INVARIANT HILBERT SCHEMES AND DESINGULARIZATIONS OF SYMPLECTIC REDUCTIONS FOR CLASSICAL GROUPS

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Abstract. Let $G \subset GL(V)$ be a reductive algebraic subgroup acting on the symplectic vector space $W = (V \oplus V^\ast)^{\otimes m}$, and let $\mu : W \to \text{Lie}(G)^\ast$ be the corresponding moment map. In this article, we use the theory of invariant Hilbert schemes to construct a canonical desingularization of the symplectic reduction $\mu^{-1}(0)/G$ for classes of examples where $G = GL(V)$, $O(V)$, or $Sp(V)$. For these classes of examples, $\mu^{-1}(0)/G$ is isomorphic to the closure of a nilpotent orbit in a simple Lie algebra, and we compare the Hilbert-Chow morphism with the (well-known) symplectic desingularizations of $\mu^{-1}(0)/G$.

Contents

1. Introduction and statement of the main results 1
2. Generalities on symplectic varieties and closures of nilpotent orbits 3
3. Case of $GL_n$ 5
4. Case of $Sp_n$ 14
References 20

1. Introduction and statement of the main results

First of all, let us recall briefly the definition of the invariant Hilbert scheme, constructed by Alexeev and Brion (see [AB05, Bri13] for more details). We work over the field of complex numbers $\mathbb{C}$. Let $G$ be a reductive algebraic group, and let $h : \text{Irr}(G) \to \mathbb{N}$ be a Hilbert function which assigns to every irreducible representation of $G$ a nonnegative integer. If $X$ is an (possibly reducible) affine $G$-variety, then the invariant Hilbert scheme $\text{Hilb}_G^h(X)$ is the moduli space that parametrizes the $G$-stable closed subschemes $Z$ of $X$ such that

$$\mathbb{C}[Z] \cong \bigoplus_{M \in \text{Irr}(G)} M^{\otimes h(M)}$$

as a $G$-module. Let us now suppose that the categorical quotient

$$X//G := \text{Spec}(\mathbb{C}[X]^G)$$

is an irreducible variety. If $h = h_0$ is the Hilbert function of the general fibers of the quotient morphism $\nu : X \to X//G$ (that is, the fibers over a nonempty open subset of $X//G$), then there exists a projective morphism

$$\gamma : \text{Hilb}_G^{h_0}(X) \to X//G,$$

called the Hilbert-Chow morphism, that sends a closed subscheme $Z \subset X$ to the point $Z//G \subset X//G$. The Hilbert-Chow morphism induces an isomorphism over the
flat locus $U \subset X\!/\!G$ of $\nu$. The main component of $\text{Hilb}^G_{h_0}(X)$ is the irreducible component defined by

$$\text{Hilb}^G_{h_0}(X)_{\text{main}} := \gamma^{-1}(U).$$

Then the restriction $\gamma : \text{Hilb}^G_{h_0}(X)_{\text{main}} \to X\!/\!G$ is a projective birational morphism, and thus $\gamma$ is a candidate for a canonical desingularization of $X\!/\!G$. It is an open problem to determine whether this restriction is always a desingularization or not.

Last, but not least, if $H$ is any algebraic subgroup of the $G$-equivariant automorphism group $\text{Aut}^G(X)$, then $H$ acts on $X\!/\!G$ and $\text{Hilb}^G_{h_0}(X)$, and the quotient morphism $\nu : X \to X\!/\!G$ and the Hilbert-Chow morphism $\gamma : \text{Hilb}^G_{h_0}(X) \to X\!/\!G$ are $H$-equivariant.

Let now $G$ be an algebraic group, let $\mathfrak{g}$ be the Lie algebra of $G$, and let $W$ be a symplectic $G$-module, that is, a $G$-module equipped with a $G$-invariant non-degenerate skew-symmetric bilinear form. Then $W$ has a $G$-equivariant moment map

$$\mu_G : W \to \mathfrak{g}^*, $$

which is defined in the usual way. To simplify the notation, we will use $\mu$ instead of $\mu_G$. The map $\mu$ being $G$-equivariant, the set-theoretic fiber $\mu^{-1}(0)$ is a $G$-stable subvariety of $W$. From now on, we suppose that $G$ is reductive. The categorical quotient $\mu^{-1}(0)/\!/G$ is called the symplectic reduction of $W$ by $G$ and plays a central role in the study of $W$. It is an important problem to determine whether $\mu^{-1}(0)/\!/G$ admits a symplectic desingularization (which is a distinguished desingularization, see Section 2 for details); a candidate for such a desingularization is given by the Hilbert-Chow morphism $\gamma : \text{Hilb}^G_{h_0}(\mu^{-1}(0))_{\text{main}} \to \mu^{-1}(0)/\!/G$.

Let us take $V$ a finite dimensional vector space, and $m$ a nonnegative integer. In this paper, we are interested in the cases where

$$W := (V \oplus V^*)^\oplus m,$$

on which $GL(V)$ acts naturally, and $G = GL(V), O(V),$ or $Sp(V)$. In this situation, we can find a classical algebraic subgroup $H \subset \text{Aut}^G(\mu^{-1}(0))$, namely

1. $H = GL_m$ for $G = GL(V)$;
2. $H = Sp_{2m}$ for $G = O(V)$; and
3. $H = SO_{2m}$ for $G = Sp(V)$.

In each case, we will see that the symplectic reduction $\mu^{-1}(0)/\!/G$ identifies with a nilpotent orbit closure in the Lie algebra $\mathfrak{h}$ of $H$, except in Case (3) for $m$ even and $m \leq \text{dim}(V)$ where it is the union of two such orbit closures. In Case (3), if $\mu^{-1}(0)/\!/G = Y_1 \cup Y_2$ is reducible, then we always consider only one component to simplify the statements (that is, $\mu^{-1}(0)/\!/G$ should be replaced by $Y_i$ and $\mu^{-1}(0)$ by $\nu^{-1}(Y_i)$). The geometry of nilpotent orbits has been extensively studied by Fu, Kraft, Namikawa, Procesi... ([KS9, KS9], [KPS], [Fu03a, Fu03b, Fu06b, Na06]). In particular, the normalizations of such closures are symplectic varieties (as defined by Beauville in [Bea00]) whose symplectic desingularizations are the so-called Springer desingularizations, obtained by collapsing the cotangent bundle over some flag varieties (see Section 2 for details).

In [Terb, Tera], we studied the invariant Hilbert scheme for classical groups acting on classical representations. We obtained classes of examples where the
Hilbert-Chow morphism is a desingularization of the categorical quotient, and further examples where it is not. In this article, we use the results of [Tera] to prove the following statements:

**Theorem A.** ([Sections 3.3 and 4.3]) With the above notation, let $G = GL(V)$, $O(V)$, or $Sp(V)$, then the Hilbert-Chow morphism $\gamma : \text{Hilb}^G_{h_0}(\mu^{-1}(0))_{\text{main}} \to \mu^{-1}(0)/G$ is a symplectic desingularization (and the unique one) if and only if

- $G = GL(V)$, $\dim(V) \geq m - 1$, and $m$ is even; or
- $G = O(V)$, and $\dim(V) \geq 2m - 1$; or
- $G = Sp(V)$, $\dim(V)$ and $m$ are even, and $\dim(V) \geq 2m - 2$.

**Theorem B.** ([Sections 3.3 and 4.3]) With the above notation, let $G = GL(V)$, $O(V)$, or $Sp(V)$, then the Hilbert-Chow morphism $\gamma : \text{Hilb}^G_{h_0}(\mu^{-1}(0))_{\text{main}} \to \mu^{-1}(0)/G$ is a desingularization that strictly dominates the symplectic desingularizations (when they exist) in the following cases:

- $G = GL(V)$ and either $\dim(V) = 1$, $m \geq 3$ or $\dim(V) = 2$, $m \geq 4$; or
- $G = O(V)$ and either $\dim(V) = 1 < m$ or $\dim(V) = 2 \leq m$; or
- $G = Sp(V)$ and either $\dim(V) = 2 < m$ or $\dim(V) = 4 \leq m$.

If $G \subset GL(V)$ is any reductive algebraic subgroup, then it is generally a difficult problem to determine whether $\text{Hilb}^G_{h_0}(\mu^{-1}(0))$ is irreducible, that is, equals its main component. In this direction, we obtain

**Proposition C.** ([Propositions 3.17 and 4.10]) With the above notation, if $G = GL(V)$ and $m \geq 2 \dim(V)$, then the invariant Hilbert scheme $\text{Hilb}^G_{h_0}(\mu^{-1}(0))$ has at least two irreducible components (and exactly two when $\dim(V) = 1$). On the other hand, if $G = O(V)$ or $Sp(V)$, and $m \geq \dim(V) = 2$, then $\text{Hilb}^G_{h_0}(\mu^{-1}(0))$ is irreducible.

In Section 2 we recall some basic facts about symplectic varieties and closures of nilpotent orbits in simple Lie algebras. The case of $GL(V)$ is treated in Section 3 and the case of $Sp(V)$ is treated in Section 4. The case of $O(V)$ is quite similar to the case of $GL(V)$, and details can be found in the thesis [Tera] §3.4 from which this article is extracted. Besides, we think that our methods also apply when $G = SL(V)$, while the case $G = SO(V)$ should be more involved.

## 2. Generalities on symplectic varieties and closures of nilpotent orbits

### 2.1. Symplectic varieties and symplectic desingularizations

Let us first recall the definitions of symplectic variety and symplectic desingularization (see [Bea00] or the survey [Paul09] for more details). Let $Y$ be a normal variety whose regular locus $Y_{\text{reg}}$ admits a *symplectic form* $\Omega$ (that is, $\Omega$ is a holomorphic 2-form which is closed and non-degenerate at every point of $Y_{\text{reg}}$) such that, for any desingularization $f : \tilde{Y} \to Y$, the 2-form $f^*(\Omega)$ extends to a 2-form on the whole $\tilde{Y}$, then we say that $Y$ is a *symplectic variety*. Moreover, if $f : \tilde{Y} \to Y$ is a desingularization such that $f^*(\Omega)$ extends to a symplectic form on $\tilde{Y}$, then we say that $f$ is a *symplectic desingularization* of $Y$. It must be emphasized that symplectic varieties do not always admit symplectic desingularizations, and when they do, there may be several of them.
As in the introduction, we denote $W = (V \oplus V^*)^{\oplus m}$, we take a reductive algebraic subgroup $G \subset GL(V)$ acting naturally on $W$, and we consider the symplectic reduction $\mu^{-1}(0)/G$. The following conjecture motivates the study (and the name!) of $\mu^{-1}(0)/G$:

**Conjecture 2.1** (Kaledin, Lehn, Sorger). With the above notation, the irreducible components $Y_1, \ldots, Y_r$ of $\mu^{-1}(0)/G$ are symplectic varieties. Moreover, if every $Y_i$ admits a symplectic desingularization, then the quotient $V^{\oplus m}/G$ is smooth.

When $G$ is a finite group, Conjecture 2.1 was proved by Kaledin and Verbitsky, but the general case remains open. Let us mention that Becker showed in [Bec09] that the converse of the second part of Conjecture 2.1 holds for $G = Sp(V)$ with $\dim(V) = 2$. In our setting, that is when $G = GL(V)$, $O(V)$, or $Sp(V)$, one easily checks that Conjecture 2.1 holds (see [Ter], §A.2) for details).

2.2. Closures of nilpotent orbits. We now recall some basic facts concerning the closures of nilpotent orbits (see [CM93, Tu03a] for more details). The following well-known result is due to Kostant, Kirillov, Souriau, and Panyushev:

**Theorem 2.2.** The normalization of the closure of an adjoint orbit in a semi-simple Lie algebra is a symplectic variety.

Let now $\mathfrak{h}$ be a simple Lie algebra of classical type. If $\mathfrak{h} = \mathfrak{sl}_m$, then every nilpotent element $f \in \mathfrak{sl}_m$ is conjugate to an element of the form $\text{diag}(J_{d_1}, \ldots, J_{d_k})$, where each $J_{d_i}$ is a Jordan block of size $d_i$, and $d = [d_1, \ldots, d_k]$ is a partition of $m$. Then there exists a one-to-one correspondence between the partitions $d = (d_1 \geq \cdots \geq d_k)$ of $m$ and the nilpotent orbits $O_d \subset \mathfrak{sl}_m$ (see [CM93, §3.1]). Now if $\mathfrak{h} = \mathfrak{sp}_{2m}$ resp. if $\mathfrak{h} = \mathfrak{so}_{2m}$, then a similar description exists (see [CM93, §5.1]), it is obtained by cutting $\mathfrak{h}$ with a $SL_{2m}$-orbit $O_d \subset \mathfrak{sl}_{2m}$, which gives a unique $Sp_{2m}$-orbit resp. a unique $O_{2m}$-orbit, if it is not empty. Let us note that an $O_{2m}$-orbit can be non-connected giving rise to two $SO_{2m}$-orbits that we will denote $O^l_d$ and $O^r_d$.

If $\mathfrak{h} = \mathfrak{sl}_m$, then $O_d$ is always normal ([KP79]). For the other classical types, the geometry of $O_d$ was studied in [KPS82]; in particular, if $\mathfrak{h} = \mathfrak{sp}_{2m}$ and $d_1 + d_2 \leq 4$ resp. if $\mathfrak{h} = \mathfrak{so}_{2m}$ and $d_1 \leq 2$, then $O_d$ is normal. In the next sections, we will be interested only by conjugacy classes of elements $f \in \mathfrak{h}$ with $f^2 = 0$. Hence, from now on, we only consider partitions $d$ such that each $d_i \leq 2$. By Theorem 2.2, the variety $O_d$ is symplectic, and we are going to describe its symplectic desingularizations (see [Fu03a, Fu06b, PN04] for details).

As before, let $\mathfrak{h}$ be a simple Lie algebra of classical type, and let $H$ be the adjoint group of $\mathfrak{h}$. We consider $f : Z \to \overline{O_d}$ a symplectic desingularization. Then, by [Fu03a, Proposition 3.1], the group $H$ acts naturally on $Z$ in such a way that $f$ is $H$-equivariant. One says that $f$ is a Springer desingularization if there exists a parabolic subgroup $P \subset H$ and a $H$-equivariant isomorphism between $Z$ and the total space of the cotangent bundle over $H/P$, denoted by $T^*(H/P)$. Then, under this isomorphism, the map $f$ becomes

$$T^*(H/P) \ni (h, x) \mapsto \text{Ad}(h).x,$$

where $u$ is the nilradical of the Lie algebra of $P$, and $H \times^P u$ denotes the quotient $(H \times u)/P$ under the (free) action of $P$ given by $(h, u) = (h \circ p^{-1}, \text{Ad}(p).u)$.

**Theorem 2.3.** ([Tu03a, Theorem 3.3]) With the above notation, if $f : Z \to \overline{O_d}$ is a symplectic desingularization, then $f$ is a Springer desingularization.
Thanks to the work of Fu and Namikawa, the Springer desingularizations of \( \overline{\mathcal{O}_d} \) are known (up to isomorphism). In particular:

- Let \( \mathfrak{h} = \mathfrak{sl}_m \) and \( \mathfrak{d} = [2^N, 1^{m-2N}] \) for some \( 0 \leq N \leq \frac{m}{2} \). We denote by \( \text{Gr}(p, \mathbb{C}^m) \) the Grassmannian of \( p \)-dimensional subspaces of \( \mathbb{C}^m \), and by \( \mathcal{T}_1^* \) resp. \( \mathcal{T}_2^* \), the cotangent bundle over \( \text{Gr}(N, \mathbb{C}^m) \) resp. over \( \text{Gr}(m-N, \mathbb{C}^m) \). By [Fu06b] \( \S 2 \), if \( N < \frac{m}{2} \), then \( \mathcal{T}_1^* \) and \( \mathcal{T}_2^* \) are the two Springer desingularizations of \( \overline{\mathcal{O}_d} \); else, \( \mathcal{T}_1^* = \mathcal{T}_2^* \) is the unique Springer desingularization of \( \overline{\mathcal{O}_d} \).

- Let \( \mathfrak{h} = \mathfrak{sp}_{2m} \) and \( \mathfrak{d} = [2^N, 1^{2(m-N)}] \) for some \( 0 \leq N \leq m \). Then \( \overline{\mathcal{O}_d} \) admits a Springer desingularization if and only if \( N = m \) ([Fu03a] Proposition 3.19). We denote by \( \text{IGr}(p, \mathbb{C}^{2m}) \) the Grassmannian of isotropic \( p \)-dimensional subspaces of \( \mathbb{C}^{2m} \), and by \( \mathcal{T}_1^* \) the cotangent bundle over \( \text{IGr}(m, \mathbb{C}^{2m}) \). By [FN04] Proposition 3.5, if \( N = m \), then \( \mathcal{T}_1^* \) is the unique Springer desingularization of \( \overline{\mathcal{O}_d} \).

- Let \( \mathfrak{h} = \mathfrak{so}_{2m} \) and \( \mathfrak{d} = [2^N, 1^{2(m-N)}] \) for some \( 0 \leq N \leq m \) with \( N \) even. If \( N = m \), then one associates to \( \mathfrak{d} \) two distinct nilpotent orbits \( \mathcal{O}_d^I \) and \( \mathcal{O}_d^H \). By [Fu03a] Proposition 3.20, the variety \( \overline{\mathcal{O}_d} \) admits a Springer desingularization if and only if \( N \in \{ m-1, m \} \). We denote by \( \text{OGr}(p, \mathbb{C}^{2m}) \) the Grassmannian of isotropic \( p \)-dimensional subspaces of \( \mathbb{C}^{2m} \). The Grassmannian \( \text{OGr}(p, \mathbb{C}^{2m}) \) is irreducible except if \( p = m \), in which case \( \text{OGr}(m, \mathbb{C}^{2m}) = \text{OG}_I \times \text{OG}_H^* \) is the union of two irreducible components (exchanged by the natural action of \( \mathbb{Z}_2 \)). We denote by \( \mathcal{T}_1^* \) resp. \( \mathcal{T}_2^* \), the cotangent bundle over \( \text{OG}_I \) resp. over \( \text{OG}_H^* \). If \( N = m-1 \), then \( \mathcal{T}_1^* \) and \( \mathcal{T}_2^* \) are the two Springer desingularizations of \( \overline{\mathcal{O}_{2m-1,12}} \) by [Fu06b] \( \S 2 \). If \( N = m \), then \( \mathcal{T}_1^* \) resp. \( \mathcal{T}_2^* \), is the unique Springer desingularization of \( \overline{\mathcal{O}_{2m}} \) resp. of \( \overline{\mathcal{O}_{2m}^H} \), by [FN04] Proposition 3.5.

3. Case of \( GL_n \)

In this section, we denote \( V \) and \( V' \) two finite dimensional vector spaces, and we take \( G = GL(V) \) and \( H = GL(V') \), both acting on

\[ W := \text{Hom}(V', V) \times \text{Hom}(V, V') \]

as follows:

\[ \forall (g, h) \in G \times H, \forall (u_1, u_2) \in W, (g, h). (u_1, u_2) := (g \circ u_1 \circ h^{-1}, h \circ u_2 \circ g^{-1}) \]

We denote by \( \mathfrak{g} \) resp. by \( \mathfrak{h} \), the Lie algebra of \( G \) resp. of \( H \), and \( N := \min \left( \left\lfloor \frac{m}{2} \right\rfloor, n \right) \), where \( n := \text{dim}(V) \), \( m := \text{dim}(V') \), and \( \lfloor . \rfloor \) is the lower integer part.

3.1. The quotient morphism. The two main results of this section are Proposition 3.3, which describes the symplectic reduction \( \mu^{-1}(0) \backslash G \), and Corollary 3.6, which gives the Hilbert function \( h_0 \) of the general fibers of the quotient morphism \( \nu : \mu^{-1}(0) \rightarrow \mu^{-1}(0) \backslash G \).

We recall that \( W \) is equipped with a \( G \)-invariant symplectic form \( \Omega \) defined by:

\[ \forall (u_1, u_2), (u'_1, u'_2) \in W, \Omega((u_1, u_2), (u'_1, u'_2)) := \text{tr}(u'_1 \circ u_2) - \text{tr}(u_1 \circ u'_2) \]

where \( \text{tr}(.) \) denotes the trace. The corresponding moment map is given by:

\[ \mu : W \rightarrow \mathfrak{g}^*, \quad (u_1, u_2) \mapsto (f \mapsto \text{tr}(u_2 \circ f \circ u_1)) \]
and thus the zero fiber of $\mu$ is the $G \times H$-stable subvariety defined by:

$$
\mu^{-1}(0) = \{(u_1, u_2) \in W \mid u_1 \circ u_2 = 0\}.
$$

Let us determine the irreducible components of $\mu^{-1}(0)$ as well as their dimensions. Let $p \in \{0, \ldots, m\}$; we define the subvariety

$$
X_p := \left\{(u_1, u_2) \in W \mid \begin{array}{l}
\text{Im}(u_2) \subset \text{Ker}(u_1) ; \\
\text{rk}(u_2) \leq \min(n, p) ; \\
\dim(\text{Ker}(u_1)) \geq \max(m - n, p).
\end{array} \right\} \subset \mu^{-1}(0),
$$

and we consider the diagram

\[
\begin{array}{ccc}
X_p & \xrightarrow{p_1} & \text{Gr}(p, V') \\
& \xleftarrow{p_2} & \\
\end{array}
\]

where the $p_i$ are the natural projections. We fix $L_0 \in \text{Gr}(p, V')$; the second projection equips $Z_p$ with a structure of homogeneous vector bundle over $\text{Gr}(p, V')$ whose fiber over $L_0$ is isomorphic to $F_p := \text{Hom}(V'/L_0, V) \times \text{Hom}(V, L_0)$. Hence, $Z_p$ is a smooth variety of dimension $p(m - p) + mn$.

**Proposition 3.1.** The irreducible components of $\mu^{-1}(0)$ are

$$
\begin{cases}
X_0, \ldots, X_m & \text{if } m \leq n; \\
X_{m-n}, \ldots, X_n & \text{if } n < m < 2n; \\
X_n & \text{if } m \geq 2n;
\end{cases}
$$

where $X_p$ is defined by (2).

**Proof.** We have

$$
\mu^{-1}(0) = \{(u_1, u_2) \in W \mid \text{Im}(u_2) \subset \text{Ker}(u_1)\} = \bigcup_{i=0}^{m} X_i.
$$

Furthermore, for every $p \in \{0, \ldots, m\}$, the morphism $p_1$ is surjective and $Z_p$ is irreducible, hence $X_p$ is irreducible.

If $m \geq 2n$, then

$$
\begin{cases}
X_0 \subset \cdots \subset X_n; \\
X_n = \cdots = X_{m-n}; \\
X_{m-n} \supset \cdots \supset X_m;
\end{cases}
$$

and thus $\mu^{-1}(0) = X_n$.

If $m < 2n$, then

$$
\begin{cases}
X_0 \subset \cdots \subset X_{\max(0, m-n)}; \\
X_{\min(m,n)} \supset \cdots \supset X_m;
\end{cases}
$$

and one easily checks that there is no other inclusion relation between the $X_p$. \(\square\)

**Corollary 3.2.** The dimension of $\mu^{-1}(0)$ is

$$
\dim(\mu^{-1}(0)) = \begin{cases}
\frac{1}{2}m^2 & \text{if } m < 2n \text{ and } m \text{ is even}; \\
\frac{1}{2}(m^2 - 1) & \text{if } m < 2n \text{ and } m \text{ is odd}; \\
2nm - n^2 & \text{if } m \geq 2n.
\end{cases}
$$
Corollary 3.4. The symplectic reduction of $W$ by $G$ is $\mu^{-1}(0)/\overline{G}$.

Proof. By Proposition 3.1 it suffices to compute the dimension of $X_p$ for some $p$.
If $p \leq n$ or $p \geq m - n$, then one may check that the map $p_1 : Z_p \to X_p$ is birational, and thus $Q(p) = \dim(X_p) = \dim(Z_p) = p(m - p) + mn$. It remains simply to study the variations of the polynomial $Q$ to obtain the result.

We recall that the quotient morphism $W \to W/\overline{G}$ is given by $(u_1, u_2) \mapsto u_2 \circ u_1 \in \text{End}(V') = \mathfrak{h}$, by classical invariant theory (see [Pro07, §9.1.4] for instance). Let us now fix $l \in \{0, \ldots, N\}$. We also fix a basis $B$ of $V$ resp. $B'$ of $V'$, and we introduce some notation that we will use in the proofs of Proposition 3.3 and Lemma 3.5.

(3) $(u_1^l, u_2^l) := \begin{pmatrix} 0_{l,m-l} & I_l & 0_{l,n-l} \\ 0_{n-l,m-l} & 0_{n-l,l} & 0_{m-l,l} \end{pmatrix} \in W$;

(4) $f_l := \begin{pmatrix} 0_{l,m-l} & I_l & 0_{l,n-l} \\ 0_{m-l,m-l} & 0_{m-l,l} \end{pmatrix} \in \mathfrak{h}$.

If $d$ is a partition of $m$, then we denote by $O_d \subset \mathfrak{h} \cong \mathfrak{gl}_m$ the corresponding nilpotent orbit (see Section 2.2).

Proposition 3.3. The symplectic reduction of $W$ by $G$ is $\mu^{-1}(0)/\overline{G} = \overline{O}_{[2N, m-2N]}$.

Proof. If $f \in \mu^{-1}(0)/\overline{G}$, then there exists $(u_1, u_2) \in \mu^{-1}(0)$ such that $f = u_2 \circ u_1$, and thus $f \circ f = (u_2 \circ u_1)^2 = u_2 \circ (u_1 \circ u_2) = 0$. Hence, the inclusion $u \subset n$.

Now, let $f \in \overline{O}_{[2N, m-2N]}$. Up to conjugation by an element of $H$, we can suppose that $f = f_l$ for some $l \leq N$, where $f_l$ is defined by (4). But then $u_2^l \circ u_1^l = f_l$ and $u_1^l \circ u_2^l = 0$, where $u_1^l$ and $u_2^l$ are defined by (3), and thus $f \in \mu^{-1}(0)/\overline{G}$.

Corollary 3.4. The symplectic reduction $\mu^{-1}(0)/\overline{G} \subset \mathfrak{h}$ is irreducible and decomposes into $N + 1$ orbits for the adjoint action of $H$:

$U_i := \overline{O}_{[2^i, m-2^i]}$, for $i = 0, \ldots, N$.

The closures of the nilpotent orbits $U_i$ are nested in the following way:

$\{0\} = U_0 \subset \cdots \subset U_N = \mu^{-1}(0)/\overline{G}$.

Hence, $\mu^{-1}(0)/\overline{G}$ is a symplectic variety (see Section 2) of dimension $2N(m - N)$ ([CM93, Corollary 6.1.4]), and whose singular locus is $\overline{U}_{N-1}$ ([KPS1 §3.2]).

By Corollary 3.2 the dimension of the general fibers of the quotient morphism $\nu$ is

$$
\begin{cases}
nm - \frac{1}{2}m^2 & \text{if } m < 2n \text{ and } m \text{ is even;} \\
nm - \frac{1}{4}(m^2 - 1) & \text{if } m < 2n \text{ and } m \text{ is odd;} \\
n^2 & \text{if } m \geq 2n.
\end{cases}
$$

If $m < 2n$, then $N = \left\lfloor \frac{m}{2} \right\rfloor$, and we denote

$$
G' := \left\{ \begin{pmatrix} M & 0_{n-N,N} \\ 0_{N,n-N} & I_N \end{pmatrix}, \ M \in GL_{n-N} \right\} \cong GL_{n-N},
$$

which is a reductive algebraic subgroup of $G \cong GL_n$.

Proposition 3.5. The general fibers of the quotient morphism $\nu : \mu^{-1}(0) \to \mu^{-1}(0)/\overline{G}$ are isomorphic to

$$
\begin{cases}
G & \text{if } m \geq 2n; \\
G/G' & \text{if } m < 2n \text{ and } m \text{ is even;}
\end{cases}
$$

where $G' \subset G$ is the subgroup defined by (6).
Proposition 3.7. (Tera, §4.4) \( \text{H is aponent of} \) ... We first suppose that \( \text{Proof.} \) ... that the invariant Hilbert scheme \( \text{Corollary 3.6.} \) The Hilbert function \( h_0 \) of the general fibers of the quotient morphism \( \nu: \mu^{-1}(0) \to \mu^{-1}(0)/G \) is given by:

\[ h_0(M) = \begin{cases} \dim(M) & \text{if } m \geq 2n; \\ \dim(MG') & \text{if } m < 2n \text{ and } m \text{ is even}; \end{cases} \]

where \( G' \subseteq G \) is the subgroup defined by (6).

If \( m < 2n \) and \( m \) is odd, then the situation is more complicated (except the case \( m = 1 \) which is trivial) because the general fibers of the quotient morphism \( \nu \) are reducible. From now on, we will only consider the cases where either \( m \geq 2n \) or \( m < 2n, m \) is even.

3.2. The reduction principle for the main component. In this section we prove our most important theoretical result, which is the reduction principle (Proposition 3.3). Let us mention that a similar reduction principle (but in a different setting) was already obtained in [Tera].

The subvariety \( \mu^{-1}(0) \subseteq W \) being \( G \times H \)-stable, it follows from [Bri13, Lemma 3.3] that the invariant Hilbert scheme

\[ \mathcal{H} := \text{Hilb}^{G}_{h_0}(\mu^{-1}(0)) \]

is a \( H \)-stable closed subscheme of \( \text{Hilb}^{G}_{h_0}(W) \). We denote by \( \mathcal{H}^{\text{main}} \) the main component of \( \mathcal{H} \). The scheme \( \text{Hilb}^{G}_{h_0}(W) \) was studied in [Tera]; let us recall

Proposition 3.7. (Tera §4.4) Let \( h_0 \) be the Hilbert function given by Corollary 3.3, and let \( H = \text{GL}(V') \) acting naturally on \( \text{Gr}(m - h_0(V), V') \times \text{Gr}(m - h_0(V'), V') \). Then there exists a \( H \)-equivariant morphism

\[ \rho: \text{Hilb}^{G}_{h_0}(W) \to \text{Gr}(m - h_0(V), V') \times \text{Gr}(m - h_0(V'), V') \]
given on closed points by \( (Z, f^1_Z, f^2_Z) \to (\text{Ker}(f^1_Z), \text{Ker}(f^2_Z)) \), where \( f^1_Z: V' \cong \text{Mor}^{G}(W, V) \to \text{Mor}^{G}(Z, V) \) and \( f^2_Z: V' \cong \text{Mor}^{G}(W, V') \to \text{Mor}^{G}(Z, V') \) are the restriction maps.
By Corollary 3.6 we have \( h_0(V) = h_0(V^*) = N \). We identify \( \text{Gr}(m - N, V^*) \) with \( \text{Gr}(N, V') \), and we denote
\[
A_i := \{(L_1, L_2) \in \text{Gr}(N, V') \times \text{Gr}(m - N, V') \mid \dim(L_1 \cap L_2) = N - i\},
\]
for \( i = 0, \ldots, N \). The \( A_i \) are the \( N + 1 \) orbits for the action of \( H \) on \( \text{Gr}(N, V') \times \text{Gr}(m - N, V') \), and
\[
A_0 = A_0 \subset A_1 \subset \cdots \subset A_N = \text{Gr}(N, V') \times \text{Gr}(m - N, V')
\]
In particular, \( A_N \) is the unique open orbit and
\[
(7) \quad A_0 = \mathcal{F}_{N, m - N} := \{(L_1, L_2) \in \text{Gr}(N, V') \times \text{Gr}(m - N, V') \mid L_1 \subset L_2\},
\]
which is a partial flag variety, is the unique closed orbit. Let
\[
\begin{align*}
&\bullet a_0 := (L_1, L_2) \in A_0, \text{ and } P \text{ the parabolic subgroup of } H \text{ stabilizing } a_0; \\
&\bullet W' := \{(u_1, u_2) \in W \mid L_2 \subset \text{Ker}(u_1) \text{ and } \text{Im}(u_2) \subset L_1\}, \text{ which is a } G \times P-
\end{align*}
\]
module contained in \( \mu^{-1}(0) \); and
\[
\bullet \mathcal{H}' := \text{Hilb}_{h_0}^G(W'), \text{ and } \mathcal{H}'^{\text{main}} \text{ its main component.}
\]
If either \( m \geq 2n \) or \( m < 2n \), \( m \) even, then \( h_0 \) coincides with the Hilbert function of the general fibers of the quotient morphism \( W' \to W'/G \) by [Tera, Proposition 4.13]; in particular, \( \mathcal{H}'^{\text{main}} \) is well-defined. We are going to prove

**Proposition 3.8.** If either \( m \geq 2n \) or \( m < 2n \), \( m \) even, and with the above notation, there is a \( H \)-equivariant isomorphism
\[
\mathcal{H}'^{\text{main}} \cong H \times P \mathcal{H}'^{\text{main}}.
\]

First of all, we need

**Lemma 3.9.** If either \( m \geq 2n \) or \( m < 2n \) with \( m \) even, then the morphism \( \rho \) of Proposition 3.7 sends \( \mathcal{H}'^{\text{main}} \) onto \( A_0 \), the \( H \)-variety defined by \( (7) \).

**Proof.** As the quotient morphism \( \nu \colon \mu^{-1}(0) \to \mu^{-1}(0)/G \) is flat over the open orbit \( U_N \), the restriction of the Hilbert-Chow morphism \( \gamma \) to \( \gamma^{-1}(U_N) \) is an isomorphism. We fix \( f_N \in U_N \), and we denote \( Q := \text{Stab}_H(f_N) \), and \( [Z_N] \) the unique point of \( \mathcal{H} \) such that \( \gamma([Z_N]) = f_N \). As \( \gamma \) is \( H \)-equivariant, \( [Z_N] \) is \( Q \)-stable. In addition, \( \rho \) is also \( H \)-equivariant, hence \( \rho([Z_N]) \) is a fixed point for the action of \( Q \). But one may check that \( \text{Gr}(N, V') \times \text{Gr}(m - N, V') \) has a unique fixed point for \( Q \), which is contained in \( A_0 \). Then, as \( A_0 \) is \( H \)-stable, we have \( \rho([Z]) \in A_0 \), for every \( [Z] \in \gamma^{-1}(U_N) \). Hence, \( \rho^{-1}(A_0) \) is a closed subscheme of \( \text{Hilb}_{h_0}^G(W) \) containing \( \gamma^{-1}(U_N) \), and the result follows. \( \square \)

The restriction \( \rho_{h_0}^{\text{main}} : \mathcal{H}^{\text{main}} \to A_0 \) is \( H \)-equivariant, hence \( \mathcal{H}^{\text{main}} \) is the total space of a \( H \)-homogeneous fiber bundle over \( A_0 \). Let \( F \) be the scheme-theoretic fiber of \( \rho_{h_0}^{\text{main}} \) over \( a_0 \). The action of \( P \) on \( \mathcal{H}^{\text{main}} \), induced by the action of \( H \), stabilizes \( F \), and there is a \( H \)-equivariant isomorphism
\[
(8) \quad \mathcal{H}^{\text{main}} \cong H \times P F.
\]
Hence, to prove Proposition 3.8 we have to determine \( F \) as a \( P \)-scheme. We start by considering \( F' \), the scheme-theoretic fiber of the restriction \( \rho_{\mathcal{H}} : \mathcal{H} \to \text{Gr}(N, V') \times \text{Gr}(m - N, V') \) over \( a_0 \), as a \( P \)-scheme. The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

**Lemma 3.10.** With the above notation, there is a \( P \)-equivariant isomorphism
\[
F' \cong \mathcal{H}',
\]
where \( P \) acts on \( \mathcal{H}' \) via its action on \( W' \).
As \( \mathcal{H}^{\text{main}} \) is an irreducible variety of dimension \( 2N(m - N) \), we deduce from \(8\) that \( F \) is an irreducible variety of dimension \( N^2 \). By Lemma \( 3.10\) the fiber \( F \) is isomorphic to a subvariety of \( \mathcal{H}^{\text{main}} \), but \( \dim(\mathcal{H}^{\text{main}}) = N^2 \), and thus there is a \( P \)-equivariant isomorphism

\[
F \cong \mathcal{H}^{\text{main}},
\]

and Proposition \( 3.8\) follows.

**Remark 3.11.** The scheme \( \mathcal{H}' \) is \( P \)-stable and identifies with a closed subscheme of \( \mathcal{H} \), hence there is an inclusion of \( H \)-schemes \( H \times \mathcal{T} \mathcal{R} \mathcal{E} \mathcal{A} \mathcal{U} \mathcal{O} \mathcal{N} \mathcal{A} \mathcal{N} \mathcal{ } \mathcal{P} \mathcal{r} \mathcal{i} \mathcal{s} \mathcal{e} \mathcal{d} \) \( \mathcal{H}' \subset \mathcal{H} \).

### 3.3. Proofs of Theorems A and B for \( GL(V) \)

Our strategy to prove Theorems A and B is the following: first we perform a reduction step (Proposition \( 3.8\)), then we use \[Tera\] §1, Theorem\] to identify \( \text{Hilb}_{\text{Gr}}(W)^{\text{main}} \), and finally we compare the Hilbert-Chow morphism \( \gamma : \text{Hilb}_{\text{Gr}}(\mu^{-1}(0))^{\text{main}} \to \mu^{-1}(0)/G \) with the Springer desingularizations of \( \mu^{-1}(0)/G \). Let us start by recalling

**Theorem 3.12.** \((Tera\) §1, Theorem\]) Let \( G = GL(V) \), let \( W = \text{Hom}(V', V) \times \text{Hom}(V, V') \), and let \( \text{h}_{W} \) be the Hilbert function of the general fibers of the quotient morphism \( W \to W//G \). We denote \( n := \dim(V) \), \( m := \dim(V') \), and by \( Y_{0} \) the blow-up of \( W//G = \text{End}(V')^{\mathbb{Z}} = \{ f \in \text{End}(V') \mid \text{rk}(f) \leq n \} \) at 0. In the following cases, the invariant Hilbert scheme \( \mathcal{H}' : = \text{Hilb}_{\text{Gr}}(W) \) is a smooth variety and the Hilbert-Chow morphism is the succession of blow-up described as follows:

- if \( n \geq 2m - 1 \), then \( \mathcal{H}' \cong W//G = \text{End}(V') \);
- if \( m > n = 1 \) or \( m = n = 2 \), then \( \mathcal{H}' \cong Y_{0} \);
- if \( m > n = 2 \), then \( \mathcal{H}' \) is isomorphic to the blow-up of \( Y_{0} \) along the transform of \( \text{End}(V')^{\mathbb{Z}} \).

Let us now consider the following diagram

\[
\begin{array}{ccc}
\mathcal{F}_{N, m-N} & \xrightarrow{p_{1}} & \text{Gr}(N, V') \\
\downarrow & & \downarrow p_{2} \\
\text{Gr}(m - N, V') & & \\
\end{array}
\]

where \( \mathcal{F}_{N, m-N} \) is defined by \(7\), \( p_{1} \) and \( p_{2} \) being the natural projections. We denote by \( V' \) the constant vector bundle over \( \mathcal{F}_{N, m-N} \) with fiber \( V' \), and by \( T_{1} \) resp. by \( T_{2} \), the pull-back of the tautological bundle over \( \text{Gr}(N, V') \) by \( p_{1} \), resp. over \( \text{Gr}(m - N, V') \) by \( p_{2} \). In particular, if \( N = \frac{m}{2} \), then \( \mathcal{F}_{N, m-N} = \text{Gr}(N, V') \) and \( T_{1} = T_{2} = T := T_{1} \) is the tautological bundle over \( \text{Gr}(N, V') \).

We deduce from Proposition \(3.8\) and Theorem \(3.12\) the following \( H \)-equivariant isomorphisms

\[
H^{\text{main}} \cong \begin{cases} 
\text{Hom}(V'/T, T), & \text{if } n \geq m - 1 \text{ and } m \text{ is even}; \\
\text{Hom}(V'/T_{2}, T_{1}), & \text{if } n = 1 \text{ and } m \geq 3; \\
\text{Bl}_{0}(\text{Hom}(V'/T_{2}, T_{1})), & \text{if } n = 2 \text{ and } m \geq 4;
\end{cases}
\]

where \( \text{Bl}_{0}(\cdot) \) denotes the blow-up along the zero section. In all these cases, \( H^{\text{main}} \) is smooth, and thus the Hilbert-Chow morphism \( \gamma : H^{\text{main}} \to \mu^{-1}(0)/G \) is a desingularization.

On the other hand, we saw in Section \(2\) that the Springer desingularizations of \( \mu^{-1}(0)/G \) are the cotangent bundles \( T_{1}^{*} := T^{*}\text{Gr}(N, V') \) and \( T_{2}^{*} := T^{*}\text{Gr}(m - N, V') \cong T^{*}\text{Gr}(N, V'') \). We then distinguish between two cases:
(1) If \( N < \frac{m}{2} \), then let us prove by contradiction that \( \gamma : \mathcal{H}^{\text{main}} \to \mu^{-1}(0)/G \) cannot be a Springer desingularization. First, we consider the isomorphism of \( G \times H \)-modules \( W \cong W^* \). Denoting \( \mathcal{H}^* := \text{Hilb}_{\text{H}}^G(\mu^{-1}(0)) \), where \( \mu^* \) is the moment map for the natural action of \( G \) on \( W^* \), there is an isomorphism of \( H \)-varieties \( \mathcal{H}^{\text{main}} \cong \mathcal{H}^{\text{star}}_{\text{main}} \). Now if we suppose that (say) \( \mathcal{H}^{\text{main}} \cong \mathcal{T}_1^* \), then we get that \( \mathcal{H}^{\text{star}}_{\text{main}} \cong \mathcal{T}_2^* \), and thus \( \mathcal{T}_1^* \cong \mathcal{T}_2^* \) as a \( H \)-variety, which is absurd.

However, one easily checks that if \( n \in \{1, 2\} \) and \( m \geq 2n + 1 \), then \( \gamma : \mathcal{H}^{\text{main}} \to \mu^{-1}(0)/G \) dominates the two Springer desingularizations \( \mathcal{T}_1^* \) and \( \mathcal{T}_2^* \) (see [Ter], §A.2.2] for details).

(2) If \( N = \frac{m}{2} \), then \( \mathcal{T}^* := \mathcal{T}_1^* = \mathcal{T}_2^* \) is the unique Springer desingularization of \( \mu^{-1}(0)/G \). Let us show that \( \gamma : \mathcal{H}^{\text{main}} \to \mu^{-1}(0)/G \) is the Springer desingularization if and only if \( n \geq m - 1 \). The implication "\( \Rightarrow \)" is given by \( \mathcal{T}^* \cong \text{Hom}(V'/T, T) \). The other implication is given by:

**Lemma 3.13.** If \( N = \frac{m}{2} \) and the Hilbert-Chow morphism \( \gamma : \mathcal{H}^{\text{main}} \to \mu^{-1}(0)/G \) is the Springer desingularization, then \( n \geq m - 1 \).

**Proof.** We suppose that \( \gamma : \mathcal{H}^{\text{main}} \to \mu^{-1}(0)/G \) is the Springer desingularization, that is, \( \mathcal{H}^{\text{main}} \cong \mathcal{T}^* \) as a \( H \)-variety. We fix \( L \in \text{Gr}(N, V') \), and we define \( P \in H, W', \) and \( \mathcal{H}^{\text{main}} \) as in Section 3.2. We have \( \mathcal{T}^* \cong H \times P \text{Hom}(V'/L, L) \), and it follows from (8) and (9) that \( \mathcal{H}^{\text{main}} \cong H \times P \mathcal{H}^{\text{main}} \). Hence, \( \mathcal{H}^{\text{main}} \cong \text{Hom}(V'/L, L) \) as a \( P \)-variety. We denote by \( \gamma' : \mathcal{H}^{\text{main}} \to W'/G \) the restriction of the Hilbert-Chow morphism. As \( \gamma' \) is projective and birational, and \( W'/G = \text{Hom}(V'/L, L) \) is smooth, Zariski’s Main Theorem implies that \( \gamma' \) is an isomorphism. It follows that the quotient morphism \( \nu' : W' \to W'/G \) is flat, and thus \( n \geq 2N - 1 \) by [Tera] Corollary 4.12.

In addition, if \( m = 4 \) and \( n = 2 \), then by (10) we have \( \mathcal{H}^{\text{main}} \cong \text{Bl}_0(T^*) \), and thus \( \gamma \) dominates the unique Springer desingularization of \( \mu^{-1}(0)/G \).

### 3.4. Reducibility of the invariant Hilbert scheme

The aim of this section is to prove Proposition C from the introduction, for \( G = GL(V) \). We suppose that \( m \geq 2n \), then \( N = n \). We fix

\[
A_n = (L'_1, L'_2) \in A_n
\]

a point of the open \( H \)-orbit of \( \text{Gr}(n, V') \times \text{Gr}(m-n, V') \), and we consider

\[
W'' := \{(u_1, u_2) \in W \mid L'_2 \subset \text{Ker}(u_1) \text{ and } \text{Im}(u_2) \subset L'_1\}
\]

\[
\cong \text{Hom}(V'/L'_2, V) \times \text{Hom}(V, L'_1),
\]

which is a \( G \)-submodule of \( W \). As \( V' = L'_1 \oplus L'_2 \), there is a natural identification \( W'' \cong \text{Hom}(L'_1, V) \times \text{Hom}(V, L'_1) \) as a \( G \)-module. Hence, the \( G \)-module \( W'' \) is symplectic and we denote by \( \mu'' : W'' \to g^* \) the corresponding \( G \)-equivariant moment map (see the beginning of Section 3.8 for details). The proof of the next lemma is analogous to the proof of [Tera, Lemma 3.7].

**Lemma 3.14.** We suppose that \( m \geq 2n \), and let \( \rho : \mathcal{H} \to \text{Gr}(n, V') \times \text{Gr}(m-n, V') \) be the morphism of Proposition 3.7. The scheme-theoretic fiber \( F''_n \) of \( \rho \) over the point \( a_n \), defined by (11), is isomorphic to the invariant Hilbert scheme \( \text{Hilb}_{n}(\mu''^{-1}(0)) \),
where $h_0$ is the Hilbert function defined by $h_0(M) = \dim(M)$, for every $M \in \text{Irr}(G)$, and $\mu'' : W'' \to g^*$ is the moment map defined above.

Remark 3.15. The Hilbert function $h_0$ of Lemma 3.14 does not generally coincide with the Hilbert function of the general fibers of the quotient morphism $\mu''^{-1}(0) \to \mu''^{-1}(0)/G$.

By Lemma 3.9, the morphism $\rho : \text{Hilb}_{h_0}^G(W) \to \text{Gr}(n, V') \times \text{Gr}(m - n, V')$ of Proposition 3.7 sends $\mathcal{H}^{\text{main}}$ onto $A_0$. Hence, to prove Proposition C for $G = GL(V)$, it is enough, by Lemma 3.14, to prove that $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$ is non-empty. We denote $V'' := L_1'$, and we equip $W'' \cong \text{Hom}(V'', V) \times \text{Hom}(V, V'')$ with the natural action of $H' := GL(V'')$. Then

$$C[W'']_2 \cong (S^2(V'') \otimes S^2(V^*)) \oplus (S^2(V''^*) \otimes S^2(V)) \oplus (\Lambda^2(V'') \otimes \Lambda^2(V^*)) \oplus (\Lambda^2(V''^*) \otimes \Lambda^2(V)) \oplus ((sl(V'') \otimes M_0) \otimes (sl(V) \otimes V_0))$$

as a $G \times H'$-module, and $V_0$ is the trivial $G$-module resp. $M_0$ is the trivial $H'$-module, and $sl(V'') := \{f \in \text{End}(V'') \mid \text{tr}(f) = 0\}$.

We denote by $I_0$ the ideal of $C[W'']$ generated by $(sl(V'') \otimes V_0) \otimes (M_0 \otimes V_0) \otimes (M_0 \otimes sl(V)) \subset C[W'']$. The ideal $I_0$ is homogeneous, $G \times H'$-stable, and contains the ideal generated by the homogeneous $H'$-invariants of positive degree of $C[W'']$. In particular, $I_0$ identifies with an ideal of $C[\mu''^{-1}(0)]$.

Proposition 3.16. Let $I_0 \subset C[W'']$ be the ideal defined above, then $I_0$ is a point of the invariant Hilbert scheme $\text{Hilb}_{h_0}^G(\mu''^{-1}(0))$ defined in Lemma 3.14.

Proof. We have to check that the ideal $I_0$ has the Hilbert function $h_0$, that is,

$$C[W'']/I_0 \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a $G$-module. To do that that, we are going to adapt the method used by Kraft and Schwarz to prove [KS Theorem 9.1]. The result [loc. cit.] was used in [Terb §2.1.3 and §3.3.2].

We denote $R := V''^* \otimes V$, which is an irreducible $G \times H'$-submodule of $W''^* \cong R \oplus R^*$. Then $R$ and $R^*$ are orthogonal modulo $I_0$, which means that the image of the $G \times H'$-submodule $R \otimes R^* \subset C[W'']$ in $C[W'']/I_0$ is isomorphic to the highest weight component of $R \otimes R^*$ (that is, $sl(V'') \otimes sl(V)$). Then, by [BrIS Lemme 4.1], any irreducible $G \times H'$-submodule of $C[R]$ is orthogonal to any irreducible $G \times H'$-submodule of $C[R^*]$, and thus the natural morphism

$$\phi : C[R]^U \otimes C[R^*]^{U'} \to (C[W'']/I_0)^{U \times U'}$$

is surjective, where $U$ resp. $U'$, denotes the unipotent radical of a Borel subgroup $B \subset G$ resp. $B' \subset H'$. Furthermore, if $T \subset B$ resp. if $T' \subset H'$, is a maximal torus, then $\phi$ is $T \times T'$-equivariant.

Now by [ProG §13.5.1] we have the following isomorphisms of $T \times T'$-algebras $C[R]^U \otimes C[R^*]^{U'} \cong C[x_1, \ldots, x_n]$, where $x_i \in \Lambda^i V'' \otimes \Lambda^i V^*$ is a highest weight vector, and $C[R^*]^{U \times U'} \cong C[y_1, \ldots, y_n]$, where $y_j \in \Lambda^j V''^* \otimes \Lambda^j V$ is a highest weight vector. Hence, there is an exact sequence

$$0 \to K_0 \to C[x_1, \ldots, x_n, y_1, \ldots, y_n] \to (C[W'']/I_0)^{U \times U'} \to 0,$$
where \( K_0 \) is the kernel of \( \phi \). One may check that the ideal \( K_0 \) is generated by the products \( x_1 y_2 \) with \( r + s > n \) (see \([KS]\) §9, Proof of Theorem 9.1(1)).

We denote \( \Lambda = \{ \epsilon_1, \ldots, \epsilon_n \} \) the weight lattice of the linear group \( GL_n \) with its natural basis, and \( \Lambda_+ \) the subset of dominant weights, that is, weights of the form \( r_1 \epsilon_1 + \ldots + r_n \epsilon_n \), with \( r_1 \geq \ldots \geq r_n \). If \( \lambda \in \Lambda_+ \), then we denote by \( S^\lambda(C^n) \) the irreducible \( GL_n \)-module of highest weight \( \lambda \). We fix \( \lambda = k_1 \epsilon_1 + \ldots + k_{t+1} \epsilon_{t+1} - \ldots - k_n \epsilon_n \in \Lambda_+ \), where each \( k_i \) is a nonnegative integer. One easily checks that the weight of the monomial

\[
\sum_{n=0}^{k_{t+1}} x_1^{k_{t+1}-k_{t+2}} x_2^{k_{t+2}-k_{t+3}} \ldots x_{t-2}^{k_{t-2}-k_{t-1}} y_1^{k_{t-1}} y_2^{k_{t-2}} \ldots y_{t-1}^{k_{t-1}} y_t^{k_{t+1}-k_{t+2}}
\]

for the action of \( T \times T' \) is \( (\lambda, \lambda') \), where \( \lambda' \) denotes the highest weight of the \( GL_n \)-module \( S^\lambda(C^n) \), and that \( \lambda \) uniquely determines this monomial. We get that the isotypic component of the \( G \)-module \( S^\lambda(V) \) in \( C[W'']/I_0 \) is the \( G \times H' \)-module \( S^\lambda(V'') \otimes S^\lambda(V) \). As \( \text{dim}(V) = \text{dim}(V'') = n \), we have \( \text{dim}(S^\lambda(V)) = \text{dim}(S^\lambda(V'')) \), for every \( \lambda \in \Lambda_+ \). In other words, each irreducible \( G \)-module \( M \) occurs in \( C[W'']/I_0 \) with multiplicity \( \text{dim}(M) \).

By Proposition 3.16 the scheme \( \text{Hilb}_{h_0}^G(\mu^{-1}(0)) \) is non-empty, and thus \( \mathcal{H} \) has an irreducible component, different from \( \mathcal{H}^{\text{main}} \), of dimension greater or equal to \( \dim(A_n) = 2(n - m) \), which implies Proposition C for \( G = GL(V) \).

3.5. **Study of the case** \( n = 1 \). We saw in Section 3.3 that \( \mathcal{H}^{\text{main}} \) is a smooth variety, and in Section 3.4 that \( \mathcal{H} \) is always reducible. In this section, we determine the irreducible components of \( \mathcal{H} \) when \( n = 1 \).

We suppose that \( m \geq 2 \) (the case \( m = 1 \) being trivial). Then \( G = G_m \) is the multiplicative group, \( \mu^{-1}(0) \vert G = \mathbb{C}^* \subset h \) by Proposition 3.3 and the morphism of Proposition 3.7 is \( \rho : \text{Hilb}_{h_0}^G(W) \to \mathbb{P}(V') \times \mathbb{P}(V'') \). The Segre embedding gives a \( H \)-equivariant isomorphism \( \mathbb{P}(V') \times \mathbb{P}(V'') \cong \mathbb{P}(\mathfrak{h}^{\mathfrak{sl}_1}) \), where \( \mathfrak{h}^{\mathfrak{sl}_1} := \{ f \in h \mid \text{rk}(f) \leq 1 \} \), and thus we can consider \( \rho' : \text{Hilb}_{h_0}^G(W) \to \mathbb{P}(\mathfrak{h}^{\mathfrak{sl}_1}) \), the morphism induced by \( \rho \).

**Proposition 3.17.** We equip the invariant Hilbert scheme \( \mathcal{H} \) with its reduced structure. If \( m > n = 1 \), then there is a \( H \)-equivariant isomorphism

\[
\mathcal{H} \cong \left\{ (f, L) \in \mathbb{C}^*_{[2,1,m-2]} \times \mathbb{P}(\mathfrak{h}^{\mathfrak{sl}_1}) \mid f \in L \right\}.
\]

In particular, \( \mathcal{H} \) is the union of two smooth irreducible components of dimension \( 2m - 2 \) defined by:

- \( C_1 := \left\{ (f, L) \in \mathbb{C}^*_{[2,1,m-2]} \times \mathbb{P}(\mathbb{C}^*_{[2,1,m-2]}) \mid f \in L \right\} = \mathcal{H}^{\text{main}} \), and the Hilbert-Chow morphism \( \gamma : \mathcal{H}^{\text{main}} \to \mathbb{C}^*_{[2,1,m-2]} \) is the blow-up of \( \mathbb{C}^*_{[2,1,m-2]} \) at 0;
- \( C_2 := \left\{ (0, L) \in \mathbb{C}^*_{[2,1,m-2]} \times \mathbb{P}(\mathfrak{h}^{\mathfrak{sl}_1}) \right\} \cong \mathbb{P}(\mathfrak{h}^{\mathfrak{sl}_1}) \), and the Hilbert-Chow morphism is the zero map.

**Proof.** By [Tera] §1,Theorem], there is a \( H \)-equivariant isomorphism

\[
\gamma \times \rho' : \text{Hilb}_{h_0}^G(W) \to \left\{ (f, L) \in \mathbb{C}^*_{[2,1,m-2]} \times \mathbb{P}(\mathfrak{h}^{\mathfrak{sl}_1}) \mid f \in L \right\}.
\]

Since \( \mathcal{H} \to \text{Hilb}_{h_0}^G(W) \), there is a \( H \)-equivariant closed embedding

\[
\gamma \times \rho' : \mathcal{H} \to \mathcal{Y} := \left\{ (f, L) \in \mathbb{C}^*_{[2,1,m-2]} \times \mathbb{P}(\mathfrak{h}^{\mathfrak{sl}_1}) \mid f \in L \right\}.
\]

One may check that \( \mathcal{Y} \) is the union of the two irreducible components \( C_1 \) and \( C_2 \), both of dimension \( 2m - 2 \). The morphism \( \gamma \times \rho' \) sends \( \mathcal{H}^{\text{main}} \) into \( C_1 \); the varieties \( \mathcal{H}^{\text{main}} \) and \( C_1 \) have the same dimension, hence \( \gamma \times \rho' : \mathcal{H}^{\text{main}} \to C_1 \) is an
isomorphism. On the other hand, we saw in Section 3.4 that \( \mathcal{H} \) admits another irreducible component, denoted by \( \mathcal{H}_2 \), of dimension at least \( 2m - 2 \), which is the dimension of \( C_2 \), and thus \( \gamma \times \rho' \) is an isomorphism between \( \mathcal{H}_2 \) and \( C_2 \).

\[ \square \]

Remark 3.18. One may check that the component \( C_2 \) of Proposition 3.17 consists of the homogeneous ideals of \( \mathbb{C}[\mu^{-1}(0)] \).

When \( m \geq 2n \geq 4 \), irreducible components of dimension greater than \( \dim(\mathcal{H}^{\text{main}}) \) may appear. For instance, if \( n = 2 \) and \( m \geq 4 \), then one may check that the irreducible component consisting of the homogeneous ideals of \( \mathbb{C}[\mu^{-1}(0)] \) is of dimension \( 4m - 5 \), whereas the main component \( \mathcal{H}^{\text{main}} \) is of dimension \( 4n - 8 \). In addition, we showed in Section 3.4 that \( \mathcal{H} \) has at least two components, but \( \mathcal{H} \) may have more components.

4. Case of \( Sp_n \)

Let \( V \) and \( V' \) be two vector spaces of dimension \( n \) (which is even) and \( m \) respectively, and let \( W := \text{Hom}(V', V) \times \text{Hom}(V, V') \). We denote \( E := V' \oplus V'^* \) on which we fix a non-degenerate quadratic form \( q \), and we take \( G = Sp(V) \) and \( H = SO(E) \). As \( G \) resp. \( H \) preserves a non-degenerate bilinear form on \( V \) resp. on \( E \), we can identify \( V \cong V^* \) as a \( G \)-module resp. \( E \equiv E^* \) as a \( H \)-module. It follows that

\[
W \cong \text{Hom}(V', V) \times \text{Hom}(V'^*, V') \\
\cong \text{Hom}(V', V) \times \text{Hom}(V'^*, V) \\
\cong \text{Hom}(E, V)
\]

as a \( G \)-module, and thus \( H \) acts naturally on \( W \). We denote by \( \mathfrak{g} \) resp. by \( \mathfrak{h} \), the Lie algebra of \( G \) resp. of \( H \).

4.1. The quotient morphism. The main results of this section are Proposition 3.1 which describes the irreducible components of the symplectic reduction \( \mu^{-1}(0)/G \), and Corollary 3.6 which gives the Hilbert function of the general fibers of the quotient morphism \( \nu : \mu^{-1}(0)/G \) for each irreducible component of \( \mu^{-1}(0)/G \). Contrary to the case of \( GL(V) \) studied in Section 3, we will see that \( \mu^{-1}(0)/G \) is reducible when \( m \leq n \) and \( m \) is even.

We have seen that \( W \) is equipped with a \( G \)-invariant symplectic form (see the beginning of Section 3.1 for details). If \( w \in \text{Hom}(E, V) \), we denote the transpose of \( w \) by \( \mathcal{W} \in \text{Hom}(V'^*, E^*) \equiv \text{Hom}(V, E) \). Then, by [Dec(9)] Proposition 3.1, the zero fiber of the moment map \( \mu : W \to \mathfrak{g}^* \) is the \( G \times H \)-stable subvariety defined by:

\[
\mu^{-1}(0) = \{ w \in W \mid \mathcal{W} \circ w = 0 \}.
\]

Remark 4.1. One may check that the biggest subgroup of \( GL(E) \) that stabilizes \( \mu^{-1}(0) \) in \( W \) is the orthogonal group \( O(E) \). However, we prefer to consider the action of \( H = SO(E) \) for practical reasons.

The proof of the next proposition is analogous to those of Proposition 3.1 and Corollary 3.2.

Proposition 4.2. The zero fiber of the moment map \( \mu : W \to \mathfrak{g}^* \) is

- an irreducible subvariety of dimension \( 2mn - \frac{1}{2}n(n + 1) \) if \( m > n \);
- the union of two irreducible components of dimension \( mn + \frac{1}{2}m(m - 1) \) if \( m \leq n \).
If \( d \) is a partition of \( 2m \), then we denote by \( O_d \) resp. by \( O_d^I \) and \( O_d^H \), the corresponding nilpotent orbit(s) of \( \mathfrak{h} \cong \mathfrak{so}_{2m} \) associated to \( d \) (see Section 2.2). The following result was proved by Becker:

**Proposition 4.3.** ([Bec09, Proposition 3.6]) The symplectic reduction of \( W \) by \( G \) is

\[
\mu^{-1}(0)/G = \begin{cases} O_{\lfloor 2m, \lfloor 2m-1, \lfloor 2m \rfloor} & \text{if } m > n; \\ O_{\lfloor 2m-1, \lfloor 2m \rfloor} & \text{if } m < n \text{ and } m \text{ is odd}; \\ O_{\lfloor 2m \rfloor} \cup O_{\lfloor 2m \rfloor}^H & \text{if } m \leq n \text{ and } m \text{ is even}. \\
\end{cases}
\]

**Corollary 4.4.** The orbits for the adjoint action of \( H \) on \( \mu^{-1}(0)/G \) are

- \( U_i := O_{\lfloor 2i, \lfloor 2(i-1) \rfloor} \), for \( i = 0, 2, \ldots, n \), if \( m > n \);
- \( U_i := O_{\lfloor 2i, \lfloor 2(i-1) \rfloor} \), for \( i = 0, 2, \ldots, m-1 \), if \( m < n \) and \( m \) is odd;
- \( U_i := O_{\lfloor 2i, \lfloor 2(i-1) \rfloor} \), for \( i = 0, 2, \ldots, m-2 \), and \( U_m := O_{\lfloor 2m \rfloor} \), \( U_{m}^I := O_{\lfloor 2m \rfloor}^I \), if \( m \leq n \) and \( m \) is even.

The closures of the nilpotent orbits \( U_i \) are nested in the following way:

\[
\begin{align*}
\{0\} &= \overline{U_0} \subset \overline{U_2} \subset \cdots \subset \overline{U_m} & \text{if } m > n; \\
\{0\} &= \overline{U_0} \subset \overline{U_2} \subset \cdots \subset \overline{U_{m-1}} & \text{if } m < n \text{ and } m \text{ is odd}; \\
\{0\} &= \overline{U_0} \subset \overline{U_2} \subset \cdots \subset \overline{U_{m-2}} = \overline{U_m^I \cap U_m^H} & \text{if } m \leq n \text{ and } m \text{ is even}.
\end{align*}
\]

If \( m > n \) or \( m \) is odd resp. if \( m \leq n \) and \( m \) is even, then the symplectic reduction \( \mu^{-1}(0)/G \) is the closure of a nilpotent orbit resp. the union of two closures of nilpotent orbits, and thus the irreducible components of \( \mu^{-1}(0)/G \) are symplectic varieties (see Section 2). If \( m > n \), then \( \mu^{-1}(0)/G \) is of dimension \( 2mn-n(n+1) \), and its singular locus is \( \overline{U_{m-2}} \). On the other hand, if \( m \leq n \), then each irreducible component of \( \mu^{-1}(0)/G \) is of dimension \( m(m-1) \), and the singular locus of \( \mu^{-1}(0)/G \) is \( \overline{U_{m-2}} \). The dimension of the irreducible components of \( \mu^{-1}(0)/G \) is given by [CM93, Corollary 6.1.4], and the singular locus of \( \mu^{-1}(0)/G \) is given by [KP82, Theorem 2].

We are now interested in the Hilbert function of the general fibers of the quotient morphism for each irreducible component of \( \mu^{-1}(0)/G \). We will distinguish between the following cases:

- If \( m > n \), then \( \mu^{-1}(0)/G \) is irreducible, and we denote by \( h_0 \) the Hilbert function of the general fibers of the quotient morphism \( \nu : \mu^{-1}(0) \to \mu^{-1}(0)/G \). By Proposition 4.2, the dimension of these fibers is \( \frac{1}{2}n(n+1) \).
- If \( m \leq n \) and \( m \) is even, then by Proposition 4.2 the zero fiber \( \mu^{-1}(0) \) is the union of two irreducible components that we denote by \( X_I \) and \( X_H \). Let \( \nu_I : X_I \to Y_I \) and \( \nu_H : X_H \to Y_H \) be the quotient morphisms. Up to the exchange of \( X_I \) and \( X_H \), we can suppose that \( Y_I = \overline{U_m^I} \) and \( Y_H = \overline{U_m^H} \). The orthogonal group \( O(E) \) acts transitively on \( U_m^I \cup U_m^H \); hence the general fibers of \( \nu_I \) and \( \nu_H \) are isomorphic. In particular, these fibers have the same Hilbert function, denoted by \( h_0 \), and the same dimension, which is \( mn - \frac{1}{2}m(m-1) \).
- If \( m < n \) and \( m \) is odd, then \( \mu^{-1}(0)/G \) is irreducible, and we denote by \( h_0 \) the Hilbert function of the general fibers of the quotient morphism \( \nu : \mu^{-1}(0) \to \mu^{-1}(0)/G \). These fibers being reducible, determining \( h_0 \) is more complicated than in the previous cases (except the case \( m = 1 \) which
is trivial). From now on, we will always exclude the case where \( m < n \) and \( m \) is odd.

If \( m < n \) and \( m \) is even, then we denote

\[
G' := \left\{ \begin{array}{c} M \\ 0_{n,m} \\ I_m \end{array} \right| M \in Sp_{n-m}, \right\} \cong Sp_{n-m},
\]

which is a reductive algebraic subgroup of \( G \cong Sp_n \). The proof of the next proposition is analogous to the proof of Proposition 3.5.

**Proposition 4.5.** If \( m > n \), then the general fibers of the quotient morphism \( \nu : \mu^{-1}(0) \to \mu^{-1}(0)/G \) are isomorphic to \( G \).

If \( m = n \), then the general fibers of the quotient morphisms \( \nu_I : X_I \to Y_I \) and \( \nu_H : X_H \to Y_H \) are isomorphic to \( G \).

If \( m < n \) and \( m \) is even, then the general fibers of \( \nu_I \) and \( \nu_H \) are isomorphic to \( G/G' \), where \( G' \subset G \) is the subgroup defined by (12).

**Corollary 4.6.** The Hilbert function \( h_0 \) defined above is given by:

\[
\forall M \in \text{Irr}(G), \ h_0(M) = \begin{cases} \text{dim}(M) & \text{if } m \geq n; \\ \text{dim}(M_{G'}) & \text{if } m < n \text{ and } m \text{ is even}; \end{cases}
\]

where \( G' \subset G \) is the subgroup defined by (12).

4.2. **The reduction principle for the main component.** In this section, we give the guidelines to prove the reduction principle when \( G = Sp(V) \) (Proposition 4.3). The strategy is the same as for \( GL(V) \) (see Section 3.3), but as the symplectic reduction \( \mu^{-1}(0)/G \) is reducible when \( m \leq n \) and \( m \) is even, it seems necessary to give some additional details.

As \( \mu^{-1}(0) \) is a \( G \times H \)-stable subvariety of \( W \), it follows from [Bri13, Lemma 3.3] that the invariant Hilbert scheme

\[
\mathcal{H} := \text{Hilb}^{G}_{h_0}(\mu^{-1}(0))
\]

is a \( H \)-stable closed subscheme of \( \text{Hilb}^{G}_{h_0}(W) \). As we aim at constructing desingularizations of the irreducible components of \( \mu^{-1}(0)/G \), we consider the two \( H \)-stable closed subschemes \( \mathcal{H}_I := \text{Hilb}^{G}_{h_0}(X_I) \) and \( \mathcal{H}_H := \text{Hilb}^{G}_{h_0}(X_H) \) instead of \( \mathcal{H} \) when \( m \leq n \) and \( m \) is even. Let us note that if we fix \( y_0 \in O(E) \setminus SO(E) \) and make \( H \) act on \( X_H \) by \((y_0 y x^{-1}) \cdot x \) for every \( y \in H \) and every \( x \in X_H \), then \( \phi : X_I \to X_H, \ x \mapsto y_0 x \) is a \( G \times H \)-equivariant isomorphism, and thus \( \mathcal{H}_I \cong \mathcal{H}_H \) as a \( H \)-scheme. We denote by \( \mathcal{H}^\text{main}_I \) resp. \( \mathcal{H}^\text{main}_H \), the main component of \( \mathcal{H}_I \) resp. \( \mathcal{H}_H \). We always have the (set-theoretic) inclusion \( \mathcal{H}_I \cup \mathcal{H}_H \subset \mathcal{H} \), but this may not be an equality. If \( m > n \), then \( \mu^{-1}(0)/G \) is irreducible, and we denote by \( \mathcal{H}^\text{main} \) the main component of \( \mathcal{H} \).

The scheme \( \text{Hilb}^{G}_{h_0}(W) \) was studied in [Terb]. In particular, we obtained

**Proposition 4.7.** ([Terb, §1.5.1]) Let \( h_0 \) be the Hilbert function given by Corollary 4.6 and let \( H = SO(E) \) acting naturally on \( Gr(2m-h_0(V^*), E) \). Then there exists a \( H \)-equivariant morphism

\[
\rho : \text{Hilb}^{G}_{h_0}(W) \to Gr(2m-h_0(V^*), E)
\]

given on closed points by \([Z] \mapsto \text{Ker}(f_Z)\), where \( f_Z : E \cong \text{Mor}^G(W,V^*) \to \text{Mor}^G(Z,V^*) \) is the restriction map.
We identify $\Gr(2m - h_0(V^*), E)$ with $\Gr(h_0(V^*), E^*)$. By Corollary 4.6 if either $m > n$ or $m \leq n$, $m$ even, then $h_0(V^*) = N = \min(m, n)$. The non-degenerate quadratic form $q$ on $E$ gives a canonical isomorphism $E \cong E^*$. In particular, $q$ identifies with a non-degenerate quadratic form on $E^*$. For $i = 0, \ldots, N$, we denote

$$A_i := \{ L \in \Gr(N, E^*) \mid q_L \text{ is of rank } i \}.$$

If $m > n$, then the $A_i$ are the $n + 1$ orbits for the action of $H$ on $\Gr(n, E^*)$. However, if $m \leq n$, then the $A_i$ are $H$-orbits for $i = 1, \ldots, m$, but the isotropic Grassmannian $A_0 = \OGr(m, E^*)$ is the union of two $H$-orbits, denoted by $\OGr^I$ and by $\OGr^H$, which are exchanged by the action of any element of $O(E) \setminus SO(E)$.

In any case, we have

$$\OGr(N, E^*) = \overline{A_0} \subset \overline{A_1} \subset \cdots \subset \overline{A_N} = \Gr(N, E^*).$$

Let us now fix some notation:

- $L_0 \in A_0$, and $P$ the parabolic subgroup of $H$ stabilizing $L_0$;
- $W' := \Hom(E/L_0, V)$, which identifies with a $G \times P$-module contained in $\mu^{-1}(0)$; and
- $\mathcal{H}' := \Hilb_{\mu_0}^0(W')$, and $\mathcal{H}^{\text{main}}$ its main component.

It must be emphasized that, if either $m > n$ or $m \leq n$, $m$ even, then the Hilbert function of the general fibers of the quotient morphism $W' \to W'//G$ coincides with the Hilbert function $h_0$ of Corollary 4.6 (in particular, $\mathcal{H}^{\text{main}}$ is well defined).

Proceeding as for Lemma 3.9 one may check that, if $m > n$ resp. if $m \leq n$ with $m$ even, then the morphism $\rho$ of Proposition 4.7 sends $\mathcal{H}^{\text{main}}$ resp. $\mathcal{H}^{\text{main}}$ and $\mathcal{H}^{\text{main}}$, onto $A_0$. More precisely, if $m \leq n$ and $m$ is even, then $\rho$ sends $\mathcal{H}^{\text{main}}$ onto one of the irreducible component of $A_0$, and $\mathcal{H}^{\text{main}}$ onto the other component. Up to the exchange of these two components, we can suppose that $\rho$ sends $\mathcal{H}^{\text{main}}$ onto $\OGr^I$, and $\mathcal{H}^{\text{main}}$ onto $\OGr^H$.

It follows that the restriction of $\rho$ equips $\mathcal{H}^{\text{main}}$, resp. $\mathcal{H}^{\text{main}}$, resp. $\mathcal{H}^{\text{main}}$, with a structure of a $H$-homogeneous fiber bundle over $A_0$ resp. over $\OGr^I$, resp. over $\OGr^H$. Hence, it is enough to determine the fiber $F_0$ over $L_0$ to determine $\mathcal{H}^{\text{main}}$, resp. $\mathcal{H}^{\text{main}}$, resp. $\mathcal{H}^{\text{main}}$. Proceeding as in Section 3.2, we obtain that $F_0$ is isomorphic to $\mathcal{H}^{\text{main}}$ as a $P$-scheme. We deduce

**Proposition 4.8.** With the above notation, we have the following $H$-equivariant isomorphisms:

- If $m > n$, then
  $$\mathcal{H}^{\text{main}} \cong H \times_P \mathcal{H}^{\text{main}}.$$

- If $m \leq n$, $m$ even, and $L_0 \in \OGr^I$ resp. $L_0 \in \OGr^H$, then
  $$\mathcal{H}^{\text{main}} \cong H \times_P \mathcal{H}^{\text{main}} \text{ resp. } \mathcal{H}^{\text{main}} \cong H \times_P \mathcal{H}^{\text{main}}.$$

### 4.3. Proofs of Theorems A and B for $Sp(V)$

In this section, we proceed as in Section 3.3 to prove Theorems A and B when $G = Sp(V)$. Before going any further, let us mention that the case $n = 2$, $m = 3$ was already handled by Becker in [Bec11]. In this situation, $\mu^{-1}(0)//G$ is a closure of a nilpotent orbit that admits two Springer desingularizations, and Becker showed that $\gamma : \Hilb_{h_0}^0(\mu^{-1}(0)) \to \mu^{-1}(0)//G$ is a desingularization that dominates them both. To obtain this result, Becker first used the existence of natural morphisms from the invariant Hilbert scheme to Grassmannians to identify $\Hilb_{h_0}^0(\mu^{-1}(0))^{\text{main}}$ with
the total space of a homogeneous line bundle over a Grassmannian, and then she showed that \( \text{Hilb}^G_{h_W}(\mu^{-1}(0)) = \text{Hilb}^G_{h_W}(\mu^{-1}(0))^{\text{main}} \) by computing the tangent space of \( \text{Hilb}^G_{h_W}(\mu^{-1}(0)) \) at every point of the main component.

Let us now recall the following result:

**Theorem 4.9.** ([Tera, §1, Theorem]) Let \( G = Sp(V) \), let \( W = \text{Hom}(E, V) \), and let \( h_W \) be the Hilbert function of the general fiber of the quotient morphism \( W \to W/G \). We denote \( n := \dim(V) \), \( e := \dim(E) \), and we denote by \( Y_0 \) the blow-up of \( W/G = \Lambda^2(E^*)^{\leq n} : = \{ Q \in \Lambda^2(E^*) \mid \text{rk}(Q) \leq n \} \) at \( 0 \). In the following cases, the invariant Hilbert scheme \( \mathcal{H}' := \text{Hilb}^G_{h_W}(W) \) is a smooth variety, and the Hilbert-Chow morphism is the succession of blow-up described as follows:

- if \( n \geq 2e - 2 \), then \( \mathcal{H}' \cong W/G = \Lambda^2(E^*) \);
- if \( e > n = 2 \) or \( e = n = 4 \), then \( \mathcal{H}' \cong Y_0 \);
- if \( e > n = 4 \), then \( \mathcal{H}' \) is isomorphic to the blow-up of \( Y_0 \) along the strict transform of \( \Lambda^2(E^*)^{\leq 2} \).

If \( m > n \), then we denote by \( T \) the tautological bundle over \( A_0 = \text{OGr}(n, E^*) \). If \( m \leq n \) and \( m \) is even, then we denote by \( T_I \) resp. by \( T_H \), the tautological bundle over \( \text{OGr}^I \) resp. over \( \text{OGr}^H \). We deduce from Proposition 4.8 and Theorem 4.9 the following \( H \)-equivariant isomorphisms

\[
\mathcal{H}^{\text{main}} = \begin{cases} 
\Lambda^2(T) & \text{if } m > n = 2; \\
\text{Bl}_0(\Lambda^2(T)) & \text{if } m > n = 4;
\end{cases}
\]

\[
\mathcal{H}^{\text{main}} = \begin{cases} 
\Lambda^2(T_e) & \text{if } n \geq 2m - 2 \text{ and } m \text{ is even}; \\
\text{Bl}_0(\Lambda^2(T_e)) & \text{if } m = n = 4;
\end{cases}
\]

(13)

where \( \bullet \) stands for \( I \) or \( II \), and \( \text{Bl}_0(.) \) denotes the blow-up along the zero section. In all these cases, the main component of the invariant Hilbert scheme is smooth, and thus the Hilbert-Chow morphism \( \gamma : \mathcal{H}^{\text{main}} \to \mu^{-1}(0)/G \) resp. \( \gamma : \mathcal{H}^{\text{main}}_I \to Y_\bullet \), is a desingularization.

It remains to compare \( \gamma \) with the Springer desingularizations (when they exist) of the irreducible components of \( \mu^{-1}(0)/G \). We saw in Section 2 that the irreducible components of \( \mu^{-1}(0)/G \) have Springer desingularizations if and only if \( m \leq n + 1 \). We then distinguish between the following cases:

1. If \( m \leq n + 1 \) and \( m \) is odd, then \( \mu^{-1}(0)/G \) admits two Springer desingularizations, which are given by the cotangent bundles \( T^*_I \) and \( T^*_H \) over \( \text{OGr}^I \) and \( \text{OGr}^H \) respectively. The natural action of the orthogonal group \( O(E) \) on \( \text{OGr}(m, E^*) \) induces an action on the cotangent bundle \( T^* \text{OGr}(m, E^*) \) that exchanges \( T^*_I \) and \( T^*_H \). On the other hand, it follows from Remark 4.1 that the group \( O(E) \) stabilizes \( \mathcal{H}^{\text{main}} \), and thus \( \gamma : \mathcal{H}^{\text{main}} \to \mu^{-1}(0)/G \) cannot be a Springer desingularization. However, if \( n \in \{ 2, 4 \} \) and \( m = n + 1 \), then one may prove that \( \gamma \) dominates the two Springer desingularizations of \( \mu^{-1}(0)/G \) (see [Bec11] Introduction) for the case \( n = 2 \), the case \( n = 4 \) being analogous.

2. If \( m \leq n \) and \( m \) is even, then \( Y_\bullet \) has a unique Springer desingularization, which is given by the cotangent bundle \( T^*_e \cong \Lambda^2(T_e) \) over \( \text{OGr}^e \). Proceeding as we did for \( GL(V) \) in Section 3.3, one may prove that \( \gamma : \mathcal{H}^{\text{main}}_e \to Y_\bullet \) is the Springer desingularization if and only if \( n \geq 2m - 2 \).
In addition, if \( m = n = 4 \), then by (13) we have \( \mathcal{H}^{\text{main}} \cong Bl_0(T^*_\bullet) \), and thus \( \gamma \) dominates the unique Springer desingularization of \( Y_\bullet \).

4.4 Study of the case \( n = 2 \). In this section, we suppose that \( m \geq n = 2 \) (the case \( m = 1 \) being trivial). We will prove that if \( m \geq 3 \) resp. if \( m = 2 \), then \( \mathcal{H} \) resp. \( \mathcal{H}_\bullet \) (where \( \bullet \) stands for \( I \) or \( II \)), is irreducible. In particular, the geometric properties of the invariant Hilbert scheme for \( G = Sp(V) \) are quite different from the case of \( G = GL(V) \) studied in Section 3. Let us recall that the case \( m = 3 \), \( n = 2 \) was treated by Becker in [Bec11]; she showed that \( \mathcal{H} \) is the total space of a line bundle over \( OGr(2,E^*) \).

We have \( G = Sp_2 = SL_2 \), and the morphism of Proposition 4.7 is \( \rho : \text{Hilb}^G_{n_0}(W) \to \text{Gr}(2,E^*) \). Denoting \( \mathfrak{h}^{\leq} := \{ f \in \mathfrak{h} \mid \text{rk}(f) \leq 2 \} \), there is a \( H \)-equivariant isomorphism

\[
\mathbb{P}(\mathfrak{h}^{\leq}) \cong \text{Gr}(2,E^*)
\]

and thus we can consider the morphism \( \rho' : \text{Hilb}^G_{n_0}(W) \to \mathbb{P}(\mathfrak{h}^{\leq}) \) induced by \( \rho \). By Proposition 4.3 we have

\[
\mu^{-1}(0) \cap G = \begin{cases} \mathcal{O}_{[2^2,12m-4]} \cup \mathcal{O}_{[2^2]} & \text{if } m \geq 3; \\
\mathcal{O}_{[2^2]} & \text{if } m = 2. 
\end{cases}
\]

Proposition 4.10. We equip all the invariant Hilbert schemes with their reduced structures. If \( m > n = 2 \), then \( \mathcal{H} = \mathcal{H}^{\text{main}} \) is a smooth variety isomorphic to

\[
Bl_0(\mathcal{O}_{[2^2,12m-4]}) := \left\{ (f,L) \in \mathcal{O}_{[2^2,12m-4]} \times \mathbb{P}(\mathcal{O}_{[2^2,12m-4]}) \mid f \in L \right\},
\]

and the Hilbert-Chow morphism \( \gamma : \mathcal{H} \to \mu^{-1}(0) \cap G \) is the blow-up of \( \mathcal{O}_{[2^2,12m-4]} \) at \( 0 \). If \( m = n = 2 \), then \( \text{Hilb}^G_{n_0}(\mu^{-1}(0)) = \mathcal{H}_I \cup \mathcal{H}_{II} \) is the union of two smooth irreducible components isomorphic to \( Bl_0(\mathcal{O}_{[2^2]}) \) and \( Bl_0(\mathcal{O}_{[2^2]}) \) respectively, and the set-theoretic intersection \( \mathcal{H}_I \cap \mathcal{H}_{II} \) is formed by the homogeneous ideals of \( C[\mu^{-1}(0)] \). Moreover, the Hilbert-Chow morphism \( \gamma : \mathcal{H}_I \to \mathcal{O}_{[2^2]} \), resp. \( \gamma : \mathcal{H}_{II} \to \mathcal{O}_{[2^2]} \), is the blow-up of \( \mathcal{O}_{[2^2]} \) resp. of \( \mathcal{O}_{[2^2]} \) at \( 0 \).

Proof. The proofs for the cases \( m = 2 \) and \( m \geq 3 \) are quite similar, and thus we will only consider the case \( m \geq 3 \) (which is simpler in terms of notation!). Using arguments similar to those used to prove Proposition 3.3.17 we obtain a closed embedding

\[
\gamma \times \rho' : \mathcal{H} \to \mathcal{Y} := \left\{ (f,L) \in \mathcal{O}_{[2^2,12m-4]} \times \mathbb{P}(\mathfrak{h}^{\leq}) \mid f \in L \right\}.
\]

One may check that \( \mathcal{Y} \) is the union of the two irreducible components \( C_1 \) and \( C_2 \) defined by:

- \( C_1 := Bl_0(\mathcal{O}_{[2^2,12m-4]}) \); and
- \( C_2 := \{(0,L) \in \mathcal{O}_{[2^2,12m-4]} \times \mathbb{P}(\mathfrak{h}^{\leq}) \} = \mathbb{P}(\mathfrak{h}^{\leq}) \).

The components \( C_1 \) and \( C_2 \) are of dimension \( 4m - 6 \) and \( 4m - 4 \) respectively. The morphism \( \gamma \times \rho' \) sends \( \mathcal{H}^{\text{main}} \) into \( C_1 \); the varieties \( \mathcal{H}^{\text{main}} \) and \( C_1 \) have the same dimension, hence \( \gamma \times \rho' : \mathcal{H}^{\text{main}} \to C_1 \) is an isomorphism.

Now it follows from [Ter98 Proposition 3.3.13] that the component \( C_2 \) identifies with the closed subset of \( \text{Hilb}^G_{n_0}(W) \) formed by the homogeneous ideals of \( C[W] \). Let us describe this identification. If \( L \in C_2 \cong \mathbb{P}(\mathfrak{h}^{\leq}) \), then we denote by \( I_L \) the ideal of \( C[W] \) generated by the homogeneous \( G \)-invariants of positive degree.
of $\mathbb{C}[W]$, and by the $G$-module $L^* \otimes V \subseteq \mathbb{C}[W]_1 \cong E \otimes V$, where $L$ is identified with a 2-dimensional subspace of $E^*$ via the isomorphism (14). Let us show that $I_L$ is a point of $\mathcal{H}$ if and only if $L \in \text{OGr}(2, E^*)$; the result will follow since $F(O_{[2,12]=4})$ identifies with $\text{OGr}(2, E^*)$ via the isomorphism (14), and since $\{(0, L) \in O_{[2,12]=4} \times F(O_{[2,12]=4})\}$ is a subvariety of $C_1$.

We denote $W' := \text{Hom}(E/L^*, V)$, then

$$\mathbb{C}[W']_2 \cong (S^2(E/L^*) \otimes S^2(V)) \oplus (\Lambda^2(E/L^*) \otimes \Lambda^2(V))$$

as a $G$-module. Let $I'_L$ be the ideal of $\mathbb{C}[W']$ generated by $\Lambda^2(E/L^*) \otimes \Lambda^2(V) \subseteq \mathbb{C}[W']_2$, then one may check (using [Terp, Proposition 3.3.13]) that

$$\mathbb{C}[W]/I_L \cong \mathbb{C}[W']/I'_L \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus \dim(M)}$$

as a $G$-module. Hence

$I_L \in \mathcal{H} \iff I_L \cap \mathbb{C}[W]'_2 \ni E_0 \otimes S^2(V)$, where $E_0$ is the trivial representation of $H$;

$\iff q_{L^*} = 0$, where $q$ is the quadratic form preserved by $H$;

$\iff L \in \text{OGr}(2, E^*)$.

\[ \square \]

Remark 4.11. In the proof of Proposition [4.10] we showed that if $m > n = 2$, then the homogeneous ideals of $\mathcal{H}$ are contained in $\mathcal{H}^{\text{main}}$. Using analogous arguments, one may check that this statement is true more generally when $m > n \geq 2$.

Acknowledgments: I am deeply thankful to Michel Brion for proposing this subject to me, for a lot of helpful discussions, and for his patience. I thank Tanja Becker for exchange of knowledge on invariant Hilbert schemes by e-mail and during her stay in Grenoble in October 2010. I also thank Bart Van Steirteghem for helpful discussions during his stay in Grenoble in Summer 2011.

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