Duality in stochastic processes from the viewpoint of basis expansions

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Received 4 February 2019
Accepted for publication 22 April 2019
Published 10 June 2019

Abstract. A new derivation method of duality relations in stochastic processes is proposed. The current focus is on the duality between stochastic differential equations and birth–death processes. Although previous derivation methods have been based on the viewpoint of time-evolution operators, the current derivation is based on basis expansions. In addition, only the tool needed for the derivation is the integration by parts, which is rather simple and understandable. The viewpoint of basis expansions enables us to obtain various dual stochastic processes. As a demonstration, dual processes based on Taylor-type and Hermite polynomials are derived.

Keywords: stochastic processes, chemical kinetics, stochastic particle dynamics
1. Introduction

Duality is a widely used concept in various research areas; for example, the Fourier transformation is an example of duality concepts, which connects an original space and a frequency domain. In stochastic processes, the duality concept has been used in order to investigate interacting particle systems (for example, see [1]). For example, a stochastic differential equation, which has a continuous-state space, is connected to a birth–death process with a discrete-state space. Sometimes statistics for stochastic differential equations is evaluated from the corresponding tractable birth–death processes, so that the duality concept has been much investigated in various contexts such as population genetics [2–4], non-equilibrium heat-conduction problems [5, 6], and simple exclusion processes [7–9].

Recently, some mathematical discussions for the derivation of dual functions and dual processes have been done; in [10], recent developments have been reviewed. Although there are discussions focusing on duality functions [11], the derivations have been mainly performed by using mathematical properties of time-evolution operators (generators); the symmetry of the generators has been used to derive dual functions [12]. In [13, 14], the discussion based on the second-quantization method (the Doi-Peliti formalism) for the time-evolution operators has also been given.
In the present paper, a new viewpoint of basis expansions is proposed in order to derive dual stochastic processes from stochastic differential equations. By focusing on the basis, it is possible to view the duality concept more flexibly. That is, we can use various types of basis expansions, and each expansion has its own dual stochastic process; consequently we can easily derive various types of dual stochastic processes. In the present paper, as an example, we focus on a simple stochastic differential equation, which is called as the stochastic logistic Ito equation in [15]. Using the stochastic logistic Ito equation, we demonstrate derivation of three different dual stochastic processes; one is the conventional dual process; one is based on the Taylor-type expansion and it gives a slightly different process from the conventional one; the final one is based on the Hermite polynomials and a completely different dual process is derived. Here, we have a comment for the usage of the orthogonal polynomials. As shown later, various orthogonal polynomials could be used to derive dual processes. Although there are a few works about the connections between the duality and orthogonality relations [21, 22], the proposition in the present paper is different from these works; the usage of the orthogonal polynomials as the basis expansions is the main theme here.

The structure of the present paper is as follows. In section 2, the brief review of the conventional duality concept is given. Section 3 focuses on the mathematical structure of duality concept; in order to understand the duality, it is enough to use the integration by parts and basis expansions. In section 4, the derivations of dual stochastic processes based on the basis expansions are demonstrated using the concrete example.

2. Brief summary of duality concept

2.1. Definition of the duality

In the present paper, we focus on the duality relation between stochastic differential equations and birth–death processes. The stochastic differential equation has a continuous-state and continuous-time; the state vector at time \( t \) is given by \( x_t \in \mathbb{R}^{D_1} \), where \( D_1 \) is the dimension of the state vector. The birth–death process, whose state vector at time \( t \) is \( n_t \in \mathbb{N}^{D_2} \), has a discrete-state and continuous-time. Note that these two processes do not need to have the same dimensions.

The process \( (x_t) \) is said to be dual to \( (n_t) \) with respect to a duality function \( D : \mathbb{R}^{D_1} \times \mathbb{N}^{D_2} \rightarrow \mathbb{R} \) if for all \( (x_t), (n_t) \) and \( t \geq 0 \) we have

\[
\mathbb{E}_{n_t}[D(x_0, n_t)] = \mathbb{E}_{x_t}[D(x_t, n_0)],
\]

where \( \mathbb{E}_{x_t} \) and \( \mathbb{E}_{n_t} \) are the expectations in the processes \( (x_t) \) starting from \( x_0 \) and \( (n_t) \) starting from \( n_0 \), respectively.

2.2. Problem settings

Here, we focus on the stochastic logistic Ito equation in [15] as an example. As denoted in [15], the stochastic logistic Ito equation is related to the stochastic Fisher and Kolmogorov–Petrovsky–Piscounov (sFKPP) equation, which plays an important role in the study of the front-propagation problems [16–19] and QCD context [20].
spatial part in the sFKPP equation is neglected in the stochastic logistic Ito equation, and the dual stochastic process has also been known. Here, we employ the notation in [14].

The target stochastic differential equation is expressed as the following Ito-type stochastic differential equation
\[
dx = -\gamma x(1-x)dt + \sigma \sqrt{x(1-x)}dW(t)
\]
for \(0 \leq x \leq 1\), where \(\gamma\) and \(\sigma\) are parameters, and \(W(t)\) expresses a Wiener process. Note that a variable transformation \(u = 1 - x\) recovers the previous discussions in [15].

The corresponding partial differential equation (the Fokker–Planck equation) is as follows:
\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} [\gamma x(1-x)p(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2 x(1-x)p(x, t)],
\]
where \(p(x, t)\) is the probability density function at time \(t\).

2.3. Conventional duality

There are some discussions for the derivation of the dual stochastic process; for example, see [12]. If focusing on the stochastic differential equations, the so-called Doi-Peliti method [23–25] achieves a systematic derivation [14]; the Doi-Peliti method is based on the second-quantization in quantum mechanics, and creation and annihilation operators play important roles in the derivation. Note that the details are not written here because the new derivation method proposed in the present paper does not need the knowledge of the Doi-Peliti method. We give only the final consequences of the discussion; finally, for the system in (2), we obtain the following dual stochastic process denoted as a birth-coagulation process for particles \(A\):

Reaction1 : \(A \rightarrow A + A\),
Reaction2 : \(A + A \rightarrow A\),

i.e.
\[
n \rightarrow n + 1 \quad \text{at rate } \gamma n,
n \rightarrow n - 1 \quad \text{at rate } \frac{\sigma^2 n(n-1)}{2},
\]
where \(n\) is the number of particles \(A\). The master equation for the birth-coagulation process is written as follows:
\[
\frac{d}{dt} P(n, t) = \gamma (n-1)P(n-1, t) - \gamma nP(n, t) + \frac{\sigma^2}{2} n(n+1)P(n+1, t) - \frac{\sigma^2}{2} (n-1)nP(n, t),
\]
where \(P(n, t)\) is the probability distribution for the state with \(n\) particles at time \(t\).

Through the dual function \(D(x, n) = x^n\), the original stochastic differential equation in (2) is connected with the birth-coagulation process in (6) as follows:
where the initial condition for the birth-coagulation process should be
\[ P(m, t = 0) = 1 \quad \text{and} \quad P(n, t = 0) = 0 \quad \text{for} \quad n \neq m, \]
and the initial condition for the stochastic differential equation should be
\[ p(x, t = 0) = \delta(x - x_0), \]
where \( x_0 \) is the initial position, and \( \delta(x) \) is the Dirac delta function. That is, once we solve the dual stochastic process in (6), we can immediately obtain the \( m \)th moment in the stochastic differential equation for arbitrary initial conditions; it is not necessary to perform simulations with different initial conditions for the stochastic differential equation. This property has been exploited in statistical physics [12], and there is also a numerical application of this property in nonlinear Kalman filtering [26].

In [14], further discussions for slightly different stochastic differential equations were given; some extensions of the duality concept are needed in order to recover probabilistic property and to deal with negative transition rates; for details, see [14]. Although these extensions have been done via some techniques for the time-evolution operators written in the creation and annihilation operators, in the following discussions in the present paper, simple ways for the same extensions will be shown.

### 3. Rewriting the duality concept from the viewpoint of basis expansions

This section gives one of the main contributions of the present paper. From the viewpoint of basis expansions, it is straightforward to understand the derivation of the duality relations between stochastic differential equations and birth–death processes. For readability, we here restrict our discussion to one variable cases. It is straightforward to extend the discussion to multivariate cases.

Here, as denoted above, we interest in the \( m \)th moment of the stochastic differential equation. Suppose that \( p(x, t) \) is the probability density distribution for the stochastic differential equation and the time-evolution for the corresponding Fokker–Planck equation is given as a time evolution operator \( \mathcal{L} \), a formal solution of the Fokker–Planck equation is written as
\[ p(x, t) = e^{\mathcal{L}t} p(x, t = 0) = e^{\mathcal{L}t} \delta(x - x_0), \]
where we suppose that the initial position of the stochastic differential equation is \( x = x_0 \). Hence, the calculation of the \( m \)th moment is rewritten as follows:
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\[ E[x^m] = \int_{-\infty}^{\infty} x^m p(x,t) \, dx \]
\[ = \int_{-\infty}^{\infty} x^m (e^{Lt} \delta(x - x_0)) \, dx \]
\[ = \int_{-\infty}^{\infty} (e^{Lt} x^m) \delta(x - x_0) \, dx \]
\[ = \int_{-\infty}^{\infty} \tilde{p}(x,t) \delta(x - x_0) \, dx \]
\[ = \tilde{p}(x_0,t), \quad (11) \]

where \( L^\dagger \) is the adjoint operator of \( L \), and \( \tilde{p}(x,t) \) is obtained as a result of the time-evolution using the adjoint operator \( L^\dagger \). Note that \( \tilde{p}(x,t) \) is not a probability density distribution in general.

The adjoint operator is easily derived from the integration by parts; for a pedagogical purpose, we here describe more details. Firstly, consider the following conventional stochastic differential equation:
\[ dx = A(x,t) \, dt + B(x,t) \, dW(t). \quad (12) \]

Then, the time-evolution operator for the corresponding Fokker–Planck equation is given as follows\[ [27]: \]
\[ L = -\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t), \quad (13) \]

where
\[ D^{(1)}(x,t) = A(x,t), \quad D^{(2)}(x,t) = \frac{1}{2} (B(x,t))^2. \quad (14) \]

Here focusing on the first term in (13), we have
\[ -\int_{-\infty}^{\infty} dx \tilde{p}(x,t) \frac{\partial}{\partial x} \left( D^{(1)}(x,t) p(x,t) \right) \]
\[ = - \left[ \tilde{p}(x,t) D^{(1)}(x,t) p(x,t) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx \left( \frac{\partial}{\partial x} \tilde{p}(x,t) \right) D^{(1)}(x,t) p(x,t) \]
\[ = \int_{-\infty}^{\infty} dx \left\{ D^{(1)}(x,t) \left( \frac{\partial}{\partial x} \tilde{p}(x,t) \right) \right\} p(x,t), \quad (15) \]

where we used the integration by parts and the fact that the probability density function \( p(x,t) \) goes to 0 when \( x \to \pm \infty \). After employing the same calculation for the second term in (13), we have the adjoint time-evolution operator \( L^\dagger \) as follows:
\[ L^\dagger = D^{(1)}(x,t) \frac{\partial}{\partial x} + D^{(2)}(x,t) \frac{\partial^2}{\partial x^2}. \quad (16) \]

From (11), it is clear that it is enough to perform a time-evolution with \( L^\dagger \) and the initial condition \( x^n \) instead of the time-evolutions for the original system with various initial conditions \( \delta(x - x_0) \), i.e. the particle starts from the position \( x_0 \); once we obtain \( \tilde{p}(x_0,t) \), the \( m \)th moment of the original stochastic differential equation for various
initial conditions is evaluated. However, note that \( \tilde{p}(x, t) \) does not correspond to a birth–death process; \( x \) is a continuous variable. In order to recover the discrete characteristics of the dual stochastic process, we need a basis expansion as follows:

\[
\tilde{p}(x, t) = \sum_{n=0}^{\infty} \tilde{P}(n, t)\phi_n(x),
\]

where \( \{\phi_n(x)\} \) are basis functions and \( \{\tilde{P}(n, t)\} \) are expansion coefficients. By using an adequate basis \( \{\phi_n(x)\} \), we can obtain the time-evolution equations for the coefficients \( \tilde{P}(n, t) \) from the adjoint time-evolution operator \( \mathcal{L}^\dagger \).

Of course, in general, \( \tilde{P}(n, t) \) is not a probability distribution, and some techniques are needed to interpret the time-evolution equations for \( \tilde{P}(n, t) \) as a dual stochastic process (birth–death processes) with discrete characteristics. We will demonstrate the techniques for the interpretation in the next section.

4. Demonstration and some techniques to recover probabilistic properties

4.1. Restatement of the problem

For readers’ convenience, here a concise statement of the problem is denoted again.

The main aim is to evaluate the \( m \)th moment \( \mathbb{E}[x^m] \) of the stochastic differential equation in (2) at time \( t \). In order to perform the evaluation, it is enough to solve the following partial differential equation:

\[
\frac{\partial}{\partial t} \tilde{p}(x, t) = -\gamma x(1-x) \frac{\partial}{\partial x} \tilde{p}(x, t) + \frac{\sigma^2}{2} x(1-x) \frac{\partial^2}{\partial x^2} \tilde{p}(x, t),
\]

where the initial condition for \( \tilde{p}(x, t) \) should be

\[
\tilde{p}(x, t = 0) = x^m.
\]

Solving (18), the \( m \)th moment with an arbitrary initial condition \( x = x_0 \) is immediately given as \( \tilde{p}(x_0, t) \).

4.2. Conventional duality

As reviewed in section 2, the conventional viewpoint of the time-evolution operator gives the dual stochastic process (the birth–death process or the birth-coagulation process) in (5). We here show that a simple power-expansion recovers the dual stochastic process immediately.

Employing the basis function

\[
\phi_n(x) = x^n,
\]

we obtain

\[
\tilde{p}(x, t) = \sum_{n=0}^{\infty} P(n, t)x^n.
\]
Inserting \((21)\) into \((18)\), we have
\[
\frac{d}{dt} \sum_{n=0}^{\infty} P(n, t)x^n = -\gamma x(1 - x) \frac{d}{dx} \sum_{n=0}^{\infty} P(n, t)x^n + \frac{\sigma^2}{2} x(1 - x) \frac{d^2}{dx^2} \sum_{n=0}^{\infty} P(n, t)x^n
\]
\[
= \gamma \sum_{n=0}^{\infty} (n - 1)P(n - 1, t)x^n - \gamma \sum_{n=0}^{\infty} nP(n, t)x^n
\]
\[
+ \frac{\sigma^2}{2} \sum_{n=0}^{\infty} (n + 1)nP(n + 1, t)x^n - \frac{\sigma^2}{2} \sum_{n=0}^{\infty} n(n - 1)P(n, t)x^n,
\]
and hence, by comparing the coefficients with the same degree in \(x^n\), we recover the time-evolution equation in \((6)\) for \(\{P(n, t)\}\).

4.3. Taylor-type basis functions

Here, the following Taylor-type basis expansion is employed:
\[
\phi_n(x) = \frac{x^n}{n!},
\]
so that,
\[
\tilde{p}(x, t) = \sum_{n=0}^{\infty} P_T(n, t) \frac{x^n}{n!},
\]
where \(\{P_T(n, t)\}\) are coefficients for the Taylor-type case.

Using the same discussion with section 4.2, we have the following time-evolution equation for the coefficients \(\{P_T(n, t)\}\):
\[
\frac{d}{dt} P_T(n, t) = \gamma n(n - 1)P_T(n - 1, t) - \gamma nP_T(n, t)
\]
\[
+ \frac{\sigma^2}{2} nP_T(n + 1, t) - \frac{\sigma^2}{2} (n - 1)P_T(n, t).
\]

Although this equation does not satisfy the probability conservation law, as discussed in [14], it is possible to recover the probabilistic characteristics and we can use Monte Carlo simulations (for example, the Gillespie algorithm [28]) in order to obtain the coefficients \(\{P_T(n, t)\}\). We briefly review the procedures to use the Monte Carlo simulations. The basic principle for the rewriting of the time-evolution equation is as follows:

(i) Separate the terms into the following two parts:

(A) Terms in which the state is changed;

(B) Terms in which the state is not changed.

(ii) Make the part (A) satisfy the probability conservation law by subtracting some terms. For the compensation, the corresponding terms are added to the part (B).

Using \((25)\), we demonstrate the procedures. Note that the first and third terms in the rhs in \((25)\) have \(P_T(n - 1, t)\) and \(P_T(n + 1, t)\), and hence the states are changed.
as $n \to n - 1$ and $n \to n + 1$ respectively. In contrast, the second and fourth terms only have $P_T(n, t)$, which means that these terms do not change the states. Hence, we have

$$
\frac{d}{dt}P_T(n, t) = \left\{ \gamma n(n - 1)P_T(n - 1, t) + \frac{\sigma^2}{2}nP_T(n + 1, t) \right\}
+ \left\{ -\gamma nP_T(n, t) - \frac{\sigma^2}{2}(n - 1)nP_T(n, t) \right\}.
$$

(26)

Focusing on terms in the first curly bracket, we need to subtract two terms in order to satisfy the probability conservation law for this part, and the subtracted terms are added to the second curly bracket;

$$
\frac{d}{dt}P_T(n, t) = \left\{ \gamma n(n - 1)P_T(n - 1, t) - \gamma(n - 1)(n - 2)P_T(n, t) \right\}
+ \left\{ -\sigma^2 nP_T(n + 1, t) + \frac{\sigma^2}{2}(n - 1)P_T(n, t) \right\}
+ \left\{ -\gamma nP_T(n, t) - \frac{\sigma^2}{2}(n - 1)P_T(n, t) \right\}
+ \gamma(n - 1)(n - 2)P_T(n, t) + \frac{\sigma^2}{2}(n - 1)P_T(n, t) \right\}.
$$

(27)

The first term in the rhs in (27) satisfies the probability conservation law, and it corresponds to the following birth–death process:

$$
n \to n + 1 \text{ at rate } \gamma(n - 1)(n - 2),
\quad n \to n - 1 \text{ at rate } \sigma^2(n - 1)/2.
$$

(28)

Of course, the second term in the rhs in (27) should be dealt with adequately; making $N$ sample paths via the Monte Carlo simulations and denoting $i$th path as $\{n_t^{(i)}\}$, we have

$$
P_T(n, t) = \frac{w_{\text{ini}}}{N} \sum_{i=0}^{N} \exp \left\{ \int_0^t \frac{d\tau}{V_T(n_t^{(i)})} \right\} \delta_{n, n_t^{(i)}},
$$

(29)

where

$$
V_T(n) = -\gamma n - \frac{\sigma^2}{2} n(n - 1) + \gamma(n - 1)(n - 2) + \frac{\sigma^2}{2}(n - 1),
$$

(30)

and $\delta_{A,B}$ is the Kronecker delta function; $w_{\text{ini}}$ is determined by the initial condition, which will be explained later. This is because the second term in the rhs in (27) does not change the state $n$; this term plays a role only as a weighting factor. Such splitting of the time-evolution equation has been discussed in [14], and the term is called the Feynman–Kac term [1].

As for the initial condition, $\bar{p}(x, t = 0) = x^m$ should be satisfied, and then the initial particle number should be set as $m$ in the Monte Carlo simulation, and we take

$$
w_{\text{ini}} = m!.
$$

(31)

In short, it is enough to make sample paths using the birth–death process in (28) and to evaluate (29); the coefficients $\{P_T(n, t)\}$ in (24) are calculated numerically via the Monte Carlo simulations.

https://doi.org/10.1088/1742-5468/ab1dd9
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Note that the birth–death process in (28) cannot be interpreted as a chemical reaction system. It has been known that the Doi-Peliti formalism, which has been used to derive the dual process in [14], is deeply related to stochastic processes for chemical reactions [29]; in this sense, the viewpoint of the basis expansion gives a new dual process (24) naturally, and it would be difficult to find the dual process via the Doi-Peliti formalism. In addition, because of the factorial $n!$ in (24), $P_T(n,t)$ becomes considerably large compared with $P(n,t)$ in the conventional dual process (6); although this is different from importance sampling methods, this could be useful to perform the Monte Carlo simulations because $P(n,t)$ takes very small value in general for large $n$, and rare event sampling usually needs some numerical techniques.

4.4. Hermite-type basis functions

As a final example, a dual stochastic process based on Hermite polynomials is derived. That is, the following basis expansion is employed here:

$$\phi_n(x) = H_n(x),$$  \hspace{1cm} (32)

where \{H_n(x)\} is the Hermite polynomials obtained from

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.\hspace{1cm} (33)$$

That is,

$$\tilde{p}(x,t) = \sum_{n=0}^{\infty} P_H(n,t)H_n(x).\hspace{1cm} (34)$$

The Hermite polynomials satisfy the following three-term recurrence relation [30]:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),\hspace{1cm} (35)$$

and hence, we have

$$xH_n(x) = \frac{1}{2} H_{n+1}(x) + nH_{n-1}(x).\hspace{1cm} (36)$$

In addition, the derivative of $H(x)$ w.r.t. $x$ gives [30]

$$\frac{d}{dx}H_n(x) = 2nH_n(x) - H_{n+1}(x).\hspace{1cm} (37)$$

Using (36) and (37), the time-evolution equation for \{P_H(n,t)\} is obtained as

$$\frac{\partial}{\partial t}P_H(n,t) = \frac{1}{2} \gamma (n-1)P_H(n-1,t) + \left( -\gamma n - \frac{\sigma^2}{2} (n-1)n \right) P_H(n,t)$$

$$+ \left( \gamma (2n+1)(n+1) + \sigma^2 n(n+1) \right) P_H(n+1,t)$$

$$+ \left( -2\gamma (n+1)(n+2) - \sigma^2 (2n+1)(n+1)(n+2) \right) P_H(n+2,t)$$

$$+ 2(\gamma + \sigma^2)(n+1)(n+2)(n+3)P_H(n+3,t)$$

$$- 2\sigma^2 (n+1)(n+2)(n+3)(n+4)P_H(n+4,t).\hspace{1cm} (38)$$

Note that the time-evolution equation in (38) does not satisfy the probability conservation law. In addition, it is not enough to employ the technique used in the Taylor
case in order to recover the probabilistic characteristics; for example, the final term in (38) changes the state $n \to n + 4$, but the sign of the term is minus and hence it should correspond to a negative transition rate.

The negative transition rate problem has also appeared in the Doi-Peliti formalism [14], and it is possible to avoid the problem as below.

A new particle $n_0$ is added to the system, and the following rules are employed:

(i) Change the sign of the term related to the state change when we have the negative transition rate. (Note that when the state is not changed, the term can be considered as a weighting (Feynman–Kac) term, and there is no need to change the sign.)

(ii) Change the number of particle $n_0$ (add one particle) for the terms which had the negative transition rate.

In order to understand the above rule, it would be easy to compare the following equation with (38):

\[
\frac{\partial}{\partial t} P_H(n, n_0, t) = \frac{1}{2} \gamma (n - 1) P_H(n - 1, n_0, t) + \left( \gamma n - \frac{\sigma^2}{2} (n - 1) n \right) P_H(n, n_0, t) \]

\[
+ \left( \gamma (2n + 1) (n + 1) + \sigma^2 n (n + 1) \right) P_H(n + 1, n_0, t) \]

\[
+ (2\gamma (n + 1) (n + 2) + \sigma^2 (2n + 1) (n + 1) (n + 2)) P_H(n + 2, n_0 - 1, t) \]

\[
+ 2(\gamma + \sigma^2) (n + 1) (n + 2) (n + 3) P_H(n + 3, n_0, t) \]

\[
+ 2\sigma^2 (n + 1) (n + 2) (n + 3) (n + 4) P_H(n + 4, n_0 - 1, t). \]  \tag{39}

Then, using the same discussion with the Taylor case, we have the following birth-death process after some calculations:

\[
\begin{align*}
n, n_0 &\to n + 1, n_0 \quad \text{at rate } \gamma n/2, \\
n, n_0 &\to n - 1, n_0 \quad \text{at rate } \gamma (2n - 1) n + \sigma^2 (n - 1) n, \\
n, n_0 &\to n - 2, n_0 + 1 \quad \text{at rate } 2\gamma (n - 1) n + \sigma^2 (2n - 3) (n - 1) n, \tag{40} \\
n, n_0 &\to n - 3, n_0 \quad \text{at rate } 2(\gamma + \sigma^2) (n - 2) (n - 1) n, \\
n, n_0 &\to n - 4, n_0 + 1 \quad \text{at rate } 2\sigma^2 (n - 3) (n - 2) (n - 1) n
\end{align*}
\]

and

\[
V_H(n) = -\frac{1}{2} \gamma n - \frac{\sigma^2}{2} n (n - 1) + \gamma (2n - 1) n \\
+ \sigma^2 (n - 1) n + 2\gamma (n - 1) n + \sigma^2 (2n - 3) (n - 1) n \\
+ 2(\gamma + \sigma^2) (n - 2) (n - 1) n + 2\sigma^2 (n - 3) (n - 2) (n - 1) n. \tag{41}
\]

In short, making $N$ sample paths from (40), the coefficients in (34) are obtained by

\[
P_H(n, t) = \frac{1}{N} \sum_{i=0}^{N} w_{ini} \left\{ \int_0^t dt' \, V_H \left( n_{i}^{(i)}(t') \right) \right\} \delta_{n, n_{i}^{(i)}(t)} (-1)^{n_{0,i}^{(i)}}, \tag{42}
\]

where $n_{i}^{(i)}$ and $n_{0,i}^{(i)}$ correspond to the particle number of $n$ and $n_0$ of $i$th path at time $t$, respectively. The initial weights $\{w_{ini}\}$ for $i$th path should be chosen carefully, as denoted later. The factor $(-1)^{n_{0,i}^{(i)}}$ means that the state change, which is caused by the
negative transition rate, gives a contribution of a factor \((-1)\) and hence the virtual particle number \(n_0\) plays a role as the change of the sign; note that \(n_{0,t=0}^{(i)} = 0\) at the initial time. In addition, the initial condition should be chosen so that
\[
\sum_{n=0}^{\infty} P_H(n, t=0) H_n(x) = x^n,
\] (43)
and, as discussed in the Taylor case, the Monte Carlo results should be adequately scaled via \(\{w_{\text{ini}}^{(i)}\}\). For example, when \(m = 1\), we start from 1 particle system in the Monte Carlo simulations, and \(w_{\text{ini}}^{(i)} = 1/2\) for all \(i\) because \(H_1(x) = 2x\). In order to evaluate the second order moment, the initial condition and the weights \(\{w_{\text{ini}}^{(i)}\}\) become a little complicated. Since \(H_0(x) = 1\) and \(H_2(x) = 4x^2 - 2\), we have \(x^2 = \frac{1}{4}H_2(x) + \frac{1}{2}H_0(x)\). Hence, we could employ the following initial settings:

- Choose the initial particle number as \(n_{t=0}^{(i)} = 0\) or \(n_{t=0}^{(i)} = 2\) randomly (i.e. with probability 1/2 respectively).
- If we choose \(n_{t=0}^{(i)} = 0\), set \(w_{\text{ini}}^{(i)} = \frac{1}{2} \times \left(\frac{1}{2}\right)^{-1}\); if \(n_{t=0}^{(i)} = 2\), set \(w_{\text{ini}}^{(i)} = \frac{1}{4} \times \left(\frac{1}{2}\right)^{-1}\). Note that the factor \((\frac{1}{2})^{-1}\) is needed in order to include the probability for the initial choice.

5. Concluding remarks

In the present paper, we give a new viewpoint for the duality between stochastic differential equations and birth–death processes. The viewpoint is based on the basis expansion, which enables us to obtain various dual stochastic processes naturally. As far as we know, the birth–death processes in (28) and (40) have not been derived yet as the dual stochastic processes for (2). Additionally, the derivation is rather simple compared with the previous one [14]; the knowledge of the creation and annihilation operators are not needed, and only the elemental calculus and the integration by parts are enough to obtain the time-evolution operator for the dual process. It was also shown that the negative transition rate problem is adequately avoided using the similar technique in the previous work [14].

In statistical physics, the duality relation has been used for calculating physical quantities such as moments; in some cases, it is possible to obtain analytical solutions for dual birth–death processes, and the analytical solutions enable us to give the time-dependent or stationary solutions for the original processes. Note that the merit of the dual processes is not restricted to the cases with analytically solvable systems. We can see the important point in (1); once the time-evolution of the dual birth–death process is evaluated, the time-evolution of the original stochastic differential equation \textit{with arbitrary initial conditions} is obtained. As explained in section 2.3, this merit is used to construct the nonlinear Kalman filter [26]. In this sense, the numerical solutions for a little complicated dual processes are enough for practical purposes. In addition, we reformulated the time-evolution equation so as to be suitable for the Monte Carlo simulations,
which will avoid the curse of dimensionality for multivariate cases. As shown in the present paper, the non-uniqueness of the dual stochastic process is explicitly clarified. It would be beneficial to find useful basis expansions for each original stochastic process in future. We expect that these extensions will open up a way to seek practical and numerical studies of the usage of the duality concepts in stochastic processes.

Acknowledgments

This work was supported by JST, PRESTO Grant No. JPMJPR18M4, Japan.

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https://doi.org/10.1088/1742-5468/ab1dd9