Nonperturbative Flavor Breaking in Topological Susceptibility at Hot QCD Criticality

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We show that the QCD topological susceptibility nonperturbatively gets a significant contribution signaled by flavor-nonuniversal quark condensates at around the pseudo-critical temperature of the chiral crossover. It implies a remarkable flavor breaking in the axial anomaly as well as the QCD theta vacuum in high temperature QCD, which are (almost) flavor universal in the vacuum. The flavor breaking is triggered by nonperturbative thermal loop corrections, which cannot be dictated by the low-energy expansion associated with the chiral symmetry breaking, such as the chiral perturbation theory. This would give an impact on the thermal history and the cosmological evolution of QCD axion including the estimate of the relic abundance as a cold dark matter candidate. Applications to heavy-ion collision-phenomenology would also be possible.

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INTRODUCTION AND SUMMARY

The topological susceptibility is a crucial probe in studying the QCD \( \theta \) vacuum structure and the axial anomaly. It is also important for QCD axion, which is postulated as an elegant solution to so-called the strong CP problem. In particular, when the QCD axial potential and mass are evaluated in the thermal history, the relic abundance as a cold dark matter today and the CP problem. In particular, when the QCD axial potential and mass are evaluated in the thermal history, the relic abundance as a cold dark matter today and the cosmological evolution of axion field in our Universe will be subject mainly to the temperature dependence of the topological susceptibility around the QCD phase transition epoch \([1]\). It would furthermore be a key quantity for hot QCD phenomena relevant to ongoing and designated heavy-ion collision experiments, such as physics induced by the topological charge fluctuation (i.e. susceptibility) closely tied with the presence of the QCD sphaleron \([2,3]\). Thus the QCD topological susceptibility has extensively been analyzed in multi-point of views with field theoretical, cosmological and astrophysical concerns.

The topological susceptibility \( \chi_{\text{top}} \) is defined as the curvature of the free energy of QCD with respect to the \( \theta \) at the QCD vacuum with \( \theta = 0 \). To our best knowledge, the key role for the \( \chi_{\text{top}} \) in real-life QCD is provided by an approximate chiral symmetry: with the approximate chiral symmetry for quarks, we can evaluate the \( \chi_{\text{top}} \) in relation to the quark condensates \( \langle \bar{q}q \rangle \) and the current quark mass \( m_q \) (e.g., see Appendix of the literature \([7]\), showing the derivation with use of the Ginzsparg-Wilson Dirac operator)

\[
\chi_{\text{top}} = \tilde{m}^2(q) \sum_q \frac{\langle \bar{q}q \rangle}{m_q}, \quad \frac{1}{\tilde{m}(q)} = \sum_q \frac{1}{m_q}, \quad (1)
\]

at the leading order of \( m_q \). This is the intriguing formula having the nonperturbative correlation in QCD between the axial anomaly along with the \( \theta \) (dictated by the left hand side) and the chiral symmetry breaking (right hand side). The mediator for this correlation, the effective mass \( \tilde{m}(q) \) reflects the flavor-singlet nature of the axial anomaly in QCD \([8]\): the axial anomaly detected by the \( \chi_{\text{top}} \) goes away if either of quarks get massless.

In the QCD vacuum with the lightest three flavors \((u,d,s)\) having the mass well below the intrinsic QCD scale \( \sim O(1) \text{ GeV} \), the quark condensates are well degenerated \((\langle \bar{u}u \rangle \simeq \langle \bar{d}d \rangle \simeq \langle \bar{s}s \rangle \simeq \Sigma)\) as observed in the recent lattice simulation, \(\langle \bar{s}s \rangle/(\langle l l \rangle) = 1.08 \pm 0.16 \) \((l = u, d, s)\) \([9]\). In that case, the generic formula in Eq. \((1)\) can be reduced to the Leutwyler-Smiluga (LS) relation \([10]\): \(\chi_{\text{top}}|_{\text{LS}} = \tilde{m}(u,d,s)\Sigma\), which is derived at the leading order in the chiral perturbation theory (ChPT). This LS relation indeed works well in the QCD vacuum since the size of the three-flavor breaking is of \( O(\frac{m_q}{m_W}) = O(10^{-2}) \), which supports the success of the ChPT also for this topological sector. The lattice QCD, with a physical pion mass realized and the continuum limit taken, have also shown consistency with the ChPT prediction \([11,13]\).

However, the situation might dramatically be altered at finite temperature: as has been observed in several analyses on hot lattice QCD \([11,15]\), thermal loop effects would cause a partial restoration of the chiral symmetry, where only the lightest \( l \) quark condensates \(\langle l l \rangle\) drop, but the strange quark’s \(\langle \bar{s}s \rangle\) still keeps nonzero (or damps more slowly than the \(\langle \bar{l}l \rangle\)). This implies a nonperturbative flavor breaking in the quark condensates, and it surely cannot be captured by the ChPT. Thus, this
and the formula Eq. 1 may allow us to realize a non-perturbative flavor breaking for the $\chi_{\text{top}}$ at the hot QCD criticality.

In this Letter, we show that the QCD topological susceptibility nonperturbatively gets a significant contribution from the flavor breaking at around the (pseudo) critical temperature of the chiral phase transition (chiral crossover). It implies an enhanced flavor breaking in the axial anomaly as well as the QCD theta vacuum in high temperature QCD.

To make a concrete demonstration, we employ a non-perturbative analysis in a linear sigma model with the lightest three flavors, based on the Cornwall-Jackiw-Tomboulis (CJT) formalism 16, and compute thermal corrections to the effective potential coming from meson loops. The linear sigma model description allows us to cross the chiral phase boundary, in which the role of the chiral order parameter is played by sigma fields, as will be seen below #1.

It turns out that the linear sigma model undergoes a chiral crossover at a pseudo critical temperature ($T_{pc}$), above which the up and down quark condensates dramatically drop, while the strange quark condensate still almost keeps the vacuum value. This chiral crossover phenomenon is consistent with the recent observation in lattice QCD with $2 + 1$ flavors 20–23. We then observe that due to the surviving strange quark, for $T/T_{pc} \sim 1.5 – 2$ the topological susceptibility is enhanced by about 15–30% (about more than 10 times larger than the vacuum case), when compared to the flavor universal limit (Fig. 2). The significance of the flavor breaking gets more eminent monotonically as the temperature increases (Fig. 1). The scaling property with respect to the temperature turns out to be consistent with the lattice QCD data 11 12 (Fig. 3).

The flavor breaking that we find in the present analysis is triggered by nonperturbative-thermal loop corrections and the nonperturbatively derived criticality of hot QCD, which cannot be dictated by the low-energy expansion associated with the chiral symmetry breaking, such as the ChPT. Our result can be tested by the lattice QCD with more accurate precision for realization of the (approximate) chiral symmetry and the continuum limit than the current status.

#1 Some nonperturbative analyses on the QCD topological susceptibility at finite temperatures have so far been done based on chiral effective models 17–19. However, no discussion on the correlation with quark condensates was made because their topological susceptibilities do not hold the flavor singlet form as in Eq. 4 (see also Eq. 5), for the flavor singlet condition, hence it seems to have been impossible to find the nonperturbative flavor breaking as addressed in the present Letter.

CHIRAL EFFECTIVE MODEL DESCRIPTION

We begin by introducing a linear sigma model with the lightest three flavors, which has recently been proposed by including a possible axial-anomaly induced-flavor breaking term 24 (Eq. 4). We parameterize the scalar- and pseudoscalar-meson nonets by a $3 \times 3$ matrix field $\Phi$ as $\Phi = (\sigma_a + i\tau_a)T_a$, where $\sigma_a$ are the scalar fields and $\pi_a$ are the pseudoscalar fields. $T_a = \lambda_a/2$ ($a = 0, 1, \ldots, 8$) are the generators of $U(3)$ normalized by $\text{tr}[T_a T_b] = 1/2\delta_{ab}$, where $\lambda_a=1,\ldots,8$ are the Gell-Mann matrices with $\lambda_0 = \sqrt{2/3} 1_{3 \times 3}$. Under the chiral $SU(3)_L \times SU(3)_R \times U(1)_A$ symmetry, $\Phi$ transforms as $\Phi \rightarrow g_A \cdot g_L \cdot \Phi \cdot g_R \dagger$, where $g_{L,R} \in SU(3)_{L,R}$ and $g_A \in U(1)_A$. The three-flavor linear sigma model is thus written as 24,

$$L = \text{tr} \left[ \partial_\mu \Phi \partial^\mu \Phi^\dagger \right] - V(\Phi),$$

with

$$V(\Phi) = V_0 + V_{\text{anom}} + V_{\text{anom}} + V_{SB} + V_{SB-\text{anom}}.$$  (3)

$V_0$ is an invariant part under the $SU(3)_L \times SU(3)_R \times U(1)_A$ symmetry,

$$V_0 = \mu^2 \text{tr}[(\Phi \Phi^\dagger)] + \lambda_1 \text{tr}[(\Phi \Phi^\dagger)^2] + \lambda_2 \text{tr}[(\Phi \Phi^\dagger)]^2.$$  (4)

The chiral $SU(3)_L \times SU(3)_R$ invariant, but $U(1)_A$ anomalous part is incorporated in $V_{\text{anom}}$:

$$V_{\text{anom}} = -B \left( \text{det}[\Phi] + \text{det}[\Phi^\dagger] \right).$$  (5)

$V_{SB}$ denotes the explicit chiral symmetry breaking term, which is originated from the current quark mass matrix $M = \text{diag}[m_u, m_d, m_s]$ in the underlying QCD Lagrangian. This $M$ acts as a spurion field that transforms in the same way as the $\Phi$ does. At the leading order of expansion in $m_q$, the $V_{SB}$ thus takes the form

$$V_{SB} = -c \text{tr}[M \Phi \Phi^\dagger + M \Phi^\dagger \Phi].$$  (6)

The parameter $c$ necessarily comes along with the current quark masses ($M$) in physical observables, which reflects the renormalization scale ambiguity in defining quark condensates.

$V_{SB-\text{anom}}$ is the axial-anomaly induced-flavor breaking term. In the minimal flavor violation limit where single $M$ is only allowed to be inserted, the $V_{SB-\text{anom}}$ is cast into the form

$$V_{SB-\text{anom}} = -k c \left[ e^{\delta} e^{\varepsilon} \right] M_3 \Phi^\dagger \Phi^\varepsilon + \text{h.c.}. $$  (7)

This $k$ coupling term plays a crucial role to realize the inverse mass hierarchy for scalar mesons below 1 GeV, as has been shown in 24.

In the QCD generating functional, the $\theta$-term can be rotated away by the $U(1)_A$ rotation with the rotation
angle \( \theta_q \) for quark fields. Then the \( \theta \)-dependence is fully transferred into the quark mass matrix, \( \bar{q}_R M q_R \) \( (\bar{q}_R M' q_L) \) with \( M = \text{diag}[m_u e^{i \theta_u}, m_d e^{i \theta_d}, m_s e^{i \theta_s}] \). It is important to note here that the \( \theta_q \) is constrained by the flavor singlet condition (for \( \theta \ll 1 \)),

\[
\theta_q = \frac{\bar{m}(u, d, s)}{m_q} \theta,
\]

so that the \( U(1)_A \) anomaly keeps the flavor singlet nature \([3]\). We shall make a matching of this QCD generating functional with the one corresponding to the present linear sigma model. We are thus allowed to eliminate the \( \theta \)-dependence in the sigma field \( \Phi (\Phi^\dagger) \), which has been introduced as the interpolating field of quark bilinear \( \bar{q}_R q_L \) \( (\bar{q}_L q_R) \), so that the \( \theta \) parameter should be entered in \( V_{SB} \) and \( V_{SB-anom} \) only via the quark mass matrix \( M \).

We assume the isospin symmetry, \( m_t = m_u = m_d \neq m_s \), so that the vacuum expectation values are taken as \( \langle \Phi \rangle = \text{diag}[\bar{\Phi}_1, \bar{\Phi}_2, \bar{\Phi}_3] \) with \( \bar{\Phi}_1 = \bar{\Phi}_2 \neq \bar{\Phi}_3 \).

All the introduced parameters \( \mu^2, \lambda_{1,2}, B, cm_{l,s}, k \), and \( \theta \) are taken to be real and positive.

**NONPERTURBATIVE FLAVOR BREAKING IN TOPOLOGICAL SUSCEPTIBILITY**

To perform nonperturbative calculation of thermal loop contributions, we work on the CJT formalism \([10]\) by using the imaginary time formalism. The meson propagators contributing to thermal loops are then treated as full propagators determined by stationary conditions for the CJT effective potential \( V_{CJT} \), simultaneously with the vacuum expectation values \( \bar{\Phi}_{1,3} \). Details are to be reported elsewhere \([25]\), instead, we shall here just show the result on the numerical analysis of the CJT formalism.

Since the CJT effective potential depends on the theta parameter \( \theta \) through the quark mass matrix \( M \) as noted above, the topological susceptibility is straightforwardly computed as

\[
\chi_{\text{top}} = -\frac{\partial^2 V_{CJT}}{\partial \theta^2} \bigg|_{\theta=0} = \left( \frac{2\langle l \rangle}{m_l} + \frac{\langle s \rangle}{m_s} \right) \bar{m}^2(u, d, s),
\]

which is precisely in accord with Eq. (1). Terms suppressed by the higher order in \( m_s \) have been omitted, as in Eq. (1). The quark condensates are calculated also through the \( V_{CJT} \) as

\[
\langle l \rangle = \frac{\partial V_{CJT}}{\partial m_l} = -2e(\bar{\Phi}_1 + 2k\bar{\Phi}_1\bar{\Phi}_3),
\]

\[
\langle s \rangle = \frac{\partial V_{CJT}}{\partial m_s} = -2e(\bar{\Phi}_3 + 2k\bar{\Phi}_3^2). 
\]

Input parameters have been chosen by following the literature \([24]\) as \( \mu^2 = 0.54 \times 10^8 \text{MeV}^2, \lambda_1 = 12.0, \lambda_3 = 20.9, cm_l = 6.04 \times 10^8 \text{MeV}^3, cm_s = 194 \times 10^5 \text{MeV}^3, B = 3.83 \times 10^4 \text{MeV}, k = 3.58 \text{GeV}^{-1} \), which well reproduce the scalar and pseudoscalar spectroscopy up to the mass scale of 1 GeV at the vacuum \([24]\).

By increasing temperature, we find the crossover "phase transition" for the chiral symmetry in \([25]\). Similar crossover phenomenon has also been observed in other three-flavor models based on the CJT formalism \([27, 28]\). The pseudo-critical temperature \( T_{pc} \) can simply be identified by \( d^2 \langle l \rangle(T)/dT^2\big|_{T=T_{pc}} = 0 \), to be \( \approx 215 \text{MeV} \) in the present analysis. This crossover phenomenon is in a qualitative sense consistent with the current result of the lattice QCD with 2 + 1 flavors \([20, 23]\). Note that the definition of our pseudo-critical temperature \( T_{pc} \) is different from the lattice QCD’s and has been estimated to be larger than the lattice QCD value \([20, 23]\); by construction of the present chiral effective model with single current quark mass matrix \( M \) only included in operators, we cannot evaluate the chiral susceptibility, through which the pseudo-critical temperature in the lattice simulation is defined.

To grasp the flavor breaking effect on the topological susceptibility in Eq. (1), we first show enhancement of the flavor breaking in the quark condensates. See Fig. 1, where we plot the ratio of quark condensates as a function of the temperature, \( \langle s \rangle(T)/\langle l \rangle(T) \). Around the pseudo-critical temperature \( T_{pc} \approx 215 \text{MeV} \), \( \langle s \rangle(T)/\langle l \rangle(T) \) is drastically enhanced and monotonically increased, so that the violation of flavor symmetry between the light quark and the strange quark condensates gets more eminent in the high temperature region.

Now we see that the significance of flavor breaking is reflected in the topological susceptibility as depicted in Fig. 2. To extract the strange quark contribution to the topological susceptibility, we compare \( \chi_{\text{top}} \) with the
As to the vacuum value of $\chi$, FIG. 2: The temperature dependence of the three-flavor universal limit, $\chi_{3fl}(T)/\chi_{3fl}(T = 0)$.

Due to the sizable strange quark condensate as seen from Fig. 1, the topological susceptibility is enhanced by about 15-30% for $T/T_{pc}^* \sim 1.5 - 2$, and increases monotonically as $T$ develops. This tendency presumably continues until $(\langle s\bar{s}\rangle(T))$ drops.

Fig. 3 shows the $\chi_{top}(T)$ normalized to the vacuum value, in comparison with the ChPT prediction up to the next-leading order (one loop) [20] and the recent lattice data with $2+1$ (+1) flavors having a physical pion mass and the continuum limit being taken [11, 12] #2. Remarkably, the predicted $T$ dependence (denoted by “CJT” in the figure) is actually slower-damping, and is overall consistent with the lattice QCD data [11, 12], which is not realized by the ChPT. This would manifest the importance of nonperturbative thermal contribution including the enhanced flavor breaking by $(\langle s\bar{s}\rangle(T))/\langle\bar{l}l\rangle(T) \gg 1$ above $T_{pc}^*$ as depicted in Fig. 1.

The predicted curve does not follow the dilute instanton gas approximation [20, 21] in whole temperature range [12], rather favors the ChPT at $T \lesssim T_{pc}^*$: This implies that the chiral symmetry for light quarks might be essential in the low-temperature region, as discussed in the literature [20]. Beyond the ChPT-governed domain, above $T_{pc}^*$ the strange quark condensate would serve as an important source to develop the topological susceptibility, as the consequence of the nonperturbative flavor breaking.

CONCLUSION

We have shown the enhanced flavor breaking in the topological susceptibility through nonuniversal quark condensates, which would be crucial to understand the hot QCD with the theta parameter, especially at around the hot QCD criticality, and hence the associated $U(1)_A$ anomaly as well. In particular, the lattice QCD measurement of strange quark condensate at around and over the pseudo critical temperature would be crucial to check how the nonperturbative flavor breaking is critically operative in the topological susceptibility. This critical phenomenon should directly be observed in the future lattice QCD with appropriate accuracy.

Our proposal has been explicitized using some specific chiral effective model, especially including an axial anomaly-induced flavor breaking term (Eq. 7). We have checked that even without this speciality, our finding is substantially unchanged, which will in detail be reported elsewhere [23].

The nonperturbative flavor breaking, especially, the significance of the strange quark condensate, in the topological susceptibility at around the pseudo critical temperature would give an impact on applications to QCD axion cosmological models, and could also be relevant to heavy-ion collision phenomenology. Those directions deserve another publication.

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