ENERGETIC VARIATIONAL APPROACHES FOR INVISCID MULTIPHASE FLOW SYSTEMS WITH SURFACE FLOW AND TENSION

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ABSTRACT. We consider the governing equations for the motion of the inviscid fluids in two moving domains and an evolving surface from an energetic point of view. We employ our energetic variational approaches to derive inviscid multiphase flow systems with surface flow and tension. More precisely, we calculate the variation of the flow maps to the action integral for our model to derive both surface flow and tension. We also study the conservation and energy laws of our multiphase flow systems. The key idea of deriving the pressure of the compressible fluid on the surface is to make use of the feature of the barotropic fluid, and the key idea of deriving the pressure of the incompressible fluid on the surface is to apply a generalized Helmholtz-Weyl decomposition on a closed surface.

1. Introduction

We are interested in a mathematical modeling of a soap bubble floating in the air and an air bubble moving in water. When we focus on a soap bubble, we can see the fluid flow in the bubble. We call the fluid flow in the bubble a surface flow. We can consider a surface flow as fluid flow on an evolving surface. In order to make

2020 Mathematics Subject Classification. 49Q20, 76-10, 35A15, 49S05.
Key words and phrases. Mathematical modeling, Energetic variational approach, Multiphase flow system, Surface flow, Surface tension, Inviscid fluid.
a mathematical model of a soap bubble floating in the air, we have to study the dependencies among fluid-flows in domains and surface flow. This paper considers the governing equations for the motion of the inviscid fluids in two moving domains and an evolving surface from an energetic point of view. We employ our energetic variational approaches to derive three inviscid multiphase flow systems with surface flow and tension.

Let us first introduce fundamental notations. Let \( t \geq 0 \) be the time variable, \( x(= t(x_1, x_2, x_3)) \), \( \xi_A(= t(\xi_A^1, \xi_A^2, \xi_A^3)) \), \( \xi_B(= t(\xi_B^1, \xi_B^2, \xi_B^3)) \), \( \xi_S(= t(\xi_S^1, \xi_S^2, \xi_S^3)) \) \( \in \mathbb{R}^3 \) the spatial variables, and \( X(= t(X_1, X_2)) \) \( \in \mathbb{R}^2 \) the spatial variable. Fix \( T > 0 \).

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with a smooth boundary \( \partial \Omega \). The symbol \( n_\Omega = n_\Omega(x) = t(n_\Omega^1, n_\Omega^2, n_\Omega^3) \) denotes the unit outer normal vector at \( x \in \partial \Omega \). Let \( \Omega_A(t)(= \{\Omega_A(t)\}_{0 \leq t < T}) \) be a bounded domain in \( \mathbb{R}^3 \) with a moving boundary \( \Gamma(t) \). Assume that \( \Gamma(t)(= \{\Gamma(t)\}_{0 \leq t < T}) \) is a smoothly evolving surface. The symbol \( n_\Gamma = n_\Gamma(x, t) = t(n_\Gamma^1, n_\Gamma^2, n_\Gamma^3) \) denotes the unit outer normal vector at \( x \in \Gamma(t) \). For each \( t \in [0, T) \), assume that \( \Omega_A(t) \subseteq \Omega \). Set \( \Omega_B(t) = \Omega \setminus \Omega_A(t) \). It is clear that \( \Omega = \Omega_A(t) \cup \Gamma(t) \cup \Omega_B(t) \) (see Figure 1). Set

\[
\begin{align*}
\Omega_{A,T} & = \bigcup_{0 < t < T} \{ \Omega_A(t) \times \{ t \} \}, \\
\Omega_{B,T} & = \bigcup_{0 < t < T} \{ \Omega_B(t) \times \{ t \} \}, \\
\Gamma_T & = \bigcup_{0 < t < T} \{ \Gamma(t) \times \{ t \} \}, \\
\Omega_T & = \Omega \times (0, T), \\
\partial \Omega_T & = \partial \Omega \times (0, T).
\end{align*}
\]

In this paper we assume that the fluids in \( \Omega_{A,T} \) and \( \Omega_{B,T} \) are barotropic compressible ones, and that the fluid on \( \Gamma_T \) is compressible or incompressible one. An incompressible fluid on the surface means one that satisfies the surface divergence-free condition. Let us state physical notations. Let \( \rho_A = \rho_A(x, t) \), \( v_A = v_A(x, t) = t(v_A^1, v_A^2, v_A^3) \), and \( p_A = p_A(x, t) \) be the density, the velocity, and the pressure of the fluid in \( \Omega_A(t) \), respectively. Let \( \rho_B = \rho_B(x, t) \), \( v_B = v_B(x, t) = t(v_B^1, v_B^2, v_B^3) \), and \( p_B = p_B(x, t) \) be the density, the velocity, and the pressure of the fluid in \( \Omega_B(t) \), respectively. Let \( \rho_S = \rho_S(x, t) \), \( v_S = v_S(x, t) = t(v_S^1, v_S^2, v_S^3) \), and \( p_S = p_S(x, t) \) be the density, the velocity, and the pressures of the fluid on \( \Gamma(t) \), respectively.

**Remark 1.1.**

(i) The symbol \( p_S \) corresponds to the pressure of the compressible fluid on \( \Gamma_T \), and \( \Pi_S \) corresponds to the pressure of the incompressible fluid on \( \Gamma_T \). We call \( p_S \), \( \Pi_S \) total pressures. Total pressure means one that includes surface pressure and tension.

(ii) We call \( v_S \) a total velocity on \( \Gamma_T \). Total velocity means that \( v_S \) can be divided into surface velocity \( u_S \) and motion velocity \( w_S \), that is, \( v_S = u_S + w_S \). The surface flow \( u_S \) is a tangential vector on \( \Gamma(t) \). The motion velocity \( w_S \) is the speed of the evolving surface \( \Gamma(t) \). Therefore, the total velocity \( v_S \) is not a necessary tangential vector on \( \Gamma(t) \). In this paper, we assume that \( w_S \) is a normal vector on \( \Gamma(t) \), and focus on the total velocity and total pressure to make our models.

Let us explain the basic assumptions of mathematical modeling of inviscid multiphase flow systems with surface flow and tension. We assume that

\[
\begin{align*}
v_B \cdot n_\Omega & = 0 & & \text{on } \partial \Omega_T, \\
v_A \cdot n_\Gamma & = v_B \cdot n_\Gamma = v_S \cdot n_\Gamma & & \text{on } \Gamma_T.
\end{align*}
\]
The condition \( v_B \cdot n_\Omega = 0 \) means that fluid particles do not go out of the domain \( \Omega \).

This paper has two purposes. The first one is to apply our energetic variational approaches to derive three multiphase flow systems. The first system is the following inviscid multiphase flow system with compressible surface flow:

\[
\begin{align*}
D_t^A \rho_A + (\text{div}v_A)\rho_A &= 0 & \text{in } \Omega_{A,T}, \\
\rho_A D_t^A v_A + \text{grad}p_A &= 0 & \text{in } \Omega_{A,T}, \\
D_t^B \rho_B + (\text{div}v_B)\rho_B &= 0 & \text{in } \Omega_{B,T}, \\
\rho_B D_t^B v_B + \text{grad}p_B &= 0 & \text{in } \Omega_{B,T}, \\
D_t^S \rho_S + (\text{div}v_S)\rho_S &= 0 & \text{on } \Gamma_T, \\
\rho_S D_t^S v_S + \text{grad}_T p_S + p_S H_T n_T + p_B n_T - p_A n_T &= 0 & \text{on } \Gamma_T,
\end{align*}
\]

(1.3)

with (1.2) and

\[
\begin{align*}
p_A &= p_A(u) = \rho_A p_A'(u) - p_A'(u) & \text{in } \Omega_{A,T}, \\
p_B &= p_B(u) = \rho_B p_B'(u) - p_B'(u) & \text{in } \Omega_{B,T}, \\
p_S &= p_S(u) = \rho_S p_S'(u) - p_S'(u) & \text{on } \Gamma_T.
\end{align*}
\]

(1.4)

Here \( p_A, p_B, p_S \) are three \( C^1 \)-functions, \( p' = p'(u) = dp/du(u) \), \( D_t^A f = \partial_t f + (v_A, \nabla) f, \) \( D_t^B f = \partial_t f + (v_B, \nabla) f, \) \( D_t^S f = \partial_t f + (v_S, \nabla) f, \) \( (v_A, \nabla) f = v_A^f \partial_x f + v_A^\delta \partial_\delta f, \) \( \text{div} v_A = \nabla \cdot v_A, \) \( \text{grad} f = \nabla f, \) \( \nabla = (\partial_1, \partial_2, \partial_3), \) \( \partial_i = \delta_i, \partial_\delta, \partial_3, \) \( \partial_i = \delta_i, \partial_\delta, \partial_3, \) \( \partial_i = \sum_j \delta_{ij} n_j^3 \partial_j f, \) \( H_T = H_T(x, t) = -\nabla_T n_T. \) Remark that \( H_T \) is the mean curvature in the direction \( n_T. \) Remark also that the conditions (1.3) mean barotropic ones, and that (1.4) correspond to the pressures derived from a thermodynamic approach (see [16]).

The second system is the following inviscid multiphase flow system with incompressible surface flow:

\[
\begin{align*}
D_t^A \rho_A + (\text{div}v_A)\rho_A &= 0 & \text{in } \Omega_{A,T}, \\
\rho_A D_t^A v_A + \text{grad}p_A &= 0 & \text{in } \Omega_{A,T}, \\
D_t^B \rho_B + (\text{div}v_B)\rho_B &= 0 & \text{in } \Omega_{B,T}, \\
\rho_B D_t^B v_B + \text{grad}p_B &= 0 & \text{in } \Omega_{B,T}, \\
D_t^S \rho_S &= 0 & \text{on } \Gamma_T, \\
\text{div}_T v_S &= 0 & \text{on } \Gamma_T, \\
\rho_S D_t^S v_S + \text{grad}_T \Pi_S + \Pi_S H_T n_T + p_B n_T - p_A n_T &= 0 & \text{on } \Gamma_T,
\end{align*}
\]

(1.5)

with (1.2) and (1.4). Remark that the condition \( \text{div}_T v_S = 0 \) means surface divergence-free one.
The third system is the following inviscid multiphase flow system with a tangential compressible surface flow:

\[
\begin{aligned}
D_t^A \rho_A + (\text{div}v_A)\rho_A &= 0 \quad \text{in } \Omega_{A,T}, \\
\rho_A D_t^B v_A + \text{grad} P_A &= 0 \quad \text{in } \Omega_{A,T}, \\
D_t^B \rho_B + (\text{div}v_B)\rho_B &= 0 \quad \text{in } \Omega_{B,T}, \\
\rho_B D_t^B v_B + \text{grad} P_B &= 0 \quad \text{in } \Omega_{B,T}, \\
D_t^S H + (\text{div}v_S)\rho_S &= 0 \quad \text{on } \Gamma_T, \\
\rho_S D_t^S v_S + \text{grad} P_S &= 0 \quad \text{on } \Gamma_T, \\
p_S H_{\Gamma T} + p_B n_\Gamma - p_A n_\Gamma &= 0 \quad \text{on } \Gamma_T,
\end{aligned}
\]

with (1.2) and (1.4).

Remark 1.2. In order to analyze or numerical simulate systems (1.3) and (1.5), we need some conditions on the surface flow \( w_S \) or the motion velocity \( w_S \) since we focus on the total velocity and pressure to make our models. System (1.7) is suitable for mathematical analysis since the motion velocity \( w_S \) is given by

\[
w_S = \frac{1}{\rho_S H_T} \{ p_A(v_A \cdot n_\Gamma) - p_B(v_B \cdot n_\Gamma) \} n_\Gamma \quad \text{on } \Gamma_T
\]

if \( \rho_S H_T \neq 0 \). In fact, by (1.2) and (1.6), we find that

\[
\rho_S H_T (n_\Gamma \cdot v_S) = p_A(\rho_A) - p_B(\rho_B)
\]

\[
= p_A(n_\Gamma \cdot v_S) = p_B(n_\Gamma \cdot v_S) \quad \text{on } \Gamma_T.
\]

From \( v_S \cdot n_\Gamma = w_S \cdot n_\Gamma \) and \( P_T = (0,0,0) \), we have (1.7).

The second purpose is to study the conservation and energy laws of systems (1.3) and (1.5). In fact, any solution to systems (1.3) and (1.5) with (1.2) and (1.4) satisfies that for \( t_1 < t_2 \),

\[
\begin{aligned}
\int_{\Omega_A(t_2)} \rho_A(x,t_2) \, dx + \int_{\Omega_B(t_2)} \rho_B(x,t_2) \, dx + \int_{\Gamma(t_2)} \rho_S(x,t_2) \, d\mathcal{H}^2_x
&= \int_{\Omega_A(t_1)} \rho_A(x,t_1) \, dx + \int_{\Omega_B(t_1)} \rho_B(x,t_1) \, dx + \int_{\Gamma(t_1)} \rho_S(x,t_1) \, d\mathcal{H}^2_x,
\end{aligned}
\]

\[
\begin{aligned}
\int_{\Omega_A(t_2)} \rho_A v_A \, dx + \int_{\Omega_B(t_2)} \rho_B v_B \, dx + \int_{\Gamma(t_2)} \rho_S v_S \, d\mathcal{H}^2_x
&= \int_{\Omega_A(t_1)} \rho_A v_A \, dx + \int_{\Omega_B(t_1)} \rho_B v_B \, dx + \int_{\Gamma(t_1)} \rho_S v_S \, d\mathcal{H}^2_x
&\quad - \int_{t_1}^{t_2} \int_{\partial \Omega} p_B n_\Omega \, d\mathcal{H}^2_x \, dt,
\end{aligned}
\]
and

\[
\int_{\Omega_A(t_2)} \frac{1}{2} \rho_A |v_A|^2 \, dx + \int_{\Omega_B(t_2)} \frac{1}{2} \rho_B |v_B|^2 \, dx + \int_{\Gamma(t_2)} \frac{1}{2} \rho_S |v_S|^2 \, d\mathcal{H}_x^2
\]

\[
= \int_{\Omega_A(t_1)} \frac{1}{2} \rho_A |v_A|^2 \, dx + \int_{\Omega_B(t_1)} \frac{1}{2} \rho_B |v_B|^2 \, dx + \int_{\Gamma(t_1)} \frac{1}{2} \rho_S |v_S|^2 \, d\mathcal{H}_x^2
\]

\[
+ \int_{t_1}^{t_2} \int_{\Omega_A(t)} (\text{div} v_A) p_A \, dx dt + \int_{t_1}^{t_2} \int_{\Omega_B(t)} (\text{div} v_B) p_B \, dx dt
\]

\[
+ \int_{t_1}^{t_2} \int_{\Gamma(t)} (\text{div} v_S) p_S \, d\mathcal{H}_x^2 \, dt.
\]

Here \(d\mathcal{H}_x^2\) denotes the 2-dimensional Hausdorff measure. Moreover, any solution to system (1.3) with (1.2) and (1.4) satisfies that for \(t_1 < t_2\),

\[
\int_{\Omega_A(t_2)} \left( \frac{1}{2} \rho_A |v_A|^2 + p_A(\rho_A) \right) \, dx + \int_{\Omega_B(t_2)} \left( \frac{1}{2} \rho_B |v_B|^2 + p_B(\rho_B) \right) \, dx
\]

\[
+ \int_{\Gamma(t_2)} \left( \frac{1}{2} \rho_S |v_S|^2 + p_S(\rho_S) \right) \, d\mathcal{H}_x^2
\]

\[
= \int_{\Omega_A(t_1)} \left( \frac{1}{2} \rho_A |v_A|^2 + p_A(\rho_A) \right) \, dx + \int_{\Omega_B(t_1)} \left( \frac{1}{2} \rho_B |v_B|^2 + p_B(\rho_B) \right) \, dx
\]

\[
+ \int_{\Gamma(t_1)} \left( \frac{1}{2} \rho_S |v_S|^2 + p_S(\rho_S) \right) \, d\mathcal{H}_x^2.
\]

We often call (1.8), (1.9), (1.10), and (1.11), the law of conservation of mass, the law of conservation of momentum, the law of conservation of energy, and the law of conservation of total energy, respectively. See Theorem 2.3 for details.

Let us state three difficulties in the derivation of our multiphase flow systems, and the key ideas to overcome these difficulties. The first difficulty is to drive the pressure of the compressible fluid on the surface \(\Gamma_T\). In order to derive the pressure terms of our compressible fluid systems, we make use of the feature of the barotropic fluids. More precisely, we assume that the pressure of the compressible fluid depends only on the density of the fluid (see (1.3)). The second difficulty is to drive the pressure of the incompressible fluid on the surface. In order to derive the surface pressure of system (1.5), we apply a generalized Helmholtz-Weyl decomposition on a closed surface (see Lemma 7.2). The third difficulty is to derive the relationship among the pressures of the fluids in the moving domains \(\Omega_A(t)\), \(\Omega_B(t)\), and surface \(\Gamma(t)\). To overcome the difficult point, we apply an energetic variational approach to derive the relationship. An energetic variational approach is a mathematical modeling method, which had been studied by Strutt [22] and Onsager [17, 18].

To derive system (1.3), we study the variation of the following action integral:

\[
\int_0^T \int_{\Omega_A(t)} \left\{ \frac{1}{2} \rho_A^e |v_A^e|^2 - p_A(\rho_A^e) \right\} \, dx dt + \int_0^T \int_{\Omega_B(t)} \left\{ \frac{1}{2} \rho_B^e |v_B^e|^2 - p_B(\rho_B^e) \right\} \, dx dt
\]

\[
+ \int_0^T \int_{\Gamma(t)} \left\{ \frac{1}{2} \rho_S^e |v_S^e|^2 - p_S(\rho_S^e) \right\} \, d\mathcal{H}_x^2 \, dt.
\]

Here \(\Delta^e\) is a variation of \(\Delta\). See Theorem 2.8 for details.
Let us state some results on mathematical modeling of inviscid fluid flow systems on surfaces. Arnol’d [1, 2] applied the Lie group of diffeomorphisms to derive an inviscid incompressible fluid system on a manifold. See also Ebin-Marsden [3]. Koba-Liu-Giga [4] employed an energetic variational approach to derive an inviscid incompressible fluid system on an evolving closed surface. Koba [5] made use of an energetic variational approach to derive an inviscid compressible fluid system on an evolving closed surface. This paper improves the methods in [4, 5] to derive the inviscid multiphase flow systems with surface flow and tension.

Finally, we introduce the results related to this paper. Serrin [6] introduced Euler’s ideas and Cauchy’s principle to derive the inviscid fluid system in a domain. Gyarmati [7], Hyon-Kwak-Liu [8], Koba-Sato [9] employed their energetic variational approaches to make and study several models for fluid dynamics in domains. Bothe-Prüss [10] applied the Boussinesq-Scriven law to make their model for multiphase flow with surface tension and viscosities. See [4, 10] for the Boussinesq-Scriven law. Koba [11] applied the first law of thermodynamics to derive their multiphase flow system with surface flow and tension. Our approach is different from one in [4, 11]. See also Slattery-Sagis-Oh [12], Gatignol-Prud’homme [13] for interfacial phenomena.

The outline of this paper is as follows: In Section 2 we first introduce flow maps in $\Omega_T$, and we state the main results of this paper. In Section 3 we study some properties of the Riemannian metrics determined by flow maps in $\Omega_T$. In Section 4 we calculate variations of flow maps to our action integrals determined by the kinetic energies $(\rho_A|v_A|^2/2,\rho_B|v_B|^2/2,\rho_S|v_S|^2/2)$. In Section 5, we apply an energetic variational approach to make mathematical models for inviscid multiphase flow. In Section 6 we investigate the conservation and energy laws of our systems. In Appendix, we give some useful tools to analyze functions on surfaces.

2. Variations of flow maps and main results

We first introduce variations $(\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)$ of flow maps $(x_A, x_B, x_S)$ in $\Omega_T$. The flow maps $(x_A, x_B, x_S)$ and its variations $(\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)$ are essential tools to make mathematical models for inviscid multiphase flow. Then we state the main results of this paper.

Let us explain the conventions used in this paper. We use the italic characters $i, j, k, \ell$ as 1, 2, 3, and the Greek characters $\alpha, \beta$ as 1, 2, that is, $i, j, k, \ell \in \{1, 2, 3\}$ and $\alpha, \beta \in \{1, 2\}$.

We first define our moving domains and surface

**Definition 2.1** (Moving domains and surface). Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^\infty$-boundary $\partial\Omega$. Let $\Omega_A(t) = \{\Omega_A(t)\}_{0 \leq t < T}$ be a $C^\infty$-bounded domain in $\mathbb{R}^3$ depending on time $t \in [0, T)$ such that $\Omega_A(t) \subseteq \Omega$. Set 

$$
\Omega_B(t) = \Omega \setminus \Omega_A(t) \text{ and } \Gamma(t) = \partial\Omega_A(t).
$$

For each $0 \leq t < T$, assume that $\Gamma(t)$ is a closed Riemannian 2-dimensional manifold. Define $\Omega_{A,T}$, $\Omega_{B,T}$, $\Gamma_T$, $\Omega_T$, and $\partial\Omega_T$ by (1.1). Set

$$
\overline{\Omega}_{A,T} = \bigcup_{0 \leq t < T} \{\overline{\Omega}_A(t) \times \{t\}\}, \quad \overline{\Omega}_{B,T} = \bigcup_{0 \leq t < T} \{\overline{\Omega}_B(t) \times \{t\}\},
$$

$$
\overline{\Gamma}_T = \bigcup_{0 \leq t < T} \{\Gamma(t) \times \{t\}\}, \quad \overline{\Omega}_T = \overline{\Omega} \times [0, T).
$$
Let $\Omega$, $\{\Omega_A(t)\}_{0 \leq t < T}$ be bounded domains satisfying the properties as in Definition 2.1. From now we fix $\Omega$ and $\Omega_A(t)$. Note that $\Omega_B(t)$ is a bounded domain in $\mathbb{R}^3$ with $C^\infty$-boundaries $\partial \Omega$ and $\Gamma(t)$ (see Figure 1).

Next we introduce function spaces in moving domains and surfaces.

**Definition 2.2** (Function spaces in moving domains and surfaces). Let $\mathcal{S} \subset \mathbb{R}^4$. Define

$$C^\infty(\mathcal{S}) = \{f : \mathcal{S} \to \mathbb{R}; f = F|_{\mathcal{S}} \text{ for some } F \in C^\infty(\mathbb{R}^4)\}.$$ 

For example,

$$C^\infty(\Omega_{A,T}) = \{f : \Omega_{A,T} \to \mathbb{R}; f = F|_{\Omega_{A,T}} \text{ for some } F \in C^\infty(\mathbb{R}^4)\},$$

$$C^\infty(\Omega_{B,T}) = \{f : \Omega_{B,T} \to \mathbb{R}; f = F|_{\Omega_{B,T}} \text{ for some } F \in C^\infty(\mathbb{R}^4)\},$$

$$C^\infty(\Gamma_T) = \{f : \Gamma_T \to \mathbb{R}; f = F|_{\Gamma_T} \text{ for some } F \in C^\infty(\mathbb{R}^4)\}.$$ 

Remark that function spaces in fixed domains and surfaces in $\mathbb{R}^3$ are usual function spaces.

Now we introduce flow maps $(\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)$ in $\Omega_T$.

**Definition 2.3** (Flow maps in $\Omega_T$).

(i) Let $\tilde{x}_A = \tilde{x}_A(\xi_A, t) = \iota^{A}(\tilde{x}^A_1(\xi_A, t), \tilde{x}^A_2(\xi_A, t), \tilde{x}^A_3(\xi_A, t))$ be in $[C^\infty(\Omega_A(0) \times [0, T))]^3$. We call $\tilde{x}_A$ a flow map in $\Omega_{A,T}$ if the following three properties hold:

**Property (I)** For $0 < t < T$,

$$\Omega_A(t) = \{x = \iota^{A}(x_1, x_2, x_3) \in \mathbb{R}^3; x = \tilde{x}_A(\xi_A, t), \xi_A \in \Omega_A(0)\},$$

$$\Omega_A(t) = \{x = \iota^{A}(x_1, x_2, x_3) \in \mathbb{R}^3; x = \tilde{x}_A(\xi_A, t), \xi_A \in \Omega_A(0)\}.$$

**Property (II)** There exists a smooth function $v_A = v_A(x, t) = \iota^{A}(v^A_1(x, t), v^A_2(x, t), v^A_3(x, t))$ such that for every $\xi_A \in \Omega_A(0)$,

$$\begin{cases}
\partial_t \tilde{x}_A(\xi_A, t) = v_A(\tilde{x}_A(\xi_A, t), t), & t \in (0, T), \\
\tilde{x}_A(\xi_A, 0) = \xi_A.
\end{cases}$$

We call $v_A$ the velocity determined by the flow map $\tilde{x}_A$.

**Property (III)** For each $0 < t < T$ and $\Lambda_A(t) \subset \Omega_A(t)$, there is $\mathfrak{M}_A \subset \Omega_A(0)$ such that

$$\Lambda_A(t) = \{x = \iota^{A}(x_1, x_2, x_3) \in \mathbb{R}^3; x = \tilde{x}_A(\xi_A, t), \xi_A \in \mathfrak{M}_A\}.$$

(ii) As in (i), we define a flow map $\tilde{x}_B$ in $\Omega_{B,T}$ and the velocity $v_B$ determined by the flow map $\tilde{x}_B$.

(iii) Let $\tilde{x}_S = \tilde{x}_S(\xi_S, t) = \iota^{S}(\tilde{x}^S_1(\xi_S, t), \tilde{x}^S_2(\xi_S, t), \tilde{x}^S_3(\xi_S, t))$ be in $[C^\infty(\Gamma(0) \times [0, T))]^3$.

We call $\tilde{x}_S$ a flow map in $\Gamma_T$ if the following three properties hold:

**Property (I)** For $0 < t < T$,

$$\Gamma(t) = \{x = \iota^{S}(x_1, x_2, x_3) \in \mathbb{R}^3; x = \tilde{x}_S(\xi_S, t), \xi_S \in \Gamma(0)\}.$$

**Property (II)** There exists a smooth function $v_S = v_S(x, t) = \iota^{S}(v^S_1(x, t), v^S_2(x, t), v^S_3(x, t))$ such that for every $\xi_S \in \Gamma(0)$,

$$\begin{cases}
\partial_t \tilde{x}_S(\xi_S, t) = v_S(\tilde{x}_S(\xi_S, t), t), & t \in (0, T), \\
\tilde{x}_S(\xi_S, 0) = \xi_S.
\end{cases}$$

We call $v_S$ the velocity determined by the flow map $\tilde{x}_S$.

**Property (III)** For each $0 < t < T$ the mapping $\tilde{x}_S(:, t) : \Gamma(0) \to \Gamma(t)$ is bijective.
Let \((\bar{x}_A, \bar{x}_B, \bar{x}_S)\) be flow maps in \(\bar{\Omega}_{T}\), and \((v_A, v_B, v_S)\) be the velocities determined by the flow maps \((\bar{x}_A, \bar{x}_B, \bar{x}_S)\) satisfying the properties as in Definition 2.3.

From now we fix \((\bar{x}_A, \bar{x}_B, \bar{x}_S)\) and \((v_A, v_B, v_S)\).

Next we introduce two types of variations. The first one is a variation of our domains and surface. The second one is a variation of flow maps in a variation of our domains and surface.

**Definition 2.4** (Variations of domains and surface). For each \(-1 < \varepsilon < 1\), let \(\Omega^\varepsilon_A(t) = \{\Omega^\varepsilon_A(t)\}_{0 \leq t < T}\) be a \(C^\infty\)-bounded domain in \(\mathbb{R}^3\) depending on time \(t \in [0, T)\) such that \(\Omega^\varepsilon_A(t) \Subset \Omega\). For each \(0 \leq t < T\), assume that \(\Gamma^\varepsilon(t)\) is a closed Riemannian 2-dimensional manifold. Set

\[
\Omega_B^\varepsilon(t) = \Omega \setminus \{\Gamma^\varepsilon(t)\} \quad \text{and} \quad \Gamma^\varepsilon(t) = \partial \Omega^\varepsilon_A(t).
\]

Write

\[
\Omega^\varepsilon_{A,T} = \bigcup_{0 < t < T} \{\Omega^\varepsilon_A(t) \times \{t\}\}, \quad \Omega^\varepsilon_{B,T} = \bigcup_{0 < t < T} \{\Omega^\varepsilon_B(t) \times \{t\}\},
\]

\[
\Gamma^\varepsilon_T = \bigcup_{0 < t < T} \{\Gamma^\varepsilon(t) \times \{t\}\}, \quad \Gamma^\varepsilon_{A,T} = \bigcup_{0 \leq t < T} \{\Omega^\varepsilon_A(t) \times \{t\}\},
\]

\[
\Gamma^\varepsilon_{B,T} = \bigcup_{0 \leq t < T} \{\Omega^\varepsilon_B(t) \times \{t\}\}, \quad \Gamma^\varepsilon_T = \bigcup_{0 \leq t < T} \{\Gamma^\varepsilon(t) \times \{t\}\}.
\]

We say that \((\Omega^\varepsilon_{A,T}, \Omega^\varepsilon_{B,T}, \Gamma^\varepsilon_T)\) is a variation of \((\Omega_{A,T}, \Omega_{B,T}, \Gamma_T)\) if \(\Omega^\varepsilon_A(0) = \Omega_A(0), \Omega^\varepsilon_B(0) = \Omega_B(0), \Gamma^\varepsilon(0) = \Gamma(0), \Omega^\varepsilon_A(t)_{|\varepsilon = 0} = \Omega_A(t), \Omega^\varepsilon_B(t)_{|\varepsilon = 0} = \Omega_B(t), \) and \(\Gamma^\varepsilon(t)_{|\varepsilon = 0} = \Gamma(t)\).

Note that \(\Omega = \Omega^\varepsilon_A(t) \cup \Gamma^\varepsilon(t) \cup \Omega^\varepsilon_B(t)\) (see Figure 1). In this paper, we assume that the dependence of \(\Omega^\varepsilon_A(t), \Omega^\varepsilon_B(t), \) and \(\Gamma^\varepsilon(t)\) is smooth with respect to the parameter \(\varepsilon\). From now the symbol \(n_{\Gamma^\varepsilon}(x,t) = \gamma n_{\Gamma^\varepsilon}^\varepsilon, n_{\Gamma^\varepsilon}^\varepsilon, n_{\Gamma^\varepsilon}^\varepsilon\) denotes the unit outer normal vector at \(x \in \Gamma^\varepsilon(t)\).

**Definition 2.5** (Flow maps in \((\Omega^\varepsilon_{A,T}, \Omega^\varepsilon_{B,T}, \Gamma^\varepsilon_T)\)). Let \(-1 < \varepsilon < 1\), and let \((\Omega^\varepsilon_{A,T}, \Omega^\varepsilon_{B,T}, \Gamma^\varepsilon_T)\) be a variation of \((\Omega_{A,T}, \Omega_{B,T}, \Gamma_T)\).

(i) Let \(\bar{x}^\varepsilon_A(\xi_A, t) = \bar{x}^\varepsilon_A(\xi_A, t) \bar{x}^\varepsilon_A(\xi_A, t) \bar{x}^\varepsilon_A(\xi_A, t)\) be in \([C^\infty(\Omega_A(0) \times [0, T))]])^3\). We call \(\bar{x}^\varepsilon_A\) a flow map in \(\Omega^\varepsilon_{A,T}\) if the following three properties hold:

**Property (I)** For \(0 < t < T\),

\[
\Omega^\varepsilon_A(t) = \{x = t(x_1, x_2, x_3) \in \mathbb{R}^3; \ x = \bar{x}^\varepsilon_A(\xi_A, t), \xi_A \in \Omega_A(0)\},
\]

\[
\Omega^\varepsilon_B(t) = \{x = t(x_1, x_2, x_3) \in \mathbb{R}^3; \ x = \bar{x}^\varepsilon_A(\xi_A, t), \xi_A \in \Omega_A(0)\}.
\]

**Property (II)** There exists a smooth function \(v^\varepsilon_A = v^\varepsilon_A(x,t) = v^\varepsilon_A(x_1, x_2, x_3)\) such that for every \(\xi_A \in \Omega_A(0)\),

\[
\begin{align*}
\partial_t \bar{x}^\varepsilon_A(\xi_A, t) &= v^\varepsilon_A(\bar{x}^\varepsilon_A(\xi_A, t), t), \ t \in (0, T), \\
\bar{x}^\varepsilon_A(\xi_A, 0) &= \xi_A.
\end{align*}
\]

We call \(v^\varepsilon_A\) the velocity determined by the flow map \(\bar{x}^\varepsilon_A\).

**Property (III)** For each \(0 < t < T\) and \(\Lambda^\varepsilon_A(t) \subset \Omega^\varepsilon_A(t)\), there is \(M_A \subset \Omega_A(0)\) such that

\[
\Lambda^\varepsilon_A(t) = \{x = t(x_1, x_2, x_3) \in \mathbb{R}^3; \ x = \bar{x}^\varepsilon_A(\xi_A, t), \xi_A \in M_A\}.
\]
(ii) As in (i), we define a \textit{flow map} \( \tilde{x}^S_B \) in \( \overline{\Omega_{B,T}} \) and the velocity \( \tilde{v}^S_B \) determined by the flow map \( \tilde{x}^S_B \).

(iii) Let \( \tilde{x}^S_B = \tilde{x}^S_B(\xi_S, t) = t(\tilde{x}^S_1(\xi_S, t), \tilde{x}^S_2(\xi_S, t), \tilde{x}^S_3(\xi_S, t)) \) be in \( [C^\infty(\Gamma(0) \times (0, T))]^3 \). We call \( \tilde{x}^S_B \) a \textit{flow map} in \( \Gamma^*_T \) if the following three properties hold:

\textbf{Property (I)} For \( 0 < t < T, \)
\[ \Gamma^*_t = \{ x = t(x_1, x_2, x_3) \in \mathbb{R}^3; \ x = \tilde{x}^S_B(\xi_S, t), \ \xi_S \in \Gamma(0) \}. \]

\textbf{Property (II)} There exists a smooth function \( \tilde{v}^S_B = \tilde{v}^S_B(x, t) = t(\tilde{v}^S_1(x, t), \tilde{v}^S_2(x, t), \tilde{v}^S_3(x, t)) \) such that for every \( \xi_S \in \Gamma(0), \)
\[ \begin{align*}
\partial_t \tilde{x}^S_B(\xi_S, t) = \tilde{v}^S_B(\tilde{x}^S_B(\xi_S, t), t), & \quad t \in (0, T), \\
\tilde{x}^S_B(\xi_S, 0) = \xi_S.
\end{align*} \]

We call \( \tilde{v}^S_B \) the \textit{velocity determined} by the flow map \( \tilde{x}^S_B \).

\textbf{Property (III)} For each \( 0 < t < T \) the mapping \( \tilde{x}^S_B(\cdot, t): \Gamma(0) \rightarrow \Gamma^*_T(t) \) is bijective.

For each \(-1 < \varepsilon < 1\), from now on, the symbol \( (\Omega^-_{A,T}, \Omega^-_{B,T}, \Gamma^-) \) is a variation of \( (\Omega^-_{A,T}, \Omega^-_{B,T}, \Gamma^-) \), \( \tilde{x}^S_A, \tilde{x}^S_B, \tilde{x}^S_S \) denote flow maps in \( (\Omega^-_{A,T}, \Omega^-_{B,T}, \Gamma^-) \), and \( (\tilde{v}_A, \tilde{v}_B, \tilde{v}_S) \) denote the velocities determined by the flow map \( \tilde{x}^S_A, \tilde{x}^S_B, \tilde{x}^S_S \) satisfying the properties as in Definitions 2.4 and 2.5.

\textbf{Definition 2.6} (Variations of flow maps in \( (\Omega^-_{A,T}, \Omega^-_{B,T}, \Gamma^-) \)). Let \( z_A = z_A(x, t) = t(z^A_1, z^A_2, z^A_3), z_B = z_B(x, t) = t(z^B_1, z^B_2, z^B_3), z_S = z_S(x, t) = t(z^S_1, z^S_2, z^S_3) \) be smooth functions in \( \mathbb{R}^3 \). We say that \( (z_A, z_B, z_S) \) is a variation of \( (\tilde{x}^S_A, \tilde{x}^S_B, \tilde{x}^S_S) \) if there are smooth functions \( y_A = y_A(\xi_A, t), y_B = y_B(\xi_B, t), y_S = y_S(\xi_S, t) \) such that for \( 0 \leq t < T, \xi_A \in \Omega^-_{A}(0), \xi_B \in \Omega^-_{B}(0), \xi_S \in \Gamma^-(0), \)
\[ \frac{d}{d\varepsilon} \bigg| _{\varepsilon=0} \tilde{x}^S_A(\xi_A, t) = y_A(\xi_A, t), \quad \frac{d}{d\varepsilon} \bigg| _{\varepsilon=0} \tilde{x}^S_B(\xi_B, t) = y_B(\xi_B, t), \quad \frac{d}{d\varepsilon} \bigg| _{\varepsilon=0} \tilde{x}^S_S(\xi_S, t) = y_S(\xi_S, t). \]

Before stating the main results of this paper, we recall the transport theorems. For each \(-1 < \varepsilon < 1\), let \( \rho_A, \rho_B, \rho_S, \rho'_A, \rho'_B, \rho'_S \) be smooth functions in \( \mathbb{R}^4 \).

\textbf{Proposition 2.7} (Continuity equations). The following two assertions hold:

(i) Assume that for every \( 0 < t < T \) and \( \Lambda \subset \Omega, \)
\[ \frac{d}{dt} \left( \int_{\Omega^-_{A}(t) \cap \Lambda} \rho_A \ dx + \int_{\Omega^-_{B}(t) \cap \Lambda} \rho_B \ dx + \int_{\Gamma^-_{t}(t) \cap \Lambda} \rho_S \ d\mathcal{H}^2_x \right) = 0. \]
Then \( (\rho_A, \rho_B, \rho_S) \) satisfies
\[ \begin{system}
D^A_t \rho_A + (\text{div}\rho_A) \rho_A = 0 & \text{in} \ \Omega^-_{A,T}, \\
D^B_t \rho_B + (\text{div}\rho_B) \rho_B = 0 & \text{in} \ \Omega^-_{B,T}, \\
D^S_t \rho_S + (\text{div}\rho_S) \rho_S = 0 & \text{on} \ \Gamma^-_T.
\end{system} \]

(ii) Let \(-1 < \varepsilon < 1\). Assume that for every \( 0 < t < T \) and \( \Lambda \subset \Omega, \)
\[ \frac{d}{dt} \left( \int_{\Omega^-_{A}(t) \cap \Lambda} \rho'_A \ dx + \int_{\Omega^-_{B}(t) \cap \Lambda} \rho'_B \ dx + \int_{\Gamma^-_{t}(t) \cap \Lambda} \rho'_S \ d\mathcal{H}^2_x \right) = 0. \]
Then \((\rho_A^\varepsilon, \rho_B^\varepsilon, \rho_S^\varepsilon)\) satisfies

\[
\begin{aligned}
D_t^{A,\varepsilon} \rho_A^\varepsilon + (\text{div} \nu_A^\varepsilon) \rho_A^\varepsilon &= 0 \quad \text{in } \Omega^\varepsilon_{A,T}, \\
D_t^{B,\varepsilon} \rho_B^\varepsilon + (\text{div} \nu_B^\varepsilon) \rho_B^\varepsilon &= 0 \quad \text{in } \Omega^\varepsilon_{B,T}, \\
D_t^{S,\varepsilon} \rho_S^\varepsilon + (\text{div} \nu_S^\varepsilon) \rho_S^\varepsilon &= 0 \quad \text{on } \Gamma^\varepsilon_T.
\end{aligned}
\]

(2.2)

Here \(D_t^{A,\varepsilon} f := \partial_t f + (\nu_A^\varepsilon, \nabla) f\), \(D_t^{B,\varepsilon} f := \partial_t f + (\nu_B^\varepsilon, \nabla) f\), and \(D_t^{S,\varepsilon} f := \partial_t f + (\nu_S^\varepsilon, \nabla) f\), and \(\text{div} \nu_S^\varepsilon = \partial_t v^1_S + \partial^2_{x^1} v^2_S + \partial^3_{x^1} v^3_S\), where \(\partial_t v^j = \sum_{j=1}^3 (\delta_{ij}^x - n^x_j n^y_j) \partial_j f\).

The proof of Proposition 2.2 can be found in [3, 8, 3, 14, 15]. We can prove Proposition 2.4 by applying Lemma 3.3.

Now we state the main results of this paper. Let \(\rho_0^A \in C^\infty(\Omega_A(0))\), \(\rho_0^B \in C^\infty(\Omega_B(0))\), and \(\rho_0^S \in C^\infty(\Gamma(0))\). Assume that \(\langle \rho_A, \rho_B, \rho_S \rangle\) satisfies (2.1) and

\[
\begin{aligned}
\rho_A|_{t=0} &= \rho_0^A \quad \text{in } \Omega_A(0), \\
\rho_B|_{t=0} &= \rho_0^B \quad \text{in } \Omega_B(0), \\
\rho_S|_{t=0} &= \rho_0^S \quad \text{on } \Gamma(0).
\end{aligned}
\]

(2.3)

For each \(-1 < \varepsilon < 1\) assume that \(\langle \rho_A^\varepsilon, \rho_B^\varepsilon, \rho_S^\varepsilon \rangle\) satisfies (2.2) and

\[
\begin{aligned}
\rho_A^\varepsilon|_{t=0} &= \rho_0^A \quad \text{in } \Omega_A(0), \\
\rho_B^\varepsilon|_{t=0} &= \rho_0^B \quad \text{in } \Omega_B(0), \\
\rho_S^\varepsilon|_{t=0} &= \rho_0^S \quad \text{on } \Gamma(0).
\end{aligned}
\]

(2.4)

Let \(p_A, p_B, p_S \in C^1(\mathbb{R})\). For each variation \((\tilde{x}_A^\varepsilon, \tilde{x}_B^\varepsilon, \tilde{x}_S^\varepsilon)\) of the flow map \((\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)\),

\[
A[\tilde{x}_A^\varepsilon, \tilde{x}_B^\varepsilon, \tilde{x}_S^\varepsilon] := \int_0^T \int_{\Omega_A(t)} \left\{ \frac{1}{2} \rho_A^\varepsilon |v_A^\varepsilon|^2 - p_A(\rho_A^\varepsilon) \right\} \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega_B(t)} \left\{ \frac{1}{2} \rho_B^\varepsilon |v_B^\varepsilon|^2 - p_B(\rho_B^\varepsilon) \right\} \, dx \, dt
\]

\[
+ \int_0^T \int_{\Gamma(t)} \left\{ \frac{1}{2} \rho_S^\varepsilon |v_S^\varepsilon|^2 - p_S(\rho_S^\varepsilon) \right\} \, d\mathcal{H}^2 \, dt.
\]

We call \(A[\tilde{x}_A^\varepsilon, \tilde{x}_B^\varepsilon, \tilde{x}_S^\varepsilon]\) and \(\langle \rho_\varepsilon |v_\varepsilon|^2 / 2, p_\varepsilon(\rho_\varepsilon) \rangle\) the action integral and energy densities for our models. Moreover, we assume that the Riemannian metrics \(\sqrt{G_A}, \sqrt{G_B}, \sqrt{G_B}, \sqrt{G_S}, \sqrt{G_S}\) are positive functions (see Section 3 for details).

**Theorem 2.8** (Variations of the flow maps to the action integral). Assume that for every \(0 \leq t < T\), \(\xi_A \in \Omega_A(0), \xi_B \in \Omega_B(0), \text{ and } \xi_S \in \Gamma(0)\),

\[
\begin{aligned}
\rho_A^\varepsilon(\tilde{x}_A^\varepsilon(\xi_A, t), t)|_{t=0} &= \rho_A(\tilde{x}_A(\xi_A, t), t), \\
\rho_B^\varepsilon(\tilde{x}_B^\varepsilon(\xi_B, t), t)|_{t=0} &= \rho_B(\tilde{x}_B(\xi_B, t), t), \\
\rho_S^\varepsilon(\tilde{x}_S^\varepsilon(\xi_S, t), t)|_{t=0} &= \rho_S(\tilde{x}_S(\xi_S, t), t),
\end{aligned}
\]

(2.5)

and

\[
\begin{aligned}
z_A(\tilde{x}_A(\xi_A, T^-), T^-) &= t(0, 0, 0), \\
z_B(\tilde{x}_B(\xi_B, T^-), T^-) &= t(0, 0, 0), \\
z_S(\tilde{x}_S(\xi_S, T^-), T^-) &= t(0, 0, 0).
\end{aligned}
\]

(2.6)
Then

\[
\frac{d}{dx} |_{x=0} A[\tilde{x}_A, \tilde{x}_B, \tilde{x}_S] = - \int_0^T \int_{\Omega_A(t)} \left( (\rho_A D_t^A v_A + \nabla p_A) \cdot z_A \right) (x, t) \, dx \, dt \\
- \int_0^T \int_{\Omega_B(t)} \left( \rho_B D_t^B v_B + \nabla p_B \right) \cdot z_B \, dx \, dt \\
- \int_0^T \int_{\Gamma(t)} \left( \rho_S D_t^S v_S + \nabla p_S + p_S H_T n_T \right) \cdot z_S \, dH^2_T dt \\
+ \int_0^T \int_{\Gamma(t)} (p_A n_T) \cdot z_A \, dH^2_T dt - \int_0^T \int_{\Gamma(t)} (p_B n_T) \cdot z_B \, dH^2_T dt \\
+ \int_0^T \int_{\partial \Omega} (p_B n_{\Omega}) \cdot z_B \, dH^2_T dt.
\]

Here \((z_A, z_B, z_S)\) is a variation of \((\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)\), \((p_A, p_B, p_S)\) is defined by (1.4), and \(f(T-) := \lim_{t \to T} f(t)\).

Applying (2.7), we make mathematical models for inviscid multiphase flow with surface flow. See Section 3 for details.

**Theorem 2.9** (Conservation and energy laws).

(i) Any solution to system (1.3) with (1.2) and (1.4) satisfies (1.8), (1.9), (1.10), and (1.11).

(ii) Any solution to system (1.5) with (1.2) and (1.4) satisfies (1.8), (1.9), and (1.10).

We prove Theorem 2.8 in Section 4 and Theorem 2.9 in Section 6.

### 3. Preliminaries

Let us first recall some properties of the Riemannian metrics determined by flow maps. Then we study the representation formulas for our energy densities.

Let \(\tilde{x}_A = \tilde{x}_A(\xi_A, t)\) be the flow map in \(\Omega_{A, T}\), and \(v_A = v_A(x, t)\) be the velocity determined by the flow map \(\tilde{x}_A\), that is, for every \(\xi_A \in \Omega_{A, 0}\) and \(0 < t < T\),

\[
\begin{aligned}
\tilde{x}_A &= \tilde{x}_A(\xi_A, t) = \mathcal{F}(\xi_A, t), \\
v_A &= v_A(x, t) = (v_{1A}(x, t), v_{2A}(x, t), v_{3A}(x, t)), \\
\partial_t \tilde{x}_A &= v_A(\tilde{x}_A(\xi_A, t), t), \\
\tilde{x}_A(\xi_A, 0) &= \xi_A.
\end{aligned}
\]

For each \(i, j \in \{1, 2, 3\}\),

\[
g^A_i = g^A_i(\xi_A, t) := \frac{\partial \tilde{x}_A}{\partial \xi^A_i} = \mathcal{F}(\xi_A, t), \\
g^A_i \cdot g^A_j = \sum_{\ell=1}^3 \frac{\partial \tilde{x}_A^i}{\partial \xi_A^\ell} \frac{\partial \tilde{x}_A^j}{\partial \xi_A^\ell},
\]

and

\[
G_A := G_A(\xi_A, t) = \det(g^A_{ij})_{3 \times 3}.
\]

For the flow map \(\tilde{x}_B\) in \(\Omega_{B, T}\), we define \(g^B_i, g^B_{ij}, G_B\) similarly to \(g^A_i, g^A_{ij}, G_A\).
For each \(-1 < \varepsilon < 1\), let \(\hat{x}_A^\varepsilon = \hat{x}_A^\varepsilon(\xi_A, t)\) be a flow map in \(\Omega^A_{A,T}\), and \(v_A^\varepsilon = v_A^\varepsilon(x, t)\) be the velocity determined by the flow map \(\hat{x}_A^\varepsilon\), that is, for every \(\xi_A \in \Omega_A(0)\) and \(0 < t < T\),

\[
\begin{cases}
\hat{x}_A^\varepsilon = \hat{x}_A^\varepsilon(\xi_A, t) = t(\hat{\xi}_A^\varepsilon(\xi_A, t), \hat{x}_A^\varepsilon(\xi_A, t), \hat{x}_A^\varepsilon(\xi_A, t)), \\
v_A^\varepsilon = v_A^\varepsilon(x, t) = t(v_A^\varepsilon(x, t), v_A^\varepsilon(x, t), v_A^\varepsilon(x, t)), \\
\partial_t \hat{x}_A^\varepsilon = v_A^\varepsilon(\hat{x}_A^\varepsilon(\xi_A, t), t), \\
\hat{x}_A^\varepsilon(\xi_A, 0) = \xi_A.
\end{cases}
\]

For each \(i, j = 1, 2, 3\),

\[
g_i^{A, \varepsilon} = g_i^{A, \varepsilon}(\xi_A, t) := \frac{\partial \hat{x}_A^\varepsilon}{\partial \xi_i^A} = t \left( \frac{\partial \hat{x}_A^{A, \varepsilon}}{\partial \xi_i^A}, \frac{\partial \hat{x}_A^{A, \varepsilon}}{\partial \xi_i^A}, \frac{\partial \hat{x}_A^{A, \varepsilon}}{\partial \xi_i^A} \right),
\]

\[
g_{ij}^{A, \varepsilon} = g_{ij}^{A, \varepsilon}(\xi_A, t) := g_i^{A, \varepsilon} \cdot g_j^{A, \varepsilon},
\]

and

\[
G_A^\varepsilon := G_A^\varepsilon(\xi_A, t) = \text{det}(g_{ij}^{A, \varepsilon})_{3 \times 3}.
\]

For each flow map \(\hat{x}_B^\varepsilon\) in \(\Omega_{B,T}\), we define \(g_i^{B, \varepsilon}, g_{ij}^{B, \varepsilon}, G_B^\varepsilon\) similarly to \(g_i^{A, \varepsilon}, g_{ij}^{A, \varepsilon}, G_A^\varepsilon\).

**Remark 3.1.** Let \(\hat{\xi} = A, B\). Since \(\left(\nabla_{\hat{\xi}} \hat{x}_A^\varepsilon\right)(\nabla_{\hat{\xi}} \hat{x}_A^\varepsilon) = (g_{ij}^{A, \varepsilon})_{3 \times 3}, \left(\nabla_{\hat{\xi}} \hat{x}_B^\varepsilon\right)(\nabla_{\hat{\xi}} \hat{x}_B^\varepsilon) = (g_{ij}^{B, \varepsilon})_{3 \times 3}\), \(\text{det}(\left(\nabla_{\hat{\xi}} \hat{x}_A^\varepsilon\right)) = \text{det}(\nabla_{\hat{\xi}} \hat{x}_A^\varepsilon), \text{ and det}(\left(\nabla_{\hat{\xi}} \hat{x}_B^\varepsilon\right)) = \text{det}(\nabla_{\hat{\xi}} \hat{x}_B^\varepsilon)\), we see that for \(f \in C(\mathbb{R}^4)\),

\[
\begin{align*}
\int_{\Omega(t)} f(x, t) \, dx &= \int_{\Omega(0)} f(\hat{x}_A^\varepsilon(\xi_A, t), t) \sqrt{G_A^\varepsilon(\xi_A, t)} \, d\xi_A, \\
\int_{\Omega(t)} f(x, t) \, dx &= \int_{\Omega(0)} f(\hat{x}_B^\varepsilon(\xi_A, t), t) \sqrt{G_B^\varepsilon(\xi_A, t)} \, d\xi_A.
\end{align*}
\]

We call \(\sqrt{G_A^\varepsilon}\) and \(\sqrt{G_B^\varepsilon}\) the Riemannian metrics determined by the flow maps \(\hat{x}_A^\varepsilon\) and \(\hat{x}_B^\varepsilon\), respectively.

Now we define the Riemannian metric determined by the flow map \(\hat{x}_S\) in \(\Gamma_T\). Let \(\hat{x}_S = \hat{x}_S(\xi_S, t)\) be the flow map in \(\Gamma_T\), and \(v_S = v_S(x, t)\) be the velocity determined by the flow map \(\hat{x}_S\), that is, for every \(\xi_S \in \Gamma(0)\) and \(0 < t < T\),

\[
\begin{cases}
\hat{x}_S = \hat{x}_S(\xi_S, t) = t(\hat{\xi}_S^\varepsilon(\xi_S, t), \hat{x}_S^\varepsilon(\xi_S, t), \hat{x}_S^\varepsilon(\xi_S, t)), \\
v_S = v_S(x, t) = t(v_S^\varepsilon(x, t), v_S^\varepsilon(x, t), v_S^\varepsilon(x, t)), \\
\partial_t \hat{x}_S = v_S(\hat{x}_S(\xi_S, t), t), \\
\hat{x}_S(\xi_S, 0) = \xi_S.
\end{cases}
\]

Since \(\Gamma(0)\) is a closed Riemannian 2-dimensional manifold, there are \(N \in \mathbb{N}, \Gamma_m \subset \Gamma(0), \Phi_m = \Phi_m(X) \in [C^\infty(\mathbb{R}^2)]^3, U_m \subset \mathbb{R}^2, \Psi_m = \Psi_m(\xi_S) \in C^\infty(\mathbb{R}^3) (m =\)
Now we use \( (3.2) \). Then for each \( \xi \in \Gamma(0) \) and \( 0 < t < T \), we set \( \hat{x}(\xi) = \hat{x}(\xi, t) = \hat{x}(\Phi_m(X), t)(= \hat{x}(\xi, t)) \).

Then
\[
\begin{aligned}
\partial_t \hat{x} &= \partial_t \hat{x}(\xi, t) = v_S(\hat{x}(\xi, t), t), \\
\hat{x}|_{t=0} &= \Phi_m(X)(= \xi).
\end{aligned}
\]

Now we write
\[
\Phi := \Phi_m \text{ if } \xi \in \Gamma_m.
\]

Then for each \( \xi \in \Gamma(0) \) and \( 0 < t < T \),
\[
\begin{aligned}
\partial_t \hat{x} &= \partial_t \hat{x}(\xi, t) = v_S(\hat{x}(\xi, t), t), \\
\hat{x}|_{t=0} &= \Phi(X)(= \xi).
\end{aligned}
\]

We also call \( \hat{x} \) a flow map in \( \Gamma_T \). For the flow map \( \hat{x} \) and \( \alpha, \beta = 1, 2 \),
\[
g_\alpha = g_\alpha(X, t) := \frac{\partial \hat{x}}{\partial \alpha} = t \left( \frac{\partial \hat{x}_1}{\partial \alpha}, \frac{\partial \hat{x}_2}{\partial \alpha}, \frac{\partial \hat{x}_3}{\partial \alpha} \right),
\]
\[
g_{\alpha\beta} = g_{\alpha\beta}(X, t) := g_\alpha \cdot g_\beta = \frac{\partial \hat{x}}{\partial \alpha} \cdot \frac{\partial \hat{x}}{\partial \beta},
\]
and
\[
G_S = G_S(X, t) := g_{11}g_{22} - g_{12}g_{21}.
\]

For each \( -1 < \varepsilon < 1 \), let \( \hat{x}_{S, \varepsilon} = \hat{x}_{S, \varepsilon}(\xi, t) \) be a flow map in \( \Gamma_{T, \varepsilon} \), and \( v_{S, \varepsilon} = v_{S, \varepsilon}(x, t) \) be the velocity determined by the flow map \( \hat{x}_{S, \varepsilon} \). Fix \( \xi \in \Gamma(0) \). Assume that \( \xi_S \in \Gamma_m \) for some \( m \), \( m \in \{1, 2, \cdots, N\} \). Since we can write \( \xi_S = \Phi_m(X) \) for some \( X = t(X_1, X_2) \in U_m \subset \mathbb{R}^2 \), we set
\[
\hat{x}_{S, \varepsilon} = \hat{x}_{S, \varepsilon}(X, t) = \hat{x}(\Phi_m(X), t)(= \hat{x}_{S, \varepsilon}(\xi, t)).
\]

Then
\[
\begin{aligned}
\partial_t \hat{x}_{S, \varepsilon} &= \partial_t \hat{x}_{S, \varepsilon}(X, t) = v_{S, \varepsilon}(\hat{x}_{S, \varepsilon}(X, t), t), \\
\hat{x}_{S, \varepsilon}|_{t=0} &= \Phi_m(X)(= \xi).
\end{aligned}
\]

Now we use \( (3.2) \). Then for each \( \xi_S \in \Gamma(0) \) and \( 0 < t < T \),
\[
\begin{aligned}
\partial_t \hat{x}_{S, \varepsilon} &= \partial_t \hat{x}_{S, \varepsilon}(X, t) = v_{S, \varepsilon}(\hat{x}_{S, \varepsilon}(X, t), t), \\
\hat{x}_{S, \varepsilon}|_{t=0} &= \Phi(X)(= \xi_S).
\end{aligned}
\]
We also call $\dot{x}_S^\varepsilon$ a flow map in $\Gamma_T$. For the flow map $\dot{x}_S^\varepsilon$ and $\alpha, \beta = 1, 2$,
\[
\varrho_\alpha^\varepsilon = \varrho_\alpha^\varepsilon(x, t) := \frac{\partial \dot{x}_S^\varepsilon}{\partial x_\alpha} = t \left( \frac{\partial \dot{x}_1^S \varepsilon}{\partial x_\alpha} \frac{\partial \dot{x}_2^S \varepsilon}{\partial x_\alpha} \frac{\partial \dot{x}_3^S \varepsilon}{\partial x_\alpha} \right),
\]
\[
\varrho_{\alpha\beta}^\varepsilon = \varrho_{\alpha\beta}^\varepsilon(x, t) := \varrho_\alpha^\varepsilon \cdot \varrho_\beta^\varepsilon = \frac{\partial \dot{x}_S^\varepsilon}{\partial x_\alpha} \cdot \frac{\partial \dot{x}_S^\varepsilon}{\partial x_\beta},
\]
and
\[
G_S^\varepsilon = G_S^\varepsilon(x, t) := \varrho_1^\varepsilon \varrho_2^\varepsilon - \varrho_1^\varepsilon \varrho_2^\varepsilon.
\]
We call $\sqrt{G_S}$ the Riemannian metric determined by the flow map $\dot{x}_S$, and $\sqrt{G_S^\varepsilon}$ the Riemannian metric determined by the flow map $\dot{x}_S^\varepsilon$. In fact, the following lemma holds.

**Lemma 3.2.** Let $\mathcal{M}_S \subset \Gamma(0)$, $0 < t < T$, and $-1 < \varepsilon < 1$. Set
\[
\Lambda_S(t) = \{ x \in \mathbb{R}^3; \ x = \dot{x}_S(\xi_S, t), \ \xi_S \in \mathcal{M}_S \},
\]
\[
\Lambda_S^\varepsilon(t) = \{ x \in \mathbb{R}^3; \ x = \dot{x}_S^\varepsilon(\xi_S, t), \ \xi_S \in \mathcal{M}_S \}.
\]
Then for each $f \in C(\mathbb{R}^3 \setminus \mathbb{R})$,
\[
\int_{\mathcal{F}(t)} f(x, t) \ d\mathcal{H}_x^2 = \sum_{m=1}^N \int_{\mathcal{U}_m} \hat{\varphi}_m \hat{f} \sqrt{G_S(X, t)} \ dX,
\]
\[
\int_{\Gamma^*(t)} f(x, t) \ d\mathcal{H}_x^2 = \sum_{m=1}^N \int_{\mathcal{U}_m} \hat{\varphi}_m \hat{f} \varepsilon \sqrt{G_S^\varepsilon(X, t)} \ dX,
\]
\[
\int_{\Lambda_S(t)} f(x, t) \ d\mathcal{H}_x^2 = \sum_{m=1}^N \int_{\mathcal{U}_m} 1_{\mathcal{M}_S \cap \Gamma_m}(\Phi_m(X)) \hat{\varphi}_m \hat{f} \sqrt{G_S(X, t)} \ dX,
\]
\[
\int_{\Lambda_S^\varepsilon(t)} f(x, t) \ d\mathcal{H}_x^2 = \sum_{m=1}^N \int_{\mathcal{U}_m} 1_{\mathcal{M}_S \cap \Gamma_m}(\Phi_m(X)) \hat{\varphi}_m \hat{f} \varepsilon \sqrt{G_S^\varepsilon(X, t)} \ dX.
\]

Here
\[
\begin{cases}
\hat{f} = \hat{f}(X, t) := f(\dot{x}_S(\Phi_m(X), t), t), \\
\hat{f}_\varepsilon = \hat{f}_\varepsilon(X, t) := f(\dot{x}_S^\varepsilon(\Phi_m(X), t), t), \\
\hat{\varphi}_m = \hat{\varphi}_m(X) := \varphi_m(\Phi_m(X)).
\end{cases}
\]

We prove Lemma 3.2 in Appendix. From now, we assume that the Riemannian metrics $\sqrt{G_A}, \sqrt{G_A^\varepsilon}, \sqrt{G_B}, \sqrt{G_B^\varepsilon}, \sqrt{G_S}, \sqrt{G_S^\varepsilon}$ are positive functions throughout this paper. From Lemma 3.2 and 3.3, and Lemma 1.3, we have the following two lemmas.

**Lemma 3.3** (Properties of the Riemannian metrics determined by flow maps). Let $f = f(x, t) \in C^\infty(\mathbb{R}^4)$. Then
\[
\int_{\Omega_A(t)} f(\text{div}u_A) \ dx = \int_{\Omega_A(0)} f(\dot{x}_A(\xi_A, t), t) \frac{d}{dt} \sqrt{G_A} \ d\xi_A,
\]
\[
\int_{\Omega_A^\varepsilon(t)} f(\text{div}u_A^\varepsilon) \ dx = \int_{\Omega_A(0)} f(\dot{x}_A^\varepsilon(\xi_A, t), t) \frac{d}{dt} \sqrt{G_A^\varepsilon} \ d\xi_A,
\]
\[
\int_{\Omega_A(t)} f(\text{div}z_A) \ dx = \int_{\Omega_A(0)} f(\dot{x}_A(\xi_A, t), t) \left( \frac{d}{\varepsilon} \right)_{\varepsilon=0} \sqrt{G_A^\varepsilon} \ d\xi_A,
\]
Lemma 3.4 (Representation formulas for the energy densities)

Here (z_A, z_B, z_S) is a variation of (\tilde{z}_A, \tilde{z}_B, \tilde{z}_S) and \((\tilde{f}, \tilde{f}_\varepsilon, \tilde{\Psi}_m)\) is defined by (3.7).

Let \(\rho_0^A \in C^\infty(\Omega(0))\), \(\rho_0^B \in C^\infty(\Omega_B(0))\), \(\rho_0^S \in C^\infty(\Gamma(0))\). Let \(p_A, p_B, p_S \in C^1(\mathbb{R})\).

Then two assertions hold:

(i) Assume that \((\rho_A, p_B, p_S)\) satisfies (2.21) and (2.23). Then

\[
\int_{\Omega_A(t)} \frac{1}{2} \rho_A |v_A|^2 \, dx = \int_{\Omega_A(0)} \frac{1}{2} \rho_0^A(\xi_A)|\partial_t \tilde{x}_A(\xi_A, t)|^2 \, d\xi_A,
\]

\[
\int_{\Omega_B(t)} \frac{1}{2} \rho_B |v_B|^2 \, dx = \int_{\Omega_B(0)} \frac{1}{2} \rho_0^B(\xi_B)|\partial_t \tilde{x}_B(\xi_B, t)|^2 \, d\xi_B,
\]

\[
\int_{\Gamma(t)} \frac{1}{2} \rho_S |v_S|^2 \, d\mathcal{H}_x = \sum_{m=1}^N \int_{U_m} \tilde{\Psi}_m \frac{1}{2} \rho_0^S(\Phi_m(X))|\partial_t \tilde{x}_S(X, t)|^2 \, dX,
\]

\[
\int_{\Omega_A(t)} p_A(\rho_A) \, dx = \int_{\Omega_A(0)} p_A \left( \frac{\rho_0^A(\xi_A)}{\sqrt{G_A}} \right) \sqrt{G_A} \, d\xi_A,
\]

\[
\int_{\Omega_B(t)} p_B(\rho_B) \, dx = \int_{\Omega_B(0)} p_B \left( \frac{\rho_0^B(\xi_B)}{\sqrt{G_B}} \right) \sqrt{G_B} \, d\xi_B,
\]

\[
\int_{\Gamma(t)} p_S(\rho_S) \, d\mathcal{H}_x = \sum_{m=1}^N \int_{U_m} \tilde{\Psi}_m p_S \left( \frac{\rho_0^S(\Phi_m(X))}{\sqrt{G_S}} \right) \sqrt{G_S} \, dX.
\]

Here \(\tilde{\Psi}_m\) is defined by (3.7).

(ii) Let \(-1 < \varepsilon < 1\). Assume that \((\rho^A_\varepsilon, \rho^B_\varepsilon, \rho^S_\varepsilon)\) satisfies (2.2) and (2.4). Then

\[
\int_{\Omega_A(t)} \frac{1}{2} \rho^A_\varepsilon |v^A_\varepsilon|^2 \, dx = \int_{\Omega_A(0)} \frac{1}{2} \rho_0^A(\xi_A)|\partial_t \tilde{x}_A(\xi_A, t)|^2 \, d\xi_A,
\]

\[
\int_{\Omega_B(t)} \frac{1}{2} \rho^B_\varepsilon |v^B_\varepsilon|^2 \, dx = \int_{\Omega_B(0)} \frac{1}{2} \rho_0^B(\xi_B)|\partial_t \tilde{x}_B(\xi_B, t)|^2 \, d\xi_B,
\]

\[
\int_{\Gamma^e(t)} \frac{1}{2} \rho^S_\varepsilon |v^S_\varepsilon|^2 \, d\mathcal{H}_x = \sum_{m=1}^N \int_{U_m} \tilde{\Psi}_m \frac{1}{2} \rho_0^S(\Phi_m(X))|\partial_t \tilde{x}_S(X, t)|^2 \, dX,
\]

\[
\int_{\Omega_A(t)} \rho_\varepsilon(\rho_\varepsilon) \, dx = \int_{\Omega_A(0)} \rho_0^A(\xi_A)|\partial_t \tilde{x}_A(\xi_A, t)|^2 \, d\xi_A,
\]

\[
\int_{\Omega_B(t)} \rho_\varepsilon(\rho_\varepsilon) \, dx = \int_{\Omega_B(0)} \rho_0^B(\xi_B)|\partial_t \tilde{x}_B(\xi_B, t)|^2 \, d\xi_B,
\]

\[
\int_{\Gamma(t)} \rho_\varepsilon(\rho_\varepsilon) \, d\mathcal{H}_x = \sum_{m=1}^N \int_{U_m} \tilde{\Psi}_m \rho_0^S(\Phi_m(X))|\partial_t \tilde{x}_S(X, t)|^2 \, dX.
\]
\[
\begin{align*}
\int_{\Omega_A(t)} p_A(\rho_A^\varepsilon) \, dx &= \int_{\Omega_A(0)} p_A \left( \frac{\rho_A^\varepsilon(\xi_A)}{\sqrt{G_A^\varepsilon}} \right) \sqrt{G_A^\varepsilon} \, d\xi_A, \\
\int_{\Omega_B(t)} p_B(\rho_B^\varepsilon) \, dx &= \int_{\Omega_B(0)} p_B \left( \frac{\rho_B^\varepsilon(\xi_B)}{\sqrt{G_B^\varepsilon}} \right) \sqrt{G_B^\varepsilon} \, d\xi_B, \\
\int_{\Gamma^+} p_S(\rho_S^\varepsilon) \, d\mathcal{H}_2^A &= \sum_{m=1}^N \int_{\Gamma^+} \hat{\Psi}_m p_S \left( \frac{\rho_S^\varepsilon(\Phi_m(X))}{\sqrt{G_S^\varepsilon}} \right) \sqrt{G_S^\varepsilon} \, dX.
\end{align*}
\]

Here \( \hat{\Psi}_m \) is defined by (3.4).

4. Variations of action integral with respect to flow maps

Let us prove one of the main results. Let \( \rho_0^A \in C^\infty(\Omega_A(0)) \), \( \rho_0^B \in C^\infty(\Omega_B(0)) \), and \( \rho_0^S \in C^\infty(\Gamma(0)) \). Assume that \( (\rho_A, \rho_B, \rho_S) \) satisfy (2.1) and (2.3). For each \(-1 < \varepsilon < 1\), assume that \( (\rho_A^\varepsilon, \rho_B^\varepsilon, \rho_S^\varepsilon) \) satisfy (2.2) and (2.4). Fix \( p_A, p_B, p_S \in C^1(\mathbb{R}) \). Assume (2.3) and (2.6) hold.

We first study some properties of the variation \( (z_A, z_B, z_S) \) of \( (\tilde{x}_A^\varepsilon, \tilde{x}_B^\varepsilon, \tilde{x}_S^\varepsilon) \) (see Definition 2.6). Let \( \xi_S \in \Gamma(0) \). Assume that \( \xi_S = \Phi(X) \), where \( \Phi \) is defined by (3.2). Set

\[
y_S = y_S(X, t) = \frac{\partial}{\partial t} \big( y_1^S, y_2^S, y_3^S \big) = \frac{\partial}{\partial t} \big( \Phi(X), t \big)(= \tilde{y}_S(\xi_S, t)).
\]

**Lemma 4.1** (Properties of variations \( (y_A, y_B, y_S, \tilde{y}_S) \)). For all \( \xi_A \in \Omega_A(0), \xi_B \in \Omega_B(0), \xi_S \in \Gamma(0) \),

\[
\begin{align*}
(y_A) &= y_A(\xi_A, T-) = (0, 0, 0), \\
(y_B) &= y_B(\xi_B, T-) = (0, 0, 0), \\
(y_S) &= y_S(\xi_S, T-) = (0, 0, 0), \\
(y_S) &= y_S(X, T-) = (0, 0, 0),
\end{align*}
\]

where \( \xi_S = \Phi(X) \).

**Proof of Lemma 4.1** We only show (4.3). Since \( \tilde{x}_S^\varepsilon(\xi_S, 0) = \xi_S \) for every \(-1 < \varepsilon < 1\) and \( \xi_S \in \Gamma(0) \), we find that

\[
y_S(\xi_S, 0) = \frac{\partial \tilde{x}_S^\varepsilon}{\partial \varepsilon}(\xi_S, 0) = \lim_{h \to 0} \frac{\tilde{x}_S^{\varepsilon+h}(\xi_S, 0) - \tilde{x}_S^\varepsilon(\xi_S, 0)}{h} = \frac{\xi_S - \xi_S}{h} = (0, 0, 0).
\]

By assumption (2.6), we see that \( \tilde{y}_S(\xi_S, T-) = (0, 0, 0) \). Thus, we have (4.3). Therefore, the lemma follows.

**Proposition 4.2** (Variations of our energies with respect to flow maps).

\[
\begin{align*}
\frac{d}{d\varepsilon} \big|_{\varepsilon=0} \int_0^T \int_{\Omega_A(t)} \frac{1}{2} \rho_A^\varepsilon |v_A^\varepsilon|^2 \, dxdt &= - \int_0^T \int_{\Omega_A(t)} (\rho_A D_t^\varepsilon v_A) \cdot z_A \, dxdt, \\
\frac{d}{d\varepsilon} \big|_{\varepsilon=0} \int_0^T \int_{\Omega_B(t)} \frac{1}{2} \rho_B^\varepsilon |v_B^\varepsilon|^2 \, dxdt &= - \int_0^T \int_{\Omega_B(t)} (\rho_B D_t^\varepsilon v_B) \cdot z_B \, dxdt, \\
\frac{d}{d\varepsilon} \big|_{\varepsilon=0} \int_0^T \int_{\Gamma^+(t)} \frac{1}{2} \rho_S^\varepsilon |v_S^\varepsilon|^2 \, dxdt &= - \int_0^T \int_{\Gamma(t)} (\rho_S D_t^\varepsilon v_S) \cdot z_S \, d\mathcal{H}_2^A \, dt,
\end{align*}
\]
\( \frac{d}{d\varepsilon} \mid _{\varepsilon=0} \int_{\Omega(t)} p_A(\rho_A^\varepsilon) \ dx = \int_{\Omega(t)} (p_A(\rho_A) - \rho_A' p_A(\rho_A)) \text{div} z_A \ dx, \)

\( \frac{d}{d\varepsilon} \mid _{\varepsilon=0} \int_{\Omega_b(t)} p_B(\rho_B^\varepsilon) \ dx = \int_{\Omega_b(t)} (p_B(\rho_B) - \rho_B' p_B(\rho_B)) \text{div} z_B \ dx, \)

\( \frac{d}{d\varepsilon} \mid _{\varepsilon=0} \int_{\Gamma(t)} ps(\rho_S^\varepsilon) \ d\mathcal{H}_x^2 = \int_{\Gamma(t)} (ps(\rho_S) - \rho_S' ps(\rho_S)) \text{div} z_S \ d\mathcal{H}_x^2, \)

\( \frac{d}{d\varepsilon} \mid _{\varepsilon=0} \int_{\Gamma(t)} 1 \ d\mathcal{H}_x^2 = \int_{\Gamma(t)} \text{div} z_S \ d\mathcal{H}_x^2. \)

**Proof of Proposition 4.2.** We first derive (4.5) and (4.6). By Lemma 3.4 we check that

\( (R.H.S.) \) of (4.12)

\[ = \int_0^T \int_{\Omega_A(\xi_t)} \rho_A^0(\xi_t) \partial_t \tilde{x}_A(\xi_t, t) \cdot \partial_t \tilde{y}_A(\xi_t, t) \ dx \ dt \]

\[ = \int_0^T \int_{\Omega_A(\xi_t)} \rho_A^0(\xi_t) \partial_t \tilde{x}_A(\xi_t, t) \cdot \partial_t \tilde{y}_A(\xi_t, t) \ dx \ dt. \]

Here we used some properties of flow maps (see Definitions 2.3, 2.5, 2.6). Using integration by parts with (4.11) and Lemma 3.4 we see that

\( (R.H.S.) \) of (4.12)

\[ = -\int_0^T \int_{\Omega_A(\xi_t)} \rho_A^0(\xi_t) \frac{d}{dt} \tilde{v}_A(\tilde{x}_A(t), t) \cdot \tilde{y}_A(\tilde{y}_A(t)) \sqrt{G_A(\tilde{x}_A(t), t)} \ dx \ dt \]

\[ = -\int_0^T \int_{\Omega_A(\xi_t)} \{\rho_A D_t^A \tilde{v}_A \}(x, t) \cdot z_A(x, t) \ dx \ dt, \]

which is (4.5). Similarly, we have (4.6).

Secondly, we show (4.7). By Lemma 3.4 we check that

\[ \frac{d}{d\varepsilon} \mid _{\varepsilon=0} \int_{\Gamma(t)} \frac{N}{2} \rho_S^\varepsilon \ |v_S^\varepsilon|^2 \ dx \ dt \]

\[ = \frac{d}{d\varepsilon} \mid _{\varepsilon=0} \int_{\Gamma(t)} \sum_{m=1}^N \Psi_m \frac{1}{2} \rho_0^S(\Phi_m(X)) \partial_t \tilde{x}_S(X, t) \cdot \partial_t \tilde{y}_S(X, t) \ dX \ dt \]

\[ = \int_0^T \sum_{m=1}^N \Psi_m \rho_0^S(\Phi_m(X)) \partial_t \tilde{x}_S(X, t) \cdot \partial_t \tilde{y}_S(X, t) \ dX \ dt. \]
Using integration by parts with (4.4) and Lemma 3.3, we see that
\[
= -\int_0^T \int_{\Gamma(t)} \{\rho S D^S_t v_S\}(x, t) \cdot z_S(x, t) \, dx \, dt.
\]
Thus, we have (4.7).

Thirdly, we derive (4.8) and (4.9). By Lemmas 3.3, 3.4, we check that
\[
\frac{d}{d\varepsilon} \int_{\Omega(t)} p_A(\rho_A^\varepsilon) \, dx = \frac{d}{d\varepsilon} \int_{\Omega_A(0)} p_A \left( \frac{\rho^A_0}{\sqrt{G_A^\varepsilon}} \right) \sqrt{G_A^\varepsilon} \, d\xi_A
\]
\[
= \int_{\Omega_A(0)} (-p'_A \left( \frac{\rho^A_0}{\sqrt{G_A^\varepsilon}} \right) + p_A \left( \frac{\rho^A_0}{\sqrt{G_A^\varepsilon}} \right) \left( \frac{d}{d\varepsilon} \varepsilon \right) \sqrt{G_A^\varepsilon}) \, d\xi_A
\]
\[
= \int_{\Omega_A(t)} (p_A(\rho_A) - \rho_A p'_A(\rho_A))(\text{div}_A) \, dx.
\]
Thus, we have (4.8). Similarly, we see (4.9).

Finally, we show (4.10). By Lemmas 3.3, 3.4, we observe that
\[
\frac{d}{d\varepsilon} \int_{\Gamma'(t)} p_S(\rho_S^\varepsilon) \, d\mathcal{H}_x^2 = \frac{d}{d\varepsilon} \int_{\gamma(t)} \sum_{m=1}^N \hat{\psi}_m \rho_S(\Phi_{m}(X)) \sqrt{G_S^\varepsilon} \, dX
\]
\[
= \int_{\Gamma(t)} (p_S(\rho_S) - \rho_S p'_S(\rho_S))(\text{div}_S \, z_S) \, d\mathcal{H}_x^2.
\]
Here
\[
K_m(X, t) := -p'_S \left( \frac{\rho^S_0(\Phi_{m}(X))}{\sqrt{G_S^\varepsilon}} \right) \rho^S_{0}(\Phi_{m}(X)) + p_S \left( \frac{\rho^S_0(\Phi_{m}(X))}{\sqrt{G_S^\varepsilon}} \right).
\]
Thus, we have (4.10). The derivation of (4.11) is left for the reader. Therefore, Proposition 4.12 is proved.

Let us attack Theorem 2.8

Proof of Theorem 2.8 From Proposition 4.2, we see that
\[
\frac{d}{d\varepsilon} \int_{\gamma(t)} A[\hat{x}_A, \hat{z}_B, \hat{\om}_S] = -\int_0^T \int_{\Gamma(t)} (\rho_A D^A_t v_A) \cdot z_A \, dx \, dt
\]
\[
- \int_0^T \int_{\Omega_B(t)} (\rho_B D^B_t v_B) \cdot z_B \, dx \, dt - \int_0^T \int_{\Gamma(t)} (\rho_S D^S_t v_S) \cdot z_S \, d\mathcal{H}_x^2 \, dt
\]
\[+
\int_0^T \int_{\Omega_A(t)} p_A \text{div}_A \, dx \, dt + \int_0^T \int_{\Omega_B(t)} p_B \text{div}_B \, dx \, dt
\]
\[+
\int_0^T \int_{\Gamma(t)} p_A \text{div}_T \, z_S \, d\mathcal{H}_x^2 \, dt
\]
where \((p_A, p_B, p_S)\) is defined by (1.3). Using integration by parts (Lemma 7.1), we have (R.H.S.) of (2.7). Therefore, Theorem 2.8 is proved. □

5. Mathematical modeling

Let us apply our energetic variational approaches to derive our multiphase flow systems (1.3), (1.5), and (1.8). We suppose that \((v_A, v_B, v_S)\) satisfy (1.2). From (1.2) we assume that

(5.1) \[
\begin{cases}
\varphi_B \cdot n\Omega = 0 \text{ on } \partial\Omega, \\
\varphi_A \cdot n\Gamma = \varphi_B \cdot n\Gamma = \varphi_S \cdot n\Gamma \text{ on } \Gamma(t).
\end{cases}
\]

Here \(\varphi_A(x) := z_A(x,t)\), \(\varphi_B(x) := z_B(x,t)\), and \(\varphi_S(x) := z_S(x,t)\). Let \(p_A, p_B, p_S \in C^1(\mathbb{R})\). For the flow maps \((\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)\) in \(\Omega_T\),

(5.2) \[
A_1[\tilde{x}_A, \tilde{x}_B, \tilde{x}_S] := \int_0^T \int_{\Omega_A(t)} \left\{ \frac{1}{2} \rho_A |v_A|^2 - p_A(\rho_A) \right\} (x,t) \, dx \, dt \\
+ \int_0^T \int_{\Omega_B(t)} \left\{ \frac{1}{2} \rho_B |v_B|^2 - p_B(\rho_B) \right\} \, dx \, dt \\
+ \int_0^T \int_{\Gamma(t)} \left\{ \frac{1}{2} \rho_S |v_S|^2 - p_S(\rho_S) \right\} \, dH^2 \, dt,
\]

(5.3) \[
A_2[\tilde{x}_A, \tilde{x}_B, \tilde{x}_S] := \int_0^T \int_{\Omega_A(t)} \left\{ \frac{1}{2} \rho_A |v_A|^2 - p_A(\rho_A) \right\} (x,t) \, dx \, dt \\
+ \int_0^T \int_{\Omega_B(t)} \left\{ \frac{1}{2} \rho_B |v_B|^2 - p_B(\rho_B) \right\} \, dx \, dt \\
+ \int_0^T \int_{\Gamma(t)} \left\{ \frac{1}{2} \rho_S |v_S|^2 \right\} \, dH^2 \, dt,
\]

(5.4) \[
A_3[\tilde{x}_A, \tilde{x}_B, \tilde{x}_S] := \int_0^T \int_{\Omega_A(t)} \frac{1}{2} \rho_A |v_A|^2 \, dx \, dt + \int_0^T \int_{\Omega_B(t)} \frac{1}{2} \rho_B |v_B|^2 \, dx \, dt \\
+ \int_0^T \int_{\Gamma(t)} \frac{1}{2} \rho_S |v_S|^2 \, dH^2 \, dt,
\]

and

(5.5) \[
A_4[\tilde{x}_A, \tilde{x}_B, \tilde{x}_S] := \int_0^T \int_{\Omega_A(t)} \rho_A(\rho_A) \, dx \, dt + \int_0^T \int_{\Omega_B(t)} \rho_B(\rho_B) \, dx \, dt \\
+ \int_0^T \int_{\Gamma(t)} \rho_S(\rho_S) \, dH^2 \, dt.
\]

Remark that the action integrals \(A_1, A_2\) and \((A_3, A_4)\) correspond to systems (1.3), (1.5), and (1.6), respectively. Now we study the variations of our action integrals with respect to flow maps. From Theorem 2.8 and Proposition 7.2, we consider the following variational problems.
Proposition 5.1 (Fundamental lemmas of calculus of variations). Fix $0 < t < T$. Then the following three assertions holds:

(i) Assume that for every $\varphi_A \in [C^\infty(\Omega_A(t))]^3$, $\varphi_B \in [C^\infty(\Omega_B(t))]^3$, $\varphi_S \in [C^\infty(\Gamma(t))]^3$ satisfying (5.1),

$$\int_{\Omega_A(t)} (\rho_A D_t^A v_A + \nabla p_A) \cdot \varphi_A \, dx + \int_{\Omega_B(t)} (\rho_B D_t^B v_B + \nabla p_B) \cdot \varphi_B \, dx$$

$$+ \int_{\Gamma(t)} (\rho_S D_t^S v_S + \nabla_T p_S + \rho_S H_T n_T - p_A n_T + p_B n_T) \cdot \varphi_S \, d\mathcal{H}^2 = 0.$$ 

Then

$$\begin{align*}
\rho_A D_t^A v_A + \nabla p_A &= 0 & \text{in } \Omega_A(t), \\
\rho_B D_t^B v_B + \nabla p_B &= 0 & \text{in } \Omega_B(t), \\
\rho_S D_t^S v_S + \nabla_T p_S + \rho_S H_T n_T - p_A n_T + p_B n_T &= 0 & \text{on } \Gamma(t).
\end{align*}$$

(5.6)

Here $(p_A, p_B, p_S)$ is defined by (1.4).

(ii) Assume that for every $\varphi_A \in [C^\infty(\Omega_A(t))]^3$, $\varphi_B \in [C^\infty(\Omega_B(t))]^3$, $\varphi_S \in [C^\infty(\Gamma(t))]^3$ satisfying (5.1) and $\text{div}_T \varphi_S = 0$ on $\Gamma(t)$,

$$\int_{\Omega_A(t)} (\rho_A D_t^A v_A + \nabla p_A) \cdot \varphi_A \, dx + \int_{\Omega_B(t)} (\rho_B D_t^B v_B + \nabla p_B) \cdot \varphi_B \, dx$$

$$+ \int_{\Gamma(t)} (\rho_S D_t^S v_S - p_A n_T + p_B n_T) \cdot \varphi_S \, d\mathcal{H}^2 = 0.$$ 

Then there exists $\Pi_S \in C^1(\Gamma(t))$ such that

$$\begin{align*}
\rho_A D_t^A v_A + \nabla p_A &= 0 & \text{in } \Omega_A(t), \\
\rho_B D_t^B v_B + \nabla p_B &= 0 & \text{in } \Omega_B(t), \\
\rho_S D_t^S v_S + \nabla_T \Pi_S + \Pi_S H_T n_T - p_A n_T + p_B n_T &= 0 & \text{on } \Gamma(t).
\end{align*}$$

(5.7)

(iii) Assume that for every $\varphi_A \in [C^\infty(\Omega_A(t))]^3$, $\varphi_B \in [C^\infty(\Omega_B(t))]^3$, $\varphi_S \in [C^\infty(\Gamma(t))]^3$ satisfying (5.1),

$$\int_{\Omega_A(t)} \rho_A D_t^A v_A \cdot \varphi_A \, dx + \int_{\Omega_B(t)} \rho_B D_t^B v_B \cdot \varphi_B \, dx + \int_{\Gamma(t)} \rho_S D_t^S v_S \cdot p_T \varphi_S \, d\mathcal{H}^2$$

$$= -\int_{\Omega_A(t)} \nabla \varphi_A \cdot \varphi_A \, dx - \int_{\Omega_B(t)} \nabla \varphi_B \cdot \varphi_B \, dx$$

$$- \int_{\Gamma(t)} (\nabla_T p_S + \rho_S H_T n_T - p_A n_T + p_B n_T) \cdot \varphi_S \, d\mathcal{H}^2.$$ 

Then

$$\begin{align*}
\rho_A D_t^A v_A + \nabla p_A &= 0 & \text{in } \Omega_A(t), \\
\rho_B D_t^B v_B + \nabla p_B &= 0 & \text{in } \Omega_B(t), \\
P_T \rho_S D_t^S v_S + \nabla_T p_S &= 0 & \text{on } \Gamma(t), \\
\rho_S H_T n_T - p_A n_T + p_B n_T &= 0 & \text{on } \Gamma(t).
\end{align*}$$

(5.8)

Proof of Proposition 5.1. Since the assertion (i) is clear, we prove (ii) and (iii).
We first show (ii). We now consider the case when \( \varphi_B = \ell(0, 0, 0) \) and \( \varphi_S = \ell(0, 0, 0) \), that is, for every \( \varphi_A \in [C^\infty(\Omega_A(t))]^3 \) satisfying (5.1),

\[
\int_{\Omega_A(t)} (\rho_A D_t^A v_A + \nabla p_A) \cdot \varphi_A \, dx = 0.
\]

This shows that

\[
\rho_A D_t^A v_A + \nabla p_A = 0 \text{ in } \Omega_A(t).
\]

Similarly, we see that

\[
\rho_B D_t^B v_B + \nabla p_B = 0 \text{ in } \Omega_B(t).
\]

Next we consider a general case, that is, for every \( \varphi_A \in [C^\infty(\Gamma(t))]^3 \) satisfying \( \text{div}_\Gamma \varphi_S = 0 \) on \( \Gamma(t) \),

\[
\int_{\Gamma(t)} (\rho_S D_t^S v_S - p_A n_\Gamma + p_B n_\Gamma) \cdot \varphi_S \, d\mathcal{H}_2^2 = 0.
\]

Since \( \text{div}_\Gamma \varphi_S = 0 \), we apply the generalized Helmholtz-Weyl decomposition (Lemma 7.2) to find that there exists \( \Pi_S \in C^1(\Gamma(t)) \) such that

\[
\rho_S D_t^S v_S - p_A n_\Gamma + p_B n_\Gamma = -\nabla_\Gamma \Pi_S - \Pi_S H_\Gamma n_\Gamma \text{ on } \Gamma(t).
\]

Thus, we see (ii).

Next we prove (iii). We now consider the case when \( \varphi_S = \ell(0, 0, 0) \), that is, for every \( \varphi_A \in [C^\infty(\Omega_A(t))]^3 \), \( \varphi_B \in [C^\infty(\Omega_B(t))]^3 \) satisfying (5.1),

\[
\int_{\Omega_A(t)} (\rho_A D_t^A v_A + \nabla p_A) \cdot \varphi_A \, dx + \int_{\Omega_B(t)} (\rho_B D_t^B v_B + \nabla p_B) \cdot \varphi_B \, dx = 0.
\]

This shows that

\[
\rho_A D_t^A v_A + \nabla p_A = 0 \text{ in } \Omega_A(t),
\]

\[
\rho_B D_t^B v_B + \nabla p_B = 0 \text{ in } \Omega_B(t).
\]

Next we consider a general case, that is, for every \( \varphi_S \in [C^\infty(\Gamma(t))]^3 \),

\[
\int_{\Gamma(t)} (P_\Gamma \rho_S D_t^S v_S + \nabla_\Gamma p_S + p_S H_\Gamma n_\Gamma - p_A n_\Gamma + p_B n_\Gamma) \cdot \varphi_S \, d\mathcal{H}_2^2 = 0.
\]

Therefore, we see that

\[
P_\Gamma \rho_S D_t^S v_S + \nabla_\Gamma p_S = 0 \text{ on } \Gamma(t),
\]

\[
p_S H_\Gamma n_\Gamma - p_A n_\Gamma + p_B n_\Gamma = 0 \text{ on } \Gamma(t).
\]

Thus, we see (iii). Therefore, Proposition 5.1 is proved.

Let us derive our multiphase flow systems.

5.1. Inviscid multiphase flow system with compressible surface flow. Let us construct system (1.3). Assume that (\( v_A, v_B, v_S \)) satisfy (1.2). Based on Proposition 2.7, we admit (2.1). Set the action integral \( A_1 \) defined by (5.2). We assume the following energetic variational principle:

\[
\frac{\delta A_1}{\delta \tilde{x}} \bigg|_{z_B n_\Gamma = 0 \text{ on } \partial \Omega, \, z_A n_\Gamma = z_B n_\Gamma = z_S n_\Gamma \text{ on } \Gamma(t)} = 0.
\]

Here \( \tilde{x} = (\tilde{x}_A, \tilde{x}_B, \tilde{x}_S) \) and \( (z_A, z_B, z_S) \) is a variation of \( (\tilde{x}_A, \tilde{x}_B, \tilde{x}_S) \). From assertion (i) of Proposition 5.1, we derive (5.6). Combining (2.1) and (5.6), we therefore have system (1.3).
5.2. Inviscid multiphase flow system with incompressible surface flow.
Let us construct system (1.3). Assume that \((v_A, v_B, v_S)\) satisfy (1.2). Based on Proposition 2.7, we admit (2.1) and \(\text{div}_T v_S = 0\) on \(\Gamma_T\). Set the action integral \(A_2\) defined by (5.3). We assume the following energetic variational principle:
\[
\frac{\delta A_2}{\delta \tilde{x}} \bigg|_{z_B n_{\Omega} = 0 \text{ on } \partial \Omega, \; z_A n_T = z_B n_T = z_S n_T} = 0.
\]
Here \(\tilde{x} = (\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)\) and \((z_A, z_B, z_S)\) is a variation of \((\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)\). From assertion (ii) of Proposition 5.1, we derive (5.7). Combining (2.1), \(\text{div}_T v_S = 0\), and (5.7), we therefore have system (1.6).

5.3. Inviscid multiphase flow system with tangential surface flow.
Let us construct system (1.3). Assume that \((v_A, v_B, v_S)\) satisfy (1.2). Based on Proposition 2.7, we admit (2.1). Set the action integrals \(A_3\) and \(A_4\) defined by (5.4) and (5.5), respectively. We assume the following energetic variational principle:
\[
\frac{\delta A_3}{\delta \tilde{x}} \bigg|_{z_B n_{\Omega} = 0 \text{ on } \partial \Omega, \; z_A n_T = z_B n_T = z_S n_T} = 0.
\]
Here \(\tilde{x} = (\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)\) and \((z_A, z_B, z_S)\) is a variation of \((\tilde{x}_A, \tilde{x}_B, \tilde{x}_S)\). From assertion (iii) of Proposition 5.1, we derive (5.8). Combining (2.1) and (5.8), we therefore have system (1.6).

6. Conservation and energy laws

Let us study the conservation and energy laws of our systems.

**Proof of Theorem 2.9.** We only show (i). We first derive (1.3). From Proposition 2.7, we find that for \(0 < t < T\),
\[
\frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A(x, t) \, dx + \int_{\Omega_B(t)} \rho_B(x, t) \, dx + \int_{\Gamma(t)} \rho_S(x, t) \, d\mathcal{H}^2_x \right) = 0.
\]
Integrating with respect to \(t\), we have (1.8).

Secondly, we derive (1.9) and (1.10). Applying system (1.3) and Lemma 3.3 we check that
\[
(6.1) \quad \frac{d}{dt} \left( \int_{\Omega_A(t)} \rho_A v_A \, dx + \int_{\Omega_B(t)} \rho_B v_B \, dx + \int_{\Gamma(t)} \rho_S v_S \, d\mathcal{H}^2_x \right)
\]
\[
= \int_{\Omega_A(t)} \rho_A D_t^A v_A \, dx + \int_{\Omega_B(t)} \rho_B D_t^B v_B \, dx + \int_{\Gamma(t)} \rho_S D_t^S v_S \, d\mathcal{H}^2_x
\]
\[
= \int_{\Omega_A(t)} (- \nabla p_A) \, dx + \int_{\Gamma_B(t)} (- \nabla p_B) \, dx
\]
\[
+ \int_{\Gamma(t)} (\nabla_T p_S - \rho_S H_T n_T + \rho_A n_T - \rho_B n_T) \, d\mathcal{H}^2_x.
\]
Using the divergence theorems (Lemma 7.1), we find that
\[
\text{R.H.S. of (6.1)} = \int_{\Omega_A(t)} \text{div}(\rho_A I_{3 \times 3}) \, dx + \int_{\Gamma_B(t)} \text{div}(\rho_B I_{3 \times 3}) \, dx
\]
\[
+ \int_{\Gamma(t)} (\nabla_T p_S - \rho_S H_T n_T + \rho_A n_T - \rho_B n_T) \, d\mathcal{H}^2_x = - \int_{\partial \Omega} \rho_B n_{\Omega} \, d\mathcal{H}^2_x.
\]
Note that \( \int_{\Gamma(t)} \frac{\partial t}{\partial t} p \, d\mathcal{H}_x^2 = - \int_{\Gamma(t)} p_S H_n \, d\mathcal{H}_x^2 \). Integrating with respect to \( t \), we have \( \text{(1.9)} \). In the same manner, we observe that

\[
\text{(6.2)} \quad \frac{d}{dt} \left( \int_{\Omega_A(t)} \frac{1}{2} \rho_A |v_A|^2 \, dx + \int_{\Omega_B(t)} \frac{1}{2} \rho_B |v_B|^2 \, dx + \int_{\Gamma(t)} \frac{1}{2} \rho_S |v_S|^2 \, d\mathcal{H}_x^2 \right)
\]

\[
= \int_{\Omega_A(t)} \rho_A D_t^A v_A \cdot v_A \, dx + \int_{\Omega_B(t)} \rho_B D_t^B v_B \cdot v_B \, dx + \int_{\Gamma(t)} \rho_S D_t^S v_S \cdot v_S \, d\mathcal{H}_x^2
\]

\[
= \int_{\Omega_A(t)} (-\nabla p_A) \cdot v_A \, dx + \int_{\Omega_B(t)} (-\nabla p_B) \cdot v_B \, dx
\]

\[
+ \int_{\Gamma(t)} (-\nabla p_S - p_S H_n - p_A H_n - p_B H_n) \cdot v_S \, d\mathcal{H}_x^2
\]

\[
= \int_{\Omega_A(t)} (\text{div} V_A) p_A \, dx + \int_{\Omega_B(t)} (\text{div} V_B) p_B \, dx + \int_{\Gamma(t)} (\text{div} V_S) p_S \, d\mathcal{H}_x^2.
\]

Note that \( v_B \cdot n_\Omega = 0 \). Integrating with respect to \( t \), we have \( \text{(1.10)} \).

Finally, we derive \( \text{(1.11)} \). Using Lemma \( 3.3 \) and \( 1.4 \), we see that

\[
\text{(6.3)} \quad \frac{d}{dt} \left( \int_{\Omega_A(t)} p_A(\rho_A) \, dx + \int_{\Omega_B(t)} p_B(\rho_B) \, dx + \int_{\Gamma(t)} p_S(\rho_S) \, d\mathcal{H}_x^2 \right)
\]

\[
= - \int_{\Omega_A(t)} p_A(\text{div} v_A) \, dx - \int_{\Omega_B(t)} p_B(\text{div} v_B) \, dx - \int_{\Gamma(t)} p_S(\text{div} v_S) \, d\mathcal{H}_x^2.
\]

Here we used the fact that

\[
\frac{d}{dt} \int_{\Omega_A(t)} p_A(\rho_A) \, dx = \int_{\Omega_A(t)} \{ (D_t^A \rho_A)p_A'(\rho_A) + p_A(\rho_A)(\text{div} v_A) \} \, dx
\]

\[
= \int_{\Omega_A(t)} \{ (-\text{div} v_A)\rho_A p_A'(\rho_A) + p_A(\rho_A)(\text{div} v_A) \} \, dx = - \int_{\Omega_A(t)} p_A(\text{div} v_A) \, dx.
\]

From \( \text{(6.2)} \) and \( \text{(6.3)} \) we see \( \text{(1.11)} \). Therefore, Theorem \( 2.9 \) is proved. \( \square \)

7. Appendix

In Appendix, we prove Lemma \( 3.2 \) and introduce several formulas for integration by parts and a generalized Helmholtz-Weyl Decomposition on a closed surface.

Proof of Lemma \( 3.2 \). Let \( N, \Gamma_m, \Phi_m, U_m, \Psi_m \) be the symbols appearing in \( \text{(3.1)} \). We only show \( \text{(3.3)} \) and \( \text{(3.5)} \). We first attack \( \text{(3.3)} \). Fix \( 0 < t < T \). For each \( m \in \{1, 2, \cdots, N\} \),

\[
\Gamma_m(t) = \{ x \in \mathbb{R}^3; x = \hat{x}_S(\Phi_m(X), t), \ X \in U_m \}.
\]

Since the mapping \( \hat{x}_S(\cdot, t) : \Gamma(0) \to \Gamma(t) \) is bijective, we see that

\[
\Gamma(t) = \bigcup_{m=1}^N \Gamma_m(t)
\]
and that the inverse mapping \( \eta_S = \eta_S(x, t) \) of \( \tilde{x}_S(\xi_S, t) \) exists, that is,

\[
\begin{align*}
\xi_S &= \eta_S(x, t), \\
\eta_S(\tilde{x}_S(\xi_S, t), t) &= \xi_S, \\
\tilde{x}_S(\eta_S(x, t), t) &= x.
\end{align*}
\]

Now we set

\[
\tilde{\Psi}_m(x, t) = \Psi_m(\eta_S(x, t)).
\]

By definition, we find that

\[
\begin{align*}
\text{supp } \tilde{\Psi}_m &\subset \Gamma_m(t), \\
||\tilde{\Psi}_m||_{L^\infty} &= 1, \\
\sum_{m=1}^N \tilde{\Psi}_m &= 1 \text{ on } \Gamma(t).
\end{align*}
\]

Therefore, we see that \( \tilde{\Psi}_m \) is a partition of unity. From \( \tilde{\Psi}_m(x, t) = \Psi_m(\eta_S(x, t)) = \Psi_m(\eta_S(\tilde{x}_S(\xi_S, t), t)) = \Psi_m(\xi_S, t) = \Psi_m(\Phi_m(X)) \), we check that

\[
\int_{\Gamma(t)} f(x, t) \, dH_x^2 = \sum_{m=1}^N \int_{\Gamma_m(t)} \tilde{\Psi}_m(x, t) f(x, t) \, dH_x^2
\]

\[
= \sum_{m=1}^N \int_{\Gamma_m(t)} \Psi_m(\eta_S(x, t)) f(x, t) \, dH_x^2
\]

\[
= \sum_{m=1}^N \int_{U_m} \Psi_m(\Phi_m(X)) f(\tilde{x}_S(\Phi_m(X), t), t) \det \left( \frac{1}{2} \left( \nabla_X \tilde{x}_S(\Phi_m(X), t) \right) \right) \, dX
\]

\[
= \sum_{m=1}^N \int_{U_m} \Psi_m(\Phi_m(X)) f(\tilde{x}_S(\Phi_m(X), t), t) \sqrt{G_S(X, t)} \, dX,
\]

which is (3.3).

Next we prove (3.5). Since \( \Lambda_S(t) \subset \Gamma(t) \), there is \( V_m \subset U_m \) such that

\[
\mathcal{M}_S \cap \Gamma_m = \{ \xi_S \in \mathbb{R}^3; \xi_S = \Phi_m(X), X \in V_m \}.
\]

It is clear that

\[
\Lambda_S(t) \cap \Gamma_m(t) = \{ x \in \mathbb{R}^3; x = \tilde{x}_S(\Phi_m(X), t), X \in V_m \}.
\]

Therefore, we see that for \( X \in U_m \)

\[
1_{V_m}(X) = 1_{\mathcal{M}_S \cap \Gamma_m}(\Phi_m(X)) = 1_{\Lambda_S(t) \cap \Gamma_m}(\tilde{x}_S(\Phi_m(X), t)).
\]
By the previous argument to derive \((3.3)\), we check that
\[
\int_{\Lambda_3(t)} f(x, t) \, d\mathcal{H}_x^2 = \int_{\Gamma(t)} 1_{\Lambda_3(t)}(x) f(x, t) \, d\mathcal{H}_x^2
\]
\[
= \sum_{m=1}^{N} \int_{\Gamma_m(t)} 1_{\Lambda_3(t) \cap \Gamma_m(t)}(x) \tilde{\psi}_m(x, t) f(x, t) \, d\mathcal{H}_x^2
\]
\[
= \sum_{m=1}^{N} \int_{\Gamma_m(t)} 1_{\Lambda_3(t) \cap \Gamma_m(t)}(x) \psi_m(\eta_S(x, t)) f(x, t) \, d\mathcal{H}_x^2
\]
\[
= \sum_{m=1}^{N} \int_{\nu_m \cap \Gamma_m} (\tilde{\psi}_m(X)) \psi_m(\Phi_m(X)) f(\tilde{x}_S(\Phi_m(X), t), t) \sqrt{G_s(X, t)} \, dX.
\]
Thus, we see \((3.3)\). Therefore, Lemma 3.2 is proved. \(\square\)

Lemma 7.1 (Formulas for integration by parts).
Fix \(0 \leq t < T\) and \(j = 1, 2, 3\). Then the following three assertions hold:
(i) For every \(f_A, g_A \in C^1(\bar{\Gamma}(t))\), \(F_A \in [C^1(\bar{\Omega}(t))]^3\),
\[
\int_{\Omega_A(t)} \text{div} F_A \, dx = \int_{\Gamma(t)} F_A \cdot n_\Gamma \, d\mathcal{H}_x^2,
\]
\[
\int_{\Omega_A(t)} (\partial_j f_A) g_A \, dx = -\int_{\Omega_A(t)} f_A(\partial_j g_A) \, dx + \int_{\Gamma(t)} f_A g_A n_\Gamma^j \, d\mathcal{H}_x^2.
\]
(ii) For every \(f_B, g_B \in C^1(\bar{\Omega}(t))\), \(F_B \in [C^1(\bar{\Omega}(t))]^3\),
\[
\int_{\Omega_B(t)} \text{div} F_B \, dx = \int_{\partial \Omega} F_B \cdot n_\Omega \, d\mathcal{H}_x^2 - \int_{\Gamma(t)} F_B \cdot n_\Gamma \, d\mathcal{H}_x^2,
\]
\[
\int_{\Omega_B(t)} (\partial_j f_B) g_B \, dx
\]
\[
= -\int_{\Omega_B(t)} f_B(\partial_j g_B) \, dx + \int_{\partial \Omega} f_B g_B n_j^\Omega \, d\mathcal{H}_x^2 - \int_{\Gamma(t)} f_B g_B n_\Gamma^j \, d\mathcal{H}_x^2.
\]
(iii) For every \(f_S, g_S \in C^1(\Gamma(t))\), \(F_S \in [C^1(\Gamma(t))]^3\),
\[
\int_{\Gamma(t)} \text{div}_\Gamma F_S \, d\mathcal{H}_x^2 = -\int_{\Gamma(t)} H_\Gamma(F_S \cdot n_\Gamma) \, d\mathcal{H}_x^2,
\]
\[
\int_{\Gamma(t)} (\partial_j^\Gamma f_S) g_S \, d\mathcal{H}_x^2 = -\int_{\Gamma(t)} f_S(\partial_j^\Gamma g_S) \, d\mathcal{H}_x^2 - \int_{\Gamma(t)} H_\Gamma f_S g_S n_\Gamma^j \, d\mathcal{H}_x^2.
\]

The proof of surface divergence theorem (the assertion (iii) of Lemma 7.1) can be founded in Simon \([20]\) and Koba \([12]\).

Lemma 7.2 (Generalized Helmholtz-Weyl decomposition). Let \(\Gamma_*\) be a smooth closed 2-dimensional surface in \(\mathbb{R}^3\). Set
\[
C_{\text{div}_\Gamma^\infty}(\Gamma_*) = \{ \varphi \in [C^\infty(\Gamma_*)]^3; \text{div}_\Gamma \varphi = 0 \}.
\]
Let \(F_* \in [C^1(\Gamma_*)]^3\). Assume that for each \(\varphi_* \in C_{\text{div}_\Gamma^\infty}(\Gamma_*)\)
\[
\int_{\Gamma_*} F_* \cdot \varphi_* \, d\mathcal{H}_x^2 = 0.
\]
Then there is $\Pi_* \in C^1(\Gamma_*)$ such that $F_* = \nabla_{\Gamma_*} \Pi_* + \Pi_* H_{\Gamma_*} n_{\Gamma_*}$. Here $n_{\Gamma_*} = n_{\Gamma_*}(x)$ is the unit outer normal vector at $x \in \Gamma_*$ and $H_{\Gamma_*} = -\text{div}_{\Gamma_*} n_{\Gamma_*}$.

The proof of Lemma 7.2 can be founded in Koba-Liu-Giga [14].

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