Delayed Feedback Control on a Class of Generalized Gyroscope Systems under Parametric Excitation

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Abstract

The nonlinear dynamics of the parametrically excited vibrations of a class of generalized gyroscope systems under delayed feedback control is investigated by the averaging method and simulations in this paper. The influence of feedback control on the stability of the trivial solution and the amplitude of the periodic vibrations is presented based on Routh-Hurwitz criterion and the Levenberg-Marquardt method respectively. It is shown that the stability of the trivial solution can be varied when feedback control and time delay are employed. The amplitudes of periodic solutions can also be modulated greatly by feedback gain and time delay. However, the influence of time delay on amplitudes is periodic. The simulations obtained by numerically integrating the original system are in good agreement with the analytical results.

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Keywords: generalized gyroscope systems; the parametrically excited vibrations; feedback control; time delay

1. Introduction

As an interesting problem in classical mechanics, the motion and the wide variety of applications of various gyroscopes have been attracting numerous researchers’ attention for decades. Using parametric stabilization near a combination resonance, Mcdonald et al. [1] examined a class of weakly damped
gyroscopic system with certain symmetries, which is subject to small periodic excitations. Feng et al. [2] studied the dynamic characteristics of a vibrating wheel micro-gyroscope and obtained a time invariant model for the micro-gyroscope which is much simpler to use in the design of micro-gyroscopes. Chen et al. [3] investigated the dynamic behaviour of a dissipative gyroscope mounted on a vibrating base qualitatively by the centre manifold theorem and the normal form theorem. Tsai et al. [4] presented a nonlinear feedback control system of the single-axis gyroscope governed by a servo motor. The closed-loop feedback is designed to account for the gimbal nutation as the gyroscope undergoing an angular velocity in the perpendicular directions. In another paper [5], they explored the dynamics of a nonlinear micro-machined gyroscope, such as spring hardening/softening, hysteresis, typical resonances and chaos theoretically and numerically. Ge et al. [6] considered the nonlinear motion of a symmetric gyro mounted on a vibrating base with particular emphasis on its long term dynamic behavior for a wide range of parameters. Tong et al. [7] discussed the motion of a symmetric gyro which is subjected to a harmonic vertical base excitation without taking into account the damping effect. McDonald [8] analyzed the local and global bifurcations for parametrically excited gyroscopic systems, using rotating shafts and pipes conveying fluid as motivating examples.

Time-delayed feedback control has become very popular among researchers focusing on changes in stability of response, undesirable bifurcations, high-amplitude vibrations, quasi-periodic motion and chaotic behavior. Hu et al. [9] studied the primary resonance and 1/3 sub-harmonic resonance of a harmonically forced Duffing oscillator under state feedback with a time delay. Tamaševičius et al [10] analyzed the delayed feedback control of periodic orbits without torsion in non-autonomous chaotic systems theoretically and experimentally. Xu et al [11] studied the stability, bifurcation and chaos of a delayed oscillator with negative damping and delayed feedback control. Qian and Tang [12] considered time-delayed feedback controller for the instability of a nonlinear beam under moving load. Nonlinear analysis of time-delay position feedback control of container cranes was given by modelling the crane-hoist-payload assembly as a double pendulum [13]. Wang et al [14] concerned the non-resonant response of a maglev system by employing centre manifold reduction and the method of multiple scales. Attilio Maccari et al [15] proposed an asymptotic perturbation method based on Fourier expansion and time rescaling to explore the state feedback control with a time delay of two coupled nonlinear oscillators with a parametric excitation. Erzgräber et al [16] demonstrated a global view on a nonlinear oscillator subject to time-delayed feedback control. Alhazza et al [17] presented a single-input single-output multimode delayed-feedback control methodology to mitigate the free vibrations of a flexible cantilever beam. Xu et al [18] discussed the effect of time delays on the saturation control of first-mode vibration of a stainless-steel beam. Ashesh Saha [19] investigated the control of friction-driven oscillations by time-delayed feedback based on a simplified model of a spring- mass-damper system with the mass resting on a friction belt moving with a constant velocity.

In this paper, the dynamical behavior of the system described by Eq. (1), with 1:1 internal resonance and only one oscillator parametrically excited, is presented to investigate the effects of delayed feedback control and parametric excitation on response of such a nonlinearly coupled system when principal parametric resonance exist.

\[
\begin{align*}
\dot{x} + \omega_1^2 x &= \varepsilon \left( -c_1 \dot{x} - \beta_1 \dot{y} - \beta_2 x^3 + F_0 x \cos \omega t + g_1 x(t - \tau) + g_2 \dot{x}(t - \tau) \right) \\
\dot{y} + \omega_2^2 y &= \varepsilon \left( -c_2 \dot{y} + \beta_3 \dot{x} - \beta_4 y^3 + g_3 y(t - \tau) + g_4 \dot{y}(t - \tau) \right).
\end{align*}
\]

where \( \varepsilon \) is a small positive parameter, \( \tau \) is positive time delay.

2. The modulation equations
For the case of principal parametric resonance and internal resonance, detuning parameters $\sigma_1$ and $\sigma_2$ are introduced such that

$$(\omega/2)^2 = \omega_1^2 + \varepsilon \sigma_1, \quad \omega_2^2 = \omega_2^2 + \varepsilon \sigma_2.$$  

By the averaging method, the approximate solutions of Eq.(1) are supposed in the following form

$$x = A_1 \cos(\omega t/2 + \theta_1), \quad y = A_2 \cos(\omega t/2 + \theta_2)$$

where the amplitudes $A_i$ and the phases $\theta_i (i = 1, 2)$ are time-dependent and governed by

$$-2\omega A_1 = c_1 A_1 \omega + F_d A_1 \sin 2\theta_1 + \beta_1 A_1 \omega \cos(\theta_1 - \theta_2) + 2g_1 A_1 \sin \frac{\omega}{2} \omega t - g_2 \omega A_1 \cos \frac{1}{2} \omega \tau$$

$$-4\omega A_2 = -3\beta_2 A_1 + 4\sigma_1 A_1 + 2g_2 \omega A_1 \sin \frac{1}{2} \omega \tau + 4g_1 A_1 \cos \frac{1}{2} \omega \tau + 2F_d A_1 \cos 2\theta_1 - 2\beta_1 A_2 \omega \cos(\theta_1 - \theta_2)$$

$$-2\omega A_2 = \omega c_2 A_2 - \omega A_1 \beta_1 \cos(\theta_1 - \theta_2) + 2g_1 A_1 \sin \frac{1}{2} \omega \tau - g_4 \omega A_2 \cos \frac{1}{2} \omega \tau$$

$$-4\omega A_2 \beta_2 = 2g_4 A_2 \omega \sin \frac{1}{2} \omega \tau - 2\omega A_1 \beta_1 \sin(\theta_1 - \theta_2) + 4g_3 A_2 \cos \frac{1}{2} \omega \tau - 4A_2 \sigma_2 - 3\beta_4 A_2^2 + 4A_2 \sigma_1$$

where the dot indicates differentiation with respect to $T = \varepsilon t$. Equation (4) is an autonomous dynamical system, the fixed points of which correspond to the periodic motions of the system described by Eq.(1). It is clearly seen that the addition of feedback control modifies the modulation equations.

3. Stability of the trivial solution

In the investigation of stability of the trivial solution, it is necessary to express Eq.(4) in Cartesian form. For this purpose, the first-order approximation can alternatively be expressed in the form

$$q_i = A_i \cos \theta_i, q_2 = A_i \sin \theta_i, q_4 = A_2 \sin \theta_2.$$  

where $q_i (i = 1, 2, 3, 4)$ are governed by the following equations.

$$\omega \dot{q}_1 = -\frac{1}{2} q_c \omega - \frac{1}{2} \beta_1 q_1 \omega - q_1 g_1 \sin \frac{1}{2} \omega \tau + \frac{1}{2} q_1 g_2 \omega \cos \frac{1}{2} \omega \tau - 3q_2 \beta_2 \left(q_1^2 + q_2^2\right)$$

$$+ q_2 \sigma_1 + \frac{1}{2} q_1 g_2 \omega \sin \frac{1}{2} \omega \tau + q_2 g_1 \cos \frac{1}{2} \omega \tau - \frac{1}{2} q_2 F_d$$

$$\omega \dot{q}_2 = -\frac{1}{2} q_2 \omega - F_d q_2 \omega - \frac{1}{2} \beta_1 q_2 \omega - q_2 g_1 \sin \frac{1}{2} \omega \tau + \frac{1}{2} q_2 g_2 \omega \cos \frac{1}{2} \omega \tau + \frac{3}{4} q_1 \left(q_1^2 + q_2^2\right) \beta_2$$

$$- q_1 \sigma_1 - \frac{1}{2} q_1 g_2 \omega \sin \frac{1}{2} \omega \tau - q_1 g_1 \cos \frac{1}{2} \omega \tau + \frac{1}{2} q_1 F_d$$

$$\omega \dot{q}_3 = -\frac{1}{2} q_2 \omega - \frac{1}{2} \beta_3 q_3 \omega - q_3 g_3 \sin \frac{1}{2} \omega \tau + \frac{1}{2} q_3 g_4 \omega \cos \frac{1}{2} \omega \tau - q_4 \sigma_2 - \frac{3}{4} q_4 \left(q_3^2 + q_4^2\right) \beta_4$$

$$+ q_4 \sigma_1 + \frac{1}{2} q_3 g_4 \omega \sin \frac{1}{2} \omega \tau + q_4 g_3 \cos \frac{1}{2} \omega \tau$$

$$\omega \dot{q}_4 = -\frac{1}{2} q_4 \omega - \frac{1}{2} \beta_3 q_4 \omega - q_4 g_3 \sin \frac{1}{2} \omega \tau + \frac{1}{2} q_4 g_4 \omega \cos \frac{1}{2} \omega \tau + q_1 \sigma_2 + \frac{3}{4} q_1 \left(q_3^2 + q_4^2\right) \beta_4$$

$$- q_2 \sigma_1 - \frac{1}{2} q_2 g_4 \omega \sin \frac{1}{2} \omega \tau - q_2 g_3 \cos \frac{1}{2} \omega \tau$$
The stability of the fixed points is determined by the eigenvalues of the corresponding Jacobian matrix of Eq.(6). The eigenvalues for the trivial solution are the roots of

\[ \lambda^4 + \delta_1 \lambda^3 + \delta_2 \lambda^2 + \delta_3 \lambda + \delta_4 = 0. \]  

(7)

According to Routh-Hurwitz criterion, the stable boundaries for the trivial solution are given in \( \sigma_1 - F_d \) plane as shown in Fig.1, where solid and dashed lines indicate saddle-node and Hopf bifurcation boundaries respectively. Under the boundaries are the stable domains, which are larger than that of the case without feedback control. In Fig. 2, the boundaries of the trivial solution are given in \( g_1 - \tau \) and \( g_3 - \tau \) planes for \( \beta_2 = \beta_4 = 1, \quad c_1 = c_2 = 1, \quad \beta_1 = \beta_3 = 10, \quad \varepsilon = 0.01, \quad \omega = 2, \quad F_d = 10, \quad \sigma_2 = 0, \quad \sigma_4 = 0 \). Within those shaded domains, the trivial solution is stable. Figs. 3 and 4 are simulation results by numerically integrating Eq.(1) to demonstrate the trivial solution is stable or unstable.

Fig. 1 The stable boundaries of the trivial solution in \( \sigma_1 - F_d \) plane, \( \tau = 1 \), (a) \( g_1 = 5 \), (b) \( g_3 = 5 \)

Fig. 2 The stable boundaries of the trivial solution in \( g_1 - \tau \) and \( g_3 - \tau \) planes

![Fig. 3 Time history of the response](image1)

(a) 0.15  
(b) 0.15

Fig. 3 Time history of the response (a) corresponding to Fig. 1(a): \( (\sigma_1, F_d) = (0,10) \), (b) corresponding to Fig. 2(b): \( (\tau, g_3) = (2,2) \)
4. Effect of feedback control on Amplitudes

The following two equations can be obtained from Eq.(4) by setting right-hand sides equal to zero and eliminating \( \sin(\theta_1 - \theta_2) \) and \( \cos(\theta_1 - \theta_2) \).

\[
8c_i A_i^2 \omega^2 \beta_i \beta_i A_i^2 c_i^2 + 32 \sigma_i A_i^2 \beta_i \beta_i A_i^2 \sigma_i - 32 \sigma_i A_i^2 \beta_i \beta_i A_i^2 \sigma_i - 4 g_i^2 A_i^4 \omega^2 \beta_i^2 - 4 F_i^2 A_i^4 \beta_i^2 \\
-24 \beta_i A_i^4 \beta_i^2 \sigma_i - 24 \beta_i A_i^4 \beta_i \beta_i A_i^2 \sigma_i - 18 \beta_i A_i^4 \beta_i \beta_i A_i^2 \beta_i A_i^2 \sigma_i + 24 \beta_i A_i^4 \beta_i \beta_i A_i^2 \sigma_i \\
+24 \beta_i A_i^4 \beta_i \sigma_i - 32 \beta_i A_i^4 \sigma_i - 24 \beta_i A_i^4 \beta_i \beta_i A_i^2 \beta_i A_i^2 \sigma_i - 4 c_i^2 A_i^4 \omega^2 \beta_i^2 + 4 \beta_i A_i^4 c_i^2 \omega^2 \\
+16 g_i A_i^4 \beta_i^2 + 16 \beta_i A_i^4 \sigma_i + 9 \beta_i^2 A_i^3 \beta_i^2 + 16 \beta_i A_i^4 \beta_i^2 A_i^2 \sigma_i - 4 \beta_i A_i^4 g_i^2 \omega^2 \\
+16 \sigma_i A_i^4 \beta_i^2 + 9 \beta_i^2 A_i^4 \beta_i^2 + 24 \sigma_i A_i^4 \beta_i \beta_i A_i^4 \beta_i A_i^2 g_i^2 - 32 \sigma_i A_i^4 \beta_i \beta_i A_i^2 g_i^3 \\
+32 g_i A_i^4 \beta_i \beta_i A_i^2 \sigma_i + 24 g_i A_i^4 \beta_i \beta_i A_i^4 \beta_i A_i^2 g_i - 32 g_i A_i^4 \beta_i \beta_i A_i^2 \sigma_i - 8 \beta_i^2 A_i^4 c_i^2 \omega^2 g_i + 32 \sigma_i A_i^4 \beta_i^2 g_i \\
-32 \beta_i A_i^4 \sigma_i g_i - 24 \beta_i A_i^4 \beta_i g_i + 32 \beta_i A_i^4 \sigma_i g_i - 8 c_i A_i^4 \omega^2 \beta_i^2 g_i - 24 \beta_i A_i^4 \beta_i^2 g_i \\
-8 c_i A_i^4 \omega^2 \beta_i \beta_i A_i^2 c_i^2 \omega^2 g_i A_i^2 \cos(\omega \tau / 2) + (16 \sigma_i A_i^4 \beta_i^2 g_i \omega - 16 \beta_i^2 A_i^4 \sigma_i g_i \omega \\
-12 \beta_i A_i^4 \beta_i g_i \omega + 16 \beta_i A_i^4 \sigma_i g_i \omega + 16 c_i A_i^4 \omega \beta_i^2 g_i + 16 \beta_i A_i^4 c_i \omega g_i - 12 \beta_i A_i^4 \beta_i^2 g_i \omega \\
+16 c_i A_i^4 \omega \beta_i \beta_i A_i^2 g_i + 12 \beta_i A_i^4 \beta_i \beta_i A_i^2 \omega + 16 \beta_i A_i^4 \sigma_i \omega g_i A_i^2 \beta_i - 16 \sigma_i A_i^4 \beta_i \beta_i A_i^2 g_i \omega \\
+16 g_i A_i^4 \omega \beta_i \beta_i A_i^2 \sigma_i + 12 \beta_i A_i^4 \beta_i g_i A_i^2 \omega - 16 g_i A_i^4 \omega \beta_i \beta_i A_i^2 g_i \sin(\omega \tau / 2) \\
-32 g_i A_i^4 \beta_i \beta_i A_i^2 g_i \cos(\omega \tau - 16 g_i A_i^4 \omega \beta_i \beta_i A_i^2 g_i \sin(\omega \tau - 16 g_i A_i^4 \beta_i \beta_i A_i^2 g_i \omega \sin(\omega \tau \\
+8 g_i A_i^4 \beta_i \beta_i A_i^2 g_i \cos(\omega \tau = 0
\]

\[
4 A_i^2 c_i^2 \omega^2 + 16 A_i^2 g_i^2 \omega^2 + 6 A_i^2 \sigma_i^2 + 16 A_i^2 \sigma_i^2 - 32 A_i^2 \sigma_i \sigma_i + 9 A_i^2 \beta_i^2 \\
- 24 A_i^4 \beta_i \sigma_i + 24 A_i^4 \sigma_i \beta_i - 4 \beta_i^3 A_i^2 \omega^2 - (16 A_i^2 c_i^2 \omega g_i^3 - 12 A_i^4 \beta_i \beta_i g_i \omega \\
- 16 A_i^2 \sigma_i g_i \omega + 16 A_i^2 \sigma_i g_i \omega \sin(\omega \tau / 2) + (32 A_i^2 \sigma_i g_i - 8 A_i^2 c_i^2 g_i \omega \\
- 32 A_i^2 \sigma_i \sigma_i - 24 A_i^2 \beta_i \beta_i g_i ) \cos(\omega \tau / 2) = 0
\]

Based on Eqs.(8) and (9), the variation of amplitude of periodic motions with time delay can be presented for different feedback gains as shown in Fig.4 since 3D-plot may be inconvenient to view. It is not difficult to find that the effect of time delay is periodic with period \( 4\pi / \omega \), which can be verified by Eq.(4).
5. Conclusions

The averaging method and numerical simulations are employed in this paper to study the principal parametric resonance of a class of gyroscope systems under feedback control. Time delay in feedback and 1:1 internal resonance are considered. According to Routh-Hurwitz criterion, the boundary curves between stable and unstable domains for the trivial solution are given in excitation and control parameter planes respectively. The influence of control gains and time delay on the amplitude of periodic motions are presented by the Levenberg-Marquardt method. It is shown that the stability of the trivial solution can be varied when feedback control and time delay are employed. The amplitudes of periodic solutions can also be modulated greatly by feedback gain and time delay. However, the influence of time delay on amplitudes is periodic. The simulations obtained by numerically integrating the original system are in good agreement with the analytical results.

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