AUTOMORPHISM GROUP OF BATRYREV CALABI-YAU
THREEFOLDS

MOHAMMAD FARAJZADEH TEHRANI

Abstract. In this paper we will prove that all Batryrev Calabi-Yau threefolds arising as resolution of generic hyperplane section of Reflexive Fano-Gorenstein fourfolds have finite automorphism group. In sequence we will review the consequence of Morrison conjecture and its weaker version, Wilson conjecture, on this class of C-Y threefolds.

1. Introduction

The construction of Calabi-Yau threefolds, that brought by far the largest amount of examples is the construction of Batyrev [B1]. He starts with a reflexive polytop \( \Delta \) in dimension four, takes the corresponding toric 4-fold \( X_\Delta \), takes a generic anti-canonical section and obtains this way a singular Calabi-Yau threefold \( Z_\Delta \). There is always a crepant resolution \( Z_\Delta \). This resolution is induced by a maximal projective crepant partial (MPCP) desingularization \( X_\Delta \) of \( X_\Delta \). So you get a smooth Calabi-Yau. Reflexive polytops in dimension four are classified and the number is 473,800,776. Among these, there exist at least 3000 different topological types of C.Y threefolds, i.e. with different Hodge numbers. In the same paper, Batyrev examined the mirror symmetry conjecture on this set of manifolds and he showed that the dual C.Y families obtained from dual polytops \( \Delta \) and \( \Delta^* \), have mirror hodge numbers. So he proposed them as potential candidate for the mirror family. This was a generalization of mirror construction for Quintic threefold, previously proposed by physicists in [CDGP].

For any C.Y threefold \( Z \), its Kahler cone \( k_Z \subset H^{1,1}(Z, \mathbb{R}) \) (not to be confused with canonical bundle \( K_Z \)) is the set of closed forms which can be represented by a Kahler form. Under the isomorphism

\[
H^2(Z, \mathbb{R}) = H^{1,1}(Z, \mathbb{R}) = \text{Pic}(Z) \otimes \mathbb{R} = N^1(Z) \otimes \mathbb{R}
\]

we see that Kahler cone is equal to ample-cone and its closure (nef cone) is dual to the closure of cone of effective curves, \( \overline{NE}(Z, \mathbb{R}) \), of \( Z \). We will show the compactified Kahler-cone by \( \overline{k}_Z \).

Due to the vital role of Kahler-cone in A-side of mirror symmetry, it is very important to understand the geometry of Kahler-cone, but determining the shape of Kahler-cone and this question that whether it is generated by a finite set of rational rays is very hard to answer. In [M1, M2], Morrison proposed the following conjecture on the shape of \( \overline{k}_X \).

Conjecture 1.1. For any C.Y threefold \( Z \), There is rational polyhedral fundamental domain for the action of automorphism group on Kahler cone. In fact if
the automorphism group is finite, Kahler cone of $Z$ is generated by finitely many rational rays (it is rational polyhedral).

This conjecture has been verified in very few cases like, for fiber-product of elliptic fibrations by Grassi-Morrison ([GM]), for desingularized Horrocks-Mumford quintics by Borcea ([Bor]) and C.Y hypersurfaces in Smooth Fano fourfolds by Kollar ([BK], appendix B). Similar version of this conjecture for K3-surfaces is proved by Kovacs([Kov]). The content of this paper is inspired from the cited work of Kollar. In section 3 we will review Kollar’s result in more details.

Let $Z$ be a C.Y threefold as before and $c_2(Z) \in H^4(Z, \mathbb{Z}) \cong N_1(Z)$ be the second Chern class. From theorem 1.1 of [Mi] we know that for any Kahler class $w \in k_Z$, $w \cdot c_2(Z) > 0$; therefore $w \cdot c_2(Z) \geq 0$ for any $w \in \overline{\mathcal{K}}_Z$, i.e. $c_2(Z)$, as a linear function on $H^2(Z, \mathbb{R})$ is non-negative on the closure of Kahler-cone. This would imply that the integral curve class in $N_1(Z)$, corresponding to second chern class, belongs to the closure of cone of effective curves; i.e. $[c_2] \in \overline{\mathcal{NE}(Z)}$.

The following theorem is due to Wilson,

**Theorem 1.1.** If $c_2$ is strictly positive on the closure of Kahler cone of a Calabi-Yau threefold $Z$ (i.e. $[c_2]$ is in the interior of $\mathcal{NE}(Z)$), or if $k_Z$ is rational polyhedral then the automorphism group of $Z$ is finite.

Based on this theorem, Wilson proposed the following weaker version of Morrison conjecture,

**Conjecture 1.2.** If $c_2$ is strictly positive on the closure of Kahler cone of Calabi-Yau threefold, then the Kahler-cone of $Z$ is generated by finitely many rational extremal rays (i.e. it is rational polyhedral).

In this paper we will prove that

**Theorem 1.2.** For all (generic) desingularized Batyrev Calabi-Yau threefolds $Z_{\Delta}$, $c_2(Z_{\Delta})$ is strictly positive on closure of Kahler-cone.

Therefore from this and theorem [11] we conclude that all Batyrev C.Y threefolds have finite automorphism group and following corollary is an immediate consequence of that,

**Corollary 1.1.** Either all Batyrev C.Y threefolds have rational polyhedral Kahler-cone or there is a counter-example to Morrison conjecture among them.

Based on Kollar example cited above (the case of $X_{\Delta}$ smooth), We believe that a similar phenomena appears here and all these C.Y manifolds have rational polyhedral Kahler cone. Therefore Morrison conjecture holds for this huge class of C.Y manifolds. Note that when $X_{\Delta}$ is smooth, then by Kollar’s result and theorem [12] Morrison conjecture is true.

In the book [CK], mirror symmetry for this class of manifolds have been studied extensively. But because it is hard to describe the Kahler-cone of an arbitrary Batyrev C.Y and also it is hard to describe the complex moduli of its mirror, authors have restricted their attention only to the toric part of Kahler-cone and toric deformations of mirror, for which they can use the combinatorial tools. Therefore the extension of their discussion to whole Kahler-moduli (similarly complex-moduli)
needs stronger results on the geometry of Kahler-cone (complex-moduli space). In sequel papers we would try to prove the Wilson conjecture which would result in validity of Morrison conjecture for Batyrev C.Y threefolds.

Acknowledgment 1.1. I would like to thank my advisor, professor Tian, and professor P.M.H.Wilson for their helps and comments.

2. Basic facts about Batyrev construction

Let $M \cong \mathbb{Z}^n$ be a free abelian group of rank $n$, then $M$ is a lattice in vector space $M_\mathbb{R} = M \otimes \mathbb{R}$ and $N = \text{Hom}(M, \mathbb{Z})$ is the dual lattice in $N_\mathbb{R} = N \otimes \mathbb{R}$. Let $\Delta$ be an integral polytop (convex hull of finitely many lattice points) in $M$ including 0 as an interior point. Any such polytop corresponds to a possibly singular toric algebraic variety $X_\Delta$ and an ample Cartier divisor $D_\Delta$ on it.

Definition 2.1. For arbitrary polytop $\Delta$ (convex set) in $M_\mathbb{Q}$ containing zero, its dual is defined by

$$\Delta^* = \{ y \in N_\mathbb{Q} \mid \langle x, y \rangle \geq -1, \text{ for all } x \in \Delta \} \subset N_\mathbb{Q}$$

where $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ is the non-degenerate pairing between $N$ and $M$.

Definition 2.2. Let $H$ be a rational affine hyperplane in $M_\mathbb{Q}$, $p \in M$ an arbitrary integral point. Assume $H$ is affinely generated by integral points $H \cap M$, i.e., there exists $n_H \in N$ and an integer $c$ such that

$$H = \{ x \in M_\mathbb{Q} \mid \langle x, n_H \rangle = c \}$$

Then the absolute value $|c - \langle p, n_H \rangle|$ is called the integral distance between $H$ and $p$.

Definition 2.3. Let $0 \in \Delta \subset M_\mathbb{Q}$ be a integral polytop. $(\Delta, M)$ is called a reflexive pair if the integral distance between zero an all codimension-one faces of $\Delta$ equals one. In this case $(\Delta^*, N)$ is also a reflexive pair.

From now on $(\Delta, \Delta^*)$ corresponds to a pair of reflexive polytops of dimension four. For such polytop $\Delta$, the corresponding fan of toric variety $X_\Delta$ is given by the cones over faces of $\Delta^*$ and similarly the fan of $X_{\Delta^*}$ is given by the cones over the faces of $\Delta$. We will show the fan of $X_\Delta$ by $\Sigma$ and the fan of $X_{\Delta^*}$ by $\Sigma^*$. For a reflexive polytop $\Delta$, $D_\Delta$ is the anti-canonical divisor.

Proposition 2.1. (theorem 4.1.9, [H]) For a four dimensional reflexive $\Delta$ as above

- $D_\Delta$ is a section of very ample line bundle $\mathcal{O}_\Delta(1)$ and $X_\Delta$ is a Fano-Gorenstein variety with canonical singularities,
- linear system $|{-K_{X_\Delta}}|$ is base-point free and its generic section $Z_\Delta$ is a C.Y model (it has a trivial canonical bundle) with at most canonical singularities.
- $X_\Delta$ is smooth in codimension-one, its singular locus is supported on a collection of toric subvarieties $V_\Theta$ corresponding to some faces $\Theta$ of $\Delta$, of dimension 0,1,2 (point, curve and surfaces singularities respectively). $Z_\Delta$ is also smooth at codimension-one, so its singular locus has curve and point
components corresponding to intersection with singular locus of $X_\Delta$. Moreover for $\Theta$ as before, if $\dim \Theta = 0$ then $Z_\Delta$ does not pass through corresponding singular point; if $\dim \Theta = 1$, $Z_\Delta \cap V_\Theta$ is a set of $d(\Theta)$ (length of $\Theta$) singular points on $Z_\Delta$ and if $\dim \Theta = 2$, $Z_\Delta \cap V_\Theta$ is an irreducible curve of singularities on $Z_\Delta$.

For a given fan $\Sigma$ in $N$, a refinement $\Sigma'$ of $\Sigma$ (subdividing cones of $\Sigma$ into smaller ones) corresponds to a map of toric varieties $X_{\Sigma'} \to X_\Sigma$. When the fan $\Sigma$ is given by cones over the face of a polytop $\Delta^*$, a triangulation of faces of $\Delta^*$ (i.e. a subdivision of faces of $\Delta^*$ into simplecies) gives a refinement of $\Sigma$, but not any triangulation results in a projective variety $X_{\Sigma'}$. We call a triangulation of faces of $\Delta^*$ projective if the corresponding variety $X_{\Sigma'}$ is projective and we call it maximal if it can not be refined further more (using more triangulations).

**Proposition 2.2.** (see [B1]) For a four-dimensional reflexive polytop $\Delta$ as before, there is a maximal projective triangulation of $\Sigma$ (not unique). Any such triangulation corresponds to a maximal projective crepant partial desingularization (MPCP) $\pi: \overline{X}_\Delta = X_{\Sigma'} \to X_\Delta$ such that

- $\overline{X}_\Delta$ is smooth in codimension three and has only $Q$-factorial singular points ($\Sigma'$ will be simplicial and $\overline{X}_\Delta$ is an orbifold).
- $K_{\overline{X}_\Delta} = \pi^* K_{X_\Delta}$ (i.e. the resolution is crepant); proper transform $\overline{Z}_\Delta$ of $Z_\Delta$ is a smooth C.Y threefold and is a section of base point free system $|\overline{K}_{\overline{X}_\Delta}|$.

Throughout the paper, by Batyrev C.Y threefolds, we mean the smooth generic C.Y threefolds $\overline{Z}_\Delta$ obtained from the procedure above.

There is a explicit formula for the hodge numbers of $\overline{Z}_\Delta$ which we are going to discuss in the rest of this section. For any such C.Y, there are two interesting Hodge numbers, $h^{1,1}$ and $h^{1,2}$ and since we are interested only in the $A$-model part of mirror symmetry we will only discuss the $h^{1,1}$ part which is the dimension of Kahler-cone of $\overline{Z}_\Delta$.

Let $\Theta$ be a face of (of arbitrary codimension-one) of $\Delta$. Corresponding to this face, there is a dual face $\Theta^*$ of dual polytop $\Delta^*$ defined by following equation,

$$\Theta^* = \{ y \in De^* \subset N \mid \langle x, y \rangle = -1 \text{ for any } x \in \Delta \}$$

If $\dim \Theta = i$ then $\Theta^* = 3 - i$. For any $i$-dimensional face $\Theta$, let $l(\Theta)$ denote the number of integral points of $\Theta$ and $l^*(\Theta)$ denote the number of interior integral points of $\Theta$, i.e. those who don’t lie on $(i-1)$-dimensional subfaces of $\Theta$. Then we have,

**Theorem 2.3.** ([B1], proposition 4.4.1) For a four dimensional reflexive polytop $\Delta$ as before

$$H^{1,1}(\overline{Z}_\Delta) = l(\Delta^*) - 5 - \sum_{\dim \Theta^* = 3} l^*(\Theta^*) + \sum_{\dim \Theta^* = 2} l^*(\Theta^*) \cdot l^*(\Theta)$$

(1)
instead of discussing the proof, we will explain the right hand side of equation (1). This would lead to a natural base for $H^{1,1}(\mathbb{Z}_\Delta)$ and this is the only thing we need for the proof of theorem 1.2.

Excluding origin, there are $k = l(\Delta^*) - 1$ integral points on $\Delta^*$, all of which lie on the boundary of $\Delta^*$. Let us call these points $\{v_1, \cdots, v_k\}$. All these points appear as vertices of a maximal triangulation of $\Delta^*$ and therefore correspond to a toric divisor of $X_{\Sigma'}$ where as before $\Sigma'$ is the fan over the refinement of $\Delta^*$, obtained via a MPCI desingularization. Let us call the corresponding divisors by $\{D_1 \cdots D_k\}$. If $v_i$ is an interior point of some 3-dimensional face $\Theta^*$ of $\Delta^*$, then $D_i$ lies over a singular 0-skeleton of $X_{\Sigma'}$ and so does not meet $\mathbb{Z}_\Delta$ at any point, because $Z_\Delta$ is disjoint from the singular 0-skeleton of $X_{\Sigma'}$. The number of such divisors is $\sum_{\dim \Theta^* = 3} l^*(\Theta^*)$ and this explains why the corresponding factor should be deducted from the the set of divisors which contribute to $H^{1,1}(\mathbb{Z}_\Delta)$. Now assume $\{v_1, \cdots, v_k\}$, $l = l(\Delta^*) - 1 - \sum_{\dim \Theta^* = 3} l^*(\Theta^*)$ are all integral points of boundary of $\Delta^*$, except those which lie in the interior of a 3-dimensional face and let $\{D_1 \cdots D_l\}$ be the corresponding divisors. Any such divisor intersects $\mathbb{Z}_\Delta$ in a non-empty divisor and the intersection divisor is connected if and only if the corresponding point $v_i$ does not lie in the interior of a 2-dimensional face. In fact if $v_i$ is an interior point of a 2-dimensional face $\Theta^*$, then $\Theta$ is 1-dimensional and $Z_\Delta \cap V_\Theta$ is made of $d(\Theta) = l^* (\Theta) + 1$ singular points; therefore $D_i \cap \mathbb{Z}_\Delta$ has $(l^* (\Theta)^1) + 1$ irreducible components. This explains the factor $\sum_{\dim \Theta^* = 2} l^*(\Theta^*) \cdot l^*(\Theta)$ in the right side of equation. So considering all irreducible components of intersection of $\mathbb{Z}_\Delta$ and toric divisors of $X_{\Sigma'}$, we get

$$l(\Delta^*) - 1 - \sum_{\dim \Theta^* = 3} l^*(\Theta^*) + \sum_{\dim \Theta^* = 2} l^*(\Theta^*) \cdot l^*(\Theta)$$

irreducible divisors in $\mathbb{Z}_\Delta$ which is still four more than the right hand side of equation (1). But any point $m \in M$ corresponds to rational function on $X_{\Sigma'}$ and by restriction, a rational function on $\mathbb{Z}_\Delta$. This way we get a four-dimensional space of linear relations between divisors obtained above which should be subtracted from the number we obtained above.

We organize what we got from above discussion in following paragraph,

- Let $\{v_1 \cdots v_a\}$ be the integral points on the boundary of $\Delta^*$ which don’t lie on the interior of a two-dimensional or 3-dimensional face. Corresponding to these points there are $a$ irreducible divisors $E_1 \cdots E_a$, $E_i = D_i \cap \mathbb{Z}_\Delta$ on $\mathbb{Z}_\Delta$.

- Let $\{u_1 \cdots u_b\}$ be the integral points on the boundary of $\Delta^*$ which lie on the interior of some two-dimensional face $\Theta$. Corresponding to $u_i$, $1 \leq i \leq b$, there are $d = d(\Theta)$ irreducible divisors $F_{i1}, \cdots, F_{id}$, where $\sum_j F_{ij} = D_i \cap \mathbb{Z}_\Delta$ on $\mathbb{Z}_\Delta$.

- Irreducible divisors $F_i$ and $F_{ij}$ generate $H^{1,1}(\mathbb{Z}_\Delta, \mathbb{Q})$ and there is a four-dimensional set of linear relations among them, So we can eliminate some four of them to get a basis of $H^{1,1}(\mathbb{Z}_\Delta, \mathbb{Q})$. Later we will need these divisors in the proof of theorem 1.2.
3. Second Chern Class of Batyrev Calabi-Yau Threefolds

We start this section with a theorem of Kollár from which we inspired the most (from appendix of [BK]).

**Theorem 3.1.** Let $X$ be a smooth Fano variety, $\dim X \geq 4$. Let $Z \subset X$ be a smooth Calabi-Yau divisor in $|-K_X|$. Then the natural map $\iota^*: \oplus \to \oplus$ (or dually the map $\iota_*: \oplus \to \oplus$) is an isomorphism of Kahler cones (respectively cone of effective curves). Moreover Kahler cone of $Z$ is rational polyhedral in this case.

The starting point of the proof is the isomorphism, $H^{1,1}(X) \cong H^{1,1}(Z)$, obtained from Lefschetz hyperplane theorem. In the case we are studying in this paper, we again have Fano toric 4-folds $X_\Delta$ but not all of them are smooth and therefore a priori we don’t have any comparison between Picard group of $X_\Delta$ and $Z_\Delta$. When we move to the a MPCP resolution $X_\Delta$, the smooth CY $Z_\Delta$ is still a section of $|K_X|$ but $X_\Delta$ is no longer fano so we again don’t have the Lefschetz theorem; some divisors on $X_\Delta$ do not intersect $X_\Delta$ and some others intersect in more than one component.

Followings is an example where this phenomena happens,

**Example 3.1.** Consider the fan $\Sigma$ given by following top dimensional cones in $\mathbb{Z}^4$, where $e_i$’s are a base of $\mathbb{Z}^4$,

1. $\sigma = (e_1, e_2, e_3, e_4, e_1 + e_2 + 3e_3 - 2e_4)$.
2. $(\pm e_1, \pm e_2, \pm e_3, e_4)$ where all signs are not $+$ at the same time.
3. $(\pm e_1, \pm e_2, \pm e_3, e_1 + e_2 + 3e_3 - 2e_4)$ where all signs are not $+$ simultaneously.

So in total we get 15 cones. This Fan is Gorenstein Fano and the corresponding polytop $\Delta$ is given by convex hull of following vertices:

1. $(e_1, e_2, e_3, -1)$ where $e_i = \mp 1$; $(-1, -1, -1, -1)$ corresponds to first item above and rest of them correspond to second group.
2. $-(e_1, e_2, e_3, \sum_{i=1}^3 e_i - 1)$; where $e_i = \pm 1$ and all of them are not $+$ simultaneously. These correspond to 3rd item above.

Let $X_\Delta$ be the corresponding toric variety. Among the cones listed above, the first one is singular and its MPCP resolution $X_\Delta = X_\sigma'$ is obtained by adding a wall ( A small resolution resulting in a rational curve $C$ above corresponding singular point). After this resolution it still has one simplicial singular cone which is a quotient singularity $\mathbb{C}^4/\mathbb{Z}_2$ and does not have MPCP desingularization. All 3-dimensional sub cones of the first item are non-singular, so we only have one singular point on corresponding affine chart. The cones in second item are all smooth. The cones in 3rd item have quotient singularity $\mathbb{C}^3/\mathbb{Z}_2$ and there is no MPCP resolution for them. So there is a total of 8 singular points on $X_\Delta$ all of type $\mathbb{C}^3/\mathbb{Z}_2$. The Calabi-Yau 3-fold $Z_\Delta \subset X_\Delta$ is smooth and does not see the MPCP-resolution $X_\Delta \to X_\Delta$.

Put $v = e_1 + e_2 + e_3 - 2e_4$, then any Weil divisor on $X$ is of the form $\sum_{i=1}^3 (a_i D_v + b_i D_{(-e_i)}) + c D_{e_4} + d D_v$ but not all of them are $\mathbb{Q}$-Cartier. The $\mathbb{Q}$-Cartier divisors are of the form $\sum_{i=1}^3 (a_i D_v + b_i D_{(-e_i)}) + c D_{e_4} + (\sum a_i - 2c) D_v$; so the parameter $d$ can not be chosen arbitrarily for $\mathbb{Q}$-Cartier divisors. Therefore $\text{Pic}(X_\Delta) \otimes \mathbb{Q} = \mathbb{Q}^3$. 
On $\overline{X}_\Delta$ all coefficients can be determined independently so $\text{Pic}(\overline{X}_\Delta) = \mathbb{Q}^4$ and in fact, the proper transform of any Weil divisor to $\overline{X}_\Delta$ is $\mathbb{Q}$-Cartier.

From the formula (1) we have $\nu^{-1}h^1(Z_\Delta) = 8 - 4 - 0 = 4$. This shows that $\text{Pic}(\overline{X}_\Delta) \otimes \mathbb{Q} \rightarrow \text{Pic}(Z_\Delta) \otimes \mathbb{Q}$ is an isomorphism, but $\text{Pic}(X_\Delta) \otimes \mathbb{Q} \rightarrow \text{Pic}(Z_\Delta) \otimes \mathbb{Q}$ is just an embedding (by proposition 3.5 of [DKH]).

Considering cone of effective curves this time we see that the rational curve $C$ obtained from resolution of $X_\sigma$ is in $\overline{\text{NE}}(\overline{X}_\Delta)$, but it can not be in the image of $\iota_*\overline{\text{NE}}(Z_\Delta)$, contrary to the Kollar’s case.

One can calculate the Euler-characteristic of this example and see that $\chi(X) = -6$. Physicists are interested in C.Y threefolds with Euler-characteristic $\pm 6$.

For any smooth C.Y threefold $Z$, the second chern class of tangent bundle $c_2(Z)$ is an integral class in $H^4(Z)$ or dually in $N_1(Z)$ and so it gives a linear map $c_2(x) \wedge - : \text{Pic}(Z) \rightarrow \mathbb{Z}$. As we mentioned before, from theorem 1.1 of [Mi] we know that this linear map is non-negative on kahler cone i.e. $[c_2(Z)] \in \overline{\text{NE}}(Z)$.

Proof. (of theorem 1.1 Wilson [W1]) In both case we get an intrinsically determined projective embedding of $Z$. In the first case when $c_2$ is strictly positive, the intersection of $kZ$ with level sets $c_2 \cdot w \equiv a$, $a \in \mathbb{Z}$, are bounded, therefore there are finitely many integral points on it. Let $a$ be the first positive integer where this set is non-empty. Then consider the ample class obtained by sum of elements of this set and the projective embedding determined by a multiple of this class into some $\mathbb{P}^n$. Thus any automorphism of $Z$ comes from a linear automorphism of $\mathbb{P}^n$. Therefore $\text{Auto}(Z)$ has finitely many components and since it is discrete, it would be finite. Similar story is true for the second case. In both cases there is a $\text{Auto}(Z)$-invariant ample class which gives the desired result. $\square$

We end with the proof of theorem 1.2 reviewing some examples and its implication on Wilson conjecture.

Proof. (of theorem 1.2) Let $\Delta$ be a four-dimensional reflexive polytop as before, and $Z_\Delta$, the smooth C.Y obtained from a MPCP resolution of $X_\Delta$. Consider the set of divisors $E_i$ and $F_{ij}$ defined at the end of section 2.

If we assume that $\overline{X}_\Delta$ is smooth, then Chern classes of tangent bundle are given by the equation,

$$c(T\overline{X}_\Delta) = \prod (1 + D_i)$$

where the product is over all its boundary divisors. $Z_\Delta$ is a smooth section of base-point free linear system $|K_{\overline{X}_\Delta}|$; therefore from the exacts sequence

$$0 \rightarrow T\overline{Z}_\Delta \rightarrow T\overline{X}_\Delta \rightarrow N_{\overline{Z}_\Delta, \overline{X}_\Delta} \rightarrow 0$$

we get,

$$c(T\overline{Z}_\Delta) = c(T\overline{X}_\Delta) / c(N_{\overline{Z}_\Delta, \overline{X}_\Delta}) = \prod (1 + D_i) / (1 + \sum D_i)$$

(3)
\[
(1 + \left(\sum D_i\right) + \left(\sum D_i \cdot D_j\right) + \cdots)(1 - \left(\sum D_i\right) + \left(\sum D_i\right)^2 + \cdots) =
\]
\[
1 + 0 + \left(\sum_{i<j} D_i \cdot D_j\right) - \left(\sum D_i\right)^2 + \left(\sum D_i\right)^2 + \cdots
\]

therefore
\[
c_2(TZ_\Delta) = c_2(TX_\Delta) = \left(\sum_{i<j} D_i \cdot D_j\right) |_{Z_\Delta}
\]
or
\[
[c_2(Z_\Delta)] = \sum C_{ij}
\]

where \( C_{ij} = Z_\Delta \cap D_i \cap D_j \) is a curve class on \( Z \). From this formula it is easy to see that \( c_2(Z_\Delta) \in \text{NE}(Z_\Delta) \), because it is given as a sum of effective classes.

In general case, \( X_\Delta \) may have isolated \( \mathbb{Q} \)-factorial singularities and it is an orbifold. So we may either try to extend the definition of tangent bundle to this case and show that exact sequence \( \text{(2)} \) still makes sense, or we can do the following easy trick. Consider a toric resolution \( Y \to X_\Delta \) whose exceptional divisors are supported on the isolated singularities of \( X_\Delta \). Then write the same exact sequence this time for \( Z_\Delta \subset Y \) to get a formula for Chern classes of \( Z_\Delta \). But then since \( Z_\Delta \) does not meet the singularities of \( X_\Delta \), we see that after reduction to \( Z_\Delta \) in formula \( \text{(3)} \), only boundary divisors of \( X_\Delta \) contribute and we get the same equation for the second Chern class of \( X_\Delta \). So the formulas given above still makes sense for general case.

Let's try to understand the curves \( C_{ij} \) in more details. As before, we divide the boundary integral vertices of \( \Delta^* \) into four sets,

1. A vertex in the interior of a 3-dimensional face.
2. A vertex in the interior of a 2-dimensional face.
3. A vertex in the interior of a 1-dimensional face.
4. An original vertex of \( \Delta^* \).

Let \( S^{ij} \) be a toric surface in \( X_\Delta \) obtained from the intersection of two vertices of type \( 1 \leq i \leq j \leq 4 \). We investigate \( S^{ij} \cap Z_\Delta \) case by case.

- \( S^{ij} \cap Z_\Delta \) : is empty since any toric divisor coming from type (1) vertex does not meet \( Z_\Delta \).
- \( S^{ij} \cap Z_\Delta , i \leq 3 \) : The intersection is empty unless the two points are connected by a line in triangulation and there are 2 possibilities for a line joining these two points: If it is contained in a 3-dim face, then the corresponding surface in resolution lies over a singular point of toric variety and so has no intersection with \( Z_\Delta \). If it is contained in 2-dimensional face, the corresponding surface lies over a curve and is of the form \( \mathbb{P}^1 \times \mathbb{P}^1 \) so that its intersection with \( Z_\Delta \) is a collection of toric rational curves.
- \( S^{ij} \cap Z_\Delta , i > 1 \) : This time there are 3 possibilities for connecting line (if exists). If it is inside a 3-dimensional face then the intersection with \( Z_\Delta \) of corresponding surface is empty. If it lies in 2-dimensional face, the situation
is as above. If it lies in a 1-dim face then the corresponding curve on $\mathbb{Z}_\Delta$ is the curve in the intersection of two branches of exceptional divisor obtained from the resolution of singular curve of $\mathbb{Z}_\Delta$.

- $S^{11} \cap \mathbb{Z}_\Delta$: These corresponds to open smooth curves of $\mathbb{Z}_\Delta$.

To prove the theorem, it is enough to show that the set of curves $C_{ij}$ appearing in the formula above includes a basis for $N_1(\mathbb{Z}_\Delta) \otimes \mathbb{Q}$, because then it would imply that $[c_2]$ is an interior point of $\overline{NE}(\mathbb{Z}_\Delta)$.

From Hard-Lefschetz theorem we know that the map

$$H^{1,1}(\mathbb{Z}_\Delta, \mathbb{Q}) \otimes H^{1,1}(\mathbb{Z}_\Delta, \mathbb{Q}) \to H^4(\mathbb{Z}_\Delta, \mathbb{Q}) \cong N_1(\mathbb{Z}_\Delta, \mathbb{Q})$$

is surjective. Therefore elements of the form $E_i \cdot E_j$, $E_i \cdot F_{ij}$ and $F_{ij} \cdot F_{kl}$ generate $N_1(\mathbb{Z}_\Delta)$. The summation in the equation of second chern class includes all these intersections except the intersection between two identical divisor. But any toric divisor on $X_\Delta$ is equivalent to some linear combination of other toric divisors and therefore the intersection between identical divisors can be written as a linear combination of former ones.

Together with Wilson’s theorem this proves that all Batyrev C.Y threefolds have finite automorphism group. Therefore either all of them have rational polyhedral Kahler cone or there is a counter-example to Morrison conjecture among these manifolds.

Note that Kollar’s results says that for smooth $X_\Delta$, $Z_\Delta$ has rational polyhedral Kahler cone and therefore its combination with what we just proved here, results in validity of Morrison conjecture. We believe that some thing similar to Kollar’s result holds for general case and therefore Morrison conjecture holds for all Batyrev C.Y threefolds. □

Remark 3.1. This theorem is no longer true for complete intersections. The example discussed in [GM] is the intersection of two hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. Its Picard number is 19 and the curve class corresponding to its second chern class lies in a two-dimensional face of boundary. They also proved that there is a rational polyhedral fundamental domain for the action of automorphism group on Kahler cone.

References

[B1] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties J. Algebraic Geom. 3 (1994), no. 3, 493535
[Bor] Borcea, Ciprian On desingularized Horrocks-Mumford quintics. J. Reine Angew. Math. 421 (1991), 2341
[BK] Borcea, Ciprian Homogeneous vector bundles and families of Calabi-Yau threefolds. II. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 8391, Proc. Sympos. Pure Math., 52, Part 2,
[CDGP] Candelas, Philip; de la Ossa, Xenia C.; Green, Paul S.; Parkes, Linda A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. Nuclear Phys. B 359 (1991), no. 1, 2174.
[CK] David A. Cox, Sheldon Katz, Mirror symmetry and algebraic geometry, mathematical surveys and monographs vol. 68.
[DKH] Danilov and Khovanskii, Newton polytop.
[GM] Antonella Grassi, David R. Morrison, 
Automorphisms and the Kahler cone of certain Calabi-Yau manifolds. Vol. 71, No. 3 Duke mathematical Journal, Sep 1993.

[Kov] Sandor J. Kovacs, The cone of curves of $K3$ surfaces Math. Ann. 300, 681-691 (1994)

[Mi] Miyaoka, Y The Chern class and Kodaira dimension of a minimal variety. Adv. Stud. Pure Math., vol. 10, pp. 449-476.

[M1] D.R. Morrison, Hodge-theoretic aspects of mirror symmetry.

[M2] D.R. Morrison Compactifications of moduli spaces inspired by mirror symmetry. arXiv:alg-geom/9304007v1

[W1] P.M.H Wilson The role of $c_2$ in the Calabi-Yau classification. Mirror symmetry, II, 381-392, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997,

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

E-mail address: mfarajza@math.princeton