Shape transition in the even-even Cerium isotopes

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Abstract

The deformation energy of the even-even nuclei of the Cerium isotopic chain is investigated by means of the Macroscopic-Microscopic method with a semiclassical shell correction. We consider axially symmetric shapes. Binding energy and two neutron separation energy are also evaluated. For the sake of clarity several important details of the calculations are also given. It turns out that all these nuclei have prolate equilibrium shape. The regions of maximum deformation are obtained around \( N = 64 \) and \( N = 102 \). There is no critical-point of quantum phase transition in this isotopic chain.

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I. INTRODUCTION

Nowadays it is well established that the majority of nuclei possess a nonzero intrinsic electric quadrupole moment (IEQM). This feature means that the charge distribution inside the nucleus deviates from the spherical symmetry. In other words, apart from very few nuclei, the surface of the nucleus is generally not spherical in its ground state. The intrinsic quadrupole electric moments (or equivalently the nuclear deformation) can be deduced from two types of measurements:

- The reduced electric quadrupole transition probability, $B(E2)$
- The static electric quadrupole moments of ground and excited states, $Q$

It turns out that in a number of cases, the two methods of measurement do not systematically lead to the same values. Important discrepancies occur for several nuclei. This is essentially due to the fact that not only different experimental techniques are used but above all, because different models can be implemented to deduce the nuclear deformation for the both cases. In Ref. [3] it is stated that deformations deduced from $B(E2)$ have a "more general character". In other words, "$B(E2)$-type" data reflect not only static nuclear deformation (permanent deviation of the nuclear shape from sphericity), but also dynamic deformation. Furthermore, $B(E2)$ measurements are model independent and thus are generally more reliable. This is corroborated by the fact that the only systematic compilation in which the deformation of the ground state is given explicitly is based on $B(E2; 0^+ \rightarrow 2^+)$ and has been published in Ref. [1].

In the present work, experimental values refer to these ones.

Theoretical approaches to the deformation energy can be divided into two categories; Dynamic calculations to find the shape of the ground state (or even of excited states) and static calculations by determining the absolute minimum (ground state) or multiple minima (shape isomers) in the potential energy surface (PES) for a given nucleus. Thus, on the one hand, we have the so-called collective models, which themselves are subdivided into two groups: The "Geometric Collective Model" also called "the Collective Bohr Hamiltonian" (CBH and its variants) and the "Algebraic Model", well known under the name of the "Interacting Boson Model" (IBM and its variants) [4]. On the other hand, "particle models" consider the nucleus as a collection of interacting nucleons (fermions). In practice, the classical $N$-body problem can be approximately solved by the usual approximation of the mean field with eventually residual interactions. In this respect, the "best" mean field is deduced after applying a variational principle in the Hartree-Fock-Bogoliubov method (HFB). In this model, the determination of the potential energy surface (PES) of the nucleus amounts to perform constrained Hartree-Fock-Bogoliubov (CHFB) calculations [5]. We will not address very complicated methods "beyond the mean field" such as the Quasiparticle Random Phase Approximation (QRPA) or the Generator-Coordinate-Method (GCM) methods which are unsuitable in practice for large scale calculations.

Because of CHFB calculations are time consuming, especially in large studies, Microscopic-Macroscopic method (Mic-Mac) constitutes a good alternative which, is up to now, implemented [6]. In the present work, we use an improved variant of this method. The word "improved" means that we use semi-classical method to avoid the well-known drawbacks (spurious dependence on two mathematical parameters) of the standard Strutinsky shell correction (see text below).

The present study is devoted to the deformation energy, equilibrium nuclear shapes and binding energy of the ground state of the even-even cerium isotopes. There are many reasons to this choice. One of them is to re-test our previous calculations. In effect, similar calculations have been already performed by us in the xenon, barium, and cerium region Ref [7]. However because the phenomenological mean potential varies smoothly with $N$ and $Z$, we have made, in the past, a rough approximation by choosing the same set of parameters for the phenomenological mean potential, for the all treated nuclei. Originally, this approximation was done only for simplifying the calculations. Here, contrarily to that study, each nucleus has its "own" mean potential with a specific set of parameters. In this way it is possible to evaluate in a rigorous way the uncertainty introduced in the previous calculations. Apart from this remark, there are several main other reasons which could justify this choice: (i) First, it should be interesting to see how the deformation energy and binding energy vary with the neutron number ($N$) for this isotopic chain. (ii) Second, the present study extends the previous calculations to all cerium isotopes up to the drip lines (34 versus 13 nuclei). (iii) Third, we also will attempt to deduce, from potential energy surface (PES) curves, the shape transition from spherical to axially deformed nuclei, looking for the so-called $X(5)$ critical-point between $U(5)$ and SU(3) symmetry limits of the IBM [8-9].

It is worth to recall briefly some information deduced from the literature for the cerium isotopes. Among the numerous studies, we only cite some of them: In 2005 Smith et al [10] have studied excited states of $^{122}$Ce up to spin $14h$ deducing a probable quadrupole deformation of about $\beta \approx 0.35$. The deformed nucleus $^{130}$Ce has been studied in 1985, using the techniques of in-beam gamma-ray spectroscopy [11]. The corresponding data have been interpreted in terms of the cranking model by assuming a prolate deformation with $\varepsilon_2 \approx 0.25$ ($\beta \approx 0.27$). High-spin states in $^{132}$Ce have been also studied by A.J. Kirwan et al. [12]. They found a superdeformed band with deformation $\beta \approx 0.4$ much more larger that the ground state deformation ($\beta \approx 0.2$). E. Michelakakis et al [13] by evaluating $\gamma$-ray transitions in
142\textit{Ce} and 144\textit{Ce} conclude that in cerium isotopes (near the beta-stable line) the onset of nuclear deformation occur between \( N = 86 \) and \( N = 88 \). "Pure" theoretical calculations have been performed in Ref. [14] and [15] with projected shell model (PSM) and Hartree-Bogoliubov ansatz in the valence space respectively for 122\textit{Ce} and 124–132\textit{Ce} for low lying yrast spectra. Good values of energy levels and reduced transition probabilities \( B(E2, 0^+ \rightarrow 2^+) \) have been obtained respectively in these two papers. Other approaches for the rich-neutron cerium isotopes have been made in Ref. [16].

A study of the shape transition from spherical to axially deformed nuclei in the even \textit{Ce} isotopes can be found in Ref. [18]–[21] with respectively, the deformation parameter (referred to as \( \delta B \)) before, the normalization is expressed by:

\[
\text{Constants in which } \beta = 0
\]

look for very high accuracy in binding energy, because this is not the purpose of the present work. Relativistic Hartree-Fock-Bogoliubov theory has been used to predict ordinary particle model:

\[\hat{H}(\beta) | \Psi_{i}(\beta) \rangle = \epsilon_{i}(\beta) | \Psi_{i}(\beta) \rangle\] (5)
where $|\Psi_i\rangle$ and $\epsilon_i$ are respectively the eigenfunctions and the associated eigenvalues of nucleons. Hamiltonian $\hat{H}$ contains four contributions which are: (i) kinetic energy, (ii) central deformed mean field, (iii) spin-orbit and (iv) Coulomb interactions. We perform analogous calculations as in Nilsson model but our deformed mean potential is of Woods-Saxon type and therefore is "more realistic". Although calculations are not self consistent, they are microscopic. It is to be noted that our Schrödinger equation has a form which is very close to the one of the Skyrme-Hartree-Fock method. 

Eq. (5) is solved by our FORTRAN program described in details in Ref [22] and improved in two successive versions [23] and [24].

C. Microscopic corrections

Microscopic corrections are defined as the sum of shell and pairing corrections which themselves are calculated separately for each kind of nucleons.

$$\delta B_{\text{micro}}(\beta) = \delta E_{\text{shell}}(N, \beta) + \delta E_{\text{shell}}(Z, \beta) + \delta P_{\text{pairing}}(N, \beta) + \delta P_{\text{pairing}}(Z, \beta)$$

In this formula the shell correction is defined by the usual Strutinsky prescription, i.e., as the difference between the sum of the single particle energies (which contains the shell effects) and an averaged (or smoothed) sum (which is free from shell effects)

$$\delta E_{\text{shell}}(N \text{ or } Z) = \sum_{i=1}^{N\text{ or } Z} \epsilon_i(\beta) - \sum_{i=1}^{N\text{ or } Z} \frac{\epsilon_i(\beta)}{N\text{ or } Z}$$

Energies $\epsilon_i(\beta)$ are deduced from Eq. (5). In our procedure, the second sum is found by means of a semi-classical way instead a Strutinsky smoothing procedure, see Ref. [27]. This avoids the well-known weakness of the standard shell correction method, namely, the dependence on two unphysical parameters which are the "smoothing" parameter and the order of the curvature correction. Moreover, it has been clearly shown that Strutinsky level density method is only an approximation of that of the semi-classical theory [26].

The "pure" pairing correlation energy is defined by:

$$P(\beta) = \sum_{i=1}^{\infty} 2\epsilon_i(\beta)\nu_i^2 - \sum_{i=1}^{N/2\text{ or } Z/2} 2\epsilon_i(\beta) - \frac{\Delta^2}{G}$$

where $\nu_i^2$, $\Delta$ and $\lambda$ are the usual occupation probabilities, gap and Fermi energy of the BCS approximation (the factor "2" is simply due to the Kramers degeneracy). Since the smooth part of pairing correlations is already contained in the liquid drop model, we have to add only the one due to the shell oscillations of the level density. This contribution is defined by means of a similar formula to Eq. (7)

$$\delta P_{\text{pairing}}(N \text{ or } Z, \beta) = P(\beta) - \overline{P(\beta)}$$

where the averaged pairing is defined as $\overline{P(\beta)} = (1/2)g_{\text{semicl}}.(\lambda\Delta^2)$. We use a simple BCS method to account for pairing correlations. To calculate Eq (7) and (8) we follow the method detailed in Ref. [27] with its FORTRAN code. The treatment of the pairing has also been explained in Ref. [7] and references quoted therein.

D. Numerical constants and prescriptions

1. Constants of the Microscopic model

For each kind of particles the mean central and the mean spin-orbit field are written as [22]:

$$V(\beta) = \frac{V_0}{1 + \exp(R_V L_V(\beta)/a_0)} \quad V_{SO}(\beta) = \lambda \left( \frac{\hbar}{2Mc} \right) \frac{V_0}{1 + \exp(R_{SO} L_{SO}(\beta)/a_0)}$$
where \( L_V(\beta) \) and \( L_{SO}(\beta) \) contains the information on the deformation. In fact, these functions contain 9 constants: \( V_{0\text{neut}}, V_{0\text{prot}}, R_V, R_{SO\text{--neut}}, R_{SO\text{--prot}}, a_0, \lambda_{\text{neut}}, \lambda_{\text{prot}} \). These quantities are taken from the "universal" parameters \(^{24}\) (see appendix B) which is an optimized set.

The Coulomb mean field is approximated by a uniform charge distribution inside a deformed surface. The volume conservation is therefore \( \text{Vol} = (4/3)\pi R_{\text{ch}}^3 \) with the simple assumption \( R_{\text{ch}} = R_V \).

2. Constants of the liquid drop model

As already stated, we have chosen the parameters of Myers and Swiatecki (see table \(^{1}\)) because this set contains a reduced number of parameters with respect to more modern formulae.

All the constants are needed in the binding energy whereas only \( a_S, C_C, \kappa \) play a role in the potential energy surface.

3. Nuclear mass excesses

Nuclear masses are deduced as mass excesses:
\[
M_{\text{excess}}(A, Z) = ZM_H + (A - Z)M_n - B(A, Z)
\]
where \( M_H = 7.298034\text{MeV} \) is the hydrogen mass excess and \( M_n = 8.071431\text{MeV} \) the neutron mass excess. This makes comparisons with experimental values easiest.

III. RESULTS

In our previous paper \(^{7}\) calculations for isotopes \(^{116-130}Ca \) showed that the equilibrium deformations (\( \beta \approx 0.25 - 0.30 \)) have always been obtained for symmetric prolate shapes (\( \gamma = 0^{\circ} \)). Results obtained in Ref.\(^{32}\) with a similar approach for the nuclei \(^{116-130}Ca \), corroborate this fact. For these reasons, we think that it is needless to account for the axial asymmetry in a "pure" static study of the equilibrium deformation. However, we have to consider prolate \( (\gamma = 0^{\circ}) \) as well as oblate \( (\gamma = 60^{\circ}) \) nuclear shapes. In this regard, it is worth remembering that oblate shape given by \( (\beta > 0, \gamma = 60^{\circ}) \) is equivalent to the set \( (\beta < 0, \gamma = 0^{\circ}) \).

A. Comparison between the different contributions entering in the potential energy surface

It could be useful to compare the importance of the different terms entering in the right hand side of Eq.\(^{10}\). In this respect, we have drawn in Fig.\(^{1}\) for axially prolate shape, the four microscopic contributions \( \delta E_{\text{shell}}(N, \beta), \delta E_{\text{shell}}(Z, \beta), \delta P_{\text{pairing}}(N, \beta), \delta P_{\text{pairing}}(Z, \beta) \) for the case of \(^{160}Ca \) as functions of \( \beta \). Following the cited order, we can say that the difference between the highest and lowest values (in the interval \( \beta \in [0, 0.7] \)) are respectively about 11.0 MeV, 10.5 MeV, 5.7 MeV, 3.5 MeV for the four corrections. Thus, these variations show that the shell corrections \( \delta E_{\text{shell}}(N, \beta), \delta E_{\text{shell}}(Z, \beta) \) are more important than \( \delta P_{\text{pairing}}(N, \beta), \delta P_{\text{pairing}}(Z, \beta) \) and have a clear minimum at respectively \( \beta = 0.35 \) and \( \beta = 0.30 \). It is well known that for each kind of nucleon the shell correction is in opposite phase with respect to the pairing correction (this means for that when \( \delta E_{\text{shell}}(N, \beta) \) increases with \( \beta, \delta P_{\text{pairing}}(N, \beta) \) decreases and vice versa).

Contrarily to these curves, the liquid drop model is strictly increasing with \( \beta \), and its minimum occurs always at the beginning \( \beta = 0 \) (spherical shape). When all the contributions are added, the minimum of the potential energy surface of the nucleus is reached at about \( \beta = 0.3 \) and is mainly due to the shell corrections. When \( \beta \) becomes more and more, larger the contribution of the liquid drop energy becomes preponderant so that the equilibrium deformation occurs generally between \( \beta = 0 \) and \( \beta = 0.4 \). Because of the convention of the sign stated before, \( \delta B_{\text{micro}} \) defined in Eq.\(^{11}\) must be negative in order to increase the binding energy of the nucleus. Since the shell corrections (for protons and neutrons) play a major role in \( \delta B_{\text{micro}} \), it is naturally expected that negative (but absolute large) values of shell correction contribute to increase the binding energy of the nucleus. In this respect, it is well known that the shell correction is essentially determined by the distribution of single-particle levels in the vicinity of the sharp Fermi level (defined here as the midway between the last occupied level and the first empty level). Following Ref.\(^{31}\), we can state that "the nuclear ground state, as well as any other relatively stable state should correspond to the lowest possible degeneracy, or, in other words, the lowest density of state near the Fermi level". This is illustrated in Fig. \(^{2}\) where the single-particle levels are drawn as function of the deformation \( \beta \) \((\gamma \text{ being fixed at } \gamma = 0^{\circ}) \). To this end we have used the FORTRAN code of Ref.\(^{22}\) and \(^{24}\). The area where the single-particle level density is low near
the Fermi level (black stars) is indicated by a circle. Thus, it is not so surprising that, it is in this region where the neutron shell correction becomes the most important, involving a minimum in the PES of the nucleus.

B. Equilibrium deformations

They are given in table for prolate as well as oblate shapes (see table legend for details). The minima of PES for the corresponding wells are denoted \( \text{minpro} \) and \( \text{minobl} \). The deformation energy is defined as the difference \( E_{\text{def}} = E_{\text{PES}}(0) - E_{\text{PES}}(\beta) \), i.e., the difference between the potential energy surface for a spherical shape and the one corresponding to the absolute minimum of PES. Permanent deformations will be in principle characterized by large values of \( E_{\text{def}} \) and are responsible of rotational spectra.

From this table, some remarks may be drawn:
(i) Two regions of prolate deformation are found. They occur around \( N = 64 \) and \( N = 102 \) with maximum deformation about \( \beta \approx 0.30 \). The deformation energy (between spherical and deformed shape) is about 6.70\( MeV \) for \( N = 64 \) and 9.30\( MeV \) for \( N = 102 \) and decreases from either side from these two nuclei.
(ii) Spherical deformation occur at and near the (magic) numbers \( N = 82 \) and \( N = 128 \) (not shown).
(iii) The deformation energy decreases from \( N = 64 \) (maximum) to \( N = 82 \) (minimum) and reincreses again to \( N = 102 \) (maximum). We have found graphically that the first inflexion point occurs between \( N = 72 \) and \( N = 74 \) and a second inflexion point is found between \( N = 90 \) and \( N = 92 \). One can consider (somewhat arbitrarily) that spherical shapes occur approximately between these two limits.
(iv) The minima of prolate equilibrium deformations are, by far, always deeper compared to the ones of the oblate minima (\( \text{minpro} \ll \text{minobl} \)). In other words cerium isotopes prefer, by far, prolate shapes. In other words, the deformation energy increases in average with the asymmetry \( \gamma \). This justifies a posteriori that, in a static study of the equilibrium deformation, it is needless to account for axial asymmetry. It is worth to remember that most of nuclei of the chart have prolate shape (see Ref. 22).
(v) Even though the experimental deformations are known only in absolute value from \( B(E2) \), a good agreement is obtained if one excepts the three "nearly magic" nuclei \(^{138-142}\text{Ce}\).

In Fig. 3 are displayed the present equilibrium deformations, experimental values, our ”old” calculations and other studies performed by different authors which are: Kern et al. 32, Hilaire and Girod 35, Gotz et al. 33, and Nix et al. 34. All calculations are based on Macro-Micro method (with different mean fields or different parameters). Except the one of Ref. 35 which uses Hartree-Fock-Bogoliubov model with Gogny force.
(jj) Near magic number \( (N = 82) \) all calculations give spherical equilibrium deformation whereas experimental results are always slightly deformed (even for \( N = 82 \)). It seems difficult to overcome this defect with a pure static approach which neglects the role of the mass parameters.
(jj) The overall tendency of these calculations is the same except the fact that HFB calculations differ significantly from the others with higher values in some regions.
(jjj) Apart from HFB calculations, theoretical values are generally quite close from each others.
(jv) Our old and new calculations give very close results (see Table II). Thus, even if it is better to choose a proper set of mean-field parameters for each nucleus, we do not commit a significant error by taking the same set of parameters for nuclei that do not differ strongly by the number of neutrons \( (N) \).

C. Mass Excesses

We list from a FORTRAN file (see Fig. 4) the results of our theoretical calculations of the binding energies and mass excesses (m-excess) for the even-even cerium isotopic chain. For the sake of completeness, experimental mass excesses and the ones of the FRDM model (see Ref. 13) are also given. We must keep in mind that only 6 parameters are used in the liquid drop model whereas 16 parameters are necessary in the FRDM model. This explains the "better quality" of the FRDM model. However, we have checked that the variations of binding energy or mass excesses from one isotope to the nearest is practically the same in our model and the one of FRDM (the deviations are about \( \pm 0.35 MeV \)). For this reason, the calculation of the two neutron separation energies (see the following subsection III D) will almost be probably the same for the two approaches even though our model is not so accurate.

D. Transitional regions in cerium isotopes

In Fig. 5 is shown the gradual transition in the potential energy surface from spherical vibrator to the axially deformed rotor when the number of neutrons \( (N) \) increases from 76 to 92. One signature of \( X(5) \) symmetry which is
a critical-point of phase/shape transitions (quantum phase transition between spherical and axial symmetries) should be a long flatness of the potential energy surface with eventually a weak barrier from prolate to oblate shapes. In this figure, for \( N > 82 \), the width of the flatness increases as one moves away from \( N = 82 \) but at the same time the difference between oblate and prolate minima and barrier between oblate and prolate shapes also increase. For example the differences between oblate and prolate energy minima and barriers for isotopes with \( N = 88, 90, 92 \) are respectively about \( 1.5\,\text{MeV}, 2.5\,\text{MeV} \) and \( 3.3\,\text{MeV} \) with energy barrier about \( 2\,\text{MeV}, 4\,\text{MeV} \) and \( 5.5\,\text{MeV} \) respectively. The width of the bottom of the well must be relativized with the height of the barrier. Thus for the case of \( N = 92 \) the width is important, i.e. about \( \Delta \beta \approx \beta_{\text{pro}} - \beta_{\text{obt}} \approx 0.26 - (-0.20) \approx 0.46 \) but the barrier is about \( 5.5\,\text{MeV} \) and therefore seems too high. The case \( N = 90 \) gives a width of \( \Delta \beta \approx 0.3 \) with a barrier of about \( 4\,\text{MeV} \). For \( N < 82 \), the case \( N = 76 \) seems to be relatively equivalent to \( N = 90 \) with a slightly smaller width and a lower height barrier. Thus it is difficult to determine clearly the existence of a \( X(5) \) critical-point. Thus, everything seems to indicate a continuous transition.

In Fig. 6 is displayed the two-neutron separation energy \( \text{TSN} \) as function of the neutron number \( N \). A clear jump is seen from \( N = 82 \) to \( N = 84 \), i.e. from one major shell to the following. Just before \( N = 82 \) and just after \( N = 84 \) the TSN varies more slowly. Far for the "jump" the curve becomes quasi-linear. Once again, no special behavior is noted around \( N = 90 \) which from Ref.\[36\] and \[37\] should constitute with \( Z \approx 62 \) the first order shape transition \( (X(5) \text{ critical-point}) \) in the rare earth region. In Ref.\[38\], it has been pointed out that "Empirical evidence of transitional symmetry at the \( X(5) \) critical-point has been observed in \( ^{150}\text{Nd}, ^{152}\text{Sm}, ^{154}\text{Gd}, \) and \( ^{156}\text{Dy} \)." One of the most important signatures of the phase transition is given by a sudden jump in the value of the energy ratio \( R_{4/2} = 4^+_1/2^+_1 \) from one nucleus to the next. We found it useful to compare the experimental values of this ratio (see Fig. 7) in the cases of the isotopic chains of \( \text{Ce} \) and \( \text{Sm} \) (The experimental values of the considered levels have been deduced from the adopted level of \( \text{ENSDF site} \[39\]). The figure shows clearly two facts. First, the important variation of \( R_{4/2} \) near of the magic number \( N = 82 \) for the both isotopic chains and then, the important difference between the behavior the two isotopic chain from \( N = 88 \) to \( N = 90 \). In effect in the case of the Samarium, there is a sudden increase of this ratio whereas this is not the case for the Cerium isotopes. This has been attributed to the \( X(5) \) critical-point symmetry of the nucleus \( ^{152}\text{Sm} \). Thus the present study confirms that cerium isotopic chain is characterized by a continuous shape/phase transition.

IV. CONCLUSION

Potential energy surfaces have been drawn for the cerium isotopic chain. All even-even nuclei between the two drip lines have been considered. To this end, we have used the microscopic macroscopic method in which the quantum corrections have been evaluated by a semi-classical procedure. The microscopic model is based on a "realistic" Schrödinger equation including a mean field of a Woods-Saxon type. The macroscopic part of the energy is evaluated from the liquid drop model using the version of Myers and Swiatecki. The following points must be remembered:

(i) All equilibrium deformations have been found prolate with an important deformation energy compared to oblate shapes.

(ii) The maximum deformations are of order \( \beta \approx 0.3 \) and are located around \( N = 64 \) and \( N = 102 \) with deformation energy about \( 6\,\text{MeV} \) and \( 9\,\text{MeV} \) respectively. The equilibrium deformations decrease as one moves away from these two nuclei.

(iii) Spherical shapes are found in the neighborhood of \( N = 82 \).

(iv) Good agreement is obtained between theoretical and experimental values if one excepts the area of the shell closure \( N = 82 \) where the latter are slightly larger.

(v) This isotopic chain possesses a continuous shape/phase transition from spherical shapes toward the axially symmetric ones.

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Appendix A: Constants of the binding energy of the liquid drop model

The constants of Eq.(1) are defined as follows:

\[ C_V = a_V \left[ 1 - \kappa I^2 \right] \] (in the volume term)
\[ C_S = a_S \left[ 1 - \kappa I^2 \right] \] (in the surface term)
\[ I = \frac{N - Z}{N + Z} \] (relative neutron excess)
\[ \varepsilon = +1(even - even), 0(odd), -1(odd - odd) \] (in the pairing term)
\[ C_C = \frac{3 e^2}{5 r_0} \] (in the Coulomb term)
\[ C_d = \frac{\pi^2}{2} \left( \frac{a_0}{r_0} \right)^2 \frac{e^2}{r_0} \] (diffuseness correction)

The last correction to the Coulomb energy takes into account that the liquid drop has not a sharp but a diffuse surface of the Woods-Saxon type \[ \ldots \]. The diffuseness parameter is \( a_0 \) and the charge radius "contains" \( r_0 \) \( (R_{ck} = r_0 A^{1/3}) \).

Appendix B: Constants of the Woods-Saxon mean potential

"Universal parameters" of the Woods-Saxon central and Spin-orbit potentials entering in Eq(9)
Central mean field Depth (MeV) Neutrons $V_{0\text{neut}} = 49.6(1 - 0.86I)$ Protons $V_{0\text{prot}} = 49.6(1 + 0.86I)$
Central mean field Radius (fm) $R_{V\text{neut}} = 1.347A^{1/3}$ $R_{V\text{prot}} = 1.275A^{1/3}$
Spin-orbit mean field SO-coupling strength $\lambda$ 35.0 36.0
Spin-orbit mean field Radius (fm) $R_{SO-\text{neut}} = 1.310A^{1/3}$ $R_{SO-\text{prot}} = 1.200A^{1/3}$
Central mean field diffuseness (fm) $a_0 = 0.70$ $a_0 = 0.70$
Spin-orbit mean field diffuseness (fm) $a_0 = 0.70$ $a_0 = 0.70$

**Figures**

**FIG. 1:** Contributions of the shell and pairing corrections for the two kind of nucleons and the one of the liquid drop model to the total potential energy surface of the nucleus $^{160}\text{Ce}$. 
FIG. 2: Single-particle energies of the microscopic model as function of deformation for prolate shapes ($\beta > 0$) for the nucleus $^{160}\text{Ce}$. Spherical spectroscopic notation is given for spherical deformation ($\beta = 0$). The circle in dotted line indicates the area of lowest level density.
FIG. 3: Theoretical equilibrium deformations for even-even cerium isotopes evaluated by different or similar approaches.

| N  | A  | Z  | bind | m-excess | exp | frdm |
|----|----|----|------|----------|-----|------|
| 58 | 116 | 58 | 914.85 | -23.94 | exp | -29.21 |
| 60 | 118 | 58 | 942.64 | -35.59 | exp | -40.57 |
| 62 | 120 | 58 | 968.86 | -45.66 | exp | -50.01 |
| 64 | 122 | 58 | 993.74 | -54.40 | exp | -57.99 |
| 66 | 124 | 58 | 1017.15 | -61.67 | exp | -64.93 |
| 68 | 126 | 58 | 1039.35 | -67.73 | exp | -70.82 |
| 70 | 128 | 58 | 1060.58 | -72.81 | exp | -75.54 |
| 72 | 130 | 58 | 1080.54 | -76.63 | exp | -79.17 |
| 74 | 132 | 58 | 1099.75 | -79.68 | exp | -81.89 |
| 76 | 134 | 58 | 1118.37 | -82.18 | exp | -84.02 |
| 78 | 136 | 58 | 1136.63 | -84.30 | exp | -85.67 |
| 80 | 138 | 58 | 1154.66 | -86.18 | exp | -87.62 |
| 82 | 140 | 58 | 1171.81 | -87.19 | exp | -88.68 |
| 84 | 142 | 58 | 1184.16 | -83.39 | exp | -84.78 |
| 86 | 144 | 58 | 1195.44 | -78.53 | exp | -80.23 |
| 88 | 146 | 58 | 1207.28 | -74.23 | exp | -75.72 |
| 90 | 148 | 58 | 1218.60 | -69.41 | exp | -70.83 |
| 92 | 150 | 58 | 1229.50 | -64.17 | exp | -65.80 |
| 94 | 152 | 58 | 1239.76 | -58.28 | exp | -59.78 |
| 96 | 154 | 58 | 1249.85 | -52.23 | exp | -52.90 |
| 98 | 156 | 58 | 1258.66 | -44.90 | exp | -45.40 |
| 100 | 158 | 58 | 1266.78 | -36.87 | exp | -37.29 |
| 102 | 160 | 58 | 1274.68 | -28.63 | exp | -28.70 |
| 104 | 162 | 58 | 1281.19 | -19.00 | exp | -19.01 |
| 106 | 164 | 58 | 1287.19 | -8.86 | exp | -8.62 |
| 108 | 166 | 58 | 1292.74 | 1.74 | exp | 2.23 |
| 110 | 168 | 58 | 1297.58 | 13.04 | exp | 13.43 |
| 112 | 170 | 58 | 1301.96 | 24.81 | exp | 25.00 |
| 114 | 172 | 58 | 1305.73 | 37.17 | exp | 36.82 |
| 116 | 174 | 58 | 1309.33 | 49.72 | exp | 49.07 |
| 118 | 176 | 58 | 1312.60 | 62.59 | exp | 61.53 |
| 120 | 178 | 58 | 1315.49 | 75.84 | exp | 74.94 |
| 122 | 180 | 58 | 1318.69 | 88.79 | exp | 87.48 |
| 124 | 182 | 58 | 1321.72 | 101.90 | exp | 99.94 |

FIG. 4: Theoretical binding energies and mass excesses of the present approach compared to the experimental mass excesses and the ones given by the FRDM model of Ref. \[18\]. All energies are expressed in MeV. The experimental data as well as the frdm results have been entered manually in the code. Asterics mean that no experimental data is available for the corresponding nucleus.
Cerium Isotopes (Z=58)

Potential energy Surface (MeV)
FIG. 5: Shape evolution for cerium isotopes from $N = 78$ to $N = 92$.

FIG. 6: Two-neutron separation energies ($S_{2N}$) along the cerium isotopic chain. This quantity is defined as $S_{2N}(A, Z, N) = \text{Bind}(A, Z, N) - \text{Bind}(A - 2, Z, N - 2)$ where the binding energy $\text{Bind}(A, Z, N)$ is given by Eq. [1]. Note that in our approach the neutron drip line (where $S_{2N} \approx 0$) can be extrapolated around $N = 128$ for Cerium isotopes.

FIG. 7: $R_{4/2}$ energy ratio as function of neutron number for Cerium and Samarium isotopes. Sudden variations are associated with magic closure shells for both chains (at $N = 82$) and with $X(5)$ critical point which occurs only for $\text{Sm}$ (at $N = 90$).
TABLE I: Parameters of the liquid drop model in the Myers and Swiatecki version [28]

| N  | A   | $\beta_{pro}$ | $\beta_{min}$ | $\beta_{obl}$ | $E_{def}$ | $|\beta_{exp}|$ |
|----|-----|---------------|---------------|---------------|----------|-------------|
|    |     | (MeV)        | (MeV)         | (MeV)         | (MeV)    |             |
| 58 | 116 | 0.30         | 0.90          | -0.21         | 3.62     | 4.80        |
| 60 | 118 | 0.32         | 0.88          | -0.23         | 4.07     | 5.87        |
| 62 | 120 | 0.32         | 1.03          | -0.23         | 4.33     | 6.19        |
| 64 | 122 | 0.31         | 1.16          | -0.23         | 4.23     | 6.68        |
| 66 | 124 | 0.30         | 1.47          | -0.21         | 4.15     | 6.17        |
| 68 | 126 | 0.29         | 1.75          | -0.21         | 3.87     | 5.43        |
| 70 | 128 | 0.27         | 1.82          | -0.21         | 3.48     | 4.67        |
| 72 | 130 | 0.25         | 2.02          | -0.2          | 3.27     | 3.34        |
| 74 | 132 | 0.20         | 1.90          | -0.17         | 2.60     | 1.97        |
| 76 | 134 | 0.16         | 1.28          | -0.14         | 1.63     | 0.93        |
| 78 | 136 | 0.10         | 0.04          | -0.07         | 0.18     | 0.19        |
| 80 | 138 | 0.00         | -1.93         | 0.00          | -1.93    | 0.00        |
| 82 | 140 | 0.00         | -3.96         | 0.00          | -3.96    | 0.00        |
| 84 | 142 | 0.00         | -2.07         | 0.00          | -2.07    | 0.00        |
| 86 | 144 | 0.15         | 0.02          | -0.06         | 0.53     | 0.50        |
| 88 | 146 | 0.19         | 0.73          | -0.11         | 2.43     | 1.99        |
| 90 | 148 | 0.23         | 1.15          | -0.14         | 3.76     | 3.15        |

TABLE II: Equilibrium deformations as well as deformation energies for the cerium isotopic chain. The columns give successively the number of neutrons (N), the mass number (A), the prolate equilibrium deformation ($\beta_{pro}$), the minimum of the prolate well ($minpro$), the oblate equilibrium deformation ($\beta_{obl}$), the minimum of the oblate well ($minobl$), the deformation energy ($E_{def}$, see text), the experimental deformation energy ($\beta_{exp}$). Note: The deformation energy is always given for the prolate equilibrium shape because no absolute minimum is obtained for oblate shape.

| Cerium (Z = 58) | N  | N = 58 | 60 | 62 | 64 | 66 | 68 | 70 | 72 | 74 | 76 | 78 | 80 | 82 |
|-----------------|----|--------|----|----|----|----|----|----|----|----|----|----|----|----|
|                 | Present $\beta$ | +0.30 | +0.32 | +0.32 | +0.31 | +0.30 | +0.29 | +0.27 | +0.25 | +0.20 | +0.16 | +0.10 | +0.00 | +0.00 |
|                 | Old $\beta$ | +0.28 | +0.30 | +0.31 | +0.31 | +0.31 | +0.30 | +0.27 | +0.24 | +0.22 | +0.18 | +0.06 | +0.11 | +0.00 |
| Present $E_{def}(MeV)$ | 4.80 | 5.87 | 6.19 | 6.68 | 6.17 | 5.43 | 4.67 | 3.34 | 1.97 | 0.93 | 0.19 | 0.00 | 0.00 |
| Old $E_{def}(MeV)$ | 4.82 | 5.77 | 6.03 | 6.31 | 7.08 | 5.36 | 4.41 | 3.35 | 2.13 | 0.77 | 0.00 | 0.24 | 0.00 |

TABLE III: New equilibrium deformations and deformation energies vs old [7].