A powerful and simple frequency formula to nonlinear fractal oscillators

Kang-Le Wang and Chun-Fu Wei

Abstract
In this work, a fractal nonlinear oscillator is successfully established by fractal derivative in a fractal space, and its variational principle is obtained by semi-inverse transform method. The variational principle can provide conservation laws in an energy form. The approximate frequency of the fractal oscillator is found by a simple fractal frequency formula. An example shows the fractal frequency formula is a powerful and simple tool to fractal oscillators.

Keywords
Fractal oscillator, fractal derivative, two-scale transform method, semi-inverse transform method, fractal frequency formula

Introduction
The oscillator is a very common phenomenon in nature such as the oscillator of the springs, the oscillator of water waves, and so on. These oscillator phenomena can be described by the oscillator equation. The general oscillator equation is given as follows

\[
\frac{d^2 v}{dt^2} + f(v) = 0
\]  

with the initial condition

\[
v(0) = A, \quad v'(0) = 0
\]  

The general oscillator equation is established in a continuous space. Consider the oscillator in a fractal space or microgravity space. Equation (1) can be described by He’s fractal derivative as follows

\[
\frac{H D_v}{D^{\delta}} \left( \frac{H D_v}{D^{\delta}} \right) + f(v) = 0
\]  

where \( \frac{H D_v}{D^{\delta}} \) is He’s fractal derivative and is defined as follows

\[
\frac{H D_v}{D^{\delta}} (t_0) = \Gamma(1+\delta) \lim_{\Delta t \to 0} \frac{v(t) - v(t_0)}{(t - t_0)^\delta}
\]
The analytical approximate solution of equation (3) has been found by many excellent methods such as Taylor series method,\textsuperscript{9,10} He’s homotopy perturbation method,\textsuperscript{11,12} variational iteration method,\textsuperscript{13–16} reduced differential transform method,\textsuperscript{17} and so on.

Recently, Prof. Ji-Huan He\textsuperscript{1} suggested the possible, simplest but most effective frequency formulation for nonlinear oscillators. Prof. He research the nonlinear oscillators as follows

\[
\frac{d^2 \xi}{dt^2} + \xi + \xi^3 + \xi^5 = 0 \quad (5)
\]

with the initial condition

\[
\xi(0) = A, \quad \xi'(0) = 0 \quad (6)
\]

We find He’s frequency formula is also valid for fractal nonlinear oscillator with He’s fractal derivative. Equation (5) can be rewritten into the following form

\[
\frac{H}{D t^a} \left( \frac{H}{D t^a} \xi \right) + \xi + \xi^3 + \xi^5 = 0 \quad (7)
\]

with the initial condition

\[
x(0) = 0, \quad \frac{H}{D t^a} \xi(0)/D t^a = 0 \quad (8)
\]

In this paper, the variational principle\textsuperscript{18–20} of the fractal nonlinear oscillator is obtained according to the semi-inverse method. The fractal two-scale transform method\textsuperscript{21–23} and fractal frequency formula\textsuperscript{1} are adopted to find the approximate analytical solution of equation (7) in a fractal space base He’s fractal derivative.

**Two-scale transform method**

The geometric illustration of the fractal two-scale transform method was given in He and Ain.\textsuperscript{23} The advantage of the fractal two-scale transform method is that it can be adopted convert approximately a fractal space into its continuous partner.

Consider a fractal differential equation as follows

\[
\frac{H}{D t^a} \left( \frac{H}{D t^a} G(x, y, t) \right) + \frac{H}{D x^b} G(x, y, t) + \frac{H}{D y^c} G(x, y, t) + H(G(x, y, t)) = 0 \quad (9)
\]

where $\frac{H}{D t^a}$, $\frac{H}{D x^b}$ and $\frac{H}{D y^c}$ are fractal derivative.

We use fractal two-scale transform method to convert equation (9) into its partner. Assume

\[
T = t^a
\]

\[
X = x^b
\]

\[
Y = y^c
\]

So, equation (9) can be written into the following form

\[
\frac{H}{D T} G(X, Y, T) + \frac{H}{D X} G(X, Y, T) + \frac{H}{D Y} G(X, Y, T) + H(G(X, Y, T)) = 0 \quad (13)
\]

Equation (13) is the partner of equation (9).
**Variational principle of fractal nonlinear oscillator**

In this work, we mainly research the fractal nonlinear oscillators as follows

\[ \frac{H^D}{Dt^2} \left( \frac{H^D \xi}{Dt^2} \right) + \xi + \xi^3 + \xi^5 = 0 \]  

(14)

with the initial condition

\[ \tilde{\xi}(0) = 0, \frac{H^D \tilde{\xi}(0)}{Dt^2} = 0 \]  

(15)

The variational principle of equation (14) can be given by semi-inverse transform method\textsuperscript{24–26} as follows

\[ J(\xi) = \int \left( \frac{1}{2} \left( \frac{H^D \xi}{Dt^2} \right)^2 - \frac{1}{2} \xi^2 - \frac{1}{4} \xi^4 - \frac{1}{6} \xi^6 \right) H^D t^2 \]  

(16)

**Fractal frequency formula**

Consider the general fractal oscillator equation as follows

\[ \frac{H^D}{Dt^2} \left( \frac{H^D \xi(t)}{Dt^2} \right) + W(\xi(t)) = 0 \]  

(17)

with the initial condition

\[ \tilde{\xi}(0) = 0, \frac{H^D \tilde{\xi}(0)}{Dt^2} = 0 \]  

(18)

We use the two-scale transform method to equation (17) and assume

\[ T = t^2 \]  

(19)

Therefore, equation (17) can be written into its partner as follows

\[ \frac{H^D \xi(T)}{DT^2} + W(\xi(T)) = 0 \]  

(20)

with the initial condition

\[ \tilde{\xi}(0) = 0, \frac{H^D \tilde{\xi}(0)}{DT} = 0 \]  

(21)

Adopt the fractal frequency formula, and the approximate frequency can be easily obtained as follows

\[ \omega = \sqrt{\frac{H^D W(\xi(T))}{D_\xi}} |_{\xi = \xi^*} \]  

(22)

If we want to improve the accuracy of the frequency, we just need to use more points instead of \( \xi = \theta/2 \).

**Main results**

Consider the fractal cubic-quintic Duffing oscillator equation as follows

\[ \frac{H^D}{Dt^2} \left( \frac{H^D \xi}{Dt^2} \right) + \xi + \xi^3 + \xi^5 = 0 \]  

(23)
with the initial condition

$$\xi(0) = \theta$$  \hfill (24) \\
$$\frac{HD\xi(0)}{Dt^a} = 0$$  \hfill (25)

We assume

$$T = t^a$$  \hfill (26)

Substituting equation (26) into equation (23), we obtain the following result

$$\frac{HD\xi}{DT^2} + \xi + \xi^3 + \xi^5 = 0$$  \hfill (27) \\
with the initial condition

$$\xi(0) = \theta$$  \hfill (28) \\
$$\frac{HD\xi(0)}{DT} = 0$$  \hfill (29)

Equation (27) has the following form

$$\frac{HD\xi}{DT^2} + W(\xi) = 0$$  \hfill (30) \\
where

$$W(\xi) = \xi + \xi^3 + \xi^5$$  \hfill (31)

The square of equation (30) frequency is given by

$$\omega^2 = \left. \frac{HDW(\xi)}{D\xi} \right|_{\xi=\xi}$$  \hfill (32) \\
$$\frac{HDW(\xi)}{D\xi} = (1 + 3\xi^2 + 5\xi^4)$$  \hfill (33)

So, we have

$$\omega = \left(1 + \frac{3}{4}\theta^2 + \frac{5}{10}\theta^4\right)^{\frac{1}{2}}$$  \hfill (34) \\
We assume $\theta = 1$, according to equation (34), and we have

$$\omega \approx 1.37178120$$  \hfill (35)

When $\theta = 1$, the exact frequency of equation (23) is given by He.$^1$

$$\omega_{\text{exact}} = 1.52359$$  \hfill (36)

The relative error is 9.6%. 

When the $\theta$ is very small, the approximate frequency is good accuracy. We may use more points instead of $\xi = \frac{\theta}{2}$ to improve the accuracy of the frequency. For example

$$
\omega = \left[ \frac{1}{4} \left( \frac{H_D \mathcal{W}(\xi)}{D_\xi} \bigg|_{\xi=0.40} + \frac{H_D \mathcal{W}(\xi)}{D_\xi} \bigg|_{\xi=0.50} + \frac{H_D \mathcal{W}(\xi)}{D_\xi} \bigg|_{\xi=0.55} + \frac{H_D \mathcal{W}(\xi)}{D_\xi} \bigg|_{\xi=0.60} \right) \right]^2
$$

(37)

When $\theta = 1$, according to equation (37), we can obtain

$$
\omega = 1.480163082
$$

(38)
The relative error is 2.85%. Equation (37) is better accuracy than equation (34). If we want to improve the accuracy of the frequency again, we can use more points instead of \( \theta \). Consequently, we can obtain the approximate analytical solution of equation (23) as follows

\[
\xi = \theta \cos \omega t^2
\]  

(39)

Therefore, the approximate analytical solution of equation (23) can be obtained as follows

\[
\xi_{app} \approx \cos(1.480163082t^2)
\]  

(40)

Figure 1 shows that the different \( \alpha \) values for equation (40). When \( \alpha = 1 \), the result obtained is the same as He. \(^1\)

In Figure 2, we show the absolute error of the exact solution and the approximate solution for \( \alpha = \theta = 1 \). We can see that our method in this paper is very accurate and effective.

**Conclusion**

In this paper, the nonlinear oscillator is described by fractal derivative in a fractal space, and its variational principle is successfully established via semi-inverse transform method. The two-scale transform method and fractal frequency formula are adopted to find the approximate frequency of fractal oscillator equation in fractal space. The example shows the frequency formula is an excellent and accuracy tool to fractal oscillators.

**Declaration of Conflicting Interests**

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**

The author(s) received no financial support for the research, authorship, and/or publication of this article.

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