Observables Generalizing Positive Operator Valued Measures

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Abstract

We discuss a generalization of POVM which is used in quantum-like modeling of mental processing.

1 Introduction

Recently quantum information was actively used in applications to cognitive science, psychology, genetics, see, e.g., [1]-[4] and references herein. As was pointed out in [4], although many mental models cannot be described by the classical probabilistic model (Kolmogorov, 1933; at least multi-Kolmogorovian framework has to be explored), the quantum probabilistic formalism also may be too special to cover all sorts of mental probabilistic nonclassicality. In [1] it was shown that there exist statistical data from cognitive psychology which cannot be represented with the aid of standard quantum observables given by Hermitian operators (the Dirac-von Neumann quantum formalism). It seems that non-Hermitian operators has to be involved in operational representation of mental observables. In [1] a quantum-like representation algorithm (QLRA) was elaborated. It can be applied to produce operator representation of observables (of any origin: physical, mental, biological), but operators are not always Hermitian. This is a consequence of the experimental fact that, for statistical data collected in cognitive psychology, the matrices of transition probabilities are not always doubly stochastic, non-doubly stochastic matrices can arise as well. And we know that the matrix of transition probabilities for two quantum observables (with nondegenerate spectra) given by Hermitian operators is always doubly stochastic. Moreover, the author of [1] was not able to describe aforementioned statistical data even by using positive operator valued measures (POVMs) which represent generalized quantum observables in quantum information theory. In this note we describe the class of observables which are produced by QLRA and study its connection with the class of “conventional
generalized observables”, POVMs. We also study the question whether the statistical data described by “new generalized observables” can also be described by POVMs. We show that the class of new generalized observables is larger than the class of POVMs.

2 New generalization of quantum mechanical formalism

Let us consider a finite dimensional Hilbert space $H$. Let $\mathcal{E} = \{e_j\}_{j=1}^n$ be an orthonormal basis:

$$\psi = \sum_j c_j e_j, c_j = c_j(\psi) \in \mathbb{C}.$$  (1)

Each $\mathcal{E}$ generates a class of (conventional) quantum observables, self-adjoint operators:

$$\hat{a}\psi = \sum_j y_j c_j(\psi)e_j,$$  (2)

where $X_a = \{y_1, ..., y_n\}, y_j \in \mathbb{R}, y_j \neq y_i$ is the range of values of $a$ (so we start with consideration of observables with nondegenerate spectra).

Let now $\mathcal{E} = \{e_j\}_{j=1}^n$ be an arbitrary basis (thus in general $\langle e_j, e_i \rangle \neq 0, i \neq j$) consisting of normalized vectors, i.e., $\langle e_j, e_j \rangle = 1$.

We generalize the Dirac-von Neumann formalism by considering observables (2) for an arbitrary $\mathcal{E}$. We also consider an arbitrary nonzero vector of $H$ as a pure quantum state. We postulate (by generalizing Born’s postulate):

$$P_\psi(a = y_j) = \frac{|c_j(\psi)|^2}{\sum_j |c_j(\psi)|^2},$$  (3)

where the coefficients $c_j(\psi)$ are given by the expansion (1).

If $\mathcal{E}$ is an orthonormal basis, then $c_j(\psi) = \langle \psi, e_j \rangle$, $\sum_j |c_j(\psi)|^2 = ||\psi||^2$ and for a normalized vector $\psi$, we obtain the ordinary Born’s rule.

Our generalization of the Dirac-von Neumann formalism is also very close to another well known (and very popular in quantum information theory) generalization of the class of quantum observables, namely, to the formalism of POVMs. To proceed in this way, we introduce projectors on the basis vectors: $\pi_j\psi = c_j(\psi)e_j$. We remark that $\pi_j^2 = \pi_j$, but in general $\pi_j^* \neq \pi_j$. We have: $|c_j(\psi)|^2 = \langle \pi_j\psi, \pi_j\psi \rangle = \langle M_j\psi, \psi \rangle$, where $M_j = \pi_j^*\pi_j$. We remark that each $M_j$ is self-adjoint and, moreover, positively defined. We also set $M = \sum_j M_j$. Then our generalization of Born’s rule can be written as:

$$P_\psi(a = y_j) = \frac{\langle M_j\psi, \psi \rangle}{\langle M\psi, \psi \rangle} = \frac{\text{Tr} \rho_\psi M_j}{\text{Tr} \rho_\psi M},$$  (4)

where $\rho_\psi = |\psi\rangle\langle \psi|$. We remark that, for an arbitrary nonzero $\psi$, the operator $\rho_\psi \geq 0$. 

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Now we generalize the conventional notion of the density operator, by considering any nonzero $\rho \geq 0$ as a generalized density operator. The corresponding generalization of Born’s postulate has the following form:

$$P_\psi(a = y_j) = \frac{\text{Tr} \rho M_j}{\text{Tr} \rho M}. \quad (5)$$

The only difference from the POVM formalism is that the operator $M \neq I$ (the unit operator).

We remark that $\langle M \psi, \psi \rangle = \sum_j |c_j(\psi)|^2 \neq 0, \psi \neq 0$. Thus (we are in the finite dimensional case) the inverse operator $M^{-1}$ is well defined.

We now proceed with our formalization and consider an arbitrary (separable) Hilbert space $H$.

**Definition 1.** A generalized quantum state is represented by an arbitrary trace class nonnegative (nonzero) operator $\rho$: $\rho \geq 0$, $0 < \text{Tr} \rho < \infty$.

**Definition 2.** A generalized quantum observable is represented by an arbitrary (so in general non normalized) positive operator valued measure $E$ on a measurable space $(X, F)$ such that $E(X) > 0$.

Thus, for a generalized quantum observable $E$, we have:

1. $E(B) \geq 0$, for any set $B \in F$, and $E(X) > 0$;
2. $E(B)^* = E(B)$, for any set $B \in F$;
3. $E(\bigcup_{j=1}^n B_j) = \sum_{j=1}^n E(B_j)$ for all disjoint sequences $\{B_j\}$ in $F$.

**Generalized Born’s rule:** Let $\rho$ and $E$ be generalized quantum state and observable, respectively. Then the probability to find the result $x$ of the $E$-measurement in a measurable set $B$ (for an ensemble represented by $\rho$) is given by

$$P_\rho(x \in B) = \frac{\text{Tr} \rho E(B)}{\text{Tr} \rho E(X)}. \quad (6)$$

We remark that $\text{Tr} \rho E(X) > 0$. To prove this, we consider the spectral expansion of the trace class operator $\rho = \sum_j q_j \psi_j \otimes \psi_j$. Here at least one $q_j > 0$. Then $\text{Tr} \rho E(X) = \sum_j q_j \langle E(X) | \psi_j, \psi_j \rangle > 0$.

We remark that probabilities with respect to non normalized (generalized) quantum state $\rho$, i.e., $\text{Tr} \rho \neq 1$, always can be rewritten as probabilities with respect to the corresponding normalized (i.e., usual) quantum state. Set

$$\rho' = \rho / \text{Tr} \rho. \quad (7)$$

Then we can scale the nominator and denominator in (6) by diving by $\text{Tr} \rho$ and obtain:

$$P_\rho(x \in B) = P_{\rho'}(x \in B) = \frac{\text{Tr} \rho' E(B)}{\text{Tr} \rho' E(X)}. \quad (8)$$

Hence, on the level of probabilities the magnitude of the trace of a generalized quantum state does not play any role. Therefore we can restrict our generalization of quantum formalism just to observables and operate with standard
quantum states.\(^1\)

The natural question arises: *Can probabilities for a (novel) generalized quantum observable be always represented as probabilities with respect to some POVM?*

By taking into account the above discussion on a possibility to express probabilities with respect to a generalized quantum state as probabilities with respect to the corresponding standard quantum state we can formulate this question in the following way. Denote the space of all density operators by the symbol \(\mathcal{D}\) and the space of all probability measures on \((X, \mathcal{F})\) by the symbol \(\mathcal{M}\). Then each POVM, \(B \rightarrow W(B), B \in \mathcal{F}\), induces the map

\[
j_W : \mathcal{D} \to \mathcal{M}, p_W(B|\rho) \equiv j_W(\rho)(B) = \text{Tr} \rho W(B). \tag{9}\]

Each (novel) generalized quantum observable, \(B \rightarrow E(B), B \in \mathcal{F}\), induces the map

\[
i_E : \mathcal{D} \to \mathcal{M}, p_E(B|\rho) \equiv j_E(\rho)(B) = \frac{\text{Tr} \rho E(B)}{\text{Tr} \rho E(X)}. \tag{10}\]

We are interested whether each map \(i_E\) can be represented as the \(j_W\)-map for some POVM \(W\).

**Example.** Consider a generalized observable acting in the qubit space and given by two operators:

\[
E_0 = 2 |0\rangle \langle 0|\quad \text{and} \quad E_1 = |1\rangle \langle 1|.
\]

Both operators are Hermitian and positive, but \(E_0 + E_1 \neq I\). Hence, the family \(E = \{E_0, E_1\}\) is a generalized observable, but not POVM. We have \(p_E(0|\rho) = \frac{2 \rho_{00}}{2 \rho_{00} + \rho_{11}}\), \(p_E(1|\rho) = \frac{\rho_{11}}{2 \rho_{00} + \rho_{11}}\). Suppose now that there exists POVM \(W = \{W_0, W_1 = I - W_0\}\) such that \(p_E(\alpha|\rho) = p_W(\alpha|\rho), \alpha = 0, 1\). Consider matrix of \(W_0\) in the basis \(|0\rangle, |1\rangle\): \(W_0 = (x_{ij})\). Take three density matrices \(\rho_1 = \text{diag}(1,0), \rho_2 = \text{diag}(0,1), \rho_3 = \text{diag}(1/2,1/2)\). Then \(p_E(0|\rho_1) = 1, p_E(0|\rho_2) = 0, p_E(0|\rho_3) = 2/3\). Hence, \(p_W(0|\rho_1) = 1 = x_{00}, p_W(0|\rho_2) = 0 = x_{11}, p_W(0|\rho_3) = 2/3 = (x_{00} + x_{11})/2 = 1/2\).

The main difference of the generalized quantum probabilities from the conventional quantum probabilities is that the new model is *nonlinear* with respect to the density operator and the conventional model is linear. Hence, it seems that quantum-like applications in cognitive science and psychology lead to nonlinear calculus of probabilities generalizing the quantum probabilistic calculus.

**References**

[1] A. Khrennikov, *Ubiquitous quantum structure: from psychology to finances*, Springer, Berlin-Heidelberg-New York, 2010.

\(^1\)However, non normalized generalizations of quantum states can appear quite naturally in some models, e.g., in prequantum classical statistical field theory, see, e.g., [5]. Here, non normalized generalizations of quantum states arise as covariance operators of subquantum random fields. And, although finally at the level of measurement we operate with their normalized versions, i.e., the standard quantum states, on the subquantum level the magnitude of the trace plays an important role.
[2] M. Asano, M. Ohya, Y. Tanaka, A. Khrennikov and I. Basieva, “On application of Gorini-Kossakowski-Sudarshan-Lindblad equation in cognitive psychology”, *Open Systems & Information Dynamics* **18**, (2011) 55-69.

[3] M. Asano, M. Ohya, Y. Tanaka, A. Khrennikov and I. Basieva, ”Quantum Uncertainty and Decision-making in Game Theory”, in *QP-PQ: Quantum Probability and White Noise Analysis* 28, WSP, Singapore, 2011, pp. 51-60.

[4] L. Accardi, A. Khrennikov, M. Ohya, “The problem of quantum-like representation in economy, cognitive science, and genetics,” in *Quantum Bio-Informatics II: From Quantum Information to Bio-Informatics*, edited by L. Accardi, W. Freudenberg, M. Ohya, WSP, Singapore, 2008, pp. 1-8.

[5] A. Khrennikov, “A pre-quantum classical statistical model with infinite-dimensional phase space”, *J. Phys. A: Math. Gen.* **38**, 9051-9073.