Note on the universality and the functoriality
of the perfect $\mathcal{F}$-locality

Lluis Puig

Institut de Mathématiques de Jussieu, lluis.puig@imj-prg.fr
6 Av Bizet, 94340 Joinville-le-Pont, France

Abstract: In [6, Proposition 18.21] we have proved some universality of the so-called localizing functor associated with a Frobenius $P$-category $\mathcal{F}$, where $P$ is a finite $p$-group, with respect to the coherent $\mathcal{F}$-localities $(\tau, \mathcal{L}, \pi)$ such that the contravariant functor $\text{Ker}(\pi)$ [6, 17.8.2] maps any subgroup of $P$ to an Abelian $p$-group. The purpose of this Note is both to move from the localizing functor to the perfect locality associated with $\mathcal{F}$ and to remove the Abelian hypothesis in the target. As a consequence, we get the functoriality for the perfect localities in the strongest form, improving [7, Theorem 9.15].

1. Introduction

1.1. Let $p$ be a prime number, $P$ a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category [6, 2.8]. In [6, Chap. 17] we introduce the localities $(\tau, \mathcal{L}, \pi)$ associated to $\mathcal{F}$ and, in particular, the perfect $\mathcal{F}$-locality $(\hat{\tau}, P, \hat{\pi})$ [6, 17.13]. The main purpose of this Note is to prove that the perfect $\mathcal{F}$-locality $(\hat{\tau}, P, \hat{\pi})$ is an initial object in a suitable category of $\mathcal{F}$-localities, namely in the full subcategory of the $p$-coherent $\mathcal{F}$-localities [7, 2.8 and 2.9]; that is to say, that there exists a unique isomorphism class of $\mathcal{F}$-locality functors [7, 2.9] from $(\hat{\tau}, P, \hat{\pi})$ to any $p$-coherent $\mathcal{F}$-locality $(\tau, \mathcal{L}, \pi)$. Our result in [6, Proposition 18.21] can be considered as a first weaker tentative towards this statement.

1.2. Let us recall our definitions. Denote by $\text{iGr}$ the category formed by the finite groups and by the injective group homomorphisms; moreover, for any finite subgroup $G$ with $P$ as a Sylow $p$-subgroup, respectively denote by $\mathcal{F}_{G,P}$ and $T_{G,P}$ the categories where the objects are all the subgroups of $P$ and, for two of them $Q$ and $R$, the respective sets of morphisms $\mathcal{F}_{G,P}(Q,R)$ and $T_{G,P}(Q,R)$ are formed by the group homomorphisms from $R$ to $Q$ induced by the conjugation by elements of $P$, and by the set $T_G(R,Q)$ of such elements, the compositions being the obvious ones.

1.3. Now, a Frobenius $P$-category $\mathcal{F}$ is a subcategory of $\text{iGr}$ containing $\mathcal{F}_P = \mathcal{F}_{P,P}$ where the objects are all the subgroups of $P$ and the morphisms fulfill the following three conditions [6, 2.8 and Proposition 2.11]

1.3.1 For any subgroup $Q$ of $P$, the inclusion functor $(\mathcal{F})_Q \rightarrow (\text{iGr})_Q$ is full.

1.3.2 $\mathcal{F}_P(P)$ is a Sylow $p$-subgroup of $\mathcal{F}(P)$.

We say that a subgroup $Q$ of $P$ is fully centralized in $\mathcal{F}$ if for any $\mathcal{F}$-morphism $\xi: Q \cdot C_P(Q) \rightarrow P$ we have $\xi(C_P(Q)) = C_P(\xi(Q))$; similarly, replacing in this condition the centralizer by the normalizer, we say that $Q$ is fully normalized.
1.3.3 For any subgroup $Q$ of $P$ fully centralized in $F$, any $F$-morphism $\varphi : Q \to P$ and any subgroup $R$ of $N_P(\varphi(Q))$ such that $\varphi(Q) \subset R$ and that $F_P(Q)$ contains the action of $F_R(\varphi(Q))$ over $Q$ via $\varphi$, there exists an $F$-morphism $\zeta : R \to P$ fulfilling $\zeta(\varphi(u)) = u$ for any $u \in Q$.

Note that, with the notation above, $F_G = F_{G,P}$ is a Frobenius $P$-category.

1.4. Then, a coherent $F$-locality is a triple $(\tau, \mathcal{L}, \pi)$ formed by a small category $\mathcal{L}$, a surjective functor $\pi : \mathcal{L} \to F$ and a functor $\tau : T_P \to \mathcal{L}$ from the transporter category $T_P = T_{P,P}$ of $P$, fulfilling the following three conditions

\[ \text{[6, 17.3, 17.8 and 17.9]} \]

1.4.1 The composition $\pi \circ \tau$ coincides with the composition of the canonical functor $\kappa_P : T_P \to F_P$ with the inclusion $F_P \subset F$.

1.4.2 For any pair of subgroups $Q$ and $R$ of $P$, $\text{Ker}(\pi_Q)$ acts regularly on the fibers of the following maps determined by $\pi$

\[ \pi_{Q,R} : \mathcal{L}(Q, R) \to F(Q, R) \]

Analogously, for any pair of subgroups $Q$ and $R$ of $P$, we denote by

\[ \tau_{Q,R} : T_P(Q, R) \to \mathcal{L}(Q, R) \]

the map determined by $\tau$, and whenever $R \subset Q$ we set $i^Q_R = \tau_{Q,R}(1)$; if $R = Q$ then we write $Q$ once for short.

1.4.4 For any pair of subgroups $Q$ and $R$ of $P$, any $x \in \mathcal{L}(Q, R)$ and any $v \in R$, we have $x \tau_R(v) = \tau_Q(\pi_{Q,R}(x)(v)) \cdot x$.

Moreover, we say that $(\tau, \mathcal{L}, \pi)$, or $\mathcal{L}$ for short, is $p$-coherent when $\text{Ker}(\pi_Q)$ is a finite $p$-group for any subgroup $Q$ of $P$. Note that, with the notation above, $T_G = T_{G,P}$ endowed with the obvious functors

\[ T_P \to T_G \text{ and } T_G \to F_G \]

becomes a coherent $F_G$-locality.

1.5. If $(\tau, \mathcal{L}, \pi)$ is a coherent $F$-locality, the subgroups $\tau_Q(Q) \subset \mathcal{L}(Q)$ when $Q$ runs over the subgroups of $P$ define an interior structure in $\mathcal{L}$ [6, 1.3] and we denote by $\widetilde{\mathcal{L}}$ the corresponding exterior quotient. Moreover, since $F$ is divisible [6, 2.3.1] and $\mathcal{L}$ fulfills condition 1.4.2, if $Q'$ and $R'$ are subgroups of $P$, and we have $R \subset Q$ and $Q' \subset R'$, denoting by $\mathcal{L}(Q', Q)_{R', R}$ the set of $y \in \mathcal{L}(Q', Q)$ such that $(\pi_{Q', Q}(y))(R) \subset R'$, we get a restriction map (possibly empty!)

\[ \iota^{Q',Q}_{R',R} : \mathcal{L}(Q', Q)_{R', R} \to \mathcal{L}(R', R) \]

fulfilling $\iota^{Q',Q}_{R',R}(y) = y \iota^Q_R$ for any $y \in \mathcal{L}(Q', Q)_{R', R}$; similarly, we get a contravariant functor from $\mathcal{L}$ to the category $\mathfrak{Gr}$ of finite groups [6, 17.8.2]

\[ \text{Ker}(\pi) : \mathcal{L} \to \mathfrak{Gr} \]

\[ 1.5.2 \]
sending any subgroup $Q$ of $P$ to $\text{Ker}(\pi_Q)$ and any $\mathcal{L}$-morphism $x: R \to Q$ to the group homomorphism

$$\mathfrak{Re}(\pi)_x : \text{Ker}(\pi_Q) \longrightarrow \text{Ker}(\pi_R)$$

fulfilling $u \cdot x = x \cdot (\mathfrak{Re}(\pi)_x(u))$ for any $u \in \text{Ker}(\pi_Q)$; actually, it follows from [6 Proposition 17.10] that $\mathfrak{Re}(\pi)$ factorizes through the exterior quotient $\tilde{L}$.

1.6. If $\mathcal{L}'$ is a second coherent $\mathcal{F}$-locality with structural functors $\tau'$ and $\pi'$, we call $\mathcal{F}$-locality functor from $\mathcal{L}$ to $\mathcal{L}'$ any functor $l: \mathcal{L} \to \mathcal{L}'$ fulfilling

$$\tau' = \text{id} \circ \tau \quad \text{and} \quad \pi' \circ \text{id} = \pi$$

the composition of two $\mathcal{F}$-locality functors is obviously an $\mathcal{F}$-locality functor; note that the equality $\tau' = \text{id} \circ \tau$ forces $\text{id}$ to be compatible with the restriction maps. Moreover, we can construct a third coherent $\mathcal{F}$-locality $\mathcal{L} \times_\mathcal{F} \mathcal{L}'$ from the corresponding category defined by the pull-back of sets

$$(\mathcal{L} \times_\mathcal{F} \mathcal{L}')(Q, R) = \mathcal{L}(Q, R) \times_{\mathcal{F}(Q, R)} \mathcal{L}'(Q, R)$$

with the obvious composition and with the structural maps

$$T_P(Q, R) \xrightarrow{\tau'_Q \cdot R} (\mathcal{L} \times_\mathcal{F} \mathcal{L}')(Q, R) \xrightarrow{\pi'_Q \cdot R} \mathcal{F}(Q, R)$$

respectively induced by $\tau$ and $\tau'$, and by $\pi$ and $\pi'$. Note that we have obvious $\mathcal{F}$-locality functors

$$\mathcal{L} \leftarrow \mathcal{L} \times_\mathcal{F} \mathcal{L}' \longrightarrow \mathcal{L}'$$

1.7. Actually, any $\mathcal{F}$-locality functor $l: \mathcal{L} \to \mathcal{L}'$ determines a natural map

$$\chi_l : \mathfrak{Re}(\pi) \longrightarrow \mathfrak{Re}(\pi')$$

conversely, it is quite clear that any subfunctor $\mathcal{L}$ of $\mathcal{L}$ determines a quotient coherent $\mathcal{F}$-locality $\mathcal{L}/\mathcal{L}$ defined by the quotient sets

$$(\mathcal{L}/\mathcal{L})(Q, R) = \mathcal{L}(Q, R)/\mathcal{L}(R)$$

for any pair of subgroups $Q$ and $R$ of $P$, and by the corresponding induced composition. We say that two $\mathcal{F}$-locality functors $l$ and $\bar{l}$ from $\mathcal{L}$ to $\mathcal{L}'$ are naturally $\mathcal{F}$-isomorphic if we have a natural isomorphism $\lambda: l \cong \bar{l}$ fulfilling $\pi' * \lambda = \text{id}_P$; in this case, $\lambda_Q$ belongs to $\text{Ker}(\pi'_Q)$ for any subgroup $Q$ of $P$ and, since $l(i_Q^P) = i_Q^P = \bar{l}(i_Q^P)$, $\lambda$ is uniquely determined by $\lambda_P$; indeed, we have

$$\lambda_P \cdot i_Q^P = i_Q^P \cdot \lambda_Q$$

Once again, the composition of a natural $\mathcal{F}$-isomorphism with an $\mathcal{F}$-locality functor or with another such a natural $\mathcal{F}$-isomorphism is a natural $\mathcal{F}$-isomorphism.
1.8. For any coherent \( F \)-locality \((\tau, L, \pi)\) recall that [6, Remark 17.11]

1.8.1 If \( Q \) is a subgroup of \( P, R \) and \( T \) are subgroups of \( N_P(Q) \) containing \( Q \), and \( x \) is an element of \( L(R, T) \) such that \( \pi_{\tau, \tau}(x) \) stabilizes \( Q \), then there is \( y \in L(Q) \) such that \( i^\tau_Q \cdot y = x \cdot i^\tau_Q \) and moreover we have

\[
y \cdot \tau_Q(w) \cdot y^{-1} = \tau_Q\left(\left(\pi_{\tau, \tau}(x)\right)(w)\right)
\]

for any \( w \in T \).

Indeed, the existence of \( y \) follows from the divisibility of \( F \) (cf. condition 1.3.1) and from condition 1.4.2 above; in particular, for any \( w \in T \), from condition 1.4.4 we have

\[
i^\tau_Q \cdot y \cdot \tau_Q(w) = x \cdot \tau_T(w) \cdot i^\tau_Q = \tau_R\left(\left(\pi_{\tau, \tau}(x)\right)(w)\right) \cdot x \cdot i^\tau_Q = i^\tau_Q \cdot \tau_Q\left(\left(\pi_{\tau, \tau}(x)\right)(w)\right) \cdot y
\]

\[1.8.2\]

and we apply again the divisibility of \( F \) and condition 1.4.2 above [6, 17.7.1]. It turns out that condition 1.8.1 is meaningful when applying the lemma below; actually, in [6, Lemma 18.8] we already claimed a wrong stronger version and therefore it seems reasonable to repeat here the good part of the argument.

**Lemma 1.9.** Let \( L \) and \( M \) be finite groups, \( P \) a Sylow \( p \)-subgroup of \( L \), \( Z \) a normal Abelian \( p \)-subgroup of \( M \) and \( \bar{\sigma} : L \to \bar{M} = M/Z \) a group homomorphism. Assume that there is a group homomorphism \( \tau : P \to \bar{M} \) lifting the restriction of \( \bar{\sigma} \) and fulfilling the following condition

1.9.1 For any subgroup \( R \) of \( P \) and any \( x \in L \) such that \( R^x \subset P \), there is \( y \in M \) such that \( \bar{\sigma}(x) = y \) and that \( \tau(u^x) = \tau(u)^y \) for any \( u \in R \).

Then, there is a group homomorphism \( \sigma : L \to M \) lifting \( \bar{\sigma} \) and extending \( \tau \). Moreover, if \( \sigma' : L \to M \) is a group homomorphism which lifts \( \bar{\sigma} \) and extends \( \tau \) then there is \( z \in Z \) such that \( \sigma'(x) = \bar{\sigma}(x)^z \) for any \( x \in L \).

**Proof:** It is clear that \( \bar{\sigma} \) determines an action of \( L \) on \( Z \) and therefore, for any \( n \in \mathbb{N} \), it makes sense to consider the cohomology groups \( H^n(L, Z) \) and \( H^n(P, Z) \). Moreover, \( M \) determines an element \( \bar{\mu} \) of \( H^2(M, Z) \) and if there is a group homomorphism \( \tau : P \to M \) lifting the restriction of \( \bar{\sigma} \) then the corresponding image of \( \bar{\mu} \) in \( H^2(P, Z) \) has to be zero; thus, since the restriction map

\[H^2(L, Z) \to H^2(P, Z)\]

\[1.9.2\]

is injective [2, Ch. XII, Theorem 10.1], we also get

\[(H^2(\bar{\sigma}, \text{id}_Z))(\bar{\mu}) = 0\]

\[1.9.3\]

and therefore there is a group homomorphism \( \sigma : L \to M \) lifting \( \bar{\sigma} \).
At this point, the difference between \( \tau \) and the restriction of \( \sigma \) to \( P \) defines a 1-cocycle \( \theta: P \to Z \) and, for any subgroup \( R \) of \( P \) and any \( x \in L \) such that \( R^x \subset P \), it follows from condition 1.9.1 that there is \( y \in M \) such that \( y = \tilde{\sigma}(x) \) and that, for any \( u \in R \), we have

\[
\theta(u^x) = \tau(u^x)^{-1}\sigma(u^x) = \tau(u^{-1})^y\sigma(u)^{\sigma(x)}
\]

consequently, since the map sending \( u \) to get a new group homomorphism \( \sigma \) is a 1-coboundary, the cohomology class \( \bar{\eta} \) defines a 1-cocycle \( \theta \) and it follows from condition 1.9.1 that there is an element \( \bar{\eta} \in H^1(L, Z) \) then, it suffices to modify \( \sigma \) by a representative of \( \bar{\eta} \) to get a new group homomorphism \( \sigma': L \to M \) lifting \( \tilde{\sigma} \) and extending \( \tau \).

Now, if \( \sigma': L \to M \) is a group homomorphism which lifts \( \tilde{\sigma} \) and extends \( \tau \), the element \( \sigma'(x)\sigma(x)^{-1} \) belongs to \( Z \) for any \( x \in L \) and thus, we get a 1-cocycle \( \lambda: L \to Z \) mapping \( x \in L \) on \( \sigma'(x)\sigma(x)^{-1} \), which vanish over \( P \); hence, it is a 1-coboundary [2, Ch. XII, Theorem 10.1] and therefore there exists \( z \in Z \) such that

\[
\lambda(x) = z^{-1}\sigma(x)z\sigma(x)^{-1}
\]

so that we have \( \sigma'(x) = \sigma(x)^z \) for any \( x \in L \). We are done.

1.10. When inductively trying to obtain a more general statement by removing the Abelian hypothesis on the \( p \)-group \( Z \), it quickly appears a lack of inductiveness in condition 1.9.1. The remark leading to this Note is that for an analogous statement respectively replacing the groups \( L \) and \( M \) by the perfect \( F \)-locality \( P \) [6, 17.13] and by a \( p \)-coherent \( F \)-locality \( L \), the lack of inductiveness disappear. In the next section, we recall the definition of the perfect \( F \)-locality and prove our main result. In the last section, we give a direct proof for the functoriality of the perfect \( F \)-locality, which replaces all the section 9 in [7] and, moreover, improves [7, Theorem 9.15].

1.11. As we explain below, our main result allows a dramatrical reduction in the proof of the existence and the uniqueness of the perfect \( F \)-locality in [7, §6]; it also supplies an easier direct proof of [7, Theorem 8.10]. Perhaps the main interest of this Note is to show that the cohomology of categories (or higher limits) are not needed when dealing with perfect \( F \)-localities: all we need from the cohomology groups is contained in Lemma 1.9 above and in [1, Proposition 3.2]. Let us recall this result.

1.12. Actually, it is possible to relativize all the above definitions to a nonempty set \( \mathcal{X} \) of subgroups of \( P \) which contains any subgroup of \( P \) admitting an \( F \)-morphism from some subgroup in \( \mathcal{X} \), and we respectively denote by \( T_p^x \), \( F^x \), \( P^x \) and \( L^x \) the full subcategories of \( T_p \), \( F \), \( P \) and \( L \)
over $\mathcal{X}$ as the set of objects. Let $U \in \mathcal{X}$ be fully normalized in $\mathcal{F}$ and assume that $CP(U) = Z(U)$; if $Q$ and $R$ are subgroups of $NP(U)$ containing $U$, it follows from condition 1.3.3 and from $[6, \text{Corollary 4.9}]$ that, denoting by $\hat{R}$ the respective images of $Q$ and $R$ in $\mathcal{F}(U)$, any element in $T_{\mathcal{F}(U)}(\hat{Q}, \hat{R})$ determines an $\mathcal{F}$-morph from $R$ to $Q$; it is clear that this correspondence defines a functor $t: \mathcal{T}_{\mathcal{F}(U)} \to \mathcal{F}$. Then, Proposition 3.2 in [1] proves the following statement.

**Proposition 1.13.** With the notation above, for any torsion contravariant functor $\mathcal{F}_U : \mathcal{F}^\times \to \mathcal{Z}_{(p)^\text{-mod}}$ vanishing outside of $\{\theta(U)\}_{\theta \in \mathcal{F}(P, U)}$ and any $n \geq 1$ the functor $t$ induces a group isomorphism

$$\mathbb{H}_n(\mathcal{F}, \mathcal{F}_U) \cong \mathbb{H}_n(\mathcal{T}_{\mathcal{F}(U)}, \mathcal{F}_U \circ t)$$

1.13.1.

2. The perfect $\mathcal{F}$-locality

2.1. As in 1.1 above, let $p$ be a prime number, $P$ a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category. Recall that any subgroup $Q$ of $\mathcal{P}$ fully centralized in $\mathcal{F}$ has a so-called centralizer $CP(Q)$ which is a Frobenius $CP(Q)$-category contained in $\mathcal{F}$ [6, Proposition 2.16]; similarly, if $(\tau, \mathcal{L}, \pi)$ is a coherent $\mathcal{F}$-locality then we also have a centralizer $C_{\mathcal{L}}(Q)$ which is a coherent $C_{\mathcal{F}}(Q)$-locality contained in $\mathcal{L}$ [6, 17.5]. Moreover, recall that, for any subgroup $Q$ of $\mathcal{P}$ fully centralized in $\mathcal{F}$, the so-called $\mathcal{C}_F(Q)$-hyperfocal subgroup $H_{C_{\mathcal{F}}(Q)}$ is the subgroup of $CP(Q)$ generated by the union of the sets $\{v\sigma(v)^{-1}\}_{v \in R}$ where $R$ runs over the set of subgroups of $CP(Q)$ and $\sigma$ over the set of $p'$-elements of $(C_{\mathcal{F}}(Q))(R)$.

**Lemma 2.2.** Let $(\tau, \mathcal{L}, \pi)$ a coherent $\mathcal{F}$-locality. For any subgroup $Q$ of $\mathcal{P}$ fully centralized in $\mathcal{F}$ such that $\text{Ker}(\pi_Q)$ is a $p'$-group, we have

$$H_{C_{\mathcal{F}}(Q)} \subseteq \text{Ker}(\tau_Q)$$

2.2.1.

**Proof:** For any subgroup $R$ of $CP(Q)$, any element $v$ of $R$ and any $p'$-element $\sigma$ of $(C_{\mathcal{F}}(Q))(R)$, we have

$$\tau_Q(v\sigma(v)^{-1}) = \tau_Q(v)\tau_Q(\sigma(v)^{-1})$$

2.2.2; but, by the very definition of $C_{\mathcal{L}}(Q)$ [6, 17.5], there is a $p'$-element $s$ in $\mathcal{L}(Q, R)$ lifting $\sigma$ and acting trivially on $Q$; in particular, it follows from the divisibility of $\mathcal{F}$ and from condition 1.3.2 that $s$ determines a $p'$-element $t$ in $\mathcal{L}(Q)$ fulfilling $i_Q^Q \cdot t = s \cdot i_Q^Q$ and acting trivially on $Q$; thus, since we are assuming that $\text{Ker}(\pi_Q)$ is a $p'$-group, $t$ is actually the trivial element of $\mathcal{L}(Q)$. Consequently, by the coherence of $\mathcal{L}$ we get (cf. condition 1.4.4)

$$\tau_{Q,R}(v^{-1})s^{-1}i_Q^Q = s^{-1}i_Q^Q \cdot \tau_Q(\sigma(v)^{-1})$$

2.2.3;
but, according to the definition of $s$, we have $s^{-1}i_Q^{Q,R} = i_Q^{Q,R}$ and therefore, since $\tau_{Q,R}(v^{-1})i_Q^{Q,R} = i_Q^{Q,R}\cdot \tau_{Q,R}(v^{-1})$, we finally get $\tau_Q(\sigma(v)^{-1}) = \tau_Q(v^{-1})$ (cf. condition 1.3.2). We are done.

2.3. We say that $(\tau, L, \pi)$ is a perfect $\mathcal{F}$-locality if it is a coherent $\mathcal{F}$-locality and for any subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$ we have [6, 17.12 and 17.13]

$$\text{Ker}(\pi_Q) \subset \tau_Q(N_P(Q)) \quad \text{and} \quad H_{C_P}(Q) = \text{Ker}(\tau_Q) \quad \text{2.3.1};$$

note that, in this case, $\tau_Q(N_P(Q))$ is a Sylow $p$-subgroup of $L(Q)$ [6, Proposition 2.11] and we have Ker$(\pi_Q) = \tau_Q(C_P(Q))$. Denoting by sc the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ — namely the set of subgroups $Q$ of $P$ such that $C_P(\varphi(Q)) \subset \varphi(Q)$ for any $\varphi$ in $\mathcal{F}(P,Q)$ [6, 4.8] — the existence and the uniqueness of a perfect $\mathcal{F}^\tau$-locality $\mathcal{P}^\tau$ has been proved by Chermak in [3] employing the classification of simple finite groups and by Oliver in [5] employing Chermak's induction, but where the whole classification can be reduced to a shorter one for $p = 2$, through the work by Glauberman and Lynd [4].

2.4. Actually, in [7] we prove the existence and the uniqueness of perfect $\mathcal{F}$-locality $\mathcal{P}$ without any classification. More precisely, we already gave in [6, Ch. 20] (and we correct in [7, Theorem 7.2]!) a proof of the existence and the uniqueness of a perfect $\mathcal{F}$-locality $\mathcal{P}$ from the existence and the uniqueness of $\mathcal{P}^\tau$, and in [7, §6] we prove the existence and the uniqueness of $\mathcal{P}^\tau$. As a matter of fact, all these results are not needed here, except for the functorial part in [7, Theorem 7.2]; more generally, we work over a nonempty set $\mathcal{X}$ chosen as in 1.12 above.

**Theorem 2.5.** With the notation above, let $(\hat{\tau}, \mathcal{P}^\tau, \hat{\pi})$ be a perfect $\mathcal{F}^\tau$-locality. For any $p$-coherent $\mathcal{F}$-locality $(\tau, L, \pi)$ there exists a unique natural $\mathcal{F}^\tau$-isomorphism class of $\mathcal{F}^\tau$-locality functors from $\mathcal{P}^\tau$ to $L^\tau$.

**Proof:** According to [7, Theorem 7.2, and 7.3], we may assume that all the subgroups in $\mathcal{X}$ are $\mathcal{F}$-selfcentralizing. From 1.5.2 let us consider the family $\{\Gamma^i(\text{Ret}(\pi))\}_{i \in \mathbb{N}}$ of contravariant functors from $L$ to $\text{Ret}$ inductively defined by

$$\Gamma^0(\text{Ret}(\pi)) = \text{Ret}(\pi) \quad \text{and} \quad \Gamma^{i+1}(\text{Ret}(\pi)) = [\text{Ret}(\pi), \Gamma^i(\text{Ret}(\pi))].$$

We clearly may assume that $\text{Ret}(\pi)$ is not trivial whereas, since the $\mathcal{F}$-locality $L$ is $p$-coherent, $\Gamma^i(\text{Ret}(\pi))$ is trivial for $i \in \mathbb{N}$ big enough; let $\ell \in \mathbb{N}$ be the biggest element such that the contravariant functor $K = \Gamma^\ell(\text{Ret}(\pi))$ is not trivial and denote by $(\tilde{\tau}, \tilde{L}, \tilde{\pi})$ the coherent $\mathcal{F}$-locality formed by the quotient category $\tilde{L} = L/K$ (cf. 1.6.3) with the corresponding structural functors

$$\tilde{\tau} : \mathcal{T}_P \to \tilde{L} \quad \text{and} \quad \tilde{\pi} : \tilde{L} \to \mathcal{F}.$$
It is quite clear that \((\tau, \hat{L}, \hat{\pi})\) remains a \(p\)-coherent \(\mathcal{F}\)-locality; hence, arguing by induction on the size of \(L\), we may assume that there exists a unique natural \(\mathcal{F}^x\)-isomorphism class of \(\mathcal{F}^x\)-locality functors \(\hat{\sigma}^x : \mathcal{P}^x \to \hat{\mathcal{L}}^x\).

We claim that there exists an \(\mathcal{F}^x\)-locality functor \(\sigma^x : \mathcal{P}^x \to \mathcal{L}^x\) lifting \(\hat{\sigma}^x\); let \(U\) be a minimal subgroup in \(\mathcal{X}\) fully normalized in \(\mathcal{F}\) \cite[Proposition 2.7]{6} and set

\[
\mathfrak{Y} = \mathcal{X} - \{\theta(U) \mid \theta \in \mathcal{F}(P, U)\}
\]

where the case \(\mathfrak{Y} = \emptyset\) is not excluded. First of all, we claim the existence of a group homomorphism

\[
\sigma_U : \mathcal{P}^x(U) \to \mathcal{L}(U)
\]

fulfilling \(\pi_U \circ \sigma_U = \hat{\pi}^x_U\) and \(\tau_U = \sigma_U \circ \hat{\tau}^x_U\); we will apply Lemma 1.9 above to the finite groups \(\mathcal{P}^x(U)\) and \(\mathcal{L}(U)\), to the Sylow \(p\)-subgroup \(\hat{\tau}^{(U)}_x(N_P(U))\) of \(\mathcal{P}^x(U)\) \cite[2.3]{2} and to the group homomorphism \(\sigma_U^x : \mathcal{P}^x(U) \to \mathcal{L}(U)\) determined by the functor \(\hat{\sigma}^x\).

Since according to Lemma 2.2 we have \(\tau_U(H_C(U)) = \{1\}\), the group homomorphism \(\tau_U : N_P(U) \to \mathcal{L}(U)\) factorizes throughout \(\tau_U^x : N_P(U) \to \mathcal{P}(U)\); that is to say, we have a unique group homomorphism

\[
\eta : \tau_U^x(N_P(U)) \to \tau_U(N_P(U))
\]

fulfilling \(\eta \circ \hat{\tau}^x_U = \tau_U\) and lifting the restriction of \(\hat{\sigma}^x_U\). We claim that \(\eta\) fulfills condition 1.9.1; let \(R\) a subgroup of \(N_P(U)\) and \(\hat{x}\) an element of \(\mathcal{P}^x(U)\) such that \(\hat{\tau}^x_U(R)^{\hat{x}} \subset \hat{\tau}^x_U(N_P(U))\); actually, according to the Alperin Fusion Theorem applied to the group \(\mathcal{P}^x(U)\), in order to show that condition 1.9.1 holds, we may assume that \(R\) contains \(U\) and that \(\hat{x}\) normalizes \(\hat{\tau}^x(R)\).

In this case, \(\xi = \hat{\pi}^x_U(\hat{x})\) belongs to \(N_{\mathcal{F}(U)}(\mathcal{F}^x(U))\) and therefore it follows from \cite[Corollary 2.13]{6} that \(\xi\) can be lifted to some element \(\zeta\) of the stabilizer \(\mathcal{F}(R)_U\) of \(U\) in \(\mathcal{F}(R)\); then, there is \(\hat{y} \in \mathcal{F}^x(R)_U\) lifting \(\zeta\) and therefore, according to 1.5.1, there exists \(\hat{y}_U \in \mathcal{P}^x(U)\) fulfilling \(\hat{\tau}^x_U(\hat{y}_U) = \hat{y}^r_U\), and normalizing \(\hat{\tau}^x(R)\); in particular, we have

\[
\hat{\pi}^x_U(\hat{y}_U) = \xi = \hat{\pi}^x_U(\hat{x})
\]

and, since \(\text{Ker}(\hat{\pi}^x_U) = \hat{\pi}^x_U(C_P(U))\) \cite[2.3]{2} and \(U \subset R\), we have \(\hat{y}_U = \hat{x} \cdot \hat{\pi}^x(U)(z)\) for some element \(z \in Z(U) \subset R\). Hence, up to replacing \(\hat{y}\) by \(\hat{y} \cdot \hat{\pi}^x(U)(z)^{-1}\), we may assume that \(\hat{y}_U = \hat{x}\).

In this situation we set \(\hat{y} = \hat{\sigma}^x_U(\hat{y})\), which belongs to \(\mathcal{L}(R)_U\), and choose an element \(y \in \mathcal{L}(R)_U\) lifting \(\hat{y}\); once again, according to 1.5.1, there is \(y_U \in \mathcal{L}(U)\) fulfilling \(i_{U}^y \cdot y_U = y^i_U\) and normalizing \(\tau_U(R)\). All this can be
summarized in the following commutative diagram

\[
\begin{array}{ccc}
\zeta \in \mathcal{F}(R) & \rightarrow & \xi \in N_{\mathcal{F}(U)}(\mathcal{F}_R(U)) \\
\uparrow \quad \hat{y} \in \mathcal{L}(R) & \rightarrow & \uparrow \quad \hat{y}_U \in N_{\mathcal{L}(U)}(\bar{\tau}_U(R)) \\
\hat{y} \in \mathcal{P}^\tau(R) & \rightarrow & \hat{x} \in N_{\mathcal{P}^\tau(U)}(\hat{\tau}(R)) \\
y \in \mathcal{L}(R) & \rightarrow & y_U \in N_{\mathcal{L}(U)}(\tau_U(R))
\end{array}
\]

it is clear that the image \( \bar{y}_U \) of \( y_U \) in \( N_{\mathcal{L}(U)}(\bar{\tau}_U(R)) \) fulfills \( \bar{i}_U \cdot \bar{y}_U = \bar{y}_U \cdot \bar{i}_U \) and therefore it coincides with \( \bar{\sigma}_U(\hat{x}) \), since the functor \( \bar{\sigma}_U \) applied to the commutative \( \mathcal{P}^\tau \)-diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\hat{y}} & R \\
\uparrow \hat{i}_U & \rightarrow & \uparrow \hat{i}_U \\
U & \xrightarrow{\hat{x}} & U
\end{array}
\]

yields the commutative \( \bar{\mathcal{L}} \)-diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\bar{y}} & R \\
\uparrow \bar{i}_U & \rightarrow & \uparrow \bar{i}_U \\
U & \xrightarrow{\bar{\sigma}_U(\hat{x})} & U
\end{array}
\]

Moreover, from the coherence of \( \mathcal{L} \), as in 1.8.2 above for any \( v \in R \) we have

\[
i_U \cdot y_U \cdot \tau_U(v) = y \cdot i_U \cdot \tau_R(v) = y \cdot \tau_R(v) \cdot i_U = \tau_R(\zeta(v)) \cdot y \cdot i_U
\]

\[
i_U \cdot \tau_U(\zeta(v)) \cdot y_U
\]

and therefore, according to 1.5.1, we get

\[
y_U \cdot \tau_U(v) \cdot (y_U)^{-1} = \tau_U(\zeta(v))
\]

but by the very definitions of \( \eta \) and of \( \zeta \) we have

\[
\tau_U(v) = \eta(\hat{\tau}^\tau(v)) \quad \text{and} \quad \tau_U(\zeta(v)) = \eta(\hat{x} \cdot \hat{\tau}^\tau(v) \cdot \hat{x}^{-1})
\]

which proves that \( \eta \) fulfills condition 1.9.1. Consequently, it follows from Lemme 1.9 that there exists a group homomorphism

\[
\sigma_U : \mathcal{P}^\tau(U) \rightarrow \mathcal{L}(U)
\]

fulfilling \( \sigma_U \circ \hat{\tau}^\tau = \tau_U \) and lifting \( \hat{\sigma}^\tau_U \), which proves our claim.
If \(|X| = 1\) then \(U = P\) and the existence of \(\sigma_U\) proves the existence of an \(\mathcal{F}^x\)-locality functor \(\sigma^x : \mathcal{P}^x \to \mathcal{L}^x\) lifting \(\bar{\sigma}^x\); now, assuming that \(|X| \neq 1\) and arguing by induction on \(|X|\), we may assume that there exists a unique natural \(\mathcal{F}^x\)-isomorphism class of \(\mathcal{F}^x\)-locality functors \(\sigma^x : \mathcal{P}^x \to \mathcal{L}^x\). Then, the composition

\[
\bar{\sigma}^x : \mathcal{P}^x \to \mathcal{L}^x \to \mathcal{L}^x
\]

and the restriction \(\bar{\sigma}^x|_{\mathcal{P}^x} : \mathcal{P}^x \to \mathcal{L}^x\) of \(\bar{\sigma}^x\) to \(\mathcal{P}^x\) are two \(\mathcal{F}^x\)-locality functors from \(\mathcal{P}^x\) to \(\mathcal{L}^x\); thus, by the induction hypothesis, they are naturally \(\mathcal{F}^x\)-isomorphic; hence, it follows from 1.7.3 above that there exists an element \(\bar{z}_p \in \text{Ker}(\bar{\pi}_Q^\mathcal{P})\) such that, for any \(Q \in \mathcal{P}\), we have \(\bar{z}_p \cdot i_Q^p = i_Q^p \cdot \bar{z}_Q\) for a suitable \(\bar{z}_Q \in \text{Ker}(\bar{\pi}_Q^\mathcal{P})\) in such a way that the family \(\{\bar{z}_Q\}_{Q \in \mathcal{P}}\) defines a natural isomorphism \(\bar{\sigma}^x|_{\mathcal{P}^x} \cong \bar{\sigma}^x\).

At this point, choose a lifting \(z_p\) of \(\bar{z}_p\) in \(\text{Ker}(\pi_Q^\mathcal{P})\); since the \(\mathcal{F}\)-locality \(\mathcal{L}\) is coherent, it follows from [6, Proposition 17.10] that \(z_p\) centralizes \(\tau_p(P)\) and therefore, for any \(Q \in \mathcal{P}\), we still have \(z_p \cdot i_Q^p = i_Q^p \cdot z_p\) for a suitable \(z_p \in \text{Ker}(\pi_Q^\mathcal{P})\) lifting \(\bar{z}_p\). Then, up to replacing \(\sigma^x\) by the functor \(\sigma^x\) sending any \(\mathcal{P}^\mathcal{P}\)-morphism \(\hat{x} : R \to Q\) to the \(\mathcal{L}^\mathcal{P}\)-morphism \((z_Q)^{-1} \cdot \sigma^x(\hat{x}) \cdot z_R\), we may assume that \(\bar{\sigma}^x|_{\mathcal{P}^x}\) and \(\bar{\sigma}^x\) coincide with each other; indeed, if \(\hat{x} = \hat{\tau}_{Q,R}(u)\) for some \(u \in \mathcal{T}_P(Q,R)\) then we get

\[
\begin{align*}
(i_Q^p(z_Q)^{-1} \cdot \sigma^x(\hat{\tau}_{Q,R}(u)) \cdot z_R) &= (z_p)^{-1} \cdot i_Q^p \cdot \tau_{Q,R}(u) \cdot z_R \\
&= (z_p)^{-1} \cdot \tau_p(u) \cdot i_R^p \cdot z_R = \tau_p(u) \cdot i_R^p \\
&= i_Q^p \cdot \tau_{Q,R}(u)
\end{align*}
\]

so that \(\sigma^x\) is also an \(\mathcal{F}^\mathcal{P}\)-locality functor.

Now, considering the direct product in 1.6, we get the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}^\mathcal{P} \times_{\mathcal{F}^\mathcal{P}} \mathcal{L}^\mathcal{P} & \longrightarrow & \mathcal{L}^\mathcal{P} \\
\uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
\]

and it is quite clear that the \(\mathcal{F}^\mathcal{P}\) - and \(\mathcal{F}^x\)-locality functors \(\sigma^\mathcal{P}\) and \(\bar{\sigma}^x\) determine “diagonal” \(\mathcal{F}^\mathcal{P}\) - and \(\mathcal{F}^x\)-locality functors

\[
\Delta_{\text{id},\sigma^\mathcal{P}} : \mathcal{P}^\mathcal{P} \to \mathcal{P}^\mathcal{P} \times_{\mathcal{F}^\mathcal{P}} \mathcal{L}^\mathcal{P} \quad \text{and} \quad \Delta_{\text{id},\bar{\sigma}^x} : \mathcal{P}^x \to \mathcal{P}^x \times_{\mathcal{F}^x} \mathcal{L}^x
\]

2.5.17.
Denote by $\mathcal{M}^\nabla$ the “image” of $\mathcal{P}^\nabla$ in $\mathcal{P}^\nabla \times_{\mathcal{F}^\nabla} \mathcal{L}^\nabla$, by $\rho^\nabla : \mathcal{P}^\nabla \to \mathcal{M}^\nabla$ the $\mathcal{F}^\nabla$-locality isomorphism obviously determined by $\Delta_{\text{id}, \sigma^\nabla}$, and by $\mathcal{M}^\times$ the subcategory of $\mathcal{P}^\times \times_{\mathcal{F}^\times} \mathcal{L}^\times$ which coincides with $\mathcal{M}^\nabla$ over $\mathfrak{Q}$ and fulfills

$$\mathcal{M}^\times (Q, V) = (\mathcal{P}^\times \times_{\mathcal{F}^\times} \mathcal{L}^\times)(Q, V)$$

for any pair of subgroups $Q$ in $\mathfrak{X}$ and $V$ in $\mathfrak{X} - \mathfrak{Q}$.

Denoting by $\hat{\mathcal{M}}^\times$ the “image” of $\mathcal{M}^\times$ in $\mathcal{P}^\times \times_{\mathcal{F}^\times} \mathcal{L}^\times$, it is quite clear that $\Delta_{\text{id}, \sigma^\times}$ determines an $\mathcal{F}^\times$-locality functor $\hat{\rho}^\times : \mathcal{P}^\times \to \hat{\mathcal{M}}^\times$ and, in order to prove the existence of an $\mathcal{F}^\times$-locality functor $\sigma^\times : \mathcal{P}^\times \to \mathcal{L}^\times$ lifting $\bar{\sigma}^\times$, it suffices to show that there exists an $\mathcal{F}^\times$-locality functor $\rho^\times : \mathcal{P}^\times \to \mathcal{M}^\times$ lifting $\bar{\rho}^\times$.

Let us consider the transporter category $T_{\mathcal{P}^\times(U)}$ of the group $\mathcal{P}^\times(U)$ (cf. 1.2); denoting by $\mathfrak{M}$ the set of subgroups of $N^\mathcal{P}(U)$ belonging to $\mathfrak{X}$, and by $T_{N^\mathcal{P}(U)}^\mathfrak{M}$ and $N_{\mathcal{F}}^\mathfrak{M}(U)$ the respective full subcategories of $T_{N^\mathcal{P}(U)}$ and $N_{\mathcal{F}}(U)$ over $\mathfrak{M}$, we claim that the full subcategory $T_{\mathcal{P}^\times(U)}^\mathfrak{M}$ of $T_{\mathcal{P}^\times(U)}$ over $\mathfrak{M}$, endowed with the functors

$$T_{N^\mathcal{P}(U)}^\mathfrak{M} \to T_{\mathcal{P}^\times(U)}^\mathfrak{M} \quad \text{and} \quad T_{\mathcal{P}^\times(U)}^\mathfrak{M} \to N_{\mathcal{F}}^\mathfrak{M}(U)$$

determined by the injective group homomorphism $\hat{\tau}_U : N^\mathcal{P}(U) \to \mathcal{P}^\times(U)$, becomes a $p$-coherent $N_{\mathcal{F}}^\mathfrak{M}(U)$-locality. Indeed, it is clear that $\hat{\tau}_U$ induces the left-hand functor and, since $\hat{\tau}_U$ is injective, the conjugation in $\mathcal{P}^\times(U)$ determines the right-hand one, which is “surjective” by statement 1.8.1; then, since we assume that all the groups in $\mathfrak{X}$ are $\mathcal{F}$-selfcentralizing, the $p$-coherence is easily checked (cf. 1.4.5).

Actually, denoting by $N_{\mathcal{P}^\times(U)}^\mathfrak{M}$ the full subcategory of $N_{\mathcal{P}^\times(U)}$ over $\mathfrak{M}$, statement 1.8.1 above induces an $N_{\mathcal{F}}^\mathfrak{M}(U)$-locality isomorphism

$$\kappa^\mathfrak{M}_U : N_{\mathcal{P}^\times(U)}^\mathfrak{M} \cong T_{\mathcal{P}^\times(U)}^\mathfrak{M}$$

Indeed, for any pair of subgroups $Q$ and $R$ in $\mathfrak{M}$, any $N_{\mathcal{P}^\times(U)}$-morphism $\hat{x} : R \to Q$ comes from a $\mathcal{P}^\times$-morphism $\hat{x}^U : R^U \to Q^U$, which stabilizes $U$ and is uniquely determined since any $\mathcal{P}^\times$-morphism is an epimorphism [6, Proposition 24.2]; in particular, by restriction $\hat{x}^U$ determines an element $\hat{r}_{U, U}^Q(U) \cdot (x^U)$ in $\mathcal{P}^\times(U)$ belonging to $T_{\mathcal{P}^\times(U)}(Q, R)$ (cf. 1.5.1). Now, it is easily checked that this correspondence defines an “injective” $N_{\mathcal{F}}^\mathfrak{M}(U)$-locality functor from $N_{\mathcal{P}^\times(U)}$ to $T_{\mathcal{P}^\times(U)}$; for the “surjectivity”, note that two subgroups
Q and \( Q' \) in \( \mathfrak{M} \) are \( N_{p^x}^{\text{m}}(U) \)-isomorphic if and only if they are \( T_{p^x}^{\text{m}}(U) \)-isomorphic, since both statements are equivalent to \( Q \) and \( Q' \) being \( N_{p^x}^{\text{m}}(U) \)-isomorphic; consequently, it suffices to show that

\[
(N_{p^x}^{\text{m}}(U))(Q) \cong (T_{p^x}^{\text{m}}(U))(Q)
\]

but, it is quite clear that (cf. 2.5.19)

\[
|(N_{p^x}^{\text{m}}(U))(Q)| = |N_Z(Q)(U)| \cdot |(N_{p^x}^{\text{m}}(U))(Q)| = |(T_{p^x}^{\text{m}}(U))(Q)|
\]

Similarly, denoting by

\[
u^x : T_F^x \rightarrow \mathcal{M}^x \quad \text{and} \quad \varpi^x : \mathcal{M}^x \rightarrow \mathcal{F}^x
\]

the structural functors of \( \mathcal{M}^x \), and by \( T_{\mathcal{M}^x(U)}^{\text{m}} \) the full subcategory over \( \mathfrak{M} \) of the transporter category \( T_{\mathcal{M}^x(U)} \) of the group \( \mathcal{M}^x(U) \), the group homomorphism \( \nu^x : N_F(U) \rightarrow \mathcal{M}^x(U) \) is still injective and, as above, it induces obvious functors

\[
u^m : T_{N_F(U)}^{\text{m}} \rightarrow T_{\mathcal{M}^x(U)}^{\text{m}} \quad \text{and} \quad \varpi^m : T_{\mathcal{M}^x(U)}^{\text{m}} \rightarrow N_{p^x}^{\text{m}}(U)
\]

once again, the right-hand functor is “surjective” by statement 1.8.1; then, \( T_{\mathcal{M}^x(U)}^{\text{m}} \) endowed with these functors becomes a \( p \)-coherent category.

Now, denoting by \( \mathfrak{N} \) the set of subgroups of \( N_F(U) \) belonging to \( \mathfrak{M} \) and by \( N_{\mathcal{M}^{\mathfrak{B}}(U)}^{\text{m}} \) the full subcategory of \( N_{\mathcal{M}^{\mathfrak{B}}}(U) \) over \( \mathfrak{N} \), we claim that we still have an “injective” \( N_{p^x}^{\text{m}}(U) \)-locality functor

\[
\mu^m : N_{\mathcal{M}^{\mathfrak{B}}}(U) \rightarrow T_{\mathcal{M}^x(U)}^{\text{m}}
\]

indeed, once again, any \( N_{\mathcal{M}^{\mathfrak{B}}}(U) \)-morphism \( x : R \rightarrow Q \) comes from an \( \mathcal{M}^{\mathfrak{B}} \)-morphism \( x^U : R \rightarrow Q \) which stabilizes \( U \) and, since \( \mathcal{M}^{\mathfrak{B}} \cong \mathcal{P}_{p^m} \), \( x^U \) is also uniquely determined by \( x \) [6, Proposition 24.2]; thus, by restriction \( x^U \) determines an element \( T_{U,R}(x^U) \) in \( \mathcal{M}^x(U) \) belonging to \( T_{\mathcal{M}^x(U)}^{\text{m}}(Q,R) \)
(cf. 1.5.1). As above, it is easily checked that this correspondence defines the announced “injective” \( N_{p^x}^{\text{m}}(U) \)-locality functor.

On the other hand, it is clear that the homomorphism \( \sigma_U \) (cf. 2.5.13) determines a “diagonal” group homomorphism

\[
\rho_U = \Delta_{\text{id}, \sigma_U} : \mathcal{P}^{x}(U) \rightarrow (\mathcal{P}^{x} \times_{\mathcal{F}^{x}} \mathcal{L}^{x})(U) = \mathcal{M}^{x}(U)
\]
which clearly induces an “injective” $N^m_X(U)$-locality functor

$$N^m_{p^\#}(U) \cong \mathcal{T}^m_{p^\#}(U) \xrightarrow{\rho^m} \mathcal{T}^m_{\mathcal{M}^\varphi(U)}$$

mapping $N^m_{p^\#}(U)$ on $\mathcal{T}^m_{\mathcal{M}^\varphi(U)}$. At this point, denoting by $\rho^m$ the restriction of the isomorphism $\rho^{\#}$ above to $N^m_{p^\#}(U)$, from 2.5.25 and 2.5.27 we get the two $N^m_X(U)$-locality functors

$$N^m_{p^\#}(U) \cong N^m_{\mathcal{M}^\varphi(U)} \xrightarrow{\mu^m} \mathcal{T}^m_{\mathcal{M}^\varphi(U)}$$

hence, since $N^m_{p^\#}(U)$ is a perfect $N^m_X(U)$-locality [6, 17.13], it follows from the induction hypothesis on $|X|$ that both functors are naturally $N^m_X(U)$-isomorphic, which according to equality 1.7.3 above only depends on an element $z_{N^m_{p(U)}}$ of $\text{Ker}(\varpi^m_{N^m_{p(U)}}) \subset \mathcal{M}^\varphi(U)$ centralizing $\rho^m(N^m_{p(U)})$.

But, since $\sigma_U$ lifts $\sigma_U^\varphi$ (cf. 2.5.13), denoting by $\hat{\rho}^m$ and $\hat{\mathcal{T}}^m_{\mathcal{M}^\varphi(U)}$ the respective compositions of $\mu^m$ and $\mathcal{T}^m_{\mathcal{M}^\varphi(U)}$ with the canonical functor $\mathcal{T}^m_{\mathcal{M}^\varphi(U)} \to \mathcal{T}^m_{\mathcal{M}^\varphi(U)}$, it is easily checked that

$$\hat{\rho}^m \circ \rho^m = \mathcal{T}^m_{\mathcal{M}^\varphi(U)} \circ \kappa^m_U$$

and therefore the image $\bar{z}_{N^m_{p(U)}}$ of $z_{N^m_{p(U)}}$ in $\mathcal{M}^\varphi(U)$ determines a natural $N^m_X(U)$-automorphism of the $N^m_X(U)$-locality functor $\mathcal{T}^m_{\mathcal{M}^\varphi(U)}$; once again, this natural $N^m_X(U)$-automorphism only depends on an element

$$\bar{s}_{N^m_{p(U)}} \in \text{Ker}(\varpi^m_{N^m_{p(U)}}) \subset \mathcal{M}^\varphi(U)$$

which can be lifted to some $s_{N^m_{p(U)}} \in \text{Ker}(\varpi^m_{N^m_{p(U)}})$; as above, since $s_{N^m_{p(U)}}$ centralizes $\rho^m(N^m_{p(U)})$ [6, Proposition 17.10], this element determines a natural $N^m_X(U)$-automorphism of the $N^m_X(U)$-locality functor $\mathcal{T}^m_{\rho^m}$ Consequentially, we can choose $z_{N^m_{p(U)}}$ in such a way that $\bar{z}_{N^m_{p(U)}} = 1$; then, up to replacing $\rho_U$ by its $z_{N^m_{p(U)}}$-conjugate, we may assume that

$$\mu^m \circ \rho^m = \mathcal{T}^m_{\rho^m} \circ \kappa^m_U$$

in particular, for any $N^m_{p^\#}(U)$-morphism $\hat{\gamma} : R \to Q$ such that $Q$ and $R$ both contain $U$, we may assume that (cf. 1.5.1)

$$\rho_U \left(\hat{\gamma} \circ \rho^m_\#(\hat{\gamma})\right) = \mu^m \left(\rho^m_\#(\hat{\gamma})\right) = r^\varphi_{\#}(\rho^m_\#(\hat{\gamma}))$$

2.5.32.
Moreover, for any $V \in \mathfrak{X} - \mathfrak{Y}$ fully normalized in $\mathcal{F}$, it follows from [6, Corollary 2.13] that there is a $\mathcal{P}^x$-isomorphism $\hat{y}_V : N_P(U) \rightarrow N_P(V) = N$ fulfilling

$$\hat{y}_V(U) = V$$

then, considering the restriction maps $\hat{y}_{V,U}^N$ and $r_{V,U}^N$ respectively corresponding to $\mathcal{P}^x$ and to $\mathcal{M}^x$ (cf. 1.5.1), and setting $y_v = \rho^\varphi(\hat{y}_V)$, we introduce the group homomorphism

$$\rho_v : \mathcal{P}^x(V) \rightarrow \mathcal{M}^x(V)$$

sending any $\hat{x} \in \mathcal{P}^x(V)$ to

$$r_{V,U}^N(y_v) \cdot \rho_v (r_{V,U}^N(\hat{y}_V))^{-1} \cdot \hat{x} \cdot r_{V,U}^N(y_v) \cdot r_{V,U}^N(\hat{y}_V)^{-1}$$

which does not depend on our choice of $\hat{y}_V$ and fulfills

$$\pi_v \circ \rho_v = \hat{\pi}_v \quad \text{and} \quad \rho_v \circ \hat{\pi}_v = \tau_v$$

since $\mathcal{P}^x$ and $\mathcal{M}^x$ are coherent. Indeed, another choice has the form $\hat{y}_V \cdot \hat{s}$ for some element $\hat{s}$ in $\mathcal{P}^x(N_P(U)_U)$; but, according to our choice of $\rho_v$ above (cf. 2.5.32), we have

$$\rho_v (r_{V,U}^N(y_v)(\hat{s})) = r_{V,U}^N(\rho^\varphi(\hat{s}))$$

Note that, for any $N_P^\varphi(V)$-morphism $\hat{y} : R \rightarrow Q$ such that $Q$ and $R$ both strictly contain $V$, from 2.5.32 and 2.5.35 we still get

$$\rho_v (r_{V,V'}^\varphi(\hat{y})) = r_{V,V'}^\varphi(\rho^\varphi(\hat{y}))$$

Going further, for any $V, V' \in \mathfrak{X} - \mathfrak{Y}$ fully normalized in $\mathcal{F}$, setting $N = N_P(V)$ and $N' = N_P(V')$, it follows from [6, condition 2.8.2] that any $\mathcal{P}^x$-morphism $\hat{x} : V \rightarrow V'$ factors as $\hat{x} = r_{V',V}^{N',N}(\hat{y}) \cdot \hat{s}$ for suitable $\hat{y}$ in $\mathcal{P}^x(N, N)_{V',V}$ and $\hat{s}$ in $\mathcal{P}^x(V)$; then, in $\mathcal{M}^x(V', V)$ we define

$$\rho_{V',V}^x(\hat{x}) = r_{V',V}^{N',N}(\rho^\varphi(\hat{y})) \cdot \rho_v(\hat{s})$$

this definition does not depend on our choice since, for such another decomposition $\hat{x} = r_{V',V}^{N',N}(\hat{y}) \cdot \hat{s}$, we get $\hat{y} = \hat{y} \cdot \hat{t}$ and $\hat{s} = r_{V'}^N(\hat{t})^{-1} \cdot \hat{s}$ for a suitable $\hat{t}$ in $\mathcal{P}^x(N)_V$, so that we have (cf. 2.5.38)

$$r_{V',V}^{N',N}(\rho^\varphi(\hat{y})) \cdot \rho_v(\hat{s}) = r_{V',V}^{N',N}(\rho^\varphi(\hat{y} \cdot \hat{t})) \cdot \rho_v(\hat{t}^{-1} \cdot \hat{s})$$

$$= r_{V',V}^{N',N}(\rho^\varphi(\hat{y})) \cdot r_{V}^{N}(\rho^\varphi(\hat{t})) \cdot \rho_v(\hat{t})^{-1} \cdot \rho_v(\hat{s})$$

$$= \rho_{V',V}^x(\hat{x})$$
In particular, if \( Q \) and \( Q' \) are a pair of subgroups of \( P \) respectively contained in \( N \) and \( N' \), and strictly containing \( V \) and \( V' \), for any \( \hat{x} \) in \( P^\circ(Q',Q)_{V',V} \) we claim that
\[
\rho_{V',V}^a \bigl( \hat{\rho}_{V',V}^Q(\hat{x}) \bigr) = \hat{\rho}_{V',V}^Q \bigl( \rho^a(\hat{x}) \bigr)
\]
indeed, assuming that \( \hat{x} \in P^\circ(Q',Q)_{V',V} \) and therefore, considering the element \( \hat{x} \) for suitable elements \( \hat{y} \in P^\circ(N',N)_{V',V} \) and \( \hat{z} \in P^\circ(V) \); consequently, setting \( Q'' = (\varpi_{N,N'}(\hat{y}^{-1}))(Q') \subset N \), we get
\[
\hat{z} = \hat{\rho}_{V',V}^Q \bigl( \hat{\rho}_{Q',Q}^Q(\hat{y}^{-1})\cdot \hat{x} \bigr)
\]
and therefore, considering the element \( \hat{s} = \hat{\rho}_{Q',Q}^Q(\hat{y}^{-1})\cdot \hat{x} \) which belongs to \( P^\circ(Q'',Q)_{V,V} \), we still get \( \hat{x} = \hat{\rho}_{Q',Q}^Q(\hat{y})\cdot \hat{s} \); hence, we obtain
\[
\hat{\rho}_{V',V}^Q \bigl( \hat{x} \bigr) = \hat{\rho}_{Q',Q}^Q(\hat{y})\cdot \hat{s}
\]
and, since \( \hat{\rho}_{V',V}^Q(\hat{s}) \) belongs to \( P^\circ(V) \), from definition 2.5.39 and from equality 2.5.38 we still obtain (cf. 1.6)
\[
\rho_{V',V}^a \bigl( \hat{\rho}_{V',V}^Q(\hat{x}) \bigr) = \hat{\rho}_{V',V}^a \bigl( \rho^a(\hat{y}) \bigr)\cdot \rho_v \bigl( \hat{\rho}_{Q',Q}^Q(\hat{s}) \bigr)
\]
indeed, assuming that
\[
\hat{x} = \hat{\rho}_{V',V}^a \bigl( \hat{x}' \bigr) = \hat{\rho}_{V',V}^a \bigl( \hat{x}' \bigr) = \hat{\rho}_{V',V}^a \bigl( \hat{x} \bigr)
\]
for suitable \( \hat{y} \in P^\circ(N',N)_{V',V} \), \( \hat{y}' \in P^\circ(N'',N)_{V'',V'} \), \( \hat{s} \in P^\circ(V) \) and \( \hat{s}' \in P^\circ(V') \), we get (cf. 2.5.39)
\[
\hat{\rho}_{V',V}^a(\hat{x}')\cdot \rho_{V',V}^a(\hat{x}) = \hat{\rho}_{V',V}^a(\hat{x}')\cdot \rho_{V',V}^a(\hat{x})
\]
indeed, assuming that
\[
\hat{x} = \hat{\rho}_{V',V}^a(\hat{x}')\cdot \hat{s} \quad \text{and} \quad \hat{x}' = \hat{\rho}_{V',V}^a(\hat{x}')\cdot \hat{s}'
\]
for suitable \( \hat{y} \in P^\circ(N',N)_{V',V} \), \( \hat{y}' \in P^\circ(N'',N')_{V''',V'} \), \( \hat{s} \in P^\circ(V) \) and \( \hat{s}' \in P^\circ(V') \), we get (cf. 2.5.39)
\[
\hat{\rho}_{V',V}^a(\hat{x}')\cdot \rho_{V',V}^a(\hat{x}) = \hat{\rho}_{V',V}^a(\hat{x}')\cdot \rho_{V',V}^a(\hat{x})
\]
Moreover, it is clear that \( \tilde{y}' \cdot \tilde{y} \) belongs to \( \mathcal{P}^\mathcal{F}(N'', N)_{\nu, V} \) and it follows easily from the very definition of \( \rho_\nu \) in 2.5.35 that the element \( s'' = s''_\nu \) in \( \mathcal{P}^\mathcal{F}(V) \) fulfills
\[
\rho_\nu(s'') = \rho_\nu((\tilde{y}')_\nu \cdot (\tilde{y})_\nu) = \rho_\nu(s') \rho_\nu(s)
\]
which proves our claim.

We are ready to consider any pair of subgroups \( V \) and \( V' \) in \( \mathfrak{A} - 2\mathfrak{F} \). We clearly have \( N = N_\nu(V) \neq V \) and it follows from [6, Proposition 2.7] that there is an \( \mathcal{F} \)-morphism \( \nu : N \to P \) such that \( \nu(V) \) is fully normalized in \( \mathcal{F} \); moreover, we choose \( \tilde{n} \in \mathcal{P}^\mathcal{F}(\nu(N), N) \) lifting \( \mathcal{F} \)-isomorphism \( \nu_* \) from \( N \) to \( \nu(N) \) determined by \( \nu \). That is to say, we may assume that

2.5.50 There is a pair \((N, \tilde{n})\) formed by a subgroup \( N \) of \( P \) which strictly contains and normalizes \( V \), and by an element \( \tilde{n} \) in \( \mathcal{P}^\mathcal{F}(\nu(N), N) \) lifting \( \nu_* \) for an \( \mathcal{F} \)-morphism \( \nu : N \to P \) such that \( \nu(V) \) is fully normalized in \( \mathcal{F} \).

We denote by \( \mathfrak{E}(V) \) the set of such pairs and we write \( \tilde{n} \) instead of \((N, \tilde{n})\), setting \( \tilde{n}N = (\tilde{n}_\nu(N), N) \), \( \tilde{n}V = (\tilde{n}_\nu(N), \tilde{n}(\tilde{n})) \) and \( n = \rho_\nu(\tilde{n}) \). Then, for any \( \mathcal{P}^\mathcal{F} \)-morphism \( \tilde{x} : V \to V' \), we consider pairs \((N, \tilde{n}) \in \mathfrak{E}(V) \) and \((N', \tilde{n}') \in \mathfrak{E}(V') \) and, since \( \tilde{n}V \) and \( \tilde{n}'V' \) are both fully normalized in \( \mathcal{F} \), we can define
\[
\rho_{\tilde{x}, V}^N(\tilde{x}) = \rho_{\tilde{x}, V}^{\tilde{n}N, \tilde{n}'N'}(\tilde{x}) = \rho_{\tilde{x}, V}^{\tilde{n}N, \tilde{n}'N'}(\tilde{x}) \cdot \rho_{\tilde{r}_{\tilde{n}, V}^N, \tilde{r}_{\tilde{n}', V}^{\tilde{n}'N'}}(\tilde{n}, \tilde{n}', \tilde{r}_{\tilde{n}, V}^N, \tilde{r}_{\tilde{n}', V}^{\tilde{n}'N'})(\tilde{n}, \tilde{n}', \tilde{n})
\]

This definition is independent of our choices; indeed, for another pair \((\tilde{N}, \tilde{n}) \in \mathfrak{E}(V) \), setting \( \tilde{N} = \langle N, \tilde{N} \rangle \) and considering a new \( \mathcal{F} \)-morphism \( \psi : \tilde{N} \to P \) such that \( \psi(V) \) is fully normalized in \( \mathcal{F} \), we can obtain a third pair \((\tilde{N}, \tilde{m}) \in \mathfrak{E}(V) \); then \( \tilde{r}_{\tilde{m}, \tilde{N}, \tilde{N}}^{\tilde{n}, \tilde{N}}(\tilde{n}, \tilde{N}) \cdot \tilde{r}_{\tilde{m}, \tilde{N}, \tilde{N}}^{\tilde{n}, \tilde{N}}(\tilde{m}, \tilde{N}) \) respectively belong to \( \mathcal{P}^\mathcal{F}(\tilde{n}N, \tilde{n}'N) \) and to \( \mathcal{P}^\mathcal{F}(\tilde{m}N, \tilde{m}'N) \); in particular, since \( \tilde{n}V \), \( \tilde{n}'V \) and \( \tilde{m}V \) are fully normalized in \( \mathcal{F} \), we get
\[
\rho_{\tilde{x}, V}^N(\tilde{x}) = \rho_{\tilde{x}, V}^{\tilde{n}N, \tilde{n}'N'}(\tilde{x}) = \rho_{\tilde{x}, V}^{\tilde{n}N, \tilde{n}'N'}(\tilde{x}) \cdot \rho_{\tilde{r}_{\tilde{n}, V}^N, \tilde{r}_{\tilde{n}', V}^{\tilde{n}'N'}}(\tilde{n}, \tilde{n}', \tilde{r}_{\tilde{n}, V}^N, \tilde{r}_{\tilde{n}', V}^{\tilde{n}'N'})(\tilde{n}, \tilde{n}', \tilde{n})
\]

Symmetrically, we can replace \((\tilde{N}', \tilde{n}') \) for another pair \((\tilde{N}', \tilde{n}') \in \mathfrak{E}(V') \).
Moreover, equality 2.5.41 still holds with this general definition; more generally, for any pair of subgroups $Q$ and $Q'$ of $P$ strictly containing $V$ and $V'$, we claim that

$$\rho^x_{V',V} (r_{Q',Q}^Q (\hat{x})) = r_{V',V}^Q (\rho^y (\hat{x}))$$  \hspace{1cm} 2.5.53

for any $\hat{x}$ in $P^x(Q', Q)_{V', V}$; indeed, setting $R = N_Q(V)$, $R' = N_{Q'}(V')$ and $\hat{y} = r_{R,R}^{Q,Q} (\hat{x})$, it is clear that we have pairs $(R, \hat{n})$ in $\mathcal{E}(V)$ and $(R', \hat{n}')$ in $\mathcal{E}(V')$, and by the very definition 2.5.51 and by equality 2.5.41 we have

$$\rho^x_{V', V} (r_{Q', Q}^Q (\hat{x})) = r_{R', R}^{Q',Q} (\rho^y (\hat{x})).$$  \hspace{1cm} 2.5.54.

Once again, for another $V'' \in \mathfrak{X} - \mathfrak{Y}$, setting $N'' = N_P(V'')$ and considering a $\hat{P}^{x,y}$-morphism $\hat{x'} : V' \to V''$, we claim that

$$\rho^x_{V',V} (\hat{x}') = \rho^y_{V'',V} (\hat{x}') \rho^x_{V',V} (\hat{x})$$  \hspace{1cm} 2.5.55;

indeed, considering a pair $(N'', \hat{n}'')$ in $\mathcal{E}(V'')$ and setting $\hat{x}'' = \hat{x}' \cdot \hat{x}$, from the very definition 2.5.51 we get

$$\rho^x_{V',V} (\hat{x}) =$$

$$r_{a_{\hat{N},N},V'} (n')^{-1} \rho^x_{a_{\hat{N},N},a_{\hat{N},N}} (r_{a_{\hat{N},N},V'} (\hat{n}')) \cdot r_{a_{\hat{N},N},V'} (\hat{n}')).$$

$$\rho^y_{V'',V} (\hat{x}') =$$

$$r_{a_{\hat{N}',N''},V''} (n'')^{-1} \rho^x_{a_{\hat{N}',N''},a_{\hat{N}',N''}} (r_{a_{\hat{N}',N''},V''} (\hat{n}'')) \cdot r_{a_{\hat{N}',N''},V''} (\hat{n}''))$$

$$\rho^x_{V',V} (\hat{x}'') =$$

$$r_{a_{\hat{N}'',N''},V''} (n'')^{-1} \rho^x_{a_{\hat{N}'',N''},a_{\hat{N}'',N''}} (r_{a_{\hat{N}'',N''},V''} (\hat{n}'')) \cdot r_{a_{\hat{N}'',N''},V''} (\hat{n}''))$$

and it follows from equality 2.5.44 that the composition of the first and the second equalities above coincides with the third one.

At this point, we are able to complete the definition of the $\mathcal{X}$-locality functor $\rho^x : \mathcal{P}^x \to \mathcal{M}^x$ lifting $\hat{\rho}^x$. For any $\mathcal{P}^x$-morphism $\hat{x} : R \to Q$ either $R$ belongs to $\mathfrak{Y}$ and we simply set $\rho^x (\hat{x}) = \rho^y (\hat{x})$, or $R$ belongs to $\mathfrak{X} - \mathfrak{Y}$ and,
denoting by $R_s$ the image of $R$ in $Q$ and by $\hat{x}_*: R \cong R_s$ the $\mathcal{P}^x$-isomorphism determined by $\hat{x}$, we set $\rho^x(\hat{x}) = i_{R_s}^Q \cdot \rho^x(\hat{x}_*)$ (cf. 2.5.50); in both cases, from 2.5.36 we get

$$\pi \circ \rho^x = \pi \quad \text{and} \quad \rho^x \circ \hat{\tau} = \tau$$ 2.5.57.

Note that if $Q$ contains $R$ then we have $\rho^x(i_{R}^Q) = i_{R_s}^Q$.

Then, we claim that for another $\mathcal{P}^x$-morphism $\hat{y}: T \to R$ we have

$$\rho^x(\hat{x} \cdot \hat{y}) = \rho^x(\hat{x}) \cdot \rho^x(\hat{y})$$ 2.5.58;

indeed, if $T$ belongs to $\mathcal{Y}$ then we just have

$$\rho^x(\hat{x} \cdot \hat{y}) = \rho^y(\hat{x} \cdot \hat{y}) = \rho^x(\hat{x}) \cdot \rho^y(\hat{y}) = \rho^x(\hat{x}) \cdot \rho^x(\hat{y})$$ 2.5.59.

If $R$ belongs to $\mathcal{X} - \mathcal{Y}$ then $\hat{y}$ is a $\mathcal{P}^x$-isomorphism and, with the notation above applied to the $\mathcal{P}^x$-morphism $\hat{x} \cdot \hat{y}: T \to Q$, we have $T_s = R_s$ and moreover $(\hat{x} \cdot \hat{y})_* = \hat{x}_* \cdot \hat{y}$; in this case, from equality 2.5.44 we get

$$\rho^x(\hat{x} \cdot \hat{y}) = i_{R_s}^Q \cdot \rho^x(\hat{x}_* \cdot \hat{y}_*) = i_{R_s}^Q \cdot \rho^x(\hat{x}_*) \cdot \rho^x(\hat{y}_*)$$

$$= i_{R_s}^Q \cdot \rho^x(\hat{x}_*) = i_{R_s}^Q \cdot \rho^x(\hat{y}_*)$$ 2.5.60.

Finally, assume that $T \in \mathcal{X} - \mathcal{Y}$ and $R \in \mathcal{Y}$, denote by $T_s$ and $T_{ss} \subset R_s$ the respective images of $T$ in $R$ and $Q$, and by $\hat{x}_{ss}: T_s \to T_{ss}$ the $\mathcal{P}^x$-isomorphism fulfilling (cf. 1.5.1)

$$\hat{x}_* \cdot i_{R_s}^{T_s} = i_{R_s}^{T_{ss}} \cdot \hat{x}_{ss}$$ 2.5.61;

then, it follows from 2.5.38 and 2.5.55 that we have

$$\rho^x(\hat{x} \cdot \hat{y}) = i_{R_s}^Q \cdot \rho^x(\hat{x}_{ss} \cdot \hat{y}_*) = i_{R_s}^Q \cdot \rho^x(\hat{x}_{ss}) \cdot \rho^x(\hat{y}_*)$$

$$= i_{R_s}^Q \cdot \rho^x(\hat{x}_{ss}) \cdot \rho^y(\hat{y}_*)$$

$$= i_{R_s}^Q \cdot \rho^y(\hat{x}_*) \cdot \rho^y(\hat{y}_*)$$

$$= i_{R_s}^Q \cdot \rho^y(\hat{x}_*) \cdot \rho^y(\hat{y}_*)$$ 2.5.62.

This proves the existence of the $\mathcal{F}^x$-locality functor $\rho^x: \mathcal{P}^x \to \mathcal{M}^x$ and therefore the existence of the $\mathcal{F}^x$-locality functor $\sigma^x: \mathcal{P}^x \to \mathcal{L}^x$ lifting $\hat{\sigma}^x$.

It remains to prove the uniqueness; this proof follows the same steps as the proof of the existence. Let $\sigma^x$ and $\tilde{\sigma}^x$ two $\mathcal{F}^x$-locality functors from $\mathcal{P}^x$ to $\mathcal{L}^x$; it is clear that they induce two $\mathcal{F}^x$-locality functors $\tilde{\sigma}^x$ and $\tilde{\sigma}^x$ from $\mathcal{P}^x$ to $\mathcal{L}^x$; hence, by induction on the size of $\mathcal{L}$, $\tilde{\sigma}^x$ and
\(\sigma' \circ \pi = \tau\) for a suitable element \(x\) of \(P\) which can be lifted to \(z \in \text{Ker}(\pi)\); moreover, according to [5, Proposition 17.10], this element centralizes \(\pi(P)\) and, as above, it determines another \(\mathcal{F}\)-locality functor naturally \(\mathcal{F}\)-isomorphic to \(\sigma' \circ \pi\). In conclusion, up to replacing \(\sigma' \circ \pi\) by a naturally \(\mathcal{F}\)-isomorphic \(\mathcal{F}\)-locality functor, we may assume that \(\tilde{\sigma} = \sigma' \circ \pi\).

Now, with the choice of \(U\) above, \(\sigma^x\) and \(\sigma' \circ \pi\) determine two group homomorphisms

\[
\sigma^x : \mathcal{P}^x(U) \rightarrow \mathcal{L}(U) \quad \text{and} \quad \sigma' \circ \pi : \mathcal{P}^x(U) \rightarrow \mathcal{L}(U) \tag{2.5.63}
\]

fulfilling \(\sigma^x \circ \tilde{\sigma} = \tau\) and both lifting \(\tilde{\sigma} : \) in particular, it follows from Lemma 1.9 that \(\sigma^x\) and \(\sigma' \circ \pi\) are conjugate to each other by an element of \(\mathcal{K}(U) \subset \mathcal{M}(U)\) centralizing \(\tau\). If \(|\mathcal{X}| = 1\) then \(U = P\) and the existence of this element already proves the uniqueness.

Assuming that \(|\mathcal{X}| \neq 1\), it is clear that \(\sigma^x\) and \(\sigma' \circ \pi\) determine two \(\mathcal{F}(\mathcal{P})\)-locality functors

\[
\sigma^\mathcal{P} : \mathcal{P}^\mathcal{P}(U) \rightarrow \mathcal{L}(U) \quad \text{and} \quad \sigma' \circ \pi : \mathcal{P}^\mathcal{P}(U) \rightarrow \mathcal{L}(U) \tag{2.5.64}
\]

and that, with obvious notation, we have \(\tilde{\sigma} = \sigma' \circ \pi\) since we already have \(\tilde{\sigma} = \sigma' \circ \pi\). But, arguing by induction on \(|\mathcal{X}|\), \(\sigma^\mathcal{P}\) and \(\sigma' \circ \pi\) are naturally \(\mathcal{F}(\mathcal{P})\)-isomorphic; according to 1.7.3, such a natural \(\mathcal{F}(\mathcal{P})\)-isomorphism is determined by an element \(z \in \text{Ker}(\pi)\); since \(\tilde{\sigma}^\mathcal{P} = \sigma' \circ \pi\), the image \(\tilde{z} \in z\) of \(z\) in \(\mathcal{L}(\mathcal{P})\) determines a natural \(\mathcal{F}(\mathcal{P})\)-automorphism of this \(\mathcal{F}(\mathcal{P})\)-locality functor.

Thus, arguing as above, we may assume that \(z\) belongs to \(\mathcal{K}(\mathcal{P}) \subset \text{Ker}(\pi)\) and still determines the identity on \(\tilde{\sigma} : \) in conclusion, up to replacing \(\sigma' \circ \pi\) by a naturally \(\mathcal{F}\)-isomorphic \(\mathcal{F}\)-locality functor, we may assume that \(\tilde{\sigma} = \sigma' \circ \pi\) and \(\tilde{\sigma} = \sigma' \circ \pi\).

In particular, for any \(\mathcal{P}^x\)-morphism \(\hat{x} : R \rightarrow Q\), we have

\[
\sigma' \circ \pi(\hat{x}) = \sigma^x(\hat{x}) \cdot \theta^x_{\hat{x}} \tag{2.5.65}
\]

for a suitable element \(\theta^x_{\hat{x}} \in \mathcal{K}(R)\); actually, since \(\sigma^x\) and \(\sigma' \circ \pi\) are \(\mathcal{P}^x\)-locality functors, it is easily checked that \(\theta^x_{\hat{x}}\) only depends on the class \(\hat{x}\) of \(\hat{x}\) in \(\mathcal{P}(Q, R) = \mathcal{F}(Q, R)\) and we can write \(\theta^x_{\hat{x}}\) instead of \(\theta^x_{\hat{x}}\); moreover, for another \(\mathcal{P}^x\)-morphism \(\hat{y} : T \rightarrow R\), we get

\[
\sigma' \circ \pi(\hat{x} \cdot \hat{y}) = \sigma^x(\hat{x}) \cdot \sigma^x(\hat{y}) = \sigma^x(\hat{x}) \cdot \theta^x_{\hat{x}} \cdot \sigma^x(\hat{y}) \cdot \theta^x_{\hat{y}} = \sigma^x(\hat{x} \cdot \hat{y}) \cdot (\theta^x_{\hat{x}} \sigma^x(\hat{y}) \cdot \theta^x_{\hat{y}}) \cdot \theta^x_{\hat{x} \cdot \hat{y}} \tag{2.5.66}
\]
That is to say, recall that the contravariant functor \( K \) determines a new contravariant functor (cf. 1.5)

\[ \tilde{K} : \tilde{L} \rightarrow \text{Ab} \subset \text{Gr} \]

let us denote by \( \tilde{K}^{x-\eta} : \tilde{L}^{x} \rightarrow \mathbb{Z}\text{-mod} \) the contravariant functor which vanishes over \( \mathfrak{Q} \) and coincides with \( \tilde{K} \) over \( \mathfrak{X} - \mathfrak{Q} \), and by \( \tilde{\sigma}^{x} : \tilde{P}^{x} = \tilde{F}^{x} \rightarrow \tilde{L}^{x} \) the functor induced by \( \sigma^{x} \); then, since \( \sigma^{\eta} = \sigma^{x} \) and \( \tilde{\sigma}^{x} = \tilde{\sigma}^{x} \), the correspondence \( \theta^{x} \) sending \( \hat{x} \) to \( \theta_{\hat{x}}^{x} \) defines an element of

\[
\mathbb{G}^{1}(\tilde{F}^{x}, \tilde{K}^{x-\eta} \circ \tilde{\sigma}^{x}) = \prod_{q}(\tilde{K}^{x-\eta} \circ \tilde{\sigma}^{x})(q(0))
\]

where \( q : \Delta_{1} \rightarrow \tilde{F}^{x} \) runs over the “functors” from \( \Delta_{1} = \{0,1\} \) to \( \tilde{F}^{x} \) or, equivalently, over the \( \tilde{F}^{x} \)-morphisms \([8, 2.3]\), and, according to equality 2.5.66, \( \theta^{x} \) is a 1-cocycle; namely, in additive notation, we get

\[
0 = ((\tilde{K}^{x-\eta} \circ \tilde{\sigma}^{x})(\hat{y}))(\theta_{\hat{x}}^{x}) - \theta_{\hat{x} \hat{y}}^{x} + \theta_{\hat{y}}^{x}
\]

Now, in order to prove that \( \sigma^{x} \) and \( \sigma^{\prime x} \) are naturally \( \tilde{F}^{x} \)-isomorphic, it suffices to prove that \( \theta^{x} \) is a 1-coboundary; in other words, it suffices to prove that, for any \( R \in \mathfrak{X} \), there exists \( z_{R} \in (\tilde{K}^{x-\eta} \circ \tilde{\sigma}^{x})(R) \subset \text{Ker}(\pi_{R}) \) in such a way that

\[
\theta_{\hat{x}}^{x} = ((\tilde{K}^{x-\eta} \circ \tilde{\sigma}^{x})(\hat{y}))(z_{R}) - z_{R}
\]

indeed, in this case, we have

\[
\sigma^{\prime x}(\hat{x}) = \sigma^{x}(\hat{x}) \cdot (z_{R}) \cdot (z_{R})^{-1} = z_{R} \cdot \sigma^{x}(\hat{x}) \cdot (z_{R})^{-1}
\]

so that the family \( \{z_{R}\}_{R \in \mathfrak{X}} \) determines a natural \( \tilde{F}^{x} \)-isomorphism \( \sigma^{x} \cong \sigma^{\prime x} \).

On the other hand, the functor \( \tilde{\sigma}^{x} : \tilde{F}^{x} \rightarrow \tilde{L}^{x} \) induces a \( \mathbb{Z}_{(p)}\tilde{F}(U) \)-module structure on \( K(U) \) and, denoting by \( d_{\tilde{\sigma}^{x}} : \tilde{F}(U) \rightarrow \mathbb{Z}_{(p)}\text{-mod} \) the contravariant functor determined by this module, which sends \( \{\text{id}_{U}\} \) to \( K(U) \) and vanishes elsewhere, it follows from Proposition 1.13 applied to the functor \( \tilde{K}^{x-\eta} \circ \tilde{\sigma}^{x} \) that, for any \( n \geq 1 \), we have a canonical group isomorphism

\[
\mathbb{H}^{n}(\tilde{F}^{x}, \tilde{K}^{x-\eta} \circ \tilde{\sigma}^{x}) \cong \mathbb{H}^{n}(\tilde{F}(U), d_{\tilde{\sigma}^{x}})
\]

2.5.72;

consequently, it suffices to prove that the restriction \( \theta_{U}^{x} \) of \( \theta^{x} \) to \( \tilde{F}(U) \) is an 1-coboundary.

But, since \( \tilde{F}(U) = \tilde{P}^{x}(U) \), any \( \tilde{F}(U) \)-morphism \( \tilde{\varphi} : \tilde{R} \rightarrow \tilde{Q} \) comes from some element \( \hat{x} \in \tilde{P}^{x}(U) \) fulfilling \( \tilde{\pi}_{\tilde{U}}(\hat{x}) = \varphi \) (cf. 1.12) and, since there is \( z \in K(U) \) such that \( \sigma^{\prime x}_{\tilde{U}} = z \cdot \sigma^{x}_{\tilde{U}} \) (cf. 2.5.63), we get

\[
\theta_{\hat{x}}^{x} = \sigma_{\tilde{U}}^{x}(\hat{x})^{-1} \cdot \sigma_{\tilde{U}}^{\prime x}(\hat{x}) = z \cdot \sigma_{\tilde{U}}^{x}(\hat{x})^{-1}
\]

2.5.73;
moreover, if $\tilde{R} \neq \{\tilde{id}_U\}$ then we are assuming that $\sigma^\varphi(\tilde{x}) = \sigma^\varphi(\hat{x})$ and therefore $z$ and $\sigma^\varphi_\varphi(\hat{x})$ centralize each other; thus, in additive notation we still get
\[
\theta^X_{\tilde{x}} = (d_{\tilde{id}_U}(\tilde{x}))(z) - z
\]
proving that $\theta^\varphi_{\tilde{x}}$ is an 1-coboundary. We are done.

2.6. It is clear that Theorem 2.5 supplies a direct proof for the uniqueness of the perfect $\mathcal{F}^\varphi$-locality; indeed, note that any two perfect $\mathcal{F}^\varphi$-localities $(\hat{\tau}^\varphi, \hat{\pi}^\varphi)$ and $(\hat{\tau}'^\varphi, \hat{\pi}'^\varphi)$ can be extended to $p$-coherent $\mathcal{F}$-localities $(\tau, \mathcal{L}, \pi)$ and $(\tau', \mathcal{L}', \pi')$ by setting
\[
\mathcal{L}(Q, R) = \mathcal{F}(Q, R) = \mathcal{L}'(Q, R)
\]
for any pair of subgroups $Q$ and $R$ of $P$ such that $R \notin X$, and therefore from Theorem 2.5 we get $\mathcal{F}^\varphi$-locality functors
\[
\mathcal{P}^x \rightarrow \mathcal{P}'^x \quad \text{and} \quad \mathcal{P}'^x \rightarrow \mathcal{P}^x
\]
such that both compositions are naturally $\mathcal{F}^\varphi$-isomorphic to the corresponding identity functors.

2.7. Theorem 2.5 also provides a dramatic simplification for the proof of the existence of the perfect $\mathcal{F}^\varphi$-locality in [7, §6]. Indeed, it follows from this theorem that in [7, diagram 6.1.2] there exists a $\mathcal{F}^\varphi$-locality functor from $\mathcal{P}^\varphi$ to $\mathcal{M}^\varphi$, which actually has to be a section of the vertical left-hand arrow in this diagram; then, the “image” $\hat{\mathcal{P}}^\varphi$ of this functor is an $\mathcal{F}^\varphi$-sublocality of $\mathcal{M}^\varphi$ isomorphic to $\mathcal{P}^\varphi$, which allows the definition of the $\mathcal{F}^\varphi$-locality $\hat{\mathcal{P}}^x$ as in [7, 6.3]; finally, the proof of the existence of the $\mathcal{F}^\varphi$-locality $\mathcal{P}^x$ is completed in [7, 6.4].

3. Functoriality of the perfect $\mathcal{F}$-locality

3.1. With the notation in 2.1 above, let us consider the perfect $\mathcal{F}$-locality $(\tau, \mathcal{P}, \pi)$. Let $P'$ be a second finite $p$-group, $\mathcal{F}'$ a Frobenius $P'$-category and $(\tau', \mathcal{P}', \pi')$ the corresponding perfect $\mathcal{F}'$-locality. If $\alpha : P' \rightarrow P$ is an $(\mathcal{F}', \mathcal{F})$-functorial group homomorphism [6, 12.1], recall that we have a so-called Frobenius functor $f_\alpha : \mathcal{F}' \rightarrow \mathcal{F}$ [6, 12.1], and denote by $t_\alpha : T_P \rightarrow T_P$ the obvious functor induced by $\alpha$.

3.2. In [7, Section 9], considering suitable quotients $\mathcal{P}$ and $\mathcal{P}'$ of $\mathcal{P}$ and $\mathcal{P}'$, we give a quite involved proof of the existence of a unique isomorphism class of functors $\tilde{g}_\alpha : \mathcal{P}' \rightarrow \mathcal{P}$ fulfilling
\[
\tilde{\tau} \circ t_\alpha = \tilde{g}_\alpha \circ \tau' \quad \text{and} \quad \pi \circ \tilde{g}_\alpha = f_\alpha \circ \pi'
\]
In this section we show that such an statement is an immediate consequence of Theorem 2.5 above and that, as a matter of fact, we can deal with the whole perfect $\mathcal{F}$- and $\mathcal{F}'$-localities without taking quotients. More generally,
choosing a nonempty set $X$ of subgroups of $P$ as in 2.3 above, we work in the relative context still considering the perfect $F^x$-locality $(\tau^x, \mathcal{P}^x, \pi^x)$.

3.3. Assuming that $\alpha(P')$ belongs to $X$, it is clear that the set $X'$ of subgroups $Q'$ of $P'$ such that $\alpha(Q')$ belongs to $X$ still fulfills the condition in 2.3 above; in particular, we can consider the perfect $F^{x'}$-locality $(\tau^{x'}, \mathcal{P}^{x'}, \pi^{x'})$ and the functors $f_\alpha$ and $t_\alpha$ clearly induce the corresponding functors

$$f_\alpha^x : F^{x'} \to F^x \quad \text{and} \quad t_\alpha^x : T^{x'} \to T^x$$

3.3.1. We claim that there is a unique isomorphism class of functors $g_\alpha^x : \mathcal{P}^{x'} \to \mathcal{P}^x$ fulfilling

$$\tau^x \circ t_\alpha^x = g_\alpha^x \circ \tau^{x'} \quad \text{and} \quad \pi^x \circ g_\alpha^x = f_\alpha^x \circ \pi^{x'}$$

3.3.2. In particular, if $P''$ is a third finite $p$-group, $\mathcal{F}''$ a Frobenius $P''$-category, $(\tau'', \mathcal{P}'', \pi'')$ the perfect $\mathcal{F}''$-locality and $\alpha' : P'' \to P'$ an $(\mathcal{F}'', \mathcal{F}')$-functorial group homomorphism such that $\alpha'(P'')$ belongs to $X'$, denoting by $\mathcal{X}''$ the set of subgroups $Q''$ of $P''$ such that $\alpha'(Q'')$ belongs to $X'$, it is clear that the functors $g_\alpha^x \circ g_{\alpha'}^{x'}$ and $g_{\alpha\circ\alpha'}^{x''}$ from $\mathcal{P}^{x''}$ to $\mathcal{P}^x$ are naturally isomorphic.

3.4. In any case, we can consider the pull-back

$$\begin{array}{ccc}
\mathcal{F}'' & \xrightarrow{f_\alpha} & \mathcal{F} \\
\uparrow \tau' & & \uparrow \tau \\
\text{Res}_{f_\alpha}(\mathcal{P}) & \xrightarrow{t_\alpha} & \mathcal{P}
\end{array}$$

3.4.1; but, according to condition 1.4.1, we have

$$f_\alpha \circ (\pi' \circ \tau') = (\pi \circ \tau) \circ t_\alpha$$

3.4.2 and therefore $\pi' \circ \tau' : T_{P'} \to \mathcal{F}'$ and $\tau \circ t_\alpha : T_{P'} \to \mathcal{P}$ determine a functor

$$\tau_\alpha : T_{P'} \to \text{Res}_{f_\alpha}(\mathcal{P})$$

3.4.3 fulfilling $\pi_\alpha \circ \tau_\alpha = \pi' \circ \tau'$ and $t_\alpha \circ \tau_\alpha = \tau \circ t_\alpha$; then, it is easily checked that the triple $(\tau_\alpha, \text{Res}_{f_\alpha}(\mathcal{P}), \pi_\alpha)$ is a $p$-coherent $\mathcal{F}'$-locality. In particular, with obvious notation, we get the $p$-coherent $\mathcal{F}^{x'}$-locality $(\tau_\alpha^{x'}, \text{Res}_{f_\alpha}(\mathcal{P})^{x'}, \pi^{x'})$ and an immediate application of Theorem 2.5 yields the following result, which proves our claim.

**Corollary 3.5.** With the notation above, there is a unique natural $\mathcal{F}^{x'}$-isomorphism class of $\mathcal{F}^{x'}$-locality functors

$$b_\alpha^{x'} : \mathcal{P}^{x'} \to \text{Res}_{f_\alpha}(\mathcal{P})^{x'}$$

3.5.1. In particular, we have a natural isomorphism

$$\text{Res}_{f_\alpha}(b_\alpha^{x'}) \circ b_{\alpha'}^{x''} \cong b_{\alpha\circ\alpha'}^{x''}$$

3.5.2.
References

[1] Carles Broto, Ran Levi and Bob Oliver, *The homotopy theory of fusion systems*, Journal of Amer. Math. Soc. 16(2003), 779-856.

[2] Henri Cartan & Samuel Eilenberg, “*Homological Algebra*”, Princeton Math. 19, 1956, Princeton University Press

[3] Andrew Chermak. *Fusion systems and localities*, Acta Mathematica, 211(2013), 47-139.

[4] George Glauberman & Justin Lynd, *Control of fixed points and existence and uniqueness of centric systems*, arxiv.org/abs/1506.01307.

[5] Bob Oliver. *Existence and Uniqueness of Linking Systems: Chermak’s proof via obstruction theory*, Acta Mathematica, 211(2013), 141-175.

[6] Llúís Puig, “*Frobenius categories versus Brauer blocks*”, Progress in Math. 274(2009), Birkhäuser, Basel.

[7] Llúís Puig, *Existence, uniqueness and functoriality of the perfect locality over a Frobenius P-category*, arxiv.org/abs/1207.0066, submitted to Algebra Colloquium.

[8] Llúís Puig, *A criterion on trivial homotopy*, arxiv.org/abs/1308.3765, submitted to Journal of Algebra.