MANIFOLDS WITH PIC1 PINCHED CURVATURE

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Abstract. Recently it has been proved [20, 16, 21] that three-dimensional complete manifolds with non-negatively pinched Ricci curvature must be flat or compact, thus confirming a conjecture of Hamilton. In this paper we generalise our work on the existence of Ricci flows from non-compact pinched three-manifolds in order to prove a higher-dimensional analogue. We construct a solution to Ricci flow, for all time, starting with an arbitrary complete non-compact manifold that is PIC1 pinched. As an application we prove that any complete manifold of non-negative complex sectional curvature that is PIC1 pinched must be flat or compact.

1. Introduction

In [20], the following result was proved by lifting a remaining additional hypothesis of bounded curvature from the work of Deruelle-Schulze-Simon [16], which in turn appealed to work of Lott [21].

Theorem 1.1 (Hamilton’s pinching conjecture, cf. [15, Conjecture 3.39]). Suppose $(M^3, g_0)$ is a complete (connected) three-dimensional Riemannian manifold with $\text{Ric} \geq \epsilon \text{scal} \geq 0$ for some $\epsilon > 0$. Then $(M^3, g_0)$ is either flat or compact.

In this paper we develop a higher-dimensional version of the Ricci flow existence theory we established in [20], in order to prove a pinching result in general dimension. In order to state the result, we need to understand a little about the notion of isotropic curvature.

Denote the space of algebraic curvature tensors on $\mathbb{R}^n$ by $\mathcal{C}_B(\mathbb{R}^n)$. Given $R \in \mathcal{C}_B(\mathbb{R}^n)$, we can extend it by complex linearity to $\mathbb{C}^n$. Although we always have the symmetries of $R$ in mind, we view it as a $(0, 4)$ tensor to avoid ambiguities of normalisation. To each two-complex-dimensional subspace $\Sigma$ of $\mathbb{C}^n$ we can then associate a complex sectional curvature. Concretely, if $v, w \in \mathbb{C}^n$ give an orthonormal basis of $\Sigma$, then the complex sectional curvature associated with $\Sigma$ is $R(v, w, \bar{v}, \bar{w})$. We say that $R \in \mathcal{C}_B(\mathbb{R}^n)$ has non-negative complex sectional curvature if all these curvatures are non-negative, whichever $\Sigma$ we choose, and we denote the cone of all such curvature tensors by $\mathcal{C}_{\text{PIC}_2}$. Because this condition is $O(n)$-invariant, we can talk of a manifold having non-negative complex sectional curvature if this property holds for the curvature tensor at every tangent space.
We can weaken the notion of curvature positivity by restricting the sections $\Sigma$ that we consider. In particular, one can ask for non-negativity of complex sectional curvature only for PIC1 sections, defined to be those $\Sigma$ that contain some nonzero vector $v$ whose conjugate $\bar{v}$ is orthogonal to the entire section $\Sigma$. The algebraic curvature tensors $R$ with non-negative complex sectional curvature for each such restricted $\Sigma$ form a cone we denote by $C_{\text{PIC1}}$.

Asking that a curvature tensor lies in $C_{\text{PIC1}}$ or $C_{\text{PIC2}}$ is asking that certain natural curvature averages are non-negative.

For $n \geq 4$, one can describe the cones $C_{\text{PIC1}}$ and $C_{\text{PIC2}}$ more explicitly as follows (cf. [5, Propositions 7.14 and 7.18]). The cone $C_{\text{PIC2}}$ is the cone consisting of curvature tensors $R$ satisfying

$$ R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0 $$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. Similarly, $C_{\text{PIC1}}$ is the cone of curvature tensors satisfying

$$ R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq 0 $$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda \in [0, 1]$.

**Theorem 1.2 (Main theorem).** Suppose $(M^n, g_0)$ is a complete manifold of non-negative complex sectional curvature with $n \geq 3$ that is pinched in the sense that

$$ R_{g_0} - \varepsilon_0 \text{scal}(R_{g_0}) \cdot I \in C_{\text{PIC1}} $$

for some $\varepsilon_0 > 0$. Then $(M, g_0)$ is either flat or compact.

To clarify, we denote by $R_g$ the curvature tensor of a Riemannian metric $g$, with (1.1) holding in every tangent space. Given $R \in C_B(\mathbb{R}^n)$, we write $\text{Ric}(R)_{ik} := R_{ijkj}$ and $\text{scal}(R) := \text{Ric}(R)_{ii}$ in order to match the usual notions of Ricci and scalar curvature exactly. Meanwhile, $I$ is the curvature tensor defined by

$$ I_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}. $$

Thus on the unit round sphere $S^n$ we would have $R_g = I$.

Explicitly, our pinching condition (1.1) is saying exactly that when we compute the complex sectional curvature corresponding to a PIC1 section then not only should it be non-negative, it should also be bounded below by $\varepsilon_0 \text{scal}(R_{g_0})$.

In every dimension, every $R \in C_{\text{PIC1}}$ automatically has non-negative Ricci curvature, and for $n = 3$ the conditions are equivalent (see, for example, [22, Section 2]). For $n = 3$, the PIC1 pinching condition (1.1) considered in the theorem is equivalent to pinched Ricci curvature. This case is already handled by the theory we developed in [20] in that dimension, and thus we may focus here on the case that $n \geq 4$.

Our main theorem extends several earlier pinching results. Brendle and Schoen [8, Theorem 7.4] proved that if $(M^n, g_0)$ is assumed, in addition to
non-negative complex sectional curvature, to satisfy a stronger PIC2 pinching condition, that is, one assumes

\[ R_{g_0} - \varepsilon_0 \text{scal}(R_{g_0}) \cdot I \in \mathcal{C}_{\text{PIC2}}, \]

for some \( \varepsilon_0 > 0 \), and additionally one assumes that the sectional curvature is uniformly bounded throughout, and that the scalar curvature is strictly positive, then \((M, g_0)\) is compact. In turn, this generalises earlier work of Chen and Zhu [12] and of Ni and Wu [24].

The strategy we use to prove Theorem 1.2 is to assume, contrary to the theorem, that our manifold is neither flat nor compact, and then to flow \((M, g_0)\) under Ricci flow for all time and to analyse its asymptotic behaviour. The rough idea that echoes what has been done in previous work starting with Chen-Zhu [12] is that parabolic blow-downs of the Ricci flow would like to settle down to an expanding gradient soliton, but that this is incompatible with the (scale-invariant) pinching condition. The main challenge that we address is to establish the following existence theorem for Ricci flow, starting with a manifold of possibly unbounded curvature, that mirrors our earlier theory from [20].

**Theorem 1.3.** For any \( n \geq 4 \) and \( \varepsilon_0 \in (0, \frac{1}{n(n-1)}) \), there exist \( a_0 > 0 \) and \( \varepsilon'_0 \in (0, \frac{1}{n(n-1)}) \) such that the following holds. Suppose \((M^n, g_0)\) is a complete non-compact manifold such that

\[ R_{g_0} - \varepsilon_0 \text{scal}(R_{g_0}) \cdot I \in \mathcal{C}_{\text{PIC1}} \]

on \( M \). Then there exists a smooth complete Ricci flow solution \( g(t) \) on \( M \times [0, +\infty) \) such that for all \( t > 0 \),

(a) \( R_{g(t)} - \varepsilon'_0 \text{scal}(R_{g(t)}) \cdot I \in \mathcal{C}_{\text{PIC1}} \);

(b) \( |R_{g(t)}| \leq a_0 t^{-1} \).

Note that the restriction \( \varepsilon_0 < \frac{1}{n(n-1)} \), together with the pinching hypothesis, implies that \( \text{scal}(R_{g_0}) \geq 0 \).

This existence theorem only requires PIC1 pinching, with no requirement for non-negative complex sectional curvature. A theory for Ricci flow starting with general open manifolds with non-negative complex sectional curvature was developed by Cabezas-Rivas and Wilking [10]. Although we do have that hypothesis at our disposal in the application, we must develop a new theory in order to obtain both long-time existence and the required curvature decay, both of which require the PIC1 pinching hypothesis. We emphasise that for the existence theory we do not make any non-collapsing assumption, we do not have any boundedness of curvature assumption, and we do not assume non-negativity of the complex sectional curvature.

**Remark 1.4.** Our PIC1 pinching theorem raises the question of whether the cone in which the curvature of \( g_0 \) is assumed to lie can be weakened from \( \mathcal{C}_{\text{PIC2}} \) to \( \mathcal{C}_{\text{PIC1}} \). Our Ricci flow existence theory already works in this more general
situation, but having some negative complex sectional curvature complicates
the required blow-down argument in the proof of Theorem 4.4.

If we combine our main result with earlier work of Böhm-Wilking [2] and
Brendle [5], we rapidly obtain the following consequence.

**Corollary 1.5.** Suppose \((M^n, g_0)\) is a complete manifold of non-negative com-
plex sectional curvature with \(n \geq 3\) that is not everywhere flat, and is pinched
in the sense of \((1.1)\) for some \(\varepsilon_0 > 0\). Then \((M, g_0)\) is diffeomorphic to a
spherical space form.

We will give the proof of Corollary 1.5 towards the end of Section 4.

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2. Pinching cones

In this section, we briefly survey some general tools used throughout this
paper that are largely taken from [6, 2]. For simplicity we work in dimension
\(n \geq 4\).

Given \(R \in \mathcal{C}_B(\mathbb{R}^n)\), define a new algebraic curvature tensor \(Q(R)\) by

\[
Q(R)_{ijkl} = R_{ijpq} R_{klpq} + 2 R_{ipkq} R_{jplq} - 2 R_{iplq} R_{jpkq}
\]

and consider the ‘Hamilton ODE’: \(\frac{d}{dt} R = Q(R)\). For the significance of this
ODE, and the geometry behind the quantity \(Q(R)\), one can refer to [2].

We are interested in cones \(C \subset \mathcal{C}_B(\mathbb{R}^n)\) that are invariant under the Hamil-
ton ODE, such as \(C_{\text{PIC}1}\) and \(C_{\text{PIC}2}\) [9, 31]. We will consider cones \(C\) with the
following properties:

\(\text{(I)}\) \(C\) is closed, convex, and \(O(n)\)-invariant;

\(\text{(II)}\) \(C\) is transversally invariant under the Hamilton ODE;

\(\text{(III)}\) Every \(R \in C \setminus \{0\}\) has positive scalar curvature;

\(\text{(IV)}\) The curvature tensor \(I\) defined in (1.2) lies in the interior of \(C\).

In (II), transversally invariant is equivalent to saying that \(Q(R)\) lies in the
interior of the tangent cone \(T_R C\) for all \(R \in C \setminus \{0\}\).

In the following, we will recall two families of cones that satisfy \(\text{(I)-(IV)},
\) adopting notation from [5] and [2] where appropriate. We start by recalling
the endomorphism \(\ell_{a,b}\) on \(\mathcal{C}_B(\mathbb{R}^n)\) introduced by Böhm and Wilking [2]. For
any \(a, b \in \mathbb{R}\) and \(R \in \mathcal{C}_B(\mathbb{R}^n)\), we define

\[
\ell_{a,b}(R) = R + b \cdot \left( \text{Ric}(R) - \frac{\text{scal}(R)}{n} g \right) \otimes g + \frac{a}{n} \text{scal}(R) \cdot g \otimes g
\]

\[
= R + b \cdot \text{Ric}(R) \otimes g + \frac{a - b}{n} \text{scal}(R) \cdot g \otimes g
\]
where $g$ is the metric ($g_{ij} = \delta_{ij}$ on $\mathbb{R}^n$) and $\otimes$ denotes the Kulkarni-Nomizu product:

$$(A \otimes B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} + A_{jk}B_{il} - A_{jl}B_{ik}.$$  

In [9], the following family of cones $\hat{C}(s)$ was introduced by Brendle and Schoen in order to prove the Differentiable Sphere Theorem, in the spirit of the work of Böhm and Wilking [2].

**Definition 2.1.** Define a family of closed, convex, $O(n)$-invariant cones $\hat{C}(s)$, $s \in (0, +\infty)$, as follows:

(a) for $s \in (0, \frac{1}{2}]$,

$$\hat{C}(s) = \left\{ \ell_{a,s}(R) : R \in C_{\text{PIC2}}, \ \text{Ric}(R) \geq \frac{p}{n} \text{scal}(R) \right\},$$

where

$$2a = \frac{2s + (n - 2)s^2}{1 + (n - 2)s^2}, \quad p = 1 - \frac{1}{1 + (n - 2)s^2}.$$  

(b) for $s \in (\frac{1}{2}, +\infty)$,

$$\hat{C}(s) = \left\{ \ell_{s,\frac{1}{2}}(R) : R \in C_{\text{PIC2}}, \ \text{Ric}(R) \geq \frac{p}{n} \text{scal}(R) \right\},$$

where

$$p = 1 - \frac{4}{n - 2 + 8s}.$$  

The importance of the cones $\hat{C}(s)$ stems from the following proposition, which can be used to show that certain Ricci flows become round.

**Proposition 2.2** (Propositions 13, 14 and 15 in [9], cf. [2]). For every $s > 0$, the cone $\hat{C}(s)$ satisfies the properties (I)-(IV). Moreover, $\hat{C}(s)$ varies continuously in $s$.

The continuity here is with respect to the Hausdorff topology once we intersect with the closed unit ball $\{R \in C_B(\mathbb{R}^n) : |R| \leq 1\}$. As we are dealing with convex cones, the complements of the cones also vary continuously.

As $s \to +\infty$, the cone $\hat{C}(s)$ converges to $\mathbb{R}_{\geq 0}I$. In particular, knowing that there exists $s_0 > 0$ such that $R \in \hat{C}(s)$ for all $s \geq s_0$ is equivalent to knowing that $R$ is the curvature tensor of a space-form with non-negative curvature. It was shown by Brendle-Huisken-Sinestrari [6] that compact ancient solutions of Ricci flow for $t \in (-\infty, 0)$ with $R_{g(t)} \in \hat{C}(s_0)$ for some $s_0 > 0$ and all $t < 0$ (on each tangent space) must have constant sectional curvature for each $t < 0$. This was later generalised by Yokota [33] to the case of complete ancient Ricci flows with possibly unbounded curvature.
We follow the idea of [6] and define for each \( s > 0 \), \( \mathcal{C}(s) \) to be the cone of all algebraic curvature tensors \( R \in \mathcal{C}_B(\mathbb{R}^n) \) satisfying
\[
R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda \mu R_{1234} + \frac{1}{s} (1 - \lambda^2)(1 - \mu^2) \cdot \text{scal}(R) \geq 0
\]
for all orthonormal four-frames \( \{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n \) and \( \lambda, \mu \in [0, 1] \). Clearly, we have \( \mathcal{C}_{\text{PIC2}} \subset \mathcal{C}(s) \subset \mathcal{C}_{\text{PIC1}} \) for all \( s > 0 \). See Appendix A for a geometric interpretation of \( \mathcal{C}(s) \), which also implies these inclusions. The cones \( \mathcal{C}(s) \) are nested, getting smaller as \( s \) increases, and their intersection is \( \mathcal{C}_{\text{PIC2}} \). Moreover, \( \mathcal{C}(s) \) converges to \( \mathcal{C}_{\text{PIC2}} \) as \( s \to \infty \). Following the idea in the proof of [6, Theorem 12], in the spirit of [2], we define
\[
\tilde{\mathcal{C}}(b, s) := \ell_{a,b}(\mathcal{C}(s)),
\]
with \( a = b + \frac{1}{2}(n - 2)b^2 \).

**Proposition 2.3.** For any \( n \geq 4 \), \( s > 0 \), \( b \in (0, \Upsilon_n) \), where
\[
(2.4) \quad \Upsilon_n := \sqrt{\frac{2n(n-2)+4-2}{n(n-2)}},
\]
the cone \( \tilde{\mathcal{C}}(b, s) \) satisfies properties (I)-(IV). Moreover, for each such \( b \), the cones \( \tilde{\mathcal{C}}(b, s) \) vary continuously in \( s \).

**Proof.** This is contained in the proof of [6, Theorem 12], which in turn appeals to [6, Proposition 10] and [2, Proposition 3.2]. For property (III) see also Lemma A.2.

\[\square\]

3. A PRIORI ESTIMATES UNDER PINCHING CONDITIONS

In this section, we will establish local a priori estimates along the Ricci flow. But first we give a result that says that \( \ell_{a,b} \) turns positive curvatures into pinched curvatures.

**Lemma 3.1.** Suppose, for \( n \geq 3 \), that \( C \subset \mathcal{C}_B(\mathbb{R}^n) \) is a convex cone with the properties that \( \text{Ric}(R) \geq 0 \) for every \( R \in C \) and that every \( R \in \mathcal{C}_B(\mathbb{R}^n) \) of non-negative curvature operator lies in \( C \). Suppose further that \( a > b \geq 0 \). Then there exists \( \delta > 0 \) depending on \( a, b, n \) and the cone \( C \) such that if \( S \in C \) and \( R = \ell_{a,b}(S) \) then
\[
R - \delta \text{scal}(R) \cdot I \in C.
\]

**Proof.** By assumption, \( S \in C \) and hence \( \text{Ric}(S) \geq 0 \). We claim that this implies that \( \text{Ric}(S) \otimes g \) has non-negative curvature operator, in which case it must be in the cone \( C \) by assumption. Indeed, a computation tells us that for a general element \( A = \alpha_{ij} e_i \otimes e_j \in \Lambda^2 \mathbb{R}^n \), we have
\[
\text{Ric}(S) \otimes g(A, A) = 4\alpha_{ij} \alpha_{kl} \text{Ric}(S)_{ik},
\]
and the right-hand side is non-negative for each \( j \) separately.
Given any symmetric bilinear form $h$ on $\mathbb{R}^n$, we can compute that

$$\text{scal}(h \otimes g) = 2(n - 1) \text{tr}(h),$$

and hence

$$\text{scal}(\text{Ric}(S) \otimes g) = 2(n - 1)\text{scal}(S)$$

and

$$\text{scal}(I) = n(n - 1).$$

Unravelling the definition of $R$ gives

$$R = S + b \text{Ric}(S) \otimes g + \frac{2(a - b)}{n} \text{scal}(S) \cdot I,$$

so

$$\text{scal}(R) = (2a(n - 1) + 1) \text{scal}(S).$$

Therefore for $\delta \in (0, \frac{2(a - b)}{n(2a(n - 1) + 1})$, we obtain

$$R - \delta \text{scal}(R) \cdot I = S + b \text{Ric}(S) \otimes g + \frac{2(a - b)}{n} \text{scal}(S) \cdot I - \delta \text{scal}(R) \cdot I$$

$$= S + b \text{Ric}(S) \otimes g + \left(\frac{2(a - b)}{n} - \delta (2a(n - 1) + 1)\right) \text{scal}(S) \cdot I$$

$$\in C$$

We assemble the specific instances of the previous lemma in the following corollary.

**Lemma 3.2.** Suppose $R = \ell_{a,b}(S)$ for some $a > b \geq 0$ and $S \in C_B(\mathbb{R}^n)$, for $n \geq 3$. Then there exists $\delta(n, a, b) > 0$ such that the following holds:

(i) $\text{Ric}(R) \geq \delta \text{scal}(R)g$ if $\text{Ric}(S) \geq 0$;

(ii) $R - \delta \text{scal}(R) \cdot I \in C_{\text{PIC1}}$ if $S \in C_{\text{PIC1}}$;

(iii) $R - \delta \text{scal}(R) \cdot I \in C_{\text{PIC2}}$ if $S \in C_{\text{PIC2}}$;

It is also convenient to record a converse statement.

**Lemma 3.3.** Suppose for $n \geq 3$ that $C$ is a closed convex cone in $C_B(\mathbb{R}^n)$ such that

(1) $I$ is in the interior of the cone $C$, and

(2) every $R \in C \setminus \{0\}$ has positive scalar curvature.

Then given $\varepsilon \in (0, \frac{1}{n(n-1)})$, for small enough $b > 0$ depending on $\varepsilon$, $n$ and $C$, we have the following: Every $R \in C_B(\mathbb{R}^n)$ with $R - \varepsilon \text{scal}(R) \cdot I \in C$ satisfies $R \in \ell_{a,b}(C)$ for $a := b + \frac{1}{2}(n - 2)b^2$.

**Proof.** Because $\varepsilon \in (0, \frac{1}{n(n-1)})$, by considering the scalar curvature of $R - \varepsilon \text{scal}(R) \cdot I \in C$ and using that each element of $C$ has non-negative scalar curvature, we deduce that $\text{scal}(R) \geq 0$. Using that every $\tilde{R} \in C \setminus \{0\}$ has positive scalar curvature, and also that the cone is closed, we deduce that $|\tilde{R}| \leq c \text{scal}(\tilde{R})$ for every $\tilde{R} \in C$ and some $c > 0$ depending only on $C$. This
ensures that if $R$ is scaled to have $|R| = 1$ then not only is the term $\varepsilon \text{scal}(R) \cdot I$ in the cone, but so is a neighbourhood of radius $r_0 > 0$ depending on $\varepsilon$ and the cone, including on $c$, but not on the specific $R$ chosen with $|R| = 1$. Thus by convexity of the cone we can add an error $S$ of magnitude $|S| < r_0$ to $R$ and still have

$$R + S = [R - \varepsilon \text{scal}(R) \cdot I] + [S + \varepsilon \text{scal}(R) \cdot I] \in C.$$ 

In particular, since $\ell_{a,b}(C)$ for $a := b + \frac{1}{2}(n - 2)b^2$ varies continuously in $b$, we conclude that for small enough $b > 0$ we have $R \in \ell_{a,b}(C)$. \hfill \Box

The following lemma translates the pinching condition into a cone condition.

**Lemma 3.4.** Suppose $R \in \mathcal{C}_B(\mathbb{R}^n)$, $\varepsilon_0 \in (0, \frac{1}{n(n-1)})$ and

$$R - \varepsilon_0 \text{scal}(R) \cdot I \in \mathcal{C}_{\text{PIC1}}.$$ 

Then $R \in \tilde{\mathcal{C}}(b, s_0)$ for some $s_0, n, \varepsilon_0 > 0$ and $0 < b, \varepsilon_0 < \Upsilon_n$.

Note that by tracing the hypothesis $\text{(3.1)}$ and using that $\varepsilon_0 \in (0, \frac{1}{n(n-1)})$ we see that $\text{scal}(R) \geq 0$ automatically in the lemma.

**Proof.** We apply Corollary $\text{A.3}$ in the appendix to $S := R - \varepsilon_0 \text{scal}(R) \cdot I \in \mathcal{C}_{\text{PIC1}}$ in place of $R$, and with $\varepsilon = \frac{\varepsilon_0}{2}$. We deduce that there exists $s_0 > 0$ depending only on $n$ and $\varepsilon_0$ such that

$$S + \frac{\varepsilon_0}{2} \text{scal}(S) \cdot I \in \tilde{\mathcal{C}}(s_0).$$

Because $\text{scal}(S) = (1 - \varepsilon_0 n(n-1)) \text{scal}(R) \leq \text{scal}(R)$, we can expand $S$ to give

$$R - \frac{1}{2} \varepsilon_0 \text{scal}(R) \cdot I \in \tilde{\mathcal{C}}(s_0)$$

where we are using that $I$ lies in the convex cone $\tilde{\mathcal{C}}(s_0)$. In fact, $I$ lies in the interior of the cone $\tilde{\mathcal{C}}(s_0)$ (even of the smaller cone $\mathcal{C}_{\text{PIC2}}$) and every $R \in \tilde{\mathcal{C}}(s_0) \setminus \{0\}$ has positive scalar curvature. This latter fact follows because $\tilde{\mathcal{C}}(s_0) \subset \mathcal{C}_{\text{PIC1}}$ and the same is true for $\mathcal{C}_{\text{PIC1}}$ e.g. by Lemma $\text{A.2}$. We can then invoke Lemma $\text{3.3}$ to conclude that for some $b \in (0, \Upsilon_n)$ (and corresponding $a := b + \frac{1}{2}(n - 2)b^2$) we have $R \in \ell_{a,b}(\tilde{\mathcal{C}}(s_0)) = \tilde{\mathcal{C}}(b, s_0)$. \hfill \Box

We now prove a local version of Hamilton’s celebrated ODE-PDE theorem.

**Theorem 3.5** (Local ODE-PDE theorem). Suppose for $n \geq 3$ that $C$ is a closed, convex, $O(n)$-invariant cone in $\mathcal{C}_B(\mathbb{R}^n)$ with the properties

1. For some $\varepsilon > 0$, if $S \in \mathcal{C}_B(\mathbb{R}^n)$ satisfies $|S - I| \leq \varepsilon$, then $S \in C$.
2. $C$ is invariant under the Hamilton ODE.

Suppose further that $(M^n, g(t))$, $t \in [0, T]$, is a smooth solution to the Ricci flow with $g(0) = g_0$ such that for some $x_0 \in M$, we have

(i) $B_{g_0}(x_0, 1) \subset M$;
(ii) $R_{g_0} \in C$ on $B_{g_0}(x_0, 1)$;

(iii) $\text{scal}(g(t)) \geq \text{scal}(g_0)$ on $\bar{B}_{g_0}(x_0, 1)$ for $t \in [0, T]$;
(iv) $\text{det}(g(t)) \geq \text{det}(g_0)$ on $\bar{B}_{g_0}(x_0, 1)$ for $t \in [0, T]$;
(v) $\text{Ric}(g(t)) \leq \text{Ric}(g_0)$ on $\bar{B}_{g_0}(x_0, 1)$ for $t \in [0, T]$.

Then there exists a smooth solution $g(t) \in \mathcal{C}_B(\mathbb{R}^n)$, $t \in [0, T]$ such that for some $\alpha \in (0, 1)$ and all $t \in [0, T]$ we have $\text{det}(g(t)) \geq \alpha \text{det}(g_0)$ on $\bar{B}_{g_0}(x_0, 1)$ for $t \in [0, T]$.

\hfill \Box
(iii) $|R_{g(t)}| \leq c_0 t^{-1}$ on $B_{g_0}(x_0, 1) \times (0, T]$ for some $c_0 < \infty$.

Then for every $l \geq 0$, there exists $S_0(n, c_0, \epsilon, l) > 0$ such that for all $t \in [0, T \wedge S_0]$, we have

$$R_{g(t)}(x_0) + t^l \cdot I \in C.$$

In this work we will apply the local ODE-PDE theorem in the case that $l = 0$ and $C = \tilde{C}(b, s_0)$ for some $s_0 > 0$ and $b \in (0, \Upsilon_n)$, where $\Upsilon_n$ was defined in \cite{24}, in order to show that the pinching condition is almost preserved locally. In that case it can be viewed as being analogous to \cite{20} Lemma 3.1] in the three-dimensional theory. For ease of reference we record the consequence we require in the following lemma.

**Lemma 3.6.** Suppose for $n \geq 4$ that $(M^n, g(t)), t \in [0, T]$, is a smooth solution to the Ricci flow with $g(0) = g_0$ such that for some $x_0 \in M$, we have

(i) $B_{g_0}(x_0, 1) \subset M$;

(ii) $R_{g_0} \in \tilde{C}(b, s_0)$ on $B_{g_0}(x_0, 1)$ for some $s_0 > 0$ and $b \in (0, \Upsilon_n)$;

(iii) $|R_{g(t)}| \leq c_0 t^{-1}$ on $B_{g_0}(x_0, 1) \times (0, T]$ for some $c_0 < \infty$.

Then there exists $S_0(n, c_0, b, s_0) > 0$ such that for all $t \in [0, T \wedge S_0]$, we have

$$R_{g(t)}(x_0) + I \in \tilde{C}(b, s_0).$$

**Proof of Theorem 3.5.** We denote by $d$ the distance $d(R, S) = |R - S|$ on $C_g(\mathbb{R}^n)$. Define a function $\varphi : B_{g_0}(x_0, 1) \times [0, T] \to \mathbb{R}$ by $\varphi(x, t) = d(R_{g(t)}(x), C)$. We first show that $\varphi$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \varphi \leq C(n)|R_{g(t)}| \varphi$$

in the barrier sense whenever $\varphi > 0$. If this is the case, then it follows from \cite{19} Theorem 1.1] that

$$\varphi(x_0, t) \leq \epsilon t^l$$

if $t \leq S_0$ is sufficiently small

since $\varphi(\cdot, 0) = 0$ on $B_{g_0}(x_0, 1)$ and $\varphi \leq d(R_{g(t)}(x), 0) \leq c_0 t^{-1}$ on $B_{g_0}(x_0, 1) \times (0, T]$. By definition of $\varphi$ we have

$$R_{g(t)}(x_0) - \varphi(x_0, t) \xi(R_{g(t)}(x_0)) \in C$$

whenever $R_{g(t)}(x_0) \notin C$, where for $S \notin C$ we write $\xi(S) = (S - \pi(S))/|S - \pi(S)|$, where $\pi$ is the projection to the cone $C$. In particular, convexity of the cone implies that if $R_{g(t)}(x_0) \notin C$ then

$$R_{g(t)}(x_0) + t^l I = [R_{g(t)}(x_0) - \varphi(x_0, t) \xi(R_{g(t)}(x_0))] + [t^l I + \varphi(x_0, t) \xi(R_{g(t)}(x_0))]$$

$$\in C,$$

by (3.3) and the definition of $\epsilon$, provided that $t \leq S_0$ is sufficiently small. This will complete the proof.

In order to prove (3.2) at an arbitrary point $(x_1, t_1)$ where $\varphi(x_1, t_1) > 0$, we must locally trivialize the tangent bundle, and hence the bundle in which $R_{g(t)}$ lives, near $(x_1, t_1)$ so that we can compare $R_{g(t)}(x)$ and $R_{g(t_1)}(x_1)$ for $(x, t)$ near
(x_1, t_1). At the fixed time t_1 we do this by radial parallel transport centred at x_1. In the time direction we adopt the time-dependent gauge transformation often referred to as the Uhlenbeck trick [14]. This allows us to compare tensors at different points. It makes the fibre metric (and derived cones, and \( \pi \) and \( \xi \)) constant in a neighbourhood of \((x_1, t_1)\), and leads to a simpler evolution equation

\[
(3.4) \quad \left( \frac{\partial}{\partial t} - \Delta \right) R = Q(R)
\]

for the curvature tensor, for \( Q \) as in \([2,1]\).

To show the evolution equation \((3.2)\) in the barrier sense, we work near our point \((x_1, t_1)\) where \( \varphi(x_1, t_1) > 0 \), or equivalently \( d(R_g(t_1)(x_1), C) > 0 \). Since \( \varphi(x, t) \) is in general only continuous, we construct a barrier as follows. Let

\[
\varphi(x, t) = \langle \xi(R_g(t_1)(x_1)), R_g(t)(x) \rangle
\]

for \((x, t)\) sufficiently close to \((x_1, t_1)\) so that the bundle trivialisation above is valid. Then \( \varphi(x_1, t_1) = \varphi(x_1, t_1) \) and convexity of the cone implies \( \varphi(x, t) \leq \varphi(x, t) \) for \((x, t)\) in the neighbourhood of \((x_1, t_1)\) where \( \varphi \) is defined. Hence, \( \varphi \) serves as a lower barrier for \( \varphi \) in the strongest possible sense.

In order to compute the evolution equation for \( \varphi \) at \((x_1, t_1)\), we first note that we always have \( |\pi(R)| \leq |R| \), and so writing \( L_0 \) for the Lipschitz constant of \( Q \) on the unit ball in \( C_B(\mathbb{R}^n) \) (which depends only on \( n \)) we can compute

\[
|Q(R) - Q(\pi(R))| \leq |R|^2 |Q(R)| - Q(\pi(R))| \leq |R|^2 L_0 \left| \frac{R}{|R|} - \frac{\pi(R)}{|R|} \right| = L_0 |R| |R - \pi(R)|.
\]

At \((x_1, t_1)\) we then have

\[
(3.5) \quad \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \varphi = \langle \xi(R_g(t_1)(x_1)), Q(R_g(t_1)(x_1)) \rangle \leq \langle \xi(R_g(t_1)(x_1)), Q(R_g(t_1)(x_1)) - Q(\pi(R_g(t_1)(x_1))) \rangle \leq L_0 |R_g(t_1)| \varphi.
\]

Here we have used the fact that \( \pi(R_g(t_1)(x_1)) \in \partial C \) so that

\[
\langle \xi(R_g(t_1)(x_1)), Q(\pi(R_g(t_1)(x_1))) \rangle \leq 0
\]

because \( C \) is invariant under the Hamilton ODE. This completes the proof. \( \square \)

Next we will show that for \( R_g(t) \) inside the pinching cones \( \dot{C}(s) \) or \( \dot{C}(b, s) \) of large magnitude, the pinching improves with time. We use an approach of Brendle-Huisken-Sinestrari [B] developed in the global setting.
Lemma 3.7. Suppose $C$ is either $\hat{C}(s)$ or $\tilde{C}(b, s)$, for $s > 0$ and $b \in (0, \Upsilon_n)$, with $n \geq 4$. Then there exists $\hat{T}(n, C) > 0$ such that the following holds:

Suppose $(M, g(t))$, $t \in [0, T]$ is a smooth solution to the Ricci flow such that for some $x_0 \in M$ and $r_0 \in (0, 1)$, we have

(i) $B_{g(t)}(x_0, 8(n-1)r_0^{-1}) \subseteq M$;

(ii) $R_{g(t)} + I \in C$ on $B_{g(t)}(x_0, 8(n-1)r_0^{-1})$;

(iii) $\text{Ric}_{g(t)} \leq (n-1)r_0^{-2}$ on $B_{g(t)}(x_0, r_0)$,

for all $t \in [0, T]$. Then for all $t \in [0, T \wedge \hat{T}]$,

$$R_{g(t)}(x_0) + (4 - t \cdot \text{scal}(R_{g(t)}(x_0))) I \in C.$$ 

Proof. For each $t \in [0, T]$ and $x \in B_{g(t)}(x_0, 8(n-1)r_0^{-1})$, we denote

$$S = S_{g(t)}(x) := R_{g(t)}(x) + (2 - t n(n-1) - t \text{scal}(R_{g(t)}(x))) \cdot I$$

and define

$$\varphi(x, t) = d\left(C, S_{g(t)}(x)\right).$$

Observe that by convexity of the cone $C$, and the facts that $R_{g(0)} + I \in C$ on $B_{g(0)}(x_0, 8(n-1)r_0^{-1})$ and $I \in C$, we have $S_{g(0)} = (R_{g(0)} + I) + I \in C$, and hence $\varphi(\cdot, 0) \equiv 0$, throughout $B_{g(0)}(x_0, 8(n-1)r_0^{-1})$.

We first show that there exist $\hat{T}, \delta > 0$ depending only on $n$ and $C$ such that $\varphi$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \varphi \leq -\delta t^{-2} \varphi^2$$

in the barrier sense at every point $(x_1, t_1)$ for which $t_1 \in (0, T \wedge \hat{T})$, $x_1 \in B_{g(t_1)}(x_0, 8(n-1)r_0^{-1})$ and $\varphi(x_1, t_1) > 0$. We want to construct a lower barrier for $\varphi$ at $(x_1, t_1)$ satisfying the desired evolution inequality.

Borrowing ideas and notation from the proof of Theorem 3.5, including the bundle trivialisation near to $(x_1, t_1)$, we define

$$\varphi(x, t) = \langle \xi(S_{g(t_1)}(x_1)), S_{g(t)}(x) \rangle$$

which satisfies $\varphi(x, t) \leq \varphi(x, t)$ for all $(x, t)$ in the neighbourhood of $(x_1, t_1)$ where the bundle trivialisation holds, with equality at $(x_1, t_1)$.

Recall the evolution equation (3.4) for $R_{g(t)}(x)$, and the evolution equation

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \text{scal}(R_{g(t)}) = 2|\text{Ric}(R_{g(t)})|^2$$

for $\text{scal}(R_{g(t)})$, [30], Proposition 2.5.4, which give us

$$\left(\frac{\partial}{\partial t} - \Delta\right) S = Q(R) - \text{scal}(R)I - 2t|\text{Ric}(R)|^2I - n(n-1)I.$$
At \((x_1, t_1)\) we thus obtain
\[
(3.6) \quad \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \varphi = \langle \xi(S), Q(R) - \text{scal}(R)I - 2t|\text{Ric}(R)|^2I - n(n-1)I \rangle.
\]

Building on [6, Lemma 5], in [33, Lemma 10], Yokota showed that provided we constrain \(\hat{T} > 0\) (and hence \(t \leq \hat{T}\)) depending on \(n\) and the cone being considered, if \(\hat{R} \in C\) and
\[
(3.7) \quad \hat{S} := \hat{R} + (1 - t \text{scal}(\hat{R})) \cdot I \in C_B(\mathbb{R}^n) \setminus C,
\]
then
\[
(3.8) \quad \langle Q(\hat{R}) - \text{scal}(\hat{R})I - 2t|\text{Ric}(\hat{R})|^2I - n(n-1)I, \xi(\hat{S}) \rangle \leq -\mu |\hat{S}|^2,
\]
for some \(\mu > 0\) depending on \(n\) and the cone. We are able to apply this estimate in the case \(\hat{R} = R + I\), at \((x_1, t_1)\), in which case \(\hat{S}\) coincides with \(S\).

Yokota also observed that \(t \text{scal}(\hat{R}) > 1\) by (3.7) because \(\hat{R} \in C\) while \(S \notin C\), and his argument implies that provided we have constrained \(\hat{T} > 0\) sufficiently, then \(|\hat{R}|, S, \text{scal}(\hat{R}), |R|\) and \(\text{scal}(R)\) are all comparable up to an \(n\)-dependent factor, and in particular can all be controlled from below by \(\frac{1}{t}\) (up to an \(n\)-dimensional constant). This comparability means that we can switch from \(\hat{R}\) to \(R\) in his conclusion (3.8), adding only controllable errors. More precisely, we obtain
\[
\langle Q(R) - \text{scal}(R)I - 2t|\text{Ric}(R)|^2I - n(n-1)I, \xi(S) \rangle \leq -\mu |S|^2 + C(1 + |S|) \leq -\frac{\mu}{2} |S|^2,
\]
where \(C\) is an \(n\)-dependent constant in front of an error term that can be absorbed because by constraining \(t \leq \hat{T}\) we can ensure that \(|S|^2\) is sufficiently large to dominate any term that is linear in \(|S|\).

In particular, we obtain
\[
(3.9) \quad \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \varphi \leq -\frac{\mu}{2} |S|^2.
\]

Next we note that because \(R + I \in C\) and \(t \text{scal}(\hat{R}) > 1\), we have
\[
\varphi(x_1, t_1) \leq |(R + I) - S| = |(t \text{scal}(\hat{R}) - 1)I| \leq t \text{scal}(\hat{R})|I| \leq C_0(n)t|S|,
\]
because, as remarked above, \(\text{scal}(\hat{R})\) and \(|S|\) differ only by some \(n\)-dependent factor. Thus there exists \(\delta > 0\) depending only on \(n\) and \(C\) such that
\[
(3.10) \quad \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \varphi \leq -\delta t^{-2} \varphi^2.
\]
at \((x_1, t_1)\).

Now we are ready to set up a maximum principle argument. By hypothesis (iii) and [20, Lemma 8.3], the function \(\eta(x, t) = d_{g(t)}(x, x_0) + \frac{2}{3}(n - 1)r_0^{-1}t\)
satisfies
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \eta \geq 0
\]
in the barrier sense whenever \( d_{g(t)}(x, x_0) \geq r_0 \). Let \( \phi \) be a smooth non-
increasing function on \( \mathbb{R} \) such that \( \phi \equiv 1 \) on \( (-\infty, 1] \), \( \phi \) vanishes outside
\( (-\infty, 2] \) and satisfies \( \phi'' \geq -10^2 \phi, |\phi'| \leq 10^2 \phi^{1/2} \). Define
\[
\Phi(x, t) = e^{-10^3 t} \phi \left( \frac{\eta(x, t)}{Ar_0^{-1}} \right), \quad A = 4(n - 1)
\]
so that \( (\partial_t - \Delta_{g(t)}) \Phi \leq 0 \) in the sense of barriers. Moreover due to the choice
of \( A \), \( \Phi(x_0, t) \equiv 1 \) for \( t \in [0, T \wedge \hat{T}] \) and \( \Phi \) vanishes outside \( B_{g(t)}(x_0, 2Ar_0^{-1}) \),
i.e., \( B_{g(t)}(x_0, 8(n - 1)r_0^{-1}) \).

Consider the function \( G = \Phi \varphi \). Considering the support of \( \Phi \), we see
that \( G \) attains its maximum on \( M \times [0, T \wedge \hat{T}] \) at some point \((x_1, t_1)\) with
\( x_1 \in B_{g(t_1)}(x_0, 8(n - 1)r_0^{-1}) \). Because we showed that \( \varphi(\cdot, 0) \equiv 0 \), throughout
\( B_{g(0)}(x_0, 8(n - 1)r_0^{-1}) \), we see that \( G(\cdot, 0) \equiv 0 \) throughout \( M \). Since our goal
is to obtain an upper bound for \( G \), we may assume \( t_1 > 0 \) and \( G(x_1, t_1) > 0 \); in particular we have both \( \varphi(x_1, t_1) > 0 \) (so our calculations above are valid)
and \( \Phi(x_1, t_1) > 0 \). We may assume \( \Phi \) and \( \varphi \) to be smooth when we apply the
maximum principle; for example see [29, Section 7] for a detailed exposition.

In this case, at \((x_1, t_1)\) we have
\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) G = \varphi \cdot \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \Phi + \Phi \cdot \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \varphi
- 2\langle \nabla \varphi, \nabla \Phi \rangle
\leq -\delta t^{-2} \varphi^2 \Phi + \frac{2|\nabla \Phi|^2}{\Phi} \varphi
\leq -\delta t^{-2} \varphi^2 \Phi + \frac{2 \cdot 10^4}{A^2 r_0^{-7}} \varphi.
\]
and thus
\[
\sup_{M \times [0, T \wedge \hat{T}]} G = G(x_1, t_1) \leq C_n r_0^2 \delta^{-1} t^2
\]
for some dimensional constant \( C_n > 0 \). Evaluating at \( x_0 \), where \( \Phi = 1 \), yields
\[
0 \leq \varphi(x_0, t) \leq Lt^2,
\]
for some constant \( L < \infty \) depending only on \( n \) and \( C \). By definition, we have
\( S - \varphi(x_0, t) \xi(S) \in C \), and so we can compute at \( x_0 \)
\[
R + (4 - t \text{scal}(R))I = S + 2I + n(n - 1)tI
= [S - \varphi(x_0, t) \xi(S)] + [2I + n(n - 1)tI + \varphi(x_0, t) \xi(S)]
\]
which must lie in $C$ for $t \in [0, T \wedge \hat{T}]$, provided we constrain $\hat{T} > 0$ depending on $n$ and $C$, where we have used the convexity of the cone $C$ and the fact that $I$ lies in the interior of $C$, along with the estimate (3.12). \hfill \Box

We now use the cone almost improving nature (i.e. Lemma 3.7) to obtain curvature estimates.

**Proposition 3.8.** Suppose $M^n$ is a non-compact (connected) manifold for $n \geq 4$, and $g(t)$, $t \in [0, T]$ is a smooth solution to the Ricci flow on $M$ so that for some $x_0 \in M$, $s_0 > 0$ and $b \in (0, \Upsilon_n)$, we have

(i) $B_{g(t)}(x_0, 1) \subseteq M$ for all $t \in [0, T]$;

(ii) $R_{g(t)} + I \in \tilde{C}(b, s_0)$ on $B_{g(t)}(x_0, 1)$ for $t \in [0, T]$.

Then there exist $C_0(n, b, s_0), S_1(n, b, s_0) > 0$ such that for $t \in (0, S_1 \wedge T]$,

$$|R_{g(t)}(x_0)| \leq C_0 t^{-1}.$$ 

Thus we obtain $C_0/t$ decay of the full curvature tensor analogous to the estimate in [20, Lemma 3.3].

**Proof.** The proof of the curvature estimate uses the Perelman-inspired point-picking argument from [29, Lemma 2.1], and initially mirrors the proof of [20, Lemma 3.3].

Suppose the conclusion is false for some $n \geq 4$, $b \in (0, \Upsilon_n)$ and $s_0 > 0$. Then for any $a_k \to +\infty$, we can find a sequence of non-compact manifolds $M^n_k$, Ricci flows $g_k(t), t \in [0, T_k]$ and $x_k \in M_k$ satisfying the hypotheses but so the curvature estimate fails with $C_0 = a_k$ in an arbitrarily short time. We may assume $a_k T_k \to 0$. By smoothness of each Ricci flow, we can choose $t_k \in (0, T_k]$ so that

(i) $B_{g_k(t_k)}(x_k, 1) \subseteq M_k$ for $t \in [0, t_k]$;

(ii) $R_{g_k(t)} + I \in \tilde{C}(b, s_0)$ on $B_{g_k(t_k)}(x_k, 1)$ for $t \in [0, t_k]$;

(iii) $|R_{g_k(t)}(x_k)| < a_k t^{-1}$ for $t \in (0, t_k)$;

(iv) $|R_{g_k(t_k)}(x_k)| = a_k t_k^{-1}$.

By (iv) and the fact that $a_k t_k \to 0$, [29, Lemma 5.1] implies that for sufficiently large $k$, we can find $\beta(n) > 0$, times $\hat{t}_k \in (0, t_k]$ and points $\tilde{x}_k \in B_{g_k(\hat{t}_k)}(x_k, \frac{3}{4} - \frac{1}{2} \beta \sqrt{a_k t_k})$ such that

$$|R_{g_k(t)}(x)| \leq 4 |R_{g_k(\hat{t}_k)}(\tilde{x}_k)| = 4 Q_k$$

whenever $d_{g_k(\hat{t}_k)}(x, \tilde{x}_k) < \frac{1}{8} \beta a_k Q_k^{-1/2}$ and $\hat{t}_k - \frac{1}{8} a_k Q_k^{-1} \leq t \leq \tilde{t}_k$ where $\hat{t}_k Q_k \geq a_k \to +\infty$.

Included in the proof of [29, Lemma 5.1] is that for each such $(x, t)$ we have $x \in B_{g_k(t)}(x_k, 1)$ i.e. the cylinder $B_{g_k(\hat{t}_k)}(\tilde{x}_k, \frac{1}{8} \beta a_k Q_k^{-1/2}) \times [\hat{t}_k - \frac{1}{8} a_k Q_k^{-1}, \tilde{t}_k]$ where (3.14) holds lies lies within the region where (ii) holds.

Consider the parabolic rescaling centred at $(\tilde{x}_k, \tilde{t}_k)$, namely $\tilde{g}_k(t) = Q_k g_k(\tilde{t}_k + Q_k^{-1} t)$ for $t \in [-\frac{1}{8} a_k, 0]$ so that
(a) \( |R_{\tilde{g}_k(0)}(\tilde{x}_k)| = 1; \)
(b) \( |R_{\tilde{g}_k(t)}| \leq 4 \) on \( B_{\tilde{g}_k(0)}(\tilde{x}_k, \frac{1}{8}a_k) \times [-\frac{1}{8}a_k, 0] \), and
(c) \( R_{\tilde{g}_k(t)} + Q_k^{-1}I \in \tilde{C}(b, s_0) \) on \( B_{\tilde{g}_k(0)}(\tilde{x}_k, \frac{1}{8}a_k) \times [-\frac{1}{8}a_k, 0] \).

If we had a uniform positive lower bound on the injectivity radii \( \text{inj}(\tilde{g}_k(0))(\tilde{x}_k) \), then Hamilton’s compactness theorem would enable us to extract a subsequence converging in the \( C^\infty \) Cheeger-Gromov sense to a complete ancient solution of Ricci flow \( g_\infty(t) \) which would be non-compact, non-flat and have bounded curvature. Moreover, we would have \( R_{g_\infty(t)} \in \tilde{C}(b, s_0) \) for all \( t \leq 0 \) so that \([33, \text{Lemma 15}]\) would apply to give a contradiction to the non-compactness of the underlying manifolds \( M_k \). To accommodate the lack of an injectivity radius lower bound, we can instead take a local limit \( \tilde{g}_\infty(t) \) of \( \tilde{g}_k(t) \) by lifting to a Euclidean ball via the exponential map of \( \tilde{g}_k(0) \) as in the proof of \([20, \text{Lemma 3.3}]\). (We only need to consider the local limit \( \tilde{g}_\infty(0) \), but we take the limit at each time for consistency with \([20]\).) Since the limit is a priori only locally defined on a ball in Euclidean space, we need to extract more information along the sequence first.

We now improve the pinching behaviour at \( t = 0 \) by using the fact that the flow is almost a complete ancient solution with bounded curvature. We first show that it becomes almost PIC2 pinched.

**Claim 3.1.** For any \( L, \varepsilon > 0 \) and \( s \geq s_0 \), there exists \( N \in \mathbb{N} \) such that for all \( k > N \), we have

\[
R_{\tilde{g}_k(t)} + \varepsilon I \in \tilde{C}(b, s)
\]
on \( B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0] \).

**Proof of claim.** Let \( \mathcal{S} \) be the set of \( s' \in [s_0, +\infty) \) so that for all \( L > 0 \) and \( \varepsilon \in (0, 1) \), we can find \( N \in \mathbb{N} \) such that for all \( k > N \), we have

\[
(3.15) \quad R_{\tilde{g}_k(t)} + \varepsilon I \in \tilde{C}(b, s')
\]
on \( B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0] \). Clearly, \( s_0 \in \mathcal{S} \) by (c) above. We want to show that \( \mathcal{S} \) is open and closed so that \( \mathcal{S} = [s_0, +\infty) \).

We first show that \( \mathcal{S} \) is closed. Let \( s_i \in \mathcal{S} \) so that \( s_i \to s_\infty \in \mathbb{R} \). By definition of the cones \( \tilde{C}(b, \tau) \) we see we have the following type of continuity of the cones with respect to \( \tau \): For any \( \varepsilon' > 0 \), there exists \( \delta > 0 \) such that if \( R \in \tilde{C}(b, s) \) for some \( |s - s_\infty| < \delta \), then \( R + \varepsilon' \text{scal}(R) \cdot I \in \tilde{C}(b, s_\infty) \).

For any given \( L > 0 \) and \( \varepsilon \in (0, 1) \) for which we want \( (3.15) \) to be true with \( s' = s_\infty \) for sufficiently large \( k \), we choose \( \varepsilon' = \frac{1}{8n(n-1)} \varepsilon \) and obtain a corresponding \( \delta > 0 \) from the above continuity. We fix \( i_0 \) sufficiently large so that \( |s_{i_0} - s_\infty| < \delta \). Since \( s_{i_0} \in \mathcal{S} \), we can apply the definition of \( s_{i_0} \in \mathcal{S} \) with \( \varepsilon \) replaced by \( \frac{\varepsilon}{4} \) to deduce that there exists \( N \in \mathbb{N} \) such that for all \( k > N \),

\[
R_{\tilde{g}_k(t)} + \frac{\varepsilon}{4} I \in \tilde{C}(b, s_{i_0})
\]
on $B_{g_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$. Hence, $\tilde{R} = R_{g_k(t)} + \frac{\varepsilon}{4} I$ satisfies
\begin{equation}
\hat{\tilde{R}} + \frac{1}{8n(n-1)} \varepsilon \cdot \text{scal}(\tilde{R}) I \in \mathcal{C}(b, s_\infty). \tag{3.16}
\end{equation}

Since
\[ \text{scal}(\tilde{R}) = \text{scal}(R_{\tilde{g}_k(t)}) + \frac{\varepsilon}{4} n(n-1), \]
we can then unwind the definition of $\tilde{R}$ to give
\begin{equation}
R_{\tilde{g}_k(t)} + \left(\frac{\varepsilon}{4} + \frac{\varepsilon^2}{8} + \frac{1}{8n(n-1)} \varepsilon \cdot \text{scal}(R_{\tilde{g}_k(t)})\right) I \in \mathcal{C}(b, s_\infty). \tag{3.17}
\end{equation}

A coarse consequence of the curvature bound $|R_{\tilde{g}_k(t)}| \leq 4$ from (b) is that $\text{scal}(R_{\tilde{g}_k(t)}) \leq 4n(n-1)$, so keeping in mind that $\varepsilon \leq 1$ we find that
\[ R_{\tilde{g}_k(t)} + \varepsilon I \in \mathcal{C}(b, s_\infty) \]
on $B_{g_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$ for sufficiently large $k$. Hence $S$ is closed.

It remains to show that $S$ is open. Let $s' \in S$. It suffices to show that for some $\sigma' > 0$, the following is true: For all $L > 0$ and $\varepsilon \in (0, 1)$, the modified curvature tensor $\tilde{R} = R_{g_k(t)} + \varepsilon I$ satisfies $\tilde{R} - \sigma' \cdot \text{scal}(\tilde{R}) I \in \mathcal{C}(b, s')$ on $B_{g_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$ for sufficiently large $k$. The result will then follow using the continuity of the cones.

Set $\varepsilon' = \frac{\varepsilon}{32}$ and let $L' > L$ be a constant to be chosen later. Since $s' \in S$, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have
\[ R_{g_k(t)} + \varepsilon' I \in \mathcal{C}(b, s') \]
on $B_{g_k(0)}(\tilde{x}_k, L') \times [-L'^2, 0]$.

Define $\sigma' := \min\{\hat{T}(n, \mathcal{C}(b, s')), \frac{1}{2n(n-1)}\}$, where $\hat{T}$ is the positive constant obtained from Lemma 3.7 with $C$ chosen to be $\mathcal{C}(b, s')$. Set $r_0 = \frac{1}{8} \sqrt{\varepsilon} \in (0, 1)$, let $(x, \tau) \in B_{g_k(0)}(\tilde{x}_k, L) \times [-L^2, 0] \subseteq B_{g_k(0)}(\tilde{x}_k, L') \times [-L'^2, 0]$ and define a rescaled Ricci flow
\[ \hat{g}_k(t) = 4r_0^2 \tilde{g}_k \left( \tau + \frac{t - \sigma'}{4r_0^2} \right), \quad t \in [0, \sigma']. \]

In order to ensure that $\hat{g}_k(t)$ is defined on the whole time interval $[0, \sigma']$, even if $\tau$ is chosen at its least value $-L^2$, we ask that $L'$ is large enough so that $L^2 + \frac{\sigma'}{4r_0^2} \leq (L')^2$. We would also like to choose $L'$ sufficiently large so that for every $t \in [0, \sigma']$ there is enough space to fit balls $B_{\hat{g}_k(t)}(x, 8(n-1)r_0^{-1})$ within $B_{\tilde{g}_k(0)}(\tilde{x}_k, L')$. By the overall curvature bound of part (b), we have $|R_{\hat{g}_k(t)}| \leq r_0^{-2}$, and this bound is enough to allow us to compare geodesic balls at different times. In particular, we have the inclusion
\[ B_{\hat{g}_k(t)}(x, 8(n-1)r_0^{-1}) \subseteq B_{\tilde{g}_k(0)}(x, L_0) \]
for some $L_0$ depending on $n, \varepsilon$ and $\sigma'$. Thus it suffices to insist that $L' \geq L + L_0$.

We conclude that, increasing $N$ if necessary (possibly depending also on $\varepsilon$, $L'$), for $k > N$, $\hat{g}_k(t), t \in [0, \sigma']$ is a Ricci flow such that for all $t \in [0, \sigma']$,.
(1) $B_{\tilde{g}_k(t)}(x, s(n-1)r_0^{-1}) \subseteq M_k$;
(2) $R_{\tilde{g}_k(t)} + \frac{1}{2} I \in \tilde{C}(b, s)$ on $B_{\tilde{g}_k(t)}(x, 8(n-1)r_0^{-1})$;
(3) $\text{Ric}_{\tilde{g}_k(t)} \leq (n-1)r_0^{-2}$ on $B_{\tilde{g}_k(t)}(x, r_0)$.

Applying Lemma 3.7 to $\tilde{g}_k(t)$ at $(x, \sigma')$ gives

$$R_{\tilde{g}_k(\sigma')(x)}(x) + \left(4 - \sigma' \cdot \text{scal}(R_{\tilde{g}_k(\sigma')(x)})\right) I \in \tilde{C}(b, s').$$

Rescaling the metric back by the same factor $4r_0^2 = \frac{\varepsilon}{16}$ then gives

$$R_{\tilde{g}_k(\tau)(x)}(x) + \left(\frac{\varepsilon}{4} - \sigma' \cdot \text{scal}(R_{\tilde{g}_k(\tau)(x)})\right) I \in \tilde{C}(b, s').$$

Rewriting using $\check{R} = R_{\tilde{g}_k(\tau)(x)} + \varepsilon I$ gives

(3.18) $$\check{R} + \left(\frac{\varepsilon}{4} + \sigma' \varepsilon(n-1) - \varepsilon\right) I - \sigma' \cdot \text{scal}(\check{R}) I \in \tilde{C}(b, s').$$

By definition, $\sigma' \leq \frac{1}{2n(n-1)}$, so the part in brackets is negative and we deduce that for the given $\varepsilon, L > 0$,

(3.19) $$\check{R} - \sigma' \cdot \text{scal}(\check{R}) I \in \tilde{C}(b, s')$$

throughout $B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$ for sufficiently large $k$, as desired. This shows that $S$ is also open and hence $S = [s_0, +\infty)$. This completes the proof of the claim.

Thanks to Claim 3.1 any local limit $\tilde{g}_\infty(0)$ (in the sense described before Claim 3.1) of the metrics $\tilde{g}_k(0)$ has curvature lying in $\tilde{C}(b, s) := \ell_{a,b}(\tilde{C}(s))$, where $a := b + \frac{1}{2}(n-2)b^2$, for all $s$ sufficiently large. By definition, being in $\tilde{C}(s)$ for all $s$ sufficiently large is equivalent to being in $C_{\text{PIC2}}$. Thus we find that $\tilde{g}_\infty(0) \in \ell_{a,b}(C_{\text{PIC2}}) \subset C_{\text{PIC2}}$ because $a > b \geq 0$, e.g. using Lemma 3.2.

Next, we will improve this control further by showing that $\tilde{g}_\infty(0)$ lies in $\tilde{C}(s)$ for all $s$ sufficiently large, which will ultimately show that any local blow up limit $\tilde{g}_\infty(0)$ is in fact a space form.

**Claim 3.2.** There exists $\bar{s}_0 > 0$ such that the following is true: for any $L, \varepsilon > 0$ and $s \geq \bar{s}_0$, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have

$$R_{\tilde{g}_k(t)} + \varepsilon I \in \tilde{C}(s)$$

on $B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$.

**Proof of claim.** Since $\tilde{C}(s) \to C_{\text{PIC2}}$ as $s \to \infty$, we have $\tilde{C}(b, s) \to \ell_{a,b}(C_{\text{PIC2}})$ by definition. Lemma 3.2 tells us that if $R \in \ell_{a,b}(C_{\text{PIC2}})$ then for $\delta > 0$ depending on $b$ and $n$, we have $R - \delta \cdot \text{scal}(R) \cdot I \in C_{\text{PIC2}}$. Thus for $\bar{s} > 0$ large enough,

$$\tilde{C}(b, s) \subset \{ S \in \mathcal{C}_B(\mathbb{R}^n) : S - \frac{\delta}{2} \cdot \text{scal}(S) \cdot I \in C_{\text{PIC2}} \}$$
for every $s \geq \hat{s}$, while for $\hat{s}_0 > 0$ small enough, by continuity of the cones $\hat{C}(s)$, and the fact that $\hat{C}(s) \to C_{\text{PIC}2}$ as $s \searrow 0$, we have

$$\{ S \in C_B(\mathbb{R}^n) : S - \frac{\hat{\delta}}{2} \text{scal}(S) \cdot I \in C_{\text{PIC}2} \} \subset \hat{C}(\hat{s}_0).$$

To summarise, for our given small $\hat{s}_0 > 0$, we have

$$(3.20) \quad \hat{C}(b, s) \subset \hat{C}(\hat{s}_0)$$

for every $s \geq \hat{s}$.

Mimicking the proof of Claim 3.1, we let $S'$ be the set of $s' \in [\hat{s}_0, +\infty)$ so that for all $L > 0, \varepsilon \in (0, 1)$, we can find $N \in \mathbb{N}$ such that for all $k > N$, we have

$$R_{\hat{g}_k(t)} + \varepsilon I \in \hat{C}(s')$$
on $B_{\hat{g}_0(0)}(\tilde{x}_k, L) \times [-L^2, 0]$. The set $S'$ is non-empty because $\hat{s}_0 \in S'$ by Claim 3.1 and (3.20). Now we can carry out the same argument as in the proof of Claim 3.1 to show that $S' = [\hat{s}_0, +\infty)$. This completes the proof of the claim.

By Claim 3.2, any local limit $\tilde{g}_\infty(0)$ of the metrics $\tilde{g}_k(0)$ is a space-form with non-negative curvature. As $|R_{\tilde{g}_k(0)}(\tilde{x}_k)| = 1$, this forces $\text{Ric}_{\tilde{g}_\infty(0)} = \alpha$ for some dimensional constant $\alpha > 0$. Now we can argue as in the proof of [20, Lemma 3.3] to draw a contradiction from the non-compactness of $M_k$ for $k$ sufficiently large. This finishes the proof.

4. Existence of Ricci flow under pinching

In this section, we will construct a smooth complete Ricci flow solution with scaling invariant estimates from metrics with pinched curvature. To do this, we will need a local existence theorem for the Ricci flow, which is one of the main goals of this section.

**Theorem 4.1.** For any $s_0 > 0, n \geq 4$ and $b \in (0, \Upsilon_n)$, there exist $a_0(n, b, s_0) > 0$ and $T(n, b, s_0) > 0$ such that the following holds. Suppose $(M^n, g_0)$ is a manifold such that

(a) $B_{g_0}(p, 4) \subset M$;
(b) $R_{g_0} \in \hat{C}(b, s_0)$ on $B_{g_0}(p, 4)$;

Then there exists a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(p, 1) \times [0, T]$ such that $g(0) = g_0$ and

(i) $R_{g(t)} + I \in \hat{C}(b, s_0)$;
(ii) $|R_{g(t)}| \leq a_0 t^{-1}$.

We will need several ingredients for the proof of Theorem 4.1. The first of these is a result of Hochard that allows us to construct a local Ricci flow on regions with bounded curvature by modifying an incomplete Riemannian
metric at its extremities in order to make it complete, without increasing the curvature too much, and without changing it in the interior.

**Proposition 4.2** (Proposition 4.2 in [20], based on [17]). For \( n \geq 2 \) there exist constants \( \alpha \in (0, 1] \) and \( \Lambda > 1 \) depending on \( n \) so that the following is true. Suppose \( (N^n, h_0) \) is a smooth manifold (not necessarily complete) that satisfies \( |R_{h_0}| \leq \rho^{-2} \) throughout, for some \( \rho > 0 \). Then there exists a smooth Ricci flow \( h(t) \) on \( N \) for \( t \in [0, \alpha \rho^2] \), with the properties that

(i) \( h(0) = h_0 \) on \( N_0 = \{ x \in N : B_{h_0}(x, \rho) \subset N \} \);

(ii) \( |R_{h(t)}| \leq \Lambda \rho^{-2} \) throughout \( N \times [0, \alpha \rho^2] \).

We also recall the shrinking balls lemma, which is one of the local ball inclusion results based on the distance distortion estimates of Hamilton and Perelman from [26, Lemma 8.3].

**Lemma 4.3** ([29, Corollary 3.3]). For \( n \geq 2 \) there exists a constant \( \beta \geq 1 \) depending only on \( n \) such that the following is true. Suppose \( (N^n, g(t)) \) is a Ricci flow for \( t \in [0, S] \) and \( x_0 \in N \) with \( B_{g(t)}(x_0, r) \subset N \) for some \( r > 0 \), and \( \text{Ric}_{g(t)} \leq a/t \) on \( B_{g_0}(x_0, r) \) for each \( t \in (0, S] \). Then \( B_{g(t)}(x_0, r - \beta \sqrt{at}) \subset B_{g_0}(x_0, r) \).

We are now in a position to prove Theorem 4.1. We stay as close as possible to the proof of [20, Theorem 5.1] as possible.

**Proof of Theorem 4.1.** We start by specifying the positive constants that will be used in the construction.

- \( \Lambda(n) > 1 \) and \( \alpha(n) \) from Proposition 4.2
- \( \beta(n) \) from Lemma 4.3
- \( C_0(n, b, s_0) \) from Proposition 3.8
- \( a_0(n, b, s_0) = \max\{1, \alpha \Lambda, \Lambda(\alpha + C_0)\} \)
- \( S_0(n, a_0, b, s_0) \) from Lemma 3.6
- \( S_1(n, b, s_0) \) from Proposition 3.8
- \( \Lambda_0(n, b, s_0) := \max\{4S_1^{-1/2}a_0^{-1/2}, 4S_0^{-1/2}a_0^{-1/2}, 8\beta, 8C_0^{-1/2}a_0^{-1/2}\} \)
- \( \mu(n, b, s_0) := \sqrt{1 + \alpha C_0^{-1}} - 1 > 0 \)

Choose \( \rho \in (0, 1) \) sufficiently small so that for all \( x \in B_{g_0}(p, 4) \), we have \( |R_{g_0}| \leq \rho^{-2} \). We have no uniform positive lower bound for \( \rho \). Apply Proposition 4.2 with \( N = B_{g_0}(p, 4) \) to find a smooth solution \( g(t) \) to the Ricci flow defined on \( B_{g_0}(p, 3) \times [0, \alpha \rho^2] \) with \( |R_{g(t)}| \leq \Lambda \rho^{-2} \) and \( g(0) = g_0 \) on \( B_{g_0}(p, 3) \). In particular, for all \( (x, t) \in B_{g_0}(p, 3) \times (0, \alpha \rho^2] \), we have \( |R_{g(t)}| \leq \Lambda \rho^{-2} \leq a_0 t^{-1} \).

Now define sequences of times \( t_k \) and radii \( r_k \) inductively as follows:

(a) \( t_1 = \alpha \rho^2 \), \( r_1 = 3 \) where \( \rho \) is obtained from above;
(b) \( t_{k+1} = (1 + \mu)^2 t_k \) for all \( k \geq 1 \);
(c) \( r_{k+1} = r_k - \Lambda_0 \sqrt{a_0 t_k} \) for all \( k \geq 1 \).

Consider the following statement:

\( \mathcal{P}(k) \): There exists a smooth Ricci flow solution \( g(t) \) defined on \( B_{g_0}(p, r_k) \times [0, t_k] \) with \( g(0) = g_0 \) such that \( |R_{g(t)}| \leq a_0 t^{-1} \).

Clearly, \( \mathcal{P}(1) \) is true. We want to show that \( \mathcal{P}(k) \) is true for every \( k \) while \( r_k > 0 \), and we do so by induction.

Suppose \( \mathcal{P}(k) \) is true for some \( k \in \mathbb{N} \), and consider a Ricci flow \( g(t) \) that provides. We want to show that \( \mathcal{P}(k+1) \) is true if \( r_{k+1} > 0 \) by extending \( g(t) \) to a longer time interval.

Let \( x \in B_{g_0}(p, r_k + \frac{3}{4} \Lambda_0 \sqrt{a_0 t_k}) \) so that

\[
B_{g_0} \left( x, \frac{1}{4} \Lambda_0 \sqrt{a_0 t_k} \right) \subset B_{g_0}(p, r_k).
\]

Consider the rescaled Ricci flow \( \tilde{g}(t) = \lambda^{-2} g(\lambda^2 t) \) for \( t \in [0, \lambda^{-2} t_k] \) where \( \lambda = \frac{1}{4} \Lambda_0 \sqrt{a_0 t_k} \) so that \( B_{\tilde{g}(0)}(x, \frac{1}{4} \Lambda_0 \sqrt{a_0 t_k}) = B_{\tilde{g}(0)}(x, 1) \) and \( \lambda^{-2} t_k = 16 \Lambda_0^{-2} a_0^{-1} \).

On the rescaled domain, the Ricci flow \( \tilde{g}(t) \) is smooth and satisfies

(i) \( R_{\tilde{g}(0)}(x) + I \in \tilde{C}(b, s_0) \)

(ii) \( |R_{\tilde{g}(t)}| \leq a_0 t^{-1} \) on \( B_{\tilde{g}(0)}(x, 1) \times (0, 16 \Lambda_0^{-2} a_0] \).

Applying Lemma \textbf{3.6} to \( \tilde{g}(t) \), we deduce that

\[
R_{\tilde{g}(t)}(x) + I \in \tilde{C}(b, s_0)
\]

for \( t \leq \min\{S_0, 16 \Lambda_0^{-2} a_0^{-1}\} = 16 \Lambda_0^{-2} a_0^{-1} \) thanks to the choice of \( \Lambda_0 \). Hence for all \( (x, t) \in B_{g_0}(p, r_k + \frac{3}{4} \Lambda_0 \sqrt{a_0 t_k}) \times [0, t_k] \), we have

\[
R_{g(t)} + \left( \frac{1}{4} \Lambda_0 \sqrt{a_0 t_k} \right)^{-2} I \in \tilde{C}(b, s_0).
\]

We now use this to obtain an improved curvature estimate on a slightly smaller ball. For \( x \in B_{g_0}(p, r_k + \frac{1}{4} \Lambda_0 \sqrt{a_0 t_k}) \), we have

\[
B_{g_0} \left( x, \frac{1}{2} \Lambda_0 \sqrt{a_0 t_k} \right) \subset B_{g_0} \left( p, r_k + \frac{3}{4} \Lambda_0 \sqrt{a_0 t_k} \right).
\]

By Lemma \textbf{4.3} and our choice of \( \Lambda_0 \), we deduce that

\[
B_{g(t)} \left( x, \frac{1}{4} \Lambda_0 \sqrt{a_0 t_k} \right) \subset B_{g_0} \left( p, r_k + \frac{3}{4} \Lambda_0 \sqrt{a_0 t_k} \right).
\]

Therefore, the rescaled Ricci flow \( \tilde{g}(t) \), \( t \in [0, 16 \Lambda_0^{-2} a_0^{-1}] \) satisfies

(I) \( B_{\tilde{g}(t)}(x, 1) \in M \) for all \( t \in [0, 16 \Lambda_0^{-2} a_0^{-1}] \);

(II) \( R_{\tilde{g}(t)} + I \in \tilde{C}(b, s_0) \) on \( B_{\tilde{g}(t)}(x, 1) \) for \( t \in [0, 16 \Lambda_0^{-2} a_0^{-1}] \) and hence Proposition \textbf{3.8} applies to show that

\[
|R_{\tilde{g}(t)}(x)| \leq C_0 t^{-1}
\]
for $0 < t \leq \min\{S_1, 16A_0^{-2}a_0^{-1}\} = 16A_0^{-2}a_0^{-1}$. Since this estimate is scaling invariant, we have improved the curvature decay of $g(t)$ from $a_0t^{-1}$ to $C_0t^{-1}$ on $B_{g_0}(p, r_{k+1} + \frac{1}{4}A_0\sqrt{a_0t_k}) \times (0, t_k]$.

Now we construct an extension of $g(t)$. Let $\Omega = B_{g_0}(p, r_{k+1} + \frac{1}{4}A_0\sqrt{a_0t_k})$ so that for $h_0 = g(t_k)$, (4.1) implies $\sup_{\Omega} |R_{h_0}| \leq \rho_0^{-2}$ where $\rho_0 = \sqrt{C_0^{-1}t_k}$. Moreover, for $x \in B_{g_0}(p, r_{k+1})$, Lemma 4.3 (using only the original $a_0t^{-1}$ curvature decay rather than the refined $C_0t^{-1}$ decay) and the choice of $\Lambda_0$ implies

$$B_{g(t_k)}(x, \rho_0) = B_{g(t_k)} \left( x, \sqrt{C_0^{-1}t_k} \right)$$

$$\subset B_{g_0}(x, \sqrt{C_0^{-1}t_k + \beta\sqrt{a_0t_k}})$$

$$\subset B_{g_0} \left( x, \frac{1}{4}\Lambda_0\sqrt{a_0t_k} \right) \subset \Omega.$$  

(4.2)

This shows that $B_{g_0}(p, r_{k+1}) \subset \Omega_{\rho_0}$ where $\Omega_{\rho_0}$ is computed using $h_0$. By applying Proposition 4.2, we find a Ricci flow $g(t)$ on $B_{g_0}(p, r_{k+1}) \times [t_k, t_{k+1}]$, extending $g(t)$ on this smaller ball, with $t_{k+1} = t_k + \alpha\rho_0^2 = (1 + \mu)^2t_k$ and

$$|R_{g(t)}| \leq \Lambda\rho_0^{-2} - \Lambda C_0t_k^{-1} \leq a_0t_{k+1}^{-1} \leq a_0t_k^{-1}$$

thanks to the choice of $a_0$. This shows that $\mathcal{P}(k + 1)$ is true if $r_{k+1} > 0$.

Since $\lim_{k \to +\infty} r_k = -\infty$ and $r_1 = 3$, there exists $i \in \mathbb{N}$ such that $r_i \geq 2$ and $r_{i+1} < 2$. Since $\mathcal{P}(i)$ holds, we now wish to estimate the corresponding $t_i$ from below.

$$2 > r_{i+1} = 3 - \Lambda\sqrt{a_0} \cdot \sum_{k=1}^{i} \sqrt{t_k}$$

(4.4)

$$\geq 3 - \Lambda\sqrt{a_0} \cdot \sum_{k=0}^{\infty} (1 + \mu)^{-k}$$

$$= 3 - \sqrt{t_i} \cdot \frac{\Lambda_0\sqrt{a_0}(1 + \mu)}{\mu}$$

which implies

$$t_i > \frac{\mu^2}{a_0\Lambda_0^2(1 + \mu)^2} =: T(n, b, s_0).$$

(4.5)

This shows that there exists a smooth Ricci flow solution $g(t)$ on $B_{g_0}(p, 2) \times [0, T]$ such that $g(0) = g_0$ and $|R_{g(t)}| \leq a_0t^{-1}$. The conclusion on pinching follows from applying Lemma 3.6 on $B_{g_0}(x, 1)$ where $x \in B_{g_0}(p, 1)$ provided that we shrink $T$ further if necessary. This completes the proof. □

We can now establish the existence of a Ricci flow on $M \times [0, +\infty)$ as claimed in Theorem 1.3 using Theorem 4.1.
Proof of Theorem 1.3. By Lemma 3.4, the pinching hypothesis (1.3) implies that $R_{g_0} \in \bar{C}(b, s_0)$ for some $s_0 > 0$ and $b \in (0, \gamma_n)$ depending only on $n$ and $\varepsilon_0$.

Fix $p \in M$. Pick $R_i \to +\infty$ and denote $h_{i,0} = R_i^{-2}g_0$. Then $R_{h_{i,0}} \in \bar{C}(b, s_0)$ for all $i$. We apply Theorem 4.1 to $h_{i,0}$ to obtain a Ricci flow solution $h_i(t)$ on $B_{h_{i,0}}(p, 1) \times [0, T]$ with

(i) $|R_{h_i(t)}| \leq a_0 t^{-1};$

(ii) $R_{h_i(t)} + I \in \bar{C}(b, s_0)$

for some $a_0(n, s_0), T(n, s_0) > 0$. Define the rescaled Ricci flow solution $g_i(t) = R_i^2h_i(R_i^{-2}t)$ on $B_{g_0}(p, R_i) \times [0, TR_i^2]$ with

\[
|g_i(0)| = g_0;
\]
\[
|R_{g_i(t)}| \leq a_0 t^{-1};
\]
\[
R_{g_i(t)} + R_i^{-2}I \in \bar{C}(b, s_0)
\]

(4.6)

on each $B_{g_0}(p, R_i) \times (0, TR_i^2]$.

By [14] Corollary 3.2 [see also [28]] and a modification of Shi’s higher order estimate given in [13] Theorem 14.16, we deduce that for any $k \in \mathbb{N}, S > 0$ and $\Omega \Subset M$, there exists $C(k, \Omega, g_0, a_0, S) > 0$ so that for sufficiently large $i$ we have

\[
\sup_{\Omega \times [0, S]} |\nabla^k R_{g_i(t)}| \leq C(k, \Omega, g_0, a_0, S).
\]

By applying the Ascoli-Arzelà theorem in coordinate charts, a subsequence converges to a smooth solution $g(t) = \lim_{i \to +\infty} g_i(t)$ of the Ricci flow on $M \times [0, +\infty)$ so that $g(0) = g_0, |R_{g(t)}| \leq a_0 t^{-1}$ and

(4.8) $R_{g(t)} \in \bar{C}(b, s_0)$

throughout $M \times [0, +\infty)$. This implies the pinching conclusion (a) of the theorem by Lemma 3.2 because $\bar{C}(b, s_0) \subset \ell_{a, b}(C_{\text{PIC1}})$, where $a := b + \frac{1}{2}(n - 2)b^2$. Moreover, $g(t)$ is a complete solution by Lemma 4.3 This completes the proof. \qed

Now we are ready to prove the main Theorem 1.2 which we restate in the following equivalent form.

**Theorem 4.4.** Suppose, for $n \geq 4$, that $(M^n, g_0)$ is a complete non-compact manifold such that

(i) $R_{g_0} - \varepsilon_0 \text{scal}(R_{g_0}) \cdot I \in C_{\text{PIC1}}$ for some $\varepsilon_0 \in (0, 1);$

(ii) $R_{g_0} \in C_{\text{PIC2}},$

then $(M, g_0)$ is flat.

Before we begin the proof, we observe the following basic local fact that we will need more than once.

**Lemma 4.5.** A manifold of dimension at least three that splits isometrically into a nontrivial product cannot be PIC1.
To clarify, being PIC1 means that the curvature tensor lies in the interior of $C_{\text{PIC1}}$, or alternatively that the complex sectional curvatures corresponding to PIC1 sections are strictly positive.

**Proof.** Consider a manifold $M_1 \times M_2$, where $\dim(M_1) \geq 1$ and $\dim(M_2) \geq 2$. Pick a unit vector $v$ in some tangent space $T_x M_1$ and two orthonormal vectors $e_1, e_2$ in a tangent space $T_y M_2$. Then the complex vectors $v$ and $\frac{1}{\sqrt{2}}(e_1 + ie_2)$ span a PIC1 section with complex sectional curvature

$$R(v, \frac{1}{\sqrt{2}}(e_1 + ie_2), v, \frac{1}{\sqrt{2}}(e_1 - ie_2)) = \frac{1}{2}(R(v, e_1, v, e_1) + R(v, e_2, v, e_2)) = 0,$$

which is not strictly positive.$\square$

**Proof of Theorem 4.4.** By working on the universal cover, we may assume $M$ to be simply connected. Suppose on the contrary, we have $\text{scal}(R_g(p)) > 0$ for some $p \in M$. By Theorem 1.3, we can find a long-time solution $g(t)$ to the Ricci flow on $M \times [0, +\infty)$ such that

(a) $R_{g(t)} - \varepsilon'_0 \text{scal}(R_{g(t)}) \cdot I \in C_{\text{PIC1}}$ for some $\varepsilon'_0 \in (0, \frac{1}{n(n-1)})$;

(b) $|R_{g(t)}| \leq a_0 t^{-1}$ for some $a_0 > 0$.

Moreover, it follows from [19 Theorem 3.1] that $g(t)$ also satisfies $R_{g(t)} \in C_{\text{PIC2}}$ for all $t > 0$. Furthermore, the strong maximum principle implies that $\text{scal}_g(t) > 0$ for all $t > 0$.

**Claim 4.1.** We have $K(g(t)) > 0$ for all $t > 0$. That is, all real sectional curvatures are strictly positive for positive times.

**Proof of Claim.** Fix a time $t_0 > 0$ at which to consider the sectional curvatures. Because $R_{g(t_0)} \in C_{\text{PIC2}}$, Cabezas-Rivas and Wilking [10 Theorem 5.1] tell us that $(M, g(t_0))$ splits isometrically as $\Sigma^k \times F^{n-k}$ where $\Sigma$ is the (closed) soul and $F$ is diffeomorphic to $\mathbb{R}^{n-k}$ and carries a complete metric $h$ with $R_h \in C_{\text{PIC2}}$. Because $\Sigma$ is closed, but $M$ is non-compact, we must have $k < n$. By (a) and the positivity of the scalar curvature, $(M, g(t_0))$ is (strictly) PIC1, which is incompatible with being a non-trivial product by Lemma 4.5 so $k = 0$ and hence $M^n$ is diffeomorphic to $\mathbb{R}^n$. Similarly, considering the de-Rham decomposition of $M$, (a) implies that $(M, g(t_0))$ is irreducible. Meanwhile, if $(M, g(t_0))$ is symmetric, then the scalar curvature is a positive constant, which by (a) implies a uniform positive lower bound for the Ricci curvature; Bonnet-Myers then forces $M$ to be compact, which is assumed not to be the case. Hence, $(M, g(t_0))$ is of positive scalar curvature, diffeomorphic to $\mathbb{R}^n$, non-symmetric and irreducible. In particular, Berger’s holonomy classification theorem implies that $\text{Hol}(M, g(t_0))$ is either $\text{SO}(n)$ or, possibly if $n$ is even, $U(n/2)$. This is because all other options would be Ricci flat or Einstein (hence compact). If $\text{Hol}(M, g(t_0)) = \text{SO}(n)$, it follows from the strong maximum principle argument in [7 Proposition 9] that $K(g(t_0)) > 0$. Indeed, if any sectional curvature were zero, then every parallel translation of that section would also have zero sectional curvature, and because the holonomy group is
SO(n) the manifold would have to be flat, violating the positivity of the scalar curvature.

If \( \text{Hol}(M, g(t_0)) = U(n/2) \), we can still deduce \( K(g(t_0)) > 0 \) as pointed out in the proof of [10, Corollary 7.6]. We include the argument for convenience. First, we note that \( g(t_0) \) is Kähler by the holonomy. Suppose \( K(\sigma) = 0 \) for some \( x \in M \) and real plane \( \sigma \subset T_x M \). If \( \sigma \) is a complex holomorphic plane, i.e. \( \sigma = \text{span}\{ v, Jv \} \) for some \( v \in T_x M \), then the strong maximum principle \([7, Proposition 9]\) implies that the holomorphic sectional curvature \( K(\sigma) \) vanishes on \( M \) since \( \text{Hol}(M, g(t_0)) = U(n/2) \) and hence \( g(t_0) \) is flat \([13, Chapter IX, Proposition 7.1]\) which contradicts the positive scalar curvature. Suppose now \( \sigma = \text{span}\{ u, v \} \) for some \( u, v \in T_x M \) where \( \{ u, v \} \) are orthonormal but \( Ju \notin \sigma \). We fix an orthonormal frame \( \{ e_i, Je_i \}_{i=1}^{n/2} \) such that defining \( u_i := \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}Je_i) \) makes \( \{ u_i \}_{i=1}^{n/2} \) a unitary frame with \( u = e_1 \) and \( v = \cos \theta \cdot Je_1 + \sin \theta \cdot e_2 \) for some \( \theta \in (0, 2\pi) \). Then \( K(\sigma) = R(u, v, u, v) = 0 \) is equivalent to
\[
\cos^2 \theta \cdot R(e_1, Je_1, e_1, Je_1) + \sin^2 \theta \cdot R(e_1, e_2, e_1, Je_1) + \sin^2 \theta \cdot R(e_1, e_2, e_1, e_2) = 0.
\]
By considering the linear transformation of \( T_x M \) that fixes each \( u_i \) for \( i \neq 2 \), but sends \( u_2 \) to \( -u_2 \), which is an element in \( U(n/2) \), we deduce by the strong maximum principle \([7, Proposition 9]\) that \( (4.9) \) holds also with the sign of the middle term reversed, and adding both gives
\[
\cos^2 \theta \cdot R(e_1, Je_1, e_1, Je_1) + \sin^2 \theta \cdot R(e_1, e_2, e_1, e_2) = 0
\]
and hence \( R(e_1, e_2, e_1, e_2) = 0 \) since \( K \geq 0 \). Similarly, since \( u_2 \mapsto \sqrt{-1}u_2 \) is an element of \( U(n/2) \), we deduce that \( R(e_1, Je_2, e_1, Je_2) = 0 \). Therefore by the Bianchi identity and Kähler symmetries we have
\[
R(u_1, u_2, u_2, u_2) = -R(e_1, Je_1, e_2, Je_2) = R(e_1, e_2, Je_2, Je_1) + R(e_1, Je_2, e_1, e_2) = -R(e_1, e_2, e_1, e_2) - R(e_1, Je_2, e_1, Je_2) = 0.
\]
Using once again the invariance under \( U(n/2) \) coming from the strong maximum principle, we deduce that the orthogonal bisectional curvature of \( g(t_0) \) vanishes. This then contradicts the positivity of the scalar curvature because the scalar curvature can be written as an average of orthogonal bisectional curvatures. For example, according to a formula of Berger \([25, (2.1)]\), one can write the scalar curvature in terms of the orthogonal Ricci curvature \( R^{\perp} \), which vanishes when the orthogonal bisectional curvature vanishes. This completes the proof of the claim. \( \square \)

Since we have \( K(g(t)) > 0 \) and \( |R_g(t)| \leq a_0 t^{-1} \) for all \( t > 0 \), a result of Gromoll-Meyer \([14, Theorem B.65]\) implies that there exists \( c_0(n, a_0) > 0 \) such
that $\text{inj}(g(t)) \geq c_0 \sqrt{t}$ for all $t > 0$. We claim that the asymptotic volume ratio is positive, i.e. for any $x \in M$,

$$\text{AVR}(g_0) := \lim_{r \to \infty} \frac{\text{Vol}_{g_0}(B_{g_0}(x, r))}{\omega_n r^n} > 0,$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. For $t_0 > 0$ sufficiently large, Lemma 4.3 implies that

$$B_{g(t_0)}(x, c_0 \sqrt{t_0}) \subset B_{g_0}(x, c_1 \sqrt{t_0})$$

for some $c_1(n, a_0) > 0$. Additionally, $g(t_0) \leq g_0$ because $\text{Ric}_{g(t)} \geq 0$, so

$$\text{Vol}_{g_0}(B_{g_0}(x, c_1 \sqrt{t_0})) \geq \text{Vol}_{g_0}(B_{g(t_0)}(x, c_0 \sqrt{t_0}))$$

(4.10)

$$\geq \text{Vol}_{g(t_0)}(B_{g(t_0)}(x, c_0 \sqrt{t_0}))$$

$$\geq c_2(n, a_0) t_0^{n/2}$$

where the last inequality follows by G"unther's theorem because $\text{inj}(g(t_0)) \geq c_0 \sqrt{t_0}$ and $|R_{g(t_0)}| \leq a_0 t^{-1}$.

Since $t_0$ is arbitrarily large, we see that $\text{AVR}(g_0) > 0$. It is well-known that the asymptotic volume ratio $\text{AVR}(g)$ is preserved under Ricci flow with $\text{Ric}_{g(t)} \geq 0$ and $|R_{g(t)}| \leq a_0 t^{-1}$; for instance see the proof of [22, Theorem 7]. Therefore, $\text{AVR}(g(t)) = \text{AVR}(g_0) > 0$ for all $t > 0$.

On the other hand, since $R_{g(t)} \in C_{\text{PIC2}}$, $g(t)$ satisfies the Hamilton’s differential Harnack inequality for all $t \geq 0$ by [24]. Together with the fact that $t \cdot \text{scal}_{g(t)}$ is uniformly bounded for all $t > 0$, we can use an argument of Schulze-Simon [27, Theorem 1.2] to deduce that $(M, g_i(t), p)$ where $g_i(t) = i^{-2} g(i^2 t)$ converges sub-sequentially in the Cheeger-Gromov sense to $(M_\infty, g_\infty(t), p_\infty)$ which is an expanding gradient soliton with the same asymptotic volume ratio as $g_0$, see also [4, Proposition 12]. By (a), the Ricci curvature of $g_\infty(t)$ is pinched and hence $g_\infty(t)$ is flat for all $t > 0$ using [23, Corollary 3.1]. This implies $\text{AVR}(g_\infty(t)) = 1$ for all $t > 0$ which forces $\text{AVR}(g_0) = 1$ and hence $R_{g_0} \equiv 0$ by the rigidity of volume comparison. This contradicts the non-flatness we have assumed at $p$. This completes the proof.

It remains to prove the corollary combining our main PIC1 Pinching Theorem [1,2] with earlier work.

Proof of Corollary 1.5. Because $(M, g_0)$ is assumed to be not everywhere flat, by the PIC1 Pinching Theorem [1,2] it must be compact. Without loss of generality we may reduce $\varepsilon_0$ so that $\varepsilon_0 \leq (0, \frac{1}{n(n-1)})$. By Lemma 3.3 applied with $C = C_{\text{PIC1}}$, the pinching hypothesis implies that there exists $b > 0$ such that for $a := b + \frac{1}{2}(n-2)b^2$ we have $R_{g_0} \in \ell_{a,b}(C_{\text{PIC1}})$. But $\ell_{a,b}(C_{\text{PIC1}})$ is invariant under the Hamilton ODE by [2] Proposition 3.2, so if we start the Ricci flow (which is always possible on a closed manifold) then for later times $t > 0$ we still have $R_{g(t)} \in \ell_{a,b}(C_{\text{PIC1}})$, and thus by Lemma 3.2 we have

$$R_{g(t)} - \delta \text{scal}(R_{g(t)}) \cdot I \in C_{\text{PIC1}}$$
for some $\delta > 0$. Because $g_0$ is assumed not to be flat, for $t > 0$ we have $\text{scal}(R_{g(t)}) > 0$ by the maximum principle, and thus $g(t)$ is strictly PIC1. The result then follows from Brendle’s PIC1 version of the sphere theorem [3, Theorem 3]. □

Appendix A. A geometric interpretation of $\check{C}(s)$

In Section 2 for each $s > 0$, $\check{C}(s)$ was defined to be the cone of all algebraic curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ satisfying

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} + \frac{1}{s} (1 - \lambda^2)(1 - \mu^2) \cdot \text{scal}(R) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and $\lambda, \mu \in [0, 1]$. In this appendix we interpret this definition in terms of complex sectional curvatures and use the insight in order to give a quantitative relationship between $C_{\text{PIC1}}$ and $\check{C}(s)$.

As in Section 1, given an algebraic curvature tensor $R \in \mathcal{C}_B(\mathbb{R}^n)$, we can extend by complex linearity and consider complex sectional curvatures of two-complex-dimensional subspaces $\Sigma \subset \mathbb{C}^n$.

Every section $\Sigma$ contains an isotropic vector $v \in \Sigma$, i.e. so that $(v, v) = 0$. To see this directly, pick any basis $\tilde{v}, \tilde{w}$ of $\Sigma$. Either $\tilde{w}$ is isotropic, in which case we set $v = \tilde{w}$, or we can solve the quadratic polynomial $(\tilde{v} + zw, \tilde{v} + zw) = 0$, and then set $v = \tilde{v} + zw$. By scaling, we may assume that $|v| = 1$.

Because $v$ is isotropic and of unit length, we can pick orthonormal $e_1, e_2 \in \mathbb{R}^n$ so that $v = \frac{1}{\sqrt{2}}(e_1 + ie_2)$. Note that $\bar{v} = \frac{1}{\sqrt{2}}(e_1 - ie_2)$ is orthogonal to $v$, e.g. $(v, \bar{v}) = (v, v) = 0$.

As stated earlier, we refer to $\Sigma$ as a PIC1 section if $\bar{v}$ is orthogonal to $\Sigma$. More generally $\bar{v}$ will lie at an angle $\theta \in [0, \pi/2]$ to $\Sigma$ and defining $\alpha = \cos \theta \in [0, 1]$ we can define a unit vector $w \in \Sigma$ orthogonal to $v$ by writing

$$w = \alpha \bar{v} + \sqrt{1 - \alpha^2} u$$

for some unit $u \in \mathbb{C}^n$ that is orthogonal to both $v$ and $\bar{v}$. Because $(v, w) = \alpha$, we see that $\alpha = 0$ precisely when $\Sigma$ is a PIC1 section. Indeed, $\alpha$ can be viewed as a measure of how far $\Sigma$ is from being a PIC1 section that depends only on $\Sigma$ (and not on our choices of vectors above) with $\alpha = 1$ being precisely the case that $\Sigma$ is a real section (that is, the complexification of a two-real-dimensional plane in $\mathbb{R}^n$, or equivalently a section that contains a vector $v$ such that $\{v, \bar{v}\}$ gives an orthonormal basis for $\Sigma$). We thus view $\alpha$ as a function on the set of sections.

A calculation then reveals that an alternative characterisation of $\check{C}(s)$ is the cone of all curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ whose complex sectional curvature satisfies

$$K^C(\Sigma) + \frac{\alpha(\Sigma)^2}{s} \text{scal}(R) \geq 0. \quad \text{(A.1)}$$
We see very clearly the inclusions $C_{\text{PIC}2} \subset \tilde{C}(s) \subset C_{\text{PIC}1}$ mentioned in Section 2.

It might be initially a little surprising that a curvature tensor in $C_{\text{PIC}1}$, which is assumed only to have non-negative complex sectional curvature for very special sections (the PIC1 sections) does, in fact, enjoy a lower bound for all complex sectional curvatures.

**Lemma A.1.** Suppose $R \in C_{\text{PIC}1} \subset C_B(\mathbb{R}^n)$. Then for all complex sections $\Sigma \subset \mathbb{C}^n$ we have

$$K^C(\Sigma) \geq -C(n) \alpha(\Sigma) \text{scal}(R).$$

Before we prove Lemma A.1, we record the following more basic control that is equivalent to every $R \in C_{\text{PIC}1} \setminus \{0\}$ having positive scalar curvature.

**Lemma A.2.** If $R \in C_{\text{PIC}1}$, then

$$|R| \leq C(n) \text{scal}(R).$$

**Proof.** Take any orthonormal basis $\{e_i\}$. Then

$$\text{scal}(R) = \sum_{i \neq j} R(e_i, e_j, e_i, e_j) = \frac{1}{2(n-2)} \sum_{i,j,k \text{ distinct}} \left[R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k)\right].$$

Each term in square brackets can be written

$$R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k) = R(e_i, e_j + ie_k, e_i, e_j - ie_k) = 2K^C(\Sigma) \geq 0,$$

where $\Sigma$ is the PIC1 section spanned by $e_i$ and $\frac{1}{\sqrt{2}}(e_j + ie_k)$. Because all these terms are non-negative, and sum to $2(n-2)\text{scal}(R)$, we have

$$0 \leq R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k) \leq 2(n-2)\text{scal}(R).$$

To control the sectional curvature of a plane spanned by $e_i, e_j$ for arbitrary $i \neq j$, we pick any $k \neq i, j$ and compute

$$2R(e_i, e_j, e_i, e_j) = \left[R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k)\right]$$

$$+ \left[R(e_j, e_i, e_j, e_i) + R(e_j, e_k, e_j, e_k)\right]$$

$$- \left[R(e_k, e_j, e_k, e_j) + R(e_k, e_i, e_k, e_i)\right].$$

(A.2)

Therefore

$$-(n-2)\text{scal}(R) \leq R(e_i, e_j, e_i, e_j) \leq 2(n-2)\text{scal}(R).$$

**Proof of Lemma A.1.** We are interested in the complex sectional curvature corresponding to $\Sigma$, which is spanned by the orthonormal basis $\{v, w\}$ of the
type considered earlier. We compute
\[
K^C(\Sigma) = R(v, w, \bar{v}, \bar{w})
\]
\[
= R(v, \alpha \bar{v} + \sqrt{1 - \alpha^2} u, \alpha v + \sqrt{1 - \alpha^2} \bar{u})
\]
\[
= \alpha^2 R(v, \bar{v}, v, v) + (1 - \alpha^2) R(v, u, \bar{v}, \bar{u})
\]
\[
+ \alpha \sqrt{1 - \alpha^2} R(v, \bar{v}, v, \bar{u}) + \alpha \sqrt{1 - \alpha^2} R(v, u, v, v)
\]
\[
\geq (1 - \alpha^2) R(v, u, \bar{v}, \bar{u}) - C(n) \alpha \text{scal}(R),
\]
where we have used Lemma A.2.

Now note that the section spanned by \(v\) and \(u\) is a \(\text{PIC1}\) section because \(\bar{v}\) is orthogonal to both \(v\) (because \(v\) is isotropic) and \(u\) (by construction).
Therefore \(R(v, u, \bar{v}, \bar{u}) \geq 0\) because \(R \in \text{C}_{\text{PIC1}}\).

We conclude that
\[
R(v, w, \bar{v}, \bar{w}) \geq -C(n) \alpha \text{scal}(R),
\]
as required. \(\square\)

**Corollary A.3.** For every \(n \geq 4\) and \(\varepsilon > 0\) there exists \(s_0 > 0\) such that for every \(R \in \text{C}_{\text{PIC1}}\) we have
\[
R + \varepsilon \text{scal}(R) \cdot I \in \mathcal{C}(s_0).
\]

**Proof.** Take an arbitrary section \(\Sigma \subset \mathbb{C}^n\). Then by Lemma A.1 we have
\[
K^C(\Sigma) \geq -C(n) \alpha(\Sigma) \text{scal}(R)
\]
\[
\geq -\varepsilon \text{scal}(R) - c_1(n, \varepsilon) \alpha(\Sigma)^2 \text{scal}(R),
\]
for some \(c_1(n, \varepsilon) > 0\), by Young’s inequality. If we write \(\hat{R} := R + \varepsilon \text{scal}(R) \cdot I\), then \(\text{scal}(\hat{R}) \geq \text{scal}(R)\), so we find that
\[
K^C_R(\Sigma) + c_1(n, \varepsilon) \alpha(\Sigma)^2 \text{scal}(\hat{R}) \geq 0,
\]
and we can choose \(s_0 = \frac{1}{c_1(n, \varepsilon)}\) to deduce that \(\hat{R} \in \mathcal{C}(s_0)\). \(\square\)

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