Similarity reduction of a three-dimensional model of the far turbulent wake behind a towed body

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Abstract. Semi-empirical three-dimensional model of turbulence in the approximation of the far turbulent wake behind a towed body in a passively stratified medium is considered. The sought-for quantities of the model are the velocity defect, kinetic turbulent energy, kinetic energy dissipation rate, averaged density defect and density fluctuation variance. The full group of transformations admitted by this model is found. The governing equations are reduced into ordinary differential equations by similarity reduction and method of the B-determining equations (BDE method). System of ordinary differential equations was solved numerically. The obtained solutions agree with experimental data.

1. Introduction

One of the examples of a three-dimensional free turbulent flow is a turbulent wake behind an axisymmetric body in a stratified medium. Sufficiently complete experimental data on the dynamics of a turbulent wake behind a towed body of revolution in a linearly stratified medium were obtained by Lin and Pao and presented in (Lin & Pao, 1979).

The turbulent wake behind an axisymmetric towed body in a linearly stratified medium was numerically simulated in (Hassid, 1980). Based on hierarchy of semi-empirical turbulence models of the second order, the numerical simulation of the dynamics of a turbulent wake in a stable stratified medium was carried out in (Chernykh & Fomina & Moshkin, 2006). A satisfactory agreement with experimental data (Lin & Pao, 1979) was obtained in (Hassid, 1980) and (Chernykh & Fomina & Moshkin, 2006).

The series of paper (Kaptsov & Efremov, 2005), (Kaptsov & Shan’ko, 2006), (Kaptsov & Efremov & Schmidt, 2008) and (Kaptsov & Efremov & Chernykh, 2009) was devoted to construction of similarity solutions for some semi-empirical turbulence models. In this paper is considered three-dimensional semi-empirical model of the far turbulent wake behind an axisymmetric towed body in a passively stratified medium (see (Chernykh & Fomina & Moshkin, 2006), (Chashechkin & Chernykh & Voropaeva, 2005) and (Voropaeva & Chernykh, 1992)).

In section 2 is defined the most general symmetry group of a given model (see (Ovsyannikov, 1982), (Olver, 1986)), which will allow us to pass to a system of degenerate elliptic equations. In section 3 we focus on the second-order solutions of the BDE (Andreev & Kaptsov & Pukhnachov & Rodionov, 1998). Obtained differential constraints will allow us to pass to a system of ordinary differential equations. In section 4 we present and discuss calculation results.
2. Similarity reduction

To calculate the characteristics of the far turbulent wake behind an axisymmetric towed body in a passively stratified medium we use the three-dimensional semi-empirical model of turbulence (see Kolmogorov, 1942), (Wilcox, 1994), (Jones & Launder, 1972), (Chernykh & Fomina & Moshkin, 2006), (Chashechkin & Chernykh & Voropaeva, 2005), (Voropaeva & Chernykh, 1992) and references therein.

\[ U_0 \frac{\partial u}{\partial x} = \frac{\partial}{\partial y} C_\rho \frac{e^2}{\epsilon} \frac{\partial u}{\partial y} + \frac{\partial}{\partial z} C_e \frac{e^2}{\epsilon} \frac{\partial u}{\partial z}, \]

(1)

\[ U_0 \frac{\partial e}{\partial x} = \frac{\partial}{\partial y} C_e \frac{e^2}{\epsilon} \frac{\partial e}{\partial y} + \frac{\partial}{\partial z} C_e \frac{e^2}{\epsilon} \frac{\partial e}{\partial z} + C_e \frac{\epsilon}{\epsilon} \left( \frac{\partial u}{\partial y} \right)^2 + C_e \frac{\epsilon}{\epsilon} \left( \frac{\partial u}{\partial z} \right)^2 - \epsilon, \]

(2)

\[ U_0 \frac{\partial e}{\partial x} = \frac{\partial}{\partial y} C_e \frac{e^2}{\epsilon} \frac{\partial e}{\partial y} + \frac{\partial}{\partial z} C_e \frac{e^2}{\epsilon} \frac{\partial e}{\partial z} + C_e \frac{\epsilon}{\epsilon} \frac{\partial u}{\partial y} + C_e \frac{\epsilon}{\epsilon} \frac{\partial u}{\partial z} - C_e \frac{e^2}{\epsilon}, \]

(3)

\[ U_0 \frac{\partial \langle \rho^1 \rangle}{\partial x} = \frac{\partial}{\partial y} C_\rho \frac{e^2}{\epsilon} \frac{\partial \langle \rho^1 \rangle}{\partial y} + \frac{\partial}{\partial z} C_\rho \frac{e^2}{\epsilon} \frac{\partial \langle \rho^1 \rangle}{\partial z} - \frac{\partial}{\partial z} C_\rho \frac{e^2}{\epsilon}, \]

(4)

\[ U_0 \frac{\partial \langle \rho^2 \rangle}{\partial x} = \frac{\partial}{\partial y} C_\rho \frac{e^2}{\epsilon} \frac{\partial \langle \rho^2 \rangle}{\partial y} + \frac{\partial}{\partial z} C_\rho \frac{e^2}{\epsilon} \frac{\partial \langle \rho^2 \rangle}{\partial z} + 2C_\rho \frac{e^2}{\epsilon} \frac{\partial \langle \rho^1 \rangle}{\partial y}, \]

\[ + 2C_\rho \frac{e^2}{\epsilon} \left( \frac{\partial \langle \rho^1 \rangle}{\partial z} - 1 \right)^2 - C_T \frac{\langle \rho^2 \rangle}{\epsilon}. \]

(5)

In these equations $U_0$ is the velocity of an incoming undisturbed flow, $u(x, y, z)$ is the velocity defect, $e(x, y, z)$ is the turbulent kinetic energy, $\epsilon(x, y, z)$ is the kinetic energy dissipation rate, $\langle \rho^1 \rangle(x, y, z)$ is the averaged density defect, and $\langle \rho^2 \rangle(x, y, z)$ is the density fluctuation variance. The quantities $C_e = 0.136$, $\sigma = 1.3$, $C_{\rho 1} = 1.44$, $C_{\rho 2} = 1.92$, $C_\rho = 0.208$, $C_{1\rho} = 0.087$, $C_T = 1.25$ are generally accepted empirical constants (Gibson & Launder, 1976), (Rodi, 1987). This model based on the three-dimensional parabolized system of averaged Navier-Stokes equations in the Oberbeck-Boussinesq approximation (Chernykh & Fomina & Moshkin, 2006), (Chashechkin & Chernykh & Voropaeva, 2005) and the algebraic relations (Rodi, 1987).

In what follows, we assume that the velocity of an incoming undisturbed flow equals unity. The marching variable $x$ in equations (1)–(5) acts as time.

Lie algebra of the model (1)–(5) is spanned by nine vector fields

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial u}, \quad X_5 = \frac{\partial}{\partial \langle \rho^1 \rangle}, \]

\[ X_6 = -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} + y \frac{\partial}{\partial \langle \rho^1 \rangle}, \]

\[ X_7 = y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} + 2e \frac{\partial}{\partial e} + 2\epsilon \frac{\partial}{\partial \epsilon} + \langle \rho^1 \rangle \frac{\partial}{\partial \langle \rho^1 \rangle} + 2\langle \rho^2 \rangle \frac{\partial}{\partial \langle \rho^2 \rangle}, \]

\[ X_8 = x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - 2e \frac{\partial}{\partial e} - 3\epsilon \frac{\partial}{\partial \epsilon}, \quad X_9 = (\langle \rho^1 \rangle - z) \frac{\partial}{\partial \langle \rho^1 \rangle} + 2\langle \rho^2 \rangle \frac{\partial}{\partial \langle \rho^2 \rangle}. \]

Next consider the linear combination of scaling vector fields $X_7$ and $X_8$

\[ Z = x \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} + \alpha z \frac{\partial}{\partial z} + (\alpha - 1)u \frac{\partial}{\partial u} + 2(\alpha - 1)e \frac{\partial}{\partial e} \]

\[ + (2\alpha - 3)\epsilon \frac{\partial}{\partial \epsilon} + \alpha \langle \rho^1 \rangle \frac{\partial}{\partial \langle \rho^1 \rangle} + 2\alpha \langle \rho^2 \rangle \frac{\partial}{\partial \langle \rho^2 \rangle}. \]
The solution of the model (1)–(5) invariant with respect to operator Z has the form of
\[ u = x^{-1} U(\xi, \eta), \epsilon = x^{-3} E(\xi, \eta), \langle \rho_1 \rangle = x^3 H(\xi, \eta), \langle \rho_2 \rangle = x^2 R(\xi, \eta), \]
where \( \xi = y/x^3 \) and \( \eta = z/x^3 \) are the similarity variables; \( U, E, G, H, \) and \( R \) are arbitrary functions. The numerical analysis of degeneration of the far turbulent wake in a passively stratified medium shows that the functions \( U, E \) and \( G \) must be presented in the form (see [Chernykh & Fomina & Moshkin, 2006], [Chashechkin & Chernykh & Voropaeva, 2005] and [Voropaeva & Chernykh, 1992])
\[ U(\xi, \eta) = U\left(\sqrt{\xi^2 + \eta^2}\right), \quad E(\xi, \eta) = E\left(\sqrt{\xi^2 + \eta^2}\right), \quad G(\xi, \eta) = G\left(\sqrt{\xi^2 + \eta^2}\right). \tag{7} \]
Substituting (6) into equations (1)–(5), we obtain the reduced system. Changing to polar coordinates \( \xi = r \cos(\phi), \eta = r \sin(\phi) \), and by virtue of (7) the reduced system becomes
\[ \frac{C_e E^2}{G} U'' + \frac{C_e}{G} \left(2E' - \frac{EG'}{G} + \frac{E}{r}\right) + \alpha r U' + (1 - \alpha) U = 0, \tag{8} \]
\[ \frac{C_e E^2}{G} E'' + \frac{C_e E'}{G} \left(2E' - \frac{EG'}{G} + \frac{E}{r}\right) + C_e E^2 U'' + \alpha E' + 2(1 - \alpha) E - G = 0, \tag{9} \]
\[ \frac{C_e E^2}{G} \frac{G''}{G} + \frac{C_e E'}{G} \left(2E' - \frac{EG'}{G} + \frac{E}{r}\right) + C_e C_4 U E' + \alpha G' - C_4 E = \frac{G^2}{E}, \tag{10} \]
\[ \frac{C_e^2}{G} \left(H_{rr} + \frac{H_{\phi\phi}}{r^2}\right) + \frac{C_e E H_r}{G} \left(2E' - \frac{EG'}{G} + \frac{E}{r}\right) + \alpha H_r + \alpha H \]
\[ + C_\rho \sin \phi \frac{E}{G} \left(\frac{EG'}{G} - 2E'\right) = 0, \tag{11} \]
\[ \frac{C_\rho^2 E^2}{G} \left(R_{rr} + \frac{R_{\phi\phi}}{r^2}\right) + \frac{C_\rho^2 E R_r}{G} \left(2E' - \frac{EG'}{G} + \frac{E}{r}\right) + \alpha R_r - \left(C_\tau \frac{G}{E} + 2\alpha\right) R \]
\[ + 2C_\rho^2 E^2 \left(H_r^2 + \frac{H_{\phi\phi}}{r^2} - 2 \sin \phi H_r - 2 \cos \phi \frac{H_{\phi}}{r} + 1\right) = 0, \tag{12} \]
where \( U = U(r), E = E(r), G = G(r), H = H(r, \phi), R = R(r, \phi) \). Here and elsewhere, subscripts denote partial derivatives, so \( H_r = \partial H/\partial r \), etc.

3. BDE method

We now apply the BDEs method to reduce the last two equations of (8)–(12) to some ordinary differential equations. The concept of BDEs that generalize the defining equations of the symmetry groups was introduced in [Kaptsov, 1995] and [Andreev & Kaptsov & Pukhnachov & Rodionov, 1998]. Consider more general equation than (11)
\[ H_{\phi\phi} + r^2 H_{rr} + A(r) H_r + B(r) H + C(r) \sin(\phi) = 0, \tag{13} \]
where \( A(r), B(r), C(r) \) are arbitrary functions. We take BDE corresponding to (13) of the form
\[ D_\phi^2 h + r^2 D_r^2 h + b_1(r, \phi) D_r h + b_2(r, \phi) h = 0. \tag{14} \]
Here and throughout \( D_\phi, D_r \) are the operators of total differentiation with respect to \( \phi \) and \( r \). The functions \( b_1(r, \phi) \) and \( b_2(r, \phi) \) are to be determined together with the function \( h \). Note that for the classical defining equations holds
\[ b_1(r, \phi) = A(r), \quad b_2(r, \phi) = B(r). \]
We seek second–order solution of (14) in the form
\[ h = H_{\phi\phi} + h_1(\phi, H, H_{\phi}) . \]  

Substituting (15) into BDE (14) leads to an equation which includes derivatives of the fourth order. We can express the derivatives \( H_{rr\phi\phi}, H_{\phi\phi\phi\phi}, H_{rr\phi\phi} \) and \( H_{\phi\phi} \) by using (13). Setting the coefficient of \( H_{rr} \) equal to zero we obtain \( b_1(r, \phi) = A(r) \).

The left-hand side of (14) is a polynomial with respect to \( H_{rr} \) and \( H_{r\phi} \). This polynomial must identically vanish. Collecting similar terms we obtain the equation
\[ 2 \left( A(r) H_r + B(r) H + C(\phi) \sin(\phi) \right) h_1 H_{\phi\phi} - 2H_{\phi} h_1 H_{2\phi} - 2h_{1\phi} H_{\phi} + B(r) - b_2(r, \phi) = 0, \]  

\[ h_{1\phi\phi\phi} = 0, h_{1\phi\phi} = 0. \]  

It is easy to show that the general solution of the equations (16), (17) is
\[ h_1(\phi, H, H_{\phi}) = h_2(\phi) H_{\phi} + h_3(\phi, H), \quad b_2(r, \phi) = B(r) - 2h'_2(\phi). \]

Substituting the functions \( b_1, b_2 \) and \( h_1 \) into BDE (14) we obtain that the left-hand side of (14) is a polynomial with respect to \( H_r \) and \( H_{\phi} \). This polynomial must identically vanish. Collecting similar terms leads to the following equations
\[ (B(r) H + C(\phi) \sin(\phi)) h_3_{HH} + (2h'_2(\phi) - B(r)) h_3 + C(\phi) (\cos(\phi) h_2(\phi) - \sin(\phi)) - h_{3\phi\phi} = 0, \]

\[ h_{3HH} = 0, 2h_{3\phi H} + h'_2(\phi) - 2h'_2(\phi) h_2(\phi) = 0. \]

The equations (17), (18) imply
\[ h_3(\phi, H) = ((h_2(\phi)^2 - h'_2(\phi))/2 + h_4) H, \]
\[ h'_2(\phi) - h'^2_2(\phi) - 2\cot(\phi) h_2(\phi) + 2(1 - h_4) = 0. \]

here \( h_4 \) is arbitrary constant.

Clearly, that the Riccati equation (19) has the partial solution
\[ h_2(\phi) = \tan(\phi) \]

for \( h_4 = 1/2 \).

Thus we find the second–order solution of the BDE (14)
\[ h = H_{\phi\phi} + \tan(\phi) H_{\phi}. \]

The corresponding differential constraint \( h = 0 \) has the general solution
\[ H = H_1(r) \sin(\phi) + H_2(r), \]  

where \( H_1 \) and \( H_2 \) are arbitrary functions.

Substitution (20) into (12) gives
\[ C_1 \rho \frac{E^2}{G} \left( R_{rr} + \frac{1}{r^2} R_{\phi\phi} \right) + \left( C_1 \frac{E}{G} \left( \frac{E}{r} + 2E' - \frac{E}{G} G' \right) + \alpha r \right) R_r - \left( C_T \frac{E}{G} + 2\alpha \right) R \]
\[ + 2C_1 \frac{E^2}{r^2 G} \left( (r H'_1 - H_1)(r H'_1 + H_1 - 2r) \sin^2(\phi) + 2r^2 (H'_1 - 1) H'_2 \sin(\phi) \right) + r^2 H'^2_2 + (H_1 - r)^2 = 0. \]  

(21)
By analogy with the case of the equation (11), consider more general equation than (21)

\[ R_{\phi\phi} + r^2 R_{rr} + K(r) R_r + L(r) R + M(r) \sin^2(\phi) + N(r) \sin(\phi) + P(r) = 0, \]  

(22)

where \( K(r), L(r), M(r), N(r) \) and \( P(r) \) are arbitrary functions. The BDE method applied to equation (22) gives rise to the following particular results:

\[ b_1(r, \phi) = K(r), b_2(r, \phi) = L(r) - 8 \sin^{-2}(2\phi), \]

\[ h = R_{\phi\phi} - 2 \cot(2\phi) R_\phi, \]

\[ N(r) = 0. \]

(23)

(24)

The formula (24) for the equation (21) takes the form

\[ (H_1' - 1)H_2' = 0. \]

Clearly, that we must explain the case

\[ H_2' = 0. \]

(25)

Integrating differential constraint \( h = 0 \) corresponding to the BDE solution (23), we find

\[ R = R_1(r) \sin^2(\phi) + R_2(r), \]

(26)

where \( R_1(r) \) and \( R_2(r) \) are arbitrary functions.

Thus in the similarity variables \( \xi \) and \( \eta \) from (20), (25) and (26) we have

\[ H(\xi, \eta) = H_3 \left( \sqrt{\xi^2 + \eta^2} \right) \eta + H_2, \]

(27)

\[ R(\xi, \eta) = R_3 \left( \sqrt{\xi^2 + \eta^2} \right) \eta^2 + R_2 \left( \sqrt{\xi^2 + \eta^2} \right), \]

(28)

where \( H_3 = H_1/\sqrt{\xi^2 + \eta^2} \) and \( R_3 = R_1/(\xi^2 + \eta^2) \).

It allows us to reduce the model (1)–(5) to the system of ordinary differential equations. Substituting equations (7), (27) and (28) into the reduced system we obtain \( H_2 = 0 \) and

\[ C_e \frac{E^2 U''}{G} + C_e \left( 2 \frac{E E'}{G} - \frac{E^2 G'}{G^2} + \frac{E^2}{\tau G} \right) U' + \alpha \tau U'' - (\alpha - 1)U = 0, \]

(29)

\[ C_e \frac{E^2 E''}{G} + C_e \left( \frac{2 E E' G'}{G^2} + \frac{E^2}{\tau G} \right) E' + \alpha \tau E'' - 2(\alpha - 1)E + G + C_e \frac{E^2 U'^2}{G} = 0, \]

(30)

\[ \frac{C_e E^2 G''}{\sigma} - \frac{C_e}{\sigma} \left( 2 \frac{E E'}{G} - \frac{E^2 G'}{G^2} + \frac{E^2}{\tau G} \right) G' + \alpha \tau G'' - (2\alpha - 3)G + \frac{C_e G^2}{E} + C_e C_1 E U'^2 = 0, \]

(31)

\[ C_\rho \frac{E^2 H_3''}{G} + C_\rho \left( \frac{2 E E'}{G} - \frac{E^2 G'}{G^2} + 3 \frac{E^2}{\tau G} \right) H_3' + \alpha \tau H_3' + C_\rho \frac{H_3 - 1}{\tau} \left( \frac{2 E E'}{G} - \frac{E^2 G'}{G^2} \right) = 0, \]

(32)

\[ C_{1\rho} \frac{E^2 R_3''}{G} + C_{1\rho} \left( \frac{2 E E'}{G} - \frac{E^2 G'}{G^2} + 5 \frac{E^2}{\tau G} \right) R_3' + \alpha \tau R_3' + \frac{2 C_{1\rho}}{\tau} \left( \frac{2 E E'}{G} - \frac{E^2 G'}{G^2} \right) + \frac{C_T \tau}{2 C_{1\rho} E} \right) R_3 + 2 C_\rho \left( \frac{2(H_3 - 1)}{\tau} + H_3 \right) H_3' = 0, \]

(33)

\[ C_{1\rho} \frac{E^2 R_4''}{G} + C_{1\rho} \left( 2 \frac{E E'}{G} - \frac{E^2 G'}{G^2} + \frac{E^2}{\tau G} \right) R_4' + \alpha \tau R_4' - \left( \frac{C_T G}{E} + 2\alpha \right) R_4 + 2 C_{1\rho} \frac{E^2 R_3}{G} + 2 C_\rho E^2 (H_3 - 1)^2 = 0, \]

(34)

where \( \tau = \sqrt{\xi^2 + \eta^2} \).
4. Calculation results

System (29)–(34) has to satisfy the conditions

\[ \begin{align*}
U' &= E' = G' = H'_3 = R'_3 = R'_4 = 0, \tau = 0, \\
U &= E = G = H_3 = R_3 = R_4 = 0, \tau \to \infty.
\end{align*} \tag{35} \tag{36} \]

Conditions (35) takes into account flow symmetry with respect to the OX axis. The boundary conditions (36) imply that all functions take zero values outside the turbulent wake.

The system (29)–(34) of ordinary differential equations satisfying boundary conditions (35) and (36) was solved numerically. Additional difficulties are caused by the fact that the coefficients of ordinary differential equations have singularities. The problem was solved by a modified shooting method and asymptotical expansion of the solution in the vicinity of the singular point (Kaptsov & Shan’ko, 2006).

![Figure 1. Profile of the function \( U/U_0 \).](image1)

![Figure 2. Profile of the function \( E/E_0 \).](image2)

![Figure 3. Profile of the function \( G/G_0 \).](image3)

![Figure 4. Profile of the function \( H \).](image4)

It has been shown (Kaptsov & Efremov, 2005) that the value of \( \alpha \) equals 1/3. The results for the problem solution are illustrated in Figs. 1–10. Figs. 1–5 show the profiles of the functions \( U/U_0, E/E_0, G/G_0, H \) and \( R/R_0 \) as \( \xi = 0 \), where subscript 0 denote axial value of the function. The functions \( U/U_0, E/E_0, G/G_0, H \) and \( R/R_0 \) are plotted in Figs. 6–10.
The function $H(0, \eta)$ characterizing the degree of fluid mixing in the turbulent wake given in Fig. 4. As can be seen, the maximum value of this function slightly differs from 0.34, which is consistent with the present notions of incomplete fluid mixing in the wakes.

In Figs. 11, 12 the axial values of the velocity defect and the turbulent energy are compared with Lin and Pao's experimental data (Lin & Pao, 1979), Hassid's computational results (Hassid, 1980) and results of numerical computations (Chernykh & Fomina & Moshkin, 2006).

We have borrowed this figures from the work (Chernykh & Fomina & Moshkin, 2006) and have put the values. We can see satisfactory agreement with Lin and Pao's experimental data here as well.

The main results of the paper are as follows. Three-dimensional semi-empirical turbulence model of the far turbulent wake behind an axisymmetric towed body in a passively stratified medium was reduced to the system of ordinary differential equations due to similarity reduction and BDE method. System of ordinary differential equations satisfying natural boundary conditions was solved numerically. The constructed solutions agree with experimental data. The considered approach can be applied to other semi-empirical models of turbulence.
Figure 8. Calculated function $G/G_0$.

Figure 9. Calculated function $H$.

Figure 10. Calculated function $R/R_0$.

Figure 11. Axial values of the velocity defect.

Figure 12. Axial values of the turbulent energy.

Acknowledgments

The author is grateful to Professor G.G. Chernykh and Professor O.V. Kaptsov for the problem statement and for useful discussions. This work was supported by the Russian Foundation for...
Basic Research (grant No. 10-01-00435) and Siberian Branch of Russian Academy of Sciences (grant No. 103).

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