NEW 2-DESIGNS OVER FINITE FIELDS
FROM DERIVED AND RESIDUAL DESIGNS

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Abstract. Based on the existence of designs for the derived and residual parameters of admissible parameter sets of designs over finite fields we obtain a new infinite series of designs over finite fields for arbitrary prime powers \(q\) with parameters \(2-(8, 4, (q^6-1)(q^3-1); q)\) as well as designs with parameters \(2-(10, 4, 85λ; 2), 2-(10, 5, 765λ; 2), 2-(11, 5, 6205λ; 2), 2-(11, 5, 502605λ; 2), \) and \(2-(12, 6, 423181λ; 2)\) for \(λ = 7, 12, 19, 21, 22, 24, 31, 36, 42, 43, 48, 49, 55, 60, 63\).

1. INTRODUCTION

A design over a finite field with parameters \(t-(n, k, λ; q)\) is a pair \((V, B)\) consisting of an \(n\)-dimensional vector space \(V\) over the finite field \(F_q\) with \(q\) elements and a set \(B\) of \(k\)-dimensional subspaces of \(V\) such that each \(t\)-dimensional subspace of \(V\) is contained in \(λ\) elements of \(B\).

Designs over finite fields are also called \(q\)-analogs of combinatorial designs or subspace designs [8, 9].

Since the first non-trivial \(t-(n, k, λ; q)\) designs for \(t > 1\) were introduced in 1987 [18] the interest in these objects has increased. Several results on parameter sets of new constructed designs over finite fields have been published [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 17], whereas until now only two infinite series for arbitrary field size \(q\) are known:

- In [17] a series of \(2-(n, 3, \frac{q^3-1}{q-1}; q)\) designs constructed for all integers \(n \geq 7\) with \(n \equiv \pm 1 \mod 6\) and all prime powers \(q\) admitting the normalizer of a Singer cycle group as a group of automorphisms.

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In [10] a series of $2-(\ell m, 3, q^{m})$ designs is given for all $m \geq 3$ and $\ell \geq 7$ with $\ell \equiv 5 \mod (6q-1)$ admitting the special linear group $\text{SL}(m,q^\ell)$ as a group of automorphisms.

In this paper we present a new infinite series of designs over finite fields for arbitrary field size. Furthermore, we use parameter sets for which designs over the binary field can be constructed using a computer aided approach to deduce new parameters by considering reduced designs over finite fields.

2. Preliminaries

In the following by $\binom{\cdot}{\cdot}_q$ we denote the set of $k$-dimensional subspaces of $V$—its cardinality is given by the $q$-binomial coefficient

$$\binom{n}{k}_q := \frac{(q^n - 1)(q^{n-1} - 1) \cdot \ldots \cdot (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdot \ldots \cdot (q - 1)}.$$

Given a $t$-$(n, k, \lambda; q)$ design $(V, B)$ the existence of designs with new parameters can be derived from a result by Suzuki [16].

Lemma 2.1. Let $(V, B)$ be a $t$-$(n, k, \lambda; q)$ design, $i$ and $j$ non-zero integers with $i + j \leq t$, $P \in \binom{\cdot}{\cdot}_1$, and $H \in \binom{\cdot}{\cdot}_1$. Then the following equation holds:

$$|\{B \in B \mid P \subseteq B \subseteq H\}| = \lambda \binom{n-i}{k-i}_q \binom{n-t}{k-t}_q.$$

Plugging different values for $i$ and $j$ into this lemma immediately yields the existence of designs for new parameters:

- By setting $i = t - 1$ and $j = 0$ the pair $(V, B)$ defines a design for reduced parameters
  $$\text{red}[t-(n, k, \lambda; q)] := (t - 1)-(n, k, \lambda \frac{q^{t-1}-1}{q-1}; q).$$

- If $K^\perp := \{x \in V \mid \langle x, y \rangle = 0 \forall y \in K\}$ denotes the dual space for some non-singular bilinear form $\langle \cdot, \cdot \rangle$ by setting $i = 0$ and $j = t$ the pair $(V, \{K^\perp \mid K \in B\})$ defines a design for the dual parameters
  $$\text{dual}[t-(n, k, \lambda; q)] := t-(n, n - k, \lambda \binom{n}{k}_q \binom{1}{1}_q; q).$$

- By taking an $(n-1)$-dimensional subspace $H \in \binom{\cdot}{\cdot}_1$ and defining $i = t - 1$ and $j = 1$ the pair $(H, \{K \in B \mid K \subseteq H\})$ is a design for the residual parameters
  $$\text{res}[t-(n, k, \lambda; q)] := (t - 1)-(n - 1, k, \lambda \frac{q^{t-1}-1}{q-1}; q).$$

Furthermore, by taking a 1-dimensional subspace $P \in \binom{\cdot}{\cdot}_1$ and considering factor spaces the pair $(V/P, \{K/S \mid K \in B, P \subseteq K\})$ yields a design for the derived parameters

$$\text{det}[t-(n, k, \lambda; q)] := (t - 1)-(n - 1, k - 1, \lambda; q).$$

The following theorem [11, Theorem 1] serves as the major construction tool for the aforementioned results of this work.

Theorem 2.2. Let $t$-$(n, k, \lambda; q)$ be parameters of designs. The existence of designs for the derived and residual parameters implies the existence of a design for the reduced parameters.
3. Extension of Suzuki’s design

In 1992 Suzuki [17] found an infinite series of 2-(n, 3, \( \frac{q^3-1}{q-1} \); q) designs constructed for all integers \( n \geq 7 \) with \( n \equiv \pm 1 \mod 6 \). By applying Theorem 2.2 and due to the existence of dual designs Suzuki’s design can be extended to a new infinite family (also see [11, Corollary 2]).

**Theorem 3.1.** For all prime powers \( q \) there exist 2-(8, 4, \( \frac{(q^3-1)(q^2-1)}{(q-1)(q^2-1)} \); q) designs.

**Proof.** In the following let \( n = 7 \). Then we get:

\[
\begin{align*}
\text{der}[3-(8, 4, \frac{q^3-1}{q-1}; q)] &= 2-(7, 3, \frac{q^3-1}{q-1}; q) \\
\text{res}[3-(8, 4, \frac{q^3-1}{q-1}; q)] &= \text{dual}[2-(7, 3, \frac{q^3-1}{q-1}; q)] = 2-(7, 4, \frac{[4]}{12}_q; q) \\
\text{red}[3-(8, 4, \frac{q^3-1}{q-1}; q)] &= 2-(8, 4, \frac{(q^3-1)(q^2-1)}{(q-1)(q^2-1)}; q)
\end{align*}
\]

For \( n = 7 \) Suzuki’s 2-(7, 3, \( \frac{q^3-1}{q-1} \); q) design and its dual 2-(7, 4, \( \frac{[4]}{12}_q \); q) design exist. Their parameters are the derived and residual parameters of 3-(8, 4, \( \frac{q^3-1}{q-1} \); 2). Theorem 2.2 implies the existence of designs over \( F_q \) for the reduced parameters 2-(8, 4, \( \frac{(q^3-1)(q^2-1)}{(q-1)(q^2-1)} \); q).

\( \square \)

4. Kramer–Mesner approach

In this section we recall the approach for a computer aided construction of designs proposed by Kramer and Mesner [13]. We use this approach to construct certain designs over the binary field which serve as initial values for the construction theorem in the next section. The designs in Table 1 can also be found [8]. For sake of completeness we recall the construction of these designs using the Kramer–Mesner approach.

Subgroups \( G \) of the general linear group \( GL(n, q) \) act on subspaces of \( V \) from the left considering subspaces as column spaces. The corresponding orbit of \( G \) on the subspace \( K \) is given by \( G(K) := \{ \alpha(K) \mid \alpha \in G \} \).

A \( t-(n, k, \lambda; \rho) \) design \((V, B)\) admits a subgroup \( G \) of \( GL(V) \) as a group of automorphisms if and only if it consists of orbits of \( G \) on the set of \( k \)-dimensional subspaces of \( V \).

In order to obtain a selection of orbits of \( G \) on \( \binom{V}{k} \) we consider the incidence matrix \( A^G_{t,k} \) whose rows are indexed by the \( G \)-orbits on the set of \( t \)-dimensional subspaces of \( V \) and whose columns are indexed by the orbits on the set of \( k \)-dimensional subspaces. The entry of \( A^G_{t,k} \) corresponding to the orbits \( G(T) \) and \( G(K) \), respectively, is defined by the number \( a^G_{T,K} := \{ K' \in G(K) \mid T \subseteq K' \} \).

Any \( t-(n, k, \lambda; \rho) \) design admitting a subgroup \( G \) of \( GL(V) \) as a group of automorphisms bijectively corresponds to a binary vector \( x \) satisfying \( A^G_{t,k}x = [\lambda, \ldots, \lambda]^t \).

The binary vector \( x \) stands for the selection of \( G \)-orbits on \( \binom{V}{k} \) whose union forms the corresponding design.

Using this construction approach we list some 2-(9, k, \( \lambda; 2 \)) designs for \( k \in \{3, 4\} \) in Table 1 which we utilize to get new designs by Theorem 2.2 in the next section. Solving the corresponding Diophantine system of equations with an LLL based algorithm [19] only takes a few seconds for the given parameters. An overview on published parameters can be found in [8].

We use the following constructions for subgroups of \( GL(n, q) \):
The result can be checked along Figure 1. Starting with the two underlined parameters we successively can deduce the four parameters in the boxes due to the

Proof. The result can be checked along Figure 1. Starting with the two underlined parameters we successively can deduce the four parameters in the boxes due to the

Table 1. 2-(9, k, λ; 2) designs for k ∈ {3, 4}

| t-(n, k, λ; q) | G               | \( |A_{t,k}^G| \) | λ                  |
|---------------|-----------------|-----------------|-------------------|
| 2-(9, 3, λ; 2)| \( N(3, 2^3) \)| 31 × 529         | 21, 22, 42, 43, 63|
|               | \( N(8, 2) \times 1 \) | 28 × 408        | 7, 12, 19, 24, 31, 36, 43, 48, 55, 60 |
|               | \( M(3, 2^3) \)  | 40 × 460        | 49                |
| 2-(9, 4, λ; 2)| \( N(9, 2) \)   | 11 × 725        | 21, 63, 84, 126, 147, 189, 210, 252, 273, 315, 336, 378, 399, 441, 462, 504, 525, 567, 588, 630, 651, 693, 714, 756, 777, 819, 840, 882, 903, 945, 966, 1008, 1029, 1071, 1092, 1134, 1155, 1197, 1218, 1260, 1281, 1323 |

- A Singer cycle of the general linear group GL(n, q) is a cyclic group of order \( q^n - 1 \) whereas its generator can be obtained from the matrix representation of any primitive element of the field \( \mathbb{F}_q^* \). The normalizer \( N(n, q) \) of the Singer cycle is given by as the semi-direct product of the Galois group \( \langle \phi \rangle \) and a Singer cycle of GL(n, q) having the order \( n(q^n - 1) \).
- If \( G \) is a subgroup of GL(n, q) by \( G \times 1 \) we mean the direct product of \( G \) with the trivial group of matrix dimension 1 such that \( G \times 1 \) is a subgroup of GL(n + 1, q).
- The set \( M(n, q) \) denotes the subgroup of GL(n, q) consisting of all monomial matrices of GL(n, q) which is the set of all invertible matrices having exactly one non-zero entry in each row and in each column.
- By lifting any matrix group \( G \) of GL(m, q') can be interpreted as a subgroup of GL(ml, q).

5. New parameters

In this section we obtain new parameters for which designs exist by iterated application of Theorem 2.2.

Theorem 5.1. The existence of designs with parameters \((t - 1)\cdot(2t + 3, t, \lambda; q)\) and \((t - 1)\cdot(2t + 3, t + 1, \lambda \frac{q^{t+3}-1}{q^t-1}; q)\) imply the existence of designs with the following parameters:

- \((t - 1)\cdot(2t + 4, t + 1, \lambda \frac{q^{t+5}-1}{q^t-1}; q)\),
- \((t - 1)\cdot(2t + 4, t + 2, \lambda \frac{(q^{t+3}-1)(q^{t+5}-1)}{(q^t-1)(q^6-1)}; q)\),
- \((t - 1)\cdot(2t + 5, t + 2, \lambda \frac{q^{t+5}-1}{(q^t-1)(q^6-1)}; q)\),
- \((t - 1)\cdot(2t + 5, t + 2, \lambda \frac{(q^{t+3}-1)(q^{t+5}-1)(q^{t+6}-1)}{(q^t-1)(q^t-1)(q^4-1)}; q)\),
- \((t - 1)\cdot(2t + 6, t + 3, \lambda \frac{(q^{t+5}-1)(q^{t+6}-1)(q^{t+7}-1)}{(q^t-1)(q^t-1)(q^4-1)}; q)\).
fact that residual and derived designs imply reduced designs and that designs with
dual parameters do exist.

**Corollary.** Dual parameters do exist.

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