Pseudo-orthogonal groups and integrable dynamical systems

in two dimensions

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Integrable systems in low dimensions, constructed through the symmetry reduction method, are studied using phase portrait and variable separation techniques. In particular, invariant quantities and explicit periodic solutions are determined. Widely applied models in Physics are shown to appear as particular cases of the method.

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I. INTRODUCTION

Integrable Hamiltonian systems play a fundamental role in the study and description of physical systems, due to their many interesting properties, both from the mathematical and physical points of view. The construction of such models represents a contribution to this field, and many of them have proved to be of an extraordinary physical interest. Let us remind here the Morse [1] and Pöschl-Teller [2] potentials in one dimension, or the Calogero [3–5] and Sutherland [6] potentials. These constructions have been also considered from
many points of view. See for instance the reviews \[7\] and \[8\].

A method to construct these systems is the use of the Marsden-Weinstein reduction procedure \[9\], or its extensions \[10\], to free Hamiltonians lying on a $N$-dimensional homogeneous space under a suitable Lie group. In this way (using an appropriate momentum map), one assures the integrability, or even the superintegrability \[11\] of the system. In the first case, there exist $N$ constants of motion in involution, one of them the Hamiltonian. The superintegrability requires more than $N$ constants of motion (not all of them in involution) and more than one subset of $N$ constants in involution. There are good reasons to suspect that any integrable system may be constructed in this way, as a reduction of a free one \[10\], so the problem to construct these systems and study their properties is a profitable and very interesting field. A related topic is the problem of separation of variables for the associated Hamilton-Jacobi (HJ) equations. As it is well known, the existence of quadratic invariants allows to classify and construct these systems \[12,13\], relating them in many occasions to subgroups of the invariance group.

A series of articles appeared in the last years, has been devoted to the study of these superintegrable systems constructed using homogeneous spaces over the pseudounitary groups $SU(p, q)$ \[14,15\]. In particular, using the maximal Abelian subalgebras (MASA) of the corresponding algebras, one can build a family of integrable systems of arbitrary dimension, and present their invariants and the coordinate systems in which the HJ equation is separated. The explicit solutions and a unifying view of the compact Cartan subalgebra case have been presented in \[16\].

Our aim in this article is to work in detail the low dimensional cases. The reason is twofold. On one side, the one-dimensional case allows an easy geometric description of the systems, through their phase portrait. The potentials we obtain are not new, but have been applied successfully in many physical models (for instance, the Pöschl-Teller and Morse potentials). They also appear in the study of quasi-exactly solvable (QES) models \[17,18\], as the case of exactly solvable systems, providing examples in which, from the quantum point of view, the corresponding Schrödinger equation can be solved algebraically (a part
of the spectrum for QES systems or an arbitrary number of states for the exactly solvable ones). On the other side, the 2-dimensional case can be studied from the point of view of variable separation, and we can solve the HJ equation in a wide class of coordinate systems, specially in the noncompact case [19,20]. The results we present here (in the 2- dimensional case) should be considered in a local context. Considerations about global behavior, which will differ from the compact to the noncompact case, will not be addressed in this work.

The article is organized as follows. In Section 2 we present a concise description of the method used to construct these Hamiltonian systems. Section 3 is devoted to the one-dimensional case, while the 2-dimensional case is studied in Section 4. In each case we present the list of all the conserved quantities for these systems in terms of the generators of the corresponding algebras, and the explicit form in the chosen coordinate system. Conclusions and further outlook of this research are discussed in Section 5.

II. INTEGRABLE HAMILTONIAN SYSTEMS AND PSEUDOUNITARY GROUPS

The results presented in this section are a summary of the contents of [14,15]. We will include some of them to set the notations which will be used in the following sections.

We will consider the free Hamiltonian \((\mu, \nu = 0, \ldots, N = p + q - 1)\):

\[
H = 4cg^{\mu\nu}p_\mu \bar{p}_\nu
\]  

(2.1)

(the bar denoting complex conjugate) defined in the Hermitian hyperbolic space (with coordinates \(y^\mu \in \mathbb{C}\), satisfying \(g_{\mu\nu} \bar{y}^\mu y^\nu = 1\), and conjugate momenta \(p_\mu\)):  

\[
SU(p, q)/SU(p - 1, q) \times U(1)
\]  

(2.2)

whose geometry is described in [21]. The real constant \(c\) is related to the sectional curvature of the Hermitian space. See also [16] for a detailed analysis of this space and its properties.

Using a maximal abelian subalgebra of \(su(p, q)\) [22], we carry a reduction procedure
In order to obtain a reduced Hamiltonian (which is not free) in the reduced space, a homogeneous $SO(p, q)$ space:

$$H = c \left( \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + V(s) \right)$$  \hspace{1cm} (2.3)$$

where $V(s)$ is a potential depending on the real coordinates $s^\mu$. The set of complex coordinates $y^\mu$ is transformed in the reduction procedure into a set of ignorable variables $x^\mu$ (which are the parameters of the transformation associated to the MASA of $u(p, q)$ used in the reduction) and the coordinates $s^\mu$ with the constraint $g_{\mu\nu} s^\mu s^\nu = 1$.

If $Y_\mu$, $\mu = 0, \ldots, N$, is a basis of the considered MASA of $u(p, q)$, formed by pure imaginary matrices, the relation between old $(y^\mu)$ and new coordinates $(x^\mu, s^\mu)$ is:

$$y^\mu = B(x)_\mu^\nu s^\nu, \quad B(x) = \exp(x^\mu Y_\mu)$$  \hspace{1cm} (2.4)$$

which assures the ignorability of the $x$ coordinates (the vector fields corresponding to the MASA are straightened out in these coordinates). The Jacobian matrix, $J$, is easily obtained. If:

$$A^\mu_\nu = \frac{\partial y^\mu}{\partial x^\nu} = (Y_\nu)_\mu^\rho y^\rho$$  \hspace{1cm} (2.5)$$

then:

$$J = \frac{\partial (y, \bar{y})}{\partial (x, s)} = \begin{pmatrix} A & B \\ \bar{A} & \bar{B} \end{pmatrix}.$$  \hspace{1cm} (2.6)$$

The Hamiltonian calculated in the new coordinates is written as:

$$H = c \left( \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + V(s) \right), \quad V(s) = p_x^T (A^\dagger KA)^{-1} p_x$$  \hspace{1cm} (2.7)$$

where $p_x$ are the constant momenta associated to the ignorable coordinates $x$ and $K$ is the matrix defined by the metric $g$.

Note that, to obtain these Hamiltonians, we need MASAs of $su(p, q)$ of dimension $N = p + q - 1$ (corresponding to MASAs of $u(p, q)$ of dimension $p + q$). We also require that these MASAs have a representation in terms of imaginary matrices that allows to write the
Hamiltonian in the form \( (2.7) \). Once we have chosen a particular MASA, we can obtain a set of invariants and also the corresponding coordinate systems in which the HJ equation separates. The MASAs of \( su(p, q) \) are classified in [22], and for low ranks are completely determined. The corresponding potentials have been obtained for \( SU(N) \) in [19], for \( SU(2, 1) \) in [20], for \( SU(2, 2) \) in [15] and for any \( SU(p, q) \), choosing as MASA one of the Cartan subalgebras, in [14]. From now on, we will always use contravariant coordinates, but we will write the indices as subscripts in order to simplify the notation and avoid the use of unnecessary brackets.

### III. ONE-DIMENSIONAL HAMILTONIANS

One-dimensional Hamiltonians are always integrable and the phase portrait gives a complete description of the allowed motions. We shall expose the main ideas in order to achieve a better understanding of the more complicate systems we will study in the next section. We have two cases: \( su(2) \) and \( su(1, 1) \).

#### A. \( su(2) \)

We will use as a basis for \( su(2) \) the operators \( X_1, X_2, X_3 \), which are given in the natural 2 × 2 matrix representation by:

\[
X_1 \rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 \rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]

in the metric \( K = \text{diag}(1, 1) \).

In the compact algebra \( su(2) \) there is only one class of MASAs, corresponding to the Cartan subalgebra (CC) [22]. A basis of a representative of this class of MASA is:

\[
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]

(3.1)

A basis of the corresponding MASA of \( u(2) \) can be chosen as:
\[
Y_0 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}.
\] (3.2)

The relation between old and new coordinates is given by the matrix \( B(x) \) in (2.4). Note that a change of basis in the corresponding MASA of \( u(2) \) changes only the parameters appearing in the potential:

\[
y_0 = s_0 e^{ix_0}, \quad y_1 = s_1 e^{ix_1}.
\] (3.3)

The Hamiltonian, following the general expression (2.7), is:

\[
H = c \left[ \frac{1}{2} (p_{s_0}^2 + p_{s_1}^2) + V(s) \right], \quad V(s) = \frac{m_0^2}{s_0^2} + \frac{m_1^2}{s_1^2}
\] (3.4)

with the constraint \( s_0^2 + s_1^2 = 1 \). Parameterizing the circle \( S^1 \) in spherical coordinates: \( s_0 = \cos \phi, \ s_1 = \sin \phi \), we get the Hamiltonian \((c = 1)\):

\[
H(\phi) = \frac{1}{2} p_\phi^2 + V(\phi), \quad V(\phi) = \frac{m_0^2}{\cos^2 \phi} + \frac{m_1^2}{\sin^2 \phi}.
\] (3.5)

We have only one second order conserved quantity, the Hamiltonian, which is equal to the Casimir of the algebra, \( C \), up to an additive constant:

\[
\hat{Q}_1 = X_2^2 + X_3^2.
\] (3.6)

The square of the generator in the compact Cartan subalgebra, \( C_1 = X_1^2 \), is constant after the reduction and \( C = C_1 + \hat{Q}_1 \).

The specific values of the real positive constants \( m_0, m_1 \) play no essential role in the qualitative description of the orbits and trajectories of this system. The potential has singularities (in the generic case) in \( \phi = 0, \pi/2, \pi, 3\pi/2 \). When \( m_0 \) or \( m_1 \) are equal to zero we have only two singularities in \( 0, \pi \text{ or } \pi/2, 3\pi/2 \), respectively.

The particles are confined inside a sector, and there, the motion is periodic, with an equilibrium point (a center in the phase space) corresponding to the unique minimum of the potential, in \( \tan \phi = \sqrt{m_1/m_0} \). The solution can be easily computed, using Hamilton equations. The potential is bounded from below, and the energy is always positive \((E \geq \)
\((m_0 + m_1)^2\). Though the use of HJ equation is not necessary in this context of one-dimensional systems, we will write down the equation in order to compare with the two dimensional cases. In fact, when we will make separation of variables there, we will find again this equation:

\[
\frac{1}{2} \left( \frac{\partial S}{\partial \phi} \right)^2 + \frac{m_0^2}{\cos^2 \phi} + \frac{m_1^2}{\sin^2 \phi} = E
\]  

(3.7)

with solution \((u = \cos^2 \phi)\):

\[
u = \frac{1}{2E} \left( b + \sqrt{b^2 - 4m_0^2E \cos 2\sqrt{2Et}} \right)
\]

(3.8)

and \(b = m_0^2 - m_1^2 + E\). This solution is obviously much simpler to find if we consider the equation of orbits in the phase portrait of this system.

For instance, if \(m_0 = 0\), \(m_1 = 1\), the solution is:

\[s_0 = \cos \phi = \sqrt{1 - \frac{1}{E} \cos 2\sqrt{2Et}}
\]

(3.9)

that is, a system with similar solutions to a harmonic oscillator, but now the frequency depends on the energy.

B. \(su(1,1)\)

The noncompact algebra \(su(1,1)\) has three nonconjugate classes of MASAs, compact Cartan subalgebra, noncompact Cartan subalgebra and a class of nilpotent maximal abelian subalgebras, (MANS) \[22\]. We will fix the metric to be:

\[
K = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

(3.10)

and the basis \(\{X_1, X_2, X_3\}\) is given in the 2 \(\times\) 2 matrix representation through the correspondence:

\[
X_1 \rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_3 \rightarrow \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
\]
I. Compact Cartan subalgebra (CC)

We choose, as representative of this class, the same matrices as in (3.2). Hence, the old and new coordinates are related in the same way they did in the $su(2)$ case (3.3), and the Hamiltonian is now:

$$H = c \left[ \frac{1}{2}(p_{s_0}^2 - p_{s_1}^2) + V(s) \right], \quad V(s) = \frac{m_0^2}{s_0^2} - \frac{m_1^2}{s_1^2} \quad (3.11)$$

with the constraint $s_0^2 - s_1^2 = 1$. This hyperbola can be described with a coordinate $\phi$ varying in the real line: $s_0 = \cosh \phi$, $s_1 = \sinh \phi$, and the Hamiltonian in these coordinates is ($c = -1$):

$$H(\phi) = \frac{1}{2}p_\phi^2 + V(\phi), \quad V(\phi) = -\frac{m_0^2}{\cosh^2 \phi} + \frac{m_1^2}{\sinh^2 \phi}. \quad (3.12)$$

The second order invariant (the Hamiltonian) is:

$$\hat{Q}_1 = X_2^2 + X_3^2 \quad (3.13)$$

and the trivial constant associated to the MASA is $C_1 = X_1^2$. Hence, the Casimir in terms of these two quantities is $C = C_1 - \hat{Q}_1$.

The HJ equation is:

$$\frac{1}{2} \left( \frac{\partial S}{\partial \phi} \right)^2 - \frac{m_0^2}{\cosh^2 \phi} + \frac{m_1^2}{\sinh^2 \phi} = E \quad (3.14)$$

and the solution depends on the values of $E$ and the parameters.

Considering different values of the parameters $m_0, m_1$ we obtain three different systems.

a) If $m_1 \neq 0$ the potential has a singularity in $\phi = 0$. It is easy to check that, if $m_1 \geq m_0$, there are no minima for the potential and all the motions are unbounded (with a turning point). The energy is always positive. The parameters $m_0, m_1$ do not modify qualitatively the phase portrait or the form of the solutions. The solution can be written as ($u = \cosh^2 \phi$):

$$u = \frac{1}{2E} \left( -b + \sqrt{b^2 + 4m_0^2E \cosh 2\sqrt{2Et}} \right) \quad (3.15)$$
and \( b = m_0^2 - m_1^2 - E \). If \( m_0 = 0, m_1 = 1 \), the solution is:

\[
    s_0(t) = \sqrt{1 + \frac{1}{E} \cosh \sqrt{2Et}}. \tag{3.16}
\]

b) If \( m_0 > m_1 > 0 \), the potential has two minima, symmetric respect to the origin where it has the singularity. The energy is bounded from below: \( E \geq -(m_0 - m_1)^2 \), the value \( E = -(m_0 - m_1)^2 \) corresponding to the equilibrium solution in the center of the phase space. The other solutions are easily calculated (\( b \) has the same value as in case a)):

i) \(-(m_0 - m_1)^2 < E < 0\)

\[
    s_0 = \frac{1}{\sqrt{2|E|}} \left[ b + \sqrt{b^2 + 4Em_0^2 \cos 2\sqrt{2|E|t}} \right]^{1/2}. \tag{3.17}
\]

ii) \( E = 0 \)

\[
    s_0 = \left[ m_0^2 - m_1^2 + 2(m_0^2 - m_1^2)t^2 \right]^{1/2}. \tag{3.18}
\]

When \( E > 0 \) we get the solution (3.16).

c) If \( m_1 = 0 \) there is no singularity in the potential, which has a minimum in \( \phi = 0 \), with periodic motions of negative energy and unbounded motions of positive or zero energy. The multiplicative constant \( m_0 \) plays no essential role for the qualitative description of the system. The solutions can be read off from case b) with \( m_1 = 0 \).

II. Noncompact Cartan subalgebra (NC)

A representative subalgebra of this class has the basis (in the metric (3.10)):

\[
    Y_1 = \begin{pmatrix}
        0 & i \\
        -i & 0
    \end{pmatrix} \tag{3.19}
\]

and we will add the matrix \( Y_0 = iI \) to get a MASA of \( u(1,1) \). The new and old coordinates are related in a slightly more complicated way:

\[
    y_0 = e^{ix_0}(s_0 \cosh x_1 + is_1 \sinh x_1), \quad y_1 = e^{ix_0}(-is_0 \sinh x_1 + s_1 \cosh x_1) \tag{3.20}
\]

and the Hamiltonian is written in the new coordinates as:
\[ H = c \left[ \frac{1}{2}(p_{s_0}^2 - p_{s_1}^2) + V(s) \right], \quad V(s) = \frac{m_0^2 - m_1^2 + 4m_0m_1s_0s_1}{1 + 4s_0^2s_1^2} \] (3.21)

and the constraint \( s_0^2 - s_1^2 = 1 \). Using again the \( \phi \) coordinate as in the previous case, we get:

\[ H(u) = \frac{1}{2}p_\phi^2 + V(\phi), \quad V(\phi) = -\frac{m_0^2 - m_1^2 + 2m_0m_1 \sinh 2\phi}{\cosh^2 2\phi}. \] (3.22)

The Casimir is written as \( C = \hat{Q}_1 - C_1 \) where \( C_1 = X_3^2 \), the square of the generator of the noncompact Cartan subalgebra, and

\[ \hat{Q}_1 = X_1^2 - X_2^2 \] (3.23)

which is equal to the Hamiltonian.

We will also write down HJ equation for future references:

\[ \frac{1}{2} \left( \frac{\partial S}{\partial \phi} \right)^2 - \frac{m_0^2 - m_1^2 + 2m_0m_1 \sinh 2\phi}{\cosh^2 2\phi} = E. \] (3.24)

If \( m_0 \) or \( m_1 \) are equal to 0, we obtain similar results to those of the case of compact Cartan MASA, an attractive potential if \( m_0^2 > m_1^2 \) and a repulsive one in the opposite case. If \( m_0m_1 \neq 0 \), the potential is qualitatively the same for all values of \( m_0 \) and \( m_1 \).

The potential \( V(\phi) \) has two extrema in \( \sinh 2\phi = m_1/m_0, -m_0/m_1 \). The first point corresponds to a minimum (a center in the phase portrait), and the potential takes the value \( V = -m_0^2 \). The second point is a maximum (a saddle point in the phase portrait), and \( V = m_1^2 \) there. The energy is bounded from below \( (E \geq -m_0^2) \) and the explicit solutions are \( (u = \sinh 2\phi) \):

i) \( -m_0^2 < E < 0 \)

\[ u = \frac{1}{|E|} \left[ m_0m_1 + \sqrt{(E + m_0^2)(m_1^2 - E)} \cos 2\sqrt{2|E|t} \right]. \] (3.25)

ii) \( E = 0 \)

\[ u = -\frac{m_0^2 - m_1^2}{2m_0m_1} + 4m_0m_1t^2. \] (3.26)

iii) \( 0 < E < m_1^2 \)
\[ u = \frac{1}{E} \left[ -m_0 m_1 + \sqrt{(E + m_0^2)(m_1^2 - E)} \cosh 2\sqrt{2|E|}t \right]. \]  \hfill (3.27)

iv) \( E = m_1^2 \)

\[ u = -m_0 + e^{2\sqrt{2}m_1 t}. \]  \hfill (3.28)

v) \( E > m_1^2 \)

\[ u = \frac{1}{E} \left[ -m_0 m_1 + \sqrt{(E + m_0^2)(E - m_1^2)} \sinh 2\sqrt{2}E t \right]. \]  \hfill (3.29)

III. Nilpotent subalgebra (NIL)

Though the simplest representative of this class of subalgebras is obtained in the skew-diagonal metric, we will use again the diagonal one, because in this way, the kinetic term is also diagonal. We will take as a basis:

\[ Y_1 = \begin{pmatrix} i & i \\ -i & -i \end{pmatrix} \]  \hfill (3.30)

which is a nilpotent matrix. As in the noncompact case we will also use \( Y_0 = iI \) to complete the basis of a \( u(1, 1) \) MASA. Old and new coordinates satisfy:

\[ y_0 = e^{ix_0}((1 + ix_1)s_0 + ix_1 s_1), \quad y_1 = e^{ix_0}(-ix_1 s_0 + (1 - ix_1) s_1). \]  \hfill (3.31)

The Hamiltonian is:

\[ H = c \left[ \frac{1}{2}(p_{s_0}^2 - p_{s_1}^2) + V(s) \right], \quad V(s) = \frac{2m_0 m_1}{(s_0 + s_1)^2} - \frac{m_1^2}{(s_0 + s_1)^4} \]  \hfill (3.32)

with the constraint (which is the same for all the subalgebras in the \( su(1, 1) \) case, as we are using the same metric): \( s_0^2 - s_1^2 = 1 \). The expression of the Hamiltonian in terms of the \( \phi \) coordinate (\( c = -1 \)) is:

\[ H(\phi) = \frac{1}{2} p_\phi^2 + V(\phi), \quad V(\phi) = m_1^2 e^{-4\phi} - 2m_0 m_1 e^{-2\phi}. \]  \hfill (3.33)

The Hamiltonian in terms of the second order operators in the enveloping algebra is again \( \{X_i, X_j\} = X_i X_j + X_j X_i \):
\[ \hat{Q}_1 = 2X_1^2 + \{X_1, X_3\} - X_2^2 \]  

(3.34)

and the trivial constant \( C_1 \) is equal to \((X_1 + X_3)^2\), with \( C = \hat{Q}_1 - C_1 \).

The HJ equation is:

\[ \frac{1}{2} \left( \frac{\partial S}{\partial \phi} \right)^2 - 2m_0m_1e^{-2\phi} + m_1^2e^{-4\phi} = E. \]  

(3.35)

We will assume \( m_1 \neq 0 \) to obtain nontrivial results. Depending on the constants \( m_0, m_1 \) we get essentially two classes of systems:

1) If \( m_0m_1 \leq 0 \) there is no extremum for the potential, and the energy is always positive. The solutions, with a unique turning point, are given by:

\[ e^{2\phi} = \frac{m_1}{E} \left[-m_0 + \sqrt{E + m_0^2 \cosh \sqrt{2E}t} \right]. \]  

(3.36)

2) If \( m_0m_1 > 0 \) the potential has a minimum, in \( \phi = (1/2) \log(m_1/m_0) \) and the energy is bounded from below, \( E > -m_0^2 \). As in the first case, the values of the parameters are not essential if they satisfy the constraints. The solutions for the different values of the energy are:

i) \(-m_0^2 < E < 0\)

\[ e^{2\phi} = \frac{m_1}{|E|} \left[m_0 + \sqrt{E + m_0^2 \cos \sqrt{2E}t} \right]. \]  

(3.37)

ii) \( E = 0 \)

\[ e^{2\phi} = \frac{m_1}{2m_0} + 4m_0m_1t^2. \]  

(3.38)

If \( E > 0 \) the solution is the same as (3.36).

This case completes the set of one-dimensional Hamiltonians obtained through a reduction procedure out of free Hamiltonians, invariant under \( SU(p, q) \), \( p + q = 2 \), and defined over a homogeneous space of the corresponding group. In Table I, we present a summary of these Hamiltonians in the one dimensional case.

In the next section we will treat the 2-dimensional Hamiltonians associated to the rank 2 algebras \( su(3) \) and \( su(2, 1) \).
IV. THE 2-DIMENSIONAL CASE

There are two pseudounitary algebras to be used to construct superintegrable Hamiltonians of dimension 2, $su(3)$ and $su(2,1)$. We will treat separately both cases.

A. $su(3)$

We will use as a basis for $su(3)$ the operators $\{X_1, \ldots, X_8\}$ which are given in the $3 \times 3$ matrix representation by:

\[
X_1 \rightarrow \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_2 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix},
X_3 \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_4 \rightarrow \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_5 \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_6 \rightarrow \begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_7 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
X_8 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.
\]

when the metric is: $K = \text{diag}(1,1,1)$.

In the compact case there is only one MASA, the Cartan subalgebra, generated by the matrices:

\[
\begin{pmatrix} i \\ -i \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ i \\ -i \end{pmatrix}.
\]

and we shall use the following basis for the corresponding MASA in $u(3)$:
\[
Y_0 = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}.
\]

(4.2)

The coordinates \(s\) are related to the coordinates \(y\) in the same way as in \(su(2), (3.3)\):

\[
y_0 = s_0 e^{ix_0}, \quad y_1 = s_1 e^{ix_1}, \quad y_2 = s_2 e^{ix_2}
\]

and the Hamiltonian has also the same form of all cases using compact Cartan subalgebras (3.4):

\[
H = \frac{1}{2} \left( p_0^2 + p_1^2 + p_2^2 \right) + V(s), \quad V(s) = \frac{m_0^2}{s_0^2} + \frac{m_1^2}{s_1^2} + \frac{m_2^2}{s_2^2}
\]

(4.4)

with the constraint \(s_0^2 + s_1^2 + s_2^2 = 1\).

In the one-dimensional case, there was only one invariant, which was the Hamiltonian. In this case, we can construct three invariants, only two of them in involution at the same time (one of them the Hamiltonian). The system is superintegrable in the sense of [12]. These invariants are [14]:

\[
R_{01} = (s_0 p_1 - s_1 p_0)^2 + \left( m_0 \frac{s_1}{s_0} + m_1 \frac{s_0}{s_1} \right)^2, \\
R_{02} = (s_0 p_2 - s_2 p_0)^2 + \left( m_0 \frac{s_2}{s_0} + m_2 \frac{s_0}{s_2} \right)^2, \\
R_{12} = (s_1 p_2 - s_2 p_1)^2 + \left( m_1 \frac{s_2}{s_1} + m_2 \frac{s_1}{s_2} \right)^2.
\]

(4.5)

The sum of these three invariants is the Hamiltonian up to an additive constant. In order to study the solutions of this problem we need construct a coordinate system in which the corresponding HJ equation separates into a system of ordinary differential equations. As in [16] we will use spherical coordinates [19], defined by:

\[
s_0 = \cos \phi_2 \cos \phi_1, \quad s_1 = \cos \phi_2 \sin \phi_1, \quad s_2 = \sin \phi_2.
\]

(4.6)

and the Hamiltonian is written as \((c = 1)\):
\[ H = \frac{1}{2} \left( p_{\phi_2}^2 + \frac{p_{\phi_1}^2}{\cos^2 \phi_2} \right) + V(\phi_1, \phi_2) \]

\[ V(\phi_1, \phi_2) = \frac{1}{\cos^2 \phi_2} \left( \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} \right) + \frac{m_2^2}{\sin^2 \phi_2} \quad (4.7) \]

where the constants \( m_0, m_1, m_2 \) are chosen to be nonnegative.

The second order conserved quantities (4.5) (we will follow the notation \( \hat{Q} \) for these operators) can be written in terms of the basis \( \{X_1, \ldots, X_8\} \):

\[ \hat{Q}_1 = X_3^2 + X_4^2, \quad \hat{Q}_2 = X_5^2 + X_6^2, \quad \hat{Q}_3 = X_7^2 + X_8^2 \quad (4.8) \]

with commutation relations (the commutator is a third order element which plays no essential role in the method):

\[ [\hat{Q}_1, \hat{Q}_2] = [\hat{Q}_2, \hat{Q}_3] = [\hat{Q}_3, \hat{Q}_1] \]

The Casimir is:

\[ C = 4C_1 + 2C_2 + 4C_3 + 3\hat{Q}_1 + 3\hat{Q}_2 + 3\hat{Q}_3 \]

where

\[ C_1 = X_1^2, \quad C_2 = \{X_1, X_2\}, \quad C_3 = X_2^2 \]

are the second order operators in the enveloping algebra of the compact Cartan subalgebra.

The Hamiltonian is

\[ H = Q_1 + Q_2 + Q_3 + \text{constant} \]

where \( Q_i \) is the expression of \( \hat{Q}_i \) in spherical coordinates [19]:

\[ Q_1 = \frac{1}{2} p^{\phi_1} + \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} \]

\[ Q_2 = \tan^2 \phi_2 \left( \frac{1}{2} p^{\phi_1} \sin^2 \phi_1 + \frac{m_0^2}{\cos^2 \phi_1} \right) + \cos^2 \phi_1 \left( \frac{1}{2} p^{\phi_2} + \frac{m_2^2}{\tan^2 \phi_2} \right) \]

\[ + \frac{1}{2} p^{\phi_1} p^{\phi_2} \sin 2\phi_1 \tan \phi_2 \]

\[ Q_3 = \tan^2 \phi_2 \left( \frac{1}{2} p^{\phi_1} \cos^2 \phi_1 + \frac{m_1^2}{\sin^2 \phi_1} \right) + \sin^2 \phi_1 \left( \frac{1}{2} p^{\phi_2} + \frac{m_2^2}{\tan^2 \phi_2} \right) \]

\[ - \frac{1}{2} p^{\phi_1} p^{\phi_2} \sin 2\phi_1 \tan \phi_2 \]

The HJ equation is:
\[ \frac{1}{2} \left( \frac{\partial S}{\partial \phi_2} \right)^2 + \frac{m_2^2}{\sin^2 \phi_2} + \frac{1}{2} \left( \frac{\partial S}{\partial \phi_1} \right)^2 + \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} = E \quad (4.9) \]

and separates into two ordinary differential equations using \( S(\phi_1, \phi_2) = S_1(\phi_1) + S_2(\phi_2) - Et \):

\[ \frac{1}{2} \left( \frac{\partial S_1}{\partial \phi_1} \right)^2 + \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} = \alpha_1, \quad (4.10) \]

\[ \frac{1}{2} \left( \frac{\partial S_2}{\partial \phi_2} \right)^2 + \frac{m_2^2}{\sin^2 \phi_2} + \frac{\alpha_1}{\cos^2 \phi_2} = \alpha_2 \quad (4.11) \]

where \( \alpha_2 = E \) and \( \alpha_1 \) are the separation constants (which are positive). These equations have the same form as those in (3.7). The solutions are easily computed and can be found as particular cases in [16]. The potential has singularities along the coordinate lines: \( \phi_1 = 0, \pi/2, \pi, 3\pi/2 \), and \( \phi_2 = \pi/2, 3\pi/2 \) in the generic case. It has a unique minimum inside each regularity domain. An analysis of the associated dynamical system (Hamilton equations) shows that all the orbits in a neighborhood of the critical point (center) are closed and hence, the corresponding trajectories are periodic (a direct consequence of the correspondence between extrema of the potential and critical points of the phase space). Let us restrict to the domain \( 0 < \phi_1, \phi_2 < \pi/2 \), where the minimum is in \( \tan \phi_1 = \sqrt{m_1/m_0}, \tan \phi_2 = \sqrt{m_2/(m_0 + m_1)} \). The value of the potential at this point is \((m_0 + m_1 + m_2)^2\), hence the energy \( E \) is bounded from below \((E \geq (m_0 + m_1 + m_2)^2)\). The explicit solutions are:

\[ \cos^2 \phi_2 = \frac{1}{2E} \left[ b_2 + \sqrt{b_2^2 - 4\alpha_1 E \cos 2\sqrt{2E}t} \right], \quad (4.12) \]

\[ \cos^2 \phi_1 = \frac{1}{2\alpha_1} \left[ b_1 + \frac{1}{\cos^2 \phi_2} \left[ b_1^2 - 4\alpha_1 m_0^2 \right]^{1/2} \left\{ \left( b_2 \cos^2 \phi_2 - 2\alpha_1 \right) \sin 2\sqrt{2\alpha_1} \beta_1 + 2\sqrt{\alpha_1} \left( b_2 - E \cos^2 \phi_2 \right) \cos^2 \phi_2 - \alpha_1 \right\}^{1/2} \cos 2\sqrt{2\alpha_1} \beta_1 \right] \quad (4.13) \]

where \( b_1 = \alpha_1 + m_0^2 - m_1^2 \) and \( b_2 = E + \alpha_1 - m_2^2 \).

Let us remark that these results reflect essentially the case \( su(2) \). In fact all systems we can construct using Cartan subalgebras can be described in a unified way as it was shown in [14,16].
B. \(su(2,1)\)

The basis we will use is formed by the set of operators \(\{X_1, \ldots, X_8\}\) which are given in the \(3 \times 3\) matrix representation by:

\[
\begin{align*}
X_1 &\rightarrow \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_2 &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix},
X_3 &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
X_4 &\rightarrow \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_5 &\rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
X_6 &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ -i & 0 & 0 \end{pmatrix}, \\
X_7 &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_8 &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}.
\end{align*}
\]

According to general results \([22]\), \(su(2,1)\) has four MASAs, two of them Cartan subalgebras (compact and noncompact), one orthogonally decomposable subalgebra, with one nilpotent element, and one nilpotent subalgebra. We will discuss these four cases in the following. Although some of these subalgebras have a simpler expression in some skewdiagonal metrics, we will always use the diagonal one:

\[
K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]  

(4.14)

because, in this way, the kinetic part is always diagonal. There are nine coordinate systems associated to \(O(2,1)\) free Hamiltonians: spherical, hyperbolic, elliptic (I and II), complex elliptic, horospheric, elliptic parabolic, hyperbolic parabolic and semicircular parabolic \([20,23]\).

Not all of them will separate our systems because these are not free. However, the appropriate systems have been computed in \([20]\) and we will use their results.
I. Compact Cartan subalgebra (CC)

The compact Cartan subalgebra has a basis formed by the same two matrices we used in \( su(3) \), and the same situation happens for the corresponding MASA in \( u(2,1) \) (4.2). The coordinates \( s \) are also related to the coordinates \( y \) as they did in the compact case \( su(3) \) (4.3).

However, the Hamiltonian reflects the noncompact character of \( su(2,1) \):

\[
H = \left( \frac{1}{2} \left( p_0^2 + p_1^2 - p_2^2 \right) + V(s) \right), \quad V(s) = \frac{m_0^2}{s_0^2} + \frac{m_1^2}{s_1^2} - \frac{m_2^2}{s_2^2}
\]

where the constraint \( s_0^2 + s_1^2 - s_2^2 = 1 \) must be satisfied.

This Hamiltonian separates in four coordinate systems, spherical, hyperbolic and elliptic I and II [20]. We will use spherical coordinates to discuss the explicit solution.

\[
s_0 = \cosh \phi_2 \cos \phi_1, \quad s_1 = \cosh \phi_2 \sin \phi_1, \quad s_2 = \sinh \phi_2.
\]

Choosing \( c = -1 \), we have the Hamiltonian in these coordinates:

\[
H = \frac{1}{2} \left( p_{\phi_2}^2 - \frac{p_{\phi_1}^2}{\cosh^2 \phi_2} \right) + V(\phi_1, \phi_2),
\]

\[
V(\phi_1, \phi_2) = -\frac{1}{\cosh^2 \phi_2} \left( \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} \right) + \frac{m_2^2}{\sinh^2 \phi_2}.
\]

Due to the form of the potential the constants \( m_0, m_1, m_2 \) can be chosen nonnegative. The potential is regular inside the domain \( 0 < \phi_1 < \pi/2, 0 < \phi_2 < \infty \). It has a saddle point: \( \tan \phi_1 = \sqrt{m_1/m_0}, \tan \phi_2 = \sqrt{m_2/(m_0 + m_1)} \), if \( m_0 + m_1 > m_2 \). However, due to the special form of the kinetic term (which is not positive definite), it is easy to check that the associated dynamical system has all the orbits in a neighborhood of the critical point (which is also a center as in the compact case) closed and again, the corresponding trajectories are periodic.

The second order operators in the enveloping algebra of this MASA are:

\[
C_1 = X_1^2, \quad C_2 = \{X_1, X_2\}, \quad C_3 = X_2^2
\]

The quadratic constants of motion lying in the enveloping algebra of \( su(2,1) \) and commuting with the elements in the compact Cartan subalgebra are:
\[ \hat{Q}_1 = X_3^2 + X_4^2, \quad \hat{Q}_2 = X_5^2 + X_6^2, \quad \hat{Q}_3 = X_7^2 + X_8^2 \]  
(4.18)

with commutation relations:

\[ [\hat{Q}_1, \hat{Q}_2] = [\hat{Q}_3, \hat{Q}_2] = -[\hat{Q}_3, \hat{Q}_1] \]

The Casimir is written in terms of these second order operators as:

\[ C = 4C_1 + 2C_2 + 4C_3 + 3\hat{Q}_1 - 3\hat{Q}_2 - 3\hat{Q}_3 \]

and the Hamiltonian is:

\[ H = -Q_1 + Q_2 + Q_3 + \text{constant} \]

where \( Q_i \) are the conserved quantities in spherical coordinates:

\[ Q_1 = \frac{1}{2} p_{\phi_1}^2 + \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} \]

\[ Q_2 = \tanh^2 \phi_2 \left( \frac{1}{2} p_{\phi_1}^2 \sin^2 \phi_1 + \frac{m_0^2}{\cos^2 \phi_1} \right) + \cos^2 \phi_1 \left( \frac{1}{2} p_{\phi_2}^2 + \frac{m_2^2}{\tanh^2 \phi_2} \right) \]

\[ -\frac{1}{2} p_{\phi_1} p_{\phi_2} \sin 2\phi_1 \tanh \phi_2 \]

\[ Q_3 = \tanh^2 \phi_2 \left( \frac{1}{2} p_{\phi_1}^2 \cos^2 \phi_1 + \frac{m_1^2}{\sin^2 \phi_1} \right) + \sin^2 \phi_1 \left( \frac{1}{2} p_{\phi_2}^2 + \frac{m_2^2}{\tanh^2 \phi_2} \right) \]

\[ +\frac{1}{2} p_{\phi_1} p_{\phi_2} \sin 2\phi_1 \tanh \phi_2 \]

The HJ equations corresponding to the Hamiltonian (4.17) are:

\[ \frac{1}{2} \left( \frac{\partial S_1}{\partial \phi_1} \right)^2 + \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} = \alpha_1, \quad \text{(4.19)} \]

\[ \frac{1}{2} \left( \frac{\partial S_2}{\partial \phi_2} \right)^2 + \frac{m_2^2}{\sinh^2 \phi_2} - \frac{\alpha_1}{\cosh^2 \phi_2} = \alpha_2. \quad \text{(4.20)} \]

The first one is the same as we got in \( su(3) \) (4.10), \( \alpha_1 \) is always positive and \( \alpha_2 = E \).

The solutions depend on the values of the parameters and energy

i) \( E < 0 \)

\[ u_2 = \frac{1}{2|E|} \left[ -b_2 + \sqrt{b_2^2 + 4\alpha_1 E \cos 2\sqrt{2|E|t}} \right]. \quad \text{(4.21)} \]

ii) \( E = 0 \)
\[ u_2 = \frac{\alpha_1}{\alpha_1 - m_2^2} + 2(\alpha_1 - m_2^2)t^2. \]  
\[ (4.22) \]

iii) \( E > 0 \)

\[ u_2 = \frac{1}{2E} \left[ b_2 + \sqrt{b_2^2 + 4\alpha_1 E \cosh 2\sqrt{2Et}} \right]. \]  
\[ (4.23) \]

where \( u_2 = \cosh^2 \phi_2, \) \( b_2 = E - \alpha_1 + m_2^2. \) The other equation can be solved as we did in the previous cases. The result is:

\[ u_1 = \frac{1}{2\alpha_1} \left[ b_1 + \frac{1}{u_2} \left[ \frac{b_1^2 - 4\alpha_1 m_0^2}{b_2^2 + 4\alpha_1 E} \right]^{1/2} \right] \left[ -(b_2 u_2 + 2\alpha_1) \sin 2\sqrt{2\alpha_1 \beta_1} \right. \\
\left. + 2\sqrt{\alpha_1} [(Eu_2 - b_2) u_2 - \alpha_1]^{1/2} \cos 2\sqrt{2\alpha_1 \beta_1} \right] \]  
\[ (4.24) \]

where \( u_1 = \cos^2 \phi_1 \) and \( b_1 = \alpha_1 + m_0^2 - m_1^2. \)

II. Noncompact Cartan subalgebra (NC)

There is only one noncompact Cartan subalgebra. A representative can be chosen according to the same criteria we used in \( su(1,1) \) \((3.19)\), keeping one element compact and the other (as in \( su(1,1) \)) noncompact:

\[
\begin{pmatrix}
2i \\
-i \\
-i
\end{pmatrix}, \quad 
\begin{pmatrix}
0 \\
0 & i \\
-i & 0
\end{pmatrix}
\]  
\[ (4.25) \]

and the basis for the corresponding MASA of \( u(2,1) \) will be:

\[
Y_0 = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 \\ i \\ i \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 \\ 0 & i \\ -i & 0 \end{pmatrix}.
\]  
\[ (4.26) \]

The coordinates are as in \( su(1,1) \):

\[
y_0 = e^{ix_0 s_0},
\]

\[ y_1 = e^{ix_1} (s_1 \cosh x_2 + i s_2 \sinh x_2), \]  
\[ (4.27) \]

\[ y_2 = e^{ix_1} (-i s_1 \sinh x_2 + s_2 \cosh x_2). \]
The Hamiltonian is:

\[ H = c \left( \frac{1}{2} \left( p_0^2 + p_1^2 - p_2^2 \right) + V(s) \right), \]
\[ V(s) = \frac{m_0^2}{s_0^2} + \frac{(m_1^2 - m_2^2)(s_1^2 - s_2^2) + 4m_1m_2s_1s_2}{(s_1^2 + s_2^2)^2} \] (4.28)

where we will take \( m_0 > 0 \) and \( m_1, m_2 \) can take any value, and the coordinates satisfy the constraint (the same for all \( su(2,1) \) MASAs, as we have chosen the same metric in all cases):

\[ s_0^2 + s_1^2 - s_2^2 = 1. \]

There are two systems of coordinates in which the associated HJ equation separates, hyperbolic and complex elliptic [20]. We will use hyperbolic coordinates, defined as:

\[ s_0 = \cosh \phi_2 \]
\[ s_1 = \sinh \phi_2 \sinh \phi_1 \]
\[ s_2 = \sinh \phi_2 \cosh \phi_1 \] (4.29)

and the new Hamiltonian is \((c = -1)\):

\[ H = \frac{1}{2} \left( p_{\phi_2}^2 - \frac{p_{\phi_1}^2}{\sinh^2 \phi_2} \right) + V(\phi_1, \phi_2), \]
\[ V(\phi_1, \phi_2) = -\frac{m_0^2}{\cosh^2 \phi_2} + \frac{1}{\sinh^2 \phi_2} \left( \frac{m_1^2 - m_2^2 - 2m_1m_2 \sinh 2\phi_1}{\cosh^2 2\phi_1} \right). \] (4.30)

Note that the potential follows the same pattern as the corresponding case in \( su(1,1) \). It is regular inside the domain: \(-\infty < \phi_1 < \infty, 0 < \phi_2 < \infty\), and has also a saddle point at: \( \sinh 2\phi_1 = -m_2/m_1 \), \( \tanh \phi_2 = \sqrt{|m_1/m_0|} \), when \( |m_1| < |m_0| \). As in the previous case, the associated dynamical system has a center and the trajectories in a neighborhood of it are periodic.

The basis for this MASA is \( \{2X_1 + X_2, X_8\} \), and the corresponding second order elements are:

\[ C_1 = (2X_1 + X_2)^2, \quad C_2 = \{2X_1 + X_2, X_8\}, \quad C_3 = X_8^2 \]

The second order conserved quantities, commuting with \( 2X_1 + X_2 \) and \( X_8 \), and belonging to the enveloping algebra of \( su(2,1) \) are:
\[\hat{Q}_1 = X_2^2 - X_7^2,\]
\[\hat{Q}_2 = X_3^2 + X_4^2 - X_5^2 - X_6^2,\]
\[\hat{Q}_3 = \{X_3, X_5\} + \{X_4, X_6\}\]  
(4.31)

with commuting relations:

\[\hat{[Q}_3, \hat{Q}_1]\] = \[\hat{[Q}_2, \hat{Q}_3]\], \[\hat{[Q}_1, \hat{Q}_2]\] = 0

The Casimir is written as:

\[C = C_1 - 3C_3 + 3\hat{Q}_1 + 3\hat{Q}_2\]

and the Hamiltonian is:

\[H = Q_1 + Q_2 + \text{constant}\]

Finally, the conserved quantities are expressed in hyperbolic coordinates by:

\[Q_1 = \frac{1}{2} p_{\phi_1}^2 - \frac{m_1^2 - m_2^2 - 2m_1m_2 \sinh 2\phi_1}{\cosh^2 2\phi_1}\]
\[Q_2 = \frac{1}{2} p_{\phi_2}^2 - \frac{m_0^2}{\cosh^2 \phi_2} - \frac{1}{\tanh^2 \phi_2} \left( \frac{1}{2} p_{\phi_1}^2 - \frac{m_1^2 - m_2^2 - 2m_1m_2 \sinh 2\phi_1}{\cosh^2 2\phi_1} \right)\]
\[Q_3 = \frac{1}{2} \sinh 2\phi_1 \left( p_{\phi_2}^2 + \frac{1}{\tanh^2 \phi_2} p_{\phi_1}^2 \right) - \frac{\cosh 2\phi_1}{\tanh \phi_2} p_{\phi_1} p_{\phi_2} + m_0^2 \tanh^2 \phi_2 \sinh 2\phi_1 - \frac{(m_1^2 - m_2^2) \sinh 2\phi_1 + 2m_1m_2}{\tanh^2 \phi_2 \cosh^2 2\phi_1}\]

The Hamilton-Jacobi equation separates into two ordinary differential equations:

\[\frac{1}{2} \left( \frac{\partial S_1}{\partial \phi_1} \right)^2 - \frac{m_1^2 - m_2^2 - 2m_1m_2 \sinh 2\phi_1}{\cosh^2 2\phi_1} = \alpha_1,\]
(4.32)
\[\frac{1}{2} \left( \frac{\partial S_2}{\partial \phi_2} \right)^2 - \frac{m_0^2}{\cosh^2 \phi_2} - \frac{\alpha_1}{\sinh^2 \phi_2} = E.\]
(4.33)

The solutions have the same form we have found before.

i) \(E < 0\)

\[u_2 = \frac{1}{2|E|} \left[ b_2 \pm \sqrt{b_2^2 - 4\alpha_1 E \cos 2\sqrt{2|E|}t} \right].\]
(4.34)

ii) \(E = 0\)
\[ u_2 = -\frac{\alpha_1}{\alpha_1 + m_0^2} + 2(\alpha_1 + m_0^2)t^2. \] (4.35)

iii) \( E > 0 \)

\[ u_2 = \frac{1}{2E} \left[ -b_2 + \sqrt{b_2^2 - 4\alpha_1 E \cosh 2\sqrt{2Et}} \right] \] (4.36)

where \( u_2 = \sinh^2 \phi_2, \ b_2 = E + \alpha_1 + m_0^2. \)

The solution for the other coordinate is obtained in the same way (with the change \( u_1 = \sinh 2\phi_1 \)). The equation for this coordinate is the same as that given in formula (3.24) and its solutions can be found in formulas (3.25-3.29). Due to the possible different signs of the energy and the constant \( \alpha_1 \), one should take care of the square roots appearing in all the formulas.

III. Orthogonally decomposable subalgebra (OD)

The orthogonally decomposable subalgebra (a representative of the class) has a basis formed by a compact element and a nilpotent one:

\[
\begin{pmatrix}
2i \\
-i \\
-i
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
i \\
-i
\end{pmatrix}
\] (4.37)

and the basis for the corresponding MASA of \( u(2,1) \) is:

\[
Y_0 = \begin{pmatrix}
i \\
0 \\
0
\end{pmatrix}, \quad Y_1 = \begin{pmatrix}
0 \\
i \\
i
\end{pmatrix}, \quad Y_2 = \begin{pmatrix}
0 \\
i \\
-i
\end{pmatrix}.
\] (4.38)

The coordinates have also a similar form to those in \( su(1,1) \) (3.31):

\[
y_0 = e^{ix_0} s_0,
\]

\[
y_1 = e^{ix_1} ((1 + ix_2)s_1 + ix_2 s_2),
\]

\[
y_2 = e^{ix_1} (-ix_2 s_1 + (1 - ix_2) s_2).
\] (4.39)

The Hamiltonian is:
\[ H = c\left(\frac{1}{2}\left(p_0^2 + p_1^2 - p_2^2\right) + V(s)\right), \]
\[ V(s) = \frac{m_0^2}{s_0^2} - \frac{m_2^2(s_1 - s_2)}{(s_1 + s_2)^3} + \frac{2m_1m_2}{(s_1 + s_2)^2} \] (4.40)

with \( s_0^2 + s_1^2 - s_2^2 = 1. \)

There are four coordinate systems associated to this subalgebra, hyperbolic, horospheric, elliptic parabolic and hyperbolic parabolic [20]. We will use again the hyperbolic ones, defined as in (4.29).

The Hamiltonian is \((c = -1)\):
\[ H = \frac{1}{2}\left(p_{\phi_2}^2 - \frac{p_{\phi_1}^2}{\sinh^2 \phi_2}\right) + V(\phi_1, \phi_2), \]
\[ V(\phi_1, \phi_2) = -\frac{m_0^2}{\cosh^2 \phi_2} - \frac{1}{\sinh^2 \phi_2} \left(m_2^2 e^{-4\phi_1} + 2m_1m_2 e^{-2\phi_1}\right) \] (4.41)

The potential is regular inside the domain: \(-\infty < \phi_1 < \infty, 0 < \phi_2 < \infty\), and, as it happens in all the \( su(2,1) \) cases, has a saddle point at: \( \phi_1 = (1/2) \log(|m_2/m_1|) \), \( \tanh \phi_2 = \sqrt{|m_1/m_0|} \), when \(|m_0| > |m_1|, m_1m_2 < 0\). The situation is the same as in all other cases in \( su(2,1)\).

The second order operators in the enveloping algebra of the MASA under consideration are given by:
\[ C_1 = (2X_1 + X_2)^2, \quad C_2 = \{2X_1 + X_2, X_2 + X_8\}, \quad C_3 = (X_2 + X_8)^2 \]

and the quadratic constants of motion:
\[ \hat{Q}_1 = X_3^2 + X_4^2 - X_5^2 - X_6^2, \]
\[ \hat{Q}_2 = (X_3 + X_5)^2 + (X_4 + X_6)^2, \] (4.42)
\[ \hat{Q}_3 = X_7^2 + 2\{X_1, X_2 + X_8\} \]

satisfy the commutation relations:
\[ [\hat{Q}_1, \hat{Q}_2] = [\hat{Q}_2, \hat{Q}_3], \quad [\hat{Q}_1, \hat{Q}_3] = 0 \]
The Casimir is given in terms of these operators by

\[ C = C_1 + 3C_2 - 3C_3 + 3\hat{Q}_1 - 3\hat{Q}_3 \]

and the Hamiltonian is:

\[ H = -Q_1 + Q_3 + \text{constant} \]

We can write the conserved quantities in hyperbolic coordinates:

\[
Q_1 = \frac{1}{\tanh^2 \phi_2} \left( \frac{1}{2} p_{\phi_1}^2 + m_0^2 \tanh^2 \phi_2 + \frac{1}{\tanh^2 \phi_2} \left( \frac{1}{2} p_{\phi_1}^2 + m_0^2 e^{-4\phi_1} \right) \right) - \frac{1}{\tanh \phi_2} p_{\phi_1} p_{\phi_2}
\]

\[
Q_2 = e^{2\phi_1} \left( \frac{1}{2} p_{\phi_2}^2 + m_0^2 \tanh^2 \phi_2 + \frac{1}{\tanh^2 \phi_2} \left( \frac{1}{2} p_{\phi_1}^2 + m_0^2 e^{-4\phi_1} \right) \right) - \frac{1}{\tanh \phi_2} p_{\phi_1} p_{\phi_2}
\]

\[
Q_3 = \frac{1}{2} p_{\phi_1}^2 + m_2^2 e^{-4\phi_1} + 2m_1 m_2 e^{-2\phi_2}
\]

The HJ equation is separated into the following equations

\[
\frac{1}{2} \left( \frac{\partial S_1}{\partial \phi_1} \right)^2 + \left( m_2^2 e^{-4\phi_1} + 2m_1 m_2 e^{-2\phi_1} \right) = \alpha_1, \quad (4.43)
\]

\[
\frac{1}{2} \left( \frac{\partial S_2}{\partial \phi_2} \right)^2 - \frac{m_0^2}{\cosh^2 \phi_2} - \frac{\alpha_1}{\sinh^2 \phi_2} = E. \quad (4.44)
\]

Equation (4.43) is integrated using \( u_2 = \sinh^2 \phi_2 \). The result is the same as in the previous case (4.33). Equation (4.43) is solved using \( u_1 = e^{2\phi_1} \) (the same change we use in the nilpotent MASA of the \( su(1, 1) \) case), and the results are essentially the same we have found above (see 3.33).

IV. Nilpotent subalgebra (NIL)

The nilpotent subalgebra has a basis formed by two nilpotent elements (one of order 2 and the other of order 3):

\[
\begin{pmatrix} 0 & i & i \\ i & i & 0 \\ -i & -i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad (4.45)
\]
and the basis for the MASA of \(u(2, 1)\) can be obtained adding to these two matrices the identity times the imaginary unit.

The new coordinates are defined through:

\[
y_0 = e^{ix_0}(s_0 + ix_2(s_1 + s_2)), \\
y_1 = e^{ix_0}\left(ix_2 s_0 + \left(1 - \frac{x_2^2}{2} + ix_1\right)s_1 + \left(-\frac{x_2^2}{2} + ix_1\right)s_2\right), \\
y_2 = e^{ix_0}\left(-ix_2 s_0 + \left(\frac{x_2^2}{2} - ix_1\right)s_1 + \left(1 + \frac{x_2^2}{2} - ix_1\right)s_2\right).
\] (4.46)

The Hamiltonian is:

\[
H = c \left(\frac{1}{2} (p_0^2 + p_1^2 - p_2^2) + V(s)\right), \\
V(s) = \frac{2m_0 m_1 + m_2^2}{(s_1^2 + s_2)^2} - \frac{4m_1 m_2 s_0}{(s_1^2 + s_2)^3} + \frac{m_1^2 (4s_0^2 - 1)}{(s_1^2 + s_2)^4}
\] (4.47)

with the constraint: \(s_0^2 + s_1^2 - s_2^2 = 1\).

We have now two separable coordinate systems: horospheric and semicircular parabolic [20]. We will use now the horospheric ones, defined by:

\[
s_0 = \phi_1 e^{\phi_2}, \quad s_1 = \cosh \phi_2 - \frac{1}{2} \phi_1^2 e^{\phi_2}, \quad s_2 = \sinh \phi_2 + \frac{1}{2} \phi_1^2 e^{\phi_2}.
\] (4.48)

The Hamiltonian is \((c = -1)\):

\[
H = \frac{1}{2} \left(p_{\phi_2}^2 - e^{-2\phi_2} p_{\phi_1}^2\right) + V(\phi_1, \phi_2), \\
V(\phi_1, \phi_2) = m_1^2 e^{-4\phi_2} - e^{-2\phi_2} \left(m_2^2 + 2m_0 m_1 + 4m_1 \phi_1 (m_1 \phi_1 - m_2)\right).
\] (4.49)

The potential has no singularity in the whole plain \((\phi_1, \phi_2)\). It has a saddle point at: \(\phi_1 = m_2 / 2m_1, \phi_2 = (1/2) \log(m_1/m_0)\), when \(m_0 m_1 > 0\). The situation is the same as in all other cases in \(su(2, 1)\).

The second order elements in the enveloping algebra of the nilpotent subalgebra are:

\[
C_1 = (X_2 + X_8)^2, \quad C_2 = \{X_2 + X_8, X_4 + X_6\}, \quad C_3 = (X_4 + X_6)^2
\]

and the constants of motion

26
\[ \hat{Q}_1 = 3(X_3 + X_5)^2 - 2\{2X_1 + X_2, X_2 + X_8\}, \]
\[ \hat{Q}_2 = \{2X_1 + X_2, X_4 + X_6\} + 6\{X_4, X_2 + X_8\} - 3\{X_7, X_3 + X_5\}, \]
\[ \hat{Q}_3 = 4X_1^2 + 3X_2^2 - 2\{X_1, X_2\} + 6X_3^2 + 6X_4^2 - 3X_7^2 - \{4X_1 - X_2, X_8\} + 3\{X_3, X_5\} + 3\{X_4, X_6\} \]

have the following commutation relations:

\[ [\hat{Q}_1, \hat{Q}_2] = [\hat{Q}_3, \hat{Q}_2], \quad [\hat{Q}_1, \hat{Q}_3] = 0 \]

The Casimir is

\[ C = -3C_1 - 3C_3 - \hat{Q}_1 + \hat{Q}_3 \]

and the Hamiltonian:

\[ H = Q_1 - Q_3 + \text{constant} \]

Finally, the second order constant of motion are given in horospherical coordinates by the following expressions:

\[ Q_1 = \frac{1}{2}p_{\phi_1}^2 + 4m_1\phi_1(m_1\phi_1 - m_2) \]
\[ Q_2 = \frac{1}{2}\phi_1 p_{\phi_1}^2 - \frac{1}{2}p_{\phi_1}p_{\phi_2} + (m_2^2 + 2m_0m_1)\phi_1 - m_1e^{-2\phi_2}(2m_1\phi_1 - m_2) + 4m_1\phi_1^2(m_1\phi_1 - m_2) \]
\[ Q_3 = (1 + e^{-2\phi_2}) \left( \frac{1}{2}p_{\phi_1}^2 + 4m_1\phi_1(m_1\phi_1 - m_2) \right) - \left( \frac{1}{2}p_{\phi_2}^2 + m_1^2e^{-4\phi_2} - (m_2^2 + 2m_0m_1)e^{-2\phi_2} \right) \]

This is the most interesting case, in the sense that the others are easily reduced to the cases in dimension 1, while this nilpotent subalgebra does not appear in the \( su(1,1) \) case. However, the solutions are still very similar to those found before. It is worth mentioning here, that all the potentials we have construct (and any potential we could construct by using this method) are always inverse quadratic potentials in the coordinates, and the solutions have always similar forms (though they depend on the specific characteristics of these potentials and the constants involved).
The HJ equation is separated into the following equations

\[
\frac{1}{2} \left( \frac{\partial S_1}{\partial \phi_1} \right)^2 + m_2^2 + 2m_0m_1 + 4m_1\phi_1(m_1\phi_1 - m_2) = \alpha_1, \quad (4.51)
\]

\[
\frac{1}{2} \left( \frac{\partial S_2}{\partial \phi_2} \right)^2 + m_1^2e^{-4\phi_2} - \alpha_1e^{-2\phi_2} = E. \quad (4.52)
\]

The change \( u_2 = e^{2\phi_2} \) allows to solve the second equation, and the other one is solved directly. The solutions are:

i) \( E < 0 \)

\[
u_2 = \frac{1}{2|E|} \left[ \alpha_1 + \sqrt{\alpha_1^2 + 4m_1^2E \cos 2\sqrt{2|E|}|t|} \right]. \quad (4.53)
\]

ii) \( E = 0 \)

\[
u_2 = \frac{m_1^2}{\alpha_1} + 2\alpha_1 t^2. \quad (4.54)
\]

iii) \( E > 0 \)

\[
u_2 = \frac{1}{2E} \left[ -\alpha_1 + \sqrt{\alpha_1^2 + 4m_1^2E \cosh 2\sqrt{2E}t} \right]. \quad (4.55)
\]

The first equation gives the value of the \( \phi_1 \) coordinate:

\[
\phi_1 = \frac{1}{2m_1} \left[ m_2 + \frac{1}{\nu_2} \left[ \frac{\alpha_1 - 2m_0m_1}{\alpha_1^2 + 4m_1^2E} \right]^{1/2} \left( (\alpha_1\nu_2 - 2m_1^2) \sin 2\sqrt{2}\beta_1 m_1 + 2m_1 [(E\nu_2 - \alpha_1)\nu_2 - m_1^2]^{1/2} \cos 2\sqrt{2}\beta_1 m_1 \right) \right]. \quad (4.56)
\]

In Table II, we present a summary of these Hamiltonians in the two dimensional case.

V. CONCLUSIONS

We have presented in this work a complete analysis of a series of one and two dimensional integrable Hamiltonians, which in the 2-dimensional case are superintegrable in the sense described in the Introduction. Though the one-dimensional case is always an integrable system, let us remark the importance and applications of the potentials described in Section 2. Regarding the two dimensional ones, we have provided them with a set of conserved
quantities which allows to study the HJ equations in several separable coordinate systems and compute in some interesting cases the explicit solutions.

The one-dimensional Hamiltonians obtained here have been extensively studied in the literature from many other points of view. See for instance [24] for a recent application of Morse potentials. As an example of these different approaches, all of them appear in the classification of quasi-exactly solvable Schrödinger operators [17,18], as particular types of these systems corresponding to the so called exactly solvable systems. Following the classification in [18], the exactly solvable potentials of Cases 1 and 2 are just the ones we have obtained associated to \( su(1,1) \) and its compact and noncompact Cartan subalgebras. The first one (3.12) is the celebrated Pöschl-Teller potential. That appearing in case 3 is the Morse potential (3.33), which we get using the nilpotent subalgebra of \( su(1,1) \). Finally the potential (3.5), associated to the Cartan subalgebra of \( su(2) \) is related to the modified harmonic oscillators appearing in [18] as cases 4 and 5. One has to take into account in this case, that QES potentials, as studied in [18], are defined in the line (or half-line), and we are working here in a sector of \( S^1 \) (see also [25] for a study of harmonic oscillators in a sector). The relation is not surprising at all if one considers that QES systems in the line are related to the complex Lie algebra \( sl(2) \) [17], and the ones we get here reflect the invariance under \( su(2) \) and \( su(1,1) \), the real forms of \( sl(2) \). In the QES setting, Schrödinger operators belong to the enveloping algebra of a Lie algebra, while in our approach, the corresponding classical Hamiltonians are second order Casimirs of the algebra, and hence, they are particular cases (exactly solvable) of the former.

Two prolongations of this study are now in progress. One of them is the use of contrac-
tions in Lie algebras to obtain other Hamiltonian systems associated to different algebras, not necessarily semisimple. In this sense, the Hamiltonians, the conserved quantities and the coordinate systems can be obtained by contraction [26,27]. The second one is to apply this approach to the quantum case, considering the Schrödinger equation with these potentials [28]. We also plan to study the links of this theory with QES systems and the possibility of considering partial integrability and partial variable separation in HJ equations [29].
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### TABLE I. One-dimensional potentials

| Algebra | MASA          | Kinetic term | Potential                                           |
|---------|---------------|--------------|-----------------------------------------------------|
| $su(2)$ | Compact Cartan| $p_\phi^2$   | $\frac{m_0^2}{\cos^2 \phi} + \frac{m_1^2}{\sin^2 \phi}$ |
| $su(1,1)$ | Compact Cartan | $p_\phi^2$   | $- \frac{m_0^2}{\cosh^2 \phi} + \frac{m_1^2}{\sinh^2 \phi}$ |
|         | Noncompact Cartan | $p_\phi^2$   | $- \frac{m_0^2 - m_1^2 + 2m_0 m_1 \sinh 2\phi}{\cosh^2 2\phi}$ |
|         | Nilpotent     | $p_\phi^2$   | $m_1^2 e^{-4\phi} - 2m_0 m_1 e^{-2\phi}$          |

### TABLE II. Two-dimensional potentials

| Algebra | MASA | Kinetic term | Potential                                                                 |
|---------|------|--------------|---------------------------------------------------------------------------|
| $su(3)$ | CC   | $\frac{1}{2} \left( p_{\phi_2}^2 + \frac{p_{\phi_1}^2}{\cos^2 \phi_2} \right)$ | $\frac{1}{\cos^2 \phi_2} \left( \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} \right) + \frac{m_2^2}{\sin^2 \phi_2}$ |
| $su(2, 1)$ | CC   | $\frac{1}{2} \left( p_{\phi_2}^2 - \frac{p_{\phi_1}^2}{\cosh^2 \phi_2} \right)$ | $- \frac{1}{\cosh^2 \phi_2} \left( \frac{m_0^2}{\cos^2 \phi_1} + \frac{m_1^2}{\sin^2 \phi_1} \right) + \frac{m_2^2}{\sinh^2 \phi_2}$ |
|         | NC   | $\frac{1}{2} \left( p_{\phi_2}^2 - \frac{p_{\phi_1}^2}{\sinh^2 \phi_2} \right)$ | $- \frac{m_0^2}{\cosh^2 \phi_2} + \frac{1}{\sinh^2 \phi_2} \left( \frac{m_1^2 - m_2^2 - 2m_0 m_1 \sinh 2\phi_1}{\cosh^2 2\phi_1} \right)$ |
|         | OD   | $\frac{1}{2} \left( p_{\phi_2}^2 - \frac{p_{\phi_1}^2}{\sinh^2 \phi_2} \right)$ | $- \frac{m_0^2}{\cosh^2 \phi_2} - \frac{m_1^2}{\sinh^2 \phi_2} \left( m_2^2 e^{-4\phi_1} + 2m_1 m_2 e^{-2\phi_1} \right)$ |
|         | NIL  | $\frac{1}{2} \left( p_{\phi_2}^2 - e^{-2\phi_2} p_{\phi_1}^2 \right)$ | $m_1^2 e^{-4\phi_2} - e^{-2\phi_2} \left( m_2^2 + 2m_0 m_1 + 4m_1 \phi_1 (m_1 \phi_1 - m_2) \right)$ |