A Combinatorial Approach to Causal Inference

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Abstract

The objective of causal inference is to learn the network of causal relationships holding between a system of variables from the correlations that these variables exhibit; a sub-problem of which is to certify whether or not a given causal hypothesis is compatible with the observed correlations. A particularly challenging setting for causal inference is in the presence of partial information; i.e. when some of the variables are hidden/latent. In this present work, we introduce the possible worlds framework as a method for deciding causal compatibility in this difficult setting. We define a graphical object called an possible worlds diagram, which compactly depicts the set of all possible observations. From this construction, we demonstrate explicitly, using several examples, how to prove causal incompatibility. In fact, we use these constructions to prove causal incompatibility where no other techniques have been able to. Moreover, we prove that the possible worlds framework can be adapted to provide a complete solution to the possibilistic causal compatibility problem. Even more, we also discuss how to exploit graphical symmetries and cross-world consistency constraints in order to implement a hierarchy of necessary compatibility tests that we prove converges to sufficiency.

Keywords: causal inference, causal compatibility, quantum non-classicality
1 Introduction

A theory of causation specifies the effects of actions with absolute necessity. On the other hand, a probabilistic theory encodes degrees of belief and makes predictions based on limited information. A common fallacy is to interpret correlation as causation; opening an umbrella has never caused it to rain, although the two are strongly correlated. Numerous paradoxical and catastrophic consequences are unavoidable when probabilistic theories and theories of causation are confused. Nonetheless, Reichenbach’s principle asserts that correlations must admit causal explanation; after all, the fear of getting wet causes one to open an umbrella.

In recent decades, a concerted effort has been put into developing a formal theory for probabilistic causation [1, 2]. Integral to this formalism is the concept of a causal network. A causal network is a directed acyclic graph which encodes hypotheses about the causal relationships among a set of random variables. A causal model is a causal network when equipped with a explicit description of the parameters which govern the causal relationships. Given a multivariate probability distribution for a set of variables and a proposed causal network, the causal compatibility problem aims to determine the existence or non-existence of a causal model for the given causal network which can explain the correlations exhibited by the variables. More generally, the objective of causal inference is to enumerate all causal network(s) compatible with an observed distribution. Perhaps unsurprisingly, causal inference has applications in a variety of academic disciplines including economics, risk analysis, epidemiology, bioinformatics, and machine learning [1, 3, 4, 5, 6].

For physicists, a consideration of causal influence is commonplace; the theory of special/general relativity strictly prohibits causal influences between space-like separated regions of space-time [7]. Famosly, in response to Einstein, Podolsky, and Rosen’s [8] critique on the completeness of quantum theory, Bell [9] derived an observational constraint, known as Bell’s inequality, which must be satisfied by all hidden variable models which respect the causal hypothesis of relativity. Moreover, Bell demonstrated the existence of quantum-realizable correlations which violate Bell’s inequality [9]. Recently, it has been appreciated that Bell’s theorem can be understood as an instance of causal inference [10]. Contemporary quantum foundations maintains two closely related causal inference research programs. The first is to develop a theory of quantum causal models in order to facilitate a causal description of quantum theory and to better understand the limitations of quantum resources [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. The second is the continued study of classical causal inference with the purpose of distinguishing genuinely quantum behaviors from those which admit classical explanations [19, 22, 12, 23, 24, 25, 26, 27, 28]. In particular, the results of [11] suggest causal networks which support quantum non-classicality are uncommon and typically large in size; therefore, systematically finding such causal networks will require the development of new algorithmic strategies. As a consequence, quantum foundations research has relied upon, and contributed to, the
techniques and tools used within the field of causal inference [19, 25, 16, 11]. This essay and its results are concerned exclusively with the latter research program of *classical* causal inference, but does not rule out the possibility of a generalization to quantum causal inference.

There are numerous algebraic methods for tackling the causal compatibility problem. For instance, when all variables in a probabilistic system are observed, checking the compatibility status between a joint distribution and a causal network is relatively easy; compatibility holds if and only if all conditional independence constraints implied by graphical *d-separation* relations hold [1, 29]. In more realistic situations there are ethical, economic, or fundamental barriers preventing access to certain variables in a probabilistic system, and it becomes necessary to hypothesize the existence of *latent/hidden* variables in order to adequately explain the correlations expressed by the *visible/observed* variables [1, 30, 19]. In the presence of latent variables, the causal inference problem becomes considerably more difficult. To simplify the problem, one can make assumptions about the nature of the latent variables and the parameters which govern them [31, 32, 33]. For instance, when the latent variables are assumed to have a known and finite cardinality\(^1\), it becomes possible to articulate the causal compatibility problem as a finite system of polynomial equality and inequality equations with a finite list of unknowns for which non-linear quantifier elimination methods, such as Cylindrical Algebraic Decomposition [34], can provide a complete solution. Unfortunately, these techniques are only computationally tractable in the simplest of situations. Other techniques from algebraic geometry have been used in simple scenarios to approach the causal compatibility problem as well [35, 36, 33]. When no assumptions about the nature of the latent variables are made, there are a plethora of methods for deriving novel equality [37, 30] and inequality [19, 38, 23, 27, 24, 39, 40, 11, 28, 41, 42] constraints that must be satisfied by any compatible distribution. The majority of these methods are unsatisfactory on the basis that the derived constraints are necessary, but not sufficient. A notable exception is the Inflation Technique [19], which produces a hierarchy of linear programs (solvable using efficient algorithms [43, 44, 45, 46, 47]) which are necessary and sufficient [48] for determining compatibility.

The purpose of this essay is to present a combinatorial approach to causal inference with latent variables called the *possible worlds framework*. This framework is inspired by the twin networks of Pearl [1], parallel worlds of Shpitser [49], and by some original drafts of the Inflation Technique paper [19]. The possible worlds framework accomplishes three things. First, we prove its conceptual advantages by revealing that a number of disparate instances of causal incompatibility become unified under the same premise. Second, we provide a closed-form algorithm for completely solving the *possibilistic* causal compatibility problem. To demonstrate the utility of this method, we provide a solution to an unsolved problem originally reported [50]. Third, we show that the possible worlds framework provides a hierarchy of tests, much like the Inflation Technique, which solves completely the *probabilistic* causal compatibility problem.

\(^1\)The cardinality of a random variable is the size of its sample space.
problem. Notably, the hierarchy of tests presented here has a rate of convergence com-
mensurate to the hierarchy of tests provided in [48]. Moreover, unlike the Inflation
Technique, if a distribution is compatible with a causal network, then the hierarchy
of tests provided here has the advantage of returning a causal model which generates
the distribution.

The essay is organized as follows: Section 2 begins with a review of the mathe-
matical formalism behind causal modeling, including a formal definition of the causal
compatibility problem, and also introduces the notations to be used throughout the
essay. Afterwards, Section 3 introduces the possible worlds framework and defines its
central object of study: an possible worlds diagram. Section 4 applies the possible
worlds framework to prove possibilistic incompatibility between several distributions
and corresponding causal networks, culminating in an algorithm for exactly solving
the possibilistic causal compatibility problem. Finally, Section 5 establishes a hier-
archy of tests which completely solve the probabilistic causal compatibility problem.
Section 6 concludes.

Appendix A summarizes relevant results from [50] needed in Section 2. Ap-
pendix B generalizes the results of [25], placing new upper bounds on the maximum
cardinality of the latent variables, required for Sections 2 and 5.

2 A Review of Causal Modeling

This review section is segmented into three portions. First, Section 2.1 defines di-
rected graphs and their properties. Second, Section 2.2 introduces the notation and
terminology regarding probability distributions to be used throughout the remainder
of this article. Finally, Section 2.3 defines the notion of a causal model and formally
introduces the causal compatibility problem.

2.1 Directed Graphs

Definition 1. A directed graph \(G\) is an ordered pair \(G = (Q, E)\) where \(Q\) is a
finite set of vertices and \(E\) is a set edges, i.e. ordered pairs of vertices \(E \subseteq Q \times Q\). If \((q, u) \in E\) is an edge, denoted as \(q \rightarrow u\), then \(u\) is a child of \(q\) and \(q\) is a parent
of \(u\). A directed path of length \(k\) is a sequence of vertices \(q(1) \rightarrow q(2) \rightarrow \cdots \rightarrow q(k)\)
connected by directed edges. For a given vertex \(q\), \(\text{pa}_G(q)\) denotes its parents and \(\text{ch}_G(q)\) its children. If there is a directed path from \(q\) to \(u\) then \(q\) is an ancestor of \(u\) and \(u\) is a descendant of \(q\); the set of all ancestors of \(q\) is denoted \(\text{an}_G(q)\) and the set of all descendants is denoted \(\text{des}_G(q)\). The definition for parents, children, ancestors and descendants of a single vertex \(q\) are applied disjunctively to sets of
A directed graph is **acyclic** if there is no directed path of length \( k > 1 \) from \( q \) back to \( q \) for any \( q \in Q \) and **cyclic** otherwise. For example, Figure 1 depicts the difference between cyclic and acyclic directed graphs.

**Definition 2.** The subgraph of \( G = (Q,E) \) induced by \( W \subset Q \), denoted \( \text{sub}_G(W) \), is given by,

\[
\text{sub}_G(W) = (W, E \cap (W \times W)),
\]

i.e. the graph obtained by taking all edges from \( E \) which connect members of \( W \).

### 2.2 Probability Theory

**Definition 3** (Probability Theory). A **probability space** is a triple \((\Omega, \Xi, P)\) where the state space \( \Omega \) is the set of all possible outcomes, \( \Xi \subseteq 2^{\Omega} \) is the set of events forming a \( \sigma \)-algebra over \( \Omega \), and \( P \) is a \( \sigma \)-additive function from events to probabilities such that \( P(\Omega) = 1 \).

**Definition 4** (Probability Notation). For a collection of random variables \( X_{i} = \{X_{1}, X_{2}, \ldots, X_{k}\} \) indexed by \( i \in I = \{1, 2, \ldots, k\} \) where each \( X_{i} \) takes values from \( \Omega_{i} \), a joint distribution \( P_{\Xi} = P_{12\ldots k} \) assigns probabilities to outcomes from \( \Omega_{\Xi} = \prod_{i \in I} \Omega_{i} \). The event that each \( X_{i} \) takes value \( x_{i} \), referred to as a **valuation** of \( X_{\Xi}^{2} \), is denoted \(^{2}\)A valuation is a particular type of event in \( \Xi \) where the random variables take on definite values.
as,
\[ P_T(x_T) = P_{12\ldots k}(x_1 x_2 \ldots x_k) = P(X_1 = x_1, X_2 = x_2, \ldots X_k = x_k). \] (4)

A point distribution \( P_T(y_I) = 1 \) for a particular event \( y_I \in \Omega_I \) is expressed using square brackets,
\[ P_T(y_I) = 1 \Leftrightarrow P_T(x_I) = [y_I](x_I) = \delta(y_I, x_I) = \prod_{i \in I} \delta(y_i, x_i). \] (5)

The set of all probability distributions over \( \Omega_I \) is denoted as \( P_I \). Let \( k_i \) denote the cardinality or size of \( \Omega_i \). If \( X_i \) is discrete, then \( k_i = |\Omega_i| \), otherwise \( X_i \) is continuous and \( k_i = \infty \).

### 2.3 Causal Models and Causal Compatibility

A **causal model** represents a complete description of the causal mechanisms underlying a probabilistic process. Formally, a causal model is a pair of objects \((G, P)\), which will be defined in turn. First, \( G \) is a directed acyclic graph \((Q, E)\), whose vertices \( q \in Q \) represent random variables \( X_Q = \{X_q \mid q \in Q\} \). The purpose of a causal network is to graphically encode the causal relationships between the variables. Explicitly, if \( q \rightarrow u \in E \) is an edge of the causal network, \( X_q \) is said to have causal influence on \( X_u \). Consequently, the causal network predicts that given complete knowledge of a valuation of the parental variables \( X_{pa_G}(u) = \{X_q \mid q \in pa_G(u)\} \), the random variable \( X_u \) should become independent of its non-descendants\(^4\) \[1\]. With observation as motivation, the **causal parameters** \( P \) of a causal model are a family of conditional probability distributions \( P_{q|pa_G(q)} \) for each \( q \in Q \). In the case that \( q \) has no parents in \( G \), the distribution is simply unconditioned. The purpose of the causal parameters are to predict a joint distribution \( P_Q \) on the configurations \( \Omega_Q \) of a causal network,
\[ \forall x_Q \in \Omega_Q, \quad P_Q(x_Q) = \prod_{q \in Q} P_{q|pa_G(q)}(x_q|x_{pa_G(q)}). \] (6)

If the hypotheses encoded within a causal network \( G \) are correct, then the observed distribution over \( \Omega_Q \) should factorize according to Equation 6. Unfortunately, as discussed in Section 1, there are often ethical, economic, or fundamental obstacles preventing access to all variables of a system. In such cases, it is customary to partition the vertices of causal network into two disjoint sets; the **visible (observed) vertices** \( V \), and the **latent (unobserved) vertices** \( L \) (for example, see Figure 2). Additionally, we denote visible parents of any vertex \( q \in V \cup L \) as \( vpa_G(q) = V \cap pa_G(q) \) and analogously for the latent parents \( lpa_G(q) = L \cap pa_G(q) \).

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\(^3\)It is seldom necessary to make the distinction between the random variable \( X_q \) and the index/vertex \( q \); this essay henceforth treats them as synonymous.

\(^4\)This is known as the local Markov property.
Figure 2: The causal network $G_2$ in this figure encodes a causal hypothesis about the causal relationships between the visible variables $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the latent variables $L = \{\ell_1, \ell_2, \ell_3\}$; e.g. $v_2$ experiences a direct causal influence from each of its parents, both visible $vpa_{G_2}(v_2) = \{v_1, v_4\}$ and latent $lpa_{G_2}(v_2) = \{\ell_1, \ell_2\}$.

Throughout this essay, visible variables and edges connecting them are colored blue whereas all latent variables and all other edges are colored red.

In the presence of latent variables, Equation 6 stills makes a prediction about the joint distribution $P_{V \cup L}(x_V, \lambda_L)^5$ over the visible and latent variables, albeit an experimenter attempting to verify or discredit a causal hypothesis only has access to the marginal distribution $P_V(x_V)$. If $\Omega_L$ is continuous,

$$\forall x_V \in \Omega_V, P_V(x_V) = \int_{\lambda_L \in \Omega_L} dP_{V \cup L}(x_V, \lambda_L)$$

(7)

If $\Omega_L$ is discrete,

$$\forall x_V \in \Omega_V, P_V(x_V) = \sum_{\lambda_L \in \Omega_L} P_{V \cup L}(x_V, \lambda_L).$$

(8)

A natural question arises; in the absence of information about the latent variables $L$, how can one determine whether or not their causal hypotheses are correct? The principle purpose of this essay is to provide the reader with methods for answering this question.

In general, other than being a directed acyclic graph, there are no restrictions placed on a causal network with latent variables. Nonetheless, [50] demonstrates that every causal network $G$ can be converted into a standard form that is observationally equivalent to $G$ where the latent variables are exogenous (have no parents) and whose children sets are isomorphic to the facets of a simplicial complex over $V^6$. Appendix A summarizes the relevant results from [50] necessary for making this claim.

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5. This essay adopts the notational convenient of using $\lambda_\ell \in \Omega_\ell$ for valuations of latent variables $\ell \in L$ to differentiate them from valuations $x_v \in \Omega_v$ of observed variables $v \in V$.

6. Appendix A.1 briefly discusses what it means for two causal networks to be observationally equivalent.
Additionally, Appendix B demonstrates that any finite distribution \( P_V \) which satisfies the causal hypotheses (i.e. Equation 7) can be generated using deterministic causal parameters for the visible variables and moreover, the cardinalities of the latent variables can be assumed finite\(^7\). Altogether, Appendices A and B suggest that without loss of generality, we can simplify the causal compatibility problem as follows:

**Definition 5** (Functional Causal Model). A (finite) **functional causal model** for a causal network \( G = (V \cup L, E) \) is a triple \((G, F_V, P_L)\) where

\[
F_V = \{ f_v : \Omega_{\text{pa}_v} \rightarrow \Omega_v \mid v \in V \} \tag{9}
\]

are deterministic functions for the visible variables \( V \) in \( G \), and

\[
P_L = \{ P_\ell : \Omega_\ell \rightarrow [0, 1] \mid \ell \in L \} \tag{10}
\]

are finite probability distributions for the latent variables \( L \) in \( G \). A functional causal model defines a probability distribution \( P_V : \Omega_V \rightarrow [0, 1], \)

\[
\forall x_V \in \Omega_V, \quad P_V(x_V) = \prod_{\ell \in L} \sum_{\lambda_\ell \in \Omega_\ell} P_\ell(\lambda_\ell) \prod_{v \in \ell} \delta(x_v, f_v(x_{\text{pa}_v(v), \lambda_{\text{pa}_v(v)}})). \tag{11}
\]

**Definition 6** (The Causal Compatibility Problem). Given a causal network \( G = (V \cup L, E) \) and a distribution \( P_V \) over the visible variables \( V \), the **causal compatibility problem** is to determine if there exists a functional causal model \((G, F_V, P_L)\) (defined in Definition 5) such that Equation 11 reproduces \( P_V \). If such a functional causal model exists, then \( P_V \) is said to be **compatible** with \( G \); otherwise \( P_V \) is incompatible with \( G \). The set of all compatible distributions on \( V \) for a causal network \( G \) is denoted \( \mathcal{M}_V(G) \).

### 3 The Possible Worlds Framework

Consider the causal network in Figure 3a denoted \( G_{3a} \). For the sake of concreteness, suppose one is promised the latent variables are sampled from a binary sample space, i.e. \( k_\mu = k_\nu = 2 \). Let \( z_\mu = P_\mu(0_\mu) \) and \( z_\nu = P_\nu(0_\nu) \). The causal hypothesis \( G_{3a} \) predicts (via Equation 11) that observable events \((x_a, x_b, x_c) \in \Omega_a \times \Omega_b \times \Omega_c\) will be distributed according to,

\[
P_{abc} = z_\mu z_\nu \left[ \text{obs}_{abc}(0_\mu, 0_\nu) \right] + z_\mu (1 - z_\nu) \left[ \text{obs}_{abc}(0_\mu, 1_\nu) \right] +
+ (1 - z_\mu) z_\nu \left[ \text{obs}_{abc}(1_\mu, 0_\nu) \right] + (1 - z_\mu)(1 - z_\nu) \left[ \text{obs}_{abc}(1_\mu, 1_\nu) \right], \tag{12}
\]

where \( \text{obs}_{abc}(\lambda_\mu, \lambda_\nu) \in \Omega_a \times \Omega_b \times \Omega_c \) is shorthand for the observed event generated by the autonomous functions \( f_a, f_b, f_c \) for each \((\lambda_\mu, \lambda_\nu) \in \Omega_\mu \times \Omega_\nu \). In the case of \( G_{3a} \),

\[
\text{obs}_{abc}(\lambda_\mu, \lambda_\nu) = (f_a(\lambda_\mu), f_b(\lambda_\mu, \lambda_\nu), f_c(f_b(f_a(\lambda_\mu), \lambda_\nu), \lambda_\nu)). \tag{13}
\]

\(^7\)We prove this result in Appendix B by generalizing the proof techniques used in [25].
For each distinct realization \((\lambda_\mu, \lambda_\nu) \in \Omega_\mu \times \Omega_\nu\) of the latent variables, one can consider an possible world wherein the values \(\lambda_\mu, \lambda_\nu\) are not sampled according to the respective distributions \(P_\mu, P_\nu\), but instead take on definite values. In this particular example, there are \(k_\mu \times k_\nu = 2 \times 2 = 4\) distinct, possible worlds. Figure 3b represents, and uniquely colors, these possible worlds. Note that the definite valuations of the latent variables in Figure 3b are depicted using squares\(^8\). Critically, regardless of the deterministic functional relationships \(f_a, f_b, f_c\), there are identifiable consistency constraints that must hold between these worlds. For example, \(a\) is determined by a function \(f_a : \Omega_\mu \rightarrow \Omega_a\) and thus the observed value for \(a\) in the yellow \((0_\mu 0_\nu)\)-world must be exactly the same as the observed value for \(a\) in the green \((0_\mu 1_\nu)\)-world. This cross-world consistency constraint is illustrated in Figure 3c by embedding each possible world into a larger diagram with overlapping \(\lambda_\mu \rightarrow a\) subgraphs. It is important to remark that not all cross-world consistency constraints are captured by this diagram; the value of \(b\) in the yellow \((0_\mu 0_\nu)\)-world must match the value of \(b\) in the orange \((1_\mu 0_\nu)\)-world if the value of \(a\) in both possible worlds is the same.

For comparison, in the original causal network \(G_{3a}\), the vertices represented random variables sampled from distributions associated with causal parameters; whereas in the possible worlds diagram of Figure 3c, every valuation, including the latent valuations are predetermined by the functional dependences \(f_a, f_b, f_c\). For example, Figure 3d populates Figure 3c with the observable events generated by the following functional dependences,

\[
\begin{align*}
  f_a(0_\mu) &= 0_a & f_a(1_\mu) &= 1_a, \\
  f_b(0_\mu 0_\nu) &= 3_b & f_b(0_\mu 1_\nu) &= 1_b & f_b(1_\mu 0_\nu) &= 2_b & f_b(1_\mu 1_\nu) &= 0_b, \\
  f_c(0_\nu 0_\mu) &= 0_c & f_c(1_\nu 0_\mu) &= 1_c & f_c(2_\nu 1_\mu 0_\nu) &= 2_c & f_c(0_\nu 1_\mu 1_\nu) &= 3_c.
\end{align*}
\]

The utility of Figure 3d is in its simultaneous accounts of Equation 14, the causal network \(G_{3a}\) and the cross-world consistency constraints that \(G_{3a}\) induces. Nonetheless, Figure 3d fails to specify the probabilities \(z_\mu, z_\nu\) associated with the latent events. In Section 4, we utilize diagrams analogous to Figure 3d to tackle the causal satisfiability problem. Before doing so, this essay needs to formally define the possible worlds framework.

**Definition 7 (The Possible Worlds Framework).** Let \(G = (\mathcal{V} \cup \mathcal{L}, \mathcal{E})\), be a causal network with visible variables \(\mathcal{V}\) and latent variables \(\mathcal{L}\). Let \(\mathcal{F}_\mathcal{V}\) be a set of functional parameters for \(\mathcal{V}\) defined exactly as in Equation 9. The **possible worlds diagram** for the pair \((G, \mathcal{F}_\mathcal{V})\) is a directed acyclic graph \(\mathcal{D}\) satisfying the following properties:

1. (Valuation Vertices) Each vertex in \(\mathcal{D}\) consists of three pieces (consult Figure 4 for clarity):

\(^8\)This diagrammatic convention is imminently explained in more depth by Definition 7 and associated Figure 4.
(a) An example causal network $\mathcal{G}_{3a}$.

(b) The possible worlds picture for $\mathcal{G}_{3a}$.

(c) Identifying consistency constraints among possible worlds for $\mathcal{G}_{3a}$.

(d) Populating an possible worlds diagram with the deterministic functions $f_a, f_b, f_c$ in Equation 14.

Figure 3: A causal network $\mathcal{G}_{3a}$ and the creation of the possible worlds diagram when $k_\mu = k_\nu = 2$.

Figure 4: A vertex of an possible worlds diagram dissected.
(a) a subscript \( q \in \mathcal{V} \cup \mathcal{L} \) corresponding to a vertex in \( \mathcal{G} \) (indicated inside a small circle in the bottom-right corner),

(b) an integer \( \omega \) corresponding to a possible valuation/outcome \( \omega_q \) of \( q \) where \( \omega_q \in \{0_q, 1_q, \ldots \} = \Omega_q \) (indicated inside the square of each vertex),

(c) and a decoration in the form of colored outlines\(^9\) indicating which worlds (defined below) the vertex is a member of\(^{10}\).

2. (Ancestral Isomorphism) For every valuation vertex \( \omega_q \) in \( \mathcal{D} \), the ancestral subgraph of \( \omega_q \) in \( \mathcal{D} \) is isomorphic to the ancestral subgraph of \( q \) in \( \mathcal{G} \) under the map \( \omega_q \mapsto q \).

\[
\text{sub}_D(\text{an}_D(\omega_q)) \simeq \text{sub}_G(\text{an}_G(q)) \tag{15}
\]

3. (Consistency) Each valuation vertex \( x_v \) of a visible variable \( v \in \mathcal{V} \) is consistent with the output of the functional parameter \( f_v \in \mathcal{F}_V \) when applied to the valuation vertices \( \text{pa}_D(x_v) \),

\[
x_v = f_v(\text{pa}_D(x_v)) \tag{16}
\]

4. (Uniqueness) For each latent variable \( \ell \in \mathcal{L} \), and for every valuation \( \lambda_\ell \in \Omega_\ell \) there exists a unique valuation vertex in \( \mathcal{D} \) corresponding to \( \lambda_\ell \). Unlike latent valuation vertices, the valuations of visible variables \( x_v \in \Omega_v \) may be repeated (or absent) from \( \mathcal{D} \) depending on the form of \( \mathcal{F}_V \). In such cases, duplicated \( x_v \)'s are always uniquely distinguished by world membership (colored outline).

5. (Worlds) A world is a subgraph of \( \mathcal{D} \) that is isomorphic to \( \mathcal{G} \) under the map \( \omega_q \mapsto q \). Let \( \text{wor}(\lambda_\ell) \subseteq \mathcal{D} \) denote the world containing the valuation \( \lambda_\ell \in \Omega_\ell \).\(^{11}\) Furthermore, for any subset \( V \subseteq \mathcal{V} \) of visible variables, let \( \text{obs}_V(\lambda_\ell) \in \Omega_V \) denote the observed event supported by \( \text{wor}(\lambda_\ell) \).

6. (Completeness) For every valuation of the latent variables \( \lambda_\ell \in \Omega_\ell \), there exists a subgraph corresponding to \( \text{wor}(\lambda_\ell) \).\(^{12}\)

It is important to remark that although an possible worlds diagram \( \mathcal{D} \) can be constructed from the pair \( (\mathcal{G}, \mathcal{F}_V) \), the two mathematical objects are not equivalent; the functional parameters \( \mathcal{F}_V \) can contain superfluous information that never appears in \( \mathcal{D} \). We return to this subtle but crucial observation in Section 5.1.

The essential purpose of the possible worlds construction is as a diagrammatic tool for calculating the observational predictions of a functional causal model. Lemma 1 captures this essence.

\(^9\)The order of the colored outlines are arbitrary.

\(^{10}\)Every valuation vertex belongs to at least one world.

\(^{11}\)The uniqueness property guarantees that each world \( \text{wor}(\lambda_\ell) \) is uniquely determined by \( \lambda_\ell \).

\(^{12}\)Sometimes it is useful to construct an incomplete possible worlds diagram; for example, Figure 10.
Lemma 1. Given a functional causal model \((G = (V \cup L, E), F_V, P_L)\) (see Definition 5), let \(D\) be the possible worlds diagram for \((G, F_V)\). The causal compatibility criterion (Equation 11) for \(G\) is equivalent to a probabilistic sum over worlds in \(D\):

\[
P_V = \sum_{\lambda_L \in \Omega_L} \prod_{\ell \in L} P_\ell(\lambda_\ell)[\text{obs}_V(\lambda_L)].
\] (17)

The remainder of this essay explores the consequences of adopting the possible worlds framework as a method for tackling the causal compatibility problem.

4 A Complete Possibilistic Solution

Section 3 introduced the possible worlds framework as a technique for calculating the observable predictions of a functional causal model by means of Lemma 1. In this section, we use the possible worlds framework to develop a combinatorial algorithm for completely solving the possibilistic causal compatibility problem.

Definition 8. Given a probability distribution \(P_V : \Omega_V \to [0, 1]\), its support \(\sigma(P_V)\) is defined as the subset of events which are possible,

\[
\sigma(P_V) = \{x_V \in \Omega_V \mid P_V(x_V) > 0\}.
\] (18)

An observed distribution \(P_V\) is said to be possibilistically compatible with \(G\) if there exists a functional causal model \((G, F_V, P_L)\) for which Equation 11 produces a distribution with the same support as \(P_V\). The possibilistic variant of the causal compatibility problem is naturally related to the probabilistic causal compatibility problem defined in Definition 6; if a distribution is possibilistically incompatible with \(G\), then it is also probabilistically incompatible. We now proceed to apply the possible worlds framework to prove possibilistic incompatibility between a number of distribution/causal network pairs.
Figure 6: The possible worlds diagram for $G_5$ (Figure 5) is incompatible with $P_{abc}^{(20)}$ (Equation 20).

4.1 A Simple Example Causal Structure

Consider the causal network $G_5$ depicted in Figure 5. For $G_5$, the causal compatibility criteria (Equation 11) takes the form,

$$P_{abc}(x_a x_b x_c) = \sum_{\lambda_\mu \in \Omega_\mu} \sum_{\lambda_\nu \in \Omega_\nu} P_{\mu}(\lambda_\mu) P_{\nu}(\lambda_\nu) \delta(x_a, f_a(\lambda_\mu)) \delta(x_b, f_b(\lambda_\mu, \lambda_\nu)) \delta(x_c, f_c(\lambda_\nu)).$$

(19)

The following family of distributions for arbitrary $x_b, y_b \in \Omega_b$,

$$P_{abc}^{(20)} = z[0_a x_b 1_c] + (1 - z)[1_a y_b 0_c], \quad 0 < z < 1,$$

(20)

are incompatible with $G_5$. Traditionally, distributions like $P_{abc}^{(20)}$ are proven incompatible on the basis that they violate an independence constraint that is implied by $G_5$ [1], namely,

$$\forall P_{abc} \in \mathcal{M}(G_5), \quad P_{ac}(x_a x_c) = P_a(x_a) P_c(x_c).$$

(21)

Intuitively, $G_5$ provides no latent mechanism by which $a$ and $c$ can attempt to correlate (or anti-correlate). We now prove the possibilistic incompatibility of the support $\sigma(P_{abc}^{(20)})$ with $G_5$ using the possible worlds framework.

Proof. Proof by contradiction; assume that a functional causal model $F = \{f_a, f_b, f_c\}$ for $G_5$ exists such that Equation 19 produces $P_{abc}^{(20)}$. Since there are two distinct valua-
tions of the joint variables $abc$ in $P^{(20)}_{abc}$, namely $0_a x_b 1_c$ and $1_a y_b 0_c$, consider each as being sampled from two possible worlds. Without loss of generality\textsuperscript{13}, let $0_\mu 0_\nu \in \Omega_\mu \times \Omega_\nu$ denote any valuation of the latent variables such that $\text{obs}_{abc}(0_\mu 0_\nu) = 0_a x_b 1_c$. Similarly, let $1_\mu 1_\nu \in \Omega_\mu \times \Omega_\nu$ denote any valuation of the latent variables such that $\text{obs}_{abc}(1_\mu 1_\nu) = 1_a y_b 0_c$. Using these observations, initialize an possible worlds diagram using $\text{wor}(0_\mu 0_\nu)$, colored green, and $\text{wor}(1_\mu 1_\nu)$, colored violet, as seen in Figure 6a. In order to complete Figure 6a, one simply needs to specify the behavior of $b$ in two of the “off-diagonal” worlds, namely $\text{wor}(0_\mu 1_\nu)$, colored orange, and $\text{wor}(1_\mu 0_\nu)$, colored yellow (see Figure 6b). Regardless of this choice, the observed event $\text{obs}_{ac}(0_\mu 1_\nu) = 0_{a0c}$ in the orange world $\text{wor}(0_\mu 1_\nu)$ predicts $P_{ac}(0_{a0c}) > 0$\textsuperscript{14} which contradicts $P^{(20)}_{abc}$. Therefore, because the proof technique did not rely on the value of $0 < z < 1$, $P^{(20)}_{abc}$ is possibilistically incompatible with $G_5$. 

4.2 The Instrumental Structure

The causal network $G_7$ depicted in Figure 7 is known as the Instrumental Scenario \cite{41,51,52}. For $G_7$, Equation 11 takes the form,

$$P_{abc}(x_a x_b x_c) = \sum_{\lambda_\mu \in \Omega_\mu} \sum_{\lambda_\nu \in \Omega_\nu} P_\mu(\lambda_\mu) P_\nu(\lambda_\nu) \delta(x_a, f_a(\lambda_\mu)) \delta(x_b, f_b(a, \lambda_\nu)) \delta(x_c, f_c(b, \lambda_\nu)).$$

(22)

The following family of distributions,

$$P^{(23)}_{abc} = z [0_a 0_b 0_c] + (1 - z) [1_a 0_b 1_c], \quad 0 < z < 1,$$

(23)

are possibilistically incompatible with $G_7$. The Instrumental scenario $G_7$ is different from $G_5$ in that there are no observable conditional independence constraints which can prove possibilistic the incompatibility of $P^{(23)}_{abc}$. Instead, the possibilistic incompatibility of $P^{(23)}_{abc}$ is traditionally witnessed by an Instrumental inequality originally

\textsuperscript{13}There is no loss of generality in choosing $0_\mu 0_\nu$ and $1_\mu 1_\nu$ (instead of $0_\mu 1_\nu$ and $1_\mu 0_\nu$) as the valuations for the worlds because the valuation “labels” associated with latent events are arbitrary. The valuations can not be $0_\mu 1_\nu$ and $1_\mu 1_\nu$ because of the cross-world consistency constraint $\text{obs}_c(0_\mu 1_\nu) = \text{obs}_c(1_\mu 1_\nu) = f_c(1_\nu)$.

\textsuperscript{14}The probabilities associated to each world by Lemma 1 can always be assumed positive, because otherwise, those valuations would be excluded from the latent sample space $\Omega_L$. 

Figure 7: The Instrumental Scenario.
(a) Worlds \( \text{wor}(0_{\mu}0_{\nu}) \) and \( \text{wor}(1_{\mu}1_{\nu}) \) are initialized by the observed events in \( \text{wor}(1_{\mu}0_{\nu}) \) and \( \text{wor}(0_{\mu}1_{\nu}) \) leads to a contradiction with Equation 23.

(b) Populating the events in \( \text{wor}(0_{\mu}0_{\nu}) \) and \( \text{wor}(1_{\mu}1_{\nu}) \) leads to a contradiction with Equation 23.

Figure 8: A possible worlds diagram for \( \mathcal{G}_7 \) (Figure 7). The worlds are colored: \( \text{wor}(0_{\mu}0_{\nu}) \) yellow, \( \text{wor}(1_{\mu}1_{\nu}) \) orange, \( \text{wor}(1_{\mu}0_{\nu}) \) violet, \( \text{wor}(0_{\mu}1_{\nu}) \) green.

derived in [51],

\[
\forall P_{abc} \in \mathcal{M}(\mathcal{G}_7), \quad P_{bc|a}(0_b0_c|0_a) + P_{bc|a}(0_b1_c|1_a) \leq 1. \tag{24}
\]

Independently of Equation 24, we now prove possibilistic incompatibility of \( P_{abc}^{(23)} \) with \( \mathcal{G}_7 \) using the possible worlds framework.

**Proof.** Proof by contradiction; assume that a functional model \( \mathcal{F}_V = \{f_a, f_b, f_c\} \) for \( \mathcal{G}_7 \) exists such that Equation 22 produces \( P_{abc}^{(23)} \) (Equation 23). Analogously to the proof in Section 4.1, there are only two distinct valuations of the joint variables \( abc \), namely 0\(_a\)0\(_b\)0\(_c\) and 1\(_a\)0\(_b\)1\(_c\). Therefore, define two worlds one where \( \text{obs}_{abc}(0_{\mu}0_{\nu}) = 0_a0_b0_c \) and another where \( \text{obs}_{abc}(1_{\mu}1_{\nu}) = 1_a0_b1_c \). Using these two worlds, an possible worlds diagram can be initialized as in Figure 8a where \( \text{wor}(0_{\mu}0_{\nu}) \) is colored yellow and \( \text{wor}(1_{\mu}1_{\nu}) \) colored orange. In order to complete the possible worlds diagram of Figure 8a, one first needs to specify how \( b \) behaves in two possible worlds: \( \text{wor}(0_{\mu}1_{\nu}) \) colored green and \( \text{wor}(1_{\mu}0_{\nu}) \) colored violet.

\[
\begin{align*}
\text{obs}_b(1_{\mu}0_{\nu}) &= f_b(1_a0_{\nu}) = ?_b, \\
\text{obs}_b(0_{\mu}1_{\nu}) &= f_b(0_a1_{\nu}) = ?_b. \tag{25}
\end{align*}
\]

By appealing to \( P_{abc}^{(23)} \), it must be that \( \text{obs}_b(1_{\mu}0_{\nu}) = \text{obs}_b(0_{\mu}1_{\nu}) = 0_b \) as no other valuations for \( b \) are in the support of \( P_{abc}^{(23)} \). Finally, the remaining ‘unknown’ observations for \( c \) in the violet world \( \text{obs}_c(1_{\mu}0_{\nu}) = f_c(0_b0_{\nu}) \), and green world \( \text{obs}_c(0_{\mu}1_{\nu}) = f_c(0_b1_{\nu}) \) are determined respectively by the behavior of \( c \) in the orange \( \text{wor}(1_{\mu}1_{\nu}) \) and yellow
Figure 9: The Bell causal network has variables $a, b$ ‘measuring’ hidden variable $\rho$ with ‘measurement settings’ $x, y$ determined independently of $\rho$.

Figure 10: An incomplete possible worlds diagram for the Bell network $G_9$ (Figure 9) initialized by the observed events $\text{obs}_{xaby}(0, 0, 0, 0) = 0, 0, 0, 0$ and $\text{obs}_{xaby}(1, 1, 1, 1) = 1, 0, 1, 1$. The worlds are colored: $\text{wor}(0, 0, 0, 0)$ green, $\text{wor}(1, 1, 1, 1)$ violet, $\text{wor}(0, 1, 1, 1)$ magenta, $\text{wor}(0, 1, 0, 0)$ yellow, and $\text{wor}(0, 0, 1, 0)$ orange.

$\text{wor}(0, 0)$ worlds as depicted in Figure 8b. Explicitly,

\[
\begin{align*}
\text{obs}_c(1, 0, 0) &= f_c(0, 0, 0) = \text{obs}_c(0, 0, 0) = 0, \\
\text{obs}_c(0, 1, 0) &= f_c(0, 1, 0) = \text{obs}_c(1, 1, 0) = 1. 
\end{align*}
\]  

(26)

Therefore the observed events in the green and violet worlds are fixed to be,

\[
\begin{align*}
\text{obs}_{abc}(1, 0, 0) &= 1, 0, 0, 0, \\
\text{obs}_{abc}(0, 1, 0) &= 0, 0, 0, 1. 
\end{align*}
\]  

(27)

Unfortunately, neither of these events are in the support of $P_{abc}^{(23)}$, which is a contradiction; therefore $P_{abc}^{(23)}$ is possibilistically incompatible with $G_7$.

Notice that unlike the proof from Section 4.1, here we needed to appeal to the cross-world consistency constraints (Equation 26) demanded by the possible worlds framework.

4.3 The Bell Structure

Consider the causal network $G_9$ depicted in Figure 9 known as the Bell network [9]. From the perspective of causal inference, Bell’s theorem [9] states that any distribution
compatible with \( \mathcal{G}_9 \) must satisfy an inequality constraint known as a Bell inequality. For example, the inequality due to Clauser, Horne, Shimony and Holt, referred to as the CHSH inequality, constrains correlations held between \( a \) and \( b \) as \( x, y \) vary \([53]\):

\[
\forall \mathcal{P}_{xaby} \in \mathcal{M}(\mathcal{G}_9), \quad S = \langle ab|0_x0_y \rangle + \langle ab|0_x1_y \rangle + \langle ab|1_x0_y \rangle - \langle ab|1_x1_y \rangle, \quad |S| \leq 2
\]

(28)

Correlations measured by quantum theory are capable of violating this inequality up to \( S = 2\sqrt{2} \) \([54]\). This violation is not maximum; it is possible to achieve a violation of \( S = 4 \) using Popescu-Rohrlich box correlations \([55]\). The following distribution is an example of a Popescu-Rohrlich box correlation,

\[
\mathcal{P}_{xaby}^{(29)} = \frac{1}{8} \left[ [0_x0_a0_b0_y] + [0_x1_a1_b0_y] + [0_x0_a0_b1_y] + [0_x1_a1_b1_y] + [1_x0_a0_b0_y] + [1_x1_a1_b0_y] + [1_x0_a0_b1_y] + [1_x1_a1_b1_y] \right].
\]

(29)

Unlike \( \mathcal{G}_7 \), there are conditional independence constraints placed on correlations compatible with \( \mathcal{G}_9 \), namely the no-signaling constraints \( \mathcal{P}_{a|xy} = \mathcal{P}_{a|x} \) and \( \mathcal{P}_{b|xy} = \mathcal{P}_{b|y} \). Because \( \mathcal{P}_{xaby}^{(29)} \) satisfies the no-signaling constraints, the incompatibility of \( \mathcal{P}_{xaby}^{(29)} \) with \( \mathcal{G}_9 \) is traditionally proven using Equation 28. We now proceed to prove its incompatibility using the possible worlds framework.

**Proof.** Proof by contradiction; assume that a functional causal model \( \mathcal{F}_V = \{f_a, f_b, f_x, f_y\} \) for \( \mathcal{G}_9 \) exists which supports \( \mathcal{P}_{xaby}^{(29)} \) and use the possible worlds framework. Unlike the previous proofs, we only need to consider a subset of the events in \( \mathcal{P}_{xaby}^{(29)} \) to initialize an possible worlds diagram. Consider the following pair of events and associated latent valuations which support them\(^{16}\),

\[
\mathbf{obs}_{xaby}(0_x0_y0_y) = 0_x0_y0_y, \quad \mathbf{obs}_{xaby}(1_x1_y1_y) = 1_x0_y1_y.
\]

(30)

Using Equation 30, initialize the possible worlds diagram in Figure 10 with worlds \( \text{wor}(0_x0_y0_y) \) colored green and \( \text{wor}(1_x1_y1_y) \) colored violet. An unavoidable contradiction arises when attempting to populate the values for \( f_a(0_x1_y) \) in the yellow world \( \text{wor}(0_x1_y1_y) \) and \( f_b(0_y1_y) \) in the magenta world \( \text{wor}(1_x1_y0_y) \). First, the observed event \( \mathbf{obs}_{xaby}(0_x1_y1_y) = 0_xa_yb_1y \) in the yellow world \( \text{wor}(0_x1_y1_y) \) must belong to the list of possible events prescribed by \( \mathcal{P}_{xaby}^{(29)} \); a quick inspection leads one to recognize that the only possibility is \( \mathbf{obs}_{a}(0_x1_y1_y) = f_a(0_x1_y) = 1_a \). An analogous argument in the magenta world \( \text{wor}(1_x1_y0_y) \) proves that \( \mathbf{obs}_{b}(1_x1_y0_y) = f_b(0_y1_y) = 0_b \). Therefore, the observed event in the orange world \( \text{wor}(0_x1_y0_y) \) must be,

\[
\mathbf{obs}_{abcd}(0_x1_y0_y) = 0_x1_y0_y.
\]

(31)

\(^{15}\)The two variable correlation is defined as \( \langle ab|x,0_y \rangle = \sum_{i,j=1}^{2} (-1)^{i+j} \mathcal{P}_{xay}(ia,ib|x,0_y) \).

\(^{16}\)Clearly, the values of \( \lambda_x \) and \( \lambda_y \) that support these worlds must be unique. Less obvious is the possibility for these worlds to share a \( \lambda_y \) value. Albeit if they do, the event \( 0_x0_y1_y \) becomes possible, contradicting \( \mathcal{P}_{xaby}^{(29)} \) as well.
Figure 11: The Triangle network $G_{11}$ involving three visible variables $V = \{a, b, c\}$ each sharing a pair of latent variables from $\mathcal{L} = \{\mu, \nu, \rho\}$.

and therefore $P_{xaby}(0_{x}1_{a}0_{b}0_{y}) > 0$ which contradicts $P^{(29)}_{xaby}$. Therefore, $P^{(29)}_{xaby}$ is possibilistically\textsuperscript{17} incompatible with $G_{9}$.

\subsection{The Triangle Structure}

Consider the causal network $G_{11}$ depicted in Figure 11 known as the Triangle network. The Triangle has been studied extensively in recent decades \cite{42, 22, 56, 57, 11, 24, 48, 19, 28}. The following family of distributions are possibilistically incompatible with $G_{11}$\textsuperscript{18},

$$
P^{(32)}_{abc} = p_{1}[1_{a}0_{b}0_{c}] + p_{2}[0_{a}1_{b}0_{c}] + p_{3}[0_{a}0_{b}1_{c}], \quad \sum_{i=1}^{3} p_{i} = 1, \quad p_{i} > 0. \quad (32)
$$

\textit{Proof}. Proof by contradiction: assume that a functional causal model $F_{V} = \{f_{a}, f_{b}, f_{c}\}$ for $G_{11}$ exists supporting $P^{(32)}_{abc}$ and use the possible worlds framework. For each distinct event in $P^{(32)}_{abc}$, consider a world in which it happens definitely. Explicitly define,

\begin{align*}
\text{obs}_{abc}(0_{\mu}0_{\nu}0_{\rho}) &= 1_{a}0_{b}0_{c}, \quad (33) \\
\text{obs}_{abc}(1_{\mu}1_{\rho}1_{\nu}) &= 0_{a}0_{b}1_{c}, \quad (34) \\
\text{obs}_{abc}(2_{\mu}2_{\rho}1_{\nu}) &= 0_{a}1_{b}0_{c}. \quad (35)
\end{align*}

corresponding to the exterior worlds in Figure 12. Consider magenta world $\text{wor}(0_{\mu}1_{\rho}1_{\nu})$ with partially specified observation $\text{obs}_{abc}(0_{\mu}1_{\rho}1_{\nu}) = ?_{a} ?_{b} 1_{c}$. Recalling $P^{(32)}_{abc}$, whenever $c$ takes value $1_{c}$, both $a$ and $b$ take the value $0$; i.e. $0_{a}0_{b}$. Therefore, it must be that the observed event in the magenta world $\text{wor}(0_{\mu}1_{\rho}1_{\nu})$ is $\text{obs}_{abc}(0_{\mu}1_{\rho}1_{\nu}) = 0_{a}0_{b}1_{c}$. An analogous argument holds for other worlds,

\begin{align*}
\text{obs}_{abc}(0_{\mu}1_{\rho}1_{\nu}) &= ?_{a} ?_{b} 1_{c} \Rightarrow \text{obs}_{abc}(0_{\mu}1_{\rho}1_{\nu}) = 0_{a}0_{b}1_{c}, \\
\text{obs}_{abc}(2_{\mu}2_{\rho}1_{\nu}) &= ?_{a} ?_{b} 1_{c} \Rightarrow \text{obs}_{abc}(2_{\mu}2_{\rho}1_{\nu}) = 0_{a}1_{b}0_{c}, \quad (36) \\
\text{obs}_{abc}(0_{\mu}2_{\rho}0_{\nu}) &= 1_{a} ?_{b} ?_{c} \Rightarrow \text{obs}_{abc}(0_{\mu}2_{\rho}0_{\nu}) = 1_{a}0_{b}0_{c}.
\end{align*}

\textsuperscript{17}The proof holds if the probabilities of the events in $P^{(29)}_{xaby}$ are any positive value.

\textsuperscript{18}The Inflation Technique first proved the incompatibility between $P^{(32)}_{abc}$ and $G_{11}$. 
Figure 12: An incomplete possible worlds diagram for the Triangle network $G_{11}$ (Figure 11) initialized by the triplet of observed events in Equation 35. The worlds are colored: $\text{wor}(0,0,0)$ brown, $\text{wor}(1,1,1)$ yellow, $\text{wor}(2,2,2)$ orange, $\text{wor}(0,1,1)$ magenta, $\text{wor}(2,2,1)$ blue, $\text{wor}(0,2,0)$ violet, and $\text{wor}(0,2,1)$ green.
Figure 13: The Evans Causal Structure $G_{13}$.

However, the conclusions drawn by Equation 36 predict the observed event in central, green world $\text{wor}(0\mu2\rho1\nu)$ must be,

$$\text{obs}_{abc}(0\mu2\rho1\nu) = 0_a0_b0_c,$$

and therefore $P_{abc}(0_a0_b0_c) > 0$ which contradicts $P_{abc}^{(32)}$. Therefore, $P_{abc}^{(32)}$ is possibilistically incompatible with $G_{11}$.

4.5 An Evans Causal Structure

Consider the causal network in Figure 13, denoted $G_{13}$. This causal network was first mentioned by Evans [50], along with two others, as one for which no existing techniques were able to prove whether or not it was saturated; that is, whether or not all distributions were compatible with it. Here it is shown that there are indeed distributions which are possibilistically incompatible with $G_{13}$ using the framework of possible worlds diagrams. As such, this framework currently stands as the most powerful method for deciding possibilistic compatibility.

Consider the family of distributions with three possible events:

$$P_{abcd}^{(38)} = p_1[0_a0_b0_c0_d] + p_2[1_a0_b1_c0_d] + p_3[0_a1_b1_c1_d], \quad \sum_{i=1}^{3} p_i = 1, p_i > 0. \quad (38)$$

Regardless of the values for $p_1, p_2, p_3$ (and $y_d \in \Omega_d$ arbitrary), $P_{abcd}^{(38)}$ is incompatible with $G_{13}$.

Proof. Proof by contradiction. First assume that a deterministic model $\mathcal{F}_V = \{f_a, f_b, f_c, f_d\}$ for $P_{abcd}^{(38)}$ exists and adopt the possible worlds framework. Let $\text{wor}(i\mu i\nu i\rho)$ for $i \in \{1, 2, 3\}$ index the possible worlds which support the events observed in $P_{abcd}$,

$$\begin{align*}
\text{obs}_{abcd}(0\mu0\nu0\rho) &= 0_a0_b0_c0_d, \\
\text{obs}_{abcd}(1\mu1\nu1\rho) &= 1_a0_b1_c0_d, \\
\text{obs}_{abcd}(2\mu2\nu2\rho) &= 0_a1_b1_c1_d.
\end{align*}$$

(39)

Only two possible worlds are necessary for achieving a contradiction. Consulting Figure 14 for details, these possible worlds are $\text{wor}(1\mu0\nu2\rho)$ colored violet and $\text{wor}(1\mu2\nu2\rho)$. 

20
Figure 14: A possible worlds diagram for $\mathcal{G}_{13}$ initialized by the distribution in Equation 38. The worlds are colored: $\text{wor}(0,0,0)$ magenta, $\text{wor}(1,1,1)$ orange, $\text{wor}(2,2,2)$ yellow, $\text{wor}(1,0,2)$ violet, and $\text{wor}(1,2,2)$ green.
colored green. Notice that the determined value for $a$ must be the same in both worlds as it is independent of $\lambda_{\nu}$:

$$x_a = f_a(1_\mu 2_\rho) = \text{obs}_a(1_\mu 0_\nu 2_\rho) = \text{obs}_a(1_\mu 2_\nu 2_\rho).$$

There are only two possible values for $x_a$ in any world, namely $x_a = 0_a$ or $x_a = 1_a$ as given by $P^{(38)}_{abcd}$. First suppose that $x_a = 0_a$. Then in the violet world $\text{wor}(1_\mu 0_\nu 2_\rho)$, the value of $b$, to be $\text{obs}_b(1_\mu 0_\nu 2_\rho) = f_b(0_a 0_\nu) = 0_b$ is completely constrained by consistency with the magenta world $\text{wor}(0_\mu 0_\nu 0_\rho)$. Therefore, $\text{obs}_{ab}(1_\mu 0_\nu 2_\rho) = 0_a 0_b$. By analogous logic, in the violet world the value of $c$ is constrained to be $\text{obs}_c(1_\mu 0_\nu 2_\rho) = f_c(0_b 1_\mu) = 0_c$ by the orange world $\text{wor}(1_\mu 1_\nu 1_\rho)$. Therefore, $\text{obs}_{abc}(1_\mu 0_\nu 2_\rho) = 0_a 0_b 0_c$, which is a contradiction because $0_a 0_b 0_c$ is an impossible event in $P^{(38)}_{abcd}$. Therefore, it must be that $x_a = 1_a$. An unavoidable contradiction follows from attempting to populate the green world $\text{wor}(1_\mu 2_\nu 2_\rho)$ in Figure 14 with the established knowledge that $\text{obs}_a(1_\mu 2_\nu 2_\rho) = 1_a$. The value of $\text{obs}_b(1_\mu 2_\nu 2_\rho) = f_b(1_a 1_\nu)$ has yet to be specified by any possible worlds, but choosing $f_b(1_a 1_\nu) = 1_b$ would yield an impossible event $\text{obs}_a(1_\mu 2_\nu 2_\rho) = 1_a 1_b$. Therefore, it must be that $f_b(1_a 1_\nu) = 0_b$ and $\text{obs}_a(1_\mu 2_\nu 2_\rho) = 1_a 0_b$. Similarly, the orange world $\text{wor}(1_\mu 1_\nu 1_\rho)$ fixes $f_c(0_b 1_\mu) = 1_c$ and therefore $\text{obs}_{abc}(1_\mu 1_\nu 2_\rho) = 1_a 0_b 1_c$. Finally, the yellow world $\text{wor}(2_\mu 2_\nu 2_\rho)$ already determines $\text{obs}_d(1_\mu 2_\nu 2_\rho) = f_d(0_c 2_\nu 2_\rho) = 1_d$ and therefore one concludes that,

$$\text{obs}_{abcd}(1_\mu 2_\nu 2_\rho) = 1_a 0_b 1_c 1_d,$$

which is an impossible event in $P^{(38)}_{abcd}$. This contradiction implies that no functional model $F_V = \{f_a, f_b, f_c, f_d\}$ exists and therefore $P^{(38)}_{abcd}$ is possibilistically incompatible with $G_{13}$.

To reiterate, there are currently no other methods known [50] which are capable of proving the incompatibility of any distribution with $G_{13}$\textsuperscript{19}. Therefore, the possible worlds framework can be seen as the state-of-the-art technique for determining possibilistic causation.

### 4.6 Necessity and Sufficiency

Throughout this section, we explored a number of proofs of possibilistic incompatibility using the possible worlds framework. Moreover, the above examples communicate a systematic algorithm for deciding possibilistic compatibility. Given a distribution $P_V$ with support $\sigma(P_V) \subset \Omega_V$, and a causal network $G = (V \cup L, E)$, the following algorithm sketch determines if $P_V$ is possibilistically compatible with $G$.

1. Let $W = |\sigma(P_V)| < |\Omega_V|$ denote the number of possible events provided by $P_V$.

\textsuperscript{19}It is worth noting we have also proven the non-saturation of the other two causal networks mention in [50] using analogous proofs.
2. For each $1 \leq i \leq W$, create a possible world $\text{wor}(\lambda^{(i)}_\mathcal{L})$ where $\lambda^{(i)}_\mathcal{L} = \{i_\ell \mid \ell \in \mathcal{L}\}$, thus defining the latent sample space $\Omega_\mathcal{L}$.

3. Attempt to complete the possible worlds diagram $\mathcal{D}$ initialized by the worlds $\left\{\text{wor}(\lambda^{(i)}_\mathcal{L})\right\}_{i=1}^W$.

4. If an impossible event $x_V \notin \sigma(P_V)$ is produced by any “off-diagonal” world $\text{wor}(...i_\ell ...j_\ell')$ where $i \neq j$, or if a cross-world consistency constraint is broken, back-track.

Upon completing the search, there are two possibilities. The first possibility is that the algorithm returns a completed, consistent, possible worlds diagram $\mathcal{D}$. Then by Lemma 1, $P_V$ is possibilistically compatible with $\mathcal{G}$. The second possibility is that an unavoidable contradiction arises, and $P_V$ is not possibilistically compatible with $\mathcal{G}$. \footnote{A simple C implementation of the above pseudo-algorithm for boolean visible variables ($|\Omega_v| = 2, \forall v \in \mathcal{V}$) can found at github.com/tcfraser/possibilistic_causality.}

5. A Complete Probabilistic Solution

In Section 4, we demonstrated that the possible worlds framework was capable of providing a complete possibilistic solution to the causal satisfiability problem. If however, a given distribution $P_V$ happens to satisfy a causal hypothesis on a possibilistic level, can the possible worlds framework be used to determine if $P_V$ satisfies the causal hypothesis on a probabilistic level as well? In this section, we answer this question affirmatively. In particular, we provide a hierarchy of feasibility tests for probabilistic compatibility which converges exactly. In addition, we illustrate that an possible worlds diagram is the natural data network for algorithmically implementing this converging hierarchy.

5.1 Symmetry and Superfluity

This aforementioned hierarchy of tests, to be explained in Section 5.3, relies on the enumeration of all probability distributions $P_V$ which admit uniform functional causal models $(\mathcal{G}, \mathcal{F}_V, \mathcal{P}_L)$ for fixed cardinalities $k_{\mathcal{V} \cup \mathcal{L}} = \{k_q = |\Omega_q| \mid q \in \mathcal{V} \cup \mathcal{L}\}$. A functional causal model is uniform if the probability distributions $P_\ell \in \mathcal{P}_L$ over the latent variables are uniform distributions; $P_\ell : \Omega_\ell \to k_{\ell}^{-1}$. Section 5.2 discusses why uniform functional causal models are worth considering, whereas in this section, we discuss how to efficiently enumerate all probability distributions $P_V$ that are uniformly generated from fixed cardinalities $k_{\mathcal{V} \cup \mathcal{L}}$.

One method for generating all such distributions is to perform a brute force enumeration of all deterministic strategies $\mathcal{F}_V$ for fixed cardinalities $k_{\mathcal{V} \cup \mathcal{L}}$. Depending on
the details of the causal network, the number of deterministic functions of this form is poly-exponential in the cardinalities $k_{\nu,\mu}$. This method is inefficient because it fails to consider that many distinct deterministic strategies produce the exact same distribution $P_\nu$. There are two optimizations that can be made to avoid regenerations of the same distribution $P_\nu$ while enumerating all deterministic strategies $F_\nu$. These optimizations are best motivated by an example using the possible worlds framework.

Consider the causal network $G_{15a}$ in Figure 15a with visible variables $\mathcal{V} = \{a, b, c\}$ and latent variables $\mathcal{L} = \{\mu, \nu\}$. Furthermore, for concreteness, suppose that $k_\mu = k_\nu = k_a = k_b = 2$ and $k_c = 4$. Finally let $F_\nu = \{f_a, f_b, f_c\}$ be such that,

\[
\begin{align*}
  f_a(0_\mu) &= 0_a, & f_a(1_\mu) &= 1_a, & f_b(0_\mu) &= 0_b, & f_b(1_\mu) &= 1_b, \\
  f_c(0_a0_\nu0_\nu) &= 2_c, & f_c(0_a0_\nu1_\nu) &= 0_c, & f_c(1_a1_\nu0_\nu) &= 3_c, & f_c(1_a1_\nu1_\nu) &= 1_c \\
  f_c(0_a1_\nu0_\nu) &= 0_c, & f_c(0_a1_\nu1_\nu) &= 1_c, & f_c(1_a0_\nu0_\nu) &= 2_c, & f_c(1_a0_\nu1_\nu) &= 3_c.
\end{align*}
\]

The possible worlds diagram $D$ for $G_{15a}$ generated by Equation 42 is depicted in Figure 15b. If the latent valuations are distributed uniformly, the probability distribution associated with Figure 15b (as given by Equation 17) is equal to,

\[
P_{abc} = \frac{1}{4}(0_00_02_0 + 0_01_10 + 1_11_10 + 1_11_11) = \frac{1}{4}(0_00_02_0 + 0_01_10 + 1_11_10 + 1_11_11).
\]

The first optimization comes from noticing that Equation 42 specifies how $c$ would respond if provided with the valuation $1_a0_\nu1_\nu$ of its parents, namely $f_c(1_a0_\nu1_\nu) = 3_c$. Nonetheless, this hypothetical scenario is excluded from Figure 15b (crossed out in the figure) because the functional model in Equation 42 never produces an opportunity for $a$ to be different from $b$. Consequently, the functional dependences in Equation 42 contain superfluous information irrelevant to the observed probability distribution in Equation 43.

Therefore, a brute force enumeration of deterministic strategies would regenerate Equation 43 several times, once for each assignment of $c$’s behavior in these superfluous scenarios. It is possible to avoid these regenerations by using an unpopulated possible worlds diagram $D$ as a data container/network and performing a brute force enumeration of all consistent valuations of $D$.

The second optimization comes from noticing that Equation 43 contains many symmetries. Notably, independently permuting the latent valuations, $\pi_\mu : 0_\mu \leftrightarrow 1_\mu$ or $\pi_\nu : 0_\nu \leftrightarrow 1_\nu$, leaves the observed distribution in Equation 43 invariant, but maps the functional dependences $F_\nu$ of Equation 42 to different functional dependences $F_{\nu'}^\pi$ and $F_{\nu}^\pi$. These symmetries are reflected as permutations of the worlds as depicted in Figures 15c, and 15d.

Analogously, it is possible to avoid these regenerations by first pre-computing the induced action on $D$, and thus an induced action on $F_\nu$, under the permutation group $S_\mathcal{L} = \prod_{\ell \in \mathcal{L}} \text{perm}(\Omega_\ell)$. Then, using the permutation group $S_\mathcal{L}$, one only needs
to generate a representative from the equivalence classes of possible worlds diagrams $D$ under $S_L$.

Importantly, the optimizations illuminated above, namely ignoring superfluous specifications and exploiting symmetries, are universal\(^{21}\); they can be applied for any causal network. Additionally, the possible worlds framework intuitively excludes superfluous cases and directly embodies the observational symmetries, making an possible worlds diagram the ideal data network for performing a search over observed distributions.

### 5.2 The Uniformity of Latent Distributions

The purpose of this section is motivate why it is always possible to approximate any functional causal model $(G, F, V, P_L)$ with another functional causal model $(G, \tilde{F}, V, \tilde{P}_L)$ which has latent events $\lambda_L \in \tilde{\Omega}_L$ uniformly distributed. Unsurprisingly, an accurate approximation of this form will require an increase in the cardinality $|\tilde{\Omega}_L| > |\Omega_L|$ of the latent variables.

**Definition 9** (Rational Distributions). A discrete probability distribution $P$ over $\Omega$ is **rational** if every probability assigned to events in $\Omega$ by $P$ is rational,

$$\forall \lambda \in \Omega, \quad P(\lambda) = \frac{n_\lambda}{d_\lambda}, \quad \text{where} \quad n_\lambda, d_\lambda \in \mathbb{Z}. \tag{44}$$

**Definition 10** (Distance Metric for Distributions). Given two probability distributions $P, \tilde{P}$ over the same sample space $\Omega$, the distance $\Delta(P, \tilde{P})$ between $P$ and $\tilde{P}$ is defined as,

$$\Delta(P, \tilde{P}) = \sum_{x \in \Omega} |P(x) - \tilde{P}(x)| \tag{45}$$

**Theorem 2.** Let $P_\ell : \Omega_\ell \to [0, 1]$ be any discrete probability distribution on $\Omega_\ell$, then there exists a rational approximation $\tilde{P}_\ell : \Omega_\ell \to [0, 1]$,

$$\forall \lambda_\ell \in \Omega_\ell, \quad \tilde{P}_\ell(\lambda_\ell) = \frac{1}{|\Omega_u|} \sum_{\omega_u \in \Omega_u} \delta(\lambda_\ell, g(\omega_u)), \tag{46}$$

where $g : \Omega_u \to \Omega_\ell$ is deterministic and $\Delta(P_\ell, \tilde{P}_\ell) \leq \frac{|\Omega_u| - 1}{|\Omega_\ell|}$.

**Proof.** The proof is illustrated in Figure 16. In the special case that $|\Omega_\ell| = 1$, the proof is trivial; $g$ simply maps all values of $\omega_u$ to the singleton $\lambda_\ell \in \Omega_\ell$. The proof follows from a construction of $g$ using inverse uniform sampling. Given some ordering $1_\ell < 2_\ell < \cdots$ of $\Omega_\ell$ and ordering $1_u < 2_u < \cdots$ of $\Omega_u$ compute the cumulative

\(^{21}\)As a special case, causal networks (causal networks where all variables are exogenous or endogenous) contain no superfluous scenarios.
(a) A causal network $G_{15a}$ with three visible variables $\mathcal{V} = \{a, b, c\}$ and two latent variables $\mathcal{L} = \{\mu, \nu\}$.

(b) A possible worlds diagram for $G_{15a}$. The crossed out vertex is excluded because it fails to satisfy the ancestral isomorphism property.

(c) The image of Figure 15b under the permutation $0_\mu \leftrightarrow 1_\mu$.

(d) The image of Figure 15b under the permutation $0_\nu \leftrightarrow 1_\nu$.

Figure 15: Every permutation $\pi_\ell : \Omega_\ell \rightarrow \Omega_\ell$ of valuations on the latent variables maps an possible worlds diagram to another possible worlds diagram with the same observed events. The worlds are colored: $\text{wor}(0_\mu,0_\nu)$ green, $\text{wor}(0_\mu,1_\nu)$ orange, $\text{wor}(1_\mu,0_\nu)$ yellow, and $\text{wor}(1_\mu,1_\nu)$ violet.
distribution function $P_{\leq \ell}(\lambda)$ = $\sum_{\lambda' \leq \lambda} P(\lambda')$. Then the function $g : \Omega_u \rightarrow \Omega_\ell$ is defined as,

$$g(\omega_u) = \min \{\lambda_\ell \in \Omega_\ell | P_{\leq \ell}(\lambda_\ell) | \Omega_u | \geq \omega_u\}.$$  

(47)

Consequently, the proportion of $\omega_u \in \Omega_u$ values which map to $\lambda_\ell \in \Omega_\ell$ has error $\varepsilon(\lambda_\ell)$,

$$\varepsilon(\lambda_\ell) = |\Omega_u| P(\lambda_\ell) - |g^{-1}(\lambda_\ell)|;$$  

(48)

where $|\varepsilon(\lambda_\ell)| \leq 1$ for all $\lambda_\ell \in \Omega_\ell$ with the exception of the minimum (1$\mu$) and maximum ($|\Omega_\ell|$) values where $|\varepsilon(\lambda_\ell)| \leq 1/2$. Therefore, the proof follows from a direct computation of the distance $\Delta(P_\ell, \tilde{P}_\ell)$,

$$\Delta(P_\ell, \tilde{P}_\ell) = \sum_{\lambda_\ell \in \Omega_\ell} |P(\lambda_\ell) - \tilde{P}(\lambda_\ell)|;$$  

(49)

$$= \sum_{\lambda_\ell \in \Omega_\ell} \left| \frac{1}{|\Omega_u|} P(\lambda_\ell) - |g^{-1}(\lambda_\ell)| \right|;$$  

(50)

$$= \frac{1}{|\Omega_u|} \sum_{\lambda_\ell \in \Omega_\ell} |\varepsilon(\lambda_\ell)|;$$  

(51)

$$\leq \frac{1}{|\Omega_u|} \left( |\Omega_\ell| - 2 + \frac{1}{2} \right);$$  

(52)

$$= \frac{|\Omega_\ell| - 1}{|\Omega_u|}. $$  

(53)

Proof. The proof relies on Theorem 2 and can be found in Appendix C.  

Theorem 3. Let $(G, F_\mathcal{V}, P_\mathcal{L})$ be a functional causal model with cardinalities $c_\ell = |\Omega_\ell|$ for the latent variables producing distribution $P_\mathcal{V}$. Then there exists a functional causal model $(\hat{G}, \hat{F}_\mathcal{V}, \hat{P}_\mathcal{L})$ with cardinalities $k_\ell = |\hat{\Omega}_\ell|$ for the latent variables producing $\hat{P}_\mathcal{V}$ where the distributions $\hat{P}_\mathcal{L} = \{U_\ell : \hat{\Omega}_\ell \rightarrow k_\ell^{-1} \ | \ \ell \in \mathcal{L}\}$ over the latent variables are uniform. In particular, the distance between $P_\mathcal{V}$ and $\hat{P}_\mathcal{V}$ is bounded by,

$$\Delta(P_\mathcal{V}, \hat{P}_\mathcal{V}) \leq \varepsilon = \sum_{n=1}^{L} \frac{1}{n!} \left( \frac{L(C - 1)}{K} \right)^n \in O \left( \frac{LC}{K} \right);$$  

(54)

where $C = \max \{c_\ell | \ell \in \mathcal{L}\}$, $K = \min \{k_\ell | \ell \in \mathcal{L}\}$, and $L = |\mathcal{L}|$ is the number of latent variables.

Proof. The proof relies on Theorem 2 and can be found in Appendix C.
5.3 A Converging Hierarchy of Compatibility Tests

In Section 5.1, we discussed how to take advantage of the symmetries of an possible worlds diagram and the superfluities within a set of functional parameters $F_V$ in order to optimally search over functional models. In Section 5.2, we discussed how to approximate any functional causal model $(G, F_V, P_L)$ using one with uniform latent probability distributions. Here we combine these insights into a hierarchy of probabilistic compatibility tests for the causal compatibility problem.

Definition 11. Given a causal network $G$, and given cardinalities\footnote{The cardinalities for the visible variables, $k_V = \{k_v = |\Omega_v| \mid v \in V\}$, are also assumed to be known.} $k_L = \{k_\ell = |\Omega_\ell| \mid \ell \in L\}$ for the latent variables, define the uniformly induced distributions, denoted as $U^{(k_L)}(G)$, as the set of all distributions $\tilde{P}_V \in M_V(G)$ which admit of a uniform functional model $(G, F_V, P_L)$ with cardinalities $k_L$.

Recall that Section 5.1 demonstrates a method, using the possible worlds framework, for efficient generation of the entirety of $U^{(k_L)}(G)$.

Lemma 4. The uniformly induced distributions $U^{(k_L)}(G)$ form an $\varepsilon$-dense set in $M_V(G)$,

$$P_V \in M_V(G) \implies \exists \tilde{P}_V \in U^{(k_L)}(G), \quad \Delta(P_V, \tilde{P}_V) \leq \varepsilon \in \mathcal{O}\left(\frac{LC}{K}\right) \quad (55)$$

where $\varepsilon$ is a function of $K = \min \{k_\ell \mid \ell \in L\}$, the number of latent variables $L = |L|$, and $C = \max \{c_\ell \mid \ell \in L\}$ where $c_\ell$ is the minimum upper bound placed on the cardinalities of the latent variable $\ell$ by Theorem 9.
Proof. Since \( c_L = \{ c_\ell \mid \ell \in \mathcal{L} \} \) are minimum upper bounds placed on the cardinalities of the latent variables by Theorem 9, any \( P_V \in \mathcal{M}_V(\mathcal{G}) \) must admit a functional causal model with cardinalities for the latent variables at most \( c_L \). Then by Theorem 3, there exists a uniform causal model producing \( \tilde{P}_V \in \mathcal{U}^{(k_c)}_{k_L}(\mathcal{G}) \), within a distance \( \varepsilon \) given by Equation 54.

Lemma 4 forms the basis of the following compatibility test,

**Theorem 5** (The Causal Compatibility Test of Order \( K \)). For a probability distribution \( P_V \) and a causal network \( \mathcal{G} \), the causal compatibility test of order \( K = \min \{ k_\ell \mid \ell \in \mathcal{L} \} \) is defined as the following question:

Does there exist a uniformly induced distribution \( \tilde{P}_V \in \mathcal{U}^{(k_c)}_{k_L}(\mathcal{G}) \) such that

\[
\Delta(P_V, \tilde{P}_V) \leq \varepsilon(K) \]  

As \( K \to \infty \), the distance tends to zero \( \varepsilon(K) \to 0 \) and the sensitivity of the test increases. If \( P_V \notin \mathcal{M}_V(\mathcal{G}) \), then \( P_V \) will fail the test for finite \( K \). If \( P_V \in \mathcal{M}_V(\mathcal{G}) \), then \( P_V \) will pass the test for all \( K \). Moreover, for fixed \( K \), the test can readily return the functional causal model behind the best approximation \( \tilde{P}_V \).

First notice that Theorem 5 achieves the same rate of convergence as [48]. Unlike the result of [48], Theorem 5 returns a functional model which approximates \( P_V \). It is interesting to remark that the distance bound \( \varepsilon \in \mathcal{O}(LC/K) \) in Equation 55 depends on \( C = \max \{ c_\ell \mid \ell \in \mathcal{L} \} \) where \( c_\ell \) is the minimum upper bound placed on the cardinalities of the latent variable \( \ell \) by Theorem 9. As conjectured in Appendix B, it is likely that there are tighter bounds that can be placed on these cardinalities for certain causal networks. Therefore, further research into lowering these bounds will improve the performance of Theorem 5.

6 Conclusion

In conclusion, this essay examined the abstract problem of causal compatibility for causal networks with latent variables. Section 3 introduced the framework of possible worlds in an effort to provide solutions to the causal compatibility problem. Central to this framework is the notion of an possible worlds diagram, which can be viewed as a hybrid between a causal network and the functional parameters of a causal model. It does not however, convey any information about the probability distributions over the latent variables.

In Section 4, we utilized the possible worlds framework to prove possibilistic incompatibility of a number of examples. In addition, we demonstrated the utility of our approach by resolving an open problem associated with one of Evans’ [50] causal

\[\text{Here } \varepsilon(K) \text{ is the value for } \varepsilon \text{ provided by Lemma 4.}\]
networks. Particularly, we have shown the causal network in Figure 13 is incompatible with the distribution in Equation 38. Section 4 concluded with an algorithm for completely solving the possibilistic causal compatibility problem.

In Section 5, we discussed how to efficiently search through the observational equivalence classes of functional parameters using an possible worlds diagram as a data network. Afterwards, we derived bounds on the distance between compatible distributions and uniformly induced ones. By combining these results, we provide a hierarchy of necessary tests for probabilistic causal compatibility which converge in the limit.

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A Simplifying Causal Structures

A.1 Observational Equivalence

From an experimental perspective, a causal model \((\mathcal{G}, \mathcal{P})\) has the ability to predict the effects of interventions; by manually tinkering with the configuration of a system, one can learn more about the underlying mechanisms than from observations alone [1]. When interventions become impossible, because experimentation is expensive or unethical for example, it becomes possible for distinct causal networks to admit the same set of compatible correlations. An important topic in the study of causal inference is the identification of observationally equivalent causal networks. Two causal networks \(\mathcal{G}\) and \(\mathcal{G}'\) are observationally equivalent or simply equivalent if they share the same set of compatible models \(\mathcal{M}_V(\mathcal{G}) = \mathcal{M}_V(\mathcal{G}')\). For example, the direct cause causal network in Figure 17a is observationally equivalent to the common cause causal network in Figure 17b. Identifying observationally equivalent causal networks is of fundamental importance to the causal compatibility problem; if a distribution \(\mathbb{P}_V\) is known to satisfy the hypotheses of \(\mathcal{G}\), and \(\mathcal{M}_V(\mathcal{G}) = \mathcal{M}_V(\mathcal{G}')\) then it will also satisfy the hypotheses of \(\mathcal{G}'\).

A.2 Exo-Simplicial Causal Structures

In general, other than being a directed acyclic graph, there are no restrictions placed on a causal network with latent variables. Nonetheless, [50] demonstrated a number of transformations on causal networks which leave \(\mathcal{M}_V(\mathcal{G})\) invariant. Two of these transformations are the subject of interest for this section. The first concerns itself with latent vertices that have parents while the second concerns itself with parent-less latent vertices that share children. Each will be taken in turn.

Definition 12 (See Defn. 3.6 [50]). Given a causal network \(\mathcal{G} = (\mathcal{V} \cup \mathcal{L}, \mathcal{E})\) with latent vertex \(\ell \in \mathcal{L}\), the exogenized causal network \(\text{exo}_\mathcal{G}(\ell)\) is formed by taking \(\mathcal{E}\) and (i) adding an edge \(p \rightarrow c\) for every \(p \in \text{pa}_\mathcal{G}(\ell)\) and \(c \in \text{ch}_\mathcal{G}(\ell)\) if not already present, and (ii) deleting all edges of the form \(p \rightarrow \ell\) where \(p \in \text{pa}_\mathcal{G}(\ell)\). If \(\text{pa}_\mathcal{G}(\ell)\) is empty, \(\text{exo}_\mathcal{G}(\ell) = \mathcal{G}\).
Lemma 6 (See Lem. 3.7 [50]). Given a causal network $G = (V \cup L, E)$ with latent vertex $\ell \in L$, then $M_V(\text{exo}_G(\ell)) = M_V(G)$.

Proof. See proof of Lem. 3.7 from [50].

The concept of exogenization is best understood with an example.

**Example 1.** Consider the causal network $G_{18a}$ in Figure 18a. In $G_{18a}$, the latent variable $\ell$ has parents $\text{pa}(\ell) = \{v_1, v_2, v_3\}$ and children $\text{ch}(\ell) = \{v_4, v_5\}$. Since the sample space $\Omega_\ell$ is unknown, its cardinality could be arbitrarily large or infinite. As a result, it has an unbounded capacity to inform its children of the valuations of its parents, e.g. $v_4$ can have complete knowledge of $v_1$ through $\ell$ and therefore adding the edge $v_1 \rightarrow v_4$ has no observational impact. Applying similar reasoning to all parents of $\ell$, i.e. applying Lemma 6, one converts $G_{18a}$ to the observationally equivalent, exogenized causal network $\text{exo}_{G_{18a}}(\ell)$ depicted in Figure 19.

Lemma 6 can be applied recursively to each latent variable $\ell \in L$ in order to transform any causal network $G$ into an observationally equivalent one wherein the latent variables have no parents (exogenous). Notice that the process of exogenization also works when latent vertices have latent parents, as is the case in Figure 18b. Also, when a latent vertex $\ell$ has no children, the process of exogenization disconnects $\ell$ from the rest of the causal network, where it can be ignored with no observational impact due to Equation 7.
Figure 19: The exogenized causal network $\text{exo}_{\ell_{18a}}(\ell)$.

The next observationally invariant transformation requires the exogenization procedure to have been applied first. In Figure 18d, $\ell_1$ and $\ell_2$ are exogenous latent variables where $\text{ch}_{\ell_{18d}}(\ell_2) \subset \text{ch}_{\ell_{18d}}(\ell_1)$. Therefore, because the sample space $\Omega_{\ell_1}$ is unspecified, it has the capacity to emulate any dependence that $v_3$ and/or $v_2$ might have on $\ell_2$. This idea is captured by Lemma 7.

**Lemma 7** (See Lem. 3.8 [50]). Let $G$ be a causal network with latent vertices $\ell, \ell' \in L$ where $\ell \neq \ell'$. If $\text{pa}_G(\ell) = \text{pa}_G(\ell') = \emptyset$, and $\text{ch}_G(\ell') \subseteq \text{ch}_G(\ell)$ then $M_V(G) = M_V(\text{sub}_G(V \cup L - \{\ell'\}))$.

**Proof.** See proof of Lem. 3.8 from [50].

An immediate corollary of Lemma 7 is that the latent variables $\{\ell \mid \ell \in L\}$, which are isomorphic to their children $\{\text{ch}(\ell) \mid \ell \in L\}$, are isomorphic to the facets of a simplicial complex over the visible variables.

**Definition 13.** An (abstract) simplicial complex, $\Delta$, over a finite set $V$ is a collection of non-empty subsets of $V$ such that:

1. $\{v\} \in \Delta$ for all $v \in V$; and
2. if $C_1 \subseteq C_2 \subseteq V$, $C_2 \in \Delta \Rightarrow C_1 \in \Delta$.

The maximal subsets with respect to inclusion are called the **facets** of the simplicial complex.

In [50], this concept led to the invention of mDAGs (or marginal directed acyclic graphs), a hybrid between a directed acyclic graph and a simplicial complex. In this work, we refrain from adopting the formalism of mDAGs and instead continue to consider causal networks as entirely directed acyclic graphs. Despite this refrain, Lemmas 6, 7 demonstrate that for the purposes of the causal satisfiability problem, the latent variables of a causal network can be assumed to be exogenous and to have children forming the facets of a simplicial complex. Causal networks which adhere to this characterization will be referred to as **exo-simplicial** causal networks. Figure 20 depicts four exo-simplicial causal networks respectively equivalent to the causal networks in Figure 18.
B Simplifying Causal Parameters

Recall that a causal model \((G, P)\) consists of a causal network \(G\) and causal parameters \(P\). Appendix A simplified the causal compatibility problem by revealing that each causal network \(G\) can be replaced with an observationally equivalent exo-simplicial causal network \(G'\) such that \(M_V(G) = M_V(G')\). The purpose of this section is to simplify the causal compatibility problem in three ways. Section B.1 demonstrates that the visible causal parameters \(\{P_{v \mid \text{pa}(v)} \mid v \in V\}\) of a causal model can be assumed to be deterministic without observational impact. Section B.2 shows that if the observed distribution is finite (i.e. \(|\Omega_V| < \infty\)), one only needs to consider finite probability distributions for the latent variables. Moreover, explicit upper bounds on the cardinalities of the latent variables can be computed.

B.1 Determinism

**Lemma 8.** If \(P_V \in M_V(G)\) and \(G\) is exo-simplicial (see Appendix A), then without loss of generality, the causal parameters \(P_{v \mid \text{pa}(v)}\) over the observed variables can be assumed to be deterministic, and consequently,

\[
\forall x_V \in \Omega_V, \quad P_V(x_V) = \prod_{\ell \in \mathcal{L}} \int_{\lambda_\ell \in \Omega_\ell} d\lambda_\ell \prod_{v \in \mathcal{L}} \delta(x_v, f_v(x_{v\text{pa}(v)}, \lambda_{\text{pa}(v)}))
\]  

**Proof.** Since \(P_V \in M_V(G)\), by definition, there exists a joint distribution \(P_{V \cup \mathcal{L}}\) (or...
density $dP_{\mathcal{V} \cup \mathcal{L}}$ admitting marginal $P_{\mathcal{V}}$ via Equation 7. Since the joint distribution satisfies Equation 6, it is possible to associate to each observed variable $X_v$ an independent random variable $E_{e_v}$ and measurable function $f_v : \Omega_{v\text{pa}(v)} \times \Omega_{1\text{pa}(v)} \times \Omega_{e_v}$ such that for all $v \in \mathcal{V}$,

$$X_v = f_v \left( X_{v\text{pa}(v)}, \Lambda_{1\text{pa}(v)}, E_{e_v} \right).$$

(57)

Therefore, by promoting each $e_v$ to the status of a latent variable in $G$ and adding an edge $e_v \rightarrow v$ to $\mathcal{E}$, each $X_v$ becomes a deterministic function of its parents. Finally, making use of the fact that $G$ is exo-simplicial, every error variable $e_v$ has its children $\text{ch}_G(e_v) = \{v\}$ nested inside the children of at least one other pre-existing latent variable. Therefore, by applying Lemma 7, $e_v$ is eliminated and one recovers the original $G$.

Essentially, Lemma 8 indicates that any non-determinism due to local noise variables $E_{e_v}$ can be emulated by the behavior of the latent variables $\mathcal{L}$.

B.2 The Finite Bound for Latent Cardinalities

In [25], it was shown that if the visible variables have finite cardinality (i.e. $k_{\mathcal{V}} = |\Omega_{\mathcal{V}}|$ is finite), then for a particular class of causal networks known as causal networks, the cardinalities of the latent variables could be assumed to be finite as well. A causal network is a causal network where all latent variables have no parents (are exogenous) and all visible variables either have no parents or no children [48]. The purpose of this section is to generalize the results of [25] to the case of exo-simplicial causal networks. Although the proof techniques presented here are similar to that of [25], the best upper bounds placed on $k_{\mathcal{L}} = |\Omega_{\mathcal{L}}|$ depends more intimately on the form of $G$. It is also anticipated that the upper bounds presented here are sub-optimal, much like [25]. It is also worth noting that the results presented here hold independently of whether or not Lemma 8 is applied.

**Theorem 9.** Let $(G, \mathcal{P})$ be a causal model with (possibly infinite) cardinalities $k_{\mathcal{L}} = \{k_\ell \mid \ell \in \mathcal{L}\}$ for the latent variables such that,

$$\forall x_{\mathcal{V}} \in \Omega_{\mathcal{V}}, \quad P_{\mathcal{V}}(x_{\mathcal{V}}) = \prod_{\ell \in \mathcal{L}} \int_{\lambda_{\ell} \in \Omega_{\ell}} dP_{\ell}(\lambda_{\ell}) \prod_{v \in \mathcal{V}} P_{v|\text{pa}(v)}(x_v|x_{v\text{pa}(v)}\lambda_{1\text{pa}(v)}),$$

(58)

produces the distribution $P_{\mathcal{V}}$. Then there exists a causal model $(G, \mathcal{P}')$ reproducing $P_{\mathcal{V}}$ with cardinalities $k_{\mathcal{L}} = \{k_\ell \mid \ell \in \mathcal{L}\}$ where each $k_\ell$ is a finite.

**Proof.** The following proof considers each latent variable $\xi \in \mathcal{L}$ independently and obtains a value for $k_\ell$ in each case. Let $\mathcal{L}' = \mathcal{L} - \{\xi\}$ denote the set of latent variables
Figure 21: A causal network $G_{21}$ that helps in visualizing the proof of Theorem 9.

With $\xi$ removed. Let $dP_{\mathcal{L}'} = \prod_{\ell \in \mathcal{L}'} dP_{\ell}$ be a probability density over $\Omega_{\mathcal{L}'}$ and consider the conditional probability distribution $P_{V|\xi}(x_{V}|\lambda_{\xi})$ given $\lambda_{\xi}$,

$$P_{V|\xi}(x_{V}|\lambda_{\xi}) = \frac{\int_{\Omega_{\mathcal{L}'}} dP_{\mathcal{L}'}(\lambda_{\mathcal{L}'}) \prod_{v \in V} P_{v|x_{pa(v)}}(x_{v}|x_{vpa(v)} \lambda_{pa(v)})}{\int_{\Omega_{\mathcal{L}'}} dP_{\mathcal{L}'}(\lambda_{\mathcal{L}'})} \tag{59}$$

Consulting Figure 21 for clarity, define the *district* $D \subseteq V$ of $\xi$ to be the maximal set of visible vertices $v$ in $G$ for which there exists an undirected path from $v$ to $\xi$ with alternating visible/latent vertices. Let $D^c = V - D$, $D = pa(D) - D$ and $D^c = pa(D^c) - D^c$. The district $D$ has the property that $P_{V|\xi}$ factorizes over $D, D^c$ \cite{50},

$$P_{V|\xi}(x_{V}|\lambda_{\xi}) = P_{D|\bar{D}\xi}(x_{D}|x_{\bar{D}}\lambda_{\xi})P_{D^c|\bar{D}^c\xi}(x_{D^c}|x_{\bar{D}^c}) \tag{60}$$

For varying $\lambda_{\xi}$, consider a vector representation $p_{\lambda_{\xi}}$ of the conditional distribution $P_{D|\bar{D}\xi}(x_{D}|x_{\bar{D}}\lambda_{\xi})$ and define $U = \{p_{\lambda_{\xi}} | \lambda_{\xi} \in \Omega_{\xi}\}$. By construction, the center of mass $p^*$ of $U$ represents $P_{D|\bar{D}\xi}(x_{D}|x_{\bar{D}})$,

$$p^* = \int_{\Omega_{\xi}} dP_{\xi}(\lambda_{\xi}) p_{\lambda_{\xi}} \tag{61}$$

$$P_{D|\bar{D}\xi}(x_{D}|x_{\bar{D}}) = \int_{\Omega_{\xi}} dP_{\xi}(\lambda_{\xi}) P_{D|\bar{D}\xi}(x_{D}|x_{\bar{D}}\lambda_{\xi}) \tag{62}$$

Therefore, by a variant of Carathodory’s theorem due to Fenchel \cite{58}, if $U$ is compact and connected, then $p^*$ can be written as a finite convex decomposition,

$$p^* = \sum_{j=1}^{\text{aff}(U)} w_{j}p_j, \quad \sum_{j} w_{j} = 1, \quad \forall i, w_i \geq 0. \tag{63}$$

where $\text{aff}(U)$ is the affine dimension of $U$. Then by letting $\Omega_{\xi} = \{0_{\xi}, 1_{\xi}, \ldots, \text{aff}(U)\xi\}$ be a finite sample space for $\xi$ distributed according to $P_{\xi}(\lambda_{\xi}) = w_{\lambda}$, by Equa-
\[ P_V(x_V) = \sum_{\lambda_\xi \in \Omega_\xi} P_\xi(\lambda_\xi)P_{V|\xi}(x_V|\lambda_\xi). \]  

(64)

Therefore, causal parameters exist reproducing \( P_V \) with cardinality \( k_\xi = \text{aff}(U) \).

What remains is to show that \( U \) is compact and to find a bound on \( \text{aff}(U) \).

Because of normalization constraints on each \( p_{\lambda_\xi} \), \( U \) is bounded. Moreover, [25] demonstrates that \( U \) can be taken to be closed as well. Again consulting Figure 21 for clarity, partition \( D \) into subsets \( A = \text{des}(\xi) \cap D \) and \( B = D - A \). This partitioning enables one to identify the following linear equality constraint placed on all points \( p_{\lambda_\xi} \):

\[
\sum_{x_A \in \Omega_A} P_{D|\xi}(x_D|x_D\lambda_\xi) = \sum_{x_A \in \Omega_A} P_{A|BD\xi}(x_A|x_Bx_D\lambda_\xi)P_{B|D\xi}(x_B|x_D\lambda_\xi) = P_{B|\overline{D}\xi}(x_B|x_{\overline{D}}),
\]

(65)

(66)

(67)

(68)

where the last equality holds because \( B \) is independent of \( \xi \) given \( \overline{D} \).24 Furthermore note that if \( U \) is not connected, it can be made connected by a scheme due to [25] which adds noisy variants of each \( p_{\lambda_\xi} \) to \( U \). Simply include a noise parameter \( \nu \in [0,1] \) such that \( \lambda'_\xi = (\lambda_\xi, \nu) \) and adjust the response functions for variables in \( A \) such that,

\[
P_{A|BD\xi}(x_A|x_Bx_D\lambda_\xi \nu) = \nu P_{A|BD\xi}(x_A|x_Bx_D\lambda_\xi) + \frac{1 - \nu}{|\Omega_A|} \]

(69)

For each degree of noise \( 0 \leq \nu \leq 1 \), Equation 69 defines a noisy model \( p_{\lambda_\xi, \nu} \) which are added to \( U \). As special cases, no noise \( \nu = 0 \) yields \( p_{\lambda_\xi, 0} = p_{\lambda_\xi} \in U \) and complete noise \( \nu = 1 \) yields \( p_{\lambda_{\xi, 1}} \) representing \( P_{B|D}(x_B|x_{\overline{D}})/|\Omega_A| \in U \) which is independent of \( \lambda_\xi \). Therefore, \( U \) is connected. Finally, the affine dimension \( \text{aff}(U) \) is at most the affine dimension of \( P_{D|\overline{D}} \) with the degrees of freedom associated with satisfying Equation 68 removed [25]. Therefore,

\[
k_\xi = \text{aff}(U) \leq \text{aff}(P_{D|\overline{D}}) - \text{aff}(P_{B|\overline{D}}) \]

(70)

\[ ^{24} \text{Every path from } b \in B \text{ to } \xi \text{ must pass through an unconditioned collider in } A \text{ and therefore the } \text{d-separation relation } B \perp \{\xi\} | D \text{ holds [1].} \]
C Proof of Theorem 3

Proof. The proof first constructs the distribution \( \tilde{P}_V \) which satisfies the error bound in Equation 54. Afterwards, a uniform functional model \((G, \tilde{F}_V, \tilde{P}_L)\) is constructed which produces \( \tilde{P}_V \). Begin by letting \( \tilde{P}_\ell \) denote the rational approximation of \( P_\ell \) for each \( \ell \in L \) as prescribed by Theorem 2. Then, let

\[
P_L(\lambda_L) = \prod_{\ell \in L} P_\ell(\lambda_\ell), \quad \tilde{P}_L(\lambda_L) = \prod_{\ell \in L} \tilde{P}_\ell(\lambda_\ell).
\]  

(71)

The joint distribution \( P_V \) and the rational approximation \( \tilde{P}_V \) are then given by,

\[
P_V(x_V) = \sum_{\lambda_L \in \Omega_L} P_L(\lambda_L) \delta(x_V, F_V(\lambda_L)),
\]

(72)

\[
\tilde{P}_V(x_V) = \sum_{\lambda_L \in \Omega_L} \tilde{P}_L(\lambda_L) \delta(x_V, F_V(\lambda_L)).
\]

(73)

The distance \( \Delta(P_V, \tilde{P}_V) \) between the visible joint distributions is no greater than the distance \( \Delta(P_L, \tilde{P}_L) \) between the latent joint distributions:

\[
\Delta(P_V, \tilde{P}_V) = \sum_{x_V \in \Omega_V} |P_V(x_V) - \tilde{P}_V(x_V)|
\]

(74)

\[
= \sum_{x_V \in \Omega_V} \left| \sum_{\lambda_L \in \Omega_L} \left\{ P_L(\lambda_L) - \tilde{P}_L(\lambda_L) \right\} \delta(x_V, F_V(\lambda_L)) \right|
\]

(75)

\[
\leq \sum_{\lambda_L \in \Omega_L} \sum_{x_V \in \Omega_V} |P_L(\lambda_L) - \tilde{P}_L(\lambda_L)| \delta(x_V, F_V(\lambda_L))
\]

(76)

\[
= \sum_{\lambda_L \in \Omega_L} |P_L(\lambda_L) - \tilde{P}_L(\lambda_L)|
\]

(77)

\[
= \Delta(P_L, \tilde{P}_L)
\]

(78)

The bound in Equation 54 will be derived using Equation 48. For convenience of notation, let the latent variables be indexed \( L = \{\ell_1, \ell_2, \ldots, \ell_L\} \) and let \( L' = \{u_1, u_2, \ldots, u_L\} \) index the corresponding uniformly distributed variables as defined in Theorem 2. Then,

\[
\Delta(P_L, \tilde{P}_L)
\]

(79)

\[
= \sum_{\lambda_L \in \Omega_L} |P_L(\lambda_L) - \tilde{P}_L(\lambda_L)|
\]

(80)

\[
= \sum_{\lambda_L \in \Omega_L} \left| \prod_{j=1}^L P_{\ell_j}(\lambda_{\ell_j}) - \prod_{j=1}^L \tilde{P}_{\ell_j}(\lambda_{\ell_j}) \right|
\]

(81)

\[
= \sum_{\lambda_L \in \Omega_L} \left| \prod_{j=1}^L \left( \tilde{P}_{\ell_j}(\lambda_{\ell_j}) + \varepsilon(\lambda_{\ell_j}) \right) \frac{\lambda_{\ell_j}}{\Omega_{u_j}} \right| - \prod_{j=1}^L \tilde{P}_{\ell_j}(\lambda_{\ell_j})
\]

(82)
Here it becomes advantageous to define helper variables $\Gamma_{0,j}$ and $\Gamma_{1,j}$ such that,

$$
\Gamma_{0,j}(\lambda_L) = \tilde{P}_\ell j(\lambda_{\ell j}), \quad \Gamma_{1,j}(\lambda_L) = \frac{\varepsilon(\lambda_{\ell j})}{|\Omega_{u j}|}.
$$

(83)

Additionally, let $b \in \{0,1\}^L$ be a binary string of length $L$. Then Equation 82 becomes,

$$
\Delta(P_L, \tilde{P}_L) \leq \sum_{\lambda_L \in \Omega_L} \left| \prod_{j=1}^{L} (\Gamma_{0,j}(\lambda_L) + \Gamma_{1,j}(\lambda_L)) - \prod_{j=1}^{L} \Gamma_{0,j}(\lambda_L) \right| \tag{84}
$$

$$
= \sum_{\lambda_L \in \Omega_L} \left| \sum_{b=1}^{2^L-1} \prod_{j=1}^{L} \Gamma_{b,j}(\lambda_L) \right| \tag{85}
$$

$$
\leq \sum_{\lambda_L \in \Omega_L} \sum_{b=1}^{2^L-1} \prod_{j=1}^{L} |\Gamma_{b,j}(\lambda_L)| \tag{86}
$$

Summing over $\Gamma_{0,j}$ yields 1 due to normalization of $\tilde{P}_\ell j(\lambda_{\ell j})$ in Equation 83. However, summing over $\Gamma_{0,j}$ yields $(|\Omega_{\ell j}| - 1)/|\Omega_{u j}|$ exactly as in Theorem 2. Therefore,

$$
\Delta(P_L, \tilde{P}_L) \leq \sum_{k_1=1}^{L} \left( \frac{|\Omega_{\ell k_1}| - 1}{|\Omega_{u k_1}|} \right) + \frac{1}{2!} \sum_{k_1=1}^{L} \sum_{k_2=1}^{L} \left( \frac{|\Omega_{\ell k_2}| - 1}{|\Omega_{u k_2}|} \right) + \cdots \tag{87}
$$

(88)

In order to simplify Equation 89, let $C, K$ be defined as,

$$
C = \max \{ |\Omega_{\ell j}| \mid 1 \leq j \leq L \}, \quad K = \min \{ |\Omega_{u j}| \mid 1 \leq j \leq L \}. \tag{90}
$$

Combining Equations 78, 89, and 90, one obtains the required result,

$$
\Delta(P_V, \tilde{P}_V) \leq \sum_{n=1}^{L} \frac{1}{n!} \left( \frac{L(C - 1)}{K} \right)^n \tag{91}
$$

To conclude the proof, one needs to prove the existence of a uniform functional model $(G, \tilde{F}_V, \tilde{P}_L)$ which reproduces $\tilde{P}_V$. To do so, substitute into Equation 73 the functional form of the rational approximations (Equation 46) from Theorem 2 for each $\ell_j \in \mathcal{L}$,

$$
\tilde{P}_V(x_V) = \prod_{j=1}^{L} \sum_{\lambda_{\ell_j} \in \Omega_{\ell_j}} \left\{ \frac{1}{|\Omega_{u j}|} \sum_{\omega_{u j} \in \Omega_{u j}} \delta(\lambda_{\ell_j}, g_j(\omega_{u j})) \right\} \delta(x_V, F_V(\lambda_{\ell_1}, \lambda_{\ell_2}, \ldots, \lambda_{\ell_L})). \tag{92}
$$
Perform the sum over all latent valuations to remove the inner delta function,

\[ \tilde{P}_V(x_V) = \prod_{j \in 1}^{L} \left[ \frac{1}{|\Omega_{u_j}|} \sum_{\omega_{u_j} \in \Omega_{u_j}} \delta(x_V, F_V(g_1(\omega_{u_1})g_2(\omega_{u_2}) \ldots g_L(\omega_{u_L}))) \right]. \]  

(93)

Finally, can recursively define the functions in \( \tilde{F}_V \) to be such that \( \tilde{F}_V(\omega_{L'}) = F_V(g(\omega_{L'})) \) and consequently Equation 93 defines the uniform functional model \((G, \tilde{F}_V, \tilde{P}_L)\) which reproduces \( \tilde{P}_V \). \( \Box \)