Tracking $f(R)$ cosmology

Mahmood Roshan and Fatimah Shojai

Department of Physics, University of Tehran, Tehran, Iran

Metric $f(R)$ gravity theories are conformally equivalent to models of quintessence in which matter is coupled to dark energy. We derive a condition for stable tracker solution for metric $f(R)$ gravity in the Einstein frame. We find that tracker solutions with $-0.361 < \omega_\phi < 1$ exist if $0 < \Gamma < 0.217$ and $\frac{df}{dR} \in (R) > 0$, where $\Gamma = \frac{1}{12a^2}$ is dimensionless function, $\omega_\phi$ is the equation of state parameter of the scalar field and $\tilde{R}$ refers to Jordan frame’s curvature scalar. Also, we show that there exists $f(\tilde{R})$ gravity models which have tracking behavior in the Einstein frame and so the curvature of space time is decreasing with time while they lead to the solutions in the Jordan frame that the curvature of space time can be increasing with time.

I. INTRODUCTION

Fourth order $f(R)$ gravity theories can be considered as a candidate for solving the major challenge of cosmology i.e. the late time accelerated expansion of the universe [1], without introducing any exotic matter sources [2](and references therein). These theories have particular features among the other modified theories of gravity. $f(R)$ modifications to GR appear in the low-energy effective actions of quantum gravity and the quantization of fields in curved spacetime. These theories suggest a completely geometric origin for both the early time inflation and the late time cosmic acceleration.

These theories are conformally related to GR with a self-interacting scalar field [3]. Although these models can rise to a natural acceleration mechanism, there exists some features in them which make their viability disputable. For examples these models predict an amount $\frac{1}{2}$ for PPN parameter $\gamma$, which is a gross violation of the experimental bound $|\gamma - 1| < 2.3 \times 10^{-5}$ [4]. Albeit chameleon $f(R)$ gravity models can pass the solar system tests but in the sense of cosmological considerations these theories are observationally indistinguishable from a cosmological constant [5].

Any way, our purpose here is to find out a condition for stable tracker solutions for metric $f(R)$ gravity models in the Einstein frame. Although we pass from fourth order gravity to scalar-tensor gravity in which equations are mathematically simpler, but its physical relevance is still controversial [6]. However, following standard procedures one should not conclude equivalently about the physical relation between the results. It has been demonstrated that passing from one frame to another can alert the physical meanings of the results (see [7] and references therein). For example the stability of solutions can be completely different in the two frames [8]. However, we pay our attention just to the Einstein frame.

Consider the general action of these theories in Jordan frame

$$S_J = \int d^4 x \sqrt{-\tilde{g}} \left[ f(\tilde{R}) \frac{\tilde{R}}{12a^2} + \mathcal{L}_m(\tilde{g}_{\mu\nu}) \right].$$

Where $\alpha = \sqrt{\frac{4\pi G}{c}}$ and all tilded quantities are in Jordan frame. Under the conformal transformation $g_{\mu\nu} = e^{2\alpha\varphi} \tilde{g}_{\mu\nu}$, where $\varphi = \frac{1}{2\alpha} \ln f'$ and prime denotes derivative with respect to $\tilde{R}$, we obtain the Einstein frame action

$$S_E = \int d^4 x \sqrt{-g} \left[ R \frac{\tilde{R}}{12a^2} \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) + \mathcal{L}_m(e^{-2\alpha\varphi} g_{\mu\nu}) \right].$$

Where $V = (\tilde{R} f' - f)/12a^2 f'^2$. We see that in the Einstein frame the scalar field couples minimally to gravity but couples conformally to matter fields via the function $e^{-2\alpha\varphi}$. For a spatially flat FRW space-time the modified Friedmann equations and the equation of motion of scalar field are given by

$$H^2 = 2\alpha^2 (\rho_\varphi + \rho_m)$$

$$\dot{H} = -3\alpha^2 [\rho_m(1 + \omega_m) + \rho_\varphi(1 + \omega_\varphi)]$$

$$\ddot{\varphi} + 3H \dot{\varphi} + V_\varphi = \alpha(1 - 3\omega_m)\rho_m$$

Where $\rho_m$ and $p_m$ are the energy density and pressure of cosmic fluid in the Einstein frame. Also, $\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$ and $p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$ represent the energy density and pressure of the scalar field.

In Einstein frame, the scalar field and the cosmic fluid satisfy the conservation equations

$$\dot{\rho}_\varphi + 3H(1 + \omega_\varphi)\rho_\varphi = \alpha \dot{\varphi}(\rho_m - 3p_m)$$

$$\dot{\rho}_m + 3H(1 + \omega_m)\rho_m = -\alpha \dot{\varphi}(\rho_m - 3p_m)$$

and the energy density of matter $\rho_m$, pressure $p_m$, cosmic time $t$ and scale factor $a$ are related to their Jordan frame counterparts through

$$\rho_m = e^{-4\alpha\varphi} \tilde{\rho}_m \quad p_m = e^{-4\alpha\varphi} \tilde{p}_m \quad dt = e^{\alpha\varphi} d\tilde{t} \quad a = e^{\alpha\varphi} \tilde{a}$$

It is clear from equation [6] that the evolution of scalar field is not determined only by its potential energy since there is a coupling to matter. Furthermore, such couplings give rise to additional forces on matter particles in
Using this equation to eliminate \( \dot{\phi} \) comes potential as follows

\[
V_{\text{eff}}(\phi) = V(\phi) + \rho^* e^{-\alpha \phi}
\]

(9)

Where \( \rho^* \) is a conserved quantity in the Einstein frame which is related to \( \rho_m \) via the relation \( \rho_m = \rho^* e^{-\alpha \phi} \).

II. TRACKING SOLUTIONS

In this section we find the condition for having tracking solutions. We confine our attention to the case \( \dot{\phi} > 0 \) which is satisfactory from the astrophysical point of view and the generalization to the case \( \dot{\phi} < 0 \) can be done with similar considerations. In the uncoupled quintessence model this condition requires that \( \omega_\phi < \omega_m \) and \( \omega_\phi \) be nearly constant. But here we have

\[
\dot{\Omega}_\phi = 3H(\omega_m - \omega_\phi)\Omega_m \Omega_{\phi \phi} + \alpha(1 - 3\omega_m)\dot{\phi}\Omega_m.
\]

(10)

where

\[
\dot{\phi}^2 = (1 + \omega_\phi)\rho_\phi = (1 + \omega_\phi)\frac{H^2}{2\alpha^2}\Omega_\phi,
\]

(11)

Using this equation to eliminate \( \dot{\phi} \), equation (10) becomes

\[
\dot{\Omega}_\phi = H\sqrt{\frac{1 + \omega_\phi}{2}}\Omega_m \Omega_{\phi \phi}^{1/2}(\pm(1 - 3\omega_m)) + 3(\omega_m - \omega_\phi)\sqrt{\frac{2\Omega_\phi}{1 + \omega_\phi}}.
\]

(12)

At tracking era \( \Omega_\phi \ll \Omega_m \), so we can ignore the second term in the above equation. In the matter dominated era, this shows that the condition of \( \Omega_\phi > 0 \) means that \( \dot{\phi} > 0 \) and so we select the plus sign. Also it is not necessary that \( \omega_\phi \) be negative and it can take any value between -1 and 1(provided that \( \Omega_\phi \) is very small such that \( \Omega_\phi < 0.1 \)). To obtain the tracker equation we can express the equation of motion of scalar field into the following form

\[
\frac{V_\phi}{V} = -3\alpha \sqrt{\frac{2(1 + \omega_\phi)}{\Omega_\phi}}\left[1 + \frac{1 - x'}{6} x\right] + \frac{2\alpha}{1 - \omega_\phi} \frac{\Omega_m}{\Omega_\phi},
\]

(13)

where \( x = \frac{1 + \omega_\phi}{1 - \omega_\phi} \) is the ratio of the kinetic to the potential energy for \( \phi \) and prime denotes derivative with respect to \( \ln a \). Therefore, for a tracker solution (\( \omega_\phi \approx \text{const} \)) the tracker condition becomes

\[
\frac{V_\phi}{V} \simeq -3\alpha \sqrt{\frac{2(1 + \omega_\phi)}{\Omega_\phi}} + \frac{2\alpha}{1 - \omega_\phi} \frac{\Omega_m}{\Omega_\phi}.
\]

(14)

By taking the time derivative of this equation for tracker solution, we get

\[
\frac{d}{dt}\left(\frac{V_\phi}{V}\right) \simeq \frac{3\alpha \sqrt{1 + \omega_\phi} \Omega_\phi}{\Omega_{\phi \phi}^{1/2}} \left(1 - \frac{\sqrt{3\Omega_{\phi \phi}^{-1/2}}}{3(1 - \omega_\phi)\sqrt{1 + \omega_\phi}}\right).
\]

(15)

Therefore, during tracking, the second term is dominated and we have \( \frac{d}{dt}\left(\frac{V_\phi}{V}\right) < 0 \) or equivalently \( \Gamma < 1 \), where \( \Gamma \) is a dimensionless function defined as \( \Gamma = \frac{V_\phi}{\rho^* V} \) and is nearly constant at tracker period. This condition is completely different from the tracker condition in the uncoupled quintessence model (\( \Gamma > 1 \)) [12]. By taking the time derivative of equation (13) and combining with Friedmann equations and equation (13) itself, we obtain the following relation

\[
\Gamma = 1 - \frac{2}{1 + \omega_\phi} \frac{\dot{\gamma}}{(6 + \dot{\gamma})^2} - \frac{1 - \omega_\phi}{2(1 + \omega_\phi)} \frac{\dot{\gamma}}{6 + \dot{\gamma}} + 3(\omega_m - \omega_\phi) \frac{1 - \Omega_\phi}{1 + \omega_\phi} \Lambda \Omega_\phi^{1/2} \frac{\Omega_{\phi \phi}}{2(6 + \dot{\gamma})} + (\omega_\phi - 1) \lambda \Omega_m \Omega_{\phi \phi}^{1/2}.
\]

(16)

Where \( \dot{x} = \frac{d\ln x}{d\ln a}, \dot{\gamma} = \dot{x} - \lambda \Omega_m \Omega_{\phi \phi}^{-1/2}, \tilde{\gamma} = \frac{d\ln y}{d\ln a} \) and \( \lambda = (1 - 3\omega_m) \sqrt{\frac{2}{1 + \omega_\phi}} \). From now we take \( \omega_m = 0 \) because we are considering the matter dominated era where metric \( f(R) \) gravity behaves like an interacting quintessence. Since \( \Omega_\phi \ll \Omega_m \) during tracking, a useful relation can be obtained for \( \Gamma \) by expanding it as a power series of \( \Omega_\phi \) i.e.

\[
\Gamma \simeq \frac{1}{2} - \frac{1 + \omega_\phi}{8} + \left[\Sigma_1 + \Sigma_2 \tilde{x}\right] \Omega_\phi^{1/2} + \left[\Sigma_3 + \Sigma_4 \tilde{x} + \Sigma_5 \tilde{x}^2 - \tilde{\gamma}\right] \Omega_\phi + O(\Omega_\phi^{3/2}).
\]

(17)

Where

\[
\Sigma_1 = 6\omega_\phi/\sqrt{2(1 + \omega_\phi)} - 3\sqrt{(1 + \omega_\phi)/2} - 3(1 + \omega_\phi),
\]

\[
\Sigma_2 = (1 - \omega_\phi) \sqrt{8(1 + \omega_\phi)} - \sqrt{(1 + \omega_\phi)/2} - 1 - \omega_\phi/2,
\]

\[
\Sigma_3 = 18\omega_\phi - 9 - (28 + 27\omega_\phi)1 + \omega_\phi/2,
\]

\[
\Sigma_4 = -\frac{\omega_\phi}{2} + \frac{3(1 + \omega_\phi)^{3/2}}{4\sqrt{2}},
\]

\[
\Sigma_5 = -\frac{3}{2} - \frac{9(1 + \omega_\phi)^{3/2}}{\sqrt{2}}.
\]

(18)

Therefore for the tracker solution (assuming \( \Gamma \) is nearly constant and \( \Omega_\phi \ll \Omega_m \)) to the first order in \( \omega_\phi \), we have

\[
\Gamma \simeq \frac{1}{2} - \sqrt{\frac{1 + \omega_\phi}{8}}.
\]

(19)

This equation shows that for interacting quintessence corresponding to the Einstein frame of \( f(R) \) gravity, tracking occurs if \( \Gamma < \frac{1}{2} \), although there exists another constraint which limits this condition.

Another constraint which limits this interval, comes from the stability requirement. We require that for any solution for which the equation of state parameter of dark energy is different from tracking parameter, \( \omega_m \), by a small amount such as \( \delta \omega \), then \( \delta \omega \) decays with time and the solution joins to the tracker solution.
III. STABILITY OF THE TRACKING SOLUTIONS

Now we want to check the stability of tracking solutions with constant $\omega_\phi$. Consider a solution which is perturbed form the tracker solution, $\omega_\phi = \omega_0$, by an amount $\delta \omega$. Then we expand equation (17) to lowest order in $\delta \omega$. It should be noted that $\delta \omega$ and $\Omega_\phi$ have the same order of magnitude and so one can neglect terms such as $\Omega_\phi \delta \omega$ or $\Omega_\phi^{3/2} \delta \omega$. However, we do not neglect the terms containing powers smaller than one in $\Omega_\phi$ and we apply the limit $\Omega_\phi \to 0$ to final solution for $\delta \omega$. By using the equations for $\Sigma_i$ and expanding them to the first order of $\delta \omega$ we have $\Sigma_i = \Sigma_i(\omega_0) + \sigma_i \delta \omega$, where

$$\sigma_1 = \frac{6}{\sqrt{2(1 + \omega_0)}} - \frac{3\omega_0}{\sqrt{2(1 + \omega_0)^{3/2}}} - \frac{3}{\sqrt{8(1 + \omega_0)}},$$

$$\sigma_2 = \frac{\omega_0 - 1}{\sqrt{32(1 + \omega_0)^{3/2}}} + \frac{1}{\sqrt{32(1 + \omega_0)}},$$

$$\sigma_3 = -\frac{27}{\sqrt{8}(1 + \omega_0)},$$

$$\sigma_4 = \frac{27}{\sqrt{8}(1 + \omega_0)} + \frac{28 + 27\omega_0}{\sqrt{8(1 + \omega_0)}}. \quad (20)$$

Finally, by using these equations we obtain

$$\delta'' \omega + \eta \delta' \omega + \xi \delta \omega = 0. \quad (21)$$

where

$$\eta = -[\Sigma_2(\omega_0)\Omega_\phi^{-1/2} + \Sigma_3(\omega_0)],$$

$$\xi = \frac{\omega_0^2 - 1}{2} [\Sigma_1(\omega_0)\Omega_\phi^{-1/2} - \frac{1}{\sqrt{32(1 + \omega_0)}}]. \quad (22)$$

the solution of the equation (21) is

$$\delta \omega \sim a^{-\eta + \sqrt{\eta^2 - 4\xi}}. \quad (23)$$

In order to have a decaying $\delta \omega$, $\eta$ should be positive even if the quantity under square root is negative. One can easily show that $\eta$ is positive, for any $\omega_0$ in the interval $-1 < \omega_0 < 1$, if $\Omega_\phi > 0.025$. When $\Omega_\phi < 0.025$ then the interval of $\omega_0$ for which $\eta$ is positive becomes tighter. We select the smallest interval corresponding to $\Omega_\phi \sim 0$. It is easy to show that this interval is

$$-0.361 < \omega_0 < 1, \quad (24)$$

By taking into account equation (19) we obtain the final condition for having tracker solutions

$$0 < \Gamma < 0.217. \quad (25)$$

The condition (25) should not to be confused with the condition $\Gamma > 1$ which has been appeared in the chameleon scalar tensor theory [10]. $f(R)$ gravity models can be considered as a chameleon theory for which $\beta$ is negative and is equal to $-1/\sqrt{6}$ [13]. In order to find tracker condition, the sign of $\beta$ is important. Since in the chameleon scalar tensor theory, this coupling constant is positive, the second term in the equation (14) is negative. By a similar calculation one can easily verify that the tracking condition is $\Gamma > 1$. Note that the condition $\phi > 0$ is again required for having increasing density parameter of dark energy.

IV. PROPERTIES OF TRACKING SOLUTIONS

In the uncoupled quintessence model $\rho_\phi \sim a^{-3(1+\omega_\phi)}$ and always $\rho_m < 0$ and $\dot{\rho}_m < 0$. But here the conservation equation of the dark energy density, equation (6), is not integrable even if equation of state parameter is nearly constant. So the tracking behavior is not clear. Furthermore, it is clear from (9) that if $\omega_\phi$ is nearly constant then $\dot{\rho}_\phi$ can be positive or negative. Also it is not obvious that whether $|\dot{\rho}_\phi| < |\rho_m|$ during tracking, like that of uncoupled quintessence model, or not.

Here, we want to clarify these ambiguities and explore the tracking period with some more details. To do this, we can write equation (6) as follows

$$\dot{\rho}_\phi = \alpha \rho_\phi \dot{\phi} \left( \frac{\Omega_m}{\Omega_\phi} - 3 \frac{2(1 + \omega_\phi)}{\Omega_\phi} \right). \quad (26)$$

Therefore, at the beginning of the matter dominated era the first term is dominated and so dark energy density is increasing. On the other hand, when $\Omega_\phi$ becomes larger then the second term is dominated and so $\rho_\phi$ is decreasing. Thus, there exist a maximum at the time evolution of $\rho_\phi$. We require that the dark energy density has not significant role in the matter dominated era and its role is important for us at late times. So, it is necessary to show that at this maximum, $\rho_\phi$ is smaller enough than the matter density. Let us assume that the dark energy density takes its maximum at the time $t^*$, then by using equation (20) we obtain

$$\Omega_m|_{t^*} = -\beta + \sqrt{\beta^2 + 4\beta} \frac{2}{\beta}, \quad (27)$$

where $\beta = 18(1 + \omega_\phi)$. Now, by using the condition (24) we have

$$0.925 < \Omega_m|_{t^*} < 0.974. \quad (28)$$

Thus, the dark energy density at $t^*$ is small compared to the matter density.

Moreover, it can be shown that $\rho_\phi$ decreases at a slower rate than $\rho_m$, so $t^*$ is smaller than the time at which the matter and dark energy densities are equal. Firstly, let us to show that in this era $|\dot{\rho}_\phi| < |\rho_m|$. 

After \( t^* \), by using equations (6) and (7), the condition \( |\dot{\rho}_\varphi| < |\dot{\rho}_m| \) can be written as

\[
(1 + \omega_\varphi) \frac{\Omega_\varphi}{1 - \Omega_\varphi} - \frac{\sqrt{2(1 + \omega_\varphi)}}{3} \Omega_\varphi^{1/2} < 1. 
\] (29)

During tracking the density parameter of dark energy is small and it can be shown that the condition (29) is satisfied for any \(-1 < \omega_\varphi < 1\). After tracking, in order to have \( |\dot{\rho}_\varphi| < |\dot{\rho}_m| \), as in the uncoupled quintessence, it is necessary that \( \omega_\varphi \) be decreasing with time. By using equations (9) and (13), it can be shown that \( \dot{\omega}_\varphi < 0 \) if

\[
\frac{1}{V} \left| \frac{dV_{eff}}{d\varphi} \right| < 3a \sqrt{\frac{2(1 + \omega_\varphi)}{\Omega_\varphi}}, 
\] (30)
after tracking \( \Omega_\varphi \) is not negligible compared to 1 so \( \sqrt{2(1 + \omega_\varphi)}/\Omega_\varphi \sim O(1) \). Consequently, equation (30) can be written as

\[
M_{pl} \frac{1}{V} \left| \frac{dV_{eff}}{d\varphi} \right| < 1. 
\] (31)
in which \( M_{pl} = (\sqrt{60})^{-1} \).

Now, we discuss the convergence to the tracker solution for which \( \omega_\varphi \) is nearly constant, in more detail. It is required that the evolution of the tracking dark energy density be insensitive to the initial conditions and any perturbation from tracking dark energy density should be decreasing with time. This is the main goal of introducing the tracking solutions [12].

Suppose that \( \varphi \) is perturbed from the tracking solution by an small amount \( \delta \varphi \). Any perturbation in the "position" and "velocity" of the scalar field \( \varphi \) produces a perturbation, \( \delta \rho_\varphi \), in the dark energy density \( \rho_\varphi \). In order to show that the fractional perturbation \( \frac{\delta \rho_\varphi}{\rho_\varphi} \) decays with time, we write the continuity equation of the dark energy density (9) as follows

\[
\rho_\varphi' + 3(1 + \omega_\varphi) \rho_\varphi = \frac{\sqrt{(1 + \omega_\varphi)}}{2} \Omega_\varphi^{1/2} \rho_m, 
\] (32)

so a small perturbation \( \delta \rho_\varphi \) will satisfy

\[
\delta \rho_\varphi' + 3(1 + \omega_\varphi) \delta \rho_\varphi = \frac{\sqrt{(1 + \omega_\varphi)}}{8} \left( \Omega_\varphi^{-1/2} - \Omega_\varphi^{1/2} \right) \delta \rho_\varphi, 
\] (33)

Now by using the above equations, one can write a differential equation for \( \chi = \frac{\delta \rho_\varphi}{\rho_\varphi} \) as follows

\[
\chi' + \frac{\sqrt{(1 + \omega_\varphi)}}{8} \left( \Omega_\varphi^{-1/2} - \Omega_\varphi^{1/2} \right) \chi = 0. 
\] (34)

In order to find out \( \Omega_\varphi \) during the tracking era, let us write equation (10) in the form

\[
\Omega_\varphi' + 3 \omega_\varphi \left( \Omega_\varphi - \Omega_\varphi^2 \right) + \frac{\sqrt{(1 + \omega_\varphi)}}{2} \left( \Omega_\varphi^{3/2} - \Omega_\varphi^{1/2} \right) = 0, 
\] (35)

Taking into account that the equation of state parameter of dark energy is nearly constant for the tracker solutions and also \( \Omega_\varphi \ll \Omega_m \) in the tracking era, this equation can be solved analytically when the term containing \( \Omega_\varphi^2 \) is ignored. For the sake of simplicity in solving equation (34), here we will ignore also the term containing \( \Omega_\varphi^2 \). With this approximation, the above equation has the following solution

\[
\sqrt{\Omega_\varphi} = \frac{\sqrt{2(1 + \omega_\varphi)}}{6\omega_\varphi} + \gamma_0 \frac{a^{-3\omega_\varphi}}{2}, 
\] (36)

Where \( \gamma_0 \) is an integration constant. By substituting this solution in the equation (32), the solution for \( \chi(a) \) is

\[
\chi(a) = \frac{\delta \rho_\varphi}{\rho_\varphi} = \frac{\chi_0}{\sqrt{\Omega_\varphi}} \frac{18 \omega_\varphi^2 - \omega_\varphi - 1}{18 \omega_\varphi^2} e^{-\frac{\gamma_0 \sqrt{2(1 + \omega_\varphi)}}{6\omega_\varphi}}. 
\] (37)
in which \( \chi_0 \) is an integration constant and \( y = a^{-3\omega_\varphi/2} \).

Also the corresponding solutions for \( \omega_\varphi = 0 \) are

\[
\sqrt{\Omega_\varphi} = \frac{\ln a}{\sqrt{8}} + \gamma_0, 
\]

\[
\chi(a) = \frac{\delta \rho_\varphi}{\rho_\varphi} = \frac{\chi_0}{\sqrt{\Omega_\varphi}} \frac{2a}{\sqrt{\pi}} e^{(\ln a^{1/4})^2}. 
\] (38)

This fractional perturbation has been plotted for various \( \omega_\varphi \); in Fig.1. We see that \( \frac{\delta \rho_\varphi}{\rho_\varphi} \) is a decreasing function with time. As mentioned before, in the uncoupled quintessence model \( \frac{\delta \rho_\varphi}{\rho_\varphi} \) is constant during the tracking period, but here this ratio quickly reaches to zero. Therefore, the evolution of the dark energy density is insensitive to the initial conditions for the dark energy density.

V. DISCUSSION

In this paper we have derived the conditions for fourth order \( f(R) \) gravity models to have tracker solutions in
the Einstein frame. The tracker solutions exist if $\omega_\phi$ is nearly constant and also i) $0 < \Gamma < 0.217$ and ii) $\dot{\phi} = \frac{d}{dt} \ln f' (\hat{\mathcal{R}}) > 0$.

For exploring the condition ii) further, let us consider only positive-definite forms of $f'(\hat{\mathcal{R}})$, because the conformal transformation $g_{\mu \nu} = e^{2\alpha_\phi} \bar{g}_{\mu \nu}$ is singular for $f'(\hat{\mathcal{R}}) = 0$. This forms of $f(\hat{\mathcal{R}})$ also are necessary in order to have positive effective gravitational coupling in the Jordan frame [16]. Also, suppose that $f''(\hat{\mathcal{R}}) > 0$ which is required for Ricci scalar stability [17]. Ricci scalar instability may appear if the matter-energy density (or equivalently the scalar curvature) is large enough compared with the matter density of the universe, for more details see [18]. By these assumptions, the condition ii) reduces to

$$\frac{d\hat{\mathcal{R}}}{dt} > 0 \quad (39)$$

Therefore, although the scalar curvature in the Einstein frame is decreasing and $f(\hat{\mathcal{R}})$ gravity model can show tracking behavior, the scalar curvature of the Jordan frame can be increasing.

Now, as an example, we want to find out a class of $f(\hat{\mathcal{R}})$ models which have constant $\Gamma$ and so can show tracker behavior in the Einstein frame. Only for two classes of potentials the function $\Gamma$ is constant, power law and exponential potentials. Exponential potentials lead to $\Gamma = 1$, which cannot pass the tracking condition. Thus, we consider the power law potentials. In this case, $V(\phi)$ takes the form

$$V(\phi) = V_0 \phi^m = V_0 \phi^n \quad (40)$$

Where $V_0$ is a positive integration constant. Taking into account the condition i) then $1 < n < 1.28$. Now, by using $V = (\hat{\mathcal{R}} f'' - f)/12\alpha^2 f'^2$ and equation (10) we get

$$\hat{\mathcal{R}} f''(\hat{\mathcal{R}}) - f(\hat{\mathcal{R}}) + U_0 f'(\hat{\mathcal{R}})^2 [\ln f'(\hat{\mathcal{R}})]^n = 0 \quad (41)$$

Where $U_0 = -12\alpha^2 V_0/(2\alpha)^n$. Let us to solve this equation for the case $\alpha \phi < 1$, or equivalently $\ln f' \ll 1$. By supposing that $f(\hat{\mathcal{R}}) = \hat{\mathcal{R}} + \Psi(\hat{\mathcal{R}})$ and neglecting the terms containing $\Psi(\hat{\mathcal{R}})$ with powers larger than one (note that $n$ is of order unity), the equation (41) reduces to

$$\hat{\mathcal{R}} \Psi'(\hat{\mathcal{R}}) - \Psi(\hat{\mathcal{R}}) + U_0 \Psi'(\hat{\mathcal{R}})^n = 0 \quad (42)$$

The solution of this differential equation is $\mu^2 (m - 1) \left( \frac{\hat{\mathcal{R}}}{\mu^2} \right)^m$ and so

$$f(\hat{\mathcal{R}}) = \hat{\mathcal{R}} - \mu^2 (1 - m) \left( \frac{\hat{\mathcal{R}}}{\mu^2} \right)^m \quad (43)$$

Where $m = \frac{n}{n-1}$ and

$$\mu^2 = \frac{-U_0}{\left[ nm (1 - m)^2 \right]^{1+m}} \quad (44)$$

Taking into account the condition i) one gets $m > 4.60$. Now we show that the condition ii) is satisfied for this model too. When $\dot{\phi} > 0$ then we expect that the slope of the effective potential which the scalar field experiences it is negative i.e. $\frac{d}{d\phi} V_{eff} < 0$. The effective potential corresponding to the model (43) is illustrated in Fig.V. By using equation (9), the condition $\frac{d}{d\phi} V_{eff} < 0$ reduces to

$$V_\phi < \alpha \rho_m \leq \alpha \rho_m(t_e) \quad (45)$$

In this model, there exists a free parameter $\mu$ which can be sufficiently small to pass this condition and so, all condition for having tracking solutions are satisfied. It is important to note that the model (43) is similar to the chameleon $f(\hat{\mathcal{R}})$ model which has been considered before in the literature [3]. But, in the case of the chameleon $f(\hat{\mathcal{R}})$ gravity models, theories of the kind (43) are compatible with observation in the range of the parameter $0 < m < 0.25$ [15]. Therefore, this class of chameleon $f(\hat{\mathcal{R}})$ gravity models cannot pass the sufficient condition for having tracker solutions.

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[1] V. Sahni and A. A. Starobinskiy, Int. J. Mod. Phys. D 9, 373 (2000); T. Padmanabhan, Phys. Rept. 380, 235 (2003).
[2] P. Sotiriou and V. Faraoni arXiv:0805.1726
[3] J. D. Barrow and S. Cotsakis, Phys. Lett. B 214, 515 (1988); K. I. Maeda, Phys. Rev. D 39, 3159 (1989).
[4] Bertotti, B., L. Iess, and P. Tortora, 2003, Nature 425, 374.
[5] Amendola, L., R. Gannouji, D. Polarski, and S. Tsujikawa, 2007, Phys. Rev. D75, 083504. Amendola, L., D. Polarski, and S. Tsujikawa, 2007, Phys. Rev. Lett. 98, 131302. Amendola, L., D. Polarski, and S. Tsujikawa, 2007, Int. J. Mod. Phys. D16, 1555.
[6] Y.M. Cho, Class. Quantum Grav. 14 (1997) 2963, S. Capozziello, R. de Ritis, A.A. Marino, Class. Quantum Grav. 14 (1997)3243.
[7] S. Capozziello, S. Nojiri, S. D. Odintsov, A. Troisi, Phys. Lett. B 639 (2006) 135143
[8] V. Gorini, A. Kamenshchik, U. Moschella, V. Pasquier, A. Starobinsky, Phys. Rev. D 72 (2005) 103518.
[9] V. Faraoni, Cosmology in scalar-tensor gravity (Kluwer Academic Publishers, Dordrecht,2004)
[10] Khoury, J., and A.Weltman, 2004a, Phys. Rev. D69, 044026. Khoury, J., and A. Weltman, 2004b, Phys. Rev. Lett. 93, 171104.
[11] P. G. Ferreira and M. Joyce, Phys. Rev. Lett. 79,4740 (1997); L. Wang, R. R. Caldwell, J. P. Ostriker and P. Steinhardt, Astrophys. J. 530, 17 (2000)
[12] P. J. Steinhardt, L. Wang, and Ivaylo Zlatev, Phys.Rev.Lett. 82, 896-899 (1999), astro-ph/9807002
P. J. Steinhardt, L. Wang, I. Zlatev, Phys.Rev. D 59, 123504 (1999), astro-ph/9812313.
[13] T. Chiba, Phys. Rev. D 66 (2002) 063514
[14] Rupam Das, Thomas W. Kephart, Robert J. Scherrer , Phys. Rev. D 74:103515,2006
[15] Faulkner, T., M. Tegmark, E. F. Bunn, and Y. Mao, 2007, Phys. Rev. Cembranos, J. A. R., 2006, Phys. Rev. D73, 064029. D76, 063505. Starobinsky, A. A., 2007, JETP Lett. 86, 157.
[16] T. P. Sotiriou and V. Faraoni, eprint gr-qc/0805.1726
[17] Dolgov, A. D., and M. Kawasaki, 2003a, Phys. Lett. B573, 1. Faraoni, V., 2006a, Phys. Rev. D74, 104017.
[18] S. Capozziello, M. Laurentis, S. Nojiri and S. D. Odintsove, eprint hep-th/808.1335