CLASSIFICATION OF QUADRUPLE GALOIS CANONICAL COVERS II

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Abstract. In this article we classify quadruple Galois canonical covers $φ$ of singular surfaces of minimal degree. This complements the work done in [GP2], so the main output of both papers is the complete classification of quadruple Galois canonical covers of surfaces of minimal degree, both singular and smooth. Our results show that the covers $X$ studied in this article are all regular surfaces and form a bounded family in terms of geometric genus $p_g$. In fact, the geometric genus of $X$ is bounded by 4. Together with the results of Horikawa and Konno for double and triple covers, a striking numerology emerges that motivates some general questions on the existence of higher degree canonical covers. In this article, we also answer some of these questions. The arguments to prove our results include a delicate analysis of the discrepancies of divisors in connection with the ramification and inertia groups of $φ$.

Introduction

Canonical covers of surfaces of minimal degree play a crucial role in a variety of contexts including classification problems, the study of the generation of the canonical ring, the study of linear systems on threefolds and the so-called mapping of the geography of surfaces of general type (see [Pu] for a detailed motivation.) Among them, Galois canonical covers of degree 4 are especially relevant for they behave very differently from both canonical double covers and canonical triple covers, as we showed in [GP2].

The classification of double canonical covers of surfaces of minimal degree was done by Horikawa (see [Ho1]). Canonical covers of degree 3 were classified by Horikawa (see [Ho2]) and Konno (see [Ko]). In [GP2], we have classified quadruple Galois canonical covers of smooth surfaces of minimal degree. In this article we complete the classification of quadruple Galois canonical covers of surfaces of minimal degree by studying those covers.

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whose image is a singular surface. We summarize the classification obtained in the next theorem. Before stating it we need an auxiliary construction:

Let $X$ be a canonical surface whose canonical bundle is base-point-free, let $W$ be a singular rational normal scroll and let $Y \xrightarrow{q} W$ be the minimal desingularization of $W$. Let $X \xrightarrow{\varphi} W$ be the canonical morphism of $X$ (a canonical cover of $W$, for short). There exists the following commutative square:

$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & X \\
\downarrow p & & \downarrow \varphi \\
Y & \xrightarrow{q} & W
\end{array} (\ast)
$$

where $\overline{X}$ is the normalization of the reduced part of $X \times_W Y$, which is irreducible, and $p$ and $\eta$ are induced by the projections from the fiber product onto each factor. Now we can state the following result, which inside the article is split up into Theorems 3.4, 3.5, 4.1 and 4.2:

**Theorem.** If $X$ is a canonical surface with base-point-free canonical bundle, $W$ is a singular surface of minimal degree, $X \xrightarrow{\varphi} W$ is the canonical morphism of $X$ and $\varphi$ is Galois of degree 4 with Galois group $G$, then $W = S(0, 2)$, $X$ is regular and there exists a commutative diagram like $(\ast)$, where $Y \xrightarrow{q} W$ is the minimal desingularization of $W$, the (normal) surface $\overline{X}$ has at worst canonical singularities, the morphism $q$ is the morphism from $X$ to its canonical model $X$ and,

1) if $q$ is crepant and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{X}$ is the product over $Y$ of two double covers branched along divisors $D_2$ and $D_1$ which are linearly equivalent to $2C_0 + 6f$ and $4C_0 + 6f$ respectively and $p$ is the natural morphism from the fiber product to $Y$;

2) if $q$ is crepant and $G = \mathbb{Z}_4$, then $p$ is the composition of two double covers $\overline{X}_1 \xrightarrow{p_1} Y$ branched along a divisor $D_2$ linearly equivalent to $4C_0 + 6f$ and $\overline{X} \xrightarrow{p_2} \overline{X}_1$, branched along the ramification of $p_1$ and $p_1^*D_1$, with $D_1$ linearly equivalent to $3f$, and with trace zero module $p_1^*O_Y(-\frac{1}{2}D_1 - \frac{1}{4}D_2)$;

3) if $q$ is noncrepant and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $q$ is the blowing-up of $X$ at two smooth points, $\overline{X}$ is the normalization of the fiber product over $Y$ of two double covers of $Y$ each branched along a divisor linearly equivalent to $4C_0 + 6f$, and $p$ is the natural map from the normalization of the fiber product to $Y$;

4) if $q$ is noncrepant and $G = \mathbb{Z}_4$, then $p$ is the composition of two double covers $\overline{X}_1 \xrightarrow{p_2} Y$, branched along a divisor $\Delta_2$, and $\overline{X} \xrightarrow{p_2} \overline{X}_1$, branched along the ramification of $p_1$ and $p_1^*D_1$ and with trace zero module $p_1^*O_Y(-\frac{1}{2}(D_1 + C_0) - \frac{1}{4}\Delta_2) \otimes O_{\overline{X}_1}(\overline{C}_0)$, where $D_1$ is a divisor on $Y$ and $\overline{C}_0 = p_1^{-1}C_0$, and either,

4.1) $D_1 \sim C_0 + 3f$ and $D_2 \sim 4C_0 + 6f$; or

4.2) $D_1 \sim 4C_0 + 9f$ and $D_2 \sim 2C_0 + 2f$. 


Conversely, if $\overline{X}$ is a normal surface with at worst canonical singularities, $Y = F_2$ and $\overline{X} \xrightarrow{p} Y$ is either

I. the fiber product over $Y$ of two double covers branched along divisors $D_2$ and $D_1$ as described in 1) above, or the normalization of the fiber product over $Y$ of two double covers branched along divisors $D_2$ and $D_1$ as described in 3) above; or

II. the composition of two double covers $\overline{X}_1 \xrightarrow{p_1} Y$, branched along a divisor $D_2$ and $\overline{X}_1 \xrightarrow{p_2} \overline{X}_1$, branched along the ramification of $p_1$ and $p_1^* D_1$ and having trace zero module $p_1^* \mathcal{O}_Y(-\frac{1}{2} D_1 - \frac{1}{4} D_2)$, where $D_1$ and $D_2$ are as described in 2) above; or

III. the composition of two double covers $\overline{X}_1 \xrightarrow{p_1} Y$, branched along a divisor $\Delta_2$, and $\overline{X} \xrightarrow{p_2} \overline{X}_1$, branched along the ramification of $p_1$ and $p_1^* D_1$ and with trace zero module $p_1^* \mathcal{O}_Y(-\frac{1}{2} (D_1 + C_0) - \frac{1}{4} \Delta_2) \otimes \mathcal{O}_{\overline{X}_1}(\overline{C}_0)$, where $\overline{C}_0 = p_1^{-1} C_0$ and $D_1$ and $\Delta_2$ are as described in 4.1) or 4.2) above,

then there exists a commutative diagram like (*) where $W$ is $S(0, 2)$, the morphism $q$ is the minimal desingularization of $W$, the morphism $\overline{q}$ is the morphism from $\overline{X}$ to its canonical model $X$, the morphism $\varphi$ is the canonical morphism of $X$ and is Galois with Galois group $G$, and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ in case I and $G = \mathbb{Z}_4$ in cases II and III.

Amidst the landscape of all quadruple Galois canonical covers of surfaces of minimal degree, covers of singular targets are significant because among them we find the exceptions to an, otherwise, beautiful and uniform picture. The existence of these exceptions adds even more complexity to the already subtle problem of studying covers of singular surfaces.

The canonical quadruple Galois covers $X \xrightarrow{\varphi} W$ of smooth surfaces of minimal degree (classified in [GP2]) and the quadruple Galois covers $\overline{X} \xrightarrow{p} Y$ in 1) and 2) of the above theorem exhibit the same structure. More precisely, we show that when the canonical covers $\varphi$ in [GP2] and the covers $p$ of 1) and 2) have Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$, then the surfaces of general type $X$ in [GP2] and the surfaces of general type $\overline{X}$ above are always a fiber product of two double covers. Moreover, the branch divisors of these double covers also follow a uniform pattern. On the other hand, if the Galois group is $\mathbb{Z}_4$, we show that the morphism $\varphi$ in [GP2] and the morphism $p$ of 1) and 2) are a composition of two double covers $p_1$ and $p_2$ such that $p_2$ is branched along the ramification of $p_1$ and the pullback of a divisor on the surface of minimal degree. Again the branch divisors of $p_1$ and $p_2$ fit always in the same pattern. Thus, after seeing the classification obtained in [GP2] and looking only at cases 1) and 2) of the present classification, one would be inclined to conjecture this: a surface $X$ (or a closely related birational model of $X$, obtained by a crepant, partial resolution of singularities) which is a quadruple Galois canonical cover of a surface of minimal degree is always either a fiber product of two double covers or a composition of two double covers $p_1$ and $p_2$ branched as described above. Cases 3) and 4) of the previous theorem are exactly the counterexamples to this tempting conjecture.

In cases 3) and 4), the morphism $\overline{q}$ is non-crepant, in contrast with cases 1) and 2). In case 3), where the Galois group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, the surface $\overline{X}$ is not a fiber product but the normalization of a fiber product. The fiber product of the two double covers is non-normal precisely because it has a double curve that eventually contracts to the vertex $w$ of $W$. In
case 4), where the Galois group is $\mathbb{Z}_4$, the morphism $p$ is still a composition of two double covers, but the cover $p_2$ is branched along a divisor which is not a pullback from $Y$. The main philosophical reason why these two exceptions occur is because the canonical divisor of $\overline{X}$ has a fixed part which contracts eventually to $w$.

Another unusual fact that emerges from the classification of quadruple Galois covers is that cyclic quadruple canonical covers of surfaces of minimal degree are never simple cyclic. Non-simple cyclic covers are not a common phenomenon for surfaces, so its existence in this context is interesting.

In this article we also construct families of examples to show the existence of all the cases that appear in the classification. We carry out as well a more detailed study of the singularities of $X$ (see Corollary 5.1). If $w$ is the vertex of $W$, we see that $\varphi^{-1}\{w\}$ consists of one point (cases 1), 2) and 4) of the above theorem) or two points (case 3)). In cases 3) and 4) the point or points lying over $w$ are smooth, i.e., $\varphi$ “unfolds” completely the singularity at $w$. In the remaining cases the singularity over $w$ stays the same or worsens: in case 1) the point lying over $w$ is an $A_k$ singularity ($A_1$ at best) and in case 2) it is a $D_4$ singularity. The behavior of the complement of the fiber of $w$ is like the behavior of the canonical covers of smooth surfaces of minimal degree studied in [GP2]: if $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, one can find covers for which the complement is smooth, and if $G = \mathbb{Z}_4$, the complement is necessarily singular, having at best $A_1$ singularities. Putting all of the above together we see that there do exist smooth quadruple Galois canonical covers of singular surfaces of minimal degree, but they necessarily belong to case 3). We also show that all cyclic quadruple canonical covers of surfaces of minimal degree (smooth or singular) are singular.

The results in this article show that quadruple Galois covers of singular surfaces of minimal degree form a bounded family in terms of both their geometric genus and their irregularity. In fact the classification results here show that $p_g \leq 4$ and $q = 0$. Together with the results of Horikawa and Konno for double and triple covers, the following striking numerology emerges for surfaces of general type that are Galois canonical covers of singular targets:

If $\deg \varphi = 2$, then $p_g \leq 6, q = 0$;
if $\deg \varphi = 3$, then $p_g \leq 5, q = 0$; and
if $\deg \varphi = 4$, then $p_g \leq 4, q = 0$.

Since the smallest projective space containing a singular scroll is $\mathbb{P}^3$, this pattern suggests that there do not exist higher degree canonical covers of singular rational normal scrolls, so we pose the following

**Question:** Let $X \xrightarrow{\varphi} W$ be a canonical cover of a singular surface of minimal degree $W$. Is $\deg \varphi \leq 4$?

There are strong hints towards a positive solution to the question above: in [GP2], Corollary 3.3, we prove that there are no regular Galois canonical covers of degree prime $p \geq 5$ of a surface of minimal degree $W$, smooth or singular. The significance of our question becomes clear once we realize the following: if the answer is positive, then, having in account our previous results for odd degree covers (see [GP1], Corollary 3.2), there will
be no canonical covers of degree odd bigger than 3 of surfaces of minimal degree, except perhaps covers of $\mathbb{P}^2$.

As it often happens in classification problems, the special cases are not easier to deal with. A posteriori we see that quadruple Galois canonical covers of singular surfaces of minimal degree represent a smaller portion if we compare them with the covers of smooth surfaces. However, the difficulties to carry out the classification in the singular target case are much greater. We glimpse them by briefly commenting on the strategy we follow and the techniques we employ. To study the cover $X \xrightarrow{\varphi} W$ of a singular scroll $W$ the first thing we do is to “desingularize” $\varphi$ using the commutative diagram (*). Once this is done, the only information available on $X$ is that, by construction, $X$ is a normal, irreducible surface. Likewise, little is known of $\varphi$, just that it is a birational morphism between $X$ and $W$. At this point, the best possible situation one can hope in order to continue the study of $X$ is that $X$ have canonical singularities and $\varphi$ be crepant, for in such case one can expect to deal with $X \xrightarrow{\varphi} Y$ in much the same way as with a canonical cover $X \xrightarrow{\bar{\varphi}} W$ of a smooth surface $W$. Thus, the crux of our argument, which is contained in the proofs of Theorems 2.5 and 2.6, is to find out if this favorable situation happens always, or, if not, if we can control the “badness” of $X$ and $\varphi$. To settle the question we have to study the possible discrepancies of $\varphi$ and the inertia groups of the ramification of $p$. It finally turns out, as the reader already knows, that $\varphi$ is not always crepant, but in the case it is not, by the work done in Theorems 2.5 and 2.6, we are able to narrow the field and say that $\varphi$ has to fulfill very concrete specifications (see Theorem 2.5, 2) and Theorem 2.6, 2)). This, after still some more involved work, especially when $\varphi$ is not crepant, makes the problem tractable at the end. Likewise, after Theorems 2.5 and 2.6 we find that $X$ is not too bad either (it has at worst rational, 2-Gorenstein singularities). This part of the tale has an even happier ending, since we eventually prove that the singularities of $X$ are indeed canonical.

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1. Background material

Convention: We work over an algebraically closed field of characteristic 0.

Notation: We will follow these conventions:

1) Throughout this article, unless otherwise stated, $W$ will be an embedded projective algebraic surface which is a cone over a smooth rational normal curve. Thus $W$ has minimal degree, for its degree is equal to its codimension in projective space plus 1. If the rational normal curve has degree $e$ we will denote $W$ as $S(0,e)$. We will denote by $w$ the vertex of $W$. 

2) Throughout this article, unless otherwise stated, $X$ will be a projective algebraic normal surface with at worst canonical singularities. We will denote by $\omega_X$ the canonical bundle of $X$.

We recall the following standard notation:

3) Let $e \geq 2$. By $F_e$ we denote the Hirzebruch surface whose minimal section have self intersection $-e$. Let $C_0$ denote the minimal section of $F_e$ and let $f$ be one of the fibers of $F_e$. Recall that $S(0,e)$ is the image of $F_e$ by the complete linear series $|C_0 + ef|$.

**Definition 1.1.** Let $X$ and $W$ be as in the previous notation. We will say that a surjective morphism $X \xrightarrow{\varphi} W$ is a canonical cover of $W$ if $X$ is surface of general type whose canonical bundle $\omega_X$ is ample and base-point-free and $\varphi$ is the canonical morphism of $X$.

In this paper we study Galois covers $\varphi$ of $W$. Since $W$ is singular, $\varphi$ is not in general flat. However the strategy will be to study an auxiliary, flat Galois cover. We recall here some known or easy facts regarding the algebra structure associated to a flat, quadruple Galois cover. For proofs of Proposition 1.2 see [Ca], [HM] and [Pa], and also [GP2], Proposition 2.4; for Proposition 1.3 and Proposition 1.4, see [GP2], Propositions 2.9 and 2.10.

**Proposition 1.2.** Let $X$ and $Y$ be two algebraic varieties and let $X \xrightarrow{p} Y$ be a flat, Galois cover of degree $4$.

1) If $G = \mathbb{Z}_4$, then $p_*\mathcal{O}_X$ splits as

$$p_*\mathcal{O}_X = \mathcal{O}_Y \oplus L_i^* \oplus L_{-1}^* \oplus L_{-i}^*$$

where the line bundles on $Y$, $\mathcal{O}_Y, L_i^*, L_{-1}^*$ and $L_{-i}^*$ are the eigenspaces of $1, i, -1$ and $-i$ respectively.

There exist effective Cartier divisors $D_{11}, D_{12}, D_{23}$ and $D_{33}$ on $Y$ such that $D_{11} + D_{23} = D_{12} + D_{33}$ and the following

\[
\begin{align*}
L_i \otimes L_i &= L_{-1} \otimes \mathcal{O}_Y(D_{11}) \\
L_i \otimes L_{-1} &= L_{-i} \otimes \mathcal{O}_Y(D_{12}) \\
L_i \otimes L_{-i} &= \mathcal{O}_Y(D_{11} + D_{23}) \\
L_{-1} \otimes L_{-1} &= \mathcal{O}_Y(D_{12} + D_{23}) \\
L_{-1} \otimes L_{-i} &= L_i \otimes \mathcal{O}_Y(D_{23}) \\
L_{-i} \otimes L_{-i} &= L_{-1} \otimes \mathcal{O}_Y(D_{33})
\end{align*}
\] (1.2.1)

and the multiplicative structure of $p_*\mathcal{O}_X$ is as follows:

\[
\begin{align*}
L_i^* \otimes L_i^* &\xrightarrow{D_{11}} L_{-1}^* \\
L_i^* \otimes L_{-1}^* &\xrightarrow{D_{12}} L_{-i}^* \\
L_i^* \otimes L_{-i}^* &\xrightarrow{D_{11} + D_{23}} \mathcal{O}_Y \\
L_{-1}^* \otimes L_{-1}^* &\xrightarrow{D_{12} + D_{23}} \mathcal{O}_Y \\
L_{-1}^* \otimes L_{-i}^* &\xrightarrow{D_{23}} L_i^* \\
L_{-i}^* \otimes L_{-i}^* &\xrightarrow{D_{33}} L_{-1}^* 
\end{align*}
\]
2) If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $p_* \mathcal{O}_X$ splits as $$p_* \mathcal{O}_X = \mathcal{O}_\mathfrak{Y} \oplus L_1^* \oplus L_2^* \oplus L_3^*,$$
where $\mathcal{O}_\mathfrak{Y}$, $L_1^*$, $L_2^*$ and $L_3^*$ are eigenspaces and there exist effective Cartier divisors $D_1$, $D_2$ and $D_3$ such that $L_i^{\otimes 2} = \mathcal{O}_\mathfrak{Y}(D_j + D_k)$ and $L_j \otimes L_k = L_i \otimes \mathcal{O}_\mathfrak{Y}(D_i)$ with $i \neq j$, $j \neq k$ and $k \neq i$, and the multiplicative structure of $p_* \mathcal{O}_X$ is as follows:

$$L_i^* \otimes L_j^* \xrightarrow{D_j + D_k} \mathcal{O}_\mathfrak{Y},$$
$$L_j^* \otimes L_k^* \xrightarrow{D_i} L_i^*.$$

**Proposition 1.3.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normal algebraic varieties.
1) If $\mathfrak{X} \xrightarrow{p_1} \mathfrak{Y}$ is the natural map onto $\mathfrak{Y}$ from the fiber product over $\mathfrak{Y}$ of two flat double covers $\mathfrak{X}_1 \xrightarrow{p_1} \mathfrak{Y}$ and $\mathfrak{X}_2 \xrightarrow{p_2} \mathfrak{Y}$, then $p$ is a Galois cover with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$.
2) If in addition $L_2^*$ and $L_1^*$ are the trace zero modules of $p_2$ and $p_1$ respectively, then

$$p_* \mathcal{O}_X = \mathcal{O}_\mathfrak{Y} \oplus L_1^* \oplus L_2^* \oplus (L_1^* \oplus L_2^*),$$

and, if $\mathfrak{Y}$ is locally Gorenstein, then $\mathfrak{X}$ is locally Gorenstein and $\omega_\mathfrak{X} = p^*(\omega_\mathfrak{Y} \otimes L_1 \otimes L_2)$.

**Proposition 1.4.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normal algebraic varieties.
1) If $\mathfrak{X} \xrightarrow{p} \mathfrak{Y}$ is the composition of a flat double cover $\mathfrak{X'} \xrightarrow{p_1} \mathfrak{Y}$ branched along a divisor $D_2$, followed by a flat double cover $\mathfrak{X'} \xrightarrow{p_2} \mathfrak{X}'$, branched along the ramification locus of $p_1$ and $p_1^* D_1$, where $D_1$ is a divisor on $\mathfrak{Y}$, then $p$ is a Galois cover with Galois group $\mathbb{Z}_4$.
2) If in addition $L_2^*$ is the trace zero module of $p_1$ and $p_1^* L_1^*$ is the trace zero module of $p_2$, then

$$p_* \mathcal{O}_X = \mathcal{O}_\mathfrak{Y} \oplus L_1^* \oplus L_2^* \oplus (L_1^* \otimes L_2^*),$$

and, if $\mathfrak{Y}$ is locally Gorenstein, then $\mathfrak{X}$ is locally Gorenstein and $\omega_\mathfrak{X} = p^*(\omega_\mathfrak{Y} \otimes L_1 \otimes L_2)$.

2. The desingularization diagram

The covers we want to describe and classify in this article are Galois canonical covers $\varphi$ of a singular scroll $W$. These covers are finite but, precisely because $W$ is singular, they are not in general flat. Flat covers are more tractable though, since their associated algebra structure is locally free. Thus the first thing we do is to “desingularize” $\varphi$, that is, we will “make” $W$ smooth and $\varphi$ flat. We construct the following desingularization diagram for $\varphi$:

**Definition 2.1.** Let $X \xrightarrow{\varphi} W$ be a canonical cover and let $Y \xrightarrow{q} W$ be the minimal desingularization of $W$. We define $\overline{X}$ as the normalization of the reduced part of $X \times_W Y$. The surface $\overline{X}$ is irreducible and fits in the following commutative diagram:

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{\varphi}} & X \\
\downarrow p & & \downarrow \varphi \\
Y & \xrightarrow{q} & W
\end{array} \quad (2.1.1)$$
where $p$ and $\overline{q}$ are induced by the projections from the fiber product onto each factor.

For our purposes, the key point of the above construction is that if one of $\varphi$ or $p$ is Galois with given Galois group $G$, so is the other:

**Lemma 2.2.** Let $X \stackrel{\varphi}{\rightarrow} W$ be a canonical cover and let $\overline{X} \stackrel{p}{\rightarrow} Y$ be as in (2.1.1). Then $\varphi$ is a Galois cover with Galois group $G$ if and only if $\overline{X} \stackrel{p}{\rightarrow} Y$ is a Galois cover with Galois group $G$.

**Proof.** We first assume that $\varphi$ is Galois and its Galois group is $G$. Since $O_X$ is integral over $O_W$, $O_{\overline{X}}$ is also an integral extension of $O_Y$. By construction $\overline{X}$ is normal. Recall that $W$ is a cone over a smooth rational normal curve and let $w$ be the singular point of $W$. Therefore $O_{\overline{W}}$ is in fact the integral closure of $O_Y$ in $\mathcal{K}(X)$, so $\overline{X} \stackrel{p}{\rightarrow} Y$ is also a Galois cover with the same Galois group $G$. The argument to show the converse is analogous. \(\square\)

We state now two lemmas about $p$ and $\overline{q}$. The first of them recalls well-known facts on rational singularities, so we state it without a proof:

**Lemma 2.3.** Let $\overline{X} \stackrel{\overline{q}}{\rightarrow} X$ be a birational morphism between two normal surfaces. If $X$ has rational singularities, then
1) $\overline{X}$ also has rational singularities, and
2) every reduced cycle of $\overline{X}$ contracted to a point by $\overline{q}$ has arithmetic genus $0$.

**Lemma 2.4.** With the notation of Definition 2.1, if $X \stackrel{\varphi}{\rightarrow} W$ is a Galois cover with group $G$, then
1) On $X$ and $\overline{X}$ there exist canonical divisors $K_X$ and $K_{\overline{X}}$ which are $G$-invariant and such that
$$K_{\overline{X}} \equiv \overline{q}^* K_X + a(F_1 + \cdots + F_k), \quad (2.4.1)$$
where $\equiv$ means numerical equivalence, $a$ is a nonnegative rational number and $F_1, \ldots, F_k$ are the components of the exceptional locus of $\overline{q}$.
2) If in addition $\overline{X}$ is locally Gorenstein, then there exist $K_X$ and $K_{\overline{X}}$ as above and such that
$$K_{\overline{X}} = \overline{q}^* K_X + a(F_1 + \cdots + F_k), \quad (2.4.2)$$
where $a$ is a nonnegative integer.

**Proof.** We consider the exceptional locus of $\overline{q}$. Recall that $Y$ is a Hirzebruch surface and let $C_0$ be its minimal section. Any curve $F_i$ in the exceptional locus of $\overline{q}$ maps onto $C_0$ by $p$.

Since the cover $p$ is Galois by Lemma 2.2, $G$ acts transitively on the set $\{F_1, \ldots, F_k\}$. Recall also that $X$ and $\overline{X}$ are both normal, $X$ has at worst canonical singularities (in particular $X$ is also locally Gorenstein) and, by Lemma 2.3, $\overline{X}$ has at worst rational singularities (in particular, $\overline{X}$ is locally $\mathbb{Q}$-Gorenstein). Then one can find $G$-equivariant canonical divisors $K_X$ and $K_{\overline{X}}$ such that
$$K_{\overline{X}} \equiv \overline{q}^* K_X + a(F_1 + \cdots + F_k)$$
Then \( a \) is a nonnegative rational number, because \( X \) has canonical singularities. This proves 1)

If, in addition, \( \overline{X} \) is locally Gorenstein in the previous formula we can write equality instead of numerical equivalence and \( a \) is an integer, for both \( K_{\overline{X}} \) and \( K_X \) are Cartier divisors. This proves 2). \( \square \)

The philosophy we follow now is this: instead of describing directly the quadruple Galois canonical covers \( X \xrightarrow{\varphi} W \) of \( W \) we will describe Galois covers \( \overline{X} \xrightarrow{\overline{\varphi}} Y \). We will describe also in a precise way the relation between \( \overline{X} \) and \( X \), that is, how one passes from \( \overline{X} \) to \( X \) and viceversa. That means to describe the morphism \( \overline{\varphi} \). We split the study of \( \overline{\varphi} \) in two theorems, depending on whether \( \{ \varphi^{-1}(w) \} \) consists of one or several points.

**Theorem 2.5.** Let \( W \) be a singular rational normal scroll and let \( X \xrightarrow{\varphi} W \) be a canonical cover. Let \( w \) be the singular point of \( W \) and let \( \overline{X}, Y, p, q \) and \( \overline{q} \) be as in Definition 2.1. If \( X \xrightarrow{\varphi} W \) is Galois of degree 4 and \( \{ \varphi^{-1}(w) \} \) is not a single point, then \( \overline{X} \) has at worst canonical singularities and one of the following happens:

1) either \( \overline{q} \) is crepant (i.e., \( K_{\overline{X}} = \overline{q}^* K_X \)); or

2) \( W = S(0,2) \), \( \varphi^{-1}\{w\} \) consists of two smooth points \( x_1 \) and \( x_2 \) and \( \overline{X} \xrightarrow{\overline{\varphi}} X \) is the blowing up of \( X \) at \( x_1 \) and \( x_2 \).

**Proof.** Recall that \( Y \) is a Hirzebruch surface \( F_e \) with \( e \geq 2 \), that \( C_0 \) is its minimal section and that we call \( F_1, \ldots, F_k \) the irreducible components of the exceptional locus of \( \overline{q} \), which are mapped each onto \( C_0 \) by \( p \). Let \( G \) be the Galois group of \( \varphi \). Since the order of \( G \) is 4 and the cardinality of \( \varphi^{-1}\{w\} \) is greater than one, the cardinality of \( \varphi^{-1}\{w\} \) is in fact 2 or 4. We treat these two cases separately:

**Case 1:** Cardinality of \( \varphi^{-1}\{w\} \) equals 4. In this case \( \varphi \) is étale at \( w \), and hence is flat, so \( X \times_W Y \) is the blowing up of \( X \) at the four points \( x_1, \ldots, x_4 \), lying over \( w \). Thus \( X \times_W Y \) is irreducible, reduced and normal, so, in this case, \( \overline{X} = X \times_W Y \). Since \( \varphi \) is étale at an analytic neighborhood of \( w \), the singularities at \( x_1, \ldots, x_4 \) are analytically isomorphic to the singularity at \( w \), i.e., they are all \( A_1 \) singularities. Thus \( \overline{\varphi} \) resolves \( x_1, \ldots, x_4 \). Therefore \( \overline{q} \) is crepant, i.e., \( K_{\overline{X}} = \overline{q}^* K_X \). Finally, since \( X \) has at worst canonical singularities, so does \( \overline{X} \) (the canonical singularities of \( \overline{X} \) correspond to the singular points of \( X \) different from \( x_1, \ldots, x_4 \)).

**Case 2:** Cardinality of \( \varphi^{-1}\{w\} \) equals 2. We call \( x_1 \) and \( x_2 \) the points in \( \varphi^{-1}\{w\} \). Given a subgroup \( G' \) of \( G \), let \( X' \) be the quotient of \( X \) by \( G' \). We also have a way of decomposing \( \varphi \), namely,

\[
X \xrightarrow{\varphi_2} X' \xrightarrow{\varphi_1} W,
\]

where \( X' \) is normal and \( \varphi_1 \) and \( \varphi_2 \) are Galois covers with Galois group \( G/G' \) and \( G' \) respectively.

Let \( \overline{X}' \) be the normalization of the reduced part of the fiber product of \( X' \) and \( Y \) over \( W \) and let \( \overline{X}' \xrightarrow{\overline{\varphi}_1} X' \) and \( \overline{X}' \xrightarrow{\overline{\varphi}_2} Y \) be the projections to each factor of the product. As it happened with \( \overline{X} \), \( \mathcal{O}_{\overline{X}'} \) is the integral closure of \( \mathcal{O}_Y \) inside \( \mathcal{K}(X') \) and therefore \( p_1 \) is a
Galois cover with group $G/G'$. Now let $\overline{X}$ be so that $\mathcal{O}_{\overline{X}}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{K}(X)$ and let $p_2$ be the morphism induced between $\overline{X}$ and $\overline{X}'$. Then $\mathcal{O}_{\overline{X}}$ is normal and integral over $\mathcal{O}_Y$, hence it is the integral closure of $\mathcal{O}_Y$ in $\mathcal{K}(X)$. By construction, so is $\mathcal{O}_X$, hence $\overline{X} = \overline{X}'$ and we get the following commutative diagram:

\[
\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{\varphi}} & X \\
\downarrow p_2 & & \downarrow \varphi_2 \\
\overline{X}' & \xrightarrow{\overline{\phi}} & X' \\
\downarrow p_1 & & \downarrow \phi_1 \\
Y & \xrightarrow{\psi} & W
\end{array}
\] (2.5.1)

with $p = p_1 \circ p_2$ and $\varphi = \varphi_1 \circ \varphi_2$. Let now $G' = \text{Stab}x_1 = \text{Stab}x_2$. Let $x'_1 = \varphi_2(x_1)$ and $x'_2 = \varphi_2(x_2)$. Since $G'$ is the stabilizer of both $x_1$ and $x_2$, then $x'_1 \neq x'_2$ and $\varphi_1$ is étale at a neighborhood of $w$. By the same arguments as in Case 1, $\overline{X}' \xrightarrow{\overline{\phi}} X'$ is the blowing up of $X'$ at $x'_1$ and $x'_2$. Then $p_1$ is also étale on an analytic neighborhood of $C_0$. Let $E_1$ and $E_2$ be the exceptional divisors of $q'$. Then $p^*_1C_0 = E_1 + E_2$, $E_i$ is isomorphic to $\mathbb{P}^1$, $\overline{X}'$ is smooth at every point of an analytic neighborhood $U$ of $E_1$ and $E_2$, $E_1 \cdot E_2 = 0$ and $E_1^2 = E_2^2 = C_0^2 = -e$. Now we examine the singularities of $\overline{X}$. By construction the exceptional locus of $\overline{q}$ is mapped by $\overline{q}$ on $x_1$ and $x_2$ and is in fact $p^{-1}(C_0)$. Thus $X - p^{-1}(C_0)$ and $X - \{x_1, x_2\}$ are isomorphic, so $\overline{X}$ has at worst canonical singularities outside $p^{-1}(C_0)$. By construction $\overline{X}$ is normal, hence locally Cohen-Macaulay, and since $U$ is smooth, $p_2$ is a flat, degree 2 morphism when restricted to $p_2^{-1}(U)$. Therefore $p_2^{-1}(U)$ is locally Gorenstein. By Lemma 2.3 $\overline{X}$ has rational singularities, hence $p_2^{-1}(U)$ has Gorenstein rational singularities, i.e., canonical singularities. This proves that $\overline{X}$ has canonical singularities.

Now we study $\overline{q}$. For that we study how $p_2$ is at $E_1$ and $E_2$. Recall that $p_2$ is a double cover branched along a divisor of $\overline{X}'$. Since $G$ acts transitively, there are only two possibilities for $E_1$ and $E_2$: either $E_1$ and $E_2$ are both in the branch locus of $p_2$ or none of them are. Now we deal with the two possibilities. First, let us assume that neither $E_1$ nor $E_2$ are in the branch locus of $p_2$. Let $F_i = p^*_2E_i$. Then $F_i$ is a Cartier divisor in $\overline{X}$ and it is a reduced curve, $F_1 \cdot F_2 = 0$ and $F_i^2 = -2e$. Since $X$ has rational singularities $F_i$ has arithmetic genus 0 by Lemma 2.3.

Recall that we have shown $\overline{X}$ is locally Gorenstein. Then from adjunction and from (2.4.2) we get

$$-2 = (K_{\overline{X}} + F_i) \cdot F_i = (\overline{q}^*K_X + aF_1 + aF_2 + F_i) \cdot F_i = -2e(a + 1),$$

with $a$ a nonnegative integer. This leads to a contradiction, for $e$ is an integer greater than or equal to 2.

Then the only possibility left is that both $E_1$ and $E_2$ are in the branch locus of $p_2$. Now let $F_i = p^{-1}_2E_i$. Then $F_i$ is isomorphic to $E_i$ and therefore to $\mathbb{P}^1$ and $2(F_1 + F_2) =$
\( p_2^*(E_1 + E_2) = p^*C_0 \). Using (2.4.2) and the commutativity of diagram (2.5.1) we obtain

\[
K_{\overline{X}} = \overline{q}^*K_X + a(F_1 + F_2) = \\
\overline{q}^*\varphi^*K_X + a(F_1 + F_2) = p^*(C_0 + ef) + a(F_1 + F_2) \quad (2.5.2)
\]

with \( a \) a nonnegative integer. On the other hand, let us denote by \( R \) the ramification divisor of \( p \). Then we have the formula

\[
K_{\overline{X}} = p^*K_X + R \sim p^*(-2C_0 - (e + 2)f) + R \quad (2.5.3)
\]

where \( \sim \) means linear equivalence. Thus from (2.5.2) and (2.5.3) we obtain

\[
R \sim p^*(3C_0 + (2e + 2)f) + (a + 1)(F_1 + F_2) . \quad (2.5.4)
\]

Since \( E_1 \) and \( E_2 \) are in the branch locus of \( p_2 \), \( F_1 \) and \( F_2 \) are in the support of \( R \). Since \( \overline{X} \) is normal, the multiplicity of \( E_1 \) and \( E_2 \) in the branch locus of \( p_2 \) is 1. Recall also that \( p_1 \) is étale at \( E_1 \) and \( E_2 \). Thus the multiplicity of \( F_1 \) and \( F_2 \) in \( R \) is also 1 and we can write \( R = R_1 + F_1 + F_2 \), where \( R_1 \) is an effective divisor not containing \( F_1 \) or \( F_2 \). Thus \( R_1 \cdot F_i \geq 0 \) and by (2.5.4)

\[
R_1 \sim p^*(2C_0 + (2e + 2)f) + (a + 1)(F_1 + F_2) .
\]

Putting these two pieces of information together we get

\[
0 \leq R_1 \cdot (F_1 + F_2) = \frac{1}{2}p^*(2C_0 + (2e + 2)f) \cdot p^*C_0 + \frac{1}{4}(a + 1)(p^*C_0)^2 = \\
2(2C_0 + (2e + 2)f) \cdot C_0 + (a + 1)C_0^2 = 4 - e(a + 1) \quad (2.5.5)
\]

Recall that \( e \) is an integer greater than or equal to 2. Then from (2.5.5) we obtain that

\[
2 \leq e \leq 4 \quad \text{and} \quad a = 0, 1.
\]

Moreover, if \( e = 3, 4 \), then \( a = 0 \) so in this case \( K_{\overline{X}} = \overline{q}^*K_X \). If \( e = 2 \), then either \( K_{\overline{X}} = \overline{q}^*K_X \) or \( K_{\overline{X}} = \overline{q}^*K_X + F_1 + F_2 \). In the latter case,

\[
R_1 \cdot (F_1 + F_2) = 0 \quad (2.5.6)
\]

hence \( \overline{X} \) is smooth at every point of \( F_1 \) and \( F_2 \), and \( F_1^2 = F_2^2 = -1 \). Then by Castelnuovo’s contractibility criterion \( X \) is smooth at \( x_1 \) and \( x_2 \) and \( \overline{q} \) is the blowing-up of \( X \) at \( x_1 \) and \( x_2 \). \( \square \)

Now we study the desingularization diagram (2.1.1) when the inverse image of \( w \) in \( X \) is a single point:
Theorem 2.6. Let \( W \) be a singular rational normal scroll and let \( X \xrightarrow{\varphi} W \) be a canonical cover. Let \( w \) the singular point of \( W \) and let \( \overline{X}, Y, p, q \) and \( \overline{\varphi} \) be as in Definition 2.1. If \( X \xrightarrow{\varphi} W \) is Galois of degree 4 and Galois group \( G \), and \( \{ \varphi^{-1}(w) \} \) is a single point, then one of the following happens:

1) either the surface \( \overline{X} \) has at worst canonical singularities, \( \overline{\varphi} \) is crepant (i.e., \( K_{\overline{X}} = \overline{\varphi}^*K_X \)) and \( W = S(0,2) \); or
2) the surface \( \overline{X} \) has at worst canonical singularities, \( \overline{\varphi} \) is crepant and \( G = \mathbb{Z}_4 \); or
3) the surface \( \overline{X} \) is locally 2-Gorenstein, has at worst rational singularities and \( W = S(0,2) \); \( G \) is \( \mathbb{Z}_4 \) and the exceptional divisor of \( \overline{\varphi} \) is a smooth line \( F \) with inertia group \( G \), \( F^2 = -\frac{1}{2} \) and \( 2K_{\overline{X}} = \overline{\varphi}^*2K_X + 4F \) for suitable canonical divisors \( K_{\overline{X}} \) and \( K_X \).

Proof. Let \( \varphi^{-1}\{ w \} = \{ x \} \). We treat separately the cases of \( G = \mathbb{Z}_4 \) and \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Case 1: \( G = \mathbb{Z}_4 \). Recall the multiplicative structure of \( p_*\mathcal{O}_{\overline{X}} \):

\[
p_*\mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus L_i^* \oplus L_{-1}^* \oplus L_{-i}^* ,
\]

where \( L_i^* \), \( L_{-1}^* \) and \( L_{-i}^* \) are eigenspaces for \( i \), \(-1\) and \(-i\) respectively and the multiplication is given by divisors \( D_{11}, D_{12}, D_{23} \) and \( D_{33} \), as described in Proposition 1.2.

We now describe further the branch locus of \( p \). Recall that \( L_{\leq 2} = \mathcal{O}_Y(D_{12} + D_{23}) \), and, since the double cover of \( Y \) corresponding to \( \mathcal{O}_Y \oplus L_{-1} \) can be taken to be normal, we may assume that \( D_{12} + D_{23} \) has no multiple components. On the other hand, \( D_{11} + D_{23} = D_{12} + D_{33} \), hence \( D_{23} \subseteq D_{33} \) and \( D_{12} \subseteq D_{11} \), so we can write \( D_{33} = D_{23} + D_{33}' \) and \( D_{12} = D_{12} + D_{11}' \), with \( D_{11}' = D_{33}' \). Thus the multiplicative structure of \( p_*\mathcal{O}_{\overline{X}} \) and the relation between the eigenspaces is summarized as follows:

\[
L_i^* \otimes L_i^* \xrightarrow{D_{11} + D_{12}} L_{-1}^* \\
L_i^* \otimes L_{-1}^* \xrightarrow{D_{12}} L_{-i}^* \\
L_i^* \otimes L_{-i}^* \xrightarrow{D_{11}' + D_{12} + D_{23}} \mathcal{O}_Y \\
L_{-1}^* \otimes L_{-1}^* \xrightarrow{D_{12} + D_{23}} \mathcal{O}_Y \\
L_{-1}^* \otimes L_{-i}^* \xrightarrow{D_{33}} L_i^* \\
L_{-i}^* \otimes L_{-i}^* \xrightarrow{D_{11}' + D_{23}} L_{-1}^*
\]

and

\[
L_i \otimes L_i = L_{-1} \otimes \mathcal{O}_Y(D_{11}' + D_{12}) \\
L_i \otimes L_{-1} = L_{-i} \otimes \mathcal{O}_Y(D_{12}) \\
L_i \otimes L_{-i} = \mathcal{O}_Y(D_{12} + D_{11}') \\
L_{-1} \otimes L_{-1} = \mathcal{O}_Y(D_{12} + D_{23}) \\
L_{-1} \otimes L_{-i} = L_i \otimes \mathcal{O}_Y(D_{23}) \\
L_{-i} \otimes L_{-i} = L_{-1} \otimes \mathcal{O}_Y(D_{11}' + D_{23}).
\]
Then the ramification of \( p \) falls only onto the components of \( D_{11}', D_{12} \) and \( D_{23} \). Since \( \overline{X} \) is normal, by computing locally the ramification lying over the generic points of each component of \( D_{11}', D_{12} \) and \( D_{23} \), we can conclude that \( D_{11} + D_{12} \) and \( D_{11}' + D_{23} \) have no multiple components either.

Now we discuss how \( D_{12} \) and \( D_{23} \) are. We will show that either \( D_{12} \) or \( D_{23} \) is a multiple of \( C_0 \). We will see it by looking at the intersection of \( D_{12} \) and \( D_{23} \) with \( C_0 + e f \). Let \( D \) be a smooth irreducible curve in \( |C_0 + e f| \) so that the pullback of \( D \) by \( p \) is also a smooth curve \( C \) in \( |q^* \varphi^* \mathcal{O}_W(1)| \). The curve \( D \) is isomorphic to \( \mathbb{P}^1 \). Since \( D \) does not meet \( C_0 \), it corresponds to a smooth hyperplane section of \( W \) which avoids \( w \). Thus, by adjunction, \( \omega_C = p^*(q^* \mathcal{O}_W(2) \otimes \mathcal{O}_D) \). Then, using relative duality and arguing in a similar fashion as in the proof of [GP2], Proposition 1.3, we conclude that, for some permutation \( \tau \) of \( \{i, -1, -i\} \), \((L_{\tau(i)} \otimes L_{\tau(-1)})|_D = q^* \mathcal{O}_W \otimes \omega_D \). Thus either \((L_i \otimes L_{-1})|_D = L_{-i}|_D \) or \((L_{-1} \otimes L_{-i})|_D = L_i|_D \) or \((L_i \otimes L_{-1})|_D = L_{-i}|_D \). In the first case we have \( D_{12} \cdot D = 0 \). This implies that \( D_{12} \) is a multiple of \( C_0 \). Likewise, in the second case we have that \( D_{23} \) is a multiple of \( C_0 \). Finally, if \((L_i \otimes L_{-1})|_D = L_{-1}|_D \), we see that \( 2(l_{11}' + D_{12} + D_{23}) \) and \( (D_{12} + D_{23}) \) have the same restriction to \( D \) and hence \( (2l_{11}' + D_{12} + D_{23}) \cdot D = 0 \). This implies that \( l_{11}', D_{12} \) and \( D_{23} \) are all multiple of \( C_0 \).

Therefore we may rename \( \{L_i, L_{-1}, L_{-i}\} \) as \( \{L_1, L_2, L_3\} \) and \( \{l_{11}', D_{12}, D_{23}\} \) as \( \{D_1, D_2, D_3\} \) so that

\[
\begin{align*}
L_1 \otimes L_1 &= L_2 \otimes \mathcal{O}_Y(D_1 + D_2) \\
L_1 \otimes L_2 &= L_3 \otimes \mathcal{O}_Y(D_2) \\
L_1 \otimes L_3 &= \mathcal{O}_Y(D_1 + D_2 + D_3) \\
L_2 \otimes L_2 &= \mathcal{O}_Y(D_2 + D_3) \\
L_2 \otimes L_3 &= L_1 \otimes \mathcal{O}_Y(D_3) \\
L_3 \otimes L_3 &= L_2 \otimes \mathcal{O}_Y(D_1 + D_3)
\end{align*}
\]

and \( D_2 \) is a multiple of \( C_0 \). Thus \( D_2 \) is either 0 or \( C_0 \), since we know that \( D_2 \) has no multiple components.

We study now the ramification of \( p \) and the canonical divisor of \( \overline{X} \). The ramification \( R_1 \) lying over \( D_2 + D_3 \) has inertia group \( \mathbb{Z}_4 \), i.e., the points of \( R_1 \) have stabilizer \( \mathbb{Z}_4 \). To compute the rest of the ramification, we work on \( U = Y - \{D_2 + D_3\} \), and there it is clear that the only ramification lies over \( D_1 \) and has inertia group \( \mathbb{Z}_2 \). Thus the ramification \( R \) of \( p \) and \( \omega_{\overline{X}} \) satisfy:

\[
4R = p^*(2D_1 + 3(D_2 + D_3)) \sim p^*(4L_2 + 2D_1 + D_2 + D_3) \\
2R \sim p^*(D_1 + 3L_2) \sim p^*(2L_1 + 2L_2 - D_2) \sim p^*(2L_3 + D_2) \\
2K_{\overline{X}} \sim p^*(2K_Y + 2L_1 + 2L_2 - D_2) \sim p^*(2K_Y + 2L_3 + D_2) \quad (2.6.1).
\]

If \( D_2 = 0 \), we have in fact that \( \omega_{\overline{X}} = p^*(\omega_Y \otimes L_3) \) and \( \overline{X} \) is Gorenstein. If \( D_2 = C_0 \), then \( \omega_{\overline{X}}^{\otimes 2} = p^*(\omega_Y^{\otimes 2} \otimes L_3^{\otimes 2} \otimes \mathcal{O}_Y(C_0)) \) and \( \overline{X} \) is 2-Gorenstein. Let \( F \) be the reduced cycle
consisting of the curves of $\overline{X}$ lying over $C_0$. Then $F$ is the exceptional locus of $\overline{q}$ and according to formula (2.4.2), there exist suitable canonical divisors $K_X$ and $K_{\overline{X}}$ such that $K_{\overline{X}} \equiv \overline{q}^* K_X + aF$, with $a$ a nonnegative rational number. Since $X$ is at worst locally 2-Gorenstein, we have $2K_{\overline{X}} = \overline{q}^* 2K_X + 2aF$, with $2a$ a nonnegative integer. Now we will determine $a$ and prove that $W = S(0, 2)$ if $a > 0$. We split the remaining of Case 1 into four subcases, according to whether $C_0$ is in the branch locus of $p$ or not and according to what $D_2$ is:

**Case 1.1:** $C_0$ is not in the branch locus of $p$. In this case $D_2 = 0$ and, as previously observed, $L_1 \otimes L_2 = L_3$ and $\overline{X}$ is Gorenstein. Then $K_{\overline{X}} = \overline{q}^* K_X + aF$ and $a$ is a nonnegative integer. Moreover $F = p^* C_0$ and is therefore a reduced Cartier divisor such that $F^2 = -4e$. By Lemma 2.3 we know that the arithmetic genus of $F$ is 0, hence by adjunction

$$-2 = ((K_{\overline{X}} + F) \cdot F) = (\overline{q}^* K_X + (a + 1)F) \cdot F = -4e(a + 1) .$$

This gives $1 = 2e(a + 1)$ but this is not possible since $a$ and $e$ are integers. Thus Case 1.1 does not occur.

**Case 1.2:** $C_0$ is in the branch locus of $p$ and $D_2 = C_0$. Then the inertia group of $F$ is $\mathbb{Z}_4$, $p^* C_0 = 4F$ and $F^2 = -\frac{4}{4}$. By the previous observation $\omega_{\overline{X}} \otimes 2$ is the pullback by $p$ of a certain line bundle on $Y$. On the other hand recall that $2K_{\overline{X}}$ is equal to $\overline{q}^* 2K_X + 2aF$ and, since $\omega_X = \varphi^* \mathcal{O}_Y(1)$ and by the commutativity of diagram (2.1.1), linearly equivalent to $p^*(2C_0 + 2ef) + 2aF$. Thus $\mathcal{O}_{\overline{X}}(2aF) = p^* N$, for certain line bundle $N$ on $Y$. Then $p^*(N \otimes 4) = p^* \mathcal{O}_Y(2aC_0)$. This implies that $N \otimes 4$ and $\mathcal{O}_Y(2aC_0)$ are numerically equivalent, and since $Y$ is a rational ruled surface, linearly equivalent. Then $N = \mathcal{O}_Y(aC_0)$, with $4a = 2a$, and $a$ integer so $a$ is in fact a nonnegative even integer. On the other hand we consider the ramification $R$ of $p$. We have

$$K_{\overline{X}} = p^* K_Y + R = p^*(-2C_0 - (e + 2)f) + R .$$

Since $K_{\overline{X}} \equiv p^*(C_0 + ef) + aF$, we obtain $R \equiv p^*(3C_0 + (2e + 2)f) + aF$. Recall that $C_0$ is in the branch locus of $p$ and since neither $D_1 + D_2$ nor $D_1 + D_3$ nor $D_2 + D_3$ has multiple components, $C_0$ belongs to the branch locus with multiplicity 1. Then we can write $R = R_1 + 3F$, where $R_1$ is a cycle that does not contain $F$ in its support, and therefore, $R_1 \cdot F \geq 0$. Now we compute exactly $R_1 \cdot F$. The cycle $R_1$ is numerically equivalent to $p^*(2C_0 + (2e + 2)f) + (a + 1)F$, then

$$R_1 \cdot F = (2C_0 + (2e + 2)f)C_0 + (a + 1)F^2 = 2 - \frac{(a + 1)e}{4} .$$

Hence $(a + 1)e \leq 8$. Then, since $a \geq 0$ and is even, and $e \geq 2$, then either $a = 0$ and $\overline{q}$ is crepant or $a = e = 2$. The first case is not possible, since then $\omega_{\overline{X}} = p^* \mathcal{O}_Y(1)$ and $L_1 \otimes L_2 = L_3$, hence $D_2 = 0$. In the second case we know that $X$ is 2-Gorenstein. Moreover, since $e = 2$, $F^2 = -\frac{4}{4} = -\frac{1}{2}$. 


Case 1.3: \( C_0 \) is in the branch locus of \( p \), \( D_2 = 0 \) and \( F \) has inertia group \( \mathbb{Z}_4 \) (last condition occurs if and only if \( C_0 \subset D_3 \)). In this case \( \overline{X} \) is locally Gorenstein. We have \( p^*C_0 = 4F \) and \( F^2 = -\frac{e}{4} \). By the previous observations \( \omega_{\overline{X}} \) is the pullback by \( p \) of a certain line bundle on \( Y \). On the other hand recall that \( K_{\overline{X}} \) is equal to \( \varphi^* K_X + aF \) (with \( a \) a nonnegative integer) and, since \( \omega_X = \varphi^* O_{W}(1) \) and by the commutativity of diagram (2.1.1), linearly equivalent to \( p^*(C_0 + ef) + aF \). Thus \( O_{\overline{X}}(aF) = p^*N \), for certain line bundle \( N \) on \( Y \). Then \( p^*(N^{\otimes 4}) = p^*O_Y(aC_0) \). This implies that \( N^{\otimes 4} \) and \( O_Y(aC_0) \) are numerically equivalent, and since \( Y \) is a rational ruled surface, linearly equivalent. Then \( N = O_Y(aC_0) \), with \( 4\alpha = a \), so \( a \) is multiple of 4. On the other hand we consider the ramification \( R \) of \( p \). We have

\[
K_{\overline{X}} = p^*K_Y + R = p^*(-2C_0 - (e + 2)f) + R.
\]

Since \( K_{\overline{X}} \sim p^*(C_0 + ef) + aF \), we obtain \( R \sim p^*(3C_0 + (2e + 2)f) + aF \). Recall that \( C_0 \) is in the branch locus of \( p \) and as argued before, \( C_0 \) belongs to the branch locus with multiplicity 1. Then we can write \( R = R_1 + 3F \), where \( R_1 \) is a cycle that does not contain \( F \) in its support, and therefore, \( R_1 \cdot F \geq 0 \). Now we compute exactly \( R_1 \cdot F \). The cycle \( R_1 \) is linearly equivalent to \( p^*(2C_0 + (2e + 2)f) + (a + 1)F \), then

\[
R_1 \cdot F = (2C_0 + (2e + 2)f)C_0 + (a + 1)F^2 = 2 - \frac{(a + 1)e}{4}.
\]

Hence \( (a + 1)e \leq 8 \). Then, since \( a \geq 0 \) and is multiple of 4 and \( e \geq 2 \), then we have \( a = 0 \).

Case 1.4: \( C_0 \) is in the branch locus of \( p \), \( D_2 = 0 \) and \( F \) has inertia group \( \mathbb{Z}_2 \) (last condition holds if and only if \( C_0 \subset D_1 \)). In this case \( \overline{X} \) is also Gorenstein. We have \( 2F = p^*C_0 \) and \( F^2 = -e \). By the previous observations \( \omega_{\overline{X}} \) is the pullback by \( p \) of a certain line bundle on \( Y \). On the other hand recall that \( K_{\overline{X}} \) is equal to \( \varphi^* K_X + aF \) (with \( a \) a nonnegative integer) and, since \( \omega_X = \varphi^* O_{W}(1) \) and by the commutativity of diagram (2.1.1), linearly equivalent to \( p^*(C_0 + ef) + aF \). Thus \( O_{\overline{X}}(aF) = p^*N \), for certain line bundle \( N \) on \( Y \). Then \( p^*(N^{\otimes 2}) = p^*O_Y(aC_0) \). This implies that \( N^{\otimes 2} \) and \( O_Y(aC_0) \) are numerically equivalent, and since \( Y \) is a rational ruled surface, linearly equivalent. Then \( N = O_Y(aC_0) \), with \( 2\alpha = a \), so \( a \) is even. On the other hand we consider the ramification \( R \) of \( p \). We have

\[
K_{\overline{X}} = p^*K_Y + R = p^*(-2C_0 - (e + 2)f) + R.
\]

Since \( K_{\overline{X}} \sim p^*(C_0 + ef) + aF \), we obtain \( R \sim p^*(3C_0 + (2e + 2)f) + aF \). Recall that \( C_0 \) is in the branch locus of \( p \) and, as argued before, \( C_0 \) belongs to the branch locus with multiplicity 1. Then we can write \( R = R_1 + F \), where \( R_1 \) is a cycle that does not contain \( F \) in its support, and therefore, \( R_1 \cdot F \geq 0 \). Now we compute exactly \( R_1 \cdot F \). The cycle \( R_1 \) is linearly equivalent to \( p^*(2C_0 + (2e + 2)f) + (a + 1)F \), then

\[
R_1 \cdot F = 2(2C_0 + (2e + 2)f)C_0 + (a + 1)F^2 = 4 - (a + 1)e.
\]

Hence \( (a + 1)e \leq 4 \). Since \( a \geq 0 \) and even and \( e \geq 2 \), then we have \( a = 0 \). This ends our argument when \( G = \mathbb{Z}_4 \).
Case 2: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $G_1$, $G_2$ and $G_3$ be the three order 2 subgroups of $G$ and let $X_i$ be the quotient of $X$ by $G_i$. As we argued in Case 2 of the proof of Theorem 2.5, associated to each subgroup $G_i$ we have a way of decomposing $\varphi$, namely,

$$ X \xrightarrow{\varphi_2^i} X_i \xrightarrow{\varphi_1^i} W, $$

where $X_i$ is normal and $\varphi_1^i$ and $\varphi_2^i$ are Galois covers with Galois group $G/G_i$ and $G_i$ respectively.

Let $\overline{X_i}$ be such that $\mathcal{O}_{\overline{X_i}}$ is the integral closure of $\mathcal{O}_Y$ in $\mathcal{K}(X_i)$. Then, arguing as in Case 2 of the proof of Theorem 2.5 we have the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi_2^i} & X_i \\
\downarrow p_2^i & & \downarrow \varphi_2^i \\
\overline{X_i} & \xrightarrow{q_1^i} & X_i \\
\downarrow p_1^i & & \downarrow \varphi_1^i \\
Y & \xrightarrow{q} & W
\end{array}
$$

where $p_1^i$ and $p_2^i$ are Galois covers with groups $G/G_i$ and $G_i$ respectively and $p = p_1^i \circ p_2^i$.

Moreover, by construction, $\overline{X_i}$ is normal, and, since $Y$ is smooth and $p_1^i$ is a double cover, it is also locally Gorenstein.

We will show now that $W = S(0,2)$. We examine the structure of the branch locus of $p$ and of $p_1^i$, $p_2^i$ and $p_3^i$. Since $Y$ is smooth, $p$ is a flat Galois cover with group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Recall that by Proposition 1.2 there exist divisors $D_1$, $D_2$ and $D_3$ in $Y$ such that $p_1^i$ is a double cover of $Y$ branched along $D_j + D_k$, where $i \neq j, j \neq k$ and $k \neq i$. Then since $\overline{X_i}$ is normal for all $i = 1, 2, 3$, no two among $D_1$, $D_2$ and $D_3$ have a common component. In particular $C_0$ is not contained in two of them, let us say, $D_2$ and $D_3$, so $C_0$ is not in the branch locus of $p_1^i$. Let $E = p_1^i \ast C_0$. Then $E$ is a reduced Cartier divisor and the exceptional locus of $q_1$ and $E^2 = -2e$. Now, $\deg \varphi_2^i(\text{disc}X + 1) \geq \text{disc}X_i + 1$ (see [CKM], 6.7.i; note the statement in [CKM] is incorrect: the morphism should be required to be finite). Now, since $X$ has canonical singularities, the discrepancy of each of the $X_i$ is greater than or equal to $-\frac{1}{2}$, so in particular $X_i$ has rational singularities. Then by Lemma 2.3, $E$ has arithmetic genus 0. On the other hand since $\overline{X_1}$ is normal and locally Gorenstein, for suitable canonical divisors $K_{\overline{X_1}}$ and $K_{X_1}$, we obtain as in (2.4.1)

$$ K_{\overline{X_1}} \equiv q_1^i K_{X_1} + bE, $$

where $b$ is a nonnegative rational number. Using adjunction we obtain

$$ -2 = (K_{\overline{X_1}} + E) \cdot E = (q_1^i K_{X_1} + (b + 1)E) \cdot E = -2(b + 1)e, $$

hence $b = -\frac{e - 1}{e}$. Resolving the singularities of $\overline{X_1}$ and composing with $q_1$ we obtain a resolution of singularities for $X_1$. Since $\text{disc}X_1 \geq -\frac{1}{2}$, then $b \geq -\frac{1}{2}$ and $e = 2$.
Now we see that \( \overline{7} \) is crepant. Let \( F \) be the reduced cycle consisting of the curves of \( \overline{X} \) lying over \( C_0 \). We will show now that \( C_0 \) is in the branch locus of \( p \). Assume it is not. Then \( F = p^*C_0 \) and, by formula (2.4.1), for suitable canonical divisors \( K_X \) and \( K_{\overline{X}} \) we have

\[
K_{\overline{X}} \equiv \overline{q}^*K_X + aF = p^*(C_0 + 2F),
\]

with \( a \) a nonnegative rational number and

\[
K_{\overline{X}} = p^*K_Y + R \sim p^*(-2C_0 - 4f) + R,
\]

where \( R \) is the ramification divisor of \( p \). Now by the same argument as the one used just before (2.6.2), \( E_i = p_i^*C_0 \), by Zariski’s main theorem, \( E_i \) is connected, and, as pointed out before, \( E_i \) has canonical singularities. If we compose a resolution of \( E_i \) with \( E_i \), then it will reduce to having one or two components, depending on whether \( E \) has one or two components. In any case, by (2.4.1), for suitable canonical divisors \( K_X \) and \( K_{\overline{X}} \) we have

\[
K_{\overline{X}} = p^*K_Y + R \sim p^*(-2C_0 - 4f) + R + F.
\]

\[
K_{\overline{X}} \equiv \overline{q}^*K_X + aF \sim p^*(C_0 + 2f) + aF, \quad (2.6.3)
\]

where \( a \) is a nonnegative rational number, \( R \) is the ramification divisor of \( p \) and \( R = R_1 + F \) with \( F \) not in the support of \( R_1 \). Since \( p^*C_0 = 2F \), this yields

\[
R_1 \equiv p^*(2C_0 + 6f) + (a + 1)F. \quad (2.6.4)
\]

Now \( (D_2 + D_3) \cdot C_0 = 2 \) implies \( R_1 \cdot F \geq 2 \). Then

\[
2 \leq R_1 \cdot F = (p^*(2C_0 + 6f) + \frac{a + 1}{2}p^*C_0) \cdot \frac{1}{2}p^*C_0 =
\]

\[
2(2C_0 + 6f) \cdot C_0 + (a + 1)C_0^2 = 4 - 2(a + 1).
\]

This implies \( a \leq 0 \), therefore \( a = 0 \), i.e., \( \overline{7} \) is crepant.

Finally we prove that \( \overline{X} \) has canonical singularities. If we compose a resolution of singularities of \( \overline{X} \) with \( \overline{7} \) we obtain a resolution of singularities of \( X \). Since \( a = 0 \), (2.6.3) becomes \( K_{\overline{X}} \equiv \overline{q}^*K_X \) and the discrepancies of the exceptional divisors of the resolution of \( \overline{X} \) are the same whether considered with respect to \( X \) or with respect to \( \overline{X} \). Since \( X \) has canonical singularities, \( \text{disc}X \geq 0 \) and so \( \text{disc}\overline{X} \geq 0 \). Thus \( \overline{X} \) has also canonical singularities. \( \square \)
3. Quadruple Galois canonical covers: crepant case

In this and the next section we achieve the classification of quadruple Galois canonical covers \( \varphi \) of singular rational normal scrolls \( W \). In the previous section we constructed a desingularization diagram for \( \varphi \) (see (2.1.1)) and, in Theorem 2.5 and Theorem 2.6, we studied in great detail one of the sides of this diagram, namely, the morphism \( \overline{q} \). In Theorem 2.5 and Theorem 2.6 we came to the following conclusion: either \( \overline{q} \) is crepant, that is, \( \overline{q}^* \omega_X = \omega_X \), or it is not, but in the latter case, we do know many things about the discrepancies of \( \overline{q} \), about what the Galois group of \( \varphi \) is and of what kind the possible singularities of \( \overline{X} \) are. Thus we will split the study of \( \varphi \) in two cases: the case in which \( \overline{q} \) is crepant and the case in which it is not. We deal with the former case in this section and we will deal with the latter case in Section 4.

Let \( \varphi, q, p \) and \( \overline{q} \) and let \( \overline{\varphi} = q \circ p = \varphi \circ \overline{q} \). If \( \overline{q} \) is crepant, we can “morally” think of \( \overline{\varphi} \) as a Galois canonical cover. Admittedly, \( \overline{\varphi} \) is not finite, so it is not a Galois cover according to our definition, but since \( \omega_X = \overline{q}^* \omega_X \), it turns out that \( \omega_X \) and, in fact, \( \overline{\varphi} \) is the canonical morphism of \( \overline{X} \). However \( \omega_X \) is not ample, so eventually \( \overline{\varphi} \) maps \( \overline{X} \) onto a “singular realization” of \( Y \), namely, the singular rational normal scroll \( W \). All this suggests that one can deal with canonical covers \( \varphi \) when \( q \) in way parallel to the study of Galois canonical covers of smooth rational normal scrolls carried out in [GP2]. To do so we start giving a definition:

**Definition 3.1.** Let \( \overline{X} \) be a normal surface of general type with canonical singularities whose canonical line bundle is base-point-free and let \( \overline{X} \xrightarrow{\overline{\varphi}} W \) be the canonical morphism of \( \overline{X} \). We say that \( \overline{\varphi} \) satisfies (3.1.1) if it factorizes as follows:

\[
\begin{align*}
X & \xrightarrow{p} Y \xrightarrow{q} W,
\end{align*}
\]

where \( p \) is finite and \( q \) is the minimal desingularization of \( W \).

In [GP2] we proved some general results concerning finite canonical covers and Galois canonical covers of smooth surfaces of minimal degree. This results hold in slightly greater generality, as they hold both for the above mentioned canonical covers of smooth surfaces and for finite covers \( p \) as in Definition 3.1. Thus we proceed now to state the versions of the required results of [GP2] for morphisms \( p \) and \( \overline{\varphi} \) like those in Definition 3.1:

**Proposition 3.2.** Let \( \overline{X} \) be a normal surface of general type with canonical singularities and base-point-free canonical bundle. Assume that the canonical morphism \( \overline{X} \xrightarrow{\overline{\varphi}} W \) satisfies (3.1.1). Let \( H = q^* \mathcal{O}_W(1) \).

1) If \( p \) has degree 4, then \( p_* \mathcal{O}_{\overline{X}} \) is a vector bundle on \( Y \) and

\[
p_* \mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus E \oplus (\omega_Y \otimes H^*)
\]

with \( E \) a vector bundle over \( Y \) of rank 2.

2) If, in addition to the hypothesis in 1), \( p_* \mathcal{O}_{\overline{X}} \) splits as a sum of line bundles, then

\[
p_* \mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus L_1^* \oplus L_2^* \oplus (\omega_Y \otimes H^*)
\]
with $L_1^* \otimes L_2^* = \omega_Y \otimes H^*$. 

3) If, in addition to the hypothesis in 1), $\overline{X}$ is regular, then

$$p_*\mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-C_0 - (e + 1)f) \oplus \mathcal{O}_Y(-2C_0 - (e + 1)f) \oplus \mathcal{O}_Y(-3C_0 - (2e + 2)f).$$

Sketch of proof. This proposition is analogous to [GP2], Propositions 1.3 and 1.6, 3) and one can go through the proofs there and adapt them to the present situation. We make explicit below the parallelism between the two settings. In our case, $Y$, which is a smooth Hirzebruch surface $F_e$ with $e \geq 2$, plays the role of $W$ in [GP2]. The morphism $p$ (which is flat and finite, so $p_*\mathcal{O}_{\overline{X}}$ is a vector bundle over $\mathcal{O}_Y$ of rank 4) plays the role of $\varphi$ in [GP2]. Finally the role played in [GP2] by the line bundle $\mathcal{O}_W(1) = \mathcal{O}_W(C_0 + mf)$ is here played by the line bundle $H = q^*\mathcal{O}_W(1) = \mathcal{O}_Y(C_0 + ef)$. Since $\varphi$ is the canonical morphism of $\overline{X}$, the canonical bundle of $\overline{X}$ is $\omega_{\overline{X}} = p^*H$, and we can use relative duality for $p$ as we did for $\varphi$ in [GP2]. The fact that $\varphi$ is induced by the complete canonical series implies that $H^0(p^*\mathcal{O}_Y(C_0 + ef)) = H^0(\mathcal{O}_Y(C_0 + ef))$. The regularity of $\overline{X}$ assumed in 3) has the same implications for the summands of $p_*\mathcal{O}_{\overline{X}}$ as the regularity of $X$ has for the summands of $\varphi_*\mathcal{O}_X$ in [GP2]. □

**Proposition 3.3.** Let $\overline{X}$ be a normal surface of general type with canonical singularities and base-point-free canonical bundle. Assume that the canonical morphism $\overline{X} \xrightarrow{\varphi} W$ satisfies (3.1.1) and that $p$ is Galois with Galois group $G$. Let $L_1$ and $L_2$ be as in Proposition 3.2, 2).

1) If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{X}$ is the fiber product over $Y$ of two double covers $\overline{X}_1 \xrightarrow{p_1} Y$ and $\overline{X}_2 \xrightarrow{p_2} Y$ and $p$ is the natural map from the fiber product to $Y$. The trace-zero modules of $p_1$ and $p_2$ are $L_1$ and $L_2$.

2) If $G = \mathbb{Z}_4$, then there are two divisors $D_1$ and $D_2$ on $Y$ such that $p$ is the composition of a flat double cover $\overline{X}_1 \xrightarrow{p_1} Y$ branched along $D_2$ followed by a flat double cover $\overline{X} \xrightarrow{p_2} \overline{X}_1$, branched along $p_1^*D_1$ and the ramification locus of $p_1$. Moreover, the trace zero module of $p_2$ is $p_1^*L_1$ and the trace zero module of $p_1$ is $L_2$.

Sketch of proof. This result is analogous to [GP2], Proposition 2.6, 2) and 2.7, 4). In our setting, $\overline{X}$ and $Y$ are normal varieties and $p$ is a flat, Galois cover, so Proposition 1.2 applies to $p$ as it does to $\varphi$ in [GP2]. Since $\varphi$ satisfies (3.1.1), then Proposition 3.2 implies that there is a splitting

$$p_*\mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus L_1^* \oplus L_2^* \oplus L_3^*,$$

with $L_1 \otimes L_2 = L_3 = \omega_Y^* \otimes H$. Then [GP2], Propositions 2.6, 1), 2.7, 1) and 2) apply to $\overline{X} \xrightarrow{p} Y$. □

Now we are ready to classify of quadruple Galois canonical covers $\varphi$ of singular rational normal scrolls $W$ when the morphism $\overline{q}$ defined in Definition 2.1 is crepant. To each cover $X \xrightarrow{q} W$ there corresponds a unique cover $\overline{X} \xrightarrow{p} Y$ and we will classify these latter covers. We will study separately the cyclic and the bidouble case:
Theorem 3.4. Let $W$ be a singular rational normal scroll and let $X \xrightarrow{\varphi} W$ be a Galois canonical cover with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $X$, $Y$, $q$, $\varphi$ and $p$ be as in (2.1.1). If $\varphi$ is crepant, then
1) $W = S(0, 2)$ (and hence, $Y = \mathbb{F}_2$);
2) $X$ has at worst canonical singularities;
3) $X$ is regular;
4) $\overline{X} \xrightarrow{\varphi} X$ is the morphism from $\overline{X}$ to its canonical model;
5) $\overline{X}$ is the fiber product over $Y$ of two double covers $p_1$ and $p_2$ branched along divisors $D_2$ and $D_1$ which are linearly equivalent to $2C_0 + 6f$ and $4C_0 + 6f$ respectively.

Conversely, let $\overline{X}$ be a normal surface with at worst canonical singularities and let $Y = \mathbb{F}_2$. If $\overline{X} \xrightarrow{p} Y$ is a fiber product of two double covers $p_1$ and $p_2$ as described in 5) above, then there exists a commutative diagram like (2.1.1) where $\varphi$ is crepant and $\varphi$ is the canonical morphism of $X$ and is Galois with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. It follows from Theorem 2.5 and Theorem 2.6 that, if $\varphi$ is crepant, then $\overline{X}$ has at worst canonical singularities, so we have 2).

Since $\varphi$ is crepant, $H^0(\omega_{\overline{X}}) = H^0(\omega_X)$ and $\omega_X$ is base-point-free, so $\overline{\varphi} = \varphi \circ \varphi$ is the canonical morphism of $\overline{X}$. Then $\overline{X} \xrightarrow{\overline{\varphi}} W$, $p$ and $q$ satisfy the hypothesis of Definition 3.1 and $Y = \mathbb{F}_2$. The morphism $\varphi$ is Galois and so is $p$, by Lemma 2.2 and both have the same Galois group, also by Lemma 2.2. Applying Proposition 3.3, 1) to $\overline{X} \xrightarrow{\overline{\varphi}} W$, we obtain that $\overline{X}$ is the fiber product over $Y$ of two double covers $\overline{X}_1 \xrightarrow{p_1} Y$ and $\overline{X}_2 \xrightarrow{p_2} Y$.

Those covers are branched along divisors $D_1$ and $D_2$. We set $L_1$ and $L_2$ line bundles on $Y$ such that $L_i = \mathcal{O}_Y(a_iC_0 + b_i f)$, $L_1^{\otimes 2} = \mathcal{O}_Y(D_2)$ and $L_2^{\otimes 2} = \mathcal{O}_Y(D_1)$ (we are using this rather strange notation so as to be consistent with the notation of Proposition 1.2). Then

$$p_* \mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus L_1^* \oplus L_2^* \oplus (L_1^* \otimes L_2^*).$$

We prove now 1) and the remaining of 5), that is, the description of $D_1$ and $D_2$ (we have already seen above that $\overline{X}$ is a fiber product of two covers). By Proposition 3.3, 1) and Proposition 3.2, 2), we know that $L_1^* \otimes L_2^* = \omega_Y \otimes H^*$, hence

$$a_1 + a_2 = 3$$
$$b_1 + b_2 = 2e + 2.$$

Since $D_1$ and $D_2$ are effective and linearly equivalent to $2(a_2C_0 + b_2f)$ and $2(a_1C_0 + b_1f)$ respectively, we have $a_i, b_i \geq 0$. We set $a_1 = 0, 1$ (in which case, $a_2 = 3, 2$). Since $\overline{\varphi}$ is induced by the complete canonical series of $\overline{X}$, $\overline{\varphi}$ is crepant, $\varphi$ is a canonical cover and $\overline{\varphi} = q \circ p = \varphi \circ \overline{\varphi}$, then $H^0(p^*\mathcal{O}_Y(C_0 + ef)) = H^0(\mathcal{O}_Y(C_0 + ef))$. Then $H^0(\mathcal{O}_Y((1 - a_1)C_0 + (e - b_1)f)) = 0$, so $b_1 \geq e + 1$ and, since $b_1 + b_2 = 2e + 2$, $b_2 \leq e + 1$. Now, let us assume $a_1 = 0$. Then $D_1 \sim 6C_0 + 2b_2 f$, and, since $e \geq 2$, then $3C_0$ is in the fixed part of $|D_2|$. This would imply that $\overline{X}$ is nonnormal, which is not possible. Thus $a_1$ can only be 1. Then $D_1 \sim 4C_0 + 2b_2 f$. If $e > 2$ or $b_2 < e + 1$, then $2C_0$ is in the fixed part of $|D_2|$, respectively.
and as before this is not possible. Thus we conclude that \( a_1 = 1, a_2 = 2, b_1 = b_2 = e + 1 \) and \( e = 2 \), and since \( L_i^{\otimes 2} = O_Y(D_j) \), we get \( D_2 \sim 2C_0 + 6f \) and \( D_1 \sim 4C_0 + 6f \).

Now we prove 3), that is, we show that \( X \) is regular. The irregularity of \( \overline{X} \) is the sum of \( h^1(O_Y), h^1(L_1^*), h^1(L_2^*) \) and \( h^1(L_1^* \otimes L_2^*) \), and those numbers are 0 for the above values of \( a_1, a_2, b_1 \) and \( b_2 \). Therefore \( \overline{X} \) is regular and, since \( \overline{X} \) and \( X \) are birational and have rational singularities, so is \( X \).

Finally we show 4). Recall that \( \overline{\varphi} \) is the canonical morphism of \( \overline{X} \). Since \( \varphi \) is finite, the curves (which are \(-2\)-curves) contracted by \( \overline{\varphi} \) are the same as the curves contracted by \( \overline{\varphi} \). Thus \( \overline{X} \xrightarrow{\overline{\varphi}} X \) is the morphism from \( \overline{X} \) to its canonical model.

Conversely, assume that \( \overline{X} \) is a normal surface with canonical singularities and that \( \overline{X} \xrightarrow{p} Y \) is the fiber product over \( Y = \mathbb{F}_2 \) of two double covers of \( Y \), one branched along a divisor linearly equivalent to \( 4C_0 + 6f \) and the other branched along a divisor linearly equivalent to \( 2C_0 + 6f \). Proposition 1.3 tells that \( p \) is a Galois cover with group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Proposition 1.3 also tells that the canonical bundle of \( \overline{X} \) is \( p^*O_Y(C_0 + 2f) \), so it is base-point-free. Moreover one sees easily using projection formula that \( H^0(\omega_{\overline{X}}) = H^0(O_Y(C_0 + 2f)) \), so the canonical morphism \( \overline{\varphi} \) of \( \overline{X} \) factors as \( \overline{\varphi} = q \circ p \), where \( Y \xrightarrow{q} W \) is the contraction of \( C_0 \). Now let \( X \) be the canonical model of \( \overline{X} \). Then \( \overline{\varphi} \) also factors as \( \overline{\varphi} = \varphi \circ \overline{\varphi} \), where \( X \xrightarrow{\varphi} W \) is the canonical morphism of \( \overline{X} \). Finally, we note that, since the canonical bundle of \( \overline{X} \) is base-point-free, \( \overline{\varphi} \) is crepant, and, since \( p \) is Galois with group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), so is \( \varphi \), by Lemma 2.2. \( \square \)

**Theorem 3.5.** Let \( W \) be a singular rational normal scroll and let \( X \xrightarrow{\varphi} W \) be a Galois canonical cover with Galois group \( \mathbb{Z}_4 \). Let \( \overline{X}, Y, q, \overline{\varphi} \) and \( p \) be as in Definition 2.1. If \( \overline{\varphi} \) is crepant, then

1) \( W = S(0, 2) \) (and hence, \( Y = \mathbb{F}_2 \));

2) \( \overline{X} \) has at worst canonical singularities;

3) \( X \) is regular;

4) \( \overline{X} \xrightarrow{\overline{\varphi}} X \) is the morphism from \( \overline{X} \) to its canonical model;

5) \( p \) is the composition of two double covers \( \overline{X} \xrightarrow{p_1} Y \) branched along a divisor \( D_2 \) linearly equivalent to \( 4C_0 + 6f \) and \( \overline{X} \xrightarrow{p_2} \overline{X} \), branched along the ramification of \( p_1 \) and \( p_2 \), with \( D_1 \) linearly equivalent to \( 3f \) and having trace-zero module \( p_1^*O_Y(-C_0 - 3f) \).

Conversely, let \( \overline{X} \) be a normal surface with at worst canonical singularities and let \( Y = \mathbb{F}_2 \). If \( \overline{X} \xrightarrow{p} Y \) is the composition of two double covers \( p_1 \) and \( p_2 \) as described in 5) above, then there exists a commutative diagram like (2.1.1) where \( \overline{\varphi} \) is crepant and \( \varphi \) is the canonical morphism of \( X \) and is Galois with Galois group \( \mathbb{Z}_4 \).

**Proof.** It follows from Theorem 2.5 and Theorem 2.6 that, if \( \overline{\varphi} \) is crepant, then \( \overline{X} \) has at worst canonical singularities, so we have 2). Let \( \overline{\varphi} = \varphi \circ \overline{\varphi} \). Then, as argued in the proof of Theorem 3.4, \( \overline{X} \xrightarrow{\overline{\varphi}} W \) is a canonical cover satisfying (3.1.1). Let \( Y, p \) and \( q \) satisfy the hypothesis of Definition 3.1. The morphism \( \varphi \) is Galois and, by Lemma 2.2, so is \( p \) and both have the same Galois group, which is \( \mathbb{Z}_4 \). Applying Proposition 3.2, 2) and
Proposition 3.3, 2) to \( \overline{X} \to W \), we obtain that \( \overline{X} \) is the composition of a flat double cover \( \overline{X}_1 \to Y \) branched along \( D_2 \) followed by a flat double cover \( \overline{X} \to \overline{X}_1 \), branched along \( p_2^* D_1 \) and the ramification locus of \( p_1 \). Moreover, the trace zero module of \( p_2 \) is \( p_1^* L_1 \) and the trace zero module of \( p_1 \) is \( L_2 \). Then, by Proposition 1.4,

\[
p_* \mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus L_1^* \oplus L_2^* \oplus (L_1^* \otimes L_2^*) .
\]

We prove now 3), that is, that \( X \) is regular. Let \( L_i = \mathcal{O}_Y(-a_i C_0 - b_i f) \). By Proposition 3.3, 2) and Proposition 3.2, 2), we know that \( \omega_Y \otimes H^* \), hence \( a_1 + a_2 = 3 \)

\[ b_1 + b_2 = 2e + 2 . \]

Let us assume that \( \overline{X} \) is irregular. Since \( L_1^{\otimes 2} \otimes L_2^* = \mathcal{O}_Y(D_1) \) and \( L_2^{\otimes 2} = \mathcal{O}_Y(D_2) \) are effective then \( a_2 \leq 2a_1, b_2 \leq 2b_1 \) and \( a_2, b_2 \geq 0 \). Then \( a_1, b_1 \geq 0 \) also. Moreover, \( a_1, b_1 \geq 1 \), otherwise we will contradict \( a_1 + a_2 = 3 \) or \( b_1 + b_2 = 2e + 2 \). Then let us examine all the possibilities for \( a_1 \). First, if \( a_1 = 1 \), then \( L_1^* = \mathcal{O}_Y(-C_0 - b_1 f) \) and \( L_2^* = \mathcal{O}_Y(-3C_0 - (2e + 2)f) \) are special, so \( L_1^* = \mathcal{O}_Y(-2C_0 - b_2 f) \) is not. This means \( b_2 \leq e \). In this case, since \( e \geq 2 \), \( 2C_0 \) is in the fixed part of \( |D_2| = |4C_0 + 2b_2 f| \), and \( \overline{X} \) will not be normal, which is not possible, so \( a_1 = 1 \) is ruled out. Second, if \( a_1 = 2 \), then \( D_1 \sim 3C_0 + (2b_1 - b_2) f, D_2 \sim 2C_0 + 2b_2 f, L_1 = \mathcal{O}_W(2C_0 + b_1 f) \) and \( L_2 = \mathcal{O}_W(C_0 + b_2 f) \). Since \( H^1(L_2^*) = H^1(L_3^*) = 0 \) and we are assuming \( \overline{X} \) to be irregular, then \( H^1(L_1^*) \neq 0 \). This implies \( b_1 \leq e \). Then \( b_1 + b_2 = 2e + 2 \) implies \( b_2 \geq e + 2 \). On the other hand, since \( \overline{X} \) is normal, \( C_0 \) has at most multiplicity 1 in the fixed part of \( |D_1| \), hence \( 2b_1 - b_2 - 5e \geq 0 \). Now since \( H^0(p_1^* \mathcal{O}_Y(C_0 + ef)) = H^0(\mathcal{O}_Y(C_0 + ef)) \), we have that \( b_2 > e \), hence \( b_1 < e + 2 \). Then we get \(-4e + 4 > 0 \). This contradicts \( e \geq 2 \) and so \( a_1 = 3 \) is also ruled out. Thus \( \overline{X} \) is regular, and so is \( X \).

We prove now 1) and the remaining of 5), that is, the description of \( D_1 \) and \( D_2 \). Since \( \overline{X} \) is regular, then Proposition 3.2, 3) tells that \( b_1 = b_2 = e + 1 \) and, either \( a_1 = 1, a_2 = 2 \) or \( a_1 = 2, a_2 = 1 \). If \( a_1 = 2 \) and \( a_2 = 1 \), then \( L_2^* = \mathcal{O}_Y(-C_0 - (e + 1)f) \), \( L_1^* = \mathcal{O}_Y(-2C_0 - (e + 1)f) \) and \( D_1 \sim 3C_0 + (e + 1)f \). Then, since \( e \geq 2 \), \( 2C_0 \) is in the fixed part of \( |D_1| \) and this contradicts the normality of \( \overline{X} \). Therefore the only possibility left is \( a_1 = 1, a_2 = 2, b_1 = b_2 = e + 1 \). In this case \( L_1^* = \mathcal{O}_W(-C_0 - (e + 1)f) \), \( L_2^* = \mathcal{O}_W(-2C_0 - (e + 1)f) \), so \( D_2 \) is linearly equivalent to \( 4C_0 + (2e + 2)f \). Arguing as before we see that the normality of \( \overline{X} \) implies \( e \leq 2 \), so in fact, \( e = 2 \). Since in this case \( D_1 \) is linearly equivalent to \( (e + 1)f \), this concludes the proof of 1) and 5).

Finally, by the same argument given in the proof of Theorem 3.4, \( \overline{X} \to X \) is the morphism from \( \overline{X} \) onto its canonical model \( X \), so we have 4).

Conversely, let \( Y = F_2 \), let \( \overline{X} \) be a normal surface with at worst canonical singularities and let \( \overline{X} \to Y \) be the composition of two double covers \( \overline{X}_1 \to Y \), branched along \( D_2 \)
and $X \xrightarrow{p_1} \overline{X}_1$, branched along $p_1^*D_1$ and the ramification divisor of $p_1$, where $D_1 \sim 3f$ and $D_2 \sim 4C_0 + 6f$ and $p_1^*\mathcal{O}_Y(-C_0 - 3f)$ is the trace zero module of $p_2$. By Proposition 1.4, $p$ is a Galois cover with group $\mathbb{Z}_4$, and by Lemma 2.2, so is $\varphi$. On the other hand, the trace zero module of $p_2$ is $\mathcal{O}_Y(-2C_0 - 3f)$, since $D_2 \sim 4C_0 + 6f$. Then Proposition 1.4 also implies that $\omega_{\overline{X}} = p^*\mathcal{O}_Y(C_0 + 2f)$. Then $\omega_{\overline{X}}$ is base-point-free and one sees easily using projection formula that $H^0(\omega_{\overline{X}}) = H^0(\mathcal{O}_Y(C_0 + 2f))$, so the canonical morphism $\overline{\varphi}$ factors as $\overline{\varphi} = q \circ p$, where $Y \xrightarrow{\varphi} W$ is the contraction of $C_0$. Now let $X$ be the canonical model of $\overline{X}$. Then $\overline{\varphi}$ also factors as $\overline{\varphi} = \varphi \circ q$, where $X \xrightarrow{\varphi} W$ is the canonical morphism of $X$. Since the canonical bundle of $X$ is base-point-free $\overline{q}$ is crepant. □

4. Quadruple Galois covers: non crepant case.

In this section we complete the classification of quadruple Galois canonical covers $\varphi$ of singular rational normal scrolls $W$. Now we are concerned with the case in which the morphism $\overline{q}$ in (2.1.1) is non-crepant. As we did in Section 3, instead of classifying directly canonical Galois covers $X \xrightarrow{\varphi} W$ we will classify their “desingularization” $\overline{X} \xrightarrow{\overline{q}} Y$ (see Definition 2.1) and we will specify the way to go from $\overline{X}$ to $X$ and vice versa, by explicitly characterizing the birational morphism $\overline{q}$.

**Theorem 4.1.** Let $W$ be a singular rational normal scroll and let $X \xrightarrow{\varphi} W$ be a Galois canonical cover with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\overline{X}$, $Y$, $q$, $\overline{q}$ and $p$ be as in (2.1.1). If $\overline{q}$ is not crepant, then

1) $W = S(0, 2)$;
2) $\overline{X}$ has at worst canonical singularities;
3) $X$ is regular;
4) $\overline{X} \xrightarrow{\overline{q}} X$ is the morphism from $\overline{X}$ to its canonical model.
5) $\overline{X}$ is the normalization of the fiber product over $Y$ of two double covers of $Y$ branched each along divisors $D_1$ and $D_2$, where $D_1 = D_1 + C_0$, $D_2 = D_2 + C_0$, $D_1 \sim D_2 \sim 3C_0 + 6f$ and all components of $D_1 + D_2 + C_0$ have multiplicity 1.

Conversely, if $\overline{X}$ has at worst canonical singularities and is the normalization of a fiber product over $Y$ as described in 5) above, then there exists a commutative diagram like (2.1.1) such that $\overline{q}$ is noncrepant and $\varphi$ is the canonical morphism of $X$ and is Galois with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** Since $\overline{q}$ is non crepant and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, 1) and 2) follow from Theorem 2.5 and Theorem 2.6. We are in fact in the situation of Theorem 2.5, 2), so $\overline{q}$ is the blowing down of two $-1$ curves, and, since $\omega_X$ is ample, $X$ is not only minimal (in the sense that $K_X$ is nef) but is also its canonical model. This shows 4).

Let us call $F_1$ and $F_2$ the two curves (two smooth lines) lying over $C_0$. Recall that, as seen in Theorem 2.5 just before (2.5.2), $p^*C_0 = 2F_1 + 2F_2$. The ramification locus $R$ of $p$ is of the form $R_1 + F_1 + F_2$, where $R_1$ contains neither $F_1$ nor $F_2$ and, by (2.5.6), $R_1 \cdot F_i = 0$. By (2.4.2) and having in account Theorem 2.5, 2), for suitable canonical divisors $K_{\overline{X}}$ and $K_X$ we have

$$K_{\overline{X}} = \overline{q}^*K_X + F_1 + F_2$$

(4.1.1).
Let $G_1$, $G_2$ and $G_3$ be the three index 2 subgroups of $G$ and let $X_i$ be the quotient of $X$ by $G_i$. As we argued in Case 2 of Theorem 2.5, associated to each subgroup $G_i$ we have a way of decomposing $\varphi$, namely,

\[ X \xrightarrow{\varphi_i} X_i \xrightarrow{\varphi_i^1} W \]

where $X_i$ is normal and $\varphi_i^1$ and $\varphi_i^2$ are Galois covers of degree 2. Let $\overline{X}_i$ be such that $\mathcal{O}_{\overline{X}_i}$ is the integral closure of $\mathcal{O}_Y$ in $\mathcal{K}(X_i)$. Then, again arguing as in Theorem 2.5 we have the following commutative diagram

\[
\begin{array}{ccc}
\overline{X} & \xrightarrow{j} & X \\
p_2 & & \varphi_2 \\
\overline{X}_i & \xrightarrow{q_i} & X_i & (4.1.2) \\
p_1 & & \varphi_i \\
Y & \xrightarrow{q} & W
\end{array}
\]

where $p_i^1$ and $p_i^2$ are Galois covers of degree 2 and $p = p_i^1 \circ p_i^2$. Moreover, by construction, $\overline{X}_i$ is normal and, since $Y$ is smooth and $p_1^1$ is a double cover, it is also Gorenstein. Let

\[ p_* \mathcal{O}_{\overline{X}} = \mathcal{O}_Y \oplus L_1^* \oplus L_2^* \oplus L_3^* , \]

with $L_i = \mathcal{O}_Y(a_iC_0 + b_i f)$. Then by Proposition 1.2, 2) there exist effective divisors $D_1$, $D_2$ and $D_3$ on $Y$ such that $D_1 + D_2 + D_3$ is the branch locus of $p$, $\overline{X}_i = \text{Spec}(\mathcal{O}_Y \oplus L_i^*)$, $L_i^{\otimes 2} = \mathcal{O}_Y(D_j + D_k)$ and $p_i^1$ is branched along $D_j + D_k$. By (4.1.1) and the commutativity of (4.1.2) we have

\[ \omega_{\overline{X}} = p^* \mathcal{O}_Y(C_0 + 2f) \otimes \mathcal{O}_{\overline{X}}(F_1 + F_2) . \]

We find out now the possible values of the $a_i$s and the $b_i$s. Let $D \simeq \mathbb{P}^1$ be a smooth general member in the linear system $|C_0 + 2f|$ on $Y = F_2$ and let $C$ be its inverse image under $p$. Then, since $C \cdot (C_0 + 2f) = 0$, by adjunction, $\omega_C = p^* \mathcal{O}_Y(2C_0 + 4f) \otimes \mathcal{O}_C = p^* \mathcal{O}_{\mathbb{P}^1}(4)$. Applying relative duality to $p|_C$ as we did in Case 1 of the proof of Theorem 2.6, we conclude (maybe renumbering $L_1$, $L_2$ and $L_3$) that $(L_1^* \otimes L_2^*) \otimes \mathcal{O}_D = L_3^* \otimes \mathcal{O}_D = \mathcal{O}_{\mathbb{P}^1}(-6)$. Then $b_3 = 6$ and $b_1 + b_2 = 6$. Then $b_1 + b_2 = b_3$ and, since $L_1 \otimes L_2 = L_3 \otimes \mathcal{O}_Y(D_3)$, then $D_3 \cdot D = 0$, so $D_3$ is a multiple of $C_0$. Then $D_3 = 0$ or $D_3 = C_0$, for $\overline{X}_1$ and $\overline{X}_2$ are normal. On the other hand, if $D_3 = 0$, then $L_1 \otimes L_2 = L_3$, so by Proposition 1.3, $\omega_{\overline{X}} = p^* \mathcal{O}_Y(C_0 + 2f)$, a contradiction. Hence we have $D_3 = C_0$ and

\[ a_1 + a_2 = a_3 + 1 \quad (4.1.3) . \]

Now, since $\overline{X}_1$ is normal, no two among $D_1$, $D_2$ and $D_3$ have common components. Thus $2R = p^*(D_1 + D_2 + D_3)$. By Proposition 1.2, $(L_1 \otimes L_2 \otimes L_3)^{\otimes 2} = \mathcal{O}_Y(2(D_1 + D_2 + D_3))$,
and, since $Y$ is a rational ruled surface, $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_Y(D_1 + D_2 + D_3)$. Thus from (2.4.2), (4.1.1) and the commutativity of (4.1.2) we have

\[
2K_{\overline{X}} \sim p^*(-4C_0 - 8f) + 2R \sim p^*((a_1 + a_2 + a_3 - 4)C_0 + 4f) \\
2K_{\overline{X}} \sim p^*(2C_0 + 4f) + 2F_1 + 2F_2 = p^*(3C_0 + 4f).
\]

Then $(a_1 + a_2 + a_3 - 4)C_0 + 4f \equiv 3C_0 + 4f$, and since $Y$ is a rational normal scroll, $(a_1 + a_2 + a_3 - 4)C_0 + 4f = 3C_0 + 4f$, so $a_1 + a_2 + a_3 = 7$. This together with (4.1.3) yields $a_1 + a_2 = 4$ and $a_3 = 3$. We use this information, together with $b_1 + b_2 = b_3 = 6$ previously obtained, to determine the $a_i$s and the $b_j$s. Since $L_1^{\otimes 2} \otimes \mathcal{O}_Y(-C_0) = \mathcal{O}_Y(D_2)$ and $L_2^{\otimes 2} \otimes \mathcal{O}_Y(-C_0) = \mathcal{O}_Y(D_1)$, and $D_1$ and $D_2$ are effective, we obtain $2a_1 - 1 \geq 0$ and $2a_2 - 1 \geq 0$, yielding $a_1, a_2 \geq 1$ and also, $b_1, b_2 \geq 0$. Let us assume $a_1 \leq a_2$. Then the only possibilities for $a_1, a_2$ are $a_1 = 1, a_2 = 3$ or $a_1 = a_2 = 2$. On the other hand $D_1$ and $D_2$ not having multiple components implies $b_1 \geq 2a_1 - 1$ and $b_2 \geq 2a_2 - 1$. Thus $a_1 = 1, a_2 = 3$ and $b_1 + b_2 = 6$ implies $b_1 = 1$ and $b_2 = 5$. By direct computation this implies $h^1(\mathcal{O}_{\overline{X}}) = 0$. Since $C_0 \cdot (C_0 + 2f) = 0$, the restriction of $p_*\mathcal{O}_{\overline{X}}$ to a general member $C$ of $|C_0 + 2f|$ is the same as the restriction of $\varphi_*\mathcal{O}_X$ to a general hyperplane section of $W$. But then [GP1], Lemma 2.3 tells that $b_1 = b_2 = 3$, so we get a contradiction. Therefore the only possibility left is $a_1 = a_2 = 2$. Then $b_1 + b_2 = 6, b_1 \geq 2a_1 - 1$ and $b_2 \geq 2a_2 - 1$ yields $b_1 = b_2 = 3$. Then by direct computation, $h^1(\mathcal{O}_{\overline{X}}) = 0$, so both $\overline{X}$ and $X$ are regular and 3) follows.

Now we prove 5). Recall that we showed $D_3 = C_0$. We know also that $L_1^{\otimes 2} \otimes \mathcal{O}_Y(-C_0) = \mathcal{O}_Y(D_2)$ and $L_2^{\otimes 2} \otimes \mathcal{O}_Y(-C_0) = \mathcal{O}_Y(D_1)$. Then, because of the values of the $a_i$s and the $b_j$s just found, we have that $D_1 \sim D_2 \sim 3C_0 + 6f$. Recall also that the normality of $\overline{X}_i$ for $i = 1, 2, 3$ implies that $D_1 + D_2 + D_3 = D_1 + D_2 + C_0$ does not have multiple components. Now we prove the statement in 5), namely, that $\overline{X}$ is the normalization of the fiber product over $Y$ of two double covers $p_1^1$ and $p_2^1$ of $Y$, branched along $D_2' = D_2 + C_0$ and along $D_1' = D_1 + C_0$ respectively. Let $\hat{X} \xrightarrow{\hat{p}} Y$ be the fiber product over $Y$ of $\overline{X}_1$ and $\overline{X}_2$. Let $U = Y - C_0, V = \overline{X} - F_1 - F_2$ and $\hat{V} = \hat{p}^{-1}(U)$. Since $b_1 + b_2 = b_3$, we have that $(L_1 \otimes L_2)|_U = L_3|_U$. Then, by the same reason argued for Proposition 3.3, 1) (for more details, see [GP2], Proposition 2.7), the restriction $V \xrightarrow{\hat{V}|_U} U$ is a fiber product of the restriction of $p_1^1$ and $p_2^1$ to $V$. Thus $V = \hat{V}$. In particular, since $\overline{X}$ is normal, so is $\hat{V}$. Let now $\hat{X}$ be the normalization of the reduced part of $\overline{X}$ and $\hat{V}$ the open set of $\hat{X}$ lying over $U$. Then, since $V = \hat{V}$ is normal, $\hat{V} = \hat{V} = V$, so $\hat{X}$ and $\overline{X}$ are birational. Moreover, $\mathcal{O}_{\hat{X}}$ is integral over $\mathcal{O}_Y$ and therefore, the integral closure of $\mathcal{O}_{\overline{X}}$ in $\mathcal{K}(X)$. Hence $\hat{X} = \overline{X}$.

Now we prove the converse. Let $Y = \mathbb{F}_2$ and let $\overline{X} \xrightarrow{\overline{p}} Y$ be the normalization of the fiber product of two double covers of $Y$, one branched along $D_1 + C_0$, and the other branched along $D_2 + C_0, D_1 \sim D_2 \sim 3C_0 + 6f$ and $D_1 + D_2 + C_0$ without multiple components. Let $\overline{X} \xrightarrow{p_2} Y$ be the double cover of $Y$ branched along $D_1' = D_1 + C_0$. In fact $\overline{X} = \text{Spec}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-2C_0 - 3f))$. Then $\overline{X}$ is normal and locally Gorenstein. Now, since $D_1 + D_2 + C_0$ have no multiple components, the double cover $p_2$ of $\overline{X}$ branched
along $p_1^*D_2$ is normal and is in fact $\bar{X}$. We denote $\overline{C_0} = p^{-1}C_0$ and, for general $f$, $\overline{f} = p_1^{-1}f = p_1^*f$. Then $p_1^*C_0 = 2\overline{C_0}$. The canonical bundle of $\bar{X}$ is

$$\omega_{\bar{X}} = p_1^*(\omega_Y \otimes \mathcal{O}_Y(2C_0 + 3f)) = p_1^*\mathcal{O}_Y(-f) = \mathcal{O}_{\bar{X}}(\overline{-f}) .$$

Then $\bar{X} = \text{Spec}(\mathcal{O}_{\bar{X}} \oplus \mathcal{O}_{\bar{X}}(-3\overline{C_0} - 3\overline{f}))$ and the canonical bundle of $\bar{X}$ is

$$\omega_{\bar{X}} = p_2^*(\omega_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(3\overline{C_0} + 3\overline{f})) = p_2^*(\mathcal{O}_{\bar{X}}(3\overline{C_0} + 2\overline{f})) = p^*(\omega_Y(C_0 + 2f)) \otimes p_2^*(\mathcal{O}_{\bar{X}}(\overline{C_0})).$$

Recall that $\bar{X}$ is smooth at every point of $\overline{C_0}$, for $D_1 \cdot C_0 = 0$. Moreover $p_2$ is étale at every point of $\overline{C_0}$ and $p_2^*\overline{C_0} = F_1 + F_2$, where $F_1$ and $F_2$ are two disjoint lines, each of them with self-intersection $-1$. Then $\bar{X}$ is smooth at every point of $F_1$ and $F_2$.

Let $L = p^*\mathcal{O}_Y(C_0 + 2f)$. Then $L$ is base-point-free. Using projection formula we compare $H^0(L)$ and $H^0(\mathcal{O}_Y(C_0 + 2f))$ and see that they are equal. This means that the morphism induced by $H^0(L)$ factorizes through $p$. On the other hand let $\overline{X} \xrightarrow{\overline{\varphi}} X$ be the contraction of $F_1$ and $F_2$. Since $\bar{X}$ is smooth at every point of $F_1$ and $F_2$, and $F_1^2 = F_2^2 = -1$, $X$ is smooth at the images $x_1$ and $x_2$ of $F_1$ and $F_2$. Since $\overline{X}$ is normal with at worst canonical singularities, then so is $X$. We also know that

$$K_{\overline{X}} = \overline{\varphi}^*K_X + F_1 + F_2 \quad (4.1.4) .$$

Since $\omega_{\overline{X}} = p^*(\omega_Y(C_0+2f)) \otimes p_2^*(\mathcal{O}_{\overline{X}}(\overline{C_0}))$ and $F_1 + F_2 = p_2^*\overline{C_0}$, then $\overline{\varphi}^*\omega_{\overline{X}} = L$. Moreover $H^0(L) = H^0(\overline{\varphi}^*\omega_{\overline{X}}) = H^0(\overline{\varphi}^*\overline{\varphi}^*\omega_{\overline{X}}) = H^0(\omega_X)$. Then, since $L$ is base-point-free, so is $\omega_X$ and the morphism induced by $H^0(L)$ also factorizes through the canonical morphism of $X$. Thus we have finally the desired commutative diagram:

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{\varphi}} & X \\
\downarrow p & & \downarrow \varphi \\
Y & \xrightarrow{q} & W
\end{array}$$

where $W$ is the cone over a conic inside $\mathbb{P}^3$, $q$ is the minimal desingularization of $W$, $\varphi$ is the canonical morphism of $X$, and, by (4.1.4), $\overline{\varphi}$ is noncrepant. Now the fact that over $Y - C_0$ the surface $\overline{X}$ is a fiber product and Proposition 1.3 imply that $\mathcal{K}(X)/\mathcal{K}(Y)$ is a Galois extension with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now, since $p$ is finite and $\overline{X}$ is normal, $p$ is Galois cover with group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and, by Lemma 2.2, so is $\varphi$. \qed

**Theorem 4.2.** Let $W$ be a singular rational normal scroll and let $X \xrightarrow{\varphi} W$ be a Galois canonical cover with Galois group $\mathbb{Z}_4$. Let $\overline{X}$, $Y$, $q$, $\overline{\varphi}$ and $p$ be as in Definition 2.1. If $\overline{\varphi}$ is not crepant, then

1) $W = S(0,2)$;
2) $\overline{X}$ has at worst canonical singularities;
3) $X$ is regular;
4) $\overline{X} \xrightarrow{q} X$ is the morphism to the canonical model of $\overline{X}$;

5) the morphism $p$ is the composition of two double covers $\overline{X} \xrightarrow{p_1} Y$, branched along a divisor $\Delta_2$, and $\overline{X} \xrightarrow{p_2} \overline{X}$, branched along the ramification of $p_1$ and $p_1^* D_1$ and with trace zero module $p_1^* O_Y(-\frac{1}{2}(D_1+C_0)-\frac{1}{4}\Delta_2) \otimes O_{\overline{X}}(C_0)$, where $C_0$ is $p_1^{-1} C_0$, and either

5.1) $D_1 \sim C_0 + 3f$, $\Delta_2 \sim 4C_0 + 6f$; or

5.2) $D_1 \sim 4C_0 + 9f$, $\Delta_2 \sim 2C_0 + 2f$

Conversely, let $X$ be a normal surface with at worst canonical singularities and let $Y = \mathbb{F}_2$. If $\overline{X} \xrightarrow{p} Y$ is the composition of two double covers $p_1$ and $p_2$ as described in 5) above, then there exists a commutative diagram like (2.1.1), where $\phi$ is the canonical morphism of $X$ and is Galois with Galois group $\mathbb{Z}_4$ and $q$ is noncrepent.

**Proof.** From Theorem 2.5 and Theorem 2.6, if $G = \mathbb{Z}_4$ and $q$ is noncrepent, $W = S(0,2)$, so we have 1), and $\phi^{-1}\{w\}$ consists of 1 or 2 points. We split the argument in two cases accordingly:

**Case 1:** Cardinality of $\phi^{-1}\{w\} = 2$. In this case, according to Theorem 2.5, $\overline{X}$ is locally Gorenstein and $\overline{q}$ is the blowing up of $X$ at $x_1$ and $x_2$, which are smooth points. Moreover $K_{\overline{X}} = \overline{q}^* K_X + F_1 + F_2$, where $F_1$ and $F_2$ are the exceptional divisors, $p^* C_0 = 2F_1 + 2F_2$, and $K_{\overline{X}}$ and $K_X$ are suitable canonical divisors. Then in particular the inertia group of $F_1$ and $F_2$ is $\mathbb{Z}_2$. On the other hand we have

$$K_{\overline{X}} \equiv \overline{q}^* K_X + F_1 + F_2 \equiv p^* (\frac{3}{2} C_0 + 2f) \quad (4.2.1).$$

Arguing as in the proof of Theorem 2.6, we have that

$$p_* O_{\overline{X}} = O_Y \oplus L_1^* \oplus L_2^* \oplus L_3^*$$

and

$$L_1 \otimes L_1 = L_2 \otimes O_Y(D_1 + D_2)$$

$$L_1 \otimes L_2 = L_3 \otimes O_Y(D_2)$$

$$L_1 \otimes L_3 = O_Y(D_1 + D_2 + D_3)$$

$$L_2 \otimes L_2 = O_Y(D_2 + D_3)$$

$$L_2 \otimes L_3 = L_1 \otimes O_Y(D_3)$$

$$L_3 \otimes L_3 = L_2 \otimes O_Y(D_1 + D_3)$$

where $D_1$, $D_2$, $D_3$ are effective divisors with neither multiple components nor common components pairwise and $D_2$ is either 0 or $C_0$. Furthermore, by (2.6.1) (note that in the proof of (2.6.1) we do not use the hypothesis of Theorem 2.6 that $\phi^{-1}\{w\}$ is a single point) we have

$$2K_{\overline{X}} \sim p^*(2K_Y + 2L_1 + 2L_2 - D_2) \sim p^*(2K_Y + 2L_3 + D_2) \quad (4.2.2).$$
Comparing (4.2.1) and (4.2.2) we conclude that $D_2 \neq 0$, otherwise $\frac{3}{2}C_0 + 2f$ would be numerically equivalent to a Cartier divisor on $Y$. Hence $D_2 = C_0$. But this would imply that the ramification lying over $C_0$ would have inertia group $\mathbb{Z}_4$, which contradicts the fact that the inertia group of $F_1$ and $F_2$ is $\mathbb{Z}_2$. Thus Case 1 does not actually occur, and we have proven that there is a unique point $x \in X$ lying over the vertex of $W$.

Case 2. Cardinality of $\varphi^{-1}\{w\} = 1$. In this case we know by Theorem 2.6, that $\overline{X}$ is locally 2-Gorenstein and has at worst rational singularities. Recall also that $e = 2$, $p^*C_0 = 4F$ with $F$ isomorphic to $\mathbb{P}^1$ and $F^2 = -\frac{1}{2}$. Let $L_1, L_2, L_3, D_1, D_2, D_3$ be as in the proof of Theorem 2.6, Case 1.2, and let $L_i = \mathcal{O}_Y(a_iC_0 + b_i f)$. Then $D_2 = C_0$ and $L_1 \otimes L_2 = L_3 \otimes \mathcal{O}_Y(C_0)$. Moreover,

$$2K_{\overline{X}} = \overline{q}^* 2K_X + 4F \sim \overline{q}^* 2K_X + p^* C_0$$

$$K_{\overline{X}} = \overline{q}^* K_X + 2F \equiv \overline{q}^* K_X + \frac{1}{2} p^* C_0$$

$$2K_{\overline{X}} \sim p^*(2K_Y + 2L_3 + C_0)$$

$$K_{\overline{X}} \equiv p^*(K_Y + L_3 + \frac{1}{2} C_0)$$

From this, we see that $\omega_Y \otimes L_3 = \mathcal{O}_Y(C_0 + 2f)$, hence $L_3 = \mathcal{O}_Y(3C_0 + 6f)$.

Therefore we have $a_1 + a_2 = 4$ and $b_1 + b_2 = 6$. We examine all possibilities for the $a_i$'s and the $b_j$'s. Recall that $L_1^{\otimes 2} \otimes L_2^{\ast}$ and $L_2^{\otimes 2}$ are effective, hence

$$2a_1 - a_2 \geq 0$$

$$2b_1 - b_2 \geq 0 \quad (4.2.3)$$

$$a_2, b_2 \geq 0$$

Then $a_1, b_1 \geq 0$ also. On the other hand, $D_2 + D_3$ has no multiple components, so in particular, the components of the fixed part of $|L_2^{\otimes 2}|$ have multiplicity 1, and hence $C_0$ appears with at most multiplicity 1 in the fixed part of $|L_2^{\otimes 2}|$. Likewise $D_1$ and $D_2$ do not have common components, and in particular $C_0$ is not a common component of $D_1$ and $D_2$. Since $D_2 = C_0$, we have that $|L_1^{\otimes 2} \otimes L_3^{\ast} \otimes \mathcal{O}_Y(-C_0)|$ does not have $C_0$ as fixed component. Since $D_3$ does not contain $C_0$ either, $|L_2 \otimes L_3 \otimes L_1^{\ast}|$ does not have $C_0$ as fixed component. This yields the inequalities

$$b_2 \geq 2a_2 - 1$$

$$2b_1 - b_2 \geq 2(2a_1 - a_2 - 1) - 1 \quad (4.2.4)$$

We start ruling out possible values for $a_1$. The value $a_1 = 0$ is not possible for in that case $a_2 = 0$, but $a_1 + a_2 = 4$. Suppose now that $a_1 = 1$. Then, by (4.2.3), $0 \leq a_2 \leq 2$, hence $a_1 + a_2 \leq 3$ and we reach again a contradiction. Now, if $a_1 = 4$, then $a_2 = 0$ and $2b_1 - b_2 \geq 13$. This is not possible, since $b_2 \geq 0$ and $b_1 + b_2 = 6$.

Therefore the only values for $a_1$ which are still possible are 2 and 3. If $a_1 = 2$, then $a_2 = 2$ and by (4.2.4), $b_2 \geq 3$ and hence $b_1 \leq 3$. Then, again by (4.2.4), we should have $b_1 = b_2 = 3$. This corresponds to 5.1) in the statement.
Finally, if $a_1 = 3$, then $a_2 = 1$, $b_2 \geq 1$ and $b_1 \leq 5$. But $2b_1 - b_2 \geq 7$ by (4.2.4), hence we should have $b_1 = 5$, $b_2 = 1$. This corresponds to 5.2) in the statement.

Now we finish the description of $p$ separately for $a_1 = 2$ and $a_1 = 3$. In case $a_1 = 2$, recall that $a_1 = a_2 = 2$ and $b_1 = b_2 = 3$ and we have

$$p_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-2C_0 - 3f) \oplus \mathcal{O}_Y(-2C_0 - 3f) \oplus \mathcal{O}_Y(-3C_0 - 6f)$$

and $D_1 \sim C_0 + 3f$, $D_2 = C_0$ and $D_3 \sim 3C_0 + 6f$.

Recall also that $C_0$ is contained neither in $D_1$ nor in $D_3$. Let $U = Y - C_0$ and $V = \overline{X} - F$.

Abusing the notation, we will call also $D_1$ and $D_3$ the restrictions of $D_1$ and $D_3$ to $U$. Likewise we call $L_1, L_2, L_3$ to the restrictions of $L_1, L_2, L_3$ to $U$.

Then, on $U$ we have:

\[
\begin{align*}
L_1 \otimes L_1 & = L_2 \otimes \mathcal{O}_Y(D_1) \\
L_1 \otimes L_2 & = L_3 \\
L_1 \otimes L_3 & = \mathcal{O}_Y(D_1 + D_3) \\
L_2 \otimes L_2 & = \mathcal{O}_Y(D_3) \\
L_2 \otimes L_3 & = L_1 \otimes \mathcal{O}_Y(D_3) \\
L_3 \otimes L_3 & = L_2 \otimes \mathcal{O}_Y(D_1 + D_3)
\end{align*}
\]

and $p|_U$ is the composition of $U' \xrightarrow{\pi_1} U$, a double cover of $U$ branched along $D_3$ and $\pi_2$, a double cover of $U'$ branched along $\pi_1^*D_1$ and the ramification of $\pi_1$.

On the other hand we consider the double cover $\overline{X} \xrightarrow{p_1} Y$ branched along $C_0 + D_3$ and the cover $\hat{X} \xrightarrow{p_2} \overline{X}$, branched along the ramification of $p_1$ and $p_1^*D_1$ and with trace zero module $p_1^*\mathcal{O}_Y(-C_0 - 3f) \otimes \overline{\mathcal{O}_X}(\overline{-C_0})$. Let $\hat{X}_{\text{norm}}$ be the normalization of $\hat{X}$. First note that the open set of $\hat{X}$ lying over $U$ is equal to $V$, which is normal since $\overline{X}$ is. On the points of $\hat{X}$ lying over $C_0$ we see only one singularity of type $A_1$: the point lying over the intersection of $D_1$ and $C_0$. Indeed, recall that $C_0 \cdot D_1 = 1$, hence the intersection is transversal and so is the intersection of $p_1^*D_1$ and the ramification of $p_1$ lying over $C_0$. Hence $\hat{X}$ is normal everywhere. By construction, the open set of $\hat{X}$ lying over $U$ is $V$, so $\overline{X}$ and $\hat{X}$ are birational. Since $\hat{X}$ is normal and integral over $Y$, $\hat{X}$ is in fact the integral closure of $\mathcal{O}_Y$ in $\mathcal{K}(X)$, so in fact $\overline{X} = \hat{X}$. Thus we have seen that $p$ is the composition of two double covers $\overline{X} \xrightarrow{p_1} Y$ branched along a divisor $\Delta_2 = D_3 + C_0 \sim 4C_0 + 6f$ and $\overline{X} \xrightarrow{p_2} \overline{X}$, branched along the ramification of $p_1$ and $p_1^*D_1$, where $D_1 \sim C_0 + 3f$. We have also seen that the trace zero module of $p_2$ is $p_1^*\mathcal{O}_Y(-C_0 - 3f) \otimes \overline{\mathcal{O}_X}(\overline{-C_0})$. This proves 5.1). Now we prove 2). We know that outside $F$, the surface $\overline{X}$ and $X$ are isomorphic so, outside $F$, $\overline{X}$ has canonical singularities by hypothesis. On the other hand, we have seen that $\overline{X} = \hat{X}$, and that the points of $F$ are smooth points of $\hat{X}$ except for one point which is an $A_1$ singularity, which is a canonical singularity. Thus 2) is proven in case 5.1).

Now, since $p$ is the composition of two double covers, and since we know its trace zero modules we can easily see that

$$\omega_{\overline{X}} = p^* \mathcal{O}_Y(C_0 + 2f) \otimes p_2^* \mathcal{O}_{\overline{X}}(\overline{C_0}) \quad (4.2.5)$$
Then $K_{\overline{X}} \cdot F = -1$ and $K_{\overline{X}}$ intersects strictly positively every other curve of $\overline{X}$, so $X$ is not only minimal but it is also the canonical model of $\overline{X}$. This ends the proof of 4) in case 5.1).

Finally, we describe $p$ if $a_1 = 3$. Then $a_1 = 3$, $a_2 = 1$, $b_1 = 5$, $b_2 = 1$ so we have

$$p_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y (-3C_0 - 5f) \oplus \mathcal{O}_Y (-C_0 - f) \oplus \mathcal{O}_Y (-3C_0 - 6f).$$

In this case, $D_2 = C_0$, $D_1 \sim 4C_0 + 9f$, $D_3 \sim C_0 + 2f$. Like before $C_0$ is contained in neither $D_1$ nor $D_3$. Then we can argue as before to show that $p$ is the composition of a double cover $\overline{X} \overset{p_2}{\to} Y$ branched along $\Delta_2 = C_0 + D_3 \sim 2C_0 + 2f$ and a double cover $\overline{X} \overset{p_1}{\to} \overline{X}$, branched along the ramification of $p_1$ and $p_1^*D_1$ and with trace zero module $p_1^*\mathcal{O}_Y(-2C_0 - 5f) \otimes \mathcal{O}_{\overline{X}_1}(\mathcal{C}_0)$. This proves the description in 5.2). Again $D_1 \cdot C_0 = 1$, and $D_3 \cdot C_0 = 0$, hence there is only one point on $F$, the one lying over $D_1 \cap C_0$, which is singular, and its singularity is of type $A_1$. Then, by the same reason as before, $\overline{X}$ is Gorenstein and therefore has canonical singularities, so we have 2) in case 5.2). The proof of 4) in this case is as well as in the case 5.1).

To prove that $\overline{X}$ is regular we need just to use the splitting of $p_* \mathcal{O}_{\overline{X}}$, which is determined by the values of the $a_i$s and the $b_j$s corresponding to 5.1) and 5.2), and compute the cohomology. Since $\overline{X}$ is birational to $X$ and both have rational singularities, $X$ is also regular and hence 3) is proven.

Now we prove the converse. Let $\overline{X}$ be a normal surface with at worst canonical singularities and let $\overline{X} \overset{p_2}{\to} Y$ be the composition of two double covers $\overline{X}_1 \overset{p_1}{\to} Y$ branched along a divisor $\Delta_2$ and $\overline{X} \overset{p_2}{\to} \overline{X}_1$, branched along the ramification of $p_1$ and $p_1^*D_1$, where $D_1$ and $\Delta_2$ satisfy condition 5.1) or 5.2) of the statement. Let $\mathcal{C}_0$ be the inverse image of $C_0$ by $p_1$ and let $F$ be the inverse image of $C_0$ by $p$. Let finally $p_1^*(\mathcal{O}_Y(C_0 + 3f)) \otimes \mathcal{O}_{\overline{X}_1}(\mathcal{C}_0)$ or $p_1^*(\mathcal{O}_Y(2C_0 + 5f)) \otimes \mathcal{O}_{\overline{X}_1}(\mathcal{C}_0)$ be the trace zero module of $p_2$ accordingly. Then one easily obtains as before the formula (4.2.5) for the canonical of $\overline{X}$. Thus $L = p^*(\mathcal{O}_Y(C_0 + 2f))$ is the free part of $\omega_{\overline{X}}$.

We compare now $H^0(L)$ and $H^0(\mathcal{O}_Y(C_0 + 2f))$. In the first place, using projection formula we see $H^0(p_1^*\mathcal{O}_Y(C_0 + 2f)) = H^0(\mathcal{O}_Y(C_0 + 2f))$. Similarly $H^0(p^*\mathcal{O}_Y(C_0 + 2f)) = H^0(p_1^*\mathcal{O}_Y(C_0 + 2f))$. Thus the morphism induced by $|L|$ factorizes as $q \circ p$, where $Y \overset{q}{\to} S(0,2)$ is the morphism induced by $|C_0 + 2f|$.

Now, since $C_0$ is a component of $\Delta_2$, by construction of $p$, $p^*C_0 = 4F$, where $F$ is a smooth line. By Stein factorization, $q \circ p$ factorizes as the composition of $\overline{X} \overset{q}{\to} Y \overset{\varphi}{\to} W$, where, by the commutativity $q \circ p = \varphi \circ q$, $q$ contracts only $F$ and $\varphi$ is finite. Since by hypothesis $\overline{X}$ is normal and $\Delta_2 \cdot (\Delta_2 - C_0) = 0$, $\Delta_2$ is smooth along $C_0$. Since in addition $D_1 \cdot C_0 = 1$, $F$ contains only one singular point of $\overline{X}$, which is of type $A_1$. Contracting $F$ give raise to a smooth point in $X$. Then, since $\overline{X}$ has canonical singularities so does $X$. Now, since $p^*C_0 = 4F$, we have $F^2 = -1/2$. By (4.2.5) we have also $\omega_{\overline{X}} \cdot F = -1$, so

$$K_{\overline{X}} = \overline{q}^*K_X + 2F$$
for suitable canonical divisors, so \( \overline{q} \) is noncrepant. Comparing this formula with (4.2.5) yields \( q^* \omega_X = p^* O_Y (C_0 + 2f) \), so \( H^0 (L) = H^0 (q^* \omega_X) = H^0 (\omega_X) \), so \( \omega_X \) is base-point-free, for so is \( L \), and in the factorization \( q \circ p = \varphi \circ \overline{q} \), \( \varphi \) is in fact the canonical morphism of \( X \).

Finally by Proposition 1.4 \( p \), and therefore \( \varphi \), are Galois covers with Galois group \( Z_4 \). \( \square \)

We finish this section summarizing the splitting of \( p_* O_{\overline{X}} \) for all the surfaces \( \overline{X} \) which appear in Theorems 3.4, 3.5, 4.1 and 4.2. Even though we already showed there that \( \overline{X} \) is regular, the reader can check this fact at once by looking at the next corollary:

**Corollary 4.3.** Let \( W = S(0,2) \), let \( X \overset{\varphi}{\rightarrow} W \) be a quadruple Galois canonical cover and let \( \overline{X} \) and \( p \) be as in Definition 2.1. Then the vector bundle \( p_* O_{\overline{X}} \) splits as follows:

1) \( p_* O_{\overline{X}} = O_Y \oplus O_Y (-3C_0 - 3f) \oplus O_Y (-3C_0 - 6f) \) if \( X \) is as in Theorem 3.4 or Theorem 3.5, i.e., if \( \overline{q} \) is crepant.

2) \( p_* O_{\overline{X}} = O_Y \oplus O_Y (2C_0 + 2f) \oplus O_Y (3C_0 - 6f) \), if \( X \) is as in Theorem 4.1 or in Theorem 4.2, 5.1.

3) \( p_* O_{\overline{X}} = O_Y \oplus O_Y (2C_0 + 2f) \oplus O_Y (3C_0 - 6f) \), if \( X \) is as in Theorem 4.2, 5.2.

**Proof.** The corollary follows from the values for \( a_i s \) and \( b_j s \) found in the proofs of Theorems 3.4, 3.5, 4.1 and 4.2. \( \square \)

5. **Singularities of quadruple Galois canonical covers and examples.**

In this section we describe further the surfaces \( X \) and \( \overline{X} \) classified in Theorems 3.4, 3.5, 4.1 and 4.2 and the morphism \( \overline{q} \). We focus especially in the study of the singularities of \( X \) and \( \overline{X} \).

**Corollary 5.1.** Let \( W = S(0,2) \) and let \( w \) be its vertex. Let \( X \overset{\varphi}{\rightarrow} W \) be a quadruple Galois canonical cover with Galois group \( G \) and let \( \overline{X} \) and \( \overline{q} \) be as in Definition 2.1.

1) If \( G = Z_2 \times Z_2 \) and \( \overline{q} \) is crepant (i.e., \( \varphi \) is as in Theorem 3.4), then there is only one point \( x \) lying over \( w \). Moreover, \( x \) is at best an \( A_1 \) singularity and in general an \( A_1 \) singularity. Moreover \( \overline{X} \overset{\overline{q}}{\rightarrow} X \) is the minimal desingularization of \( x \) if \( x \) is of type \( A_1 \) and a partial desingularization of \( x \) (that consists of two consecutive blowing ups) otherwise.

2) If \( G = Z_2 \times Z_2 \) and \( \overline{q} \) is noncrepant (i.e., \( \varphi \) is as Theorem 4.1), then there are only two points \( x_1 \) and \( x_2 \) lying over \( w \), they are smooth and \( \overline{X} \overset{\overline{q}}{\rightarrow} X \) is the blowing up of \( X \) at \( x_1 \) and \( x_2 \).

3) If \( G = Z_4 \) and \( \overline{q} \) is crepant (i.e., \( \varphi \) is as Theorem 3.5), then

3.1) there is only one point \( x \) lying over \( w \). Moreover, \( x \) is a \( D_4 \) singularity and \( \overline{X} \overset{\overline{q}}{\rightarrow} X \) is the blowing up of \( X \) at \( x \);

3.2) \( X - \{x\} \) is singular and the mildest possible set of singularities of \( X - \{x\} \) consists of 9 \( A_1 \) singularities.

4) If \( G = Z_4 \) and \( \overline{q} \) is noncrepant (i.e., \( \varphi \) is as Theorem 4.2), then

4.1) there is only one point \( x \) lying over \( w \) and \( x \) is smooth.
4.2) The morphism $\overline{X} \xrightarrow{\pi} X$ is the contraction of a smooth line $F$, which is $p^{-1}C_0$. The line $F$ consists of smooth points of $\overline{X}$ and an $A_1$ singularity and its self-intersection is $F^2 = -\frac{1}{2}$. Moreover $K_{\overline{X}} = \pi^*K_X + 2F$.

4.3) $X - \{x\}$ is singular and the mildest possible set of singularities of $X - \{x\}$ consists of $9$ $A_1$ singularities.

Proof. First we prove 1). We use the description of the branch divisors of $p$ given in Theorem 3.4, 5). Since $D_1 \sim 4C_0 + 6f$, $C_0$ is a component of $D_1$, so we can write $D_1 = D_1' + C_0$. Since $\overline{X}$ is normal, $C_0$, $D_1'$ and $D_2$ have no common components. Since $C_0 \cdot (3C_0 + 6f) = 0$, $D_1$ does not meet $C_0$ so $\overline{X}_2$ is smooth at the points (which are in the ramification locus of $p_2$) lying over $C_0$. On the other hand $D_2 \cdot C_0 = 2$. Then $D_2$ can meet $C_0$ transversally or not. In the second case $D_2$ can be smooth at the intersection point with $C_0$ or can have an $A_k$ singularity. All this means that the inverse image of $C_0$ by $p$ is a $-2$ cycle $Z$ consisting of one smooth $-2$-line or two lines meeting at one point $\overline{x}$, which is either a smooth point of $\overline{X}$ or an $A_k$ singularity, and the points of $Z$ are smooth points of $\overline{X}$ except maybe $\overline{x}$. In any case contracting $Z$, as $\overline{q}$ does, gives rise to a unique point $x$ lying over $w$, which is a singularity of type $A_{k+2}$ if $\overline{x}$ is of type $A_k$, is of type $A_2$ if $\overline{x}$ is smooth but $D_2$ and $C_0$ do not meet transversally and is of type $A_1$ if $D_2$ and $C_0$ meet transversally. This can be easily seen resolving $\overline{x}$ if necessary and looking at the total transform of $Z$.

Part 2) was already proven in Theorem 2.5 so we now prove 3.1). As argued before, since $\overline{X}$ is normal, $D_2 = D_2' + C_0$ and $D_2$ does not meet $C_0$. On the other hand $D_1 \sim 3f$, and since $\overline{X}$ is normal $D_1$ consists of $3$ distinct fibers of $F_2$. Thus, if we call $E = p_1^{-1}C_0$, all points of $E$ are smooth points of $\overline{X}_1$ and, near $E$ $p_2$ is branched along $E + p_1^*D_1$. Such curve is smooth at all points of $E$ except at $3$ distinct points which are $A_1$ singularities. Then if we call $F = p^{-1}C_0$, $F$ is a smooth line with $F^2 = -1/2$ lying over $C_0$ and all the points of $F$ are smooth points of $\overline{X}$ except three distinct points $\overline{x}_1$, $\overline{x}_2$ and $\overline{x}_3$ which are $A_1$ singularities. Now $\overline{X} \xrightarrow{\pi} X$ contracts only $F$ and therefore gives rise to a single point $x$ lying over $w$, and $x$ is a singularity of type $D_4$. The last claim is immediate once we resolve $\overline{X}$ at $\overline{x}_1$, $\overline{x}_2$ and $\overline{x}_3$, since the total transform of $F$ is the $-2$-cycle which appears in the minimal desingularization of a $D_4$ singularity. In fact, $\overline{q}$ is a partial desingularization of $x$ consisting in blowing up $X$ at $x$ once.

Now we prove 4.1). The argument is similar to 3.1). The description of the branch divisors of $p$ given in Theorem 4.2, 5.1) and 5.2) and the fact that $\overline{X}$ is normal implies that $\Delta_2 = D_3 + C_0$ and $D_1 + D_3 + C_0$ does not have multiple components. Since $\Delta_2 \cdot C_0 = 0$, $\overline{X}_1$ is smooth along $\overline{C}_0 = p_1^{-1}C_0$. Since $D_1 \cdot C_0 = 1$, in both 5.1) and 5.2), then $D_1$ and $C_0$ meet transversally at a point. Since, near $\overline{C}_0$, $p_2$ is branched at $\overline{C}_0 + p_1^*D_1$, there is only one singular point $\overline{x}$ of $\overline{X}$ lying on $F = p^{-1}C_0$, and $\overline{x}$ is an $A_1$ singularity. On the other hand, $F$ is a smooth line with $F^2 = -\frac{1}{2}$ as in 3.1) Again we resolve $\overline{x}$ and the total trasform $T$ is a cycle with self-intersection $-1$ consisting of two smooth lines meeting transversally at one point and with self-intersections $-1$ and $-2$. Then the contraction of $T$ is a smooth point, and so $\overline{X} \xrightarrow{\pi} X$ contracts $F$ to a unique point $x$ lying over $w$, and $x$
is smooth. Finally $K_{\overline{X}} = \overline{\Omega}^* K_X + 2F$ follows from (4.2.5), since $p^* \mathcal{O}_Y (C_0 + 2f) = \overline{\Omega}^* K_X$ and $p_2^* \overline{\Omega} = 2F$.

Finally, we prove 3.2) and 4.2). The mildest singularities in $\overline{X} - F$ occur when $D_2 - C_0$ and $D_1$ in Theorem 3.5, 5) and $\Delta_2 - C_0$ and $D_1$ in Theorem 4.2, 5.1) and 5.2) meet transversally. Since $(D_2 - C_0) \cdot D_1 = (\Delta_2 - C_0) \cdot D_1 = 9$, if the intersection is transversal $\overline{X} - F$ has 9 singular points which are of type $A_1$ and so has $X - \{x\}$.

We end this section showing the existence of surfaces $X$ like those classified in Theorems 3.4, 3.5, 4.1 and 4.2. We use the notation of Corollary 5.1.

**Proposition 5.2.** There exist families of quadruple canonical covers $X \xrightarrow{\varphi} W$ as in Theorem 3.4 with $X - \{x\}$ smooth and $x$ an $A_1$ singularity.

*Proof:* These families were constructed in [GP1], Example 3.7. □

**Proposition 5.3.** There exist families of quadruple canonical covers as in Theorem 3.5 and Theorem 4.2 with $X - \{x\}$ smooth except for 9 $A_1$ singularities

*Proof.* According to the converse part of Theorem 3.5 and Theorem 4.2 we just have to construct two double covers $\overline{X} \xrightarrow{p_2} \overline{X}_1$ and $\overline{X}_1 \xrightarrow{p_1} Y$ ($Y = F_2$) branched along suitable divisors. For surfaces as in Theorem 3.5 we choose $D_1 \sim 3f$ consisting of three distinct fibers and we choose $D_2 = D'_2 + C_0$ with $D'_2 \sim 3C_0 + 6f$ and so that $D_1 + D'_2 + C_0$ has no multiple components. Then the cover $\overline{X} \xrightarrow{p} Y$ constructed according to Theorem 3.4 is normal. We choose furthermore $D_1$ and $D'_2$ so that $D'_2$ is smooth and $D_1$ and $D'_2$ meet transversally. All this can be achieved because $D'_2 \sim 3C_0 + 6f$ is base-point-free. Then as argued in Corollary 5.1 $\overline{X} - F$ has only 9 singularities, which are of type $A_1$, and has therefore canonical singularities, and so has $X$ according to Theorem 3.5. Note that these are the examples we would obtain in [GP2], Proposition 3.11 if we allowed $m = e = 2$. Allowing $D_1 + D_2$ to have worse singularities one can construct $X$ with worse singularities.

To construct surfaces as in Theorem 4.2 again we construct $p_1$ and $p_2$ following the guidelines in the converse part of Theorem 4.2, choosing $D_1$ and $\Delta_2$ so that $D_1 + D_3 + C_0$ has no multiple components and so that $D_1$ and $D_3$ are smooth and meet transversally. This can be achieved in both cases 5.1) and 5.2) of Theorem 4.2, because $C_0 + 3f$, $3C_0 + 6f$, $4C_0 + 9f$ and $C_0 + 2f$ are base-point-free. This assures us that $\overline{X} - F$ has only 9 singular points, which are of type $A_1$, so in particular $\overline{X}$ has canonical singularities, and, according to Theorem 4.2, so does $X$. Allowing $D_1 + \Delta_2$ to have worse singularities, one can construct $X$ with worse singularities. □

**Proposition 5.4.** There are families of smooth quadruple canonical covers $X \xrightarrow{\varphi} W$ as in Theorem 4.1.

*Proof.* Let $Y = F_2$. By the converse part in Theorem 4.1 we construct $\overline{X}$ as the normalization of the fiber product of two double covers $\overline{X}_1 \xrightarrow{p_1} Y$ and $\overline{X}_2 \xrightarrow{p_2} Y$, branched respectively along divisors $D'_1$ and $D'_2$, where $D'_i = D_i + C_0$ and $D_1 \sim D_2 \sim 3C_0 + 6f$. We also take $D_1 + D_2 + C_0$ without multiple components. This can be done because $3C_0 + 6f$ is base-point-free. We also take $D_1$ and $D_2$ smooth and intersecting transversally, which
again can be achieved because $3C_0 + 6f$ is base-point-free. From the description made in the proof of Theorem 4.1 of the normalization of the fiber product as a composition of two double covers, the second cover is branched along the pullback of $D_2$ so we see that $\overline{X}$ is smooth. We also know by Theorem 4.1 that the canonical morphism of $\overline{X}$ only contracts the inverse image of $C_0$ to two smooth points in $X$, so $X$ is a smooth surface. If we allow worse singularities in $D_1 + D_2$, then examples of singular $X$ can be constructed. □

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