Cosmological Solutions on Compactified $AdS_5$

with a Thermal Bulk

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Abstract

This paper is an investigation of the effects of a thermal bulk fluid in brane world models compactified on $AdS_5$. Our primary purpose is to study how such a fluid changes the bulk dynamics and to compare these effects with those generated by matter localized to the branes. We find an exact cosmological solution for a thermally excited massless bulk field, as well as perturbative solutions with matter on the brane and in the bulk. We then perturb around these solutions to find solutions for a massive bulk mode in the limit where the bulk mass ($m_B$) is small compared to the AdS curvature scale and $T < m_B$. We find that without a stabilizing potential there are no physical solutions for a thermal bulk fluid. We then include a stabilizing potential and calculate the shift in the radion as well as the time dependence of the weak scale as a function of the bulk mass. It is shown that, as opposed to a brane fluid, the bulk fluid contribution to the bulk dynamics is controlled by the bulk mass.
I. INTRODUCTION AND MOTIVATION

The possibility that we are living on an isolated defect has led to new paradigms which are now being explored. While the low energy effective theory of these models correctly reproduces the standard model physics, we expect that the cosmology should be quite dissimilar at large enough temperatures. In particular, the cosmology of models which solve the hierarchy problem through warped compactification, such as the Randall Sundrum model [1] (RSI), are especially interesting because we expect to see deviations from the standard cosmology as we approach the weak scale [2–14]. The cosmology of these models has been discussed previously including the effects of matter density on the branes. What makes the cosmology of these models particularly interesting is that we expect five dimensional gravity to become strongly coupled at the TeV scale. Above the TeV scale, it is believed that the theory should still have a description in terms of a local quantum field theory, via the holographic principle [17]. In this paper we will continue the exploration of the cosmology of (RSI) by studying the effects of a thermal fluid in the bulk. We will study temperatures low enough so that, $T < m_{kk}M_5/k$ (where $M_5$ is the five dimensional Planck scale and $k$ is the $AdS$ curvature scale) and a weakly coupled ($k < M_5$) five dimensional gravity description is still valid. Since our results only depend upon the geometry they are also applicable to models where the hierarchy is stabilized via supersymmetry and generated via the geometry [15]. However, we will couch our results in terms of (RSI).

The primary motivation for this work stems from the fact that there must be some nontrivial bulk physics in the RSI model, as well as in the supersymmetric models mentioned above. Thus, we would expect that at some point in its thermal history, more specifically at the TeV scale, a thermal bulk will be relevant. In its purest form, RSI entails the trapping of the standard model particles on a brane of negative tension positioned at an orbifold fixed point. At some finite proper distance in the extra dimension there is a positive tension brane where the graviton is localized. In this choice of coordinates, the small overlap of the graviton wave function with the standard model brane leads to the apparent weakness
of gravity. This minimal model must be augmented to account for the stabilization of configuration. That is, in the minimal RSI, the distance between the branes parameterizes a flat direction in field space. The introduction of new physics in the bulk can lift this flat direction. For instance, the Goldberger-Wise (GW) mechanism is a natural way to lift the degeneracy by introducing a bulk scalar with a non-trivial vacuum profile. Thus, in general, we may expect that in addition to gravitons propagating in the bulk, there will be other fields which may play a role in the bulk physics. In fact, it is phenomenologically viable to have particles with standard model gauge charges propagate in the extra dimension \cite{16}. If this were the case then bulk thermal effects would be enhanced due to the large number of species which would equilibrate.

In this work we seek to understand how the cosmology changes once the bulk thermal effects are taken into account. We would expect that at long distances the bulk fluid will have the same effect as a fluid trapped to the brane. Though we might be concerned by the fact that, since the lapse function is changing exponentially, the effect of the bulk fluid at long distances could differ from those of a brane fluid. It has been pointed out that the visible and Planck brane matter densities \((\rho, \rho*)\) contribute to the expansion rate as \cite{7}

\[
H^2 = \frac{8\pi G_N}{3} (\rho_* + \rho e^{-4kb_0}),
\]

where \(b_0\) is the proper distance between the branes. Given that the natural scale of the Planck brane is \(M_{pl}\), this puts strong constraints on the brane matter density, as pointed out in \cite{7}. Indeed, this constraint implies that the Planck brane must be in its vacuum state for all intents and purposes. Thus, depending on the bulk matter density profile, a bulk energy density could lead to over-closing the universe.

We would also expect a bulk fluid to effect the bulk dynamics quite differently from a brane fluid. A brane fluid generates a non-trivial radion potential, which in the absence of a stabilizing mechanism, tends to push the branes apart, unless one imposes the unphysical fine-tuning between the visible brane energy density \((\rho)\) and Planck brane density \((\rho_*)\),

\[
\rho_* = -\rho e^{-4kb_0} \quad \text{\cite{2-4}}.
\]

Here we calculate the radion potential generated by a bulk thermal...
fluid [18], and determine its effect on the above fine tuning relation. We then include a stabilizing potential to see how a bulk fluid shifts the radion expectation value and the weak scale as functions of time.

To address these issues we will solve the coupled bulk fluid Einstein equations. We start by finding an exact solution with fluid in the bulk and vacuum branes. In general, this is rather intractable problem, since the sources on the right hand side of the Einstein equation are complicated highly non-linear functions of the metric ansatz. However, we will be able to find an exact solution in the particularly simple case of a massless field at temperatures too low to excite higher Kaluza-Klein modes. We then find a realistic solution with matter in the bulk, as well as on the branes, which is valid when the energy densities are small compared to the bulk curvature (i.e. visible brane temperatures less than a TeV). We then perturb around the massless solution to find more interesting solutions which include the effects of a massive bulk field. We show that without a stabilizing mechanism, there are no physical solutions. Furthermore, once stabilization is taken into account, all the non-trivial bulk dynamics, e.g. the evolution of the weak scale, are driven by the bulk field mass.

II. EXACT SOLUTION WITH A BULK FLUID

The five dimensional action is given by

\[ L = \int d^5 x \sqrt{-G}(-M_5^3 R - \Lambda + L_B) + \int d^4 x \sqrt{-g} L_{TeV} + \sqrt{-g_*} L_{Pl}, \]

where \( g \) and \( g_* \) are the induced metrics on the visible and Planck branes, respectively. \( M_5 \) is the five dimensional Planck scale and \( L_B \) is the bulk field Lagrangian which describes the stress energy of the bulk fluid. The fifth dimension is compactified on \( S_1/Z_2 \), with the Planck and visible branes placed at the orbifold fixed points, \( y = 0, 1 \), respectively. For this section we will leave the branes empty, and thus, their only contribution to the action are due to their tensions \( V_* \) and \( V \).

We will search for solutions which have a stable (time independent) fifth dimension. We thus make the following ansatz for the metric.
\[ ds^2 = e^{-2A_0(y)} dt^2 - a(t)^2 e^{-2A_0(y)} dx^2 - b_0^2 dy^2. \]  

We could rescale the \( y \) coordinate to eliminate the constant \( b_0 \), but choose not to for bookkeeping purposes. The bulk stress energy tensor is

\[ T_{mn} = \Lambda g_{mn} + g_{mp} T^p_n, \]

with \( T^p_n = \text{diag}(\rho_B(t, y), -\tilde{P}_B(t, y), -P_5(t, y)) \).

The Bianchi identity leads to

\[ A_0'(y)(\rho_B(t, y) - 3P_B(t, y) + 4P_5(t, y)) - P_5'(t, y) = 0. \]

\[ \dot{\rho}_B(t, y) + 3 \frac{\dot{a}}{a}(t)(P_B(t, y) + \rho_B(t, y)) = 0. \]

Primes denote derivative with respect to \( y \), and dots denote derivative with respect to the time-like coordinate.

The \( G_{00} \), \( G_{ii} \) and \( G_{55} \) Einstein equations in the bulk are

\[ \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{b_0^2} e^{-2A_0(y)} \left( 2(A_0'(y))^2 - A_0''(y) \right) = \frac{\kappa^2}{3} e^{2A_0(y)} (\rho_B(t, y) + \Lambda), \]

\[ \frac{3}{b_0^2} e^{-2A_0(y)} (2(A_0'(y))^2 - A_0''(y)) - \left( \left( \frac{\dot{a}}{a} \right)^2(t) + 2 \frac{\ddot{a}}{a}(t) \right) = \kappa^2 e^{-2A_0(y)} (P_B(t, y) - \Lambda), \]

\[ 6(A_0(y))' - 3b_0^2 e^{2A_0(y)} \left( \left( \frac{\dot{a}}{a} \right)^2(t) + \frac{\ddot{a}}{a}(t) \right) = \kappa^2 b_0^2(P_5(t, y) - \Lambda), \]

and we have defined \( \kappa^2 = \frac{1}{2M_5^3} \). Combining the \( G_{00} \) and \( G_{ii} \) equations, we see that the bulk energy density and pressure must be of the highly restricted form

\[ \rho_B(t, y) = \dot{\rho}(t)e^{2A_0(y)}, \quad P_5(t, y) = \dot{P}_5(t)e^{2A_0(y)} \quad P_B(t, y) = \dot{P}_B(t)e^{2A_0(y)}. \]

This restriction results from our simple choice of ansatz. Whether or not a bulk fluid will indeed yield this form will be discussed later.

To solve this system of equations we first make linear ansatz for \( A_0 \), since the geometry should reduce to AdS in the limit of vanishing bulk energy-momentum, which leads to the
usual result $|A_0 - kb_0 y|$ with $k = -\kappa^2 \Lambda / 6$, as in the RSI solution. This result has been derived using the $G_{00}$ and $G_{ii}$ equations. We have not, to this point, imposed the $G_{55}$ equation which for this simple ansatz is identical to the Bianchi identity

$$2\dot{P}_5(t) + \dot{\rho}_B(t) - 3\dot{P}_B(t) = 0, \quad (11)$$

With the result that

$$\dot{\rho}_B(t) = \frac{3}{\kappa^2} \left( \frac{\dot{a}}{a} \right)^2(t), \quad (12)$$

$$\dot{P}_B(t) = -\frac{1}{\kappa^2} \left( \left( \frac{\dot{a}}{a} \right)^2(t) + 2 \frac{\ddot{a}}{a}(t) \right), \quad (13)$$

$$\dot{P}_5(t) = -\frac{3}{\kappa^2} \left( \left( \frac{\dot{a}}{a} \right)^2(t) + \frac{\ddot{a}}{a}(t) \right), \quad (14)$$

which is seen to obey (1). A related solution was found previously in [10,11]. Note that the fluid tends to build up near the visible brane. It is straightforward to see that the usual cosmology on the brane results from this solution.

We must now deal with the fact that our ansatz leads to the constraint (10) on the form of the bulk pressures and energy density. One simple case for which we will have a solution is vacuum domination, as discussed in [10,11], where the constraints (10) are obviously satisfied. Here we are more interested in the more inevitable case where the energy density arises from thermal fluctuations. In this case we must calculate the local pressure and energy density using our solution to determine if the constraint is indeed satisfied. We will postpone this calculation until we have included the effects of matter on the branes.

Before closing this section we note that the jump conditions

$$3[A'_0(0)]_- = \kappa^2 V, \quad -3[A'_0(1)]_- = -\kappa^2 V, \quad (15)$$

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1From here on, the absolute value, implied by the orbifolding, will be dropped for notational simplicity.
lead to the usual fine tunings. One tuning is needed to fix the flatness of the brane and the other to enforce the zero force condition between the branes,

$$V_* = 6k/\kappa^2, \quad V = -V_*.$$  

(16)

The fact that there are two fine-tunings is a consequence of the lack of a radion potential for this solution, which one might have expected to be generated by the existence of the bulk fluid. This result is clearly related to the constraint (10). We shall explore this point further below.

### III. SOLUTIONS WITH MATTER ON THE BRANE AND IN THE BULK

Solutions with matter on the brane were previously found in [7,12,8]. Here we augment these results to include the effects of a massless bulk fluid. The derivation is similar to that in [7] but is included for completeness.

The previous ansatz is not general enough to allow for a solution with brane matter. We thus introduce the brane stress energy tensors

$$T^A_B = \frac{\delta(y)}{b_0} \text{diag}(V_* + \rho, V_* - p, V_* - p, V_* - p, 0) + \frac{\delta(1-y)}{b_0} \text{diag}(-V + \rho, -V - p, V - p, V - p, 0),$$  

(17)

and we modify the ansatz, as follows

$$ds^2 = \exp^{-2A_0(y)} (1 + 2f(t,y)) dt^2 - a(t)^2 \exp^{-2A_0(y)} (1 + 2g(t,y)) dx^2 - b_0^2 dy^2, \quad (18)$$

and treat both the bulk and brane matter as perturbing sources around the standard RSI solution. We will calculate to leading non-trivial order in the perturbations $\rho \sim O(\epsilon)$.

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2In [12] a solution was found which is accurate to all orders in the brane matter density. However, the solution is only valid as long as the energy density on the brane is small enough that it can still be assumed that the $G_{55}$ Einstein equation is automatically satisfied. This will be true only as long as the matter energy density is small compared to the scales in the stabilizing field potential.
Since $\dot{\rho} \sim \dot{a}\rho$, it will often be the case that we will be able to drop time derivatives of the fluctuations. The solutions will again restrict the bulk profile of the matter density to be of the form (11).

The $G_{05}$ Einstein equation is now no longer automatically satisfied, and is given by

$$\ddot{f}(t, y) + \frac{\dot{a}}{a}(t) (f'(t, y) - g'(t, y)) = 0. \quad (19)$$

In this equation we must keep the time derivative of the perturbation, since it is leading order. While the (00), (ii) and (55) linearized Einstein equations are given by

$$3 \left( \frac{\dot{a}}{a} \right)^2 e^{2A_0(y)} - 6(A_0'(y))^2 + 12A_0'(y)f'(t, y) - 3f''(t, y) = b_0^2 \kappa^2 (e^{-2A_0(y)} \rho_B(t, y)), \quad (20)$$

$$- 4k b_0 (2f'(t, y) + g'(t, y)) + \frac{1}{b_0^2} (2f''(t, y) + g''(t, y)) - e^{2A_0(y)} \left( \frac{\ddot{a}}{a}(t) + \left( \frac{\dot{a}}{a} \right)^2 \right) = \kappa^2 (P_B(t, y)), \quad (21)$$

$$- 3b_0^2 e^{2A_0(y)} \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a}(t) \right) - 3A_0'(y)(3f'(t, y) + g'(t, y)) = b_0^2 \kappa^2 P^5(y, t). \quad (22)$$

The $y$ component of the contracted Bianchi identities is still of the form (I) and the energy conservation equations for the brane as well as bulk matter take on the form of (I).

The jump conditions are

$$[f'(0)]_- = \frac{\kappa^2 b_0}{3} \rho_\star(t), \quad [f'(1)]_- = \frac{\kappa^2 b}{3} \rho(t). \quad (23)$$

$$[g'(0)]_- = \kappa^2 b_0 (P_\star(t) + \frac{2}{3} \rho_\star(t)), \quad [g'(1)]_- = -\kappa^2 b (P(t) + \frac{2}{3} \rho(t)). \quad (24)$$

We may now solve for $f(t, y)$ and $g(t, y)$ as follows. As before the leading order Einstein equations lead to $A_0(y) = kb_0 y$. Following our exact solution in the previous section, we assume the matter has the form (III). Then integrating the next to leading order $G_{00}$ equation leads to

$$f(t, y) = A(t)e^{4kb_y} - \frac{1}{4k^2} \left[ \left( \frac{\dot{a}}{a} \right)^2 - \frac{\kappa^2}{3} \dot{B}(t) \right] e^{2kb_y}. \quad (25)$$

Imposing the jump conditions at 0 and 1, yields the relations...
\[8k^2 A(t) - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{k\kappa^2}{3} \rho_*(t), \tag{26}\]

\[-8k^2 A(t)e^{4kb} + e^{2kb} \left(\frac{\dot{a}}{a}\right)^2 = -\frac{k\kappa^2}{3} \rho(t), \tag{27}\]

which leads to the FRW-like equation for the expansion rate

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3} \left[\frac{\dot{\rho}_B}{k}(t)(1 - e^{-2kb}) + \rho_*(t) + \rho(t)e^{-4kb}\right]. \tag{28}\]

Where we have used the relation \(8\pi G_N = \frac{\kappa^2}{1 - e^{-2kb_0}}\), which can be read off of the effective four dimensional action. To leading order, \(\frac{\dot{a}}{a}\) is the expansion rate on the visible brane. Notice that this result agrees with the contribution to the expansion rate that one would get by averaging the bulk energy over the extra dimension. As with the Planck brane matter there is no warp factor suppression, which may at first be alarming. To determine whether or not this result presents a problem in these models we need to know how the bulk energy density is normalized. Obviously, if it scales like \(M_p\), the cosmology is not viable. The correct normalization will be calculated in section four.

We may now solve for \(g(t,y)\) by assuming an equation of state \(\rho_B = w_B P_B\) and imposing the \(G_{05}\) equation in conjunction with the time component of the Bianchi identity.

\[
\frac{\dot{a}}{a}g(t,y) = -\frac{e^{2kb_0y}}{4k^2} \left[\frac{d}{dt} \left(\frac{\dot{a}}{a} - \frac{\kappa^2}{3} \dot{\rho}_B(t)\right) + \frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} - \frac{\kappa^2}{3} \dot{\rho}_B(t)\right)\right] + e^{4kb_0y}(\dot{A}(t) + \dot{a}A(t)) \tag{29}\]

Imposing the jump conditions for \(g(y)\), reproduces the usual energy conservations for the branes

\[
\dot{\rho}(t) + 3\frac{\dot{a}}{a}(t) (P(t) + \rho(t)) = 0
\]

\[
\dot{\rho}_*(t) + 3\frac{\dot{a}}{a}(t) (P_*(t) + \rho_*(t)) = 0. \tag{30}\]

Thus, to the order we are working, the branes expand at the same rate.

Similarly the \(G_{ii}\) Einstein equation leads to the FRW like relation for the acceleration
\[
\frac{\ddot{a}}{a}(t) = - \frac{4\pi G_N}{3} \left[ 3(P_*(t) + P(t)e^{-4kb_0} + \frac{\dot{P}_B}{k}(t)(1 - e^{-2kb}) + \rho_*(t) + \rho(t)e^{-4kb} + \frac{\dot{\rho}_B}{k}(t)(1 - e^{-2kb_0}) \right].
\]

(31)

The \(G_{55}\) equation leads to the requirement

\[
4A(t) + \frac{a}{\dot{a}} \dot{A}(t) = 0.
\]

(32)

However, upon inspection we see that \(A(t)\) is independent of the bulk energy density and pressure and that the constraint from the \(G_{55}\) equation is just the usual unphysical relation, which results from the lack of a stabilizing potential, \(\rho_*(t) = -e^{-4kb_0}\rho(t)\). This is just a sign that the bulk fluid has generated no radion potential. We shall see that this is a consequence of our simple ansatz for the metric, which greatly restricts the form of the bulk energy density, and that this solution will only be valid for a massless bulk field at temperatures low enough as not to excite higher KK modes.

**IV. THE BULK ENERGY DENSITY**

The solution found in the previous section necessitated that the bulk energy density and pressure have the particular form (II). As we mentioned previously, while the constraint is satisfied for a bulk cosmological constant we are more interested in the case of a thermal fluid, as this is the case of phenomenological relevance. We thus would like to calculate the thermal expectation value

\[
\langle T_{AB}(t, y) \rangle = Tr[T_{AB}(t, y)e^{-\beta_0 H}],
\]

(33)
in our background solution to test for consistency. We have assumed that the expansion rate is small enough that we have an approximate time-like killing vector, and that local thermodynamic equilibrium can be established. Furthermore, we normalize the time-like killing vector such that the Hamiltonian generates time translations on the visible brane. That is to say that, \(T\) will correspond to the proper temperature on the brane. Requiring
maximization of the proper entropy leads to the relation $\sqrt{g_{00}}T = \text{const.}$ Thus, the proper temperature measured by a local observer will grow with the warp factor as one moves away from the visible brane. This growth prevents the flow of heat between regions of differing gravitational potential. We will further assume that the temperature is low enough so that the modes of interest are not strongly interacting.

Let us first consider the field of interest to be, for now, a scalar, whose Hamiltonian will be

$$H = \frac{1}{2} e^{-2A_0(y)} (\dot{\phi}^2 + (\nabla \phi)^2) + \frac{1}{2b_0^2} (\phi')^2 e^{-4A_0(y)} + \frac{1}{2} m^2 \phi^2 e^{-4A_0(y)}. \quad (34)$$

We then perform the usual Kaluza-Klein decomposition

$$\phi(x, y) = \sum_n \phi_n(x) \frac{\psi_n(y)}{\sqrt{b_0}}, \quad (35)$$

where $\psi_n(y)$ satisfy

$$-\frac{1}{b^2} \frac{d}{dy} \left( e^{-4A_0(y)} \frac{d\psi_n}{dy}(y) \right) + m^2 e^{-4A_0(y)} \psi_n = m_n^2 e^{-2A_0(y)} \psi_n(y), \quad (36)$$

and the inner product is defined as

$$\int_{-1}^{1} dy \psi_n(y) \psi_m(y) e^{-2A_0(y)} = \delta_{nm}. \quad (37)$$

The eigenmodes are linear combinations of Bessel functions, and were computed in [23]. The relevant components of the stress energy tensor are

$$T_{00} = \sum_n \frac{1}{2b_0} \psi_n(y)^2 T_{00}^n + \frac{1}{2} \phi_n^2(x) e^{2A_0(y)} \frac{d}{dy} \left( \frac{\psi_n(y) \psi'_n(y)}{b_0^3} e^{-4A_0(y)} \right)$$

$$T_{ii} = \sum_n \frac{a^2(t)}{2b_0} \psi_n(y)^2 T_{ii}^n + \frac{1}{2} \phi_n^2(x) e^{2A_0(y)} \frac{d}{dy} \left( \frac{\psi_n(y) \psi'_n(y)}{b_0^3} e^{-4A_0(y)} \right)$$

$$T_{55} = \sum_n \frac{1}{2} \phi_n^2(x) e^{2A_0(y)} \frac{d}{dy} \left( \frac{\psi_n(y) \psi'_n(y)}{b_0} e^{-4A_0(y)} \right) + \frac{1}{2} \phi_n^2(x) \left( \frac{\psi'_n(y)}{b_0} \right)^2.$$

$T^n_{00}$ and $T^n_{ii}$ are the four dimensional energy density and pressure for the $n$th mode. In these expressions we have dropped mixing terms which will not contribute to the thermal average.
and have used the free field equations of motion to simplify $T_{55}$, $T_{05}$, which would give rise to a net flow onto the branes, has vanishing thermal expectation value. Consider now the contribution from one, say the lowest, Kaluza-Klein mode $n = 0$ in the limit of vanishing bulk mass (a fine tuned case when the bulk is not supersymmetric). In this limit, the lowest mode is massless with a constant wave function. We find

$$\rho_B(y,t) = e^{2A_0(y)}\psi_0^2(y)\frac{\pi^2}{30b_0}T^4,$$

where again $T$ is the proper temperature, as measured by a visible brane observer. Note that, as one would expect, the local energy density is weighted by the wave function and now has a profile which will only be consistent with our solution if it is trivial. The normalized wave function is given by $\psi_0^2(y) = kb_0(1 - e^{-2kb_0})$, and we see from eq.(38) that, up to exponentially suppressed terms, the zero mode contributes to the energy density exactly like a field trapped to the brane, with its normalization set by the AdS curvature scale. Thus, the solutions discussed in the previous sections are valid for a massless bulk field in the limit where the temperature is too small to excite the higher Kaluza-Klein modes. It is straightforward in this case to see that the bulk pressure in the direction transverse to the branes vanishes and that bulk pressure just reduces to the four dimensional pressure in the same way that the energy density does.

Before going on to the more interesting massive case, we point out that a massless bulk field is interesting in that it leads to no deviations from standard cosmology, which seems rather surprising at first. The lowest mode of a massless bulk field generates no bulk dynamics at all. Furthermore, its contribution to the expansion rate is \emph{exactly} the same as in the standard cosmology. Contrast this to a massless field trapped to the brane which has non-linear contributions to the expansion rate, and which generates a shift in the lapse function.
V. THE MASSIVE BULK FIELDS

Thus, we would like to extend our previous results to the case of a massive mode. Again, to make the problem tractable, we will assume that we are only exciting the lowest eigenmode. In principle it is possible to treat the sum of several modes without conceptual change. We will find solutions to Einstein’s equations by perturbing around the massless solution via an expansion in the small parameter $m_B/k$. We will further assume that the bulk energy density is small compared to the brane tension, as we did in the previous section. This will allow us to drop time derivatives since they are higher order in $\rho$. Writing $\psi_n(y) = C(1 + m_B^2 \delta(y))$ leads to

$$\delta''(y) - 4kb_0 \delta'(y) = b_0^2(1 - \frac{m_0^2}{m_B^2} e^{2kb_0 y}),$$

whose normalized solutions are given by

$$\psi(y) = kb_0 \left(1 + \frac{m_B^2}{8k^2} (3 + 2e^{2kb_0 y} - e^{2kb_0(2y-1)}(1 - e^{-2kb_0}) - 4kb_0(1 + y)),\right)$$

where $m_0$ is the massive of the lowest lying mode, $m_B$ is the bulk mass and exponentially suppressed terms have been dropped. We impose the boundary conditions $\psi'(0) = \psi'(1) = 0$, since we have not included any interactions of this field on the brane. This assumption is rather unnatural since quantum effects will always induce operators localized to the brane [21,22]. However, it is not difficult to modify this analysis to include these effects, which only complicate the algebra and do not lead to any conceptual changes. Imposing the boundary conditions we find that the mass of the mode is given by $m_0^2 = \frac{m_B^2}{2}(1 - e^{-2kb_0})$. This case is again a fine tuned case since the natural scale for the bulk mass is order $M_{pl}$, but is sufficient for our purposes. The average energy density is again just the canonical four dimensional energy density for a thermal fluid, as a consequence of the topological constraint arising from [33].

Since the bulk profile is now no longer trivial, we will have to generalize our naive ansatz. For later convenience we choose the slightly different form
\[ ds^2 = e^{-2\alpha(y,t)} dt^2 - a_0^2(t) e^{-2\beta(y,t)} dx^2 - b(t, y)^2 dy^2. \] (41)

Where

\[ \alpha(y, t) = A_0(y) + \delta\alpha(y, t) \]
\[ \beta(y, t) = A_0(y) + \delta\beta(y, t) \]
\[ b(y, t) = b_0 + \delta b(y, t). \]

All the variations are of order \( O(\frac{m^2}{\kappa^2 \sqrt{\rho}}) \), so that their time derivatives will be suppressed. We may neglect the brane matter since their effects are controlled by a different expansion parameter, and the linearized results from the previous section may be carried over directly.

It has been pointed out in [8] that the Einstein equations simplify quite a bit, if we choose to work in terms of the variables

\[ \eta = \delta\beta'(y, t) - (A'_0(y))^2 \frac{\delta b(y, t)}{b_0} \quad \sigma(y, t) = \delta\alpha'(y, t) - \delta\beta'(y, t), \] (42)

which are invariant under gauge transformations which leave the branes invariant. In terms of these variables, the \( G_{00} \) and \( G_{55} \) Einstein equations are

\[ 4A'_0(y) \eta(y, t) - \eta'(y, t) = -\frac{e^{2A_0(y)}}{3} \kappa^2 \delta T_{00}(y, t), \] (43)
\[ A'_0(y)(4\eta(y, t) + \sigma(y, t)) = \kappa^2 \delta T_{55}(y, t), \] (44)

and the combination \( G_{00} + b_0^2 e^{2A_0(y)} G_{ii} \) is given by

\[ \sigma'(y, t) - 4A'_0(y) \sigma(y, t) = -\kappa^2 e^{2A_0(y)} b_0^2 (\delta T_{00}(y, t) \frac{\delta T_{ii}(y, t)}{a_0(t)^2}). \] (45)

We note that the average of the first two equations vanishes as a consequence of the topological constraint imposed by the fact that we are working on a compact manifold. This is a manifestation of the point that to linear order in the perturbations there are no corrections to the averaged Einstein equations [7]. We have the further simplification that both the energy density and bulk pressure have the same \( y \) dependence so that the solutions to the first two equations are identical up to an overall factor. We find
\[ \eta(y, t) = \frac{-b_0 k^2 m_B^2}{12k^2} \rho_4(t) \left( - \left( e^{kb_0 (1+3y)}k b_0 y \left( e^{kb_0 (1+y)} + 2 \cosh(k b_0 (y - 1)) \right) \left( \coth(b_0 k) - 1 \right) \right) + e^{b_0 k + 4b_0 k y} \left( \coth(b_0 k) - 1 \right) \sinh(b_0 k (y - 1)) \sinh(b_0 k y) + e^{2b_0 k y} \left( 2 + e^{2b_0 k} \right) b_0 k \cscsh(b_0 k)^2 \sinh(b_0 k y) \right), \]

and

\[ \sigma(y, t) = -3(1 + \frac{P_4(T)}{\rho_4(T)}) \eta(y, t). \] (46)

In deriving these expressions we have dropped the contribution from the derivative turns in the expression for the stress energy, as they only contribute at higher orders in the mass expansion. We may now check to see if the \( G_{55} \) equation is satisfied. This is the equation which accounts for the induced radion potential and leads to, in the case of brane matter, the unphysical constraint \( \rho = -e^{-4kb_0} \rho_* \). To the order we are working \( T_{55} \) vanishes and the constraint equation leads to the condition

\[ \rho_4(T) = 3P_4(T), \] (47)

which is of course not possible for a massive thermal fluid. Thus, we see that, at least with no external stabilization mechanism, there are no physical solutions with a thermal bulk fluid.

It is simple to include the effects of matter on the branes since their effect is controlled by a separate expansion parameter. Since the Einstein equations are local and the constraint on the brane matter is independent of the fifth dimension, there is no way to even fine tune the relation between the brane and bulk matter to get a solution.

VI. INCLUSION OF A GW MECHANISM

To determine the effects of a bulk thermal fluid, we now include an additional scalar \( \xi \), which is responsible for stabilizing the space. This stabilization is accomplished by including a bulk as well as brane potentials for \( \xi \). An exact solution to this system without matter was found in [24]. Here we will perturb around this solution, and following [8], choose the simple bulk potential \( V(\xi) = \frac{1}{2} m_\xi^2 \xi^2 \) and brane potentials \( V_i(\xi) = m_i(\xi - v_i)^2 \). The solution to the unperturbed coupled Einstein scalar field equations are approximately given by
\[ \xi_0(y) = v_0 e^{-\epsilon k_{b_0} y} \quad A_0(y) = k b_0 y - \frac{\kappa^2}{12} v_0^2 \left( 1 - e^{-2\epsilon k_{b_0} y} \right), \]  

(48)

where \( \epsilon \approx \frac{m_r^2}{4k^2} \). A large hierarchy, without fine-tuning, is obtained by choosing \( e^{-k_{b_0}} = (v_1/v_0)^{\epsilon^{-1}} \). The \( G_{00} \) and \( G_{ii} \) Einstein equations are unchanged, except for the fact that gauge invariant variable is now

\[ \eta(y, t) = \delta\alpha'(y, t) - A'_0(y) \frac{\delta b(y, t)}{b_0} - \frac{\kappa^2}{3} \xi'_0(y) \delta \xi(y, t), \]  

(49)

where \( \delta \xi \) is the deviation induced by the mass of the bulk matter. Note that there are still no jumps on the branes. That is, in these Einstein equations there are no implied delta functions induced by the shift in the fields, as they have all canceled once the leading order equations have been used.

The \( G_{55} \) equation is now given by

\[ A'_0(y)(4\eta(y, t) + \sigma(y, t)) + \frac{\kappa^2}{3} \left( \xi''_0(y) \delta \xi(y, t) - \xi'_0(y) \delta \xi'(y, t) + (\xi'_0(y))^2 \frac{\delta b(y, t)}{b_0} \right) = 0 + O(m_B^4/k^4). \]  

(50)

Given that we have already solved the other Einstein equations we can solve the entire system by making the simplifying assumption that the brane potentials are stiff, so that the fluctuations of \( \xi \) on the branes can be taken to be zero. We may then use residual gauge invariance to eliminate \( \delta \xi \) in the bulk as well. Doing so allows us to calculate the shift in the radion due to the bulk field. We find

\[ 2 \int_0^1 dy \frac{\delta b(y, t)}{b_0} = \frac{m_B^2}{2k^2 v_0^2} e^{2k_{b_0}(2+\epsilon)} \left( \rho_4(T) - 3P_4(T) \right) \approx \frac{m_B^2}{18k^2 m_r^2} \frac{\rho_4(T) - 3P_4(T)}{kb_0 M_{Pl}^2 e^{-2k_{b_0}}}. \]  

(51)

\( m_r \) is the mass of the radion which in the limit we are working is approximately given by \( m_r^2 = \frac{4}{3} \kappa^2 (v_0 k)^2 e^{-2(k_{b_0}+\epsilon)} \) \[\text{[23]}\]. Note that the bulk fluid again behaves like a brane fluid, which couples conformally to the radion.

The more interesting quantity is the change in the lapse function as this quantity tells us how the weak scale changes with time. Writing

\[ M_W(t)/M_W(t_0) \approx e^{-2 \int_0^1 \delta\alpha'(y, t) dy} \]  

(52)
We find

\[
M_W(t)/M_W(t_0) \approx 1 - \frac{m_B^2}{18k^2 M^2_{pl} k^2 e^{-4k b_0}} \left(1 + \frac{3P_4(T)}{2\rho_4(T)}\right) + \frac{\delta b(t)}{b_0}.
\]  

(53)

Notice that in the limit where \(m_0 < T < m_1\) (which can be attained by fine tuning the bulk mass), and where our results are still valid, the contribution from the radion shift is subleading.

\section*{VII. CONCLUSIONS AND OUTLOOK}

In this paper we have calculated perturbations away from the vacuum \(AdS_5\) compactified geometry due to the presence of a thermal bulk in the limit where \(T < m_{KK}\). We have shown that a bulk fluid contributes to the expansion rate in a similar fashion to a fluid which is confined to the visible brane and that the induced dynamics of the fifth dimension are controlled by the mass of the bulk field. The existence of the bulk fluid does nothing to ameliorate the usual fine tuning between the brane matter when there is no stabilizing potential. In fact, without a stabilizing potential there are no physical solutions, at least that can be found by perturbing around the vacuum solution, when a thermal bulk fluid is introduced. Upon introducing a stabilizing potential we found that the shift in the weak scale due to the existence of the bulk fluid is parametrically suppressed relative to the contribution from a brane by \(m_B^2/k^2\). This stems from the fact that all the bulk dynamics induced by the bulk fluid are controlled by this parameter. Indeed, we may conclude that the onset of dynamics in the fifth dimension, due to a bulk field, is triggered at a temperature near the mass of the first mode which has a non-trivial vacuum profile.

Our results are only valid in the limit where \(T < m_{KK}^{(2)}\), where \(m_{KK}^{(2)}\) is the mass of the second Kaluza Klein excitation, so that contributions from higher excitations are suppressed. It would be interesting to find solutions where the mass of the first mode approaches the TeV scale, so that there is no need for fine tuning. This would entail solving the Einstein equations with the full Bessel function wave function source. Furthermore, one could then
also include more Kaluza-Klein excitations. This would allow for a study at temperatures high enough that the free energy begins to scale like \( T^5 \). Presumably, as the temperature increases further, the local energy density near the Planck brane will approach the Planck scale (this can’t be seen from our results since our calculation of the energy density was only valid for smaller temperatures where we could do perturbation theory). Once such energy densities are reached, black-hole formation seems inevitable and the change in geometry to AdS-Schwarzschild marks a phase transitions in the dual conformal field theory.

ACKNOWLEDGMENTS

The author is indebted to Walter Goldberger, Ben Grinstein, Rich Holman and Ted Jacobson for helpful discussions. This work was supported in part by the Department of Energy under grant numbers DOE-ER-40682-143.
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