Let $P^s$ be the $s$-dimensional complex projective space, and let $X, Y$ be two non-empty open subsets of $P^s$ in the Zariski topology. A hypersurface $H$ in $P^s \times P^s$ induces a bipartite graph $G$ as follows: the partite sets of $G$ are $X$ and $Y$, and the edge set is defined by $u \sim v$ if and only if $(u, v) \in H$. We say that $H \cap (X \times Y)$ is $(s, t)$-grid-free provided that $G$ contains no complete bipartite subgraph that has $s$ vertices in $X$ and $t$ vertices in $Y$. We conjecture that every $(s, t)$-grid-free hypersurface is equivalent, in a suitable sense, to a hypersurface whose degree in $y$ is bounded by a constant $d = d(s, t)$, and we discuss possible notions of the equivalence.

We establish the result that if $H \cap (X \times P^2)$ is $(2, 2)$-grid-free, then there exists $F \in \mathbb{C}[x, y]$ of degree $\leq 2$ in $y$ such that $H \cap (X \times P^2) = \{F = 0\} \cap (X \times P^2)$. Finally, we transfer the result to algebraically closed fields of large characteristic.

1 Introduction

The Turán number $\text{ex}(n, F)$ is the maximum number of edges in an $F$-free graph on $n$ vertices. The first systematic study of $\text{ex}(n, F)$ was initiated by Turán [Tur41], who solved the case when $F = K_t$ is a complete graph on $t$ vertices. Turán’s theorem states that, on a given vertex set, the $K_t$-free graph with the most edges is the complete and balanced $(t - 1)$-partite graph, in that the part sizes are as equal as possible.

For general graphs $F$, we still do not know how to compute the Turán number exactly, but if we are satisfied with an approximate answer, the theory becomes quite simple: Erdős and Stone [ES46] showed that if the chromatic number $\chi(F) = t$, then $\text{ex}(n, F) = \text{ex}(n, K_t) + o(n^2) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2)$. When $F$ is not bipartite, this gives an asymptotic result for the Turán number. On the other hand, for all but few bipartite graphs $F$, the order of $\text{ex}(n, F)$ is not known. Most of the research on this problem focused on two classes of graphs: complete bipartite graphs and cycles of even length. A comprehensive survey is by Füredi and Simonovits [FS13].
Suppose $G$ is a $K_{s,t}$-free graph with $s \leq t$. The Kővari–Sós–Turán theorem \cite{KST54} implies an upper bound $\text{ex}(n, K_{s,t}) \leq \frac{1}{2} \sqrt{t - 1} \cdot n^{2 - 1/s} + o(n^{2 - 1/s})$, which was improved by Füredi \cite{Furi96a} to

$$
\text{ex}(n, K_{s,t}) \leq \frac{1}{2} \sqrt{t - s + 1} \cdot n^{2 - 1/s} + o(n^{2 - 1/s}).
$$

Despite the lack of progress on the Turán problem for complete bipartite graphs, there are certain complete bipartite graphs for which the problem has been solved asymptotically, or even exactly. The constructions that match the upper bounds in these cases are all similar to one another. Each of the constructions is a bipartite graph $G$ based on an algebraic hypersurface $\mathbb{F}_p^s$. Both partite sets of $G$ are $\mathbb{F}_p^s$ and the edge set is defined by: $\pi \sim \pi'$ if and only if $(\pi, \pi') \in H$. In short, $G = (\mathbb{F}_p^s, \mathbb{F}_p^s, H(\mathbb{F}_p))$, where $H(\mathbb{F}_p)$ denotes the $\mathbb{F}_p$-points of $H$. Note that $G$ has $n := 2p^s$ vertices.

In the previous works of Erdős, Rényi and Sós \cite{ERS66}, Brown \cite{Bro66}, Füredi \cite{Furi96a}, Kollár, Rónyai and Szabó \cite{KRS96} and Alon, Rónyai and Szabó \cite{ARS99}, various hypersurfaces were used to define $K_{s,t}$-free graphs. Their equations were

$$
\begin{align*}
(x_1 y_1 + x_2 y_2 = 1, & \quad \text{for } K_{2,2}; \quad (\text{1a}) \\
(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 1, & \quad \text{for } K_{3,3}; \quad (\text{1b}) \\
(N_s \circ \pi_s)(x_1 + y_1, x_2 + y_2, \ldots, x_s + y_s) = 1, & \quad \text{for } K_{s,t} \text{ with } t \geq s! + 1; \quad (\text{1c}) \\
(N_{s-1} \circ \pi_{s-1})(x_2 + y_2, x_3 + y_3, \ldots, x_s + y_s) = x_1 y_1, & \quad \text{for } K_{s,t} \text{ with } t \geq (s - 1)! + 1, \quad (\text{1d})
\end{align*}
$$

where $\pi_s: \mathbb{F}_p^s \to \mathbb{F}_p^s$ is an $\mathbb{F}_p$-linear isomorphism and $N_s(\alpha)$ is the field norm, $N_s(\alpha) := o((p^s - 1)/(p - 1))$.

Clearly, the coefficients in (1a) and (1b) are integers and even independent of $p$. With some work, one can show that both (1c) and (1d) are polynomial equations of degree $\leq s$ with coefficients in $\mathbb{F}_p$. Therefore each equation in (1) can be written as $F(\pi, \pi') := F(x_1, \ldots, x_s, y_1, \ldots, y_s) = 0$ for some $F(\pi, \pi') \in \mathbb{F}_p[\pi, \pi']$ of bounded degree. The previous works directly count the number of $\mathbb{F}_p$ solutions to $F(\pi, \pi') = 0$ and yield $|H(\mathbb{F}_p)| = \Theta(p^{2s-1}) = \Theta(n^{2-1/s})$, for each prime $p$.

**Definition 1.** Given two sets $P_1$ and $P_2$, a set $V \subset P_1 \times P_2$ is said to contain an $(s, t)$-grid if there exist $S \subset P_1, T \subset P_2$ such that $s = |S|$, $t = |T|$ and $S \times T \subset V$. Otherwise, we say that $V$ is $(s, t)$-grid-free.

Observe that every $F(\pi, \pi')$ derived from (1) is symmetric in $x_i$ and $y_i$ for all $i$. We know that $(\pi, \pi') \in H$ if and only if $(\pi, \pi') \in H$ for all $\pi, \pi' \in \mathbb{F}_p^s$. The resulting bipartite graph $G = (\mathbb{F}_p^s, \mathbb{F}_p^s, H(\mathbb{F}_p))$ would be an extremal $K_{s,t}$-free graph if $H(\mathbb{F}_p)$ had been $(s, t)$-grid-free.

So which graphs are $K_{s,t}$-free with a maximum number of edges? The question was considered by Zoltán Füredi in his unpublished manuscript \cite{Fur88}, asserting that every $K_{2,2}$-free graph with $q$ vertices (for $q \geq q_0$) and $\frac{1}{2}q(q + 1)^2$ edges is obtained from a projective plane via a polarity with $q + 1$

2 An algebraic hypersurface in a space of dimension $n$ is an algebraic subvariety of dimension $n - 1$. The terminology from algebraic geometry used throughout the article is standard, and can be found in \cite{Sha13}.

3 We need $p \equiv 3 \pmod{4}$ for (1b) to get the correct number of $\mathbb{F}_p$ points on $H$. If $p \equiv 1 \pmod{4}$, then the right hand side of (1b) should be replaced by a quadratic non-residue in $\mathbb{F}_p$. 

2
The constructions for finally in Section 5, we consider algebraically closed fields of large characteristic.

Lazebnik [HLL15]. It was recently resolved by Hou, Lappano and Williford [DLW07].

graphs. The upper bound $\operatorname{ex}(n, C_2) = O(n^{1+1/t})$ first established by Bondy–Simonovits [BS74] has been matched only for $t = 2, 3, 5$. The $t = 2$ case was already mentioned above because $C_4 = K_{2,2}$. The constructions for $t = 3, 5$ are also algebraic (see [Ben66, FNV06] for $t = 3$ and [Ben66, Wen91] for $t = 5$). Also, a conjecture in a similar spirit about algebraic graphs of girth eight was made by Dmytrenko, Lazebnik and Williford [DLW07]. It was recently resolved by Hou, Lappano and Lazebnik [HLLL17].

The paper is organized as follows. In Section 2 we flesh out the informal conjecture above, in Section 3 we briefly discuss the $s = 1$ case, in Section 4 we partially resolve the $s = t = 2$ case, and finally in Section 5 we consider algebraically closed fields of large characteristic.
2 Conjectures on the \((s, t)\)-grid-free case

Given a field \(F\), we denote by \(F[x, y]\) the set of homogeneous polynomials in \(F[x, y]\) and by \(F_{\text{hom}}[x, y]\) the set of polynomials in \(F[x, y]\) that are separately homogeneous in \(x\) and \(y\).

We might be tempted to guess the following instance of the informal conjecture.

**False Conjecture A.** If \(H\) is almost-\((s, t)\)-grid-free, then there exists \(F(x, y) \in \mathbb{C}_{\text{hom}}[x, y]\) of degree \(d\) in \(\mathbb{C}\) for some \(d = d(s, t)\) such that \(H = \{ F = 0 \}\).

Unfortunately, Conjecture A is false because of the following example.

**Example 1.** Consider \(H_0 := \{ x_0 y_0 + x_1 y_1 + x_2 y_2 = 0 \}\) and \(H_1\) defined by
\[
x_0 y_0^d + x_1 y_0^{d-1} y_1 + x_2 \left( y_0^{d-1} y_2 + y_0^d f(y_1/y_0) \right) = 0,
\]
where \(f\) is a polynomial of degree \(d\). One can check that both \(H_0\) and \(H_1 \setminus \{ y_0 = 0 \}\) are \((2, 2)\)-grid-free, whereas equation (2) can be of arbitrary large degree in \(\mathbb{C}\).

Behind Example 1 is the birational automorphism \(\sigma: \mathbb{P}^2 \dasharrow \mathbb{P}^2\) defined by
\[
\sigma(y_0 : y_1 : y_2) := \left( y_0^d : y_0^{d-1} y_1 : y_0^{d-1} y_2 + y_0^d f(y_1/y_0) \right).
\]
Note that \(\text{id} \times \sigma\) is a biregular map\(^4\) from \(H_1 \setminus \{ y_0 = 0 \}\) to \(H_0 \setminus \{ y_0 = 0 \}\). Composition with the automorphism increased the degree of \(H_0\) in \(\mathbb{C}\) while preserving almost-\((2, 2)\)-grid-freeness. Here is another example illustrating the relationship between birational automorphisms and \((s, t)\)-grid-free hypersurfaces.

**Example 2.** Define \(H_2 := \{ x_0 y_1 y_2 + x_1 y_0 y_2 + x_2 y_0 y_1 = 0 \}\). One can also check that \(H_2 \setminus \{ y_0 y_1 y_2 = 0 \}\) is \((2, 2)\)-grid-free. Behind this example is the standard quadratic transformation \(\sigma\) from \(\mathbb{P}^2\) to itself given by \(\sigma(y_0 : y_1 : y_2) = (y_1 y_2 : y_0 y_2 : y_0 y_1)\). Note that \(\text{id} \times \sigma\) is a biregular map from \(H_2 \setminus \{ y_0 y_1 y_2 = 0 \}\) to \(H_0 \setminus \{ y_0 y_1 y_2 = 0 \}\).

Let \(\text{Cr}(\mathbb{P}^s)\) be the group of birational automorphisms on \(\mathbb{P}^s\), also known as the Cremona group. Evidently, the almost-\((s, t)\)-grid-freeness is invariant under \(\text{Cr}(\mathbb{P}^s) \times \text{Cr}(\mathbb{P}^s)\).

**Proposition 1.** If \(V_1 \subset \mathbb{P}^s \times \mathbb{P}^s\) is an almost-\((s, t)\)-grid-free set, then so is \(V_2 := (\sigma_X \times \sigma_Y) V_1\) for all \(\sigma_X, \sigma_Y \in \text{Cr}(\mathbb{P}^s)\).

**Remark 1.** Though little is known about the structure of the Cremona group in 3 dimensions and higher, the classical Noether–Castelnuovo theorem says that the Cremona group \(\text{Cr}(\mathbb{P}^2)\) is generated by the group of projective linear transformations and the standard quadratic transformation. The proof of this theorem, which is very delicate, can be found in [AC02, Chapter 8].

We say that sets \(V_1, V_2 \subset \mathbb{P}^s \times \mathbb{P}^s\) are *almost equal* if there exist nonempty Zariski-open sets \(X, Y \subset \mathbb{P}^s\) such that \(V_1 \cap (X \times Y) = V_2 \cap (X \times Y)\). We believe that the only obstruction to Conjecture A is the Cremona group.

\(^4\)A biregular map is a regular map whose inverse is also regular.
Conjecture B. Suppose $H$ is an irreducible hypersurface in $\mathbb{P}^s \times \mathbb{P}^s$. If $H$ is almost-$(s,t)$-grid-free, then there exist $\sigma \in \text{Cr} (\mathbb{P}^s)$ and $F(\overline{\sigma}, \overline{y}) \in C_{\text{hom}}[\overline{\sigma}, \overline{y}]$ of degree $\leq d$ in $\overline{y}$ for some $d = d(s,t)$ such that $H$ is almost equal to $\{F \circ (\text{id} \times \sigma) = 0\}$.

Remark 2. The conjecture is false if the irreducibility of $H$ is dropped. Take $H_0$ and $H_1$ from Example II and set $f(y) = y^d$ in [2], where $d$ can be arbitrarily large. Because both $H_0$ and $H_1$ are almost-$(2,2)$-grid-free, we know that $H_0 \cup H_1$ is almost-$(2,3)$-grid-free. However, one can show that for any $\sigma \in \text{Cr} (\mathbb{P}^s)$, the degree of $(\text{id} \times \sigma)(H_0 \cup H_1)$ in $\overline{y}$ is $\geq d$.

In fact, we believe in an even stronger conjecture.

Conjecture C. Suppose $H$ is an irreducible hypersurface in $\mathbb{P}^s \times \mathbb{P}^s$. Let $X,Y$ be nonempty Zariski-open subsets of $\mathbb{P}^s$. If $H \cap (X \times Y)$ is $(s,t)$-grid-free, then there exist $Y' \subset \mathbb{P}^s$, a biregular map $\sigma: Y \to Y'$ and $F(\overline{\sigma}, \overline{y}) \in C_{\text{hom}}[\overline{\sigma}, \overline{y}]$ of degree $\leq d$ in $\overline{y}$ for some $d = d(s,t)$ such that $H \cap (X \times Y) = \{F \circ (\text{id} \times \sigma) = 0\} \cap (X \times Y)$.

We prove Conjecture [C] if $s = 1$ and if $s = t = 2, Y = \mathbb{P}^2$ (see Section [II] and [IV] respectively).

One special case is when $H \cap (\mathbb{A}^s \times \mathbb{A}^s)$ is $(s,t)$-grid-free. In this case, $H$ can be seen as an affine algebraic hypersurface in $2s$-dimensional affine space. The group of automorphisms of $\mathbb{A}^s$, denoted by $\text{Aut} (\mathbb{A}^s)$, is a subgroup of the Cremona group. In this special case, we make a stronger conjecture.

Conjecture D. Suppose $H$ is an irreducible affine hypersurface in $\mathbb{A}^s \times \mathbb{A}^s$. If $H$ is $(s,t)$-grid-free, then there exist $\sigma \in \text{Aut} (\mathbb{A}^s)$ and $F(\overline{\sigma}, \overline{y}) \in C[\overline{\sigma}, \overline{y}]$ of degree $\leq d$ in $\overline{y}$ for some $d = d(s,t)$ such that $H = \{F \circ (\text{id} \times \sigma) = 0\}$.

Remark 3. An automorphism $\sigma \in \text{Aut} (\mathbb{A}^s)$ is elementary if it has a form

$$\sigma: (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_s) \mapsto (x_1, \ldots, x_{i-1}, cx_i + f, x_{i+1}, \ldots, x_s),$$

where $0 \neq c \in \mathbb{C}, f \in \mathbb{C}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s]$. The tame subgroup is the subgroup of $\text{Aut} (\mathbb{A}^s)$ generated by all the elementary automorphisms, and the elements from this subgroup are called tame automorphisms, while non-tame automorphisms are called wild. In Example II, we used a tame automorphism to make a counterexample to Conjecture [A]. It is known [Jut42] [vdK53] that all the elements of $\text{Aut} (\mathbb{A}^2)$ are tame. However, in the case of 3 dimensions, the following automorphism constructed by Nagata (see [Nag72]):

$$\sigma(x,y,z) = (x + (x^2 - yz)z, y + 2(x^2 - yz)x + (x^2 - yz)z, z)$$

was shown [SU03] [SU04] to be wild. See also [Kur10]. We note that the question on the existence of wild automorphisms remains open for higher dimensions.

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5 Let $R_1 = P_1/Q_1, R_2 = P_2/Q_2$ be rational functions such that $\sigma^{-1} = (1 : R_1 : R_2)$ and set $d_1 = \text{deg} P_1 = \text{deg} Q_1, d_2 = \text{deg} P_2 = \text{deg} Q_2$. On one hand, $H_0 := (\text{id} \times \sigma)H_0$ is defined by $x_0 + x_1R_1 + x_2R_2 = 0$, and so $\deg_H H_0 \geq d_2$. On the other hand, $H_1$ is defined by $x_0 + x_1R_1 + x_2(R_2 + R_2^t) = 0$ and so $\deg_H H_1 = dd_1 - d_2 - \deg G$, where $G = \text{gcd} (Q_1^tQ_2, P_1Q_1^tQ_2, Q_1^tP_2 + P_1^tQ_2)$. Since $G = \text{gcd} (Q_1^tQ_2, Q_1^tP_2 + P_1^tQ_2)$, it follows that $G$ divides $Q_2$. Hence we estimate that $\deg G \leq 2d_2$ and $\deg_H H_1 \geq dd_1 - d_2 \geq d - d_2$. So, $\deg_H (H_0 \cup H_1) \geq d$. 

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5
3 Results on the \((1,t)\)-grid-free case

As for the \(s=1\) case, one is able to fully characterize \((1,t)\)-grid-free hypersurfaces. We always assume that \(H\) is a hypersurface in \(\mathbb{P}^1 \times \mathbb{P}^1\) and \(X,Y\) are nonempty Zariski-open subsets of \(\mathbb{P}^1\) throughout this section.

**Theorem 2.** Suppose \(H = \{ F = 0 \}\), where \(F(\mathbf{x}, \mathbf{y}) \in \mathbb{C}_{\hom}[\mathbf{x}, \mathbf{y}]\). Let

\[
F(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) g(\mathbf{y}) h_1(\mathbf{x}, \mathbf{y})^{r_1} h_2(\mathbf{x}, \mathbf{y})^{r_2} \cdots h_n(\mathbf{x}, \mathbf{y})^{r_n}
\]

be the factorization of \(F\) such that \(h_1, h_2, \ldots, h_n\) are distinct irreducible polynomials depending on both \(\mathbf{x}\) and \(\mathbf{y}\). Let \(d_i\) be the degree of \(h_i\) in \(\mathbf{y}\). Then \(H \cap (X \times Y)\) is \((1,t)\)-grid-free if and only if \(\{ f = 0 \} \cap X = \emptyset\) and \(\{|g = 0\} \cap Y| + d_1 + d_2 + \cdots + d_n < t\).

**Proof.** Clearly, if \(H \cap (X \times Y)\) is \((1,t)\)-grid-free, then \(\{ f = 0 \} \cap X = \emptyset\). For fixed \(\mathbf{u} \in \mathbb{P}^1\), consider the following \(n + n + n + n + \binom{n}{2}\) systems of equations in \(\mathbf{y}\):

\[
\deg h_i(\mathbf{u}, \mathbf{y}) < d_i, \quad i = 1, 2, \ldots, n; \\
\mathbf{y} \in \mathbb{P}^1 \setminus Y \text{ and } h_i(\mathbf{u}, \mathbf{y}) = 0, \quad i = 1, 2, \ldots, n; \\
h_i(\mathbf{u}, \mathbf{y}) = \partial_\mathbf{y} h_i(\mathbf{u}, \mathbf{y}) = 0, \quad i = 1, 2, \ldots, n; \\
h_i(\mathbf{u}, \mathbf{y}) = g(\mathbf{y}) = 0, \quad i = 1, 2, \ldots, n; \\
h_i(\mathbf{u}, \mathbf{y}) = h_j(\mathbf{u}, \mathbf{y}) = 0, \quad i \neq j.
\]

Since \(h_i\)'s are irreducible and distinct, Bézout’s theorem tells us that each of these systems has no solution in \(\mathbb{P}^1\) for a generic \(\mathbf{u}\). So for a generic \(\mathbf{u} \in \mathbb{P}^1\), \(F(\mathbf{u}, \mathbf{y}) = 0\) has exactly \(M := |\{g = 0\} \cap Y| + d_1 + d_2 + \cdots + d_n\) distinct solutions in \(Y\). The conclusion follows as \(M\) is the maximal number of distinct solutions.

The informal conjecture thus holds when \(s=1\) as Theorem 2 implies:

**Corollary 3.** If \(H \cap (X \times Y)\) is \((1,t)\)-grid-free, then there exists \(F(\mathbf{x}, \mathbf{y}) \in \mathbb{C}_{\hom}[\mathbf{x}, \mathbf{y}]\) of degree \(< t\) in \(\mathbf{y}\) such that \(H \cap (X \times Y) = \{ F = 0 \} \cap (X \times Y)\).

**Proof.** Let \(H = \{ F = 0 \}\), and let \(f, g\) and \(h_1, h_2, \ldots, h_n\) be the factors of \(F\) as in [2]. Suppose \(m := |\{g = 0\} \cap Y|\) and \(m = \{|g = 0\} \cap Y| = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}\). Let \(g_i(\mathbf{y}) \in \mathbb{C}_{\hom}[\mathbf{y}]\) be linear such that \(\{g_i = 0\} = \{\mathbf{v}_i\}\). By Theorem 2, \(\bar{F}(\mathbf{x}, \mathbf{y}) := g_1(\mathbf{y}) g_2(\mathbf{y}) \cdots g_m(\mathbf{y}) h_1(\mathbf{x}, \mathbf{y}) h_2(\mathbf{x}, \mathbf{y}) \cdots h_n(\mathbf{x}, \mathbf{y})\) is of degree \(m + d_1 + d_2 + \cdots + d_n < t\) in \(\mathbf{y}\). Clearly, \(\bar{F} = F\) on \(X \times Y\).

Conjectures [3], [4] and [5] follow from the corollary in the \(s=1\) case. The birational map \(\sigma\) becomes trivial in those conjectures since \(\operatorname{Cr}(\mathbb{P}^1)\) consists only of projective linear transformations.
4 Results on the $(2, 2)$-grid-free case

Throughout the section we assume that $H$ is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ and $X$ is a nonempty Zariski-open subset of $\mathbb{P}^2$.

**Theorem 4.** If $H \cap (X \times \mathbb{P}^2)$ is $(2, 2)$-grid-free, then there exists $F(\overline{x}, \overline{y}) \in \mathbb{C}_\text{hom}[\overline{x}, \overline{y}]$ of degree $\leq 2$ in $\overline{y}$ such that $H \cap (X \times \mathbb{P}^2) = \{ F = 0 \} \cap (X \times \mathbb{P}^2)$.

The theorem resolves Conjecture C for $s = t = 2, Y = \mathbb{P}^2$. Note that the birational map $\sigma : Y \to Y'$ in the conjecture becomes trivial since biregular automorphisms of $\mathbb{P}^2$ are linear.

Our argument uses a reduction to an intersection problem of plane algebraic curves. The key ingredient is a theorem by Moura [Mou04] on the intersection multiplicity of plane algebraic curves.

**Theorem 5** (Moura [Mou04]). Denote by $I_\sigma(C_1, C_2)$ the intersection multiplicity of algebraic curves $C_1$ and $C_2$ at $\overline{x}$. For a generic point $\overline{y}$ on an irreducible algebraic curve $C_1$ of degree $d_1$ in $\mathbb{P}^2$,

$$\max_{C_2} \{ I_\sigma(C_1, C_2) : C_1 \nsubseteq C_2, \deg C_2 \leq d_2 \} = \begin{cases} \frac{1}{2}(d_2^2 + 3d_2) & \text{if } d_1 > d_2; \\ d_1d_2 - \frac{1}{2}(d_1^2 - 3d_1 + 2) & \text{if } d_1 \leq d_2. \end{cases}$$

**Corollary 6.** For a generic point $\overline{y}$ on an algebraic curve $C$ in $\mathbb{P}^2$, any algebraic curve $C'$ with $\overline{y} \in C'$ intersects with $C$ at another point unless $C$ is irreducible of degree $\leq 2$.

**Proof.** Suppose $C$ has more than one irreducible components. Let $C_1$ and $C_2$ be any two of them. Since $C_1 \cap C_2$ is finite, we can pick a generic point $\overline{y}$ on $C_1 \setminus C_2$. Now any algebraic curve $C'$ containing $\overline{y}$ intersects $C$ at another point on $C_2$. So, $C$ is irreducible.

Let $d$ and $d'$ be the degrees of $C$ and $C'$ respectively. By Theorem 5 one can check that $I_\sigma(C, C') < dd'$ for a generic point $\overline{y} \in C$ for all $d > 2$. From Bézout’s theorem, we deduce that $C$ intersects $C'$ at another point unless $d \leq 2$. \qed

In our proof of Theorem 4 we think of $H$ as a family of algebraic curves in $\mathbb{P}^2$, each of which is indexed by $\overline{y} \in X$ and is defined by $C(\overline{y}) := \{ (\overline{y}, \overline{y}) \in \mathbb{P}^2 : (\overline{y}, \overline{y}) \in H \}$. We call algebraic curve $C(\overline{y})$ the section of $H$ at $\overline{y}$. A hypersurface $H$ is $(2, 2)$-grid-free if and only if $C(\overline{y})$ and $C(\overline{y'})$ intersect at most 1 point for all distinct $\overline{y}, \overline{y'} \in X$. The last piece that we need for our proof is a technical lemma on generic sections of irreducible hypersurfaces.

**Lemma 7.** Suppose $H_1$ and $H_2$ are two different irreducible hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ defined by $h_1(\overline{x}, \overline{y}), h_2(\overline{x}, \overline{y}) \in \mathbb{C}_\text{hom}[\overline{x}, \overline{y}] \setminus (\mathbb{C}_\text{hom}[\overline{x}] \cup \mathbb{C}_\text{hom}[\overline{y}])$ respectively. Denote the section of $H_i$ at $\overline{y}$ by $C_i(\overline{y})$ for $i = 1, 2$. For generic $\overline{y} \in \mathbb{P}^2$, $C_1(\overline{y})$ and $C_2(\overline{y})$ share no common irreducible components, and moreover, each $C_i(\overline{y})$ is a reduced algebraic curve.$^6$

$^6$The algebraic curve $C_i(\overline{y})$ is reduced in the sense that its defining equation $h_i(\overline{y}, \overline{y})$ is square-free.
Proof of Theorem 4 assuming Lemma 7. Suppose $H \cap (X \times \mathbb{P}^2)$ is $(2,2)$-grid-free. Take an arbitrary $\pi \in X$ and consider algebraic curve $C(\pi)$ in $\mathbb{P}^2$. We claim that every $\pi \in C(\pi)$ is an intersection of $C(\pi)$ and $C(\pi')$ for some $\pi' \in X \setminus \{\pi\}$. Define $D(\pi) := \{F(\pi, \pi') = 0\} \cap X$. Since $\mathbb{P}^2 \setminus X$ is Zariski-closed, the set $D(\pi)$ is either empty or infinite. However, $\pi \in D(\pi)$ and the claim is equivalent to $|D(\pi)| \geq 2$.

Now pick a generic $\pi \in C(\pi)$. We know that point $\pi$ is an intersection of $C(\pi)$ and $C(\pi')$ for some $\pi' \in X \setminus \{\pi\}$ and it is the only intersection because $H \cap (X \times \mathbb{P}^2)$ is $(2,2)$-grid-free. We apply Corollary 8 to $C(\pi)$ and $C(\pi')$ and get that $C(\pi)$ is irreducible of degree $\leq 2$.

Suppose $H$ is defined by $F(\pi, \eta) \in \mathbb{C}_{\text{hom}}[\pi, \eta]$ and $F(\pi, \eta) = f(\pi)g(\eta)h_1(\pi, \eta)^{r_1}h_2(\pi, \eta)^{r_2} \cdots h_n(\pi, \eta)^{r_n}$ is the factorization of $F$ such that $h_1, h_2, \ldots, h_n$ are distinct irreducible polynomials in $\mathbb{C}_{\text{hom}}[\pi, \eta] \setminus (\mathbb{C}_{\text{hom}}[\pi] \cup \mathbb{C}_{\text{hom}}[\eta])$. The set $\{f = 0\} \cap X$ is either empty or infinite. So, for $H \cap (X \times \mathbb{P}^2)$ to be $(2,2)$-grid-free we must have $\{f = 0\} \cap X = \emptyset$. Similarly, we know that $\{g = 0\} = \emptyset$, that is, $g(\eta)$ is a nonzero constant.

Without loss of generality, we may assume that $f(\pi) = g(\eta) = 1$ and that $F(\pi, \eta)$ is square-free, that is, $r_1 = r_2 = \cdots = r_n = 1$. Let $C_i(\pi)$ be the section of $H_i := \{h_i = 0\}$ at $\pi$ for $i = 1, 2, \ldots, n$. From Lemma 7 we know that, for a generic $\pi \in X$, $\pi_i(\pi)$ and $\pi_j(\pi)$ have no common irreducible components for all $i \neq j$. Therefore $C(\pi) = \cup_{i=1}^n C_i(\pi)$ has at least $n$ irreducible components, and so $n = 1$. Now $C(\pi) = C_1(\pi) = \{h_1(\pi, \eta) = 0\}$ for all $\pi \in X$. By Lemma 7, $h_1(\pi, \eta)$ is square-free for generic $\pi$. This and the fact that $C(\pi)$ is irreducible of degree $\leq 2$ imply that $\deg h_1(\pi, \eta) \leq 2$ for a generic $\pi \in X$, and so $\deg \eta h_1(\pi, \eta) \leq 2$.

Proof of Lemma 7. Let $d_1, d_2$ be the degrees of $h_1, h_2$ in $\eta$ respectively. Suppose on the contrary that $C_1(\pi)$ and $C_2(\pi)$ share common irreducible components for a generic $\pi \in \mathbb{P}^2$. So, $h_1(\pi, \eta)$ and $h_2(\pi, \eta)$ have a common divisor in $\mathbb{C}[\eta]$. Therefore there exist two nonzero polynomials $g_1^\pi(\eta) \in \mathbb{C}_{\text{hom}}[\eta]$ of degree $< d_2$ and $g_2^\pi(\eta) \in \mathbb{C}_{\text{hom}}[\eta]$ of degree $< d_1$ such that

$$h_1(\pi, \eta)g_1^\pi(\eta) + h_2(\pi, \eta)g_2^\pi(\eta) = 0. \quad (4)$$

By treating the coefficients of $g_1^\pi(\eta)$ and $g_2^\pi(\eta)$ as variables, we can view equation (4) as a homogeneous system of $M := \left(\binom{d_1 + d_2 + 1}{2}\right)$ linear equations involving $N := \left(\binom{d_1 + 1}{2}\right) + \left(\binom{d_2 + 1}{2}\right)$ variables. Note that the coefficient in the $i$th equation of the $j$th variable, say $c_{ij}$, is a polynomial of $\pi$, that is, $c_{ij} = c_{ij}(\pi)$ for some $c_{ij}(\pi) \in \mathbb{C}[\pi]$ that depends on $h_1, h_2$ only. Because the system of linear equations has a nontrivial solution and clearly $M > N$, the rank of its coefficient matrix $(c_{ij}(\pi))$ is $< N$. Using the determinants of all $N \times N$ minors of matrix $(c_{ij}(\pi))$, we can rewrite the statement that matrix $(c_{ij}(\pi))$ is of rank $< N$ as $L := \binom{M}{N}$ polynomial equations of entries in the matrix, say

$$P_k(c_{ij}(\pi)) = 0, \quad \text{for all } k \in [L], \quad (5)$$

where $P_k(c_{ij}(\pi))$ is a polynomial of $\pi$ independent of $\pi$. Since (5) holds for a generic $\pi \in \mathbb{P}^2$, we have

$$P_k(c_{ij}(\pi)) = 0 \text{ in } \mathbb{C}[\pi], \quad \text{for all } k \in [L], \quad (6)$$

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Reversing the argument above, we can deduce that the rank of matrix $(e_{ij}(\pi))$, over the quotient field $\mathbb{C}(\pi)$, is $< N$, and so there exist two nonzero polynomials $g_1(\pi, g) \in \mathbb{C}(\pi)_{\text{hom}}[\pi]$ of degree $< d_2$ and $g_2(\pi, g) \in \mathbb{C}(\pi)_{\text{hom}}[\pi]$ of degree $< d_1$ such that

$$h_1(\pi, g)g_1(\pi, g) + h_2(\pi, g)g_2(\pi, g) = 0. \quad (7)$$

Multiplying $(7)$ by the common denominator of $g_1(\pi, g)$ and $g_2(\pi, g)$, we get two nonzero polynomials $g_1(\pi, g) \in \mathbb{C}[\pi, g]$ of degree $< d_2$ in $\mathbb{C}$ and $g_2(\pi, g) \in \mathbb{C}[\pi, g]$ of degree $< d_1$ such that

$$h_1(\pi, g)g_1(\pi, g) + h_2(\pi, g)g_2(\pi, g) = 0, \quad (8)$$

which is impossible as $\gcd(h_1, h_2) = 1$ and $\deg g_1(\pi, g) = d_1 > \deg g_2(\pi, g)$.

It remains to prove that $C_1(\pi)$ is reduced for generic $\pi$. Because $h_1(\pi, g) \notin \mathbb{C}_{\text{hom}}[\pi]$, the polynomial $h_1'(\pi, g) := \partial h_1(\pi, g)/\partial y_0$ might be assumed to be nonzero. Again, we assume, on the contrary, that $h_1(\pi, g)$ is not square-free for a generic $\pi \in \mathbb{P}^2$. This implies that $h_1(\pi, g)$ and $h_1'(\pi, g)$ have a common divisor. The same linear-algebraic argument, applied to $h_1$ and $h_1'$ instead of $h_1$ and $h_2$, then yields a contradiction. \hfill $\Box$

We can adapt the proof of Theorem 4 to the case when $\mathbb{P}^2 \setminus Y$ is finite. In this case, we obtain a weaker result though.

**Proposition 8.** Suppose $\mathbb{P}^2 \setminus Y = \{\pi_1, \pi_2, \ldots, \pi_n\}$. If $H \cap (X \times Y)$ is $(2, 2)$-grid-free, then either

1. there exists $F(\pi, g) \in \mathbb{C}_{\text{hom}}[\pi, g]$ of degree $\leq 2$ in $\pi$ such that $H \cap (X \times Y) = \{F = 0\} \cap (X \times Y)$, or
2. there exists $i \in [n]$ such that $\mathbb{P} \times \{\pi_i\} \subset H$.

**Sketch of a proof.** We follow the proof of Theorem 4 up to the point where we apply Corollary 4. Note that $X_i := \{\pi \in \mathbb{P}^2 : \pi \in C(\pi)\}$ is Zariski-closed for all $i \in [n]$. If none of those $X_i$’s equals $\mathbb{P}^2$, then for a generic $\pi \in \mathbb{P}^2$, $C(\pi)$ does not pass through any of the points in $\mathbb{P}^2 \setminus Y$. The rest of the proof of Theorem 4 still holds and we end in the first case. Otherwise $X_i = \mathbb{P}^2$ for some $i \in [n]$, which corresponds to the second case. \hfill $\Box$

## 5 Fields of finite characteristic

A standard model-theoretic argument allows us to transfer statements over fields of characteristic 0 to the fields of large characteristic.

**Theorem 9.** Let $\phi$ be a sentence in the language of rings. The following are equivalent.

1. $\phi$ is true in complex numbers.
2. $\phi$ is true in every algebraically closed field of characteristic zero.
3. $\phi$ is true in all algebraically closed fields of characteristic $p$ for all sufficiently large prime $p$.
The theorem is an application of the compactness theorem and the completeness of the theory of algebraically closed field of fixed characteristic. We refer the readers to [Mar02, Section 2.1] for further details of the theorem and related notions.

As quantifiers over all polynomials are not part of the language of rings, one has to limit the degree of hypersurface $H$ and the complexity of the open set $X$ in Theorem 4. We now formulate the analog over the fields of large characteristic.

**Theorem 10.** Let $K$ be an algebraically closed field of large characteristic, let $H$ be a hypersurface in $\mathbb{P}^2(K) \times \mathbb{P}^2(K)$ of bounded degree, and let $X$ be a Zariski-open subset of $\mathbb{P}^2(K)$ of bounded complexity (i.e. $X$ is a Zariski-open subset of $\mathbb{P}^2(K)$ that can be described by some first order predicate in the language of rings of bounded length). If $H$ is $(2, 2)$-grid-free in $X \times \mathbb{P}^2(K)$, then there exists $F(\overline{x}, \overline{y}) \in \mathbb{K}_{\text{hom}}[\overline{x}, \overline{y}]$ of degree $\leq 2$ in $\overline{y}$ such that $H \cap (X \times \mathbb{P}^2) = \{ F = 0 \} \cap (X \times \mathbb{P}^2)$.

The proof essentially rewrites Theorem 4 as a sentence in the language of rings to which Theorem 9 is applicable. We skip the tedious but routine proof.

6 Acknowledgement

The second author would like to thank Hong Wang for her suggestion on the choice of terminology in algebraic geometry.

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