No interesting sequential groups

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Abstract

We prove that it is consistent with ZFC that no sequential topological groups of intermediate sequential orders exist. This shows that the answer to a 1981 question of P. Nyikos is independent of the standard axioms of set theory. The model constructed also provides consistent answers to several questions of D. Shakhmatov, S. Todorčević and Uzcátegui. In particular, we show that it is consistent with ZFC that every countably compact sequential group is Fréchet-Urysohn.

1 Introduction

A number of areas in mathematics benefit from viewing continuity through the lens of convergence. To investigate the effects of convergence on topological structure several classes of spaces have been introduced and studied by set-theoretic topologists. These range from various generalized metric spaces to sequential ones. As a result of these efforts a vast body of classification results and metrization theorems have been developed.

A popular theme has been the study of convergence in the presence of an algebraic structure such as a topological group (see [2] and [10] for a bibliography). One of the first results of this kind, the classical metrization theorem

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by Birkhoff and Kakutani states that every first countable Hausdorff topological group is metrizable. This establishes a rather unexpected connection between the local and the global properties generally unrelated to each other.

It has been demonstrated by a number of authors, however, that various shades of convergence are, in general, different from each other, even when an algebraic structure is involved (see [1], [12], [17], [19]). A common thread among the majority of these results is the necessity of set-theoretic assumptions beyond ZFC to construct counterexamples.

A celebrated solution of Malykhin’s problem about the metrizability of countable Fréchet groups by Hrušák and Ramos-Garsía [8] is a beautiful validation of the significance of set-theoretic tools in the study of convergence.

A question that is only slightly more recent than Malykhin’s problem was asked by P. Nyikos in [12] and deals with the sequential order in topological groups. Recall that a space $X$ is sequential if whenever $A \subseteq X$ is not closed, there exists a convergent sequence $C \subseteq A$ that converges to a point outside $A$. This is a rather indirect way of saying that convergent sequences determine the topology of $X$ without supplying any ‘constructive means’ of describing the closure operator. Such a description is provided by the concept of the sequential closure of $A$: $[A]' = \text{“limits of all convergent sequences in } A\text{”}$. The next natural step is to recursively define iterated sequential closures $[A]_{\alpha}$, $\alpha \leq \omega_1$ as $[A]_{\alpha+1} = [[A]_{\alpha}]'$ and $[A]_{\alpha} = \bigcup \{ A_{\beta} \mid \beta < \alpha \}$ for limit $\alpha$. It is a quick observation that in sequential spaces $[A]_{\omega_1} = \overline{A}$ and in fact, this property characterizes the class of sequential spaces.

For many spaces it takes only countably many iterations to get the closure of any set. The smallest ordinal $\alpha \leq \omega_1$ such that $[A]_{\alpha} = \overline{A}$ for every $A \subseteq X$ is called the sequential order of $X$ which is written $\alpha = \text{so}(X)$. As a simple illustration of these concepts, sequential spaces are those for which the sequential order is defined and Fréchet (or Fréchet-Urysohn) ones are those whose sequential order is 1.

Simple examples of spaces of intermediate (i.e. strictly between 1 and $\omega_1$) sequential orders are plentiful but they all seem to have one common feature: different points of the space have different properties in terms of the sequential closure. This led P. Nyikos to ask the following question.

**Question 1** ([12]). Do there exist topological groups of intermediate sequential orders?

This question and some of its stronger versions were also asked by D. Shakhmatov in [16] (Questions 7.4 (i)–(iii)).
A weak version of this question (for homogeneous and semitopological groups, i.e. groups in which the multiplication is continuous in each factor separately) had been answered affirmatively in ZFC (see [6] and [14]).

A consistent positive answer for topological groups was first given in [18] using CH. In [19] it was shown that under CH groups of every sequential order exist.

In this paper we use some of the techniques developed by Hrušák and Ramos-García for their solution of Malykhin’s problem to show that extra set-theoretic assumptions are necessary. To be more precise, we construct a model of ZFC in which all sequential groups are either Fréchet or have sequential order $\omega_1$. For countable groups, the result can be viewed as a consistent metrization statement: it is consistent with the axioms of ZFC that all countable sequential groups of sequential order less than $\omega_1$ are metrizable.

As an aside, we show how the same model provides a consistent answer to a question of D. Shakhmatov about the structure of countably compact sequential groups.

2 Preliminaries

We use standard set-theoretic terminology, see [10]. By a slight abuse of notation we sometimes treat sequences as sets of points in their range. Basic facts about topological groups can be found in [2]. All spaces are assumed to be regular.

Following [8] define Laver-Mathias-Prikry forcing $\mathbb{L}_F$ associated to a free filter $F$ on $\omega$ as the set of those trees $T \in \omega^{<\omega}$ for which there is an $s_T \in T$ (the stem of $T$) such that for all $s \in T$, $s \subseteq s_T$ or $s_T \subseteq s$ and such that for all $s \in T$ with $s \supseteq s_T$ the set $\text{succ}_T(s) = \{ n \in \omega \mid s \cup n \in T \} \in F$ ordered by inclusion.

Full details of proofs of various properties of $\mathbb{L}_F$ can be found in [8], here we only present the statements directly used in the arguments in this paper.

A central role in the techniques of [8] is played by the concept of an $\omega$-hitting family. Recall that a family $\mathcal{H} \subseteq [\omega]^\omega$ is called $\omega$-hitting [7] if given $\langle A_n \mid n \in \omega \rangle \subseteq [\omega]^\omega$ there is an $H \in \mathcal{H}$ such that $H \cap A_n$ is infinite for all $n \in \omega$.

The following two statements from [4] supply all the necessary information to establish the preservation of $\omega$-hitting families by ccc forcings and their iterations.
Lemma 1 ([4]). Finite support iterations of ccc forcings strongly preserving \( \omega \)-hitting strongly preserve \( \omega \)-hitting.

As noted in [8] a forcing that strongly preserves \( \omega \)-hitting preserves \( \omega \)-hitting. Moreover, as the lemma below implies, these two concepts are equivalent for the forcings used in the arguments below so the definition of strong preservation is omitted.

Proposition 1 ([4]). Let \( \mathcal{I} \) be an ideal on \( \omega \) and let \( \mathcal{F} = I^* \) be the dual filter. Then the following are equivalent.

1. For every \( X \in \mathcal{I}^+ \) and every \( \mathcal{J} \leq_K \mathcal{I} \upharpoonright X \) the ideal \( \mathcal{J} \) is not \( \omega \)-hitting.
2. \( L_\mathcal{F} \) strongly preserves \( \omega \)-hitting.
3. \( L_\mathcal{F} \) preserves \( \omega \)-hitting.

The Katětov order \( \leq_K \) on ideals used above is defined by putting \( \mathcal{I} \leq_K \mathcal{J} \) whenever \( \mathcal{I} \) and \( \mathcal{J} \) are ideals on \( \omega \) and there exists an \( f : \omega \to \omega \) such that \( f^{-1}(I) \in \mathcal{J} \) for every \( I \in \mathcal{I} \).

Let \( X \) be a topological space, \( \text{nwd}(X) \) be the ideal of nowhere dense subsets of \( X \), and \( \text{nwd}^*(X) \) be the filter of dense open subsets of \( X \). Let \( \mathcal{I}_x \) be the ideal dual to the filter of open neighborhoods of \( x \in X \). Finally, the \( \pi \)-character \( \pi \chi(x, X) \) is defined as the smallest cardinality of a family \( \mathcal{U} \) of open subsets of \( X \) such that every neighborhood of \( x \) contains a \( U \in \mathcal{U} \). It is a well known fact that \( \pi \)-character and character coinide in topological groups thus every nonmetrizable topological group has an uncountable \( \pi \)-character due to Birkhoff-Kakutani theorem.

The next proposition is a direct restatement of Proposition 5.3 (a) [8].

Proposition 2 ([8]). Let \( X \) be a countable regular space and \( x \in X \). If \( \pi \chi(x, X) > \omega \) then \( \Vdash_{L_{\text{nwd}^*(X)}} \text{"} \hat{A}_{\text{gen}} \in \mathcal{I}_x^+ \land \mathcal{I}_x \upharpoonright \hat{A}_{\text{gen}} \text{ is } \omega \text{-hitting}" \)

Lemmas 6.4 and 6.5 in [8] are stated for a countable Fréchet group \( G \) whereas a careful reading of their proofs reveals that Fréchetness can be replaced by a weaker condition below.

(C) For every countable family \( \{ N_i \mid i \in \omega \} \) of nowhere dense subsets of \( G \) there exists a nontrivial convergent sequence \( C \subseteq G \) such that \( C \to 1_G \) and \( C \cap N_i \) is finite for every \( i \in \omega \).
To state the lemma we need two more definitions from [8]. The definitions encapsulate the connection between the algebraic structure of an abstract group $G$ and its topology.

**Definition 1 ([8])**. Let $G$ be an abstract group and let $X \subseteq G \setminus \{1_G\}$. A subset $A \subseteq G$ is called $X$-large if for every $b \in X$ and $a \in G$ either $a \in A$ or $b \cdot a^{-1} \in A$.

**Definition 2 ([8])**. A family $\mathcal{C}$ of subsets of an abstract group $G$ is $\omega$-hitting w.r.t. $X$ if given a family $\langle A_n \mid n \in \omega \rangle$ of $X$-large sets there is a $C \in \mathcal{C}$ such that $C \cap A_n$ is infinite for all $n \in \omega$.

**Lemma 2 ([8])**. Let $G$ be a topological group that satisfies $(C)$. Then

$$\forces_{\text{mod}^* (G)} \text{“} \mathcal{C} \text{ is } \omega\text{-hitting w.r.t. } \dot{\text{gen}} \text{”}$$

where $\mathcal{C} = \mathcal{I}^1_{1_G}$ is the ideal consisting of sequences converging to $1_G$.

The last two lemmas from [8] deal with the preservation of $\omega$-hitting w.r.t. $X$. As before, [8] notes that strong preservation implies preservation, thus the definition of strong preservation of $\omega$-hitting w.r.t. $X$ is omitted.

**Lemma 3 ([8])**. $\sigma$-centered forcings strongly preserve $\omega$-hitting w.r.t. $X$.

**Lemma 4 ([8])**. Finite support iterations of ccc forcings that strongly preserve $\omega$-hitting w.r.t. $X$ strongly preserve $\omega$-hitting w.r.t. $X$.

Recall that a space $X$ is called $T_5$ (or hereditarily normal) if every subspace of $X$ is normal. For every $x \in X$, the pseudocharacter $\psi(X, x)$ of $x$ in $X$ is defined as the smallest cardinality of a family $\mathcal{U}$ of open neighborhoods of $x$ such that $\bigcap \mathcal{U} = \{x\}$. The pseudocharacter of $X$ is then $\psi(X) = \sup \{ \psi(X, x) \mid x \in X \}$.

The following result from [5] is an elegant extension of Katětov’s product lemma to topological groups.

**Theorem 1 ([5])**. Let $G$ be a $T_5$ topological group. If there exists a nontrivial convergent sequence in $G$ then $\psi(G) = \omega$. 
3 Convergence and scaffolds

While convergent sequences determine the topology of a sequential space, a more precise description of the closure operator will be needed later. This description is supplied by the idea of a scaffold, defined below. This is not the only, or even the most efficient way of studying the sequential closure, simply one that suits the approach below.

**Definition 3.** Let \( X \) be a topological space, \( S \subseteq X \) and \( S \subseteq 2^S \). Then \( (S, S) \) is an \( \alpha \)-scaffold (or simply scaffold), \( \text{ht}(S) \), \( \text{ht}_S(x) \) for \( x \in S \) and \( \text{cor} \ S \) are defined recursively as:

(S.0) If \( S = \{x\} \), where \( x \in X \), and \( S = \{S\} \). Then \( (S, S) \) is a 0-scaffold \( \text{ht}(S) = 0, \text{ht}_S(x) = 0, \) and \( x = \text{cor} \ S \).

(S.1) Suppose there exist an \( x \in S \), a disjoint collection \( \langle U_n \mid n \in \omega \rangle \) of open subsets of \( X \), and \( \alpha_n \)-scaffolds \( (S_n, S_n) \) such that \( \text{cor} \ S_n \rightarrow x, S_n \subseteq U_n, x \notin \bigcup_n \) where \( \langle \alpha_n \mid n \in \omega \rangle \) is non-decreasing and \( \alpha = \min \{\beta \mid \beta > \alpha_n \ \text{for each} \ n \in \omega \} \). Suppose also that \( S = \{S\} \cup \langle S_n \mid n \in \omega \rangle \). Then \( (S, S) \) is an \( \alpha \)-scaffold, \( \text{ht}(S) = \alpha, \text{ht}_S(x) = \alpha, \text{ht}_S(x') = \text{ht}_{S_n}(x') \) for \( x' \in S_n \) and \( \text{cor} \ S = x \).

As one would expect, most proofs involving scaffolds proceed by tedious induction arguments on the scaffold’s height. Given a scaffold \( (S, S) \) it will be convenient to define some subsets of \( S \) and \( S \) to simplify the notation. Put \( S_{[\beta]} = \{ s \in S \mid \text{ht}_S(s) = \beta \}, S_{[\beta]} = \{ T \in S \mid \text{ht}(T) = \beta \} \).

A subset \( S \) of \( X \) will be called \( a(n) \) \((\alpha)\)-scaffold if there exists a \( S \in 2^S \) (called the stratification of \( S \)) such that \( (S, S) \) is an \( (\alpha) \)-scaffold.

While a stratification is not unique, \( \text{cor} \ S, \text{ht}_S(x), \text{ht}(S) \), as well as \( S_{[\beta]} \) and similar subsets, are independent of the choice of \( S \). This is most easily observed by noting that all of the ordinals and subsets in the list above can be expressed in terms of the Cantor-Bendixon rank (see [9]).

The utility of scaffolds is illustrated by the lemma below.

**Lemma 5.** Let \( X \) be a regular space, \( x \in [A]_\alpha \subseteq X \). Then there exists a \( \beta \)-scaffold \( (S, S) \), such that \( \beta \leq \alpha, S \subseteq X, S_{[0]} \subseteq A \), and \( x = \text{cor} \ S \).

If \( S \) is a stratification of \( S \) and \( S' \in S \) put \( S|_{S'} = \{ T \in S \mid T \subseteq S' \} \). Then \( S' \) is a scaffold and \( (S', S|_{S'}) \) is an \( \text{ht}(S') \)-scaffold. For every stratification \( S \) of \( S \), the collection \( S^- = \{ S' \in S, S' \neq S \mid S' \subseteq S'' \in S \Rightarrow S'' = S' \} \) or \( S'' = \)}
Lemma 6. Let \((S, S)\) be a scaffold in some space \(X\). Then there exists a mapping \(o : S \rightarrow \tau\), where \(\tau\) is the topology on \(X\), such that \(T \subseteq o(T)\) and \(T' \subseteq T''\) if and only if \(o(T') \subseteq o(T'')\).

We will call such a mapping (or, sometimes, just its set of values) an open stratification of \(S\).

The scaffold whose existence is provided by Lemma 5 is not always suitable for the purposes of the argument and sometimes has to be ‘thinned’. The definition below makes this idea precise.

Definition 4. Let \(S\) be an \(\alpha\)-scaffold, \(S\) be a stratification of \(S\) and \(S' \subseteq S\). Call \(S'\) an \(S\)-proper subscaffold (or simply a proper subscaffold if \(S\) is of no importance) of \(S\) and write \(S' \leq_S S\) if \(\text{cor}_S = \text{cor}_{S'}\) and there exists a (unique) stratification \(S'\) of \(S'\) such that \(S'^- \neq \emptyset\) whenever \(S^- \neq \emptyset\), and each \(\beta\)-scaffold \(B' \in S'^-\) is an \(S|_B\)-proper subscaffold of some \(\beta\)-scaffold \(B \in S^\sim\).

The lemma below introduces a general procedure for picking a subscaffold inside a given scaffold. It has a standard inductive proof which is therefore omitted.

Lemma 7. Let \((S, S)\) be a scaffold in some space \(X\), \(\tau\) be the topology of \(X\), and \(r : X \rightarrow \tau\) be a neighborhood assignment such that \(x \in r(x)\) for every \(x \in X\). Then there exists an \(S\)-proper subscaffold \((S', S')\) of \((S, S)\) and its open stratification \(o' : S' \rightarrow \tau\) such that \(o'(T) \subseteq r(\text{cor} T)\). Moreover, if \(o : S \rightarrow \tau\) is any open stratification of \(S\), the stratification \(S' = \{ o(T) \cap S' \mid T \in S, o(T) \cap S' \neq \emptyset \}\), and a subfamily of \(\{ o(T) \mid T \in S \}\) forms an(other) open stratification of \(S'\).

The following definition and the subsequent lemma describe how scaffolds can be used to gauge the sequential order of a space.

Definition 5. An \(\alpha\)-scaffold \(S\) is called semiloose (in \(X\) where \(X\) is a topological space such that \(S \subseteq X\)) if for every infinite convergent sequence \(C \subseteq S\) such that \(C \to s \in S\) there exists an infinite subsequence \(C' \subseteq C\) so that \(\text{ht}_S(s) = \min\{ \beta \mid \beta > \text{ht}_S(x)\} \) for all but finitely many \(x \in C'\).
Note in the lemma below that to establish a lower bound on the sequential order a closed scaffold is needed.

Lemma 8. Let $X$ be a regular space, $A \subseteq X$, $x \in [A]_{\alpha}$, and $x \notin [A]_{\beta}$ for any $\beta < \alpha$. Then there exists a semiloose $\alpha$-scaffold $(S, S)$ in $X$ such that $x = \text{cor } S$ and $S_{[0]} \subseteq A$. If $(S, S)$ is a closed semiloose $\alpha$-scaffold in $X$ then $\text{cor } S \in [S_{[0]}]_{\alpha}$ and $\text{cor } S \notin [S_{[0]}]_{\beta}$ for any $\beta < \alpha$.

$S$ is called loose if for every $s \in S$ there exists a convergent sequence $C_s \subseteq S$ such that $C \subseteq^* C_s$ for some $s \in S$ for any convergent sequence $C \subseteq S$. Note that loose and semiloose coincide for finite $\alpha$’s. We will not use loose scaffolds below, they are defined here merely to justify the choice of terminology.

## 4 Scaffolds in topological groups

To a large extent, the central part of the argument below aims to establish property (C) for sequential topological groups satisfying some conditions. In [8] it is noted that (C) holds for Fréchet spaces without isolated points due to a lemma in [3]. This property does not hold for arbitrary sequential groups (a quick example is provided by the free (Abelian) topological group over a convergent sequence) so some additional restrictions are necessary.

The following definition is introduced to facilitate the study of sequential order in sequential topological groups. 1-scaffolds (i.e. convergent sequences) are treated separately as they form an important special case.

Definition 6. Let $G$ be a topological group, $C = \langle c_n \mid n \in \omega \rangle \cup \{c\} \subseteq G$ be a convergent sequence in $G$, $c_n \in U_n$ be disjoint open subsets of $G$, and $c \notin \overline{U_n}$ for any $n \in \omega$ (i.e. $\{U_n \mid n \in \omega\} \cup \{G\}$ is an open stratification of $C$). Let $(S, S)$ be a scaffold in $G$ and $S^- = \{S_n \mid n \in \omega\}$. Suppose $c_n \cdot S_n \subseteq U_n \cdot \text{cor } S$ for every $n \in \omega$. Define the scaffold

$$C \otimes S = \bigcup \{c_n \cdot S_n \mid n \in \omega\} \cup \{c \cdot \text{cor } S\}$$

Also put $C \odot S = C \otimes S \setminus \text{cor}(C \otimes S) = \bigcup \{c_n \cdot S_n \mid n \in \omega\}$.

Note that the definition of $C \otimes S$ depends on the indexing of the elements of $S^-$ and $C$, as well as the choice of $U_n$’s. The latter is rarely a problem since any argument involving $\otimes$’s is usually preceded and followed by passing to
appropriate subscaffolds. In particular, in order to satisfy $c_n \cdot S_n \subseteq U_n \cdot \text{cor } S$, observe that $c_n \cdot \text{cor } S \subseteq U_n \cdot \text{cor } S$, and construct open nested $V_n$'s such that $\text{cor } S \subseteq V_n$ and $c_n \cdot V_n \subseteq U_n \cdot \text{cor } S$. Now pick increasing $n(i) \in \omega$ so that $\text{cor } S_{n(i)} \subseteq V_i$. ‘Thin’ each $S_{n(i)}$ using Lemma 7 to obtain proper subscaffolds $S_i' \subseteq S_{n(i)}$ such that $S_i' \subseteq V_i$. Put $S' = \bigcup \{S_i' \mid i \in \omega\} \cup \{\text{cor } S\}$. Now $C \otimes S'$ is defined. Note that for every proper subscaffold $S'' \subseteq S'$ of $S'$ defined as above, the product $C \otimes S''$ is also defined.

The indexing dependence can be mitigated by requiring that both factors be ordered in the type of $\omega$ and the products be taken ‘in order’. The definition of proper subscaffold can be adjusted to inherit the order, as well.

The next lemma shows that $\otimes$ does not introduce new convergent sequences.

**Lemma 9.** Let $G$ be a topological group, $(S, S)$ and $C$ be as in Definition 7. Let $C' = \{c_{n(i)} \cdot s_i \mid i \in \omega\} \subseteq C \otimes S$ be a convergent sequence. Then $\{s_i \mid i \in \omega\}$ is a convergent sequence.

The remarks following Definition 7 fully apply to the general case defined below. Disambiguating measures suggested there (e.g. ordering of $S$ and $S_n$'s) are assumed to be taken but are not explicitly mentioned.

**Definition 7.** Let $G$ be a topological group. Let $(S, S)$ be an $\alpha$-scaffold, $\alpha > 1$, and $\{(S_n, S_n) \mid n \in \omega\}$ be a countable collection of scaffolds in $G$, the sequence of $\text{ht}(S_n)$'s is nondecreasing and $\{\text{cor } S_n \mid n \in \omega\} = \{x\}$ for some $x \in G$. Let also $\{C_n \mid n \in \omega\}$ list all $C_n \in S$ such that $\text{ht}(C_n) = 1$. Suppose $C_n \otimes S_n$ are defined for every $n \in \omega$ and the natural (i.e. taken from some open stratification of $S$) choice of open $U_n$ as in Definition 7. Put

$$S \otimes \{S_n \mid n \in \omega\} = (S \setminus S_{[0]} \cdot x \cup \bigcup \{C_n \otimes S_n \mid n \in \omega\}$$

The next lemma will not be used explicitly in what follows. Rather it presents an induction hypothesis that can be used to justify the claim that $\otimes$-products result in scaffolds in the general case.

**Lemma 10.** Let $(S', S')$ and $(S'', S'')$ be scaffolds and $\text{ht}(S') \leq \text{ht}(S'')$. Let $\{S_n \mid n \in \omega\}$ be a family of (ordered) scaffolds so that the sequence of $\text{ht}(S_n)$'s is nondecreasing. Let $\{m_i \mid i \in \omega\} \subseteq \omega$ and $\{n_i \mid i \in \omega\} \subseteq \omega$ be arbitrary increasing sequences so that both $S' \otimes \{S_{n_i} \mid i \in \omega\}$ and $S'' \otimes \{S_{m_i} \mid i \in \omega\}$ are defined. Then both $S' \otimes \{S_{n_i} \mid i \in \omega\}$ and $S'' \otimes \{S_{m_i} \mid i \in \omega\}$ are
scaffolds and \( \text{ht}(S' \otimes \{ S_n, \mid i \in \omega \}) \leq \text{ht}(S'' \otimes \{ S_m, \mid i \in \omega \}) \). Moreover, \( \text{ht}(S' \otimes \{ S'_n, \mid i \in \omega \}) = \text{ht}(S' \otimes \{ S_n, \mid i \in \omega \}) \) whenever \( \{ S'_n, \mid n \in \omega \} \) is a sequence of proper subscaffolds of \( S_n \)'s.

The lemma and the corollary that follow express the idea that one can only raise the height by following a convergent sequence in a scaffold.

**Lemma 11.** Let \((S, S)\) be a scaffold in some space \(X\). Let \(C = \{c_n, \mid n \in \omega \} \subseteq S \) be a converging sequence in \(S\) such that \(c_n \rightarrow c \in S\) and \(c_n \in T_n \subseteq S\) where \(T_n \)'s are disjoint. Then \(c = \text{cor} \, T\) for some \(T \subseteq S\) and \(T\) contains all but finitely many \(T_n \)'s.

**Corollary 1.** Let \((S, S)\) and \(\{ (S_n, S_n) \mid n \in \omega \}\) be such that \(T = S \otimes \{ S_n, \mid n \in \omega \}\) is well-defined. Let \(C = \{c_i, \mid i \in \omega \} \subseteq S \otimes \{ S_n, \mid n \in \omega \}\) be a converging sequence such that \(c_i \in C_n \otimes S_n\) (here we reuse the notation from Definition 7) for increasing \(n_i \)'s and \(c_i \rightarrow c \in T\). Then \(c = s \cdot x\) where \(x\) is the same as in Definition 7 and \(\text{ht}_S(s) \geq 2\).

Recall that \(b\) is the smallest cardinality of an unbounded family in \(\omega^\omega\). The lemma and the corollary below are probably folklore, however, the author could not find a reference for the general form of this fact. For a proof of a similar statement about 2-scaffolds, see, for example [13].

**Lemma 12.** Let \((S, S)\) be an \(\alpha\)-scaffold, \(U\) be a collection of open subsets of \(X\) such that \(|U| < b\). Then there exists an \(S' \leq S\) such that every open neighborhood \(U \subseteq U\) of \(\text{cor} \, S'\) contains all but finitely many \(T \in S'\), where \(S'\) is the stratification of \(S'\) that witnesses \(S' \leq S\).

**Proof.** Suppose the statement is true for all \(\beta\)-scaffolds with \(\beta < \alpha\). Let \(S^- = \{ S_n, \mid n \in \omega \}\) and \(S|_{S_n} = \{ S_m, \mid m \in \omega \}\).

Modify \((S_n, S|_{S_n})\) if necessary using the inductive hypothesis, and for each \(U \in U\) construct a function \(f_U : \omega \rightarrow \omega\) such that \(\text{cor} \, S_n \subseteq U\) implies \(S^m_n \subseteq U\) for \(m > f_U(n)\). Let \(f : \omega \rightarrow \omega\) be a function that dominates \(\{ f_U, \mid U \in U\}\). Put \(S' = \bigcup \{ S|_{S^m_n} \mid m > f(n)\} \cup \{ S_n, \mid n \in \omega \}\cup S'\) where \(S^m_n = \bigcup \{ S^m_n \mid m > f(n)\} \cup \text{cor} \, S_n\) and \(S' = \bigcup S^m_n \cup \text{cor} \, S\). It is straightforward to check that \(S'\) is a stratification of \(S'\) that witnesses \(S' \leq S\) and that \(S' = \{ S'_n, \mid n \in \omega \}\).

Let now \(\text{cor} \, S \in U \subseteq U\) and pick an \(n' \in \omega\) be such that \(f(n) > f_U(n)\) and \(\text{cor} \, S_n \subseteq U\) for all \(n > n'\). If \(n > n'\) it follows from the definition of \(S^m_n\) that \(S^m_n \subseteq U\).
Corollary 2. Let \( Y \) be a regular space, \( X \subseteq Y \) be such that \( \psi(X) < b \), \((S, S)\) be an \( \alpha\)-scaffold. Then there exists an \( S' \leq_S S \) closed in \( X \).

Proof. Suppose the statement is true for \( \beta\)-scaffolds such that \( \beta < \alpha \), so assume that \( S \) and its stratification \( S \) are such that each element of \( S^- \) is closed in \( X \). Note that a proper subscaffold of a scaffold closed in \( X \) is also closed in \( X \).

Using the regularity of \( Y \) and \( \psi(X, \text{cor } S) < b \) pick a family \( \mathcal{U} \) of open subsets of \( Y \) such that \( |\mathcal{U}| < b \), \( \text{cor } S \in U \) for every \( U \in \mathcal{U} \) and \( \bigcap \{ U^Y \mid U \in \mathcal{U} \} \cap X = \{ \text{cor } S \} \). Apply Lemma 12 to construct \( S' \leq_S S \) and \( S' \).

Suppose \( S' \ni x \in X \setminus S' \) for some \( x \in X \). Let \( U \in \mathcal{U} \) be such that \( \text{cor } S \in U \subseteq \overline{U} \subseteq Y \setminus \{ x \} \). Since all but finitely many elements of \( S^- \) are subsets of \( U \), \( x \in \overline{T} \) for some \( T' \in S^- \). Now \( T' \subseteq T \) for some \( T \in S^- \). This (and \( S' \leq_S S \)) implies that \( T' \) is closed in \( X \) so \( x \in T' \subseteq S' \) contrary to the assumption above. \( \square \)

Lemma 13. Let \( S \) be an \( \alpha\)-scaffold and let \( \mathcal{A} \) be a cover of \( S_{[0]} \). Then either there is an \( A_S \in \mathcal{A} \) such that \( A_S \cap \text{cor } S \ni \text{cor } S \) or there exists a countable family \( \mathcal{A}_S \subseteq [\mathcal{A}]^{\omega} \) such that any \( \mathcal{A}' \subseteq \mathcal{A} \) with the property \( |A' \cap \mathcal{A}| = \omega \) for every \( A'' \in \mathcal{A}_S \) satisfies \( \bigcup \{ \text{cor } S \cap B \mid B \in A' \} \ni \text{cor } S \).

Proof. Suppose the statement is true for all \( \beta\)-scaffolds such that \( \beta < \alpha \). For each \( T \in S^- \) pick either an \( A_T \in \mathcal{A} \) such that \( \text{cor } T \in \overline{T_{[0]} \cap A_T} \) or a countable family \( \mathcal{A}_T \subseteq [\mathcal{A}]^{\omega} \) such that any \( \mathcal{A}' \subseteq \mathcal{A} \) with the property \( |A' \cap \mathcal{A}| = \omega \) for every \( A'' \in \mathcal{A}_T \) satisfies \( \bigcup \{ \text{cor } T \cap B \mid B \in A' \} \ni \text{cor } T \).

If there is a single \( B \) such that \( B = A_T \) for infinitely many \( T \)'s as above, put \( A_S = A_T \). If there are infinitely many different \( A_T \)'s with the property above put \( A_S = \{ \{ A_T \mid T \in S^- \text{ and } A_T \text{ is defined as above } \} \}. Otherwise put \( A_S = \bigcup \{ A_T \mid T \in S^- \text{ and } A_T \text{ is defined as above } \} \). \( \square \)

Recall that a continuous map \( p : X \rightarrow Y \) is called hereditarily quotient if \( p \) is quotient (i.e. \( U \subseteq X \) is open if and only if \( p^{-1}(U) \) is open) and for any \( A \subseteq X \), the restriction \( p|_{p^{-1}(A)} : p^{-1}(A) \rightarrow A \) is also quotient. It is straightforward that every open map is hereditarily quotient. The following lemma presents a well known property of such maps.

Lemma 14 (9). Let \( p : X \rightarrow Y \) be a hereditarily quotient map. Then \( \text{so}(Y) \leq \text{so}(X) \).
Since the natural quotient map from a group onto its quotient is open, we have that the sequential order of a group cannot be raised by taking a quotient. Note that for groups there is a more direct proof of this result, by ‘lifting’ every scaffold in $G/N$ to $G$.

**Corollary 3.** Let $G$ be a sequential topological group, $N \subseteq G$ be a closed normal subgroup of $G$. Then $so(G/N) \leq so(G)$.

**Lemma 15.** Let $G$ be a sequential topological group, $so(G) < \omega_1$, and every scaffold in $G$ have a closed proper subscaffold. Let $Y \subseteq G$ be a subgroup, $X = U \cap Y$ for some open $U \subseteq G$. Suppose for every semiloose $\beta$-scaffold $S \subseteq G$, where $\beta \leq so(G)$ there exists a semiloose $\beta$-scaffold $S'$ in $G$ such that $S' \subseteq Y$. Then for every $g \in X$ there exists a sequence of points of $X$ converging to $g$.

**Proof.** Let $g \in G$ be arbitrary. Since $so(G) < \omega_1$ and $g \in X$, there exists a scaffold $(S, \mathcal{S})$ such that $S \subseteq G$, $S[0] \subseteq X$, and $g = \text{cor} S$. Define $h(g)$ to be the smallest $\text{ht}(S)$ among all such scaffolds. Let $(S, \mathcal{S})$ be a scaffold witnessing $\text{ht}(S) = h(g)$. Using induction, replacing parts of $S$, and going to proper subscaffolds, if necessary, we can assume that $\text{ht}_S(q) = h(q)$ for every $q \in S$. Suppose $h(g) = \text{ht}(S) > 1$.

Consider a limit $\alpha = so(G)$ (the nonlimit case is similar) and choose a countable family $\{ S_n \mid n \in \omega \}$ of semiloose $\alpha_n$-scaffolds such that $S_n \subseteq Y$, $\text{cor} S_n = 1_G$ and $\alpha_n = \text{ht}(S_n)$ is increasing so that $\lim \alpha_n = \alpha$. Pick a closed proper subscaffold $T \subseteq S \otimes \{ S_n \mid n \in \omega \}$ and let $A = T[0]$. Since $U$ is open in $G$ and $S[0] \subseteq U$, we may assume $A \subseteq U$. Since $Y$ is a group, $A \subseteq Y$ so $A \subseteq X$. In addition, $g \in A$ and $T \setminus X = S \setminus X = \{ s \in S \mid \text{ht}_S(s) \geq 1 \}$.

Let $\beta$ be the smallest ordinal such that $t \in [A]_\beta$ for some $t \in T \setminus X$. Since there is a sequence of points of $X$ converging to $t$, $\text{ht}_S(t) = 1$ by the choice of $S$. Let $C = \langle c_i \mid i \in \omega \rangle$ be such a sequence. If $c_i \in C_{n_i} \otimes S'_{n_i}$ for an increasing $\langle n_i \mid i \in \omega \rangle$ (here we borrow the terminology of Definition 7) then by Corollary 1 $\text{ht}_S(t) > 1$ contrary to the choice of $S$. Thus we may assume that $C \subseteq C_n \otimes S'_{n}$ for some $n \in \omega$.

A similar argument and induction on $\text{ht}_{C_n \otimes S'_{n}}(c)$ shows that $t \in [A \cap C_n \otimes S'_{n}]_\beta$. Now, the condition that each $S'_n$ (therefore, each $C_n \otimes S'_n$ by Lemma 9) is semiloose implies that $\beta \geq \alpha_n$.

Let $\gamma < \alpha$. The argument above shows that (due to $\langle \alpha_n \mid n \in \omega \rangle$ increasing) $[A]_\gamma \setminus X$ is finite. Thus $g \notin [A]_\gamma$ (otherwise $\text{ht}_S(g) = 1$) contradicting $so(G) = \alpha$. 

\[ \square \]
The aim of the next lemma is to establish Property (C) for some sequential groups.

**Lemma 16.** Let $G$ be a sequential topological group, $\text{so}(G) < \omega_1$, and every scaffold in $G$ has a closed proper subscaffold. Let $X \subseteq G$ be a dense subgroup. Suppose for every semiloose $\beta$-scaffold $S \subseteq G$, where $\beta \leq \text{so}(G)$ there exists a semiloose $\beta$-scaffold $S' \in G$ such that $S' \subseteq X$. Let $\{ N_i \mid i \in \omega \} \subseteq 2^G$ be a family of nowhere dense subsets of $G$. Then there exists a nontrivial $C \subseteq X$ such that $C \rightarrow 1_G$, and $C \cap N_i$ is finite for every $i \in \omega$.

**Proof.** We can assume that each $N_i$ is closed in $G$. Suppose there is no nontrivial convergent sequence $C \subseteq X$, $C \rightarrow 1_G$ such that each $C \cap N_i$ is finite.

Suppose first that there is no such $C \subseteq G$. Pick a nontrivial $C' = \{ c_n \mid n \in \omega \} \rightarrow 1_G$, put $Y = G$, $U = G \setminus \bigcup \{ N_i \mid i \leq n \}$. Now $c_n \in U$, so one can apply Lemma 15 to find a converging sequence $C_n \rightarrow c_n$ such that $C_n \setminus \{ c_n \} \subseteq G \setminus \bigcup \{ N_i \mid i \leq n \}$. Thinning out the resulting set, if necessary, we can assume that $T = \bigcup \{ C_n \mid n \in \omega \} \cup \{ 1_G \}$ is a closed 2-scaffold.

Suppose $\text{so}(G)$ is a limit ordinal (the nonlimit case is essentially the same). Let now $\{ S_n \mid n \in \omega \}$ be such that each $S_n \subseteq G$ is a closed semiloose $\alpha_n$-scaffold where $\alpha_n \rightarrow \text{so}(G)$ is increasing, and $\text{cor} S_n = 1_G$. Passing to proper subscaffolds if necessary, assume that $S = T \otimes \{ S_n \mid n \in \omega \}$ is defined and is a closed scaffold, and for each $n \in \omega$, the set $C_n \otimes S_n \subseteq G \setminus \bigcup \{ N_i \mid i \leq n \}$. The last property and the assumption at the beginning of the previous paragraph imply that every sequence of points in $S$ converging to $1_G$ contains an infinite subsequence of $C'$. This and the closedness and semilooseness of $C_n \otimes S_n$ show that $1_G \notin [T_{\beta}]_\alpha$ whenever $\beta < \alpha_n$ for some $n \in \omega$ contradicting $\alpha_n \rightarrow \text{so}(G)$.

Therefore, we can pick a nontrivial $C' = \{ c_n \mid n \in \omega \} \rightarrow 1_G$ such that $c_n \in G \setminus \bigcup \{ N_i \mid i \leq n \}$ for every $n \in \omega$. Putting $Y = X$, $U = G \setminus \bigcup \{ N_i \mid i \leq n \}$, and applying Lemma 15 once again we can find a convergent sequence $C_n \rightarrow c_n$ so that $C_n \setminus \{ c_n \} \subseteq X \setminus \bigcup \{ N_i \mid i \leq n \}$. Now the argument in the preceding two paragraphs can be repeated to produce the desired sequence. 

Intuitively, the iteration argument is set up to eliminate the unwanted groups in the extension by destroying the appropriate witnesses in the intermediate stages. It is thus important to keep the size of the witness small (countable). In the case of Fréchet groups dealt with in [8], the groups are countable already. In the case of general sequential groups, the size of the
group must be reduced first. The following definition formalizes one obstacle to such a reduction.

**Definition 8.** Let $H$ be a topological group. Call $H$ $\omega_1$-collapsible if for every closed normal subgroup $N$ of $H$, $\psi(H/N) \leq \omega_1$ implies $H/N$ is metrizable.

**Lemma 17.** Let $H$ be a separable $\omega_1$-collapsible topological group. Then every subspace of $H$ of size at most $\omega_1$ is Lindelöf.

**Proof.** Let $Y \subseteq H$ be an arbitrary subset of $H$ of size at most $\omega_1$. Let $U = \{U_\alpha \mid \alpha \in \omega_1\}$ be a cover of $Y$ by open subsets of $H$. For every $y \in Y$ pick an open subset $1_H \in V_y$ of $H$ so that $y \cdot V_y \cdot V_y \subseteq U_\alpha$ for some $\alpha \in \omega_1$. Select a countable dense subgroup $X \subseteq H$. Construct a family $V = \{V^\alpha \mid \alpha \in \omega_1\}$ of open neighborhoods of $1_H$ so that $V_y \in V$ for every $y \in Y$ and for each $V \in V$ and $x \in X$ there is a $W \in V$, $W \cdot W^{-1} \subseteq V$, $x \cdot W \cdot x^{-1} \subseteq V$. Put $N = \bigcap V$. Now $N$ is a closed subgroup of $H$ that commutes with every $x \in X$. This, the assumption that $X$ is dense, and the continuity of $c_a : H \to H$, where $c_a(x) = x \cdot a \cdot x^{-1}$, in conjunction with the closedness of $N$ imply the (algebraic) normality of $N$.

Let $p : H \to H/N$ be the corresponding quotient map. It follows from the definition of $\omega_1$-collapsible and the construction of $N$ that $H/N$ is separable metric. The sets $p(V_y \cdot N)$, $y \in Y$ form an open cover of $p(Y)$ which has a countable subcover $\{p(V_{y_i} \cdot N) \mid i \in \omega\}$. It is straightforward to see that $\{V_{y_i} \cdot N \mid i \in \omega\}$ is a countable open refinement of $U$. \qed

The following lemma will be used to pick small witnesses in some cases. It is stated for proper forcing notions for the sake of generality. Its applications in this paper are limited to ccc notions of forcing only.

**Lemma 18.** Let $V$ be a model of CH, $\mathbb{P} \in V$ be a proper notion of forcing, and $G$ be $\mathbb{P}$-generic over $V$. Let $H \in V[G]$ be an $\omega_1$-collapsible topological group and $X, G \in V$ satisfy the following properties. $X \subseteq G$, $G$ is a topological group algebraically isomorphic to a subgroup of $H$ (below we treat $G$ as if it were an actual subgroup of $H$). $X$ is countable subgroup, dense in both $G$ and $H$. Furthermore, for every subset $A \subseteq X$, $A \in V$ the closures $\overline{A^G} = \overline{A^H} \cap G$. Then $G$ is hereditarily Lindelöf.

**Proof.** Let $Y \subseteq G$ be any subspace and let $U$ be any open (in $G$) cover of $Y$. Using CH, the regularity of $G$, and the density of $X$ in $G$, we may assume,
after refining \( \mathcal{U} \), if necessary, that \( \mathcal{U} = \{ \Omega_\alpha = G \setminus X \setminus U_\alpha \mid \alpha \in \omega_1 \} \) for some \( U_\alpha \subseteq X \) open in \( X \).

The assumptions about \( H \) and \( G \) imply that in \( V[\mathcal{G}] \) each \( \Omega_\alpha \cap Y \) is open in \( Y \) as a subspace of \( H \). Lemma 17 implies that there exists a countable (in \( V[\mathcal{G}] \)) subcover \( \mathcal{U}' \subseteq \mathcal{U} \) of \( Y \). Using the property of proper forcings that countable sets of ordinals in the extension are subsets of countable sets of ordinals in the ground model (see [11], Lemma 8.7, for example) and the properness of \( \mathbb{P} \) concludes the proof. \( \square \)

5 Main theorem

The next lemma is a generalization of Proposition 5.3 (b) from [8]. The original preservation lemma was stated for Fréchet spaces and could not be reused due to the lack of an appropriate version of Lemma 5.1([8]) for the general case needed here (see the discussion at the beginning of section 4).

Lemma 19. Let \( X \subseteq G \), where \( G \) is sequential and \( X \) has no isolated points as a subspace of \( G \). Then \( L_{\text{wtd}^*}(X) \) strongly preserves \( \omega \)-hitting.

Proof. We reuse some of the original notation of Proposition 5.3 (b) in [8]. Let \( Y \in \text{wtd}(X)^+ \) and suppose there is an \( \omega \)-hitting ideal \( \mathcal{J} \leq \mathcal{K} \text{wtd}(X) \mid Y \) witnessed by \( f : Y \rightarrow \omega \). Put \( U = \text{Int}(Y) \neq \emptyset \) and \( Z = U \cap Y \).

Put \( N_i = f^{-1}(i) \in \text{wtd}(X) \), \( \mathcal{A} = \{ N_i \mid i \in \omega \} \). For each \( x \in Z \) pick a scaffold \( S^x \subseteq G \) such that \( \text{cor} S^x = x \), \( S^x_n = \{ S^x_n \mid n \in \omega \} \) for some stratification \( S^x \) of \( S^x \) and \( (S^x_n)[0] \subseteq Z \setminus \bigcup \{ N_i \mid i \leq n \} \). Note that \( \mathcal{A} \) is a cover of \( S^x_0 \) so by the choice of \( S^x_n \), Lemma 13 implies the existence of a countable \( \mathcal{A}_x \subseteq [\mathcal{A}]^\omega \) such that whenever \( \mathcal{A}' \subseteq \mathcal{A} \) satisfies \( |\mathcal{A}' \cap \mathcal{A}''| = \omega \) for every \( \mathcal{A}'' \in \mathcal{A}_x \), the point \( x = \text{cor} S^x \in \bigcup \{ S^x_n \cap B \mid B \in \mathcal{A}' \} \).

Since \( \mathcal{J} \) is \( \omega \)-hitting there is a \( J \in \mathcal{J} \) such that \( J \cap f(\bigcup \mathcal{A}'') \) is infinite for every \( x \in Z, \mathcal{A}'' \in \mathcal{A}_x \). This implies \( f^{-1}(J) \) is dense in \( Z \) which is dense in \( U \), a contradiction. \( \square \)

The next definition provides a description of a potential witness of a sequential group with an intermediate sequential order in the final extension.

Definition 9. Let \( X \) be a non-metrizable topological group defined on \( \omega \), \( \overline{\mathcal{S}} \subseteq [[X]^\omega]^\omega \). Call \( (X, \overline{\mathcal{S}}) \) a consequential pair if \( X \) can be embedded as a subgroup in a sequential group \( G \) such that \( \text{so}(G) < \omega_1 \), for every semiloose
\(\beta\)-scaffold in \(G\) there exists a semiloose \(\beta\)-scaffold in \(X\), and \(\mathcal{S}\) lists every \(\mathcal{T}\) that can be represented as \(\mathcal{T} = \{ S \cap X \mid S \in \mathcal{S} \}\) for some scaffold \((S, \mathcal{S})\) in \(G\). If, in addition to the properties above, every scaffold in \(G\) has a proper closed subscaffold, we will call \((X, \mathcal{S})\) a strong consequential pair.

Note that such a \(G\) is unique up to an isomorphism, since the second element of the pair uniquely determines both the algebraic structure, as well as the topology of \(G\). We will say that \(X\) extends to \(G\).

**Theorem 2.** It is consistent with ZFC that every sequential group of sequential order \(< \omega_1\) is Fréchet and that every countable Fréchet group is metrizable.

**Proof.** Let the ground model \(V \models \text{CH}\), and suppose \(\{ A_\alpha \mid \alpha \in S_1^2 \}\) witnesses \(\Diamond(S_1^2)\) in \(V\). Construct a finite support iteration \(P_{\omega_2} = \langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle\) so that whenever \(\alpha \in S_1^2\) and \(A_\alpha\) codes a \(P_\alpha\)-name for a strong consequential pair, \(\dot{Q}_\alpha\) is a \(P_\alpha\)-name for \(L_{\text{nwd}^*(r)}\). Otherwise, let \(\dot{Q}_\alpha\) be a \(P_\alpha\)-name for \(L_{\text{nwd}^*(q)}\). Let \(G_{\omega_2}\) be \(P_{\omega_2}\)-generic over \(V\).

Assume that in \(V[G_{\omega_2}]\) there is a sequential group \(H\) such that \(1 < \text{so}(H) < \omega_1\) or \(H\) is countable non-metrizable and Fréchet. The Fréchet case is handled almost identically to \([8]\) so here we only consider the sequential groups of intermediate sequential orders. In this case there exists a separable \(H\) as above.

Suppose first that in \(V[G_{\omega_2}]\) there exists a closed normal subgroup \(N \subseteq H\) such that \(\psi(H/N) \leq \omega_1\) and \(H/N\) is not metrizable. By Corollary \([3]\), \(\text{so}(H/N) < \omega_1\). If \(\text{so}(H/N) = 1\), since \(H\) is separable, there is a non-metrizable countable Fréchet group in \(V[G_{\omega_2}]\). Thus we only consider the case of \(1 < \text{so}(H/N) < \omega_1\). As pointed out in \([8]\), \(b = \omega_2\) in \(V[G_{\omega_2}]\) so \(\psi(H/N) \leq \omega_1 < b\). Corollary \([2]\) implies that every scaffold in \(H/N\) has a proper closed subscaffold. Pick a countable dense subgroup \(X\) of \(H/N\) such that \(X\) contains a semiloose \(\beta\)-scaffold for every semiloose \(\beta\)-scaffold in \(H/N\) (this only requires adding countably many witnesses to \(X\)). Assume \(X = \omega\) and put

\[\mathfrak{S} = \{ \mathcal{S} \mid X \mid (S, \mathcal{S})\text{ is a scaffold in }H/N\text{ for some }S \subseteq H/N\} .\]

Now a standard argument implies the existence of a club \(C \subseteq S_1^2\) relative to \(S_1^2\) such that for all \(\alpha \in C\), \(V[G_\alpha] \models (X, \mathfrak{S}_\alpha)\) is a strong consequential pair, where \(\mathfrak{S}_\alpha = \mathfrak{S} \cap V[G_\alpha]\) and \(X\) extends to a group of intermediate sequential order.
Alternatively, suppose for every closed normal $N \subseteq H$ such that $\psi(H/N) \leq \omega_1$ the group $H/N$ is metrizable, i.e. $H$ is $\omega_1$-collapsible. Just as above, pick a countable dense subgroup $X$ of $H$ such that $X$ contains a semiloose $\beta$-scaffold for every semiloose $\beta$-scaffold in $H$ and assume $X = \omega$. Put

$$\mathfrak{S} = \{ S \mid X \mid (S, S) \text{ is a scaffold in } H \text{ for some } S \subseteq H \}$$

and

$$\mathfrak{T} = \{ (A, S) \mid A \subseteq X, (S, S) \text{ is a scaffold such that } S \upharpoonright X \in \mathfrak{S}, \mathfrak{A} \ni \text{cor } S \}$$

(this $\mathfrak{T}$ codes the closures of subsets of $X$). As before, conclude that there exists a club $C \subseteq S_2^1$ relative to $S_2^1$ such that for all $\alpha \in C$, $V[G_\alpha] \models X \subseteq G$, $(X, \mathfrak{S}_\alpha)$ is a consequential pair, and

$$\mathfrak{C}_\alpha = \{ (A, S) \mid A \subseteq X, (S, S) \text{ is a scaffold such that } S \upharpoonright X \in \mathfrak{S}, A^G \ni \text{cor } S \}$$

where $\mathfrak{S}_\alpha = \mathfrak{S} \cap V[G_\alpha]$ and $\mathfrak{C}_\alpha = \mathfrak{T} \cap V[G_\alpha]$. Embed $X$ in $G$ as in Definition 9. Since the group operation and the closures of subsets of $X$ in $G$ are ‘coded’ by $\mathfrak{S}_\alpha$ and $\mathfrak{C}_\alpha$, the properties above together with those of $H$, $V[G_\alpha]$, and $P_{\omega_2}$ imply that the conditions of Lemma 15 are satisfied and $G$ is hereditarily Lindelöf and, therefore, $T_5$. Now Theorem 1 implies that $G$ has a countable pseudocharacter. Applying Lemma 2 shows that $V[G_\alpha] \models (X, \mathfrak{S}_\alpha)$ is a strong consequential pair.

The choice of $C$ implies that if $\tau$ is the topology of $X$ inherited from $H$ in $V[G_{\omega_2}]$ then for any $\alpha \in C$, $\tau_\alpha = \tau \cap V[G_\alpha]$ where $\tau_\alpha$ is the topology on $X$ ‘induced’ by $\mathfrak{S}_\alpha$.

According to Proposition 2 at some stage $\alpha \in C$ a set $A_{gen}$ would have been added such that $V[G_{\alpha+1}] \models A_{gen} \in \mathcal{I}^+_1(\tau_\alpha)$ and the ideal $\mathcal{I}^+_1(\tau_\alpha) \upharpoonright A_{gen}$ is $\omega$-hitting. Suppose there exists an open neighborhood $U$ of $1_G$ in $X$ such that $U \cdot U \cap A_{gen} = \emptyset$. Then, just as in 8, the set $A = X \setminus U$ is $A_{gen}$-large. Lemma 16 supplies the conditions necessary for the conclusion of Lemma 2 to hold. Thus in $V[G_{\alpha+1}]$ the ideal $\mathcal{I}^+_1(\tau_\alpha)$ is $\omega$-hitting w.r.t. $A_{gen}$ so it follows from Lemmas 3 and 4 that $\mathcal{I}^+_1(\tau_\alpha)$ is $\omega$-hitting w.r.t. $A_{gen}$ in $V[G_{\omega_2}]$. Hence there is a $C_1 \in [A]^{\omega_2}$ such that $C_1$ converges to $1_G$ in $\tau_\alpha$. Now $C_1 \in S_\alpha \subseteq \mathfrak{S}_\alpha$ so $C_1$ is a subsequence of $A$ that converges to $1_G$ in $\tau$ contradicting $A \cap U = \emptyset$.

Since $H$ is sequential in $V[G_{\omega_2}]$ there exists a scaffold $(S, S)$ in $H$ such that $S_{[0]} \subseteq A$ and cor $S = 1_G$. By Lemma 19 and Lemma 1 the ideal $\mathcal{I}^+_1(\tau_\alpha) \upharpoonright A_{gen}$ is $\omega$-hitting in $V[G_{\omega_2}]$ so there exists an $I \in \mathcal{I}^+_1(\tau_\alpha)$ such that $I \cap C_1$ is infinite for every $C_1 \in S_{[0]}$ contradicting cor $S = 1_G \in \mathfrak{S}_{[0]}$. □
6 Remarks and open questions

The results about topological groups with various convergence properties obtained so far indicate some important implications set theoretic combinatorics has concerning the existence of such groups. Much less is known about more subtle interactions between convergence and group-theoretic properties of the space. It seems worthwhile to repeat a question asked in [8] here:

**Question 2.** Is it consistent with ZFC that some countable topologizable group admits a non-metrizable Fréchet group topology while another does not?

The next question is a recast of Question 2 to sequential groups although it is open for uncountable groups as well. Using the techniques of [18] it is possible to construct sequential group topologies with intermediate sequential orders on any countable topologizable group using CH.

**Question 3.** Is it consistent with ZFC that some (countable) topologizable group admits a sequential topology with intermediate sequential order while another does not?

The following two questions do not have any counterparts for Fréchet groups.

**Question 4.** Is it consistent with ZFC that groups of intermediate sequential order $\alpha$ exist for some $\alpha \in \omega_1$ but not for all of them? Only finite $\alpha$? Only infinite ones?

A more specific version of Question 3 asks about the influence the size of a group has on its convergence properties.

**Question 5.** Is it consistent with ZFC that there is an uncountable group of intermediate sequential order but there is no countable such?

In [16] D. Shakhmatov repeats his question from 1990 (Question 7.5):

**Question 6.** Is a countably compact sequential group Fréchet-Urysohn?

Here is a short argument showing that the model constructed in this paper (or, indeed, Hrušák-Ramon-García’s original model) provides a consistent positive answer to Question 6.
Lemma 20. Let $H$ be a countably compact sequential non Fréchet group in $V[G_{\omega_1}]$. Then $H$ is $\omega_1$-collapsible.

Proof. By picking a countable subgroup containing an appropriate witness we can assume that $H$ is separable. If $H$ is not $\omega_1$-collapsible, there exists a closed normal subgroup $N \subseteq H$ such that $\psi(H/N) = \omega_1 < b$ and $H/N$ is separable, sequential and not metrizable. Since there are no countable nonmetrizable Fréchet groups in $V[G_{\omega_1}]$, the group $H/N$ is not Fréchet. Thus $H/N$ contains a semiloose 2-scaffold. Corollary 2 implies that $H/N$ contains a closed semiloose 2-scaffold. But every closed semiloose 2-scaffold contains a closed copy of an infinite discrete space contradicting countable compactness of $H/N$. 

Another standard argument shows that there is a club $C \subset S^2_1$ relative to $S^2_1$ such that for every $\alpha \in C$ the model $V[G_\alpha]$ contains $X \subseteq G$ as in Lemma 18 such that $G$ is sequential, non Fréchet, and countably compact. Lemma 18 together with Theorem 1 imply that such $G$ has a countable pseudocharacter. This leads to a contradiction, just as in the proof of the lemma above.

To provide some motivation for our final question, recall that a (countable) space $X$ is analytic if its topology (viewed as a set of characteristic functions of open subsets of $X$ in $2^X$ endowed with the standard product topology) is a continuous image of the irrationals. In [21] S. Todorčević and C. Uzcátegui ask the following questions.

Question 7. What are the possible sequential orders of analytic sequential groups?

Question 8. Is there an uncountable family of pairwise nonhomeomorphic analytic sequential spaces of sequential order $\omega_1$?

The results of this paper show that it is consistent with ZFC that the only possible sequential orders of analytic sequential groups are 1 and $\omega_1$ (a free topological group over a convergent sequence is analytic and has a sequential order of $\omega_1$). Admittedly, such a consistent answer is somewhat contrary to the spirit of viewing the results about analytic spaces as ‘effective’ versions of their general counterparts but it does indicate the only possibility for a ‘true’ ZFC answer.

To answer recall that a space is $k_\omega$ if it is a quotient image of a countable sum of compact spaces. A direct construction immediately shows that any
countable $k_\omega$ (or indeed, any countable space whose topology is dominated by countably many first countable subspaces) is analytic (indeed, Borel). Now [15] shows that there are exactly $\omega_1$ nonhomeomorphic $k_\omega$ countable group topologies and [20] proves that all such topologies (other than the discrete) have sequential order $\omega_1$. Is this the best possible result in this direction (at least for group topologies)?

As the results above indicate, the model constructed in this paper tends to trivialize the structure of sequential groups. The final question is about the classification of countable such groups. As the answer to Question 8 shows, it has some relevance to the general structure of analytic sequential groups.

**Question 9.** Is it consistent with ZFC that all countable sequential groups are $k_\omega$ or metrizable? Analytic?

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