A Remarkable New Identity Satisfied by the Dirac Matrices of a Bilocal Field Theory

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Abstract

In 1925 Elie Cartan described ‘triality’ \cite{4,5} as a symmetry between SO(8; \mathbb{C}) vectors and the two types of Spin(8; \mathbb{C}) spinor. It is known that the reduced generators of the Clifford algebra \mathbb{C}_8 defined on the real, eight-dimensional Euclidean space \mathbb{E}_8 satisfy an identity that guarantees the existence of matrix representations (acting on the vector and spinor bundles of \mathbb{E}_8) of triality.

Analogously, let \mathbb{E}_{4,4} denote a real eight-dimensional pseudo-Euclidean vector space that is endowed with an indefinite inner product with signature (+, +, +, −; −, −, −, +). As a normed vector space, \mathbb{E}_{4,4} \cong M_{3,1} \times M_{3,1}^\ast, where \text{M}_{3,1} and \text{M}_{3,1}^\ast denote real four-dimensional Minkowski spacetimes, with opposite signatures. The reduced generators (i.e., the Dirac matrices) of the pseudo Clifford algebra \mathbb{C}_{4,4} defined on \mathbb{E}_{4,4} satisfy an identity \cite{10,11} that guarantees the existence of invertible linear mappings between each of the two types of S0(4, 4; \mathbb{R}) spinor and the S0(4, 4; \mathbb{R}) vector, thereby realizing matrix representations of triality that act on the vector and spinor bundles of the spacetime \mathbb{E}_{4,4}.

In this note we generalize this identity (see Eq.[13]).

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I. INTRODUCTION AND NOTATION

In 1925 Elie Cartan described ‘triality’ \[4\], \[5\] as a symmetry between three types of geometrical objects that may be defined on real, eight-dimensional \(\mathbb{R}^8\) and transform linearly under either \(\text{SO}(8; \mathbb{C})\) or \(\text{Spin}(8; \mathbb{C})\), namely a symmetry between \(\text{SO}(8; \mathbb{C})\) vectors and the two types of \(\text{Spin}(8; \mathbb{C})\) spinor (semi-spinors of the first type and semi-spinors of the second type, in the terminology of Cartan).

Analogously, let \(E_{4,4}\) denote a real eight-dimensional pseudo-Euclidean vector space that is endowed with an indefinite inner product with signature \((+,-,+,+; -,-,+,+)\) (see Gray \[6\]). As a normed vector space, \(E_{4,4} \cong M_{3,1} \times \ast M_{3,1}\), where \(M_{3,1}\) denotes a real four-dimensional Minkowski spacetime manifold that is endowed with the pseudo-Euclidean metric \(\eta_{3,1} = \text{diag}(1, 1, 1, -1)\), and \(\ast M_{3,1}\) denotes a real four-dimensional Minkowski spacetime that is endowed with the pseudo-Euclidean metric \(\text{diag}(-1, -1, -1, 1) = -\eta_{3,1}\). \(M_{3,1} \times \ast M_{3,1}\) may be regarded as a classical phase space of a single relativistic point particle, or a spacetime that carries a bilocal Minkowski field theory (appropriate restrictions on the automorphism groups of \(E_{4,4} \cong M_{3,1} \times \ast M_{3,1}\) are implied).

The reduced generators (i.e., the Dirac matrices) of the pseudo Clifford algebra \(\mathbb{C}_{4,4}\) defined on \(E_{4,4}\) satisfy an identity \[10\], \[11\] that guarantees the existence of invertible linear mappings between each of the two types of \(\text{SO}(4,4; \mathbb{R})\) spinor and the \(\text{SO}(4,4; \mathbb{R})\) vector, thereby realizing matrix representations of triality that act on the vector and spinor bundles of the spacetime \(E_{4,4}\). In this note we generalize this remarkable identity Eq.\[11\] to Eq.\[13\]. Simple applications of this formalism are given in Sections \[IV\] and \[V\].

\(E_{4,4}\) is an orientable differentiable manifold that, of course, admits a global, right-handed Cartesian atlas (as well as many other “curvilinear” and general coordinate systems). Let \(x \in E_{4,4}\) and let the 8 scalars \(x^A \in \mathbb{R}\), \(A, B, ... = 1, 2, ..., 8\), denote the Cartesian coordinates of \(x\) with respect to a global, right-handed Cartesian atlas. Let \(T_x(E_{4,4})\) denote the tangent space at \(x\). \(T_x(E_{4,4})\) is isomorphic to \(E_{4,4}\). The right-handed frame \(\{ \frac{\partial}{\partial x^A} : A = 1, \ldots , 8 \}\) that is adapted to these coordinates is orthogonal and pseudo-normal with respect to the metric defined below, and comprises a basis of \(T_x(E_{4,4})\). This coordinate system and frame are simply called a “canonical frame”. A vector field \(V\) at \(x\), \(V_x = V^A(x) \frac{\partial}{\partial x^A} \in T_x(E_{4,4})\), has contravariant components \(V^A(x)\) with respect to a canonical frame. Here the \(A, B, ... = 1, ... , 8\) are to regarded as \(T_x(E_{4,4})\) vector indices, and not as indices that enumerate the
II. DIRAC MATRICES ON $\mathbb{E}_{4,4}$

A. Representations of $SO(8; \mathbb{C})$

There is a well known relationship between Clifford algebras $C_n$ and the spinor representations of the classical complex orthogonal groups; see, for example, Boerner, *The Representations of Groups*. In particular, the Clifford algebra $C_8$ may be defined as the algebra generated by a set of eight elements $e_j, j, k = 1, \ldots, 8$, that anticommute with each other and have unit square $e_j e_k + e_k e_j = 2 \delta_{jk} I_{16 \times 16}$, where $I_{16 \times 16} = 16 \times 16$ unit matrix. The scaled commutators $\frac{1}{4} (e_j e_k - e_k e_j)$ computed from an irreducible 16-dimensional representation of the $e_j$ are the infinitesimal generators of a reducible 16-dimensional representation of $\text{Spin}(8; \mathbb{C})$, which is the universal double covering of the special orthogonal group $SO(8; \mathbb{C})$. This 16-dimensional representation of is fully reducible to the direct sum of two inequivalent irreducible $8 \times 8$ spin representations of the infinitesimal generators of $\text{Spin}(8; \mathbb{C})$. The fundamental irreducible vector representation of $SO(8; \mathbb{C})$ is also $8 \times 8$. The Dynkin diagram for $D_4 \cong SO(8; \mathbb{C})$ is symmetrical and pictured in Figure 1. The central node corresponds to the adjoint representation. The three outer nodes correspond to the vector representation (left-most node), type 1 spinor and type 2 spinor representations of $\text{Spin}(8; \mathbb{C})$. The “left-handed” and “right-handed” $\text{Spin}(8; \mathbb{C})$ spinors have counter parts that are denoted $\psi^{(1)}$ and $\psi^{(2)}$ in this paper, and transform, respectively, under two inequivalent real $8 \times 8$ irreducible spinor representations of $SO(4, 4; \mathbb{R})$ that we have called $D_{(1)}$ (type 1) and $D_{(2)}$ (type 2).

$SO(4, 4; \mathbb{R})$ is a real form of the classical complex orthogonal group $SO(8, \mathbb{C})$. $O(4, 4; \mathbb{R})$ (respectively, $SO(4, 4; \mathbb{R})$) may be defined as the group of all real matrices (respectively, with unit determinant) that preserve the norm squared of $V_x \in T_x(\mathbb{E}_{4,4})$, which is the quadratic
form
\[(V^8)^2 + (V^1)^2 + (V^2)^2 + (V^3)^2 - [(V^4)^2 + (V^5)^2 + (V^6)^2 + (V^7)^2].\]

\(O(4, 4; \mathbb{R})\) is a pseudo-orthogonal Lie group that possess two connected components \([2],[8]\), with \(SO(4, 4; \mathbb{R})\) being the identity component (the connected component containing the identity matrix). \(\text{Spin}(4, 4, \mathbb{R})\), alternatively denoted \(SO(4, 4; \mathbb{R})\), is the 2-to-1 covering group of \(SO(4, 4; \mathbb{R})\).

\(\mathbb{E}_{4,4}\) may be endowed with both \(SO(4, 4; \mathbb{R})\)-invariant and \(SO(4, 4; \mathbb{R})\)-invariant pseudo-Euclidean metrics that may each be represented in terms of an \(8 \times 8\) matrix with real matrix elements.

In a canonical \(\mathbb{E}_{4,4}\) frame the \(SO(4, 4; \mathbb{R})\)-invariant pseudo-Euclidean metric tensor \(G\) (respectively, inverse \(G^{-1}\)) has components \(G_{AB}\) (respectively, \((G^{-1})^{AB} = G^{BA} = G^{AB}\)) that are given by

\[G_{AB} = G^{AB} = \begin{pmatrix} \eta_{3,1} & 0 \\ 0 & -\eta_{3,1} \end{pmatrix}\]

The indefinite inner product is realized as \(T_x(\mathbb{E}_{4,4}) \times T_x(\mathbb{E}_{4,4}) \ni (V_x, V'_x) \mapsto < V_x, V'_x > = G_{AB} V^A_x V^B_x \in \mathbb{R}\).

**B. Spinor representations of \(\overline{SO}(4, 4; \mathbb{R})\)**

There exist two inequivalent real \(\overline{SO}(4, 4; \mathbb{R})\) basic 8-component spinor representations of \(\overline{SO}(4, 4; \mathbb{R})\). They are defined in Eqs.\([47]\) and simply denoted as \(D_{(1)}\) (type 1) and \(D_{(2)}\) (type 2). The \(\overline{SO}(4, 4; \mathbb{R})\) invariant metric, denoted \(\sigma\), is invariant under the action of both \(D_{(1)}\) and \(D_{(2)}\). Let \(S_x^{(j)}(\mathbb{E}_{4,4})\), \(j = 1, 2\), denote the two distinct basic real 8-component spinor vector spaces at \(x\), endowed with respective automorphism groups \(D_{(j)}\). As vector spaces each is isomorphic to \(\mathbb{E}_{4,4}\). (Thus, as vector spaces, both of the \(S_x^{(j)}(\mathbb{E}_{4,4})\) and \(T_x(\mathbb{E}_{4,4})\) are each isomorphic to \(\mathbb{E}_{4,4}\) but with different automorphism groups.) A spinor element \(\psi_{(j)} \in S_x^{(j)}(\mathbb{E}_{4,4})\) has components \(\psi_{(j)}^a \in \mathbb{R}\). In this note, for simplicity, we do not distinguish the spinor index on \(\psi_{(1)}\) from that on \(\psi_{(2)}\) [using a convention such as \(\psi_{(1)\dot{a}}\) and \(\psi_{(2)}^\dot{a}\) for spinor components, for example].

The disjoint union of tangent spaces \(T_x(\mathbb{E}_{4,4})\) at all points \(x \in \mathbb{E}_{4,4}\) gives the \(SO(4, 4; \mathbb{R})\) tangent bundle \(T(\mathbb{E}_{4,4})\) over \(\mathbb{E}_{4,4}\). In this case, it is a trivial bundle \(\mathbb{E}_{4,4} \times T_x(\mathbb{E}_{4,4}) \xrightarrow{\pi} \mathbb{E}_{4,4}\).
with the natural projection $\pi$ of the first factor in the Cartesian product. Clearly there also exist two distinct trivial 16-dimensional real basic 8-component spinor bundles $S^{(1)}(E_{4,4})$ and $S^{(2)}(E_{4,4})$, each with base space $E_{4,4}$ but with fibers $S_{x}^{(1)}(E_{4,4})$ and $S_{x}^{(2)}(E_{4,4})$, respectively. For each of the three bundles we denote the natural projection of the first factor in the Cartesian product by $\pi$.

$E_{4,4}$ may be endowed with a $SO(4,4;\mathbb{R})$ invariant metric $\sigma^{[10]}$ that we represent as

$$\sigma = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

where $0$ denotes the $4 \times 4$ zero matrix and $1$ denotes the $4 \times 4$ unit matrix. The matrix elements of $\sigma$ are denoted $\sigma_{ab} = \sigma_{ba}$, where $a, b, \ldots = 1, \ldots, 8$ are $SO(4,4;\mathbb{R})$ spinor indices (elaborated in Eqs. [47] through [53] below). Note that $\sigma^{2}$ is equal to the unit matrix, so that the eigenvalues of $\sigma$ of are $\pm 1$. Since the trace of $\sigma$ is zero, these eigenvalues occur with equal multiplicity.

The $SO(4,4;\mathbb{R})$ invariant (pseudo) norm-squared $\|\psi_{(j)}\|^2$ of basic real 8-component spinors $\psi_{(j)} \in S_{x}^{(j)}(E_{4,4})$ is the $SO(4,4;\mathbb{R})$-invariant quadratic form $\psi_{(j)}^a \sigma_{ab} \psi_{(j)}^b$. We define an oriented spinor basis $e_a$ of $S_{x}^{(1)}(E_{4,4})$ normalized according to

$$\langle e_a, e_b \rangle = \sigma_{ab}$$

(the oriented spinor basis of $S_{x}^{(2)}(E_{4,4})$ also satisfies Eq. [3]), so that

$$\langle \psi_{(1)}, \psi_{(1)} \rangle = \langle \psi_{(1)}^a e_a, \psi_{(1)}^b e_b \rangle = \psi_{(1)}^a \sigma_{ab} \psi_{(1)}^b = \tilde{\psi}_{(1)}^a \sigma \psi_{(1)}$$

where the tilde denotes transpose. For brevity we employ the shorthand $u \in S^{(1)}(E_{4,4})$ and $u^a \in S^{(1)}(E_{4,4})$ to denote $e_a u^a \in S^{(1)}(E_{4,4})$, with similar conventions implied for $T(E_{4,4})$ and $S^{(2)}(E_{4,4})$.

We also define an oriented vector basis $\epsilon_A$ of $T_{x}(E_{4,4})$ normalized according to

$$\langle \epsilon_A, \epsilon_B \rangle = G_{AB}.$$  

These two sets of basis vectors are related by Eq. [54] below.

The basic spinor representation of the pseudo-orthogonal group $SO(4,4;\mathbb{R})$ may be constructed from the irreducible generators $t^A$, $A = 1, \ldots, 8$, of the pseudo-Clifford algebra $C_{4,4}$ [2], [3], [9]. Following Brauer and Weyl we call such irreducible $C_{2n-2}$ generators “reduced Brauer-Weyl generators” [3]. We begin the construction of a representation of
\( S0(4, 4; \mathbb{R}) \) by defining eight real \( 8 \times 8 \) matrix reduced Brauer-Weyl generators \( \tau^A, \overline{\tau}^A, A, B, \ldots = 1,...,8 \), of the pseudo-Clifford algebra \( C_{4,4} \) that anticommute and have square \( \pm 1 \). (The \( \tau^A \) matrices play the role of the Dirac Matrices on \( \mathbb{E}_{4,4} \).) We realize this by requiring that the tau matrices satisfy (the tilde denotes transpose)

\[
\sigma \tau^A = \overline{\tau}^A = \overline{\tau}^A \sigma \quad (5)
\]

and

\[
\tau^A \tau^B + \tau^B \tau^A = 2 \mathbb{I}_{8 \times 8} \, G^{AB} = \tau^A \tau^B + \tau^B \tau^A, \quad (6)
\]

where \( \mathbb{I}_{8 \times 8} \) denotes the \( 8 \times 8 \) unit matrix. Denoting the matrix elements of \( \tau^A \) by \( \tau^A_{ab} \), we may write Eq.\([5]\) as

\[
\overline{\tau}^A_{ab} = \tau^A_{ba}, \quad (7)
\]

where we have used \( \sigma \) to lower the spinor indices. In general, \( \sigma \) (respectively, \( \sigma^{-1} \)) will be employed to lower (respectively, raise) lower case Latin indices (i.e. a \( S0(4, 4; \mathbb{R}) \) spinor index of either type).

The following identity is occasionally useful. Let \( \psi \in S^{(1)}(\mathbb{E}_{4,4}) \) be an arbitrary real eight component type-1 spinor field (a section of the type-1 spinor bundle \( S^{(1)}(\mathbb{E}_{4,4}) \)). Consider

\[
\overline{\psi} \sigma \tau^A \tau_B \psi = \left( \overline{\psi} \sigma \tau^A \tau_B \psi \right)^T = \left( \overline{\psi} \sigma \tau_B \tau^A \psi \right), \text{ by Eq.}\,[5]
\]

\[
= \frac{1}{2} \overline{\psi} \sigma \left( \tau^A \tau_B + \tau_B \tau^A \right) \psi
\]

\[
= \delta^A_B \overline{\psi} \sigma \psi, \text{ using Eq.}\,[6]. \quad (8)
\]

We adopt a real irreducible \( 8 \times 8 \) matrix representation of the tau matrices that is adapted to the \( X^8 \)-axis, in which \( \tau^8 = \mathbb{I}_{8 \times 8} = \overline{\tau}^8 \). Then, by Eq.\,[6], \( \overline{\tau}^A = -\tau^A \) for \( A = 1, \ldots, 7 \). Hence, again by Eq.\,[6], \( (\tau^A)^2 \) is equal to \( -\mathbb{I}_{8 \times 8} \) for \( A = 1,2,3 \) and is equal to \( +\mathbb{I}_{8 \times 8} \) for \( A = 4,5,6,7,8 \). The Appendix displays one possible representation.

### III. THE NEW IDENTITY

The tau matrices verify an important identity \([10], [11]\) that encodes triality: Let \( M \) be any \( 8 \times 8 \) matrix satisfying (recall that the tilde denotes transpose)

\[
\overline{\sigma} \, \widetilde{M} = \sigma \, M \quad (9)
\]
(i.e., $\sigma M$ is a symmetric matrix) and moreover transforming under $SO(4, 4; \mathbb{R})$ according to

$$M \mapsto D_{(1)} M D_{(1)^{-1}}$$

(see Eq. 11, below). Then $[10], [11]$

$$\tau_A M \tau^A = \mathbb{I}_{8 \times 8} \text{tr}(M),$$

where, as above, $\mathbb{I}_{8 \times 8}$ denotes the $8 \times 8$ unit matrix. This is a remarkable identity because this linear combination of eight terms involving an arbitrary real $8 \times 8$ symmetric matrix $\sigma M$ is proportional to the unit matrix, and there are 36 linearly independent real $8 \times 8$ matrices $M$ such that $\sigma M$ is a symmetric matrix (these are given below in Eq. 14).

This is a special case of another simple, but also remarkable, general identity that we record as

**Theorem III.1** Let $M$ be an arbitrary $8 \times 8$ matrix that transforms under $SO(4, 4; \mathbb{R})$ according to $M \mapsto D_{(1)} M D_{(1)^{-1}}$. $M$ has matrix elements $M^a_b$. Note that $M - \sigma^{-1}(\sigma M)$ is twice $\sigma^{-1}$ times the anti-symmetric part of $\sigma M$. The generalization of Eq. 11 is

$$\tau_{(\mu)} M \tau^\mu = -\tau_{(\mu)} \text{tr}(\tau^\mu M) + 2 \left( \mathbb{I}_{8 \times 8} \text{tr}(M) + M - \sigma^{-1}(\sigma M) \right)$$  \hspace{1cm} (12)

or

$$(\tau_{(\mu)})^a_b (\tau^\mu)^c_d = - (\tau_{(\mu)})^a_d (\tau^\mu)^c_b + 2 (\delta^c_d \delta^a_b - \delta^a_d \delta^c_b - \sigma^{ac} \sigma_{bd})$$  \hspace{1cm} (13)

The Proof of Theorem III.1 is straightforward. Firstly, if $\sigma M$ is symmetric then Eq. 12 devolves to Eq. 11. What if $\sigma M$ has no symmetry? Eqs. 12, 13 are linear in $M$. Expand $M$ in terms of a linear combination of the 64 basis $8 \times 8$ matrices comprised of the $36 = 35 + 1$ basis matrices $M_s \in S_{8 \times 8}$ such that $\sigma M_s$ is symmetric, plus the $28 = 7 + 21$ basis matrices $M_a \in A_{8 \times 8}$ such that $\sigma M_a$ is anti-symmetric, and verify the theorem component by component. The set of $35 + 1$ matrices $S_{8 \times 8}$ is given by

$$S_{8 \times 8} = \left\{ \tau^{(A)} \tau^{(B)} \tau^{(C)} \right\}_{\{A,B,C\} \in \{1, \ldots, 7\} \& A > B > C}, \mathbb{I}_{8 \times 8} \right\},$$

and each element of this set clearly verifies Theorem III.1.

The $7 + 21$ matrices $M_a \in A_{8 \times 8}$ such that $\sigma M_a$ is anti-symmetric are given by

$$A_{8 \times 8} = \left\{ \tau^{(A)} \right\}_{A \in \{1, \ldots, 7\}}, \tau^{(A)} \tau^{(B)} \right\}_{\{A,B\} \in \{1, \ldots, 7\} \& A > B}}.$$  \hspace{1cm} (15)
Each of $M_a \in \{\tau^{(A)}\}_{A \in \{1,\ldots,7\}}$ satisfies $\tau(\mu) M_a \tau(\mu) = -4 M_a$ as well as $\tau(\mu) \text{tr} (\tau(\mu) M_a) = +8 M_a$. Each of $M_a \in \{\tau^{(A)}\}_{\{A,B\} \in \{1,\ldots,7\} \& A > B}$ satisfies $\tau(\mu) M_a \tau(\mu) = +4 M_a$ as well as $\tau(\mu) \text{tr} (\tau(\mu) M_a) = 0$. Therefore each element of $A_{8 \times 8}$ satisfies Eqs. [12,13] and the Theorem [III.1] is proven. □

IV. BILOCAL TETRAD

Let $u = u(x^a) \in S^{(1)}(\mathbb{E}_{4,4})$ be a real eight component type-1 spinor field (a section of the type-1 spinor bundle $S^{(1)}(\mathbb{E}_{4,4})$). $u$ is called the “unit field” for reasons that are explained in Section [VI]. In a quantum theory the $u^a$ satisfy commutation relations rather than anti-commutation relations because of triality. We assume that $\langle u, u \rangle = \tilde{u} \sigma u > 0$ everywhere on $\mathbb{E}_{4,4} = \pi \left( S^{(1)}(\mathbb{E}_{4,4}) \right)$.

For brevity a vielbein set of 8 independent vector fields is simply referred to as a tetrad (vierbein). In this Section and the next we replace the indices $A, B, \ldots = 1, \ldots, 8$ with the indices $(\mu), (\nu), \ldots$, where $\mu, \nu, \ldots = 1, \ldots, 8$, in order to display this information in a more conventional form. Summarizing, $\alpha, \beta, \ldots, \mu, \nu, \ldots, a, b, \ldots = 1, \ldots, 8$. We also employ $\alpha_4, \beta_4, \ldots, \mu_4, \nu_4, \ldots = 1, \ldots, 4$.

Let $\psi \in S^{(2)}(\mathbb{E}_{4,4})$ denote a real eight component type-2 spinor field that realizes the bilocal Cartesian coordinates $x \equiv (x^\alpha, x^{4+\alpha})$ of $\pi \left( S^{(2)}(\mathbb{E}_{4,4}) \right) \cong M_{3,1} \times \ast M_{3,1}$. The Cartesian coordinates $x^\alpha = \{x\}_\alpha$ are assumed to be $C^\infty$ functions of $\psi$, $x^\alpha = x^\alpha(\psi)$ such that $\text{det} \left( \frac{\partial x^\alpha}{\partial \psi} \right) \neq 0$, so that the inverse $\psi^a = \psi^a(x^\alpha)$ always exists. We abuse notation and write $u = u(x^\alpha) = u(x^\alpha(\psi)) = u(\psi^a)$. The mass dimension of $\psi$, $[\psi]$, is -1: $[\psi] = \text{LENGTH} = 1/\text{MASS} = [\text{Planck length}]$.

We define a spacetime tetrad $E^{(\mu)}$ with components $E_{\alpha}^{(\mu)}$ as

$$E_{\alpha}^{(\mu)} = \frac{1}{\sqrt{u \sigma u}} \tilde{u} \sigma \tau(\mu) \frac{\partial}{\partial x^\alpha} \psi. \quad (16)$$

Remark: Let $f : \mathbb{E}_{4,4} \to \mathbb{R}$ and $\frac{\partial f}{\partial x^\alpha} = f_{,\alpha}$. Let $r : \mathbb{E}_{4,4} \to \mathbb{R}^+$. The tetrad $E^{(\mu)}$ may be made to transform covariantly under the local projective transformation

$$u \mapsto u' = r(x^\alpha) u$$
$$\psi \mapsto \psi' = r(x^\alpha) \psi \quad (17)$$
by replacing the gradient operator $\frac{\partial}{\partial x^\alpha}$ with

$$D_\alpha = I_{8\times8} \frac{\partial}{\partial x^\alpha} - \frac{1}{\tilde{u} \sigma u} \tau_{(\mu)} \frac{\partial u}{\partial x^\alpha} \otimes \tilde{u} \sigma \tau_{(\mu)}$$

(18)

because $D'_\alpha \psi' = (r \psi + r, \psi) - \frac{1}{r^2} \tilde{u} \sigma u \tau_{(\mu)} (ru, \psi + ru, \psi) \otimes r \tilde{u} \sigma \tau_{(\mu)} (r \psi) = r D_\alpha \psi$, since $\frac{1}{r^2} \tilde{u} \sigma u \tau_{(\mu)} (ru, \psi + ru, \psi) = r \psi$, using Eq. [11] or Eq. [12].

Therefore, if

$$E_\alpha(\mu) = \frac{1}{\sqrt{\tilde{u} \sigma u}} \tilde{u} \sigma \tau_{(\mu)} D_\alpha \psi.$$  

(19)

then

$$E_\alpha(\mu) \mapsto E'_\alpha(\mu) = r(x) E_\alpha(\mu)$$

(20)

under the local projective transformation Eq.[17].

This local projective transformation generates a conformal transformation of the metric tensor. ■

**Lemma IV.1** The inverse of the tetrad has components $E^\alpha_{(\mu)}$

$$E^\alpha_{(\mu)} = \frac{1}{\sqrt{\tilde{u} \sigma u}} \frac{\partial x^\alpha}{\partial \psi_{(\mu)}} u.$$  

(21)

Proof:

$$E^\alpha_{(\mu)} E^\beta_{(\nu)} = \left( \frac{1}{\sqrt{\tilde{u} \sigma u}} \frac{\partial x^\alpha}{\partial \psi_{(\mu)}} u \right) \left( \frac{1}{\sqrt{\tilde{u} \sigma u}} \tilde{u} \sigma \tau_{(\mu)} \frac{\partial}{\partial x^\beta} \psi \right)$$

$$= \frac{1}{\tilde{u} \sigma u} \frac{\partial x^\alpha}{\partial \psi} \tau_{(\mu)} u \tilde{u} \sigma \tau_{(\mu)} \frac{\partial}{\partial x^\beta} \psi$$

$$= \frac{\partial x^\alpha}{\partial \psi} \frac{\partial x^\beta}{\partial x^\beta} \text{ by Eq. [11] or Eq. [12]}$$

$$= \delta^\alpha_\beta$$

(22)

QED ■

Since a matrix commutes with its inverse we also have

$$E^\alpha_{(\mu)} E^\alpha_{(\nu)} = \delta^\alpha_{(\nu)}.$$  

(23)
Let’s look at an example. We make the self-consistent assumption that there exists a constant spacetime tetrad \( E^{(\mu)} \) with constant components \( E_{\alpha}^{(\mu)} \), which might verify \( E_{\alpha}^{(\mu)} = \delta^{(\mu)}_{\alpha} \), for example. Pick a constant unit field \( u \) that satisfies \( \tilde{u} \sigma u > 0 \), define

\[
\psi = \frac{1}{\sqrt{\tilde{u} \sigma u}} \tau^{(\nu)} u \ E^{(\nu)}_{\beta} x^\beta
\]

(compare with the twistor type) and compute

\[
E^{(\mu)}_{\alpha} = \frac{1}{\sqrt{\tilde{u} \sigma u}} \tilde{u} \sigma \tau^{(\mu)} \frac{\partial}{\partial x^\alpha} \psi
\]

\[
= \frac{1}{\tilde{u} \sigma u} \tilde{u} \sigma \tau^{(\mu)} \frac{\partial}{\partial x^\alpha} \left( \tau^{(\nu)} u \ E^{(\nu)}_{\beta} x^\beta \right)
\]

\[
= \frac{1}{\tilde{u} \sigma u} \left( \tilde{u} \sigma \tau^{(\mu)} \tau^{(\nu)} u \right) E^{(\nu)}_{\alpha}
\]

\[
= E^{(\mu)}_{\alpha} \text{ using the identity of Eq.[8]}
\]

The pseudo-Riemannian metric associated to this tetrad field is

\[
g_{\alpha\beta} = E^{(\mu)}_{\alpha} \eta_{(\mu)(\nu)} E^{(\nu)}_{\beta}
\]

\[
= \frac{1}{\tilde{u} \sigma u} \tilde{u} \sigma \tau^{(\mu)} \frac{\partial}{\partial x^\alpha} \psi \eta_{(\mu)(\nu)} \tilde{u} \sigma \tau^{(\nu)} \frac{\partial}{\partial x^\beta} \psi
\]

\[
= \frac{1}{\tilde{u} \sigma u} \left( \tilde{u} \sigma \tau^{(\mu)} \frac{\partial}{\partial x^\alpha} \psi \right) \eta_{(\mu)(\nu)} \tilde{u} \sigma \tau^{(\nu)} \frac{\partial}{\partial x^\beta} \psi
\]

\[
= \frac{1}{\tilde{u} \sigma u} \frac{\partial}{\partial x^\alpha} \tilde{u} \sigma \tau^{(\mu)} u \eta_{(\mu)(\nu)} \tilde{u} \sigma \tau^{(\nu)} \frac{\partial}{\partial x^\beta} \psi
\]

\[
= \frac{1}{\tilde{u} \sigma u} \frac{\partial}{\partial x^\alpha} \tilde{u} \sigma \left( \tau^{(\mu)} u \tilde{u} \sigma \tau^{(\nu)} \right) \frac{\partial}{\partial x^\beta} \psi
\]

\[
= \omega^{\alpha}_{\beta} \text{ using Eq.[11] or Eq.[12]}
\]

\[
= \omega^{a}_{b} \text{ using the coordinate-transform of } \sigma_{ab}
\]

\[
= \omega^{a}_{b} \text{ using Eq.[11] or Eq.[12]}
\]

\[
= \omega^{a}_{b}
\]

which is not an induced metric but, as one may expect, is the coordinate-transform of \( \omega_{ab} \).

The \( 4 + 4 = 8 \) dimensional spacetime \( \pi \left( S^{(1)}(\mathbb{E}_{4,4}) \right) \) endowed with this metric has zero curvature.

**V. SCHWINGER REAL REPRESENTATION OF QED**

Julian Schwinger \[15\] has given a representation of charged fermion field operators for an electron in terms of real anti-commuting \( 8 \)-component spinor fields. Therefore it may be of
interest to evaluate, using the new identity Eq.[12] and the above tetrad, and with arbitrary Schwinger spinor (bilocal) fields $F$ and $H$, the operator

$$H \gamma^\mu \frac{\partial}{\partial x^\mu} F = H \gamma^\mu \frac{1}{\sqrt{u} \sigma u} \frac{\partial x^\alpha}{\partial \psi} \tau(\mu) u \frac{\partial}{\partial x^\alpha} F$$

$$= \frac{1}{\sqrt{u} \sigma u} H \gamma^\mu \frac{\partial F}{\partial \psi} \tau(\mu) u$$

$$= \left[ - \left( \tau(\mu) \right)^a_d \left( \tau(\mu) \right)^c_b + 2 \left( \delta^a_d \delta^c_b + \delta^a_b \delta^c_d - \sigma^{ac} \sigma_{bd} \right) \right] H \frac{u^b}{\sqrt{u} \sigma u} \frac{\partial F}{\partial \psi^a},$$

which may easily be further reduced.

**VI. ALGEBRAIC SIGNIFICANCE OF THE SPINOR $u$, THE UNIT FIELD**

Let $u \in S^{1}(E_{4,4})$ be a type-1 spinor field (a section of the type-1 spinor bundle $S^{1}(E_{4,4})$), with $< u, u > = \bar{u} \sigma u > 0$ everywhere on the base space $E_{4,4}$, but being otherwise arbitrary. $u$, may be called a “unit field”. One may define a special $E_{4,4}$ frame field $\mathfrak{g}$ in terms of $u$ and the tau matrices as follows. Let $M$ be the real $8 \times 8$ matrix defined by

$$M = \frac{1}{\bar{u} \sigma u} u \otimes \bar{u} \sigma = \frac{1}{\bar{u} \sigma u} u \bar{u} \sigma$$

$$M^a_b = \frac{1}{\bar{u} \sigma u} u^a u^c \sigma_{cb}$$

Then $M$ obeys Eq.[9] and transforms under $SO(4,4;\mathbb{R})$ according to Eq.[10]. Using Eq.[11] or Eq.[12] to evaluate $\tau_A M \tau^A$ yields

$$\mathbb{I}_{8 \times 8} = \frac{1}{\bar{u} \sigma u} \tau_A \left( u \bar{u} \sigma \right) \tau^A = \left( \frac{1}{\sqrt{u} \sigma u} \tau_A u \right) \left( \frac{1}{\sqrt{u} \sigma u} \tilde{u} \sigma \tau^A \right)$$

This is a resolution of the identity on $E_{4,4}$. Alternatively this relation may be interpreted as a completeness condition verified by the $E_{4,4}$ orthogonal frame $\mathfrak{g}$ whose components $\mathfrak{g}^A_a$ are given by

$$\mathfrak{g}^A_a = \frac{1}{\sqrt{u} \sigma u} \tau^a \tau^b u^b$$

and its inverse is

$$\mathfrak{g}_a = \frac{1}{\sqrt{u} \sigma u} u^c \sigma_{cb} \tau^b_{A}$$

Accordingly Eq.[29] may be expressed in index notation as

$$\{ \mathbb{I}_{8 \times 8} \}^a_b = \delta^a_b = \mathfrak{g}^A_a \mathfrak{g}^A_b$$
Since a matrix commutes with its inverse we also have

\[ \delta^A_B = \bar{\varphi}^A_a \bar{\varphi}^a_B. \] (33)

We have defined an oriented spinor basis \( e_a \) of \( S_x^{(1)}(\mathbb{E}_{4,4}) \) in Eq[3] and an oriented vector basis \( \epsilon_A \) of \( T_x(\mathbb{E}_{4,4}) \) in Eq[4]. The two are related by

\[ \epsilon_A = e_a \bar{\varphi}^a_A \quad \text{and} \quad e_a = \epsilon_A \bar{\varphi}^a_A \] (34)

A. Split octonion algebra over \( \mathbb{R}, \mathcal{O}_s(\mathbb{R}) \)

Let \( \mathcal{O}_s(\mathbb{R}) \) denote the split octonion algebra over \( \mathbb{R} \) [17], [16], [11], [12], [13].

A nonassociative alternative multiplication of the oriented spinor basis \( e_a \) (respectively, oriented vector basis \( \epsilon_A \)) may be defined [11] that endows the real vector space \( S_x^{(1)}(\mathbb{E}_{4,4}) \) (respectively, \( T_x(\mathbb{E}_{4,4}) \)) with the structure of a normed nonassociative algebra with multiplicative unit that is isomorphic to the split octonion algebra over \( \mathbb{R}, \mathcal{O}_s(\mathbb{R}) \). This is accomplished by specifying the multiplication constants \( m_{ab}^c \) (respectively, \( m_{AB}^C \)) of the algebra, which verify

\[ e_a e_b = e_c m_{ab}^c \]
\[ \epsilon_A \epsilon_B = \epsilon_C m_{AB}^C \] (35)

The set of multiplication constants \( m_{ab}^c \) (respectively, \( m_{AB}^C \)) is defined by [11]

\[ m_{ab}^c = \bar{\varphi}^a_A \tau_A^a c_b \]
\[ m_{AB}^C = \bar{\varphi}^C_a \tau_A^a c_b \bar{\varphi}^b_B. \] (36)

It has been shown that the nonassociative product defined by Eq.35 (respectively, Eq.36) of the spinor basis \( e_a \) (respectively, of the vector basis \( \epsilon_A \)) endows the respective real vector space with the structure of the split octonion algebra over the reals [11]. This is explicit in the multiplication table below, which employs the representation of the tau matrices given in Appendix 3 and \( u^a = \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, 1, 0, 0) \), which is an eigenvector of \( \sigma \) with eigenvalue +1.
B. Multiplicative identity

An element $\Psi \in O_s(\mathbb{R})$ may be realized as

$$\Psi = e_a \psi^a = \epsilon_A \hat{\psi}^A$$

$$\hat{\psi}^A = \hat{\mathcal{S}}_A^a \psi^a \iff \psi^a = \hat{\mathcal{S}}_A^a \hat{\psi}^A.$$  \hspace{1cm} (37)

The normalized fiducial unit field $O_s(\mathbb{R}) \ni \sqrt{\langle u, u \rangle} \ u = \sqrt{\langle u, u \rangle} \ \epsilon_a u^a = \frac{1}{\sqrt{\langle u, u \rangle}} \ \tau_8^a \ u^b = e_a \hat{\mathcal{S}}_8^a = \epsilon_8 = \textbf{multiplicative identity}$ element of the split octonion algebra $O_s(\mathbb{R})$ \hspace{1cm} (38)

Multiplicative identity $= \frac{1}{\sqrt{\langle u, u \rangle}} \ \epsilon_a u^a = \frac{1}{\sqrt{\langle u, u \rangle}} \ u = \epsilon_8.$

VII. CONCLUDING REMARK

The reduced generators (i.e., the Dirac matrices) of the pseudo Clifford algebra $\mathbb{C}_{4,4}$ defined on $\mathbb{E}_{4,4}$ satisfy a remarkable identity Eq.[13] that defines invertible linear mappings between each of the two types of $SO(4,4;\mathbb{R})$ spinor and the $SO(4,4;\mathbb{R})$ vector, thereby admitting matrix representations of triality on this spacetime $\mathbb{E}_{4,4}$. The trialities are given below in Eqs[55] and [56].
VIII. APPENDIX 1: TRANSFORMATION UNDER ACTION OF $\overline{S0(4,4;\mathbb{R})}$

The special Lorentz transformation properties of the theory may be determined by constructing a real reducible $16 \times 16$ matrix representation of $\overline{S0(4,4;\mathbb{R})}$ utilizing the irreducible generators $t^A$, $A = 1, \ldots, 8$ of the (pseudo-) Clifford algebra $C_{4,4}$. Following Lord’s general procedure [9] we define the irreducible generators $t^A$ as

$$t^A = \begin{pmatrix} 0 & \tau^A \\ \tau^A & 0 \end{pmatrix}. \quad (39)$$

Let $g \in \overline{S0(4,4;\mathbb{R})}$. The $16 \times 16$ basic spinor representation of $S0(4,4;\mathbb{R})$ is reducible into the two real $8 \times 8$ inequivalent irreducible spinor representations $D^{(1)}_g$ and $D^{(2)}_g$ of $\overline{S0(4,4;\mathbb{R})}$. The reduced generators of the two real $8 \times 8$ spinor representations $D^{(1)}_g$ and $D^{(2)}_g$ of $\overline{S0(4,4;\mathbb{R})}$ follow from the calculation of the infinitesimal generators

$$t^{AB} = \left( \begin{array}{cc} \tau^A \tau^B - \tau^B \tau^A & 0 \\ 0 & \tau^A \tau^B - \tau^B \tau^A \end{array} \right)$$

$$= 4 \left( \begin{array}{cc} D^{(1)}_{AB} & 0 \\ 0 & D^{(2)}_{AB} \end{array} \right), \quad (40)$$

of the 16-component spinor representation of $\overline{S0(4,4;\mathbb{R})}$. We see, as is in fact well known from the general theory, that the 16-component spinor representation of $\overline{S0(4,4;\mathbb{R})}$ is the direct sum of two (inequivalent) real $8 \times 8$ irreducible spinor representations $D^{(1)}_g = D^{(1)}(g)$ and $D^{(2)}_g = D^{(2)}(g)$ of $\overline{S0(4,4;\mathbb{R})} \ni g$ that are generated by $D^{(1)}_{AB}$ and $D^{(2)}_{AB}$ respectively, where

$$4 D^{(1)}_{AB} = \tau^A \tau^B - \tau^B \tau^A \quad (41)$$

and

$$4 D^{(2)}_{AB} = \tau^A \tau^B - \tau^B \tau^A \quad (42)$$

For completeness we remark that the generators of the two spinor types are images of
the projection operators

\[ \chi_\pm = \frac{1}{2} (1 \pm t^9) \]
\[ \chi_+ = \left( \begin{array}{cc} I_{8\times8} & 0 \\ 0 & 0 \end{array} \right) \]
\[ \chi_- = \left( \begin{array}{cc} 0 & 0 \\ 0 & I_{8\times8} \end{array} \right) , \]

(43)

where

\[ t^9 = t^1 t^2 t^3 t^4 t^5 t^6 t^7 t^8 = \begin{pmatrix} \tau^0 & 0 \\ 0 & \tau^0 \end{pmatrix} \].

(44)

Here

\[ \tau^0 = \tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 = \tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 \]

(45)

and

\[ \tau^0 = \tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 = -\tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 = -\tau^0 \]

(46)

The representation of the tau matrices is irreducible. \( \tau^0 \) has square equal to \(+I_{8\times8}\) and commutes with each of the \( \tau^A \) matrices (and therefore with all of their products). Therefore we conclude that \( \tau^0 = \pm I_{8\times8} \) in any irreducible representation.

Let \( \omega_{AB} = -\omega_{BA} \in \mathbb{R} \), \( A, B = 1, \ldots, 8 \), enumerate a set of 28 real parameters that coordinatize \( g = g(\omega) \in S0(4, 4; \mathbb{R}) \). Also, let \( L = L(g) \in SO(4, 4; \mathbb{R}) \) have matrix elements \( L^A_B \), \( \omega^\sharp \) denote the real \( 8 \times 8 \) matrix with matrix elements \( \omega^A_B = G^{AC} \omega_{CB} \), \( \omega_1 = \frac{1}{2} \omega_{AB} D_{(1)}^{AB} \) and \( \omega_2 = \frac{1}{2} \omega_{AB} D_{(2)}^{AB} \). We find that

\[ D_{(1)} = D_{(1)}(g) = \exp \left( \frac{1}{2} \omega_1 \right) \]
\[ D_{(2)} = D_{(2)}(g) = \exp \left( \frac{1}{2} \omega_2 \right) \]
\[ L^A_B = L^A_B(g) = \left\{ \exp \left( \omega^\sharp \right) \right\}^A_B \]

(47)

where, under the action of \( \widetilde{S0}(4, 4; \mathbb{R}) \),

\[ \widetilde{D}_{(1)}^{AB} \sigma = -\sigma D_{(1)}^{AB} \Rightarrow \widetilde{D}_{(1)}(g) = \sigma D_{(1)}^{-1} \]

(48)

\[ \widetilde{D}_{(2)}^{AB} \sigma = -\sigma D_{(2)}^{AB} \Rightarrow \widetilde{D}_{(2)}(g) = \sigma D_{(2)}^{-1} \]

(49)

\[ L^A_C G_{AB} L^B_D = G_{CD} = \left\{ \tilde{L}_{CG} \right\}_C_D \]

(50)
The canonical 2-1 homomorphism $S_0(4, 4; R) \to S_0(4, 4; R) : g \mapsto L(g)$ is given by

$$8L_B^A = \text{tr} \left( D(1)^{-1} \tau^A D(2) \tau^B \right) G_{CB},$$

where $\text{tr}$ denotes the trace. Note that $D(1)(\omega) = D(2)(\omega)$ when $\omega_{AB} = 0$, i.e., when one restricts $S_0(4, 4; R)$ to

$$S_0(3, 4; R) = \left\{ g \in S_0(4, 4; R) \mid g = \begin{pmatrix} \exp \left( \frac{1}{4} \omega_{AB} D(1)^{AB} \right) & 0 \\ 0 & \exp \left( \frac{1}{4} \omega_{AB} D(2)^{AB} \right) \end{pmatrix} \text{ and } \omega_{AB} = 0 \right\}$$

This is one of the real forms of Spin$(7, C)$.

**IX. APPENDIX 2: TRIALITY AND $S_0(4, 4; R)$ COVARIANT MULTIPLICATIONS**

Let $V_1, V_2,$ and $V_3$ be vector spaces over $R$. A duality is a nondegenerate bilinear map $V_1 \times V_2 \to R$. A triality is a nondegenerate trilinear map $V_1 \times V_2 \times V_3 \to R$. A triality may be associated with a bilinear map that some authors call a “multiplication” [1] by dualizing, $V_1 \times V_2 \to *V_3 \cong V_3$.

Let $u$ denote the unit field and let $\psi(1) \in S_x^{(1)}(E_{4,4})$ and $\psi(2) \in S_x^{(1)}(E_{4,4})$. Under the action of $S_0(4, 4; R)$ we assume that $u \mapsto \bar{u} = D(1) \ u$, $\psi(1) \mapsto \bar{\psi}_1 = D(1) \ \psi(1)$ and $\psi(2) \mapsto \bar{\psi}_2 = D(2) \ \psi(2)$. Consider the following two multiplications that possess covariant transformation laws under the action of $S_0(4, 4; R) \Rightarrow S_0(4, 4; R)$. The first multiplication $m_1^A : E_{4,4} \times E_{4,4} \to E_{4,4}$ is defined by

$$Q^A = \frac{1}{\sqrt{\bar{u} \sigma u}} \bar{u} \sigma \bar{u} \tau^A \psi(2).$$

For fixed $u^a$, $Q^A \in E_{4,4}$ depends on 8 real parameters arranged into the type-2 spinor $\psi(2)$.

The second multiplication $m_2^{AB} : E_{4,4} \times E_{4,4} \to V_3$ has an image in $V_3 \cong E_{4,4} \times E_{4,4}$, and depends on 8 real parameters (for fixed $u^a$) arranged into the type-1 spinor $\psi(1)$:

$$Q^{AB} = \frac{1}{\sqrt{\bar{u} \sigma u}} \bar{u} \sigma \bar{u} \tau^A \tau^B \psi(1).$$
For fixed \( u \), \( Q^{AB} \) possesses only 8 degrees of freedom corresponding to the 8 independent degrees of freedom of \( \psi_{(1)} \), so we also refer to this map as a “multiplication.”

Eq. \([56]\) may be easily be solved for the components \( \psi_{(1)}^a = \psi_{(1)}^a (Q^{AB}) \). Consider

\[
\begin{align*}
\frac{1}{\sqrt{u \sigma u}} (\tau_B) (Q^{AB} \tau_{(A)} u) &= \frac{1}{u \sigma u} (\tau_B) (\tau_{(A)} u) \left( \tilde{u} \sigma \tau^A \tau^B \psi_{(1)} \right) \\
&= \frac{1}{u \sigma u} (\tau_B) (\tau_{(A)} u \tilde{u} \sigma \tau^{(A)}) \tau^B \psi_{(1)} \\
&= (\tau_B) \tau^B \psi_{(1)} \text{ by Eq.}\left[11\right] \text{ or Eq.}\left[12\right] \\
&= (\tau_B \tau^B) \psi_{(1)} \\
&= 8 \psi_{(1)}
\end{align*}
\]

Similarly,

\[
\psi_{(2)} = \frac{1}{\sqrt{u \sigma u}} \tau_A u Q^A. \tag{58}
\]

In this paragraph Greek indices run from 1 to 4, \( \alpha, \beta, \ldots, \mu, \nu, \ldots \) = 1, \ldots, 4, while Latin continue to run from 1 to 8, \( A, B, \ldots, a, b, \ldots \) = 1, \ldots, 8. It is convenient to define a \( \text{SO}(3, 1; \mathbb{R}) \)-invariant symplectic structure \( \Omega \) on \( \mathbb{E}_{4,4} \) (and a complex structure on the split octonion algebra) by

\[
\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{59}
\]

where 0 denotes the 4 x 4 zero matrix and 1 denotes the 4 x 4 unit matrix. The \( Q^{AB} \) may be represented in terms of an arbitrary antisymmetric \( M_{3,1} \) rank 2 tensor \( F^\beta_\alpha = -F^\alpha_\beta \) and two \( \text{SO}(3, 1; \mathbb{R}) \) scalars \( x_4 \) and \( x_8 \) according to

\[
Q^{AB} = \left\{ \begin{pmatrix} F^\alpha_\beta & *F^\alpha_\beta \\ *F^\beta_\alpha & F^\beta_\alpha \end{pmatrix} \right\}^{AB}_{AB} + Q^{48} \Omega^{AB} + Q^{88} G^{AB}, \tag{60}
\]

where \( *F^\alpha_\beta \) is dual to \( F^\alpha_\beta \) and defined by \( *F^{\mu \nu} = -\frac{1}{2} \epsilon^{\alpha \beta \mu \nu} F^\alpha_\beta \). Note that \( Q_{[AB]} = \frac{1}{2} (Q^{AB} - Q^{BA}) \) is independent of \( Q^{88} \).

Clearly, in order for Eq.\([60]\) to possess physical significance the action of \( \text{SO}(4, 4; \mathbb{R}) \) must be restricted to \( \text{SO}(3, 1; \mathbb{R}) \) in a manner that links transformations of \( x^5, x^6, x^7, x^8 \) to \( x^1, x^2, x^3, x^4 \).
A. Covariance of maps under $S_0(4; \mathbb{R})$

Let $u \mapsto \overline{u} = D_{(1)} u$, $\psi_{(1)} \mapsto \overline{\psi}_1 = D_{(1)} \psi_{(1)}$ and $\psi_{(2)} \mapsto \overline{\psi}_2 = D_{(2)} \psi_{(2)}$ under $S_0(4; \mathbb{R})$.

Consider the transformation law for the $Q_A \mapsto \overline{Q}_A$:

$\overline{Q}_A = \tilde{u} \sigma \overline{\tau}_A \overline{\psi}_2 = \overline{D}_{(1)} \ u \sigma \overline{\tau}_A \overline{D}_{(2)} \psi$

$= \tilde{u} \sigma \overline{D}_{(1)}^{-1} \overline{\tau}_A \overline{D}_{(2)} \psi$

$= L^A_B \tilde{u} \sigma \overline{\tau}_B \psi = L^A_B Q^B,$

which follows from Eq.[51]. Also $Q^{AB} \mapsto \overline{Q}^{AB}$:

$\overline{Q}^{AB} = \tilde{u} \sigma \overline{\tau}_A \overline{\tau}_B \overline{\psi}_1 = \overline{D}_{(1)} \ u \sigma \overline{\tau}_A \overline{\tau}_B \overline{D}_{(1)} \psi_{(1)}$

$= \tilde{u} \overline{D}_{(1)} \sigma \overline{\tau}_A \overline{D}_{(2)} \overline{D}_{(2)} \overline{\tau}_B \overline{D}_{(1)} \psi_{(1)}$

$= \tilde{u} \sigma \left( \overline{D}_{(1)}^{-1} \overline{\tau}_A \overline{D}_{(2)} \right) \left( \overline{D}_{(2)}^{-1} \overline{\tau}_B \overline{D}_{(1)} \right) \psi_{(1)}$

$= L^A_C L^B_D \tilde{u} \sigma \overline{\tau}_C \overline{\tau}_D \psi_{(1)} = L^A_C L^B_D Q^{CD},$

which follows from Eq.[51] and Eq.[52]. In summary, under the action of $S_0(4; \mathbb{R})$,

$$u \mapsto \overline{u} = D_{(1)} u$$

$$\psi_{(1)} \mapsto \overline{\psi}_1 = D_{(1)} \psi_{(1)}$$

$$\psi_{(2)} \mapsto \overline{\psi}_2 = D_{(2)} \psi_{(2)}.$$  

$$Q^A \mapsto \overline{Q}^A = L^A_B Q^B$$

$$Q^{AB} \mapsto \overline{Q}^{AB} = L^A_C L^B_D Q^{CD} = \{L \overline{Q} \overline{\tau}\}^{AB}$$  \hspace{1cm} (61)

X. APPENDIX 3: IRREDUCIBLE REPRESENTATION OF THE $\tau$ MATRICES

We adopt a real irreducible $8 \times 8$ matrix representation of the tau matrices (see the Appendix) in which $\overline{\tau}^8 = I_{8\times8} = \tau^8$. Then by Eq.[6] $\overline{\tau}^A = -\tau^A$ for $A = 1, \ldots, 7$. Hence, again by Eq.[6], $(\tau^A)^2$ is equal to $-I_{8\times8}$ for $A = 1,2,3$ and is equal to $I_{8\times8}$ for $A = 4,5,6,7,8$.

A particular irreducible representation of the tau matrices is
\[
\tau^1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \quad \tau^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\tau^3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \quad \tau^4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\tau^5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \quad \tau^6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\tau^7 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \quad \tau^8 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
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