SECTOR INSTABILITY IN QUASI-VISCOUS DISK ACCRETION

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ABSTRACT

A first-order correction in the $\alpha$-viscosity parameter of Shakura & Sunyaev has been introduced in the standard inviscid and thin accretion disk. A linearized time-dependent perturbative study of the stationary solutions of this “quasi-viscous” disk reveals the development of a secular instability on large spatial scales. This qualitative feature is equally manifest for two different types of perturbative treatment: a standing wave on subsonic scales, as well as a radially propagating wave. The stability of the flow is restored when the viscosity disappears.

Subject headings: accretion, accretion disks — hydrodynamics — instabilities — methods: analytical

1. INTRODUCTION

The compressible, inviscid, and thin disk flow has by now become an established model in accretion studies (Abramowicz & Zurek 1981; Fukue 1987; Chakrabarti 1989; Nakayama & Fukue 1989; Chakrabarti 1990; Kafatos & Yang 1994; Dyn & Kafatos 1995; Pariev 1996; Molteni et al. 1996; Lu et al. 1997; Das 2002; Das et al. 2003; Ray 2003a; Barai et al. 2004; Das 2004; Abraham et al. 2006; Chaudhury et al. 2006; Das et al. 2007; Goswami et al. 2007). This is a particularly expedient and simple physical system to study, especially as regards the rotating flow in the innermost regions of the disk, in the vicinity of a black hole. Steady global solutions of inviscid axisymmetric accretion onto a black hole have been meticulously studied over the years, and at present there exists an extensive body of literature devoted to the subject, with special emphasis on the transonic nature of solutions, the multitransonic character of the flow, the formation of shocks, and the stability of global solutions under time-dependent linearized perturbations.

Having stressed the usefulness of the inviscid model among researchers in accretion astrophysics, it must also be recognized that this model has its limitations. It is easy to understand that while the presence of angular momentum leads to the formation of an accretion disk in the first place, a physical mechanism must also be found for the outward transport of angular momentum, which should then make possible the inward drift of the accreting matter into the potential well of the accretor. Viscosity has been known all along to be just such a physical means to effect infall, although the exact prescription for viscosity in an accretion disk is still a matter of much debate (Frank et al. 2002). What is well appreciated, however, is that the viscous prescription should be compatible with an enhanced outward transport of angular momentum. The very well known $\alpha$ parameterization of Shakura & Sunyaev (1973) is based on this principle.

And so it transpires that on global scales, and especially on the very largest scales of the disk, the inviscid model will encounter difficulties in that without an effective outward transport of angular momentum, the accretion process cannot be sustained globally. To address this adverse issue, what is being introduced in this paper is the “quasi-viscous” disk model. This model involves prescribing a very small first-order viscous correction in the $\alpha$-viscosity parameter of Shakura & Sunyaev (1973) about the zeroth-order inviscid solution. In doing this, a viscous generalization of the inviscid flow can be logically extended to capture the important physical properties of accretion disks on large length scales, without compromising the fundamentally simple and elegant features of the inviscid model. This is the single most appealing aspect of the quasi-viscous disk model vis-à-vis many other standard models of axisymmetric flows that involve viscosity (Shakura & Sunyaev 1973; Liang & Thomson 1980; Pringle 1981; Matsumoto et al. 1984; Muchotrzeb-Czerny 1986; Abramowicz et al. 1988; Narayan & Yi 1994; Chakrabarti 1996a, 1996b, 1996c; Chen et al. 1997; Peitz & Appl 1997; Frank et al. 2002; Afshordi & Paczynski 2003; Umurhan et al. 2006).

While many previous works have also taken up the question of the stability of viscous thin disk accretion (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Livio & Shaviv 1977; Kato 1978; Umurhan & Shaviv 2005), the specific objective of this paper is to study the stability of stationary quasi-viscous inflow solutions under the influence of a time-dependent and linearized radial perturbation. It has already been shown in some earlier works that viscous solutions are stable under a perturbation of this kind (Ray 2003a; Chaudhury et al. 2006). What has been found through this particular work is that with the merest presence of viscosity (i.e., to first order in $\alpha$, which itself is much less than unity) about the stationary inviscid solutions, instabilities develop exponentially on large length scales. This has disturbing implications, because all physically meaningful inflow solutions will have to pass through these length scales, connecting the outer boundary of the flow with the surface of the accretor (or the event horizon, if the accretor is a black hole).

The perturbative treatment has been executed in two separate ways, both as a standing wave and as a high-frequency traveling wave, and in both cases the perturbation displays growth behavior. It need not always be true that standing and traveling waves will simultaneously exhibit the same qualitative properties as far as stability is concerned. Many instances in fluid dynamics bear this out. In the case of binary fluids, standing waves indicate instability, as opposed to traveling waves (Cross & Hohenberg 1993; Bhattacharya & Bhattacharjee 2005), while the whole physical picture is quite the opposite for the fluid dynamical problem of...
the hydraulic jump (Bohr et al. 1993; Singha et al. 2005; Ray & Bhattcharjee 2007). Contrary to all this, the axisymmetric stationary quasi-viscous flow is greatly affected by both a standing wave and by a traveling wave. This provides convincing evidence of its unstable character, and it is very much in consonance with similar conclusions drawn from some earlier studies. For high-frequency radial perturbations, Chen & Taam (1993) have found that inertial acoustic modes are locally unstable, with a greater degree of growth for the outward-traveling modes than for the inward ones. On the other hand, Kato et al. (1988) have revealed a growth in the amplitude of a nonpropagating perturbation at the critical point, which, however, stabilizes in the inviscid regime.

This kind of instability, one that manifests itself only if some dissipative mechanism (viscous dissipation in the case of the quasi-viscous rotational flow) is operative, is called secular instability (Chandrasekhar 1987). It should be very much instructive here to furnish a parallel instance of the destabilizing influence of viscous dissipation in a system undergoing rotation: that of the effect of viscous dissipation in a Maclaurin spheroid (Chandrasekhar 1987). In studying ellipsoidal figures of equilibrium, Chandrasekhar (1987) has discussed that a secular instability develops in a Maclaurin spheroid when the stresses are derived from an ordinary viscosity that is defined in terms of a coefficient of kinematic viscosity (as the parameterization is for an accretion disk) and when the effects arising from viscous dissipation are considered as small perturbations on the inviscid flow, to be taken into account in the first order only. It is exactly in this spirit that the quasi-viscous approximation has been prescribed for the thin accretion disk, although, unlike a Maclaurin spheroid, an astrophysical accretion disk is an open system.

Curiously enough, the geometry of the fluid flow also seems to have a bearing on the issue of stability. The same kind of study that has been done here with viscosity in a rotational flow had also been done earlier for a viscous spherically symmetric accreting system. In that treatment (Ray 2003b), viscosity was found to have a stabilizing influence on the system, causing a viscosity-dependent decay in the amplitude of a linearized standing wave perturbation. This is quite in keeping with the understanding that the respective roles of viscosity are at variance with each other in the two distinctly separate cases of spherically symmetric flows and thin disk flows. While viscosity contributes to the resistance against infall in the former case, in the latter it aids the infall process.

Finally, an important aspect of the time-dependent perturbative analysis that may be emphasized is that although the flow has been considered to be driven by the Newtonian potential, none of the physical conclusions of this work will be qualified in any serious way upon using any of the pseudo-Newtonian potentials (Paczyński & Wiita 1980; Nowak & Wagoner 1991; Artemova et al. 1996), which are regularly invoked in accretion-related literature to describe rotational flows onto a Schwarzschild black hole, even while preserving the Newtonian construct of space and time. This is especially true of the flow on large scales, where all pseudo-Newtonian potentials converge to the Newtonian limit, and therefore the conclusions of this perturbative treatment will have a similar bearing on pseudo-Schwarzschild flows.

2. THE QUASI-VISCOUS AXISYMMETRIC FLOW

For the thin disk, under the condition of hydrostatic equilibrium along the vertical direction (Matsumoto et al. 1984; Frank et al. 2002), two of the relevant flow variables are the drift velocity, $v$, and the surface density, $\Sigma$. In the thin disk approximation, the latter has been defined by vertically integrating the volume density, $\rho$, over the disk thickness, $H(r)$. This gives $\Sigma \propto \rho H$, and in terms of $\Sigma$, the continuity equation is set down as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma v r) = 0. \quad (1)$$

For a flow driven by the Newtonian potential, $V(r) = -GM/r$, assumption of hydrostatic equilibrium in the vertical direction will give the condition $H = r(c_s^2/r)^{1/2}$, in which the local speed of sound, $c_s$, and the local Keplerian velocity, $v_K$, are, respectively, defined through $c_s^2 = \gamma K r^{-1}$ and $v_K = GM/r^2$, where the constants $\gamma$ and $K$ are derived from the application of a polytropic equation of state, $P = K \rho^{\gamma}/r$, in terms of which the speed of sound may be given as $c_s^2 = \gamma P/\rho$. Written explicitly, the disk height is, therefore, expressed as

$$H = \left(\frac{\gamma K}{GM}\right)^{1/2} \rho^{(\gamma - 1)/2} r^{3/2}, \quad (2)$$

and, using this result, the continuity equation could then be recast as

$$\frac{\partial}{\partial t} \left(\rho^{(\gamma + 1)/2} \right) + \frac{1}{r^{5/2}} \frac{\partial}{\partial r} \left(\rho^{(\gamma + 1)/2} v r^{5/2} \right) = 0. \quad (3)$$

The condition for the balance of specific angular momentum in the flow is given by (Frank et al. 2002)

$$\frac{\partial}{\partial t} (\Sigma r^2 \Omega) + \frac{1}{r} \frac{\partial}{\partial r} [\Sigma v r^2 \Omega] = \frac{1}{2 \pi r^2} \frac{\partial G}{\partial r}, \quad (4)$$

where $\Omega$ is the local angular velocity of the flow, while the torque is given as

$$G = 2\pi r \nu \Sigma r^2 \frac{\partial \Omega}{\partial r}, \quad (5)$$

where $\nu$ is the kinematic viscosity associated with the flow. With the use of the continuity equation, as equation (1) gives it, and going by the Shakura & Sunyaev (1973) prescription for the kinematic viscosity, $\nu = \alpha c_s H$, it is easy to reduce equation (4) to the form (Frank et al. 2002; Narayan & Yi 1994)

$$\frac{1}{v} \frac{\partial}{\partial t} (r^2 \Omega) + \frac{\partial}{\partial r} (r^2 \Omega) = \frac{1}{\nu r H} \frac{\partial}{\partial r} \left(\frac{\alpha \rho H c_s^2 r}{\Omega_K} \frac{\partial \Omega}{\partial r}\right), \quad (6)$$

where $\Omega_K$ is defined from $v_K = r \Omega_K$.

Going back to equation (3), we define a new variable, $f = \rho^{(\gamma + 1)/2} v r^{5/2}$, whose steadiness, as it is very easy to see from equation (3), can be closely identified with the constant matter flux rate. In terms of this new variable, equation (3) can be modified as

$$\frac{\partial}{\partial t} \left(\rho^{(\gamma + 1)/2} \right) + \frac{1}{r^{5/2}} \frac{\partial f}{\partial r} = 0, \quad (7)$$

while equation (6) can be rendered as

$$\frac{1}{v} \frac{\partial}{\partial t} (r^2 \Omega) + \frac{\partial}{\partial r} (r^2 \Omega) = \frac{\gamma K}{GM} \frac{\partial}{\partial r} \left(\frac{f^2 \Omega_K \partial \Omega}{\rho v^2} \right). \quad (8)$$

The inviscid disk model is given by the requirement that $r^2 \Omega = \lambda$, in which $\lambda$ is the constant specific angular momentum.
The quasi-viscous disk that is being proposed here will introduce a first-order correction in terms involving $\alpha$, the Shakura & Sunyaev (1973) viscosity parameter, about the constant angular momentum solution. Mathematically this is represented by the prescription of an effective specific angular momentum,

$$\lambda_{\text{eff}}(r) = r^2 \Omega = \lambda + \alpha r^2 \tilde{\Omega}, \quad (9)$$

where the form of $\tilde{\Omega}$ is determined from equation (8), under the stipulation that the dimensionless $\alpha$-viscosity parameter is much smaller than unity. This smallness of the quasi-viscous correction induces only very small changes on the constant angular momentum background, and therefore, neglecting all orders of $\alpha$ higher than the first and ignoring any explicit time variation of the viscous correction term, where the latter is a standard method adopted also for Keplerian flows (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Pringle 1981; Frank et al. 2002), the dependence of $\tilde{\Omega}$ on $v$ and $\rho$ is obtained as

$$\tilde{\Omega} = -\frac{2\lambda}{r^2} \frac{\gamma K}{GM} \left( \frac{f^2 \Omega_K}{\rho^2 v^3 r^3} + \int \frac{f^2 \Omega_K}{\rho^2 v^3 r^3} \frac{1}{\partial \rho} \partial r \right). \quad (10)$$

The effect of the absence and the presence of a small viscous correction to the inviscid background flow is schematically shown in Figures 1 and 2, respectively.

The stationary solution of equation (7) can be easily obtained as a first integral, and with the help of equation (2), it can be set down as

$$2 \pi \left( \frac{\gamma K}{GM} \right)^{1/2} \rho^{(\gamma + 1)/2} v r^{5/2} = -\dot{m}, \quad (11)$$

where $\dot{m}$ is the conserved matter inflow rate. The minus sign arises because for inflows, $v$ is negative. Further, under station-ary conditions, equation (9) can be written in a modified form as

$$\lambda_{\text{eff}}(r) = \lambda - 2\alpha \lambda \frac{c_s^2}{\rho v_K}, \quad (12)$$

From equation (11), with $\rho$ approaching a constant asymptotic value on large length scales, the drift velocity, $v$, can be seen to go asymptotically as $r^{-3/2}$. If we bear in mind that for inflows, $v < 0$, the asymptotic dependence of the effective angular momentum can be shown to be

$$\lambda_{\text{eff}}(r) \sim \lambda + 2\alpha \lambda \left( \frac{r_s}{r} \right)^3, \quad (13)$$

where $r_s$ is a scale of length that, to an order of magnitude, is given by $r_s^3 \sim GM\dot{m}[c_s^2(\infty)\rho(\infty)]^{-1}$. This asymptotic behavior is entirely to be expected, because the physical role of viscosity is to transport angular momentum to large length scales of the accretion disk.

Finally, the equation for radial momentum balance in the flow will also have to be modified under the condition of quasi-viscous dissipation. This has to be done according to the scheme outlined in equation (9) by which the centrifugal term, $\lambda_{\text{eff}}^2(r)/r^3$, of the radial momentum balance equation must be corrected up to first order in $\alpha$. This finally leads to the result

$$\frac{\partial v}{\partial t} + \frac{v}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} + V'(r) - \frac{\lambda^2}{r^3} - 2\alpha \frac{\lambda}{r^3} \frac{c_s^2}{\rho v_K} = 0, \quad (14)$$

where $\tilde{\Omega}$ is given by equation (10) and $P$ is expressed as a function of $\rho$ with the help of a polytropic equation of state, as has been mentioned earlier. The steady solution of equation (14) is given as

$$v \frac{dv}{d\rho} + \frac{dP}{\rho} + V'(r) - \frac{\lambda^2}{r^3} + 4\alpha \frac{\lambda^2}{r^3} \frac{c_s^2}{\rho v_K} = 0, \quad (15)$$
whose first integral cannot be obtained analytically because of the $\alpha$-dependent term. In the inviscid limit, however, the integral is easily obtained. This case will be governed by conserved conditions, and its solutions have been well documented in accretion literature (Chakrabarti 1989; Das 2002; Das et al. 2003). They will either be open solutions passing through saddle points or closed paths about center-type points. The slightest presence of viscous dissipation, however, will radically alter the nature of solutions seen in the inviscid limit, and it may be easily understood that solutions forming closed paths about center-type points will, under conditions of small viscous correction, be of the spiraling kind (Liang & Thomson 1980; Matsumoto et al. 1984; Afshordi & Paczyński 2003). This state of affairs is appreciated very easily seen in the inviscid limit, and it may be easily understood that so-
dissipation, however, will radically alter the nature of solutions

Immediately, after having understood the qualitative nature of the station-
ary flows in the quas-viscous accretion disk, as given by equa-
tions (11), (12), and (15), it will now become possible for us to carry out a real-time linear stability analysis about the stationary solutions with the help of equations (3) and (14).

3. AN EQUATION FOR A TIME-DEPENDENT PERTURBATION ON STATIONARY SOLUTIONS

About the stationary solutions of the flow variables, $v$ and $p$, a time-depen-
dent perturbation is introduced according to the scheme $v(r, t) = v_0(r) + \delta v(r, t)$, $p(r, t) = p_0(r) + \delta p(r, t)$, and $f(r, t) = f_0(r) + \delta f(r, t)$, in all of which the subscript “0” implies stationary values, where $f_0$ especially, as can be seen from equation (3), is a constant. This constant, as is immediately evi-
dent from a look at equation (11), is very much connected to the mat-
ner flow rate, and therefore the perturbation $f'$ is to be seen as a distur-
bance on the steady, constant background accretion rate. For spherically symmetric flows, this Eulerian perturbation scheme has been applied by Pettersen et al. (1980) and Theuns & David (1992), while for inviscid axisymmetric flows, the same method has been used equally effectively by Ray (2003a, 2003b) and Chaudhury et al. (2006).

The definition of $f$ leads to a linearized dependence among $f'$, $v'$, and $\rho'$ as

$$f' = \frac{\gamma + 1}{\gamma - 1} \frac{\rho'}{\rho_0} + \frac{v'}{v_0},$$

while from equation (3), an exclusive dependence of $\rho'$ on $f'$ is obtained as

$$\frac{\partial \rho'}{\partial t} + \beta^2 \frac{v_0 p_0}{f_0} \frac{\partial f'}{\partial r} = 0,$$  

where $\beta^2 = 2(\gamma + 1)^{-1}$. Combining equations (16) and (17) renders the velocity fluctuations as

$$\frac{\partial v'}{\partial t} = \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} + \frac{v_0}{f_0} \frac{\partial f'}{\partial r} \right),$$

which, upon a further partial differentiation with respect to time, gives

$$\frac{\partial^2 v'}{\partial r^2} = \frac{v_0}{f_0} \left[ \frac{\partial^2 f'}{\partial t^2} + \frac{v_0}{f_0} \frac{\partial f'}{\partial t} \right].$$  

From equation (14), the linearized fluctuating part can be extracted as

$$\frac{\partial v'}{\partial t} + \frac{\partial}{\partial r} \left( v_0 v' + c_s^2 \frac{\rho'}{\rho_0} \right) + 4 \alpha \lambda^2 \frac{\sigma}{r^2} \left[ 2 \frac{f'}{f_0} - 2 \frac{\rho'}{\rho_0} - 3 \frac{v'}{v_0} + \frac{1}{\sigma} \int \frac{\sigma}{\partial r} \left( \frac{f'}{f_0} \right) \, dr \right] = 0,$$

where $\sigma = c_s^2 / (v_0 v_K)$ and $c_s$ is the local speed of sound in the steady state. Taking a partial derivative of equation (20) with respect to $t$ and substituting equations (17), (18), and (19) for all the first- and second-order derivatives of $v'$ and $\rho'$ delivers the result

$$\frac{\partial^2 f'}{\partial r^2} + \frac{2}{\partial r} \left( v_0 \frac{\partial f'}{\partial r} \right) - \frac{1}{\partial t} \left( v_0 \frac{\partial f'}{\partial t} \right) = 0,$$

entirely in terms of $f'$. This is the equation of motion for a per-
turbation imposed on the constant mass flux rate, $f_0$, and it is im-
portant to note here that the choice of a driving potential, Newtonian
or pseudo-Newtonian, has no explicit bearing on the form of the equation.

4. LINEAR STABILITY ANALYSIS OF STATIONARY SOLUTIONS

With a linearized equation of motion for the perturbation having been derived, a solution of the form $f'(r, t) = g(t) \exp (-i \omega t)$ is applied to it. From equation (21), this will give

$$\omega \frac{g_{\omega}}{\omega} + 2 \omega \frac{d}{dr} \left( v_0 g_{\omega} \right) - \frac{1}{\partial t} \left( v_0 \left( \frac{\partial g_{\omega}}{\partial r} - \beta^2 c_s^2 \frac{\partial g_{\omega}}{\partial r} \right) \right) = 0,$$

$$+ 4 \alpha \lambda^2 \frac{\sigma}{v_0 r^2} \left[ -i \omega g_{\omega} + \frac{3 \gamma - 1}{\gamma + 1} \frac{v_0}{f_0} \frac{dg_{\omega}}{dr} + \frac{i \omega}{\sigma} \int \frac{\sigma}{\partial r} \left( \frac{g_\omega}{dr} \right) \, dr \right] = 0.$$

The perturbation may now be treated as a standing wave spatially confined between two suitably chosen boundaries and also as a radially propagating high-frequency wave. These two distinct cases will be taken up separately in what follows, to see how stationary solutions are affected by the perturbation.

4.1. Standing Waves

It can be easily appreciated that with viscous dissipation present in the accretion system, multiply valued solutions in the phase portrait are a distinct possibility (Liang & Thomson 1980; Abramowicz & Kato 1989; Afshordi & Paczyński 2003). This, however, is physically not feasible for a fluid flow, and there-
fore, for all solutions that are multiply valued about a critical point, it should be necessary to have the inner branch of a solution fitted to its outer branch via a shock, which is standard practice in general fluid dynamical studies (Bohr et al. 1993). The outer branch of this discontinuous solution will connect the shock with the outer boundary of the disk itself, and in the phase portrait of the flow, there will exist an entire family of these outer solutions that will arguably be subsonic. It would be worthwhile at this point to remember that for this kind of a thin disk system, the local speed of acoustic propagation would go as $\beta c_s$, and the bulk
flow velocity, $v_0$, of subsonic solutions would be less than this value. Over the entire subsonic range, at two chosen points, the perturbation can be spatially constrained by requiring it to be a standing wave, which dies out at the two chosen boundaries. The outer point could suitably be chosen to be at the outer boundary of the flow itself, where, by virtue of the boundary condition on the steady flow, the perturbation would naturally decay away. The inner boundary of the standing wave is chosen to be infinitesimally close to the shock front, through which the flow will be discontinuous, and the perturbation will die out in its neighborhood. Between these two points, for solutions that are entirely subsonic, it will necessarily multiply equation (22) by $v_0 g_\omega$ and then carry out an integration by parts. The requirement that all integrated “surface terms” vanish at the two boundaries of the standing wave will give a quadratic dispersion relation in $\omega$, which reads as

$$A\omega^2 + B\omega + C = 0,$$  \hspace{1cm} (23)

in which the coefficients of each successive term are given by

$A = \int v_0 g_\omega^2 dr$, \hspace{1cm} $B = -4i\alpha^2 \int \left( \frac{g_\omega}{r^3} \right) \left( \frac{dg_\omega}{dr} \right) dr,$ \hspace{1cm} $C = \int v_0 (v_0^2 - \beta^2 c_0^2) \left( \frac{dg_\omega}{dr} \right)^2 dr$

$+ 2\alpha^2 (3\gamma - 1) \int \frac{\beta^2 \sigma g_\omega v_0}{r^3} \frac{dg_\omega}{dr} dr.$

It is possible now to find a solution for $\omega$, but of greater immediate significance is the fact that this solution will yield an $\alpha$-dependent real part of the temporal component of the perturbation, which goes as

$$\text{Re}(-i\omega) = 2\alpha \left[ \int v_0 g_\omega^2 \xi(r) dr \right]^{-1} \sim \alpha \xi(r),$$  \hspace{1cm} (24)

where $\xi(r)$ itself is expressed as

$$\xi(r) = \frac{\lambda^2}{v_0 g_\omega r^3} \int g_\omega \frac{d\sigma}{dr} dr \sim \frac{\lambda^2 c_0^2}{v_0 r_k r^3} = \frac{\lambda^2}{\mathcal{M}^2 r_k r^3},$$  \hspace{1cm} (25)

in which the Mach number $\mathcal{M} = v_0/c_0$. It is globally true that $v_k \sim r^{-1/2}$. In this situation, under the assumption that the Mach number has a power-law dependence on the radial distance, $\mathcal{M}^2 \sim r^{-2}$, it is possible to write

$$\frac{d(\ln \xi)}{d(\ln r)} = \frac{5}{2}.$$  \hspace{1cm} (26)

On large length scales, with $c_0$ approaching a constant ambient value and with $v_0 \sim r^{-5/2}$ from the continuity condition, it is easily deduced that $\xi(r) \sim r^{5/2}$. This indicates that on large length scales, the amplitude of the spatially constrained standing wave perturbation grows in time; i.e., the subsonic solutions on which this perturbation has been imposed display unstable behavior. It is easy to see that when $\alpha$ vanishes, all stationary solutions restricted by the condition $v_0 < \beta c_0$ are stable under a standing wave perturbation, which displays a purely oscillatory behavior with no growth in amplitude. For the inviscid disk (with $\alpha = 0$), this has been shown by Ray (2003a). Regarding this point, it will be instructive to mention again that Kato et al. (1988) have pointed to the existence of a nonpropagating growing perturbation localized at the critical point, but, interestingly enough, they have also found that this instability disappears in inviscid transonic flows.

4.2. Radially Propagating Waves

The perturbation is now made to behave in the manner of a radially traveling wave whose wavelength is suitably constrained to be small; i.e., it is to be smaller than any characteristic length scale in the system. A perturbative treatment of this nature has been carried out before on spherically symmetric flows (Peterson et al. 1980) and on axisymmetric flows (Ray 2003a; Chaudhury et al. 2006). In both these cases the radius of the accretor was chosen as the characteristic length scale in question and the wavelength of the perturbation was required to be much smaller than this length scale. In this study of an axisymmetric flow driven by a Newtonian potential, the radius of the accreting star, $r_*$, could be a choice for such a length scale. As a result, the frequency, $\omega$, of the waves should be large.

An algebraic rearrangement of terms in equation (22) leads to an integro-differential equation of the form

$$\mathcal{P} \frac{d^2 g_\omega}{dr^2} + Q \frac{dg_\omega}{dr} - R g_\omega + T \int g_\omega \frac{d\sigma}{dr} dr = 0,$$  \hspace{1cm} (27)

whose coefficients are given by

$$\mathcal{P} = v_0^2 - \beta^2 c_0^2,$$

$$Q = 3v_0 \frac{d v_0}{dr} - \frac{1}{v_0} \frac{d}{dr} \left( v_0^2 \beta^2 c_0^2 \right) - 2i\omega v_0 - 2\alpha^2 (3\gamma - 1) \frac{\beta^2 \sigma}{r^3},$$

$$R = 2i\omega \frac{d v_0}{dr} + \omega^2,$$ \hspace{1cm} $T = 4i\omega^2 \lambda^2 / v_0^2 r^3.$

At this stage, bearing in mind the constraint that $\omega$ is large, the spatial part of the perturbation, $g_\omega(r)$, is prescribed as $g_\omega(r) = \exp(s)$, where the function $s$ itself is represented as a power series of the form

$$s(r) = \sum_{n=-1}^{\infty} \frac{k_n(r)}{\omega^n}. $$  \hspace{1cm} (28)

The integral term in equation (27), can, through some suitable algebraic substitutions, be recast as

$$\int g_\omega \frac{d\sigma}{dr} dr = \int \exp(s) \frac{d\sigma}{ds} ds = g_\omega S,$$

where $S$ itself is given by another power series as

$$S = \sum_{m=1}^{\infty} (-1)^{m+1} m^m \sigma^m. $$  \hspace{1cm} (29)

Following this, all the terms in equation (27) can be expanded with the help of the power series for $g_\omega(r)$. Under the assumption (whose self-consistency will be justified soon) that to a leading order

$$S \sim \frac{d\sigma}{ds} \sim \frac{d\sigma}{dr} \left( \frac{\omega}{A} \frac{dA}{dr} \right)^{-1},$$

the three successive highest order terms (in decreasing order) involving $\omega$ are obtained as $\omega^2$, $\omega$, and $\omega^6$. The coefficients of each of these terms are to be collected first and then individually summed up. This is to be followed by setting each of these sums
separately to zero, which will yield for \(\omega^2, \omega,\) and \(\omega^0,\) respectively, the conditions

\[
(v_0^2 - \beta^2\sigma_0^2) \left( \frac{d^2 s_{k-1}}{dr^2} \right) - 2iv_0 \frac{ds_{k-1}}{dr} - 1 = 0,
\]

(30)

\[
(v_0^2 - \beta^2\sigma_0^2) \left( \frac{d^2 k_{-1}}{dr^2} + 2 \frac{dk_{-1}}{dr} \right) + \left[ 3\sigma_0 \frac{d\sigma_0}{dr} - \frac{1}{v_0} \frac{d}{dr} \left( v_0\beta^2\sigma_0^2 \right) - 2\alpha \lambda^2 (3\gamma - 1) \right] \frac{dk_{-1}}{dr} - 2iv_0 \frac{dk_{-1}}{dr} = 0.
\]

(31)

\[
(v_0^2 - \beta^2\sigma_0^2) \left[ \frac{d^2 k_0}{dr^2} + 2 \frac{dk_0}{dr} \right] + \left[ 3\sigma_0 \frac{d\sigma_0}{dr} - \frac{1}{v_0} \frac{d}{dr} \left( v_0\beta^2\sigma_0^2 \right) - 2\alpha \lambda^2 (3\gamma - 1) \right] \frac{dk_0}{dr} - 2iv_0 \frac{dk_0}{dr} + 4i\alpha \lambda^2 \frac{1}{v_0\sigma_0} \frac{d\sigma_0}{dr} \frac{dk_0}{dr}^{-1} = 0.
\]

(32)

Out of these, the first two, i.e., equations (30) and (31), deliver the solutions

\[
k_{-1} = \int \frac{i}{v_0 \pm \beta \sigma_0} \, dr,
\]

(33)

\[
k_0 = -\frac{1}{2} \ln(v_0\beta\sigma_0) \pm \alpha \lambda^2 (3\gamma - 1) \int \beta \sigma_0 \frac{v_0 \pm \beta \sigma_0}{v_0\sigma_0 R^3 (v_0^2 - \beta^2\sigma_0^2)} \, dr,
\]

(34)

respectively.

The two foregoing expressions give the leading terms in the power series of \(g_s(r).\) While dwelling on this matter, it will also be necessary to show that all successive terms of \(s(r)\) will self-consistently follow the condition \(\omega^{-n}[k_n(r)] \gg \omega^{-\alpha(n+1)}[k_{n-1}(r)];\) i.e., the power series given by \(g_s(r)\) will converge very quickly with increasing \(n.\) In the inviscid limit, this requirement can be shown to be very definitely true, considering the behavior of the first three terms in \(k_n(r)\) from equations (33), (34), and (32). These terms can be shown to go asymptotically as \(k_{-1} \sim r, k_0 \sim r,\) and \(k_1 \sim r^{-1}\) given the condition that \(v_0 \sim r^{-5/2}\) on large length scales, while \(\sigma_0\) approaches its constant ambient value. With the inclusion of viscosity as a physical effect, it can be seen from equations (33) and (34), respectively, that while \(k_{-1}\) remains unaffected, \(k_0\) acquires an \(\alpha\)-dependent term that goes asymptotically as \(r.\) This in itself is an indication of the extent to which viscosity might alter the inviscid conditions. However, since \(\alpha\) has been chosen to be very much less than unity, and since the wavelength of the traveling waves is also very small, implying that \(\omega \gg (v_0 \pm \beta\sigma_0)r,\) the self-consistency requirement still holds. Therefore, as far as gaining a quantitative understanding of the effect of viscosity is concerned, it should be quite sufficient to truncate the power series expansion of \(s(r)\) after considering the two leading terms only. With the help of these two terms, an expression for the perturbation may be set down as

\[
f'(r, t) \approx \frac{A_t}{\sqrt{\beta v_0\sigma_0}} \exp \left[ \pm \alpha \lambda^2 (3\gamma - 1) \right]
\]

\[\times \int \beta \sigma_0 \frac{v_0 \pm \beta \sigma_0}{v_0\sigma_0 R^3 (v_0^2 - \beta^2\sigma_0^2)} \, dr \exp \left( \int \frac{i\omega}{v_0 \pm \beta \sigma_0} \, dr \right) e^{-i\omega t},
\]

(35)

which should be seen as a linear superposition of two waves with arbitrary constants \(A_+\) and \(A_-\). Both of these two waves move with a velocity \(\beta\sigma_0\) relative to the fluid, one against the bulk flow and the other along with it, while the bulk flow itself has a velocity \(v_0.\) It should be immediately evident that all questions pertaining to the growth or decay in the amplitude of the perturbation will be crucially decided by the real terms delivered from \(k_0.\) The viscosity-dependent term is especially crucial in this regard. For the choice of the lower sign in the real part of \(f'\) in equation (35), i.e., for the outgoing mode of the traveling wave solution, it can be seen that the presence of viscosity causes the amplitude of the perturbation to diverge exponentially on large length scales, where \(\sigma_0 \approx c_f(\infty)\) and \(v_0 \sim r^{-5/2},\) where \(-t_0\) is positive for inflows. The inwardly traveling mode also displays similar behavior, albeit to a quantitatively lesser degree. It is an easy exercise to see that stability in the system would be restored for the limit of \(\alpha = 0,\) and this particular issue has been discussed by Ray (2003a) and Chaudhury et al. (2006). The exponential growth behavior of the amplitude of the perturbation, therefore, is exclusively linked to the presence of viscosity. Going back to the work of Chen & Taam (1993), it can be seen that the inertial acoustic modes of short-wavelength radial perturbations are locally unstable throughout the disk, with the outward-traveling modes growing faster than the inward-traveling modes in most regions of the disk, all of which is very much in keeping with what equation (35) indicates here.

With the help of equation (17), it should be easy to express the density fluctuations in terms of \(f'\) as

\[
\frac{\rho'}{\rho_0} = \beta \frac{v_0}{\sigma_0} \frac{dx}{dt} \frac{f'}{f_0},
\]

(36)

and likewise, the velocity fluctuations may be set down from equation (16) as

\[
\frac{v'}{v_0} = \left( 1 - \frac{v_0}{\sigma_0} \frac{dx}{dt} \right) \frac{f'}{f_0}.
\]

(37)

In a unit volume of the fluid, the kinetic energy content is

\[
E_{\text{kin}} = \frac{1}{2} (\rho_0 + \rho') (v_0 + v')^2,
\]

(38)

while the potential energy per unit volume of the fluid is the sum of the gravitational energy, the rotational energy, and the internal energy. For a quasi-viscous disk, to first order in \(\alpha,\) this sum is given by

\[
E_{\text{pot}} = (\rho_0 + \rho') \left[ V(r) + \frac{2\sigma_0}{2r^2} \right] + \rho_0 \epsilon
\]

\[+ \rho' \frac{\partial}{\partial \rho_0} (\rho_0 \epsilon) + \frac{1}{2} \rho^2 \frac{\partial^2}{\partial \rho_0^2} (\rho_0 \epsilon),
\]

(39)

where \(\epsilon\) is the internal energy per unit mass (Landau & Lifshitz 1987). In equation (39) the effective angular momentum for the quasi-viscous disk will have to be set up as a first-order correction about the inviscid conditions. Following this, a time-dependent perturbation has to be imposed about the stationary values of \(\epsilon\) and \(\rho.\) All first-order terms involving time dependence in equations (38) and (39) will vanish after time-averaging. In this situation the
leading contribution to the total energy in the perturbation comes from the second-order terms, which are all summed as

\[
E_{\text{pert}} = \frac{1}{2} \rho_0 v^2 + v_0 \rho_0 v' + \frac{1}{2} \rho_0^2 \frac{\partial^2}{\partial r^2} (\rho_0 \epsilon) - 2 \alpha \lambda^2 \frac{\rho_0}{r^2} \left[ \left( \frac{\rho'}{\rho_0} \right)^2 + \left( \frac{v'}{v_0} \right)^2 + 6 \left( \frac{v'}{v_0} \right)^2 \right] - 2 \alpha \lambda^2 \frac{\rho_0}{r^2} + 3 \frac{v' \rho_0}{v_0} - 6 \frac{v' \rho_0}{v_0} + \frac{1}{\sigma \rho_0} \int \sigma \frac{d}{dr} \left( \frac{v'}{v_0} \right) dr + \frac{1}{\sigma} \int \left[ \frac{v'}{v_0} - 2 \frac{\rho'}{\rho_0} - 3 \frac{v'}{v_0} \right] \frac{d}{dr} \left( \frac{v'}{v_0} \right) dr \right].
\]

In the preceding expression, all terms involving \( \rho' \) and \( v' \) can be written in terms of \( f' \) with the help of equations (36) and (37), in both of which, to a leading order, \( s \simeq \omega k \sqrt{r} \). This is to be followed by a time-averaging over \( f'^2 \), which contributes a factor of \( 1/2 \). The total energy flux in the perturbation is obtained by multiplying \( E_{\text{pert}} \) by the propagation velocity \( (v_0 \pm \beta c_{\rho_0}) \) and then integrating over the area of the cylindrical face of the accretion disk, which is \( 2\pi r H \). Under the thin disk approximation, \( H \ll r \), this will make it possible to derive an estimate for the energy flux as

\[
F(r) \simeq \frac{\pi \beta^2 A_2}{f_0} \sqrt{\frac{\gamma K}{GM}} \left[ \pm 1 + \frac{1 - \beta^2 (2 - \mu)}{2 \beta (M + \beta)} \right] \times \left( 1 - \frac{2 \alpha \lambda^2 \psi}{\beta^2 M^2 r_0 v_0 K r^2} \right) \times \exp \left[ \pm 2 \alpha \lambda^2 (3 - 1) \int \frac{\beta c_{\rho_0} (v_0 \pm \beta c_{\rho_0})}{v_0 + \beta c_{\rho_0}} \frac{d}{dr} \right],
\]

in which \( M \) is the Mach number, as defined earlier, and

\[
\mu = \frac{\rho_0}{c_{\rho_0}} \frac{\partial^2 (\rho_0 \epsilon)}{\partial r^2},
\]

\[
\psi = 2 \beta \left[ \left( \beta^2 - 1 \right)^2 M^2 \pm M \beta (\beta^2 - 4) + \beta^2 \right] \times \left( 1 \pm 2 \beta M + \beta^2 \mu \right)^{-1}.
\]

When \( M \to 0 \) on large length scales, \( \psi \) converges to a finite value. However, on these same length scales, what will not converge are the two terms involving \( \alpha \) in equation (41). Under the asymptotic conditions on \( v_0 \) and \( c_{\rho_0} \) discussed earlier, one term will diverge exponentially as \( r \), while another will have a power-law growth behavior of \( r^\delta \). The quasi-viscous disk will, therefore, be most pronouncedly unstable on large scales under the passage of a linearized radially propagating high-frequency perturbation. It is easy to check that under inviscid conditions, with \( \alpha = 0 \), and for an adiabatic perturbation with \( \mu = 1 \), the disk will immediately revert to stable behavior (Ray 2003a).

5. CONCLUDING REMARKS

The instability of the quasi-viscous disk raises some doubts, the primary one of which is on the possibility of a long time evolution of the disk toward a stationary end. Since the quasi-viscous disk is a dissipative system, i.e., one in which its energy is allowed to drain away, there cannot be any occasion to look for a particular solution, and a selection criterion thereof, on the basis of energy minimization, as can be done for an idealized inviscid flow, such as the Bondi solution in spherical symmetry (Bondi 1952; Garlick 1979; Ray & Bhattacharjee 2002). This, of course, will also affect the flow rate, which, in this treatment, has been perturbed and has been found to be unstable. As a result of all this, the whole system has been left without any well-defined criterion by which it could guide itself toward a steady state, transonic or otherwise.

One very important physical role of viscosity in an accretion disk is that it determines the distribution of matter in the disk. The manner in which viscosity redistributes an annulus of matter in a Keplerian flow around an accretor is very well known, with the inner region of this disk system drifting in because of dissipation, and consequently, through the conservation of angular momentum and its outward transport, making it necessary for the outer regions of the matter distribution to spread even further outward (Pringle 1981; Frank et al. 2002). This state of affairs is qualitatively not altered in any way for the quasi-viscous flow, except for the fact that since viscosity is very weak in this case, the outward transport of angular momentum can be conspicuous only on very large scales. It may rightly be conjectured that the instability that develops on the large subsonic scales of a quasi-viscous disk is intimately connected with the cumulative transfer of angular momentum on these very length scales. The accumulation of angular momentum in this region may create an abrupt centrifugal barrier against any further smooth inflow of matter. This adverse effect, on the other hand, could disappear if there should be some other means of transporting angular momentum from the inner regions of the disk. Astrophysical jets could readily afford such a means, insofar as jets actually cause a physical drift of angular momentum vertically away from the plane of the disk, instead of along it.

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