CONVERGENCE RATE OF FREE BOUNDARY OF NUMERICAL SCHEME FOR AMERICAN OPTION

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Dedicated to Professor Lishang Jiang on the occasion of his 80th birthday

Abstract. Based on the optimal estimate of convergence rate $O(\Delta x)$ of the value function of an explicit finite difference scheme for the American put option problem in [6], an $O(\sqrt{\Delta x})$ rate of convergence of the free boundary resulting from a general compatible numerical scheme to the true free boundary is proven. A new criterion for the compatibility of a generic numerical scheme to the PDE problem is presented. A numerical example is also included.

1. Introduction. Convergence rate of a numerical scheme is always an interesting and challenging problem. Using convergence rates, one can estimate and control computing time, compare different schemes, etc. Therefore, it is significant to find best possible convergence rate. Problems from finance are no exception.

An American put option is a contract which gives the holder a right to sell an underlying asset at any time before an expiration date for a certain price. For an American option, the optimal exercise strategy plays an important role in both the pricing process and its actual application. In the Black-Scholes framework, an American put option is modelled as a free boundary problem of a variational
inequality, where the free boundary is the optimal exercise boundary. As there is no closed form solution for American option pricing, numerical calculations are the only available approaches to pricing the options, as well as to seeking the optimal exercise boundaries.

By PDE methods, Liang, Hu, Jiang and Bian obtained in [10] the explicit finite difference scheme (EFDS) convergence rate estimate $O((\Delta x)^{2/3})$ for the value function of the American put option. The rate estimate is improved to $O(\Delta x)$ in [6] by Hu, Liang and Jiang, which is optimal. In [6], the convergence rate of the free boundary with a modified definition is also considered. However, the proposed scheme is difficult to implement because the definition of the modified free boundary includes an apriori unknown constant. To overcome a technical difficulty, in [11] a new definition of a slightly shifted free boundary for a more general model with jump is given and it is also proved that this slightly shifted free boundary is actually within an error of $\sqrt{\Delta x}$ to the actual free boundary.

In this paper, we use the optimal convergence rate for the value function to study the convergence rate of the free boundary. We consider a generic finite difference scheme and propose a new compatibility criterion. We shall establish an error estimates of $\sqrt{\Delta x}$ of the numerical free boundary to the actual free boundary. More properties of the numerical free boundaries are also obtained.

This paper is organized as follows: In Section 2, we introduce the model, the EFDS, and a criterion for compatibility of numerical schemes. In Section 3, some properties of the scheme are collected and proved. In Section 4 we prove a non-degeneracy result: the second order derivative of the true solution, as well as the numerical solution, experience a jump across the free boundary. In Section 5, we use the non-degeneracy and error estimate to prove our main result: the distance between the free boundary of the numerical solution and the true free boundary is $O(\sqrt{\Delta x})$. A numerical example is given in Section 6.

2. Modeling and numerical scheme. In the Black-Scholes theory, the underlying asset price $\{S_t\}_{t \geq 0}$ is assumed to be a geometric Brownian motion (cf. [5, 7]): $S_t = S_0 e^{\tilde{r} t + \sigma W_t}$, where $\tilde{r}$ is the growth rate of the underlying asset, $\sigma > 0$ measures the volatility, and $\{W_t\}_{t \geq 0}$ is the standard Brownian motion. Assume that the interest rate is a constant $r > 0$, strike price is $K$ and expiration date is $T$. It is well-known (c.f. [15, Ch. 7]) that the no arbitrage price of the American put option at time $t$ with asset price $S$ is the solution $V(S, t)$ of the variational inequality

\[
\begin{aligned}
\min \{\mathcal{L}[V], V - P\} = 0 & \quad \text{in } (0, \infty) \times (0, T), \\
V(\cdot, T) = P(\cdot) & \quad \text{on } (0, \infty) \times \{T\},
\end{aligned}
\]

(2.1)

where

\[
\mathcal{L}[V] = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, \quad P(S) = \max\{K - S, 0\}.
\]

It is well-known that no closed-form solutions are available for the variational inequality (2.1) (e.g. see [4]). In 1979, Cox and Rubinstein (see [3]) was the first to propose the Binomial Tree Method (BTM), a numerical scheme derived directly from the Black-Scholes theory by replacing the Brownian motion by (discrete) random walk; this is one of the most popular approaches for numerical evaluation and theoretical investigation of many mathematical finance problems, including the American put option.
Under the dimensionless variables
\[ u(x, t) = \frac{V(S, \tau)}{K}, \quad x = \log \frac{S}{K}, \quad t = \frac{\sigma^2}{2}(T - \tau), \quad \mu = \frac{2r}{\sigma^2}, \]
problem (2.1) is transformed to
\[
\begin{align*}
\min \{ L[u], u - g \} &= 0 & \text{in } \mathbb{R} \times (0, \infty), \\
u(x, 0) &= g^+(x), & \text{on } \mathbb{R} \times \{0\},
\end{align*}
\]
where \( g^+ = \max\{g, 0\} \) and
\[ L[\phi] = \phi_t - \phi_{xx} - (\mu - 1)\phi_x + \mu\phi, \quad g(x) = 1 - e^x. \]
Here we remark that the solution satisfies \( u \geq 0 \) so \( \min \{ L[u], u - g^+ \} = 0. \)

Denote by \( h \) the time mesh step and \( k \) the space mesh size, we claim that a generic EFDS for (2.2) can be written as
\[
\begin{align*}
u^h(x, t) &= \max\{B[u^h(\cdot, t - h)](x), g(x)\}, & x \in \mathbb{R}, t \in (0, \infty), \\
v^h(x, t) &= g^+(x), & x \in \mathbb{R}, t \in (-h, 0],
\end{align*}
\]
where \( B \) is defined by
\[ B[\psi](x) = a_0\psi(x) + a_+\psi(x + k) + a_-\psi(x - k), \]
where \( a_0, a_+, a_- \) are constants depending on specific discretization. We illustrate this with the following examples:

**Example 1.** In [6], the following discretization is used:
\[
Lu \approx \frac{u(x, t + h) - u(x, t)}{h^2} - \frac{u(x + k, t) - 2u(x, t) + u(x - k, t)}{k^2} - (\mu - 1)\frac{u(x + k, t) - u(x - k, t)}{2k} + \mu u(x, t + h).
\]
This corresponds to
\[
a_\pm = \frac{h}{k^2} \frac{1 \pm \frac{1}{2}(\mu - 1)k}{(1 + \mu h)}, \quad a_0 = \frac{k^2 - 2h}{k^2(1 + \mu h)}. \tag{2.5}
\]

**Example 2.** A binomial tree model (BTM) corresponds to \( a_0 = 0 \), e.g.,
\[
a_0 = 0, \quad a_\pm = \frac{1 \pm \frac{1}{2}(\mu - 1)k}{2(1 + \mu h)}, \quad h = \frac{1}{2}k^2. \tag{2.6}
\]

**Example 3.** Writing \( L \) as \( L[\phi] = \phi_t - e^{-\mu x}[e^{(\mu + 1)x}(e^{-x}\phi)_x]_x, \)
\[
Lu \approx \frac{u(x, t + h) - u(x, t)}{h^2} - \frac{1}{k^2} \left( e^{\frac{\mu + 1}{k}x}u(x + k, t) - e^{\frac{\mu - 1}{k}x}u(x - k, t) \right) \left( e^{\frac{\mu - 1}{k}x}u(x + k, t) - e^{\frac{\mu + 1}{k}x}u(x - k, t) \right)
\]
This gives the numerical scheme (2.3), (2.4) with
\[
a_0 = 1 - \frac{2h}{k^2} \cosh \left( \frac{\mu + 1}{2}k \right), \quad a_\pm = \frac{h}{k^2} e^{\frac{\mu - 1}{2}k}. \tag{2.7}
\]

For a general EFDS, we require the Courant-Friedrichs-Lax condition
\[ a_0 \geq 0, \quad a_+ \geq 0, \quad a_- \geq 0. \]
For the compatibility of an EFDS, we propose the following new criterion. First note that the linear equation \( \phi_t = \phi_{xx} + (\mu - 1)\phi_x - \mu \phi \) has three solutions, \( e^x, \ e^{-\mu x}, \ e^{-\mu t} \).

Clearly, these three special solutions determine completely and characterize fully the corresponding operator \( L \). We denote by \( e^{\lambda x}, \ e^{-\nu x}, \ e^{-\gamma t} \) the special solutions of \( \phi(x, t + h) = B[\phi(\cdot, t)](x) \). Then \( (\lambda, \nu, \gamma) \) are unique roots of

\[
\begin{cases}
\lambda = \frac{1}{k} \ln \frac{(1 - a_0)^2 - 4a_+ a_- + (1 - a_0)}{2a_+}, \\
\nu = \frac{1}{k} \ln \frac{(1 - a_0)^2 - 4a_+ a_- + (1 - a_0)}{2a_-}, \\
\gamma = \frac{1}{h} \ln \frac{1}{a_0 + a_+ + a_-}.
\end{cases}
\]

To obtain expected exponents \( (\lambda, \nu, \gamma) \), it is necessary and sufficient to take

\[
a_0 = 1 - (1 - e^{-\gamma h}) \cosh \frac{(\nu + \lambda)k}{2} - e^{2\nu k} \sinh \frac{\lambda k}{2} \sinh \frac{\nu k}{2}, \quad a_\pm = (1 - e^{-\gamma h}) e^{\pm \frac{1}{2}(\nu - \lambda)k} \frac{4 \sinh \frac{\lambda k}{2} \sinh \frac{\nu k}{2}}{4 \sinh \frac{\lambda k}{2} \sinh \frac{\nu k}{2}}.
\]

We say that the discretization scheme (2.3), (2.4), (2.10) is compatible if

\[
\lim_{h \to 0, k \to 0} (\lambda, \nu, \gamma) = (1, \mu, \mu).
\]

Clearly, such definition of compatibility can be extended to higher order and operators in multi-space dimension.

In the current situation, the CFL stability condition can be written as

\[
h \leq \frac{1}{\mu} \ln \frac{1 + e^{(\nu + \lambda)k}}{e^{\nu k} + e^{\lambda k}}.
\]

If we use discretization (2.7) we have

\[
\lambda = 1, \quad \nu = \mu, \quad \gamma = \frac{1}{h} \ln \left( 1 - \frac{4h}{k^2} \sinh \frac{\mu k}{2} \sinh \frac{k}{2} \right) = \mu + O(h + k^2).
\]

If we use discretization (2.5) or (2.6), we can show that

\[
\gamma = \mu + O(h), \quad \lambda = 1 + O(k^2), \quad \nu = \mu + O(k^2).
\]

We define the Modified Binomial Tree Model (MBTM) by setting \( (\lambda, \mu, \gamma) = (1, \mu, \mu) \) and \( a_0 = 0 \), i.e.

\[
a_0 = 0, \quad a_+ = \frac{1}{e^{\lambda k} + e^{-\lambda k}}, \quad a_- = \frac{1}{e^{\nu k} + e^{-\nu k}}, \quad h = \frac{1}{\mu} \ln \frac{1 + e^{(1+\mu)k}}{e^{\lambda k} + e^{\nu k}}.
\]

Clearly, the difference between our MBTM and the classical Cox-Rubinstein’s BTM is the same as the difference between continuously compounded interest rate and simply compounded interest rate.
Remark 2.1. It has been proved that BTM is equivalent to EFDS with \( h = \frac{1}{2} k^2 \); for European option, see Xu, Qian and Jiang [14]; for American option, see Qian, Xu, Jiang and Bian [13]. Jiang and Dai [8, 9] studied American options on BTM in a partial differential equation framework, and proved the uniform convergence of the BTM for American options and their approximated free boundaries in the sense of viscosity solutions (without convergence rate).

In the sequel, we shall assume that \( B \) is given by (2.4) with \((a_0, a_+, a_-)\) given by (2.10). We always assume that the stability condition (2.12) and the compatibility condition (2.11) are satisfied.

3. Preliminary. It is well-known that the variational problem (2.2) is well-posed. We quote the following for readers’ convenience:

Lemma 3.1. The variational inequality (2.2) admits a unique solution and the solution satisfies \( u_0 \geq 0 \). In addition, define the optimal exercise boundary \( x = s(t) \) by

\[
    s(t) = \inf \{ x \mid u(x,t) > g(x) \} \quad \forall t > 0. \tag{3.1}
\]

Then \( u(x,t) > g^+(x) \) when \( x > s(t) \) and \( u(x,t) = g(x) \) when \( x \leq s(t) \). Moreover, \( s(\cdot) \in C^\infty((0, \infty)), s' < 0 < s'' \) and

\[
    \lim_{t \to 0^+} \left( \frac{s(t)}{\sqrt{2t}} + \sqrt{-\ln(4\pi \mu^2 t)} \right) = 0, \quad \lim_{t \to \infty} s(t) = -\frac{\mu}{\mu + 1}.
\]

These results can be found in [1] and [2]; see also [12] and [7, §6.5].

Before prove our convergence result, we state a comparison principle.

Lemma 3.2 (Comparison Principle). Let \( u^h \) be the solution of (2.3) where \( B \) is defined by (2.4) with non-negative constants \( a_0, a_+, a_- \). Suppose \( \tilde{u} \) satisfy

\[
    \begin{cases}
    \tilde{u}(x,t) - \max \{ B[\tilde{u}(\cdot, t-h)], g \} \geq 0 \quad (or \leq 0) & \text{in } \mathbb{R} \times (0, \infty), \\
    \tilde{u}(x,t) - g^+(x) \geq 0 \quad (or \leq 0) & \text{in } \mathbb{R} \times (-h, 0),
    \end{cases} \tag{3.2}
\]

Then \( \tilde{u} - u^h \geq 0 \) \((or \leq 0)\) in \( \mathbb{R} \times (-h, \infty) \).

The proof follows by a mathematical induction on \( n \) for \( t = nh \).

Proposition 1 (Convergence Theorem). Let \( u^h \) be solution of (2.3) where \( B \) is given by (2.4) and (2.10). Assume (2.12) and

\[
    \lambda = 1 + O(h), \quad \nu = \mu + O(h), \quad \gamma = \mu + O(k). \quad \tag{3.3}
\]

Then the error to the solution of problem (2.2) is \( O(k) \), i.e.

\[
    |u - u^h| < C_0 k, \quad \tag{3.4}
\]

where \( C_0 \) is a positive constant which depends only on the given data.

Proof. Our finite difference operator is slight different from the one in [6]. Here we only need to show that our solution \( u^h \) is \( O(k) \) distance away from the solution \( \tilde{u}^h \) of the finite difference solution given in [6]:

\[
    \begin{cases}
    \tilde{u}^h(\cdot, t+h) = \max \{ B[\tilde{u}^h(\cdot, t-h)], g \} \quad \text{in } \mathbb{R} \times (0, \infty), \\
    \tilde{u}^h(\cdot) = g^+ \quad \text{in } \mathbb{R} \times (-h, T),
    \end{cases} \tag{3.5}
\]
where $\tilde{B}$ is given by (2.4) and (2.5). We denote the constants by $(\tilde{a}_0, \tilde{a}_1)$. Our assumption on $(a_0, a_1)$ implies that

$$|\gamma - \tilde{\gamma}| = O(k), \quad |a_+ - \tilde{a}_+| + |a_- - \tilde{a}_-| = O(h).$$

For any bounded and Lipschitz in $x$ continuous function $\phi$, denote $L_h\phi = \phi(\cdot, t) - B[\phi(\cdot, t - h)]$ and $\tilde{L}_h\phi = \phi(\cdot, t) - \tilde{B}[\phi(\cdot, t - h)]$. Using $a_0 + a_+ + a_- = e^{-\gamma h}$ and $\tilde{a}_0 + \tilde{a}_+ + \tilde{a}_- = e^{-\gamma h}$ we obtain

$$L_h[\phi] - \tilde{L}_h[\phi] = (e^{-\gamma h} - e^{-\gamma h})\phi(x, t) + [\tilde{a}_+ - a_+]|\phi(x + k, t) - \phi(x, t)| + |\tilde{a}_- - a_-|\|\phi(x - k, t) - \phi(x, t)\| = O(1)hk\|\phi\|_{L^\infty} + O(1)hk\|\phi_x\|_{L^\infty} = O(1)hk.$$

Now we show that $\tilde{u}_h + Ck$ is an upper solution of problem (2.3). One can show that $\|\tilde{u}_h^k\|_{L^\infty} \leq \|g_x\|_{L^\infty} \leq 1$ and $\|u_x^k\|_{L^\infty} \leq 1$ (c.f. (3.6) below). Since $\tilde{L}_h[\tilde{u}_h] \geq 0$, we have

$$L_h[\tilde{u}_h + Ck] = L_h[\tilde{u}_h] + L_h[Ck] \geq L_h[\tilde{u}_h^k] - \tilde{L}_h[\tilde{u}_h] + Ck(1 - e^{-\gamma h}) = O(hk) + Ck(1 - e^{-\gamma h}) > 0,$$

if we take $C$ large enough (independent of $h$). Thus by comparison, $\tilde{u}_h + Ck \geq u^h$. Similarly, $u^h + Ck \geq \tilde{u}_h^k$. Hence, $|u^h - \tilde{u}_h^k| = O(k)$. It is proven in [6] that $|\tilde{u}_h - u| = O(k)$. Hence we have $|u - u_h| \leq |u - \tilde{u}_h| + |u_h - \tilde{u}_h| = O(k)$. \hfill $\square$

**Proposition 2.** The solution $u^h$ of problem (2.3) satisfies, for each $\delta > 0$,

$$u^h(x, t + \delta) \geq u^h(x, t) \quad u^h(x, t) \geq u^h(x + \delta, t) \geq u(x) - (1 - e^{-\delta}). \quad (3.6)$$

**Proof.** The monotone increasing property of $u$ in $t$ follows from the comparison principle first for $0 < \delta \leq h$, and then step-by-step for all $\delta > 0$. Similarly, one can show that $u$ is decreasing in $x$. To prove the second inequality in (3.6), we first note that

$$\min_{x \in \mathbb{R}^1} \{g^+(x + \delta) - g^+(x)\} = -\max_{x < -\delta} e^x(e^{\delta} - 1) \geq -(1 - e^{\delta}).$$

The function $w(x, t) = u^h(x + \delta, t) + (1 - e^{\delta})$ satisfies

$$w(x, t) = \max\{g^+(x + \delta), B[u(\cdot, t - h)](x + \delta)\} + (1 - e^{-\delta}) \geq \max\{g^+(x, B[w(\cdot, t - h)]\} \quad \forall x \in \mathbb{R}, t > 0.$$

It is also clear that,

$$w(x, t) = g^+(x + \delta) + (1 - e^{-\delta}) \geq g^+(x) = u^h(x, t) \quad \text{for } t \in (-h, 0].$$

It then follows by comparison principle that $w \geq u^h$.

We now define the approximate free boundary $x = s^h(t)$.

**Definition 3.1.** Let $u^h$ be the solution of (2.3). Define

$$s^h(t) = \inf\{x \mid u^h(x, t) > g(x)\} \quad \forall t > 0. \quad (3.7)$$

**Proposition 3.** For every positive $t$ and $\delta$, $s^h(t) \geq s^h(t + \delta) > \ln \frac{1 - e^{-\nu_k}}{1 - e^{-\nu_k} e^{-\epsilon}}$.

**Proof.** The monotonicity of $s^h$ follows from the monotonicity of $u^h$ in $t$. For boundedness, we argue as follows. Since $u^h_0 \geq 0$ and $u^h_0 \leq 0$, the limit

$$(u^h_0(x), s^h_0) = \lim_{t \to \infty} (u^h(x, t), s^h(t))$$

for $0 < x < T$.
Proof. Let \( t > \) boundary defined in \((s^h_n, \infty)\) with bounded solution implies that
\[
u_n(s^h_n + nk + \vartheta) = A(\vartheta)e^{-\nu(s^h_n + nk + \vartheta)}, \quad \forall \vartheta \in [0, k), \quad n = 0, 1, 2, \ldots.
\]
Using \( u_n(s^h_k - k + \vartheta) = g(s^h_k - k + \vartheta), \quad \forall \vartheta \in [0, k), \) and (2.8), we derive
\[
A(\vartheta)e^{-\nu(s^h_k + \vartheta)}(1 - a_0) - a_+ e^{-\nu k} = a_-(1 - e^{-s^h_n-k+\vartheta}), \quad \forall \vartheta \in [0, k),
\]
then \( A(\vartheta) = e^{-\nu k}e^{\nu(s^h_k + \vartheta)}(1 - e^{-s^h_n+\vartheta-k}), \quad \forall \vartheta \in [0, k). \) Thus \( s^h_n = \ln \frac{1 - e^{-\nu k}}{1 - e^{-s^h_n+k-\vartheta}}. \)

4. Non-degeneracy property. At the free boundary \( x = s(t), \) \( u - g = 0 \) and \((u - g)_x = 0 \) (indeed also \((u - g)_t = 0 \)). The following shows that \( u - g \) grows quadratically as \( x \) moves to the right from the free boundary; we call it the non-degeneracy property.

Proposition 4. Let \( u \) be the solution of (2.2), \( x = s(t) \) be the free boundary defined in (3.1), and \( \hat{\mu} := \min\{1, \mu\}/2. \) Then
\[
u(x, t) - g(x) \geq \hat{\mu} [x - s(t)]^2 \quad \forall x \geq s(t), \quad t > 0. \tag{4.1}
\]
Proof. Let \( t > 0 \) be fixed. Set
\[
\phi(y) = \frac{\mu e^y + e^{-\mu y}}{\mu + 1} - 1, \quad w(x) = u(x, t) - g(x) - \phi(x - s(t)).
\]
Then \( w(s(t)) = 0, w_x(s(t)) = 0, \) and for \( x > s(t), \)
\[
w''(x) + (\mu - 1)w' - \mu w = u_{xx} + (\mu - 1)u_x - \mu u = u_t \geq 0.
\]
This implies that \( e^{-\mu x}[e^{(\mu+1)x}e^{-x} w]_x \geq 0. \) After integration we obtain \( w(x) \geq 0 \) for every \( x \geq s(t). \) Thus, \( u(x, t) - g(x) \geq \phi(x - s(t)) \) when \( x \geq s(t). \) As \( \phi(0) = 0, \phi'(0) = 0 \) and \( \phi''(y) \geq 2\hat{\mu} \) for \( y > 0, \) we have \( \phi(y) \geq \hat{\mu}y^2 \) for \( y > 0. \) The assertion of the Lemma thus follows.

Proposition 5. Let \( u^h \) be the solution of (2.3), \( x = s^h(t) \) be the approximate free boundary defined in (3.7), \( \hat{\nu} = \lambda \min\{\lambda, \nu\}/2 \) and \( \Lambda = 2(1 + \lambda + \nu) e^{(\lambda-1)} |s^h_0|. \) Then for each \( t > 0 \) and \( x = s^h(t + h) + nk \leq 0 \) with integer \( n \geq 1, \)
\[
u^h(x, t) - g(x, t) \geq (\hat{\nu} - |1 - \lambda|\lambda)(x - s^h(t + h))^2. \tag{4.2}
\]
Proof. Let \( t > 0 \) be fixed. Define
\[
\phi(y) := \frac{\nu e^{\lambda y} + \lambda e^{-\nu y}}{\lambda + \nu} - 1, \\
\quad w(y) := u^h(s^h(t + h) + y, t) - [1 - e^{-s^h(t+h)+\lambda y}] - \phi(y), \\
\quad J(y) := e^{-\nu y + \lambda y}[e^{\lambda}w(y) - e^{-\lambda(y-k)}w(y-k)].
\]
Since \( s^h(t) \geq s^h(t + h) \) we have \( u^h(s^h(t + h) + y, t) = g(s^h(t + h) + y) = 1 - e^{-s^h(t+h)+y} \) for every \( y \leq 0. \) Hence,
\[
w(0) = 0, \quad J(0) > e^{s^h(t+h)}[e^{(\lambda-1)k} - 1]. \tag{4.3}
\]
For \( x \geq s^h(t + h), \) \( u^h(x, t) \leq u^h(x, t + h) = B[u(, t)](x), \) so for \( y \geq 0, \)
\[
B[w](y) - w(y) = B[u^h(, t)](s^h(t + h) + y) - u^h(s^h(t + h) + y, t) \geq 0.
\]
Using the definition equations of \( \lambda \) and \( \nu \) in (2.8), we have

\[
a_+ = \frac{(1 - a_0)e^{\nu k}}{1 + e^{\lambda k + \nu k}} > 0, \quad a_- = \frac{(1 - a_0)e^{\lambda k}}{1 + e^{\lambda k + \nu k}} > 0.
\]

Consequently, for \( y \geq 0 \),

\[
J(y + k) - J(y) = \frac{1 + e^{\lambda k + \nu k}}{1 - a_0}e^{-\nu y} \left\{ B[w](y) - w(y) \right\} \geq 0.
\]

The implies that \( J(y + k) \geq J(y) \) for every \( y \geq 0 \). In particular, for every integer \( i \geq 0 \), \( J(ik) \geq J(0) \), so by the definition of \( J \),

\[
e^{-\lambda k}w(ik) - e^{-\lambda(i-1)k}w((i-1)k) \geq e^{-(\lambda+\nu)ik} J(0).
\]

Adding up and using (4.3) we then obtain, for every integer \( n \geq 1 \),

\[
w(nk) \geq e^{\lambda nk} \sum_{i=1}^{n} e^{-(\lambda+\nu)ik} J(0) = \frac{e^{\lambda nk} - e^{-\nu nk}}{e^{(\lambda+\nu)k} - 1} J(0)
\]

\[
\geq \frac{\left[ e^{\lambda nk} - e^{-\nu nk} \right] e^{s^h(t+k)}}{e^{(\nu+\lambda)k} - 1} \left( e^{(\lambda-1)k} - 1 \right).
\]

It then follows from the definition of \( w \) that for \( y = nk \) with integer \( n \geq 0 \),

\[
u^k(s^h(t+h) + y, t) - g(s^h(t+h) + y) - \phi(y) = w(y) + [1 - e^{s^h(t+h) + \lambda y}] - g(s^h(t+h) + y)
\]

\[
\geq \frac{\left[ e^{\lambda y} - e^{-\nu y} \right] e^{s^h(t+h)}}{e^{(\lambda+\nu)k} - 1} \left( e^{(\lambda-1)k} - 1 \right) + e^{s^h(t+h)}y[1 - e^{(\lambda-1)y}]
\]

\[
= e^{s^h(t+h)}y \left\{ e^{\vartheta_1 + \vartheta_2 - \vartheta_3} - e^{\vartheta_4} \right\} \left( \lambda - 1 \right) y
\]

\[
\geq -|\lambda - 1| (1 + \lambda + \nu) e^{s^h(t+h) + y + |\lambda - 1| y} y(y + k),
\]

by the Mean value theorem, where

\[
\vartheta_1 \in [-\nu + 1, y], \quad \vartheta_2 \in [\min\{\lambda - 1, 0\} k, \max\{0, \lambda - 1\} k],
\]

\[
\vartheta_3 \in [0, (\lambda + \nu) k], \quad \vartheta_4 \in [\min\{0, \lambda - 1\} y, \max\{0, \lambda - 1\} y].
\]

Finally, since \( \phi(0) = \phi'(0) = 0 \) and \( \phi''(y) \geq \lambda \min\{\lambda, \nu\} = 2\nu \) for \( y \geq 0 \), we have \( \phi(y) \geq \nu y^2 \) for \( y > 0 \). Thus, for \( x = s^h(t + h) + y \leq 0 \) with \( y = nk \) for some integer \( n \geq 1 \) we have

\[
u^k(x, t) - g(x) \geq \phi(y) - |\lambda - 1|(1 + \lambda + \nu) e^{x + |\lambda - 1| y} y(y + k).
\]

\[
\geq \left( \nu - 2|\lambda - 1|(1 + \nu + \mu) e^{(\lambda-1)y} \right) y^2.
\]

This completes the proof.

\[\square\]

5. **Convergence rate of the free boundary.** Here we prove the convergence rate of the approximate free boundary \( x = s^h(t) \) to the actual free boundary \( x = s(t) \).

**Theorem 1** (Main Theorem). Assume (2.7) and (3.3). Then the difference of the approximate free boundary \( s^h(t) \) and the real one \( s(t) \) has the estimate:

\[
|s^h(t) - s(t)| \leq C_1 \sqrt{k}, \quad (5.1)
\]

for some positive constant \( C_1 \) which does not depend on \( h \) and \( k \).
Proof. Fix $t > 0$. If $s^h(t) \geq s(t)$, then by Proposition 1, definition of $s^h$, and Proposition 4, we obtain
\[
C_0 k \geq u(s^h(t), t) - u^h(s^h(t), t) = u(s^h(t), t) - g(s^h(t)) \geq \hat{\mu}(s^h(t) - s(t))^2.
\]
This implies that $s^h(t) \leq s(t) + \sqrt{C_0/\hat{\mu}} k$.

On the other hand, if $s^h(t) < s(t) - k$, then, as $s^h(t + h) \leq s^h(t)$, we can write $s(t) = s^h(t + h) + y + \vartheta k$ with $\vartheta \in [0, 1)$ and $y = nk$ for some integer $n \geq 1$. By Proposition 1, definition of $s$, and Proposition 5 we have
\[
C_0 k \geq u^h(s^h(t + h) + y, t) - u(s^h(t + h) + y, t) = u^h(s^h(t + h) + y, t) - g(s^h(t + h) + y) \geq \{\hat{\nu} - |1 - \lambda|\sqrt{k}\} y^2.
\]
Therefore $|y| \leq \sqrt{C_0/|\hat{\nu} - |1 - \lambda|\sqrt{k}}$. This implies that $s(t) - s^h(t) \leq s(t) - s^h(t + h) \leq y + k \leq k + \sqrt{C_0/|\hat{\nu} - |1 - \lambda|\sqrt{k}}$.

Combining both cases, we obtain the assertion of the theorem. \qed

6. A numerical example. We give a numerical example for Example 3. Here we choose $\mu = 0.5$ and consider the solution in time interval $[0, 0.1]$. Calculation interval is $[-5, 5]$. As an American put option has no closed form solution, we treat the approximation solution for $h = h^* = \Delta t = 1.5 \times 10^{-7}$, as “exact solution” and $S^h(t)$ to be a benchmark for comparing to the others.

| $h = \Delta t$      | $2.5 \times 10^{-7}$ | $6.25 \times 10^{-7}$ | $1.56 \times 10^{-7}$ | $3.9 \times 10^{-7}$ | $9.8 \times 10^{-7}$ |
|---------------------|----------------------|-----------------------|-----------------------|----------------------|----------------------|
| Max Error           | 0.038                | 0.016                 | 0.0084                | 0.0045               | 0.0022               |

Figure 1. The free boundary $x = S^h(t)$ for different $h$. 
We found that the errors are smaller than $h^{1/4}$. According to the theoretical result, the error is bounded by $O(h^{1/4})$, and this is mostly attributed to the singularity of the solution near $(x,t) = (0,0)$. On the scheme, the first step is already of size $h$ away from 0.

The corresponding figures of approximated free boundary with different “$h$”s vs time $t$ are shown in the Figure 1.

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