On the formal principle for curves on projective surfaces

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Abstract
We prove that the formal completion of a complex projective surface along a rigid smooth curve with trivial normal bundle determines the birational equivalence class of the surface.

1 Introduction
In this paper, we investigate pairs \((X, Y)\) of complex varieties where \(Y\) is a compact subvariety of the complex variety \(X\). We are particularly interested in the analytic classification of such pairs when \(X\) is a smooth projective surface.

Definition 1.1 A pair \((X, Y)\) satisfies the formal principle if for any other pair \((X', Y')\) such that the formal completion \(\mathcal{Y}\) of \(X\) along \(Y\) is formally isomorphic to the formal completion \(\mathcal{Y}'\) of \(X'\) along \(Y'\) then the germ of \(X\) along \(Y\) is biholomorphic to the germ of \(X'\) along \(Y'\).

If \(Y\) is a smooth compact curve on a smooth surface \(X\) with non-zero self-intersection then the pair \((X, Y)\) satisfies the formal principle. Indeed, if \(Y^2 < 0\) then [7, Section 4, Satz 6] implies that \((X, Y)\) satisfies the formal principle. When \(Y^2 > 0\), then the result is implied by [6], see also the discussion in [13, Section 4]. The case of zero self-intersection is in sharp contrast. To the best of our knowledge, the first example of a pair \((X, Y)\) for which the formal principle does not hold is due to V.I. Arnold: it consists of a germ of surface \(X\) containing an elliptic curve of zero self-intersection and non-torsion normal bundle obtained through the suspension of a germ of non-linearizable biholomorphism, see [2]. Even if one restricts to neighbor-
hoods of elliptic curves with trivial normal bundles, the analytic classification differs considerably from the formal classification, see [15, Theorem 5].

There are many more works investigating the formal principle. We invite the reader to consult the recent paper [11] and the surveys in [13] and [8, Section VII.4] to get a view of different directions of research on the subject.

1.1 Projective formal principle

The results just mentioned provide an abundance of pairs for which the formal principle does not hold. They are based on local analytic construction and do not globalize. Indeed, Neeman in [17, Article 1, Theorem 6.12] shows that a smooth elliptic curve \( Y \) with trivial normal bundle on a projective surface \( X \) either is a fiber of a fibration, or \( X \) is birationally equivalent to \( \mathbb{P}(E) \), the projectivization of the unique rank two vector bundle over \( Y \) obtained as a non-trivial extension of the trivial line-bundle by itself, and \( Y \) corresponds to the natural section \( Y \to \mathbb{P}(E) \).

Taking into account this result, it seems natural to consider the following restricted version of the formal principle.

**Definition 1.2** A pair \((X, Y)\) satisfies the projective formal principle if \( X \) is a projective variety and for any other pair \((X', Y')\) such that \( X' \) is projective and the formal completion \( \mathcal{Y} \) of \( X \) along \( Y \) is formally isomorphic to the formal completion \( \mathcal{Y}' \) of \( X' \) along \( Y' \) then the germ of \( X \) along \( Y \) is biholomorphic to the germ of \( X' \) along \( Y' \). Furthermore, we will say that \((X, Y)\) satisfies the birational formal principle when, under the same assumptions as above, there exists a birational map between \( X \) and \( X' \) which sends, biregularly, a neighborhood of \( Y \) to a neighborhood of \( Y' \).

1.2 Smooth curves on projective surfaces

Our first main result says that the projective formal principle holds for smooth curves with trivial normal bundle on projective surfaces.

**Theorem A** Let \((S, C)\) be a pair where \( S \) is a smooth projective surface and \( C \) is a smooth curve contained in \( S \). If the normal bundle of \( C \) in \( S \) is trivial then \((S, C)\) satisfies the projective formal principle. Moreover, if \( C \) is not a fiber of a fibration in \( S \) then \((S, C)\) satisfies the birational formal principle.

When the curve \( C \) is a fiber of a fibration on a smooth surface \( S \) (projective or just a germ of), a result by Hirschowitz [10], see also [13, Theorem 2.2], says that the pair \((S, C)\) satisfies the formal principle. Theorem A adds nothing to this statement. The real content of it is when \( C \) is not a fiber of a fibration. Its proof is built on the existence of natural closed rational 1-forms with polar set equal to \( C \) [5, Theorem B] (a result which uses basic Hodge Theory for compact Kähler manifolds) to guarantee the convergence of any given formal isomorphism, and we use a result by Ueda to extend the (now convergent) isomorphism to birational maps.
1.3 On the Ueda type of hypersurfaces on projective manifolds

Our second main result is a common generalization of [5, Theorem B], [17, Theorem 5.6], and [14, Theorem 2.3]. Although [5, Theorem B] is sufficient to prove Theorem A, the result below seems to be of independent interest, and it might prove to be useful to investigate the projective formal principle in different situations.

**Theorem B** Let $D$ be a connected divisor on a compact Kähler manifold $X$. Assume the existence of a line bundle $\mathcal{L} \in \text{Pic}^{\text{tor}}(X)$ and of a positive integer $k$ such that $\mathcal{O}_X(kD)|D| \simeq \mathcal{L}|D|$. Then the following assertions hold true:

1. If $\mathcal{O}(kD)|kD + D_{\text{red}} \simeq \mathcal{L}|kD + D_{\text{red}}$ then, perhaps after replacing $X$ by a degree two étale covering, there exists a global closed logarithmic 1-form $\omega$ with purely imaginary periods such that

$$\text{Res} \omega|_U = D$$

where $U$ is a sufficiently small neighborhood of $|D|$.

2. If $\mathcal{O}(kD)|kD \simeq \mathcal{L}|kD$ then there exists a global closed meromorphic 1-form $\omega$ with coefficients in $\mathcal{L}^*$, without residues, and with polar divisor equal to $kD + D_{\text{red}}$.

In the statement of Theorem B, $\text{Pic}^{\text{tor}}(X)$ denotes the group of isomorphism classes of line-bundles on $X$ with torsion Chern class. Also, in Item (1) of the statement above, the periods of $\omega$ are the integrals of $\omega$ along closed real curves not intersecting the polar locus of $\omega$. Throughout the paper, starting in the statement above, we will say by abuse of language that a meromorphic form on a projective manifold with coefficients in a line-bundle $\mathcal{L} \in \text{Pic}^{\text{tor}}(X)$ is closed, if it is $\nabla$-closed for the unique flat unitary connection $\nabla$ on $\mathcal{L}$.

2 Ueda theory

In this section, we recall the definitions of Ueda type and Ueda class of smooth divisors with topologically trivial normal bundle. We adopt the point of view presented by Neeman in [17], and try to be coherent with notations used in [5].

2.1 Ueda line bundle

Let $Y$ be a smooth irreducible compact hypersurface on a complex manifold $U$. Assume that the normal bundle $N_Y$ is topologically torsion and that $Y$ and $U$ share the same homotopy type. Let us assume the existence of a flat unitary connection on $N_Y$. This condition is automatically fulfilled if $Y$ is Kähler. The monodromy representation $\rho_Y : \pi_1(Y) \to S^1 \subset \mathbb{C}^*$ of this unitary connection induces a representation of $\pi_1(U)$ into $S^1$ since we are assuming that $Y$ and $U$ have the same homotopy type. We will denote by $\widetilde{N}_Y$ the (flat unitary) line bundle on $U$ determined by the extension of $\rho_Y$ to $\pi_1(U)$. Notice that $\widetilde{N}_Y$ extends $N_Y$ in the sense that $\widetilde{N}_Y|Y = N_Y$.

The line-bundle $\mathcal{O}_U(Y)$ is another extension of $N_Y$ to $U$. The Ueda line bundle is, by definition, the line-bundle $\mathcal{U} = \mathcal{O}_U(Y) \otimes \widetilde{N}_Y^*$. 
2.2 Ueda type and Ueda class

Let \( I \subset \mathcal{O}_U \) be the ideal sheaf defining \( Y \) in \( U \). We will denote the \( k \)-th infinitesimal neighborhood of \( Y \) in \( U \) by \( Y(k) \), i.e.

\[
Y(k) = \text{Spec}(\mathcal{O}_U/I^{k+1}).
\]

The Ueda type of \( Y \) (\( \text{utype}(Y) \)) is equal to \( \infty \) if \( \mathcal{U}(Y(\ell)) \simeq \mathcal{O}_Y(\ell) \) for every non-negative integer \( \ell \), otherwise is the smallest positive integer \( k \) such that \( \mathcal{U}(Y(k)) \not\simeq \mathcal{O}_Y(k) \). In other words, the Ueda type is infinite if and only if the restriction of \( \mathcal{U} \) to \( Y \), the formal completion of \( \mathcal{U} \) along \( Y \), is trivial.

If \( \text{utype}(Y) = k < \infty \) then the Ueda class of \( Y \) is defined as the element in the cohomology group \( H^1(Y, I^k/I^{k+1}) \) mapped to \( \mathcal{U}|_Y(k) \in \text{Pic}(Y(k)) \) through the truncated exponential sequence

\[
H^1(Y, I^k/I^{k+1}) \overset{\exp}{\longrightarrow} \text{Pic}(Y(k)) \overset{\text{restriction}}{\longrightarrow} \text{Pic}(Y(k - 1)).
\]

Indeed, under the assumption that \( \mathcal{U}|_Y(k-1) \in \text{Pic}(Y(k-1)) \) is trivial, Neeman shows that map on the left is injective \([17, \text{Remark 1.7}]\) and one gets a well-defined class in \( H^1(Y, I^k/I^{k+1}) \).

For a more concrete interpretation of the Ueda type and Ueda class in terms of defining equations for \( Y \) and the associated cocycles, we invite the reader to consult Ueda’s original definition in \([20]\) and the discussion carried out in \([5, \text{Section 2.1}]\). Here we point out the following characterization of neighborhoods where \( \text{utype}(Y) = \infty \).

**Lemma 2.1** Notation as above. The Ueda type of \( Y \) is infinite if, and only if, there exists a (formal) closed logarithmic \( 1 \)-form \( \omega \in H^0(Y, \Omega^1_Y(\log Y)) \) with \( \text{Res}(\omega) = Y \) and purely imaginary periods.

**Proof** If \( \mathcal{U} \) is trivial then there exists a covering \( \{U_i\} \) of \( Y \) and (formal) functions \( f_i \in \mathcal{O}_Y(U_i) \) such that \( Y \cap U_i = \{f_i = 0\} \) and \( f_i = \lambda_{ij} \cdot f_j \) over \( U_i \cap U_j \). The sought \( 1 \)-form \( \omega \) is then defined over \( U_i \) as the logarithmic derivative of \( f_i \). Clearly, \( \text{Res}(\omega) = Y \) and it has purely imaginary periods.

Reciprocally, suppose there exists \( \omega \) with purely imaginary periods and with \( \text{Res}(\omega) = Y \). Then, over a simply connected open covering \( \{U_i\} \) of \( Y \), we can set

\[
f_i = \exp\left( \int \omega|_{U_i} \right).
\]

Since the periods of \( \omega \) are purely imaginary, over \( U_i \cap U_j \) the quotient \( f_i/f_j \) is a complex number of modulus one. This shows that \( \mathcal{O}_U(Y)|_Y \) is isomorphic to \( \tilde{N}_Y|_Y \). The triviality of \( \mathcal{U}|_Y \) follows. \( \square \)

2.3 Hypersurfaces of infinite type

By definition, if \( \text{utype}(Y) = \infty \), then the restriction of \( \mathcal{U} \) to the completion of \( U \) along \( Y \) is trivial, i.e. \( \mathcal{U} \) is trivial on a formal neighborhood of \( Y \). The theorem below, due
to Ueda (cf. [20, Theorem 3]), gives sufficient conditions to the triviality of \( U \) on an Euclidean neighborhood of \( Y \) in \( U \). Although Ueda states his result only for curves on surfaces, his proof works as it is to establish the more general result below.

**Theorem 2.2** Let \( Y \) be a smooth compact connected Kähler hypersurface of a complex manifold \( U \) with topologically torsion normal bundle. If \( \text{utype}(Y) = \infty \) and \( N_Y \) is a torsion line-bundle then \( Y \) is a (multiple) fiber of a fibration.

### 2.4 Hypersurfaces of finite type and closed formal differential forms

As above, let \( \mathcal{Y} \) be the formal completion of \( U \) along \( Y \).

**Lemma 2.3** If \( \text{utype}(Y) < \infty \) then every formal holomorphic function on \( \mathcal{Y} \) is constant.

**Proof** Aiming at a contradiction, assume the existence of a non-constant formal function \( f \in H^0(\mathcal{Y}, \mathcal{O}_Y) \). The function \( f \) is necessarily constant along the hypersurface \( Y \). The formal function \( g = f - f(Y) \) provides a (non necessarily reduced) equation for \( Y \). The logarithmic differential of \( g \) is a closed logarithmic 1-form with residue divisor \( Y \) and purely imaginary periods. Lemma 2.1 gives the sought contradiction. \( \square \)

**Lemma 2.4** If \( \text{utype}(Y) < \infty \) then the restriction morphism

\[
\ker \left\{ d : H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}}) \to H^0(\mathcal{Y}, \Omega^2_{\mathcal{Y}}) \right\} \to H^0(Y, \Omega^1_Y)
\]

from closed formal holomorphic 1-forms on \( \mathcal{Y} \) to \( H^0(Y, \Omega^1_Y) \) is injective.

**Proof** A closed formal 1-form is locally the differential of a formal holomorphic function. If \( \omega \neq 0 \) is a closed formal 1-form which vanishes when restricted to \( Y \), i.e. is in the kernel of the morphism above, then its local primitives are locally constant along \( Y \). Thus, if we choose such primitives vanishing along \( Y \) then they patch together to give a unique formal holomorphic function \( f \) vanishing along \( Y \) and such that \( \omega = df \). Lemma 2.3 implies the result. \( \square \)

### 2.5 Meromorphic functions on neighborhoods of curves of finite Ueda type

A surprising phenomenon, discovered by Ueda, is that the complex geometry of neighborhoods of curves of finite Ueda type shares features with the complex geometry of neighborhoods of ample curves, see [20, Corollary of Theorem 1].

**Theorem 2.5** Let \( C \) be a smooth curve on a smooth surface \( U \). If \( C^2 = 0 \) and \( \text{utype}(C) < \infty \), then \( C \) has a fundamental system of strictly pseudoconcave neighborhoods.

Although the field of formal meromorphic functions on the completion of \( U \) along \( C \) is of infinite transcendence degree over the complex numbers ([9, Section 5]), Theorem 2.5 combined with a result by Andreotti [1, Theorem 4] guarantee the oppositive behavior for the field of germs of meromorphic functions on neighborhoods of \( C \).
Corollary 2.6 Let $C$ and $U$ be as in Theorem 2.5. The transcendence degree of the field of germs of (convergent) meromorphic functions on neighborhoods of $C$ is at most two.

Theorem 2.5 has strong consequences when $C$ is a curve in a smooth projective surface. We collect some of them in the statement below.

Corollary 2.7 Let $C$ be a smooth curve on a smooth projective surface $S$. If $C^2 = 0$ and $\text{ute}(C) < \infty$, then the following assertions hold true:

1. $S - C$ is holomorphically convex and after the contraction of finitely many curves it becomes a Stein space;
2. The morphism $\iota_* : H_1(C, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ induced by the inclusion of $\iota : C \to S$ is surjective;
3. Every meromorphic function defined on an Euclidean neighborhood of $C$ extends to a global rational function. More generally, if $E$ is a locally free sheaf of $\mathcal{O}_S$-modules then every meromorphic section of $E$ defined on an Euclidean neighborhood of $C$ extends to a global rational section.
4. If $\omega$ is a closed holomorphic 1-form defined on a neighborhood $C$ then $\omega$ extends to a global holomorphic 1-form defined on $S$.

Proof Theorem 2.5 implies that $S - C$ satisfies the assumptions of [16, Theorem 1], Item 1 follows. Item 2 is the content of [21, Lemma 4] and Item 3 is the content of [21, Lemma 5]. Finally, Item 4 is the content of [21, Theorem 3].

3 Existence of closed rational 1-forms

This section is devoted to the proof of Theorem B from the introduction.

3.1 Algebraic fundamental group

We start things off with a generalization of Item (2) of Corollary 2.7.

Proposition 3.1 Let $D$ be an effective compact and connected divisor on a compact Kähler manifold $X$ with $D^2 = 0$ in $H^4(X, \mathbb{Q})$. Assume that the natural morphism $\iota_* : \pi_1^{\text{alg}}(|D|) \to \pi_1^{\text{alg}}(X)$ induced by the inclusion of $\iota : |D| \to X$ of the support of $D$ into $X$ is not surjective. Then, perhaps after replacing $X$ by a degree two étale covering, there exists a global closed logarithmic 1-form $\omega$ with purely imaginary periods such that

$$\text{Res} \omega|_U = D$$

where $U$ is a sufficiently small neighborhood of $|D|$.
Proof If \( \iota_* \) is not surjective then, by definition, there exists a finite group \( \Gamma \) and a surjective morphism \( \rho : \pi_1(X) \to \Gamma \) such that \( k = [\Gamma : \rho(\iota_*\pi_1(|D|))] > 1 \).

Let \( r : Y \to X \) be the Galois covering determined by the kernel of \( \rho \). It is a finite étale covering of degree equal to the cardinality of \( \Gamma \). Since we are assuming that \( X \) is compact Kähler, the same holds true for \( Y \).

Consider the divisor \( r^*D \). It is a compact divisor on \( Y \). The action of \( \Gamma \) on \( Y \) preserves \( r^*D \) and therefore acts on the set of its connected components. Since the subgroup \( \rho(\iota_*\pi_1(|D|)) \subset \Gamma \) can be identified with the subgroup which acts trivially on the set of connected components of \( r^*D \), it follows that \( r^*D \) has exactly \( k > 1 \) connected components.

Assume first \( k = 2 \). In this case, \( r^*D = D_1 + D_2 \) where \( D_1 \) and \( D_2 \) are connected disjoint divisors with \( D_1^2 = D_2^2 = 0 \). Let \( \Theta \) be a Kähler form on \( Y \), and consider the bilinear form on \( H^{1,1}(Y) \) defined by \( \alpha \cdot \beta = \int_Y \alpha \wedge \beta \wedge \Theta^{\dim X-2} \) for any \( \alpha, \beta \in H^{1,1}(Y) \). Hodge index theorem (see for instance [22, Section 6.3.2]) implies this bilinear form has signature \( (1, h^{1,1}(Y) - 1) \). Since \( D_1^2 = D_1 \cdot D_2 = D_2^2 = 0 \), it follows that a multiple of \( m_1D_1 \) of \( D_1 \) is numerically equivalent to a multiple \( m_2D_2 \) of \( D_2 \). Therefore the line bundle \( \mathcal{O}_Y(m_1D_1 - m_2D_2) \) has trivial Chern class, and as such supports a flat unitary connection. As explained in [18, Proposition 3.2], this connection uniquely determines a closed logarithmic 1-form \( \omega \) with purely imaginary periods and \( \text{Res}(\omega) = m_1D_1 - m_2D_2 \). The sought 1-form is \( \frac{1}{m_1}\omega \).

If \( k \geq 3 \) the existence of a fibration \( f : Y \to C \) mapping the connected components of \( |r^*D| \) to distinct points follows from [19, Theorem 2.1] (although stated only for projective manifolds, the result is also true for compact Kähler manifolds [18, Theorem 2]). The foliation defined by this fibration is unique, and as such, must be preserved by the action of \( \Gamma \) on \( Y \). It follows the existence of a fibration on \( X \) with \( |D| \) equal to the support of one of its fibers. The existence of \( \omega \) follows easily by pulling back a suitable logarithmic 1-form on the basis of the fibration. \( \square \)

Corollary 3.2 Let \( X \) be a compact Kähler manifold and let \( Y \) be a smooth compact hypersurface of \( X \) with numerically trivial normal bundle. If \( \text{utype}(Y) < \infty \) then the natural morphism

\[
\pi^\text{alg}_1(Y) \longrightarrow \pi^\text{alg}_1(X)
\]

is surjective.

Proof Let us prove the contrapositive assertion. For that assume that the morphism in question is not surjective. Proposition 3.1 implies the existence of 1-form \( \omega \) with purely imaginary periods and \( \text{Res}(\omega) = Y \). Lemma 2.1 implies \( \text{utype}(Y) = \infty \) as wanted. \( \square \)

3.2 Hodge theory for unitary flat line-bundles

Let \( X \) be a compact Kähler manifold and \( \mathbb{L} \) be a rank one local system with unitary monodromy. The line-bundle \( \mathcal{L} = \mathbb{L} \otimes \mathcal{O}_X \) comes endowed with a canonical flat
unitary holomorphic connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_X$ characterized by the property that the local system of flat sections of $\nabla$ is equal to $\mathbb{L} \otimes 1 \subset \mathcal{L}$.

Harmonic theory on compact Kähler manifolds adapts to the study of harmonic forms with coefficients on $\mathcal{L}$. In particular, the following consequence of the $\partial \bar{\partial}$-lemma also holds in this more general context, see for instance [4, (3.3)].

**Lemma 3.3** The homomorphisms

$$H^q(\nabla) : H^q(X, \Omega^p_X \otimes \mathcal{L}) \to H^q(X, \Omega^{p+1}_X \otimes \mathcal{L})$$

are zero.

The twisted De Rham complex $(\Omega^\bullet_X \otimes \mathcal{L}, \nabla)$ provides a resolution of $\mathbb{L}$, and therefore one deduces from Lemma 3.3, as in the case of constant coefficients, a Hodge decomposition

$$H^i(X, \mathbb{L}) = \bigoplus_{p+q=i} H^p(X, \Omega^q_X \otimes \mathcal{L}).$$

Furthermore, complex conjugation of harmonic forms yields (sesqui-linear) isomorphisms

$$H^q(X, \Omega^p_X \otimes \mathcal{L}) \to H^p(X, \Omega^q_X \otimes \mathcal{L}^*).$$

We point out also that if $\mathcal{L}$ is a line-bundle with trivial Chern class on a compact Kähler manifold $X$ and we consider the unique flat unitary connection $\nabla$ on it, then any global section $\omega$ of $H^0(X, \Omega^q_X \otimes \mathcal{L})$ is automatically $\nabla$-closed since Stoke’s Theorem implies

$$\int_X \nabla(\omega) \wedge \nabla(\omega) \wedge \Theta^{\text{dim} X - (q+1)} = 0$$

for any Kähler form $\Theta$.

### 3.3 Restriction of cohomology classes

**Proposition 3.4** Let $D$ be an effective connected divisor on a compact Kähler manifold $X$ with $D^2 = 0$ in $H^4(X, \mathbb{Q})$. Let $(\mathcal{L}, \nabla)$ be a line bundle endowed with a flat unitary connection on $X$. If the restriction morphism

$$H^1(X, \mathcal{L}) \to H^1(|D|, \mathcal{L})$$

is not injective then, perhaps after replacing $X$ by a degree two étale covering, there exists a global closed logarithmic 1-form $\omega$ with purely imaginary periods such that

$$\text{Res} \omega|_U = D$$
where $U$ is a sufficiently small neighborhood of $|D|$.

**Proof** Let $\mathbb{L}$ be the local system of flat sections of $\mathcal{L}$. Functoriality of Hodge decomposition implies that the restriction morphism

$$H^1(X, \mathbb{L}) \rightarrow H^1(|D|, \mathbb{L})$$

is also not injective. Let $\alpha$ be a non-zero element in its kernel.

If $\rho : \pi_1(X) \rightarrow \mathbb{C}^*$ is the monodromy representation of the local system $\mathbb{L}$ then any cohomology class $0 \neq \alpha \in H^1(X, \mathbb{L})$ corresponds to a non-abelian representation $\hat{\rho} : \pi_1(X) \rightarrow \text{Aff}(\mathbb{C})$ with linear part given by $\rho$.

If $\alpha$ restricts to zero in $H^1(|D|, \mathbb{L})$ then the composition

$$\pi_1(|D|) \xrightarrow{i_*} \pi_1(X) \xrightarrow{\hat{\rho}} \text{Aff}(\mathbb{C})$$

has abelian image.

Since $\text{Aff}(\mathbb{C})$ is a linear algebraic group and finitely generated subgroups of linear algebraic groups are residually finite (Malcev’s Theorem), the morphism $\hat{\rho}$ factors through the canonical morphism $\pi_1(X) \rightarrow \pi_1^{\text{alg}}(X)$. Consequently, the induced composition $\pi_1^{\text{alg}}(|D|) \rightarrow \pi_1^{\text{alg}}(X) \rightarrow \text{Aff}(\mathbb{C})$ also has abelian image, while the image of the rightmost arrow coincides with the image of $\hat{\rho}$ and, in particular, is non-abelian. Thus the natural morphism

$$\pi_1^{\text{alg}}(|D|) \rightarrow \pi_1^{\text{alg}}(X)$$

is not surjective. We apply Proposition 3.1 to conclude. \hfill \Box

### 3.4 Proof of Theorem B

Since $\mathcal{L}$ is a line bundle with zero first Chern class, there exists a unique unitary flat connection $\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega^1_X$ on $\mathcal{L}$. Let $\mathbb{L}$ be the local system of flat sections of $\mathcal{L}$ with respect to $\nabla$.

Assume $\mathcal{O}(kD)|_{kD} \simeq \mathcal{L}|_{kD}$. In terms of a sufficiently fine open covering $\{U_i\}$ of $X$, this implies the existence of holomorphic functions $f_i \in \mathcal{O}_X(U_i)$ defining $D_i|_{U_i}$, complex numbers $\lambda_{ij}$ of modulus 1, and holomorphic functions $r_{ij} \in \mathcal{O}_X(U_i \cap U_j)$ such that

$$f_i^k = (\lambda_{ij} + f_j^k \cdot r_{ij}) \cdot f_j^k$$

over the non-empty intersections $U_i \cap U_j$. This identity implies that the functions $a_{ij}$ defined by the formula

$$a_{ij} = \frac{1}{f_i^k} - \frac{1}{\lambda_{ij}} \cdot \frac{1}{f_j^k} \quad (3.1)$$

are holomorphic on $U_i \cap U_j$. From its definition, it is clear that the collection $\{a_{ij}\}$ determines an element of $H^1(X, \mathcal{L}^*)$.  

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Lemma 3.3 implies that class of \( \{d a_{ij}\} \) in \( H^1(X, \Omega^1_X \otimes \mathcal{L}^*) \) is zero. Therefore, perhaps after refining the open covering \( \{U_i\} \), we can write over \( U_i \cap U_j \).

\[
d a_{ij} = \alpha_i - \frac{1}{\lambda_{ij}} \cdot \alpha_j
\]

for suitable holomorphic 1-forms \( \alpha_i \in \Omega^1_X(U_i) \).

Therefore the 1-forms

\[
\omega_i = d \left( \frac{1}{f_k} \right) - \alpha_i
\]

satisfy

\[
\omega_i = \frac{1}{\lambda_{ij}} \cdot \omega_j
\]

and hence define a global rational 1-form \( \omega \) with coefficients in \( \mathcal{L}^* \).

It remains to verify that \( \omega \) is closed. For that, let \( \Theta \) be a Kähler form and observe that \( d \omega \) is clearly holomorphic. Stoke’s Theorem implies

\[
\int_X d \omega \wedge \overline{d \omega} \wedge \Theta^{\dim X - 2} = \lim_{\varepsilon \to 0} \int_{\partial T_{\varepsilon}} \omega \wedge \overline{d \omega} \wedge \Theta^{\dim X - 2}
\]

where \( T_{\varepsilon} \) is an \( \varepsilon \)-small tubular neighborhood of \( |D| \). Since the right hand side is clearly equal to zero, it follows that \( \omega \) is closed.

If we further assume that \( O(kD)_{kr} \simeq \mathcal{L}_{kr} \) then the restriction of \( \{a_{ij}\} \) to the support of \( D \) is zero. We have two possibilities: the cohomology class determined by \( \{a_{ij}\} \) in \( H^1(X, \mathcal{L}^*) \) is zero, or not. If it is zero then we can assume that \( \{a_{ij}\} = 0 \). We construct the sought logarithmic 1-form by taking the logarithmic differential of Eq. (3.1). If instead, the class of \( \{a_{ij}\} \) is not zero in \( H^1(X, \mathcal{L}^*) \) then we deduce that the restriction morphism \( H^1(X, \mathcal{L}^*) \to H^1(|D|, \mathcal{L}^*) \) is not injective. We apply Proposition 3.4 to conclude.

\[\square\]

### 3.5 Bound on the Ueda type of hypersurfaces

Let \( Y \) be a smooth hypersurface of a projective manifold \( X \). Define \( \text{Pic}^{\text{tor}}(Y/X) \) as the cokernel of the restriction morphism

\[
\text{Pic}^{\text{tor}}(X) \longrightarrow \text{Pic}^{\text{tor}}(Y).
\]

Theorem B admits the following immediate consequence.

**Corollary 3.5** Let \( Y \) be a smooth hypersurface of a compact Kähler manifold \( X \). Assume \( N_Y \) is numerically trivial and that the order of the normal bundle of \( Y \) in \( \text{Pic}^{\text{tor}}(Y/X) \) is a finite integer \( k \). Then \( \text{utype}(Y) \leq k \) or \( \text{utype}(Y) = \infty \).
Proof Since, by assumption, the order of the normal bundle of $Y$ in $\text{Pic}^\text{tor}(Y/X)$ is equal to $k$, there exists a line-bundle $L \in \text{Pic}^\text{tor}(X)$ such that $N_Y^k \cong L$, or equivalently, $O_X(kY)|_Y \cong L|_Y$.

If $\text{utype}(Y) > k$ then $O_X(kY)|_{(k+1)Y} \cong L|_{(k+1)Y}$ and Item (1) of Theorem B implies $\text{utype}(Y) = \infty$. □

This result generalizes [17, Theorem 5.1].

4 Convergence of formal isomorphisms

This section is devoted to the proof of Theorem A from the Introduction.

4.1 Formal diffeomorphisms preserving closed differential forms

The convergence of formal diffeomorphism will be implied by the following simple application of Artin’s approximation theorem.

Lemma 4.1 Let $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a formal biholomorphism. Suppose the existence of two pairs $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ of convergent exact meromorphic 1-forms on $(\mathbb{C}^2, 0)$ satisfying the following properties.

1. Both $\alpha_1 \wedge \beta_1$ and $\alpha_2 \wedge \beta_2$ are not identically zero.
2. There exist constants $\mu, \nu \in \mathbb{C}^\ast$ such that $\phi^* \alpha_2 = \mu \alpha_1$ and $\phi^* \beta_2 = \nu \beta_1$

Then $\phi$ is the restriction to $(\mathbb{C}^2, 0)$ of a germ of biholomorphism, i.e. $\phi$ is convergent.

Proof Let $a_1, a_2, b_1, b_2$ be meromorphic first integrals of $\alpha_1, \alpha_2, \beta_1, \beta_2$ respectively. By suitably choosing the constants of integration, we may assume that $\phi^* a_2 = \mu a_1$ and $\phi^* b_2 = \nu b_1$. Moreover, if $0$ is not a pole of $a_i$ then we choose $a_1$ and $a_2$ such that $a_1(0) = a_2(0) = 0$. Similarly for $b_i$.

Consider the system of equations

$$
\begin{cases}
\mu \cdot a_1(x_1, y_1) - a_2(x_2, y_2) = 0, \\
\nu \cdot b_1(x_1, y_1) - b_2(x_2, y_2) = 0.
\end{cases}
$$

(4.1)

Our assumptions imply that $(x_2, y_2) = \phi(x_1, y_1)$ is a formal solution for the system (4.1). For every $N \in \mathbb{N}$, Artin’s approximation theorem [3], implies the existence of a convergent solution $(x_2, y_2) = \varphi_N(x_1, y_1)$ such that the Taylor expansion of $\varphi_N$ coincides with the one of $\phi$ up to order $N$. For $N \geq 1$, the application $\varphi_N$ is a diffeomorphism.

Consider the formal biholomorphism $\psi_N = \phi \circ \varphi_N^{-1} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. Note that $\psi_N$, and all its iterates, satisfy

$$
\psi_N^* \alpha_2 = \alpha_2 \quad \text{and} \quad \psi_N^* \beta_2 = \beta_2.
$$

(4.2)
Since $\psi_N$ is tangent to the identity up to order $N$, there exists a formal vector field $v_N$ on $(\mathbb{C}^2, 0)$ vanishing up to order at least $N$, such that $\psi_N = \exp[1](v_N)$, i.e. $\psi_N$ is the flow of $v_N$ at time one.

If $N \geq 2$ then the formal vector field $v_N$ has no linear term and the expansion of its formal flow $(t, x, y) \mapsto (\exp[t](v_N))(x, y)$, can be written as

$$(t, x, y) \mapsto \sum_{i,j} p_{ij}(t)x^iy^j$$

where $p_{ij} \in \mathbb{C}[t]$ are polynomials. Therefore, the validity of Eq. (4.2) for all iterates of $\psi_N$ implies also the validity of Eq. (4.2) for the formal bilomorphisms $(x, y) \mapsto (\exp[t](v))(x, y)$ when $t \in \mathbb{C}$ is arbitrary. Consequently $L_{v_N} \alpha_2 = L_{v_N} \beta_2 = 0$, where $L_v = i_v d + di_v$ is the Lie derivative along $v$. As both $\alpha_2$ and $\beta_2$ are closed 1-forms, we deduce that both $i_{v_N} \alpha_2 = s_N$ and $i_{v_N} \beta_2 = t_N$ belong to $\mathbb{C}$. Let $w_a$ and $w_b$ be meromorphic vector fields such that $\alpha_2(w_a) = 1, \alpha_2(w_b) = 0, \beta_2(w_a) = 0$, and $\beta_2(w_b) = 1$. Since $\alpha_2 \wedge \beta_2 \neq 0$, the vector fields $w_a$ and $w_b$ are uniquely determined by these equations. Notice also that

$$v_N = s_N \cdot w_a + t_N \cdot w_b.$$ 

Since the order of $v_N$ is at least $N$ we deduce that $v_N = 0$ for $N$ sufficiently large. If $N \gg 0$ then $\psi_N$ is the identity and we conclude that $\phi$ coincides with the convergent biholomorphism $\varphi_N$. \qed

4.2 Proof of Theorem A

Let $S$ be a smooth projective surface and let $C \subset S$ be a smooth curve with trivial normal bundle. Let $C'$ and $C''$ be another pair with the same properties. Let $\mathcal{C}$ be the completion of $S$ along $C$, and let $\mathcal{C}'$ be the completion of $S'$ along $C'$. Assume the existence of a formal biholomorphism $\hat{\varphi} : \mathcal{C} \to \mathcal{C}'$.

If $C$ is a fiber of a fibration on $S$ then the same holds for $C'$ and the existence of a germ of biholomorphism between neighborhoods of $C$ and $C'$ follows from the main result of [10].

Assume from now on that $C$ is not a fiber of a fibration. Theorem B implies that utype($C$) = 1 and uclass($C$) $\in H^1(C, \mathcal{I}_C/\pi^C)$ $\cong H^1(C, \mathcal{O}_C)$. Moreover, there exists a cohomology class $a \in H^1(S, \mathcal{O}_S)$ such that uclass($C$) $\in H^1(C, \mathcal{O}_C)$ is given by the restriction of $a$ to $C$. Note that the class $a$ is the class of the cocycle $\{a_{ij}\}$ appearing in the proof of Theorem B in Equation (3.1). Let $\alpha \in H^0(S, \Omega^1_S)$ be a global holomorphic 1-form such that $\overline{\alpha}$ coincides with the cohomology class $a \in H^1(S, \mathcal{O}_S)$. Note that the pull-back of $\alpha$ to $C$ is non-zero. In particular, the foliation defined by $\alpha$ is generically transverse to $C$.

Let also $\omega \in H^0(S, \Omega^1_S(2C))$ be the closed rational 1-form with $(\omega)_{\infty} = 2C$ given by Theorem B. The $\mathbb{C}$-vector space generated by the twisted 1-form $\omega$ is not unique.
The ambiguity, of course, comes from the inclusion

$$H^0(S, \Omega^1_S) \rightarrow H^0(S, \Omega^1_S(2C)).$$

In order to choose canonically a one dimensional subspace $C \cdot \omega \subset H^0(S, \Omega^1_S(2C))$ consider the representation

$$\pi_1(S) \rightarrow \mathbb{C}$$

$$\gamma \mapsto \int \gamma \omega.$$

Although $\omega$ has poles, it has no residues, and the representation above is unambiguously defined. This representation defines an element of $H^1(S, \mathbb{C})$ which is not canonically determined due to the ambiguity $H^0(S, \Omega^1_S)$. However its class in $H^1(S, \mathbb{C})/H^0(S, \Omega^1_S)$ is unique and by construction is given by the cohomology class $a \in H^1(S, \mathcal{O}_S)$. Thus we can choose $C \cdot \omega$ inside the vector space $H^0(S, \Omega^1_S(2C))$ by imposing that the periods of $\omega$ are proportional to the periods of $\bar{\alpha}$, by Lemma 2.4 this condition determines a one-dimensional subspace of $H^0(S, \Omega^1_S(2C))$. Since the foliation defined by $\omega$ leaves the curve $C$ invariant while the foliation defined by $\alpha$ does not, the wedge product of $\alpha$ and $\omega$ does not vanish identically.

If $\alpha'$ and $\omega'$ are the analogue 1-forms on $S'$ then $\hat{\varphi}^* \alpha'$ is proportional to $\alpha$ thanks to Lemma 2.4. Similarly, $\hat{\varphi}^* \omega'$ is proportional to $\omega$. We apply Lemma 4.1 to conclude that $\hat{\varphi}$ is the restriction to $C'$ of a germ of biholomorphism between the germ of $S$ along $C$ and the germ of $S'$ along $C'$. Furthermore, we can apply Item (3) of Corollary 2.7 to guarantee the existence of a rational map $\varphi : S \dashrightarrow S'$ such that $\varphi|_{C'} = \hat{\varphi}$. Finally, $\varphi$ must be birational (i.e. of degree one) since otherwise the pre-image of $C$ would have two distinct irreducible components (all of them supporting divisors of zero self-intersection) contradicting Item (1) of Corollary 2.7.

**Remark 4.2** Theorem A also holds for a smooth curve $C$ with torsion normal bundle of order $k$ in a projective surface $S$ and utype$(C) \geq k$. The proof is essentially the same. If utype$(C) > k$ then Theorem B implies utype$(C) = \infty$. Consequently, $kC$ is a fiber of a fibration and the result follows from [10]. If instead utype$(C) = k$ then the arguments used in the proof of Theorem A can be repeated almost word-by-word: the only difference is that the closed rational 1-form $\omega$ will have in this case poles of order $k + 1$ instead of poles of order 2.

### 4.3 Divisors with trivial normal bundle

Theorem A admits the following version for reduced divisors with trivial normal bundle.

**Theorem 4.3** Let $(S, D)$ be pair where $S$ is a smooth projective surface and $D$ is a reduced divisor with trivial normal bundle, i.e. $\mathcal{O}_S(D)|_D \simeq \mathcal{O}_D$. Then $(S, D)$ satisfies the projective formal principle.
The proof is the same as the proof of Theorem A. The only difference is that we do not have available Theorem 2.5 for reduced divisors (or even for singular curves) and we are not able to conclude the birational formal principle when $D$ does not move in a fibration.

**Problem 4.4** Establish versions of Ueda’s Theorem 2.5 where the smooth curve $C$ is replaced by an arbitrary effective divisor with trivial, torsion, or unitary flat normal bundle.

Partial results toward a positive solution to Problem 4.4 have been obtained by Ueda [21] and Koike [12].

**4.4 Beyond curves with trivial normal bundle**

It is conceivable that variants of the arguments used to prove Theorem A will lead to a more general statement. Probably, the main obstruction to extending the argumentation is our deficient understanding of the following question.

**Question 4.5** Let $Y \subset X$ be a smooth hypersurface on a projective manifold $X$ with numerically trivial normal bundle and utype$(C) < \infty$. If the order of the normal bundle of $Y$ in $\text{Pic}^{0\text{r}}(Y/X)$ is finite, does there exist a foliation or a web on $X$ canonically attached to the pair $(X, Y)$?

A similar question was already raised in [5, Question 4.6].

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**References**

1. Andreotti, A.: Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves. Bull. Soc. Math. France 91, 1–38 (1963)
2. Arnold, V.I.: Bifurcations of invariant manifolds of differential equations, and normal forms of neighborhoods of elliptic curves. Funkcional. Anal. i Priložen. 10(4), 1–12 (1976)
3. Artin, M.: On the solutions of analytic equations. Invent. Math. 5, 277–291 (1968)
4. Beauville, A.: Annulation du $H^1$ pour les fibrés en droites plats, Complex algebraic varieties (Bayreuth: Lecture Notes in Math., vol. 1507, pp. 1–15. Springer, Berlin (1990)
5. Claudon, B., Loray, F., Pereira, J.V., Touzet, F.: Compact leaves of codimension one holomorphic foliations on projective manifolds. Ann. Sci. Éc. Norm. Sup. (4) 51, 1389–1398 (2018)
6. Commichau, M., Grauert, H.: Das formale Prinzip für kompakte komplexe Untermannigfaltigkeiten mit 1-positivem Normalenbündel, Ann. of Math. Stud., vol. 100, pp. 101–126. Princeton University Press, Princeton (1981)
7. Grauert, H.: Über Modifikationen und exceptionelle analytische Mengen. Math. Ann. 146, 331–368 (1962)
8. Grauert, H., Peternell, T., Remmert, R. (eds.): Several complex variables. VII, Encyclopaedia of Mathematical Sciences, vol. 74, Springer-Verlag, Berlin, 1994, Sheaf-theoretical methods in complex analysis, A reprint of Current problems in mathematics. Fundamental directions. Vol. 74 (Russian), Vseross. Inst. Nauchn. i Tekhn. Inform. (VINIITI), Moscow
9. Hironaka, H., Matsumura, H.: Formal functions and formal embeddings. J. Math. Soc. Jpn. 20, 52–82 (1968)
10. Hirschowitz, A.: Sur les plongements du type déformation. Comment. Math. Helv. 54(1), 126–132 (1979)
11. J.-M. Hwang, *An application of Cartan’s equivalence method to Hirschowitz’s conjecture on the formal principle*, arXiv e-prints (2019). arXiv:1903.09490
12. Koike, T.: Ueda theory for compact curves with nodes. Indiana Univ. Math. J. 66(3), 845–876 (2017)
13. Kosarew, S.: On some new results on the formal principle for embeddings, Proceedings of the conference on algebraic geometry (Berlin, 1985), Teubner-Texte Math., vol. 92, pp. 217–227. Teubner, Leipzig (1986)
14. Loray, F., Pereira, J.V., Touzet, F.: Representations of quasi-projective groups, flat connections and transversely projective foliations. J. Éc. Polytech. Math. 3, 263–308 (2016)
15. Loray, F., Thom, O., Touzet, F.: Two dimensional neighborhoods of elliptic curves: formal classification and foliations, version 2: one reference added (2018)
16. Narasimhan, R.: The Levi problem for complex spaces. II. Math. Ann. 146, 195–216 (1962)
17. Neeman, A.: Ueda theory: theorems and problems. Mem. Am. Math. Soc. 81(415), 123 (1989)
18. Pereira, J.V.: Fibrations, divisors and transcendental leaves. J. Algebraic Geom. 15(1), 87–110 (2006). (With an appendix by Laurent Meersseman)
19. Totaro, B.: The topology of smooth divisors and the arithmetic of abelian varieties. Mich. Math. J. 48, 611–624 (2000). (Dedicated to William Fulton on the occasion of his 60th birthday)
20. Ueda, T.: On the neighborhood of a compact complex curve with topologically trivial normal bundle. J. Math. Kyoto Univ. 22(4), 583–607 (1982)
21. Ueda, T.: Neighborhood of a rational curve with a node. Publ. Res. Inst. Math. Sci. 27(4), 681–693 (1991)
22. Voisin, Claire: Théorie de Hodge et géométrie algébrique complexe, Cours Spécialisés [Specialized Courses], vol. 10. Société Mathématique de France, Paris (2002)

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