A threshold approach to connected domination

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Abstract

A connected dominating set in a graph is a dominating set of vertices that induces a connected subgraph. We introduce and study the connected-domishold graphs, defined as graphs that admit non-negative real weights associated to their vertices such that a set of vertices is a connected dominating set if and only if the sum of the corresponding weights exceeds a certain threshold.

More specifically, we show that connected-domishold graphs form a non-hereditary class of graphs properly containing two well known classes of chordal graphs: the block graphs and the trivially perfect graphs. We characterize connected-domishold graphs in terms of thresholdness of their minimal separator hypergraphs and show, conversely, that connected-domishold split graphs can be used to characterize threshold hypergraphs. Graphs every connected induced subgraph of which is connected-domishold are characterized in terms of forbidden induced subgraphs and in terms of properties of the minimal separator hypergraph. As a side result, our approach leads to new polynomially solvable cases of the minimum-weight connected domination problem.

Keywords: connected dominating set; connected domination; connected-domishold graph; forbidden induced subgraph characterization; split graph; chordal graph; 1-Sperner hypergraph; threshold hypergraph; threshold Boolean function

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1 Introduction

1.1 Background

Threshold concepts have been a subject of investigation for various discrete structures, including graphs [16,19,44], Boolean functions [18,21,28,31,49,51], and hypergraphs [33,54]. A common theme of these studies is to understand necessary and sufficient conditions so that a given combinatorial structure defined over some finite ground set $U$ admits non-negative real weights associated to elements of $U$ such that a subset of $U$ satisfies a certain property, say $\pi$, if and only if the sum of the corresponding weights exceeds a certain threshold. This framework captures several graph classes studied in the literature, including threshold graphs [19,39,44], domishold graphs [2], and total domishold graphs [15,16]. A similar approach, in which a subset of $U$ satisfies property $\pi$ if and only if the sum of the corresponding weights equals a certain threshold, can be used to define the classes of equistable

* A part of this work appeared as an extended abstract in [17].
graphs \[45,50\] and equidominating graphs \[50\]. An even more general framework, encompassing also the knapsack problem, was introduced in \[47\]. Therein, it is only assumed that a subset of \( U \) satisfies property \( \pi \) if and only if the sum of the corresponding weights belongs to a set \( T \) of thresholds given by a membership oracle. If the weights are known and integer, then a dynamic programming approach can be employed to find a subset of \( U \) with the desired property of either maximum or minimum cost (according to a given cost function on the elements of the ground set) in time \( O(|U|M) \) and with \( M \) calls of the membership oracle, where \( M \) is a given upper bound for \( T \). Clearly, in the special case of the unit costs and for the first threshold framework described above, a minimum-sized subset of \( U \) satisfying property \( \pi \) can be found by a simple greedy algorithm starting with the empty set and adding the elements in order of non-decreasing weight until the threshold is met (or exceeded).

In general, the advantages of the above framework depend both on the choice of property \( \pi \) and on the constraints (if any) imposed on the structure of the set of thresholds \( T \). For example, if \( U \) is the vertex set of a graph, property \( \pi \) denotes the property of being an independent (stable) set in a graph, and \( T \) is restricted to be an interval unbounded from below, we obtain the class of threshold graphs \[19\], which is very well understood and admits many characterizations and linear time algorithms for recognition and several optimization problems (see, e.g., \[44\]). If \( \pi \) denotes the property of being a dominating set and \( T \) is an interval unbounded from above, we obtain the class of domishold graphs \[2\], which enjoys similar properties as the class of threshold graphs. On the other hand, if \( \pi \) is the property of being a maximal stable set and \( T \) is restricted to consist of a single number, we obtain the class of equistable graphs \[50\], for which the recognition complexity is open (see, e.g., \[43\]), no structural characterization is known, and several NP-hard optimization problems remain intractable \[47\].

Notions and results from the theory of Boolean functions \[21\] and hypergraphs \[3\] can be useful for the study of graph classes defined within the above framework. For instance, the characterization of hereditarily total domishold graphs in terms of forbidden induced subgraphs given in \[16\] is based on the facts that every threshold Boolean function is 2-asummable \[18\] and that every dually Sperner hypergraph is threshold \[15\]\(^1\) Moreover, the fact that threshold Boolean functions are closed under dualization and can be recognized in polynomial time (when given by a complete DNF) \[51\] leads to efficient algorithms for recognizing total domishold graphs and for finding a minimum total dominating set in a given total domishold graph \[15\]. The relationship also goes the other way around, for instance, total domishold graphs can be used to characterize threshold hypergraphs and threshold Boolean functions \[16\].

1.2 Aim and motivation

The aim of this paper is to further exploit and explore this fruitful interplay between graphs, hypergraphs, and (equivalently) Boolean functions. We do this by studying the class of connected-domishold graphs, a new class of graphs that can be defined in the above framework, as follows: A connected dominating set (CD set for short) in a connected graph \( G \) is a set \( S \) of vertices of \( G \) that is dominating, that is, every vertex of \( G \) is either in \( S \) or has a neighbor in \( S \), and connected, that is, the subgraph of \( G \) induced by \( S \) is connected. The ground set \( U \) is the vertex set of a connected graph \( G = (V,E) \), property \( \pi \) is the property of being a connected dominating set in \( G \), and \( T \) is an interval unbounded from above.

Our motivations for studying the notion of connected domination in the above threshold framework are twofold. First, connected domination is one of the most basic of the many variants of domination, with applications in modeling wireless networks, see, e.g., the books \[25,34,35\] and recent papers \[1,7,12,13,26,30,37,56–58,61\]. The connected dominating set problem is the problem of finding a

\(^{1}\)In \[15,16\], the hereditarily total domishold graphs were named hereditary total domishold graphs. We prefer to adopt the grammatically more correct term “hereditarily total domishold”.

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minimum connected dominating set in a given connected graph. This problem is NP-hard (and hard to approximate) for general graphs and remains intractable even under significant restrictions, for instance, for the class of split graphs. On the other hand, as outlined above, the problem is polynomially solvable in the class of connected-domishold graphs equipped with weights, as in the definition. This motivates the study of connected-domishold graphs and, in particular, identification of subclasses of connected-domishold graphs. This may lead to new classes of graphs where the connected dominating set problem (or its weighted version) is polynomially solvable.

Second, despite the large variety of graph domination concepts studied in the literature (see, e.g., [34, 35]), so far only few “threshold-like” graph classes were studied with respect to notions of domination: the classes of domishold and equidominating graphs (corresponding to the usual domination), the class of equistable graphs (corresponding to independent domination), and the class of total domishold graphs (corresponding to total domination). These graph classes differ significantly with respect to their structural and algorithmic properties. For instance, while the class of domishold graphs is a highly structured hereditary subclass of cographs, the classes of equistable and of total domishold graphs are not contained in any nontrivial hereditary class of graphs and are not understood from a structural point of view. As mentioned above, the class of total domishold graphs can be used to characterize threshold hypergraphs. Hence, in a sense, the class of total domishold graphs is as rich as the class of threshold hypergraphs. These results, differences, and challenges provide ample motivation for the study of structural and algorithmic properties of connected-domishold graphs.

1.3 The definition

Since a disconnected graph $G$ does not have any connected dominating sets, we restrict our attention to connected graphs in the following definition.

**Definition 1.1.** A connected graph $G = (V,E)$ is said to be connected-domishold (CD for short) if there exists a pair $(w,t)$ where $w : V \to \mathbb{R}_+$ is a weight function and $t \in \mathbb{R}_+$ is a threshold such that for every subset $S \subseteq V$, $w(S) := \sum_{x \in S} w(x) \geq t$ if and only if $S$ is a connected dominating set in $G$. Such a pair $(w,t)$ will be referred to as a connected-domishold (CD) structure of $G$.

We emphasize that the class of connected-domishold graphs is not the intersection of the classes of connected and domishold graphs. In fact, the two classes are incomparable: the 4-vertex cycle is connected and domishold [2] but not connected-domishold, see Example 1.3 below; the 4-vertex path is connected-domishold but not domishold. The hyphen in the name is present to remind the reader of this fact.

**Example 1.2.** The complete graph of order $n$ is connected-domishold. Indeed, any nonempty subset $S \subseteq V(K_n)$ is a connected dominating set of $K_n$, and the pair $(w,1)$ where $w(x) = 1$ for all $x \in V(K_n)$ is a CD structure of $K_n$.

**Example 1.3.** The 4-cycle $C_4$ is not connected-domishold: Denoting its vertices by $v_1,v_2,v_3,v_4$ in the cyclic order, we see that a subset $S \subseteq V(C_4)$ is CD if and only if it contains an edge. Therefore, if $(w,t)$ is a CD structure of $C_4$, then $w(v_i) + w(v_{i+1}) \geq t$ for all $i \in \{1,2,3,4\}$ (indices modulo 4), which implies $w(V(C_4)) \geq 2t$. On the other hand, $w(v_1) + w(v_3) < t$ and $w(v_2) + w(v_4) < t$, implying $w(V(C_4)) < 2t$.

1.4 Overview of results

Our results can be divided into four interconnected parts and can be summarized as follows:

A class of graphs is said to be hereditary if it is closed under vertex deletion.
1) **Characterizations in terms of derived hypergraphs (resp., derived Boolean functions); a necessary and a sufficient condition.**

In a previous work [16, Proposition 4.1 and Theorem 4.5], total domishold graphs were characterized in terms of thresholdness of a derived hypergraph and a derived Boolean function. We give similar characterizations of connected-domishold graphs. The characterizations lead to a necessary and a sufficient condition for a graph to be connected-domishold, respectively, expressed in terms of properties of the derived hypergraph (equivalently: of the derived Boolean function).

2) **The case of split graphs. A characterization of threshold hypergraphs.**

While the classes of connected-domishold and total domishold graphs are in general incomparable, we show that they coincide within the class of connected split graphs. Building on this equivalence, we characterize threshold hypergraphs in terms of the connected-domishold property of a derived split graph. We also give examples of connected split graphs showing that neither of the two conditions for connected-domishold graphs mentioned above (one necessary and one sufficient) characterizes this property.

3) **The hereditary case.**

We observe that, contrary to the classes of threshold and domishold graphs, the class of connected-domishold graphs is not hereditary. This motivates the study of so-called hereditarily connected-domishold graphs, defined as graphs every connected induced subgraph of which is connected-domishold. As our main result, we give several characterizations of the class of hereditarily connected-domishold graphs. The characterizations in terms of forbidden induced subgraphs implies that the class of hereditarily connected-domishold graphs is a subclass of the class of chordal graphs properly containing two well known classes of chordal graphs, the class of block graphs and the class of trivially perfect graphs.

4) **Algorithmic aspects via vertex separators.**

Finally, we build on all these results together with some known results on connected dominating sets and minimal vertex separators in graphs to study some algorithmic aspects of the class of connected-domishold graphs and their hereditary variant. We identify a sufficient condition, capturing a large number of known graph classes, under which the CD property can be efficiently recognized. We also show that the same condition, when applied to classes of connected-domishold graphs, results in classes of graphs for which the minimum-weight connected dominating set problem (which is \textbf{NP}-hard even on split graphs) is polynomially solvable. This includes the classes of hereditarily connected-domishold graphs and $F_2$-free split graphs (see Fig. 1), thus leading to new polynomially solvable cases of the problem.

![Figure 1: Graph $F_2$.](image)

**Structure of the paper.** In Section 2 we state the necessary definitions and preliminary results on graphs, hypergraphs, and Boolean functions. In Section 3, we give characterizations of connected-domishold graphs in terms of thresholdness of derived hypergraphs and Boolean functions. Connected-domishold split graphs are studied in Section 4 where their relation to threshold hypergraphs is also
observed. The main result of the paper, Theorem 5.3 is stated in Section 5 where some of its consequences are also derived. Section 6 discusses the algorithmic aspects of connected-domishold graphs and Section 7 concludes with a proof of Theorem 5.3

2 Preliminaries

2.1 Graphs

All graphs in this paper will be finite, simple and undirected. The (open) neighborhood of a vertex \( v \) is the set of vertices in a graph \( G \) adjacent to \( v \), denoted by \( N_G(v) \) (or simply \( N(v) \) if the graph is clear from the context); the closed neighborhood of \( v \) is denoted by \( N_G[v] \) and defined as \( N_G(v) \cup \{v\} \). The degree of a vertex \( v \) in a graph \( G \) is the cardinality of its neighborhood. The complete graph, the path and the cycle of order \( n \) are denoted by \( K_n \), \( P_n \) and \( C_n \), respectively. A clique in a graph is a subset of pairwise adjacent vertices, and an independent (or stable) set is a subset of pairwise non-adjacent vertices. A universal vertex in a graph \( G \) is a vertex adjacent to all other vertices. For a set \( S \) of vertices in a graph \( G \), we denote by \( G[S] \) the subgraph of \( G \) induced by \( S \). For a set \( X \) of graphs, we say that a graph is \( X \)-free if it does not contain any induced subgraph isomorphic to a member of \( X \).

The main notion that will provide the link between threshold Boolean functions and hypergraphs is that of separators in graphs. A separator in a graph \( G = (V, E) \) is a set \( S \subseteq V(G) \) such that \( G - S \) is not connected. A separator is minimal if it does not contain any other separator. For a pair of non-adjacent vertices \( u, v \) in a graph \( G \), a \( u, v \)-separator is a set \( S \subseteq V(G) \setminus \{u, v\} \) such that \( u \) and \( v \) are in different components of \( G - S \). A \( u, v \)-separator is said to be minimal if it does not contain any other \( u, v \)-separator. Note that every minimal separator of \( G \) is a minimal \( u, v \)-separator for some non-adjacent vertex pair \( u, v \), but not vice versa. The minimal separators are exactly the minimal \( u, v \)-separators that do not contain any other \( x, y \)-separator; for this reason they are often referred to as the inclusion minimal separators. The connection between the CD graphs and the derived hypergraphs and Boolean functions will be developed in Section 3 using the following characterization of CD sets due to Kanté et al. [36].

**Proposition 2.1** (Kanté et al. [36]). In every connected graph \( G = (V, E) \) that is not complete, a subset \( D \subseteq V \) is a CD set if and only if \( D \cap S \neq \emptyset \) for every minimal separator \( S \) in \( G \).

In other words, the CD sets of a graph \( G \) are exactly the transversals of the hypergraph of the minimal separators of \( G \) (see Section 2.3 and Definition 3.2 for definitions of these notions).

A graph \( G \) is chordal if it does not contain any induced cycle of order at least 4, and split if it has a split partition, that is, a partition of its vertex set into a clique and an independent set. One of our proofs (the proof of Theorem 5.3) will rely on the following property of chordal graphs.

**Lemma 2.2** (Kumar and Veni Madhavan [12]). If \( S \) is a minimal separator of a chordal graph \( G \), then each connected component of \( G - S \) has a vertex that is adjacent to all the vertices of \( S \).

For graph theoretic notions not defined above, see, e.g., [60].

2.2 Boolean functions

Let \( n \) be positive integer. Given two vectors \( x, y \in \{0, 1\}^n \), we write \( x \leq y \) if \( x_i \leq y_i \) for all \( i \in [n] := \{1, \ldots, n\} \). A Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is positive (or: monotone) if \( f(x) \leq f(y) \) holds for every two vectors \( x, y \in \{0, 1\}^n \) such that \( x \leq y \). A literal of \( f \) is either a variable, \( x_i \), or the negation of a variable, denoted by \( \overline{x}_i \). An implicant of a Boolean function \( f \) is a conjunction \( C \) of literals such that \( f(x) = 1 \) for all \( x \in \{0, 1\}^n \) for which \( C \) takes value 1 (we also say that \( C \) implies \( f \)). An implicant is said to be prime if it is not implied by any other implicant. If \( f \) is positive, then none of the variables appearing in any of its prime implicants appears negated. Every \( n \)-variable positive
Boolean function \( f \) can be expressed with its complete DNF (disjunctive normal form), defined as the disjunction of all prime implicants of \( f \).

A positive Boolean function \( f \) is said to be **threshold** if there exist non-negative real weights \( w = (w_1, \ldots, w_n) \) and a non-negative real number \( t \) such that for every \( x \in \{0, 1\}^n \), \( f(x) = 0 \) if and only if \( \sum_{i=1}^n w_ix_i \leq t \). Such a pair \((w, t)\) is called a **separating structure** of \( f \). Every threshold Boolean function admits an integral separating structure (see [21, Theorem 9.5]). A positive Boolean function \( f(x_1, \ldots, x_n) \) is threshold if and only if its dual function \( f^d(x) = \overline{f(\overline{x})} \) is threshold [21]; moreover, if \((w_1, \ldots, w_n, t)\) is an integral separating structure of \( f \), then \((w_1, \ldots, w_n, \sum_{i=1}^n w_i - t - 1)\) is a separating structure of \( f^d \).

Threshold Boolean functions have been characterized in [18] and [28], as follows. For \( k \geq 2 \), a positive Boolean function \( f : \{0, 1\}^n \to \{0, 1\} \) is said to be \( k \)-summable if, for some \( r \in \{2, \ldots, k\} \), there exist \( r \) (not necessarily distinct) true points of \( f \), say, \( x^1, x^2, \ldots, x^r \), and \( r \) (not necessarily distinct) false points of \( f \), say \( y^1, y^2, \ldots, y^r \), such that \( \sum_{i=1}^r x^i = \sum_{i=1}^r y^i \). (A false point of \( f \) is an input vector \( x \in \{0, 1\}^n \) such that \( f(x) = 0 \); a true point is defined analogously.) Function \( f \) is said to be \( k \)-asummable if it is not \( k \)-summable, and it is **asummable** if it is \( k \)-asummable for all \( k \geq 2 \).

**Theorem 2.3** (Chow [18], Elgot [28], see also [21, Theorem 9.14]). A positive Boolean function \( f \) is threshold if and only if it is asummable.

The problem of determining whether a positive Boolean function given by its complete DNF is threshold is solvable in polynomial time, using dualization and linear programming (see [51] and [21, Theorem 9.16]). The algorithm tests if a polynomially sized derived linear program has a feasible solution, and in case of a yes instance, the solution found yields a separating structure of the given function. Using, e.g., Karmarkar’s interior point method for linear programming [38], one can assure that a rational solution is found. This results in a rational separating structure, which can be easily turned into an integral one. We summarize this result as follows.

**Theorem 2.4.** There exists a polynomial time algorithm for recognizing threshold Boolean functions given by the complete DNF. In case of a yes instance, the algorithm also computes an integral separating structure of the given function.

**Remark 2.5.** The existence of a “purely combinatorial” polynomial time recognition algorithm for threshold Boolean functions (that is, one not relying on solving an auxiliary linear program) is an open problem [21].

A similar approach as the one outlined above shows that every connected-domishold graph has an integral CD structure; we will often use this fact throughout the paper. For further background on Boolean functions, we refer to the comprehensive monograph [21].

### 2.3 Hypergraphs

A **hypergraph** is a pair \( \mathcal{H} = (V, E) \) where \( V \) is a finite set of **vertices** and \( E \) is a set of subsets of \( V \), called **hyperedges** [3]. When the vertex set or the hyperedge set of \( \mathcal{H} \) will not be explicitly given, we will refer to them by \( V(\mathcal{H}) \) and \( E(\mathcal{H}) \), respectively. A transversal (or: hitting set) of \( \mathcal{H} \) is a set \( S \subseteq V \) such that \( S \cap e \neq \emptyset \) for all \( e \in E \). A hypergraph \( \mathcal{H} = (V, E) \) is threshold if there exist a weight function \( w : V \to \mathbb{R}_+ \) and a threshold \( t \in \mathbb{R}_+ \) such that for all subsets \( X \subseteq V \), it holds that \( w(X) \leq t \) if and only if \( X \) contains no hyperedge of \( \mathcal{H} \) [33]. Such a pair \((w, t)\) is said to be a **separating structure** of \( \mathcal{H} \).

To every hypergraph \( \mathcal{H} = (V, E) \), we can naturally associate a positive Boolean function \( f_\mathcal{H} : \{0,1\}^V \to \{0,1\} \), defined by the positive DNF expression

\[
f_\mathcal{H}(x) = \bigvee_{e \in E} \bigwedge_{u \in e} x_u
\]
for all $x \in \{0,1\}^V$. Conversely, to every positive Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ given by a positive DNF $\phi = \bigvee_{j=1}^m \bigwedge_{i \in C_j} x_i$, we can associate a hypergraph $H(\phi) = (V,E)$ as follows: $V = [n]$ and $E = \{C_1, \ldots, C_m\}$. It follows directly from the definitions that the thresholdness of hypergraphs and of Boolean functions are related as follows.

**Proposition 2.6.** A hypergraph $H = (V,E)$ is threshold if and only if the positive Boolean function $f_H$ is threshold. A positive Boolean function given by a positive DNF $\phi = \bigvee_{j=1}^m \bigwedge_{i \in C_j} x_i$ is threshold if and only if the hypergraph $H(\phi)$ is threshold.

Applying Theorem 2.3 to the language of hypergraphs gives the following characterization of threshold hypergraphs. For $k \geq 2$, a hypergraph $H = (V,E)$ is said to be $k$-summable if, for some $r \in \{2, \ldots, k\}$, there exist $r$ (not necessarily distinct) subsets $A_1, \ldots, A_r$ of $V$ such that each $A_i$ contains a hyperedge of $H$, and $r$ (not necessarily distinct) subsets $B_1, \ldots, B_r$ of $V$ such that each $B_i$ does not contain a hyperedge of $H$ and such that for every vertex $v \in V$, we have:

$$|\{i : v \in A_i\}| = |\{i : v \in B_i\}|.$$  \hspace{1cm} (1)

We say that a hypergraph $H$ is $k$-summable if it is not $k$-summable and it is asummable if it is $k$-asummable for all $k \geq 2$.

**Corollary 2.7.** A hypergraph $H$ is threshold if and only if it is asummable.

Recall that a hypergraph $H = (V,E)$ is said to be Sperner (or: a clutter) if no hyperedge of $H$ contains another hyperedge, that is, if for every two distinct hyperedges $e$ and $f$ of $H$, it holds that $\min\{|e \setminus f|, |f \setminus e|\} \geq 1$. Chiarelli and Milanić defined in [15,16] the notion of dually Sperner hypergraphs as the hypergraphs such that the inequality $\min\{|e \setminus f|, |f \setminus e|\} \leq 1$ holds for every pair of distinct hyperedges $e$ and $f$ of $H$. It was proved in [15,16] that dually Sperner hypergraphs are threshold; they were applied in the characterizations of total domishold graphs and their hereditary variant. More recently, Boros et al. introduced in [9] the following restriction of dually Sperner hypergraphs.

**Definition 2.8 (Boros et al. [9]).** A hypergraph $H = (V,E)$ is said to be 1-Sperner if for every two distinct hyperedges $e$ and $f$ of $H$, it holds that $\min\{|e \setminus f|, |f \setminus e|\} = 1$.

Note that a hypergraph is 1-Sperner if and only if it is both Sperner and dually Sperner. In particular, for Sperner hypergraphs the notions of dually Sperner and 1-Sperner hypergraphs coincide. Since a hypergraph $H$ is threshold if and only if the Sperner hypergraph obtained from $H$ by keeping only its inclusion-wise minimal hyperedges is threshold, the fact that dually Sperner hypergraphs are threshold is equivalent to the following fact, proved constructively by Boros et al. in [9] using a composition result for 1-Sperner hypergraphs developed therein.

**Theorem 2.9.** Every 1-Sperner hypergraph is threshold.

### 3 Connected-domishold graphs via hypergraphs and Boolean functions

In a previous work [16, Proposition 4.1 and Theorem 4.5], total domishold graphs were characterized in terms of thresholdness of a derived hypergraph and a derived Boolean function. In this section we give similar characterizations of connected-domishold graphs.

For completeness, we first recall some relevant definitions and a result from [16]. A total dominating set in a graph $G$ is a set $S \subseteq V(G)$ such that every vertex of $G$ has a neighbor in $S$. Note that only graphs without isolated vertices have total dominating sets. A graph $G = (V,E)$ is said to be total domishold (TD for short) if there exists a pair $(w,t)$ where $w : V \rightarrow \mathbb{R}_+$ is a weight function and
$t \in \mathbb{R}_+$ is a threshold such that for every subset $S \subseteq V$, $w(S) := \sum_{x \in S} w(x) \geq t$ if and only if $S$ is a total dominating set in $G$. A pair $(w, t)$ as above will be referred to as a total domishold (TD) structure of $G$. The minimal neighborhood hypergraph of a graph $G$ is the hypergraph denoted by $\text{MNH}(G)$ and defined as follows: the vertex set of $\text{MNH}(G)$ is $V(G)$ and the hyperedge set consists precisely of the minimal neighborhoods in $G$, that is, of the inclusion-wise minimal sets in the family of neighborhoods $\{N(v) : v \in V(G)\}$. Note that a set $S \subseteq V(G)$ is a total dominating set in $G$ if and only if it is a transversal of $\text{MNH}(G)$.

**Proposition 3.1** (Chiarelli and Milanič [16]). For a graph $G = (V, E)$, the following are equivalent:

1. $G$ is total domishold.
2. Its minimal neighborhood hypergraph $\text{MNH}(G)$ is threshold.

The constructions of the derived hypergraph and the derived Boolean function used in our characterizations of connected-domishold graphs in terms of their thresholdness are specified by the following two definitions.

**Definition 3.2.** Given a graph $G$, the minimal separator hypergraph of $G$ is the hypergraph $\text{MSH}(G) = (V(G), S(G))$, where $S(G) = \{S : S \subseteq V(G) \text{ and } S \text{ is a minimal separator in } G\}$.

Given a finite non-empty set $V$, we denote by $\{0, 1\}^V$ the set of all binary vectors with coordinates indexed by $V$. Given a graph $G = (V, E)$ and a binary vector $x \in \{0, 1\}^V$, its support set is the set denoted by $S(x)$ and defined by $S(x) = \{v \in V : x_v = 1\}$. In the following definition, we associate a Boolean function to a given $n$-vertex graph $G$. In order to avoid fixing a bijection between its vertex set and the set $[n]$, we will consider the corresponding Boolean function as defined on the set $\{0, 1\}^V$, where $V = V(G)$. Accordingly, a separating structure of such a Boolean function can be seen as a pair $(w, t)$ where $w : V \to \mathbb{R}_+$ and $t \in \mathbb{R}_+$ such that for every $x \in \{0, 1\}^V$, we have $f(x) = 0$ if and only if $\sum_{v \in S(x)} w(v) \leq t$.

**Definition 3.3.** Given a graph $G = (V, E)$, its minimal separator function is the positive Boolean function $f_{\text{ms}}^G : \{0, 1\}^V \to \{0, 1\}$ that takes value 1 precisely on vectors $x \in \{0, 1\}^V$ whose support set contains some minimal separator of $G$.

The announced characterizations of connected-domishold graphs in terms of their minimal separator hypergraphs and minimal separator functions are given in the following proposition. The proof is based on two ingredients: the characterization of the connected dominating sets of a given (non-complete) graph given by Proposition 2.1 and the fact that threshold Boolean functions are closed under dualization.

**Proposition 3.4.** For a connected graph $G = (V, E)$, the following are equivalent:

1. $G$ is connected-domishold.
2. Its minimal separator hypergraph $\text{MSH}(G)$ is threshold.
3. Its minimal separator function $f_{\text{ms}}^G$ is threshold.

Moreover, if $G$ is not a complete graph, and $(w, t)$ is an integral separating structure of $f_{\text{ms}}^G$ or of $\text{MSH}(G)$, then $(w, w(V) - t)$ is a CD structure of $G$.

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\footnote{In [16], the minimal neighborhood hypergraph of $G$ was named reduced neighborhood hypergraph (of $G$) and denoted by $\mathcal{RN}(G)$. We changed the terminology in analogy with the term “minimal separator hypergraph”, which will be introduced shortly.}
**Proof.** We will consider two cases, depending on whether $G$ is a complete graph or not.

**Case 1:** $G$ is complete.

In this case all the three statements hold. Recall that every complete graph is CD (see Example 1.2). Since complete graphs have no minimal separators, the set of hyperedges of the minimal separator hypergraph $\text{MSH}(G)$ is empty. Hence by Corollary 2.7 the hypergraph $\text{MSH}(G)$ is threshold. The absence of (minimal) separators also implies that the minimal separator function $f_G^\text{ms}$ is constantly equal to 0 and hence threshold.

**Case 2:** $G$ is not complete.

First we will show the equivalence between statements 1 and 3. Since a positive Boolean function $f$ is threshold if and only if its dual function $f^d(x) = \overline{f(\overline{x})}$ is threshold, it suffices to argue that $G$ is connected-domishold if and only if $(f_G^\text{ms})^d$ is threshold.

We claim that for every $x \in \{0,1\}^V$, we have $(f_G^\text{ms})^d(x) = 1$ if and only if $S(x)$, the support set of $x$, is a connected dominating set of $G$. Let $x \in \{0,1\}^V$ and let $S$ be the support set of $x$. By definition, $(f_G^\text{ms})^d(x) = 1$ if and only if $f_G^\text{ms}(\overline{x}) = 0$, which is the case if and only if $V \setminus S$ does not contain any minimal separator of $G$. This is in turn equivalent to the condition that $S$ is a transversal of the minimal separator hypergraph of $G$, and, by Proposition 2.1, to the condition that $S$ is a connected dominating set of $G$. Therefore, $(f_G^\text{ms})^d(x) = 1$ if and only if $S$ is a connected dominating set of $G$, as claimed.

Now, if $G$ is connected-domishold, then it has an integral connected-domishold structure, say $(w,t)$, and $(w,t-1)$ is a separating structure of the dual function $(f_G^\text{ms})^d$, which implies that $(f_G^\text{ms})^d$ is threshold. Conversely, if the dual function is threshold, with an integral separating structure $(w,t)$, then $(w,t+1)$ is a connected-domishold structure of $G$. This establishes the equivalence between statements 1 and 3.

Next, we show the equivalence between statements 2 and 3. Note that the complete DNF of $f_G^\text{ms}$, the minimal separator function of $G$, is given by the expression $\bigvee_{S \subseteq S(G)} \bigwedge_{x \in S} x_u$. It now follows directly from the definitions of threshold Boolean functions and threshold hypergraphs that function $f_G^\text{ms}(x)$ is threshold if and only if hypergraph $\text{MSH}(G)$–the hyperedges of which are exactly the elements of $S(G)$–is threshold.

Finally, if $(w,t)$ is an integral separating structure of $f_G^\text{ms}$, then $(w,w(V) - t - 1)$ is a separating structure of $(f_G^\text{ms})^d$ and hence $(w,w(V) - t)$ is a connected-domishold structure of $G$.

Recall that every 1-Sperner hypergraph is threshold (Theorem 2.9) and every threshold hypergraph is asummable (Corollary 2.7). Thus, in particular, every threshold hypergraph is 2-asummable. Applying these relations to the specific case of the minimal separator hypergraphs leads to the following.

**Definition 3.5.** We say that a graph $G$ is 1-Sperner with respect to separators if its minimal separator hypergraph $\text{MSH}(G)$ is 1-Sperner. Similarly, we say that $G$ is 2-asummable with respect to minimal separators if its minimal separator hypergraph $\text{MSH}(G)$ is 2-asummable.

**Corollary 3.6.** For every connected graph $G$, the following holds:

1. If $G$ is 1-Sperner w.r.t. separators, then $G$ is connected-domishold.
2. If $G$ is connected-domishold, then $G$ is 2-asummable w.r.t. separators.

We will show in Section 4.1 that neither of the two statements in Corollary 3.6 can be reversed. On the other hand, we will prove in Section 5 that all the three properties become equivalent in the hereditary setting.
4 Connected-domishold split graphs

The following examples show that for general connected graphs, the CD and TD properties are incomparable:

- The path $P_6$ is connected-domishold (it has a unique minimal connected dominating set, formed by the four vertices of degree two) but it is not total domishold (see, e.g., [16]).
- The graph in Fig. 2 is TD but not CD. The graph is total domishold: it has a unique minimal total dominating set, namely $\{v_1, v_4, v_5, v_8\}$. On the other hand, the graph is not connected-domishold. This can be shown by observing that it is not 2-asummable w.r.t. separators and applying Corollary 3.6. To see that the minimal separator hypergraph of $G$ is 2-summable, note that condition (1) is satisfied if we take $k = r = 2$ and $A_1 = \{v_2, v_7\}$, $A_2 = \{v_3, v_6\}$, $B_1 = \{v_2, v_3\}$, and $B_2 = \{v_6, v_7\}$.

Interestingly, we will show in Section 5 that if the CD and TD properties are required also for all induced subgraphs, then the corresponding graph classes are comparable (see Corollary 5.7). In the rest of this section, we will prove that the two properties coincide in the class of connected split graphs and examine some consequences of this result. Recall that a graph is split if and only if its vertex set has a partition into a clique and an independent set. Foldes and Hammer characterized split graphs as exactly the graphs that are $\{2K_2, C_4, C_5\}$-free [29]. In particular, this implies that a split graph can be disconnected only if it has an isolated vertex.

Lemma 4.1. Let $G$ be a connected graph and let $G'$ be the graph obtained from $G$ by adding to it a universal vertex. Then, $G$ is connected-domishold if and only if $G'$ is connected-domishold.

Proof. Let $V(G') = V(G) \cup \{u\}$, where $u$ is the added vertex. Suppose that $G$ is connected-domishold and let $(w, t)$ be a CD structure of $G$. Since the set of connected dominating sets of $G'$ consists of all connected dominating sets of $G$ together with all subsets of $V(G')$ containing $u$, we can obtain a CD structure, say $(w', t')$, of $G'$ by setting $w'(x) = w(x)$ for all $x \in V(G)$, $w'(u) = t$, and $t' = t$. Therefore, $G'$ is connected-domishold.

Conversely, if $(w', t')$ is a CD structure of $G'$, then $(w, t)$ where $t = t'$ and $w$ is the restriction of $w'$ to $V(G)$ is a CD structure of $G$. This is because a set $X \subseteq V(G)$ is a connected dominating set of $G$ if and only if it is a connected dominating set of $G'$. Therefore, if $G'$ is connected-domishold then so is $G$.

Recall that given a connected graph $G$, we denote by $MSH(G)$ (resp., $MNH(G)$) its minimal separator (resp., minimal neighborhood) hypergraph.

Lemma 4.2. Let $G$ be a connected split graph without universal vertices. Then $MSH(G) = MNH(G)$.

Proof. Fix a split partition of $V(G)$, say $V(G) = K \cup I$ where $K$ is a clique, $I$ is an independent set, and $K \cap I = \emptyset$. Clearly, the hypergraphs $MSH(G)$ and $MNH(G)$ have the same vertex set. To show that the hyperedge sets are also the same, we proceed in two steps.

Figure 2: A TD graph that is not CD.
First, we show that $E(MSH(G)) \subseteq E(MNH(G))$, that is, that every minimal separator is a minimal neighborhood. To this end, it suffices to show that every minimal separator $S$ in $G$ is a neighborhood, that is, a set of the form $S = N(v)$ for some $v \in V(G)$. This is indeed enough, because if a minimal separator $S$ in $G$ satisfies $S = N(v)$ for some $v \in V(G)$, but $N(v)$ properly contains some other neighborhood, say $N(u)$, then the fact that $N(u)$ is a separator in $G$ (for instance, it is a $u,v$-separator) would imply that $S$ is not a minimal separator.

Let $S$ be a minimal separator in $G$. Then, $S$ is a minimal $u,v$-separator for some non-adjacent vertex pair $u,v$; in particular, $S \subseteq V(G) \setminus \{u,v\}$. We claim that $N(u) \subseteq S$ or $N(v) \subseteq S$. Suppose that this is not the case. Then, there exist a neighbor of $u$, say $u'$, such that $u' \not\in S$, and a neighbor of $v$, say $v'$, such that $v' \not\in S$. Since $\{u,v,u',v'\} \subseteq V(G) \setminus S$ and $u$ and $v$ are in different components of $G - S$, vertices $u'$ and $v'$ are distinct and non-adjacent. Thus, at least one of $u'$ and $v'$, say $u'$, is in $I$. This implies that $u \in K$ and therefore $v \in I$, which implies that $v' \in K$ and hence $(u,v,v')$ is a $u,v$-path in $G - S$, a contradiction. This shows that $N(u) \subseteq S$ or $N(v) \subseteq S$, as claimed. Since each of $N(u)$ and $N(v)$ is a $u,v$-separator, the fact that $S$ is a minimal $u,v$-separator implies that $S \in \{N(u), N(v)\}$. This completes the proof of the inclusion $E(MSH(G)) \subseteq E(MNH(G))$.

It remains to show that $E(MNH(G)) \subseteq E(MSH(G))$. Let $S$ be a minimal neighborhood in $G$. Then $S = N(v)$ for some $v \in V(G)$. Since $v$ is not universal, the set $V(G) \setminus N[v]$ is non-empty. Therefore $S$ is a $v,w$-separator for any $w \in V(G) \setminus N[v]$; in particular, $S$ is a separator in $G$. Suppose for a contradiction that $S$ is not a minimal separator in $G$. Then $S$ properly contains some minimal separator, say $S'$, in $G$. By the first part of the proof, $S'$ is of the form $S' = N(z)$ for some $z \in V(G)$. However, since $N(z)$ is a neighborhood properly contained in $S = N(v)$, this contradicts the fact that $S$ is a minimal neighborhood.

**Theorem 4.3.** A connected split graph is connected-domishold if and only if it is total domishold.

**Proof.** If $G$ is complete, then $G$ is both connected-domishold and total domishold. So we may assume that $G$ is not complete. More generally, we show next that we may assume that $G$ does not have any universal vertices. Suppose that $G$ has a universal vertex, say $u$, and let $G' = G - u$. By [16] Proposition 3.3, $G$ is TD if and only if $G'$ is TD. If $G'$ is not connected, then $\{u\}$ is the only minimal connected dominating set of $G$ and hence $G$ is connected-domishold in this case. Furthermore, $G$ is also total domishold: since $G'$ is a disconnected $2K_2$-free graph, $G'$ has an isolated vertex. Therefore, by [16], $G'$ is TD, and hence so is $G$. If $G'$ is connected, then by Lemma 4.1 $G$ is CD if and only if $G'$ is CD. Therefore, the problem of verifying whether the CD and the TD properties are equivalent for $G$ reduces to the same problem for $G'$. An iterative application of the above argument eventually reduces the graph to either a graph where both properties hold or to a connected graph without universal vertices.

Now, let $G$ be a connected split graph without universal vertices. By Proposition 3.4, $G$ is connected-domishold if and only if its minimal separator hypergraph $MSH(G)$ is threshold. By Proposition 3.1, $G$ is total domishold if and only if its minimal neighborhood hypergraph $MNH(G)$ is threshold. Therefore, to prove the theorem it suffices to show that $MSH(G) = MNH(G)$. But this was established in Lemma 4.2.

**Theorem 4.4.** Let $\mathcal{H} = (V, E)$ be a hypergraph with $\emptyset \not\subseteq E$. Then $\mathcal{H}$ is threshold if and only if its split-incidence graph is connected-domishold.

**Proof.** Since $\emptyset \not\subseteq E$, the split-incidence graph of $\mathcal{H}$ is connected. It was shown in [16] that a hypergraph is threshold if and only if its split-incidence graph is total domishold. The statement of the theorem now follows from Theorem 4.3.
It might be worth pointing out that in view of Remark 2.5 and Theorem 4.4, it is an open problem of whether there is a “purely combinatorial” polynomial time algorithm for recognizing connected-domishold split graphs. (Further issues regarding the recognition problem of CD graphs are discussed in Section 6.1.)

4.1 Examples related to Corollary 3.6

We now show that neither of the two statements in Corollary 3.6 can be reversed. First we exhibit an infinite family of CD split graphs that are not 1-Sperner w.r.t. separators.

Example 4.5. Let \( n \geq 4 \) and let \( G = K^*_n \) be the graph obtained from the complete graph \( K_n \) by gluing a triangle on every edge. Formally, \( V(G) = \{u_1, \ldots, u_n\} \cup \{v_{ij} : 1 \leq i < j \leq n\} \) and \( E(G) = \{u_iu_j : 1 \leq i < j \leq n\} \cup \{u_iv_{jk} | 1 \leq j < k \leq n \text{ and } i \in \{j,k\}\}. \) The graph \( G \) is a CD graph: setting \( w(x) = \begin{cases} 1, & \text{if } x \in \{u_1, \ldots, u_n\}; \\ 0, & \text{otherwise.} \end{cases} \)

and \( t = n - 1 \) results in a CD structure of \( G. \) On the other hand, \( G \) is not 1-Sperner w.r.t. separators. Since every pair of the form \( \{u_i, u_j\} \) with \( 1 \leq i < j \leq n \) is a minimal separator of \( G, \) the minimal separator hypergraph contains \( \{u_1, u_2\} \) and \( \{u_3, u_4\} \) as hyperedges and is therefore not 1-Sperner.

Next, we show that there exists a split graph that is 2-asummable w.r.t. separators but is not CD. As observed already in [16], the fact that not every 2-asummable positive Boolean function is threshold can be used to construct split graphs \( G \) such that \( MNH(G) \) is 2-asummable and \( G \) is not total domishold. Using the result of Theorem 4.3 and its proof, this implies the existence of split graphs \( G \) that are 2-asummable w.r.t. separators but not CD. For the sake of self-containment, we describe an example of such a construction in some detail.

Example 4.6. Based on an example due to Gabelman [31], Crama and Hammer proposed in the proof of [27], Theorem 9.15] and example of a 9-variable 2-asummable positive Boolean function \( f \) that is not threshold. From this function we can derive a split graph \( G = (V,E) \) on 71 vertices, as follows. Let \( V = K \cup I \) where \( K = \{v_1, \ldots, v_9\} \) is a clique and \( I = V(G) - K \) is an independent set. To define the edges between \( K \) and \( I, \) we first associate a non-negative integer weight to each vertex, as follows:

\[
\begin{aligned}
w(v_1) &= 14, \\
w(v_2) &= 18, \\
w(v_3) &= 24, \\
w(v_4) &= 26, \\
w(v_5) &= 27, \\
w(v_6) &= 30, \\
w(v_7) &= 31, \\
w(v_8) &= 36, \\
w(v_9) &= 37, \\
w(v) &= 0 \quad \text{for all } v \in I.
\end{aligned}
\]

\( S \) be the set of all subsets \( S \) of \( K \) such that \( w(S) \geq 82 \) and let \( S_1 = \{v_1, v_6, v_9\}, \) \( S_2 = \{v_2, v_5, v_8\}, \) and \( S_3 = \{v_3, v_4, v_7\}. \) (Note that \( w(S_i) = 81 \) for all \( i \in [3]. \)) Let \( H \) be the hypergraph with vertex set \( K \) and hyperedge set given by the inclusion-wise minimal sets in \( S \cup \{S_1, S_2, S_3\}. \) It can be verified that \( H \) has precisely 62 hyperedges (including \( S_1, S_2, \) and \( S_3)) \footnote{The following is the list of sets (omitting commas and brackets) of indices of the elements of the 62 inclusion-wise minimal hyperedges of \( H): 169, 179, 189, 258, 259, 268, 269, 278, 279, 289, 347, 348, 349, 357, 358, 359, 367, 368, 369, 378, 379, 389, 456, 457, 458, 459, 467, 468, 469, 478, 479, 489, 567, 568, 569, 578, 579, 589, 678, 679, 689, 789, 1234, 1235, 1236, 1237, 1238, 1239, 1245, 1246, 1247, 1248, 1249, 1256, 1257, 1267, 1345, 1346, 1356, 2345, 2346, 2356.} \) The edges of \( G \) between vertices of \( I \) and \( K \) are defined so that set of the neighborhoods of the \( 62 \) vertices of \( I \) is exactly the set of hyperedges of \( H. \)

To show that \( G \) is not CD, it suffices, by Proposition 3.3, to show that the minimal separator hypergraph is not threshold. In the proof of Theorem 9.15 in [27] it is shown that the function \( f \) is not threshold, by showing that \( f \) is 3-smumable. This corresponds to the fact that the hypergraph of minimal separators of \( G \) is 3-smumable, as can be observed by noticing that condition 1 is satisfied for \( k = r = 3 \) and for the sets \( A_i = S_i \) for all \( i \in [3] \) and \( B_1 = \{v_1, v_7, v_8\}, B_2 = \{v_2, v_4, v_5\}, \) and \( B_3 = \{v_3, v_5, v_6\}. \) On the other hand, the fact that \( f \) is 2-asummable implies that \( G \) is 2-asummable w.r.t. separators.
5 The hereditary case

In this section we state the main result of this paper, Theorem 5.3, which gives several characterizations of graphs all connected induced subgraphs of which are CD, and derive some of its consequences. We start with an example showing that, contrary to the classes of threshold and domishold graphs, the class of connected-domishold graphs is not hereditary. We assume notation from Example 1.3.

Example 5.1. The graph $G$ obtained from $C_4$ by adding to it a new vertex, say $v_5$, and making it adjacent exactly to one vertex of the $C_4$, say to $v_4$, is CD: the (inclusion-wise) minimal CD sets of $G$ are $\{v_1,v_4\}$ and $\{v_3,v_4\}$, hence a CD structure of $G$ is given by $w(v_2) = w(v_5) = 0$, $w(v_1) = w(v_3) = 1$, $w(v_4) = 2$, and $t = 3$.

This motivates the following definition:

Definition 5.2. A graph $G$ is said to be hereditarily connected-domishold (hereditarily CD for short) if every connected induced subgraph of $G$ is connected-domishold.

In general, for a property $\Pi$ of connected graphs, a graph is said to be hereditarily $\Pi$ if every connected induced subgraph of it satisfies $\Pi$. Characterizations of classes of hereditarily $\Pi$ graphs where $\Pi$ denotes the property that the graph has a connected dominating set inducing a graph with a certain property $\Pi'$ were given, for various choices of property $\Pi'$, by Michalak in [46]. In [53], Pržulj et al. gave characterizations of hereditarily $\Pi$ graphs where $\Pi$ denotes the property that the graph has a dominating pair of vertices (that is, a pair of vertices such that every path between them is dominating). The class of hereditarily connected-domishold graphs corresponds to the case when $\Pi$ is the property of being connected-domishold. Moreover, we will say that a graph $G$ is hereditarily 1-Sperner with respect to separators if every connected induced subgraph of $G$ is 1-Sperner w.r.t. separators. The property of $G$ being hereditarily 2-asummable with respect to separators is defined analogously.

Theorem 5.3. For every graph $G$, the following are equivalent:

1. $G$ is hereditarily CD.
2. $G$ is hereditarily 2-asummable w.r.t. separators.
3. $G$ is hereditarily 1-Sperner w.r.t. separators.
4. $G$ is an $\{F_1, F_2, H_1, H_2, \ldots\}$-free chordal graph, where the graphs $F_1, F_2$, and a general member of the family $\{H_i\}$ are depicted in Fig. 3.

Figure 3: Graphs $F_1$, $F_2$, and $H_i$. (i ≥ 1)

Theorem 5.3 will be proved in Section 7. In the rest of this section, we examine some of the consequences of the forbidden induced subgraph characterization of hereditarily CD graphs given by Theorem 5.3. The diamond and the kite (also known as the co-fork or the co-chair) are the graphs depicted in Fig. 4.

The equivalence between items 1 and 4 in Theorem 5.3 implies that the class of hereditarily CD graphs is a proper generalization of the class of kite-free chordal graphs.
Corollary 5.4. Every kite-free chordal graph is hereditarily CD.

Corollary 5.4 further implies that the class of hereditarily CD graphs generalizes two well known classes of chordal graphs, the class of block graphs and the class of trivially perfect graphs. A graph is said to be a block graph if every block (maximal connected subgraph without cut vertices) of it is complete. The block graphs are well known to coincide with the diamond-free chordal graphs. A graph \( G \) is said to be trivially perfect \[32\] if for every induced subgraph \( H \) of \( G \), it holds \( \alpha(H) = |C(H)| \), where \( \alpha(H) \) denotes the independence number of \( H \) (that is, the maximum size of an independent set in \( H \)), and \( C(H) \) denotes the set of all maximal cliques of \( H \). Trivially perfect graphs coincide with the so-called quasi-threshold graphs \[62\], and are exactly the \( \{P_4, C_4\} \)-free graphs \[32\].

Corollary 5.5. Every block graph is hereditarily CD. Every trivially perfect graph is hereditarily CD.

Another class of graphs contained in the class of hereditarily CD graphs is the class of graphs defined similarly as the hereditarily CD graphs but with respect to total dominating sets. These so-called hereditarily total domishold graphs (abbreviated hereditarily TD graphs) were studied in \[16\], where characterizations analogous to those given by Theorem 5.3 were obtained, including the following characterization in terms of forbidden induced subgraphs.

Theorem 5.6. For every graph \( G \), the following are equivalent:

1. \( G \) is hereditarily total domishold.
2. No induced subgraph of \( G \) is isomorphic to a graph in Fig. \[7\]

![Forbidden induced subgraphs for the class of hereditarily total domishold graphs.](image)

Theorems 5.3 and 5.6 imply the following.

Corollary 5.7. Every hereditarily TD graph is hereditarily CD.
Proof. It suffices to verify that each of the forbidden induced subgraphs for the class of hereditarily connected-domishold graphs contains one of the graphs from Fig. 5 as induced subgraph. A cycle $C_k$ with $k \geq 4$ contains (or is equal to) one of $C_4, C_5, C_6, P_6$. The graphs $F_1$ and $F_2$ are contained in both sets of forbidden induced subgraphs. Finally, each graph of the form $H_i$ where $i \geq 1$ contains $2K_3$ as induced subgraph.

Since a graph is split if and only if it is $\{2K_2, C_4, C_5\}$-free and each of the forbidden induced subgraphs for the class of hereditarily total domishold graphs other than $F_2$ contains either $2K_2$, $C_4$, or $C_5$ as induced subgraph, Corollary 5.7 implies the following.

Corollary 5.8. Every $F_2$-free split graph is hereditarily CD.

Fig. 6 shows a Hasse diagram depicting the inclusion relations among the class of hereditarily connected-domishold graphs and several well studied graph classes. All definitions of graph classes depicted in Fig. 6 and the relations between them can be found in [22], with the exception of the hereditarily CD and hereditarily TD graphs. The fact that every co-domishold graph is hereditarily TD and that every hereditarily TD graph is $(1,2)$-polar chordal was proved in [16]. The remaining inclusion and non-inclusion relations can be easily verified using the forbidden induced subgraph characterizations of the depicted graph classes [11, 22, 33].

Figure 6: A Hasse diagram depicting the inclusion relations within several families of perfect graphs, focused around the class of hereditarily connected-domishold graphs.
6 Algorithmic aspects via vertex separators

In this section, we discuss some algorithmic issues related to connected-domishold graphs and their hereditary counterpart.

6.1 The recognition problems

We start with the computational complexity aspects of the problems of recognizing whether a given graph is CD, resp. hereditarily CD. For general graphs, the computational complexity of recognizing connected-domishold graphs is not known. We now show that the hypergraph approach outlined in Section 3 leads to a sufficient condition for the problem to be polynomially solvable in a large number of graph classes. The condition is expressed using the notion of minimal vertex separators. Recall that a separator in a graph \( G = (V, E) \) is a set \( S \subseteq V(G) \) such that \( G - S \) is not connected and that a separator is minimal if it does not contain any other separator. Recall also that a \( u, v \)-separator (for a pair of non-adjacent vertices \( u, v \)) is a set \( S \subseteq V(G) \setminus \{u, v\} \) such that \( u \) and \( v \) are in different components of \( G - S \) and that a \( u, v \)-separator is minimal if it does not contain any other \( u, v \)-separator. A minimal vertex separator in \( G \) is a minimal \( u, v \)-separator for some non-adjacent vertex pair \( u, v \).

A sufficient condition for the polynomial time solvability of the recognition problem for CD graphs in a class of graphs \( \mathcal{G} \) is that there exists a polynomial poly such that every connected graph \( G \in \mathcal{G} \) has at most \( \text{poly}(|V(G)|) \) minimal vertex separators. This is the case for chordal graphs, which have at most \( |V(G)| \) minimal vertex separators [55], as well as for many other classes of graphs, including permutation graphs, circle graphs, circular-arc graphs, chordal bipartite graphs, trapezoid graphs, cocomparability graphs of bounded dimension, distance-hereditary graphs, and weakly chordal graphs (see, e.g., [10, 40, 48]). For a polynomial poly, let \( \mathcal{G}_{\text{poly}} \) be the class of graphs with at most \( \text{poly}(|V(G)|) \) minimal vertex separators. Since every minimal separator is a minimal vertex separator, every connected graph \( G \in \mathcal{G}_{\text{poly}} \) has at most \( \text{poly}(|V(G)|) \) minimal separators.

It is known that the set of all minimal vertex separators of a given connected \( n \)-vertex graph can be enumerated in output-polynomial time. More precisely, Berry et al. [4] have developed an algorithm solving this problem in time \( \mathcal{O}(n^3|\Sigma|) \) where \( \Sigma \) is the set of all minimal vertex separators of \( G \), improving on earlier (independently achieved) running times of \( \mathcal{O}(n^5|\Sigma|) \) due to Shen and Liang [59] and Kloks and Kratsch [41]. Based on these results, we derive the following.

**Theorem 6.1.** For any polynomial poly, there is a polynomial time algorithm to determine whether a given connected graph \( G \in \mathcal{G}_{\text{poly}} \) is connected-domishold. In case of a yes instance, the algorithm also computes an integral CD structure of \( G \).

**Proof.** Let \( G = (V, E) \in \mathcal{G}_{\text{poly}} \) be a connected graph that is the input to the algorithm.

The algorithm proceeds as follows. If \( G \) is complete, then \( G \) is connected-domishold and an integral CD structure of \( G \) is returned, say \((w, t)\) with \( w(x) = 1 \) for all \( x \in V(G) \) and \( t = 1 \). Assume now that \( G \) is not complete. First, using the algorithm of Berry et al. [4], we compute in time \( \mathcal{O}(|V(G)|^3\text{poly}(|V(G)|)) \) the set \( \Sigma \) of all minimal vertex separators of \( G \). Next, the minimal separator hypergraph, \( MSH(G) \), is computed by comparing each pair of sets in \( \Sigma \) and discarding the non-minimal ones. Since \( MSH(G) \) is Sperner, there is a bijective correspondence between the hyperedges of \( MSH(G) \) and the prime implicants of the minimal separator function \( f_G^{ms} \); this yields the complete DNF of \( f_G^{ms} \). Finally, we run the algorithm given by Theorem 2.4 on the complete DNF of \( f_G^{ms} \). If \( f_G^{ms} \) is not threshold, then we conclude that \( G \) is not connected-domishold. Otherwise, the algorithm returned an integral separating structure, say \((w, t)\), of \( f_G^{ms} \). In this case we return \((w, w(V) - t)\) as a CD structure of \( G \).

It is clear that the algorithm runs in polynomial time. Its correctness follows from Proposition 3.4.

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5Minimal vertex separators are sometimes referred to as minimal separators. According to the definition of a minimal separator adopted in this paper, every minimal separator of \( G \) is a minimal vertex separator, but not vice versa.
Let  be the largest hereditary graph class such that a connected graph  is connected-domishold if and only if it is total domishold. By Theorem 4.3 class  is a generalization of the class of split graphs. Since there is a polynomial time algorithm for recognizing total domishold graphs [15][16], there is a polynomial time algorithm to determine whether a given connected graph  is connected-domishold. This motivates the following question (which we leave open):

**Question.** What is the largest hereditary graph class  such that a connected graph  is connected-domishold if and only if it is total domishold?

A polynomial time recognition algorithm for the class of hereditarily CD graphs can be derived from the characterization of hereditarily CD graphs in terms of forbidden induced subgraphs given by Theorem 5.3

**Proposition 6.2.** There exists a polynomial time algorithm to determine whether a given graph  is hereditarily CD. In the case of a yes instance, an integral CD structure of  can be computed in polynomial time.

**Proof.** One can verify in linear time that  is chordal [33] and verifying that  is also  -free can be done in time . Therefore, we only have to show that we can check in polynomial time that  does not contain an induced subgraph of the form  for each  . Observe that for all  the graph  contains an induced subgraph isomorphic to , the union of two diamonds (see Fig. 3 and Fig. 4). In time, we can enumerate all induced subgraphs  of  isomorphic to . For each such subgraph  we have to verify whether it can be extended to an induced subgraph of the form  for some . We do this as follows. Let  and  be the connected components (diamonds) of . Furthermore, let  be the two vertices of degree 2 in  and similarly let  be the two vertices of degree 2 in  . Now we can verify that  is not contained in any induced subgraph of  isomorphic to  (for some ) by checking for each pair , with , that  and  belong to different components of . This can be done in polynomial time and consequently the recognition of hereditarily CD graphs is a polynomially solvable problem.

The second part of the theorem follows from Theorem 6.1 since every hereditarily CD graph is chordal and chordal graphs are a subclass of for the polynomial .

It might seem conceivable that a similar approach as the one used in Theorem 6.1 could be used to develop an efficient algorithm for recognizing connected-domishold graphs in classes of graphs with only polynomially many minimal connected dominating sets. However, it is not known whether there exists an output-polynomial time algorithm for the problem of enumerating minimal connected dominating sets. In fact, as shown by Kanté et al. [36], even when restricted to split graphs, this problem is equivalent to the well-known TRANS-ENUM problem in hypergraphs, the problem of enumerating the inclusion-minimal transversals of a given hypergraph. The TRANS-ENUM problem has been intensively studied but it is still open whether there exists an output-polynomial time algorithm for the problem (see, e.g., the survey [27]).

### 6.2 The minimum-weight connected dominating set problem

The Minimum-Weight Connected Dominating Set (MWCDS) problem takes as input a connected graph  together with a cost function , and the task is to compute a connected dominating set of minimum total cost, where the cost of a set  is defined, as usual, as . This NP-hard problem [35] has been studied extensively due to its many applications in networking (see, e.g., [7][25][61]). The problem is not only NP-hard but also hard to approximate, even for split graphs. This can be seen as follows: Let  be a Sperner hypergraph with  and let  be its split-incidence graph. Then  is a connected split graph
without universal vertices, hence $MSH(G) = MNH(G)$ by Lemma 4.2. It can be seen that the hyper-edge set of $MNH(G)$ is exactly $E$, and therefore Proposition 2.1 implies that the problem of finding a minimum connected dominating set in $G$ is equivalent to the HITTING SET problem in hypergraphs, the problem of finding a minimum transversal of a given hypergraph. This latter problem is known to be equivalent to the well-known SET COVER problem and hence inapproximable to within a factor of $(1 - \epsilon)\log |V|$, unless $P = \text{NP}$ [23]. It follows that the MWCDS problem is hard to approximate to within a factor of $(1 - \epsilon)\log |V(G)|$, even in the class of split graphs.

This computational intractability of the MWCDS problem motivates the question of identifying restrictions on the input instances under which the problem can be solved efficiently. In this section, we show that the MWCDS problem is polynomially solvable in the class of hereditarily CD graphs, and consequently in the class of $F_2$-free split graphs. This is done by further exploiting the connections with vertex separators and Boolean functions.

First, we recall the following known results about: (i) the relation between the numbers of prime implicants of a threshold Boolean function and its dual, and (ii) the complexity of dualizing threshold Boolean functions. These results were proved in the more general context of regular Boolean functions (as well as for other generalizations, see, e.g., [8]).

**Theorem 6.3.** Let $f$ be an $n$-variable threshold Boolean function having exactly $p$ prime implicants. Then:

1. (Bertolazzi and Sassano [6], Crama [20], see also [21, Theorem 8.29]) The dual function $f^d$ has at most $q$ prime implicants, where $q$ is the total number of variables in the complete DNF of $f$.

2. (Crama and Hammer [21, Theorem 8.28] and Peled and Simeone [52]) There is an algorithm running in time $O(n^2p)$ that, given the complete DNF of $f$, computes the complete DNF of the dual function $f^d$.

We remark that the algorithm by Crama and Hammer [21] is already presented as having time complexity $O(n^2p)$, while the one by Peled and Simeone [52] is claimed to run in time $O(np)$. However, since $f^d$ can have $O(np)$ prime implicants, the total size of the output is of the order $O(n^2p)$. The time complexity $O(np)$ of the algorithm by Peled and Simeone relies on the assumption that the algorithm outputs the prime implicants of the dual function one by one, each time overwriting the previous prime implicant (with a constant number of operations per implicant on average).

The above relation between the numbers of prime implicants of a threshold Boolean function and its dual implies that classes of connected-domishold graphs with only polynomially many minimal separators are exactly the same as the classes of connected-domishold graphs with only polynomially many minimal connected dominating sets. More precisely:

**Lemma 6.4.** Let $G = (V, E)$ be an $n$-vertex connected-domishold graph that is not complete. Let $\nu_c$ (resp. $\nu_s$) denote the number of minimal connected dominating sets (resp. of minimal separators) of $G$. Then $\nu_s \leq (n - 2)\nu_c$ and $\nu_c \leq (n - 2)\nu_s$.

**Proof.** By Proposition 3.3, the minimal separator function $f_{G}^{ms}$ is threshold. Function $f_{G}^{ms}$ is an $n$-variable function with exactly $\nu_c$ prime implicants in its complete DNF. Recall from the proof of Proposition 3.3 that the dual function $(f_{G}^{ms})^d$ takes value 1 precisely on the vectors $x \in \{0, 1\}^V$ whose support is a connected dominating set of $G$. Therefore, the prime implicants of $(f_{G}^{ms})^d$ are in bijective correspondence with the minimal connected dominating sets of $G$ and the number of prime implicants of $(f_{G}^{ms})^d$ is exactly $\nu_c$. Since every minimal separator of $G$ has at most $n - 2$ vertices, Theorem 6.3 implies that $\nu_c \leq (n - 2)\nu_s$, as claimed.

Conversely, since $f_{G}^{ms} = ((f_{G}^{ms})^d)^d$, the inequality $\nu_s \leq (n - 2)\nu_c$ can be proved by a similar approach, provided we show that every minimal connected dominating set of $G$ has at most $n - 2$ vertices. But this is true since if $D$ is a connected dominating set of $G$ with at least $n - 1$ vertices,
connected dominating sets of $G$ \(\subseteq D\) for some \(u \in V(G)\), then a smaller connected dominating set \(D'\) of \(G\) could be obtained by fixing an arbitrary spanning tree \(T\) of \(G[D]\) and deleting from \(D\) an arbitrary leaf \(v\) of \(T\) such that \(N_G(u) \neq \{v\}\). (Note that since \(G\) is connected but not complete, it has at least three vertices, hence \(T\) has at least two leaves.) This completes the proof. \(\square\)

We now have everything ready to derive the main result of this section. Recall that for a polynomial \(\text{poly}\), we denote by \(G_{\text{poly}}\) the class of graphs with at most \(\text{poly}(|V(G)|)\) minimal vertex separators.

**Theorem 6.5.** For any nonzero polynomial \(\text{poly}\), the set of minimal connected dominating sets of an \(n\)-vertex connected-domishold graph from \(G_{\text{poly}}\) has size at most \(O(n \cdot \text{poly}(n))\) and can be computed in time \(O(n \cdot \text{poly}(n) \cdot (n^2 + \text{poly}(n)))\). In particular, the MWCDS problem is solvable in polynomial time in the class of connected-domishold graphs from \(G_{\text{poly}}\).

**Proof.** Let \(\text{poly}\) and \(G\) be as in the statement of the theorem and let \(\mathcal{C}(G)\) be the set of minimal connected dominating sets of \(G\). If \(G\) is complete, then \(\mathcal{C}(G) = \{\{v\} : v \in V(G)\}\) and thus \(|\mathcal{C}(G)| = n = O(n \cdot \text{poly}(n))\) (since the polynomial is nonzero). Otherwise, we can apply Lemma 6.4 to derive \(|\mathcal{C}(G)| \leq (n - 2) \cdot \text{poly}(n)\).

A polynomial time algorithm to solve the MWCDS problem for a given connected-domishold graph \(G \in G_{\text{poly}}\) with respect to a cost function \(c : V(G) \to \mathbb{R}^+\) can be obtained as follows. First, we may assume that \(G\) is not complete, since otherwise we can return a set \(\{v\}\) where \(v\) is a vertex minimizing \(c(v)\). We use a similar approach as in the proof of Theorem 6.4. Using the algorithm of Berry et al. [4], we compute in time \(O(n^3 \cdot \text{poly}(n))\) the set \(\Sigma\) of all minimal vertex separators of \(G\). We can assume that each minimal vertex separator has its elements listed according to some fixed order of \(V(G)\) (otherwise, we can sort them in time \(O(n \cdot \text{poly}(n))\) using, e.g., bucket sort). The minimal separator hypergraph, \(\text{MSH}(G)\), is then computed by comparing each pair of sets in \(\Sigma\) and discarding the non-minimal ones; this can be done in time \(O(n \cdot \text{poly}(n)^2)\). The minimal separator hypergraph directly corresponds to the complete DNF of the minimal separator function \(f_{\text{MS}}^G\).

The next step is to compute the complete DNF of the dual function \((f_{\text{MS}}^G)^d\). By Theorem 6.3, this can be done in time \(O(n^2 \cdot \text{poly}(n))\). Since each term of the DNF is a prime implicant of \((f_{\text{MS}}^G)^d\) and the prime implicants of \((f_{\text{MS}}^G)^d\) are in bijective correspondence with the minimal connected dominating sets of \(G\), we can read off from the DNF all the minimal connected dominating sets of \(G\). The claimed time complexity follows.

Once the list of all minimal connected dominating sets is available, a polynomial time algorithm for the MWCDS problem on \((G, c)\) follows immediately. \(\square\)

In the case of chordal graphs, we can improve the running time by using one of the known linear-time algorithms for listing the minimal vertex separators of a given chordal graph due to Kumar and Veni Madhavan [42], Chandran and Grandoni [14], and Berry and Pogorelcnik [5].

**Theorem 6.6.** Every \(n\)-vertex connected-domishold chordal graph has at most \(O(n^2)\) minimal connected dominating sets, which can be enumerated in time \(O(n^3)\). In particular, the MWCDS problem is solvable in time \(O(n^3)\) in the class of connected-domishold chordal graphs.

**Proof.** Let \(G\) be an \(n\)-vertex connected-domishold chordal graph. The theorem clearly holds for complete graphs, so we may assume that \(G\) is not complete. Since \(G\) is chordal, it has at most \(n\) minimal vertex separators [55]; consequently, \(G\) has at most \(n\) minimal separators. Since \(G\) is connected-domishold, it has at most \(n(n - 2)\) minimal connected dominating sets, by Lemma 6.4.

The minimal connected dominating sets of \(G\) can be enumerated as follows. First, we compute all the \(O(n)\) minimal vertex separators of \(G\) in time \(O(n + m)\) (where \(m = |E(G)|\)) using one of the known algorithms for this problem on chordal graphs [5, 14, 42]. Assuming again that each minimal vertex separator has its elements listed according to some fixed order of \(V(G)\), we then eliminate those that are not also minimal separators in time \(O(n^3)\), by directly comparing each of the \(O(n^2)\) pairs for inclusion.
The list of \( \mathcal{O}(n) \) minimal separators of \( G \) yields its minimal separator function, \( f^\text{ms}_{G} \). The list of minimal connected dominating sets of \( G \) can be obtained in time \( \mathcal{O}(n^3) \) by dualizing \( f^\text{ms}_{G} \) using one of the algorithms given by Theorem 6.3. The MWCD problem can now be solved in time \( \mathcal{O}(n^3) \) by evaluating the cost of each of the \( \mathcal{O}(n^2) \) minimal connected dominating sets and outputting one of minimum cost.

**Corollary 6.7.** The MWCD problem is solvable in time \( \mathcal{O}(n^3) \) in the class of hereditarily CD graphs and in particular in the class of \( F_2 \)-free split graphs.

**Proof.** By Theorem 6.3 every hereditarily CD graph is chordal so Theorem 6.6 applies. The statement for \( F_2 \)-free split graphs follows from Corollary 5.8.

We conclude this section with two remarks, one related to Theorem 6.6 and one related to Theorems 6.4 and 6.5.

**Remark 6.8.** The bound \( \mathcal{O}(n^2) \) given by Theorem 6.6 on the number of minimal connected dominating sets in an \( n \)-vertex connected-domishold chordal graph is sharp. There exist \( n \)-vertex connected-domishold chordal graphs with \( \Theta(n^2) \) minimal connected dominating sets. For instance, let \( S_n \) be the split graph with \( V(S_n) = K \cup I \) where \( K = \{u_1, \ldots, u_n\} \) is a clique, \( I = \{v_1, \ldots, v_n\} \) is an independent set, \( K \cap I = \emptyset \), and for each \( i \in [n] \), vertex \( u_i \) is adjacent to all vertices of \( I \) except \( v_i \). Since every vertex in \( I \) has a unique non-neighbor in \( K \), we infer that \( S_n \) is \( F_2 \)-free. Therefore, by Corollary 5.8 \( S_n \) is a (hereditarily) connected-domishold graph. Note that every set of the form \( \{u_i, u_j\} \) where \( 1 \leq i < j \leq n \) is a minimal connected dominating set of \( S_n \). It follows that \( S_n \) has at least \( \Theta(n^2) \) minimal connected dominating sets.

**Remark 6.9.** Theorems 6.4 and 6.5 motivate the question of whether there is a polynomial \( poly \) such that every connected CD graph \( G \) has at most \( poly(|V(G)|) \) minimal vertex separators. As shown by the following family of graphs, this is not the case. For \( n \geq 2 \), let \( G_n \) be the graph obtained from the disjoint union of \( n \) copies of the \( P_4 \), say \( (x_i, a_i, b_i, y_i) \) for \( i = 1, \ldots, n \), by identifying all vertices \( x_i \) into a single vertex \( x \), all vertices \( y_i \) into a single vertex \( y \), and for each vertex \( z \) other than \( x \) or \( y \), adding a new vertex \( z' \) and making it adjacent only to \( z \). It is not difficult to see that \( G_n \) has exactly two minimal CD sets, namely \( \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\} \cup \{v\} \) for \( v \in \{x, y\} \). A CD structure of \( G_n \) is given by \((w, t)\) where \( t = 4n + 1 \), \( w(x) = w(y) = 1 \), \( w(a_i) = w(b_i) = 2 \) for all \( i \in \{1, \ldots, n\} \) and \( w(z) = 0 \) for all other vertices \( z \). Therefore, \( G_n \) is CD. However, \( G_n \) has \( 4n + 2 \) vertices and \( 2^n \) minimal \( x, y \)-separators, namely all sets of the form \( \{c_1, \ldots, c_n\} \) where \( c_i \in \{a_i, b_i\} \) for all \( i \).

7 Proof of Theorem 5.3

**Theorem 5.3** (restated). For every graph \( G \), the following are equivalent:

1. \( G \) is hereditarily CD.
2. \( G \) is hereditarily 2-asummable w.r.t. separators.
3. \( G \) is hereditarily 1-Sperner w.r.t. separators.
4. \( G \) is an \( \{F_1, F_2, H_1, H_2, \ldots\} \)-free chordal graph, where the graphs \( F_1, F_2 \), and a general member of the family \( \{H_i\} \) are depicted in Fig. 3.

**Proof.** The implications \([3] \Rightarrow [1] \Rightarrow [2]\) follow from Corollary 3.6.

For the implication \([2] \Rightarrow [4]\), we only need to verify that none of the graphs in the set \( F := \{C_k : k \geq 4\} \cup \{F_1, F_2\} \cup \{H_i : i \geq 1\} \) is 2-asummable w.r.t. separators. Let \( F \in F \). Suppose first that \( F \) is a cycle \( C_k \) for some \( k \geq 4 \), let \( u_1, u_2, u_3, u_4 \) be four consecutive vertices on the cycle. Let
Claim 1. Either \( N(x) \cap \{a', b'\} = \emptyset \) or \( N(y) \cap \{a', b'\} = \emptyset \). Similarly, either \( N(x') \cap \{a, b\} = \emptyset \) or \( N(y') \cap \{a, b\} = \emptyset \).

Proof. If \( N(x) \cap \{a', b'\} \neq \emptyset \) and \( N(y) \cap \{a', b'\} \neq \emptyset \), then there exists an \((x, y)\)-path in \( G - S \), contrary to the fact that \( S \) is an \((x, y)\)-separator. The other statement follows similarly.

Claim 2. \(|N(C') \cap Z| \leq 1\) and \(|N(C) \cap Z'| \leq 1\).

Proof. If \(|N(C') \cap Z| > 1\), then \( Z \subseteq N(C') \). Since \( C' \cap S = \emptyset \), this implies that \( x \) and \( y \) are in the same connected component of \( G - S \), a contradiction. The other statement follows by symmetry.

Claim 2 implies that \( Z \neq Z' \). Up to symmetry, it remains to analyze five cases, depending whether the sets \( C, C', Z, Z' \) have vertices in common (where possible) or not. In what follows we use the notation \( u \sim v \) (resp. \( u \nsim v \)) to denote the fact that two vertices \( u \) and \( v \) are adjacent (resp. non-adjacent).

Case 3: \(|C \cap Z'| = |Z \cap Z'| = 1\).

Without loss of generality, we may assume that \( a = a' \). Since \( C \cap Z = \emptyset \) and \( a = a' \) it follows that \( x' \notin Z \), implying \( Z \cap Z' = \{y'\} \). Without loss of generality, we may assume that \( y' = y \). But the fact that \( y \sim a \) implies \( y' \sim x' \), leading to a contradiction.

The case \(|C' \cap Z| = |Z \cap Z'| = 1\) is symmetric to Case 3.

Case 4: \(|Z \cap Z'| = 1\) and \( C \cap Z' = C' \cap Z = \emptyset \).

Without loss of generality, we may assume that \( x = x' \). Since \( a, b \notin S' \) and \( S' \) separates \( x' \) and \( y' \), we conclude that \( N(y') \cap \{a, b\} = \emptyset \). By symmetry, \( N(y) \cap \{a', b'\} = \emptyset \), and consequently, \( y' \notin S \) and \( y \notin S' \). Since \( S \) separates \( x \) and \( y \) and \( \{a', b', y'\} \cap S = \emptyset \), we have \( N(y) \cap \{a', b', y'\} = \emptyset \), and, similarly, \( N(y') \cap \{a, b, y\} = \emptyset \).

We must have \( y \sim y' \) since otherwise \( G \) contains either an induced \( C_4 \) on the vertex set \( \{y, a, a', y'\} \) (if \( a \sim a' \)) or an induced \( C_5 \) on the vertex set \( \{y, a, x = x', a', y'\} \) (otherwise).

To avoid an induced copy of \( H_1 \) on the vertex set \( \{y, a, b, x = x', a', b', y'\} \), we may assume, without loss of generality, that \( a \sim a' \).
Suppose first that $b \sim b'$. Then also $a \sim b'$ or $a' \sim b$, since otherwise $\{a, a', b', b\}$ would induce a copy of $C_4$. But now (depending if we have one edge or both) the vertex set $\{y, a, b, a', b', y'\}$ induces a copy of either $F_1$ or of $F_2$. Therefore, $b \sim b'$.

Suppose that $a' \sim b$. But now, either the vertex set $\{y, a, b, x = x', a', b'\}$ induces a copy of $F_2$ (if $a \sim b'$), or the vertex set $\{y, a, b, a', b', y'\}$ induces a copy of $F_1$ (if $a' \sim b'$). Therefore, $a' \sim b$, and by symmetry, $a \sim b'$. But now, the vertex set $\{y, a, b, x = x', a', b'\}$ induces a copy of $F_1$, a contradiction.

**Case 5:** $|C \cap Z'| = |C' \cap Z| = 1$ and $Z \cap Z' = \emptyset$.

Without loss of generality, we may assume that $a = x'$ and $a' = x$. By Claim 1 it follows that $b \sim y'$. The fact that $y \sim x = a'$ implies $y \notin S'$ (since $S'$ is a clique) and consequently also $y \sim y'$ (otherwise $x' = a$ and $y'$ would be in the same component of $G - S'$). To avoid an $(x, y)$-path in $G - S$, we conclude that $y \sim b'$. Now, the vertices $\{a = x', a' = x, b', b, y', y'\}$ induce either a copy of $F_1$ (if $b \sim b'$) or of $F_2$ (otherwise). In either case, we reach a contradiction.

**Case 6:** $|C \cap Z'| = 1$ and $C' \cap Z = Z \cap Z' = \emptyset$.

Without loss of generality, we may assume that $a = x'$. By Claim 2 we have $|N(C') \cap Z| \leq 1$. Thus, we may assume that $x \notin N(C')$. Consequently, $N(x) \cap \{a, b\} = \emptyset$ and therefore also $x \sim y'$, for otherwise we would have a $C_4$ induced by $\{a, a', y', x\}$. To avoid an $(x', y)$-path in $G - S'$, we conclude that $b \sim y'$. Moreover, we also have $N(b) \cap \{a', b'\} = \emptyset$, since otherwise the vertex set $\{x, a = x', a', b', b', y'\}$ induces either a copy of $F_1$ (if $|N(b) \cap \{a', b'\}| = 1$) or of $F_2$ (otherwise). If $y \sim y'$, then, to avoid an induced $C_4$ on $\{y, a = x', a', y\}$, we conclude that $y \sim a'$. But now we have a copy of $F_1$ induced by $\{x, b, a = x', a', y, y\}$, a contradiction. Thus, $y \sim y'$, implying also $N(y) \cap \{a', b'\} = \emptyset$, since otherwise the vertex set $\{b, a = x', y, a', b', y'\}$ induces either a copy of $F_1$ (if $|N(y) \cap \{a', b'\}| = 1$) or of $F_2$ (otherwise).

Since neither of the vertices $a'$, $b'$ and $y'$ is adjacent to $b$ and $S$ is a clique containing $b$, we conclude that $\{a', b', y'\} \cap S = \emptyset$. In particular, if $K$ denotes the component of $G - S$ containing $a'$, this implies $b', y' \in V(K)$. By Lemma 2.2 there exists a vertex $w \in V(K)$ that dominates $S$. Since $S$ separates $x$ from $y$, we have $\{x, y\} \notin K$; without loss of generality, we may assume that $y \notin V(K)$.

Let $P = (w_1, w_2, \ldots, w_k)$ be a shortest $\{a', b', y'\} - w$ path in $K$ where $w_1 \in \{a', b', y'\}$ and $w_k = w$. Since $w_1$ is not adjacent to $b$ but $w_k$ is, we have $k > 1$.

Suppose that $k = 2$. If $w \sim y'$, then the vertex set $\{b, a = x', w, a', b', y'\}$ induces either a copy of $F_1$ (if $|N(w) \cap \{a', b'\}| = 1$) or of $F_2$ (otherwise), a contradiction. Hence $w \sim y'$. To avoid an induced $C_4$ on the vertices $\{w, a = x', a', y'\}$, we conclude that $w \sim a'$. But now, the vertex set $\{y, a = x', b, w, a', y'\}$ induces a copy of $F_1$, a contradiction. Therefore, $k \geq 3$.

To avoid an induced copy of a cycle of order at least 4, we conclude that vertex $a = x'$ dominates $P$. If $y' \sim w_2$ then also $a' \sim w_2$ and $b' \sim w_2$ (or otherwise we would have an induced $C_4$ on the vertex set $\{a = x', w_2, b', y'\}$ or $\{a = x', w_2, a', y'\}$) but that gives us an induced $F_1$ on the vertex set $\{y', a', a = x', w_2, w_3, w_4\}$ (where $w_4 = b$ if $k = 3$). Therefore, $y' \sim w_2$. Without loss of generality, we may assume that $w_1 = a'$. But now, the vertex set $\{y', a' = w_1, b, a = x', w_2, w_3\}$ induces a copy of either $F_1$ (if $b' \sim w_2$) or of $F_2$ (otherwise), a contradiction.

The case $|C' \cap Z| = 1$ and $C \cap Z' = Z \cap Z' = \emptyset$ is symmetric to Case 6.

**Case 7:** $C' \cap Z = C \cap Z' = Z \cap Z' = \emptyset$.

We will analyze this case depending on the number of edges between $\{x, y\}$ and $\{x', y'\}$. Let $k := |\{xx', xy', yx', yy'\} \cap E(G)|$. Clearly $k \in \{0, 1, 2, 3\}$, since $k = 4$ would imply an induced $C_4$ $\{x, y, x', y'\}$.

**Case 7.1:** $k = 3$.

Without loss of generality, we may assume that $\{xx', xy', yx', yy'\} \cap E(G) = \{xx', yx', yy'\}$. To avoid an induced $C_4$ on the vertex set $\{a, x, x', y\}$, we have $x' \sim a$ and, for a similar reason, $y \sim a'$.
To avoid an \((x,y)\)-path in \(G - S\), we have \(x \sim a'\), and similarly, \(y' \sim a\). But now, the vertex set \(\{x, a, x', y, a', y'\}\) induces either a copy of \(F_1\) (if \(a \sim a'\)) or of \(F_2\) (otherwise), a contradiction.

**Case 7.2:** \(k = 2\).

Up to symmetry, we have two subcases:

**Case 7.2.1:** \(\{xx', xy', yx', yy'\} \cap E(G) = \{xx', yy'\}\).

By Claim 1, vertices \(x', y'\) cannot be both adjacent to \(a\). Without loss of generality, we may assume that \(x' \sim a\).

Suppose that \(a \sim a'\). If \(y \sim a'\) then \(x \sim a'\) and \(\{y, a, x, x', a'\}\) induces a \(C_5\), a contradiction. Therefore \(y \sim a'\). If \(y' \sim a'\) then the subgraph of \(G\) induced by \(\{a, x, x', a', y'\}\) contains an induced \(C_4\) or \(C_5\) (depending on whether \(x \sim a'\) or not), a contradiction. Therefore \(y' \sim a\). But now, the graph \(G'[a, x, x', a', y', y]\) contains an induced \(C_5\) or \(C_6\) (depending on whether \(x \sim a'\) or not). This shows that \(a \sim a'\).

To avoid an induced \(C_4\) on the vertex set \(\{x, x', a', a\}\), we infer that \(x \sim a'\); consequently, by Claim 1, \(a' \sim y\). To avoid an induced \(C_4\) on the vertex set \(\{a, a', y', y\}\), we further infer that \(a \sim y'\); but now, the vertex set \(\{x', x, a', a, y, y'\}\) induces a copy of \(F_1\), a contradiction.

**Case 7.2.2:** \(\{xx', xy', yx', yy'\} \cap E(G) = \{xx', yx'\}\).

First observe that in this case \(x' \in S\), since otherwise \(S\) would not separate \(x\) from \(y\). In particular, since \(S\) is a clique, this implies \(a \sim x'\) and \(b \sim x'\). Furthermore, this implies that \(N(y') \cap \{a, b\} = \emptyset\), for otherwise \(S'\) would not separate \(x'\) and \(y'\). Replacing \(C = \{a, b\}\) with \(\tilde{C} = \{x', b\}\), we can now use the same arguments as in Case 6 to obtain a contradiction.

**Case 7.3:** \(k = 1\).

Without loss of generality, we may assume that \(\{xx', xy', yx', yy'\} \cap E(G) = \{xx'\}\).

We will analyze this case depending on the number \(\ell\) of edges between \(C\) and \(C'\). Formally, let \(\ell := |\{aa', ab', ba', bb'\} \cap E(G)|\).

**Case 7.3.1:** \(\ell = 0\).

First observe that \(y \sim a'\), since otherwise \(x \sim a'\) and the subgraph \(G'[\{y, a, x, x', a'\}]\) would either contain an induced \(C_4\) (if \(b \sim x'\)) or would be isomorphic to \(C_5\) (otherwise). Similar arguments imply that \(y \sim b', y' \sim a\), and \(y' \sim b\). Furthermore, in order to avoid an induced \(H_1\) on the vertex set \(\{y, a, b, x, a', b', y'\}\), vertex \(x\) can not be adjacent to both \(a'\) and \(b'\). By symmetry, \(x'\) can not be adjacent to both \(a\) and \(b\). Without loss of generality, we may assume that \(x' \sim b\) and \(x \sim b'\). If \(x \sim a'\), then also \(x' \sim a\), since otherwise we would have an induced \(H_1\) on the vertex set \(\{y, a, b, x, x', a'\}\), but now we have an induced \(F_1\) on the vertex set \(\{a, b, x, x', b', a'\}\). Therefore \(x \sim a'\), and by symmetry \(x' \sim a\). But now, \(G\) contains an induced copy of \(H_2\) on the vertex set \(\{y, a, b, x, x', a', b', y'\}\), a contradiction.

**Case 7.3.2:** \(\ell = 1\).

Without loss of generality, we may assume that \(\{aa', ab', ba', bb'\} \cap E(G) = \{ba'\}\).

To avoid an induced \(C_4\) on the vertex set \(\{x, x', a', b\}\), we may assume without loss of generality that \(x \sim a'\). By Claim 1, this implies that \(y \sim a'\) and \(y \sim b'\). To avoid an induced \(F_1\) on the vertex set \(\{y, a, b, x, x', a'\}\), we infer that \(N(x') \cap \{a, b\} = \emptyset\). Claim 1 implies that \(y' \sim a\) and \(y' \sim b\). If \(x' \sim b\), then \(x' \sim a\), but now an induced \(C_4\) arises on the vertex set \(\{x', a, b, a'\}\). Therefore, \(x' \sim b\).

Furthermore, to avoid an induced \(F_2\) on the vertex set \(\{a, x, b, x', a', b', y'\}\), we infer that \(\{ax', b'b\} \cap E(G) \neq \emptyset\). Without loss of generality, we may assume that \(ax' \in E(G)\). But now, we get an induced \(F_1\) on the vertex set \(\{a, b, x, x', a', b', y'\}\), a contradiction.

**Case 7.3.3:** \(\ell = 2\).
To avoid an induced $C_4$ on the vertex set $\{a, b, a', b'\}$, we infer that the two edges in $\{aa', ab', ba', bb'\} \cap E(G)$ must share an endpoint. Without loss of generality, we may assume that $\{aa', ab', ba', bb'\} \cap E(G) = \{ba', bb'\}$.  

Suppose first that $x' \sim b$. Then $y' \sim a$ and $y' \sim b$. To avoid an induced $F_2$ on the vertex set $\{x, x', b, a', b', y\}$, we infer that $N(x) \cap \{a', b'\} \neq \emptyset$ and consequently $y \sim a'$ and $y \sim b'$. If $x \sim b'$, then the vertex set $\{a, b, x, a', b', y'\}$ induces a copy of either $F_1$ (if $x \sim a'$) or of $F_2$ (otherwise), a contradiction. Therefore, $x \sim b'$, and a similar argument shows that also $x \sim a'$. But now $G$ contains an induced $F_2$ on the vertex set $\{x, x', a', b', y'\}$, a contradiction.

Suppose now that $x' \sim b$ and notice that this implies $x' \sim a$, or otherwise we would have an induced $C_4$ on the vertex set $\{b, a', x', b'\}$. To avoid an induced $C_4$ on the vertex set $\{x, x', a', b\}$, we infer that $x \sim a'$. This implies that $y \sim a'$ and $y \sim b'$. But now we have an induced $F_1$ on the vertex set $\{y, a, b, x, a', x'\}$, a contradiction.

Case 7.3.4: $\ell = 3$.

Without loss of generality, we may assume that $\{aa', ab', ba', bb'\} \cap E(G) = \{aa', ba', bb'\}$.  

To avoid an induced $C_4$ on the vertex set $\{x, x', a', b\}$, we may assume without loss of generality that $x \sim a'$ and consequently (by Claim 2.2) $y \sim a'$ and $y \sim b'$. To avoid an induced $F_1$ on the vertex set $\{y, a, b, a', b', y'\}$, we infer that $N(y') \cap \{a, b\} \neq \emptyset$, implying $x' \sim a$ and $x' \sim b$. But now we have an induced $F_1$ on the vertex set $\{a, b, b', y, a', x'\}$, a contradiction.

Case 7.3.5: $\ell = 4$.

To avoid an induced $C_4$ on the vertex set $\{a, x, x', a'\}$, we may assume without loss of generality that $x \sim a'$ and by Claim 2.2 this implies $N(y) \cap \{a', b'\} = \emptyset$.  

If $y' \sim a$ and $y' \sim b$, then the vertex set $\{y, a, b, a', b', y'\}$ induces a copy of $F_2$, a contradiction. We may assume without loss of generality that $y' \sim b$ and by Claim 2.2 $x' \sim a$ and $x' \sim b$. But now we have an induced $F_2$ the vertex set $\{a, b, b', y, a', x'\}$, a contradiction.

Case 7.4: $k = 0$.

Similarly as in Case 2.2, we will analyze several subcases depending on the number $\ell$ of edges between $C$ and $C'$.

Case 7.4.1: $\ell = 1$.

Without loss of generality, we may assume that $\{aa', ab', ba', bb'\} \cap E(G) = \{ba'\}$.  

Notice that $y$ can not be adjacent to both $a'$ and $b'$, since otherwise $x \sim a'$ and $x \sim b'$ (by Claim 2.2), and consequently an induced copy of $F_1$ arises on the vertex set $\{x, a, b, y, a', b'\}$. By a symmetric argument, we infer that $y$ can not be adjacent to both $a$ and $b$. Similar arguments applied to $x$ and $x'$ imply that $\{x, y\} \cap S' = \emptyset$ and $\{x', y'\} \cap S = \emptyset$. Let $K$ be the component of $G - S'$ such that $b \in V(K)$. Then $\{a, b, y\} \subseteq V(K)$. Without loss of generality, we may assume that $y' \notin V(K)$. By Lemma 2.2 there exists a vertex $w' \in V(K)$ that dominates $S'$. Clearly, $w' \notin \{a, b, y, a', b'\}$. Moreover, since $y' \notin V(K)$, we have $w' \neq y'$ and $w' \sim y'$. Let $K'$ be the component of $G - S$ such that $a' \in V(K')$. Then $\{a', b', y'\} \subseteq V(K')$. Without loss of generality, we may assume that $y \notin V(K')$. By Lemma 2.2 there exists a vertex $w \in V(K')$ that dominates $S$. Clearly, $w \notin \{a, b, a', b', y'\}$. Moreover, since $y \notin V(K')$, we have $w \neq y$ and $w \sim y$.

Notice that $w \neq w'$, since otherwise the vertex set $\{y, a, b, w = w', a', b'\}$ would induce a copy of $F_1$. Moreover, we have $w \sim w'$, $w \sim y'$, and $y \sim w'$, since otherwise we could use the same arguments as in Cases 2.2 or 3 with $Z = \{x, y\}$ and $Z' = \{x', y'\}$ replaced with $\tilde{Z} = \{w, y\}$ and $\tilde{Z}' = \{w', y'\}$, respectively, to derive a contradiction.

Since $y \notin V(K')$, we infer that $y \sim a'$ and $y \sim b'$. Similarly, since $y' \notin V(K)$, we have $y' \sim a$ and $y' \sim b$. If $w' \sim a$, then also $w' \sim b$, since otherwise the vertex set $\{w', a, b, a'\}$ would induce a $C_4$. But now we have an induced $F_1$ on the vertex set $\{y, a, b, w', a', b'\}$. Therefore, $w' \sim a$, and by a
symmetric argument also \( w \sim b' \). Further notice that if \( b \sim w' \) and \( a' \sim w \), then we have an induced \( F_1 \) on the vertex set \( \{a, b, w, w', a', b'\} \). Hence, we may assume without loss of generality that \( b \sim w' \).

Let \( P' = (w'_1 = w, w'_2, \ldots, w'_p) \) be a shortest \( w' - \{a, b, y\} \) path in \( K' \), and similarly, \( P = (w_1 = w, w_2, \ldots, w_q) \) be a shortest \( w - \{a', b', y'\} \) path in \( K' \). Since \( w' \not\in \{a, b, y\} \) and \( w \not\in \{a', b', y'\} \), we have \( p \geq 2 \) and \( q \geq 2 \).

Suppose first that \( p = 2 \). Since \( w'_1 = w' \sim a \) and \( w \sim b \), we must have \( w'_2 = y \). But now, the vertex set \( \{w'_1, y, b, a'\} \) induces a \( C_4 \), a contradiction. Therefore, \( p \geq 3 \).

Since \( w' \sim b \), we infer that vertex \( a' \) dominates \( P' \) since otherwise \( G \) would contain an induced copy of \( C_3 \) for some \( j \geq 4 \).

Suppose that \( w'_{p-1} \sim a \). To avoid an induced \( C_4 \) on the vertex set \( \{a', w'_{p-1}, a, b\} \), we infer that \( w'_{p-1} \sim b \). We must have \( p = 3 \) since if \( p \geq 4 \), then the vertex set \( \{a, b, w'_{p-1}, a', w'_{p-2}, w'_3\} \) induces a copy of \( F_1 \). But now, an induced copy of \( F_1 \) arises either on the vertex set \( \{a, b, w_2, a', w'_{p-3}\} \) (if \( b' \sim w'_2 \)), or the vertex set \( \{a, b, w_2, a', b', y'\} \) (otherwise), a contradiction. Therefore, \( w'_{p-1} \sim a \).

Suppose that \( w'_{p-1} \sim y \). In this case, the vertex set \( \{a, b, y, w'_{p-1}, a', w'_{p-2}\} \) induces a copy of either \( F_1 \) (if \( w'_{p-1} \sim b \)) or of \( F_2 \) (otherwise), a contradiction. Therefore, \( w'_{p-1} \sim y \). Consequently, \( w'_p = b \).

Suppose that \( w \sim a' \). If in addition \( w \sim w'_p \), then also \( w \sim w'_p \) (since otherwise the vertex set \( \{w'_{p-2}, w'_{p-1}, w'_p = b, w, a\} \) would induce a \( C_4 \), but now, the vertex set \( \{w'_{p-2}, w'_{p-1}, a', w'_p = b, w, a\} \) induces a copy of \( F_2 \), a contradiction. Therefore, \( w \sim w'_{p-1} \). Let \( w' \) be the neighbor of \( w \) on \( P' \), minimizing \( i \). Since \( w' = w' \sim w \), we have \( i > 2 \). Moreover, since \( w \sim w'_{p-1} \), we have \( i < p \). But now, the vertex set \( \{w'_{i-1}, a', w'_i, w, w'_p = b, a\} \) induces either a copy of \( F_1 \) (if \( w_i \sim b \)) or of \( F_2 \) (otherwise), a contradiction. Therefore, \( w \sim a' \).

Since \( w \sim a' \), we can now apply symmetric arguments as for \( P' \) to deduce that \( w_q = a' \) and that \( b \) dominates \( P \).

Suppose first that \( V(P) \cap V(P') = \emptyset \). To avoid an induced \( C_4 \) on the vertex set \( \{w'_{p-2}, a' = w_q, b = w'_p, w'_{p-2}\} \), we infer that \( w'_{p-2} \sim w_q-2 \). Suppose that \( w'_{p-1} \sim w_q-1 \). Then also \( w'_{p-1} \sim w_q-2 \) (since otherwise we would have an induced \( C_4 \) on the vertex set \( \{w'_{p-1}, w_q-2, w_q-1, w_q = a'\} \)) and by a symmetric argument also \( w'_{p-2} \sim w_q-1 \). But now, we have an induced \( F_1 \) on the vertex set \( \{w'_{p-2}, w_q = a', w'_{p-1}, w'_p = b, w_q-1, w_q-2\} \). Thus, \( w'_{p-1} \sim w_q-1 \). Moreover, we have either \( w'_{p-1} \sim w_q-1 \) or \( w'_{p-1} \sim w'_p \), since otherwise an induced \( F_2 \) arises on the vertex set \( \{w_q, w_{q-1}, w_{q-2}, w'_p, w'_{p-1}, w'_p\} \). Without loss of generality, assume that \( w'_{p-2} \sim w_q-1 \). However, an induced copy of \( F_1 \) arises on vertex set \( \{w'_{p-2}, a' = w_q, w_{q-1}, w_{q-2}, b = w'_p, w_{q-3}\} \) (where if \( q = 2 \) we define \( w_{-1} = a \)). This contradiction shows that \( V(P) \cap V(P') \neq \emptyset \).

Since \( w_q = a' \) and due to the minimality of \( P \), we have \( N(a') \cap V(P) = \{w_{q-1}\} \). On the other hand, since \( a' \) dominates \( P \), we have \( N(a') \cap V(P') = V(P') \). Therefore \( \emptyset \neq V(P) \cap V(P') = V(P) \cap (N(a') \cap V(P')) = (N(a') \cap V(P)) \cap V(P') = \{w_{q-1}\} \cap V(P') \subseteq \{w_{q-1}\} \), which yields \( V(P) \cap V(P') = \{w_{q-1}\} \). A symmetric argument implies that \( V(P) \cap V(P') = \{w_{q-1}\} \); in particular, \( w_{q-1} = w'_{p-1} \). To avoid an induced \( C_4 \) on the vertex set \( \{w'_{p-2}, a' = w_q, b = w'_p, w'_{p-2}\} \), we infer that \( w'_{p-2} \sim w_{q-2} \). But now, an induced copy of \( F_1 \) arises on vertex set \( \{w'_{p-3}, w'_{p-2}, a' = w_q, w'_{p-1}, b = w'_p, w_{q-2}\} \) (where if \( p = 2 \) we define \( w'_{-1} = b' \)). This contradiction completes the proof of Case 1.

Case 7.4.2: \( \ell = 2 \).

To avoid an induced \( C_4 \) on the vertex set \( \{a, b, a', b'\} \), we infer that the two edges in \( \{a'a, ab', ba', bb'\} \cap E(G) \) must share an endpoint. Without loss of generality, we may assume that \( \{a'a, ab', ba', bb'\} \cap E(G) = \{ba', bb'\} \). Let \( K \) be the component of \( G - S' \) such that \( b \in V(K) \). Without loss of generality, we may assume that \( y' \not\in V(K) \).

If \( b \) dominates \( S' \), then replacing \( Z' = \{x', y'\} \) with \( \tilde{Z}' = \{b, y'\} \), we can use the same arguments as in Case 1 to obtain a contradiction. Therefore, \( b \) does not dominate \( S' \), and there exists a vertex \( c' \in S' \) non-adjacent to \( b \). In particular, \( c' \not\in \{x, y\} \). Since \( S' \) is a clique, we have \( a' \sim c' \) and \( b' \sim c' \).

To avoid an induced \( C_4 \) on the vertex set \( \{c', a', b, a\} \), we infer that \( c' \sim a \).
By Lemma 2.2, there exists a vertex \( w \in V(K) \) such that \( w \) dominates \( S' \). By the above, \( w \neq b \); also, since \( a \sim a' \), we have \( w \neq a \). Thus \( w \notin C \). If \( w = x \), then \( G \) contains an induced copy of \( F_0 \) on the vertex set \( \{a, b, x, a', b', y'\} \), a contradiction. Therefore, \( w \neq x \), and with a similar argument we obtain \( w \neq y \); thus \( w \notin Z \). But now we get a contradiction using the same arguments as in Case 6 replacing \( C' = \{a', b'\} \) with \( C' = \{c', d'\} \) and \( Z' = \{x', y'\} \) with \( Z' = \{w, y'\} \). (Indeed, all the assumptions of that case are fulfilled: vertices \( w \) and \( y' \) are in different connected components of \( G - S' \) and each of \( w, y' \) dominates \( S' \); the above arguments imply that \( C' \cap Z = C \cap Z' = \emptyset \); there exists exactly one edge between \( C \) and \( C' \).)

Case 7.4.3: \( \ell \in \{3, 4\} \).

By Claim 4 no generality is lost in assuming that \( N(x) \cap \{a', b'\} = \emptyset \) and \( N(x') \cap \{a, b\} = \emptyset \). But now, the vertex set \( \{x, a, b, a', b', x'\} \) induces either a copy of \( F_1 \) (if \( \ell = 3 \)) or a copy of \( F_2 \) (if \( \ell = 4 \)), a contradiction.

Case 7.4.4: \( \ell = 0 \).

First, observe that \( \{a', b'\} \not\subseteq N(x) \). Indeed, if \( \{a', b'\} \subseteq N(x) \), then Claim 1 implies that \( y \sim a' \) and \( y \sim b' \). Also, to avoid an induced \( C_4 \) on the vertex set \( \{x, a', b', y\} \), we infer that \( y' \sim a \), and, by symmetry, that \( y' \sim b \). But now, an induced \( H_1 \) arises on the vertex set \( \{y, a, b, x, a', b', y'\} \), a contradiction. By symmetry, we also have \( \{a', b'\} \not\subseteq N(y) \), \( \{a, b\} \not\subseteq N(x') \), and \( \{a, b\} \not\subseteq N(y') \). In particular, this implies that \( \{x, y\} \cap S' = \emptyset \), and \( \{x', y'\} \cap S = \emptyset \).

Let \( K \) be the component of \( G - S \) such that \( S \setminus S' \subseteq V(K) \). Since \( x' \) and \( y' \) are in different connected components of \( G - S' \), we may assume without loss of generality that \( y' \notin V(K) \). By Lemma 2.2 there exists a vertex \( w' \in V(K) \) that dominates \( S' \). Since neither of \( a, b, x, y \) dominates \( S' \), we have \( w' \notin \{a, b, x, y\} \). We may assume that \( w' \notin S \) since otherwise we could use the same arguments as in Case 6 with \( C = \{a, b\} \) and \( Z' = \{x', y'\} \) replaced with \( C = \{a, w'\} \) and \( Z' = \{w, y'\} \), respectively, to derive a contradiction. Let \( K' \) be the component of \( G - S \) such that \( S' \setminus S \subseteq V(K') \). Since \( x \) and \( y \) are in different connected components of \( G - S \), we may assume without loss of generality that \( y \notin V(K') \). By Lemma 2.2 there exists a vertex \( w \in V(K') \) that dominates \( S \). Similarly as above, we infer that \( w \notin \{a', b', x', y'\} \), and we may assume that \( w \notin S' \).

Notice that \( w \neq w' \) since otherwise \( G \) would contain an induced \( H_1 \) on the vertex set \( \{y, a, b, w = w', a', b', y'\} \). Furthermore, we may assume that \( \{yw', wy', wy'\} \cap E(G) = \emptyset \) since otherwise we could use the same arguments as in Cases 1, 2 or 3 with \( Z = \{x, y\} \) and \( Z' = \{x', y'\} \) replaced with \( \bar{Z} = \{w, y\} \) and \( \bar{Z}' = \{w, y'\} \), respectively. Moreover, similar arguments as above for \( x \) and \( x' \) imply that \( \{a', b'\} \not\subseteq N(w) \) and \( \{a, b\} \not\subseteq N(w') \).

Let \( P = (w_1 = w, w_2, \ldots, w_p) \) be a shortest \( w'-(S \setminus S') \) path in \( K \), and let \( P' = (w'_1 = w, w'_2, \ldots, w'_q) \) be a shortest \( w-(S' \setminus S) \) path in \( K' \). We may assume that \( w_p = b \) and \( w'_q = a' \). Since \( w' \notin S \), we have \( p \geq 2 \); similarly, \( q \geq 2 \). Moreover, if \( p = q = 2 \) then \( w' \sim b \) and \( w \sim a' \), which implies an induced \( C_4 \) on the vertex set \( \{w, a', w', b\} \). Hence, from now on we assume that \( p \geq 3 \).

Suppose that \( q = 2 \). Then, \( w \sim a' \), and hence \( w \sim b' \). Notice that we may assume that \( w_{p-1} \neq w \) since otherwise \( a' \) would have to be adjacent to all \( w_j \) for \( j \in \{1, \ldots, p - 1\} \) (to avoid an induced \( C_j \) with \( j \geq 4 \)), but then we would have an induced \( H_1 \) on the vertex set \( \{y, a, b, w = w_{p-1}, w_{p-2}, w_{p-3}, a'\} \) (notice that since \( w_1 = w' \sim w = w_{p-1} \), we must have \( p > 3 \), hence \( w_{p-3} \) is well defined). To avoid a long induced cycle, we infer that \( w_{p-1} \neq y \), and also that \( w_{p-1} \sim w \) and \( w_{p-1} \sim a' \). Moreover, \( w_{p-1} \sim y \) since otherwise the vertex set \( \{y, a, b, w = w_{p-1}, a'\} \) induces a copy of either \( F_1 \) (if \( a \sim w_{p-1} \)) or of \( F_2 \) (otherwise). Notice that \( w_{p-1} \notin S \) since otherwise it would be contained in \( S \setminus S' \), contrary to the minimality of \( P \). Therefore, \( \{y, w_{p-1}, a'\} \) is a \( y-a' \) path avoiding \( S \), contrary to the fact that \( y \) and \( a' \) are in different connected components of \( G - S \). This contradiction implies that \( q \geq 3 \).

Suppose that \( w_{p-1} = w \). By the minimality of \( P \), we have \( a \sim w_j \) and \( b \sim w_j \) for every \( j \in \{1, \ldots, p - 1\} \). Also, since \( w_1 = w' \sim w = w_{p-1} \), we have \( p \geq 4 \). If \( a' \sim w_3 \), then we obtain an induced copy of \( H_i \) for some \( i \geq 1 \) on the vertex set \( \{y, a, b, w = w_{p-1}, w_{p-2}, \ldots, w_j, w_{j-1}, w_{j-2}, a'\} \), where
$j \in \{3, \ldots, p\}$ is the maximum index such that $a' \sim w_j$. Therefore, $a' \sim w_3$, and to avoid a long induced cycle, also $a' \sim w_j$ for $j > 4$. A similar argument shows that $b' \sim w_j$ for $j \geq 3$. If $a' \sim w_2$ and $b' \sim w_2$, then we obtain an induced copy of some $H_i$ on the vertex set $V(P) \cup \{a, b, y, a', b', y'\}$. If $a' \sim w_2$ and $b' \sim w_2$ (or vice-versa), then an induced copy of some $H_i$ arises on the vertex set $V(P) \cup \{a, b, y, a', b'\}$, and if $a' \sim w_2$ and $b' \sim w_2$, then an induced copy of some $H_i$ arises $V(P) \cup \{a, b, y, a', b', y'\} \setminus \{w_2\}$. This contradiction shows that we may assume that $w_{p-1} \neq w$.

By similar arguments as above, we may assume that $w_{q-1}' \neq w'$.

Suppose first that $V(P) \cap V(P') = \emptyset$. Let $r \in \{1, \ldots, p-1\}$ be the maximum index such that $a' \sim w_r$. Similarly, let $s \in \{1, \ldots, q-1\}$ be the maximum index such that $b \sim w_s'$. To avoid an induced cycle $C_j$ for some $j \geq 4$, we infer that $a' \sim w_i$ for all $i \in \{1, \ldots, r\}$ and $b \sim w_i'$ for all $i \in \{1, \ldots, s\}$.

Suppose that $r = p-1$. We consider several cases according to the value of $s$.

- Suppose that $s = q - 1$. Then $w_{p-1} \sim w_{q-1}'$ (otherwise the vertex set $\{b, w_{q-1}', w_{q-1}, w_{p-1}\}$ would induce an $C_4$). If $p \geq 4$, an induced $F_1$ arises on the vertex set $\{b, w_{q-1}', a', w_{p-1}, w_{p-2}, w_{p-3}\}$. Hence, $p = 3$. If $q \geq 4$, then the vertex set $\{w_{q-3}', w_{q-2}', w_{q-1}', w_q' = a', w_2', b = w_3\}$ induces either a copy of $F_1$ (if $w_2 \sim w_{q-2}'$) or $F_2$ (otherwise). Hence, $q = 3$. Notice that, since $y$ and $P'$ must be in different components of $G - S$, and $w_2 \not\in S$ by minimality of $P$, we have $y \sim w_2$ and by a symmetric argument also $y' \sim w_2'$. To avoid an induced $F_1$ or $F_2$ on the vertex set $\{a, b = w_3, w = w_1', w_2', w_3', w_3' = a'\}$, we have $a \sim w_2$. But now we have an induced $H_1$ on the vertex set $\{y, a, b, w_2, w_1 = w_1', a', b'\}$ (if $w_2 \sim b'$) or $\{y, a, b, w_2, a', b', y'\}$ (otherwise).

- Suppose now that $s = q - 2$. Then, $w_{p-1} \sim w_{q-2}'$ and $w_{p-1} \sim w_{q-1}'$ (otherwise the vertex set $\{b, w_{q-2}', w_{q-1}', w_q', w_{p-1}\}$ would contain an induced $C_4$ or $C_5$). If $q > 3$, then an induced $F_1$ arises on the vertex set $\{w_{q-3}', w_{q-2}', w_{q-1}', b, w_{p-1}, a'\}$. Therefore, $q = 3$, but now the vertex set $\{y, a, b, w = w_1', w_2, w_{p-1}\}$ induces a copy of either $F_1$ (if $a \sim w_{p-1}$) or of $F_2$ (otherwise).

- Suppose that $s < q - 2$. To avoid an induced cycle $C_j$ for some $j \geq 4$, we infer that $w_{p-1} \sim w_i'$ for all $i \in \{s, s + 1, \ldots, q\}$. But now, the path $P = (w = w_1', \ldots, w_s', w_{p-1}, a')$ is a $w-(S' \setminus S)$ path in $K'$ shorter than $P'$, a contradiction.

Suppose that $r < p-1$ and $s < q - 1$. To avoid an induced cycle $C_j$ for some $j \geq 4$, we infer that $w_s' \sim w_{p-1}$ and $w_r \sim w_{p-1}'$. The minimality of $P$ implies that $w_s' \sim w_j$ for $j \in \{1, \ldots, p-3\}$. Notice that $a \sim w_{p-1}$ since otherwise we can find an induced member of $H_1$ on the vertex set $F$ where

$$F = \begin{cases} 
\{y, a, b, w_{p-1}, \ldots, w_r, w_{r-1}, w_{r-2}, a'\}, & \text{if } r \geq 3; \\
\{y, a, b, w_{p-1}, \ldots, w_2, w_1 = w', b', a'\}, & \text{if } r = 2 \text{ and } w_2 \sim b'; \\
\{y, a, b, w_{p-1}, \ldots, w_2, y', b', a'\}, & \text{if } r = 2 \text{ and } w_2 \sim b'; \\
\{y, a, b, w_{p-1}, \ldots, w_1 = w', y', b', a'\}, & \text{if } r = 1.
\end{cases}$$

The above case analysis will be referred to as building an $H_i$ from $D$, where $D$ is the vertex set $\{y, a, b, w_{p-1}\}$ inducing a diamond common to all four cases.

Moreover, $w = w_1' \sim w_{p-1}$ since otherwise we can build an $H_i$ either from $\{w = w_1', a, b = w_p, w_{p-1}\}$ (if $w \sim w_{p-2}$), or from $\{w = w_1', b = w_p, w_{p-1}, w_{p-2}\}$ (otherwise). Since $w_s' \sim w_{p-1}$ and $w = w_1' \sim w_{p-1}$, we have $s \geq 2$. But now, we can build an $H_i$ either from $\{w_{s-1}', w_s', b = w_p, w_{p-1}\}$ (if $w_s' \sim w_{p-2}$), or from $\{w_1', b = w_p, w_{p-1}, w_{p-2}\}$ (otherwise).

Suppose now that $V(P) \cap V(P') \neq \emptyset$. Let $i \in \{1, \ldots, q\}$ be the maximum index such that $w_i' \in V(P)$. Let $j \in \{1, \ldots, p\}$ be the index such that $w_j = w_i$. Notice that $i \geq 2$ and $j \geq 2$. Consider the path $Q = (w = w_1', \ldots, w_j = w, w_{j+1}, \ldots, w_i')$. Let $D = (y, a, b, w = w_1')$, and $D' = \{y', a', b, w = w_1\}$. Vertices $w$ and $w'$ will be referred to as the roots of $D$ and $D'$, while vertices $y$ and $y'$ will be referred to as the tips of $D$ and $D'$. Notice that each of $D$ and $D'$ induces a diamond, and $Q$ is a path connecting the two roots. Moreover, we now show that the tips of $D$ and $D'$ do not have any neighbors on $Q$. By symmetry, it is enough to argue for the tip of $D$, that is, for $y$. 27
Since \( S \) separates \( y \) from \( S' \setminus S \), vertex \( y \) has no neighbors on \( P' \); in particular, it has no neighbors in the set \( \{ w_1', \ldots, w'_s \} \). Moreover, if \( y \) has a neighbor in the set \( \{ w_{j-1}, \ldots, w_1 \} \), say \( y \sim w_r \) for some \( r \in \{ 2, \ldots, j-1 \} \), then the fact that \( G \) is chordal and \( w_p = b \sim y \) implies that \( y \) is adjacent to \( w_s \) for all \( s \in \{ r+1, \ldots, p \} \). However, this would imply that \( y \sim w_j = w'_i \), a contradiction.

We may also assume that \( Q \) is an induced path; otherwise, we replace \( Q \) with a shortest \( y'y' \) path in \( G[V(Q)] \). The above considerations show that the subgraph of \( G \) induced by \( D \cup D' \cup V(Q) \) contains a subgraph (not necessarily induced) isomorphic to a member of \( H_i \) (consisting thus of two diamonds and a path connecting them via their roots) with the following properties:

(i) each of the two diamonds is induced,

(ii) there are no edges connecting a vertex from one diamond with a vertex with another diamond, except perhaps edges incident with their roots (if the two roots coincide) or the unique edge on the path connecting the two roots (if the path connecting the two diamonds is of length 1),

(iii) the path connecting the two diamonds is induced, and

(iv) the tips of the diamonds do not have any neighbors on the path.

Let us call a subgraph satisfying the properties (i)–(iv) a weakly induced \( H_i \). Among all subgraphs of \( G[D \cup D' \cup V(Q)] \) isomorphic to a weakly induced \( H_i \), choose one, say \( H \), minimizing the number of vertices. To complete the proof, we will now show that \( H \) is an induced subgraph of \( G \), in particular, the subgraph of \( G \) induced by \( V(H) \) is isomorphic to a member of \( H_i \). Let us denote the vertices of \( H \) as in Fig. 7 below. Let \( H' \) be the subgraph of \( G \) induced by \( V(H) \).

![Figure 7: A weakly induced \( H_n \)](image)

Suppose that \( H \neq H' \). Then, there is an edge in \( H' \) that is not present in \( H \). The only possible edges that can be present in \( H' \) but not in \( H \) are those connecting one of the vertices \( x_2, x_3, z_2, z_3 \) with one of the vertices in the set \( \{ y_2, \ldots, y_{n-1} \} \). Without loss of generality, assume that \( x_2 \) has a neighbor in the set \( \{ y_2, \ldots, y_{n-1} \} \). Since \( G \) is chordal, so is \( H' \), and hence if \( x_2 \sim y_j \) for some \( j \in \{ 2, \ldots, n-1 \} \), then \( x_2 \sim y_{j'} \) for all \( j' \in \{ 2, \ldots, j \} \). Let \( j \in \{ 2, \ldots, n-1 \} \) be the maximum index such that \( x_2 \sim y_j \). If \( j \geq 3 \), then we can replace \( H \) with the graph obtained from the (induced) diamond formed by \( \{ x_2, y_{j-2}, y_{j-1}, y_j \} \) together with the (induced) path \( (y_j, \ldots, y_n) \) and the (induced) diamond \( \{ y_n, z_1, z_2, z_3 \} \). This would result in a weakly induced \( H_i \) with a smaller number of vertices as \( H \), contrary to the choice of \( H \). Therefore, \( j = 2 \). If \( x_3 \sim y_2 \), then a smaller weakly induced \( H_i \) than \( H \) would be obtained by taking the (induced) diamond formed by \( \{ x_2, x_3, y_1, y_2 \} \) together with the (induced) path \( (y_2, \ldots, y_n) \) and the (induced) diamond \( \{ y_n, z_1, z_2, z_3 \} \). Therefore, \( x_3 \sim y_2 \). Since the tips of the diamonds do not have any neighbors on the connecting path, we infer that \( x_1 \sim y_2 \). But now, a weakly induced \( H_i \) smaller than \( H \) can be obtained by taking the (induced) diamond formed by \( \{ x_1, x_2, x_3, y_2 \} \) together with the (induced) path \( (y_2, \ldots, y_n) \) and the (induced) diamond \( \{ y_n, z_1, z_2, z_3 \} \). This contradiction shows that we must in fact have \( H = H' \) and completes the proof of Case 4.

This completes the proof of Case 4 and with it the proof of Theorem 5.3.
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