Arithmetic properties of the Genocchi numbers and their generalization

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Abstract
This note is devoted to establish some new arithmetic properties of the generalized Genocchi numbers $G_{n,a}$ ($n \in \mathbb{N}, a \geq 2$). The resulting properties for the usual Genocchi numbers $G_n = G_{n,2}$ are then derived. We show for example that for any even positive integer $n$, the Genocchi number $G_n$ is a multiple of the odd part of $n$.

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1 Introduction and Notations

Throughout this note, we let $\mathbb{N}^*$ denote the set of positive integers. For a given prime number $p$, we let $\vartheta_p$ denote the usual $p$-adic valuation. For a given rational number $r$, we let $\text{num}(r)$ and $\text{den}(r)$ respectively denote the numerator and the denominator of $r$; precisely, if $d$ is the smallest positive integer satisfying $dr \in \mathbb{Z}$ then we have: $\text{den}(r) := d$ and $\text{num}(r) := dr$. Next, for given positive integers $a$ and $n$, we let $\pi_a(n)$ denote the greatest positive divisor of $n$ which is coprime with $a$. Then, it is immediate that:

$$\pi_a(n) = \prod_{\substack{p \text{ prime} \atop p \nmid a}} p^{\vartheta_p(n)}.$$ 

In the particular case $a = 2$, $\pi_a(n)$ is called the odd part of $n$; it is the greatest odd positive divisor of $n$.

Additionally, we need to extend the congruences in $\mathbb{Z}$ to $\mathbb{Q}$ as follows:
For $a, b \in \mathbb{Q}$ and $n \in \mathbb{N}^*$, we write $a \equiv b \pmod{n}$ if the numerator of the rational number $(a - b)$ is a multiple of $n$.

In the same context, an equivalent meaning of the congruence $a \equiv b \pmod{n}$ consists to say that for any prime divisor $p$ of $n$, we have $\vartheta_p(a - b) \geq \vartheta_p(n)$. We can easily check that some properties (but not all) of the congruences in $\mathbb{Z}$ become valid in $\mathbb{Q}$. For example, we can sum side to side several congruences in $\mathbb{Q}$ with a same modulus. Also, if $a, b, c \in \mathbb{Q}$ and $n \in \mathbb{N}^*$ such that $\text{den}(c)$ is coprime with $n$ then we have:

$$a \equiv b \pmod{n} \implies ac \equiv bc \pmod{n}.$$ 

However, we cannot multiply (side to side) several congruences in $\mathbb{Q}$ with a same modulus (contrary to the congruences in $\mathbb{Z}$).

Further, we denote by $\mathbb{Q}[[t]]$ the ring of formal power series in $t$ with coefficients in $\mathbb{Q}$. If an element $f$ of $\mathbb{Q}[[t]]$ is represented as $f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}$ (where $a_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$), we call the $a_n$’s the differential coefficients of $f$ (because each $a_n$ is the $n$th derivative of $f$ at 0). If the $a_n$’s are all integers, we say that $f$ is an IDC-series (IDC abbreviates the expression “with Integral Differential Coefficients”). Many usual functions are IDC-series; we can cite for example the functions $x \mapsto e^x$, $x \mapsto \sin x$, $x \mapsto \cos x$, $x \mapsto \ln(1 + x)$, and so on. We can easily see that the sum and the product of two IDC-series become an IDC-series. Other important properties of the IDC-series are given by the author [3] who used them to establish his generalization of the Genocchi theorem (see below). One of those properties is given by the following proposition:

**Proposition 1.1** ([3, Corollary 2.2]). Let $f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}$ be an IDC-series with $a_0 \neq 0$. Then the formal power series $\frac{a_0}{f'(a_0 t)}$ is also an IDC-series.

Showing that a given function is an IDC-series is, in general, not easy. For example, the statement that the function $t \mapsto \frac{2t}{e^t + 1}$ is an IDC-series constitute a profound theorem due to Genocchi [5]. The differential coefficients of the last function (which thus are all integers) are called the Genocchi numbers and denoted by $G_n$ ($n \in \mathbb{N}$); so we have:

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{+\infty} G_n \frac{t^n}{n!}.$$ 

For a study of the Genocchi numbers, the reader can consult [1, 2, 3, 4, 5, 8].

Very recently, the author [3] has generalized the Genocchi numbers by considering, for a given integer $a \geq 2$, the function $t \mapsto \frac{at}{e^{(a-1)t} + e^{(a-2)t} + \ldots + e^t + 1}$ and its expansion into a power series:

$$\frac{at}{e^{(a-1)t} + e^{(a-2)t} + \ldots + e^t + 1} = \sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}.$$
So, for $a = 2$, we simply obtain the usual Genocchi numbers; that is $G_{n,2} = G_n$ ($\forall n \in \mathbb{N}$). The main result of [3] then states that the $G_{n,a}$’s are all integers, which generalizes the Genocchi theorem. The Genocchi numbers as well as their generalization are ultimately related to the Bernoulli numbers $(B_n)_{n \in \mathbb{N}}$, which can be defined by the exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{+\infty} B_n \frac{t^n}{n!}$$

(see e.g., [1, 7]). Among others, we have the following connection between the generalized Genocchi numbers and the Bernoulli numbers:

**Proposition 1.2 ([3, Proposition 4.1]).** For any positive integers $a$ and $n$, with $a \geq 2$, we have:

$$G_{n,a} = \sum_{k=0}^{n-1} \binom{n}{k} B_k a^k.$$ 

Furthermore, the Bernoulli numbers have many arithmetic properties. The most famous is certainly that given by the von Staudt-Clausen theorem (see e.g., [6, 7]), recalled below:

**Theorem 1.3** (von Staudt-Clausen). For any even positive integer $n$, we have that:

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbb{Z}.$$ 

Because it is known that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_n = 0$ for any odd integer $n \geq 3$, we can derive from the von Staudt-Clausen theorem the following immediate corollary which is useful for our purpose:

**Corollary 1.4.** For any natural number $n$ and any prime number $p$, we have that:

$$\vartheta_p(B_n) \geq -1.$$ 

The purpose of this note consists to establish arithmetic properties of the numbers $G_{n,a}$ (in addition to their integrality, previously proven in [3]). Precisely, we obtain for the $G_{n,a}$’s a nontrivial simple divisor and a nontrivial congruence modulo $a$.

2 The results and the proofs

We begin by establishing nontrivial divisors for the generalized Genocchi numbers.

**Theorem 2.1.** Let $a \geq 2$ be an integer. Then for all positive integer $n$, the integer $G_{n,a}$ is a multiple of the integer $\pi_a(n)$. 

3
Proof. By applying Proposition 1.1 to the IDC-series $t \mapsto e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1$, we find that the function $t \mapsto \frac{ae^t}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1}$ is an IDC-series. Now, let us explicitly express the differential coefficients of $f$ in terms of the generalized Genocchi numbers. We have:

$$
\frac{a}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} = \frac{1}{t} \cdot \frac{ae^t}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} = \frac{1}{t} \sum_{n=1}^{+\infty} G_{n,a} \frac{n^t}{n!} = \sum_{n=1}^{+\infty} \frac{G_{n,a}}{n} \cdot \frac{t^{n-1}}{(n-1)!}.
$$

Then, by substituting $t$ by $at$, we get

$$
f(t) = \sum_{n=1}^{+\infty} \frac{a^{n-1}G_{n,a}}{n} \cdot \frac{t^{n-1}}{(n-1)!}.
$$

So, since $f$ is an IDC-series, we derive that $\frac{a^{n-1}G_{n,a}}{n} \in \mathbb{Z}$ $(\forall n \in \mathbb{N}^*)$; that is

$$
n \mid \frac{a^{n-1}G_{n,a}}{n} \quad (\forall n \in \mathbb{N}^*).
$$

Next, since $\pi_a(n)$ is a divisor of $n$ which is coprime with $a$, we derive that $\pi_a(n) \mid a^{n-1}G_{n,a}$ and $\pi_a(n)$ is coprime with $a^{n-1}$. Then, by Gauss’s lemma, we conclude that $\pi_a(n)$ divides $G_{n,a}$, as required. \[\square\]

By taking $a = 2$ in Theorem 2.1, we immediately deduce the following corollary which concerns the usual Genocchi numbers:

Corollary 2.2. For all positive integer $n$, the Genocchi number $G_n$ is a multiple of the odd part of $n$. \[\square\]

We now turn to study the remainders of the generalized Genocchi numbers $G_{n,a}$ modulo $a$. The main result in this direction is the following:

Theorem 2.3. For any integers $a$ and $n$, both greater than 1, we have

$$
G_{n,a} \equiv 1 - \frac{n}{2} a \pmod{a}.
$$

Proof. Let $a$ and $n$ be two integers, both greater than 1. According to Proposition 1.2, we have that:

$$
G_{n,a} = \sum_{k=0}^{n-1} \binom{n}{k} B_k a^k = 1 - \frac{n}{2} a + \sum_{k=2}^{n-1} \binom{n}{k} B_k a^k. \quad (2.1)
$$

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$$
Next, by using Corollary 1.4, we have for any $k \in \{2,3,\ldots,n-1\}$ and any prime divisor $p$ of $a$ (so $\vartheta_p(a) \geq 1$):

$$
\vartheta_p(B_k a^k) = \vartheta_p(B_k) + (k-1)\vartheta_p(a) + \vartheta_p(a) \\
\geq -1 + (k-1) + \vartheta_p(a) \\
= k - 2 + \vartheta_p(a) \\
\geq \vartheta_p(a),
$$

showing that $B_k a^k \equiv 0 \pmod{a}$ ($\forall k \in \{2,3,\ldots,n-1\}$). By substituting these congruences into (2.1), we conclude to the required congruence $G_{n,a} \equiv 1 - \frac{2}{a} \pmod{a}$. □

From Theorem 2.3, we immediately derive the following corollary:

**Corollary 2.4.** Let $a \geq 2$ be an integer.

- If $a$ is odd then we have for any positive integer $n$:
  
  $$
  G_{n,a} \equiv 1 \pmod{a}.
  $$

- If $a$ is even then we have for any integer $n \geq 2$:
  
  $$
  G_{n,a} \equiv \begin{cases} 
  1 & \text{if } n \text{ is even} \\
  1 + \frac{a}{2} & \text{if } n \text{ is odd}
  \end{cases} \pmod{a}.
  $$

□

**Remark 2.5.** By applying Corollary 2.4 for $a = 2$, we obtain the well-known result stating that the usual Genocchi numbers of even positive orders are all odd.

We conclude this note by the following corollary which is an immediate consequence of Corollary 2.4 above.

**Corollary 2.6.** For any integers $a$ and $n$, both greater than 1, we have:

$$
\gcd(G_{n,a}, a) \in \{1,2\}.
$$

Besides, the equality $\gcd(G_{n,a}, a) = 2$ holds if and only if $a \equiv 2 \pmod{4}$ and $n$ is odd. □

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