Nontrivial Periodic Minimizer for Landau-Brazovskii Model with Constraint

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Abstract

We study the Landau-Brazovskii model with constraint, which is in the form of a second order variational problem on the real line. By reducing to handy situations, we find a nontrivial periodic minimal solution. Moreover, the proof is kept as simple and self-contained as possible in our specific case.

Keywords: critical point; calculus of variation; Landau-Brazovskii; minimal energy; minimizer;

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1 Introduction

1.1 A digression on block copolymer

Block copolymer, a synthesized polymer material, has found many applications in industry. It is consisting of multiple sequences of monomer alternating in series with different monomer blocks. The combination of different polymers endows the polymer material with rich properties, which are the key to their important applications. An example of such property manipulation can be seen in poly(urethane) foams, which are used in bedding and upholstery. Poly(urethane), a multiblock copolymer, is characterized by high-temperature resilience and low-temperature flexibility. Another important use of block copolymers is in industrial melt-adhesives. By combining polystyrene with polymers which exhibit rubber-like and adhesive properties, sturdy adhesives can be formed which are activated by heat. The structure of this copolymer utilizes polystyrene blocks on the outside and the rubber block on the inside. When heat is applied, the polystyrene parts melt and allow for limited liquid-like flow. The middle section causes adhesion and after
the temperature drops, the strength of polystyrene is restored. This property, made possible by the combination of polystyrene with other polymers, makes this block copolymer an important adhesive.

The importance of block copolymer has drawn attention to mathematicians. Various mathematical models are developed to explore the properties of block copolymer. Landau-Brazovskii model, among the most popular models, is the simplest yet desirable in capturing the nature of copolymer and thus gained much acceptance. It will be our protagonist in this paper.

1.2 Model specification

Landau-Brazovskii is formulated as below,

$$I_{\infty} (\varphi) = \int_{\mathbb{R}} \left\{ \frac{\xi^2}{2} [\varphi''(t) + \varphi(t)]^2 + \frac{\tau}{2} \varphi^2(t) - \frac{\gamma}{6} \varphi^3(t) + \frac{1}{24} \varphi^4(t) \right\} dt, \quad (1)$$

which models the energy of block copolymer in terms of specific controlling parameters $\xi > 0$, $\tau, \gamma \in \mathbb{R}$.

Block copolymer is determined by its molecular arrangement. Each molecular structure or state, by the language of the model, is represented by $\varphi$. For the purpose of application, the structure or $\varphi$ is required to meet certain practical restrictions,

(i) $\varphi(t)$ is a periodic function with period, say, $\tau = \tau(\varphi) > 0$;

(ii) The structure $\varphi(t)$ is evenly distributed,

$$\int_0^\tau \varphi(t) \, dt = 0;$$

(iii) The block copolymer should behave steadily around the state $\varphi(t)$, to be precise, structure $\varphi(t)$ should not slide easily into another state when the controlling parameters are slightly perturbed, therefore, it is necessary for $\varphi(t)$ to have the minimal energy in some sense.

1.3 Criterion for minimizer

The model posed in the last subsection comes down to a minimizing problem with constraints. But it is not quite the minimizing problem which can be solved straightforwardly using the direct method in the calculus of variation. For one thing, the energy functional is defined on the real line and the period of $\varphi(t)$ is allowed to vary, hence the compactness of the minimizing sequence
is thus lost. On the other hand, the criterion for minimizer is not clearly, since there are functions for which \( I_{\infty}(\varpi) = -\infty \), in which case a straightforward minimization does not make sense. Hence the criterion for minimizer needs to be specifically understood. One way to overcome this difficulty is to minimize the average energy rather than the energy itself. Specifically, consider the problem

$$\lim \inf_{T \to +\infty} \frac{1}{T} I_T(f; \varpi) \to \min,$$

to be noted, it will be shown in Proposition 3 below that the functional \( I_1 \) on \( \mathbb{R} \) may be replaced by one on the half real line \( \mathbb{R}^+ \). But this optimal criterion is much too loose for practical purposes, since it will never fail to find functions which may have different behaviors on compacts but still reach the same minimal average energy level. Another criterion for infinite horizon problems founds its source in optimal control problem in economics, it was introduced by [5] and [13]. It is referred to as "overtaking optimality criterion". As [3], we shall adapt this optimal criterion, such a minimizer will have minimal energy on every compact intervals and minimal average energy on the whole real line, these will be made clear in definition 8. This specification of minimizer has the advantage that its mathematical properties are physically desirable.

The Landau-Brazovskii model (1) has been employed by many chemists and physicists to simulate the block copolymer. However, few literatures have been devoted to the exploration of mathematical nature of the model. In fact, this is a difficult problem and needs in-depth investigation. [3] may be the first effort in this direction, the authors proved existence of a global periodic minimizer of (1) without constaints by localizing. Later [4] and [6] studied the constrained version of model (1). Both [3] and [4] assume a general functional form. A.J.Zaslavski also extended the studies of [3][4] and made an investigation into the structure and turpike properties of the optimal solutions, one may refer to [6] and references therein. In particular, (1) is related to a class of fourth order differential equations, for further reference in this respect, the readers are sent to [11, 9, 10, ?].

In [8], the model (1) with symmetric double well potential is studied and the existence of global periodic minimizer is shown. The author also proved the symmetric property of the minimizer. In addition, without the presence of symmetric property, a non-existence result is given there. In this paper, we study the constrained version of [8]'s model, but we do not assume any symmetric properties of (1). The proof is given following that of [6], however, we are interested in the existence of nontrivial periodic minimizer, a sufficient condition for the existence is given. Since our model takes a
specific functional form, the proof can be as simple and self-contained as possible. We shall see that the condition given in [3], which ensures the existence of nontrivial period minimizer, is also the key to the existence of nontrivial period minimizer of the constrained problem.

1.4 Notations and preparations

Before going further, we introduce some notions.

\begin{equation}
\begin{align*}
f (x, y, z) &= \frac{\xi^2}{2} z^2 - \xi^2 y^2 + \frac{\xi^2 + \tau}{2} x^2 - \frac{\gamma}{6} x^3 + \frac{1}{24} x^4, \\
&= \sum_{n=0}^{3} \frac{1}{2n} \xi_{n} x^{2n},
\end{align*}
\end{equation}

the function is determined by controlling parameters \( \xi, \tau, \gamma \). Denote the energy on a bounded interval \([T_1, T_2]\) by

\[ I_{T_1, T_2} (f; \varpi) = \int_{T_1}^{T_2} f (\varpi, \varpi', \varpi'') \, dt, \]

Note the integrand is independent of \( t \), we have

\[ I_{T_1, T_2} (f; \varpi) = I_{0, T_2 - T_1} (f; \varpi). \]

Therefore, for simplicity of notions, we always use \( I_T (f; \varpi) \) to represent the integration on any bounded interval of length \( T > 0 \). This convention will be adopted as appropriate through out this paper.

Corresponding to \( I_T (f; \varpi) \), we denote by \( J_T (f; \varpi) \) the average energy on bounded intervals of length as \( T > 0 \),

\[ J_T (f; \varpi) = \frac{1}{T} I_T (f; \varpi). \]

We have the minimization problem

\[ \zeta_T (f; x, y) = \inf \{ J_T (f; \varpi) | \varpi \in \mathcal{A}_T (x, y) \}, \]

where

\begin{align*}
\mathcal{A}_T (x, y) &= \{ \varpi \in W^{2,1} (0, T) | \mathcal{V}_\varpi (0) = x, \mathcal{V}_\varpi (T) = y \}, \\
\mathcal{V}_\varpi (s) &= (\varpi (s), \varpi' (s)) \in \mathbb{R}^2.
\end{align*}

Note that \( \mathcal{V}_\varpi (s) \) denotes the vector formed by the value of \( \varpi \) and its derivative at the point \( s \). The notion will be used frequently.

\[ E = W^{2,1}_{loc} (\mathbb{R}^+) \cap W^{1,\infty} (\mathbb{R}^+) \]
Minimization problem $\mathbb{P}^+$ on infinite interval $\mathbb{R}^+$ is denoted by,

$$
\psi^+_f = \inf \left\{ \liminf_{T \to \infty} J_T (f; \varpi) | \varpi \in E \right\},
$$

$\Theta (f)$ is the set of minimizers and $\tilde{\Theta} (f)$ is set of periodic minimizers. Minimization problem with constraint $\mathbb{P}_T (f, a; x, y)$ on finite interval

$$
\zeta_T (f, a; x, y) = \inf \left\{ J_T (f; \varpi) | \varpi \in \mathcal{A}_T (a; x, y) \right\},
$$

where

$$
\mathcal{A}_T (a; x, y) = \{ \varpi \in W^{2,1} (0, T) | [\varpi]_T = a, \mathcal{V}_\varpi (0) = x, \mathcal{V}_\varpi (T) = y \},
$$

$$
[\varpi]_T = \frac{1}{T} \int_0^T \varpi (t) \, dt.
$$

Minimization problem on infinite interval $\mathbb{R}^+$ with constraint $\mathbb{P}^+ (f; a)$ is denoted by

$$
\psi^+_f (a) = \inf \left\{ \liminf_{T \to \infty} J_T (f; \varpi) | \varpi \in E, [\varpi] = a \right\},
$$

where

$$
[\varpi] = \liminf_{T \to \infty} [\varpi]_T = \liminf_{T \to \infty} \frac{1}{T} \int_0^T \varpi (t) \, dt.
$$

$\Theta (f; a)$ is the set of minimizers and $\tilde{\Theta} (f; a)$ is set of periodic minimizers.

**Remark 1** It is shown in [4] that, for any $\varpi$ in the domain of $\mathbb{P} (f; a)$, there is $\tilde{\varpi} \in E$ such that $\lim_{T \to \infty} J_T (f; \tilde{\varpi}), \lim_{T \to \infty} [\tilde{\varpi}]_T$ exist and

$$
\lim_{T \to \infty} J_T (f; \tilde{\varpi}) = \liminf_{T \to \infty} J_T (f; \varpi),
$$

$$
[\tilde{\varpi}] = \lim_{T \to \infty} [\tilde{\varpi}]_T = [\varpi].
$$

In view of this observation, we may replace $\mathbb{P} (f)$ (respectively $\mathbb{P} (f; a)$) with the following

$$
\psi^+_f = \inf \left\{ \lim_{T \to \infty} J_T (f; \varpi) | \varpi \in E, \text{ and } \lim_{T \to \infty} J_T (f; \varpi) \text{ exists} \right\}.
$$

(respectively,

$$
\psi^+_f (a) = \inf \left\{ \lim_{T \to \infty} J_T (f; \varpi) | \varpi \in E, \text{ and } \lim_{T \to \infty} J_T (f; \varpi), \text{ [\varpi] exist, [\varpi] = a} \right\}.
$$

)
Definition 2 Assume \( \{\varpi_n\}_{n=0}^{\infty} \) is a sequence of functions in \( \mathcal{P}(f, a) \) or \( \mathcal{P}(f) \), \( \{A_n\}_{n=0}^{\infty} \) is any sequence of positive values and \( \{k_n\}_{n=0}^{\infty} \) of positive integers, define
\[
\alpha_0 = \frac{A_0}{2}, \quad \alpha_n = k_n A_n + \alpha_{n-1}, \quad n \geq 0.
\]
\[
J_n = \left[ 0, A_n \right], \quad n \geq 0.
\]
\[
L_{-1} = \left[ -\frac{A_0}{2}, \frac{A_0}{2} \right], \quad L_n = (\alpha_{n-1}, \alpha_n], \quad n \geq 0.
\]
\[
\hat{\varpi}_n(x) = \varpi_n(l_n(x - \alpha_n)),
\]
\[
l_n(x) = x - \left[ \frac{x}{A_n} \right] A_n,
\]
where \( l_n(x) \) maps \( \mathbb{R} \) into \( J_n \), the functions \( \hat{\varpi} \) is called a mixture of \( \{\varpi_n\}, \{A_n\}, \{k_n\} \), precisely,
\[
\hat{\varpi} = \text{Mix} \left( \{\varpi_n\}, \{A_n\}, \{k_n\} \right),
\]
\[
\hat{\varpi}(x) = \begin{cases} 
\hat{\varpi}_n(x) & x \in L_n, \quad n \geq 0, \\
\hat{\varpi}_0(x) & x \in L_{-1}, \\
\hat{\varpi}_n(x + \alpha_n + \alpha_{n-1}) & x \in (-\alpha_n, -\alpha_{n-1}], \quad n \geq 1
\end{cases}.
\]

This method of mixture was introduced by [4], the conclusion in remark \( \text{I} \) is proved by the method of mixture. Similarly, the method of mixture is also employed to prove \( \text{(4)} \)
\[
\psi_f = \psi_f^+, \quad \psi_f(a) = \psi_f^+(a),
\]
where \( \psi_f^+, \psi_f^+(a) \) respectively are the minimum for unconstrained and constrained minimization problem on \( \mathbb{R}^+ \), and \( \psi_f, \psi_f(a) \) are those on \( \mathbb{R} \). Since in \( \text{[6]} \), the proof for \( \text{[4]} \) is not provided, we give it here in detail.

Proposition 3 For all \( a \in \mathbb{R} \), \( \psi_f(a) = \psi_f^+(a) \). We also have \( \psi_f = \psi_f^+ \).

Proof. We only prove the first equality, the other being similar. Let \( u(t) \) be an optimal solution to the problem on the whole real line,
\[
\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(u, u', u'') \, dt = \psi_f
\]
Let \( \alpha_n \) be an increasing sequence of positive numbers,
\[
\lim_{n \to +\infty} \frac{1}{2\alpha_n} \int_{-T}^{T} f(u, u', u'') \, dt = \psi_f.
\]
Denote
\[ \beta_n = \frac{1}{2\alpha_n} \int_{-\alpha_n}^{\alpha_n} |f(u, u', u'')| \, dt, \]
let \( k_n \) be such that
\[ \frac{\alpha_{n+1}}{\sum_{l=0}^{n} k_l \alpha_l} \to 0, \quad \frac{\beta_{n+1}}{\sum_{l=0}^{n} k_l \beta_l} \to 0 \]
\[ A_m = 2(\alpha_n + 1) \]
\[ v_m(t) = \begin{cases} u(t) & t \in [-\alpha_n, \alpha_n] \\ u(t) = u'(t) = 0 & t = \pm(\alpha_n + 1) \end{cases} \]
\( v_m(t) \) is a polynomial of degree 3 on \([-\alpha_n - 1, -\alpha_n]\) and on \((\alpha_n, \alpha_n + 1)\).
\[ v = \text{Mix} \left( \{v_m\}, \{A_m\}, \{k_m\} \right)^+. \]

Then
\[ \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(v, v', v'') \, dt = \psi_f, \]

hence
\[ \psi_f \geq \psi_f^+. \]

For the converse, Let \( u(t) \) be an optimal solution to the problem on the positive real line,
\[ \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(u, u', u'') \, dt = \psi_f^+, \]

Denote
\[ \beta_n = \frac{1}{2\alpha_n} \int_{0}^{2\alpha_n} |f(u, u', u'')| \, dt, \]
let \( k_n \) be such that
\[ \frac{\alpha_{n+1}}{\sum_{l=0}^{n} k_l \alpha_l} \to 0, \quad \frac{\beta_{n+1}}{\sum_{l=0}^{n} k_l \beta_l} \to 0 \]
\[ A_m = 2(\alpha_n + 1) \]
\[ v_m(t) = \begin{cases} u(t) & t \in [-\alpha_n, \alpha_n] \\ u(t) = u'(t) = 0 & t = \pm(\alpha_n + 1) \end{cases} \]
\( v_m(t) \) is a polynomial of degree 3 on \([-\alpha_n - 1, -\alpha_n]\) and on \((\alpha_n, \alpha_n + 1)\).
\[ v = \text{Mix} \left( \{v_m\}, \{A_m\}, \{k_m\} \right). \]
Then
\[ \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(v, v', v'') \, dt = \psi_f^+, \]
hence
\[ \psi_f \leq \psi_f^+. \]

\[ \blacksquare \]

Remark 4: In view of this proposition, we need only to consider the problem on the positive half real line. In the following, we shall abandon the notion \( \psi_f^+ \) (resp. \( \psi_f^+ (a) \)) and use \( \psi_f \) (resp. \( \psi_f (a) \)) to indicate the minimizing problems in question.

Definition 5: The differential \( \partial \vartheta (x^*) \) of a convex function \( \vartheta (x) \) at \( x = x^* \) is defined as the set
\[ \{ \lambda \in \mathbb{R} : \vartheta(z) \geq \vartheta(x^*) + \lambda (z - x^*), \forall z \in \mathbb{R} \}. \]

If there exists \( \lambda \in \mathbb{R} \) such that,
\[ \vartheta(z) > \vartheta(x) + \lambda (z - x), \forall z \neq x, \]
then \( x \) is called the exposed point of \( \vartheta \).

2 Main result and open problem

We consider only the case of zero mean constraint (i.e., \( [\omega] = 0 \)), other case being similar.

Recall that \( e \) is an extreme point of a convex set \( K \), if \( e \) does not belong to the segment (excluding the end points) connecting any two points \( e_1, e_2 \in K \). The extreme point to a convex function is defined as extreme point to its graph.

Theorem 6: Let \( f \) be the energy configuration \( (2) \) determined by parameters \( \xi > 0, \tau, \gamma \in \mathbb{R} \) such that \( 0 \) is an exposed point of \( \psi_f (x) \), and
\[ \psi_f (0) < m_f = \inf \{ f(t, 0, 0) \mid t \in \mathbb{R} \}. \]
then there is a nontrivial periodic solution to the constrained minimization problem \( \mathbb{P}(f; 0) \).
Before diving into the proof of the theorem, we would like to put down a few remarks which we formulated into the following open problems.

**Open problems**

The model (1) has been put forward to help simulate copolymer, thus it is expected to have certain conditions for the existence of periodic minimizer that are easy to verify. The result of Theorem 6 may be succinct itself. However, being exposed point is a property that is difficult to validate both from theoretical and numerical perspective. Our problem is whether we can find an alternative condition for being an exposed point. For this, we have the following conjecture.

Let \( h(x) \) be any potential function with double well (not necessarily symmetric), the potential term in (1) is an example of such \( h(x) \),

\[
h(x) = \frac{\tau}{2} x^2 - \frac{\gamma}{6} x^3 + \frac{1}{24} x^4,
\]

Consider the functional

\[
I_T(\varpi) = \int_0^T \frac{\xi^2}{2} \left[ \varpi''(t) + \varpi(t) \right]^2 + h(\varpi(t)) \, dt,
\]

where \( \varpi \in E \) is periodic in \( t \) (the period is different from \( \varpi \) to \( \varpi \)), recalling

\[
E = W^{2,1}_{loc}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+).
\]

If we write

\[
h^*(x) = \frac{\xi^2}{2} x^2 + h(x)
\]

and denote by \( \bar{h}^*(x) \) the convex hull of \( h^*(x) \). Then is it true that

\[
\psi_f^+(a) = \bar{h}^*(a),
\]

where \( \psi_f^+(a) \) is similarly defined as

\[
\psi_f^+(a) = \inf \left\{ \liminf_{T \to \infty} \frac{1}{T} I_T(\varpi) \mid \varpi \in E, [\varpi] = a \right\}
\]

and

\[
[\varpi] = \liminf_{T \to +\infty} \frac{1}{T} \int_0^T \varpi(t) \, dt.
\]

If the above conjecture is true, then the condition of exposed point can be dropped. If it is not true, then are there any other alternative conditions for being an exposed point which are easy to verify?
3 Preliminaries

Theorem 7 The problem on the entire real line has the same minimum as that on the half real line. Therefore, it is enough to consider the problem on the half line.

Definition 8 The function \( \bar{\varpi} \in E \) is a strongly optimal solution to \( P(f) \) if

(i) For any bounded interval \( [T_1, T_2] \),

\[
\zeta_T(f, \nu_{T_1}, \nu_{T_2}) = \frac{1}{T} \int_0^T f(\bar{\varpi}(t), \bar{\varpi}'(t), \bar{\varpi}''(t)) \, dt,
\]

(ii) \( \psi_f = \liminf_{T \to \infty} J_T(f, \bar{\varpi}) \).

the set of all strongly optimal solutions is denoted by \( \Xi(f) \), and that which are periodic is denoted by \( \check{\Xi}(f) \).

[2] showed \( \Xi(f) \) and \( \check{\Xi}(f) \) are all nonempty. If no confusion arises, all minimizers below would mean strongly optimal solutions.

Theorem 9 \( \psi_f(x) \) is a convex function of \( x \).

Let \( x \in \mathbb{R}^2 \), define

\[
\pi_f(x) = \inf \left\{ \liminf_{T \to \infty} [I_T(f; \varpi) - T\psi_f] \middle| \varpi \in W^{2,1}_{\text{loc}}(\mathbb{R}^+), \nu(0) = x \right\}
\]

Theorem 10 For \( T > 0, x, y \in \mathbb{R}^2 \). There are \( \pi_f(x), \theta_T(f, x, y) \) such that \( \zeta_T(f, x, y) \) may be decomposed as follows,

\[
T\zeta_T(f, x, y) = T\psi_f + \pi_f(x) - \pi_f(y) + \theta_T(f, x, y),
\]

where \( \pi_f(x) \) is continuous w.r.t. \( x \), \( \theta_T(f, x, y) \) is a nonnegative continuous function of \( (T, x, y) \), moreover,

\[
\min_{y \in \mathbb{R}^2} \theta_T(f, x, y) = 0, \text{ for any } x \in \mathbb{R}^2.
\]

Here we list some properties that will be referred to in later proofs.

Property A The mapping \( T \mapsto [I_T(f; \varpi) - T\psi_f] \) is bounded on \( \mathbb{R}^+ \).

Remark 11 It is a by-product from the proof of [3], that any function \( \varpi \in E \) satisfying definition \( \Xi(f) \) (i) possesses Property A. This observation was later refined by [7].
**Property B** For any bounded interval $[T_1, T_2]$,

$$I_{T_1,T_2} (f; \varpi) \leq (T_2 - T_1) \psi_f + \pi_f (\mathcal{V}_\varpi (T_1)) - \pi_f (\mathcal{V}_\varpi (T_2)).$$

**Proposition 12** Let $T > 0$, and $\varpi \in W^{2,1} ([0, T])$. Then $I_T (f; \varpi) < \infty$ if and only if $\varpi \in W^{2,2} ([0, T])$.

**Proof.** Assume $\varpi \in W^{2,2} ([0, T])$, it easy to see $I_T (f; \varpi) < \infty$. On the other hand, $\varpi \in W^{2,1} ([0, T])$ and $I_T (f; \varpi) < \infty$ imply $\varpi \in W^{2,2} ([0, T])$. 

The above lemma tells us, the space $W^{2,1} ([0, T])$ is enough for our problem though $W^{2,2} ([0, T])$ seems more natural.

**Theorem 13** ([4]) The function $\psi_f (x)$ is convex w.r.t. $x$. In particular, $\psi_f (x)$ is continuous.

The theorem has an important consequence which is fundamental to the proof of the main result. We state it in the following lemma.

For the function $f (x, y, z)$ in (2), and $\lambda \in \mathbb{R}$, we define the lagrangian,

$$f_\lambda (x, y, z) = f (x, y, z) - \lambda x.$$

**Lemma 14** For the function $\psi_f (x)$, the following relations hold

$$\psi_{f_\lambda} = \psi_f (\eta) - \lambda \eta, \forall \lambda \in \partial \psi_f (\eta), \forall \eta \in \mathbb{R}. \tag{6}$$

$$\Theta (f; \eta) \subset \Theta (f_\lambda), \forall \lambda \in \partial \psi_f (\eta), \forall \eta \in \mathbb{R}. \tag{7}$$

Moreover, if $\eta$ is an exposed point of $\psi_f (x)$, then

$$\Theta (f; \eta) = \Theta (f_\lambda). \tag{8}$$

**Proof.** Since $\psi_f (x)$ is convex, we may define its conjugate function

$$\psi^*_f (z) = \sup_{x \in \mathbb{R}} [xz - \psi_f (x)]. \tag{9}$$
By remark 1, we have

\[- \psi_f \lambda = - \inf_{\varpi} \left\{ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f_{\lambda} (\varpi, \varpi', \varpi'') \, dt \right\} \]

\[- \inf_{\varpi} \left\{ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [f (\varpi, \varpi', \varpi'') - \lambda \varpi] \, dt \right\} \]

\[- \inf_{\varpi} \left\{ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f (\varpi, \varpi', \varpi'') \, dt - \lambda \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \varpi \, dt \right\} \]

\[= \sup_{\varpi} \left\{ \lambda \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \varpi \, dt - \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f (\varpi, \varpi', \varpi'') \, dt \right\} \]

\[= \sup_{\xi \in \mathbb{R}} \sup_{[\varpi] = \xi} \left\{ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [\lambda \xi - f (\varpi, \varpi', \varpi'')] \, dt \right\} \]

\[= \sup_{\xi \in \mathbb{R}} \{ \lambda \xi - \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f (\varpi, \varpi', \varpi'') \, dt \} \]

\[= \sup_{\xi \in \mathbb{R}} \{ \lambda \xi - \psi_f (\xi) \} = \psi_f^* (\lambda). \]

hence

\[\psi_{f_{\lambda}} = - \psi_f^* (\lambda) = \inf_{\xi \in \mathbb{R}} \{ \psi_f (\xi) - \xi \lambda \}.\]

The convexity of \( \psi_f (\xi) \) implies, \( \forall \eta \in \mathbb{R}, \forall \lambda \in \partial \psi_{f_{\lambda}} (\eta), \)

\[\psi_f (z) - \lambda z \geq \psi_f (\eta) - \lambda \eta, \forall z \in \mathbb{R}, \quad \text{(9)}\]

that is, if \( \lambda \in \partial \psi_{f_{\lambda}} (\eta), \) then

\[\inf_{\xi \in \mathbb{R}} \{ \psi_f (\xi) - \xi \lambda \} = \psi_f (\eta) - \lambda \eta.\]

Thus (6) holds and,

\[\psi_{f_{\lambda}} = \psi_f (\eta) - \lambda \eta, \forall \lambda \in \partial \psi_{f_{\lambda}} (\eta), \forall \eta \in \mathbb{R}.\]

Suppose \( \varpi \in \Theta (f, \eta), \)

\[\psi_f (\eta) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f (\varpi, \varpi', \varpi'') \, dt.\]

Noting \([\varpi] = \eta, \) we obtain

\[\psi_f (\eta) - \lambda \eta = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [f (\varpi, \varpi', \varpi'') - \lambda \varpi] \, dt, \quad \text{(10)}\]
the left hand side is nothing else than $\psi_{f\lambda}$, therefore (7) is verified.

In addition, $\eta$ being an exposed point of $\psi_f(x)$ implies the equality in (9) holds only when $z = \eta$. Hence every $\varpi$ solving (10) should verify $[\varpi] = \eta$, that is

$$\Theta (f; \eta) \subset \Theta (f\lambda),$$

showing that (8) is valid in the case of exposed point. ■

Remark 15 The proof of the lemma also indicates

$$\Theta (f\lambda) = \bigcup \{ \Theta (f; \eta), \lambda \in \partial \psi_f (\eta) \}, \forall \lambda \in \mathbb{R}. \quad (11)$$

The following theorem is implied in [3], and explicitly stated in [7], since the proof is not provided there, we give it here in detail.

Theorem 16 Let $\lambda_n$ be a bounded sequence and $\tilde{\varpi}_n$ the optimal periodic solution to problem $P (f_{\lambda_n})$. If the sequence of minimal energy $\psi_{f\lambda_n}$ is bounded. Then there exists a positive constant $C > 0$, such that

$$\| \tilde{\varpi}_n \|_{W^{1,\infty}(\mathbb{R})} \leq C.$$

Proof.

$$f_{\lambda_n}(x, y, z) = \frac{\xi^2}{2} z^2 - \xi^2 y^2 - \lambda_n x + \frac{\tau - \xi^2}{2} x^2 - \frac{\gamma}{6} x^3 + \frac{1}{24} x^4$$

$$= \frac{\xi^2}{2} z^2 - \xi^2 y^2 + \left( x - \frac{\lambda_n}{2} \right)^2 + \frac{\tau - \xi^2}{2} x^2 - \frac{\gamma}{6} x^3 + \frac{1}{24} x^4 - \frac{\lambda_n^2}{4}$$

$$\geq \frac{\xi^2}{2} z^2 - \xi^2 y^2 + \frac{\tau - \xi^2}{2} x^2 - \frac{\gamma}{6} x^3 + \frac{1}{24} x^4 - \frac{\lambda_n^2}{4}$$

$$\geq \frac{\xi^2}{2} z^2 - \xi^2 y^2 + c_1 x^4 - c_2,$$

where $c_1, c_2$ are independent of $n$ and depends on $\xi, \tau$ and $\gamma$ only. Now, substituting $\varpi$ into these functionals, we have, for $\forall \varpi \in W^{2,1} ([T_1, T_2]), W^{2,2} ([T_1, T_2])$

$$I_{T_1, T_2} (f_{\lambda_n}, \varpi)$$

$$= \int_{T_1}^{T_2} f_{\lambda_n} \left( \varpi, \varpi', \varpi'' \right) dt$$

$$\geq \int_{T_1}^{T_2} \left\{ \frac{\xi^2}{2} |\varpi''|^2 - \xi^2 |\varpi'|^2 + c_1 |\varpi|^4 - c_2 \right\},$$
an application of the interpolation inequality shows, there are positive constants \( \tilde{a}_1 \) and \( \tilde{a}_2 \) such that

\[
I_{T_1,T_2}(f_{\lambda_n}, \varpi) \geq \int_{T_1}^{T_2} \left\{ \tilde{a}_1 |\varpi''|^2 + \tilde{a}_2 |\varpi|^4 - \tilde{c}_2 \right\}, \quad \forall \varpi \in W^{2,2}([T_1,T_2]).
\]

Note in the last inequality we employed \( x^\alpha - x + 1 \geq 0 \) (\( \alpha > 1, \forall x \geq 0 \)). The Sobolev imbedding theorem yields

\[
I_{T_1,T_2}(f_{\lambda_n}, \varpi) \geq \int_{T_1}^{T_2} \left\{ a_1 |\varpi'|^2 + a_2 |\varpi|^2 - a_3 \right\}, \quad \forall \varpi \in W^{2,1}([T_1,T_2]),
\]

where \( a_1, a_2 \) and \( a_3 \) are positive constants, note they are independent of \( n \).

\[
\lim_{T_2 \to \infty} \frac{1}{T_2 - T_1} I_{T_1,T_2}(f_{\lambda_n}, \varpi) = \min
\]

then there is \( T' > T_1 \) satisfying

\[
I_{T',T'+1}(f_{\lambda_n}, \varpi) \leq \min + 1,
\]

the process can be proceeded to obtain a sequence \( T_n \to \infty \),

\[
I_{T_n,T_{n+1}}(f_{\lambda_n}, \varpi) \leq \min + 1,
\]

by [3, p171, Remark] and [1],

\[
|\nabla \varpi(t)| \leq M.
\]

Lemma 17 ([7]) Suppose that \( \tilde{\varpi} \) is a periodic, nontrivial minimizer for \( \mathbb{P}(f) \), \( \tau \) being its minimal period. By an appropriate shift of variable, we may suppose

\[
\tilde{\varpi}(0) = \min_{s \in \mathbb{R}} \tilde{\varpi}(s),
\]

then, there is \( \tilde{\tau} \in (0, \tau) \), \( \tilde{\varpi} \) is strictly increasing on \( [0, \tilde{\tau}] \) while strictly decreasing on \( [\tilde{\tau}, \tau] \).
Lemma 18 ([7]) Suppose \( \varpi \in E \) possessing Property A, then for any \( \varsigma \in \Omega (\varpi) \), there is \( \bar{\varpi} \in E \) possessing Property B such that

\[
\{ V_{\bar{\varpi}} (s) | s \in \mathbb{R} \} \subset \Omega (\varpi), V_{\bar{\varpi}} (0) = \varsigma,
\]

where

\[
\Omega (\varpi) = \{ \nu \in \mathbb{R}^2 | \exists s_n \to \infty, V_{\varpi} (s_n) \to \nu \}.
\]

We are now in a position to prove the main theorem.

4 Existence of minimizer

This whole section is devoted to the proof of the main result.

Assume that 0 is an extreme point of \( \psi_f (x) \), then there exists a sequence of exposed points \( \{ \theta_n \} \) of \( \psi_f (x) \) tending to 0. Without loss of generality, we may suppose that the sequence \( \{ \theta_n \} \) is non-increasing and \( \theta_n \neq 0, \forall n \). Since each \( \theta_n \) is an exposed point, we have an optimal periodic solution \( \tilde{\varpi}_n \) to the constrained minimization problem \( \mathbb{P} (f, \theta_n) \). Denote by \( \tau_n \) the minimal positive period of \( \tilde{\varpi}_n \).

The crux of the problem concentrates on the sequence of periods \( \tau_n \). In fact, if we show \( \{ \tau_n \} \) is bounded, then we can easily derive a minimizer from the minimizing sequence \( \{ \tilde{\varpi}_n \} \). The condition (5) not only guarantees the boundedness of \( \{ \tau_n \} \), but also ensures us a nontrivial optimal solution. To prove the theorem, we employ an argument of contradiction. That is, if \( \{ \tau_n \} \) is unbounded, then we will reach the conclusion that, for \( \forall \epsilon > 0 \), there is \( 0 \geq \varrho \geq -\epsilon \) satisfying \( \psi_f (\varrho) = f (\varrho, 0, 0) \), this obviously contradicts condition (5). To make the it easier to follow, we carry out the proof in several steps.

Lemma 19 The sequence \( \{ \tilde{\varpi}_n \}_{n \geq 1} \) does not admit a subsequence \( \{ \tilde{\varpi}_{n_k} \}_{k \geq 1} \) that are all constant.

Proof. Suppose otherwise that \( \{ \tilde{\varpi}_{n_k} \} \) are all constant functions. Since \( [\tilde{\varpi}_{n_k}] = \theta_{n_k} \), then \( \tilde{\varpi}_{n_k} (t) \equiv \theta_{n_k} \). Each function \( \tilde{\varpi}_{n_k} (t) \) is an optimal solution of \( \mathbb{P} (f, \theta_{n_k}) \), hence

\[
\psi_f (\theta_{n_k}) = \frac{1}{\tau_{n_k}} \int_0^{\tau_{n_k}} f (\tilde{\varpi}_{n_k}, \tilde{\varpi}_{n_k}', \tilde{\varpi}_{n_k}'' \theta_{n_k}) \, dt = f (\theta_{n_k}, 0, 0).
\]

But the convex function \( \psi_f (x) \) is continuous at \( x = 0 \), therefore

\[
\psi_f (0) = f (0, 0, 0),
\]
this contradicts \(5\). Thus we may assume all \(\{\tilde{\omega}_n(t)\}_{n \geq 1}\) are not constant functions.

By lemma 14, there is \(\lambda_n \in \partial \psi f(\theta_n)\) such that \(\tilde{\omega}_n \in \tilde{M}(f, \lambda_n)\) and \(f \lambda_n = f(u, u', u'') - \lambda_n u\). Since \(\theta_n\) is non-increasing and \(\psi f(x)\) is a convex function which is finite everywhere, so \(\{\lambda_n\}\) must be non-increasing and has a lower bound, then \(\{\lambda_n\}\) has a limit point \(\lambda^*\). By definition of subdifferential,

\[
\psi_f (z) \geq \psi_f (\theta_n) + \lambda_n (z - \theta_n), \forall z.
\]

therefore

\[
\psi_f (z) \geq \psi_f (0) + \lambda^* z, \forall z,
\]

namely, \(\lambda^* \in \partial \psi_f (0)\).

Lemma 20 The sequence \(\{\tilde{\omega}_n\}\) is locally bounded in \(W^{4,2}(\mathbb{R})\), that is, for any compact interval \([a, b]\), there is a constant \(C > 0\) depending only on \(b - a\), such that

\[
\|\tilde{\omega}_n\|_{W^{4,2}([a, b])} \leq C, \forall n. \tag{12}
\]

Proof. By theorem 16, there is a constant \(C > 0\), such that

\[
\|\tilde{\omega}_n\|_{W^{1,\infty}(\mathbb{R})} \leq C, \forall n. \tag{13}
\]

Indeed, the function \(f(x, y, z)\) allows lower bound of the form

\[
f(x, y, z) \geq a_1 |x|^4 - a_2 |y|^2 + a_3 |z|^2 - a_4,
\]

where \(a_1, a_2, a_3, a_4 \in \mathbb{R}\) are all positive constants. Noting that \(f_{\lambda_n}(x, y, z) = f(x, y, z) - \lambda_n x\) and \(\lambda_n\) is bounded, then \(f_{\lambda_n}\) also admits the lower bound

\[
f_{\lambda_n}(x, y, z) \geq \tilde{a}_1 |x|^4 - \tilde{a}_2 |y|^2 + \tilde{a}_3 |z|^2 - \tilde{a}_4,
\]

where \(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4 \in \mathbb{R}\) are all positive constants independent of \(n\). Therefore

\[
f_{\lambda_n}(\tilde{\omega}_n, \tilde{\omega}'_n, \tilde{\omega}''_n) \geq \tilde{a}_1 |x|^4 - \tilde{a}_2 |y|^2 + \tilde{a}_3 |z|^2 - \tilde{a}_4.
\]

This inequality combining (13) gives (12). Furthermore, since \(\tilde{\omega}_n\) solves the E-L equation

\[
\tilde{\omega}''' + \tilde{\omega}'' + \tilde{\omega} + h'(\tilde{\omega}_n) = 0.
\]

By (8) and the boundedness of \(\lambda_n, \tilde{\omega}_n\) must be bounded in \(W^{4,2}([a, b])\), namely, there is a constant \(C > 0\) \((C\) depends only on the length \(b - a\) of the interval\), satisfying

\[
\|\tilde{\omega}_n\|_{W^{4,2}([a, b])} \leq C, \forall n. \tag{14}
\]
Lemma 21 If $\tau_n \to \infty$, then $\forall \varepsilon > 0, \exists \hat{v}_n^* \in \hat{\Xi}(f_{\lambda^n})$, $\liminf_{s \to \infty} \hat{v}_n^*(s) \geq -\varepsilon$.

Proof. Since the functional $I_{t_1, t_2}(f, \bar{\omega})$ is independent of the time variable $t$, we may assume

$$\hat{\omega}_n(0) = \min_{s \in \mathbb{R}} \hat{\omega}_n(s), \forall n.$$  

Then $\hat{\omega}_n(0) < \theta_n$. By lemma [14] there is $\bar{\tau}_n \in (0, \tau_n)$, $\hat{\omega}_n$ is strictly increasing on $(0, \bar{\tau}_n)$ and strictly decreasing on $(\bar{\tau}_n, \tau_n)$. Consider the set

$$\kappa_n = \{ s \in (0, \tau_n) | \hat{\omega}_n(s) \geq \theta_n - \varepsilon \},$$

by virtue of monotonic structure of $\hat{\omega}_n$, $\kappa_n$ may be written as

$$\kappa_n = [\bar{\kappa}_n, \bar{\tau}_n] \cup [\bar{\tau}_n, \hat{\kappa}_n].$$

Simple calculations show, ($|\kappa_n|$ denotes the length of $\kappa_n$)

$$\theta_n = \frac{1}{\tau_n} \int_0^{\tau_n} \hat{\omega}_n(s) \, ds$$

$$= \frac{1}{\tau_n} \left( \int_{\kappa_n} \hat{\omega}_n(s) \, ds + \int_{(0, \tau_n) \setminus \kappa_n} \hat{\omega}_n(s) \, ds \right)$$

$$\leq C \frac{|\kappa_n|}{\tau_n} + (\theta_n - \varepsilon) \left(1 - \frac{|\kappa_n|}{\tau_n}\right),$$

this implies

$$\liminf_{n \to \infty} \frac{|\kappa_n|}{\tau_n} > 0.$$  

Since otherwise

$$\liminf_{n \to \infty} \frac{|\kappa_n|}{\tau_n} = 0$$

would lead to the contradictory inequality $0 \leq -\varepsilon$.

However, $\tau_n \to \infty$. Hence $|\kappa_n| \to \infty$ and either the length of $[\bar{\kappa}_n, \bar{\tau}_n]$ or $[\bar{\tau}_n, \hat{\kappa}_n]$ tends to infinity as $n \to \infty$. Without no loss of generality, suppose

$$\bar{\tau}_n - \bar{\kappa}_n \to \infty.$$  \hfill (15)

Define

$$\hat{v}_n(t) = \hat{\omega}_n(t + \bar{\kappa}_n),$$  \hfill (16)

Then $\{\hat{v}_n\}$ is locally bounded in $W^{1,2}(\mathbb{R})$ by boundedness (12) of $\{\hat{\omega}_n\}$. Furthermore, since $\hat{\omega}_n \in \hat{\Xi}(f_{\lambda^n})$, we have, for any bounded interval $[t_1, t_2]$,

$$J_{t_2-t_1} (f_{\lambda^n}, \hat{\omega}_n) = \zeta_{t_2-t_1} (f_{\lambda^n}, \mathcal{V}_{\hat{\omega}_n}(t_1), \mathcal{V}_{\hat{\omega}_n}(t_2)).$$
Whence $\tilde{v}_n(t)$ must satisfy the minimal relation

$$J_{t_2-t_1}(f_{\lambda_n}, \tilde{v}_n) = \zeta_{t_2-t_1}(f_{\lambda_n}, \mathcal{V}_{\tilde{v}_n}(t_1), \mathcal{V}_{\tilde{v}_n}(t_2)).$$  (17)

Since $\{\tilde{v}_n\}$ is bounded in $W^{4,2}([t_1, t_2])$, we may suppose $\tilde{v}_n$ converges weakly in $W^{4,2}([t_1, t_2])$ to some element $\tilde{v}^* \in W^{4,2}([t_1, t_2])$. Note the definition of $\tilde{v}_n$ relies on $\varepsilon$, to emphasize this dependence, we will write $\tilde{v}^*$ as $\tilde{v}^*_\varepsilon$. By Sobolev (compact) imbedding theorem,

$$W^{4,2}([t_1, t_2]) \hookrightarrow C^3([t_1, t_2]).$$  (18)

That is, $\tilde{v}_n \to \tilde{v}^*$ in $C^3([t_1, t_2])$. Sending $n$ in (17) to infinity, we obtain

$$J_{t_2-t_1}(f_{\lambda^*}; \tilde{v}^*_\varepsilon) = \zeta_{t_2-t_1}(f_{\lambda^*}, \mathcal{V}_{\tilde{v}^*_\varepsilon}(t_1), \mathcal{V}_{\tilde{v}^*_\varepsilon}(t_2)).$$

therefore, proposition 11 implies that $\tilde{v}^*_\varepsilon \in \tilde{\Xi}(f_{\lambda^*})$.

For any $s > 0$, we have $s \subset [0, \tilde{\tau}_n - \tilde{\kappa}_n]$ as long as $n$ is large enough. Since $\tilde{v}_n$ is strictly increasing on $[0, \tilde{\tau}_n - \tilde{\kappa}_n]$, one must have, by (18),

$$\tilde{v}^*_\varepsilon(s) = \lim_{n \to \infty} \tilde{v}_n(s) \geq \lim_{n \to \infty} (\theta_n - \varepsilon) = -\varepsilon,$$

hence

$$\lim_{s \to \infty} \tilde{v}^*_\varepsilon(s) \geq -\varepsilon$$

We may take this inequality one step further by showing that the limit $\lim_{s \to \infty} \tilde{v}^*_\varepsilon(s)$ actually exists and $\varrho \geq -\varepsilon$.

Note $\tilde{v}^*_\varepsilon \in \tilde{\Xi}(f_{\lambda^*})$ implies it is bounded on the real line. Moreover, $\tilde{v}^*_\varepsilon(s) = \lim_{n \to \infty} \tilde{v}_n(s)$ is non-decreasing on $[0, \tilde{\tau}_n - \tilde{\kappa}_n]$, so $\lim_{s \to \infty} \tilde{v}^*_\varepsilon(s)$ exists and it is denoted by $\varrho$,

$$\varrho = \lim_{s \to \infty} \tilde{v}^*_\varepsilon(s) \geq -\varepsilon.$$  (19)

\[\blacksquare\]

**Lemma 22** The function $\tilde{v}^*_\varepsilon$ is differentiable and $\lim_{s \to +\infty} \tilde{v}^*_\varepsilon'(s) = 0$.

**Proof.** Take $\sigma \geq 1$, define

$$\xi_n(s) = \tilde{v}^*_\varepsilon(s + n), s \in [0, \sigma].$$

Since $\tilde{v}^*_\varepsilon \in \tilde{\Xi}(f)$, $V_{\xi_n}(s)$ is bounded on $[0, \sigma]$. By the same reasoning as lemma 20, we obtain that $\xi_n(s)$ is bounded in $W^{4,2}([0, \sigma])$, one may suppose
\( \xi_n(s) \) converges weakly in \( W^{4,2}[0,\sigma] \) to \( \xi(s) \), the convergence is also valid in the sense of \( C^3([0,\sigma]) \) by Sobolev imbedding theorem, therefore
\[
\xi(s) = \lim_{n \to \infty} \xi_n(s) = \lim_{n \to \infty} \tilde{\nu}_c^*(s + n) = \varrho, \forall s \in [0,\sigma],
\]
namely, \( \xi(s) \equiv \varrho, \forall s \in [0,\sigma] \).

Now it is easy to see that \( \xi_n'(s) \to 0, s \in [0,\sigma] \). For otherwise, there exist \( \delta > 0 \) and \( s_{n_k} \in [0,\sigma] \),
\[
\xi_n'(s_{n_k}) = |\xi_n'(s_{n_k})| \geq \delta. \tag{20}
\]
Assume with no loss that \( s_{n_k} \to \tilde{s} \in [0,\sigma] \). For any \( 0 \leq s_1 < s_2 \leq \sigma \),
\[
|\xi_n'(s_1) - \xi_n'(s_2)| = \left| \int_{s_1}^{s_2} \xi_n''(t) \, dt \right| \\
\leq \int_{s_1}^{s_2} |\xi_n''(t)| \, dt \\
\leq (s_2 - s_1)^{1-\frac{1}{q}} \left( \int_{s_1}^{s_2} |\xi_n''(t)|^q \, dt \right)^{\frac{1}{q}},
\]
This indicates that \( \xi_n'(s) \) is compact in \( C([0,\sigma]) \), therefore
\[
\xi_n'(s_{n_k}) \to \xi'(\tilde{s}),
\]
but \( \xi'(\tilde{s}) = 0 \), which contradicts (20).

Now, since \( \xi_n'(s) \to 0, s \in [0,\sigma] \). Hence
\[
\tilde{\nu}_c''(z) = \xi_n'(z - \sigma(z)) \to 0, z \to +\infty,
\]
where
\[
\sigma(z) = \left\lfloor \frac{z}{\sigma} \right\rfloor = \max \{ n \in \mathbb{N} : n\sigma \leq z < (n+1)\sigma \}.
\]

**Lemma 23** For \( \varrho \) defined early, \( \psi_f(\varrho) = f(\varrho,0,0) \).

**Proof.** By (19),
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{\nu}^*(s) \, ds = \lim_{T \to \infty} \tilde{\nu}^*(T) = \varrho,
\]
but \( \tilde{\nu}^* \) is an optimal solution to \( P(f_{\lambda^*}, \varrho) \), and
\[
\psi_{f_{\lambda^*}} = \psi_f(\varrho) - \lambda^* \varrho. \tag{21}
\]
Hence

$$\psi_f (\varrho) = f (\varrho, 0, 0).$$

Indeed, since $\tilde{\upsilon}^* \in \Xi (f_{\lambda^*})$, we deduce from remark (11) that, there exists a constant $\tilde{C} > 0$ independent of $T$ such that

$$|T\psi_{f_{\lambda^*}} - I_T (f_{\lambda^*} ; \tilde{\upsilon}^*)| \leqslant \tilde{C}, \forall T.$$  

Lemma (18) then ensures an element $\tilde{\upsilon}^* \in E$ possessing Property B and

$$\{ \nu_{\tilde{\upsilon}^*} (s) | s \in \mathbb{R} \} \subset \Omega (\tilde{\upsilon}^*) = \{ (\varrho, 0) \},$$

which implies $\tilde{\upsilon}^* \equiv \varrho$ and

$$\psi_{f_{\lambda^*}} = f_{\lambda^*} (0, 0, 0) = f (\varrho, 0, 0) - \lambda^* \varrho,$$

combining (21), we arrive at

$$\psi_f (\varrho) = f (\varrho, 0, 0).$$

\[\square\]

Lemma 24 \ $\varrho \leqslant 0.$

**Proof.** Previous arguments show $\tilde{\upsilon}^* \in \Theta (f, \varrho) \cap \Theta (f_{\lambda^*})$, it can be deduced from (11) that $\lambda^* \in \partial \psi_f (\varrho)$. Since $\lambda_n$ is non-increasing and tends to $\lambda^*$, and $\lambda^* \in \partial \psi_f (0)$, then we should have $\varrho \leqslant 0$. \[\square\]

Lemma 25 \ $\tau_n \nrightarrow 0.$

**Proof.** By (14) and Sobolev imbedding theorem, one may suppose that

$$\tilde{\omega}_n (t) \Rightarrow \tilde{\omega} (t), \tilde{\omega}_n' (t) \Rightarrow \tilde{\omega}' (t), \tilde{\omega}_n'' (t) \Rightarrow \tilde{\omega}'' (t).$$

Assume $\tau_n \to 0$, take any $t \in \mathbb{R}$, note that

$$\frac{1}{\tau_n} \int_{t}^{t+\tau_n} \tilde{\omega}_n (s) = \theta_n,$$

an application of mean value theorem gives,

$$\tilde{\omega}_n (s_n) = \theta_n, s_n \in (t, t + \tau_n),$$

sending $n \to \infty$, we have

$$\tilde{\omega}_n (t) \Rightarrow \tilde{\omega} (t) = 0, \forall t \in \mathbb{R}.$$
however, \( \tilde{\omega}_n \) is an optimal solution, hence

\[
\psi_f (\theta_n) = \frac{1}{\tau_n} \int_0^{\tau_n} f \left( \tilde{\omega}_n, \tilde{\omega}_n', \tilde{\omega}_n'' \right) \to f (0, 0, 0),
\]

whence we obtain by sending \( n \to \infty \),

\[
\psi_f (0) = f (0, 0, 0),
\]

which contradicts (5). ■

**Final proof of the main theorem.** The proceeding lemmas show that, if \( \tau_n \to \infty \), then for any \( \varepsilon > 0 \), there is \( -\varepsilon \leq \varrho \leq 0 \), such that \( \psi_f (\varrho) = f (\varrho, 0, 0) \), this is an obvious contradiction to (5). Hence \( \tau_n \) is bounded and has at least one limit point \( \tau^* \in (0, \infty) \). Assume without loss of generality \( \tau_n \to \tau^* \). Since \( \omega_n \) is bounded in \( W^{4,2}_{\text{loc}} (\mathbb{R}) \), there is a subsequence

\[
\omega_{n_k} \rightharpoonup \omega^* (W^{4,2}_{\text{loc}} (\mathbb{R})),
\]

then \( \omega_{n_k} \) also converges uniformly on compacts, hence \( \omega^* \) is a periodic function, denote its period by \( \tau^* \). Therefore \( \omega^* \) must be a solution to the minimization problem \( \mathbb{P} (f_{\lambda^*}) \) with

\[
\int_0^{\tau^*} \omega^* (s) \, ds = 0,
\]

whence \( \omega^* (s) \) is also optimal to \( \mathbb{P} (f, 0) \), it is not trivial by (5). ■

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**References**

[1] A.J. Zaslavski. The existence of periodic minimal energy configurations for one-dimensional infinite horizon variational problems arising in continuum mechanics. *Journal of Mathematical Analysis and Applications*, 194:459–476, 1995.

[2] A. Leizarowitz. Infinite horizon autonomous systems with unbounded cost. *Applied Mathematics and Optimization*, 13:19–43, 1985.

[3] A. Leizarowitz and V.J. Mizel. One dimensional infinite-horizon variational problems arising in continuum mechanics. *Archive for Rational Mechanics and Analysis*, 106:161–194, 1989.
[4] B.D.Coleman, M.Marcus, and V.J.Mizel. On the thermodynamics of periodic phases. *Archive for Rational Mechanics and Analysis*, 117:321–347, 1992.

[5] David Gale. On optimal development in a multi-sector economy. *The Review of Economic Studies*, 34:1–18, 1967.

[6] M.Marcus and A.J.Zaslavski. On a class of second order variational problems with constraints. *Israel Journal of Mathematics*, 111:1–28, 1999.

[7] M.Marcus and A.J.Zaslavski. The structure of extremals of a class of second order variational problems. *Annales de l’Institut Henri Poincar. Analyse Non Linaire*, 16(5):593–629, 1999.

[8] Yuanlong Ruan. Notes on a class of one-dimensional landau-brazovsky models. *Archiv der Mathematik*, 93(1):77–86, 2009.

[9] Yuanlong Ruan. Heteroclinic solutions for the extended Fisher-Kolmogorov equation. *Journal of Mathematical Analysis and Applications*, 407.1 (2013): 119-129.

[10] Yuanlong Ruan. Heteroclinic Solutions for Nonautonomous EFK Equations. *Abstract and Applied Analysis*, Vol. 2013. Hindawi Publishing Corporation, 2013.

[11] Yuanlong Ruan. Periodic and homoclinic solutions of a class of fourth order equations. *JOURNAL OF MATHEMATICS*, 4.3 (2011): 2011.

[12] Peletier, Lambertus A., and William C. Troy. Spatial patterns: higher order models in physics and mechanics Vol. 45. Springer 2012.

[13] C. C. von Weizsacker. Existence of optimal programs of accumulation for an infinite horizon. *Econ. Studies*, 32:85–104, 1965.