Phase Separation and Emergent Structures in an Active Nematic

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We consider a phenomenological continuum model for an active nematic fluid and show a universal, model independent, instability which renders the homogeneous nematic state unstable to order fluctuations. Using numerical and analytic tools we show that, for low energy excitations in the vicinity of a critical point, this instability leads to a phase separation in which the ordered regions form bands with the direction of nematic order being perpendicular to the direction of density gradient. We argue that this mechanism of phase separation, which we term self-regulation, is a universal feature of active fluids.

I. INTRODUCTION

Active fluids encompass diverse systems ranging from bacterial colonies [1–3] to herds of animals [3] and bird flocks [5]. These systems are unified by the fact that they are composed of “microscopic” entities that consume energy and dissipate it to do work on their environment [6–8]. Depending on the symmetries of the microscopic particles and the interactions among them, these systems can be classified as isotropic (ex: self-propelled spherical colloidal particles [9]), polar (ex: self-chemotactic bacteria [10]) or nematic active fluids (ex: microtubule-motor-protein suspensions [11, 12], vibrated granular rods [13]). In this work, we consider a minimal description of an active nematic fluid with the goal of identifying universal mechanisms for the formation of emergent structures on long length scales.

Active nematics in general fall into two broad categories. The first is the self-propelled nematic, composed of self-propelled particles whose interactions have a nematic symmetry. This system has mixed symmetry in that the microscopic entity is polar (due to self-propulsion) but the interactions and therefore the macrodynamics is nematic and has been extensively studied in the literature [14–20]. The second category is a pure active nematic composed of shakers, i.e., nematogens that do not undergo any persistent motion along any one direction of their body axis. Physical realizations of pure active nematics include the microtubule suspensions mentioned above [11], symmetric vibrated rods [13], rapidly reversing strains of myxobacteria [21, 22] and melanocytes which are also thought to effectively behave as “shakers” [23, 24]. This latter class of active nematics are the focus of the study presented in this paper.

Active nematics were first considered in the seminal work of Ramaswamy and collaborators [25–29], who demonstrated that this system exhibits giant number fluctuations and these fluctuations render the system intrinsically phase separated. Subsequently, extensive studies have been carried out within the context of a “nematic Viscek” model [16, 29–37] that have delineated in detail the large scale dynamics of active nematics.

Our work builds on these findings by considering a minimal complete theory for an active nematic numerically and analytically. In particular, the equations we consider are phenomenological. Therefore, the parameters of the theory are independent of any particular microscopic model and are varied independently. We show that the curvature driven mass flux identified in [23] causes the homogeneous nematic state to be unstable and leads the system to phase separate into high density and low density bands. We focus on the regime where this phenomenon is universal (independent of particular models or parameter choices), namely low energy excitations near the critical point associated with the isotropic-nematic transition. The mechanism that leads to the formation of this band structure is identified and shown to be of the same origin as those that lead to phase separation in isotropic and polar active fluids identified earlier [38, 42].

The layout of the paper is as follows. First, we introduce the continuum hydrodynamic theory of a generic active nematic and discuss the features that render this system inherently out of equilibrium. Then, we map out the domain of linear stability of the homogeneous nematic state and identify the mechanism that destabilizes it. Also, we report a numerical study of these hydrodynamic equations that tracks the evolution of the fully nonlinear dynamics and characterize the phase separated end state and the emergent structure of the resulting band. Finally, the primary results in this work are summarized and discussed in the context of active fluids of various symmetries.
II. THE CONTINUUM THEORY

Let us consider a collection of particles, in two spatial dimensions, that are active and interact via purely nematic aligning interactions. In the presence of a medium that acts as a momentum sink, their microdynamics can be reliably captured by overdamped equations of motion such as in [31, 33]. In the limit of length scales long compared to the particle size and time scales long compared to the interaction times, it is fruitful to consider a mean field macroscopic description of the system, given by the dynamics of conserved quantities and broken symmetry variables. In the case of an overdamped active nematic of shakers, the relevant variables are the density of active units \( \rho = \langle \sum_\alpha \delta (\mathbf{r} - \mathbf{r}_\alpha) \rangle \) and the nematic order parameter \( Q_{ij} = \rho S_{ij} = \langle \sum_\alpha (\hat{u}_{\alpha i} \hat{u}_{\alpha j} - \frac{1}{2} \delta_{ij}) \delta (\mathbf{r} - \mathbf{r}_\alpha) \rangle \), where \( \{\mathbf{r}_\alpha, \hat{u}_\alpha\} \) are the positions and orientations of the shaker particles and \( \langle \rangle \) denotes coarse-graining by averaging over microscopic lengths and times. Given the microdynamics, this coarse-graining procedure can be carried out through systematic approximations as considered in [15, 29, 33]. Alternately, one can construct the macroscopic equations based on purely symmetry considerations as in [25, 29]. In this work, we take the latter route, which gives us the advantage of liberating the parameters of the hydrodynamic theory from the constraints of a particular underlying microscopic model. The dynamical equations are generically of the form

\[
\partial_t \rho = D \nabla^2 \rho + D_Q \nabla \cdot \mathbf{Q} \tag{1a}
\]

\[
\partial_t Q_{ij} = D_c \left[ \alpha (\rho) - \beta \mathbf{Q} \cdot \mathbf{Q} \right] Q_{ij} + D_h \nabla^2 Q_{ij} + D_\rho \partial_x \left( \partial_x Q_{ij} + \partial_y Q_{ik} - \delta_{ij} \partial_y Q_{kl} \right) + D_\rho \partial_y \partial_x - \frac{1}{2} \delta_{ij} \nabla^2 \rho \tag{1b}
\]

(\text{where } A \cdot B \text{ refers to the contraction } A_{ij} B_{ij} \text{ and } \nabla \nabla \text{ is the tensor } \partial / \partial x_j \partial / \partial x_j).\]

The primary physics of the above equations is as follows. In the case of an equilibrium nematic, the density, a conserved quantity, has a dynamics generically of the form \( \partial_t \rho = -\frac{1}{\gamma} \nabla^2 \delta F / \delta \rho \), where \( F \) is the free energy functional whose extremum is the equilibrium state and \( \gamma \) is a relaxation time. By retaining every symmetry allowed term in the free energy, this yields a dynamics for the density given by \( \partial_t \rho = -\nabla \cdot \mathbf{J} \), with \( J_i = -D_{\rho i} (\rho, Tr (Q^2)) \nabla_i \rho - D_{\rho i}^1 (\rho, Tr (Q^2)) \nabla_i Tr (Q^2) \). Here, the diffusion tensors \( D_{\rho i} \) have the symmetries of the underlying nematic, i.e., \( D_{ij} = A \delta_{ij} + B (\hat{n}_i \hat{n}_j - \frac{1}{2} \delta_{ij}) \), where \( \hat{n} \) is the nematic director and \( A \) and \( B \) are potentially arbitrary functions of the two scalars in the theory, namely the density and the magnitude of orientational order \( \rho S \equiv \sqrt{2 Tr Q^2} \). But, in the case of an active nematic, we have no extremization principle and hence the flux \( J \) is liberated from the above constraints and hence is generically of the form

\[
J_i = - D_{\rho i}^0 (\rho, Tr (Q^2)) \nabla_i \rho - D_{\rho i}^1 (\rho, Tr (Q^2)) \nabla_i Tr (Q^2) - D_{\rho i}^2 (\rho, Tr (Q^2)) \nabla_i Q_{kj}.
\]

The new term here is a mass flux that arises because of the activity of the individual units and the resulting anisotropic forces produced by it. This mass flux term, produced by inhomogeneities in the orientational order, is the central feature that makes an active nematic an inherently non-equilibrium system. In the present study we take the simplest form for the mass flux that captures the role of curvature, namely \( J_i = -D \nabla_i \rho - D_Q \nabla_i Q_{kj} \), where \( D \) and \( D_Q \) are constants independent of the dynamical fields. This results in Eq. (1a) for the dynamics of the density field.

The nematic order parameter \( Q_{ij} \) has a dynamics analogous to an equilibrium nematic. The first terms on the right hand side of Eq. (1b) give rise to a second order phase transition from a disordered isotropic state to an ordered nematic state when \( \alpha (\rho) \) changes sign. \( D_h \) and \( D_a \) are related to the Frank elastic constants associated with bend and splay deformations. The term proportional to \( D_\rho \) is a kinetic term arising due to the inherent anisotropy in diffusive processes in a nematic. The dynamics of these coupled fields density and nematic order parameter is the focus of the investigation reported here.

In the following, we non-dimensionalize our equations by picking the units of time to be given by \( 1/D_r \), the rotational diffusion time and our length scale to be \( \sqrt{D / D_r} \), a diffusion length. Also, we choose \( \alpha (\rho) = (1 - \rho) \) and \( \beta (\rho) = \frac{1}{\rho \gamma} \), thereby setting the density at which the isotropic nematic transition occurs to 1 and ensuring that when \( \rho >> 1 \) the nematic order saturates to a finite value. Further, to minimize the number of independent parameters in our study, we make a one elastic constant approximation, namely \( D_a = D_h = D_E \). The dimensionless dynamical equations then become

\[
\partial_t \rho = \nabla^2 \rho + D_Q \nabla \cdot \mathbf{Q} \tag{2a}
\]

\[
\partial_t Q_{ij} = \left[ \alpha - \beta \mathbf{Q} \cdot \mathbf{Q} \right] Q_{ij} + D_E \nabla^2 Q_{ij} + D_Q \left( \partial_x Q_{ij} + \partial_y Q_{ik} - \delta_{ij} \partial_y Q_{kl} \right) \tag{2b}
\]

These simplified equations are analyzed analytically
and numerically in the sections below.

III. LINEAR STABILITY ANALYSIS

The dynamical equations Eq. (2) admit two homogeneous steady states, an isotropic state when \( \rho < 1 \) and a uniaxial nematic state when \( \rho \geq 1 \). Let us focus on the high density ordered state and without loss of generality let us consider a nematic state where the direction of broken symmetry is along the \( x \) axis of our coordinate system. Then, small fluctuations about this homogeneous steady state can be parametrized as \( \rho = \rho_0 + \delta \rho (r, t), \ Q_{xx} = \frac{1}{2} \rho_0 \delta Q_{xx} (r, t) \) and \( Q_{xy} = 0 + \delta Q_{xy} (r, t) \) where \( \delta Q_{xx} = \sqrt{2(\rho_0 - 1) \rho_0 + 1} \). Introducing a spatial Fourier transform \( \tilde{X} (k, t) = \int d \mathbf{r} e^{i k \cdot \mathbf{r}} X (r, t) \), the linearized dynamics of fluctuations in density and order parameter takes the form

\[
\partial_t \begin{bmatrix} \delta \rho \\ \delta Q_{xx} \\ \delta Q_{xy} \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} D_\rho k^2 \cos(2\phi) - C_0 \\ \frac{1}{2} D_\rho k^2 \sin(2\phi) \\ 2 D_E k^2 \end{bmatrix} \begin{bmatrix} \delta \rho \\ D_Q k^2 \cos(2\phi) - 2 D_E k^2 + 2 \alpha_0 \\ 0 \\ 2 D_E k^2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \alpha_0 \\ 0 \end{bmatrix} 
\]

where \( \phi \) is the angle between the director and the spatial gradient vector \( \mathbf{k} \), \( \alpha_0 = (\rho_0 - 1) \) and \( C_0 = \sqrt{2(\rho_0 - 1) \rho_0 + 1} \).

While we can readily analyze the cubic characteristic equation of this linear system (see Appendix A), the physics at play is best exposed by considering spatial fluctuations in a direction orthogonal to the mean nematic director, i.e., consider \( \phi = 90^\circ \). In this case the director fluctuations \( \delta Q_{xy} \) decouple and are always diffusive and stable. The relevant dynamics of the system is now in the \((\rho, Q_{xx})\) subspace. The eigenvalues associated with density and magnitude of order fluctuations take the form

\[
\lambda_{\pm} = -\frac{1}{2} \left( 2 \alpha_0 + (2 D_E + D) k^2 \pm \sqrt{(2 \alpha_0 + (2 D_E + D) k^2)^2 - 4 k^2 (2 D \alpha_0 - D_Q C_0 + k^2 (-\frac{1}{2} D_\rho D_Q + 2 D D_E))} \right)
\]

Clearly \( \lambda_+ \) is always stable and the associated fluctuations decay to homogeneity. \( \lambda_- \) on the other hand, in the long wavelength limit, takes the form \( \lambda_- \sim -\frac{1}{2} k^2 \left( - D_Q \frac{C_0}{\alpha_0} + 2 \right) + O(k^4) \) and hence becomes unstable whenever \( D_Q > \frac{2 \alpha_0}{C_0} = \left( 2 (\rho_0^2 - 1) \right) \left( \frac{\rho_0 + 1}{\rho_0^2 + \rho_0 - 1} \right) \). Fig. 1 shows a plot of this threshold value of activity as a func-
tion of the mean density of the system. Note that the threshold goes to zero as the order disorder transition is approached and hence in the vicinity of the critical point, arbitrarily small values of the activity destabilize the homogeneous nematic. The vanishing of the threshold for the onset of this instability is independent of the detailed form of \( \alpha(\rho) \) and is a universal feature of the dynamics of active nematics.

The physics of this instability of the homogeneous nematic state can be understood as follows. The order parameter Eq. (2b) has a second order mean field transition from an isotropic to a nematic state that is controlled by the density of the system. This density, in the context of conventional nematics is indeed an external control parameter. But, in the case of active nematics, the dynamics of the density is in turn affected by the magnitude of order in the system through the non-equilibrium mass flux term proportional to \( \nabla \phi \). This non-equilibrium term arises due to the presence of the microscopic active forces that drive the system. The coupling between density and order renders the system self-regulating in that the order in the system is determined locally by the dynamics rather than being controlled globally by the mean density and thereby destabilizes the homogeneous state.

We can of course analyze the eigenmodes for arbitrary directions of spatial fluctuations and we do so perturbatively in the wavevector in Appendix A. We find that a range of wavevectors in a sector \( [\phi_{\text{min}}, 90^\circ] \) with respect to the nematic director go unstable, depending on the specific values of the different parameters. Fluctuations perpendicular to the director (i.e., \( \phi = 90^\circ \)) are, however, dominant for most parameters and are the only relevant dynamics close to the critical density. In the remainder of this presentation we explore Eq. (2) numerically in the unstable regime and characterize the resulting inhomogeneous end states.

### IV. EMERGENT STRUCTURES - PHASE-SEPARATED BANDS

In order to understand the consequences of the self-regulation instability on the formation of emergent structures in an active nematic fluid, we solved Eq. (2) numerically. The integration was performed using an Euler method, forward time centered space (FTCS) scheme on a grid with periodic boundaries. Typical values used for the time step and spatial resolution were 0.01 diffusion times and 0.4 diffusion lengths. The system size ranged from 200 to 1000 diffusion lengths on a side. To further reduce the number of parameters and to best illustrate the principal mechanisms at play, we first report the results for the case \( D_Q = D_\rho \), and \( D_E = 1 \). The consequence of varying \( D_\rho \) and \( D_E \) independently is discussed in Appendix B.

![FIG. 2. (color online)(a) A plot of the density (scale bar in the inset) of a typical system (\( \rho_0 = 1.01, D_Q = 1.3, D_E = 1.50 \)) that has phase-separated into bands. The vectors show the magnitude (by the length) and direction of nematic ordering. The green regions are a band of high density (repeating over the boundary) with nematic ordering along the band. The axes show the position in the system in dimensionless ‘diffusion lengths’. (b) A profile of the density taken perpendicular to the direction of ordering in the bands.](image)

The primary finding of our numerical analysis is as follows: The typical inhomogeneous state we find when the activity \( D_Q \) is greater than the threshold for onset of the self-regulation instability is shown in Fig. 2. The system develops inhomogeneous bands of alternating low and high density. The high density regions are nematic with the director lying orthogonal to the direction of the density gradient. These bands are not a pattern in that there is no characteristic length scale associated with them; and the size and number of bands is determined by initial conditions until one band grows to the size of the system. One way to understand these structures is to consider the reduced set of equations that describe the dynamics of the density and the magnitude of ordering namely

\[
\partial_t \rho = (\partial_x^2 + \partial_y^2) \rho + (\partial_x^2 - \partial_y^2) \rho S
\]

\[
\partial_t \rho S = - (\alpha(\rho) + \beta(\rho) \rho^2 S^2) \rho S + (\partial_x^2 - \partial_y^2) \rho + (\partial_x^2 + \partial_y^2) \rho S
\]

where we have assumed that the mean nematic order lies along the \( x \) axis of our coordinate system and we have set all the coefficients to 1 to display the structure of the equation most clearly. These reduced equations admit a stationary solution homogeneous in \( x \), and having a profile of high density, high order region embedded between low density isotropic regions along the \( y \) direction as has been shown in [29]. Here we present a complementary analysis.
First note that as shown in Fig. 3(a), the density of the nematic band is independent of the value of the homogeneous density of the system but instead is determined entirely by the strength of the activity $D_Q$. Secondly note that as shown in Fig. 3(b), we measure the magnitude of nematic order in the bands and find that $(\rho S)_{\text{band}} = \rho_h \sqrt{\frac{2(\rho_h - 1)}{\rho_h + 1}}$, i.e., $S$ is related to the density by the same mean field relation in the homogeneous theory, but now the mean density is replaced by $\rho_h$, the density in the band. Note that $\rho_h$ is such that the nematic state is no longer unstable to the self-regulation instability at this value of the activity. This suggests that the formation of this banded structure can be viewed as the system phase separating into a high density nematic state and a low density isotropic state both of which are stable, reminiscent of gas-liquid coexistence.

This simple picture of phase separation is modulated by fluctuations in the system that lead to the bands themselves becoming unstable and leading to complex dynamical structures as has been elucidated in [34]. This physics is beyond the scope of the minimal analysis presented here, which focuses on densities close to the critical density where, for our model, in mean field, the bands are indeed stable.

V. UNIVERSITY OF SELF-REGULATION

We have shown that the homogeneous ordered state of an active nematic is unconditionally unstable near the critical density $\rho_c$ for the onset of order. This instability arises because of the dynamic coupling between the density and order through the curvature driven mass flux. Near the critical density, the system phase separates into a high density nematic and a low density isotropic fluid with the direction of order being perpendicular to the direction of the density gradient.

This kind of phase separation phenomena are ubiquitous in model systems of active fluids. In active polar fluids, composed of self-propelled particles with polar aligning interactions, theory and simulations show that the system phase separates into high density and low density stripes with the high density regions forming propagating solitary waves [32, 42, 44–46]. These propagating waves have been observed in model experimental systems [10, 47]. In the case of active isotropic fluids composed of self propelled spheres with no aligning interactions, athermal phase separation into a dense phase and a dilute phase has been extensively reported and discussed [38, 40, 41, 43, 49]. This phenomenon has been observed in experiments of diffusophoretic Janus colloids as well [50]. The underlying physical mechanism that leads to this phase separation is the presence of a self-replenishing velocity along one direction of the body axis of a particle and the consequent persistent collisions that result among such active particles [40].

From a completely macroscopic point of view, however, the phenomenon of phase separation in active fluids of different symmetries can be unified as follows. Within a Ginzburg Landau approach, an equilibrium system near a critical point is described by a free energy that is a functional of an order parameter $\alpha$. When $\alpha \leq \alpha_c$, the free energy is a minimum when the order parameter is zero, while when $\alpha \geq \alpha_c$ the minimum and hence the equilibrium state is one where the order parameter is finite. The control pa-

![FIG. 3. (color online) (a) For fixed activity, several initial densities in the unstable region are chosen. The final densities in the banded state ($\rho_h$ and $\rho_l$) are found to be independent of the initial density, as the points for different initial densities fall on the same curve. (b) The measured value of order ($\rho S$) as a function of $\rho_h$. The solid line is the mean field prediction $\rho_h \sqrt{\frac{2(\rho_h - 1)}{\rho_h + 1}}$. Excellent agreement is found substantiating the picture that the bands are just phase coexistence between a high density nematic and a low density isotropic state.](image-url)
rameter in these theories is a density or a temperature that is tuned externally. What happens in the case of active fluids is that the control parameter is itself a dynamical variable that is coupled to the order parameter of the system. Let us explicate this statement by considering specific examples.

In the active nematic system studied in this work, the order parameter is the nematic ordering tensor $Q_{ij}$. As mentioned above, dynamical equation associated with this order parameter is indeed the same as that for an equilibrium nematic and can be schematically written as $\partial_t Q_{ij} = -\frac{1}{\gamma} \frac{\delta F[\rho, Q]}{\delta Q_{ij}}$. The key active feature of the dynamics of this fluid is that the density, an externally tunable control parameter in the equilibrium case is now itself dynamical and coupled to $Q_{ij}$ through the curvature induced mass flux. In the context of a polar active fluid, the order parameter is a vector $P$ that measures orientational ordering in the velocity field of the self-propelled entities \cite{42,43}. The dynamics of this system can generically be written as $\partial_t P = -\frac{1}{\gamma} \frac{\delta F[\rho, P]}{\delta P} + N[\rho, P]$, i.e., a part composed of exactly what we would have in the equilibrium case and a new truly nonequilibrium piece $N$. As has been shown in \cite{42,46}, the nonequilibrium piece is irrelevant to the observed phase separation behavior, which arises again due to the fact that the control parameter, the density is now dynamically coupled to the ordering. Finally in the case of the isotropic fluid, the order parameter is a density while the control parameter is a chemical potential (as in a gas-liquid system). As has been shown in the works of \cite{38,48,49}, the self-propulsion of the active particles serves as a chemical potential in the case of an active isotropic fluid. But, the local density determines the local self-propulsion speed and therefore the effective control parameter, and hence the analogy of the control parameter being rendered dynamically coupled to the order parameter extends to this class of active fluids as well.

The consequence of the above feature is that even though the dynamics of the order parameter in the system is the relaxational dynamics familiar from the near equilibrium context, due to the dynamics of the control parameter, the state of the system is determined locally by the dynamics rather than globally by the macroscopic parameters of the system. Hence active fluids are intrinsically phase separated. This universal feature of the dynamics of active fluids we call dynamical self-regulation \cite{20,42}.

VI. SUMMARY

In this work we consider a minimal model for an active nematic fluid and show using analytical and numerical analysis that the fluid undergoes phase separation beyond a given threshold for activity and that the threshold vanishes as the critical density for the order disorder transition is approached. In the coexistence region, we show that the system forms bands of nematic ordered regions where the ordering is orthogonal to the direction of the density gradient, i.e., the nematic is oriented parallel to the interface. We identify the macroscopic mechanism for this phase separation and relate it to those found in the context of polar and isotropic active fluids.

Appendix A: Generalized Linear Stability Analysis

In Section III we considered the stability of the homogeneous, ordered state with respect to spatial fluctuations that were orthogonal to the direction of nematic ordering (i.e., the angle between the spatial gradient vector ($k$) and the nematic director ($\hat{n}$) fixed at $\phi = 90^\circ$). In the following we consider arbitrary spatial fluctuations and show that the lowest threshold value for $D_Q$ and the direction of the fastest growing wavevector lie at $\phi = 90^\circ$, when the density is close to the critical value. Far away from the critical density, there is a dependence on the particular parameters which are characterized below as well.

The dynamics of arbitrary spatial fluctuations about the homogeneous ordered state is characterized by the linear system of equations Eq. (1). As we are interested in the dynamics on the longest length scales, let us consider roots of the cubic characteristic equation for the linear system in the long wavelength limit. The perturbative solution to order $k^2$ is found to be

$$\lambda_{\pm} = -\frac{k^2}{4\alpha_0} \left( C_0 \cos(2\phi)D_Q + 2\alpha_0(1+2D_E) \right) \pm \sqrt{\left( C_0 \cos(2\phi)D_Q + 2\alpha_0(1-2D_E) \right)^2 + 8\alpha_0^2 \sin^2(2\phi)D_QD_\rho} + O(k^4)$$
and

$$\lambda_3 = -2\alpha_0 + k^2 \left( -2D_E + \frac{C_0}{2\alpha_0} D_Q \cos(2\phi) \right) + O(k^4)$$

We can readily establish that $\lambda_-$ is the only eigenvalue that changes sign in the small wavevector limit and thereby leads to an instability. The threshold for the onset of this instability is identified as

$$D_Q > \frac{4\alpha_0 D_E}{\alpha_0 \sin^2(2\phi) D_\rho - 2C_0 \cos(2\phi) D_E}. \tag{A1}$$

Thus, for a fixed activity $D_Q$ a range of wavevectors become unstable.

The content of the above threshold condition can be explicated as follows. First let us fix $\phi$, the direction of the spatial fluctuation and characterize $D_Q(\phi)$, the value of activity at which spatial fluctuations in this direction go unstable. This is plotted in Fig. 4. Whenever $g_0 \equiv \frac{D_\rho}{2D_E} \frac{\alpha_0}{C_0} < 1$, which is true for densities close to the critical density for the onset of order, $D_Q(\phi)$ is a monotonic function with a minimum at $\phi = 90^\circ$, i.e., spatial fluctuations orthogonal to the direction of ordering go unstable first, i.e., at the lowest value of activity. On the other hand, when $g_0 \geq 1$, the spatial fluctuation that goes unstable first as we ramp up activity from zero is now $\phi \sim \frac{1}{2} \arccos(1/g_0)$. This happens far from the critical density at which point we should expect that universality is no longer applicable and the dynamics becomes dependent on the details of the parameters of the system.

Another fruitful representation of the instability condition Eq. (A1) is to consider fixed values of parameters $D_Q$, $D_\rho$ and $D_E$, and identify the range of wavevector directions $\phi$ that are unstable to fluctuations at any given value of mean density $\rho$. This is illustrated in Fig. 5. The range of spatial directions associated with fluctuations that destabilize the system is determined by the ratio $g_0$ defined earlier. For $g_0 > 1$, the wavevectors close to $\phi = 90^\circ$ are unstable. When we move to smaller values of $g_0$ wavevectors in a much wider sector of spatial directions become unstable. As the activity $D_Q$ becomes large, the instability extends to higher densities far from the critical density as well.

Even though a large number of modes go unstable depending on the details of the parameters, the physics of the system in the unstable region will be controlled by the fastest growing wavevector. We find that the fastest growing mode is along the $\phi = 90^\circ$ whenever $D_E + \frac{C_0}{2\alpha_0} D_Q \geq \alpha_0 \left( \frac{\alpha_0}{C_0} D_\rho + \frac{1}{2} \right)$. This relationship is always satisfied for densities close to the critical density. Far from the critical density, there exists values of parameters for which the fastest growing mode shifts to

$$\cos(2\phi) = -\frac{\xi}{\gamma} \left[ 1 + \sqrt{1 + \frac{\gamma}{\xi^2}} \right]$$
where $\gamma = 8 \frac{D_e}{D_Q} \left( \frac{C_0}{Q} \right)^2 - 1$ and $\xi = 4 \frac{D_e - 2Q}{C_e D_Q}$. Since the body of the work focuses on the universal regime close to the critical density and focuses on low energy excitations for which our quadratic in gradients theory is convergent and well behaved, this shift in the fastest growing wavevector and its consequence to the emergent structure is not probed.

**Appendix B: Generalized Numerical Analysis:**

**Role of specific parameters**

**The role of Frank Elasticity**: Most of the initial conditions chosen in the region where the homogeneous nematic state is unstable converge to the stationary phase separated banded state described in the body of the paper. The density contrast is entirely determined by $D_Q$ and is insensitive to the strength of the Frank elasticity (Fig. 6(a)). But, when the Frank elastic constant is small enough, we find that the coarsening of the system is inhibited and the resulting state has frustrated bands that are unable to coarsen to system size (Fig. 6(b)). This effect vanishes as the penalty for director fluctuations increases.

![FIG. 6.](image)

(a) The plot shows the density in the high density stripes ($\rho_h$) for different values of the kinetic term $D_{\rho}$ and activity ($D_Q$), illustrating the insensitivity of $\rho_h$ to $D_{\rho}$. (b) The top figure shows two bands which are the length of the longer dimension of the system, and formed for parameters $D_{\rho} = 0.1$, $D_Q = 1.2$, $D_E = 1.5$. The figure on bottom shows the system for the same parameters, except with $D_{\rho} = 1.0$. In this figure one band is repeating across the system about two times.

Varying the kinetic term, $D_{\rho}$, while fixing the curvature-induced mass flux, $D_Q$, does not alter the densities of the phase-separated regions which form (see Fig 7(a)). Though the properties of the phases are insensitive to this term, the value of $D_{\rho}$ affects the kinetics of the phase separation process. This is best illustrated as follows. Consider a system initialized in the homogeneous state with the ordering along the x-direction. When $D_{\rho}$ is low, the system forms bands along the x-direction and stays there. At higher $D_{\rho}$, the bands turn and repeat multiple times across the system so as to have a wavelength which is longer than either dimension of the system, as seen in Fig 7(b). This is the steady-state which the system finds in the majority of the simulations. So, we conclude that $D_{\rho}$ plays a primarily kinetic role in aiding the dynamics to find the phase-separated end state with the longest wavevector along the direction of ordering.

![FIG. 7.](image)

(a) The plot shows the density in the high density stripes ($\rho_h$) for different values of the kinetic term $D_{\rho}$, and activity ($D_Q$), illustrating the insensitivity of $\rho_h$ to $D_{\rho}$. (b) The top figure shows two bands which are the length of the longer dimension of the system, and formed for parameters $D_{\rho} = 0.1$, $D_Q = 1.2$, $D_E = 1.5$. The figure on bottom shows the system for the same parameters, except with $D_{\rho} = 1.0$. In this figure one band is repeating across the system about two times.

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