ON FILTERED MULTIPLICATIVE BASES OF SOME ASSOCIATIVE ALGEBRAS

V. BOVDI, A. GRISHKOV, S. SICILIANO

Abstract. We deal with the existing problem of filtered multiplicative bases of finite-dimensional associative algebras. For an associative algebra \( A \) over a field, we investigate when the property of having a filtered multiplicative basis is hereditated by homomorphic images or by the associated graded algebra of \( A \). These results are then applied to some classes of group algebras and restricted enveloping algebras.

1. Introduction

Let \( A \) be an associative algebra over a field \( F \) and denote by \( \mathfrak{J}(A) \) the Jacobson radical of \( A \). An \( F \)-basis \( \mathcal{B} \) of \( A \) is called multiplicative if \( \mathcal{B} \cup \{0\} \) is a semigroup under the product of \( A \). If one also has that \( \mathcal{B} \cap \mathfrak{J}(A) \) is an \( F \)-basis of \( \mathfrak{J}(A) \), then \( \mathcal{B} \) is said to be a filtered multiplicative basis (shortly, f.m.b.) of \( A \). Filtered multiplicative bases arise in the representation theory of associative algebras and were introduced by H. Kupisch in [9].

In [3], R. Bautista, P. Gabriel, A. Roiter and L. Salmeron proved that if a finite-dimensional associative algebra \( A \) has finite representation type over an algebraically closed field \( F \), then \( A \) has an f.m.b. This implies that the number of isomorphism classes of algebras of finite representation type of a given dimension is finite and reduces the classification of these algebras to a combinatorial problem. In the same paper [3] it was asked when a group algebra has an f.m.b. and such a problem (not necessarily for group algebras) has been subsequently considered by several authors: see e.g. [1], [2], [5], [6], [8], [10], [15], [18]. In particular, it is still an open problem whether a group algebra \( FG \) has an f.m.b. in the case when \( F \) is a field of odd characteristic \( p \) and \( G \) is a nonabelian \( p \)-group (see [11], Question 5).

Moreover, in [7] the same problem was investigated in the setting of restricted enveloping algebras \( \mathfrak{u}(L) \), where \( L \) is in the class \( \mathfrak{F}_p \) of finite-dimensional and \( p \)-nilpotent restricted Lie algebras over a field of positive characteristic \( p \). In particular, we characterized commutative restricted enveloping algebra having an f.m.b., and showed that if \( L \) has nilpotency class 2 and \( p > 2 \) then \( \mathfrak{u}(L) \) does not have any f.m.b.

The aim of the present paper is to provide some further contribution on the problem of existence of an f.m.b. for an associative algebra. First, we deal with the conditions under which the property of having a multiplicative basis is inherited.
by homomorphic images. This result is then used to establish when a restricted 
enveloping algebra $u(L)$ has an f.m.b., where $L \in \mathfrak{F}_2$ has nilpotency class 2 over a 
field of characteristic 2, thereby complementing the previous results in [7]. Next, 
we show that if a finite-dimensional associative algebra $A$ admits an f.m.b., then 
so does its graded algebra associated to the filtration given by the powers of the 
Jacobson radical. The combination of such a result with [7] allows to conclude that 
if $F$ is a field of odd characteristic $p$ and $G$ is a finite $p$-group of nilpotency class 2, 
then the group algebra $FG$ has no f.m.b., which provides a partial answer to the 
question 5 in [11].

In the sequel we will use freely the notation and results from the books [4, 16].

2. Preliminaries

Let $A$ be a finite-dimensional associative algebra over a field $F$ having an f.m.b. 
$\mathfrak{b}_s(A)$. Then the following simple properties hold (see [5]):

(F1) $\mathfrak{b}_s(A) \cap \mathfrak{j}^n(A)$ is an $F$-basis of $\mathfrak{j}^n(A)$ for every $n \geq 1$;

(F2) if $u, v \in \mathfrak{b}_s(A) \setminus \mathfrak{j}^k(A)$ and $u \equiv v \pmod{\mathfrak{j}^k(A)}$ then $u = v$;

(F3) if another $F$-algebra $B$ admits an f.m.b. then so does $A \otimes_F B$.

We denote by $A^-$ the restricted Lie algebra associated to $A$ via the Lie bracket 
$[x, y] = xy - yx$ for every $x, y \in A$ and $p$-map given by ordinary $p$-exponentiation. 
For a subset $S$ of $A$ we denote by $\langle S \rangle$ and $\langle S \rangle_F$, respectively, the associative sub-
algabracia and the $F$-vector subspace spanned by $S$.

Let $L$ be a restricted Lie algebra over a field $F$ of positive characteristic $p$ with 
a $p$-map $[p]$. We denote by $\omega(L)$ the augmentation ideal of $u(L)$, that is, the 
associative ideal generated by $L$ in $u(L)$. The restricted ideals of $L$ given by 
$$
D_m(L) = L \cap \omega^m(L), \quad (m \geq 1)
$$
are called the dimension subalgebras of $L$ (see [14]). Similarly to the dimension 
subgroups (in the context of modular group algebras), these subalgebras can be 
explicitly described as $D_m(L) = \sum_{p^i \geq m} \gamma_i(L)[p]^i$, where $\gamma_i(L)[p]^i$ is the 
restricted subalgebra of $L$ generated by the set of $p^i$th powers of the ith term of the lower central 
series of $L$. The center of $L$ will be denoted by $Z(L)$. For a subset $S$ of $L$ we 
will denote by $\langle S \rangle_p$ the restricted subalgebra generated by $S$. A restricted Lie 
algabra $H$ is said to be nilcyclic if $H = \langle x \rangle_p$ for some $p$-nilpotent element $x$ of $H$.

It is well-known that if $L$ is finite-dimensional and $p$-nilpotent then $\omega(L)$ is 
nilpotent (see [10], Corollary 3.7 of Chapter 1). Clearly, in this case $\omega(L)$ coincides 
with $\mathfrak{j}(u(L))$ and $u(L) = F \cdot 1 \oplus \omega(L)$, so that $u(L)$ is a local basic $F$-algabra. In this 
algabra, if $u(L)$ has an f.m.b. $\mathfrak{b}_s(u(L))$, then we can assume without loss of generality that $1 \in \mathfrak{b}_s(u(L))$. For each $x \in L$, the largest subscript $m$ such that $x \in D_m(L)$ is 
called the height of $x$ and denoted by $\nu(x)$. The combination of Theorem 2.1 and 
Theorem 2.3 from [14] yields the following.

Lemma 1. Let $L \in \mathfrak{F}_p$ be a restricted Lie algabra over a field $F$, and let $\{x_i\}_{i \in I}$ 
be an ordered basis of $L$ chosen such that 
$$
D_m(L) = \langle x_i \mid \nu(x_i) \geq m \rangle_F, \quad (m \geq 1).
$$

Then for each positive integer $n$ the following statements hold:

(i) $\omega(L)^n = \langle x \mid \nu(x) \geq n \rangle_F$, where $x = x_1^{\alpha_1} \cdots x_l^{\alpha_l}$,
$$
\nu(x) = \sum_{j=1}^l \alpha_j \nu(x_{i_j}), \quad i_1 < \cdots < i_l \quad \text{and} \quad 0 \leq \alpha_j \leq p - 1.
$$
The set \{ y \mid \nu(y) = n \} is an \( F \)-basis of \( \omega(L)^n \) modulo \( \omega(L)^{n+1} \).

If \( S \) is a subset of a \( p \)-nilpotent restricted Lie algebra then the minimal positive integer \( n \) such that \( z^{[p]}^n = 0 \) for every \( z \in S \) is called the exponent of \( S \) and denoted by \( e(S) \).

3. Homomorphic images and restricted enveloping algebras

Let \( A \) be an associative algebra over a field \( F \) and let \( \mathfrak{bs}(A) \) be an \( F \)-basis of \( A \). A subset \( P \subset A \) is called \( \mathfrak{bs}(A) \)-regular if for every \( x \in P \) one has that either \( x \in \mathfrak{bs}(A) \) or \( x = a - b \) for some \( a, b \in \mathfrak{bs}(A) \).

**Theorem 1.** Let \( A = \langle g_1, \ldots, g_m \rangle \) be a finitely generated associative algebra over a field \( F \) and let \( \mathfrak{bs}(A) = \{ a_i \mid i \in I \} \) be a multiplicative basis of \( A \) such that \( \{ g_1, \ldots, g_m \} \subseteq \mathfrak{bs}(A) \). Let \( \psi : A \rightarrow B \) be a surjective homomorphism of \( A \) onto an \( F \)-algebra \( B \) and \( H = \{ \psi(g_i) \mid i = 1, \ldots, m \} \). Then the following statements hold:

(i) If there exists \( J \subseteq I \) such that \( \mathfrak{bs}(B) = \{ b_i = \psi(a_i) \mid i \in J \} \) is a multiplicative basis of \( B \) containing \( H \), then \( \mathfrak{RET}(\psi) \) has a \( \mathfrak{bs}(A) \)-regular \( F \)-basis.

(ii) If \( \mathfrak{RET}(\psi) \) has a \( \mathfrak{bs}(A) \)-regular \( F \)-basis, then there exists \( J \subseteq I \) such that \( \mathfrak{bs}(B) = \{ b_i = \psi(a_i) \mid i \in J \} \) is a multiplicative basis of \( B \).

**Proof.** (i) For every \( i, j \in I \) we have either \( b_i b_j = 0 \) or \( b_i b_j = b_k \) for some \( k \in J \). Denote by \( K \) the \( F \)-vector space spanned by the set

\[ Z = \{ a_i \in \mathfrak{bs}(A) \mid \psi(a_i) = 0 \} \cup \{ a_j - a_k \mid \psi(a_j) = \psi(a_k), \ a_j, a_k \in \mathfrak{bs}(A) \} . \]

Clearly \( K \subset \mathfrak{RET}(\psi) \). Let us prove that \( K = \mathfrak{RET}(\psi) \).

Let \( v = \sum_{i \in X} \alpha_i a_i \in \mathfrak{RET}(\psi) \setminus K \), such that \( \alpha_i \neq 0 \) for all \( i \) in the finite subset \( X \subseteq I \). Let us choose the element \( v \) such that the cardinality \( |X \setminus J| \) is minimal. If \( X \subseteq J \) then \( \{ b_i = \psi(a_i) \mid i \in X \} \) is an \( F \)-linear dependent subset of the algebra \( B \), a contradiction. Hence there exists \( i \in X \setminus J \) and \( a_i = w(g_1, \ldots, g_m) \), where \( w(x_1, \ldots, x_m) \) is a monomial in the free associative algebra \( F\langle x_1, \ldots, x_m \rangle \). It follows that

\[ b_i = \psi(a_i) = w(\psi(g_1), \ldots, \psi(g_m)) . \]

Since \( \mathfrak{bs}(B) = \{ b_i \mid i \in J \} \) is a multiplicative basis and \( \psi(g_1), \ldots, \psi(g_m) \in \mathfrak{bs}(B) \), we get \( w(\psi(g_1), \ldots, \psi(g_m)) = b_j \) for some \( j \in J \). Therefore \( \psi(a_i) = \psi(a_j) \) and \( a_i - a_j \in K \). Fix the natural numbers \( i, j \) and put \( X_0 = (X \setminus \{ i \}) \cup \{ j \} \). Then

\[ v + \alpha_i(a_j - a_i) = \sum_{s \in X_0} \alpha_s a_s \in \mathfrak{RET}(\psi) \setminus K \]

where \( \alpha_i = \alpha_i \) and \( |X_0 \setminus J| = |X \setminus J| - 1 \), a contradiction. Hence \( K = \mathfrak{RET}(\psi) \) and, in particular, \( \mathfrak{RET}(\psi) \) has a basis which is a \( \mathfrak{bs}(A) \)-regular set.

(ii) Assume that \( \mathfrak{RET}(\psi) \) has a basis \( K \) which is a \( \mathfrak{bs}(A) \)-regular set. By the Zorn’s Lemma we can assume that \( I \) is a well-ordered set.

Put \( I_0 = \{ i \in I \mid a_i \in \mathfrak{RET}(\psi) \} \) and

\[ I_1 = \{ \ i \in I \setminus I_0 \mid a_i - a_j \in \mathfrak{RET}(\psi) \text{ for some } j > i \} . \]

Define the function \( p : I_1 \rightarrow I \) by

\[ I_1 \ni i \mapsto \min \{ \ j \in I \setminus I_0 \mid j > i, a_i - a_j \in \mathfrak{RET}(\psi) \} . \]

Put \( I_2 = I_1 \setminus p(I_1) \).

We split the proof in several steps:

**Step 1:** If \( i, j \in I_1 \) and \( i < j \) then \( p(i) \neq p(j) \).
Let $p(i) = p(j)$. Clearly $a_i - a_{p(i)}, a_j - a_{p(j)} \in \mathfrak{Rr}(\psi)$ and

$$a_i - a_j = (a_i - a_{p(i)}) - (a_j - a_{p(j)}) \in \mathfrak{Rr}(\psi),$$

so that $p(i) \leq j < p(j) = p(i)$, a contradiction.

**Step 2**: For every $i \in I_1$ we define the corresponding $i$-ray

$$R(i) = \{ p_0(i) < p_1(i) < p_2(i) < \cdots \mid p_k(i) \in I \},$$

where $p_0(i) = i$, $p_1(i) = p(i)$ and, moreover, $p_{n+1}(i) = p(p_n(i))$ if $p_n(i) \in I_1$ while $p_{n+1}(i)$ is not defined if $p_n(i) \notin I_1$.

Note that every $i$-ray is contained in a unique maximal $j$-ray. Moreover, a $j$-ray is maximal for $j \in I_1$ if and only if $j \in I_2$. The former part follows from the fact that the well-ordered set $I$ does not contain any infinite decreasing chain, and the latter one is trivial as for every $j \in p(I_1)$ with $p(i) = j$ we have $R(j) \subset R(i)$.

**Step 3**: For every two different maximal rays $R(i)$ and $R(j)$ we have $R(i) \cap R(j) = \emptyset$ and for every $i \in I_1$ there exists a unique minimal $f(i) \in I_2$ such that $i \in R(f(i))$. Moreover, for $i, j \in I_1$ we have $\psi(a_i) = \psi(a_j)$ if and only if $f(i) = f(j)$. In particular, as $f(f(i)) = f(i)$, for every $i \in I_1$ we have that $a_i - a_{f(i)} \in \mathfrak{Rr}(\psi)$.

Let us prove that the $\mathfrak{bs}(A)$-regular set

$$K_1 = \{ a_i \mid i \in I_0 \} \cup \{ a_i - a_{p(i)} \mid i \in I_1 \}$$

is a basis of $\mathfrak{Rr}(\psi)$.

**Step 4**: $(K_1)_F = \mathfrak{Rr}(\psi)$. As $\mathfrak{Rr}(\psi)$ has a $\mathfrak{bs}(A)$-regular basis $K$ it is enough to prove that if $a_i - a_j \in \mathfrak{Rr}(\psi)$ with $j > i \notin I_0$ then $a_i - a_j \in (K_1)_F$. By Step 3 we have $f(i) = f(j) = k$ and so

$$i, j \in R(k) = \{ k < k_1 < k_2 < \cdots < k_s = i < \cdots < j = k_{s+t} < \cdots \}.$$ 

It follows that $a_i - a_j = \sum_{l=s}^{s+t-1}(a_{k_l} - a_{k_{l+1}})$ and $a_{k_l} - a_{k_{l+1}} \in K_1$ for every $s \leq l < s + t$, yielding the claim.

**Step 5**: The set $K_1$ is $F$-linearly independent.

Let $\sum_{i \in I_0} \alpha_i a_i + \sum_{j \in I_1} \beta_j (a_j - a_{p(j)}) = 0$, where $\alpha_i, \beta_j \in F$ for every $i \in I_0$ and $j \in I_1$. As $I_0 \cap I_1 = \emptyset$ we have $\alpha_i = 0$ for all $j \in I_0$.

Suppose that $\beta_j \neq 0$ for some $s \in I_1$. Put $j = \max \{ s \mid \beta_s \neq 0 \}$. Then $p(j) > i$ for every $i$ such that $\beta_i \neq 0$. It follows that $\beta_j = 0$, a contradiction.

**Step 6**: $\mathcal{B} = \{ b_i = \psi(a_i) \mid i \in I = I \setminus (I_0 \cup p(I_1)) \}$ is a multiplicative basis of $B$.

Observe that if $b_i = \psi(a_i) \in B$ and $i \notin I_3$, then either $i \in I_0$ and $b_i = 0$ or $i \in p(I_1)$ and so $\psi(a_i) = b_i = \psi(a_{f(i)}) = b_{f(i)} \in \mathcal{B}$ with $f(i) \notin I_0 \cup p(I_1)$. As a consequence, $\mathcal{B}$ is a $F$-basis of $B$.

Suppose now that $\sum_{i \in I_3} \beta_i a_i = 0$ for some $\beta_i \in F$. Then $\sum_{i \in I_3} \beta_i a_i \in \mathfrak{Rr}(\psi)$ and so, by Steps 4 and 5, we get

$$\sum_{i \in I_3} \beta_i a_i = \sum_{j \in I_0} \alpha_j a_j + \sum_{s \in I_1} \gamma_s (a_s - a_{p(s)}).$$

Since $I_0 \cap (I_1 \cup I_3) = \emptyset$ we have $\alpha_j = 0$ for every $i \in I_0$. Let $t = \max \{ s \mid \gamma_s \neq 0 \}$.

As $j < p(j) \leq p(t)$ for every $j$ such that $\beta_j \neq 0$ and $p(t) \notin I_3$, relation (1) forces $\gamma_t = 0$, a contradiction. Thus $\mathcal{B}$ is a $F$-basis of $B$. It remains to show that $\mathcal{B}$ is multiplicative. Let $i, j \in I_3$. Then there exists $k \in I$ such that $a_i a_j = a_k$. If $k \in I_0$ then one has $b_ib_j = 0$. On the other hand, if $k \in I_3$ then $b_ib_j \in \mathcal{B}$. Finally, if $k \notin (I_0 \cup I_3)$ then $k \in p(I_1)$, so that $b_k = b_{f(k)} \in \mathcal{B}$. \[\square\]
Remark 1. Suppose that $A$ is a finitely generated associative algebra over a field $F$ having a multiplicative basis $\mathfrak{bs}(A)$. Assume that we can choose a minimal set of generators $\{a_1, \ldots, a_n\}$ of $A$ such that $\{a_1, \ldots, a_n\} \subseteq \mathfrak{bs}(A)$. For a set $X = \{x_1, \ldots, x_n\}$ we denote by $F(X)$ the free $F$-associative algebra over $X$ and by $X^*$ the free monoid on $x_1, \ldots, x_n$. Clearly, there exists an homomorphism $\psi : F(X) \to A$ such that $\psi(x_i) = a_i$ and $\mathfrak{Ret}(\psi)$ has an $X^*$-regular $F$-basis (by Theorem 7).

Let $\mathfrak{L}$ be the relatively free nilpotent restricted Lie algebra of class 2 on the set $\{x, y\}$ over a field of characteristic 2. Denote by $\mathfrak{c}(s)$ the nilcyclic restricted Lie algebras of exponent $s$ and put $\mathfrak{h}(s) = \mathfrak{L}/I$, where $I$ is the restricted ideal of $\mathfrak{L}$ generated by $x^{[2]^s}, y^{[2]^s}$ and $[x, y]^{[2]^s}$. For every $m, n \geq 0$ and $s > 0$ in the sequel we will use the restricted Lie algebra

$$L(m, n; s) = \mathfrak{c}(s) \oplus \cdots \oplus \mathfrak{c}(s) \oplus \mathfrak{h}(s) \oplus \cdots \oplus \mathfrak{h}(s).$$

The restricted enveloping algebra of $L(m, n; s)$ admits an f.m.b. Indeed we have

Lemma 2. For every $m, n \geq 0$ and $s > 0$ the associative algebra $u(L(m, n; s))$ has a filtered multiplicative basis.

Proof. Since we have

$$u(L(m, n; s)) \cong u(\mathfrak{c}(s)) \otimes_F \cdots \otimes_F u(\mathfrak{c}(s)) \otimes_F u(\mathfrak{h}(s)) \otimes_F \cdots \otimes_F u(\mathfrak{h}(s)),$$

by virtue of (F3) and Theorem 1 of [7] it enough to show that $u(\mathfrak{h}(s))$ has an f.m.b. Let $\mathfrak{L}$ be the relatively free nilpotent restricted Lie algebra of class 2 on the set $\{x, y\}$ and $I$ the restricted ideal of $\mathfrak{L}$ generated by $x^{[2]^s}, y^{[2]^s}$ and $[x, y]^{[2]^s}$. Consider the unique associative homomorphism $\hat{\pi} : u(\mathfrak{L}) \to u(\mathfrak{h}(s))$ extending the canonical map $\pi : \mathfrak{L} \to \mathfrak{L}/I = \mathfrak{h}(s)$. As $\mathfrak{L}^{[2]^s} \subseteq Z(\mathfrak{L})$, it is clear that $\mathfrak{Ret}(\hat{\pi}) = Iu(L)$ is spanned by the elements of the form $x^2 \omega_1, y^2 \omega_2, ((xy)^2)^2 + (yx)^2)^2 \omega_3$, where the $\omega_i$ are monomials in $x, y$. Consequently, by Theorem 1 we see that $u(\mathfrak{h}(s))$ has a multiplicative basis $\mathfrak{bs}(u(\mathfrak{h}(s)))$ with $\mathfrak{bs}(u(\mathfrak{h}(s))) \setminus \{1\} \subseteq \omega(\mathfrak{h}(s))$. Finally, as $\mathfrak{h}(s)$ is finite-dimensional and $p$-nilpotent we have $\omega(\mathfrak{h}(s)) = \mathfrak{J}(u(\mathfrak{L}))$, so that $\mathfrak{bs}(u(\mathfrak{h}(s)))$ contains an $F$-basis of $\mathfrak{J}(u(\mathfrak{h}(s)))$.

We say that an associative algebra $A$ over a field of characteristic 2 is of Heisenberg type if there exist $m, n \geq 0, s > 0$, and an f.m.b. $\mathfrak{B}$ of $u(L(m, n, s))$ such that $A \cong u(L(m, n, s))/J$ for some ideal $J$ of $u(L(m, n, s))$ having a $\mathfrak{B}$-regular basis.

Let $L$ be a finite-dimensional $p$-nilpotent restricted Lie algebra over a field of characteristic $p > 0$. In [7] we proved that if $L$ is abelian then $u(L)$ has a filtered multiplicative basis if and only if it is a direct sum of cyclic restricted subalgebras. Moreover, we showed that if $L$ has nilpotent class 2 and $p > 2$ then $u(L)$ does not have any filtered multiplicative basis. Here we prove the following:

Theorem 2. If $L \in \mathfrak{F}_p$ has nilpotency class 2 then $u(L)$ has a filtered multiplicative basis if and only if $p = 2$ and $u(L)$ is of Heisenberg type.

Proof. Suppose that $u(L)$ has an f.m.b. $\mathfrak{bs}(u(L))$ such that $1 \in \mathfrak{bs}(u(L))$. By Theorem 3 of [7] the ground field must have characteristic 2. Let

$$\Gamma = \mathfrak{bs}(u(L)) \setminus (\omega(L)^2 \cup \{1\}) = \{g_1, \ldots, g_t\}.$$
Then $\Gamma$ is a minimal set of generators of $u(L)$ as a unitary associative $F$-algebra and, moreover, by property (F1) and Lemma [1] for every $i = 1, \ldots, t$ there exists $c_i \in L$ such that $c_i \equiv g_i \pmod{\omega^2(L)}$. As $L$ is not commutative, by Lemma 2 of [2] there exist $1 \leq i, j \leq t$ such that $[c_i, c_j] \notin \mathcal{D}_3(L)$. If $1 \leq k \leq t$ with $k \neq i, j$, as an easy consequence of Lemma [1] we deduce the following Facts:

(a) if $c_i [c_j, c_k] \in \omega(L)^3$ then $[c_j, c_k] \in \mathcal{D}_3(L)$;
(b) if $c_j [c_i, c_k] \in \omega(L)^3$ then $[c_i, c_k] \in \mathcal{D}_3(L)$.

(c) $c_k [c_i, c_j] + c_j [c_i, c_k] \notin \omega(L)^4$;
(d) $c_k [c_i, c_j] + c_i [c_j, c_k] \notin \omega(L)^4$;
(e) $c_i [c_j, c_k] + c_j [c_i, c_k] + c_k [c_i, c_j] \notin \omega(L)^4$.

Consider the following six elements:

$m_1 = g_i g_j g_k = c_i c_j c_k \pmod{\omega(L)^4}$;
$m_2 = g_i g_k g_j = c_i c_j c_k + c_i [c_j, c_k] \pmod{\omega(L)^4}$;
$m_3 = g_j g_i g_k = c_i c_j c_k + c_k [c_i, c_j] \pmod{\omega(L)^4}$;
$m_4 = g_k g_i g_j = c_i c_j c_k + c_j [c_i, c_k] + c_j [c_i, c_k] \pmod{\omega(L)^4}$;
$m_5 = g_k g_j g_i = c_i c_j c_k + c_i [c_j, c_k] \pmod{\omega(L)^4}$.

Consequently, by property (F2) we get

$$\dim_F \left(\left(\langle m_1, \ldots, m_6 \rangle + \omega^4(L)\right) / \omega^4(L)\right) \leq 4,$$

so that we must have $m_s = m_t$ for some $s \neq t$. By Facts (a) and (b) we immediately have that

$$\{m_1, m_2, m_3\} \cap \{m_4, m_5, m_6\} = \emptyset.$$

We claim that $[c_i, c_k] \in \mathcal{D}_3(L)$. Suppose the contrary. Notice that, by Fact (b), we have $m_2 \neq m_3$ and $m_3 \neq m_4$. Now we distinguish two cases:

Case 1: $[c_i, c_k] \in \mathcal{D}_3(L)$. Then property (F2) yields $m_1 = m_2$ and $m_4 = m_6$ and, moreover, by Lemma [1] we have $c_i [c_j, c_k] \in \omega(L)^4$. It follows that

$$\dim_F (m_1, \ldots, m_6) = 4$$

and so $m_1 = m_5$ or $m_3 = m_6$. In both cases we conclude that $c_j [c_i, c_k] \notin \omega(L)^4$, contradicting Fact (b).

Case 2: $[c_j, c_k] \notin \mathcal{D}_3(L)$. Then $c_i [c_j, c_k] \notin \omega(L)^3$, so that $m_1 \neq m_2$ and $m_4 \neq m_6$. It follows that $m_1 = m_3$ and $m_3 = m_5$ and, in turn,

$$c_i [c_j, c_k] + c_j [c_i, c_k] \in \omega(L)^4,$$

which is impossible by Lemma [1].

Therefore $[c_i, c_k] \in \mathcal{D}_3(L)$ and in a similar way one can show that $[c_j, c_k] \in \mathcal{D}_3(L)$, as well. It follows that $g_i g_k \equiv g_k g_i \pmod{\omega(L)^3}$ and $g_j g_k \equiv g_k g_j \pmod{\omega(L)^3}$ and then, by Lemma [1] and (F2), $g_k$ commutes both with $g_i$ and $g_j$. Thus, for any $g_i \in \Gamma$ one has that either $g_i$ is in the center $Z(u(L))$ of $u(L)$ or there exists a unique $g_j \in \Gamma$ which does not commute with $g_i$. We can then reindex the elements of $\Gamma \setminus \{1\}$ in such a way that $[g_i g_j] \neq 0$ for $i = 1, \ldots, r$ and all the other commutators are zero. Consider the restricted Lie algebra $L(m, n; s)$, where $n = t - 2m$ and $s$ is the exponent of $\Gamma$ in $u(L)^{-1}$. For every $i = 1, \ldots, m$ let $x_i, y_i$ be generators
of $i$th copy of $h_{(s)}$ and for every $j = 1, \ldots, n$ let $z_j$ be a generator of the $j$th copy of $c_{(s)}$. For every $i = 1, \ldots, m$ one has
\[
g_{2i-1}^2 g_{2i} \equiv c_{2i-1}^2 c_{2i} \equiv c_{2i} c_{2i-1}^2 \equiv g_{2i} g_{2i-1}^2 \pmod{(\omega(L))^4}
\]
and so property (F2) forces $[g_{2i-1}^2, g_{2i}] = 0$. Thus $g_{2i-1}^2 \in Z(u(L))$ and in a similar way one can prove that $g_{2i} \in Z(u(L))$, as well. It follows that $[g_{2i-1}, g_{2i}] \in Z(u(L))$ for every $i = 1, \ldots, m$. The just proved properties assure the existence a unique surjective restricted homomorphism $\phi : L(m, n; s) \rightarrow u(L)$ such that $\phi(x_i) = g_{2i-1}$ and $\phi(y_i) = g_{2i}$ for every $i = 1, \ldots, m$ and $\phi(z_j) = g_{2m+j}$ for every $j = 1, \ldots, n$. Let $\phi : u(L(m, n; s)) \rightarrow u(L)$ denote the unique algebra homomorphism extending $\phi$. Then Theorem 1 allows to conclude that $A \cong u(L(m, n; s))/J$, where $J = \ker(\phi)$ is an ideal of $u(L)$ having a $\mathcal{B}$-regular $F$-basis, proving the necessity part.

The sufficiency part is an immediate consequence of Lemma 2 and Theorem 1.

We conclude this section with some open problems. We say that an associative algebra $A$ over a field of characteristic 2 is of strong Heisenberg type if there exist $m, n \geq 0, s > 0$, and an f.m.b. $\mathcal{B}$ of $u(L(n, m; s))$ such that $A \cong u(L(m, n; s))/J$ for some ideal $J$ of $L(m, n; s)$ having a $\mathcal{B}$-regular basis. As $u(L(m, n; s))/J \cong u(L(m, n; s))/Ju(L(m, n; s))$, it is clear that in such a case $A$ is of Heisenberg type.

If the following problem has a positive answer then the conclusion of Theorem 2 would be considerably improved:

**Problem 1.** Let $L \in \mathcal{F}_p$ of nilpotency class 2 over a field of characteristic 2 and suppose that $u(L)$ has an f.m.b. Is $u(L)$ of strong Heisenberg type?

Likely, the characterization of the restricted Lie algebras $L$ such that $u(L)$ is of Heisenberg type could be a delicate task involving the isomorphism problem for restricted Lie algebras:

**Problem 2.** Characterize the restricted Lie algebras $L$ whose restricted enveloping algebra $u(L)$ is of Heisenberg type.

Finally, we suspect that the following problem could have a positive answer:

**Problem 3.** Suppose that $L \in \mathcal{F}_p$ is not abelian and $u(L)$ has an f.m.b. Is it true that $p = 2$ and $L$ is nilpotent of class 2?

4. **Associated graded algebras and group algebras**

For an associative algebra $A$, in this section we will consider the associated graded algebra
\[
gr(A) = \bigoplus_{i \geq 0} \mathfrak{J}^i(A)/\mathfrak{J}^{i+1}(A),
\]
associated to the filtration given by the powers of the Jacobson radical $\mathfrak{J}(A)$ of $A$.

**Theorem 3.** Let $A$ be a finite-dimensional associative algebra over a field $F$. If $A$ has an f.m.b., then $gr(A)$ has an f.m.b.

**Proof.** Let $bs(A)$ be an f.m.b. of $A$. Put
\[
bs(A)_i = \left(bs(A) \cap \mathfrak{J}^i(A)\right) \setminus \mathfrak{J}^{i+1}(A), \quad (i = 0, 1, \ldots, n - 1)
\]
where \( n \) is the nilpotency class of \( \mathfrak{z}(A) \). Then, in view of [5], the images of the elements of \( \mathfrak{bs}(A) \) in \( A/\mathfrak{z}^{i+1}(A) \) form an \( F \)-basis \( \mathfrak{bs}(A)_i \) for the vector space \( \mathfrak{z}^i(A)/\mathfrak{z}^{i+1}(A) \), where \( i = 0, \ldots, n-1 \). As a consequences, the set
\[
\mathfrak{bs}(A) := \bigoplus_{i=0}^{n-1} \mathfrak{bs}(A)_i
\]
is an \( F \)-basis of \( \text{gr}(A) \). Of course one has \( \mathfrak{z}^{i}(\text{gr}(A)) = \bigoplus_{j \geq 1} \mathfrak{z}^i(A)/\mathfrak{z}^{i+1}(A) \). Now, let \( \overline{b}_i = b_i + \mathfrak{z}^{i+1}(A) \in \mathfrak{bs}(A)_i \) and \( \overline{b}_j = b_j + \mathfrak{z}^{j+1}(A) \in \mathfrak{bs}(A)_j \), where \( b_i, b_j \in \mathfrak{bs}(A) \). If \( b_ib_j \in \mathfrak{z}^{i+j+1}(A) \) then \( \overline{b}_ib_j = 0 \) in \( \text{gr}(A) \). Suppose then that \( b_ib_j \notin \mathfrak{z}^{i+j+1}(A) \). Since \( \mathfrak{bs}(A) \) is a f.m.b. of \( A \) one has \( b_ib_j \in \mathfrak{bs}(A) \cap \mathfrak{z}^{i+j}(A) \), so that \( \overline{b}_ib_j \in \mathfrak{bs}(A)_{i+j} \). Therefore \( \mathfrak{bs}(A) \) is an f.m.b of \( \text{gr}(A) \), yielding the claim.

For every prime \( p \) we will indicate by \( F_p \) the field with \( p \) elements. If \( g \) and \( h \) are two elements of a group then we will denote by \((g, h)\) their group commutator. We recall that a finite \( p \)-group \( G \) is said to be powerful if either \( p = 2 \) and \( G' \subseteq G^4 \) or \( p > 2 \) and \( G' \subseteq G^p \). Here \( G' \) is the derived subgroup of \( G \) and \( G^k \) denotes the subgroup of \( G \) generated by the elements \( g^k, \ g \in G \).

**Corollary 1.** Let \( FG \) be the group algebra of a finite \( p \)-group \( G \) over the field \( F \) of positive characteristic \( p \). Denote by \( \mathfrak{L}(G) \) the restricted Lie algebra associated with \( G \). Then the following statement hold:

(i) if \( FG \) possesses an f.m.b. then so does \( u(\mathfrak{L}(G) \otimes_{F_p} F) \);

(ii) if \( p > 2 \) and \( G \) is nilpotent of class 2 then \( FG \) does not have any f.m.b.

**Proof.** (i) We first recall the construction of \( \mathfrak{L}(G) \) by means of the Zassenhaus-Jennings-Lazard series of \( G \). For every \( n \in \mathbb{N} \) the \( n \)th dimension subgroup of \( G \) is defined by setting
\[
\mathfrak{D}_n(G) = G \cap (1 + \omega^n(\text{FG})) = \prod_{ip^j \geq n} \gamma_i(G)p^j,
\]
where \( \omega(\text{FG}) \) is the augmentation ideal of \( \text{FG} \) and the \( \gamma_i(G) \) are the terms of the descending central series of \( G \). Then the \( F_p \)-vector space
\[
\mathfrak{L}(G) = \bigoplus_{n \in \mathbb{N}} \mathfrak{D}_n(G)/\mathfrak{D}_{n+1}(G)
\]
has the structure of a restricted Lie algebra with respect to the Lie bracket and \( p \)-map defined by the following conditions:
\[
[g\mathfrak{D}_{i+1}(G), h\mathfrak{D}_{i+1}(G)] = (g, h)\mathfrak{D}_{i+j+1}(G),
\]
\[
(g\mathfrak{D}_{i+1}(G))^p = g^p\mathfrak{D}_{pi+1}(G).
\]
(For details we refer the reader to Chapter VIII of [12].) Now, as \( G \) is a \( p \)-group we clearly have \( \mathfrak{z}(\text{FG}) = \omega(\text{FG}) \) and then, by a well-known theorem of Quillen in [13], \( \text{gr}(\text{FG}) \) is isomorphic as an \( F \)-algebra to the restricted enveloping algebra \( u(\mathfrak{L}(G) \otimes_{F_p} F) \). Consequently, Theorem [3] allows to conclude that \( u(\mathfrak{L} \otimes_{F_p} F) \) has an f.m.b., as required.

(ii) If \( G \) is nilpotent of class 2 then it is clear that its associated restricted Lie algebra \( \mathfrak{L}(G) \) is nilpotent of class at most 2. Now, if \( \mathfrak{L}(G) \) is abelian, as \( p > 2 \) and \( \gamma_2(G) = 1 \), from [2] and [3] it follows that \( \gamma_2(G) \subseteq \mathfrak{D}_2(G) = G^p \). Therefore \( G \) is powerful and so, in view of Theorem 1 of [6], the group algebra \( FG \) cannot have
an f.m.b. On the other hand, if $\mathcal{L}(G)$ has nilpotence class 2, then by Theorem 3 of [7] the restricted enveloping algebra $u(\mathcal{L}(G) \otimes_{F_0} F)$ does not have any filtered multiplicative basis. Hence, from the part (1) the claim follows at once.

The previous result gives a partial answer to question 5 in [11]. Note also that a possible positive solution of Problem 3 combined with Corollary 1 would settle completely to question 5 in [11], as well. Finally, it is worth remarking that, in general, the converse of Theorem 3 is false. For instance, consider the following example:

**Example.** Let $F$ be a field of positive characteristic $p$ containing an element $\alpha$ which is not a $p$-th root in $F$. Consider the abelian restricted Lie algebra

$$L_\alpha = Fx + Fy + Fz$$

with $x^p = \alpha z$, $y^p = z$, and $z^p = 0$. Note that $J(u(L))$ coincides with the augmentation ideal $\omega(L)$ of $u(L)$. Consider the restricted Lie algebra

$$\text{gr}(L) = \bigoplus_{n \in \mathbb{N}} \mathcal{D}_n(L)/\mathcal{D}_{n+1}(L) \quad (n \in \mathbb{N}).$$

It is easy to see that $\text{gr}(L)$ is isomorphic to the direct sum of three cyclic restricted Lie algebras and so $u(\text{gr}(L))$ has an f.m.b. (see [7], Theorem 1). Moreover, by Theorem 2.2 of [17] one has $u(\text{gr}(L)) \cong \text{gr}(u(L))$, hence $\text{gr}(u(L))$ has an f.m.b.

On the other hand, for what was showed in [7] (see the example on page 607), in this case $u(L)$ cannot have any filtered multiplicative basis.

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VICTOR BOVDI, ALEXANDER GRISHKOV
IME, USP, Rua do Matao, 1010 – Citade Universitária, CEP 05508-090, Sao Paulo, Brazil
E-mail address: \{vbovdi, shuragri\}@gmail.com

SALVATORE SICILIANO,
Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento, Via Provinciale Lecce–Arnesano, 73100–LECCE, ITALY
E-mail address: salvatore.siciliano@unisalento.it