Calabi-Yau 4-folds and toric fibrations

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ABSTRACT

We present a general scheme for identifying fibrations in the framework of toric geometry and provide a large list of weights for Calabi–Yau 4-folds. We find 914,164 weights with degree $d \leq 150$ whose maximal Newton polyhedra are reflexive and 525,572 weights with degree $d \leq 4000$ that give rise to weighted projective spaces such that the polynomial defining a hypersurface of trivial canonical class is transversal. We compute all Hodge numbers, using Batyrev’s formulas (derived by toric methods) for the first and Vafa’s formulas (obtained by counting of Ramond ground states in $N = 2$ LG models) for the latter class, checking their consistency for the 109,308 weights in the overlap. Fibrations of $k$-folds, including the elliptic case, manifest themselves in the $N$ lattice in the following simple way: The polyhedron corresponding to the fiber is a subpolyhedron of that corresponding to the $k$-fold, whereas the fan determining the base is a linear projection of the fan corresponding to the $k$-fold.

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1 Introduction

It used to seem obvious from the critical dimension of superstrings and the apparent dimension of space-time that only complex manifolds with dimensions up to 3 play a role in string theory. The second string revolution, however, has changed this picture: It was shown that strong coupling phenomena may increase the effective dimension of space-time to 11 dimensions [1], thereby providing a geometrical interpretation of a number of string dualities in the context of M-theory. Geometrization of the $SL(2, \mathbb{Z})$ symmetry of type IIB strings may even lead to 12 dimensions [2]. Whether or not there are situations with an effectively 12-dimensional space-time, F-theory compactification already has an impressive record as a way to talk about compactifications of type IIB strings and providing the missing geometrizations of duality symmetries [2–5]. In a related development, non-perturbative physics of 3-dimensional compactifications seems to have the potential to teach us a lot about issues like SUSY breaking in 4 dimensions [4], which we may recover in some decompactification limit.

For all these issues it is important to have a number of generic examples of Calabi–Yau 4-folds at one’s disposal and, in particular for applications in F-theory, to know how to identify elliptic fibrations in an easy way. The purpose of the present paper is twofold: On the one hand, we provide large classes of 4-folds in a systematic way. In addition we give a detailed discussion of the fibration structure in the toric context. From our experience with K3 fibrations [6] we expect that many families of toric Calabi–Yau hypersurfaces will have members that admit elliptic fibrations in appropriate regions of the quantum moduli space (or, in technical terms, for a triangulation of the fan that is compatible with the reflexive intersection in the $N$ lattice that provides the fiber).

Toric methods are known to physicists mainly because of the work of Batyrev [7], which appeared in a situation where it had become increasingly clear that complete intersections in products of (weighted) projective spaces were not general enough to grasp phenomena like mirror symmetry [8–11]. In the context of toric geometry, which provides a natural extension of previous constructions, mirror symmetry manifests itself as the elementary duality (or polarity) of polytopes. The classification of toric Calabi-Yau hypersurfaces is equivalent to the enumeration of reflexive polytopes, a problem that can be stated in simple combinatorial terms.

The link between the polytopes that generate the fan defining a toric variety and weights that admit transversal polynomials of appropriate degree for Calabi-Yau hypersurfaces in weighted projective spaces turned out to be very simple, at least in low dimensions: Just take the maximal Newton polytope (MNP), which consists of all exponent vectors of monomials whose degree is equal to the sum of weights. Its dual, which always contains a simplex that encodes the weighted projective space we started with, turns out to be integer for all MNPs in up to 4 dimensions, implying (by definition) that they are reflexive. In the case of 4-dimensional hypersurfaces, however, we will see that only about 20% of our MNPs are reflexive.

Even without transversality weight systems may lead to reflexive polyhedra in the way we just described, and any reflexive polyhedron is a subpolyhedron of an MNP defined by one or several weight systems. This observation is one of the keys to an approach for the classification of reflexive polyhedra [12, 13]. In the present context we used it to create large lists of weight systems that lead to reflexive polyhedra. Here the fact that transversality and reflexivity are practically unrelated properties in more than 4 dimensions becomes even more
apparent: Among the 914,164 weight systems we constructed that lead to reflexive polyhedra, less than one percent allow for transversal polynomials!

Fibrations provide a beautiful example of how algebraic structures in a toric variety manifest themselves in terms of linear structures in the \( N \) lattice. As we will see, we can identify the fiber as a variety corresponding to some subpolytope \( \Delta^*_{\text{fiber}} \) of \( \Delta^*_{\text{CY}} \) and the base as a variety whose toric description is given in terms of a fan \( \Sigma_{\text{fiber}} \) that is a projection of \( \Sigma_{\text{CY}} \) along the direction of the sublattice supporting \( \Delta^*_{\text{fiber}} \). This makes it very easy to look for elliptic fibrations simply by looking for two dimensional integer subpolytopes containing the interior point. An even simpler approach would start from a multiply weighted space such that one of the weight systems leads to the elliptic fiber. Let us also mention here that our approach is particularly useful for discussing the degeneration of fibers.

In section 2 we discuss the relation between weighted projective spaces and their toric resolutions. We also present formulas and strategies for the calculation of Hodge numbers. In section 3 we discuss toric fibrations and the toric description of the base manifold. In section 4 we present our numerical results on weight systems leading to Calabi–Yau fourfolds.

While we were finishing the present work there appeared a preprint [14] that partly overlaps with it.

2 Weighted projective spaces vs. toric varieties

We assume that the reader is familiar with basic notions of toric geometry, such as the definitions of cones and fans (see, e.g., [15] or [16]). We use standard notation, denoting the dual lattices by \( M \) and \( N \), their real extensions by \( M_{\mathbb{R}} \) and \( N_{\mathbb{R}} \), and the fan in \( N_{\mathbb{R}} \) by \( \Sigma \). To each one-dimensional cone in \( \Sigma \) with primitive generator \( v_k \) we assign a homogeneous coordinate \([17] z_k, k = 1, \cdots, N \). From the resulting \( \mathbb{C}^N \) we remove the exceptional set

\[
Z_\Sigma = \bigcup_I \{(z_1, \cdots, z_N) : z_i = 0 \forall i \in I\}
\]  

(1)

where the union \( \bigcup_I \) is taken over all sets \( I \subseteq \{1, \cdots, N\} \) for which \( \{v_i : i \in I\} \) does not belong to a cone in \( \Sigma \). Then our toric variety \( V_\Sigma \) is given by the quotient of \( \mathbb{C}^N \setminus Z_\Sigma \) by a group which is the product of a finite abelian group and \((\mathbb{C}^*)^{N-n}\) acting by

\[
(z_1, \cdots, z_N) \sim (\lambda_{w_1}^{w_1} z_1, \cdots, \lambda_{w_N}^{w_N} z_N) \quad \text{if} \quad \sum_k w_k^j v_k = 0
\]

(2)

\((N-n)\) of these linear relations are independent). Whenever \( \Sigma \) is simplicial, the corresponding variety \( V_\Sigma \) will have only quotient singularities.

Given a collection of positive integers \((w^1, \cdots, w^{n+1})\), called weights, there are two extreme ways of building a toric variety. In both cases one starts by taking \( n+1 \) vectors \( v_i \) in \( \mathbb{R}^n \) such that any \( n \) of them are linearly independent and \( \sum w^i v_i = 0 \). A convenient choice is

\[
v_i = e_i, \quad i = 1, \cdots, n, \quad v_{n+1} = -\frac{1}{w^{n+1}} \sum_1^n w^i e_i
\]

(3)
(or a multiple of these vectors). The $M$ lattice is the lattice freely generated by the $v_i$.

The weighted projective space $\mathbb{WP}(w_1, \ldots, w_{n+1})$ is the toric variety determined by the unique fan whose one-dimensional cones are $v_1, \ldots, v_{n+1}$. The weights $w_i$ define a grading of monomials by $d(\prod z_i^{a_i}) = \sum w_i a_i$. For the construction of Calabi–Yau hypersurfaces one considers quasi-homogeneous polynomials of degree $d = \sum w_i$. A polynomial $W$ is said to be transverse if the set of equations $\partial W/\partial z_i = 0$ is solved only by $z_i = 0 \forall i$. This condition ensures that the hypersurface has no singularities in addition to those coming from the singularities of the ambient space. These weight systems were classified for $n \leq 4$ in [9, 10].

A sigma model on such a singular variety may be constructed as a particular phase of the low energy limit of an $N = 2$ supersymmetric gauged linear sigma model [18]. A gauged linear sigma model with a single gauge field always contains a Landau–Ginzburg phase. The numbers of chiral primary fields of a given charge in the $N = 2$ superconformal field theory that is the low energy limit of a gauged linear sigma model do not change when going from one phase to another. Therefore we can use Vafa’s formulas [19] for charge degeneracies in $N = 2$ superconformal Landau–Ginzburg models to calculate what mathematicians call “physicists’ Hodge numbers”.

Another, a priori quite different approach, consists in considering the (maximal) Newton polyhedron $\Delta$ associated with the most general polynomial $W$ of degree $d = \sum w_i$. This is just the convex hull of the points $(a_1, \ldots, a_{n+1})$ in $\mathbb{Z}^{n+1}$ determined by the exponents occurring in the monomials of $W$. By construction $\Delta$ lies in the hyperplane $\sum w_i a_i = d$ and contains the point $1 = (1, \ldots, 1)$. After changing to $n$-dimensional integer coordinates, with $1 \to 0$, we may identify the resulting lattice with the $M$ lattice. If the origin $0$ of $M$ is in the interior of $\Delta$ (in this case we say that the weight system has the “interior point property”), the dual polyhedron

$$\Delta^* = \{ y \in \mathbb{R}^n : \langle y, x \rangle \geq -1 \forall x \in \Delta \}$$

(4)

is bounded. If, furthermore, all vertices of $\Delta^*$ are in $N$, $\Delta$ is said to be reflexive. The integer generators $v_1, \ldots, v_{n+1}$ of above may be identified with the points dual to the intersection of the planes $x_1 = 0, \ldots, x_{n+1} = 0$ with the hyperplane given by the degree condition (before the change of coordinates). $v_1, \ldots, v_{n+1}$ are points, but not necessarily vertices of $\Delta^*$. If they are, then the coordinate hyperplanes correspond to facets (codim 1 faces) of $\Delta$, i.e. the points of $\Delta$ affinely span these hyperplanes. In this case we say that a weight system has the “span property”.

It was shown in [13] that transversality always implies the interior point property and that for $n \leq 4$ the interior point property implies reflexivity. The fact that transversality implies reflexivity of $\Delta$ for $n \leq 4$ had been checked by computer [20,21]. Note, however, that the proof of [13] also applies to the much larger class of all abelian orbifolds [11] and the MNPs on the respective sublattices that arise by dividing out phase symmetries that still admit transversal polynomials.

Denoting the various sets of weight systems (in obvious notation) by $T$, $I$ and $R$, the following relations between the different types hold:

$n = 2, 3 : T = I = R$,
$n = 4 : T \subset I = R$,
$n > 4 : T \subset I, R \subset I$, no further relations.
The statement $T = I$ for $n = 2, 3$ is only known due to explicit constructions of the corresponding sets of 3 or 95 weights, respectively [13, 22]. In more than 2 dimensions the span property is independent of whether the weight system belongs to $T$, $I$ or $R$.

For a reflexive polyhedron $\Delta$ we now consider the fan $\Sigma$ over some triangulation of the faces of $\Delta^*$. A Calabi–Yau hypersurface in $\mathcal{V}_\Sigma$ is given by the zero locus of

$$p = \sum_{x \in \Delta \cap M} a_x \prod_{k=1}^N z_k^{(\nu_k, x) + 1}. \quad (5)$$

If $\Sigma$ is defined by a maximal triangulation of $\Delta^*$, the generic hypersurface of this type is smooth for $n \leq 4$ [7]. Also by [7], the Hodge numbers $h_{11}$ and $h_{1,n-2}$ are known, and in [23] the remaining Hodge numbers of the type $h_{1i}$ were calculated. For a hypersurface of dimension $n - 1 \geq 3$ these formulas can be summarised as

$$h_{1i} = \delta_{i1} \left( l(\Delta^*) - n - 1 - \sum_{\text{codim}\theta^* = 1} l^*(\theta^*) \right) + \delta_{n-2,i} \left( l(\Delta) - n - 1 - \sum_{\text{codim}\theta = 1} l^*(\theta) \right)$$

$$+ \sum_{\text{codim}\theta^* = i+1} l^*(\theta^*) l^*(\theta) \quad (6)$$

for $1 \leq i \leq n - 2$, where $l$ denotes the number of integer points of a polyhedron and $l^*$ denotes the number of interior integer points of a face.

For Calabi–Yau 4-folds there is a linear relation among the Hodge numbers that has been obtained using index theorems in [14, 24]. The same relation can, in fact, be obtained as a simple consequence of a sum rule for charge degeneracies of Ramond ground states that has been derived from modular invariance of the elliptic genus for arbitrary $N = 2$ superconformal field theories [25]:

$$\text{tr}(-)^F J_0^2 = \frac{c}{36} \text{tr}(-)^F = -\frac{d}{12} \chi, \quad (7)$$

where $J_0$ is the (left-moving) $U(1)$ charge, $c = 3d$ is the central charge and the trace extends over the Ramond ground states (we need to be careful with the sign of the Euler characteristic because the Hodge numbers $h_{pq}$ of the $\sigma$ model on a Calabi-Yau manifold and the charge degeneracies $n_{pq}$ of Ramond ground states of charge $(Q_L, Q_R) = (p - \frac{d}{2}, q - \frac{d}{2})$ are related by $h_{p,q} = n_{d-p,q}$). For a Calabi–Yau manifold of arbitrary dimension this implies

$$\sum_{p, q=0}^d (-)^{p+q} (p - \frac{d}{2})^2 h_{p,q} = -\frac{d}{12} \chi. \quad (8)$$

In 4 dimensions this equation is equivalent to

$$h_{22} = 44 + 4h_{11} - 2h_{12} + 4h_{13}, \quad (9)$$

where we used Poincaré and Hodge duality of the Hodge diamond and omitted the contribution $20h_{02} - 52h_{01}$ on the r.h.s. that vanishes for toric Calabi-Yau hypersurfaces. For the Euler characteristic of 4-folds we thus find

$$\chi = 6(8 + h_{1,1} - h_{1,2} + h_{1,3}) \quad (10)$$
In 2 dimensions the above sum rule uniquely determines the Hodge diamond of the K3 surface, whereas it is trivially satisfied (and therefore no so well-known) for CY 3-folds.

As an example for different cases that can occur in different dimensions consider the weight system \((1, \cdots, 1, 2)\). The weighted projective space \(\mathbb{WP}_{(1, \cdots, 1, 2)}\) can be represented by the vectors

\[
v_1 = (1, 0, \cdots, 0), \cdots, v_n = (0, \cdots, 0, 1), v_{n+1} = (-1/2, \cdots, -1/2)
\]

in the lattice \(N\) consisting of points in \(\mathbb{R}^n\) with coordinates that are either all integer or all half integer. Independently of \(n\), \(\mathbb{WP}_{(1, \cdots, 1, 2)}\) has precisely one pointlike singularity, located at \(z_1 = \cdots = z_n = 0\) and determined by the \(\mathbb{Z}_2\) quotient \((z_1, \cdots, z_{n+1}) \sim (-z_1, \cdots, -z_n, z_{n+1})\). In terms of toric geometry, this singularity corresponds to the fact that the simplex spanned by the vertex corresponding to expressions \(z_{n+1}^{(n+1)/2} z_i\) with \(i = 1, \cdots, n\).

Let us now consider what happens for various values of \(n\), \(W\) can be chosen as the Fermat polynomial

\[
z_1^{n+2} + \cdots + z_n^{n+2} + z_{n+1}^{(n+2)/2},
\]

so that the maximal Newton polytope \(\Delta\) is a simplex, whereas for odd \(n\) the vertex corresponding to \(z_{n+1}^{(n+2)/2}\) is replaced by \(z_{n+1}^{(n+1)/2} z_i\) with \(i = 1, \cdots, n\).

Let us now consider what happens for various values of \(n\):

- \(n = 2\): The vertices of \(\Delta^*\) are just \(v_1, v_2\) and \(v_3, v_4\) is in the interior of the edge (facet) \(\overline{v_1v_2}\). Whether we blow up \(\mathbb{WP}_{(1, 1, 2)}\) by the divisor corresponding to \(v_4\) does not matter because this divisor does not intersect the hypersurface.

- \(n = 3\): The monomials \(z_3^2 z_i\) correspond to a plane in the \(M\) lattice whose dual is \(v_5\). Thus \(\mathcal{V}_2\) is the blow-up of \(\mathbb{WP}_{(1, 1, 1, 2)}\).

- \(n = 4\): Once again \(W\) is of Fermat type, but now \(v_6\) lies outside \(\Delta^*\). The variety \(\mathcal{V}_2 = \mathbb{WP}_{(1, 1, 1, 1, 2)}\) has the \(\mathbb{Z}_2\) singularity, but the generic hypersurface of degree 6 does not intersect it. The blow-up of \(\mathbb{WP}_{(1, 1, 1, 1, 2)}\) corresponds to a different reflexive polyhedron leading to a hypersurface with different Hodge numbers.

- \(n = 5\): The monomials \(z_5^2 z_i\) correspond to a plane in the \(M\) lattice whose dual is the point \(v_7/2 \in N_{\mathbb{R}}\) which is not in \(N\). \(\Delta^*\) is no longer reflexive. As there is no Fermat type monomial in \(z_6\), any degree 7 hypersurface in \(\mathbb{WP}_{(1, 1, 1, 1, 1, 2)}\) intersects the singular point \(z_1 = \cdots = z_6 = 0\). Vafa’s formulas give \(h_{11} = 1, h_{12} = 0\) and \(h_{13} = 455\). They certainly do not correspond to the blow-up of \(\mathbb{WP}_{(1, 1, 1, 1, 1, 2)}\): Whereas the convex hull \(\tilde{\Delta}^*\) of \(v_1, \cdots, v_7\) is reflexive and the corresponding variety even smooth, \(l(\tilde{\Delta}^*)\) being 8, eq. (6) tells us that \(\tilde{h}_{11} = 8 - 6 = 2\). We note, however, that eq. (6) can be applied to non-integer polyhedra as well. In the present case the results of inserting \(\Delta, \Delta^*\) into this formula coincide with those of Vafa’s formulas. We do not know whether this is always true.
3 Fibrations

The aim of this section is to give a general recipe for identifying fibrations of hypersurfaces of holonomy $SU(n - 1)$ in $n$-dimensional toric varieties where the generic fiber is an $n' - 1$ dimensional variety of holonomy $SU(n' - 1)$. In other words, it will apply to elliptic fibrations of K3 surfaces, CY threefolds, CY fourfolds, etc., to K3 fibrations of CY $k$-folds with $k \geq 3$, to threefold fibrations of fourfolds, and so on. The main message is that the structures occurring in the fibration are reflected in structures in the $N$ lattice: The fiber, being an algebraic subvariety of the whole space, is encoded by a polyhedron $\Delta^\star_{\text{fiber}}$ which is a subpolyhedron of $\Delta^\star_{\text{CY}}$, whereas the base, which is a projection of the fibration along the fiber, can be seen by projecting the $N$ lattice along the linear space spanned by $\Delta^\star_{\text{fiber}}$. The details given in the following are somewhat technical; the reader is advised to check the various steps with some explicit example, e.g. the one given later.

Assume that $\Delta^\star$ contains a lower-dimensional reflexive subpolyhedron $\Delta^\star_{\text{fiber}} = (N_{\text{fiber}})_R \cap \Delta^\star_{\text{CY}}$ with the same interior point. This allows us to define a dual pair of exact sequences

$$0 \rightarrow N_{\text{fiber}} \rightarrow N_{\text{CY}} \rightarrow N_{\text{base}} \rightarrow 0 \quad (14)$$

and

$$0 \rightarrow M_{\text{base}} \rightarrow M_{\text{CY}} \rightarrow M_{\text{fiber}} \rightarrow 0. \quad (15)$$

Using the same arguments as in [6], we can convince ourselves that the image of $\Delta^\star_{\text{CY}}$ under $M_{\text{CY}} \rightarrow M_{\text{fiber}}$ is dual to $\Delta^\star_{\text{fiber}}$. Let us also assume that the image $\Sigma_{\text{base}}$ of $\Sigma_{\text{CY}}$ under $\pi : N_{\text{CY}} \rightarrow N_{\text{base}}$ defines a fan in $N_{\text{base}}$. This is certainly not true for arbitrary triangulations of $\Delta^\star$. Constructing fibrations, one should rather build a fan $\Sigma_{\text{base}}$ from the images of the one-dimensional cones in $\Sigma_{\text{CY}}$ and try to construct a triangulation of $\Sigma_{\text{CY}}$ and thereby of $\Delta^\star_{\text{CY}}$ that is compatible with the projection. It would be interesting to know whether this is always possible whenever the intersection of a reflexive polyhedron with a linear subspace of $N_R$ is again reflexive.

The set of one-dimensional cones in $\Sigma_{\text{base}}$ is the set of images of one-dimensional cones in $\Sigma_{\text{CY}}$ that do not lie in $N_{\text{fiber}}$. The image of a primitive generator $v_i$ of a cone in $\Sigma_{\text{CY}}$ is the origin or a positive integer multiple of a primitive generator $\tilde{v}_j$ of a one-dimensional cone in $\Sigma_{\text{base}}$. Thus we can define a matrix $r^j_i$, most of whose elements are 0, through $\pi v_i = r^j_i \tilde{v}_j$ with $r^j_i \in \mathbb{N}$ if $\pi v_i$ lies in the one-dimensional cone defined by $\tilde{v}_j$ and $r^j_i = 0$ otherwise. Our base space is the multiply weighted space determined by

$$(\tilde{z}_1, \ldots, \tilde{z}_N) \sim (\lambda^w_j \tilde{z}_1, \ldots, \lambda^w_j \tilde{z}_N), \quad j = 1, \ldots, \tilde{N} - \tilde{n} \quad (16)$$

where the $\lambda^w_j$ are any integers such that $\sum_i w^i_j \tilde{v}_i = 0$. The projection map from $V_\Sigma$ (and, as we will see, from the Calabi–Yau hypersurface) to the base is given by

$$\tilde{z}_i = \prod_j z_j^{r^j_i}. \quad (17)$$

This is well defined: $z_j \rightarrow \lambda^w_k z_j$ leads to $\tilde{z}_i \rightarrow \lambda^w_k r^j_i \tilde{z}_j$ which is among the good equivalence relations because applying $\pi$ to $\sum w^i_k v_j = 0$ gives $\sum w^i_k r^j_i \tilde{v}_i = 0$. 

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A generic point in the base space will have \( \tilde{z}_i \neq 0 \) for all \( i \), implying \( z_i \neq 0 \) for all \( v_i \notin \Delta^*_\text{fiber} \). The choice of a specific point in \( V_{\Sigma_{\text{base}}} \) and the use of all equivalence relations except for those involving only \( v_i \in \Delta^*_\text{fiber} \) allows to fix all \( z_i \) except for those corresponding to \( v_i \in \Delta^*_\text{fiber} \). Thus the preimage of a generic point in \( V_{\Sigma_{\text{base}}} \) is indeed a variety in the moduli space determined by \( \Delta^*_\text{fiber} \).

What we have seen so far is just that \( V_{\Sigma} \) is a fibration over \( V_{\Sigma_{\text{base}}} \) with generic fiber \( V_{\Sigma_{\text{fiber}}} \) (this is actually the statement of an exercise on p. 41 of ref. [15]) and how this fibration structure manifests itself in terms of homogeneous coordinates. Now we also want to see how this can be extended to hypersurfaces. To this end note that if \( v_k \in \Delta^*_\text{fiber} \) then \( \langle v_k, x \rangle \) only depends on the equivalence class \( [x] \in M_{\text{fiber}} \) of \( x \) under

\[
x \sim y \quad \text{if} \quad x - y \in M_{\text{base}}.
\]

Thus we may rewrite eq. (5) as

\[
p = \sum_{[x] \in \Delta^*_\text{fiber} \cap M_{\text{fiber}}} a'_[x] \prod_{v_k \in \Delta^*_\text{fiber}} z_k^{\langle v_k, [x] \rangle + 1} \quad \text{with} \quad a'_[x] = \sum_{x \in [x]} a_x \prod_{v_k \notin \Delta^*_\text{fiber}} z_k^{\langle v_k, x \rangle + 1}.
\]

In each coordinate patch for \( V_{\Sigma_{\text{base}}} \) this is just an equation for the fiber with coefficients that are polynomial functions of coordinates of the base space.

There are two different occasions upon which the fiber may degenerate: The fiber being a hypersurface in \( V_{\Sigma_{\text{fiber}}} \), it can either happen that \( V_{\Sigma_{\text{fiber}}} \) itself degenerates or that the coefficients of the equation determining the fiber hit a singular point in the moduli space. While we do not know any way to read off the occurrence of the second case from the toric data, we can surely see the first case: Whenever a one-dimensional cone (with primitive generator \( \tilde{v}_i \)) in \( \Sigma_{\text{base}} \) is the image of more than one one-dimensional cone in \( \Sigma \), the fiber becomes reducible over the divisor \( \tilde{z}_i = 0 \) determined by \( v_i \). Different components of the fiber correspond to different equations \( z_j = 0 \) with \( \pi v_j = r^*_j \tilde{v}_i \). The intersection patterns of the different components of the reducible fibers are crucial for understanding enhanced gauge symmetries [1, 26] and deserve further study.

Let us consider as an example the well known class of Calabi–Yau hypersurfaces (threefolds) of degree \( 6n + 12 \) in \( \mathbb{P}^{(1,1,n,2n+4,3n+6)} \) [27]. If \( 12/n \) is integer, the corresponding Newton polyhedron will be a simplex and the dual polyhedron \( \Delta^* \) can be described as the convex hull of the points

\[
(1,0,0,0), \ (0,1,0,0), \ (0,0,1,0), \ (0,0,0,1), \ (-1,-n,-2n-4,-3n-6) \quad (20)
\]

in \( N \simeq \mathbb{Z}^4 \). The elliptic fiber is determined by \( \Delta^*_\text{fiber} \) with vertices

\[
(0,0,1,0), \ (0,0,0,1), \ (0,0,-2,-3) \quad \text{in} \quad N_{\text{fiber}} = N \cap \{x_1 = x_2 = 0\}.
\]

Of course it is just the torus given by the Weierstrass equation in \( \mathbb{P}^{(1,2,3)} \). The projection of \( N \) to \( N_{\text{base}} = N/N_{\text{fiber}} \) is realised as \( \pi : (x_1, x_2, x_3, x_4) \to (x_1, x_2) \) (“throwing away the last two coordinates of each point”). \( \Delta^* \) gets projected to the convex hull of \( (1,0), (0,1) \) and \( (-1,-n) \). Each of these points provides a one-dimensional cone in \( \Sigma_{\text{base}} \), which clearly is the fan of the Hirzebruch surface \( \mathbb{F}^n \) (compare, for example, with [15], p. 7). All other integer points are of the form \( (0,-l) \) with integer \( 0 \leq l \leq n/2 \). They correspond to the one-dimensional cone generated by \( (0,-1) \) and give examples for nontrivial values of the \( r^*_j \) defined above.
Let us get even more specific and consider the case \( n = 4 \). Neglecting 4 points in the interiors of facets of \( \Delta^* \) (the corresponding divisors do not intersect the Calabi–Yau hypersurface and are therefore irrelevant for the present discussion), we can arrange the integer points of \( \Delta^* \) according to their images

\[
\bar{v}_1 := (1, 0), \quad \bar{v}_2 := (0, 1), \quad \bar{v}_3 := (-1, -4), \quad \bar{v}_4 := (0, -1), \quad 0 := (0, 0)
\]

in \( \Sigma_{\text{base}} \):

\[
\pi v = r \bar{v}_1 \quad \text{for} \quad v_1 := (1, 0, 0, 0),
\]

\[
\pi v = r \bar{v}_2 \quad \text{for} \quad v_2 := (0, 1, 0, 0),
\]

\[
\pi v = r \bar{v}_3 \quad \text{for} \quad v_3 := (-1, -4, -12, -18),
\]

\[
\pi v = r \bar{v}_4 \quad \text{for} \quad v_4 := (0, -1, -3, -4), \quad v_5 := (0, -1, -4, -6), \quad v_6 := (0, -2, -6, -9),
\]

\[
\pi v = 0 \quad \text{for} \quad v_7 := (0, 0, 1, 0), \quad v_8 := (0, 0, 1, 0), \quad v_9 := (0, 0, -2, -3).
\]

The projection to the base is given by

\[
\bar{z}_1 = z_1, \quad \bar{z}_2 = z_2, \quad \bar{z}_3 = z_3, \quad \bar{z}_4 = z_4 z_5 z_6^2.
\]

There are many linear relations among the \( v_i \). One of them is \( 2v_7 + 3v_8 + v_9 = 0 \), ensuring that

\[
(z_1, \cdots, z_6, z_7, z_8, z_9) \sim (z_1, \cdots, z_6, \lambda^2 z_7, \lambda^3 z_8, \lambda z_9).
\]

With respect to this relation, \( p \) is quasi-homogeneous of degree 6. With linear redefinitions of \( z_7, z_8, z_9 \), one can bring \( p \) into Weierstrass form

\[
p = z_8^2 - z_7^3 + f(z_1, \cdots, z_6) z_7 z_8^2 + g(z_1, \cdots, z_6) z_8^3.
\]

Let us also briefly discuss strategies for finding fibrations. One strategy is to take reflexive polyhedra and look for lower-dimensional reflexive polyhedra that are contained in them. In particular, looking for elliptic fibrations, one might just intersect \( \Delta^* \) with any two-dimensional plane spanned by integer points of \( \Delta^* \). Checking \( \binom{N}{2} \) pairs of points w.r.t. whether the plane spanned by them carries a reflexive subpolyhedron is no challenge to present day computer power, even for large numbers of polyhedra.

An even simpler approach for constructing large numbers of fibrations could make use of the following observation: If \( \Delta', \Delta'' \) are reflexive polyhedra in lattices \( M', M'' \) respectively, then

\[
\Delta := \Delta' \times \Delta'' = \{(x', x'') : x' \in \Delta', \ x'' \in \Delta''\}
\]

is also a reflexive polyhedron. Its dual is

\[
\Delta^* = \{(\lambda y', (1-\lambda)y'') : \ y' \in \Delta', \ y'' \in \Delta'', \ 0 \leq \lambda \leq 1\}
\]

and the set \( \Sigma(k) \) of \( k \)-dimensional cones in \( \Sigma \) is given by

\[
\Sigma(k) = \{(v', v'') : \ v' \in \Sigma(k'), \ v'' \in \Sigma(k''), \ k' + k'' = k\}.
\]

Of course \( \Sigma \) is a fibration. The equation \( p = 0 \) defines a Calabi-Yau \( (k-1) \)-fold which may be interpreted either as a fibration with base \( \Sigma \) and generic fiber a hypersurface in \( \Sigma \) or as a fibration with base \( \Sigma' \) and generic fiber a hypersurface in \( \Sigma' \). In this way one can immediately construct almost 3 million elliptic fibration fourfolds by combining one of the 16 two-dimensional reflexive polyhedra with reflexive polyhedra coming from one of the 184,026 weight systems of [13].
4 Scans for fourfolds

We performed systematic searches for the two different types of Calabi–Yau fourfold models described by a single weight system as analysed in section 2. In both cases we used improved versions of the same basic strategy, namely to check for all partitions \( w^1, \cdots, w^{n+1} \) of \( d = 6, 7, \cdots \) whether they allow for transverse polynomials or reflexive polyhedra.

For the transverse weights, the preselection that we imposed to reduce the number of partitions is, in the language of [28], “compatibility with at most one unresolved pointer to an unknown weight”. With this method we obtained the complete list of 525572 weight systems of degree up to 4000. Based on the statistics given in table 1 we cannot give an estimate of the total number, since \( 2 \times 10^5/d \) is a good approximation to the average number of transversal weights per degree, which would lead to a divergent sum. It is well-known, however, that this set is finite [28]. A complete enumeration is impossible with our present approach (this can be inferred from the rate at which our program slows down with growing \( d \) and the fact that the degrees of the 3462 Fermat weights range up to \( d = 3263442 \)), but the numbers in table 1 do not seem to exclude the possibility of a complete classification along the lines of refs. [9,10].

The situation is quite different for 4-folds coming from reflexive weights, which are far more numerous. Here we are limited by disk space rather than by calculation time. We have the complete list of 914164 weights with degree \( d \leq 150 \), only 6918 of which are transversal. Together with the weights that are both transversal and reflexive we have thus accumulated more than \( 10^6 \) reflexive weights. Various sublists, as well as the files with the complete results are available on the internet.\(^1\)

A central goal of our computer studies was to investigate the relation between transversality and reflexivity of the maximal Newton polyhedra. Whereas transversality always implies reflexivity in up to 4 dimensions, it turned out that only about 20% of the five dimensional polyhedra defined by our transversal weights are reflexive. This number shows only little dependence (in the form of a slight decrease) on the degree. Since the gauged linear \( \sigma \) models based on these weights have a Landau–Ginzburg phase we could use Vafa’s formulas to compute all charge degeneracies of Ramond ground states. For the 109308 reflexive weights in this class we could thus check the coincidence of these numbers with Batyrev’s result for the cohomology of the Calabi–Yau hypersurfaces in the toric varieties defined by the reflexive polytopes. The subtle connections between the two types of Hodge numbers are discussed in [23]. Going from the weighted projective space to the variety defined by a maximal triangulation of the reflexive polyhedron we do not resolve all of the singularities of the ambient space. As we saw in our example at the end of section 2, even in cases when there exists an “obvious” way of blowing up the embedding variety, this need not be the one leading to the same Hodge numbers.

After the general statistics of transversal and reflexive weights is tables 1 and 2 we provide lists of weights that may be of particular interest because of small Hodge numbers \( h_{11} \) or \( h_{13} \), or because of a negative Euler number, which is desirable in the context of SUSY breaking [4]. In table 3 we give the possible negative Euler number that arise in our lists. The value \( \chi = -30 \) only occurs for non-reflexive weights, and our smallest value \( \chi = -252 \) (which is not divisible by 24) occurs at degree 108 in the reflexive case and only at degree 484 in the transversal case.

\(^1\)The URL is http://tph.tuwien.ac.at/~kreuzer/CY
In table 4 we give the numbers of weights of various types that we find with $h_{11} \leq 7$ or $h_{13} \leq 7$. In table 5 we give all 30 transversal and 113 reflexive weights with $\chi < 0$ and $d \leq 150$; there is an overlap of 26 weights in this table that are both reflexive and transversal. Up to degree 4000 we found 174 more transversal weights with negative Euler number, so that altogether we have 291 weights with $\chi < 0$ (the smallest values of $h_{11}$ and $h_{13}$ in this list are both 22).

In tables 6 and 7 we list all our weights with $h_{11} = 1$ or $h_{13} = 1$; Fermat weights all have $\chi > 0$ and do not contribute to any of our tables of special weights, except for 8 weights with $6 \leq d \leq 18$ that give $h_{11} = 1$. Some “first occurrences” are:
The first non-reflexive transversal weight system is $(1, 1, 1, 1, 1, 2)$.
The first non-transversal reflexive weight system is $(1, 1, 1, 1, 1, 3)$.
The first degree that does not admit any weight that is transversal and reflexive is 11.
The first degree for which no transversal weight system exists is 1733 (there are only 29 such degrees $d$ in the range $d \leq 4000$).

When looking for models with very specific features it may be useful to go beyond the class of models considered here. In particular if we are interested in elliptic fibrations it will be more economic to generate reflexive polytopes in terms of combined weight systems [12] since the complete set of relevant weights is already known [13] and the fibrations structure is encoded (and can be pre-selected) in a rather simple and explicit way. Note also that the 3 million elliptic fibrations that were mentioned at the end of the last section are known to be all connected in a web with singular transitions that respect the fibration structure because the same is true for the polytopes of which we are taking direct products [29].

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Tables

Table 1: There are 109308 reflexive weights among the 525572 transversal weights with \( d \leq 4000 \).

| d   | trans | ref. | d   | trans | ref. | d   | trans | ref. | d   | trans | ref. |
|-----|-------|------|-----|-------|------|-----|-------|------|-----|-------|------|
| 100 | 11798 | 3578 | 1100| 18333 | 3917 | 2100| 11328 | 2292 | 3100| 6765  | 1208 |
| 200 | 28457 | 6685 | 1200| 16351 | 3261 | 2200| 9694  | 1773 | 3200| 7449  | 1369 |
| 300 | 31075 | 6716 | 1300| 14427 | 3045 | 2300| 8944  | 1627 | 3300| 6811  | 1300 |
| 400 | 29229 | 6163 | 1400| 15334 | 3196 | 2400| 10807 | 2314 | 3400| 6476  | 1282 |
| 500 | 26792 | 5798 | 1500| 12907 | 2450 | 2500| 7385  | 1773 | 3500| 6209  | 1230 |
| 600 | 26578 | 5649 | 1600| 13570 | 2764 | 2600| 9190  | 1897 | 3600| 6597  | 1329 |
| 700 | 22367 | 4665 | 1700| 12432 | 2454 | 2700| 8470  | 1700 | 3700| 6061  | 1218 |
| 800 | 22139 | 4725 | 1800| 11594 | 2400 | 2800| 8134  | 1618 | 3800| 6288  | 1249 |
| 900 | 20704 | 4478 | 1900| 11030 | 2118 | 2900| 7975  | 1523 | 3900| 5394  | 1046 |
| 1000| 17475 | 3605 | 2000| 10273 | 1961 | 3000| 6827  | 1248 | 4000| 5903  | 1105 |

\[ \sum 236614 \quad 52062 \quad 136251 \quad 27566 \quad 88754 \quad 17344 \quad 63953 \quad 12336 \]

Table 2: There are 914051 reflexive, 6918 reflexive and transversal, and 25435 general transversal weights with \( d \leq 150 \).

| d   | R    | RT   | T    | d   | R    | RT   | T    | d   | R    | RT   | T    |
|-----|------|------|------|-----|------|------|------|-----|------|------|------|
| 10  | 9    | 8    | 11   | 60  | 11489| 490  | 1480 | 110 | 84095| 509  | 2235 |
| 20  | 164  | 63   | 109  | 70  | 19046| 375  | 1397 | 120 | 14038| 955  | 3482 |
| 30  | 835  | 209  | 422  | 80  | 30747| 586  | 2030 | 130 | 137806| 456  | 2005 |
| 40  | 2485 | 252  | 684  | 90  | 45815| 744  | 2529 | 140 | 178688| 656  | 2845 |
| 50  | 5724 | 356  | 1093 | 100 | 62779| 495  | 2043 | 150 | 220331| 764  | 3070 |

\[ \sum 9217 \quad 888 \quad 2319 \quad \sum 169876 \quad 2690 \quad 9479 \quad \sum 734958 \quad 3340 \quad 13637 \]

Table 3: Negative values of the Euler number (the value \( \chi = -30 \) only occurs for non-reflexive weights).

| -6  | -12 | -18 | -24 | -30 | -36 | -42 | -48 | -60 | -66 | -72 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| -84 | -90 | -96 |     | -120|     |     |     | -132| -138| -144|
| -168| -180| -192| -198|     |     |     |     |     |     |     |
| -240| -252|     |     |     |     |     |     |     |     |     |

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Table 4: Numbers of weights with small $1 \leq h_{11} \leq 5$ or $1 \leq h_{13} \leq 5$ in the list of all reflexive weights with $d \leq 150$, and reflexive (RT) or general (T) transversal weights with $d \leq 4000$.

| $h_{11}$ | $R_{150}$ | $RT_{4000}$ | $T_{4000}$ | $h_{13}$ | $R_{150}$ | $RT_{4000}$ | $T_{4000}$ |
|----------|-----------|-------------|------------|----------|-----------|-------------|------------|
| 1        | 8         | 8           | 33         | 1        |          |             |            |
| 2        | 33        | 27          | 106        | 2        |          |             |            |
| 3        | 101       | 66          | 255        | 3        |          |             |            |
| 4        | 168       | 88          | 411        | 4        |          |             |            |
| 5        | 267       | 111         | 508        | 5        |          |             |            |
| 6        | 501       | 183         | 800        | 6        |          |             |            |
| 7        | 617       | 158         | 789        | 7        |          |             |            |

Table 5: Weights for 4-folds with negative Euler number and $d \leq 150$: Degrees of reflexive and transversal weights are in boldface, $V$ and $\overline{V}$ denote the numbers of vertices of $\Delta$ and $\Delta^*$, and non-reflexive weights have no entry for the number $\overline{P}$ of lattice points of $\Delta^*$.

| $d$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $P$ | $V$ | $\overline{P}$ | $\overline{V}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | $h_{22}$ | $\chi$ |
|-----|-------|-------|-------|-------|-------|-------|-----|-----|----------------|----------------|----------|----------|----------|----------|-------|
| 70  | 7     | 7     | 7     | 13    | 17    | 19    | 84  | 8   | 38            | 7             | 26       | 108      | 72       | 220      | −12    |
| 77  | 7     | 7     | 7     | 17    | 19    | 20    | 93  | 8   | 38            | 7             | 26       | 135      | 81       | 202      | −120   |
| 80  | 5     | 5     | 5     | 14    | 22    | 29    | 199 | 8   | 42            | 7             | 24       | 210      | 177      | 428      | −6     |
| 80  | 5     | 5     | 5     | 16    | 18    | 31    | 199 | 8   | 42            | 7             | 24       | 210      | 177      | 428      | −6     |
| 80  | 10    | 13    | 13    | 13    | 15    | 16    | 38  | 6   | 48            | 6             | 38       | 96       | 26       | 108      | −144   |
| 84  | 7     | 7     | 7     | 12    | 18    | 33    | 133 | 6   | 38            | 6             | 22       | 165      | 111      | 246      | −144   |
| 84  | 11    | 11    | 11    | 12    | 18    | 21    | 43  | 6   | 38            | 6             | 28       | 91       | 31       | 98       | −144   |
| 85  | 5     | 5     | 5     | 16    | 23    | 31    | 214 | 8   | 42            | 7             | 24       | 240      | 192      | 428      | −96    |
| 85  | 5     | 5     | 5     | 17    | 21    | 32    | 214 | 8   | 42            | 7             | 24       | 240      | 192      | 428      | −96    |
| 88  | 8     | 8     | 8     | 19    | 22    | 23    | 93  | 8   | 38            | 7             | 26       | 135      | 81       | 202      | −120   |
| 88  | 8     | 11    | 12    | 19    | 19    | 19    | 44  | 9   | 70            | 8             | 63       | 108      | 29       | 196      | −48    |
| 90  | 5     | 5     | 5     | 18    | 24    | 33    | 231 | 6   | 42            | 6             | 22       | 272      | 209      | 424      | −198   |
| 90  | 7     | 7     | 7     | 19    | 20    | 30    | 84  | 8   | 38            | 7             | 26       | 108      | 72       | 220      | −12    |
| 90  | 9     | 9     | 9     | 10    | 16    | 37    | 113 | 6   | 39            | 6             | 22       | 144      | 91       | 208      | −138   |
| 90  | 9     | 9     | 9     | 13    | 20    | 30    | 84  | 8   | 38            | 7             | 26       | 108      | 72       | 220      | −12    |
| 90  | 9     | 9     | 9     | 14    | 19    | 30    | 84  | 8   | 38            | 7             | 26       | 108      | 72       | 220      | −12    |
| 90  | 9     | 9     | 9     | 17    | 22    | 24    | 84  | 8   | 38            | 7             | 26       | 108      | 72       | 220      | −12    |
| $d$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $P$ | $V$ | $\bar{P}$ | $\bar{V}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | $h_{22}$ | $\chi$ |
|-----|-------|-------|-------|-------|-------|-------|-----|-----|--------|--------|--------|--------|--------|--------|-----|
| 90  | 9     | 10    | 14    | 19    | 19    | 19    | 48  | 6   | 75     | 6      | 65     | 108    | 33     | 220    | −12  |
| 90  | 9     | 12    | 13    | 13    | 13    | 30    | 55  | 7   | 38     | 7      | 27     | 92     | 42     | 136    | −90  |
| 90  | 11    | 11    | 11    | 15    | 18    | 24    | 42  | 6   | 42     | 6      | 30     | 101    | 30     | 82     | −198 |
| 91  | 11    | 13    | 13    | 13    | 16    | 25    | 55  | 9   | 34     | 8      | 23     | 90     | 43     | 128    | −96  |
| 92  | 7     | 7     | 7     | 22    | 23    | 26    | 84  | 8   | 38     | 7      | 26     | 108    | 72     | 220    | −12  |
| 95  | 10    | 13    | 15    | 19    | 19    | 19    | 42  | 10  | 37     | 9      | 30     | 72     | 30     | 140    | −24  |
| 96  | 7     | 7     | 7     | 19    | 24    | 32    | 93  | 8   | 38     | 7      | 26     | 135    | 81     | 202    | −120 |
| 96  | 7     | 12    | 19    | 19    | 19    | 20    | 42  | 10  | 84     | 9      | 75     | 126    | 27     | 200    | −96  |
| 96  | 8     | 11    | 19    | 19    | 19    | 20    | 42  | 9   | 76     | 8      | 69     | 108    | 27     | 212    | −24  |
| 98  | 7     | 7     | 14    | 20    | 24    | 26    | 79  | 8   | 39     | 7      | 27     | 108    | 69     | 212    | −24  |
| 99  | 7     | 7     | 7     | 22    | 23    | 33    | 93  | 8   | 38     | 7      | 26     | 135    | 81     | 202    | −120 |
| 99  | 9     | 9     | 9     | 17    | 22    | 33    | 93  | 8   | 38     | 7      | 26     | 135    | 81     | 202    | −120 |
| 99  | 9     | 9     | 9     | 19    | 20    | 33    | 93  | 8   | 38     | 7      | 26     | 135    | 81     | 202    | −120 |
| 99  | 9     | 9     | 9     | 19    | 23    | 30    | 93  | 8   | 38     | 7      | 26     | 135    | 81     | 202    | −120 |
| 99  | 9     | 10    | 19    | 19    | 19    | 23    | 41  | 9   | 91     | 8      | 81     | 135    | 26     | 202    | −120 |
| 99  | 9     | 11    | 11    | 11    | 15    | 42    | 108 | 6   | 42     | 6      | 24     | 141    | 85     | 198    | −144 |
| 99  | 9     | 11    | 19    | 20    | 20    | 20    | 41  | 9   | 91     | 8      | 81     | 135    | 26     | 202    | −120 |
| 99  | 9     | 14    | 14    | 14    | 15    | 33    | 54  | 6   | 42     | 6      | 29     | 102    | 41     | 120    | −100 |
| 100 | 7     | 7     | 7     | 23    | 25    | 31    | 93  | 8   | 38     | 7      | 26     | 135    | 81     | 202    | −120 |
| 102 | 6     | 6     | 6     | 17    | 32    | 35    | 214 | 8   | 42     | 7      | 24     | 240    | 192    | 428    | −96  |
| 102 | 9     | 12    | 17    | 17    | 17    | 30    | 56  | 8   | 34     | 8      | 25     | 82     | 43     | 152    | −36  |
| 102 | 12    | 17    | 17    | 18    | 21    | 44  | 7   | 34     | 7      | 26     | 81     | 32     | 114    | −90  |
| 104 | 8     | 13    | 20    | 21    | 21    | 21    | 39  | 6   | 93     | 6      | 83     | 147    | 24     | 178    | −192 |
| 105 | 7     | 7     | 14    | 24    | 26    | 27    | 85  | 9   | 42     | 10     | 30     | 126    | 74     | 208    | −84  |
| 105 | 7     | 15    | 17    | 22    | 22    | 22    | 45  | 12  | 84     | 9      | 74     | 126    | 30     | 208    | −84  |
| 105 | 10    | 15    | 17    | 21    | 21    | 21    | 42  | 9   | 37     | 8      | 30     | 72     | 30     | 140    | −24  |
| 105 | 11    | 15    | 15    | 15    | 21    | 28    | 55  | 9   | 34     | 8      | 23     | 90     | 43     | 128    | −96  |
| 108 | 7     | 7     | 7     | 24    | 27    | 36    | 104 | 6   | 38     | 6      | 23     | 165    | 92     | 174    | −252 |
| 110 | 5     | 5     | 10    | 22    | 26    | 42    | 184 | 8   | 43     | 7      | 25     | 200    | 165    | 404    | −12  |
| 110 | 9     | 15    | 20    | 22    | 22    | 22    | 44  | 9   | 47     | 8      | 36     | 90     | 32     | 136    | −84  |
| 110 | 10    | 11    | 21    | 21    | 21    | 26    | 41  | 9   | 91     | 8      | 81     | 135    | 26     | 202    | −120 |
| 110 | 10    | 15    | 19    | 22    | 22    | 22    | 42  | 9   | 37     | 8      | 30     | 72     | 30     | 140    | −24  |
| 110 | 11    | 11    | 16    | 20    | 22    | 30    | 52  | 9   | 34     | 8      | 24     | 80     | 42     | 148    | −36  |
| 112 | 7     | 7     | 14    | 16    | 24    | 44    | 119 | 6   | 39     | 6      | 23     | 147    | 100    | 242    | −96  |
| $d$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $P$ | $V$ | $\overline{P}$ | $\overline{V}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | $h_{22}$ | $\chi$ |
|-----|-------|-------|-------|-------|-------|-------|-----|-----|--------------|--------------|---------|---------|---------|---------|-----|
| 112 | 7     | 16    | 20    | 23    | 23    | 23    | 42  | 9   | 92           | 8            | 81      | 144     | 27      | 188     | −168 |
| 112 | 8     | 13    | 21    | 21    | 21    | 28    | 42  | 9   | 76           | 8            | 69      | 108     | 27      | 212     | −24  |
| 112 | 11    | 11    | 16    | 22    | 24    | 28    | 39  | 6   | 39           | 6            | 29      | 82      | 29      | 112     | −96  |
| 114 | 6     | 14    | 19    | 25    | 25    | 25    | 53  | 10  | 88           | 9            | 77      | 135     | 38      | 234     | −72  |
| 114 | 9     | 15    | 19    | 19    | 19    | 33    | 55  | 7   | 38           | 7            | 27      | 92      | 42      | 136     | −90  |
| 114 | 15    | 18    | 19    | 19    | 19    | 24    | 43  | 7   | 38           | 7            | 28      | 91      | 31      | 98      | −144 |
| 115 | 5     | 5     | 10    | 22    | 31    | 42    | 194 | 9   | 45           | 9            | 27      | 220     | 174     | 408     | −66  |
| 115 | 5     | 5     | 10    | 23    | 29    | 43    | 194 | 9   | 45           | 9            | 27      | 220     | 174     | 408     | −66  |
| 115 | 10    | 15    | 23    | 23    | 23    | 23    | 42  | 9   | 37           | 8            | 30      | 72      | 30      | 140     | −24  |
| 120 | 5     | 5     | 10    | 24    | 32    | 44    | 206 | 6   | 43           | 6            | 23      | 242     | 187     | 400     | −144 |
| 120 | 7     | 7     | 14    | 22    | 30    | 40    | 79  | 8   | 39           | 7            | 27      | 108     | 69      | 212     | −24  |
| 120 | 8     | 8     | 16    | 27    | 30    | 31    | 85  | 9   | 42           | 10           | 30      | 126     | 74      | 208     | −84  |
| 120 | 8     | 14    | 15    | 15    | 15    | 53    | 111 | 6   | 49           | 6            | 29      | 147     | 86      | 210     | −144 |
| 120 | 8     | 15    | 23    | 23    | 23    | 28    | 39  | 6   | 93           | 6            | 83      | 147     | 24      | 178     | −192 |
| 120 | 10    | 17    | 17    | 18    | 24    | 34    | 39  | 7   | 57           | 7            | 47      | 92      | 29      | 164     | −48  |
| 120 | 11    | 11    | 20    | 22    | 24    | 32    | 38  | 6   | 43           | 6            | 31      | 91      | 28      | 98      | −144 |
| 120 | 12    | 13    | 13    | 16    | 26    | 40    | 50  | 7   | 39           | 7            | 28      | 83      | 39      | 146     | −48  |
| 121 | 10    | 11    | 21    | 21    | 21    | 37    | 41  | 9   | 91           | 8            | 81      | 135     | 26      | 202     | −120 |
| 125 | 4     | 13    | 25    | 25    | 25    | 33    | 87  | 10  | 72           | 8            | 56      | 144     | 68      | 252     | −72  |
| 126 | 9     | 9     | 18    | 20    | 28    | 42    | 79  | 8   | 39           | 7            | 27      | 108     | 69      | 212     | −24  |
| 126 | 9     | 9     | 18    | 22    | 26    | 42    | 79  | 8   | 39           | 7            | 27      | 108     | 69      | 212     | −24  |
| 126 | 11    | 11    | 14    | 16    | 63    | 113 | 6   | 39           | 6            | 22      | 144     | 91      | 208     | −138 |
| 126 | 13    | 13    | 18    | 21    | 26    | 35    | 34  | 8   | 55           | 9            | 43      | 90      | 23      | 128     | −96  |
| 128 | 7     | 7     | 14    | 30    | 32    | 38    | 79  | 8   | 39           | 7            | 27      | 108     | 69      | 212     | −24  |
| 130 | 7     | 7     | 7     | 18    | 26    | 65    | 199 | 8   | 42           | 7            | 24      | 210     | 177     | 428     | −6   |
| 130 | 10    | 10    | 13    | 20    | 24    | 53    | 100 | 9   | −              | 9            | 26      | 120     | 80      | 228     | −36  |
| 132 | 7     | 7     | 14    | 27    | 33    | 44    | 85  | 9   | 42           | 10           | 30      | 126     | 74      | 208     | −84  |
| 132 | 8     | 8     | 8     | 31    | 33    | 44    | 93  | 8   | 38           | 7            | 26      | 135     | 81      | 202     | −120 |
| 132 | 11    | 11    | 12    | 20    | 22    | 56    | 97  | 6   | 43           | 6            | 25      | 126     | 77      | 200     | −96  |
| 132 | 11    | 12    | 23    | 23    | 23    | 40    | 38  | 6   | 104          | 6            | 92      | 165     | 23      | 174     | −252 |
| 132 | 12    | 12    | 12    | 19    | 33    | 44    | 93  | 8   | 38           | 7            | 26      | 135     | 81      | 202     | −120 |
| 132 | 12    | 12    | 12    | 23    | 33    | 40    | 93  | 8   | 38           | 7            | 26      | 135     | 81      | 202     | −120 |
| 132 | 12    | 13    | 13    | 13    | 15    | 66    | 108 | 6   | 42           | 6            | 24      | 141     | 85      | 198     | −144 |
| 132 | 12    | 15    | 22    | 22    | 22    | 39    | 54  | 6   | 42           | 6            | 29      | 102     | 41      | 120     | −144 |
| $d$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $P$ | $V$ | $\overline{P}$ | $\overline{V}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | $h_{22}$ | $\chi$ |
|-----|-------|-------|-------|-------|-------|-------|-----|-----|-------------|-------------|--------|--------|--------|--------|-----|
| 132 | 12    | 17    | 17    | 22    | 30    | 34    | 34  | 6   | 62          | 6           | 52     | 100    | 24     | 148    | −96 |
| 133 | 7     | 7     | 21    | 31    | 33    | 34    | 90  | 8   | 49          | 7           | 37     | 135    | 78     | 234    | −72 |
| 134 | 7     | 7     | 7     | 22    | 24    | 67    | 199 | 8   | 42          | 7           | 24     | 210    | 177    | 428    | −6  |
| 135 | 4     | 15    | 27    | 27    | 27    | 35    | 87  | 10  | 72          | 8           | 56     | 144    | 68     | 252    | −72 |
| 135 | 7     | 7     | 14    | 30    | 32    | 45    | 85  | 9   | 42          | 10          | 30     | 126    | 74     | 208    | −84 |
| 135 | 9     | 9     | 18    | 26    | 31    | 42    | 85  | 9   | 42          | 10          | 30     | 126    | 74     | 208    | −84 |
| 135 | 15    | 19    | 20    | 27    | 27    | 27    | 38  | 6   | 48          | 6           | 38     | 96     | 26     | 108    | −144|
| 136 | 7     | 7     | 14    | 31    | 34    | 43    | 85  | 9   | 42          | 10          | 30     | 126    | 74     | 208    | −84 |
| 136 | 8     | 8     | 8     | 21    | 23    | 68    | 214 | 8   | 42          | 7           | 24     | 240    | 192    | 428    | −96 |
| 136 | 16    | 17    | 17    | 24    | 28    | 34    | 40  | 7   | 35          | 7           | 27     | 73     | 30     | 126    | −48 |
| 138 | 6     | 6     | 12    | 23    | 44    | 47    | 194 | 9   | 45          | 9           | 27     | 220    | 174    | 408    | −66 |
| 138 | 6     | 22    | 23    | 29    | 29    | 29    | 49  | 6   | 104         | 6           | 91     | 165    | 34     | 214    | −192|
| 140 | 5     | 13    | 28    | 28    | 28    | 38    | 83  | 6   | 83          | 6           | 64     | 168    | 64     | 220    | −192|
| 140 | 7     | 7     | 20    | 21    | 30    | 55    | 123 | 7   | 49          | 7           | 32     | 150    | 102    | 280    | −48 |
| 140 | 10    | 19    | 19    | 26    | 28    | 38    | 38  | 6   | 64          | 6           | 52     | 104    | 28     | 156    | −96 |
| 140 | 11    | 11    | 20    | 30    | 33    | 35    | 40  | 7   | 49          | 7           | 39     | 83     | 28     | 146    | −48 |
| 140 | 20    | 20    | 20    | 21    | 24    | 35    | 51  | 8   | −           | 7           | 27     | 90     | 39     | 128    | −96 |
| 142 | 7     | 7     | 7     | 23    | 27    | 71    | 214 | 8   | 42          | 7           | 24     | 240    | 192    | 428    | −96 |
| 143 | 10    | 13    | 13    | 19    | 26    | 62    | 98  | 10  | −           | 10          | 31     | 120    | 76     | 232    | −30 |
| 144 | 7     | 7     | 14    | 32    | 36    | 48    | 93  | 6   | 39          | 6           | 24     | 147    | 83     | 178    | −192|
| 144 | 9     | 9     | 18    | 28    | 32    | 48    | 93  | 6   | 39          | 6           | 24     | 147    | 83     | 178    | −192|
| 144 | 9     | 16    | 16    | 21    | 34    | 48    | 57  | 6   | 63          | 6           | 48     | 114    | 42     | 176    | −96 |
| 144 | 16    | 16    | 18    | 21    | 32    | 41    | 48  | 8   | −           | 9           | 27     | 84     | 37     | 132    | −72 |
| 144 | 16    | 17    | 17    | 18    | 34    | 42    | 33  | 6   | 69          | 6           | 57     | 112    | 23     | 140    | −144|
| 145 | 5     | 5     | 15    | 28    | 39    | 53    | 204 | 8   | 54          | 7           | 35     | 232    | 183    | 452    | −36 |
| 145 | 5     | 5     | 15    | 29    | 37    | 54    | 204 | 8   | 54          | 7           | 35     | 232    | 183    | 452    | −36 |
| 145 | 5     | 14    | 29    | 29    | 29    | 39    | 83  | 6   | 83          | 6           | 64     | 168    | 64     | 220    | −192|
| 150 | 7     | 7     | 7     | 24    | 30    | 75    | 231 | 6   | 42          | 6           | 22     | 272    | 209    | 424    | −198|
| 150 | 10    | 14    | 17    | 17    | 17    | 75    | 111 | 6   | 49          | 6           | 29     | 147    | 86     | 210    | −144|
| 150 | 11    | 11    | 25    | 30    | 33    | 40    | 39  | 7   | 54          | 7           | 42     | 92     | 27     | 136    | −90 |
| 150 | 13    | 13    | 15    | 20    | 39    | 50    | 51  | 8   | 49          | 8           | 38     | 85     | 38     | 178    | −6  |
| 150 | 21    | 24    | 25    | 25    | 25    | 30    | 42  | 6   | 42          | 6           | 30     | 101    | 30     | 82     | −198|
Table 6: Weights for 4-folds with $h_{11} = 1$ (for reflexive $\Delta$ this implies $V = \overline{V} = 6$).

| $d$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $P$ | $V$ | $\overline{P}$ | $\overline{V}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | $h_{22}$ | $\chi$ |
|-----|-------|-------|-------|-------|-------|-------|-----|-----|--------------|--------------|-------|-------|-------|-------|-----|
| 6   | 1     | 1     | 1     | 1     | 1     | 1     | 462 | 6   | 7            | 6            | 1      | 0      | 426   | 1752  | 2610 |
| 7   | 1     | 1     | 1     | 1     | 1     | 2     | 496 | 10  | -            | 7            | 1      | 0      | 455   | 1868  | 2784 |
| 8   | 1     | 1     | 1     | 1     | 2     | 2     | 483 | 6   | 7            | 6            | 1      | 0      | 443   | 1820  | 2712 |
| 9   | 1     | 1     | 1     | 1     | 2     | 3     | 575 | 10  | -            | 7            | 1      | 0      | 523   | 2140  | 3192 |
| 10  | 1     | 1     | 1     | 1     | 1     | 5     | 1128 | 6   | 8            | 6            | 1      | 0      | 976   | 3052  | 5000 |
| 10  | 1     | 1     | 1     | 2     | 2     | 3     | 489 | 10  | -            | 7            | 1      | 0      | 447   | 1836  | 2736 |
| 12  | 1     | 1     | 1     | 1     | 1     | 2     | 1167 | 6   | 8            | 6            | 1      | 0      | 1099  | 4084  | 6108 |
| 12  | 1     | 1     | 1     | 2     | 3     | 4     | 603 | 6   | 7            | 6            | 1      | 0      | 547   | 2236  | 3336 |
| 14  | 1     | 1     | 1     | 2     | 2     | 7     | 1081 | 6   | 8            | 6            | 1      | 0      | 935   | 3788  | 5664 |
| 15  | 1     | 1     | 2     | 3     | 3     | 5     | 492 | 10  | -            | 7            | 1      | 0      | 447   | 1836  | 2736 |
| 16  | 1     | 1     | 1     | 2     | 3     | 8     | 1226 | 9   | -            | 7            | 1      | 0      | 1059  | 4284  | 6408 |
| 16  | 1     | 1     | 2     | 3     | 4     | 5     | 509 | 13  | -            | 9            | 1      | 0      | 463   | 1900  | 2832 |
| 18  | 1     | 1     | 1     | 2     | 2     | 3     | 984 | 6   | 8            | 6            | 1      | 0      | 851   | 3452  | 5160 |
| 18  | 1     | 2     | 3     | 3     | 4     | 5     | 309 | 14  | -            | 9            | 1      | 0      | 283   | 1180  | 1752 |
| 20  | 1     | 2     | 3     | 4     | 5     | 5     | 314 | 10  | -            | 7            | 1      | 0      | 267   | 1196  | 1776 |
| 21  | 1     | 1     | 3     | 4     | 5     | 7     | 564 | 12  | -            | 9            | 1      | 0      | 511   | 2092  | 3120 |
| 21  | 1     | 2     | 3     | 3     | 5     | 7     | 378 | 13  | -            | 8            | 1      | 0      | 343   | 1420  | 2112 |
| 22  | 1     | 1     | 2     | 3     | 4     | 11    | 1095 | 12  | -            | 8            | 1      | 0      | 946   | 3832  | 5730 |
| 22  | 1     | 2     | 3     | 4     | 5     | 7     | 358 | 20  | -            | 13           | 1      | 0      | 326   | 1352  | 2010 |
| 24  | 2     | 3     | 3     | 4     | 5     | 7     | 187 | 13  | -            | 9            | 1      | 0      | 171   | 732   | 1080 |
| 26  | 1     | 2     | 2     | 3     | 5     | 13    | 855 | 12  | -            | 8            | 1      | 0      | 739   | 3004  | 4488 |
| 28  | 1     | 1     | 3     | 4     | 5     | 14    | 1148 | 12  | -            | 8            | 1      | 0      | 991   | 4012  | 6000 |
| 28  | 1     | 3     | 4     | 5     | 7     | 8     | 300 | 17  | -            | 13           | 1      | 0      | 273   | 1140  | 1692 |
| 30  | 1     | 2     | 3     | 4     | 5     | 15    | 759 | 9   | -            | 7            | 1      | 0      | 656   | 2672  | 3990 |
| 30  | 2     | 3     | 4     | 5     | 7     | 9     | 189 | 16  | -            | 12           | 1      | 0      | 172   | 736   | 1086 |
| 36  | 1     | 2     | 3     | 5     | 7     | 18    | 896 | 13  | -            | 10           | 1      | 0      | 773   | 3140  | 4692 |
| 40  | 1     | 3     | 4     | 5     | 7     | 20    | 685 | 14  | -            | 10           | 1      | 0      | 591   | 2412  | 3600 |
| 42  | 2     | 3     | 4     | 5     | 7     | 21    | 418 | 13  | -            | 9            | 1      | 0      | 361   | 1492  | 2220 |
| 42  | 4     | 5     | 6     | 7     | 9     | 11    | 93  | 19  | -            | 16           | 1      | 0      | 84    | 384   | 558  |
| 46  | 1     | 4     | 5     | 6     | 7     | 23    | 599 | 19  | -            | 13           | 1      | 0      | 517   | 2116  | 3156 |
| 50  | 2     | 3     | 5     | 7     | 8     | 25    | 419 | 17  | -            | 12           | 1      | 0      | 361   | 1492  | 2220 |
| 60  | 3     | 4     | 5     | 7     | 11    | 30    | 316 | 14  | -            | 11           | 1      | 0      | 271   | 1132  | 1680 |
Table 7: Weights for 4-folds with $h_{13} = 1$ (in this case all polytopes are simplices).

| $d$  | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $P$ | $V$ | $\overline{P}$ | $\overline{V}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | $h_{22}$ | $\chi$ |
|------|-------|-------|-------|-------|-------|-------|-----|-----|-------------|-------------|---------|---------|---------|---------|------|
| 2415 | 105   | 230   | 279   | 462   | 534   | 805   | 7   | 6   | -   | 6 | 273 0 1 | 1140 1692 |
| 2484 | 96    | 265   | 276   | 597   | 621   | 629   | 7   | 6   | -   | 6 | 273 0 1 | 1140 1692 |
| 2520 | 180   | 215   | 336   | 364   | 585   | 840   | 7   | 6   | -   | 6 | 283 0 1 | 1180 1752 |
| 2565 | 105   | 194   | 410   | 431   | 570   | 855   | 7   | 6   | -   | 6 | 326 0 1 | 1352 2010 |
| 2700 | 123   | 240   | 263   | 540   | 675   | 859   | 7   | 6   | -   | 6 | 273 0 1 | 1140 1692 |
| 2700 | 270   | 300   | 369   | 475   | 486   | 800   | 7   | 6   | 407 6 | 373 0 1 | 1540 2292 |
| 2777 | 335   | 395   | 397   | 407   | 476   | 767   | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3024 | 268   | 336   | 384   | 467   | 689   | 880   | 7   | 6   | -   | 6 | 447 0 1 | 1836 2736 |
| 3108 | 123   | 279   | 296   | 597   | 777   | 1036  | 7   | 6   | -   | 6 | 273 0 1 | 1140 1692 |
| 3120 | 240   | 260   | 481   | 576   | 715   | 848   | 7   | 6   | -   | 6 | 447 0 1 | 1836 2736 |
| 3125 | 434   | 500   | 520   | 521   | 525   | 625   | 7   | 6   | 462 6 | 426 0 1 | 1752 2610 |
| 3216 | 171   | 237   | 536   | 609   | 670   | 993   | 7   | 6   | -   | 6 | 463 0 1 | 1900 2832 |
| 3234 | 385   | 390   | 474   | 539   | 552   | 894   | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3240 | 391   | 407   | 450   | 540   | 558   | 894   | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3240 | 396   | 397   | 461   | 474   | 648   | 864   | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3241 | 391   | 461   | 463   | 475   | 556   | 895   | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3276 | 396   | 403   | 455   | 546   | 576   | 900   | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3360 | 240   | 260   | 517   | 672   | 775   | 896   | 7   | 6   | -   | 6 | 447 0 1 | 1836 2736 |
| 3432 | 184   | 312   | 429   | 655   | 812   | 1040  | 7   | 6   | -   | 6 | 463 0 1 | 1900 2832 |
| 3456 | 150   | 415   | 432   | 551   | 756   | 1152  | 7   | 6   | -   | 6 | 343 0 1 | 1420 2112 |
| 3528 | 387   | 432   | 504   | 588   | 637   | 980   | 7   | 6   | 483 6 | 443 0 1 | 1820 2712 |
| 3528 | 432   | 441   | 504   | 516   | 631   | 1004  | 7   | 6   | 483 6 | 443 0 1 | 1820 2712 |
| 3582 | 350   | 404   | 454   | 597   | 782   | 995   | 7   | 6   | -   | 6 | 523 0 1 | 2140 3192 |
| 3584 | 392   | 448   | 456   | 467   | 782   | 1039  | 7   | 6   | -   | 6 | 523 0 1 | 2140 3192 |
| 3600 | 352   | 450   | 464   | 525   | 784   | 1025  | 7   | 6   | -   | 6 | 523 0 1 | 2140 3192 |
| 3696 | 184   | 336   | 439   | 693   | 924   | 1120  | 7   | 6   | -   | 6 | 463 0 1 | 1900 2832 |
| 3750 | 521   | 600   | 624   | 625   | 630   | 750   | 7   | 6   | 462 6 | 426 0 1 | 1752 2610 |
| 3780 | 268   | 420   | 480   | 567   | 945   | 1100  | 7   | 6   | -   | 6 | 447 0 1 | 1836 2736 |
| 3780 | 335   | 420   | 480   | 689   | 756   | 1100  | 7   | 6   | -   | 6 | 447 0 1 | 1836 2736 |
| 3780 | 391   | 525   | 540   | 630   | 651   | 1043  | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3780 | 461   | 462   | 540   | 553   | 756   | 1008  | 7   | 6   | -   | 6 | 455 0 1 | 1868 2784 |
| 3888 | 288   | 400   | 436   | 605   | 863   | 1296  | 7   | 6   | -   | 6 | 447 0 1 | 1836 2736 |
| 3960 | 360   | 396   | 400   | 890   | 891   | 1023  | 7   | 6   | 603 6 | 547 0 1 | 2236 3336 |
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