Fekete-Szegö inequality for Classes of \((p, q)\)-Starlike and \((p, q)\)-Convex Functions

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Abstract: In the present paper, the new generalized classes of \((p, q)\)-starlike and \((p, q)\)-convex functions
are introduced by using the \((p, q)\)-derivative operator. Also, the \((p, q)\)-Bernardi integral operator for
analytic function is defined in the open unit disc \(U = \{z \in \mathbb{C} : |z| < 1\}\). Our aim for these classes is
to investigate the Fekete-Szegö inequalities. Moreover, Some special cases of the established results are
discussed. Further, certain applications of the main results are obtained by applying the \((p, q)\)-Bernardi
integral operator.

Keywords: \((p, q)\)-starlike functions, \((p, q)\)-convex functions, Fekete-Szegö inequality, \((p, q)\)-Bernardi
integral operator.

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1 Introduction

The \(q\)-analysis is a generalization of the ordinary analysis without using the limit notation. The first
application and usage of the \(q\)-calculus was introduced by Jackson [11] and [12]. Moreover, several
applications in various fields of Mathematics and physics (see for details [22], [26]). Recently, there is an
extension of \(q\)-calculus, denoted by \((p, q)\)-calculus which obtained by substituting \(q\) by \(q/p\) in \(q\)-calculus.
The \((p, q)\)-integer was considered by Chakrabarti and Jagannathan [5], see also, [2], [3] and [19]. The two
important geometric properties of analytic functions are starlikeness and convexity. So that, there are
many publications in Geometric Function Theory by using the $q$-differential operator, for example. A generalization of starlike functions $S^*$ were investigated by Ismail et al. [10]. Further, Close-to-convexity of a certain family of $q$- Mittag-Leffler functions were studied by [27]. Also, the coefficient inequality $q$-starlike functions were discussed by [30]. Recently, Coefficient estimates of $q$-starlike and $q$-convex functions were studied by [28]. Further, new subclasses of analytic functions associated with $q$-differential operators were introduced and discussed, see for example [1], [9], [21], [15], [23], [24] and [30]. Motivated by an emerging idea of $(p, q)$-analysis as a generalization of $q$-analysis, in this paper, we extend the idea of $q$-starlikeness and $q$-convexity to $(p, q)$-starlikeness and $(p, q)$-convexity, then we will obtain the Fekete-Szegö inequalities for these classes. Also, we will apply these results on the introduced $(p, q)$-Bernardi integral operator.

We recall some basic notations and definitions from $(p, q)$-calculus, which are used in this paper.

The $(p, q)$-derivative of the function $f$ is defined as [29]:

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z} \quad (z \neq 0; \ 0 < q < p \leq 1);$$

(1.1)

From equation (1.1), it is clear that if $f$ and $g$ are the two functions, then

$$D_{p,q} (f(z) + g(z)) = D_{p,q}f(z) + D_{p,q}g(z)$$

(1.2)

and

$$D_{p,q} (cf(z)) = cD_{p,q}f(z),$$

(1.3)

where $c$ is constant.

We note that $D_{p,q}f(z) \rightarrow f'(z)$ as $p \rightarrow 1$ and $q \rightarrow 1^-$, where $f'$ is the ordinary derivative of the function $f$.

In particular, using equation (1.1), the $(p, q)$-derivative of the function $h(z) = z^n$ is as follows:

$$D_{p,q}h(z) = [n]_{p,q}z^{n-1},$$

(1.4)

where $[n]_{p,q}$ denotes the $(p, q)$-number and is given as:

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \quad (0 < q < p \leq 1).$$

(1.5)

Since, we note that $[n]_{p,q} \rightarrow n$ as $p \rightarrow 1$ and $q \rightarrow 1^-$, therefore in view of equation (1.4), $D_{p,q}h(z) \rightarrow h'(z)$ as $p \rightarrow 1$ and $q \rightarrow 1^-$, where $h'(z)$ denotes the ordinary derivative of the function $h(z)$ with respect to $z$.

Also, the $(p, q)$-integral of the function $f$ on $[0, z]$ is defined as [14]:

$$\int_0^z f(t)d_{p,q}t = (p-q)z \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}z\right),$$

where $\left|\frac{q}{p}\right| < 1$ and $0 < q < p \leq 1$.

In particular, the $(p, q)$-integral of the function $h(z) = z^n$ is given by

$$\int_0^z h(t)d_{p,q}t = \frac{z^{n+1}}{[n+1]_{p,q}},$$

(1.6)

where $n \neq -1$ and $[n]_{p,q}$ is given by equation (1.5).

Again, since $[n+1]_{p,q} \rightarrow n + 1$ as $p \rightarrow 1$ and $q \rightarrow 1^-$, therefore for the same choices of $p$ and $q$, equation (1.6) reduces to $\int_0^z h(t)dt = \frac{z^{n+1}}{n+1}$, which is the ordinary integral of the function $h(z)$ on $[0, z]$. 

2
In this paper, we consider the class $A$ consisting of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1.7)

and analytic in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

Also, using equations (1.2), (1.3) and (1.4), we get the $(p,q)$-derivative of the function $f$, given by equation (1.7) as:

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1} \hspace{1cm} (0 < q < p \leq 1)$$  \hspace{1cm} (1.8)

where $[n]_{p,q}$ is given by equation (1.5).

For the analytic functions $f$ and $g$ in $U$, we say that the function $g$ is subordinate to $f$ in $U$ \[17\], and write

$$g(z) \prec f(z) \hspace{1cm} \text{or} \hspace{1cm} g \prec f$$

if there exists a Schwarz function $w$, which is analytic in $U$ with

$$w(0) = 0 \text{ and } |w(z)| < 1,$$

such that

$$g(z) = f(w(z)) \hspace{1cm} (z \in U).$$  \hspace{1cm} (1.9)

Ma-Minda \[16\] defined the classes of starlike and convex functions, denoted by $S^*(\phi)$ and $C(\phi)$, respectively, by using the subordination principle between certain analytic functions. These subclasses are defined as follows:

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$$  \hspace{1cm} (1.10)

and

$$C(\phi) = \left\{ f \in A : \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z) \right\},$$  \hspace{1cm} (1.11)

where the function $\phi(z)$ is analytic in $U$ with $\Re(\phi(z)) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$. It is clear that $S^*(\phi)$ and $C(\phi)$ are the subclasses of $A$.

The classes of $q$-starlike and $q$-convex functions, denoted by $S_q^*(\phi)$ and $C_q(\phi)$, respectively, are defined by using the subordination principle as \[4\]:

$$S_q^*(\phi) = \left\{ f \in A : zD_qf(z) \prec \phi(z) \right\}$$  \hspace{1cm} (1.12)

and

$$C_q(\phi) = \left\{ f \in A : D_q(zf_qf(z)) \prec \phi(z) \right\},$$  \hspace{1cm} (1.13)

where the function $\phi(z)$ is analytic in $U$ with $\Re(\phi(z)) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$. These classes are the subclasses of $A$.

The Fettke-Szegö problem \[7\] is to find the coefficients estimates for second and third coefficients of functions in any class of analytic function having a specified geometric property. In this paper, we introduce the classes of $(p,q)$-starlike and $(p,q)$-convex functions by using the $(p,q)$-derivative in terms of the subordination principle. Also, we find the Fekete-Szegö inequalities which is obtained by the maximizing the absolute value of the coefficient $|a_3 - a_2^2|$ for the functions belonging to these classes, see for example \[6\], \[8\], \[13\], \[25\] and \[28\]). Further, the $(p,q)$-Bernardi integral operator for analytic functions, is defined in the open unit disc $U$ to discuss the application of the results established in this paper.

3
2 Main Results

First, we define the classes of \((p, q)\)-starlike functions and \((p, q)\)-convex functions, denoted by \(S^{*}_{p,q}(\phi)\) and \(C_{p,q}(\phi)\), respectively, in terms of the subordination principle by taking the \((p, q)\)-derivative in place of \(q\)-derivative in the respective definitions of the classes of \(q\)-starlike and \(q\)-convex functions.

The respective definitions of the classes \(S^{*}_{p,q}(\phi)\) and \(C_{p,q}(\phi)\) are as follows:

**Definition 2.1.** The function \(f \in A\) is said to be \((p, q)\)-starlike if it satisfies the following subordination:

\[
\frac{zD_{p,q}f(z)}{f(z)} \prec \phi(z) \quad (0 < q < p \leq 1),
\]

where the function \(\phi(z)\) is analytic in \(U\) with \(\Re(\phi(z)) > 0\), \(\phi(0) = 1\) and \(\phi'(0) > 0\).

**Definition 2.2.** The function \(f \in A\) is said to be \((p, q)\)-convex if it satisfies the following subordination:

\[
\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)} \prec \phi(z) \quad (0 < q < p \leq 1),
\]

where the function \(\phi(z)\) is analytic in \(U\) with \(\Re(\phi(z)) > 0\), \(\phi(0) = 1\) and \(\phi'(0) > 0\).

![Figure 1: The class \(S^{*}_{0,2,0.5}(1+z/1-z)\) for the complex number \(z = x + iy, \ x, y \in \mathbb{R}\).](image)

**Remark 2.1.** We note that, for \(p = 1\) the classes \(S^{*}_{p,q}(\phi)\) and \(C_{p,q}(\phi)\), reduce to the classes \(S^{*}_{q}(\phi)\) and \(C_{q}(\phi)\), which are defined by equations (1.12) and (1.13), respectively. Again, for \(p = 1\) and \(q \rightarrow 1^-\), the classes \(S^{*}_{p,q}(\phi)\) and \(C_{p,q}(\phi)\) reduce to the classes \(S^{*}(\phi)\), defined by equation (1.10) and \(C(\phi)\), defined by equation (1.11), respectively.

First of all, we need to mention the following lemma [16]:

**Lemma 2.1.** If \(p(z) = 1 + c_1z + c_2z^2 + \ldots\) is a function with \(\Re(p(z)) > 0\) and \(\mu \in \mathbb{C}\), then

\[
|c_2 - \mu c_1^2| \leq 2 \max \{1; |2\mu - 1|\}.
\]

The result is sharp for giving two choices of the function \(p(z)\) as follows:
Figure 2: The class $C_{0.2.0.5} \left( \frac{1 + z}{1 - z} \right)$ for the complex number $z = x + iy$, $x, y \in \mathbb{R}$.

\[
p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}.
\]

Now, we investigate the Fétete-Szegő inequality of the class $S_{p,q}^*(\phi)$ in the following result:

**Theorem 2.1.** Let $\phi(z) = 1 + b_1 z + b_2 z^2 + \ldots$, with $b_1 \neq 0$. If $f$, given by equation (1.7), belongs to the class $S_{p,q}^*(\phi)$, then

\[
|a_3 - \mu a_2^2| \leq \frac{|b_1|}{|3|_{p,q} - 1} \max \left\{ 1; \left| \frac{b_2}{b_1} + \frac{b_1}{|2|_{p,q} - 1} \left( 1 - \frac{|3|_{p,q} - 1}{|2|_{p,q} - 1} \mu \right) \right| \right\},
\]

where $b_1, b_2, \ldots \in \mathbb{R}$, $\mu \in \mathbb{C}$ and $0 < q < p \leq 1$. The result is sharp.

**Proof.** Let $f \in S_{p,q}^*(\phi)$, then in view of Definition 2.1, the function $f$ satisfies the subordination (2.1). Thus, by using equation (1.9), there is a Schwarz function $w$ such that

\[
\frac{z^{D_{p,q}f}(z)}{f(z)} = \phi(w(z)).
\]

We define the function

\[
p(z) = 1 + c_1 z + c_2 z^2 + \ldots
\]

in terms of the function $w(z)$ as:

\[
p(z) = \frac{1 + w(z)}{1 - w(z)},
\]

which gives

\[
w(z) = \frac{p(z) - 1}{p(z) + 1}.
\]

Using equations (2.3) and (2.6), we get

\[
\phi(w(z)) = \phi \left( \frac{c_1 z + c_2 z^2 + \ldots}{2 + c_1 z + c_2 z^2 + \ldots} \right) = \phi \left( \frac{1}{2} \left[ c_1 z + \left( c_2 - \frac{1}{2} c_1 \right) z^2 + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) z^3 + \ldots \right] \right).
\]

Since $\phi(z) = 1 + b_1 z + b_2 z^2 + \ldots$, therefore, equation (2.7) gives

\[
\phi(w(z)) = 1 + \frac{b_1 c_1}{2} z + \left[ \frac{b_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{b_2 c_1^2}{4} \right] z^2 + \ldots.
\]
Now, using equations \((1.7)\) and \((1.8)\), we get
\[
\frac{zD_{p,q}f(z)}{f(z)} = \frac{z + \sum_{n=2}^{\infty} [r_{p,q} a_n z^n]}{z + \sum_{n=2}^{\infty} a_n z^n} = 1 + ([2]_{p,q} - 1) a_2 z + \big( ([3]_{p,q} - 1) a_3 - ([2]_{p,q} - 1) a_2^2 \big) z^2 + \ldots \tag{2.9}
\]

Using equations \((2.8)\) and \((2.9)\) in equation \((2.4)\), then comparing the coefficients of \(z\) and \(z^2\) from the both sides of the resultant equation and simplifying, we get
\[
a_2 = \frac{b_1 c_1}{2([2]_{p,q} - 1)} \tag{2.10}
\]
and
\[
a_3 = \frac{b_1}{2([3]_{p,q} - 1)} \left[ c_2 - \frac{1}{2} \left( 1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \right) c_1^2 \right]. \tag{2.11}
\]

Next, for \(\mu \in \mathbb{C}\), using equations \((2.10)\) and \((2.11)\), we have
\[
a_3 - \mu a_2^2 = \frac{b_1}{2([3]_{p,q} - 1)} \left[ c_2 - \frac{1}{2} \left( 1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q} - 1 - \mu}{[2]_{p,q} - 1} \right) \right) c_1^2 \right]. \tag{2.12}
\]
If we take
\[
v = \frac{1}{2} \left( 1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q} - 1 - \mu}{[2]_{p,q} - 1} \right) \right), \tag{2.13}
\]
then, from equation \((2.12)\), we get
\[
|a_3 - \mu a_2^2| = \frac{|b_1|}{2([3]_{p,q} - 1)} |c_2 - vc_1^2| \tag{2.14}
\]

Hence, by applying Lemma 2.1, equation \((2.14)\), gives the Feteke-Szegő inequality, given by equation \((2.3)\), for the class \(S^*_p(\phi)\).

Further, our result is sharp, that is, the equality holds, when \(p(z) = p_1(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + \ldots\) and equation \((2.4)\), gives
\[
\frac{zD_{p,q}f(z)}{f(z)} = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \phi(z) = 1 + b_1 z + b_2 z^2 + \ldots \tag{2.15}
\]

Then, by comparing equations \((2.8)\) and \((2.15)\), we have \(c_1 = 2\) and \(c_2 = 2\), then equation \((2.12)\) gives the equality sign in the place of inequality in assertion \((2.3)\).

Similarly, for \(p(z) = p_2(z) = \frac{1 + z^2}{1 - z^2} = 1 + 2z^2 + \ldots\), equation \((2.4)\) gives
\[
\frac{zD_{p,q}f(z)}{f(z)} = \phi \left( \frac{p_2(z) - 1}{p_2(z) + 1} \right) = \phi(z^2) = 1 + b_1 z^2 + \ldots \tag{2.16}
\]

Then, by comparing equations \((2.8)\) and \((2.16)\), we have \(c_1 = 0\) and \(c_2 = 2\) and hence equation \((2.12)\) gives the equality sign in the place of inequality in assertion \((2.3)\).

Taking \(p = 1\) and \(q \longrightarrow 1^-\) in Theorem 2.1, we get the following corollary:

**Corollary 2.1.** Let \(\phi(z) = 1 + b_1 z + b_2 z^2 + \ldots\), with \(b_1 \neq 0\). If \(f\) given by equation \((1.7)\) belongs to the class \(S^*_p(\phi)\), then
\[
|a_3 - \mu a_2^2| \leq \frac{|b_1|}{2} \max \left\{ 1; \frac{b_2}{b_1} + b_1 (1 - 2\mu) \right\},
\]
where \(b_1, b_2, \ldots \in \mathbb{R}\) and \(\mu \in \mathbb{C}\). The result is sharp.

**Remark 2.2.** For \(p = 1\), inequality \((2.3)\), gives the Feteke-Szegő inequality \([4]\) for the class \(S^*_p(\phi)\) .
Next, we investigate the Feteke-Szegő inequality for the class $C_{p,q}(\phi)$ in the following result:

**Theorem 2.2.** Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 \neq 0$. If $f$, given by equation (1.7), belongs to the class $C_{p,q}(\phi)$, then

\[
|a_3 - \mu b_2^2| \leq \frac{|b_1|}{|3|_{p,q}([3]_{p,q} - 1)} \max \left\{ 1; \frac{|b_2|}{|b_1|} + \frac{|b_1|}{|2|_{p,q} - 1} \left( 1 - \frac{|3|_{p,q}([3]_{p,q} - 1)}{|2|_{p,q}([2]_{p,q} - 1)} \right) \right\},
\]

(2.17)

where $b_1, b_2, \ldots \in \mathbb{R}$, $\mu \in \mathbb{C}$ and $0 < q < p \leq 1$. The result is sharp.

**Proof.** Let $f \in C_{p,q}(\phi)$, then in view of Definition 2.2 the function $f$ satisfies the subordination (2.2), thus, by using equation (1.9), there is a Schwarz function $w$ such that

\[
\frac{D_{p,q}(zD_{p,q}(f(z)))}{D_{p,q}(f(z))} = \phi(w(z)),
\]

(2.18)

where $w$ is given by equation (2.6) and $\phi(w(z))$ is given by equation (2.8).

Using equations (1.7) and (1.8), we get

\[
\frac{D_{p,q}(zD_{p,q}(f(z)))}{D_{p,q}(f(z))} = \frac{z + \sum_{n=2}^{\infty} |n|_{p,q} a_n z^n}{z + \sum_{n=2}^{\infty} |n|_{p,q} a_n z^n} = 1 + [2]_{p,q}([2]_{p,q} - 1) a_2 z + \left( [3]_{p,q}([3]_{p,q} - 1) a_3 - [2]_{p,q}([2]_{p,q} - 1) a_2 \right) z^2 + \ldots
\]

(2.19)

Comparing the coefficients of $z$ and $z^2$ in equations (2.8) and (2.19) and simplifying, we obtain

\[
a_2 = \frac{b_1 c_1}{2[2]_{p,q}([2]_{p,q} - 1)}
\]

(2.20)

and

\[
a_3 = \frac{b_1}{2[3]_{p,q}([3]_{p,q} - 1)} \left[ c_2 - \frac{1}{2} \left( 1 - \frac{b_2}{b_1} \right) \left( 1 - \frac{|3|_{p,q}([3]_{p,q} - 1)}{|2|_{p,q}([2]_{p,q} - 1)} \right) c_1^2 \right].
\]

(2.21)

Next, for $\mu \in \mathbb{C}$, equations (2.20) and (2.21), give

\[
a_3 - \mu a_2^2 = \frac{b_1}{2[3]_{p,q}([3]_{p,q} - 1)} \left[ c_2 - \frac{1}{2} \left( 1 - \frac{b_2}{b_1} \right) \left( 1 - \frac{|3|_{p,q}([3]_{p,q} - 1)}{|2|_{p,q}([2]_{p,q} - 1)} \right) c_1^2 \right].
\]

(2.22)

If we take

\[
v = \frac{1}{2} \left( 1 - \frac{b_2}{b_1} \right) \left( 1 - \frac{|3|_{p,q}([3]_{p,q} - 1)}{|2|_{p,q}([2]_{p,q} - 1)} \right),
\]

(2.23)

then using equations (2.22) and (2.23), we get

\[
|a_3 - \mu a_2^2| = \frac{|b_1|}{2[3]_{p,q}([3]_{p,q} - 1)} |c_2 - vc_1^2|.
\]

(2.24)

Now, by applying Lemma 2.1, equation (2.24), gives the Feteke-Szegő inequality, given by equation (2.17) for the class $C_{p,q}(\phi)$.

Further, our result is sharp, when $p(z) = p_1(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + \ldots$ and equation (2.18), gives

\[
\frac{D_{p,q}(zD_{p,q}(f(z)))}{D_{p,q}(f(z))} = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \phi(z) = 1 + b_1 z + b_2 z^2 + \ldots
\]

(2.25)

Then, by comparing equations (2.8) and (2.25), we have $c_1 = 2$ and $c_2 = 2$ and hence equation (2.22) gives the equality sign in the place of inequality in assertion (2.17).
Similarly, when \( p(z) = p_2(z) = \frac{1 + z^2}{1 - z^2} = 1 + 2z^2 + \ldots \), equation (2.18) gives
\[
\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)} = \phi \left( \frac{p_2(z) - 1}{p_2(z) + 1} \right) = \phi(z^2) = 1 + b_1z^2 + \ldots,
\]
then, by comparing equations (2.8) and (2.26), we have \( c_1 = 0 \) and \( c_2 = 2 \) and hence equation (2.22) gives the equality sign in the place of inequality in assertion (2.17).

Taking \( p = 1 \) and \( q \to 1^- \) in Theorem 2.2, we get the following corollary [3]:

**Corollary 2.2.** Let \( \phi(z) = 1 + b_1z + b_2z^2 + \ldots \), with \( b_1 \neq 0 \). If \( f \) given by equation (1.7) belongs to the class \( \mathcal{C}(\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \frac{|b_1|}{6} \max \left\{ 1; \left| \frac{b_2}{b_1} + b_1 \left( 1 - \frac{3}{2}\mu \right) \right| \right\},
\]
where \( b_1, b_2, \ldots \in \mathbb{R} \) and \( \mu \in \mathbb{C} \). The result is sharp.

**Remark 2.3.** For \( p = 1 \), inequality (2.17) gives the Fekete-Szegő inequality for the class \( \mathcal{C}_q(\phi) \) [4].

In the next section, we discuss the coefficient bounds of the first and third coefficients of the functions belonging to the classes \( \mathcal{S}_{p,q}^* (\phi) \) and \( \mathcal{C}_{p,q}(\phi) \).

## 3 Coefficient bounds

In this section, we estimate the coefficient bounds for the coefficients of \( z \) and \( z^2 \) of \((p,q)\)-starlike and \((p,q)\)-convex functions.

First, we need to mention the following lemma [16]:

**Lemma 3.1.** If \( p(z) = 1 + c_1z + c_2z^2 + \ldots \) is a function with \( \Re(p(z)) > 0 \), then
\[
|c_2 - vc_1^2| \leq \begin{cases} -4v + 2, & \text{if } v \leq 0; \\ 2, & \text{if } 0 \leq v \leq 1; \\ 4v - 2, & \text{if } v \geq 1. \end{cases}
\]

Also, the above upper bound is sharp, and it can be improved as follows when \( 0 < v < 1 \):
\[
|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \left( 0 < v \leq \frac{1}{2} \right)
\]
and
\[
|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \left( \frac{1}{2} \leq v < 1 \right).
\]

Now, we establish the following result for estimation of the coefficient bound for the functions belonging to the class \( \mathcal{S}_{p,q}^* (\phi) \):
Theorem 3.1. Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let

$$\sigma_1 = \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 - b_1)}{(3|p,q| - 1)b_1^2}, \quad (3.4)$$
$$\sigma_2 = \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 + b_1)}{(3|p,q| - 1)b_1^2}, \quad (3.5)$$
$$\sigma_3 = \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2b_2}{(3|p,q| - 1)b_1^2}. \quad (3.6)$$

If $f$, given by equation (1.7), belongs to the class $S^*_{p,q}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b_2}{3|p,q| - 1} + \frac{b_1^2}{(2|p,q| - 1)\mu} \left( \frac{1}{2|p,q| - 1} - \frac{1 - |2|p,q| - 1 - \mu}{2|p,q| - 1} \right) |a_2|^2, & \text{if } \mu \leq \sigma_1; \\
\frac{b_1}{3|p,q| - 1}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
\frac{b_2}{(2|p,q| - 1)\mu} \left( \frac{1}{2|p,q| - 1} - \frac{1 - |2|p,q| - 1 - \mu}{2|p,q| - 1} \right), & \text{if } \mu \geq \sigma_2.
\end{cases} \quad (3.7)$$

Further, if $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(2|p,q| - 1)\mu}{(3|p,q| - 1)b_1^2} \left( 1 - \frac{|3|p,q| - 1 - \mu}{2|p,q| - 1} \right) |a_2|^2 \leq \frac{b_1}{3|p,q| - 1}. \quad (3.8)$$

and if $\sigma_3 \leq \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(2|p,q| - 1)\mu}{(3|p,q| - 1)b_1^2} \left( 1 - \frac{|3|p,q| - 1 - \mu}{2|p,q| - 1} \right) |a_2|^2 \leq \frac{b_1}{3|p,q| - 1}. \quad (3.9)$$

Proof. For $v \leq 0$, equation (2.13) gives

$$\mu \leq \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 - b_1)}{(3|p,q| - 1)b_1^2}.$$

Let $\frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 - b_1)}{(3|p,q| - 1)b_1^2} = \sigma_1$, then from the above relation, we have $\mu \leq \sigma_1$.

Let $p(z)$ be a function, given by equation (2.5), with $\Re(p(z)) > 0$ and $f(z)$, given by equation (1.7), be a member of the class $S^*_{p,q}(\phi)$, then equation (2.14) holds. Thus using Lemma 3.1 for $v \leq 0$ in equation (2.14), we get

$$|a_3 - \mu a_2^2| \leq \frac{b_1}{2(|3|p,q| - 1)}(-4v + 2),$$

which on using equation (2.13), gives

$$|a_3 - \mu a_2^2| \leq \frac{b_1}{3|p,q| - 1} \left( \frac{b_2}{b_1} + \frac{b_1}{2|p,q| - 1} \left( 1 - \frac{|3|p,q| - 1 - \mu}{2|p,q| - 1} \right) \right), \quad (3.10)$$

where $\mu \leq \sigma_1$.

Simplifying the right hand side of inequality (3.10), we get the first inequality of assertion (3.7).

Again, if we take $0 \leq v \leq 1$, then equation (2.14), gives
\[
\sigma_1 \leq \mu \leq \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 + b_1)}{([3]_{p,q} - 1)b_1^2},
\]
where \(\sigma_1\) is given by equation (3.4).

Let \(\frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 - b_1)}{([3]_{p,q} - 1)b_1^2} = \sigma_2\), then from the above relation, we have \(\sigma_1 \leq \mu \leq \sigma_2\).

Now, using Lemma 3.1 for \(0 \leq v \leq 1\) in equation (2.14), we get
\[
|a_3 - \mu a_2^2| \leq \frac{b_1}{[3]_{p,q} - 1},
\]
which gives the second inequality of assertion (3.7).

Next, if we take \(v \geq 1\), then equation (2.13), gives that \(\mu \geq \sigma_2\).

Now, using Lemma 3.1, for \(v \geq 1\) in equation (2.14), we get
\[
|a_3 - \mu a_2^2| \leq \frac{b_1}{2([3]_{p,q} - 1)}(4v - 2),
\]
which on using equation (2.13), gives
\[
|a_3 - \mu a_2^2| \leq \frac{b_1}{3[3]_{p,q} - 1} \left( -\frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{3[3]_{p,q} - 1 - \mu}{2[2]_{p,q} - 1} \right) \right).
\]
(3.11)

Inequality (3.11) gives the third inequality of assertion (3.7).

Further, if \(0 < v \leq \frac{1}{2}\), then using equation (2.13), we have
\[
0 < \frac{1}{2} \left( 1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{3[3]_{p,q} - 1 - \mu}{2[2]_{p,q} - 1} \right) \right) \leq \frac{1}{2},
\]
which on simplifying, gives
\[
\sigma_1 < \mu \leq \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2b_2}{([3]_{p,q} - 1)b_1^2},
\]
(3.12)
where \(\sigma_1\) is given by equation (3.4).

Let \(\frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2b_2}{([3]_{p,q} - 1)b_1^2} = \sigma_3\), then from relation (3.12), we have \(\sigma_1 < \mu \leq \sigma_3\).

Now, using equations (2.10) and (3.4), we get
\[
|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2| = |a_3 - \mu a_2^2| + \left( \mu - \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2b_2}{([3]_{p,q} - 1)b_1^2} \right) \frac{b_1|c_1|^2}{4([2]_{p,q} - 1)^2},
\]
(3.13)
which on using equation (2.14), we get
\[
|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 = \frac{b_1}{2([3]_{p,q} - 1)} \left( |c_2 - vc_1|^2 + \frac{1}{2} \left( 1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{3[3]_{p,q} - 1 - \mu}{2[2]_{p,q} - 1} \right) \right) |c_1|^2 \right).
\]
Using equation (2.13) in equation (3.14), we get
\[
|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 = \frac{b_1}{[3]_{p,q} - 1} \left( \frac{1}{2} \left( |c_2 - vc_1|^2 + v|c_1|^2 \right) \right).
\]
which in view of inequality (3.2), gives

\[ |a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 \leq \frac{b_1}{|3|_{p,q} - 1}. \]  

(3.15)

Now, using inequality (3.15) in equation (3.13), we get

\[ |a_3 - \mu a_2^2| + \left( \mu - \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 - b_1)}{|3|_{p,q} - 1} \right) |a_2|^2 \leq \frac{b_1}{|3|_{p,q} - 1}, \]

where \( \sigma_1 < \mu \leq \sigma_3. \)

Simplifying the above inequality, we obtain the assertion (3.8).

Similarly, if \( \frac{1}{2} < v < 1, \) then using equation (2.13), we get \( \sigma_3 \leq \mu < \sigma_2, \) where \( \sigma_2 \) and \( \sigma_3 \) are given by equations (3.5) and (3.6), respectively.

Now, using equations (2.10) and (3.5), we get

\[ |a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 = |a_3 - \mu a_2^2| + \left( \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 + b_1)}{|3|_{p,q} - 1} - \mu \right) \frac{c_1^2}{2|p,q| - 1} \]

(3.16)

Using equation (2.14) in equation (3.16), we get

\[ |a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 = \frac{b_1}{2|3|_{p,q} - 1} \left( |c_2 - vc_1|^2 + \frac{1}{2} \left( 1 + \frac{b_2}{b_1} + \frac{b_1}{2|p,q| - 1} \left( 1 - \frac{|3|_{p,q} - 1}{|3|_{p,q} - 1} \right) \right) |c_1|^2 \right), \]

(3.17)

which, on using equation (2.13), gives

\[ |a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 = \frac{b_1}{2|3|_{p,q} - 1} \left( \frac{1}{2} \left( |c_2 - vc_1|^2 + (1 - v)|c_1|^2 \right) \right). \]

(3.18)

Now, since \( \frac{1}{2} < v < 1, \) therefore using inequality (3.3) of Lemma 3.1, equation (3.18) gives

\[ |a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 \leq \frac{b_1}{|3|_{p,q} - 1}. \]

(3.19)

Using inequality (3.19) in equation (3.16), we get

\[ |a_3 - \mu a_2^2| + \left( \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 + b_1)}{|3|_{p,q} - 1} - \mu \right) |a_2|^2 \leq \frac{b_1}{|3|_{p,q} - 1}, \]

where \( \sigma_3 \leq \mu < \sigma_2. \)

Finally, on simplifying the above inequality, we obtain the assertion (3.9).

Taking \( p = 1 \) in Theorem 3.1, we get the following corollary for the class \( S^*_p(\phi): \)

**Corollary 3.1.** Let \( \phi(z) = 1 + b_1z + b_2z^2 \ldots \) with \( b_1 > 0 \) and \( b_2 \geq 0. \) Let

\[ \sigma_1 = \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 - b_1)}{|3|_{p,q} - 1} \],

(3.20)

\[ \sigma_2 = \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2(b_2 + b_1)}{|3|_{p,q} - 1} \],

(3.21)

\[ \sigma_3 = \frac{(2|p,q| - 1)b_1^2 + (2|p,q| - 1)^2b_2}{|3|_{p,q} - 1} \].

(3.22)
If \( f \), given by equation (1.7), belongs to the class \( S^*_p(\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b_2}{[3]_q - 1} + \frac{b_1^2}{[2]_q - 1} \left( \frac{1}{[3]_q - 1} - \frac{\mu}{[2]_q - 1} \right), & \text{if } \mu \leq \sigma_1; \\
\frac{b_1}{[3]_q - 1}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
-\frac{b_2}{[3]_q - 1} - \frac{b_1^2}{[2]_q - 1} \left( \frac{1}{[3]_q - 1} - \frac{\mu}{[2]_q - 1} \right), & \text{if } \mu \geq \sigma_2.
\end{cases}
\] (3.23)

Further, if \( \sigma_1 < \mu \leq \sigma_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{(2)_q - 1)^2}{(3)_q - 1} \left[ b_1 - b_2 - \frac{b_1^2}{[2]_q - 1} \left( 1 - \frac{[3]_q - 1}{[2]_q - 1} \right) |a_2|^2 \right] \leq \frac{b_1}{[3]_q - 1} \] (3.24)

and if \( \sigma_3 \leq \mu < \sigma_2 \), then

\[
|a_3 - \mu a_2^2| + \frac{(2)_q - 1)^2}{(3)_q - 1} \frac{b_1 + b_2 + \frac{b_1^2}{[2]_q - 1} \left( 1 - \frac{[3]_q - 1}{[2]_q - 1} \right) |a_2|^2 \frac{b_1}{[3]_q - 1} \] (3.25)

Next, we obtain the coefficient bound for the functions belonging to the class \( C_{p,q}(\phi) \):

**Theorem 3.2.** Let \( \phi(z) = 1 + b_1 z + b_2 z^2 \ldots \) with \( b_1 > 0 \) and \( b_2 \geq 0 \). Let

\[
\rho_1 = \frac{[2]_p,q([2]_{p,q} - 1)b_2^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 - b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2},
\] (3.26)

\[
\rho_2 = \frac{[2]_p,q([2]_{p,q} - 1)b_2^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 + b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2},
\] (3.27)

\[
\rho_3 = \frac{[2]_p,q([2]_{p,q} - 1)b_2^2 + ([2]_{p,q}[2]_{p,q} - 1)^2b_2}{[3]_{p,q}([3]_{p,q} - 1)b_1^2}.
\] (3.28)

If \( f \), given by equation (1.7), belongs to the class \( C_{p,q}(\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b_2}{[3]_{p,q}([3]_{p,q} - 1)} + \frac{b_1^2}{[2]_{p,q} - 1} \left( \frac{1}{[3]_{p,q}([3]_{p,q} - 1)} - \frac{\mu}{[2]_{p,q}([2]_{p,q} - 1)} \right), & \text{if } \mu \leq \rho_1; \\
\frac{b_1}{[3]_{p,q}([3]_{p,q} - 1)}, & \text{if } \rho_1 \leq \mu \leq \rho_2; \\
-\frac{b_2}{[3]_{p,q}([3]_{p,q} - 1)} - \frac{b_1^2}{[2]_{p,q} - 1} \left( \frac{1}{[3]_{p,q}([3]_{p,q} - 1)} - \frac{\mu}{[2]_{p,q}([2]_{p,q} - 1)} \right), & \text{if } \mu \geq \rho_2.
\end{cases}
\] (3.29)

Further, if \( \rho_1 < \mu \leq \rho_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{[2]_{p,q}([2]_{p,q} - 1)^2}{[3]_{p,q}([3]_{p,q} - 1)b_1^2} \left[ b_1 - b_2 - \frac{b_1^2}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q}([3]_{p,q} - 1)}{[2]_{p,q}([2]_{p,q} - 1)} \mu \right) |a_2|^2 \right] \leq \frac{b_1}{[3]_{p,q}([3]_{p,q} - 1)},
\] (3.30)

and if \( \rho_3 \leq \mu < \rho_2 \), then

\[
|a_3 - \mu a_2^2| + \frac{[2]_{p,q}([2]_{p,q} - 1)^2}{[3]_{p,q}([3]_{p,q} - 1)b_1^2} \left[ b_1 + b_2 + \frac{b_1^2}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q}([3]_{p,q} - 1)}{[2]_{p,q}([2]_{p,q} - 1)} \mu \right) |a_2|^2 \right] \leq \frac{b_1}{[3]_{p,q}([3]_{p,q} - 1)}.\] (3.31)
Proof. For \( v \leq 0 \), equation (2.23) gives
\[
\mu \leq \frac{[2]_{p,q} ([2]_{p,q} - 1) b_1^2 + ([2]_{p,q} [2]_{p,q} - 1)^2 (b_2 - b_1)}{[3]_{p,q} ([3]_{p,q} - 1) b_1^2}.
\]
Let \( [2]_{p,q} ([2]_{p,q} - 1) b_1^2 + ([2]_{p,q} [2]_{p,q} - 1)^2 (b_2 - b_1) = \rho_1 \), then from the above relation we have \( \mu \leq \rho_1 \).

Let \( p(z) \) be a function given by equation (2.5) with \( \Re (p(z)) > 0 \) and \( f(z) \), given by equation (1.7), be a member of the class \( C_{p,q}(\phi) \), then equation (2.24) holds. Thus, using Lemma 3.1, for \( v \leq 0 \), in equation (2.24), we get
\[
|a_3 - \mu a_2^2| \leq \frac{b_1}{2 [3]_{p,q} ([3]_{p,q} - 1)} (-4v + 2),
\]
which on using equation (2.23), gives
\[
|a_3 - \mu a_2^2| \leq \frac{b_1}{[3]_{p,q} ([3]_{p,q} - 1) ([2]_{p,q} ([2]_{p,q} - 1) \mu)} \left( \frac{b_2}{b_1} + \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q} ([3]_{p,q} - 1)}{[2]_{p,q} ([2]_{p,q} - 1) \mu} \right) \right), \tag{3.32}
\]
where \( \mu \leq \rho_1 \).

Inequality (3.32) gives the first inequality of assertion (3.29).

Again, if we take \( 0 \leq v \leq 1 \), then equation (2.23) gives
\[
\rho_1 \leq \mu \leq \frac{[2]_{p,q} ([2]_{p,q} - 1) b_1^2 + ([2]_{p,q} [2]_{p,q} - 1)^2 (b_2 + b_1)}{[3]_{p,q} ([3]_{p,q} - 1) b_1^2}.
\]
Let \( [2]_{p,q} ([2]_{p,q} - 1) b_1^2 + ([2]_{p,q} [2]_{p,q} - 1)^2 (b_2 + b_1) = \rho_2 \), then \( \rho_1 \leq \mu \leq \rho_2 \), where \( \rho_1 \) is given by equation (3.26).

Now, using Lemma 3.1, for \( 0 \leq v \leq 1 \), in equation (2.24), we get
\[
|a_3 - \mu a_2^2| \leq \frac{b_1}{[3]_{p,q} ([3]_{p,q} - 1)},
\]
which gives the second inequality of assertion (3.29).

Next, if we take \( v \geq 1 \), then equation (2.23) gives that \( \mu \geq \rho_2 \).

Now, using Lemma 3.1, for \( v \geq 1 \) in equation (2.24), we get
\[
|a_3 - \mu a_2^2| \leq \frac{|b_1|}{2 [3]_{p,q} ([3]_{p,q} - 1)} (4v - 2),
\]
which on using equation (2.23) gives
\[
|a_3 - \mu a_2^2| \leq \frac{b_1}{[3]_{p,q} ([3]_{p,q} - 1)} \left( -\frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q} ([3]_{p,q} - 1)}{[2]_{p,q} ([2]_{p,q} - 1) \mu} \right) \right), \tag{3.33}
\]
where \( \mu \geq \rho_2 \).

Simplifying the right hand side of inequality (3.33), we get the third inequality of assertion (3.29).

Further, if \( 0 < v \leq \frac{1}{2} \), then using equation (2.23), we have
\[
0 < \frac{1}{2} \left( 1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q} ([3]_{p,q} - 1)}{[2]_{p,q} ([2]_{p,q} - 1) \mu} \right) \right) \leq \frac{1}{2},
\]
which on simplifying, gives

$$\rho_1 < \mu \leq \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2b_2}{[3]_{p,q}([3]_{p,q} - 1)b_1^2}. \quad (3.34)$$

Let \( \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2b_2}{[3]_{p,q}([3]_{p,q} - 1)b_1^2} = \rho_3 \), then from (3.34), we have \( \rho_1 < \mu \leq \rho_3 \), where \( \rho_1 \) is given by (3.26).

Now, using equations (2.20) and (3.26), we get

$$|a_3 - \mu a_2^2| + (\mu - \rho_1)|a_2|^2 = |a_3 - \mu a_2^2| + \left( \mu - \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 - b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2} \right) \frac{b_1^2|c_1|^2}{4[2]_{p,q}([2]_{p,q} - 1)^2}, \quad (3.35)$$

which on using equation (2.24), we get

$$|a_3 - \mu a_2^2| + (\mu - \rho_1)|a_2|^2 = b_1 \frac{1}{2[3]_{p,q}([3]_{p,q} - 1)} \left( |\epsilon_2 - v\epsilon_1|^2 + \frac{1}{2} \left( 1 - \frac{b_2}{b_1} \right) \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q}([3]_{p,q} - 1)}{[2]_{p,q}([2]_{p,q} - 1)\mu} \right) |c_1|^2 \right), \quad (3.36)$$

Again, using equation (2.23) in equation (3.36), we have

$$|a_3 - \mu a_2^2| + (\mu - \rho_1)|a_2|^2 = b_1 \frac{1}{2[3]_{p,q}([3]_{p,q} - 1)} \left( |\epsilon_2 - v\epsilon_1|^2 + v|c_1|^2 \right), \quad (3.37)$$

which in view of inequality (3.2) gives

$$|a_3 - \mu a_2^2| + (\mu - \rho_1)|a_2|^2 \leq b_1 \frac{1}{2[3]_{p,q}([3]_{p,q} - 1)}. \quad (3.38)$$

Now, using equation (2.20) and inequality (3.37) in equation (3.35), we get

$$|a_3 - \mu a_2^2| + \left( \mu - \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 - b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2} \right) |a_2|^2 \leq b_1 \frac{1}{[3]_{p,q}([3]_{p,q} - 1)}. \quad (3.39)$$

Simplifying the above inequality, we obtain the assertion (3.30).

Similarly, if \( \frac{1}{2} \leq v < 1 \), then using equation (2.23), we get \( \rho_3 < \mu < \rho_2 \).

Now, using equations (2.20) and (3.27), we get

$$|a_3 - \mu a_2^2| + (\rho_2 - \mu)|a_2|^2 = \frac{|a_3 - \mu a_2^2| + \left( \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 + b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2} - \mu \right) \frac{b_1^2|c_1|^2}{4[2]_{p,q}([2]_{p,q} - 1)^2}}{2[3]_{p,q}([3]_{p,q} - 1)} \left( |\epsilon_2 - v\epsilon_1|^2 + \frac{1}{2} \left( 1 + \frac{b_2}{b_1} \right) \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q}([3]_{p,q} - 1)}{[2]_{p,q}([2]_{p,q} - 1)\mu} \right) |c_1|^2 \right), \quad (3.38)$$

Using equation (2.24) in equation (3.38) and then simplifying, we get

$$|a_3 - \mu a_2^2| + (\rho_2 - \mu)|a_2|^2 = b_1 \frac{1}{2[3]_{p,q}([3]_{p,q} - 1)} \left( |\epsilon_2 - v\epsilon_1|^2 + \frac{1}{2} \left( 1 + \frac{b_2}{b_1} \right) \frac{b_1}{[2]_{p,q} - 1} \left( 1 - \frac{[3]_{p,q}([3]_{p,q} - 1)}{[2]_{p,q}([2]_{p,q} - 1)\mu} \right) |c_1|^2 \right), \quad (3.39)$$

which on using equation (2.23), gives

$$|a_3 - \mu a_2^2| + (\rho_2 - \mu)|a_2|^2 = \frac{|b_1|}{2[3]_{p,q}([3]_{p,q} - 1)} \left( \frac{1}{2} \left( |\epsilon_2 - v\epsilon_1|^2 + (1 - v)|c_1|^2 \right) \right). \quad (3.39)$$
Now, since $\frac{1}{2} \leq v < 1$, therefore using inequality (3.3) of Lemma 3.1 in equation (3.39), we get
\[
|a_3 - \mu a_2^2| + (\rho_2 - \mu)|a_2|^2 \leq \frac{b_1}{|3|_{p,q}(|3|_{p,q} - 1)}.
\] (3.40)

Using inequality (3.40) in equation (3.38), gives
\[
|a_3 - \mu a_2^2| + \left(\frac{2|q|(|2|q - 1)b_1^2 + (|2|q^2 - 1)^2(b_2 - b_1)}{|3|_{p,q}(|3|_{p,q} - 1)b_1^2} - \mu\right)|a_2|^2 \leq \frac{b_1}{|3|_{p,q}(|3|_{p,q} - 1)},
\]
where $\rho_3 \leq \mu < \rho_2$.

Finally, on simplifying the above inequality, we obtain assertion (3.31).

For $p = 1$, Theorem 2.2, gives the following corollary for the class $C_q(\phi)$:

**Corollary 3.2.** Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let
\[
\rho_1 = \frac{|2|q|(|2|q - 1)b_1^2 + (|2|q^2 - 1)^2(b_2 - b_1)}{|3|_{q}(|3|_{q} - 1)b_1^2},
\]
\[
\rho_2 = \frac{|2|q|(|2|q - 1)b_1^2 + (|2|q^2 - 1)^2(b_2 + b_1)}{|3|_{q}(|3|_{q} - 1)b_1^2},
\]
\[
\rho_3 = \frac{|2|q|(|2|q - 1)b_1^2 + (|2|q^2 - 1)^2b_2}{|3|_{p,q}(|3|_{p,q} - 1)b_1^2}.
\]

If $f$, given by equation (1.7), belongs to the class $C_q(\phi)$, then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b_2}{|3|_{q}(|3|_{q} - 1)} + \frac{b_1^2}{|2|_q - 1} \left(\frac{1}{|3|_{q}(|3|_{q} - 1)} - \frac{\mu}{|2|_q^2(|2|_q - 1)}\right), & \text{if } \mu \leq \rho_1; \\
\frac{b_1}{|3|_{q}(|3|_{q} - 1)}, & \text{if } \rho_1 \leq \mu \leq \rho_2; \\
\frac{-b_2}{|3|_{q}(|3|_{q} - 1)} + \frac{b_1^2}{|2|_q - 1} \left(\frac{1}{|3|_{q}(|3|_{q} - 1)} - \frac{\mu}{|2|_q^2(|2|_q - 1)}\right), & \text{if } \mu \geq \rho_2.
\end{cases}
\] (3.44)

Further, if $\rho_1 < \mu \leq \rho_3$, then
\[
|a_3 - \mu a_2^2| + \frac{|2|q|(|2|q - 1)^2}{|3|_{q}(|3|_{q} - 1)b_1^2} \left(b_1 - b_2 - \frac{b_1^2}{|2|_q - 1} \left(1 - \frac{|3|_{q}(|3|_{q} - 1)}{|2|_q^2(|2|_q - 1)}\right)\right)|a_2|^2 \leq \frac{b_1}{|3|_{q}(|3|_{q} - 1)}.\] (3.45)

and if $\rho_3 \leq \mu < \rho_2$, then
\[
|a_3 - \mu a_2^2| + \frac{|2|q|(|2|q - 1)^2}{|3|_{q}(|3|_{q} - 1)b_1^2} \left(b_1 + b_2 + \frac{b_1^2}{|2|_q - 1} \left(1 - \frac{|3|_{q}(|3|_{q} - 1)}{|2|_q^2(|2|_q - 1)}\right)\right)|a_2|^2 \leq \frac{b_1}{|3|_{q}(|3|_{q} - 1)}.\] (3.46)

In the next section, we discuss some applications of the results, established in Sections 1. and 2. .

4 Application

We recall that the Bernardi integral operator $F_c$ is given by [2]:
\[
F_c(f(z)) = \frac{1 + c}{z^c} \int_0^z t^{c-1} f(t)dt \quad (f \in \mathcal{A}, \ c > -1).
\]
Now, in view of above equation, we introduce the \((p,q)\)-Bernardi integral operator \(L(z)\) as:

\[
L(z) := F_{c,p,q}(f(z)) = \frac{[1 + c]_{p,q}}{z^\beta} \int_0^z t^{e-1} f(t) d_{p,q} t \quad c = 0, 1, 2, 3, \ldots \quad (4.1)
\]

Let \(f \in A\), then using equations (1.6) and (1.8), we obtain the following power series for the function \(L\) in the open unit disc \(U = \{ z \in \mathbb{C} : |z| < 1 \}:

\[
L(z) = z + \sum_{n=2}^\infty \frac{[1 + c]_{p,q}}{[n + c]_{p,q}} a_n z^n \quad (c = 1, 2, 3, \ldots; 0 < q < p \leq 1; f \in A). \quad (4.2)
\]

It is clear that \(L(z)\) is analytic in open disc \(U\).

We note that, by taking \(p = 1\) in equation (4.1), we get \(q\)-Bernardi integral operator [18].

Let

\[
L_n = \frac{[1 + c]_{p,q}}{[n + c]_{p,q}} \quad n \geq 1. \quad (4.3)
\]

Now, applying the Theorem 2.1 to the function \(L(z)\), defined by equation (4.2), we get the following application of the theorem:

**I.** Let \(\phi(z) = 1 + b_1 z + b_2 z^2 + \ldots\) with \(b_1 \neq 0\). If \(\mathcal{L}\), given by equation (4.3), belongs to the class \(\mathcal{S}_{p,q}^*(\phi)\), then

\[
|a_3 - \mu a_2| \leq \frac{|b_1|}{|3|_{p,q} L_3 - 1} \max \left\{ \frac{b_3}{b_1}, \frac{b_2}{b_1} + \frac{b_1}{2|q|_{p,q} L_2 - 1} \right\} \left( 1 - \frac{|3|_{p,q} L_3 - 1}{|2|_{p,q} L_2 - 1} \right) \phi^2(z),
\]

where \(L_2\) and \(L_3\) are given by equation (4.3), \(b_1, b_2, \ldots \in \mathbb{R}, \mu \in \mathbb{C}, 0 < q < p \leq 1\).

Next, applying the Theorem 2.2 to the function \(L(z)\), defined by equation (4.2), we get the following application of the theorem:

**II.** Let \(\phi(z) = 1 + b_1 z + b_2 z^2 + \ldots\) with \(b_1 \neq 0\). If \(\mathcal{L}\), given by equation (4.2), belongs to the class \(\mathcal{C}_{p,q}^*(\phi)\), then

\[
|a_3 - \mu a_2| \leq \frac{|b_1|}{|3|_{p,q} L_3 (|2|_{p,q} L_2 - 1)} \max \left\{ \frac{b_3}{b_1}, \frac{b_2}{b_1} + \frac{b_1}{2|q|_{p,q} - 1} L_2 \right\} \left( 1 - \frac{|3|_{p,q} L_3 (|3|_{p,q} L_3 - 1)}{|2|_{p,q} L_2 (|2|_{p,q} L_2 - 1)} \right) \phi(z),
\]

where \(L_2\) and \(L_3\) are given by equation (4.3), \(b_1, b_2, \ldots \in \mathbb{R}, \mu \in \mathbb{C}, 0 < q < p \leq 1\).

Further, applying the Theorem 3.1 to the function \(L(z)\), defined by equation (4.2), we get the following application of the theorem:

**III.** Let \(\phi(z) = 1 + b_1 z + b_2 z^2 + \ldots\) with \(b_1 > 0\) and \(b_2 \geq 0\). Let

\[
\sigma_1 = \frac{([2]_{p,q} L_2 - 1)b_2^2 + ([2]_{p,q} L_2 - 1)^2(b_2 - b_1)}{([3]_{p,q} L_3 - 1)b_1^2},
\]

\[
\sigma_2 = \frac{([2]_{p,q} L_2 - 1)b_2^2 + ([2]_{p,q} L_2 - 1)^2(b_2 + b_1)}{([3]_{p,q} L_3 - 1)b_1^2},
\]

\[
\sigma_3 = \frac{([2]_{p,q} L_2 - 1)b_2^2 + ([2]_{p,q} L_2 - 1)^2b_2}{([3]_{p,q} L_3 - 1)b_1^2}.
\]
If $L$, given by equation (4.2), belongs to the class $S_{p,q}^*(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b_2}{|3|_{p,q} - 1} + \frac{b_2^2}{|2|_{p,q} - 1} \left( \frac{1}{|3|_{p,q} - 1} - \frac{\mu}{|2|_{p,q} - 1} \right), & \text{if } \mu \leq \sigma_1; \\
\frac{b_1}{|3|_{p,q} - 1}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
\frac{-b_2}{|3|_{p,q} - 1} - \frac{b_2^2}{|2|_{p,q} - 1} \left( \frac{1}{|3|_{p,q} - 1} - \frac{\mu}{|2|_{p,q} - 1} \right), & \text{if } \mu \geq \sigma_2.
\end{cases}$$  

(4.4)

Further, if $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| \leq \left( \frac{2|p,q|L_2 - 1}{|3|_{p,q}L_3 - 1} \frac{1}{b_2^2} \left( 1 - \frac{|3|_{p,q}L_3 - 1}{|2|_{p,q}L_2 - 1} \frac{1}{\mu} \right) \right) |a_2|^2 \leq \frac{b_1}{|3|_{p,q}L_3 - 1},$$

and if $\sigma_3 \leq \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| \leq \left( \frac{2|p,q|L_2 - 1}{|3|_{p,q}L_3 - 1} \frac{1}{b_2^2} \left( 1 - \frac{|3|_{p,q}L_3 - 1}{|2|_{p,q}L_2 - 1} \frac{1}{\mu} \right) \right) |a_2|^2 \leq \frac{b_1}{|3|_{p,q}L_3 - 1},$$

where $L_2$ and $L_3$ are given by equation (4.3).

Finally, applying the Theorem 3.1 to the function $L(z)$, defined by equation (4.2), we get the following application of the theorem:

IV. Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let

$$\rho_1 = \frac{2|p,q|(2|p,q|L_2 - 1)b_2^2 + (2|p,q|L_2|2|p,q|L_2 - 1)^2(b_2 - b_1)}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)b_2^2},$$

$$\rho_2 = \frac{2|p,q|L_2(2|p,q|L_2 - 1)b_2^2 + (2|p,q|L_2|2|p,q|L_2 - 1)^2(b_2 + b_1)}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)b_2^2},$$

$$\rho_3 = \frac{2|p,q|L_2(2|p,q|L_2 - 1)b_2^2 + (2|p,q|L_2|2|p,q|L_2 - 1)^2b_2}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)b_2^2}.$$

If $L$, given by equation (4.2), belongs to the class $C_{p,q}^*(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b_2}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)} + \frac{b_2^2}{|2|_{p,q}L_2 - 1} \left( \frac{1}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)} - \frac{\mu}{|2|_{p,q}L_2(|2|_{p,q}L_2 - 1)} \right), & \text{if } \mu \leq \rho_1; \\
\frac{b_1}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)}, & \text{if } \rho_1 \leq \mu \leq \rho_2; \\
\frac{-b_2}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)} - \frac{b_2^2}{|2|_{p,q}L_2 - 1} \left( \frac{1}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)} - \frac{\mu}{|2|_{p,q}L_2(|2|_{p,q}L_2 - 1)} \right), & \text{if } \mu \geq \rho_2.
\end{cases}$$

Further, if $\rho_1 < \mu \leq \rho_3$, then

$$|a_3 - \mu a_2^2| \leq \left( \frac{2|p,q|L_2(2|p,q|L_2 - 1)^2}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)b_2^2} \left( b_1 - b_2 - \frac{b_2^2}{|2|_{p,q}L_2 - 1} \left( 1 - \frac{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)}{|2|_{p,q}L_2(|2|_{p,q}L_2 - 1)} \frac{1}{\mu} \right) \right) \right) |a_2|^2 \leq \frac{b_1}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)}.$$

and if $\rho_3 \leq \mu < \rho_2$, then

$$|a_3 - \mu a_2^2| \leq \left( \frac{2|p,q|L_2(2|p,q|L_2 - 1)^2}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)b_2^2} \left( b_1 + b_2 + \frac{b_2^2}{|2|_{p,q}L_2 - 1} \left( 1 - \frac{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)}{|2|_{p,q}L_2(|2|_{p,q}L_2 - 1)} \frac{1}{\mu} \right) \right) \right) |a_2|^2 \leq \frac{b_1}{|3|_{p,q}L_3(|3|_{p,q}L_3 - 1)},$$

where $L_2$ and $L_3$ are given by equation (4.3).
5 Conclusion

In our results, by using the \((p,q)\)-derivative operator, the generalized classes of \((p,q)\)-starlike and \((p,q)\)-convex functions were introduced which are a generalization of starlike and convex functions. Also, the \((p,q)\)-Bernardi integral operator for analytic functions were defined in the open unit disc \(U = \{z \in \mathbb{C} : |z| < 1\}\). In our main results, the Fekete-Szegő inequalities. For the validity of our results can be applicable for our introduced \((p,q)\)-Bernardi integral operator. Moreover, Some special cases of the results were established. Further, certain applications of the main results for the \((p,q)\)-starlike and \((p,q)\)-convex functions were obtained by applying the \((p,q)\)-Bernardi integral operator.

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