CONCERNING THE SLOPE OF THE CEPHEID PERIOD–LUMINOSITY RELATION

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ABSTRACT

We discuss the impact of possible differences in the slope of the Cepheid period–luminosity (PL) relation on the determination of extragalactic distances in the context of recent studies that suggest changes in this slope. We show that the Wesenheit function \( W = V - R \times (V - I) \), widely used for the determination of the Cepheid distances, is expected to be highly insensitive to changes in the slope of the underlying (monochromatic) PL relations. This occurs because the reddening trajectories in the color–magnitude plane are closely parallel to lines of constant period. As a result \( W \)-based PL relations have extremely low-residual dispersion, which is because differential (and the total line of sight) reddening is eliminated in the definition of \( W \) and the residual scatter due to a star’s intrinsic color/position within the Cepheid is also largely insensitive to \( W \). Basic equations are presented and graphically illustrated, showing the insensitivity of \( W \) to changes in the monochromatic PL relations.

Key words: Cepheids – galaxies: distances and redshifts – stars: distances

1. INTRODUCTION

There has recently been some concern raised over the possibility that the slope of the Cepheid period–luminosity (PL) relation is not universal. Ngeow & Kanbur (2005), Kanbur et al. (2007), Koen & Siluyele (2007), Ngeow et al. (2008), and other studies cited therein suggest that the optical PL relation shows a break around 10 days. These results are controversial and are still being debated in the literature (see, for instance, Benedict et al. 2008; Pietrzynski et al. 2007). Given the impact that Cepheids have traditionally had on the determination of extragalactic distances in the context of recent studies that suggest changes in this slope, it does need to be said that for the extragalactic distance scale the vast majority of studies have used Cepheids whose periods are all longer than 10 days.

2. THE PLC

We begin by noting that, for all stars, Stephan’s law in combination with simple spherical geometry gives rise to a two-parameter mapping of radius and effective temperature into total luminosity. By the definition of effective temperature \( (T_{\text{eff}}) \) we have

\[
L = 4\pi R^2 \sigma T_{\text{eff}}^4. \tag{1}
\]

Allowing a photometric color \( (V-I) \), say) to stand in for temperature, and a monochromatic magnitude \( (M_V, \text{say}) \) to substitute for bolometric luminosity, only one more observable is needed to complete the transfer from theory to observation. For Cepheids, their (easily observed) periods can stand in for their (hard to observe) radii; where lines of constant radius in the color–magnitude plane functionally substitute for lines of constant period. The physically motivated equation above then becomes the following equation based exclusively on observables, while necessarily retaining the form of an equation with two (and only two) independent variables: in this case \( log(P) \) and \( (V-I) \). Thus,

\[
M_V = \alpha \log(P) + \beta (V-I) + \gamma. \tag{2}
\]

It is important to note that both of these above equations describe and cover the infinite plane of the color–magnitude diagram. There is no PL relation in either of these equations. More forcefully stated, Equation (2) is not the PL relation for Cepheids. Any general correlation of period and luminosity will only come through a constraint of these equations. That constraint, in the context of Cepheids, is imposed by the existence of an instability strip within which the natural period of oscillation is excited, and manifest as a periodic change of luminosity. The instability strip, expressed as an additional and external mathematical constraint on the above equations, produces the observed PL relations by narrowing down the range of luminosities seen at any given period. The slope of the red and blue boundaries to the instability strip as they cut lines of constant period determines the slope of the PL relation. This was first illustrated and emphasized in Madore & Freedman (1991), their Figure 3.

Before advancing further, we introduce the Wesenheit function \( W = V - R_{VI} \times (V-I) \). This specific combination of magnitude and color is explicitly constructed so as to mathematically guarantee that their sum will be numerically independent of the total amount of interstellar extinction. With foreknowledge that (by definition) \( A_V = R_{VI} \times E(V-I) \), and again having the added definitions that \( V = V_0 + A_V \) and \( (V-I) = (V-I)_0 + E(V-I) \), with simple algebra

\[
W = V - R_{VI} \times (V-I),
\]

then

\[
= V_0 + A_V - R_{VI} \times (V-I)_0 + R_{VI} \times E(V-I),
\]
which upon regrouping terms becomes

\[ W = [V_o - R_{VI} \times (V - I)_o] + [A_V - R_{VII} \times E(V - I)]. \]

By construction and definition the last two terms sum to zero, leaving

\[ W = V_o - R_{VII} \times (V - I)_o = W_o. \]

In other words, \( W \), as constructed from observed (reddened) colors and (extincted) magnitudes, is numerically equivalent to \( W_o \), as it would be calculated from intrinsic colors and intrinsic magnitudes. This occurs, it must be noted, without any explicit knowledge of the reddenings and/or extinctions involved. Simply put, \( W \) is reddening independent.

The geometrical interpretation of \( W \) is illuminating. As Figure 1 illustrates, lines of constant \( W \) fall, by construction, identically along reddening trajectories having a slope of \( R_{VII} = A_V / E(V - I) \). Whether a star is at position B because it was reddened from its intrinsic position at A, or whether it is there because that is its intrinsic color and magnitude, is irrelevant: \( W \) collapses the color–magnitude diagram in such a way that the distinction usually made by reddening is moot. All stars along a given reddening trajectory will have the same value of \( W \). Stars with a given value of \( W \) may have any value of reddening.

We now interface \( W \) to Equation (2). First of all

\[ M_W = M_V - R_{VII} \times (V - I)_o, \]

and so

\[ M_W = \alpha \log(P) + (\beta - R_{VII}) \times (V - I)_o + \gamma. \]

Clearly, if \((\beta - R_{VII}) = 0.0\) then \( W \) will be not only reddening-free but it will also be a dispersionless function of period alone, with slope = \( \alpha \). Indeed, it appears observationally that \( \beta \) and \( R_{VII} \) must be quite similar in magnitude given the fact that the dispersion in \( W \) is found to be very small (i.e., on the order of \( \pm 0.08 \) mag).

3. PERIOD–LUMINOSITY RELATIONS

3.1. The Mathematical Representation

Let us take the mean ridge-line relation between magnitude and color (defining the instability strip that is to act as the constraint on the period–luminosity color (PLC)) to be represented by \( M_V(\text{mean}) = S \times (V - I)(\text{mean}) \). It then follows from Equation (2) that

\[ M_V(\text{mean}) = \alpha[S/(S - \beta)] \log(P) - \gamma[S/(S - \beta)], \]

and

\[ W(\text{mean}) = \alpha[(S - R_{VII})/(S - \beta)] \log(P) - \gamma[(S - R_{VII})/(S - \beta)]. \]

Scatter in these mean (PL) relations will be determined by the color width of the instability strip multiplied by (the projection factor) \( \beta \), in the case of monochromatic PL relations, and multiplied by \((\beta - R)\) in the case of the Wesenheit functions. As noted above, if \( \beta \) and \( R \) are of similar magnitude then the scatter in the Wesenheit function will be less than the scatter in the corresponding PL relations by the factor \((\beta - R)/\beta\). Should \( \beta \) equal \( R \) then \( W \) would be dispersionless.

If the instability strip (the constraint imposed upon Equation (2)) moves in a way that changes the slope of the mean PL relation through a change in \( \Delta S \) this will result in a change in the slope of the mean \( W - \log P \) relation, \( \Delta A_W = \alpha(R - \beta)/(S - \beta)^2 \times \Delta S \). For the same tilt in the instability strip \( \Delta S \), the slope in the monochromatic PL relation will change by \( \Delta A = -\alpha\beta/(S - \beta)^2 \times \Delta S \). This means that a 10% change (say) in the slope of the \( V \)-band PL relation mapped to \( W \) would be reduced by a factor of \(-(R - \beta)/\beta\). This is the same factor that results in the decreased scatter observed in \( W \) as compared with the scatter at fixed period in the \( V \)-band PL relation. In the extreme case where \( \beta \) equals \( R \) then \( W \) would not change its slope at all for any change in tilt of the instability strip.

3.2. The Graphical Representation

The central portion of Figure 1 shows a \( V - (V - I) \) color–magnitude diagram with two possible realizations of the Cepheid instability strip cutting across lines of constant period as laid down by the underlying PL–color relation. The strip to the left is significantly more vertical that the strip to the right. Both instability strips share the same zero point in this realization, but they could differ without changing the conclusions of this paper. Fiducial points are marked in both strips by open and filled circles. The filled circles in the more highly inclined strip mark the blue and red borders of the instability strip at two distinct periods. The open circles in the left-hand strip mark the same color boundaries at precisely the same two periods. The downward-sloping dashed lines crossing the entire figure are lines of constant period.

The dashed horizontal lines carry the individual data points in the tilted instability strip over to the right and into its corresponding PL relation. Data from the steep instability strip map to the left PL relation. Clearly, by tilting the instability strip one generates PL relations with different slopes. However, by
projecting the instability strips along lines of constant period, as is done by following the broken lines to the upper left panel, one finds a dispersionless PL relation as defined by \( W_\beta \) which, in addition to having no scatter, is incapable of distinguishing between stars originating either in the vertical or the tilted instability strip.

The solid lines emanating from the filled data points in the tilted instability strip show the trajectories of reddening lines. Their slight divergence from lines of constant period results in a composite \( W-\log P \) relation that has some residual scatter and a slight dependence of its slope on the originating slant of the instability strip. Unlike the gross differences seen in the (left and right) monochromatic PL. relations the divergence in the \( W_\beta \) relations can hardly be resolved and require an exploded view as given in Figure 2 to see the impact.

### 3.3. The Numerical Representation

It has been claimed (Sandage & Tammann 2008, their Table 1) that variations in the slope of the \( V \)-band PL relation are found at the 5% level around a median value of about \(-2.75\) among nearby galaxies (excluding the poorly populated Sextans A/B data point) in which Cepheids have been discovered. If one were to rely exclusively on \( V \)-band PL relations for Cepheid distances that might be cause for concern. But such is not the case. Virtually, all modern studies of the distances to galaxies as gauged by optical observations of the Cepheids use \( W \).

For the sake of the following calculation we assume that \( R_{VI} = 2.45 \) (as in Freedman et al. 2001) and that \( \beta = 3.0 \) (as derived, for example, from the sample of Galactic Cepheids observed by Benedict et al. 2008, illustrated in Figure 3). In this case, the reduction factor \( V \) to \( W \) in the slope change as given above is \(- (R - \beta) / \beta = -(3.0 - 2.45) / 2.45 = -0.3\). That is, a 5% change in slope seen in the \( V \)-band PL relation translates

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**Figure 2.** Expanded view of the \( W_\beta-\log P \) relation as generated by two instability strips of different tilt. The slight offset in slopes between the lines of constant period and reddening lines leads to dispersion in \( W \) and a (very) slight relative tilt of the two relations. Both the dispersion and change in slope collapse to zero linearly with the difference \((\beta - R_V)\).

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**Figure 3.** Galactic Cepheid calibrators from Benedict et al. (2008). Upper panels: \( V \)- and \( I \)-band PL relations. Lower left panel: \( W(VI) \) PL relation. Lower right panel: the correlation of \( V \) and \((V-I)\) residuals from their respective PL and PC relations. The solid line is the trajectory expected for differential reddening; the dashed line is the best-fit regression. The difference between the two is not statistically significant; see W. L. Freedman & B. F. Madore (2009, in preparation) for details.
into a 1.5% change in slope of the mean $W$–log $P$ relation. Had $\beta$ and $R$ been numerically identical then there would have been absolutely no impact of a change in the disposition of the instability strip on the slope of the mean $W$–log $P$ relation. And indeed this would be almost exactly the case if we had adopted the value of $\beta = 2.43$ as advocated by Sandage & Tammann (2008) based on a study of LMC Cepheids. In that case, the 5% slope changes in the optical would reduce to a less that two-tenths of a percent change in the slope in $W$. We do not believe that either $\beta$ or $R$ are currently known to 10% precision themselves, so the exercise must be taken cautiously, being illustrative rather than definitive. The point stands however that the slope of $W$ is clearly going to be relatively impervious to possible changes in the slopes at optical wavelengths.

4. CONCLUSIONS

Not only does the scatter intrinsic to the $W$–log $P$ relation collapse with respect to the optical PL relations, but the slope of the $W$–log $P$ relation is also very insensitive to changes in slope of the originating optical PL relations. Changes in the tilt of the instability strip that naturally explain and will give rise to changes in slopes of the monochromatic PL relations are greatly diminished in their overall impact in the projection of data into the $W$–log $P$ plane. The Wesenheit function is defined to be reddening-free; however, its slope is additionally and coincidently very resistant to changes in the slopes of the originating monochromatic, optical PL relations. The demonstration by Ngeow & Kanbur (2005) that the Wesenheit function for LMC Cepheids does not show any evidence for a break in slope over the full range of observed periods is a predictable and universal feature of $W$. It naturally follows from the closely coincident slopes of the lines of constant period and reddening trajectories.

Since $W$ is already the default method for determining true moduli to nearby galaxies when using Cepheids observed at optical wavelengths, we conclude that slope changes in the V-band PL relations, confirmed or not, will have minimal impact on the Wesenheit function, or the distance scale based upon it. This point has already been made by Ngeow & Kanbur (2006) where they suggest that slope changes may affect determinations of $H_0$ at the 1%–2% level.

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The referee points out that Koen et al. (2007) also find a linear Wesenheit/PLC relation in the LMC, but note that with more data nonlinearities could be measurable.