THE LOCUS OF LOG CANONICAL SINGULARITIES

FLORIN AMBRO

CONTENTS

0. Introduction 1
1. Log varieties and (relative effective) log pairs 2
2. The locus of log canonical singularities 4
3. Kawamata's lc centers and perturbation trick 9
4. Appendix on seminormal varieties 10
References 13

0. INTRODUCTION

Given a log variety \((X, B_X)\), there is a naturally defined closed subscheme \(LCS(X, B_X) \subset X\) [Sho, 3.14], called the locus of log canonical singularities. The defining ideal sheaf, introduced by V. Shokurov (unpublished), is the algebraic counterpart of the multiplier ideal sheaves associated with singular hermitian metrics.

Although \(LCS(X, B_X)\) was initially introduced to measure how far the log variety was from being log terminal (some authors called it the non-Kawamata log terminal locus), it was realized recently that \(LCS(X, B_X)\) is worth studying in its own, being an intermediate step for inductive arguments in higher dimensional algebraic geometry. This technique was the main ingredient in the proof of some of the basic theorems of the (Log) Minimal Model Program [KMM, Ch. 2-4].

Our goal is to investigate the LCS locus in the ambient variety, and relate its singularities to those of the ambient space. As a first step in this direction, V. Shokurov proved [Sho, 3.6-8] that \(LCS(X, B_X)\) is normal if \((X, B_X)\) has pure log terminal singularities, and it has normal components intersecting normally (hence seminormal) if \((X, B_X)\) is strictly log terminal. The latter was generalized by J. Kollár [Kol, 17.5] to the log terminal case.

In this paper, as conjectured by V. Shokurov, we show that \(LCS(X, B_X)\) is seminormal if \((X, B_X)\) has log canonical singularities.

The natural category in which to study the LCS locus is that of (relative effective) log pairs \(\pi : (X, D) \to S\), where \(D\) is effective over \(S\), that is the negative part of \(D\) is \(\pi\)-exceptional. A log variety is a log pair \((X, D)\) with \(\pi = id_X\).
We recall in Section 1 the basic definitions. In Section 2 we introduce the LCS ideal sheaf for log pairs, which is a slight modification of V. Shokurov’s definition. The ideal sheaves are isomorphic for log varieties [Remark 2.7]. The main technical result is Theorem 2.6, which is a generalization of [Sho, 3.6] (see also [Kol1, 17.4]), and follows closely their proof. As a corollary, we obtain the contraction which, via a formal seminormality result, implies the seminormality of the LCS locus. In Section 3 we use Kawamata’s technique to show that any finite union of lc centers is seminormal, under some restrictions. Section 4 is an appendix on seminormality.

Acknowledgments. I would like to thank Professor V. Shokurov for setting up the problem and also for his valuable support.

1. Log varieties and (relative effective) log pairs

A variety is a reduced scheme of finite type over a fix field $k$. We have to assume $\text{char}(k) = 0$, since we use Kawamata-Viehweg vanishing as a main technical tool.

We first define the basic objects of this paper.

**Definition 1.1.** (i). A relative effective log pair $\pi: (X, D) \to S$ is a normal variety $X$ equipped with an $\mathbb{R}$-Weil divisor $D = \sum d_iD_i$ ($d_i \in \mathbb{R}$), and a morphism $\pi: X \to S$ such that

(a) $K_X + D$ is an $\mathbb{R}$-Cartier Weil divisor.

(b) $D$ is relative effective, that is the components $D_i$ of $D$ with negative coefficients are $\pi$-exceptional (codim$(\pi(D_i), S) \geq 2$).

(c) $\pi$ is a contraction, that is $\mathcal{O}_S = \pi_*\mathcal{O}_X$.

(ii). A log variety is a log pair $\pi: (X, D) \to S$ such that $S = X, \pi = \text{id}_X$.

Then the second condition is equivalent to $D$ being an effective divisor.

We call $D$ the pseudo-boundary of the log pair, and we also call $K + D$ a log divisor, since its sections correspond to rational differentials with poles along $D$. Recall that $K_X$ is a $\mathbb{Z}$-Weil divisor on $X$, uniquely defined up to linear equivalence in its class, called the canonical class.

For simplicity of terminology, we will drop the adjective “relative effective” and if there is no danger of confusion, we will also drop $\pi$ and $S$ from the notation, so we will say that $(X, D)$ is a log pair.

1.2. Given a log pair $\pi: (X, D) \to S$, a desingularization $\mu: Y \to X$ determines canonically a log pair $\varphi: (Y, D_Y) \to S$ [Sho, pp.114].

\[
\begin{array}{ccc}
(Y, D_Y) & \xrightarrow{\mu} & (X, D) \\
\downarrow \varphi & & \downarrow \pi \\
S & \xrightarrow{\pi} & S
\end{array}
\]

Consider the following equality of Weil divisors:

\[
K_Y + \mu^{-1}D + \sum E_i = \mu^*(K_X + D) + \sum a_iE_i,
\]
where the sum runs over the $\mu$-exceptional divisor of $Y$, $\mu^{-1}D$ is the proper transform of the Weil divisor $D$. The above formula determines uniquely the coefficients $a_i = a(E_i; X, D)$, which are called the log discrepancy of the exceptional divisors $E_i$. They are independent of the ambient resolution in which $E_i$ seats. We also extend the definition to non-exceptional divisors $E_i$, declaring $a(E_i; X, D) = 1 - e$, where $e$ is the coefficient of $E_i$ in $D$.

If we denote $D_Y = \mu^{-1}D + \sum (1 - a_i)E_i$, $\varphi = \pi \circ \mu$, then $\varphi: (Y, D_Y) \to S$ becomes a log pair such that

$$K_Y + D_Y = \mu^*(K_X + D).$$

We say that $\mu$ is a crepant morphism of log pairs if the above equality holds. Note that $(X, D)$ and $(Y, D_Y)$ have the same log discrepancies, and they should be viewed as being equivalent.

**Example 1.** Given a log variety $(X, B)$ and a resolution of singularities $\mu: Y \to X$, it is easy to see that $\mu: (Y, B_Y) \to X$ is a log pair, while $B_Y$ may have negative coefficients. This is the main example of log pairs appearing in the study of log varieties.

**Definition 1.3.** A log pair $(X, D)$ is log canonical if all log discrepancies are nonnegative. In particular, $d_i \leq 1 \forall i$ (we say that $D$ is a subboundary in this case).

**Definition 1.4.**

(i). A log pair $(X, D)$ has (log) nonsingular support if $X$ is a nonsingular variety and $D = \sum d_i D_i$ is a Weil divisor such that $\cup_{d_i \neq 0} D_i$ is a union of smooth divisors intersecting transversely.

(ii). A log resolution of a log pair $(X, D)$ is a log pair induced on a resolution of singularities $(Y, D_Y)$ which has nonsingular support.

**Example 1.5.** Assume $(X, D)$ is a log pair with nonsingular support. Then $(X, D)$ is log canonical iff $d_i \leq 1$, $\forall i$ (see the proof of 2.2).

**Example 1.6.** Let $X$ be a toric variety and $B_X = \sum B_i$ be the complement of the embedded torus. Then $(X, B_X)$ is a Calabi-Yau log variety:

$$K_X + B_X = 0.$$ 

Moreover, $(X, B_X)$ is log canonical (see [Reid, 4.8], or [Ale, 3.1]).

**Remark 1.7.** Although log canonicity involves all possible prime divisors with center on $X$, it is enough to check it on a log resolution $\mu: (Y, D_Y) \to (X, D)$. From definition, $(X, D)$ is log canonical iff $(Y, D_Y)$ is log canonical which, in turn, is equivalent to $D_Y$ being a subboundary.
2. The locus of log canonical singularities

**Notation 2.1.** For an \( \mathbb{R} \)-Weil divisor \( D = \sum d_i D_i \), \( d_i \in \mathbb{R} \) on a normal variety \( X \) we define

(i). the coherent divisorial sheaf \( \mathcal{O}_X(D) \subset \mathcal{R}_X = K(X) \) defined as
\[
H^0(U, \mathcal{O}_X(D)) = \{ f \in K(X); (f) + D|_U \geq 0 \}, \quad U \subseteq X.
\]

(ii). If \( \mathcal{F} \) a coherent sheaf on \( X \), and \( \mathcal{O}_X(D) \) is an invertible sheaf, we denote \( \mathcal{F}(D) := \mathcal{F} \otimes \mathcal{O}_X(D) \).

(iii). the round up (down) of \( D \), \( \lceil D \rceil = \sum \lceil d_i \rceil D_i \) \( \lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i \).

(iv). the positive (negative) part of \( D \), \( D^+ = \sum_{d_i > 0} d_i D_i \), \( D^- = \sum_{d_i > 0} d_i D_i \), so the decomposition \( D = D^+ + D^- \) holds.

Note the identities

a) \( \mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor) \).

b) \( \lceil -D \rceil = -\lfloor D \rfloor \).

c) \( \lceil -(D^+) \rceil = -\sum_{d_i \geq 1} \lfloor d_i \rfloor D_i \).

We declare that taking the positive (negative) part of a divisor has precedence over all other operations. For example, we will write \( -D^- \) for \( -(D^-) \).

**Definition-Proposition 2.2.** Let \( (X, D) \) be a log pair and let \( \mu : Y \to X \) a log resolution with \( D^Y \) the corresponding pseudo-boundary on \( Y \). Then the coherent ideal sheaf on \( X \)
\[
\mathcal{I}(X, D) = \mu_* \mathcal{H}om_Y(\mathcal{O}_Y((D^Y)^+), \mathcal{O}_Y) = \mu_* \mathcal{O}_Y(\lceil -(D^Y)^+ \rceil)
\]
is independent of the log resolution. The induced subscheme of \( X \), denoted \( LCS(X, D) \), is called the **locus of log canonical singularities of the log pair** \((X, D)\).

**Remark 2.3.** The above definition is a slight modification of V. Shokurov’s definition of the LCS ideal (see also [Ko2, 2.16]). He defined the coherent sheaf \( \mathcal{I}(X, D) = \mu_* \mathcal{O}_Y(\lceil -D^Y \rceil) \), which is isomorphic to an ideal sheaf only if \( D \) is effective, in which case \( \mathcal{I}(X, D) \simeq \mathcal{I}(X, D) \) (see Remark 2.7). It is interesting that \( \mathcal{I}(X, D) \) has good vanishing properties by the very definition, which is not the case for the actual ideal \( \mathcal{I}(X, D) \).

**Proof.** Using Hironaka’s hut, it is enough to check that \((X, D)\) has nonsingular support, and \( \tau : Y \to X \) is a sequence of blow-ups with nonsingular centers, then
\[
\tau_* \mathcal{O}_Y(\lceil -D^Y \rceil) = \mathcal{O}_X(\lceil -D^+ \rceil).
\]

Indeed, let \( \{E_i\}_{i=1}^t \) be the exceptional locus of \( \tau \), with \( m_i = cod_X(\tau(E_i)) \). Then \( K_Y = \tau^* K_X + \sum_{i=1}^t \alpha_i E_i \) and \( D^Y = \tau^* D - \sum_{i=1}^t \alpha_i E_i \), where \( \alpha_i \geq m_i - 1 \). We claim that \( \lceil D^Y \rceil \leq \tau^* \lfloor D \rfloor \).

Indeed, if \( \tau(E_k) \) lies in \( D_1, \ldots, D_s \) \((s \leq t)\) only, then \( m_k \geq s \) and \( \lceil \sum_{j=1}^s d_j \rceil \leq \sum_{j=1}^s d_j \).
\[\sum_{j=1}^{s} [d_j] + s - 1 \leq \sum_{j=1}^{s} [d_j] + \alpha_k, \text{ that is } |\tau^* D| \leq \tau^* [D] + \sum_{i=1}^{t} \alpha_i E_i.\]

Now \([D^+ Y]^* = [D Y]^* \leq (\tau^* [D])^* \leq \tau^* (\lfloor D \rfloor^*) = \tau^* (\lfloor D^+ \rfloor).\]

This last inequality, together with the fact that \(D\) has simple normal crossings support implies at once that \(\tau_* \mathcal{O}_Y(-\lfloor (D^+ Y)^* \rfloor) = \mathcal{O}_X(-\lfloor D^+ \rfloor).\)

We have a dichotomy: either \(\text{LCS}(X, D) = \emptyset\), in which case we say that \((X, D)\) has Kawamata log terminal singularities (klt for short), or \(\text{LCS}(X, D) \neq \emptyset\) is a proper subscheme of \(X\). We are mainly interested in the second case.

**Example 2.** Let \((X, D)\) be a log pair with log nonsingular support. If we write \(D = \sum d_i D_i\), and \(E = \sum_{d_i \geq 1} [d_i] D_i\), then

\[\text{LCS}(X, D) = (E, \mathcal{O}_E).\]

**Example 3.** Let \(S = \mathbb{A}^2\) and let \(C : (y^2 - x^3 = 0) \subset S\) a curve with a cusp at the origin \(P\). Then

(i). \(\text{LCS}(S, tC) = \emptyset\) if \(0 \leq t < \frac{5}{6}\).

(ii). \(\text{LCS}(S, \frac{5}{6}C) = \{P\}\) and \((S, \frac{5}{6}C)\) is log canonical.

(iii). \(\text{LCS}(S, tC) = C\) as a set and \((S, tC)\) is not log canonical for \(t > \frac{5}{6}\).

**Example 4.** Let \(L, H \subset \mathbb{P}^3\) be a line and a plane intersecting in a point. Let \(H_1, H_2, H_3\) be three general planes passing through the line \(L\). Let \(B = H + \frac{2}{3}(H_1 + H_2 + H_3)\). Then \((\mathbb{P}^3, B)\) is a log variety with log canonical singularities, and

\[\text{LCS}(\mathbb{P}^3, B) = L \cup H.\]

This is an example of non pure dimensional LCS locus.

**Example 5.** Let \(L_1, L_2, L_3 \subset \mathbb{P}^3\) be three lines passing through a point \(P\) such that \(\dim T_P(L_1 \cup L_2 \cup L_3) = 3\). Let \(H_{i1}, H_{i2}, H_{i3}\) be generic planes passing through the line \(L_i\), for \(1 \leq i \leq 3\). Let \(B = \frac{2}{3} \sum H_{ij}\). Then \((\mathbb{P}^3, B)\) has log canonical singularities, and

\[\text{LCS}(\mathbb{P}^3, B) = L_1 \cup L_2 \cup L_3.\]

The same is true if we consider \(n\) general lines passing through a point in \(\mathbb{P}^m\).

**Remark 2.4.** Let us denote \(X^{(i)} = \{\eta \in X; \text{codim}(\eta, X) = j\}\). Assume the log pair \((X, D)\) is log canonical. It is interesting that the \(\text{LCS}(X, D)\) is smooth in \(X^{(1)}\), it has at most ordinary double points in \(X^{(2)}\), and it can have triple ordinary points in \(X^{(3)}\) (similarly for any \(n > 3\)). So we could naturally ask if the triple ordinary points are the only type of singularities appearing in \(X^{(3)}\). Although they are seminormal, the answer might be negative!
Proposition 2.5. Let \( \tau : (Y, D_Y) \to (X, D_X) \) be a birational contraction of log pairs and assume that \( \tau \) is crepant, that is
\[
\tau^*(K_X + D_X) = K_Y + D_Y.
\]
Then \( \tau \) induces a dominant morphism between the LCS schemes
\[
\tau' : LCS(Y, D_Y) \to LCS(X, D_X),
\]
Proof. Let \( \mu : Z \to Y \) be a log resolution. Then \( \tau \circ \mu : Z \to X \) is also a log resolution, and \((D_X)^\mu = (D_Y)^\mu\). Therefore \( I(X, D_X) = \tau_*I(Y, D_Y) \). This easily implies that \( O_{LCS(X, D_X)} \to O_{LCS(Y, D_Y)} \) is injective, hence \( \tau' \) is dominant.

Theorem 2.6. Let \( \pi : (X, D) \to S \) be a log pair such that \(- (K_X + D)\) is \(\pi\)-nef and \(\pi\)-big.

(i). The following sequence is exact:
\[
0 \to \pi_*\mathcal{I}(X, D) \to \mathcal{O}_S \to \pi_*O_{LCS(X, D)} \to 0
\]
(ii). \( R^1\pi_*\mathcal{I}(X, D) = 0 \) if \( \pi \) is a birational contraction, or if \( R^1\pi_*\mathcal{O}_X = 0 \).

Proof. We assume first that \((X, D)\) has log nonsingular support. Denoting \( I = I(X, D) \) and \( E = \lceil -D^- \rceil \) we have
\[
\mathcal{O}_X(\lceil -D \rceil) = \mathcal{I} \otimes \mathcal{O}_X(E).
\]
Since \(- (K_X + D)\) is \(\pi\)-nef and \(\pi\)-big, and \((X, D)\) is log nonsingular, Kawamata-Viehweg vanishing implies \( R^j\pi_*\mathcal{O}_X(\lceil -D \rceil) = 0 \), \( \forall j \geq 1 \), that is
\[
R^1\pi_*\mathcal{I}(E) = 0, \ \forall j \geq 1.
\]

Look at the following commutative diagram with exact rows:
\[
\begin{array}{c}
0 \to \pi_*\mathcal{I}(X, D) \to \mathcal{O}_S \to \pi_*O_{LCS(X, D)} \to 0 \\
|j_0| \downarrow \quad \downarrow i_1 \quad \downarrow j_2 \\
0 \to \pi_*\mathcal{I}(E) \to \pi_*\mathcal{O}_X(E) \to \pi_*O_{LCS(E)} \to R^1\pi_*\mathcal{I}(E) = 0
\end{array}
\]
Since \( E \) is effective \(\pi\)-exceptional, \( j_1 \) is an isomorphism. Moreover, \( i_2 \) is surjective due to vanishing, hence \( j_2 \) is surjective. But \( j_2 \) is injective, hence \( j_0, j_1, j_2 \) are all isomorphisms and \( i_1 \) is surjective. This proves the first part.

For the second, let us assume that \( \pi \) is birational. Consider now the following commutative diagram
\[
\begin{array}{c}
0 \to R^1\pi_*\mathcal{I} \to R^1\pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_Y \to 0 \\
|j| \downarrow \\
R^1\pi_*\mathcal{I}(E) = 0 \to R^1\pi_*\mathcal{O}_X(E)
\end{array}
\]
where the top row is exact from the above argument. We claim that \( j \) is injective. Indeed, \( \pi_*\mathcal{O}_E(E) \) surjects onto \( Ker(j) \) and \( \pi_*\mathcal{O}_E(E) = 0 \) [KMM, 1-3-2]. Therefore the morphism \( R^1\pi_*\mathcal{I} \to R^1\pi_*\mathcal{O}_Y(E) \) is injective too, hence
Now, for the general case, let $\mu : (Y, D_Y) \to (X, D)$ be a log resolution and denote $\nu = \pi \circ \mu$ and $E = \lceil -(D_Y^-) \rceil$. We have the following diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \pi_* I(X, D) & \longrightarrow & \pi_* O_X & \longrightarrow & \pi_* O_{LCS(X, D)} \\
& & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & \nu_* I(Y, D^Y) & \longrightarrow & \nu_* O_Y & \longrightarrow & \nu_* O_{LCS(Y, D^Y)} & \longrightarrow & 0
\end{array}
\]

where the bottom row is exact from the previous step. With the same argument as above, we obtain that all the vertical arrows are isomorphisms. In particular, the last arrow of the top arrow is surjective. Finally, the exact sequence of lower terms of the spectral sequence

\[
E_2^{p,q} = R^p \pi_* R^q \mu_* I(Y, D^Y) \Longrightarrow E^{p+q} = R^{p+q} \nu_* I(Y, D^Y)
\]
gives the injection $R^1 \pi_* I(X, D) \hookrightarrow R^1 \nu_* I(Y, D^Y)$, hence we are done from the previous case.

**Remark 2.7.** Note that if $D$ is effective, $j_0$ is an isomorphism between $I(X, D)$ and $I'(X, D)$.

**Proposition 2.8.** Let $\tau : (Y, D_Y) \to (X, D_X)$ be a crepant birational contraction of log pairs

\[
\begin{array}{ccc}
(Y, D_Y) & \xrightarrow{\tau} & (X, D_X) \\
\downarrow \varphi & & \downarrow \pi \\
S & & 
\end{array}
\]

Then $\varphi_* O_{LCS(Y, D_Y)} = \pi_* O_{LCS(X, D_X)}$, that is $\pi_* O_{LCS(X, D_X)}$ is a birational invariant of log pairs. In particular, if $(X, B)$ is a log variety and $\tau : Y \to X$ is a resolution, then $\tau' : LCS(Y, B^Y) \to LCS(X, B)$ is a contraction, that is $O_{LCS(X, B)} = \tau'_* O_{LCS(Y, B^Y)}$.

**Proof.** Let $\mu : Z \to Y$ be a log resolution. Then $\tau \circ \mu : Z \to X$ is also a log resolution, and $(D_X)^Z = (D_Y)^Z$. We then apply the previous theorem for $\tau \circ \mu : Z \to X$. \qed

**Corollary 2.9** (Connectedness Lemma [Sho, 5.7], [Kol1, 17.4]). Assume $\pi : (X, D) \to S$ is a log pair such that $- (K_X + D)$ is $\pi$-nef and $\pi$-big. Then $LCS(X, D) \cap \pi^{-1}(s)$ is connected for every $s \in S$. 

Proof. The surjection
\[ O_S \to \pi_*O_{\text{LCS}(X,D)} \to 0. \]
easily implies the connectivity of the fibers. \qed

Lemma 2.10. Let \((X, B)\) be a log variety. Assume there is a log resolution \(\mu : (Y, B^Y) \to (X, B)\) such that \(\text{LCS}(Y, B^Y)\) is a reduced scheme. Then \(\text{LCS}(X, B)\) is seminormal.

Proof. Note that \(\text{LCS}(Y, B^Y)\) is reduced iff \(\lfloor (B^Y)^+ \rfloor\) is a reduced divisor. Then \(\text{LCS}(Y, B^Y)\) is a simple normal crossings divisor with the induced reduced structure, which is seminormal by 4.6. But \(\text{LCS}(X, B)\) is a contraction of \(\text{LCS}(Y, B^Y)\), so we can apply 4.5. \qed

Corollary 2.11. Assume \((X, B)\) is a log variety with log canonical singularities. Then \(\text{LCS}(X, B)\) is a seminormal variety.

Theorem 2.12 (V. Shokurov). Let \((X, B)\) be a log variety and assume that \(B\) is effective. Let \(\pi : X \to S\) be a proper morphism and let \(L\) be a Cartier divisor on \(X\) such that
\[ L = K_X + B + H \]
where \(H\) is a \(\pi\)-nef and \(\pi\)-big \(\mathbb{R}\)-Cartier divisor. Then
\[ R^j \pi_*(\mathcal{I}(X, B)(L)) = 0, \forall j \geq 1. \]

Proof. Let \(\mu : (Y, B^Y) \to (X, B)\) be a log resolution and denote \(\nu = \pi \circ \mu\). We can assume that there is an effective \(\mathbb{R}\)-divisor \(F\), and a \(\nu\)-ample \(\mathbb{R}\)-divisor \(A\) on \(Y\) such that \(\mu^*H = F + A\) and \([\tau_B^Y - F] = [\tau_B^Y]\). Since \(-(K_Y + B^Y)\) is \(\mu\)-nef and \(\mu\)-big, Kawamata-Viehweg vanishing gives \(R^j \mu_* \mathcal{O}_Y([\tau_B^Y]) = 0, \forall j \geq 1, \) hence, by the projection formula,
\[ R^j \mu_* \mathcal{O}_Y([\tau_B^Y] + \mu^*L) = 0, \forall j \geq 1. \]

Therefore the Grothendieck spectral sequence degenerates and the following isomorphism holds
\[ R^j \pi_*(\mu_* \mathcal{O}_Y([\tau_B^Y])(L)) \simeq R^j \nu_* \mathcal{O}_Y([\tau_B^Y] + \mu^*L), \forall j \geq 0. \]
But \(\mu^*L - K_Y - B^Y - F\) is \(\nu\)-ample, so Kawamata-Viehweg vanishing gives
\[ R^j \nu_* \mathcal{O}_Y([\tau_B^Y] + \mu^*L) = 0, \forall j \geq 1. \]

Finally, since \(B\) is effective, \(\mu_* \mathcal{O}_Y([-B^Y]) = \mathcal{I}'(X, D) \simeq \mathcal{I}(X, D)\), so we obtain our vanishing. \qed
3. Kawamata’s lc centers and perturbation trick

Kawamata’s perturbation technique applies for log varieties \((X, B)\) with the following property \([\text{Ka2}]\): there is another log variety \((X, B^o)\) such that

(i). \((X, B^o)\) has Kawamata log terminal singularities,
(ii). \(B^o < B\).

We will assume this throughout this section.

**Definition 3.1** ([\text{Ka1}]). Let \((X, B)\) be a log variety. Then any codimension one component \(E\) of \(\text{LCS}(Y, B^Y)\), for any resolution \(\mu: Y \to X\), is called a log canonical (lc) place. The image on \(X\) of a log canonical place is called a log canonical (lc) center.

Log canonical centers are building blocks of the LCS locus. Note that the LCS locus is the union of all the lc centers. All the irreducible components of the LCS locus are lc centers.

**Proposition 3.2.** Assume \((X, B)\) is a log canonical variety. Then any finite union of lc centers is a seminormal variety. In particular, every irreducible component of \(\text{LCS}(X, B)\) is seminormal.

**Proof.** (cf. [\text{Ka1}, 1.5]) Let \(W_1, \ldots, W_k\) be lc centers for \((X, B)\), \(W = \cup_i W_i\), and let \(E_{ij}\) be all corresponding lc places on a log resolution \(\mu: Y \to X\) such that \(\mu(E_{ij}) = W_i\). Let \(H_i \supseteq W_i(1 \leq i \leq k)\) be generic effective divisors. Define

\[B_\epsilon = (1 - \epsilon)B + \epsilon B^o + \sum a_iH_i, \quad 0 < \epsilon \ll 1, a_i \in \mathbb{R}.\]

Then \(K_X + B_\epsilon = K_X + B + \epsilon(\sum a_iH_i - (B - B^o))\) is an \(\mathbb{R}\)-Cartier divisor. Let \(a_i (1 \leq i \leq k)\) be the smallest positive numbers satisfying the inequalities

\[a(E_{ij}; X, B_\epsilon) \leq a(E_{ij}; X, B), \quad \forall i, j, \forall \epsilon.\]

\((a_i = \min_j \frac{\nu(E_{ij}; B - B^o)}{\nu(E_{ij}; M)}, \text{ where } \nu(E; M) \text{ denotes the coefficient of } E \subset Y \text{ in } \mu^*(M))\)

Then \(\text{LCS}(X, B_\epsilon) = W\). Indeed, \(\text{LCS}(X, B_\epsilon) \subseteq \text{LCS}(X, B)\) for small \(\epsilon\), and if \(E \in \text{LCS}(Y, B^Y)\), \(\mu(E) \not\subseteq W\), then \(a(E; X, B_\epsilon) = a(E; B - \epsilon(B - B^o)) < a(E; B) \leq 0\).

Therefore \((X, B_\epsilon)\) is log canonical in the generic points of \(W = \text{LCS}(X, B_\epsilon)\), and even if it has worse singularities in proper points of \(W\), \([B^Y_\epsilon]\) is a reduced divisor on \(Y\) for \(\epsilon\) small enough. Therefore \(\text{Proposition 2.11}\) gives the seminormality of \(W\). \(\Box\)

**Proposition 3.3.** ([\text{Ka1}, 1.5]) Assume the log variety \((X, B)\) has log canonical singularities. Let \(W_1, W_2\) be two lc centers of \((X, B)\) on \(X\). Then every irreducible component of \(W_1 \cap W_2\) is a lc center for \((X, B)\). In particular, there are minimal (with respect to inclusion) centers.
**Lemma 3.4** ([Ka1]). Let \((X, B)\) be a log canonical variety, and let \(W \subseteq LCS(X, B)\) be a minimal lc center. Then there is a log canonical variety \((X, B')\) and a log resolution \(\mu : (Y, B'^Y) \to (X, B')\) such that \(E = LCS(Y, B'^Y)\) is a smooth prime divisor. In particular, there is an induced contraction \(\nu : E \to W\), hence \(W\) is normal.

**Proof.** We can assume \(W = LCS(X, B)\) using the argument of 3.2. The variety \((X, B)\) stays log canonical because \(W\) is minimal. Let \(\mu : (Y, B'^Y) \to (X, B)\) be a log resolution, and let \(LCS(Y, B'^Y) \subseteq \cup_{1 \leq i \leq k} E_i\), where \(\mu(E_i) = W\).

We have to decrease the log discrepancies of all but one of the \(E_i\)'s and the following trick was kindly suggested by V. Shokurov. Let \(\{m_i\}_i\) be a bounded family of integer vectors with integers entries, and let \(A \) be an ample Cartier divisor on \(Y\) and \(H\) a nef and big Cartier divisor on \(X\). After taking a high multiple of \(A\), we can assume that \(|A - \sum m_i E_i|\) is a free linear system for all vectors \(\{m_i\}_i\) in our bounded family. Since \(\mu^*H\) is nef and big on \(Y\), there is \(N \in \mathbb{N}\) such that \(\mu^*(NH) = E + A\), with \(E\) effective. Therefore after scaling the family \(\{m_i\}_i\) and \(E\) with \(N\), there is an effective \(\mathbb{Q}\)-Cartier divisor \(M \sim_N H\) such that

\[ \mu^*M = E + \sum m_i E_i + F, \forall \{m_i\}_i, \]

where \(E\) is an effective \(\mathbb{Q}\)-divisor (same for all \(\{m_i\}_i\)), and \(F\) is a \(\mathbb{Q}\)-free effective divisor, not containing \(E_i\)'s in its support. Define

\[ B'_\epsilon = (1 - \epsilon)B + \epsilon B^o + \epsilon a M, \quad 0 < \epsilon \ll 1, a \in \mathbb{R}. \]

Then \(B'^Y = B^Y + \epsilon(aE + \sum am_i E_i - \mu^*(B - B^o)) + \epsilon a F\). Let \(a\) be the smallest positive number satisfying the inequalities

\[ a(E_i; X, B) \leq a(E_i; X, B), \forall i, \forall \epsilon, \]

that is \(a = \min_i \frac{\mu(E_i; B - B^o)}{m_i + \nu(E_i; B)}\). Since we have a family, we can assume that the equality holds for exactly one \(E_i\). We just take now \(B' = B'_\epsilon\).

4. **Appendix on seminormal varieties**

We say that a morphism \(f : Y \to X\) is a *quasi-isomorphism* if it is a universal homeomorphism such that \(k(f(x)) \xrightarrow{\text{cts}} k(x)\) for all Grothendieck points \(x \in X\). Note that any quasi-isomorphism is birational. A *universal homeomorphism* is a morphism \(f : Y \to X\) such that for any base change \(X' \to X\), the induced morphism \(Y \times_X X' \to X'\) is a (topologically) homeomorphism.

**Definition 4.1.** (i). Let \(f : Y \to X\) be a dominant morphism of preschemes such that \(f_* O_Y\) is a quasi-coherent \(O_X\)-algebra (this always happens in applications, for example if \(f\) is quasi-compact and quasi-separated). The *seminormalization* of \(f\) is an integral quasi-isomorphism \(sn_f : X^{sn,f} \to X\) which factors \(f\), and is maximal with respect to this property. That is, if \(g : Z \to X\) is another integral quasi-isomorphism
factoring \( f \), then there is a unique morphism \( X^{sn,f} \rightarrow Z \) making the following diagram commutative:

\[
\begin{array}{c}
Y \\
\downarrow f \\
X^{sn,f} \\
\downarrow sn_f \\
X \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow g \\
\rightarrow \\
\downarrow \\
Z \\
\end{array}
\]

(ii). We say that \( f \) is seminormal, or that \( X \) is seminormal in \( Y \), if \( sn_f \) is an isomorphism.

Sketch of proof. [See [AB, Tr]] We have to prove existence only. Let \( C \) be the integral closure of \( O_X \) in \( f_*O_Y \), which is a quasi-coherent \( O_X \)-algebra (EGA II.6.3.4). Define \( C^{sn} \) as follows

\[
H^0(U,C^{sn}) = \{ s \in H^0(U,C); \forall x \in U, s_x \in O_x + R(C_x) \}, \quad U \subseteq X
\]

where \( R(A) \) denotes the radical of the ring \( A \), that is the intersection of all its maximal ideals.

Then it follows that \( C^{sn} \) is a quasi-coherent \( O_X \)-algebra and

\[
\text{sn}_f : \text{Spec}_X(C^{sn}) \rightarrow X
\]

satisfies the required universal property. For a detailed proof with “quasi-isomorphism” replaced by “universal homeomorphism”, see [AB]. \( \square \)

4.2. (Functoriality) Let \( f_1 : Y_1 \rightarrow X_1 \) and \( f_2 : Y_2 \rightarrow X_2 \) be two dominant morphisms such that there are two morphisms \( \alpha : X_1 \rightarrow X_2, \beta : Y_1 \rightarrow Y_2 \) with \( f_2 \circ \beta = \alpha \circ f_1 \). Then there is a unique morphism \( \alpha^* : X_1^{sn,f_1} \rightarrow X_2^{sn,f_2} \) such that the following diagram is commutative:

\[
\begin{array}{c}
Y_1 \\
\downarrow \\
X_1^{sn,f_1} \\
\downarrow \text{sn}_{f_1} \\
X_1 \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow \\
X_2^{sn,f_2} \\
\downarrow \text{sn}_{f_2} \\
X_2 \\
\end{array}
\quad \begin{array}{c}
\rightarrow \\
\downarrow f_2 \\
Y_2 \\
\downarrow \\
\rightarrow f_1 \\
Y_1 \\
\end{array}
\]

Indeed, giving \( \alpha^* \) is the same as giving a map \( X_1^{sn,f_1} \rightarrow Z = X_1^{sn,f_1} \times_{X_2} X_2^{sn,f_2} \). But \( Z \rightarrow X_1 \) is an integral quasi-isomorphism factoring \( f_1 \), hence the existence and uniqueness of \( \alpha^* \) follows from the universal property of \( \text{sn}_{f_1} \).

Using the above functoriality and chasing diagrams, it is easy to see that seminormal morphisms behave well under composition. If \( f : Z \rightarrow Y \) and \( g : Y \rightarrow X \) are dominant morphisms, then \( g, f \) \( \Rightarrow \) \( g \circ f \) \( \Rightarrow \), and \( g \circ f \) \( \Rightarrow \) \( g \) \( sn \).
4.3. (Constructions) We say that \( f : Y \to X \) is a contraction (fiber space) if the natural morphism \( \mathcal{O}_X \to f_*\mathcal{O}_Y \) is an isomorphism (is an algebraically closed extension). Contractions and fiber spaces are example of seminormal morphisms. Indeed, \( C^{\text{sn}} = C = \mathcal{O}_X \) in this case.

4.4. Let’s fix a field \( k \) of any characteristic. From now on we consider varieties only, i.e. algebraic reduced \( k \)-schemes (possibly with more than one irreducible component). Let \( X \) be a variety with normalization \( \pi : \bar{X} \to X \), which is a birational finite morphism. Define the seminormalization of \( X \) to be the seminormalization of \( X \) in \( \bar{X} \). Then \( \text{sn}_X : X^{\text{sn}} \to X \) is a finite quasi-isomorphism which is maximal in the following sense:

For any quasi-isomorphism \( Z \xrightarrow{g} X \) from a variety \( Z \), there is a unique morphism \( \sigma : X^{\text{sn}} \to Z \) such that \( g \circ \sigma = \text{sn}_X \).

There is a functor associating to any variety \( X \) its seminormalization \( X^{\text{sn}} \), and to any morphism \( f : X \to Y \) its unique extension \( f^{\text{sn}} : X^{\text{sn}} \to Y^{\text{sn}} \).

\[
\begin{array}{ccc}
X^{\text{sn}} & \xrightarrow{f^{\text{sn}}} & Y^{\text{sn}} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

This follows from the general functoriality, since any morphism lifts to normalizations.

**Proposition 4.5.** Let \( f : Y \to X \) be a contraction, or more generally, a seminormal morphism. Then \( Y \) seminormal implies that \( X \) is seminormal.

**Proof.** Indeed, \( \text{sn}_Y \) is an isomorphism and \( f = \text{sn}_X \circ (f^{\text{sn}} \circ \text{sn}_Y^{-1}) \), so \( f \) factors through the quasi-isomorphism \( \text{sn}_X \). But the seminormalization of \( X \) in \( Y \) is \( X \), hence \( \text{sn}_X \) is an isomorphism.

**Lemma 4.6.** Let \( D \) be the support of a reduced normal crossing divisor on a nonsingular variety \( X \). Then \( D \) is seminormal.

**Proof.** Since a local ring \( \mathcal{O} \) is seminormal iff its completion \( \mathcal{O}^- \) is seminormal \([GT, 5.3]\), we can assume \( \mathcal{O}_{D,P} = k[X_1, \ldots, X_n]/(X_1 \cdots X_s), \ s \leq n \). It is easy to see that this ring is seminormal.

**Remark 4.7.** By Serre’s criterion, a variety \( X \) is normal iff

(i). \( X \) is nonsingular in codimension 1 and

(ii). \( X \) is \( S_2 \)-saturated, that is \( \mathcal{O}_X = j_*\mathcal{O}_{X-Z} \) for every closed subset \( Z \) of \( X \) codimension at least 2.

Similarly, an \( S_2 \)-saturated variety \( X \) is seminormal iff \( X \) is seminormal in codimension 1 \([GT, 2.6]\). Moreover, the codimension 1 seminormal singularities are classified. They basically look like the origin on the \( n \) coordinate axes in \( \mathbb{A}^n \).
REFERENCES

[Ale] V. Alexeev, Log canonical singularities and complete moduli of stable pairs, Preprint (1996).

[AB] A. Andreotti, E. Bombieri, Sugli omeomorfismi delle varietà algebriche, Ann. Scuola Norm. Sup. Pisa (3), 23 (1969), 431–450.

[GT] S. Greco, C. Traverso, On seminormal schemes, Comp. Math. 40(3), (1980), 325–365.

[Ka1] Y. Kawamata, On Fujita’s freeness conjecture for 3-folds and 4-folds, Duke preprint alg-geom/9510004.

[Ka2] Y. Kawamata, Subadjunction of log canonical divisors II, Duke preprint alg-geom/9712014.

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, Introduction to the minimal model program, Algebraic Geometry, Sendai, Advanced Studies in Pure Math. 10 (1987), 283–360.

[Kol1] J. Kollár et al., Flips and abundance for algebraic threefolds, Astérisque 211 (1992), 1–258.

[Kol2] J. Kollár, Singularities of pairs, Duke preprint alg-geom/9601026.

[Reid] M. Reid, Young person’s guide to canonical singularities, Algebraic Geometry (Bowdoin,1985). Proc. Sympos. Pure Math. 46:1, Amer. Math. Soc., Providence, RI (1987), 345–414.

[Sho] V. Shokurov, 3-Fold log flips, Russian Acad. Sci. Izv. Math. 40:1 (1993), 95–202.

[Tr] C. Traverso, Seminormality and Picard group, Ann. Sc. Norm. Sup. Pisa 24 (1970), 585–595.

DEPARTMENT OF MATHEMATICS, THE JOHNS HOPKINS UNIVERSITY, 3400 N. CHARLES, BALTIMORE MD 21218

E-mail address: ambro@chow.mat.jhu.edu