HOMOLOGICAL DIMENSIONS OF KÖTHE ALGEBRAS

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ABSTRACT. Given a metrizable Köthe algebra $\lambda(P)$, we compute the global dimension, the weak global dimension, the bidimension, and the weak bidimension of $\lambda(P)$ in terms of the Köthe set $P$.

Köthe sequence spaces play a significant rôle in modern functional analysis. On the one hand, the class of Köthe spaces is rather large and contains many important spaces of smooth functions and distributions [15]. On the other hand, Köthe spaces are often used to provide various examples and counterexamples in the theory of topological vector spaces.

As was observed in [1], many Köthe spaces can be viewed as topological algebras under pointwise multiplication. The study of homological properties of Köthe algebras was initiated by the author in [7, 8, 9]. In particular, necessary and sufficient conditions for a Köthe algebra to be biprojective (or, equivalently, biflat) were obtained, and homological dimensions of biprojective Köthe algebras were computed in some special cases.

In this note we complete the study of homological dimensions of metrizable Köthe algebras by computing the global dimension, the weak global dimension, the bidimension, and the weak bidimension of a Köthe algebra $\lambda(P)$ in terms of the Köthe set $P$. In fact, a great deal of this work was done in [7, 8, 9]. However, the following two questions were left unanswered: (1) Does there exist a biprojective Köthe algebra $\lambda(P)$ whose bidimension equals 2, while the global dimension equals 1?; and (2) What can be said about homological dimensions of nonbiprojective Köthe algebras? It is the aim of this paper to answer the above questions.

1. Preliminaries

Throughout, all vector spaces and algebras are assumed to be over the field $\mathbb{C}$ of complex numbers. All algebras are assumed to be associative, but not necessarily unital. The unitization of an algebra $A$ is denoted by $A_+$. By a $\hat{\otimes}$-algebra we mean an algebra $A$ endowed with a complete locally convex topology in such a way that the product map $A \times A \rightarrow A$ is jointly continuous. Note that the above map uniquely extends to a continuous linear map.

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$A \hat{\otimes} A \to A$, $a \otimes b \mapsto ab$, where the symbol $\hat{\otimes}$ stands for the completed projective tensor product (whence the name “$\hat{\otimes}$-algebra”). If the topology on $A$ can be determined by a family of submultiplicative seminorms (i.e., a family $\{\| \cdot \|_\lambda : \lambda \in \Lambda \}$ of seminorms such that $\|ab\|_\lambda \leq \|a\|_\lambda \|b\|_\lambda$ for all $a, b \in A$), then $A$ is said to be locally $m$-convex (or an Arens-Michael algebra). A Fréchet algebra is a $\hat{\otimes}$-algebra $A$ whose underlying locally convex space is a Fréchet space (unlike some authors, we do not assume $A$ to be locally $m$-convex).

Let $I$ be any set, and let $P$ be a set of nonnegative real-valued functions on $I$. For $p \in P$, we shall write $p_i$ for $p(i)$. Recall that $P$ is a Köthe set on $I$ if the following axioms are satisfied:

$$\forall i \in I \exists p \in P : p_i > 0;$$
$$\forall p, q \in I \exists r \in P : \max\{p_i, q_i\} \leq r_i \quad \forall i \in I.$$

(P1)

(P2)

Given a Köthe set $P$, the Köthe space $\lambda(P)$ is defined as follows:

$$\lambda(P) = \left\{ x = (x_i) \in C^I : \|x\|_p = \sum_{i} |x_i|p_i < \infty \quad \forall p \in P \right\}.$$

This is a complete locally convex space with the topology determined by the family of seminorms $\{\| \cdot \|_p : p \in P \}$. Clearly, $\lambda(P)$ is a Fréchet space if and only if $P$ contains an at most countable cofinal subset.

For each $i \in I$ denote by $e_i$ the function on $I$ which is 1 at $i$, 0 elsewhere. Obviously, $x = \sum_{i} x_ie_i$ for each $x \in \lambda(P)$.

Given a Köthe set $P$ on $I$, each $n$-tuple $(p^1, \ldots, p^n) \in P^n$ determines a function on $I^n$ by $(i_1, \ldots, i_n) \mapsto p^1_{i_1} \cdots p^n_{i_n}$. The set of all such functions will be denoted by $P^{\times n}$. Clearly, $P^{\times n}$ is a Köthe set on $I^n$. By [3], there exists a topological isomorphism

$$\lambda(P)^{\otimes n} = \lambda(P) \hat{\otimes} \cdots \hat{\otimes} \lambda(P) \xrightarrow{\sim} \lambda(P^{\times n});$$

(1)

e_{i_1} \otimes \cdots \otimes e_{i_n} \mapsto e_{i_1,\ldots,i_n}.

It is easy to see that the topology on $\lambda(P^{\times n})$ is determined by the family of seminorms $\{\| \cdot \|_{(p^1,\ldots,p^n)} : p \in P \}$. We will denote the above seminorm simply by $\| \cdot \|_p$; this should not cause any confusion.

If $P, Q$ are Köthe sets, we write $P \prec Q$ if for each $p \in P$ there exist $q \in Q$ and $C > 0$ such that $p_i \leq Cq_i$ for all $i \in I$. This is equivalent to say that $\lambda(Q) \subset \lambda(P)$, and the embedding of $\lambda(Q)$ into $\lambda(P)$ is continuous. If $P \prec Q$ and $Q \prec P$ (i.e., if $\lambda(P) = \lambda(Q)$ topologically), we write $P \sim Q$. We set

$$P \cdot Q = \{pq = (p_iq_i)_{i \in I} : p \in P, q \in Q\},$$

$$P^{[\alpha]} = \{p^{[\alpha]} = (p_i^{[\alpha]})_{i \in I} : p \in P\} \quad (\alpha > 0).$$

Note that $P^{[2]} \sim P \cdot P$ by (P2).

It is easy to see that $P \prec P^{[2]}$ if and only if for each $a, b \in \lambda(P)$ the pointwise product $ab$ belongs to $\lambda(P)$, and for each $p \in P$ there exist $q \in P$ and $C > 0$
such that $\|ab\|_p \leq C\|a\|_q\|b\|_q$ for every $a, b \in \lambda(P)$. Hence the above condition implies that $\lambda(P)$ is a $\otimes$-algebra under pointwise multiplication. Note that this condition is satisfied automatically whenever $p_i \geq 1$ for each $p \in P$ and each $i \in I$; moreover, in this case $\lambda(P)$ is locally $m$-convex. Algebras of the form $\lambda(P)$ (where $P$ is any Köthe set satisfying $P \prec P[2]$) are called Köthe algebras.

**Example 1.** The Banach algebra $\ell^1(I)$ is clearly a Köthe algebra.

**Example 2.** The algebra $\mathbb{C}^I$ endowed with the direct product topology is a Köthe algebra. To see this, it suffices to set $P$ to be the family of all nonnegative functions with finite support. It is also clear that $\mathbb{C}^I$ is locally $m$-convex.

**Example 3.** Fix a real number $0 < R \leq \infty$ and a nondecreasing sequence $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ of positive numbers with $\lim_n \alpha_n = \infty$. The power series space $\Lambda_R(\alpha)$ is the set of all complex sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $\|x\|_r = \sum_n |x_n|^r \alpha_n < \infty$ for all $0 < r < R$. Evidently, $\Lambda_R(\alpha)$ is a metrizable Köthe space. If $R \geq 1$, then $\Lambda_R(\alpha)$ satisfies condition $P \prec P[2]$ and is therefore a Köthe algebra. Moreover, since the seminorms $\|\cdot\|_r$ are submultiplicative for $r \geq 1$, we see that $\Lambda_R(\alpha)$ is locally $m$-convex provided that $R > 1$.

Here are two special cases of Example 3.

**Example 4.** If $\alpha_n = \log n$, then $\Lambda_\infty(\alpha)$ is topologically isomorphic to the space of rapidly decreasing sequences

$$s = \left\{ x = (x_n) \in \mathbb{C}^N : \|x\|_k = \sum_n |x_n| n^k < \infty \quad \forall k \in \mathbb{N} \right\}.$$

**Example 5.** If $\alpha_n = n$, then $\Lambda_R(\alpha)$ is topologically isomorphic to the space of functions holomorphic on the disc $\mathbb{D}_R = \{ z \in \mathbb{C} : |z| < R \}$ of radius $R$. Under this identification, the multiplication in $\Lambda_R(\alpha)$ corresponds to the “component-wise” product of the Taylor expansions of holomorphic functions (the Hadamard product; see [14]). The resulting topological algebra is denoted by $\mathcal{H}(\mathbb{D}_R)$.

We now recall some basic facts from the homology theory of $\otimes$-algebras. For details, see [3, 4, 14]. Some details on weak homological dimensions can also be found in [12, 9].

Let $A$ be a $\otimes$-algebra. By a left $A\otimes$-module we mean a left $A$-module endowed with a complete locally convex topology in such a way that the action $A \times X \to X$ is jointly continuous. If $X$ and $Y$ are left $A\otimes$-modules, then the space of all continuous $A$-module morphisms from $X$ to $Y$ is denoted by $A\mathcal{H}(X, Y)$. Right $A\otimes$-modules and $A\otimes$-bimodules are defined similarly. The category of left $A\otimes$-modules (respectively, right $A\otimes$-modules, $A\otimes$-bimodules) and continuous $A$-module morphisms will be denoted by $A\mathcal{M}$ (respectively, $\mathcal{M}$-$A$, $\mathcal{M}$-$A\otimes$).

If $X$ is a right $A\otimes$-module and $Y$ is a left $A\otimes$-module, then their $A$-module tensor product $X \otimes_A Y$ is defined to be the completion of the quotient $(X \otimes Y)/N$, where $N \subset X \otimes Y$ is the closed linear span of all elements of the form $x \cdot a \otimes y -$
$x \otimes a \cdot y$ ($x \in X$, $y \in Y$, $a \in A$). As in pure algebra, the $A$-module tensor product can be characterized by a universal property (see [3] for details).

A chain complex $C_\bullet = (C_n, d_n)_{n \in \mathbb{Z}}$ in $A\text{-mod}$ is admissible if it splits in the category of topological vector spaces. A left $A\otimes$-module $P$ is projective if the functor $A_\bullet \otimes \hom(P,-)$ is exact in the sense that for every admissible chain complex $C_\bullet$ in $A\text{-mod}$ the complex $A_\bullet \otimes \hom(P,C_\bullet)$ of vector spaces is exact. Projective right $A\otimes$-modules and projective $A\otimes$-bimodules are defined similarly. A $\otimes$-algebra $A$ is biprojective if $A$ is projective in $A\text{-mod}$. A resolution of $X \in A\text{-mod}$ is a chain complex $P_\bullet = (P_n, d_n)_{n \geq 0}$ together with a morphism $\varepsilon : P_0 \to X$ such that $0 \leftarrow X \xrightarrow{\varepsilon} P_\bullet$ is an admissible complex. If $P_n$ is projective for each $n \geq 0$, then $(P_\bullet, \varepsilon)$ is a projective resolution. It is known that the category $A\text{-mod}$ has enough projectives, i.e., every $X \in A\text{-mod}$ has a projective resolution. The homological dimension of $X \in A\text{-mod}$ is the minimum integer $n = \text{dh}_A X$ with the property that $X$ has a projective resolution $(P_\bullet, \varepsilon)$ with $P_i = 0$ for all $i > n$. If no such $n$ exists, we set $\text{dh}_A X = \infty$. The global dimension of $A$ is defined by

$$\text{dg} A = \sup \{ \text{dh}_A X : X \in A\text{-mod} \}.$$  

The bidimension of $A$ is defined to be the homological dimension of $A_+$ in $A\text{-mod}$-$A$. We always have $\text{dg} A \leq \text{db} A$. Algebras $A$ with $\text{db} A = 0$ are called contractible. Equivalently, $A$ is contractible if and only if $A$ is biprojective and unital.

Now let $A$ be a Fréchet algebra, and let $A\text{-mod}(\text{Fr})$ denote the full subcategory of $A\text{-mod}$ consisting of left Fréchet $A$-modules. The categories of right Fréchet $A$-modules and of Fréchet $A$-bimodules will be denoted by $\text{mod}$-$A(\text{Fr})$ and $A\text{-mod}$-$A(\text{Fr})$, respectively. By using [3, Theorem III.1.27], it is easy to see that a left Fréchet $A$-module $P$ is projective if and only if $P$ is projective in $A\text{-mod}(\text{Fr})$ (in the sense that the functor $A_\bullet \otimes \hom(P,-)$ is exact on $A\text{-mod}(\text{Fr})$). Together with [3, Theorem III.5.4], this implies that for each $X \in A\text{-mod}(\text{Fr})$ the homological dimension $\text{dh}_A X$ does not depend on whether we compute it in $A\text{-mod}$ or in $A\text{-mod}(\text{Fr})$. The same is true of $\text{db} A$ (as for $\text{dg} A$, we do not know the answer; see also Remark 2 below).

A left Fréchet $A$-module $F$ is flat if the functor $(-) \otimes_A F$ is exact on $\text{mod}$-$A(\text{Fr})$, i.e., if for every admissible chain complex $C_\bullet$ in $\text{mod}$-$A(\text{Fr})$ the complex $C_\bullet \otimes_A F$ of vector spaces is exact. A Fréchet algebra $A$ is biflat if $A$ is flat in $A\text{-mod}$-$A(\text{Fr})$. A resolution $(P_\bullet, \varepsilon)$ of $X$ in $A\text{-mod}(\text{Fr})$ is a flat resolution if $P_n$ is flat for each $n \geq 0$. The weak homological dimension of $X \in A\text{-mod}(\text{Fr})$ is the minimum integer $n = w.dh_A X$ with the property that $X$ has a flat resolution $(P_\bullet, \varepsilon)$ with $P_i = 0$ for all $i > n$. If no such $n$ exists, we set $w.dh_A X = \infty$. The weak global dimension of $A$ is defined by

$$w.dg A = \sup \{ w.dh_A X : X \in A\text{-mod}(\text{Fr}) \}.$$  

The weak bidimension of $A$ is defined to be the weak homological dimension of $A_+$ in $A\text{-mod}$-$A(\text{Fr})$. We always have $w.dg A \leq w.db A$. Algebras $A$ with $w.db A = 0$ are called amenable.
Since each projective Fréchet $A$-module is flat, we have $w.dh_A X \leq dh_A X$ for every $X \in A\text{-mod}(\text{Fr})$. Consequently, $w.dg A \leq dg A$ and $w.db A \leq db A$ for each Fréchet algebra $A$.

Suppose that $X$ is a right Fréchet $A$-module and $Y$ is a left Fréchet $A$-module. The space $\text{Tor}^A_n(X,Y)$ is defined to be the $n$th homology of the complex $X \otimes_A Q_\bullet$, where $Q_\bullet$ is a flat resolution of $Y$ in $A\text{-mod}(\text{Fr})$. Equivalently, $\text{Tor}^A_n(X,Y)$ is the $n$th homology of the complex $P_\bullet \otimes_A Y$, where $P_\bullet$ is a flat resolution of $X$ in $\text{mod}-A(\text{Fr})$. The above definitions do not depend on the particular choice of $P_\bullet$ and $Q_\bullet$. We have $w.dh_A Y \leq n$ if and only if for each $X \in \text{mod}-A(\text{Fr})$ $\text{Tor}^A_{n+1}(X,Y) = 0$ and $\text{Tor}^A_n(X,Y)$ is Hausdorff. There is a similar characterization of $\text{dh}_A Y$ in terms of Ext spaces, but we will not use it in the sequel.

**Remark 1.** When dealing with flat modules and weak dimensions, we deliberately restrict ourselves to Fréchet modules and Fréchet algebras. For an explanation, see [9, Remark 2.5] and [10, Remark 7.2].

### 2. Biprojective Köthe algebras

Let $\lambda(P)$ be a Köthe algebra. Throughout we will use the following conditions $(U), (N), (B)$, and $(M)$ on the Köthe set $P$ ("$U$" is for "unital", "$N$" is for "nuclear", "$B$" is for "biprojective" or "biflat", and "$M$" is for "matrices"):  

- $(U)$ $\forall p \in P \sum_i p_i < \infty$.
- $(N)$ $\forall p \in P \exists q \in P \exists \alpha \in \ell^1 : p \leq \alpha q$.
- $(B)$ $P \sim P^{[2]}$.
- $(M)$ There exist complex matrices $\alpha = (\alpha_{ij})_{i,j \in I}$ and $\beta = (\beta_{ij})_{i,j \in I}$ such that:
  - $(M1)$ $\alpha_{ij} + \beta_{ij} = 1 \quad (i,j \in I);$  
  - $(M2)$ $\forall p \in P \ \exists C > 0 \ \exists q \in P \ \forall j \in \mathbb{N} \ \sup_i |\alpha_{ij}| p_i p_j \leq C q_j$;  
  - $(M3)$ $\forall p \in P \ \exists C > 0 \ \exists q \in P \ \forall i \in \mathbb{N} \ \sup_j |\beta_{ij}| p_j p_i \leq C q_i$.

Clearly, condition $(U)$ means exactly that $\lambda(P)$ is unital. By the Grothendieck-Pietsch criterion, $(N)$ is equivalent to $\lambda(P)$ being nuclear. Condition $(B)$ holds if and only if $\lambda(P)$ is biprojective [9, Theorem 3.5]. Moreover, if $\lambda(P)$ is metrizable, then $(B)$ is equivalent to $\lambda(P)$ being biflat [9, Theorem 5.2]. Finally, if $(B)$ holds, then $(N) \Leftrightarrow (M) \Leftrightarrow \text{db} \lambda(P) \leq 1$ [9, Corollary 4.4 and Theorem 4.7]. By [9, Proposition 6.11], $(U)$ is also equivalent to $\lambda(P)$ being contractible. Therefore $(U)$ implies $(B)$, $(N)$, and $(M)$.

Let $P$ be a Köthe set on $I$. For each $p \in P$ we define a function $\bar{p} : I \rightarrow \mathbb{R}_+$ by $\bar{p}_i = \min\{p_i, 1\}$. Clearly, $\bar{P} = \{\bar{p} : p \in P\}$ is a Köthe set.

**Lemma 1.** If $P \prec P^{[2]}$, then $\bar{P} \prec P \cdot \bar{P}$.

**Proof.** Given $p \in P$, choose $q \in P$ and $C \geq 1$ such that $p \leq C q^2$ and $p \leq q$. Fix any $i \in I$. If $q_i < 1$, then $p_i < 1$, so that $\bar{p}_i = p_i$, $\bar{q}_i = q_i$, whence $\bar{p}_i \leq C q_i q_i$. If
Theorem 3. Let $q_i \geq 1$, then $\bar{q}_i = 1$, and $\bar{p}_i \leq 1 \leq Cq_i = Cq_\bar{q}_i$. Therefore $\bar{p} \leq Cq\bar{q}$, which proves the claim.

Corollary 2. Let $\lambda(P)$ be a Köthe algebra. Then for each $a \in \lambda(P)$ and each $x \in \lambda(\bar{P})$ the pointwise product $a \cdot x$ is in $\lambda(\bar{P})$. Moreover, for each $\bar{p} \in \bar{P}$ there exist $q \in \bar{P}$ and $C > 0$ such that $\|a \cdot x\|_\bar{p} \leq C\|a\|_q \|x\|_\bar{q}$ for every $a \in \lambda(P)$, $x \in \lambda(\bar{P})$. Therefore $\lambda(\bar{P})$ is a $\lambda(P)$-$\bar{\otimes}$-module under pointwise multiplication.

We now state the main result of this section.

Theorem 3. Let $A = \lambda(P)$ be a Köthe algebra satisfying (B) and (N). Suppose that $\text{d}_{\text{h}}A \lambda(\bar{P}) \leq 1$. Then $\text{d}_{\text{b}}A \leq 1$.

To prove Theorem 3, we need some preparation. Let $A$ be a $\bar{\otimes}$-algebra, and let $X$ be a left $A$-$\bar{\otimes}$-module. Following [3], we set $X_\Pi = A \bar{\otimes}_A X$ and define

$$\varpi_X : X_\Pi \rightarrow X, \ a \bar{\otimes} x \mapsto a \cdot x \ (a \in A, x \in X).$$

Suppose now that $A$ is biprojective. Then, by a result of Helemskii [3, Section V.2], $\text{d}_{\text{h}}A \leq 1$ if and only if the “diagonal” map

$$A \bar{\otimes} X_\Pi \rightarrow (A_+ \bar{\otimes} X_\Pi) \oplus (A \bar{\otimes} X), \ a \bar{\otimes} x \mapsto (a \bar{\otimes} x, a \bar{\otimes} \varpi_X(x))$$

is a coretraction in $A$-$\text{mod}$.

To apply the above result to $A = \lambda(P)$ and $X = \lambda(\bar{P})$, we first have to describe $X_\Pi$ explicitly.

Lemma 4. Let $A = \lambda(P)$ be a Köthe algebra satisfying (B), and let $X = \lambda(\bar{P})$. Then $X_\Pi$ is isomorphic to $A$ in $A$-$\text{mod}$. Under this identification, the canonical map $\varpi_X : X_\Pi \rightarrow X$ becomes the identity embedding of $\lambda(P)$ into $\lambda(\bar{P})$.

Proof. By [3, Lemma 6.4],

$$X_\Pi \cong \left\{ x = (x_i) \in \prod_i (e_iX) : \|x\|_{p,q} = \sum_i \|x_i\|_p q_i < \infty \ \forall p, q \in \bar{P} \right\},$$

and, under this identification, the canonical map $\varpi_X$ takes each $x = (x_i) \in X_\Pi$ to $\sum_i x_i \in X$. Since $e_iX = \mathbb{C}e_i$, it follows that

$$X_\Pi \cong \left\{ a = (a_i) \in \mathbb{C}^I : \|a\|_{p,q} = \sum_i |a_i| \bar{p}_i q_i < \infty \ \forall p, q \in \bar{P} \right\} = \lambda(\bar{P} \cdot P).$$

Therefore we need only prove that $\bar{P} \cdot P \sim P$. Since $P^{[2]} \prec P$ and $\bar{P} \prec P$, we have $P \cdot P \prec P$. For the converse, take any $p \in P$ and choose $C \geq 1$ and $q \in P$ such that $p \leq Cq^2$ and $p \leq q$. Fix any $i \in I$. If $q_i < 1$, then $\bar{q}_i = q_i$, and so $p_i \leq Cq_i q_i$. If $q_i \geq 1$, then $\bar{q}_i = 1$, and so $p_i \leq q_i \leq Cq_i q_i$. Therefore $p \leq Cq \bar{q}$, so that $P \prec \bar{P} \cdot P$, and, finally, $P \cdot P \sim P$, as required.

Proof of Theorem 3. Set $X = \lambda(\bar{P})$. Identifying $X_\Pi$ with $A$ by Lemma 4, we see that the canonical map (3) becomes

$$A \bar{\otimes} A \rightarrow (A_+ \bar{\otimes} A) \oplus (A \bar{\otimes} X), \ a \bar{\otimes} b \mapsto (a \bar{\otimes} b, a \bar{\otimes} b).$$

(4)
Since $A$ is biprojective (see [7, Theorem 3.5]) and $\text{dh}_A X \leq 1$, it follows that $(\mathcal{P})$ is a coretraction in $A$-mod. Therefore there exists a continuous linear map $\varphi: A \to A \hat{\otimes} A$ and an $A$-module morphism $\psi: A \hat{\otimes} X \to A \hat{\otimes} A$ such that
\begin{equation}
 a \otimes b = a \cdot \varphi(b) + \psi(a \otimes b) \quad (a, b \in A).
\end{equation}
Since $\varphi$ and $\psi$ are continuous, for each $p \in P$ there exist $q \in Q$ and $C > 0$ such that
\begin{equation}
\|\varphi(a)\|_{p,p} \leq C\|a\|_q \quad (a \in A); \quad \|\psi(a \otimes x)\|_{p,p} \leq C\|a\|_q\|x\|_q \quad (a \in A, x \in X).
\end{equation}
Using the isomorphism $A \hat{\otimes} A \cong \lambda(\mathcal{P}{\times}{2})$ (see (1)), we may represent each $\varphi(e_j)$ as
\begin{equation}
 \varphi(e_j) = \sum_{k,l} \lambda_{kli} e_k \otimes e_l \quad (j \in I).
\end{equation}
Then (9) implies that
\begin{equation}
\sum_{k,l} |\lambda_{kli}| p_k p_l \leq Cq_j \quad (j \in I).
\end{equation}
Now fix any $i \in I$ and define
\begin{equation}
 \psi_i: X \to A \hat{\otimes} A, \quad \psi_i(x) = \psi(e_i \otimes x) \quad (x \in X).
\end{equation}
Since $\psi$ is an $A$-module morphism, we have
\begin{equation}
 e_j \cdot \psi_i(x) = \delta_{ji} \psi_i(x) \quad (i, j \in I),
\end{equation}
and so $\text{Im} \psi_i \subset e_i \cdot A \hat{\otimes} A = \mathbb{C}e_i \otimes A$. Therefore for each $i \in I$ there exists a linear map $f_i: X \to A$ such that
\begin{equation}
 \psi(e_i \otimes x) = \psi_i(x) = e_i \otimes f_i(x) \quad (x \in X).
\end{equation}
Setting $a = e_i$ and $b = e_j$ in (7), we see that
\begin{equation}
\|f_i(e_j)\|_{p_i} \leq Cq_i \bar{q}_j \leq Cq_i \quad (i, j \in I).
\end{equation}
Let
\begin{equation}
 f_i(e_j) = \sum_k \mu_{kij} e_k \quad (i, j \in I).
\end{equation}
Then (11) is equivalent to
\begin{equation}
\sum_k |\mu_{kij}| p_k p_i \leq Cq_i \quad (i, j \in I).
\end{equation}
Setting $a = e_i$ and $b = e_j$ in (3) and taking into account (8), (10), and (12), we see that
\begin{equation}
 e_i \otimes e_j = e_i \cdot \sum_{k,l} \lambda_{kli} e_k \otimes e_l + e_i \otimes \sum_k \mu_{kij} e_k,
\end{equation}
which is equivalent to
\begin{equation}
 e_j = \sum_l \lambda_{dlj} e_l + \sum_k \mu_{kij} e_k = \sum_k (\lambda_{ikj} + \mu_{kij}) e_k \quad (i, j \in I).
\end{equation}
Now set
\[ \alpha_{ij} = \lambda_{ij}, \quad \beta_{ij} = \mu_{ij} \quad (i, j \in I). \]
Then (14) implies that
\[ \alpha_{ij} + \beta_{ij} = 1 \quad (i, j \in I), \]
i.e., (M1) holds. Next, (9) implies that
\[ |\alpha_{ij}| p_ip_j \leq Cq_j \quad (i, j \in I), \]
i.e., (M2) holds. Finally, (13) implies that
\[ |\beta_{ij}| p_jp_i \leq Cq_i \quad (i, j \in I), \]
i.e., (M3) holds. Thus \( P \) satisfies (M), and so \( \text{db} \lambda(P) \leq 1 \) by [7, Theorem 4.7].

Combining Theorem 3 with our earlier results obtained in [4] and [9] yields a complete classification of biprojective Köthe algebras by their homological dimensions \( \text{dg} \) and \( \text{db} \). Before formulating the result, let us recall some notation.

Let \( P \) be a Köthe set on \( I \). The Köthe space \( \lambda_{\infty}(P) \) is defined by
\[ \lambda_{\infty}(P) = \left\{ a = (a_i) \in C^I : \|a\|_p^\infty = \sup_i |a_i| p_i < \infty \quad \forall p \in P \right\}. \]
This is a complete locally convex space with the topology determined by the family of seminorms \( \{\|\cdot\|_p^\infty : p \in P\} \). Clearly, \( \lambda(P) \subset \lambda_{\infty}(P) \), and the embedding is continuous. By the Grothendieck-Pietsch criterion, \( \lambda(P) \) is nuclear if and only if \( \lambda(P) = \lambda_{\infty}(P) \) topologically, which is equivalent to condition (N). If \( \lambda(P) \) is a \( \widehat{\otimes} \)-algebra under pointwise multiplication (i.e., if \( P \prec P^{[2]} \)), then so is \( \lambda_{\infty}(P) \), and the algebra embedding \( \lambda(P) \subset \lambda_{\infty}(P) \) makes \( \lambda_{\infty}(P) \) into a \( \lambda(P) \)-\( \widehat{\otimes} \)-module.

Given a \( \widehat{\otimes} \)-algebra \( A \), we consider \( \mathbb{C} \) as an \( A \)-\( \widehat{\otimes} \)-module by letting \( A \) act on \( \mathbb{C} \) trivially. In other words, \( \mathbb{C} = A+/A. \)

**Theorem 5.** Let \( A = \lambda(P) \) be a Köthe algebra satisfying (B). Then
\[
\text{dg}A = \text{db}A = \begin{cases} 
0, & P \text{ satisfies (U)}. \\
1, & P \text{ satisfies (N) and (M), but does not satisfy (U). In this case, } \text{dh}_A \mathbb{C} = 1. \\
2, & P \text{ satisfies (N), but does not satisfy (M). In this case, } \text{dh}_A \lambda(P) = 2. \\
2, & P \text{ does not satisfy (N). In this case, } \text{dh}_A \lambda_{\infty}(P) = 2. 
\end{cases}
\]

**Proof.** By [4, Theorem 3.5], condition (B) is equivalent to \( A \) being biprojective. Hence (B) implies that \( \text{db}A \leq 2 \) [4, Proposition 2.5.8], and so \( \text{dg}A \leq 2 \). The rest follows from Theorem 3 and [4] Theorems 4.3 and 4.7 together with [7, Proposition 6.11].
3. Nonbiprojective Köthe algebras

In this section, we show that the homological dimensions dg, db, w.dg, and w.db of a nonbiprojective metrizable Köthe algebra are infinite. First we need a lemma.

Lemma 6. Let $P$ be a Köthe set. Suppose that $P^{[k]} \not< P^{[l]}$ for some $k, l \in \mathbb{R}$, $0 < k < l$. Then $P^{[2]} \not< P$.

Proof. Set $r = l/k$. Then $P^{[r]} \not< P$, and, by induction, $P^{[r^n]} \not< P$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ such that $\alpha = r^n \geq 2$. Then for each $p \in P$ there exist $C \geq 1$ and $q \in P$ such that $p^\alpha \leq Cq$ and $p \leq q$. Now fix any $i \in I$. If $p_i \geq 1$, then $p_i^2 \leq p_i^\alpha \leq Cq$. If $p_i < 1$, then $p_i^2 < p_i \leq Cq$. Thus $p_i^2 \leq Cq$, which proves the claim. □

Theorem 7. Let $A = \lambda(P)$ be a metrizable Köthe algebra not satisfying (B). Then for each odd $n \in \mathbb{N}$ we have $\text{Tor}_n^A(\mathbb{C}, \mathbb{C}) \neq 0$. Moreover, the latter space is not Hausdorff. As a corollary, $dg = db = w.dg = w.db = w.db_A \mathbb{C} = \infty$.

Proof. By [3, 2.3.3], the spaces $\text{Tor}_n^A(\mathbb{C}, \mathbb{C})$ are the homology of the chain complex

$$0 \leftarrow \mathbb{C} \leftarrow A \leftarrow A \otimes A \leftarrow \cdots \leftarrow A \otimes^{n+1} A \leftarrow \cdots,$$

the differential being given by

$$d(a_0 \otimes \cdots \otimes a_n) = a_0a_1 \otimes \cdots \otimes a_n$$

$$+ \sum_{k=1}^{n-2} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n$$

$$+ (-1)^{n-1} a_0 \otimes \cdots \otimes a_{n-1} a_n.$$ Identifying $A \otimes^n$ with $\lambda(P^{\times n})$ (see [3]), we see that for each $n$ the differential $d: \lambda(P^{\times (n+1)}) \rightarrow \lambda(P^{\times n})$ acts by the formula

$$d(e_{i_0 \ldots i_n}) = \delta_{i_0 i_1} e_{i_1 \ldots i_n} + \sum_{k=1}^{n-2} (-1)^k \delta_{i_k i_{k+1}} e_{i_0 \ldots \hat{i_k} \ldots i_n} + (-1)^{n-1} \delta_{i_{n-1} i_n} e_{i_0 \ldots \hat{i_n} \ldots i_2 \ldots i_n}, \quad (15)$$

where, as usual, the notation $\hat{i_k}$ indicates that $i_k$ is omitted. For notational convenience, set $e_i^n = e_{i\ldots i}$ with the subscript “$i$” repeated $n$ times. Then (15) implies that

$$d(e_i^{n+1}) = \begin{cases} e_i^n, & n \text{ is odd;} \\ 0, & n \text{ is even.} \end{cases} \quad (16)$$

For each $n \in \mathbb{N}$ and each $i \in I$ set

$$E_i^n = \left\{ \sum \alpha_{k_1 \ldots k_n} e_{k_1 \ldots k_n} \in \lambda(P^{\times n}) : \alpha_{i \ldots i} = 0 \right\}.$$ It follows from (15) that $d(e_{k_0 \ldots k_n}) \in E_i^n$ unless $(k_0, \ldots, k_n) = (i, \ldots, i)$. Therefore

$$d(E_{n+1}^i) \subset E_n^i. \quad (17)$$
Suppose that $n$ is odd, and assume, towards a contradiction, that $\text{Tor}_n^A(\mathbb{C}, \mathbb{C})$ is Hausdorff. This is equivalent to say that the image of $d$: $\lambda(P^{x(n+1)}) \to \lambda(P^{\times n})$ is closed. By the Open Mapping Theorem, $d$ is an open map onto its image. Therefore for each $p \in P$ there exist $q \in P$ and $C > 0$ such that for each $y \in \text{Im} \ d$ there exists $x \in d^{-1}(y)$ satisfying $\|x\|_p \leq C\|y\|_q$.

Now fix any $i \in I$, set $y = e_i^n$ (which belongs to $\text{Im} \ d$ by (16)), and find $x \in d^{-1}(y)$ as above. Since $\lambda(P^{x(n+1)}) = \mathbb{C}e_i^{n+1} \oplus E_{n+1}$, we may decompose $x$ as $x = \alpha e_i^{n+1} + z$, where $z \in E_{n+1}$ and $\alpha \in \mathbb{C}$. Applying $d$ and using (16) and (17), we see that $\alpha = 1$. Therefore

$$p_i^{n+1} \leq \|x\|_p \leq C\|y\|_q = Cq_i^n \quad (i \in I),$$

and so $P^{[n+1]} \prec P[n]$. By Lemma 3, this implies that $P[2] \prec P$, and, finally, $P[2] \sim P$, i.e., (B) holds. The resulting contradiction shows that $\text{Tor}_n^A(\mathbb{C}, \mathbb{C})$ is not Hausdorff, as required. \hfill $\square$

The next theorem summarizes what we know about homological dimensions of metrizable Köthe algebras.

**Theorem 8.** Let $A = \lambda(P)$ be a metrizable Köthe algebra. Then

$$w.dg \ A = w.db \ A = \begin{cases} 0, & P \text{ satisfies (U).} \\ 1, & P \text{ satisfies (B) and (N), but does not satisfy (U). In this case, } \text{w.dh}_A \mathbb{C} = 1. \\ 2, & P \text{ satisfies (B), but does not satisfy (N). In this case, } \text{w.dh}_A \lambda(P) = 2. \\ \infty, & P \text{ does not satisfy (B). In this case, } \text{w.dh}_A \mathbb{C} = \infty. \end{cases}$$

$$dg \ A = db \ A = \begin{cases} 0, & P \text{ satisfies (U).} \\ 1, & P \text{ satisfies (B), (N), and (M), but does not satisfy (U). In this case, } \text{dh}_A \mathbb{C} = 1. \\ 2, & P \text{ satisfies (B) and (N), but does not satisfy (M). In this case, } \text{dh}_A \lambda(P) = 2. \\ \infty, & P \text{ does not satisfy (B). In this case, } \text{dh}_A \mathbb{C} = \infty. \end{cases}$$

**Proof.** As was already mentioned (see the proof of Theorem 3), condition (B) implies that $db \ A \leq 2$. Hence all the dimensions $dg \ A$, $w.dg \ A$, and $w.db \ A$ are $\leq 2$. Now (18) follows from Theorem 7 and from [3, Theorems 5.2 and 6.10, Proposition 6.11], while (19) follows from Theorems 3 and 3. \hfill $\square$

**Remark 2.** It easily follows from Theorem 8 that for a metrizable Köthe algebra $A = \lambda(P)$ the global dimension of $A$ does not depend on whether we consider $A$ as a Fréchet algebra or as a $\mathbb{C}$-algebra.
In this section we compute homological dimensions of the Köthe algebras discussed in Examples 1–5. We will see, in particular, that every combination of (U), (N), (B), (M) described in (18) and (19) is possible.

**Example 6.** The algebra $\ell^1(I)$ satisfies (B), but does not satisfy (N). Therefore for each $d \in \{ dg, db, wdg, wdb \}$ we have $d(\ell^1(I)) = 2$; moreover, we have $dh_{\ell^1(I)} = 2$. For $dg$, $db$, and $dh_{\ell^1(I)}$, this is an old result by Helemskii [2] (see also [3, V.2.16]); for w.db, the result is due to Selivanov [12].

**Example 7.** The algebra $C^I$ satisfies (U), and so $dg C^I = db C^I = 0$. If $C^I$ is metrizable (i.e., if $I$ is at most countable), then $w.dg C^I = w.db C^I = 0$. These results are due to Helemskii [3, Theorem IV.5.27]. Moreover, he proved [loc. cit.] that each commutative Arens-Michael algebra $A$ with $dg A = 0$ is topologically isomorphic to $C^I$ for some $I$.

**Example 8.** The algebra $\Lambda_R(\alpha)$ (see Example 3) satisfies (B) if and only if either $R = 1$ or $R = \infty$ [1, Example 3.5]. Therefore for each $1 < R < \infty$ and each $d \in \{ dg, db, wdg, wdb \}$ we have $d(\Lambda_R(\alpha)) = dh_{\Lambda_R(\alpha)} C = w.dh_{\Lambda_R(\alpha)} C = \infty$. In particular, $d(\mathcal{H}(\mathbb{D}_R)) = \infty$ whenever $1 < R < \infty$.

Before giving further examples, we would like to note that condition (M) is satisfied automatically for many natural Köthe spaces. In particular, if $I = \mathbb{N}$, and if $p_i \leq p_{i+1}$ for each $p \in P$ and each $i \in \mathbb{N}$, then (M) follows from (B) [1, Corollary 7.5].

**Example 9.** The algebra $\Lambda_\infty(\alpha)$ satisfies (B) and hence (M) (see above). Clearly, $\Lambda_\infty(\alpha)$ does not satisfy (U). The Grothendieck-Pietsch criterion implies that $\Lambda_\infty(\alpha)$ satisfies (N) if and only if $sup_n (\log n)/\alpha_n < \infty$ (see, e.g., [3, 29.6 and 28.16]). Therefore for each $d \in \{ dg, db, wdg, wdb \}$ we have

$$d(\Lambda_\infty(\alpha)) = \begin{cases} 1 & \text{if } sup_n (\log n)/\alpha_n < \infty \\ 2 & \text{otherwise.} \end{cases}$$

In particular, $d(s) = d(\mathcal{H}(\mathbb{C})) = 1$.

**Example 10.** The algebra $\Lambda_1(\alpha)$ satisfies (B). The Grothendieck-Pietsch criterion implies that $\Lambda_1(\alpha)$ satisfies (N) if and only if $lim_n (\log n)/\alpha_n = 0$ (see, e.g., [1, 29.6 and 28.16]). However, the latter condition implies that $\Lambda_1(\alpha)$ satisfies (U). Therefore for each $d \in \{ dg, db, wdg, wdb \}$ we have

$$d(\Lambda_1(\alpha)) = \begin{cases} 0 & \text{if } lim_n (\log n)/\alpha_n = 0 \\ 2 & \text{otherwise.} \end{cases}$$

In particular, $d(\mathcal{H}(\mathbb{D}_1)) = 0$.  

4. Examples
Example 11. Let $I = \mathbb{N} \times \mathbb{N}$. For each $i, j, k \in \mathbb{N}$ we define

$$p^{(k)}_{ij} = \begin{cases} 
2^{(kj)}(i + j)^k & (i \leq k), \\
(i + j)^k & (i > k).
\end{cases}$$

Set $p^{(k)} = (p^{(k)}_{ij})_{i,j \in \mathbb{N}}$, and consider the Köthe set $P = \{p^{(k)}\}_{k \in \mathbb{N}}$. As was shown in [4, Theorem 7.9], $P$ satisfies $(B)$ and $(N)$, but does not satisfy $(M)$. Note that, since $p^{(k)}_{ij} \geq 1$ for all $i, j, k$, we have $\lambda(\bar{P}) = \ell^1$. Therefore $\mathrm{dg} \lambda(P) = \mathrm{db} \lambda(P) = \mathrm{dh} \lambda(P) = 2$, while $\mathrm{w.dg} \lambda(P) = \mathrm{w.db} \lambda(P) = \mathrm{w.dh} \lambda(P) \mathbb{C} = 1$.

5. REMARKS ON QUASIBIPROJECTIVITY

It is interesting to compare Theorem 7 with recent results of Selivanov [13] on quasibiprojective Banach algebras. By definition [loc. cit.], a $\hat{\otimes}$-algebra $A$ is quasibiprojective if $A = A^2$ and the map

$$\pi_A : A \hat{\otimes} A \to A \hat{\otimes} A, \quad a \hat{\otimes} b \mapsto a \hat{\otimes} b,$$

is a retraction in $A\text{-mod-}A$. The latter condition means that there exists an $A\hat{\otimes}$-bimodule morphism $\rho_A : A \hat{\otimes} A \to A \hat{\otimes} A$ such that $\pi_A \rho_A = 1$. Each quasibiprojective algebra is biprojective, but the converse is false. For example, Selivanov proved that all sequence algebras $\ell^p$ ($1 \leq p < \infty$) are quasibiprojective, but are not biprojective unless $p = 1$. A similar assertion holds for the convolution algebras $L^p(G)$, where $G$ is an infinite compact group. The Schatten ideals $S^p(H)$ (where $H$ is an infinite-dimensional Hilbert space) are quasibiprojective whenever $1 \leq p \leq 2$, but are not biprojective unless $p = 1$. More examples of quasibiprojective Banach algebras can be found in [13, Theorem 3.16]. As for general $\hat{\otimes}$-algebras, it is easy to show (by using [3, Lemma 5.1]) that all Köthe algebras $\lambda(P)$ are quasibiprojective.

Selivanov proved that, if $A$ is a quasibiprojective, non-biprojective Banach algebra, then $\mathrm{dg} A = \mathrm{db} A = \infty$ (see [13, Theorem 3.14]). In fact, an easy modification of his argument shows that $\mathrm{w.dg} A = \mathrm{w.db} A = \infty$ as well (Selivanov, private communication). It is natural to ask whether or not Selivanov’s theorem can be extended to Fréchet algebras. If yes, then the last statement of our Theorem 7 (except for the equality $\mathrm{w.dh} A \mathbb{C} = \infty$) would be an easy consequence of this general result. However, we do not know whether this general result is true. The difficulty is that Selivanov’s argument heavily relies on some geometric properties of Banach spaces which do not hold for nonnormable Fréchet spaces.

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