We show that a non-trivial topological effect breaks the conformal invariance of pure Yang-Mills theory. Thus it is possible that classic particle-like solutions exists in pure Yang-Mills theory. We find a static, non-singular solution in source-free SU(2) Yang-Mills theory in four-dimensional Minkowski space. This solution is a stable soliton characterized by non-trivial topology and imaginary $A_{\mu}^a < 0$. It yields hermitian Hamilton, and finite, positive energy.
\[ T_{\mu
u} \rightarrow T_{\mu
u} + \partial^{\lambda}(F_{\mu\lambda}^a A_{\nu}^a) = \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta}^a F^{\alpha\beta a} - F_{\mu\beta}^a F^{\mu\beta a}. \] (3)

Eq. (3) leads to \( T_{\mu}^\mu = 0 \) in four-dimensional spacetime. However, for the case of non-trivial topology presence (eq. (1)), this surface term cannot be added, since it obviously changes physical energy-momentum. So that \( T_{\mu}^\mu = 0 \) cannot be yielded in this case.

When Yang-Mills fields are static, the Coleman-Deser equation is

\[ 2 \int d^3 \vec{x} T_i^i = \int d^3 \vec{x} \left( \frac{1}{2} F_{ij}^a F_{ij}^a - F_{i0}^a F_{i0}^a - \frac{1}{2} \right) = 0 \] (4)

can be obtained even though the non-trivial topology is present, since we need only static Yang-Mills equations and requirement of finite energy to obtain it. This equation provides some constrains for static solution of Yang-Mills equations. For trivial topology, the term \( F_{i0}^a F_{i0}^a \) vanishes due to static Yang-Mills equations. It leads to the trivial solution \( F_{i0}^a = 0 \). Meanwhile, the static Hamilton density

\[ H = \frac{1}{4} F_{ij}^a F_{ij}^a - \frac{1}{2} F_{i0}^a F_{i0}^a \] (5)

implies that only \( F_{i0}^a F_{i0}^a < 0 \) is allowed. It indicates that the static solution exists only for imaginary \( A^a_0 \). It means that \( A^a_0 \) have to be continued to complex plane analytically. The complex solutions were studied by Dolan in Euclidian space. He obtained some Abelian-like solutions with zero action. In this present paper, we will actually study imaginary solution of \( A_0 \) with in Minkowski space. It will yield finite, no-zero energy. In the static pure Yang-Mills theory, at least, this imaginary \( A^a_0 \) is allowed by all fundamental principle, such as positivity of energy, hermitian of Hamilton, etc.

In the following we will try to find an analytic, static, non-singular soliton solution of source-free SU(2) Yang-Mills equations. We can define “static dual” in Minkowski space,

\[ \tilde{F}_{ij}^a = i \epsilon_{ijk} F_{ko}^a, \quad \tilde{F}_{k0}^a = - \frac{i}{2} \epsilon_{ijk} F_{ij}^a. \] (6)

where \( a = 1, 2, 3 \). The “static dual” in the above equation satisfy

\[ \tilde{\tilde{F}}_{ij}^a = \tilde{F}_{ij}^a, \quad \tilde{\tilde{F}}_{i0}^a = \tilde{F}_{i0}^a. \] (7)

It is easily to check that, the static Yang-Mills equations and Coleman-Deser equation (4) will be automatically satisfied if field strengths are “static self-dual”

\[ F_{\mu\nu}^a = \tilde{F}_{\mu\nu}^a. \] (8)

In addition, the “static self-dual” also yielded minimum energy of the system.1

For obtaining an explicit analytic solution, in this paper we take a spherical symmetry ansatz for gauge fields

\[ \begin{cases} A_{ia} = \frac{f(r)}{g(r)} \epsilon_{ian} \hat{x}_n, \\ A_{0a} = \frac{i}{g(r)} \phi(r) \hat{x}^a, \end{cases} \] (9)

Than eqs. (8) and (9) lead to

1The solutions of eq. (8) may be named as “static instanton”. However, it must be pointed out that the usual static instanton is not well-defined: The instanton is related to tunneling effects of quantum mechanics which exists only in Euclidean space (imaginary time). In the case absence time, there exists no tunneling effects, so that solutions of eq (8) is different from traditional instanton.
\[
\begin{aligned}
&\begin{cases}
f' = -\phi(1 - 2f), \\
\phi' = -\frac{2}{r^2}f(1 - f),
\end{cases}
\end{aligned}
\]
with \( f' = \frac{d}{dr}f \), \( \phi' = \frac{d}{dr}\phi \). To obtain the above equation, the identity,
\[
\epsilon_{jan}\tilde{x}_i\tilde{x}_n - \epsilon_{ian}\tilde{x}_j\tilde{x}_n + \epsilon_{ijn}\tilde{x}_a\tilde{x}_n = \epsilon_{ija},
\]
has been used. The eq. (10) can reduce to static Liouville equation \([7]\). Its solutions are well-known. An analytic, non-singular solution is,
\[
\begin{aligned}
f &= \frac{1}{2}(1 - \frac{\kappa r}{\sinh(\kappa r)}), \\
\phi &= \frac{1}{2r}(1 - \frac{\kappa}{\tanh(\kappa r)}),
\end{aligned}
\]
where \( \kappa \) is a positive integral constant with mass-dimension. The solutions (12) lead to their asymptotic behaviour as follows
\[
\begin{aligned}
f &\xrightarrow{r \to 0} \frac{1}{12}\kappa^2 r^2, \\
f &\xrightarrow{r \to \infty} \frac{1}{2} - e^{-\kappa r}, \\
\phi &\xrightarrow{r \to 0} \frac{1}{6}\kappa^2 r, \\
\phi &\xrightarrow{r \to \infty} \frac{\kappa}{2} + \frac{1}{2r}.
\end{aligned}
\]
Thus it is suprised that \( A_0 \to \) constant instead of zero at \( r \to \infty \) (it is different from instanton and monopole), but while all \( F_{\mu\nu} \) still fall off as \( r^{-2} \).

Due to “static self-dual” of field strength, the topological mass (energy) of the soliton is
\[
M = \int d^3\tilde{x}H = -\int d^3\tilde{x}\tilde{F}_0^a\tilde{F}_0^a = -\int d^3\tilde{x}\partial_i(A_0^a\tilde{F}_0^a)
= -\int d^3\tilde{x}\partial_i(A_0^a\partial_iA_0^a) = \frac{\pi}{g^2}\kappa.
\]
Therefore, the integral constant \( \kappa \) can be interpreted as an unit of energy. Moreover, we can see that the topological effect in our solution is characterized by the topological mass, which is determined by asymptotic behaviour of \( A_0^a \) at \( r \to \infty \). The energy density distribution function is
\[
\rho(r) = -\tilde{F}_0^a\tilde{F}_0^a = g^{-2}\{\phi\phi'' + \frac{2}{r}\phi\phi' + \phi'^2\}.
\]
The energy density distribution is shown in fig. 1. Effective radius of the soliton (soliton size) is usually defined as half maximum width of spectral distribution. Then From the fig. 1 we can see that the effective radius of this soliton is \( r_0 = 1.3\kappa^{-1} \). It is interesting that \( r_0M \) is \( \kappa \)-independent. The fig. 1 also shows that this soliton is stable for any boundary conditions, i.e., fixed \( \kappa \).

We have shown that there exist static topological solution with finite energy in pure Yang-Mills theory. This fact indicates that the massless gauge particles in Yang-Mills theory can become static, massive particles due to non-linear self-interaction. It is interesting to compare with U(1) electromagnetic theory, in which the photon can not be static and massive forever. In our solution, the zero component of Yang-Mills field is imaginary. However, it does not cause any problem for static pure Yang-Mills theory. In fact, the ansatz (11) naturally satisfies the Coulomb gauge condition, \( \partial_iA_i = 0 \). Then for the static case, Lorentz condition \( \partial_\mu A^\mu = 0 \) allows \( A_0 \) to be arbitrary time-independent function (no matter what it is real or imaginary).

The confinement of gauge particles is an active subject (specially in quantum chromodynamics). Although the solution (12) yields finite energy, the confinement of gauge particles is still allowed here. There are two different mechanisms to achieve the confinement: The first one is to take a limit of solution (12), i.e., \( \kappa \to \infty \). It is obvious that, in this limit topological mass \( M \to \infty \) and effective radius \( r_0 \to 0 \). It means that the static point-like particles will generate infinite energy. The second one is that there exist some other singular solutions for eq. (12). For example, the solution
\[
f = r\phi = \frac{\lambda}{r + 2\lambda}
\]
leads gauge fields to be singular at \( r = 0 \). Thus energy obtained from solution (16) is infinite. These two mechanisms can be distinguished by asymptotic behaviour of \( \phi \) (or \( A_0 \)) at \( r \to \infty \). In the first one, \( \phi \to \infty \) and \( \phi' \to 0 \) at \( r \to \infty \) when \( \kappa \to \infty \). Meanwhile, in the second one, \( \phi \to r^{-2} \to 0 \) at \( r \to \infty \). The above discussion indicates that both of confinement and no-confinement are allowed in pure Yang-Mills theory. Whether or not presence of confinement is determined by asymptotic behaviour of \( A_0^a \) at \( r \to \infty \).

To conclude, we obtain a static spherical symmetry no-singular soliton solution in source-free Yang-Mills theory when a non-trivial topology is present. This topological soliton is stable, and has finite, positive and minimum mass (energy). Value of the topological mass can be fixed by effective radius of the solution. At the limit of point-particle, the gauge fields will be confinement (with infinite energy). The topological effect is determined by asymptotic behaviour of \( A_0^a \) at \( r \to \infty \).

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