THE DA VIES METHOD FOR HEAT KERNEL UPPER BOUNDS OF NON-LOCAL DIRICHLET FORMS ON ULTRA-METRIC SPACES*

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Abstract We apply the Davies method to give a quick proof for the upper estimate of the heat kernel for the non-local Dirichlet form on the ultra-metric space. The key observation is that the heat kernel of the truncated Dirichlet form vanishes when two spatial points are separated by any ball of a radius larger than the truncated range. This new phenomenon arises from the ultra-metric property of the space.

Key words heat kernel; ultra-metric; Davies method

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1 Introduction

We are concerned with the heat kernel estimate for the non-local Dirichlet form on the ultrametric space. Let \((M,d,\mu)\) be an ultra-metric measure space; that is, \(M\) is locally compact and separable, \(d\) is an ultra-metric, and \(\mu\) is a Radon measure with full support in \(M\).

Recall that a metric \(d\) is called an ultra-metric if, for any points \(x,y,z \in M\),

\[
d(x, y) \leq \max\{d(x, z), d(z, y)\}.
\]

For any \(x \in M\) and \(r > 0\) the metric ball \(B(x, r)\) is defined by

\[
B(x, r) := \{y \in M, d(x, y) \leq r\}.
\]

It is known that any two metric balls are either disjoint or that one contains the other (see [1, 2]). Thus any ball is both closed and open so that its boundary is empty.

An ultra-metric space is totally disconnected, so the process described by the heat kernel is a pure jump process. In this article, we consider the the Dirichlet form \(E\) only having a non-local part without the killing part in the Beurling-Deny decomposition (see [6, p.120]). Let \(E\) be the energy form given by

\[
E(f,g) = \iint_{M \times M \setminus \text{diag}} (f(x) - f(y))(g(x) - g(y))d\mu(x, y), \tag{1.1}
\]
where \( j \) is a symmetric Radon measure on \((M \times M) \setminus \text{diag}\). For simplicity, we denote \( E(f, f) \) by \( E(f) \). We need to specify the domain of \( E \). Let \( D \) be the space defined by

\[
D = \left\{ \sum_{i=0}^{n} c_i 1_{B_i} : n \geq 1, c_i \in \mathbb{R}, B_i \text{ are compact disjoint balls} \right\}.
\]  

(1.2)

Let the space \( F \) be determined by

\[
F = \text{the closure of } D \text{ under norm } \sqrt{E(u) + \|u\|^2_2}.
\]  

(1.3)

Then \((E, F)\) is a regular Dirichlet form in \( L^2 := L^2(M, \mu) \) if the measure \( j \) satisfies that \( j(B, B^c) < \infty \) for any ball \( B \) (see [2, Theorem 2.2]).

Recall that the indicator function \( 1_B \) for any ball \( B \) belongs to the space \( F \) if \( j(B, B^c) < \infty \), because

\[
E(1_B) = 2j(B, B^c) < \infty
\]

(see [2, formula (4.1)]).

In the sequel, we will fix some numbers \( \alpha > 0 \) and \( \beta > 0 \), and some value \( R_0 \in (0, \text{diam } M] \), which contains the case when \( R_0 = \text{diam } M = \infty \). The letter \( C \) is universal positive constant which may vary at each occurrence.

We list some conditions to be used later on.

- **Condition** (TJ): there exists a transition function \( J(x, dy) \) such that

\[
dj(x, y) = J(x, dy)d\mu(x),
\]

and for any ball \( B := B(x, r) \) with \( x \in M \) and \( r \in (0, R_0) \),

\[
J(x, B^c) \leq \frac{C}{r^\beta}.
\]

- **Condition** (DUE): the heat kernel \( p_t(x, y) \) exists and satisfies

\[
p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}}
\]

for all \( t \in (0, R_0^3) \) and \( \mu \)-almost all \( x, y \in M \).

- **Condition** (wUE): the heat kernel \( p_t \) exists and satisfies

\[
p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left( 1 + \frac{d(x, y) \wedge R_0}{t^{1/\beta}} \right)^{-\beta}
\]

for all \( t \in (0, R_0^3) \) and \( \mu \)-almost all \( x, y \in M \).

Our aim in this article is to obtain the following:

**Theorem 1.1** Let \((E, F)\) be given by (1.1), (1.3) on an ultra-metric measure space. Then

\[
(TJ) + \text{(DUE)} \Rightarrow \text{(wUE)}.
\]  

(1.4)

The result (1.4) is not new (see [2, Lemma 5.2, Theorem 12.1 and Subsection 12.3]), and was proved in [2, Theorem 2.8] by using a very complicated idea that was developed in several articles [7–11] (After this article was finished, we became aware of the fact that a much simpler proof was recently presented in [2]). What is new in this article is that we apply the Davies method developed in [3, 5] (see also [13, 14], [12, p1152] on general metric measure spaces including fractals) to give a much simpler proof for the implication (1.4). The key observation...
here is that the heat kernel for the truncated Dirichlet form vanishes at any time when points are separated by a ball of a radius larger than the truncated range (see Lemma 2.7 below). This new phenomenon arises from the ultra-metric property of the space \( M \) (General metric spaces do not admit this nice property). It would be interesting to generalize Theorem 1.1 by using the Davies method to the more general case when the condition (DUE) becomes

\[
p_t(x, y) \leq \frac{C}{V(x, \phi^{-1}(t))},
\]

where \( \phi \) is an increasing function on \([0, \infty)\), and \( V(x, r) = \mu(B(x, r)) \) is the volume of the ball \( B(x, r) \) that is also sensitive to the center \( x \).

2 Heat Kernel for the Truncated Dirichlet Form

We need to consider the truncated form and then estimate the corresponding heat kernel. We will do this by using the Davies method.

For any \( \rho > 0 \), let \( \mathcal{E}(\rho) \) be defined by

\[
\mathcal{E}(\rho)(u, v) = \int_M \int_{B(x, \rho)} (u(x) - u(y))(v(x) - v(y))dj(x, y).
\]

It is known that if

\[
\sup_{x \in M} \int_{B(x, r)^c} J(x, dy) < \infty, \quad r \in (0, R_0),
\]

then \( (\mathcal{E}(\rho), \mathcal{F}) \) is closable and its closure form \( (\mathcal{E}(\rho), \mathcal{F}(\rho)) \) is a regular Dirichlet form (see [10, Proposition 4.2]). Condition (TJ) implies condition (2.2), so it also implies that the form \( (\mathcal{E}(\rho), \mathcal{F}(\rho)) \) is regular (see [2, Theorem 2.2]).

Proposition 2.1 If conditions (TJ), (DUE) hold, then the following functional inequality (called Nash inequality) holds: there exists a constant \( C_N > 0 \) (subscript \( N \) means that this constant comes from “Nash” inequality) such that, for any \( u \in \mathcal{F} \cap L^1 \),

\[
\|u\|_2^{1+\nu} \leq C_N \left( \mathcal{E}(\rho)(u) + K_0 \|u\|_2^2 \right) \|u\|_1^{2\nu},
\]

where \( \nu := \beta/\alpha \) and \( K_0 := \rho^{-\beta} + R_0^{-\beta} \).

Proof As (DUE) holds, we have, by [3, Theorem (2.1)], that

\[
\|u\|_2^{2(1+\nu)} \leq C \left( \mathcal{E}(\rho)(u) + R_0^{-\beta} \|u\|_2^2 \right) \|u\|_1^{2\nu}.
\]

On the other hand, by condition (TJ) and the symmetry of \( j \), we have

\[
\mathcal{E}(u) - \mathcal{E}(\rho)(u) = \int_M \int_M (u(x) - u(y))^2 \mathbf{1}_{\{d(x, y) > \rho\}}dj(x, y)
\leq \int_M \int_M 4u^2(x) \mathbf{1}_{\{d(x, y) > \rho\}}dj(x, y)
\leq 4 \int_M u^2(x) dj(x) \sup_{x \in M} \int_{B(x, \rho)^c} J(x, dy)
\leq C \rho^{-\beta} \|u\|_2^2
\]

(note, by this inequality, that \( \mathcal{F}(\rho) = \mathcal{F} \)). Plugging this into (2.4), we obtain (2.3). \( \square \)

We now need the following:
Proposition 2.2  Let \( B \) be any ball of radius \( r > 0 \). If \( 0 < \rho \leq r \), then
\[
\mathcal{E}^{(\rho)}(e^{-\psi} f, e^{\psi} g) = \mathcal{E}^{(\rho)}(f, g)
\] (2.5)
for any \( f, g \in \mathcal{F}^{(\rho)} \cap L^\infty \), where \( \psi = \lambda 1_B \) with \( \lambda \in \mathbb{R} \).

Proof  As \( \psi = \lambda 1_B \in \mathcal{F} \), we see that \( e^{-\psi} - 1 \in \mathcal{F}^{(\rho)} \cap L^\infty \), by using the Markov property of \((\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})\), and hence both functions \( e^{\psi} g \) and \( e^{-\psi} g \) belong to \( \mathcal{F}^{(\rho)} \cap L^\infty \) if \( g \in \mathcal{F}^{(\rho)} \cap L^\infty \).

As \( 0 < \rho \leq r \), we see that \( B(x, \rho) \subset B(x, r) = B \) if \( x \in B \), whereas \( B(x, \rho) \subset B^c \) if \( x \in B^c \), because two balls \( B(x, \rho) \) and \( B \) are disjoint by using the ultra-metric property. Thus
\[
\psi(x) = \psi(y) = \lambda \text{ if } x \in B, y \in B(x, \rho),
\]
\[
\psi(x) = \psi(y) = 0 \text{ if } x \in B^c, y \in B(x, \rho).
\]

It follows that
\[
\mathcal{E}^{(\rho)}(e^{-\psi} f, e^{\psi} g) = \int_M \int_{B(x, \rho)} (e^{-\psi(x)} f(x) - e^{-\psi(y)} f(y)) (e^{\psi(x)} g(x) - e^{\psi(y)} g(y)) \, dj
\]
\[
= \int_B \int_{B(x, \rho)} + \int_{B^c} \int_{B(x, \rho)} \cdots
\]
\[
= \int_B \int_{B(x, \rho)} (e^{-\lambda} f(x) - e^{-\lambda} f(y)) (e^{\lambda} g(x) - e^{\lambda} g(y)) \, dj
\]
\[
+ \int_{B^c} \int_{B(x, \rho)} (f(x) - f(y)) (g(x) - g(y)) \, dj
\]
\[
= \int_M \int_{B(x, \rho)} (f(x) - f(y)) (g(x) - g(y)) \, dj = \mathcal{E}^{(\rho)}(f, g),
\]
thus proving (2.5). \( \square \)

Corollary 2.3  Let \( B \) be any ball of radius \( r > 0 \). If \( 0 < \rho \leq r \), then
\[
\mathcal{E}^{(\rho)}(e^{-\psi} f, e^{\psi} f^{2p-1}) \geq \frac{1}{p} \mathcal{E}^{(\rho)}(f^{p})
\] (2.6)
for any \( f \in \mathcal{F}^{(\rho)} \cap L^\infty \) and any \( p \geq 1 \), where \( \psi = \lambda 1_B \) with \( \lambda \in \mathbb{R} \).

Proof  Using the elementary inequality
\[
(a - b)(a^{2p-1} - b^{2p-1}) \geq \frac{2p-1}{p^2} (a^p - b^p)^2 \geq \frac{1}{p} (a^p - b^p)^2,
\]
for any non-negative numbers \( a, b \) and any \( p \geq 1 \), we obtain, by letting \( a = f(x), b = f(y) \), that
\[
\mathcal{E}^{(\rho)}(e^{-\psi} f, e^{\psi} f^{2p-1}) = \mathcal{E}^{(\rho)}(f, f^{2p-1})
\]
\[
= \int_{M \times B(x, \rho)} (f(x) - f(y))(f^{2p-1}(x) - f^{2p-1}(y)) \, dj
\]
\[
\geq \frac{1}{p} \int_{M \times B(x, \rho)} (f^p(x) - f^p(y))^2 \, dj
\]
\[
= \frac{1}{p} \mathcal{E}^{(\rho)}(f^p),
\]
thus proving (2.6). \( \square \)

Remark 2.4  Inequality (2.6) is an enhancement of the previous similar results in [3, formula (3.11)] and [13, formula (2.8)] (see also [12, formula (3.11)]) in the setting of the ultra-metric space.
For any \( f \in \mathcal{F} \cap L^\infty \) with \( \|f\|_2 = 1 \), denote by \( \{Q_t\}_{t \geq 0} \) the heat semigroup of \((\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})\). Let
\[
Q_t f := e^{\rho t} Q (e^{-\rho} f)
\]
(2.7)
be the perturbed semigroup of \( \{Q_t\}_{t \geq 0} \).

**Proposition 2.5** Let \( B \) be any ball of radius \( r > 0 \). If \( 0 < \rho < r \), then
\[
\frac{d}{dt} \|Q_t f\|_{2p} \leq -\frac{C_N}{p} \|Q_t f\|_{2p}^{\frac{1}{2}+2\rho p} \|Q_t f\|_{p}^{-2\rho p} + \frac{K_0}{p} \|Q_t f\|_p
\]
(2.8)
for any non-negative \( f \in \mathcal{F}^{(\rho)} \cap L^\infty \) and any \( p \geq 1 \), where \( \psi = \lambda 1_B \) with \( \lambda \in \mathbb{R} \) as before, and \( C_N \) is the same as in (2.3).

**Proof** Note that \( Q_t f \in \mathcal{F}^{(\rho)} \cap L^\infty \). Then, using (2.6) with \( f \) being replaced by \( f_t \),
\[
\frac{d}{dt} \|f_t\|_{2p} = 2p \int f_t^{2p-1} (\partial Q_t f)_{\psi} \mu - 2p \left( e^{\rho t} f_t^{2p-1}, Q_t f \right) - 2p e^{\rho t} (e^{\rho t} f_t^{2p-1}, e^{-\rho} f_t)
\]
\[
\leq -2 \mathcal{E}^{(\rho)}(f_t^p) \leq -C_N \|f_t\|_{2p}^{2p(1+\nu)} \|f_t\|_p^{-2p\nu} + K_0 \|f_t\|_2^{2p}.
\]
(2.9)
Plugging this into (2.9), we obtain
\[
2p \|f_t\|_{2p} \frac{d}{dt} \|f_t\|_{2p} = \frac{d}{dt} \|Q_t f\|_{2p} \leq -2C_N \|f_t\|_{2p}^{2p(1+\nu)} \|f_t\|_p^{-2p\nu} + 2K_0 \|f_t\|_2^{2p},
\]
which gives (2.8) after dividing \( 2p \|f_t\|_{2p}^{2p-1} \) on both sides.

For any integer \( k \geq 1 \), we define the function \( w_k(t) \) for \( t > 0 \) by
\[
w_k(t) := \sup_{s \in (0,t]} \left\{ s^{(2k-1-1)/(2k)} \|f_s\|_{2p} \right\}.
\]
(2.10)
Clearly, \( w_k(t) \) is non-decreasing in \( t > 0 \). Note that, by (2.8) with \( p = 1 \),
\[
\frac{d}{dt} \|f_t\|_2 \leq K_0 \|f_t\|_2,
\]
and thus
\[
\|f_t\|_2 \leq \exp(K_0 t) \|f\|_2.
\]
This gives that for any \( t > 0 \),
\[
w_1(t) = \sup_{s \in (0,t]} \|f_s\|_2 \leq \sup_{s \in (0,t]} \{ \exp(K_0 s) \|f\|_2 \} \leq \exp(K_0 t) \|f\|_2.
\]
(2.11)
We will estimate \( w_k \) by iteration.

**Proposition 2.6** Let \( w_k \) be defined as in (2.10). Then for any non-negative \( f \in \mathcal{F}^{(\rho)} \cap L^\infty \),
\[
w_{k+1}(t) \leq C_1 \exp(2K_0 t) \|f\|_2
\]
(2.12)
for any \( t > 0 \), where \( C_1 \) is some universal constant depending only on \( \nu, C_N \) (but independent of \( t, \rho, r \) and \( \lambda \)), and \( K_0 = \rho^{-\beta} + R_0^{-\beta} \) as before.
Proof For any integer \( k \geq 1 \), denote
\[ u_k(s) = \| f_s \|_{2^k} \quad (s > 0) \]
for simplicity. Applying (2.8) with \( p = 2^k \), we have
\[ u_{k+1}'(s) \leq - \frac{C_N^{-1}}{2^k} u_{k+1}(s) \left[ 1 + 2^{k+1} \nu \right] - \frac{K_0}{2^k} u_{k+1}(s) \quad (s > 0). \]
As \( w_k(s) \geq s^{(2^{k-1} - 1)/(2^k)} u_k(s) \), by definition (2.10), we obtain
\[ u_{k+1}' \leq - \frac{C_N^{-1}}{2^k} u_{k+1}(s) \left[ 1 + 2^{k+1} \nu \right] w_k(s) + K_0 2^{-k} u_{k+1}(s). \]
Applying Proposition A1 in Appendix with \( u(s) = u_{k+1}(s), \theta = 2^{k+1} \nu \) and \( b = C_N^{-1} 2^{-k}, p = 2^k, \)
\( w = u_k \), and \( K = K_0 2^{-k}, a = 1 \), we obtain
\[ u_{k+1}(s) \leq \left\{ \frac{2}{2^{k+1} \nu} \cdot \frac{2^k}{C_N^{-1} 2^{-k}} \right\}^{-1} \left\{ s^{(2^{k+1} - 2)/(2^{k+2})} \right\} \left\{ s^{(2^{k-1} - 1)/(2^k)} \exp \left\{ K_0 4^{-k} s \right\} w_k(s) \right\} \]
that is,
\[ s^{(2^{k+1} - 2)/(2^{k+2})} u_{k+1}(s) \leq \left\{ \frac{2^k}{C_N^{-1} \nu} \right\} \left\{ \exp \left\{ K_0 4^{-k} s \right\} w_k(s) \right\} \]
for any \( s > 0 \).

Therefore, for any \( t > 0, k > 0 \),
\[ w_{k+1}(t) = \sup_{s \in (0,t]} \left\{ s^{(2^{k+1} - 2)/(2^{k+2})} u_{k+1}(s) \right\} \leq \left\{ \frac{2^k}{C_N^{-1} \nu} \right\} \sup_{s \in (0,t]} \left\{ \exp \left\{ K_0 4^{-k} s \right\} w_k(s) \right\} \leq \left\{ \frac{2^k}{C_N^{-1} \nu} \right\} \exp \left\{ K_0 2^{-k} t \right\} w_k(t) =: (Da)^{2^{-k}} w_k(t), \quad (2.13) \]
where \( D := (C_N^{-1} \nu)^{-1/(2^k)} \exp (K_0 t) \) and \( a := 2^{1/(2^k)} > 1 \). For the second inequality, we use \( 2^{-k} \leq 1 \).

By iteration, we see from (2.13) and (2.11) that for any \( k \geq 1 \),
\[ w_{k+1}(t) \leq (Da)^{2^{-k}} w_k(t) \leq (Da)^{2^{-k}} \cdot (Da^{k-1})^{2^{-k-1}} w_{k-1}(t) \leq \cdots \leq D^\ell w_1(t), \]
where \( C_1 = \max \{ 1, (C_N^{-1} \nu)^{-1/(2^k)} \} a^2 \), thus proving (2.12). \( \square \)

We now estimate the heat kernel \( q_t^{(p)}(x, y) \) of the truncated Dirichlet form \( (E^{(p)}, F^{(p)}) \).
Lemma 2.7 Assume that conditions (TJ), (DUE) hold. Then, for any ball \( B \) of radius \( r \in (0, R_0) \) and for any \( 0 < \rho \leq r \), the heat kernel \( q_t^{(\rho)}(x, y) \) of \((\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})\) exists and satisfies

\[
q_t^{(\rho)}(x, y) = 0
\]  

(2.14)

for \( \mu \)-almost all \( x \in B, y \in B^c \) and for all \( t > 0 \).

Proof Let \( 0 \leq f \in \mathcal{F} \cap L^\infty \) with \( \|f\|_2 = 1 \). By (2.12), we have

\[
t^{(2^{k-1}-1)/(2^k \nu)} \|f_t\|_{2^k} \leq C_1 \exp(2K_0t),
\]

which gives, by letting \( k \to \infty \), that

\[
\|Q_t^\psi f\|_\infty = \|f_t\|_\infty \leq \frac{C_1}{t^{1/(2\nu)}} \exp(2K_0t)
\]

(2.15)

for any \( t > 0 \). It follows that

\[
\left\|Q_t^\psi\right\|_{1 \to \infty} := \sup_{\|f\|_1=1} \|Q_t^\psi f\|_{\infty} \leq \frac{C_1}{t^{1/(2\nu)}} \exp(2K_0t).
\]

As \( Q_t^{-\psi} \) is the adjoint of the operator \( Q_t^\psi \), we see that

\[
\|Q_t^\psi\|_{1 \to \infty} = \sup_{\|f\|_1=1} \|Q_t^\psi f\|_{2^k} \leq \|Q_t^{-\psi}\|_{2^k \to \infty} \leq \frac{C_1}{t^{1/(2\nu)}} \exp(2K_0t),
\]

and thus

\[
\|Q_t^\psi\|_{1 \to \infty} \leq \|Q_t/2\|_{1 \to \infty} \leq \frac{C_1^2}{(l/2)^{1/\nu}} \exp(2K_0t).
\]

From this, we conclude that the heat kernel \( q_t^{(\rho)}(x, y) \) of \((\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})\) exists and satisfies

\[
q_t^{(\rho)}(x, y) \leq \frac{C}{t^{1/\nu}} \exp(2K_0t + \lambda(1_B(y) - 1_B(x))) .
\]

(2.16)

Therefore, for \( \mu \)-almost all \( x \in B, y \in B^c \) and for all \( t > 0 \),

\[
q_t^{(\rho)}(x, y) \leq \frac{C}{t^{1/\nu}} \exp(2K_0t - \lambda).
\]

Letting \( \lambda \to \infty \), we have \( q_t^{(\rho)}(x, y) = 0 \). The proof is complete. \( \square \)

Lemma 2.7 says that the heat kernel \( q_t^{(\rho)}(x, y) \) of the truncated Dirichlet form will vanish for any \( t > 0 \) when points \( x \) and \( y \) are separated by any ball of radius \( r > \rho \). This means that the Hunt process associated with the truncated Dirichlet form never goes farther away than the truncated range \( \rho \). This lemma will give the strong tail estimate of the heat semigroup \( \{P_t\} \) of the form \((\mathcal{E}, \mathcal{F})\). This new phenomenon arises from the ultra-metric property of the space \( M \). General metric spaces do not admit this nice property. For example, for \( \delta \in (0, 1) \), the Dirichlet form \((\mathcal{E}^\delta, \mathcal{F}^\delta)\) is regular on \( \mathbb{R}^d \) (see [4, (2.14)–(2.16)]). Let \( Z^\delta \) be a symmetric Markov process associated with \((\mathcal{E}^\delta, \mathcal{F}^\delta)\). The generator of \( Z^\delta \) is

\[
\mathcal{L}^\delta u(x) = \lim_{\varepsilon \to 0} \int_{|y| \leq |x| \leq \varepsilon} (u(y) - u(x))J_\delta(x, y)dy,
\]

where \( J_\delta(x, y) \) is a symmetric non-negative function (see [4, (3.4)]). \( Z^\delta \) has a quasi-continuous heat kernel \( q_t^{(\delta)}(x, y) \) defined on \([0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \). For \( \delta_0 > 0 \), there exist \( c = c(\delta_0) > 0 \) such that

\[
q_t^{(\delta)}(x, y) \geq ct^{-d/2} \text{ for every } t > \delta_0 \text{ and quasi everywhere } x, y \text{ with } |x - y|^2 \leq t \text{ (see [4, Theorem 3.4]).}
\]

\( \square \) Springer
Proposition 2.8 Assume that conditions (TJ), (DUE) hold. Then, for any ball $B$ of radius $r \in (0, R_0)$ and for any $t > 0$,

$$P_t 1_{B^c} \leq \frac{Ct}{r^\beta} \text{ in } B$$

(2.17)

for some universal constant $C > 0$ (independent of $t, B$).

Proof By Lemma 2.7, we have that for $\mu$-almost all $x \in B$ and all $t > 0$,

$$Q_t 1_{B^c}(x) = \int_{B^c} q_t^{(\rho)}(x, y) d\mu(y) = 0,$$

if $0 < \rho \leq r$. Applying Proposition A2 in Appendix for $\rho = \frac{r}{2}, \Omega = M, f = 1_{B^c}$, we see that

$$P_t 1_{B^c} \leq Q_t 1_{B^c} + 2t \sup_{x \in M} J(x, B(x, \rho)^c) \leq C \frac{t}{r^\beta} \text{ in } B.$$ 

The proof is complete. □

We are in a position to prove Theorem 1.1.

Proof Fix points $x_0, y_0 \in M$ and fix $t \in (0, R_0^3)$. Without loss of generality, assume that $d(x_0, y_0) \geq 1$; otherwise (wUE) follows directly from (DUE) and nothing is proved. Let $r = \frac{d(x_0, y_0)}{2}$. Noting that $P_t 1_{B^c}$ is monotone decreasing in $r$, we have from (2.17), for $r > 0, t > 0$, that

$$P_t 1_{B^c} \leq \frac{Ct}{(r \wedge R_0)^\beta}.$$ 

(2.18)

It follows from condition (DUE) and (2.18), for $\mu$-almost all $x \in B(x_0, r), y \in B(y_0, r)$, that we have

$$p_{2t}(x, y) = \int_M p_t(x, z)p_t(z, y) d\mu(z)$$

$$\leq \int_{B(x_0, r)^c} p_t(x, z)p_t(z, y) d\mu(z) + \int_{B(y_0, r)^c} p_t(x, z)p_t(z, y) d\mu(z)$$

$$\leq \sup_{z \in M} p_t(z, y) \int_{B(x_0, r)^c} p_t(x, z) d\mu(z) + \sup_{z \in M} p_t(x, z) \int_{B(y_0, r)^c} p_t(z, y) d\mu(z)$$

$$\leq 2 \frac{C}{t^{5/3}} \cdot \frac{Ct}{(r \wedge R_0)^\beta}.$$ 

Therefore, for $\mu$-almost all $x_0, y_0 \in M$ and $t \in (0, R_0^3)$, we conclude that

$$p_{2t}(x_0, y_0) \leq \frac{C}{t^{5/3}} \cdot \frac{t}{(d(x_0, y_0) \wedge R_0)^\beta}.$$ 

This inequality, together with (DUE), will imply (wUE). The proof is complete. □

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Appendix

The following was first shown in [3, Lemma 3.21], and then was modified in [13, Lemma 2.6] (see also [12, Lemma 3.4]):

**Proposition A1** Let $w: (0, \infty) \to (0, \infty)$ be a non-decreasing function and suppose that $u \in C^1((0, \infty); (0, \infty))$ satisfies that for all $t \geq 0$,

$$u'(t) \leq -b t^{p-2} w^\theta(t) u^{1+\theta}(t) + Ku(t)$$

for some $b > 0$, $p > 1$, $\theta > 0$ and $K > 0$. Then

$$u(t) \leq \left( \frac{2 \rho^a}{\theta b} \right)^{1/\theta} t^{-(p-1)/\theta} e^{K p^{-a} t} w(t)$$

for any $a \geq 1$.

The following result was proved in [10, Proposition 4.6]:

**Proposition A2** For an open set $\Omega \subset M$, let $\{P_t^\Omega\}$ and $\{Q_t^\Omega\}$ be the heat semigroups of $(\mathcal{E}, \mathcal{F}(\Omega))$ and $(\mathcal{E}^{(d)}, \mathcal{F}^{(d)}(\Omega))$, respectively. Then, for any $t > 0$,

$$P_t^\Omega f \leq Q_t^\Omega f + 2t \sup_{x \in M} J(x, B(x, \rho)^c) \|f\|_\infty.$$