Elastic neutrino – electron scattering of solar neutrinos and potential effects of magnetic and electric dipole moments

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Abstract

We consider elastic neutrino – electron scattering of solar neutrinos with magnetic moments and electric dipole moments, where the solar neutrino state at the scattering site is determined by the evolution in matter and solar magnetic fields of the initial electron neutrino state. We present the general cross section for an arbitrary superposition of active and sterile neutrino types with positive and negative helicities, with particular emphasis on the effect of transverse polarization, which gives rise to an azimuthal asymmetry as a function of the recoil electron momentum. Within our physically motivated approximation, we perform a general CP analysis and show that in the 1-Dirac and 2-Majorana neutrino cases no CP-violating effects are present, which means that it is not possible to distinguish between magnetic and electric dipole moments in these cases. We also study the consequences of neutrino energy averaging on the cross section and stress that in the 2-Majorana neutrino case this averaging leads to a suppression of the transverse neutrino polarization effects.

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I. INTRODUCTION

Recently, the physics of neutrino oscillations [1] has become one of the most active fields of research in particle physics. One of the reasons is the experimental evidence for neutrino oscillations found in atmospheric neutrino measurements [2]. The other important problem in this field is the longstanding solar neutrino deficit which also finds a natural explanation in terms of neutrino oscillations, whether by vacuum oscillations or by the MSW effect [3] (for recent works see [4,5]). Concerning the solar neutrino puzzle, it has been noticed long time ago that neutrino magnetic moments (MM) and/or electric dipole moments (EDM) of order $10^{-11}$ $\mu_B$, where $\mu_B$ is the Bohr magneton, and a sizable magnetic field in the solar interior can contribute to a solution of this problem [3-8] (for reviews see also Ref. [9]).

A particular attractive scenario in this context, which combines the matter effect with the effect of solar magnetic fields and neutrino MMs or EDMs, is given by resonant spin–flavour precession (RSFP) [10–12], which allows for good fits of the solar neutrino data (for recent papers see [13–19]). This scenario is possible even without neutrino mixing. However, very little is known about magnetic fields in the solar interior and one has to resort to plausible assumptions, a problem in all attempts to solve the solar neutrino puzzle with neutrino MMs and EDMs.

If neutrinos have MMs and/or EDMs then the electromagnetic interaction will give a contribution to elastic neutrino–electron scattering in addition to the weak interactions [20], which is enhanced at low energies. This observation has been used to derive laboratory limits on neutrino MM/EDMs [21]. Independently, such limits can also be obtained by taking advantage of the solar neutrino flux [22,23]. Furthermore, if MM/EDMs of solar neutrinos and solar magnetic fields are important for a solution of the solar neutrino puzzle, then the solar neutrino flux on earth will have some transverse polarization in general. It has been stressed that such a polarization leads to a weak-electromagnetic interference cross section in elastic neutrino–electron scattering, which could be observed via an azimuthal asymmetry in the distribution of the electron recoil momenta in the plane orthogonal to the incoming neutrino momentum [24–29]. A suitable experiment to find such an effect could be the HELLAZ experiment [30].

It is the purpose of this paper to present a general and consistent study of elastic neutrino–electron scattering of solar neutrinos, thereby incorporating the twofold effects of the neutrino MM/EDMs: 1. They contribute to the evolution of the initial electron neutrino state with negative helicity to the state which is detected on earth by elastic neutrino–electron scattering. 2. Neutrino MM/EDMs contribute to elastic neutrino–electron scattering by providing a pure electromagnetic cross section and a weak-electromagnetic interference cross section. We will assume an arbitrary number of neutrinos, including neutrinos of the sterile type, and derive the cross section for an arbitrary superposition of neutrino flavours or types and helicities. We will meticulously distinguish between Dirac and Majorana neutrinos. We will employ the following approximation in our physical scenario: we neglect all neutrino masses in the elastic neutrino–electron cross section, but retain the usual term quadratic in the neutrino masses in the neutrino evolution equation [3]. This has to be kept in mind when assessing the results of this paper.

Since we neglect neutrino masses in scattering, the most adequate basis for our consideration is the flavour basis of neutrino states, though physically, as we will show, no basis is
preferred. We, therefore, put particular emphasis on the formulation of the MM and EDM matrices in the flavour basis and stress that the most useful entity in this context is the MM/EDM matrix $\lambda = \mu - id$, where $\mu$ and $d$ are the MM and EDM matrices, respectively. Considering $\lambda$ and the neutrino mixing matrix $U_L$, we make a general discussion of the independent, physical phases in our problem. We will show that there are no such phases in the 1-Dirac and 2-Majorana neutrino cases, from which it follows that in these cases no distinction between MM and EDM within our approximation is possible. Finally, we discuss decoherence effects as a consequence of neutrino energy averaging, which is of importance for the weak-electromagnetic interference cross section and neutrino mass-squared differences larger than around $10^{-10}$ eV$^2$. This effect is caused by vacuum oscillations between sun and earth and by the inevitable energy averaging due to finite energy and angle resolution in the detection of the recoil electron in $\nu e^{-}$ scattering.

The paper is organized as follows. In Section II we discuss the MM and EDM interaction for Dirac and Majorana neutrinos. The evolution equation of the solar neutrino state in matter and magnetic fields is explained in Section III. Section IV treats the formulation of the neutrino density matrix, which is applied in Section V in the calculation of the elastic neutrino – electron cross section. Section VI contains the applications of Sections III and V for the 1-Dirac and 2-Majorana cases and a general discussion of the physical phases and the effect of CP invariance in our problem. Section VII discusses the decoherence effect due to neutrino energy averaging and in Section VIII we present our conclusions.

II. NEUTRINO MAGNETIC MOMENTS AND ELECTRIC DIPOLE MOMENTS

If Dirac neutrinos are furnished with magnetic moments (MM) and electric dipole moments (EDM), the interaction with the electromagnetic field is described by the Hamiltonian

$$\mathcal{H}_{\text{em}}^D = \frac{1}{2} \bar{\nu} (\mu + id \gamma_5) \sigma^{\alpha\beta} \nu F_{\alpha\beta} = \frac{1}{2} \bar{\nu}_R \chi^{\alpha\beta} \nu_L F_{\alpha\beta} + \text{h.c.}.$$  \hspace{1cm} (2.1)

We assume for the time being that we consider neutrino flavours (or types if take into account sterile fields $\nu_s$). Thus, $\nu^T = (\nu_e, \nu_\mu, \nu_\tau, \nu_s, \ldots)$ is the vector of the flavour eigenfields. Hermiticity of the Hamiltonian (2.1) requires that the MM and EDM matrices are both hermitian:

$$\mu^\dagger = \mu, \quad d^\dagger = d.$$  \hspace{1cm} (2.2)

Therefore, the diagonal elements of $\mu$ and $d$ are real for Dirac neutrinos. With the decomposition $\nu = \nu_L + \nu_R$, where $\nu_L$ and $\nu_R$ are left and right-chiral fields, respectively, the second part of Eq.(2.1) follows, where the MM and EDM matrices are condensed in the non-hermitian matrix

$$\lambda = \mu - id \quad \text{with} \quad \mu = \frac{1}{2} (\lambda + \lambda^\dagger), \quad d = \frac{i}{2} (\lambda - \lambda^\dagger).$$  \hspace{1cm} (2.3)

The Hamiltonian (2.1) can be rewritten in any basis. One might, e.g., want to formulate in the basis of neutrino mass eigenfields. In the most general case one has to perform separate unitary rotations.
\[ \nu_L = S_L \nu'_L \quad \text{and} \quad \nu_R = S_R \nu'_R \]  

(2.4)
on the left and right-chiral fields, respectively, in which case the matrix (2.3) transforms as

\[ \lambda' = S_R^\dagger \lambda S_L. \]  

(2.5)

According to Eq.(2.3), the MM and EDM matrices in the new basis are obtained by

\[ \mu' = \frac{1}{2} \left\{ S_R^\dagger \mu S_L + S_L^\dagger \mu S_R - i(S_R^\dagger dS_L - S_L^\dagger dS_R) \right\}, \]

\[ d' = \frac{1}{2} \left\{ S_R^\dagger dS_L + S_L^\dagger dS_R + i(S_R^\dagger \mu S_L - S_L^\dagger \mu S_R) \right\}, \]  

(2.6)

respectively. Note that the matrix \( \lambda \) is the object which transforms simply under a basis change, not the MM and EDM matrices.

For Majorana neutrinos we have the Hamiltonian [31]

\[ H^M_{\text{em}} = -\frac{1}{4} \nu_T C^{-1}(\mu + id\gamma_5)\sigma^{\alpha\beta} \nu \sigma_{\alpha\beta} = -\frac{1}{4} \nu_T C^{-1} \lambda \sigma^{\alpha\beta} \nu_L \sigma_{\alpha\beta} + \text{h.c.}, \]  

(2.7)

where \( C \) is the charge conjugation matrix and \( \nu = \nu_L + (\nu_L)^c \). The superscript \( c \) denotes the charge conjugate field. By the anticommutation properties of the fermionic fields \( \nu_L \), it follows that

\[ \mu^T = -\mu, \quad d^T = -d. \]  

(2.8)

Thus the MM and EDM matrices are antisymmetric and hermitian, and, therefore, imaginary. Then, \( \lambda \) (2.3), which is defined as in the Dirac case, is antisymmetric as well. The factor \(-1/4\) in the Hamiltonian (2.7) has been chosen because it would appear by rewriting the Dirac form (2.1) in purely left-handed fields, i.e., in Majorana from. Since the right-handed component of a Majorana field is the charge conjugate of the left-handed component, left and right basis rotations are related by

\[ S_R = S_L^*, \]  

(2.9)

and in the new basis we have

\[ \mu' = i \left\{ \text{Im}(S_L^T \mu S_L) - \text{Re}(S_L^T dS_L) \right\}, \quad d' = i \left\{ \text{Im}(S_L^T dS_L) + \text{Re}(S_L^T \mu S_L) \right\}. \]  

(2.10)

In which basis the MM and EDM matrices are the “physically relevant” ones depends on the situation. Obviously, if one could experimentally distinguish neutrino mass eigenstates, \( \mu \) and \( d \) in the mass eigenbasis would be considered physical and diagonal MMs and EDMs could be distinguished in the Dirac case. For transition moments the freedom of making phase rotations, i.e., basis transformations with \( S_L = S_R \) being a diagonal matrix of phase factors, blurs the distinction between transition MMs and EDMs even for Dirac neutrino mass eigenstates. For Majorana neutrinos in the mass eigenbasis, however, there is only the freedom of making a transformation with \( S_L \) being a diagonal sign matrix. In practice one cannot measure neutrino mass eigenstates and we will consider the situation that in elastic neutrino – electron scattering all neutrino masses are neglected, but neutrino masses enter in the usual quadratic way in the evolution equation of the neutrino state with background matter and magnetic fields. In Section [V] we will come back to the question of physically observable quantities related to MMs and EDMs in this context.
III. THE EVOLUTION EQUATION OF THE SOLAR NEUTRINO STATE

If neutrinos possess a sizeable MM and/or EDM it is possible that solar neutrinos acquire a transverse polarization on their way to the surface of the sun if a large solar magnetic field exists \[24,25,28,29,27\]. The evolution of the neutrino state produced in the core of the sun under the influence of the solar magnetic field and matter effects is governed by the Schrödinger-like equation \[3,31,10–12,32,33\]

\[
\frac{d}{dz} \begin{pmatrix} \varphi_- \\ \varphi_+ \end{pmatrix} = \begin{pmatrix} V_L + \frac{1}{2\omega} M^\dagger M & -B_+ \lambda^\dagger \\ -B_- \lambda & V_R + \frac{1}{2\omega} MM^\dagger \end{pmatrix} \begin{pmatrix} \varphi_- \\ \varphi_+ \end{pmatrix} \equiv H_{\text{eff}} \begin{pmatrix} \varphi_- \\ \varphi_+ \end{pmatrix}. \tag{3.1}
\]

In this equation, \(\varphi_-\) and \(\varphi_+\) denote the vectors of neutrino flavour wave functions corresponding to negative and positive helicity, respectively, and \(\omega\) denotes the neutrino energy. The elements of \(\varphi_\pm\) are ordered according to \(\alpha = e, \mu, \tau\), followed by an arbitrary number of sterile neutrinos, in general. The matter potential \(V_L\) is given by

\[
V_L = \sqrt{2} G_F \text{diag}(n_e - n_n/2, -n_n/2, -n_n/2, 0, \ldots), \tag{3.2}
\]

where \(n_e\) (\(n_n\)) is the electron (neutron) density in the sun. \(M\) denotes the neutrino mass matrix in the flavour basis. With the diagonal matrix \(\hat{m}\) of neutrino masses and the unitary diagonalizing matrices \(U_L\) and \(U_R\), we have the relations

\[
\hat{m} = U_R^\dagger M U_L \Rightarrow M^\dagger M = U_L \hat{m}^2 U_L^\dagger \quad \text{and} \quad MM^\dagger = U_R \hat{m}^2 U_R^\dagger. \tag{3.3}
\]

Furthermore, we use the definition

\[
B_\pm = B_x \pm iB_y. \tag{3.4}
\]

Note that in Eq.(3.1) the neutrino propagates along the \(z\)-axis and in our approximation only \(B_x\) and \(B_y\) – the components of the solar magnetic field orthogonal to the neutrino momentum – contribute to the neutrino evolution.

Let us list the important differences between Dirac and Majorana neutrinos with respect to the quantities appearing in the evolution equation (3.1):

- Dirac neutrinos: \(V_R = 0\), \(M\) arb., \(U_R\) arb., \(\lambda\) arb.,

- Majorana neutrinos: \(V_R = -V_L\), \(M^T = M\), \(U_R = U_L^\ast\), \(\lambda^T = -\lambda\). \tag{3.5}

In this table arb. stands for arbitrary.

The neutrinos are produced as electron neutrinos in the sun at the coordinate \(z_0\) and are detected on earth at \(z_1\). Hence we express the initial condition as

\[
\varphi_-(z_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \varphi_+(z_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{3.6}
\]

and the neutrino state at the detector is formally given by
\[
\begin{pmatrix}
  a_- \\
  a_+
\end{pmatrix} \equiv \begin{pmatrix}
  \varphi_-(z_1) \\
  \varphi_+(z_1)
\end{pmatrix} = P \exp \left\{ -i \int_{z_0}^{z_1} dz H_{\text{eff}}(z) \right\} \begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix}.
\] (3.7)

In this equation \( P \) denotes path ordering. For a given magnetic field along the neutrino path in the sun, the neutrino state described by the vectors \( a_{\mp} \) can in principle be obtained by solving Eq.(3.1), as function of neutrino MMs, EDMs, masses and mixing parameters.

The densities \( n_e \) and \( n_n \) are provided by the Solar Standard Model. The flavour vectors \( a_{\mp} \) have to be used in the calculation of the elastic neutrino – electron cross section of solar neutrinos.

### IV. THE DENSITY MATRIX AND POLARIZATION VECTORS

In the calculation of the neutrino scattering cross section we need the density matrix of the initial neutrino state

\[
\rho^{\alpha\beta} = \sum_{r,s=\pm} u_r(k) \bar{u}_s(k) \rho_{rs}^{\alpha\beta}.
\] (4.1)

The elements of the density matrix obey the relations \( \sum_{\alpha,r} \rho_{rr}^{\alpha\alpha} = 1 \) and \( \rho_{rs}^{\alpha\beta} = (\rho_{sr}^{\beta\alpha})^* \). In the case of full coherence they are connected with the coefficients \( a_{\mp}^{\alpha} \) (3.7) by

\[
\rho_{rs}^{\alpha\beta} = a_r^{\alpha} a_s^{\beta*}.
\] (4.2)

In the Dirac representation of the gamma matrices we have for the 4-spinor \( u \) for a massless neutrino

\[
u_{\pm}(k) = \sqrt{\omega} \begin{pmatrix}
  \chi_+ \\
  \pm \chi_-
\end{pmatrix},
\] (4.3)

where \( \omega \) is the energy of the initial neutrino. Now we choose the \( z \)-axis along the direction of the initial neutrino momentum. Hence the neutrino 4-momentum is \( k = (\omega, 0, 0, \omega)^T \) and the 2-spinors of positive and negative helicity are given by

\[
\chi_+ = \begin{pmatrix}
  1 \\
  0
\end{pmatrix}, \quad \chi_- = \begin{pmatrix}
  0 \\
  1
\end{pmatrix},
\] (4.4)

respectively. Using Eqs.(4.3) and (4.4) it is easy to verify that the density matrix can be written in the following two equivalent ways:

\[
\rho^{\alpha\beta} = \frac{1}{2} \left\{ (1 + \gamma_5) \rho^{\alpha\beta}_{++} + (1 - \gamma_5) \rho^{\alpha\beta}_{--} + \frac{1 + \gamma_5}{2} \rho^{\alpha\beta}_{+-} + \frac{1 - \gamma_5}{2} \rho^{\alpha\beta}_{-+} \right\} k^\mu
\] (4.5)

\[
= \frac{1}{2} \left\{ \rho^{\alpha\beta}_{++} + \rho^{\alpha\beta}_{--} + \gamma_5 (\rho^{\alpha\beta}_{++} - \rho^{\alpha\beta}_{--}) + \gamma_5 \rho^{\alpha\beta}_{\mp} \right\} k^\mu,
\] (4.6)

with the generalized polarization 4-vectors orthogonal to the neutrino momentum.
\[ s_{\alpha\beta} = \begin{pmatrix} 0 \\ \rho_{+\beta} \\ i\rho_{+\beta} \\ 0 \end{pmatrix}, \quad s_{\alpha\beta} = \begin{pmatrix} 0 \\ -\rho_{-\beta} \\ i\rho_{-\beta} \\ 0 \end{pmatrix}, \quad s_{\alpha\beta} = s_{\alpha\beta} - s_{\alpha\beta} = \begin{pmatrix} 0 \\ \rho_{+\beta} + \rho_{-\beta} \\ i(\rho_{+\beta} - \rho_{-\beta}) \\ 0 \end{pmatrix}. \] (4.7)

The longitudinal component of the polarization can be read off from Eq.(4.6) and is given by \( \rho_{\alpha\beta} \). The polarization vectors of a flavour mixed neutrino state are complex in general with \( (s_{\alpha\beta})^* = s_{\beta\alpha} \) and \( (s_{\alpha\beta})^* = s_{\beta\alpha} \). Only in the case \( \alpha = \beta \), the vector \( s_{\alpha\alpha} \) is real. In Ref. [29] an expression similar to (4.5) is used for the density matrix. In the case of a single neutrino flavour Eq.(4.6) reduces to the expressions given in Refs. [26,34].

V. ELASTIC – NEUTRINO ELECTRON SCATTERING WITH ARBITRARY NEUTRINO POLARIZATION

In this section we present the weak, electromagnetic and interference cross sections for the process

\[ \nu(k) + e^- (p) \rightarrow \nu(k') + e^- (p'). \] (5.1)

We work in the restframe of the initial electron and use the following notation for the 4-momenta:

\[ k = \begin{pmatrix} \omega \\ \vec{k} \end{pmatrix}, \quad p = \begin{pmatrix} m_e \\ 0 \end{pmatrix}, \quad k' = \begin{pmatrix} \omega - T \\ \vec{k}' \end{pmatrix}, \quad p' = \begin{pmatrix} m_e + T \\ \vec{p}' \end{pmatrix}, \] (5.2)

where \( m_e \) is the electron mass, neutrino masses are neglected in the cross sections \( (k^2 = k'^2 = 0) \) and \( T = E_e - m_e \) is the recoil energy of the scattered electron. For the angle \( \theta = \angle (\vec{p}', \vec{k}) \) of the recoil electron, momentum conservation implies

\[ \cos \theta = \frac{\omega + m_e}{\omega} \sqrt{\frac{T}{T + 2m_e}}, \] (5.3)

and the electron recoil energy \( T \) is bounded by \( 0 \leq T \leq T_{\text{max}} \) with \( T_{\text{max}} = 2\omega^2/(2\omega + m_e) \).

The cross section for elastic neutrino – electron scattering consists of three terms

\[ \frac{d^2\sigma}{dT d\phi} = \frac{d^2\sigma_w}{dT d\phi} + \frac{d^2\sigma_{\text{em}}}{dT d\phi} + \frac{d^2\sigma_{\text{int}}}{dT d\phi}, \] (5.4)

where \( \phi \) is the azimuthal angle which is measured in the plane orthogonal to the momentum of the initial neutrino. The first and the second term are the pure weak and electromagnetic terms, respectively, and the third term is the interference term between the weak and the electromagnetic amplitude.
A. The weak cross section

The weak interaction of neutrinos with electrons is described by the effective Hamiltonian

\[ \mathcal{H}_w = \frac{G_F}{\sqrt{2}} \sum_{\alpha} \bar{\nu}_\alpha \gamma_\lambda (1 - \gamma_5) \nu_\alpha \bar{e} \gamma^\lambda (g_V^\alpha - g_A^\alpha \gamma_5) e, \]

where \( G_F \) is the Fermi constant and

\[ g_V^\mu,\tau = 2 \sin^2 \Theta_W + 1/2, \quad g_A^\mu,\tau = -1/2, \]

\[ g^\mu = 0, \quad g_A^\mu = 0, \]

with the weak mixing angle \( \Theta_W \). To write down the weak cross section it is useful to define the left and right-handed constants

\[ g_L^\alpha = \frac{1}{2} (g_V^\alpha + g_A^\alpha), \quad g_R^\alpha = \frac{1}{2} (g_V^\alpha - g_A^\alpha), \]

respectively. Note that for active neutrinos, independent of the flavour \( \alpha \), we have \( g_R^\alpha = \sin^2 \Theta_W \), but, of course, \( g_R^\alpha = 0 \) for sterile neutrinos.

With these constants, the weak cross sections for Dirac and Majorana neutrinos are given by

\[ \frac{d^2 \sigma_D}{dT d\phi} = \sum_{\alpha} |a_-^\alpha|^2 \frac{d^2 \sigma(\nu_\alpha e^-)}{dT d\phi}, \]

\[ \frac{d^2 \sigma_M}{dT d\phi} = \sum_{\alpha} \left( |a_-^\alpha|^2 \frac{d^2 \sigma(\nu_\alpha e^-)}{dT d\phi} + |a_+^\alpha|^2 \frac{d^2 \sigma(\bar{\nu}_\alpha e^-)}{dT d\phi} \right), \]

respectively, where

\[ \frac{d^2 \sigma(\nu_\alpha e^-)}{dT d\phi} = \frac{G_F^2 m_e}{\pi^2} \left( (g_L^\alpha)^2 + (g_R^\alpha)^2 \left( 1 - \frac{T}{\omega} \right)^2 - g_L^\alpha g_R^\alpha \frac{m_e T}{\omega^2} \right), \]

\[ \frac{d^2 \sigma(\bar{\nu}_\alpha e^-)}{dT d\phi} = \frac{G_F^2 m_e}{\pi^2} \left( (g_R^\alpha)^2 + (g_L^\alpha)^2 \left( 1 - \frac{T}{\omega} \right)^2 - g_L^\alpha g_R^\alpha \frac{m_e T}{\omega^2} \right) \]

are the cross sections for elastic scattering of neutrinos and antineutrinos of flavour \( \alpha \) off electrons, respectively. \( |a_-^\alpha|^2 \) \( |a_+^\alpha|^2 \) is the probability of finding a left-handed (right-handed) neutrino of flavour \( \alpha \) in the initial state. The right-handed states correspond to antineutrinos in the Majorana case, whereas for Dirac neutrinos they do not interact weakly and hence the respective term is absent in Eq.\((5.8)\).

B. The electromagnetic cross section

The electromagnetic cross section has the same form for Dirac and Majorana neutrinos:
\[ \frac{d^2 \sigma_{em}}{dT d\phi} = \frac{\alpha^2}{2m_e^2 \mu_B^2} \left( \frac{1}{T} - \frac{1}{\omega} \right) \left( a_+^\dagger \lambda^\dagger \lambda a_- + a_-^\dagger \lambda^\dagger \lambda a_+ \right) , \]  
\[ (5.12) \]

where \( \alpha = e^2/4\pi \) and \( \mu_B = e/2m_e \). The matrix \( \lambda \) is given in Eq.\((2.3)\) and we have

\[ \lambda^\dagger \lambda = \mu^2 + \bar{\mu}^2 + d^2 - i[\mu, d] \]
\[ \lambda \lambda^\dagger = \mu^2 + \bar{\mu}^2 + i[\mu, d]. \]  
\[ (5.13) \]

Eq.\((5.12)\) reduces to the well known result in the single flavour Dirac case [20], for special cases with several flavours see Refs. [35,23].

C. The interference cross section

If the polarization of the initial neutrino possesses a component transverse to its momentum, an interference term between the weak and electromagnetic amplitude appears in the cross section [24,25]. This term shows a dependence on the azimuthal angle \( \phi \). With the definitions

\[ g^\alpha = g^\alpha_V \left( 2 - \frac{T}{\omega} \right) + g^\alpha_A \frac{T}{\omega} , \quad \bar{g}^\alpha = g^\alpha_V \left( 2 - \frac{T}{\omega} \right) - g^\alpha_A \frac{T}{\omega} \]
\[ (5.14) \]

and \( \hat{k} = \vec{k}/|\vec{k}| \), we find for Dirac and Majorana neutrinos, respectively,

\[ \frac{d^2 \sigma^D_{int}}{dT d\phi} = \frac{G_F \alpha}{2\sqrt{2}\pi m_e T \mu_B} \sum_{\alpha,\beta} \text{Re} \left[ \left( \vec{p}' \mu_{\alpha\beta} + (\hat{k} \times \vec{p}') d_{\alpha\beta} \right) g^\alpha \bar{s}^\beta \right] , \]
\[ (5.15) \]
\[ \frac{d^2 \sigma^M_{int}}{dT d\phi} = \frac{G_F \alpha}{2\sqrt{2}\pi m_e T \mu_B} \sum_{\alpha,\beta} \text{Re} \left[ \left( \vec{p}' \mu_{\alpha\beta} + (\hat{k} \times \vec{p}') d_{\alpha\beta} \right) \left( g^\alpha \bar{s}^\beta - \bar{g}^\alpha \bar{s}^\beta \right) \right] . \]
\[ (5.16) \]

With \( \vec{p}' = (p'_{x}, p'_{y}, p'_{z})^T \) and \( \hat{k} = (0, 0, 1)^T \), one has \( \hat{k} \times \vec{p}' = (-p'_{y}, p'_{x}, 0)^T \) and, using the explicit form of the polarization vectors \((4.7)\) given by

\[ \bar{s}^{\beta \alpha} = \begin{pmatrix} a_{-}^{\beta} a_{-}^{\alpha*} \\ i a_{+}^{\beta} a_{+}^{\alpha*} \end{pmatrix} , \quad \bar{s}^{\beta \alpha} = \begin{pmatrix} -a_{-}^{\beta} a_{-}^{\alpha*} \\ i a_{-}^{\beta} a_{+}^{\alpha*} \end{pmatrix} \]
\[ (5.17) \]

for full coherence, we obtain

\[ \frac{d^2 \sigma^D_{int}}{dT d\phi} = F \text{Re} \left[ a_+^\dagger \lambda g a_- (p'_{x} - i p'_{y}) \right] , \]
\[ (5.18) \]
\[ \frac{d^2 \sigma^M_{int}}{dT d\phi} = F \text{Re} \left[ a_+^\dagger (\lambda \bar{g} \lambda) a_- (p'_{x} - i p'_{y}) \right] , \]
\[ (5.19) \]

where we have defined \( F = G_F \alpha/2\sqrt{2}\pi m_e T \mu_B \) and the diagonal matrices \( g = \text{diag}(g^\alpha) \) and \( \bar{g} = \text{diag}(\bar{g}^\alpha) \).

For a single flavour Dirac neutrino, Eq.\((5.13)\) reduces to the expression given in Ref. [26].
VI. ELASTIC NEUTRINO – ELECTRON SCATTERING OF SOLAR NEUTRINOS

A. A single Dirac neutrino

For a single Dirac neutrino the Hamiltonian governing the evolution equation (3.1) has the form

$$H_{\text{eff}} = \left( \begin{array}{cc} V_L + \frac{m^2}{2\omega} & -B_+ (\mu + id) \\ -B_- (\mu - id) & -\frac{m^2}{2\omega} \end{array} \right) = \text{diag}(1, e^{-i\delta}) H'_{\text{eff}} \text{diag}(1, e^{i\delta})$$

(6.1)

with

$$\mu + id = \sqrt{\mu^2 + d^2} e^{i\delta},$$

(6.2)

where $H'_{\text{eff}}$ differs from $H_{\text{eff}}$ by $\sqrt{\mu^2 + d^2}$ instead of $\mu \pm id$. This leads to

$$a_+ = e^{-i\delta} a'_+, \quad a_- = e^{-i\beta} a_{-},$$

(6.3)

where $a'_+$ and also $a_-$ do not depend on $\delta$ but only on $\sqrt{\mu^2 + d^2}$.

In this case we have

$$a_+^\dagger \lambda g a_- = \sqrt{\mu^2 + d^2} (a'_+)^* a_- g^e \quad \text{and} \quad a_-^\dagger \lambda^\dagger \lambda a_- + a_+^\dagger \lambda \lambda a_+ = \mu^2 + d^2.$$  

(6.4)

Therefore, one cannot distinguish between the MM and EDM of a single Dirac neutrino in elastic neutrino – electron scattering of solar neutrinos.

As a function of the azimuthal angle, the interference cross section (5.18) has a maximum when $p'_x + ip'_y$ and $a_+^\dagger \lambda^\dagger \lambda a_- = a_-^\dagger \lambda^\dagger \lambda a_-$ are aligned in the complex plane. In general, with matter effects in addition to the magnetic field interaction, there is no obvious meaning of the phase of the latter quantity. However, if we can neglect matter effects and the direction of the magnetic field is fixed, i.e.,

$$B_+(z) = B(z) e^{i\beta},$$

(6.5)

where $\beta$ does not depend on $z$, the solution of the evolution equation (3.1) for a single Dirac neutrino is given by

$$a_- = \cos \left( \sqrt{\mu^2 + d^2} \int_{z_0}^{z_1} dz B(z) \right) \quad \text{and} \quad a_+ = ie^{-i(\delta + \beta)} \sin \left( \sqrt{\mu^2 + d^2} \int_{z_0}^{z_1} dz B(z) \right).$$

(6.6)

We have left out the irrelevant phase stemming from $m^2/2\omega$. Consequently, we obtain

$$2 a_+^\dagger \lambda g a_- = -ie^{i\beta} \sqrt{\mu^2 + d^2} g^e \sin \left( 2 \sqrt{\mu^2 + d^2} \int_{z_0}^{z_1} dz B(z) \right),$$

(6.7)

and the maximum of the interference cross section at

$$\left( \begin{array}{c} p'_x \\ p'_y \end{array} \right) \propto \left( \begin{array}{c} B_y \\ -B_x \end{array} \right)$$

(6.8)

defines an azimuthal angle orthogonal to the direction of the transverse magnetic field. Thus, in such a situation the interference cross section allows to determine the direction of the magnetic field.
B. Two Majorana neutrinos

Now we come to the second simplest case, which is the case of two Majorana neutrinos with flavours $e$ and $x$. For two Majorana flavours the matrix $\lambda$ is simply given by

$$\lambda = \Lambda \epsilon = |\Lambda| e^{-i\delta} \epsilon \quad \text{with} \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (6.9)

Discussing first the electromagnetic cross section (5.12), Eq. (6.9) readily gives

$$a_\pm^\dagger \lambda^\dagger \lambda a_\mp + a_\mp^\dagger \lambda \lambda^\dagger a_\pm = |\Lambda|^2.$$  \hspace{1cm} (6.10)

Due to conservation of probability and the simple form of the matrix $\lambda$ (6.9), the coefficients $a_\mp$ do not occur and the electromagnetic cross section depends thus only on the single electromagnetic moment in the problem. The term (6.10) has the same structure as in the 1-Dirac case (see Eq. (5.4)).

Now we will perform a thorough discussion of all possible phases appearing in $U_L$ and $\lambda$ and we will show that they play no role in the two flavour case, in the framework of our approximation. The matrix $U_L$ (3.3) can be written as

$$U_L = e^{i\hat{\sigma}} V e^{i\hat{\rho}},$$  \hspace{1cm} (6.11)

where $\hat{\sigma}$ and $\hat{\rho}$ are diagonal phase matrices and $V$ corresponds to the KM matrix, which is real for two flavours [36]. Note that the phases $\rho_j$ play no role in our problem for any number of flavours because these so-called Majorana phases [37] drop out in the effective Hamiltonian of the evolution equation (3.1). Furthermore, a phase transformation with $e^{i\hat{\sigma}}$ upon $\lambda$ has the effect

$$e^{i\hat{\sigma}} \lambda e^{i\hat{\rho}} = \tilde{\Lambda} \epsilon \quad \text{with} \quad \tilde{\Lambda} = \Lambda e^{i(\sigma_e + \sigma_x)} = |\Lambda| e^{2i\gamma} \quad \text{and} \quad \gamma = (-\delta + \sigma_e + \sigma_x)/2.$$  \hspace{1cm} (6.12)

These observations suggest to perform the phase transformation

$$P^\dagger H_{\text{eff}} P = H'_{\text{eff}} \quad \text{with} \quad P = \text{diag} \left( e^{i\hat{\sigma} e^{-i\gamma}}, e^{-i\hat{\rho} e^{i\gamma}} \right),$$  \hspace{1cm} (6.13)

where

$$H'_{\text{eff}} = \begin{pmatrix} V_L + \frac{1}{2\mu} V \hat{m}^2 V^T & B_+ |\Lambda| \epsilon \\ -B_- |\Lambda| \epsilon & -V_L + \frac{1}{2\mu} V \hat{m}^2 V^T \end{pmatrix}.$$  \hspace{1cm} (6.14)

depends only on the orthogonal matrix $V$ and $|\Lambda|$. From the relation (6.13) it follows that

$$a_- = e^{i\hat{\sigma}} e^{-i\gamma} a'_- , \quad a_+ = e^{-i\hat{\rho}} e^{i\gamma} a'_+ ,$$  \hspace{1cm} (6.15)

where $a'_+$ is obtained by evolution with $H'_{\text{eff}}$ and is independent of any of the phases discussed above.

With Eqs. (6.9) and (6.15) we arrive at

$$a_\pm^\dagger (\lambda g + \bar{g} \lambda) a_- = |\Lambda| \left\{ - (a'_+)^* a'_- (g e + \bar{g} x) + (a'_e)^* a'_x (\bar{g} e + g x) \right\}.$$  \hspace{1cm} (6.16)
Thus, no phase of $U_L$ and $\lambda$ remains in the interference and electromagnetic cross sections of elastic neutrino–electron scattering of solar neutrinos. In particular, no distinction is possible between the transition MM and EDM, irrespective of the basis where they are defined (see also Eqs. (7.12) and (7.13)). This statement is valid in the framework of the approximation of neglecting neutrino masses in the scattering process and the validity of the evolution equation (3.1).

Let us briefly mention three special cases within the 2-Majorana neutrino case. If we have no neutrino mixing, only the first term in Eq. (6.16) contributes, because only $a_e^-$ and $a_x^+$ are non-zero. This is the minimal scenario for RSFP. Neutrino mixing also drops out of $H_{\text{eff}}$ (3.14), if we require $m_1 = m_2$. This case can be conceived as stemming from a Zeldovich–Konopinski–Mahmoud neutrino $\nu = \nu_{eL} + (\nu_{xL})^c$ [38], which has a conserved lepton number. If in addition to $m_1 = m_2$ we set $V_L = 0$ and require a fixed direction of the transverse magnetic field (see Eq. (6.5)), then we have an interference cross section analogous to the 1-Dirac neutrino case, where in the expression (5.7) the quantity $g^e$ is replaced by $g^e + \bar{g}^x$ and $\sqrt{\mu^2 + d^2}$ by $|\Lambda|$. Again, with the help of the interference cross section, one could in principle determine the direction of the magnetic field.

C. Phase counting

Having discussed at length the 1-Dirac and 2-Majorana neutrino cases, where we have shown that there are no physical phases, we now proceed to the general phase counting for $n$ neutrinos. We will assume that the charged lepton mass matrix is diagonal and positive. Our focus is on the neutrino flavour fields or states in the left-handed sector because we have formulated elastic neutrino–electron scattering with these entities.

In the case of Dirac neutrinos we have no interaction of the right-handed neutrinos. Therefore, in the physical situation under consideration, the matrix $U_R$ (3.3) is unphysical and can be rotated away and the mass term has the form

$$\mathcal{H}_\text{mass}^D = \bar{\nu}_R M \nu_L + \text{h.c.} \quad \text{with} \quad M = \hat{m} U_L^\dagger.$$  \hfill (6.17)

We are left with the following phase freedom:

$$\nu_L = e^{i\sigma_L} \nu_L', \quad \nu_R = e^{i\sigma_R} \nu_R' \quad \Rightarrow \quad \lambda \rightarrow e^{-i\sigma_R} \lambda e^{i\sigma_L}, \quad M \rightarrow e^{-i\sigma_R} M e^{i\sigma_L} \quad \text{or} \quad U_L \rightarrow e^{-i\sigma_L} U_L e^{i\sigma_R}. \hfill (6.18)$$

The vectors $\varphi_\mp$ (3.1) and, therefore, also $a_\pm$, transform in the same way as the fields $\nu_{L,R}$. Obviously, the cross sections (7.12) and (7.18) are invariant under the phase transformation (6.18). In general, $U_L$ is of the form (6.11), where $V$ is of the KM type with $(n-1)(n-2)/2$ phases [30]. The phases $\hat{\sigma}$ drop out of $M^\dagger M$ and $MM^\dagger$, and with $\sigma_L = \hat{\sigma}$ we shift the phases $\hat{\sigma}$ to $\lambda$ (see Eq. (6.18)). The remaining phase freedom in $\sigma_R$ is used to remove $n$ phases from the $n^2$ phases in $\lambda$. Thus, we are left with $(n-1)(n-2)/2 + n(n-1) = (3n-2)(n-1)/2$ independent phases in the problem.

\footnote{This statement disagrees with the result of Ref. [29].}
In the Majorana case\footnote{In order to have a MM/EDM for Majorana neutrinos, we need $n \geq 2$.} we have the mass term

$$H_{\text{mass}}^M = -\frac{1}{2} \nu_L^T C^{-1} M \nu_L + \text{h.c.} \quad \text{with} \quad M = U_L^* \hat{m} U_L^\dagger$$

(6.19)

and the phase transformation

$$\nu_L = e^{i\sigma_L} \nu'_L \quad \Rightarrow \quad \lambda \to e^{i\sigma_L} \lambda e^{i\sigma_L}, \quad M \to e^{i\sigma_L} M e^{i\sigma_L} \quad \text{or} \quad U_L \to e^{-i\sigma_L} U_L.$$ \hspace{1cm} (6.20)

Again the electromagnetic cross section (5.12) and the interference cross section (5.19) are invariant under this phase transformation, the phases $\hat{\rho}$ – the Majorana phases – cancel, and the phases $\hat{\sigma}$ can be transferred by the transformation (6.20) to $\lambda$, which is antisymmetric and, therefore, has $n(n-1)/2$ phases. Now there is no freedom to remove phases from the MM/EDM matrix as in the Dirac case, but one phase of $\lambda$ can still be eliminated by by redefining $\nu_L$ with a common phase of the type of $\gamma$ in Eq.(6.13). Finally, we arrive at $(n-1)(n-2)/2 + n(n-1)/2 - 1 = n(n-2)$ independent phases.

In summary, we have found the following numbers of independent, physical phases in the problem we are studying:

$$\begin{align*}
\text{Dirac case:} & \quad (3n-2)(n-1)/2 \\
\text{Majorana case:} & \quad n(n-2)
\end{align*}$$

\hspace{1cm} \text{physical phases.} \quad (6.21)

Eq.(6.21) explains why in the 1-Dirac and 2-Majorana neutrino cases we have no phases left in the elastic neutrino – electron cross section. It is interesting to observe that phases can be shifted from the mixing matrix $U_L$ to the MM/EDM matrix $\lambda$ and vice versa. In our physical setting the phases in the MM/EDM matrix are not physical in the sense of the phase transformations (6.18) and (6.20) and the transformed MMs and EDMs (2.6) and (2.10) and, therefore, a distinction between MM and EDM is thus unphysical as well.

**D. Invariance of the cross section under general basis transformations**

Up to now we have considered only the freedom of performing phase transformations on the neutrino fields. However, since we neglect neutrino masses in the cross section, we could use any basis for its calculation. Let us first consider Dirac neutrinos and the general basis transformation (2.3). Then the transformed MM/EDM matrix is given by Eq.(2.3) and the transformed flavour coefficients are obtained by

$$a_- = S_L a'_- \quad \text{and} \quad a_+ = S_R a'_+.$$ \hspace{1cm} (6.22)

We also have to take into account that in the weak Hamiltonian (5.5) the transformed matrices
appear, where the matrices $g_{V,A}$ are diagonal matrices of the coupling constants (5.6). Taking this set of transformed quantities, we can immediately rewrite the cross sections (5.12) and (5.18) in terms of the primed quantities. The same can be done with the weak cross section (5.8), if we notice that it is a function of the 4 expressions $a^\dagger g_{V,A}g_{V,A}a$ (see Eq.(5.10)).

For Majorana neutrinos, we have $S_R = S_L^*$. In addition, one can easily check that in the calculation of the antineutrino part (5.11) of the weak cross section (5.9) one gets $g_{V,A}^\dagger$ and the same applies to the $\bar{g}$ term in the interference cross section (5.19). These observations lead to invariance of the Majorana neutrino cross section under the general basis transformation (2.4). Consequently, in our physical problem there is no preferred basis, neither for Dirac nor for Majorana neutrinos.

E. CP invariance

Let us now study the effect of CP invariance on the phases discussed above. We focus on the MM and EDM matrices in the flavour basis and then compare with the mass basis. It will turn out that all the phases counted in the previous subsection are effects of CP violation.

In the flavour basis, CP invariance for Dirac neutrinos is expressed as invariance of the Lagrangian under the CP transformation

$$\text{CP: } \begin{align*}
\nu_L &\to -e^{i\alpha_L} C\nu_L^*, \\
\nu_R &\to -e^{i\alpha_R} C\nu_R^*, \\
\ell &\to -e^{i\alpha} C\ell^*, \\
F_{\alpha\beta} &\to -\varepsilon(\alpha)\varepsilon(\beta) F_{\alpha\beta},
\end{align*}$$

(6.24)

where $\varepsilon(\alpha)$ is 1 for $\alpha = 0$ and $-1$ for $\alpha = 1, 2, 3$, $\ell$ is the vector of the charged lepton fields and $\alpha_L, \alpha_R$ denote diagonal phase matrices. For simplicity, we have left out space-time arguments of the fields. It is straightforward to check that invariance under this transformation, using the Hamiltonians (2.1) and (6.17) and assuming non-vanishing neutrino masses, implies

$$\text{CP invariance } \Rightarrow \quad e^{i\alpha_R} \lambda^* e^{-i\alpha_L} = \lambda, \quad e^{i\alpha_R} M^* e^{-i\alpha_L} = M \quad \text{or} \quad e^{i\alpha_L} U_L^* e^{-i\alpha_R} = U_L.$$  

(6.25)

Consequently, we can define phase-rotated quantities

$$\nu_L = e^{i\alpha_L/2} \nu'_L, \quad \nu_R = e^{i\alpha_R/2} \nu'_R \quad \Rightarrow \quad \lambda' = e^{-i\alpha_R/2} \lambda e^{i\alpha_L/2}, \quad U'_L = e^{-i\alpha_L/2} U_L e^{i\alpha_R/2}$$

(6.26)

such that

$$U'_L^* = U'_L \quad \text{and} \quad \lambda'^* = \lambda'.$$

(6.27)

As expected, CP invariance ensues the existence of phase-transformed fields such that the mixing matrix $U'_L$ and the MM/EDM matrix $\lambda'$ are both real. Decomposing $\lambda'$ into MM and EDM matrices we obtain thus
\[
\mu' = \frac{1}{2}(\lambda' + \lambda'^T) \text{ real, symmetric,} \\
\quad d' = \frac{i}{2}(\lambda' - \lambda'^T) \text{ imaginary, antisymmetric.} \quad (6.28)
\]

If we go from the primed basis into the neutrino mass basis, we have \( \tilde{\lambda} = \lambda' U_L' \) as MM/EDM matrix, which is again real. Thus we have a decomposition of \( \tilde{\lambda} \) into MM and EDM matrices with properties analogous to (6.28), though the MMs and EDMs are related in a complicated way via Eq.(2.6) with \( S_L = U_L' \) and \( S_R = 1 \).

Let us now specialize this discussion to the physical situation of neglecting neutrino masses in \( \nu e^- \) scattering of solar neutrinos, such that neutrino masses enter only via the terms \( M^\dagger M \) and \( MM^\dagger \) in the evolution equation (3.1). Eq.(6.25) leads to the following condition for CP invariance:

\[
M^\dagger M = e^{i\alpha_L} M^T M^* e^{-i\alpha_L} \quad \text{and} \quad MM^\dagger = e^{i\alpha_R} M^* M^T e^{-i\alpha_R}. \quad (6.29)
\]

With the form of \( M \) given in Eq.(6.17), the second relation is trivially fulfilled and the first relation translates into

\[
U_L \hat{m}^2 U_L^\dagger = e^{i\alpha_L} U_L^* \hat{m}^2 U_L^T e^{-i\alpha_L}. \quad (6.30)
\]

The CP phases (6.24) of the right-handed fields do not occur in this condition. Hence, a phase transformation of \( \nu_R \) like in Eq.(6.18) can be used to remove \( n \) phases from \( \lambda \), which introduces a change \( \alpha_R \rightarrow \alpha_R - 2\sigma_R \) in the CP transformation. Therefore, if, after performing the transformation (6.26) on \( \nu_L \), the mixing matrix has the form \( U_L' = V e^{i\hat{\rho}} \) with \( V \) real and the MM/EDM matrix has the form \( e^{i\hat{\beta}} \lambda' \) with \( \lambda' \) real, where \( \hat{\rho} \) and \( \hat{\beta} \) are arbitrary diagonal phase matrices, the Lagrangian is not invariant under CP in general. However, the phases \( \hat{\rho} \) and \( \hat{\beta} \) do not lead to physical consequences in elastic \( \nu e^- \) scattering of solar neutrinos.

Coming to CP invariance in the case of Majorana neutrinos, we use the same CP transformation (6.24), except that the line with \( \nu_R \) has to be dropped. The invariance of the mass term (6.19) requires

\[
\hat{e}^{i\alpha_L} Me^{i\alpha_L} = -M^*. \quad (6.31)
\]

Assuming now for simplicity not only non-zero but also non-degenerate neutrino masses, one can show with Eqs.(6.19) and (6.31) that the following conditions hold for CP invariance:

\[
U_L^* = i e^{-\alpha_L} U_L \hat{\varepsilon} \quad \text{and} \quad \hat{e}^{i\alpha_L} \lambda e^{i\alpha_L} = -\lambda^*, \quad (6.32)
\]

where \( \hat{\varepsilon} \) is a diagonal sign matrix, which is not determined by our manipulations. Making the phase redefinition (6.26) of the left-handed neutrino field, Eq.(6.32) leads to

\[
U_L' = e^{-i\alpha_L/2} U_L = R e^{-i\varepsilon \pi/4} \quad (6.33)
\]

for the mixing matrix, where \( R \) is a real orthogonal matrix, and to

\[
\lambda' = \hat{e}^{i\alpha_L/2} \lambda e^{i\alpha_L/2} \quad \Rightarrow \quad \lambda'^* = -\lambda' \quad (6.34)
\]

for the MM/EDM matrix. Consequently, with Eq.(6.34) we obtain
\[ \mu' = \lambda' \text{ imaginary, antisymmetric, } \quad d' = 0, \quad (6.35) \]

independent of the sign matrix \( \varepsilon \).

Let us now relate the CP transformation in the flavour basis with that in the mass basis. The mass eigenfields obtained by \( \nu_L = U_L \tilde{\nu}_L \) have the matrix

\[ U_L^\dagger e^{i\alpha_L} U_L^* = i\varepsilon \quad (6.36) \]

instead of \( e^{i\alpha_L} \) (6.24), where \( U_L \) is given in Eq. (6.33). Hence, in the mass basis the CP transformation is given by

\[ \tilde{\nu}_L \to -i\varepsilon C \tilde{\nu}_L^*. \quad (6.37) \]

These CP signs \( \varepsilon_j \) – the CP parities – then enter also in the MM/EDM matrix in the mass basis given by

\[ \tilde{\lambda} = e^{-i\varepsilon \pi/4} R^T \lambda' R e^{-i\varepsilon \pi/4} = ((R^T \lambda' R)_{jk} e^{-i(\varepsilon_j + \varepsilon_k)\pi/4}) \quad (6.38) \]

If \( \varepsilon_j = \varepsilon_k \), the phase factor in expression on the right-hand side of Eq. (6.38) is \( \pm i \), whereas for \( \varepsilon_j = -\varepsilon_k \) it is 1. In the first case of equal CP parities, \( \tilde{\lambda}_{jk} \) represents a transition EDM, whereas for opposite CP parities this quantity is a transition MM [40, 29]. Therefore, for Majorana neutrinos and CP invariance in the primed flavour basis one has \( d' = 0 \), whereas in the mass basis either \( \tilde{\mu}_{jk} \) or \( \tilde{d}_{jk} \) is zero (or both are zero). This is completely different from the Dirac case where in both bases the same properties (6.28) hold.

Let us now come to our physical approximation for elastic neutrino – electron scattering with Majorana neutrinos. Eq. (6.31) implies for the relevant term

\[ M^\dagger M = e^{i\alpha_L} M^T M^* e^{-i\alpha_L} \quad \text{or} \quad U_L \hat{m}^2 U_L^\dagger = e^{i\alpha_L} U_L^* \hat{m}^2 U_T e^{-i\alpha_L}. \quad (6.39) \]

Obviously, the phase matrix \( e^{-i\varepsilon \pi/4} \) drops out. Thus, in the case of CP invariance in our physical scenario, the CP parities are irrelevant. Moreover, after performing the phase transformation Eq. (6.26) on \( \nu_L \), Eq. (6.39) is fulfilled for \( U_L' = R e^{i\tilde{\beta}} \) with an arbitrary diagonal phase matrix \( \tilde{\beta} \). Furthermore, there is the freedom to redefine \( \nu_L \) with a common phase which can be used to remove one phase from the MM/EDM matrix. Therefore, if the mixing matrix has the form \( R e^{i\tilde{\beta}} \) and the MM/EDM matrix has the form \( e^{i\gamma} \lambda' \) with \( \lambda' = -\lambda' \), CP is violated at the level of the Lagrangian in general. However, neither the phases \( \tilde{\beta} \) and \( \gamma \) nor the CP parities \( \varepsilon \) lead to any physical consequences in our scenario.

VII. DECOHERENCE EFFECTS AS A CONSEQUENCE OF NEUTRINO ENERGY AVERAGING

A. Decoherence effects in the solar neutrino state

In this section we consider the effect of neutrino oscillations and averaging over the neutrino energy in order to assess effective coherence or incoherence of the solar neutrino state arriving at the earth. We use the arguments presented, e.g., in Ref. [11].
The neutrino state undergoes only vacuum oscillations between the sun and the earth. Therefore, denoting the values of $\varphi_{\pm}$ at the edge of the sun by $b_{\pm}$, we can write $a_{\pm}$ as

$$a_{\pm} = U_L \exp \left( -i m^2 L / 2 \omega \right) U_L^{\dagger} b_{\pm}, \quad a_{\mp} = U_R \exp \left( -i m^2 L / 2 \omega \right) U_R^{\dagger} b_{\mp},$$

(7.1)

respectively. Here $L \approx 1.5 \times 10^{11}$ m is the distance between the sun and the earth. Now the crucial point is that, according to the quadratic appearance of $a_{\pm}$ in the cross sections (5.8), (5.9), (5.12), (5.18) and (5.19), the following phase factors are important:

$$e^{\pm i \varphi_{jk}} \quad \text{with} \quad \varphi_{jk} = 2\pi \frac{L}{\ell_{jk}} = \frac{\Delta m^2_{jk} L}{2\omega},$$

(7.2)

where $\Delta m^2_{jk} = m_j^2 - m_k^2 > 0$ and $\ell_{jk} = 4\pi \omega / \Delta m^2_{jk}$ is an oscillation length. The phases (7.2) vary with energy as

$$\delta \varphi_{jk} = \frac{\Delta m^2_{jk} L \delta \omega}{2 \omega} = 2\pi \frac{L}{\ell_{jk}} \frac{\delta \omega}{\omega}.$$

(7.3)

Hence, integration over energy intervals such $\delta \omega \gg \omega \ell_{jk} / L \forall j, k$ leads to an averaging of the oscillations, which can formally be expressed as

$$\langle e^{\pm i \varphi_{jk}} \rangle = \delta_{jk},$$

(7.4)

where $\delta_{jk}$ is the Kronecker delta.

Numerically, we have

$$\frac{\ell_{jk}}{L} \approx 2.5 \quad \frac{\omega(\text{MeV})}{\Delta m^2_{jk}(\text{eV}^2) / L(\text{m})} \approx 1.7 \times 10^{-11} \quad \frac{\omega(\text{MeV})}{\Delta m^2_{jk}(\text{eV}^2)} \approx 4.5 \times 10^{-12}$$

(7.5)

where in the last step we have used $\omega \approx 0.27$ MeV, the average energy of the $pp$-neutrinos, which are most suitable for measuring the azimuthal asymmetry in $\nu e^-$ scattering [28]. If we consider, for example, $\Delta m^2 \sim 10^{-8}$ eV$^2$ allowed by the RSFP scenario [11, 12, 13, 14, 16, 17], we find $\ell / L \sim 5 \times 10^{-4}$, where $\ell$ is the oscillation length corresponding to $\Delta m^2$. Therefore, to avoid the averaging (7.4) associated with the vacuum oscillations, one would have to measure the neutrino energy with an accuracy better than $\delta \omega / \omega \sim 10^{-4}$, which seems rather impossible.

Actually, even if we concentrate on the solar $^7$Be line with $\omega = 862.27$ keV [12], the natural line broadening by the high temperatures in the center of the sun with $\delta \omega = 1.63 \text{ keV}$ [12] is sufficient to cause considerable averaging. In this case we obtain $(L / \ell)(\delta \omega / \omega) \approx \Delta m^2 / 7.8 \times 10^{-9}$ eV$^2$ and, therefore, a decoherence effect for $\Delta m^2 \gtrsim 10^{-8}$ eV$^2$ [43].

The energy averaging of the vacuum oscillations is equivalent to consider the neutrino state arriving at the earth as an incoherent mixture of mass eigenstates. In the case of total incoherence the density matrix is a diagonal matrix in the mass basis: $\rho_{rs} = \text{diag}(a_j^r a_j^{s \ast})$ with $r, s = \pm$, where $j$ numbers the neutrino mass eigenstates.
B. The energy-averaged cross sections

In this and the next subsection we assume that ω represents an average neutrino energy or the center value of an energy interval of length δω over which the averaging takes place. Furthermore, we assume that δω ≪ ω holds and that the averaging condition δϕ_{jk} ≫ 2π holds for all neutrino masses m_j ≠ m_k. Performing the averaging procedure (7.4) in the weak, electromagnetic and interference cross sections, it turns out that the averaged cross sections are written in a simpler way by using the coefficients

\[ \tilde{b}_- = U_R^1 b_-, \quad \tilde{b}_+ = U_R^1 b_+ , \]  

and the matrix

\[ \tilde{\lambda} = U_R^1 \lambda U_L , \]  

which represent the flavour coefficients \( b_\mp \) (7.1) and the matrix \( \lambda \) (2.3), respectively, transformed into in the mass basis. Eqs.(7.6) and (7.7) refer to the Dirac case, but in the Majorana case one only has to replace \( U_R \) by \( U_R^* \) (see Eq.(3.5)). In the following we use the notation \( \tilde{b}_T^\pm = (b_T^j)^\pm \), i.e., we label flavour indices with α and mass indices with j. Using Eqs.(7.6) and (7.7), after neutrino energy averaging or for effective total coherence loss between neutrino mass eigenstates, the cross sections (5.9), (5.12) and (5.19) for Majorana neutrinos take the shape

\[ \langle d^2 \sigma^M_w \rangle = \sum_{\alpha,j} |U_{L\alpha j}|^2 \left( |\tilde{b}_-|^2 \frac{d\sigma(\nu_\alpha e^-)}{dT d\phi} + |\tilde{b}_+|^2 \frac{d\sigma(\bar{\nu}_\alpha e^-)}{dT d\phi} \right) , \]  

\[ \langle d^2 \sigma^M_{em} \rangle = \frac{\alpha^2}{2m_e^2 \mu_B^2} \left( 1 - \frac{1}{\omega} \right) \sum_j \left( |\tilde{b}_-|^2 (\tilde{\lambda}^\dagger \tilde{\lambda})_{jj} + |\tilde{b}_+|^2 (\tilde{\lambda} \tilde{\lambda}^\dagger)_{jj} \right) , \]  

\[ \langle d^2 \sigma^M_{int} \rangle = F \text{Re} \left[ \sum_j b_T^j b_T^j \left( \bar{\tilde{\lambda}} U_L^\dagger (g - \bar{g}) U_L \right)_{jj} (p_x' - ip_y') \right] , \]

respectively. The corresponding expressions for Dirac neutrinos are obtained from Eqs.(7.8) and (7.10) by dropping the \( \sigma(\bar{\nu}_e e^-) \) and \( \bar{g} \) terms, respectively.

It is interesting to note that for total incoherence of the neutrino mass eigenstates only the axial part of the weak interaction contributes to the interference cross section for Majorana neutrinos. Inserting Eq.(5.14) into the cross section Eq.(7.10), we obtain

\[ \langle d^2 \sigma^M_{int} \rangle = 2F \frac{T}{\omega} \text{Re} \left[ \sum_{\alpha,j,k} b_T^{j*} b_T^j \bar{\lambda}_{jk} U_{L\alpha k} U_{L\alpha j} g_A^\alpha (p_x' - ip_y') \right] . \]  

The dependence on the electron recoil energy of this expression is very different from the corresponding term (5.19) in the case of full coherence and the Dirac terms with and without coherence (see Eqs.(5.18) and (7.10) without the \( \bar{g} \) term), because the recoil energy T drops out of the product \( FT \).
C. Decoherence in the 2-Majorana neutrino case

Now we consider in detail the effect of decoherence for the two flavours $e$ and $x = \mu, \tau, s$ of Majorana neutrinos. For this purpose we will refer to the discussion in Section VI B. We have proved in this section that all phases of the problem are unphysical. Therefore, we use the mixing matrix (compare with Eq.(6.11))

$$V = \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

(7.12)

where $c \equiv \cos \theta$, $s \equiv \sin \theta$ and the quantity $|\Lambda|$ for the transition MM/EDM (see Eqs.(6.9) and (6.12)). With the averaged Majorana interference cross section (7.11) and

$$\tilde{\lambda} = |\Lambda|V^T \epsilon V = |\Lambda| \epsilon,$$

(7.13)

we arrive at the final result

$$\left\langle \frac{d^2 \sigma_{\text{int}}^M}{dT d\phi} \right\rangle = F |\Lambda| \sin 2\theta \frac{T}{\omega} (g_A^e - g_A^\mu) \Re \left[ (|b_{1}^+|^2 - |b_{1}^-|^2) (p_x' - i p_y') \right].$$

(7.14)

Note that the vectors $b_{\pm}' = (b_{\pm}^j)'$ represent the neutrino state at the edge of the sun: by Eq.(7.6) they are related to the flavour vectors $a_{\pm}'$ which are obtained by evolution with the Hamiltonian $H_{\text{eff}}'(6.14)$ – analogous to $a_{\pm}'(6.15)$ – and are independent of any phases initially in $U_L$ and $\lambda$.

The expression (7.14) is proportional to the mixing angle $\sin 2\theta$. This shows that the question if the solar neutrino state on earth is to be considered as a coherent or effectively incoherent admixture of mass eigenstates has a strong effect on the interference cross section, whereas this question has no bearing on the electromagnetic cross section in the 2-Majorana case. Large values for $\sin 2\theta$ are disfavoured in the RSFP scenario [13,18] and by the non-observation of electron antineutrinos in Super-Kamiokande [14,18,45,46] and hence the asymmetry is suppressed. These arguments suggest that a significant asymmetry measured in an experiment is unlikely to result from a 2-Majorana neutrino scenario, except for very small mass-squared differences ($\Delta m^2 < 10^{-11}$, see Eq.(7.5)). Of course, it could result from Dirac diagonal moments. In this case the states of negative and positive helicity belong to the same mass eigenvalue and no averaging due to oscillations is possible.

VIII. CONCLUSIONS

In this paper we have considered elastic neutrino – electron scattering of solar neutrinos, taking into account the possibility that neutrinos have MMs and EDMs. We have presented the most general cross section for an initial neutrino state, which can be an arbitrary superposition of different neutrino types – including sterile neutrinos – with arbitrary helicities. Consistency requires that the neutrino superposition which undergoes the elastic neutrino – electron scattering is considered as the result of an evolution of the initial electron neutrino state with negative helicity generated in the core of the sun. The neutrino MMs and
EDMs enter into this evolution equation (3.1) as well as into the cross section. Only by taking this twofold effect of the MM/EDM matrix (2.3) into account, the final results for the pure electromagnetic cross section (5.12) and the weak-electromagnetic interference cross section, i.e., (5.18) for Dirac and (5.19) for Majorana neutrinos, are invariant under phase transformations (6.18) and (6.20) of the neutrino fields.

In this context we have to mention our approximation: We have neglected neutrino masses in the cross section, but in the evolution equation (3.1) we have taken into account the usual quadratic dependence of the effective Hamiltonian on the neutrino masses, as required by the effect of background matter [3]. We have formulated the cross section and the evolution equation in the flavour basis. However, we want to stress that we are not obliged to stick to this basis. Since we neglect neutrino masses in the cross section, we could choose rotated neutrino fields according to Eq.(2.4). The final result for the cross section would not depend on the transformation matrices \( S_L(S_R) \) (see Subsection VI D). Thus, with our physically motivated approximation there is no preferred basis of neutrino fields.

Of particular importance is the weak-electromagnetic interference cross section: if it were non-zero, it would indicate that the solar neutrinos have acquired some amount of transverse polarization due to MMs and EDMs and a magnetic field in the solar interior. Such an interference cross section would show up in an azimuthal asymmetry of the momentum distribution of the recoil electron, in the plane orthogonal to the direction of the incoming neutrino [24].

In our physical scenario we have shown that in the 1-Dirac neutrino case the azimuthal asymmetry does not allow to distinguish between MM and EDM but is a function of \( \sqrt{\mu^2 + d^2} \) (6.4) as is the pure electromagnetic cross section. In the 2-Majorana neutrino case the same holds for the transition moments (see Eq.(6.16)) and, in addition, none of the phases in the neutrino mixing matrix \( U_L \) is physical either. We have also made a general counting of the physical, independent phases in our framework for \( n \) neutrino flavours or types in the Dirac and Majorana cases. We have pointed out that, by phase transformations on the neutrino fields, phases can be shifted from \( U_L \) to \( \lambda \) and vice versa such that in order to obtain phase convention-independent quantities one has to combine elements from both matrices according to the phase transformations (6.18) and (6.20). We want to stress that the entity which transforms under basis transformations in correspondence with the neutrino fields is the matrix \( \lambda = \mu - id \), but not the separate MM and EDM matrices \( \mu \) and \( d \), respectively. Furthermore, in the flavour basis we are working with, for Majorana neutrinos the so-called Majorana phases drop out trivially, because in our approximation the neutrino mass matrix appears only in the evolution equation (3.1) as \( M^\dagger M \) and \( MM^\dagger \) (see Eqs.(3.3) and (6.11)). The same holds, in the case of CP invariance, for the CP parities of the neutrino mass eigenfields.

Finally, we have shown that averaging over small neutrino energy intervals, which is inevitable through realistic neutrino detection, has a drastic effect on the weak-electromagnetic interference cross section. This effect comes about because of neutrino oscillations in vacuum between the sun and the earth [11] and is operative at least for neutrino mass-squared differences larger than about \( 10^{-9} \div 10^{-10} \) eV\(^2\), having in mind solar neutrino energies below 1 MeV. In the 2-Majorana neutrino case the averaged interference cross section is then proportional to \( \sin^2\theta \), where \( \theta \) is the mixing angle in \( U_L \). Thus, in a 2-Majorana RSFP scenario without mixing, the averaged interference cross section is zero. However, mixing in
the general 2-Majorana RSFP scenario tends to be suppressed anyway according to the non-observation of solar $\bar{\nu}_e$’s in Super-Kamiokande. An observation of a significant azimuthal asymmetry could be an indication of very small mass-squared differences or of Dirac diagonal moments.

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