XXZ Bethe states as highest weight vectors of the $sl_2$ loop algebra at roots of unity

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(March 22, 2022)

Abstract

We prove some part of the conjecture that regular Bethe ansatz eigenvectors of the XXZ spin chain at roots of unity are highest weight vectors of the $sl_2$ loop algebra. Here $q$ is related to the XXZ anisotropic coupling $\Delta$ by $\Delta = (q + q^{-1})/2$, and it is given by a root of unity, $q^{2N} = 1$, for a positive integer $N$. We show that regular XXZ Bethe states are annihilated by the generators $\bar{x}^+_k$'s, for any $N$. We discuss, for some particular cases of $N = 2$, that regular XXZ Bethe states are eigenvectors of the generators of the Cartan subalgebra, $\bar{h}_k$'s. Here the loop algebra $U(L(sl_2))$ is generated by $\bar{x}_k^+$ and $\bar{h}_k$ for $k \in \mathbb{Z}$, which are the classical analogues of the Drinfeld generators of the quantum loop algebra $U_q(L(sl_2))$. A representation of $U(L(sl_2))$ is called highest weight if it is generated by a vector $\Omega$ which is annihilated by the generators $\bar{x}_k^+$'s and such that $\Omega$ is an eigenvector of the $\bar{h}_k$'s. We also discuss the classical analogue of the Drinfeld polynomial which characterizes the irreducible finite-dimensional highest weight representation of $U(L(sl_2))$.

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I. INTRODUCTION

The XXZ spin chain is one of the most important exactly solvable quantum systems\(^1,2\). The Hamiltonian under the periodic boundary conditions is given by

$$H_{XXZ} = -J \sum_{j=1}^{L} \left( \sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right). \quad (1.1)$$

Here the parameter \(\Delta\) expresses the anisotropic coupling between adjacent spins, and it is related to the \(q\) parameter of the quantum group \(U_q(sl_2)\) by the relation \(\Delta = (q + q^{-1})/2\).

Recently it was shown that when \(q\) is a root of unity, the Hamiltonian commutes with the generators of the \(sl_2\) loop algebra\(^3\). The symmetry of the XXZ Hamiltonian is enhanced at the root of unity. The spectra of the XXZ spin chain have been studied numerically\(^4,5\), and some generalizations of the \(sl_2\) loop algebra symmetry have also been derived for some trigonometric vertex models\(^6\). Furthermore, the spectral degeneracy associated with the \(sl_2\) loop algebra has been shown for the eight-vertex model and some elliptic IRF models\(^7,8\). However, it has been a conjecture that the Bethe ansatz eigenvectors are of highest weight with respect to the \(sl_2\) loop algebra.

In this paper we show several rigorous results supporting the highest-weight conjecture. Based on the algebraic Bethe ansatz method, we show some part of the conjecture that a regular Bethe ansatz eigenvector of the XXZ spin chain should give a highest weight vector of the \(sl_2\) loop algebra in the sense of the Drinfeld realization of the quantum loop algebra \(U_q(L(sl_2))\). Here the regular Bethe ansatz eigenvector is defined by such an eigenvector that is constructed from the Bethe ansatz wave-function where all the rapidities are finite.

Let us consider the classical analogue of the Drinfeld realization of \(U_q(L(sl_2))\)\(^9,10\). The \(sl_2\) loop algebra \(U(L(sl_2))\) is generated by the classical analogues of the Drinfeld generators, \(\bar{x}_k^\pm\) and \(\bar{h}_k\) for \(k \in \mathbb{Z}\). In the sense of the Drinfeld realization, a representation of the loop algebra \(U(L(sl_2))\) is highest weight if it is generated by a vector \(\Omega\) which is annihilated by the generator \(\bar{x}_k\) for any integer \(k\) and such that \(\Omega\) is a common eigenvector of generators of the Cartan subalgebra, \(\bar{h}_k\)’s. For the case of any \(N\), we show that a regular XXZ Bethe state \(|R\rangle\) at the root of unity is annihilated by the generators \(\bar{x}_k^\pm\)’s: \(\bar{x}_k^\pm |R\rangle = 0\), for all \(k \in \mathbb{Z}\). For some particular case of \(N = 2\), we show that the regular XXZ Bethe state \(|R\rangle\) is a common eigenvector of the generators \(\bar{h}_k\)’s of the Cartan subalgebra. Thus, the proof of the highest-weight conjecture is partially completed for the case of \(N = 2\). For the vacuum state, we introduce a method for calculating the eigenvalues of the Cartan generators \(\bar{h}_k\)’s. Some version of the highest-weight conjecture has been addressed in Ref.\(^3\) from a different viewpoint. It is also addressed in Ref.\(^5\).

We also discuss a method for calculating the classical analogue of the Drinfeld polynomial. Through some examples of the Drinfeld polynomial, we suggest a conjecture that the irreducible representation of \(U(L(sl_2))\) generated by a regular XXZ Bethe state with \(R\) down spins on \(L\) sites should be given by \(2^{(L-2R)/N}\), when \(L\) and \(R\) are integral multiples of \(N\). The result is consistent with the analytic computation of the degeneracy shown for the eight-vertex model and some elliptic SOS models associated with the eight-vertex model\(^7,8\). In association with the \(sl_2\) loop algebra, the Drinfeld polynomial is also addressed in Ref.\(^5\) with a different method.
II. SL₂ LOOP ALGEBRA AND THE DRINFELD REALIZATION

A. Loop algebra symmetry of the XXZ spin chain

We review some results on the \( sl_2 \) loop algebra symmetry of the XXZ spin chain. We first introduce the quantum group \( U_q(sl_2) \). The generators \( S^\pm \) and \( S^Z \) satisfy the following relations

\[
[S^+, S^-] = \frac{q^{2S^Z} - q^{-2S^Z}}{q - q^{-1}}, \quad [S^Z, S^\pm] = \pm S^\pm
\]

(2.1)

Here the parameter \( q \) is generic. The comultiplication is given by

\[
\Delta(S^\pm) = S^\pm \otimes q^{-S^Z} + q^{S^Z} \otimes S^\pm, \quad \Delta(S^Z) = S^Z \otimes I + I \otimes S^Z
\]

(2.2)

We consider the \( L \)th tensor product of spin 1/2 representation \( V^\otimes L \). The representations of the generators \( S^\pm \) and \( S^Z \) are given by

\[
q^{S^Z} = q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2}
\]

\[
S^\pm = \sum_{j=1}^{L} q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \otimes \sigma^\pm_j \otimes q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2}
\]

(2.3)

Considering the automorphism of the \( U_q(L(sl_2)) \), we may introduce the following operator

\[
T^\pm = \sum_{j=1}^{L} q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \otimes \sigma^\pm_j \otimes q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2}
\]

(2.4)

Let us now define the operators \( S^{\pm(N)} \) and \( T^{\pm(N)} \) by

\[
S^{\pm(N)} = (S^\pm)^{N}/[N]!, \quad T^{\pm(N)} = (T^\pm)^{N}/[N]!
\]

(2.5)

Here we have used the \( q \)-integer \([n] = (q^n - q^{-n})/(q - q^{-1})\) and the \( q \)-factorial \([n]! = [n][n-1] \cdots [1] \). We also note that \( q \) is not a root of unity. Then, we can show the following expressions

\[
S^{\pm(N)} = \sum_{1 \leq j_1 < \cdots < j_N \leq L} q^{\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{N}{2}\sigma^Z} \otimes \sigma^\pm_{j_1} \otimes q^{\frac{(N-2)}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{(N-2)}{2}\sigma^Z}
\]

\[
\otimes \sigma^\pm_{j_2} \otimes q^{\frac{N-4}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{N-4}{2}\sigma^Z} \otimes q^{-\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{N}{2}\sigma^Z}.
\]

(2.6)

For \( T^{\pm(N)} \) we have

\[
T^{\pm(N)} = \sum_{1 \leq j_1 < \cdots < j_N \leq L} q^{-\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{N}{2}\sigma^Z} \otimes \sigma^\pm_{j_1} \otimes q^{\frac{(N-2)}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{(N-2)}{2}\sigma^Z}
\]

\[
\otimes \sigma^\pm_{j_2} \otimes q^{-\frac{(N-4)}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{(N-4)}{2}\sigma^Z} \otimes q^{\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{N}{2}\sigma^Z}.
\]

(2.7)

Here we recall that \( q \) is generic.
Let the symbol $\tau_{6V}(v)$ denotes the (inhomogeneous) transfer matrix of the six-vertex model. We now take the parameter $q$ a root of unity. We consider the limit of sending $q$ to a root of unity: $q^{2N} = 1$. Then we can show the (anti) commutation relations\(^3\) in the sector of $S^Z \equiv 0 \pmod N$

$$S^{\pm(N)} \tau_{6V}(v) = q^N \tau_{6V}(v) S^{\pm(N)}, \quad T^{\pm(N)} \tau_{6V}(v) = q^N \tau_{6V}(v) T^{\pm(N)}$$

(2.8)

Here we recall that $S^Z$ denotes the Z-component of the total spin operator.

Since the XXZ Hamiltonian $H_{XXZ}$ is given by the logarithmic derivative of the (homogeneous) transfer matrix $T_{6V}(v)$, we have in the sector $S^Z \equiv 0 \pmod N$

$$[S^{\pm(N)}, H_{XXZ}] = [T^{\pm(N)}, H_{XXZ}] = 0.$$ \hspace{1cm} (2.9)

Thus, the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ commute with the XXZ Hamiltonian in the sector $S^Z \equiv 0 \pmod N$.

Let us now consider the algebra generated by the operators $S^{\pm(N)}$ and $T^{\pm(N)}$.\(^3\) When $q$ is a primitive $2N$th root of unity, or a primitive $N$th root of unity with $N$ odd, we consider the identification in the following:\(^3\):

$$E_0^+ = S^{+(N)}, \quad E_0^- = S^{-(N)}, \quad E_1^+ = T^{-(N)}, \quad E_1^- = T^{+(N)}, \quad H_0 = -H_1 = \frac{2}{N} S^Z. \hspace{1cm} (2.10)$$

It is shown in Ref.\(^3\) that the operators $E_j^\pm, H_j$ for $j = 0, 1$, satisfy the defining relations of the algebra $U(L(sl_2))$.

Noticing the automorphism

$$\theta(E_0^+) = E_1^+, \quad \theta(H_0) = H_1 \hspace{1cm} (2.11)$$

we may take the identification in the following:

$$E_0^+ = T^{-(N)}, \quad E_0^- = T^{+(N)}, \quad E_1^+ = S^{+(N)}, \quad E_1^- = S^{-(N)}, \quad H_0 = -H_1 = \frac{2}{N} S^Z. \hspace{1cm} (2.12)$$

B. Classical analogue of the Drinfeld realization

Let us review briefly the Drinfeld realization of the quantum loop algebra\(^9,10\). The quantum affine algebra $U_q(sl_2)$ is isomorphic to the associative algebra over $\mathbb{C}$ with generators $x_k^\pm (k \in \mathbb{Z}), h_k (k \in \mathbb{Z} \setminus \{0\}), K^\pm$, central element $C^\pm$, and the following relations:

$$CC^{-1} = C^{-1} C = KK^{-1} = K^{-1} K = 1$$

$$[h_k, h_\ell] = \delta_{k,-\ell} \frac{1}{k} [2k] \frac{C^k - C^{-k}}{q-q^{-1}}$$

$$K h_k = h_k K, \quad K x_k^\pm K^{-1} = q^{\pm 2} x_k^\pm$$

$$[h_k, x_\ell^\pm] = \pm \frac{1}{k} [2k] C^{\mp(k+|k|)/2} x_{k+\ell}^\pm$$

$$x_{k+1}^\pm x_k^\mp - q^{\pm 2} x_k^\pm x_{k+1}^\mp = q^{\pm 2} x_k^\pm x_{k+1}^\mp - x_{k+1}^\pm x_k^\mp$$

4
\[
[x^+_k, x^-_\ell] = \frac{1}{q - q^{-1}} (C^{k-\ell} \psi_{k+\ell} - \phi_{k+\ell})
\]
\[
\sum_{k=0}^\infty \bar{\psi}_k u^k = K \exp \left( (q - q^{-1}) \sum_{k=1}^\infty h_k u^k \right)
\]
\[
\sum_{k=0}^\infty \bar{\phi}_{-k} u^{-k} = K^{-1} \exp \left( -(q - q^{-1}) \sum_{k=1}^\infty h_{-k} u^{-k} \right)
\]
\[(2.13)\]

Let us consider the classical analogue of the Drinfeld realization\(^{10}\). Putting \(q = \exp \epsilon\), we take the limit of \(\epsilon\) to zero: \(q = 1 + \epsilon + \cdots\). Hereafter we assume the trivial center: \(C^{\pm 1} = 1\).

We define the classical analogs of the Drinfeld generators as follows.

\[
K = q^\hbar_0
\]
\[
h_k = \bar{h}_K + O(\epsilon) \quad (k \in \mathbb{Z} \setminus \{0\})
\]
\[
x^+_k = \bar{x}^+_k + O(\epsilon) \quad (k \in \mathbb{Z})
\]
\[(2.14)\]

From the last two relations of the set of defining relations of eqs. (2.13), we have the following relations

\[
\psi_k = 2\epsilon \bar{h}_k + O(\epsilon^2) \quad (k \geq 1)
\]
\[
\phi_{-k} = -2\epsilon \bar{h}_{-k} + O(\epsilon^2) \quad (k \geq 1).
\]
\[(2.15)\]

Then, from the classical limit of the quantum loop algebra, we have the following relations:

\[
[\bar{h}_k, \bar{h}_\ell] = 0
\]
\[
[\bar{h}_k, x^+_\ell] = \mp 2x^+_k x^+_\ell
\]
\[
\bar{x}^+_k \bar{x}^+_\ell - \bar{x}^+_\ell \bar{x}^+_k = \bar{x}^+_k \bar{x}^+_\ell - \bar{x}^+_\ell \bar{x}^+_k
\]
\[
[\bar{x}^+_k, \bar{x}^-_\ell] = \bar{h}_{k+\ell}
\]
\[(2.16)\]

Here \(k, \ell \in \mathbb{Z}\).

Let us consider the classical limit of the isomorphism of the Drinfeld realization of the quantum loop group to the quantum affine algebra \(U_q(\hat{sl}_2)\). It is the isomorphism between the \(sl_2\) loop algebra and the classical limit of the quantum affine algebra given by the following:

\[
E^+_1 \mapsto \bar{x}^+_0
\]
\[
E^+_0 \mapsto \bar{x}^-_1 \quad E^-_0 \mapsto \bar{x}^+_1
\]
\[
-H_0 = H_1 \mapsto \bar{h}_0
\]
\[(2.17)\]

Applying the isomorphism to the identification (2.12), we have the correspondence:

\[
\bar{x}^+_0 = T^{-(N)} , \quad \bar{x}^-_0 = T^{+(N)} , \quad \bar{x}^+_1 = S^{+(N)} , \quad \bar{x}^-_1 = S^{-(N)} , \quad \bar{h}_0 = \frac{2}{N} S^Z
\]
\[(2.18)\]

Let us now give the definition of a highest weight representation for the loop algebra \(U(L(sl_2))\). They are given by the following:
A representation of the sl$_2$ loop algebra is highest weight if it is generated by a vector $\Omega$ which is annihilated by the generator $\bar{x}_k^+$ for all $k \in \mathbb{Z}$ and such that $\Omega$ is an eigenvector of the Cartan generator $\bar{h}_k$ for $k \in \mathbb{Z}$.

Thus, a vector $\Omega$ is highest weight if it is annihilated by $\bar{x}_k^+$ for all integer $k$, and it is an eigenvector of the generator $\bar{h}_k$ for all integer $k$.

### III. FORMULAS OF ALGEBRAIC BETHE ANSATZ

#### A. R matrix and L operator

Let us summarize some formulas of the algebraic Bethe ansatz$^{11-13}$. We define the $R$ matrix of the XXZ spin chain by

$$
R(z - w) = \begin{pmatrix}
  f(w - z) & 0 & 0 & 0 \\
  0 & g(w - z) & 1 & 0 \\
  0 & 1 & g(w - z) & 0 \\
  0 & 0 & 0 & f(w - z)
\end{pmatrix}
$$

(3.1)

where $f(z - w)$ and $g(z - w)$ are given by

$$
f(z - w) = \frac{\sinh(z - w - 2\eta)}{\sinh(z - w)} \quad g(z - w) = \frac{\sinh(-2\eta)}{\sinh(z - w)}
$$

(3.2)

Hereafter we shall write $f(t_1 - t_2)$ by $f_{12}$ for short.

We now introduce $L$ operators for the XXZ spin chain

$$
L_n(z) = \begin{pmatrix}
  L_n(z)_1^1 & L_n(z)_1^2 \\
  L_n(z)_2^1 & L_n(z)_2^2
\end{pmatrix} = \begin{pmatrix}
  \sinh (zI_n + \eta \sigma_n^z) & \sinh 2\eta \sigma_n^+ \\
  \sinh 2\eta \sigma_n^- & \sinh (zI_n - \eta \sigma_n^z)
\end{pmatrix}
$$

(3.3)

Here $I_n$ and $\sigma_n^a$ ($n = 1, \ldots, L$) are acting on the $n$th vector space $V_n$. The $L$ operator is an operator-valued matrix which acts on the auxiliary vector space $V_0$. The symbols $\sigma^\pm$ denote $\sigma^+ = E_{12}$ and $\sigma^- = E_{21}$, and $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices.

In terms of the $R$ matrix and $L$ operators, the Yang-Baxter equation is expressed as

$$
R(z - t) \ (L_n(z) \otimes L_n(t)) = (L_n(t) \otimes L_n(z)) \ R(z - t)
$$

(3.4)

We define the monodromy matrix $T$ by the product: $T(z) = L_L(z) \cdots L_2(z)L_1(z)$, and the transfer matrix $\tau_{6V}(z)$ by the trace: $\tau_{6V}(z) = \text{Tr} T(z)$. The matrix elements of $T(z)$

$$
T(z) = \begin{pmatrix}
  A(z) & B(z) \\
  C(z) & D(z)
\end{pmatrix}
$$

(3.5)

satisfy the commutation relations derived from the Yang-Baxter equations such as $B_1B_2 = B_2B_1$ and

$$
A_1B_2 = f_{12}B_2A_1 - g_{12}B_1A_2, \quad D_1B_2 = f_{21}B_2D_1 - g_{21}B_1D_2
$$

(3.6)

Here we recall $B_1$ denotes $B(t_1)$. Furthermore we can show.$^{13}$
\[ C_0 B_1 \cdots B_n = B_1 B_2 \cdots B_n C_0 \]
\[ + \sum_{j=1}^{n} B_1 \cdots B_{j-1} B_{j+1} \cdots B_n g_{0j} \left\{ A_0 D_j \prod_{k \neq j} f_{0k} f_{kj} - A_j D_0 \prod_{k \neq j} f_{k0} f_{jk} \right\} \]
\[ - \sum_{1 \leq j < k \leq n} B_0 B_1 \cdots B_{j-1} B_{j+1} \cdots B_{k-1} B_{k+1} \cdots B_n \]
\[ \times g_{0j} g_{0k} \{ A_j D_k f_{jk} \prod_{\ell \neq j, k} f_{j\ell} f_{\ell k} + A_k D_j f_{kj} \prod_{\ell \neq j, k} f_{k\ell} f_{\ell j} \} \quad (3.7) \]

Let us denote by \( |0\rangle \) the vector where all the spins are up. Then we can show

\[ A(z)|0\rangle = a(z)|0\rangle, \quad D(z)|0\rangle = d(z)|0\rangle \quad (3.8) \]

where \( a(z) = \sinh^L (z + \eta) \) and \( d(z) = \sinh^L (z - \eta) \).

Making use of the commutation relations (3.6) one can show that the vector \( B_1 B_2 \cdots B_R |0\rangle \) is an eigenvector of the XXZ spin Hamiltonian if the rapidities \( t_1, t_2, \ldots, t_R \) satisfy the Bethe ansatz equations

\[ \frac{a_j}{d_j} = \prod_{k \neq j} \left( \frac{f_{kj}}{f_{jk}} \right) \quad \text{for} \quad j = 1, \cdots R. \quad (3.9) \]

### B. Formulas with infinite rapidities

Let us normalize the operators \( A(z) \)'s as follows

\[ \hat{A}(z) = A(z)/n(z) \quad \hat{B}(z) = B(z)/(g(z)n(z)) \]
\[ \hat{D}(z) = D(z)/n(z) \quad \hat{C}(z) = B(z)/(g(z)n(z)) \].

(3.10)

Here the normalization factor \( n(z) \) is given by \( n(z) = \sinh^L z \). Then, we can show the following:

\[ \hat{A}(\pm \infty) = q^{\pm S_z} \quad \hat{B}(\infty) = -T^- \quad \hat{B}(-\infty) = -S^- \]
\[ \hat{D}(\pm \infty) = q^{\mp S_z} \quad \hat{C}(\infty) = -S^+ \quad \hat{C}(-\infty) = -T^+ \]

(3.11)

Here, we recall that \( S^\pm \) and \( S^z \) are defined on the \( L \)th tensor product of spin 1/2 representations. Hereafter, we shall write \( \hat{C}(\infty) \)'s simply as \( \hat{C}_\infty \)'s.

We now derive quite a useful formula. Sending \( z_0 \) to infinity in eq. (3.7), we have the following:

\[ \hat{C}_\infty B_1 \cdots B_M |0\rangle = \sum_{j=1}^{M} B_1 \cdots B_{j-1} B_{j+1} \cdots B_M |0\rangle \]
\[ \times e^{t_j} \left( q^{(L/2)-(M-1)} d_j \prod_{k \neq j} f_{kj} - q^{-(L/2)+(M-1)} a_j \prod_{k \neq j} f_{jk} \right) \quad (3.12) \]

where \( t_1, \ldots, t_M \) are given arbitrary. Applying the formula (3.12) \( N \) times, we have an important formula in the following:
We may assume they satisfy the Bethe ansatz equations (3.9), then we have

$$\left(\hat{C}_\infty\right)^N B_1 \cdots B_M |0\rangle$$

$$= \sum_{S_N \subseteq \Sigma_M} \sum_{P \in S_N} \left( \prod_{j \in \Sigma_M \setminus S_N} B_j \right) |0\rangle \exp(\sum_{j \in S_N} t_j)$$

$$\prod_{n=1}^N \left( q^{(L/2)-(R-n)} d_{jP_n} \prod_{k \neq jP_1,\ldots,jP_n} f_{kjP_n} - q^{-(L/2)+(R-n)} a_{jP_n} \prod_{k \neq jP_1,\ldots,jP_n} f_{kjP_n} \right)$$

(3.13)

Here the set $\Sigma_M$ is given by $\Sigma_M = \{1, 2, \ldots, M\}$ and $S_N$ denote the subset of $\Sigma_M$ with $N$ elements. The symbol $S_N$ denotes the symmetry group of $N$ elements. The symbol $P \in S_N$ means that $P$ is a permutation of $N$ letters. Here we recall that $t_1, \ldots, t_M$ are given arbitrary.

**IV. PROOF OF THE ANNIHILATION OF REGULAR XXZ BETHE VECTORS BY THE DRINFELD GENERATORS**

**A. The $N$th power of $C$ operators acting on regular Bethe states**

Let us now discuss the proof for the annihilation property: when $t_1, \ldots, t_R$ are finite and they satisfy the Bethe ansatz equations (3.9), then we have

$$S^{+(N)} B_1 \cdots B_R |0\rangle = 0$$

(4.1)

We may assume $R > N$, otherwise it is trivial. Making use of the Bethe ansatz equations (3.9) we have the following formula:

$$\prod_{n=1}^N \left( q^{(L/2)-(R-n)} d_{jP_n} \prod_{k \neq jP_1,\ldots,jP_n} f_{kjP_n} - q^{-(L/2)+(R-n)} a_{jP_n} \prod_{k \neq jP_1,\ldots,jP_n} f_{kjP_n} \right)$$

$$= \left( \prod_{n=1}^N a_{jP_n} \right) \prod_{n=1}^N \left( \prod_{k \notin S_N} f_{kjP_n} \right) \prod_{n=1}^N \left( q^{(L/2)-(R-n)} \prod_{k \neq jP_1,\ldots,jP_n} f_{kjP_n + jP_{n+1}} - q^{-(L/2)+(R-n)} \prod_{\ell=1}^{n-1} f_{jP_\ell jP_n} \right)$$

(4.2)

By induction on $N$, we can show the following formula:

$$\sum_{P \in S_N} \prod_{n=1}^N x_n \prod_{\ell=1}^{n-1} f_{jP_\ell jP_n} - y_n \prod_{\ell=1}^{n-1} f_{jP_\ell jP_n} = \prod_{n=1}^N (x_n - y_n) \times \sum_{P \in S_N} \prod_{n=1}^N \prod_{\ell=1}^{n-1} f_{jP_\ell jP_n}$$

(4.3)

Applying the formula (4.3) we obtain

$$\left(\hat{C}_\infty\right)^N B_1 \cdots B_R |0\rangle$$

$$= \sum_{S_N \subseteq \Sigma_R} \left( \prod_{\ell \in \Sigma_R \setminus S_N} B_\ell \right) |0\rangle \exp(\sum_{j \in S_N} t_j) \left( \prod_{j \in S_N} a_j \right) \prod_{j \in S_N} \left( \prod_{k \notin S_N} f_{jk} \right)$$

$$\prod_{n=1}^N \left( q^{(L/2)-(R-n)} - q^{-(L/2)+R-n} \right) \sum_{P \in S_N} \prod_{n=1}^N \prod_{\ell=1}^{n-1} f_{jP_\ell jP_n}$$

(4.4)
Noting the relation

$$\hat{C}(\infty)^N = (-1)^N \left(S^+\right)^N,$$  \quad (4.5)

we divide the both hand sides of (4.4) by the $q$ factorial $[N]!$, and we have for the case of generic $q$

$$S^{+(N)} B_1 B_2 \cdots B_R |0\rangle$$

$$= (-1)^N \sum_{S_N \subseteq \Sigma_R} \sum_{P \in S_N} \left( \prod_{t \in \Sigma_R \backslash S_N} B_t \right) |0\rangle \exp(\sum_{j \in S_N} t_j) \left( \prod_{j \in S_N} a_j \right) \prod_{k \in S_N} \left( \prod_{\ell \in S_N} f_{jk} \right)$$

$$\times \left[ \frac{L}{2} - R + N \right]_q (q - q^{-1})^N \left( \sum_{P \in S_N} \prod_{n=1}^N \prod_{\ell=1}^{n-1} f_{j_P \ell_P n} \right) \quad (4.6)$$

Here, we have defined the $q$-binomial by

$$[m]_q! = \frac{[m]!}{[m-n]! [n]!}, \quad \text{for} \quad m \geq n.$$  \quad (4.7)

**B. The case of roots of unity**

Let us now consider the case when $q$ is a root of unity. When $q$ is a $2N$th root of unity or an $N$th root of unity with $N$ odd, we can show the following equality

$$\sum_{P \in S_N} \prod_{n=1}^N \prod_{\ell=1}^{n-1} f_{j_P \ell_P n} = 0 \quad (4.8)$$

Let us define a function $F(z_1, \cdots, z_N)$ by

$$F(z_1, \cdots, z_N) = \sum_{P \in S_N} \prod_{n=1}^N \prod_{\ell=1}^{n-1} f_{j_P \ell_P n}$$  \quad (4.9)

Then we can show that there is no pole for $F(z_1, \cdots, z_N)$ with respect to any variable $z_j$, and also that $F(z_1, \cdots, z_N)$ vanishes when one of the variables is sent to infinity: $z_j = \infty$ for some $j$. Therefore we have the equality (4.8). It follows from (4.6) and (4.8) that the action of the operator $S^{+(N)}$ on the regular Bethe ansatz state $B_1 \cdots B_R |0\rangle$ is given by zero.

By a similar method with $S^{+(N)}$, we can show that the operator $T^{+(N)}$ makes the regular Bethe ansatz eigenstate vanishes. Thus, we have obtained

$$S^{+(N)} B_1 \cdots B_M |0\rangle = 0, \quad T^{+(N)} B_1 \cdots B_M |0\rangle = 0.$$  \quad (4.10)
V. EIGENVALUES OF THE CARTAN GENERATORS ON REGULAR BETHE STATES

We discuss a method by which we can calculate the eigenvalue of the generator $\hat{h}_k$ on a regular Bethe state. For an illustration, we consider the case of $N = 2$.

Let us calculate the eigenvalue of the operator $S^+(2)T^-(2)$ on a regular Bethe state with $R$ down-spins: $|R⟩ = B_1 \cdots B_R|0⟩$. Here we recall that $R$ denotes the number of regular rapidities. We set $M = R + 2$. We assume that $t_1, \ldots, t_R$ satisfy the Bethe ansatz equations, while $z_{R+1}$ and $z_{R+2}$ are sent to infinity, later. We now consider the formula (??) for the case of $N = 2$ in the following:

$$\left(\hat{C}_\infty\right)^2 B_1 B_2 \cdots B_M|0⟩ = \sum_{S_2 \subseteq \Sigma_M} \sum_{P \in S_2} \prod_{\ell \in \Sigma_M \setminus S_2} B_\ell|0⟩ \exp(\sum_{j \in S_2})$$

$$\times \prod_{n=1}^2 \left( a_{j,n} q^{-L/2} \prod_{k \neq j_1, \ldots, j_{n-1}} (q f_{j_{n-1}, k}) - d_{j,n} q^{L/2} \prod_{k \neq j_1, \ldots, j_n} (q^{-1} f_{j_{n}, k}) \right)$$

(5.1)

Here we recall $T^- = -\hat{T}(\infty)$ and $S^+ = -\hat{C}(\infty)$. In eq. (5.1) we have four cases for the set $S_2$: $S_2 = \{R + 1, R + 2\}$, $\{j, R + 2\}$, $\{j, R + 1\}$ and $\{j_1, j_2\}$. Here, $j, j_1, j_2 \in \{1, 2, \ldots, R\}$. For an illustration, we highlight the term with $S_2 = \{R + 1, R + 2\}$ in the following:

$$\left(\hat{C}_\infty\right)^2 \prod_{\ell \in \Sigma_M} B_\ell|0⟩$$

$$= \prod_{\ell = 1}^R B_\ell|0⟩ e^{z_{R+1} + z_{R+2}} \sum_{P \in S_2} \prod_{n=1}^2 \left( a_{j,n} q^{-L/2} \prod_{k \neq j_1, \ldots, j_{n-1}} (q f_{j_{n-1}, k}) - d_{j,n} q^{L/2} \prod_{k \neq j_1, \ldots, j_n} (q^{-1} f_{j_{n}, k}) \right)$$

+ \ldots.

(5.2)

After sending $z_{R+1}$ to infinity, we take the limit of sending $z_{R+2}$ to infinity. Then, we have the following:

$$\lim_{z_{R+2} \to \infty} \left( \lim_{z_{R+1} \to \infty} \left(\hat{C}_\infty\right)^2 B_1 \cdots B_R \hat{B}_{R+1} \hat{B}_{R+2}|0⟩ \right)$$

$$= \prod_{\ell = 1}^R B_\ell|0⟩ \left[ L C_2 + [3] \sum_{k=1}^R e^{4t_k} \right] - [2] L \sum_{k=1}^R e^{2t_k} + [2] \sum_{j<k} e^{2(t_j + t_k)}$$

$$+ \sum_{j=1}^R \left[ \hat{B}_\infty \prod_{\ell \neq j} B_\ell|0⟩ \left( -(q^2 + q^{-2}) \sum_{k \neq j} e^{2t_k} + (L - 1) - e^{2t_j} \right) - \frac{\delta \hat{B}_\infty}{\delta \epsilon_{R+2}} \prod_{\ell \neq j} B_\ell|0⟩ \right]$$

$$\times \left[ \frac{L}{2} - R - 2\left[2] (q - q^{-1}) e^{t_j} \prod_{k \neq j} f_{j_k} \right.$$}

$$+ \sum_{1 \leq j_1 < j_2 \leq R} \left( \hat{B} \infty \right)^2 \prod_{\ell = 1; \ell \neq j_1, j_2}^R B_\ell|0⟩ e^{t_{j_1} + t_{j_2}} (q - q^{-1})^2$$

$$\times \left( \frac{L}{2} - R - 2\left[2] (L - 3)[2] a_{j_1} a_{j_2} \prod_{k \neq j_1, j_2} f_{j_1 k, f_{j_2, k}} \right).$$

(5.3)

Here, $\epsilon_{R+2} = \exp(-2z_{R+2})$. We note that $q$ is generic, so far.
We multiply the normalization factor $[2]^2$ with (5.3). Then we take the limit of sending $q$ to a root of unity. If $L/2 - R \equiv 0 \pmod{2}$, then all the unwanted terms vanish. We have

$$S^{(2)}T^{-(2)} B_1 \cdots B_R |0\rangle = \left( L C_2 + [3] \sum_{k=1}^{R} e^{4\pi i k} \right) B_1 \cdots B_R |0\rangle \quad (5.4)$$

VI. CLASSICAL ANALOGUE OF THE DRINFELD POLYNOMIAL

A. Definition

Let us denote the eigenvalue of the generator $\bar{h}_k$ on $\Omega$ as

$$\bar{h}_k \Omega = \bar{d}^+_k \Omega \quad \text{and} \quad \bar{h}_{-k} \Omega = \bar{d}^-_k \Omega \quad (6.1)$$

for $k \geq 1$ and $k \in \mathbb{Z}$.

The classical analogue of the Drinfeld polynomial $P(u)$ satisfies

$$\sum_{k=1}^{\infty} \bar{d}_k u^k = -u \frac{P'(u)}{P(u)}, \quad \sum_{k=1}^{\infty} \bar{d}_{-k} u^{-k} = \deg P - u \frac{P'(u)}{P(u)}. \quad (6.2)$$

B. Some useful formulas

Recall that $\bar{x}^\pm_k$ and $\bar{h}_k$ denote the classical analogues of the Drinfeld generators $x^\pm_k$ and $h_k$, respectively. Then, we can show

$$\bar{x}^+_k = \frac{1}{2^k} \left( \text{ad}_{\bar{h}_1} \right)^k \bar{x}^+_0, \quad \bar{x}^-_k = \frac{1}{2^k} \left( \text{ad}_{\bar{h}_{-1}} \right)^k \bar{x}^-_0 \quad (6.3)$$

for $k > 0$. Here $\bar{h}_1$ is given by

$$\bar{h}_1 = [\bar{x}^+_0, \bar{x}^-_1], \quad \bar{h}_{-1} = (-1)[\bar{x}^+_1, \bar{x}^+_0], \quad (6.4)$$

From the second identification (2.12) we have

$$\bar{h}_1 = [S^{(N)}, T^{-(N)}], \quad \bar{h}_{-1} = [S^{-(N)}, T^{+(N)}] \quad (6.5)$$

In the second identification (2.12), we have

$$\bar{x}^+_0 = S^{+(N)} \quad (6.6)$$

Thus, we can express the generators $\bar{x}^+_k$ in terms of $S^{\pm(N)}$ and $T^{\pm(N)}$ for any integer $k$.

The equation (6.3) can also be formulated as

$$[\bar{h}_1, \bar{x}_k] = 2\bar{x}^+_k \quad (6.7)$$

Making use of eqs. (6.7) and (4.10), we can recursively show that the generator $\bar{x}^+_k$ vanishes on regular XXZ Bethe ansatz eigenvectors at roots of unity. Here, we have assumed that
the action of $\bar{h}_1$ on a regular Bethe ansatz state can be calculated with the method of §5. For $N = 2$, it is finished as shown in §5. Precisely speaking, however, it is a conjecture for $N > 2$.

In order to calculate the evaluation parameters of the Drinfeld polynomial, we can use the following:

$$\bar{h}_{k+1} = \frac{1}{2}[[\bar{h}_k, \bar{x}_0^+], \bar{x}_1^-]$$

(6.8)

C. An example of the classical analogue of the Drinfeld polynomial

For $L = 6$ and $N = 3$, the Drinfeld polynomial of the degenerate eigenspace for $|0\rangle$ is given by

$$P(u) = (1 - a_1 u)(1 - a_2 u)$$

(6.9)

where $a_1$ and $a_2$ are given by $10 \pm 3\sqrt{11}$. The roots are distinct and the degree of $P$ is two, so that the dimension is given by $2^2 = 4$.

VII. DISCUSSION

Through the classical analogue of the Drinfeld polynomial, we can calculate the dimension of the multiplets of the $sl_2$ loop algebra.

The dimensions of the $sl_2$ multiplets should be consistent with that of the XYZ model: $2^{(L-2M)/N}$? In fact, we can show that when $L = 2M$ and the rapidities $t_1, \ldots, t_M$ are finite-valued solutions to the Bethe ansatz equations we have

$$S^{\pm(N)}B_1 \cdots B_M|0\rangle = T^{\pm(N)}B_1 \cdots B_M|0\rangle = 0$$

(7.1)

More details of computation of the classical analogues of the Drinfeld polynomials should be reported elsewhere.

Acknowledgements

This work is partially supported by the Grant-in-Aid (No. 14702012).
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