On $p$-ary Bent Functions and Strongly Regular Graphs

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Abstract

Our main result is a generalized Dillon-type theorem, giving graph-theoretic conditions which guarantee that a $p$-ary function in an even number of variables is bent, for $p$ a prime number greater than 2. The key condition is that the component Cayley graphs associated to the values of the function are strongly regular, and either all of Latin square type, or all of negative Latin square type. Such a Latin or negative Latin square type bent function is regular or weakly regular, respectively. Its dual function has component Cayley graphs with the same parameters as those of the original function. We also give a criterion for bent functions involving structure constants of association schemes. We prove that if a $p$-ary function with component Cayley graphs of feasible degrees determines an amorphic association scheme, then it is bent. Since amorphic association schemes correspond to strongly regular graph decompositions of Latin or negative Latin square type, this result is equivalent to our main theorem. We show how to construct bent functions from orthogonal arrays and give some examples.

Keywords: bent function, strongly regular graph, Latin square type graph, amorphic association scheme
1 Introduction

Bent functions over a finite field can be thought of as maximally non-linear functions. They can be defined using Walsh transforms, but can also be studied using the combinatorics of their level sets, the parameters of certain associated Cayley graphs, and the algebras generated by the adjacency matrices of these graphs. Dillon [D74] characterized bent Boolean functions as those whose supports form combinatorial structures known as difference sets of elementary Hadamard type. An alternative and closely related characterization is that a Boolean function is bent if and only if its Cayley graph is strongly regular with parameters $(\nu, k, \lambda, \mu)$ satisfying $\lambda = \mu$ (Bernasconi, Codenotti, and VanderKam [BCV01]). These theorems do not generalize in an obvious way for primes $p$ greater than 2. The Cayley graphs associated with a bent $p$-ary function are not necessarily strongly regular (see, for example, §10.3).

We consider $p$-ary functions over the finite field $GF(p)$ with $p$ elements, where $p$ is a prime number greater than 2. A function $f : GF(p)^{2m} \rightarrow GF(p)$ determines a collection of component Cayley graphs corresponding to the values of $f$. When $f$ is even, these graphs are undirected. We will usually also assume that $f$ vanishes at 0 (a weak assumption, since adding a constant to a bent function results in another bent function). The component Cayley graphs are regular, with degrees determined by the sizes of the level sets of $f$. Our main result is a generalization of the theorems of Dillon and Bernasconi, Codenotti, and VanderKam in one direction. We prove that if the component Cayley graphs of $f$ are all strongly regular and are either all of Latin square type with feasible degrees, or all of negative Latin square type with feasible degrees, then $f$ is bent. The feasibility conditions are simply conditions arising from the possible sizes of the level sets of a bent function.

The proof of our main theorem uses an expression for the Walsh transform of $f$ in terms of the eigenvalues of the component Cayley graphs. When the component Cayley graphs are strongly regular and of feasible Latin or negative Latin square type, we obtain formulas for the eigenvalues and their multiplicities, which we use to calculate the values of the Walsh transform and to show that $f$ is bent.

As a consequence of the proof outlined above, we also find that functions of feasible Latin square type are regular, and functions of feasible negative Latin square type are $(-1)$-weakly regular. In each case, the component Cayley graphs of the dual function are strongly regular, with the same pa-
rameters as those of the original function. The proof of this duality theorem uses a relationship between the component functions of the dual function and the Fourier transforms of the component functions of the original function.

The papers of Gol’fand, Ivanov, and Klin [GIK94], van Dam [vD03], and van Dam and Muzychuk [vDM10] describe the close relationship between graphs of Latin and negative Latin square type and amorphic association schemes. We say that a $p$-ary function $f$ is amorphic if its level sets determine an amorphic association scheme. In a previous paper, [CJMPW16], we showed that for any prime $p$ greater than 2, there are $(p+1)!/2$ bent amorphic functions of two variables with algebraic normal form homogeneous of degree $p − 1$ and such that the level sets corresponding to nonzero elements of $GF(p)$ all have size $p − 1$. In this paper, we generalize part of our previous result by proving that if an even $p$-ary function of $2m$ variables with level sets of feasible sizes is amorphic, then it is bent. The key to the proof is a criterion for a function to be bent involving sums of structure constants of an association scheme. We also use a result of Ito, Munemasa, and Yamada [IMY91] describing the structure constants of an amorphic association scheme.

In light of the relationship between Latin and negative Latin square type graphs and amorphic association schemes described in [GIK94], [vD03], and [vDM10], we see that a $p$-ary function is amorphic if and only if its component Cayley graphs are strongly regular and either all of Latin square type, or all of negative Latin square type. Thus, our criterion for bent functions involving eigenvalues and our criterion involving structure constants lead to equivalent theorems, proven by different methods.

The existence of amorphic bent $p$-ary functions of $2m$ variables follows from the existence of $p$-class amorphic association schemes with corresponding graphs of appropriate degrees. Such amorphic association schemes can be constructed from orthogonal arrays of size $(N + 1) \times N^2$, where $N = p^m$. Orthogonal arrays of these dimensions exist by a construction of Bush [B52]. We describe a construction of amorphic bent functions of Latin square type from orthogonal arrays and give examples for $n = 2$ and 4, and $p = 3, 5, \text{and } 7$. We also give examples of amorphic bent functions of negative Latin square type, and of bent functions whose component Cayley graphs are not all strongly regular.

It would be interesting to find a combinatorial generalization of the theorems of Dillon and Bernasconi, Codenotti, and VanderKam in the other direction, giving simple graph-theoretic properties that the component Cay-
ley graphs of a bent $p$-ary function must possess. In the case $p = 3$, Tan, Pott, and Feng [TPF10] show that if $f : GF(3^{2m}) \to GF(3)$ is an even weakly regular bent function with $f(0) = 0$, then the component Cayley graphs of $f$ are strongly regular and either all of Latin square or all of negative Latin square type. Using the theory of quadratic residues, Chee, Tan, and Zhang [CTZ12] and Feng, Wen, Xiang, and Yin [FWXY13] generalize this result for $p$ a prime greater than 2. However, their decompositions for $p > 3$ are not related to the component graphs studied in this paper.

Our paper is structured as follows. In Section 2, we study the level sets of a $p$-ary function. We use a result of Kumar, Scholtz, and Welsh to calculate the possible sizes of the level sets of an even bent $p$-ary function of $2m$ variables which vanishes at 0. The component Cayley graphs and the corresponding component functions of a $p$-ary function are defined in Section 3. We explain why the eigenvalues of these graphs are the values of the Fourier transforms of the component functions. In Section 4, we state a criterion for a $p$-ary function to be bent, involving eigenvalues of the component Cayley graphs. We give formulas for the eigenvalues of the component Cayley graphs of a function of feasible Latin or negative Latin square type in Section 5. We revisit the theorems of Dillon and Bernasconi, Codenotti, and VanderKam in this context. In Section 6, we prove that $p$-ary functions of feasible Latin or negative Latin square type are bent. We describe their dual functions in Section 7. In Section 8, we discuss $p$-ary functions that determine association schemes. We state a structure constant criterion for such a function to be bent. Using this criterion, we prove that amorphic functions with component Cayley graphs of feasible degrees are bent. In Section 9, we describe how to construct amorphic bent functions of Latin square type from orthogonal arrays. Section 10 is devoted to examples, most of which were constructed with the aid of computers. We conclude with some questions and ideas for further study.

2 Sizes of level sets of bent functions

In this section, we study even $p$-ary functions of an even number of variables, vanishing at 0. We obtain necessary, but not sufficient, conditions for such a function to be bent by considering the sizes of its level curves. We refer to these conditions as feasibility conditions. The feasibility conditions are derived from the possible sizes of the level sets in the bent case, which we
calculate using a result of Kumar, Scholtz, and Welsh [KSW85]. We also state the feasibility conditions, equivalently, in terms of the degrees of a function’s component Cayley graphs, in §3.3.

We fix, once and for all, an ordering for $GF(p^n)$. That ordering will be used for all vectors whose coordinates are indexed by $GF(p)^n$, all matrices whose entries are indexed by $GF(p)^n \times GF(p)^n$, and all vertices of associated component Cayley graphs defined below.

We routinely identify the elements of $GF(p)$ with \{0, 1, 2, \ldots, p - 1\}.

2.1 The Walsh transform

Let $p$ be a prime number, and let $\zeta$ be the $p$th root of unity given by

$$
\zeta = e^{2\pi i / p}.
$$

Let $n$ be a positive integer. A $p$-ary function $f : GF(p)^n \rightarrow GF(p)$ determines a well-defined complex-valued function $\zeta^f : GF(p)^n \rightarrow \mathbb{C}$. The Walsh or Walsh-Hadamard transform of $f$ is defined to be the function $W_f : GF(p)^n \rightarrow \mathbb{C}$ given by

$$
W_f(x) = \sum_{y \in GF(p)^n} \zeta^{f(y) - \langle x, y \rangle},
$$

where $\langle , \rangle$ is the usual inner product on $GF(p)^n$.

2.2 The Fourier transform

If $g : GF(p)^n \rightarrow \mathbb{C}$ is a complex-valued function on $GF(p)^n$, the Fourier transform of $g$ is the function $\hat{g} : GF(p)^n \rightarrow \mathbb{C}$ given by

$$
\hat{g}(x) = \sum_{y \in GF(p)^n} g(y)\zeta^{-\langle x, y \rangle}.
$$

Thus, the Walsh transform of $f : GF(p)^n \rightarrow GF(p)$ is the Fourier transform of $\zeta^f : GF(p)^n \rightarrow \mathbb{C}$.

2.3 Bent functions

Let $f : GF(p)^n \rightarrow GF(p)$ be a $p$-ary function. We say that $f$ is bent if

$$
|W_f(x)| = p^{\frac{n}{2}},
$$

5
for all $x$ in $GF(p)^n$.

We note that if $p$-ary functions $f_1$ and $f_2$ differ by a constant element of $GF(p)$, then $f_1$ is bent if and only if $f_2$ is bent, so from now on we will assume that $f(0) = 0$.

We will also assume that $f$ is even, i.e., $f(x) = f(-x)$, for all $x$ in $GF(p)^n$. When $f$ is even, the component Cayley graphs of $f$ are undirected (see §3.2).

2.4 Level sets of a $p$-ary function

The level sets of $f$, for $1 \leq i \leq p - 1$, are the sets

$$D_i = \{ x \in GF(p)^n \mid f(x) = i \}.$$  

For consistency with our later discussions of component Cayley graphs and association schemes, we define

$$D_p = \{ x \in GF(p)^n \mid x \neq 0 \text{ and } f(x) = 0 \} \quad \text{and} \quad D_0 = \{ 0 \}.$$  

Thus, the level set $f^{-1}(0)$ is the union of $D_p$ and $\{ 0 \}$. In our discussion of the sizes of the level sets of $f$, it is convenient to express our results in terms of the sizes of the sets $D_i$, because these sizes are the degrees of the component Cayley graphs, which we define in §3.2 below.

2.5 Level sets of a bent Boolean function

In the Boolean ($p = 2$) case, the Walsh transform takes integer values. If $f: GF(2)^n \to GF(2)$ is bent, $n$ must be even, since $W_f(0) = \pm 2^n$. Setting $m = \frac{n}{2}$, we find that the only possible sizes for $D_1$ are

$$|D_1| = 2^{2m-1} \pm 2^{m-1}.$$  

2.6 Kumar–Scholtz–Welsh theorem

The description of the Walsh transform in the next theorem is a key step in our calculation of the sizes of level sets of even bent $p$-ary functions. This theorem follows directly from a result of Kumar, Scholtz, and Welsh [KSW85, Property 7], but we include a proof for the sake of completeness.
Theorem 2.1 (Kumar-Scholtz-Welsh). Suppose that \( f : \text{GF}(p)^{2m} \to \text{GF}(p) \) is an even bent function, where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Then, for every \( x \) in \( \text{GF}(p)^{2m} \),

\[
W_f(x) = \pm \zeta^j p^m
\]

for some integer \( j \) with \( 0 \leq j \leq p - 1 \).

Proof. Fix \( x \) in \( \text{GF}(p)^{2m} \), and let \( W = W_f(x) \). It is sufficient to show that \( p^{-m}W \) is a root of unity in \( \mathbb{Q}(\zeta) \), since the only roots of unity in \( \mathbb{Q}(\zeta) \) are those of the form \( \pm \zeta^j \) (see, e.g., [BS66, p. 157] or [C07, Corollary 3.5.12]).

We will first show that \( p^{-m}W \) is an element of \( \mathbb{Z}[\zeta] \), and then use a theorem of Kronecker [K1857], which states that an element of \( \mathbb{Z}[\zeta] \), all of whose conjugates have magnitude 1, is a root of unity in \( \mathbb{Q}(\zeta) \) (for a more accessible source, see [C07, Corollary 3.3.10]).

For \( \alpha \) in \( \mathbb{Z}[\zeta] \), let \( \langle \alpha \rangle \) denote the principal ideal generated by \( \alpha \) in \( \mathbb{Z}[\zeta] \). Note that the ideal \( \langle p \rangle \) has a factorization as \( \langle p \rangle = \langle 1 - \zeta \rangle^{p-1} \), and the ideal \( \langle 1 - \zeta \rangle \) in \( \mathbb{Z}[\zeta] \) is prime (see, e.g., [BS66, p. 157] or [W82, Lemma 1.4]). Let \( \overline{W} \) be the complex conjugate of \( W \). Since \( f \) is bent, \( W \overline{W} = p^{2m} \). Thus, the ideal in \( \mathbb{Z}[\zeta] \) generated by \( W \overline{W} \) has a factorization into prime ideals as \( \langle 1 - \zeta \rangle^{2m(p-1)} \). Suppose that \( \langle W \rangle = \langle 1 - \zeta \rangle^k \) and \( \langle \overline{W} \rangle = \langle 1 - \zeta \rangle^\ell \) for some integers \( k \) and \( \ell \). Since \( \langle 1 - \overline{\zeta} \rangle = \langle 1 - \zeta \rangle \), we also have \( \langle W \rangle = \langle 1 - \zeta \rangle^k = \langle 1 - \zeta \rangle^\ell \), so \( k = \ell \). Therefore, \( \langle W \rangle = \langle \overline{W} \rangle = \langle p^m \rangle \). It follows that \( W = up^m \) for some unit \( u \) of magnitude 1 in \( \mathbb{Z}[\zeta] \).

The conjugates of \( u \) are the images of \( u \) under the elements of the Galois group of \( \mathbb{Q}(\zeta) \). This Galois group consists of the \( p - 1 \) automorphisms \( \sigma_k \) of \( \mathbb{Q}(\zeta) \), which are determined by the equations \( \sigma_k(\zeta) = \zeta^k \), for \( 1 \leq k \leq p - 1 \). It is straightforward to show that \( \sigma_k(W_f(x)) = W_{kf}(kx) \). It can also be shown that \( kf \) is bent, for \( k \) in \( \{1, 2, \ldots, p - 1\} \), for example, by using the balanced derivative criterion of [8.4]. Thus, all the conjugates of \( u \) under the actions of the maps \( \sigma_k \), i.e., all the images \( \sigma_k(u) \), have magnitude 1. It follows from the theorem of Kronecker [K1857] mentioned above, that \( u \) is a root of unity in \( \mathbb{Q}(\zeta) \). Therefore \( u = \pm \zeta^j \) for some \( j \) with \( 0 \leq j \leq p - 1 \). \( \square \)

2.7 Feasible sizes of level sets of \( p \)-ary functions

In this section, we calculate the possible sizes of level sets of even bent \( p \)-ary functions of \( 2m \) variables, vanishing at 0. At the end of this section, we state feasibility conditions for a function to be bent, based on these sizes. As a
first step toward this goal, we prove the following corollary of the theorem of Kumar, Scholtz, and Welsh.

**Corollary 2.2.** Suppose that \( f : GF(p)^{2m} \to GF(p) \) is an even bent function such that \( f(0) = 0 \), where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Then

\[
W_f(0) = \pm p^m.
\]

Furthermore, the level sets \( D_i \), for \( 1 \leq i \leq p - 1 \), are all the same size, i.e.,

\[
|D_1| = |D_2| = \cdots = |D_{p-1}|.
\]

**Proof.** Let \( k_i = |D_i| = |f^{-1}(i)| \) for \( 1 \leq i \leq p - 1 \), and let \( k_p = |D_p| = |f^{-1}(0)| - 1 \). Notice that since \( f \) is even and \( f(0) = 0 \), \( k_i \) must be an even integer, for \( 1 \leq i \leq p \).

From the definition of the Walsh transform,

\[
W_f(0) = 1 + k_1 \zeta + k_2 \zeta^2 + \cdots + k_{p-1} \zeta^{p-1} + k_p. \tag{2.1}
\]

By the result of Kumar, Scholtz, and Welsh,

\[
W_f(0) = \pm \zeta^j p^m, \tag{2.2}
\]

for some \( j \) such that \( 0 \leq j \leq p - 1 \). Since \( 1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1} = 0 \), Equation (2.1) can be rewritten as

\[
W_f(0) = \sum_{i=1}^{p-1} (k_i - 1 - k_p) \zeta^i.
\]

The roots of unity \( \zeta, \zeta^2, \ldots, \zeta^{p-1} \) are linearly independent over \( \mathbb{Q} \). It follows from Equation (2.2) that if \( 1 \leq j \leq p - 1 \), then \( k_i - 1 - k_p = 0 \), for \( i \neq j \). But this is impossible, since \( k_i \) and \( k_p \) are both even. Therefore \( j = 0 \), and \( W_f(0) = \pm p^m \).

Furthermore, we must have \( k_i - 1 - k_p = -W_f(0) \) for \( 1 \leq i \leq p - 1 \). Therefore \( k_1 = k_2 = \cdots = k_{p-1} \), i.e.,

\[
|D_1| = |D_2| = \cdots = |D_{p-1}|.
\]

\( \square \)

We now calculate the possible sizes of level sets of even bent \( p \)-ary functions of \( 2m \) variables, vanishing at 0.
Proposition 2.3. Suppose that $f : GF(p)^{2m} \rightarrow GF(p)$ is an even bent function such that $f(0) = 0$, where $p$ is a prime number greater than 2, and $m$ is a positive integer. Then the possible sizes of the sets $D_i$ are

$$|D_i| = (N - 1) \frac{N}{p},$$

for $1 \leq i \leq p - 1$, and

$$|D_p| = (N - 1) \left( \frac{N}{p} + 1 \right),$$

where $N = W_f(0) = \pm p^m$.

Proof. By Corollary 2.2, the Walsh transform of $f$ at 0 is $W_f(0) = \pm p^m$. Also by Corollary 2.2, the sizes of the level sets $D_1, D_2, \ldots, D_{p-1}$ are all equal. Let $k = |D_i|$, for $1 \leq i \leq p - 1$, and let $k_p = |D_p|$. Let $N = W_f(0)$. Since

$$\{0\} \cup D_1 \cup D_2 \cup \cdots \cup D_p = GF(p)^{2m},$$

the constants $k$ and $k_p$ are related by the equation.

$$1 + (p - 1)k + k_p = p^{2m} = N^2. \quad (2.3)$$

By the definition of the Walsh transform at 0,

$$1 + k \left( \zeta + \zeta^2 + \cdots + \zeta^{p-1} \right) + k_p = N. \quad (2.4)$$

Since $\zeta + \zeta^2 + \cdots + \zeta^{p-1} = -1$, Equation (2.4) can be rewritten as

$$k_p = k + N - 1.$$

Substituting into Equation (2.3) we find that

$$pk + N = N^2.$$

Hence,

$$k = (N - 1) \frac{N}{p}$$

and

$$k_p = (N - 1) \left( \frac{N}{p} + 1 \right).$$
Remark 2.4. A straightforward calculation shows that if \( f \) satisfies the hypotheses of Proposition 2.3 then \( p \) divides the norm-squared of the “signature” \(|f^{-1}(0)|, |f^{-1}(1)|, \ldots, |f^{-1}(p-1)|\) of the function \( f \), i.e., \( p \) divides the quantity
\[
|\{0\} \cup D_p|^2 + \sum_{i=1}^{p-1} |D_i|^2.
\]

We have described the possible sizes of the level sets of an even bent function of \( 2m \) variables in the Boolean case, in §2.5 and in the \( p \)-ary case, for \( p \) a prime number greater than 2, in Proposition 2.3. These results lead to the following feasibility conditions for a function to be bent.

Let \( f : GF(p)^{2m} \to GF(p) \) be an even function with \( f(0) = 0 \), where \( p \) is a prime number, and \( m \) is a positive integer. Let \( D_i = f^{-1}(i) \) for \( 1 \leq i \leq p-1 \), and let \( D_p = f^{-1}(0) \setminus \{0\} \). We say that the level sets of \( f \) are of feasible sizes if, for \( 1 \leq i \leq p \),
\[
|D_i| = (N - 1)r_i,
\]
(2.5)
where
\[
N = \pm p^m, \quad r_i = \frac{N}{p} \quad \text{for} \quad 1 \leq i \leq p - 1, \quad \text{and} \quad r_p = \frac{N}{p} + 1.
\]

A function whose level sets are not of feasible sizes cannot be bent. In the next section, we will state these feasibility conditions, equivalently, in terms of the degrees of a function’s component Cayley graphs (see §3.3).

3 Cayley graphs of \( p \)-ary functions

In this section, we describe a collection of regular graphs \( \{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\} \), the component Cayley graphs, associated to a \( p \)-ary function \( f \). We define the component functions \( f_i \) of \( f \) to be the indicator functions of the sets \( D_i \) described above. The eigenvalues of the adjacency matrix of \( \Gamma_i \) are the values of the Fourier transform of \( f_i \). The adjacency matrices of the component Cayley graphs commute.

3.1 Component functions of a \( p \)-ary function

Suppose that \( f : GF(p)^n \to GF(p) \) is a \( p \)-ary function. Recall that we define \( D_i = f^{-1}(i) \), for \( 1 \leq i \leq p - 1 \), and \( D_p = f^{-1}(0) \setminus \{0\} \). The component
functions $f_i : GF(p)^n \to C$ of $f$ are defined to be the indicator functions of the sets $D_i$, given by

$$f_i(x) = \begin{cases} 1 & \text{if } x \in D_i, \\ 0 & \text{otherwise}, \end{cases}$$

for $1 \leq i \leq p$.

### 3.2 Component Cayley graphs of a $p$-ary function

Let $f : GF(p)^n \to GF(p)$ be an even function with $f(0) = 0$, where $p$ is a prime number greater than 2, and $n$ is a positive integer. The function $f$ determines a graph decomposition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$ of the complete graph on the vertex set $GF(p)^n$. For $1 \leq i \leq p - 1$, there is an edge in $\Gamma_i$ between distinct vertices $x$ and $y$ in $GF(p)^n$ if $f(x - y) = i$, i.e., if $x - y \in D_i$. There is an edge in $\Gamma_p$ between distinct vertices $x$ and $y$ in $GF(p)^n$ if $f(x - y) = 0$, i.e., if $x - y \in D_p$. Note that these graphs may be considered undirected, since $f$ is even, so $f(x - y) = f(y - x)$. The graph $\Gamma_i$ is the Cayley graph of the pair $(GF(p)^n, D_i)$. We refer to the graphs $\Gamma_i$ as the component Cayley graphs or simply the Cayley graphs of $f$. We can also regard $\Gamma_i$ as the Cayley graph of the component function $f_i$. The graph $\Gamma_i$ is regular of degree $|D_i|$, i.e., every vertex is of degree $|D_i|$.

For example, let $f : GF(3)^2 \to GF(3)$ be given by $f(x_0, x_1) = -x_0^2 + x_1^2$. The component Cayley graphs $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ of $f$ are shown in Figure 1.

![Figure 1: The Cayley graphs of the 3-ary function $-x_0^2 + x_1^2$.](image)
3.3 Feasible degrees of component Cayley graphs

Suppose that \( f : GF(p)^{2m} \rightarrow GF(p) \) is an even function such that \( f(0) = 0 \), where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. In Equation (2.5) of Section 2.7 we described the feasible sizes of the level sets of \( f \). If the level sets of \( f \) are not of these sizes, \( f \) cannot be bent. We now restate these feasibility conditions in terms of degrees of graphs. We say that the component Cayley graphs \( \Gamma_i \) of \( f \) are of feasible degrees if the degrees of these graphs correspond to the feasible sizes of level sets, i.e.,

\[
\text{degree}(\Gamma_i) = (N - 1)r_i,
\]

where

\[
N = \pm p^m, \quad r_i = \frac{N}{p} \quad \text{for} \quad 1 \leq i \leq p - 1, \quad \text{and} \quad r_p = \frac{N}{p} + 1.
\]

If the degrees of the graphs \( \Gamma_i \) are not of these sizes, the function \( f \) cannot be bent.

3.4 Adjacency matrices

Let \( \Gamma \) be a matrix with vertex set \( V \) of size \( \nu \). The adjacency matrix of \( \Gamma \), with respect to a fixed ordering of the vertices, is the \( \nu \times \nu \) matrix \( A \) whose rows and columns are indexed by the elements of \( V \), such that

\[
A_{xy} = \begin{cases} 
1 & \text{if } x \neq y \text{ and } (x, y) \text{ is an edge of } \Gamma, \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( f : GF(p)^n \rightarrow GF(p) \) be an even function such that \( f(0) = 0 \). Let \( A(i) \) be the adjacency matrix of the component Cayley graph \( \Gamma_i \) with respect to the ordering of \( GF(p)^n \) fixed in [2]. We will show below that the matrices \( A(1), A(2), \ldots, A(p) \) commute, since they share a common basis of eigenvectors.

3.5 Hadamard vectors

Let \( \nu = p^n \). For each vector \( x \) in \( GF(p)^n \), we define a vector \( h(x) \) in \( \mathbb{C}^\nu \), using the same fixed ordering of \( GF(p)^n \) as in [2] by

\[
h(x)_y = \zeta^{-\langle x, y \rangle}.
\]
We call the vectors $h(x)$ generalized Hadamard or simply Hadamard vectors. The vector $h(0)$ is the all 1’s vector. By the following lemma, the remaining vectors $h(x)$, where $x \neq 0$, span the subspace of $\mathbb{C}^\nu$ orthogonal to $h(0)$.

**Lemma 3.1.** The $\nu$ Hadamard vectors $h(x)$ in $\mathbb{C}^\nu$ are orthogonal and linearly independent over $\mathbb{C}$.

**Proof.** Let $H$ be the matrix whose columns are the Hadamard vectors $h(x)$, for $x$ in $GF(p)^n$. It is straightforward to show that $H^\dagger H = \nu I$, where $\nu = p^n$, and $I$ is the $\nu \times \nu$ identity matrix. □

The matrix $H$ whose columns are the vectors $h(x)$ is sometimes called a generalized Hadamard or Butson matrix.

### 3.6 Eigenvalues corresponding to Hadamard vectors

Suppose that $\Gamma$ is a graph with vertex set $V$ of size $\nu$, and $A$ is the adjacency matrix of $\Gamma$. The set of eigenvalues of $A$ is called the *spectrum* of the graph $\Gamma$. We sometimes refer to the eigenvalues of $A$ as eigenvalues of $\Gamma$.

Let $f: GF(p)^n \rightarrow GF(p)$ be a even function with $f(0) = 0$, with component Cayley graphs $\Gamma_i$ and corresponding adjacency matrices $A(i)$. We will show that the Hadamard vectors $h(x)$ of Lemma 3.1 form a basis of common eigenvectors of the matrices $A(i)$ over $\mathbb{C}$, and the values of the Fourier transforms $\hat{f}_i$ of the component functions $f_i$ are the eigenvalues of these matrices.

**Lemma 3.2.** The Hadamard vector $h(x)$ is an eigenvector of $A(i)$ corresponding to the eigenvalue $\hat{f}_i(x)$, for each $x$ in $GF(p)^n$, and for $1 \leq i \leq p$.

**Proof.** The entry in position $y$ in the product $A(i)h(x)$ is

$$(A(i)h(x))_y = \sum_t A(i)_{yt}h(x)_t$$

$$= \sum_t f_i(t - y)\zeta^{-(x,t)}$$

$$= \sum_z f_i(z)\zeta^{-(x,z+y)}$$

$$= \left(\sum_z f_i(z)\zeta^{-(x,z)}\right)\zeta^{-(x,y)}$$

$$= \hat{f}_i(x)h(x)_y.$$
Thus, 
\[ A(i)h(x) = \hat{f}_i(x)h(x), \]
i.e., the vector \( h(x) \) is an eigenvector of \( A(i) \) corresponding to the eigenvalue \( \hat{f}_i(x) \).

As an immediate corollary of the previous lemma, we see that the adjacency matrices \( A(i) \) commute.

**Corollary 3.3.** If \( f : GF(p)^n \to GF(p) \) is an even \( p \)-ary function with \( f(0) = 0 \), the adjacency matrices \( A(1), A(2), \ldots, A(p) \) of the component Cayley graphs of \( f \) commute.

We will often use the notation \( \lambda_i(x) \) to denote the eigenvalue of \( A(i) \) corresponding to the Hadamard eigenvector \( h(x) \). Thus,
\[ \lambda_i(x) = \hat{f}_i(x), \quad (3.2) \]
for \( 1 \leq i \leq p \) and for all \( x \) in \( GF(p)^n \).

## 4 Eigenvalue criterion for bent functions

In this section, we characterize even bent \( p \)-ary functions in terms of eigenvalues of their component Cayley graphs.

**Proposition 4.1.** Let \( f : GF(p)^n \to GF(p) \) be an even function such that \( f(0) = 0 \), where \( p \) is a prime number, and \( n \) is a positive integer. Then the Walsh transform of \( f \) satisfies
\[ W_f(x) = 1 + \sum_{i=1}^{p} \zeta^i \lambda_i(x), \]
for all \( x \) in \( GF(p)^n \), where \( \lambda_i(x) \) is the eigenvalue from Equation (3.2) above.

**Proof.** Recall that we defined \( D_i = f^{-1}(i) \) for \( 1 \leq i \leq p - 1 \), and \( D_p = f^{-1}(0) \setminus \{0\} \). As above, we denote by \( f_i \) the component function of \( f \) defined in Equation (3.1), and by \( \hat{f}_i \) its Fourier transform. The Walsh transform of
$f$ can be written in terms of the Fourier transforms of the functions $f_i$ as

$$W_f(x) = \sum_{y \in GF(p)^n} \zeta^{f(y)} \zeta^{-(y,x)}$$

$$= \zeta^0 + \sum_{i=1}^{p-1} \sum_{y \in D_i} \zeta^i \zeta^{-(y,x)} + \sum_{y \in D_p} \zeta^{-(y,x)}$$

$$= 1 + \sum_{i=1}^{p-1} \sum_{y \in GF(p)^n} f_i(y) \zeta^{-(y,x)} + \sum_{y \in GF(p)^n} f_p(y) \zeta^{-(y,x)}$$

$$= 1 + \zeta \hat{f}_1(x) + \zeta^2 \hat{f}_2(x) + \cdots + \zeta^{p-1} \hat{f}_{p-1}(x) + \hat{f}_p(x).$$

Since $\lambda_i(x) = \hat{f}_i(x)$, (see Equation (3.2) above), this completes the proof. □

From the previous result, we obtain the following characterization of an even bent $p$-ary function in terms of eigenvalues of its component Cayley graphs.

**Proposition 4.2.** Let $f : GF(p)^n \to GF(p)$ be an even function such that $f(0) = 0$, where $p$ is a prime number, and $n$ is a positive integer. Then $f$ is bent if and only if

$$|1 + \sum_{i=1}^{p} \zeta^i \lambda_i(x)| = p^{\frac{n}{2}}$$

for all $x$ in $GF(p)^n$, where $\lambda_i(x)$ is the eigenvalue from Equation (3.2) above.

## 5 Feasible Latin and negative Latin square type functions

In this section, we consider even $p$-ary functions whose component Cayley graphs are strongly regular and either all of Latin square type with feasible degrees, or all of negative Latin square type with feasible degrees. We describe the eigenvalues of the component Cayley graphs of such functions. In order to illustrate how our main result is related to the theorems of Dillon and Bernasconi, Codenotti, and VanderKam, we recast their theorems in this context. We begin with some background material on strongly regular graphs and graphs of Latin and negative Latin square type (for further details see, for example, Godsil and Royle [GR01, Chapter 10]).
5.1 Strongly regular graphs

Let Γ be a $k$-regular graph on $\nu$ vertices (every vertex has degree $k$). The graph Γ is called strongly regular if there exist nonnegative integers $\lambda$ and $\mu$ such that if $x$ and $y$ are neighbors in Γ, there are $\lambda$ common neighbors of $x$ and $y$, and if $x$ and $y$ are not neighbors in Γ, there are $\mu$ common neighbors of $x$ and $y$. The constants $(\nu, k, \lambda, \mu)$ are called the parameters of the graph Γ.

A strongly regular graph on the vertex set $GF(p)^n$ with parameters $(\nu, k, \lambda, \mu)$ corresponds to a symmetric partial difference set with parameters $(\nu, k, \lambda, \mu)$ (see, for example, [JM17, Chapter 6]). Thus, many of our statements about strongly regular graphs could be rephrased in terms of symmetric partial difference sets.

5.2 Eigenvalues of strongly regular graphs

The eigenvalues of the adjacency matrix of a strongly regular graph and their multiplicities can be expressed in terms of the parameters of the graph by the following well-known formulas. A strongly regular graph Γ with parameters $(\nu, k, \lambda, \mu)$ has eigenvalues $k$, $\theta$, and $\tau$, where the eigenvector $k$ corresponds to the all 1’s vector,

$$\theta = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2},$$

and

$$\tau = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}.$$

From these equations we see that

$$\theta + \tau = \lambda - \mu \quad \text{and} \quad \theta \tau = -k + \mu. \quad (5.1)$$

The multiplicities of $\theta$ and $\tau$ on the space of vectors in $\mathbb{C}^\nu$ orthogonal to the all 1’s vector are given by

$$m_\theta = \frac{(\nu - 1)\tau + k}{\tau - \theta} \quad \text{and} \quad m_\tau = \frac{(\nu - 1)\theta + k}{\theta - \tau}.$$
5.3 Latin and negative Latin square type graphs

We say that a strongly regular graph $\Gamma$ is of *Latin square type* if there exist integers $N > 0$ and $r > 0$ such that the parameters of $\Gamma$ are

$$(\nu, k, \lambda, \mu) = (N^2, (N-1)r, N + r^2 - 3r, r^2 - r).$$  (5.2)

A strongly regular graph $\Gamma$ is of *negative Latin square type* if there exist integers $N < 0$ and $r < 0$ such that the parameters of $\Gamma$ are given by Equation (5.2).

If $\Gamma$ is a strongly regular graph of Latin square type then the eigenvalues of $\Gamma$ are

$$k = (N-1)r, \quad \theta = N - r, \quad \text{and} \quad \tau = -r,$$  (5.3)

where $\theta$ has multiplicity $m_\theta = (N-1)r$ on the subspace of $\mathbb{C}^\nu$ orthogonal to the all 1’s vector, and $\tau$ has multiplicity $m_\tau = (N-1)(N-r+1)$.

If $\Gamma$ is a strongly regular graph of negative Latin square type, then the eigenvalues of $\Gamma$ are

$$k = (N-1)r, \quad \theta = -r, \quad \text{and} \quad \tau = N - r,$$  (5.4)

where $\theta$ has multiplicity $m_\theta = (N-1)(N-r+1)$ on the subspace of $\mathbb{C}^\nu$ orthogonal to the all 1’s vector, and $\tau$ has multiplicity $m_\tau = (N-1)r$.

If $N = \pm p^m$ and $r = \frac{N}{p}$, the eigenvalues $k$, $\theta$, and $\tau$ are distinct, except in the case that $m = 1$ and $N = p$. In this case $r = 1$, so $\theta = k = p - 1$, $\lambda = p - 2$, $\mu = 0$, and $\tau = -1$. In this case, when there are only two distinct eigenvalues, the graph $\Gamma$ is not connected, and consists of $p$ copies of the complete graph $K_p$.

5.4 Latin and negative Latin square type functions

Let $f : GF(p)^n \to GF(p)$ be a $p$-ary function, where $p$ is prime number, and $n$ is a positive integer. We say that $f$ is of *Latin square type* if $f$ is even with $f(0) = 0$, and its component Cayley graphs are all strongly regular and of Latin square type. Similarly, we say that $f$ is of *negative Latin square type* if $f$ is even with $f(0) = 0$, and its component Cayley graphs are all strongly regular and of negative Latin square type.

We will sometimes use the abbreviations LST and NLST for Latin square type and negative Latin square type, respectively.
5.5 Feasible LST and NLST functions

Let \( f : GF(p)^{2m} \to GF(p) \) be a \( p \)-ary function, where \( p \) is a prime number, and \( m \) is a positive integer. We say that \( f \) is of feasible Latin square type if it is of Latin square type and the parameters of the component Cayley graph \( \Gamma_i \) are

\[
(\nu, k, \lambda, \mu) = (N^2, (N-1)r_i, N + r_i^2 - 3r_i, r_i^2 - r_i),
\]

where \( N = p^m \), \( r_i = \frac{N}{p} \) for \( 1 \leq i \leq p-1 \), and \( r_p = \frac{N}{p} + 1 \). Similarly, we say that \( f \) is of feasible negative Latin square type if it is of negative Latin square type and the parameters of the component Cayley graphs are given by Equation (5.5), where \( N = -p^m \), \( r_i = \frac{N}{p} \) for \( 1 \leq i \leq p-1 \), and \( r_p = \frac{N}{p} + 1 \).

Remark 5.1. If \( m = 1 \), there are no functions of feasible negative Latin square type for \( p \geq 5 \). If there were such a function, the formula above would give a value of \( \lambda = -p + 4 \), which is impossible since \( \lambda \geq 0 \).

5.6 Eigenvalues of feasible LST and NLST graphs

From the formulas of the previous two sections, we can calculate the eigenvalues of the component Cayley graphs of \( f \) and their multiplicities, in the case that \( f \) is of feasible Latin or negative Latin square type. In this section, we capture a more subtle feature of how these eigenvalues interact, which is key to proving our main result and the subsequent duality theorem.

Let \( f : GF(p)^{2m} \to GF(p) \) be an even function such that \( f(0) = 0 \), where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Let \( A(i) \) be the adjacency matrix of the component Cayley graph \( \Gamma_i \). Recall that we denote by \( \lambda_i(x) \) the eigenvalue of \( A(i) \) corresponding to the Hadamard vector \( h(x) \) defined in §3.5.

Proposition 5.2. Let \( f : GF(p)^{2m} \to GF(p) \) be a \( p \)-ary function, where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Suppose that \( f \) is of feasible Latin or negative Latin square type. Then for each nonzero \( x \) in \( GF(p)^{2m} \), there exists a unique distinguished index \( j \) in \( \{1, 2, \ldots, p-1, p\} \) such that

\[
\lambda_j(x) = N - r_j,
\]

while for all the remaining values of \( i \) such that \( 1 \leq i \leq p \) and \( i \neq j \),

\[
\lambda_i(x) = -r_i.
\]
Proof. Let $\nu = p^{2m}$, let $I$ be the $\nu \times \nu$ identity matrix, and let $J$ be the $\nu \times \nu$ all 1’s matrix. Recall that for $x$ in $GF(p)^n$, the Hadamard vectors $h(x)$ are orthogonal vectors in $\mathbb{C}^\nu$, and $h(0)$ is the all 1’s vector. Thus, if $x$ is a nonzero point in $GF(p)^n$, then $Jh(x) = 0$. The adjacency matrices $A(i)$ of the component graphs $\Gamma_i$ of $f$ satisfy

$$I + \sum_{i=1}^{p} A(i) = J.$$ 

Multiplying on the right by $h(x)$, where $x \neq 0$, gives

$$h(x) + \sum_{i=1}^{p} A(i)h(x) = \left(1 + \sum_{i=1}^{p} \lambda_i(x)\right)h(x) = 0,$$

It follows that if $x \neq 0$,

$$1 + \sum_{i=1}^{p} \lambda_i(x) = 0. \tag{5.6}$$

Since the component graphs $\Gamma_i$ are of Latin or negative Latin square type with $N = \pm p^m$, $r_i = \frac{N}{p}$ for $1 \leq i \leq p - 1$, and $r_p = \frac{N}{p} + 1$, the eigenvalues of $\Gamma_i$ are $k_i = (N - 1)r_i$, $N - r_i$, and $-r_i$. When $x \neq 0$, the eigenvalue $\lambda_i(x)$ must take the value $N - r_i$ or $-r_i$, for $1 \leq i \leq p$.

Let $a$ be the number of the eigenvalues in the set $\{\lambda_i(x) \mid 1 \leq i \leq p - 1\}$ which take the value $N - r_i$. Similarly, let $b = 1$ if $\lambda_p(x) = N - r_p$, and let $b = 0$ otherwise. We wish to show that one of the numbers $a$ and $b$ is 1 and the other is 0. Let $r$ be the common value of $r_i$ for $1 \leq i \leq p - 1$, i.e., $r = \frac{N}{p}$. Then $r_p = r + 1$. Substituting into Equation (5.6) we obtain

$$0 = 1 + a(N - r) + (p - 1 - a)(-r) + b(N - r_p) + (1 - b)(-r_p)$$
$$= 1 + a(N - r) + (p - 1 - a)(-r) + b(N - r - 1) + (1 - b)(-r - 1)$$
$$= 1 + (a + b)N + p(-r) - 1$$
$$= (a + b)N - N.$$

Thus, $a + b = 1$, so there is exactly one index $j$, with $1 \leq j \leq p$, such that $\lambda_j(x) = N - r_j$. For the remaining values of $i \neq j$, $\lambda_i(x) = -r_i$. \qed
5.7 Dillon and Bernasconi–Codenotti–VanderKam theorems

Suppose that \( f : GF(2)^{2m} \to GF(2) \) is a Boolean function such that \( f(0) = 0 \), where \( m \) is a positive integer. We say that the Cayley graph \( \Gamma \) of \( f \) is the component Cayley graph \( \Gamma_1 \). Recall from §2.5 that if \( f \) is bent, then the only possible sizes for the set \( D_1 = f^{-1}(1) \) (and hence the only possible degrees of the Cayley graph \( \Gamma \)) are \( |D_1| = 2^{2^m-1} \pm 2^{m-1} \). These are the feasible degrees of the Cayley graph of a Boolean function of \( 2m \) variables that vanishes at 0. Dillon’s criterion \([D74]\) for bent Boolean functions was stated in the language of difference sets. We state an essentially equivalent version in terms of strongly regular graphs.

**Theorem 5.3** (Dillon). Let \( f : GF(2)^{2m} \to GF(2) \) be a function with \( f(0) = 0 \). Then \( f \) is bent if and only if its Cayley graph is strongly regular of feasible Latin or negative Latin square type.

Bernasconi, Codenotti, and VanderKam \([BCV01]\) proved that a function \( f : GF(2)^{2m} \to GF(2) \) with \( f(0) = 0 \) is bent if and only if the Cayley graph \( \Gamma \) of \( f \) is strongly regular with parameters \((2^{2m}, k, \lambda, \lambda)\), for some \( \lambda \), where \( k = |D_1| \). From the discussion above, we see that \( k = 2^{2^m-1} \pm 2^{m-1} \) and \( \lambda = 2^{2^{m-2}} \pm 2^{m-1} \).

In the next section, we show that the theorems of Dillon and Bernasconi, Codenotti, and VanderKam can be generalized in one direction (Theorem \[T.1\]). In \[T.3\] we give examples to show that converse of Theorem \[T.1\] does not hold, since the component Cayley graphs of a \( p \)-ary bent function are not necessarily strongly regular.

### 6 Feasible Latin and negative Latin square type functions are bent

We now prove our main result.

**Theorem 6.1.** Let \( f : GF(p)^{2m} \to GF(p) \) be an even function such that \( f(0) = 0 \), where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. If the component Cayley graphs of \( f \) are all strongly regular and are either all of feasible Latin square type, or all of feasible negative Latin square type, then \( f \) is bent.
Proof. By the hypotheses of the theorem, the component Cayley graphs \( \Gamma_i \) of \( f \) are strongly regular with parameters
\[
(N^2, (N - 1)r_i, N + r_i^2 - 3r_i, r_i^2 - r_i),
\]
where \( N = p^m \) if the graphs are all of feasible Latin square type, \( N = -p^m \) if the graphs are all of feasible negative Latin square type, \( r_i = \frac{N}{p} \), for \( 1 \leq i \leq p - 1 \), and \( r_p = \frac{N}{p} + 1 \).

In order to show that \( f \) is bent, we wish to show that the magnitude of the Walsh transform \( W_f(x) \) is \( |N| = p^m \), for each \( x \) in \( GF(p)^{2m} \).

The Walsh transform at \( x = 0 \) is easily calculated, by counting the number of times \( f \) takes each value \( i \), as
\[
W_f(0) = \sum_{y \in GF(p)^{2m}} \zeta^{f(y)}
= 1 + \sum_{i=1}^{p-1} \zeta^i(N - 1)r_i + (N - 1)r_p
= 1 + (N - 1) \left( \frac{N}{p} \right) \sum_{i=1}^{p-1} \zeta^i + (N - 1) \left( \frac{N}{p} + 1 \right)
= 1 - (N - 1) \left( \frac{N}{p} \right) + (N - 1) \left( \frac{N}{p} + 1 \right)
= N.
\]
Thus, \( W_f(0) = \pm p^m \).

We will now apply the characterization of bent functions in terms of eigenvalues (Proposition 4.2) to show that \( |W_f(x)| = p^m \) for all nonzero \( x \) in \( GF(p)^{2m} \). Recall from Proposition 5.2 that for each nonzero \( x \) in \( GF(p)^{2m} \), there exists a unique distinguished value of \( j \) in \( \{1, 2, \ldots, p - 1, p\} \) such that \( \lambda_j(x) = N - r_j \), while for all the remaining values of \( i \) such that \( 1 \leq i \leq p \) and \( i \neq j \), we have \( \lambda_i(x) = -r_i \). Continuing with this notation and using Proposition 4.1, we find that
\[
W_f(x) = 1 + \sum_{i=1}^{p} \zeta^i \lambda_i(x)
= 1 + \zeta^j(N - r_j) + \sum_{i \neq j} \zeta^i(-r_i)
\]
21
\[
= 1 + \zeta^j N + \sum_{i=1}^{p} \zeta^i (-r_i)
\]
\[
= 1 + \zeta^j N - \left( \frac{N}{p} \right) \sum_{i=1}^{p-1} \zeta^i - \left( \frac{N}{p} + 1 \right)
\]
\[
= 1 + \zeta^j N + \left( \frac{N}{p} \right) - \left( \frac{N}{p} + 1 \right)
\]
\[
= \zeta^j N.
\]
Therefore \( f \) is bent. \( \square \)

As an immediate consequence of the proof of the theorem above, we obtain the following corollary.

**Corollary 6.2.** Let \( f : GF(p)^{2m} \rightarrow GF(p) \) be a \( p \)-ary function, where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Suppose that \( f \) is of feasible Latin or negative Latin square type. Then \( W_f(0) = p^m \) in the feasible Latin square type case, and \( W_f(0) = -p^m \) in the feasible negative Latin square type case. Furthermore, if \( x \neq 0 \), then

\[
W_f(x) = \zeta^j W_f(0),
\]

where \( j \) is the distinguished index described in Proposition 5.2 such that \( \lambda_j(x) \) has the form \( N - r_j \).

This corollary gives us a dual \( p \)-ary function \( f^* \), satisfying

\[
W_f(x) = \zeta^{f^*(x)} W_f(0).
\]

The properties of \( f^* \) are described in the next section.

### 7 Dual functions

In this section, we prove that a bent function \( f \) of feasible Latin or negative Latin square type (as defined in [5.5]) has a dual \( f^* \) whose component Cayley graphs have the same parameters as those of \( f \). The main idea of the proof is to relate the component functions \( f_i^* \) of \( f^* \) to the component functions \( f_i \) of \( f \) by means of the equation

\[
f_i^*(x) = \frac{1}{N} \hat{f}_i(x) + \frac{r_i}{N} - r_i \delta_0(x), \quad (7.1)
\]
where \( \hat{f}_i \) is the Fourier transform of \( f_i \), and \( \delta_0 \) is the delta function centered at 0 (see Equation (7.4)). Since the eigenvalues of the component Cayley graphs of a \( p \)-ary function are given by the Fourier transforms of the component functions, Equation (7.1) allows us to calculate the eigenvalues of the component Cayley graphs of \( f^* \) and show that these graphs are strongly regular with the desired parameters.

### 7.1 Regular and weakly regular bent functions

A bent function \( f : GF(p)^n \to GF(p) \) is said to be **regular** if there exists a dual function \( f^* : GF(p)^n \to GF(p) \) such that

\[
W_f(x) = \zeta^{f^*(x)}p^{n/2}
\]

for all \( x \) in \( GF(p)^n \). Similarly, a bent function \( f : GF(p)^n \to GF(p) \) is said to be **weakly regular** or \( \mu \)-**weakly regular** if there exists a constant \( \mu \) in \( \mathbb{C} \) with magnitude 1 and a dual function or \( \mu \)-weakly regular dual function \( f^* : GF(p)^n \to GF(p) \) such that

\[
W_f(x) = \mu^{f^*(x)}p^{n/2}
\]

for all \( x \) in \( GF(p)^n \). It is known that the dual \( f^* \) of a regular or weakly regular bent function \( f \) is also bent. If \( f \) is regular, so is \( f^* \), and if \( f \) is weakly regular, so is \( f^* \). If \( f \) is an even function, then \( f^* \) is also even. See [JM17, §6.4] for further background on duality.

### 7.2 Regularity and feasible LST and NLST functions

We show that a function \( f : GF(p)^{2m} \to GF(p) \) of feasible Latin or negative Latin square type (as defined in §5.5) is regular or weakly regular, respectively.

Recall that in the feasible Latin or negative Latin square case, the parameters of the component Cayley graph \( \Gamma_i \) are

\[
(\nu, k, \lambda, \mu) = (N^2, (N - 1)r_i, N + r_i^2 - 3r_i, r_i^2 - r_i),
\]

where

\[
N = \pm p^m, \quad r_i = \frac{N}{p} \quad \text{for} \quad 1 \leq i \leq p - 1, \quad \text{and} \quad r_p = \frac{N}{p} + 1. \quad (7.2)
\]
When \( N = p^m \), the graph \( \Gamma_i \) is of Latin square type, and when \( N = -p^m \), it is of negative Latin square type.

Note that in the case \( p = 3 \) and \( m = 1 \), it is possible to have a strongly regular graph decomposition of \( \text{GF}(p)^{2m} \) that is both Latin and negative Latin square type (see Example 9.4), but only the negative Latin square type graph decomposition is feasible.

As above, we denote the eigenvalue of \( \Gamma_i \) corresponding to the Hadamard eigenvector \( h(x) \) by \( \lambda_i(x) \) (see Sections 3.5 and 3.6).

**Proposition 7.1.** Let \( f : \text{GF}(p)^{2m} \to \text{GF}(p) \) be a \( p \)-ary function, where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. If \( f \) is of feasible Latin square type, then \( f \) is a regular bent function. If \( f \) is of feasible negative Latin square type, then \( f \) is a \((-1)\)-weakly regular bent function.

**Proof.** The function \( f \) is bent, by Theorem 6.1. We define \( f^*(0) = 0 \). By Proposition 5.2, for every nonzero \( x \) in \( \text{GF}(p)^{2m} \), there is a unique distinguished index \( j \) in \( \{1, 2, \ldots, p\} \) such that \( \lambda_j(x) = N - r_j \). If \( j \neq p \), we define \( f^*(x) = j \), and if \( j = p \), we define \( f^*(x) = 0 \). By Corollary 6.2, \( W_f(x) = \zeta^{f^*(x)p^m} \) in the Latin square type case, and \( W_f(x) = -\zeta^{f^*(x)p^m} \) in the negative Latin square type case. Thus, \( f \) is regular in the Latin square type case and \((-1)\)-weakly regular in the negative Latin square type case, and \( f^* \) is the dual of \( f \). \( \square \)

### 7.3 Level sets of dual functions

We describe the level sets of the dual function of \( f \), when \( f \) is of feasible Latin or negative Latin square type, using Proposition 5.2 on distinguished indices of eigenvalues.

Suppose that \( f : \text{GF}(p)^{2m} \to \text{GF}(p) \) is an even bent function with \( f(0) = 0 \) such that \( f \) is regular or weakly regular. Let \( f^* : \text{GF}(p)^{2m} \to \text{GF}(p) \) be the dual function of \( f \). Recall that we define \( D_i = f^{-1}(i) \), for \( 1 \leq i \leq p - 1 \), and \( D_p = f^{-1}(0) \setminus \{0\} \). Similarly we define the corresponding sets for the dual function:

\[
D_i^* = (f^*)^{-1}(i), \text{ for } 1 \leq i \leq p - 1, \text{ and } D_p^* = (f^*)^{-1}(0) \setminus \{0\}. \quad (7.3)
\]

Recall also that the eigenvalues of the adjacency matrix \( A(i) \) of the component Cayley graph \( \Gamma_i \) of \( f \) are the values \( \hat{f}_i(x) \) of the Fourier transform of \( f_i \) for \( x \) in \( \text{GF}(p)^{2m} \). More concisely,

\[
\lambda_i(x) = \hat{f}_i(x),
\]
for $1 \leq i \leq p$ and for all $x$ in $GF(p)^{2m}$. If the graphs $\Gamma_i$ are strongly regular, and all of feasible Latin square type or all of feasible negative Latin square type, then the eigenvalues of $A(i)$ are $k_i = (N - 1)r_i$, $N - r_i$, and $-r_i$, where $N$ and $r_i$ are as in Equation (7.2).

Proposition 7.2. Let $f : GF(p)^{2m} \to GF(p)$ be a feasible Latin or negative Latin square type $p$-ary function, where $p$ is a prime number greater than 2, and $m$ is a positive integer. Then the set $D_i^*$ is given by

$$D_i^* = \{ x \in GF(p)^{2m} \setminus \{0\} \mid \hat{f}_i(x) = N - r_i \},$$

for $1 \leq i \leq p$. Furthermore, the cardinality $k_i^*$ of $D_i^*$ is

$$k_i^* = |D_i^*| = (N - 1)r_i = |D_i| = k_i.$$

Proof. The description of $D_i^*$ follows directly from the description of the dual function $f^*$ in the proof of Proposition 7.1 (noting that $\lambda_i(x) = \hat{f}_i(x)$). For every nonzero $x$ in $GF(p)^{2m}$, there is a unique distinguished index $i$ in $\{1, 2, \ldots, p\}$ such that $\lambda_i(x) = N - r_i$, by Proposition 5.2. The multiplicity of $N - r_i$ as an eigenvalue of $\Gamma_i$ on the orthogonal complement of the all 1’s vector is given by $(N - 1)r_i$ (see §5.3). □

### 7.4 Component functions of dual functions

We formulate an expression for the $i$th component function of a dual function $f^*$, in terms of the Fourier transform of the $i$th component function of the original function $f$.

The $i$th component function of $f^*$ is the function $f_i^* : GF(p)^{2m} \to \mathbb{C}$ given by

$$f_i^*(x) = \begin{cases} 1 & \text{if } x \in D_i^*, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq p$.

The following result is an immediate corollary of Proposition 7.2.

Corollary 7.3. Let $f : GF(p)^{2m} \to GF(p)$ be a feasible Latin or negative Latin square type $p$-ary function, where $p$ is a prime number greater than 2, and $m$ is a positive integer. Then $f_i^*(0) = 0$, and

$$f_i^*(x) = 1 \quad \text{if and only if } \quad \hat{f}_i(x) = N - r_i,$$

for $x$ in $GF(p)^{2m} \setminus \{0\}$ and $1 \leq i \leq p$, where $N$ and $r_i$ are as in Equation (7.2).
In order to relate the component functions of the dual \( f^* \) to the component functions of \( f \), we introduce a delta function and its Fourier transform. We define the delta function at 0 on \( GF(p)^{2m} \) to be the function \( \delta_0 : GF(p)^{2m} \to \mathbb{C} \) given by

\[
\delta_0(x) = \begin{cases} 
1 & \text{if } x = 0, \\
0 & \text{otherwise.}
\end{cases}
\] (7.4)

We denote by \( \iota \) the constant function \( \iota : GF(p)^{2m} \to \mathbb{C} \) given by \( \iota(x) = 1 \), for \( x \) in \( GF(p)^{2m} \).

The next lemma follows directly from the definition of the Fourier transform.

**Lemma 7.4.** The Fourier transform of the delta function at 0 on \( GF(p)^{2m} \) is given by

\[
\hat{\delta}_0 = \iota,
\]

where \( \iota \) is the function defined above, with constant value 1 on \( GF(p)^{2m} \). The Fourier transform of \( \iota \) is given by

\[
\hat{\iota} = p^{2m}\delta_0.
\]

The following proposition expresses the component functions of the dual \( f^* \) in terms of the Fourier transforms of the component functions of \( f \).

**Proposition 7.5.** Let \( f : GF(p)^{2m} \to GF(p) \) be a feasible Latin or negative Latin square type \( p \)-ary function, where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Then the \( i \)th component function \( f_i^* \) of the dual of \( f \) satisfies

\[
f_i^*(x) = \frac{1}{N} \hat{f}_i(x) + \frac{r_i}{N} - r_i\delta_0(x),
\] (7.5)

for \( 1 \leq i \leq p \), where \( N \) and \( r_i \) are as in Equation (7.2), and \( \hat{f}_i \) is the Fourier transform of the \( i \)th component function of \( f \).

**Proof.** Recall that \( \hat{f}_i(x) = \lambda_i(x) \), where \( \lambda_i(x) \) is the eigenvalue of the adjacency matrix of \( \Gamma_i \) corresponding to the Hadamard eigenvector \( h(x) \). Thus, by Equations (5.3) and (5.4) of §5.3, \( \hat{f}_i(0) = (N - 1)r_i \), and \( \hat{f}_i(x) \) equals either \( N - r_i \) or \( -r_i \) for \( x \neq 0 \). By Proposition 7.2, for \( x \neq 0 \), \( \hat{f}_i(x) = N - r_i \).
if and only if $x \in D_i^*$, where $D_i^*$ is as in Equation (7.3). Therefore,

$$\frac{1}{N} \hat{f}_i(x) + \frac{r_i}{N} - r_i \delta_0(x) = \begin{cases} \frac{1}{N}(N-1)r_i + \frac{r_i}{N} - r_i & \text{if } x = 0, \\ \frac{1}{N}(N-r_i) + \frac{r_i}{N} - 0 & \text{if } x \in D_i^*, \\ \frac{1}{N}(-r_i) + \frac{r_i}{N} - 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in D_i^* \\ 0 & \text{otherwise}, \end{cases} = f_i(x).$$

\[\square\]

### 7.5 Eigenvalues for dual functions

We calculate the eigenvalues of the component Cayley graphs of dual functions, using the Fourier transforms of the corresponding component functions described in Proposition 7.5 above.

We first review some basic properties of inverse Fourier transforms, which we will need in the proof of the next proposition. Recall that the Fourier transform of a function $g: GF(p)^n \to \mathbb{C}$ is the function $\hat{g}: GF(p)^n \to \mathbb{C}$ given by

$$\hat{g}(x) = \sum_{y \in GF(p)^n} g(y) \zeta^{-\langle x,y \rangle}.$$  

The inverse Fourier transform of a function $h: GF(p)^n \to \mathbb{C}$ is the function $\check{h}: GF(p)^n \to \mathbb{C}$ given by

$$\check{h}(x) = \frac{1}{p^n} \sum_{y \in GF(p)^n} h(y) \zeta^{\langle x,y \rangle}.$$  

The Fourier transform and its inverse satisfy

$$\check{\hat{g}}(x) = g(x).$$

If $g$ is an even function, i.e., if $g(-x) = g(x)$ for all $x$ in $GF(p)^n$, then

$$\check{g}(x) = \frac{1}{p^n} \hat{g}(x).$$

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Thus, if \( g \) is even, 
\[
\hat{g}(x) = p^n g(x).
\] (7.6)

The next proposition is dual to Proposition 7.5 in the sense that it expresses the Fourier transforms of the component functions of the dual function \( f^* \) in terms of the component functions of the original function \( f \).

**Proposition 7.6.** Let \( f : \text{GF}(p)^{2m} \to \text{GF}(p) \) be a feasible Latin or negative Latin square type \( p \)-ary function, where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Then the Fourier transform of the \( i \)th component function \( f_i^* \) of the dual of \( f \) satisfies
\[
(f_i^*)^\wedge(x) = N f_i(x) + N r_i \delta_0(x) - r_i,
\]
for \( 1 \leq i \leq p \), where \( N \) and \( r_i \) are as in Equation (7.2).

**Proof.** Taking the Fourier transform of each term of Equation (7.5) of Proposition 7.5, we obtain
\[
(f_i^*)^\wedge(x) = \frac{1}{N} \hat{f}_i(x) + \frac{r_i}{N} \hat{\iota}(x) - r_i \hat{\delta}_0(x),
\]
where \( \iota \) is the constant function with value 1 on \( \text{GF}(p)^{2m} \). By Equation (7.6), \( \hat{f}_i(x) = p^{2m} f_i(x) = N^2 f_i(x) \). The Fourier transforms of \( \iota \) and \( \delta_0 \) are given by Lemma 7.4 as \( \hat{\iota}(x) = N^2 \delta_0(x) \) and \( \hat{\delta}_0(x) = 1 \). Therefore
\[
(f_i^*)^\wedge(x) = N f_i(x) + N r_i \delta_0(x) - r_i.
\]

\[\square\]

From Proposition 7.6, we obtain the eigenvalues of the component Cayley graphs of the dual function.

**Corollary 7.7.** Let \( f : \text{GF}(p)^{2m} \to \text{GF}(p) \) be a feasible Latin or negative Latin square type \( p \)-ary function, where \( p \) is a prime number greater than 2, and \( m \) is a positive integer. Then the eigenvalues of the component Cayley graph \( \Gamma_i^* \) of the dual of \( f \) and their multiplicities are the same as those of \( \Gamma_i \), for \( 1 \leq i \leq p \). Specifically,

1. the eigenvalue \( k_i^* = (N - 1)r_i \) corresponds to the all 1’s eigenvector;

2. the eigenvalue \( N - r_i \) occurs with multiplicity \( (N - 1)r_i \) on the vector space orthogonal to the all 1’s vector, and
3. the eigenvalue $-r_i$ occurs with multiplicity $(N-1)(N+1-r_i)$ on the vector space orthogonal to the all 1’s vector,

where $N$ and $r_i$ are as in Equation (7.2).

Proof. Let $f_i^*$ be the $i$th component function of the dual of $f$. The eigenvalues of $\Gamma_i^*$ are the values $(f_i^*)^\wedge(x)$ of the Fourier transform of $f_i^*$, for $x$ in $GF(p)^{2m}$. By Proposition 7.6, the Fourier transform of $f_i^*$ is given by

$$(f_i^*)^\wedge(x) = Nf_i(x) + Nr_i\delta_0(x) - r_i.$$ 

Thus,

$$(f_i^*)^\wedge(x) = \begin{cases} Nr_i - r_i & \text{if } x = 0, \\
N - r_i & \text{if } x \in D_i, \\
-r_i & \text{if } x \in GF(p)^{2m} \setminus \{0\} \cup D_i, \end{cases}$$ 

so $\Gamma_i^*$ has the same eigenvalues as $\Gamma_i$. The multiplicity of $N - r_i$ on the vector space orthogonal to the all 1’s vector is $|D_i| = (N-1)r_i$. The multiplicity of $-r_i$ on the vector space orthogonal to the all 1’s vector is

$$|GF(p)^{2m} \setminus \{0\} \cup D_i| = N^2 - 1 - (N-1)r_i = (N-1)(N+1-r_i),$$

Therefore, the multiplicities of the eigenvalues of $\Gamma_i^*$ are the same as those of $\Gamma_i$. \qed

7.6 Duality theorem

In this section we prove that the dual of a feasible Latin or negative Latin square type function is also a feasible Latin or, respectively, negative Latin square type function.

Theorem 7.8. Let $f: GF(p)^{2m} \to GF(p)$ be an even function such that $f(0) = 0$, where $p$ is a prime number greater than 2, and $m$ is a positive integer. Suppose that the component Cayley graphs $\Gamma_i$ of $f$ are all strongly regular and are either all of feasible Latin square type, or all of feasible negative Latin square type. Then the component Cayley graphs $\Gamma_i^*$ of the dual function $f^*$ are also all strongly regular, and the parameters of $\Gamma_i^*$ are the same as the parameters of $\Gamma_i$. 

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Proof. Recall that a simple regular graph which is not complete or edgeless and which has exactly two distinct eigenvalues corresponding to eigenvectors orthogonal to the all 1’s vector must be a strongly regular graph (see, for example, Brouwer and Haemers, [BH11, Theorem 9.1.2]). Thus, it follows directly from Corollary 7.7 that the component Cayley graphs \( \Gamma^*_i \) are all strongly regular, for \( 1 \leq i \leq p \), since each has at most 3 eigenvalues. Of the parameters \((\nu, k^*_i, \lambda^*_i, \mu^*_i)\) for \( \Gamma^*_i \), the parameters \( \nu = N^2 \) and \( k^*_i = (N - 1)r_i \) are known, (where, as above, \( N = \pm p^m \), \( r_i = \frac{N}{p} \) for \( 1 \leq i \leq p - 1 \), and \( r_p = \frac{N}{p} + 1 \)). We will calculate the parameters \( \mu^*_i \) and \( \lambda^*_i \) from the eigenvalues of \( \Gamma^*_i \), using Corollary 7.7. We denote the two distinct eigenvalues of \( \Gamma^*_i \) on the vector space orthogonal to the all 1’s vector by \( \theta_i \) and \( \tau_i \) (where by convention \( \theta_i > \tau_i \), although this order is not needed here). By Corollary 7.7, one of these two eigenvalues is \( N - r_i \), and the other is \(-r_i\). Therefore, by Equation 5.1,
\[
\mu^*_i = k^*_i + \theta^*_i \tau^*_i = r^2_i - r_i,
\]
and
\[
\lambda^*_i = \mu^*_i + \theta^*_i + \tau^*_i = N + r^2_i - 3r_i.
\]
It follows that the graphs \( \Gamma^*_i \) are either all of feasible Latin square type or all of feasible negative Latin square type. \( \square \)

8 Amorphic bent functions

In this section we consider even \( p \)-ary functions which determine association schemes. We give a criterion for such a function to be bent, in terms of structure constants of its association scheme. We show that a function which determines an amorphic association scheme and whose component Cayley graphs are of feasible degrees must be bent.

It is well-known that a \( p \)-ary function \( f: GF(p)^n \rightarrow GF(p) \) is bent if and only if the derivative functions given by \( D_bf(x) = f(x+b) - f(x) \) are balanced, for all nonzero \( b \) in \( GF(p)^n \) (i.e., \( D_bf \) takes all values equally often). In the case that \( f \) determines an association scheme, we show that the number of times \( D_bf \) takes each value in \( GF(p) \) can be expressed in a natural way in terms of structure constants of the association scheme. The structure constants of amorphic association schemes were described by Ito, Munemasa, and Yamada [IMY91]. By summing the appropriate structure constants, we show that \( D_bf \) is balanced if \( f \) is amorphic.
8.1 Association schemes

Let $V$ be a finite set. A binary relation $R$ on $V$ is a subset of $V \times V$. The dual of a relation $R$ is the set $R^* = \{(y, x) \in V \times V \mid (x, y) \in R\}$.

Let $\{R_0, R_1, \ldots, R_p\}$ be a set of disjoint binary relations on $V$ whose union is $V \times V$, such that $R_0 = \{(x, x) \in V \times V \mid x \in V\}$, and such that for each $i$ there is a $j$ for which $R_i^* = R_j$. For $0 \leq i, j, k \leq p$ and for $(x, y)$ in $R_k$, let

$$\rho_{ij}^k(x, y) = |\{z \in V \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}|.$$

We say that the collection $(V, R_0, R_1, \ldots, R_p)$ forms a $p$-class association scheme if the numbers $\rho_{ij}^k(x, y)$ are independent of which pair $(x, y)$ we choose in $R_k$ (hence depend only on $i$, $j$, and $k$). The numbers $\rho_{ij}^k$ are called the structure constants or intersection numbers of the association scheme. If, in addition, $R_i^* = R_i$ for all $i$, then we say that the association scheme is symmetric. A symmetric association scheme determines a collection of undirected graphs $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$ on the vertex set $V$.

We are interested in the case in which $V = GF(p)^n$ and the relations $R_i$ correspond to the component Cayley graphs $\Gamma_i$ of a $p$-ary function. Let $f : GF(p)^n \to GF(p)$ be an even function such that $f(0) = 0$, where $p$ is a prime number greater than 2, and $n$ is a positive integer. Associated with $f$ is a set of binary relations $\{R_0, R_1, \ldots, R_p\}$ on $V$ given by

$$R_0 = \{(x, x) \in V \times V \mid x \in V\},$$

$$R_i = \{(x, y) \in V \times V \mid f(x - y) = i\}$$

for $1 \leq i \leq p - 1$, and

$$R_p = \{(x, y) \in V \times V \mid f(x - y) = 0 \text{ and } x \neq y\}.$$

Note that, by the assumption that $f$ is even, these relations are all self-dual. Furthermore,

$$(x, y) \in R_i \text{ if and only if } x - y \in D_i,$$

where $D_0 = \{0\}$, $D_i = f^{-1}(i)$ for $1 \leq i \leq p - 1$, and $D_p = f^{-1}(0) \setminus \{0\}$. The relations $R_i$, for $1 \leq i \leq p$, correspond to the component Cayley graphs $\Gamma_i$ of $f$.

Recall that we denote by $A(i)$ the adjacency matrix of the component Cayley graph $\Gamma_i$ of $f$. Let $A(0) = I$, the $\nu \times \nu$ identity matrix, where $\nu = p^n$. The matrices $A(0), A(1), \ldots, A(p)$ can also be thought of as the adjacency
matrices of the relations \( R_0, R_1, \ldots, R_p \). The sum of these adjacency matrices is the all 1’s matrix \( J \). The condition that \( f \) determines a \( p \)-class symmetric association scheme with structure constants \( \rho^k_{ij} \) is equivalent to the condition that there exist nonnegative integers \( \rho^k_{ij} \) such that
\[
A(i)A(j) = \sum_{k=0}^{p} \rho^k_{ij} A(k)
\]
for \( 0 \leq i, j, k \leq p \). In this case, the matrices \( A(0), A(1), \ldots, A(p) \) generate a Bose-Mesner algebra (see, e.g., [JM17, §6.7.1]). The structure constants can be calculated from the adjacency matrices by the following formula (see [CVL80, Chapter 17]):
\[
\rho^k_{ij} = \left( \frac{1}{p^n |D_k|} \right) tr(A(i)A(j)A(k)),
\]
for \( 1 \leq i, j, k \leq p \), where \( tr \) denotes the matrix trace, and the sets \( D_k \) are as above.

### 8.2 Amorphic association schemes and functions

Let \( V \) be a finite set, and let \( \mathcal{R} = \{R_0, R_1, \ldots, R_p\} \) be a set of disjoint binary relations on \( V \) whose union is \( V \times V \). A set of disjoint binary relations \( \mathcal{T} = \{T_0, T_1, \ldots, T_m\} \) whose union is \( V \times V \) is called a fusion of \( \mathcal{R} \) if each \( T_i \) is a union of elements of \( \mathcal{R} \). An association scheme \((V, R_0, R_1, \ldots, R_p)\) is called amorphic if for each fusion \( \mathcal{T} \) of \( \mathcal{R} \), the collection \((V, T_0, T_1, \ldots, T_m)\) is also an association scheme. A 2-class association scheme is trivially amorphic.

Consider an even function \( f : GF(p)^n \to GF(p) \) such that \( f(0) = 0 \). Let \( V = GF(p)^n \), and let \( \{R_0, R_1, \ldots, R_p\} \) be the binary relations determined by \( f \), as described above. We call \( f \) amorphic if \((V, R_0, R_1, \ldots, R_p)\) is an amorphic association scheme.

### 8.3 van Dam and Gol’fand–Ivanov–Klin theorems

There is a close relationship between amorphic association schemes and strongly regular graphs of Latin and negative Latin square type.

A theorem of Gol’fand, Ivanov, and Klin from [GIK94] (which we learned of from van Dam and Muzychuk [vDM10]), states that the graphs determined
by a $p$-class amorphic association scheme, with $p \geq 3$, are all strongly regular, and are either all of Latin square type, or all of negative Latin square type.

Van Dam [vD03, Theorem 3] proved the converse: a decomposition of a complete graph into strongly regular graphs, all of Latin square type, or all of negative Latin square type, determines an amorphic association scheme. Thus, $p$-ary functions whose component Cayley graphs are strongly regular and all of Latin square type or all of negative Latin square type are amorphic functions.

### 8.4 Balanced derivative criterion for bent functions

Let $f : GF(p)^n \rightarrow GF(p)$ be a $p$-ary function. The derivative function $D_b f : GF(p)^n \rightarrow GF(p)$ is defined by

$$D_b f(x) = f(x + b) - f(x).$$

If $f$ is linear, then $D_b f$ is constant. The following result, which is well-known (see, e.g., [JM17, Proposition 6.3.9]), implies that bent functions are in some sense maximally non-linear.

**Proposition 8.1.** The function $f$ is bent if and only if $D_b f$ is balanced for all $b \neq 0$, i.e., if $D_b f$ takes each value in $GF(p)$ equally often.

### 8.5 Structure constant criterion for bent functions

In this section, we consider even $p$-ary functions whose component Cayley graphs determine symmetric $p$-class association schemes. We state a criterion for such a function to be bent, involving sums of structure constants of these schemes.

Let $f : GF(p)^n \rightarrow GF(p)$ be an even function with $f(0) = 0$. Suppose that $f$ determines an association scheme with structure constants $\rho^k_{ij}$, as described in §8.1. Let $D_i = f^{-1}(i)$, for $1 \leq i \leq p - 1$, and let $D_p = f^{-1}(0) \setminus \{0\}$.

**Proposition 8.2.** Suppose that $b \in D_i$, for some $i$ such that $1 \leq i \leq p$. The number of times $D_b f$ takes the value $j$, for $1 \leq j \leq p - 1$, is

$$\left( \sum_{k=0}^{p} \rho^j_{i+k (mod p),k} \right) + \rho^j_{p,p-j}.$$
Proof. Suppose that \( x \in D_k \) and \( x + b \in D_m \), for some \( m \) and \( k \) such that \( 0 \leq m, k \leq p \). Then \( f(x + b) - f(x) = j \) if and only if \( m - k \, (\text{mod } p) = j \). Either \( m = j + k \, (\text{mod } p) \), or \( m = p \) and \( k = p - j \).

Recall that the condition that \((x, y) \in R_t\) is equivalent to the condition that \(x - y \in D_t\). Therefore, \((b, 0) \in R_i\) and

\[
\rho^i_{m,k} = |\{ z \in GF(p)^n \mid (b, z) \in R_m \text{ and } (z, 0) \in R_k \}| = |\{ z \in GF(p)^n \mid b - z \in D_m \text{ and } z \in D_k \}|.
\]

Let \( x = -z \). Since \( f \) is even, \( z \in D_k \) if and only if \(-z \in D_k \). Therefore,

\[
\rho^i_{m,k} = |\{ x \in GF(p)^n \mid b + x \in D_m \text{ and } x \in D_k \}|.
\]

Summing over all pairs of indices \((m, k)\) such that \(m - k \, (\text{mod } p) = j\), we obtain the total number of \(x\) in \(GF(p)^n\) for which \(f(x + b) - f(x) = j\):

\[
\sum_{k=0}^{p} \rho^i_{j+k \, (\text{mod } p), k} + \rho^i_{p,p-j}.
\]

□

From Proposition 8.2, we obtain the following criterion for a function to be bent, in terms of structure constants of an association scheme.

Proposition 8.3. Let \( f : GF(p)^n \rightarrow GF(p) \) be an even function with \( f(0) = 0 \) that determines an association scheme with structure constants \( \rho^i_{ij} \). Then \( f \) is bent if and only if

\[
\left( \sum_{k=0}^{p} \rho^i_{j+k \, (\text{mod } p), k} \right) + \rho^i_{p,p-j} = p^{n-1} \]

for \(1 \leq i \leq p\) and \(1 \leq j \leq p - 1\).

8.6 Ito–Munemasa–Yamada theorem

If a \(p\)-ary function determines an amorphic association scheme, we can use a theorem of Ito, Munemasa, and Yamada [IMY91] (which is formulated in more modern notation by Van Dam and Muzychuk in [vDM10, Corollary 1]) to obtain the structure constants of this amorphic association scheme.
Let $f : GF(p)^{2m} \rightarrow GF(p)$ be an even function with $f(0) = 0$, where $p$ is a prime number greater than 2, and $m$ is a positive integer. Suppose that $f$ determines an amorphic association scheme. By the theorem of Gol’fand, Ivanov, and Klin [GIK94] mentioned in Section 8.3, the component Cayley graphs $\Gamma_i$ of $f$ are strongly regular, and are either all of Latin square type, or all of negative Latin square type. Therefore, each graph $\Gamma_i$ has parameters of the form

$$(\nu, k, \lambda, \mu) = (N^2, (N−1)r_i, N + r_i^2 − 3r_i, r_i^2 − r_i),$$

where $N$ equals $p^m$ in the Latin square case and $−p^m$ in the negative Latin square case, and each $r_i$ is an integer with the same sign as $N$.

The structure constants $\rho_{ij}^i$ of an association scheme satisfy

\begin{align*}
(a) & \quad \rho_{ii}^i = N + r_i^2 − 3r_i \text{ if } 1 \leq i \leq p, \\
(b) & \quad \rho_{jj}^i = (r_j − 1)r_j \text{ if } i \text{ and } j \text{ are distinct and } 1 \leq i, j \leq p, \\
(c) & \quad \rho_{ij}^i = \rho_{ji}^i = (r_i − 1)r_j \text{ if } i \text{ and } j \text{ are distinct and } 1 \leq i, j \leq p, \text{ and} \\
(d) & \quad \rho_{jk}^i = \rho_{kj}^i = r_jr_k \text{ if } i, j, \text{ and } k \text{ are distinct and } 1 \leq i, j, k \leq p.
\end{align*}

The proof below is included for completeness, and is only for the special case of interest to us, when the component Cayley graphs of $f$ are of feasible degrees. In this case, $r_i = \frac{N}{p}$ for $1 \leq i \leq p − 1$, and $r_p = \frac{N}{p} + 1$.

**Proof.** When the component Cayley graphs of $f$ are of feasible degrees, these structure constant formulas may be derived from Proposition 5.2 and Equation (8.1). We note that

$$tr(A(i)A(j)A(k)) = \sum_{x \in GF(p)^n} \lambda_i(x) \lambda_j(x) \lambda_k(x),$$

where $\lambda_i(x)$ is the eigenvalue of $\Gamma_i$ corresponding to the Hadamard vector $h(x)$ (see §3.2). For $x \neq 0$, each eigenvalue $\lambda_i(x)$ is either of the form $N − r_i$.
or $-r_i$. For each fixed $x \neq 0$, there is exactly one value of $i$ such that $\lambda_i(x)$ is of the form $N - r_i$. Furthermore, the multiplicities of each eigenvalue are also known. Let $i$, $j$, and $k$ be distinct integers such that $1 \leq i, j, k \leq p$. By a straightforward counting argument, we find that

$$tr(A(i)^3) = N^2(N - 1)r_i(N + r_i^2 - 3r_i),$$

$$tr(A(i)^2A(j)) = N^2(N - 1)r_i(r_i - 1)r_j,$$

and

$$tr(A(i)A(j)A(k)) = N^2(N - 1)r_ir_jr_k.$$

Substituting these three cases into Equation (8.1), we obtain the desired structure constants. \hfill \Box

### 8.7 Proof that feasible amorphic functions are bent

In this section, we use our structure constant criterion for bent functions to prove that if an even $p$-ary function of $2m$ variables, vanishing at 0, with component Cayley graphs of feasible degrees is amorphic, then it is bent. As usual, the feasible degrees are those specified in §3.3.

**Theorem 8.5.** Suppose that $f: GF(p)^{2m} \rightarrow GF(p)$ is an even function such that $f(0) = 0$, where $p$ is a prime number greater than 2, and $m$ is a positive integer, such that the component Cayley graphs of $f$ are of feasible degrees and $f$ determines an amorphic association scheme. Then $f$ is bent.

**Remark 8.6.** Theorem 8.5 is equivalent to Theorem 6.1, due to the results of van Dam [vD03] and Gol'fand, Ivanov, and Klin [GIK94] discussed in §8.3, but we include both theorems because the proofs are different.

**Proof.** We will use the structure constant criterion of Proposition 8.3 to show that $f$ is bent: we will show that

$$\left(\sum_{k=0}^{p} p^s_{t+k (mod p),k} \right) + p^s_{p,p-t} = p^{2m-1}$$

for $1 \leq s \leq p$ and $1 \leq t \leq p - 1$. To evaluate this sum, we use the theorem of Ito, Munemasa, and Yamada (Theorem 8.4). We consider the following four cases: $s = p$; $s \neq p$ and $s = t$; $s \neq p$ and $s = p - t$; and $s \neq p$, $s \neq t$, and $s \neq p - t$ (this last case cannot occur for $p = 3$).
Terms of types (a) and (b) in Theorem 8.4 do not occur in the sum of Equation (8.2). In the following chart, we indicate how many times terms of each of the remaining types occur in the sum, in each of the four cases. There are a total of \( p + 2 \) terms in each column. In the chart, we use the convention that \( i, j, \) and \( k \) represent distinct values in the set \( \{1, 2, \ldots, p-1\} \). We also use the notation \( r = \frac{N}{p} \).

| Term type and value | Number of occurrences |
|---------------------|-----------------------|
|                     | \( s = p \) | \( s = t \neq p \) | \( s = p - t \neq p \) | \( s \neq p, t, p - t \) |
| \( \rho_{ij} = \rho_{ji} = 1 \) | 0 | 1 | 1 | 0 |
| \( \rho_{ij} = \rho_{ji} = 0 \) | 0 | 1 | 1 | 2 |
| \( \rho_{ij}^p = \rho_{ji}^p = 0 \) | 2 | 0 | 0 | 0 |
| \( \rho_{ij}^p = \rho_{ji}^p = r^2 \) | 2 | 0 | 0 | 0 |
| \( \rho_{ij}^p = \rho_{ji}^p = r^2 - r \) | 0 | 1 | 1 | 2 |
| \( \rho_{ij}^p = \rho_{ji}^p = r^2 - 1 \) | 0 | 1 | 1 | 0 |
| \( \rho_{ij}^p = \rho_{ji}^p = r^2 + r \) | 0 | 1 | 1 | 2 |
| \( \rho_{ij}^p = \rho_{ji}^p = r^2 \) | 0 | \( p - 3 \) | \( p - 3 \) | \( p - 4 \) |
| \( \rho_{ij}^p = \rho_{ji}^p = r^2 \) | \( p - 2 \) | 0 | 0 | 0 |

We see that for each column, the number of occurrences of each term type multiplied by the value sums to \( pr^2 = p^{2m-1} \). For example, for the \( s = p \) column, we obtain \( 2r^2 + (p - 2)r^2 = pr^2 \). Therefore, \( f \) is bent.

8.8 Examples

To illustrate some properties of amorphic bent functions, we include an example from [CJMPW16]. Additional properties of the functions in this example are given in Examples 10.4 and 10.5. We show a sample sum of structure constants of the type of Propositions 8.2 and 8.3. We also give an example of a bent function which is not amorphic.

Example 8.7. ([CJMPW16]) Let \( f : GF(3)^2 \rightarrow GF(3) \) be an even bent function with \( f(0) = 0 \). Then \( f(x_0, x_1) \) is equivalent to either \(-x_0^2 + x_1^2\) or \(x_0^2 + x_1^2\) under the action of \( GL(2, GF(3)) \) on \((x_0, x_1)\). In each case, the function \( f \) determines an amorphic association scheme.

1. In the case of \(-x_0^2 + x_1^2\), the structure constants \( \rho_{ij}^k \) are given in the following arrays.
2. In the case of \( x_0^2 + x_1^2 \), the component Cayley graph \( \Gamma_3 \) is empty, and the structure constants \( \rho_{ij}^k \) are given in the following arrays.

\[
\begin{array}{c|cccc}
\rho_{ij}^0 & 0 & 1 & 2 & 3 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 \\
2 & 0 & 0 & 2 & 0 \\
3 & 0 & 0 & 0 & 4 \\
\end{array}
\begin{array}{c|cccc}
\rho_{ij}^1 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 2 \\
3 & 0 & 0 & 2 & 2 \\
\end{array}
\begin{array}{c|cccc}
\rho_{ij}^2 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 2 & 0 \\
2 & 1 & 0 & 1 & 0 \\
3 & 0 & 2 & 0 & 2 \\
\end{array}
\begin{array}{c|cccc}
\rho_{ij}^3 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
2 & 0 & 1 & 0 & 1 \\
3 & 1 & 1 & 1 & 1 \\
\end{array}
\]

**Example 8.8.** Consider the structure constants for \(-x_0^2 + x_1^2\) given above. The expression in Proposition 8.2 for \( i = 1 \) and \( j = 2 \), is

\[
\rho_{2,0}^1 + \rho_{0,1}^1 + \rho_{1,2}^1 + \rho_{2,3}^1 + \rho_{3,1}^1 = 3.
\]

The other sums from Proposition 8.2 can be calculated from the arrays above in a similar manner.

**Example 8.9.** Consider the function \( f : GF(5)^2 \rightarrow GF(5) \) given by

\[
f(x_0, x_1) = 3x_0^4 + 2x_0^2 + 2x_0x_1.
\]

It can be checked that \( f \) is bent with

\[
D_1 = \{(1, 3), (2, 0), (3, 0), (4, 2)\}
\]

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\[ D_2 = \{(1, 1), (2, 4), (3, 1), (4, 4)\} \]
\[ D_3 = \{(1, 4), (2, 3), (3, 2), (4, 1)\} \]
\[ D_4 = \{(1, 2), (2, 2), (3, 3), (4, 3)\} \]
\[ D_5 = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (2, 1), (3, 4), (4, 0)\} \]

However, graphs \( \Gamma_1, \Gamma_2, \) and \( \Gamma_4 \) have 6 distinct eigenvalues, \( \Gamma_3 \) has 2 distinct eigenvalues, and \( \Gamma_5 \) has 7 distinct eigenvalues, so the graphs are not all strongly regular. By the theorem of Gol’fand, Ivanov, and Klin \[GIK94\] (see §8.3), \( f \) is not amorphic.

We give more examples in §10.

9 Orthogonal arrays and bent functions

Orthogonal arrays are closely related to strongly regular graphs of Latin square type, and may be used to construct amorphic association schemes. See \[GR01\] §10.4 and \[vDM10\] for further background on these topics. We will describe how to construct bent functions using orthogonal arrays.

9.1 Orthogonal arrays

Let \( S \) be a set of size \( N \). An orthogonal array of size \( r \times N^2 \) with entries in \( S \) consists of \( r \) rows of \( N^2 \) entries from \( S \), such that for any two rows, the \( N^2 \) ordered pairs determined by the columns are all distinct. Such an array is denoted \( OA(r, N) \).

We are primarily interested in orthogonal arrays of size \((N + 1) \times N^2\), where \( S = GF(p)^m \), \( N = p^m \), and \( p \) is prime.

**Example 9.1.** Let \( S = GF(3) \), \( r = 4 \), and \( N = 3 \). The following is an \( OA(4, 3) \) with entries in \( S \).

\[
OA = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\
\end{bmatrix}
\]
9.2 Latin square type graphs from orthogonal arrays

We can form a graph $\Gamma$ from an orthogonal array $OA(r, N)$ as follows. The vertices of $\Gamma$ are the columns of the array. Two distinct vertices are connected by an edge exactly when the columns have the same entry in one row. It is well-known that the graph $\Gamma$ is either complete (in the case $r = N + 1$) or strongly regular of Latin square type. We include a proof for the convenience of the reader. We then give examples in which we construct graphs from orthogonal arrays and use these graphs to construct amorphic bent functions.

**Lemma 9.2.** The graph $\Gamma$ determined by an $OA(r, N)$ is either complete or strongly regular of Latin square type with parameters

$$(N^2, (N - 1)r, N + r^2 - 3r, r^2 - r).$$

*Proof.* Consider any column $v$ of the array. The $i$th entry of $v$ occurs in exactly $N - 1$ other locations in row $i$. By the definition of an orthogonal array, two columns can agree in at most one row. Thus, vertex $v$ has $(N - 1)r$ neighbors.

If $r = N + 1$, then each vertex $v$ has $N^2 - 1$ neighbors, and $\Gamma$ is complete.

Suppose that $v$ and $w$ are neighbors, i.e., $v$ and $w$ are distinct columns, with equal entries in some row $i$. A column $u$ which is a neighbor of both $v$ and $w$ agrees with each of $v$ and $w$ in exactly one row. There are $N - 2$ neighbors $u$ that have the same entry in row $i$ as $v$ and $w$. Any other neighbor $u$ of $v$ and $w$ must agree with $v$ in some row $j \neq i$ and with $w$ in some row $k \neq i, j$. There are $(r - 1)(r - 2)$ such ordered pairs $(j, k)$, each corresponding to exactly one neighbor $u$ of $v$ and $w$. Thus there are $N + r^2 - 3r$ common neighbors of $v$ and $w$.

Finally, suppose that $\Gamma$ is not complete, and that $v$ and $w$ are distinct columns which are not adjacent. A neighbor $u$ of $v$ and $w$ must agree with $v$ in some row $j$ and with $w$ in some row $k \neq j$. There are $r(r - 1)$ such ordered pairs $(j, k)$, each corresponding to exactly one neighbor $u$ of $v$ and $w$. □

**Example 9.3.** The graph $\Gamma$ determined by the orthogonal array $OA = OA(4, 3)$ of Example 9.1 is a complete graph on 9 vertices.

**Example 9.4.** We may partition the orthogonal array $OA$ of Example 9.1 into two smaller orthogonal arrays:

$$OA_1 = \begin{array}{ccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 
\end{array}$$
and
\[ \mathcal{O}_2 = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \end{pmatrix}. \]

We identify the vertices of the graphs \( \Gamma_1 \) and \( \Gamma_2 \) corresponding to \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) with the vertices of the graph \( \Gamma \) corresponding to \( \mathcal{O}_1 \) in the obvious way. The graphs \( \Gamma_1 \) and \( \Gamma_2 \) are both strongly regular with parameters \((9, 4, 1, 2)\). They are of Latin square type with \( N = 3, r = 2 \), and of negative Latin square type with \( N = -3, r = -1 \). They form a strongly regular decomposition of the complete graph on 9 vertices. The negative Latin square type decomposition is feasible. By van Dam’s theorem (see §8.3), this graph decomposition determines an amorphic association scheme.

Let us use the first two entries in each column of \( \mathcal{O}_1 \) to identify the 9 vertices of these graphs with elements of \( \mathbb{GF}(3)^2 \). The neighbors of \((0, 0)\) in \( \Gamma_1 \) form the set
\[ D_1 = \{(0, 1), (0, 2), (1, 0), (2, 0)\}. \]
The neighbors of \((0, 0)\) in \( \Gamma_2 \) form the set
\[ D_2 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}. \]
Now, let us define an even function \( f: \mathbb{GF}(3)^2 \to \mathbb{GF}(3) \) by setting
\[ f(x) = \begin{cases} 0 & \text{if } x = (0, 0), \\ 1 & \text{if } x \in D_1, \\ 2 & \text{if } x \in D_2. \end{cases} \]
The graphs \( \Gamma_1 \) and \( \Gamma_2 \) are component Cayley graphs of \( f \). The third component Cayley graph of \( f \) is empty. The function \( f \) is amorphic, and consequently bent. It can be shown that \( f \) is given by
\[ f(x_0, x_1) = x_0^2 + x_1^2. \]
See Example 10.5 for more properties of this function.

**Example 9.5.** Similarly, if we partition the orthogonal array \( \mathcal{O}_1 \) of Example 9.1 into three arrays, consisting of the first row, the second row, and the last two rows, we obtain sets \( D_1 = \{(0, 1), (0, 2)\}, D_2 = \{(1, 0), (2, 0)\}, \) and \( D_3 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}. \) The corresponding graph decomposition of the complete graph on 9 vertices is a strongly regular decomposition. All
three graphs are of Latin square type. The graphs $\Gamma_1$ and $\Gamma_2$ have parameters $(9, 2, 1, 0)$, and the graph $\Gamma_3$ has parameters $(9, 4, 1, 2)$. The sets $D_1$, $D_2$, and $D_3$ determine the amorphic bent function given by

$$g(x_0, x_1) = -x_0^2 + x_1^2.$$ 

See Example 10.4 for more properties of this function.

9.3 Bent functions from orthogonal arrays

In this section, we show how to construct an amorphic bent $p$-ary function of Latin square type on $2m$ variables, for any prime number $p$ greater than 2 and any positive integer $m$. We start with an orthogonal array of appropriate dimensions, and use a generalization of the procedure in the previous example. The technique for constructing amorphic association schemes from orthogonal arrays is known (see [vDM10, §5]), but we include details in the case of interest for completeness.

Thanks to a construction of Bush [B52], it is known that it is possible to construct an orthogonal array of type $OA(N + 1, N)$ when $N = p^m$, for every prime number $p$. If the entries are in $GF(p^m)$, we may construct our orthogonal array in such a way that all entries in the first column are equal to the 0 element of $GF(p^m)$.

Let $p$ be a prime number greater than 2. Consider a partition of the $N + 1$ rows of an $OA(N + 1, N)$ with entries in $GF(p^m)$ into $p - 1$ sets of $r = \frac{N}{p}$ rows, and one set of $r_p = \frac{N}{p} + 1$ rows. We denote the corresponding orthogonal subarrays by $OA_1, OA_2, \ldots, OA_p$. This partition determines a strongly regular decomposition $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ of the complete graph on $p^{2m}$ vertices, consisting of $p$ strongly regular graphs of Latin square type. (Once again, we identify the vertices of each graph $\Gamma_i$ with the vertices of the complete graph on $GF(p)^{2m}$ determined by the original $OA(N + 1, N)$.) By van Dam’s theorem [vD03] (see §8.3), this graph decomposition determines an amorphic association scheme.

Let $D_i$ be the set of all neighbors of 0 in the graph $\Gamma_i$ corresponding to $OA_i$. By our assumption that the first column of our array consists of 0 entries, a vertex $v$ is in $D_i$ if and only if it has a 0 entry in one of the rows in $OA_i$. Therefore, $D_i$ is symmetric, i.e., if $x \in D_i$ then $-x \in D_i$. Define an
even function \( f: \mathbb{GF}(p)^{2m} \to \mathbb{GF}(p) \) by setting

\[
f(x) = \begin{cases} 
0 & \text{if } x = 0, \\
i & \text{if } x \in D_i \text{ and } 1 \leq i \leq p - 1, \\
0 & \text{if } x \in D_p.
\end{cases}
\]

Then the component Cayley graphs of \( f \) are the strongly regular Latin square type graphs \( \Gamma_i \). By Theorem 6.1 or Theorem 8.5 the function \( f \) is amorphic and bent.

### 10 Examples

In this section we give examples of amorphic bent functions of Latin and negative Latin square type, together with their duals. We also give examples of 5-ary bent functions whose component Cayley graphs are not all strongly regular.

#### 10.1 Examples constructed from orthogonal arrays

In this section we provide three examples of amorphic bent \( p \)-ary functions of Latin square type, which were constructed from orthogonal arrays using a computer. In each case, an orthogonal array of size \((p^2 + 1) \times p^4\) was constructed by the method of Bush, with symbols in \( \mathbb{GF}(p^2) \), which were then replaced by entries from \( \mathbb{GF}(p)^2 \). The algebraic form of the resulting function was found using [CJMPW16, Theorem 53 and Corollary 6].

**Example 10.1.** The following amorphic 3-ary bent function on \( \mathbb{GF}(3)^4 \) was constructed from a \( 10 \times 81 \) orthogonal array, using a computer:

\[
f(x_0, x_1, x_2, x_3) = 2x_0x_3 + x_1x_2 + x_0^2x_1x_2 + 2x_0x_1^2x_3.
\]

The component Cayley graphs \( \Gamma_i \) of \( f \) are all strongly regular of Latin square type. The parameters of \( \Gamma_1 \) and \( \Gamma_2 \) are \((81, 24, 9, 6)\) and the parameters of \( \Gamma_3 \) are \((81, 32, 13, 12)\). The function \( f \) is regular with dual

\[
f^*(x_0, x_1, x_2, x_3) = x_0x_3 + 2x_1x_2 + x_0x_2^2x_3 + 2x_1x_2x_3^2.
\]
Example 10.2. The following amorphic 5-ary bent function on $GF(5)^4$ was constructed from an orthogonal array, using a computer:

$$4x_0^3x_3 + 3x_0^2x_1x_2 + x_0x_1^2x_3 + 3x_1^3x_2 + x_0^4x_1^3x_2 + 3x_1^3x_4x_3$$

The component Cayley graphs $\Gamma_i$ are all strongly regular of Latin square type. The parameters of $\Gamma_i$, for $1 \leq i \leq 4$, are $(625, 120, 35, 20)$ and the parameters of $\Gamma_5$ are $(625, 144, 43, 30)$. The function is regular with dual $2x_1x_2^3 + 4x_0x_2^2x_3 + 2x_1x_2x_3^2 + x_0x_3^3 + 2x_0x_2x_3^3 + 4x_1x_2x_3^4$.

Example 10.3. The following amorphic 7-ary bent function on $GF(7)^4$ was constructed from an orthogonal array, using a computer:

$$6x_0^5x_3 + 4x_0^4x_1x_2 + x_0^3x_1^2x_3 + 6x_0^2x_1^3x_2 + 5x_0x_1^4x_3 + 4x_1^5x_2 + 5x_0^6x_1^5x_2 + 4x_0^5x_1^6x_3$$

The component Cayley graphs $\Gamma_i$ are all strongly regular of Latin square type. The parameters of $\Gamma_i$, for $1 \leq i \leq 6$, are $(2401, 336, 77, 42)$ and the parameters of $\Gamma_7$ are $(2401, 384, 89, 56)$. The function is regular with dual $2x_0x_2^4x_3 + 6x_0x_2^2x_3^3 + x_0x_3^5 + 3x_1x_2^5 + x_1x_2^3x_3^2 + 3x_1x_2x_3^4 + 3x_0x_2x_3^5 + 2x_1x_2^5x_3$.

10.2 Examples on $GF(3)^2$

Every even bent function $f : GF(3)^2 \to GF(3)$ with $f(0) = 0$ is equivalent to the function of Example 10.4 or of Example 10.3 below under the action of $GL(2, GF(3))$ on the variables $(x_0, x_1)$ (see [CJMPW16, Proposition 10]). Example 10.4 is amorphic of Latin square type, and Example 10.3 is amorphic of negative Latin square type.

Example 10.4. The function $f : GF(3)^2 \to GF(3)$ given by

$$f(x_0, x_1) = -x_0^2 + x_1^2$$

is an even bent function with $f(0) = 0$. The component Cayley graphs $\Gamma_1$ and $\Gamma_2$ are strongly regular with parameters $(9, 2, 1, 0)$. The component Cayley graph $\Gamma_3$ is strongly regular with parameters $(9, 4, 1, 2)$. The graphs $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ are all of Latin square type. The function $f$ is amorphic and regular, with dual $f^*(x_0, x_1) = x_0^3 - x_1^2$. 

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Example 10.5. The function $f: \mathbb{GF}(3)^2 \to \mathbb{GF}(3)$ given by

\[ f(x_0, x_1) = x_0^2 + x_1^2 \]

is an even bent function with $f(0) = 0$. The component Cayley graphs $\Gamma_1$ and $\Gamma_2$ are strongly regular with parameters $(9, 4, 1, 2)$. The component Cayley graph $\Gamma_3$ is empty. The graphs $\Gamma_1$ and $\Gamma_2$ are of Latin square type with $N = 3$ and $r = 2$ and of negative Latin square type with $N = -3$ and $r = -1$. However, only the negative Latin square type parameters satisfy the feasibility condition $r = \frac{N}{p}$, where $p = 3$. The function $f$ is amorphic and $(-1)$-weakly regular, with dual $f^*(x_0, x_1) = -x_0^2 - x_1^2$.

10.3 Examples on $\mathbb{GF}(5)^2$

In a previous paper, [CJMPW16, Proposition 14 and Example 65], we classified all even bent functions $g: \mathbb{GF}(5)^2 \to \mathbb{GF}(5)$ with $g(0) = 0$ into eleven equivalence classes under the action of $\text{GL}(2, \mathbb{GF}(5))$ on the variables $(x_0, x_1)$. Using additional computer calculations, it can be shown that the three functions of Example 10.6 represent the only equivalence classes whose functions are of amorphic Latin square type, i.e., those whose component Cayley graphs are all strongly regular of Latin square type. It can also be shown that there is no even bent function $g: \mathbb{GF}(5)^2 \to \mathbb{GF}(5)$ with $g(0) = 0$ whose component Cayley graphs are all strongly regular of feasible negative Latin square type (see Remark 5.1). The remaining examples in this section are not amorphic. In these examples, some or all of the component Cayley graphs are not strongly regular.

Example 10.6. The following three functions $g_i: \mathbb{GF}(5)^2 \to \mathbb{GF}(5)$ are even bent functions with $g_i(0) = 0$:

\[
\begin{align*}
g_1(x_0, x_1) &= x_0^3 x_1 + 2x_1^4, \quad g_2(x_0, x_1) = -x_0 x_1^3 + x_1^4, \quad g_3(x_0, x_1) = -x_0^3 x_1 + x_1^4.
\end{align*}
\]

The component Cayley graphs $\Gamma_i$, for $1 \leq i \leq 4$, are strongly regular with parameters $(25, 4, 3, 0)$. The component Cayley graph $\Gamma_5$ is strongly regular with parameters $(25, 8, 3, 2)$. The graphs $\Gamma_i$, for $1 \leq i \leq 5$, are all of Latin square type. The functions $g_i$ are amorphic and regular, with duals

\[
\begin{align*}
g_1^*(x_0, x_1) &= 2x_0^4 - x_0 x_1^3, \quad g_2^*(x_0, x_1) = x_0^4 + x_0^3 x_1, \quad g_3^*(x_0, x_1) = x_0^4 + x_0 x_1^3.
\end{align*}
\]
Example 10.7. The function \( g: GF(5)^2 \to GF(5) \) given by
\[
g(x_0, x_1) = -x_0^2 + 2x_1^2
\]
is an even bent function with \( g(0) = 0 \). The function \( g \) is (-1)-weakly regular, with dual \( g^*(x_0, x_1) = -x_0^2 + 3x_1^2 \). The degree of \( \Gamma_i \), for \( 1 \leq i \leq 4 \), is \( k_i = 6 \). The graph \( \Gamma_5 \) is empty. It can be shown, by checking the number of distinct eigenvalues of each graph, that the component Cayley graphs \( \Gamma_i \) are not strongly regular. The unions \( \Gamma_1 \cup \Gamma_4 \) and \( \Gamma_2 \cup \Gamma_3 \) are strongly regular of negative Latin square type with parameters \((25, 12, 5, 6)\).

Example 10.8. The function \( g: GF(5)^2 \to GF(5) \) given by
\[
g(x_0, x_1) = -x_0x_1 + x_1^2
\]
is an even bent function with \( g(0) = 0 \). The function \( g \) is regular, with dual \( g^*(x_0, x_1) = x_0^2 + x_0x_1 + 3x_0^2x_1 \). The graph \( \Gamma_5 \) is strongly regular of Latin square type with parameters \((25, 8, 3, 2)\). The degree of \( \Gamma_i \), for \( 1 \leq i \leq 4 \), is \( k_i = 4 \). It can be shown, by checking the number of distinct eigenvalues of each graph, that the component Cayley graphs \( \Gamma_1, \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \) are not strongly regular. The unions \( \Gamma_1 \cup \Gamma_4 \) and \( \Gamma_2 \cup \Gamma_3 \) are strongly regular of Latin square type with parameters \((25, 8, 3, 2)\).

Example 10.9. The function \( g: GF(5)^2 \to GF(5) \) given by
\[
g(x_0, x_1) = 2x_0x_1^3 + x_1^4 - x_1^2
\]
is an even bent function with \( g(0) = 0 \). The function \( g \) is regular, with dual \( g^*(x_0, x_1) = x_0^4 + x_0^4 + 3x_0^3x_1 \). None of the component Cayley graphs \( \Gamma_i \) is strongly regular. Moreover, no union of the component Cayley graphs \( \Gamma_i \cup \Gamma_j \) for \( i \neq j \) (and hence no union of the form \( \Gamma_i \cup \Gamma_j \cup \Gamma_k \) for \( i, j, \) and \( k \) distinct) is strongly regular. The degrees of the component Cayley graphs are \( k_i = 4 \), for \( 1 \leq i \leq 4 \), and \( k_5 = 8 \).

10.4 Examples on \( GF(3)^4 \)
Recall that in Example 10.1 we gave a bent function on \( GF(3)^4 \) whose component Cayley graphs are all strongly regular of Latin square type. We now give two examples of bent functions on \( GF(3)^4 \) whose component Cayley graphs are all strongly regular of negative Latin square type.
Example 10.10. The function \( f : GF(3)^4 \to GF(3) \) given by
\[
f(x_0, x_1, x_2, x_3) = -x_0^2 - x_1^2 + x_2 x_3
\]
is an even bent function with \( f(0) = 0 \). The component Cayley graphs \( \Gamma_1 \) and \( \Gamma_2 \) are strongly regular with parameters \((81, 30, 9, 12)\). The component Cayley graph \( \Gamma_3 \) is strongly regular with parameters \((81, 20, 1, 6)\). The graphs \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \) are all of negative Latin square type. The function \( f \) is amorphic and \((-1)\)-weakly regular, with dual
\[
f^*(x_0, x_1, x_2, x_3) = x_0^2 + x_1^2 - x_2 x_3.
\]
In this example, \( D_1^* = D_2, D_2^* = D_1, \) and \( D_3^* = D_3 \).

Example 10.11. The function \( f : GF(3)^4 \to GF(3) \) given by
\[
f(x_0, x_1, x_2, x_3) = x_0^2 + x_1^2 + x_0 x_2 + 2x_2 x_3
\]
is an even bent function with \( f(0) = 0 \). The component Cayley graphs \( \Gamma_1 \) and \( \Gamma_2 \) are strongly regular with parameters \((81, 30, 9, 12)\). The component Cayley graph \( \Gamma_3 \) is strongly regular with parameters \((81, 20, 1, 6)\). The graphs \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \) are all of negative Latin square type. The function \( f \) is amorphic and \((-1)\)-weakly regular, with dual
\[
f^*(x_0, x_1, x_2, x_3) = 2x_0^2 + 2x_1^2 + x_0 x_3 + x_2 x_3 + 2x_3^2.
\]
In this example, we know of no simple relationship between the sets \( D_1, D_2, \) and \( D_3 \) and the sets \( D_1^*, D_2^*, \) and \( D_3^* \).

10.5 Ideas for further study
We conclude with some questions and ideas for further study.

1. Can we find a way to construct all amorphic bent functions of the type of Theorem 8.5? Can we count them?

2. Consider equivalence classes of \( p \)-ary functions under the action of \( GL(n, GF(p)) \) on coordinates. Do there exist non-equivalent bent functions which determine isomorphic association schemes? Which of our amorphic examples have isomorphic association schemes?

3. Find examples of functions that are not bent, whose level sets determine association schemes that are not amorphic.
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