DEFORMATION OF FINITE DIMENSIONAL $C^*$-QUANTUM
GROUPOIDS

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Abstract. In this work we prove, in a self contained way, that any finite dimensional $C^*$-quantum groupoid can be deformed in order that the square of the antipode is the identity on the base. We also prove that for any $C^*$-quantum groupoid with non abelian base, there is uncountably many $C^*$-quantum groupoids with the same underlying algebra structure but which are not isomorphic to it. In fact, the $C^*$-quantum groupoids are closed in an analog of the procedure presented by D.Nikshych ([N] 3.7) in a more general situation.
1. **Flipper (separating) projections**

1.1. **Notations.** In what follows, \( N \) is a finite dimensional von Neumann algebra, so \( N \) is isomorphic to a sum of matrix algebras \( \bigoplus_{\gamma} M_{n_{\gamma}} \), we denote the family of minimal central projections of \( N \) by \( \{ p^{\gamma} \} \), we denote a given family of matrix units for \( N \) by \( \{ e_{ij}^{\gamma} \}_{1 \leq i, j \leq n_{\gamma}} \).

Let’s denote by \( tr_N \) the canonical (algebraic) trace on \( N \) and \( E_{Z(N)}(N) \) the canonical conditional expectation \( E : N \mapsto Z(N) \) (the center of \( N \)), so one has: \( tr_N \circ E_{Z(N)} = tr_N \), and \( E_N \) is faithful. An easy calculation gives for any \( n \in N : E_{Z(N)}(n) = \sum_{\gamma} \frac{1}{n_{\gamma}} \sum_{i,j} n_{\gamma} e_{ij}^{\gamma} e_{ji}^{\gamma} \)

We shall denote the opposite von Neumann algebra of \( N \) by \( N^o \), so this is \( N \) with the opposite multiplication, hence a matrix unit of \( N^o \) is given by the transposed of \( N \)'s: \( \{ e_{ji}^{\gamma} \}_{1 \leq i, j \leq n_{\gamma}} \). We shall not in general distinguish \( N \) and \( N^o \), except when the multiplication occurs.

1.2. **Definitions.** i) The application \( m : N \otimes N \mapsto N \) is the multiplication map defined for any \( x = \sum_{i} x_i \otimes y_i \) by \( m(x) = \sum_{i} x_i y_i \).

ii) The elements \( f \) in \( N \otimes N \) such that for the multiplication of \( N^o \otimes N \), one has: \( f(a^o \otimes 1) = f(1 \otimes a) \) for any \( a \in N \) and \( m(f) = 1 \), are called the separating elements of \( N \).

1.3. **Example (cf [Val1],[Val2]).** The element \( e = \sum_{\gamma} \sum_{i,j} \frac{1}{n_{\gamma}} e_{ij}^{\gamma} e_{ji}^{\gamma} \) is the only orthogonal projection of \( N^o \otimes N \) such that, for any \( n \in N : e(n^o \otimes 1) = e(1 \otimes n) \) and if \( e(1 \otimes n) = 0 \) then \( n = 0 \). As an element of \( N \otimes N \), \( e \) is also known as the symmetric separating element of \( N \).

1.4. **Lemma.** For any \( y \) in \( N \otimes N \), and for the multiplication of \( N^o \otimes N \), one has \( m(ey) = m(y) \).

Proof: For any \( x, z \) in \( N \), one has:

\[
m(e(x^o \otimes z)) = m(\sum_{\gamma} \sum_{i,j} \frac{1}{n_{\gamma}} (xe_{ij}^{\gamma})^o \otimes e_{ji}^{\gamma} z) = m(\sum_{\gamma} \sum_{i,j} \frac{1}{n_{\gamma}} xe_{ij}^{\gamma} e_{ji}^{\gamma} z) = m(x \otimes z)
\]

the lemma follows □

1.5. **Proposition.** Let \( f \) be any element of \( N \otimes N \), the following assertions are equivalent:

1) \( f \) is a separating element of \( N \),

2) \( f \) is any element of \( N \otimes N \) such that in \( N^o \otimes N \): \( fe = f \) and \( ef = e \),
3) \( f \) is a (not necessarily orthogonal) projection of \( N^o \otimes N \) with the same direction than \( e \)

4) there exists an element \( g \) in \( N \) such that \( E_{Z(N)}(g) = 1 \) and \( f = (1 \otimes g)e \) in \( N^o \otimes N \) (or equivalently in \( N \otimes N \)).

Proof: Let \( f = \sum f_i^o \otimes f'_i \) and \( h = \sum h_i^o \otimes h'_i \) verifying 1) then in \( N^o \otimes N \), one has:

\[
fh = f(\sum h_i^o \otimes h'_i) = \sum f(h_i^o \otimes h'_i) = \sum f(h_i^o \otimes 1)(1^o \otimes h'_i)
\]

\[
= \sum f(1^o \otimes h_i)(1^o \otimes h'_i) = \sum f(1^o \otimes h_i)(1^o \otimes h'_i) = f(1^o \otimes \sum h_i h'_i) = f
\]

hence 1) implies 2).

Let \( f = \sum f_i^o \otimes f'_i \) be any element verifying 2) then in \( N^o \otimes N \):

\[
f^2 = fefe = f(ef)e = fe^2 = fe = f
\]

so \( f \) is an algebraic projection, and as \( fe = f \) (resp. \( ef = e \)), one has: \( \ker e \subset \ker f \) (resp. \( \ker f \subset \ker e \)) and \( f \) verifies 3).

Let \( f = \sum f_i^o \otimes f'_i \) be any element verifying 3), then one has:

\[
f = fe = \sum (f_i^o \otimes f'_i)e = \sum (f_i^o \otimes f'_i)e = \sum (1^o \otimes f'_i)(f_i^o \otimes 1)e
\]

\[
= \sum (1^o \otimes f'_i)(1^o \otimes f_i)e = (1^o \otimes \sum f'_i f_i)e
\]

So there exists a \( g \) in \( N \) such that \( f = (1^o \otimes g)e \) and one has:

\[
e = ef = e(1 \otimes g)e = (\sum \sum \frac{1}{n_{\gamma}} e_{i,j}^o \otimes e_{j,i}^o)(1 \otimes g)e
\]

\[
= \sum \sum \frac{1}{n_{\gamma}} e_{i,j}^o \otimes e_{j,i}^o (1 \otimes e_{j,i}^o)(1 \otimes e_{j,i}^o)e
\]

\[
= \sum \sum \frac{1}{n_{\gamma}} (1 \otimes e_{j,i}^o)(1 \otimes e_{j,i}^o)e
\]

One deduces that \( \sum \sum \frac{1}{n_{\gamma}} e_{j,i}^o e_{i,j}^o = 1 \) and \( f \) verifies 4).

To end let \( f = \sum f_i^o \otimes f'_i \) be any element verifying 4) then for any \( a \) in \( N \), one has:

\[
f(1 \otimes a) = (1 \otimes g)e(1 \otimes a) = (1 \otimes g)e(\hat{a} \otimes 1) = f(\hat{a} \otimes 1)
\]

Moreover: one easily checks that: \( m(f) = m((1 \otimes g)e) = \sum \frac{1}{n_{\gamma}} e_{j,i}^o e_{i,j}^o = 1 \). So \( f \) verifies 1).
1.6. Remarks. What distinguishes $e$ among all separating elements of $N$ is the fact that it is an automadjoint (orthogonal) projection for the natural $C^*$-structure of $N^0 \otimes N$. When $N$ is not abelian, there always exist separating elements different from $e$: for example if $n_0$ is such that $n_{0,0} \neq 1$ then $f = (1 \otimes g)e$, where $g = 1 + e_{i,1}^0 - \sum_{i \neq 1} e_{i,i}^0$, is such an element. The element $g$ in 4) needs not to be invertible, for example if $N = M_2(\mathbb{C})$ (the $C^*$-algebra of complexe $2 \times 2$ matrices) with its usual matrix unit $(e_{i,j})$ then $g = 2e_{1,1}$ works.

1.7. Lemma and notations. If $g$ is any strictly positive element of $N$, let’s define a new involution denoted by $\ast_g$ on the underlying algebra of $N$ by the formulae: $x^\ast_g = gx^\ast g^{-1}$, then with this new $\ast$-involution one gives to this algebra a new structure of finite $C^*$-algebra we shall denote $N_g$.

Proof: Let’s prove that $N_g$ is actually a $C^*$-algebra. As any finite dimensional $C^*$-algebra, $N$ is isomorphic to a finite sum of matrix algebras over $\mathbb{C}$, then one can suppose that: $N = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ which acts naturally on $H = \mathbb{C}^n$. Let’s denote by $\langle, \rangle$ the natural scalar product of $\mathbb{C}^n$, if one defines a new pairing on $\mathbb{C}^n$ by the formula $\langle \xi, \eta \rangle = \langle g^{-\frac{1}{2}}\xi \eta, g^{-\frac{1}{2}}\eta \rangle$, for any $\xi, \eta \in \mathbb{C}^n$, it’s easy to see that it is a new scalar product on $\mathbb{C}^n$. Let $x$ be any element in $N$, then for any $\xi, \eta \in \mathbb{C}^n$, one has:

$$\langle x\xi, \eta \rangle_g = \langle g^{-\frac{1}{2}}x\xi, g^{-\frac{1}{2}}\eta \rangle = \langle x\xi, g^{-1}\eta \rangle = \langle g^{-\frac{1}{2}}x\xi, g^{-\frac{1}{2}}g^\ast g^{-1}\eta \rangle$$

and so $x^\ast_g$ appears to be the adjoint of $x$ for this new scalar product and $N_g$ is a $C^*$-algebra.

1.8. Proposition. Let $g$ be any strictly positive element of $N$, such that $E_{Z(N)}(g) = 1$, and $f$ any element of $N \otimes N$, then the following assertions are equivalent:

i) $f$ is the separating element of $N$ associated to $g$ by proposition 1.5.

ii) $f$ is an orthogonal projection of $(N^0)_{(g^\frac{1}{2})^0} \otimes N_{g^\frac{1}{2}}$ with the same direction than $e$.

Proof: If i) is true by proposition 1.5, $f$ is an idempotent of $N^0 \otimes N$ with the same direction than $e$. Let’s prove that it is self adjoint for $(N^0)_{(g^\frac{1}{2})^0} \otimes N_{g^\frac{1}{2}}$, using the separability property for $e$, one has:

$$f^\ast_s = ((1 \otimes g)e)^\ast_s = (g^\frac{1}{2}\otimes g^\frac{1}{2})((1 \otimes g)e)((g^{-\frac{1}{2}})^0 \otimes g^{-\frac{1}{2}})$$

$$= (g^\frac{1}{2}\otimes g^\frac{1}{2})e(1 \otimes g)((g^{-\frac{1}{2}})^0 \otimes g^{-\frac{1}{2}}) = (1 \otimes g)e((g^{-\frac{1}{2}})^0 \otimes g^\frac{1}{2})$$

$$= (1 \otimes g)e = f,$$

hence, ii) is true. Now let’s suppose that $f$ verifies ii), then by proposition 1.5, there exists a $k$ in $N$ such that $f = (1 \otimes k)e$, as $f$ is self adjoint, one has:

$$f = f^\ast_s = ((1 \otimes k)e)^\ast_s = (g^\frac{1}{2}\otimes g^\frac{1}{2})((1 \otimes k)e)((g^{-\frac{1}{2}})^0 \otimes g^{-\frac{1}{2}})$$

$$= (g^\frac{1}{2}\otimes g^\frac{1}{2})e(1 \otimes k^\ast)((g^{-\frac{1}{2}})^0 \otimes g^{-\frac{1}{2}}) = (1 \otimes g)e((g^{-\frac{1}{2}})^0 \otimes k^\ast g^{-\frac{1}{2}})$$

$$= (1 \otimes g)e(1 \otimes g^{-\frac{1}{2}}k^\ast g^{-\frac{1}{2}}),$$
so we can deduce that:

\[ 1 = E_{Z(N)}(g) = m(f) = m((1 \otimes g)e(1 \otimes g^{-\frac{1}{2}}k^*g^{-\frac{1}{2}})) = m((1 \otimes g)e)g^{-\frac{1}{2}}k^*g^{-\frac{1}{2}} = E_{Z(N)}(g)g^{-\frac{1}{2}}k^*g^{-\frac{1}{2}} = g^{-\frac{1}{2}}k^*g^{-\frac{1}{2}}, \]

hence \( k = g^* = g \) and \( f \) verifies i). □

2. Deformation of \( C^* \)-quantum groupoids

Let us recall the definition of a \( C^* \)-quantum groupoid (or a weak Hopf \( C^* \)-algebra):

2.1. Definition. (G. Böhm, K. Szlachányi, F. Nill) [BoSz], [BoSzNi]

A weak Hopf \( C^* \)-algebra is a collection \((A, \Delta, \kappa, \epsilon)\) where: \( A \) is a finite-dimensional \( C^* \)-algebra, \( \Delta : A \to A \otimes A \) is a generalized coproduct, which means that: \((\Delta \otimes i)\Delta = (i \otimes \Delta)\Delta, \kappa \) is an antipode on \( A \), i.e., a linear application from \( A \) to \( A \) such that 
\[(\kappa \otimes *)^2 = i \text{ (where } * \text{ is the involution on } A), \kappa(xy) = \kappa(y)\kappa(x) \text{ for every } x, y \in A \text{ with} \]
\[(\kappa \otimes \kappa)\Delta = \varsigma\Delta\kappa \text{ (where } \varsigma \text{ is the usual flip on } A \otimes A). \]

We suppose also that \((m(\kappa \otimes i) \otimes i)(\Delta \otimes i)\Delta(x) = (1 \otimes x)\Delta(1) \) (where \( m \) is the multiplication of tensors, i.e., \( m(a \otimes b) = ab \)), and that \( \epsilon \) is a counit, i.e., a positive linear form on \( A \) such that \((\epsilon \otimes i)\Delta = (i \otimes \epsilon)\Delta = i, \) and for every \( x, y \in A: \)
\[(\epsilon \otimes \epsilon)((x \otimes 1)\Delta(1)(1 \otimes y)) = \epsilon(xy). \]

2.2. Results. (L. Vainerman, D. Nikshych [NV1], [NV2], [BoSzNi]) If \((A, \Delta, \kappa, \epsilon)\) is a weak Hopf \( C^* \)-algebra, then the following assertions are true:

0) The sets

\[ A_t = \{ x \in A/\Delta(x) = \Delta(1)(x \otimes 1) = (x \otimes 1)\Delta(1) \} \]
\[ A_s = \{ x \in A/\Delta(x) = \Delta(1)(1 \otimes x) = (1 \otimes x)\Delta(1) \} \]

are sub \( C^* \)-algebras of \( A \); we call them respectively target and source Cartan subalgebra of \((A, \Delta)\).

1) The application \( \epsilon_t = m(i \otimes \kappa)\Delta \) takes values in \( A_t \) and \( \epsilon_s = m(\kappa \otimes i)\Delta \) takes values in \( A_s \). We will call target counit the application \( \epsilon_t \) and we call source counit \( \epsilon_s \).

2) The \( C^* \)-algebra \( A \) has a unique projection \( p \), called the Haar projection, characterized by the relations:
\[ \kappa(p) = p, \epsilon_t(p) = 1 \text{ and for every } a \in A, ap = \epsilon_t(a)p. \]

3) There exists a unique faithful positive linear form \( \phi \), called the normalized Haar measure of \((A, \Delta, \kappa, \epsilon)\), satisfying the following three properties:
\[ \phi \circ \kappa = \phi, (i \otimes \phi)(\Delta(1)) = 1 \text{ and, for every } x, y \in A:
\]
\[ (i \otimes \phi)((1 \otimes y)\Delta(x)) = \kappa((i \otimes \phi)(\Delta(y)(1 \otimes x))). \]

4) \((\kappa \otimes i)\Delta(1)\) is a separating element for \( N = A_t \).

5) One says that the collection \((A, \Delta, \kappa, \epsilon)\) is a weak Kac algebra if it is a \( C^* \)-quantum groupoid the antipode of which is involutive, this is equivalent to the fact that \( \phi \) is a trace.
2.3. Remark. The $C^*$-quantum groupoids we consider are not weak Kac algebras in general, even there exist $C^*$-quantum groupoids for which $\kappa^2 \neq \text{Id}$ on $A_t$ or $A_s$, but one can deform the initial structure in order that $\kappa^2(x) = x$, for any $x$ in $A_t$ or $A_s$. This is the object of what follows.

2.4. Proposition. If $(A, \Delta, \kappa, \epsilon)$ is a $C^*$-quantum groupoid, there exists a unique invertible element $q \in A_t$, such that for any matrix unit $(e_{j,i}^\gamma)$ of $A_t$:

1) $\Delta(1) = \sum \sum \frac{1}{n_{\gamma}} \kappa^{-1}(e_{i,j}^\gamma q) \otimes e_{j,i}^\gamma$

2) $q = \kappa^2(q)$ and $E_{Z(A_t)}(q) = 1$

3) For any $x \in A_tA_s$ (the $C^*$-algebra generated by $A_t$ and $A_s$), one has: $\kappa^2(x) = q^{-1}\kappa(q)x\kappa(q^{-1})$

4) $q$ is positive.

Proof: Let’s use the notations of proposition 1.5 in the case of $N = A_t$: as $f = (\kappa \otimes i)\Delta(1)$ is a separating element for $N = A_t$, there exists a unique $q$ in $A_t$, such that $f = (1 \otimes q)e$ and for which 1) is true. Let’s denote by $h$ any element in $A$ such that $hq = 0$, as $\epsilon$ is a co-unity for $A$, one has:

\[
h = h(\epsilon \otimes i)(\Delta(1)) = (\epsilon \otimes i)((1 \otimes h)\Delta(1))
\]

\[
= (\epsilon \circ \kappa^{-1} \otimes i)(\kappa \otimes i)((1 \otimes h)(\kappa \otimes i)(\Delta(1)))
\]

\[
= (\epsilon \circ \kappa^{-1} \otimes i)((1 \otimes h)q)e = 0
\]

Hence $q$ is invertible in $A$. From proposition 1.5 4) one has: $\sum \sum \frac{1}{n_{\gamma}} e_{i,j}^\gamma q e_{j,i}^\gamma = 1$.

As $\Delta(1)$ is selfadjoint one has:

\[
\sum \sum \frac{1}{n_{\gamma}} \kappa^{-1}(e_{i,j}^\gamma q) \otimes e_{j,i}^\gamma = \sum \sum \frac{1}{n_{\gamma}} \kappa^{-1}(e_{i,j}^\gamma q)^* \otimes e_{i,j}^\gamma
\]

As the family $(e_{i,j}^\gamma)$ is a base for $A_t$, one has for any $\gamma$ and $i, j$: $\kappa^{-1}(e_{i,j}^\gamma q) = \kappa^{-1}(e_{j,i}^\gamma q)^*$

So for any $x$ in $A_t$:

\[\kappa^{-1}(xq) = \kappa^{-1}(x^*q)^*\]

applying this to $x = 1$ and $x = q$ one also has: $\kappa^{-1}(q) = \kappa^{-1}(q)^*$, and $\kappa^{-1}(q^2) = \kappa^{-1}(q^2)^*$. So using the fact that $(\kappa \circ \star)^2 = 1$, one obtains:

\[\kappa^{-1}(q) = \kappa^{-1}(q^2)\kappa^{-1}(q^{-1}) = \kappa^{-1}(q^2)^*\kappa^{-1}(q^{-1})\]

\[= (\kappa^{-1}(q)\kappa^{-1}(q^*)^*)\kappa^{-1}(q^{-1})\]

\[= \kappa^{-1}(q^*)\kappa^{-1}(q)^*\kappa^{-1}(q^{-1}) = \kappa^{-1}(q^*)\kappa^{-1}(q)\kappa^{-1}(q^{-1}) = \kappa^{-1}(q^*)\kappa^{-1}(q^{-1})\]

\[= \kappa(q),\]

hence $\kappa^2(q) = q$ and $q$ verifies 2).
The fact that for any $x$ in $A_t$ one has $\kappa^{-1}(xq) = \kappa^{-1}(x^*q)^*$ implies that:

\[
\kappa^2(x) = \kappa(\kappa(x)) = \kappa(\kappa^{-1}(x^*)) = \kappa(\kappa^{-1}(x^*)^*\kappa^{-1}(q)\kappa^{-1}(q^{-1})) = q^{-1}\kappa(\kappa^{-1}(x^*))^*\kappa^{-1}(q) = q^{-1}\kappa(\kappa^{-1}(x^*)^*) = q^{-1}\kappa(\kappa^{-1}(x^*)^*) = q^{-1}\kappa(\kappa^{-1}(x^*)^*) = q^{-1}\kappa(\kappa^{-1}(xq)) = q^{-1}xq
\]

Now let $y$ be any element of $A_s$, and let $z$ be the element in $A_t$ such that $y = \kappa(z)$, then, as $A_s$, and $A_t$ commute, for any $x$ in $A_t$, one has:

\[
\kappa^2(xy) = \kappa^2(y)\kappa^2(x) = \kappa(\kappa^2(z))\kappa^2(x) = \kappa(q^{-1}\kappa(z)q^{-1}xy) = \kappa(q)q^{-1}xyq\kappa(q^{-1}) = \kappa(q)q^{-1}xyq\kappa(q^{-1}),
\]

so $q$ verifies 3).

By an other calculus, one has:

\[
\kappa^{-1}(q) = \kappa^{-1}(q)^* = (\kappa^{-1}(q^*)^{-1}\kappa^{-1}(q)^* \kappa^{-1}(q^*) = (\kappa^{-1}(q))^{-1}\kappa^{-1}(q^*) = \kappa^{-1}(q^*),
\]

So $q = q^*$, if $|q|$ denotes the module of $q$ in his polar decomposition, as a consequence $q$ and $|q|^{1/2}$ commute, hence $\kappa^2(|q|^{1/2}) = |q|^{1/2}$. But one has: $(\kappa \circ *)^2 = 1$, so $\kappa((|q|^{1/2})^*) = |q|^{1/2} = \kappa((|q|^{1/2}))$, this implies that $\kappa(|q|^{1/2})$ is hermitian. Now one has: $\kappa(|q|) = \kappa(|q|^{1/2})\kappa(|q|^{1/2}) = \kappa(|q|^{1/2})\kappa(|q|^{1/2}) \geq 0$.

One can view $A_s$ as a sub-$C^*$-algebra of the linear operators of a finite Hilbert space $H$, and $A_t$ as acting on $\oplus \mathbb{C}^{n_\gamma}$ in such a way that the $e_{i,j}^\gamma$’s form the canonical matrix unit of $\mathcal{L}(\mathbb{C}^{n_\gamma})$. Let’s denote by $\eta_i^\gamma$ the canonical base of $\mathbb{C}^{n_\gamma}$. As $\Delta(1)$ is positive, then for any $\eta$ in $H$ and any $e_k^o$, one has:

\[
0 \leq (\Delta(1)(\eta \otimes \epsilon_k^0), \eta \otimes \epsilon_k^0) \\
\leq \sum_{\gamma} \sum_{i,j} \frac{1}{n_\gamma} (\kappa^{-1}(e_{i,j}^\gamma q) \eta \otimes \epsilon_j^0, \eta \otimes \epsilon_k^0) \\
\leq \frac{1}{n_\gamma^o} \sum_j (\kappa^{-1}(e_{j,i}^0 q) \eta \otimes \epsilon_j^0, \eta \otimes \epsilon_k^0) \\
\leq \frac{1}{n_\gamma^o} \sum_j (\kappa^{-1}(e_{j,k}^0 q) \eta, \eta)(\epsilon_j^0, \epsilon_k^0) \\
\leq \frac{1}{n_\gamma^o} (\kappa^{-1}(e_{k,k}^0 q) \eta, \eta),
\]
One deduces that \( \kappa^{-1}(e_{k,k}^0 q) \geq 0 \), and \( \kappa(q) = \kappa^{-1}(q) = \sum_{\gamma} \kappa^{-1}(e_{k,k}^0 q) \geq 0 \). As \( q \) and \( |q| \) commute, \( \kappa(q) \) and \( \kappa(|q|) \) are commuting each other, positive and also:

\[
\kappa(q)^2 = \kappa(q^2) = \kappa(q^* q) = \kappa(|q|^2) = \kappa(|q|)^2,
\]

hence one has \( \kappa(q) = \kappa(|q|) \), so \( q = |q| \geq 0 \) and \( q \) verifies 4). \( \square \)

2.5. Lemma. Let \( q \) be the canonical element of \( A_t \) defined in proposition 2.4. If the base \( A_t \) is not commutative then the set of strictly positive elements \( k \) in \( A_t \), such that \( \kappa^2(k) = k \) and \( E_{Z(A)}(k^{-1} q) = 1 \) is uncountable, more precisely the set of the \( k^{-1} q \)'s spectra is uncountable.

Proof: Due to proposition 2.4 3), a strictly positive element \( k \) of \( A_t \) such that \( \kappa^2(k) = k \) is just a strictly positive element of commuting with \( q \), let’s use the notations of 1.1 for \( N = A_t \), for any \( \gamma \), \( q^\gamma \) and \( k^\gamma \) appear to be two strictly positive matrices in \( M_n(\mathbb{C}) \) which commute, up to conjugacy, these are two diagonal matrices with all diagonal elements strictly positive, let’s say \((k_i^\gamma)_{i=1..n}\) and \((q_i^\gamma)_{i=1..n}\). The equality \( E_{Z(N)}(k^{-1} q) = 1 \) just means that \( \sum_i (k_i^\gamma)^{-1} q_i^\gamma = 0 \). The lemma follows. \( \square \)

2.6. Theorem. Let \( (A, \Delta, \kappa, \epsilon) \) be any \( C^* \)-quantum groupoid, and let \( k \) be any positive invertible element of \( A_t \) given by lemma 2.4. Let’s give the algebra \( A \) a new involution \( *_k \), a new coproduct \( \Delta_k \), a new antipode \( \kappa_k \) and a new counit \( \epsilon_k \) by the following formulas:

\[
A \ni a \mapsto a^*_k = ka^* k^{-1}, \quad \Delta_k(a) = \Delta(a)(1 \otimes k^{-1}), \quad \kappa_k(a) = k \kappa(x) k^{-1}, \quad \text{and} \quad \epsilon_k(a) = \epsilon(ak).
\]

If one denotes by \( A_k \), the algebra \( A \) together with the new involution \( *_k \), then the 4-tuple \( (A_k, \Delta_k, \kappa_k, \epsilon_k) \) is a new \( C^* \)-quantum groupoid, with the same underlying Cartan subalgebras than \( (A, \Delta, \kappa, \epsilon) \) and \( k^{-1} q \) is the canonical element associated by proposition 2.4 to \( (A_k, \Delta_k, \kappa_k, \epsilon_k) \).

Proof: By lemma 2.7, \( A_k \) is a \( C^* \)-algebra. For any \( a, b \in A \), using the notations of proposition 2.4 and the fact that \( \kappa^{-1} \otimes i \) is multiplicative viewed as an application from \( N^o \otimes N \) to \( N \otimes N \), one has:

\[
\Delta_k(a) \Delta_k(b) = \Delta(a)(\Delta(1)(1 \otimes k^{-1})\Delta(1))\Delta(b)(1 \otimes k^{-1})
\]
\[
= \Delta(a)(\kappa^{-1} \otimes i)(f(1 \otimes k^{-1})f)\Delta(b)(1 \otimes k^{-1})
\]
\[
= \Delta(a)(\kappa^{-1} \otimes i)(f(1 \otimes k^{-1} q) e)\Delta(b)(1 \otimes k^{-1})
\]
\[
= \Delta(a)(\kappa^{-1} \otimes i)(f(1 \otimes E_{Z(N)}(k^{-1}q))\Delta(b)(1 \otimes k^{-1})
\]
\[
= \Delta(a)(\kappa^{-1} \otimes i)(f)\Delta(b)(1 \otimes k^{-1}) = \Delta(a)\Delta(1)\Delta(b)(1 \otimes k^{-1})
\]
\[
= \Delta_k(ab),
\]
so $\Delta_k$ is multiplicative and obviously linear. As $A_s$ and $A_t$ are commuting, then one also has:

$$\Delta_k(a^*) = \Delta_k(ka^*k^{-1}) = \Delta(ka^*k^{-1})(1 \otimes k^{-1}) = \Delta(k)\Delta(a^*)\Delta(k^{-1})(1 \otimes k^{-1})$$

$$= (k \otimes 1)\Delta(a^*)(k^{-1} \otimes k^{-1}) = (k \otimes k)(1 \otimes k^{-1})\Delta(a^*)(k^{-1} \otimes k^{-1})$$

$$= (k \otimes k)(\Delta(a)(1 \otimes k^{-1}))^*(k^{-1} \otimes k^{-1}) = (k \otimes k)\Delta_k(a)^*(k^{-1} \otimes k^{-1})$$

$$= \Delta_k(a)^*,$$

hence $\Delta_k$ is a *-morphism. In an other hand, we have:

$$(\Delta_k \otimes i)\Delta_k(a) = (\Delta_k \otimes i)(\Delta(a)(1 \otimes k^{-1})) = (\Delta \otimes i)(\Delta(a))(1 \otimes k^{-1} \otimes k^{-1})$$

$$= (i \otimes \Delta)(\Delta_k(a)(1 \otimes k))(1 \otimes k^{-1} \otimes k^{-1})$$

$$= (i \otimes \Delta)(\Delta_k(a))(1 \otimes \Delta(1))(1 \otimes k \otimes 1)(1 \otimes k^{-1} \otimes k^{-1})$$

$$= (i \otimes \Delta)(\Delta_k(a))(i \otimes \Delta)(1 \otimes 1)(1 \otimes 1 \otimes k^{-1})$$

$$= (i \otimes \Delta_k)(\Delta_k(a))$$

So $\Delta_k$ is a coproduct. As $A_t$ and $A_s$ commute, one has:

$$\zeta(\kappa_k \otimes \kappa_k)\Delta_k(a) = (k \otimes k)\zeta(\kappa \otimes \kappa)\Delta_k(a)(k^{-1} \otimes k^{-1})$$

$$= (k \otimes k)\zeta((\kappa \otimes \kappa)(\Delta(a)(1 \otimes k^{-1}))(k^{-1} \otimes k^{-1})$$

$$= (k \otimes k)(\kappa(k^{-1}) \otimes 1)\zeta((\kappa \otimes \kappa)\Delta(a))(k^{-1} \otimes k^{-1})$$

$$= (k \otimes k)(\kappa(k^{-1}) \otimes 1)\Delta(\kappa(a))(k^{-1} \otimes k^{-1})$$

$$= (\kappa(k^{-1}) \otimes k)\Delta(\kappa(a))(k^{-1} \otimes k^{-1})$$

$$= (\kappa(k^{-1}) \otimes k)\Delta(\kappa_k(a))(1 \otimes k^{-1})$$

$$= (\kappa(k^{-1}) \otimes k)\Delta(1)(\kappa_k(a))(1 \otimes k^{-1})$$

But, as $\kappa^{-1} \otimes i$ is multiplicative from $A^o \otimes A$ to $A \otimes A$, structure, and using the fact that $\kappa^2(k) = k$, one has:

$$(\kappa(k^{-1}) \otimes k)\Delta(1) = (\kappa^{-1}(k^{-1}) \otimes k)\Delta(1) = (\kappa^{-1} \otimes i)((\kappa^{-1} \otimes k)(\kappa \otimes i)(\Delta(1))$$

$$= (\kappa^{-1} \otimes i)(\kappa \otimes i)(\Delta(1)) = \Delta(1)$$

Replacing this in the former list of equalities, one obtains:

$$\zeta(\kappa_k \otimes \kappa_k)\Delta_k(a) = \Delta(1)(\kappa_k(a))(1 \otimes k^{-1}) = \Delta_k(\kappa_k(a))$$
Hence $\kappa$ is an antipode. For every $a$ in $A$, one has:

$$
k_k((\kappa_k(a^{*k})^{*k}) = k\kappa(\kappa_k(a^{*k})^{*k})k^{-1} = k\kappa(\kappa_k(a^{*k})^{*k}k^{-1})k^{-1}
= k\kappa(k^{-1}\kappa_k(a^{*k})^{*k})k^{-1}
= k\kappa(k^{-1}\kappa_k(a^{*k})^{*k}k^{-1} = k\kappa(\kappa_k(a^{*k})^{*k})k^{-1}
= k\kappa(\kappa_k(a^{*k})^{*k})(k^{-1})k^{-1} = k\kappa(\kappa_k(a^{*k})^{*k})k^{-1}
= k\kappa(\kappa_k(a^{*k})^{*k})k^{-1} = k\kappa((k^{-1})^{*k})a\kappa(\kappa_k(a^{*k})^{*k})k^{-1} = k(k^{-1}ak)k^{-1}
= a.
$$

Let’s now use the co-unity $\epsilon$ and the fact that $k$ commutes with $A$:

$$(\epsilon_k \otimes i)\Delta_k(a) = (\epsilon \otimes i)(\Delta(a)(k \otimes k^{-1})) = (\epsilon \otimes i)\Delta(ak)k^{-1} = akk^{-1} = a$$

and more obviously:

$$(i \otimes \epsilon_k)\Delta_k(a) = (i \otimes \epsilon)(\Delta(a)(1 \otimes k^{-1}k)) = (i \otimes \epsilon)\Delta(a) = a,$$

hence $\epsilon_k$ is also a co-unity, and for every $a, b$ in $A$, one has:

$$\epsilon_k(xy) = \epsilon(xyk) = \epsilon(xkk^{-1}yk) = (\epsilon \otimes \epsilon)((x \otimes 1)\Delta(1)(k \otimes k^{-1}yk))
= (\epsilon_k \otimes \epsilon_k)((x \otimes 1)\Delta(1)(1 \otimes k^{-1}y))
= (\epsilon_k \otimes \epsilon)(x \otimes 1)\Delta_k(1)(1 \otimes y))$$

Then for every $a$ in $A$, one has:

$$\epsilon_k(a^{*k}a) = \epsilon(a^{*k}ak) = \epsilon(ka^{*k}k^{-1}ak) = \epsilon(ka^{*k}k^{-\frac{1}{2}}k^{-\frac{1}{2}}ak)
= \epsilon((k^{-\frac{1}{2}}ak)(k^{-\frac{1}{2}}ak))
\geq 0.$$

Finally, using lemma 1.4 for any $x$ in $A$, one has:

$$(m(\kappa_k \otimes i) \otimes i)(\Delta_k \otimes i)\Delta_k(x)
= (m(\kappa_k \otimes i) \otimes i)((\Delta \otimes i)\Delta(x)(1 \otimes k^{-1} \otimes k^{-1}))
= (m \otimes i)((k \otimes 1 \otimes 1)(\kappa \otimes i \otimes i)((\Delta \otimes i)\Delta(x))(k^{-1} \otimes k^{-1} \otimes k^{-1}))
= (k \otimes 1)(m \otimes i)((\kappa \otimes i \otimes i)((\Delta \otimes i)\Delta(x))(k^{-1} \otimes 1 \otimes 1))(k^{-1} \otimes k^{-1})$$
but $\kappa \otimes i \otimes i$ is multiplicative from $A \otimes A \otimes A$ to $A_0 \otimes A \otimes A$, so we can write, using lemma \[\ref{lemma1.4}\] that:

$$
(m \otimes i)((\kappa \otimes i \otimes i)((\Delta \otimes i)\Delta(x))(k^{-1} \otimes 1 \otimes 1))
= (m \otimes i)((\kappa \otimes i \otimes i)((\Delta \otimes i)\Delta(x))(\Delta(1) \otimes 1))(k^{-1} \otimes 1 \otimes 1))
= (m \otimes i)((\kappa \otimes i \otimes i)(\Delta(1) \otimes 1)(\kappa \otimes i \otimes i)((\Delta \otimes i)\Delta(x)))
= (m \otimes i)((\kappa \otimes i \otimes i)((\Delta \otimes i)\Delta(x)))
= (m \otimes i)((\kappa \otimes i \otimes i)(\Delta \otimes i)\Delta(x)))
= (1 \otimes x)\Delta(1),
$$

Hence replacing this in the former equality and using the fact that $k$ commutes with $A_s$, one has:

$$
(m(\kappa_k \otimes i) \otimes i)(\Delta_k \otimes i)\Delta_k(x)
= (k \otimes 1)(1 \otimes x)\Delta(1)(k^{-1} \otimes k^{-1})
= (1 \otimes x)\Delta(1)(1 \otimes k^{-1})
= (1 \otimes x)\Delta_k(1)
$$

So $(A_k, \Delta_k, \kappa_k, \varepsilon_k)$ is a $C^*$-quantum groupoid. As for any $x$ in $A$, one has $\Delta_k(x) = \Delta(x)(1 \otimes k^{-1})$, then it’s obvious that the Cartan subalgebras are the same with this new structure. Finally there is a single element of $A_t$ verifying proposition \[\ref{proposition2.3}\] 1), but, as $\Delta_k(1) = \Delta(1)(1 \otimes k^{-1})$, an easy computation gives that $k^{-1}q$ verifies proposition \[\ref{proposition2.3}\] for $\Delta_k$, the theorem follows.

\[\square\]

2.7. Corollary. Every finite dimensional $C^*$-quantum groupoid can be deformed in such a way that the antipode becomes involutive on the Cartan subalgebras.

Proof: It suffices to apply the last theorem for $k = q$.

\[\square\]

2.8. Corollary. Any $C^*$-quantum groupoid, the Cartan subalgebras of which are not abelian, can be deformed in such a way that the antipode becomes non-involutive on the Cartan subalgebras; there is uncountably many non isomorphic $C^*$-quantum groupoids with its underlying algebra structure.

Proof: Let $(A, \Delta, \kappa, \varepsilon)$ be any $C^*$-quantum groupoid, for which $A_t$ is not abelian, then obviously there exits a $k'$ in $A_t$ given by lemma \[\ref{lemma2.5}\] such that $k'^{-1}q$ is not in the center of $A_t$, then by theorem \[\ref{theorem2.6}\] the antipode $\kappa_{k'}$ is not involutive on $A_t$.

As the canonical element associated with $(A_k, \Delta_k, \kappa_k, \varepsilon_k)$ by proposition \[\ref{proposition2.3}\] is $k^{-1}q$, the spectrum of $k^{-1}q$ is an invariant of the isomorphism class of $(A_k, \Delta_k, \kappa_k, \varepsilon_k)$, one can conclude using lemma \[\ref{lemma2.5}\].

\[\square\]
2.9. **Remark.** As the deformation moves the natural involution of $A$, this leads to a natural question: for any $C^*$-quantum groupoid, the Cartan subalgebras of which are not abelian, is there uncountably many non isomorphic $C^*$-quantum groupoids with its underlying $C^*$-algebra structure?

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