Boson stars and oscillatons in an inflationary universe

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Spherically symmetric gravitationally bound, oscillating scalar lumps (boson stars and oscillatons) are considered in Einstein’s gravity coupled to massive scalar fields in $1+D$ dimensional de Sitter-type inflationary space-times. We show that due to inflation bosons and oscillatons lose mass through scalar radiation, but at a rate that is exponentially small when the expansion rate is slow.

I. INTRODUCTION

Current physical field theories explaining the presently known fundamental interactions are remarkably successful to account for phenomena both on particle physics (i.e. $10^{-18}$m) and on cosmological ($10^{26}$m) scales, usually incorporate various fundamental scalar fields (Higgs bosons, axion, supersymmetric partners of fermions, inflaton, waterfield etc.). It has been observed since a long time ago that self-interacting scalar fields admit at least up to $D=4$ spatial dimensions spatially localized, oscillating lumps, nowadays called oscillatons\textsuperscript{[1–3]}. Oscillatons are reminiscent of spatially localized, time-periodic breathers known in one dimensional sine-Gordon theory. There is, however, a fundamental difference, in that oscillatons are only \textit{approximatively} time-periodic since they radiate, thereby losing continuously their energy, albeit very slowly\textsuperscript{[4,5]}. Moreover oscillon-type objects exist in a broad class of field theories containing in addition to scalar, vector and possibly other fields\textsuperscript{[7–8]}. A prominent example admitting classical oscillons is the bosonic sector of the Standard Model\textsuperscript{[9–11]}. Einstein’s gravity coupled to a free, massive real Klein-Gordon field, also possesses oscillon-type solutions, which go under the name of oscillatons\textsuperscript{[12–17]}. An important problem is to investigate the possible pertinence of oscillatons for cosmology where according to current theories some scalars fields are necessarily involved. In recent investigations the evolution of oscillatons in an inflating background has been investigated\textsuperscript{[18–23]}. It has been found that an inflating background also induces a radiative tail in oscillatons, \textit{distinct} from the one present in flat space-times, leading to an additional leakage of energy. However this decay rate is also exponentially suppressed, so inflating oscillatons may survive for an exponentially long time provided that the horizon is far larger than the width of an oscillon. As recently shown, due to quantum effects, the decay rate of oscillatons becomes power law-like and in many (but not in all) cases dominates over the classical decay rate\textsuperscript{[24]}. There are indications that oscillatons can be copiously produced from thermal initial conditions and a substantial fraction of the energy of the system will be stored by them\textsuperscript{[18,23,25]}. Therefore it is important to investigate the physical role of such long living localized objects. In a cosmological setting it is necessary to take into account the influence of gravity, i.e. instead of simply considering oscillatons in a background metric, to investigate solutions of the \textit{coupled} Einstein-scalar system -- oscillatons -- with an expanding, asymptotically homogeneous metric.

In this paper we undertake a thorough investigation of localized, oscillaton-type solutions in Einstein’s gravity coupled to a massive Klein-Gordon field, when space-time undergoes inflation, driven by a negative pressure cosmological fluid -- i.e. including a cosmological constant, $\Lambda > 0$. (For recent reviews on inflation theory see e.g.\textsuperscript{[26–28]}) Our results show that once admitting massive scalar fields, both in present day cosmology and in the early Universe, spatially localized, very long living oscillatons exist. The inflaton field generates an energy density of the order of $(10^{16} GeV)^4$ and negative pressure which can be described by an effective (and slowly varying) cosmological parameter. Using the standard slow roll approximation for the inflaton, the metric during the inflationary era can be approximated very well with a de Sitter one. The order of magnitude of the cosmological constant during the inflationary epoch is $\Lambda \approx 10^{-11}$ (in Planck units). If other massive scalar fields are present during inflation such as the waterfall-field in hybrid inflationary scenarios\textsuperscript{[29,30]}, oscillaton production is expected to be significant. We find that oscillatons may survive on cosmological time-scales only if the mass of the scalar field is larger than $10^{12}/c^2$ (in Planck units). The mass loss of an oscillon during Hubble time is completely determined by the ratio $\mu_c/\mu_\Lambda$, where $\mu_\Lambda$ is the energy density due to $\Lambda$ and $\mu_c$ is that of the oscillon in its center. For example when $\mu_c/\mu_\Lambda \approx 55$ the mass loss of such a “waterfall” oscillon is of the order of 1% during Hubble time. This mass loss is exponentially decreasing for increasing values of the ratio $\mu_c/\mu_\Lambda$.

Another interesting scenario for oscillaton production is during reheating, assuming that the inflaton field undergoes oscillations near the minimum of its potential. These oscillatons store a considerable amount of energy and due to parametric resonances induced by quantum effects (for oscillatons this has been shown in Ref.\textsuperscript{[24]}) they may influence the fluctuation spectrum. The present value of $\Lambda \approx 10^{-122}$ (in Planck units) being extremely small, its effect on the mass loss of generic oscillatons is completely negligible on a cosmological time
scale. For example if the mass of the scalar field is of the order of $m \approx 10^{-15}$eV and the amplitude of the oscillaton is approximatively $10^{-17}$, the effect of the cosmological constant results in a roughly 1% mass loss during the lifetime of the Universe.

We also consider (spherically symmetric) boson stars in this setting, i.e. when a complex scalar field is coupled to gravity with a nonzero cosmological constant. Adopting our small amplitude expansion techniques developed for oscillons and oscillatons we obtain boson star solutions in a systematic way. Remarkably, this method yields very good results even for near maximal mass boson stars. Spatially well localized boson star-type solutions with a static metric, familiar in the case $\Lambda = 0$, exist, however, radiating boson stars with a time dependent metric loosing continually their mass due to emission of scalar waves. We obtain their mass loss using our methods developed for oscillons resp. oscillatons, with the result that this mass loss is exponentially small and it is negligible for sufficiently small values of $\Lambda$.

II. FIELD EQUATIONS

We consider a real or complex scalar field coupled to gravity. The total Lagrangian density is

$$\mathcal{L} = \frac{1}{16\pi} \mathcal{L}_G + \mathcal{L}_M ,$$

where the Einstein Lagrangian density in case of a cosmological constant $\Lambda$ is

$$\mathcal{L}_G = \sqrt{-g} (R - 2\Lambda) .$$

We work in a $D + 1$ dimensional curved space-time with metric $g_{ab}$ and assume metric signature $(-+++...)$. Unless explicitly noted, we use Planck units $G = c = \hbar = 1$. In some cases it is advantageous to use natural units, where $c = \hbar = 1$ (but $G \neq 1$) and all quantities are measured in various powers of electron volts. Because of the different notation in the potential term we consider the real and the complex field case separately. Variation of the action with respect to $g^{ab}$ yields Einstein equations,

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} .$$

A. Boson stars

We consider a complex scalar field $\Phi$ with a self-interaction potential $U(\Phi^*\Phi)$. For a free field with mass $m$ the potential is $U(\Phi^*\Phi) = m^2 \Phi^*\Phi$. The Lagrangian density belonging to the scalar field is

$$\mathcal{L}_M = -\sqrt{-g} \left[ g^{ab} \Phi^*_a \Phi^*_b + U(\Phi^*\Phi) \right] .$$

Variation of the action with respect to $\Phi^*$ yields the wave equation

$$g^{ab} \Phi_{,ab} - \Phi U'(\Phi^*\Phi) = 0 ,$$

where the prime denotes derivative with respect to $\Phi^*\Phi$. The stress-energy tensor in this case is

$$T_{ab} = \frac{1}{2} \left\{ \Phi^*_a \Phi^*_b + \Phi^*_b \Phi^*_a - g_{ab} \left[ g^{cd} \Phi^*_c \Phi^*_d + U(\Phi^*\Phi) \right] \right\} .$$

We shall assume that the self-interaction potential, $U(\Phi^*\Phi)$, has a minimum $U(\Phi^*\Phi) = 0$ at $\Phi = 0$, and expand its derivative as

$$U'(\Phi^*\Phi) = m^2 \left[ 1 + \sum_{k=1}^{\infty} u_k (\Phi^*\Phi)^k \right] ,$$

where $m$ is the scalar field mass and $u_k$ are constants.

In order to get rid of the $8\pi$ factors in the equations we introduce a rescaled scalar field and potential by

$$\phi = \sqrt{8\pi} \Phi , \quad \hat{U}(\phi^*\phi) = 8\pi U(\Phi^*\Phi) .$$

Then

$$\hat{U}'(\phi^*\phi) = m^2 \left[ 1 + \sum_{k=1}^{\infty} \hat{u}_k (\phi^*\phi)^k \right] ,$$

with

$$\hat{u}_k = \frac{u_k}{(8\pi)^k} .$$

B. Oscillatons

In this case we consider a real scalar field $\Phi$ with a self-interaction potential $U(\Phi)$. For a free field with mass $m$ the potential is $U(\Phi) = m^2 \Phi^2/2$. The Lagrangian density belonging to the scalar field is

$$\mathcal{L}_M = -\sqrt{-g} \left( \frac{1}{2} \Phi^a \Phi^a + U(\Phi) \right) .$$

Variation of the action with respect to $\Phi$ yields the wave equation

$$g^{ab} \Phi_{,ab} - U'(\Phi) = 0 ,$$

where the prime now denotes derivative with respect to $\Phi$. The stress-energy tensor is

$$T_{ab} = \Phi_{,a} \Phi_{,b} - g_{ab} \left( \frac{1}{2} \Phi^c \Phi^c + U(\Phi) \right) .$$

We shall assume that $U(\Phi)$ has a minimum $U(\Phi) = 0$ at $\Phi = 0$, and expand its derivative as

$$U'(\Phi) = m^2 \left( \Phi + \sum_{k=2}^{\infty} u_k \Phi^k \right) ,$$
where \( m \) is the scalar field mass and \( u_k \) are constants.

In order to get rid of the \( 8\pi \) factors in the equations we introduce a rescaled scalar field and potential by

\[
\phi = \sqrt{8\pi} \Phi, \quad \mathcal{U}(\phi) = 8\pi U(\Phi).
\]

Then

\[
\mathcal{U}'(\phi) = m^2 \left( \phi + \sum_{k=2}^{\infty} u_k \phi^k \right),
\]

with

\[
\bar{u}_k = \frac{u_k}{(8\pi)^(k-1)/2}.
\]

C. Scaling properties

In both the real and the complex case, if the \( \phi(x^c) \) and \( g_{ab}(x^c) \) solves the field equations with a potential \( \mathcal{U} \) and cosmological constant \( \Lambda \), then

\[
\hat{\phi}(x^c) = \phi(\gamma x^c) \quad \text{and} \quad \hat{g}_{ab}(x^c) = g_{ab}(\gamma x^c)
\]

for any positive constant \( \gamma \), is a solution with a rescaled potential \( \gamma^2 \mathcal{U} \) and rescaled cosmological constant \( \gamma^2 \Lambda \). This scaling property may be used to set the scalar field mass to any prescribed value, for example to make \( m = 1 \).

If \( D = 1 \) then, by definition, the Einstein tensor is traceless, and from the trace of the Einstein equations it follows that the potential \( \mathcal{U} \) is constant. Hence we assume that \( D > 1 \).

D. Spherically symmetric \( D + 1 \) dimensional space-time

We consider a spherically symmetric \( D + 1 \) dimensional space-time using isotropic coordinates \( x^\mu = (t, r, \theta_1, \ldots, \theta_{D-1}) \). In this coordinate system the metric is diagonal and its spatial part is in conformally flat form, with components

\[
g_{tt} = -A, \quad g_{rr} = B,
\]

\[
g_{\theta_i \theta_i} = r^2 B, \quad g_{\theta_i \theta_a} = r^2 B \prod_{k=1}^{n-2} \sin^2 \theta_k,
\]

where \( A \) and \( B \) are functions of temporal coordinate \( t \) and radial coordinate \( r \). In case of oscillations, the periodic change in the acceleration of the constant radius observers is much smaller in isotropic coordinates than in the more commonly used Schwarzschild coordinate system [10, 35]. It is also more convenient to study the Newtonian limit of boson stars using isotropic coordinates [36]. The components of the Einstein tensor are

\[
G_{tt} = D - 1 \left[ \frac{D}{AB^2} (B,t)^2 \right. - \frac{A}{r^{D-1} B^{D+1}} \left( \frac{r^{D-1} B}{B^{D+1}} \right)_t, \]

\[
G_{rr} = D - 1 \left[ \frac{(D-2)}{4r^2 A^{D-2} B^2} \left( r^2 A^{D-2} B \right)_r - \frac{1}{A^2 B^2} \left( \frac{B}{A^2} - 1 \right) - \frac{D-2}{r^2} \right],
\]

\[
G_{tr} = D - 1 \left[ \frac{1}{2 A^2} \left( \frac{B}{A} \right)_t \right],
\]

\[
G_{\theta_i \theta_i} = r^2 G_{rr} + \frac{r^2 B}{2 A^2} \left( \frac{A}{r A^2 B} \right)_r + \frac{D-2}{2} r^3 B^2 \left( \frac{B}{r A^2 B} \right)_r.
\]

The potential independent term in the wave equation [13] and [12] is

\[
g^{\alpha \beta} \phi_{, \alpha \beta} = \frac{\phi_{,rr}}{B} - \frac{\phi_{,tt}}{A} - \frac{\phi_{,t} (B/D)}{2B^2 (A)} + \frac{\phi_{,r}}{2r^2 A^{D-2} B^{D-1}} (r^{D-2} A B^{D-2})_r.
\]

E. de Sitter space-time

If the cosmological constant \( \Lambda \) is positive, far from the central boson star or oscillation the space-time should approach the de Sitter metric. In static Schwarzschild coordinates on a \( D + 1 \) dimensional space-time this metric can be written as

\[
ds^2 = -(1 - H^2 r^2) dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega_{D-1}^2,
\]

where \( d\Omega_{D-1}^2 \) is the metric on the unit sphere \( S^{D-1} \). The Hubble constant \( H \) is related to the cosmological constant \( \Lambda \) by

\[
H^2 = \frac{2\Lambda}{D(D-1)}.
\]

In isotropic coordinates the de Sitter metric takes the form

\[
ds^2 = -\left( 1 - \frac{H^2 r^2}{4} \right)^2 dt^2 + \frac{dr^2}{\left( 1 + \frac{H^2 r^2}{4} \right)^2} + r^2 d\Omega_{D-1}^2,
\]

where the two radial coordinates are related by

\[
\tilde{r} = \frac{r}{1 + \frac{H^2 r^2}{4}}.
\]
The cosmological horizon, which is at $r = 1/H$ in the Schwarzschild coordinate system, is at $r_h = 2/H$ in isotropic coordinates.

### III. SMALL-AMPLITUDE EXPANSION

The small-amplitude expansion procedure has been applied successfully to describe the core region of flat background oscillons [44, 45, 46]. The method was generalized in [47] and [48] to the case when the scalar field is coupled to Einstein gravity, and in [49] to a very similarly behaving scalar-dilaton system. In this section we expand oscillators in the case when the cosmological constant is positive.

Newtonian boson stars have already been investigated in the paper of Ruffini and S. Bonazzola [32], obtaining a system of two coupled differential equations, which nowadays are generally called Schrödinger-Newton equations in the literature [40–43]. The same system was obtained as weak gravity limit of general relativistic boson stars in [44] and [45]. The leading order results of the small-amplitude expansion of boson stars, that we present in this section, when restricted for $\Lambda = 0$, also yield the same equations.

The repulsive effect of the cosmological constant is taken into account in a similar way as in [20–22]. Since we intend to investigate localized objects, where at large distances the scalar field is negligible, by the generalized Birkhoff’s theorem [46–48], the metric should approach the Schwarzschild-de Sitter metric. This also means that in the asymptotic de Sitter region the expansion rate (Hubble parameter) is constant.

#### A. Choice of coordinates

We are looking for spatially localized bounded solutions of the field equations for which $\phi$ is small and the metric is close to de Sitter. The smaller the amplitude of a boson star or an oscillaton is, the larger its spatial extent becomes. Therefore, we introduce a new radial coordinate $\rho$ by

$$\rho = \varepsilon mr, \tag{29}$$

where $\varepsilon$ denotes the small-amplitude parameter. Motivated by the scaling property (18) we have also included a $m$ factor into the definition of $\rho$.

Long-living localized solutions are expected to exist only if their size is considerably smaller than the distance to the cosmological horizon. We introduce a rescaled Hubble constant $h$ by

$$H = \varepsilon^2 mh, \tag{30}$$

and assume that even $h$ is reasonably small. This way we ensure that the typical size of boson stars or oscillatons, which is of the order $1/\varepsilon m$ using the $r$ coordinates, remains smaller than the radius of the cosmological horizon, $r_h = 2/H = 2/(\varepsilon^2 mh)$.

We will see in Section III E that the energy density $\mu$ of the scalar field at the central region of small amplitude oscillons or boson stars is proportional to $\varepsilon^4$. On the other hand, the cosmological constant can be interpreted as a fluid with energy density,

$$\mu_{\Lambda} = \frac{\Lambda}{8\pi} \tag{31}$$

and pressure $p_{\Lambda} = -\mu_{\Lambda}$. Using (26) and (30) $\mu_{\Lambda}$ can be written as

$$\mu_{\Lambda} = \frac{D(D-1)}{16\pi} \varepsilon^4 m^2 h^2. \tag{32}$$

The smallness of $h$ guaranties that $\mu_{\Lambda}$ remains smaller than energy density $\mu$ of the scalar field even in the small-amplitude limit.

We expand $\phi$ and the metric functions in powers of $\varepsilon$ as

$$\phi = \sum_{k=1}^{\infty} \varepsilon^{2k} \phi_{2k}, \tag{33}$$

$$A = \frac{1 - \varepsilon^2 h^2 \rho^2}{1 + \varepsilon^2 h^2 \rho^2} \sum_{k=1}^{\infty} \varepsilon^{2k} A_{2k}, \tag{34}$$

$$B = \frac{1}{1 + \varepsilon^2 h^2 \rho^2} \sum_{k=1}^{\infty} \varepsilon^{2k} B_{2k}. \tag{35}$$

The first terms in $A$ and $B$ represent the de Sitter background, according to (27). Since we intend to use asymptotically de Sitter coordinates, we look for functions $\phi_{2k}$, $A_{2k}$ and $B_{2k}$ that tend to zero when $\rho$ is large. One could initially include odd powers of $\varepsilon$ in the expansions (33)-(35). However, for oscillators it can be shown by the method presented below, that the coefficients of those terms necessarily vanish when we are looking for configurations that remain bounded in time. It can also be verified directly, that the small-amplitude expansion of boson stars do not include odd powers of $\varepsilon$. This is in contrast to flat background oscillons, where the leading order behavior of the amplitude is proportional to $\varepsilon$.

The frequency of boson stars and oscillatons also depends on their amplitude. The smaller the amplitude is, the closer the frequency becomes to the threshold $m$. Hence we introduce a rescaled time coordinate $\tau$ by

$$\tau = \omega t, \tag{36}$$

and expand the square of the $\varepsilon$ dependent factor $\omega$ as

$$\omega^2 = m^2 \left(1 + \sum_{k=1}^{\infty} \varepsilon^{2k} \omega_{2k}\right). \tag{37}$$

It is possible to allow odd powers of $\varepsilon$ into the expansion of $\omega^2$, but the coefficients of those terms turn out
to be zero when solving the equations arising from the small-amplitude expansion. There is a considerable freedom in choosing different parametrizations of the small-amplitude states, changing the actual form of the function \( \omega \). The physical parameter is not \( \varepsilon \) but the frequency of the periodic states that will be given by \( \omega \). Similarly to the asymptotically flat case in \([33]\) and to the dilaton model in \([39]\), it turns out that for spatial dimensions \( 2 < D < 6 \) the parametrization of the small-amplitude states can be fixed by setting \( \omega = m \sqrt{1 - \varepsilon^2} \).

The field equations we intend to solve using the \( \tau \) and \( \rho \) coordinates are the Einstein equations \([3]\), substituting the Einstein tensor components from \([20, 24]\), together with the wave equation \([5]\) or \([12]\). We note that these equations are not independent. The \((\tau, \rho)\) component of the Einstein equations is a constraint, and the wave equation is a consequence of the contracted Bianchi identity.

### B. Boson stars

In case of boson stars the metric is static, hence \( A_k \) and \( B_k \) are time independent. The complex scalar field \( \phi \) oscillates with frequency \( \omega \),

\[
\phi = \psi e^{i \omega t},
\]

where \( \psi \) is real and depends only on the radial coordinate \( \rho \). The small-amplitude expansion coefficients of \( \phi \) satisfy

\[
\phi_k = \psi_k e^{i \omega t} = \psi_k e^{i \tau},
\]

where \( \psi_k \) are real functions of \( \rho \). Substituting the expansion \([33, 35]\) into the field equations, if \( D \neq 2 \), then to leading \( \varepsilon^4 \) order the \((\rho, \rho)\) component gives

\[
B_2 = \frac{A_2}{2 - D}. \tag{40}
\]

Since for \( D = 2 \) there is no solution representing a localized object we assume \( D > 2 \). Then the \((\tau, \tau)\) component and the wave equation yield to leading order a coupled system of differential equations for \( A_2 \) and \( \psi_2 \),

\[
\frac{d^2 A_2}{d\rho^2} + \frac{D - 1}{\rho} \frac{d A_2}{d\rho} = 2 \frac{D - 2}{D - 1} \psi_2^2, \tag{41}
\]

\[
\frac{d^2 \psi_2}{d\rho^2} + \frac{D - 1}{\rho} \frac{d \psi_2}{d\rho} = \psi_2 (A_2 - \omega^2 - \hbar^2 \rho^2). \tag{42}
\]

Introducing the functions \( s \) and \( S \) by

\[
s = \omega_2 - A_2, \quad S = \psi_2 \sqrt{2 \frac{D - 2}{D - 1}}, \tag{43}
\]

equations \([41]\) and \([42]\) can be written into the form

\[
\frac{d^2 S}{d\rho^2} + \frac{D - 1}{\rho} \frac{d S}{d\rho} + (s + \hbar^2 \rho^2) S = 0, \tag{44}
\]

\[
\frac{d^2 s}{d\rho^2} + \frac{D - 1}{\rho} \frac{d s}{d\rho} + S^2 = 0. \tag{45}
\]

Since the same equations will be obtained for small-amplitude oscillations in the next subsection, we postpone the discussion of this system to Subsection III.

Proceeding to higher orders, the \( \varepsilon^6 \) components give a system of lengthy differential equations linear in \( A_4, B_1, \psi_4 \) and their derivatives. These equations have nonlinear source terms containing \( A_2, B_2 \) and \( \psi_2 \). They will also involve the first coefficient \( u_1 \) from the series expansion of the potential \( U \). To leading order, the structure of small-amplitude boson stars only depends on the mass parameter \( m \).

### C. Oscillatons

In this case the scalar field is real, and the metric components are also oscillating. We are looking for solutions that remain bounded as time passes. It turns out, that these configurations are necessarily periodically oscillating in time.

From the \( \varepsilon^2 \) components of the field equations follows that \( B_2 \) only depends on \( \rho \) and that

\[
\phi_2 = p_2 \cos(\tau + \delta), \tag{46}
\]

where two new functions, \( p_2 \) and \( \delta \) are introduced, depending only on \( \rho \). From the \( \varepsilon^4 \) part of the \((\tau, \rho)\) component of the Einstein equations it follows that \( B_4 \) can remain bounded as time passes only if \( \delta \) is a constant. Then by a shift in the time coordinate we set

\[
\delta = 0. \tag{47}
\]

This shows that the scalar field oscillates simultaneously, with the same phase at all radii.

From the \( \varepsilon^4 \) component of the difference of the \((\rho, \rho)\) and \((\theta_1, \theta_1)\) Einstein equations follows that \( A_2 \) also depends only on the radial coordinate \( \rho \), showing that the metric is static to order \( \varepsilon^4 \). This is the main advantage of using isotropic coordinates over Schwarzschild coordinates. But even in this system the coefficients \( A_4 \) and \( B_4 \) will already contain \( \cos(2\tau) \) terms. From the \( \varepsilon^6 \) component of the \((\tau, \rho)\) Einstein equations follows that

\[
B_4 = b_4 - \frac{p_2^4}{4(D - 1)} \cos(2\tau), \tag{48}
\]

where \( b_4 \) is a function of \( \rho \). The \((\rho, \rho)\) component shows that if \( D \neq 2 \) then

\[
B_2 = \frac{A_2}{2 - D}. \tag{49}
\]

If \( D = 2 \) then \( A_2 = 0 \), and there are no nontrivial localized regular solutions for \( B_2 \) and \( p_2 \), so we assume \( D > 2 \) from now. The \((\tau, \tau)\) component yields

\[
\frac{d^2 A_2}{d\rho^2} + \frac{D - 1}{\rho} \frac{d A_2}{d\rho} = D - 2 \frac{p_2^2}{D - 1}. \tag{50}
\]
The wave equation provides an equation that determines the time dependence of \( \phi_1 \). In order to keep \( \phi_1 \) bounded, the resonance terms proportional to \( \cos \tau \) must vanish, yielding
\[
\frac{d^2 p_2}{dp^2} + \frac{D - 1}{\rho} \frac{dp_2}{dp} = p_2(A_2 - \omega_2 - \hbar^2 \rho^2). \tag{51}
\]

Equations \( 50 \) and \( 51 \) do not depend on the coefficients \( \bar{u}_k \) of the potential \( \bar{U}(\phi) \). This means that the leading order small-amplitude behavior of oscillatons is always the same as for the Klein-Gordon case.

Introducing the functions \( s \) and \( S \) by
\[
s = \omega_2 - A_2, \quad S = p_2 \sqrt{\frac{D - 2}{D - 1}}, \tag{52}
\]
equations \( 50 \) and \( 51 \) can be written into the form \( 44 \) and \( 45 \) already obtained at the small-amplitude expansion of boson stars.

D. Schrödinger-Newton equations

The equations describing both small-amplitude boson stars and oscillatons on an expanding background has been written into a form, which for \( \hbar = 0 \) reduces to the time-independent Schrödinger-Newton (SN) equations
\[
\frac{d^2S}{d\rho^2} + \frac{D - 1}{\rho} \frac{dS}{d\rho} + (s + \hbar^2 \rho^2)S = 0, \tag{53}
\]
\[
\frac{d^2S}{d\rho^2} + \frac{D - 1}{\rho} \frac{dS}{d\rho} + S^2 = 0. \tag{54}
\]

If \( S(\rho) \) and \( s(\rho) \) are solutions of \( 53 \) and \( 54 \), then the transformed functions
\[
\tilde{S}(\rho) = \lambda^2 S(\lambda \rho), \quad \tilde{s}(\rho) = \lambda^2 s(\lambda \rho) \tag{55}
\]
solve the equations
\[
\frac{d^2 \tilde{S}}{d\rho^2} + \frac{D - 1}{\rho} \frac{d\tilde{S}}{d\rho} + (\tilde{s} + \hbar^2 \rho^2)\tilde{S} = 0, \tag{56}
\]
\[
\frac{d^2 \tilde{s}}{d\rho^2} + \frac{D - 1}{\rho} \frac{d\tilde{s}}{d\rho} + \tilde{S}^2 = 0, \tag{57}
\]
which are obtained from the SN equations by replacing the constant \( \hbar \) by \( \hbar = \lambda^2 \hbar \).

If the scalar field, and consequently \( S \), tends to zero for \( \rho \to \infty \) then \( 54 \) gives the following approximation for \( s \),
\[
s = s_0 + s_1 \rho^{2-D} \tag{58}
\]
Since we are looking for solutions for which \( A_2 = \omega_2 - s \) also approaches zero for large \( \rho \), necessarily \( \omega_2 = s_0 \). In order to make the solution of the SN equations \( 53-54 \) unique we demand
\[
\omega_2 = s_0 = -1. \tag{59}
\]
Setting \( \omega_k = 0 \) for \( k \geq 3 \) this also fixes the connection between the frequency \( \omega \) and the small-amplitude parameter \( \varepsilon \), giving
\[
\omega = m \sqrt{1 - \varepsilon^2}. \tag{60}
\]
Substituting into \( 53 \) we get
\[
\frac{d^2 S}{d\rho^2} + \frac{D - 1}{\rho} \frac{dS}{d\rho} + (\hbar^2 \rho^2 - 1 + s_1 \rho^{2-D})S = 0, \tag{61}
\]
determining the large \( \rho \) behavior of \( S \).

E. Energy density

The energy density of the scalar field is \( \mu = T_{\mu \nu} u^\mu u^\nu \), where the unit timelike vector \( u^\mu \) has the components \((1/\sqrt{A}, 0, \ldots, 0)\). In terms of the rescaled scalar field \( \phi \), the energy density of oscillatons can be written as
\[
\mu = \frac{1}{16\pi} \left[ \frac{1}{A} \left( \frac{d\phi}{dt} \right)^2 + \frac{1}{B} \left( \frac{d\phi}{dr} \right)^2 + 2\bar{U}(\phi) \right], \tag{62}
\]
and for boson stars
\[
\mu = \frac{1}{16\pi} \varepsilon^4 \frac{1}{D - 2} m^2 S^2. \tag{64}
\]
For oscillatons we obtain the same expression for \( \mu \) using \( 43 \), \( 39 \) and \( 42 \). Measuring the scalar field mass \( m \) in eV/c\(^2\) units, the energy density at the symmetry center for \( D = 3 \) can be written as
\[
\mu_c = \varepsilon^4 \frac{\left( \frac{m^2 c^2}{eV} \right)^2}{1.436 \times 10^{39} \text{kg/m}^2}. \tag{65}
\]
(We use Roman m for meters in order to distinguish from the scalar field mass \( m \).) Here we have used the numerically obtained central value of \( S \), which is \( S_c = 1.0215 \) for \( D = 3 \).

The effective energy density \( \mu_A \) corresponding to the present value of the cosmological constant is given in ordinary units by \( 108 \) in Appendix D. Comparing to \( 65 \), one can see that the energy density of the scalar field is much higher than \( \mu_A \) unless \( \varepsilon \) or \( m \) is extremely small. Using \( 62 \), the ratio of \( \mu_A \) to the central density \( \mu_c \) can be written as
\[
\frac{\mu_A}{\mu_c} = \frac{D(D - 2)}{S_c^2} \hbar^2, \tag{66}
\]
providing an important physical interpretation for the rescaled Hubble constant \( \hbar \). In Table 1 we give the value of \( S_c \) and the coefficient of \( \hbar^2 \) in \( \mu_A/\mu_c \) for the relevant dimensions. We will see in the following sections that \( \hbar \) turns out to be the essential parameter determining the energy loss rate of boson stars and oscillatons.
TABLE I: The numerically calculated values of the function $S$ at the center and the coefficient of $h^2$ in $\mu_A/\mu_c$ for $D = 3$, 4 and 5 spatial dimensions.

|       | $D = 3$ | $D = 4$ | $D = 5$ |
|-------|---------|---------|---------|
| $S_1$ | 3.50553 | 7.69489 | 10.4038 |
| $S_t$ | 4.95134 | 88.2419 | 23.3875 |

TABLE II: The numerical values of the constants $s_1$ and $S_t$ for $h = 0$ in 3, 4 and 5 spatial dimensions.

|       | $D = 3$ | $D = 4$ | $D = 5$ |
|-------|---------|---------|---------|
| $s_1$ | 3.50553 | 7.69489 | 10.4038 |
| $S_t$ | 4.95134 | 88.2419 | 23.3875 |

F. Outer core region

If $2 < D < 6$, and the cosmological constant is zero, i.e. $h = 0$, then assuming that $s_0 = -1$ the SN equations have a unique localized nodeless solution. Solutions with nodes have higher energy and are unstable. For large $\rho$ the function $S$ satisfies the equation

$$\frac{d^2S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + (-1 + s_1 \rho^{2-D})S = 0,$$

which has exponentially decaying solutions

$$S = S_t \left( \frac{e^{-\rho}}{\rho^{D-1}/2} \right) \left[ 1 + O \left( \frac{1}{\rho} \right) \right], \quad D = 3,$$

$$S = S_t \left( \frac{e^{-\rho}}{\rho^{D-1}/2} \right) \left[ 1 + O \left( \frac{1}{\rho} \right) \right], \quad D > 3.$$

The numerically determined values of the constants $s_1$ and $S_t$ for the case $h = 0$ are given in Table II.

The radial dependence of the function $S$ for a large value of $h$ is compared to the $h = 0$ flat background case. The function $S$ for $h = 0$ decays exponentially, while for $h > 0$ curve starts to oscillate. For $h = 0.1016$ the first change in signature is at $\rho = 8.04$.

Assuming that $h$ is small, we calculate the amplitude $\alpha$ by WKB analysis. Our aim is to continue the function $S$ given by (58) and (59) into the very large distances where the cosmological term $h^2 \rho^2$ dominates. We will express the amplitude $\alpha$ in terms of the constant $S_t$. To the function which is exponentially decaying when going outwards at the outer core region belongs an oscillating tail with a certain amplitude and phase. Adding the mode which exponentially grows when going outwards does not change the function noticeably in the outer core region but adds an oscillating tail with a phase shift $\pi/2$. The resulting tail amplitude is minimal if this second mode is not present. Hence the WKB method will give the minimum amplitude $\alpha = \alpha_{\text{min}}$. In Sec. IV E we will compare the WKB amplitude to results obtained by direct numerical integration of the SN equations. Besides indicating the correctness of the WKB results, this will also give information on how large $h$ can be chosen such

\[ S = \frac{\alpha}{\rho^2} \cos \left( \frac{h \rho^2}{2} + \beta \right), \quad (71) \]
that the WKB result can be still considered a valid approximation.

IV. WKB FORMALISM

A. Schrödinger form

Introducing a new radial coordinate
\[ y = h \rho \]
(72)
we transform \([61]\) into a form corresponding to a Schrödinger equation with \( h \) playing the role of the Planck’s constant \( \hbar \).

\[ h^2 \left( \frac{d^2 S}{dy^2} + \frac{D - 1}{y} \frac{dS}{dy} \right) + \left( y^2 - 1 + \frac{h^{D-2}}{y^{D-2} S_1} \right) S = 0. \]
(73)
Introducing
\[ S = y^{(1-D)/2} Z, \]
(74)
we can transform this into a one-dimensional form,
\[ h^2 \frac{d^2 Z}{dy^2} + p^2 Z = 0, \]
(75)
where
\[ p^2 = y^2 - 1 + \frac{h^{D-2}}{y^{D-2} S_1} - \frac{h^2}{4y^2} (D - 1)(D - 3). \]
(76)
The outer domain of the core region is represented now by \( 0 < y \ll 1 \), where
\[ Z = S_1 y^{1-s_1/2} y^{s_1/2} e^{-y/h} \left[ 1 + O \left( \frac{h}{y} \right) \right], \quad D = 3, \]
(77)
\[ Z = S_1 y^{(D-1)/2} e^{-y/h} \left[ 1 + O \left( \frac{h}{y} \right) \right], \quad D > 3. \]
(78)

B. WKB approximation

Introducing a new function \( f \) by
\[ Z = e^{if/h} \]
(79)
and substituting into \([65]\) we get
\[ ih \frac{d^2 f}{dy^2} - \left( \frac{df}{dy} \right)^2 + p^2 = 0. \]
(80)
We write \( f \) as a power series in \( h \),
\[ f = \sum_{k=0}^{\infty} f_k h^k. \]
(81)

Differently from the standard WKB approach, now \( p^2 \) also has to be expanded in \( h \),
\[ p^2 = \sum_{k=0}^{\infty} p_k^2 h^k, \]
(82)
where the nonvanishing coefficients are
\[ p_0^2 = y^2 - 1, \quad p_2^2 = -\frac{(D - 1)(D - 3)}{4y^2}, \]
(83)
\[ p_{D-2}^2 = \frac{s_1}{y^{D-2}}. \]
(84)
Substituting into \([80]\), to leading order we get
\[ \left( \frac{df_0}{dy} \right)^2 = p_0^2, \quad \frac{df_0}{dy} = \pm p_0. \]
(85)
The order \( h \) components yield
\[ i \frac{d^2 f_0}{dy^2} + p_1 = 2 \frac{df_0}{dy} \frac{df_1}{dy} \]
(87)
C. WKB for \( D > 3 \)

If \( D > 3 \) then \( p_1 = 0 \),
\[ f_1 = \frac{i}{2} \ln \left( \frac{df_0}{dy} \right) = \frac{i}{2} \ln \left( \pm p_0 \right), \]
(88)
and we get the standard WKB result,
\[ Z = \frac{A_+}{\sqrt{|p_0|}} \exp \left( \pm \frac{i}{h} \int_1^y p_0 dy \right), \quad 0 < y < 1, \]
(89)
\[ Z = \frac{B_+}{\sqrt{|p_0|}} \exp \left( \pm \frac{i}{h} \int_1^y p_0 dy \right), \quad y > 1. \]
(90)
The general solution for \( 0 < y < 1 \) is a sum of two terms with proportionality constants \( A_+ \) and \( A_- \), and similarly for the \( y > 1 \) case with constants \( B_+ \) and \( B_- \). For the \( 0 < y < 1 \) case \( |p_0| = \sqrt{1 - y^2} \). Substituting \([83]\), it is possible to perform the integral in the exponentials. For \( 0 < y < 1 \),
\[ \frac{i}{h} \int_1^y p_0 dy = -\frac{1}{h} \int_1^y \sqrt{1 - y^2} dy = -\frac{1}{2h} \left( y \sqrt{1 - y^2} + \arcsin y - \frac{\pi}{2} \right), \]
(91)
while for \( y > 1 \),
\[ \frac{i}{h} \int_1^y p_0 dy = \frac{i}{h} \int_1^y \sqrt{y^2 - 1} dy = \frac{i}{2h} \left[ y \sqrt{y^2 - 1} - \ln \left( y + \sqrt{y^2 - 1} \right) \right]. \]
(92)
In the region $0 < y \ll 1$, substituting the expansion of (91), to leading order (89) takes the form

$$Z = A_{\pm} \exp \left[ \frac{1}{\hbar} \left( y - \frac{\pi}{4} \right) \right].$$  

(93)

Comparing with (78), follows that for $D > 3$

$$A_+ = S_+ h^{(D-1)/2} \exp \left( -\frac{\pi}{4\hbar} \right), \quad A_- = 0.$$  

(94)

**D. WKB for $D = 3$**

Because of the term proportional to $\hbar$ in $p^2$, the $D = 3$ case has to be treated separately. Then (87) takes the form

$$i \frac{d}{dy} [\ln (\pm p_0)] \pm \frac{s_1}{2y_p} = \frac{df_1}{dy}.$$  

(95)

Using (83),

$$i \frac{d}{dy} \left[ \ln \left( \pm \sqrt{y^2 - 1} \right) \right] \pm \frac{s_1}{2y}\sqrt{1 - y^2} = \frac{df_1}{dy}.$$  

(96)

Integrating, for $0 < y < 1$,

$$f_1 = \frac{i}{2} \ln \left( \sqrt{y^2 - 1} \right)$$  

(97)

$$\pm \frac{s_1}{2} \left[ \ln y - \ln \left( 1 + \sqrt{1 - y^2} \right) \right] + c_1,$$

and for $y > 1$,

$$f_1 = \frac{i}{2} \ln \left( \sqrt{y^2 - 1} \right)$$  

(98)

$$\pm \frac{s_1}{2} \arctan \left( \frac{1}{\sqrt{y^2 - 1}} \right) + c_2.$$

Substituting into (78), for $0 < y < 1$,

$$Z = \frac{A_+}{\sqrt{|p_0|}} \left( \frac{y}{1 + \sqrt{1 - y^2}} \right)^{\pm s_1/2} \exp \left( \pm \frac{i}{\hbar} \int_1^y p_0 dy \right),$$  

(99)

and for $y > 1$,

$$Z = \frac{B_+}{\sqrt{p_0}} \exp \left\{ \frac{\pm is_1}{2} \left[ \arctan \left( \frac{1}{\sqrt{y^2 - 1}} \right) - \frac{\pi}{2} \right] \right\} \times \exp \left( \frac{\pm i}{\hbar} \int_1^y p_0 dy \right).$$  

(100)

For $0 < y \ll 1$ the expansion of (99) gives

$$Z = A_+ \left( \frac{y}{2} \right)^{\pm s_1/2} \exp \left[ \frac{\pm i}{\hbar} \left( y - \frac{\pi}{4} \right) \right].$$  

(101)

Comparing with (77), follows that for $D = 3$

$$A_+ = 2^{s_1/2} S_+ h^{1-s_1/2} \exp \left( -\frac{\pi}{4\hbar} \right), \quad A_- = 0.$$  

(102)

**E. Connection formulae**

The next step is to relate the amplitudes $A_{\pm}$ in the $y < 1$ region to the amplitudes $B_{\pm}$ for $y > 1$. The expressions (89) and (90) for $D > 3$ are just the standard WKB solutions, so one can apply the appropriate connection formulae to express $B_{\pm}$ in terms of $A_{\pm}$. In our case we need the one where the field is exponentially increases when going away from the turning point (see for example (34.18) of (94))

$$\frac{1}{\sqrt{p_0}} \exp \left( \frac{1}{\hbar} \int_1^y |p_0| dy + \frac{i\pi}{4} \right)$$  

$$\rightarrow \frac{1}{\sqrt{|p_0|}} \exp \left( \frac{1}{\hbar} \int_1^y |p_0| dy \right),$$  

(103)

from $y > 1$ to $y < 1$. This is equivalent to

$$A_- = 0, \quad B_+ = A_+ \exp \left( \pm \frac{i\pi}{4} \right).$$  

(104)

The expressions (99) and (100) for $D = 3$ differ from the standard WKB solutions by factors involving $s_1$. However, since both of these factors take the value 1 at the $y = 1$ turning point, the formulae (104) relating the amplitudes at the two sides hold in this case too. A detailed derivation of (104) is given in Appendix A.

**F. Schrödinger-Newton tail amplitude**

For $y > 1$ the expression (92) can be approximated by

$$i \frac{1}{\hbar} \int_1^y p_0 dy = iy^2/2\hbar.$$  

(105)

In the $D > 3$ case to leading order (90) yields

$$Z = B_+ \sqrt{y} \exp \left( \pm iy^2/2\hbar \right).$$  

(106)

Substituting $B_+$ from (104) gives

$$Z = \frac{A_+}{\sqrt{y}} \cos \left( \frac{y^2}{2\hbar} + \frac{\pi}{4} \right), \quad D > 3.$$  

(107)

For $D = 3$, to leading order (100) yields

$$Z = \frac{B_+}{\sqrt{y}} \exp \left[ \pm i \left( \frac{y^2}{2\hbar} + \frac{\pi s_1}{4} \right) \right],$$  

(108)

and substituting $B_+$ from (104) gives

$$Z = \frac{A_+}{\sqrt{y}} \cos \left( \frac{y^2}{2\hbar} + \frac{\pi s_1}{4} + s_1 + 1 \right), \quad D = 3.$$  

(109)

Using (74) we can express the original $S$ variable in the SN equations using the coordinate $\rho = y/\hbar$, obtaining the same asymptotic formula as in (77),

$$S = \frac{\alpha}{\rho^2} \cos \left( \frac{h\rho^2}{2} + \beta \right),$$  

(110)
where the amplitude is

$$\alpha = \frac{A_+}{h^{D/2}} \quad (111)$$

for any $D$, and the phase is

$$\beta = \begin{cases} \frac{\pi}{4} (s_1 + 1) & \text{if } D = 3, \\ \frac{\pi}{4} & \text{if } D > 3. \end{cases} \quad (112)$$

Substituting the value of $A_+$ from (111) and (112) gives our final WKB result for the minimal amplitude

$$\alpha_{\min} = \begin{cases} \frac{S_c}{\sqrt{h}} \left( \frac{2}{h} \right) \frac{\omega}{\sqrt{\pi}} \exp \left( -\frac{\pi}{4h} \right) & \text{if } D = 3, \\ \frac{S_c}{\sqrt{h}} \exp \left( -\frac{\pi}{4h} \right) & \text{if } D > 3. \end{cases} \quad (113)$$

It is instructive to compare the above analytical result to the minimal amplitude obtained by the numerical solution of the SN equations (53)-(54). For a chosen $h$ one has to minimize the oscillating tail for those solutions for which the field $s$ tends to $s_0 = -1$ asymptotically, while $S$ has no nodes in the core region. The solutions depend very strongly on the central values $s_c$ and $S_c$. The results for various $h$ for $D = 3$ spatial dimensions are listed in Table III. The amplitudes $\alpha_{\min}$ for $D = 3, 4, 5$ dimensions are plotted on Fig. 2, and the relative difference from the analytically obtained WKB value is displayed on Fig. 3. From this figure we read off that the WKB analysis gives less than 10% error, if $h < 0.06$ in case of $D = 3$, and if $h < 0.1$ when $D = 4$ or 5.

![FIG. 2: The curves represent the WKB tail amplitude, while the points the numerically calculated values, as a function of $h$, for $D = 3, 4, 5$ spatial dimensions.](image1)

![FIG. 3: Relative difference of the numerically calculated minimal tail amplitude from the analytically calculated WKB values.](image2)

### G. Scalar field tail amplitude

For oscillations, using (53), (54), (52) and (110), the real scalar in the tail region behaves like

$$\phi = \epsilon^2 \phi_2 = \epsilon^2 p_2 \cos \tau = \epsilon^2 S \sqrt{\frac{D-1}{D-2}} \cos \tau$$

$$= \frac{\phi_A}{r^{\frac{2}{D}}} \cos(\omega t) \cos \left( \frac{h \epsilon^2 m^2 r^2}{2} + \beta \right), \quad (114)$$

where the amplitude is

$$\phi_A = \frac{\epsilon^2}{m^2} \sqrt{\frac{D-1}{D-2}} \alpha_{\min}. \quad (115)$$
Applying the identity
\[
\cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)],
\]
it is apparent that (114) is a sum of an ingoing and outgoing wave, both with amplitude \( \phi_A/2 \). One might be tempted to add an ingoing wave with an opposite amplitude to obtain a purely outgoing solution, and to conclude that the remaining outgoing component has the amplitude \( \phi_A/2 \). However, an ingoing wave contains a \( \cos(\hbar \varepsilon m^2 r^2/2 + \beta) \) component, which would change the amplitude of the WKB mode which is increasing when going away from the turning point. What one can do instead is adding a standing wave proportional to \( \sin(\hbar \varepsilon m^2 r^2/2 + \beta) \), corresponding to the suppressed WKB mode. If the time dependence is also phase shifted, then we obtain the minimal amplitude outgoing wave,
\[
\phi = \frac{\phi_A}{\sqrt{2}} \cos \left( \frac{\hbar \varepsilon m^2 r^2}{2} + \beta - \omega t \right). \tag{117}
\]

In case of boson stars, using (88), (89), (90) and (110), the complex scalar in the tail region behaves like
\[
\phi = \varepsilon^2 \phi_2 = \varepsilon^2 \psi e^{i\tau} = \varepsilon^2 S \sqrt{D - 1 \over D - 2} e^{i\tau} = \frac{\phi_A}{\sqrt{2} R} e^{\omega t} \cos \left( \frac{\hbar \varepsilon m^2 r^2}{2} + \beta \right). \tag{118}
\]
Now one can obtain the minimal amplitude outgoing wave by adding a term
\[
- i \frac{\phi_A}{\sqrt{2} R} e^{i\omega t} \sin \left( \frac{\hbar \varepsilon m^2 r^2}{2} + \beta \right), \tag{119}
\]

obtaining
\[
\phi = \frac{\phi_A}{\sqrt{2} R} \exp \left[ - i \left( \frac{\hbar \varepsilon m^2 r^2}{2} + \beta - \omega t \right) \right]. \tag{120}
\]

H. Mass loss rate

Even if \( \rho \gg 1/\hbar \) in the oscillating tail region, according to (34) and (35), for small \( \varepsilon \) the metric functions \( A \) and \( B \) are still very close to 1. The mass loss rate of the system can be calculated by applying the expression (B14) for the time derivative of the mass function from Appendix B.

\[
\frac{dM}{dt} = - \frac{\pi \hbar \varepsilon^2 m^3 \phi_A^2}{8 \Gamma \left( \frac{2}{D} \right)}. \tag{123}
\]

Here, since we are interested in the leading \( \varepsilon \) order result, we substituted \( \omega = m \), and for large \( r \) we neglected the term arising when taking the derivative of the \( r^{-D/2} \) factor. For large radius \( \hat{m} \) agrees with the total mass \( M \) of the oscillon or boson star. Using (113) and (115) for the mass loss rate and substituting \( h = H/(m \varepsilon^2) \), for \( D > 3 \) we obtain
\[
\frac{dM}{dt} = - \frac{\pi \hbar \varepsilon^2 (D - 1) S^2 \varepsilon^6 \varepsilon^{-D}}{8 m^D - 3 (D - 2) \Gamma \left( \frac{2}{D} \right)} \exp \left( - \frac{\pi m \varepsilon^2}{2 H} \right). \tag{124}
\]

If \( D = 3 \) there is an extra term involving \( s_1 \),
\[
\frac{dM}{dt} = - \frac{1}{2} S^2 \varepsilon^3 \left( \frac{2 m \varepsilon^2}{H} \right) \exp \left( - \frac{\pi m \varepsilon^2}{2 H} \right). \tag{125}
\]

For small-amplitude configurations the parameter \( \varepsilon \) can be expressed by the mass \( M \) using (114). Because of the complexity of the resulting expression we only give the result for \( D = 3 \), when \( M = \varepsilon s_1/(2 m) \)
\[
\frac{dM}{dt} = - 4 S^2 \varepsilon^3 \left( \frac{m \varepsilon^2}{H s_1} \right) \exp \left( - \frac{2 m \varepsilon^2 M^2}{H s_1^2} \right). \tag{126}
\]

For specific \( m \) and \( H \) this can be integrated numerically in order to obtain the change of the mass as a function of time.

Using the leading order expression (B14) for the total mass and dividing by \( M \) we can obtain the relative mass loss rate. The Hubble time \( T_H = 1/H \) describes the time-scale of the expansion of the universe. The value of \( T_H \) obtained from the present value of the cosmological constant is given by (D7) in Appendix D. Dividing by both \( M \) and \( H \) we obtain the relative mass loss extrapolated for a period corresponding to the Hubble time. For \( D > 3 \),
\[
\frac{T_H}{M} \frac{dM}{dt} = - \frac{S^2}{(D - 2) s_1 h} \exp \left( - \frac{\pi}{2 h} \right), \tag{127}
\]
and for \( D = 3 \),
\[
\frac{T_H}{M} \frac{dM}{dt} = - \frac{S^2}{s_1 h} \left( \frac{2}{h} \right) \exp \left( - \frac{\pi}{2 h} \right), \tag{128}
\]

where the values of the constants \( s_1, S \) are given in Table II. This result is independent on the units used to measure the time and mass, and it depends on \( H, m \) and \( \varepsilon \) only through the rescaled cosmological constant \( h = H/(m \varepsilon^2) \). According to (34) the square of the parameter \( h \) is proportional to the ratio of the central density \( \mu_c \) to the effective energy density corresponding to the cosmological constant \( \mu_A \). On fig. 6 and on Table II we give \( T_H/M \) as a function of the density ratio \( \mu_r = \mu_c/\mu_A \).
large \( \varepsilon \) the expression (129) can be applied. Apart from the amplitude parameter \( \varepsilon \) the mass loss (125) depends only on the ratio \( H/m \). It can be checked numerically that for a concrete choice of \( H/m \) there is only one \( \varepsilon \) value where the two type of mass loss rates are equal, which we denote by \( \varepsilon_c \). For low \( H/m \) values the function \( \varepsilon_c \) can be expanded as

\[
\varepsilon_c = c_3 \left( \frac{H}{m} \right)^{1/3} (1 + \varepsilon_1 + \varepsilon_2 + \ldots) ,
\]

with

\[
\varepsilon_1 = \frac{R}{3c_3^2} \left( \frac{H}{m} \right)^{1/3} , \quad \varepsilon_2 = \frac{c_4 R}{9c_3^3} \left( \frac{H}{m} \right)^{2/3} ,
\]

where the newly introduced two constants are

\[
c_3 = \left( \frac{2c_2}{\pi} \right)^{1/3} , \quad c_4 = \frac{2}{\pi} (2s_1 + 5) .
\]

On Fig. 5 we plot \( \varepsilon_c \) as a function of \( H/m \). To illustrate how well the above analytic approximation works we plot for relatively large values of \( H/m \), although the solutions for \( \epsilon > \epsilon_{\text{max}} \approx 0.5 \) are unstable. The quantity \( H/m \) is

![FIG. 5: The values of \( \varepsilon_c \), for which the oscillaton mass loss rates (125) and (129) are equal are plotted as a function of the ratio of the Hubble constant \( H \) and the scalar field mass \( m \) (solid line). Above the line the asymptotically flat result (129) is dominant, while below (125) induced by the cosmological constant gives a larger contribution. The first three orders of the analytical approximation given by (131) are also shown.](image)

V. MASS LOSS RATE OF OSCILLATONS IN THE \( \Lambda = 0 \) CASE

If the cosmological constant is zero and the space-time is asymptotically flat then the metric of boson stars is static and their mass is independent of time. In contrast, oscillatons are very slowly losing mass by scalar field radiation even in the \( \Lambda = 0 \) case [16]. This mass loss is nonperturbatively small in the amplitude parameter \( \varepsilon \). According to [16], for \( D = 3 \) this mass loss rate depends only on the amplitude parameter, and it is given by

\[
\frac{dM}{dt} = -\frac{c_1}{\varepsilon^2} \exp \left( -\frac{c_2}{\varepsilon} \right) ,
\]

where the numerical values of the constants are

\[
c_1 = 30.0 , \quad c_2 = 22.4993 .
\]

For small \( \varepsilon \) this is exponentially small, while the mass loss induced by the cosmological constant, given by (129), tends to zero only polynomially when \( \varepsilon \to 0 \). This implies that if \( \Lambda \neq 0 \) then for small \( \varepsilon \) the mass loss (125) originating from the cosmological constant dominates, while for

![FIG. 4: The relative mass loss during a Hubble time period, \( \frac{\Delta M}{M_0} \), as a function of \( \frac{\mu}{\mu_A} \) for \( D = 3 \), 4 and 5 spatial dimensions.](image)

![TABLE IV: The rescaled cosmological constant \( \hbar \) and the density ratio \( \frac{\mu}{\mu_A} \) belonging to various relative mass loss during a Hubble time period.](image)
VI. POSSIBLE EFFECTS IN THE CURRENT UNIVERSE

From the definition [50] of the rescaled cosmological constant we can express the square of the small-amplitude parameter \( \varepsilon \). The amplitude of the scalar field is proportional to \( \varepsilon^2 \) to leading order. Measuring the cosmological constant in 1/s units and the scalar field mass in eV/c² units, [50] can be written as

\[
\varepsilon^2 = \frac{6.58 \times 10^{-16}}{h} \frac{H_0}{m} \frac{eV}{mc^2}.
\]

According to (127) and (128), the part of the mass lost during a Hubble time period only depends on the value of the rescaled Hubble constant \( h \). For example, we can look for configurations where this mass loss is one percent. According to Table IV for the physically interesting 3 + 1 dimensional case this corresponds to the value of \( h_{0.01} = 0.07985 \). Using the present value of the Hubble constant, which is given by (106),

\[
\varepsilon^2_{0.01} = 1.60 \times 10^{-32} \frac{eV}{mc^2}.
\]

Oscillatons or boson stars are generally stable if the amplitude parameter \( \varepsilon \), which for larger amplitudes can be defined from the frequency \( \omega \) by \( \varepsilon = \sqrt{1 - \omega^2/m^2} \), is smaller than \( \varepsilon_{\text{max}} \approx 0.5 \). From (130) it is apparent that significant mass loss for the larger amplitude configurations with \( \varepsilon \approx \varepsilon_{\text{max}} \) can happen only for extremely small scalar field masses. Very small \( \varepsilon \) values are not excluded, but since for \( D = 3 \), according to (150), the total mass of the oscillaton or boson star is proportional to \( \varepsilon \), these configurations tend to have extremely small masses. Similarly, since by (64) the radius is inversely proportional to \( \varepsilon \), small \( \varepsilon \) values belong to spatially extended configurations. It is obvious that the influence of the cosmological constant is larger for large radius objects. In Table V we list the values of \( \varepsilon \), the total mass and the radius for various choices of the scalar field mass \( m \), assuming that the mass-loss extrapolated for a Hubble time period is 1%. Keeping the same scalar mass, for larger \( \varepsilon \), and consequently, for larger total masses and smaller radiuses, the mass loss rate is smaller than 1%. For all configurations listed in Table V the \( \Delta = 0 \) oscillaton mass loss rate given by (129) is negligible, many orders of magnitude smaller than 1%.

To show some concrete examples, we examine three specific choices for the scalar field mass. Firstly, for an axion with \( m = 10^{-5} eV \) the mass of the oscillaton for which \( \frac{dM}{dt} = 0.01 \) is about that of a very small asteroid, and its radius is about 15 astronomical units. Secondly, if \( m = 9.55 \times 10^{-18} eV/c^2 \), then the total mass is about a solar mass, \( M_\odot = 2.00 \times 10^{30} \text{kg} \), which is distributed in a region of radius of 238 light-years. Thirdly, for \( m = 10^{-25} \) the corresponding mass is about the mass of the Milky Way (including dark matter), and the radius is \( 2 \times 10^6 \) light-years, which is about four times the radius of the stellar disk. According to Table IV and (133), for all the above configurations the central density is \( \mu_c = 54.55 \mu_A = 3.7 \times 10^{-22} \text{kg/m}^3 \).

The maximal amplitude boson stars or oscillatons, with \( \varepsilon \approx \varepsilon_{\text{max}} \) generally have much bigger masses and smaller sizes than the states with 1% mass loss discussed in the previous paragraphs. For example, the \( \varepsilon = 0.5 \) oscillaton for the axion case has a mass \( M = 2.3 \times 10^{25} \text{kg} \) (about four Earth masses) and radius \( r_{0.5} = 18 \text{cm} \). Only considering the mass loss (125) induced by the cosmological constant, the maximal amplitude boson stars or oscillatons have extremely long lifetime. For the axion case, using (135), we get \( h = 5.1 \times 10^{-28} \), and the relative mass loss during a Hubble time period is of the order \( 10^{-10^{27}} \), certainly negligible. For boson stars formed by complex fields this is the only mechanism to lose mass, consequently, close to maximal mass boson stars have practically constant mass. However oscillatons lose mass even in the \( \Lambda = 0 \) case by (129) which gives \( \frac{dM}{dt} = -3.1 \times 10^{10} \) for \( \varepsilon = 0.5 \), showing that this state is unstable on the cosmological time-scale. However, the mass loss decreases exponentially with \( \varepsilon \). An initially maximal mass oscillaton created in the early universe loses mass relatively quickly, and its amplitude parameter decreases to about \( \varepsilon = 0.31 \) during a Hubble time period, where according to (129) the relative mass loss rate is \( \frac{dM}{dt} = -0.14 \). For more details and different scalar field masses see Table VIII of [32].

| \( mc^2/eV \) | \( \varepsilon_{0.01} \) | \( M/\text{kg} \) | \( r_{0.5}/\text{m} \) |
|---|---|---|---|
| \( 10^{10} \) | \( 1.27 \times 10^{-21} \) | \( 5.90 \times 10^{-13} \) | \( 6.95 \times 10^4 \) |
| \( 10^5 \) | \( 4.01 \times 10^{-19} \) | \( 1.87 \times 10^{-3} \) | \( 2.20 \times 10^7 \) |
| \( 1 \) | \( 1.27 \times 10^{-16} \) | \( 5.90 \times 10^4 \) | \( 6.95 \times 10^6 \) |
| \( 10^{-5} \) | \( 4.01 \times 10^{-14} \) | \( 1.87 \times 10^{12} \) | \( 2.20 \times 10^{12} \) |
| \( 10^{-10} \) | \( 1.27 \times 10^{-11} \) | \( 5.90 \times 10^{19} \) | \( 6.95 \times 10^{14} \) |
| \( 10^{-15} \) | \( 4.01 \times 10^{-9} \) | \( 1.87 \times 10^{27} \) | \( 2.20 \times 10^{17} \) |
| \( 10^{-20} \) | \( 1.27 \times 10^{-6} \) | \( 5.90 \times 10^{34} \) | \( 6.95 \times 10^{19} \) |
| \( 10^{-25} \) | \( 4.01 \times 10^{-4} \) | \( 1.87 \times 10^{42} \) | \( 2.20 \times 10^{22} \) |
| \( 10^{-30} \) | \( 1.27 \times 10^{-1} \) | \( 5.90 \times 10^{49} \) | \( 6.95 \times 10^{24} \) |

TABLE V: Oscillaton or boson star configurations for which 1% of the mass is lost during Hubble time, assuming the present value of the cosmological constant. For each choice of the scalar field mass \( m \) the amplitude parameter \( \varepsilon \), the total mass \( M \) and the radius containing 95% of the mass is given.

VII. INFLATIONARY ERA

The energy density of the inflaton field during the inflationary epoch can be estimated as \( \mu_\lambda = (10^{16} \text{GeV})^4 \) (see e.g. [26]). This expression is valid in natural units, where \( c = h = 1 \) (but \( G \neq 1 \)) and the energy is measured in electron volts. In ordinary units this corresponds to the mass density \( \mu_\lambda = 2.3 \times 10^{22} \text{kg/m}^3 \). Assuming a de
Sitter geometry, the Hubble constant is

\[ H = \sqrt{\frac{8\pi G \mu_A}{3}} = 3.6 \times 10^{37} \frac{1}{s} = 2.4 \times 10^{13} \text{GeV}, \quad (137) \]
in ordinary and natural units. By (28) this belongs to a cosmological constant

\[ \Lambda = \frac{3H^2}{c^2} = 4.3 \times 10^{58} \frac{1}{\text{m}^2} = 1.1 \times 10^{-11}, \quad (138) \]
in ordinary and Planck units, respectively.

If in addition to the inflaton field there is another scalar field \( \chi \) on this de Sitter background, and it is massive, then it is likely to form localized oscillaton configurations (or boson stars in case of a complex field). This second field may be, for example, the waterfall field in a hybrid inflation theory (29, 30). If the mass parameter of the field may be, for example, the waterfall field in a hybrid inflation theory (29, 30). If the mass parameter of the field \( m \), then we can apply (135) to relate the small-amplitude parameter \( \varepsilon \) to the rescaled Hubble constant \( h \).

The parameter \( h \) is the essential parameter determining the mass loss rate of the object. For example, if the mass loss extrapolated for a Hubble time period is assumed to be 1\%, then \( h = h_{0.01} = 0.07985 \), and for the inflationary era we obtain

\[ \varepsilon_{0.01}^2 = 3.0 \times 10^{14} \frac{\text{GeV}}{mc^2}. \quad (139) \]

Since oscillatons and boson stars are stable only if the amplitude parameter satisfies \( \varepsilon < \varepsilon_{\text{max}} \approx 0.5 \), states with 1\% mass loss can only exist for \( m > 1.2 \times 10^{15} \text{GeV}/c^2 \). For objects formed from fields with \( m < 1.2 \times 10^{15} \text{GeV}/c^2 \), necessarily \( \varepsilon < \varepsilon_{\text{max}} < \varepsilon_{0.01} \). Then from (135) it follows that \( h > h_{0.01} \), which implies that the mass loss during the Hubble time is always larger than 1\%. Oscillatons and boson stars formed from scalar fields with mass \( m < 10^{15} \text{GeV}/c^2 \) are very short living during the inflationary era. Another way to see why this lower limit is necessary is to calculate the central density \( \mu_c \) by (65), and observe that for small scalar field mass \( \mu_c < \mu_A \), and hence long living oscillatons are not expected to exist.

For scalar field masses \( m > 1.2 \times 10^{15} \text{GeV}/c^2 \) we list the properties of some oscillatons with 1\% mass loss in Table VII. All these states have the central density \( \mu_A = 54.55 \mu_A = (2.7 \times 10^{16} \text{GeV})^4 = 1.3 \times 10^{46} \text{kg}/\text{m}^3 \), and the \( \Lambda = 0 \) mass loss is much smaller than 1\% for them.

The mass loss rate induced by the cosmological constant decreases very quickly when considering higher amplitude configurations. The behavior of the \( \Lambda = 0 \) oscillaton mass loss is just the opposite. The two kind of relative mass loss extrapolated for a Hubble time period of some maximal amplitude states belonging to \( \varepsilon = \varepsilon_{\text{max}} \approx 0.5 \) are given in Table VII.

### VIII. REHEATING

Oscillatons are likely to form after the end of inflation, in the reheating era, when the inflaton field is oscillating around the potential minimum. These oscillatons are likely to influence the efficiency of how the inflaton’s energy is transferred to other fields. Since oscillatons are concentrated energy lumps, their influence on the formation of inhomogeneities may also be important. At the reheating stage the equation of state can be well approximated by a pressureless fluid, and hence the effective cosmological constant can be taken to be zero. For the lifetime of oscillatons in this era one can apply the considerations given in (32) which are valid in the \( \Lambda = 0 \) asymptotically flat case. The mass of the inflaton field when it is near the vacuum value can be estimated as \( m = 10^{13} \text{GeV}/c^2 \) (see e.g. (28)). The maximal amplitude stable oscillaton which can be formed by this field, when \( \varepsilon = \varepsilon_{\text{max}} \approx 0.5 \), has the total mass \( M = 0.023 \text{kg} \) and radius \( r_{95} = 1.8 \times 10^{-28} \text{m} \). The mass and radius of smaller amplitude oscillatons formed by this field can be easily obtained by using the fact that the total mass is proportional to \( \varepsilon \) and the radius is proportional to \( 1/\varepsilon \).

If the cosmological constant is zero, then a natural time scale is the oscillation period, \( T_\omega = 2\pi/\omega \). According to (60), for small \( \varepsilon \) we can replace \( \omega \) by \( m \). Taking the total mass from (315), the part of the mass lost during an oscillation period is

\[ T_\omega \frac{dM}{M} dt = \frac{4\pi c_1}{s_1 \varepsilon^3} \exp \left( \frac{-c_2}{\varepsilon} \right). \quad (140) \]

This expression is independent of the scalar field mass \( m \) and on the units that we use for measuring time. Even for
the maximal amplitude oscillaton with $\varepsilon = 0.5$ this gives a very tiny value, $\frac{dM}{dt} = -2.5 \times 10^{-17}$. This shows that oscillatons formed in the reheating period perform a large number of oscillations before the mass loss by emitting classical scalar field radiation becomes apparent. However, since their oscillation period is extremely short, $T_\omega = 4.1 \times 10^{-37}s$, they lose about half of their mass in $8.3 \times 10^{-21}s$. On the other hand, since this mass loss depends exponentially on $\varepsilon$, even during a period corresponding to the lifetime of the universe the amplitude parameter does not decrease much below $\varepsilon = 0.166$, where $\frac{dM}{dt} = -0.13$.

IX. CONCLUSIONS

We have constructed spherically symmetric, spatially localized time dependent solutions in a large class of scalar theories coupled to Einstein’s theory of gravitation and a positive cosmological constant, $\Lambda$. Examples include boson star-type objects (for a complex field) and oscillatons (for a real field), considered previously for vanishing cosmological constant. A positive constant has important qualitative effects on boson stars, in that due to the repulsion induced by $\Lambda$ only radiating solutions exist, leading to a mass loss. For sufficiently small values of $\Lambda$ this effect is negligible, so boson stars with almost unchanged mass may be present in our Universe. Even when $\Lambda = 0$ oscillatons radiate slowly, and the consequent mass loss is maximal for large amplitudes. There is an additional mass loss of oscillatons induced by a nonzero $\Lambda > 0$, which dominates for small amplitudes (corresponding to large sizes). The mass loss of an oscillaton during a Hubble time period, is completely determined by the ratio $\mu_\gamma/\mu_\Lambda$, where $\mu_\Lambda$ is the energy density due to $\Lambda$ and $\mu_\gamma$ is that of the oscillaton in its center. Our computations are valid in spatial dimensions $2 < D < 6$, although the case $D = 3$ has been given special attention. An important cosmological application of our results is for “waterfall”-type oscillatons appearing in hybrid inflation models.

Appendix A: Connection formulae

In this appendix we derive the formulae (104) connecting the amplitudes in the $y < 1$ and $y > 1$ regions. We follow the method described in [50]. Since the expressions for $Z$ obtained by the WKB method are singular at $y = 1$, it is necessary to find a solution of (75) which is valid in a region around this point. Let us introduce a new radial coordinate by

$$y = 1 + x. \quad \text{(A1)}$$

We approximate $p^2$ for small $h$ by

$$p^2 \approx y^2 - 1 = x(x + 2) \approx 2x. \quad \text{(A2)}$$

Introducing a rescaled coordinate $z$ by

$$x = -\frac{h^{2/3}}{2^{1/3}}, \quad \text{(A3)}$$

(75) takes the form

$$\frac{d^2Z}{dz^2} - zZ = 0. \quad \text{(A4)}$$

The solution can be written in terms of Airy functions

$$Z = a \text{Ai} z + b \text{Bi} z. \quad \text{(A5)}$$

For $z \gg 0$ this has the asymptotic form

$$Z = \frac{1}{\sqrt{\pi} z^{1/4}} \left[ \frac{a}{2} \exp \left( -\frac{2}{3} z^{3/2} \right) + b \exp \left( \frac{2}{3} z^{3/2} \right) \right], \quad \text{(A6)}$$

and for $z \ll 0$

$$Z = \frac{1}{\sqrt{\pi} (-z)^{1/4}} \left\{ a \sin \left[ \frac{2}{3} (-z)^{3/2} + \frac{\pi}{4} \right] + b \cos \left[ \frac{2}{3} (-z)^{3/2} + \frac{\pi}{4} \right] \right\}. \quad \text{(A7)}$$

We consider two matching regions surrounding $y = 1$, where $|z| \gg 1$ and at the same time $|x| \ll 1$. Both of these can hold simultaneously if $h$ is small enough. First we continue through the matching region with $x < 0$. Substituting $y = 1 + x$ into (91) and expanding in power series around $x = 0$, we obtain

$$\frac{i}{h} \int_{1}^{y} p_0 dy = \frac{1}{3h} (-2x)^{3/2} + \mathcal{O} \left( x^{5/2} \right). \quad \text{(A8)}$$

This result can also be obtained by simply approximating $p_0$ by $\sqrt{2x}$. In this region the influence of the $s_1$ term is subleading, so for $3 \leq D \leq 5$ dimensions we obtain both from (89) and (99)

$$Z = \frac{A_+}{(-2x)^{1/4}} \exp \left[ \frac{1}{3h} (-2x)^{3/2} \right]. \quad \text{(A9)}$$

Comparing with (A6), we get

$$a = 0, \quad b = \frac{\sqrt{\pi}}{(2h)^{1/6}} A_+. \quad \text{(A10)}$$

Continuing into the second matching region with $x > 0$, (A7) yields

$$Z = \frac{b}{2\sqrt{\pi} (-z)^{1/4}} \left\{ \exp \left( \frac{i\pi}{4} \right) \exp \left[ \frac{2}{3} (-z)^{3/2} \right] + \exp \left( -\frac{i\pi}{4} \right) \exp \left[ -\frac{2}{3} (-z)^{3/2} \right] \right\}. \quad \text{(A11)}$$

For small positive $x$ (92) gives

$$\frac{i}{h} \int_{1}^{y} p_0 dy = \frac{i}{3h} (2x)^{3/2} + \mathcal{O} \left( x^{5/2} \right). \quad \text{(A12)}$$
To leading order both (90) and (100) can be written as
\[ Z = \frac{B_\pm}{(2\pi)^{1/4}} \exp \left[ \pm \frac{i}{3} \hat{r} (2x)^{3/2} \right]. \]  
(A13)
Comparing with (A11) gives
\[ B_\pm = b(2\pi)^{1/6} \exp \left( \pm \frac{i\pi}{4} \right). \]  
(A14)
Substituting b from (A10) yields the desired formulae.

Appendix B: Mass in spherically symmetric asymptotically de Sitter space-times  

Since the scalar field tends to zero exponentially, at large distances the metric approaches the static Schwarzschild-de Sitter metric. Considering \( D + 1 \) dimensional space-times, in Schwarzschild area coordinates the Schwarzschild-de Sitter metric has the form
\[ ds^2 = -F(\hat{r})dt^2 + \frac{1}{F(\hat{r})}dr^2 + \hat{r}^2d\Omega_{D-1}^2, \]  
(B1)
where
\[ F(\hat{r}) = 1 - \frac{r_0^{D-2}}{\hat{r}^{D-2}} - H^2\hat{r}^2 \]  
(B2)
and \( r_0 \) is a constant related to the mass \( M \) by
\[ M = \frac{(D - 1)\pi \frac{\hat{r}}{2}}{8\pi \Gamma \left( \frac{D}{2} \right)} r_0^{D-2}. \]  
(B3)
In general spherically symmetric space-times there is a naturally defined radius function \( \hat{r} \), defined in terms of the area of the symmetry spheres. In Schwarzschild coordinates \( \hat{r} = \hat{r} \), while in the isotropic coordinates \( \hat{r} = r\sqrt{B} \). In terms of this radius function one can define the Misner-Sharp energy (or local mass) function \( \hat{m} \) [51, 52], which can be defined for arbitrary dimensions by
\[ \hat{m} = \frac{(D - 1)\pi \frac{\hat{r}}{2}}{8\pi \Gamma \left( \frac{D}{2} \right)} \hat{r}^{D-2} \left( 1 - g^{ab}\hat{r}_a\hat{r}_b - H^2\hat{r}^2 \right). \]  
(B4)
For the Schwarzschild-de Sitter metric \( \hat{m} = M \). At infinity the mass function \( \hat{m} \) agrees with the Abbott-Deser mass [53, 54], which is a Killing energy defined in static asymptotically de Sitter space-times.
The Misner-Sharp energy \( \hat{m} \) also agrees with a conserved energy that can be defined using the Kodama vector. The Kodama vector \( K^a = \epsilon^{ab}\hat{r}_b \), defined by
\[ K^t = \frac{\hat{r}_r}{\sqrt{AB}}, \quad K^r = -\frac{\hat{r}_t}{\sqrt{AB}}, \]  
(B6)
It can be checked that, in general, the Kodama vector is divergence free, \( K^a_{;a} = 0 \). Since contracting with the Einstein tensor, \( G^{ab}K_{a;b} = 0 \), the current
\[ J_a = T_{ab}K^b \]  
(B7)
is also divergence free, \( J^a_{;a} = 0 \), it defines a conserved charge. Integrating on a constant \( t \) hypersurface with a future oriented unit normal vector \( \hat{n}^a \), the conserved charge is
\[ E = \frac{2\pi \frac{\hat{r}}{2}}{\Gamma \left( \frac{D}{2} \right)} \int_0^\infty \hat{r}^{D-1} \sqrt{B} \hat{n}^a J_a dr \]  
(B8)
\[ = \frac{2\pi \frac{\hat{r}}{2}}{\Gamma \left( \frac{D}{2} \right)} \int_0^\infty \hat{r}^{D-1} \frac{\hat{r}}{A} (T_{tt}\hat{r}_t - T_{tr}\hat{r}_r) dr. \]
It can be checked that the derivative of the mass function \( \hat{m} \) can be expressed in terms of the current \( J_a \),
\[ \hat{m}_{,a} = \frac{2\pi \frac{\hat{r}}{2}}{\Gamma \left( \frac{D}{2} \right)} \epsilon_{ab}J^b. \]  
(B9)
For the radial derivative follows that
\[ \hat{m}_{,r} = \frac{2\pi \frac{\hat{r}}{2}}{\Gamma \left( \frac{D}{2} \right)} \frac{\hat{r}^{D-1}}{A} (T_{tt}\hat{r}_t - T_{tr}\hat{r}_r), \]  
(B10)
which, comparing with (B8), gives \( E = \hat{m} \).
The time derivative of the mass function is
\[ \hat{m}_{,t} = \frac{2\pi \frac{\hat{r}}{2}}{\Gamma \left( \frac{D}{2} \right)} \frac{\hat{r}^{D-1}}{B} (T_{tt}\hat{r}_t - T_{rr}\hat{r}_r). \]  
(B11)
This equation describes the mass loss caused by the outward energy current of the massive scalar field. For large radii the function \( \hat{m} \) tends to the total mass \( M \) of the oscillon or boson star. Substituting the small-amplitude expansions (34) and (35) into the definition (B4) of the mass function \( \hat{m} \), to leading order in \( \varepsilon \) we obtain
\[ \hat{m} = -\varepsilon^{A-D} \frac{\pi \frac{\hat{r}}{2}}{8\pi m^{D-2} \Gamma \left( \frac{D}{2} \right)} \frac{d\hat{m}_{,a}}{ds} \]  
(B12)
We have seen in Subsections III B and III C that for both the boson star and oscillaton case \( B_2 = A_2/(2 - D) \) and \( A_2 = -1 - s \), giving
\[ \hat{m}_{,t} = -\varepsilon^{A-D} \frac{\pi \frac{\hat{r}}{2}}{8\pi m^{D-2} \Gamma \left( \frac{D}{2} \right)} \left( \frac{D - 1}{2m} \right) \frac{ds}{d\rho}. \]  
(B13)
Since the asymptotic behavior of \( s \) is given by (68),
\[ M = \varepsilon^{A-D} \frac{(D - 1) \pi \frac{\hat{r}}{2}}{8\pi m^{D-2} \Gamma \left( \frac{D}{2} \right)} \frac{ds}{d\rho}. \]  
(B14)
For \( 3 + 1 \) dimensional space-time
\[ M_{(D=3)} = \varepsilon^{\frac{8}{2m}}. \]  
(B15)
For small $h$ the numerical value of the constant $s_1$ can be approximated by the $h = 0$ value given in Table [11]. Measuring the $mc^2$ belonging to the scalar field in electron volts and the mass of the boson star or oscillaton in kilograms,

$$M_{(D=3)} = 4.657 \times 10^{20} kg \frac{eV}{mc^2} \varepsilon.$$  \hfill (B16)

**Appendix C: Proper mass and core radius**

The proper mass inside a sphere of radius $r$ is defined by the $D$ dimensional volume integral of the energy density $\mu$,

$$M_p(r) = \frac{2\pi^\frac{D}{2}}{\Gamma \left( \frac{D}{2} \right)} \int_0^r dr \mu B^{D/2} r^{D-1}.$$  \hfill (C1)

Substituting the leading order expression (B4) for $\mu$, and the leading order value $B = 1$ from (B3), using the rescaled radial coordinate $\rho = \varepsilon mr$,

$$M_p(\rho) = \frac{\varepsilon^{D-\pi} r^D (D-1)}{8\pi(D-2)m^{D-2}\Gamma \left( \frac{D}{2} \right)} \int_0^\rho d\rho \rho^{D-1} S^2.$$  \hfill (C2)

The integral can be performed using the SN equation (D3),

$$M_p(\rho) = -\frac{\varepsilon^{D-\pi} r^D (D-1)\rho^{D-1}}{8\pi(D-2)m^{D-2}\Gamma \left( \frac{D}{2} \right)} \frac{ds}{dp}.$$  \hfill (C3)

To leading order in $\varepsilon$ the calculated proper mass $M_p$ agrees with the mass function $m$ given by (B14). However at higher order the proper mass is expected to be larger by an amount corresponding to the binding energy.

Oscillatons and boson stars do not have a definite outer surface. A natural definition for their size is to take the radius $r_n$ inside which $\rho_n$ percent of the mass-energy can be found. It is usual to take, for example, $n = 95$ or 99.9. The mass-energy inside a given radius $r$ can be defined either by the integral (C1) or by taking the local mass function $m$ in (B14). To leading order in $\varepsilon$ both definitions give (C3). The rescaled radius $\rho_n$ can be defined by

$$\frac{M_p(\rho_n)}{M_p(\infty)} = \frac{n}{100}.$$  \hfill (C4)

The numerical values of $\rho_n$ for various $n$ in $D = 3, 4, 5$ dimensions are listed in Table VIII. The physical radius is

$$r_n = \frac{\rho_n}{\varepsilon m}.$$  \hfill (C5)

In ordinary units, measuring the scalar field $mc^2$ in electron volts and $r_n$ in meters (Roman m),

$$r_n = 1.97 \times 10^{-7} \frac{m \rho_n}{\varepsilon mc^2} \frac{eV}{mc^2}.$$  \hfill (C6)

| $D = 3$ | $D = 4$ | $D = 5$ |
|---------|---------|---------|
| $\rho_{90}$ | 2.240 | 1.778 | 1.317 |
| $\rho_{95}$ | 3.900 | 3.013 | 2.284 |
| $\rho_{99}$ | 4.714 | 3.455 | 2.652 |
| $\rho_{99.9}$ | 5.675 | 4.410 | 3.478 |
| $\rho_{99.99}$ | 7.239 | 5.692 | 4.634 |

**TABLE VIII:** The radius inside which given percentage of the mass-energy is contained for various spatial dimensions.

We note that for complex field boson stars the current

$$j^a = \frac{i}{2} g^{ab} (\Phi^* \Phi_b - \Phi^*_b \Phi)$$  \hfill (C7)

can be used to define a conserved quantity. The conserved particle number $N$ can be defined by integrating the time component $j^0$ for a spacelike slice. To leading order in $\varepsilon$ the function $j^0$ is proportional to the density $\mu$ given in (B4). Hence, to this order, the use of the particle number instead of the mass-energy yields the same result for the radius of boson stars.

**Appendix D: Present value of the cosmological constant**

For the present observational value of the Hubble constant we take

$$H_0 = 70.4 \frac{\text{km}}{s Mpc} = 2.28 \times 10^{-18} \frac{1}{s}.$$  \hfill (D1)

according to the combined 7-year WMAP data [57]. To this corresponds a critical density

$$\mu_{cr} = \frac{3H_0^2}{8\pi G} = 9.31 \times 10^{-24} \frac{kg}{m^3}.$$  \hfill (D2)

The fraction of this belonging to the dark energy provided by the cosmological constant is given by $\Omega_\Lambda = 0.728$, according to WMAP results [57]. Consequently, the energy density belonging to the cosmological constant is

$$\mu_\Lambda = \mu_{cr} \Omega_\Lambda = 6.78 \times 10^{-24} \frac{kg}{m^3}.$$  \hfill (D3)

According to (B4), this corresponds to a cosmological constant of

$$\Lambda = \frac{8\pi G \mu_\Lambda}{c^2} = 1.26 \times 10^{-52} \frac{1}{m^2}.$$  \hfill (D4)

In Planck units, since the Planck length is $l_P = 1.616252 \times 10^{-35} m$,

$$\Lambda = 3.30 \times 10^{-122}.$$  \hfill (D5)

Since we are interested in the effect of the cosmological constant, we neglect the ordinary and dark matter
content of the universe. Then the space-time can be described by a de Sitter metric with Hubble constant $H$ given by (20). In Planck, ordinary and natural units

$$H = 1.05 	imes 10^{-61} = 1.95 	imes 10^{-18} \frac{1}{s} = 1.28 \times 10^{-33} \text{eV}.$$ (D6)

The Hubble time is then

$$T_H = \frac{1}{H} = 5.14 \times 10^{17}s = 16.3\text{Gyr}. \quad (D7)$$

Even though the ordinary and dark matter content have been neglected, this value is still not too far from the age of the universe, which according to WMAP data is 13.75Gyr assuming the $\Lambda$CDM model [57].

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