Some Norm Estimates for Semimartingales

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Abstract

In this paper we introduce a new type of norms for semimartingales, under both linear and nonlinear expectations. Our norm is defined in the spirit of quasimartingales, and it characterizes square integrable semimartingales. This work is motivated by our study of zero-sum stochastic differential games [21], whose value process is conjectured to be a semimartingale under a class of probability measures. As a by product, we establish some a priori estimates for doubly reflected BSDEs without imposing the Mokobodski’s condition directly.

Key words: Semimartingale, quasimartingale, G-expectation, second order backward SDEs, doubly reflected backward SDEs, Doob-Meyer decomposition.

AMS 2000 subject classifications: 60H10, 60H30.

1 Introduction

For an optimization problem under volatility uncertainty, the value process can be characterized as the unique solution of a second order backward SDE, introduced by Cheridito, Soner, Touzi, and Victoir [2] and Soner, Touzi, and Zhang [26], and as the unique viscosity solution of a path dependent HJB equation, introduced by Ekren, Keller, Touzi, and Zhang [6] and Ekren, Touzi, and Zhang [7, 8, 9]. See also the closely

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related concepts $G$-martingale introduced by Peng [18] and $G$-BSDE studied by Hu, Ji, Peng, and Song [13]. This value process is a supermartingale (or, in general case, a $g$-supermartingale as introduced in Peng [17]), under the associated non-dominated class of mutually singular probability measures. In Pham and Zhang [21], we studied a zero sum stochastic differential game and characterized the game value process as the unique viscosity solution of a path dependent Bellman-Isaacs equation. It is natural to conjecture that, under certain technical conditions, this value process should be a semimartingale under the underlying class of probability measures, which will enable us to characterize the value process as the solution to an extended second order BSDE with a non-convex generator. This requires a systematic study on square integrable semimartingales, namely semimartingales whose martingale part and total variation of its finite variation part are square integrable, under both linear and nonlinear expectations.

Our first goal of this paper is to introduce a norm which characterizes square integrable semimartingales, under a fixed (linear) probability measure. Our norm is strongly motivated from the definition of quasimartingales. The main feature is that the norm involves only the semimartingale itself, without involving directly its decomposition. This is important in applications because the semimartingale under consideration is typically a value process, e.g. in [21], and thus has a representation. We prove that a progressively measurable process is a square integrable semimartingale if and only if it has finite norm in our sense.

We next extend our norm to semimartingales under nonlinear expectations, typically the $G$-expectation of Peng [18]. As observed in Soner, Touzi and Zhang [24], a $G$-martingale is a supermartingale under a class of probability measures. It is clear that a $G$-supermartingale satisfies the same property. However, a $G$-submartingale is in general neither a supermartingale nor a submartingale under each probability measure. We show that, any progressively measurable process with finite norm under $G$-expectation in our sense has to be a semimartingale under each probability measure.

Our long term goal is to apply our norm, or its variations if necessary, to study the structure of general $G$-semimartingales and to study the semimartingale property of the viscosity solution of path dependent Bellman-Isaacs equations. The latter can also be viewed as regularity of viscosity solutions of path dependent PDEs. We hope to address these issues in future research.

As a by product, we also provide a tractable sufficient condition for the well-
posedness of doubly reflected backward SDEs (DRBSDE, for short) without imposing
directly the Mokobodski’s condition. There are typically two approaches in the litera-
ture. One is to assume the Mokobodski’s condition, namely there exists a square inte-
grable semimartingale between the two given barriers, see e.g. Cvitanic and Karatzas
[3] and Peng and Xu [19], and the other is to use local solutions, see e.g. Hamadene
and Hassani [11] and Hamadene, Hassani and Ouknine [12]. The latter approach,
while easy to verify its conditions, does not yield any norm estimates. We remark
that such estimates are important in applications, for example when one considers
discritization of DRBSDEs, see e.g. Chassagneux [1]. Inspired by our norm for square
integrable semimartingales, we introduce a norm for the barriers of the DRBSDEs,
which is equivalent to the Mokobodski’s condition but is easier to verify. Moreover,
we provide the sensitivity analysis of DRBSDEs with respect to its barriers. Such an
estimate seems new in the literature and is important in numerical discretization of
DRBSDEs.

The rest of the paper will be organized as follows. In next section we introduce the
norm for semimartingales under a fixed probability measure and obtain the estimates.
In Section 3 study DRBSDEs by introducing a norm in the same spirit. In Section
4 we extend the norm to the $G$-framework. Finally in Appendix we provide some
additional results.

2 Norm Estimates for Semimartingales

Let $T > 0$ be fixed, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space on $[0, T]$, and $\mathbb{D}(\mathbb{F})$ be
the space of $\mathbb{F}$-progressively measurable càdlàg processes. We shall always assume

\[ \mathbb{F} \text{ is right continuous and its } \mathbb{P} \text{-augmentation } \bar{\mathbb{F}} \text{ is a Brownian filtration,} \]

and consequently, any $\mathbb{F}$-martingale $M$ is continuous, $\mathbb{P}$-a.s. (2.1)

We note that the filtration $\mathbb{F}$ is not necessarily complete under $\mathbb{P}$. The removal of
the completeness requirement will be important in Section 4 below. However, the
following simple lemma, see e.g. [25], shows that we may assume all the processes
involved in this section are $\mathbb{F}$-progressively measurable.

Lemma 2.1 For any $\bar{\mathbb{F}}$-progressively measurable process $X$, there exists a unique
$(dt \times d\mathbb{P}$-a.s.) $\mathbb{F}$-progressively measurable process $\bar{X}$ such that $\bar{X} = X$, $dt \times d\mathbb{P}$-a.s.
Moreover, if $X$ is càdlàg, $\mathbb{P}$-a.s., then so is $\bar{X}$. 

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We recall that a semimartingale $Y \in D(F)$ has the following decomposition:

$$Y_t = Y_0 + M_t + A_t,$$  \hspace{1cm} (2.2)

where $M$ is a martingale, $A$ has finite variation, and $M_0 = A_0 = 0$. Now given a process $Y \in D(F)$, we are interested in the following questions:

(i) Is $Y$ a semimartingale?

(ii) Do we have appropriate norm estimates for $Y$, $M$, and $A$?

The first question was answered by Bichteler-Dellacherie, see e.g. \cite{22} and Appendix of this paper for some further discussion. The main goal of this section is to answer the second question. As explained in Introduction, the latter question is natural and important for our study of semimartingales under nonlinear expectations.

## 2.1 Some preliminary results

We first note that, when $Y$ is a supermartingale or submartingale, it is well known that $Y$ is a semimartingale and the following norm estimates hold. Since the arguments will be important for our general case, we provide the proof for completeness.

**Lemma 2.2** Let (2.1) hold. There exist universal constants $0 < c < C$ such that, for any $Y$ in the form of (2.2) with monotone $A$, it holds

$$c\|Y\|_{F,0}^2 \leq \mathbb{E}^P \left[ |Y_0|^2 + \langle M \rangle_T + |A_T|^2 \right] \leq C\|Y\|_{F,0}^2. \hspace{1cm} (2.3)$$

where, for any $Y \in D(F)$,

$$\|Y\|_{F,0}^2 := \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right]. \hspace{1cm} (2.4)$$

**Proof.** The first inequality is obvious. We shall only prove the second inequality. By otherwise using the standard stopping techniques, we may assume without loss of generality that $\mathbb{E}^P[\sup_{0 \leq t \leq T} |Y_t|^2 + \langle M \rangle_T + |A_T|^2] < \infty$.

Apply Itô’s formula and recall (2.1) that $M$ is continuous, we have

$$Y_T^2 = Y_0^2 + \langle M \rangle_T + 2 \int_0^T Y_t dM_t + 2 \int_0^T Y_{t-} dA_t + \sum_{0 < t \leq T} |\Delta Y_t|^2. \hspace{1cm} (2.5)$$

Note that

$$\mathbb{E}^P \left[ \left( \int_0^T |Y_t|^2 d\langle M \rangle_t \right)^{\frac{1}{2}} \right] \leq \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t| \langle M \rangle_T^{\frac{1}{2}} \right] \leq \frac{1}{2} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \langle M \rangle_T \right] < \infty.$$
Then $Y_t dM_t$ is a true martingale, and thus, for any $\varepsilon > 0$, it follows from (2.5) and the monotonicity of $A$ that

$$E^P[\langle M \rangle_T] \leq E^P[\langle M \rangle_T + \sum_{0 \leq t \leq T} |\Delta Y_t|^2] = E^P[Y_T^2 - Y_0^2 - 2 \int_0^T Y_t \, dA_t]$$

(2.6)

$$\leq E^P[Y_T^2 + |Y_0|^2 + 2 \sup_{0 \leq t \leq T} |Y_t||A_T|] \leq C\varepsilon^{-1}\|Y\|_{F,0}^2 + \varepsilon E^P[|A_T|^2].$$

Moreover, note that $A_T = Y_T - Y_0 - M_T$. Then (2.6) leads to

$$E^P[|A_T|^2] \leq C\|Y\|_{F,0}^2 + C E^P[\langle M \rangle_T] \leq C\varepsilon^{-1}\|Y\|_{F,0}^2 + C\varepsilon E^P[|A_T|^2].$$

Set $\varepsilon := \frac{1}{2C}$ for the above $C$, we obtain $E^P[|A_T|^2] \leq C\|Y\|_{F,0}^2$. This, together with (2.6), proves the second inequality.

The next lemma is a discrete version of Lemma 2.2. Since the arguments are very similar, we omit the proof.

**Lemma 2.3** Let $0 = \tau_0 \leq \cdots \leq \tau_n = T$ be a sequence of stopping times. In the setting of Lemma 2.2, if $A_{\tau_i} \in F_{\tau_{i-1}}$, then

$$cE^P[\max_{0 \leq i \leq n} |Y_{\tau_i}|^2] \leq E^P[|Y_0|^2 + \langle M \rangle_T + |A_T|^2] \leq C E^P[\max_{0 \leq i \leq n} |Y_{\tau_i}|^2].$$

(2.7)

### 2.2 Square integrable semimartingales

In this subsection we characterize the norm for square integrable semimartingales. For $0 \leq t_1 < t_2 \leq T$, let $\bigvee_{t_1}^{t_2} A$ denote the total variation of $A$ over the interval $(t_1, t_2]$.

**Definition 2.4** We say a semimartingale $Y$ in the form of (2.2) is a square integrable semimartingale if

$$E^P[|Y_0|^2 + \langle M \rangle_T + \left( \bigvee_{0}^{T} A \right)^2] < \infty.$$  

(2.8)

We remark that (2.8) is the norm used in standard literature for semimartingales, see e.g. [22]. Clearly, for a square integrable semimartingale $Y$, we have $\|Y\|_{F,0} < \infty$. However, when $A$ is not monotone, in general the left side of (2.8) cannot be dominated by $C\|Y\|_{F,0}^2$. See Example 5.1 below.
Our goal is to characterize square integrable semimartingales through the process $Y$ itself, without involving $M$ and $A$ directly. In many applications, see e.g. [21], we may have a representation formula for the process $Y$, but in general it is difficult to obtain representation formulae for $M$ and $A$. So conditions imposed on $Y$ are more tractable than those on $M$ and $A$. We introduce the following norm:

$$\|Y\|_{\mathbb{F}}^2 := \|Y\|_{\mathbb{F},0}^2 + \sup_{\pi} \mathbb{E}^P\left[ \left( \sum_{i=0}^{n-1} \mathbb{E}_t^P(Y_{t_{i+1}} - Y_{t_i}) \right)^2 \right],$$  \hspace{1cm} (2.9)

where the supremum is over all partitions $\pi : 0 = \tau_0 \leq \cdots \leq \tau_n = T$ for some stopping times $\tau_0, \ldots, \tau_n$.

**Remark 2.5** Our norm $\| \cdot \|_{\mathbb{F}}$ is strongly motivated from the definition of quasimartingale: a process $Y \in \mathbb{D}(\mathbb{F})$ is a called a quasimartingale if

$$\sup_{\pi} \mathbb{E}^P\left[ \sum_{i=0}^{n-1} \mathbb{E}_t^P(Y_{t_{i+1}} - Y_{t_i}) \right] < \infty,$$  \hspace{1cm} (2.10)

where $\pi$ is a deterministic partition of $[0,T]$. We refer to Rao [23] and Dellacherie and Meyer [3] for the theory of quasimartingales, see also Meyer and Zheng [15]. It is clear that our definition imposes stronger condition on $Y$: $\|Y\|_{\mathbb{F}} < \infty$ implies $Y$ is a quasimartingale.

The following a priori estimate is the main technical result of the paper.

**Theorem 2.6** There exist universal constants $0 < c < C$ such that, for any square integrable semimartingale $Y_t = Y_0 + M_t + A_t$,

$$c\|Y\|_{\mathbb{F}}^2 \leq \mathbb{E}^P\left[ |Y_0|^2 + \langle M \rangle_T + \left( \int_0^T A \right)^2 \right] \leq C\|Y\|_{\mathbb{F}}^2.$$  \hspace{1cm} (2.11)

**Proof.** (i) We first prove the left inequality. Let $\pi : 0 = \tau_0 \leq \cdots \leq \tau_n = T$ be an arbitrary partition, and denote $\Delta A_{\tau_{i+1}} := A_{\tau_{i+1}} - A_{\tau_i}$. Then

$$\mathbb{E}^P\left[ \left( \sum_{i=0}^{n-1} |\mathbb{E}_t^P(Y_{\tau_{i+1}} - Y_{\tau_i})| \right)^2 \right] = \mathbb{E}^P\left[ \left( \sum_{i=0}^{n-1} |\mathbb{E}_t^P(A_{\tau_{i+1}} - A_{\tau_i})| \right)^2 \right]$$

$$\leq \mathbb{E}^P\left[ \left( \sum_{i=0}^{n-1} |\mathbb{E}_t^P(|\Delta A_{\tau_{i+1}}|)| \right)^2 \right] = \mathbb{E}^P\left[ \left( \sum_{i=0}^{n-1} |\mathbb{E}_t^P(|\Delta A_{\tau_{i+1}}|)| - |\Delta A_{\tau_{i+1}}| + \sum_{i=0}^{n-1} |\Delta A_{\tau_{i+1}}| \right)^2 \right]$$

$$\leq C\mathbb{E}^P\left[ \left( \sum_{i=0}^{n-1} |\mathbb{E}_t^P(|\Delta A_{\tau_{i+1}}|)| - |\Delta A_{\tau_{i+1}}| \right)^2 \right] + C\mathbb{E}^P\left[ \left( \int_0^T A \right)^2 \right].$$  \hspace{1cm} (2.12)
Note that
\[
\sum_{i=0}^{j} \left[ \mathbb{E}_{\tau_i}^{P}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}| \right] \cdot j = 0, \cdots, n - 1, \text{ is a martingale.}
\]

Then
\[
\mathbb{E}^{P} \left[ \left( \sum_{i=0}^{n-1} \mathbb{E}_{\tau_i}^{P}(|\Delta A_{\tau_{i+1}}|) \right)^2 \right] = \mathbb{E}^{P} \left[ \sum_{i=0}^{n-1} \left( \mathbb{E}_{\tau_i}^{P}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}| \right)^2 \right]
\]
\[
\leq C \mathbb{E}^{P} \left[ \sum_{i=0}^{n-1} \left( \mathbb{E}_{\tau_i}^{P}(|\Delta A_{\tau_{i+1}}|) \right)^2 \right] \leq C \mathbb{E}^{P} \left[ \sum_{i=0}^{n-1} \left( \mathbb{E}_{\tau_i}^{P}(|\Delta A_{\tau_{i+1}}| + |\Delta A_{\tau_{i+1}}|)^2 \right) \right]
\]
\[
\leq C \mathbb{E}^{P} \left[ \sum_{i=0}^{n-1} |\Delta A_{\tau_{i+1}}|^2 \right] \leq C \mathbb{E}^{P} \left[ \left( \sum_{i=0}^{n-1} |\Delta A_{\tau_{i+1}}|^2 \right)^2 \right] \leq C \mathbb{E}^{P} \left[ \left( \bigvee_{0}^{T} A \right)^2 \right].
\]
This, together with (2.12) and the left inequality of (2.3), proves the left inequality of (2.11).

(ii) We now prove the right inequality. First, for any \( \varepsilon > 0 \), following the arguments in Lemma 2.2 one can easily show that
\[
\mathbb{E}^{P}[(M)_{T}] \leq C \varepsilon^{-1} \|Y\|_{P,0}^{2} + \varepsilon \mathbb{E}^{P} \left[ \left( \bigvee_{0}^{T} A \right)^2 \right].
\]
(2.13)

We claim that
\[
\mathbb{E}^{P} \left[ \left( \bigvee_{0}^{T} A \right)^2 \right] \leq C \|Y\|_{P}^{2} + C \mathbb{E}^{P}[(M)_{T}].
\]
(2.14)

This, together with (2.13) and by choosing \( \varepsilon \) small enough, implies the right inequality of (2.11) immediately.

We prove (2.14) in four steps.

**Step 1.** Let \( \pi : 0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = T \) be an arbitrary partition. Note that
\[
\mathbb{E}^{P}_{\tau_1}[Y_{\tau_{i+1}}] - Y_{\tau_i} = \mathbb{E}^{P}_{\tau_1}[A_{\tau_{i+1}}] - A_{\tau_i}.
\]
Then
\[
\sum_{i=0}^{n-1} \left[ A_{\tau_{i+1}} - \mathbb{E}^{P}_{\tau_i}[A_{\tau_{i+1}}] \right] = A_T - \sum_{i=0}^{n-1} \left( \mathbb{E}^{P}_{\tau_i}[A_{\tau_{i+1}}] - A_{\tau_i} \right)
\]
\[
= Y_T - Y_0 - M_T - \sum_{i=0}^{n-1} \left( \mathbb{E}^{P}_{\tau_i}[Y_{\tau_{i+1}}] - Y_{\tau_i} \right).
\]
By the definition of $\|Y\|_p$ (2.9), we see that
\[
\mathbb{E}^P \left[ \left( \sum_{i=0}^{n-1} \left[ A_{t_{i+1}} - \mathbb{E}_{t_i}^P [A_{t_{i+1}}] \right] \right)^2 \right] \leq C\|Y\|_p^2 + CE^P[\langle M \rangle_T].
\]

Note that
\[
\sum_{i=0}^{j-1} \left[ A_{t_{i+1}} - \mathbb{E}_{t_i}^P [A_{t_{i+1}}] \right], \quad j = 1, \ldots, n, \quad \text{is a martingale.}
\]

Then
\[
\mathbb{E}^P \left[ \sum_{i=0}^{n-1} \left[ A_{t_{i+1}} - \mathbb{E}_{t_i}^P [A_{t_{i+1}}] \right]^2 \right] \leq C\|Y\|_p^2 + CE^P[\langle M \rangle_T]. \tag{2.15}
\]

**Step 2.** In this step we assume $A_t = \int_0^t a_s dK_s$, where $K$ is a continuous non-decreasing process and $a$ is a simple process. That is,
\[
a = a_{t_0} 1_{(t_0)} + \sum_{i=0}^{n-1} a_{t_i} 1_{(t_i, t_{i+1})} \quad \text{for some} \quad 0 = t_0 < \cdots < t_n = T, \quad a_{t_i} \in \mathcal{F}_{t_i}.
\]

Then, denoting $\alpha_i := \text{sgn} (a_{t_i}) \in \mathcal{F}_{t_i}$,
\[
\sqrt{T} A = \int_0^T |a_t| dK_t = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \alpha_i a_t dK_t = \sum_{i=0}^{n-1} \alpha_i [A_{t_{i+1}} - A_{t_i}]
\]
\[
= \sum_{i=0}^{n-1} \alpha_i (A_{t_{i+1}} - \mathbb{E}_{t_i}^P [A_{t_{i+1}}]) + \sum_{i=0}^{n-1} \alpha_i (\mathbb{E}_{t_i}^P [A_{t_{i+1}}] - A_{t_i}).
\]

Note that
\[
\sum_{i=0}^{j} \alpha_i (A_{t_{i+1}} - \mathbb{E}_{t_i}^P [A_{t_{i+1}}]), \quad j = 0, \ldots, n - 1, \quad \text{is a martingale.}
\]

Then
\[
\mathbb{E}^P \left[ \left( \sqrt{T} A \right)^2 \right] \leq C\mathbb{E}^P \left[ \sum_{i=0}^{n-1} \left| A_{t_{i+1}} - \mathbb{E}_{t_i}^P [A_{t_{i+1}}] \right|^2 \right] + \left( \sum_{i=0}^{n-1} \left| \mathbb{E}_{t_i}^P [A_{t_{i+1}}] - A_{t_i} \right| \right)^2.
\]

By (2.15) and the definition of $\|Y\|_p$ (2.9) we obtain (2.14).

**Step 3.** We now prove (2.14) for general continuous process $A$. Denote $K_t := \sqrt{T} A$. Since $A$ is continuous, $K$ is also continuous. Moreover $dA_t$ is absolutely continuous.
with respect to $dK_t$ and thus $dA_t = a_t dK_t$ for some $a$. By [14], Chapter 3 Lemma 2.7, for every $\varepsilon > 0$ there exists a simple process $\{a^\varepsilon\}$ such that

$$
\mathbb{E}^P \left[ \left( \int_0^T |a^\varepsilon_t - a_t| dK_t \right)^2 \right] \leq \varepsilon.
$$

(2.16)

Denote

$$A^\varepsilon_t := \int_0^t a^\varepsilon_s dK_s, \quad Y^\varepsilon_t := Y_0 + M_t + A^\varepsilon_t.$$

Then by Step 2 we see that

$$
\mathbb{E}^P \left[ \left( \bigvee_0^T A^\varepsilon \right)^2 \right] \leq C\|Y^\varepsilon\|^2_P + C\mathbb{E}^P(\langle M \rangle_T).
$$

(2.17)

Note that

$$\bigvee_{0}^{T} A \leq \bigvee_{0}^{T} A^\varepsilon + \bigvee_{0}^{T} |A^\varepsilon - A| \leq \bigvee_{0}^{T} A^\varepsilon + \int_{0}^{T} |a^\varepsilon_t - a_t| dK_t.$$

Then

$$
\mathbb{E}^P \left[ \left( \bigvee_0^T A \right)^2 \right] \leq C\mathbb{E}^P \left[ \left( \bigvee_0^T A^\varepsilon \right)^2 \right] + C\varepsilon.
$$

(2.18)

On the other hand, apply the left inequality of (2.11) on $Y^\varepsilon - Y = A^\varepsilon - A$, we get

$$
\|Y^\varepsilon - Y\|^2_P \leq C\mathbb{E}^P \left[ \left( \bigvee_0^T (A^\varepsilon - A) \right)^2 \right] \leq C\mathbb{E}^P \left[ \left( \int_{0}^{T} |a^\varepsilon_t - a_t| dK_t \right)^2 \right] \leq C\varepsilon.
$$

Then

$$\|Y^\varepsilon\|^2_P \leq C\|Y\|^2_P + C\varepsilon.$$

Plug this and (2.18) into (2.17), we get

$$
\mathbb{E}^P \left[ \left( \bigvee_0^T A \right)^2 \right] \leq C\|Y\|^2_P + C\mathbb{E}^P(\langle M \rangle_T) + C\varepsilon.
$$

Since $\varepsilon$ is arbitrary, we obtain (2.14).

**Step 4.** We now prove (2.14) for the general case. Since $A$ has finite variation, we can decompose $A = A^c + A^d$, where $A^c$ is the continuous part and $A^d$ is the part
with pure jumps. Since $Y$ is càdlàg and $M$ is continuous, $A$ and $A^d$ are càdlàg. We denote $Y^c_t = Y_0 + M_t + A^c_t$. From Step 3 we have

$$\mathbb{E}^P \left[ |Y_0|^2 + \langle M \rangle_T + \left( \int_0^T A^c_t \right)^2 \right] \leq C \|Y^c\|_P^2.$$

Note that

$$\|Y^c\|_P \leq \|Y\|_P + \|A^d\|_P$$

and apply the left inequality of (2.11) on $A^d$ we see that

$$\|A^d\|_P^2 \leq C \mathbb{E}^P \left[ \left( \int_0^T A^d_t \right)^2 \right].$$

Then

$$\mathbb{E}^P \left[ |Y_0|^2 + \langle M \rangle_T + \left( \int_0^T A^c_t \right)^2 \right] \leq C \|Y\|_P^2 + C \mathbb{E}^P \left[ \left( \int_0^T A^d_t \right)^2 \right].$$

Moreover, note that

$$\int_0^T A \leq \int_0^T A^c + \int_0^T A^d.$$

Thus, to prove (2.14), it suffices to show that

$$\mathbb{E}^P \left[ \left( \int_0^T A^d_t \right)^2 \right] \leq C \|Y\|_P^2. \quad (2.19)$$

To this end, we first note that

$$\int_0^T A^d = \sum_{0 < t \leq T} |\Delta A_t| = \sum_{0 < t \leq T} |\Delta Y_t|. \quad (2.20)$$

Define, for each $n$,

$$D_n := \sum_{0 < t \leq T} |\Delta Y_t| 1_{\{ |\Delta Y_t| \geq \frac{1}{n} \}},$$

and, $\tau^0_n := 0$, and for $m \geq 0$, by denoting $Y^*_t := Y^*_T$ for $t \geq T$,

$$\tau^*_m := \inf \left\{ t > \tau^*_m : |\Delta Y_t| \geq \frac{1}{n} \right\} \wedge (T + 1).$$
We remark that we use $T + 1$ instead of $T$ here so that $\Delta Y_T$ will not be counted repeatedly at below. By the right continuity of $\mathbb{F}$ we see that $\tau^n_i$ are stopping times.

It is clear that

$$D_n \uparrow \sum_{0 \leq t \leq T} |\Delta Y_t| \text{ as } n \to \infty, \text{ and } \sum_{i=1}^m |\Delta Y^n_{\tau^n_i}| \uparrow D_n \text{ as } m \to \infty.$$  

Therefore, to obtain (2.19) it suffices to show that

$$\mathbb{E}^\mathbb{P}\left[\left(\sum_{i=1}^m |\Delta Y^n_{\tau^n_i}|\right)^2\right] \leq \|Y\|_\mathbb{P}^2 \text{ for all } n, m. \tag{2.21}$$

We now fix $n, m$. Since $\mathbb{F}$ is a Brownian filtration, all $\mathbb{F}$-stopping time is predictable, see e.g. [20], Corollary 5.7. Then for each $\tau^n_i$, there exist $\{\tau^n_{i,j}, j \geq 1\}$ such that $\tau^n_{i,j} < \tau^n_i$ and $\tau^n_{i,j} \uparrow \tau^n_i$ as $j \to \infty$. By definition of $\|Y\|_\mathbb{P}$ (2.9), we have

$$\mathbb{E}^\mathbb{P}\left[\left(\sum_{i=1}^m \mathbb{E}^\mathbb{P}_{\tau^n_{i-1} \vee \tau^n_{i,j}} [Y^n_{\tau^n_i}] - Y^n_{\tau^n_{i-1} \vee \tau^n_{i,j}}\right)^2\right] \leq \|Y\|_\mathbb{P}^2. \tag{2.22}$$

Send $j \to \infty$, since $\mathbb{F}$ is continuous, we see that

$$\lim_{j \to \infty} \mathbb{E}^\mathbb{P}_{\tau^n_{i-1} \vee \tau^n_{i,j}} [Y^n_{\tau^n_i}] - Y^n_{\tau^n_{i-1} \vee \tau^n_{i,j}} = Y^n_{\tau^n_i} - Y^n_{\tau^n_{i-1}} = \Delta Y^n_{\tau^n_i}.$$  

Then, noting that $\mathbb{E}^\mathbb{P}[\sup_{0 \leq t \leq T} |\Delta Y_t|^2] < \infty$, by applying the Dominated Convergence Theorem we obtain (2.21) from (2.22). This implies (2.19), which in turn implies (2.14).

As a direct consequence of the above a priori estimates, we have

**Theorem 2.7** A process $Y \in \mathbb{D}(\mathbb{F})$ is a square integrable semimartingale if and only if $\|Y\|_\mathbb{P} < \infty$.

**Proof.** By Theorem 2.6, it suffices to prove the if part. Assume $\|Y\|_\mathbb{P} < \infty$. From Remark 2.5, $Y$ is a quasimartingale. By Rao’s theorem, see e.g. [22], Chapter III Theorem 18, $Y = M + A$, where $M$ is a local martingale and $A$ is a predictable process with paths of locally integrable variation. By the standard stopping technique and by Theorem 2.6 again, it is easy to see that $Y$ is indeed a square-integrable semimartingale. 

\[\Box\]
3 Doubly Reflected BSDEs

In this section we assume \( \mathcal{F} \) is generated by a standard Brownian motion \( B \) and augmented with all the \( \mathbb{P} \)-null sets. We consider the following Doubly Reflected Backward SDE (DRBSDE, for short) with \( \mathcal{F} \)-progressively measurable solution triplet \((Y, Z, A)\):

\[
\begin{align*}
Y_t &= \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + A_T - A_t; \\
L \leq Y \leq U, \quad [Y_t - L_t]dK_t^+ = [U_t - Y_t]dK_t^- = 0.
\end{align*}
\]

Here \( Y \in \mathbb{D}(\mathcal{F}) \) and \( A \) has finite variation with orthogonal decomposition \( A = K^+ - K^- \). We say \((Y, Z, A)\) satisfying (3.1) is a local solution if

\[
\sup_{0 \leq t \leq T} |Y_t| + \int_0^T |Z_t|^2 dt + \sqrt{T} A < \infty, \quad \mathbb{P}\text{-a.s.}
\]

and a solution if

\[
\|(Y, Z, A)\|^2 := \mathbb{E}_0^\mathbb{P}\left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt + \left( \sqrt{\int_0^T A} \right)^2 \right] < \infty.
\]

Throughout this section, we assume the following standing assumptions:

**Assumption 3.1**

(i) \( \xi \) is \( \mathcal{F}_T \)-measurable, \( f(\cdot, 0, 0) \) is \( \mathcal{F} \)-progressively measurable, and

\[
I^2_0 := I^2_0(\xi, f) := \mathbb{E}_0^\mathbb{P}\left[ |\xi|^2 + \left( \int_0^T |f_t(0, 0)| dt \right)^2 \right] < \infty.
\]

(ii) \( f \) is uniformly Lipschitz continuous in \((y, z)\);

(iii) \( L, U \in \mathbb{D}(\mathcal{F}) \); \( L \leq U \); \( L_T \leq \xi \leq U_T \); and

\[
\|(L, U)\|_{\mathbb{P}, 0}^2 := \|L^+\|_{\mathbb{P}, 0}^2 + \|U^-\|_{\mathbb{P}, 0}^2 < \infty.
\]

**Remark 3.2** In the standard BSDE literature, one requires \( \mathbb{E}_0^\mathbb{P}\left[ \int_0^T \left| f(t, 0, 0) \right|^2 dt \right] < \infty \). Our condition (3.4) is slightly weaker. In fact, most estimates in the BSDE literature can be improved by replacing \( \mathbb{E}_0^\mathbb{P}\left[ \int_0^T \left| f(t, 0, 0) \right|^2 dt \right] \) with \( \mathbb{E}_0^\mathbb{P}\left[ \left( \int_0^T |f(t, 0, 0)| dt \right)^2 \right] \), and the arguments are rather standard. We refer to the Appendix of the monograph [4] for interested readers.

It is well known that Assumption 3.1 does not yield the wellposedness of DRBSDE (3.1). At below is a simple counterexample.
Example 3.3 Let $L = U$ be deterministic, càdlàg, and $\int_0^T L = \infty$. Then DRBSDE (3.1) with $\xi = L_T$ and $f = 0$ has no solution.

Proof. Assume there is a solution $(Y, Z, A)$. Since $L \leq Y \leq U$, one must have $Y = L$, which leads to $Z = 0$ and $A = L$. But this contradicts with the assumption that $L$ has infinite variation.

In the literature, there are two approaches for wellposedness of DRBSDEs. We first report a result from Hamadene, Hassani and Ouknine [12]:

Lemma 3.4 Let Assumption 3.1 hold. Assume further the following separation condition:

$$L_t < U_t \quad \text{and} \quad L_{t-} < U_{t-} \quad \text{for all} \; t.$$ (3.6)

Then (3.1) admits a local solution.

The condition (3.6) is mild and easy to verify, but it does not yield any a priori estimates. We remark that [12] takes a slightly different form of DRBSDEs. But that is mainly for the sake of uniqueness. One can easily check that a local solution in [12] is a local solution in our sense, so the existence in Lemma 3.4 is valid.

We next report a result from Peng and Xu [19], following the original work Cvitanic and Karatzas [3]:

Lemma 3.5 Let Assumption 3.1 hold. Assume further the following Mokobodski’s type of condition:

there exists a square integrable semimartingale $Y^0$ such that $L_t \leq Y_t^0 \leq U_t$. (3.7)

Then DRBSDE (3.1) admits a unique solution and the following estimate holds:

$$\| (Y, Z, A) \|^2 \leq C \left[ I_0^2 + \| Y^0 \|^2_P \right].$$ (3.8)

However, in those works there is no discussion on the sufficient conditions for the existence of such $Y^0$. Our goal in this section is to provide a tractable equivalent condition. In light of the norm $\| . \|_P$ (2.9), we introduce the following norm for the barriers $(L, U)$:

$$\| (L, U) \|^2_P := \| (L, U) \|^2_{P,0} + \sup_{\pi} E^P \left( \sum_{i=0}^{n-1} \left( \mathbb{E}_{\tau_i}^P (L_{\tau_{i+1}}) - U_{\tau_i})^+ + [L_{\tau_i} - \mathbb{E}_{\tau_i}^P (U_{\tau_{i+1}})]^+ \right)^2 \right).$$ (3.9)
where the supremum is again taken over all partitions \( \pi : 0 = \tau_0 \leq \cdots \leq \tau_n = T \).

Our main result of this section is:

**Theorem 3.6** Let Assumption 3.1 hold. Then the following are equivalent:

(i) The DRBSDE (3.1) admits a unique solution \((Y, Z, A)\);

(ii) the Mokobodski condition (3.7) holds;

(iii) \( \|(L, U)\|_P < \infty \).

Moreover, in this case we have the estimate:

\[
\|(Y, Z, A)\|^2 \leq C[I_0^2 + \|(L, U)\|_P^2].
\] (3.10)

In addition, we have the following estimates for the difference of two DRBSDEs:

**Theorem 3.7** Assume \((\xi_i, f^i, L^i, U^i), i = 1, 2,\) satisfy all the conditions in Theorem 3.6 and let \((Y^i, Z^i, A^i)\) denote the solution to the corresponding DRBSDE (3.1). Denote \(\delta Y := Y^1 - Y^2\), and similarly for the other notations. Then

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 + |\delta A_t|^2 + \int_0^T |\delta Z_t|^2 dt \right]
\leq C \mathbb{E}^P \left[ |\delta \xi|^2 + \left( \int_0^T |\delta f(t, Y^1_t, Z^1_t)| dt \right)^2 \right] + C \sum_{i=1}^2 \left[ I_0(\xi_i, f^i) + \|(L^i, U^i)\|_P \right] \left( \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta L_t|^2 + |\delta U_t|^2 \right] \right)^+. \tag{3.11}
\]

These two theorems will be proved in the rest of this section. We first note that

**Remark 3.8** (i) In the case that there is only one barrier \(L\), we may view it as \(U = \infty\). One can check straightforwardly that \(\|(L, U)\|_P = \|L^+\|_{P,0}\). Then Theorems 3.6 and 3.7 reduce to standard results for reflected BSDEs with one barrier, see El Karoui et al [10].

(ii) In the case \((L^1, U^1) = (L^2, U^2)\), the last term in (3.11) vanishes and Peng and Xu [19] has already obtained the estimate.

### 3.1 Proof of Theorem 3.7

As usual we start with some a priori estimates.
Lemma 3.9 Assume \((\xi_i, f^i, L^i, U^i), i = 1, 2\), satisfy Assumption 3.1. If the corresponding DRBSDE (3.1) has a solution \((Y^i, Z^i, A^i)\), then

\[
\mathbb{E}^p \left[ \sup_{0 \leq t \leq T} [|\delta Y_t|^2 + |\delta A_t|^2] + \int_0^T |\delta Z_t|^2 dt \right] \leq CI^2,
\]

(3.12)

where, recalling the norm \(\|(Y, Z, A)\|\) defined by (3.3),

\[
I^2 := \mathbb{E}^p \left[ [\delta \xi]^2 + \left( \int_0^T |\delta f(t, Y_t^1, Z_t^1)| dt \right)^2 \right] + \sum_{i=1}^2 \|Y^i, Z^i, A^i\| \left( \mathbb{E}^p \left[ \sup_{0 \leq t \leq T} [|\delta L_t|^2 + |\delta U_t|^2] \right] \right)^{1/2}.
\]

(3.13)

Proof. Let \(\lambda > 0\) be a constant which will be specified later. Applying Itô’s formula on \(e^{\lambda t}|\delta Y_t|^2\) we have

\[
e^{\lambda t}|\delta Y_t|^2 + \lambda \int_t^T e^{\lambda s}|\delta Y_s|^2 ds + \int_t^T e^{\lambda s}|\delta Z_s|^2 ds = e^{\lambda T}|\delta \xi|^2 + 2 \int_t^T e^{\lambda s} \delta Y_s \left( f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \right) ds + 2 \int_t^T e^{\lambda s} \delta Y_s \delta A_s
\]

\[-2 \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s.
\]

(3.14)

For any \(\varepsilon > 0\), note that

\[
2 \int_t^T e^{\lambda s} |\delta Y_s| \left| f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \right| ds
\]

\[\leq C \int_t^T e^{\lambda s} |\delta Y_s| \left( |\delta f(s, Y_s^1, Z_s^1)| + |\delta Y_s| + |\delta Z_s| \right) ds
\]

\[\leq C \left[ \sup_{t \leq s \leq T} |\delta Y_s| \int_t^T e^{\lambda s} |\delta f(s, Y_s^1, Z_s^1)| ds + \int_t^T e^{\lambda s} [|\delta Y_s|^2 + |\delta Y_s| |\delta Z_s|] ds \right]
\]

\[\leq \varepsilon \sup_{t \leq s \leq T} |\delta Y_s|^2 + \frac{1}{2} \int_t^T e^{\lambda s} |\delta Z_s|^2 ds + C \varepsilon^{-1} \left( \int_t^T e^{\lambda s} |\delta f(s, Y_s^1, Z_s^1)| ds \right)^2.
\]

(3.15)
and, with the orthogonal decompositions $A^i = K^{i,+} - K^{i,-}$,

$$2 \int_t^T e^{\lambda s} \delta Y_s \delta A_s$$

$$= 2 \int_t^T e^{\lambda s} \left( Y^1_{s^-} dK^{1,+}_s - Y^1_{s^-} dK^{1,-}_s - Y^2_{s^-} dK^{2,+}_s + Y^2_{s^-} dK^{2,-}_s \right)$$

$$\leq 2 \int_t^T e^{\lambda s} \left( L^1_{s^-} dK^{1,+}_s - L^1_{s^-} dK^{1,-}_s - L^2_{s^-} dK^{2,+}_s + L^2_{s^-} dK^{2,-}_s \right)$$

$$= 2 \int_t^T e^{\lambda s} \left( \delta L_{s^-} dK^{1,+}_s - \delta U_{s^-} dK^{1,-}_s - \delta L_{s^-} dK^{2,+}_s + \delta U_{s^-} dK^{2,-}_s \right)$$

$$\leq 2e^{\lambda(T-t)} \sup_{0 \leq s \leq T} \left[ ||\delta L_s|| + ||\delta U_s|| \right] \left[ \sqrt{A^1} + \sqrt{A^2} \right]. \quad (3.16)$$

Plug (3.15) and (3.16) into (3.14), we obtain

$$e^{\lambda T} |\delta Y_t|^2 + \lambda \int_t^T e^{\lambda s} |\delta Y_s|^2 ds + \int_t^T e^{\lambda s} |\delta Z_s|^2 ds$$

$$\leq e^{\lambda T} |\delta \xi|^2 + \epsilon \sup_{t \leq s \leq T} |\delta Y_s|^2 + \frac{1}{2} \int_t^T e^{\lambda s} |\delta Z_s|^2 ds$$

$$+ C \int_t^T e^{\lambda s} |\delta Y_s|^2 ds + C \epsilon^{-1} \left( \int_t^T e^{\lambda s} |\delta f(s, Y^1_s, Z^1_s)| ds \right)^2$$

$$+ 2e^{\lambda(T-t)} \sup_{0 \leq s \leq T} \left[ ||\delta L_s|| + ||\delta U_s|| \right] \left[ \sqrt{A^1} + \sqrt{A^2} \right] - 2 \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s.$$

Set $\lambda = C$ for the above $C$, we get

$$e^{\lambda T} |\delta Y_t|^2 + \frac{1}{2} \int_t^T e^{\lambda s} |\delta Z_s|^2 ds$$

$$\leq e^{\lambda T} |\delta \xi|^2 + \epsilon \sup_{t \leq s \leq T} |\delta Y_s|^2 + C \epsilon^{-1} \left( \int_t^T e^{\lambda s} |\delta f(s, Y^1_s, Z^1_s)| ds \right)^2 \quad (3.17)$$

$$+ 2e^{\lambda(T-t)} \sup_{0 \leq s \leq T} \left[ ||\delta L_s|| + ||\delta U_s|| \right] \left[ \sqrt{A^1} + \sqrt{A^2} \right] - 2 \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s.$$

Take expectation on both sides, we have

$$\sup_{0 \leq t \leq T} \mathbb{E}^P[|\delta Y_t|^2] + \mathbb{E}^P \left[ \int_0^T |\delta Z_s|^2 ds \right] \leq C \left[ 1 + \epsilon^{-1} \right] I^2 + \epsilon \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right]. \quad (3.18)$$
Moreover, by (3.17) we have

\[
\sup_{0 \leq t \leq T} e^{\lambda t} |\delta Y_t|^2 \\
\leq e^{\lambda T} |\delta \xi|^2 + \varepsilon \sup_{0 \leq t \leq T} |\delta Y_t|^2 + C\varepsilon^{-1} \left( \int_0^T e^{\lambda t} |\delta f(t, Y^1_t, Z^1_t)| dt \right)^2
\]

(3.19)

\[
+ 2e^{\lambda T} \sup_{0 \leq t \leq T} \left[ |\delta L_t| + |\delta U_t| \right] \left[ \sqrt{A^1} + \sqrt{A^2} \right] + 2 \sup_{0 \leq t \leq T} \left| \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s \right|. 
\]

Apply the Burkholder-Davis-Gundy Inequality and note that \( \lambda = C \), we get

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} \left| \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s \right| \right] \leq C \mathbb{E}^P \left[ \left( \int_0^T |\delta Y_t \delta Z_t|^2 dt \right)^{\frac{1}{2}} \right]
\]

(3.20)

\[
\leq C \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t| \left( \int_0^T |\delta Z_t|^2 dt \right)^{\frac{1}{2}} \right]
\]

\[
\leq \sqrt{\varepsilon} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C\varepsilon^{-\frac{1}{2}} \mathbb{E}^P \left[ \int_0^T |\delta Z_t|^2 dt \right].
\]

Take expectation on both sides of (3.19), and apply (3.20) and then (3.18), we obtain

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] \leq C[1 + \varepsilon^{-1}] I^2 + C\varepsilon \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right]
\]

\[
+ C\sqrt{\varepsilon} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C\varepsilon^{-\frac{1}{2}} \mathbb{E}^P \left[ \int_0^T |\delta Z_t|^2 dt \right]
\]

\[
\leq C\left[ \sqrt{\varepsilon} + \varepsilon(1 + \varepsilon^{-\frac{1}{2}}) \right] \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C[1 + \varepsilon^{-\frac{1}{2}}] [1 + \varepsilon^{-1}] I^2
\]

\[
\leq C\sqrt{\varepsilon} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C\varepsilon^{-\frac{1}{2}} I^2
\]

Set \( \varepsilon := \frac{1}{4C^2} \) for the above \( C \). Then

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] \leq CI^2.
\]

Plug this into (3.18), we get

\[
\mathbb{E}^P \left[ \int_0^T |\delta Z_t|^2 dt \right] \leq CI^2.
\]

Finally, notice that

\[
\delta A_t = \delta Y_0 - \delta Y_t - \int_0^t [f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)] ds + \int_0^t \delta Z_s dB_s.
\]
One can easily get the estimate for $\delta A$.

**Proof of Theorem 3.7.** This is a direct consequence of Lemma 3.9 and Theorem 3.6.

We emphasize that in next subsection, we shall prove Theorem 3.6 by using Lemma 3.9 but without using Theorem 3.7. So there is no danger of cycle proof.

### 3.2 Proof of Theorem 3.6

Again, we start with a priori estimate. Recall Lemma 3.4.

**Lemma 3.10** Let Assumption 3.1 and $(3.6)$ hold, and $f = 0$. Then the local solution $(Y, Z, A)$ of DRBSDE $(3.1)$ satisfies $(3.10)$.

**Proof.** Without loss of generality, we assume $\|(L, U)\|_P < \infty$. We proceed in three steps.

**Step 1.** We first assume $(Y, Z, A)$ is a solution of $(3.1)$ and $Y$ is continuous. Then $K^+$ and $K^-$ are also continuous. Apply Itô’s formula on $|Y_t|^2$, by the minimum condition in $(3.1)$ we have,

$$d|Y_t|^2 = 2Y_tZ_t dB_t + |Z_t|^2 dt - 2Y_t d K^+_t + 2Y_t d K^-_t$$

then for any $\varepsilon > 0$,

$$E^P \left[ |Y_T|^2 + \int_t^T |Z_s|^2 ds \right] = E^P \left[ |\xi|^2 + 2 \int_t^T L^- d K^+_s - 2 \int_t^T U^- d K^-_s \right]$$

$$\leq E^P \left[ |\xi|^2 + 2 \sup_{0 \leq s \leq T} L^-_s K^+_T + 2 \sup_{0 \leq s \leq T} U^-_s K^-_T \right]$$

$$\leq E^P \left[ |\xi|^2 + C\varepsilon^{-1} \sup_{0 \leq s \leq T} \left( |L^+_s|^2 + |U^-_s|^2 \right) + \varepsilon \left( |K^+_T|^2 + |K^-_T|^2 \right) \right]$$

$$\leq E^P \left[ |\xi|^2 \right] + C\varepsilon^{-1} \|(L, U)\|_{P,0}^2 + \varepsilon E^P \left[ \left( \bigvee_{0}^{T} A \right)^2 \right].$$

Following standard arguments, in particular by applying the Burkholder-Davis-Gundy Inequality on $(3.21)$, we have

$$E^P \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] \leq C E^P \left[ |\xi|^2 \right] + C\varepsilon^{-1} \|(L, U)\|_{P,0}^2 + C \varepsilon E^P \left[ \left( \bigvee_{0}^{T} A \right)^2 \right].$$

(3.22)
We claim that
\[
\mathbb{E}^p \left[ (\int_0^T A)^2 \right] \leq C \mathbb{E}^p \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] + C \| (L, U) \|_p^2. \tag{3.23}
\]
Combine (3.22) and (3.23) and set \( \varepsilon \) small, we prove (3.10) immediately.

To prove (3.23), we define a sequence of stopping times: \( \tau_0 := 0 \) and, for \( i \geq 0 \),
\[
\begin{align*}
\tau_{2i+1} & := \inf \{ t \geq \tau_{2i} : K_t^+ > K_{\tau_{2i}}^+ \} \land T, \\
\tau_{2i+2} & := \inf \{ t \geq \tau_{2i+1} : K_t^- > K_{\tau_{2i+1}}^- \} \land T.
\end{align*} \tag{3.24}
\]
Then \( dK_t^+ = 0 \) on \([\tau_{2i}, \tau_{2i+1}]\) and \( dK_t^- = 0 \) on \([\tau_{2i+1}, \tau_{2i+2}]\), and thus
\[
\begin{align*}
Y_t & = Y_{\tau_{2i+1}} - \int_{\tau_{2i+1}}^{t} Z_s dB_s - (K_{\tau_{2i+1}}^- - K_t^-), \quad t \in [\tau_{2i}, \tau_{2i+1}]; \\
Y_t & = Y_{\tau_{2i+2}} - \int_{\tau_{2i+2}}^{t} Z_s dB_s - (K_{\tau_{2i+2}}^+ - K_t^+), \quad t \in [\tau_{2i+1}, \tau_{2i+2}].
\end{align*} \tag{3.25}
\]
Since \( L \) and \( U \) are right continuous and \( K \) is continuous, by the minimum condition in (3.1), we have
\[
Y_{\tau_{2i}} = U_{\tau_{2i}} 1_{\{\tau_{2i}<T\}} + \xi 1_{\{\tau_{2i}=T\}} \quad \text{and} \quad Y_{\tau_{2i+1}} = L_{\tau_{2i+1}} 1_{\{\tau_{2i+1}<T\}} + \xi 1_{\{\tau_{2i+1}=T\}}. \tag{3.26}
\]
In particular, on \( \{\tau_{2i} < T\} \), we have \( Y_{\tau_{2i}} = U_{\tau_{2i}} > L_{\tau_{2i}} \), then \( Y_t > L_t \) for \( t \) in a right neighborhood of \( \tau_{2i} \) and thus \( dK_t^+ = 0 \). This implies that \( \tau_{2i+1} > \tau_{2i} \) on \( \{\tau_{2i} < T\} \).

Similarly, \( \tau_{2i+2} > \tau_{2i+1} \) on \( \{\tau_{2i+1} < T\} \). Moreover, as in [12], we see that
\[
\text{for a.s. } \omega, \ \tau_n(\omega) = T \text{ for } n \text{ large enough.} \tag{3.27}
\]
Indeed, denote \( \tau^* := \lim_{n \to \infty} \tau_n \). If \( \tau^* < T \), then \( \tau_n < \tau^* \) for all \( n \) and we get
\[
L_{\tau^*} = \lim_{n \to \infty} L_{\tau_{2i+1}} = \lim_{n \to \infty} Y_{\tau_{2i+1}} = Y_{\tau^*} = \lim_{n \to \infty} Y_{\tau_{2i}} = \lim_{n \to \infty} U_{\tau_{2i}} = U_{\tau^*}.
\]
This contradicts with (3.6).

For each \( i \), by (3.25) and (3.26),
\[
\begin{align*}
0 & \leq \mathbb{E}^p_{\tau_{2i}} [K_{\tau_{2i+1}}^-] - K_{\tau_{2i}}^- + \mathbb{E}^p_{\tau_{2i}} [Y_{\tau_{2i+1}}] - Y_{\tau_{2i}} \\
& = \mathbb{E}^p_{\tau_{2i}} \left[ L_{\tau_{2i+1}} 1_{\{\tau_{2i+1}<T\}} + \xi 1_{\{\tau_{2i+1}=T\}} \right] - U_{\tau_{2i}} 1_{\{\tau_{2i}<T\}} - \xi 1_{\{\tau_{2i}=T\}} \\
& = \left[ \mathbb{E}^p_{\tau_{2i}} [L_{\tau_{2i+1}}] - U_{\tau_{2i}} \right] 1_{\{\tau_{2i}<T\}} + \mathbb{E}^p_{\tau_{2i}} \left[ L_{\tau_{2i+1}} - \xi 1_{\{\tau_{2i}<T=\tau_{2i+1}\}} \right] \\
& \leq \left[ \mathbb{E}^p_{\tau_{2i}} [L_{\tau_{2i+1}}] - U_{\tau_{2i}} \right]^+.
\end{align*}
\]
Then for any $n$,
\[
\mathbb{E}^p \left[ \left( \sum_{i=0}^{n} \mathbb{E}_{\tau_{2i}}^p [K_{\tau_{2i+1}}^- - K_{\tau_{2i}}^-] \right)^2 \right] \leq \|(L, U)\|^2_{\mathbb{F}}.
\]

Send $n \to \infty$, we get
\[
\mathbb{E}^p \left[ \left( \sum_{i \geq 0} \mathbb{E}_{\tau_{2i+1}}^p [K_{\tau_{2i+1}}^- - K_{\tau_{2i}}^-] \right)^2 \right] \leq \|(L, U)\|^2_{\mathbb{F}}. \tag{3.28}
\]

Similarly,
\[
\mathbb{E}^p \left[ \left( \sum_{i \geq 0} \mathbb{E}_{\tau_{2i+2}}^p [K_{\tau_{2i+2}}^+ - K_{\tau_{2i+1}}^+] \right)^2 \right] \leq \|(L, U)\|^2_{\mathbb{F}}. \tag{3.29}
\]

Denote
\[
\hat{Y}_{\tau_n} := Y_{\tau_n} - \sum_{i \leq \frac{n}{2}} \mathbb{E}_{\tau_{2i}}^p [K_{\tau_{2i+1}}^- - K_{\tau_{2i}}^-] + \sum_{i \leq \frac{n-1}{2}} \mathbb{E}_{\tau_{2i+1}}^p [K_{\tau_{2i+2}}^+ - K_{\tau_{2i+1}}^+] \tag{3.30}
\]

By (3.28) and (3.29), we have
\[
\mathbb{E}^p \left[ \max_{n \geq 0} |\hat{Y}_{\tau_n}|^2 \right] \leq C \mathbb{E}^p \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] + C \|(L, U)\|^2_{\mathbb{F}}. \tag{3.31}
\]

Note that
\[
\hat{Y}_{\tau_n} = Y_0 + \int_0^{\tau_n} Z_s dB_s + \sum_{i \leq \frac{n}{2}} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^p [K_{\tau_{2i+1}}^-]] - \sum_{i \leq \frac{n-1}{2}} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^p [K_{\tau_{2i+2}}^+]]
\]
is a martingale. By (3.31), we have
\[
\mathbb{E}^p \left[ \sum_{i \leq \frac{n}{2}} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^p [K_{\tau_{2i+1}}^-]]^2 + \sum_{i \leq \frac{n-1}{2}} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^p [K_{\tau_{2i+2}}^+]]^2 \right]
\]
\[
= \mathbb{E}^p \left[ \left( \sum_{i \leq \frac{n}{2}} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^p [K_{\tau_{2i+1}}^-]] - \sum_{i \leq \frac{n-1}{2}} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^p [K_{\tau_{2i+2}}^+]] \right)^2 \right]
\]
\[
= \mathbb{E}^p \left[ \left( \hat{Y}_{\tau_n} - Y_0 - \int_0^{\tau_n} Z_s dB_s \right)^2 \right] \leq C \mathbb{E}^p \left[ \sup_{i \geq 0} |\hat{Y}_{\tau_i}|^2 + \int_0^{\tau_n} |Z_t|^2 dt \right]
\]
\[
\leq C \mathbb{E}^p \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] + C \|(L, U)\|^2_{\mathbb{F}}.
\]
Send \( n \to \infty \) and

\[
\mathbb{E}^P \left[ \sum_{i \geq 0} [K_{2i+1}^+ - \mathbb{E}^P_{T_{2i}} [K_{2i+1}^-]]^2 + \sum_{i \geq 0} [K_{2i+2}^+ - \mathbb{E}^P_{T_{2i+1}} [K_{2i+2}^-]]^2 \right]
\]

\[
= \mathbb{E}^P \left[ \left( \sum_{i \geq 0} [K_{2i+1}^- - \mathbb{E}^P_{T_{2i}} [K_{2i+1}^-]] \right)^2 + \left( \sum_{i \geq 0} [K_{2i+2}^- - \mathbb{E}^P_{T_{2i+1}} [K_{2i+2}^-]] \right)^2 \right]
\]

\[
\leq C \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] + C \|(L, U)\|_{\mathbb{F}}^2.
\] (3.32)

This, together with (3.27), (3.28) and (3.29), implies further that

\[
\mathbb{E}^P \left[ |K_T^+|^2 + |K_T^-|^2 \right] = \mathbb{E}^P \left[ \left( \sum_{i \geq 0} [K_{2i+1}^- - \mathbb{E}^P_{T_{2i}} [K_{2i+1}^-]] \right)^2 + \left( \sum_{i \geq 0} [K_{2i+2}^- - \mathbb{E}^P_{T_{2i+1}} [K_{2i+2}^-]] \right)^2 \right]
\]

\[
\leq C \mathbb{E}^P \left[ \left( \sum_{i \geq 0} [K_{2i+1}^- - \mathbb{E}^P_{T_{2i}} [K_{2i+1}^-]] \right)^2 + \left( \sum_{i \geq 0} \left[ \mathbb{E}^P_{T_{2i}} [K_{2i+1}^-] - K_{2i}^- \right] \right)^2 \right]
\]

\[
+ C \mathbb{E}^P \left[ \left( \sum_{i \geq 0} [K_{2i+2}^- - \mathbb{E}^P_{T_{2i+1}} [K_{2i+2}^-]] \right)^2 + \left( \sum_{i \geq 0} \left[ \mathbb{E}^P_{T_{2i+1}} [K_{2i+2}^-] - K_{2i+1}^- \right] \right)^2 \right]
\]

\[
\leq C \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] + C \|(L, U)\|_{\mathbb{F}}^2.
\]

This proves (3.23) and hence (3.10).

Step 2. We next assume \((Y, Z, A)\) is a local solution but \(Y\) is still continuous. Let \(\tau_i\) be defined by (3.24). Then (3.23)-(3.27) still hold. This implies

\[
Y_{\tau_{2i}} \geq -U_{\tau_{2i}} 1_{\{\tau_{2i} < T\}} - |\xi| 1_{\{\tau_{2i} = T\}} \geq - \left[ \sup_{0 \leq t \leq T} U_t^- + |\xi| \right],
\]

\[
Y_{\tau_{2i}} \leq \mathbb{E}^P_{\tau_{2i}} [Y_{\tau_{2i+1}}] \leq \mathbb{E}^P_{\tau_{2i}} \left[ L_{\tau_{2i+1}}^+ 1_{\{\tau_{2i+1} < T\}} + |\xi| 1_{\{\tau_{2i+1} = T\}} \right] \leq \mathbb{E}^P_{\tau_{2i}} \left[ \sup_{0 \leq t \leq T} L_t^+ + |\xi| \right].
\]

Then

\[
\max_{i \geq 0} |Y_{\tau_{2i}}| \leq \left[ \sup_{0 \leq t \leq T} U_t^- + |\xi| \right] \vee \left[ \sup_{0 \leq s \leq T} \mathbb{E}^P_{s} \left[ \sup_{0 \leq t \leq T} L_t^+ + |\xi| \right] \right]
\]

\[
\leq \sup_{0 \leq s \leq T} \mathbb{E}^P_{s} \left[ \sup_{0 \leq t \leq T} \left[ L_t^+ + U_t^- \right] + |\xi| \right].
\]
Thus

\[
\mathbb{E}^P \left[ \max_{i \geq 0} |Y_{2i}|^2 \right] \leq \mathbb{E}^P \left[ \left( \sup_{0 \leq s \leq T} \mathbb{E}_s^P \left[ \sup_{0 \leq t \leq T} [L_t^+ + U_t^-] + |\xi| \right] \right)^2 \right] \\
\leq C \mathbb{E}^P \left[ \left( \sup_{0 \leq s \leq T} [L_s^+ + U_s^-] + |\xi| \right)^2 \right] \leq C \mathbb{E}^P [||\xi||^2] + C \|(L, U)\|_{\mathcal{P}, 0}^2. \tag{3.33}
\]

Now for any \(n\), define

\[
\hat{\tau}_n := \inf \left\{ t : \sup_{0 \leq s \leq t} |Y_s| + \int_0^t |Z_s|^2 ds + \sqrt{t} A \geq n \right\} \wedge T. \tag{3.34}
\]

Then

\[
\mathbb{E}^P \left[ \sup_{0 \leq t < \hat{\tau}_n} |Y_t|^2 + \int_0^{\hat{\tau}_n} |Z_t|^2 dt + \left( \sqrt{\hat{\tau}_n} A \right)^2 \right] < \infty. \tag{3.35}
\]

Define

\[
\tilde{\tau}_n := \inf \{ \tau_{2i} : \tau_{2i} \geq \hat{\tau}_n \}.
\]

Then by (3.1) and (3.33), we have

\[
Y_t = Y_{\tilde{\tau}_n} + \int_t^{\tilde{\tau}_n} Z_s dB_s - (K_{\tilde{\tau}_n}^- - K_t^-), \quad Y_t \leq U_t, \quad [U_t - Y_t] dK_t^- = 0, \quad t \in [\tilde{\tau}_n, \tilde{\tau}_n]; \\
\mathbb{E}^P [|Y_{\tilde{\tau}_n}|^2] \leq C \mathbb{E}^P [||\xi||^2] + C \|(L, U)\|_{\mathcal{P}, 0}^2.
\]

By standard arguments for Reflected BSDEs with one barrier, see e.g. [10],

\[
\mathbb{E}^P [|Y_{\tilde{\tau}_n}|^2] \leq C \mathbb{E}^P [||\xi||^2] + C \|(L, U\|_{\mathcal{P}, 0}^2 + C \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |U_t^-|^2 \right] \leq C \mathbb{E}^P [||\xi||^2] + C \|(L, U)\|_{\mathcal{P}, 0}^2.
\]

This, together with (3.35), implies that

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq \tilde{\tau}_n} |Y_t|^2 + \int_0^{\tilde{\tau}_n} |Z_t|^2 dt + \left( \sqrt{\tilde{\tau}_n} A \right)^2 \right] < \infty.
\]

Then by Step 1, we obtain

\[
\mathbb{E}^P \left[ \sup_{0 \leq t \leq \tilde{\tau}_n} |Y_t|^2 + \int_0^{\tilde{\tau}_n} |Z_t|^2 dt + \left( \sqrt{\tilde{\tau}_n} A \right)^2 \right] \leq C \mathbb{E}^P [|Y_{\tilde{\tau}_n}|^2] + C \|(L, U)\|_{\mathcal{P}, 0}^2 \\
\leq C \mathbb{E}^P [||\xi||^2] + C \|(L, U)\|_{\mathcal{P}, 0}^2.
\]
Note that \( \hat{\tau}_n = T \) when \( n \) is large enough. Send \( n \to \infty \) and apply the Monotone Convergence Theorem, we prove (3.10).

**Step 3.** Finally we allow \( Y \) to be discontinuous.. Let

\[
\begin{align*}
\bar{Y}_t := Y_t - \sum_{0 < s \leq t} \Delta Y_s, \quad \bar{K}_t^+ := K_t^+ - \sum_{0 < s \leq t} \Delta K_s^+, \quad \bar{K}_t^- := K_t^- - \sum_{0 < s \leq t} \Delta K_s^-,
\end{align*}
\]

\[
\bar{A}_t := \bar{K}_t^+ - \bar{K}_t^-, \quad \bar{L}_t := L_t - \sum_{0 < s \leq t} \Delta K_s^+, \quad \bar{U}_t := U_t + \sum_{0 < s \leq t} \Delta K_s^-, \quad \bar{\xi} := \xi - \sum_{0 < s \leq T} \Delta Y_s.
\]

Then it is clear that \( \bar{Y} \) is continuous, \((\bar{L}, \bar{U})\) satisfies (3.6), and \((\bar{Y}, Z, \bar{A})\) is a local solution to DRBSDE (3.1) with coefficients \((\bar{\xi}, 0, \bar{L}, \bar{U})\). By Step 2, we have

\[
\| (Y, Z, \bar{A}) \|^2 \leq C \mathbb{E}^\mathbb{P}\left[|\bar{\xi}|^2\right] + C\| (\bar{L}, \bar{U}) \|^2 \mathbb{P}.
\]

One can check straightforwardly that

\[
\| (Y, Z, A) \|^2 \leq C\| (Y, Z, \bar{A}) \|^2 + C \mathbb{E}^\mathbb{P}\left[\left(\sum_{0 \leq t \leq T} [\Delta K_t^+ + \Delta K_t^-]\right)^2\right];
\]

\[
\mathbb{E}^\mathbb{P}[|\bar{\xi}|^2] \leq C \mathbb{E}^\mathbb{P}[|\xi|^2] + C \mathbb{E}^\mathbb{P}\left[\left(\sum_{0 \leq t \leq T} [\Delta K_t^+ + \Delta K_t^-]\right)^2\right];
\]

\[
\| (\bar{L}, \bar{U}) \|^2 \leq \| (L, U) \|^2 \mathbb{P}.
\]

Then

\[
\| (Y, Z, A) \|^2 \leq C \mathbb{E}^\mathbb{P}[|\xi|^2] + C\| (L, U) \|^2 \mathbb{P} + C \mathbb{E}^\mathbb{P}\left[\left(\sum_{0 \leq t \leq T} [\Delta K_t^+ + \Delta K_t^-]\right)^2\right]. \tag{3.36}
\]

Note that, when \( \Delta K_t^+ > 0 \), by the minimum condition of (3.1) we see that \( Y_t^- = L_t^- \). Since \( K^+ \) and \( K^- \) are orthogonal, we have \( \Delta Y_t = -\Delta K_t^+ \). Thus \( L_t \leq Y_t = Y_t^- - \Delta K_t^+ = L_t^- - \Delta K_t^+ \). This implies that \( \sum_{0 < t \leq T} \Delta K_t^+ \leq \sum_{0 < t \leq T} [\Delta L_t^-] \). Similarly we have \( \sum_{0 < t \leq T} \Delta K_t^- \leq \sum_{0 < t \leq T} [\Delta U_t^+] \). Following the arguments for (2.19), one can easily prove that

\[
\mathbb{E}^\mathbb{P}\left[\left(\sum_{0 < t \leq T} [\Delta L_t^- + [\Delta U_t^+]]\right)^2\right] \leq C\| (L, U) \|^2 \mathbb{P}.
\]

Then (3.10) follows from (3.36) immediately.

**Proofs of Theorem 3.6.** First, by Lemma 3.5 we know (ii) implies (i). On the other hand, if (i) holds true, then \( Y^0 := Y \) is clearly a square integrable semimartingale between \( L \) and \( U \). That is, (i) and (ii) are equivalent.
Next, assume (ii) holds true. Since \( L \leq Y^0 \leq U \), then for any partition \( \pi : 0 = \tau_0 < \cdots < \tau_n = T \),

\[
L^+ + U^- \leq (Y^0)^+ + (Y^0)^- = |Y^0|;
\]

\[
\left[ \mathbb{E}^P_\tau[L_{\tau_{i+1}} - U_{\tau_i}] \right]^+ + \left[ L_{\tau_i} - \mathbb{E}^P_\tau[U_{\tau_{i+1}}] \right]^+
\leq \left[ \mathbb{E}^P_\tau[Y^0_{\tau_{i+1}} - Y^0_{\tau_i}] \right]^+ + \left[ Y^0_{\tau_i} - \mathbb{E}^P_\tau[Y^0_{\tau_{i+1}}] \right]^+ = \left| \mathbb{E}^P_\tau[Y^0_{\tau_{i+1}}] - Y^0_{\tau_i} \right|.
\]

This implies immediately that \( \|(L, U)\|_P \leq \|Y^0\|_P \), and thus (iii) holds.

It remains to prove that (iii) implies (ii). We first assume (3.6) holds. Then it follows from Lemma 3.4 that DRBSDE (3.1) with \( f = 0 \) admits a local solution \((Y^0, Z^0, A^0)\). Applying Lemma 3.10 we see that \( \|(Y^0, Z^0, A^0)\| \leq C[I_0 + \|(L, U)\|_P] \). This implies (3.7).

In the general case, denote \( U^n := U + \frac{1}{n} \). Then \((L, U^n)\) satisfies (3.6). By the above arguments, DRBSDE (3.1) with coefficients \((\xi, 0, L, U^n)\) has a unique solution \((Y^n, Z^n, A^n)\) satisfying

\[
\|(Y^n, Z^n, A^n)\|_2 \leq C\mathbb{E}^P[|\xi|^2] + C\|(L, U^n)\|_P^2.
\]

It is obvious that \( \|(L, U^n)\|_P \leq \|(L, U)\|_P \). Then

\[
\|(Y^n, Z^n, A^n)\|_2 \leq C\mathbb{E}^P[|\xi|^2] + C\|(L, U)\|_P^2.
\]

Now for \( m > n \), applying Lemma 3.9 we have

\[
\mathbb{E}^P\left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^2 + |A^n_t - A^m_t|^2 + \int_0^T |Z^n_t - Z^m_t|^2 dt \right]
\leq C\left[\|(Y^n, Z^n, A^n)\| + \|(Y^m, Z^m, A^m)\|\right] \left[ \frac{1}{n} - \frac{1}{m} \right]
\leq \frac{C}{n} \left( \mathbb{E}^P[|\xi|^2] \right)^{\frac{1}{2}} + \|(L, U)\|_P.
\]

Send \( n \to \infty \), we obtain limit processes \((Y^0, Z^0, A^0)\). Following standard arguments we see that \( Y^0 \) satisfies the requirement in (ii).

4 Semimartingales under G-expectation

In this section we introduce a nonlinear expectation, which is a variation of the G-expectation proposed by Peng [18], and we shall still call it G-expectation. Let
(\(\Omega, \mathcal{F}, \mathbb{F}\)) be a filtered space such that \(\mathbb{F}\) is right continuous and \(\mathcal{P}\) be a family of probability measures. For each \(\mathbb{P} \in \mathcal{P}\) and \(\mathbb{F}\)-stopping time \(\tau\), denote
\[
\mathcal{P}(\tau, \mathbb{P}) := \{ \mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_\tau \}. \tag{4.1}
\]
Throughout this section, we shall always assume

**Assumption 4.1**

(i) \((2.1)\) holds for every \(\mathbb{P} \in \mathcal{P}\);

(ii) \(\mathcal{N}_\mathbb{P} \subset \mathcal{F}_0\), where \(\mathcal{N}_\mathbb{P}\) is the set of all \(\mathcal{P}\)-polar sets, that is, all \(E \in \mathcal{F}\) such that \(\mathbb{P}(E) = 0\) for all \(\mathbb{P} \in \mathcal{P}\).

(iii) For any \(\mathbb{P} \in \mathcal{P}\), \(\mathbb{F}\)-stopping time \(\tau\), \(\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\tau, \mathbb{P})\), and any partition \(E_1, E_2 \in \mathcal{F}_\tau\) of \(\Omega\), the probability measure \(\bar{\mathbb{P}}\) defined below also belongs to \(\mathcal{P}(\tau, \mathbb{P})\):
\[
\bar{\mathbb{P}}(E) := \mathbb{P}_1(E \cap E_1) + \mathbb{P}_2(E \cap E_2), \quad \text{for all } E \in \mathcal{F}. \tag{4.2}
\]

We provide below an important example for such \(\mathcal{P}\), which induces the \(G\)-expectation of Peng [18], and we refer to [25] for more examples.

**Example 4.2** Let \(\Omega := \{ \omega \in C([0, T], \mathbb{R}) : \omega_0 = 0 \}\), \(B\) the canonical process, \(\mathbb{F}\) the right limit of the filtration generated by \(B\), \(\mathbb{P}_0\) the Wiener measure. Let \(0 \leq \underline{\sigma} < \overline{\sigma}\) be two constants. For each bounded \(\mathbb{F}\)-progressively measurable process \(\sigma\), denote \(X^\alpha_t := \int_0^t \alpha_s dB_s\), \(\mathbb{P}_0\)-a.s. Then the following class \(\mathcal{P}\) satisfies Assumption 4.1:
\[
\mathcal{P} := \{ \mathbb{P}^\sigma : \sigma \leq \underline{\sigma} \leq \overline{\sigma} \} \quad \text{where} \quad \mathbb{P}^\sigma := \mathbb{P}_0 \circ (X^\alpha)^{-1}.
\]

**4.1 Definitions**

We first define

**Definition 4.3** We say an \(\mathbb{F}\)-progressively measurable process \(Y\) is a \(\mathcal{P}\)-martingale (resp. \(\mathcal{P}\)-supermartingale, \(\mathcal{P}\)-submartingale, \(\mathcal{P}\)-semimartingale) if it is a \(\mathbb{P}\)-martingale (resp. \(\mathbb{P}\)-supermartingale, \(\mathbb{P}\)-submartingale, \(\mathbb{P}\)-semimartingale) for all \(\mathbb{P} \in \mathcal{P}\).

We next define the \(G\)-expectation and conditional \(G\)-expectation. For any \(\mathcal{F}\)-measurable random variable \(\xi\) such that \(\mathbb{E}^\mathbb{P}[|\xi|] < \infty\) for all \(\mathbb{P} \in \mathcal{P}\), its \(G\)-expectation is defined by
\[
\mathbb{E}^G[\xi] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P}[\xi]. \tag{4.3}
\]
The conditional $G$-expectation is more involved. For any $F$-stopping time $\tau$, denote
\[
E^{G,P}_\tau[\xi] := \operatorname{ess sup}_{P' \in \mathcal{P}(\tau, F)} E^{P'}_\tau[\xi], \quad P\text{-a.s.}
\tag{4.4}
\]
We note that, by Lemma 2.1, we may take the convention that $E^{G,P}_\tau[\xi]$ is $F_\tau$-measurable. When the family $\{E^{G,P}_\tau[\xi], P \in \mathcal{P}\}$ can be aggregated, that is, there exists an $F_\tau$-measurable random variable, denoted as $E^G_\tau[\xi]$, such that
\[
E^G_\tau[\xi] = E^{G,P}_\tau[\xi], \quad P\text{-a.s. for all } P \in \mathcal{P},
\tag{4.5}
\]
we call $E^G_\tau[\xi]$ the conditional $G$-expectation of $\xi$. We refer to Soner, Touzi and Zhang [25] for the detailed study on the aggregation issue. Following standard arguments, we have the following time consistency (or say, Dynamic Programming Principle), whose proof is provided in the Appendix for completeness:

**Lemma 4.4** Under Assumption 4.1, for any $\tau_1 \leq \tau_2$ and any $P \in \mathcal{P}$, we have
\[
E^{G,P}_{\tau_1}[\xi] = \operatorname{ess sup}_{P' \in \mathcal{P}(\tau_1, P)} E^{P'}_{\tau_1}[E^{G,P}_{\tau_2}[\xi]], \quad P\text{-a.s.}
\]

We finally define

**Definition 4.5** We say an $F$-progressively measurable process $Y$ is a $G$-martingale (resp. $G$-supermartingale, $G$-submartingale) if, for any $P \in \mathcal{P}$ and any $F$-stopping times $\tau_1 \leq \tau_2$,
\[
Y_{\tau_1} = (\text{resp.} \geq, \leq) E^{G,P}_{\tau_1}[Y_{\tau_2}], \quad P\text{-a.s.}
\]
We remark that a $P$-martingale is also called a symmetric $G$-martingale in the literature, see e.g. [27].

### 4.2 Characterization of $P$-semimartingales

The following result is immediate:

**Proposition 4.6** Let Assumption 4.1 hold.

(i) A $P$-martingale (resp. $P$-supermartingale, $P$-submartingale) must be a $G$-martingale (resp. $G$-supermartingale, $G$-submartingale).

(ii) If $Y$ is a $G$-martingale (resp. $G$-supermartingale, $G$-submartingale) and $M$ is a $P$-martingale, then $Y + M$ is a $G$-martingale (resp. $G$-supermartingale, $G$-submartingale).

(iii) A $G$-supermartingale is a $P$-supermartingale. In particular, a $G$-martingale is a $P$-supermartingale.
Proof. (i) and (ii) are obvious. To prove (iii), let $Y$ be a $G$-supermartingale. Then for any $\tau_1 \leq \tau_2$ and any $\mathbb{P} \in \mathcal{P}$,

$$Y_{\tau_1} \geq \mathbb{E}_{\tau_1}^G[Y_{\tau_2}] \geq \mathbb{E}_{\tau_1}^P[Y_{\tau_2}], \quad \mathbb{P}\text{-a.s.}$$

That is, $Y$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}$, and thus is a $\mathcal{P}$-supermartingale. □

We next study $\mathcal{P}$-semimartingales. In light of Theorem 2.7, we define a new norm:

$$\|Y\|_{\mathcal{P}} := \sup_{\mathbb{P} \in \mathcal{P}} \|Y\|_{\mathbb{P}}. \quad (4.6)$$

The following result is a direct consequence of Theorems 2.6 and 2.7.

**Theorem 4.7** Let Assumption 4.1 hold. If $\|Y\|_{\mathcal{P}} < \infty$, then $Y$ is a $\mathcal{P}$-semimartingale. Moreover, for any $\mathbb{P} \in \mathcal{P}$ and for the decomposition

$$Y_t = Y_0 + M^P_t + A^P_t, \quad \mathbb{P}\text{-a.s.} \quad (4.7)$$

we have

$$\mathbb{E}^P \left[ (M^P)_T + (\bigvee_0^T A^P)^2 \right] \leq C \|Y\|_{\mathcal{P}}^2.$$

The norm $\| \cdot \|_{\mathcal{P}}$ is defined through each $\mathbb{P} \in \mathcal{P}$. The following definition relies on the $G$-expectation directly:

$$\|Y\|_{G}^2 := \mathbb{E}^G \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] + \sup_{\pi} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^P \left[ \left( \sum_{i=0}^{n-1} \mathbb{E}_{\tau_i}^G(Y_{\tau_{i+1}} - Y_{\tau_i}) \right)^2 \right]. \quad (4.8)$$

**Remark 4.8** (i) If the involved conditional $G$-expectations exist, then we may simplify the definition of $\|Y\|_{G}$:

$$\|Y\|_{G}^2 := \mathbb{E}^G \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] + \sup_{\pi} \mathbb{E}^G \left[ \left( \sum_{i=0}^{n-1} \mathbb{E}_{\tau_i}^G(Y_{\tau_{i+1}} - Y_{\tau_i}) \right)^2 \right].$$

(ii) In general $\| \cdot \|_{G}$ does not satisfy the triangle inequality and thus is not a norm. (iii) For $G$-submartingales $Y^1, Y^2$, the triangle inequality holds:

$$\|Y^1 + Y^2\|_{G} \leq \|Y^1\|_{G} + \|Y^2\|_{G}.$$

However, in general $Y^1 + Y^2$ may not be a $G$-submartingale anymore. □
Nevertheless, $\|Y\|_G$ involves the process $Y$ only. The following estimate is the main result of this section.

**Theorem 4.9** Assume Assumption 4.1 holds. Then there exists a universal constant $C$ such that $\|Y\|_P \leq C \|Y\|_G$.

**Proof.** Without loss of generality, we assume $\|Y\|_G < \infty$. For any $\mathbb{P} \in \mathcal{P}$ and any partition $\pi : 0 = \tau_0 \leq \cdots \leq \tau_n = T$, denote

$$N_{\tau_i} := \sum_{j=0}^{i-1} \left[ \mathbb{E}^{G,\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) - Y_{\tau_j} \right].$$

Then

$$Y_{\tau_i} - N_{\tau_i} = Y_0 + \sum_{j=0}^{i-1} \left[ Y_{\tau_{j+1}} - \mathbb{E}^{G,\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) \right]$$

$$= Y_0 + \sum_{j=0}^{i-1} \left[ Y_{\tau_{j+1}} - \mathbb{E}^{\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) \right] - \sum_{j=0}^{i-1} \left[ \mathbb{E}^{G,\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) - \mathbb{E}^{\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) \right].$$

Note that

$$\sum_{j=0}^{i-1} \left[ Y_{\tau_{j+1}} - \mathbb{E}^{\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) \right]$$

is a $\mathbb{P}$-martingale,

$$\sum_{j=0}^{i-1} \left[ \mathbb{E}^{G,\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) - \mathbb{E}^{\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) \right]$$

is nondecreasing and is $\mathcal{F}_{\tau_{i-1}}$-measurable.

Applying Lemma 2.3 we obtain

$$\mathbb{E}^{\mathbb{P}} \left[ \left( \sum_{j=0}^{n-1} \left[ \mathbb{E}^{G,\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) - \mathbb{E}^{\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) \right] \right)^2 \right] \leq C \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq i \leq n} \|Y_{\tau_i}\|^2 + \|N_{\tau_i}\|^2 \right] \leq C \|Y\|_G^2.$$

This, together with the definition of $\cdot \|_G$, implies that

$$\mathbb{E}^{\mathbb{P}} \left[ \left( \sum_{j=0}^{n-1} \|\mathbb{E}^{\mathbb{P}}_{\tau_j} (Y_{\tau_{j+1}}) - Y_{\tau_j}\|^2 \right) \right] \leq C \|Y\|_G^2.$$

Since $\pi$ is arbitrary, we get $\|Y\|_P \leq C \|Y\|_G$. Finally, since $\mathbb{P} \in \mathcal{P}$ is arbitrary, we prove the result. \qed

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4.3 Doob-Meyer Decomposition for $G$-submartingales

As a special case of Theorem 4.7, we have the following decomposition for $G$-submartingales.

**Proposition 4.10** Assume Assumption 4.1 holds. If $Y$ is a $G$-submartingale satisfying $\|Y\|_P < \infty$ (in particular if $\|Y\|_G < \infty$), then all the results in Theorem 4.7 hold.

** Remark 4.11** Unlike Lemma 2.2 for $G$-submartingales in general we do not have $\|Y\|_P \leq C \sup_{P \in \mathcal{P}} \|Y\|_{P,0}$. See Example 5.2 below.

Now let $Y$ be as in Proposition 4.10, and consider its decomposition (4.7). Let $A^P = L^P - K^P$ be the orthogonal decomposition. We have the following conjecture:

**Conjecture (Doob-Meyer decomposition)**: The family $\{K^P, P \in \mathcal{P}\}$ satisfies the following property:

$$
-K_t^P = \operatorname{ess} \sup_{P' \in \mathcal{P}(t,P)} \mathbb{E}^{P'}_t \left[ -K^P_T \right].
$$

(4.9)

In particular, if the families $\{M^P, K^P, L^P, P \in \mathcal{P}\}$ can be aggregated into $\{M, K, L\}$, then $-K$ is a $G$-martingale, and we have the following desired Doob-Meyer decomposition for $G$-submartingales:

$$
Y_t = Y_0 + [M_t - K_t] + L_t,
$$

where $M - K$ is a $G$-martingale and $L$ is nondecreasing.

(4.10)

We refer again to [25] for the issue of aggregation. In particular, we can always aggregate the families $\{M^P, K^P, L^P, P \in \mathcal{P}\}$ when the class $\mathcal{P}$ is separable, in the sense of [25]. This conjecture looks natural, but it is quite subtle. Our estimates in this section are rather preliminary. We hope to address the issue more thoroughly in some future research.

5 Appendix

We first provide an example such that $\|Y\|_{P,0} < \infty$ but $\|Y\|_P = \infty$.  

Example 5.1 Fix $\mathbb{P}$. Let $K$ be an $\mathbb{F}$-progressively measurable continuous increasing process such that $K_0 = 0$ and $\mathbb{E}^\mathbb{P}[K_T^2] = \infty$. Define the sequence of stopping times: $\tau_0 := 0$ and, for $n \geq 1$, $\tau_n := \inf\{t \geq 0 : K_t = n\} \wedge T$. Since $K_T < \infty$, $\tau_n = T$ for $n$ large enough, a.s. We now define the process $Y_t$ as follows: $Y_0 := 0$, and for $n \geq 0$,

$$Y_t := \begin{cases} Y_{\tau_2n} - K_t + K_{\tau_2n}, & t \in (\tau_{2n}, \tau_{2n+1}] ; \\ Y_{\tau_{2n}+1} - K_t, & t \in (\tau_{2n+1}, \tau_{2n+2}]. \end{cases} \quad (5.1)$$

Then \(|Y|_\mathbb{P}, 0 < \infty \) but \(|Y|_\mathbb{F} = \infty\).

Proof. It is easy to check that $-1 \leq Y_t \leq 0$ and $\int_0^T Y = K_T$. Then $|Y|_{\mathbb{P}, 0} \leq 1$ and $\mathbb{E}^\mathbb{P}\left( (\int_0^T Y)^2 \right) = \infty$. By Theorem 2.7 we get $|Y|_\mathbb{P} = \infty$.

We next provide a $G$-submartingale such that $\sup_{\mathbb{P} \in \mathbb{P}} |Y|_{\mathbb{P}, 0} < \infty$, but $|Y|_\mathbb{P} = \infty$.

Example 5.2 Fix $\mathbb{P}$. Let $K$ be as in Example 5.1 such that $-K$ is a $G$ martingale and $\mathbb{E}^G[K_T^2] = \infty$, instead of $\mathbb{E}^\mathbb{P}[K_T^2] = \infty$. Then the process $Y$ defined in Example 5.1 satisfies all the requirements.

Proof. By the proof of Example 5.1 clearly $\sup_{\mathbb{P} \in \mathbb{P}} |Y|_{\mathbb{P}, 0} < \infty$, but $|Y|_\mathbb{P} = \infty$. Moreover, on $(\tau_{2n}, \tau_{2n+1}]$, $dY_t = dK_t$ and thus is a $G$-martingale; and on $(\tau_{2n+1}, \tau_{2n+2}]$, $dY_t = dK_t$, then $Y$ is increasing and thus is a $G$-submartingale. So $Y$ is a $G$-submartingale on $[0, T]$.

We finally prove Lemma 4.4.

Proof of Lemma 4.4. First we have

$$\mathbb{E}^{G, \mathbb{P}}_{t_1}[\xi] = \underset{P' \in \mathcal{P}(t_1, P)}{\sup} \mathbb{E}^{P'}_{t_1}[\xi] = \underset{P' \in \mathcal{P}(t_1, P)}{\sup} \mathbb{E}^{P'}_{t_1}[\mathbb{E}^{P'}_{t_2}[\xi]] \leq \underset{P' \in \mathcal{P}(t_1, P)}{\sup} \mathbb{E}^{P'}_{t_1}[\mathbb{E}^{G, P'}_{t_2}[\xi]].$$

To prove the other inequality, for each $P' \in \mathcal{P}(t_1, P)$, we recall from Neveu [16] that there exists a sequence $\mathbb{P}^n \in \mathcal{P}(\tau_2, \mathbb{P}')$ such that

$$\mathbb{E}^{P^n}_{t_2}[\xi] \to \underset{P' \in \mathcal{P}(t_2, P')}{{\cal E}^{\mathbb{P}'}}[\xi], \quad n \to \infty.$$  

We claim that we may construct $\mathbb{P}^n \in \mathcal{P}(\tau_2, \mathbb{P}') \subset \mathcal{P}(\tau_1, \mathbb{P})$ such that

$$\mathbb{E}^{P^n}_{t_2}[\xi] = \max_{1 \leq m \leq n} \mathbb{E}^{P_m}_{t_2}[\xi], \mathbb{P}'-\text{a.s.} \quad (5.2)$$
Then clearly $\mathbb{E}_{\tau_2}^\tilde{P}[\xi] \uparrow \sup_{\tilde{P} \in \mathcal{P}(\tau_2, P')} \mathbb{E}_{\tau_2}^\tilde{P}[\xi]$, $\mathbb{P}$-a.s.. Thus,

$$
\mathbb{E}_{\tau_1}^P \left[ \sup_{\tilde{P} \in \mathcal{P}(\tau_2, P')} \mathbb{E}_{\tau_2}^\tilde{P}[\xi] \right] = \lim_{n \to \infty} \mathbb{E}_{\tau_1}^P \left[ \mathbb{E}_{\tau_2}^{\tilde{P}_n}[\xi] \right]
= \lim_{n \to \infty} \mathbb{E}_{\tau_1}^{\tilde{P}_n} \left[ \mathbb{E}_{\tau_2}^{\tilde{P}_n}[\xi] \right] = \lim_{n \to \infty} \mathbb{E}_{\tau_1}^{\tilde{P}_n}[\xi] \leq \mathbb{E}_{\tau_1}^{G,P}[\xi]$, $\mathbb{P}$-a.s.

By the arbitrariness of $\tilde{P} \in \mathcal{P}(\tau_1, \mathbb{P})$, we prove the lemma.

It remains to prove (5.2). Indeed, for $\mathbb{P}^1, \mathbb{P}^2 \in \mathcal{P}(\tau_2, \mathbb{P})$, define

$$
\tilde{\mathbb{P}}^2(E) := \mathbb{P}^1(E \cap E^+) + \mathbb{P}^2(E \cap E^-), \text{ for any } E \in \mathcal{F},
$$

where $E^+ := \{ \mathbb{E}_{\tau_2}^{\mathbb{P}^1}[\xi] \geq \mathbb{E}_{\tau_2}^{\mathbb{P}^2}[\xi] \}$ and $E^- := \{ \mathbb{E}_{\tau_2}^{\mathbb{P}^1}[\xi] < \mathbb{E}_{\tau_2}^{\mathbb{P}^2}[\xi] \}$. Clearly $E^+, E^- \in \mathcal{F}_{\tau_2}$ and form a partition of $\Omega$. Then it follows from Assumption 4.1 (iii) that $\tilde{\mathbb{P}}^2 \in \mathcal{P}(\tau_2, \mathbb{P}')$. Moreover, for any $E \in \mathcal{F}_{\tau_2}$:

$$
\mathbb{E}_{\tau_2}^{\tilde{P}^2}[\xi 1_E] = \mathbb{E}_{\tau_2}^{\mathbb{P}^1}[\xi 1_E] = \mathbb{E}_{\tau_2}^{\mathbb{P}^1}[\xi 1_{E \cap E^+}] + \mathbb{E}_{\tau_2}^{\mathbb{P}^2}[\xi 1_{E \cap E^-}] \\
= \mathbb{E}_{\tau_2}^{\mathbb{P}^1} \left[ \mathbb{E}_{\tau_2}^{\mathbb{P}^1}[\xi] 1_{E \cap E^+} \right] + \mathbb{E}_{\tau_2}^{\mathbb{P}^2} \left[ \mathbb{E}_{\tau_2}^{\mathbb{P}^2}[\xi] 1_{E \cap E^-} \right] \\
= \mathbb{E}_{\tau_2}^{\mathbb{P}^1} \left[ \mathbb{E}_{\tau_2}^{\mathbb{P}^1}[\xi \vee \mathbb{E}_{\tau_2}^{\mathbb{P}^2}[\xi]] 1_E \right].
$$

Thus $\mathbb{E}_{\tau_2}^{\tilde{P}^2}[\xi] = \mathbb{E}_{\tau_2}^{\mathbb{P}^1}[\xi \vee \mathbb{E}_{\tau_2}^{\mathbb{P}^2}[\xi]]$. Repeat the arguments we prove (5.2) and hence the lemma.

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