Space-fractional versions of the negative binomial and Polya-type processes

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Abstract
In this paper, we introduce a space fractional negative binomial (SFNB) process by subordinating the space fractional Poisson process to a gamma process. Its one-dimensional distributions are derived in terms of generalized Wright functions and their governing equations are obtained. It is a Lévy process and the corresponding Lévy measure is given. Extensions to the case of distributed order SFNB process, where the fractional index follows a two-point distribution, is analyzed in detail. Finally, the connections of the SFNB process to a space fractional Polya-type process is also pointed out.

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1 Introduction

The aim of this paper is to introduce a new fractional version of the well-known negative binomial (NB) process. The latter is widely applied in many different fields, mainly for its property of overdispersion (see [7] and the references therein). A first fractional version of the NB process has been introduced by Vellaisamy and Maheshwari in [14] and we refer to it as a time-fractional variant, since it is defined by means of the so-called “time-fractional Poisson process”, as we will see soon after some necessary preliminaries. Other and different fractional versions have been defined in [2] and [4], in terms of the fractional gamma process; see [14] for the differences among these approaches.

Henceforth, \( \mathbb{Z}_+ = \{0, 1, \ldots \} \) denotes the set of nonnegative integers. Let \( X \) be

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a negative binomial r.v. with parameters $\gamma > 0$ and $\eta \in (0, 1)$, with distribution
\[
\mathbb{P}(X = n) = \binom{n + \gamma - 1}{n} \eta^n (1 - \eta)^\gamma, \quad n \in \mathbb{Z}_+,
\]  
(1)
which we denote by $NB(\gamma, \eta)$. Let $\{N(t, \lambda)\}_{t \geq 0}$ be a homogeneous Poisson process with rate $\lambda > 0$, and $\{\Gamma(t)\}_{t \geq 0}$ be an independent gamma process, where $\Gamma(t) \sim G(\alpha, pt)$, the gamma distribution with scale parameter $\alpha^{-1}$ and shape parameter $pt > 0$. Then the process $\{Q(t, \lambda)\}_{t \geq 0} = \{N(\Gamma(t), \lambda)\}_{t \geq 0}$ is called a negative binomial process and $Q(t, \lambda) \sim NB(pt, \eta)$, for $t > 0$, where $\eta = \lambda/(\alpha + \lambda)$ (see [7] and [14]).

Let $\{D_\beta(t)\}_{t \geq 0}$, $0 < \beta < 1$, be a $\beta$-stable subordinator. Then the space-fractional Poisson process (SFPP) defined as $\{N_\beta(t, \lambda)\}_{t \geq 0} = \{N(D_\beta(t), \lambda)\}_{t \geq 0}$ has been studied in detail in [12]. Its one-dimensional distributions are given by
\[
\overline{g}_\beta(n|t, \lambda) = \mathbb{P}[N_\beta(t, \lambda) = n] = \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{(-\lambda^\beta t)^k}{k!} \frac{\Gamma(\beta k + 1)}{\Gamma(\beta k + 1 - n)} , \quad t \geq 0. 
\]  
(2)
It solves the fractional difference-differential equation ([12] eq. (2.4)) defined by
\[
\frac{\partial}{\partial t} \overline{g}_\beta(n|t, \lambda) = -\lambda^\beta (1 - B)^{\beta} \overline{g}_\beta(n|t, \lambda), \quad \beta \in (0, 1],
\]  
(3)
where $B$ is the backward shift operator defined by $Bu(n) = u(n - 1)$ for any function $u : \mathbb{N} \to \mathbb{R}$.

We now introduce here the space fractional negative binomial (SFNB) process in terms of the space-fractional Poisson process as follows:

**Definition 1** Let $\{N_\beta(t, \lambda)\}_{t \geq 0}$ be the SFPP. Then the process defined by
\[
\overline{Q}_\beta(t, \lambda) = N_\beta(\Gamma(t), \lambda),
\]  
(4)
where $\{\Gamma(t)\}_{t \geq 0}$ is an independent gamma process, is called the SFNB process.

Note that we follow an approach similar to the one used in [14], where the time-fractional NB process was defined as $Q_\beta(t, \lambda) = N_\beta(\Gamma(t), \lambda)$, where $\{N_\beta(t, \lambda)\}_{t \geq 0}$ is the time-fractional Poisson process (see [8], [11]).

The one-dimensional distribution of $\overline{Q}_\beta(t, \lambda)$ can be written as
\[
\overline{g}_\beta(n|\alpha, pt, \lambda) := \mathbb{P}[\overline{Q}_\beta(t, \lambda) = n] = \int_{0}^{+\infty} \overline{g}_\beta(n|z, \lambda)g(z|\alpha, pt)dz ,
\]  
(5)
where $\overline{g}_\beta(n|z, \lambda)$ is the probability mass function (pmf) of SFPP defined in [2] and $g(z|\alpha, pt)$ is the density of $G(\alpha, pt)$, defined by
\[
g(z|\alpha, pt) = \frac{\alpha^{pt}}{\Gamma(pt)} z^{pt-1} e^{-\alpha z}, \quad \alpha, p, z, t > 0.
\]
We will derive the explicit expression of the distribution (5), in terms of the generalized Wright function which is defined as
\[ p \Psi_q \left[ \frac{z}{(b_j, \beta_j)_1.q} \right] := \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \]
for \( z, a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j \in \mathbb{R}, i = 1, \ldots, p, j = 1, \ldots, q. \)

Moreover, by applying the results in [3], we will obtain the space-fractional differential equation satisfied by the distribution (5). It will be expressed in terms of the shift operator
\[ e^{cD_x} f(x) := \sum_{n=0}^{\infty} \frac{e^{c n D_x^n}}{n!} f(x) = f(x + c), \]
where \( D_x = d/dx \), which is defined for any analytic function \( f : \mathbb{R} \to \mathbb{R} \) and \( c \in \mathbb{R}. \)

In the last section, we present two different generalizations of the previous results: the first one is obtained by taking the fractional index \( \beta \) random instead of constant in \((0,1)\), as before. In particular, we will consider the case of a two-valued discrete random variable. The second extension is obtained by considering the following subordination \( W_\beta(t, u) = N_\beta(u, \Gamma^d(t)) \), where \( \{N_\beta(\cdot, \Gamma^d(t))\}_{t \geq 0} \) is the space-fractional Poisson process with \( \lambda \) replaced by the process \( \Gamma^d(t) \) and \( d \) is a positive constant. We call \( \{W_\beta(t)\}_{t \geq 0} \) space-fractional “Polya-type” process, in analogy to [14]. For all the above mentioned processes, we give the one-dimensional distributions and the corresponding governing equations.

First, we present some preliminary definitions and results that will be used later. We start with the following result on the differential equation satisfied by the density of the gamma process proved in [3].

**Lemma 2** The one-dimensional density \( g(x|\alpha, pt) \) of the gamma process \( \{\Gamma(t)\}_{t \geq 0} \) satisfies the following Cauchy problem, for \( x, t \geq 0 \),
\[
\begin{cases}
\frac{\partial}{\partial t} g(x|\alpha, pt) = -\alpha(1 - e^{-\frac{c}{\alpha}})g(x|\alpha, pt) \\
g(x|\alpha, 0) = \delta(x) \\
\lim_{|x| \to +\infty} g(x|\alpha, pt) = 0,
\end{cases}
\]
where \( e^{-\frac{c}{\alpha}} \) is the partial derivative version of the shift operator defined in (6), for \( c = -1/p \), and \( \delta(x) \) is the Dirac delta function.
An alternative differential equation satisfied by the one-dimensional distributions of the SFNB process is given in terms of fractional derivatives and thus we recall the following.

**Definition 3 (Riemann-Liouville fractional derivative)** Let \( m \in \mathbb{Z}_+ \setminus \{0\} \) and \( \nu \geq 0 \). If \( f(t) \in AC^{m-1}[0,T], 0 \leq t \leq T \), then the (left-hand) Riemann-Liouville (R-L) fractional derivative \( \partial_t^\nu f(t) \) is defined by

\[
\partial_t^\nu f(t) := \begin{cases} 
\frac{1}{\Gamma(m - \nu)} \frac{d^m}{dt^m} \int_0^t \frac{f(s)}{(t-s)^{\nu+1}} ds, & m - 1 < \nu < m, \\
\frac{d^m}{dt^m} f(t), & \nu = m,
\end{cases}
\]

(8)

where \( AC^n[0,T] \) denotes the space of absolutely continuous functions whose \((n-1)\)-th derivatives are also continuous on \([0,T]\).

We need also the following result which is Lemma 4.2 of [14].

**Lemma 4** The density of the gamma process \( \Gamma(t) \sim G(\alpha, pt) \) satisfies the following fractional differential equation, for any \( \nu \geq 0 \):

\[
\partial_t^\nu g(y|\alpha, pt) = p\partial_t^{\nu-1}(\log(\alpha y) - \psi(pt))g(y|\alpha, pt), \quad y > 0, \quad \nu \geq 0,
\]

(9)

\[
g(y|\alpha, 0) = 0,
\]

(10)

where \( \psi(x) := \Gamma'(x)/\Gamma(x) \) is the digamma function and \( \partial_t^\nu (\cdot) \) is R-L derivative defined in (8).

**2 Space fractional negative binomial process**

We present now our first result on the SFNB process defined in Definition 1.

**Theorem 5** The one-dimensional distribution of the process \( \{\bar{Q}_\beta(t, \lambda)\}_{t \geq 0} \) is given by

\[
\delta_{\beta}(n|\alpha, pt, \lambda) = \frac{(-1)^n}{n!} \frac{1}{\Gamma(pt)} 2^{\Psi_1} \left[ -\frac{\lambda^\beta}{\alpha} \frac{(1,\beta)(pt,1)}{(1-n,\beta)} \right], \quad n \in \mathbb{Z}^+, \ t \geq 0
\]

(11)

and satisfies the following space-fractional equation

\[
(e^{-\frac{\lambda}{\alpha} B_t} - 1)\delta_{\beta}(n|\alpha, pt, \lambda) = \frac{\lambda^\beta}{\alpha} (I - B)^\beta \delta_{\beta}(n|\alpha, pt, \lambda),
\]

(12)

with initial condition \( \delta_{\beta}(n|\alpha, 0, \lambda) = 1_{\{n=0\}} \) and \( B \) is the backward operator.
Proof. Formula (11) can be derived by applying (5) together with equation (2):

\[ \delta_\beta(n|\alpha, pt, \lambda) = \frac{(-1)^n}{n!} \frac{\alpha^pt}{\Gamma(pt)} \sum_{r=0}^{\infty} \frac{(-\lambda^\beta)^r}{r!} \frac{\Gamma(\beta r + 1)}{\Gamma(\beta r + 1 - k)} \int_0^{+\infty} z^{r+pt-1} e^{-\alpha z} dz \]

\[ = \frac{(-1)^n}{n!} \frac{1}{\Gamma(pt)} \sum_{r=0}^{\infty} \frac{(-\lambda^\beta/\alpha)^r}{r!} \frac{\Gamma(\beta r + 1)\Gamma(r + pt)}{\Gamma(\beta r + 1 - n)}. \]

The series converges for \(|\lambda^\beta/\alpha| < 1\), by Theorem 1.5, p.56 in [6] with \(\Delta = \beta - (\beta + 1) = -1, \delta = |\beta|\beta = 1\).

In order to prove equation (12) we apply formula (2), as follows

\[ e^{-\frac{1}{\lambda}\beta} \delta_\beta(n|\alpha, pt, \lambda) = \int_0^{+\infty} \frac{\beta_\beta(n|z, \lambda)e^{-\frac{1}{\lambda}\beta}}{\beta_\beta(n|z, \lambda)} g(z|\alpha, pt) dz \]

\[ = \frac{\beta_\beta(n|\alpha, pt, \lambda)}{1} + \frac{1}{\alpha} \int_0^{+\infty} \frac{\beta_\beta(n|z, \lambda)}{\partial z} g(z|\alpha, pt) dz \]

\[ = \frac{\beta_\beta(n|\alpha, pt, \lambda)}{1} - \frac{1}{\alpha} \int_0^{+\infty} \frac{\partial}{\partial z} \frac{\beta_\beta(n|z, \lambda)}{\lambda} g(z|\alpha, pt) dz \]

\[ = \frac{\beta_\beta(n|\alpha, pt, \lambda)}{1} + \frac{\lambda^\beta}{\alpha} (I - B)^\beta \int_0^{+\infty} \frac{\beta_\beta(n|z, \lambda)}{\lambda} g(z|\alpha, pt) dz, \]

using (3). Note that integration by parts is used to obtain the third step above. This proves the result.

Remark 6
(i) Note that for \(\beta = 1\) equation (12) reduces to formula (69) in [6].

(ii) Formula (11) represents a proper distribution as can be ascertained by the following steps:

\[ \sum_{n=0}^{\infty} \delta_\beta(n|\alpha, pt, \lambda) = \frac{1}{\Gamma(pt)} \sum_{j=0}^{\infty} \frac{(-\lambda^\beta)^j}{j!\alpha^j} \Gamma(j + pt) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\beta j + 1)}{\Gamma(\beta j + 1 - n)} \]

\[ = \frac{1}{\Gamma(pt)} \sum_{j=0}^{\infty} \frac{(-\lambda^\beta)^j}{j!\alpha^j} \Gamma(j + pt)(1 - \lambda)^{\beta j} = 1 \]

Also, when \(\beta = 1\), we can check that (11) reduces to the distribution of the negative binomial distribution. Indeed, we can write

\[ \delta_1(n|\alpha, pt, \lambda) = \frac{(-1)^n}{n!} \frac{1}{\Gamma(pt)} \sum_{j=n}^{\infty} \frac{(-\lambda)^j}{(j-n)!} \Gamma(j + pt) \]

\[ = \frac{\Gamma(n + pt)}{n!\Gamma(pt)} \left( \frac{\lambda}{\alpha} \right)^n \sum_{l=0}^{\infty} \left( -\frac{\lambda}{\alpha} \right)^l \frac{(n + l + pt - 1)}{l} \]
\[
\left( \frac{\lambda}{\alpha + \lambda} \right)^n \left( \frac{\alpha}{\alpha + \lambda} \right)^{pt} \left( \frac{n + pt - 1}{n} \right),
\]

which is the distribution of \( Q(t, \lambda) \sim NB(pt, \lambda/\alpha + \lambda) \).

An alternative fractional pde satisfied by the distribution \( 11 \) can be obtained by applying Lemma 4.

**Theorem 7** The distribution of the SFNB process \( \bar{\delta}_\beta (n|\alpha, pt, \lambda) \) satisfies the following fractional pde:

\[
\frac{1}{p} \partial_t^\gamma \bar{\delta}_\beta (n|\alpha, pt, \lambda) = \partial_t^{-1} [\log (\alpha - \psi (pt)) \bar{\delta}_\beta (n|\alpha, pt, \lambda)] + \int_0^{\infty} (\log y) \bar{\delta}_\beta (n|y, \lambda) \partial_t^{-1} g (y|\alpha, pt) dy
\]

with \( \gamma \geq 0 \) and \( \bar{\delta}_\beta (n|\alpha, 0, \lambda) = 1_{\{n=0\}} \).

**Proof.** By considering (9)-(10) we get

\[
\partial_t^\gamma \bar{\delta}_\beta (n|\alpha, pt, \lambda) = \partial_t^\gamma \int_0^{\infty} \bar{p}_\beta (n|y, \lambda) g (y|\alpha, pt) dy
\]

which proves the result. \( \blacksquare \)

The SFNB process \( 11 \) is defined by time-changing the process \( \{N_\beta (t, \lambda)\}_{t \geq 0} \) by means of the gamma subordinator, but \( \{\bar{N}_\beta (t, \lambda)\}_{t \geq 0} \) is itself a subordinator and the Laplace transform of its probability mass function can be derived from (2.12) of \( 12 \), i.e.

\[
\sum_{n=0}^{\infty} e^{-un} \bar{p}_\beta (n|z, \lambda) = e^{-\lambda^\beta z (1-e^{-u})^\beta}.
\]

As a consequence, \( \{\bar{Q}_\beta (t, \lambda)\}_{t \geq 0} \) is a subordinator with Laplace transform of the distribution equal to

\[
\mathbb{E}e^{-u\bar{Q}_\beta (t, \lambda)} = \sum_{n=0}^{\infty} e^{-un} \bar{\sigma}_\beta (n|\alpha, pt, \lambda) = \int_0^{+\infty} e^{-\lambda^\beta z (1-e^{-u})^\beta} g (z|\alpha, pt) dz
\]

\[
= \exp \left\{ -pt \ln \left( 1 + \frac{\lambda^\beta (1-e^{-u})^\beta}{\alpha} \right) \right\}.
\]
Thus the Laplace exponent of $\{Q_{\beta}(t, \lambda)\}_{t \geq 0}$ is given by

$$
\psi(u) := -\frac{1}{t} \ln \left( E e^{-uQ_{\beta}(t, \lambda)} \right) = p \ln \left( 1 + \frac{\lambda^\beta (1 - e^{-u})^\beta}{\alpha} \right)
$$

and the discrete Lévy measure can be derived as follows.

**Theorem 8** The discrete Lévy measure of the SFNB process is given by

$$
\nu_{\beta}(\cdot) = p \sum_{k=1}^{\infty} (-1)^k \delta_{\{k\}}(\cdot) \sum_{r=0}^{\infty} \frac{(-\lambda^\beta/\alpha)^r}{r} \frac{\beta r}{k},
$$

where $\delta_{\{k\}}(\cdot)$ is the Dirac measure concentrated at $k$.

**Proof.** Since the SFNB process is a subordinated process (see (4)), we can apply Theorem 30.1, p. 197 in [13], which gives a calculation rule for the Lévy measure in the case of subordinated Lévy processes. Let $\mu_{\Gamma}$ be the Lévy measure of the gamma process $\{\Gamma(t)\}_{t \geq 0}$ (i.e. $\mu_{\Gamma}(s) = ps^{-1}e^{-\alpha s}$), thus we get

$$
\nu_{\beta}(\cdot) = \int_0^{+\infty} \sum_{k=1}^{\infty} \tau_{\beta}(k) s, \lambda) \delta_{\{k\}}(\cdot) \mu_{\Gamma}(s) ds \quad \text{(using (2))}
$$

$$
= p \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \delta_{\{k\}}(\cdot) \sum_{r=0}^{\infty} \frac{(-\lambda^\beta/\alpha)^r}{r} \frac{\Gamma(\beta r + 1)}{\Gamma(\beta r + 1 - k)} \int_0^{+\infty} s^{r-1} e^{-\alpha s} ds
$$

$$
= p \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \delta_{\{k\}}(\cdot) \sum_{r=1}^{\infty} \frac{(-\lambda^\beta/\alpha)^r}{r} \frac{\Gamma(\beta r + 1)}{\Gamma(\beta r + 1 - k)}.
$$

Note that in the last step we have used the fact that the inner sum is zero for $r = 0$, since $\Gamma(1 - k) = \infty$ for any $k \geq 1$.

It is easy to check that, in the special case $\beta = 1$, formula (13) coincides with the well-known Lévy measure of the NB process:

$$
\nu_{1}(\cdot) = p \sum_{k=1}^{\infty} (-1)^k \delta_{\{k\}}(\cdot) \sum_{r=0}^{\infty} \frac{(-\lambda/\alpha)^r}{r} \binom{r}{k}
$$

$$
= p \sum_{k=1}^{\infty} (-1)^k \delta_{\{k\}}(\cdot) \sum_{l=0}^{\infty} \frac{(-\lambda/\alpha)^{l+k}}{l+k} \frac{l+k}{k}
$$

$$
= p \sum_{k=1}^{\infty} \frac{(\lambda/\alpha)^k}{k} \delta_{\{k\}}(\cdot) \sum_{l=0}^{\infty} (-\lambda/\alpha)^{l+k-1} \binom{l+k-1}{l}
$$

$$
= p \sum_{k=1}^{\infty} \frac{\delta_{\{k\}}(\cdot) \left( \frac{\lambda}{\alpha} \right)^k}{k}. \quad \text{(13)}
$$
3 Generalizations of the previous results

3.1 SFNB process of distributed order

A generalization of the results presented in section 1 can be obtained by considering the case where the fractional index $\beta$ in the equation (12) satisfied by the distribution of the process (4) is itself random. Distributed-order fractional equations have been already treated in the literature, in the case of superdiffusions and relaxation equations; see, for example, [5], [9], [10], [1]. Here, we will assume the random index $\beta$ follows two-valued discrete distribution, namely,

$$f(\beta) = a_1 \delta(\beta - \beta_1) + a_2 \delta(\beta - \beta_2), \quad \beta_1, \beta_2 \in (0, 1),$$

where $\delta(\cdot)$ is the Dirac delta function and $a_1, a_2 \geq 0$ with $a_1 + a_2 = 1$. Thus, and also in view of (12), we are interested in the solution of the following equation

$$(e^{-\frac{\delta\beta_t}{\alpha}} - 1)\overline{\delta}_{\beta_1, \beta_2}(n|\alpha, pt, \lambda)f(\beta)d\beta = \left(\int_0^1 \frac{\lambda^\beta}{\alpha}(I - B)^\beta f(\beta)d\beta\right)\overline{\delta}_{\beta_1, \beta_2}(n|\alpha, pt, \lambda)$$

(15)

(with initial condition $\overline{\delta}_{\beta}(n|\alpha, 0, \lambda) = 1_{\{n=0\}}$), which, under the assumption (13), can be rewritten as follows

$$(e^{-\frac{\delta\beta_t}{\alpha}} - 1)\overline{\delta}_{\beta_1, \beta_2}(n|\alpha, pt, \lambda) = \left[\frac{a_1}{\alpha}(I - B)^{\beta_1} + \frac{a_2}{\alpha}(I - B)^{\beta_2}\right]\overline{\delta}_{\beta_1, \beta_2}(n|\alpha, pt, \lambda).$$

(16)

**Theorem 9** The solution to equation (16) is given by

$$\overline{\delta}_{\beta_1, \beta_2}(n|\alpha, pt, \lambda) = \frac{(-1)^n}{n!} \frac{1}{\Gamma(pt)} \sum_{r=0}^{\infty} \frac{(-a_1 \lambda^\beta / \alpha)^r}{r!} 2\overline{\Psi}_1 \left[-\frac{a_2 \lambda^\beta}{\alpha}, 0; \frac{1}{1-n + \beta_1 + \beta_2} (pt+r, 1)\right],$$

(17)

for $n \in \mathbb{Z}^+$, $t \geq 0$.

**Proof.** We start by deriving from (16) the equation satisfied by the probability generating function $G_{\beta_1, \beta_2}(u|\alpha, pt, \lambda) := \sum_{n=0}^{\infty} u^n \overline{\delta}_{\beta_1, \beta_2}(n|\alpha, pt, \lambda)$:

$$(e^{-\frac{\delta\beta_t}{\alpha}} - 1)G_{\beta_1, \beta_2}(u|\alpha, pt, \lambda) = \sum_{n=0}^{\infty} u^n \left[\frac{a_1}{\alpha}(I - B)^{\beta_1} + \frac{a_2}{\alpha}(I - B)^{\beta_2}\right] \overline{\delta}_{\beta_1, \beta_2}(n|\alpha, pt, \lambda)$$

(18)

$$= \sum_{n=0}^{\infty} u^n \left[\frac{a_1}{\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\beta_1}{j} \overline{\delta}_{\beta_1, \beta_2}(n-j|\alpha, pt, \lambda) + \frac{a_2}{\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\beta_2}{j} \overline{\delta}_{\beta_1, \beta_2}(n-j|\alpha, pt, \lambda)\right]$$

$$= \sum_{j=0}^{\infty} (-u)^j \binom{\beta_1}{j} \sum_{n=j}^{\infty} u^n \overline{\delta}_{\beta_1, \beta_2}(n-j|\alpha, pt, \lambda).$$
Thus, we get the following function

\[ G_{\beta_1,\beta_2}(u|\alpha, pt, \lambda) = \left[ 1 + \frac{a_1 \lambda^{\beta_1}}{\alpha} (1-u)^{\beta_1} + \frac{a_2 \lambda^{\beta_2}}{\alpha} (1-u)^{\beta_2} \right]^{-pt} \]  

Indeed it is

\[ e^{-\frac{1}{\beta}t} \left[ 1 + \frac{a_1 \lambda^{\beta_1}}{\alpha} (1-u)^{\beta_1} + \frac{a_2 \lambda^{\beta_2}}{\alpha} (1-u)^{\beta_2} \right]^{-pt} \]

\[ = G_{\beta_1,\beta_2}(u|\alpha, pt, \lambda) + \left[ \frac{a_1 \lambda^{\beta_1}}{\alpha} (1-u)^{\beta_1} + \frac{a_2 \lambda^{\beta_2}}{\alpha} (1-u)^{\beta_2} \right] G_{\beta_1,\beta_2}(u|\alpha, pt, \lambda). \]

Now we only need to prove that the coefficients in the series expansion of (19) coincide with (17): for \( u \) such that \( \frac{a_1 \lambda^{\beta_1}}{\alpha} (1-u)^{\beta_1} + \frac{a_2 \lambda^{\beta_2}}{\alpha} (1-u)^{\beta_2} < 1 \), we can write

\[ G_{\beta_1,\beta_2}(u|\alpha, pt, \lambda) = \sum_{l=0}^{\infty} \binom{pt+l-1}{l} (-1)^l \left[ \frac{a_1 \lambda^{\beta_1}}{\alpha} (1-u)^{\beta_1} + \frac{a_2 \lambda^{\beta_2}}{\alpha} (1-u)^{\beta_2} \right]^l \]

\[ = \sum_{l=0}^{\infty} \binom{pt+l-1}{l} \left( -\frac{1}{\alpha} \right)^l \sum_{r=0}^{l} \binom{l}{r} a_1^{\lambda^{\beta_1}r} a_2^{\lambda^{\beta_2}r (1-u)^{\beta_2 (l-r)}} \]

\[ = \sum_{l=0}^{\infty} \binom{pt+l-1}{l} \left( -\frac{1}{\alpha} \right)^l \sum_{r=0}^{l} \binom{l}{r} a_1^{\lambda^{\beta_1}r} a_2^{\lambda^{\beta_2}r (1-u)^{\beta_2 (l-r)}} \sum_{n=0}^{\infty} \binom{\beta_1 r + \beta_2 (l-r)}{n} (-u)^n. \]

Thus, we get

\[ \delta_{\beta_1,\beta_2}(n|\alpha, pt, \lambda) = \frac{(-1)^n}{n!} \frac{1}{\Gamma(pt)} \sum_{l=0}^{\infty} \Gamma(pt+l) \left( -\frac{1}{\alpha} \right)^l \sum_{r=0}^{l} \binom{l}{r} \frac{a_1^{\lambda^{\beta_1}r} a_2^{\lambda^{\beta_2}r (1-u)^{\beta_2 (l-r)}}}{r! (l-r)!} \frac{\Gamma(\beta_1 r + \beta_2 (l-r) + 1)}{\Gamma(\beta_1 r + \beta_2 (l-r) + 1 - n)} \]

\[ = \frac{(-1)^n}{n!} \frac{1}{\Gamma(pt)} \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\Gamma(pt+l) (-1)^l a_1^{\lambda^{\beta_1}r} a_2^{\lambda^{\beta_2}r \Gamma(\beta_1 r + \beta_2 (l-r) + 1)}}{\alpha^l (l-r)!} \frac{\Gamma(\beta_1 r + \beta_2 (l-r) + 1)}{\Gamma(\beta_1 r + \beta_2 (l-r) + 1 - n)} \]

(20)
where the inner series converges for $|a_1 \lambda^{\beta_1}/\alpha| < 1$, since
$$\Delta = \beta_1 - 1 - \beta_1 = -1$$
and $\delta = \beta_1^{-\beta_1} = 1$. In order to prove that (17) represents a proper distribution, we evaluate the sum of (20) over all $n \in \mathbb{Z}^+$:

$$
\sum_{n=0}^{\infty} \mathcal{B}_{\beta_1, \beta_2}(n; \alpha, pt, \lambda)
= \frac{1}{\Gamma(pt)} \sum_{l=0}^{\infty} \frac{(-a_2 \lambda^{\beta_2})^l \alpha^l}{(p + l + r) \Gamma(pt + l + r)} \sum_{r=0}^{\infty} \frac{(-a_1 \lambda^{\beta_1})^r \alpha^r}{(p + l + r) \Gamma(pt + l + r)} (\frac{\beta_1 + \beta_2}{n})
= \frac{1}{\Gamma(pt)} \sum_{l=0}^{\infty} \frac{(-a_2 \lambda^{\beta_2})^l \alpha^l}{\Gamma(pt + l + r)} \Gamma(pt + l + r) (1 + 1)^\beta_1 + \beta_2 = \frac{\Gamma(pt)}{\Gamma(pt)} = 1.
$$

It is easy to check that, for $a_1 = 1, a_2 = 0$ and $\beta_1 = \beta$, formula (17) reduces to (11).

### 3.2 Connections to a space-fractional “Polya-type” process

Another generalization of the SFNB process can be obtained by considering the following process

$$\overline{W}_\beta(t, u) = \overline{N}_\beta(u, \Gamma^d(t)). \quad (21)$$

Here $\{\overline{N}_\beta(t, \Gamma^d(t))\}_{t \geq 0}$ is the space-fractional Poisson process with random parameter represented by the process $\Gamma^d(t)$, where $d$ is a positive constant and $\{\Gamma(t)\}_{t \geq 0}$ is an independent gamma process with parameters $\frac{1}{\alpha}, pt$. The SFNB process can be recovered form (21) for $d = 1/\beta$ and $u = 1/\alpha$. Its distribution can be written as follows:

$$
\mathbb{P}\{\overline{W}_\beta(t, u) = n\} = \overline{\tau}_\beta(n; \alpha, pt, d)
= \int_0^{\infty} \mathbb{P}\{\overline{N}_\beta(u, \alpha^d) = n\} \frac{1}{\lambda^{pt} \Gamma(pt)} y^{pt-1} e^{-\frac{1}{\lambda^d} dy}
= \frac{1}{\lambda^{pt} \Gamma(pt)} \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{(-u)^k}{k!} \frac{\Gamma(\beta k + 1)}{\Gamma(\beta k + 1 - n)} \int_0^{\infty} y^{pt + \beta dk - 1} e^{-\frac{1}{\lambda^d} dy}
= \frac{1}{\Gamma(pt)} \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{(-u \lambda^{\beta d})^k \Gamma(\beta k + 1) \Gamma(pt + \beta dk)}{\Gamma(\beta k + 1 - n)}
$$

\[10\]
When $d = 1/\beta$ and $u = 1/\alpha$, we get

\[
\mathbb{P}\left\{ W_\beta(t, 1/\alpha) = n \right\} = \frac{(-1)^n}{n!} \frac{1}{\Gamma(pt)} \frac{\lambda^{n-1}}{(1-n,\beta)} \Gamma\left[\frac{\lambda^{(1,\beta)(pt,1)}}{(1-n,\beta)}\right].
\]

which is the distribution of the SFNB process.

The one-dimensional distribution of the space-fractional Polya-type process $\tilde{\eta}_\beta(n|u, \lambda, pt, d)$ satisfies the following pde:

\[
\frac{1}{\beta} \partial_\nu^\nu \tilde{\eta}_\beta(n|u, \lambda, pt, d) = \partial_\nu^{n-1} \left( \log(\alpha/\psi(pt)) \tilde{\eta}_\beta(n|u, \lambda, pt, d) \right) + \int_0^\infty \left( \log(y) \tilde{\eta}_\beta(n|u, y^d) \partial_\nu^{n-1} g(y|\alpha, pt) dy, \right.
\]

which follows using

\[
\tilde{\eta}_\beta(n|u, \lambda, pt, d) = \int_0^\infty P_\beta(n|u, y^d) g(y|\alpha, t) dy
\]

and the fractional pde satisfied by $g(y|\alpha, t)$ given in Lemma 4.

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