EXISTENCE OF DENSITIES FOR STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Abstract. In this work, we prove a version of Hörmander’s theorem for a stochastic evolution equation driven by a trace-class fractional Brownian motion with Hurst exponent $\frac{1}{2} < H < 1$ and an analytic semigroup on a given separable Hilbert space. In contrast to the classical finite-dimensional case, the Jacobian operator in typical solutions of parabolic stochastic PDEs is not invertible which causes a severe difficulty in expressing the Malliavin matrix in terms of an adapted process. Under a Hörmander’s bracket condition and some algebraic constraints on the vector fields combined with the range of the semigroup, we prove the law of finite-dimensional projections of such solutions has a density w.r.t Lebesgue measure. The argument is based on rough path techniques and a suitable analysis on the Gaussian space of the fractional Brownian motion.

1. Introduction

Hörmander’s theorem is one of the central aspects of Probability theory with many applications to the theory of partial differential equations, ergodic theory, stochastic filtering and numerical analysis of stochastic processes. Let $X$ be a $d$-dimensional SDE written in Stratonovich form

$$dX_t = V_0(X_t)dt + \sum_{j=1}^n V_j(X_t) \circ dW^j_t,$$

where $V_0, \ldots, V_n$ are smooth vector fields and $(W^i)_{i=1}^n$ is a standard $n$-dimensional Brownian motion. It is well known that if $\{V_j(x); j = 1, \ldots, n\}$ is $\mathbb{R}^d$, then the law of the solution of (1.1) (at a given time $t$) has a smooth density w.r.t Lebesgue measure. Based on the fundamental work of Hörmander, we know that much weaker conditions on the vector fields, the so-called parabolic Hörmander’s bracket condition, also produce smoothness of the law of $X_t$. This phenomenon is called hypoellipticity.

The main tool in proving hypoellipticity for finite-dimensional SDEs is based on Malliavin calculus. More precisely, let $\mathcal{M}_t$ be the Malliavin matrix

$$\mathcal{M}_t = (\langle DX^i_t, DX^j_t \rangle_{L^2([0,T];\mathbb{R}^n)})_{1 \leq i,j \leq d},$$

at a time $t > 0$, where $DX^i_t$ is the Gross-Sobolev-Malliavin derivative of $X^i_t$ w.r.t the Brownian motion. In order to get suitable integrability of the Malliavin matrix associated with $X_t$, the key idea is to connect $\mathcal{M}_t$ with the Jacobian $J_{s,t}; s \leq t$ of the SDE constructed as follows. Denote by $\Phi_t$ the (random) solution map to (1.1) so that $X_t = \Phi_t(x_0)$. It is well-known that under mild integrability assumption, we do have a flow of smooth maps, namely a two parameter family of maps $\Phi_{s,t}$ such that $X_t = \Phi_{s,t}(x_s)$ for every $s \leq t$ and such that $\Phi_{t,u} o \Phi_{s,t} = \Phi_{s,u}$ and $\Phi_0 = \Phi_0$. For a given initial condition $x_0$, we then denote by $J_{s,t}$ the derivative of $\Phi_{s,t}$ evaluated at $X_s$.

Under rather weak assumptions, the Jacobian is invertible and this fact allows us to write

$$\mathcal{M}_t = J_{0,t} C_t J_{0,t}^*, \quad (*)$$

where

$$C_t = \int_0^t J_{0,s}^{-1} V(X_s) V^*(X_s) (J_{0,s}^{-1})^* ds,$$

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and $V$ is the $d \times n$-matrix-valued function obtained by concatenating the vector fields $V_j$ for $j = 1, \ldots, n$. By representation $1.2$, the invertibility of $\mathcal{M}_t$ is equivalent to the invertibility of the so-called reduced Malliavin matrix $\mathcal{C}_t$ given by the following quadratic form

$$\langle \mathcal{C}_t \xi, \xi \rangle = \sum_{j=1}^{n} \int_0^t \langle \xi, J_{0,t}^{-1} V_j(X_s) \rangle^2 ds; \xi \in \mathbb{R}^d.$$ 

Then, Itô’s formula and Norris’s lemma $(29)$ combined with the parabolic Hörmander’s bracket condition allow us to conclude hypoellipticity for finite-dimensional SDEs of the form $(1.1)$. The analysis of the hypoellipticity phenomena for stochastic partial differential equations (henceforth abbreviated by SPDE) is much harder. The main technical problem with the generalization of Hörmander’s theorem to parabolic SPDEs is the fact that the Jacobian $J_{0,t}$ is typically not invertible regardless the type of noise. The existence of densities for finite-dimensional projections of SPDEs driven by Brownian motion was firstly tackled by Baudoin and Teichmann $2$ where the linear part of the SPDE generates a group of bounded linear operators on a Hilbert space. In this case, the Jacobian becomes invertible. Shamarova $33$ studies the existence of densities for a stochastic evolution equation driven by Brownian motion in 2-smooth Banach spaces. Recently, based on a pathwise Fubini theorem for rough path integrals, Gerashimovics and Hairer $17$ overcome the lack of invertibility of the Jacobian for SPDEs driven by Brownian motion. They show that the Malliavin matrix is invertible on every finite-dimensional subspace and jointly with a purely pathwise Norris type lemma, they prove that laws of finite-dimensional projections of SPDE solutions driven by Brownian motion admit smooth densities w.r.t Lebesgue measure. In contrast to $2$, the authors are able to prove existence and smoothness of densities for truly parabolic systems generated by semigroups and SPDEs driven by Brownian motion under a priori integrability conditions on the Jacobian.

The goal of this paper is to prove the existence of densities for finite-dimensional projections for a SPDE driven by a trace-class fractional Brownian motion (henceforth abbreviated by FBM) with Hurst exponent $\frac{1}{2} < H < 1$. The novelty of our work is to handle the infinite-dimensional case jointly with the fractional case which requires a new set of ideas. For FBM driving noise with $H > 1/2$ and under ellipticity assumptions on the vector fields $\{V_i; 0 \leq i \leq n\}$, the existence and smoothness of the density for SDEs are shown by Hu and Nualart $20$ and Nualart and Saussereau $27$. The hypoelliptic case for $H > 1/2$ is treated by Baudoin and Hairer $11$ based on previous papers of Nualart and Saussereau $28$ and Hu and Nualart $20$. When $\frac{1}{4} < H < \frac{1}{2}$, the integrability of the Jacobian given by Cass, Litterer and Lyons $7$ yields smoothness of densities in the elliptic case. The hypoelliptic case was treated in a series of works by Cass and Friz $5$, Cass, Friz and Victoir $10$ and culminating with Cass, Hairer, Litterer and Tindel $17$ who provide smoothness of densities for a wide class of Gaussian noises including FBM with $\frac{1}{4} < H < \frac{1}{2}$.

1.1. Main result. In this article, we aim to provide a version of Hörmander’s theorem for a SPDE of the form

$$(1.3) \quad dX_t = (A(X_t) + F(X_t))dt + G(X_t)dB_t,$$

where $(A, \text{dom}(A))$ is the infinitesimal generator of an analytic semigroup $\{S(t); t \geq 0\}$ on a separable Hilbert space $E$, $B$ is a trace-class FBM taking values on a separable Hilbert space $U$ with Hurst parameter $\frac{1}{2} < H < 1$ and $F, G$ are smooth coefficients. Let $\mathcal{T} : E \to \mathbb{R}^d$ be a bounded and surjective linear operator. The goal is to prove, under Hörmander’s bracket conditions, that the law of $\mathcal{T}(X_t)$ has a density w.r.t Lebesgue for every $t > 0$. In this article, we obtain the proof of this result under the additional assumption that the analytic semigroup has a dense range in $E$ at a given time $t > 0$ which is satisfied in many concrete examples (see Remark $55$). Moreover, in order to overcome the lack of invertibility of the Jacobian operator, some algebraic constraints on the vector fields combined with the range of the semigroup are imposed (Assumption B). To the best of our knowledge, this is the first result of hypoellipticity (existence of densities) for SPDEs driven by FBM. The result is build on a carefully analysis of the Itô map (solution map)
defined on a suitable abstract Wiener space associated with a trace-class FBM \( B \) with parameter \( \frac{1}{4} < H < 1 \) and taking values on suitable space of increments. By means of rough path techniques, it is shown that \( B \mapsto X(B) \) is Fréchet differentiable and hence differentiable in the sense of Malliavin calculus. Even though the noise \( B \) is more regular than Brownian motion (in the sense of Hölder regularity), the rough path formalism in the sense of Gubinelli [14, 15] allows us to obtain better estimates for the Itô map compared to the classical approach [34] or other frameworks based on fractional calculus [24].

Let us define

\[ G_0(x) := A(x) + F(x); x \in \text{dom}(A^\infty), \]

where \( \text{dom}(A^\infty) = \bigcap_{n \geq 1} \text{dom}(A^n) \) is equipped with the projective limit topology associated with the graph norm of \( \text{dom}(A) \). Given the SPDE (1.3), let \( \mathcal{V}_k \) be a collection of vector fields given by

\[ \mathcal{V}_k := \{ G_i; i \geq 1 \}, \quad \mathcal{V}_{k+1} := \mathcal{V}_k \cup \{ [G_j, V]; V \in \mathcal{V}_k \text{ and } j \geq 0 \}, \]

where \( G_i(x) \) are some orthonormal basis \( (e_i)_{i=1}^\infty \) of \( U \) and \([\cdot, \cdot]\) denotes the Lie bracket (see (5.3)) between smooth vector fields on \( \text{dom}(A^\infty) \). We also define the vector spaces \( \mathcal{V}_k(x_0) := \text{span}\{ V(x_0); V \in \mathcal{V}_k \} \) and we set

\[ \mathcal{D}(x_0) := \bigcup_{k \geq 1} \mathcal{V}_k(x_0), \]

for each \( x_0 \in \text{dom}(A^\infty) \). Let us now state the main result of this work.

**Theorem 1.1.** Let \( X^{\alpha} \) be the SPDE solution of (1.3) with a given initial condition \( x_0 \in \text{dom}(A^\infty) \). For a given \( t \in (0, T] \), assume that \( \mathcal{D}(x_0) \) and \( S(t)E \) are dense subsets of \( E \). Under assumptions H1-A1-A2-A3-B1-B2-C1-C2-C3, if \( T : E \rightarrow \mathbb{R}^d \) is a bounded linear surjective operator, then the law of \( T(X_t^{\alpha}) \) has a density w.r.t Lebesgue measure in \( \mathbb{R}^d \).

The remainder of this paper is organized as follows. In Section 2, we establish some preliminary results on the Gaussian space of trace-class FBM and the associated Malliavin calculus. Section 3 and 4 present the main technical results concerning regularity of the Itô map in the sense of Malliavin calculus and the existence of the right-inverse of the Jacobian, respectively. Section 5 presents the proof of Theorem 1.1.

### 2. Preliminaries

**2.1. Fractional powers of infinitesimal generators.** In this work, we make extensive use of the regularizing effects of an analytic semigroup. Throughout this article, \( E \) is a given separable Hilbert space and \( (A, \text{dom}(A)) \) is the infinitesimal generator of an analytic semigroup \( \{S(t); t \geq 0\} \) on \( E \) satisfying the following property: there exist constants \( \lambda, M > 0 \) such that

\[ \|S(t)\| \leq Me^{-\lambda t} \text{ for all } t \geq 0. \]

In this case, we can define the fractional power \( \{(-A)^\alpha, \text{Dom}((-A)^\alpha)\} \) for any \( \alpha \in \mathbb{R} \) (see Sections 2.5 and 2.6 in [30]). To keep notation simple, we denote \( E_\alpha := \text{Dom}((-A)^\alpha) \) for \( \alpha > 0 \) equipped with the norm \( \|x\|_\alpha := \|(-A)^\alpha x\|_E \) which is equivalent to the graph norm of \( (-A)^\alpha \). If \( \alpha < 0 \), let \( E_\alpha \) be the completion of \( E \) w.r.t to the norm \( \|x\|_\alpha := \|(-A)^\alpha x\|_E \). If \( \alpha = 0 \), we set \( E_0 = E \). Then, \( (E_\alpha)_{\alpha \in \mathbb{R}} \) is a family of separable Hilbert spaces such that \( E_\delta \hookrightarrow E_\alpha \) whenever \( \delta \geq \alpha \). Moreover, \( S(t) \) may be extended to \( E_\alpha \) as bounded linear operators for \( \alpha < 0 \) and \( t \geq 0 \). Moreover, \( S(t) \) maps \( E_\alpha \) to \( E_\delta \) for every \( \alpha \in \mathbb{R} \) and \( \delta \geq 0 \). We also denote \( \| \cdot \|_{\beta \rightarrow \alpha} \) as the norm operator of the space of bounded linear operators \( \mathcal{L}(E_\delta, E_\alpha) \) from \( E_\beta \) to \( E_\alpha \) and, with a slight abuse of notation, we set \( \| \cdot \| = \| \cdot \|_{0 \rightarrow 0} \). The space of bounded multilinear operators from the \( n \)-fold space \( E_\alpha^n \) to \( E_\alpha \) is equipped with the usual norm \( \| \cdot \|_{(n), \alpha \rightarrow \alpha} \) for \( \alpha \geq 0 \).
2.2. Preliminaries on the gaussian space of fractional Brownian motion. Let us start our
discussion by recalling some elementary facts on the fractional Brownian motion (FBM). The FBM
with Hurst parameter $0 < H < 1$ is a centered Gaussian process with covariance

$$R_H(t, s) := \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}).$$

Throughout this paper, we fix $1/2 < H < 1$. Let $\beta = \{\beta_t; 0 \leq t \leq T\}$ be a FBM defined on a complete
probability space $(\Omega, F, \mathbb{P})$. Let $E$ be the set of all step functions on $[0, T]$ equipped with the inner
product

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_H := R_H(t, s).$$

One can check (see e.g Chapter 5 in [26] or Chapter 1 in [25]) for every $\varphi, \psi \in E$, we have

$$\langle \varphi, \psi \rangle_H = \alpha_H \int_0^T \int_0^T |r - u|^{2H-2} \varphi(r) \psi(u) du dr,$$

where $\alpha_H := H(2H - 1)$. Let $\mathcal{H}$ be the reproducing kernel Hilbert space associated with FBM, i.e.,
the closure of $E$ w.r.t (2.1). The mapping $\mathbb{1}_{[0,t]} \mapsto \beta_t$ can be extended to an isometry between $\mathcal{H}$
and the first chaos $\{\beta(\varphi); \varphi \in \mathcal{H}\}$. We shall write this isometry as $\beta(\varphi)$.

Let us define the following kernel

$$K_H(t, s) := c_H |s|^{1/2-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{-\frac{1}{2}} du; s < t,$$

where $c_H = \left( \frac{H(2H-1)}{\beta(2H-2, H-1/2)} \right)^{1/2}$ and beta denotes the Beta function. From (2.2), we have

$$\frac{\partial K_H}{\partial t}(t, s) = c_H \left( \frac{1}{s} \right)^{H-\frac{3}{2}} (t - s)^{H-\frac{1}{2}}.$$ 

Consider the linear operator $K_H^*: E \rightarrow L^2([0, T]; \mathbb{R})$ defined by

$$(K_H^* \varphi)(s) := \int_s^T \varphi(t) \frac{\partial K_H}{\partial t}(t, s) dt; 0 \leq s \leq T.$$ 

We observe $(K_H^* \mathbb{1}_{[0,t]})(s) = K_H(t, s) \mathbb{1}_{[0,t]}(s)$. It is well-known (see e.g [25]) that $K_H^*$ can be extended
to an isometric isomorphism between $\mathcal{H}$ and $L^2([0, T]; \mathbb{R})$. Moreover,

$$\beta(\varphi) = \int_0^T (K_H^* \varphi)(t) dt; \varphi \in \mathcal{H},$$

where

$$w_t := \beta((K_H^*)^{-1}(\mathbb{1}_{[0,t]}))$$

is a real-valued Brownian motion. From (2.3), we can represent

$$\beta_t = \int_0^t K_H(t, s) dw_s; 0 \leq t \leq T,$$

and (2.4) implies both $\beta$ and $w$ generate the same filtration. Lastly, we recall that $\mathcal{H}$ is a linear space
of distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider
the linear space $|\mathcal{H}|$ as the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}$ such that

$$\|f\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |f(t)||f(s)||t - s|^{2H-2} ds dt < \infty,$$
for a constant $\alpha_H > 0$. The space $|H|$ is a Banach space with the norm $\|\cdot\|$ and isometric to a subspace of $H$ which is not complete under the inner product $\langle\cdot,\cdot\rangle$. Moreover, $E$ is dense in $|H|$. The following inclusions hold true

\begin{equation}
L^{|H|}([0, T]; \mathbb{R}) \hookrightarrow |H| \hookrightarrow H,
\end{equation}

where

\begin{equation}
\langle f, g \rangle_H = \alpha_H \int_0^T \int_0^T |u - v|^{2H - 2} f(u)g(v)dudv,
\end{equation}

for $f, g \in L^{|H|}([0, T]; \mathbb{R})$. Moreover, there exists a constant $C$ such that

\begin{equation}
\|f\|^2_H = C \int_0^T |f(s)|^2 ds,
\end{equation}

where $I^{-\frac{1}{2}}_{T-}$ is the right-sided fractional integral given by

\begin{equation}
I^{-\frac{1}{2}}_{T-} f(x) := \frac{1}{\Gamma(H - \frac{1}{2})} \int_0^T f(s) s - x)^H_1 ds; 0 \leq x \leq T.
\end{equation}

For further details, we refer the reader to Lemma 1.6.6 and (1.6.14) in [25].

2.3. Malliavin calculus on Hilbert spaces. Throughout this article, we fix a self-adjoint, non-negative and trace-class operator $Q : U \rightarrow U$ defined on a separable Hilbert space $U$. Then, there exists an orthonormal basis $\{e_i; i \geq 1\}$ of $U$ and eigenvalues $\{\lambda_i; i \geq 1\}$ such that

\begin{equation}
Qe_i = \lambda_i e_i; i \geq 1,
\end{equation}

where trace $Q = \sum_{k=1}^{\infty} \lambda_k < +\infty$. We assume that $\lambda_k > 0$ for every $k \geq 1$. Let $U_0 := Q^{1/2}(U)$ be the linear space equipped with the inner product

\begin{equation}
\langle u_0, v_0 \rangle_0 := \langle Q^{-1/2} u_0, Q^{-1/2} v_0 \rangle_U; u_0, v_0 \in U_0,
\end{equation}

where $Q^{-1/2}$ is the inverse of $Q^{1/2}$. Then, $(U_0, \langle \cdot, \cdot \rangle_0)$ is a separable Hilbert space with an orthonormal basis $\{\sqrt{\lambda_k} e_k; k \geq 1\}$.

Let $W$ be a $Q$-Brownian motion given by

\begin{equation}
W_t := \sum_{k \geq 1} \sqrt{\lambda_k} e_k w_t^k; t \geq 0,
\end{equation}

where $(w_k^0)_{k \geq 1}$ is a sequence of independent real-valued Brownian motions. Let $(\beta^k)_{k \geq 1}$ be a sequence of independent FBM’s, where $\beta^k$ is associated with $w^k$ via $\beta^k_t := \int_0^t \sqrt{K_H(t, s)} dw_s^k; 0 \leq t \leq T$.

We then set

\begin{equation}
B_t := \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta^k_t; 0 \leq t \leq T.
\end{equation}

For separable Hilbert spaces $E_1$ and $E_2$, let us denote $L_2(E_1; E_2)$ as the space of all Hilbert-Schmidt operators from $E_1$ to $E_2$ equipped with the usual inner product. Let $\mathcal{F}$ be the sigma-field generated by $\{B(\varphi); \varphi \in H \otimes L_2(U_0, \mathbb{R})\}$ where $B : H \otimes L_2(U_0, \mathbb{R}) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ is the linear operator defined by
of the form
\[ \Phi^i := \Phi(\sqrt{\lambda_i}e_i); i \geq 1. \]
We recall the tensor product \( \mathcal{H} \otimes \mathcal{L}_2(U_0, \mathbb{R}) \) is isomorphic to \( \mathcal{L}_2(U_0, \mathcal{H}) \). The elements of \( \mathcal{H} \otimes \mathcal{L}_2(U_0, \mathbb{R}) \) are described by
\[ \sum_{m,j=1}^{\infty} a_{m,j} \sqrt{\lambda_m} e_m \otimes h_j, \]
where \((a_{m,j})_{m,j} \in \ell^2(\mathbb{N}^2), (h_j)\) is an orthonormal basis for \( \mathcal{H} \) and we denote
\[ e \otimes h : y \in U_0 \mapsto (e, y)_{U_0} h. \]
It is easy to check that \( \mathbb{E} \left[ B(\Phi)B(\Psi) \right] = \langle \Phi, \Psi \rangle_{\mathcal{L}_2(U_0, \mathcal{H})} \) for every \( \Phi, \Psi \in \mathcal{L}_2(U_0, \mathcal{H}) \). In this case, \( \left( \Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}_2(U_0, \mathcal{H}) \right) \) is the Gaussian space associated with the isonormal Gaussian process \( B \).

For Hilbert spaces \( E_1 \) and \( E_2 \), let \( C^\infty_p(E_1; E_2) \) be the space of all functions \( f : E_1 \to E_2 \) such that \( f \) and all its derivatives have polynomial growth. Let \( \mathcal{P} \) be the set of all cylindrical random variables of the form
\[ F = f(B(\varphi_1), \ldots, B(\varphi_m)), \]
where \( f \in C^\infty_p(\mathbb{R}^m; \mathbb{R}) \) and \( \varphi_i \in \mathcal{L}_2(U_0, \mathcal{H}); i = 1, \ldots, m \) for some \( m \geq 1 \). The Malliavin derivative of an element of \( F \in \mathcal{P} \) of the form (2.10) over the Gaussian space \( \left( \Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}_2(U_0, \mathcal{H}) \right) \) is defined by
\[ DF := \sum_{k=1}^m \frac{\partial}{\partial x_k} f(B(\varphi_1), \ldots, B(\varphi_m)) \varphi_k. \]
We observe
\[
\langle DF, h \rangle_{\mathcal{L}_2(U_0, \mathcal{H})} = \sum_{k=1}^m \frac{\partial}{\partial x_k} f(B(\varphi_1), \ldots, B(\varphi_m)) \langle \varphi_k, h \rangle_{\mathcal{L}_2(U_0, \mathcal{H})} = \frac{d}{de} f(B(\varphi_1) + e(\varphi_1, h)_{\mathcal{L}_2(U_0, \mathcal{H})} \ldots, B(\varphi_m) + e(\varphi_m, h)_{\mathcal{L}_2(U_0, \mathcal{H})}) \Big|_{e=0}. \]
For a given separable Hilbert space \( E \), let \( \mathcal{P}(E) \) be the set of all cylindrical \( E \)-valued random variables of the form
\[ F = \sum_{j=1}^n F_j h_j, \]
where \( F_j \in \mathcal{P} \) and \( h_j \in E \) for \( j = 1, \ldots, n \) and \( n \geq 1 \). We then define
\[ DF := \sum_{j=1}^n DF_j \otimes h_j. \]
A routine exercise yields the following result.

**Lemma 2.1.** The operator \( D : \mathcal{P}(E) \subset L^p(\Omega; E) \to L^p(\Omega; \mathcal{L}_2(U_0, \mathcal{H}) \otimes E) \) is closable and densely defined for every \( p \geq 1 \).
For an integer \( k \geq 1 \) and \( p \geq 1 \), let \( \mathbb{D}^{1,p}(E) \) be the completion of \( \mathcal{P}(E) \) w.r.t the semi-norm

\[
\|F\|_{\mathbb{D}^{1,p}(E)} := \left[ \mathbb{E}\|F\|_p^p + \mathbb{E}\|D F\|_{\mathcal{L}_2(U_0;H) \otimes E}^p \right]^{1/p}.
\]

Let us now devote our attention to some criteria for checking when a given functional \( F : \Omega \to E \) belongs to the Sobolev spaces \( \mathbb{D}^{1,p}(E) \) for \( p > 1 \). In the sequel, \( \text{loc} \) denotes localization in the sense of [26].

**Lemma 2.2.** For a given \( p > 1 \), assume that \( F \in L^p_{\text{loc}}(\Omega; E) \) and \( \langle F, x \rangle_E \in \mathbb{D}^{1,p}(\mathbb{R}) \) for every \( x \in E \).

If there exists \( \xi \in L^p_{\text{loc}}(\Omega; \mathcal{L}_2(U_0;H) \otimes E) \) such that

\[
(2.11) \quad \langle D(F, u)_E, h \rangle_{\mathcal{L}_2(U_0;H)} = \langle \xi(u), h \rangle_{\mathcal{L}_2(U_0;H)},
\]

for every \( u \in E \) and \( h \in \mathcal{L}_2(U_0;H) \), then \( F \in \mathbb{D}^{1,p}(E) \) and \( DF = \xi \).

**Proof.** Consider the Gaussian space \( (\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}_2(U_0;H)) \), take a localizing sequence \((\Omega_n, F_n) \in \mathcal{F} \times \mathbb{D}^{1/2}(\mathbb{R})\) such that \( F_n = \langle F, u \rangle_E \) on \( \Omega_n \) and \( \Omega_n \uparrow \Omega \) as \( n \to +\infty \). Then, apply Theorem 3.3 given by [31]. \( \square \)

In view of the Hölder path regularity of the underlying noise, it will be useful to play with Fréchet and Malliavin derivatives. In this case, it is convenient to realize \( \mathbb{P} \) as a Gaussian probability measure defined on a suitable Hölder-type separable Banach space equipped with a Cameron-Martin space which supports possibly infinitely many independent FBM. Let \( C_0^\infty(\mathbb{R}_+) \) be the space of smooth functions \( w : [0, \infty) \to \mathbb{R} \) satisfying \( w(0) = 0 \) and having compact support. Given \( \gamma \in (0, 1) \) and \( \delta \in (0, 1) \), we define for every \( w \in C_0^\infty(\mathbb{R}_+) \), the norm

\[
\|w\|_{\mathcal{W}\gamma,\delta} := \sup_{t,s \in \mathbb{R}_+} \frac{|w(t) - w(s)|}{|t - s|^{\gamma}(1 + |t| + |s|)^\delta}.
\]

Let \( \mathcal{W}\gamma,\delta \) be the completion of \( C_0^\infty(\mathbb{R}_+) \) w.r.t \( \| \cdot \|_{\mathcal{W}\gamma,\delta} \). We also write \( \mathcal{W}_{T}\gamma,\delta \) when we restrict the arguments to the interval \([0, T]\). It should be noted that \( \| \cdot \|_{\mathcal{W}_{T}\gamma,\delta} \) is equivalent to the \( \gamma \)-Hölder norm on \([0, T]\) given by

\[
|f(0)| + |f|_\gamma,
\]

where

\[
|f|_\gamma := \sup_{0 \leq t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\gamma}, \quad \frac{1}{2} < \gamma < 1.
\]

Moreover, \( \mathcal{W}_{T}\gamma,\delta \) is a separable Banach space. Let \( \lambda = (\lambda_i)_{i=1}^\infty \) be the sequence of strictly positive eigenvalues of \( Q \). In addition to trace \( Q = \sum_{i \geq 1} \lambda_i < \infty \), let us assume \( \sum_{i \geq 1} \sqrt{\lambda_i} < \infty \). Let \( \mathcal{W}_{\lambda, T}\gamma,\delta \) be the vector space of functions \( g : \mathbb{N} \to \mathcal{W}_{T}\gamma,\delta \) such that

\[
\|g\|_{\mathcal{W}_{\lambda, T}\gamma,\delta} := \sum_{i=1}^\infty \sqrt{\lambda_i} \|g_i\|_{\mathcal{W}_{T}\gamma,\delta} < \infty.
\]

Clearly, \( \mathcal{W}_{\lambda, T}\gamma,\delta \) is a normed space.

**Lemma 2.3.** \( \mathcal{W}_{\lambda, T}\gamma,\delta \) is a separable Banach space equipped with the norm \( \| \cdot \|_{\mathcal{W}_{\lambda, T}\gamma,\delta} \).
Proof. Let \( \|g_n - g_m\|_{W_{T}^{\gamma,\delta}} \to 0 \) as \( n, m \to +\infty \). Then, for \( \epsilon > 0 \), there exists \( N(\epsilon) \) such that

\[
\sum_{i=1}^{\infty} \sqrt{\lambda_i} \|g^i_n - g^i_m\|_{W_{T}^{\gamma,\delta}} < \epsilon,
\]

for every \( n, m > N(\epsilon) \). Since \( W_{T}^{\gamma,\delta} \) is complete, then there exists \( g : \mathbb{N} \to W_{T}^{\gamma,\delta} \) defined by \( g^i := \lim_{n \to \infty} g^i_n \) in \( W_{T}^{\gamma,\delta} \) for each \( i \geq 1 \). By construction, we observe that

\[
\sum_{i=1}^{\infty} \sqrt{\lambda_i} \|g^i\|_{W_{T}^{\gamma,\delta}} \leq \sum_{i=1}^{\infty} \sqrt{\lambda_i} \|g^i - g^i_n\|_{W_{T}^{\gamma,\delta}} + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \|g^i_n\|_{W_{T}^{\gamma,\delta}}
\]
\[
\leq \sum_{i=1}^{\infty} \frac{\epsilon}{\sqrt{\lambda_i}} + \sup_{j \geq 1} \|g_j\|_{W_{\infty}^{\gamma,\delta,\infty}}
\]
\[
\leq \left( \sum_{i=1}^{\infty} \lambda_i \right)^{1/2} \sqrt{2\epsilon} + \sup_{n \geq 1} \|g_n\|_{W_{\infty}^{\gamma,\delta,\infty}} < \infty.
\]

For separability, let \( \bigoplus_{j=1}^{\infty} W_{T}^{\gamma,\delta} \_2 = \{ f : \mathbb{N} \to W_{T}^{\gamma,\delta}; \|f\|_2 < \infty \} \) be the \( l_2 \)-direct sum of the Banach spaces \( W_{T}^{\gamma,\delta} \), where

\[
\|f\|_2 = \left( \sum_{j=1}^{\infty} \|f^j\|_{W_{T}^{\gamma,\delta}}^2 \right)^{1/2}.
\]

Since \( \text{trace } Q < \infty \), then

(2.12) \( \| \cdot \|_{W_{\infty}^{\gamma,\delta,\infty}} \leq (\text{trace } Q)^{1/2} \| \cdot \|_2 \).

Of course, \( \cup_{n \geq 1} \bigoplus_{j=1}^{n} W_{T}^{\gamma,\delta} \subset \bigoplus_{j=1}^{\infty} W_{T}^{\gamma,\delta} \) and clearly \( \cup_{n \geq 1} \bigoplus_{j=1}^{n} W_{T}^{\gamma,\delta} \) is a dense subset of \( \bigoplus_{j=1}^{\infty} W_{T}^{\gamma,\delta} \).

Since \( W_{T}^{\gamma,\delta} \) is separable, the previous argument shows \( \bigoplus_{j=1}^{\infty} W_{T}^{\gamma,\delta} \_2 \) is separable and hence (2.12) implies \( W_{\infty}^{\gamma,\delta,\infty} \) is separable as well.

\[\square\]

Lemma 2.4. If \( \gamma \in (\frac{1}{2}, H) \) and \( \gamma + \delta \in (H, 1) \), then there exists a Gaussian probability measure \( \mu_{\gamma,\delta}^{\infty} \) on \( W_{\infty}^{\gamma,\delta,\infty} \). Therefore, there exists a separable Hilbert space \( H \) continuously imbedded into \( W_{\infty}^{\gamma,\delta,\infty} \) such that \( (W_{\infty}^{\gamma,\delta,\infty}, H, \mu_{\gamma,\delta}^{\infty}) \) is an abstract Wiener space.

Proof. From Lemma 4.1 in [19], we know there exists a probability measure \( \mu_{\gamma,\delta} \) on \( W_{T}^{\gamma,\delta} \) such that the canonical process is a FBM with Hurst parameter \( \frac{1}{2} < H < 1 \) as long as \( \gamma \in (\frac{1}{2}, H) \) and \( \gamma + \delta \in (H, 1) \). Let \( W_{T}^{\gamma,\delta,\infty} := \prod_{j \geq 1} W_{T}^{\gamma,\delta} \) be the countable product of the Banach spaces \( W_{T}^{\gamma,\delta} \) equipped with the product topology which makes \( W_{T}^{\gamma,\delta,\infty} \) as a topological vector space. Let \( \mu_{\gamma,\delta}^{\infty} \) be the product probability measure \( \otimes_{j \geq 1} \mu_{\gamma,\delta} \) over \( W_{T}^{\gamma,\delta,\infty} \) equipped with the usual product sigma-algebra. Then, \( \mu_{\gamma,\delta}^{\infty} \) is a Gaussian probability measure (see e.g. Example 2.3.8 in [4]). Moreover, we observe

\[
\mu_{\gamma,\delta}^{\infty}(W_{T}^{\gamma,\delta,\infty}) = 1.
\]

Indeed, by construction, we can take a sequence of \( \mu_{\gamma,\delta} \)-independent FBMs \( \beta^i; i \geq 1 \). By using the modulus of continuity of FBM, it is well-known that \( \mathbb{E}_{\mu_{\gamma,\delta}} \|\beta\|_{W_{T}^{\gamma,\delta}} = \mathbb{E}_{\mu_{\gamma,\delta}} \|\beta^i\|_{W_{T}^{\gamma,\delta}} < \infty \) for every \( i \geq 1 \). Therefore
\[ E_{\mu_{\gamma,\delta}} \sum_{i=1}^{\infty} \lambda_i \| \beta_i \|_{W^{\gamma,\delta}_{T}} = E_{\mu_{\gamma,\delta}} \| \beta_1 \|_{W^{\gamma,\delta}_{T}} \sum_{i=1}^{\infty} \lambda_i < \infty, \]

and this proves that \( \mu_{\gamma,\delta} \) is a Gaussian probability measure on the Banach space \( W^{\gamma,\delta}_{\lambda,T} \). As a conclusion, this shows that we have an abstract Wiener space structure for \( \mu_{\gamma,\delta} \).

\[ \square \]

In the sequel, with a slight abuse of notation, we define \( K^*_H : \mathcal{E} \otimes L_2(U_0, \mathbb{R}) \to L^2([0, T]; L_2(U_0, \mathbb{R})) \) as follows:

\[ K^*_H(h \otimes \varphi)(s) := \int_s^T h(t) \frac{\partial}{\partial t} K_H(t, s) dt \varphi; \; h \in \mathcal{E}, \varphi \in L_2(U_0, \mathbb{R}). \]

Clearly,

\[ (K^*_H(h_1 \otimes \varphi_1), K^*_H(h_2 \otimes \varphi_2))_{L^2([0, T]; L_2(U_0, \mathbb{R}))} = \langle (h_1 \otimes \varphi_1), (h_2 \otimes \varphi_2) \rangle_{\mathcal{H} \otimes L_2(U_0, \mathbb{R})}, \]

for every \( h_1, h_2 \in \mathcal{E} \) and \( \varphi_1, \varphi_2 \in L_2(U_0, \mathbb{R}) \) and hence we can extend \( K^*_H \) to an isometric isomorphism from \( \mathcal{H} \otimes L_2(U_0, \mathbb{R}) \) to \( L^2([0, T]; L_2(U_0, \mathbb{R})) \). Let us also denote \( K_H : L^2([0, T]; L_2(U_0, \mathbb{R})) \to \mathcal{H} \) by

\[ K_H f(t) := \int_0^t K_H(t, s) f(s) ds; 0 \leq t \leq T, i \geq 1, \]

for \( f \in L^2([0, T]; L_2(U_0, \mathbb{R})) \). Here, \( \mathcal{H} := \text{Range } K_H \) is the Hilbert space equipped with the norm

\[ \| K_H f \|_{\mathcal{H}}^2 := \int_0^T \| f(s) \|_{L_2(U_0, \mathbb{R})}^2 ds = \sum_{i=1}^{\infty} \| f(\sqrt{\lambda_i} e_i) \|_{L^2([0, T]; \mathbb{R})}^2 \]

where \( \mathcal{H} := \text{Range } K_{H,1} \) and

\[ K_{H,1} g(t) := \int_0^t K_H(t, s) g(s) ds; 0 \leq t \leq T, \]

for \( g \in L^2([0, T]; \mathbb{R}) \). We recall (see Th 3.6 [22]) there exists a constant \( C \) such that

\[ \| K_{H,1} g \|_{W^{\gamma,\delta}_T} \leq C \| g \|_{L^2([0, T]; \mathbb{R})}, \]

for every \( g \in L^2([0, T]; \mathbb{R}) \). Therefore, Cauchy-Schwartz inequality yields

\[ \| K_H f \|_{W^{\gamma,\delta}_{\lambda,T}} \leq (\text{trace } Q)^{1/2} \| K_H f \|_{\mathcal{H}}, \]

for every \( f \in L^2([0, T]; L_2(U_0, \mathbb{R})) \). Let us set \( \mathbb{P} = \mu_{\gamma,\delta} \) and \( \Omega = W^{\gamma,\delta}_{\lambda,T} \). Summing up the above computations, we conclude \( \mathcal{H} \) is the Cameron-Martin space associated with \( \mathbb{P} \) in Lemma [22] namely

\[ (2.13) \quad \int_{\Omega} \exp(i(\omega, z)_{\Omega,\Omega^*}) \mathbb{P}(d\omega) = e^{-\frac{1}{2} \| z \|^2_{\Omega^*}}; z \in \Omega^*, \]

where \( \Omega^* \) is the topological dual of \( \Omega \).

By applying Prop. 4.1.3 in [22] (see also [18]), we arrive at the following result. Let \( \mathcal{R}_H := K_H \circ K^*_H \) be the injection of \( L_2(U_0; \mathcal{H}) \) into \( \Omega \). We observe \( \mathcal{R}_H : L_2(U_0; \mathcal{H}) \to \Omega \) is a bounded operator with dense range.

**Corollary 2.1.** If a random variable \( Y : \Omega \to \mathbb{R} \) is Fréchet differentiable along directions in the Cameron-Martin space \( \mathcal{H} \), then

\[ h \mapsto Y(\omega + \mathcal{R}_H(h)) \]

is Fréchet differentiable for each \( \omega \in \Omega \). Moreover, \( Y \in D^{1,2}_{loc}(\mathbb{R}) \) and
\[ \nabla Y(\cdot)(R_f h) = \langle DY, h \rangle_{L_2(U_0, \mathcal{H})}, \]
for every \( h \in L_2(U_0, \mathcal{H}) \).

3. Malliavin differentiability of solutions

In this section, we discuss differentiability in the sense of Malliavin calculus (on the probability space defined on Lemma 2.4) of SPDE mild \( F \)-valued \( \mathbb{F} \)-adapted solutions of

\[ dX_t = (A(X_t) + F(X_t))dt + G(X_t)dB_t, \]
in a separable Hilbert space \( E \). Here, \( \mathbb{F} \) is the filtration generated by an \( U \)-valued FBM \( B \) of the form

\[ B_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i \beta^i_t; 0 \leq t \leq T. \]

We will assume \( \sum_{i=1}^{\infty} \lambda_i < \infty \) and additional regularity conditions: \( \sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty \) and \( \lambda_i > 0 \) for all \( i \geq 1 \). The coefficients \( F : E \to E \) and \( G : E \to \mathcal{L}(U; E) \) will satisfy suitable minimal regularity conditions (see Assumption H1) to ensure well-posedness of (3.1). Let us define \( G_i(x) := G(x)(e_i) \) for an orthonormal basis \( (e_i)_{i \geq 1} \) of \( U \). Then, we view the solution as

\[ X_t = S(t)x_0 + \int_0^t S(t-s)F(X_s)ds + \int_0^t S(t-s)G(X_s)dB_s, \]
where the \( dB \) differential is understood in Young’s sense \[34, 15\]

\[ \int_0^t S(t-s)G(X_s)dB_s = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(t-s)G_i(X_s)d\beta^i_s, \]
where the convergence of the sum is understood \( \mathbb{P}\text{-a.s} \) in \( E \) in the sense of Lemma 3.2 below. The solution of (3.2) will take values on \( \mathbb{E} \) for suitable \( \alpha > 0 \).

In order to prove Fréchet differentiability, it is crucial to play with linear SPDE solutions living in Banach spaces which are “sensible” to the Hölder-type norm of the noise space \( \mathcal{W}^{\gamma, \delta}_{H, 2} \). For this purpose, we make use of the algebraic/analytic formalism developed by \[14\] in the framework of rough paths. Even though we are working with a regular noise \( 1/2 < \gamma < 1 \), the techniques developed by \[14\] \[15\] allow us to derive better estimates than the classical approach of \[34\] or fractional calculus given by \[24\].

3.1. Algebraic integration. For completeness of presentation, let us summarize the basic objects of \[14\] \[15\] which will be important to us. At first, we fix some notation. For a given normed space \( V \) equipped with a norm \( \| \cdot \|_V \), \( \mathcal{C}_k(V) \) is the set of all continuous functions \( g : [0, T]^k \to V \) such that \( g_{t_1 \ldots t_k} = 0 \) whenever \( i \leq k-1 \). We define \( \delta : \mathcal{C}_n(V) \to \mathcal{C}_{n+1}(V) \) by

\[ (\delta F)_{t_1 \ldots t_{n+1}} := \sum_{j=1}^{n+1} (-1)^j F_{t_1 \ldots t_j \ldots t_{n+1}}; F \in \mathcal{C}_n(V), \]
where \( \hat{i} \) means that this particular argument is omitted. We are mostly going to use the two special cases: If \( F \in \mathcal{C}_1(V) \), then

\[ (\delta F)_{ts} = F_t - F_s; (t, s) \in [0, T]^2. \]
If \( F \in \mathcal{C}_2(V) \), then

\[ (\delta F)_{tsu} = -F_{su} + F_{tu} - F_{ts}; (t, s, u) \in [0, T]^3. \]
We measure the size of the increments by H"older regularity defined as follows: For \( f \in C_2(V) \) and \( \mu \geq 0 \), let us define

\[
\| f \|_{\mu,V} := \sup_{s,t \in [0,T]} \frac{\| f_{s,t} \|_V}{|t-s|^{\mu}},
\]

and the sets \( C^\mu_2(V) := \{ f \in C_2(V); \| f \|_{\mu,V} < \infty \} \), \( C^\mu_1(V) := \{ f \in C_1(V); \| \delta f \|_{\mu,V} < \infty \} \). In the same way, for \( h \in C_3(V) \), we set

\[
\| h \|_{\gamma,\rho,V} := \sup_{s,u,t \in [0,T]} \frac{\| h_{s,u,t} \|_V}{|t-u|^{\rho}|s-u|^{\gamma}},
\]

\[
\| h \|_{\mu,V} := \inf \left\{ \sum_i \| h_i \|_{\mu_i,\rho_i,V}; h = \sum_i h_i, 0 < \rho_i < \mu \right\},
\]

where the last infimum is taken over all sequences \( \{ h_i \in C_3(V) \} \) such that \( h = \sum_i h_i \) and for all choices of numbers \( \rho_i \in (0, \mu) \). Then, \( \| \cdot \|_{\mu,V} \) is a norm on the space \( C_3(V) \), and we set

\[
C^\mu_3(V) := \{ h \in C_3(V); \| h \|_{\mu,V} < \infty \}.
\]

Let us denote \( \mathcal{Z}C_k(V) := C_k(V) \cap \text{Ker} \delta |_{C_k(V)} \) and \( BC_k(V) := C_k(V) \cap \text{Range} \delta |_{C_{k-1}(V)} \). We have \( \mathcal{Z}C_{k+1}(V) = BC_{k+1}(V) \) for \( k \geq 1 \).

The convolutional increments will be defined as follows. Let \( S_n = \{ (t_1, \ldots, t_n); T \geq t_1 \geq t_2 \geq \ldots t_n \geq 0 \} \). For a Banach space \( V \), \( \hat{C}_n(V) \) denotes the space of continuous functions from \( S_n \) to \( V \). We also need a modified version of basic increments distorted by the semigroup as follows: Let \( \delta : \hat{C}_n(E) \to \hat{C}_{n+1}(E) \) given by

\[
(\delta F)_{t_1 \ldots t_{n+1}} := (\delta F)_{t_1 \ldots t_{n+1}} - a_{t_1 t_2} F_{t_2 \ldots t_n},
\]

where \( a_{t_1 t_2} := S(t_1 - t_2) - \text{Id} \) for \( (t_1, t_2) \in S_2 \).

**H"older-type space of increments.** We need to define H"older-type subspaces of \( \hat{C}_k \) for \( 1 \leq k \leq 3 \) associated with \( E_\alpha; \alpha \in \mathbb{R} \). For \( \mu \geq 0 \) and \( g \in \hat{C}_2(E_\alpha) \), we define the norm

\[
\| g \|_{\mu,\alpha} := \sup_{t,s \in S_2} \frac{|g_{t,s}|_\alpha}{|t-s|^{\mu}}
\]

and the spaces

\[
\hat{C}^\mu_2(E_\alpha) := \{ g \in \hat{C}_2(E_\alpha); \| g \|_{\mu,\alpha} < \infty \},
\]

\[
\hat{C}^\mu_1(E_\alpha) := \{ f \in \hat{C}_1(E_\alpha); \| \delta f \|_{\mu,\alpha} < \infty \},
\]

\[
C^\mu_1(E_\alpha) := \{ f \in \hat{C}_1(E_\alpha); \| \delta f \|_{\mu,\alpha} < \infty \}.
\]

We denote \( \hat{C}^{0,\alpha}_1 := \hat{C}_1(E_\alpha) \) equipped with the norm

\[
\| f \|_{0,\alpha} := \sup_{0 \leq t \leq T} |f_t|_\alpha.
\]

We also equip \( C^{\mu,\alpha}_1 \) and \( \hat{C}^{\mu,\alpha}_1 \) with the norms given, respectively, by

\[
\| f \|_{C^{\mu,\alpha}_1} := \| f \|_{0,\alpha} + \| \delta f \|_{\mu,\alpha},
\]

\[
\| f \|_{\hat{C}^{\mu,\alpha}_1} := \| f \|_{0,\alpha} + \| \delta f \|_{\mu,\alpha}.
\]

We observe that
Lemma 3.1. \( \hat{C}^\mu \to C^\mu \), for every \( \mu \in (0, 1) \) due to the following estimate: For \( \lambda \geq \mu \), we have

\[
\| \delta f \|_{\mu,0} \leq \| \hat{\delta} f \|_{\mu,\lambda} + C|T|^\lambda \mu \| f \|_{0,\lambda},
\]
for every \( f \in \hat{C}^\mu \) (see Lemma 2.4 in [13]).

Let us now consider the 3-increment spaces. If \( h \in \hat{C}_3(E_\alpha) \), we define

\[
\| h \|_{p,\rho,\alpha} := \sup_{t,u,s \in S_3} \frac{|h_{txs}|}{|t-u|^\gamma |u-s|^\rho},
\]
\[
\| h \|_{\mu,\alpha} := \inf \left\{ \sum_i \| h_i \|_{\mu,\rho,\alpha}; h = \sum_i h_i, 0 < \rho_i < \mu \right\},
\]
where the last infimum is taken over all sequences \( h_i \) such that \( h = \sum_i h_i \) and for all choices of the numbers \( \rho_i \in (0, \mu) \). One can check \( \| \|_{\mu,\alpha} \) is a norm and we define

\[
\hat{C}^\mu_{3,\alpha} := \{ h \in \hat{C}_3(E_\alpha); \| h \|_{\mu,\alpha} < \infty \}.\]

We also need Hölder-type spaces for operator-valued increments. For \( \mu \geq 0 \) and \( \alpha, \beta \in \mathbb{R} \), we set

\[
\hat{C}^\mu_2 \mathcal{L}^{\beta,\alpha} := \hat{C}^\mu_2 \left( \mathcal{L}(E_\beta; E_\alpha) \right) = \{ f : S_2 \to \mathcal{L}(E_\beta; E_\alpha); \| f \|_{\mu,\beta-\alpha} < \infty \},
\]
where

\[
\| f \|_{\mu,\beta-\alpha} := \sup_{t,s \in S_2} \frac{\| f_{ts} \|_{\beta-\alpha}}{|t-s|^\mu}.
\]

In order to work with the convolution sewing map (see [13]), we define

\[
\mathcal{Z} \hat{C}^\mu_3 := \hat{C}^\mu_3 \cap \ker \hat{\delta} |_{\mathcal{C}_j}; j = 2, 3.
\]

We recall \( \mathcal{Z} \hat{C}_j := \text{Ker} \hat{\delta} |_{\mathcal{C}_j}; j \geq 1 \). Let us define \( \mathcal{E}^{\mu,\alpha}_2 := \cap_{\gamma \leq \mu \wedge \alpha} \hat{C}^{\mu-\epsilon,\alpha+\epsilon}_2 \), where \( \epsilon \in [0, \mu] \wedge [0, 1] \).

Infinite-dimensional regularized noise: We define

\[
X^{x,i}_s := S(t-s)(\delta x^i)_s \sqrt{\lambda_i}; (t, s) \in S_2,
\]
for \( x = (x^i)_{i \geq 1} \in W^{1,\gamma}_T \) and \( 1/2 < \gamma < H < 1, \gamma + \delta \in (H, 1) \). Let us now collect some important properties of the regularized noise.

**Lemma 3.1.** The following properties hold true: \( X^{x,i} \in \hat{C}^2_3 \mathcal{L}^{\beta,\alpha} \) for \( i \geq 1 \) and for every \( (\alpha, \beta) \in \mathbb{R}^2 \) such that \( \beta \geq \alpha \). Moreover, there exists a constant \( C \) which depends on \( (\alpha, \beta) \) such that

\[
\sup_{(t,s) \in S_2} \| X^{x,i}_{ts} \|_{\beta-\alpha} \leq C \sqrt{\lambda_i} \| x^i \|_{W^{1,\gamma}_T},
\]
for every \( i \geq 1 \). Moreover, the following algebraic condition holds

\[
(\delta X^{x,i})_{tsu} = (X^{x,i} a)_{tsu}; (t, s, u) \in S_3,
\]
where \( a_{su} = S(s-u) - Id; (s, u) \in S_2 \).
Lemma 3.2. Let us fix \( x = (x^i)_{i \geq 1} \in \mathcal{H}^{\gamma, \delta, \infty}_{\Lambda, T} \) where \( 1/2 < \gamma < H < 1, \gamma + \delta \in (H, 1) \). Assume \( z = (z^i)_{i \geq 1} \) satisfies \( \sup_{i \geq 1} \|z^i\|_{x_1^i, \delta} < \infty \) for \( \eta + \gamma > 1 \). Then

\[
\mathcal{J}_{t_1, t_2}(\hat{dx}z) := \sum_{i=1}^{\infty} \sqrt{\lambda_i} X_{t_1, t_2}^i z_t^i + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \Lambda^i (X_{t_1, t_2}^{x_i, \hat{z}^i})_{t_1, t_2}
\]

satisfies:

(i) There exists a constant \( C \) such that

\[
\|\hat{\delta} \mathcal{J}(\hat{dx}z)\|_{\gamma, \alpha} \leq C \|x\|_{W^{\gamma, \delta, \infty}_{\Lambda, T}} \sup_{i \geq 1} \{ \|z^i\|_{0, \delta} + \|\hat{z}^i\|_{0, \beta} \},
\]

for \( \alpha \leq \beta \).

(ii)

\[
\mathcal{J}_{t_1, t_2}(\hat{dx}z) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_{t_2}^{t_1} S(t_1 - u) z_t^i du dx^i \text{ in } E_\alpha,
\]

for each \( (t_1, t_2) \in S_2 \).

Proof. The proof is a straightforward application of Lemma 3.1 above and Lemma 3.2, Th. 3.5 and Cor 3.6 in [15]. We omit the details. \( \square \)
3.2. The Itô map. For a given \( y_0 = \psi \in E \), the Itô map \( x \mapsto y \) is defined as the solution of the equation
\[
y_t = S(t-s)y_s + \int_s^t S(t-u)F(y_u)du + \mathcal{J}_s(\hat{dx}G(y)); (t,s) \in S_2,
\]
which can be rewritten in terms of the increment operator \( \hat{\delta} \)
\[
(\hat{\delta}y)_s = \int_s^t S(t-u)F(y_u)du + \mathcal{J}_s(\hat{dx}G(y)); y_0 = \psi.
\]

Next, we list the basic assumptions needed for the existence and uniqueness of the SPDE solution. Before that, let us check that we may choose the correct set of parameters.

**Lemma 3.3.** For given \( \frac{1}{2} < H < 1 \) and \( 1/2 > \kappa > 1/4 \), there exist \( \hat{\gamma}, \kappa_0 \) satisfying \( \hat{\gamma} > \kappa_0 > \kappa > \frac{1}{4} \) with \( \hat{\gamma} + \kappa > 1 \) and \( \hat{\gamma} - \kappa \geq \kappa_0 \) such that
\[
X^{x,i} \in C^{\hat{\gamma},-\kappa}_2 \cap C^{\kappa_0}_2 L^{\kappa,\kappa},
\]
for every \( i \geq 1 \).

**Proof.** From Lemma 3.1 and the definition of the spaces \( \mathcal{W}_I^{\hat{\gamma},\delta} \), there exists a constant \( C \) (which does not depend on \( i \geq 1 \)) such that
\[
\|X^{x,i}\|_{H^{-\epsilon,0} \to \kappa} \leq C \sqrt{\lambda_0} \|x^i\|_{\mathcal{W}_T^{\hat{\gamma},\delta}},
\]
\[
\|X^{x,i}\|_{H^{-\eta,\kappa} \to \kappa} \leq C \sqrt{\lambda_1} \|x^i\|_{\mathcal{W}_T^{\eta,\delta}},
\]
for every \( \kappa > 0 \), \( \epsilon \in (0,H) \), \( \eta \in (0,H) \) and \( \delta > 0 \) such that \( H - \epsilon - \delta \in (H,1) \) and \( H - \eta + \delta \in (H,1) \). For a given \( \frac{1}{2} < H < 1 \) and \( \frac{1}{2} > \kappa > \frac{1}{4} \), choose \( \epsilon = \epsilon(\kappa, H) \in (0,H) \) in such way that
\[
H - \epsilon + \kappa > 1.
\]
Choose \( \eta = \eta(\epsilon, H) \) in such way that
\[
\eta > \frac{1}{2} + \epsilon \quad \text{and} \quad H - \kappa > \eta.
\]
Of course, (3.11) implies \( \frac{1}{2} + \epsilon < \eta < H - \kappa \). Choose \( \delta \) accordingly to these conditions. We then set \( \hat{\gamma} = H - \epsilon, \kappa_0 = H - \eta \) where \( \epsilon \) and \( \eta \) satisfy (3.10) and (3.11). Then, by construction \( \hat{\gamma} + \kappa = H - \epsilon + \kappa > 1 \) due to (3.10) and \( \hat{\gamma} > \kappa_0 > \kappa > \frac{1}{4} \) due to (3.11). Moreover, \( \eta - \epsilon > \frac{1}{2} > \kappa > \frac{1}{4} \) so that
\[
\hat{\gamma} - \kappa_0 > \frac{1}{2} > \kappa > \frac{1}{4}.
\]
Finally, we stress the choice of \( \epsilon \) and \( \eta \) does not depend on the index \( i \geq 1 \). This concludes the proof. \( \square \)

Let us assume the following regularity assumptions on \( F,G \):

**Assumption H1:** For \( 1/2 > \kappa > 1/4 \), we assume that \( F,G_i : E_\kappa \to E_\kappa \) is Lipschitz (uniformly in \( i \geq 1 \)) and they have linear growth: there exists a constant \( C \) such that
\[
|G_i(x)|_\kappa \leq C(1 + |x|_\kappa), \quad |F(x)|_\kappa \leq C(1 + |x|_\kappa); x \in E_\kappa,
\]
for every \( i \geq 1 \). Furthermore, we suppose that \( F,G_i \) can also be seen as maps from \( E \) to \( E \), and when considered as such, it holds that \( F,G_i \) are Lipschitz (uniformly in \( i \geq 1 \)).
In the sequel, recall $C_{1}^{\kappa,\kappa}$ is the subspace of $C_{1}(E_{\kappa})$ such that

$$\|z\|_{C_{1}^{\kappa,\kappa}} = \|z\|_{0,\kappa} + \|\delta z\|_{\kappa,\kappa} < \infty.$$  

In what follows, $x \in W_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ where $\tilde{\gamma} + \delta \in (H,1)$, $\frac{1}{2} < \tilde{\gamma} < H$, 

(3.12) \[ \tilde{\gamma} > \kappa > \frac{1}{4}, \]

and $\tilde{\gamma} + \kappa > 1, \tilde{\gamma} - \kappa \geq \kappa_{0}$. By Lemma 3.3, $X^{x}$ satisfies (3.12). By using Assumption H1, the following result is a straightforward application of Theorem 4.3 in [15].

**Proposition 3.1.** Under Assumption H1 and the choice of indices (3.12), for each $\psi \in E_{\kappa}$ there exists a unique global solution to (3.8) in $C_{1}^{\kappa,\kappa}$. 

By noticing (see Lemma 2.4) that $(\beta_{i})_{i \geq 1} \in W_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ a.s., Proposition 3.1 yields the following result.

**Proposition 3.2.** Under Assumption H1 and the choice of indices in (3.12), for each initial condition $x_{0} \in E_{\kappa}$, there exists a unique adapted process $X$ which is solution to (3.7).

3.3. Fréchet differentiability. Let us now devote our attention to the Fréchet differentiability of the Itô map

$$\Phi : W_{\lambda,T}^{\tilde{\gamma},\delta,\infty} \rightarrow C_{1}^{\kappa,\kappa} \quad x \mapsto y,$$

where $y$ is the mild solution of (3.8) driven by $x \in W_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ and the indices $\tilde{\gamma}, \delta, \kappa_{0}, \kappa$ satisfy (3.12). Then, the Fréchet derivative is a mapping

$$\nabla \Phi : W_{\lambda,T}^{\tilde{\gamma},\delta,\infty} \rightarrow \mathcal{L}(W_{\lambda,T}^{\tilde{\gamma},\delta,\infty} : C_{1}^{\kappa,\kappa}).$$

The importance of Fréchet differentiability lies on the following argument: Once we have Fréchet differentiability of the Itô map $x \mapsto y$, we shall use the Fréchet derivative chain rule to infer that $(X_{t},h)_{E}$ is Fréchet differentiable along the direction of the Cameron-Martin space $H$ for a given $h \in E$ and $t \in [0,T]$. Hence, Corollary 2.1 implies

$$\langle X_{t},h \rangle_{E} \in \mathbb{D}^{1,2}_{loc}(\mathbb{R}).$$

Then, we must use Lemma 2.2 and try to conclude a representation. We follow the idea contained in the work of Nualart and Saussereau [28]. At first, we list a set of assumptions on the vector fields which will be important in this section.

**Assumption A1:** The vector fields, $G_{i}, F : E_{\kappa} \rightarrow E_{\kappa}$ are Fréchet differentiable and also differentiable when considering from $E$ to $E$. Moreover,

$$\sup_{i \geq 1} \sup_{x \in E_{\kappa}} \|\nabla G_{i}(x)\|_{\kappa \rightarrow \kappa} + \sup_{x \in E_{\kappa}} \|\nabla F(x)\|_{\kappa \rightarrow \kappa} < \infty,$$

$$\sup_{i \geq 1} \sup_{x \in E} \|\nabla G_{i}(x)\| + \sup_{x \in E} \|\nabla F(x)\| < \infty.$$ 

**Assumption A2:**

$$\sup_{i \geq 1} \sup_{g \in E} \|\nabla^{(2)} G_{i}(g)\|_{(2),q \rightarrow q} + \sup_{f \in E} \|\nabla^{(2)} F(f)\|_{(2),k \rightarrow \kappa} < \infty,$$

for $q \in \{0, \kappa\}$ and there exists a constant $C$ such that

$$\sup_{i \geq 1} \|\nabla G_{i}(f) - \nabla G_{i}(g)\| + \sup_{i \geq 1} \|\nabla^{(2)} G_{i}(f) - \nabla^{(2)} G_{i}(g)\|_{(2),a \rightarrow 0} \leq C \|f - g\|_{E},$$

for every $f, g \in E.$
At first, it is necessary to investigate flow properties of linear equations. We start with the following corollary whose proof is entirely analogous to Proposition 3.3.1, so we omit the details.

**Corollary 3.1.** Suppose $F, G$ satisfy Assumptions A1-H1 and let us fix $(x, y) \in W_{\Lambda, T}^\gamma, \delta, \infty \times \hat{C}_1^{\alpha, \kappa}$ and 

$$t_0 \in [0, T].$$

Then, for every $\eta \in \hat{C}_1^{\alpha, \kappa}$,

$$v_t = \eta_t + \int_{t_0}^t S(t-s)\nabla F(y_s)v_s ds + \mathcal{H}_{ts}(d\hat{x} \nabla G(y)v)$$

admits a unique solution in $v \in \hat{C}_1^{\alpha, \kappa}$ on the interval $[t_0, T]$.

The following lemma plays a key role on the Fréchet differentiability of the Itô map.

**Lemma 3.4.** Let $[s_0, t_0]$ be a subset of $[0, T]$, let $Z_t = \sum_{i \geq 1} \sqrt{\lambda_i} \int_{s_0}^t S(t-s)z_i^s dx_i^s; s_0 \leq t \leq t_0$ where $x \in W_{\Lambda, T}^\gamma, \delta, \infty$ and assume $\sup_{t \geq 1} \|z^t\|_{0, \eta} + \sup_{t \geq 1} \|\hat{\delta} z^t\|_{\zeta, \eta - \alpha} < \infty$ on the interval $[s_0, t_0]$ for some $\eta \geq 0$ where $0 \leq \alpha \leq \min(\zeta, \eta), 0 \leq \zeta \leq \gamma$ and $\gamma + \zeta > 1$. Then, there exists a constant $C$ which depends on $\eta$ and $\gamma$ such that

\begin{equation}
\|\hat{\delta} Z\|_{\eta, \gamma} \leq C\|x\|_{W_{\Lambda, T}^\gamma, \delta, \infty} \sup_{s \geq 1} \|z^s\|_{0, \eta} + |t_0 - s_0|^{\gamma - \zeta} \sup_{s \geq 1} \|\hat{\delta} z^s\|_{\zeta, \eta - \alpha},
\end{equation}

\begin{equation}
\|\hat{\delta}^2 Z\|_{\eta, \gamma} \leq C\|x\|_{W_{\Lambda, T}^\gamma, \delta, \infty} \sup_{s \geq 1} \|z^s\|_{0, \eta} + |t_0 - s_0|^{\gamma - \zeta} \sup_{s \geq 1} \|\hat{\delta} z^s\|_{\zeta, \eta - \alpha},
\end{equation}

on the interval $[s_0, t_0]$.

\textbf{Proof.} In the sequel, $C$ is a constant which may differ from line to line. To keep notation simple, without loss of generality, we set $s_0 = 0, t_0 = T$. We observe $(\hat{\delta} Z)_ts = \sum_{i \geq 1} \sqrt{\lambda_i} \int_s^t S(t-u)z^u_i dx^u_i$.

From the proof of Lemma 3.2, we know that

$$\int_s^t S(t-u)z^u_i dx^u_i = X^{x,i,z_i}_s + \hat{\Lambda}(X^{x,i}\hat{\delta} z^i)_ts; (t, s) \in S_2,$$

where $X^{x,i} \in \hat{C}_2^{\alpha, \eta}$ due to Lemma 3.1. Then, checking the proof of Lemma 3.2, we have $X^{x,i}\hat{\delta} z^i \in \mathcal{Z}\hat{C}_3^{\alpha + \gamma, \theta}$ for $\theta \leq \eta - \alpha$. Now,

\begin{equation}
\left| \sum_{i \geq 1} \sqrt{\lambda_i} \int_s^t S(t-u)z^u_i dx^u_i \right|_\eta \leq \sum_{i \geq 1} \sqrt{\lambda_i} \left| \int_s^t S(t-u)z^u_i dx^u_i \right|_\eta \leq C \sum_{i \geq 1} \sqrt{\lambda_i} \|X^{x,i}_t\|_{\eta \eta} |z^i_1| \eta
\end{equation}

\begin{equation}
+ \sum_{i \geq 1} \sqrt{\lambda_i} \left| \hat{\Lambda}(X^{x,i}\hat{\delta} z^i)_ts \right|_\eta \leq C\|x\|_{W_{\Lambda, T}^\gamma, \delta, \infty} \|t - s\|^{\gamma} \sup_{s \geq 1} \|z^s\|_{0, \eta} + \sum_{i \geq 1} \sqrt{\lambda_i} \left| \hat{\Lambda}(X^{x,i}\hat{\delta} z^i)_ts \right|_\eta; (t, s) \in S_2.
\end{equation}

By applying the “convolution” Sewing lemma (Th 3.5 in [15]), there exists a constant $C_{\zeta + \gamma}$ such that

$$\|\hat{\Lambda} X^{x,i}\hat{\delta} z^i\|_{\zeta + \gamma, \theta + \epsilon} \leq C_{\zeta + \gamma, \epsilon} \|X^{x,i}\hat{\delta} z^i\|_{\zeta + \gamma, \theta},$$

for every $\epsilon \in [0, \zeta + \gamma] \cap [0, 1)$. Take $\theta = \eta - \alpha$ and $\epsilon = \alpha$. Then,

\begin{equation}
\left| \hat{\Lambda}(X^{x,i}\hat{\delta} z^i)_ts \right|_\eta \leq C_{\zeta + \gamma, \epsilon} \|X^{x,i}\hat{\delta} z^i\|_{\zeta + \gamma, \theta} |t - s|^{\gamma - \epsilon}.
\end{equation}

On the other hand, $(X^{x,i}\hat{\delta} z^i)$ is a 3-increment, where
\[
\|X^{x,i}\delta z^i\|_{\zeta+\tilde{\gamma},\eta-\alpha} = \inf \left\{ \sum_j \|h_j\|_{\rho_j,\zeta+\tilde{\gamma}-\rho_j,\eta-\alpha} : X^{x,i}\delta z^i = \sum_j h_j, 0 < \rho_j < \zeta + \tilde{\gamma} \right\},
\]

and the last infimum is taken over all sequences \(h_j\) such that \(X^{x,i}\delta z^i = \sum_j h_j\) and for all choices of the numbers \(\rho_j \in (0, \zeta + \tilde{\gamma})\). Here, we recall for any 3-increment \(f\), we have

\[
\|f\|_{\rho_j,\zeta+\tilde{\gamma}-\rho_j,\eta-\alpha} = \sup_{t,u,s \in S_t} \frac{|f_{tu}|_{\eta-\alpha}}{|t-u|_{\rho_j}|u-s|_{\zeta+\tilde{\gamma}-\rho_j}}.
\]

Take \(h_i = X^{x,i}\delta z^i\) and \(\rho_j = \tilde{\gamma}\). By definition, \(X^{x,i}\delta z^i = \sum_{t,u,s} h_{i}\), and then

\[
\|X^{x,i}\delta z^i\|_{\zeta+\tilde{\gamma},\eta-\alpha} \leq \sup_{t,u,s \in S_t} \frac{|X^{x,i}\delta z^i|_{\eta-\alpha}}{|t-u|_{\rho_j}|u-s|_{\zeta+\tilde{\gamma}-\rho_j}} \leq \sup_{t,u,s \in S_t} \frac{|X^{x,i}\delta z^i|_{\eta-\alpha}}{|t-u|_{\tilde{\gamma}}|u-s|_{\tilde{\gamma}}} \leq C\|x^i\|_{W^{3,\infty}_{3,\infty}} \|\delta z^i\|_{\zeta,\eta-\alpha}.
\]

Then, (3.16) yields

\[
(3.17) \quad \sum_{i \geq 1} \sqrt{\lambda_i} \left| \hat{\Lambda}(X^{x,i}\delta z^i)_{ts} \right|_{\eta} \leq C_{\zeta+\tilde{\gamma},\alpha} |t-s|^{\zeta+\tilde{\gamma}-\alpha} \|x\|_{W_{\lambda,T}^{3,\infty}} \sup_{i \geq 1} \|\delta z^i\|_{\zeta,\eta-\alpha}.
\]

Finally, we shall plug (3.17) into (3.15) and conclude the proof of (3.13). By observing (3.17) and (3.15), we conclude (3.14).

**Lemma 3.5.** Assume that hypotheses H1-A1-A2 hold true. Let \(y\) be the solution of (3.8) with initial condition \(\psi\) and driven by \(x \in W_{\lambda,T}^{3,\infty}\). Then, the mapping

\[
L : W_{\lambda,T}^{3,\infty} \times \hat{C}_1^{\kappa,\kappa} \to \hat{C}_1^{\kappa,\kappa},
\]

defined by

\[
(x,y) \mapsto L(x,y)_t := y_t - S_t \psi - \int_0^t S(t-s)F(y_s)ds - J_{t0}(\tilde{d}(x)G(y))
\]
is Fréchet differentiable. In particular, for each \((x,y) \in W_{\lambda,T}^{3,\infty} \times \hat{C}_1^{\kappa,\kappa}\) and \((q,v) \in W_{\lambda,T}^{3,\infty} \times \hat{C}_1^{\kappa,\kappa}\), we have

\[
(3.18) \quad \nabla_1 L(x,y)(q)_t = -J_{t0}(\tilde{d}qG(y)),
\]

\[
(3.19) \quad \nabla_2 L(x,y)(v)_t = v_t - \int_0^t S(t-s)\nabla F(y_s)v_sds - J_{t0}(\tilde{d}x \nabla G(y)v); 0 \leq t \leq T.
\]

Moreover, for each \(x \in W_{\lambda,T}^{3,\infty}\), the mapping \(\nabla_2 L(x, \Phi(x)) : \hat{C}_1^{\kappa,\kappa} \to \hat{C}_1^{\kappa,\kappa}\) is an homeomorphism.

**Proof.** In the sequel, \(C\) is a constant which may differ form line to line. By the very definition,

\[
L(x+h,y+v)_t - L(x,y)_t = v_t - \int_0^t S(t-u)[F(y_u + v_u) - F(y_u)]du,
\]
Then, for \( i \geq 1 \),

\[
\begin{align*}
F(y_u + v_u) &= \nabla F(y_u)v_u + z_u(y, v), \\
G_i(y_u + v_u) &= \nabla G_i(y_u)v_u + c^i_u(y, v), \\
G_i(y_u + v_u) &= G_i(y_u) + c^i_u(y, v),
\end{align*}
\]

where

\[
z_u(y, v) := \left( \int_0^1 (1-r) \nabla^{(2)} F(y_u + rv_u)dr \right) (v_u, v_u), \\
c^i_u(y, v) := \left( \int_0^1 (1-r) \nabla^{(2)} G_i(y_u + rv_u)dr \right) (v_u, v_u),
\]

for \( i \geq 1 \) and \( 0 \leq u \leq t \). Therefore,

\[
\begin{align*}
L(x + h, y + v)t - L(x, y)t - \nabla_1 L(x, y)(h)_t - \nabla_2 L(x, y)(v)_t &= R_1(y, v)t + R_2(y, v)t + R_3(y, v)t,
\end{align*}
\]

where

\[
\begin{align*}
R_1(y, v)_t &= -\int_0^t S(t-u)z_u(y, v)du, \\
R_2(y, v)_t &= -\sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t S(t-u)c^i_u(y, v)dx^i_u, \\
R_3(y, v)_t &= -\sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t S(t-u)c^i_u(y, v)dh^i_u.
\end{align*}
\]

We need to check

\[
\| R_1(y, v) + R_2(y, v) + R_3(y, v) \|_{\hat{c}^{i, \kappa}} = o(\| h \|_{W^{i, \infty}} + \| v \|_{C^{i, \kappa}})^{1/2}.
\]

The first term is easy. Indeed, if the second order derivative of \( F \) is bounded, then the norm of the bilinear form \( z_u(y, v) \) can be estimated as follows \( \| z_u(y, v) \|_{(2, \kappa) \to \kappa} \leq C \| v \|_{\kappa} \leq C \| v \|_{C^{i, \kappa}} \). Therefore,

\[
\| R_1(u, v) \|_{\hat{c}^{i, \kappa}} \leq C \| v \|_{C^{i, \kappa}}.
\]

Then,

\[
\frac{\| R_1(u, v) \|_{\hat{c}^{i, \kappa}}}{(\| h \|_{W^{i, \infty}} + \| v \|_{C^{i, \kappa}})^{1/2}} \leq \frac{\| R_1(u, v) \|_{\hat{c}^{i, \kappa}}}{(\| v \|_{C^{i, \kappa}})^{1/2}} \leq C \| v \|_{\hat{c}^{i, \kappa}}.
\]

Let us now estimate \( R_2(y, v) \). At first, since \( R_2(y, v)_0 = 0 \), then

\[
\| R_2(y, v) \|_{\hat{c}^{i, \kappa}} \leq (2 + T^\kappa) \| \delta R_2(y, v) \|_{\kappa, \kappa}.
\]

where

\[
-(\delta R_2(y, v))_t = \sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t S(t-u)c^i_u(y, v)dx^i_u = J_{\hat{c}^{i, \kappa}}(\tilde{d}xc(y, v)) = \| \delta R_2(y, v) \|_{\kappa, \kappa} = \| J(\tilde{d}xc(y, v)) \|_{\kappa, \kappa}.
\]

By Lemma 3.4, there exists a constant \( C \) such that

\[
\| J(\tilde{d}xc(y, v)) \|_{\kappa, \kappa} \leq C \| x \|_{W^{\kappa, \infty}} \left\{ \sup_{i \geq 1} \| c^i(y, v) \|_{0, \kappa} + \sup_{i \geq 1} \| \delta c^i(y, v) \|_{0, 0} \right\}.
\]

By definition,
\((\hat{\delta}c^i(y, v))_{ts} = c^i_t(y, v) - c^i_s(y, v) + c^i_s(y, v) - S(t - s)c^i_s(y, v); (t, s) \in S_2\).

By viewing \(\nabla(2)G_i : E_\kappa \times E_\kappa \to E_\kappa\) as a bounded bilinear form where \(\kappa > 0\), we observe \(c^i_s(y, v) \in E_\kappa\) and this little gain of spatial regularity allows us to estimate

\[
(3.24) \quad \| (\hat{\delta}c^i(y, v))_{ts} \|_E \leq \| (\hat{\delta}c^i(y, v))_{ts} \|_E + \| (S(t - s) - Id)c^i_s(y, v) \|_E,
\]

where (see e.g Th 6.13 in [30])

\[
\| (S(t - s) - Id)c^i_s(y, v) \|_E \leq C|t - s|^\kappa |c^i_s(y, v)|_E
\]

(3.25)

and this estimate (3.25) is due to the boundedness \(\text{sup}_{i \geq 1} \sup_{a \in E_\kappa} \| \nabla(2)G_i(a) \|_{(2), \kappa, \kappa} < \infty\).

For each \(i \geq 1\) and \(u \in [0, t]\), we observe \(\int_0^t (1 - r)\nabla(2)G_i(y_u + rv_u)dr : E \times E \to E\) is a bounded bilinear form, so that we shall estimate

\[
\| c^i_u(y, v) - c^i_r(y, v) \|_E \leq C\|v_u - v_r\|_E\|v_u\|_E + C\|v_u - v_r\|_E\|v_r\|_E
\]

\[
+ \int_0^1 (1 - r)\nabla(2)G_i(y_u + rv_u) - \nabla(2)G_i(y_r + rv_r)\|_{(2), 0 \to 0} dr \|v_u - v_r\|_E^2.
\]

By using the Lipschitz property on the bilinear form \(\nabla(2)G_i\), we have

\[
\int_0^1 (1 - r)\| \nabla(2)G_i(y_u + rv_u) - \nabla(2)G_i(y_r + rv_r)\|_{(2), 0 \to 0} dr \leq C \int_0^1 (1 - r)\| y_u - y_r\|_E dr
\]

\[
+ \int_0^1 (1 - r) r\| v_u - v_r\|_E dr \leq C\| y_u - y_r\|_E + C\| v_u - v_r\|_E.
\]

Now, we observe \(\hat{C}^i_{\kappa, \kappa} \hookrightarrow C^i_{1, 0}\) (see (3.14) and \(E_\kappa \hookrightarrow E\). Therefore,

\[
(3.26) \quad \frac{\| c^i_u(y, v) - c^i_r(y, v) \|_E}{|u - r|^\kappa} \leq C2\| v\|_{E_\kappa, \kappa} + C\| v\|_{C^i_{1, \kappa, \kappa}}.
\]

By assumption, \(\sup_{i \geq 1} \sup_{p \in E_\kappa} \| \nabla^2 G_i(p) \|_{(2), \kappa, \kappa} < \infty\) and hence

\[
(3.27) \quad \sup_{i \geq 1} \| c^i(y, v) \|_{0, \kappa} \leq C\| v\|_{E_\kappa, \kappa}.
\]

Plugging (3.27), (3.26), (3.24) and (3.21) into (3.22), we conclude from (3.22) that \(R_2(y, v) \|_{c^i_{1, \kappa, \kappa}} \leq C\| v\|_{E_\kappa, \kappa}^2\).

Let us now estimate \(R_3(y, v)\). Similar to (3.22), from Lemma 5.4 we know there exists a constant \(C\) such that

\[
(3.28) \quad \| F(\hat{d}x(y, v)) \|_{\kappa, \kappa} \leq C\| h\|_{W^{\gamma, \delta, \kappa}} \left\{ \sup_{i \geq 1} \| c^i(y, v) \|_{0, \kappa} + \sup_{i \geq 1} \| \hat{\delta}c^i(y, v) \|_{\kappa, 0} \right\}.
\]

Clearly, Assumption A1 yields

\[
(3.29) \quad \sup_{i \geq 1} \| c^i(y, v) \|_{0, \kappa} \leq C\| v\|_{0, \kappa} \leq C\| v\|_{E_\kappa, \kappa}.
\]

Similar to (3.21) and (3.25), we observe
\begin{equation}
\|(\delta e^i(y,v))_{ts}\|_E \leq \|(\delta e^i(y,v))_{ts}\|_E + \|(S(t-s) - \text{Id})e^i_s(y,v)\|_E,
\end{equation}
where
\begin{equation}
\|(S(t-s) - \text{Id})e^i_s(y,v)\|_E \leq C|t-s|^\kappa \|v\|_{0,\kappa}; (t,s) \in S_2.
\end{equation}

The boundedness and the Lipschitz property on $\nabla G_i$ (Assumption A2) allow us to estimate
\begin{equation}
\|e^i_u(y,v) - e^i_{u'}(y,v)\|_E \leq C\|v_u - v_{u'}\|_E + C\|v_{u'}\|_E \left\{ \|y_u - y_{u'}\|_E + \|v_u - v_{u'}\|_E \right\}.
\end{equation}
Then, (3.3) yields
\begin{equation}
\frac{\|e^i_u(y,v) - e^i_{u'}(y,v)\|_E}{|u - u'|^\kappa} \leq C\|v\|_{\ell_1} + \|v\|_{\ell_1} \left\{ \|y\|_{\ell_1} + \|v\|_{\ell_1} \right\}.
\end{equation}

By using (3.28), (3.29), (3.30), (3.31), and (3.32), we infer
\begin{equation}
\|\mathcal{J}(dx_e(y,v))\|_{\kappa,\kappa} = O(\|h\|_{W^{\gamma,\delta}_{\ell_1}} \times \|v\|_{\ell_1}).
\end{equation}

One can easily check $(x,y) \mapsto \nabla_1 L(x,y) \in L(W^{\gamma,\delta}_{\ell_1}; \hat{C}^{\kappa,\kappa})$ and $(x,y) \mapsto \nabla_2 L(x,y) \in L(\hat{C}^{\kappa,\kappa}; \hat{C}^{\kappa,\kappa})$ are both continuous. Summing up all the above steps, we conclude $L$ is Fréchet differentiable and formulas (3.18) and (3.19) hold true. It remains to check $\nabla_2 L(x, \Phi(x))$ is a $\hat{C}^{\kappa,\kappa}$-homeomorphism. By open mapping theorem, this is an immediate consequence of Corollary 3.1 (which proves it is an isomorphism). The continuity can be easily checked so we left the details of this point to the reader. \hfill \Box

By applying implicit function theorem, $x \mapsto \Phi(x)$ is continuously Fréchet differentiable and the following formula holds true
\begin{equation}
\nabla \Phi(x) = -\nabla_2 L(x, \Phi(x))^{-1} \circ \nabla_1 L(x, \Phi(x)); x \in W^{\gamma,\delta}_{\ell_1,\ell_2}.
\end{equation}
The inverse operator yields $\nabla_2 L(x, \Phi(x))(\nabla_2 L(x, \Phi(x))^{-1}(v)) = v$ so that
\begin{align*}
\nabla_2 L(x, \Phi(x))^{-1}(v)_t = v_t + \int_0^t S(t-u) \nabla F(\Phi(x)_u) \nabla_2 L(x, \Phi(x))^{-1}(v)_u du \\
+ \sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t S(t-u) \nabla G_i(\Phi(x)_u) \nabla_2 L(x, \Phi(x))^{-1}(v)_u dx^i_u; 0 \leq t \leq T,
\end{align*}
for each $v \in \hat{C}^{\kappa,\kappa}_1$. Therefore, for each $x, h \in W^{\gamma,\delta}_{\ell_1,\ell_2}$, $\nabla \Phi(x)(h)$ is the unique solution of
\begin{align*}
\nabla \Phi(x)(h)_t = &\sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t S(t-u) G_i(\Phi(x)_u) dh^i_u + \sum_{j \geq 1} \sqrt{\lambda_j} \int_0^t S(t-u) \nabla G_j(\Phi(x)_u) \nabla \Phi(x)(h)_u dx^j_u \\
+ &\int_0^t S(t-u) \nabla F(\Phi(x)_u) \nabla \Phi(x)(h)_u du; 0 \leq t \leq T.
\end{align*}

Now, by Corollary 3.1 for each $u \in (0, T), x \in W^{\gamma,\delta}_{\ell_1,\ell_2}$ and $i \geq 1$, the mapping $t \mapsto \Psi^i_{t,u}(x)$ given by
\begin{equation}
\Psi^i_{t,u}(x) := S(t-u) G_i(\Phi(x)_u) + \int_u^t S(t-\ell) \nabla G_j(\Phi(x)_\ell) \Psi^i_{\ell,u}(x) dx^j_{\ell},
\end{equation}
(3.35) \[ + \int_u^t S(t - \ell)\nabla F(\Phi(x)_\ell)\Psi^i_{t,u}(x)d\ell, \]

where $\Psi^i_{t,u}(x) = 0$ for $u > t$, it is a well-defined element of $\hat{\mathcal{C}}^{\kappa,\kappa}$ over $[u,T]$. Let us denote $\Gamma^i_{x,u,u'}(t) := \Psi^i_{t,u}(x) - \Psi^i_{t,u'}(x)$ for $0 \leq u' \leq u \leq t \leq T$. It is simple to check that

\[
\Gamma^i_{x,u,u'}(t) = S(t-u)\left[\Psi^i_{u,u}(x) - \Psi^i_{u,u'}(x)\right] + \sum_{j \geq 1} \sqrt{\lambda_j} \int_u^t S(t-\ell)\nabla G_j(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)d\ell + \int_u^t S(t-\ell)\nabla F(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)d\ell.
\]

The following technical lemma is important to derive an alternative representation for $\nabla \Phi(x)$. 

**Lemma 3.6.** If Assumptions H1-A1-A2 hold true, then for each $x \in W^{1,\infty}_{\alpha,T}$, there exists a positive constant $C$ which only depends on $\|x\|_{W^{1,\infty}_{\alpha,T}}$ and $\|\delta \Phi(x)\|_{\kappa,\kappa}$ such that

\[
|\Gamma^i_{x,u,u'}(t)|_{\kappa} \leq C|\Psi^i_{u,u}(x) - \Psi^i_{u,u'}(x)|_{\kappa},
\]

for every $0 \leq u' < u \leq t \leq T$ and $i \geq 1$.

**Proof.** Fix $0 \leq u' < u \leq T$, $i \geq 1$, $0 \leq \alpha \leq \min\{\kappa, \eta\}$ for $\eta \geq 0$. Let us denote $\phi^i_{x,u,u'} = [\Psi^i_{u,u}(x) - \Psi^i_{u,u'}(x)]$. In the sequel, $C$ is a constant which may differ from line to line. Of course,

\[
\|\delta \Gamma^i_{x,u,u'}\|_{\kappa,\eta} \leq \|\delta S(-u)\phi^i_{x,u,u'}\|_{\kappa,\eta} + \sum_{j \geq 1} \sqrt{\lambda_j} \int_u^t S(-\ell)\nabla G_j(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)d\ell + \int_u^t S(-\ell)\nabla F(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)d\ell \leq I_1 + I_2 + I_3.
\]

At first, we observe $S(t-u)\phi^i_{x,u,u'} - S(t-s)S(s-u)\phi^i_{x,u,u'} = 0$ so that $I_1 = 0$.

By Lemma 3.4 (see (3.14)), we observe there exists a constant $C$ such that

\[
I_2 \leq C\|x\|_{W^{1,\infty}_{\alpha,T}}\sup_{j \geq 1} \|z^{ij}_{x,u,u'}\|_{0,\eta}|T-u|^\alpha + |T-u|^\frac{\alpha}{2} \sup_{j \geq 1} \|\delta z^{ij}_{x,u,u'}\|_{\kappa,\eta-\alpha},
\]

where $z^{ij}_{x,u,u'}(\ell) = \nabla G_j(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)$. Let us take $\eta = \kappa = \alpha$. We observe

\[
|z^{ij}_{x,u,u'}(\ell)|_{\kappa} \leq \|\nabla G_j(\Phi(x)_\ell)\|_{\kappa,\kappa}\|\Gamma^i_{x,u,u'}(\ell)|_{\kappa},
\]

so that the boundedness assumption on the gradient $\nabla G_j$ yields

(3.36) \[
\|z^{ij}_{x,u,u'}\|_{0,\kappa} \leq C\|\Gamma^i_{x,u,u'}\|_{0,\kappa} \leq C\|\Gamma^i_{x,u,u'}\|_{\kappa,\kappa}.
\]

Triangle inequality yields

\[
\|\delta z^{ij}_{x,u,u'}\|_{E} \leq \|\nabla G_j(\Phi(x)_s) - \nabla G_j(\Phi(x)_s)\|_{0,0} + \|\nabla G_j(\Phi(x)_s)\|_{0,0} + \|\delta \Gamma^i_{x,u,u'}(t)\|_{E} + \|\delta \Gamma^i_{x,u,u'}(t)\|_{E} + \|\|\text{Id} - S(t-s)\nabla G_j(\Phi(x)_s)\Gamma^i_{x,u,u'}(s)\|_{E} =: I_4 + I_5 + I_6.
\]

where $\nabla G_j(\Phi(x)_s)\Gamma^i_{x,u,u'}(s) \in E_{\kappa}$. We observe

(3.37) \[
I_6 \leq C|t-s|^{\kappa}\|\Gamma^i_{x,u,u'}\|_{0,\kappa}.
\]
The imbedding \( \mathcal{C} \) yields
\[
I_5 \leq C|t - s|^{\kappa} \{ \| \hat{\delta} \Gamma_{x,u,u}^{i} \|_{\kappa,\kappa} + \| \Gamma_{x,u,u'}^{i} \|_{0,\kappa} \} = C|t - s|^{\kappa} \| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa}.
\]
We observe
\[
I_4 \leq C \| \delta \Phi(x) \|_{\kappa,\kappa}|t - s|^{\kappa} \| \Gamma_{x,u,u'}^{i} \|_{0,\kappa}.
\]
Summing up \( (3.39) \), \( (3.38) \) and \( (3.37) \), we have
\[
(3.39) \quad \| \hat{\delta} \Gamma_{x,u,u}^{i} \| \leq C \left( 1 + \| \delta \Phi(x) \|_{\kappa,\kappa} \right) \| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa}.
\]
This shows that
\[
I_2 \leq C \| x \|_{W_{\gamma,\kappa}^{\gamma,\infty}} \left\{ \| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa}|T - u|^{\gamma - \kappa} + |T - u|^{\gamma - \kappa} \left( 1 + \| \delta \Phi(x) \|_{\kappa,\kappa} \right) \| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa} \right\}.
\]
We notice that
\[
I_3 \leq C \sup_{u \leq s < t \leq T} \frac{\int_{s}^{t} S(t - \ell) \nabla F(\Phi(x)) \Gamma_{x,u,u'}^{i}(\ell) \ell^{\kappa}}{|t - s|^{\kappa}} = C \| \Gamma_{x,u,u'}^{i} \|_{0,\kappa}|T - u|^{1 - \kappa}.
\]
Summing up the above inequalities, we have
\[
\| \hat{\delta} \Gamma_{x,u,u}^{i} \|_{\kappa,\kappa} \leq C \| x \|_{W_{\gamma,\kappa}^{\gamma,\infty}} \left\{ C|T - u|^{\gamma - \kappa} \left( 1 + \| \delta \Phi(x) \|_{\kappa,\kappa} \right) \| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa} \right\}
\]
\[
+ C \| \Gamma_{x,u,u'}^{i} \|_{0,\kappa}|T - u|^{1 - \kappa}.
\]
Therefore,
\[
\| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa} \leq \| S(\cdot - u) \varphi_{x,u,u'}^{i} \|_{0,\kappa} + C \| x \|_{W_{\gamma,\kappa}^{\gamma,\infty}} \left\{ C|T - u|^{\gamma - \kappa} \left( 1 + \| \delta \Phi(x) \|_{\kappa,\kappa} \right) \| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa} \right\}
\]
\[
+ C \| \Gamma_{x,u,u'}^{i} \|_{0,\kappa}|T - u|^{1 - \kappa},
\]
where \( \| S(\cdot - u) \varphi_{x,u,u'}^{i} \|_{0,\kappa} \leq C \| \varphi_{x,u,u'}^{i} \|_{\kappa} \). Finally, by working on a small interval and performing a standard patching argument, the estimate \( (3.42) \) allows us to conclude
\[
\| \Gamma_{x,u,u'}^{i} \|_{\kappa,\kappa} \leq C_{x,y,T} \| \varphi_{x,u,u'}^{i} \|_{\kappa},
\]
where \( C_{x,y,T} = g(\| x \|_{W_{\gamma,\kappa}^{\gamma,\infty}}, \| \delta \Phi(x) \|_{\kappa,\kappa}, T) \) for a function \( g: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) growing with its arguments. This implies
\[
\| \Gamma_{x,u,u'}^{i}(t) \|_{\kappa} = \| \Psi_{u,u}^{i}(x) - \Psi_{u,u'}^{i}(x) \|_{\kappa} \leq C_{x,y,T} \| \Psi_{u,u}^{i}(x) - \Psi_{u,u'}^{i}(x) \|_{\kappa}.
\]
\[\square\]

We are now in position to state the main result of this section. Let \( \mathcal{C}_{\alpha,\lambda}^\infty \) be the subset of \( W_{\lambda,T}^{\gamma,\infty} \) composed by functions \( g: \mathbb{N} \rightarrow \mathcal{C}_{\alpha}^\infty \).
Theorem 3.1. Under Assumptions (H1-A1-A2), the Itô map \( x \mapsto \Phi(x) \) is continuously Fréchet differentiable and for each \( x, h \in \mathcal{W}^{7,δ,∞}_{λ,T} \), \( \nabla \Phi(x)(h) \) is the unique solution of the equation (3.34). In addition, the following representation formula holds true

\[
\nabla \Phi(x)(t) = \sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t \Phi_{t,u}^i(x) dh_u^i \in E_κ; 0 \leq t \leq T,
\]

for each \( (x, h) \in \mathcal{W}^{7,δ,∞}_{λ,T} \times C^{0,∞}_{0,λ}. \)

Proof. The fact that \( x \mapsto \Phi(x) \) is continuously Fréchet differentiable and it satisfies (3.34) are consequences of (3.33). Obviously,

\[
\sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t \Phi_{t,u}^i(x) dh_u^i = \sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t S(t-u)G_i(\Phi(x)_u) dh_u^i 
+ \sum_{i \geq 1} \sqrt{\lambda_i} \int_0^t \int_0^t S(t-\ell)\nabla F(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) d\ell dh_u^i 
+ \sum_{i \geq 1} \sqrt{\lambda_i} \sum_{j \geq 1} G_j(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) dx_\ell^j dh_u^i; 0 \leq t \leq T.
\]

Let us fix \( i \geq 1 \) and \( x \in \mathcal{W}^{7,δ,∞}_{λ,T} \). By Lemma 3.0 and noticing

\[
\Phi_{t,u}^i(x) - \Phi_{t,u'}^i(x) = G_i(\Phi(x)_u) - S(u-u')G_i(\Phi(x)_{u'}) - \sum_{j \geq 1} \sqrt{\lambda_j} \int_{u'}^u S(u-\ell)\nabla G_j(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) dx_\ell^j 
- \int_{u'}^u S(u-\ell)\nabla F(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) d\ell; 0 \leq u' < u \leq T; i \geq 1,
\]

we clearly have \( u \mapsto \Phi_{t,u}^i(x) \) is continuous, so that we shall apply Fubini’s theorem to get

\[
\int_0^t \int_u^t S(t-\ell)\nabla F(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) d\ell dh_u^i = \int_0^t \int_0^t S(t-\ell)\nabla F(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) dh_u^i d\ell,
\]

\[
\int_0^t \int_u^t S(t-\ell)\nabla G_j(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) dx_\ell^j dh_u^i = \int_0^t \int_0^t S(t-\ell)\nabla G_j(\Phi(x)_\ell) \Phi_{\ell,u}^i(x) dx_\ell^j d\ell; 0 \leq t \leq T; i \geq 1.
\]

Therefore,

\[
\sqrt{\lambda_i} \int_0^t \Phi_{t,u}^i(x) dh_u^i = \sqrt{\lambda_i} \int_0^t S(t-u)G_i(\Phi(x)_u) dh_u^i 
+ \int_0^t S(t-\ell)\nabla F(\Phi(x)_\ell) \sqrt{\lambda_i} \int_0^t \Phi_{\ell,u}^i(x) d\ell dh_u^i 
+ \sum_{j \geq 1} \sqrt{\lambda_j} \int_0^t S(t-\ell)\nabla G_j(\Phi(x)_\ell) \sqrt{\lambda_i} \int_0^t \Phi_{\ell,u}^i(x) dx_\ell^j d\ell; 0 \leq t \leq T.
\]

At this point, in order to complete the proof of representation (3.43), we only need to check

\[
\sup_{i \geq 1} \sup_{0 \leq t \leq T} \| \Phi_{t,u}^i \|_{0,κ} < \infty.
\]

Since \( \Phi_t^i \) is the solution of the linear equation (3.33), a completely similar argument as detailed in the proof of Lemma 3.0 yields
for each \(0 \leq u \leq t \leq T\), where \(C_{x,y,T} = g(\|x\|_{\mathcal{W}_{\lambda,T}^{\gamma,\delta}},\|\Phi(x)\|_{\kappa,T})\) for a function \(g : \mathbb{R}^3_+ \to \mathbb{R}_+\) growing with its arguments. This completes the proof.

Let us now check Malliavin differentiability. Let us fix \(t \in [0,T]\), \(g \in E\) and now look the mapping \(\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty} \ni x \mapsto \langle \Phi(x),g \rangle_E \in \mathbb{R}\). We can represent \(\Phi(x)_t = \tau_t(\Phi(x))\), where \(\tau_t : \mathcal{C}^{1,\kappa}_1 \to E\) is the evaluation map which is a bounded linear operator for every \(t \in [0,T]\). Then, the Fréchet derivative of \(x \mapsto \Phi(x)_t\) is equal to the linear operator

\[
\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty} \ni f \mapsto \nabla \Phi(x)(f)_t \in E_{\kappa} \subset E.
\]

Similarly, the Fréchet derivative of \(x \mapsto \langle \Phi(x)_t,g \rangle_E\) is equal to

\[
f \mapsto \langle \nabla \Phi(x)(f)_t, g \rangle_E.
\]

We must find an \(L_2(U_0, \mathcal{H})\)-valued random element \(\omega \mapsto a(\omega)\) such that

\[
\langle \nabla \Phi(\cdot)(\mathcal{R}_H h)_t, g \rangle_E = \langle \langle a(\cdot), h \rangle_{L_2(U_0,\mathcal{H})} \rangle_{L_2(U_0, \mathcal{H})} a.s,
\]

for each \(h \in L_2(U_0, \mathcal{H})\). If this is the case, then \(a = \mathcal{D} \langle X_t, g \rangle_E\) a.s. The following result is a straightforward consequence of the definition of \(\mathcal{R}_H\).

**Lemma 3.7.** If \(h \in \mathcal{C}^{\infty}_0\) and \(\varphi \in L_2(U_0; \mathbb{R})\), then

\[
\mathcal{R}_H(h \otimes \varphi) \in \mathcal{C}^{\infty}_0.
\]

**Corollary 3.2.** Under the probability space given in Lemma 2.4, the random variable \(\langle X_t, g \rangle_E \in \mathcal{D}^{1,2}_0(\mathbb{R})\) and \(\mathcal{D} \langle X_t, g \rangle_E \in L_2(U_0; \mathcal{H})\) is the Hilbert-Schmidt linear operator defined by

\[
\mathcal{D} \langle X_t, g \rangle_E(\sqrt{\lambda} e_i) := \langle \sqrt{\lambda} \Psi_t^{i}, g \rangle_E a.s,
\]

for every \(t \in [0,T]\) and \(g \in E\).

**Proof.** Let us fix \(t \in [0,T]\) and \(g \in E\). By Lemma 2.4 we shall represent \(X_t(\omega) = \Phi(\omega)_t; (\omega, t) \in \mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty} \times [0,T]\). Since \(H \subset \mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}\), then

\[
f \mapsto \langle X_t(f), g \rangle_E = \langle \Phi(f)_t, g \rangle_E
\]

is Fréchet differentiable at all vectors \(f \in \mathcal{H}\). In this case, Corollary 2.1 yields \(\langle X_t, g \rangle_E \in \mathcal{D}^{1,2}_0(\mathbb{R})\) and

\[
\langle \nabla \Phi(\cdot)(\mathcal{R}_H v)_t, g \rangle_E = \langle \mathcal{D} \langle X_t, g \rangle_E, v \rangle_{L_2(U_0, \mathcal{H})} \text{ locally in } \Omega,
\]

for each \(v \in L_2(U_0; \mathcal{H})\). Let us take \(v = (h \otimes \varphi) \in \mathcal{C}^{\infty}_0 \otimes L_2(U_0, \mathbb{R})\). By using (3.43)

\[
\langle \nabla \Phi(\mathcal{R}_H v)_t, g \rangle_E = \sum_{i \geq 1} \sqrt{\lambda_i} \langle \int_0^t \Psi^{i}_{t,u}(x) d(\mathcal{R}_H v^i)_u, g \rangle_E.
\]

We observe

\[
(\mathcal{R}_H v^i)_u = \int_0^u \frac{\partial K_H}{\partial u}(u,s)K^*_H(h \otimes \varphi)_s(e_i)ds.
\]

Therefore,

\[
\int_0^t \Psi^{i}_{t,u}(x) d(\mathcal{R}_H v^i)_u = \sqrt{\lambda_i} \int_0^t \Psi^{i}_{t,u}(x) \left( \int_0^u \frac{\partial K_H}{\partial u}(u,s)K^*_H(h \otimes \varphi)_s(e_i)ds \right) du
\]
for a given $\mathbf{E}$

Then, $D(3.47)$

At first, we observe the postulated object

Theorem 3.2. If Assumptions H1-A1-A2 hold true, then

\[
\langle \sqrt{\lambda_i} \Psi_{t,u}(x) \rangle := \langle \sqrt{\lambda_i} \Psi_{t,u}^i, g \rangle_E \ a.s.
\]

We observe (3.47) provides a well-defined Hilbert-Schmidt operator from $U_0$ to $H$ because

\[
\sum_{i=1}^{\infty} \lambda_i \int_0^T |K_H^*(\langle \Psi_{t,u}^i(\omega), g \rangle_E)|^2 ds = \sum_{i=1}^{\infty} \lambda_i \int_0^T | \int_0^T \partial K_H(u, s) \langle \Psi_{t,u}^i(\omega), g \rangle_E | du|^2 ds \\
\leq \int_0^T \left( \int_0^T | \int_0^T \partial K_H(u, s) | du \right)^2 ds \|g\|^2_{L^2} \sup_{i \geq 1} \|\Psi_{t,u}^i(\omega)\|^2_{L^2}\text{Tr} \langle Q \rangle < \infty,
\]

for each $\omega \in \Omega$. This concludes the proof.

We are now able to state the main result of this section.

**Theorem 3.2.** If Assumptions H1-A1-A2 hold true, then $X_t \in D_{loc}^2(E)$ for each $t \in [0,T]$ and the following formula holds

\[
(3.48) \ D_X X_t = S((t-s)G(X_s)) + \int_s^t S((t-r)\nabla F(X_r)) D_X X_r dr + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_s^t S((t-r)\nabla G_i(X_r)) D_X X_r d\beta_r,
\]

where $D_X X_t = 0$ for $s > t$.

**Proof.** At first, we observe the postulated object $D_X X_t$ takes values on $\mathcal{H} \otimes \mathcal{L}_2(U_0; \mathbb{R}) \otimes E \equiv \mathcal{L}_2(U_0; \mathcal{H} \otimes E)$. Let us compute

\[
\langle D_X X_t, g \rangle_E, v \rangle_{\mathcal{L}_2(U_0; H)}
\]

for a given $g \in E$ and $v = (\varphi \otimes h) \in \mathcal{L}_2(U_0; H)$. By definition,

\[
\langle D_X X_t, g \rangle_E, v \rangle_{\mathcal{L}_2(U_0; H)} = \sum_{i=1}^{\infty} \langle \sqrt{\lambda_i} \Psi_{t,u}^i, g \rangle_E, (\varphi \otimes h)(\sqrt{\lambda_i} e_i) \rangle_{\mathcal{H}}
\]

\[
= \sum_{i=1}^{\infty} \varphi(e_i) \lambda_i \langle \Psi_{t,u}^i, g \rangle_E, h \rangle_{\mathcal{H}}
\]

Let us define a Hilbert-Schmidt operator $\Psi_{t,u}(\omega) : U_0 \rightarrow L_0^1([0,T]; E) \hookrightarrow \mathcal{H} \otimes E$ as follows
\[ \Psi_{t,\omega}(\sqrt{\lambda_i e_i}) := \sqrt{\lambda_i}\Psi_{t,\omega}; \omega \in \Omega. \]

By (2.6), there exists a constant \( C \) such that
\[
\sum_{i=1}^{\infty} \|\Psi_{t,\omega}(\sqrt{\lambda_i e_i})\|^2_{\mathcal{H}\otimes E} \leq C \sum_{i=1}^{\infty} \lambda_i \|\Psi_{t,\omega}\|_0^2 \leq C \sup_{i \geq 1} \|\Psi_{t,\omega}\|_{0,\kappa,\text{trace}} Q < \infty \ a.s.
\]

We claim that \( X_t \in D^{1,2}_{loc}(E) \) and
\[
\mathbf{D} X_t = \Psi_{t,\omega} \ a.s.
\]

Indeed, we observe \( \Psi_t \) satisfies
\[
\Psi_{t,s} = S(t-s)G(X_s) + \int_s^t S(t-r)\nabla F(X_r)\Psi_{r,s} dr + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_s^t S(t-r)\nabla G_i(X_r)\Psi_{r,s} d\beta_r \ a.s,
\]

where \( \Psi_{t,s} = 0 \) for \( t < s \). Moreover,
\[
\left\langle \mathbf{D}(X_t, g)_{E, v} \right\rangle_{\mathcal{L}_2(U_0; \mathcal{H})} = \sum_{i=1}^{\infty} \varphi(e_i) \lambda_i \left\langle (\Psi_{t,\omega})_{E, h} \right\rangle_{\mathcal{H}}
\]
\[
= \sum_{i=1}^{\infty} \left\langle \Psi_{t,\omega} \sqrt{\lambda_i e_i}, g \right\rangle_{E, h} \varphi(e_i) \sqrt{\lambda_i} h \right\rangle_{\mathcal{H}}
\]
\[
= \left\langle \mathbf{D} X_t g, v \right\rangle_{\mathcal{L}_2(U_0; \mathcal{H})} \ a.s.
\]

By applying Lemma 2.2 and Corollary 3.2 we conclude the proof. \( \square \)

4. THE RIGHT INVERSE OF THE SPDE JACOBIAN

In this section, we investigate the existence of the right inverse of the Jacobian SPDE operator under some algebraic constraints on the vector fields combined with the range of the semigroup. From now on, it will be useful to make clear the dependence on the initial conditions of (3.1). Let us write \( X_{y} \) as the solution of (3.1) for an initial condition \( y \in E_\kappa \). In previous section, we made use of the \( \hat{C}_{\kappa,\kappa} \)-topology to get differentiability of \( X_{x_0} \) (in Malliavin’s sense) for each initial condition at \( x_0 \in E_\kappa \).

Even though we are interested in establishing the existence of densities \( T(X_{x_0}^\omega) \) for initial conditions on \( \text{dom}(A^\infty) \subset E_\kappa \), it is important to work with the solution map \( E \to \hat{C}^{\alpha,0} \) given by
\[
y \mapsto X^y \in \hat{C}^{\alpha,0}_1,
\]
for some \( \alpha > 1 - H \). One drawback to keep the flow from \( E_\kappa \) to \( \hat{C}^{\alpha,\kappa}_1 \) is that \( X^{x_0} \) does not belong to \( C^{\alpha,\kappa}_1 \) and the best we can get is \( X^{x_0} \in \hat{C}^{\alpha,0}_1 \) a.s. For this purpose, we need to impose further regularity assumptions as described in Th 3.2 in [24], which we list here for the sake of preciseness:

**Assumption A3:** There exists \( \gamma_1, \gamma_2 \in (0, (2H - 1) \wedge \frac{3}{4}) \) and \( c_1 \) such that
\[
\|S(r)G(x)\| \leq \frac{c_1}{r^{\gamma_1}}(1 + \|x\|_E),
\]
for every \( x, y \in E \). Furthermore, for \( \alpha > 1 - H, \alpha < \frac{1}{2}(1 - \gamma_1) \wedge \frac{1}{4}(1 - \gamma_2) \), assume there exist constants \( c_2 > 0, \eta \in [0, 1 - \alpha) \) and \( \tilde{\beta} \in (\alpha, \frac{1}{2}) \) such that

\[
\| \nabla_x S(r)(G(x) - G(y)) \| \leq \frac{c_1}{r^{\gamma_2}} \| x - y \|_E,
\]

for every \( x, y \in E \). The proof of this fact is quite standard and the main arguments do not differ too much from the classical Brownian motion driving case (see e.g. Th. 3.9 in [16]), so we left the details to the reader. Moreover, for a given \( t \in \mathbb{R} \)

\[
0 \leq \sup_{0 \leq s \leq t} \frac{\| f(t) - f(s) \|_E}{|t - s|^{1 + \alpha}} < \infty.
\]

Therefore, under Assumptions H1 and A3, the uniqueness of the flow described in Th. 3.2 in [24] and Assumption A3, we shall take \( \kappa \in \left( \frac{1}{2}, \frac{1}{2} \right) \) with \( \kappa = \alpha + \epsilon \) and \( 0 < \epsilon < \alpha \) such that

\[
C^{\alpha, \epsilon}_{1-} \subset W^{\alpha, \infty}(0, T; E),
\]

where \( W^{\alpha, \infty}(0, T; E) \) is the space of all measurable functions \( f : [0, T] \rightarrow E \) such that

\[
\| f \|_{\alpha, \infty} := \left( \| f \|_{0, 0} + \sup_{0 \leq s \leq t} \int_0^t \frac{\| f(t) - f(s) \|_E}{|t - s|^{1 + \alpha}} \right) < \infty.
\]

Remark 4.1. Recall that infinitesimal generators of analytic semigroups are sectorial. Then, it is known (see e.g Corollary 2.1.7 in [22]) that \( S(t) \) is one-to-one for every \( t \geq 0 \). We also observe the left-inverse linear operator \( S(-t) \) of \( S(t) \) defined on the subspace \( S(t)E \) is, in general, unbounded.

Example 4.1. Let \( E = L^2(0, 1) \) with Dirichlet boundary conditions. Take the orthonormal basis

\[
e_n(x) = \sqrt{2}\sin(\pi nx); 0 < x < 1,
\]

with eigenvalues \( \lambda_n = \pi^2n \). Then, the heat semigroup generated by the Laplacian \( A = \Delta \) is given by

\[
\| \nabla_x S(r)(G(x) - G(y)) \| \leq \frac{c_2}{r^{\gamma_2}} \| x - y \|_E,
\]

for every \( r \in (0, T], 0 < s < r, x, y \in E \) and \( i \geq 1 \).
\[ S(t)f = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle f, e_n \rangle E e_n, \]
for \( f \in E \). This is an analytic semigroup whose left-inverse is equal to
\[ S(-t)g = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle g, e_n \rangle E e_n, \]
for \( g \in S(t)E \).

In order to obtain a right-inverse operator-valued process for the Jacobian, we need to assume the following regularity conditions. In the sequel, we denote \( S^-(t) := S(t); t \geq 0 \) where \( S(t) \) stands the left-inverse linear operator on \( S(t)E \).

**Assumption B1**: Let \( \alpha > 1 - H \) be a constant as defined in Assumption A3. For each path \( f \in C_{1,0}^{\alpha,0} \),
\[ \sup_{i \geq 1} \left\{ \| S^{-} \nabla G_{i}(f) S \|_{0,0}^{\alpha,0} + \| \delta S^{-} \nabla G_{i}(f) S \|_{0,0}^{\mu,0} \right\} < \infty, \]
for \( \mu + \tilde{\gamma} > 1 \), where \( \frac{1}{2} < \tilde{\gamma} < H \) satisfies (3.12).

**Assumption B2**: For each path \( f \in C_{1,0}^{\alpha,0} \), \( \| S^{-} \nabla F(f) S \|_{0,0}^{\alpha,0} < \infty \).

In Assumptions B1-B2, we assume
\[ (4.3) \quad \nabla F(w)z \in S(T)E \]
for every \( w, z \in E \) and \( i \geq 1 \).

**Remark 4.2.** Since \( S(T)z = S(t)S(T-t)z \) for every \( 0 \leq t \leq T \) and \( z \in E \), then \( S(T)E \subset S(t)E_{\beta} \) for every \( 0 \leq t \leq T \) and \( \beta \geq 0 \).

**Remark 4.3.** We stress we implicitly assume in Assumptions B1-B2 that \( \nabla F(f_i)S(t)x \in S(t)E \) and \( \nabla G_{i}(f_{i})S(t)x \in S(t)E \) for every \( t \geq 0 \), \( x \in E \) and \( i \geq 1 \). This property holds true under \( \mu, \tilde{\gamma} \) due to Remark 1.3. In this case, taking into account that \( S \) is a differentiable semigroup, then (see e.g Prop 3.12 in [21]) we have \( \nabla F(w)z \in \cap_{n=1}^{\infty} \text{dom}(A^{\alpha}) \) and \( \nabla G_{i}(w)z \in \cap_{n=1}^{\infty} \text{dom}(A^{\alpha}) \) for every \( w, z \in E \) and \( i \geq 1 \).

In the sequel, we freeze an initial condition \( y \in E_{\alpha} \). Let us now investigate the existence of an operator-valued process \( J_{0 \rightarrow t}^{+}(y) \) such that
\[ J_{0 \rightarrow t}(y)J_{0 \rightarrow t}^{+}(y) = \text{Id a.s.} \quad 0 \leq t \leq T, \]
where \( \text{Id} \) is the identity operator on \( S(t)E \). We start the analysis with the following equation
\[ (4.4) \quad U_{t}(y) = - \int_{0}^{t} \left[ \text{Id} + U_{r}(y) \right] S(-r) \nabla F(X_{r}^{\beta}) S(r) dr \]
\[- \sum_{i=1}^{\infty} \sqrt{\lambda_{i}} \int_{0}^{t} \left[ \text{Id} + U_{r}(y) \right] S(-r) \nabla G_{i}(X_{r}^{\beta}) S(r) d\beta_{i}^{\beta} \].

Let \( C_{1}^{\mu,0} \) be the linear space of \( L(E; E) \)-valued functions \( r \mapsto f_{r} \) such that
\[ \| f \|_{C_{1}^{\mu,0}} := \| f \|_{0,0} + \| \delta f \|_{\mu,0} < \infty. \]
Lemma 4.1. Under Assumptions B1-B2, there exists a unique adapted solution $U(y)$ of (4.4) such that $U(y) \in C^\mu_{1,0} \text{ a.s for } \mu + \gamma > 1$ and $0 < \mu < \gamma$.

Proof. For a given $g \in W^{1,\infty}_{\lambda,T}$ and $w \in C^\mu_{1,0}$, let us define $\Gamma : C^\mu_{1,0} \to C^\mu_{1,0}$ by

$$
\Gamma(U)_t := - \int_0^t [\text{Id} + U_r] S(-r) \nabla F(w_r) S(r) dr - \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t [\text{Id} + U_r] S(-r) \nabla G_i(w_r) S(r) dg^i_r.
$$

We claim that $\Gamma$ is a contraction map on a small interval $[0,T]$. Indeed, for $U,V \in C^\mu_{1,0}$, if $q_t = \Gamma(U)_t - \Gamma(V)_t$, then

$$
q_t = \int_0^t [V_r - U_r] S(-r) \nabla F(w_r) S(r) dr + \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t [V_r - U_r] S(-r) \nabla G_i(w_r) S(r) dg^i_r =: q^1_t + \sum_{i=1}^\infty q^{2,i}_t.
$$

Assumption B2 implies the existence of a constant $C_F$ such that

$$
\|q^1_t\|_{0,0} \leq \sup_{0 \leq s \leq T} \|q^1_t\| \leq \int_0^T \left\| [V_r - U_r] S(-r) \nabla F(w_r) S(r) \right\|_{0,0} dr 
\leq C_F T \|U - V\|_{0,0}
$$

(4.5)

$$
\frac{\|q^1_t - q^1_s\|}{|t-s|^{\mu}} \leq C_F \|U - V\|_{0,0} |t-s|^{1-\mu} \leq C_F T^{1-\mu} \|U - V\|_{0,0}.
$$

(4.6)

Then,

$$
\|\delta q^1\|_{\mu,0} \leq C_F T^{1-\mu} \|U - V\|_{0,0}
$$

(4.7)

Young-Loeve’s inequality yields

$$
\left\| \sum_{i=1}^\infty (q^{2,i}_t - q^{2,i}_s) \right\| \leq \frac{1}{2^\mu+\gamma} \sum_{i=1}^\infty \left\| [\delta V - U] S(-s) \nabla G_i(w_s) S(\cdot) \right\|_{\mu,0} \|g^i\|_{W^{\mu,s}_\infty} |t-s|^{\mu+\gamma} \sqrt{\lambda_i} + \sum_{i=1}^\infty \left\| [V_s - U_s] S(-s) \nabla G_i(w_s) S(\cdot) \right\|_{\mu,0} \|g^i\|_{W^{\mu,s}_\infty} |t-s|^{\mu+\gamma} \sqrt{\lambda_i}
$$

(4.7)

where by linearity, we have
\[
\| \delta [V - U] S^{-1} \nabla G_t(w) S \|_{\mu, 0 \to 0} \leq \| S^{-1} \nabla G_t(w) S \|_{0, 0 \to 0} \| \delta (V - U) \|_{\mu, 0 \to 0} \\
+ \| V - U \|_{0, 0 \to 0} \| \delta S^{-1} \nabla G_t(w) S \|_{\mu, 0 \to 0} \\
(4.8)
\]
for a constant \( C_G \) coming from Assumption B1. Summing up (4.7) and (4.8), we have
\[
(4.9) \quad \left\| \sum_{i=1}^{\infty} \delta q^{2i} \right\|_{\mu, 0 \to 0} \leq C_G \| V - U \|_{C^{1,0} \to 0} T^\tilde{\gamma} \| g \|_{W^{1,\tilde{\gamma}}_{\tilde{\mu}, \tilde{T}}} + C_G \| V - U \|_{C^{1,0} \to 0} \| g \|_{W^{1,\tilde{\gamma}}_{\tilde{\mu}, \tilde{T}}} T^{\tilde{\gamma} - \mu},
\]
where we recall \( \tilde{\gamma} > \mu \). In addition, (4.7) yields
\[
(4.10) \quad \left\| \sum_{i=1}^{\infty} q^{2i} \right\|_{0, 0 \to 0} \leq C_G \| V - U \|_{C^{1,0} \to 0} T^{\mu + \tilde{\gamma}} \| g \|_{W^{1,\tilde{\gamma}}_{\tilde{\mu}, \tilde{T}}} + C_G \| V - U \|_{C^{1,0} \to 0} \| g \|_{W^{1,\tilde{\gamma}}_{\tilde{\mu}, \tilde{T}}} T^{\tilde{\gamma}}.
\]
Summing up (4.5), (4.6), (4.9) and (4.10), we conclude
\[
(4.11) \quad \| \Gamma(U) - \Gamma(V) \|_{C^{1,0} \to 0} \leq \left[ C_F (T^{1 - \mu} + T) + C_G \| g \|_{W^{1,\tilde{\gamma}}_{\tilde{\mu}, \tilde{T}}} (2T^{\tilde{\gamma} - \mu} + T^{\mu + \tilde{\gamma}}) \right] \| U - V \|_{C^{1,0} \to 0}.
\]
By making \( T \) small in (4.11), we conclude there exists a unique fixed point for \( \Gamma \) on small interval \([0, T]\) whose size does not depend on the initial condition. The construction of a global unique solution from the solution in \([0, T]\) is standard and it is left to the reader for sake of conciseness. This pathwise argument clearly provides a unique adapted process \( U \) realizing (4.4). \[\square\]

Now, we set \( R_t(y) = U_t(y) + \text{Id} \) and we observe that
\[
R_t(y) = \text{Id} - \int_0^t R_s(y) S(-s) \nabla F(X^y_s) S(s) ds \\
(4.12)
- \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t R_s(y) S(-s) \nabla G_i(X^y_s) S(s) d\beta^i_s; 0 \leq t \leq T.
\]
We arrive at the following result which will play a key role in representing the Malliavin matrix.

**Proposition 4.1.** If Assumptions H1-A1-A2-A3-B1-B2 hold, then for each initial condition \( y \in E_k \), the Jacobian \( J_{0 \to t}(y) \) admits a right-inverse adapted process \( J^+_{0 \to t}(y) \) which satisfies
\[
J^+_{0 \to t}(y) = S(-t) - \int_0^t J^+_{0 \to s}(y) \nabla F(X^y_s) S(s - t) ds \\
(4.13)
- \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t J^+_{0 \to s}(y) \nabla G_i(X^y_s) S(s - t) d\beta^i_s; 0 \leq t \leq T.
\]
**Proof.** The candidate is \( J^+_{0 \to t}(y) := R_t(y) S^{-1}(t) \) defined on \( S(t)E \). At first, we observe
\[
S(s)S(-t) = S(s - t) \text{ on } S(t)E \subset S(t - s)E,
\]
for every $s < t$. Then, (4.13) is well-defined in view of Assumptions B1-B2. Let us check it is the right-inverse. Let

$$V_t(y) = \int_0^t S(-s)\nabla F(X^y_s)S(s)[\text{Id} + V_s(y)]ds$$

(4.14)

$$+ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(-s)\nabla G_i(X^y_s)S(s)[\text{Id} + V_s(y)]d\beta^i_s; 0 \leq t \leq T.$$ 

By following a similar proof of Lemma 4.1, we can safely state there exists a unique adapted solution $\{P_s(y)\}$ such that

$$\int_0^t S(-s)\nabla F(X^y_s)S(s)P_s(y)ds = \text{Id} + \int_0^t S(-s)\nabla F(X^y_s)S(s)[\text{Id} + V_s(y)]d\beta^i_s,$$

(4.15)

and therefore $J_{0 \rightarrow t}(y) = S(t)P_t(y)$. Equations (4.12), (4.15) and integration by parts in Hilbert spaces yield

$$\langle P_t(y)R_t(y)w, w' \rangle_E = \langle R_t(y)w, P^*_t(y)w' \rangle_E = \langle w, w' \rangle_E + \int_0^t \langle dR_s(y)w, P^*_s(y)w' \rangle_E$$

$$+ \int_0^t \langle R_s(y)w, dP^*_s(y)w' \rangle_E,$$

for each $w, w' \in E$, where $P^*$ is the adjoint. To keep notation simple, we set $I_1 = \int_0^t \langle dR_s(y)w, P^*_s(y)w' \rangle_E$ and $I_2 = \int_0^t \langle R_s(y)w, dP^*_s(y)w' \rangle_E$. We observe

$$I_1 = -\int_0^t \langle P_s(y)R_s(y)S(-s)\nabla F(X^y_s)S(s)w, w' \rangle_E ds$$

$$- \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t \langle P_s(y)R_s(y)S(-s)\nabla G_i(X^y_s)S(s)w, w' \rangle_E d\beta^i_s.$$ 

In addition, Assumption B1 allows us to represent

$$I_2 = \int_0^t \langle S(-s)\nabla F(X^y_s)S(s)P_s(y)R_s(y)w, w' \rangle_E ds$$

$$+ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t \langle S(-s)\nabla F(X^y_s)S(s)P_s(y)R_s(y)w, w' \rangle_E d\beta^i_s.$$ 

This shows that

$$P_t(y)R_t(y) = \text{Id} + \int_0^t S(-s)\nabla F(X^y_s)S(s)(P_s(y)R_s(y)w)ds$$

$$+ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(-s)\nabla G_i(X^y_s)S(s)P_s(y)R_s(y)d\beta^i_s.$$ 

(4.16)
\[- \int_0^t P_s(y)R_s(y)S(-s)\nabla F(X^y_s)S(s)ds \]
\[- \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t P_s(y)R_s(y)S(-s)\nabla G_i(X^y_s)S(s)d\beta^y_s.\]

We now observe there exists a unique solution of (4.16). To see this, let \( Q_t(y) = P_t(y)R_t(y) - \text{Id} \) and from (4.16), we have

\[
Q_t(y) = \int_0^t S(-s)\nabla F(X^y_s)S(s)Q_s(y)ds + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(-s)\nabla G_i(X^y_s)S(s)Q_s(y)d\beta^y_s
\]
(4.17)
\[- \int_0^t Q_s(y)S(-s)\nabla F(X^y_s)S(s)ds - \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t Q_s(y)S(-s)\nabla G_i(X^y_s)S(s)d\beta^y_s.\]

The same argument of the proof of Lemma (4.1) yields the existence of a unique solution of equation (4.17). This obviously implies that (4.16) admits only one solution. Since \( \text{Id} \) solves (4.16), we do have \( P_t(y)R_t(y) = \text{Id} \) for every \( t \in [0,T] \) and we conclude \( J_{0 \to t}(y)J_{0 \to t}^+(y) = S(t)P_t(y)R_t(y)S^{-1}(t) = \text{Id} \ a.s. \)

5. Existence of densities under Hörmander’s bracket condition

In this section, we examine the existence of the densities for random variables of the form \( T(X^t) \) for a bounded linear operator \( T : E \to \mathbb{R}^d \) for a given \( t \in (0,T) \). Throughout this section, we fix a set of parameters \( \kappa, \kappa_0, \gamma, \delta, \lambda \) as described in (3.12). In order to state a Hörmander’s bracket condition, we need to work with smooth vector fields \( F, G_i; i \geq 1 \).

Let

\[
dom(A^n) := \{ h \in E; h \in \text{dom}(A^{n-1}) \text{ and } A^{n-1}h \in \text{dom}(A) \},
\]

\[
\|h\|_{\text{dom}(A^n)}^2 := \sum_{i=0}^{n} \|A^ih\|_E^2,
\]

\[
\text{dom}(A^\infty) := \cap_{n=1}^{\infty} \text{dom}(A^n).
\]

We observe \( \text{dom}(A^\infty) \) is a Fréchet space equipped with the family of seminorms \( \| \cdot \|_{\text{dom}(A^n)}; n \geq 0 \).

In the sequel, for each \( t \in [0,T] \), we equip \( S(t)E \) with the following inner product

\[
(S(t)x, S(t)y)_{S(t)E} := \langle x, y \rangle_E; \ x, y \in E.
\]

Notice that this is a well-defined inner product due to the injectivity of the semigroup. One can easily check \( S(t)E \) is a separable Hilbert space equipped with the norm associated with (5.1). Moreover, for each \( x_0 \in E_\alpha \) and \( t \in [0,T] \), \( J_{0 \to t}^+(x_0) : S(t)E \to E \) admits an adjoint as a bounded linear operator from \( E \) to \( S(t)E \). Indeed, let \( J_{0 \to t}^*(x_0) : E \to S(t)E \) be the linear operator defined by

\[
y \mapsto J_{0 \to t}^+(x_0)y := S(t)R_t^*(x_0)y.
\]

Then,

\[
\langle J_{0 \to t}^+(x_0)S(t)x, y \rangle_E = \langle R_t(x_0)S(-t)S(t)x, y \rangle_E
\]
\[
= \langle x, R^*_t(x_0)y \rangle_E + \langle S(t)x, J_{0 \to t}^{+, +}(x_0)y \rangle_{S(t)E},
\]

where \( \|J_{0 \to t}^{+, +}(x_0)y\|_{S(t)E} = \|R^*_t(x_0)y\|_E \leq \|R^*_t(x_0)\|_E \cdot \|y\|_E \). This proves our claim. We observe \( R_t^*(x_0) = \text{Id} + U_t^*(x_0) \), where
whenever $V$ for each $r$ (5.3) $V$: $V$ (5.2) so that $A$ vector field 

Definition 5.1. A vector field $V$ on an open subset $U \subset M$ of a Fréchet space $M$ is a smooth map $V: U \to M$.

Let us recall the concept of Lie brackets between two vector fields $V_1, V_2 : \text{dom}(A^\infty) \to \text{dom}(A^\infty)$

(5.3) $[V_1, V_2](r) := \nabla V_2(r)V_1(r) - \nabla V_1(r)V_2(r),$

for each $r \in \text{dom}(A^\infty)$. We observe $[V_1, V_2] : \text{dom}(A^\infty) \to \text{dom}(A^\infty)$ is a well-defined vector field whenever $V_1, V_2$ are vector fields on $\text{dom}(A^\infty)$. Moreover, $\frac{1}{1} < \kappa < 1$ implies $\text{dom}(A) \subset \text{dom}(-A^\kappa)$, so that $\text{dom}(A^\infty) \subset E_\kappa$.

Assumption C1: $G : E \to L^2(U_0; S(T)E)$ satisfies:

(i) $x \mapsto G_i(x)$ is an $S(T)\text{dom}(A)$-valued continuous mapping for each $i \geq 1$. Moreover,

(ii) $E \int_0^T \|G(X^x_\cdot)\|_{L^2(U_0; S(T)E)}^2dr < \infty.$

Assumption C2: $F_i : E \to \text{dom}(A^\infty)$ are smooth mappings with bounded derivatives for every $i \geq 1$ with the property that

$$\sup_{\ell \geq 1} \sup_{y \in E} \|\nabla^n G_\ell(y)\|_{\text{dom}(A^\infty)} < \infty,$$

for every $n, m \geq 1$. There exists a constant $C$ such that

$$\|G_\ell(y)\|_{\text{dom}(A)} \leq C(1 + \|y\|_{\text{dom}(A)}), \ y \in \text{dom}(A),$$

for every $\ell \geq 1$. Moreover, $F, G_i, i \geq 1 : \text{dom}(A^k) \to \text{dom}(A^k)$ are $C^\infty$-bounded for every $k, i \geq 1$. 

$$U_i^*(x_0) = -\int_0^t (S(-r)\nabla F(X^x_\cdot S(r))^* (\text{Id} + U_i^*(x_0)) dr$$

$$- \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t (S(-r)\nabla G_i(X^x_\cdot S(r))^* (\text{Id} + U_i^*(x_0)) dB_i^r,$n

so that

$$R_i^*(x_0) = \text{Id} - \int_0^t (S(-r)\nabla F(X^x_\cdot S(r))^* R_i^*(x_0) dr$$

(5.2)

$$- \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t (S(-r)\nabla G_i(X^x_\cdot S(r))^* R_i^*(x_0) dB_i^r.$$

In other words,

$$J_i^a(x_0) = S(t) - \int_0^t S(t)(S(-r)\nabla F(X^x_\cdot S(r))^* R_i^*(x_0) dr$$

$$- \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t S(t)(S(-r)\nabla G_i(X^x_\cdot S(r))^* R_i^*(x_0) dB_i^r.$$

$U_i^*(x_0) = -\int_0^t (S(-r)\nabla F(X^x_\cdot S(r))^* (\text{Id} + U_i^*(x_0)) dr$

$$- \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t (S(-r)\nabla G_i(X^x_\cdot S(r))^* (\text{Id} + U_i^*(x_0)) d\beta_i^r,$$

so that

$$R_i^*(x_0) = \text{Id} - \int_0^t (S(-r)\nabla F(X^x_\cdot S(r))^* R_i^*(x_0) dr$$

(5.2)

$$- \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t (S(-r)\nabla G_i(X^x_\cdot S(r))^* R_i^*(x_0) dB_i^r.$$

In other words,

$$J_i^a(x_0) = S(t) - \int_0^t S(t)(S(-r)\nabla F(X^x_\cdot S(r))^* R_i^*(x_0) dr$$

$$- \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^t S(t)(S(-r)\nabla G_i(X^x_\cdot S(r))^* R_i^*(x_0) dB_i^r.$$
Assumption C3: For every \( n, p \geq 1 \), \( \nabla^n G_p(x)v \in S(T)\text{dom}(A) \) and \( \nabla^n F(x)v \in S(T)\text{dom}(A) \) for every \( x \in \text{dom}(A) \) and \( v \in \text{dom}^n(A) \).

Under Assumption C2, if we assume that \( x_0 \in \text{dom}(A^\infty) \), then we can construct a solution process with \( \alpha \)-Hölder continuous trajectories in \( \text{dom}(A^\infty) \). This is true because the Picard approximation procedure converges in every Hilbert space \( \text{dom}(A^m) \), and the topology of \( \text{dom}(A^\infty) \) is the projective limit of the ones on \( \text{dom}(A^m) \). We summarize this fact into the following remark.

**Remark 5.1.** Under Assumption C2, for each initial condition \( x_0 \in \text{dom}(A) \), \( [G_0, V](X_t^{x_0}) \in S(t)E \) has a unique strong solution. If \( x_0 \in \text{dom}(A^\infty) \), then we can construct a solution of \( (5.7) \) taking values on \( \text{dom}(A^\infty) \) and such that

\[
\|\delta X^{x_0}\|_{\alpha, \text{dom}(A^m)} < \infty,
\]

for every \( m \geq 1 \).

**Remark 5.2.** Assumption C3 plays a role in constructing the argument towards the existence of densities which requires

\[
[G_0, V](X_t^{x_0}) \in S(t)E
\]

in order to belong to the domain of \( \mathbf{J}^+_0(x_0) \) for every \( V \in \mathbb{V}_m; m \geq 0 \) (see \( [5.10] \)), where \( G_0 \) is the vector field given by \( (5.5) \).

The following elementary remark is useful.

**Lemma 5.1.** If \( V : E \to \text{dom}(A^\infty) \) is a smooth mapping with bounded derivatives, then

\[
\sup_{y \in E} \|\nabla^n V(y)\|_{(n), 0 \to 0} < \infty.
\]

**Proof.** The \( n \)-th Fréchet derivative of \( V \) viewed as a map from \( E \) to \( \text{dom}(A) \) is given by \( \nabla^n V : E \to \mathcal{L}_n(E^n; \text{dom}(A)) \), where

\[
\|\nabla^n V(x)(h_1, \ldots, h_n)\|_{\text{dom}(A)} \leq \|\nabla^n V(x)\|_{(n), E \to \text{dom}(A)} \|h_1\|_E \times \ldots \times \|h_n\|_E.
\]

Then,

\[
\|\nabla^n V(x)(h_1, \ldots, h_n)\|_E \leq \|\nabla^n V(x)(h_1, \ldots, h_n)\|_{\text{dom}(A)} \leq \|\nabla^n V(x)\|_{(n), E \to \text{dom}(A)} \|h_1\|_E \times \ldots \times \|h_n\|_E
\]

\[
\leq \sup_{y \in E} \|\nabla^n V(y)\|_{(n), E \to \text{dom}(A)} \|h_1\|_E \times \ldots \times \|h_n\|_E,
\]

and hence \( \|\nabla^n V(x)\|_{(n), 0 \to 0} \leq \sup_{y \in E} \|\nabla^n V(y)\|_{(n), E \to \text{dom}(A)} < \infty \) for every \( x \in E \). \qed

Let us now investigate the existence of densities for the SPDE \( (5.1) \). We start with some preliminary results.

**Lemma 5.2.** Under Assumptions H1-A1-A2-A3-B1-B2-C1-C2, for each \( x_0 \in \text{dom}(A) \), we have

\[
(5.4) \quad \mathbf{D}_r X_t^{x_0} = J_{0 \to t}(x_0)J_{0 \to r}^+(x_0)G(X_r^{x_0}) \text{ a.s.,}
\]

for every \( r < t \). Therefore,

\[
(5.5) \quad \mathbf{D}_r \mathcal{T}(X_t^{x_0}) = \mathcal{T}(J_{0 \to t}(x_0)J_{0 \to r}^+(x_0)G(X_r^{x_0})) \text{ a.s.,}
\]

for every \( r < t \).
Proof. On one hand, Remark 5.1 and (5.48) yields

\[(5.6) \quad D_rX^x_t = G(X^x_t) + \int_r^t \nabla F(X^x_t)D_rX^x_t \, dl + \sum_{i=1}^\infty \int_r^t \nabla G_i(X^x_t)D_rX^x_t \, dl_t,\]

for \(0 \leq r < t\). On the other hand, Assumption C2 implies that (4.2) has a strong solution for \(y = x_0 \in \text{dom}(A)\) and for each \(v = G_j(X^x_0)\). Having said that, let us fix \(0 \leq r < t\) and a positive integer \(j \geq 1\). The fact that \(G_j(E) \subseteq S(T)E\) and Remark 5.2 yield

\[
G_j(X^x_0) + \int_r^t \nabla F(X^x_t)J_{0 \rightarrow t}(x_0)J_{0 \rightarrow t}(x_0)G_j(X^x_t) \, dl + \sum_{i=1}^\infty \int_r^t \nabla G_i(X^x_t)J_{0 \rightarrow t}(x_0)J_{0 \rightarrow t}(x_0)G_j(X^x_t) \, dl_t
\]

\[
= G_j(X^x_0) + \left( \int_r^t \nabla F(X^x_t)J_{0 \rightarrow t}(x_0) \, dl + \sum_{i=1}^\infty \int_r^t \nabla G_i(X^x_t)J_{0 \rightarrow t}(x_0) \, dl_t \right) J_{0 \rightarrow t}(x_0)G_j(X^x_t)
\]

By invoking (3.49), (4.44), Lemma 3.6, (5.6) and Assumption C1(i), we know that both \((r, t) \mapsto D_rX^x_t\) and \((r, t) \mapsto J_{0 \rightarrow t}(x_0)G_j(X^x_t)\) are jointly continuous a.s on the simplex \(\{(r, t); 0 \leq r < t \leq T\}\). This fact combined with the uniqueness of the SPDE solution of (5.6) (for each fixed \(r\)) implies that they are indistinguishable

\[
(D_rX^x_t)(\sqrt{x_0J_{0 \rightarrow t}}) = J_{0 \rightarrow t}(x_0)J_{0 \rightarrow t}(x_0)G_j(X^x_t) \quad \text{a.s.,}
\]

for each \(j \geq 1\). Assumption C1 (ii) implies

\[
r \mapsto J_{0 \rightarrow t}(x_0)J_{0 \rightarrow t}(x_0)G(X^x_t) \in \mathcal{L}_2(U_0; \mathcal{L}(\mathcal{H} \otimes \mathcal{H})) \quad \text{a.s.,}
\]

for every \(t \in [0, T]\). Summing up the above arguments, we shall conclude (5.47) holds true. The chain rule yields representation (5.5).

In what follows, let us denote

\[
(5.7) \quad \gamma_t := \left( \langle DT_i(X^x_t), DT_j(X^x_t) \rangle_{\mathcal{L}_2(U_0; \mathcal{H})} \right)_{1 \leq i, j \leq d},
\]

where \(\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_d) : E \to \mathbb{R}^d\). In order to investigate non-degeneracy of the Malliavin derivative, it is convenient to work with a reduced Malliavin operator. Let us define the self-adjoint linear operator \(C_t : E \to E\) by the following quadratic form

\[
\langle C_t y, y \rangle_E := \alpha \beta \int_0^t \int_0^t \langle J^+_0(x_0)G_r(X^x_t), y \rangle_E \langle J^+_0(x_0)G_r(X^x_t), y \rangle_E |u - v|^{2H-2} \, du \, dv
\]

\[
(5.8)
\]

\[
\leq \sum_{i=1}^\infty \left\| \langle J^+_0(x_0)G_r(X^x_t), y \rangle_E \right\|_H^2 = \sum_{i=1}^\infty \left\| \langle G_r(X^x_t), J^+_0(x_0) \rangle_{S^+(\mathcal{H})} \right\|_H^2
\]

for \(y \in E\) and \(0 < t \leq T\). In (5.8), the norm in \(\mathcal{H}\) is computed over \([0, t]\). We observe \(C_t\) is a well-defined bounded linear operator due to Assumption C1 (ii) and \(\frac{\alpha}{\beta} < 2\).

By applying Lemma 5.2 and (5.4), we arrive at the following representation.

Lemma 5.3. Under Assumptions H1-A2-A2-A3-B1-B2-C1-C2, for each \(x_0 \in \text{dom}(A)\), we have

\[
\gamma_t = (\mathcal{T} \circ J_{0 \rightarrow t}(x_0))^* C_t (\mathcal{T} \circ J_{0 \rightarrow t}(x_0))^*.
\]
Let us define

\begin{equation}
G_0(x) := A(x) + F(x); x \in \text{dom}(A^\infty).
\end{equation}

Given the SPDE (5.1), let \( \mathcal{V}_k \) be a collection of vector fields given by

\begin{equation}
\mathcal{V}_0 = \{ G_i; i \geq 1 \}, \quad \mathcal{V}_{k+1} := \mathcal{V}_k \cup \{ [G_j, U]; U \in \mathcal{V}_k \text{ and } j \geq 0 \}.
\end{equation}

We also define the vector spaces \( \mathcal{V}_k(x_0) := \text{span}\{ V(x_0); V \in \mathcal{V}_k \} \) and we set

\[ \mathcal{D}(x_0) := \cup_{k \geq 1} \mathcal{V}_k(x_0), \]

for each \( x_0 \in \text{dom}(A^\infty) \).

Note that under Assumption C2, all the Lie brackets in (5.10) are well-defined as vector fields \( \text{dom}(A^\infty) \to \text{dom}(A^\infty) \).

**Proposition 5.1.** If Assumptions H1-A1-A2-A3-B1-B2-C1-C2-C3 hold true, then for each \( x_0 \in \text{dom}(A^\infty) \), we have

\begin{equation}
J_{0 \to t}^+(x_0)V(X_t^{x_0}) = V(x_0) + \int_0^t J_{0 \to s}^+(x_0)[G_0, V](X_s^{x_0})ds
+ \sum_{\ell=1}^\infty \sqrt{\lambda_\ell} \int_0^t J_{0 \to s}^+(x_0)[G_\ell, V](X_s^{x_0})d\beta_\ell^s; 0 \leq t \leq T,
\end{equation}

where \( V \in \mathcal{V}_n \) for \( n = 0, 1, 2, \ldots \).

**Proof.** At first, we take \( V \in \mathcal{V}_0 \). Assumptions C2-C3 yield \( V(X_s^{x_0}) \in S(T)E_a, [G_0, V](X_s^{x_0}) \in S(T)E \), and \([G_\ell, V](X_s^{x_0}) \in S(T)E \) a.s. Moreover, change of variables for Young integrals yields

\begin{equation}
V(X_t^{x_0}) = V(x_0) + \int_0^t \nabla V(X_s^{x_0})G_0(X_s^{x_0})ds + \sum_{\ell=1}^\infty \sqrt{\lambda_\ell} \int_0^t \nabla V(X_s^{x_0})G_\ell(X_s^{x_0})d\beta_\ell^s,
\end{equation}

where \( G_0(X_s^{x_0}) = A(X_s^{x_0}) + F(X_s^{x_0}); 0 \leq s \leq T \). We observe Young-Loeve’s inequality and A1-A2-A3 allow us to state the Young integral in (5.12) is well-defined. Recall the Lie bracket \([G_0, V](X_s^{x_0}) = \nabla V(X_s^{x_0})G_0(X_s^{x_0}) - \nabla G_0(X_s^{x_0})V(X_s^{x_0}) \), so that we can actually rewrite

\[ V(X_t^{x_0}) = V(x_0) + \int_0^t \left( \nabla G_0(X_s^{x_0})V(X_s^{x_0}) + [G_0, V](X_s^{x_0}) \right)ds + \sum_{\ell=1}^\infty \sqrt{\lambda_\ell} \int_0^t \nabla V(X_s^{x_0})G_\ell(X_s^{x_0})d\beta_\ell^s, \]

where \( \nabla G_0(X_s^{x_0})V(X_s^{x_0}) = A(V(X_s^{x_0})) + \nabla F(X_s^{x_0})V(X_s^{x_0}); 0 \leq s \leq T \). This implies that \( V(X_s^{x_0}) \) can be written as the mild solution of

\begin{align*}
V(X_t^{x_0}) &= S(t)V(x_0) + \int_0^t S(t-s)(\nabla F(X_s^{x_0})V(X_s^{x_0}) + [G_0, V](X_s^{x_0}))ds \\
&\quad + \sum_{\ell=1}^\infty \sqrt{\lambda_\ell} \int_0^t S(t-s)\nabla V(X_s^{x_0})G_\ell(X_s^{x_0})d\beta_\ell^s,
\end{align*}

so that

\begin{align*}
S(-t)V(X_t^{x_0}) &= V(x_0) + \int_0^t S(-s)(\nabla F(X_s^{x_0})V(X_s^{x_0}) + [G_0, V](X_s^{x_0}))ds
\end{align*}
(5.13) 
\[ + \sum_{t=1}^{\infty} \sqrt{\lambda_t} \int_0^t S(-s)\nabla V(X_s^{x_0})G_t(X_s^{x_0})d\beta_s, 0 \leq t \leq T. \]

The adjoint operator \( J_{0\to t}^\ast(x_0) \) yields

\[ \langle J_{0\to t}^\ast(x_0)V(X_t^{x_0}), y \rangle_E = \langle V(X_t^{x_0}), J_{0\to t}^\ast(x_0)y \rangle_{S(t)E} = \langle S(-t)V(X_t^{x_0}), R_t^\ast(x_0)y \rangle_E \]

for a given \( y \in E \). Hence, integration by parts yields

\[ \langle J_{0\to t}^\ast(x_0)V(X_t^{x_0}), y \rangle_E = \langle V(x_0), y \rangle_E + \int_0^t \langle dS(-s)V(X_s^{x_0}), R_s^\ast(x_0)y \rangle_E \]
\[ + \int_0^t \langle S(-s)V(X_s^{x_0}), dR_s^\ast(x_0)y \rangle_E; 0 \leq t \leq T. \]

By combining (5.13) and (5.2), we conclude that (5.11) holds true for

(5.16) \[ G \subseteq G \]

(5.15) \[ G \]

(5.14)

\[ \sup_{t \geq 1} \| \delta \nabla V(X_t^{x_0})G_t(X_t^{x_0}) \|_{a,0} < \infty \ a.s. \]

At first, we observe if \( W : \text{dom}(A^\infty) \to \text{dom}(A^\infty) \) is smooth, then

\[ \nabla[G_0, W](x)(h) = \nabla^2 W(x)(h, Ax) + \nabla W(x)A(h) + \nabla^2 W(x)(h, F(x)) + \nabla W(x)\nabla F(x)h \]

\[ - A\nabla W(x)h - \nabla^2 F(x)(h, W(x)) - \nabla F(x)\nabla W(x)h; h \in \text{dom}(A^\infty), \]

(5.15)

\[ \nabla[G_p, W](x)(h) = \nabla^2 W(x)(h, G_p(x)) + \nabla W(x)\nabla G_p(x)(h) - \nabla^2 G_p(x)(h, W(x)) - \nabla G_p(x)\nabla W(x)(h), \]

for \( h \in \text{dom}(A^\infty) \) and \( p \geq 1 \). If \( V = [G_0, G_p] \), we observe

\[ \nabla V(X_t^{x_0})G_t(X_t^{x_0}) = -A\nabla G_p(X_t^{x_0})G_t(X_t^{x_0}) - \nabla^2 F(X_t^{x_0})(G_p(X_t^{x_0}), G_t(X_t^{x_0})) \]
\[ - \nabla F(X_t^{x_0})\nabla G_p(X_t^{x_0})G_t(X_t^{x_0}) + \nabla^2 G_p(X_t^{x_0})(AX_t^{x_0}, G_t(X_t^{x_0})) + \nabla G_p(X_t^{x_0})AG_t(X_t^{x_0}) \]
\[ + \nabla^2 G_p(X_t^{x_0})(F(X_t^{x_0}), G_t(X_t^{x_0})) + \nabla G_p(X_t^{x_0})\nabla F(X_t^{x_0})G_t(X_t^{x_0}) \]
\[ = \sum_{i=1}^{7} I_{1,p_\ell}(t). \]

Since \( F, G_t : E \to \text{dom}(A^\infty) \) has bounded derivatives of all orders (by Assumption C2), we shall use Lemma 5.7 to get

\[ \| I_{1,p_\ell}(t) - I_{1,p_\ell}(s) \|_E \leq \| \nabla G_p(X_t^{x_0})G_t(X_t^{x_0}) - \nabla G_p(X_s^{x_0})G_t(X_s^{x_0}) \|_{\text{dom}(A)} \]
\[ + \| A\nabla G_p(X_t^{x_0})G_t(X_t^{x_0}) - A\nabla G_p(X_s^{x_0})G_t(X_s^{x_0}) \|_E \]
\[ \leq \sup_{y \in E} \| \nabla G_p(y) \|_{E \to \text{dom}(A)} \| G_t(X_t^{x_0}) - G_t(X_s^{x_0}) \|_E \]
\[
\|I_{2,p,t}(t) - I_{2,p,t}(s)\|_E \leq \|\nabla^2 F(X_{t}^{x_0})(G_p(X_{t}^{x_0}), G_t(X_{t}^{x_0})) - \nabla^2 F(X_{t}^{x_0})(G_p(X_{t}^{x_0}), G_t(X_{t}^{x_0}))\|_E \\
+ \|\nabla^2 F(X_{t}^{x_0})(G_p(X_{t}^{x_0}), G_t(X_{t}^{x_0})) - \nabla^2 F(X_{t}^{x_0})(G_p(X_{t}^{x_0}), G_t(X_{t}^{x_0}))\|_E \\
+ \|\nabla^2 F(X_{t}^{x_0})(G_p(X_{t}^{x_0}), G_t(X_{t}^{x_0})) - \nabla^2 F(X_{t}^{x_0})(G_p(X_{t}^{x_0}), G_t(X_{t}^{x_0}))\|_E \\
\leq \|\nabla^2 F(x_0) - \nabla^2 F(x_0)\|_{(2),0\rightarrow0} \|G_p(x_0)\|_E \|G_t(x_0)\|_E \\
+ \|\nabla^2 F(x_0)\|_{(2),0\rightarrow0} \|G_p(x_0)\|_E \|G_t(x_0)\|_E \\
+ \|\nabla^2 F(x_0)\|_{(2),0\rightarrow0} \|G_t(x_0) - G_t(x_0)\|_E \|G_p(x_0)\|_E \\
\leq C \delta X_{t}^{x_0} \|E(1 + \|X^{x_0}\|_{0,0})^2 \\
+ 2C \sup_{y \in E} \|\nabla^2 F(y)\|_{(2),0\rightarrow0} \|G_p(y)\|_E \|G_t(y)\|_E \|\delta X_{t}^{x_0}\|_E,
\]

\[
\|I_{3,p,t}(t) - I_{3,p,t}(s)\|_E \leq \|\nabla^2 F(X_{t}^{x_0}) G_p(X_{t}^{x_0}) G_t(X_{t}^{x_0}) - \nabla^2 F(X_{t}^{x_0}) G_p(X_{t}^{x_0}) G_t(X_{t}^{x_0})\|_E \\
+ \|\nabla^2 F(X_{t}^{x_0}) G_p(X_{t}^{x_0}) G_t(X_{t}^{x_0}) - \nabla^2 F(X_{t}^{x_0}) G_p(X_{t}^{x_0}) G_t(X_{t}^{x_0})\|_E \\
+ \|\nabla^2 F(X_{t}^{x_0}) G_p(X_{t}^{x_0}) G_t(X_{t}^{x_0}) - \nabla^2 F(X_{t}^{x_0}) G_p(X_{t}^{x_0}) G_t(X_{t}^{x_0})\|_E \\
\leq C \sup_{y \in E} \|\nabla^2 F(y)\|_{(2),0\rightarrow0} \|G_p(y)\|_E \|\delta X_{t}^{x_0}\|_E(1 + \|X^{x_0}\|_{0,0}),
\]

\[
\|I_{4,p,t}(t) - I_{4,p,t}(s)\|_E \leq \|\nabla^2 G_p(X_{t}^{x_0}) (AX_{t}^{x_0}, G_t(X_{t}^{x_0})) - \nabla^2 G_p(X_{t}^{x_0}) (AX_{t}^{x_0}, G_t(X_{t}^{x_0}))\|_E \\
+ \|\nabla^2 G_p(X_{t}^{x_0}) (AX_{t}^{x_0}, G_t(X_{t}^{x_0})) - \nabla^2 G_p(X_{t}^{x_0}) (AX_{t}^{x_0}, G_t(X_{t}^{x_0}))\|_E \\
+ \|\nabla^2 G_p(X_{t}^{x_0}) (AX_{t}^{x_0}, G_t(X_{t}^{x_0})) - \nabla^2 G_p(X_{t}^{x_0}) (AX_{t}^{x_0}, G_t(X_{t}^{x_0}))\|_E \\
\leq \|\nabla^2 G_p(y)\|_{(3),0\rightarrow0} \|\delta X_{t}^{x_0}\|_E \|X_{t}^{x_0}\|_{0,\text{dom}(A)}(1 + \|X^{x_0}\|_{0,0}) \\
+ \sup_{y \in E} \|\nabla^2 G_p(y)\|_{(2),0\rightarrow0} \|\delta X_{t}^{x_0}\|_{\text{dom}(A)}(1 + \|X^{x_0}\|_{0,0}) \\
+ \sup_{y \in E} \|\nabla^2 G_p(y)\|_{(2),0\rightarrow0} \|\delta X_{t}^{x_0}\|_E(1 + \|X^{x_0}\|_{0,\text{dom}(A)}),
\]

\[
\|I_{5,p,t}(t) - I_{5,p,t}(s)\|_E \leq \|\nabla G_p(X_{t}^{x_0}) A G_t(X_{t}^{x_0}) - \nabla G_p(X_{t}^{x_0}) A G_t(X_{t}^{x_0})\|_E \\
+ \|\nabla G_p(X_{t}^{x_0}) A G_t(X_{t}^{x_0}) - \nabla G_p(X_{t}^{x_0}) A G_t(X_{t}^{x_0})\|_E \\
\leq C \sup_{y \in E} \|\nabla^2 G_p(y)\|_{(2),0\rightarrow0} \|\delta X_{t}^{x_0}\|_E \|G_t(x_0)\|_{\text{dom}(A)} \\
+ C \sup_{y \in E} \|\nabla G_p(y)\|_E \|\nabla G_t(y)\|_{\text{dom}(A)} \|\delta X_{t}^{x_0}\|_E, \\
\leq C \sup_{y \in E} \|\nabla^2 G_p(y)\|_{(2),0\rightarrow0} \|\delta X_{t}^{x_0}\|_E(1 + \|X^{x_0}\|_{0,\text{dom}(A)}).
\]
and adapted to our infinite-dimensional setting. For a given rough path theory. For sake of completeness, we recall the following concepts borrowed from [14]

\[ \| \phi \|_{E \rightarrow \text{dom}(A)} = \| \phi \|_E. \]

\[ \| I_{t,p} \|_{\delta Y} \leq C \sup_{y \in E} \| \nabla G_p(y) \|_{E \rightarrow \text{dom}(A)} \| \delta X_{t} \|_E. \]

\[ \| I_{t,p} \|_{\delta Y} \leq C \sup_{y \in E} \| \nabla G_p(y) \|_{E \rightarrow \text{dom}(A)} \| \delta X_{t} \|_E. \]

\[ \| I_{t,p} \|_{\delta Y} \leq C \sup_{y \in E} \| \nabla G_p(y) \|_{E \rightarrow \text{dom}(A)} \| \delta X_{t} \|_E. \]

5.1. Doob-Meyer-type decomposition. Let us now turn our attention to a Doob-Meyer decomposition in the framework of integral equations involving a trace-class FBM. This will play a key step in the proof of the existence of density of Theorem 1.1. We recall the parameters\( \tilde{\gamma} \) and \( \tilde{\delta} \).

This shows that \( (5.14) \) holds true for vector fields of the \( [G_i, G_p]; p = 1, 2, \ldots \). A similar computation also shows \( (5.14) \) for vector fields of the form \( [G_j, G_p]; j, p = 1, 2, \ldots \). This shows that \( (5.11) \) holds for vector fields \( V \in V_1 \).

Let \( \mathcal{G} \in C_1^\infty(U) \) be the set of all sequences of real-valued functions on \([0, T]\), \( (f_i)_i \equiv 1 \) such that \( \| f_i \|_{\beta} < \infty \) for \( 0 < \beta \leq 1 \). Let \( Y' : [0, T] \rightarrow U^* \) be a \( U^* \)-valued path such that \( (Y'^i)_i \equiv 1 \in C_{1, \infty}^\beta \) where \( Y'^i = Y'(e_i); i \geq 1 \). We then observe if

\[ \delta Y_{ts} = Y'_s \delta G_{ts} + R_{ts}, \quad s < t, \]

\[ \delta Y_{ts} = Y'_s \delta G_{ts} + R_{ts}, \quad s < t, \]

then, \( \delta Y_{ts} = \int_{s}^{t} Y'_r dG_r = \sum_{i=1}^{\infty} \sqrt{x_i} \int_{s}^{t} Y'^i_r dY^i_r \) is a well defined Young integral, where the remainder is characterized by

\[ \delta Y_{ts} = Y'_s \delta G_{ts} + R_{ts}, \quad s < t, \]

then, \( \delta Y_{ts} = \int_{s}^{t} Y'_r dG_r = \sum_{i=1}^{\infty} \sqrt{x_i} \int_{s}^{t} Y'^i_r dY^i_r \) is a well defined Young integral, where the remainder is characterized by

\[ \delta Y_{ts} = Y'_s \delta G_{ts} + R_{ts}, \quad s < t, \]

then, \( \delta Y_{ts} = \int_{s}^{t} Y'_r dG_r = \sum_{i=1}^{\infty} \sqrt{x_i} \int_{s}^{t} Y'^i_r dY^i_r \) is a well defined Young integral, where the remainder is characterized by

\[ \delta Y_{ts} = Y'_s \delta G_{ts} + R_{ts}, \quad s < t, \]

then, \( \delta Y_{ts} = \int_{s}^{t} Y'_r dG_r = \sum_{i=1}^{\infty} \sqrt{x_i} \int_{s}^{t} Y'^i_r dY^i_r \) is a well defined Young integral, where the remainder is characterized by

\[ \delta Y_{ts} = Y'_s \delta G_{ts} + R_{ts}, \quad s < t, \]

then, \( \delta Y_{ts} = \int_{s}^{t} Y'_r dG_r = \sum_{i=1}^{\infty} \sqrt{x_i} \int_{s}^{t} Y'^i_r dY^i_r \) is a well defined Young integral, where the remainder is characterized by
\[ R_{ts}^Y = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_s^t (Y^j_s - Y^j_t) \, dg_j, \]

and \( \| R^2 \|_{2\gamma, x} < \infty \) due to Young-Loève inequality. The class of all pairs \( (Y, Y') \) of the form (5.17) constitutes a subset of controlled paths which we denote it by \( S_{G}^2([0, T]; U^*) \). Next, we recall the following concept of truly rough (see [13, 12]).

**Definition 5.3.** For a fixed \( s \in (0, T] \), we call a \( \frac{1}{2} \)-rough path \( G : [0, T] \rightarrow U \), "rough at time \( s \)" if

\[ \forall v^* \in U^* \text{ non-null } : \limsup_{t \downarrow s} \left| \langle v^*, \delta G(t) \rangle \right| / |t - s|^{\frac{1}{2}} = +\infty. \]

If \( G \) is rough on some dense subset of \([0, T]\), then we call it truly rough.

**Lemma 5.4.** The \( U \)-valued trace-class FBM given by (2.2) is truly rough.

**Proof.** The proof follows the same lines of Example 2 in [13] together with the law of iterated logarithm for Gaussian processes as described by Th 7.2.15 in [23]. We left the details to the reader. \( \Box \)

The following result is given by Th. 6.5 in Friz and Hairer [12].

**Theorem 5.1.** Assume that \( G \) is a truly rough path. Let \( (Y, Y') \) and \( (\tilde{Y}, \tilde{Y}') \) be controlled paths in \( S_{G}^2([0, T]; U^*) \) and let \( N, \tilde{N} \) be a pair of real-valued continuous paths. Assume that

\[ \int_0^1 Y \, dG + \int_0^1 N \, dt = \int_0^1 \tilde{Y} \, dG + \int_0^1 \tilde{N} \, dt \]

on \([0, T]\). Then, \( (Y, Y') = (\tilde{Y}, \tilde{Y}') \) and \( N = \tilde{N} \) on \([0, T]\).

5.2. **Proof of Theorem 1.1.** We are now in position to prove the main result of this paper.

**Proof.** Fix \( x_0 \in \text{dom}(A^\infty) \) and \( \gamma \in (0, T] \). By Lemma 5.3, we have

\[ \gamma_{\gamma} = (T \circ J_{0-t}(x_0))^* \mathcal{C}_t (T \circ J_{0-t}(x_0))^* \]

so that it is sufficient to prove that \( \gamma_{\gamma} \) is positive definite a.s. For this purpose, we start by noticing that

\[ \langle \gamma_{\gamma} x, x \rangle_E = \left\langle \mathcal{C}_t (T \circ J_{0-t}(x_0))^* x, (T \circ J_{0-t}(x_0))^* x \right\rangle_E : x \in \mathbb{R}^d. \]

We observe that \( (T \circ J_{0-t}(x_0))^* \) is one-to-one. By assumption, \( \text{Ker} T^* = \{0\} \) and clearly \( \text{Ker} J_{0-t}^*(x_0) = \{0\} \). Indeed, if \( y \in \text{Ker} J_{0-t}^*(x_0) \), then for every \( x \in \mathbb{R}^d \)

\[ \langle y, S(t)x \rangle_E = \langle y, J_{0-t}(x_0)J_{0-t}^*(x_0)S(t)x \rangle_E = \langle J_{0-t}^*(x_0)J_{0-t}(x_0)y, S(t)x \rangle_{S(t)} = 0. \]

This implies \( y \in (S(t))^{-1} = \{0\} \) (the orthogonal complement in \( E \)). Therefore, it is sufficient to check

\[ \mathcal{C}_t \text{ is positive definite a.s.} \]

Similar to the classical Brownian motion case, we argue by contradiction. Let us suppose there exists \( \varphi_0 \neq 0 \) such that

\[ \mathbb{P}\{\mathcal{C}_t \varphi_0, \varphi_0 \}_{E} = 0 \]
Take $\varphi \in E$. By (5.8), we have

$$(5.20) \quad \langle \mathcal{G}_t \varphi, \varphi \rangle_E = \alpha_H \sum_{\ell=1}^{\infty} \int_0^t \int_0^t \langle \mathbf{J}_{0-r}^{-\alpha}(x_0) \mathcal{G}_r(X^{x_0}), \varphi \rangle_E \langle \mathbf{J}_{0-r}^{-\alpha}(x_0) \mathcal{G}_r(X^{x_0}), \varphi \rangle_E |u - v|^{2H-2} \, du \, dv.$$ 

Let us define

$$K_s = \text{span}[\mathbf{J}_{0-\alpha}^+(x_0) \mathcal{G}_r(X^{x_0}); 0 \leq r \leq s, \ell \in \mathbb{N}] ; 0 < s \leq T,$$

and we set $K_{0+} = \cap_{k>0} K_s$. The Brownian filtration $\mathbb{F}$ allows us to make use of the Blumenthal zero-one law to infer that $K_{0+}$ is deterministic$^3$ a.s. Let $N > 0$ be a natural number and let $N_s$ be the (possibly infinite) dimension of the quotient space $\frac{K}{K_{0+}}$. Consider the non-decreasing adapted process $\{ \min\{N, N_s\}, 0 < s \leq T \}$ and the stopping time

$$S = \inf \{ 0 < s \leq T; \min \{ N, N_s \} > 0 \}.$$

One should notice that $S > 0$ a.s. If $S = 0$ on a set $A$ of positive probability, then for every $\epsilon > 0$ there exists $0 < s \leq T$ such that

$$\epsilon > s > 0 \text{ and } \min\{N, N_s\} > 0,$$

on $A$. This means that we should have $N_s > 0$ for every $s \in (0, T]$ on $A$. This implies that with a positive probability the dimension of $\frac{K}{K_{0+}}$ is strictly positive which is a contradiction.

We now claim that $K_{0+}$ is a proper subset of $E$. Otherwise, $K_{0+} = E$ which implies $K_s = E$ for every $0 < s \leq T$. In this case, if $\varphi \in E$ is such that $\langle \mathcal{G}_t \varphi, \varphi \rangle_E = 0$ with positive probability, then $\langle \mathbf{J}_{0-\alpha}^+(x_0) \mathcal{G}_r(X^{x_0}), \varphi \rangle_E = 0$ for every $r \in [0, \ell]$ and $\ell \in \mathbb{N}$ with positive probability which in turn would imply that $\varphi \in \frac{K_s}{K_{0+}} = E^\perp$ so that $\varphi = 0$. This contradicts (5.19). Now we are able to select a non-null $\varphi \in E^\perp$ such that $K_{0+} \subset \text{ker}\varphi$. At first, we observe $\varphi(K_s) = 0$ for every $0 \leq s < S$ so that

$$(5.21) \quad \langle \mathbf{J}_{0-\alpha}^+(x_0) \mathcal{G}_r(X^{x_0}), \varphi \rangle_E = 0 \forall \ell \geq 1 \text{ and } 0 \leq s < S.$$ 

We claim

$$(5.22) \quad \langle \mathbf{J}_{0-\alpha}^+(x_0) V(X^{x_0}), \varphi \rangle_E = 0 \text{ for every } 0 \leq s < S, V \in \mathcal{V}_k, k \geq 0,$$

where we observe $V$ in (5.22) takes values on $S(T) E$. We show (5.22) by induction. For $k = 0$, (5.21) implies (5.22). Let us assume (5.22) holds for $k - 1$. Let $V \in \mathcal{V}_{k-1}$. Then, we have

$$0 = \langle \mathbf{J}_{0-\alpha}^+(x_0) V(X^{x_0}), \varphi \rangle_E$$

$$= \langle V(x_0), \varphi \rangle_E + \int_0^s \langle \mathbf{J}_{0-\alpha}^+(x_0) [G_0, V](X^{x_0}), \varphi \rangle_E d\ell$$

$$+ \sum_{\ell=1}^{\infty} \sqrt{\lambda_\ell} \int_0^s \langle \mathbf{J}_{0-\alpha}^+(x_0) [G_\ell, V](X^{x_0}), \varphi \rangle_E d\beta^\ell_\ell,$$

where $\langle V(x_0), \varphi \rangle_E = 0$ by the induction hypothesis. By Theorem 5.1 we must have

$$\langle \mathbf{J}_{0-\alpha}^+(x_0) [G_\ell, V](X^{x_0}), \varphi \rangle_E = \langle \mathbf{J}_{0-\alpha}^+(x_0) [G_\ell, V](X^{x_0}), \varphi \rangle_E = 0,$$

for every $0 \leq r \leq s$ and $0 \leq s < S$ and $\ell \geq 1$. This proves (5.22). Clearly, (5.22) implies

$$(5.23) \quad \varphi(\mathcal{V}(x_0)) = 0 \text{ for every non-negative integer } k,$$

$^3$We say that a random subset $A \subset E_n$ is deterministic a.s when all random elements $a \in A$ are constant a.s.
and hence the Hörmander’s bracket condition implies $\varphi = 0$. By Th 2.1.1 in [29], we then conclude the proof. \hfill \Box

**Remark 5.3.** The assumption that $S(t)E$ is dense in $E$ seems a bit restrictive but it covers a rather general class of examples. For instance, if $(A, \text{dom}(A))$ is a densely defined self-adjoint operator such that

$$
\sup_{x \in \text{dom}(A) \setminus \{0\}} \frac{\langle x, Ax \rangle_E}{\|x\|^2_E} < \infty,
$$

then $(A, \text{dom} A)$ is the generator of a self-adjoint analytic semigroup (see Th 7.3.4 and Example 7.4.5 in [5]). Since analytic semigroups are one-to-one, $S(t)$ is one-to-one for every $t \geq 0$ and hence, $S(t)E$ is dense in $E$ for every $t \geq 0$. The heat semigroup on $L^2$ has dense range (see [11]). More generally, assume there exists a separable Hilbert space $W$ densely and continuously embedded into $E$ with compact embedding. Assume that

- $A : W \to W^*$ is continuous and its restriction to $W$, $A_E : \text{dom}(A_E) \to E$ where $\text{dom}(A_E) = \{u \in W : Au \in E\}$ and $A_Eu = Au; u \in \text{dom}(A_E)$, is a self-adjoint operator.
- There exists $\lambda \in \mathbb{R}$ and $\eta > 0$ such that

$$
(Au, u)_{W^*,W} + \lambda \|u\|_W^2 \geq \eta \|u\|_W^2,
$$

for each $u \in W$.

Then, $S(t)E$ is dense in $E$ for every $t \in [0, T]$. See e.g [3] for further details.

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