A CONTINUOUS FRAMEWORK FOR FAIRNESS

PHILIPP HACKER AND EMIL WIEDEMANN

ABSTRACT. Increasingly, discrimination by algorithms is perceived as a societal and legal problem. As a response, a number of criteria for implementing algorithmic fairness in machine learning have been developed in the literature. This paper proposes the Continuous Fairness Algorithm (CFAt) which enables a continuous interpolation between different fairness definitions. More specifically, we make three main contributions to the existing literature. First, our approach allows the decision maker to continuously vary between concepts of individual and group fairness. As a consequence, the algorithm enables the decision maker to adopt intermediate “worldviews” on the degree of discrimination encoded in algorithmic processes, adding nuance to the extreme cases of “we’re all equal” (WAE) and “what you see is what you get” (WYSIWYG) proposed so far in the literature. Second, we use optimal transport theory, and specifically the concept of the barycenter, to maximize decision maker utility under the chosen fairness constraints. Third, the algorithm is able to handle cases of intersectionality, i.e., of multi-dimensional discrimination of certain groups on grounds of several criteria. We discuss three main examples (college admissions; credit application; insurance contracts) and map out the policy implications of our approach. The explicit formalization of the trade-off between individual and group fairness allows this post-processing approach to be tailored to different situational contexts in which one or the other fairness criterion may take precedence.

1. INTRODUCTION

Suppose a classification decision needs to be made for a number of individuals that fall into different groups. We call the agent or institution taking this decision the decision maker. For instance, the decision maker could be a financial institution deciding whom to grant a certain loan. Other relevant scenarios are college admissions, eligibility for parole in criminal justice, hiring or insurance decisions, see [20] for a thorough discussion. In most of the scenarios we are interested in, decision makers increasingly employ algorithmic methods to reach their decisions. We therefore seek to establish a novel framework that subjects the decision maker to fairness constraints which can be operationalized in a setting of algorithmic decision making[1] these constraints facilitate a trade-off between individual and group fairness (thereby minimizing bias in algorithmic decision making) while preserving as much utility for the decision maker as possible.

For each individual, certain observable data is available, like credit history, degree certificates, scores quantitatively reflecting an interviewer’s judgment, etc. Part of this data might have been collected or processed through procedures biased against certain groups of individuals. Evidence is mounting that bias indeed haunts algorithmic decision procedures, particularly those based on machine learning methods (cf. [4,17]). Moreover, even if the data was handled correctly, it might allow for inferences about an individual’s ethnicity, gender, or other criteria about membership in legally protected groups. Finally, it might contain private information inappropriate to be disclosed to the decision maker, or which should not be made available to the decision maker for some other reason. In other words, in a number of settings, full disclosure of the observable data to the decision maker is likely to result in unfair decisions.

1 We would like to point out that our framework also covers cases of non-algorithmic decision making and thus applies widely to decisions implicating fairness.
Instead, in our framework an impartial agent, like a regulatory body, will transform the observable “raw data” into a fair representation of the data (cf. [31] for the terminology), which can then be used by the decision maker to take its decision. For instance, if the observable data is biased against one group of individuals, the data would be “corrected” by being transported to a fair representation that mitigates, or even fully removes, this bias.

The map from an individual’s observable data to the fair representation is the main subject of this work. It should satisfy, as closely as possible, a number of requirements reflecting the “fairness” of the process while maintaining decision maker utility to a maximum:

(1) Individual fairness: Similar observable data should be mapped to similar fair data.
(2) Group fairness: The fair data of an individual should not allow for any inference on the individual’s group membership (statistical parity).
(3) Monotonicity: If one individual scores better in the raw data, then they should also score better, or equally well, in the fair data.
(4) Decision maker utility: The fair data should conceal as little information as possible to the decision maker.

These requirements will be defined more rigorously below (see Section 2). They have been well-studied except for monotonicity, and it is known [14] that they cannot all be perfectly satisfied at the same time; this is a general problem in fairness constraints in algorithmic decision making [10, 19]. For instance, if groups have different raw score distributions, it will be necessary to attribute different fair scores to two individuals with similar raw scores if they belong to different groups in order to achieve group fairness; however, this will violate monotonicity and individual fairness. If, as will typically be the case, different groups exhibit different statistics in the raw data, then indeed the only way to satisfy requirements (1–3) simultaneously is to assign the same fair score to every individual (cf. [19]). This, of course, would neither be “fair” in any meaningful sense, nor would it be of any use to the decision maker, as they would be completely “blindfolded” in making their decision. Correspondingly, requirement (4) would be violated to an extreme extent in this case.

Our general discussion is close to the works of Dwork et al. [12] and of Feldman et al. [13]. However, as the model will show in greater detail, we will introduce a novel framework which greatly facilitates the mathematical analysis of the “raw-to-fair” map. One of the main novelties of this paper is thus to work in a continuous rather than a discrete setting, in the Euclidean space \( \mathbb{R}^n \). We would like to stress at the outset that our framework is continuous in a dual sense. First, our probability measures, i.e., the distributions of (raw and fair) scores across groups, are continuous (absolutely continuous with respect to Lebesgue measure, to be precise). This not only allows for highly fine-grained scoring, but it is also a mathematical prerequisite for the existence of a unique optimal transport map that we will use in our model to maximize decision maker utility. The choice of a continuous probability measure is justified as long as the number of individuals in question is sufficiently large, and as long as it is unlikely that many of the individuals have exactly the same raw score. Continuous distributions are frequently used in economics and admit a richer mathematical theory. In this paper, specifically, we will exploit some elements of the mathematical theory of optimal transport. Working with continuous distributions is generally not a significant restriction, as every continuous distribution can be arbitrarily approximated by discrete ones in Wasserstein distance (i.e. in the weak topology), and vice versa.

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Footnote:

2 In particular this requires that the raw data contains sufficiently fine degrees of evaluation. If this is granted, then adding some stochastic noise to the evaluation of the raw data will remove undesired “concentration” effects, if necessary.
Second, another type of continuity resides in the possibility to variably choose a parameter, $\theta$, that allows us to skew the model more towards individual or toward group fairness, as conditions warrant. Friedler et al. [14] consider not only the observable data and the decision finally made by the decision maker, but also a so-called construct space, which can be thought to contain the “actual” or “true” properties of every individual. The (possibly inaccurate or biased) evaluation of the “true” data is then modelled by a map from construct space to observable space. Construct space and the construct-to-observable map are, by definition, not measurable or observable in any way. Rather, the relation between construct and observable space must be postulated in an axiomatic fashion. To this end, the authors of [14] propose two extreme “worldviews”: WAE (“we’re all equal”) and WYSIWYG (“what you see is what you get”). In the first case, it is assumed that any differences between groups in the distribution of the raw data in the observed space are due to discrimination, incorrect data handling, or other exogenous factors, and that these differences should therefore not be visible in construct space; insofar as it is relevant for the decision, different groups are assumed to have the same ”true” distributions of scores (which differs from the observable distributions). In the second case, the assumption is that the observable data truly reflects the properties of the individuals and can thus be immediately used for decision making. In this worldview, effects of bias or inaccurate collection of data are either flatly denied or tolerated. This assumption therefore collapses the distinction between construct and observed space. Both assumptions, however, are extreme cases: either groups are postulated to be perfectly equal in construct space, or data in the observed space is assumed to be perfectly correct.

A main contribution of this paper is to add nuance to these distinct worldviews. We suggest a framework that allows to continuously interpolate between WAE and WYSIWYG. Indeed, in the mathematical theory of optimal transport, the technique of displacement interpolation is used to continuously move from one probability distribution to another in a particularly natural way, and we apply this tool in the context of fair representations. To this end, we introduce a parameter, $\theta$, that allows us to continuously move from WAE to WYSIWYG. As we will see, this implies that we can equally move from a maximal fulfillment of individual (WYSIWYG: $\theta = 0$) to a maximal fulfillment of group fairness (WAE: $\theta = 1$). The resulting Continuous Fairness Algorithm (CFA $\theta$) allows us to formalize the trade-off between these fairness concepts, and therefore to adapt the framework on a case-by-case basis to different decision making contexts in which different fairness constraints may be normatively desirable. We would like to stress that our model is able to guarantee monotonicity within groups, but not for members belonging to different groups; this is also the source of the violation of individual fairness in our transportation exercise. Furthermore, since our model operates with a continuous distribution, the need for arbitrary randomization in order to achieve a fair distribution is minimized.$^3$

Quite obviously, one key question for the construction of a fair representation is the choice of the target representation for $\theta = 1$: among all possible target measures, one ought to find the one that optimizes individual fairness and decision maker utility. Indeed, the requirement, stemming from the desire to achieve statistical parity, that all the raw score distributions of the different groups be mapped onto one single representation (in this extreme case) does not say anything about what this distribution should look like. Dwork et al. [12] (at 221) choose the distribution of the privileged group as the target distribution. By contrast, we show that there is a potentially more convincing target distribution which occupies a “middle ground” between the distributions of all the different

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$^3$ This is due to the fact that a necessary precondition for the need to randomize is that different individuals have exactly the same fair score; this is excluded in a continuous setting, and can be neglected in the discrete setting if the evaluation procedure is sufficiently fine-grained, see the discussion above.
groups (the so-called barycenter with respect to Wasserstein-2 distance). Choosing the barycenter as the target distribution has two important advantages: first, it does not impose the distribution of one “privileged” or majority group onto the other groups. Second, it is the distribution that is closest to all the raw distributions in a least square sense; therefore, it preserves decision maker utility to a maximum if we use optimal transport theory to map all raw distributions onto the barycenter. The existence of such an intermediate distribution, the barycenter, is mathematically highly nontrivial and was only proved in 2011 by [1] under certain assumptions that, fortunately, are plausible for many scenarios of algorithmic decision making.

The approach that is probably closest to ours is [13] (at 11-15). In contrast to their paper, which uses a ”median” distribution (i.e. the barycenter in the Wasserstein-1 metric), our approach introduces the barycenter in Wasserstein-2 (which operates with the notion of least squares in Wasserstein-2 distance), and uses the unique optimal transport map from raw to fair scores. Quadratic cost and distance functions seem plausible if one assumes that the utility of output data declines ever more sharply (and not only linearly) the more it is removed from a sufficiently precise approximation of the truth; in other words, if the distance between an output and ground truth becomes too large, the data is almost of no use to the decision maker any more. This seems realistic particularly when important differential consequences are attached to different outputs (as in our examples); arguably, it is only in such cases of important distinctions that a decision maker will resort to an algorithmic model in the first place. The barycenter allows us not only to maximize decision maker utility under fairness constraints, but also to vary between different fairness measures (individual vs. group/WYSIWYG vs. WAE), depending on the concrete decision-making framework our algorithm is applied to. Further important differences between our approach and the one adopted in [13] are that we are able to handle high-dimensional raw scores; and cases of intersectionality (see next paragraph).

To summarize, our CFAθ has a number of advantageous features. First, in the WAE case (θ = 1), group fairness is fulfilled as well as within-group monotonicity. Second, individual fairness and decision maker utility are optimized by the choice of the barycenter as the target distribution, and the optimal transport toward it. We may, however, also choose θ differently so that the resulting distribution does not correspond to full group fairness, but still fulfills the so-called 80 percent rule, an important threshold for US disparate impact doctrine ([4], at 701 et seq.) that is increasingly gaining traction in EU anti-discrimination law, too. It requires that the probability of a member of a disprivileged group being positively labeled is at least 80 percent of the respective probability of a member of the privileged group, see [13, 29]. Hence, θ can be consciously chosen to force compliance with existing antidiscrimination legislation. As the discussion of the policy implications of our model shows, the choice of θ is a deeply normative one and can be adapted to different situations in which individual or rather group fairness should be the primary goal. Third, we are able to handle the problem of intersectionality, i.e. the phenomenon that an individual may belong to several (protected) groups at the same time. As a recent judgment by the Court of Justice of the European Union has shown, intersectionality presents a pressing problem in real-life decisions. As mentioned, in our approach to it, in contrast to [12], we do not move the data distribution of a discriminated group to that of the privileged one, but rather we transport the distributions of

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4 In the one-dimensional case, a kind of barycenter has been used by Feldman et al. [13], but with respect to the Wasserstein-1 distance.

5 These conditions include absolute continuity of the raw distribution, which can be arbitrarily approximated by discrete distributions in Wasserstein space, as noted above; and a quadratic cost (or utility) function, a condition that can be fulfilled by initially transforming the utility function of the decision maker appropriately.

6 CJEU case C-443/15 Parris ECLI:EU:C:2016:897.
all groups to respective intermediate representations that are chosen by displacement interpolation (the \( \theta \) score). Importantly, we may choose \( \theta \) differently for different groups; this allows us to treat particularly disadvantaged groups (for example those that fulfill several protected criteria at the same time) “fairer” (with respect to group fairness) than other ones, if so desired.

Importantly, in all of these data corrections by the CF \( \theta \), we take the raw score as a given output of a prior classification task; therefore, the full force of machine learning can be unleashed to calculate the raw score which is only transformed into a fair score after its elaboration. While being a stand-alone procedure, our approach may nevertheless be fruitfully complemented by fairness or data collection/quality constraints applied to the calculation of the raw score itself (see below, Section 4).

The remainder of the paper is organized as follows: Section 2 introduces the mathematical model and establishes its fundamental optimality properties. Readers unfamiliar with mathematical notation may jump right to Section 3 which discusses a number of examples. Section 4 places the model within the broader framework of related work. Section 5 offers policy implications. Section 6 discusses limitations and future research questions.

2. The Model

Let \( X \) be a set of individuals that may have certain traits indexed by \( 1, \ldots, N \). We are given a group membership map \( g : X \to \{0;1\}^N \) whose \( i \)-th component indicates whether or not an individual carries trait number \( i \). This induces a partition of \( X \) via

\[
X = \bigcup_{k \in \{0;1\}^N} g^{-1}(k)
\]

into at most \( 2^N \) groups \( X_k := g^{-1}(k) \) (note that some of these could be empty). Let us call the number of (non-empty) groups \( G \).

Various qualitative data may be collected from individuals in the process of decision-making. This includes qualitative data such as personal interviews, expert opinions, letters of motivation, etc. Obviously, it should not be the objective of an abstract theory of fairness to design a map from this data to a quantitative ranking that reflects the decision maker’s preferences. Rather, we assume such a map as given, and therefore our starting point is a score function \( S : X \to \mathbb{R}^n \). The score, which may be composed of \( n \) partial scores for different categories, is assumed to express the decision maker’s utility function in the sense that if the decision maker had the full observable data at their disposal, then they would always prefer an individual with a higher score in each category to one with a lower one.

The restrictions of \( S \) to \( X_k \) are denoted \( S_k \). On \( \mathbb{R}^n \), we introduce probability measures \( \mu \) and \( \{\mu_k\}_{k=1}^G \) that encode the score distribution within the entire set of individuals, and in the \( k \)-th group, respectively. This means that \( \mu(B) \) is the proportion of individuals with a score contained in a subset \( B \subset \mathbb{R}^n \), and likewise \( \mu_k(B) \) denotes the proportion of individuals with score in \( B \) in the \( k \)-th group. If we index the individuals themselves by a vector in \( \mathbb{R}^n \), i.e. \( X = \mathbb{R}^n \), then we can write

\[
w_k := |X_k| = |g^{-1}(k)|
\]

\( ^7 \) In many cases, \( n = 1 \) may be sufficient. Indeed, in most real-life applications, at some point all the information about the individuals must be brought into a linear ordering.

\( ^8 \) We use the term “probability measure” in the sense of “normed measure”; no randomness is insinuated by this terminology.
for the proportion of members of group $k$ in the total population $X$, and

$$\mu_k = dx^n \circ S_k^{-1}.$$  

where $dx^n$ is the Lebesgue measure (i.e., the Euclidean volume, or, in case $n = 1$, the length). Accordingly, we have

$$\mu = dx^n \circ S^{-1} = \sum_{k=1}^{G} w_k \mu_k.$$  

Our decisive continuity assumption can be stated as follows:

**Assumption A1.** The measures $\mu$ and $\mu_k$, $k = 1, \ldots, G$, are absolutely continuous with respect to Lebesgue measure and have finite variance, with densities $f \in L^1(\mathbb{R}^n)$ and $f_k \in L^1(\mathbb{R}^n)$.

The fair representations for each group will be maps $T_k$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. They transform a “raw score” into a “fair score” for group $k$. The resulting fair score for an individual $x \in X_k$ is then given by $T_k(S(x))$. The raw-to-fair map $T_k$ will transport the measure $\mu_k$ to the pushforward measure

$$\nu_k = \mu_k \circ T_k^{-1}$$

representing the “fair score distribution” for group $k$.

We will ask the converse question: Given a target distribution $\nu_k$, how can we choose $T_k$ so that $\nu_k = \mu_k \circ T_k^{-1}$? Any such map $T_k$ is called a transport map from $\mu_k$ to $\nu_k$. In general, there are many such transport maps. In the sequel we will describe how to choose $\nu_k$ and corresponding transport maps $T_k$ in order to guarantee a maximal amount of fairness in the sense of requirements (1)–(4), using an algorithm which we call (CFA-$\theta$). Once these data have been constructed, the decision maker will be presented the distribution

$$\bar{\nu} := \sum_{k=1}^{G} w_k \nu_k$$

and will make a decision based on this distribution; an individual $x \in X_k$ will thus be classified through their fair score $T_k(S(x))$.

We first need to impose target distributions $\nu_k$. To this end, let us discuss the extreme worldviews WYSIWYG and WAE, which will form the endpoints of our interpolation, within our model:

**WYSIWYG.** When the “raw” score is deemed a true and fair representation of reality, nothing needs to be done, hence we set $\nu_k = \mu_k$ for all $k = 1, \ldots, G$ (thereby maximising decision maker utility), and accordingly $\nu = \mu$. Monotonicity then forces $T_k = id$ for all $k$, thus optimising individual fairness at the same time. Of course, if $\mu_k$ are different, then group fairness is violated.

**WAE.** Under the hypothesis that any differences between the $\mu_k$ emerge solely from undesired exogenous factors, such as bias, and should be removed, there should exist a single target distribution $\nu$ independent of $k$ so that $T_k$ transports $\mu_k$ to $\nu$. This produces statistical parity, hence group fairness is optimised, whereas the other three requirements will typically be violated. The problem is then to find the common fair distribution $\nu$ and the transport maps $T_k$ minimising these violations.

**Definition 2.1 (Optimal transport map).** Let $\mu, \nu$ be two probability measures on $\mathbb{R}^n$ with finite variance. An optimal transport map is a transport map between $\mu$ and $\nu$ that minimises the cost
The term “barycenter” is motivated by analogy with the Euclidean case, where the center of mass of points \( x_k \) with weights \( w_k \) is precisely the least square approximation of these weighted points, i.e. the minimiser of the expression \( \sum_k w_k |x - x_k|^2 \). In the special case of two groups...
with score distributions $\mu_1, \mu_2$ and weights $w_1, w_2$, respectively, the barycenter is given by the displacement interpolation:

$$v = \mu_1 \circ (T^{w_2})^{-1},$$

where $T$ is the optimal transport map between $\mu_1$ and $\mu_2$ and $T^{w_1}$ is defined as in Definition 2.3.

We will need some elementary properties of the barycenter (we assume from now on that every $\mu_k$ is invertible, it is possible to write $3.8$ in [1]), then the map

$$T_k : x \mapsto T_k(x - z_k) + \bar{z}$$

is defined as in Definition 2.3. As translations are invertible, it is possible to write $S_k^{\ast} : x \mapsto S_k(x - z_k) + \bar{z}$.

Proof. We denote the expectation of a probability measure by $E$. If $\nu$ is the barycenter of measures $\{\mu_k\}$ with weights $\{w_k\}$, then the barycenter of the translations $\{(\mu_k)_{\bar{z}}\}$ with the same weights is given by $\nu$, where $\bar{z} = \sum_{k=1}^G w_k z_k$.

Let $T_k$ be the optimal transport map from $\mu_k$ to $v$ (whose existence is guaranteed by Proposition 3.8 in [1]), then the map

$$T_k^* : x \mapsto T_k(x - z_k) + \bar{z}$$

is a transport map from $(\mu_k)_{\bar{z}}$ to $\nu$, and therefore

$$\sum_{k=1}^G w_k W_2^2((\mu_k)_{\bar{z}}, \lambda) < \sum_{k=1}^G w_k W_2^2((\mu_k)_{\bar{z}}, \nu).$$

(2.4)

We compute

$$\sum_{k=1}^G w_k \int_{\mathbb{R}^n} |T_k^*(x) - x|^2 d(\mu_k)_{\bar{z}} = \sum_{k=1}^G w_k \int_{\mathbb{R}^n} |T_k(x - z_k) - (x - z_k) + \bar{z} - z_k|^2 d(\mu_k)_{\bar{z}}$$

$$= \sum_{k=1}^G w_k \int_{\mathbb{R}^n} |T_k(x) - x + \bar{z} - z_k|^2 d\mu_k$$

(2.5)

On the other hand, let $S_k^* : x \mapsto S_k(x - z_k) + \bar{z}$ be the optimal transport map from $(\mu_k)_{\bar{z}}$ to $\lambda$. As translations are invertible, it is possible to write

$$\sum_{k=1}^G w_k \int_{\mathbb{R}^n} |S_k^*(x) - x|^2 d\mu_k + \sum_{k=1}^G w_k |\bar{z} - z_k|^2 + \sum_{k=1}^G w_k (\bar{z} - z_k) \cdot (E[v] - E[\mu_k]).$$

Lemma 2.5. Let $\mu$ be a probability measure on $\mathbb{R}^n$, then the translation of $\mu$ by a vector $z \in \mathbb{R}^n$ is defined as

$$v_z(B) = v(B - z)$$

for any measurable $B \subset \mathbb{R}^n$. It holds that

i) If $v$ is the barycenter of measures $\{\mu_k\}$ with weights $\{w_k\}$, $k = 1, \ldots, G$, then the barycenter of the translations $\{(\mu_k)_{\bar{z}}\}$ with the same weights is given by $v_z$, where $\bar{z} = \sum_{k=1}^G w_k z_k$.

ii) If $T$ is the optimal transport map from $\mu$ to $v$, then $T + z$ is the optimal transport map from $\mu$ to $v_z$.

iii) The barycenter of measures with zero expectation has itself zero expectation.
Continuous Fairness Algorithm (CFA).

As a generalization, we may also pick different values of \( \theta \) between \( \mu_k \) and \( \nu \). Then any minimizer must indeed have the property \( \mathbb{E}(\nu - \mathbb{E}[\nu]) = 0 \). Choosing \( \nu = -E[\nu] \), we discover

\[
\sum_{k=1}^{G} W_2^2(\mu_k, \nu) = \sum_{k=1}^{G} W_2^2(\mu_k, \nu) + |z|^2 + 2z \cdot E[\nu],
\]

where we used \( E[\mu_k] = 0 \). Choosing \( z = -E[\nu] \), we discover

\[
\sum_{k=1}^{G} W_2^2(\mu_k, \nu - E[\nu]) = \sum_{k=1}^{G} W_2^2(\mu_k, \nu) - |E[\nu]|^2,
\]

so that any minimizer must indeed have the property \( E[\nu] = 0 \). \( \square \)

We are finally ready to define our continuous fairness algorithm and prove its optimality.

**Continuous Fairness Algorithm (CFA|\( \theta \)).** Given \( \mu_k \) satisfying Assumption A1 and \( w_k \) the corresponding weights, let \( \nu \) be the barycentre. Then choose \( \nu^k \) to be the displacement interpolation \( \mu_k^\theta \) between \( \mu_k \) and \( \nu \), and \( T_k \) to be the optimal transport map between \( \mu_k \) and \( \nu_k \) (cf. Theorem 2.2).

For \( \theta = 0 \) and \( \theta = 1 \), we reobtain the algorithms described above for WYSIWyG and WAE, respectively. As a generalization, we may also pick different values of \( \theta \) for different \( k \). In this case, we would choose \( \nu^\theta \) as the displacement interpolation \( \mu_k^\theta \) between \( \mu_k \) and \( \nu \).

The question remains how the algorithm (CFA|\( \theta \)) performs with regard to our fairness requirements (1)-(4). Before we answer these questions, we have to formulate these requirements in a more precise way.

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10 This may be significant in order to handle intersectionality, as explained in the introduction.
Consider the fairness criteria in order:

(1) As discussed above, individual fairness will typically be in conflict with group fairness. As a consequence, it only makes sense to consider individual fairness within groups (see also [12] at 220). Individual fairness is usually [12][14] formulated in terms of continuity or Lipschitz continuity. In our notation, this would mean that there exists a Lipschitz constant \( L > 0 \) such that
\[
|T_k(x) - T_k(y)| \leq L|x - y| \quad \text{for all } x, y \in [0, 1] \text{ and } k = 1, \ldots, G. \tag{2.7}
\]
However, this will be impossible to satisfy in general\(^{11}\) although under certain assumptions, like the convexity of the support of the target measure, it can be guaranteed (Theorem 1.27 in [2]). Instead, we aim to characterize individual fairness in a least square sense: To this end, we consider the quantity
\[
\sum_{k=1}^{G} w_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{2} |T_k(x) - T_k(y)|^2 d\mu_k(x) d\mu_k(y). \tag{2.8}
\]
Clearly, it is non-negative, and is zero if and only if \( T_k \) takes a constant value \( \mu_k \)-almost everywhere, for all \( k = 1, \ldots, G \). This last observation implies that it is unreasonable to minimize this quantity, as the minimizing configuration would assign the same fair score to every individual. Not only would such a classification be completely useless for the decision maker, but it is also questionable if it would be “fair” in any meaningful sense of the word. Indeed, being treated similarly as a much less qualified member of one’s group is hardly any fairer than being treated completely differently than a similarly qualified one. This indicates that in (2.7), the Lipschitz modulus \( L \) should not be minimal, but as close as possible to one. In our least-square framework, therefore, instead of minimizing (2.8), we should rather minimize
\[
E_{\text{ind}} := \sum_{k=1}^{G} w_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{2} |(T_k(x) - T_k(y)) - (x - y)|^2 d\mu_k(x) d\mu_k(y), \tag{2.9}
\]
where we call \( E_{\text{ind}} \) the individual fairness error.

(2) The requirement of statistical parity imposes that the fair representations conceal any information on group membership. This means that the events “an individual belongs to group \( k' \)” and “an individual has score \( s' \)” should be stochastically independent. More precisely, we demand for each \( k \in \{1, \ldots, G\} \) and every measurable \( B \subset \mathbb{R}^n \)
\[
w_k \nu_k(B) = w_k \sum_{l=1}^{G} w_l \nu_l(B). \tag{2.10}
\]
Let now \( \nu_1, \ldots, \nu_G \) be any other set of positive numbers with \( \sum_{k=1}^{G} = 1 \). Then multiplying (2.10) with \( \nu_k \) and summing over \( k \) yields
\[
\sum_{k=1}^{G} (\nu_k - w_k) \nu_k = 0.
\]
For any measurable \( B \subset \mathbb{R}^n \), consider the vector \( \nu := (\nu_1(B), \ldots, \nu_G(B)) \). Since the choice of the \( \nu_k \) was arbitrary, the vector \( (\nu_1 - w_1, \ldots, \nu_G - w_G) \) can run through a relatively open subset of the linear subspace \( \{x : \sum_{k=1}^{G} x_k = 0\} \subset \mathbb{R}^G \), and hence \( \nu \) is orthogonal to this subspace. But the orthogonal complement of this subspace is the span of the vector \((1, \ldots, 1), \)

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\(^{11}\) The main obstruction to continuity of transport maps is the possible non-connectedness of the support of the target measure (cf. Theorem 1.27 in [2]). This is the case if, within one group, there are further subgroups with very different score statistics.
so we conclude that all components of \( \tilde{\nu} \) are equal. Since \( B \) was chosen arbitrarily, it follows that all the measures \( \nu_k, k = 1, \ldots, G \), are identical.

(3) Again, the monotonicity requirement can only be fulfilled within groups. As \( \mathbb{R}^n \) admits no linear ordering for \( n \geq 2 \), we restrict ourselves to the case \( n = 1 \). Monotonicity can be formulated as

\[
T_k(x) \leq T_k(y) \quad \text{whenever } x \leq y,
\]

for every \( k = 1, \ldots, G \).

(4) Finally, we assume that the decision maker utility is given as

\[
U = -\frac{1}{2} \sum_{k=1}^{G} w_k \int_{\mathbb{R}} |T_k(x) - x|^2 dx,
\]

where the negative sign reflects that the utility is higher if the squared difference of the raw score and the fair score is smaller. Note that vendor utility is maximized when \( T_k = id \), i.e. when the raw score and the fair score are identical.

Recall the situation \( \theta = 1 \), which reflects the WAE worldview. Then all raw distributions are mapped to the same fair distribution \( \nu \). We have argued in (2) above that this is necessary (and sufficient) for perfect group fairness in the sense of statistical parity. We have the following optimality result:

**Theorem 2.6.** Let \( \theta = 1 \). Among all possible choices of the target map \( \nu \) and of transport maps \( T_k \) from \( \mu_k \) to \( \nu \), the ones specified in CFA\( \theta \) maximize decision maker utility, minimize the individual fairness functional (2.9), and, in the case \( n = 1 \), ensure monotonicity within groups.

**Proof.** By definition, the barycenter \( \nu \) of the \( \mu_k \) with weights \( w_k \) minimizes the functional

\[
\nu \mapsto \sum_{k=1}^{G} w_k W_2^2 (\mu_k, \nu)
\]

among all probability measures on \( \mathbb{R}^n \) (cf. Theorem 2.4). On the other hand, by (2.2) and (2.3), this functional is equal to the negative utility

\[
\sum_{k=1}^{G} w_k \int_{\mathbb{R}} |T_k(x) - x|^2 d\mu_k(x)
\]

if and only if \( T_k \) is the unique optimal transportation map from \( \mu_k \) to \( \nu \). This already establishes maximal utility for our choice of \( \nu, T_k \). Note also that this choice is the unique one with this property.

By Brenier’s Theorem 2.2, the optimal transport maps \( T_k \) are cyclically monotone, which implies monotonicity in the one-dimensional setting. So it remains to show that (2.9) is also minimized by our choices of \( \nu, T_k \). Notice first that \( E_{\text{ind}} \) is invariant under translations of the measures \( \mu_k \) and \( \nu \). Thus, given \( \{\mu_k\} \) and \( \{w_k\} \) for \( k = 1, \ldots, G \), the minimal value of \( E_{\text{ind}} \) is attained replacing \( \mu_k \) by \( \bar{\mu}_k \), where \( \bar{\mu}_k \) is the translated version of \( \mu_k \) with zero expectation, and minimizing only with among measures \( \bar{\nu} \) with zero expectation.

Denoting by \( E \) the expectation of a measure, we then compute for the individual fairness error of \( \bar{\mu}_k, w_k, \bar{\nu} \), and \( \bar{T}_k \) any transport maps from \( \bar{\mu}_k \) to \( \bar{\nu} \):
$E_{ind} = \sum_{k=1}^{G} w_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{2} \left[ (T_k(x) - T_k(y)) - (x - y) \right]^2 d\mu_k(x) d\bar{\mu}_k(y) = \sum_{k=1}^{G} w_k \left[ \int_{\mathbb{R}^n} |T_k(x) - x|^2 d\mu_k(x) - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (T_k(x) - x) \cdot (T_k(y) - y) d\mu_k(x) d\bar{\mu}_k(y) \right] = -2U - \sum_{k=1}^{G} w_k |E[\bar{v}] - E[\bar{\mu}]|^2 = -2U.$

But we have already seen that the negative utility is minimized by choosing $\bar{v}$ as the barycenter of the $\bar{\mu}_k$ with weights $w_k$, and $T_k$ the optimal transport maps from $\mu_k$ to $\bar{v}$. This choice is consistent with our requirement that $E[\bar{v}] = 0$, since the barycenter of measures with expectation zero has itself expectation zero by virtue of Lemma 2.5ii).

Translating back again and recalling the invariance of $E_{ind}$ under such translations, we see that the translated measure $\nu := \bar{v}_k$, where $\bar{v}_k = \sum_{k=1}^{G} w_k E[\bar{\mu}_k]$, and the translated transport maps $T_k := T_k + E[\mu_k]$ minimizes $E_{ind}$ for our original measures $\mu_k$ among all possible choices of $\nu$ and $T_k$. The statement of the Theorem then follows by observing that $\nu$ thus chosen is precisely the barycenter of the $\mu_k$ with weights $w_k$ by Lemma 2.5i), and the $T_k$ are the optimal transport maps from $\mu_k$ to $\nu$ by Lemma 2.5i).

$\square$

3. Examples

In this section, we consider three examples that highlight different features of our fairness framework: college admissions; credit decisions; and insurance contracts.

**College Admissions.** We start by looking at college admissions decisions, a common example in the literature on algorithmic fairness, see [14]. As explained in greater detail in the model, our approach consists in transforming a raw score that individuals receive into a fair score that fulfills certain fairness requirements while simultaneously optimizing the utility of the decision maker, in this case the college, under the fairness constraints. The raw score that applicants receive will, in the case of colleges admissions, be some score that aims to predict college performance based on a number of input data, such as admissions tests; high school GPA; previous job or educational experience etc. If we care about group fairness, we will be interested in how the individual raw scores are distributed in different groups, and to what extent our Continuous Fairness Algorithm CFA$\theta$ will remedy inequality between these distributions.

For the sake of analytical clarity, let us consider a drastically simplified example. We assume that there are only two applicant groups of interest (for example, two different racial or gender groups), each consisting of an equal, large number of members. The college will admit the top-ranked 50 percent of applicants. Individuals receive raw scores that run continuously from 0 (worst) to 1 (best). Raw scores in Group A are distributed according to a compactly supported probability density function on the interval from 0 to 0.5 (e.g. a normal distribution centered at 0.25, with tails cut off at 0 and 0.5); similarly, raw scores in Group B are distributed according to a density function of the same shape from 0.5 to 1, see Figure 1. Thus, even the worst member of Group B scores better than the best member of Group A. In this extreme case, if the college was to base its admissions decision on the raw scores, it would accept all members of Group B and no members of Group A. While maintaining individual fairness within groups and also between members of different groups, this outcome violates group fairness to an extreme extent.
To apply our fairness framework, we first have to find the intermediate distribution that constitutes the barycenter. This is the distribution that minimizes the sum of the squares of the distances to both raw group distributions (in the Wasserstein metric). In our case, it is easy to see that it is the distribution of the same shape as the raw distributions which runs from 0.25 to 0.75 and which peaks at a score of 0.50. Note, however, that the characterization of the barycenter will not be as easy in more general circumstances, when the two (or more) distributions are no longer merely translates of each other.

By choosing $\theta$, we can now vary to what extent we would like to transport the raw distributions of each group toward this intermediate distribution. If we choose $\theta = 1$, our CFA$\theta$ maps the raw distributions of both groups completely onto the intermediate distribution (cf. Figure 2). If we present this novel distribution to the college, and the college applies its decision rule of accepting the top-ranking 50 percent, it will accept all those with fair scores of 0.5 or more. In this case, group fairness is fully safeguarded as it is impossible to deduce group membership from the fair score: the fair distribution is exactly the same for both groups. This is equivalent with the observation that an equal percentage of applicants is expected from each group (statistical parity). It can

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12 If the group size is odd, the college will have to randomize its admission decision for the lowest ranking pair of individuals at or above score 0.5. Of course this issue is not visible in our continuum framework.
be understood as a reconstruction of the equal distribution of scores among groups in the construct space as defined by [14] (WAE). Within groups, our use of an optimal transport map from the raw to the fair distribution guarantees monotonicity: all those that received a higher raw score than other group members receive a higher, or equal, fair score. Furthermore, the optimality constraint on the transport ensures that decision maker utility is maximized: everyone gets a fair score that is as close as possible, in the least square sense, to their raw score. However, monotonicity is violated between groups: members of the lower-ranking half of Group B each had a higher raw score than the members of the upper-ranking half of Group A, but the ranking is inverted for the fair scores. By implication, individual fairness between members of different groups is violated; for example, the top-ranking member of Group A had a similar raw score to the lowest-ranking member of Group B. However, they are now at opposite ends of the spectrum of fair scores. This is exactly the price one has to pay for group fairness, and the reason why affirmative action continues to be so contentious.

This does not exhaust the possibilities our algorithm offers, however. Rather, it admits for any degree of approximation between the raw scores, ranging from zero approximation ($\theta = 0$: fair scores = raw scores) to the full approximation just discussed ($\theta = 1$: fair scores of both groups distributed along the barycenter). For example, if $\theta = 1/2$, each of the distributions is transported “halfway” toward the barycentre distribution, as shown in Figure 3.

As noted in the introduction, $\theta$ can also be chosen so that the “80 percent rule” is fulfilled. This would mean that the probability of a member of Group A of being admitted is at least 80 percent of the respective probability of a member of Group B. In every case, the college must base its admissions decision on the fair distribution that results from the choice of a specific $\theta$. As the section on policy implications will highlight in greater detail, it is thus crucial to choose $\theta$ wisely.

We note in passing that the preceding discussion easily carries over to private companies or public agencies making hiring decisions; this is another potentially highly relevant field for algorithmic fairness since cases of algorithmic bias have already been documented in this realm, see, e.g., [21].

Credit Applications. Our second major example concerns a decision maker such as a bank that decides on applications for credit. This is an area where algorithmic decision making has already become firmly rooted, with algorithmically determined credit rating scores such as the FICO score in the US and the SCHUFA score in Germany being in wide use. A number of Fintech startups
are basing their loan decisions entirely on algorithmic models. At the same time, scholars have criticized the credit rating system as opaque and biased, see [22]. Let us therefore consider an example in which individual credit applicants are assigned a credit rating raw score based on a number of factors such as their financial history, available collateral etc. The decision rule for the institution consists in establishing a cut-off threshold at a certain score below which loan applications will be rejected. Above this score, applicants receive loan offers. The better their scores, the better the loan conditions they are offered.

Again, the CFAθ forces the institution to base its decision on a modified, fair score instead of the raw score. Just like in the previous example, the choice of a concrete θ is paramount to determine to what extent different distributions in the various applicant groups should be approximated to the barycenter distribution. Importantly, let us assume that we have two relevant, binary categories (for example, race: black and white; gender: male and female) that create four different groups. If, for example, there is evidence of particular raw score bias against blacks and against male applicants separately, but moreover of even greater bias against black male applicants, we can choose a higher θ for the transformation of the scores belonging to the latter group. This pushes these particularly biased raw scores closer to the intermediate distribution than the less biased raw scores of the black and white female and white male applicants. Hence, our framework accommodates different degrees of bias arising from intersectionality.

Insurance Contracts. A final example stems from insurance contracts. Among standard consumer contracts, insurance contracts are probably the ones that are based on the most refined statistical models. If they were not, insurance companies would be out of business fairly soon. Increasingly, insurance companies are offering personalized insurance contracts whose conditions, including premiums, are determined by a complex set of factors (see, e.g., [15]). Let us therefore imagine that an insurance company seeks to distill a risk score from the input data. In the case of car insurance, this may include accident history; driving style data; car model; age group etc.

Again, there may be significant differences in the distribution of the raw risk scores among relevant groups. What makes the case of insurance so special, at least in the EU, is the fact that the Court of Justice of the European Union, in its landmark Test Achats decision in 2011, ruled that gender may not be a determining factor in insurance pricing. While we cannot offer a comprehensive analysis of the legal implications of this judgment, it is clear that, for example in the case of car insurance, insurers may comply with the ruling by not offering women cheaper premiums than men despite statistical evidence that the former are safer drivers, see [23] at 3. Therefore, insurance companies should even have an intrinsic interest in transforming raw risk scores such that sex differences are erased from the data. Our algorithm provides for a straightforward way of achieving this: we only have to take the distributions for the male and the female group, respectively; calculate the barycenter; and fully transport the scores onto that distribution by setting θ = 1. Depending on how much leeway the law leaves to insurance companies, they (or an

13 See, e.g., https://www.kreditech.com/.
14 This pattern is basically found, for example, in the empirical study by Ayres and Siegelman [3], at 309 et seq., for offers made by car dealers to members of the respective groups. In other settings, of course, women may be the group discriminated against.
15 CJEU case C-236/09 Test-Achat ECLI:EU:C:2011:100, para. 28–34; on this, see [25].
16 See, e.g., the guidelines by the European Commission stating that “[t]he use of risk factors which might be correlated with gender [...] remains possible, as long as they are true risk factors in their own right” (European Commission, Guidelines on the application of Council Directive 2004/113/EC to insurance, in the light of the judgment of the Court of Justice of the European Union in Case C-236/09 (Test-Achats), OJ 2012 C 11/1, para. 17). Thus, the legal admissibility of correlated factors crucially depends on whether these factors can plausibly be related to the risks covered by
impartial agent) may also merely approximate male and female score distributions to one another, instead of making them identical. Here again, it may be of interest to fulfill the “80 percent rule”.

4. Relation to Other Work

The field of algorithmic fairness proliferates and is generating a staggering number of definitions of algorithmic fairness. Without laying claim to completeness, we see definitions as falling into four different categories that can be roughly attributed to different stages in the algorithmic process. First, there are data reconfiguration approaches (also called “data massaging”, in [31]) that seek to transform input data into a fairer representation (pre-processing approaches); an example is [31]. A second set of approaches seeks to control the algorithmic process from the input to the output (in-processing approaches). One subset of this group establishes constraints via a distance metric, generally to ensure individual fairness. Here, examples are [12][14]. The distance metric, in their approaches, measures distances between individuals or groups in the observed space. Then, a constraint is introduced to ensure that the output, in what they [14] call the decision space, does not significantly increase distances between groups or individuals. Another subset similarly seeks to restrain the mapping from input to output data, but by positing equal output probabilities at the group level, such as statistical parity. Examples include again [12][14] with their group fairness definition, but also [5][10][11][19] (calibration, statistical parity). Third, another type of data reconfiguration approach transforms the output data (post-processing approaches), in our case a raw score delivered by an algorithm, or some other decision procedure. Fourth, a final category looks at the relationship between predictions, i.e., output, and real outcomes (reality check approaches); examples include [5][10][19] with measures such as predictive parity, conditional use accuracy equality and error rate balance, as well as [18] with equalized odds.

As should have become clear during the discussion of our model, we combine a post-processing data reconfiguration with a process control approach. In this, our approach is related to the ones employed by [12][14][31]. However, as opposed to [12][14], we do not operate with a general distance metric, and do not include a general Lipschitz condition to guarantee that distances stay approximately the same in Wasserstein space (however, we can guarantee Lipschitz continuity under some constraints, see the discussion around (2.7)); rather, we measure individual fairness in a least square sense, and aim to ensure fairness by monotonicity, which in turn is achieved through optimal transport. Our raw score can, however, be understood as a functional equivalent of the distance metric in [12] (at 215, 224), with the difference that we generally (although not necessarily) envision the raw score already as the result of an algorithmic process. At the group level, for $\theta = 1$, our model enforces group fairness, understood as statistical parity, and can be understood as a reconstruction of the equal distribution of scores among groups in the construct space as defined by [14]. We provide a specific measure for this equal distribution, however, by the construct of the barycenter. Furthermore, by varying $\theta$ via displacement interpolation, we are able to continuously navigate between the extreme worldviews (WAE, WYSIWYG) presented in [14]. To the extent that we choose statistical parity as a measure for group fairness, our model is also similar to [32], who introduces an equal acceptance rate. Finally, in so far as we can guarantee a fulfillment of the “80 percent rule”, our approach is akin to [13][29].

Another paper that introduces a post-processing constraint that allows for varying degrees of group fairness is [30]. It does not, however, operate with the notion of the barycenter. Rather, they use a cumulative distribution function to guarantee (subject to a significance parameter) that a ranking includes at least a certain proportion of members from a protected group in its top ranks. the insurance. In machine learning contexts, where specific factors may not always be reconstructable from the output (particularly in deep neural networks), insurers can “play it safe” by approximating male and female scores.
One advantage of that measure is that it enforces only a minimum constraint: it leaves the ranking intact if (in the raw ranking) more members of the protected group are included than specified in the proportion constraint. This could be relevant particularly if the disadvantaged group has one low-scoring subgroup, but also one high-scoring subgroup that performs better than the advantaged group. Our framework will achieve a similar result if one splits the disadvantaged group into a high- and a low-scoring one, and applies a low $\theta$ to the higher scoring one. This would preserve the high scores of this subgroup, while one may transport the low scores of the other subgroup towards the barycenter by means of a higher $\theta$.

Our model does not yet capture the fourth category of approaches, those investigating the relationship between prediction and real outcome. This remains an aspiration for future extension of the model. We note, however, that we could apply our transformation to the raw scores calculated in the training phase of a machine learning model; in this way, it could extend from the post- to the pre-processing phase.

5. Policy Implications

Because of the proliferation of fairness definitions (see above, Section 4), regulators seeking to implement fairness constraints face a serious selection problem. Against this background, our model offers a convenient and flexible framework to switch between different goals within one single model. In order to have a solid basis on which to discuss the policy implications of our framework, we shall briefly revisit our main findings in non-technical terms. The Continuous Fairness Algorithm $\text{CFA}_\theta$ essentially does three things. First, it defines an intermediate score distribution (the barycenter) as the target distribution onto which all the different raw score distributions of the different, relevant groups (different gender; different races etc.) can be mapped. Second, it introduces a parameter, $\theta$, that governs the extent to which the raw score distributions of each group are transported toward the target distribution. Third, by using an optimal transport map to achieve the transformation from the raw to the fair distribution, we maximize utility for the decision maker: given the chosen fair distribution, the fair score of an individual remains as close as possible to its raw score in the least squares sense, and therefore retains as much useful information for the decision maker as possible. By choosing $\theta$, we therefore choose the optimal shape of the fair score distribution that the decision maker is presented; by using optimal transport, we achieve the best possible transformation of every single raw score onto that fair distribution. The more we approximate the barycenter distribution, the more group fairness is fulfilled, but the less individual fairness between members of different groups can be guaranteed.

Against this background, we would like to briefly discuss two policy implications: a substantive and a procedural one.

The Choice of $\theta$. On the substantive side, it is obvious that the choice of $\theta$ is crucial for a model. Ideally, the debate about what $\theta$ to choose for different situations of algorithmic decision making ought to be governed by a broad democratic discourse. Despite the formality of the model, its core features, we believe, can be discussed outside of a technocratic framework and may even become the subject of discussion in legislative bodies. As a contribution to this discourse, we would like to offer the following suggestions for the choice of an adequate $\theta$.

First, the more we can be sure that the raw scores, and the raw distributions, capture ground truth and are free from bias and other exogenous distortions, the less there is a need to transform the raw data into a fair representation; $\theta$ may then assume a relatively smaller value than in cases in which we suspect significant bias. Conversely, if we have reason to believe that the training data
are ridden with bias, and if ground truth is not available, a high $\theta$ score seems attractive (see also [29], at 2).

Second, since the choice of $\theta$ governs the trade-off between more group fairness (high $\theta$) and more individual fairness (low $\theta$), as discussed in the introduction, we suppose that different $\theta$ values will be appropriate for different situations. In fact, it is the beauty of the model that it allows us to adapt to different areas of algorithmic decision making depending on whether we want to strengthen group or individual fairness. As a tentative suggestion, we could say that group fairness, i.e., statistical parity, could be a greater concern in areas that a) primarily form the basis of future life opportunities (e.g., high school or college admissions); or that b) directly relate to basic needs (such as access to housing; to justice; or to health insurance). By contrast, individual fairness could be a greater motivation in areas where, primarily, past performance or events are evaluated (e.g., job applications, perhaps also credit decisions).

Clad in academic metaphors, group fairness could be more important in situations similar to the granting of scholarships (enabling future work), and individual fairness for awards (recognizing past achievements). Evidently, many areas will combine elements of both patterns; this is precisely where the advantage of our model lies that allows for intermediate degrees combining individual and group fairness.

We would also like to stress that some areas may necessitate yet other fairness metrics. For example, in criminal justice sentencing decisions, the different costs of false positives and false negative decisions for the concerned individuals suggests a focus on error rate balance, (see, for the controversy surrounding the COMPAS algorithm, [5, 10, 19]). The error rate balance is a statistical measure at the group level, but is not equivalent to statistical parity. One important future extension of our model, therefore, will be to include this measure as well.

**Designing Procedures to Implement CFA $\theta$.** On the procedural side, models of algorithmic fairness call for new types of regulatory implementations [16, 27]. We are aware that this raises several challenges for real-world implementation. To illustrate them, let us briefly return to the example of the college admissions decision. Our model necessitates four separate procedural steps: first, the computation of the raw score predicting college performance; second, its transformation into a fair score; third, the application of the decision rule to this fair representation; and, fourth, the identification of the successful applicants on the basis of their fair scores.

We take the first step, the calculation of the raw scores, as a given. This is precisely where machine learning models enter the scene. However, we do want to note that, already at this step, bias minimization techniques could apply (see, e.g., [8] and the brief overview in section “Relation to Other Work”).

For the second step, at a minimum, an impartial, trusted party is needed to perform the transformation of the raw into the fair representation. Moreover, ideally based on legislative guidance, this party also needs to determine the value of $\theta$. One key policy question will be whether private companies should take on the role of an impartial party, as in financial auditing; or whether a government agency should be endowed with this task (see [29], [28], at 98, for a discussion). Both solutions raise questions of conflicts of interest, capture etc.

After the transformation of the scores, the college needs to know the fair distribution, i.e., the number of individuals scoring the different fair scores. Based on this, it may calculate the fair score threshold for acceptance to fill its empty slots. The application of the decision rule, by the decision maker, based on the fair rather than the raw data seems straightforward to us.

Finally, one has to identify those individuals that have received a fair score above the threshold. There are two ways in which this can be done, each with their own costs and benefits. As a first option, the college may be handed, along with the fair distribution, a list of applicants that
contains the respective fair scores. This has the advantage of making only one intervention in the college’s decision procedure necessary: the transformation of the raw into the fair scores. The college must then accept the individuals based on their fair score. The disadvantage of providing the college with information about the identity of the applicants, and their respective fair score, is that the college may attempt to circumvent the fairness procedure if group membership can be deduced from the applicants’ identifiers, for example their names. It may then be tempted to base its admissions decision not on the fair score, but on an attempted reconstruction of the raw score, more directly on group membership, or on a mixture of both. Providing the college with full information therefore smooths the decision procedure, but leaves room for cheating by the decision maker. Therefore, some degree of monitoring of the final decisions seems necessary in this case, too.

In the alternative, the impartial party may keep the list identifying individual applicants and tell the college which applicants have qualified given the threshold that the college has calculated based on the fair distribution. This option may be preferable if a) the raw scores are not known to the college; and if b) there is reason to suspect that the college might violate fairness rules. However, this procedure is more intrusive for the college and costlier for the impartial party.

In the end, implementing a framework for fair decision making in the algorithmic context will provide quite a challenge in the real world. With our model, we hope to facilitate the most relevant trade-offs involved, and to make transparent the different design choices that policymakers face.

6. LIMITATIONS AND FUTURE RESEARCH

In closing, we would like to point out three limitations of our model and suggest avenues for future research. First, the distinguishing characteristic of our model is the continuous framework we operate with. However, real-world data typically comes in discrete format. There are two related ways of bridging this gap. First, as was noted in the introduction, continuous measures can be arbitrarily approximated by discrete measures in Wasserstein space. Second and more generally, there are a number of numerical methods for optimal transport problems that can be used to calculate approximations for our model, see [24] and references therein.

Second, our model presupposes the existence of a raw score that can then be mapped onto a fair score. Many important real-world algorithms feature such scores, for example the COMPAS algorithm used for criminal justice sentencing decisions; the FICO and SCHUFA credit rating algorithms; and certain types of insurance algorithms (see [9] at 201). However, not all algorithmic decision making procedures need to operate with explicit raw scores. Some may tie a decision rule directly to a combination of input data (via some performance metric), without the explicit calculation of a raw score. In these models, knowledge about the ranking of the individuals is only implicitly, not explicitly, represented in the (machine learning) model. This is not the end of the story, however. On the one hand, other fairness constraints may be applied directly to the collection and processing of the input data. Also, a raw score may be reconstructed from the output decisions. On the other hand, more importantly, societies are free to decide that only such models in which bias mitigation techniques, such as our model, can be effectively implemented may be used in certain contexts. In other words, if fairness constraints cannot be applied to models, this should be a problem for those wishing to use these models, rather than for those that are judged by them.

Third, the real-world implementation of our framework is complicated if decision making is based on data that changes over time, such as insurance premiums responding to evolving behavior. Here, suitable mechanisms for a continuous implementation of our fairness framework into the decision making environment need to be found.
Finally, as mentioned in the policy implications, an important avenue for future research consists in including yet other important fairness definitions beyond individual and group fairness, such as error rate balance, into our framework. This would allow us to apply it to other types of algorithms, for example the COMPAS algorithm.

REFERENCES

[1] M. Agueh and G. Carlier. Barycenters in the Wasserstein space. *SIAM J. Math. Anal.* **43** (2011), 904–924.
[2] L. Ambrosio and N. Gigli. A user’s guide to optimal transport. *Modelling and optimisation of flows on networks.* Springer, Berlin Heidelberg, 2013. 1–155.
[3] I. Ayres and P. Siegelmann. Race and Gender Discrimination in Bargaining for a New Car, *Am. Econ. Rev.* **85** (1995), 304–321.
[4] S. Barocas and A. Selbst. Big Data’s Disparate Impact, *Cal. L. Rev.* **104** (2016), 671–732.
[5] R. Berk et al. Fairness in Criminal Justice Risk Assessments: The State of the Art. [arXiv:1703.09207] 2017.
[6] Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C. R. Acad. Sci. Paris* **305** (1987), 805–808.
[7] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44** (1991), 375–417.
[8] T. Calders and S. Verwer. Three naive Bayes approaches for discrimination-free classification, *Data Mining and Knowledge Discovery* **21** (2010), 277–292.
[9] D. Chen et al. Enhancing Transparency and Control When Drawing Data-Driven Inferences About Individuals, *Big Data* **5** (2017), 197–212.
[10] A. Chouldechova. Fair prediction with disparate impact: A study of bias in recidivism prediction instruments. [arXiv:1703.00056] 2017.
[11] A. Datta et al. Automated experiments on ad privacy settings, *Proceedings on Privacy Enhancing Technologies* **1** (2015), 92–112.
[12] C. Dwork et al. Fairness through awareness. *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, ACM, 2012.
[13] M. Feldman et al. Certifying and removing disparate impact, *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, ACM, 2015, 259–268.
[14] S. A. Friedler et al. On the (im)possibility of fairness. [arXiv:1609.07236] 2016.
[15] L. Guan (2016). From physical to cyber: Escalating protection for personalized auto insurance, *Proceedings of the 14th ACM Conference on Embedded Network Sensor Systems* 42–55.
[16] Ph. Hacker. Personal data, exploitative contracts, and algorithmic fairness: autonomous vehicles meet the internet of things. *International Data Privacy Law*, 2017.
[17] Ph. Hacker. Personalized Law and Machine Bias, Working Paper (on file with authors).
[18] M. Hardt, E. Price, and N. Srebro. Equality of opportunity in supervised learning. In: *29 Advances in Neural Information Processing Systems* (2016), 3315–3323.
[19] J. Kleinberg et al. Inherent trade-offs in the fair determination of risk scores. [arXiv:1609.05807] 2016.
[20] J. A. Kroll et al. Accountable algorithms. *U. Pa. L. Rev.* **165** (2016), 633–705.
[21] S. Lowry and G. Macpherson. A Blot on the Profession, *British Medical Journal* **296** (1988), 657–658.
[22] S. Pasquale. The Black Box Society. The Secret Algorithms That Control Money and Information. Harvard University Press, Cambridge, MA, 2015.
[23] Ch. Reed et al. Responsibility, Autonomy and Accountability: Legal Liability for Machine Learning. Queen Mary School of Law Legal Studies Research Paper No. 243/2016 (2016), https://ssrn.com/abstract=2853462.
[24] F. Santambrogio. Optimal transport for applied mathematicians. Calculus of variations, PDEx, and modeling. *Progress in Nonlinear Differential Equations and their Applications* **87**. Birkhäuser/Springer, Cham, 2015. xxvii+353 pp.
[25] Ch. Tobler. Case C-236/09. Association belge des Consommateurs Test-Achats ASBL, Yann van Vugt, Charles Basselier v. Conseil des ministres, Judgment of the Court of Justice (Grand Chamber) of 1 March 2011, *Common Market Law Review* **48** 2041–2060.
[26] A. Tutt. An FDA for Algorithms, *Administrative Law Review* **69** (2017), 83–125.
[27] M. Veale and R. Binns. Fairer Machine Learning in the Real World: Mitigating Discrimination Without Collecting Sensitive Data, *Big Data & Society* (forthcoming), https://ssrn.com/abstract=3060763.
[28] S. Wachter et al. Why a right to explanation of automated decision-making does not exist in the general data protection regulation, *International Data Privacy Law* 7 (2017), 76-99.
[29] M. Zafar. Fairness Constraints: Mechanisms for Fair Classification, [arXiv:1507.05259](https://arxiv.org/abs/1507.05259) 2017.
[30] M. Zehlike et al. FA* IR: A Fair Top-k Ranking Algorithm, 26th ACM International Conference on Information and Knowledge Management, [arXiv:1706.06368](https://arxiv.org/abs/1706.06368) 2017.
[31] R. Zemel et al. Learning fair representations. *Proceedings of the 30th International Conference on Machine Learning (ICML-13)*, 2013.
[32] I. Zliobaite. On the relation between accuracy and fairness in binary classification, [arXiv:1505.05723](https://arxiv.org/abs/1505.05723) 2015.

**Philipp Hacker:** LAW DEPARTMENT, HUMBOLDT UNIVERSITY OF BERLIN, UNTER DEN LINDEN 6, 10099 BERLIN, GERMANY; A.SK FELLOW, WZB SOCIAL SCIENCES CENTRE; RESEARCH FELLOW, CENTRE FOR LAW, ECONOMICS AND SOCIETY/CENTRE FOR BLOCKCHAIN TECHNOLOGIES, UCL  
*E-mail address:* philipp.hacker@rewi.hu-berlin.de

**Emil Wiedemann:** INSTITUTE OF APPLIED MATHEMATICS, LEIBNIZ UNIVERSITY HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY  
*E-mail address:* wiedemann@ifam.uni-hannover.de