Pseudo Hermitian formulation of Black-Scholes equation

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Abstract

We show that the non Hermitian Black-Scholes Hamiltonian and its various generalizations are \( \eta \)-pseudo Hermitian. The metric operator \( \eta \) is explicitly constructed for this class of Hamiltonians. It is also shown that the effective Black-Scholes Hamiltonian and its partner form a pseudo supersymmetric system.

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I. INTRODUCTION

During the past few years there has been a great interest in studying problems of finance using various tools of physics [1]. In particular, different problems of finance have been studied from the point of view of quantum physics [2–5]. For example, various options have been modelled using quantum mechanical potentials [6], option pricing with stochastic volatility has been studied using the path integral technique [7]. Also, quantum mechanics has been used to analyze option pricing, stock market returns [8–10], Black-Scholes (BS) equation [11, 12] etc. Supersymmetry formalism has been employed to obtain new solvable diffusion processes [13].

The BS equation (and its various generalizations) plays a dominant role in option pricing. The solutions of the BS equation may be found by mapping it into a Schrödinger like equation. Then various quantities like the pricing kernel or the option price can be obtained using the solutions of the Schrödinger like equation. It may be pointed out that from the point of view of quantum mechanics the BS Hamiltonian is non Hermitian. On the other hand during the last decade non Hermitian quantum mechanics has been studied extensively. A feature of such systems is that Schrödinger equation with many non Hermitian potentials admit real eigenvalues [14]. Subsequently it was shown that this unusual feature may be attributed to \( \mathcal{PT} \) symmetry [14] or more generally to \( \eta \)-pseudo Hermiticity [15]. Here our objective is to show that the BS Hamiltonian and its various generalizations are \( \eta \)-pseudo Hermitian and we shall determine the explicit form of the metric operator \( \eta \) for each case. We shall also show that the effective BS Hamiltonian together with its partner Hamiltonian [16] form a pseudo supersymmetric system.

II. \( \eta \)-PSEUDO HERMITICITY OF BS HAMILTONIAN

The BS equation for option pricing with constant volatility is given by [2]

\[
\frac{\partial C}{\partial t} = -\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS \frac{\partial C}{\partial S} + rC
\]

where \( C, S, \sigma \) and \( r \) denotes the price of the option, the stock price, the volatility of the stock price and the risk-free spot interest rate respectively. Now under the transformations [2]

\[
C(S, t) = e^{\xi} \psi(S) \quad \text{and} \quad S(x) = e^{\xi}, \quad -\infty < x < \infty
\]
the BS equation (1) becomes

\[ H_{BS} \psi = \epsilon \psi \]

\[ H_{BS} = -\frac{\sigma^2}{2} \frac{d^2}{dx^2} + \left( \frac{\sigma^2}{2} - r \right) \frac{d}{dx} + r \]

(3)

where \( H_{BS} \) is called the BS Hamiltonian. It is well known that the BS Hamiltonian in Eq.(3) can be brought to the Schrödinger form [2, 3]. To show this we use the similarity transformation

\[ \rho H_{BS} \rho^{-1} = h_{BS} \]

(4)

where

\[ \rho = \exp \left[ -\left( \frac{1}{2} - \frac{r}{\sigma^2} \right) x \right], \quad h_{BS} = -\frac{\sigma^2}{2} \frac{d^2}{dx^2} + \frac{1}{2} \frac{\sigma^2}{2} + r \]

(5)

The Hamiltonian \( h_{BS} \) in Eq.(5) can be interpreted as a Schrödinger Hamiltonian of a particle of mass \( \frac{1}{\sigma^2} \) moving in a constant potential \( \frac{1}{2\sigma^2} \left( \frac{\sigma^2}{2} + r \right)^2 \). It is important to note that the BS Hamiltonian in Eq.(3) is non Hermitian while the Hamiltonian \( h_{BS} \) in Eq.(5) is Hermitian.

We shall now show that the BS Hamiltonian \( H_{BS} \) is in fact \( \eta \)-pseudo Hermitian.

We recall that a Hamiltonian \( H \) is said to be \( \eta \)-pseudo Hermitian if

\[ H^\dagger = \eta H \eta^{-1} \]

(6)

where \( \eta \) is a Hermitian operator. It has been shown that eigenvalues of a \( \eta \)-pseudo Hermitian Hamiltonian are either completely real or occur in complex conjugate pairs [15]. In the context of financial modeling the BS equation usually has real eigenvalues and consequently it is of interest to examine the BS Hamiltonian from the point of view of pseudo Hermiticity.

Let us now define the metric operator \( \eta \) as

\[ \eta = \rho^2 = \exp \left[ -\left( 1 - \frac{2r}{\sigma^2} \right) x \right] \]

(7)

Then it can be verified that \( \eta = \eta^\dagger \) and

\[ H_{BS}^\dagger = \eta H_{BS} \eta^{-1} \]

(8)

so that the BS Hamiltonian is \( \eta \)-pseudo Hermitian. Two important properties, namely, the completeness relation and the scalar product get modified for non Hermitian systems. In the present case they are given by

\[ \sum_n |\psi_n \rangle \langle \psi_n| \eta = 1, \quad \langle \psi_m | \psi_n \rangle_\eta = \langle \psi_m | \eta \psi_n \rangle = \int \eta(x) \psi_m^*(x) \psi_n(x) dx = \delta_{mn} \]

(9)
The above relations may be used to determine the pricing kernel. The pricing kernel 
\( p(x, \tau, x') \) is defined as the conditional probability that the stock which has a value \( e^x \) at 
time \( t \) will have a value \( e^{x'} \) at time \( T = t + \tau \). The pricing kernel for the BS Hamiltonian is 
then given by

\[
p(x, \tau, x') = \langle x | e^{-\tau H_{BS}} | x' \rangle = \sum_n \eta(x') e^{-\tau \epsilon_n} \psi_n^*(x') \psi_n(x) \tag{10}
\]

Then the option price is given by

\[
C(x, t) = \int \eta(x') p(x, \tau, x') g(x') \, dx' \tag{11}
\]

where \( g(x) \) is the pay off function.

A. \( \eta \)-Pseudo Hermiticity of the Generalized BS Hamiltonian

Sometimes the BS Hamiltonian can be generalized by including a security dependent 
potential \( V(x) \). The resulting generalized Hamiltonian which satisfies the martingale condition 
is given by [2]

\[
H = \sigma^2 \frac{d^2}{dx^2} + \left( \frac{\sigma^2}{2} - V(x) \right) \frac{d}{dx} + V(x) \tag{12}
\]

For an interpretation of the potential \( V(x) \) from the point of view of finance we refer the 
reader to ref [2]. Now, the generalized BS Hamiltonian in (12) is again non Hermitian. This 
can be seen from the fact that

\[
H^\dagger = -\frac{\sigma^2}{2} \frac{d^2}{dx^2} - \left( \frac{\sigma^2}{2} - V(x) \right) \frac{d}{dx} + V'(x) + V(x) \neq H \tag{13}
\]

The similarity transformation which transforms \( H \) into the Schrödinger form is given by

\[
\rho = \exp \left[ \frac{1}{\sigma^2} \int^x V(y) dy - \frac{1}{2} x \right] \tag{14}
\]

\[
h = \rho H \rho^{-1} = -\frac{\sigma^2}{2} \frac{d^2}{dx^2} + \frac{1}{2} V' + \frac{1}{2 \sigma^2} \left( V + \frac{1}{2} \sigma^2 \right)^2
\]

To show the \( \eta \)-pseudo Hermiticity of the generalized BS Hamiltonian \( H \) we define the metric 
operator \( \eta \) as in the last section i.e,

\[
\eta = \rho^2 = \exp \left[ \frac{2}{\sigma^2} \int^x V(y) dy - x \right] \tag{15}
\]

Then \( \eta = \eta^\dagger \) and after some calculations it can be shown that

\[
\eta H \eta^{-1} = H^\dagger \tag{16}
\]

so that the generalized BS Hamiltonian is \( \eta \)-pseudo Hermitian.
B. \( \eta \)-pseudo Hermiticity of the effective BS Hamiltonians

Path dependent options such as the Down-and-out barrier option, Out-and-out barrier option or the Double-knock-out barrier option can be analyzed by adding a potential term to the BS Hamiltonian (3) and the effective Hamiltonian is given by [2, 8]

\[
H_{\text{eff}} = H_{BS} + V(x) \tag{17}
\]

where \( H_{BS} \) is given by (3). Clearly for a real potential \( V(x) \) we have

\[
H_{\text{eff}}^\dagger = H_{BS}^\dagger + V(x) = -\frac{\sigma^2}{2} \frac{d^2}{dx^2} - \left( \frac{\sigma^2}{2} - r \right) \frac{d}{dx} + r + V(x) \neq H_{\text{eff}} \tag{18}
\]

Since \( V(x) \) is real, it is clear that in this case the metric operator is given by (7) i.e.,

\[
\eta = \exp \left[ - \left( 1 - \frac{2r}{\sigma^2} \right) x \right]
\]

\[
\eta H_{\text{eff}} \eta^{-1} = H^\dagger \tag{19}
\]

In other words, the effective Hamiltonian \( H_{\text{eff}} \) is \( \eta \)-pseudo Hermitian.

III. FACTORIZATION OF EFFECTIVE HAMILTONIANS

It may be noted that factorization approach to effective Hamiltonians is often useful to find new exactly solvable processes [13, 16]. To use such a technique it is necessary to write the effective Hamiltonian as a combination of two operators \( A \) and \( B \):

\[
H_{\text{eff}} = BA + \delta
\]

\[
B = \frac{\sigma}{\sqrt{2}} \left[ -\frac{d}{dx} + W(x) + \left( \frac{1}{2} - \frac{r}{\sigma^2} \right) \right]
\]

\[
A = \frac{\sigma}{\sqrt{2}} \left[ \frac{d}{dx} + W(x) - \left( \frac{1}{2} - \frac{r}{\sigma^2} \right) \right]
\]

\[
\delta = \frac{1}{2\sigma^2} \left( \frac{\sigma^2}{2} - r \right)^2 + r
\]

where \( V(x) = \frac{\sigma^2}{2} [W^2(x) - W''(x)] \). Clearly \( A \) and \( B \) are not Hermitian conjugates of each other. However, it can be shown that

\[
B = \eta^{-1} A^\dagger \eta \equiv A^\# \tag{21}
\]
so that $A$ and $A^\#$ are $\eta$-pseudo adjoints of each other. Consequently the effective BS Hamiltonian may be written as

$$H_{\text{eff}} = A^\# A + \delta$$  \hspace{1cm} (22)

From Eq.\,(21) it is possible to introduce the concept of the partner of the effective Hamiltonian \[16\]. Reversing the order of the operators in Eq.\,(22) we find

$$H_{\text{eff}, P} = AA^\# + \delta = H_{\text{eff}} + V_P$$  \hspace{1cm} (23)

where $V_P = \frac{\sigma^2}{2} [W^2(x) + W'(x)]$. The Hamiltonian in Eq.\,(23) may be called the partner of the effective Hamiltonian. We shall now show that the effective Hamiltonians $H_{\text{eff}}$ and $H_{\text{eff}, P}$ are related by pseudo supersymmetry. To this end let us now consider an operator $Q$ defined by

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (24)

so that its pseudo adjoint is given by

$$Q^\# = \eta^{-1} Q^\dagger \eta = \begin{pmatrix} 0 & 0 \\ A^\# & 0 \end{pmatrix}$$  \hspace{1cm} (25)

Then it can be easily shown that

$$\{ Q, Q^\# \} = \mathcal{H}, \quad [Q, \mathcal{H}] = [Q^\#, \mathcal{H}] = 0$$  \hspace{1cm} (26)

where $\mathcal{H}$ is given by

$$\mathcal{H} = \begin{pmatrix} H_{\text{eff}, P} - \delta & 0 \\ 0 & H_{\text{eff}} - \delta \end{pmatrix}$$  \hspace{1cm} (27)

Furthermore, it may be shown that

$$\eta \mathcal{H} \eta^{-1} = \mathcal{H}^\dagger$$  \hspace{1cm} (28)

so that $\mathcal{H}$ is $\eta$-pseudo Hermitian. The relations (26) constitute the $\eta$-pseudo supersymmetry algebra. Finally it may be noted that in case $r \to \frac{\sigma^2}{2}$, the operators $A$ and $B$ defined in \[20\] become

$$A = \frac{\sigma}{\sqrt{2}} \left[ \frac{d}{dx} + W(x) \right]$$

$$B = \frac{\sigma}{\sqrt{2}} \left[ -\frac{d}{dx} + W(x) \right] = A^\dagger$$  \hspace{1cm} (29)
Also in this case \( \eta = 1 \) and the pseudo supersymmetry algebra reduces to classical supersymmetry algebra. This is a consequence of the fact that in this limit the effective Hamiltonian \cite{17} becomes Hermitian because of the absence of the first order derivative term. An explicit example can be found in ref \cite{16}.

IV. CONCLUSIONS

Here we have shown that the quantum BS Hamiltonian and it’s various generalizations are \( \eta \)-pseudo Hermitian. The metric operator \( \eta \) has also been found for each of these BS Hamiltonians. It has also been shown that the effective BS Hamiltonian and its partner Hamiltonian form a pseudo supersymmetric system.

It may be recalled that here we have considered real potentials. However, in some situations (for example, to treat Asian options) it may be necessary to use complex potentials in the effective BS Hamiltonian \cite{17} and it would be of interest to examine \( \mathcal{PT} \) symmetry or \( \eta \)-pseudo Hermiticity of such systems. Finally we feel it would also be of interest to investigate higher dimensional systems like the Merton-German Hamiltonians \cite{2} from the point of view of \( \eta \)-pseudo Hermiticity.

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