APPROXIMATING THE GROUND STATE OF GAPPED QUANTUM SPIN SYSTEMS

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ABSTRACT. We consider quantum spin systems defined on finite sets \( V \) equipped with a metric. In typical examples, \( V \) is a large, but finite subset of \( \mathbb{Z}^d \). For finite range Hamiltonians with uniformly bounded interaction terms and a unique, gapped ground state, we demonstrate a locality property of the corresponding ground state projector. In such systems, this ground state projector can be approximated by the product of observables with quantifiable supports. In fact, given any subset \( X \subset V \) the ground state projector can be approximated by the product of two projections, one supported on \( X \) and one supported on \( X^c \), and a bounded observable supported on a boundary region in such a way that as the boundary region increases, the approximation becomes better. Such an approximation was useful in proving an area law in one dimension, and this result corresponds to a multi-dimensional analogue.

1. INTRODUCTION

The intense interest in entangled states for purposes of quantum information and computation during the past decade has stimulated new investigations in the structure of ground states of quantum spin systems. It was soon found that the focus on entanglement, and the mathematical structures related to this concept, also provide a new and useful way to investigate important physical properties relevant for condensed matter physics \([1, 10]\). In particular, progress was made in our understanding of constructive approximations of the ground states of quantum spin models, and about the computational efficiency of numerical algorithms to compute ground state properties and to simulate the time evolution of such systems \([18, 19]\).

It is now understood that there is a relationship between the amount of entanglement in a state and the degree of difficulty or the amount of resources needed to construct a good approximation of it \([17]\). To make this more precise one has to use a quantitative measure of entanglement. For pure states the \textit{entanglement entropy} is a very natural such measure for bi-partite entanglement. For a two-component system with components \( A \) and \( B \) (e.g., \( A \) and \( B \) label a partition of the spin variables into two sets), the entanglement entropy of the system in a pure state \( \psi \) with respect to this partition is defined as the von Neumann entropy of the density matrix describing the restriction of \( \psi \) to subsystem \( A \). Mathematically, this density matrix, \( \rho_A \), is obtained as the partial trace of the orthogonal projection onto the state \( \psi \) calculated by tracing out the degrees of freedom in \( B \).

It was conjectured that under rather general assumptions, the entanglement entropy associated with \( A \) satisfies an \textit{Area Law}. By Area Law one means that if \( A \) is associated with a subvolume of a fixed system in a ground state then, with a suitable definition of boundary, the entanglement entropy of \( A \) should be bounded by a constant times the size of the boundary of \( A \) (e.g., the surface area in the case of a three-dimensional volume).

One way to understand this conjecture is to observe that it holds almost trivially for the so-called Matrix Product States (MPS), aka Finitely Correlated States \([3]\) and their generalizations in higher dimensions known as PEPS \([16]\). It is therefore plausible that to prove the Area Law it is key to
have a good handle on MPS-like approximations of a general ground state. By using such a strategy Hastings obtained a proof of the Area Law for one-dimensional systems [5].

The goal of this paper is to prove a generalization to arbitrary dimensions of the approximation result Hastings used in his proof of the Area Law in one dimension. This is Theorem 2.2 below, which demonstrates that the ground state projector of certain gapped quantum spin systems can be approximated by a product of three projections with known supports. We describe this result in detail in the next section.

2. A Ground State Approximation Theorem

We begin with a brief description of the quantum spin systems that will be considered in this work. Let \( \mathcal{V} \) be a countable, locally finite set equipped with a metric \( d \). In most examples, \( \mathcal{V} = \mathbb{Z}^n \) and \( d \) is, for example, the Euclidean metric. Since our arguments do not make any use of the structure of the underlying lattice, we present our models in this more general setting. Most of our analysis applies on finite subsets \( V \subset \mathcal{V} \). More specifically, to each \( x \in \mathcal{V} \), we will associate a finite-dimensional Hilbert space \( \mathcal{H}_x \); the dimension of \( \mathcal{H}_x \) will be denoted by \( n_x \). Set \( M_{n_x} \) to be the complex \( n_x \times n_x \) matrices defined over \( \mathbb{C} \). We denote by \( \mathcal{H}_V = \bigotimes_{x \in V} \mathcal{H}_x \) the Hilbert space of states over \( V \), and similarly \( \mathcal{A}_V = \bigotimes_{x \in V} M_{n_x} \) is the algebra of local observables. For any finite subsets \( X \subset Y \subset \mathcal{V} \), the local algebra \( \mathcal{A}_X \) can be embedded in \( \mathcal{A}_Y \) by using that any \( A \in \mathcal{A}_X \) corresponds to \( A \otimes 1 \in \mathcal{A}_Y \). Thus, the algebra of quasi-local observables, \( \mathcal{A}_V \), can be defined as the norm-closure of the union of all local algebras; the union taken over all finite subsets of \( \mathcal{V} \).

Moreover, we say that an observable \( A \in \mathcal{A}_V \) is supported in \( X \subset \mathcal{V} \), denoted by \( \text{supp}(A) = X \), if \( X \) is the minimal set for which \( A \) can be written as \( A = \tilde{A} \otimes 1 \) with \( \tilde{A} \in \mathcal{A}_X \).

A quantum spin model is then defined by its interaction and the corresponding local Hamiltonians. An interaction is function \( \Phi \) from the set of finite subsets of \( \mathcal{V} \) into \( \mathcal{A}_V \) with the property that given any finite \( V \subset \mathcal{V} \), \( \Phi(V) = \Phi(V)^* \in \mathcal{A}_V \). Given such an interaction, one can associate a family of local Hamiltonians, parametrized by the finite subsets of \( \mathcal{V} \), defined by setting

\[
H_V = \sum_{X \subset V} \Phi(X)
\]

for finite \( V \subset \mathcal{V} \). Since each local Hamiltonian \( H_V \) is self-adjoint, it generates a one parameter group of automorphisms, which we will denote by \( \tau^V_t \), that is often called the finite volume dynamics. This dynamics is defined by setting

\[
\tau^V_t(A) = e^{itH_V} A e^{-itH_V}, \quad A \in \mathcal{A}_V.
\]

When the subset \( V \) on which the dynamics is defined has been fixed, we will often write \( \tau_t \) to ease the notation.

Theorem 2.2 below holds for certain quantum spin models. Our first assumption pertains to the underlying set \( \mathcal{V} \). We must assume that there exists a number \( \mu_0 > 0 \) for which

\[
\kappa_{\mu_0} = \sup_{x \in \mathcal{V}} \sum_{y \in \mathcal{V}} e^{-\mu_0 d(x,y)} < \infty.
\]

Next, we make the assumptions on the interactions we consider precise.

A1: We consider interactions that are of finite range. Specifically, we assume that there exists a number \( R > 0 \), called the range of the interactions, for which \( \Phi(X) = 0 \) if the diameter of \( X \) exceeds \( R \). Here, for any finite \( X \subset \mathcal{V} \), the diameter of \( X \) is

\[
diam(X) = \max \{ d(x,y) : x, y \in X \}
\]

where \( d \) is the metric on \( \mathcal{V} \).

A2: We will only consider uniformly bounded interactions. Hence, there is a number \( J > 0 \) such that for any finite \( X \subset \mathcal{V} \), \( \| \Phi(X) \| \leq J \).
A3: Let $\chi_\Phi$ be the characteristic function defined over the set of finite subsets of $\mathcal{V}$, i.e. for any finite $X \subset \mathcal{V}$, $\chi_\Phi(X) = 1$ if $\Phi(X) \neq 0$, and $\chi_\Phi(X) = 0$ otherwise. We assume the following quantity is finite

$$N_\Phi = \sup_{x \in \mathcal{V}} \sum_{X \subset \mathcal{V}, x \in X} |X| \chi_\Phi(X) < \infty$$

where the sum above is taken over finite subsets $X \subset \mathcal{V}$ and $|X|$ is the cardinality of $X$. Another relevant quantity, which often appears in our estimates, is

$$C_\Phi = \sup_{x \in \mathcal{V}} \sum_{X \subset \mathcal{V}, x \in X} \chi_\Phi(X) \leq N_\Phi.$$  

A4: We will assume that a given local Hamiltonian $H_\mathcal{V}$ has a unique, normalized ground state which we denote by $\psi_0$. $P_0$ will be the projection onto this ground state in $\mathcal{H}_\mathcal{V}$, and we will label by $\gamma$ the length of the gap to the first excited state.

Much progress has been made recently in proving locality estimates for general quantum spin systems, see e.g. [6, 11]. The following estimate was proven in [14].

**Theorem 2.1** (Lieb-Robinson Bound). Let $\mathcal{V}$ be a countable, locally finite set equipped with a metric $d$ for which (2.3) holds. Let $\Phi$ be an interaction on $\mathcal{V}$ that satisfies assumptions A1, A2, and A3. For every $\mu \geq \mu_0$, there exists numbers $c$ and $v$ such that given any finite $V \subset \mathcal{V}$ and any observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $X, Y \subset V$ and $X \cap Y = \emptyset$, the bound

$$\| [\tau_t^V(A), B] \| \leq c \|A\| \|B\| \min\{|\partial_B X|, |\partial_B Y|\} e^{-\mu(d(X,Y) - v|t|)}$$

holds for all $t \in \mathbb{R}$.

Here for any finite $X \subset \mathcal{V}$, the set

$$\partial B X = \{x \in X : \text{there exists } Y \subset \mathcal{V} \text{ with } x \in Y, Y \cap \mathcal{V} \setminus X \neq \emptyset, \text{ and } \Phi(Y) \neq 0\},$$

the number $d(X,Y) = \min\{d(x,y) : x \in X, y \in Y\}$, and we stress that both $c$ and $v$ are independent of $V \subset \mathcal{V}$.

Before we state Theorem 2.2 some further notation is necessary. Let $\Phi$ be an interaction which satisfies assumptions A1 - A4 above. Again, most of our analysis applies on finite sets $V \subset \mathcal{V}$, and often the particular finite set under consideration is the one whose existence is guaranteed by A4. In general, for finite sets $X \subset V \subset \mathcal{V}$, we will denote by $\partial X$ the R-boundary of $X$ in $\mathcal{V}$, i.e.,

$$\partial X = \{x \in X : \text{there exists } y \in V \setminus X, \text{ with } d(x,y) \leq R\}.$$ 

We note that with $X$ fixed, the set $\partial X$ is independent of $V$, for $V$ sufficiently large, and so we suppress this in our notation. To state our main result, we fix an additional subset $\mathcal{X} \subset V$. The following sets, defined relative to this fixed set $\mathcal{X}$ and dependent on a length-scale $\ell > R$, will also play an important role. Set

$$\mathcal{X}_{\text{int}} = \mathcal{X}^{\mathcal{V}}_{\text{int}}(\ell) = \{x \in \mathcal{X} : \text{for all } y \in \partial \mathcal{X}, d(x,y) \geq \ell\},$$

$$\mathcal{X}_{\text{bd}} = \mathcal{X}^{\mathcal{V}}_{\text{bd}}(\ell) = \{x \in V : \text{there exists } y \in \partial \mathcal{X}, d(x,y) < \ell\},$$

and

$$\mathcal{X}_{\text{ext}} = \mathcal{X}^{\mathcal{V}}_{\text{ext}}(\ell) = \{x \in V \setminus \mathcal{X} : \text{for all } y \in \partial \mathcal{X}, d(x,y) \geq \ell\}.$$ 

The sets $\mathcal{X}_{\text{int}}$, $\mathcal{X}_{\text{bd}}$, and $\mathcal{X}_{\text{ext}}$ correspond respectively to the $\ell$-interior, the $\ell$-border, and the $\ell$-exterior of $\mathcal{X}$ in $\mathcal{V}$.

We caution that the notation $\partial \mathcal{X}_{\text{ext}}$ here...
refers to the boundary of this set as a subset of $V$, as indicated above; not as a subset of $V$. It will also be important for us that there exists a number $C > 0$, independent of both $\ell$ and $V$, for which
\[
\max \{|\partial X_{\text{int}}|, |\partial X_{\text{bd}}|, |\partial X_{\text{ext}}|\} \leq C \ell |\partial X|.
\]
The above inequality constitutes the main structural assumption on the set $V$ and the reference set $X \subset V$. We may now state our main result.

**Theorem 2.2.** Let $V$ be a countable, locally finite set equipped with a metric $d$ for which (2.3) holds, and moreover, let $\Phi$ be an interaction on $V$ which satisfies assumptions $A1 - A4$, take $V \subset V$ to be the finite set from $A4$, and suppose $X \subset V$ satisfies (2.13). For any $\ell > R$, there exists two projections $P_{X_{\text{bd}}} \in A_{X_{\text{bd}}}$ and $P_{X_{\text{c}}} \in A_{V \setminus X}$ and an observable $P_{X_{\text{bd}}} \in A_{X_{\text{bd}}(3\ell)}$ with $\|P_{X_{\text{bd}}}\| \leq 1$ such that
\[
\|P_{X_{\text{bd}}} P_{x} P_{X_{\text{c}}} - P_{0}\| \leq KC_{\Phi} N_{\Phi} |\partial X|^2 \ell^{7/2} e^{-\ell/2}\xi.
\]
Here $\xi$ is defined by
\[
\frac{2}{\xi} = \frac{\mu \gamma^2}{\mu^2 \epsilon^2 + \gamma^2},
\]
for $\mu \geq 2 \mu_0$.

In many situations, it is useful to have a non-negative approximation of the ground state projector. For this reason, we also state the following corollary.

**Corollary 2.3.** Under the assumptions of Theorem 2.2, the estimate
\[
\|P_{x} P_{x} P_{X_{\text{bd}}} P_{X_{\text{c}}} P_{x} P_{X_{\text{c}}} - P_{0}\| \leq KC_{\Phi} N_{\Phi} |\partial X|^2 \ell^{7/2} e^{-\ell/2}\xi
\]
also holds.

Since all the observables involved have norm bounded by 1, Corollary 2.3 is an immediate consequence of Theorem 2.2.

With the methods discussed in [14], Theorem 2.2 can easily be generalized to other types of interactions. For example, the finite range condition $A1$ can be replaced with an exponentially decaying analogue. Moreover, the uniform bound on the local interactions terms, i.e. $A2$, can also be lifted. One could also consider systems where the underlying sets satisfied a bound of the form (2.13) with a larger power of $\ell$. This would increase the power of $\ell$ in the statement of Theorem 2.2, but not effect the exponential decay. We give the argument in the context described above for convenience of presentation. It would be very interesting to see that the methods developed here can be used to prove an area law, similar to Hastings’ result [5], in dimensions greater than 1, but this is beyond the scope of the present work.

3. **Basic Set-Up**

Before beginning the proof of this theorem, we will introduce the basic objects to be analyzed. Let $V \subset V$ be the finite set described in $A4$, take $\ell > R$, and fix $X \subset V$ satisfying (2.13). Note that $H_V$, as defined in (2.1), can be written as a sum of three local Hamiltonians
\[
H_V = H_{X_{\text{int}}} + H_{X_{\text{bd}}} + H_{X_{\text{ext}}},
\]
where $H_{X_{\text{bd}}}$ is as in (2.1) with $V = X_{\text{bd}}$, and for $Z \in \{X_{\text{int}}, X_{\text{ext}}\}$,
\[
H_{Z} = \sum_{X \in \{V, V \setminus Z \neq \emptyset\}} \Phi(X),
\]
corresponds to local Hamiltonians that include boundary terms. Without loss of generality, we may subtract a constant, namely $\langle \psi_0, H_V \psi_0 \rangle$, and thereby assume that the ground state energy of each of these Hamiltonians is zero. In this case, each term in (3.1) has zero ground state expectation. It
is not clear, however, that each of these terms, when applied to the ground state, have a quantifiably small norm. To achieve small norms, we introduce non-local versions of these observables.

Generally, given any self-adjoint Hamiltonian $H$ on $\mathcal{H}_V$, a local observable $A \in \mathcal{A}_V$, and $\alpha > 0$, we may define a new observable

$$
(A)_\alpha = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \tau_t(A) e^{-\alpha t^2} dt,
$$

where here $\tau_t(A) = e^{itH} A e^{-itH}$. It is easy to see that for every $\alpha > 0$, $\| (A)_\alpha \| \leq \| A \|$. In the present context, we will define the $\langle \cdot \rangle_\alpha$ operation with respect to the Hamiltonian $H_V$ and its corresponding dynamics $\tau^V_t$. Observe that these non-local observables still sum to the total Hamiltonian, i.e.,

$$
H_V = (H^b_{\text{int}})_\alpha + (H_{\text{bd}})_\alpha + (H^b_{\text{ext}})_\alpha,
$$

They also have zero ground state expectation, i.e.,

$$
\langle \psi_0, (H^b_{\text{int}})_\alpha \psi_0 \rangle = \langle \psi_0, (H_{\text{bd}})_\alpha \psi_0 \rangle = \langle \psi_0, (H^b_{\text{ext}})_\alpha \psi_0 \rangle = 0.
$$

We will prove, in Proposition 5.2, that each of these smeared-out Hamiltonians, when applied to the ground state, has a norm bounded by an exponentially decaying quantity. The cost of this small norm estimate is a loss of locality, which is expressed in terms of the support of these Hamiltonians.

Equipped with the Lieb-Robinson bound in Theorem 2.1, we begin the next step in the proof of Theorem 2.2. Here we approximate each of $(H^b_{\text{int}})_\alpha$, $(H_{\text{bd}})_\alpha$, and $(H^b_{\text{ext}})_\alpha$ with local observables $M_X(\alpha)$, $M_{X_{\text{bd}}(2\ell)}(\alpha)$ and $M_X(\alpha)$ respectively. These observables will be constructed in such a way that they not only have a quantifiable support but also a small vector norm, when applied to the ground state.

To make these approximating Hamiltonians explicit, we introduce local evolutions. Define three different dynamics, each acting on $\mathcal{A}_V$, by setting

$$
\tau^X_t = e^{itH_{X}} A e^{-itH_{X}},
\tau^{X_{\text{bd}}(2\ell)}_t = e^{itH_{X_{\text{bd}}(2\ell)}} A e^{-itH_{X_{\text{bd}}(2\ell)}},
\tau^X_{V\setminus X} = e^{itH_{V\setminus X}} A e^{-itH_{V\setminus X}},
$$

for any local observable $A \in \mathcal{A}_V$ and any $t \in \mathbb{R}$. The set $X_{\text{bd}}(2\ell)$ is as defined in (2.11) with the length scale doubled, and the local Hamiltonians used in the evolutions above are as in (2.1). The approximating Hamiltonians are given by

$$
M_X(\alpha) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \tau^X_t (H^b_{\text{int}}) e^{-\alpha t^2} dt,
M_{X_{\text{bd}}(2\ell)}(\alpha) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \tau^{X_{\text{bd}}(2\ell)}_t (H_{\text{bd}}) e^{-\alpha t^2} dt,
M_X(\alpha) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \tau^X_{V\setminus X} (H^b_{\text{ext}}) e^{-\alpha t^2} dt.
$$

The observables defined above, i.e. in (3.7), have been chosen such that $\text{supp}(M_X(\alpha)) = X$, $\text{supp}(M_{X_{\text{bd}}(2\ell)}(\alpha)) = X_{\text{bd}}(2\ell)$, and $\text{supp}(M_X(\alpha)) = V \setminus X$. To ease notation, we will often write $M_{X_{\text{bd}}(\alpha)} = M_{X_{\text{bd}}(2\ell)(\alpha)}$ and suppress the exact size of the support.

The following technical estimate summarizes the results mentioned above.

**Lemma 3.1.** Let $\mathcal{V}$ be a countable, locally finite set equipped with a metric $d$ for which (2.3) holds, and let $\Phi$ be an interaction on $\mathcal{V}$ which satisfies assumptions A1 - A4. For the finite set $V \subset \mathcal{V}$ from A4 and any set $X \subset V$ satisfying (2.13) for all $\ell > R$, the estimate

$$
\| H_V - (M_X(\alpha) + M_{X_{\text{bd}}(\alpha)} + M_X(\alpha)) \| \leq K |\partial X|^{3/2} e^{-\frac{\xi}{4}},
$$

is obtained for some $K$ depending only on $\delta$. The constants $h, H_{\text{ext}}$, and $K$ depend only on $\mathcal{V}$, $\Phi$, $\delta$, $\ell_0$, and $R$.
holds along the parametrization $2\alpha \epsilon t = \mu v^2$ where $\mu \geq 2\mu_0$. The numbers $\xi$ and $\epsilon$ are defined in terms of the gap $\gamma$ and the quantities $\mu$ and $v$ from the Lieb-Robinson estimates as

$$0 < \frac{2}{\xi} = (1 - \epsilon)\mu = \frac{\gamma^2}{\mu^2v^2 + \gamma^2\mu}.$$  

Along the same parametrization, the bound

$$\max \{ \|M_X(\alpha)\psi_0\|, \|M_{X_{bd}}(\alpha)\psi_0\|, \|M_{X^c}(\alpha)\psi_0\| \} \leq K|\partial X|^3/2e^{-\gamma^2/4\alpha},$$

also holds.

These estimates play an important role in the proof of Theorem 2.2. We prove this lemma in Section 5. For now, we use it to prove Theorem 2.2.

4. Proof of Theorem 2.2

The first step in verifying the bound claimed in (2.14) is to find an explicit approximation to the ground state projector $P_0$. With this in mind, define

$$P_\alpha = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{iH_V t} e^{-\alpha t^2} dt,$$

for any $\alpha > 0$. Clearly, $P_\alpha \in \mathcal{A}_V$ and for any vectors $f, g \in \mathcal{H}_V$, the spectral theorem implies

$$\langle f, (P_\alpha - P_0)g \rangle = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\alpha t^2} \int_{0}^{\infty} e^{i\lambda t} d(f, E_\lambda g) dt - \langle f, P_0 g \rangle$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} e^{-\alpha t^2} dt d(f, E_\lambda g)$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\chi^2}{4\alpha}} d(f, E_\lambda g),$$

where we have denoted by $E_\lambda$ the spectral projection corresponding to $H_V$. This readily yields that

$$\|P_\alpha - P_0\| \leq e^{-\frac{\chi^2}{4\alpha}},$$

where $\gamma$ is the gap of the Hamiltonian $H_V$.

Using the operators introduced in (3.7), we define an analogous ground state approximate by setting

$$\tilde{P}_\alpha = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{i(M_X + M_{X_{bd}} + M_{X^c}) t} e^{-\alpha t^2} dt.$$  

Here we have dropped the dependence of $M_X$, $M_{X_{bd}}$, and $M_{X^c}$ on $\alpha$. Clearly,

$$\|\tilde{P}_\alpha - P_0\| \leq \|\tilde{P}_\alpha - P_\alpha\| + \|P_\alpha - P_0\|.$$

The final term above we have bounded in (4.3). To bound the first term in (4.5), we introduce the function

$$F_\lambda(\lambda) = e^{i\lambda(M_X + M_{X_{bd}} + M_{X^c}) t} e^{i(1-\lambda)H_V t}.$$  

One easily calculates that

$$F_\lambda'(\lambda) = -i t e^{i\lambda(M_X + M_{X_{bd}} + M_{X^c}) t} \{ H_V - (M_X + M_{X_{bd}} + M_{X^c}) \} e^{i(1-\lambda)H_V t}.$$  

Using Lemma 3.1, we conclude that, if \(2\alpha\epsilon\ell = \mu\nu^2\), then

\[
\left\| \tilde{P}_\alpha - P_\alpha \right\| \leq \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \| F_t(1) - F_t(0) \| \ e^{-\alpha t^2} \ dt
\]

\[
\leq \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \| H_V - (M_X + M_{Xbd} + M_X^c) \| \ |t| e^{-\alpha t^2} \ dt
\]

(4.8)

\[
\leq K|\partial X|\ell^2 e^{-\frac{\ell}{2\epsilon}}.
\]

In the bound above, and for the rest of this section, the number \(K\) which appears in our estimates will depend on the parameters of the quantum spin models, but not on \(\ell\). It may change from line to line, but we do not indicate this in our notation.

Inserting both (4.3) and (4.8) into (4.5) yields

\[
\left\| \tilde{P}_\alpha - P_0 \right\| \leq K|\partial X|\ell^2 e^{-\frac{\ell}{2\epsilon}},
\]

again along the parametrization \(2\alpha\epsilon\ell = \mu\nu^2\).

Set

\[
\delta = \ell^{3/2} e^{-\frac{\ell}{2\epsilon}}.
\]

The projections \(P_X\) and \(P_{X^c}\), which appear in the statement of Theorem 2.2, are defined to be the spectral projection corresponding to the matrix \(M_X\), respectively \(M_{X^c}\), onto those eigenvalues less than \(\delta\). By definition, \(P_X \in A_X\) and \(P_{X^c} \in A_{X^c}\) as claimed. Moreover, using Lemma 3.1, it is easy to see that

\[
\max_{Z \in \{X, X^c\}} \| (1 - P_Z) \psi_0 \| \leq \max_{Z \in \{X, X^c\}} \frac{1}{\delta} \| M_Z \psi_0 \| \leq K|\partial X|e^{-\frac{\ell}{2\epsilon}},
\]

(4.11)

by construction; hence this choice of \(\delta\).

With this bound, we can insert these projections into our previous estimates. In fact,

\[
\left\| \tilde{P}_\alpha P_X P_{X^c} - P_0 \right\| \leq \left\| (\tilde{P}_\alpha - P_0) P_X P_{X^c} \right\| + \| P_0 (1 - P_X P_{X^c}) \|.
\]

The first term above is estimated using (4.9) above. Since

\[
(1 - P_X P_{X^c}) = \frac{1}{2} \left\{ (1 - P_X)(1 + P_{X^c}) + (1 - P_{X^c})(1 + P_X) \right\},
\]

it is clear that

\[
\| P_0 (1 - P_X P_{X^c}) \| \leq \| P_0 (1 - P_X) \| + \| P_0 (1 - P_{X^c}) \| \leq 2K|\partial X|e^{-\frac{\ell}{2\epsilon}},
\]

(4.14)

using (4.11). Therefore, we now have that

\[
\left\| \tilde{P}_\alpha P_X P_{X^c} - P_0 \right\| \leq K|\partial X|\ell^2 e^{-\frac{\ell}{2\epsilon}},
\]

(4.15)

again along the parametrization \(2\alpha\epsilon\ell = \mu\nu^2\).

In order to define the final observable \(P_{X_{bd}}\), we write

\[
\tilde{P}_\alpha P_X P_{X^c} = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{i(M_X + M_{Xbd} + M_{X^c})t} e^{-i(M_X + M_{Xbd})t} e^{i(M_X + M_{X^c})t} P_X P_{X^c} e^{-\alpha t^2} \ dt.
\]

Since the supports of \(M_X\) and \(M_{X^c}\) are disjoint, it is clear that

\[
e^{i(M_X + M_{X^c})t} P_X P_{X^c} - P_X P_{X^c} = \]

(4.16)

\[
= \frac{1}{2} (e^{iM_Xt} P_X - P_X)(e^{iM_{X^c}t} P_{X^c} + P_{X^c}) + \frac{1}{2} (e^{iM_{X^c}t} P_{X^c} - P_{X^c})(e^{iM_Xt} P_X + P_X).
\]

(4.17)
Moreover, for $Z \in \{X, X^c\}$, we have that
\[
|\langle f, (e^{iM_Z t} - 1)P_Z g \rangle| = \left| \int_0^\delta (e^{i\lambda t} - 1) d\langle f, E_X^P Z g \rangle \right| \leq \delta |t| \|f\| \|g\|,
\]
and therefore,
\[
\|e^{i(M_X+M_{X^c}) t} P_X P_{X^c} - P_X P_{X^c} \| \leq 2\delta |t|.
\]
If we now define the operator,
\[
\hat{\mathcal{P}}_\alpha = \sqrt{\alpha} \int_{-\infty}^{\infty} e^{i(M_X+M_{X^c}) t} e^{-i(M_X+M_{X^c}) t} e^{-\alpha t^2} dt,
\]
then we have just demonstrated that
\[
\|\hat{\mathcal{P}}_\alpha P_X P_{X^c} - P_0\| \leq \|\hat{\mathcal{P}}_\alpha P_X P_{X^c} - \hat{\mathcal{P}}_\alpha P_X P_{X^c}\| + \|\hat{\mathcal{P}}_\alpha P_X P_{X^c} - P_0\|
\]
\[
\leq 2\delta \sqrt{\alpha} \int_{-\infty}^{\infty} |t| e^{-\alpha t^2} dt + K|\partial X|^2 e^{-\frac{t}{2}}
\]
\[
\leq K|\partial X|^2 e^{-\frac{t}{2}}.
\]

The proof of Theorem 2.2 is complete if we show that the operator $\hat{\mathcal{P}}_\alpha$, defined in (4.20) above, is well approximated by a local observable. In fact, below we introduce an observable $P_{X_{bd}} \in A_{X_{bd}(3\ell)}$ with $\|P_{X_{bd}}\| \leq 1$ for which
\[
\|P_{X_{bd}} - \hat{\mathcal{P}}_\alpha\| \leq K|\partial X|^2 e^{-\frac{t}{2}}.
\]
From this estimate, Theorem 2.2 easily follows as
\[
\|P_{X_{bd}} P_X P_{X^c} - P_0\| \leq \left\| \left(P_{X_{bd}} - \hat{\mathcal{P}}_\alpha \right) P_X P_{X^c} \right\| + \|\hat{\mathcal{P}}_\alpha P_X P_{X^c} - P_0\|
\]
\[
\leq K|\partial X|^2 e^{-\frac{t}{2}}.
\]
We need only prove that such an observable $P_{X_{bd}}$ exists.

The existence of this observable is simple. We just take the normalized partial trace of $\hat{\mathcal{P}}_\alpha$ over the complimentary Hilbert space, i.e., the one associated with $X_{bd}(3\ell) \subset V$. To prove the estimate in (4.22), it is convenient to calculate this partial trace as an integral over the group of unitaries, technical details on this may be found in [2] and also [14]. We recall that given an observable $A \in \mathcal{A}_V$ and a set $Y \subset V$, one may define an observable
\[
\langle A \rangle_Y = \int_{\mathcal{U}(Y^c)} U^* A U \mu(dU),
\]
where $\mathcal{U}(Y^c)$ denotes the group of unitary operators over the Hilbert space $\mathcal{H}_{Y^c}$ and $\mu$ is the associated, normalized Haar measure. It is easy to see that for any $A \in \mathcal{A}_V$, the observable $\langle A \rangle_Y$ has been localized to $Y$ in the sense of supports, i.e., $\langle A \rangle_Y \in \mathcal{A}_Y$.

Let us define
\[
P_{X_{bd}} = \langle \hat{\mathcal{P}}_\alpha \rangle_{X_{bd}(3\ell)}.
\]
Clearly $P_{X_{bd}} \in A_{X_{bd}(3\ell)}$ and $\|P_{X_{bd}}\| \leq 1$ as desired.

The difference between $\hat{\mathcal{P}}_\alpha$ and $P_{X_{bd}}$ can be expressed in terms of a commutator, in fact,
\[
\hat{\mathcal{P}}_\alpha - P_{X_{bd}} = \int_{\mathcal{U}(X_{bd}(3\ell)^c)} U^* \left[ \hat{\mathcal{P}}_\alpha, U \right] \mu(dU).
Thus, to show that the difference between $\hat{\mathcal{P}}_\alpha$ and $P_{\mathcal{X}_{\text{bd}}}$ is small (in norm), we need only estimate the commutator of $\hat{\mathcal{P}}_\alpha$ with an arbitrary unitary supported in $\mathcal{X}_{\text{bd}}(3\ell)^C$.

This follows from a Lieb-Robinson estimate. Note that

$$\mathcal{A}$$

$$
(4.27) \quad \left[\hat{\mathcal{P}}_\alpha, U\right] = \sqrt{\frac{i\alpha}{\pi}} \int_{-\infty}^{\infty} \left[ e^{i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t} e^{-i(M_M+M_{X_c})t}, U \right] e^{-\alpha t^2} dt.
$$

To estimate the integrand, we define the function

$$\mathcal{B}$$

$$
(4.28) \quad f(t) = \left[ e^{i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t} e^{-i(M_M+M_{X_c})t}, U \right].
$$

A short calculation demonstrates that

$$\mathcal{C}$$

$$
(4.29) \quad f'(t) = i \left[ e^{i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t} M_{\mathcal{X}_{\text{bd}}} e^{-i(M_M+M_{X_c})t}, U \right].
$$

The form of the derivative appearing in (4.29) suggests that we define the evolution

$$\mathcal{D}$$

$$
(4.30) \quad \alpha_t(A) = e^{i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t} M_{\mathcal{X}_{\text{bd}}} e^{-i(M_M+M_{X_c})t} A e^{-i(M_M+M_{X_c})t} \quad \text{for any} \quad A \in \mathcal{A}_V.
$$

With this in mind, we rewrite

$$\mathcal{E}$$

$$
(4.31) \quad f'(t) = i \left[ e^{i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t} M_{\mathcal{X}_{\text{bd}}} e^{-i(M_M+M_{X_c})t}, U \right] = i \left[ \alpha_t(M_{\mathcal{X}_{\text{bd}}}) e^{i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t} e^{-i(M_M+M_{X_c})t}, U \right] = i \alpha_t(M_{\mathcal{X}_{\text{bd}}}) f(t) + i \left[ \alpha_t(M_{\mathcal{X}_{\text{bd}}}), U \right] e^{i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t} e^{-i(M_M+M_{X_c})t}.
$$

Written as above, the function $f$ can be bounded using norm-preservation, see [11] and [13] for details. For example, let $V(t)$ be the unitary evolution that satisfies the time-dependent differential equation

$$\mathcal{F}$$

$$
(4.32) \quad i \frac{d}{dt} V(t) = V(t) \alpha_t(M_{\mathcal{X}_{\text{bd}}}) \quad \text{with} \quad V(0) = 1.
$$

Explicitly, one has that

$$\mathcal{G}$$

$$
(4.33) \quad V(t) = e^{i(M_M+M_{X_c})t} e^{-i(M_M+M_{\mathcal{X}_{\text{bd}}}+M_{X_c})t}.
$$

Considering now the product

$$\mathcal{H}$$

$$
(4.34) \quad g(s) = V(s) f(s)
$$

it is easy to see that

$$\mathcal{I}$$

$$
(4.35) \quad g'(s) = V'(s) f(s) + V(s) f'(s) = i V(s) \left[ \alpha_s(M_{\mathcal{X}_{\text{bd}}}), U \right] V(s)^*.
$$

Thus

$$\mathcal{J}$$

$$
(4.36) \quad V(t) f(t) = g(t) - g(0) = \int_0^t g'(s) ds,
$$

and therefore,

$$\mathcal{K}$$

$$
(4.37) \quad f(t) = i V(t)^* \int_0^t V(s) \left[ \alpha_s(M_{\mathcal{X}_{\text{bd}}}), U \right] V(s)^* ds.
$$

The bound

$$\mathcal{L}$$

$$
(4.38) \quad \| f(t) \| \leq \int_0^{|t|} \| \left[ \alpha_s(M_{\mathcal{X}_{\text{bd}}}), U \right] \| ds.
$$

readily follows.
Unfortunately, the Lieb-Robinson velocity associated to the dynamics $\alpha_t(\cdot)$ grows with $\ell$. For this reason, we estimate (4.38) by comparing back to the original dynamics. Consider the interpolating dynamics

\begin{equation}
\tag{4.39}
\hat{h}_s(r) = \alpha_r (\tau_{s-r}(A)) ,
\end{equation}

for any local observable $A$ and $0 \leq r \leq s$. Here $\tau_r(A) = e^{iH_V r}A e^{-iH_V r}$. With $s$ fixed, it is easy to calculate

\begin{equation}
\tag{4.40}
h'_s(r) = i\alpha_{s-r} ([(M_X + M_{Xbd} + M_{Xc}) - H_V, \tau_r(A)]) .
\end{equation}

We conclude then that

\begin{equation}
\tag{4.41}
\|\alpha_s(M_{Xbd}) - \tau_s(M_{Xbd})\| = \left\| \int_0^s h'_s(r) \, dr \right\|
\leq 2 \|M_{Xbd}\| \|H_V - (M_X + M_{Xbd} + M_{Xc})\| \; s
\leq 2 \left(JC\Phi|X_{bd}\right) \left(K|\partial X|\ell^{3/2} e^{-\frac{s}{7}}\right) s
\leq K|\partial X|^2 \ell^{3/2} e^{-\frac{s}{7}} ,
\end{equation}

where, for the second inequality above, we used Lemma 3.1 and the bound

\begin{equation}
\tag{4.42}
\|M_{Xbd}\| \leq \sum_{x \in X_{bd}} \sum_{x \in X} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \left\| x_{tXbd} (\Phi(X)) \right\| e^{-\alpha t^2} \, dt
\leq JC\Phi|X_{bd}| .
\end{equation}

From (4.38), it is clear that

\begin{equation}
\tag{4.43}
\|f(t)\| \leq \int_0^{\ell} \|\tau_s(M_{Xbd}), U\| \, ds + \int_0^{\ell} \|\alpha_s(M_{Xbd}) - \tau_s(M_{Xbd}), U\| \, ds .
\end{equation}

The first term above, we bound, using Theorem 2.1 as follows

\begin{equation}
\tag{4.44}
\int_0^{\ell} \|\tau_s(M_{Xbd}), U\| \, ds \leq C|\partial X_{bd}(2\ell)| \|M_{Xbd}\| \int_0^{\ell} e^{-\mu(d(X_{bd}(2\ell), U) - \nu|s|)} \, ds
\leq K|\partial X|^2 \ell^2 e^{-\mu} \frac{1}{\mu v} \left(e^{\mu v|\ell|} - 1\right) .
\end{equation}

For the second, we apply (4.41). Thus, we have found that

\begin{equation}
\tag{4.45}
\|\hat{P}_\alpha, U\| \leq \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \|f(t)\| e^{-\alpha t^2} \, dt
\leq K|\partial X|^2 \ell^2 e^{-\mu} \frac{1}{\mu v} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{\mu v|t|} e^{-\alpha t^2} \, dt
+ 2K|\partial X|^2 \ell^{5/2} e^{-\frac{s}{7}} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} t^2 e^{-\alpha t^2} \, dt
\leq K|\partial X|^2 \ell^2 e^{-\frac{s}{7}} .
\end{equation}

Combining this with (4.26), we obtain that

\begin{equation}
\tag{4.46}
\left\| P_{Xbd} - \hat{P}_\alpha \right\| \leq K|\partial X|^2 \ell^2 e^{-\frac{s}{7}} ,
\end{equation}

and the proof of Theorem 2.2 is now complete.
5. A Technical Estimate

In this section, we will prove Lemma 3.1. For convenience, we restate the result.

**Lemma 3.1.** Let $\mathcal{V}$ be a countable, locally finite set equipped with a metric $d$ for which (2.3) holds, and let $\Phi$ be an interaction on $\mathcal{V}$ which satisfies assumptions A1 - A4. For the finite set $V \subset \mathcal{V}$ from A4 and any set $X \subset V$ satisfying (2.13) for all $\ell > R$, the estimate

$$\|H_V - (M_X(\alpha) + M_{X_{\text{bd}}}(\alpha) + M_X^c(\alpha))\| \leq K|\partial X|\ell^3/2e^{-\xi},$$

holds along the parametrization $2\alpha \ell = \mu \ell^2$ when $\mu \geq 2\mu_0$. The numbers $\xi$ and $\epsilon$ are defined in terms of the gap $\gamma$ and the quantities $\mu$ and $\nu$ from the Lieb-Robinson estimates as

$$0 < \frac{2}{\xi} = (1 - \epsilon)\mu = \frac{\gamma^2}{\mu^2 \nu^2 + \gamma^2 \mu}.$$

Our bounds on the prefactor $K$ depend on various parameters, e.g. $J^2, C_\Phi, N_\Phi, \mu, \nu, \gamma, R, \kappa \mu/2$, but it is independent of $\ell, V$, and $X$. Along the same parametrization, the bound

$$\max \{\|M_X(\alpha)\psi_0\|, \|M_{X_{\text{bd}}}(\alpha)\psi_0\|, \|M_X^c(\alpha)\psi_0\|\} \leq K|\partial X|\ell^3/2e^{-\xi}.$$

also holds.

To prove Lemma 3.1, we begin with a few simple propositions. Let $V \subset \mathcal{V}$ be the finite set described in A4, take $\ell > R$, and fix $X \subset V$ satisfying (2.13). As we demonstrates in Section 3, see also Section 2, the local Hamiltonian corresponding to $\Phi$ in $V$ can be written as

$$H_V = H^b_{X_{\text{int}}} + H_{X_{\text{bd}}} + H^b_{X_{\text{ext}}}$$

where the sets $X_{\text{int}}, X_{\text{bd}},$ and $X_{\text{ext}}$, and the corresponding local Hamiltonians, each depend on a length scale $\ell > R$. We begin with a basic commutator estimate.

**Proposition 5.1.** Let $\Phi$ be an interaction on $\mathcal{V}$ which satisfies assumptions A1 - A4. Let $V \subset \mathcal{V}$ be the finite set described in A4, take $\ell > R$, and fix $X \subset V$ satisfying (2.13). One has that

$$\max \{\|[H_V, H^b_{X_{\text{int}}}]\|, \|[H_V, H_{X_{\text{int}}}]\|, \|[H_V, H^b_{X_{\text{ext}}}]\|\} \leq 4CJ^2C_\Phi N_\Phi |\partial X|\ell,$$

where the quantities $C_\Phi$ and $N_\Phi$ are as introduced in (2.6) and (2.5).

**Proof.** Define

$$C_{\text{int}} = [H_V, H^b_{X_{\text{int}}}], \quad C_{\text{bd}} = [H_V, H_{X_{\text{bd}}}], \quad \text{and} \quad C_{\text{ext}} = [H_V, H^b_{X_{\text{ext}}}].$$

From the definitions, (2.10), (2.11), and (2.12), and the fact that $\ell > R$, it is clear that $C_{\text{int}} = [H_{X_{\text{bd}}}, H^b_{X_{\text{int}}}], C_{\text{ext}} = [H_{X_{\text{bd}}}, H^b_{X_{\text{ext}}}], \text{and} -C_{\text{bd}} = C_{\text{int}} + C_{\text{ext}}$. Clearly,

$$C_{\text{int}} = [H_{X_{\text{bd}}}, H^b_{X_{\text{int}}}] = \sum_{X \subset V \cap \partial X_{\text{int}} \neq \emptyset} [H_{X_{\text{bd}}}, \Phi(X)],$$

and therefore,

$$\|C_{\text{int}}\| \leq \sum_{x \in \partial X_{\text{int}} \subset V} \sum_{x \in X} \|[H_{X_{\text{bd}}}, \Phi(X)]\|$$

$$\leq \sum_{x \in \partial X_{\text{int}} \subset V} \sum_{x \in X} \sum_{z \in V} \sum_{z \in Y} \|[\Phi(Y), \Phi(X)]\|$$

$$\leq 2J^2 \sum_{x \in \partial X_{\text{int}}} \sum_{x \in X} \sum_{z \in V} \sum_{z \in Y} \chi_\Phi(Y)\chi_\Phi(X)$$

$$\leq 2J^2 C_\Phi N_\Phi |\partial X_{\text{int}}|.$$
A similar estimate applies to $C_{\text{ext}}$. Using (2.13), the proof is complete.

As is discussed in Section 3, the first step in the proof of Theorem 2.2 is to introduce the smearing operation $(\cdot)_\alpha$, see (3.3). One consequence of this definition is

\begin{equation}
H_V = (H^b_{X_{\text{int}}})_\alpha + (H^b_{X_{bd}})_\alpha + (H^b_{X_{\text{ext}}})_\alpha.
\end{equation}

When applied to the ground state, each of the terms above can be estimated, in norm, in terms of the system’s gap. This is the content of Proposition 5.2 below.

**Proposition 5.2.** Let $\Phi$ be an interaction on $V$ which satisfies assumptions A1 – A4. For any finite set $X \subset V \subset \mathcal{V}$, $\alpha > 0$, and $A \in \{H^b_{X_{\text{int}}}, H^b_{X_{bd}}, H^b_{X_{\text{ext}}}\}$,

\begin{equation}
\| (A)_\alpha \psi_0 \| \leq \frac{\|[H_V, A]\|}{\gamma} e^{-\frac{\gamma^2}{4\alpha}}.
\end{equation}

**Proof.** Let $A \in \{H^b_{X_{\text{int}}}, H^b_{X_{bd}}, H^b_{X_{\text{ext}}}\}$. As previously discussed, see (3.5), $0 = \langle \psi_0, A \psi_0 \rangle = \langle \psi_0, (A)_\alpha \psi_0 \rangle$. Setting $C_A = [H_V, A]$, we find that

\begin{align}
\| (A)_\alpha \psi_0 \| &\leq \frac{1}{\gamma} \| H_V (A)_\alpha \psi_0 \| \\
&= \frac{1}{\gamma} \| [H_V, (A)_\alpha] \psi_0 \| \\
&= \frac{1}{\gamma} \| (C_A)_\alpha \psi_0 \|.
\end{align}

The value of $\| (C_A)_\alpha \psi_0 \|$ can be estimated using the spectral theorem. For any vector $f$, one has that

\begin{align}
\langle f, (C_A)_\alpha \psi_0 \rangle &= \frac{\alpha}{\pi} \int_{-\infty}^{\infty} e^{-\alpha t^2} \langle f, \tau^V_t (C_A) \psi_0 \rangle dt \\
&= \frac{\alpha}{\pi} \int_{-\infty}^{\infty} e^{-\alpha t^2} \langle f, e^{it H_V} C_A \psi_0 \rangle dt \\
&= \frac{\alpha}{\pi} \int_{-\infty}^{\infty} e^{-\alpha t^2} \int_{-\infty}^{\infty} e^{it \lambda} d\langle f, E_\lambda C_A \psi_0 \rangle dt \\
&= \int_{-\infty}^{\infty} \frac{\alpha}{\pi} \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{it \lambda} dt d\langle f, E_\lambda C_A \psi_0 \rangle \\
&= \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{4\alpha}} d\langle f, E_\lambda C_A \psi_0 \rangle.
\end{align}

In the third equality above we have introduced the notation $E_\lambda$ for the spectral projection corresponding to the self-adjoint operator $H_V$, and we also used that $\langle \psi_0, C_A \psi_0 \rangle = 0$. The last equality is a basic result concerning Fourier transforms of gaussians (and re-scaling), (see e.g. (5.59) in [13]). We conclude then that

\begin{equation}
\| \langle f, (C_A)_\alpha \psi_0 \rangle \| \leq e^{-\frac{\gamma^2}{4\alpha}} \| f \| \| C_A \|,
\end{equation}

and with $f = (C_A)_\alpha \psi_0$ we find that

\begin{equation}
\| (C_A)_\alpha \psi_0 \| \leq e^{-\frac{\gamma^2}{4\alpha}} \| C_A \|.
\end{equation}

Putting everything together, we have shown that

\begin{equation}
\| (A)_\alpha \psi_0 \| \leq \frac{1}{\gamma} \| (C_A)_\alpha \psi_0 \| \leq \frac{e^{-\frac{\gamma^2}{4\alpha}}}{\gamma} \| C_A \|.
\end{equation}
The next step in the proof of Theorem 2.2 is to make a strictly local approximation of the smeared terms appearing in (5.9) above. We defined these local approximations in Section 3, see (3.7), and Proposition 5.3 below provides an explicit estimate.

**Proposition 5.3.** Let $\mathcal{V}$ be a countable, locally finite set equipped with a metric $d$ for which (2.2) holds, and let $\Phi$ be an interaction on $\mathcal{V}$ which satisfies assumptions A1 - A4. Let $V \subset \mathcal{V}$ be the finite set described in A4, take $\ell > R$, and fix $X \subset V$ satisfying (2.13). The estimate

$$ \max \left \{ \left \| (H_{X_{\text{int}}}^b)_{\alpha} - M_X(\alpha) \right \|, \left \| (H_{\text{ext}}^b(t))_{\alpha} - M_{\text{ext}}(2\ell)(\alpha) \right \|, \left \| (H_{X_{\text{int}}}^b)_{\alpha} - M_X(\alpha) \right \| \right \} $$

(5.16)

$$ \leq 2CJ^2C_\Phi N_\Phi |\partial \mathcal{X}| \left( 4\sqrt{\frac{2\epsilon \mu}{\pi}} e^{\beta/2} + c \kappa / 2 e^{\delta R} \right) e^{-\frac{t}{\xi}}. $$

holds along the parametrization $2\alpha \epsilon \ell = \mu v^2$ where $\mu \geq 2\mu_0$. Here

$$ 0 < \frac{2}{\xi} = (1 - \epsilon)\mu = \frac{\gamma^2}{\mu^2 v^2 + \gamma^2 \mu}. $$

**Proof.** We will prove the estimate for $(H_{X_{\text{int}}}^b)_{\alpha} - M_X(\alpha)$, the other bounds follow similarly. Note that

$$ \left \| (H_{X_{\text{int}}}^b)_{\alpha} - M_X(\alpha) \right \| \leq \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \left \| \tau_t(H_{X_{\text{int}}}^b) - \tau_t^X(H_{X_{\text{int}}}^b) \right \| e^{-\alpha t^2} dt. $$

(5.17)

To bound the integral above, we introduce a parameter $T > 0$. For $|t| > T$, we use that

$$ \left \| \tau_t \left( H_{X_{\text{int}}}^b \right) - \tau_t^X \left( H_{X_{\text{int}}}^b \right) \right \| \leq \int_0^{|t|} \left \| \frac{d}{ds} \left( \tau_s \left( H_{X_{\text{int}}}^b \right) - \tau_s^X \left( H_{X_{\text{int}}}^b \right) \right) \right \| ds $$

(5.18)

$$ \leq 2 \left \| [H_V, H_{X_{\text{int}}}^b] \right \||t|. $$

From this, it follows readily that

$$ \sqrt{\frac{\alpha}{\pi}} \int_{|t| > T} \left \| \tau_t(H_{X_{\text{int}}}^b) - \tau_t^X(H_{X_{\text{int}}}^b) \right \| e^{-\alpha t^2} dt \leq \frac{8CJ^2C_\Phi N_\Phi |\partial \mathcal{X}|}{\sqrt{\alpha \pi}} \ell e^{-\alpha T^2}, $$

(5.19)

where we used Proposition 5.1.

For $|t| \leq T$, the estimate below is an immediate consequence of Lemma 3.1 in [14]:

$$ \left \| \tau_t(H_{X_{\text{int}}}^b) - \tau_t^X(H_{X_{\text{int}}}^b) \right \| \leq \int_0^{|t|} \left \| [H_V - H_X, \tau_s^X \left( H_{X_{\text{int}}}^b \right) \right \| ds. $$

(5.20)

The above commutator may be written as

$$ [H_V - H_X, \tau_s^X \left( H_{X_{\text{int}}}^b \right) \right] = \sum_{X \subset V \atop X \cap Y \neq \emptyset, X \cup Y \neq \emptyset} \sum_{Y \subset V \atop Y \cap X_{\text{int}} \neq \emptyset} \left[ \Phi(X), \tau_s^X \left( \Phi(Y) \right) \right]. $$

(5.21)

Estimating this, we find that

$$ \left \| [H_V - H_X, \tau_s^X \left( H_{X_{\text{int}}}^b \right) \right] \right \| \leq \sum_{x \in \partial X} \sum_{y \in X_{\text{int}}} \sum_{X \subset V \atop X \subset V \cap Y \neq \emptyset} \sum_{x \in X} \sum_{y \in Y} \left[ \Phi(x), \tau_s^X \left( \Phi(y) \right) \right] $$

(5.22)

$$ \leq cJ^2 \sum_{x \in \partial X} \sum_{y \in X_{\text{int}}} \sum_{X \subset V \atop X \subset V \cap Y \neq \emptyset} \sum_{x \in X} \sum_{y \in Y} \chi_X(X)(\chi_Y(Y) \min \left \{ |\partial_x X|, |\partial_y Y| \right \}) e^{-\mu(d(X,Y) - v|s|)}, $$

where we have used the Lieb-Robinson bound, i.e. Theorem 2.1 with $\mu \geq \mu_0$ from (2.3). Since $\Phi$ has a finite range $R$, it is clear that $d(x,y) - 2R \leq d(X,Y)$. This implies that

$$ \left \| [H_V - H_X, \tau_s^X \left( H_{X_{\text{int}}}^b \right) \right] \right \| \leq cJ^2C_\Phi N_\Phi e^{2\mu R} \sum_{x \in \partial X} \sum_{y \in X_{\text{int}}} e^{-\mu(d(x,y) - v|s|)}, $$

(5.23)
Now, for each fixed $x \in \partial \mathcal{X}$, $d(x, y) \geq \ell - 2R$. Summing on all $y$ yields

\begin{equation}
\| [H_V - H_X, \tau^X (H^b_{\text{inh}})] \| \leq c \kappa_{\mu/2} J^2 C \Phi N \Phi e^{3\mu R} |\partial \mathcal{X}| e^{-\mu (\ell^2 - v|s|)},
\end{equation}

for $\mu \geq 2 \mu_0$. Comparing back to (5.21), this bound demonstrates that

\begin{equation}
\| \tau_t (H^b_{\text{inh}}) - \tau^X (H^b_{\text{inh}}) \| \leq c \kappa_{\mu/2} J^2 C \Phi N \Phi e^{3\mu R} |\partial \mathcal{X}| e^{-\mu \ell/2} \left( e^{\mu |t|} - 1 \right),
\end{equation}

and hence, for any $T > 0$, we have that

\begin{equation}
\sqrt{\frac{\alpha}{\pi}} \int_{-T}^{T} \| \tau_t (H^b_{\text{inh}}) - \tau^X (H^b_{\text{inh}}) \| e^{-\alpha t^2} \, dt \leq c \kappa_{\mu/2} J^2 C \Phi N \Phi \frac{|\partial \mathcal{X}|}{\mu^2} e^{-\mu \ell/2} \int_{-T}^{T} e^{\mu |t|} e^{-\alpha t^2} \, dt
\end{equation}

(5.27)

Adding our results for large and small $T$, it is clear that

\begin{equation}
\| (H^b_{\text{inh}})_\alpha - M_X (\alpha) \| \leq 2 J^2 C \Phi N \Phi \frac{|\partial \mathcal{X}|}{\mu^2} e^{-\alpha T^2} \left( \frac{4 \epsilon \mu T}{\sqrt{\alpha \pi}} + c \kappa_{\mu/2} e^{3\mu R} \right) \exp \left[ \alpha T^2 - \frac{\mu}{2} \left( \ell - \frac{\mu T^2}{2\alpha} \right) \right].
\end{equation}

We now choose a parametrization which illustrates the decay. Let $\alpha$ satisfy

\[
\frac{\mu^2 T^2}{2\alpha} = \epsilon \ell \quad \text{with} \quad 0 < \epsilon = \left( 1 + \frac{\gamma^2}{\mu^2 \ell^2} \right)^{-1} < 1,
\]

and choose $T$ to satisfy the equation

\begin{equation}
\alpha T^2 - \frac{\mu}{2} \left( \ell - \frac{\mu T^2}{2\alpha} \right) = 0.
\end{equation}

In this case, $2\alpha T^2 = (1 - \epsilon) \mu \ell$, and thus

\begin{equation}
\| (H^b_{\text{inh}})_\alpha - M_X (\alpha) \| \leq 2 J^2 C \Phi N \Phi \frac{|\partial \mathcal{X}|}{\mu^2} \exp \left( \frac{2 \epsilon \mu T}{\pi} \ell^{3/2} + c \kappa_{\mu/2} e^{3\mu R} \right).
\end{equation}

Equipped with these estimates, we are now ready to prove Lemma 3.1.

**Proof.** (of Lemma 3.1) Using (5.9) and Proposition 5.3, we find that

\begin{equation}
\| H_V - (M_X (\alpha) + M_{\text{inh}} (\alpha) + M_{\text{inh}} (\alpha)) \|
\end{equation}

\begin{equation}
\leq 6 C \kappa_{\mu/2} J^2 C \Phi N \Phi \frac{|\partial \mathcal{X}|}{\mu^2} \left( \sqrt{\frac{2 \epsilon \mu}{\pi}} \ell^{3/2} + c \kappa_{\mu/2} e^{3\mu R} \right) e^{-\frac{\gamma}{2}},
\end{equation}

along the parametrization $2\alpha \epsilon \ell = \mu \ell^2$. This is the bound claimed in (5.1).

To see that (5.3) is true, note that, for example,

\begin{equation}
\| M_X (\alpha) \psi_0 \| \leq \| (H^b_{\text{inh}})_\alpha \psi_0 \| + \| (H^b_{\text{inh}})_\alpha - M_X (\alpha) \psi_0 \|.
\end{equation}

Using Propositions 5.1 and Proposition 5.2, it is clear that

\begin{equation}
\| (H^b_{\text{inh}})_\alpha \psi_0 \| \leq \| [H_V, H^b_{\text{inh}}] \| e^{-\frac{\gamma}{4}},
\end{equation}

(5.33)

\begin{equation}
\| (H^b_{\text{inh}})_\alpha - M_X (\alpha) \psi_0 \| \leq 4 C J^2 C \Phi N \Phi \frac{|\partial \mathcal{X}|}{\mu^2} e^{-\frac{\gamma}{4}},
\end{equation}

(5.33)
along the parametrization $2\alpha \ell = \mu v^2$. Combining this with Proposition 5.3, we have that

$$
\| M_\lambda(\alpha) \psi_0 \| \leq 2C \ell J^2 C \Phi \| \partial_\lambda \| \left( \frac{2\ell}{\gamma} + \frac{4}{\mu v} \sqrt{\frac{2e\mu}{\pi} \ell^{3/2}} + \frac{e\kappa/2}{\mu v} \right) e^{-\frac{\ell}{\xi}}.
$$

A similar bound applies to both $\| M_{\lambda_b}(\alpha) \psi_0 \|$ and $\| M_{\lambda_C}(\alpha) \psi_0 \|$. This completes the proof of the Lemma [5.1].

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