THE CONVEX REAL PROJECTIVE ORBIFOLDS WITH RADIAL OR TOTALLY GEODESIC ENDS: A SURVEY OF SOME PARTIAL RESULTS

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Abstract. A real projective orbifold has a radial end if a neighborhood of the end is foliated by projective geodesics that develop into geodesics ending at a common point. It has a totally geodesic end if the end can be completed to have the totally geodesic boundary.

We will prove a homeomorphism between the deformation space of convex real projective structures on an orbifold $O$ with radial or totally geodesic ends with various conditions with the union of open subspaces of strata of the subset $\text{Hom}_E(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))/\text{PGL}(n + 1, \mathbb{R})$ of the PGL$(n + 1, \mathbb{R})$-character variety for $\pi_1(O)$ given by corresponding end conditions for holonomy representations.

Lastly, we will talk about the openness and closedness of the properly (resp. strictly) convex real projective structures on a class of orbifold with generalized admissible ends.

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1. Introduction

1.1. Preliminary. Let $H$ be a closed upper-half space $\{x \in \mathbb{R}^n | x_n \geq 0\}$ with boundary $\partial H = \{x \in \mathbb{R}^n | x_n = 0\}$. An orbifold $O$ is a second countable Hausdorff space where each point $x$ has a neighborhood with a chart $(U, G, \phi)$ consisting of

- a finite group $G$ acting on $U$ an open subset of $H$,
- $\phi : U \rightarrow \phi(U)$ inducing a homeomorphism $U/G \rightarrow \phi(U)$ to a neighborhood $\phi(U)$ of $x$.

Also, these charts are compatible in some obvious sense as explained by Satake and Thurston. Such a triple $(U, G, \phi)$ is called a model of a neighborhood of $O$. The orbifold boundary $\partial O$ is the set of points with only models of form $(U, G, \phi)$ where $U$ meets $\partial H$. A closed orbifold is a compact orbifold with empty boundary. The orbifolds in this paper are the quotient space of a manifold under the action of a finite group. (See Chapter 13 of Thurston [57] or more modern Moerdijk [53].)
We will study properly convex real projective structures on such orbifolds. A properly convex real projective orbifold is the quotient $\Omega/\Gamma$ of a properly convex domain $\Omega$ in an affine space in $\mathbb{RP}^n$ by a group $\Gamma$, $\Gamma \subset \text{PGL}(n+1, \mathbb{R})$, of projective automorphisms acting on $\Omega$ properly but maybe not freely. Let $\pi_1(O)$ denote the orbifold-fundamental group of $O = \Omega/\Gamma$, which is isomorphic to $\Gamma$. Given an orbifold $O$, a properly convex real projective structure on $O$ is a diffeomorphism $f : O \to \Omega/\Gamma$ for a properly convex real projective orbifold of form $\Omega/\Gamma$ as above.

Finite volume complete hyperbolic manifolds are examples since we can use the Klein model and identify $\Omega$ as the model space where $\Gamma$ is the hyperbolic isometry group, which acts projectively on $\Omega$. (See Example 2.5.)

For closed $n$-dimensional orbifolds, these structures are somewhat well studied by Benoist [7] generalizing the previous work for surfaces by Goldman [39] and Choi-Goldman [26]. The work of Cooper, Long, and Thistlethwaite [31] and [30] showed the existence of deformations for some hyperbolic 3-orbifolds that are explicitly computable. Currently, there seems to be more interest in this field due to these and other developments. (For a recent survey, see [29].)

A strongly tame orbifold is an orbifold that has a compact suborbifold whose complement is homeomorphic to a disjoint union of closed $(n-1)$-orbifolds times intervals. The theory is mostly applicable to strongly tame orbifolds that are not manifolds, and is most natural in this setting. In fact, the theory is mostly adopted for Coxeter orbifolds, i.e., orbifolds based on convex polytopes with faces silvered, and also, orbifolds that are doubles of these. (See Section 1.3.)

One central example to keep in mind is the tetrahedron with silvered sides and edge orders equal to 3. This orbifold admits a complete hyperbolic structure. Also, it admits deformations to convex real projective orbifolds. The deformation space of real projective structures is homeomorphic to a four cell. (See Choi [24] and Marquis [52].) We can also take the double of this orbifold. The deformation space is 5-dimensional and can be explicitly computed in [21]. (See Chapter 7 of [19].) Except for ones based on tetrahedra, complete hyperbolic Coxeter 3-orbifolds with all edge orders 3 have at least six dimensional deformation spaces by Theorem 1 of Choi-Hodgson-Lee [28].

Another well-known prior example is due to Tillmann: This is a complete hyperbolic orbifold on a complement of two-points $p,q$ in the 3-sphere where the singularities are two simple arcs connecting $p$ to itself and $q$ to itself forming a link of index 1 and another simple arc connecting $p$ and $q$. These arcs have $\mathbb{Z}_3$ as the local group. Heard, Hodgson, Martelli, and Petronio [44] labelled this orbifold $2h_{1,1}$. The dimension is computed to be 2 by Porti and Tillman [56]. (See Figure 1.)

For all these examples, we know that some horospherical ends deform to lens-type radial ones and vice versa. We can also obtain totally geodesic ends by “cutting off” some radial ends. We call the phenomenon “cusp opening”. Benoist [10] first found such phenomena for a Coxeter 3-orbifold. We also had the many numerical and theoretical results for above Coxeter 3-orbifolds which we plan to write more explicitly in a later paper [27]. Also, Greene [41] found many such examples using explicit computations. Recently, Ballas, Danciger, and Lee [4] found these phenomena using cohomology arguments for complete finite-volume hyperbolic 3-manifolds as explained in their MSRI talks in 2015. Some of the computations are available from the authors.
1.2. Main results. We concentrate on studying the ends that are well-behaved, i.e., ones that are foliated by lines or are totally geodesic. In this setting we wish to study the deformation spaces of the convex real projective structures on orbifolds with some boundary conditions using the character varieties. Our main aim is

- to identify the deformation space of convex real projective structures on an orbifold $O$ with certain boundary conditions with the union of some open subsets of strata of $\text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))/\text{PGL}(n+1, \mathbb{R})$ defined by conditions corresponding to the boundary conditions.

This is an example of the so-called Ehresmann-Thurston-Weil principle [63]. (See Canary-Epstein-Green [14], Goldman [38], Lok [51], Bergeron-Gelander [11], and Choi [23].) The precise statements are given in Theorems 6.7 and 6.8.

See Definition 5.1 for the condition (IE) and (NA). We use the notion of strict convexity with respect to ends as defined in Definition 5.3. Our main result is the following as a corollary of Theorem 6.12:

Corollary 1.1. Let $O$ be a noncompact strongly tame SPC $n$-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(O)$ are trivial. Then $\text{hol}$ maps the deformation space $C\text{Def}_{E,\text{uc}}(O)$ of SPC-structures on $O$ homeomorphically to a union of components of

$$\text{rep}_{E,\text{uc}}^\text{st}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).$$

The same can be said for $S\text{Def}_{E,\text{uc}}(O)$.

These terms will be defined more precisely later on in Sections 2.1.8 and 6.1.1. Roughly speaking, $C\text{Def}_{E}(O)$ (resp. $S\text{Def}_{E}(O)$) is the deformation spaces of properly convex (resp. strictly properly convex) real projective structures with conditions on ends that each end holonomy group fixes a point.

$$\text{rep}_E^\text{st}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$$

is the space of characters each of whose end holonomy group fixes a point.

$C\text{Def}_{E,\text{uc}}(O)$ (resp. $S\text{Def}_{E,\text{uc}}(O)$)
is the deformation space of properly convex (resp. strictly properly convex) real projective structures with conditions on ends that each end has a lens-cone neighborhood or a horospherical one, and each end holonomy group fixes a unique point.

\[ \text{rep}^\pi_{E,\text{ce}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \]

is the space of characters each of whose end holonomy group fixes a unique point and acts on a lens-cone or a horosphere.

Our main examples satisfy this condition: Suppose that a strongly tame properly convex 3-orbifold \( \mathcal{O} \) with radial ends admits a finite volume complete hyperbolic structure and has radial ends only and any end neighborhood contains singularities of dimension 1 of order 3 or 6, and end orbifolds have base spaces homeomorphic to disks or spheres. The theory simplifies by Corollary 6.6, i.e., each end is always of lens-type or horospherical, so that

\[ \text{SDef}_{E,\text{u,ce}}(\mathcal{O}) = \text{SDef}_{E}(\mathcal{O}). \]

Corollary 6.13 applies to these cases, and the space under hol maps homeomorphically to a union of components of

\[ \text{rep}^E(\pi_1(\mathcal{O}), \text{PGL}(4, \mathbb{R})). \]

For a strongly tame Coxeter orbifold \( \mathcal{O} \) of dimension \( n \geq 3 \) admitting a complete hyperbolic structure, hol is a homeomorphism from

\[ \text{SDef}_{E,\text{u,ce}}(\mathcal{O}) = \text{SDef}_{E}(\mathcal{O}) \]

to a union of components of

\[ \text{rep}^E(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \]

by Corollary 6.14. For this theory, we can consider a Coxeter orbifold based on a convex polytope admitting a complete hyperbolic structure with all edge orders equal to 3. More specifically, we can consider a hyperbolic ideal simplex or a hyperbolic ideal cube with such structures. (See Choi-Hodgson-Lee [28] for examples of 6-dimensional deformations.)

One question is whether we can remove the stability condition on the target character varieties. (For closed orbifolds, see Theorem 4.1 of [29] essentially following from Benoist [9].) We plan to prove this for many interesting orbifolds, such as strongly tame orbifolds.

1.3. Remarks. We give some remarks on our results here: The theory here is by no means exhaustive final words. We have a somewhat complicated theory of ends [15], [16], [17], [18], which is used in this paper. Instead of publishing some of these and [19]. We will try to refine and generalize the theory and put into a monograph [20] in a near future. Our boundary condition is very restrictive in some sense. With this it could be said that the above theory is not so surprising.

Ballas, Cooper, Leitner, Long, and Tillman have different restrictions on ends and they are working with manifolds. The associated end neighborhoods have nilpotent holonomy groups. (See [30], [33], [49], [50], [48], and [3]). They are currently developing the theory of ends and the deformation theory based on this assumption. Of course, we expect to benefit and thrive from many interactions between the theories as they happen in multitudes of fields.

Originally, we developed the theory for orbifolds as given in papers of Choi [24], Choi, Hodgson, and Lee [28], and [19]. However, the recent examples of Ballas
[1], [2], and Ballas, Danciger, and Lee [4] can be covered using fixing sections. Also, differently from the above work, we can allow ends with hyperbolic holonomy groups.

1.4. Outline. In Section 2, we will go over real projective structures. We will define end structures of convex real projective orbifolds. We first discuss totally geodesic ends and define lens condition for these totally geodesic ends. Then we define radial ends and radial foliation marking for radial ends. We define end orbifolds, and horospherical ends. We define the space of characters and the deformation spaces of convex real projective structures on orbifolds with radial or totally geodesic ends. Finally, we discuss the local homeomorphisms between the subsets of deformation spaces and those of character varieties. Here, we are not yet concerned with convexity.

In Section 3, we will discuss the known facts about the convex real projective orbifolds including the Vinberg duality result.

In Section 4, we will discuss the end theory. We introduce pseudo-ends and pseudo-end neighborhoods, and pseudo-end fundamental groups. We introduce admissible groups and admissible ends. We introduce the lens-conditions for radial ends and totally geodesic ends.

In Section 5, we will relate the relative hyperbolicity of the fundamental groups of strongly tame properly convex real projective orbifolds with the “relative” strict convexity of the real projective structures. (See Section 5.2.) In Section 5.1, we define the stable properly convex real projective orbifolds relative to ends. We will show that under mild conditions properly convex strongly tame real projective orbifolds with generalized admissible ends have stable holonomy groups.

In Section 6, we will state our main results. In Section 6.2, we will state that a holonomy homomorphism map from a deformation space with end conditions to the character variety is a homeomorphism to a union of open subsets of strata of the character variety. This map is injective. In Section 6.3, we will say about the closedness.

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2. Preliminary

2.1. Basic definitions.

2.1.1. Topological notation. Define \( \text{bd}A \) for a subset \( A \) of \( \mathbb{RP}^n \), \( n \geq 2 \), (resp. in \( \mathbb{S}^n \)) to be the topological boundary in \( \mathbb{RP}^n \) (resp. in \( \mathbb{S}^n \)) and define \( \partial A \) for a manifold or orbifold \( A \) to be the manifold or orbifold boundary and \( A^o \) denote the manifold interior. The closure \( \text{Cl}(A) \) of a subset \( A \) of \( \mathbb{RP}^n \) (resp. of \( \mathbb{S}^n \)) is the topological closure in \( \mathbb{RP}^n \) (resp. in \( \mathbb{S}^n \)).
2.1.2. The Hausdorff metric. Recall the standard elliptic metric $d$ on $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Given two sets $A$ and $B$ of $\mathbb{RP}^n$ (resp. of $\mathbb{S}^n$),
\[ d(A, B) := \inf \{d(x, y) | x \in A, y \in B \}. \]
We can let $A$ or $B$ be points as well obviously.

The Hausdorff distance between two convex subsets $K_1, K_2$ of $\mathbb{RP}^n$ (resp. of $\mathbb{S}^n$) is defined by
\[ d^H(K_1, K_2) = \inf \{ \epsilon \geq 0 \mid \text{Cl}(K_1) \subset N_\epsilon(\text{Cl}(K_2)), \text{Cl}(K_2) \subset N_\epsilon(\text{Cl}(K_1)) \} \]
where $N_\epsilon(A)$ is the $\epsilon$-neighborhood of $A$ under the standard metric $d$ of $\mathbb{RP}^n$ (resp. of $\mathbb{S}^n$) for $\epsilon > 0$. $d^H$ gives a compact Hausdorff topology on the set of all compact subsets of $\mathbb{RP}^n$ (resp. of $\mathbb{S}^n$). (See p. 281 of [55].)

We say that a sequence of sets $\{K_i\}$ geometrically converges to a set $K$ if $d^H(K_i, K) \to 0$. If $K$ is assumed to be closed, then the geometric limit is unique.

**Lemma 2.1.** Suppose that a sequence $\{K_i\}$ of compact convex domains geometrically converges to a compact convex domain $K$ in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Then
\[ d^H(\partial K_i, \partial K) \to 0. \]

2.1.3. Real projective structures. Given a vector space $V$, we let $\mathbb{P}(V)$ denote the space obtained by taking the quotient space of $V - \{O\}$ under the equivalence relation
\[ v \sim w \text{ for } v, w \in \mathbb{R}^{n+1} - \{O\} \text{ iff } v = sw, \text{ for } s \in \mathbb{R} - \{0\}. \]
We let $[v]$ denote the equivalence class of $v \in V - \{O\}$. Recall that the projective linear group $\text{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{RP}^n$, i.e., $\mathbb{P}(\mathbb{R}^{n+1})$, in a standard manner.

Let $\mathcal{O}$ be a noncompact strongly tame $n$-orbifold where the orbifold boundary is not necessarily empty.

- A real projective orbifold is an orbifold with a geometric structure modelled on $(\mathbb{RP}^n, \text{PGL}(n+1, \mathbb{R}))$. (See Thurston [58], [23] and Chapter 6 of [25].)
- $\mathcal{O}$ has a universal cover $\hat{\mathcal{O}}$ where the deck transformation group $\pi_1(\mathcal{O})$ acts on.
- The underlying space of $\mathcal{O}$ is homeomorphic to the quotient space $\hat{\mathcal{O}}/\pi_1(\mathcal{O})$.
- A real projective structure on $\mathcal{O}$ gives us a so-called development pair $(\text{dev}, h)$ where
  - $\text{dev} : \hat{\mathcal{O}} \to \mathbb{RP}^n$ is an immersion, called the developing map,
  - and $h : \pi_1(\mathcal{O}) \to \text{PGL}(n+1, \mathbb{R})$ is a homomorphism, called a holonomy homomorphism, satisfying
  \[ \text{dev} \circ \gamma = h(\gamma) \circ \text{dev} \text{ for } \gamma \in \pi_1(\mathcal{O}). \]

Let $\mathbb{R}^{n+1*}$ denote the dual of $\mathbb{R}^{n+1}$. Let $\mathbb{RP}^{n*}$ denote the dual projective space $\mathbb{P}(\mathbb{R}^{n+1*})$. $\text{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{RP}^{n*}$ by taking the inverse of the dual transformation. Then a representation $h : \pi_1(\mathcal{O}) \to \text{PGL}(n+1, \mathbb{R})$ has a dual representation $h^* : \pi_1(\mathcal{O}) \to \text{PGL}(n+1, \mathbb{R})$ sending elements of $\pi_1(\mathcal{O})$ to the inverse of the dual transformation of $\mathbb{R}^{n+1*}$.

The complement of a codimension-one subspace of $\mathbb{RP}^n$ can be identified with an affine space $\mathbb{R}^n$ where the geodesics are preserved. We call the complement an affine subspace. A convex domain in $\mathbb{RP}^n$ is a convex subset of an affine subspace. A properly convex domain in $\mathbb{RP}^n$ is a convex domain contained in a precompact subset of an affine subspace.
A **convex real projective orbifold** is a real projective orbifold projectively diffeomorphic to the quotient $\Omega/\Gamma$ where $\Omega$ is a convex domain in an affine subspace of $\mathbb{R}P^n$ and $\Gamma$ is a discrete group of projective automorphisms of $\Omega$ acting properly. If an open orbifold has a convex real projective structure, it is covered by a convex domain $\Omega$ in $\mathbb{R}P^n$. Equivalently, this means that the image of the developing map $\text{dev}(\tilde{O})$ for the universal cover $\tilde{O}$ of $O$ is a convex domain for the developing map $\text{dev}$ with associated holonomy homomorphism $h$. Here we may assume $\text{dev}(\tilde{O}) = \Omega$, and $O$ is projectively diffeomorphic to $\text{dev}(\tilde{O})/h(\pi_1(O))$. In our discussions, since $\text{dev}$ is an imbedding and so is $h$, $\tilde{O}$ will be regarded as an open domain in $\mathbb{R}P^n$ and $\pi_1(O)$ as a subgroup of $\text{PGL}(n+1, \mathbb{R})$ in such cases.

**Remark 2.2.** Given a vector space $V$, we denote by $S(V)$ the quotient space of $(V - \{O\})/\sim$ where $v \sim w$ iff $v = sw$ for $s > 0$.

We will represent each element of $\text{PGL}(n+1, \mathbb{R})$ by a matrix of determinant $\pm 1$; i.e., $\text{PGL}(n+1, \mathbb{R}) = \text{SL}_\pm(n+1, \mathbb{R})/\langle \pm I \rangle$. Recall the covering map $S^n = S(\mathbb{R}^{n+1}) \rightarrow \mathbb{R}P^n$. For each $q \in \text{PGL}(n+1, \mathbb{R})$, there is a unique lift in $\text{SL}_\pm(n+1, \mathbb{R})$ preserving each component of the inverse image of $\text{dev}(\tilde{O})$ under $S^n \rightarrow \mathbb{R}P^n$. We will use this representative.

### 2.1.4. End structures.

A **strongly tame** $n$-orbifold is one where the complement of a compact set is diffeomorphic to a union of $(n-1)$-dimensional orbifolds times intervals. Of course it can be compact.

Let $O$ be a strongly tame $n$-orbifold. Each end has a neighborhood diffeomorphic to a closed $(n-1)$-orbifold times an interval. The **fundamental group** of an end is the fundamental group of such an end neighborhood. It is independent of the choice of the end neighborhood by Proposition 4.1.

An end $E$ of a real projective orbifold $O$ is **totally geodesic** or **of type T** if the following hold:

- The end has an end neighborhood homeomorphic to a closed connected $(n-1)$-dimensional orbifold $B$ times a half-open interval $(0, 1]$.
- $B$ completes to a compact orbifold $U$ diffeomorphic to $B \times [0, 1]$ in an ambient real projective orbifold.
- The subset of $U$ corresponding to $B \times \{0\}$ is the added boundary component.
- Each point of the added boundary component has a neighborhood projectively diffeomorphic to the quotient orbifold of an open set $V$ in an affine half-space $P$ so that $V \cap \partial P \neq \emptyset$ by a projective action of a finite group.

The completion is called a **compactified end neighborhood** of the end $E$. The boundary component is called the **ideal boundary component** of the end. Such ideal boundary components may not be uniquely determined as there are two projectively nonequivalent ways to add boundary components of elementary annuli (see Section 1.4 of [22]). Two compactified end neighborhoods of an end are **equivalent** if they contain a common compactified end neighborhood.

We also define as follows:

- The equivalence class of the chosen compactified end neighborhood is called a **marking** of the totally geodesic end.
- We will also call the ideal boundary the **end orbifold** of the end.

**Definition 2.3.** A **lens** is a properly convex domain $L$ so that $\partial L$ is a union of two smooth strictly convex open disks. A properly convex domain $K$ is a **generalized**
lens if \( \partial K \) is a union of two open disks one of which is strictly convex and smooth and the other is allowed to be just a topological disk. A lens-orbifold is a compact quotient orbifold of a lens by a properly discontinuous action of a projective group \( \Gamma \). Thus, for two boundary components \( A \) and \( B \), \( A/\Gamma \) and \( B/\Gamma \) are homotopy equivalent to \( L/\Gamma \) by the obvious inclusion maps.

**Lens condition**: The ideal boundary is realized as a totally geodesic suborbifold in the interior of a lens-orbifold in some ambient real projective orbifold of \( \mathcal{O} \).

If the lens condition is satisfied for an \( \mathcal{T} \)-end, we will call it the \( \mathcal{T} \)-end of lens-type.

Let \( \tilde{\mathcal{O}} \) denote the universal cover of \( \mathcal{O} \) with the developing map \( \text{dev} \). An end \( E \) of a real projective orbifold is radial or of type \( R \) if the following hold:

- The end has an end neighborhood \( U \) foliated by properly imbedded projective geodesics.
- Choose a map \( f : \mathbb{R} \times [0, 1] \to \mathcal{O} \) so that \( f|\mathbb{R} \times \{t\} \) for each \( t \) is a geodesic leaf of such a foliation of \( U \). Then \( f \) lifts to \( \tilde{f} : \mathbb{R} \times [0, 1] \to \tilde{\mathcal{O}} \) where \( \text{dev} \circ \tilde{f}|\mathbb{R} \times \{t\} \) for each \( t, t \in [0, 1] \), maps to a geodesic in \( \mathbb{R}P^n \) ending at a point of concurrency common for \( t \).

The foliation is called a radial foliation and leaves radial lines of \( E \). Two such radial foliations \( F_1 \) and \( F_2 \) of radial end neighborhoods of an end are equivalent if the restrictions of \( F_1 \) and \( F_2 \) in an end neighborhood agree. A radial foliation marking is an equivalence class of radial foliations. The marking will give us an ideal boundary component on the end.

**Definition 2.4.** A real projective orbifold with radial or totally geodesic ends is a strongly tame orbifold with a real projective structure where each end is an \( \mathcal{R} \)-end or a \( \mathcal{T} \)-end with a marking given for each.

Let \( \mathbb{R}P^n_x \) denote the space of concurrent lines to a point \( x \) where \( \mathbb{R}P^n_x \) is projectively diffeomorphic to \( \mathbb{R}P^{n-1} \). The real projective transformations fixing \( x \) induce real projective transformations of \( \mathbb{R}P^n_x \). Radial lines in an \( \mathcal{R} \)-end neighborhood are equivalent if they agree outside a compact subset. The space of equivalent classes of radial lines in an \( \mathcal{R} \)-end neighborhood is an \( (n-1) \)-orbifold by the properness of the radial lines. The end orbifold associated with an \( \mathcal{R} \)-end is defined as the equivalence space of radial lines in \( \mathcal{O} \). The equivalence space of radial lines in an \( \mathcal{R} \)-end has the local structure of \( \mathbb{R}P^{n-1} \) since we can lift a local neighborhood to \( \tilde{\mathcal{O}} \), and these radial lines lift to lines developing into concurrent lines. The end orbifold has a unique induced real projective structure of one dimension lower.

**Example 2.5.** Let \( \mathbb{R}^{n+1} \) have standard coordinates \( x_0, x_1, \ldots, x_n \), and let \( B \) be the subset in \( \mathbb{R}P^n \) corresponding to the cone given by

\[
x_0 > \sqrt{x_1^2 + \cdots + x_n^2}.
\]

The Klein model gives a hyperbolic space as \( B \subset \mathbb{R}P^n \) with the isometry group \( \mathrm{PO}(1, n) \), a subgroup of \( \mathrm{PGL}(n+1, \mathbb{R}) \) acting on \( B \). Thus, a complete hyperbolic orbifold is projectively diffeomorphic to a real projective orbifold of \( B/\Gamma \) for \( \Gamma \) in \( \mathrm{PO}(1, n) \). The interior of a finite-volume hyperbolic \( n \)-orbifold with rank \( (n-1) \) horospherical ends and totally geodesic boundary forms an example of a properly convex strongly tame real projective orbifold with radial or totally geodesic ends.
For horospherical ends, the end orbifolds have Euclidean structures. (Also, we could allow hyperideal ends by attaching radial ends. Section 3.1.1 in [15].)

**Example 2.6.** For examples, if the end orbifold of an R-end \( E \) is a 2-orbifold based on a sphere with three singularities of order 3, then a line of singularity is a leaf of a radial foliation. End orbifolds of Tillman’s orbifold and the the double of a tetrahedral reflection orbifold are examples. A double orbifold of a cube with edges having orders 3 only has eight such end orbifolds. (See Proposition 4.6 of [16] and their deformations are computed in [28]. Also, see Ryan Greene [41] for the theory.)

2.1.5. **Horospherical ends.** An ellipsoid in \( \mathbb{RP}^n = \mathbb{P}(\mathbb{R}^{n+1}) \) (resp. in \( \mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1}) \)) is the projection \( C - \{O\} \) of the null cone \( C := \{ x \in \mathbb{R}^{n+1} | B(x, x) = 0 \} \) for a nondegenerate symmetric bilinear form \( B: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R} \). Ellipsoids are always equivalent by projective automorphisms of \( \mathbb{RP}^n \). An ellipsoid ball is the closed contractible domain in an affine subspace \( A \) of \( \mathbb{RP}^n \) (resp. \( \mathbb{S}^n \)) bounded by an ellipsoid contained in \( A \). A horoball is an ellipsoid ball with a point \( p \) of the boundary removed. An ellipsoid with a point \( p \) on it removed is called a horosphere.

The vertex of the horosphere or the horoball is defined as \( p \).

Let \( U \) be a horoball with a vertex \( p \) in the boundary of \( B \). A real projective orbifold that is real projectively diffeomorphic to an orbifold \( U/\Gamma_p \) for a discrete subgroup \( \Gamma_p \subset \text{PO}(1,n) \) fixing a point \( p \in \text{bd}B \) is called a horoball orbifold. A horospherical end is an end with an end neighborhood that is such an orbifold.

2.1.6. **Deformation spaces and the space of holonomy homomorphisms.** An isotopy \( i: \mathcal{O} \to \mathcal{O} \) is a self-diffeomorphism so that there exists a smooth orbifold map \( J: \mathcal{O} \times [0,1] \to \mathcal{O} \), so that

\[
i_t: \mathcal{O} \to \mathcal{O} \text{ given by } i_t(x) = J(x,t)
\]

are self-diffeomorphisms for \( t \in [0,1] \) and \( i = i_1, i_0 = I_\mathcal{O} \). We will extend this notion strongly.

- Two real projective structures \( \mu_0 \) and \( \mu_1 \) on \( \mathcal{O} \) with R-ends or T-ends with end markings are isotopic if there is an isotopy \( i \) on \( \mathcal{O} \) so that \( i^*(\mu_0) = \mu_1 \) where \( i^*(\mu_0) \) is the induced structure from \( \mu_0 \) by \( i \) where we require for each \( t \)
  - \( i_t^*(\mu_0) \) has a radial end structure for each radial end,
  - \( i_t \) sends the radial end foliation for \( \mu_0 \) from an R-end neighborhood to the radial end foliation for real projective structure \( \mu_t = i_t^*(\mu_0) \) with corresponding R-end neighborhoods,
  - \( i_t \) extends to diffeomorphisms of the compactifications of \( \mathcal{O} \) using the radial foliations and the totally geodesic ideal boundary components for \( \mu_0 \) and \( \mu_t \).

We define \( \text{Def}_E(\mathcal{O}) \) as the deformation space of real projective structures on \( \mathcal{O} \) with end marks; more precisely, this is the quotient space of the real projective structures on \( \mathcal{O} \) satisfying the above conditions for ends of type R and T under the isotopy equivalence relations. We put on \( \mathcal{O} \) a radial foliation on each end neighborhood of type R and attach an ideal boundary component for each end neighborhood of type T to obtain a new compactified orbifold \( \overline{\mathcal{O}} \). We introduce the equivalence relation based on isotopies and end neighborhood structures and ideal boundary
components. We may assume that the developing maps extend to the smooth maps of the universal cover \( \hat{\mathcal{O}} \) of \( \mathcal{O} \). The topology of such a space is defined by the compact open \( C^2 \)-topology for the space of developing maps \( \text{dev[\hat{\mathcal{O}}]} \). (See [23], [14] and [38] for more details.)

2.1.7. The end restrictions. To discuss the deformation spaces, we introduce the following notions. The end will be either assigned an \( R \)-type or a \( T \)-type.

- An \( R \)-type end is required to be radial.
- A \( T \)-type end is required to have totally geodesic properly convex ideal boundary components of lens-type or be horospherical.

A strongly tame orbifold will always have such an assignment in this paper, and finite-covering maps will always respect the types. We will fix the types for ends of our orbifolds in consideration.

2.1.8. Character spaces of relevance. Since \( \mathcal{O} \) is strongly tame, the fundamental group \( \pi_1(\mathcal{O}) \) is finitely generated. Let \( \{g_1, \ldots, g_m\} \) be a set of generators of \( \pi_1(\mathcal{O}) \).

As usual \( \text{Hom}(\pi_1(\mathcal{O}), G) \) for a Lie group \( G \) has an algebraic topology as a subspace of \( G^m \). This topology is given by the notion of algebraic convergence

\[
\{h_1\} \rightarrow h \text{ if } h_1(g_j) \rightarrow h(g_j) \in G \text{ for each } j, j = 1, \ldots, m.
\]

A conjugacy class of representation is called a character in this paper.

The \( \text{PGL}(n+1, \mathbb{R}) \)-character variety \( \text{rep}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \) is the quotient space of the homomorphism space

\[
\text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]

where \( \text{PGL}(n+1, \mathbb{R}) \) acts by conjugation

\[
h(\cdot) \mapsto gh(\cdot)g^{-1} \text{ for } g \in \text{PGL}(n+1, \mathbb{R}).
\]

Similarly, we define

\[
\text{rep}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R})) := \text{Hom}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R}))/\text{SL}_{\pm}(n+1, \mathbb{R})
\]

as the \( \text{SL}_{\pm}(n+1, \mathbb{R}) \)-character variety.

A representation or a character is stable if the orbit of it or its representative is closed and the stabilizer is finite under the conjugation action in

\[
\text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \text{ (resp. } \text{Hom}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R}))).
\]

By Theorem 1.1 of [45], a representation \( \rho \) is stable if and only if it is irreducible and no proper parabolic subgroup contains the image of \( \rho \). The stability and the irreducibility are open conditions in the Zariski topology. Also, if the image of \( \rho \) is Zariski dense, then \( \rho \) is stable. \( \text{PGL}(n+1, \mathbb{R}) \) acts properly on the open set of stable representations in \( \text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \). Similarly, \( \text{SL}_{\pm}(n+1, \mathbb{R}) \) acts so on \( \text{Hom}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R})). \) (See [45] for more details.)

A representation of a group \( G \) into \( \text{PGL}(n+1, \mathbb{R}) \) or \( \text{SL}_{\pm}(n+1, \mathbb{R}) \) is strongly irreducible if the image of every finite index subgroup of \( G \) is irreducible. Actually, many of the orbifolds have strongly irreducible and stable holonomy homomorphisms by Theorem 5.4.

An eigen-1-form of a linear transformation \( \gamma \) is a linear functional \( \alpha \) in \( \mathbb{R}^{n+1} \) so that \( \alpha \circ \gamma = \lambda \alpha \) for some \( \lambda \in \mathbb{R} \). We recall the lifting of Remark 2.2.
to be the subspace of representations $h$ satisfying

**The vertex condition for $R$-ends:** $h|\pi_1(\widetilde{E})$ has a nonzero common eigenvector for positive eigenvalues for the lift of $h(\pi_1(\widetilde{E}))$ in $\text{SL}_\pm(n+1, \mathbb{R})$ for each $R$-type $p$-end fundamental group $\pi_1(\widetilde{E})$.

**The hyperplane condition for $T$-ends:** $h|\pi_1(\widetilde{E})$ acts on a hyperplane $P$ for each $T$-type $p$-end fundamental group $\pi_1(\widetilde{E})$ discontinuously and cocompactly on a lens $L$, a properly convex domain with $L^o \cap P = L \cap P \neq \emptyset$ or a horoball tangent to $P$.

- We denote by
  $$\text{Hom}^s(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$
  the subspace of stable and irreducible representations, and define
  $$\text{Hom}_E^s(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$
  to be
  $$\text{Hom}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \cap \text{Hom}^s(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$

- We define
  $$\text{Hom}_{E,u}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$
  to be the subspace of representations $h$ where
  - $h|\pi_1(\widetilde{E})$ has a unique common eigenspace of dimension 1 in $\mathbb{R}^{n+1}$ with positive eigenvalues for its lift in $\text{SL}_\pm(n+1, \mathbb{R})$ for each $p$-end fundamental group $\pi_1(\widetilde{E})$ of $R$-type and
  - $h|\pi_1(\widetilde{E})$ has a common null-space $P$ of eigen-1-forms uniquely satisfying the following:
    * $\pi_1(\widetilde{E})$ acts properly on a lens $L$ with $L \cap P$ nonempty interior in $P$ or
    * $H - \{p\}$ for a horosphere $H$ tangent to $P$ at $p$
  for each $p$-end fundamental group $\pi_1(\widetilde{E})$ of the end of $T$-type.

**Remark 2.7.** The above condition for type $T$ generalizes the principal boundary condition for real projective surfaces.

Suppose that there are no ends of $T$-type. Since each $\pi_1(\widetilde{E})$ is finitely generated and there is only finitely many conjugacy classes of $\pi_1(\widetilde{E})$,

$$\text{Hom}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

is a closed semi-algebraic subset.

$$\text{Hom}_{E,u}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

is an open subset of this closed semi-algebraic subset by Lemma 2.8.

**Lemma 2.8.** Let $V$ be a semialgebraic subset of $\text{PGL}(n+1, \mathbb{R})^n$. For each $(g_1, \ldots, g_n) \in V$, we arbitrarily choose a maximal eigenspace $E_i(g_i) \subset \mathbb{R}^{n+1}$ corresponding to the eigenvalue $\lambda(g_i)$ where $\bigcap_{i=1}^n E_i(g_i) \neq \{0\}$ on every point of semialgebraic subset $V$.

We assume that for each $i$, $(g_1, \ldots, g_n) \in V \mapsto E_i(g_i) \subset \mathbb{R}^{n+1}$ has a nonzero continuous section on $V$. Then the dimension function of the intersection $\bigcap_{i=1}^n E_i(g_i)$ is upper semi-continuous in $V$. 

Proof. Since the limit subspace of $E_i(g)$ is contained in an eigenspace of $g$, this follows. □

When there are ends of $\mathcal{T}$-type,

\[ \text{Hom}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \text{ and } \text{Hom}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \]

are the unions of open subsets of this closed semi-algebraic subsets since we have to consider the lens condition.

We define

- $\text{rep}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$
- $\text{rep}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$

(2.2)

Note that when there are no $\mathcal{T}$-type ends, $\text{rep}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ is a closed subset of

$\text{rep}^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ and $\text{rep}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$

is an open subset of

$\text{rep}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$.

Note that elements of Def$_E(O)$ have characters in

\[ \text{rep}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R})). \]

Denote by Def$_{E,u}(O)$ the subspace of Def$_E(O)$ of equivalence classes of real projective structures with characters in

\[ \text{rep}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})). \]

Also, we denote by Def$_E^s(O) \subset \text{Def}_E(O)$ and Def$_{E,u}^s(O) \subset \text{Def}_{E,u}(O)$ the subspaces of equivalence classes of real projective structures with stable and irreducible characters.
2.2. Oriented real projective structures. Recall that \( \text{SL}_\pm(n+1, \mathbb{R}) \) is isomorphic to \( \text{GL}(n+1, \mathbb{R})/\mathbb{R}^+ \). Then this group acts on \( S^n = S(\mathbb{R}^{n+1}) \). We let \([v]\) denote the equivalence class of \( v \in \mathbb{R}^{n+1} - \{O\} \). There is a double covering map \( S^n \to \mathbb{RP}^n \) with the deck transformation group generated by \( A \). This gives a projective structure on \( S^n \). The group of projective automorphisms is identified with \( \text{SL}_\pm(n+1, \mathbb{R}) \).

An \((S^n, \text{SL}_\pm(n+1, \mathbb{R}))\)-structure on \( O \) is said to be an oriented real projective structure on \( O \). We define \( \text{Def}_{S^n}(O) \) as the deformation space of \((S^n, \text{SL}_\pm(n+1, \mathbb{R}))\)-structures on \( O \). Again, we can define the radial end structures and totally geodesic ideal boundary for oriented real projective structures and also horospherical end neighborhoods in obvious ways. They correspond in the direct way in the following theorem also.

**Theorem 2.9.** There is a one-to-one correspondence between the space of real projective structures on an orbifold \( O \) with the space of oriented real projective structures on \( O \). Moreover, a real projective diffeomorphism of real projective orbifolds is an \((S^n, \text{SL}_\pm(n+1, \mathbb{R}))\)-diffeomorphism of oriented real projective orbifolds and vice versa.

**Proof.** Straightforward. See p. 143 of Thurston [58]. \( \square \)

2.3. The local homeomorphism theorems. For technical reasons, we will be assuming \( \partial O = \emptyset \) in most cases. Here, we are not yet concerned with convexity of orbifolds. The following map \( \text{hol} \), the so-called Ehresmann-Thurston map, is induced by sending \((\text{dev}, h)\) to the conjugacy class of \( h \) as isotopies preserve \( h \):

**Theorem 2.10** ([19]). Let \( O \) be a noncompact strongly tame real projective \( n \)-orbifold with radial ends or totally-geodesic ends of lens-type with markings and given types \( R \) or \( T \). Assume \( \partial O = \emptyset \). Then the following map is a local homeomorphism:

\[
\text{hol} : \text{Def}^{\ast}_{E, u}(O) \to \text{rep}^{\ast}_{E, u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).
\]

Also, we define

\[
\text{rep}^{\ast}_{E}(\pi_1(O), \text{SL}_\pm(n+1, \mathbb{R})), \text{rep}^{\ast}_{E, u}(\pi_1(O), \text{SL}_\pm(n+1, \mathbb{R}))
\]

similarly to Section 2.1.6.

By lifting \((\text{dev}, h)\) by the method of Section 2.2, we obtain that

\[
\text{hol} : \text{Def}^{\ast}_{E, u}(O) \to \text{rep}^{\ast}_{E, u}(\pi_1(O), \text{SL}_\pm(n+1, \mathbb{R}))
\]

is a local homeomorphism.

**Remark 2.11.** The restrictions of end types are necessary for this theorem to hold. (See Goldman [38], Canary-Epstein-Green [14], Choi [23], and Bergeron-Gelander [11] for many versions of similar results.)

3. Convex real projective structures

3.1. Metrics. Let \( \Omega \) be a properly convex open domain. A line or a subspace of dimension-one in \( \mathbb{R}P^n \) has a two-dimensional homogenous coordinate system. Define a metric by defining the distance for \( p, q \in \Omega \),

\[
d_{\Omega}(p, q) = \log ||[o, s, q, p]||
\]
where $o$ and $s$ are endpoints of the maximal segment in $\Omega$ containing $p,q$ where $o,q$ separates $p,s$ and $[o,s,q,p]$ denotes the cross ratio.

Given a properly convex real projective structure on $\mathcal{O}$, there is a Hilbert metric which we denote by $d_{\mathcal{O}}$ on $\tilde{\mathcal{O}}$. Since the metric $d_{\mathcal{O}}$ is invariant under the deck transformation group, we obtain a metric $d_{\mathcal{O}}$ on $\mathcal{O}$.

Assume that $K_i \rightarrow K$ geometrically for a sequence of properly convex domains $K_i$ and a properly convex domain $K$. Suppose that two sequences of points \{x_i\} and \{y_i\} converge to $x,y \in K^0$ respectively. Since the end of a maximal segments always are in $\partial K_i$ and $\partial K_i \rightarrow \partial K$ by Lemma 2.1, we obtain

\begin{equation}
\lim_{\mathcal{O}} d_{K_i^o}(x_i, y_i) = d_{K^o}(x, y)
\end{equation}

holds.

3.2. Convexity and convex domains.

**Proposition 3.1** (Kuiper [47], Koszul [46], Vey [60]).

- A strongly tame real projective orbifold is properly convex if and only if each developing map sends the universal cover to a properly convex open domain bounded in an affine subspace of $\mathbb{RP}^n$.
- If a strongly tame convex real projective orbifold is not properly convex, then its holonomy homomorphism is virtually reducible.

**Proposition 3.2** (Corollary 2.13 of Benoist [9]). Suppose that a discrete subgroup $\Gamma$ of $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_\pm(n, \mathbb{R})$) acts on a properly convex $(n-1)$-dimensional open domain $\Omega$ in $\mathbb{RP}^{n-1}$ (resp. $\mathbb{S}^{n-1}$) so that $\Omega/\Gamma$ is compact. Then the following statements are equivalent.

- Every subgroup of finite index of $\Gamma$ has a finite center.
- Every subgroup of finite index of $\Gamma$ has a trivial center.
- Every subgroup of finite index of $\Gamma$ is irreducible in $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_\pm(n, \mathbb{R})$).
  That is, $\Gamma$ is strongly irreducible.
- The Zariski closure of $\Gamma$ is semisimple.
- $\Gamma$ does not contain a normal infinite nilpotent subgroup.
- $\Gamma$ does not contain a normal infinite abelian subgroup.

**Theorem 3.3** (Theorem 1.1 of Benoist [9]). Let $\Gamma$ be a discrete subgroup of $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_\pm(n, \mathbb{R})$) with a trivial virtual center. Suppose that a discrete subgroup $\Gamma$ of $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_\pm(n, \mathbb{R})$) acts on a properly convex $(n-1)$-dimensional open domain $\Omega$ so that $\Omega/\Gamma$ is a compact orbifold. Then every representation of a component of $\text{Hom}(\Gamma, \text{PGL}(n, \mathbb{R}))$ (resp. $\text{Hom}(\Gamma, \text{SL}_\pm(n, \mathbb{R}))$) containing the inclusion representation also acts on a properly convex $(n-1)$-dimensional open domain cocompactly and properly discontinuously.

In general, a join of two convex sets $C_1$ and $C_2$ in an affine subspace $A$ of $\mathbb{RP}^n$ is defined as

\[
[tv_1 + (1-t)v_2]|v_i \in C_{C_i}, i = 1, 2, 0 \leq t \leq 1
\]

where $C_{C_i}$ is a cone in $\mathbb{RP}^{n+1}$ corresponding to $C_i, i = 1, 2$. The join is denoted by $C_1 \ast C_2$ in this paper.

Given subspaces $V_1, \ldots, V_m \subset \mathbb{RP}^n$ (resp. $\subset \mathbb{S}^n$) that are from linear independent subspaces in $\mathbb{R}^{n+1}$ and a subset $C_i \subset V_i$ for each $i$, we define a strict join of
choices of $C_i$ sets.

Let $\Omega$ be a closed $(n-1)$-dimensional properly convex projective orbifold and let $\Omega$ denote its universal cover in $\mathbb{RP}^{n-1}$ (resp. in $\mathbb{S}^{n-1}$). Then

(i) $\Omega$ is projectively diffeomorphic to the interior of a strict join $K_1 \cdots K_{l_0}$ where $K_i$ is a properly convex open domain of dimension $n_i \geq 0$ corresponding to a convex open cone $C_i \subset \mathbb{R}^{n_i+1}$.

(ii) $\Omega$ is the image of the interior of $C_1 \oplus \cdots \oplus C_{l_0}$.

(iii) The fundamental group $\pi_1(\Sigma)$ is virtually isomorphic to a cocompact subgroup of $\mathbb{Z}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}$ for $l_0 - 1 + \sum_{i=1}^{l_0} n_i = n$ with following properties:

- Each $\Gamma_i$ has the property that each finite index subgroup has a trivial center.

- Each $\Gamma_j$ acts on $K_j^o$ cocompactly and the Zariski closure is a semisimple Lie group in $\text{PGL}(n_j+1, \mathbb{R})$ (resp. in $\text{SL}_+(n_j+1, \mathbb{R})$), and acts trivially on $K_m$ for $m \neq j$.

- The subgroup corresponding to $\mathbb{Z}^{l_0-1}$ acts trivially on each $K_j$.

3.3. The duality. We start from linear duality. Let us choose the origin $O$ in $\mathbb{R}^{n+1}$. Let $\Gamma$ be a group of linear transformations $\text{GL}(n+1, \mathbb{R})$. Let $\Gamma^*$ be the affine dual group defined by $\{g^{-1}|g \in \Gamma\}$ acting on the dual space $\mathbb{R}^{n+1*}$. Suppose that $\Gamma$ acts on a properly convex cone $C$ in $\mathbb{R}^{n+1}$ with the vertex $O$.

An open convex cone $C^*$ in $\mathbb{R}^{n+1*}$ is dual to an open convex cone $C$ in $\mathbb{R}^{n+1}$ if $C^* \subset \mathbb{R}^{n+1*}$ equals

$$\{\phi \in \mathbb{R}^{n+1*} | \phi|\text{Cl}(C) - \{O\} > 0\}.$$ 

$C^*$ is a cone with vertex as the origin again. Note $(C^*)^* = C$.

Let $\mathbb{R}_+$ denote the set of positive real numbers. Now $\Gamma^*$ acts on $C^*$. Also, if $\Gamma$ acts cocompactly on $C$ if and only if $\Gamma^*$ acts on $C^*$ cocompactly. A central dilatation extension $\Gamma'$ of $\Gamma$ is the subgroup of $\text{GL}(n+1, \mathbb{R})$ generated by $\Gamma$ and a dilatation $sI$ by a scalar $s \in \mathbb{R}_+ - \{1\}$ with the fixed $O$. The dual of $\Gamma'$ is a central dilatation extension of $\Gamma^*$.

Given a subgroup $\Gamma$ in $\text{PGL}(n+1, \mathbb{R})$, an affine lift in $\text{GL}(n+1, \mathbb{R})$ is any subgroup that maps to $\Gamma$ isomorphically under the projection. Given a subgroup $\Gamma$ in $\text{PGL}(n+1, \mathbb{R})$, the dual group $\Gamma^*$ is the image in $\text{PGL}(n+1, \mathbb{R})$ of the dual of any affine lift of $\Gamma$.

A properly convex open domain $\Omega$ in $\mathbb{P}(\mathbb{R}^{n+1})$ (resp. in $\mathbb{S}(\mathbb{R}^{n+1})$) is dual to a properly convex open domain $\Omega^*$ in $\mathbb{P}(\mathbb{R}^{n+1*})$ (resp. in $\mathbb{S}(\mathbb{R}^{n+1*})$) if $\Omega$ corresponds to an open convex cone $C$ and $\Omega^*$ to its dual $C^*$. We say that $\Omega^*$ is dual to $\Omega$. We also have $(\Omega^*)^* = \Omega$ and $\Omega$ is properly convex if and only if so is $\Omega^*$.

We call $\Gamma$ a dividing group if a central dilatational extension acts cocompactly on $C$. $\Gamma$ is dividing if and only if so is $\Gamma^*$. (See [62] and [8]).
Theorem 3.5 (Vinberg [61]). Let $\mathcal{O}$ be a properly convex real projective orbifold of form $\Omega/\Gamma$. Let $\mathcal{O}^* = \Omega^*/\Gamma^*$ be a properly convex real projective orbifold. Then $\mathcal{O}^*$ is diffeomorphic to $\mathcal{O}$.

Given a convex domain $\Omega$ in an affine subspace $A \subset \mathbb{R}^n$, a supporting hyperplane $h$ at a point $x \in \text{bd}\Omega$ is a hyperplane so that a component of $A - h$ contains the interior of $\Omega$.

A hyperspace is an element of $\mathbb{RP}^n$ since it is represented as a 1-form, and an element of $\mathbb{RP}^n$ can be considered as a hyperspace in $\mathbb{RP}^n$. The following definition applies to $\Omega \subset \mathbb{RP}^n$ (resp. $\mathbb{S}(\mathbb{R}^{n+1})$) and $\Omega^* \subset \mathbb{RP}^n$ (resp. $\mathbb{S}(\mathbb{R}^{n+1})$).

Given a properly convex domain $\Omega$, we define the augmented boundary of $\Omega$

$$\text{bd}^A\Omega := \{(x,h) | x \in \text{bd}\Omega, h \text{ is a supporting hyperplane of } \Omega \}.$$  

Remark 3.6. For open properly convex domains $\Omega_1$ and $\Omega_2$, we have

(3.2) $\Omega_1 \subset \Omega_2$ if and only if $\Omega_2^* \subset \Omega_1^*$.

The following standard results are proved in Section 3 of [16]. We will call the homeomorphism below as the duality map.

Proposition 3.7. Suppose that $\Omega \subset \mathbb{RP}^n$ (resp. $\mathbb{S}(\mathbb{R}^{n+1})$) and its dual $\Omega^* \subset \mathbb{RP}^n$ (resp. $\mathbb{S}(\mathbb{R}^{n+1})$) are properly convex domains.

(i) There is a proper quotient map $\Pi_{A^*} : \text{bd}^A\Omega \to \text{bd}\Omega$ given by sending $(x,h)$ to $x$.

(ii) Let $\Gamma$ act on properly discontinuously $\Omega$ if and only if so acts $\Gamma^*$ on $\Omega^*$.

(iii) There exists a homeomorphism $D : \text{bd}^A\Omega \leftrightarrow \text{bd}^A\Omega^*$ given by sending $(x,h)$ to $(h,x)$.

(iv) Let $A \subset \text{bd}^A\Omega$ be a subspace and $A^* \subset \text{bd}^A\Omega^*$ be the corresponding dual subspace $D(A)$. If a group $\Gamma$ acts properly discontinuously on $A$ if and only if $\Gamma^*$ so acts on $A^*$.

Given a convex domain $\Omega$, we denote by $R_p(\Omega)$ the space of directions of open rays in $\Omega$ from $p$ in $S^{n-1}_p$.

Proposition 3.8. Let $\Omega^*$ be the dual of a properly convex domain $\Omega$. Then

(i) $\text{bd}\Omega$ is $C^1$ and strictly convex if and only if $\text{bd}\Omega^*$ is $C^1$ and strictly convex.

(ii) $\Omega$ is a horospherical orbifold if and only if so is $\Omega^*$.

(iii) Let $p \in \text{bd}\Omega$. Then $D$ sends in a one-to-one and onto manner

$$\{(p,h) | h \text{ is a supporting hyperplane of } \Omega \text{ at } p\}$$

to $\{(h^*,p^*) | h^* \in D = p^* \cap \text{bd}\Omega^*\}$ where $D$ is a properly convex set in $\text{bd}\Omega$.

(iv) $\text{bd}\Omega^*$ contains a properly convex domain $D = P \cap \text{bd}\Omega^*$ open in a totally geodesic hyperplane $P$ if and only if $\text{bd}\Omega$ contains a vertex $p$ with $R_p(\Omega)$ a properly convex domain. Moreover, $D^*$ and $R_p(\Omega)$ are properly convex and are projectively diffeomorphic to dual domains in $\mathbb{RP}^n$.

4. The end theory

We will now discuss in detail the end theory. The following is simply the notions useful in relative hyperbolic group theory as can be found in Bowditch [12].
4.1. p-ends, p-end neighborhoods, and p-end fundamental groups. Let $\mathcal{O}$ be a real projective orbifold with the universal cover $\tilde{\mathcal{O}}$ and the covering map $p_\mathcal{O}$. Each end neighborhood $U$, diffeomorphic to $S^1 \times (0,1)$, of an end $E$ lifts to a connected open set $\tilde{U}$ in $\tilde{\mathcal{O}}$. A subgroup $\Gamma_{\tilde{U}}$ of $\Gamma$ acts on $\tilde{U}$ where
\[ p_\mathcal{O}^{-1}(U) = \bigcup_{g \in \pi_1(\mathcal{O})} g(\tilde{U}). \]

Each component $\tilde{U}$ is said to be a \textit{proper pseudo-end neighborhood}.

- An \textit{exiting sequence} of sets $U_1, U_2, \cdots$ in $\tilde{\mathcal{O}}$ is a sequence so that for each compact subset $K$ of $\mathcal{O}$ there exists an integer $N$ satisfying $p_\mathcal{O}^{-1}(K) \cap U_i = \emptyset$ for $i > N$.
- A \textit{pseudo-end sequence} is an exiting sequence of proper pseudo-end neighborhoods
\[ \{U_i | i = 1, 2, 3, \ldots \}, \] where $U_{i+1} \subset U_i$ for every $i$.

- Two pseudo-end sequences $\{U_i\}$ and $\{V_j\}$ are \textit{compatible} if for each $i$, there exists $J$ such that $V_j \subset U_i$ for every $j, j > J$ and conversely for each $j$, there exists $I$ such that $U_i \subset V_j$ for every $i, i > I$.
- A \textit{compatibility class} of a proper pseudo-end sequence is called a \textit{pseudo-end} of $\tilde{\mathcal{O}}$. Each of these corresponds to an end of $\mathcal{O}$ under the universal covering map $p_\mathcal{O}$.
- For a pseudo-end $\tilde{E}$ of $\tilde{\mathcal{O}}$, we denote by $\Gamma_{\tilde{E}}$ the subgroup $\Gamma_{\tilde{U}}$ where $U$ and $\tilde{U}$ is as above. We call $\Gamma_{\tilde{E}}$ a \textit{pseudo-end fundamental group}. We will also denote it by $\pi_1(\tilde{E})$.
- A \textit{pseudo-end neighborhood} $U$ of a pseudo-end $\tilde{E}$ is a $\Gamma_{\tilde{E}}$-invariant open set containing a proper pseudo-end neighborhood of $\tilde{E}$. A proper pseudo-end neighborhood is an example.

(From now on, we will replace “pseudo-end” with the abbreviation “p-end”.)

**Proposition 4.1.** The p-end fundamental group $\Gamma_{\tilde{E}}$ is independent of the choice of $U$.

**Proof.** Given $U$ and $U'$ that are end-neighborhoods for an end $E$, let $\tilde{U}$ and $\tilde{U}'$ be p-end neighborhoods for a p-end $\tilde{E}$ that are components of $p^{-1}(U)$ and $p^{-1}(U')$ respectively. Let $\tilde{U}''$ be the component of $p^{-1}(U'')$ that is a p-end neighborhood of $\tilde{E}$. Then $\Gamma_{\tilde{U}''} \subseteq \Gamma_{\tilde{U}}$ since both are subgroups of $\Gamma$. Any $G$-path in $U$ in the sense of Bridson-Haefliger [13] is homotopic to a $G$-path in $U''$ by a translation in the $I$-factor. Thus, $\pi_1(U') \to \pi_1(U)$ is surjective. Since $\tilde{U}$ is connected, any element $\gamma$ of $\Gamma_{\tilde{U}}$ is represented by a $G$-path connecting $x_0$ to $\gamma(x_0)$. (See Example 3.7 in Chapter III, $G$ of [13]) Thus, $\Gamma_{\tilde{U}}$ is isomorphic to the image of $\pi_1(U) \to \pi_1(\mathcal{O})$. Since $\Gamma_{\tilde{U}''}$ is surjective to the image of $\pi_1(U'') \to \pi_1(\mathcal{O})$, it follows that $\Gamma_{\tilde{U}''}$ is isomorphic to $\Gamma_{\tilde{U}}$. \(\square\)

Let $\mathcal{O}$ be a strongly tame real projective orbifold. We give each end of $\mathcal{O}$ a marking. We fix a developing map $\text{dev} : \tilde{\mathcal{O}} \to \mathbb{RP}^n$ in this subsection.

A \textit{ray} from a point $v$ of $\mathbb{RP}^n$ is a segment with endpoint equal to $v$ oriented away from $v$.

Let $E$ be an R-end of $\mathcal{O}$.
• Let $\tilde{E}$ denote a p-R-end corresponding to $E$ and $U$ denote a p-R-end neighborhood of $\tilde{E}$ with a radial foliation $\mathcal{F}$ induced from the end marking on an end neighborhood of $E$.

• Two radial leaves of equivalent radial foliations of proper p-end neighborhoods of $\tilde{E}$ are equivalent if they agree on a proper p-end neighborhood of $\tilde{E}$.

• Let $x$ be the common end point of the images under the developing map of leaves of $\mathcal{F}$. We call $x$ the p-end vertex of $\tilde{O}$. $x$ will be denoted by $v_E$ if its associated p-end neighborhood corresponds to a p-end $\tilde{E}$.

• Let $S^{n-1}$ denote the space of equivalence classes of rays from $v_E$ diffeomorphic to an $(n-1)$-sphere where $\pi_1(\tilde{E})$ acts as a group of projective automorphisms. Here, $\pi_1(\tilde{E})$ acts on $v_E$ and sends leaves to leaves in $U_1$.

• We denote by $R_{v_E}(\tilde{O}) = \tilde{S}_E$ as the following space
  $$\{ [l] \mid l \subset \tilde{O}, \text{dev}(l) \text{ is a ray from } v_E \},$$
which is an $(n-1)$-dimensional open manifold.

• The map $\text{dev}$ induces an immersion
  $$\tilde{S}_E \to S^{n-1}_v.$$ 
  Also, $\Gamma_E$ projectively acts on $\tilde{S}_E$ by $g([l]) = [g(l)]$ for each leaf $l$ and $g \in \Gamma_E$.

• Recall that $\tilde{S}_E/\Gamma_E$ is diffeomorphic to the end orbifold denoted by $S_E$.
  Thus, $S_E$ has a convex real projective structure. (However, the projective structure and the differential topology on $S_E$ does depend on the end markings.)

Given a T-end of $\tilde{O}$ and an end neighborhood $U$ of the product form $S_E \times [0,1)$ with a compactification by a totally geodesic orbifold $S_E$, we take a component $U_1$ of $p^{-1}(U)$ and a convex domain $\tilde{S}_E$ developing into a totally geodesic hypersurface $P$ under $\text{dev}$. Here $\tilde{E}$ is the p-end corresponding to $E$ and $U_1$. $\Gamma_E$ acts on $U_1$ and hence on $\tilde{S}_E$. Again $\tilde{S}_E/\Gamma_E$ is projectively diffeomorphic to the end orbifold to be denote by $S_E$ again. We call $\tilde{S}_E$ the p-ideal boundary component of $\tilde{O}$. Generalizing further an open subset $U$ of $\tilde{O}$ containing a proper p-end-neighborhood of $\tilde{E}$, where $\pi_1(\tilde{E})$ acts on, is said to be a p-end neighborhood.

4.2. **The admissible groups.** If every subgroup of finite index of a group $\Gamma$ has a finite center, $\Gamma$ is said to be a virtual center-free group. An admissible group $G$ acting on projective $S^{n-1}$ is a finite extension of the finite product $Z^{l-1} \times \Gamma_1 \times \cdots \times \Gamma_l$ for infinite hyperbolic or trivial groups $\Gamma_i$ with following properties:

- $G$ acts on a properly convex domain of form $K_1 \ast \cdots \ast K_l$ for a strictly convex domain $K_j \subset S^{n-1}$ for each $j$, $j = 1, \ldots, l$, in an $n_j$-dimensional subspace of $S^{n-1}$, $0 \leq n_j \leq n-1$,

- $\Gamma_j$ is the restriction of $G$ to each $K_j$ and extended on $S^{n-1}$ to act trivially on $K_m$ for $m \neq j$ where $K_j/\Gamma_j$ is an orbifold of dimension $n_j$.

- $Z^{l-1}$ acts trivially on each $K_j$ and is a virtual center of $G$.

This is strictly stronger than the conclusion of Proposition 3.4 of Benoist and is needed for now. Here, we conjecture that we do not need the stronger condition but the conclusion of Proposition 3.4 is enough assumption for everything in this paper. (See also Example 5.5.3 of [54] as pointed out by M. Kapovich.)
We have $l = 1$ if and only if the end fundamental group is hyperbolic or is trivial. If our orbifold has a complete hyperbolic structure, then end fundamental groups are virtually free abelian.

![Diagram](image)

**Figure 2.** The universal covers of horospherical and lens shaped ends. The radial lines form cone-structures.

### 4.3. The admissible ends

Let $\mathcal{O}$ be a convex real projective orbifold with the universal cover $\tilde{\mathcal{O}}$.

- A *cone* over a point $x$ and a set $A$ in an affine subspace of $\mathbb{RP}^n$ (resp. in $S^n$), $x \notin \text{Cl}(A)$ is the set given by $x \ast A$ in $\mathbb{RP}^n$ (resp. in $S^n$).
- Take a cone $C := \{x\} \ast L$ over a lens $L$ and a point $x$, $x \notin \text{Cl}(L)$, so that every maximal segment $l$ from $x$ in $C$ ends in one component $\partial_1 L$ of $\partial L$ and meets $\partial_1 L$ and $\partial_2 L$ exactly once. A *lens-cone* is $C - \{x\}$.
- Take a cone $C := \{x\} \ast L$ over a generalized lens $L$ with the same properties as above where a nonsmooth component has to be in the boundary of the cone. A *generalized lens-cone* is $C - \{x\}$.
- For two components $A_1$ and $A_2$ of $\partial L$ for $L$ as above in the lens-cone, $A_1$ is called a *top hypersurface* if it is in $\text{bd}(\{x\} \ast L)$ and $A_2$ is then called a *bottom hypersurface*.
- A (generalized) *lens* of a (generalized) lens-cone $C$ is the lens-shaped domain $A$ so that $C = \{x\} \ast A - \{x\}$ for a point $x \notin \text{Cl}(A)$ and with the properties in the first item.
- A *totally-geodesic subdomain* is a convex domain in a hyperspace.
- A *cone-over* a totally-geodesic open domain $A$ is $\{x\} \ast A - \{x\}$ for the cone $\{x\} \ast A$ over a point $x$ not in the hyperspace.

(See Figure 2.) We will also call a real projective orbifold with boundary to be

- a *lens-cone* or
- a *lens*, provided it is compact,
if it is covered by such domains and is diffeomorphic to a closed \((n - 1)\)-orbifold times an interval.

We introduce some relevant adjectives: Let \(S_E\) be an \((n - 1)\)-dimensional end orbifold corresponding to a p-end \(\tilde{E}\), and let \(\mu\) be a holonomy homomorphism

\[
\pi_1(\tilde{E}) \rightarrow \text{PGL}(n + 1, \mathbb{R}) \text{ (resp. } \text{SL}_\pm(n + 1, \mathbb{R}))
\]

restricted from that of \(O\).

- Suppose that \(\mu(\pi_1(\tilde{E}))\) acts on a (generalized) lens-shaped domain \(K\) in \(\mathbb{R}P^n\) (resp. in \(S^n\)) with boundary a union of two open \((n - 1)\)-cells \(A_1\) and \(A_2\) and \(\pi_1(\tilde{E})\) acts properly on \(A_1\) and \(A_2\). Then \(\mu\) is said to be a (generalized) lens-shaped representation for \(\tilde{E}\).
- \(\mu\) is a totally-geodesic representation if \(\mu(\pi_1(\tilde{E}))\) acts on a totally-geodesic subdomain.
- If \(\mu(\pi_1(\tilde{E}))\) acts on a horoball \(K\), then \(\mu\) is said to be a horospherical representation. In this case, it follows \(\text{bd}K - \partial K = \{v_E\}\) for the p-end vertex \(v_E\) of \(\tilde{E}\).
- If \(\mu(\pi_1(\tilde{E}))\) acts on a strictly joined domain, then \(\mu\) is said to be a strictly joined representation.

Let \(C'\) be a generalized lens and \(L := \{v_E\} \ast C' - \{v_E\}\) be a generalized lens-cone over \(C'\). A concave p-end-neighborhood is an imbedded p-end neighborhood of form \(L - C'\).

**Definition 4.2.** (Admissible ends) Let \(O\) be a real projective orbifold with the universal cover \(\tilde{O}\). Let \(E\) be an R-end of \(O\) and \(\tilde{E}\) be the corresponding p-end with the p-end fundamental group \(\pi_1(\tilde{E})\).

- We say that the radial end \(E\) of \(O\) is of lens-type if \(\tilde{E}\) has a p-end neighborhood that is a lens-cone of form \(L \ast \{v_E\} - \{v_E\}\) and \(\pi_1(\tilde{E})\) acts on for its lens \(L\).
- \(E\) is of generalized lens-type if \(\tilde{E}\) has a concave p-end neighborhood. Equivalently, a p-end neighborhood of \(\tilde{E}\) is the interior of a generalized lens-cone of form \(L \ast \{v_E\} - \{v_E\}\) and \(\pi_1(\tilde{E})\) acts on the generalized lens \(L\).

A p-R-end \(\tilde{E}\) is admissible if the p-end fundamental group acts on \(\tilde{S}_E\) as an admissible group and if \(\tilde{E}\) is a horospherical or lens-type p-R-end.

A T-end \(E\) is of lens-type if \(E\) satisfies the lens-condition that the ideal boundary end orbifold \(S_E\) has a lens-neighborhood \(L\) in an ambient real projective orbifold containing \(O\). For a component \(C_1\) of \(L - S_E\) inside \(O\), \(C_1 \cup S_E\) is said to be the one-sided end neighborhood of \(S_E\). Given a p-end \(\tilde{E}\), the orbifold \(S_E\) is covered by a domain \(\tilde{S}_E\) in the boundary of p-end neighborhood corresponding \(\tilde{E}\) in a hyperspace.

A T-end is admissible if it is of lens-type and the p-end fundamental group for a p-end \(\tilde{E}\) acts on the ideal boundary \(\tilde{S}_E\) as an admissible group.

A p-end is admissible in a generalized sense if it is admissible or is a generalized lens-type p-R-end.

**Example 4.3.** A model of a lens-type R-end can be made by a positive diagonal group acting on the standard simplex \(T\) in \(\mathbb{R}P^3\). We take a vertex \(v\) to be \([1, 0, 0, 0]\) and we choose an abelian group \(G\) of rank 3 acting on \(T\) and properly and freely on the interior \(F\) of the side of \(T\) opposite \(v\). We choose \(G\) so that the eigenvalue
at $v$ is not the largest or the smallest one for $g \in \Delta$ for the Zariski closure $\Delta$ of $G$. $\{v\} * F - \{v\})/G$ is an end neighborhood of an ambient orbifold. The existence of lens follows by considering orbits of points under $\Delta$. (This follows by Theorem 5.1 of [16] since we can show that the uniform middle eigenvalue condition holds. See also Ballas [1], [2], and Ballas-Danciger-Lee [4] which include many graphics for ends.)

Example 2.6 give these examples by Proposition 4.6 of [16] or more generally by Theorem 6.5; that is, we show that these have to be admissible lens-type R-ends or horospherical R-ends. (Note also that these properties of the examples will hold during deformations as we will show later in Section 6.2.)

5. The relative hyperbolicity of $\pi_1(O)$ and the strict convexity

5.1. SPC-structures and its properties.

Definition 5.1. For a strongly tame orbifold $O$,

(IE) $O$ or $\pi_1(O)$ satisfies the infinite-index end fundamental group condition (IE) if $[\pi_1(O) : \pi_1(E)] = \infty$ for the end fundamental group $\pi_1(E)$ of each end $E$.

(NA) $O$ or $\pi_1(O)$ satisfies the nonannular property (NA) if

$$\pi_1(\tilde{E}_1) \cap \pi_1(\tilde{E}_2)$$

is finite for two distinct p-ends $\tilde{E}_1, \tilde{E}_2$ of $\tilde{O}$, and a free abelian group of rank 2 is conjugate to a subgroup of $\pi_1(E)$ for some end $E$.

(NA) implies that $O$ contains no essential torus and also that $\pi_1(E)$ contains every element $g \in \pi_1(O)$ normalizing $\langle h \rangle$ for an infinite order $h \in \pi_1(E)$ for an end fundamental group $\pi_1(E)$ of an end $E$. These conditions are satisfied by complete hyperbolic manifolds with cusps and are group theoretical properties with respect to the end groups.

Definition 5.2. An SPC-structure or stable properly-convex real projective structure on an $n$-orbifold is a real projective structure so that the orbifold is projectively diffeomorphic to a quotient orbifold of a properly convex domain in $\mathbb{RP}^n$ by a discrete group of projective automorphisms that is stable and irreducible.

Definition 5.3. Suppose that $O$ has an SPC-structure. Let $\tilde{U}$ be the inverse image in $\tilde{O}$ of the union $U$ of some choice of a collection of disjoint end neighborhoods of $O$ with compact Cl($U$). If every straight arc and every non-$C^1$-point in bd$\tilde{O}$ are contained in the closure of a component of $\tilde{U}$, then $O$ is said to be strictly convex with respect to the collection of the ends. And $O$ is also said to have a strict SPC-structure with respect to the collection of ends.

Notice that the definition depends on the choice of $U$. However, we will show that if each component $U$ is required to be of lens-type or horospherical, then we show that the definition is independent of $U$ in [19].

Theorem 5.4. Let $O$ be a noncompact strongly tame properly convex real projective manifold with generalized admissible ends and satisfies (IE) and (NA). Then any finite-index subgroup of the holonomy group is strongly irreducible and is not contained in a proper parabolic subgroup of PGL($n + 1, \mathbb{R}$) (resp. SL$_{\pm}(n + 1, \mathbb{R})$).

For proof, see Section 4 of [19].
5.2. **Bowditch’s method.** There are results proved by Cooper, Long, and Tillman [33] and Crampon and Marquis [34] similar to below. However, the ends have to be horospherical in their work. We will use Bowditch’s result [12] to show

**Theorem 5.5.** Let \( O \) be a noncompact strongly tame strict SPC-orbifold with generalized admissible ends \( E_1, \ldots, E_k \) and satisfies (IE) and (NA). Assume \( \partial O = \emptyset \). Let \( \tilde{U} \) be the inverse image \( U_i \) in \( \tilde{O} \) for a mutually disjoint collection of neighborhoods \( U_i \) of the ends \( E_i \) for \( i = 1, \ldots, k \). Then

- \( \pi_1(O) \) is relatively hyperbolic with respect to the end fundamental groups
  \[ \pi_1(E_1), \ldots, \pi_1(E_k). \]

Hence \( O \) is relatively hyperbolic with respect to \( U_1 \cup \cdots \cup U_k \).
- If \( \pi_1(E_{l+1}), \ldots, \pi_1(E_k) \) are hyperbolic for some \( 1 \leq l \leq k \) (possibly some of the hyperbolic ones), then \( \pi_1(O) \) is relatively hyperbolic with respect to the end fundamental group \( \pi_1(E_1), \ldots, \pi_1(E_l) \).

For definitions and results on relative hyperbolicity of metric spaces, see Bowditch [12] or Farb [37].

The idea for proof is as follows: Let \( U \) be a union of end neighborhoods of \( O \) diffeomorphic to an orbifold times an interval. \( O - U \) is a compact orbifold with boundary. We contract \( CL(C) \cap bd \tilde{O} \) for each component \( C \) of \( p^{-1}(U) \) to a singleton to obtain a quotient space \( X \). Then \( X \) is homeomorphic to a compact metric space, i.e., a compactum. We demonstrate that the axioms of Bowditch are satisfied by analyzing the triples of points in \( X \) in [19].

5.3. **Converse.** The converse to Theorem 5.5 is as follows:

**Theorem 5.6.** Let \( O \) be a noncompact strongly tame properly convex real projective orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume \( \partial O = \emptyset \). Suppose that \( \pi_1(O) \) is a relatively hyperbolic group with respect to the admissible end groups \( \pi_1(E_1), \ldots, \pi_1(E_k) \). Then \( O \) is strictly SPC with respect to the admissible ends \( E_1, \ldots, E_k \).

Let \( U \) be as in the above section, and let \( \tilde{U} = p^{-1}(U) \subset \tilde{O} \). To give some idea of the proof, we take any segment in \( bd \tilde{O} \) not contained in any component of \( CL(\tilde{U}) \cap bd \tilde{O} \). Then we find a triangle \( T \) with \( \partial T \subset bd \tilde{O} \) using a sequence of points converging to an interior point of the segment. Also, we construct so that \( \partial T \) is not in the closure of any \( p \)-end neighborhood.

Let us recall standard definitions in Section 3.1 of Drutu-Sapir [36]. An **ultrafilter** \( \omega \) is a finite additive measure on \( P(\mathbb{N}) \) of \( \mathbb{N} \) so that each subset has either measure 0 or 1 and all finite sets have measure 0. If a property \( P(n) \) holds for all \( n \) from a set with measure 1, we say that \( P(n) \) holds \( \omega \)-almost surely.

Let \( (X, d_X) \) be a metric space. Let \( \omega \) be a nonprincipal ultrafilter over the set \( \mathbb{N} \) of natural numbers. For a sequence \( (x_i)_{i \in \mathbb{N}} \) of points of \( X \), its \( \omega \)-**limit** is \( x \in X \) if for every neighborhood \( U \) of \( x \) the property that \( x_i \in U \) holds \( \omega \)-almost surely.

An **ultraproduct** \( \prod X_n/\omega \) of a sequence of sets \( (X_n)_{n \in \mathbb{N}} \) is the set of the equivalence classes of sequences \( (x_n) \) where \( (x_n) \sim (y_n) \) if \( x_n = y_n \) holds for \( \omega \)-almost surely.

Given a sequence of metric spaces \( (X_n, d_n) \), consider the ultraproduct \( \prod X_n \) and an observation point \( e = (e_n) \). Let \( D(x, y) = \lim_\omega d_n(x_n, y_n) \). Let \( \prod e X_n/\omega \) denote...
the set of equivalence classes of sequences of bounded distances from $e$. The $\omega$-limit $\lim^\omega(X_n)_e$ is the metric space obtained from $\prod_e X_n/\omega$ by identifying all pair of points $x, y$ with $D(x, y) = 0$.

Given an ultrafilter $\omega$ over the set $\mathbb{N}$ of natural numbers, an observation point $e = (e_i)_{i \in \mathbb{N}}$, and sequence of numbers $\delta = (\delta_i)_{i \in \mathbb{N}}$ satisfying $\lim_\omega \delta_i = \infty$, the $\omega$-limit $\lim^\omega(X, d_X/\delta_i)_e$ is called the asymptotic cone of $X$. (See [42], [43] and Definitions 3.3 to 3.8 in [36].) We denote it by $\text{Con}^\omega(X, e, \delta)$.

For a sequence $(A_n)$ of subsets $A_n$ of $X$, we denote by $\lim^\omega(A_n)$ the subset of $\text{Con}^\omega(X, e, \delta)$ that consists of all elements $(x_n)$ where $x_n \in A_n$ $\omega$-almost surely. The asymptotic cone is always complete and $\lim^\omega(A_n)$ is closed.

Next, we choose a nonprincipal ultrafilter $\omega$ and a sequence $l_i \to \infty$. We use the $\omega$-limit $\tilde{O}$ of $\frac{1}{l_i}d_{\tilde{O}}$ on $\tilde{O}$ with a constant base point $e_i = e \in \tilde{O}$. This turns out to be a tree-graded space in the sense of Drutu and Sapir [36]. Let $T^\omega_i$ be $T^\omega$ with the metric $\frac{1}{l_i}d_{\tilde{O}}|T^\omega$, which is a hex metric of de la Harpe [35].

The inverse image in $U$ of $\tilde{O}$ of the union of disjoint end neighborhoods of $O$. A piece in the limit tree-graded space is a limit of a sequence of components of $U$ by Proposition 7.26 of [36]. The sequence $\{T^\omega_i \subset \tilde{O}\}$ converges to a triangle $T^\omega$ with the hex metric and we show that $T^\omega$ is not contained in a piece by a geometric argument. However, a triangle with a hex metric cannot be divided into more than one piece.

5.4. Strict SPC-structures deform to strict SPC-structures. By above Theorems 5.5 and 5.6, the property of strictness of the SPC-structures is topological. Hence, the strictness is a stable property among the set of the SPC-structures.

**Theorem 5.7.** Let $O$ denote a noncompact strongly tame strict SPC-orbifold with admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$. Let

$$E_1, \ldots, E_n, E_{n+1}, \ldots, E_k$$

be the ends of $O$ where $E_{n+1}, \ldots, E_k$ are some or all of the hyperbolic ends.

- Given a deformation through SPC-structures with generalized admissible ends of a strict SPC-orbifold with respect to admissible ends $E_1, \ldots, E_k$ to an SPC-structure with generalized admissible end, the SPC-structures remain strictly SPC with respect to $E_1, \ldots, E_k$.
- Given a deformation through SPC-structures with generalized admissible ends of a strict SPC-orbifold with respect to $E_1, \ldots, E_n$ to an SPC-structure with generalized admissible end, the SPC-structures remain strictly SPC with respect to admissible ends $E_1, \ldots, E_n$.

6. The openness and closedness in character varieties

We will now begin to discuss the main aim of this paper. This is to identify the deformation spaces of convex real projective structures on a strongly tame orbifold $O$ with end conditions with parts of character varieties of $\pi_1(O)$ with corresponding conditions on holonomy groups of ends. We mention that the uniqueness condition below simplifies the theory greatly. Otherwise, we need to use the sections picking the vertices and the totally geodesic planes fixed by the holonomy group of each end.
6.1. The semi-algebraic properties of $\text{rep}^\ast(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ and related spaces. We will now recall Section 2.1.8 and make it more precise.

A parabolic subalgebra $\mathfrak{p}$ is an algebra in a semisimple Lie algebra $\mathfrak{g}$ whose complexification contains a maximal solvable subalgebra of $\mathfrak{g}$ (p. 279–288 of [59]). A parabolic subgroup $P$ of a semisimple Lie group $G$ is the full normalizer of a parabolic subalgebra.

We recall from Section 2.1.8. Since $O$ is the interior of a compact orbifold, there exists a finite set of generators $g_1, \ldots, g_m$ with finitely many relators. First, $\text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ can be identified with a semi-algebraic subset of $\text{PGL}(n+1, \mathbb{R})^m$ corresponding to the relators. Each end of $O$ is assigned to be an $R$-type end or a $T$-type end.

Let $\text{Hom}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ denote the subspace of

$$\text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$$

where the holonomy of each p-end fundamental group fixes a point of $\mathbb{RP}^n$ for an end of type $R$ or acts on a subspace $P$ of codimension-one and on a lens meeting $P$ satisfying the lens-condition or a horoball tangent to $P$ for an end of type $T$. Since there are only finitely many p-end fundamental groups up to conjugation by elements of $\pi_1(O)$, we obtain that

$$\text{Hom}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$$

is a closed semi-algebraic subset provided that is no $T$-end. If there are $T$-ends, then we obtain a union of open subsets of closed semi-algebraic subsets.

Since each end fundamental group is finitely generated, the conditions of having a common 1-dimensional eigenspace for each of a finite collection of finitely generated subgroups is a semi-algebraic condition.

Let $\rho \in \text{Hom}_E(\pi_1(E), \text{PGL}(n+1, \mathbb{R}))$ where $E$ is a horospherical end. Then $\rho(\pi_1(E))$ is virtually abelian by Theorem 1.1 of [15]. Define

$$\text{Hom}_E,\text{par}(\pi_1(E), \text{PGL}(n+1, \mathbb{R}))$$

to be the space of representations where an abelian group of finite index goes into a parabolic subgroup in a copy of $\text{PO}(n, 1)$. By Lemma 6.1,

$$\text{Hom}_E,\text{par}(\pi_1(E), \text{PGL}(n+1, \mathbb{R}))$$

is a closed semi-algebraic set.

**Lemma 6.1.** Let $G$ be a finite extension of a finitely generated free abelian group $\mathbb{Z}^m$. Then $\text{Hom}_E,\text{par}(G, \text{PGL}(n+1, \mathbb{R}))$ is a closed algebraic set.

**Proof.** Let $P$ be a maximal parabolic subgroup of a copy of $\text{PO}(n+1, \mathbb{R})$ that fixes a point $x$. Then $\text{Hom}(\mathbb{Z}^m, P)$ is a closed semi-algebraic set.

$$\text{Hom}_E,\text{par}(\mathbb{Z}^m, \text{PGL}(n+1, \mathbb{R}))$$

equals a union

$$\bigcup_{g \in \text{PGL}(n+1, \mathbb{R})} \text{Hom}(\mathbb{Z}^m, gPg^{-1}),$$

another closed semi-algebraic set. Now $\text{Hom}_E,\text{par}(G, \text{PGL}(n+1, \mathbb{R}))$ is a closed semi-algebraic subset of

$$\text{Hom}_E,\text{par}(\mathbb{Z}^m, \text{PGL}(n+1, \mathbb{R})).$$

$\square$
Let $E$ be an end orbifold of $\mathcal{O}$. Given
\[ \rho \in \text{Hom}_E(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})), \]
we define the following sets:

- Let $E$ be an end of type $\mathcal{R}$. Let
  \[ \text{Hom}_{E,\text{RL}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \]
  denote the space of representations $h$ of $\pi_1(E)$ where $h(\pi_1(E))$ acts on a lens-cone \( \{p\} \ast L \) for a lens $L$ and $p \notin \text{Cl}(L)$ of a p-end $\tilde{E}$ corresponding to $E$ and the lens $L$ itself. Thus, it is an open subspace of the above semi-algebraic set $\text{Hom}_E(\pi_1(E), \text{PGL}(n + 1, \mathbb{R}))$.

- Let $E$ denote an end of type $\mathcal{T}$. Let
  \[ \text{Hom}_{E,\text{TL}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \]
  denote the space of totally geodesic representations $h$ of $\pi_1(E)$ satisfying the following condition:
  - $h(\pi_1(E))$ acts on an lens $L$ and a hyperspace $P$ where
  - $L^o \cap P \neq \emptyset$ and
  - $L/h(\pi_1(E))$ is a compact orbifold with two strictly convex boundary components.

\[ \text{Hom}_{E,\text{TL}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \]
again an open subset of the semi-algebraic set
\[ \text{Hom}_E(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})). \]
(This follows by the proof of Theorem 8.1 of [15].)

Let $R_E : \text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \ni h \to h|\pi_1(E) \in \text{Hom}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R}))$ be the restriction map to the p-end fundamental group $\pi_1(E)$ corresponding to the end $E$ of $\mathcal{O}$.

A representative set of p-ends of $\tilde{\mathcal{O}}$ is the subset of p-ends where each end of $\mathcal{O}$ has a corresponding p-end and a unique corresponding p-end. Let $\mathcal{R}_\mathcal{O}$ denote the representative set of p-ends of $\tilde{\mathcal{O}}$ of type $\mathcal{R}$, and let $\mathcal{T}_\mathcal{O}$ denote the representative set of p-ends of $\tilde{\mathcal{O}}$ of type $\mathcal{T}$. We define a more symmetric space:

\[ \text{Hom}_{\mathcal{O},\text{cc}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \]
to be
\[ \text{Hom}^*(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \cap 
\left( \bigcup_{E \in \mathcal{R}_\mathcal{O}} R_E^{-1} \left( \text{Hom}_{\mathcal{O},\text{RL}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \cup \text{Hom}_{\mathcal{O},\text{RL}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \right) \right) \cap 
\left( \bigcup_{E \in \mathcal{T}_\mathcal{O}} R_E^{-1} \left( \text{Hom}_{\mathcal{O},\text{TL}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \cup \text{Hom}_{\mathcal{O},\text{TL}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \right) \right). \]

Hence, this is a union of open subsets of semi-algebraic sets in
\[ X := \text{Hom}_{\mathcal{O},\text{cc}}^*(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})). \]
(We don’t claim that the union is open in $X$. These definitions allow for changes between horospherical ends to lens-type radial ones and totally geodesic ones.)
Let $\text{Hom}^s_{E,u}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$ denote the subspace of 
$\text{Hom}^s_{E}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$
where each element $h$ satisfies the following properties:

- $h|\pi_1(\tilde{E})$ fixes a unique point of $\mathbb{RP}^n$ corresponding to the common eigenspace of positive eigenvalues for lifts of elements of the p-end fundamental group $\pi_1(\tilde{E})$ of $\mathcal{R}$-type (recall Remark 2.2) and
- $h|\pi_1(\tilde{E})$ has a common null-space $P$ of an eigen-1-forms which is unique under the condition that
  - $\pi_1(\tilde{E})$ acts properly on a lens $L$ with $L \cap P$ with nonempty interior in $P$ or
  - $H - \{p\}$ for a horosphere $H$ tangent to $P$ at $p$
for each p-end fundamental group $\pi_1(\tilde{E})$ of the end of $\mathcal{T}$-type.

We obtain the union of open subsets of semi-algebraic subsets since we need to consider finitely many generators of the fundamental groups of the ends again by Lemma 2.8.

**Remark 6.2.** The lens condition is equivalent to the condition here. We repeat it here to put these into set theoretical terms.

Since

$\text{rep}^s_{E,u}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$

is the Hausdorff quotient of the above set with the conjugation $\text{PGL}(n + 1, \mathbb{R})$-action, this is the union of open subsets of semi-algebraic subset by Proposition 1.1 of [45].

We define

$\text{Hom}^s_{E,u,ce}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$

to be the subset

$\text{Hom}^s_{E,u}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R})) \cap
\left( \bigcap_{E \in \mathcal{R}_{\mathcal{O}}} R^{-1}_E (\text{Hom}_{E,\text{zar}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \cup \text{Hom}_{E,\mathbb{R}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R}))) \right) \cap
\left( \bigcap_{E \in \mathcal{T}_{\mathcal{O}}} R^{-1}_E (\text{Hom}_{E,\text{zar}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R})) \cup \text{Hom}_{E,\mathbb{R}}(\pi_1(E), \text{PGL}(n + 1, \mathbb{R}))) \right)$. 

Similarly to the above,

$\text{rep}^s_{E,u,ce}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$

is a union of open subsets of strata in

$\text{rep}^s_{E,u}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$.

**Example 6.3.** The uniqueness as above holds automatically for convex real projective orbifolds where a set of orbifold singularities contains a leaf of a radial foliation of an $\mathcal{R}$-end neighborhood as in Example 2.6. By Theorem 6.5, we obtain that these have lens-shaped radial ends only.

**Lemma 6.4.** Suppose that $O$ is a strongly tame real projective orbifold with radial ends. Suppose the end fundamental group of an end $E$ is
- virtually generated by finite order elements and
- is virtually abelian or is hyperbolic.
Suppose that the end orbifold $\Sigma_E$ is convex. Then the end $E$ is either properly convex lens-type radial end or is horospherical.

**Theorem 6.5.** Suppose that $O$ is a strongly tame properly convex real projective orbifold with radial ends. Suppose that each end fundamental group is

- virtually generated by finite order elements and
- is virtually abelian or is hyperbolic.

Then the holonomy is in

$$\text{Hom}^E_{\Sigma, u, ce}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R})).$$

We need the end classification results from [16], [17], and [18].

We immediately obtain:

**Corollary 6.6.** Let $M$ be a real projective orbifold with radial ends. Suppose that $M$ admits finite-volume hyperbolic 3-orbifold with ends that are horospherical. Suppose that the end orbifold is either a small orbifold with cone points of orders 3 or a disk orbifold with corner reflectors orders 6. Then the ends must be of lens-type $R$-ends or horospherical $R$-ends.

**Proof.** Each end fundamental group is virtually abelian of rank 2. Hence, the end orbifold is finitely covered by a 2-torus. By the classification of real projective 2-torus [6], and the existence of order 3 or 6 singularities, the torus must be properly convex or complete affine. By the proof of Proposition 4.6 of [16], the end is either horospherical or of lens-type. (Here we just need that the end orbifold be convex.)

This result may be generalized to higher dimensions but we lack the formulation.

6.1.1. **Main theorems.** We now state our main results:

- We define $\text{Def}^E_{\Sigma, u, ce}(O)$ to be the subspace of $\text{Def}^E(O)$ with real projective structures with generalized admissible ends and stable irreducible holonomy homomorphisms.
- We define $C\text{Def}^E_{\Sigma, u, ce}(O)$ to be the subspace of $\text{Def}^E_{\Sigma, u}(O)$ consisting of SPC-structures with generalized admissible ends.
- We define $S\text{Def}^E_{\Sigma, u, ce}(O)$ to be the subspace of $\text{Def}^E_{\Sigma, u, ce}(O)$ consisting of strict SPC-structures with admissible ends.

We remark that these spaces are dual to the same type of the spaces but switching the $R$-end with $T$-ends and vice versa by Proposition 3.7.

**Theorem 6.7.** Let $O$ be a noncompact strongly tame $n$-orbifold with generalized admissible ends. Assume $\partial O = \emptyset$. Suppose that $O$ satisfies (IE) and (NA). Then the subspace

$$C\text{Def}^E_{\Sigma, u, ce}(O) \subset \text{Def}^E_{\Sigma, u, ce}(O)$$

is open.

Suppose further that every finite-index subgroup of $\pi_1(O)$ contains no nontrivial infinite nilpotent normal subgroup. Then hol maps $C\text{Def}^E_{\Sigma, u, ce}(O)$ homeomorphically to a union of components of

$$\text{rep}^E_{\Sigma, u, ce}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R})).$$

**Theorem 6.8.** Let $O$ be a strict SPC noncompact strongly tame $n$-dimensional orbifold with admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$. Then
Suppose further that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup. Then hol maps the deformation space $\text{SDef}_{E,u,ce}(\mathcal{O})$ of strict SPC-structures on $\mathcal{O}$ with admissible ends homeomorphically to a union of components of

$$\text{rep}_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})).$$

We prove Theorems 6.7 and 6.8 by dividing into the openness result in Section 6.2 and the closedness result in Section 6.3.

### 6.2. Openness

We will show the following by proving Theorem 6.10.

**Theorem 6.9.** Let $\mathcal{O}$ be a noncompact strongly tame real projective $n$-orbifold and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. In $\text{Def}_{E,u,ce}(\mathcal{O})$, the subspace $\text{CDef}_{E,u,ce}(\mathcal{O})$ of SPC-structures with generalized admissible ends is open, and so is $\text{SDef}_{E,u,ce}(\mathcal{O})$.

We are given a properly real projective orbifold $\mathcal{O}$ with ends $E_1, \ldots, E_{e_1}$ of $\mathcal{R}$-type and $E_{e_1+1}, \ldots, E_{e_1+e_2}$ of $\mathcal{T}$-type. Let us choose representative $p$-ends $\tilde{E}_1, \ldots, \tilde{E}_{e_1}$ and $\tilde{E}_{e_1+1}, \ldots, \tilde{E}_{e_1+e_2}$. Again, $e_1$ is the number of $\mathcal{R}$-type ends, and $e_2$ the number of $\mathcal{T}$-type ends of $\mathcal{O}$.

We define a subspace of $\text{Hom}_{E}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$ to be as in Section 6.1.

Let $\mathcal{V}$ be an open subset of

$$\text{Hom}_{E}^{\mathcal{V}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$$

invariant under the conjugation action of $\text{PGL}(n + 1, \mathbb{R})$ so that the following hold:

- one can choose a continuous section $s_{\mathcal{V}}^{(1)} : \mathcal{V} \to (\mathbb{R}P^n)^{e_1}$ sending a holonomy homomorphism to a common fixed point of $\Gamma_{\tilde{E}_i}$ for $i = 1, \ldots, e_1$ and
- $s_{\mathcal{V}}^{(1)}$ satisfies

  $$s_{\mathcal{V}}^{(1)}(gh(\cdot)g^{-1}) = g \cdot s_{\mathcal{V}}^{(1)}(h(\cdot))$$

for $g \in \text{PGL}(n + 1, \mathbb{R})$.

$s_{\mathcal{V}}^{(1)}$ is said to be a fixed-point section. In these cases, we say that $\mathcal{R}$-end structures are determined by $s_{\mathcal{V}}^{(1)}$.

Again we assume that for the open subset $\mathcal{V}$ of

$$\text{Hom}_{E}^{\mathcal{V}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$$

the following hold:

- one can choose a continuous section $s_{\mathcal{V}}^{(2)} : \mathcal{V} \to (\mathbb{R}P^n)^{e_2}$ sending a holonomy homomorphism to a common dual fixed point of $\pi_1(\tilde{E}_i)$ for $i = e_1 + 1, \ldots, e_1 + e_2$, and
- $s_{\mathcal{V}}^{(2)}$ satisfies

  $$s_{\mathcal{V}}^{(2)}(gh(\cdot)g^{-1}) = (g^*)^{-1} \circ s_{\mathcal{V}}^{(2)}(h(\cdot))$$

for $g \in \text{PGL}(n + 1, \mathbb{R})$, and $s_{\mathcal{V}}^{(2)}$ is said to be a dual fixed-point section. In this case, we say that $\mathcal{T}$-end structures are determined by $s_{\mathcal{V}}^{(2)}$.

We define $s_{\mathcal{V}} : \mathcal{V} \to (\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^n)^{e_2}$ as $s_{\mathcal{V}}^{(1)} \times s_{\mathcal{V}}^{(2)}$ and call it a fixing section. Let $\mathcal{V}$ and $s_{\mathcal{V}} : \mathcal{V} \to (\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^n)^{e_2}$ be as above.
We define $\text{Def}_E^{s,ce}(O)$ to be the subspace of $\text{Def}_E^{s,ce}(O)$ of real projective structures with generalized admissible ends and stable irreducible holonomy homomorphisms in $\mathcal{V}$.

We define $\text{CDef}_E^{s,ce}(O)$ to be the subspace consisting of SPC-structures with generalized admissible ends and holonomy homomorphisms in $\mathcal{V}$ in $\text{Def}_E^{s,ce}(O)$.

We define $\text{SDef}_E^{s,ce}(O)$ to be the subspace of consisting of strict SPC-structures with admissible ends and holonomy homomorphisms in $\mathcal{V}$ in $\text{Def}_E^{s,ce}(O)$.

**Theorem 6.10.** Let $O$ be a noncompact strongly tame real projective $n$-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$.

Choose an open $\text{PGL}(n+1, \mathbb{R})$-conjugation invariant set

$$\mathcal{V} \subset \text{Hom}^e_\mathbb{R}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})),$$

and a fixing section $s_V : \mathcal{V} \rightarrow (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^n)^{e_2}$.

Then $\text{CDef}_E^{s,ce}(O)$ is open in $\text{Def}_E^{s,ce}(O)$, and so is $\text{SDef}_E^{s,ce}(O)$.

By Theorems 6.9 and 2.10, we obtain:

**Corollary 6.11.** Let $O$ be a noncompact strongly tame real projective $n$-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$.

Then

$$\text{hol} : \text{CDef}_E^{s,ce}(O) \rightarrow \text{rep}_E^{s,ce}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$$

is a local homeomorphism.

Furthermore, if $O$ has a strict SPC-structure with admissible ends, then so is

$$\text{hol} : \text{SDef}_E^{s,ce}(O) \rightarrow \text{rep}_E^{s,ce}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).$$

We just give a heuristic idea for proof of Corollary 6.11 for $\text{SDef}_E^{s,ce}(O)$ here as in [19]. We begin by taking a properly convex open cone $C$ in $\mathbb{R}^{n+1}$ corresponding to $\bar{O}$. Let $O$ be a properly convex real projective orbifold with generalized admissible ends. Let $\Gamma = h(\pi_1(O))$ in $\text{PGL}(n+1, \mathbb{R})$ be the holonomy group. Then $\Gamma$ acts on $C$. There is a Koszul-Vinberg function on $C$. The Hessian will be a $\Gamma$-invariant metric. Thus, $O$ has an induced metric $\mu$ by using sections. For each p-end $\bar{E}$, we obtain a p-end neighborhood $U$. We approximate the p-end neighborhood $U$ very close to $\bar{O}$ in the Hausdorff metric $d^H$. Let $C_U \subset \mathbb{R}^{n+1}$ denote the open cone associated with $U$. We choose a Koszul-Vinberg Hessian metric on $C_U$ approximating that of $C$.

Let $h_t$ be a parameter of representations in the appropriate character variety with $h_0 = h$. Let $\Gamma_t = h_t(\pi_1(O))$ where $\Gamma_0 = \Gamma$.

By Theorem 2.10, a convex real projective structure on $O$ with radial or totally geodesic ends has the holonomy homomorphism $h_t$ and developing maps $\text{dev}_t : O \rightarrow \mathbb{RP}^n$. $\text{dev}_0$ can be considered as the inclusion $O \rightarrow \mathbb{RP}^n$. $\text{dev}_t$ may not be an inclusion in general. However, by the generalized admissibility of the ends, $\text{dev}_t|U$ is an imbedding for $0 < t < \epsilon$ for sufficiently small $\epsilon > 0$. Here, a key point is that $U_t$ can be chosen to be properly convex when deforming. If $\Gamma_t|\pi_1(E)$ is not virtually abelian, it cannot have any horospherical representation. Thus, by the stability of the lens condition, we have openness for the associated end neighborhood. If $\pi_1(E)$ is virtually abelian, $\Gamma_0|\pi_1(E)$ can be horospherical. If $\Gamma_t|\pi_1(E)$ is horospherical for $t > 0$, then this case is straightforward. We consider the case when $\Gamma_t|\pi_1(E)$, $t > 0$, then this case is straightforward.
becomes diagonalizable by our lens condition for holonomy homomorphisms. By the classification of Benoist [5] of projective structures with diagonalizable holonomy groups, we have brick decompositions for the transversal orbifold structure $\Sigma_{E,t}$ for $E$ and $\text{dev}_t$. The transversal projective structure on $\Sigma_{E,t}$ has to have only one brick. Otherwise, we can find a fundamental domain of a finite cover of $\Sigma_{E,t}$ which under $\text{dev}_t$ is noninjective. Since during the deformation the brick number doesn’t change, we obtain a contradiction that $\text{dev}_t|F_t$ cannot converge to a map with compact image as $t \to 0$. (In other words, a sequence of real projective structures with more than one bricks cannot converge to a complete affine structure.) Therefore, $\Sigma_{E,t}$ is properly convex for $\text{dev}_t$, $t > 0$. Since the holonomy is for the lens type ones, we will have lens-type ends. (See Proposition 6.5 of [19].) Let $C_{U_t} \subset \mathbb{R}^{n+1}$ denote the convex open cone corresponding to $U_t = \text{dev}_t(U) \subset \mathbb{R}^n$.

The affine space $\mathbb{R}^{n+1}$ is compactified as $\mathbb{RP}^{n+1}$. We show that $\text{Cl}(C_{U_t})$ changes in a continuous manner under the Hausdorff metric in $\mathbb{RP}^{n+1}$ containing $\mathbb{R}^{n+1}$. Also, the Hessian metric $\mu$ in the complement of the union of end neighborhoods varies continuously to Hessian metrics $\mu_t$. Then we patch these metrics together to obtain a Hessian metric for $\mathcal{O}$. The existence of Hessian metrics and by generalized admissibility of ends, we can show the proper convexity. (The author learned that Cooper, Long, and Tillman [32] came up with the similar arguments in slightly different settings. Also, they use different topology using developing maps of ends neighborhood. This makes thing simpler and maybe more clear.)

### 6.3. The closedness of convex real projective structures.

We recall

$$\text{rep}_E^*(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

the subspace of stable irreducible characters of

$$\text{rep}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

which is shown to be the union of open subsets of semi-algebraic subsets in Section 6.1, and denote by $\text{rep}_{E,u,ce}^*(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ the subspace of stable irreducible characters of $\text{rep}_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$, an open subset of a semialgebraic set.

In this section, we will need to discuss $\mathbb{S}^n$ but only inside a proof.

**Theorem 6.12.** Let $\mathcal{O}$ be a noncompact strongly tame SPC $n$-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(\mathcal{O})$ are trivial. Then the following hold:

- **The deformation space** $\text{CDef}_{E,u,ce}(\mathcal{O})$ of SPC-structures on $\mathcal{O}$ with generalized admissible ends maps under $\text{hol}$ homeomorphically to a union of components of $\text{rep}_{E,u,ce}^*(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$.
- **The deformation space** $\text{SDef}_{E,u,ce}(\mathcal{O})$ of strict SPC-structures on $\mathcal{O}$ with admissible ends maps under $\text{hol}$ homeomorphically to the union of components of $\text{rep}_{E,u,ce}^*(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$.

We will give some general idea behind the proofs here and show only the closedness. We will use the developing maps to $\mathbb{S}^n$. Given a sequence $\mu_i$ of properly convex real projective structures on $\mathcal{O}$ satisfying some boundary conditions, let $h_i$ denote the corresponding holonomy homomorphism with developing maps $\text{dev}_i$. 


Let $K_i \subset S^n$ denote the closure of the images $\text{dev}_i(\tilde{O})$. We may choose a subsequence so that $h_i \to h$ for a representation $h : \pi_1(O) \to \text{PGL}(n+1, \mathbb{R})$ and $K_i \to K$ for a compact convex domain $K$.

If $K_i$ geometrically converges to a convex domain $K$ with empty interior, then $h$ is reducible since $K$ is contain in a proper subspace. Hence, $h$ is not in the target character space.

Suppose that $K_i$ geometrically converges to a convex domain $K$ with nonempty interior $K^\circ$ by choosing a subsequence. Suppose that $K^\circ$ is not properly convex. Then $K$ contain a pair of antipodal points. We take the maximal great sphere $S_i$ in $K$ for $i \geq 0$. The limiting holonomy group acts on $S_i$ and hence is reducible. Thus, $h$ is not in the target character subspace.

Hence, $K$ is properly convex, and $K^\circ/\Gamma$ is a properly convex real projective orbifold $O'$. We can show that $O'$ is diffeomorphic to $O$. By [40], we can show that $h_i$ converges to a faithful representation $h : \pi_1(O) \to \text{SL}_\pm(n+1, \mathbb{R})$.

6.4. Nicest cases. Theorems 6.5 and 6.12 imply the following:

**Corollary 6.13.** Let $O$ be a strongly tame SPC $n$-dimensional real projective orbifold with only radial ends and satisfies (IE) and (NA). Suppose that each end fundamental group is generated by finite order elements and is virtually abelian or hyperbolic. Assume $\partial O = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(O)$ are trivial. Then $\text{hol}$ maps the deformation space $C\text{Def}_E(O)$ of SPC-structures on $O$ homeomorphically to a union of components of $\text{rep}^E_{\text{u,ce}}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$.

The same can be said for $S\text{Def}_E(O)$.

These types of deformations from structures with cusps to ones with lens-type ends are realized in our main examples as stated in Section 1.2. We need the restrictions on the target space since the convexity of $O$ is not preserved under the hyperbolic Dehn surgery deformations of Thurston, as pointed out by Cooper at ICERM in September 2013.

Strongly tame properly convex Coxeter orbifolds admitting complete hyperbolic structures will satisfy the premise. Also, $2h_{1,1}$ and the double of the simplex orbifold do also.

For Coxeter orbifolds, this simplifies further.

**Corollary 6.14.** Let $O$ be a strongly tame Coxeter $n$-dimensional real projective orbifold, $n \geq 3$, with only radial ends admitting a complete hyperbolic structure. Then $S\text{Def}_E,\text{u,ce}(O)$ is homeomorphic to the union of components of $\text{rep}^E(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$.

Finally, $S\text{Def}_E,\text{u,ce}(O) = S\text{Def}_E(O)$.

We give a sketch of the proof. Consider a component $C$ of $\text{rep}^E_{\text{u,ce}}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$ corresponding to a component of $S\text{Def}_E,\text{u,ce}(O)$. Let $C'$ be the inverse image of $C$ in $\text{Hom}^E_{\text{u,ce}}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))$. 

Then we claim that $C'$ is open in

$$\text{Hom}_E^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) :$$

Let $h : \pi_1(O) \to \text{PGL}(n+1, \mathbb{R})$ be a representation in $C'$. By Theorem 2.10, there is a neighborhood $J$ of $h$ realized by orbifold $O_b$ diffeomorphic to $O$ for each $b \in J$. Each end is convex since a compact projective Coxeter $(n-1)$-orbifold, $n-1 \geq 2$, admitting a Euclidean structure, is always convex by Vinberg [62]. By Lemma 6.4, the end is properly convex of lens-type or is horospherical. Hence, $h$ is in

$$\text{Hom}_E^{\text{u,ce}}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).$$

The closedness follows as in Section 6.3.

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