Path integral representation of the quantum evolution in dynamical systems with a symmetry for the non-zero momentum level reduction

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Abstract

For the case of reduction onto the non-zero momentum level, in the problem of the path integral quantization of a scalar particle motion on a smooth compact Riemannian manifold with the given free isometric action of the compact semisimple Lie group, the path integral representation of the matrix Green's function, which describes the quantum evolution of the reduced motion, has been obtained. The integral relation between the path integrals representing the fundamental solutions of the parabolic differential equation defined on the total space of the principal fiber bundle and the linear parabolic system of the differential equations on the space of the sections of the associated covector bundle has been derived.

1 Introduction

There is a number of remarkable properties in dynamical systems with a symmetry. The main property of these systems manifests itself in a relationship between an original system and another system (a reduced one) obtained from the original system after “removing” the group degrees of freedom.

One of the system of this class is the dynamical system which describes a motion of a scalar particle on a smooth compact finite-dimensional Riemannian manifold with a given free isometric smooth action of a semisimple compact Lie group. In fact, the original motion of the particle takes place on the total space of a principal fiber bundle, and the reduced motion — on the orbit space of this bundle.

This system bears close resemblance to the gauge field models, where the reduced evolution is given on the orbit space of a gauge group action. That is why a great deal of attention has been devoted to the quantization of the finite-dimensional system related to the particle motion on a manifold with a group action $[1, 2, 3, 4]$.

In gauge theories, the motion on the orbit space is described in terms of the gauge fields that are restricted to a gauge surface. Moreover, a description of this motion is only possible by means of dependent variables.

Such a description is used in a heuristic method of the path integral quantization of the gauge fields proposed by Faddeev and Popov $[5]$. However, at
present, it is not even quite clear how to define correctly the path integral measure on the space of the gauge fields. Therefore, in order to establish the final validity of the method it would be desirable to carry out its additional investigation from the standpoint of a general approach developed in the integration theory. There is a hope that it gives us an answer on yet unsolved questions of the Faddeev–Popov method.

As a first step in that direction, it was studied the path integral reduction in the aforementioned finite-dimensional dynamical system \[6\]. We have used the methods of the stochastic process theory for definition of a path integral measure and in order to study the path integral transformation under the reduction. That is, we dealt with diffusion on a manifold with a given group action and with the path integral representation of the solution of the backward Kolmogorov equation.

Path integral reduction is based on the separation of the variables or, in other words, on the factorization of the original path integral measure into the ‘group’ measure and the measure that is given on the orbit space. In our papers, it was fulfilled with the help of the nonlinear filtering stochastic differential equation. Note that a similar approach to the measure factorization was developed in \[7\]. Also, the questions related to the factorization have been studied in \[8\].

As a result of the reduction, the integral relation between the wave functions of the corresponding ‘quantum’ evolutions (the reduced and original diffusions) was obtained.

It was found that the Hamilton operator of the reduced dynamical system (the differential generator of the stochastic process) has an extra potential term. This term comes from from the reduction Jacobian.

In \[9\], the path integral reduction has been considered in the case when the reduced motion is described in terms of the dependent variables.

As in gauge theories, we have suggested that the principal bundle is a trivial one. Then, in the principal fiber bundle, there is a global cross-section. The cross-section may be determined with the choice of the special gauge surface. The evolution on this gauge surface serves for description of the corresponding reduced evolution on the orbit space.

In this paper we will study the case of the non-zero momentum level reduction in the path integral for the discussed finite-dimensional dynamical system. The path integral, which describes the evolution of the reduced motion on the orbit space, will be represent the fundamental solution of the linear parabolic system of the differential equations.

2 Definitions

In our papers \[6\], we have considered the diffusion of a scalar particle on a smooth compact Riemannian manifold \[\mathcal{P}\]. The backward Kolmogorov equation for the original diffusion was as follows

\[
\begin{aligned}
\left\{
&\frac{\partial}{\partial t} + \frac{1}{2} \mu^2 \kappa \Delta \mathcal{P}(p_a) + \frac{1}{\mu^2 \kappa m} V(p_a) \right\} \psi_{b}(p_a,t_a) = 0 \\
&\psi_{b}(p_a,t_b) = \phi_{0}(p_b), \\
&\quad (t_b > t_a),
\end{aligned}
\]  

(1)

where \(\mu^2 = \frac{\hbar}{m}\), \(\kappa\) is a real positive parameter, \(\Delta \mathcal{P}(p_a)\) is a Laplace–Beltrami operator on \(\mathcal{P}\), and \(V(p)\) is a group–invariant potential term. In a chart with
the coordinate functions $Q^A = \varphi^A(p)$, $p \in \mathcal{P}$, the Laplace–Beltrami operator is written as

$$\Delta \varphi(Q) = G^{-1/2}(Q) \frac{\partial}{\partial Q^A} G^{AB}(Q) G^{1/2}(Q) \frac{\partial}{\partial Q^B},$$

with $G = \text{det}(G_{AB})$, $G_{AB}(Q) = G(\frac{\partial}{\partial Q^A} \cdot \frac{\partial}{\partial Q^B})$.

In accordance with the theory developed by Daletskii and Belopolskaya [10], the solution of (11) is given by the global semigroup which is a limit (under the refinement of the subdivision of the time interval) of a superposition of the local semigroups

$$\psi_{t_a}(p_a, t_a) = U(t_b, t_a)\phi_0(p_a) = \lim_{q \to 0} \tilde{U}_{\eta}(t_a, t_1) \cdots \tilde{U}_{\eta}(t_{n-1}, t_b)\phi_0(p_a). \quad (2)$$

Each local semigroup is determined by the path integrals with the integration measures defined by the local representatives $\eta^A(t)$ of the global stochastic process $\eta(t)$ (See 11). The local stochastic process $\eta^A(t)$ are given by the solutions of the following stochastic differential equations:

$$d\eta^A(t) = \frac{1}{2 \mu^2} G^{-1/2}(G^{1/2} G^{AB}) dt + \mu \sqrt{G} X^A_M(\eta(t)) dw^M(t), \quad (3)$$

where the matrix $X^A_M$ is defined by the identity $\sum_{K=1}^n X^A_K X^K_B = G^{AB}$. (We denote the Euclidean indices by over–barred indices.)

Therefore, the behavior of the global semigroup (2) is completely defined by these stochastic differential equations. The global semigroup can be written symbolically as follows

$$\psi_{t_a}(p_a, t_a) = E\left[\phi_0(\eta(t_b)) \exp\left\{ \frac{1}{\mu^2 \kappa m} \int_{t_a}^{t_b} V(\eta(u))du \right\} \right]$$

$$= \int_{\Omega^\eta} d\mu^\eta(\omega) \phi_0(\eta(t_b)) \exp\{\ldots\}, \quad (4)$$

where $\eta(t)$ is a global stochastic process on a manifold $\mathcal{P}$. $\Omega^\eta = \{\omega(t) : \omega(t_a) = 0, \eta(t) = p_a + \omega(t)\}$ is the path space on this manifold. The path integral measure $\mu^\eta$ is defined by the probability distribution of a stochastic process $\eta(t)$.

2.1 Geometry of the problem

Since in our case, there is a free isometric smooth action of a semisimple compact Lie group $G$ on the original manifold $\mathcal{P}$, this manifold can be viewed as a total space of the principal fiber bundle $\pi : \mathcal{P} \to \mathcal{P}/G = \mathcal{M}$.

At the first step of the reduction procedure, we have transformed the original coordinates $Q^A$ given on a local chart of the manifold $\mathcal{P}$ for new coordinates $(Q^A, \alpha^\alpha)$ $(A = 1, \ldots, N_P, N_P = \dim \mathcal{P}; \alpha = 1, \ldots, N_G, N_G = \dim G)$ related to the fiber bundle. In order to meet a requirement of a one-to-one mapping between $Q^A$ and $(Q^A, \alpha^\alpha)$, we are forced to introduce the additional constraints, $\chi^\alpha (Q^A) = 0$.

These constraints define the local submanifolds in the manifold $\mathcal{P}$. On the assumption that these local submanifolds (local sections) can be ‘glued’ into the global manifold $\Sigma$, we come to a trivial principal fiber bundle $P(\mathcal{M}, G)$. 

3
We note that this bundle is locally isomorphic to the trivial bundle \( \Sigma \times G \rightarrow \Sigma \). It allows us to use the coordinates \( Q^*A \) for description of the evolution on the manifold \( \mathcal{M} \).

If we replace the coordinate basis \( \left( \frac{\partial}{\partial Q^a} \right) \) for a new coordinate basis \( \left( \frac{\partial}{\partial Q^a} \right)=\left( \frac{\partial}{\partial \sigma^a} \right) \), we get the following representation for the original metric \( \bar{G} \) on the manifold \( \mathcal{P} \):

\[
\left( \begin{array}{cc}
G_{CD}(Q^*)(P_\perp) & G_{CD}(Q^*)(P_\perp)K^D_{\mu}\partial_{\sigma^a}(a) \\
G_{CD}(Q^*)(P_\perp)K^D_{\sigma^a}(a) & \gamma_{\mu\nu}(Q^*)\partial_{\sigma^a}(a)\partial_{\sigma^b}(a)
\end{array} \right).
\]

To obtain this expression we have used the right action of the group \( G \) on a manifold \( \mathcal{P} \). It was given by functions \( F^A(Q,a) \), performing an action, and their derivatives: \( F^C_B(Q,a)\equiv \frac{\partial\epsilon^C_{\sigma^a}(Q,a)}{\partial Q^a} \). For example, \( G_{CD}(Q^*)\equiv G_{CD}(F(Q^*,e)) \) is defined as

\[
G_{CD}(Q^*) = F^M_C(Q^*,a)F^N_B(Q^*,a)G_{MN}(F(Q^*,a)),
\]

(\( \epsilon \) is an identity element of the group \( G \)). In [5], the Killing vector fields \( K^\mu_\beta \) for the Riemannian metric \( G_{AB}(Q) \) are also taken on the submanifold \( \Sigma \equiv \{ \chi^* = 0 \} \), i.e. the components \( K^\mu_\beta \) depend on \( Q^* \). By \( \gamma_{\mu\nu} \), defined as \( \gamma_{\mu\nu} = K^\mu_\beta G_{AB}K^\nu_\beta \), we denote the metric given on the orbit of the group action.

The operator \( P_\perp(Q^*) \), which projects the vectors onto the tangent space to the gauge surface \( \Sigma \), has the following form:

\[
(P_\perp)^A_B = \delta^A_B - \chi^\alpha_B(\chi^{**})^{-1}\beta\chi^{**}_B(\chi^{**})^A_B,
\]

\( (\chi^{**})^A_B \) is a transposed matrix to the matrix \( \chi^\alpha_B \equiv \frac{\partial\chi^\alpha}{\partial Q^a} \), \( (\chi^{**})^A_B = G^{AB}\chi^{**}_B(\chi^{**})^A_B \).

The pseudoinverse matrix \( \bar{G}^{AB}(Q^*,a) \) to the matrix [5] is determined by the equality

\[
\bar{G}^{AB}\bar{G}^{BC} = \left( \begin{array}{cc}
(P_\perp)^A_C & 0 \\
0 & \delta^\beta_\beta
\end{array} \right).
\]

It follows that \( \bar{G}^{AB} \) is equal to

\[
\left( \begin{array}{cc}
G^{EFG}\eta^{CG}_{CG}(\Phi^{-1})^{D}_{\alpha\beta}G^{CD}G^{EF}G^{DG} & G^{EFG}\eta^{CG}_{CG}(\Phi^{-1})^{D}_{\alpha\beta}G^{CD}
\\
G^{EFG}\eta^{CG}_{CG}(\Phi^{-1})^{D}_{\alpha\beta}G^{CD} & G^{EFG}\eta^{CG}_{CG}(\Phi^{-1})^{D}_{\alpha\beta}G^{CD}
\end{array} \right).
\]

The matrix \( (\Phi^{-1})^\alpha_{\beta} \) is inverse to the Faddeev – Popov matrix \( \Phi \), which is given by

\[
(\Phi)^\alpha_{\beta}(Q) = K^\alpha_\beta(Q)\frac{\partial\chi^\alpha}{\partial Q^a}.
\]

In [5],

\[
N^A_C = \delta^A_C - K^A_\alpha(\Phi^{-1})^\alpha\chi^\alpha_C
\]

is a projection operator with the following properties:

\[
N^A_BN^B_C = N^A_C, \quad N^A_BK^B_C = 0, \quad (P_\perp)^A_BN^C_A = (P_\perp)^C_B, \quad N^A_B(P_\perp)^C_A = N^C_B.
\]

The matrix \( \bar{u}^\alpha_\beta(a) \) is an inverse matrix to matrix \( \bar{u}^\alpha_\beta(a) \). The det \( \bar{u}^\alpha_\beta(a) \) is a density of a right invariant measure given on the group \( G \).
The determinant of the matrix \( [5] \) is equal to
\[
\text{det}(\bar{G}) = \text{det} G_{AB}(Q^\ast) \text{det} \gamma_{\alpha\beta}(Q^\ast) (\text{det} \chi^\top)^{-1}(Q^\ast) (\text{det} \bar{u}^\ast_{\alpha}(a))^2 \\
\times (\text{det} \Phi^\ast_{\beta}(Q^\ast))^2 \text{det}(P_{\perp})^{G^\ast}_{AB}(Q^\ast) \\
= \text{det} \left( (P_{\perp})^{G^\ast}_{L} G^\ast_{DC}(P_{\perp})^{G^\ast}_{AB} \right) \text{det} \gamma_{\alpha\beta} (\text{det} \bar{u}^\ast_{\alpha})^2,
\]
where the “horizontal metric” \( G^H \) is defined by the relation \( G^H_{\alpha\beta} = \Pi^H \Pi^C G_{DB} \), in which \( \Pi^H_\alpha = \delta^H_\alpha - K^\mu_\alpha \gamma^{\mu\nu} K^D_\nu G_{DB} \) is the projection operator. (From the definition of \( \Pi^H_\alpha \) it follows that \( \Pi^H_\alpha N^\alpha_1 = \Pi^H_\alpha \) and \( \Pi^H_\beta N^\beta_1 = N^\beta_1 \).)

Note also that \( \text{det} \bar{G} = \text{det} G_0 \) on the surface \( \Sigma \). On this surface \( \text{det}(P_{\perp})^{G^\ast}_{L} = 1 \).

### 2.2 The semigroup on \( \Sigma \) and its path integral representation

Transition to the bundle coordinates on \( \mathcal{P} \) leads to the replacement of the local stochastic process \( \eta^A_t \) by the process \( \bar{u}^A_t \) instead of the stochastic differential equation for the process \( \eta^A_t \) we get the system of equations for the processes \( Q^A_t \) and \( a^\ast_\alpha \):

\[
dQ^A_t = \mu^2 \kappa \left( -\frac{1}{2} G^{EM} N^C_E N^B_M H^A_{CB} + j^A_1 + j^A_{II} \right) dt + \mu \sqrt{\kappa} \bar{N}^A \bar{X}^M dw^M_t, \tag{7}
\]

\[
da^\ast_\alpha = -\frac{1}{2} \beta^2 \kappa \left[ G^{RS} \bar{N}^B_R (Q^\ast) \lambda^A_{BM} \varepsilon^a_\alpha + G^{BP} \lambda^A_{BM} K^B_\nu \varepsilon^a_\alpha - G^{CA} N^A C^M \lambda^a_{AM} \varepsilon^a_\alpha \\
- G^{MB} \lambda^a_{BM} \varepsilon^a_\alpha \frac{\partial}{\partial Q^\ast} (\varepsilon^a_\alpha) \right] dt + \mu \sqrt{\kappa} \bar{v}^a_\alpha \bar{X}^B_M dw^M_t. \tag{8}
\]

In these equations, \( \bar{v} \equiv \bar{v}(a) \), and the other coefficients depend on \( Q^\ast \).

In equation \( (4) \), \( \Gamma^L_{CD} \) are the Christoffel symbols defined by the equality

\[
G^H_{AB} \Gamma^L_{CD} = \frac{1}{2} (G^H_{AC,D} + G^H_{AD,C} - G^H_{CD,A}), \tag{9}
\]

in which by the derivatives we mean the following: \( G^H_{AC,D} = \frac{\partial G^H_{AC}}{\partial Q^D} \bigg|_{Q=Q^\ast} \). Also, by \( j^A_1 \) we have denoted the mean curvature vector of the orbit space, and by \( j^A_{II}(Q^\ast) \) — the projection of the mean curvature vector of the orbit onto the submanifold \( \Sigma \). This vector can be defined as

\[
j^A_{II}(Q^\ast) = -\frac{1}{2} G^{EM} N^A_E N^B_M \left[ \gamma^{\alpha\beta} G_{CD}(\bar{\nabla} K_{\alpha\beta} C) \right] (Q^\ast) \\
= -\frac{1}{2} N^A_C \left[ \gamma^{\alpha\beta} (\bar{\nabla} K_{\alpha\beta} C) \right] (Q^\ast), \tag{10}
\]

where

\[
(\bar{\nabla} K_{\alpha\beta} C)(Q^\ast) = K^A_{\alpha}(Q^\ast) \frac{\partial}{\partial Q^A} K^C_{\beta}(Q) \bigg|_{Q=Q^\ast} + K^A_{\alpha}(Q^\ast) K^B_{\beta}(Q^\ast) \bar{F}^C_{AB}(Q^\ast)
\]

\(^1\)This phase space transformation of the stochastic processes does not change the path integral measures in the evolution semigroups.
with
\[ \Gamma^C_{AB}(Q^*) = \frac{1}{2} \left( G^{CE}(Q^*) \left( \frac{\partial}{\partial Q^* A} G_{EB}(Q^*) + \frac{\partial}{\partial Q^* B} G_{EA}(Q^*) - \frac{\partial}{\partial Q^* B} G_{AB}(Q^*) \right) \right). \]

Note also that in equation (8), \( \Lambda^\alpha_B = (\Phi^{-1})^\alpha_B \), \( \Lambda^{\beta \alpha}_{AM} = \frac{\partial}{\partial q^\alpha_{AM}} (\Lambda^\beta) \), and \( K^B_{GP} = \frac{\partial}{\partial q^B} (K^B_P) \).

The superposition of the local semigroup \( \hat{U}_\zeta \), together with a subsequent limiting procedure, gives the global semigroup determined on the submanifold \( \Sigma \).

Our next transformation in the path integral reduction procedure, performed in [9], was related to the factorization of the path integral measure generated by the process \( \zeta \). First of all, it was done in the path integrals for the local evolution semigroups. In each semigroup, we have separated the local evolution, given on the orbit of the group action, from the evolution on the orbit space. Then, we extended the factorization onto the global semigroup by taking an appropriate limit in the superposition of new-obtained local semigroups.

In case of the reduction onto non-zero momentum level, that is when \( \lambda \neq 0 \), it has led us to the integral relation between the path integrals for the Green’s functions defined on the global manifolds \( \Sigma \) and \( P \):

\[
G^\lambda_{pq}(Q^*_b, t_b; Q^*_a, t_a) = \int_G G_P(p_b \theta, t_b; p_a, t_a) D^\lambda_{qp}(\theta) d\mu(\theta), \quad (Q^* = \pi_\Sigma(p)). \tag{11}
\]

Here \( D^\lambda_{pq}(a) \) are the matrix elements of an irreducible representation \( \Lambda^\lambda \) of a group \( G \), \( \sum_q D^\lambda_{pq}(a) D^\lambda_{qk}(b) = D^\lambda_{pb}(ab) \).

The Green’s function \( G_P(Q_b, t_b; Q_a, t_a) \) is defined\(^2\) by semigroup (4):

\[
\psi(Q_a, t_a) = \int_G G_P(Q_b, t_b; Q_a, t_a) \varphi_0(Q_b) d\nu_P(Q_b)
\]

\[(d\nu_P(Q) = \sqrt{G(Q)} dQ^1 \cdots dQ^{N_r}).\]

The probability representation of the kernel \( G_P(Q_b, t_b; Q_a, t_a) \) of the semigroup (4) (the path integral for \( G_P \)) may be obtained from the path integral (4) by choosing \( \varphi_0(Q) = G^{-1/2}(Q) \delta(Q - Q^*) \) as an initial function.

The Green’s function \( G^\lambda_{pq} \) is presented by the following path integral

\[
G^\lambda_{pq}(Q^*_b, t_b; Q^*_a, t_a) = \\
\int_{\xi_S(t_a) = Q^*_a}^{\xi_S(t_b) = Q^*_b} \left\{ \exp \frac{1}{\mu^2 K M} \int_{t_a}^{t_b} V(\xi_S(u)) du \right\} \\
\times \sum_{\xi_S(t_a) = Q^*_a}^{\xi_S(t_b) = Q^*_b} \left\{ \exp \frac{1}{\mu^2 K M} \int_{t_a}^{t_b} \tilde{V}(\xi_S(u)) du \right\} \\
\left\{ \frac{1}{2} \mu^2 K \left[ \gamma_{\nu}(\xi_S(u))(J_\nu)_{pq} \right] \right\} \\
- (G^{RS}_{AB} \Lambda^\beta_R \Lambda^\beta_B K_{GP} - G^{CA} \Lambda^\alpha_M \Lambda^\beta_{AM}) (J_\beta)_{pq} \right\} du \\
+ \mu \sqrt{\Lambda^\beta_C \Lambda^\beta_A} (J_\beta)_{pq} \prod_{R, M} \tilde{K}^B_{RM} d\nu_M(u), \tag{12}
\]

\(^2\)We have assumed that equation (4) has a fundamental solution.
The measure in this path integral is generated by the global stochastic process $\xi(t)$ given on the submanifold $\Sigma$. This process is described locally by equations (12).

In equation (12), $\xi(t)$ is a multiplicative stochastic integral. This integral is a limit of the sequence of time-ordered multipliers that have been obtained as a result of breaking of a time interval $[s,t]$, $[s = t_0 \leq t_1 \cdots \leq t_n = t]$. The time order of these multipliers is indicated by the arrow directed to the multipliers given at greater times. We note that, by definition, a multiplicative stochastic integral represents the solution of the linear matrix stochastic differential equation.

On the right-hand side of (12), by $(J_\lambda)_{pq}^\lambda \equiv \left( \frac{\partial D_\lambda^\lambda (a)}{\partial a_e} \right)_{a=e}$ we denoted the infinitesimal generators of the representation $D_\lambda^\lambda(a)$:

$$\tilde{L}_\mu D_\lambda^\lambda(a) = \sum_q (J_\lambda)_{pq}^\lambda D_\lambda^\lambda(a_q)$$

($\tilde{L}_\mu = \tilde{v}_\mu^\lambda(a) \tilde{\partial}_a$ is a right-invariant vector field).

The differential generator (the Hamiltonian operator) of the matrix semigroup with the kernel (12) is

$$\frac{1}{2} \mu^2 \kappa \left\{ \left[ G^{CD} R^A_{\lambda N_D} R^B_{\lambda N_D} \frac{\partial^2}{\partial Q^A \partial Q^B} - G^{CD} N^A_D N^B_D \frac{\partial}{\partial Q^A} \right] - 2 \left( j^A_I + j^A_H \right) \frac{\partial}{\partial Q^A} + \frac{2 \tilde{\chi}}{(\mu^2 \kappa)^2} \right\} (I)_{pq} + 2 N^A_C G^{CP} \lambda^\lambda (J_\lambda)_{pq}^\lambda \frac{\partial}{\partial Q^A}$$

$$- \left( G^{RS} R^B_{RS} \lambda^\lambda_A - G^{RP} \lambda^\lambda_A \lambda^\lambda_B \lambda^\lambda_B - G^{G,A} N^M_D \lambda^\lambda_{AM} \right) (J_\lambda)_{pq}$$

$$+ G^{SB} \lambda^\lambda_B \lambda^\lambda_Q (J_\lambda)_{pq}^\lambda (J_\lambda)_{pq}^\lambda$$

(13)

where $(I)_{pq}$ is a unity matrix.

The operator acts in the space of the sections $\Gamma(\Sigma, V^*_\lambda)$ of the associated covector bundle with the scalar product $\langle \psi_n, \psi_m \rangle$

$$\langle \psi_n, \psi_m \rangle = \int_{\Sigma} \langle \psi_n, \psi_m \rangle_{V^*_\lambda} \det^{1/2} \left( P_\perp^D \right) G^{PH}_{DC} \left( P_\perp^E \right) \det^{1/2} \gamma_{\alpha\beta}$$

$$\times dQ^{*1} \wedge \cdots \wedge dQ^{*N_P}.$$  

(14)

$\Gamma(\Sigma, V^*_\lambda)$ is isomorphic to the space of the equivariant functions on $\mathcal{P}$. The isomorphism between the functions $\tilde{\psi}_n(p)$, such that

$$\tilde{\psi}_n(pg) = D^\lambda_{mn}(g) \tilde{\psi}_m(p),$$

is given by the following equality: $\tilde{\psi}_n(F(Q^*,e)) = \psi_n(Q^*)$.

Another form of this scalar product is as follows

$$\langle \psi_n, \psi_m \rangle = \int_{\Sigma} \langle \psi_n, \psi_m \rangle_{V^*_\lambda} \det \Phi_\lambda^A \prod_{\alpha=1}^N \delta(\chi^\alpha(Q^*)) \det^{1/2} G_{AB} \, dQ^{*1} \wedge \cdots \wedge dQ^{*N_P}.$$
3 Girsanov transformation

In the case of the reduction onto the zero momentum level, our goal is to obtain the description of true evolution on the orbit space $M$ in terms of the evolution given on an additional gauge surface $\Sigma$. By true evolution we mean such a diffusion on $M$ which has the Laplace—Beltrami operator as a differential generator.

A required correspondence between the diffusion on $M$ and the diffusion on $\Sigma$ can be achieved only in that case when the stochastic process $\tilde{\xi}_\Sigma$ related to the diffusion on $\Sigma$ is described by the stochastic differential equations, which look as equations (7), but without the "$j_{II}$-term" in the drift:

$$dQ_t^* = \mu^2 \kappa \left( \frac{1}{2} G^{EM}_{E} N^C_{E} N^B_{M} H Y^A_{M} + j^A_{I} \right) dt + \mu \sqrt{\kappa N^C_{E}} \tilde{X}^C_{M} dw^M_t. \tag{15}$$

Note that in case of the reduction onto the zero-momentum level, the differential generator of the process $\tilde{\xi}_\Sigma$ could be transformed into the Laplace—Beltrami operator (a differential generator of the process on $M$), if we succeeded in finding the independent variables that parametrize $\Sigma$.

In the same way, in order to come to the correct description of the reduced diffusion on $M$ for the reduction onto the non-zero momentum level, we should properly transform the semigroup, given by the kernel (12).

In the path integral (12), such a transformation, in which we perform the transition to the process $\tilde{\xi}_\Sigma$ with the local stochastic differential equations (15) from the process $\xi_\Sigma$ defined by the equation (7), is known as the Girsanov transformation. In spite of the fact that in the equations (12) and (15), the diffusion coefficients are degenerated, the Girsanov transformation formula can be nevertheless derived by making use of the Itô’s differentiation formula for the composite function. It is necessary only to take into account the predefined ambiguities, which exist in the problem.

When we deal with the system of the linear parabolic differential equations, as in our case, the multiplicative stochastic integral should be also involved in the Girsanov transformation. Assuming a new form of this integral for the process $\tilde{\xi}_\Sigma$, we compare the differential generators for the processes $\xi_\Sigma$ and $\tilde{\xi}_\Sigma$. The existence and uniqueness solution theorem for the the system of the differential equations allows us to determine a new multiplicative stochastic integral for the process $\tilde{\xi}_\Sigma$.

After lengthy calculation which we omit for brevity and because of its resemblance to the calculation performed in [9, 11] for $\lambda = 0$ case, we come to the following expression for the multiplicative stochastic integral:

$$\hat{\exp}(\ldots)_{pq}^\lambda (\tilde{\xi}_\Sigma(t)) = \hat{\exp} \int_{t_0}^t \left\{ \frac{1}{2} \mu^2 \kappa \left[ \gamma^{\nu\sigma} (J_{\sigma})_{pq}^\lambda (J_{\nu})_{pq}^\lambda \right] 
- G^{RB}_{LK} (P_{\perp})_{B}^I (P_{\perp})_{L}^A (J_{I})_{AB}^\lambda (J_{I})_{AB}^\lambda 
- \left( G^{\alpha\beta}_{RB} \tilde{\gamma}^R_{AB} \Lambda^\beta_{AB} \Lambda^\alpha R_{\nu\sigma}^B - \left( G^{\alpha\beta}_{CM} N^C_{E} N^B_{M} \Lambda^\beta_{A,M} \right) (J_{\beta})_{pq}^\lambda \right] du 
+ \mu \sqrt{\kappa} \left[ G^{HL}_{KL} (P_{\perp})_{A}^I (J_{I})_{AB}^\lambda (J_{I})_{AB}^\lambda \tilde{X}^B_{M} dw^M_t \right] \right\} \tag{16}.$$
where the integrand \( \tilde{\lambda} \) coincides with the path integral reduction Jacobian for the \( \lambda = 0 \) case. It was obtained in [11] in the following way.

We first rewrote the exponential of the Jacobian for getting rid of the stochastic integral: the stochastic integral was replaced by an ordinary integral taken with respect to the time variable. It was made with the help of the Itô’s identity. Then it was obtained the geometrical representation of the Jacobian:

\[
\exp(\ldots)_{pq}(\xi(t)) = \exp \int_{t_a}^{t_b} \left[ -\frac{1}{2} \mu^2 \kappa (L_{\lambda}^\Sigma(P_\lambda)_{\lambda 1} G_{LK}(P_\lambda)_{\lambda 2}) \right] dt,
\]

\[
+ \mu \sqrt{\kappa} G_{LK}(P_\lambda)_{\lambda 1} \lambda^a \partial_a \tilde{x}_\lambda du^\mu I_{pqr}^\lambda
\]

\[
\times \exp \int_{t_a}^{t_b} \left\{ -\frac{1}{2} \mu^2 \kappa \left[ \gamma^{\alpha\nu}(J_\alpha)_{\nu q} (J_\nu)_{\nu q} - 2 \Pi^a J_{\lambda 1} \Lambda^\alpha_{\lambda}(J_\lambda)_{\lambda q} \right]
\]

\[
- \left( G^{RS} \Gamma^B_{RS} \Lambda^\beta_{\beta B} + G^{RP} \Lambda^\beta_{\beta M} K^B_{\sigma \beta} - G^{CA} N^M_{CA} \Lambda^\beta_{\beta M} \right) (J_\beta)_{\beta q} \right\} du
\]

\[
+ \mu \sqrt{\kappa} \Pi^\mu_{\lambda} \Lambda^\mu_{\mu (J_\mu)_{\mu q}} \tilde{x}_\lambda du^\lambda \right\}\right)^n (17).
\]

The first factor of (17) coindices with the path integral reduction Jacobian for

The first factor of (17) coindices with the path integral reduction Jacobian for the \( \lambda = 0 \) case. It was obtained in [11] in the following way.

We first rewrote the exponential of the Jacobian for getting rid of the stochastic integral: the stochastic integral was replaced by an ordinary integral taken with respect to the time variable. It was made with the help of the Itô’s identity. Then it was obtained the geometrical representation of the Jacobian:

\[
\left( \gamma(Q'_{(t_b)}) \right)^{\frac{1}{2}} \exp \left\{ -\frac{1}{8} \mu^2 \kappa \int_{t_a}^{t_b} \tilde{J} dt \right\},
\]

where the integrand \( \tilde{J} \) is equal to

\[
\tilde{J} = R_P - H R - G - \frac{1}{4} F^2 - ||j||^2. \quad (19)
\]

In this expression, \( R_P \) is a scalar curvature of the original manifold \( \mathcal{P} \), \( H R \) is a scalar curvature of the manifold with the degenerated metric \( G^{AB} \). More exactly,

\[
H R \equiv G^{A'C'} N^B_{AC} N^C_{B} H R_{SEC} \left( J_\mu \right)_{\mu B} N^B_{AC} N^C_{B} H R_{SEC} \left( J_\mu \right)_{\mu B},
\]

where \( N^S_{AC} N^E_{B} H R_{SEC} \) is equal to

\[
\left( \frac{\partial}{\partial Q_{CS}} N^S_{AC} H R_{SEC} + \frac{\partial}{\partial Q^{CE}} H R_{SEC} \right) \left( \frac{\partial}{\partial Q^{KS}} H R_{SEC} \right) H R_{SEC} \left( J_\mu \right)_{\mu B},
\]

\( R_G \) is the scalar curvature of the orbit:

\[
R_G \equiv \frac{1}{2} \gamma^{\mu\nu} e^\sigma_{\mu\nu} e^\alpha_{\sigma\nu} + \frac{1}{4} \gamma^{\mu\nu} \gamma^{\alpha\beta} e^\gamma_{\mu\nu} e^\delta_{\gamma\beta}.
\]

By \( F^2 \) we denote the following expression:

\[
F^2 \equiv (G^{ES} N^E_{B} ) (G^{MP} N^P_{AB} ) \gamma_{\mu\nu} F^\mu_{P} F^\nu_{AB},
\]

in which the curvature \( F_{EP} \) of the connection \( A^E_{EP} = \gamma^{\mu\nu} K^R_{\mu} G_{RP} \) \( ^3 \) is given by

\[
F_{EP}^\alpha = \frac{\partial}{\partial Q^{CE}} A^\alpha_{EP} - \frac{\partial}{\partial Q^{CE}} A^\alpha_{EP} + e^\sigma_{\nu\sigma} A^\nu_{EP} A^\sigma_{EP}.
\]

\( ^3 \)In the case of the reduction, this connection is naturally defined on the principal fiber bundle.
The last term of (19), the “square” of the fundamental form of the orbit, is
\[ \|j\|^2 = G_{AB}^T \gamma^\alpha \gamma^\beta j_\alpha^B j_\beta^A, \]
where
\[ j_\alpha^B(Q^*) = -\frac{1}{2} G_{PS}^T N_P^B N_S^E (\Delta_E \gamma_{\alpha}^D)(Q^*) \]
with
\[ \Delta_E \gamma_{\alpha}^D = \left( \frac{\partial}{\partial x^E} \gamma_{\alpha}^D - c^\gamma_{\alpha E} A^E_{\gamma} \gamma_{\beta}^D - c^\gamma_{\beta E} A^E_{\gamma} \gamma_{\alpha}^D \right). \]
To obtain \( j_\alpha^B(Q^*) \) we have projected the second fundamental form \( j_\alpha^C(Q) \) of the orbit onto the direction which is parallel with the orbit space. In other words, we calculated the following expression:
\[ \tilde{G}_p^\alpha \tilde{G} (\Pi_Y^E(Q)(\nabla_{K_\alpha} K_\theta)^D \frac{\partial}{\partial x^D}, \frac{\partial}{\partial x^E}), \]
where \( \tilde{G} \) was the metric of the manifold \( \mathcal{P} \).

Therefore, the Girsanov transformation allows us to rewrite the integral relation (11) as follows
\[ (\gamma(Q^*_p) \gamma(Q^*_m))^{-1/4} \tilde{G}_{pq}^\lambda(Q^*_p, t_b; Q^*_m, t_a) = \int_{\tilde{G}} G_P(p_b, t_b; p_a, t_a) D_{pq}^I(\theta) d\mu(\theta), \]
where the Green’s function \( \tilde{G}_{pq}^\lambda \) is given by the following path integral
\[ \tilde{G}_{pq}^\lambda(Q^*_p, t_b; Q^*_m, t_a) = \int_{\tilde{G}} d\mu \exp \left\{ \int_{t_a}^{t_b} \left( \frac{1}{2} \mu^2 \tilde{J} - \int_{\Gamma^D} \left( \frac{\tilde{\xi}(u)}{\mu^2 \tilde{K} \tilde{J}} - \frac{1}{8} \mu^2 \tilde{J} \right) du \right) \right\} \]
\[ \times \exp \left\{ \int_{t_a}^{t_b} \left\{ -\frac{1}{2} \mu^2 \tilde{K} \left[ \gamma_{\alpha \nu}^{\lambda}(J_\alpha)^p_{\nu} (J_\beta)^q_{\beta} - 2 \Pi_C^E j_{11}^E (J_\alpha)^q_{pq} \right. \right. \right. \]
\[ - \left( G_{RS}^T \nabla^\beta_{RS} A_B^\beta + G_{RP}^T A_B^\beta K_\sigma^B - G_C^A N_A^M A_{A,M}^B \right) (J_\beta)^q_{pq} \right. \]
\[ + \mu \sqrt{\Pi} C (J_\beta)^q_{pq} \tilde{L} \tilde{B} (du^M) \right\}. \]

(20)
\( \tilde{G}_{pq}^\lambda \) is the kernel of the evolution semigroup which describes the true reduced evolution on the orbit space \( \mathcal{M} \). This semigroup acts in the space of sections \( \Gamma(\Sigma, V^*_\lambda) \) of the associated covector bundle \( P \times G V^*_\lambda \) with the following scalar product:
\[ (\psi, \psi) = \int_{\Sigma} (\psi, \psi) \mid dQ^{1/2} \left( (P_\perp)_{A}^{D} G^{B} \left( P_{\perp} \right)_{B}^{C} \right) \]
\[ \times dQ^{*1} \wedge \ldots \wedge dQ^{* N_P}. \]

(21)
The differential generator of the matrix semigroup with the kernel \( \tilde{G}_{pq}^\lambda \) is
\[ \frac{1}{2} \mu^2 \tilde{K} \left\{ \left[ G_{CD}^P N_A^M A_B^A \frac{\partial^2}{\partial Q^{*1} \partial Q^{*B}} - G_{CD}^P N_A^M A_B^A \Gamma_{EM}^A \frac{\partial}{\partial Q^{*A}} \right] \right\} \]
\[ + 2 j_{11}^A \frac{\partial}{\partial Q^{*A}} + \frac{2 \tilde{V}}{\mu^2 \tilde{K}^2 m} - \frac{1}{4} \int (I_\lambda)_{pq} + 2 N_C^A G^A \Lambda_{I}^D (J_\alpha)^q_{pq} \frac{\partial}{\partial Q^{*A}} \]
\[ - \left( G_{RS}^T \nabla^\beta_{RS} A_B^\beta + G_{RP}^T A_B^\beta K_\sigma^B - G_C^A N_A^M A_{A,M}^B \right) (J_\beta)^q_{pq} \]
\[ + A_C^\nu \gamma_{\alpha \nu} \left( \nabla_{K_\alpha} K_\theta \right)^E (J_\alpha)^q_{pq} + G^S B_{\alpha}^\nu (J_\alpha)^q_{pq} \frac{\partial}{\partial Q^{*A}} \right\}. \]

(22)
The first term of the last line in (22) comes from $-2\Pi^C_{LM} \Lambda^p_{\lambda} \Lambda^q_{\nu}$ - term of the multiplicative stochastic integral given in (20). Its derivation is based on the following relations:

$$\Pi^C_{LM} \Lambda^p_{\lambda} \Lambda^q_{\nu} = \Lambda^p_{\lambda} - A^p_{\lambda}, \quad A^p_{\lambda} \gamma^{\mu\nu} [\nabla K, K]_{\lambda}^C = 0.$$ 

In the next section we will obtain another representation for the multiplicative stochastic integral. For this purpose, it is sufficient to consider the transformation of the differential operator (22), since there exists a quite definite relationship between the integrand of the path integral and the corresponding differential generator.

4 The horizontal Laplacian

It is well-known that the horizontal Laplacian $\Delta^E$

$$\left(\Delta^E\right)_{pq}^\lambda = \sum^n_{k=1} \left(\nabla^E_{X_k} X_i\right)_k - \nabla^E_{\nabla^M_{X_k} X_i}\right)_{pq}^\lambda$$

$$= \Delta^M_{pq} \Gamma^\lambda_{pq} + 2h_{ij} \left(\Gamma^E_{pq}\right)^\lambda_{ij} \partial_i - h_{ij} \left[\partial_i \left(\Gamma^E_{pq}\right)^\lambda_{ij} - \left(\Gamma^E_{pq}\right)^\lambda_{ij} \partial_i + \left(\Gamma^E_{pq}\right)^\lambda_{ij} \partial_i \right],$$
determined on the space of the sections of the associated vector bundle $E = P \times_G V_\lambda$, is an invariant operator which can be considered as a generalization of the Laplace–Beltrami operator given on the base manifold $\mathcal{M}$. It would be naturally to expect that in the case of description of the evolution by means of dependent variables, there is also a corresponding operators which may be refer to as the horizontal Laplacian.

For the covector bundle, such an operator may be given by the following expression:

$$\left(\Delta^{\cdot E}\right)_{pq}^\lambda = \sum^n_{k=1} \left(\nabla^{\cdot E}_{Y_k} Y_{\lambda} - \nabla^{\cdot E}_{\nabla^M_{Y_k} Y_{\lambda}}\right)_{pq}^\lambda$$

$$= \Delta^M_{pq} \Gamma^\lambda_{pq} - 2G^{LM} N^E_{LM} \left(\Gamma^E_{pq}\right)^\lambda_{ijk} \partial_i \partial_j - G^{LM} N^E_{LM} \left[\partial_i \left(\Gamma^E_{ij} \right)^\lambda_{pq} - \left(\Gamma^E_{ij} \right)^\lambda_{pq} \partial_i + \left(\Gamma^E_{ij} \right)^\lambda_{pq} \partial_i \right],$$
in which $Y^A_M = Y^A_M X^P_M$ is defined by the equality $\sum_{\lambda_{pq}} Y^A_M Y^B_M = G^{PR} N^A_R N^B_R$ and where $\left(\Gamma^E_{ij}\right)_{pq}$ is defined as $A^p_{\lambda} \left(J_{\alpha}\right)_{pq}^\lambda$.

The covariant derivative $\nabla^{\cdot E}_{\alpha}$ is defined as

$$\nabla^{\cdot E}_{\alpha, p} = N^D_B \left(\Gamma_{pq}^{\alpha} \partial_{\alpha} - \Lambda^p_{\lambda} \partial_{\alpha} \right)_{pq}^\lambda$$

and

$$\nabla^{\cdot E}_{\alpha, B} = H^{\cdot E}_{AB} \partial_{\alpha}. $$

The horizontal Laplacian $\Delta^{\cdot E}$ can be also written as follows

$$G^{LM} N^E_{LM} \left(\frac{\partial^2}{\partial Q^E \partial Q^C} + \partial_{\alpha}^E \partial_{\alpha}^C (N^B_E) \partial_{\alpha}^B \partial_{\alpha}^D - H^{\cdot E}_{AB} N^D_B \partial_{\alpha}^D\right)_{pq}^\lambda.$$
\[
\left(-2A^\alpha_E \frac{\partial}{\partial Q^C} - A^\alpha_B \frac{\partial}{\partial Q^D}(N^B_C) - \frac{\partial}{\partial Q^E}(A^\alpha_C) + \frac{1}{2} \Gamma^B_E C^A_B A^\alpha_D \right)(J_\alpha)^\lambda_q \left\{ (A^\alpha_B J_\beta)^\lambda_{pq} \right\}.
\]

(23)

It turns out, that operator (23) is intrinsically related to the operator (22). First note that diagonal parts of these operator (without taking into account the potential terms \(\tilde{V}\) and \(\tilde{J}\) in (22)) are equal. It may be checked with the help of the following identity

\[
-\frac{1}{2} N^A_E \Gamma^{\alpha'}_{CD} N^D_C N^P_B G^{C'B'} + \frac{1}{2} N^A_E N^L_M N^M_A G^{L'M'}
\]

\[
= -\frac{1}{2} G^{EM} N^C_E N^B_M \Gamma^{A}_{CB} + jA.
\]

The off-diagonal matrix elements of the operators (22) and (23), that include the generator \((J_\alpha)^\lambda_{pq}\), are also equal. In order to show this, in the operator (22), one should rewrite such terms in the following way

\[
- \left( -G^{RS} \tilde{A}^P_{RS} A^\alpha_B \Gamma^{B}_{\sigma P} + G^{RP} A^\sigma_A \Gamma^{B}_{\sigma P} - G^{CA} N^M_L A^\alpha_M - \gamma^{\mu \sigma} N^\alpha_{P} K^{A}_{\mu} K^{P}_{\sigma A} \right),
\]

(24)

where \(G^{RS} = G^{RS} - K^{R}_{\gamma} \gamma^{\mu} K^{S}_{\mu}\), and the analogous terms of the operator (23) as follows

\[
- G^{PQ} N^E_P N^B_Q N^C_E A^\alpha_C - G^{PQ} N^E_P N^C_E \frac{\partial}{\partial Q^E}(A^\alpha_C) + G^{PQ} N^A_P N^C_E \Gamma^{B}_{AC} N^D_B A^\alpha_D.
\]

(25)

Replacing the term, which involve the derivative of \(A^\alpha_E\), with the expression

\[
G^{LM} N^E_L N^B_M \frac{\partial}{\partial Q^E}(A^\alpha_E) = N^E_R \Gamma^{R}_{ES} \gamma^{\alpha \sigma} K^{\sigma}_{\sigma} + G^{PB} N^E_P \gamma^{\alpha \sigma} K^{\sigma}_{\sigma} \Gamma^{R}_{EB} G^{RS}
\]

\[
+ N^E_{\gamma} \gamma^{\alpha \sigma} K^{P}_{\sigma} + G^{LM} N^E_L \gamma^{\alpha \sigma} K^{C}_{\sigma} G^{CD} K^{D}_{\sigma E}
\]

and making use of the identity

\[
N^A_E \Gamma^{\alpha'}_{CD} N^D_C N^P_B G^{C'D'} = N^A_E N^L_M N^M_A G^{LM} - G^{CT} N^C_E N^A_U + G^{CR} A^\beta_C N^R_T K^{\beta}_{BR} + G^{LM} \Gamma^{D}_{LM} N^A_D,
\]

one can arrive at the equality of the transformed expressions. It will be noted that in the expression obtained as a result of the transformation of (25), besides of the necessary terms, that are equal to the corresponding terms coming from (24), there are redundant terms. But, it can be verified that these terms are mutually cancelled. It follows from the calculation in which one should takes into account the Killing identity, the equality

\[
\gamma^{\beta \nu}(\triangle_{K_{\beta}} K_{\beta})^T A^\alpha_P = 0,
\]

which is obtained from the identity

\[
-\gamma^{\beta \nu}(\triangle_{K_{\beta}} K_{\beta})^T = \frac{1}{2} G^{PT} N^E_P \left( \gamma^{\mu \nu} \frac{\partial}{\partial Q^E} \gamma^{\mu \nu} \right),
\]

and the condition \(c^\beta_{\alpha} = 0\), which is valid for the structure constants of the semisimple Lie groups.
Except for the potential terms, the only distinction between (22) and (23) consists of the terms that involve the product of two group generators. But, since
\[ G^{LM} N_L^E N_M^P A_p^E A_p^P = G^{EP} A_p^E A_p^P - \gamma^{\mu\nu}, \]
we can present the operator (22) as
\[ \frac{1}{2} \mu^2 \kappa \left[ (\triangle^E)_{pq}^\lambda + \gamma^{\mu\nu} (J_\mu)_{pq}^\lambda (J_\nu)_{pq}^\lambda \right] + \left( \frac{1}{\mu^2 \kappa m} V - \frac{1}{8} \mu^2 \kappa J \right) (I^\lambda)_{pq}. \]

5 The path integral for the matrix Green’s function \( \tilde{G}^\lambda_{pq} \)

Now we can rewrite the multiplicative stochastic integral in the path integral (20). We already know that \( (J_\alpha)_{pq}^\lambda \) terms of the drift in the integrand of the multiplicative stochastic integral are equal to the corresponding terms (25) of the operator (23). These terms can be rewritten as follows
\[
-G^{PQ} N_Q^A N_B^E N_A^C N_B^{CE} C_D^{AC} - G^{PQ} N_Q^A N_B^E N_A^C \frac{\partial}{\partial Q^E} (A_C^0) \\
+ G^{PQ} N_Q^A N_C^B H^{AB} N_B^{DE} A_D^{AC},
\]
and also as
\[-G^{PQ} N_Q^E N_B^B \nabla^H (N_B^E A_C^0).\]

The coefficient \( \Pi_K^\alpha A^\beta_C \) of the diffusion term of the integrand may be written in the form
\[ \Pi_K^\alpha A^\beta_C = A^\alpha_K - A^\beta_K. \]

Thus, we obtain the following path integral representation of the matrix Green’s function \( \tilde{G}^\lambda_{pq} \): 
\[
\tilde{G}^\lambda_{pq}(Q_0^a, t_b; Q_0^a, t_a) = \int d\tilde{\epsilon} \mu^2 \exp \left\{ \int_{t_a}^{t_b} \left\{ \frac{V}{\mu^2 \kappa} - \frac{1}{8} \mu^2 \kappa J \right\} du \right\}
\times \exp \left\{ \frac{1}{2} \mu^2 \kappa \left[ \gamma^{\mu\nu} (J_\mu)_{pq}^\lambda (J_\nu)_{pq}^\lambda - G^{PQ} N_Q^E N_B^B \nabla^H (N_B^E A_C^0) (J_\alpha)_{pq}^\lambda \right] du 
- \mu \sqrt{\kappa} N_B^B A_D^{AC} (J_\alpha)_{pq}^\lambda \tilde{X}_M^{\alpha} du_{AB} \right\}.
\]

In \( (Q_0^a, t_b) \)-variables this Green’s function satisfies the forward Kolmogorov equation with the operator
\[ \hat{H}_\kappa \equiv \frac{\hbar \kappa}{2m} \left[ (\Delta^E)_{pq}^\lambda + \gamma^{\mu\nu} (J_\mu)_{pq}^\lambda (J_\nu)_{pq}^\lambda \right] - \frac{\hbar \kappa}{8m} \left[ J \right]_{pq}^\lambda + \frac{V}{\hbar \kappa} I_{pq}^\lambda, \]
where the horizontal Laplacian \( (\Delta^E)_{pq}^\lambda \) is
\[
(\Delta^E)_{pq}^\lambda = \sum_{E=1}^{n_p} \left( \nabla^E_{Y^A_{pq}^{\alpha A}} \nabla^E_{Y^B_{pq}^{\beta B}} - \nabla^E_{Y^M_{pq}^{\alpha A}} \nabla^E_{Y^M_{pq}^{\beta B}} \right)_{pq}^\lambda
= \Delta_{pq}^\lambda + 2 G^{LM} N_L^E N_M^C (G^E)^{\lambda}_{pq} \partial Q^C
- G^{LM} N_L^E N_M^C \left[ \partial Q^C (N_B^E (G^E)^{\lambda}_{pq}) - (G^E)^{\lambda}_{pq} \right]_{pq}^\lambda + H_{pq}^E N_B^C (G^E)^{\lambda}_{pq} + H_{pq}^{CD} + (G^E)^{\lambda}_{pq}. \]
The Laplace operator $\triangle_M$ is

$$\triangle_M = G^{CD} N_C^A N_D^B \frac{\partial^2}{\partial Q^* A \partial Q^* B} - G^{CD} N_C^E N_D^M H^A_{EM} \frac{\partial}{\partial Q^* A} + 2 j^A_I \frac{\partial}{\partial Q^* A}.$$  

At $\kappa = i$ the forward Kolmogorov equation becomes the Schrödinger equation with the Hamilton operator $\hat{H}_E = -\frac{i}{\hbar} \hat{H}_E |_{\kappa = i}$. The operator $\hat{H}_E$ acts in the Hilbert space of the sections of the associated vector bundle $E = P \times_G V$. The scalar product in this space has the same volume measure as in (21).

6 Conclusion

In this paper, we have considered the transformation of the path integral obtained as a result of the reduction of the finite-dimensional dynamical system with a symmetry. We have dealt with the reduction, which in the constrained dynamical system theory is called the reduction onto the non-zero momentum level.

Because of exploiting the dependent variables for the description of the local reduced motion, we were forced to consider only the trivial principal fiber bundles. Thereby, our consideration is a global one only for the trivial principal bundle. For the nontrivial principal fiber bundle, that may be related to the dynamical system with a symmetry, the dependent variable description of the evolution is valid in a some local domain.

Although for the nontrivial principal fiber bundles, there is a method [12] which allows us to extend the local evolution to a global one, but in general this problem remains unsolved, especially for the reason of a possible existence of the non-trivial topology of the orbit space.

In conclusion, we note that besides of the application of the obtained path integral representation (and the integral relation) in the quantization of the finite-dimensional dynamical systems with a symmetry, this representation may be useful for a quantum description (in the Schrödinger’s approach) of the excited modes in the gauge fields models.

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