Bell Non-Locality in Many-Body Quantum Systems with Exponential Decay of Correlations

Carlos H. S. Vieira1 · Cristhiano Duarte2,3,4 · Raphael C. Drumond5 · Marcelo Terra Cunha1

Received: 29 May 2021 / Accepted: 27 September 2021 / Published online: 16 October 2021
© The Author(s) under exclusive licence to Sociedade Brasileira de Física 2021, corrected publication 2022

Abstract

Using Bell inequalities as a tool to explore non-classical physical behaviours, in this paper we analyse what one can expect to find in many-body quantum physics. Concretely, framing the usual correlation scenarios as a concrete spin lattice, we want to know whether or not it is possible to violate a Bell inequality restricted to this scenario. Using clustering theorems, we are able to show that a large family of quantum many-body systems behave almost-locally, violating Bell inequalities (if so) only by a non-significant amount. We also provide examples, explain some of our assumptions via counter-examples and present all the proofs for our results. We hope the paper is self-contained.

Keywords Quantum Foundations · Many-Body Quantum Systems · Bell Non-Locality · Clustering Theorems · Quantum Correlations.

1 Introduction

Quantum physics features correlations showing no parallel with classical physics. Bell non-locality and contextuality are the most prominent examples. The former can be understood as a phenomenon in which the statistics obtained from local measurements acting on distant parts of a quantum system cannot be replicated by any model of (local) classical variables [1]. In other words, the statistics shown by this type of local experiments cannot be reproduced from deterministic strategies, even if aided by shared randomness [2].

The fact that local deterministic strategies fail to frame scenarios exhibiting non-local data is usually detected through violations of so-called Bell inequalities [1, 3, 4]: linear combinations of expected values of correlations from local measurements with a bound calculated under the assumption of Bell locality. A violation of such inequalities witnesses the presence of Bell non-locality in the system (for a review, see [3]).

Ultimately, nonlocality only manifests itself when considering a scenario involving multiple physical systems, whether they are black boxes in the device-independent scenario or real quantum systems. In particular, non-locality in many-body quantum systems has been extensively explored [5–17], see [18] for a review. For instance, it has been discussed in the literature how to use non-locality measurements as an indicator of quantum phase transitions (QPT’s) in several many-body systems models [5, 6, 10, 12]. In all of these works, Bell correlations between spin pairs, measured through CHSH inequality [4], were used to characterize QPTs. Surprisingly, it was observed that such inequality was not violated in any of these models [5, 6, 10, 12].

As a matter of fact, considering only the overlap between many-body quantum systems and the use of CHSH violation as a marker for quantumness, it is remarkable how rich non-locality is. On the one hand, in Ref. [9] the authors showed for translationally invariant lattices, pairs of spins do not exhibit any violation of CHSH inequality, even though the
global state may be highly entangled. On the other hand, it is known that for simple lattices with no translational symmetry it is, indeed, possible to get CHSH violations for some pairs of sites [11, 14].

Detection of multipartite non-locality is another example of the exchange between many-body physics and foundations of quantum mechanics. Although it is known that it is mathematically hard to characterize non-local effects in more complex Bell scenarios [19], recent work has shown that it is possible to detect multipartite non-locality by simpler Bell inequalities, involving only two-body correlators [20–26]. In particular, Ref. [24] demonstrated that physically relevant states, such as the ground state of some spin models in many-body systems, exhibit non-locality for these types of Bell inequality. In Ref. [27], it was remarked that some observables from many-body systems, like energy, can be used as a witness to non-locality. From these tools, it was possible to witness non-locality in a Bose–Einstein condensate of 480 atoms [28] and in a thermal ensemble of $5 \times 10^5$ atoms [29].

This work is placed exactly at this intersection between foundations of physics and many-body quantum mechanics. As a matter of fact, in here we investigate general non-local features in spin lattices. More precisely, we will show two situations in which regions of the system cannot show expressive non-locality when measurements are made in sufficiently distant regions of the lattice: ground states of gapped Hamiltonians and thermal equilibrium states of these lattices for high temperature. We also analyze how violations of Bell inequalities can arise from the interactions of the spins in a lattice, when the initial state is product.

The paper is organized as follows: In Section 2, we present our main results followed by a short discussion. In Section 3, we give a short review on the necessary aspects of non-locality and the clustering theorems for many-body Hamiltonians. In Section 4, we give the proofs of the results enunciated in Section 2, before conclusions are shown together with a discussion of future lines of research, in Section 5.

## 2 Results

This section contains the main results of our work. Every definition, proposition and result are followed or come right after a short motivation or justification. This way we feel this section can stand by itself.

However, we are bridging between two quite well-established fields, so that we are building our findings upon some common knowledge and jargon coming from many-body quantum systems and foundations of quantum mechanics. If the reader is not comfortable with the presentation, we refer them to Section 3 where we present the basics necessary for a better hold of our results.

### 2.1 Main Results

The simplest Bell scenario is one in which two causally separated agents, Alice and Bob, have available two dichotomic measurements each. Outcomes are supposed to be $+1$ or $-1$. Alice has access to $A_0$, $A_1$, and Bob has access to $B_0$, $B_1$ [3]. Up to relabelling, the only non-trivial Bell inequality for this scenario is the CHSH inequality [4]:

$$
\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2.
$$

In this Bell scenario, every system exhibiting an aggregated statistics verifying the inequality in (1) is called local and the correlations presented by it can be explained by a local theory [3]. Non-local quantum features are already manifest even at this simple scenario, as we know this inequality can be violated—even at this simple scenario, as we know this inequality can be violated—within quantum theory—by a particular choice of measurements and states, with the maximum violation reaching $2 \sqrt{2}$ [30].

We can realize this Bell experiment via a quantum spin system. Let $\Omega$ be a lattice representing the location of a finite set of spins, and let $\mathcal{H}_\Omega$ be the Hilbert space associated with that lattice. Additionally, consider that the spins interact with each other, this interaction given by a Hamiltonian operator $H$ acting on $\mathcal{H}_\Omega$. We will assume that the interactions are short-ranged; that is, the range of the interactions is small compared to the size of the lattice. In this experiment, Alice has her action restricted to a region $X \subset \Omega$ of the system, while Bob has his action restricted to a region $Y \subset \Omega$, as illustrated in Fig. 1. Denote by $r$ the distance between $X$ and $Y$, and by $|Z|$ the number of sites in a region $Z \subset \Omega$. Alice’s measurements are operators acting on the lattice with support in the region $X$, while Bob’s measurements are also operators acting on the lattice but supported in $Y$. Finally, assume also that the norm of these operators is upper-bounded by 1.

In this setting, the expected values of these measurements are given by $\langle A_i B_j \rangle = \text{Tr}(\rho A_i B_j)$ where $\rho$ is the state of the whole spin system. Thus, denoting $A = \{A_0, A_1\}$ and $B = \{B_0, B_1\}$ we can define the following quantities.

$$
B_{\text{CHSH}}^{X,Y}(\rho, A, B) := \sup_{A,B} B_{\text{CHSH}}^{X,Y}(\rho, A, B),
$$

where we are optimizing over all operators $A_i, B_j$ acting on $X$ and $Y$ with $\|A_i\|, \|B_j\| \leq 1$. Therefore, if $B_{\text{CHSH}}^{X,Y}(\rho) \leq 2$, the state $\rho$ is local for this Bell experiment.

It is remarkable that the amount of (non-)locality of a state $\rho$ in a Bell experiment can be quantified and seen as a resource. In the framework of general quantum resource
theories, the authors of [31] work out a generic resource measure connecting resourceful objects with operational tasks. The more resourceful an object is, the better it performs in a fixed (sub-channel) discrimination task. Their results strongly, and almost entirely, relying on the convexity of the resourceless states. In this sense, it is possible to translate their findings to our scenario: states exhibiting more non-locality can be far more useful within an operational perspective than their local counterpart.

Our goal is to use clustering theorems to recover almost-local behaviours for many-body quantum systems. We want to guarantee that when the two parts $X, Y$ are far away from each other, regardless of the rest of the system, possible violations of CHSH are vanishingly small. For doing so, we define the following class of states.

**Definition 1** Given two disjoint regions $X, Y \subset \Omega$ and a real number $\epsilon > 0$, a quantum state $\rho$ acting on $\mathcal{H}_\Omega$ is $\epsilon$-local with respect to CHSH and with relation to these two regions if

$$E_{\text{CHSH}}^{X,Y}(\rho) \leq 2 + \epsilon.$$  

(3)

It is important to note that the notion of $\epsilon$-locality defined above is linked to the $X, Y$ regions. What we are going to show, though, is that there are important classes of $\epsilon$-local states, with $\epsilon \ll 1$ regardless of regions, as long as they are sufficiently separated from each other. Actually, that is a quite natural assumption in Bell experiments, as assuming the agents are far from each other ensures that there is no direct causal influence on the correlations.

The above discussion motivates the definition of a state states with exponential clustering of correlation [32–34]:

**Definition 2** A quantum state $\rho$ acting on $\mathcal{H}_\Omega$ shows exponential clustering of correlations if there are two positive constants $C, \lambda$, so that for any two disjoint regions $X, Y$ and any pair of operators $A, B$ supported at $X, Y$, respectively, we have

$$\left| \langle AB \rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho \right| \leq \|A\|\|B\|\|X|Ce^{-\lambda r}.$$  

(4)

**Remark:** For sake of simplicity and to improve the readability, we are always assuming that $|X| \leq |Y|$. The general case is obtained by changing from $|X|$ to $\min(|X|, |Y|)$. As we are more interested in the distance between the subsets, we will stick to our assumption without any loss of generality.

Two important classes of states that have exponential clustering of correlations are the ground state of a gapped Hamiltonian (see Theorem 1) and the thermal quantum states at inverse temperature less than a fixed $\beta^*$ (see Theorem 2). In fact, theorems of type 1, 2 are usually called by clustering theorems [35].

As we will see in the following proposition, exponential clustering of correlations (Definition 2) implies $\epsilon$-locality (Definition 1). In fact, it does not take much mathematical effort to demonstrate such an implication (see section 4.1). However, such a hypothesis is quite common in the condensed matter literature. Going further, as mentioned above, several interesting states satisfy exponential clustering of correlations, which justifies the relevance of the results that follow.

**Proposition 1** Given two disjoint regions $X, Y \subset \Omega$ and $\rho$ a quantum state with exponential clustering of correlations, then $\rho$ is $\epsilon$-local for CHSH with respect this two regions, where $\epsilon = 4|X|Ce^{-\lambda r}$.

Recall that the constants $C, \lambda$ do not depend on the regions. Therefore, by distancing Bob from Alice, so that
$r$ becomes increasingly larger, $\epsilon$ will be as close to zero as you want.

As mentioned earlier, Theorem 1 ensures exponential clustering of correlations for the ground state of a gapped Hamiltonian. So, from Theorem 1 and Proposition 1 we get our first main result.

\textbf{Result 1} If $\rho$ is the ground state of a gapped Hamiltonian of the lattice, then there are $C, \lambda > 0$ such that given $X, Y \subset \Omega$ disjoint regions we have that $\rho$ is $\epsilon$-local state for CHSH with respect these two regions, where $\epsilon = 4|X||Y|C e^{-\lambda r}$.

For the same reasons already presented, we will have a small $\epsilon$ if the distance between the parts is large, as expected in a Bell experiment.

We also can use Theorem 2 and Proposition 1 to show a similar property for thermal states. However, a thermal state has additional properties that allow us to show a stronger result.

\textbf{Result 2} Let $\rho(\beta)$ be a thermal state acting on the lattice with a inverse temperature $\beta$ less than a fixed $\beta^*$, and let $A$ be a set of operators acting on $X \subset \Omega$. There is $r^* > 0$ such that given $Y \subset \Omega$ with $r \geq r^*$ we have $B_{\text{CHSH}}(\rho(\beta), A, B) \leq 2$ for every set of operators $B$ acting on $Y$.

Broadly speaking, Result 2 is saying that for every choice of measurements for Alice, if Bob is far enough, we cannot measure the distance (see Theorem 3). This theorem bounds that the propagation of correlations in the lattice is when we start from a product state. Applying Theorem 3 and using similar ideas, it represents the maximum effective velocity of propagation of the information across the lattice [36]. Therefore, we can enunciate the following result.

\textbf{Result 3} Suppose the initial state of the system is a product state, i.e. $\rho(0) = \otimes_{x \in \Omega} \rho_x$. Then, there are $C, v, \lambda > 0$ such that given two disjoint regions $X, Y \subset \Omega$ then $\rho(t)$ is $\epsilon$ -local for CHSH with respect these two regions, where $\epsilon = 4|X||Y|C e^{\lambda vt} - 1) e^{-\lambda x^2}$.

The constant $v$ is called the Lie–Robinson velocity, and it represents the maximum effective velocity of propagation of the information across the lattice [36]. Therefore, we conclude that we will have an effective local behaviour for a time of the order $\sqrt{\lambda}$.

So far, we have used CHSH as a tool for non-locality detection. However, some of the previous results can be promptly generalized to more complex scenarios with richer Bell inequalities. So, let us consider a scenario where $N$ spatially separated agents share a quantum state. Each party $i$ chooses one out of the $m$ possible $M_i$ dichotomic measurements and performs it on her/his part of the shared quantum state. The reason for restricting it to dichotomic measurements comes from the fact that in this case a Bell inequality can be written through correlators [3].

A Bell inequality for this scenario involves the sum of correlators between many parts at the same time. However, as discussed in the introduction, there is an interest in Bell inequalities with correlators of at most two bodies. These inequalities are simpler, and from them, it will be possible to better visualize our results. We will start from these inequalities, and at the end of this section, we will return to the general case.

A general Bell inequality involving correlators of one and two bodies can be written as:

$$\sum_{i,j=1}^{N,M} a_k^{(i)} E_k^{(i)} + \sum_{i,j=1}^{N,M} \beta_{kl}^{(i)} E_k^{(i)} E_l^{(j)} \leq \Delta_C,$$

where $E_k^{(i)}$ is the $k\text{-th}$ measurement of agent $i$ and $a_k^{(i)}$, $\beta_{kl}^{(i)}$, $\Delta_C$ are real constants, with $\Delta_C$ the local bound. Again, every state whose aggregated statistics respects this inequality, for all choices of measurements, is called local.

This family of Bell inequalities is already capable of signalling non-locality for physically relevant states [24]. It is a fact that in a bipartite scenario where all measurements are dichotomic, all inequalities can be written in this way. On the other hand, in a multipartite scenario, this class of inequalities is important due to the ease of implementation in many-body systems models [28, 29].

Again, we can perform this Bell experiment on a quantum spin system. Now, each agent $i$ has their action restricted to a region $X_i$ of the system, as illustrated in Fig. 2. We will indicate by $r_{ij}$ the distance between the regions $X_i$ and $X_j$. As before, the measurements from the agent $i$ are operators acting on the lattice with support in the region $X_i$ and with norm less than or equal to 1. Let us denote by $B^{\text{CHSH}}_i$ the set of measurements operators from agent $i$, that is $\{E_1^{(i)}, \ldots, E_{M_i}^{(i)}\}$. Similarly to Eq. (2), we define

$$B_{2\text{Body}}^{X_1 \ldots X_N}(\rho, E^{(1)}, \ldots, E^{(N)}) = \sum_{i,k=1}^{N,M} a_k^{(i)} E_k^{(i)} \rho + \sum_{i,j=1}^{N,M} \beta_{kl}^{(i)} E_k^{(i)} E_l^{(j)} \rho,$$

where $B_{2\text{Body}}^{X_1 \ldots X_N}(\rho) = \sup_{E^{(1)}, \ldots, E^{(N)}} B_{2\text{Body}}^{X_1 \ldots X_N}(\rho, E^{(1)}, \ldots, E^{(N)})$.

If $B_{2\text{Body}}^{X_1 \ldots X_N}(\rho) > \Delta_C$, then the state $\rho$ shows non-locality in this configuration of Bell’s experiment. The generalization for Proposition 1 is the following.

\textbf{Proposition 2} Let $\rho$ be a quantum state acting on $\mathcal{H}_\Omega$ showing exponential clustering of correlations. Then, there exist $C, \lambda > 0$ such that for every $X_1, \ldots, X_N \subset \Omega$ disjoint regions we have
such that for every $\Delta_1, \ldots, \Delta_i$. Using Proposition 2 together with the Theorem 1, the following generalization from Result 1 is obtained.

**Result 4** If $\rho$ is the ground state of a gapped Hamiltonian of the lattice, then there are $C, \lambda > 0$ such that for every $X_1, \ldots, X_N \subset \Omega$ disjoint regions we have

$$E_{2\text{body}}^{X_1,\ldots,X_N}(\rho) \leq \Delta_C + C \sum_{i \neq j} \sum_{k,l=1}^N \min \{|X_i|, |X_j|\} |\rho_{kl}^{(ij)}| e^{-\lambda t_{ij}}.$$  

So, if all parts are far enough, then $E_{2\text{body}}^{X_1,\ldots,X_N}(\rho)$ will also have an upper bound as close as we want to the local bound. Consequently, we will not be able to see substantial violations of any Bell inequality that only involves correlations of one and two bodies in this kind of states.

Analogously, using Proposition 2 together with the clustering theorem for thermal states, that is Theorem 2, we get the following result.

**Result 5** If $\rho(\beta)$ is a thermal state acting on the lattice with inverse temperature $\beta$ less than a fixed $\beta^*$, then there exist $C, \lambda > 0$ such that for every $X_1, \ldots, X_N \subset \Omega$ disjoint regions we have

$$E_{2\text{body}}^{X_1,\ldots,X_N}(\rho(\beta)) \leq \Delta_C + C \sum_{i \neq j} \sum_{k,l=1}^N \min \{|X_i|, |X_j|\} |\rho_{kl}^{(ij)}| e^{-\lambda t_{ij}}.$$  

Again, if all parts are far apart from each other, we have the same conclusion for at most small violations.

As a final result for two-body Bell’s inequalities, we have the generalization of Result 3.

**Result 6** Suppose that the initial state of the system is a product state, i.e. $\rho(0) = \otimes_{x \in \Omega} \rho_x$. Then, there are $C, \lambda, \nu > 0$ such that, for every $X_1, \ldots, X_N \subset \Omega$ disjoint regions, we have

$$E_{2\text{body}}^{X_1,\ldots,X_N}(\rho(t)) \leq \Delta_C + C(e^{\lambda t} - 1) \sum_{i \neq j} \sum_{k,l=1}^N |X_i||X_j| |\rho_{kl}^{(ij)}| e^{-\lambda t_{ij}/2}.$$  

As we discussed, a general Bell inequality in this scenario can involve a sum of correlators of many bodies. Actually, we can write it arbitrarily by:

$$\sum_{n=1}^N \sum_{i_1, \ldots, i_n} \gamma_{k_1, \ldots, k_n} \left< E_{k_1}^{(i_1)} \cdots E_{k_n}^{(i_n)} \right> \leq \Delta_C.$$  

Fig. 2  Multipartite Bell experiment in a spin system
Thus, analogous to the previous constructions let us define:

$$B_{\text{Bell}}^{X_1,\ldots,X_N}(\rho, E^{(1)}, \ldots, E^{(N)})$$

$$= \sum_{n=1}^{N} \sum_{i \neq \ldots \neq n}^{N} \sum_{k_1, \ldots, k_n=0} \tau_{k_1, \ldots, k_n}^{i_1, \ldots, i_n} \left( \langle E_{i_1} \rangle \ldots \langle E_{i_n} \rangle \right);$$

(7)

$$\Delta_{\text{Bell}}^{X_i,\ldots,X_i}(\rho) = \sup_{E^{(1)},\ldots,E^{(N)}} B_{\text{Bell}}^{X_i,\ldots,X_i}(\rho, E^{(1)}, \ldots, E^{(N)}).$$

Generalization of the previous results can be obtained, but first we need to extend the notion of exponential clustering of correlations to when we are considering the correlations of many bodies at the same time. The next proposition shows us that the assumptions in Definition 2 are enough to extend the notion of exponential clustering of correlations to the case of correlations between many parts.

**Proposition 3** If $\rho$ is a quantum state acting in $\mathcal{H}_\Omega$ with exponential clustering of correlations, then for any set of disjoint regions $X_1, \ldots, X_n \subset \Omega$ and any set of operators $E_1, \ldots, E_n$ supported at $X_1, \ldots, X_n$, respectively, we have

$$\left| \langle E_1 \ldots E_n \rangle - \langle E_1 \rangle \ldots \langle E_n \rangle \right| \leq \|E_1\| \ldots \|E_n\| (n-1)|X|Ce^{-\lambda r},$$

where $C, \lambda > 0$ are the same constants as from the definition of exponential clustering of correlations, $|X| = \max\{|X_1|, \ldots, |X_n|\}$ and $r = \min r_j$, with $r_j$ being the distance between the regions $X_i$ and $X_j$.

With this proposition and the same ideas as before, we can generalize Results 4 and 5. Before that, in order not to overcharge the notation we will denote by $\Gamma$ the following sum of constants:

$$\Gamma = \sum_{n=1}^{N} \sum_{i \neq \ldots \neq n}^{N} \sum_{k_1, \ldots, k_n=0} \tau_{k_1, \ldots, k_n}^{i_1, \ldots, i_n} (n-1)\left| \tau_{k_1, \ldots, k_n}^{i_1, \ldots, i_n} \right|.$$  

(8)

**Result 7** If $\rho$ is the ground state of a gapped Hamiltonian of the lattice, then there exist $C, \lambda > 0$ such that for every $X_1, \ldots, X_N \subset \Omega$ disjoint regions we have

$$\Delta_{\text{Bell}}^{X_i,\ldots,X_i}(\rho) \leq \Delta_{C} + C|X|\Gamma e^{-\lambda r}.$$  

**Result 8** If $\rho(\beta)$ is a thermal state acting on the lattice with inverse temperature $\beta$ less than a fixed $\beta^*$, then there exist $C, \lambda > 0$ such that for every $X_1, \ldots, X_N \subset \Omega$ disjoint regions we have

$$\Delta_{\text{Bell}}^{X_i,\ldots,X_i}(\rho(\beta)) \leq \Delta_{C} + C|X|\Gamma e^{-\lambda r}.$$  

Thus, if the experiment is carried out in such a way that all the parts are away from each other, no Bell inequality will be significantly violated for these two families of states.

### 2.2 Summary of Results

Summing up, this section contains our results divided into three categories.

First, we explored non-locality for spin lattices based on the CHSH inequality. We have seen in Result 1 that if Alice and Bob’s actions are restricted to distant regions on the lattice, then the ground state of a gapped Hamiltonian is unable to significantly violate CHSH. Additionally, in Result 2, we saw that thermal states have an even more restricted behaviour: if one party fixes their measurements, there is a minimum distance above which it is not possible to see any CHSH violation. In Result 3, we saw how non-local correlations are created in time when the initial system is a product state.

The second bit is a generalization of the first three results to a scenario with more parties and more measurements for each part. In this case, we first analyse Bell inequalities that only involve correlators of one and two bodies. The conclusions are the same as those obtained for the previous cases, the exception being Result 5 where it is no longer possible to conclude non-violation.

Finally, we dealt with general Bell inequalities. For this, we adopted some simplifications, one of which was to look only at the minimum distance between observables. With that, we enunciate the generalizations of Results 4 and 5.

We invite the reader to check out Sects. 3 and 4. They contain all the mathematical details and in-depth proofs for the results we approached above.

### 3 Preliminaries

#### 3.1 Bell Inequalities

Broadly speaking, in our work we are investigating non-local aspects of spin lattices via Bell inequalities. Centring our attention on the latter, in this section we cover the basics of what we mean by a non-local correlation.

Correlation scenarios are usually formulated in a device-independent language [3]. Think of it as a collection of $N$ black-boxes. Each box comes with $m$ buttons on the top and $o$ light bulbs at the bottom. Whenever a button is pressed, one light bulb goes off as a response to this action. The entire formalism is coined to hidden the inner physical mechanism of each box. As we do not have access to the physical details producing an outcome given
that a certain button was pressed, the only description for this \((N, m, o)\)-scenario is via the aggregated joint statistics

\[
\bar{p} = \{p(ab...c|xy...z)\} \in \mathbb{R}^{(om)^n}. \tag{9}
\]

Each \(p(ab...c|xy...z)\) simply means the joint probability of getting outcome \(a\) out of the first box when the \(x\) button was pressed, and outcome \(b\) out of the second box when the \(y\) button was pressed, ..., and outcome \(c\) out of the \(N\)th-box when button \(z\) was pressed. See Fig. 3.

The definition of the local set of correlations is motivated in various ways. Particularly, we refer the reader to the modern [37]. For sake of simplicity, we will go with an alternative one. If we assume that each of these boxes is independent of one another, Eq. (9) would reflect it and factorize as:

\[
p(ab...c|xy...z) = p(a|x) \times p(b|y) \times ... \times p(c|z). \tag{10}
\]

When correlations across the boxes are detected, and Eq. (10) does not hold true, intuitively we assume that what is happening is that there is an exogenous variable, say \(\lambda\), we are not accounting for, but that in its presence the independence would manifest:

\[
p(ab...c|xy...z, \lambda) = p(a|x, \lambda) \times p(b|y, \lambda) \times ... \times p(c|z, \lambda). \tag{11}
\]

When this is the case, the behaviour we want to look at is nothing but the average of Eq. (11), in fact:

\[
p(ab...c|xy...z) = \int_{\Lambda} p(ab...c|xy...z, \lambda) d\mu(\lambda)
= \int_{\Lambda} p(a|x, \lambda) \times p(b|y, \lambda) \times ... \times p(c|z, \lambda) d\mu(\lambda). \tag{12}
\]

Equation (12) is the very mathematical expression of what we mean by a correlation to be local. For the finite case, it suffices to consider discrete variables, and by doing so, we replace the integral for a sum:

\[
p(ab...c|xy...z) = \sum_{\lambda} p(ab...c|xy...z, \lambda)q(\lambda)
= \sum_{\lambda} p(a|x, \lambda)p(b|y, \lambda)...p(c|z, \lambda)q(\lambda), \tag{13}
\]

where \(q(\lambda)\) is a probability vector.

In a given \((N, m, o)\)-scenario, we say that \(p(ab...c|xy...z)\) is local whenever it verifies Eq. (13). Basically, it says that there is a hidden variable we do not have access to that explains the correlation across the boxes, or, more succinctly, that there is a classical explanation for such correlations.

For a fixed \((N, m, o)\)-scenario, the set of local correlations is a polytope, and as such, it can be described through its facets [38]. That is to say that every local correlation must satisfy a finite set of linear inequalities. Whenever one of these inequalities is violated, we know for sure that the correlation we are looking at is not attainable with a local model. Because of his influential work on locality, these dividing inequalities are usually known as Bell inequalities [3].

The simplest correlation scenario, i.e. the \((2,2,2)\)-scenario, has a single Bell inequality, the CHSH inequality [4]. Satisfying CHSH inequality is a necessary and sufficient condition for a behaviour to be local. Note that in more complex scenarios, the local polytope is a multi-faceted object and that in order to attest that a certain correlation is local, we must check all of the facet defining inequalities [3].

### 3.2 Many-Body Quantum Systems and Clustering Theorems

To define our quantum spin system, let \(\Omega\) be a finite set of sites that will be called lattice. Let \(d\) be a metric in \(\Omega\), which gives the distance between the sites in the lattice. We associate with each site \(x\) in \(\Omega\) a finite-dimensional Hilbert space \(\mathcal{H}_x\) and for each \(X \subset \Omega\), the Hilbert space
associated is given by the tensor product $\mathcal{H}_X = \bigotimes_{x \in X} \mathcal{H}_x$.
The algebra of observables in $X$ is denoted by $\mathcal{L}(\mathcal{H}_X)$. The support of an operator $A \in \mathcal{L}(\mathcal{H}_X)$ is given by $\text{support}(A) = \{x \in X : 4A < 0\}$, where $4A \in \mathcal{L}(\mathcal{H}_X)$; that is, the support of an operator is given by the smallest set such that the operator acts as an identity in the complement of that set. An interaction for such a system is a map $h$ from the set of subsets of $\Omega$ to $\mathcal{L}(\mathcal{H}_X)$ such that $h(X)$ has support in $X$. The Hamiltonian is given by $H = \sum_{X \subseteq \Omega} h(X)$. The dynamics of the model is given by $A(t) = e^{-itH} A e^{itH}$.

Lastly, let $R$ be the maximal distance for the interactions. This is the general construction of a finite quantum spin system. The additional assumption is that the interactions are short-ranged; that is, $R$ is small compared with the size of the lattice.

For this system with specific additional assumptions, there are classes of states that have exponential clustering of correlations. The first case is the ground state of a gapped Hamiltonian [32, 34].

**Theorem 1** (Clustering theorem for gapped ground state) Let $\rho$ be the ground state of a system with a spectral gap $\Delta E > 0$ above the ground energy. Then, there exist constants $C, \lambda > 0$ such that, given two disjoint regions $X, Y \subseteq \Omega$ at a distance $r$ from each other and two operators $A, B \in \mathcal{L}(\mathcal{H}_\Omega)$ with support in $X, Y$, respectively, we have the following bound.

$$\left| \langle AB \rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho \right| \leq \|A\| \|B\| |X| C e^{-\lambda r}.$$  

The coefficients $C$ and $\lambda$ are independent of $A$ and $B$. Actually, $C, \lambda$ depends only on the geometry of the lattice, the maximum interaction energy and the spectral gap. For this reason, if we move Alice away from Bob in Result 1, $e$ will approach 0. The same argument applies to Results 4 and 7.

Thermal equilibrium states, or thermal states, for short, provide a second class of very important state with exponential clustering of correlations. A thermal state, or Gibbs state, of a Hamiltonian $H$ at inverse temperature $\beta$ is given by

$$\rho(\beta) = \frac{e^{-\beta H}}{Tr(e^{-\beta H})}. \quad (14)$$

There exists a universal inverse critical temperature $\beta^*$, which is, in particular, independent of the system size, below which correlations decay exponentially. This parameter essentially depends only on the typical energy of interaction and the spatial dimension of the lattice (see [33] for more details). With this assumption, the following theorem is shown in Ref. [33].

**Theorem 2** (Clustering Theorem for thermal states) Let $\rho(\beta)$ be the thermal state at inverse temperature $\beta < \beta^*$. There are constants $C(\beta), \lambda(\beta)$ such that if $X, Y$ are two disjoint regions on the lattice at distance $r$ and $A, B$ operator acting in $X, Y$, respectively, we have

$$\left| \langle AB \rangle_{\rho(\beta)} - \langle A \rangle_{\rho(\beta)} \langle B \rangle_{\rho(\beta)} \right| \leq \|A\| \|B\| |X| Y C(\beta) e^{-\lambda(\beta) r}.$$  

Once again, distancing $X$ from $Y$ does not result in changes to $C(\beta), \lambda(\beta)$. Thanks to this and the fact that $\rho(\beta)$ is full rank, it was possible to find a minimum distance in Result 2 which it is not possible to violate CHSH. An analogous argument applies to Results 5 and 8.

Additionally, note that the hypothesis $\beta < \beta^*$ plays an important role in our result. As discussed in [33] (Section III A), the $(\beta^*)^{-1}$ constant can be seen as an upper bound on physical critical temperatures. This is important since close to phase transitions the correlation length usually diverges. Therefore, $(\beta^*)^{-1}$ defines an interesting temperature above which far reaching correlation lengths are ruled out.

It is important to emphasize that Lieb–Robinson’s bounds are fundamental in the proof of the two previous theorems [36]. In this seminal paper, Lieb and Robinson prove that there is a bound for the maximal effective velocity for the propagation of information in a quantum spin system with short-range interactions. Another application of Lieb–Robinson bounds is in the propagation of correlations [39]. It is easy to see that if we start with a product state, there will be no correlations between the parts of the system at time $t = 0$. What was shown in [39] is that there is a bound to how much correlation can be created in time. This bound grows exponentially with time but decreases exponentially with distance. Indeed, they prove the following result [39].

**Theorem 3** (Propagation of Correlations) Let $X, Y$ be disjoint regions of $\Omega$ with $r = d(X, Y)$. Let $A, B \in \mathcal{L}(\mathcal{H}_\Omega)$ have support in $X, Y \subseteq \Omega$, respectively, and $\rho(0) = \bigotimes_{x \in \Omega} \rho_x$ be the initial state of the lattice. Then,

$$\left| \langle AB \rangle_{\rho(0)} - \langle A \rangle_{\rho(0)} \langle B \rangle_{\rho(0)} \right| \leq \|A\| \|B\| |X| Y C(\beta^*)^r e^{-\lambda r/2}.$$  

## 4 Proofs

This section contains the proofs of our main results. Results that are known in the literature are not discussed here. Our readers might want to check [32–34, 39] to find proofs for Theorems 1, 2 and 3.

### 4.1 Proposition 1

**Proposition 1.** Given two disjoint regions $X, Y \subseteq \Omega$ and $\rho$ a quantum state with exponential clustering of correlations, then
\( \rho \) is \( e \)-local for CHSH with respect these two regions, where \( e = 4|X|Ce^{-\beta r} \).

From Definition 2, for each pair of measurements \( A_i, B_j \) from Alice and Bob, respectively, we have

\[
\left| \langle A_iB_j \rangle - \langle A_i \rangle \langle B_j \rangle \right| \leq \|A_i\|\|B_j\||X|Ce^{-\beta r}.
\]  (15)

But, as \( A_i \) and \( B_j \) have spectrum in \([-1, 1]\), we have \( \|A_i\| \) and \( \|B_j\| \) less or equal to 1. Given that and also applying the modulus function property \( (x \leq |x|) \) to Eq. (15), we have:

\[
\langle A_iB_j \rangle \leq \langle A_i \rangle \langle B_j \rangle + |X|Ce^{-\beta r}.
\]  (16)

Replacing in (2):

\[
\hat{B}_{CHSH}^{XY}(\rho, A, B) \leq \langle A_0 \rangle \langle B_0 \rangle + \langle A_0 \rangle \langle B_1 \rangle + \langle A_1 \rangle \langle B_0 \rangle - \langle A_1 \rangle \langle B_1 \rangle + 4|X|Ce^{-\beta r}.
\]  (17)

Let us denote

\[
\hat{B}_{CHSH}^{XY}(\rho, A, B) = \langle A_0 \rangle \langle B_0 \rangle + \langle A_0 \rangle \langle B_1 \rangle + \langle A_1 \rangle \langle B_0 \rangle - \langle A_1 \rangle \langle B_1 \rangle .
\]  (18)

Therefore,

\[
\hat{B}_{CHSH}^{XY}(\rho, A, B) \leq \hat{B}_{CHSH}^{XY}(\rho, A, B) + 4|X|Ce^{-\beta r}.
\]  (19)

The term \( \hat{B}_{CHSH}^{XY}(\rho, A, B) \) defined in Eq. (18) represents an uncorrelated system and then can be simulated by a classical system. For this reason, this quantity must respect the CHSH inequality, so that \( \hat{B}_{CHSH}^{XY} \leq 2 \). Indeed,

\[
\hat{B}_{CHSH}^{XY}(\rho, A, B) = \langle A_0 \rangle + \langle A_1 \rangle (B_0) + \langle A_0 \rangle - \langle A_1 \rangle \langle B_1 \rangle \leq |\langle A_0 \rangle + \langle A_1 \rangle | + |\langle A_0 \rangle - \langle A_1 \rangle | = 2 \max \{|\langle A_0 \rangle|, |\langle A_1 \rangle|\} \leq 2.
\]  (20)

Now, putting all these elements together in ineq. (17) we get

\[
\hat{B}_{CHSH}^{XY}(\rho, A, B) \leq 2 + 4|X|Ce^{-\beta r}.
\]  (21)

Finally, note that the term \( 4|X|Ce^{-\beta r} \) is independent of \( A, B \) because of that we can optimize over all pairs \( A, B \) without having to care for this term:

\[
\hat{B}_{CHSH}^{XY}(\rho) = \sup_{A, B} \hat{B}_{CHSH}^{XY}(\rho, A, B) \leq 2 + 4|X|Ce^{-\beta r}.
\]  (22)

Summing up, Eq. (22) says that \( \rho \) is an \( e \)-local state for CHSH with respect to \( X, Y \) where the appropriate \( e \) is given by \( 4|X|Ce^{-\beta r} \).

### 4.2 Result 2

**Result 2.** Let \( \rho(\beta) \) be a thermal state acting on the lattice with a inverse temperature \( \beta \) less than a fixed \( \beta^* \), and let \( A \) be a set of operators acting on \( X \subset \Omega \). There is \( r^* > 0 \) such that given \( Y \subset \Omega \) with \( r \geq r^* \) we have \( \hat{B}_{CHSH}^{XY}(\rho(\beta), A, B) \leq 2 \) for every set of operators \( B \) acting on \( Y \).

To find a proof for Result 2, start recalling that Theorem 2 guarantees us that the thermal states show exponential clustering of correlations. That is to say that they satisfy the hypothesis of Proposition 1. So, from Eq. (19):

\[
\hat{B}_{CHSH}^{XY}(\rho(\beta), A, B) \leq \hat{B}_{CHSH}^{XY}(\rho(\beta), A, B) + 4|X|Ce^{-\beta r},
\]  (23)

where, now, the expected values are obtained with the Gibbs state.

Once again, we have an upper bound for CHSH as close as we want to the local bound, as long as we consider the parts sufficiently distant from one another. Nonetheless, we can go beyond that and guarantee non-violation. For this, we will use the thermal state property to be full rank.

Alice’s measurements are dichotomic, so that for each \( i \in \{0, 1\} \) there is a POVM \( \{E_i^{(1)}, E_i^{(1)}\} \) such that \( A_i = E_i^{(1)} - E_i^{(-1)} \). It is known that if Alice’s pair of measurements commute, they will not violate CHSH [3]. Therefore, if \( A_i = \pm \), being \( 1 \) the identity operator, then for every quantum state CHSH inequality will not be violated. Hence, suppose \( A_0 \) and \( A_1 \) different from \( \pm \). Thus, neither \( E_i^{(1)} = 0 \) nor \( E_i^{(-1)} = 0 \) holds true. Other than that, as \( \rho(\beta) \) is a full rank matrix and also a density matrix, it follows that \( \rho(\beta) \) is definite positive. Additionally, we also have that \( E_i^{(1)} \) and \( E_i^{(-1)} \) are positive semi-definite non-null. Then, \( Tr(\rho(\beta)E_i^{(1)}) \) and \( Tr(\rho(\beta)E_i^{(-1)}) \) are strictly positive, and smaller than 1.

For this reason,

\[
\langle A_i \rangle = Tr(\rho(\beta)E_i^{(1)}) - Tr(\rho(\beta)E_i^{(-1)})
\]  (24)

and

\[
-\langle A_i \rangle = -Tr(\rho(\beta)E_i^{(1)}) + Tr(\rho(\beta)E_i^{(-1)})
\]  (25)

Using this fact in (20):

\[
\hat{B}_{CHSH}^{XY}(\rho(\beta), A, B) \leq 2 \max \{|\langle A_0 \rangle|, |\langle A_1 \rangle|\} < 2,
\]  (26)

which shows that there is \( \delta > 0 \) such that:
\[ E_{\text{CHSH}}^{XY}(\rho(\beta), \mathbf{A}, \mathbf{B}) \leq 2 - \delta. \]  

Therefore, replacing in (19), we have:
\[ E_{\text{CHSH}}^{XY}(\rho(\beta), \mathbf{A}, \mathbf{B}) \leq 2 - \delta + 4|X|C e^{-2\delta}. \]  

Note that so far we have not used anything about Bob’s operators. If Bob is far, or to be more exact, if \( r^* = \frac{1}{\delta} \ln \left( \frac{4|X|C}{\delta} \right) \), then \( E_{\text{CHSH}}^{XY}(\rho(\beta), \mathbf{A}, \mathbf{B}) \leq 2 \). Thus, we will not see any violation of CHSH for thermal states as long as we take the measurements far enough.

### 4.3 Result 3

**Result 3.** Suppose the initial state of the system is a product state, i.e. \( \rho(0) = \bigotimes_{x \in \Omega_1} \rho_1 \). Then, there are \( C, \lambda, \gamma > 0 \) such that, given two disjoint regions \( X, Y \subseteq \Omega \), then \( \rho(t) \) is \( \epsilon \)-local for CHSH with respect these two regions, where \( \epsilon = 4|X||Y|C(e^{\lambda t} - 1)e^{-\gamma t} \).

The proof of this result is completely analogous to that of Proposition 1. Indeed, for Theorem 3 we have:
\[ \langle A_i B_j \rangle \leq \langle A_i \rangle \langle B_j \rangle + |X||Y|C(e^{\lambda t} - 1)e^{-\gamma t}/2. \]  

Thus, applying the same ideas used from expression (16), the result is concluded.

### 4.4 Proposition 2

**Proposition 2.** Let \( \rho \) be a quantum state acting on \( \mathcal{H}_\Omega \) showing exponential clustering of correlations. Then, there exist \( C, \lambda, \gamma > 0 \) such that for every \( X_1, \ldots, X_N \subseteq \Omega \) disjoint regions we have
\[ E_{\text{2Body}}^{X_1 \cdots X_N}(\rho) \leq \Delta_C + \sum_{i \neq j} \sum_{k,l=1}^{M_M} |X_i||\rho_{kl}^{(i)}|e^{-\gamma t}. \]

The proof we discuss in this section follows the same argument we used to prove our first proposition. Using the definition of a state with exponential clustering of correlations, for each pair of measurements from different agents, we get:
\[ \langle E_k^{(i)} E_l^{(j)} \rangle - C|X| \gamma e^{-\gamma t} \leq \langle E_k^{(i)} E_l^{(j)} \rangle \leq \langle E_k^{(i)} \rangle \langle E_l^{(j)} \rangle + C|X| \gamma e^{-\gamma t}, \]

for all \( i,j \in \{1, \ldots, N\} \) with \( i \neq j \), where \( |X_i| = \min \{ |X_i|, |X_j| \} \). Thus, given \( \mu \geq 0 \):
\[ \mu \langle E_k^{(i)} E_l^{(j)} \rangle \leq \mu \langle E_k^{(i)} \rangle \langle E_l^{(j)} \rangle + \mu C|X| \gamma e^{-\gamma t} = \mu \langle E_k^{(i)} \rangle \langle E_l^{(j)} \rangle + |\mu| C|X| \gamma e^{-\gamma t}. \]

On the other hand, if \( \mu < 0 \):
\[ \mu \langle E_k^{(i)} E_l^{(j)} \rangle \leq \mu \langle E_k^{(i)} \rangle \langle E_l^{(j)} \rangle - \mu C|X| \gamma e^{-\gamma t} = \mu \langle E_k^{(i)} \rangle \langle E_l^{(j)} \rangle + |\mu| C|X| \gamma e^{-\gamma t}. \]

So, for every \( \mu \in \mathbb{R} \):
\[ \mu \langle E_k^{(i)} E_l^{(j)} \rangle \leq \mu \langle E_k^{(i)} \rangle \langle E_l^{(j)} \rangle + |\mu| C|X| \gamma e^{-\gamma t}. \]

Applying this inequality in Eq. (6), we have:
\[ E_{\text{2Body}}^{X_1 \cdots X_N}(\rho, E^{(1)}, \ldots, E^{(N)}) \leq \sum_{i=1}^{N \times M_M} e^{(i)} \langle E_k^{(i)} \rangle \]

\[ + \sum_{i \neq j} \sum_{k,l=1}^{M_M} |X_i||\rho_{kl}^{(i)}|e^{-\gamma t}. \]

Define:
\[ E_{\text{2Body}}^{X_1 \cdots X_N}(\rho, E^{(1)}, \ldots, E^{(N)}) = \sum_{i=1}^{N \times M_M} e^{(i)} \langle E_k^{(i)} \rangle \]

\[ + \sum_{i \neq j} \sum_{k,l=1}^{M_M} |X_i||\rho_{kl}^{(i)}|e^{-\gamma t}. \]

So again, \( E_{\text{2Body}}^{X_1 \cdots X_N}(\rho, E^{(1)}, \ldots, E^{(N)}) \) represents an uncorrelated system and as such, it must respect the local bound. Therefore,
\[ E_{\text{2Body}}^{X_1 \cdots X_N}(\rho, E^{(1)}, \ldots, E^{(N)}) \leq \Delta_C + \sum_{i \neq j} \sum_{k,l=1}^{M_M} |X_i||\rho_{kl}^{(i)}|e^{-\gamma t}. \]

Note the right side of the inequality does not depend on which measurements have been taken. Therefore, taking the supremum over them, we have:
\[ E_{\text{2Body}}^{X_1 \cdots X_N}(\rho) \leq \Delta_C + \sum_{i \neq j} \sum_{k,l=1}^{M_M} |X_i||\rho_{kl}^{(i)}|e^{-\gamma t}. \]  

### 4.5 Result 6

**Result 6.** Suppose that the initial state of the system is a product state, i.e. \( \rho(0) = \bigotimes_{x \in \Omega_1} \rho_x \). Then, there are \( C, \lambda, \gamma > 0 \) such that for every \( X_1, \ldots, X_N \subseteq \Omega \) disjoint regions we have
Thus, applying the same ideas used from inequality (30), the result is concluded.

### 4.6 Proposition 3

**Proposition 3.** If \( \rho \) is a quantum state acting in \( \mathcal{H}_\Omega \) with exponential clustering of correlations, then, for any set of disjoint regions \( X_1, \ldots, X_N \subseteq \Omega \) and any set of operators \( E_1, \ldots, E_n \) supported at \( X_1, \ldots, X_n \), respectively, we have

\[
\left| \langle E_1 \cdots E_n \rangle - \langle E_1 \rangle \cdots \langle E_n \rangle \right| \leq \| E_1 \| \cdots \| E_n \| (n-1) |X| Ce^{-\lambda r}.
\]

where \( C, \lambda > 0 \) are the same constants from the definition of exponential clustering of correlations, \( |X| = \max \{ |X_1|, \ldots, |X_n| \} \), and \( r = \min r^\rho \) with \( r^\rho \) being the distance between the regions \( X_i \) and \( X_j \).

As \( \rho \) is a state with exponential clustering of correlation, we have that:

\[
\left| \langle E_i \rangle \cdots \langle E_j \rangle - \langle E_i \rangle \cdots \langle E_j \rangle \right| \leq C |X| e^{-\lambda r} \leq C |X| e^{-\lambda r}, \tag{39}
\]

for all \( i, j \in \{1, \ldots, n\} \). But, more than that, as the minimum distance between the supports of the observables is \( r \), given \( \{ E_1, \ldots, E_n \} \), we have that \( \{ E_1, \ldots, E_n \} \) is an observable supported in \( X_1 \cup \cdots \cup X_n \) and the distance between this larger region and the \( X_i \) is still at least \( r \). Thus, again by the definition of a state with exponential clustering of correlations:

\[
\left| \langle E_i \cdots E_j \rangle - \langle E_i \rangle \cdots \langle E_j \rangle \right| \leq (k - 1) |X| e^{-\lambda r}. \tag{40}
\]

From this result, we will show by induction that for every \( k \in \{1, \ldots, n\} \):

\[
\left| \langle E_1 \cdots E_k \rangle - \langle E_1 \rangle \cdots \langle E_k \rangle \right| \leq (k - 1) |X| e^{-\lambda r}. \tag{41}
\]

The case \( k = 1 \) is trivial, corresponding to \( \langle E_i \rangle = \langle E_i \rangle \), while the case \( k = 2 \) rewrites expression (39). Suppose then, by induction, that the result is true for \( k = m - 1 < n \), that is:

\[
\left| \langle E_1 \cdots E_{m-1} \rangle - \langle E_1 \rangle \cdots \langle E_{m-1} \rangle \right| \leq (m - 2) |X| e^{-\lambda r}. \tag{42}
\]

Multiplying these inequalities by \( \langle E_{m} \rangle \), we have:

\[
- (m - 2) \left| \langle E_{m} \rangle \right| |X| e^{-\lambda r} \leq \left| \langle E_{m} \rangle \cdots \langle E_{m} \rangle \right| \leq (m - 2) |X| e^{-\lambda r}. \tag{43}
\]

Recalling that \( \left| \langle E_{m} \rangle \right| \leq 1 \), it follows:

\[
- (m - 2) |X| e^{-\lambda r} \leq \left| \langle E_{m} \rangle \cdots \langle E_{m} \rangle \right| \leq |X| e^{-\lambda r}. \tag{44}
\]

From (40), we have

\[
- |X| e^{-\lambda r} \leq \langle E_{m} \rangle \cdots \langle E_{m} \rangle \leq |X| e^{-\lambda r}. \tag{45}
\]

Thus, adding (44) and (45), we get

\[
- (m - 1) |X| e^{-\lambda r} \leq \langle E_{m} \rangle \cdots \langle E_{m} \rangle \leq (m - 1) |X| e^{-\lambda r}. \tag{46}
\]

That is,

\[
\left| \langle E_{m} \rangle \cdots \langle E_{m} \rangle \right| \leq (m - 1) |X| e^{-\lambda r}. \tag{47}
\]

And so, we concluded the result by induction.

### 4.7 Result 7

**Result 7.** If \( \rho \) is the ground state of a gapped Hamiltonian of the lattice, then there exist \( C, \lambda > 0 \) such that for every \( X_1, \ldots, X_N \subseteq \Omega \) disjoint regions we have

\[
B_{XY}^X \leq \Delta_C + C |X| e^{-\lambda r}.
\]

From Theorem 1 and Proposition 3, we have:

\[
\gamma_{k_1, \ldots, k_n}^{(i_1, \ldots, i_n)} \langle E_{k_1}^{(i_1)} \cdots E_{k_n}^{(i_n)} \rangle \leq C |X| e^{-\lambda r} (n - 1) \left| \gamma_{k_1, \ldots, k_n}^{(i_1, \ldots, i_n)} \right|.
\]

Then, applying this inequality in (7), we have

\[
B_{XY}^X (\rho, E^{(1)}, \ldots, E^{(N)}) \leq (N - 1) \sum_{n=1}^{N} \sum_{i_1 \neq \ldots \neq i_n} \sum_{k_1, \ldots, k_n} \left| \gamma_{k_1, \ldots, k_n}^{(i_1, \ldots, i_n)} \right|.
\]

Let us define:
\[ \tilde{B}_{\text{Bell}}^{X_1, ..., X_N}(\rho, E^{(1)}, ..., E^{(N)}) = \sum_{n=1}^{N} \sum_{i_1 \neq \ldots \neq i_n=1}^{M_i \ldots M_n} \sum_{k_1 \ldots k_n=0}^{C} \gamma_{k_1, \ldots, k_n}(E_{k_1}^{(1)} \ldots E_{k_n}^{(n)}). \]  

(49)

So again, \( \tilde{B}_{\text{Bell}}^{X_1, ..., X_N}(\rho, E^{(1)}, ..., E^{N}) \) represents an uncorrelated system and as such the local bound must be preserved. Therefore,

\[ B_{\text{Bell}}^{X_1, ..., X_N}(\rho, E^{(1)}, ..., E^{N}) \leq \Delta_C + C |X| e^{-\lambda r}, \]  

(50)

where

\[ \Gamma = \sum_{n=1}^{N} \sum_{i_1 \neq \ldots \neq i_n=1}^{M_i \ldots M_n} \sum_{k_1 \ldots k_n=0}^{C} (n-1) \left| \gamma_{k_1, \ldots, k_n}(E_{k_1}^{(1)} \ldots E_{k_n}^{(n)}) \right|. \]  

(51)

The right-hand side of the inequality (50) does not depend on what measurements have been taken. Therefore, taking the supremum over them, we have

\[ B_{\text{Bell}}^{X_1, ..., X_N}(\rho) \leq \Delta_C + C |X| e^{-\lambda r}. \]  

(52)

### 4.8 Result 8

**Result 8.** If \( \rho(\beta) \) is a thermal state acting on the lattice with inverse temperature \( \beta \) less than a fixed \( \beta^* \), then there exist \( C, \lambda > 0 \) such that for every \( X_1, ..., X_N \subset \Omega \) disjoint regions we have

\[ B_{\text{Bell}}^{X_1, ..., X_N}(\rho(\beta)) \leq \Delta_C + C |X| e^{-\lambda r}. \]

The proof of this result is the same as that of the previous one, the only change is in the use of Result 2 in place of Theorem 1.

### 5 Conclusion

In this paper, we investigated non-local aspects of many-body quantum systems. More precisely, using clustering theorems, we demonstrated that relevant classes of quantum states are unable to exhibit significant non-locality.

First, exploring the CHSH scenario for spin lattices, we were able to show that for agents acting only on distant regions of the lattice, the ground state of gapped Hamiltonians can at most show exponentially bounded violations of any Bell inequality. Second, we also managed to prove that for thermal states, this behaviour is even more restrictive, as there is a minimum distance between regions that screens-off any non-local effect. Finally, we discussed how come non-local correlations evolve in time when the initial state is a simple product state.

Scenarios with more parties and more measurements were also investigated, first focusing only on Bell inequalities involving one-body and two-body correlators. A more complete generalization is given in Results 7 and 8.

It is natural to ask why we have assumed gapped systems to begin with. Remarkably enough, there is an example in the literature showing this is an assumption we needed to demand. In Ref. [11], the ground state of a lattice with short-range interactions is considered, and it is observed that pairs of distant sites do have a violation of CHSH close to the quantum bound. We hope the mathematical toolbox we have provided here can be used to explain why there is this discrepancy between gapped and un-gapped systems when considering non-local aspects.

To emphasize the non-triviality of our results, we draw attention to how counter-intuitive many-body quantum systems can be. As an example, in Ref. [40] (Proposition 1) they built a state having an overlap arbitrarily close to a product state. Despite that, the reduced density matrix of a specific region \( A \) fulfills a so-called volume law in terms of their entanglement content, i.e. \( S(\rho_A) = \mathcal{O}(|A|) \). Therefore, having a considerable amount of entanglement. Simply put, in Ref. [40] the authors were able to find a multipartite state extremely close to being a product but whose entanglement scales with the volume of the lattice. Our results point towards a different direction though, as we were able to formally proof that this type of peculiar behaviour does not occur for Bell non-locality in physically relevant classes of states.

Another point worth mentioning is the relationship between entanglement and non-locality. For a long time, it was thought that Bell’s inequality violation varied monotonically with the amount of entanglement [41]. However, in recent results, it has been shown that in some scenarios, violation and entropy of entanglement are practically independent [42], and in more extreme cases which may even be inversely proportional [43]. More concretely, in [42] (Corollary 1) it is shown that for any \( \delta > 0 \), there is \( n \) large enough and one Bell inequality in a \( (2, n) \) scenario such that the ratio between the quantum violation and the classic bound is greater than \( \sqrt{n} \log(n) \) for a state with entropy of entanglement \( E(|\psi|) \leq \delta \). Therefore, although entanglement is necessary to obtain a violation of Bell’s inequalities, the amount of entanglement can be irrelevant.

We would like to highlight that it is also possible to obtain robust non-local violations for the states we worked out here. This apparent dichotomy arises from a choice of a different set of physical assumptions. In fact, in [21] investigating an antiferromagnetic Heisenberg model on thermal equilibrium, the authors found a Bell inequality involving only
two-body correlators (5) that is robustly violated up to high temperatures.

Speaking of further works, considering open quantum systems rather than closed ones as we did here, in Ref. [44, 45] the authors generalized both the clustering theorem for gapped ground states (Thm. 1) and the propagation of correlations (Thm. 3). Because these were basic results we built our results upon, we believe that it is also possible to restate Results 1, 3, 4, 6, and 7 into the framework of open quantum systems. For being more realistic, functioning also a toy model for quantum memories, we believe that the local aspects of shown by our results can become even more pronounced in this new scenario.

In conclusion, the main message we wanted to put out with this work is that under certain physical assumptions, a large family of quantum systems with many parts behave as classical, local systems. We hope this paper can make a bridge between rather abstract foundations of quantum physics and more palpable many-body quantum physics. This interchange benefits both areas.

Acknowledgements Many thanks to R. Rabelo for all the stimulating discussion.

Funding This paper is a result of the Brazilian National Institute of Science and Technology on Quantum Information. This work is supported by the Brazilian agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico, Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, FAEP, and by the National Research, Development and Innovation Office of Hungary (NKFIH) through the Quantum Information National Laboratory of Hungary and through the grant FK 135220. This project/research was supported by grant number FQXi-RFP-IPW-1905 from the Foundational Questions Institute and Fetzer Franklin Fund, a donor advised fund of Silicon Valley Community Foundation. CD was supported by a fellowship from the Grand Challenges Initiative at Chapman University.

Data Availability All the necessary data and material are included in the main text.

Conflicts of Interest/Competing Interests The authors declare that there is no conflict of interest whatsoever with the publication and in the elaboration of this work.

References

1. J.S. Bell, On the einstein podolskyrosen paradox. Physics Physique Fizika 1(3), 195–200 (1964)
2. A. Fine, Hidden variables, joint probability, and the bell inequalities. Phys. Rev. Lett. 48, 291–295 (1982)
3. N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, S. Wehner, Bell nonlocality. Rev. Mod. Phys. 86, 419–478 (2014)
4. J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. 23, 880–884 (1969)
5. F. Altintas, R. Eryigit, Correlation and nonlocality measures as indicators of quantum phase transitions in several critical systems. Ann. Phys. (New York) 327, 12 (2012)
6. J. Batle, M. Casas, Nonlocality and entanglement in the xy model. Phys. Rev. A 82, 062101 (2010)
7. J. Batle, M. Casas, Nonlocality and entanglement in qubit systems. J. Phys. A: Math. Theor. 44(44), 445304 (2011)
8. S. Campbell, M. Paternostro, Multipartite nonlocality in a thermalized Ising spin chain. Phys. Rev. A 82(4), 042324 (2010)
9. T.R. de Oliveira, A. Saguiu, M.S. Sarandy, Nonviolation of bell’s inequality in translation invariant systems. EPL (Europhysics Letters) 100(6), 60004 (2012)
10. D.L. Deng, C. Wu, J.L. Chen, S.J. Gu, S. Yu, C.H. Oh, Bell nonlocality in conventional and topological quantum phase transitions. Phys. Rev. A 86, 032305 (2012)
11. J.C. Getelina, T.R. de Oliveira, J.A. Hoyos, Violation of the bell inequality in quantum critical random spin-1/2 chains. Physics Letters A 382(39), 2799–2804 (2018)
12. L. Justino, T.R. de Oliveira, Bell inequalities and entanglement at quantum phase transitions in the XXZ model. Phys. Rev. A 85, 052128 (2012)
13. E. Oudot, J.D. Bancal, P. Sekatski, N. Sangouard, Bipartite nonlocality with a many-body system. New J. Phys. 21(10), 103043 (2019)
14. Z. Sun, Y. Wu, H. Huang, B. Wang, Bell inequality and nonlocality in an exactly solvable two-dimensional ising–heisenberg spin systems. Solid State Commun. 185, 30–34 (2014)
15. Z.-Y. Sun, X. Guo, M. Wang, Multipartite quantum nonlocality in two-dimensional transverse-field ising models on t x n square lattices. The European Physical Journal B 92(4), 75 (2019)
16. Z.Y. Sun, M. Wang, Y.Y. Wu, B. Guo, Multipartite nonlocality and boundary conditions in one-dimensional spin chains. Phys. Rev. A 99(4), 042323 (2019)
17. Z. Wang, S. Singh, M. Navascués, Entanglement and nonlocality in infinite 1d systems. Phys. Rev. Lett. 118(23), 230401 (2017)
18. G.D. Chiara, A. Sanpera, Genuine quantum correlations in quantum many-body systems: a review of recent progress. Rep. Prog. Phys. 81(7), 074002 (2018)
19. L. Babai, L. Fortnow, C. Lund, Non-deterministic exponential time has two-prover interactive protocols. Comput. Complex. 1(1), 3–40 (1991)
20. M. Fadel, J. Tura, Bounding the set of classical correlations of a many-body system. Phys. Rev. Lett. 119, 230402 (2017)
21. I. Frérot, T. Roscilde, Detecting many-body bell nonlocality by solving ising models. Phys. Rev. Lett. 126, 140504 (2021)
22. A. Piga, A. Aloy, M. Lewenstein, I. Frérot, Bell correlations at ising quantum critical points. Phys. Rev. Lett. 123(17), 170604 (2019)
23. J. Tura, R. Augusiak, A. Sainz, B. Lucke, C. Klempt, M. Lewenstein, A. Acín, Nonlocality in many-body quantum systems detected with two-body correlators. Ann. Phys. 362, 370–423 (2015)
24. J. Tura, R. Augusiak, A.B. Sainz, T. Vertesi, M. Lewenstein, A. Acín, Detecting nonlocality in many-body quantum states. Science 344(6189), 1256–1258 (2014)
25. J. Tura, A.B. Sainz, T. Vertesi, A. Acín, M. Lewenstein, R. Augusiak, Translationally invariant multipartite bell inequalities involving only two-body correlators. J. Phys. A: Math. Theor. 47(42), 424024 (2014)
26. Z. Wang, S. Singh, M. Navascués, Entanglement and nonlocality in infinite 1d systems. Phys. Rev. Lett. 118, 230401 (2017)
27. J. Tura, G. De las Cuevas, R. Augusiak, M. Lewenstein, A. Acín, J.I. Cirac, Energy as a detector of nonlocality of many-body spin systems. Phys. Rev. X 7, 021005 (2017)
28. R. Schmied, J.D. Bencal, B. Allard, M. Fadel, V. Scarani, P. Treutlein, N. Sangouard, Bell correlations in a bose-einstein condensate. Science 352(6284), 441–444 (2016)
29. N.J. Engelsen, R. Krishnakumar, O. Hosten, M.A. Kasevich, Bell correlations in spin-squeezed states of 500000 atoms. Phys. Rev. Lett. 118, 140401 (2017)
30. B.S. Tsirelson, Quantum generalizations of bell’s inequality. Letters in Mathematical Physics 4(2), 93–100 (1980)
31. A.F. Ducuara, P. Skrzypczyk, Operational interpretation of weight-based resource quantifiers in convex quantum resource theories. Phys. Rev. Lett. 125(11), 110401 (2020)
32. M.B. Hastings, T. Koma, Spectral gap and exponential decay of correlations. Communications in Mathematical Physics 265(3), 781–804 (2006)
33. M. Kliesch, C. Gogolin, M.J. Kastoryano, A. Riera, J. Eisert, Locality of temperature. Phys. Rev. X 4, 031019 (2014)
34. B. Nachtergaele, R. Sims, Lieb-robinson bounds and the exponential clustering theorem. Communications in Mathematical Physics 265(1), 119–130 (2006)
35. K. Fredenhagen, A remark on the cluster theorem. Communications in Mathematical Physics 97(3), 461–463 (1985)
36. E.H. Lieb, D.W. Robinson, The finite group velocity of quantum spin systems. Comm. Math. Phys. 28(3), 251–257 (1972)
37. E.G. Cavalcanti, R. Lal, On modifications of reichenbach’s principle of common cause in light of bell’s theorem. J. Phys. A Math. Theor. 47(42), 424018 (2014)
38. A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency (Springer-Verlag, Algorithms and Combinatorics, 2004)
39. B. Nachtergaele, Y. Ogata, R. Sims, Propagation of correlations in quantum lattice systems. Journal of Statistical Physics 124(1), 1–13 (2006)
40. H. Wilming, M. Goihl, I. Roth, J. Eisert, Entanglement-ergodic quantum systems equilibrate exponentially well. Phys. Rev. Lett. 123(20), 200604 (2019)
41. R.F. Werner, M.M. Wolf, Bell inequalities and entanglement. Quantum Inf. Comput. 1(3), 1–25 (2001)
42. M. Junge, C. Palazuelos, Large Violation of Bell Inequalities with Low Entanglement. Communications in Mathematical Physics 306(3), 695–746 (2011)
43. G. Vallone, G. Lima, E.S. Gomez, G. Canas, J.A. Larsson, P. Mataloni, A. Cabello, Bell scenarios in which nonlocality and entanglement are inversely related. Phys. Rev. A 89(1), 012102 (2014)
44. M.J. Kastoryano, J. Eisert, Rapid mixing implies exponential decay of correlations. J. Math. Phys. 54(10), 102201 (2013)
45. D. Poulin, Lieb-Robinson bound and locality for general markovian quantum dynamics. Phys. Rev. Lett. 104(19), 190401 (2010)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.