frac{1}{10} \) leads to the overestimation of the number of clusters.

**Exponential distribution**

Let \( p_{[s,t]} = P([s,t]) \) and \( e_{[s,t]} = \int_{s}^{t} \lambda e^{x} \, dx \). Then

\[
p_{[s,t]} = e^{-s} - e^{-t}, \quad e_{[s,t]} = \frac{(1+s)e^{-s} - (1+t)e^{-t}}{e^{-s} - e^{-t}} = s + 1 - \frac{t - s}{e^{t-s} - 1}
\]

Take any convex partition of \( \mathbb{R} \), \( A = \{[0,t_1], [t_1, t_2], \ldots \} \). Let \( p_i = p_{[t_{i-1}, t_i]} \) and \( e_i = e_{[t_{i-1}, t_i]} \). Then

\[
\Delta(A) = \frac{R^2}{2} \sum_{i=1}^{\infty} p_i e_i^2 + \sum_{i=1}^{\infty} p_i \ln p_i
\]

Assume that this sum is finite. Let \( s \in [t_{n-1}, t_n] \) and \( A' = (A \setminus \{[t_{n-1}, t_n]\}) \cup \{[t_{n-1}, s], [s, t_n]\} \) and let \( p_{n,1}, e_{n,1}, p_{n,2}, e_{n,2} \) be defined as \( p_i, e_i \), but for the intervals \([t_{n-1}, s] \) and \([s, t_n] \) respectively. Then

\[
\Delta(A') - \Delta(A) = \frac{R^2}{2} (p_{n,1} e_{n,1}^2 + p_{n,2} e_{n,2}^2 - p_{n,1} e_{n,1}^2 + p_{n,2} e_{n,2}^2) = \frac{p_{n,1} p_{n,2}}{p_n} \left| e_{n,1} - e_{n,2} \right|^2.
\]

Note that \( p_{n,1} e_{n,1} + p_{n,2} e_{n,2} = p_n e_n \). Using this we can compute that

\[
p_{n,1} e_{n,1}^2 + p_{n,2} e_{n,2}^2 - p_{n,1} e_{n,1}^2 = \frac{p_{n,1} p_{n,2}}{p_n} \left| e_{n,1} - e_{n,2} \right|^2.
\]

Choose \( s \) so that \( p_{n,1} = p_{n,2} = \frac{1}{2} p_n \). Then it can be computed that \( s = t_{n-1} + \ln 2 - \ln(1 - e^{t_{n-1} - t_n}) \). Recall that

\[
e_{n,2} = s + 1 - \frac{t_n - s}{e^{t_n-s-1}}.
\]

Since \( e_{n,1} \in [t_{n-1}, s] \), \( t_n - s > t_n - t_{n-1} - \ln 2 \) and \( x \mapsto x/(e^x - 1) \) is a decreasing function, we obtain that

\[
|e_{n,2} - e_{n,1}| = e_{n,2} - e_{n,1} > e_{n,2} - s = 1 - \frac{t_n - s}{e^{t_n-s-1}} > 1 - \frac{t_n - t_{n-1} - \ln 2}{e^{t_n-t_{n-1}-\ln 2-1}}.
\]
Hence if \( t_n - t_{n-1} > 3 \) then \(|e_{n,2} - e_{n,1}| > \frac{1}{2} \). By (A.35) and (A.36) we get that for \( t_n - t_{n-1} > 3 \)
\[
\Delta(A') - \Delta(A) = \frac{R^2}{2} \cdot \frac{1}{4} p_n |e_{n,2} - e_{n,1}|^2 - p_n \ln 2 > p_n \left( \frac{R^2}{32} - \ln 2 \right).
\]
It means that for \( \Sigma < (32 \ln 2)^{-1} \) we increase the value of the function \( \Delta \) by dividing every segment of length larger than 3. As a result, no finite convex partition can be a maximiser of the function \( \Delta \) in this case.

**Mixture of two normals**

Let \( g_a(x) = \frac{1}{\sqrt{2\pi}} \left( e^{-(x-a)^2/2} + e^{-(x+a)^2/2} \right) \) be the density of a mixture of two normal distributions, \( N(a,1) \) and \( N(-a,1) \). We prove that for \( a > 1 \) this distribution is bi-modal. It is easy to compute its derivatives:
\[
g_a'(x) = -\frac{1}{2\sqrt{2\pi}} (e^{-(x-a)^2/2} + e^{-(x+a)^2/2})
\]
\[
g_a''(x) = -\frac{1}{2\sqrt{2\pi}} \left( e^{-(x-a)^2/2} + e^{-(x+a)^2/2} - (x-a)^2 e^{-(x-a)^2/2} - (x+a)^2 e^{-(x+a)^2/2} \right).
\]
Hence \( g_a'(0) = 0 \) and \( g_a''(0) = -\frac{1}{2\sqrt{2\pi}} e^{-a^2/2} (2 - 2a^2) > 0 \), which means that 0 is a local minimum of \( g_a \). Moreover the equation \( g_a'(x) = 0 \) is equivalent to
\[
U_a(x) := e^{2ax} - \frac{a+x}{a-x}
\]
Let us look for the solutions of this equation on \( x \in (0, \infty) \). It is clear that there are no solutions for \( x \geq a \). It is straightforward to verify that \( U_a(0) = 0 \), \( U_a(a^-) = -\infty \). Moreover \( U_a'(x) = 0 \) is for \( x \in (0, a) \) equivalent to \( V_a(x) := (x-a)^3 e^{2ax} = 1 \). We have
\[
V_a'(x) = 2(x-a)e^{2ax}(1 + a(x-a))
\]
and hence \( V_a'(x) = 0 \) has exactly one solution for \( x \in (0, a) \) (which is \( \frac{a^2 - 1}{a} \)). Since \( V_a(0) = a^3 > 1 \) and \( V_a(a) = 0 \) we deduce that \( V_a(x) = 1 \) has exactly one solution in \( (0, a) \), and so the equation \( U_a'(x) = 0 \). It is straightforward to verify that \( U_a'(0) > 0 \) and therefore \( U_a(x) = 0 \) has exactly one solution for \( x \in (0, a) \).

It follows that \( g_a \) has exactly one zero on \((0, \infty)\); by symmetry there is also exactly one zero on \((-\infty, 0)\), so there are 3 zeros in total. Since we know that \( x = 0 \) is the local minimum of \( g_a \) and \( \lim_{x \to \pm \infty} g_a(x) = 0 \) it follows that \( g_a \) is bimodal.

**Proof of Theorem 4.3**

Take \( d = 1 \) and \( \alpha = \tau = \sum = 1 \). Let \( y_1, \ldots, y_n \in \mathbb{R}^d \). Take any partition \( \mathcal{J} \) of \([n]\). Let \( J_n \in \mathcal{J} \) be the cluster containing \( n \) and assume that \(|J_n| \geq 2 \). Let \( \mathcal{J}_{n,\{n\}} \) be obtained by creating a singleton out of \( n \), i.e. \( \mathcal{J}_{n,\{n\}} = \mathcal{J} \setminus \{J_n\} \cup \{J_n \setminus \{n\}, \{n\} \} \). By (2.2) it is easy to show that the quotient \( P(\mathcal{J}_{n,\{n\}}) \mid y_1, \ldots, y_n) / P(\mathcal{J} \mid y_1, \ldots, y_n) \) is equal to
\[
\frac{1}{|J_n| - 1} \left( \frac{|J_n| + 1}{2|J_n|} \right) \exp \left\{ \frac{y_n^2}{4} + \frac{(\sum y_{J_n \setminus \{n\}})^2}{2|J_n|} - \frac{(\sum y_{\{n\}})^2}{2(|J_n| + 1)} \right\}.
\]
(A.37)
The exponent in the formula above is equal to

\[ y_n^2 \frac{|J_n| - 1}{4(|J_n| + 1)} - y_n \frac{\sum y_{J_n\setminus\{n\}}}{|J_n| + 1} + \frac{(\sum y_{J_n\setminus\{n\}})^2}{2|J_n||J_n| + 1}, \]  

(A.38)

which is a convex quadratic function of \( y_n \). Now, since \(|J_n| \geq 2\), it follows that

\[ \frac{|J_n| - 1}{4(|J_n| + 1)} \geq \frac{1}{12} \quad \text{and} \quad \left\lfloor \frac{\sum y_{J_n\setminus\{n\}}}{|J_n| + 1} \right\rfloor \leq |y_{J_n\setminus\{n\}}|. \]  

(A.39)

Now let \( L = 2 \cdot 18^4 \) and \( \bar{x}_m = 18^m \). We show that if

\[ n \leq L^{m+1}, \quad y_n \geq \bar{x}_m, \quad \text{and} \quad |y_1, \ldots, y_{n-1}| \leq \bar{x}_{m-1} \]  

(*)

then \( h_{J_n}(y_1, \ldots, y_n) > 1 \) (regardless of \( J_n \)) and hence in MAP partition for \([n]\) based on data \((y_i)_{i=1}^n\) singleton \([n]\) forms a separate cluster. Assume (*). Note that if \( n \leq L^{m+1} \) and \( |y_1, \ldots, y_{n-1}| \leq \bar{x}_{m-1} \) then by (A.37), (A.38) and (A.39) we obtain that

\[ h_{J_n}(y_1, \ldots, y_n) \geq \frac{1}{L^{m+1}} \sqrt{\frac{1}{2}} \exp \left\{ \frac{1}{12} y_n^2 - \bar{x}_{m-1} y_n \right\} =: l(y_n). \]

Now as we can easily compute zeros of quadratic function, \( l(y_n) \geq 1 \) is implied by

\[ y_n \geq 6(\bar{x}_{m-1} + \sqrt{\bar{x}_{m-1}^2 + \frac{1}{3}(m + 1) \ln L + (\ln 2)/2}). \]

It can be easily proved by induction that \( 3\bar{x}_{m-1}^2 > \frac{1}{6}((m + 1) \ln L + (\ln 2)/2) \) for \( m \geq 2 \) (note that the left-hand side is geometric with respect to \( m \), while the right-hand side is linear) and therefore

\[ 6(\bar{x}_{m-1} + \sqrt{\bar{x}_{m-1}^2 + \frac{1}{3}(m + 1) \ln L + (\ln 2)/2}) < 18\bar{x}_{m-1} = \bar{x}_m \]

and as \( y_n \geq \bar{x}_m \) we have that \( h_{J_n}(y_1, \ldots, y_n) > 1 \).

Note that if \((y_n)_{n=1}^\infty \) is a sequence whose terms belong to \( \{\bar{x}_m : m \in \mathbb{N}\} \) then if for some \( m \in \mathbb{N} \)

\[ n \leq L^{m+1}, \quad y_n \geq \bar{x}_m, \quad \text{and} \quad y_1, \ldots, y_{n-1} < y_n \]  

(*)'

then condition (*) holds with some \( m' \geq m \) (the one that satisfies \( \bar{x}_{m'} = y_n \)). Indeed, if (*)' is satisfied and \( y_n = \bar{x}_{m'} \) then as \( y_1, \ldots, y_{n-1} < y_n \) we have \( y_1, \ldots, y_{n-1} \leq \bar{x}_{m'-1} \). Moreover \( \bar{x}_{m'} = y_n = \bar{x}_m \) and hence \( m' \geq m \) and \( n \leq L^{m+1} \leq L^{m'+1} \) and hence (*) is satisfied.

We now give an example of probability weights \((p_m)_{m \in \mathbb{N}}\) such that the following probability distribution \( P = \sum_{m=1}^{\infty} p_m \delta_{x_m} \) has a finite fourth moment and if \((X_n)_{n=1}^\infty \overset{iid}{\sim} P\) then (*)' happens almost surely infinitely many times. Let \( q = L^{-1} \) and \( p_m = (1 - q)q^{m-1} \). It is straightforward to check that in this case \( P \) has finite fourth moment, as

\[ \sum_{m=1}^{\infty} p_m x_m^4 = (1 - L^{-1}) \sum_{m=1}^{\infty} \frac{(18^m)^4}{(2 \cdot 18^m)^{m-1}} = 18^4(1 - L^{-1}) \sum_{m=1}^{\infty} \frac{1}{2} < \infty. \]

Now let \( s_m = \sum_{i=1}^{m} p_i = 1 - q^m \). Then \( s_m^{L^m} \to 0 \). Let

\[ n_m = \sum_{i=0}^{m} L^i = \frac{L^{m+1} - 1}{L - 1} < L^{m+1} \]
and $A_m = \{\max_{n_{m-1} \leq i < n_m} X_i \geq \tilde{x}_m\}$. Then the probability of $A_m$ is equal to $1 - s_{m-1}^{L_m}$ which converges to $1 - e^{-L}$. By the Borel-Cantelli Lemma, it follows that almost surely infinitely many of the events $A_m$ happens. Let $(x_n)_{n=1}^\infty$ be a realisation of $(X_n)_{n=1}^\infty$ and let $\{m_k\}_{k=1}^\infty$ be an increasing sequence of all indices $m$ for which $A_m$ hold. Now let $$\hat{n}_m = \min\{n_{m-1} \leq n < n_m: x_n = \max_{n_{m-1} \leq i < n_m} x_i\}.$$ Then $x_{\hat{n}_m k} \geq \tilde{x}_{mk}$ for $k \in \mathbb{N}$. Let $(k_i)_{i=1}^\infty$ be a sequence such that $x_{\hat{n}_k i} < x_{\tilde{n}_k i}$ for $k < k_i$ (such subsequence exists since $\tilde{x}_m \to \infty$). Note that $x_{\tilde{n}_k i} \geq \tilde{x}_{mk}$, $\hat{n}_m k_i < n_{mk_i} < L^{mk_i+1}$ and also

$$x_l < x_{\hat{n}_m k_i} \quad \text{if } m(l) = m_{k_i},$$

$$x_l \leq x_{\hat{n}_m(i)} < x_{\tilde{n}_m k_i} \quad \text{if } m(l) = m_k \text{ for some } k < k_i,$$

$$x_l < \tilde{x}_{m(l)} \leq x_{\tilde{n}_m k_i} \quad \text{otherwise},$$

where $m(l) = \min\{m \in \mathbb{N}: n_m > l\}$. From this it follows that for every $i \in \mathbb{N}$ condition (**) is satisfied with $n = \hat{n}_n$ and $m = m_k$. This proves that almost surely the MAP partition creates a new cluster out of a new observation infinitely many times.

**Proof of Theorem 4.4**

Let $X_1, X_2, \ldots \overset{iid}{\sim} P = \text{Exp}(1)$ and $\mathcal{J}_n$ be the MAP partition computed on the basis of $X_1, \ldots, X_n$. We can assume that every value of $X_i$ is unique and hence by Proposition 1 we obtain that convex hulls of sets in $\mathcal{J}_n$ are pairwise disjoint. Let $M_n = \max\{X_1, \ldots, X_n\}$ and $\mathcal{T}_n$ be a partition of $[0, M_n]$ into $|\mathcal{J}_n|$ segments that induce $\mathcal{J}_n$, i.e. for every $J \in \mathcal{J}_n$ there exist $I \in \mathcal{T}_n$ such that $\{x_j: j \in J\} \subset I$.

Suppose, contrary to our claim, that the sequence $|\mathcal{J}_n|$ is bounded by some $K \in \mathbb{N}$. In order to use the results regarding the behaviour of $\Delta$ function in the exponential case (Section 3.2) we need to ensure that there exist a sequence of segments $\mathcal{T}_n \in \mathcal{T}_n$ and subsequence $|n_k\rangle_{k=1}^\infty$ such that

(i) $\lim_{k \to \infty} |\mathcal{T}_{n_k}| = \infty$, where $|\cdot|$ is segment length,

(ii) $L := \limsup_{k \to \infty} \inf |\mathcal{T}_{n_k}| < \infty$.

We now construct such a sequence. Let $I_n^1$ be the sequence of the longest segments in $\mathcal{T}_n$ (i.e. $\text{diam} I_n^1 = \max_{I \in \mathcal{T}_n}$). Since almost surely $M_n \to \infty$ and the number of clusters within the MAP partitions is assumed to be bounded, it follows that $|I_n^1| \to \infty$. If $\limsup_{k \to \infty} \inf I_{n_k}^1 < \infty$, set $\mathcal{T}_n = I_n^1$ and $n_k = k$. Otherwise proceed inductively: having constructed the sequence $(I_n^1)_{n=1}^\infty$ and subsequence $(n_k)_{k=1}^\infty$ such that $\lim_{k \to \infty} |I_{n_k}^1| = \infty$ do as follows: if $\limsup_{k \to \infty} \inf I_{n_k}^1 < \infty$ set $\mathcal{T}_n = I_n^1$ and $n_k = n_k$. If not, let $I_n^{i+1}$ be the sequence of the longest segments to the left of $I_n^i$ in $\mathcal{T}_n$ (i.e. $|I_n^{i+1}| = \max\{|I|: I \in \mathcal{T}_n, \sup I \leq \inf I_{n_k}^i\}$). By the assumption about bounded number of clusters we obtain that $\lim_{k \to \infty}|I_{n_k}^{i+1}| = \infty$. Note that this procedure has to stop after at most $K$ iterations, because by construction there are at most $K - i$ segments to the left of $I_n^i$. Therefore requirement (ii) is bound to be finally satisfied.

Note that, because of (i) and (ii) we can deduce that $\liminf_{k \to \infty} P(\mathcal{T}_{n_k}) \geq P((L, \infty)) =: p > 0$.

Let $\mathcal{J}_n$ be the cluster in $\mathcal{J}_n$ induced by $\mathcal{T}_n$, i.e. $\mathcal{J}_n = \{i \leq n: X_i \in \mathcal{T}_n\}$. Let $(\mathcal{I}_n, \mathcal{T}_n)$ be a partition of $\mathcal{T}_n$ into two equally probable segments, which induces partition $(\mathcal{J}_n, \mathcal{J}_n')$ of $\mathcal{J}_n$. Let $\mathcal{J}_n'$ be obtained from $\mathcal{J}_n$ by replacing $\mathcal{J}_n$ by two sets $\mathcal{J}_n'$ and
\( \mathcal{J}_n^\delta \). Then
\[
\frac{P(\mathcal{J}_n | x)}{P(\mathcal{J}_n | x)} = C \frac{a_{n,1}! a_{n,2}!}{a_n!} \left( \frac{a_n}{a_{n,1}a_{n,2}} \right)^{3/2} \frac{R_{\delta n}}{R_{a_{n,1}a_{n,2}}} \exp \{ D_n \}^{1/2},
\]
where \( a_n, a_{n,1}, a_{n,2} \) are the sizes of \( \mathcal{J}_n, \mathcal{J}_n^1, \mathcal{J}_n^2 \) respectively, \( R_n = \sqrt{R^2 + U^2/n} \) and
\[
D_n = a_{n,1} \left\| R_{a_{n,1}} R^2 x_{\mathcal{J}_n^1} \right\|^2 + a_{n,2} \left\| R_{a_{n,2}} R^2 x_{\mathcal{J}_n^2} \right\|^2 - a_n \left\| R_{\delta n} R^2 x_{\mathcal{J}_n^\delta} \right\|^2.
\]
By Lemma 4.7 we have
\[
a_n/n - P(\mathcal{T}_n) \to 0, \quad a_{n,1}/n - P(\mathcal{T}_n^1) \to 0, \quad a_{n,2}/n - P(\mathcal{T}_n^2) \to 0
\]
Since \( \liminf_{k \to \infty} P(\mathcal{T}_{nk}) \geq p > 0 \) it follows that \( a_{nk} \to \infty \) as \( k \to \infty \). By Stirling formula and the Strong Law of Large Numbers
\[
n^{-1/2} \left( \frac{P(\mathcal{J}_{nk} | x)}{P(\mathcal{J}_{nk} | x)} \right) \approx n^{-1/2} \frac{a_{nk,1}! a_{nk,2}!}{a_{nk}!} \exp \left\{ \frac{R^2}{2} \frac{a_{nk,1} x_{\mathcal{J}_{nk}^1}^2}{a_{nk}} + \frac{a_{nk,2}}{a_{nk}} \frac{x_{\mathcal{J}_{nk}^2}^2}{a_{nk}} - \frac{a_{nk}}{a_{nk}} \left\| \frac{x_{\mathcal{J}_{nk}^\delta}^2}{a_{nk}} - \frac{a_{nk}}{a_{nk}} \right\| \right\} \approx 1
\]
By Corollary A.11 (with \( \delta = p/2 \)) we know that \( x_{\mathcal{J}_{nk}^1}^1 \) and \( x_{\mathcal{J}_{nk}^2}^2 \) approximate \( \mathbf{X}(X | X \in \mathcal{T}_n) \) and \( \mathbf{X}(X | X \in \mathcal{T}_n^1) \). Since \( \mathcal{T}_n \) becomes arbitrarily large, its length finally exceeds 3 and previous considerations (cf. Section 3.2) lead to the conclusion, that \( P(\mathcal{J}_{nk} | A_{nk} \neq 0) \) is small enough \( k \). This is a contradiction that proves our assertion.

**Proof of Lemma 4.5**

Note that we may assume that \( P(A) > 0 \) for \( A \in \mathcal{A} \). Indeed, if \( P(A) = 0 \) then \( \Delta(\{A\}) = 0 \) (by a natural convention that \( 0 \ln 0 = 0 \)) and on the other hand if \( X_1, X_2, \ldots \sim P \) then almost surely \( X_i \notin A \) for \( i \in \mathbb{N} \).

We abuse the notation slightly and denote \( P_{J^A} = |J^A|/n \) for \( A \in \mathcal{A} \). By the law of large numbers the sequence \( (X_n)_{n=1} \) almost surely satisfies \( p_{J^A} \to P(A) > 0 \). By Stirling formula
\[
\prod_{J \in J_n^A} (np_J)! \approx \prod_{J \in J_n^A} \left( \frac{np_J}{e} \right)^{np_J} \sqrt{2\pi np_J} = \sqrt{2\pi n} |J_n^A| \prod_{J \in J_n^A} p_J \cdot \left( \frac{n}{e} \prod_{J \in J_n^A} p_{J}^{p_J} \right)^n
\]
from which it follows by the Strong Law of Large Numbers that
\[
\sqrt{n} \sum_{J \in J_n^A} \ln (np_J)! \approx \sqrt{n} \sum_{J \in J_n^A} \ln \left( \frac{np_J}{e} \right)^{np_J} \frac{2}{2} \prod_{J \in J_n^A} p_{J} \approx \frac{2}{2} \prod_{J \in J_n^A} p_{J}^{p_{J}}. \quad \text{Note that since } J_n^A \text{ has at most } |A| \text{ elements,}
\]
\[
\lim_{n \to \infty} \sqrt{\prod_{J \in J_n^A} |J|^{\delta/2} \det R_{J}} = 1 \quad \text{and} \quad \lim_{n \to \infty} \sqrt{n} \prod_{J \in J_n^A} |J|^{(d+2)/2} \det R_{J} = 1.
\]
Let $r > 0$. Lemma A.10.

\[ \frac{1}{n} \sum_{J \in \mathcal{J}_n^A} |J|^{-1} R_{|J|} R^2 \mathbf{X}_J^2 \approx \sum_{J \in \mathcal{J}_n^A} p_J R \mathbf{X}_J^2 \approx \sum_{A \in \mathcal{A}} p_A R \mathbb{E}\{X | X \in A\}^2. \quad (A.42) \]

Applying (A.40), (A.41) and (A.42) together with (2.3) to the formula (2.2) for $\mathcal{J}^A$ completes the proof of the Lemma.

**Proof of Lemma 4.8**

From Corollary 4.2 (a) we know $\min\{np_J : J \in \mathcal{J}_n\} \to \infty$. By applying Stirling formula to each factor $(np_J)!$ and taking into account that by Corollary 4.2 (b) the number of factors is bounded, we obtain that

\[ \prod_{J \in \mathcal{J}_n} (np_J)! \approx \prod_{J \in \mathcal{J}_n} \left( \frac{np_J}{e} \right)^{np_J} 2\pi np_J = \left( \frac{n}{e} \right)^n 2\pi n^{\mathcal{J}_n^A - 1} \sqrt{\prod_{J \in \mathcal{J}_n} p_J} \left( \prod_{J \in \mathcal{J}_n} p_J^{p_J} \right)^n. \]

By definition the elements of $\hat{\mathcal{J}}_n$ are convex and hence by Lemma 4.7 the frequencies $p_J$ for $J \in \mathcal{J}_n$ approximate the respective probabilities of sets in $\hat{\mathcal{J}}_n$ uniformly. Hence, as $\mathcal{J}_n$ is bounded almost surely, it follows that

\[ \sqrt{\prod_{J \in \mathcal{J}_n} (np_J)!} \approx \frac{1}{2} \prod_{J \in \mathcal{J}_n} p_J^{p_J} \approx \frac{1}{2} \prod_{A \in \hat{\mathcal{J}}_n} p_A^p. \]

By applying a similar argument to the remaining part of formula (2.2), the result follows by Corollary A.11, which is an easy consequence of Lemma A.10 (here we also use Corollary 4.2 (a)).

**Lemma A.10.** If $P$ satisfies $(\ast)$ and for $X \sim P$ we have $\mathbb{E}\|X\|^2 < \infty$ then $P$ satisfies

\[ \lim_{n \to \infty} \sup_{C \in K} \|\mathbb{E}_n X 1_{X \in C} - \mathbb{E} X 1_{X \in C}\| = 0 \quad \text{almost surely.} \quad (***) \]

where $\mathbb{E}_n f(X) = \int_K f(X) dP_n = \frac{1}{n} \sum_{i=1}^n f(X_i)$.

**Proof.** Let $x^{(i)} (i \leq d)$ be the $i$-th coordinate of vector $x$. We now prove that for every $r > 0$

\[ \lim_{n \to \infty} \sup_{C \in K \cap [-r,r]^d} |\mathbb{E}_n X^{(i)} 1_{X \in C} - \mathbb{E} X^{(i)} 1_{X \in C}| = 0. \quad (A.43) \]

Fix $r > 0$ and $C \in K \cap [-r,r]^d$. For $m \in \mathbb{N}$ and $-m \leq k \leq m - 1$ let $C^m_k = C \cap [rk/m, (r(k + 1)/m) \times \mathbb{R}^{d-1})$. Then

\[ |\mathbb{E} X^{(i)} 1_{X \in C} - \sum_{k=-m}^{m-1} r k/m P(C^m_k)| \leq \frac{r}{m} P(C) \leq \frac{r}{m}. \quad (A.44) \]

It follows from the same reasoning

\[ |\mathbb{E}_n X^{(i)} 1_{X \in C} - \sum_{k=-m}^{m-1} r k/m P_n(C^m_k)| \leq \frac{r}{m} \quad \text{for every } n \in \mathbb{N}. \quad (A.45) \]
Now choose $\varepsilon > 0$ and $m > r/\varepsilon$. Note that $C_k^n$ are convex sets (as intersections of two convex sets) and hence by $(\ast)$ we may choose $N$ so that for $n > N$ and any convex $C''$ we have that $|P_n(C') - P(C'')| < \varepsilon/(2m)$ and hence

$$\left| \sum_{k=-m}^{m-1} \frac{k}{m} P(C_k^n) - \sum_{k=-m}^{m-1} \frac{k}{m} P_n(C_k^n) \right| \leq \sum_{k=-m}^{m-1} \left| \frac{k}{m} P(C_k^n) - P_n(C_k^n) \right| < r\varepsilon. \quad (A.46)$$

By combining (A.44), (A.45) and (A.46) we obtain that $\|E_n X^{(1)} 1_{X \in C} - E X^{(1)} 1_{X \in C}\| < (2 + r)\varepsilon$ for $n > N$ and since the choice of $N$ does not depend on $C$, (A.43) follows.

We now prove that almost surely

$$\lim_{n \to \infty} \sup_{C \in K} \|E_n X^{(1)} 1_{X \in C} - E X^{(1)} 1_{X \in C}\| = 0. \quad (A.47)$$

The same result for the remaining coordinates of $(\ast\ast)$ follows in the same way, from which follows the statement of the Lemma. Note that the function $r \mapsto E|X^{(1)}|1_{X \in [-r,r]}d$ is decreasing to 0 as $r$ goes to infinity. By the Strong Law of Large Numbers almost surely $\lim_{n \to \infty} E_n|X^{(1)}|1_{X \notin [-K,K]}d = E|X^{(1)}|1_{X \notin [-K,K]}d$ for every $K \in \mathbb{N}$.

Fix $C \in K$ and $\varepsilon > 0$. Since $\lim_{K \to \infty} E|X^{(1)}|1_{X \notin [-K,K]}d = 0$ it follows that there exist $K \in \mathbb{N}$ such that $E|X^{(1)}|1_{X \notin [-K,K]}d < \varepsilon$ and $\lim_{n \to \infty} E_n X^{(1)}|1_{X \notin [-K,K]}d < \varepsilon$. The latter means that there exist $n_1$ such that $E|X^{(1)}|1_{X \notin [-K,K]}d < \varepsilon$ for every $n > n_1$. By (A.43) there exist $n_2 \in \mathbb{N}$ such that for every $n > n_2$

$$|E_n X^{(1)} 1_{X \in C} - E X^{(1)} 1_{X \in C}| < \varepsilon.$$

Therefore for $n > \max\{n_1, n_2\}$ we get

$$|E_n X^{(1)} 1_{X \in C} - E X^{(1)} 1_{X \in C}| < |E_n X^{(1)} 1_{X \in C \cap [-K,K]}d - E X^{(1)} 1_{X \in C \cap [-K,K]}d| +$$

$$+ E_n X^{(1)} 1_{X \notin [-K,K]}d + E|X^{(1)}|1_{X \notin [-K,K]}d < 3\varepsilon.$$

Because $n_1, n_2$ do not depend on $C$, (A.47) follows, which finishes the proof of the Lemma.

**Corollary A.11.** If $P$ satisfies $(\ast)$ and for $X \sim P$ we have $E\|X\|^2 < \infty$ then for every $\delta > 0$ we have

$$\lim_{n \to \infty} \sup_{C \in K} \sup_{P(C) > \delta} \|E_n(X | X \in C) - E(X | X \in C)\| = 0 \quad \text{almost surely.}$$

**Proof.** This is a straightforward consequence of Lemma A.10 and the definition $E_n(X | X \in C) = E_n X 1_{X \in C}/P_n(C)$.

**Proof of Remark 4.11**

Assume that $(\mathcal{F}, d)$ is a (pseudo)metric space. We prove that $(F_K(\mathcal{F}), \tilde{d})$ is also a (pseudo)metric space. Take any $A = \{A^{(1)}, \ldots, A^{(\ell)}\} \in F_K(\mathcal{F})$ and $B = \{B^{(1)}, \ldots, B^{(\ell)}\} \in F_K(\mathcal{F})$. By definition

$$\tilde{d}(A, B) = \min_{\sigma \in \Sigma_K} \max_{1 \leq k \leq K} d(A^{(k)}, B^{(\sigma(k))}) \geq 0,$$
since \(d(A^{(i)}, B^{(j)}) \geq 0\) for any \(i, j \leq K\) (as in the definition we assume that \(A^{(i)} = \emptyset\) and \(B^{(j)} = \emptyset\) for \(i > k\) or \(j > l\) respectively). Let \(C = \{C^{(1)}, \ldots, C^{(l)}\} \in F_K(\mathcal{F})\) and let \(\sigma_1, \sigma_2\) and \(\sigma_3\) be permutations of \([K]\) that satisfy

\[
\overline{d}(A, B) = \max_{i \leq K} d(A^{(i)}, B^{(\sigma_1(i))}) \quad \text{and} \quad \overline{d}(B, C) = \max_{i \leq K} d(B^{(i)}, C^{(\sigma_2(i))})
\]

Note that \(d(A^{(i)}, B^{(\sigma_1(i))}) + d(B^{(\sigma_1(i))}, C^{(\sigma_2(\sigma_1(i))}) \geq d(A^{(i)}, C^{(\sigma_2(\sigma_1(i))})\) and hence

\[
\overline{d}(A, B) + \overline{d}(B, C) = \max_{i \leq K} d(A^{(i)}, B^{(\sigma_1(i))}) + \max_{i \leq K} d(B^{(\sigma_1(i))}, C^{(\sigma_2(\sigma_1(i))}) \geq \max_{i \leq K} \left( d(A^{(i)}, B^{(\sigma_1(i))}) + d(B^{(\sigma_1(i))}, C^{(\sigma_2(\sigma_1(i))}) \right) \geq \max_{i \leq K} d(A^{(i)}, C^{(\sigma_2(\sigma_1(i))}) \geq \overline{d}(A, C)
\]

and the triangle inequality follows. This means that \(\overline{d}\) is a pseudometric on \(F_K\).

Now assume that \((\mathcal{F}, d)\) is finitely compact. Let \((A_n)_{n=1}^{\infty}\) be a sequence in \(F_K(\mathcal{F})\) and let \(A_n = \{A_n^{(1)}; A_n^{(2)}, \ldots, A_n^{(k_n)}\}\). As the sequence \((k_n)_{n=1}^{\infty}\) is bounded by \(K\) we may choose a subsequence \(A_{n_k}\) and \(K \in \mathbb{N}\) such that \(|A_{n_k}| = K\) for every \(k \in \mathbb{N}\).

Consider the sequence \((A_{n_k}^{(i)})_{k=1}^{\infty}\). This sequence is bounded (as \((A_n)_{n=1}^{\infty}\) is bounded). Therefore it has a subsequence \((A_{n_k}^{(i)})_{k=1}^{\infty}\) converging in \(d\) to \(A^{(i)} \in \mathcal{F}\). Now we consider \((A_{n_k}^{(j)})_{k=1}^{\infty}\) and again we choose a subsequence \((A_{n_k\ell_m})_{m=1}^{\infty}\) converging in \(d\) to \(A^{(2)} \in \mathcal{F}\).

By iterating this procedure \(K\) times we obtain a family \(\hat{A} = \{A^{(1)}, \ldots, A^{(K)}\}\) of ‘limiting’ sets. It is easy to verify that the final subsequence of \((A_n)_{n=1}^{\infty}\) converges in \(\overline{d}\) to \(\hat{A}\), which finishes the proof.

### Proof of Theorem 4.12

Take any \(E = \{E^{(1)}, \ldots, E^{(k)}\} \in \mathcal{E}\) and assume that it is a limit of \((\hat{A}_{n_k})_{k=1}^{\infty}\) in \(\overline{d}_P\). By Theorem 4.10 the sequence \((\hat{A}_{n_k})_{k=1}^{\infty}\) converges to \(E\) also in \(\overline{d}_P\). Since for every \(k \in \mathbb{N}\) every pair of sets in the family \(\hat{A}_{n_k}\) has at most one point in common (Proposition 1) then by the continuity of \(P\) with respect to the Lebesgue measure every pair of sets within \(\hat{A}_{n_k}\) has an intersection of \(P\) measure 0. Therefore by the continuity of the intersection with respect to \(d_P\) (Doob (1994), Chapter III, Formula (13.3)) we get that \(P(E^{(i)} \cap E^{(j)}) = 0\) for \(1 \leq i < j \leq K\).

To prove that \(E\) is a \(P\)-partition it is left to show that \(P(\bigcup \mathcal{E}) = 1\) (we denote \(\bigcup \mathcal{E} = \bigcup_{E \in \mathcal{E}} E\)). Suppose this is not the case. It means that \(E_0 = \mathbb{R}^d \setminus \bigcup \mathcal{E}\) is an open set with positive probability. Therefore it includes a ball \(B'\) of positive probability. Since \(B'\) is a convex set, we get \(P_{B'} \rightarrow P_{B'} > 0\) and therefore there exist \(n' \in \mathbb{N}\) such that \(X_{n'} \in B'\). This is not possible, since \(X_{n'} \in \bigcup \hat{A}_n\) for every \(n \geq n'\) and therefore \(X_{n'} \in \bigcup \mathcal{E}\), which is a contradiction.

By Lemma 4.8 and the continuity of \(\Delta\) with respect to the metric \(\overline{d}_P\) we obtain:

\[
\sqrt{Q_{X_{1:n}}(\hat{J}_n)} \approx \exp(\Delta(\hat{A}_n)) \approx \exp(\Delta(\mathcal{E})). \tag{A.48}
\]

Now take any finite \(P\)-partition \(A\). We can assume that each \(X_{n}\) belongs to exactly one of the sets in \(A\), \(P_{\hat{J}A} \rightarrow P_A\) and \(\overline{d}_{\hat{J}A} \rightarrow E(X \mid X \in A)\) for \(A \in A\) (it just requires adding a countable number of conditions on the infinite iid sequence with distribution
is almost surely finite. Note that Lemma A.12.

Proofs for Section 5 (Discussion)

Lemma A.12. Let $\alpha > 0$, $G_0 = N(0, T)$, $F_T = N(\theta, \Sigma)$ for $\theta \in \mathbb{R}^d$. Let $X_1, X_2, \ldots$ be an infinite sample from the Gaussian DPMM, i.e. a sequence of random variables, obtained by the following construction defined by Equation (2.1). Then almost surely

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|X_i\|^4 < \infty.$$  

Proof. It is a well known fact (Sethuraman (1994)) that an infinite sample from the Gaussian DPMM may be performed by the following procedure

$$p = (p_1, p_2, \ldots) \sim SB(\alpha)$$

$$\theta = (\theta_1, \theta_2, \ldots) \overset{iid}{\sim} N(0, T)$$

$$X_1, X_2, \ldots | p, \theta \overset{iid}{\sim} \sum_{i=1}^{\infty} p_i N(\theta_i, \Sigma)$$

where $SB(\alpha)$ is the so called stick-breaking construction (i.e. $p_n = V_n \prod_{i=1}^{n-1} (1 - V_i)$, where $V_1, V_2, \ldots \overset{iid}{\sim} \text{Beta}(1, \alpha)$). Therefore

$$E(\|X_1\|^4 | p, \theta) = \sum_{n=1}^{\infty} p_n Q(\theta_n)$$  

(A.50)

where by definition $Q(\theta) = E\|X\|^4$ when $X \sim N(\theta, \Sigma)$. Note that $Q$ is a multivariate polynomial in the coefficients of $\theta$; it is given by the formula

$$Q(\theta) = \sum_{k=1}^{d} (\theta_k^4 + 6\sigma_{k,k} \theta_k^2 + 3\sigma_{k,k}^2) + \sum_{k \neq l} (2\sigma_{k,l} + 4\sigma_{k,l} \theta_{(k)} \theta_{(l)} + \theta_{(k)}^2 + \sigma_{k,k} \sigma_{l,l} + \sigma_{k,k} \theta_{(l)}^2 + \sigma_{l,l} \theta_{(k)}^2)$$

where $\theta_{(k)}$ is the $k$-th coefficient of $\theta$ and $[\sigma_{k,l}]_{k,l \leq d} = \Sigma$. We show that for $\sum_{i=1}^{n} p_i Q(\theta_i)$ is almost surely finite. Note that

$$\mathbb{P}\left( \sum_{n=1}^{\infty} p_i Q(\theta_i) < \infty \right) = \int_{\mathbb{R}^\infty} \mathbb{P}\left( \sum_{n=1}^{\infty} p_i Q(\theta_i) < \infty | p = \overline{p} \right) dSB(\overline{p})$$  

(A.51)

Note that given $p = \overline{p} = (\overline{p}_n)_{n=1}^{\infty}$, where $\sum_{n=1}^{\infty} p_n = 1$ and $\overline{p}_n \in [0, 1]$, the series $\sum_{n=1}^{\infty} p_n Q(\theta_n)$ is a sequence of independent random variables, which is almost surely bounded by the Kolmogorov’s Two Series Theorem (Durrett (2010), Theorem 2.5.3). Indeed, when $\theta \sim N(0, T)$ then all mixed moments of $\theta$ are finite (i.e. $E \prod_{i=1}^{d} \theta^w_i < \infty$ for every $w_1, \ldots, w_d \in \mathbb{N}$) and therefore $E Q(\theta) < \infty$ and $\text{Var}Q(\theta) < \infty$, so

$$\sum_{n=1}^{\infty} E_{p = \overline{p}} \overline{p}_n Q(\theta_n) = E_{p = \overline{p}} Q(\theta) \sum_{n=1}^{\infty} \overline{p}_n = E Q(\theta) < \infty,$$

$$\sum_{n=1}^{\infty} \text{Var}_{p = \overline{p}} \overline{p}_n Q(\theta_n) = \text{Var}_{p = \overline{p}} Q(\theta) \sum_{n=1}^{\infty} \overline{p}_n^2 < \text{Var}Q(\theta) < \infty$$

41
Therefore \( P(\sum_{i=1}^{\infty} p_i Q(\theta_i) < \infty | p = \bar{p}) = 1 \) and (A.51) implies \( P(\sum_{i=1}^{\infty} p_i Q(\theta_i) < \infty) = 1 \). Finally, conditioned on \( p = \bar{p} \) and \( \theta = \bar{\theta} \), the sequence \( X_1, X_2, \ldots \) is a sequence of independent random variables and therefore if \( \sum_{i=1}^{\infty} p_i Q(\theta_i) < \infty \) then by the Strong Law of Large Numbers and (A.50), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|X_i\|^2 = E(\|X_1\|^2 | p = \bar{p}, \theta = \bar{\theta}) < E(\|X_1\|^4 | p = \bar{p}, \theta = \bar{\theta})^{\frac{1}{2}} < \infty.
\]

This means that

\[
P(\lim \sup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|X_i\|^2 < \infty) = \int_{\mathbb{P}^{\mathbb{R}^J} \times \mathbb{P}^{\mathbb{R}^J}} P(\lim \sup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|X_i\|^2 < \infty | p = \bar{p}, \theta = \bar{\theta}) dSB(\bar{p}) dG_0^{\infty}(\bar{\theta}) \geq \int_{\mathbb{P}^{\mathbb{R}^J} \times \mathbb{P}^{\mathbb{R}^J}} P(\sum_{n=1}^{\infty} p_n Q(\bar{\theta}_n) < \infty) dSB(\bar{p}) dG_0^{\infty}(\bar{\theta}) = P(\sum_{n=1}^{\infty} p_n Q(\theta_n) < \infty) = 1
\]

and the proof follows. \( \Box \)

**A comment on extending the results to the finite Dirichlet mixture models**

The **finite Dirichlet zero-mean Gaussian mixture model** is a model of the form

\[
p = (p_1, \ldots, p_K) \sim \text{Dir}(\alpha, \ldots, \alpha).
\]

\[
\theta = (\theta_1, \ldots, \theta_K) \overset{iid}{\sim} \mathcal{N}(0, T).
\]

\[
x = (x_1, \ldots, x_n) \mid \theta, p \overset{iid}{\sim} \sum_{k=1}^{K} p_k \mathcal{N}(\theta_k, \Sigma).
\]

Given the vector \( p \) of probabilities of belonging to respective clusters, the marginal probability on partition \( \mathcal{J} \) of indices is

\[
P(\mathcal{J} \mid p) = \sum_{\tau : \mathcal{J} \rightarrow \{1, \ldots, K\}} \prod_{J \in \mathcal{J}} p_{\tau(J)}^{[J]},
\]

where the sum is over all injective functions \( \tau \) from \( \mathcal{J} \) to \( \{1, \ldots, K\} \). By properties of the Dirichlet distribution we have that for every \( \tau : \mathcal{J} \rightarrow \{1, \ldots, K\} \)

\[
E \prod_{J \in \mathcal{J}} p_{\tau(J)}^{[J]} = \frac{\Gamma(K\alpha)}{\Gamma(K\alpha + n)} \prod_{J \in \mathcal{J}} \frac{\Gamma(\alpha + |J|)}{\Gamma(\alpha)} = \frac{1}{(K\alpha)^n} \prod_{J \in \mathcal{J}} a_{\{J\}}^{[J]},
\]

where \( a_{\{J\}}^{(b)} = \frac{\Gamma(a+b)}{\Gamma(a)} = a(a+1) \ldots (a+b-1) \). There are \( K^{[\mathcal{J}]} = K(K-1) \ldots (K-|\mathcal{J}|+1) \) injections from \( \mathcal{J} \) to \( \{1, \ldots, K\} \), therefore

\[
P(\mathcal{J}) = E P(\mathcal{J} \mid p) = \frac{K^{[\mathcal{J}]} \prod_{J \in \mathcal{J}} a_{\{J\}}^{[J]}}{(K\alpha)^n} \prod_{J \in \mathcal{J}} (1 + \alpha)^{|\mathcal{J}|-1}.
\]
We will call the resulting distribution the finite Chinese Restaurant Process and write $\mathcal{J} \sim \text{CRP}^K(\alpha_n)$. It is easy to see that $\text{CRP}^K(\frac{1}{K})_n$ converges to $\text{CRP}(\alpha)_n$, which is a well-known fact (see Neal (2000)). We can now consider a finite CRP based zero-mean Gaussian model, defined by the following scheme

$$
\begin{align*}
\mathcal{J} & \sim \text{CRP}^K(\alpha)_n \\
\theta & = (\theta_j)_{j \in \mathcal{J}} | \mathcal{J} \overset{iid}{\sim} \mathcal{N}(0, T) \\
\mathbf{x}_J & = (x_j)_{j \in J} | \mathcal{J}, \theta \overset{iid}{\sim} \mathcal{N}(\theta_J, \Sigma) \quad \text{for } J \in \mathcal{J}.
\end{align*}
$$

This leads to the following formula for the posterior in the exactly same way, as in the proof of Remark 2.1.

**Remark A.13.** The conditional probability of partition $\mathcal{J}$ in the finite zero-mean Gaussian model, given the observation vector $\mathbf{x} = (x_j)_{j=1}^{n}$, is proportional to

$$
K^{||\mathcal{J}||}C^{||\mathcal{J}||} \prod_{J \in \mathcal{J}} \frac{(1 + \alpha)^{|J| - 1}}{|J|^{d/2} \det R_{J,J}} \cdot \exp \left\{ \frac{1}{2} \sum_{J \in \mathcal{J}} |J| \cdot ||R_{J,J}^{-1}R_{J,J}||^2 \right\} =: Q_\alpha(\mathcal{J})_K,
$$

where the notation is the same as in Remark 2.1.

We denote the MAP partition in this model by $\hat{\mathcal{J}}^K_n(x_1, \ldots, x_n)$. We now discuss the applicability of the results from the article to this partition.

**Proposition 1 for finite mixture model**

In this case the proof from Proposition 1 remains unchanged. It used the fact that the Chinese Restaurant Process prior on partitions depends only on the number of observations within each cluster and hence it is not changing when we replace clusters $I, J$ with $\hat{I}, \hat{J}$ such that $|\hat{I}| = |I|$, $|\hat{J}| = |J|$ and $\hat{I} \cup \hat{J} = I \cup J$. This is also the case with the finite Chinese Restaurant Process.

**Proposition 2 for finite mixture model**

In our proof, Lemma A.3 and Lemma A.4 were used only for Proposition A.5, but the latter is obvious for the finite mixture model since when the number of clusters is bounded by $K$ then the number of observations in the largest cluster is at least $n/K$. Therefore Corollary A.6 also holds for the finite mixture model.

As for the proof of Proposition A.7, in the Equation (A.13) the factor $\frac{m|\mathcal{M}|}{(\alpha + M)!}$ should be replaced by

$$
K_{||\mathcal{M}||}^{[m]}K_{||\mathcal{M}||}^{[M]} \cdot \frac{\alpha + m + M}{(\alpha + m)(\alpha + M)} \cdot \frac{(1 + \alpha)^{(m)}(1 + \alpha)^{(M)}}{(1 + \alpha)^{(m+M)}} < 2 \frac{(1 + \alpha)^{(m)}(1 + \alpha)^{(M)}}{(1 + \alpha)^{(m+M)}}
$$

Note that

$$
\frac{(1 + \alpha)^{(m)}(1 + \alpha)^{(M)}}{(1 + \alpha)^{(m+M)}} = \frac{(1 + \alpha)(2 + \alpha)\ldots(M + \alpha)}{(m + 1 + \alpha)(m + 2 + \alpha)\ldots(m + M + \alpha)} = \\
\frac{1 + \alpha}{m + 1 + \alpha} \cdot \frac{2 + \alpha}{m + 2 + \alpha} \cdot \ldots \cdot \frac{M + \alpha}{m + M + \alpha} < \\
\frac{1 + [\alpha]}{m + 1 + [\alpha]} \cdot \frac{2 + [\alpha]}{m + 2 + [\alpha]} \cdot \ldots \cdot \frac{M + [\alpha]}{m + M + [\alpha]} = \\
\frac{(M + [\alpha])!(m + [\alpha])!}{[\alpha]!(m + M + [\alpha])!} < \frac{(M + [\alpha])!(m + [\alpha])!}{(m + M + [\alpha])!}.
$$

43
By Stirling inequality (A.17) we get that

\[
\frac{(M + [\alpha])(m + [\alpha])}{(m + M + [\alpha])} < \frac{2\pi e^2 \sqrt{(M + [\alpha])(m + [\alpha])}}{\sqrt{2\pi} \sqrt{m + M + [\alpha]}} \left(\frac{(M + [\alpha])}{m + M + [\alpha]}\right)^{M + [\alpha]} \left(\frac{(m + [\alpha])}{m + M + [\alpha]}\right)^{m + [\alpha]} = \sqrt{2\pi e^{2-[\alpha]}} \sqrt{\frac{m + [\alpha]}{m + M + [\alpha]}} \left(\frac{m + [\alpha]}{m + M + [\alpha]}\right)^{m + [\alpha]} < \sqrt{2\pi e^{2-[\alpha]}} \left(\frac{m + [\alpha]}{m + M + [\alpha]}\right)^{m + [\alpha]}.
\]

Therefore we can transform (A.19) into

\[
\liminf_{n \to \infty} \min \frac{\mathbb{P}(\mathcal{J}_n | x)}{\mathbb{P}(\mathcal{J}_n | x)} = \liminf_{n \to \infty} \frac{C'\left(\frac{m + M}{m M^*}\right)^{d/2} (m + [\alpha])^{\alpha + 1/2}}{(m + M + [\alpha])^{\alpha}} = 0.
\]

Indeed, note that as \(m/M \to 0\), we have \(C'\left(\frac{m + M}{m M^*}\right)^{d/2} (m + [\alpha])^{\alpha + 1/2} \to 0\), so even if \(m \to \infty\) so that \((m + [\alpha])^{\alpha + 1/2} \to \infty\) we have \((m + [\alpha])^{\alpha + 1/2} \left(\frac{m + [\alpha]}{m + M + [\alpha]}\right)^m \to 0\). Therefore the proof of Proposition 2 does not require major changes.

Finally in the proof of Proposition 2 the only place where the prior is important is Equation (A.21), which now become

\[
\frac{(1 + \alpha)^{(a+b-1)}(1 + \alpha)^{(M-1)}}{(1 + \alpha)^{(b-1)(1 + \alpha)^{(a+M-1)}}} = \frac{(b + \alpha)^{(a)}}{\left(1 + \alpha\right)^{(a+M-1)}} \leq \frac{b + \alpha}{M + \alpha} \to 0.
\]

The rest of the proof of Proposition 2 remains unchanged.

**Proposition 3 for finite mixture model**

Proposition 3 was an easy consequence of Theorem 4.12. It still holds for finite mixture model if we restrict our attention to the \(P\)-partitions of the observation space with at most \(K\) sets (which is the number of clusters assumed by the model).

The place of the prior distribution on the space of partitions in the proof of Theorem 4.12 was in Lemma 4.5 and Lemma 4.8. Proof of Lemma 4.5 remains unchanged, provided that

\[
\sqrt[n]{\prod_{J \in \mathcal{J}_n} (1 + \alpha)^{(|J|-1)}} \approx \frac{n}{e} \prod_{A \in \mathcal{A}} p_A^a,
\]

where \(\mathcal{A}\) is a partition of the observation space with at most \(K\) sets (otherwise clearly
$Q_n(J_n^A) = 0$). Note that
\[
\prod_{J \in J_n^A} (1 + \alpha)^{(|J| - 1)} \leq \prod_{J \in J_n^A} (1 + \lceil \alpha \rceil)^{(\lceil |J| - 1 \rceil)} = \prod_{J \in J_n^A} \frac{(|J| + \lceil \alpha \rceil - 1)!}{|\alpha|!} = 
\]
\[
= \left(\lceil \alpha \rceil \right)^{-K} \prod_{J \in J_n^A} (|J| + \lceil \alpha \rceil - 1)! \approx 
\]
\[
\approx \left(\lceil \alpha \rceil \right)^{-K} \prod_{J \in J_n^A} \sqrt{2\pi(|J| + \lceil \alpha \rceil - 1)} \left(\frac{|J| + \lceil \alpha \rceil - 1}{e}\right)^{|J| + \lceil \alpha \rceil - 1} = 
\]
\[
e^{-n - |A|\lceil \alpha \rceil - 1} \left(\lceil \alpha \rceil \right)^{-K} \prod_{A \in A} \sqrt{2\pi(|J_n^A| + \lceil \alpha \rceil - 1)} \left(\frac{|J_n^A| + \lceil \alpha \rceil - 1}{e}\right)^{|J_n^A| + \lceil \alpha \rceil - 1}.
\]

Since $|J_n^A|/n \to p_A$ it is easy now to deduce that
\[
\sqrt{\prod_{J \in J_n^A} (1 + \alpha)^{(\lceil |J| - 1 \rceil)}} \approx e^{-n \sum_{\alpha} p_A (1 + \alpha)^{\lceil \alpha \rceil - 1} \prod_{A \in A} p_A^n}.
\]

The proof of Lemma 4.8 can be easily modified in the same way.
Supplement B
Simulations

This supplementary material provides computer simulations for considerations presented in Section 3.1 of the paper together with a short simulation study of the clustering properties of the MAP for the finite mixture input. The experiments were performed on a 64-bit Linux machine with R version 3.2.3 (R Core Team (2014)). Sampling from the posterior was performed by using MCMC methods, i.e. running a Markov Chain for which the posterior probability is a stationary distribution. Our Markov Chain was the one used in Algorithm 2 from Neal (2000). The choice of the MAP was always performed by the following procedure: firstly 100 MCMC steps were recorded after 100 burn-in period (the initial partition is a single cluster). Then the posterior probability of every resulting partition (up to the norming constant) was computed and the one with the highest output was chosen as the MAP.

Sampling from multivariate normal and computing values of its distribution function were performed using `mvtnorm` package (Genz et al. (2016)).

Uniform distribution on an interval
The experiment involved creating a sample of size 200 from Unif\([-1,1]\] distribution and constructing the MAP partition. This procedure was performed for all possible combinations of parameters \((\Sigma, T, \alpha) \in \{1,.1,.01,.001\} \times \{1,.1\}^2\). The results are presented in Table 1.

The analysis of Table 1 yields to the following conclusions. Firstly, there is an apparent connection between the observed number of clusters in the MAP and the value predicted in Section 3.1, which is \(1/\sqrt{3\Sigma}\). Secondly, decreasing values of \(\alpha\) and \(T\) leads to the smaller number of clusters in the estimated MAP, but their impact is significantly smaller than the impact of \(\Sigma\). In case of \(\alpha\) it is easily justified by the formula (2.2), where we have the factor \(\alpha \#\text{clusters}\) in the prior weight. The role of \(T\) is more difficult to explain as it occurs in two factors: as \(U \#\text{clusters}\) and in \(R_m = \sqrt{\Sigma^{-1} + T^{-1}}/m\). In the later it is divided by the cluster size, so the intuition is that there is indeed positive correlation between the value of \(T\) and the number of clusters.

Exponential distribution
We sample an iid sequence from Exp(1) of size 2000. Then we construct the MAP division of first \(k\) observations for \(k = 100, 200, 300, \ldots, 2000\). The parameters of the model were \(\alpha = T = 1, \Sigma = (32\log 2)^{-1}\). The results are presented in Figure 5, where each row corresponds to different value of \(k\) and within a row the partition is indicated by colors.

Figure 5 is consistent with the considerations presented in Section 3.2 regarding the number of clusters in the MAP. It suggests that at some stage a group of extremal observations will create a new cluster and therefore the number of clusters in the MAP tends to infinity.
Figure 5: Clustering in the MAP partition of the first $k = 100, 200, 300, \ldots, 2000$ observations in the iid sample from $\text{Exp}(1)$. Each row corresponds to a different value of $k$; different clusters are denoted by different colors.

**Mixture of two normals**

In this numerical experiment firstly we sample an iid sequence $X_1, \ldots, X_{2000} \sim N(0, 1)$ and an iid sequence of Rademacher random variables $R_1, \ldots, R_{2000} \sim \frac{1}{2} (\delta_1 + \delta_{-1})$. Note that for every $a \in \mathbb{R}$ the distribution of the random variable $Z_{a,i} = X_i + aY_i$ is the mixture of two normals with means $a$ and $-a$. The MAP partition of the vector $Z_{a,1}, \ldots, Z_{a,k}$ is computed for $a \in \{.1, .5, .8, 1\}$ and $k \in \{500, 1000, 1500, 2000\}$. The parameters of the mode were $\alpha = \Sigma = T = 1$. Note that the choice of $\Sigma$ parameter is consistent with the variance within mixtures. The results are shown on Figure 6.

Section 3.3 predicts that with this choice of $a$ parameters, a single cluster partition has larger posterior probability than splitting the observations into two clusters of equal size. Figure 6 does not fully support these predictions. However this is due to the imperfection of the MAP MCMC approximation. Indeed, Figure 7 shows the comparison of the logarithms of posterior probabilities of the sample estimators of MAP and single cluster partition. In all cases the single cluster partition has higher posterior probability.
### Table 1: Results of numerical experiments with all combinations of $\Sigma$, $T$, $\alpha$ ∈ $\{1, 0.1, 0.01\} \times \{1, 0.1\}$

| $\Sigma$ | $T$ | $\alpha$ | sizes of clusters | # clusters | est  |
|----------|-----|----------|-------------------|------------|------|
| 1        | 1   | 1        | 200               | 1          | 0.58 |
| 1        | 1   | 0.1      | 200               | 1          | 0.58 |
| 1        | 1   | 0.01     | 200               | 1          | 0.58 |
| 1        | 0.1 | 1        | 180, 15, 3, 1, 1  | 5          | 0.58 |
| 1        | 0.1 | 0.1      | 200               | 1          | 0.58 |
| 1        | 0.1 | 0.01     | 200               | 1          | 0.58 |
| 1        | 0.01| 1        | 200               | 1          | 0.58 |
| 1        | 0.01| 0.1      | 200               | 1          | 0.58 |
| 1        | 0.01| 0.01     | 200               | 1          | 0.58 |
| 0.01     | 1   | 1        | 110, 90           | 2          | 1.83 |
| 0.01     | 1   | 0.1      | 106, 94           | 2          | 1.83 |
| 0.01     | 1   | 0.01     | 105, 95           | 2          | 1.83 |
| 0.01     | 0.1 | 1        | 105, 95           | 2          | 1.83 |
| 0.01     | 0.1 | 0.1      | 121, 79           | 2          | 1.83 |
| 0.01     | 0.1 | 0.01     | 110, 90           | 2          | 1.83 |
| 0.01     | 0.01| 1        | 116, 84           | 2          | 1.83 |
| 0.01     | 0.01| 0.1      | 103, 97           | 2          | 1.83 |
| 0.01     | 0.01| 0.01     | 200               | 1          | 1.83 |
| 0.01     | 1   | 1        | 52, 40, 30, 30, 28, 18, 2 | 7 | 5.77 |
| 0.01     | 1   | 0.1      | 35, 34, 31, 30, 20, 19 | 7 | 5.77 |
| 0.01     | 1   | 0.01     | 59, 52, 51, 38    | 4          | 5.77 |
| 0.01     | 0.1 | 1        | 46, 42, 37, 36, 22, 12, 4, 1 | 8 | 5.77 |
| 0.01     | 0.1 | 0.1      | 49, 48, 43, 37, 23 | 5 | 5.77 |
| 0.01     | 0.1 | 0.01     | 51, 45, 43, 31, 30 | 5 | 5.77 |
| 0.01     | 0.01| 1        | 56, 44, 37, 34, 27, 1, 1 | 7 | 5.77 |
| 0.01     | 0.01| 0.1      | 55, 44, 38, 35, 28 | 5 | 5.77 |
| 0.01     | 0.01| 0.01     | 51, 48, 45, 43, 13 | 5 | 5.77 |
| 0.001    | 1   | 1        | 20, 17, 16, 14, 13, 12, 12, 11, 11, 10, 8, 5 | 16 | 18.26 |
| 0.001    | 1   | 0.1      | 20, 19, 18, 16, 16, 15, 14, 14, 13, 12, 9, 7 | 14 | 18.26 |
| 0.001    | 1   | 0.01     | 21, 20, 17, 17, 17, 16, 14, 14, 13, 12, 11, 10, 9, 9 | 14 | 18.26 |
| 0.001    | 0.1 | 1        | 20, 18, 17, 16, 16, 15, 14, 13, 12, 9, 7, 6, 5, 3 | 16 | 18.26 |
| 0.001    | 0.1 | 0.1      | 21, 20, 18, 15, 15, 13, 13, 12, 11, 10, 9, 9, 8, 7 | 16 | 18.26 |
| 0.001    | 0.1 | 0.01     | 25, 21, 17, 17, 16, 16, 15, 14, 13, 13, 12, 11, 10 | 13 | 18.26 |
| 0.001    | 0.01| 1        | 27, 25, 24, 23, 23, 21, 19, 14, 13, 13, 10, 11 | 11 | 18.26 |
| 0.001    | 0.01| 0.1      | 21, 20, 20, 19, 18, 18, 17, 16, 13, 12, 10, 9, 7 | 13 | 18.26 |
| 0.001    | 0.01| 0.01     | 33, 28, 25, 22, 20, 20, 16, 14, 11, 11 | 10 | 18.26 |

Table 1: Results of numerical experiments with all combinations of $\Sigma$, $T$, $\alpha$ ∈ $\{1, 0.1, 0.01\} \times \{1, 0.1\}$

Column ‘sizes of clusters’ presents the sizes of the clusters created in the MAP, sorted in decreasing order. Column ‘est’ is equal to $1/(\sqrt{3\Sigma})$ and should approximate the number of clusters, which is given in column ‘# clusters’.
Figure 6: Clustering in the MAP partition of the first $k = 100, 500, 1000, 1500, 2000$ observations (in columns) in the iid sample from the mixture of two normal distributions $\mathcal{N}(a, 1)$ and $\mathcal{N}(-a, 1)$ (in rows). Different clusters are denoted by different colors.
Figure 7: Comparison of posterior probabilities of the approximation of the MAP (as in Figure 6) and the single cluster. In all cases single cluster has higher posterior probability.