Twisted Gravitational Waves in the Presence of a Cosmological Constant

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We find exact nonlinear solutions of general relativity (GR) that represent twisted gravitational waves (TGWs) in the presence of a cosmological constant. A TGW is a nonplanar wave propagating along a fixed spatial direction with a null Killing wave vector that has a nonzero twist tensor. The solutions all turn out to have wave fronts with negative Gaussian curvature. Among the classes of solutions presented in this paper, we find a unique class of simple conformally flat TGWs that is due to the presence of a negative cosmological constant. The properties of this special solution are studied in detail.

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I. INTRODUCTION

A twisted gravitational wave is a free nonlinear unidirectional radiative solution of general relativity (GR) such that its null propagation vector $k$ has a nonzero twist tensor. Imagine a gravitational field described by the metric $ds^2 = g_{\mu\nu}(t - z, x, y) dx^\mu dx^\nu$ in $x^\mu = \ldots$

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(t, x, y, z) coordinates that represents a gravitational wave propagating along the z direction. Throughout this paper, we use units such that \( c = G = 1 \); moreover, the signature of the spacetime metric is +2 and greek indices run from 0 to 3, while latin indices run from 1 to 3. Let \( k = \partial_t + \partial_z \) be the null propagation vector of the wave; then,

\[
k_\mu k^\mu = 0, \quad k_\mu ;_\nu + k_{\nu ;\mu} = 0.
\]

(1)

It follows from these relations that \( k_{\mu ;\nu} k^{\nu} = 0 \), so that the spacetime under consideration admits a null geodesic Killing vector field that is nonexpanding and shearfree.

Spacetimes that admit a covariantly constant null vector field \( k \) with \( k_\mu k^\mu = 0 \) and \( k_{\mu ;\nu} = 0 \) represent plane-fronted gravitational waves with parallel rays (pp-waves). These were first discovered in 1925 by Brinkmann \[4\] and have since been the subject of detailed investigations \[5–9\]. As discussed in Ref. \[1\], plane gravitational waves have at least five Killing vector fields and form a subclass of pp-waves. From

\[
k_{\mu ;\nu} = \frac{1}{2} (k_{\mu ;\nu} + k_{\nu ;\mu}) + \frac{1}{2} (k_{\mu ;\nu} - k_{\nu ;\mu}),
\]

(2)

we note that if the twist tensor

\[
T_{\mu \nu} = \frac{1}{2} (k_{\mu ;\nu} - k_{\nu ;\mu}) = k_{[\mu ;\nu]}
\]

(3)
of the gravitational wave under discussion in Eq. (1) vanishes, then our assumptions in Eq. (1) lead to \( k_{\mu ;\nu} = 0 \) and hence we have a pp-wave. On the other hand, if \( T_{\mu \nu} \neq 0 \), we then have twisted waves that are nonplanar; that is, they have nonuniform wave fronts with nonzero Gaussian curvature.

To illustrate these ideas in more detail, let us assume a spacetime metric of the form

\[
ds^2 = -\gamma_0 (dt^2 - dz^2) + \gamma_1 dx^2 + 2\gamma_2 dx dy + \gamma_3 dy^2,
\]

(4)

where \( \gamma_\mu = \gamma_\mu(t - z, x, y) \). The \((t, x, y, z)\) coordinate system is physically admissible \[10\] if \( \gamma_0 > 0, \gamma_1 > 0, \gamma_3 > 0 \) and \( \Delta := \gamma_1 \gamma_3 - \gamma_2^2 > 0 \). In these coordinates, \( k^\mu = (1, 0, 0, 1) \) is the null propagation Killing vector field. It is useful to introduce the retarded and advanced null coordinates \( u = t - z \) and \( v = t + z \), respectively, and write metric (4) as

\[
ds^2 = -\gamma_0 du dv + \gamma_1 dx^2 + 2\gamma_2 dx dy + \gamma_3 dy^2,
\]

(5)
where \(k = 2 \partial_\nu\). The wave front corresponds to hypersurfaces of constant \(u = u_0\), in which case the metric reduces to

\[
d\sigma^2 = ds^2\big|_{u=u_0} = \gamma_1(u_0, x, y) \, dx^2 + 2\gamma_2(u_0, x, y) \, dx \, dy + \gamma_3(u_0, x, y) \, dy^2.
\]

The Gaussian curvature of this surface vanishes for \(pp\)-waves and is nonzero for twisted gravitational waves (TGWs). The formula for the Gaussian curvature is given in Appendix A. For metric (4), the null propagation vector \(k\) is normal to the wave front, namely,

\[
k_\mu = \gamma_0(u, x, y) \, (-1, 0, 0, 1) = -\gamma_0(u, x, y) \frac{\partial u}{\partial x^\mu}.
\]

It follows that in this case, the wave’s twist tensor (3) is given by

\[
T_{\mu\nu} = \frac{1}{2} \left( \frac{\partial \gamma_0}{\partial x^\mu} \frac{\partial u}{\partial x^\nu} - \frac{\partial \gamma_0}{\partial x^\nu} \frac{\partial u}{\partial x^\mu} \right).
\]

If \(\gamma_0\) is only a function of \(u\), or if it is a constant independent of coordinates, then \(T_{\mu\nu} = 0\) and we have a \(pp\)-wave; otherwise, \(T_{\mu\nu} \neq 0\) and we have a TGW. Using Eq. (7), Eq. (8) can be written as

\[
T_{\mu\nu} = \frac{1}{2\gamma_0} \left( \frac{\partial \gamma_0}{\partial x^\nu} k_\mu - \frac{\partial \gamma_0}{\partial x^\mu} k_\nu \right),
\]

where

\[
\frac{\partial \gamma_0}{\partial x^\mu} k_\mu = 0,
\]

since \(2 \partial_\nu \gamma_0 = \gamma_0, t + \gamma_0, z = 0\). Equations (9) and (10) imply that the twist scalar \(\omega\) vanishes in this case

\[
\omega^2 := \frac{1}{2} T_{\mu\nu} T^{\mu\nu} = 0.
\]

This result is in agreement with the theorem that \(\omega = 0\) if and only if the null geodesic congruence is hypersurface-orthogonal.

The TGWs under consideration here belong to the Kundt class of solutions of GR. As demonstrated in Appendix A of Ref. [2], it is possible to write our TGW metric (5) in Kundt’s form. The Kundt solutions have been extensively studied and the solutions presented in this paper are probably known in some form in other coordinate systems.

No reasonable astronomical source of TGWs is known. Gravitational radiation emitted by known astronomical sources are expected to have expanding nearly spherical wave fronts far from the source. Therefore, TGWs have been tentatively interpreted in terms of running cosmological waves. Observations of distant supernovae have led to the discovery of
the accelerating expansion of the universe. The standard cosmological models that take this
acceleration into account involve a positive cosmological constant \( \Lambda \). Thus we might expect
that TGWs should be compatible with the existence of a cosmological constant. However,
in previous work on TGWs [1–3], the cosmological constant was set equal to zero in order
to make the field equations tractable. It is therefore important to look for TGWs in the
presence of a nonzero cosmological constant. This issue will be addressed in the present
paper. We seek TGW solutions of the gravitational field equations in vacuum but with a
cosmological constant \( \Lambda \), namely,

\[
R_{\mu\nu} = \Lambda g_{\mu\nu},
\]

where \( R_{\mu\nu} := R^\rho_{\mu\rho\nu} \) is the Ricci tensor. To render the resulting differential equations
manageable, we assume a solution of the form (5) such that the gravitational potentials are
all functions of the dimensionless variable

\[
w = s u + p x + q y,
\]

where \( s, p \) and \( q \) are in general nonzero constant parameters of dimensions \( 1/\text{length} \). We
must specifically assume that \( s \neq 0 \) throughout; otherwise, the solution is static and cannot
represent a wave. Moreover, it is clear that by a simple coordinate translation we can add
any constant to \( w \); henceforth, such constants will be ignored throughout with no loss in
generality. There was a preliminary indication in previous work that such solutions may
accommodate a cosmological constant [11]. We assume throughout that \( \gamma_0' := d\gamma_0/dw \neq 0 \);
otherwise, \( T_{\mu\nu} = 0 \) and the solution would represent \( pp \)-waves. The gravitational field
equations (12) are worked out explicitly in Appendix B for metric (5) when the metric
coefficients are all functions of \( w \) defined in Eq. (13).

In Section II, we present the general solution of the field equations (12) for metric (4)
where the gravitational potentials are only functions of \( w \); furthermore, we assume that
there is no cross term \( (\gamma_2 = 0) \) and \( q = 0 \) in Eq. (13) for the sake of simplicity. These
latter restrictions are in turn removed in Sections III and IV, respectively. That is, we keep
\( q = 0 \), but extend our results to the case where a cross term is present \( (\gamma_2 \neq 0) \) in Section
III and in Section IV, we return to the setting of Section II with no cross term, but with
\( q \neq 0 \) in Eq. (13). In the presence of both positive and negative \( \Lambda \), we find classes of TGW
solutions that depend on the solution of an ordinary differential equation for \( A(w) := \ln \gamma_0 \).
Moreover, from the results of Sections II and IV, a simple unique conformally flat TGW
solution is found for negative $\Lambda$ such that $g_{\mu\nu} = w^{-2} \eta_{\mu\nu}$ and $p^2 + q^2 = -\Lambda/3$. The physical properties of this solution are investigated in detail in Section V. A discussion of our results is contained in Section VI.

II. TGWS IN THE PRESENCE OF $\Lambda$

Consider a metric of the form

$$ds^2 = -e^{A(w)}(dt^2 - dz^2) + e^{B(w)} dx^2 + e^{C(w)} dy^2,$$

where

$$w = s u + p x.$$  

This spacetime contains three Killing vector fields, namely, $\partial_t + \partial_z$, $p \partial_t - s \partial_z$, and $\partial_y$. The field equations (12) in this case reduce to the following five equations:

$$p^2 A'(A' + 2 C') + 4 \Lambda e^B = 0,$$

where $A' := dA/dw$, etc.,

$$2 A'' + A'^2 = A'(B' + C'),$$

$$2 C'' + C'^2 = A'^2 + B' C',$$

$$2 A'' + 2 C'' + C'^2 = A'(B' + C') + B'C',$$

$$2 A'(B' + C') = B'^2 + C'^2 + 2 B'' + 2 C'',$$

cf. Appendix B. These results can also be obtained from the field equations in the presence of $\Lambda$ given in Appendix A of Ref. [1].

Equation (19) is equivalent to the sum of Eqs. (17) and (18). After dividing both sides of Eq. (17) by $A' \neq 0$, the resulting equation can be simply integrated and we get

$$A' = 2 k_A e^{\frac{1}{2} (B + C - A)},$$

where $k_A \neq 0$ is a dimensionless integration constant. Furthermore, the sum of Eqs. (19) and (20) can be integrated once and the result is

$$B' - A' = k_B e^{-\frac{1}{2} (B + C)};$$
similarly, the difference between Eqs. (17) and Eq. (18) can also be integrated and we find

\[ A' - C' = k_C e^{-\frac{1}{2} (2A - B + C)} \],

(23)

where \(k_B\) and \(k_C\) are dimensionless constants of integration.

Let us now start with Eq. (21) and define a function \(F(w)\),

\[ F := e^{\frac{1}{2} A}, \quad F' = k_A e^{\frac{1}{2} (B+C)} \quad A' = 2 \frac{F'}{F}, \]

(24)

in terms of which \(B'\) and \(C'\) can be written using Eqs. (22) and (23) as

\[ B' = 2 \frac{F''}{F} + \frac{k_A k_B}{F}, \quad C' = 2 \frac{F'^2}{F^2} - \frac{k_C}{F^2} e^{\frac{1}{2} (B-C)}. \]

(25)

We note that for \(A \in (-\infty, \infty)\), \(F \in (0, \infty)\). To calculate \(C'\) in terms of \(F\), we go back to Eq. (17) and find

\[ A' C' = 2 \left( 2 \frac{F''}{F} - 2 \frac{F'^2}{F^2} - \frac{k_A k_B}{F} \right). \]

(26)

Substituting this result in Eq. (16), we get

\[ 2 \frac{F''}{F} - 4 \frac{F'^2}{F^2} - \frac{k_A k_B}{F} + \lambda e^B = 0, \]

(27)

where

\[ \lambda := \frac{\Lambda}{p^2} \]

(28)

is in this case the dimensionless reduced cosmological constant.

Let us now return to Eq. (24) and note that

\[ F'' = \frac{1}{2} k_A (B' + C') e^{\frac{1}{2} (B+C)}. \]

(29)

Next, using Eq. (25) we find

\[ B' + C' = 4 \frac{F'}{F} + \frac{k_A k_B}{F'} - \frac{k_C}{F^2} e^{\frac{1}{2} (B-C)}. \]

(30)

Substituting this relation in Eq. (29) results in

\[ 2 F'' - 4 \frac{F'^2}{F} - k_A k_B = -\frac{k_A k_C}{F^2} e^B, \]

(31)

where Eq. (24) has been employed as well. Assuming that \(k_C \neq 0\), we can find \(\exp(B)\) from Eq. (31) and substitute it in Eq. (27) to find a second order ordinary differential equation for the function \(F\). Indeed, we get

\[ F \left( 2 F'' - \beta \right) (1 + \alpha F^3) - F'^2 (1 + 4 \alpha F^3) = 0, \]

(32)
where
\[ \alpha := -\frac{\lambda}{k_A k_C}, \quad \beta := k_A k_B. \] (33)

The first integral of Eq. (32) can be determined by writing \(2 F'' = d(F'^2)/dF\) and integrating the resulting equation for \(F'^2\). We find
\[ F'^2 = F(1 + \alpha F^3) \left[ k_0 - \frac{1}{3} \beta \ln(\alpha + F^{-3}) \right], \] (34)
where \(k_0\) is a new dimensionless integration constant. In terms of \(F\), the metric functions are
\[ e^A = F^2, \quad e^B = \frac{F'^2}{k_A k_C} \left( \frac{3F}{1 + \alpha F^3} \right), \quad e^C = \frac{k_C}{k_A} \left( \frac{3F}{1 + \alpha F^3} \right)^{-1}. \] (35)

The Gaussian curvature of wave front with constant \(u\) is given in Appendix A. In this particular case, it can be simply obtained from formula (B2) of Appendix B of Ref. [1], namely,
\[ K_G = -\frac{1}{4} e^{-B} [2 C_{xx} - (B_x - C_x)C_2], \] (36)
or, since the gravitational potentials are all functions of \(w\),
\[ K_G = -\frac{p^2}{4} e^{-B} [2 C'' - (B' - C')C']. \] (37)

Using Eq. (18), we find
\[ K_G = -\frac{p^2}{4} A'^2 e^{-B} < 0. \] (38)
Therefore, in general, our TGWs have wave fronts with negative Gaussian curvature.

In the following subsections, we will consider some special parameter values.

A. \( \Lambda = 0 \)

It is a consequence of Eq. (16) that for \( \Lambda = 0 \) we have \( A' = -2C' \), since \( A' \neq 0 \) by assumption. Moreover, \( A = -2C \) plus a constant that can be absorbed in a redefinition of the \(y\) coordinate. It then follows from Eq. (21) that
\[ e^B = \frac{1}{k_A^2} F F'^2. \] (39)
Furthermore, Eq. (23) implies \( k_C = 3k_A \) in this case. The wave front has negative Gaussian curvature given by \( K_G = -k_A^2 p^2/F^3 \), where \( F \) is a solution of the differential equation
\[ 2 F'' - \beta = \frac{F'^2}{F}. \] (40)
This equation and its first integral can be obtained from Eqs. (32) and (34) for $\alpha = 0$, respectively. Let us note that for $k_B = 0$, we have $\beta = 0$ and $\sqrt{F}$ depends linearly on $w$. In this special case, the spacetime metric takes the form

$$ds^2 = w^4(-dt^2 + dz^2 + dx^2) + w^{-2}dy^2,$$  \hspace{1cm} (41)

which is of Petrov type D and essentially coincides with the metric discussed in Refs. \[2\] \[3\].

B. $k_C = 0$

If we assume that $k_C = 0$, the difference between $A$ and $C$ must be a constant that can be absorbed in the redefinition of the $y$ coordinate. Therefore, we set $A = C$. It then follows from Eq. (21) that

$$e^{\frac{A}{2}B} = \frac{1}{k_A} \frac{F'}{F}.$$  \hspace{1cm} (42)

Substituting this relation in Eq. (16), we find

$$\Lambda = 3K_G = -3 k_A^2 p^2 < 0.$$  \hspace{1cm} (43)

The metric coefficients are determined in this case from the differential equation for $F$, namely,

$$2 F'' - \beta = 4 \frac{F'^2}{F},$$  \hspace{1cm} (44)

which also follows from Eq. (32) by writing it as

$$2 F'' - \beta = \frac{F'^2}{F} \frac{1 + 4 \alpha F^3}{1 + \alpha F^3},$$  \hspace{1cm} (45)

and formally letting $\alpha$ go to infinity. The first integral of Eq. (44) is given by

$$F'^2 = \beta_0 F^4 - \frac{1}{3} \beta F,$$  \hspace{1cm} (46)

where $\beta_0$ is an integration constant. Finally, let us mention that if $k_B = 0$ as well, then $\beta = 0$ and via constant rescalings of spacetime coordinates and parameters $(s, p)$, the spacetime metric can be rendered conformally flat; that is,

$$ds^2 = w^{-2} \eta_{\mu \nu} dx^\mu dx^\nu,$$  \hspace{1cm} (47)

where $(\eta_{\mu \nu}) = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor. A simple generalization of this conformally flat TGW is derived at the end of Section IV.
III. ADDITION OF A CROSS TERM

Let us now continue the general approach adopted in Section II with the addition of a cross term and look for TGW solutions with metrics of the form

$$ds^2 = -e^{A(w)}(dt^2 - dz^2) + e^{B(w)} dx^2 + 2 h(w)dx dy + e^{C(w)} dy^2,$$

where the coordinate admissibility condition is in this case

$$f := e^{B+C} - h^2 > 0.$$  \hspace{1cm} (49)

We note that $w = s u + p x$; hence, the simple coordinate transformation $y \mapsto -y$ changes the overall sign of the cross term $h$, but otherwise leaves the metric invariant.

For the explicit determination of the field equations, we introduce the standard null coordinates and write $dt^2 - dz^2 = du dv$ in Eq. (48) and then work out the consequences of $R_{\mu\nu} = \Lambda g_{\mu\nu}$, cf. Appendix B. As before, $R_{uv} = R_{vx} = R_{vy} = 0$ are trivially satisfied by symmetry. We then have four inhomogeneous equations depending on the $\Lambda$ term, namely, $R_{xy} = \Lambda h$, $R_{yy} = \Lambda \exp(C)$, $R_{uv} = -(\Lambda/2) \exp(A)$ and $R_{xx} = \Lambda \exp(B)$, as well as three homogeneous equations $R_{ux} = R_{uy} = R_{uu} = 0$. The results of this section depend crucially on the assumption that

$$h(w) \neq 0.$$  \hspace{1cm} (50)

Let us start with the inhomogeneous field equations. Using Eq. (49) and its derivative, $R_{xy} = \Lambda h$ and $R_{yy} = \Lambda \exp(C)$ lead to the same equation which can be expressed as

$$C' f' - 2 f (C'' + C'^2 + C' A') = 4\lambda f^2 e^{-C},$$

where $\lambda$ is the reduced cosmological constant defined in Eq. (28). In the same way, from $R_{uv} = -(\Lambda/2) \exp(A)$ we get

$$A' f' - 2 f (A'' + A'^2 + C' A') = 4\lambda f^2 e^{-C}.$$  \hspace{1cm} (52)

Finally, $R_{xx} = \Lambda \exp(B)$ implies

$$[C' f' - 2 f (C'' + C'^2 + C' A')] e^C - 2 f^2 \left(2 A'' + A'^2 - \frac{f'}{f} A'\right) e^{-B} = 4\lambda f^2.$$  \hspace{1cm} (53)

If in this equation the part proportional to $\exp(-B)$ vanishes, we recover Eq. (51). Therefore Eqs. (51) and (53) imply

$$2 A'' + A'^2 - \frac{f'}{f} A' = 0$$  \hspace{1cm} (54)
which can be simply integrated. The result is
\[ f = \ell_0 A'^2 e^A, \]  
where \( \ell_0 > 0 \) is a constant of integration. Moreover, subtracting Eq. (51) from Eq. (52) results in
\[ f = \ell_1 (A' - C')^2 e^{2(A+C)} , \]  
where \( \ell_1 > 0 \) is another constant of integration. From Eqs. (55) and (56), we find
\[ \pm \sqrt{\frac{\ell_0}{\ell_1}} A' e^{-3A/2} = (A' - C') e^{-(A-C)} , \]  
which can be simply integrated to yield
\[ e^C = \pm \frac{2}{3} \sqrt{\frac{\ell_0}{\ell_1}} e^{-A/2} + \ell_2 e^A , \]  
where \( \ell_2 \) is an integration constant. Employing Eq. (56) in Eq. (51), we find
\[ \left( \frac{C''(A')}{A'} \right)' \left( \frac{C'}{A'} - 1 \right)^{-3} = 2 \lambda \ell_1 A' e^{2A+C} . \]  
Calculating \( C'/A' \) via Eq. (58) and substituting the result in Eq. (59), we get a formula for the reduced cosmological constant, namely,
\[ \lambda = -\frac{3}{4} \frac{\ell_2}{\ell_0} . \]  

It remains to investigate the homogeneous equations. Let us start with \( R_{uy} = 0 \). As before, employing Eq. (49) and its derivative in \( R_{uy} = 0 \) leads to simplifications that turn this field equation into
\[ 2 f \left[ (C'' + C'^2) h - h'' \right] - f' \left( C' h - h' \right) = 0 , \]  
which can be simply integrated to yield
\[ f = \ell_3 (C' h - h')^2 e^{2C} , \]  
where \( \ell_3 > 0 \) is a constant of integration. From Eqs. (56) and (62), one can derive a differential equation whose solution is
\[ e^A = \sqrt{\frac{\ell_3}{\ell_1}} h + \ell_4 e^C , \]
where we have written \( h \) instead of \( \pm h \) or \( \mp h \), since the overall sign of \( h \) can be changed via the coordinate transformation \( y \mapsto -y \), and \( \ell_4 \) is an integration constant. Next, using Eq. (58) in Eq. (63), we find

\[
h = L_0 e^{-A/2} + L_1 e^A, \quad L_0 := \pm \frac{2}{3} \ell_4 \sqrt{\ell_0 \ell_3}, \quad L_1 := \sqrt{\ell_1 \ell_3 (1 - \ell_2 \ell_4)}.
\]  

(64)

Let us now consider \( R_{ux} = 0 \), which reduces in the same way to

\[
(f + h^2) \left[ 2(C'' + C'^2) - C' \frac{f'}{f} \right] + hh' \frac{f'}{f} - 2 hh'' - A' f' + 2 A'' f = 0.
\]  

(65)

Substituting Eq. (61) in Eq. (65) and using Eq. (54), we find after some algebra

\[
2(C'' + C'^2) - C' \frac{f'}{f} = A'^2.
\]  

(66)

This result is in fact a simple consequence of Eqs. (54) and (58).

The last field equation to consider is then \( R_{uu} = 0 \). In the same manner as before, this field equation reduces to

\[
f'' - \frac{1}{2} f'^2 - A' f' = B' C' e^{B + C} - h'^2.
\]  

(67)

From Eq. (69) and its derivative, we find

\[
B' = \frac{f' + 2 hh'}{f + h^2} - C',
\]  

(68)

so that the field equation under consideration takes the form

\[
f'' - \frac{1}{2} f'^2 - A' f' = C'(f' + 2 hh') - C'^2(f + h^2) - h'^2.
\]  

(69)

It is useful at this point to introduce the function \( F(w) \), given as in Eq. (24) of Section II by \( F = \exp(A/2) \). Then,

\[
f = 4 \ell_0 F'^2, \quad A' = 2 \frac{F'}{F}.
\]  

(70)

Employing Eq. (58) for \( C \) and Eq. (64) for \( h \), field Eq. (69) reduces, after much algebra, to

\[
\frac{F'''}{F'} + W_1(F) \frac{F''}{F} + W_2(F) \frac{F'^2}{F^2} + W_3(F) = 0,
\]  

(71)

where \( W_i, i = 1, 2, 3 \), are given by

\[
W_1 = - \frac{1 + 4 \tilde{\lambda} F^3}{1 + \lambda F^3}, \quad W_2 = \left( \frac{1 - 2 \tilde{\lambda} F^3}{1 + \lambda F^3} \right)^2, \quad W_3 = \frac{9 \ell_1}{8 \ell_0 \ell_3} \left( \frac{F}{1 + \lambda F^3} \right)^2.
\]  

(72)
Here, $\tilde{\lambda}$ is proportional to the cosmological constant and is given by

$$\tilde{\lambda} := \mp 2 \lambda \sqrt{\ell_0 \ell_1}. \quad (73)$$

Let us introduce $\Psi$ given by

$$\Psi := \frac{1}{2} F^{\gamma^2}; \quad (74)$$

then, Eq. (71) can be written as

$$F^2 \frac{d^2 \Psi}{dF^2} + W_1(F) F \frac{d\Psi}{dF} + W_2(F) \Psi + F^2 W_3(F) = 0. \quad (75)$$

In principle, from this linear inhomogeneous second-order ordinary differential equation we can determine $A(w)$ and hence the other metric functions for these TGWs with a cosmological constant. That is, given an appropriate solution of Eq. (75) for $A(w)$, Eqs. (49), (55), (58) and (64) can be used to find the corresponding spacetime metric.

The Gaussian curvature of the wave front for these TGWs can be calculated using the formula given in Appendix A. With $w = s u + p x$, where $u = u_0$ is a constant, we find

$$K_G = \frac{p^2 e^C}{4 f^2} \left[ C' f' - 2 f (C'' + C'^2) \right], \quad (76)$$

which simplifies via Eq. (66) and the result is

$$K_G = - \frac{p^2 A^2 e^C}{4 f} < 0, \quad (77)$$

so that, as before, the Gaussian curvature of the wave front is negative.

Returning to Eq. (75), let us note that this equation simplifies considerably in the absence of a cosmological constant (i.e., $\tilde{\lambda} = 0$). In fact, the general solution of this equation can be expressed as

$$\Psi = (\mu_1 + \mu_2 \ln F) F - \frac{\ell_1}{8 \ell_0 \ell_3} F^4, \quad (78)$$

where $\mu_1$ and $\mu_2$ are integration constants and $F^{\gamma^2} = 2 \Psi$.

**IV. A SIMPLE GENERALIZATION**

We consider a metric of the form

$$ds^2 = -e^{A(w)} du dv + e^{B(w)} dx^2 + e^{C(w)} dy^2, \quad (79)$$
where $x$ and $y$ are now treated on the same footing, namely,

$$w = s u + p x + q y . \quad (80)$$

This spacetime contains three Killing vector fields as well; that is, $\partial_t + \partial_z$, $p \partial_t - s \partial_x$ and $q \partial_t - s \partial_y$. The field equations presented in Appendix B contain three trivial ones that simply vanish by symmetry, namely, $R_{uv} = R_{vx} = R_{uy} = 0$. The others include four homogeneous and three inhomogeneous field equations, the latter involving the cosmological constant. The four homogeneous equations, namely, $R_{xy} = 0$, $R_{uy} = 0$, $R_{ux} = 0$ and $R_{uu} = 0$ can be expressed as

$$2 A'' + A'^2 = A'(B' + C') , \quad (81)$$
$$2 B'' + B'^2 + 2 A'' = A'(B' + C') + B' C' , \quad (82)$$
$$2 C'' + C'^2 + 2 A'' = A'(B' + C') + B' C' \quad (83)$$

and

$$2 B'' + B'^2 + 2 C'' + C'^2 = 2 A'(B' + C') , \quad (84)$$

respectively. Inspection of Eqs. (82) and (83) reveals the symmetry between $B$ and $C$, so that $2 B'' + B'^2 = 2 C'' + C'^2$; then the other equations imply

$$2 A'' + A'^2 = 2 B'' + B'^2 = 2 C'' + C'^2 = A'(B' + C') , \quad 2 A'' = B' C' . \quad (85)$$

Substituting $2 A'' = B' C'$ in Eq. (81), we find

$$\left(A' - B'\right)\left(A' - C'\right) = 0 . \quad (86)$$

Thus either $A' = B'$ or $A' = C'$; however, the symmetry between $B$ and $C$ implies that it is sufficient to consider one of these; therefore, we choose the case $A' = B'$. Moreover, we can henceforth simply set

$$A = B , \quad (87)$$

since the constant of integration can always be absorbed in the redefinition of the advanced null coordinate $v$. With $A = B$, Eq. (81) can now be integrated and we find

$$e^C = \hat{k}^2 A'^2 , \quad (88)$$
where \( \hat{k}_C^2 \) is an integration constant. Using this relation in \( 2C'' + C'^2 = 2A'' + A'^2 \) results in an ordinary differential equation for \( A(w) \),

\[
4A'' - 2A' A'' - A'^3 = 0.
\] (89)

Next, the three inhomogeneous equations, namely, \( R_{uu} = -(\Lambda/2) \exp(A) \), \( R_{xx} = \Lambda \exp(B) \) and \( R_{yy} = \Lambda \exp(C) \), all reduce to the same equation when we employ Eqs. (85) and (87), namely,

\[
(4A'' + A'^2) e^{-A} = \Sigma_0, \quad \Sigma_0 = -\frac{4 \Lambda + 3 q^2 \hat{k}_C^{-2}}{p^2}.
\] (90)

Remarkably, Eq. (90) turns out to be a first integral of Eq. (89) and \( \Sigma_0 \) is simply an integration constant. It is possible to integrate Eq. (90) once and the result is

\[
A'^2 = \frac{1}{3} \Sigma_0 e^A + \Pi_0 e^{-\frac{1}{3}A},
\] (91)

where \( \Pi_0 \) is another integration constant. Equation (91) can be solved by quadrature; that is,

\[
\int e^{\frac{1}{3}A} \frac{d\zeta}{\sqrt{\Pi_0 + \frac{1}{3} \Sigma_0 \zeta^6}} = \pm \frac{1}{4} w.
\] (92)

Finally, the wave front has negative Gaussian curvature. Using the result given in Appendix A or formula (B2) of Appendix B of Ref. [1], we find for Gaussian curvature of the wave front \( (u = \text{constant}) \),

\[
K_G = -\frac{1}{4} A'^2 \left( p^2 e^{-B} + q^2 e^{-C} \right) < 0.
\] (93)

In connection with the possibility of the addition of a cross term in this case, we mention that the analytic treatment of the problem appears to be prohibitively complicated. This conclusion is based on a close inspection of the field equations given in Appendix B.

### A. Conformally Flat Solution

Let us assume that \( C = A \) in Eq. (88), so that the spacetime metric is conformally flat. This means via Eqs. (90) and (91) that \( \Sigma_0 = 3 \hat{k}_C^{-2}, \Pi_0 = 0 \) and

\[
p^2 + q^2 = -\frac{4}{3} \hat{k}_C^2 \Lambda,
\] (94)
which is possible if the cosmological constant is \textit{negative}. By constant rescalings of the spacetime coordinates as well as the parameters \((s, p, q)\), we can recast this solution into the form
\[ ds^2 = \Omega^2 \eta_{\mu\nu} \, dx^\mu \, dx^\nu, \]
where \(\Omega^{-1} = s \, u + p \, x + q \, y\) such that
\[ p^2 + q^2 = -\frac{1}{3} \Lambda. \]  
(95)

For \(q = 0\), this solution reduces to the simple conformally-flat solution \[17\] we found in Section II. It is convenient to introduce an angle \(\theta\), \(0 \leq \theta < 2 \pi\), such that
\[ p = (-\Lambda/3)^{1/2} \cos \theta, \quad q = (-\Lambda/3)^{1/2} \sin \theta. \]  
(96)

We show in Appendix C that the TGW spacetime under discussion here is indeed the \textit{unique} solution of \(R_{\mu\nu} = \Lambda \, g_{\mu\nu}\) that is conformally flat and represents a unidirectional gravitational wave.

Conformally flat plane wave spacetimes have been the subject of extensive investigations, see, for instance, Ref. \[8\], p. 603. The corresponding energy-momentum tensor is of the null fluid type, which can be interpreted either in terms of null dust or a pure (null) electromagnetic radiation field \[12, 13\]. An example of the latter situation has been discussed in detail, in connection with the phenomenon of cosmic jets \[14\], in Section IV of Ref. \[15\]. Conformally flat Kundt solutions with a cosmological constant are treated in Ref. \[9\], Section 18.3.3, p. 344.

\section{V. Conformally Flat TGWS Due to a Negative \(\Lambda\)}

The purpose of this section is to investigate in detail the Petrov type O solution that we found in the previous section, namely,
\[ g_{\mu\nu} = \Omega^2 \, \eta_{\mu\nu}, \quad \Omega^{-1} = s \, u + \varpi \, \cos \theta \, x + \varpi \, \sin \theta \, y, \]  
(97)

where \(u = t - z\) is the retarded null coordinate, \(s \neq 0\) and \(\theta\) are constant parameters and \(\varpi > 0\) is given by
\[ \varpi := \left(-\frac{\Lambda}{3}\right)^{1/2}. \]  
(98)

For a conformally flat spacetime, the Weyl curvature tensor \(C_{\mu\nu\rho\sigma}\) vanishes. Therefore, a Ricci-flat spacetime representing a nonlinear gravitational wave cannot be conformally flat; otherwise, the Riemann curvature tensor would completely vanish. Thus the existence of
our solution is purely due to the presence of the cosmological constant $\Lambda \neq 0$. Indeed, for $\Lambda = 0$, our metric in $(u, v, x, y)$ coordinates under rescalings of the spacetime coordinates via $(u, v, x, y) \mapsto (u^{-1}, -s^2 v, s x, s y)$ reduces to
\[
ds^2 = -du dv + u^2 (dx^2 + dy^2),
\]
which is simply flat as demonstrated at the end of Appendix B of Ref. \[16\].

The curvature tensor for the spacetime under consideration reduces to
\[
R_{\mu\nu\rho\sigma} = \frac{1}{3}\Lambda (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}).
\]
Furthermore, the wave front in this case has constant negative Gaussian curvature
\[
K_G = \frac{1}{3} \Lambda < 0,
\]
 cf. Appendix A.

### A. Timelike Geodesics

It is interesting to investigate the motion of free test particles in this gravitational field. Null geodesics are conformally invariant; hence, null geodesics in this conformally flat TGW spacetime are the same as those in Minkowski spacetime. We therefore concentrate on timelike geodesics. There are three Killing vector fields in this spacetime, namely,
\[
\partial_t + \partial_z, \quad \varpi \cos \theta \partial_t - s \partial_x, \quad \varpi \sin \theta \partial_t - s \partial_y.
\]
Thus there are three constants of timelike geodesic motion that can be obtained from projecting the 4-velocity vector of a free massive test particle, $\dot{x}^\mu = dx^\mu / d\eta$, on the Killing vector fields. Here $\eta$ is the proper time along the timelike geodesic world line. We have
\[
\Omega^2 (\dot{t} - \dot{z}) = C_v, \quad \Omega^2 (\varpi \cos \theta \dot{t} + s \dot{x}) = C_1, \quad \Omega^2 (\varpi \sin \theta \dot{t} + s \dot{y}) = C_2,
\]
where $C_v, C_1$ and $C_2$ are constants of the motion. Furthermore, the 4-velocity is a timelike unit vector; hence, $\Omega^2 \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1$.

It is convenient to take advantage of the circumstance that the geodesic equations of motion can be simply obtained from a Lagrangian of the form $(ds/d\eta)^2$. Therefore, we find
\[
\frac{d}{d\eta} (\Omega^2 \dot{t}) = -s \Omega, \quad \frac{d}{d\eta} (\Omega^2 \dot{x}) = \varpi \cos \theta \Omega, \quad \frac{d}{d\eta} (\Omega^2 \dot{y}) = \varpi \sin \theta \Omega.
\]
etc. Let us write
\[
\frac{d}{d\eta}(\Omega^{-1}) = \frac{d}{d\eta}(s u + \varpi \cos \theta x + \varpi \sin \theta y) = s \frac{C_v}{\Omega^2} + \varpi \cos \theta \dot{x} + \varpi \sin \theta \dot{y},
\] (105)
where Eq. (103) has been used. Next, multiplying both sides of this equation with \(\Omega^2 > 0\) and employing Eq. (104), we find
\[
\frac{d^2\Omega}{d\eta^2} + \varpi^2 \Omega = 0,
\] (106)
which has the general solution
\[
\Omega(\eta) = \Omega_0 \cos[\varpi (\eta - \eta_0)],
\] (107)
where \(\eta_0\) and \(\Omega_0 = \Omega(\eta_0)\) are constants of integration. It is clear that as proper time \(\eta\) increases monotonically from \(\eta_0\), \(\Omega(\eta)\) decreases monotonically and eventually approaches the singular value of zero at \(\eta = \eta_0 + \pi/(2 \varpi)\).

It is now straightforward to use our result for \(\Omega(\eta)\) in Eqs. (103) and (105) to find \(\dot{x}^\mu(\eta)\). Integrating these results, we determine \(x^\mu(\eta)\) for a timelike geodesic, which may be expressed as
\[
x^\mu(\eta) - x^\mu(\eta_0) = \frac{C^\mu}{\varpi^2 \Omega_0^2} \tan[\varpi (\eta - \eta_0)] - \frac{D^\mu}{\varpi^2 \Omega_0} \left(1 - \frac{1}{\cos[\varpi (\eta - \eta_0)]}\right),
\] (108)
where
\[
C_0 = \frac{s^2}{\varpi} C_v + C_1 \cos \theta + C_2 \sin \theta, \quad C_3 = C_0 - \varpi C_v,
\] (109)
\[
C_1 = -s C_v \cos \theta + \frac{\varpi}{s} (C_1 \sin \theta - C_2 \cos \theta) \sin \theta,
\] (110)
\[
C_2 = -s C_v \sin \theta - \frac{\varpi}{s} (C_1 \sin \theta - C_2 \cos \theta) \cos \theta
\] (111)
and
\[
D_0 = D_3 = -s, \quad D_1 = \varpi \cos \theta, \quad D_2 = \varpi \sin \theta.
\] (112)
The integration constants in these equations are related via
\[
\frac{1}{\Omega_0^2} = s u(\eta_0) + \varpi [x(\eta_0) \cos \theta + y(\eta_0) \sin \theta],
\] (113)
where \(u(\eta_0) = t(\eta_0) - z(\eta_0)\), and
\[
\varpi^2 \Omega_0^2 = (s^2 - \varpi^2) C_v^2 + 2 \varpi C_v (C_1 \cos \theta + C_2 \sin \theta) - \frac{\varpi^2}{s^2} (C_1 \sin \theta - C_2 \cos \theta)^2,
\] (114)
which follows from \(\Omega^2 \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1\).
1. Cosmic Jet

In certain dynamic spacetime regions, geodesics tend to line up, as measured by static fiducial observers, and thus produce a cosmic jet whose speed asymptotically approaches the speed of light \[14, 15\]. For plane gravitational wave spacetimes, this cosmic jet property was first demonstrated in Ref. \[15\] and further studied in Ref. \[16\]. A plane gravitational wave admits parallel null rays, so that the four principal null directions of the Weyl tensor coincide and are all parallel to the direction of propagation of the plane wave and hence perpendicular to the uniform wave front. With respect to the static observers in these spacetimes, timelike geodesics exhibit the cosmic jet property, where the jet motion is parallel to the direction of motion of the plane wave \[15, 16\]. However, the nonuniformity of the wave front in the case of nonplanar TGWs implies that the resulting cosmic jet direction is oblique with respect to the direction of wave propagation \[16\]. It is interesting to investigate this property for the case under consideration here. To this end, imagine a congruence of timelike geodesics in our conformally flat TGW spacetime. We are interested in the motion of a member of this congruence at time \(\eta\) with respect to a static observer spatially at rest in this spacetime. The natural tetrad frame of these static fiducial observers is given by

\[
e^{\alpha \hat{\mu}} = \frac{1}{\Omega} \delta^\alpha_\mu,
\]

where in \(1/\Omega = s u + \varpi \cos \theta x + \varpi \sin \theta y\), the spatial coordinates \(x, y\) and \(z\) are constants. These fiducial observers exist so long as \(\Omega \neq 0\). Projecting \(\dot{x}^\alpha = (\dot{t}, \dot{x}, \dot{y}, \dot{z})\) upon \(e^\alpha \hat{\mu}\) at \(x^\alpha(\eta)\) results in the instantaneous relation

\[
\dot{x}^\alpha e^\alpha \hat{\mu} = \Omega (\dot{t}, \dot{x}, \dot{y}, \dot{z}) = U^{\hat{\mu}} := \Gamma(1, V_x, V_y, V_z),
\]

where \(U^{\hat{\mu}}\) is the 4-velocity of the timelike geodesic as measured by the fiducial static observer. We find

\[
U^{\hat{\mu}} = \frac{1}{\varpi \Omega_0} \frac{C_\mu + \Omega_0 D_\mu \sin[\varpi (\eta - \eta_0)]}{\cos[\varpi (\eta - \eta_0)]}.
\]

It follows that as \(\eta \to \eta_0 + \pi/(2 \varpi)\), \(\Gamma \to \infty\) and the oblique cosmic jet is characterized by

\[
V_x \to \frac{C_1 + \Omega_0 D_1}{C_0 + \Omega_0 D_0}, \quad V_y \to \frac{C_2 + \Omega_0 D_2}{C_0 + \Omega_0 D_0}, \quad V_z \to \frac{C_3 + \Omega_0 D_3}{C_0 + \Omega_0 D_0}.
\]

One can check using Eq. (114) that indeed \(V_x^2 + V_y^2 + V_z^2 \to 1\) as the cosmic jet develops.

Tidal effects of our conformally flat TGWs are studied in the next subsection.
B. Jacobi Equation

Imagine a static observer that is at rest in space in our conformally flat TGW spacetime. The observer carries an orthonormal tetrad frame $e^\mu_\alpha$ along its world line and uses this frame to set up a geodesic (Fermi) normal coordinate system in its neighborhood. The Fermi system is discussed in Appendix D. We are interested in the motion of nearby geodesics with respect to the accelerated static observer that permanently occupies the origin of the Fermi coordinate system. This analysis is carried out in several steps.

1. Tetrad of the Static Observer

The static fiducial observer has a natural tetrad frame $e^\mu_\alpha = \Omega^{-1} \delta^\mu_\alpha$, where $\Omega \neq 0$. The world line of such an observer is given by $\bar{x}^\mu = (t, x_0, y_0, z_0)$, where

$$\tau = \frac{1}{s} \left[ \ln(st + \omega \cos \theta x_0 + \omega \sin \theta y_0 - s z_0) - \ln(\omega \cos \theta x_0 + \omega \sin \theta y_0 - s z_0) \right]$$

is the proper time of the static observer and we have assumed that $\tau = 0$ at $t = 0$. Such observers are not geodesic; in fact, they are accelerated with

$$\mathcal{A}^\mu = \frac{De^\mu_0}{d\tau} = -\omega \cos \theta \ e^\mu_1 - \omega \sin \theta \ e^\mu_2 + s \ e^\mu_3.$$  \hspace{1cm} (120)

The static observer carries the spatial frame of the tetrad along its world line for measurement purposes. It is straightforward to check that the spatial frame $e^\mu_i$, for $i = 1, 2, 3$, is indeed Fermi-Walker transported; that is, its components satisfy the equation of Fermi-Walker transport,

$$\frac{dS^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} e^\alpha_0 S^\beta = (\mathcal{A} \cdot S) e^\mu_0 - (e^0_\alpha \cdot S) \mathcal{A}^\mu,$$  \hspace{1cm} (121)

where $S^\mu$ is a vector that is Fermi–Walker transported along $e^\mu_0$.

2. Spacetime Curvature as Measured by Static Observers

Suppose that the static observer with orthonormal tetrad $e^\mu_\alpha$ measures the spacetime curvature in our conformally flat TGW spacetime. The components of the Riemann curvature tensor as measured by the observer are given by

$$R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(\tau) := R_{\mu\nu\rho\sigma} e^\mu_\alpha e^\nu_\beta e^\rho_\gamma e^\sigma_\delta.$$  \hspace{1cm} (122)
These constitute the projection of the Riemann curvature tensor upon the tetrad frame of the static observer. Taking advantage of the symmetries of the Riemann tensor, this quantity can be represented by a $6 \times 6$ matrix $\mathcal{R} = (\mathcal{R}_{IJ})$, where the indices $I$ and $J$ range over the set $(01, 02, 03, 23, 31, 12)$. Thus we can write

$$\mathcal{R} = \begin{bmatrix} \mathcal{E} & \mathcal{B} \\ \mathcal{B}^\dagger & \mathcal{S} \end{bmatrix},$$

where $\mathcal{E}$ and $\mathcal{S}$ are symmetric $3 \times 3$ matrices and $\mathcal{B}$ is traceless. The tidal matrix $\mathcal{E}$ represents the “electric” components of the curvature tensor as measured by the static observer, whereas $\mathcal{B}$ and $\mathcal{S}$ represent its “magnetic” and “spatial” components, respectively. In the case under consideration, $e^\mu_\dot{\alpha} = \Omega^{-1} \delta^\mu_\alpha$, so that Eq. (100) implies

$$\mathcal{E} = -\frac{1}{3} \Lambda I, \quad \mathcal{B} = 0, \quad \mathcal{S} = \frac{1}{3} \Lambda I,$$

where $I$ is the $3 \times 3$ identity matrix $I = \text{diag}(1, 1, 1)$. These results should be contrasted with the Weyl curvature of the Ricci-flat TGWs in Ref. [2]. The absence of the gravitomagnetic component of the Riemann tensor as measured by static fiducial observers is a peculiar feature of this propagating TGW that has no Weyl curvature. This point can be further illustrated via the Bel tensor in this case.

The super-energy-momentum tensor of a gravitational field is proportional to the symmetric and traceless quantity [17, 18]

$$\mathcal{T}_{\dot{\alpha}\dot{\beta}} = \tilde{T}_{\mu\nu\rho\sigma} e^\mu_\dot{\alpha} e^\nu_\dot{\beta} e^\rho_0 e^\sigma_0,$$

where $\tilde{T}_{\mu\nu\rho\sigma}$ is the natural gravitational analog of the energy-momentum tensor of the electromagnetic field

$$\tilde{T}_{\mu\nu\rho\sigma} = \frac{1}{2} \left( R_{\mu\xi\rho\zeta} R^\xi_{\sigma} - R_{\mu\xi\sigma\zeta} R^\xi_{\rho} \right) - \frac{1}{4} g_{\mu\nu} R_{\alpha\beta\rho\gamma} R^{\alpha\beta}_{\sigma\gamma}$$

and was introduced by Bel in 1958 [19]. In general, Bel’s tensor, $\tilde{T}_{\mu\nu\rho\sigma}$, is symmetric and traceless in its first pair of indices and symmetric in its second pair of indices. In a Ricci-flat spacetime, Bel’s tensor reduces to the completely symmetric and traceless Bel-Robinson tensor.

For the conformally flat TGW with Riemann curvature (100), the Bel tensor is given by

$$\tilde{T}_{\mu\nu\rho\sigma} = \left( \frac{\Lambda}{3} \right)^2 \Omega^4 \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right),$$
which is traceless in its second pair of indices as well. In this case, the corresponding symmetric and traceless super-energy-momentum tensor as measured by the static fiducial observers is proportional to

\[ T_{\hat{\alpha}\hat{\beta}} = \left( \frac{\Lambda}{3} \right)^2 \left( 2 \eta_{\hat{\alpha} \hat{\delta}} \eta_{\hat{\beta} \hat{\delta}} + \frac{1}{2} \eta_{\hat{\alpha} \hat{\beta}} \right), \]  

which has the peculiar character of a perfect fluid at rest with energy density \( \Lambda^2/6 \) and pressure \( \Lambda^2/18 \). The super-Poynting vector vanishes in this case in contrast to the Ricci-flat TGWs discussed in Ref. [2]. This circumstance illustrates the limitation of the super-momentum concept in the absence of Weyl curvature.

3. Tidal Equations

The equation for the motion of a timelike geodesic relative to our fiducial static observer within the framework of the Fermi coordinate system \((T, X)\) can be written as

\[ \frac{d^2 X^i}{dT^2} + \mathcal{A}^i + (\mathcal{E}^i_j + \mathcal{A}^i \mathcal{A}_j) X^j = 0, \]  

where the contribution of relative velocity has been neglected, see Appendix D. Here, \( \mathcal{A}^1 = -\varpi \cos \theta, \mathcal{A}^2 = -\varpi \sin \theta, \mathcal{A}^3 = s \) and \( \mathcal{E}_{ij} = R_{0i0j} = \varpi^2 \delta_{ij} \). Thus Eq. (129) can be expressed as

\[ \frac{d^2 X^1}{dT^2} + \varpi^2 (1 + \cos^2 \theta) X^1 + \varpi^2 \sin \theta \cos \theta X^2 - s \varpi \cos \theta X^3 = \varpi \cos \theta, \]  

\[ \frac{d^2 X^2}{dT^2} + \varpi^2 \sin \theta \cos \theta X^1 + \varpi^2 (1 + \sin^2 \theta) X^2 + -s \varpi \sin \theta X^3 = \varpi \sin \theta, \]  

\[ \frac{d^2 X^3}{dT^2} - s \varpi \cos \theta X^1 - s \varpi \sin \theta X^2 + (s^2 + \varpi^2) X^3 = -s. \]  

It proves convenient to define \( P_i, i = 1, 2, 3 \), as follows:

\[ P_1 = \varpi \cos \theta X^1 + \varpi \sin \theta X^2 - s X^3, \quad P_2 = -\sin \theta X^1 + \cos \theta X^2, \]  

\[ P_3 = s \cos \theta X^1 + s \sin \theta X^2 + \varpi X^3. \]  

Then, Eqs. (130)–(132) can be written in terms of the new quantities as

\[ \frac{d^2 P_1}{dT^2} + (s^2 + 2 \varpi^2) P_1 = s^2 + \varpi^2, \]  

\[ \frac{d^2 P_2}{dT^2} + \varpi^2 P_2 = 0, \quad \frac{d^2 P_3}{dT^2} + \varpi^2 P_3 = 0. \]  

It is now straightforward to write down the general solution of the tidal equations in this case. That is,

\[
P_1 = \frac{s^2 + \omega^2}{s^2 + 2\omega^2} + \xi_1 \cos(\sqrt{s^2 + 2\omega^2} T + \varphi_1),
\]

\[
P_2 = \xi_2 \cos(\omega T + \varphi_2), \quad P_3 = \xi_3 \cos(\omega T + \varphi_3),
\]

where \(\xi_i\) and \(\varphi_i\), for \(i = 1, 2, 3\), are integration constants. Let us note that we can write

\[
P_1 = D_i X^i, \quad P_2 = N_i X^i, \quad P_3 = E_i X^i,
\]

where \(D_i\) are given by Eq. (112), and \(N_i\) and \(E_i\) are defined here via Eqs. (133) and (134). That is, for \(i = 1, 2, 3\),

\[
(D_i) = (\omega \cos \theta, \omega \sin \theta, -s), \quad (N_i) = (-\sin \theta, \cos \theta, 0), \quad (E_i) = (s \cos \theta, s \sin \theta, \omega),
\]

which are three spatially orthogonal vectors. It follows that

\[
X^i = \frac{D_i}{s^2 + 2\omega^2} + \frac{D_i}{s^2 + 2\omega^2} \xi_1 \cos(\sqrt{s^2 + 2\omega^2} T + \varphi_1)
\]

\[
+ N_i \xi_2 \cos(\omega T + \varphi_2) + \frac{E_i}{s^2 + 2\omega^2} \xi_3 \cos(\omega T + \varphi_3).
\]

The transverse character of linearized gravitational waves in GR is well known. Twisted gravitational waves that are Ricci-flat exhibit in addition a longitudinal component as well [1]. For a general discussion of the corresponding longitudinal component in the presence of Weyl curvature tensor, see Refs. [20, 21]. However, the Weyl conformal curvature tensor vanishes for our special solution; for a general discussion of the Jacobi equation in this case, see Ref. [22]. The longitudinal component in the absence of Weyl curvature is given by \(X^3\) in the present case, which is along the direction of wave propagation and can be obtained from Eq. (141) for \(i = 3\). This longitudinal feature is illustrated in Figure 1.

VI. DISCUSSION

Three classes of TGW solutions in the presence of a cosmological constant have been presented in Sections II–IV. These are generally implicit, as each GR solution depends upon the solution of an ordinary differential equation for \(A(w)\), where \(w\) depends linearly on spacetime coordinates \((t, x, y, z)\). Among the new solutions, there is a simple unique conformally flat solution with \(g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}\) that has a conformal factor \(\Omega\) given by \(\Omega^{-1} = \)
FIG. 1: Plot of $\varpi X^i$ versus $\varpi T$ for $i = 1$ (blue dot), $i = 2$ (red dash-dot) and $i = 3$ (solid black), where $X^i$ is given by Eq. 141. Initial conditions at $T = 0$ are chosen such that $(X^1, X^2, X^3) = (1, 0.5, 0)$ and $dX^i/dT = 0$ for $i = 1, 2, 3$. Thus, $\varphi_1 = \varphi_2 = \varphi_3 = 0$. Moreover, $s = \sqrt{2} \varpi$ and $\theta = 45^\circ$, so that $\xi_1 = 3(\sqrt{2} - 1)/4$, $\xi_2 = -\sqrt{2}/(4 \varpi)$ and $\xi_3 = 3/2$.

$s u + p x + q y$, where $(s, p, q)$ are constants subject to $p^2 + q^2 = -\Lambda/3$. The wave front for this simple TGW has constant negative Gaussian curvature determined by the cosmological constant $\Lambda$, namely, $K_G = \Lambda/3$. This special explicit solution for negative cosmological constant has been studied in detail in the previous section. That is, the timelike geodesics of this solution have been worked out and the deviation of these geodesics relative to the world lines of static observers in this spacetime have been examined in connection with measurements of static fiducial observers.

All of the known TGWs, regardless of the presence of the cosmological constant, have wave fronts with negative Gaussian curvature. It is not known whether this is a general feature of TGWs or occurs due to the formal simplicity of the solutions that have been found thus far.
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Appendix A: Gaussian Curvature of the Wave Front

Consider a TGW spacetime with metric of the form
\[ ds^2 = -\gamma_0 dt^2 + \gamma_1 dx^2 + 2\gamma_2 dx dy + \gamma_3 dy^2 + \gamma_0 dz^2 , \tag{A1} \]
where \( \gamma_\mu = \gamma_\mu(u,x,y) \) and \( u := t - z \). The metric of the wave front is given by
\[ d\sigma^2 = \gamma_1(u,x,y) dx^2 + 2\gamma_2(u,x,y) dx dy + \gamma_3(u,x,y) dy^2 , \tag{A2} \]
where \( u \) is a constant in this case. It is possible to show that for any \((t,x,y,z)\),\n\[
^{(s)}R_{xyxy} = ^{(\sigma)}R_{xyxy} , \tag{A3}
\]
where \(^{(s)}R_{xyxy}\) is a component of the Riemann curvature tensor for metric (A1), while \(^{(\sigma)}R_{xyxy}\) is the corresponding component for metric (A2).

For metric (A2), the Gaussian curvature \(K_G\) is given by
\[
K_G = \frac{^{(\sigma)}R_{xyxy}}{\Delta} , \quad \Delta = \gamma_1 \gamma_3 - \gamma_2^2 > 0 . \tag{A4}
\]
From
\[
\gamma_3 ^{(\sigma)}R_{xyxy} = \Delta ^{(\sigma)}R_{xyxy} , \tag{A5}
\]
we get the simple relation
\[
K_G = \frac{1}{\gamma_3} ^{(\sigma)}R_{xyxy} . \tag{A6}
\]
Using the standard formula for the Riemann tensor, we find
\[
K_G = \frac{1}{2\Delta} \left( 2\gamma_2,xy - \gamma_1,yy - \gamma_3,xx \right) - \frac{\gamma_3}{4\Delta^2} \left( 2\gamma_1,x \gamma_2,y - \gamma_1,x \gamma_3,x - \gamma_2^2 \right) \tag{A7}
\]
\[
- \frac{\gamma_1}{4\Delta^2} \left( 2\gamma_2,x \gamma_3,y - \gamma_1,y \gamma_3,y - \gamma_2^2 \right) + \frac{\gamma_2}{4\Delta^2} \left[ \gamma_1,x \gamma_3,y - 2\gamma_1,y \gamma_3,x \right.
\]
\[
+ \left. (2\gamma_2,x - \gamma_1,yy) - \gamma_3,x \right] ,
\]
where a comma denotes partial differentiation.

In the special case where \(\gamma_2 = 0, \gamma_1 = e^B\) and \(\gamma_3 = e^C\), Eq. (A7) reduces to formula (B2) of Appendix B of Ref. [1].
Appendix B: Gravitational Field Equations

The purpose of this appendix is to present the gravitational field equations (12) for metric (5) when condition (13) is satisfied. It follows from the admissibility conditions for the coordinates that the metric can be written as

\[ ds^2 = -e^A du dv + e^B dx^2 + 2 h dx dy + e^C dy^2, \]  

(B1)

where \( A, B, C \) and \( h \) are functions of \( w = s u + p x + q y \) and \( f(w) := \exp(B + C) - h^2 > 0 \). The metric is invariant under the exchange of \((x, B, p)\) with \((y, C, q)\), respectively. This invariance is then reflected in the field equations \( R_{\mu\nu} = \Lambda g_{\mu\nu} \). In this connection, it is convenient to introduce

\[ H_A = h' - h A', \quad H_B = h' - h B', \quad H_C = h' - h C', \]  

(B2)

where \( h' := dh/dw \), etc. The homogeneous field equations are then given by \( R_{uu} = 0 \), \( R_{ux} = 0 \), \( R_{uy} = 0 \) and \( R_{xy} = 0 \), which can be expressed as

\[ (f + h^2)(H_B + H_C)^2 - 2 f H_B H_C - 2 f h[A'(H_B + H_C) - (H'_B + H'_C)] \]  

\[ - f h^2(B' - C')^2 - f^2 [2 B'' + B'^2 + 2 C'' + C'^2 - 2 A'(B' + C')] = 0, \]  

(B3)

\[ q e^B [h(H_B + H_C)H_B + 2 f H'_B + f(B' - C')H_B] + p [h^2(H_B + H_C)H_C \]  

\[ - f h(A'H_B + A'H_C + B'H_C + C'H_B - 2 H'_B) \]  

\[ - f^2 (2 C'' + C'^2 + 2 A'' - A'B' - A'C' - B'C') = 0, \]  

(B4)

\[ p e^C [h(H_B + H_C)H_C + 2 f H'_C - f(B' - C')H_C] + q [h^2(H_B + H_C)H_B \]  

\[ - f h(A'H_B + A'H_C + B'H_C + C'H_B - 2 H'_B) \]  

\[ - f^2 (2 B'' + B'^2 + 2 A'' - A'B' - A'C' - B'C') = 0 \]  

(B5)

and

\[ h q^2 e^B [h(H_B + H_C)B' + f [(B' - C')B' + 2 A'B' + 2 B''] \]  

\[ + h p^2 e^C [h(H_B + H_C)C' + f [-(B' - C')C' + 2 A'C' + 2 C''] \]  

\[ + 2 pq [-2 fhh'' - h^2 h'(H_B + H_C) + f h(B' + C')H_A \]  

\[ + f^2 (2 A'' + A'^2 - A'B' - A'C') = 0, \]  

(B6)
respectively. Furthermore, the three inhomogeneous field equations, namely, $R_{uv} = -(\Lambda/2) \exp(A)$, $R_{xx} = \Lambda \exp(B)$ and $R_{yy} = \Lambda \exp(C)$ are given by

$$-4 f^2 \Lambda = p^2 e^C [h(H_B + H_C)A' + f [2 A'' + 2 A^2 + A'C' - A'B']] + q^2 e^B [h(H_B + H_C)A' + f [2 A'' + 2 A^2 + A'B' - A'C']] - 2 pq [h^2(H_B + H_C)A' + 2 fh'A' + f h(2 A'' + 2 A^2 - A'B' - A'C')] ,$$

(B7)

and

$$-4 f^2 \Lambda = q^2 e^B [h(H_B + H_C)B' + f (2 B'' + B^2 + 2 A'B' - B'C')] + p^2 e^{-B} [h^3(H_B + H_C)C' + 2 fh(hC'' + h'C' + 2 h'A' - hA'B' - hB'C')]
+ f^2 (4 A'' + 2 A^2 + 2 C'' + C^2 - 2 A'B' - B'C')] - 2 pq [hh'(H_B + H_C) + 2 fh'' + fh'(2 A' - B' - C')] ,$$

(B8)

and

$$-4 f^2 \Lambda = p^2 e^C [h(H_B + H_C)C' + f (2 C'' + C^2 + 2 A'C' - B'C')] + q^2 e^{-C} [h^3(H_B + H_C)B' + 2 fh(hB'' + h'B' + 2 h'A' - hA'C' - hB'C')]
+ f^2 (4 A'' + 2 A^2 + 2 B'' + B^2 - 2 A'C' - B'C')] - 2 pq [hh'(H_B + H_C) + 2 fh'' + fh'(2 A' - B' - C')] .$$

(B9)

The field equations employed in this paper can be obtained as special cases of the results given in this appendix.

**Appendix C: Conformally Flat TGW Solution of $R_{\mu\nu} = \Lambda g_{\mu\nu}$**

We look for solutions of the gravitational field equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with a conformally flat TGW metric of the form

$$ds^2 = e^{A(u,x,y)} (-du \, dv + dx^2 + dy^2) .$$

(C1)

The corresponding field equations consist of three equations with source $\Lambda$, namely,

$$-2 R_{uv} = R_{xx} = R_{yy} = \Lambda e^{A(u,x,y)} ,$$

(C2)

which can be written out explicitly for metric [C1] as

$$A_{xx} + A_{yy} + (A_x)^2 + (A_y)^2 = -2 \Lambda e^A ,$$

(C3)
respectively. Furthermore, there are four nontrivial source-free field equations

\[ R_{uu} = R_{ux} = R_{uy} = R_{xy} = 0, \quad (C6) \]

which can be expressed as

\[ 2A_{,uu} = (A_u)^2, \quad (C7) \]

\[ 2A_{,ux} = A_{,u} A_{,x}, \quad 2A_{,uy} = A_{,u} A_{,y}, \quad 2A_{,xy} = A_{,x} A_{,y}, \quad (C8) \]

respectively.

Let us subtract Eq. (C3) from Eqs. (C4) and (C5) to get

\[ 2A_{,xx} = (A_x)^2, \quad 2A_{,yy} = (A_y)^2. \quad (C9) \]

Then, Eq. (C3) reduces to

\[ (A_x)^2 + (A_y)^2 = -\frac{4}{3} \Lambda e^A. \quad (C10) \]

Next, by virtue of Eq. (C7) we have

\[ \left(e^{-\frac{1}{2}A}\right)_{,uu} = \frac{1}{4} [2A_{,uu} - (A_u)^2] e^{-\frac{1}{2}A} = 0, \quad (C11) \]

which implies that exp \((-A/2)\) is a linear function of \(u\). Similarly, it follows from Eq. (C9) that \(\exp(-A/2)\) depends linearly upon \(x\) and \(y\) as well. Thus, we can write

\[ e^{-\frac{1}{2}A} = s u + p x + q y \quad (C12) \]

plus a constant that can always be removed by a simple coordinate translation. The integration constant \(s\) is arbitrary, while Eq. (C10) implies

\[ p^2 + q^2 = -\frac{1}{3} \Lambda. \quad (C13) \]

The remaining field Eqs. (C8) are all satisfied by this unique class of conformally flat solutions with constant parameters \((s, p, q)\) subject to restriction (C13). This gravitational field disappears in the absence of a negative cosmological constant.
Appendix D: Deviation Equation in Fermi Coordinates

Consider an arbitrary static observer with proper time $\tau$ following a world line $\bar{x}^\mu(\tau)$. Let $e^\mu_\alpha(\tau)$ be a Fermi-Walker transported tetrad along $\bar{x}^\mu(\tau)$. At each event on the observer’s path, we imagine the set of all spacelike geodesics that are orthogonal to the world line at $\bar{x}^\mu(\tau)$ and form a spacelike hypersurface. Let $x^\mu$ be an event on this hypersurface that can be connected to $\bar{x}^\mu(\tau)$ with a unique spacelike geodesic of proper length $\varsigma$. We assign to event $x^\mu$ Fermi coordinates $X^\hat{i} = (T, X^\hat{i})$, where

$$ T := \tau, \quad X^\hat{i} := \varsigma \sigma^\mu(\tau) e^\mu_\hat{i}(\tau). \quad \text{(D1)} $$

The unit spacelike vector tangent at $\bar{x}^\mu(\tau)$ to the unique spacelike geodesic connecting $\bar{x}^\mu(\tau)$ with $x^\mu$ is denoted by $\sigma^\mu$; hence, $\sigma^\mu(\tau) e^\mu_0(\tau) = 0$. It is clear that the reference observer occupies the spatial origin of the Fermi coordinate system.

We wish to study the timelike geodesic equation in the Fermi coordinate system. In this way, we can determine the motion of a free test particle relative to the fiducial static observer. Neglecting the relative velocity, the reduced geodesic equation can be expressed as

$$ \frac{d^2 X^\hat{i}}{dT^2} + \mathcal{A}^\hat{i}(T) + [\mathcal{E}_{\hat{i}\hat{j}}(T) + \mathcal{A}_{\hat{i}}(T) \mathcal{A}_{\hat{j}}(T)] X^\hat{j} = 0, \quad \text{(D2)} $$

where $\mathcal{A}^\hat{i}$ is the 4-acceleration of the fiducial static observer projected upon its frame and $\mathcal{E}_{\hat{i}\hat{j}}$ are the corresponding components of the tidal matrix, namely,

$$ \mathcal{A}^\hat{i}(T) = \frac{D e^\mu_0}{dT} e^\mu_\hat{i} = A^\mu e^\mu_\hat{i}, \quad \mathcal{E}_{\hat{i}\hat{j}}(T) = R_{\hat{i}0\hat{j}0} e^\mu_0 e^\nu_\hat{i} e^\rho_0 e^\sigma_\hat{j}. \quad \text{(D3)} $$

For background material on the equations of motion in Fermi coordinates, we refer to Refs. [23–25] and the references cited therein. The Fermi coordinate system is generally admissible in a certain cylindrical spacetime domain around $\bar{x}^\mu(\tau)$.

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