Adding a uniton via the DPW method

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Abstract

In this paper we describe how the operation of adding a uniton arises via the DPW method of obtaining harmonic maps into compact Riemannian symmetric spaces out of certain holomorphic one forms. We exploit this point of view to investigate which unitons preserve finite type property of harmonic maps. In particular, we prove that the Gauss bundle of a harmonic map of finite type into a Grassmannian is also of finite type.

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1 Introduction

In [12], Uhlenbeck observed that, in the setting of harmonic maps of $\mathbb{R}^2$ into a compact Riemannian symmetric space $G/K$, the harmonic map equations amount to the flatness of a family of connections depending on an auxiliary parameter $\lambda \in S^1$. This zero-curvature formulation yields an action of a certain loop group on the space of harmonic maps; underlying this dressing action is the existence of Iwasawa-type decompositions of the loop groups and loop algebras concerned (cf. [1, 3, 7, 8, 12].)

In [7], the authors introduced a systematic procedure of obtaining harmonic maps into a compact Riemannian symmetric space $G/K$ from holomorphic 1-forms with values in the subspace

$$\Lambda_{-1, \infty} = \{ \xi \in \mathfrak{g}^\mathbb{C} | \lambda \xi \text{ extends holomorphically to } |\lambda| < 1 \},$$

where $\mathfrak{g}^\mathbb{C}$ denotes the Lie algebra of $G^\mathbb{C}$, and proved that this correspondence between $\Lambda_{-1, \infty}$-valued holomorphic 1-forms $\mu$, the holomorphic potentials, and harmonic maps is equivariant with respect to loop group actions by dressing and by gauge transformations on the 1-forms $\mu$.

Another well known operation for generating new harmonic maps from a given one was introduced by Uhlenbeck [12] and is called adding a uniton. In the case of harmonic maps into Grassmannians, this procedure corresponds to the forward replacements and backward replacements of Burstall and Wood [6].
In this paper we describe how the operation of adding a uniton arises via gauge transformations on the holomorphic 1-forms $\mu$ and we exploit this point of view to investigate which unitons preserve finite type property. Recall that harmonic maps of finite type are obtained by integrating a pair of commuting Hamiltonian vector fields on certain finite-dimensional subspaces of loop algebras (cf. [2, 4]) and play a fundamental role in the genus one case, i.e., harmonic maps from a two-torus – for example, it was shown in [2] that any non-conformal harmonic map of a two-torus into a rank one symmetric space $G/K$ is of finite type. In particular, we prove that the Gauss bundle of a harmonic map of finite type into a Grassmannian is also of finite type. Finally we show that these unitons preserving finite type can be added by taking a limit of dressing transformations as in the completion procedure studied by Bergvelt and Guest in [1]. To obtain these results one has to enlarge the loop groups involved in DPW procedure, as in [3], and the space of holomorphic potentials $\mu$ (see Section 4).

2 Extended solutions

We start by summarizing briefly the relevant definitions and results concerning the well known correspondence between harmonic maps and extended solutions, referring the reader to the seminal paper [12] for details.

Let $G$ be a compact (connected) semisimple matrix Lie group, with identity $e$ and Lie algebra $\mathfrak{g}$. Equip $G$ with a bi-invariant metric. Let $G^{\mathbb{C}}$ be the complexification of $G$, with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (thus $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$). Consider the based loop group

$$\Omega G = \{ \gamma : S^1 \to G \ (\text{smooth}) \ | \ \gamma(1) = e \}$$

and the corresponding infinite-dimensional Lie algebra

$$\Omega \mathfrak{g} = \{ \gamma : S^1 \to \mathfrak{g} \ (\text{smooth}) \ | \ \gamma(1) = 0 \}.$$ 

A smooth map $\Phi : \mathbb{C} \to \Omega G$ is called an extended solution if it satisfies

$$\Phi^{-1} d\Phi = (1 - \lambda^{-1}) \alpha' + (1 - \lambda) \alpha''$$

for each $\lambda \in S^1$, where $\alpha'$ is a $\mathfrak{g}^{\mathbb{C}}$-valued $(1,0)$-form on $\mathbb{C}$ with complex conjugate $\alpha''$. Observe that, for each $z \in \mathbb{C}$, $\lambda \mapsto \Phi(z)(\lambda)$ is holomorphic on $\mathbb{C}^*$.

**Theorem 1.** [12] a) If $\Phi : \mathbb{C} \to \Omega G$ is an extended solution, then the map $\phi : \mathbb{C} \to G$ defined by $\phi(z) = \Phi(z)(-1)$ is harmonic. b) If $\phi : \mathbb{C} \to G$ is harmonic, then there exists an extended solution $\Phi : \mathbb{C} \to \Omega G$ such that $\phi(z) = \Phi(z)(-1)$, for all $z \in \mathbb{C}$. This is unique up to multiplication on the left by an element $\gamma \in \Omega G$ such that $\gamma(-1) = e$.

**Remark 1.** Let $G/K$ be a symmetric space with automorphism $\tau$ and base point $x_0 = eK$. Define a map $\iota : G/K \to G$ by $\iota(g \cdot x_0) = \tau(g)g^{-1}$. It is well known that $\iota$ is a totally geodesic embedding, the Cartan embedding of $G/K$ into $G$, so that if $\varphi : \mathbb{C} \to G/K$ is harmonic then $\iota \circ \varphi : \mathbb{C} \to G$ is also, and we can apply Theorem 1 to $\iota \circ \varphi$. 
3 Holomorphic potentials and extended framings

In [1] the authors introduced a systematic procedure of obtaining harmonic maps into a symmetric space from certain holomorphic 1-forms. The main ingredients are the existence of various loop factorizations and the concept of extended framing. Next we will recall this construction.

Let $N = G/K$ be a symmetric space with automorphism $\tau$ and associated symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let $\phi : \mathbb{C} \to N$ be a smooth map and take a lift $\psi : \mathbb{C} \to G$ of $\phi$, that is, we have $\phi = \pi \circ \psi$ where $\pi : G \to G/K$ is the coset projection. Corresponding to the symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ there is a decomposition of $\alpha = \psi^{-1}d\psi$, $\alpha = \alpha_\ell + \alpha_m$. Let $\alpha_m = \alpha'_m + \alpha''_m$ be the type decomposition of $\alpha_m$ into $(1,0)$-form and $(0,1)$-form of $\mathbb{C}$.

Consider the loop of 1-forms $\alpha_\lambda = \lambda^{-1}\alpha'_m + \alpha_\ell + \lambda\alpha''_m$. We may view $\alpha_\lambda$ as a $\Lambda_\tau\mathfrak{g}$-valued 1-form, where

$$\Lambda_\tau\mathfrak{g} = \{ \xi : S^1 \to \mathfrak{g} \text{ (smooth)} \mid \tau(\xi(\lambda)) = \xi(-\lambda) \text{ for all } \lambda \in S^1 \}.$$  \hspace{1cm} (2)

It is well known that $\phi$ is harmonic if, and only if, $d + \alpha_\lambda$ is a loop of flat connections on the trivial bundle $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^n$. Hence, if $\phi$ is harmonic, we can define a smooth map $\Psi : \mathbb{C} \to \Lambda_\tau G$, where $\Lambda_\tau G$ is the infinite-dimensional Lie group corresponding to the loop Lie algebra $\mathfrak{g}$.

$$\Lambda_\tau G = \{ \gamma : S^1 \to G \text{ (smooth)} \mid \tau(\gamma(\lambda)) = \gamma(-\lambda) \text{ for all } \lambda \in S^1 \},$$

such that $\Psi^{-1}d\Psi = \alpha_\lambda$. The smooth map $\Psi$ is called an extended framing (associated to $\phi$). Our harmonic map is recovered from $\Psi$ via $\phi = \pi \circ \Psi_1$ (here we are using the notation $\Psi_1(z) = \Psi(z)(1)$).

Consider the subspace $\Lambda_{-1,\infty}$ as in [1].

**Definition 1.** A holomorphic 1-form $\mu$ on $\mathbb{C}$ with values in $\Lambda_{-1,\infty}$ is called a holomorphic potential. The space of all holomorphic potentials is denoted by $\mathcal{P}$. If $\mu \in \mathcal{P}$ takes values in $\Lambda_{-1,\infty}^+ = \Lambda_\tau \mathfrak{g}^C \cap \Lambda_{-1,\infty}$, we say that $\mu$ is a $\tau$-twisted holomorphic potential. The space of all $\tau$-twisted holomorphic potentials is denoted by $\mathcal{P}_\tau$. Thus $\mu = \sum_{k \geq -1} \lambda^k \mu^k \in \mathcal{P}_\tau$ if $\mu_{\text{even}}$ is a $\mathfrak{g}^C$-valued 1-form and $\mu_{\text{odd}}$ is a $\mathfrak{m}^C$-valued 1-form.

Fix an Iwasawa decomposition of the reductive group $K^C$: $K^C = KB$ where $B$ is a solvable Lie subgroup. Consider the following infinite-dimensional twisted Lie groups:

$$\Lambda_\tau G^C = \{ \gamma : S^1 \to G^C \text{ (smooth)} \mid \tau(\gamma(\lambda)) = \gamma(-\lambda) \text{ for all } \lambda \in S^1 \};$$
$$\Lambda_{B,\tau}^+ G^C = \{ \gamma \in \Lambda_\tau G^C \mid \gamma \text { extends holomorphically to } |\lambda| < 1; \gamma(0) \in B \}.$$  \hspace{1cm}

We have the following twisted loop group decomposition:

**Theorem 2.** [1] Multiplication $\Lambda_\tau G \times \Lambda_{B,\tau}^+ G^C \to \Lambda_\tau G^C$ is a diffeomorphism onto.

So, let $\mu$ be a $\tau$-twisted holomorphic potential; since $\mu$ is holomorphic, its $(0,1)$-part vanishes; then $\mu = \xi dz$ for some holomorphic function $\xi : \mathbb{C} \to \mathfrak{g}^C$ and it satisfies the Maurer-Cartan equation $d\mu + \frac{1}{2}[\mu \wedge \mu] = 0$, that is, $d\mu = d + \mu$ is a flat connection. We can integrate to obtain a unique map $\Psi_\mu : \mathbb{C} \to \Lambda_\tau G^C$ such that $\Psi_\mu^{-1}d\Psi_\mu = \mu$ and $\Psi_\mu(0) = e$. If we factorize $\Psi_\mu$ pointwise according to Theorem [2] we obtain a smooth map $\Phi_\mu : \mathbb{C} \to \Lambda_\tau G$ such that $\Psi_\mu = \Phi_\mu b$, with $b : \mathbb{C} \to \Lambda_{B,\tau}^+ G^C$.
Theorem 3. \[7\] \( \Phi_\mu : \mathbb{C} \to \Lambda_r G \) is an extended framing.

4 Holomorphic potentials and extended solutions

Since in our study of harmonic maps we will use extended solutions instead of extended framings and, as in \[3\], we will consider an action on the space of all extended solutions of germs at zero of maps \( \mathbb{C} \to G^C \), in this section we reformulate the DPW construction of harmonic maps according to our conveniences. In particular, we have to enlarge the space of holomorphic potentials. For completeness, we give detailed proofs of the main results.

Fix \( 0 < \varepsilon < 1 \). Let \( C_\varepsilon \) and \( C_{1/\varepsilon} \) denote the circles of radius \( \varepsilon \) and \( 1/\varepsilon \) centered at \( 0 \in \mathbb{C} \); define open subsets of \( \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \} \) by

\[ I_\varepsilon = \{ \lambda \in \mathbb{P}^1 \mid |\lambda| < \varepsilon \}, \quad I_{1/\varepsilon} = \{ \lambda \in \mathbb{P}^1 \mid |\lambda| > 1/\varepsilon \}, \quad E^\varepsilon = \{ \lambda \in \mathbb{P}^1 \mid \varepsilon < |\lambda| < 1/\varepsilon \}; \]

put \( I^\varepsilon = I_\varepsilon \cup I_{1/\varepsilon} \) and \( C^\varepsilon = C_\varepsilon \cup C_{1/\varepsilon} \) so that \( \mathbb{P}^1 = I^\varepsilon \cup C^\varepsilon \cup E^\varepsilon \). Consider the infinite-dimensional Lie groups

\[ \Lambda^\varepsilon G = \{ \gamma : C^\varepsilon \to G^C \mid \text{(smooth)} \mid \gamma(1/\lambda) = \gamma(1) / \lambda \} \]

\[ \Omega^\varepsilon_G = \{ \gamma \in \Lambda^\varepsilon G \mid \gamma \text{ extends holomorphically to } \gamma : E^\varepsilon \to G^C \} \]

and the corresponding infinite-dimensional Lie algebras

\[ \Lambda^\varepsilon g = \{ \gamma : C^\varepsilon \to g^C \mid \text{(smooth)} \mid \gamma(1/\lambda) = \gamma(1) / \lambda \} \]

\[ \Omega^\varepsilon_G = \{ \gamma \in \Lambda^\varepsilon g \mid \gamma \text{ extends holomorphically to } \gamma : E^\varepsilon \to g^C \} \]

To express the extended solution equation in terms of \( \Lambda^\varepsilon G \) we shall make use of the following Iwasawa-type decomposition of \( \Lambda^\varepsilon G \):

Theorem 4. \[9\] Multiplication \( \Omega^\varepsilon_E G \times \Lambda^\varepsilon_I G \to \Lambda^\varepsilon G \) is a diffeomorphism. In particular, any \( \gamma \in \Lambda^\varepsilon G \) may be written uniquely in the form \( \gamma = \gamma_E \gamma_I \), where \( \gamma_E \in \Omega^\varepsilon_E G \) and \( \gamma_I \in \Lambda^\varepsilon_I G^C \).

Remark 2. The limiting case of Theorem \[3\] as \( \varepsilon \to 0 \) is the more familiar decomposition \( \Omega G \times \Lambda^*_r G^C \to \Lambda^*_r G^C \) where \( \Lambda^*_r G^C = \{ \gamma : S^1 \to G^C \mid \gamma \text{ is smooth} \} \) and

\[ \Lambda^*_r G^C = \{ \gamma \in \Lambda^*_r G^C \mid \gamma \text{ extends holomorphically to } |\lambda| < 1 \}. \]

This result is due to Pressely-Segal \[11\].

We also have to enlarge our class of potentials: fix \( 0 < \varepsilon < 1 \) and consider the subspace of \( \Lambda^\varepsilon g \) defined by \( \Lambda^{\varepsilon,\infty} \in \{ \xi \in \Lambda^\varepsilon g \mid \lambda \xi \text{ extends holomorphically to } I_\varepsilon \} \); each element \( \xi \in \Lambda^{\varepsilon,\infty} \) can be written as \( \xi = (\xi_+, \xi_-) \), where \( \xi_+ : C_\varepsilon \to g^C \) extends meromorphically to \( I_\varepsilon \) with at most a simple pole at 0 and \( \xi_- : C_{1/\varepsilon} \to g^C \) is defined by \( \xi_-(\lambda) = \xi_+(1/\lambda) \).
Definition 2. A 1-form $\mu = (\mu_+, \mu_-)$ on $\mathbb{C}$ with values in $\Lambda_{-1,\infty}^\varepsilon$ such that $\mu_+$ is holomorphic is called a $\varepsilon$-holomorphic potential. The space of all $\varepsilon$-holomorphic potentials is denoted by $\mathcal{P}_\varepsilon$.

Remark 3. The space of holomorphic potentials $\mathcal{P}$ can be interpreted as the limiting case of Definition 2 as $\varepsilon \to 1$.

Let $\mu$ be a holomorphic $\varepsilon$-potential, so that $d\mu = d + \mu$ is a flat connection. This means that we can integrate to obtain a unique map $\Psi_\mu : \mathbb{C} \to \Lambda^\varepsilon G$, with $\Psi_\mu^{-1} d\Psi_\mu = \mu$ and $\Psi_\mu(0) = e$. We call $\Psi_\mu$ a complex extended solution. If we factorize $\Psi_\mu$ according to Theorem 4, we obtain a map $\Phi_\mu : \mathbb{C} \to \Omega_{\mathbb{C}}^\varepsilon G$ such that $\Psi_\mu = \Phi_\mu b$, with $b : \mathbb{C} \to \Lambda_1^\varepsilon G$. Note that, since $\Psi_\mu(0) = e$, we also have $\Phi_\mu(0) = e$ and $b(0) = e$.

Theorem 5. $\Phi_\mu : \mathbb{C} \to \Omega_{\mathbb{C}}^\varepsilon G \subset \Omega G$ is a extended solution.

Proof. With respect to the Iwasawa decomposition, the Lie algebra $\Lambda^\varepsilon g$ splits into a direct sum of Lie subalgebras:

$$\Lambda^\varepsilon g = \Omega_{\mathbb{C}}^\varepsilon g \oplus \Lambda_1^\varepsilon g.$$  \hspace{1cm} (3)

Since $\Phi_\mu = \Psi_\mu b^{-1}$, we have

$$\Phi_\mu^{-1} d\Phi_\mu = \text{Ad}_b(\mu) - db b^{-1}. \hspace{1cm} (4)$$

But $b$ takes values in $\Lambda_1^\varepsilon G$, that is

$$b(z) = b_0(z) + b_1(z) + b_2(z) + \ldots$$  \hspace{1cm} (5)

for all $\lambda \in I_\varepsilon$, so that $db b^{-1}$ takes values in $\Lambda_1^\varepsilon g$, then by (3) and (4)

$$\Phi_\mu^{-1} d\Phi_\mu = (\text{Ad}_b(\mu))_{\Omega_{\mathbb{C}}^\varepsilon g}. \hspace{1cm} (6)$$

Now, $\mu$ is a 1-form on $\mathbb{C}$ with values in $\Lambda_{-1,\infty}^\varepsilon$ and $\Lambda_1^\varepsilon G$ acts on $\Lambda_{-1,\infty}^\varepsilon$; in $C_\varepsilon$ we can write

$$\mu = \sum_{k \geq -1} \mu_k \lambda^k;$$

hence

$$(\text{Ad}_b(\mu))_{\Omega_{\mathbb{C}}^\varepsilon g} = (\lambda^{-1} - 1) \text{Ad}_{b_0}(\mu_{-1}) + (\lambda - 1) \overline{\text{Ad}_{b_0}(\mu_{-1})}. \hspace{1cm} (7)$$

Since $\mu_{-1}$ is a $g^\mathbb{C}$-valued holomorphic 1-form we can write $\mu_{-1} = \xi_{-1} dz$ for some holomorphic function $\xi_{-1} : \mathbb{C} \to g^\mathbb{C}$; hence

$$\Phi_\mu^{-1} d\Phi_\mu = (1 - \lambda^{-1}) d\alpha' + (1 - \lambda) d\alpha'', \hspace{1cm} (8)$$

with

$$\alpha' = -\text{Ad}_{b_0}(\mu_{-1}) = -\text{Ad}_{b_0}(\xi_{-1}) dz, \hspace{1cm} (9)$$

that is, $\Phi_\mu$ is an extended solution. \qed

Then any $\varepsilon$-holomorphic potential $\mu$ gives rise to a harmonic map $\phi_\mu : \mathbb{C} \to G$ with $\phi_\mu(z) = \Phi_\mu(z)(-1)$ and $\phi_\mu(0) = \Phi_\mu(0)(-1) = e$.

Remark 4. Again, by taking the limiting case of Theorem 5 as $\varepsilon \to 1$ we see that any holomorphic potential $\mu \in \mathcal{P}$ gives rise to a harmonic map $\phi_\mu : \mathbb{C} \to G$.  

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Remark 5. Set \( \alpha_\lambda = (1-\lambda^{-1})\alpha' + (1-\lambda)\alpha'' \). Let \( d = \partial + \bar{\partial} \) and \( d_{\alpha_\lambda} = \partial_{\alpha_\lambda} + \bar{\partial}_{\alpha_\lambda} \), respectively, be the type decompositions of the connections \( d \) (the trivial connection) and \( d_{\alpha_\lambda} = d + \alpha_\lambda \), respectively, on \( \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^n \). Considering the \((0,1)\)-parts of equations (4) and (5), we obtain \(-(1-\lambda)\alpha'' = \bar{\partial}b b^{-1} \). This means that the gauge transformation \( b \) gauges the holomorphic structure \( \bar{\partial}_{\alpha_\lambda} \) on \( \mathbb{C}^n \) to the trivial holomorphic structure \( \bar{\partial} \) on \( \mathbb{C}^n \), that is, \( b^{-1}\bar{\partial}_{\alpha_\lambda} b = \bar{\partial} \). In particular, \( b : (\mathbb{C}^n, \partial) \to (\mathbb{C}^n, \bar{\partial}_{\alpha_\lambda}) \) is a holomorphic isomorphism.

Remark 6. Denote by \( \tilde{P} \) the space of all 1-forms on \( \mathbb{C} \) satisfying the following conditions: a) \( \mu \) takes values in \( \Lambda_{-1,\infty} \); b) \( d\mu = d + \mu \) is a flat connection; c) the \((0,1)\)-part of \( \mu \) takes values in \( \Lambda_+G^\mathbb{C} \), the Lie algebra of \( \Lambda_+G^\mathbb{C} \). Again we can integrate and apply the Iwasawa decomposition of Remark 2 to obtain a smooth map \( \Phi : \mathbb{C} \to \Omega G \). Moreover, the arguments given to prove Theorem 5 carry over directly to this case to prove that \( \Phi \mu \) is also an extended solution. This space of potentials will be useful for us in section 7.

Any harmonic map from \( \mathbb{C} \) to \( G \) is obtained, up to left multiplication by a constant, from a holomorphic potential \( \mu \in \mathbb{P} \):

Theorem 6. Let \( \phi : \mathbb{C} \to G \) be a harmonic map such that \( \phi(0) = e \). Then there exists a holomorphic potential \( \mu \in \mathbb{P} \) such that \( \phi = \phi_\mu \).

Proof. Let \( \phi : \mathbb{C} \to G \) be a harmonic map such that \( \phi(0) = e \). Let \( \Phi \) be an extended solution associated to \( \phi \), with \( \Phi^{-1} d\Phi = (1 - \lambda^{-1})\alpha' + (1 - \lambda)\alpha'' \). Consider the \( \Lambda_+G^\mathbb{C} \)-valued \((0,1)\)-form \( \theta = -(1-\lambda)\alpha'' \). The \( \bar{\partial} \)-problem \( \bar{\partial} b b^{-1} = \theta, b(0) = e \) can be solved over \( \mathbb{C} \) (see appendix of 7). Let \( b : \mathbb{C} \to \Lambda_+G^\mathbb{C} \) be the unique solution of this problem and put \( \Psi = \Phi b \). We have \( \Psi^{-1} d\Psi = \text{Ad}_{\Phi^{-1}} (\Phi^{-1} d\Phi) + b^{-1} d\lambda \). By construction we see that \( \mu = \Psi^{-1} d\Psi \) is a \( \Lambda_{-1,\infty} \)-valued 1-form of type \((1,0)\) (in particular, due to Maurer-Cartan equation, \( \mu \) is holomorphic) such that \( \phi = \phi_\mu \). \( \square \)

5 Extended framings vs. extended solutions

In the previous sections we have seen two procedures of obtaining harmonic maps from a potential \( \mu \): via extended solutions we get an harmonic map \( \phi : \mathbb{C} \to G \); via extended framings (if \( \mu \) is a twisted potential) we get an harmonic map \( \tilde{\phi} : \mathbb{C} \to G/K \). We show in this section that (when \( \mu \) is a twisted potential) \( \phi \) and \( \tilde{\phi} \) are essentially the same harmonic map.

Fix a twisted potential \( \mu \in \mathbb{P}_\tau \) and integrate to obtain a complex extended solution \( \Psi : \mathbb{C} \to \Lambda G^\mathbb{C} \). This map has a unique factorization \( \Psi = \Phi \tilde{\phi} \), with \( \Phi : \mathbb{C} \to \Omega G \) an extended solution and \( \tilde{\phi} : \mathbb{C} \to \Lambda_+G^\mathbb{C} \). The corresponding harmonic map into \( G \) is given by \( \phi = \Phi^{-1} \) (again, we are using the notation \( \Phi_\lambda(z) = \Phi(z)(\lambda) \)). On the other hand, we can view \( \Psi \) as a map from \( \mathbb{C} \) to \( \Lambda_+G^\mathbb{C} \) and use the decomposition of Theorem 2 to write \( \Psi = \Phi \tilde{\phi} \), where \( \Phi : \mathbb{C} \to \Lambda_+G \) is an extended framing and \( \tilde{\phi} : \mathbb{C} \to \Lambda_+^+ \). The corresponding harmonic map into the symmetric space \( G/K \) is given by \( \phi = \pi \circ \tilde{\phi} \). Let \( \iota : G/K \to G \) be the Cartan embedding. It happens that \( \phi \) has values in \( \iota(G/K) \) and \( \phi = \iota \circ \tilde{\phi} \). In fact: we have \( \iota \circ \pi \circ \tilde{\phi} = \tau(\tilde{\phi}) \tilde{\phi}^{-1} = \tilde{\phi}^{-1} \tilde{\phi}^{-1} \); however, by the uniqueness of
the decomposition $\Psi = \Phi b$, we have $\Phi = \hat{\Phi}\hat{b}^{-1}$ and $b = \hat{\Phi}_1\hat{b}$, so that $\iota \circ \pi \circ \hat{\Phi}_1 = \hat{\Phi}_1\hat{b}^{-1} = \Phi_1$; hence $\Phi$ and $\hat{\Phi}$ produce the same harmonic map into $G/K$.

6 Dressing actions and gauge transformations

Another consequence of Iwasawa-type decomposition of Theorem 4 is that it allows us to define a natural action $\#_\varepsilon$ of $\Lambda^* G$ on $\Omega^p E$: if $g \in \Omega^p E G$ and $h \in \Lambda^* G$, then $h \#_\varepsilon g = (hg)_E$. Applying this action pointwise, we obtain from an extended solution $\Phi : \mathbb{C} \to \Omega G$ a new map $h \#_\varepsilon \Phi : \mathbb{C} \to \Omega G$ defined by $(h \#_\varepsilon \Phi)(z) = h \#_\varepsilon \Phi(z)$, for all $z \in \mathbb{C}$.

Theorem 7. [3, 7, 8] $h \#_\varepsilon \Phi$ is an extended solution.

We will recall now from [3] how these actions vary with $\varepsilon$. For $0 < \varepsilon < \varepsilon' < 1$ we have injections $\Lambda^* G \subset \Lambda^* G$ and $\Omega^p E G \subset \Omega^p E G$. Similarly, for $0 < \varepsilon < 1$, we have $\Omega_{\text{hol}} G \subset \Omega^p E G$, where $\Omega_{\text{hol}} G = \bigcap_{0 < \varepsilon < 1} \Omega^p E G$. It is easy to see that

$$\Omega_{\text{hol}} G = \{ \gamma : \mathbb{C}^* \to G^C | \gamma \text{ is holomorphic}, \gamma(1) = e \text{ and } \overline{\gamma(\lambda)} = \gamma(1/\lambda) \}.$$ 

We have:

Theorem 8. [3] For $0 < \varepsilon < \varepsilon' < 1$, $\gamma \in \Lambda^* G \subset \Lambda^* G$, and $g \in \Omega^p E G \subset \Omega^p E G$, we have $\gamma \#_\varepsilon g = \gamma \#_{\varepsilon'} g \in \Omega^p E G$.

Corollary 1. [3] The action of each $\Lambda^* G$ preserves $\Omega_{\text{hol}} G$ and, for $0 < \varepsilon < \varepsilon' < 1$, $\gamma \in \Lambda^* G \subset \Lambda^* G$ and $g \in \Omega_{\text{hol}} G$, we have $\gamma \#_\varepsilon g = \gamma \#_{\varepsilon'} g$.

It follows that we can take a direct limit as $\varepsilon \to 0$ and so obtain an action on $\Omega_{\text{hol}} G$ of the group of germs at zero of maps $\mathbb{C} \to G^C$. Henceforth, we write $\gamma \# g$ for this action on $\Omega_{\text{hol}} G$.

On the other hand, the holomorphic gauge group

$$\mathcal{G}^\varepsilon = \{ h = (h_+, h_-) : \mathbb{C} \to \Lambda^* G \text{ such that } \partial h_+ = 0 \}$$

acts on the space $\mathcal{P}^\varepsilon$ of $\varepsilon$-holomorphic potentials by gauge transformations: if $\mu \in \mathcal{P}^\varepsilon$ and $h \in \mathcal{G}^\varepsilon$, then $h \cdot \mu = \text{Ad}_h(\mu) - dh h^{-1} \in \mathcal{P}^\varepsilon$. It happens that the correspondence between holomorphic potentials and extended solutions $\mu \to \Phi_\mu$ is equivariant with respect to these actions:

Theorem 9. [3, 7] If $h \in \mathcal{G}^\varepsilon$, then $\Phi_{h_{\mu}} = h(0)\# \Phi_\mu$.

Remark 7. The limiting case of Theorem 9 as $\varepsilon \to 1$ can be stated as follows:

Consider the dressing action $\#$ of $\Lambda G^C$ on $\Omega G$ corresponding to the Iwasawa decomposition of $\mathbb{R}$ let $\mathcal{G}$ be the holomorphic gauge group of all $h : \mathbb{C} \to \Lambda G^C$ such that $\partial h = 0$; let $\mu \in \mathcal{P}$ be a holomorphic potential. Then $\Phi_{h_{\mu}} = h(0)\# \Phi_\mu$.

7 Adding a uniton and gauge transformations

Another well known operation for generating new extended solution from a given one was introduced by Uhlenbeck [12] and is called adding a uniton. In this section we describe how this
operation arises via gauge transformations on the holomorphic potential. More precisely: consider a holomorphic potential $\mu \in \mathcal{P}$ with associated extended solution $\Phi_{\mu}$; denote by $S_{\mu}$ the set of all gauge transformations $h : \mathbb{C} \to \Lambda G^c$ such that $h \cdot \mu = \text{Ad}_{h}(\mu) - dh \cdot h^{-1}$ is in $\mathcal{P}$ (see Remark [10]; each element $h$ of $S_{\mu}$ gives rise to a new extended solution $\Phi_{h,\mu}$; in particular, we have $\mathcal{G} \subset S_{\mu}$ and $\Phi_{h,\mu} = h(0)\#\Phi_{\mu}$ if $h \in \mathcal{G}$; next we describe the elements of $h \in S_{\mu}$ whose action on $\mu$ corresponds to the operation of adding a uniton to $\Phi_{\mu}$.

**Theorem 10.** [12] Let $\Phi : \mathbb{C} \to \Omega U(n)$ be an extended solution, $\phi : \mathbb{C} \to U(n)$ the corresponding harmonic map. Write $\alpha_0 = \frac{1}{2} \phi^{-1} d\phi = A_z dz + A_{\bar{z}} d\bar{z}$. Let $\hat{\ell}$ be a subbundle of $\mathbb{C}^n$ with Hermitian projection $\hat{\pi} : \mathbb{C}^n \to \hat{\ell}$ satisfying the uniton conditions: a) $\hat{\pi}^+ A_z \hat{\pi} = 0$; b) $\hat{\pi}^+ (\partial \hat{\pi} + A_{\bar{z}} \hat{\pi}) = 0$. Then $\tilde{\Phi} : \mathbb{C} \to \Omega U(n)$ given by $\tilde{\Phi}_\lambda = \Phi_{\lambda}(\hat{\pi} + \lambda \hat{\pi}^+)$ is an extended solution.

This operation of obtaining new harmonic maps from a given one is called *adding a uniton* in [12]. Note that the second uniton condition means that $\hat{\ell}$ is a holomorphic subbundle of $\mathbb{C}^n$ with respect to the holomorphic structure $\bar{\partial}_{\alpha_0}$.

We recall from [12] how to add a uniton to a harmonic map into a Grassmannian:

Let $G_k(\mathbb{C}^n)$ be the complex Grassmannian of $k$-planes in $\mathbb{C}^n$. The unitary group $U(n)$ acts transitively on $G_k(\mathbb{C}^n)$ with stabilizers conjugate to $U(k) \times U(n-k)$. Fix a complex $k$-plane $V_0 \in G_k(\mathbb{C}^n)$ with stabilizer $K$ and let $\pi_0$ be the Hermitian projection onto $V_0$. Let $\tau$ be the involution of $U(n)$ given by conjugation by $Q_0 = \pi_0 - \pi_0^\perp$. The identity component of the fixed set of $\tau$ is $K$ so that $G_k(\mathbb{C}^n)$ is a symmetric space with involution $\tau$. The corresponding Cartan embedding $\iota_k : G_k(\mathbb{C}^n) \to U(n)$ is given by $\iota_k(V) = Q_0(\pi_V - \pi_V^\perp)$, where $\pi_V$ denotes the Hermitian projection onto the $k$-plane $V$.

**Theorem 11.** [12] Suppose that $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is a harmonic map and $\Phi$ an extended solution associated to $\phi = \iota_k \circ \psi$. Let $\hat{\ell}$ be a subbundle of $\mathbb{C}^n$, with Hermitian projection $\hat{\pi} : \mathbb{C}^n \to \hat{\ell}$, satisfying the uniton conditions, and such that $[\phi, \hat{\pi}] = 0$. Then $\tilde{\Phi} : \mathbb{C} \to \Omega U(n)$ given by $\tilde{\Phi}_\lambda = \Phi_{\lambda}(\hat{\pi} + \lambda \hat{\pi}^+)$ is an extended solution associated to a harmonic map $\tilde{\psi}$ into a Grassmannian $G_k(\mathbb{C}^n)$: $\tilde{\Phi}_{-1} = \iota_k \circ \tilde{\psi}$.

Let $\ell$ be a holomorphic subbundle of $(\mathbb{C}^n, \bar{\partial})$, with Hermitian projection $\pi$. Fix a holomorphic potential $\mu \in \mathcal{P}$. Suppose that $\pi^+ \mu_{-1} \pi = 0$. Define $\gamma_\ell : \mathbb{C} \to \Omega G$ by

$$
\gamma_\ell = \pi + \lambda^{-1} \pi^+.
$$

This map $\gamma_\ell$ gauges the connection $d_{\mu}$ to the connection associated to the 1-form

$$
\gamma_\ell : \mu = \left( \pi + \lambda^{-1} \pi^+ \right) \left( \sum_{k \neq -1} \mu_k \lambda^k \right) \left( \pi + \lambda \pi^+ \right) - \left( 1 - \lambda^{-1} \right) d\pi \left( \pi + \lambda \pi^+ \right).
$$

The coefficient in $\lambda^{-2}$ on the right hand of this equality is zero since $\pi^+ \mu_{-1} \pi = 0$. Moreover, since $\ell$ is a holomorphic subbundle of $\mathbb{C}^n$, we have $\bar{\partial} \pi \pi = 0$; hence the $(0,1)$-part of the coefficient in $\lambda^{-1}$ on the right hand of this equality also vanishes. Then $\gamma_\ell \cdot \mu \in \mathcal{P}$ and $\gamma_\ell \in S_{\mu}$.

**Theorem 12.** $\Phi_{\gamma_\ell \mu}$ is obtained from $\Phi_{\mu}$ by adding a uniton. More precisely: consider the Iwasawa decomposition of $\Psi_{\mu}$, $\Psi_{\mu} = \Phi_{\mu} b$, and the subbundle $\ell = b_0 \ell$, where $b_0(z) = b(z)(0)$, with Hermitian
projection \( \check{\pi} : \mathbb{C}^n \to \hat{\ell} \); then \( \hat{\ell} \) satisfies the unitor conditions of Theorem \([17]\) and

\[
\Phi_{\gamma, \mu} = (\check{\pi}_0 + \lambda^{-1}\check{\pi}^\perp_0 )\Phi_{\mu}(\check{\pi} + \lambda \check{\pi}^\perp),
\]

where \( \check{\pi}_0 \) is the Hermitian projection onto the fibre of \( \hat{\ell} \) at \( z = 0 \).

**Proof.** First note that \( \Psi_{\gamma, \mu} = (\check{\pi}_0 + \lambda^{-1}\check{\pi}^\perp_0 )\Phi_{\mu}(\pi + \lambda \pi^\perp) \), i.e., \( \Psi_{\gamma, \mu}^{-1}d\Psi_{\gamma, \mu} = \gamma_{\ell} \cdot \mu \) and \( \Psi_{\gamma, \mu}(0) = e \). Then, to find \( \Phi_{\gamma, \mu} \), we have to factorize \( \Psi_{\gamma, \mu} \) according to the Iwasawa decomposition. We have

\[
\Psi_{\gamma, \mu} = (\check{\pi}_0 + \lambda^{-1}\check{\pi}^\perp_0 )\Phi_{\mu}(\pi + \lambda \pi^\perp) = (\check{\pi}_0 + \lambda^{-1}\check{\pi}^\perp_0 )\Phi_{\mu}(\pi + \lambda \pi^\perp)(\hat{\pi} + \lambda \hat{\pi}^\perp) = (\pi_0 + \lambda^{-1}\pi^\perp_0 )\Phi_{\mu}(\pi + \lambda \pi^\perp)(\hat{\pi} + \lambda \hat{\pi}^\perp) = (\hat{\pi} + \lambda \hat{\pi}^\perp). \tag{11}
\]

We claim that \( \hat{b} = (\hat{\pi} + \lambda \hat{\pi}^\perp)b(\pi + \lambda \pi^\perp) \) takes values in \( \Lambda_+ G^C \). In fact, \( \hat{b} \) is holomorphic at \( \lambda = 0 \) if \( \hat{\pi}^\perp b_0 \pi = 0 \) (this is the coefficient in \( \lambda^{-1} \)). But \( \hat{\pi}^\perp b_0 \pi = 0 \) if and only if \( \hat{\ell} = b_0 \ell \), whence \( \hat{\pi}^\perp b_0 \pi \) vanishes automatically. The invertibility follows by applying the same argument to \( \hat{b}^{-1} \). Whence \( \hat{b} \) takes values in \( \Lambda_+ G^C \). From \([11]\) we conclude now that \( \Phi_{\gamma, \mu} = (\pi_0 + \lambda^{-1}\pi^\perp_0 )\Phi_{\mu}(\pi + \lambda \pi^\perp) \) (in particular, \( \hat{\ell} \) satisfies the unitor conditions.)

Reciprocally, any unitor can be added via the action of some loop of the form \([10]\) on the holomorphic potential \( \mu \):

**Theorem 13.** Suppose that the subbundle \( \hat{\ell} \) with Hermitian projection \( \check{\pi} : \mathbb{C}^n \to \hat{\ell} \) satisfies the unitor conditions with respect to \( \Phi_{\mu} \). Set \( \ell = b_0^{-1} \hat{\ell} \). Then: a) \( \ell \) is holomorphic with respect to the trivial holomorphic structure \( \hat{\partial} \); b) \( \pi^\perp_{\mu - 1} \pi = 0 \) and c) we have \( \Phi_{\gamma, \mu} = (\pi_0 + \lambda^{-1}(\pi^\perp_0 )\Phi_{\mu}(\pi + \lambda \pi^\perp) \).

**Proof.** a) We have seen in Remark \([5]\) that \( b : (\mathbb{C}^n, \partial) \to (\mathbb{C}^n, \hat{\partial}_{\alpha_0}) \) is a holomorphic isomorphism; hence, \( \ell \) is a holomorphic subbundle of \( (\mathbb{C}^n, \partial) \), since \( \hat{\ell} \) is a holomorphic subbundle of \( (\mathbb{C}^n, \hat{\partial}_{\alpha_0}) \). b) Equation \( \pi^\perp_{\mu - 1} \pi = 0 \) follows directly from the first unitor condition and \([9] \). c) This statement follows directly from Theorem \([12]\).

## 8 Harmonic maps into Grassmannians and subbundles

As an application of the ideas of section \([7]\) in section \([9]\) we will be able to improve and clarify some results presented in the unpublished second author’s doctoral thesis \([10]\) concerning unitons preserving finite type property of harmonic maps. Before we shall recall from \([6]\) some relevant facts about harmonic maps into Grassmannians:

Let \( N = G/K \) be a symmetric space with involution \( \tau \). Denote by \( \mathfrak{g} \) and \( \mathfrak{k} \) the Lie algebras of \( G \) and \( K \), respectively, and consider the corresponding symmetric decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) into \( \pm 1 \)-eigenspaces of the derivation of \( \tau \). Recall that, for each \( x = g \cdot x_0 \), the surjective map \( \mathfrak{g} \to T_z N \) given by \( x \mapsto \left. \frac{d}{dt} \right|_{t=0} \exp t \xi \cdot x \) has the Lie algebra \( \text{Ad}_g \mathfrak{k} \) as kernel and so restricts to an isomorphism \( \text{Ad}_g \mathfrak{m} \to T_z N \). The inverse map \( \beta_x : T_z N \to \text{Ad}_g \mathfrak{m} \) defines a \( \mathfrak{g} \)-valued 1-form \( \beta \) on \( N \), the Maurer-Cartan form of \( N = G/K \). We denote by \([\mathfrak{m}]\) the subbundle of the trivial bundle \( \mathfrak{g} = N \times \mathfrak{g} \) defined by \([\mathfrak{m}]_{g \cdot x_0} = \text{Ad}_g(\mathfrak{m}) \).
If $N$ is actually the (matrix) group manifold $G$, acting on itself by right translations, then $\beta$ is just the (left) Maurer-Cartan form $\theta$ of $G$. Moreover:

**Lemma 1.** If $\psi : \mathbb{C} \to G/K$ is a smooth map and $\iota$ is the Cartan embedding of $G/K$ into $G$, then, with $\phi = \iota \circ \psi$ we have $(\phi^{-1} d\phi)\phi^* \theta = -2\psi^* \beta$.

**Proof.** In fact: let $X \in T_{x} G/K$; then $X = \frac{d}{dt} \big|_{t=0} \exp t \beta(X) \cdot x$ so that

$$\iota^*(X) = \frac{d}{dt} \big|_{t=0} \tau(\exp t \beta(X)) \iota(x) \exp \left(-t \beta(X)\right) = \iota(x) (\text{Ad}_{\iota(x)^{-1}} \tau(\beta(X)) - \beta(X)) = -2\iota(x) \beta(X);$$

hence $\iota^* \theta = \iota^{-1} d\iota = -2\beta$, and from this equality, by taking the pullback by $\psi$, we see that the statement of this lemma holds. \hfill \Box

Consider now the case of the Grassmannian $G_k(\mathbb{C}^n)$. Denote by $\mathfrak{u}(n)$ the Lie algebra of $U(n)$ and take $V_0 \in G_k(\mathbb{C}^n)$ as base point. The associated symmetric decomposition $\mathfrak{u}(n) = \mathfrak{t} \oplus \mathfrak{m}$ is given by

$$\mathfrak{t}^\mathbb{C} = \text{Hom}(V_0, V_0) \oplus \text{Hom}(V_0, V_0^\perp),$$

$$\mathfrak{m}^\mathbb{C} = \text{Hom}(V_0, V_0^\perp) \oplus \text{Hom}(V_0^\perp, V_0).$$

Let $T \to G_k(\mathbb{C}^n)$ be the tautological subbundle of $G_k(\mathbb{C}^n) \times \mathbb{C}^n$ whose fibre at $V \in G_k(\mathbb{C}^n)$ is $V$ itself. With respect to the usual Hermitian structure of $G_k(\mathbb{C}^n)$, the Maurer-Cartan form of $G_k(\mathbb{C}^n)$ restricted to the bundle of $(1,0)$-vectors gives an isomorphism $\beta^{(1,0)} : T^{(1,0)} G_k(\mathbb{C}^n) \to \text{Hom}(T, T^*).$

Identify a smooth map $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ with the smooth complex subbundle $\overline{\psi}$ of the trivial bundle $\mathbb{C}^n$ given by setting the fibre at $z$ equal to $\psi(z)$ for all $z \in \mathbb{C}$. Conversely any rank $k$ subbundle of $\mathbb{C}^n$ induces a map $\mathbb{C} \to G_k(\mathbb{C}^n)$. Denote the Hermitian projection onto a vector subbundle $\overline{\psi}$ by $\pi_{\overline{\psi}}$ and define vector bundle morphisms $A'_\psi, A''_\psi : \psi \to \psi^\perp$ called the $\partial$- and $\bar{\partial}$-second fundamental forms of $\psi$ in $\mathbb{C}^n$ by $A'_\psi(v) = \pi_{\psi^\perp}(\frac{\partial v}{\partial z})$ and $A''_\psi(v) = \pi_{\psi^\perp}(\frac{\partial v}{\partial \bar{z}})$, for $v$ a smooth section of $\psi$. Note that $A'_\psi$ is minus the adjoint of $A''_{\psi^\perp}$. The second fundamental forms of $\psi$ in $\mathbb{C}^n$, $A'_\psi$ and $A''_\psi$, represent, via $\beta^{(1,0)}$, the $(1,0)$-components of the partial derivatives $\partial \psi$ and $\bar{\partial} \psi$ (see [6]):

$$\psi^* \beta \left( \frac{\partial}{\partial z} \right) = \psi^* \beta^{(1,0)} \left( \frac{\partial}{\partial z} \right) + \psi^* \beta^{(0,1)} \left( \frac{\partial}{\partial \bar{z}} \right) = A'_\psi + A''_{\psi^\perp}. \quad (12)$$

Comparing Lemma 1 with (12) we obtain the following formula to the derivative of $\phi = \iota \circ \psi$ in terms of the second fundamental forms of $\psi$:

$$\frac{1}{2} \phi^{-1} \partial \phi = -(A'_\psi + A''_{\psi^\perp}). \quad (13)$$

In this setting, the harmonicity equations can be reformulated as follows (cf. [6]): we give each subbundle of $\mathbb{C}^n$ the connection induced from the trivial connection of $\mathbb{C}^n$ and corresponding Koszul–Malgrange holomorphic structure; a smooth map $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is harmonic if and only if $A'_\psi$ is holomorphic; this holds if and only if $A''_{\psi^\perp}$ is anti-holomorphic.

Now, given two holomorphic bundles $E, F$ over a Riemann surface and a holomorphic bundle morphism $A : E \to F$, there are unique holomorphic subbundles $\text{Ker} A$ and $\text{Im} A$ of $E$ and $F$,
respectively, that coincide with $\text{Ker} A$ and $\text{Im} A$ almost everywhere. Thus we can define the $\partial$- and $\bar{\partial}$-Gauss bundles of a harmonic map $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ by $G^{(1)}(\psi) = \text{Im} A'_{\psi}$ and $G^{(-1)}(\psi) = \text{Im} A''_{\psi}$, respectively. These bundles also represent harmonic maps (cf. [4]). Iterating this construction we set $G^{(0)}(\psi) = \psi$, and for $i = 1, 2, \ldots$, $G^{(i)}(\psi) = g_i G^{(i-1)}(\psi)$, $G^{(-i)}(\psi) = g_i^{-1} G^{(-i-1)}(\psi)$. The bundle $G^{(i)}(\psi)$ is called the $i^{th}$-Gauss bundle of $\psi$.

**Remark 8.** The harmonic map $G^{(-1)}(\psi)$ can be obtained by adding the unitor $\ell = \text{ker} A'_{\psi, i}$ to $\psi$.

### 9 Units preserving finite type

**Definition 3.** [4] A harmonic map $\phi : \mathbb{C} \to G$ is of finite type if it can be obtained from an extended solution $\Phi_\mu : \mathbb{C} \to \Omega G$ whose holomorphic potential $\mu = \xi dz$ is constant of the form $\xi = \lambda^{d-1} \eta$, for some odd $d \in \mathbb{N}$, with

$$\eta \in \Omega_d g = \{ \eta \in \Omega g \mid \eta = \sum_{|k| \leq d} \eta_k \lambda^k \}.$$  

It is clear that:

**Theorem 14.** Fix a harmonic map $\phi_\mu : \mathbb{C} \to G$ of finite type, with $\mu = \lambda^{d-1} \eta dz$ and $\eta \in \Omega_d g$. Fix a constant subspace of $\mathbb{C}^n$ and let $\pi_0 : \mathbb{C}^n \to \ell_0$ be the corresponding hermitian projection. If $\gamma_{t_0} = \pi_0 + \lambda^{d-1} \pi_0^1$ is an automorphism of finite type. W e claim that this set is also of finite type.

Consider the vector subspace $\ell_0 = \text{ker} \eta_{-d}$ of $\mathbb{C}^n$, with Hermitian projection $\pi_0$; in this case, we have $\pi_0^1 \mu_{-d} \pi_0 = 0$, whence $\gamma_{t_0} \in S_\mu$. A more concrete example of unitons preserving finite type is given by the following theorem:

**Theorem 15.** If $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is of finite type, then, for each integer $r$, $G^{(r)}(\psi)$ is of finite type.

**Proof.** Fix $\psi(0)$ as base point of $G_k(\mathbb{C}^n)$, let $K$ be the isotropic subgroup of $U(n)$ associated to $\psi(0)$ and $\tau$ the corresponding automorphism. Let $\mu = \xi dz \in P_\tau$ be a constant $\tau$-twisted holomorphic potential of the form $\xi = \lambda^{d-1} \eta$, for some odd $d \in \mathbb{N}$, with $\eta = \sum_{|k| \leq d} \eta_k \lambda^k \in \Omega_d g$, associated to $\psi$. Since $d$ is odd and $\eta$ is twisted, we have $\eta_{-d} \in \mathfrak{m}^C$. According to the decomposition of $\mathfrak{m}^C$ into $(1,0)$-vectors and $(0,1)$-vectors, $\mathfrak{m}^C = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, with $\mathfrak{m}^+ = \text{Hom}(\psi(0), \psi(0)^\perp)$ and $\mathfrak{m}^- = \text{Hom}(\psi(0)^\perp, \psi(0))$, write $\eta_{-d} = \eta_+ \mathfrak{m}^+ + \eta_- \mathfrak{m}^-$. Fix the vector subspace $\ell_0 = \text{ker} \eta_{-d}$. Clearly we have $\gamma_{t_0} \in S_\mu$; thus $\Phi_n_{t_0, \mu}$ gives rise to a harmonic map of finite type. We claim that this harmonic map is precisely the Gauss bundle $G^{(-1)}(\psi)$.

Let $\Psi_\mu$ be the complex extended solution associated to $\mu$. Factorize $\Psi_\mu$ according to Remark 2. $\Psi_\mu = \Phi \Phi$. Fix an Iwasawa decomposition $K^C = KB$ and factorize $\Psi_\mu$ according to Theorem 2. $\Psi_\mu = \Phi \Phi$. Since $i$ $b_0$ takes values in $B \subset K^C$, $ii) \Phi$ frames $\psi$, that is, $\psi = \Phi_1 \cdot \psi(0)$, and $iii) b_0 = \Phi_1 b_0$ (cf. Section 4), we conclude that $b_0 : \mathbb{C} \to GL(n, \mathbb{C})$ also frames $\psi$, that is, $\psi = b_0 \cdot \psi(0)$. In particular, $\psi^*[\mathfrak{m}^C]_z = \text{Ad}_{b_0(z)}(\mathfrak{m}^C)$ and

$$\text{Hom}(\psi^* T, \psi^* T^\perp)_z = \text{Ad}_{b_0(z)}(\mathfrak{m}^+), \quad \text{Hom}(\psi^* T^\perp, \psi^* T)_z = \text{Ad}_{b_0(z)}(\mathfrak{m}^-) \quad (14)$$
for each $z \in \mathbb{C}$. On the other hand, from (9) and (13) follows that

$$A'_\psi + A'_{\psi \perp} = \text{Ad}_{b_0}(\eta_{-d}).$$

Hence, from (14) and (15) we conclude that $A'_{\psi \perp} = \text{Ad}_{b_0}(\eta_{-d})$. Thus,

$$\ker A'_{\psi \perp} = \ker \text{Ad}_{b_0}(\eta_{-d}) = b_0 \ell_0.$$

Taking account Remark 8 and Theorem 12, this establishes the claim.

Iterating this argument and reversing orientation we have the result.

However, unitons do not always preserve finite type: let $\phi : \mathbb{C} \to G_k(\mathbb{C}^n)$ be a (non-constant) harmonic map of finite type and $\delta : \mathbb{C} \to G_s(\mathbb{C}^m)$ a (non-constant) holomorphic map. Then $\psi = \phi \oplus \delta : \mathbb{C} \to G_{k+s}(\mathbb{C}^n \oplus \mathbb{C}^m)$ is a harmonic map which is obtained from $\phi$ by adding the uniton $\delta$. If $A'_\delta$ has singular points, the same happens to $A'_\psi = A'_\phi \oplus A'_\delta$; in this case, $\psi$ can not be of finite type, since equation (15) ensures that the second fundamental forms $A'_{\psi \perp}$ and $A''_{\psi \perp}$ associated to a harmonic map $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ of finite type have no singular points (points where the rank of $\text{Im}A'_{\psi}$ drops).

Remark 9. Let $\{\gamma_a\}$ be a curve in $\Lambda^*_\gamma G$ and $\Phi$ an extended solution. Suppose $\lim_{a \to 0} \gamma_a = \gamma$ with $\gamma \in \Lambda^* G$. The extended solution $\tilde{\Phi} = \lim_{a \to 0} (\gamma_a \# \Phi)$ need not to coincide with $(\gamma \Phi)_E$; thus, $\tilde{\Phi}$ is not, apriori, obtained by applying a dressing transformation to $\Phi$. Following [1], we shall refer to the procedure of obtaining $\tilde{\Phi}$ from $\Phi$ as completion. In [12], Uhlenbeck suggested that any uniton can be added this way. Bergvelt and Guest [1] settled negatively this conjecture and studied in detail this procedure of completion in the case of curves in $\Lambda^*_\gamma G$ defined by $\gamma_a = \pi_V + \xi_a \pi_{V^\perp}$, where $V$ is a constant subspace of $\mathbb{C}^n$, $a \in \mathbb{C}$, and $\xi_a$ is the rational function in $\lambda$ given by

$$\xi_a(\lambda) = \frac{\bar{a}\lambda - 1}{\lambda - a}.$$ 

We observe that if $\Phi_\mu$ is an extended solution associated to the holomorphic potential $\mu \in \mathcal{P}$, and $V$ is a constant subspace of $\mathbb{C}^n$ such that $\pi_{V^\perp}^{\perp} \mu V = 0$ everywhere, then $\lim_{a \to 0} \gamma_a \cdot \mu = \gamma \cdot \mu \in \mathcal{P}$, with $\gamma = \pi_V + \lambda^{-1} \pi_{V^\perp}$. Hence, in this case, the completion procedure amounts to add a uniton.

In particular, we see that the unitons preserving finite type in Theorem 14 can be added via completion.

References

[1] M.J. Bergvelt and M.A. Guest, *Action of loop groups on harmonic maps*. Trans. Amer. Math. Soc. **326** (1991), 861–886.

[2] F.E. Burstall, D. Ferus, F. Pedit, and U. Pinkall, *Harmonic Tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. **138** (1993), 173–212.

[3] F.E. Burstall and F. Pedit, *Dressing orbits of harmonic maps*, Duke Math. J. **80** (1995), 353–382.
Adding a uniton via the DPW method

[4] F.E. Burstall and F. Pedit, *Harmonic maps via Adler-Konstant-Symes theory*, Harmonic maps and Integrable Systems (A.P. Fordy and J.C. Wood, eds), Aspects of Mathematics E23, Vieweg, 1994, pp. 221–272. CMP 94:09

[5] F.E. Burstall, J. H. Rawnsley, *Twistor Theory for Riemannian Symmetric Spaces*, Lectures Notes in Math. 1424 Berlin, Heidelberg: 1990.

[6] F.E. Burstall and J.C. Wood, *The construction of harmonic maps into complex Grassmannians*, J. Diff. Geom. 23 (1986), 255–297.

[7] J. Dorfmeister, F. Pedit and H. Wu, *Weiestrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. 6 (1998), 633-668.

[8] M.A. Guest and Y. Ohnita, *Group actions and deformations for harmonic maps*, J. Math. Soc. Japan 45 (1993), 671–710.

[9] I. McIntosh, *Global solutions of the elliptic 2D periodic Toda lattice*, Nonlinearity 7 (1994), 85-108.

[10] R. Pacheco, *Harmonic maps and loop groups*, Ph.D. thesis, University of Bath, 2004.

[11] A.N. Pressley and G.B. Segal, *Loop Groups*, Oxford University Press, 1986.

[12] K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom. 30 (1989), 1–50.