The longitudinal-transverse bending of multilayered concrete rods taking into account the influence of gravity

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Abstract. The paper considers the problem of longitudinal-transverse bending of a multilayered concrete rod taking into account the influence of gravity. The law of deformation of each layer of the rod is adopted in the form of approximation by a polynomial of the second order. The main purpose in this work is to study the influence of gravity on the ability of the rod to withstand the applied loads. In this work, the achievement by the maximum deformation the limit allowable value in one or several layers of the rod under tension or compression, corresponding to the points of transition to the descending branch in the deformation diagram, is accepted as a criterion for the conditional limit state. The approximate solutions of these problems are found by the Bubnov-Galerkin method.

1. Introduction
Active development of mass housing construction requires the expansion of industrial nomenclature of building materials and structures, the use of new technologies in construction and creation of new durable and cost-effective building structures. At present, reinforced concrete rod elements are the most effective elements of load-bearing structures for frames of housing and industrial elements. Reinforced concrete has been for the last decades and will remain in the long term the basis of any construction object due to its physical and mechanical properties, durability and technical and economic efficiency of production and use of products made of it, as well as the availability of sufficient raw materials for the production of cement and concretes. The corresponding solutions are obtained for structures operating under flat bending conditions in the works [1-4]. The problem of determining the strength of structures under combined action of several force factors, including gravity, often appears at solving practical problems.

2. Methods
The relation of stress and strain in the \(i\)-th layer of the rod is taken in the form of a quadratic polynomial [5]

\[
\sigma_i = A_{i1} \varepsilon + A_{i2} \varepsilon^2,
\]

where \(\sigma_i\) are the stresses, \(\varepsilon\) is a strain, \(A_{i1}, A_{i2}\) are experimentally determined coefficients characterizing the material properties of the \(i\)-th layer of the rod.
Assume that the rod has the form of a cross-section, shown in figure 1, where \( h_{01}, b_{02}, b_{03}, h_{01}, h_{03} \) are parameters of cross-section of the rod, \( h_i \) is a height of the \( S_i \) layer in cross-section of the rod.

The coefficients \( A_{ij}, A_{ji} \) can be determined by using the traditional limit characteristics of concrete (\( \sigma_{s_i}, \sigma_{c_i} \) are strength limits of concrete at tension and compression of samples of the \( i \)-th layer, \( E_i \) is a modulus of concrete elasticity of the \( i \)-th layer), following the methodology given in [5]

\[
\varepsilon_{s_i}^* = \frac{\sigma_{s_i}}{E_i}, \quad \varepsilon_{c_i}^* = 2 \frac{\sigma_{c_i}}{E_i}, \quad A_{ij} = E_i, \quad A_{ji} = \frac{1}{4} \frac{E_i^2}{\sigma_{c_i}},
\]

or the least squares method if the diagrams of concrete tension and compression are available.

The values of the coefficients \( A_{ij}, A_{ji} \) which are obtained by the formulas (2) and the least squares method on the basis of diagrams of concrete deformation [6], are given in the tables 1, 2.

Comparisons of deformation diagrams constructed using the approximating formulas (1), which coefficients are determined by the least squares method and the ratios (2), with experimental diagrams.
are given in figure 2. The figure shows that experimental diagrams sufficiently approximate the solution at deformations close to the limit.

Table 1. Coefficients of the second order polynomial obtained from limit values

| Concrete grade | $E$, MPa | $\sigma^*, \text{MPa}$ | $\varepsilon^*, \%$ | $A_1$, MPa | $A_2$, MPa | $\bar{A}$, % |
|----------------|---------|------------------|------------------|---------|---------|----------|
| B10            | 17 800  | 7.48             | 0.68             | 0.0084  | 0.0038  | 17 800   |
|                |         |                  |                  | 10 589  | 572     | 38,89    |
| B30            | 32 200  | 21.8             | 1.44             | 0.135   | 0.0045  | 32 200   |
|                |         |                  |                  | 11 890  | 367     | 38,24    |
| B50            | 38 600  | 35.9             | 1.84             | 0.186   | 0.0047  | 38 600   |
|                |         |                  |                  | 10 375  | 766     | 44,63    |

Table 2. Coefficients of the second order polynomial obtained by the least squares method.

| Concrete grade | $A_1$, MPa | $A_2$, MPa | $\varepsilon^*, \%$ | $\varepsilon^*, \%$ | $\bar{A}$, % |
|----------------|-------------|-------------|------------------|------------------|----------|
| B10            | 12 488      | 5 078 524  | 0.084            | 0.0038           | 33,26    |
| B30            | 32 351      | 11 660 358 | 0.135            | 0.0045           | 38,14    |
| B50            | 56 490      | 21 305 730 | 0.186            | 0.0047           | 39,29    |

The loss of exploitation qualities of reinforced concrete structures is characterized by ultimate deformations ($\varepsilon_i^*$) under compression and ($\varepsilon_i^*$) tension, as noted in many experimental essays [1,7]. Their excess usually leads to intensive processes of cracking, which is not considered in this paper. Then, as a criterion of conditional limit state in the $i$-th layer we take the achievement of the maximum deformation of limit allowable value under tension or compression, thus the ratios (1) are based on the segment $-\varepsilon_i^* \leq \varepsilon \leq \varepsilon_i^*$.

We choose a Cartesian coordinate system with the $x$ axis directed along the rod axis with a certain reference to the cross section and with the origin at one of the ends of the rod.

Consider that it is true for internal force factors

$$N = \sum_{i=1}^{n} \int \sigma_i dS, \quad M_y = \sum_{i=1}^{n} \int \sigma_i z dS,$$

and the equilibrium equations take place

$$\frac{d^2 M_y}{dx^2} = q_x - \frac{dm_y}{dx}, \quad \frac{dN}{dx} = -q_z,$$

where $N$ is a projection of the internal force vector on the $x$ axis, $M_y$ is a projection of the internal moment vector on the $y$ axis, $q_x$, $q_z$ are projections of the distributed load vector applied to the axis of the rod, $m_y$ is a projection of the distributed moment vector on the $y$ axis.

Consider that the Kirchhoff-Liav hypotheses are valid, then the ratios take place

$$\varepsilon(x, z) = \varepsilon_0(x) + z\kappa_y(x),$$

$$\varepsilon_0(x) = \frac{du_0}{dx}, \quad \kappa_y = \frac{d^2 w_0}{dx^2},$$

where $u_0$, $w_0$ are components of the displacement vector of the points of the axial line of the rod accordingly along the $x$ and $z$ axes.

If there is a case of the rod pinched at both ends, then it is true

$$u_0(0) = w_0(0) = u_0(l) = w_0(l) = 0,$$
\[
\frac{dv_0}{dx} \bigg|_{x=0} = \frac{dv_0}{dx} \bigg|_{x=l} = 0. \tag{8}
\]

From equations (1)-(6), we obtain a system of differential equations in relation to displacements with boundary conditions (7)-(8).

The solution of these equations can be found by the Bubnov-Galerkin method \[8, 9\]. Following this method, we can determine the functions \(u_0(x), w_0(x)\) as

\[
u_0(x) = \sum_{k=1}^{N} B_k y_k(x), \quad w_0(x) = \sum_{k=1}^{N} C_k k^2 x^2, \tag{9}\]

where \(B_k, D_k\) are coefficients that determine the solution of the problem.

We satisfy the conditions (6), (7) considering that

\[
y_k(x) = r_k(x) = \cos \frac{k-1}{l} \pi x - \cos \frac{k+1}{l} \pi x. \tag{10}\]

If we substitute the displacements (9) in obtained differential equations and denote the left parts of the corresponding equations as \(L_i(x)\), \(L_i(x)\), setting as the basis functions in the Bubnov-Galerkin method the same functions as in the expansion (10)

\[
f_k(x) = \cos \frac{k-1}{l} \pi x - \cos \frac{k+1}{l} \pi x, \quad k = 1...N, \tag{11}\]

we get a system of \(2N\) algebraic equations in relation to \(2N\) unknowns \(B_k, C_k, k = 1...N\),

\[
\int_0^l L_i(x) f_k(x) dx = 0, \quad i = 1...3, \quad k = 1...N. \tag{12}\]

The value of the required coefficients \(B_k, C_k, k = 1...N\) is determined from the indicated system by any numerical method \[10\]. The value of bending moments and longitudinal force is determined by the ratios (3).

The indicated solution of the problem is convenient to find using a system of symbolic calculations, such as Maple, which allows to obtain approximate analytical solutions \[8\], due to rather cumbersome formulas at increase of the number of approximations \(N\) and errors when rounding free members.

3. Results

Consider a rod pinched at both ends (figure 3). Assume that the rod is affected by distributed loads

\[
\overline{q}_1 = q_1 e_1, \quad \overline{q}_2 = q_2 e_2, \quad \overline{q}_{3k} = q_{3k} e_1 + q_{3k} e_3, \tag{13}\]

where \(\overline{q}_1\) is a longitudinal load distributed along the surface of the lower side of the rod, \(\overline{q}_2\) is a transverse load distributed along the surface of the upper side of the rod, \(\overline{q}_{3k}\) is a vector of the distributed load of mass forces operating in the \(k\)-th section of the rod.

Then, we obtain for projections of loads along the rod length

\[
q_i = \int_{\bar{h}(0)}^{\bar{h}(z)} q_i dz + \sum_{j=1}^{k} \int_{-\bar{h}(j)}^{h(z)} q_{i,j} dz, \quad g = \int_{\bar{h}(0)}^{\bar{h}(z)} q_i dz + \sum_{j=1}^{k} \int_{-\bar{h}(j)}^{h(z)} q_{i,j} dz, \tag{14}\]

where \(b_j(z)\) is an equation of the curve bounding the section contour in the \(i\)-th layer.

The indicated loads will generate a distributed moment

\[
m_y = -\int_{\bar{h}(0)}^{\bar{h}(0)} xq_i dz + \int_{\bar{h}(0)}^{\bar{h}(z)} q_i dz - \sum_{j=1}^{k} \int_{-\bar{h}(j)}^{h(z)} (xq_{i,j} - zq_{i,j}) dz. \tag{15}\]

If we assume that the load projections (12) have the form

\[
q_1 = t_{11} + t_{12}x + t_{13}x^2, \quad q_2 = t_{21} + t_{22}x + t_{23}x^2, \quad q_{3k} = g \rho r \cos \alpha, \quad q_{3k} = g \rho r \sin \alpha, \tag{16}\]

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where \( g \) is gravity acceleration, \( \alpha \) is an angle that determines the gravity direction in relation to the axis of the rod, \( \rho_i \) is density of the \( i \)-th layer of the concrete rod.

![Figure 3. A rod pinched at both ends](image)

If we assume that the rod length is \( l=7 \) m and the first layer of the rod is made of concrete grade B30, the second – B10, the third – B50, and the parameters of the rod section (figure 1) have the form

\[
\begin{align*}
  h_0 &= 0, h_1 = \frac{1}{5} m, h_2 = \frac{2}{5} m, h_3 = \frac{3}{5} m, h_{01} = \frac{1}{10} m, h_{02} = \frac{1}{2} m, \\
  b_{01} &= \frac{3}{10} m, b_{02} = \frac{4}{10} m, b_{03} = \frac{3}{10} m,
\end{align*}
\]

we take the concrete density, as there are no experiments to measure the density of these concrete grades, for all layers equal to \( \rho_1 = \rho_2 = \rho_3 = 2500 \text{ kg/m}^3 \) [11], the gravity acceleration \( g = -9.81 \text{ m/s}^2 \), \( N=5 \), the coefficients determining the loads (16) equal to

\[
\begin{align*}
  t_{11} &= -10 \frac{kN}{m^2}, t_{12} = -0.1 \frac{kN}{m^2}, t_{13} = -0.05 \frac{kN}{m^2}, \\
  t_{21} &= -12 \frac{kN}{m^2}, t_{22} = -0.1 \frac{kN}{m^2}, t_{23} = -0.05 \frac{kN}{m^2}.
\end{align*}
\]

In order to estimate the bearing capacity of the rod, we determine the maximum and minimum deformation in every rod section for each layer. Considering that the expression for deformation (5) is a linear function in relation to \( z \), the maximum and minimum deformation in every section for each layer should be searched as a maximum or minimum at the vertices of the broken line bounding the specified layer.

In the case when \( \alpha = 0 \), the solution corresponding to this problem has the form (figures 4-7).
In the case when \( \alpha = \frac{\pi}{6} \), the solution corresponding to this problem has the form (figures 8-11).

In the case when \( \alpha = \frac{\pi}{2} \), the solution corresponding to this problem has the form (figures 12-15).
4. Discussion
As we can see from the diagrams of maximum and minimum deformations in the cross-section of the rod (figures 6, 7, 10, 11, 14, 15, 18, 19) and tables 1 and 2, the deformations are closer to the limit values under tension ($\varepsilon > 0$). Therefore, we content ourselves with checking the case $\varepsilon \leq \varepsilon_t^*$ at estimation of the ability of the rod to withstand the applied loads. Consider that the rod has lost its
bearing capacity if in any section the maximum deformation in the cross-section layer exceeds the limit deformation corresponding to the given material of the layer under tension $\varepsilon^*$ (tables 1, 2).

Figures 6, 18 show that the maximum deformation in each layer and section in the rod, where gravity is directed along the axis of the rod ($\alpha = 0$), practically does not differ from the case when gravity is completely absent ($\rho_1 = \rho_2 = \rho_3 = 0$). It can be noticed that with an increase of the angle $\alpha$, the gravity effect on the maximum deformation also increases in the cross sections of the rod (figures 6, 10, 14, 18). The greatest deformations occur when the force of gravity acts perpendicular to the axis of the rod ($\alpha = \frac{\pi}{2}$), in which case the maximum deformations exceed similar deformations in the cross sections of the rod more than twice, but without taking into account the gravity effect (figures 14, 18).

Comparing the maximum deformations in the cross section layers of the rod with limit values (tables 1, 2), it is obvious that the indicated deformations do not exceed the limit values for every layer in each cross section in all cases, except the case if $\alpha = \frac{\pi}{2}$ (figure 14). In this case, the loss of bearing capacity occurs in the first layer of the rod near the pinched parts.

5. Conclusion
The given examples show that it is important to take into account the gravity effect when calculating the ability of multilayered rods made of heavy concretes to withstand the applied loads.

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