SLOWLY CHANGING VECTORS AND THE ASYMPTOTIC
FINITE-DIMENSIONALITY OF AN OPERATOR SEMIGROUP

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Abstract. Let $X$ be a Banach space and let $T : X \to X$ be a linear power bounded operator. Put $X_0 = \{ x \in X \mid T^n x \to 0 \}$. We prove that if $X_0 \neq X$ then there exists $\lambda \in \text{Sp}(T)$ such that, for every $\varepsilon > 0$, there is $x$ such that $\|Tx - \lambda x\| < \varepsilon$ but $\|T^n x\| > 1 - \varepsilon$ for all $n$. The technique we develop enables us to establish that if $X$ is reflexive and there exists a compactum $K \subset X$ such that $\lim \inf_{n \to \infty} \rho(T^n x, K) < \alpha(T) < 1$ for every norm-one $x \in X$ then codim $X_0 < \infty$. The results hold also for a one-parameter semigroup.

1. Introduction

In this article, $X$ is a complex Banach space, $T : X \to X$ is a linear operator whose all powers are bounded by a constant $C < \infty$. We use the notations: $B_X$ is the unit ball in $X$, $\Gamma$ is the unit circle in $\mathbb{C}$, $X_0 = \{ x \in X \mid T^n x \to_{n \to \infty} 0 \}$.

A vector $x$ is called an $\varepsilon$-almost eigenvector (or simply an $\varepsilon$-eigenvector) if there exists $\lambda \in \mathbb{C}$ such that $\|Tx - \lambda x\| < \varepsilon$. These vectors exist for each $\lambda \in \text{Sp}(T) \cap \Gamma$. In Section 1, we study $\varepsilon$-eigenvectors that do not shorten much under the iterations $T^n$.

Definition 1. Suppose that $\varepsilon > 0$. Call a vector $x \in X$ $\varepsilon$-slow if
\[
\exists \lambda \in \Gamma \mid \|Tx - \lambda x\| < \varepsilon \quad \text{and} \quad \|T^n x\| > 1 - \varepsilon \quad \forall n = 0, 1, 2 \ldots
\]

For example, the eigenvectors $x$, $Tx = \lambda x$, $\lambda \in \Gamma$, are slow.

Example 1. $T : l_2 \to l_2$ is the right shift, $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. This is an isometry; hence, every norm-one $\varepsilon$-eigenvector is $\varepsilon$-slow.

Example 1*. $T : l_2 \to l_2$ is the left shift, $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. The spectrum includes $\Gamma$ but $T^n x \to 0$ for every $x$ and there are no slow vectors.

If a vector $x$ is $\varepsilon$-slow then $T^n x$ are $C\varepsilon$-slow for each $n$, since $T(T^n x) - \lambda T^{n+1} x = T^n(T x - \lambda x)$. Thus, the angle between the (complex) lines $T^n x$ and $T^{n+1} x$ is slow not only for $n = 0$ but for all $n \in \mathbb{N}$ (i.e., the vector changes slowly under the iterations).

Remark. Our terminology is by no means connected with the terms “slow vector” and “slow variable” of the classical theory of dynamical systems. In the title, we call slow vectors slowly changing.

Definition 2. An operator $T$ has slow vectors if, for every $\varepsilon > 0$, there exist $\varepsilon$-slow vectors. An operator $T$ has many slow vectors if $\text{dim } X = \infty$ and, for every $\varepsilon > 0$ and $n < \infty$, there exist $n$-dimensional subspaces in $X$ whose unit spheres consist of $\varepsilon$-slow vectors.

If the powers $\|T^n\|$ are bounded below, i.e., there exists a number $c$ such that $c\|x\| \leq \|T^n x\|$ for every $x$, then there are many slow vectors. Indeed, if $\|x\| = 1$ and $x$ is a $c\varepsilon$-eigenvector then $\frac{x}{\|x\|}$ is $\varepsilon$-slow.
If \( X_0 = X \) then it is clear that there are no slow vectors. It turns out that the condition \( X_0 = X \) is the only obstacle to the existence of slow vectors; if \( \text{codim} \, X_0 = \infty \) then there are many slow vectors (Theorem 1.1).

In Section 2, slow vectors are used in the study of the asymptotic properties of \( T^n \).

It is known that if there exists an attracting compact set \( K \), i.e., such that
\[
\forall x \in B_X \lim_{n \to \infty} \rho(T^n x, K) = 0
\]
then \( X = X_0 \oplus L \), where \( L \) is a finite-dimensional invariant subspace in \( X \). This was proved in [1] for Markov semigroups in \( L_1 \). For an arbitrary Banach space, this was established in [2, 3]. In [4] it was proved that for the splitting \( X = X_0 \oplus L \), \( \dim L < \infty \), it suffices that a compact set \( K \) attract only sometimes, i.e.,
\[
\forall x \in B_X \liminf_{n \to \infty} \rho(T^n x, K) = 0.
\]

A semigroup \( T^n : X \to X \) or a one-parameter semigroup \( T_t : X \to X \) is called asymptotically finite-dimensional [5] if \( \text{codim} \, X_0 < \infty \). In [6, 1.3.33] the question is posed whether a semigroup is asymptotically finite-dimensional if
\[
\forall x \in B_X \liminf_{n \to \infty} \rho(T^n x, K) < \alpha(T) < 1. \tag{\ast}
\]
Clearly, the condition (\ast) follows from the above-listed analogous conditions.

In Section 2 of the article, we prove that if \( X \) is reflexive then an operator satisfying (\ast) has few slow vectors and \( \text{codim} \, X_0 < \infty \) (Theorem 2.3). Thus, we give a partial positive answer to the question of [6]. Theorem 2.3 is easily generalized to the case of a one-parameter semigroup of \( \{T_t : X \to X, t \geq 0\} \).

Note that it is in the reflexive case that the condition \( \text{codim} \, X_0 < \infty \) for a bounded semigroup implies the splitting \( X_0 \oplus L \) [7].

For nonreflexive \( X \), the author does not know the answer to the question of [6, 1.3.33].

2. Slow Vectors

**Theorem 1.1.** Suppose that \( X \) is a Banach space, \( T : X \to X \), \( \|T^n\| < C \). If \( X_0 \neq X \) then \( T \) has slow vectors. If \( \text{codim} \, X_0 = \infty \) then \( T \) has many slow vectors.

**Proof.** Without loss of generality, by passing to the equivalent norm \( \|x\| := \sup_x \{\|T^n x\|\} \), we may assume that \( \|T\| \leq 1 \).

The scheme of the proof is as follows: (a) \( X_0 = 0 \Rightarrow \) slow vectors exist; (b) \( X_0 = 0 \Rightarrow \) there are many slow vectors; (c) \( \text{codim} \, X_0 = \infty \Rightarrow \) there are many slow vectors.

(a) Introduce the norm \( \|x\|_p := \lim_{n \to \infty} \|T^n x\| \) on \( X \). In this norm, \( T \) is an isometry. The norms \( \| \| \) and \( \| \| \) need not be equivalent but
\[
\|T^k x\| \sim_{k \to \infty} \|T^k x\|_p \tag{1.1}
\]
for all \( x \). If the powers of \( T \) are not bounded below then the space \( (X, \| \|_p) \) is incomplete. Let \( \hat{X} \) be the completion of \( X \) in the norm \( \| \|_p \). Extend the isometry \( T \) of \( (X, \| \|_p) \) to \( \hat{X} \) and denote the extension by \( \hat{T} \). Take \( \lambda \in \text{Sp}(\hat{T}) \cap \Gamma \). Suppose that \( \varepsilon > 0 \) and \( \hat{x} \in \hat{X} \) is a \( \| \|_p \)-one \( \varepsilon \)-eigenvector corresponding to \( \lambda \). Involving the fact that \( \hat{T} : \hat{X} \to \hat{X} \) is an isometry, we get
\[
\|\hat{T}\hat{x} - \lambda \hat{x}\|_p < \varepsilon \quad \text{and} \quad \forall n \|\hat{T}^n \hat{x}\|_p = \|\hat{x}\|_p = 1 > 1 - \varepsilon.
\]
The set $X$ is dense in $\hat{X}$. If a vector $x \in X$ is sufficiently $\| \cdot \|_p$-close to $\hat{x}$ then all strict inequalities of the last formula remain valid. Thus, there exists a vector $x \in X$ such that

$$
\| Tx - \lambda x \|_p < \varepsilon \quad \text{and} \quad \forall n \| T^n x \|_p > 1 - \varepsilon.
$$

Thus, $x$ is an $\varepsilon$-slow vector of $T$ in the norm $\| \cdot \|_p$. Of course, $x$ need not be slow in the initial norm, since $\| Tx - \lambda x \|$ may be large. However, applying the $\| \cdot \|_p$-isometry $T^k$ to (1.2), we infer

$$
\forall k \| T(T^k x) - \lambda T^k x \|_p < \varepsilon \quad \text{and} \quad \forall n \| T^n(T^k x) \|_p > 1 - \varepsilon.
$$

By (1.1), starting from some $k = k_0$, inequalities (1.3) also hold for the norm $\| \cdot \|$, i.e., starting from some $k$, the vector $T^k x$ is as well slow in the initial norm of $X$.

(b) Take a number $\lambda \in \Gamma$ in the spectrum of the isometry $\hat{T}$ of $(X, \| \cdot \|_p)$; $\lambda$ has many $\varepsilon$-eigenvectors (there exist even infinite-dimensional spheres of $\varepsilon$-eigenvectors (see [8, Chapter IV, Theorems 5.33, 5.9]).

Let $l < \infty$ and let $W$ be an $l$-dimensional subspace in $(X, \| \cdot \|_p)$ whose $\| \cdot \|_p$-unit sphere $S$ consists of $\varepsilon$-eigenvectors. Perturbing $W$ slightly, we may assume that $W \subset X$. All vectors in $S$ satisfy (1.2) and (1.3). By (1.1), for each $x \in S$, all the vectors $T^{k_0} x$ are slow vectors for the operator $T : X \to X$ starting from some $k_0$. The ellipsoid $S$ is compact; therefore, $k_0$ may chosen common for all $x \in S$. Thus, $X$ includes $(l - 1)$-dimensional ellipsoids of the form $T^{k_0}(S)$ consisting of small vectors.

(c) Consider the quotient space $X/X_0$. Its elements are $[x] := x + X_0$. The norm $\| [x] \|$ is as follows: $\| [x] \| = \rho(x, X_0) = \inf \{ \| x - x_0 \|, x_0 \in X_0 \}$. We have $T(X_0) \subset X_0$; therefore, the operator $[T] : X/X_0 \to X/X_0$ is defined. Clearly, $[T]^n = [T^n]$. It is easy to see that if $[x] \neq [0]$ then $[T]^n[x] \neq 0$. By (b), $[T]$ has many slow vectors, i.e., for each $l$, in $X/X_0$, there are $l$-dimensional ellipsoids of slow vectors $[x]$ for $[T]$ (moreover, we may assume that these ellipsoids have the form $[S]$, where $S$ is an ellipsoid in $X$):

$$
\forall n \geq 0 \| [T^n][x] \| > 1 - \varepsilon \quad \text{and} \quad \| [T][x] - \lambda [x] \| < \varepsilon \forall [x] \in [S].
$$

Since $T^k x_0 \to 0$ for all $x_0 \in X_0$, we have

$$
\forall n \geq 0 \| T^n(T^k x) \| > 1 - \varepsilon \quad \text{and} \quad \| T(T^k x) - \lambda T^k x \| < k \to \infty \varepsilon
$$

for every $x \in [x] = x + X_0 \in [S]$.

The compactness of $S$ enables us to assert now that, starting from some $k$, the ellipsoids $T^k S \subset X$ consist of slow vectors. \hfill \Box

In what follows, we will need some properties of slow vectors.

Denote by $S_{m, \lambda}$ the operator $\frac{1}{m+1} \sum_{i=0}^m T_{\lambda i}$, the Cesaro mean of $T/\lambda$.

**Lemma 1.2.** Suppose that $\delta > 0$, $m \in \mathbb{N}$. If $T$ has slow vectors then there exists $\lambda \in \Gamma$ such that

$$
\exists x \in B_X \mid \| S_{m, \lambda} x - x \| < \delta \quad \text{and} \quad \forall n \| T^n(S_{m, \lambda} x) \| > 1 - \delta.
$$

If $T$ has many slow vectors then there exist subspaces $W \subset X$ of an arbitrarily large dimension whose unit spheres $S$ consist of vectors $x$ satisfying the inequality of (1.5).

**Proof.** Involving a compactness of $\Gamma$, consider $\lambda \in \Gamma$ to which there correspond slow vectors. If $\varepsilon := \varepsilon(\delta, m)$ is very small and $x$ is an $\varepsilon$-slow vector corresponding
vectors and so \( \text{codim} \) \( S \). We may assume that \( \lambda \) is a balanced set.

**Remark.** Geometrically, Lemma 1.2 means that, for each \( m \), there exist spheres of an arbitrarily large dimension that almost do not flatten under the mappings \( T^n(S_m, \alpha) \) for any \( n \).

3. **ASymptotic Finite-Dimensionality in the Reflexive Case**

Throughout the section, we suppose that \( T \colon X \to X \) satisfies (\(*\) ). We may assume that \( T \) is a balanced set.

**Lemma 2.1.** Suppose that \( x \in B_X \). For each \( k \), there exist vectors \( a_1, \ldots, a_k \in K \), numbers \( m_1 > m_2 > \cdots > m_k \), and \( t_1, \ldots, t_k \), \( |t_i| \leq \alpha^{-1} \), such that

\[
\|T^{m_1}x - [t_1T^{m_2}a_1 + t_2T^{m_3}a_2 + \cdots + t_{k-1}T^{m_k}a_{k-1} + t_ka_k]\| \leq \alpha^k. \tag{2.1}
\]

**Proof.** We write down some inequalities for \( k = 1, 2, 3 \). The first is condition (\(*\) ), and the validity of each subsequent inequality is guaranteed by an application (\(*\) ) to the preceding inequality multiplied by \( \alpha \):

\[
\exists n_1 \mid \|T^{n_1}x - t_1a_1\| \leq \alpha, \quad |t_1| \leq 1,
\]

\[
\exists n_2 \mid \|T^{n_2}(T^{n_1}x - t_1a_1) - t_2a_2\| \leq \alpha^2, \quad |t_2| \leq \alpha,
\]

\[
\exists n_3 \mid \|T^{n_3}(T^{n_2}(T^{n_1}x - t_1a_1) - t_2a_2) - t_3a_3\| \leq \alpha^3, \quad |t_3| \leq \alpha^2, \ldots.
\]

To finish, it remains to remove parenthesis and put \( m_j = n_j + \cdots + n_k \).

The sum of the numbers \( |t_i| \) in (2.1) does not exceed \( h := \sum_{i=1}^{k} \alpha^i = \frac{1}{1-\alpha} \). Hence, the convex hull \( \hat{K} \) of \( \bigcup_{i=0}^{\infty} T^i(hK) \) attracts \( B_X \), i.e.,

\[
\forall x \in B_X \quad \forall \varepsilon > 0 \quad \exists n \in \mathbb{N} \quad \exists a \in \hat{K} \quad \|T^n x - a\| < \varepsilon. \tag{2.2}
\]

We now show that \( T \) cannot act by multiplication by a scalar on the subspaces \( X \) whose dimension is rather high.

**Theorem 2.2.** \( \dim \ker(T - \lambda I) < \infty \) for all \( \lambda \in \Gamma \).

**Proof.** Choose a finite \( (1 - \alpha)\)-net of \( k \) vectors for \( K \) and consider the subspace \( Y \) spanned by the net. By the Kreǐn–Krasnosel’skii–Milman Theorem [9], in every subspace \( Z \subset X \) such that \( \dim Z > \dim Y \), there exists a norm-one vector \( z \) such that \( \rho(z, Y) = 1 \). By (\(*\) ), \( \rho(T^n z, Y) < \alpha + (1 - \alpha) = 1 \) for some \( n \). Therefore, \( Tz \) cannot have the form \( \lambda \cdot z \).

**Theorem 2.3.** Suppose that \( X \) is reflexive. They \( T \) cannot have many slow vectors and so codim \( X_0 < \infty \).

**Proof.** It suffices to prove that to no \( \lambda \in \Gamma \) there correspond many slow vectors. We may assume that \( \lambda = 1 \).

By the Statistical Ergodic Theorem (see, for example, [10, \S 2]), the operator means \( S_{m,1} = S_m = \frac{1}{m+1} \sum_{j=0}^{m} T^j \) converge to the projection \( P \) of \( X \) onto \( \ker(I - T) \). By Theorem 2.2, \( \dim \ker(I - T) < \infty \).

On a compact set \( K \), the convergence of \( S_m - P \) to zero is uniform (for example, by Arzelà’s Theorem). The operators \( S_m \) commute with \( T \); therefore, the convergence \( (S_m - P) \to 0 \) is also uniform on \( K \). Hence, starting from some \( m \), \( \|S_m(a) - P(a)\| \) is sufficiently small for all \( a \in K \), for example, less than \( \frac{1}{3} \). But by (2.2), for every \( x \in B \) and large \( n \), the vector \( T^n x \) is close to \( \hat{K} \). Therefore,

\[
\exists m \forall x \in B_X \quad \|(S_m - P)T^n x\| = \|T^n(S_m x) - P x\| \leq_{n \to \infty} 1/3.
\]
This implies, for example, that, under $T^n \circ S_m$, every $k$-dimensional sphere such that $k > \dim \ker(I - T)$, “flattens” three times for large $n$ along some radius $x$ ($x$ must be chosen so that $Px = 0$).

This, by Lemma 1.2 and the remark thereto, means that the number $\lambda = 1$ does not have many slow vectors.

The inequality $\operatorname{codim} X_0 < \infty$ follows now from Theorem 1.1. □

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