Review

A review of $\mu$-$\tau$ flavor symmetry in neutrino physics

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Abstract

Behind the observed pattern of lepton flavor mixing is a partial or approximate $\mu$-$\tau$ flavor symmetry—a milestone on our road to the true origin of neutrino masses and flavor structures. In this review article we first describe the features of $\mu$-$\tau$ permutation and reflection symmetries, and then explore their various consequences on model building and neutrino phenomenology. We pay particular attention to soft $\mu$-$\tau$ symmetry breaking, which is crucial for our deeper understanding of the fine effects of flavor mixing and CP violation.

Keywords: flavor mixing, $\mu$-$\tau$ symmetry, neutrino mass and oscillation, particle physics

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. A brief history of the neutrino families

Soon after the French physicist Henri Becquerel first discovered the radioactivity of uranium in 1896 [1], some nuclear physicists began to focus their attention on the beta decays $(A,Z) \rightarrow (A,Z+1) + e^-$, where the energy spectrum of the outgoing electrons was expected to be discrete as constrained by the energy and momentum conservations. Surprisingly, the British physicist James Chadwick observed a continuous electron energy spectrum of the beta decay in 1914 [2], and such a result was firmly established in the 1920s [3]. At that time there were two typical points of view towards resolving this discrepancy between the observed and expected energy spectra of electrons: (a) the Danish theorist Niels Bohr intended to give up the energy conservation law, and later his idea turned out to be wrong; (b) the Austrian theorist Wolfgang Pauli preferred to add in a new particle, which marked the birth of a new science. In 1930 Pauli pointed out that a light, spin-1/2 and neutral particle — known as the electron antineutrino today— appeared in the beta decay and carried away some energy and momentum in an invisible way, and thus the energy spectrum of electrons in the process $(A,Z) \rightarrow (A,Z+1) + e^- + \bar{\nu}_e$ was continuous. Three years later the Italian theorist Enrico Fermi took this hypothesis seriously and developed an effective field theory of the beta decay [4], which made it possible to calculate the reaction rates of nucleons and electrons or positrons interacting with neutrinos or antineutrinos.

In 1936 the German physicist Hans Bethe pointed out that an inverse beta decay mode of the form $\bar{\nu}_e + (A,Z) \rightarrow (A,Z-1) + e^+$ should be a feasible way to verify the existence of electron antineutrinos produced from fission bombs or reactors [5]. This bright idea was elaborated by the Italian theorist Bruno Pontecorvo in 1946 [6], and it became more realistic with the development of the liquid scintillation counting techniques in the 1950s. For example, the invisible incident $\bar{\nu}_e$ triggers the reaction $\bar{\nu}_e + p \rightarrow n + e^+$, in which the emitted positron annihilates with an electron and the daughter nucleus is captured in the detector. Both events can be
observed since they emit gamma rays, and the corresponding flashes in the liquid scintillator are separated by some microseconds. The American experimentalists Frederick Reines and Clyde Cowan did the first reactor antineutrino experiment and confirmed Pauli’s hypothesis in 1956 [7]. Such a discovery motivated Pontecorvo to speculate on the possibility of lepton number violation and neutrino-antineutrino transitions in 1957 [8]. His viewpoint was based on a striking conjecture made by Fermi’s doctoral student Ettore Majorana in 1937: a massive neutrino could be its own antiparticle [9]. Whether Majorana’s hypothesis is true or not remains an open question in particle physics.

In 1962 the muon neutrino—a puzzling sister of the electron neutrino—was first observed by the American experimentalists Leon Lederman, Melvin Schwartz and Jack Steinberger in a new accelerator-based experiment [10]. Their discovery more or less motivated the Japanese theorists Ziro Maki, Masami Nakagawa and Shoichi Sakata to think about lepton flavor mixing and $\nu_e \leftrightarrow \nu_\mu$ transitions [11]. The tau neutrino, another sister of the electron neutrino, was finally observed at the Fermilab in 2001 [12]. Today we are left with three lepton families consisting of the charged members ($e$, $\mu$, $\tau$) and the neutral members ($\nu_e$, $\nu_\mu$, $\nu_\tau$), together with their antiparticles. Table 1 shows the total lepton number ($L$) and individual flavor numbers ($L_e$, $L_\mu$, $L_\tau$) assigned to all the known leptons and antileptons in the standard theory of electroweak interactions. So far the nonconservation of $L_e$, $L_\mu$ and $L_\tau$ has been observed in a number of neutrino oscillation experiments.

The standard electroweak theory about charged leptons and neutrinos was first formulated by the American theorist Steven Weinberg in 1967 [13]. In this seminal paper the neutrinos were assumed to be massless, and hence there should be no lepton flavor conversion. Just one year later, a preliminary experimental evidence for finite neutrino masses and lepton flavor mixing appeared thanks to the first observation of solar neutrinos and their deficit as compared with the prediction of the standard solar model [14]. The point was that the observed deficit of solar electron neutrinos could easily be attributed to $\nu_e \rightarrow \nu_\mu$ and $\nu_e \rightarrow \nu_\tau$ oscillations [15]—a pure quantum phenomenon which would not take place if every neutrino were massless and the lepton flavor were conserved. Hitherto the flavor oscillations of solar, atmospheric, accelerator and reactor neutrinos (or antineutrinos) have all been established [16], and thus a nontrivial extension of the standard theory of electroweak interactions is unavoidable in order to explain the origin of nonzero but tiny neutrino masses and the dynamics of significant lepton flavor mixing.

At present the most popular and intriguing idea for neutrino mass generation is the seesaw mechanism, which attributes the tiny masses of three neutrinos to the existence of a few heavy degrees of freedom and lepton number violation. There are three typical seesaw mechanisms on the market:

- Type-II seesaw—one heavy Higgs triplet is added into the standard theory and the lepton number is violated by its interactions with both the lepton doublet and the Higgs doublet [22–26];
- Type-III seesaw—three heavy triplet fermions are introduced into the standard theory and the lepton number is violated by their Majorana mass term [27, 28].

After the heavy degrees of freedom are integrated out, all the three seesaw mechanisms are convergent to a unique effective Majorana neutrino mass operator [29]. Although a given seesaw scenario can qualitatively explain why the neutrinos may have tiny masses, it is not powerful enough to determine the flavor structures of charged leptons and neutrinos. Hence a combination of the seesaw idea and possible flavor symmetries is desirable so as to achieve some testable quantitative predictions in the lepton sector.

In comparison with three lepton families, there exist three quark families consisting of the up-type quarks ($u$, $c$, $t$) and the down-type quarks ($d$, $s$, $b$), together with their antiparticles. All these leptons and quarks constitute the flavor part of particle physics, and their mass spectra, flavor mixing properties and CP violation are the central issues of flavor dynamics. In this review article we shall focus on the lepton flavors, especially the $\mu$–$\tau$ flavor symmetry in the neutrino sector and its striking impacts on the phenomenology of neutrino physics.

### 1.2. The $\mu$–$\tau$ Flavor Symmetry Stands Out

Since Noether’s theorem was first published in 1918 [30], symmetries have been playing a very crucial role in understanding the fundamental laws of Nature. In fact, symmetries are so powerful that they can help simplify the complicated problems, classify the intricate systems, pin down the conservation laws and even determine the dynamics of interactions. In elementary particle physics there are many successful examples of this kind, such as the continuous space-time translation symmetries, the $SU(3)_c$ quark flavor symmetry and the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge symmetries. Historically these examples led us to the momentum and energy
conservation laws, the quark model and the standard model (SM) of electroweak and strong interactions, respectively [16]. The original symmetries in a given theory may either keep exact or be broken, so as to make our description of the relevant phenomena consistent with the experimental observations. For instance, the electromagnetic $U(1)_{em}$ gauge symmetry, the strong $SU(3)_c$ gauge symmetry and the continuous space-time translation symmetries are all exact; but the $SU(3)_q$ quark flavor symmetry, the electroweak $SU(2)_L \times U(1)_Y$ gauge symmetry and the P and CP symmetries in weak interactions must be broken. That is why exploring new symmetries and studying possible symmetry-breaking effects have been one of the main streams in particle physics, normally from lower energies to higher energies.

Although the SM has proved to be very successful in describing the phenomena of strong, weak and electromagnetic interactions, it by no means a complete theory. The flavor part of this theory is particularly unsatisfactory, because it involves many free parameters but still cannot provide any solution to a number of burning problems, such as the origin of neutrino masses and lepton flavor mixing, the baryon number asymmetry of the Universe and the nature of cold or warm dark matter. To go beyond the SM in this connection, flavor symmetries are expected to serve for a powerful guideline.

Flavor symmetries can be either Abelian or non-Abelian, either local or global, either continuous or discrete, and either spontaneously broken or explicitly broken. All these possibilities have been extensively explored in the past few decades, so as to explain the observed lepton and quark mass spectra and the observed flavor mixing patterns [31–34]. Given the peculiar pattern of lepton flavor mixing which is suggestive of a constant unitary matrix with some special entities (e.g. $1/\sqrt{2}$, $1/\sqrt{3}$ or $1/\sqrt{6}$), a lot of attention has been paid to the global and discrete flavor symmetry groups in the model building exercises. The advantages of such a choice are obvious at least in the following aspects: (a) the model does not involve any Goldstone bosons or additional gauge bosons which may mediate harmful flavor-changing-neutral-current processes; (b) the discrete group may come from some string compactifications or be embedded in a continuous symmetry group; (c) the model contains no family-dependent D-terms contributing to the sfermion masses if it is built in a supersymmetric framework. Although many discrete flavor symmetry groups have been taken into account in building viable neutrino mass models, it remains unclear which one can finally stand out as the unique basis of the true flavor dynamics of leptons and quarks.

But it turns out to be clear that any promising discrete flavor symmetry group in the neutrino sector has to accommodate the simplest $\mu$–$\tau$ flavor symmetry—a sort of $Z_2$ transformation symmetry with respect to the $\nu_{\mu}$ and $\nu_{\tau}$ neutrinos, because the latter has convincingly revealed itself through the currently available neutrino oscillation data (as one can see in the next section). In other words, a partial or approximate $\mu$–$\tau$ flavor symmetry must be behind the observed pattern of the 3 × 3 Pontecorvo–Maki–Nakagawa–Sakata (PMNS) lepton flavor mixing matrix $U$ [35], and thus it may serve as an especially useful low-energy window to look into the underlying structures of lepton flavors at either the electroweak scale or superhigh-energy scales. It is just this observation that motivates us to review what we have learnt from

- **the $\mu$–$\tau$ permutation symmetry**—the neutrino mass term is unchanged under the transformations

$$\nu_\mu \rightarrow \nu_\tau, \quad \nu_\tau \rightarrow \nu_\mu.$$ (1.1)

- **the $\mu$–$\tau$ reflection symmetry**—the neutrino mass term keeps unchanged under the transformations

$$\nu_\mu \rightarrow \nu_\mu^c, \quad \nu_\tau \rightarrow \nu_\tau^c, \quad \nu_\mu \rightarrow \nu_\tau^c,$$ (1.2)

where the superscript ‘c’ denotes the charge conjugation of the relevant neutrino field, and to explore their interesting implications and consequences on various aspects of neutrino phenomenology.

In particular, we stress that slight or soft breaking of the $\mu$–$\tau$ flavor symmetry is expected to help resolve the octant of the largest neutrino mixing angle $\theta_{23}$ and even the quadrant of the CP-violating phase $\delta$ in the standard parametrization of the PMNS matrix $U$ [16], which consists of three rotation angles ($\theta_{12}, \theta_{13}, \theta_{23}$) and one ($\delta$ in the Dirac case) or three ($\theta, \mu, \sigma$ in the Majorana case) CP-violating phases. It may also offer a straightforward link between the neutrino mass spectrum and the lepton flavor mixing pattern. All these issues are very important on the theoretical side and highly concerned in the ongoing and upcoming neutrino experiments.

The remaining parts of this review paper are organized in the following way. In section 2 we begin with a brief introduction to the phenomenology of lepton flavor mixing and neutrino oscillations, followed by a short description of the main outcomes of various solar, atmospheric, reactor and accelerator neutrino oscillation experiments. In the three-flavor scheme a global analysis of current experimental data leads us to a preliminary pattern of lepton flavor mixing, which exhibits an approximate $\mu$–$\tau$ flavor symmetry. Section 3 is devoted to an overview of the $\mu$–$\tau$ flavor symmetry of the Majorana neutrino mass matrix and its connection with the flavor mixing parameters. Two kinds of symmetries, the $\mu$–$\tau$ permutation symmetry and the $\mu$–$\tau$ reflection symmetry, will be classified and discussed. The typical and instructive ways to slightly break the $\mu$–$\tau$ flavor symmetry are introduced. In particular, the effects of $\mu$–$\tau$ symmetry breaking induced by radiative corrections are described in some detail. The contribution of the charged-lepton sector to lepton flavor mixing is also discussed. In section 4 we turn to some larger discrete flavor symmetry groups to illustrate how the $\mu$–$\tau$ flavor symmetry can naturally arise as a residual symmetry. Both the bottom–up approach and the top-down approach will be described in this connection. In section 5 we concentrate on the strategies of model building with the help of the $\mu$–$\tau$ permutation or reflection symmetry. A combination of the seesaw mechanism and the $\mu$–$\tau$ flavor symmetry is taken into account, and the relationship between the Friedberg–Lee symmetry and the $\mu$–$\tau$ flavor symmetry is explored. We also comment on the model-building exercises associated with the light sterile neutrinos. Section 6 is devoted to some phenomenological consequences of the $\mu$–$\tau$ flavor symmetry on some interesting topics.
in neutrino physics, including neutrino oscillations in terrestrial matter, flavor distributions of the Ultrahigh-energy (UHE) cosmic neutrinos at neutrino telescopes, a possible connection between the leptogenesis and low-energy CP violation, and a likely unified flavor texture of leptons and quarks. The concluding remarks and an outlook are finally made in section 7.

2. Behind the lepton flavor mixing pattern

To see why an approximate $\mu$-$\tau$ flavor symmetry is behind the observed pattern of the PMNS lepton flavor mixing matrix $U$, let us first introduce some basics about neutrino mixing and flavor oscillations and then discuss current experimental constraints on the structure of $U$.

2.1. Lepton flavor mixing and neutrino oscillations

Just similar to quark flavor mixing, lepton flavor mixing can also take place provided weak charged-current interactions and Yukawa interactions coexist in a simple extension of the SM. The standard form of weak charged-current interactions and Yukawa interactions coexist in a simple extension of the SM. The standard form of weak charged-current interactions of the charged leptons and neutrinos reads

$$-\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} \sum_{\alpha} \left[ \bar{\nu}_\alpha \gamma^\mu \nu_{\alpha L} W^-_{\mu} + \text{h.c.} \right],$$

where $\alpha$ runs over $e, \mu$ and $\tau$, and the superscript $''\nu''$ denotes the flavor eigenstate of a charged lepton. The leptonic Yukawa interactions are expected to be responsible for the mass generation of both charged leptons and neutrinos after spontaneous electroweak symmetry breaking, although the origin of neutrino masses is very likely to involve some new degrees of freedom and lepton number violation [36]. Without going into details of a specific neutrino mass model, here we assume massive neutrinos to be the Majorana particles and write out the effective lepton mass terms as follows:

$$-\mathcal{L}_m = \frac{1}{2} \bar{\nu}_\alpha \left( M_\alpha \right)_{\alpha \beta} \nu_\beta + \text{h.c.},$$

with $M_\alpha$ being symmetric and $M_L$ being arbitrary. Given the unitary matrices $O_L$ and $O_R$, $M_L$ and $M_R$ can be diagonalized through the transformations

$$O^T_L M_L O_L = D_L = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

$$O^T_R M_R O_R = D_R = \begin{pmatrix} m_1' & 0 & 0 \\ 0 & m_2' & 0 \\ 0 & 0 & m_3' \end{pmatrix}. \quad (2.3)$$

Then it is possible to reexpress $\mathcal{L}_m$ in terms of the mass eigenstates of charged leptons and neutrinos:

$$-\mathcal{L}_m = \frac{1}{2} \bar{\nu}_\alpha \left( D_L \right)_{\alpha \beta} \nu_\beta + \text{h.c.},$$

In doing so, one must consistently reexpress $\mathcal{L}_{ee}$ in terms of the relevant mass eigenstates:

$$-\mathcal{L}_{ee} = \frac{g}{\sqrt{2}} \left( \bar{\nu}_\mu \gamma^\mu \nu_\mu \right) U_{1\mu} \nu_1 + \text{h.c.},$$

where $U = O^T_L O_R$ is just the unitary PMNS matrix which describes the strength of lepton flavor mixing in weak interactions.

In a commonly chosen basis where the flavor eigenstates of three charged leptons are identified with their mass eigenstates, the flavor eigenstates of three neutrinos can be expressed as

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{1e} & U_{1\mu} & U_{1\tau} \\ U_{2e} & U_{2\mu} & U_{2\tau} \\ U_{3e} & U_{3\mu} & U_{3\tau} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (2.6)$$

The nine elements of $U$ can be parameterized in terms of three rotation angles and three CP-violating phases. For example, $U = VP$ with $V = O_{32} O_{13} O_{12}$ and $P = \text{Diag}(e^{i\theta}, e^{i\phi}, 1)$, where

$$U_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix},$$

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}. \quad (2.7)$$

and $O_R = \text{Diag}(1, 1, e^{i\delta})$ with $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$ (for $ij = 12, 13, 23$). To be more explicit,

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ V_{\nu 1} & V_{\nu 2} & c_{13}e^{i\delta} \\ V_{\nu 1} & V_{\nu 2} & c_{13}e^{i\delta} \end{pmatrix}, \quad (2.8)$$

in which

$$V_{\nu 1} = -s_{12}c_{13} - c_{12}s_{13}s_{23}e^{i\delta},$$

$$V_{\nu 2} = c_{12}c_{13} - s_{12}s_{13}s_{23}e^{i\delta},$$

$$V_{\nu 3} = s_{13}c_{23} - c_{13}s_{23}e^{i\delta},$$

$$V_{\nu 2} = -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}. \quad (2.9)$$

Without loss of generality, the three mixing angles are all arranged to lie in the first quadrant, while $\delta$ may vary from 0 to $2\pi$. The fact that the elements of $V$ in its first row and third column are very simple functions of the flavor mixing angles makes the latter have straightforward relations with the

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3 Whether $U$ is really unitary or not actually depends on the mechanism of neutrino mass generation. In the canonical seesaw mechanism [17–21], for instance, the mixing between light and heavy Majorana neutrinos leads to tiny unitarity-violating effects for $U$ itself. However, the unitarity of $U$ has been tested at the percent level [37, 38], and thus it makes sense to assume $U$ to be exactly unitary for the time being.
amplitudes of solar \( \theta_{12} \), reactor \( \theta_{13} \) and atmospheric \( \theta_{23} \) neutrino oscillations, respectively [39]. In this parametrization \( \delta \) is usually referred to as the ‘Dirac’ phase, while \( \rho \) and \( \sigma \) are the Majorana phases which have nothing to do with neutrino oscillations. If massive neutrinos were the Dirac particles, one would simply forget the Majorana phase matrix \( P \). Throughout this review we mainly concentrate on the Majorana neutrinos, because they are well motivated from a theoretical point of view. Then the symmetric Majorana neutrino mass matrix can be reconstructed in terms of the neutrino masses \( m_i \) (for \( i = 1, 2, 3 \)) and the PMNS matrix \( U \) in the chosen flavor basis:

\[
M = \begin{pmatrix}
M_{ee} & M_{e\mu} & M_{e\tau} \\
M_{\mu e} & M_{\mu\mu} & M_{\mu\tau} \\
M_{\tau e} & M_{\tau\mu} & M_{\tau\tau}
\end{pmatrix} = U D_{\text{m}} U^T. \tag{2.10}
\]

In a specific neutrino mass model which is able to fix the texture of \( M_\nu \), one may reversely obtain some testable predictions for the neutrino masses and flavor mixing parameters.

\( \mathcal{L}_{\text{osc}} \) in equation (2.5) tells us that a \( \nu_\alpha \) neutrino flavor can be produced from the \( W^+ + \nu_\beta \rightarrow \nu_\alpha \) interaction, and a \( \nu_\beta \) neutrino flavor can be detected through the \( \nu_\beta + W \rightarrow \beta^- \) interaction (for \( \alpha, \beta = e, \mu, \tau \)). The effective Hamiltonian responsible for the propagation of \( \nu_\beta \) in vacuum is expressed as

\[
\mathcal{H}_{\text{eff}} = \frac{1}{2E} m_i m_j^{\prime} = \frac{1}{2E} V D_{\text{m}}^{2} \bar{V}^t, \tag{2.11}
\]

where \( E \gg m_i \) is the neutrino beam energy, and the Majorana phase matrix \( P \) has been cancelled. Thanks to a quintessentially quantum-mechanical effect, the \( \nu_\beta \rightarrow \nu_\beta \) oscillation happens if the \( \nu_\beta \) beam travels a proper distance \( L \). The probability of such a flavor oscillation is given by \[40\]

\[
P(\nu_\beta \rightarrow \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i<j} \text{Re} \phi_{ij}^{\prime} \sin^2 \Delta_{ij} + 8 \text{Im} \phi_{ij}^{\prime} \prod_{i<j} \sin \Delta_{ij}, \tag{2.12}
\]

where \( \Delta_{ij} \equiv \Delta m_{ij}^2 L/(4E) \) and \( \phi_{ij}^{\prime} \equiv U_{i\alpha} U_{j\beta}^* U_{i\alpha}^* U_{j\beta} \) (for \( i, j = 1, 2, 3 \) and \( \alpha, \beta = e, \mu, \tau \)). The unitarity of the PMNS matrix \( U \) leads us to

\[
\text{Im} \phi_{ij}^{\prime} = J \sum_{\gamma} \epsilon_{i\gamma} \sum_{k} \epsilon_{j\gamma}, \tag{2.13}
\]

with \( J = c_{12} s_{12} c_{13} s_{13} c_{23} s_{23} \sin \delta \) being the so-called Jarlskog invariant [41], which is a universal measure of lepton CP and T violation in neutrino oscillations. The probability of \( \pi_{\alpha} \rightarrow \pi_{\beta} \) oscillations can be directly read off from equation (2.12) by making the replacement \( U \rightarrow U^* \). There are in general two categories of neutrino oscillation experiments: \textit{appearance} (\( \alpha = \beta \)) and \textit{disappearance} (\( \alpha \neq \beta \)). Both the solar neutrino oscillations (\( \nu_e \rightarrow \nu_e \)) and the reactor antineutrino oscillations (\( \bar{\nu}_e \rightarrow \bar{\nu}_e \)) are of the disappearance type. In comparison, the atmospheric muon-neutrino (or muon-antineutrino) oscillations are essentially of the disappearance type, and the accelerator neutrino (or antineutrino) oscillations can be of either type.

Given equation (2.5), the reactions \( \nu_\alpha + e^- \rightarrow \nu_\alpha + e^- \) and \( \bar{\nu}_\alpha + e^- \rightarrow \bar{\nu}_\alpha + e^- \) can take place via the charged-current interactions. That is why the behavior of neutrino (or antineutrino) flavor conversion in a dense medium may be modified by the coherent forward \( \nu_\alpha e^- \) or \( \bar{\nu}_\alpha e^- \) scattering. This matter effect is also referred to as the Mikheyev–Smirnov–Wolfenstein (MSW) effect [42, 43]. In this case equations (2.11)–(2.13) have to be replaced by their counterparts in matter [44].

2.2. Current neutrino oscillation experiments

The fact that neutrinos are massive and lepton flavors are mixed has been firmly established in the past two decades, thanks to a number of solar, atmospheric, reactor and accelerator neutrino (or antineutrino) oscillation experiments [16]. Let us briefly go over some of them in the following, before we discuss what is behind the observed pattern of lepton flavor mixing.

2.2.1. Solar neutrino oscillations. The solar \( ^{8}\text{B} \) neutrinos were first observed in the Homestake experiment in 1968, but the measured flux was only about one third of the value predicted by the standard solar model (SSM) [14, 45]. This anomaly was later confirmed by other experiments, such as GALLEX/ GNO [46, 47], SAGE [48], Super-Kamiokande [49] and SNO [50]. The SNO experiment was particularly crucial because it provided the first model-independent evidence for the flavor conversion of solar \( \nu_\ell \) neutrinos into \( \nu_\ell \) and \( \nu_\ell \) neutrinos.

The target material of the SNO detector is heavy water, which allows the solar \( ^{8}\text{B} \) neutrinos to be observed via the charged-current (CC) reaction \( \nu_\ell + D \rightarrow e^- + p + p \), the neutral-current (NC) reaction \( \nu_\ell + D \rightarrow \nu_\ell + p + n \) and the elastic-scattering process \( \nu_\ell + e^- \rightarrow \nu_\ell + e^- \) (for \( \alpha = e, \mu, \tau \)) [50]. The neutrino fluxes extracted from these three channels can be expressed as \( \phi_{\text{CC}} = \phi_{\nu_e} + \phi_{\nu_\mu} + \phi_{\nu_\tau} \) and \( \phi_{\text{NC}} = \phi_{\nu_e} + 0.155 \phi_{\nu_\tau} \), where \( \phi_{\nu_\tau} \) denotes a sum of the fluxes of \( \nu_\mu \) and \( \nu_\tau \) neutrinos. If there were no flavor conversion, \( \phi_{\nu_\mu} = 0 \) and \( \phi_{\text{CC}} = \phi_{\nu_e} = \phi_{\text{NC}} \) would hold. The SNO data yielded \( \phi_{\nu_e} = 2.35^{+0.22}_{-0.22} \text{(stat)}^{+0.13}_{-0.10} \text{(syst)} \) [51], \( \phi_{\nu_\mu} = 0.09^{+0.06}_{-0.06} \text{(stat)}^{+0.08}_{-0.08} \text{(syst)} \) and \( \phi_{\nu_\tau} = 4.94^{+0.38}_{-0.34} \text{(stat)}^{+0.21}_{-0.22} \text{(syst)} \). The phenomenon of flavor conversion (i.e. \( \phi_{\nu_\mu} \neq 0 \)) and the SSM prediction for \( \phi_{\nu_\tau} \) could be a crucial hint that the SSM could be in trouble.

The observed deficit of solar \( ^{8}\text{B} \) neutrinos is attributed to \( \nu_\ell \rightarrow \nu_\ell \) and \( \nu_\ell \rightarrow \nu_\ell \) transitions modified by significant matter effects in the core of the Sun. The survival probability of \( ^{8}\text{B} \) neutrinos may roughly approximate to \( P(\nu_\ell \rightarrow \nu_\ell) \approx 0.32 [52] \), leading us to \( \theta_{12} \approx 34^\circ \).

Note that the recent Borexino experiment has done a real-time measurement of the mono-energetic solar \( ^{8}\text{Be} \) neutrinos and observed a remarkable deficit corresponding to \( P(\nu_\ell \rightarrow \nu_\ell) = 0.56^{+0.1}_{-0.1} [53] \). This result can approximately be interpreted as a vacuum oscillation effect, since the low-energy \( ^{8}\text{Be} \) neutrino oscillation is not very sensitive to matter effects in the Sun [52]. So one is left with the averaged survival probability \( P(\nu_\ell \rightarrow \nu_\ell) \approx 1 - 0.5 \sin^2 2\theta_{12} \approx 0.56 \) for solar \( ^{7}\text{Be} \) neutrinos, from which \( \theta_{12} \approx 35^\circ \) can be obtained.
Such a result is apparently consistent with the aforementioned value of $\theta_{13}$ extracted from the data of solar $^8\text{B}$ neutrinos.

### 2.2.2. Atmospheric neutrino oscillations.

It is well known that the atmospheric $\nu_\mu$, $\nu_\tau$, $\nu_e$ and $\bar{\nu}_\mu$ events are produced in the Earth’s atmosphere by cosmic rays, mainly through the decay modes $\pi^+ \rightarrow \mu^+ + \nu_\mu$, with $\mu^+ \rightarrow e^+ + \nu_\mu + \bar{\nu}_e$, and $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$, with $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$. If there were nothing wrong with the atmospheric neutrinos that enter and excite an underground detector, they would have an almost perfect spherical symmetry (i.e. the downward- and upward-going neutrino fluxes should be equal, $F_\mu(\theta) = F_\mu(\pi - \theta)$ and $F_\mu(\theta) = F_\mu(\pi - \theta)$ with respect to the zenith angle $\theta$). In 1998 the Super-Kamiokande Collaboration observed an approximate up–down flux asymmetry for atmospheric $\nu_\mu$ and $\bar{\nu}_\mu$ events and a significant up–down flux asymmetry for atmospheric $\nu_\tau$ and $\bar{\nu}_\tau$ events [54]. Such a striking anomaly could naturally be attributed to $\nu_\mu \rightarrow \nu_\tau$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_\tau$ oscillations for those upward-going $\nu_\mu$ and $\bar{\nu}_\mu$ events, since the detector itself was insensitive to $\nu_e$ and $\bar{\nu}_e$ events. This observation was actually the first model-independent discovery of neutrino flavor oscillations, and it marked an important turning point in experimental neutrino physics.

In 2004 the Super-Kamiokande Collaboration did a careful analysis of the disappearance probability of $\nu_\mu$ and $\bar{\nu}_\mu$ events as a function of the neutrino flight length $L$ over the neutrino energy $E$, and observed a clear dip in the $L/E$ distribution as the first direct evidence for atmospheric neutrino oscillations [55]. Such a dip is consistent with the sinusoidal probability of neutrino flavor oscillations but incompatible with exotic new physics, such as the neutrino decay and neutrino decoherence scenarios.

It is a great challenge to observe the atmospheric $\nu_\mu \rightarrow \nu_\tau$ oscillation because this requires the neutrino beam energy greater than a threshold of 3.5 GeV, such that a tau lepton can be produced via the charged-current interaction of incident $\nu_\tau$ with the target nuclei in the detector. The Super-Kamiokande data are found to be best described by neutrino oscillations that include the $\nu_\tau$ appearance in addition to the overwhelming signature of the $\nu_\mu$ disappearance. In particular, a neural network analysis of the zenith-angle distribution of multi-GeV contained events has recently demonstrated the $\nu_\tau$ appearance effect at the 3.8σ level [56].

### 2.2.3. Accelerator neutrino oscillations.

If the atmospheric $\nu_\mu$ and $\bar{\nu}_\mu$ flavors oscillate, a fraction of the accelerator-produced $\nu_\mu$ and $\bar{\nu}_\mu$ events may also disappear on their way to a remote detector. This expectation has been confirmed by two long-baseline neutrino oscillation experiments: K2K [57] and MINOS [58]. Both of them observed a reduction of the $\nu_\mu$ flux and a distortion of the $\nu_\mu$ energy spectrum, implying $\nu_\mu \rightarrow \nu_\mu$ oscillations. The most amazing result obtained from the atmospheric and accelerator neutrino oscillation experiments is $\sin^2 2\theta_{23} \simeq 1$ or equivalently $\theta_{23} \simeq 45^\circ$, which hints at a likely $\mu$–$\tau$ flavor symmetry in the lepton sector.

Today the most important accelerator neutrino oscillation experiment is the T2K experiment, which has discovered the $\nu_\mu \rightarrow \nu_\tau$ appearance oscillations and carried out a precision measurement of the $\nu_\mu \rightarrow \nu_\mu$ disappearance oscillations. Since its preliminary data were first released in 2011, the T2K experiment has proved to be very successful in establishing the $\nu_\mu$ appearance out of a $\nu_\mu$ beam at the 7.3σ level and constraining the neutrino mixing parameters $\theta_{13}$, $\theta_{23}$ and $\delta$ [59–61]. The point is that the leading term of $P(\nu_\mu \rightarrow \nu_\tau)$ is sensitive to $\sin^2 2\theta_{13}\sin^2 2\theta_{23}$ and its sub-leading term is sensitive to $\delta$ and terrestrial matter effects [62]. Figure 1 is an illustration of the allowed region of $\sin^2 2\theta_{13}$ as a function of the CP-violating phase $\delta$, as constrained by the present T2K data [61]. One can observe an unsuppressed value of $\theta_{13}$ in this plot, together with a preliminary hint of $\delta$ around $-\pi/2$. The latter is also suggestive of a possible $\mu$–$\tau$ reflection symmetry, as one will see in section 3.

In comparison with K2K, MINOS and T2K, the accelerator-based OPERA experiment was designed to search for the $\nu_\tau$ appearance in a $\nu_\mu$ beam. After several years of data taking, the OPERA Collaboration reported five $\nu_\tau$ events in 2015. These events have marked a discovery of the $\nu_\mu \rightarrow \nu_\tau$ appearance oscillations with the 5.1σ significance [63].

### 2.2.4. Reactor antineutrino oscillations.

Since the first observation of the $\bar{\nu}_e$ events emitted from the Savannah River reactor in 1956 [7], nuclear reactors have been playing a special role in neutrino physics. In particular, $\theta_{12}$ and $\theta_{23}$ have been measured in the KamLAND [64] and Daya Bay [65, 66] reactor antineutrino experiments, respectively.

Given the average baseline length $L=180$ km, the KamLAND experiment was sensitive to the $\Delta m^2_{21}$-driven $\pi^+ \rightarrow \bar{\nu}_e$ oscillations, and the reactor antineutrino oscillations are the first neutrino oscillation experiment with $\delta = 0$ and $\sin^2 2\theta_{13} \simeq 1$.
The Daya Bay experiment was designed to probe the smallest lepton flavor mixing angle $\theta_{13}$ with an unprecedented sensitivity $\sin^2 2\theta_{13} \sim 1\%$ by measuring the $\Delta m^2_{31}$-driven $\nu_\tau \to \nu_\tau$ oscillation with a baseline length $L \approx 2$ km. In 2012 the Daya Bay Collaboration announced a 5.2$\sigma$ discovery of $\theta_{13} = 0$ and obtained $\sin^2 2\theta_{13} = 0.092 \pm 0.016 \text{(stat)} \pm 0.005 \text{(syst)}$ [65]. Some similar but less significant results were also achieved in the RENO [68] and Double Chooz [69] reactor antineutrino oscillation experiments.

The Daya Bay experiment has also measured the energy dependence of the $\overline{\nu}_\tau$ disappearance and seen a nearly full oscillation cycle against $\overline{\nu}_e$ [70]. The updated result $\sin^2 2\theta_{13} = 0.090^{+0.008}_{-0.009}$ is obtained in the three-flavor framework. A combination of the Daya Bay measurement of $\theta_{13}$ and the T2K measurement of a relatively strong $\nu_\mu \to \nu_e$ appearance signal [61] drives a slight but intriguing preference for $\delta \sim -\pi/2$, as shown in Figure 1. In addition, the relative large $\theta_{13}$ is so encouraging that the next-generation precision experiments should be able to determine the neutrino mass ordering and the CP-violating phase $\delta$ in the foreseeable future [71].

2.3. The observed pattern of the PMNS matrix

In the three-flavor scheme there are six independent parameters which govern the behaviors of neutrino oscillations: two neutrino mass-squared differences $\Delta m^2_{21}$ and $\Delta m^2_{31}$, three flavor mixing angles $\theta_{12}, \theta_{13}$ and $\theta_{23}$, and the Dirac CP-violating phase $\delta$. Those successful atmospheric, solar, accelerator and reactor neutrino oscillation experiments discussed above allow us to determine $\Delta m^2_{21}$, $\Delta m^2_{31}$, $\theta_{12}$, $\theta_{13}$ and $\theta_{23}$ to a good degree of accuracy. The ongoing and future neutrino oscillation experiments are expected to fix the sign of $\Delta m^2_{31}$ and pin down the value of $\delta$.

A global analysis of the available data on solar (SNO, Super-Kamiokande, Borexino), atmospheric (Super-Kamiokande), accelerator (MINOS, T2K) and reactor (KamLAND, Daya Bay, RENO) neutrino (or antineutrino) oscillations has been done by several groups [72–74]. Here we quote the main results obtained by Capozzi et al [72] in Table 2. Some immediate comments are in order.

- The unfixed sign of $\Delta m^2_{31}$ implies two possible neutrino mass orderings: normal ($m_1 < m_2 < m_3$) or inverted ($m_3 < m_1 < m_2$).

Table 2. The best-fit values, together with the 1$\sigma$ and 3$\sigma$ intervals, for the six three-flavor neutrino oscillation parameters from a global analysis of current experimental data [72].

| Parameter | Best fit | 1$\sigma$ range | 3$\sigma$ range |
|-----------|----------|----------------|----------------|
| $\Delta m^2_{31}$ ($10^{-5}$ eV$^2$) | 7.54 | 7.32–7.80 | 6.99–8.18 |
| $\Delta m^2_{21}$ ($10^{-3}$ eV$^2$) | 2.47 | 2.41–2.53 | 2.26–2.65 |
| $\sin^2 \theta_{13}$ | 0.08 | 0.91–3.25 | 2.59–3.59 |
| $\sin^2 \theta_{23}$ | 0.24 | 0.15–2.54 | 1.76–2.95 |
| $\sin^2 \theta_{12}$ | 0.37 | 0.41–4.70 | 3.74–6.26 |
| $\delta/\pi$ | 1.39 | 1.12–1.77 | 0.00–2.00 |

The notations $\delta m^2 \equiv m_1^2 - m_2^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ have been used in [72]. Their relations with $\Delta m^2_{31}$ and $\Delta m^2_{12}$ are $\Delta m^2_{31} = \delta m^2$ and $\Delta m^2_{12} = \Delta m^2 + \delta m^2/2$.5

4 The notations $\delta m^2 \equiv m_1^2 - m_2^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ have been used in [72]. Their relations with $\Delta m^2_{31}$ and $\Delta m^2_{12}$ are $\Delta m^2_{31} = \delta m^2$ and $\Delta m^2_{12} = \Delta m^2 + \delta m^2/2$.5

Figure 2. A schematic illustration of the fermion mass spectrum of the SM at the electroweak scale, where the three neutrino masses are assumed to have a normal ordering.

and could accomplish a terrestrial test of the large-mixing-angle MSW solution to the long-standing solar neutrino problem. In fact, it succeeded in doing so in 2003 [64], with an impressive determination of $\theta_{12} \simeq 34^\circ$. A very striking sinusoidal behavior of $P(\nu_\tau \to \nu_\tau)$ against $L/E$ was also observed by the KamLAND Collaboration later on [67].
In short, the sign of \( \Delta m^2_{31} \), the octant of \( \theta_{23} \) and the value of \( \delta \) remain unknown. Whether these three open issues are potentially correlated with one another is an intriguing question.

Combining equations (2.8) and (2.9) with table 2, we obtain the remarkable result

\[
|U_{\alpha\beta}| \simeq |U_{11}|, \quad |U_{22}| \simeq |U_{22}|, \quad |U_{33}| \simeq |U_{33}|
\]

(2.14)

to a reasonably good degree of accuracy. This result becomes more transparent when the allowed ranges of the nine PMNS matrix elements are explicitly given at the 3\( \sigma \) level:

\[
|U| \simeq \begin{bmatrix} 0.79 - 0.85 & 0.50 - 0.59 & 0.13 - 0.17 \\ 0.19 - 0.56 & 0.41 - 0.74 & 0.60 - 0.78 \\ 0.19 - 0.56 & 0.41 - 0.74 & 0.60 - 0.78 \end{bmatrix}
\]

(2.15)

in the \( \Delta m^2_{31} > 0 \) case; or

\[
|U| \simeq \begin{bmatrix} 0.89 - 0.85 & 0.50 - 0.59 & 0.13 - 0.17 \\ 0.19 - 0.56 & 0.40 - 0.73 & 0.61 - 0.79 \\ 0.20 - 0.56 & 0.41 - 0.74 & 0.59 - 0.78 \end{bmatrix}
\]

(2.16)

in the \( \Delta m^2_{31} < 0 \) case. In either case the pattern of \( U \) is significantly different from that of quark flavor mixing described by the Cabibbo–Kobayashi–Maskawa (CKM) matrix \( V_{\text{CKM}} \) [81, 82]. The latter is close to the identity matrix because its maximal flavor mixing angle is the Cabibbo angle \( \theta_C \simeq 13^\circ \) [16].

In fact, the equality \( |U_{\alpha\beta}| = |U_{\beta\alpha}| \) (for \( i = 1, 2, 3 \)) exactly holds if either of the following two sets of conditions can be satisfied [83]:

\[
|U_{\alpha\beta}| = |U_{\beta\alpha}| \iff \begin{cases} \theta_{23} = \frac{\pi}{4}, & \theta_{13} = 0 ; \\ \theta_{23} = \frac{\pi}{4}, & \delta = \pm \frac{\pi}{2} . \end{cases}
\]

(2.17)

The possibility of \( \theta_{13} = 0 \) has been ruled out, but \( \theta_{23} = \pi/4 \) and \( \delta = -\pi/2 \) are both allowed at the 1\( \sigma \) or 2\( \sigma \) level (and \( \delta = \pi/2 \) is allowed at the 3\( \sigma \) level) as shown in table 2. That is why we claim that there must be an approximate \( \mu-\tau \) flavor symmetry behind the observed pattern of \( U \). In this case the \( \mu-\tau \) symmetry is expected to be a good starting point for model building, no matter what larger flavor groups it belongs to. On the experimental side, it is imperative to measure \( \theta_{23} \) and \( \delta \) as accurately as possible, so as to fix the strength of \( \mu-\tau \) symmetry breaking.

At this point it is also worth mentioning that three of the six unitarity triangles of \( U \) (the so-called Majorana triangles) in the complex plane, defined by the three orthogonality relations

\[
\begin{align*}
\langle \nu_1, \nu_2, \nu_3 \rangle &= 0, \\
\langle \nu_2, \nu_3, \nu_1 \rangle &= 0, \\
\langle \nu_3, \nu_1, \nu_2 \rangle &= 0,
\end{align*}
\]

will become the isosceles triangles provided \( \theta_{23} = \pi/4 \) and \( \delta = \pm \pi/2 \) are satisfied. This interesting property, which can be intuitively seen in figure 3 [84], is another reflection of the \( \mu-\tau \) flavor symmetry as an instructive guideline for the study of leptonic CP violation. We shall go into details of this flavor symmetry and its breaking mechanisms in the subsequent sections.

3. An overview of the \( \mu-\tau \) flavor symmetry

The approximate \( \mu-\tau \) flavor symmetry displayed in the PMNS matrix \( U \) is also expected to show up in the mass matrix of either charged leptons or neutrinos. From a model-building point of view, one needs to know what forms of \( M_L \) and \( M_N \) can give rise to the observed pattern of lepton flavor mixing as well as the correct mass spectra of \( (e, \mu, \tau) \) and \( (\nu_1, \nu_2, \nu_3) \). Since \( U = O_L O_R \) is a measure of the mismatch between \( O_L \) and \( O_R \), which have been used to diagonalize \( M_L M_L^T \) and \( M_N \) in equation (2.3), we prefer to work in the basis where \( O_L \) equals the identity matrix (i.e. \( M_L \) itself is simply a diagonal matrix). In this basis the neutrino mixing effects are completely determined by the structure of \( M_N \). The latter can be reconstructed in terms of \( U \) and the neutrino masses as shown in equation (2.10). The six independent elements of \( M_N \) turn out to be

\[
M_{\alpha\beta} \equiv (m_{\alpha\beta})_{\alpha\beta} = \sum_{i=1}^{3} m_i U_{i\alpha} U_{i\beta}.
\]

(3.1)
where $\alpha$ and $\beta$ run over $e, \mu, \tau$. Of course, $M_\nu$ totally consists of nine independent physical parameters.

With the help of the standard parametrization of $U$ in equations (2.7)-(2.9), we express the six effective neutrino mass terms as follows:

$$
\langle m \rangle_{ee} = m_{1e}^2 e_{1e}^2 + m_{2e}^2 e_{2e}^2 + m_{3e}^2 e_{3e}^2,
$$

$$
\langle m \rangle_{\mu\mu} = m_{1\mu}^2 (s_{1\mu}^2 c_{1\mu} + c_{1\mu} s_{1\mu}^2)^2 + m_{2\mu}^2 (c_{1\mu} s_{1\mu}^2 - s_{1\mu} c_{1\mu}^2)^2 + m_{3\mu}^2 s_{1\mu}^2 c_{1\mu}^2,
$$

$$
\langle m \rangle_{\tau\tau} = m_{1\tau}^2 (s_{1\tau}^2 c_{1\tau} + c_{1\tau} s_{1\tau}^2)^2 + m_{2\tau}^2 (c_{1\tau} s_{1\tau}^2 - s_{1\tau} c_{1\tau}^2)^2 + m_{3\tau}^2 s_{1\tau}^2 c_{1\tau}^2,
$$

$$
\langle m \rangle_{e\mu} = - m_{1e} c_{1\mu} (s_{1\mu} c_{1\tau} + s_{1\tau} c_{1\mu}) + m_{2e} c_{1\mu} (c_{1\mu} s_{1\tau} + s_{1\mu} c_{1\tau}) + m_{3e} s_{1\mu} c_{1\mu}
$$

$$
\langle m \rangle_{e\tau} = m_{1e} c_{1\tau} (s_{1\mu} c_{1\mu} + s_{1\mu} c_{1\tau}) - m_{2e} c_{1\tau} (c_{1\mu} s_{1\mu} + s_{1\mu} c_{1\tau}) + m_{3e} s_{1\mu} c_{1\mu},
$$

$$
\langle m \rangle_{\mu\tau} = - m_{1\mu} c_{1\tau} (s_{1\mu} c_{1\mu} + s_{1\mu} c_{1\tau}) + m_{2\mu} c_{1\tau} (c_{1\mu} s_{1\mu} + s_{1\mu} c_{1\tau}) + m_{3\mu} s_{1\mu} c_{1\mu}
$$

(3.2)

where $m_1 = m_{1e} e_{1e}$, $m_2 = m_{2e} e_{2e}$ and $s_1 = s_{1\mu} e_{1\mu}$ are defined for the sake of simplicity. The numerical profiles of $\langle m \rangle_{e\mu}$ versus the lightest neutrino mass (i.e. $m_1$ in the normal hierarchy (NH) case or $m_3$ in the inverted hierarchy (IH) case) are illustrated in figure 4 [85], where the best-fit values of two neutrino mass-squared differences and three flavor mixing angles presented in table 2 are input while the Dirac and Majorana phases are allowed to vary in the intervals $[0, 2\pi]$ and $[0, \pi]$, respectively. It is obvious that the relations $\langle m \rangle_{e\mu} \approx |\langle m \rangle_{\mu\tau}|$ and $\langle m \rangle_{e\tau} \approx |\langle m \rangle_{e\mu}|$ hold in most of the parameter space. Instead of regarding this observation as the reflection of a kind of flavor ‘anarchy’ [86], we treat it seriously as an indication of the $\mu-\tau$ symmetry which can always be embedded in a much larger flavor symmetry group.

The $\mu-\tau$ flavor symmetry itself is so powerful that it constrains the texture of $M_\nu$ by establishing some equalities or linear relations among its six independent elements. In other words, the $\mu-\tau$ symmetry has been identified as the minimal (and thus most convincing) flavor symmetry in the neutrino sector to give rise to $|\langle m \rangle_{e\mu}| = |\langle m \rangle_{\mu\tau}|$ and $|\langle m \rangle_{e\tau}| = |\langle m \rangle_{e\mu}|$. It is therefore expected to be a successful bridge between neutrino phenomenology and model building. In this section we shall mainly describe the salient features of $\mu-\tau$ permutation and reflection symmetries and outline some typical ways to softly break them. The larger flavor symmetry groups and model-building exercises will be discussed in sections 4 and 5, respectively.

3.1. The $\mu-\tau$ permutation symmetry

Let us begin with the $\mu-\tau$ permutation symmetry which has been defined in equation (1.1) [87-90]. Historically, the discussion about this simple flavor symmetry was motivated by the experimental facts $\theta_{23} \sim \pi/4$ and \sin\^2 2$\theta_{13} < 0.18$ [91], which point to $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ as an ideal possibility. In this case the PMNS matrix $U$ is greatly simplified and may take the following form in a chosen phase convention:

$$
U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} c_{12} & \sqrt{2} s_{12} & 0 \\ -s_{12} & c_{12} & -\kappa \\ -\kappa s_{12} & \kappa c_{12} & 1 \end{pmatrix},
$$

(3.3)

where $\kappa = \pm 1$ will manifest itself in the corresponding neutrino mass matrix $M_\nu$. The latter can be easily reconstructed as follows:

$$
M_{\nu} = \frac{1}{2} \begin{pmatrix} 2m_{11} & -\sqrt{2} m_{12} & -\kappa \sqrt{2} m_{12} \\ \cdots & m_{22} + m_3 & \kappa (m_{22} - m_3) \\ \cdots & \cdots & m_{22} + m_3 \end{pmatrix},
$$

(3.4)

in which

$$
\begin{align*}
& m_{11} = m_{1e}^2 + m_{1\mu}^2 + m_{1\tau}^2, \\
& m_{12} = (m_{1e} - m_{1\mu}) (s_{1\mu} c_{1\tau} + s_{1\tau} c_{1\mu}) \\
& m_{22} = m_{2e}^2 + m_{2\mu}^2 + m_{2\tau}^2.
\end{align*}
$$

(3.5)

It is clear that the elements of $M_{\nu}$ satisfy the relations $M_{\nu_{\mu\tau}} = \kappa M_{\nu_{\mu\tau}}$ and $M_{\nu_{\mu\mu}} = M_{\nu_{\tau\tau}}$. So $M_{\nu}$ is invariant under the $\mu-\tau$ permutation operation $\nu_\mu \leftrightarrow \kappa \nu_\tau$, which is represented by the transformation matrix

$$
S^\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \kappa \\ 0 & \kappa & 0 \end{pmatrix},
$$

(3.6)

where $S^\pm$ corresponds to $\kappa = \pm 1$. So the most general neutrino mass matrix respecting the $\mu-\tau$ permutation symmetry can be parameterized as

$$
M_{\nu} = \begin{pmatrix} A & B & \kappa B \\ B & C & D \\ \kappa B & D & C \end{pmatrix}.
$$

(3.7)

where $A, B, C$ and $D$ are in general complex. For simplicity, here these four parameters are all taken to be real [32]. Both $\kappa = \pm 1$ lead to $\theta_{23} = \pi/4$, as the phases or signs of the elements of $U$ can always be rearranged to assure its mixing angles to lie in the first quadrant. This kind of rephasing is implemented by redefining the phases of relevant lepton fields. In the following we shall focus on the $\kappa = +1$ case. Note that the $\mu-\tau$ permutation symmetry has no definite prediction for $\theta_{13}$. In fact,

$$
\tan 2\theta_{13} = \frac{2\sqrt{2} B}{C + D - A},
$$

(3.8)

and the three neutrino mass eigenvalues of $M_{\nu}$ are given by

$$
\begin{align*}
& m_1 = \frac{1}{2} \left[ A + C + D - \sqrt{(A - C - D)^2 + 8B^2} \right], \\
& m_2 = \frac{1}{2} \left[ A + C + D + \sqrt{(A - C - D)^2 + 8B^2} \right], \\
& m_3 = C - D.
\end{align*}
$$

(3.9)

Depending on the specific values of the relevant free parameters, the neutrino mass spectrum can be either normal ($\Delta m_{31}^2 > 0$) or inverted ($\Delta m_{31}^2 < 0$).
The neutrino mass matrix $M_\nu$ in equation (3.7) will become more predictive if its free parameters are further constrained. Let us consider two simple but instructive cases for illustration:

- If $C + D - A = B$ is assumed, then equation (3.8) leads us to the prediction $\sin \theta_{12} = 1/\sqrt{3}$ or equivalently $\theta_{12} \simeq 35.3^\circ$.

In this case we are left with the well-known tri-bimaximal (TB) mixing pattern of massive neutrinos [92, 93]:

$$ U_{\text{TB}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & -\sqrt{3} \\ -1 & \sqrt{2} & \sqrt{3} \end{pmatrix}. $$ (3.10)
together with the mass eigenvalues
\[ m_1 = C + D - 2B, \]
\[ m_2 = C + D + B \]
and
\[ m_3 = C - D. \]

- If \( C + D - A = 0 \) is taken, then equation (3.8) yields
\[ \theta_{12} = \theta_{23} = \pi/4. \]
In this case one arrives at another special pattern of the neutrino mixing matrix—the so-called bi-maximal (BM) flavor mixing pattern [94, 95]:

\[
U_{\text{BM}} = \frac{1}{2} \begin{pmatrix}
\sqrt{2} & \sqrt{2} & 0 \\
-1 & 1 & -\sqrt{2} \\
-1 & 1 & \sqrt{2}
\end{pmatrix},
\] (3.11)
as well as the mass eigenvalues
\[ m_1 = C + D - \sqrt{2}B, \]
\[ m_2 = C + D + \sqrt{2}B \]
and
\[ m_3 = C - D. \]

Considering other constraints on the free parameters of \( M_\nu \) in equation (3.7), one may similarly derive other neutrino mixing patterns which maintain \( \theta_{23} = \pi/4 \) and \( \theta_{13} = 0 \) but predict different values of \( \theta_{12} \) (e.g. the golden-ratio [96] and hexagonal [97, 98] patterns). In such examples all the three flavor mixing angles are constants, which have nothing to do with the neutrino masses. It is possible to link \( \theta_{12} \) with the ratio of \( m_1 \) to \( m_2 \) in the following way [99, 100]:

\[
\tan \theta_{12} = \sqrt{\frac{m_1}{m_2}},
\] (3.12)

if \( A = 0 \) (i.e. \( \langle m \rangle_{ee} = 0 \)) holds. On the other hand, taking \( C = D \) will lead to \( m_3 = 0 \), as one can see from equation (3.9). Such a special neutrino mass spectrum is not in conflict with the present experimental data, and it may have some interesting consequences in neutrino phenomenology [84].

At this point let us point out that the canonical \( \mu-\tau \) permutation symmetry can be generalized in a way described by the transformation matrix [101]

\[
S(\theta, \phi) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos 2\theta & \sin 2\theta e^{-2i\phi} \\
0 & \sin 2\theta e^{2i\phi} & -\cos 2\theta
\end{pmatrix},
\] (3.13)
Obviously, this symmetry will be reduced to the canonical one if \( \theta = \pi/4 \) and \( \phi = 0 \) are taken. The requirement of \( M_\nu \), being invariant under \( S(\theta, \phi) \), imposes the following constraints on its elements:

\[
M_{ee} - e^{i2\theta} M_{ep} \tan \theta = 0, \\
(e^{-i\theta} M_{ee} - e^{-i\phi} M_{\mu\mu}) + 2M_{\mu e} \cos 2\theta = 0. 
\] (3.14)

Such an \( M_\nu \) will result in a flavor mixing matrix \( U \) consisting of \( \theta_{13} = 0 \) and \( \theta_{23} = \theta \). The point is that \( S(\theta, \phi) \) has an eigenvector

\[
u = \begin{pmatrix} 0 \\ e^{-i\phi} \sin \theta \\ -e^{-i\phi} \cos \theta \end{pmatrix} 
\] (3.15)

with the eigenvalue \(-1\). Hence the relation \( S(\theta, \phi)\nu = -\nu \) together with \([S(\theta, \phi)]^\dagger M_\nu [S(\theta, \phi)]^\ast = M_\nu \) will lead us to \( S(\theta, \phi)(M_\nu u^\ast) = -(M_\nu u^\ast) \), indicating that \( M_\nu u^\ast \) is proportional to \( u \). One may therefore draw the conclusion that \( u \) constitutes one column of \( U \). On the other hand, it is straightforward to show that the reconstructed \( M_\nu \) via equation (2.10) in terms of \( U \) featuring \( \theta_{13} = 0 \) satisfies equation (3.14). In other words, \( M_\nu \) always assumes an \( S(\theta, \phi) \) symmetry of the form given in equation (3.13) for the case of \( \theta_{13} = 0 \). But in what follows we shall focus on the canonical \( \mu-\tau \) permutation symmetry, simply because it is much simpler and favored by the experimental result \( \theta_{23} \approx \pi/4 \).

Finally, it is worth pointing out that there is a remarkable variant of the \( \mu-\tau \) permutation symmetry—namely, the so-called \( \mu-\tau \) permutation antisymmetry [102, 103]. When the latter is concerned, the neutrino mass matrix satisfies the transformation

\[ S^\dagger M_\nu^{\text{AS}} S = -M_\nu^{\text{AS}}, \] (3.16)

and thus it has a special form

\[
M_\nu^{\text{AS}} = \begin{pmatrix} 0 & B & -B \\ B & C & 0 \\ -B & 0 & -C \end{pmatrix} 
\] (3.17)

The unitary matrix used to diagonalize this special mass matrix is given by

\[
U^{\text{AS}} = \frac{1}{2N} \begin{pmatrix} 2B^\ast \\ C^\ast + iN \end{pmatrix} \begin{pmatrix} 2B^\ast \\ C^\ast - iN \end{pmatrix} - \begin{pmatrix} 2B \\ C^\ast + iN \end{pmatrix} + iN \begin{pmatrix} 2B \\ C^\ast - iN \end{pmatrix}, \] (3.18)

where \( N = \sqrt{2|B|^2 + |C|^2} \). In this case \( \theta_{12} \) and \( \theta_{23} \) are both equal to \( \pi/4 \), and a finite value of \( \theta_{13} \) is allowed. The three mass eigenvalues of \( M_\nu^{\text{AS}} \) are \( iM, -iM \) and 0, respectively. To fit current neutrino oscillation data, one has to introduce proper perturbations to \( M_\nu^{\text{AS}} \) so as to break its \( \mu-\tau \) permutation antisymmetry and arrive at an acceptable neutrino mass spectrum [103]. Note that an arbitrary Majorana neutrino mass matrix can always be decomposed into two parts: the first part respects the \( \mu-\tau \) permutation symmetry, and the second part possesses the \( \mu-\tau \) permutation antisymmetry. In such a treatment the second part may serve as a perturbation term to characterize the \( \mu-\tau \) symmetry breaking effects. This point will become clearer in section 3.3.

To summarize, the \( \mu-\tau \) permutation symmetry was motivated by the early experimental data of neutrino oscillations and has played an important role in understanding the lepton flavor structures. The discovery of a relatively large value of \( \theta_{13} \) [66] requires one to go beyond the original model-building scope in this respect, either by taking into account much larger \( \mu-\tau \) symmetry breaking effects [104, 105] or by paying particular attention to the \( \mu-\tau \) reflection symmetry. The latter approach is more attractive because it can both produce a nonzero \( \theta_{13} \) and predict \( \delta = \pm \pi/2 \) in the first place (i.e. in the symmetry limit).

3.2. The \( \mu-\tau \) reflection symmetry

The \( \mu-\tau \) reflection symmetry was originally put forward by Harrison and Scott [106], who were inspired by the approximate \( \mu-\tau \) universality in the neutrino oscillation data. They started from the possibility of \( |U_{\mu i}| = |U_{\nu i}| \) (for \( i = 1, 2, 3 \)), switched off two Majorana phases and arrived at the following parametrization of \( U \) in a proper phase convention:

\[
U = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1' & v_2' & v_3' \end{pmatrix}. 
\] (3.19)

where the elements \( u_i \) have been arranged to be real. Thanks to its unitarity, \( U \) contains four independent real parameters which can be chosen as \( u_1, u_2, |v_1| \) and \( \gamma_1 \equiv \text{arg}(v_1) \). For instance, the phase difference \( \gamma_21 \equiv \gamma_2 - \gamma_1 \) is determined via

\[
\cos \gamma_{21} = \frac{-u_{32}u_{13}}{2|v_{21}|} = \frac{-u_{32}u_{13}}{\sqrt{(1 - u_{22}^2)(1 - u_{11}^2)}}. 
\] (3.20)

The Jarlskog invariant [41] of this neutrino mixing pattern turns out to be

\[
|J| = |\text{Im}(u_1u_2^\ast v_1v_2)| = |u_{12}|v_{12}^\ast \sin \gamma_{21} = \frac{1}{2} u_{12} v_{12}^\ast (1 - u_{12}^2 - u_{22}^2) = \frac{1}{2} |U_{\mu 1}U_{\nu 2}U_{\nu 3}|. 
\] (3.21)

On the other hand, \( J \) is given by

\[
|J| = c_{12}c_{13}c_{23}s_{13}s_{23}2|\sin \delta| = \frac{1}{2} |U_{\mu 1}U_{\nu 2}U_{\nu 3}\sin \delta|\sin 2\theta_{23}. 
\] (3.22)

in the standard parametrization of \( U \), as pointed out below equation (2.13). If \( \theta_{13} = 0 \) or equivalently \( U_{\nu 3} = 0 \), one will immediately obtain \( |J| = \sin 2\theta_{23}|\sin \delta| = 1 \) from equations (3.21) and (3.22). This interesting result implies that the condition \( |U_{\mu l}| = |U_{\nu l}| \) must yield \( \theta_{23} = \pi/4 \) and \( \delta = \pm \pi/2 \)—the same conclusion has been drawn in equation (2.17). Current neutrino oscillation data have given \( \theta_{13} \approx 9^\circ \) [66] and provided a preliminary but noteworthy hint \( \delta \sim \pm \pi/2 \) [61]. That is why the special mixing pattern in equation (3.19), which is invariant under the \( \mu-\tau \) reflection symmetry transformation (namely, \( (S^\ast U)^\dagger = U \)), is phenomenologically appealing.

Now let us turn to the neutrino mass matrix. It is easy to show that an \( M_\nu \) obeying the condition

\[
(S^\dagger M_\nu S)^\dagger = M_\nu 
\] (3.23)
can lead us to the neutrino mixing matrix in equation (3.19). In other words, \(M_\nu\) is required to be invariant with respect to the \(\mu-\tau\) reflection transformations defined by equation (1.2)\(^6\). To be explicit, one may parametrize this kind of \(M_\nu\) as follows:

\[
M_\nu = \begin{pmatrix}
A & B & B' \\
B & C & D \\
B^* & D & C^*
\end{pmatrix}
\]

(3.24)

where \(A\) and \(D\) are two real parameters.

Note that it is possible to obtain a neutrino mass matrix of the above form without invoking the \(\mu-\tau\) reflection symmetry. In [109, 110] the authors started from a particular neutrino mass matrix at a seesaw scale,

\[
M_\nu^0 = m \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

(3.25)

and allowed it to run down to a much lower scale via the one-loop renormalization-group equations (RGEs) in the supersymmetry framework [111, 112]. Then the resulting neutrino mass matrix at the electroweak scale reads as

\[
M_\nu = (I + \Delta)M_\nu^0 (I + \Delta^*)
\]

\[
= m \begin{pmatrix}
1 + 2 \delta_{ee} & \delta_{e\mu} + \delta_{e\tau} & \delta_{e\mu} + \delta_{e\tau}^* \\
\delta_{e\mu} + \delta_{e\tau} & 1 + \delta_{\mu\mu} + \delta_{\mu\tau} & 2 \delta_{\mu\mu} \\
\delta_{e\mu} + \delta_{e\tau}^* & 2 \delta_{\mu\mu} & 1 + \delta_{\tau\mu} + \delta_{\tau\tau}^* \\
\end{pmatrix}
\]

(3.26)

with \(I\) being the identity matrix and \(\Delta\) representing the Hermitian radiative correction matrix,

\[
\Delta = \begin{pmatrix}
\delta_{ee} & \delta_{e\mu} & \delta_{e\tau} \\
\delta_{e\mu}^* & \delta_{\mu\mu} & \delta_{\mu\tau} \\
\delta_{e\tau}^* & \delta_{\mu\tau} & \delta_{\tau\tau}
\end{pmatrix}
\]

(3.27)

It is obvious that the texture of \(M_\nu\) in equation (3.26) is the same as that in equation (3.24), but the former as a model-building example crucially depends on the special form of \(M_\nu^0\) and the relevant RGEs. We shall subsequently concentrate on the pattern of \(M_\nu\) in equation (3.24) and assume it to result from the \(\mu-\tau\) reflection symmetry.

The correspondence between the flavor mixing matrix in equation (3.19) and the neutrino mass matrix in equation (3.24) (i.e. the former as a consequence of the latter) was first noticed in [113]. It can be verified by taking \(M_\nu = UD_\nu U^\dagger\) in equation (3.23),

\[
S^+U^*D_\nu U^{\dagger} S^+ = UD_\nu U^\dagger,
\]

(3.28)

where \(D_\nu\) has been defined in equation (2.3). Hence \(S^+U^*\) is identical to \(U\) up to a diagonal phase matrix \(X\)—namely, \(S^+U^* = UX\), in which \(X_{ij}\) is either an arbitrary phase factor for \(m_i = 0\) or \(\pm 1\) for \(m_i \neq 0\). We are therefore led to a particular neutrino mixing matrix of the form

\[U = \begin{pmatrix}
\sqrt{X_{11}} \nu_1 \\
\sqrt{X_{22}} \nu_2 \\
\sqrt{X_{33}} \nu_3
\end{pmatrix}
\]

(3.29)

It is clear that the relation \([U^\dagger]\! U = |U|\) holds, and this pattern of \(U\) will have the same form as the one given in equation (3.19) if \(X_{ii} = 1\) (for \(i = 1, 2, 3\)) is taken. Provided \(X_{ii}\) happens to be \(-1\), one may redefine the corresponding neutrino mass eigenstate \(\nu_i\) as \(i\nu_i\) to absorb the imaginary factor in the first row of \(U\) and assure the second and third rows of \(U\) to satisfy \(U_{ii} = U^\tau_{ii}\). In the meantime the mass eigenvalue \(m_i\) should be replaced by \(m_i' = -m_i\), implying that \(\nu_i'\) has a Majorana phase equal to \(\pi/2\).

In short, as for the Majorana neutrinos, the flavor mixing matrix resulting from the mass matrix \(M_\nu\) in equation (3.24) can also be parameterized as that given by equation (3.19), but the relevant Majorana phases are fixed to be 0 or \(\pi/2\). One may see this interesting conclusion from another angle. Let us convert \(M_\nu\) in equation (3.24) into a real matrix by a unitary transformation in the (2,3) plane [114]:

\[
U'_{23}M_\nu U'_{23} = \begin{pmatrix}
A & \sqrt{2} \text{Im}B & \sqrt{2} \text{Re}B \\
\cdots & D - \text{Re}C & \text{Im}C \\
\cdots & \cdots & D + \text{Re}C
\end{pmatrix}
\]

(3.30)

with

\[
U'_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2} & 0 & 0 \\
0 & i & 1 \\
0 & -i & 1
\end{pmatrix}
\]

(3.31)

The mass matrix in equation (3.30) can subsequently be diagonalized by a real orthogonal matrix \(O\) to get real mass eigenvalues, and the resulting neutrino mixing matrix \(U = U'_{23}O\) takes the form given by equation (3.19) with special Majorana phases. To conclude, the \(\mu-\tau\) reflection symmetry can not only predict \(\theta_{23} = \pi/4\) and \(\delta = \pm \pi/2\) but also constrain the corresponding Majorana phases to be 0 or \(\pi/2\).

It is worth stressing that equation (3.24) is not the only form of \(M_\nu\), which is able to generate both \(\theta_{23} = \pi/4\) and \(\delta = \pm \pi/2\) in the elements of a neutrino mass matrix just need to satisfy the following relation to achieve this goal [102, 115–118]:

\[
\sum_{\alpha=e}^{\tau} M_{\alpha\nu} M_{\nu\alpha}^* = \sum_{\alpha=e}^{\tau} M_{\alpha\nu} M_{\nu\alpha}^*.
\]

(3.32)

There are some other solutions to this equation, besides that given by equation (3.24). For example,

\[
M_{\nu\nu} = M_{\mu\nu}, \quad M_{\mu\mu} = M_{\tau\nu}, \quad M_{\mu\tau} = M_{\tau\tau}, \quad M_{\mu\mu} = M_{\tau\tau}, \quad M_{\mu\tau} = M_{\tau\tau}, \quad (3.33)
\]

leading to a special neutrino mass matrix of the form

\[
M_\nu = \begin{pmatrix}
A & B & B^* \\
B & C & A \\
B^* & A & C^*
\end{pmatrix}
\]

(3.34)

Note that this matrix is not equivalent to that given by equation (3.24) even if \(D = A\) is taken, because here \(A\) is a complex parameter instead of a real one. Note also that the present matrix does not possess a transparent flavor symmetry
(e.g. the $\mu$-$\tau$ reflection symmetry), and thus it is difficult to be derived from an underlying flavor model. That is why we pay particular attention to the pattern of $M_\nu$ in equation (3.24), which is much more favored from the model-building point of view.

3.3. Breaking of the $\mu$-$\tau$ permutation symmetry

Now that the $\mu$-$\tau$ permutation symmetry gives rise to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$, the observed result of $\theta_{13}$ ($\approx 9^\circ$) and a possible deviation of $\theta_{23}$ from $\pi/4$ definitely signify the breaking of this interesting flavor symmetry. One nevertheless should discuss the symmetry-breaking effects at the mass matrix level, in light of the fact that the symmetry is realized for the mass matrix rather than for the flavor mixing matrix. To quantify the strength of $\mu$-$\tau$ symmetry breaking, it is necessary to introduce a characteristic measure of such effects. In fact, the most general perturbation to a Majorana neutrino mass matrix $M_\nu^{(0)}$ with the $\mu$-$\tau$ permutation symmetry can be decomposed into a symmetry-conserving part and a symmetry-violating part:

$$M_\nu^{(1)} = \frac{1}{2} \left( \begin{array}{ccc} 2\delta_{ee} & \delta_{em} & \delta_{et} \\
\delta_{em} & \delta_{mm} + \delta_{er} & 2\delta_{mt} \\
\delta_{et} & 2\delta_{mt} & \delta_{tt} \\
\end{array} \right) + \frac{1}{2} \left( \begin{array}{ccc} 0 & \delta_{mm} - \delta_{er} & \delta_{et} - \delta_{ee} \\
\delta_{mm} - \delta_{er} & 0 & \delta_{tt} - \delta_{ee} \\
\delta_{et} - \delta_{ee} & \delta_{tt} - \delta_{ee} & 0 \\
\end{array} \right)$$

(3.35)

whose parameters are small in magnitude. Because the symmetry-conserving part can be absorbed via a redefinition of the original elements of $M_\nu^{(0)}$, we are left with the full neutrino mass matrix $M_\nu = M_\nu^{(0)} + M_\nu^{(1)}$ in the following form [101]:

$$M_\nu = \begin{pmatrix} A' & B'(1 + \epsilon_1) & B'(1 - \epsilon_1) \\
B'(1 + \epsilon_1) & C'(1 + \epsilon_2) & D' \\
B'(1 - \epsilon_1) & D' & C'(1 - \epsilon_2) \end{pmatrix}$$

(3.36)

with

$$A' = A + \delta_{ee}, \quad B' = B + \frac{\delta_{em} + \delta_{er}}{2},$$

$$D' = D + \delta_{ee}, \quad C' = C + \frac{\delta_{mm} + \delta_{er}}{2},$$

$$\epsilon_1 = \frac{\delta_{mm} - \delta_{er}}{2B'}, \quad \epsilon_2 = \frac{\delta_{mm} - \delta_{er}}{2C'}.$$  

(3.37)

So it is convenient to use the dimensionless parameters

$$\epsilon_1 = \frac{M_{ee} - M_{ee}}{M_{mm} + M_{ee}}, \quad \epsilon_2 = \frac{M_{mm} - M_{ee}}{M_{mm} + M_{ee}}$$

(3.38)

to measure the effects of $\mu$-$\tau$ symmetry breaking. If $|\epsilon_1|$ and $|\epsilon_2|$ are both small enough (e.g. $\lesssim 0.2$), then one may argue that the neutrino mass matrix $M_\nu$ possesses an approximate $\mu$-$\tau$ permutation symmetry.

The small parameters $\epsilon_1$ and $\epsilon_2$ can be linked to the observable quantities through a reconstruction of $M_{\nu_{ij}}$ in terms of the neutrino mass and flavor mixing parameters. Note that $|\epsilon_{1,2}|$ are not rephasing-invariant—namely, they are sensitive to the phase transformations

$$\nu_e \rightarrow e^{i\phi} \nu_e, \quad \nu_\mu \rightarrow e^{i\phi} \nu_\mu, \quad \nu_\tau \rightarrow e^{i\phi} \nu_\tau.$$  

(3.39)

In this case the values of $|\epsilon_{1,2}|$ will accordingly change:

$$|\epsilon_1| \rightarrow \frac{|M_{ee} - M_{ee} e^{i\phi_{13}}|}{|M_{ee} + M_{ee} e^{i\phi_{13}}|},$$

$$|\epsilon_2| \rightarrow \frac{|M_{mm} - M_{ee} e^{i\phi_{13}}|}{|M_{mm} + M_{ee} e^{i\phi_{13}}|},$$

(3.40)

where $\phi_{13,2} \equiv \phi_1 - \phi_2$. Hence the values of $|\epsilon_{1,2}|$ are not fully physical. Nevertheless, $\phi_{13,2}$ is supposed to be strongly suppressed in the presence of an approximate $\mu$-$\tau$ permutation symmetry. So one expects that this small phase difference should not change the main features of $|\epsilon_{1,2}|$. Based on this reasonable argument, we neglect $\phi_{13,2}$ for the time being and shall take account of it when necessary. Figure 5 illustrates the possible values of $|\epsilon_{1,2}|$ against the lightest neutrino mass, where the relevant neutrino oscillation parameters used to reconstruct $M_{\nu_{ij}}$ take values in their $1\sigma$ ranges given by table 2. We see that $|\epsilon_1|$ and $|\epsilon_2|$ are simultaneously small when the neutrino mass spectrum has an inverted hierarchy, and $|\epsilon_2|$ is unacceptably large when $m_1$ is very small (i.e. when the neutrino mass hierarchy is normal). The numerical results shown in figure 5 can be understood by using the approximate expressions of $|\epsilon_{1,2}|$ to be given below. Thanks to the smallness of $\theta_{13} \sim 0.15$ and $\Delta \theta_{23} \equiv \theta_{23} - \pi/4 \in [-0.09, +0.09]$, it is possible to obtain

$$\epsilon_1 \sim -\Delta \theta_{23} = \frac{m_1 \theta_{13} - m_2 \theta_{13}}{m_2 - m_1},$$

$$\epsilon_2 \sim 2 \Delta \theta_{23} = \frac{4m_2 \Delta \theta_{23} + 2m_2 \theta_{13}}{m_2 + m_3},$$

(3.41)

where the notations introduced in equations (3.2) and (3.5) have been used (in particular, here $\theta_{13} \equiv \theta_{13}^{\text{obs}}$). For the sake of simplicity, we consider the properties of $|\epsilon_{1,2}|$ in three typical cases [119]:

1. $m_1 \ll m_2 \ll m_3$. Given this highly hierarchical neutrino mass spectrum, we obtain

$$|\epsilon_1| \sim \frac{m_1 \theta_{13}}{m_2 \theta_{13}^{\text{obs}}} \approx 1.9, \quad |\epsilon_2| \sim 2 |\Delta \theta_{23}|.$$  

(3.42)

implying that $M_\nu$ itself does not really exhibit an approximate $\mu$-$\tau$ permutation symmetry.

2. $m_2 \approx m_3 \gg m_1$. Because of the near equality of $m_1$ and $m_3$ in this case, the Majorana phases $\rho$ and $\sigma$ become relevant:

$$\epsilon_1 \sim -\Delta \theta_{23} = \frac{c_{12}^2 + s_{12}^2 e^{i(\sigma - \rho)} c_{13}^2 s_{12}^2}{[1 - e^{i(\sigma - \rho)} c_{12}^2 s_{12}^2]} \theta_{13},$$

$$\epsilon_2 \sim 2 \Delta \theta_{23} = \frac{1 - e^{i(\sigma - \rho)} c_{12}^2 s_{12}^2}{c_{12}^2 + s_{12}^2 e^{i(\sigma - \rho)} c_{13}^2 s_{12}^2} \theta_{13}.$$  

(3.43)
Note that the coefficient of $\tilde{\theta}_{13}$ in $e_1$ is inversely proportional to that in $e_2$. So $|e_1|$ will be much larger than 1 for $\sigma - \rho \sim 0$, and $|e_2|$ will have a too large value for $\sigma - \rho \sim \pi/2$. A careful analysis indicates that $|e_1|$ and $|e_2|$ have no chance to be small enough at the same time.

- $m_2 \simeq m_3 \gg m_1$. In this case let us consider four sets of special but typical values of $\rho$ and $\sigma$. Given $(\rho, \sigma) = (0, 0)$ or $(\pi/2, \pi/2)$, for example, $|e_1|$ will be much larger than 1 owing to a nearly complete cancellation between the two components of $m_{12}$ as one can see in equation (3.5). If $(\rho, \sigma) = (0, \pi/2)$ or $(\pi/2, 0)$, then $e_{1,2}$ can approximate to

$$e_1 \sim -\Delta \theta_{23} + \frac{\exp(i\phi - s_{12}^2) \exp(i\phi_{13})}{2c_{12}s_{12}} \tilde{\theta}_{13},$$

$$e_2 \sim 2(1 \pm c_{12}^2 \pm s_{12}^2) \Delta \theta_{23} \pm 4c_{12}s_{12}\tilde{\theta}_{13},$$

which are allowed to be simultaneously small.

We conclude that a quasi-degenerate neutrino mass spectrum allows $M_\nu$ to show an approximate $\mu$-$\tau$ permutation symmetry. This conclusion can also be put in another way: given $m_1 \simeq m_2 \simeq m_3$, the Majorana neutrino mass matrix with an approximate $\mu$-$\tau$ permutation symmetry is able to generate a phenomenologically viable pattern of neutrino mixing—a pattern compatible with current experimental data, as one will see later on. A similar approach has been discussed in [120] to explore the gap between $U_{TB}$ and the experimentally-allowed pattern as well as what such a gap implies on the underlying texture of $M_\nu$. Because $U_{TB}$ can be obtained by invoking the condition $M_{ee} + M_{\mu\mu} = M_{\mu\tau} + M_{\tau\mu}$ for $M_\nu$ on the basis of the $\mu$-$\tau$ permutation symmetry, one needs $e_1$, $e_2$ and an extra small parameter to describe the actual departure of $M_\nu$ from $M_\nu^{TB} \equiv U_{TB}D_\nu U_{TB}^T$. Hence a similar conclusion has been reached [120]: only in the $m_1 \simeq m_2 \simeq m_3$ case can $M_\nu$ possess an approximate $\mu$-$\tau$ symmetry and give rise to a viable neutrino mixing pattern which is very close to the form of $U_{TB}$. In this sense $U_{TB}$ itself is accidental, at least from a point of view of model building, although its predictions for $\theta_{12}$ and $\theta_{23}$ are quite close to the experimental values.

Instead of reconstructing $M_\nu$ in terms of neutrino masses and flavor mixing parameters to constrain the sizes of $e_{1,2}$, now we follow a more straightforward way to study the texture of $M_\nu$ with an approximate $\mu$-$\tau$ permutation symmetry (i.e. both $e_1$ and $e_2$ are assumed to be small from the beginning) and explore the direct dependence of $\tilde{\theta}_{13}$ and $\Delta \theta_{23}$ on $e_{1,2}$. There are two good reasons for doing so: (a) it is important to identify what kind of symmetry breaking can give rise to the viable phenomenological consequences in a model-building exercise; (b) some specific symmetry-breaking patterns are able to predict a few interesting correlations among the physical parameters and thus may bridge the gap between the unknown and known parameters. Let us first derive the expressions of $\tilde{\theta}_{13}$ and $\Delta \theta_{23}$ arising from the most general symmetry breaking pattern. In the standard parametrization the unitary matrix used for diagonalizing a Majorana neutrino mass matrix takes the form $U = P_{TB}V_P$, where $V$ and $P$ have been given in equations (2.6)–(2.9), and $P_{TB} = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$. Although $\phi_1$, $\phi_2$ and $\phi_3$ have no physical meaning, they are necessary in the diagonalization process and thus should not be ignored from here. When the $\mu$-$\tau$ permutation symmetry is exact, $\phi_2$ is automatically equal to $\phi_3$. In the presence of small symmetry-breaking effects, $\phi_{32}$ becomes a finite but small quantity. By taking $M_\nu = UD_UD^T$ in equation (3.36) and doing perturbation expansions for those small quantities, one can obtain the following relations which connect $\phi_{32}$, $\Delta \theta_{23}$ and $\tilde{\theta}_{13}$ with the perturbation parameters $e_{1,2}$:

$$m_{12}\Delta \theta_{23} + m_{13}\tilde{\theta}_{13} - m_{23}\tilde{\theta}_{13} - \frac{m_{12}^2}{2}(2e_1 + i\phi_{32}) = 0,$$

$$m_{22-3}\Delta \theta_{23} + m_{23}\tilde{\theta}_{13} - \frac{m_{23}^2 + \frac{m_{23}^2}{2}}{2}(e_2 + i\phi_{32}) = 0,$$

in which $m_{22+3} = m_{32} \pm m_3$ are defined. After solving these equations in a straightforward way, we obtain $\tilde{\theta}_{13}^*$ and $\Delta \theta_{23}$ as functions of $e_{1,2}$ [101]:

$$\tilde{\theta}_{13}^* = (2\Delta m^2_{31})^{-1}\left[2m_3 m_{12}c_{12}\tilde{e}_1 + 2m_1 m_{12}c_{12}^2 \tilde{e}_1^* + m_{22-3}c_{12}^2\tilde{e}_2 + m_{13}\tilde{e}_3^* \right] + (2\Delta m^2_{32})^{-1}\left[2m_3 m_{12}s_{12}\tilde{e}_1 + 2m_1 m_{12}s_{12}^2 \tilde{e}_1^* - m_{22-3}s_{12}^2\tilde{e}_2 - m_{13}\tilde{e}_3^* \right],$$

$$\Delta \theta_{23} = \text{Re}\{\frac{(2\Delta m^2_{31})^{-1}}{2}[2m_3 c_{12} s_{12}(m_{12} \tilde{e}_1 s_{12} + m_3 \tilde{e}_3^*) + m_{22-3} c_{12}^2 s_{12}^2 \tilde{e}_2 + m_{13} s_{12}^2 \tilde{e}_3^*] + \frac{(2\Delta m^2_{32})^{-1}}{2}[2m_3 c_{12} s_{12}(m_{12} \tilde{e}_1 c_{12}^2 + m_3 \tilde{e}_3^*) - m_{22-3} c_{12}^2 s_{12}^2 \tilde{e}_2 + m_{13} c_{12}^2 \tilde{e}_3^*] \}.$$
With the help of these results, one may get a ball-park feeling of the dependence of $\theta_{13}$ and $\Delta \theta_{23}$ on $\epsilon_{1,2}$ in some particular cases to be discussed below. One will see that the values of $\theta_{13}$ and $\Delta \theta_{23}$ are strongly correlated with the neutrino mass spectrum when the strength of $\mu-\tau$ permutation symmetry breaking (i.e. the size of $\epsilon_{1,2}$) is specified.

In the assumption of CP conservation, equation (3.46) is simplified to

$$\theta_{13} = \frac{2m_{\nu_{12}}^2}{2(m_3 + m_1)} + \frac{2m_{\nu_{12}}^2}{2(m_3 + m_2)} - \frac{2m_{\nu_{12}}^2 - 2m_{\nu_{12}}^2}{2(m_3 + m_2)},$$

$$\Delta \theta_{23} = \frac{2m_{\nu_{12}}^2}{2(m_3 + m_1)} - \frac{2m_{\nu_{12}}^2 - 2m_{\nu_{12}}^2}{2(m_3 + m_2)},$$

in which the signs ‘$\mp$’ correspond to $m_{1,2} = \pm m_{1,2}$ for $\rho, \sigma = 0$ or $\pi/2$. Some more specific discussions and comments are in order.

1. As for the neutrino mass spectrum with a vanishingly small $m_1$, the expression of $\theta_{13}$ can be further simplified to

$$\theta_{13} \sim \frac{1}{2} \sqrt{\tau} \ c_{12} s_{12} (2 \epsilon_1 - \epsilon_2) \simeq 0.04 (2 \epsilon_1 - \epsilon_2).$$

where $\tau = \Delta m_{31}^2 / \Delta m_{32}^2 \approx 0.03$. Given $|\epsilon_{1,2}| \lesssim 0.2$, it is impossible to get the observed value of $\theta_{13}$ from the above expression.

2. In the $m_1 \approx m_2 \gg m_3$ case, $\theta_{13}$ and $\Delta \theta_{23}$ are sensitive to the combination of $\rho$ and $\sigma$. If $\rho$ and $\sigma$ are equal, $\theta_{13}$ will be highly suppressed:

$$\theta_{13} \sim \frac{1}{4} r c_{12} s_{12} (2 \epsilon_1 - \epsilon_2) \simeq 0.004 (2 \epsilon_1 - \epsilon_2).$$

Otherwise, $\theta_{13}$ turns out to be

$$\theta_{13} \sim \frac{1}{2} \cos 2 \theta_{12} \sin 2 \theta_{13} (2 \epsilon_1 - \epsilon_2),$$

which is still unable to fit its experimental value.

3. When the neutrino mass spectrum is nearly degenerate, one may consider the following four special but typical cases.

- $(\rho, \sigma) = (0, 0)$. In this case $\theta_{13}$ approximates to

$$\theta_{13} \sim \frac{2m_0^2}{2m_3} c_{12} s_{12} \epsilon_2,$$

where $m_0$ denotes the overall neutrino mass scale. The factor $m_0^2 / \Delta m_{31}^2$ can enhance the value of $\theta_{13}$ as $m_0$ goes up, but $m_0 \lesssim 0.1$ eV is expected to hold in light of the present cosmological upper bound on the sum of three neutrino masses (e.g. $m_1 + m_2 + m_3 \approx 3 m_0 < 0.23$ eV at the 95% confidence level [121]). So $\theta_{13}$ is at most 0.03. Such a result is certainly unacceptable.

- $(\rho, \sigma) = (0, \pi/2)$. In this case we obtain

$$\theta_{13} \sim \frac{2m_0^2}{2m_3} c_{12} s_{12} (2c_1^2 \epsilon_1 + s_1^2 \epsilon_2),$$

$$\Delta \theta_{23} \sim \frac{2m_0^2}{2m_3} c_{12} s_{12} (2c_1^2 \epsilon_1 + s_1^2 \epsilon_2).$$

Thanks to the enhancement factor $m_0^2 / \Delta m_{31}^2$, $\theta_{13}$ is easy to reach the measured value. Moreover, there is a correlation between $|\Delta \theta_{23}|$ and $\theta_{13}$, i.e. $|\Delta \theta_{23}| \sim \theta_{13} \epsilon_{12} \epsilon_{12} \approx 6^\circ$. This relatively large $|\Delta \theta_{23}|$ will be tested in the near future.

- $(\rho, \sigma) = (\pi/2, 0)$. In this case the results are

$$\theta_{13} \sim \frac{2m_0^2}{2m_3} c_{12} s_{12} (2c_1^2 \epsilon_1 + s_1^2 \epsilon_2),$$

$$\Delta \theta_{23} \sim \frac{2m_0^2}{2m_3} c_{12} s_{12} (2c_1^2 \epsilon_1 + s_1^2 \epsilon_2).$$

The correlation between $|\Delta \theta_{23}|$ and $\theta_{13}$ predicts $|\Delta \theta_{23}| \sim \theta_{13} \epsilon_{12} \epsilon_{12} \approx 13^\circ$, which is too large to be acceptable.

- $(\rho, \sigma) = (\pi/2, \pi/2)$. This special case is disfavored because $\theta_{13}$ is extremely suppressed by the factor $\Delta m_{31}^2 / m_0^2$ as one can see from

$$\theta_{13} \sim \frac{\Delta m_{31}^2}{4m_0^2} c_{12} s_{12} \epsilon_1.$$

To summarize, in the CP-conserving case considered above only the example of a quasi-degenerate neutrino mass spectrum with $(\rho, \sigma) = (0, \pi/2)$ is still allowed by current experimental data.

When CP violation is taken into account, one has to deal with some more free parameters. But this possibility is certainly more realistic and more interesting, because it is related to the asymmetry between matter and antimatter in a variety of weak interaction processes including neutrino oscillations. Here let us take two typical examples for illustration.

(A) In the first example we assume the Majorana phases to take trivial values (i.e. 0 or $\pi/2$) and the symmetry-breaking parameters to be purely imaginary (i.e. $\epsilon_{1,2} = i |\epsilon_{1,2}|$). Then equation (3.46) leads us to the results $\Delta \theta_{23} = 0$ and

$$\theta_{13} \sim \frac{2m_0^2}{2m_3} c_{12} s_{12} \epsilon_2,$$

which implies $\delta = \pm \pi/2$. These results are actually the same as those predicted by the $\mu-\tau$ reflection symmetry, simply because the perturbation under consideration is so special that the overall neutrino mass matrix $M$ as given in equation (3.36) respects this flavor symmetry.
Note that \( \theta_{13} \) has the same expressions as those shown in equations (3.48)–(3.50) when \( m_1 \) or \( m_3 \) is vanishingly small. If the quasi-degenerate neutrino mass spectrum is concerned, then

\[
(\rho, \sigma) = (0, 0) : \theta_{13} \approx \frac{\Delta m^2_{31}}{8 m^2_6} c_{12} s_{13} (2|\epsilon_1| - |\epsilon_2|),
\]

\[
(\rho, \sigma) = (0, \pi/2) : \theta_{13} \approx \frac{2 m^2_2}{\Delta m^2_{31}} c_{12} s_{13} (2|\epsilon_1| - |\epsilon_2|),
\]

\[
(\rho, \sigma) = (\pi/2, 0) : \theta_{13} \approx \frac{2 m^2_2}{\Delta m^2_{31}} c_{12} s_{13} (2|\epsilon_1| - |\epsilon_2|),
\]

\[
(\rho, \sigma) = (\pi/2, \pi/2) : \theta_{13} \approx \frac{r}{2} c_{12} s_{13} (2|\epsilon_1| - |\epsilon_2|) \tag{3.56}
\]

can be obtained. Similar to the results achieved in the CP-conserving case, an acceptable value of \( \theta_{13} \) is only obtainable from equation (3.56) with \((\rho, \sigma) = (0, \pi/2) \) or \((\pi/2, 0) \). It is worth mentioning that \( \theta_{13} \) is always proportional to \( 2|\epsilon_1| - |\epsilon_2| \) in this case, because it is the combination \( 2\epsilon_1 - \epsilon_2 \) that is invariant under the phase transformations in equation (3.39) when \( \epsilon_{1,2} \) are imaginary.

(B) In the second example we relax \( \rho \) and \( \sigma \) to see whether \( \theta_{13} \) is possible to fit current experimental data in the \( m_1 \approx m_2 \gg m_3 \) case, where

\[
\theta_{13} = \sqrt{\cos^2 2\theta_1 [1 - \cos 2(\sigma - \rho)]^2 + \sin^2 2(\sigma - \rho)} \times \frac{r}{2} c_{12} s_{13} (2|\epsilon_1| - |\epsilon_2|). \tag{3.57}
\]

The maximal value of \( \theta_{13} \) turns out to be \( \theta_{13}^{\text{max}} \approx 0.252 |\epsilon_1| - |\epsilon_2| \) at \( \sigma - \rho = \pi/4 \). It is therefore hard to make \( \theta_{13} \) compatible with the data unless \( \epsilon_{1,2} \) have a relative phase of \( \pi \) and take their upper limit \( \sim 0.2 \) (for a valid perturbation expansion). In this extreme case \( \Delta \theta_{23} \approx -0.03 \cos^{-1} (\epsilon_{11}) \lesssim 2\nu \). Of course, the chance for this extreme case to happen is rather small. So it is fair to say that the \( m_3 \to 0 \) limit seems not be favored by \( m_\nu \) in equation (3.36) even when CP violation is taken into account. Finally, it is worth emphasizing that \( \delta \) may take any value in the interval \([0, 2\pi)\) regardless of the strength of \( \mu-\tau \) symmetry breaking. This point can be interpreted as follows. The vanishing of \( \theta_{13} \) and \( \Delta \theta_{23} \) is protected by the \( \mu-\tau \) permutation symmetry, so their realistic magnitudes are subject to the small effects of symmetry breaking. In contrast, \( \delta \) is not well defined when the \( \mu-\tau \) permutation symmetry is exact, and it turns out to have physical meaning only after this symmetry is broken and the resulting neutrino mass matrix \( M_\nu = M^{(0)}_\nu + M^{(1)}_\nu \) contains the nontrivial phases. Hence \( \delta \) is sensitive to the phases of \( \epsilon_{1,2} \). As shown in equation (3.50), \( \delta = \pm \pi/2 \) can emerge if \( \epsilon_{1,2} \) are purely imaginary. Note that a finite value of \( \delta \) may arise even when \( \epsilon_{1,2} \) are real, if \( M^{(0)}_\nu \) itself involves the nontrivial Majorana phases [101]. Here let us take \( \epsilon_2 = 2\epsilon_1 \) as an example which resembles the symmetry breaking induced by the RGE running effects, as one can see later. In this special case \( \delta \) is given by

\[
\tan \delta = \frac{m_2 \sin 2\sigma - m_3 \sin 2\rho}{m_1 \cos 2\rho - m_2 \cos 2\sigma - m_3 \rho}, \tag{3.58}
\]

which is not directly associated with \( \epsilon_{1,2} \) and may be large no matter whether the neutrino mass spectrum exhibits a normal or inverted hierarchy.

The above discussions are subject to the smallness of \( \mu-\tau \) symmetry breaking—namely, the symmetry-breaking terms are relatively small as compared with the entries of \( M^{(0)}_\nu \). In a given neutrino mass model the elements of \( M^{(0)}_\nu \) may not be at the same order of magnitude, and some of them are even possible to vanish at the tree level. Provided the higher-order contributions break the flavor symmetry, then they may play a dominant role in those originally vanishing entries to make the predictions of the resulting neutrino mass matrix \( M_\nu \) compatible with current neutrino oscillation data. Of course, there is nothing wrong with this situation, but it motivates us to reconsider the physical implications of an approximate \( \mu-\tau \) permutation symmetry. Though a concrete model may more or less correlate the symmetry-breaking terms appearing in different elements of \( M_\nu \), we just treat them as independent parameters in our subsequent analysis. A good example of this kind is a hierarchical neutrino mass matrix which can lead us to the mass spectrum \( m_1 < m_2 \ll m_3 \). In this case the \( \mu, \mu \) and \( \tau \tau \) entries of \( M_\nu \) are expected to be much larger than the others in magnitude by a factor \( \sim \sqrt{1/r} = \sqrt{\Delta m^2_{31}/\Delta m^2_{21}} \). If these large entries arise from a simple flavor symmetry and those smaller ones come from the symmetry-breaking effects, a typical form of \( M_\nu \) may be [122]

\[
M_\nu = m_0 \begin{pmatrix} d e & c e & b e \\ c e & 1 + a e & -1 \\ b e & -1 & 1 + e \end{pmatrix}, \tag{3.59}
\]

where \( \epsilon \) denotes a small perturbation (or symmetry-breaking) parameter, and \( a, b, c \) and \( d \) are all the real coefficients of \( O(1) \). A straightforward calculation yields the flavor mixing angles and neutrino mass eigenvalues as follows:

\[
\theta_{13} \approx \frac{1}{2\sqrt{2}} (b - c) \epsilon, \\
\Delta \theta_{23} \approx \frac{1}{4} (a - 1) \epsilon, \\
\tan 2\theta_{12} \approx \frac{2\sqrt{2} (b + c)}{a + 1 - 2d} ; \tag{3.60}
\]

and

\[
m_1 \approx \frac{1}{4} em_0 (2d + a + 1 - \Delta), \\
m_2 \approx \frac{1}{4} em_0 (2d + a + 1 + \Delta), \\
m_3 \approx 2m_0. \tag{3.61}
\]
in which \( \Delta = \sqrt{(2d - a - 1)^2 + 8(b + c)^2}. \) The small perturbation parameter \( \epsilon \) is found to be

\[
\epsilon \simeq \frac{4\sqrt{c}}{\sqrt{\Delta(2d + a + 1)}}.
\] (3.62)

We see that \( \theta_{13} \) and \( \Delta\theta_{23} \) are proportional to \( b - c \) and \( a - 1 \), respectively. Moreover, \( \theta_{13} \) is about \( \sqrt{c} \equiv \sqrt{\Delta m^2_{31}/\Delta m^2_{31}} \) up to an \( O(1) \) factor, in agreement with the experimental result. So the smallness of \( \theta_{13} \) is connected to the hierarchy between \( \Delta m^2_{31} \) and \( \Delta m^2_{31} \) in this scenario.

If \( m_1 \approx m_2 \gg m_3 \) holds, \( M_\nu \) is expected to exhibit a different hierarchical structure, in which the \( ee, \mu\mu, \mu\tau \) and \( \tau\tau \) entries should be much larger than the others. To illustrate, a simple but instructive example of this kind is

\[
M_\nu = m_0 \begin{pmatrix}
  1 + de & \epsilon & be \\
  c\epsilon & 1 + ae & 1 \\
  be & 1 + c\epsilon & a + 1
\end{pmatrix}.
\] (3.63)

The results for three flavor mixing angles are similar to those given in equation (3.60), but the neutrino mass eigenvalues turn out to be

\[
\begin{align*}
m_1 &\approx 2m_0 + \frac{1}{4} \epsilon m_0 (2d + a + 1 - \Delta), \\
m_2 &\approx 2m_0 + \frac{1}{4} \epsilon m_0 (2d + a + 1 + \Delta), \\
m_3 &\approx \frac{1}{2} \epsilon m_0 (a + 1).
\end{align*}
\] (3.64)

In this case \( \epsilon \approx 2r/\Delta \), leading to a high suppression of \( \theta_{13} \) and \( \Delta\theta_{23} \). The point is that \( m_1 \) is equal to \( m_2 \) at the leading order, and their degeneracy is lifted at the next-to-leading order. So \( \epsilon \) should be at the order of \( (m_3 - m_1)/(m_2 + m_3) \approx 0.01 \). To get around this problem, one may assume \( a = -1, b = -c \) and \( d = 0 \), such that \( r = \Delta = 0 \) holds and \( \epsilon \) may not be strongly suppressed. In this situation it is possible to obtain an appreciable value of \( \theta_{13} \), but lifting the degeneracy of \( m_1 \) and \( m_2 \) requires a higher-order perturbation.

Last but not least, there is a well-known form of \( M_\nu \) which yields an inverted neutrino mass spectrum with \( m_3 = -m_2 \) at the tree level:

\[
M_\nu = m_0 \begin{pmatrix}
  1 + de & \epsilon & be \\
  c\epsilon & 1 + ae & 1 \\
  be & 1 + c\epsilon & a + 1
\end{pmatrix},
\] (3.65)

in which the most general next-to-leading-order terms have been included. This particular texture of \( M_\nu \) can be attributed to an \( L_e - L_\mu - L_\tau \) flavor symmetry [123]—namely, the leading-order entries have a vanishing \( L_e - L_\mu - L_\tau \) charge but the others do not. After a straightforward calculation, the phenomenological consequences of this neutrino mass matrix can be summarized as follows:

\[
\theta_{13} \approx \frac{(a - b)\epsilon}{2\sqrt{2}}, \\
\Delta\theta_{23} \approx \frac{(c - d)\epsilon}{2}, \\
\tan 2\theta_{12} \approx \frac{4\sqrt{2}}{(2 + a + b - 2e)\epsilon};
\] (3.66)

and

\[
\begin{align*}
m_1 &\approx \frac{1}{4} [2e + 2 + a + b]m_0, \\
m_2 &\approx \frac{1}{4} [2e + 2 + a + b]m_0 + 4\sqrt{2}m_0, \\
m_3 &\approx \frac{1}{2} (a + b - 2e)m_0.
\end{align*}
\] (3.67)

Similar to the previous case, the symmetry-breaking parameter \( \epsilon \) has to be exceedingly small unless \( a = -b \) and \( e = -1 \) are assumed. But such an assumption will lead to \( \tan 2\theta_{12} \approx \sqrt{2}/\epsilon \), implying an unacceptably large value of \( \theta_{13} \) for \( \epsilon \lesssim 0.2 \). Namely, the pattern of \( M_\nu \) in equation (3.65) is not favored by current experimental data. We have certainly assumed CP conservation in the above examples. When CP violation is taken into account, some of the model parameters become complex [124] and the procedure for diagonalizing a complex and hierarchical neutrino mass matrix can be found in [125].

3.4. Breaking of the \( \mu\tau \) reflection symmetry

Let us proceed to discuss possible ways of breaking the \( \mu\tau \) reflection symmetry. Without involving any model details, the most general correction to a neutrino mass matrix with the \( \mu\tau \) reflection symmetry is in the form as given by equation (3.35).

It can be divided into two parts: the part respecting the \( \mu\tau \) reflection symmetry and the part violating this interesting symmetry. Namely,

\[
M_\nu^{(1)} = \frac{1}{2} \begin{pmatrix}
  2\text{Re}\delta_{ee} & \delta_{e\mu} + \delta_{e\tau}^* & \delta_{e\mu} + \delta_{e\tau} \\
  \delta_{e\mu} + \delta_{e\tau} & 2\text{Re}\delta_{\mu\tau} & \delta_{e\mu} + \delta_{e\tau} \\
  \delta_{e\mu} + \delta_{e\tau} & 2\text{Re}\delta_{\mu\tau} & \delta_{e\mu} + \delta_{e\tau}
\end{pmatrix}
\] (3.68)

in a way similar to equation (3.35). As the first part of \( M_\nu^{(1)} \) can be absorbed into the original neutrino mass matrix given in equation (3.24), which is now defined as \( M_\nu^{(0)} \) in the \( \mu\tau \) reflection symmetry limit, the full neutrino mass matrix \( M_\nu = M_\nu^{(0)} + M_\nu^{(1)} \) can be parametrized as

\[
M_\nu = \begin{pmatrix}
  \tilde{A}(1 + i\epsilon_3) & \tilde{B}(1 + \epsilon_1) & \tilde{B}^*(1 - \epsilon_3^*) \\
  \vdots & \tilde{C}(1 + \epsilon_2) & \tilde{D}(1 + i\epsilon_4) \\
  \vdots & \vdots & \tilde{C}^*(1 - \epsilon_2^*)
\end{pmatrix}
\] (3.69)
with
\[ \tilde{A} = A + \text{Re} \delta_{ee}, \quad \tilde{B} = B + \frac{\delta_{ee} + \delta_{ee}^*}{2}, \]
\[ \tilde{D} = D + \text{Re} \delta_{ee}, \quad \tilde{C} = C + \frac{\delta_{ee} + \delta_{ee}^*}{2}; \]  \hspace{1cm} (3.70)

and
\[ \epsilon_1 = \frac{M_{ee} - M_{ee}'}{M_{ee} + M_{ee}'} = \frac{\delta_{ee} - \delta_{ee}^*}{2B}, \]
\[ \epsilon_2 = \frac{M_{ee} - M_{ee}'}{M_{ee} + M_{ee}'} = \frac{\delta_{ee} - \delta_{ee}^*}{2C}, \]
\[ \epsilon_3 = \frac{\text{Im} M_{ee}'}{\text{Re} M_{ee}} = \frac{\text{Im} \delta_{ee}}{A}, \]
\[ \epsilon_4 = \frac{\text{Im} M_{ee}'}{\text{Re} M_{ee}} = \frac{\text{Im} \delta_{ee}}{D}. \]  \hspace{1cm} (3.71)

Hence these dimensionless quantities characterize the strength of $\mu-\tau$ reflection symmetry breaking. They should be small (e.g. $\lesssim 0.2$) so that $M_\nu$ in equation (3.69) may possess an approximate $\mu-\tau$ reflection symmetry. Note that rephasing the neutrino fields actually allows us to remove $\epsilon_{3,4}$, so we shall omit $\epsilon_{3,4}$ without loss of generality in the following discussions. As pointed out in section 3.2, the $\mu-\tau$ reflection symmetry of a neutrino mass matrix $M_\nu^{(0)}$ requires the corresponding neutrino mixing matrix $U^{(0)} = P_\nu^{(0)}V^{(0)}P_\nu^{(0)}$ to have a special pattern in which $\theta_{23}^{(0)} = \pi/4$, $\delta^{(0)} = \pm \pi/2$, $\rho^{(0)} = 0$ or $\pi/2$ and $\sigma^{(0)} = 0$ or $\pi/2$ hold. Moreover, the three phases of $P_\nu^{(0)}$ have to satisfy $\phi_1^{(0)} = \phi_2^{(0)} + \phi_3^{(0)} = 0$ although they have no physical meaning. When this flavor symmetry is slightly broken, the resulting flavor mixing parameters will depart from the above values. Let us define the relevant deviations as $\Delta \phi_1 = \phi_1 - 0$, $\Delta \phi_2 = \phi_2 + \phi_3/2 - 0$, $\Delta \theta_{23} = \theta_{23} - \pi/4$, $\Delta \rho = \rho - \rho^{(0)}$ and $\Delta \sigma = \sigma - \sigma^{(0)}$, and suppose that all of them are small quantities governed by $\epsilon_{1,2}$. The explicit relations of these quantities with $\epsilon_{1,2}$ can be established by doing a perturbation expansion for $M_\nu = U^{(0)}D^{(0)}U^{(0)}$. After some calculations, one arrives at

\[ m_{12}^2 \Delta \rho + m_{23}^2 \Delta \sigma - m_{11}^2 \Delta \phi_1, \]
\[ m_{12}^2 \Delta \rho + m_{23}^2 \Delta \sigma - 2m_{12} \Delta \theta_{23} = -m_{12} \Delta \phi, \]
\[ m_{12}^2 (m_{12}^2 c_{12}^2 s_{13}^2 - m_{12} s_{12}^2 s_{13}) \Delta \rho - m_{23}^2 (m_{12}^2 s_{12}^2 - s_{13}^2) \Delta \sigma \]
\[ = -\frac{1}{2}(m_{12}^2 + m_{11}^2 + m_{23}^2) \Delta \theta_{23} + \frac{1}{2} m_{11}^2 s_{13} \delta, \]
\[ m_{12}^2 (m_{12}^2 + m_{23}^2) \Delta \rho - m_{23}^2 (m_{12}^2 c_{12}^2 - 2 s_{13}^2) \Delta \sigma \]
\[ = -m_{12} \Delta \theta_{23} + m_{12} \delta, \]
\[ = -\frac{1}{2}(m_{22}^2 + m_{32}^2) (\epsilon_2 - 2 \Delta \phi), \]  \hspace{1cm} (3.72)

where $m_{11,23}$ and $m_{22,32}$ stand for $m_{11}$ or $m_{33}$ and $m_{22} \pm m_{33}$ respectively. Moreover, $m_\tau$ (or $m_\mu$) and $s_3$ are equal to $\pm m_\mu$ (or $\pm m_\tau$) and $\pm s_3$, respectively, for $\rho^{(0)} = 0$ or $\pi/2$ (or $\sigma^{(0)} = 0$ or $\pi/2$) and $\delta^{(0)} = \pm \pi/2$. Note that the above four equations actually correspond to the conditions of $\mu-\tau$ reflection symmetry breaking defined in equation (3.71). After solving these equations, one will see the clear dependence of $\Delta \theta_{23}$, $\Delta \rho$ and $\Delta \sigma$ on the perturbation parameters $R_{1,2} \equiv \text{Re}(\epsilon_{1,2})$ and $I_{1,2} \equiv \text{Im}(\epsilon_{1,2})$. For clarity, we express their results in the following parametrizations:

\[ \Delta \theta_{23} = c_{23} R_1 + s_{23} R_2, \]
\[ \Delta \rho = c_{23} R_1 + s_{23} R_2, \]
\[ \Delta \sigma = c_{23} R_1 + s_{23} R_2. \]  \hspace{1cm} (3.73)

The lengthy expressions of these 16 coefficients are listed in the appendix for reference.

Since the Majorana phases cannot be pinned down in a foreseeable future, we only discuss the properties of $\Delta \theta_{23}$ and $\Delta \rho$ in several typical schemes regarding the neutrino mass spectrum.

(1) As for $m_1 < m_2 \ll m_3$, the expressions of $\Delta \theta_{23}$ and $\Delta \rho$ can approximate to

\[ \Delta \theta_{23} \approx (2m_2^2)^{-1} \left[ -2m_2^2 c_{12}^2 s_{13}^2 R_2 - 4m_2^2 \sin^2 \theta_{13} \sin^2 \theta_{13} \right], \]
\[ \Delta \rho \approx (2m_2^2)^{-1} \left[ -2m_2^2 c_{12}^2 \sin^2 \theta_{13} \sin^2 \theta_{13} \right]. \]  \hspace{1cm} (3.74)

In this case the largest coefficient for $\Delta \theta_{23}$ is $|c_{23}^2| \approx 0.5$. On the other hand, $\Delta \theta_{23}$ is almost insensitive to $R_1$ and $I_{1,2}$. In contrast, $\rho$ is more sensitive to the symmetry breaking because of an enhancement factor $1/\sin^2 \theta_{13}$. Numerically, the coefficients $|c_{23}^2|_{(1,1/2)}$ (or $|c_{23}^2|_{(2,1/2)}$) take values of $O(1)$ (or $O(0.1)$) around $m_1 \approx 0.001 \text{ eV}$.

(2) The results in the $m_1 \approx m_3 \gg m_2$ case strongly depend on the values of $\rho^{(0)}$ and $\sigma^{(0)}$. When these two phases are equal, one will have

\[ \Delta \theta_{23} \approx -s_{23} R_1 - c_{23} R_1 I_{1,2} \left( 1 - \frac{3}{4} I_{1,2} \right) - \frac{1}{2} R_2, \]
\[ \Delta \rho \approx (4c_{23}^2 s_{13}^2)^{-1} \left[ -c_{23}^2 s_{13}^2 R_2 + \frac{3}{4} I_{1,2} \right]. \]  \hspace{1cm} (3.75)

where $r \equiv \Delta m_{21}^2/\Delta m_{31}^2 \approx 0.03$. Among the relevant coefficients, $|c_{23}^2|_{(1/2)}$ and $|c_{23}^2|_{(1/2)}$ are of $O(1)$, and the others are much smaller. If $\rho^{(0)} = 0$ and $\sigma^{(0)} = \pi/2$, the expressions of $\Delta \theta_{23}$ and $\Delta \rho$ turn out to be

\[ \Delta \theta_{23} \approx -4c_{23}^2 R_1 - c_{23}^2 R_2 (4I_{1} - I_2), \]
\[ \Delta \rho \approx (c_{23}^2 R_2)^{-1} [c_{23}^2 R_2 (4I_1 - I_2)] - \frac{1}{4} (4I_1 - I_2). \]  \hspace{1cm} (3.76)
In this case $|\rho_{1}^{\theta}| \sim 1$ is the largest coefficient for $\Delta \theta_{23}$ while the other three are smaller by at least one order of magnitude. $\delta$ is very unstable against the symmetry breaking as its coefficients can easily exceed 10 due to the enhancement factor $1/r$. (3) When the mass spectrum $m_1 \simeq m_2 \simeq m_3$ is concerned, the effects of $\mu$-$\tau$ reflection symmetry breaking in $M_\nu$ may be significantly magnified, as can be seen from the corresponding results of $\Delta \theta_{23}$ and $\Delta \delta$. In the $\rho^{(0)} = \sigma^{(0)} = 0$ case one obtains

$$\Delta \theta_{23} \simeq 2 \frac{m_2^2}{\Delta m_{31}^2} \left[ 2 s_{12}^2 R_1 - r c_{12} s_{13} \delta (2 l_1 - l_2) + R_2 \right],$$

$$\Delta \delta \simeq (c_{12} s_{13})^{-1} \left\{ - \frac{1}{12} \left[ r c_{12}^2 + (c_{12}^2 - s_{12}^2) s_{13}^2 \right] R_1 + \frac{2 m_1^2}{\Delta m_{31}^2} \left[ r c_{12}^2 s_{12} - (c_{12}^2 - s_{12}^2) s_{13}^2 \right] R_2 - \frac{m_1^2}{\Delta m_{31}^2} (c_{12} - s_{12}) (2 l_1 - l_2) \right\}. \quad (3.77)$$

Given $m_1 \sim 0.1$ eV, for example, $|\rho_{1}^{\theta}|$ can reach 6 while $|\rho_{12,13}^{\theta}|$ are much smaller. $\Delta \delta$ is highly sensitive to $l_{1,2}$ whose relevant coefficients are of $\mathcal{O}(10)$, but it is insensitive to $R_{1,2}$ whose relevant coefficients are of $\mathcal{O}(0.1)$. On the other hand, $(\rho^{(0)}, \sigma^{(0)}) = (0, \pi/2)$ will lead $\Delta \theta_{23}$ and $\Delta \delta$ to

$$\Delta \theta_{23} \simeq 2 \frac{m_2^2}{\Delta m_{31}^2} \left[ 2 s_{12}^2 R_1 + 2 (c_{12}^2 - s_{12}^2) c_{12} s_{13} l_1 \right] + s_{12}^4 R_1 + c_{12} s_{13} l_2 \right],$$

$$\Delta \delta \simeq \frac{m_1^2}{\Delta m_{31}^2} \left[ 8 \frac{m_1^2}{\Delta m_{31}^2} r c_{12}^2 s_{12} (2 c_{12}^2 R_1 - s_{12}^2 R_2) + 4 (c_{12}^2 - s_{12}^2) l_1 + 2 c_{12} l_2 \right]. \quad (3.78)$$

In this case $|\rho_{12,13}^{\theta}|$ are respectively of $\mathcal{O}(1)$ and $\mathcal{O}(0.1)$, and all the coefficients associated with $\Delta \delta$ may be enhanced to 100.

In short, $\Delta \delta$ and $\Delta \theta_{23}$ tend to be enlarged when the degeneracy of the neutrino mass grown and $\rho^{(0)}$ takes a different value from $\sigma^{(0)}$. Furthermore, $\delta$ is in general more unstable than $\theta_{23}$ with respect to the effects of $\mu$-$\tau$ reflection symmetry breaking. In particular, the coefficients associated with $\Delta \delta$ can reach a value about 100 provided $m_1 \simeq m_2 \simeq m_3$ and $\rho^{(0)} \neq \sigma^{(0)}$. Furthermore, $\Delta \theta_{23}$ is more sensitive to $R_{1,2}$ while $\Delta \delta$ is more sensitive to $l_{1,2}$. With the help of such observations, one may break the $\mu$-$\tau$ reflection symmetry in a proper way whenever necessary in a model-building exercise.

### 3.5. RGE-induced $\mu$-$\tau$ symmetry breaking effects

If a certain flavor symmetry is associated with the neutrino mass generation and lepton flavor mixing at a superhigh energy scale $\Lambda_{FS}$, it will be broken due to the RGE evolution of $M_\nu$ from $\Lambda_{FS}$ down to the electroweak scale $\Lambda_{EW} \sim 10^2$ GeV. In this case the significant difference between $m_\mu$ and $m_\tau$ is just a source of symmetry breaking which can be transmitted to $M_\nu$ via the RGE running effects. From a model-building point of view, it is in general natural to introduce a kind of flavor symmetry at an energy scale much higher than $\Lambda_{EW}$ (e.g. the seesaw scale). So the RGE-triggered symmetry breaking effects should be taken into account when such a model is confronted with the available experimental data at low energies [126]. It is therefore worthwhile to explore how the $\mu$-$\tau$ flavor symmetry is broken via the RGEs. At the one-loop level the RGE running behavior of $M_\nu$ is described by [127–130]

$$\frac{\mathrm{d}M_{\nu}}{\mathrm{d}r} = C(Y'_{i}Y_{j}) M_{\nu} + CM_{\nu}(Y'_{i}Y_{j}) + \alpha M_{\nu} \quad (3.79)$$

with $t \equiv (1/16\pi^2)\ln(\mu/\Lambda_{FS})$, in which $\mu$ denotes an energy scale between $\Lambda_{EW}$ and $\Lambda_{FS}$, $C = 1$ and

$$\alpha \simeq \frac{6}{5} \theta^2 \sim 6 \theta^2 + 6 y_r^2. \quad (3.80)$$

within the minimal supersymmetry standard model (MSSM). In equation (3.79) the $\alpha$-term just provides an overall rescaling factor which will be referred to as $L_\alpha$, while the other two terms may change the structure of $M_\nu$. In the basis chosen for equation (3.79), the Yukawa coupling matrix of three charged leptons is diagonal: $Y = \text{Diag} [\chi_\nu, \chi_\mu, \chi_\tau]$. For $\chi_\nu \ll \chi_\mu \ll \chi_\tau$, it is reasonable to neglect the contributions of both $\chi_\nu$ and $\chi_\tau$ in the subsequent discussions. So we are left with $\chi_\tau^2 = (1 + \tan^2 \beta)m_h^2/\lambda^2$ in the MSSM with $\nu \simeq 246 \text{ GeV}$ being the vacuum expectation value of the Higgs fields. There are normally two ways to proceed: (a) one may follow the method described in [131–133] to derive the differential RGEs of neutrino masses and flavor mixing parameters from equation (3.79); (b) one may first integrate equation (3.79) to arrive at the RGE-corrected neutrino mass matrix at $\Lambda_{EW}$ and then diagonalize the latter to obtain neutrino masses and flavor mixing parameters. Here we follow the second approach so as to see how $M_\nu$ is modified by the RGE running effects in a transparent way. After integrating equation (3.79), we get [134, 135]

$$M'_{\nu} = I_\alpha M_{\nu} I_{\tau} \quad (3.81)$$

at $\Lambda_{EW}$, where $I_{\tau} \equiv \text{Diag} \{1, 1, 1 - \Delta_{\tau}\}$ and

$$L_\alpha = \exp \left( \int_{\Lambda_{FS}}^{\Lambda_{EW}} \alpha \mathrm{d}r \right), \quad \Delta_{\tau} = \int_{\Lambda_{EW}}^{\Lambda_{FS}} y_{\tau}^2 \mathrm{d}r. \quad (3.82)$$

Given $\Lambda_{FS} \sim 10^{14}$ GeV, for example, $L_\alpha$ changes from 0.9 to 0.8 and $\Delta_{\tau}$ changes from 0.002 to 0.044 when $\tan \beta$ varies from 10 to 50.

Now let us consider a specific neutrino mass matrix respecting the $\mu$-$\tau$ permutation symmetry at $\Lambda_{FS}$ such as the one given in equation (3.4) with $\kappa = +1$. With the help of equation (3.81), one may calculate the RGE corrections to this matrix at $\Lambda_{EW}$ and express the result in a way parallel to equation (3.4):
\[ M'_{\nu} \approx I_{\alpha} \left[ M_{\nu} - \Delta \begin{pmatrix} M_{\mu} \cr M_{\tau} \cr 2M_{\mu} \end{pmatrix} \right] \] (3.83)

It becomes obvious that the term proportional to \( \Delta \) measures the strength of \( \mu-\tau \) symmetry breaking. One may diagonalize \( M'_{\nu} \) via the unitary transformation \( U' = P' Y' P'_v \), where \( Y' \) is an analogue of \( Y \) shown in equation (2.8), and \( P'_\alpha \) (or \( P'_\beta \)) is an analogue of \( P_\alpha \) (or \( P_\beta \)) as given above equation (3.45). Note that \( P'_\alpha \) should not be ignored in this treatment so as to keep everything consistent, although its phases have no physical meaning. After a lengthy but straightforward calculation, we obtain the neutrino masses

\[ \begin{align*}
m'_{11} &\approx I_{\alpha} (1 - s_{\Delta_{\nu}}^2) m_1, \\
m'_{12} &\approx I_{\alpha} (1 - c_{\Delta_{\nu}}^2) m_2, \\
m'_{13} &\approx I_{\alpha} (1 - \Delta_{\nu}) m_3,
\end{align*} \] (3.84)

at \( \Lambda_{\text{EW}} \); and the flavor mixing angles [136]

\[ \begin{align*}
\theta'_{12} &\approx \theta_{12} + \frac{1}{2} c_{12} s_{12} \frac{m_1^2 + m_2^2}{\Delta m_{21}^2} \Delta_{\nu}, \\
\theta'_{13} &\approx c_{12} s_{12} \frac{m_3^2 - m_2^2}{\Delta m_{31}^2} \Delta_{\nu}, \\
\theta'_{23} &\approx \frac{\pi}{4} + \frac{m_1^2 + m_2^2}{2 \Delta m_{21}^2} c_{12}^2 \Delta_{\nu},
\end{align*} \] (3.85)

at \( \Lambda_{\text{EW}} \). In addition, the two Majorana phases at the electroweak scale are given by

\[ \begin{align*}
\rho' &\approx \rho - \frac{m_3 m_2 \sin 2(\rho - \sigma) c_{12}}{\Delta m_{21}^2} \Delta_{\nu}, \\
\sigma' &\approx \sigma - \frac{m_3 m_2 \sin 2(\rho - \sigma) s_{12}}{\Delta m_{21}^2} \Delta_{\nu},
\end{align*} \] (3.86)

Note that the validity of these interesting analytical results depends on the prerequisite that none of the physical parameters evolves violently from \( \Lambda_{\text{FS}} \) down to \( \Lambda_{\text{EW}} \) to make the perturbation expansions invalid. In this sense it is more general to discuss the running effects of relevant flavor parameters by deriving their differential RGEs from equation (3.79). But equations (3.84)-(3.86) remain very useful for us to see some salient features of radiative corrections to the neutrino masses and flavor mixing parameters. Some more comments and discussions are in order.

- If the neutrino mass ordering is inverted or quasi-degenerate, the radiative correction to \( \theta_{12} \) tends to be appreciable because it may be enhanced by a factor of \( O(100) \) in either of these two cases, although \( \chi_1 \) itself is strongly suppressed even in the MSSM. Nevertheless, a proper choice of \( \rho - \sigma \sim \pm \pi i/2 \) can give rise to a significant cancellation in \( |m_1 + m_2| \) and thus suppress the aforementioned enhancement and stabilize the running behavior of \( \theta_{12} \) [137, 138]. If the neutrino mass ordering is normal, \( \theta'_{12} - \theta_{12} \sim c_{12} s_{12} \Delta_{\nu}/2 \) is expected to be small. The higher the scale \( \Lambda_{\text{FS}} \) is, the smaller the corresponding value of \( \theta_{12} \) will be as compared with its value at \( \Lambda_{\text{EW}} \), because it always increases when running down in the MSSM scheme [139].

- As for \( \theta_{13} \), the RGE running effects are small no matter whether the neutrino mass ordering is normal or inverted. To generate \( \theta_{13} \sim 9^\circ \) at \( \Lambda_{\text{FS}} \) from \( \theta_{13} = 0 \) at \( \Lambda_{\text{FS}} \), the value of \( \tan \beta \) has to be larger than 50 unless the absolute neutrino mass scale is unreasonably higher than 0.1 eV [140].

- In this case the bottom-quark Yukawa coupling eigenvalue \( y_b \) would lie in the non-perturbation regime at a superhigh energy scale—a result which is definitely unacceptable [141]. Namely, the RGE-induced \( \mu-\tau \) permutation symmetry breaking is hard to fit the observed value of \( \theta_{13} \), and thus one has to introduce a nonzero \( \theta_{13} \) at \( \Lambda_{\text{FS}} \). In the latter case the analytical approximations obtained in equations (3.84)-(3.86) remain valid\(^8\), simply because \( \theta_{13} \) itself does not significantly affect other parameters [133].

- An appreciable deviation of \( \theta_{23} \) from \( \pi/4 \) can be produced to fit the present experimental data when the neutrino masses are nearly degenerate. In particular, such a deviation will be positive in the MSSM scheme if \( |\Delta m_{31}^2| > 0 \) holds, as one can see from equation (3.85). This interesting observation offers a potential correlation between the octant of \( \theta_{23} \) and the neutrino mass hierarchy, and two numerical examples will be presented below [142].

Finally it is worth pointing out that a finite and meaningful value of \( \delta \) will be generated along with a nonzero value of \( \theta_{13} \) after the exact permutation symmetry at \( \Lambda_{\text{FS}} \) is broken via the RGE running effects. Of course, the real seeds of such a radiative generation of \( \delta \) must be the nontrivial Majorana phases \( \rho \) and \( \sigma \) at \( \Lambda_{\text{FS}} \) [143, 144].

The above approach also applies to the RGE-activated \( \mu-\tau \) reflection symmetry breaking. Consider that a neutrino mass matrix \( M_{\nu} \) exhibiting the \( \mu-\tau \) reflection symmetry is generated at a high scale \( \Lambda_{\text{FS}} \). Then the corresponding neutrino mass matrix \( M_{\nu}' \) at the electroweak scale \( \Lambda_{\text{EW}} \) can be deduced through equation (3.81) when the RGE running effects are taken into account. It is possible to reparametrize \( M_{\nu}' \) in the form given by equation (3.69) with \( \epsilon_2 = 2 \epsilon_1 = \Delta_{\nu} \), as well as \( \epsilon_{3,4} = 0 \) [114]. Therefore, we can derive the RGE-induced \( \Delta \theta_{23}, \Delta \delta, \Delta \rho \) and \( \Delta \sigma \) from equation (3.73) by taking \( R_3 = 2 R_1 = \Delta_{\nu} \) and \( I_{1,2} = 0 \). While the result of \( \Delta \theta_{23} \) is the same as that shown in equation (3.85), the result of \( \Delta \delta \) appears as

\[ \Delta \delta = \frac{m_3 (m_3 - m_2) c_{12} s_{12}}{m_3 - m_2} \Delta_{\nu} \left( \frac{m_1^2 s_{13}^2 + m_2^2 s_{13}}{m_2 - m_3} \right) \left( m_2 + m_3 c_{12}^2 s_{13} \right) \] (3.87)

For illustration, we show the possible values of \( \Delta \theta_{23}, \Delta \delta, \Delta \rho \) and \( \Delta \sigma \) against the lightest neutrino mass in four typical cases

\[^8\text{In this case the expression of } \theta'_{13} \text{ in equation (3.85) should be modified as } \theta'_{13} \approx \theta_{13} + c_{12} s_{12} m_1 - m_2 |\Delta_{\nu}| / \Delta m_{31}^2.\]
Figure 6. The possible values of $\Delta \theta_{23}$, $\Delta \delta$, $\Delta \rho$ and $\Delta \sigma$ against the lightest neutrino mass in four typical cases. We have taken $\Lambda_{FS} \sim 10^{14}$ GeV for the flavor symmetry scale. The sign ‘+’ or ‘−’ corresponds to $(\rho^{(0)}, \sigma^{(0)}) = (0, 0)$ or $(\rho^{(0)}, \sigma^{(0)}) = (0, \pi/2)$.

in figure 6. Case (a): $(\rho^{(0)}, \sigma^{(0)}) = (0, 0)$ and $\tan \beta = 50$ with a normal neutrino mass ordering; case (b): $(\rho^{(0)}, \sigma^{(0)}) = (0, 0)$ and $\tan \beta = 30$ with an inverted mass ordering; case (c): $(\rho^{(0)}, \sigma^{(0)}) = (0, \pi/2)$ and $\tan \beta = 10$ with a normal neutrino mass ordering; and case (d): $(\rho^{(0)}, \sigma^{(0)}) = (0, \pi/2)$ and $\tan \beta = 10$ with an inverted mass ordering. We have taken $\delta^{(0)} = -\pi/2$ in our numerical calculations. As one can see, the RGE running effects may be significant in the $m_1 \simeq m_2 \gg m_3$ case and especially in the $m_1 \simeq m_2 \simeq m_3$ case with $\sigma^{(0)} \neq \rho^{(0)}$. Moreover, $\Delta \theta_{23}$ tends to be larger than $\Delta \delta$ when $\rho^{(0)}$ and $\sigma^{(0)}$ take the same values, or vice versa.

We have discussed the RGE-induced corrections to a Majorana neutrino mass matrix which initially possesses the $\mu$-$\tau$ permutation or reflection symmetry. Now we turn to radiative corrections to $|U_{\mu\ell}| = |U_{\tau\ell}|$, since these equalities are a straightforward consequence of the $\mu$-$\tau$ symmetry. In other words, the observed effects of $\mu$-$\tau$ symmetry breaking in the PMNS matrix $U$ can be attributed to the RGE-induced corrections to $|U_{\mu\ell}| = |U_{\tau\ell}|$ when the neutrino masses and flavor mixing parameters run from $\Lambda_{FS}$ down to $\Lambda_{EW}$. In this case it is possible to get the correct octant of $\theta_{23}$ and even the correct quadrant of $\delta$ [142]. Let us illustrate this striking point with the help of the one-loop RGEs in the MSSM framework. Note that the framework of the SM for the RGE running is less interesting in this connection for two simple reasons: (a) it is difficult to make the deviation of $\theta_{23}$ from $45^\circ$ appreciable even if the neutrinos have a nearly degenerate mass spectrum; (b) the SM itself largely suffers from the vacuum-stability problem for the measured value of the Higgs mass ($\simeq 125$ GeV) as the flavor symmetry scale $\Lambda_{FS}$ is above $10^{10}$ GeV [145].

For simplicity, here we start from $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ at $\Lambda_{FS} \sim 10^{14}$ GeV to fit the observed pattern of the PMNS matrix $U$ at $\Lambda_{EW} \sim 10^2$ GeV by taking account of the RGE evolution. Hence we are not subject to the special values of the Majorana phases as constrained by the $\mu$-$\tau$ reflection symmetry imposed on the neutrino mass matrix $M_\nu$. Namely, we mainly concentrate on the radiative breaking of the equalities $|U_{\mu\ell}| = |U_{\tau\ell}|$ without going into details of the model-building issues.

We first define three $\mu$-$\tau$ ‘asymmetries’ of the PMNS matrix $U$ and then figure out their expressions in the standard parametrization as follows:

$$\Delta_1 \equiv |U_{\mu 1}|^2 - |U_{\mu 1}|^2 = (\cos^2 \theta_{13} \cos \delta) (\sin^2 \theta_{13} + \sin^2 \theta_{12}) \cos 2\theta_{23} - \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos \delta,$$

$$\Delta_2 \equiv |U_{\tau 2}|^2 - |U_{\mu 2}|^2 = (\sin^2 \theta_{23} + \cos^2 \theta_{12}) \cos 2\theta_{23} + \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos \delta,$$

$$\Delta_3 \equiv |U_{\tau 3}|^2 - |U_{\mu 3}|^2 = \cos^2 \theta_{13} \cos 2\theta_{23}. \quad (3.88)$$

It is clear that the three asymmetries satisfy the sum rule $\Delta_1 + \Delta_2 + \Delta_3 = 0$, and they vanish when the exact $\mu$-$\tau$ flavor symmetry holds (i.e., when $\theta_{13} = 0$ and $\theta_{23} = \pi/4$, or when $\delta = \pm \pi/2$ and $\theta_{23} = \pi/4$). The present best-fit results of neutrino mixing parameters listed in table 2 indicate that the possibility of $\delta = -\pi/2$ and $\theta_{23} = \pi/4$ is slightly more favored than either the possibility of $\delta = \pi/2$ and $\theta_{23} = \pi/4$ or that of $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. Hence we infer that the natural condition for all the three $\Delta_i$ to vanish should be $\delta = -\pi/2$ and $\theta_{23} = \pi/4$ at $\Lambda_{FS} \sim 10^{14}$ GeV, and the observed pattern of lepton flavor mixing at $\Lambda_{EW} \sim 10^2$ GeV should be a consequence of the most ‘economic’ $\mu$-$\tau$ symmetry breaking triggered by the RGE running effects from $\Lambda_{FS}$ to $\Lambda_{EW}$ [142]. Explicitly, the one-loop RGEs of $\Delta_i$ are given by
\[
\frac{d\Delta_i}{dt} = -y^2_i [\xi_2(\Delta_3^2 + |U_{22}|^2 \Delta_2 + |U_{23}|^2 \Delta_3) + \xi_3(\Delta_3^2 + |U_{22}|^2 \Delta_2 + |U_{23}|^2 \Delta_3)] \\
+ \xi_2(\Delta_3^2 + |U_{22}|^2 \Delta_2 + |U_{23}|^2 \Delta_3) \cos \Phi_{12} \\
+ \xi_3(\Delta_3^2 + |U_{22}|^2 \Delta_2 + |U_{23}|^2 \Delta_3) \cos \Phi_{13} \\
+ \mathcal{J}(\sin \Phi_{12} - \sin \Phi_{13}), 
\]

\[
\frac{d\Delta_3}{dt} = y^2_i [\xi_3(\Delta_3^2 + |U_{22}|^2 \Delta_2 + |U_{23}|^2 \Delta_3)] \\
- \xi_2(\Delta_3^2 + |U_{22}|^2 \Delta_2 + |U_{23}|^2 \Delta_3) \cos \Phi_{12} \\
+ \xi_3(\Delta_3^2 + |U_{22}|^2 \Delta_2 + |U_{23}|^2 \Delta_3) \cos \Phi_{13} \\
+ \mathcal{J}(\sin \Phi_{12} - \sin \Phi_{13}), 
\]

(3.89)

where \( \xi_j \equiv (m_j^2 + m_{\nu}^2)/\Delta m_{ij}^2 \), \( \zeta_i \equiv 2m_{\nu}m_i/\Delta m_{ij}^2 \) as well as \( \sin \Phi_j \equiv \text{Re}(U_{ij}U_{ij}^*)/|U_{ij}U_{ij}^*|^2 \) and \( \cos \Phi_j \equiv \text{Im}(U_{ij}U_{ij}^*)/|U_{ij}U_{ij}^*|^2 \).

The two Majorana CP-violating phases \( \rho \) and \( \sigma \) affect the evolution of \( \Delta_i \) via \( \cos \Phi_i \) and \( \sin \Phi_i \). Given the fact |\( \Delta m_{ij}^2 | \sim 30\Delta m_{32}^2 | \) with \( \Delta m_{32}^2 = 7.5 \times 10^{-3} \text{eV}^2 \) [72], we expect \( \xi_{31} \gg |\xi_{32}| \approx |\xi_{33}| \) to hold in most cases. But this does not necessarily mean that \( \Delta_3 \) should be more stable against radiative corrections than \( \Delta_1 \) and \( \Delta_2 \), because their running behaviors also depend on the initial inputs of \( |U_{ij}|^2 \).

An appreciable deviation of \( \theta_3 \) from \( \pi/4 \), which is sensitive or equivalent to an appreciable deviation of \( \Delta_3 \) from zero, requires a sufficiently large value of \( t \beta \). Hence the resulting octant of \( \theta_3 \) is controlled by the neutrino mass ordering (i.e. the sign of \( \Delta m_{23}^2 \) or \( \Delta m_{32}^2 \), or equivalently the sign of \( \xi_{31} \) or \( \xi_{32} \)). Since the signs of \( \xi_j \) and \( \zeta_i \) are always the same, one may adjust the evolving direction of \( \Delta_3 \) without much fine-tuning of the other relevant parameters.

We proceed to present two numerical examples to illustrate radiative corrections to the equalities \( |U_{ij}| = |U_{ij}| \) corresponding to the best-fit results of neutrino oscillation parameters given in [72] and [73]. We start from \( \Delta = 0 \) (i.e. \( \theta_3 = 45^\circ \) and \( \delta = 270^\circ \)) at \( \Lambda_{\text{FS}} \sim 10^{14} \text{GeV} \) and run them down to \( \Lambda_{\text{EW}} \sim 10^2 \text{GeV} \) via the RGEs that have been given in equation (3.89). Table 3 summarizes the typical inputs and outputs of these two examples. Some discussions are in order.

(1) In example I with the inverted neutrino mass ordering, the best-fit results of \( \Delta m_{23}^2, \Delta m_{31}^2, \theta_1, \theta_2, \theta_3, \delta \) and the neutrino mass ordering can be successfully reproduced from the proper inputs at \( \Lambda_{\text{FS}} \) given in [72] and [73]. We start from \( \Delta = 0 \) (i.e. \( \theta_3 = 45^\circ \) and \( \delta = 270^\circ \)) at \( \Lambda_{\text{FS}} \sim 10^{14} \text{GeV} \) and run them down to \( \Lambda_{\text{EW}} \sim 10^2 \text{GeV} \) via the RGEs that have been given in equation (3.88). One has to adjust the initial values of \( \rho \) and \( \sigma \) at \( \Lambda_{\text{FS}} \) in a careful way to control the running behaviors of six neutrino oscillation parameters, so that their best-fit results at \( \Lambda_{\text{EW}} \) can be correctly reproduced. Nevertheless, we find that it is really possible to resolve the octant of \( \theta_3 \) and the quadrant of \( \delta \) through the radiative breaking of \( |U_{ij}| = |U_{ij}| \).
by inputting proper values of the absolute neutrino mass
m$_1$ and the MSSM parameter tan $\beta$. Once such unknown
parameters are measured or constrained to a better degree of
accuracy in the future, it will be possible to examine whether the quantum corrections can really accommodate the observed effects of $\mu$-$\tau$ symmetry breaking.

3.6. Flavor mixing from the charged-lepton sector

So far we have taken the charged-lepton mass matrix $M_l$ to be
diagonal. If a specific texture of the neutrino mass matrix $M_\nu$ fails
to generate the observed pattern of lepton flavor mixing, how-
ever, a non-diagonal form of $M_l$ should be taken into account.
Of course, $M_l$ is usually non-diagonal in a realistic flavor-sym-
metry model. In a grand unified theory (GUT), for example, $M_l$
is always associated with the down-type quark mass matrix $M_d$
and thus both of them should be non-diagonal. In all these con-
texts it is meaningful to consider the contribution of $M_l$ to the
overall lepton flavor mixing matrix $U = O_l^T O_\nu$ [146–161], where
the phases of $O_l$ and $O_\nu$ deserve some particular attention.
To clarify the issue, we follow the formulas given in [162] but adopt
a slightly different parametrization $O = U_{23} U_{13} U_{12} P \tilde{P}$, for $O_l$ or
$O_\nu$, where $P$ = Diag [$e^{i\alpha_1}$, $e^{i\alpha_2}$, $e^{i\alpha_3}$] and

\[
U_{12} = \begin{pmatrix}
  c_{12} & s_{12} & 0 \\
  -s_{12} & c_{12} & 0 \\
  0 & 0 & 1
\end{pmatrix},
\]

\[
U_{13} = \begin{pmatrix}
  c_{13} & 0 & s_{13} \\
  0 & 1 & 0 \\
-\bar{s}_{13} & 0 & c_{13}
\end{pmatrix},
\]

\[
U_{23} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & c_{23} & s_{23} \\
  0 & -\bar{s}_{23} & c_{23}
\end{pmatrix}.
\]

with the definition $\tilde{s}_{ij} = s_{ij} e^{i\delta_{ij}}$ (for $ij = 12, 13, 23$). In comparison, we recall the standard parametrization $O = P O_{23} O_{13} O_{12} P$, with $P$ = Diag [$e^{i\beta_1}$, $e^{i\beta_2}$, $e^{i\beta_3}$] and

\[
\delta_{12} = \phi_2 - \phi_1, \quad \delta_{13} = \beta + \phi_3 - \phi_1, \quad \delta_{23} = \phi_3 - \phi_2,
\]

\[
\alpha_1 = \phi_1 + \rho, \quad \alpha_2 = \phi_2 + \sigma, \quad \alpha_3 = \phi_3.
\]

To be more specific, the lepton flavor mixing matrix $U$
in the new parametrization reads

\[
U = O_l^T O_\nu = P_{\alpha}^T U_{12}^T U_{13}^T U_{12} U_{13} U_{12} P_{\alpha},
\]

\[
= P_{\alpha} U_{23} U_{13} U_{12} P_{\alpha},
\]

where the phase matrix $P_{\alpha}$ is irrelevant in physics as it can be
rotated away via a redefinition of the phases of three
charged-lepton fields. In most cases of the model-building
exercises, the three angles of $O_l$ and the (1, 3) angle of $O_\nu$ are

\[\text{significantly small}^{9}.\] To the leading order of such small
parameters, the three mixing angles of $U$ can approximate to [162]

\[
\begin{align*}
\delta_{12} & \approx \tilde{\delta}_{12} - \bar{\theta}_{12}^3 \nu_{12}^2 \bar{\nu}_{23}^2 + \bar{\theta}_{12}^3 \nu_{12}^2 \nu_{23} \\
\delta_{13} & \approx \tilde{\delta}_{13} - \bar{\theta}_{13} \nu_{13}^2 \bar{\nu}_{23} \\
\delta_{23} & \approx \tilde{\delta}_{23} - \bar{\theta}_{23} \nu_{23}^2 \nu_{12}.
\end{align*}
\]

(3.93)

A particularly interesting case is that $\theta_{13}$ and $\theta_{12}$ are both
negligibly small as compared with $\bar{\theta}_{12}$, leading us to the
approximate results

\[
\delta \approx \delta_{12} - \bar{\theta}_{12}^3 - \pi,
\]

\[
\delta_{13} \approx \theta_{12}^3 \nu_{13}^2
\]

\[
\delta_{23} \approx \delta_{12} + \theta_{12}^3 \nu_{23}^2 \cos \delta.
\]

(3.94)

In obtaining equation (3.94), we have used equation (3.91) as
well as $\delta_{13} \approx \delta_{12}$ and $\delta_{12} \approx \delta_{12}$. Given the approximate
$\mu$-$\tau$ flavor symmetry, $\bar{\theta}_{12}$ is supposed to be around $\pi/4$. If $\bar{\theta}_{12}$
approximates to the Cabibbo angle of quark flavor mixing
$\theta_{13} \approx 0.22^{10}$, then equation (3.94) can give rise to an interesting
relation $\theta_{13} \approx \theta_{13} \sqrt{2} \approx 0.15$, which is in good agreement
with the present experimental data. Moreover, equation (3.94)
implies

\[
\delta_{12} \approx \delta_{12} + \theta_{12}^3 \nu_{12}^2 \cos \delta.
\]

(3.95)

If $O_\nu \approx U_{BM}$ as given in equation (3.11), one will arrive at
$s_{12} < (1 - \theta_{13}^3 \sqrt{2}$ from equation (3.95). This lower bound
is apparently inconsistent with the experimental result of $\theta_{12}$. If $O_\nu \approx U_{BM}$ as given in equation (3.10), however, the situa-
tion will be different because $\bar{\theta}_{12}$ itself is already close to the
observed value of $\theta_{12}$. In this case $\delta$ is required to approach
$\pm \pi/2$ in order to suppress the contribution from the second
term in equation (3.95).

Although the pattern of lepton flavor mixing appears strik-
ingly different from that of quark flavor mixing, they have
been speculated to have a potential link. In this respect a
viable GUT model may offer an ideal context where the rel-
ate quarks and leptons reside in the same representations
such that their respective Yukawa coupling matrices can be
naturally correlated.\footnote{This assumption makes sense in some
GUT models, where $M_l$ and $M_d$ are usually related to each other
in a similar way.} For instance, the aforementioned relation
$\theta_{13} \approx \theta_{13} \sqrt{2}$ makes a possible unification of quarks and
leptons quite appealing. Such a relation will come out if $M_l$
respects the $\mu$-$\tau$ permutation symmetry and $O_\nu$ has a CKM-like
structure (i.e. $\tilde{\theta}_{12}$ is very close to the Cabibbo angle $\theta_C$
while $\theta_{13}$ and $\theta_{23}$ are negligibly small). Moreover, the well-
known Gatto–Sartori–Tonin (GST) relation $\theta_{13} \approx \sqrt{\sin \beta \sin \gamma}$
[164] tempts us to attribute the Cabibbo angle to the down-
type quark mass matrix $M_d$. The latter happens to be related to
$M_l$ in the GUT framework [165]. Let us take the SU(5)

\[\text{10}\]
GUT model, which can be embedded in the SO(10) GUT model, as an example. Accordingly, each family of quarks and leptons are grouped into the SU(5) representations \(5\) and \(10\) in the manner

\[
\begin{pmatrix}
  d' & d'' & e \\
  d'' & d'' & e \\
  e & e & e
\end{pmatrix}, \quad
\begin{pmatrix}
  0 & u'' & u' & d' \\
  0 & u' & u' & u'' \\
  0 & 0 & 0 & e
\end{pmatrix}
\]

with the subscripts ‘r’, ‘b’ and ‘g’ being the color indices of quarks. Since the right-handed neutrino fields \(N_i\) are the SU(5) singlets, their masses can be much larger than the electroweak symmetry breaking scale. In the minimal SU(5) GUT scenario whose Higgs sector only contains the 5-dimensional representations \(H_5\) and \(H_{\bar{5}}\), the Yukawa interaction terms include \(Y_{5}^{ij}10_{i} H_{5}\), \(Y_{\bar{5}}^{ij}N_{i} H_{\bar{5}}\) and \(Y_{5}^{ij}\bar{5}_{i} H_{5}\) with \(i\) and \(j\) being the family indices [33]. After the SU(5) symmetry is broken down to the SM gauge symmetry, the Yukawa coupling matrices of quarks and leptons take the following forms:

\[
\mathcal{L}_{\text{mass}} = Y_{5}^{ij}Q_{i}u_{j} H_{5} + Y_{\bar{5}}^{ij}d_{i}H_{\bar{5}} + Y_{5}^{ij}d_{i}N_{j}H_{5},
\]

where \(Y_{ij}^{5}\) can finally be achieved provided \(M\) possesses the \(\mu\)-\(\tau\) flavor symmetry. But \(\delta_{12}\) and \(\delta_{13}\) remain undetermined, preventing us from calculating \(\delta\) and \(\theta\) based on equation (3.94). One may follow the idea of spontaneous CP breaking to find a way out. To be specific, the CP symmetry is imposed on the model in the beginning to forbid the presence of CP violation, and it is spontaneously broken by some specific fields so that the resulting CP-violating phases are under control. A good example of this kind has been given in [174], where a scalar field acquires an imaginary vacuum expectation value and thus breaks the CP symmetry to the maximum level. This vacuum expectation value leads to \(\delta_{12} = \pm \pi/2\), while \(\delta_{13}\) remains vanishing since the neutrino sector has no link to the relevant scalar field. So we arrive at \(\delta = \pm \pi/2\) for the PMNS matrix \(U\), and \(s_{12}\) should take a value which is not far away from the experimental value of \(s_{12}\). In such an explicit model-building exercise one may construct \(O_{e} = U_{\text{TB}}\) to make things easy.

4. Larger flavor symmetry groups

Before introducing some concrete models to show how to realize the \(\mu\)-\(\tau\) flavor symmetry, let us first formulate the connection between neutrino mixing and flavor symmetries from a group-theoretical angle and then outline the general strategy for generating a particular neutrino mixing pattern. In addition, we shall briefly describe some recent works on combining the flavor symmetries and generalized CP (GCP) symmetries. Such a new treatment can not only pave the way for realizing the \(\mu\)-\(\tau\) flavor symmetry but also lay the foundation for embedding it in a larger flavor symmetry group. The latter will become relevant especially when one is ambitious to nail down the PMNS matrix \(U\) completely, since the \(\mu\)-\(\tau\) symmetry itself can only fix a part of the texture of \(U\).
4.1 Neutrino mixing and flavor symmetries

As shown in section 3.1, \( \theta_3 = 0 \) and \( \theta_3 = \pi/4 \) lead the third column of \( U \) to the form \( (0, -1, 1)^T/\sqrt{2} \), implying a \( Z_2 \) or \( \mu-\tau \) permutation symmetry of \( M_l \). And the reverse is also true. In fact, a link between the pattern of neutrino mixing and a certain flavor symmetry of \( M_l \) like this always exists, irrespective of any particular form of \( U \) in a sense as follows. In the basis where \( M_l \) is diagonal, \( U \) is identified as the unitary matrix used to diagonalize \( M_l \), i.e., \( U^\dagger M_l U = D_l \). Hence the relation \( M_l u_i^* = m_i u_i \) is implied, where \( u_i \) denotes the \( i \)-th column of \( U \) and \( m_i \) is the \( i \)-th neutrino mass eigenvalue corresponding to \( u_i \). Then it is easy to check that \( M_l \) must be invariant under \( S_l \) (for \( i = 1, 2, 3 \)):

\[
S_l^i M_l S_l^i = S_l^i (m_i u_i u_i^T - m_i u_i u_i^T) = m_i u_i u_i^T + m_i u_i u_i^T + m_i u_i u_i^T = U D_l U^T = M_l,
\]

(4.1)

where \( S_l \) are defined in terms of \( u_i \) via

\[
S_l = u_i u_i^T - (u_i u_i^T + u_i u_i^T).
\]

(4.2)

All the three \( S_l \) are unitary and commute with one another. Thanks to the relation \( S_l S_l = S_l \), only two of them are independent. Furthermore, each of \( S_l \) has order 2 as indicated by \( S_l^2 = 1 \). Summarizing these features, we reach the conclusion that \( M_l \) always has the Klein symmetry \( K_4 = Z_2 \times Z_2 \) [175–177]. The symmetry operators can be explicitly constructed in terms of \( u_i \) in an approach described by equation (4.2). For instance,

\[
\begin{align*}
S_{1 TB}^1 &= \frac{1}{3} \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \\
S_{2 TB}^2 &= \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \\
S_{3 TB}^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.
\end{align*}
\]

(4.3)

when \( U \) takes the well-known flavor mixing pattern \( U_{TB} \) as given in equation (3.10). Note that \( S_{3 TB}^3 \) is nothing but the \( \mu-\tau \) permutation operation in equation (3.6), up to a sign rearrangement.

The above conclusion can be put another way. If \( M_l \) assumes a flavor symmetry described by \( S_l \), then the normalized invariant eigenvector of \( S_l (\text{defined by } S_l u_i = u_i) \) will constitute one column of \( U \) [175–177]12. One may verify this observation as follows. The invariance of \( M_l \) with respect to \( S_l \) yields

\[
M_l u_i^* = S_l^i M_l S_l^i u_i = S_l M_l u_i = M_l u_i,
\]

(4.4)

which in turn implies that \( M_l u_i^* = m_i u_i \) is identical to \( u_i \) up to a coefficient (i.e., the neutrino mass eigenvalue \( m_i \)):

\[
M_l u_i^* = m_i u_i = u_i M_l u_i = m_i.
\]

(4.5)

12 But which column it will occupy is a matter of convention and can be determined upon some phenomenological considerations.

It should be noted that the resulting flavor mixing pattern is independent of \( m_i \). This point means that one may obtain a constant pattern of neutrino mixing directly from the flavor symmetry by bypassing a concrete form of \( M_l \). To do so, however, one has to know all the three symmetry operators \( S_l \). If only a single \( S_l \) is known, then \( U \) can be partially determined. For example, when \( M_l \) respects a symmetry defined by \( S_{1 TB}^1 \) or \( S_{2 TB}^2 \) given in equation (4.3), one column of \( U \) will be \( (2, -1, 1)^T/\sqrt{6} \) or \( (1, 1, 1)^T/\sqrt{3} \), respectively. We are therefore led to the so-called TM1 or TM2 mixing pattern [175, 178–183]:

\[
\begin{align*}
U_{TM1} &= \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} c + \sqrt{2} c & \sqrt{2} s + \sqrt{2} s \ \\ -1 & \sqrt{2} c - \sqrt{2} s & \sqrt{2} s + \sqrt{2} s \ \\ -c - \sqrt{2} s & \sqrt{2} s & \sqrt{2} s + \sqrt{2} s \end{pmatrix}, \\
U_{TM2} &= \frac{1}{\sqrt{6}} \begin{pmatrix} 2c & \sqrt{2} & 2s \\ -c + \sqrt{3} s & \sqrt{2} & \sqrt{2} s + \sqrt{2} s \\ -c & \sqrt{2} s & \sqrt{2} s + \sqrt{2} s \end{pmatrix}.
\end{align*}
\]

(4.6)

where \( \theta \equiv \sin \theta e^{i \phi} \) has been defined. In equation (4.6) the location of the invariant eigenvector of \( S_{1 TB}^1 \) or \( S_{2 TB}^2 \) has been specified. The model-building details and phenomenological consequences of these two particular neutrino mixing patterns will be elaborated in section 5.3. Note that an \( S_l \) can only limit one column of \( U \) up to the associated Majorana phase. The reason lies in the fact that when one \( u_i \) is rephased as a whole, it remains an invariant eigenvector of \( S_l \). As for the charged leptons, the diagonal Hermitian matrix \( M_l M_l^\dagger \) possesses the symmetry

\[
T^\dagger M_l M_l^\dagger T = M_l M_l^\dagger,
\]

(4.7)

in which \( T \equiv \text{Diag} (e^{i \phi _1}, e^{i \phi _2}, e^{i (\phi _3 + \phi _2)}) \) is a diagonal phase matrix. Conversely, the Hermitian matrix \( M_l M_l^\dagger \) must be diagonal if it satisfies the phase transformation in equation (4.7). It is worth pointing out that the above discussions keep valid in the basis of non-diagonal \( M_l \). But in this case one needs to make an appropriate transformation of those basis-dependent quantities. For example, the symmetry operator which is imposed on \( M M_l^\dagger \) will become \( T = O_l T O_l^\dagger \) with \( O_l \) being the unitary matrix used to diagonalize \( M M_l^\dagger \).

Although it is quite possible that the \( S_l \) and \( T \) symmetries just arise by accident, we choose to identify them as the residual symmetries from a larger flavor symmetry group \( G_F \) in the following [175–177]. In other words, the subgroup \( G_l \) (or \( G_l \)) generated by \( S_l \) (or \( T \)) remains intact in the neutrino (or charged-lepton) sector when \( G_F \) itself is broken down:

\[
G_F \mapsto \begin{cases} G_l = \langle S_l \rangle \text{ for } M_l, \\ G_l = \langle T \rangle \text{ for } M M_l^\dagger. \end{cases}
\]

(4.8)

From the bottom-up point of view, one may get a hold of \( G_F \) by demanding that \( G_l \) and \( G_l \) close to a group [184]. In order for \( G_F \) to be finite, there must exist an \( n \) such that \( T^n = 1 \). As already remarked, we need a non-degenerate diagonal \( T \) to guarantee the diagonality of \( M M_l^\dagger \), implying \( n \geq 3 \).
For the sake of simplicity, let us tentatively set $n = 3^{13}$. This means that $T$ has 1, $\omega$ and $\omega^2$ as its diagonal entries, where $\omega = e^{2\pi i/3}$. But the ordering of these three entries has to be specified, because different options may result in different $G_F$. Given $G_F = \{T = \text{Diag}(1, \omega, \omega^2)\}$ and $G_{Z_2} = \{S_1^{TB}, S_2^{TB}\}$, for example, we can arrive at $G_F = S_4$ (the permutation group of four objects). Furthermore, it has been shown that $S_4$ (or a group containing $S_4$ as a subgroup) is the unique discrete group that is able to naturally accommodate the flavor mixing pattern $U_{TB}$ from the group-theoretical arguments [177, 185]. Indeed, it is likely that merely one $S_4$ symmetry is the surviving symmetry from $G_F$. In this situation $G_F$ is found to be successively $S_4$, $A_4$ (the alternating group of four objects) or $S_3$ (the permutation group of three objects) when the generator of $G_F$ is only one of $S_1^{TB}, S_2^{TB}$ and $S_3^{TB}$. Interestingly, the $A_4$ group that has been popularly considered for realizing $U_{TB}$ [186, 187] actually does not contain a subgroup for the last column of $U_{TB}$. But an accidental symmetry in such models may elevate the $A_4$ symmetry to $S_4$, making $U_{TB}$ available. Roughly speaking, the group generated by $T$ and $S_i$ tends to be small when $S_i$ have some regular forms like those illustrated in equation (4.3). If the neutrino mixing pattern and the relevant $S_i$ lack any regularity, the resulting $G_F$ will become much larger and even not finite. That is why the observed pattern of lepton flavor mixing is more likely to originate from a reasonable flavor symmetry than that of quark flavor mixing [177]. In order to accommodate the experimental value of $\theta_{13}$, which more or less lowers the regularity of $U$, a much larger flavor symmetry group has to be invoked (an example of this kind is $\Delta(6N^3)$ [189] with $N$ being a big number) [190–193].

The aforementioned procedure of constructing $G_F$ from certain $G_i$ and $G_l$ can be reversed: assuming a flavor symmetry group $G_F$ for the charged leptons and neutrinos, one may take some of its subgroups as the residual symmetries and derive the associated flavor mixing pattern [188]. Such an exercise can be done in a purely group-theoretical way with no need of building an explicit dynamic model. First of all, we need to divide the elements of $G_F$ into two categories—one with the elements of order 2 and the other with the elements of order $\geq 3$. Then we may identify three elements $S_i$ which satisfy $S_i S_j = S_j S_i = S_k$ (for $i = j = k$ in the former category as the generators of $G_l$, and fix an element $T$ in the latter category as the generator of $G_F$). In the basis where $T$ is diagonal, $U$ will be composed of the normalized invariant eigenvectors of $S_i$. If there are not three such order-2 elements, then only one order-2 element can be identified as the residual symmetry for $M_l$, i.e. $G_v = Z_2$ in which case $U$ can be partly determined. In fact, some authors have considered the possibility of reducing $G_v$ from $Z_2 \times Z_2$ to $Z_2$ so as to leave $\theta_{13}$ as a free parameter [194–197].

4.2. Model building with discrete flavor symmetries

Now let us recapitulate some key issues concerning an application of the group-theoretical arguments to some model-building exercises. The most important issue in building a flavor-symmetry model is how to break $G_F$ while preserving the desired residual symmetries. To serve this purpose, one may introduce a new type of scalar fields—flavons, which do not carry any quantum number of the SM gauge symmetry but can constitute some nontrivial representations of $G_F$. They break a given flavor symmetry via acquiring their appropriate vacuum expectation values (VEVs). In the canonical (type-I) seesaw mechanism the Lagrangian responsible for generating the lepton masses may take a general form as

$$-\mathcal{L}_{\text{mass}} = \sum_{i} y_i^{\nu} T^T l^T H_u \frac{\phi_{\gamma i}^{\nu}}{\Lambda} + \sum_{i} y_i^{\nu} T^T N^\nu H_u \frac{\phi_{\gamma i}^{\nu}}{\Lambda} + \sum_{i} y_i^{\nu} N^\nu N^\nu \frac{\phi_{\gamma i}^{\nu}}{\Lambda} + \text{h.c.,}$$

in which $l$ and $N$ stand respectively for the right-handed charged-lepton and neutrino fields, and the superscripts $\rho$, $\sigma$ and $\gamma$ denote the representations occupied by the fermion and flavon fields. Since the flavor symmetry breaks in different manners in the charged-lepton and neutrino sectors, the flavon fields for these two sectors are distinct. But when $\phi_{\gamma i}$ and $\phi_{\gamma N}$ have the same transformation properties, they can be identified as the same field. In order to separate $\phi_{\gamma i}$ from $\phi_{\gamma N}$, an extra symmetry is usually needed (e.g. an auxiliary $Z_{\phi_{\gamma N}}$ symmetry under which $\phi_{\gamma i}$ and $\phi_{\gamma N}$ are odd while the other fields are even). Furthermore, there must be enough independent flavon fields so that the Yukawa coupling parameters present in each sector can fit the lepton masses. Note that $\Lambda$ represents a high energy scale above which the new fields may become active, and the ratio of a flavon’s VEV to $\Lambda$ (denoted as $\epsilon = \langle \phi_{\gamma} \rangle / \Lambda$) is typically assumed to be a small quantity such that the magnitude of a Yukawa coupling parameter can be regulated by the power of $\epsilon$—the spirit of the Froggatt–Nielson mechanism for explaining the observed quark mass hierarchy [198]. This point will become relevant when one needs to either produce a hierarchical mass matrix like $M_l$ or discuss the soft breaking of a flavor symmetry by higher-order terms. In this connection it is necessary to know what forms the flavon VEVs $\langle \phi_{\gamma i,N} \rangle$ should take in order to break all the flavor symmetries apart from $G_i$ and $G_v$. One can easily verify that $M_{\nu} M_{\nu}^{\dagger}$ will be invariant under $T$ if $\langle \phi_{\gamma i} \rangle$ satisfies the condition [188]

$$T^\gamma \langle \phi_{\gamma i} \rangle = \langle \phi_{\gamma i} \rangle.$$

(4.10)

Similarly, the effective neutrino mass matrix resulting from the seesaw mechanism will be unchanged with respect to $S_i$ provided [188]

$$S_i^\gamma \langle \phi_{\gamma i} \rangle = \langle \phi_{\gamma i} \rangle, \quad S_i^\gamma \langle \phi_{\gamma N} \rangle = \langle \phi_{\gamma N} \rangle.$$

(4.11)

If a flavon field has no way to offer the proper VEV required by equation (4.10) or (4.11), it has to be forbidden to acquire a VEV and therefore contributes nothing to the lepton masses. With these observations in mind, we are well prepared to assign suitable representations for the fermion and flavon fields in order to build a phenomenologically viable model.
The 24 elements of $\sigma$, being diagonal, only 14. There are five irreducible representations for $S_4$, only two flavors can mix to help give rise to a diagonal $M_{\text{mix}}$ and $M_{\text{mix}}$ whose dimensions are $3 \times 3$. As for $\sigma_1, \sigma_2, \sigma_3$ and $\omega$, whose dimensions are self-explanatory. Once the flavor symmetry is specified, a two-dimensional representation, one column of $\sigma_1$ will have a representation.

When an order-3 element is chosen as $T$, the flavor mixing pattern $U_{\text{mix}}$ shown in equation (3.11) will arise.

Table 4. The explicit forms of $T, S_i, (\phi_i)$ and $(\phi_i^*)$ in each irreducible representation with $T$ being diagonal [188].

| $'$ | 1 | 2 | 3 | 3' |
|-----|---|---|---|----|
| $T$ | 1 | Diag $\{ \omega, \omega^2 \}$ | Diag $\{ 1, \omega, \omega^2 \}$ | Diag $\{ 1, \omega, \omega^2 \}$ |
| $(\phi_i)$ | 1 | $(0, 0)^T$ | $(1, 0, 0)^T$ | $(1, 0, 0)^T$ |
| $S_1$ | -1 | $\sigma_1$ | $S_1^{\text{TB}}$ | $-\sigma_1^{\text{TB}}$ |
| $S_2$ | 1 | Diag $\{ 1, 1 \}$ | $S_2^{\text{TB}}$ | $S_2^{\text{TB}}$ |
| $S_3$ | -1 | $\sigma_1$ | $S_3^{\text{TB}}$ | $-\sigma_3^{\text{TB}}$ |
| $(\phi_i)$ | 0 | $(1, 1)^T$ | $(0, 0, 0)^T$ | $(1, 1, 1)^T$ |

where $y_i = y_1/\sqrt{3} + \sqrt{2} y_2^3/\beta$, $y_2 = y_1/\sqrt{3} - \sqrt{2} y_2^3/\beta$ and $y_3 = y_1/\sqrt{3} - \sqrt{2} y_2^3/\beta$ containing the proper Clebsch–Gordan coefficients. Note that there are three free parameters in equation (4.12), exactly enough to fit the three charged-lepton masses. On the other hand, $(\phi_i^*)$, $(\phi_i^*)^*$ and $(\phi_i^*)^*$ will result in a Dirac neutrino mass matrix of the form

$$M_D = \langle H_0 \rangle \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & y_{23} \\ y_{13} & y_{23} & y_{22} \end{pmatrix},$$

in which the four independent elements are given by $y_{11} = y_1/\sqrt{3} + \sqrt{2} y_2^3/\beta$, $y_{12} = y_1/\sqrt{3} - \sqrt{2} y_2^3/\beta$, $y_{22} = y_1/\sqrt{3} - \sqrt{2} y_2^3/\beta$ and $y_{23} = y_1/\sqrt{3} - \sqrt{2} y_2^3/\beta$. One can see that the texture of $M_D$ has the same (2, 3) permutation symmetry as the one shown in equation (3.7), and $y_{22} + y_{23} = y_{11} + y_{12}$ holds. In addition, the texture of the heavy Majorana neutrino mass matrix $M_N$ is identical to that of $M_D$ with the replacements $y_{1,2,3} \rightarrow y_{1,2,3}^\nu$. Then it is straightforward to follow the canonical seesaw formula $M_L = M_D M_N^T M_D^T$ to show that the neutrino mixing pattern $U_{\text{mix}}$ can be achieved. Finally, let us stress that the allowed flavon VEVs and the neutrino mixing pattern will change if just one order-2 element is preserved as the residual symmetry.

As illustrated in table 4, the flavon VEVs have to align in some particular directions to preserve the desired residual symmetries. Whether such special alignments can be naturally achieved matters, as it has something to do with whether the corresponding flavor-symmetry model is convincing or not. In this regard the most popular method of deriving special flavon VEVs is the so-called F-term alignment mechanism [199]. The latter invokes the supersymmetry, allowing one to take advantage of the $U(1)$-R-symmetry—$U(1)_R$ (under which the superpotential terms should carry a total charge of 2) [15]. The charge assignments for this symmetry generally go as follows: the superfields for the SM fermions carry a charge of 1, while the superfields for Higgs and flavons are neutral. Hence the superpotential for generating lepton masses is not affected by this symmetry, but the flavon fields themselves cannot form the superpotential terms any more. To constrain the flavon VEVs, we may introduce the driving fields (denoted by $\psi$) [199] which transform trivially with respect to the SM gauge symmetry but nontrivially under the relevant flavor symmetry. In particular, they carry a charge of 2 of the $U(1)_R$ symmetry, implying that they must appear linearly in the superpotential terms.
terms. To be more explicit, the superpotential involving both \( \psi \) and the flavon fields takes the form
\[
W = \psi(M_1^2 + M_2\phi + c\phi^2 + \cdots),
\]
where \( M_1 \) and \( M_2 \) are two dimension-one parameters, and \( c \) is dimensionless. If the supersymmetry remains intact when the flavor symmetry suffers breakdown, the flavon potentials will be given by
\[
V(\phi) = \sum_a \left| \frac{\partial W}{\partial \psi_a} \right|^2,
\]
where \( a \) is used to distinguish different components of the driving fields in multi-dimensional representations. By definition, the flavon VEVs are taken to minimize the potential \( V(\phi) \):
\[
- F^n_{\psi a} = \left| \frac{\partial W}{\partial \psi_a} \right| = 0.
\]

For illustration, let us explain how the VEV alignments of \( \phi_i^1 \) and \( \phi_i^3 \) in table 4\(^{16} \) can be derived with the help of three driving fields \( \psi^1, \psi^2 \) and \( \psi^3 \). The related superpotential appears as
\[
W = \psi^1(c_1\phi_1^3 \phi_i^3 + c_2\phi_2^3 \phi_i^3 - M^2) + \psi^2(c_3\phi_1^3 \phi_i^3 + c_4\phi_2^3 \phi_i^3 + c_5\phi_3^3 \phi_i^3).
\]
Note that a term like \( \phi^3(M'^3 \phi_i^3) \) has been forbidden by the \( Z_2^{\text{aux}} \) symmetry mentioned below equation (4.9). The conditions for the flavon VEVs \( \langle \phi_i^1 \rangle = (v_1, v_2, v_3)^T \) and \( \langle \phi_i^3 \rangle = (v_1', v_2', v_3')^T \) turn out to be
\[
\begin{aligned}
&c_1(v_1^2 + 2v_2v_3) + c_2(v_1'^2 + 2v_2'v_3') - M^2 = 0, \\
&c_1(v_1v_2' - v_2v_1') + c_2(v_2v_3' - v_3v_2') = 0, \\
&c_1(v_3v_1' - v_1v_3') = 0,
\end{aligned}
\]
as well as
\[
\begin{aligned}
&c_3 \left( \frac{v_1^2 + 2v_2v_3}{v_1^2 + v_2v_3} \right) + c_4 \left( \frac{v_1^2 + 2v_2v_3}{v_2^2 + v_2v_3} \right) \\
&+ c_5 \left( \frac{v_2v_3 + v_1v_3'}{-v_3v_1' - v_1v_3'} \right) = 0.
\end{aligned}
\]
The VEV alignments illustrated in table 4 (i.e. \( v_{1,2,3} = v_{1,2,3}' = 0 \)) apparently satisfy these equations. In particular, the scale of \( v_1 \) and \( v_1' \) is determined by \( M \)—the only dimensional parameter in equation (4.17), which is supposed to be located at a high energy scale. However, a driving field in the representation 1 may not be welcome in some specific models, in which case the mass term like that in equation (4.17) will be absent and the trivial vacuum (i.e. all the flavon VEVs are vanishing) offers a solution to the conditions for minimizing \( V(\phi) \).

\(^{16}\)A possible way of realizing the VEV alignments of \( \phi_i^1 \) and \( \phi_i^3 \) in table 4 can be found in [200].

To find a way out, the negative and supersymmetry-breaking flavon mass terms \( -m_2^2|\phi|^2 \) are usually invoked to drive the flavon VEVs from the trivial ones [199].

Last but not least, let us point out that there exists a class of models where no residual symmetries survive. Such models are named the indirect models while those featuring residual symmetries are referred to as the direct models [201]. This classification is based on how the symmetries for \( M_1M_1' \) and \( M_2 \) come about. In the direct models \( M_1M_1' \) and \( M_2 \) are constrained by \( G_l \) and \( G_r \)—the remnants of \( G_F \), respectively. The relevant flavon VEVs should preserve these residual symmetries. In the indirect models the special forms of \( M_1M_1' \) and \( M_2 \) arise accidentally, and the flavon VEVs do not necessarily keep invariant under any symmetry. In a realistic indirect model \( M_1 \) and \( M_2 \) are typically taken to be diagonal. On the other hand, the Yukawa interaction term of the neutrinos takes the form
\[
y_i(\mathcal{T}^{\nu}_L G_F^\nu_H^c S^\nu_3),
\]
which can be expressed as
\[
y_i(\mathcal{T}^{\nu}_L v_1 + \mathcal{T}^{\nu}_L v_2 + \mathcal{T}^{\nu}_L v_3)NH e^\nu_i A_1^\nu,
\]
after the flavon fields have developed their VEVs as \( \langle \phi_i^1 \rangle = (v_1, v_2, v_3)^T \) and \( \langle \phi_i^3 \rangle = (v_1', v_2', v_3')^T \). The effective Majorana neutrino mass matrix turns out to be [33]
\[
M_\nu = \sum_i \frac{v_i^2(\mu^2)^{\nu}}{M_i N_i} \langle \phi_i^3 \rangle \langle \phi_i^3 \rangle^T,
\]
where \( M_i \) (for \( i = 1, 2, 3 \)) denote the right-handed neutrino masses. Interestingly, the resulting PMNS matrix \( U \) has a special structure: its three columns are proportional to \( \langle \phi_1^3 \rangle, \langle \phi_2^3 \rangle \) and \( \langle \phi_3^3 \rangle \), provided the latter are orthogonal to one another. One is therefore encouraged to choose these column vectors to align with the columns of the desired PMNS matrix \( U \) [202]. In order to achieve the flavor mixing pattern \( U_{\text{TB}} \), for instance, the following alignments for \( \langle \phi_i^3 \rangle \) are favored:
\[
\begin{aligned}
\langle \phi_1^3 \rangle &= v_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \\
\langle \phi_2^3 \rangle &= v_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
\langle \phi_3^3 \rangle &= v_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.
\end{aligned}
\]

These VEVs either stay invariant or change their signs under \( S_i^{\text{TB}} \), but none of \( S_i^{\text{TB}} \) is preserved by all the three VEVs. Since \( \langle \phi_i^3 \rangle \) appear quadratically in \( M_i \), the sign changes are actually irrelevant and thus \( M_i \) acquires the \( S_i^{\text{TB}} \) symmetries accidentally. To summarize, in the indirect models the flavor symmetry is mainly responsible for constraining the Yukawa coupling structure of the neutrinos to have the form given in equation (4.20). It can also assist the generation of special flavon VEVs like those shown in equation (4.23).
4.3. Generalized CP and spontaneous CP violation

As discussed above, the residual-symmetry approach lacks the predictive power for the Majorana phases. In addition, obtaining a reasonable value of $\theta_{13}$ at the leading order either needs a large $G_F$ which will complicate the theory or reduces $G_F$ from $Z_2 \times Z_2$ to $Z_2$ which will be less predictive. It is found that the GCP symmetry [107, 108] can constrain the Majorana phases and allow a finite $\theta_{13}$ to be predicted. That is why the GCP has recently attracted a lot of attention in the model-building exercises, especially since $\theta_{13}$ was experimentally determined [203].

A theoretical reason for the introduction of the GCP concept lies in the fact that the canonical CP symmetry is not always consistent with the discrete non-Abelian flavor symmetry. To consistently define the CP symmetry in the context of a discrete flavor symmetry, a nontrivial condition has to be fulfilled as one can see later on. For the sake of simplicity and without loss of generality, let us consider a scalar multiplet $\Phi = (\phi, \phi^*)^T$ on which $G_F$ acts as

$$\Phi \rightarrow \rho(g)\Phi, \quad g \in G_F. \quad (4.24)$$

Here $\rho$ is used to denote the representation constituted by $\Phi$ which is not necessarily irreducible. On the other hand, a CP symmetry is expected to act on $\Phi$ in the form

$$\Phi \rightarrow X\Phi^*, \quad (4.25)$$

in which $X$ is unitary to keep the kinetic term $|\partial\Phi|^2$ invariant. Accordingly, a successive implementation of the CP, $g \in G_F$ and inverse CP symmetries will transform $\Phi$ through the following path:

$$\Phi \rightarrow X\Phi^* \rightarrow X\rho(g)^*\Phi^* \rightarrow X\rho(g)^*X^{-1}\Phi. \quad (4.26)$$

The consistency requirement suggests that $X\rho(g)^*X^{-1}$ should be a transformation belonging to $G_F$. We are therefore required to combine the CP symmetry with $G_F$ as follows [107, 108]:

$$X\rho(g)^*X^{-1} = \rho(g'), \quad g' \in G_F. \quad (4.27)$$

Thanks to the preservation of the multiplicity rules, we just need to assure the consistency of $X$ with the generators of $G_F$. The canonical CP symmetry (i.e. $X = 1$) always follows when $\rho$ is a real representation. But in a theory involving the complex representations $X = 1$ itself may not satisfy the consistency condition, in which case a GCP symmetry has to be invoked. Note that the GCP transformation may interchange different representations (especially those which are complex conjugate to each other). That is why one chooses to define CP on a vector space spanned by $\phi$ together with its complex conjugate counterpart $\phi^*$. In principle, different real representations can also be connected by a GCP transformation [108]. Hence $\rho$ has to contain all the representations connected by the GCP transformations. Moreover, all the matrices allowed by equation (4.27) constitute a representation of the automorphism group $\text{Aut}(G_F)$ of $G_F$. Let us consider two elements $a_1$ and $a_2$ of $\text{Aut}(G_F)$ whose representation matrices are $X_1$ and $X_2$, respectively. The representation matrix of $a_2a_1$ is $X_2WX_1$, as determined by the reasoning [108]:

$$\begin{align*}
\rho(g'^*) = X_2 \rho(g') X_2^{-1} = X_2 W \rho(g') W X_2^{-1} \\
= X_2 W X_2 \rho(g') X_2^{-1} W X_2^{-1},
\end{align*} \quad (4.28)$$

where $W$ is the matrix interchanging the complex conjugate components of $\Phi$ (i.e. $\Phi^* = W\Phi$), and it has the properties

$$W^2 = 1, \quad g \rightarrow WgW^*. \quad (4.29)$$

Note that for an $X$ satisfying equation (4.27), $g X g^*$ is also a solution but it does not supply any additional physical implications. Hence the GCP transformations of physical interest are given by $\text{Aut}(G_F)$ modding out those equivalent ones and form a group called $H_{\text{CP}}$. Consequently, the group $G_{\text{CP}}$ constituted by $G_F$ and $H_{\text{CP}}$ is isomorphic to their semi product [107, 108]

$$G_{\text{CP}} = G_F \times H_{\text{CP}}. \quad (4.30)$$

And the multiplication rule for any two elements of $G_{\text{CP}}$ is given by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2^*, h_1 h_2), \quad (4.31)$$

where $g_2^* = h_2g_1^{-1}$.

Now let us look at some physical implications of the GCP. By definition, the GCP symmetries constrain $M_i M_j$ and $M_i$ in the following way:

$$X^i M_j M_i^* X = (M_i M_j^*)^*, \quad X^i M_i^* X^* = M_i^*. \quad (4.32)$$

Since $M_i M_j^*$ and $M_i$ are diagonalized by $O_l$ and $O_\nu$, respectively, one may derive the following relations from equation (4.32):

$$X^i D_{2l} X_l = D_{2l}, \quad X^i D_{2\nu} X_\nu = D_{2\nu}, \quad (4.33)$$

in which

$$X_l = O_l^i X^{i \nu}_\nu, \quad X_\nu = O_{2\nu}^i X^{i \nu}_\nu. \quad (4.34)$$

It becomes clear that $X_l$ is a diagonal phase matrix. In comparison, $X_\nu$ is also diagonal but its finite entries are $\pm 1$. These results allow us to obtain a PMNS matrix with the property.
\[ X^\dagger UX = U^*. \] (4.35)

It is easy to check that this relation will lead us to the trivial CP phases [204, 205]. Hence it is necessary to break the GCP symmetry in order to accommodate CP violation. A phenomenologically interesting and economical way of breaking \( G_F \times H_{CP} \) is to preserve the residual symmetries \( G_F \) and \( G_e \times H_{CP}^e = Z_2 \times H_{CP} \) in the charged-lepton and neutrino sectors [107], respectively, as illustrated in figure 7. This goal can be achieved by requiring the related flavon VEVs to satisfy the conditions

\[ T(\phi) = \langle \phi \rangle, \quad S(\phi) = X(\phi)^\dagger = \langle \phi \rangle, \] (4.36)

in which \( T, S \) and \( X \) are the representation matrices for the generators of \( G_F, Z_2 \) and \( H_{CP}^e \), respectively. Furthermore, the commutation between \( Z_2 \) and \( H_{CP}^e \) requires

\[ XS^*X^{-1} = S. \] (4.37)

In light of \( S^2 = 1 \), we may diagonalize \( S \) by a unitary matrix \( O_S \) (i.e. \( O_S^\dagger SO_S = D_S = \pm \text{Diag}(-1, 1, -1) \)). One can see that the first and third eigenvalues of \( S \) are degenerate. In this case it is possible to redefine \( O_S \) by carrying out a complex \((1, 3)\) rotation. In combination with the condition given in equation (4.37), this freedom allows us to have \( O_S^\dagger O_S^* = X \). Taking account of the invariance of \( M_\nu \) under \( Z_2 \times H_{CP}^e \),

\[ S^\dagger M_\nu S^* = M_\nu, \quad X^\dagger M_\nu X^* = M_\nu^*, \] (4.38)

one finds

\[ D_S^\dagger O_S^\dagger M_\nu O_S^* D_S = O_S^\dagger M_\nu O_S^* D_S, \]

\[ O_S^\dagger M_\nu O_S^* = (O_S^\dagger M_\nu O_S^*). \] (4.39)

This means that \( O_S^\dagger M_\nu O_S^* \) can be diagonalized by a real orthogonal matrix

\[ R(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}. \] (4.40)

The PMNS matrix turns out to be [107]

\[ U = U_l^\dagger O_{3\theta} R(\theta) P_\nu, \] (4.41)

up to possible permutations of its rows or columns. Here \( U_l \) is determined by \( G_F \), whereas \( R \) is a diagonal matrix (with its entries being \( \pm 1 \) or \( i \)) to keep the three neutrino mass eigenvalues positive. In such an approach \( \theta \) is the only free parameter associated with the flavor mixing angles and CP-violating phases [107], and thus we are led to a few testable correlations among them. Of course, the value of \( \theta \) can be determined by confronting it with the experimental result of \( \theta_{13} \).

In the above direct-model approach, the values of CP-violating phases are purely determined by the group structure of \( G_F \times H_{CP} \). So we are left with little room for manoeuvre once the flavor symmetry has been specified. Furthermore, this approach typically leads the CP phases to some simple or special numbers such as 0, \( \pm \pi/2 \) or \( \pi \) [206–209], although nontrivial CP phases may arise as the flavor symmetry becomes larger [210, 211]. Alternatively, the indirect models can provide various nontrivial CP phases which are easier to control. In such models the canonical CP symmetry is consistent with \( G_F \) and all the parameters are real. A nontrivial CP phase is introduced by the complex VEV of a flavon field \( \phi \) which transforms trivially with respect to \( G_F \) [212–215].

In order to control the phase of \( \langle \phi \rangle \), one may introduce an extra \( Z_2 \) symmetry under which \( \phi \) carries a single charge [216]. When the \( F \)-term alignment mechanism [199] is used to shape the flavon VEVs, there will be a superpotential term relevant to \( \phi: \psi(\phi^\dagger M_\nu^* \pm M_\nu^2) \), where \( \psi \) denotes a driving field which is neutral under \( Z_2 \) and \( G_F \). Therefore, \( \langle \phi \rangle \) is required to satisfy the condition

\[ -F_\phi = \left| \frac{\langle \phi \rangle^*}{\Lambda^2 \pm M_\nu^2} \right| = 0, \] (4.42)

which may lead to any nontrivial phase for \( \langle \phi \rangle \) with the help of an adjustable \( n \) [217–219].

5. Realization of the \( \mu-\tau \) flavor symmetry

We proceed to discuss how to realize the \( \mu-\tau \) flavor symmetry in different physical contexts. First of all, let us make some general remarks.

(1) \( M M_\nu^\dagger \) and \( M_\nu \) cannot simultaneously possess the same \( \mu-\tau \) flavor symmetry. Otherwise, the resulting lepton flavor mixing pattern would only contain a finite value of \( \theta_{12} \) and thus be unrealistic. That is why one usually takes \( M_\nu \) to be diagonal and attributes the effects of lepton flavor mixing purely to \( M_\nu \), in a specific model-building exercise. Of course, one may impose a flavor symmetry on \( M_\nu \) but keep \( M_\nu \) diagonal, or allow both of them to be structurally non-diagonal and unparallel. Different basis choices are equivalent in principle, but one basis is likely to be advantageous over another in practice when building an explicit model to determine or constrain the flavor structures of both charged leptons and neutrinos. To illustrate this phenomenological point, we assume \( M M_\nu^\dagger \) to have the \( \mu-\tau \) permutation symmetry in a form like that given in equation (3.7), leading to the eigenvalues \( m_{\alpha}^2 \) (for \( \alpha = e, \mu, \tau \)) whose expressions are analogous to those given in equation (3.9). In this case one has to tolerate some significant cancellations among the parameters of \( M M_\nu^\dagger \) in order to generate a strongly hierarchical charged-lepton mass spectrum (i.e. \( m_e \ll m_\mu \ll m_\tau \), as shown in figure 2 [16]). It is certainly possible to conjecture that there exists a flavor symmetry which guarantees the exact cancellations and thus \( m_\mu = m_\tau = 0 \) at the lowest order, and it is the effect of flavor symmetry breaking that gives rise to nonzero \( m_\mu \) and \( m_\tau \), together with the associated flavor mixing parameters. Assuming the strength of symmetry breaking to be \( \epsilon \sim \mathcal{O}(0.1) \), one may expect that the finite values of \( m_\mu, \theta_{13} \) and \( \Delta \theta_{23} = \theta_{23} - \pi/4 \) emerge at the \( \mathcal{O}(\epsilon) \) level while nonzero \( m_\mu \) arises at the \( \mathcal{O}(\epsilon^2) \) level. To achieve such results, however, a delicate arrangement of the relevant perturbation terms is indispensable [220]. In
contrast, the neutrino mass spectrum should not exhibit a strong hierarchy unless $m_1$ is negligibly small. Another point is also noteworthy: although the lepton flavor mixing matrix is a constant matrix in the symmetry limit, it may be strongly correlated with the mass spectrum of charged leptons or neutrinos (or both of them) after the flavor symmetry is broken. When discussing the effects of $\mu$-$\tau$ flavor symmetry breaking, one therefore prefers to work in the basis of $M_l$ being diagonal such that it is more straightforward to infer some information about the neutrino mass spectrum from the experimental observations of $\theta_{13}$ and $\theta_{23}$. In other words, let us simply concentrate on the $\mu$-$\tau$ permutation or reflection symmetry of $M_l$ and its possible breaking schemes in the basis where $M_l$ itself is diagonal.

(2) A $U(1)_L \times U(1)_R \times U(1)_V$ flavor symmetry or its subgroup should be invoked in the charged-lepton sector to make $M_l M_l^T$ diagonal. Nevertheless, the left-handed charged-lepton and neutrino fields reside in the same $\text{SU}(2)_L$ doublets, so the symmetry placed on one sector should equally act on the other sector. Hence the major challenge in model building is to preserve a symmetry in one sector but break it properly in the other sector. To serve this purpose, additional Higgs (or flavon) fields furnishing nontrivial representations of the flavor symmetry are generally needed. In this connection the Higgs (or flavon) fields, which are neutral with respect to the symmetry for keeping the diagonality of $M_l M_l^T$ but transform nontrivially under the $\mu$-$\tau$ flavor symmetry, can be employed to break the would-be degeneracy between $m_\mu$ and $m_\tau$.

(3) A number of flavor-symmetry models, which are phenomenologically viable, have typically used the canonical seesaw mechanism to explain the smallness of three neutrino masses. Accordingly, at least two right-handed neutrinos are introduced in addition to the SM fields. No matter whether they assume the $\mu$-$\tau$ flavor symmetry or not, the effective Majorana neutrino mass matrix $M_l$ will have this symmetry provided the left-handed neutrinos respect the same symmetry. This interesting point can be explicitly seen from the seesaw formula $M_l = M_D M_N^T M_D^T$ [17–21] by taking the form of $M_D$ as

$$M_D = \langle H \rangle \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix},$$

(5.1)

where $M_N$ is an arbitrary symmetric matrix. However, one might wish that the right-handed neutrinos should also obey some flavor symmetries in order to reduce the number of free parameters [221].

5.1. Models with the $\mu$-$\tau$ permutation symmetry

In this section we aim to use a series of models, proposed by Grimus and Lavoura [113, 222, 223], to illustrate the essential ingredients for building a phenomenologically viable model based on the $\mu$-$\tau$ flavor symmetry. The main idea of their models is that the $U(1)_V \times U(1)_V \times U(1)_V$ symmetry is conserved by the Yukawa interactions but softly broken by the dimension-three Majorana mass terms of right-handed neutrinos. Thus let us introduce three right-handed neutrinos $N_i$ (for $\alpha = e, \mu, \tau$) and allow each of them to carry an associated lepton number. To properly break the flavor symmetry, three Higgs doublets $H_i$ (for $i = 1, 2, 3$) are needed and each of them develops a VEV $v_i$ (for $\sqrt{v_1^2 + v_2^2 + v_3^2} \approx 246$ GeV). In addition to the lepton-number symmetries, two $Z_2$-type flavor symmetries are also enforced [222]:

$$Z_2^{\mu \tau} : \begin{align*} L_\mu & \leftrightarrow L_\tau, \\ \mu_R & \leftrightarrow \tau_R, \\ N_\mu & \leftrightarrow N_\tau, \\ H_1 & \rightarrow - H_2; \\ \mu_R & \leftrightarrow \tau_R, H_2, H_3 \text{change sign.} \end{align*}$$

Here $Z_2^{\mu \tau}$ is the $\mu$-$\tau$ permutation symmetry interchanging all the fermion fields of $\mu$ and $\tau$. Note that $H_3$ is odd under the $Z_2^{\mu \tau}$ symmetry, and thus it will break this symmetry when obtaining a VEV. Such a requirement is actually necessary for obtaining $m_\mu = m_\tau$, as one will see later on. In addition, there is an auxiliary symmetry $Z_{2}^{\text{aux}}$ which separates $H_1$ from $H_2$ and $H_3$. Because of this symmetry, $H_2$ and $H_3$ are responsible for the mass generation of $\mu$ and $\tau$ while $H_1$ only couples to $e_L$ as well as $N_\ell$. Given the above setting, the Lagrangian for generating lepton masses is expressed as [224]

$$- \mathcal{L}_\text{mass} = y_1 \bar{e}_L e_R H_1 + y_2 (\bar{e}_L \mu_R + \bar{\tau}_R \mu_R) H_2 + y_3 (\bar{\tau}_R \mu_R - \bar{\tau}_R \tau_R) H_3 + \frac{1}{2} \bar{N}_\mu N_\mu \tilde{H}_1 + \text{h.c.},$$

(5.3)

where $\tilde{H}_1 \equiv i \sigma_2 H_1$. After the electroweak symmetry breaking, $M_l$ and $M_D$ turn out to be diagonal:

$$M_l = \begin{pmatrix} y_1 v_1 & 0 & 0 \\ 0 & y_2 v_2 + y_3 v_3 & 0 \\ 0 & 0 & y_3 v_3 \end{pmatrix},$$

$$M_D = \begin{pmatrix} y_1 v_1 & 0 & 0 \\ 0 & y_2 v_2 & 0 \\ 0 & 0 & y_3 v_3 \end{pmatrix}.$$  

(5.4)

Note that it is the coexistence of $H_2$ and $H_3$ that renders $m_\mu = m_\tau$. The heavy Majorana neutrino mass matrix $M_N$ breaks the lepton-number symmetries in a soft way but maintains the other symmetries, so it also possesses the $\mu$-$\tau$ permutation symmetry. Taking account of the seesaw formula, one therefore arrives at the light Majorana neutrino mass matrix $M_l$ which shares the same symmetry.

Several comments on this simple but instructive model-building approach are in order.

(a) In such a model the right-handed neutrinos necessarily suffer the $\mu$-$\tau$ permutation symmetry. The reason is that $U(1)_V$ and $U(1)_V$ do not commute with $Z_2^{\mu \tau}$. As a result of the $U(1)_V \times U(1)_V$ symmetry, $L_\mu$ (or $L_\tau$) is only allowed to have couplings to $\mu_R$ and $N_\mu$ (or $\tau_R$ and $N_\mu$). In order to keep $\mathcal{L}_\text{mass}$ invariant under $Z_2^{\mu \tau}$, $\mu_R$ and $\tau_R$ have to transform into each other as $L_\mu$ and $L_\tau$ do. So do $N_\mu$ and $N_\tau$. Furthermore, $U(1)_V$, $U(1)_V$, and $Z_2^{\mu \tau}$ may be unified in
a larger non-Abelian group which proves to be the two-dimensional orthogonal group $O_2$. (b) Another issue concerns the introduction of so many Higgs doublets. Although the rates of lepton-flavor-changing processes can be under control \cite{226}, one has to overcome some other phenomenological problems brought by so many electroweak-scale scalars (see, e.g. \cite{227–229} for the discussions on some consequences of the multi-Higgs fields introduced in a number of flavor-symmetry models). To isolate the flavor symmetry issues from the electroweak issues, one can use the flavons which emerge at a high energy scale $\Lambda$ to substitute for the extra Higgs doublets. As an example, we only keep $H_1$ while replacing $H_2$ and $H_3$ with two flavon fields $\phi_2$ and $\phi_3$. And $\phi_3$ (or $\phi_2$) has the same transformation properties as $H_2$ (or $H_3$) with respect to $Z_2^\mu$ and $Z_2^{\mu\tau}$. Correspondingly, the Lagrangian relevant for lepton masses can be obtained from that in equation (5.3) with the replacements
\begin{align}
(T_{\mu\tau\mu\tau} + T_{\tau\tau\tau\tau})H_2 &\to (T_{\mu\tau\mu\tau} + T_{\tau\tau\tau\tau})H_2 \phi_2^\mu \Lambda, \\
(T_{\mu\tau\mu\tau} - T_{\tau\tau\tau\tau})H_3 &\to (T_{\mu\tau\mu\tau} - T_{\tau\tau\tau\tau})H_3 \phi_3^\mu \Lambda.
\end{align}

(c) From the point of view of naturalness, the strong mass hierarchy of charged leptons needs some explanations. As for the model under discussion, the smallness of $m_3$ can be easily explained with the help of the Froggatt–Nielsen mechanism \cite{198}. If one assigns a U(1)$_{EN}$ charge $n$ to $e_R$, then the Yukawa interaction term for the electron will become
\begin{equation}
\gamma_1 T_{\mu\tau} e_R H \left( \langle \phi \rangle / \Lambda \right)^n,
\end{equation}
where $\phi$ is a flavon field carrying a U(1)$_{EN}$ charge of $-1$. Given $\epsilon \equiv \langle \phi \rangle / \Lambda$ as a very small quantity, $m_3$ is suppressed by $\epsilon^n$. However, this model does not permit the mechanism to explain the smallness of $m_\mu$ as compared with $m_\tau$. One finds that a relation $y_3(v_3) \approx - y_\mu(v_\mu)$ with an accuracy of about 10% is required for obtaining the realistic values of $m_3$ and $m_\mu$. Although a fine-tuning of this amount is not unacceptable, it is better to find a more natural way to obtain this kind of approximate relation. The idea is based on a new symmetry $K$ which connects $H_2$ and $H_3$ as follows \cite{230}:
\begin{equation}
\mu_R \to -\mu_R, \quad H_2 \to H_3.
\end{equation}

Hence the coefficients $y_2$ and $y_3$ in equation (5.3) should be opposite to each other. Furthermore, the $K$ symmetry can constrain the scalar potential to such a form that $v_2 = v_3$ is most likely to take place \cite{230}, in which case $m_\mu$ is exactly vanishing. When this symmetry is softly broken in the scalar potential, it is natural for us to expect a slight difference between $v_2$ and $v_3$. And thus we arrive at a finite value of $m_\mu$ which is much smaller than that of $m_\tau$ \cite{230}.

(d) The model under discussion can be slightly modified to accommodate the $\mu$-$\tau$ reflection symmetry \cite{230}. For this purpose, $Z_2^{\mu\tau}$ is replaced by the $\mu$-$\tau$ reflection symmetry while the other symmetries keep unchanged. Note that the $\mu$-$\tau$ reflection symmetry can be viewed as a GCP symmetry described by
\begin{equation}
F_{\alpha \beta} \to X_{\alpha \beta} F_{\alpha \beta}^c,
\end{equation}
where $F$ stands for all the fermion fields (i.e. $L$, $N$ and the right-handed charged-lepton fields), $X = S^\pm$ as given by equation (3.6)\footnote{A generalization of the $\mu$-$\tau$ reflection symmetry can be found in \cite{231}, where $X$ takes a form different from $S^\pm$.}, and the Greek subscripts run over $\epsilon$, $\mu$ and $\tau$. On the other hand, the actions of this symmetry on the three Higgs fields are supposed to be
\begin{equation}
H_1 \to H_1^\ast, \quad H_2 \to -H_2^\ast, \quad H_3 \to -H_3^\ast.
\end{equation}
The resulting $\mathcal{L}_{\text{mass}}$ appears as
\begin{align}
-\mathcal{L}_{\text{mass}} &= y_1 T_{\mu\tau} e_R H_1 + (y_2 T_{\mu\tau} e_R + y_3^* T_{\tau\tau} e_R) H_2 \\
&\quad + (y_3 T_{\tau\tau} e_R - y_2 T_{\mu\tau} e_R) H_3 + y_4 T_{\tau\tau} N H_1 \\
&\quad + (y_5 T_{\mu\tau} N_R + y_6 T_{\tau\tau} N_R) H_1 + \text{h.c.},
\end{align}
where $y_1$ and $y_3$ are real by definition. As a result, the expressions of $M_1$ and $M_2$ are changed to
\begin{align}
M_1 &= \begin{pmatrix}
y_{\nu_1} & 0 & 0 \\
0 & y_{\nu_2} + y_{\nu_3} & 0 \\
0 & 0 & y_{\nu_2}^* \mp y_{\nu_3}^* 
\end{pmatrix}, \\
M_2 &= \begin{pmatrix}
y_{\nu_1} & 0 & 0 \\
0 & y_{\nu_3} & 0 \\
0 & 0 & y_{\nu_3}^* 
\end{pmatrix}.
\end{align}

It should be noted that the phase difference between $v_2$ and $v_3$ cannot be $\pm \pi/2$. Otherwise, one would be led to the unrealistic result $m_\mu = m_\tau$. One may easily understand this observation by taking account of the fact that the $\mu$-$\tau$ reflection symmetry still holds when $v_2$ is real and $v_3$ is purely imaginary. If $M_1$ and $M_2$ take the forms in equations (3.24) and (5.11) respectively, then the seesaw formula assures the texture of $M_\nu$ to respect the $\mu$-$\tau$ reflection symmetry.

5.2. Models with the $\mu$-$\tau$ reflection symmetry

Now we turn to the model-building issues based on the $\mu$-$\tau$ reflection symmetry. First of all, let us consider an interesting model developed by Mohapatra and Nishi for the purpose of explaining the smallness of $m_\mu$ and $m_\tau$ reflection symmetry still holds when $v_2$ is real and $v_3$ is purely imaginary. If $M_1$ and $M_2$ take the forms in equations (3.24) and (5.11) respectively, then the seesaw formula assures the texture of $M_\nu$ to respect the $\mu$-$\tau$ reflection symmetry.

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\[ T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}. \]  

(5.12)

where \( \theta \) is an arbitrary constant. As compared with \( \text{U}(1)_{\nu} \times \text{U}(1)_{\tau} \times \text{U}(1)_{\tau} \), \( \text{U}(1)_{\nu-\tau} \) has a quite appealing consequence: it commutes with the \( \mu-\tau \) reflection symmetry as indicated by \( S^+ T S^+ = T \). This means that the whole flavor symmetry may be simply a direct product of \( \text{U}(1)_{\nu-\tau} \) and the \( \mu-\tau \) reflection symmetry. In order to lift the would-be degeneracy between \( m_\nu \) and \( m_\tau \), two extra Higgs doublets \( H_{L2} \), carrying the \( \text{U}(1)_{\nu-\tau} \) charges of \( \pm 2 \) are introduced in addition to the SM Higgs doublet \( H_0 \). Given this framework, the Yukawa coupling terms relevant for \( M_1 \) include [232]

\[ \lambda_0 \overline{\nu}_\tau H_0 + \lambda_{12} \overline{\nu}_\mu H_{L2} + \lambda_{2--} \overline{\nu}_\tau H_{L2}. \]  

(5.13)

The invariance of these terms with respect to the \( \mu-\tau \) reflection symmetry requires \( \lambda_0 = \text{Re} (\lambda_0) \) and \( \lambda_{12} = \lambda_{2--} \) as well as the following transformation properties for the Higgs fields:

\[ H_0 \rightarrow H'_{0}, \quad H_{L2} \rightarrow H^{*}_{L2}. \]  

(5.14)

Apparently, the \( \mu-\tau \) reflection symmetry will be broken if \( \langle H_{L2} \rangle \neq \langle H_0 \rangle \) holds. In particular, the relation \( \langle H_{L2} \rangle \ll \langle H_0 \rangle \) is required in order to generate \( m_\tau \ll m_\mu \). Such a relation can be induced in the scalar potential by a scalar field which changes its sign under the \( \mu-\tau \) reflection symmetry [233]. It should be pointed out that the unwanted terms \( \overline{\nu}_\mu \overline{\nu}_\mu H_0 \) and \( L_\tau \overline{\nu}_\tau H_0 \), which may render \( M_{11} M_{11}^\dagger \) non-diagonal, have been forbidden by some additional symmetries (e.g. a \( Z_2^{\text{aux}} \) symmetry under which only \( \mu \tau \) and \( H_{L2} \) change signs). Note that the Yukawa coupling terms in equation (5.13) will give rise to an accidental \( \text{U}(1)_1 \times \text{U}(1)_2 \) symmetry under which the related fields transform as

\[ L_\nu \rightarrow L_\nu e^{i\theta}, \quad \overline{\nu}_R \rightarrow \overline{\nu}_R e^{i\theta}, \quad \mu \tau \rightarrow \mu \tau e^{i\theta}. \]  

(5.15)

Although the \( \text{U}(1)_{\nu-\tau} \) symmetry will be spontaneously broken by \( (H_{L2})_0 \), the \( \text{U}(1) \times \text{U}(1) \) symmetry promises \( M_{11} M_{11}^\dagger \) to be diagonal. On the other hand, the Yukawa coupling terms of the neutrinos are given by [232]

\[ y_{11} \overline{\nu}_\nu N_1 H_0 + y_2 \overline{\nu}_\nu N_2 H_0 + y_3 \overline{\nu}_\nu N_3 H_0, \]  

(5.16)

where \( y_0 = \text{Re} (y_0) \) and \( y_2 = \bar{y}_2 \), as required by the \( \mu-\tau \) reflection symmetry. However, the resulting \( M_\nu \) cannot be realistic unless \( M_\nu \) contains the \( \text{U}(1)_{\nu-\tau} \) symmetry breaking terms. The latter can be achieved by including \( \phi_1 \) and \( \phi_2 \) which carry the respective \( \text{U}(1)_{\nu-\tau} \) charges of 1 and 2 but transform trivially under the \( \mu-\tau \) reflection symmetry. Consequently, the mass or Yukawa-coupling terms contributing to \( M_\nu \) are obtained as

\[ M_{11} \overline{\nu}_\nu N_1 + M_{21} \overline{\nu}_\nu N_2 + y_{11} \overline{\nu}_\nu N_1 \phi_1 + y_{12} \overline{\nu}_\nu N_2 \phi_1 + y_{22} \overline{\nu}_\nu N_2 \phi_2, \]  

(5.17)

where \( M_{11} \) and \( M_{21} \) are real mass parameters, whereas \( y_{12} = y_{13}^* \) and \( y_{22} = y_{23}^* \) hold. After \( \phi_1 \) and \( \phi_2 \) develop their real VEVs, the texture of \( M_\nu \) turns out to exhibit the \( \mu-\tau \) reflection symmetry. Given the form of \( M_0 \) in equation (5.16), the texture of \( M_\mu \) resulting from the seesaw formula will also share the \( \mu-\tau \) reflection symmetry.

It has recently been pointed out that the \( \mu-\tau \) reflection symmetry is not a necessary condition for obtaining the equality \( |U_{\mu i}| = |U_{\tau i}| \) (for \( i = 1, 2, 3 \)) for the PMNS matrix [234, 235]. In fact, this equality may arise in a more general context to be discussed below. Such a statement is based on the observation that an \( O_1 \) of the form in equation (3.19) together with a real orthogonal \( O_i \) always results in a PMNS matrix featuring \( U_{\mu i} = U_{\tau i}^* \). Moreover, it is found that \( O_{13} \) or \( O_{\mu i} \) will satisfy the above criteria when the residual symmetries \( G_\mu \) and \( G_\tau \) are real and able to completely fix the neutrino mixing pattern [234, 235]. Consider that \( G_\tau \) is generated by a real \( T \) in which case \( O_1 \) can be identified as the unitary matrix used to diagonalize \( T \). To obtain \( O_\mu \), we first determine the eigenvalues of \( T \) with the help of [234]

\[ \lambda_1^2 - \chi^2 + \chi \lambda_1 - 1 = 0, \]  

(5.18)

where \( \chi \) is the trace of \( T \) and \( |\lambda_1| = 1 \) for \( i = 1, 2, 3 \) holds. In addition to the eigenvalue \( \lambda_1 = 1 \), one finds the other two eigenvalues

\[ \lambda_{2,3} = \frac{1}{2} \left( \chi \pm \sqrt{(\chi - 1)^2 - 4} \right). \]  

(5.19)

Apart from the special case of \( \chi = -1 \), \( \chi \), and \( \lambda_3 \) are generally complex and conjugate to each other. For the general case, one can choose the eigenvectors corresponding to \( \lambda_2 \) and \( \lambda_3 \) to be complex conjugate to each other and the eigenvector for \( \chi = 1 \) to be real, making \( O_1 \) of the form in equation (3.19). In comparison, the generators \( S_1 \) and \( S_2 \) of \( G_\mu = Z_2 \times Z_2 \) are order 2 and thus only have the eigenvalues 1 and -1. Since we have required \( S_1 \) and \( S_2 \) themselves to be real, their common eigenvectors can be chosen to be real as well. Then we arrive at \( O_\mu = O_1 P_1 \) with \( O_1 \) being the real orthogonal matrix diagonalizing \( S_1 \) and \( S_2 \) simultaneously. Note that the undetermined diagonal phase matrix \( P_1 \) arises from the fact that the \( S_1 \) and \( S_2 \) symmetries can only determine \( O_\mu \) up to the Majorana phases. Putting all these pieces together, a PMNS matrix \( U = O_1^* O_2 P_2 \) fulfilling the relation \( |U_{\mu i}| = |U_{\tau i}| \) is finally available. Some immediate comments are in order.

1. If \( G_\mu \) only contains one \( Z_2 \) factor, then \( O_2 \) cannot be completely fixed—an example of this kind is the TM1 or TM2 mixing pattern given in equation (4.6). In this case the reality of \( O_2 \) is lost and the relation \( |U_{\mu i}| = |U_{\tau i}| \) cannot hold any more. Therefore, \( G_\mu \) and \( G_\tau \) are required to be able to pin down the lepton flavor mixing pattern in a realistic model-building exercise.

2. When one considers unifying \( G_\mu \) and \( G_\tau \) in a larger discrete symmetry \( G_F \), a subgroup of \( O(3) \) (e.g. \( A_4 \), \( S_4 \), or \( A_5 \)) will be a promising candidate in which \( G_\mu \) and \( G_\tau \) are inherently real [234, 235]. Nevertheless, none of \( A_5 \), \( S_4 \), or \( A_5 \) can give us a satisfactory \( \theta_1 \) at the lowest order [234]. Hence higher-order contributions have to be invoked in these symmetry-based models.
(3) This approach may serve as a complementarity to the \(\mu-\tau\) reflection symmetry approach but cannot take over from it. Although both of them can predict \(\theta = \pm \pi/2\) and \(\theta_{23} = \pi/4\), the former is also able to determine \(\theta_{12}\) and \(\theta_{13}\) but the latter fails in this connection. On the other hand, only the \(\mu-\tau\) reflection symmetry dictates the Majorana phases to take trivial values (i.e. 0 or \(\pm \pi/2\)).

(4) In fact, it is possible to generalize the theorem formulated here to recover the \(\mu-\tau\) reflection symmetry such that the Majorana phases can be fixed. Suppose that \(M_\mu\) itself, instead of \(G_\nu\), should be real in which case \(O_\nu\) is a real orthogonal matrix without being subject to an undetermined \(P\). The resulted neutrino mixing matrix will have the form in equation (3.19) with the Majorana phases being trivial. It is not surprising to find that such results agree with those predicted by the \(\mu-\tau\) reflection symmetry, simply because the neutrino mass matrix will take the form in equation (3.24) when one returns to the \(M_\ell M_\ell^T\)-diagonal basis by an \(O_\ell\) transformation of the form in equation (3.19) [236–239]. This scenario provides an alternative way of getting at some interesting consequences of the \(\mu-\tau\) reflection symmetry [235].

5.3. On the TM1 and TM2 neutrino mixing patterns

As emphasized before, a general neutrino mass matrix with the \(\mu-\tau\) permutation symmetry is unable to give a definite prediction for \(\theta_{12}\) unless some further conditions are placed on it. The most popular example in this regard might be \(M_\mu = M_\nu + M_\mu\), a condition that can lead us to the flavor mixing pattern \(U_{TB}\) [92, 93]. The latter yields \(\theta_{12} = \arctan(1/\sqrt{2}) \approx 35.3^\circ\), which is compatible with the experimental value to a reasonably good degree of accuracy. Before the size of \(\theta_{13}\) was reasonably measured in the Daya Bay experiment [66], the \(U_{TB}\) pattern was widely believed to be a good description of neutrino mixing and many discrete flavor symmetry groups (most notably, A4 [186, 187]) had been employed to build the explicit models for deriving this special flavor mixing matrix. However, the Daya Bay discovery of a relatively large \(\theta_{13}\) requires a remarkable modification of \(U_{TB}\) [240]. The simplest alternative turns out to be the TM1 or TM2 flavor mixing pattern given in equation (4.6) [175, 178–183]. They owe their names to the fact that they keep the first and second columns of the \(U_{TB}\) pattern, respectively. In fact, one has [178, 182]

\[
U_{TM1} = U_{TB} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -se^{-i\phi} \\ 0 & -se^{i\phi} & c \end{pmatrix},
\]

\[
U_{TM2} = U_{TB} \begin{pmatrix} c & 0 & se^{-i\phi} \\ 0 & 1 & 0 \\ -se^{i\phi} & 0 & c \end{pmatrix},
\]

where \(c \equiv \cos \theta\) and \(s \equiv \sin \theta\). These two patterns are attractive from a phenomenological point of view. First of all, the observed \(\theta_{13}\) can be easily accommodated thanks to the free parameter \(\theta\). Second, the value of \(s_{12}^2\) predicted by \(U_{TB}\) is only slightly modified in the TM1 or TM2 pattern:

\[
s_{12}^2 = \frac{1}{3} \frac{1 - 3s_{13}^2}{1 - s_{13}^2} \approx 0.318 \quad \text{(for TM1)},
\]

\[
s_{12}^2 = \frac{1}{3} \frac{1 - 3s_{13}^2}{1 - s_{13}^2} \approx 0.341 \quad \text{(for TM2)}. \tag{5.21}
\]

Note that the value of \(s_{12}^2\) given by the TM1 mixing matrix is in better agreement with the experimental result than that given by \(U_{TB}\) or the TM2 mixing matrix. Third, \(\theta_{23}\) is correlated with \(\theta_{13}\) and the phase parameter \(\phi\) as follows:

\[
s_{23}^2 \approx \frac{1}{2} \left(1 - 2\sqrt{2}s_{13} \cos \varphi\right) \quad \text{(for TM1)},
\]

\[
s_{23}^2 \approx \frac{1}{2} \left(1 + \sqrt{2}s_{13} \cos \varphi\right) \quad \text{(for TM2)}. \tag{5.22}
\]

The dependence of \(s_{23}^2\) on \(\varphi\), with the value of \(\theta_{13}\) as an input, is illustrated in figure 8. One can see that \(\theta_{23}\) may stay close to \(\pi/4\) if \(\varphi\) takes a value around \(\pi/2\) or \(3\pi/2\). In addition, the correlation between \(\theta_{23}\) and \(\varphi\) is stronger in the TM1 case and thus easier to be tested. Finally, the Jarlskog invariant of CP violation in the lepton sector [41] is found to be

\[
J = \frac{1}{6} \sqrt{2 - 6s_{13}^2} s_{13} \sin \varphi \quad \text{(for TM1)},
\]

\[
J = \frac{1}{6} \sqrt{2 - 3s_{13}^2} s_{13} \sin \varphi \quad \text{(for TM2)}, \tag{5.23}
\]

whose maximum magnitude is around 3.8% at \(\varphi = \pi/2\) or \(3\pi/2\). By comparing the above expression of \(J\) with

Figure 8. The possible values of \(s_{12}^2\) and \(\sin \delta/\sin \varphi\) against \(\varphi\) in the TM1 (red lines) or TM2 (blue lines) neutrino mixing case.
\[ \mathcal{J} = c_1 s_1 c_2 s_1 s_2 s_3 \sin \delta \] in the standard parametrization of \( U \) given below equation (2.13), we obtain the Dirac CP-violating phase \( \delta \) as follows:

\[
\begin{align*}
\sin \delta &= \frac{1 - s^2_{13}}{\sqrt{1 - 6 s^2_{13} - 4 s^2_{13} \cos 2 \varphi}} \quad \text{(for TM1),} \\
\sin \delta &= \frac{1 - s^2_{13}}{\sqrt{1 - 3 s^2_{13} - 2 s^2_{13} \cos 2 \varphi}} \quad \text{(for TM2).} \\
\end{align*}
\]

(5.24)

As shown in figure 8, \( \delta \) is approximately equal to \( \varphi \), especially around \( \varphi = \pi/2 \) and \( 3\pi/2 \).

Theoretically, exploring some appropriate physical contexts that can justify these two particular mixing patterns makes sense. Besides the residual-symmetry approach discussed in section 4.1 [209, 241–246], the Friedberg–Lee (FL) symmetry [247–255] can also lead to the TM1 and TM2 mixing patterns in a natural way. The FL symmetry in the neutrino sector is defined in the sense that the neutrino mass operators should be invariant under the following translational transformations for the neutrino fields:

\[ \nu_\alpha \to \nu_\alpha + \eta_a \xi, \]  

(5.25)

in which \( \alpha \) runs over \( e, \mu \) and \( \tau \), \( \xi \) is a spacetime-independent Grassmann-algebra element which anti-commutes with the neutrino fields, and \( \eta_a \) are the complex coefficients. The FL symmetry dictates the neutrino mass terms to be of the form

\[
-\mathcal{L}_{\text{mass}} = a (\eta^*_a \eta^*_a \phi^*_d \phi_d - \eta^*_a \phi^*_d \phi_d) + b (\eta^*_a \phi^*_d \phi_d - \eta^*_a \phi^*_d \phi_d) + c (\eta^*_a \phi^*_d \phi_d - \eta^*_a \phi^*_d \phi_d) + \text{h.c.} 
\]

(5.26)

with \( a, b \) and \( d \) being the arbitrary complex numbers. When proposing the original version of this effective flavor symmetry, Friedberg and Lee assumed the neutrino fields to transform in a universal way (i.e. \( \eta_a = \eta_a = \eta_a \)) [247] in which case the neutrino mass matrix appears as [251]

\[
M_\nu = \begin{pmatrix}
 b + d & -b & -d \\
 -b & a + b & -a \\
 -d & -a & a + d
\end{pmatrix}.
\]

(5.27)

This matrix can be transformed into the following form by using the \( U_{TB}\) matrix:

\[
U_{TB}^{\dagger}M_\nu U_{TB} = \frac{1}{2} \begin{pmatrix}
 3(b + d) & 0 & \sqrt{3}(d - b) \\
 0 & 4a + b + d
\end{pmatrix}
\]

(5.28)

which can be further diagonalized by a complex (1, 3) rotation\(^{18}\). So the unitary matrix used to diagonalize \( M_\nu \) has the same form as \( U_{TM1} \) in equation (5.20). However, a potential problem associated with \( M_\nu \) in equation (5.28) is its prediction \( m_3 = 0 \), which is in conflict with \( \Delta m^2_{31} > 0 \). A simple way out is to add a universal term \( m_0 (\mu \nu^c_3 + \tau \nu^c_3 + \mu \nu^c_3) \) to the above neutrino mass operators [247], leading to \( m_2 = m_0 \). But this 18 Taking \( b = d \), one is led to the \( \mu-\tau \) permutation symmetry and thus the flavor mixing pattern \( U_{TB} \).

term explicitly breaks the FL symmetry, and hence it needs a convincing explanation. Another way out is to modify the FL symmetry by assuming \( \eta_2 = -2\eta_2 = -2\eta_2 \) [256], and the associated neutrino mass matrix reads

\[
M'_\nu = \begin{pmatrix}
 2d + b & 2b & 2d \\
 2d & a + 4b & a + 4d \\
 2d & -a & a + 4d
\end{pmatrix}.
\]

(5.29)

Similarly, \( U_{TB} \) can transform \( M'_\nu \) into the form

\[
U_{TB}^{\dagger}M'_\nu U_{TB} = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 3(b + d) & \sqrt{6}(d - b) \\
 0 & \sqrt{6}(d - b) & 2(a + b + d)
\end{pmatrix}
\]

(5.30)

which can be further diagonalized by a complex (2, 3) rotation. Hence the resultant neutrino mixing matrix will be the TM2 pattern. In particular, \( m_1 = 0 \) comes out from this ansatz, implying that such a modified FL symmetry does not need to be broken. It is worth pointing out that a combination of the modified FL symmetry and the \( \mu-\tau \) reflection symmetry can pin down all the physical parameters of massive neutrinos. Note that if \( \eta_2 = \eta_2 \) allows for this kind of operation which otherwise would not make sense. In this specific scenario the neutrino mass matrix maintains its form in equation (5.29) and is subject to some further conditions such as \( b = d^* \) and \( \text{Im}(a) = 0 \). The neutrino masses and flavor mixing quantities can be determined in terms of three real parameters: \( \text{Re}(a), \text{Re}(b) \) and \( \text{Im}(b) \). While the above results for \( (m_\nu, m_\nu, m_\nu) \) and \( (\theta_{12}, \theta_{13}, \theta_{23}) \) still hold, the CP phases will take specific values (\( \varphi = \pi/2 \) or \( 3\pi/2 \) and \( \rho, \sigma = 0 \) or \( \pi/2 \)). To summarize, the modified FL symmetry can lead us to some definite predictions which are compatible with current neutrino oscillation data, but the FL symmetry itself remains puzzling and needs a further study.

Finally, let us emphasize that the TM1 and TM2 mixing patterns can also be derived via the indirect model approach. To achieve the TM1 or TM2 mixing matrix based on an indirect model which has been introduced at the end of section 4.2 for realizing \( U_{TB} \), one has to modify the VEV alignments in equation (4.23) or construct \( M_\nu \) and \( \phi^c_\nu \) with off-diagonal elements. A possible solution inspired by equation (5.29) is to change \( \langle \phi^c_\nu \rangle \) and \( \langle \phi^c_\nu \rangle \) to the following forms [257]:

\[
\langle \phi^c_\nu \rangle = v_1 \begin{pmatrix}
 1 \\
 0 \\
 0
\end{pmatrix}, \quad \langle \phi^c_\nu \rangle = v_2 \begin{pmatrix}
 1 \\
 0 \\
 0
\end{pmatrix}.
\]

(5.31)

It is obvious that the effective neutrino mass matrix obtained from equation (4.22) with such VEV alignments will have the same form as that given in equation (5.29). In this case one neutrino remains massless, although the seesaw mechanism with three right-handed neutrinos has been taken into account. The observation that the TM1 or TM2 mixing pattern can be viewed as \( U_{TB} \) multiplied by a complex (1, 3) or (2, 3) rotation matrix from its right-hand side suggests another possible solution: one may keep the regular VEV alignments and (or) \( M_\nu \) \( M_\nu \) and \( \phi^c_\nu \) \( \phi^c_\nu \) with off-diagonal elements.
off-diagonal terms in an approximately diagonal $M_N$ [214, 258]. To be specific, $M_N$ may have the form

$$M_N = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{pmatrix} \quad \text{(for TM1)},$$

$$M_N = \begin{pmatrix} M_{11} & 0 & M_{13} \\ 0 & M_2 & 0 \\ M_{13} & 0 & M_{33} \end{pmatrix} \quad \text{(for TM2)},$$

(5.32)

where the off-diagonal terms may be traced back to the symmetry-breaking effects and thus relatively small. In this situation one can go back to the mass basis of right-handed neutrinos by means of a complex $(2,3)$ or $(1,3)$ rotation which will in turn change the Yukawa coupling matrix of the neutrinos. Then it is easy to check that the seesaw formula allows us to arrive at a particular texture of $M_\nu$, which finally leads us to the TM1 or TM2 neutrino mixing matrix.

### 5.4. When the sterile neutrinos are concerned

A ‘sterile’ neutrino $\nu_s$, as its name suggests, does not carry any quantum number of the SM gauge symmetry and thus does not directly take part in the standard weak interactions. Although there has not been any convincing evidence for sterile neutrinos, their existence is either theoretically motivated or experimentally hinted. A good example of this kind is the heavy right-handed neutrinos introduced for implementing the type-I seesaw mechanism [214]. In this context, the Majorana neutrino mass matrix for $\nu_s$, $\nu_\mu$, $\nu_\tau$, and $\nu_\tau$ can be expressed as [269]

$$M_\nu = \begin{pmatrix} m_{ee} & m_{e\mu} & m_{e\tau} & m_{e\nu_4} \\ m_{\mu e} & m_{\mu\mu} & m_{\mu\tau} & m_{\mu\nu_4} \\ m_{\tau e} & m_{\tau\mu} & m_{\tau\tau} & m_{\tau\nu_4} \\ m_{\nu_4 e} & m_{\nu_4\mu} & m_{\nu_4\tau} & m_{\nu_4\nu_4} \end{pmatrix}.$$  

Here the $3 \times 3$ submatrix of $M_\nu$ for the active neutrinos has already been assumed to have the $\mu-\tau$ permutation symmetry. But it is unnecessary to assume $m_{\nu_4} = m_{\nu_3}$ if the origin of sterile neutrinos is different from that of active ones. For instance, the active neutrino sector can be viewed as a consequence of the type-II seesaw mechanism [22–26] while the other elements of $M_\nu$ might originate from the canonical seesaw mechanism. A general $4 \times 4$ neutrino mass matrix can be diagonalized by a $4 \times 4$ unitary matrix $U$ to give 4 real mass eigenvalues $m_i$ (for $i = 1, 2, 3, 4$). And $U_4$ contains 6 rotation angles $\theta_{ij}$ (for $ij = 12, 13, 23, 14, 24, 34$), 3 Dirac phases and 3 Majorana phases. A quite popular parametrization of $U_4$ reads as follows [273]:

$$U_4 = O_{34}U_{23}U_{13}O_{23}U_{12}P,$$

(5.34)

where $O_{ij}$ is a real orthogonal rotation matrix with the mixing angle $\theta_{ij}$ and $U_4$ denotes a unitary rotation matrix with both the mixing angle $\theta_{ij}$ and the phase parameter $\gamma_{ij}$ as illustrated in equation (3.90). In addition, $P = \text{Diag} \{e^{i\theta_{14}}, e^{i\theta_{24}}, 1, e^{i\theta_{34}}\}$ denotes the Majorana phase matrix. This parametrization has an advantage that the upper-left $3 \times 3$ submatrix of $U_4$ can simply reduce to the PMNS matrix $U$ of the active neutrinos in the limit of vanishing active-sterile mixing.

Now we proceed to study the implications of the neutrino mass matrix in equation (5.33) by assuming that it can explain the experimental data concerning the active neutrinos. First of all, we intend to find out the possible values of $m_\nu$ and $\theta_{ij}$. To serve this purpose, let us reconstruct the elements of $M_\nu$ in terms of $U_4$ and the neutrino masses [274]:

$$m_{\mu\mu} \simeq -m_{\nu_1\nu_2} (s_{13}^2 c_{23} + c_{13} s_{23}) + m_{\nu_1\nu_2} (c_{13} c_{23} - s_{13} s_{23})$$

$$m_{\mu\tau} \simeq m_{\nu_1\nu_2} (s_{13} s_{23} + c_{13} c_{23}) + m_{\nu_1\nu_2} (c_{13} c_{23} - s_{13} s_{23})$$

$$m_{\mu\nu_4} \simeq m_{\nu_1\nu_2} (s_{13} c_{23} + c_{13} s_{23}) + m_{\nu_1\nu_2} (c_{13} c_{23} - s_{13} s_{23})$$

$$m_{\tau\tau} \simeq m_{\nu_1\nu_2} (s_{13} c_{23} + c_{13} s_{23}) + m_{\nu_1\nu_2} (c_{13} c_{23} - s_{13} s_{23})$$

$$m_{\tau\nu_4} \simeq m_{\nu_1\nu_2} (s_{13} c_{23} + c_{13} s_{23}) + m_{\nu_1\nu_2} (c_{13} c_{23} - s_{13} s_{23})$$

(5.35)

where $m_\nu \equiv m_{\nu_1 + 2 \nu_2}$, $m_\tau \equiv m_{\nu_3 + 2 \nu_4}$ and $m_4 \equiv m_{\nu_4 + 2 \nu_4}$. In obtaining these relations, we have made use of the smallness of $\theta_{14}$ as implied by the unitarity of $U$ which has been tested at the percent level [37, 38]. And we are allowed to identify $\theta_{12}$, $\theta_{13}$, $\theta_{23}$ and $\theta_{13}$ in $U_4$ with their counterparts in $U$ without loss of much accuracy. Taking account of the $\mu-\tau$ symmetry conditions $m_{\nu_1} = m_\tau$, and $m_{\nu_2} = m_{\nu_4}$, we arrive at

$$\delta_{34} = \sqrt{2} \left( \frac{m_3 - m_2}{m_3} \Delta \theta_{23} - m_3 s_{13} + m_1 s_{13} - s_{13} \delta_{34} - s_{24} \right),$$

$$m_4 = \sqrt{2} \frac{m_3 \Delta \theta_{23} + m_1 s_{13} - m_3 s_{13}}{s_{14} (\delta_{34} - \delta_{24}).}$$

(5.36)

where $m_{11}$, $m_{12}$ and $m_{32}$ have already been defined in equation (3.5), and $\Delta \theta_{23} \equiv \theta_{23} - \pi/4$. There are so many free parameters that no definite conclusions can be drawn. For the sake of simplicity and illustration, let us consider the

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19 In addition, possible keV sterile neutrinos could be a good candidate for warm dark matter in the Universe [263–265].
In the $m_1 < m_2 \ll m_3$ case, $M_1$ can be parameterized in the form

$$M_1 = m \begin{pmatrix} ee^2 & de^2 & de^2 & e \\ de^2 & ee & ce & ae \\ de^2 & ce & ee & be \\ e & ae & be & 1 \end{pmatrix},$$

where $\epsilon \simeq 0.1$, and $a, b, c, d$ and $e$ are all the $\mathcal{O}(1)$ coefficients. The relatively heavy $v_4$ as compared with $v_i$ (for $i = 1, 2, 3$) can be integrated out, like the treatment for the heavy Majorana neutrinos in the seesaw mechanism, leaving us a $3 \times 3$ neutrino mass matrix of the form in equation (3.59). According to our previous analysis, we can naturally obtain the phenomenologically acceptable results without having to tune the relevant coefficients.

(b) The overall neutrino mass matrix that can result in $m_1 \simeq m_2 \gg m_3$ together with $(\rho, \sigma) = (0, 0)$ appears as

$$M = m \begin{pmatrix} 2ce & de^2 & de^2 & e \\ de^2 & ce & ce & ae \\ de^2 & ce & ce & be \\ e & ae & be & 1 \end{pmatrix},$$

The resulting active neutrino mass matrix is similar to that given in equation (3.63) after the sterile neutrino is integrated out. As discussed before, the fine-tuning conditions $a + b - 2d \simeq 0$ and $(a + b)^2 - 2 \simeq 0$ will be required to fit the experimental data. It is therefore difficult to realize such a scenario from the viewpoint of model building.

(c) The $m_1 \simeq m_2 \simeq m_3$ case with $(\rho, \sigma) = (0, 0)$, which allows a relatively light $v_4$, can be viewed as a consequence of the following mass matrix:

$$M = m \begin{pmatrix} fe^2 + 1 & ee^2 & ee^2 & e \\ ee^2 & de + 1 & -de & ae \\ ee^2 & de & de + 1 & be \\ e & ae & be & c \end{pmatrix}.$$  

Like the mass matrix given in equation (5.40), the present texture of $M_1$ can naturally fit the experimental results regarding the active neutrino mixing and flavor oscillations. Of course, some higher-order terms in the above ansätze of the neutrino mass matrix (e.g. those located at the $\mu\tau$ entries) can be neglected in the first place so as to get much more and much simpler predictions.

6. Some consequences of the $\mu\tau$ symmetry

The exact or approximate $\mu\tau$ flavor symmetry may have some important phenomenological consequences in neutrino physics and cosmology. Here let us look at a few interesting examples of this type, such as neutrino oscillations in matter, flavor distributions of ultrahigh-energy (UHE) cosmic neutrinos, cosmic matter-antimatter asymmetry via leptogenesis, and fermion mass matrices with a common $Z_2$ symmetry.
6.1. Neutrino oscillations in matter

It is known that the behaviors of neutrino mass states in matter can be quite different from those in vacuum, simply because the matter-induced coherent forward scattering effects modify the genuine neutrino masses and flavor mixing parameters when a neutrino beam travels through a normal medium (e.g., the Sun or Earth) [42, 43]. If the PMNS matrix $U$ in vacuum possesses the $\mu$-$\tau$ flavor symmetry, one can show that the effective PMNS flavor mixing matrix $\tilde{U} = VP$ in matter must possess the same symmetry [20]. In other words, the matter effects respect the $\mu$-$\tau$ permutation or reflection symmetry in neutrino oscillations [106, 275]. Let us go into details of this observation in two different ways in the following.

In the basis where the mass and flavor eigenstates of three charged leptons are identical, we denote the effective neutrino mass matrix in matter as $\tilde{M}$. No matter whether massive neutrinos are of the Dirac or Majorana nature, the effective Hamiltonian responsible for the propagation of a neutrino in matter can be expressed in a similar way as that in equation (2.11):

$$\tilde{H}_{\text{eff}} = \frac{1}{2E} \tilde{M}_i \tilde{M}_\nu = \frac{1}{2E} \tilde{D}_i V D^\dagger V^\dagger + D_m,$$

where $\tilde{D}_i = \text{Diag}(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$ with $\tilde{m}_i$ (for $i = 1, 2, 3$) being the effective neutrino masses in matter, and $D_m = \text{Diag}[A, 0, 0]$ with $A = 2\sqrt{2} G_F N_E$ denoting the charged-current contribution to the coherent $\nu^c e^c$ forward scattering (here $N_E$ stands for the background number density of electrons [42]).

Without loss of generality, it is always possible to redefine the phases of three neutrino mass eigenstates such that $V_{\nu_i}$ and $\tilde{V}_{\nu_i}$ (for $i = 1, 2, 3$) are all real and positive [39]. To be explicit,

$$V = O_{23} \tilde{O}_\delta O_{13} O_{12} = \begin{pmatrix} c_{12} s_{13} & s_{12} c_{13} & s_{13} \\ -s_{12} c_{23} e^{-i\delta} - c_{12} s_{13} s_{23} & c_{12} c_{23} e^{-i\delta} - s_{12} s_{13} c_{23} & c_{13} s_{23} \\ s_{12} s_{23} & -c_{12} s_{23} & c_{23} \end{pmatrix},$$

with

$$V_{\mu\tau} = -s_{12} c_{23} e^{-i\delta} - c_{12} s_{13} s_{23},$$
$$V_{\mu2} = c_{12} c_{23} e^{-i\delta} - s_{12} s_{13} c_{23},$$
$$V_{\mu1} = s_{12} s_{23} c_{23},$$
$$V_{\mu3} = c_{13} s_{23} c_{23},$$
and $O_\delta = \text{Diag}[1, e^{-i\delta}, 1]$. Such a phase convention is apparently different from that taken in equation (2.8). In matter the effective matrix $\tilde{V}$ takes the same phase convention as $V$ in equation (6.2). In this particular basis $\tilde{V}_{\mu i} = |V_{\nu i}|$ and $\tilde{V}_{\mu i} = |\tilde{V}_{\nu i}|$ (for $i = 1, 2, 3$) hold, and thus it will be suitable for discussing the $\mu$-$\tau$ flavor symmetry. Thanks to equation (6.1), we simply obtain

$$\sum_i m_i^2 |\tilde{V}_{\mu i}^*| = \sum_i m_i^2 V_{\nu i} |V_{\nu i}^*| + \delta e\alpha_{23} A,$$

where $\alpha$ and $\beta$ run over $e, \mu$ and $\tau$. If $V$ possesses the $\mu$-$\tau$ symmetry, then it is straightforward to show that $\tilde{V}$ must have the same symmetry. Let us consider two distinct possibilities.

(A) The $\mu$-$\tau$ permutation symmetry. In this limit $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ hold, and thus the phase parameter $\delta$ in $V$ can be removed. We are then left with a real $V$ with $V_{\nu_3} = 0$, $V_{\nu_1} = -V_{\nu_2}$, $V_{\nu_2} = -V_{\nu_3}$, and $V_{\nu_3} = V_{\nu_2}$ from equation (6.5), corresponding to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ in the parametrization of $\tilde{V}$ similar to equation (6.2). Note that it is impossible to simultaneously have $\tilde{V}_{\nu_1} = V_{\nu_1}$, $\tilde{V}_{\nu_2} = V_{\nu_2}$ and $\tilde{V}_{\nu_3} = V_{\nu_3}$ (or $\tilde{V}_{\mu_1} = -\tilde{V}_{\mu_1}$, $\tilde{V}_{\mu_3} = -\tilde{V}_{\mu_3}$, and $\tilde{V}_{\mu_3} = -\tilde{V}_{\mu_3}$), because they violate the unitarity requirement of $\tilde{V}$. We conclude that $\tilde{V}$ can have the same $\mu$-$\tau$ permutation symmetry as $V$ does.

(B) The $\mu$-$\tau$ reflection symmetry. In this more interesting limit $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ hold, leading to $V_{\nu_1} = V_{\nu_2}^*$, $V_{\nu_3} = V_{\nu_3}$, and $V_{\nu_3} = V_{\nu_3}$ as both $V_{\nu_3}$ and $V_{\nu_3}$ are real in the phase convention of equation (6.2), one may also take $V_{\nu_3} = V_{\nu_3}$ and thus $V_{\nu_3} = V_{\nu_3}$ (for $i = 1, 2, 3$) in this case [106]. As a result, equation (6.4) leads us to the relations

$$\sum_i m_i^2 |\tilde{V}_{\mu i}| = \sum_i m_i^2 |\tilde{V}_{\mu i}|,$$

and hence it is straightforward to obtain $\tilde{V}_{\nu i} = V_{\nu i}$ from equation (6.6) for arbitrary $m_i^2$. Given the parametrization of $\tilde{V}$ similar to equation (6.2), the above equalities give rise to $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$. That is why $V$ possesses the $\mu$-$\tau$ reflection symmetry as $V$ does.

Now we turn to another way to show that the matter effects respect the $\mu$-$\tau$ reflection symmetry in neutrino oscillations. Taking the parametrization of $V$ in equation (6.2), we find

$$\tilde{M}_i \tilde{M}_\nu = O_{23} \tilde{O}_\delta O_{13} O_{12} D^2 \tilde{O}_{13} O_{12} + D_m O_\delta^2 O_{13} O_{12},$$

with $O_{12} O_{13} O_{12}$ being a real orthogonal matrix which consists of three rotation angles ($\theta_{12}, \theta_{13}, \theta_{23}$) and has been defined to diagonalize the real bracketed part in equation (6.7). Up to the freedom of diagonal phase matrices $P_L$ and $P_R$ denoted
below, which are actually irrelevant to neutrino oscillations, the effective neutrino mixing matrix in matter turns out to be

\[ V \equiv \tilde{O}_{23} O'_{13} \tilde{O}_{13} = P_1 O_{23} O'_{13} \tilde{O}_{13} P_R \]
\[ = P_1 O_{23} O'_{13} \tilde{O}_{13} P_R, \]  \hspace{1cm} (6.8)

where \( \tilde{O}_{13} \) and \( \tilde{O}_{13} \) have been identified with \( \tilde{O}_{12} \) and \( \tilde{O}_{13} \), respectively. In comparison, \( \tilde{\theta}_{23} \) and \( \tilde{\delta} \) must arise from a mixture of \( \theta_{23} \) and \( \delta \). In a way analogous to equation (2.13), we define the effective Jarlskog invariant \( \mathcal{F} \) in matter and calculate it with the help of equation (6.8). After a straightforward calculation, we arrive at

\[ \mathcal{F} = \tilde{c}_{12} \tilde{\delta} \tilde{c}_{13} \tilde{\delta}_{23} \sin \tilde{\delta} \]
\[ = \tilde{c}_{12} \tilde{\delta} \tilde{c}_{13} \tilde{\delta}_{23} \sin \delta, \]  \hspace{1cm} (6.9)

in which the formula proportional to \( \sin \tilde{\delta} \) is derived from the definition of \( V \) in equation (6.8), while the one proportional to \( \sin \delta \) is obtained from the last equality of equation (6.8).

Note that the free parameter \( \tilde{\theta}_{23} \) does not enter \( \mathcal{F} \), although it mixes with \( \theta_{23} \) and \( \delta \) in the expressions of \( \tilde{\theta}_{13} \) and \( \tilde{\delta} \). Thus equation (6.9) leads us to the interesting Toshev relation [276]

\[ \sin^2 \tilde{\theta}_{23} \sin \tilde{\delta} = \sin^2 \theta_{23} \sin \delta, \]  \hspace{1cm} (6.10)

which links \( \tilde{\theta}_{23} \) and \( \tilde{\delta} \) in matter to \( \theta_{23} \) and \( \delta \) in vacuum. In addition to the parametrization of \( V \) in equation (6.8) or equation (6.2), one may also derive the Toshev-like relation from two other parametrizations of \( V \) or \( \tilde{V} \) [277].

Given \( \tilde{\theta}_{23} = \pi/4 \) and \( \tilde{\delta} = \pm \pi/2 \) for the PMNS flavor mixing matrix in vacuum, the Toshev relation in equation (6.10) tells us that sin \( 2\tilde{\theta}_{23} \sin \tilde{\delta} = \pm 1 \) must hold in matter. As a result, we are left with \( \tilde{\theta}_{23} = \theta_{23} = \pi/4 \) and \( \tilde{\delta} = \delta = \pm \pi/2 \). We draw the conclusion that the \( \mu-\tau \) reflection symmetry keeps unchanged for neutrino oscillations in matter. Finally, it is obvious that these discussions are not applicable for the \( \mu-\tau \) permutation symmetry case, in which \( \delta \) and \( \delta \) are irrelevant in physics because of \( \tilde{\theta}_{13} = \theta_{13} = 0 \).

### 6.2. Flavor distributions of UHE cosmic neutrinos

An extremely important topic in high-energy neutrino astronomy is to search for the point-like neutrino sources that may help resolve a long-standing problem—the very origin of UHE cosmic rays. The reason is simply that the UHE protons originating in a cosmic accelerator (e.g. the gamma ray burst or active galactic nuclei [40]) unavoidably interact with their ambient photons via \( p + \gamma \rightarrow \Delta^+ \rightarrow \pi^+ + n \) or their ambient protons through \( p + p \rightarrow \pi^+ + X \) with \( X \) being other particles, producing a large amount of energetic charged pions whose decays can therefore produce copious UHE cosmic muon neutrinos and electron neutrinos or their antiparticles. The so-called neutrino telescope is such a kind of deep underground ice or water Cherenkov detector which can record those rare events induced by the UHE cosmic neutrinos. Today the most outstanding neutrino telescope is the km³-volume IceCube detector at the South Pole, which has successfully identified thirty-seven extraterrestrial neutrino candidate events with deposited energies ranging from 30 TeV to 2 PeV [278, 279]. Among them, the three PeV events represent the highest-energy neutrino interactions ever observed. But where such extraordinary PeV neutrinos came from remains quite mysterious.

Given a distant astrophysical source of either \( p\gamma \) or \( pp \) collisions, it has the same \( \nu_e + \pi^- \) flavor distribution \( \Phi^e : \Phi^\mu : \Phi^\tau = 1 : 2 : 0 \), where \( \Phi^\alpha \equiv \Phi^\alpha_i + \Phi^\alpha_n \) with \( \Phi^\alpha_i \) and \( \Phi^\alpha_n \) being the fluxes of \( \nu_i \) and \( \pi^- \) for \( (\alpha = e, \mu, \tau) \) at the source. Such an initial flavor distribution is expected to change to \( \Phi^T_\alpha : \Phi^T_\mu : \Phi^T_\tau = 1 : 1 : 1 \) at a neutrino telescope, where \( \Phi^T_\alpha \equiv \Phi^T_{\nu_i} + \Phi^T_{\nu_n} \) is similarly defined, because the UHE cosmic neutrinos may oscillate many times on the way to the Earth and finally reach a flavor democracy [280] if the PMNS neutrino mixing matrix \( U \) satisfies the \( |U_{\mu}| = |U_{\tau}| \) condition (for \( i = 1, 2, 3 \) [83]). As a consequence, the effects of \( \mu-\tau \) symmetry breaking can modify the ratio \( \Phi^T_\alpha : \Phi^T_\mu : \Phi^T_\tau = 1 : 1 : 1 \) in a nontrivial way.

To be more explicit, the flavor distribution of UHE cosmic neutrinos at a terrestrial neutrino telescope is given by

\[ \Phi^T_\alpha = \sum_{\beta} [\Phi^S_{\nu_\beta} P(\nu_\beta \rightarrow \nu_\alpha) + \Phi^S_{\mu} P(\mu \rightarrow \nu_\alpha)] \]
\[ = \sum_{\beta} [U_{\alpha\beta}^2 |U_{\beta i}|^2 \Phi^S_{\nu_i} + U_{\alpha\mu}^2 |U_{\mu i}|^2 \Phi^S_{\nu_i}], \]  \hspace{1cm} (6.11)

where we have used

\[ P(\nu_\beta \rightarrow \nu_\alpha) = P(\mu \rightarrow \nu_\alpha) = \frac{1}{2} \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2. \]  \hspace{1cm} (6.12)

Note that equation (6.12) holds for a very simple reason: the galactic distance that the non-monochromatic UHE cosmic neutrinos have traveled far exceeds the observed neutrino oscillation lengths, and thus \( P(\nu_\beta \rightarrow \nu_\alpha) \) are averaged over many oscillations and finally become energy- and distance-independent. Given \( \Phi^S_{\nu_i} : \Phi^S_{\mu} : \Phi^S_{\tau} = 1 : 2 : 0 \) and the unitarity of \( U \), we immediately obtain

\[ \Phi^T_\alpha = \frac{\Phi_0}{3} \frac{1}{2} \sum_i |U_{\alpha i}|^2 (|U_{i i}|^2 + 2 |U_{\alpha i}|^2 ) \]
\[ = \frac{\Phi_0}{3} \left[ 1 + \sum_i |U_{i i}|^2 (|U_{\mu i}|^2 - |U_{\tau i}|^2 ) \right]. \]  \hspace{1cm} (6.13)

where \( \Phi_0 \equiv \Phi^S_{\nu_e} + \Phi^S_{\nu_\mu} + \Phi^S_{\nu_\tau} \) denotes the total flux of neutrinos and antineutrinos of all flavors. That is why \( \Phi^T_\alpha : \Phi^T_\mu : \Phi^T_\tau = 1 : 1 : 1 \) can be achieved provided the equalities \( |U_{\mu}| = |U_{\tau}| \) (for \( i = 1, 2, 3 \)) hold. In fact, the difference between \( \Phi^T_\mu \) and \( \Phi^T_\tau \) is a straightforward signature of the \( \mu-\tau \) symmetry breaking:

\[ \Phi^T_\mu - \Phi^T_\tau = \frac{\Phi_0}{3} \sum_i (|U_{\mu i}|^2 - |U_{\tau i}|^2)^2. \]  \hspace{1cm} (6.14)

This exact and parametrization-independent result is very useful for us to probe the leptonic flavor mixing structure via the detection of UHE cosmic neutrinos at neutrino telescopes [281].
In the standard parametrization of $U$, let us define $e \equiv \theta_{23} - \pi/4$ to measure a part of the $\mu$-$\tau$ symmetry-breaking effects. Then we arrive at
\[
\Phi^T_\mu : \Phi^T_\tau : \Phi^T_{\nu_e} = (1 + D_e) : (1 + D_\mu) : (1 + D_\tau)
\]
with $D_e = -\Delta$, $D_\mu = \Delta + \bar{\Delta}$ and $D_\tau = \Delta - \bar{\Delta}$, where the first- and second-order perturbation terms $\Delta$ [282] and $\bar{\Delta}$ [283] are expressed as
\[
\Delta \approx \frac{1}{2} \sin^2 2\theta_{12} \sin e - \frac{1}{4} \sin 4\theta_{12} \sin \theta_{13} \cos \delta,
\]
\[
\bar{\Delta} \approx (4 - \sin^2 2\theta_{12}) \sin^2 e \pm \sin^2 2\theta_{12} \sin^2 \theta_{13} \cos^2 \delta
\]
\[+ \sin 4\theta_{12} \sin e \sin \theta_{13} \cos \delta,
\]
respectively. So a difference between $\Phi^T_\mu$ and $\Phi^T_\tau$ (i.e. a difference between $D_e$ and $D_\mu$) is actually an effect of the second-order $\mu$-$\tau$ symmetry breaking. It is easy to show $\Delta \geq 0$ for arbitrary values of $\delta$, but $\bar{\Delta}$ may be either positive or negative (or vanishing in the $\mu$-$\tau$ symmetry limit). When the $3\sigma$ intervals of $\theta_{12}, \theta_{13}, \theta_{23}$ and $\delta$ in table 2 are taken into account, the upper bounds of $|\Delta|$ and $\bar{\Delta}$ can both reach about 0.1, implying that the flavor democracy of $\Phi^T_\mu, \Phi^T_\tau$ and $\Phi^T_{\nu_e}$ may maximally be broken at the same level.

It is convenient to define three working observables at neutrino telescopes and connect them to the $\mu$-$\tau$ symmetry-breaking quantities:
\[
R_e \equiv \frac{\Phi^T_\mu}{\Phi^T_{\nu_e} + \Phi^T_\tau} \approx \frac{1}{2} - \frac{3}{2} \Delta,
\]
\[
R_\mu \equiv \frac{\Phi^T_\mu}{\Phi^T_\nu + \Phi^T_\tau} \approx \frac{1}{2} + \frac{3}{4} (\Delta + \bar{\Delta}),
\]
\[
R_\tau \equiv \frac{\Phi^T_\tau}{\Phi^T_\mu + \Phi^T_\nu} \approx \frac{1}{2} - \frac{3}{4} (\Delta - \bar{\Delta}).
\]

The small departure of $R_\alpha$ (for $\alpha = e, \mu, \tau$) from 1/2 is therefore a clear measure of the overall effects of $\mu$-$\tau$ symmetry breaking.

Now we turn to the flavor distribution of UHE cosmic neutrinos at a neutrino telescope by detecting the $\pi$ flux from a very distant astrophysical source via the famous Glashow-resonance (GR) channel $\pi e \rightarrow W^- \rightarrow$ anything [284], which happens over a narrow energy interval around $E^{GR}_\pi \approx M_W^2/2m_e \approx 6.3$ PeV, and its cross section is about two orders of magnitude larger than those of $\sigma N$ interactions of the same $\pi$ energy [285]. A measurement of the GR phenomenon is also important for another reason: it may serve as a sensitive discriminator of UHE cosmic neutrinos originating from $p\gamma$ and $pp$ collisions [286–289]. Let us assume that a neutrino telescope is able to observe both the GR-mediated $\pi$ events and their $\mu$ and $\tau$ events of charged-current interactions in the vicinity of $E^{GR}_\pi$. Then their ratio $R_{GR} \equiv \Phi^T_{\nu_\mu}/\Phi^T_{\nu_\tau}$ can tell us something about the lepton flavor mixing.

To see this point more clearly, we start from the initial flavor distribution of UHE neutrinos at a cosmic accelerator:
\[
\Phi^0_\mu : \Phi^0_{\nu_\mu} : \Phi^0_{\nu_e} : \Phi^0_{\nu_\tau} : \Phi^0_{\nu_\mu} = 1 : 1 : 2 : 0 : 0
\]
($pp$ collisions) or $1 : 0 : 1 : 0 : 0$ ($p\gamma$ collisions). Thanks to neutrino oscillations, the $\pi$ flux at a neutrino telescope can be calculated by using the following formula:
\[
\Phi^T_\pi = \sum_i \beta_i |U_{\alpha i}|^2 |U_{\beta i}|^2 \Phi^0_{\beta \pi},
\]
(6.18)

Since the expression of $\Phi^0_\pi$ has been given in equation (6.13), it is straightforward to obtain $R_{GR}$ for two different astrophysical sources:
\[
R_{GR}(pp) \approx \frac{1}{2} - \frac{3}{2} \Delta - \frac{1}{2} \bar{\Delta},
\]
\[
R_{GR}(p\gamma) \approx \frac{\sin^2 2\theta_{12}}{4} - \frac{4 + \sin^2 2\theta_{12} \Delta}{4} - \frac{\sin^2 2\theta_{12} \bar{\Delta}}{4} + \frac{1 + \cos^2 2\theta_{12}}{2} \sin^2 \theta_{13}.
\]

(6.19)

It is clear that the deviation of $R_{GR}(pp)$ from 1/2 and that of $R_{GR}(p\gamma)$ from $\sin^2 2\theta_{12}/4$ are both governed by the effects of $\mu$-$\tau$ symmetry breaking, which can maximally be of $O(1)$.

As discussed in the literature [46, 288], the IceCube detector running at the South Pole has a discovery potential to probe $R_{GR}(pp)$. In comparison, it is more challenging to detect the GR-mediated UHE $\pi$ events originating from the pure $p\gamma$ collisions at a cosmic accelerator.

In the above discussions we have neglected the uncertainties associated with the initial neutrino fluxes at a given astrophysical source. A careful analysis of the flavor ratios of UHE cosmic neutrinos originating from $pp$ or $p\gamma$ collisions actually leads us to the ratio $\Phi^T_\mu : \Phi^T_{\nu_\mu} : \Phi^T_{\nu_\tau} \approx 1 : 2(1 - \eta) : 0$ with $\eta \approx 0.08$ [290]. If this uncertainty is taken into account, then equation (6.13) will accordingly change to
\[
\Phi^T_\alpha \approx \frac{\Phi^0_\alpha}{3} \left[ 1 + \sum_i |U_{\alpha i}|^2 (|U_{\beta i}|^2 - |U_{\alpha i}|^2) - 2\eta \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 \right]
\]
(6.20)

for $\alpha = e, \mu, \tau$. This result implies that the $\eta$-induced correction is in general comparable with (or even larger than) the effect of $\mu$-$\tau$ symmetry breaking. On the other hand, there may exist the uncertainties associated with the identification of different flavors at the neutrino telescope. Given the IceCube detector for example, the experimental error for determining the ratio of the muon track to the non-muon shower is typically $\xi \sim 20\%$ [291], depending on the event numbers. Such an estimate means that the ratio $R_\mu$ in equation (6.17) may in practice be contaminated by $\xi$, and this contamination is very likely to overwhelm the $\mu$-$\tau$ symmetry breaking effect $(\Delta + \bar{\Delta})$ [281].

We have not considered other complexities and uncertainties associated with the origin of UHE cosmic neutrinos, such as their energy dependence, the effect of magnetic fields and possible new physics [292]. It remains too early to say that we have correctly understood the production mechanism of UHE cosmic rays and neutrinos from a given cosmic accelerator. But progress in determining the neutrino mixing parameters...
is so encouraging that it may finally allow us to well control the error bars from particle physics (e.g. the effect of $\mu$-$\tau$ symmetry breaking) and thus focus on the unknowns from astrophysics (e.g. the initial flavor composition of UHE cosmic neutrinos [293]). Needless to say, any constraint on the flavor distribution of UHE cosmic neutrinos to be achieved from a neutrino telescope will be greatly useful in diagnosing the astrophysical sources and in understanding the properties of neutrinos themselves.

6.3. Matter-antimatter asymmetry via leptogenesis

Besides interpreting why the masses of the three known neutrinos are tiny in a qualitatively natural way, the canonical seesaw mechanism has another attractive feature—it can also provide a natural explanation of the observed baryon-antibaryon asymmetry of the Universe via the leptogenesis mechanism [294]: the CP-violating, lepton-number-violating and out-of-equilibrium decays of the hypothetical heavy Majorana neutrinos $N_i$ may give rise to a lepton-antilepton asymmetry; and the latter is subsequently converted to the baryon-antibaryon asymmetry thanks to the sphaleron process [295]. The amount of the lepton-antilepton asymmetry depends on baryon asymmetry thanks to the sphaleron process [295].

The CP-violating asymmetries in the mass basis of $N_i$ may give rise to a lepton-antilepton asymmetry; and the latter is subsequently converted to the baryon-antibaryon asymmetry thanks to the sphaleron process [295]. The amount of the lepton-antilepton asymmetry depends on the CP-violating asymmetries $\epsilon_i$ between the decays of $N_i$ and their CP-conjugate processes. If the masses of $N_i$ have a strong hierarchy (i.e. $M_1 \ll M_2 \ll M_3$), then only $\epsilon_1$ is expected to really matter at a temperature lower than $M_1$. As a good approximation [296],

$$\epsilon_1 = \frac{3}{16\pi(H)^2} \sum_{i=1}^{3} \text{Im}[\langle Y_i \gamma_i \rangle_{Y} M_i],$$

(6.21)

where $Y_\nu$ denotes the Yukawa coupling matrix of the neutrinos in the mass basis of $N_i$. With the help of $M_\nu = UD_i U^T$ in equation (2.10) and the seesaw formula $M_\nu = Y_\nu D_\nu^1 Y_\nu^C (H)^2$ in the chosen basis, where $D_\nu = \text{Diag}(M_1, M_2, M_3)$ and $(H) = v\sqrt{2} \simeq 174$ GeV, it is convenient to parametrize $Y_\nu$ in the following way [297]:

$$Y_\nu = \frac{1}{(H)} U^T D_i R \sqrt{D}_N,$$

(6.22)

in which $R$ is a complex orthogonal matrix satisfying $RR^T = R^T R = 1$. Inserting equation (6.22) into equation (6.21), we arrive at

$$\epsilon_1 = -\frac{3}{16\pi(H)^2} \frac{M_1}{\sum_{i=1}^{3} m_i |R_{1i}|^2} \text{Im}[(\Delta m_{31}^2 R_{21}^2 + \Delta m_{31}^2 R_{21}^2)/(\sum_{i=1}^{3} m_i |R_{1i}|^2)],$$

(6.23)

where the orthogonality condition $R_{21}^2 + R_{31}^2 + R_{11}^2 = 0$ has been used. Equation (6.23) shows that $\epsilon_1$ is essentially independent of the PMNS matrix $U$, implying that in general there is no direct link between CP violation in $N_i$ decays and that in low-energy neutrino oscillations [298]. If a certain flavor symmetry is taken into account to reduce the number of free parameters associated with the heavy and (or) light neutrinos, then $R$ will get constrained to a simpler form so that the above expression of $\epsilon_1$ can be simplified to some extent [299]. For instance, a model which can predict a constant neutrino mixing pattern is expected to result in a real diagonal $R$ and thus a vanishing $\epsilon_1$ [300–304]. This point is particularly transparent in the so-called ‘form-dominance’ scenarios where $Y_\nu$ is simply taken as $U$ multiplied by a diagonal matrix [305].

In fact, it is easy to see that the $\mu$-$\tau$ permutation symmetry may have some interesting implications on the leptogenesis picture [306–312]. In the limit of such a flavor symmetry the Yukawa coupling matrix of the neutrinos appears as [306, 307]

$$Y_\nu = \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_2 \end{array} \right),$$

(6.24)

and the heavy Majorana neutrino mass matrix $M_N$ has the same form as the light Majorana neutrino mass matrix $M_\nu$ given in equation (3.7). One can make the transformation

$$U^T M_N U_N = D_N,$$

(6.25)

where $U_N = O_M U^T_{12}$ with $O_M$ being the maximal mixing matrix:

$$O_M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

In this mass basis $Y_\nu$ becomes

$$Y_\nu = Y_U U_N = U U^T Y_U = U U^T_{12} D_Y O_M U^T_{12},$$

(6.26)

in which $U = O_M U_{12}$ is just the PMNS matrix of the three light Majorana neutrinos consisting of $\theta_{31} = \pi/4$. Comparing equation (6.22) with equation (6.26), we see that $R$ is now subject to the (1, 2) rotation subspace [306, 307]. In this case $\epsilon_1$ is given by

$$\epsilon_1 = -\frac{3}{16\pi(H)^2} \frac{M_1}{(H)^2} \frac{\text{Im}[(\Delta m_{31}^2 R_{21}^2 + \Delta m_{31}^2 R_{21}^2)/(\sum_{i=1}^{3} m_i |R_{1i}|^2)]}{m_1 |R_{11}|^2 + m_2 |R_{22}|^2},$$

(6.27)

With the help of this result and $R_{21}^2 + R_{31}^2 + R_{11}^2 = 1$, it is easy to show that $|\epsilon_1|$ has an upper bound

$$|\epsilon_1| \leq \frac{3}{16\pi} \frac{M_1}{(H)^2} |m_2 - m_1|.$$  

(6.28)

Note that such an upper bound of $\epsilon_1$ can be used to set a lower bound on $M_1$ if the leptogenesis mechanism works well in explaining the cosmological matter-antimatter asymmetry. In comparison with the upper bound of $\epsilon_1$ obtained in a more general case, where $m_2$ in equation (6.28) ought to be replaced by $m_3$ [313], the present upper bound can be lowered by at least a factor $\sqrt{(\Delta m_{21}^2)/(\Delta m_{31}^2)}$. When the effect of $\mu$-$\tau$ flavor symmetry breaking is concerned, the situation will become more complicated and thus a careful analysis has to be done [307].

In the literature the so-called minimal seesaw model [314] has received some special attention due to its simplicity and stronger predictive power, so its connection to the leptogenesis
mechanism deserves some discussions. Because there are only two heavy Majorana neutrinos in this model, $M_N$ and $\tilde{Y}$, may read as follows [306, 307]:

$$M_N = \begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} y_{12} & y_{13} \\ y_{22} & y_{23} \\ y_{32} & y_{33} \end{pmatrix}. \tag{6.29}$$

The transformation $Q_M^T M_N Q_M = D_N$, where

$$Q_M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{6.30}$$

leads us to the mass eigenvalues $M_2 = M_{22} + M_{23}$ and $M_3 = M_{32} - M_{33}$. In the mass basis the Yukawa coupling matrix $\tilde{Y}$ becomes

$$Y_e = \tilde{Y} Q_M = \frac{1}{\sqrt{2}} \begin{pmatrix} 2y_{12} & 0 \\ y_{22} + y_{23} & y_{23} - y_{22} \\ y_{32} + y_{33} & y_{33} - y_{32} \end{pmatrix}. \tag{6.31}$$

It is interesting to see that the two columns of $Y_e$ are orthogonal to each other, and thus $Y_e^\dagger Y_e$ must be diagonal and $\epsilon_1$ must be vanishing. So a finite value of $\epsilon_1$ has to come from the $\mu$-$\tau$ permutation symmetry breaking. Let us take a simple example to illustrate the effect of $\mu$-$\tau$ symmetry breaking and its connection to CP violation in the decays of two heavy Majorana neutrinos. To be explicit, we assume $M_2$ and $\tilde{Y}$ to be real and introduce the symmetry breaking term as

$$\tilde{Y}_\nu = \tilde{Y}_\nu + \begin{pmatrix} -y_{12}' & y_{12}' \\ 0 & 0 \end{pmatrix}, \tag{6.32}$$

in which $y_{12}'$ is complex so as to accommodate CP violation. In the mass basis we obtain

$$Y_\nu' = Y_\nu' Q_M = \begin{pmatrix} r_a & r_b \\ a & -b \end{pmatrix}, \tag{6.33}$$

where $a \equiv (y_{22} + y_{33})\sqrt{2}$, $b \equiv (y_{22} - y_{33})\sqrt{2}$, $r_1 \equiv \sqrt{2}y_{12}/a$ and $r_2 \equiv \sqrt{y_{12}' y_{12}}/b$ are defined. The parameter $r_2$ is therefore responsible for the generation of $\theta_{13}$ and CP violation. It is found that the experimental data at low energies can be reproduced when one assumes $r_1 \simeq 1$, $|r_2| \simeq \sqrt{2} \theta_{13}$, $3a^2(M_2^2 - M_3^2) = m_2$ and $2b^2(M_2^2 + M_3^2) = m_3$ [315]. In this case the CP-violating asymmetry between the decay of the lighter heavy Majorana neutrino (with mass $M_3$) and its CP-conjugate process can approximate to

$$\epsilon = \frac{\hbar^2}{16\pi} \cdot \frac{m_2}{m_3} \text{Im}(r_2^*) \sim 10^{-4} \sin 2\phi, \tag{6.34}$$

where $\phi \equiv \arg(r_2)$ and $b \sim O(1)$ has been taken into account. This estimate implies that the strength of $\mu$-$\tau$ symmetry breaking required by the observed value of $\theta_{13}$ is sufficient for a successful leptogenesis mechanism to work in such a minimal seesaw scenario [307].

When the $\mu$-$\tau$ reflection symmetry is imposed on both the Yukawa coupling matrix $\tilde{Y}$ and the heavy Majorana neutrino matrix $M_N$, one has

$$\tilde{Y}_\nu = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, \tag{6.35}$$

with $y_{11}$ being real, and the form of $M_N$ is the same as that given in equation (3.24). In this case $M_N$ can be diagonalized by a unitary matrix $U_N$, whose complex conjugate is of the form shown in equation (3.19). Given the Yukawa coupling matrix $Y_e = \tilde{Y}_e U_N$ in the mass basis of $N_i$, a straightforward calculation shows that $Y_e^\dagger Y_e$ is actually a real matrix which prohibits CP violation [113]. To make the leptogenesis idea viable, one may choose to softly break the $\mu$-$\tau$ reflection symmetry [316]. Another way out is to keep the $\mu$-$\tau$ reflection symmetry but take into account the so-called flavor effects to make the leptogenesis mechanism work [323]. If the mass of the lightest heavy Majorana neutrino $N_1$ lies in the range $10^9$ GeV $< M_1 < 10^{15}$ GeV, the $\tau$ leptons can be in thermal equilibrium, making them distinguishable from the $e$ and $\mu$ flavors. In this situation the decay of $N_1$ to the $\tau$ flavor should be treated separately from the decays of $N_1$ to the other two flavors [317, 318]. Accordingly, the flavored CP-violating asymmetries associated with the decays of $N_i$ are given by

$$\epsilon_i^\tau = -\frac{1}{8\pi} \cdot \frac{M_i}{(H^2)^2} \sum_{j=1}^3 \text{Im} \left[ \frac{3}{2} (Y_{\nu_{ij} i} Y_{\nu_{ij} j} Y_{\nu_{i j} j} Y_{\nu_{i j} j} M_{i j}^2) - (Y_{\nu_{ij} i} Y_{\nu_{ij} j} Y_{\nu_{i j} j} Y_{\nu_{i j} j} M_{i j}^2) \right], \tag{6.36}$$

in which $\alpha$ runs over $e, \mu$ and $\tau$. After a straightforward calculation, one can obtain $\epsilon_1^\tau = 0$ and $\epsilon_2^\tau = -\epsilon_3^\tau$ as a direct consequence of the $\mu$-$\tau$ reflection symmetry. This result confirms the vanishing of the overall CP-violating asymmetry (i.e. $\epsilon_1 = \epsilon_2^\mu + \epsilon_3^\mu = 0$). In spite of $\epsilon_1 = 0$, the lepton-antilepton asymmetry produced from the decays of $N_1$ is still likely to survive if such processes took place in the temperature range $10^9$ GeV $< T = M_1 < 10^{15}$ GeV in which the flavor effects should take effect [232]. When the temperature of the Universe dropped below $10^9$ GeV, however, the $\mu$ leptons would also be in thermal equilibrium. In this case the simple leptogenesis scenario under discussion would not work anymore if $M_1 < 10^9$ GeV held.

6.4. Fermion mass matrices with the $Z_2$ symmetry

Being capable of relating the smallness of quark flavor angles to the smallness of quark mass ratios in a way similar to the GST relation, the Fritzsch texture-zero ansatz is a simple and instructive example for studying the possible underlying flavor structures of three quark families [319, 320]:

$$M_q = \begin{pmatrix} 0 & c_q & 0 \\ c_q^* & 0 & b_q \end{pmatrix}, \tag{6.37}$$
where \( q = u \) (up) or \( d \) (down), and the Hermiticity has been assumed to reduce the number of free parameters. Given the strong quark mass hierarchies as shown in figure 2, \( M_q \) is expected to have a strongly hierarchical structure (i.e. \( |c_q| \ll |b_q| \ll |a_q| \)). Unfortunately, such a simple texture has been ruled out mainly because its phenomenological consequences cannot simultaneously fit the small size of \( V_{ub} \) and the large value of \( m_t \). One way out is to introduce an additional free parameter to make the \((2, 2)\) entry of \( M_q \) nonzero \([321–323]\):

\[
M_q = \begin{pmatrix}
0 & c_q & 0 \\
c_q^* & d_q & b_q \\
0 & b_q^* & a_q
\end{pmatrix}.
\]

(6.38)

Then it is easy to check that this new ansatz, which may have a more or less weaker hierarchy than its original counterpart, can be in good agreement with the experimental data on quark flavor mixing and CP violation \([324, 325]\). Although the Fritzsch-like equation in equation (6.38) is apparently different from the pattern of \( M_c \) with a \( \mu-\tau \) flavor symmetry (i.e. the former contains a few texture zeros while the latter is characterized by some linear relations or equalities of different entries), it is possible to find a flavor basis where \( M_q \) may essentially have the same form as \( M_c \). Here we consider a special but \( \phi \) signifies the strong \((2, 3)\) and \( \theta \), \( 0.1813 \in \text{equation (6.39)} \) should be in order to avoid a significant cancellation. For the \( \mu - \tau \) permutation symmetry, the \( (2, 3) \) permutation symmetry can be broken by allowing

\[
\begin{align*}
\tilde{M}_{12}^f &= \tilde{M}_{12}^f, \\
\tilde{M}_{22}^f &= \tilde{M}_{33}^f, \\
\phi_1^f &\neq \phi_3^f.
\end{align*}
\]

(6.41)

If the symmetry breaking only comes from the phase parameters (i.e. \( \phi_1^f = \phi_3^f \)), then \( \theta_{13}^{\text{CKM}} \) can fit its experimental value by taking \( \phi_1^u - \phi_3^d \sim 0.08 \), where \( \phi_1^u \equiv \phi_1^d - \phi_2^d \) is defined. On the other hand, \( \theta_{13}^{\text{CKM}} \) can be calculated via the correlation \( \theta_{13}^{\text{CKM}} \sim \theta_{12}^{\text{CKM}} \), but the result is somewhat smaller than the observed value \([325]\). In this case one may either give up \( \theta_{13} \) = 0 or break the \((2, 3)\) permutation symmetry of \( \tilde{M}_q \) in a slightly stronger way. The former approach may not make a realization of the GST relation self-evident. Following the latter approach, we can obtain nonzero \( \theta_{13} \) by allowing \( \tilde{M}_{12}^f = \tilde{M}_{12}^q \) but keeping \( \tilde{M}_{22}^q = \tilde{M}_{33}^q \). Note that one may simply take \( \tilde{M}_{13} = 0 \) and arrive at the following zero texture of fermion mass matrices:

\[
M_f = P_f^t \tilde{M}_f P_f = P_f^t \begin{pmatrix} B_1 & 0 \\ B_t & C_t \end{pmatrix} P_f.
\]

(6.42)

Then it is easy to show that \( \theta_{13} \) is vanishingly small and \( \theta_{13} \) can be given by \( \theta_{13} \approx \sqrt{m_u m_d/m_\nu} \), which is large enough to generate a phenomenologically acceptable value of \( \theta_{13}^{\text{CKM}} \). Since the fermion mass matrices in equation (6.42) have the same structure as those quark mass matrices in equation (6.38) \( a_q = d_q \) is taken, a universal treatment of lepton and quark flavor mixing issues seems quite natural in this sense.

We proceed to discuss some phenomenological consequences of equation (6.39) in the lepton sector. Above all, the experimental result \( \theta_{13} \approx \pi/4 \) signifies the strong \((2, 3)\) permutation symmetry breaking. If the symmetry is only broken by \( \phi_2^f \neq \phi_3^f \), then the phase difference \( \phi_1^u - \phi_3^d \) has to be very close to \( \pm \pi/2 \) in order to avoid a significant cancellation between the charged-lepton and neutrino sectors. To be explicit, we focus on the texture in equation (6.42) to obtain more definite predictions. We have \( \theta_{12} \approx \sqrt{m_u m_d/m_\nu} \approx 0.07 \) and \( \theta_{13} \) is highly suppressed due to \( m_\tau \ll m_\mu \). So \( \theta_{13} \) should take the dominant responsibility for \( \theta_{13} \approx \pi/4 \), implying \( \theta_{13} \approx \theta_{12} \) as a good approximation. Taking tan \( \theta_{12} \approx \sqrt{m_\mu/m_\tau} \) in a parallel way, one can roughly fix the three neutrino masses:

\( m_1 \sim 0.008 \text{ eV}, \quad m_2 \sim 0.011 \text{ eV} \) and \( m_3 \sim 0.051 \text{ eV} \). In addition, \( \theta_{13}' \approx \sqrt{m_\mu/m_\tau} \approx 0.18 \), and hence the correct value of \( \theta_{13} \) can be achieved after the contribution from \( \theta_{13}' \) (ranging between \(-0.05 \) and \( 0.05 \), as a function of \( \phi_2^u - \phi_3^d \) is properly included. Finally, it is also possible to obtain \( \delta \approx \pm \pi/2 \).

At this point it is worth mentioning the idea of ‘flavor democracy’ \([333, 334]\) or ‘universal strength of Yukawa couplings’ \([335]\) for the mass matrices of leptons and quarks, which can be regarded as an extreme case of \( M_f \) shown in equation (6.39) after they receive some proper perturbations.
respecting the $Z_2$ permutation symmetry. While the charged-lepton mass hierarchy, the quark mass hierarchy and small quark flavor mixing effects are easily understood in such an approach, one has to constrain the neutrino mass matrix $M_\nu$ to be nearly diagonal so as to obtain significant flavor mixing effects in the lepton sector [336]. It is also found that a combination of the flavor democracy idea and the seesaw mechanism allows one to build a phenomenologically viable model [337].

To summarize, the universal texture of fermion mass matrices shown in equation (6.39), which possesses the $(2, 3)$ or $Z_2$ permutation symmetry—an analogue of the $\mu$-$\tau$ permutation symmetry, can serve as a starting point to describe the flavor structures of quarks and leptons and to identify their similarities and differences. Its variation with some symmetry-breaking effects in equation (6.42) contains both the texture zeros and the linear equality of different entries [338, 339]. Such an ansatz is predictive and can essentially fit current experimental data. For simplicity, here we have only discussed the possibility that all the fermion mass matrices share the same texture but their respective parameters are independent of one another. The latter can get correlated if the GUT framework is taken into account (see, e.g. [340, 341], where the (2, 3) permutation symmetry is combined with the SU(5) and SO(10) GUT models, respectively).

7. Summary and outlook

In the past two decades we have witnessed a booming period in neutrino physics—an extremely important part of particle physics and cosmology. Especially since the first discovery of atmospheric neutrino oscillations at the Super-Kamiokande detector in 1998, quite a lot of significant breakthroughs have been made in experimental neutrino physics, as recognized by both the 2015 Nobel Prize in Physics and the 2016 Breakthrough Prize in Fundamental Physics. On the one hand, the striking and appealing phenomena of atmospheric, solar, reactor and accelerator neutrino (or antineutrino) oscillations have all been observed in a convincing way, and the oscillation parameters $\Delta m^2_{21}, |\Delta m^2_{31}|, \theta_{12}, \theta_{13}$ and $\theta_{23}$ have been determined to an impressive degree of accuracy. On the other hand, the unusual geo-antineutrino events and extraterrestrial PeV neutrino events have also been observed, and the sensitivities to neutrino masses in the beta decays, $0\nu 2\beta$ decays and cosmological observations have been improved to a great extent. On the theoretical side, a lot of efforts have been made towards a deeper understanding of the origin of tiny neutrino masses, the secret of flavor mixing and CP violation and a unified picture of leptons and quarks at a much larger framework beyond the SM. Moreover, one has explored various implications of massive neutrinos on the cosmological matter-antimatter asymmetry, warm dark matter and many violent astrophysical or astronomical processes. All these have demonstrated neutrino physics to be one of the most important and exciting frontiers in modern science.

What the present review article has concentrated on is the ‘minimal flavor symmetry’ behind the observed lepton flavor mixing pattern—a partial (or approximate) $\mu$-$\tau$ symmetry which is definitely favored by current experimental data, and its various phenomenological implications in neutrino physics. We have discussed both the $\mu$-$\tau$ permutation symmetry and the $\mu$-$\tau$ reflection symmetry, and pointed out a few typical ways to slightly break such a symmetry. Some larger discrete flavor symmetry groups, in which the intriguing $\mu$-$\tau$ symmetry can naturally arise as a residual symmetry, have been briefly described. Both the bottom–up approach and the top-down approach have been illustrated in this connection, in order to bridge the gap between model building attempts and neutrino oscillation data. To be more explicit, we have summarized the basic strategies of model building with the help of the $\mu$-$\tau$ symmetry, either in the presence or in the absence of the seesaw mechanism. The phenomenological consequences of the $\mu$-$\tau$ flavor symmetry on some interesting and important topics, such as neutrino oscillations in matter, radiative corrections to the equalities $|U_{\mu\alpha}| = |U_{\tau\alpha}|$ from a superhigh-energy scale down to the electroweak scale, flavor distributions of the UHE cosmic neutrinos at a neutrino telescope, a possible connection between the leptogenesis and low-energy CP violation through the seesaw mechanism and a unified flavor structure of leptons and quarks, have also been discussed. Therefore, we hope that this review could serve as a new milestone in the development of neutrino phenomenology, in order to make so many theoretical ideas of ours much deeper and more convergent.

There are still quite a number of open fundamental questions about massive neutrinos and lepton flavor issues. The
immediate ones include how small the absolute neutrino mass scale is, whether the neutrino mass spectrum is normal as those of the charged leptons and quarks, whether the antiparticles of massive neutrinos are just themselves, how large the CP-violating phase \( \delta \) can really be, which octant the largest flavor mixing angle \( \theta_{23} \) belongs to, whether there are light and (or) heavy sterile neutrinos, what the role of neutrinos is in dark matter, whether the observed matter-antimatter asymmetry of the Universe is directly related to CP violation in neutrino oscillations, etc. Motivated by so many challenging and exciting questions, we are on our way to discovering a new physics world in the coming decades.

Last but not least, let us make some concluding remarks with the help of the Fritzsch–Xing ‘pizza plot’ shown in figure 10. This picture provides a brief summary of 28 fundamental parameters associated with the SM itself and neutrino masses, lepton flavor mixing angles and CP-violating phases. Among them, the five parameters of strong and weak CP violation deserve some special attention. In the quark sector the strong CP-violating phase \( \theta \) remains a mystery, but the weak CP-violating phase \( \delta \) has been determined to a good degree of accuracy. In the lepton sector none of the CP-violating phases has been directly measured, although a very preliminary hint \( \delta_l \sim -\pi/2 \) has been achieved from a comparison between the present T2K and Daya Bay data. The true value of \( \delta_l \) is expected to be determined in the future long-baseline neutrino oscillation experiments. If massive neutrinos are really the Majorana particles, however, it will be extremely challenging to probe or constrain the Majorana CP-violating phases \( \rho \) and \( \sigma \) which can only show up in some extremely rare processes involving lepton number nonconservation.

Perhaps some of the flavor puzzles cannot be resolved unless we finally find out the fundamental flavor theory. But the latter cannot be formulated without a lot of phenomenological and experimental inputs. As Leonardo da Vinci once stressed, ‘Although nature commences with reason and ends in experience, it is necessary for us to do the opposite. That is, to commence with experience and from this to proceed to investigate the reason’.

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**Appendix**

As discussed in section 3.4, the delicate effects of \( \mu-\tau \) reflection symmetry breaking are characterized by the quantities \( \Delta \theta_{23} \), \( \Delta \delta \), \( \Delta \rho \) and \( \Delta \sigma \). Their expressions in our analytical approximations can be parametrized as

\[
\Delta \theta_{23} = c_{\theta 1} R_1 + c_{\theta 2} R_2 + c_{\theta 2} R_3 + c_{\theta 2} R_2,
\]

\[
\Delta \delta = c_{\delta 1} R_1 + c_{\delta 2} R_1 + c_{\delta 2} R_2 + c_{\delta 2} R_2,
\]

\[
\Delta \rho = c_{\rho 1} R_1 + c_{\rho 2} R_1 + c_{\rho 2} R_2 + c_{\rho 2} R_2,
\]

\[
\Delta \sigma = c_{\sigma 1} R_1 + c_{\sigma 2} R_1 + c_{\sigma 2} R_2 + c_{\sigma 2} R_2.
\]

(A.1)

where the relevant coefficients are given by

\[
c_{\theta 1}^0 = 2 (m_2^2 - m_1^2)^2 + m_3^2 T_\alpha^2 \Omega_\beta,
\]

\[
c_{\theta 1}^0 = 4m_1 m_1 T_\alpha^2 \Omega_\beta,
\]

\[
c_{\rho 1}^0 = -m_1 - 2m_1 + T_\alpha^2 \Omega_\beta,
\]

(A.2)

and

\[
c_{\theta 1}^1 = 2 \left[ m_1^2 (m_1 + m_2) m_2^2 - 3 T
\]

\[
+ 2m_1 - y_{m_1} + y_{m_2} - 3 \right] \Omega_\beta,
\]

\[
c_{\theta 1}^2 = 2 \left[ (m_1 + m_2 - m_3) T_\alpha^2 \Omega_\beta
\]

\[
- 2m_1 - y_{m_1} + y_{m_2} - 3 \right] \Omega_\beta,
\]

\[
c_{\rho 1}^2 = [ -m_1^2 (m_1 + m_2) m_2^2 + 3 T
\]

\[
- 2m_1^2 m_2^2 - 2m_2 m_1 T_\alpha^2 \Omega_\beta
\]

\[
+ 2m_1 - y_{m_1} + y_{m_2} - 3 \right] T_\alpha^2 \Omega_\beta.
\]

(A.3)

21 See some recent reviews on the future prospects of leptonic CP violation [342], neutrino oscillation experiments [80, 343], searches for the 0\( \nu \)–2\( \nu \) decays [344], electromagnetic properties of massive neutrinos [345] and sterile neutrino physics [265].

22 Here we use \( \delta_l \) to denote the Dirac CP phase in the standard parametrization of the PMNS matrix \( U \), so as to distinguish it from its analogue \( \delta \) in the CKM quark flavor mixing matrix.
and
\[
c^\text{el}_i = 8m_1m_3[\bar{m}_1^2(m_1 + m_2)T] \\
- (\bar{m}_1 - m_3)(\bar{m}_2 - m_3)m_1m_3[\bar{m}_1^2\bar{s}_1^3\Omega], \\
c^\text{el}_i = -8m_1m_3m_1m_3[\bar{m}_1 - m_3](\bar{m}_2 - m_3) \\
- 2m_1m_1 + s_1^3\bar{s}_1^2\bar{t}_1^2\Omega, \\
c^\text{el}_2 = 4m_1m_3[\bar{m}_1^2(m_1 + m_2)(m_1 + m_3)T] \\
- m_1m_1 + m_2m_3 + m_2m_3 + m_1m_3T \Omega, \\
c^\text{el}_2 = (\bar{m}_1 - m_3)(\bar{m}_2 - m_3)m_2m_3 + 2[2m_1m_3\bar{s}_1^2 \\
+ (\bar{m}_1 + m_3)(\bar{m}_1^2 + m_2^2) - m_1\bar{s}_1^2 - m_3\bar{s}_1^2] T \Omega, \quad (A.4)
\]

In equations (A.2)–(A.5), \( T = \tan 2\theta_12 \), and \( \Omega_\theta, \Omega_\delta, \rho_1 \) and \( \delta_1 \) are defined through the equations
\[
\Omega_\theta^{-1} = 2(\bar{m}_1 - m_2)(\bar{m}_1^2 - m_3), \\
\Omega_\delta^{-1} = 2(\bar{m}_1 + m_3)(\bar{m}_1 - m_3)(\bar{m}_2 - m_3)T\bar{s}_1^3, \\
\rho_1^{-1} = 4m_1m_3c_1s_1(\Omega_\theta T\bar{s}_1^3)^{-1}, \\
\delta_1^{-1} = 4m_1m_3c_1\bar{s}_1^2(\Omega_\delta T\bar{s}_1^3)^{-1}. \quad (A.6)
\]

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