Non-Abelian Born–Infeld action, geometry and supersymmetry

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Abstract
In this work, we propose a new non-Abelian generalization of the Born–Infeld Lagrangian. It is based on a geometrical property of the Abelian Born–Infeld Lagrangian in its determinantal form. Our goal is to extend the Abelian second-type Born–Infeld action to the non-Abelian form preserving this geometrical property, which permits us to compute the generalized volume element as a linear combination of the components of metric and the Yang–Mills energy–momentum tensors. Under the BPS-like condition, the action proposed reduces to that of the Yang–Mills theory, independently of the gauge group. New instanton-wormhole solution and static and spherically symmetric solution in curved spacetime for an SU(2) isotopic ansatz are solved and the $N = 1$ supersymmetric extension of the model is performed.

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1. Introduction
In 1934, Born and Infeld [1, 3] introduced the most relevant version of nonlinear electrodynamics with, among others, the following main properties.

(i) Geometrically the BI Lagrangian density is one of the simplest non-polynomial Lagrangian densities that are invariant under the general coordinate transformations.
(ii) The BI electrodynamics is the only causal spin-1 theory [6, 8] apart from the Maxwell theory. The vacuum is characterized with $F_{\mu\nu} = 0$ and the energy density is definite semi-positive.
(iii) The BI theory conserves helicity [7] and solves the problem of the self-energy of the charged particles [1, 3, 23].

Recently, interest has increased in this nonlinear electromagnetic theory since it turned out to play an important role in the development of string theory, as was very well described in the pioneering work of Barbashov and Chernikov [5]. The nonlinear electrodynamics of the Born–Infeld Lagrangian, shown in [24], describes the low-energy process on D-branes.
which are non-perturbative solitonic objects that arise for the natural D-dimensional extension of the string theory. The structure of the string theory was improved significantly with the introduction of the D-branes, because many physically realistic models can be constructed, for example, the well-known ‘brane-world’ scenario that naturally introduces BI electrodynamics into gauge theories. From the point of view of gravity and supergravity theories, the precise form of Born–Infeld electrodynamics on the D-brane in arbitrary background is not yet known with certainty, principally in the case of SU(N) gauge fields [25].

With the recent advent of the physics of D-branes, the solitons in the non-perturbative spectrum of the string theory, it has been realized that their low-energy dynamics can be properly described by the so-called Dirac–Born–Infeld (DBI) action [38, 39]. Since single branes are known to be described by the Abelian DBI action, one might expect naturally that multiple brane configurations would be a non-Abelian generalization of the Born–Infeld action. Specifically in the case of superstring theory one has to deal with a supersymmetric extension of DBI actions and when the number of D-branes coincides there is a symmetry enhancement [40] and the Abelian DBI action should be generalized to its non-Abelian counterpart. Several possibilities for extending the Abelian BI action to the case of non-Abelian gauge symmetry have been discussed in the literature (e.g. [11]). Basically, as the starting point of all these attempts to pass to the non-Abelian case of the BI action is the Abelian BI action in its standard form, they all differ in the way that the group trace operation is defined. In the superstring-branes context the basic requirements that any candidate for the NBI action will fulfil are as follows:

(i) it will not contain odd powers of the $F$ field strength (with this requirement one can make contact with the tree level of the open superstring action);
(ii) the action will linearize by the BPS conditions and to equations of motion which coincide with those arising by imposing the vanishing of the $\beta$-function for background fields in the open superstring theory [25];
(iii) if the action is linearized under BPS conditions it should be connected to the possibility of supersymmetrizing the Born–Infeld theory.

With these significant reasons, it is interesting to generalize the Born–Infeld Lagrangian towards non-Abelian electromagnetic fields.

In this work, a new non-Abelian generalization of the Born–Infeld action is presented. This new non-Abelian Born–Infeld action fulfils the requirements given above and, for instance, is an admissible strong candidate for effective action for superstrings and D-branes; besides the fundamental importance such non-Abelian generalization has in the context of gravitation theory and nonlinear electrodynamics. It is based on a geometrical property of the Abelian Born–Infeld Lagrangian in its determinantal form. We extend the Abelian Born–Infeld action, in a similar form, as was suggested by Hull et al [15], to its non-Abelian counterpart naturally preserving this geometrical identity. This fact permits us to compute the generalized volume element of the action as a linear combination of the components of metric and the Yang–Mills energy–momentum tensors. We show that the Lagrangian proposed as a candidate for the non-Abelian Born–Infeld theory gives a very wide spectrum of gravitational exact solutions, and also in the case of flat $O(4)$ configurations the structure of the proposed action fulfils the energy-BPS considerations and the topological bound given by Minkowski’s inequality [27]: the action proposed reduces automatically to the Yang–Mills form under BPS-like conditions. This means that the Tseytlin prescription of symmetrized trace clearly is not the only one that takes the linear form under BPS considerations [28]. The new non-Abelian generalization of the Born–Infeld Lagrangian presented here is consistent not only from the BPS point of view, but also from first principles: independence of the gauge group and conservation of its structure.
in all types of configurations. The plan of this paper is as follows: in section 2, we describe from a geometrical point of view the non-Abelian Born–Infeld (NBI) action. In section 3, the determinant of the NBI action is computed and the minimum requirements for the NBI Lagrangian are enumerated. Sections 4, 5 and 6 explicitly are devoted to analyse the structure of the energy–momentum tensor from topological considerations, and the comparison with other prescriptions and the conditions under the NBI action is simplified. In sections 7 and 8, the dynamical equations derived from the new Lagrangian proposed and static spherically symmetric solution in curved spacetime for an $SU(2)$ isotopic ansatz and an Euclidean $SU(2)$ instanton-wormhole are solved. In section 9, the $N = 1$ supersymmetric extension of the non-Abelian Born–Infeld action proposed is successfully performed, and finally remarks and conclusions are given in section 10.

Our convention is as in [2] with signatures of the metric, Riemann and Einstein tensors all positive (+++). The internal indices (gauge group) are denoted by $a, b, c, \ldots$, spacetime indices by Greek letters $\mu, \nu, \rho, \ldots$ and the tetrad indices by capital Roman letters $A, B, C, \ldots$.

2. Geometrical identity and natural non-Abelian generalization of the Born–Infeld action

As was shown in [15, 27], if $(\mathcal{M}, g_{\mu\nu})$ is a Riemannian 4-manifold, then the action of the BI theory in this manifold is

$$S_{BI} = \int \frac{b^2}{4\pi} \left( \sqrt{-g} - \sqrt{|g|} \det(g_{\mu\nu} + \mathcal{F}_{\mu\nu}) \right) dx^4$$

(1)

where we define $g \equiv \det g_{\mu\nu}$ and $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$. As in [27], $B_{\mu\nu}$ is a background 2-form which is not necessarily constant but $d\mathcal{F} = 0$. Note that from the antisymmetric property of the indices of the tensors, we have

$$\det(g_{\mu\nu} + \mathcal{F}_{\mu\nu}) = \det(g_{\mu\nu} - \mathcal{F}_{\mu\nu}).$$

$$[\det(g_{\mu\nu} + \mathcal{F}_{\mu\nu})]^2 = g \det(g_{\mu\nu} + \mathcal{F}_{\mu\lambda} \mathcal{F}_{\nu}^{\lambda}).$$

Thus, we can write equation (1) in the following form:

$$S_{BI} = \int \frac{b^2}{4\pi} \left( \sqrt{-g} - \frac{4}{\sqrt{|g|}} \det(g_{\mu\nu} + \mathcal{F}_{\mu\lambda} \mathcal{F}_{\nu}^{\lambda}) \right) dx^4.$$  

(2)

From this form of the determinant, the natural non-Abelian generalization of the Born–Infeld action is

$$S_{NBI} = \int \frac{b^2}{4\pi} \left( \sqrt{-g} - \frac{4}{\sqrt{|g|}} \det(g_{\mu\nu} + \mathcal{F}_{\mu\lambda} \mathcal{F}_{\nu}^{\lambda}) \right) dx^4.$$  

(3)

From the expansion of the determinant in equation (3a) (we keep $B_{\mu\nu} = 0$ for simplicity)

$$S_{NBI} = \frac{b^2}{4\pi} \int \sqrt{-g} dx^4 \left\{ 1 - \frac{4}{\sqrt{|g|}} \left[ \frac{\gamma^2}{2} + \frac{\gamma^3}{3} + \frac{1}{8} (\mathcal{M})^2 - \frac{1}{4} \mathcal{M}^4 \right] \right\}$$

(3a)

where we define

$$M_{\mu\nu} \equiv F_{\mu\lambda} F_{\nu}^{\lambda}; \quad \gamma \equiv \left( 1 + \frac{F_{\mu\nu} F_{\mu\lambda} F_{\nu}^{\lambda}}{4} \right); \quad \mathcal{M}_{\mu\nu} \equiv M_{\mu\nu} - \frac{g_{\mu\nu}}{4} F_{\mu\alpha} F_{\nu}^{\alpha};$$

$$\mathcal{M}_{\mu\nu} \equiv \mathcal{M}_{\nu\mu}; \quad \mathcal{M}_{\mu\nu} \equiv \mathcal{M}^2; \quad (\mathcal{M}_{\mu\nu})^2 \equiv (\mathcal{M}^2)^2$$

(3b)
and note that all quantities in the fourth root in $S_{\text{NBI}}$ are adimensionalized over the Born–Infeld absolute field $b$. We can see that only one general invariant $S \equiv -\frac{1}{4} F_{\mu \nu}^a F_{a}^{\mu \nu}$ of the electromagnetic field is the basic block for the extension of the Born–Infeld Lagrangian to its non-Abelian counterpart (not particular trace prescriptions). The pseudoscalar invariant $P \equiv -\frac{1}{4} F_{\mu \nu} \tilde{F}^{\mu \nu}$ is restored only when we pass to the $U(1)$ gauge group, in contrast with the non-Abelian generalization (dependent on the gauge group) of the Born–Infeld action proposed by Hagiwara [4], which has three invariants of the nine possible gauge invariants for $SU(2)$ Yang–Mills fields [10].

In summary, from the Abelian Born–Infeld action (2) as the starting point, its non-Abelian generalization is performed in a natural form (3) and we can see that: (1) in the NBI action proposed, only $F_{\mu \nu}^a F_{a}^{\mu \nu}$ is the basic block for its construction and (2) the full form of the action (3a) is independent of the gauge group.

3. Requirements for the non-Abelian generalization and explicit computation of the determinant

By analogy with the Abelian case, the Lagrangian will satisfy the following properties [16].

(i) The ordinary Yang–Mills theory in the limit $b \to \infty$ should be found.

(ii) The electric component, $F_{\mu 0}^a$, non-Abelian electromagnetic tensor should be bounded for (i) when the magnetic components vanish.

(iii) The action will be invariant under diffeomorphisms.

(iv) The action will be real.

The proposed non-Abelian Born–Infeld action (3a) fulfils all the properties described above, as we will see in the explicit computation of the determinant. For tensors represented by matrices or hypermatrices, the algebraic invariant associated with a matrix $a$ can be obtained as traces of the powers of the given matrix. According to the Cayley–Hamilton theorem [20–22], only a finite number of these powers are linearly independent and, therefore, only a finite number of invariants are linearly independent. A more convenient set of invariants is given by the discriminants which are suitable combinations of traces and constructed in terms of alternating products with the unit matrix $\mathbb{I}$.

In order to perform the explicit computation of the determinant, we define the following:

$$\langle a \rangle \equiv \text{tr}(a) = a_i^i, \quad \langle a' \rangle \equiv \text{tr}(a') = (a')_i^i. \quad (4)$$

The relation between the traces of different powers of $a$ and its determinant in $D = 4$ is

$$\det(a) = \frac{1}{4!} \left[ \langle a \rangle^4 - 6 \langle a \rangle^2 (a^2) + 8 \langle a \rangle (a^3) + 3 (a^2)^2 - 6 (a^4) \right]. \quad (5)$$

If $a = (g_{\mu \nu} + F_{\mu \lambda}^a F_{\nu}^{\lambda} )$, the second factor in the square root in (3a) takes the familiar form

$$\det \left( g_{\mu \nu} + F_{\mu \lambda}^a F_{\nu}^{\lambda} \right) = g \left[ \gamma^4 - \frac{\gamma^2}{2} \mathcal{M}^2 - \frac{\gamma}{3} \mathcal{M}^3 + \frac{1}{8} (\mathcal{M}^2)^2 - \frac{1}{4} \mathcal{M}^4 \right]. \quad (6)$$

Now we can easily see that the proposed NBI action (3a) fulfils the four requirements given above.

1 Here the zero index corresponds to the time.

2 The polynomial under the root should start with terms such as $1 - \frac{\langle \mathcal{P}_{\mu \nu} \rangle^2}{b^2} + \cdots$ when the magnetic components are zero.
4. Energy–momentum tensor

From the determinant form of the NBI Lagrangian we can obtain the energy–momentum tensor, varying equation (3a) with respect to $g_{\mu\nu}$ as usual,

$$\frac{1}{2} T_{\mu\nu} \equiv \frac{1}{\sqrt{g}} \frac{\delta L}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{g}} \frac{\delta (\sqrt{g}L_{\text{NBI}})}{\delta g^{\mu\nu}}. \quad (7)$$

We obtain in matrix form

$$T_{\mu\nu} = \frac{b^2}{4\pi} \left\{ g_{\mu\nu} - \frac{\sqrt{g} + M}{2\sqrt{g}} \left[g_{\mu\nu} - \frac{1}{2} \sigma^{\rho\sigma} \left(F^\rho_{\mu\sigma} F_{\nu\rho} - g_{\rho\sigma} g_{\mu\nu}\right)\right]\right\} \quad (8)$$

another useful quantity that is the trace of the energy–momentum tensor

$$T^\mu_\mu = \frac{b^2}{4\pi} \left\{ g^{\mu\mu} - \frac{\sqrt{g} + M}{2\sqrt{g}} \left[g^{\mu\mu} - \frac{1}{2} \sigma^{\rho\sigma} \left(F^\rho_{\mu\sigma} F_{\mu\rho} - g_{\rho\sigma} g^{\mu\mu}\right)\right]\right\}. \quad (9)$$

From the expression for the energy–momentum tensor, we can find the conditions under which it nullifies, i.e. $T_{\mu\nu} = 0$,

$$F^a_{\mu\lambda} F^{\lambda}_{\nu a} = g_{\mu\nu} \frac{F^a_{\rho\sigma} F^{\rho\sigma}}{4}. \quad (10)$$

This shows that in the combined system

$$I = I(F) + \int_M d^4x \sqrt{g} R(g) \quad (11)$$

this action is extremized on an Einstein manifold with an (anti-)self-dual field configuration. Note also that expression (10) is the condition under which the self-dual configuration saturates Minkowski’s inequality for an arbitrary gauge group [27].

Some remarks are given as follows.

(i) It was shown in [27] that in a general curved background metric the Born–Infeld action is bounded by a topological quantity, and the bound is realized when the gauge field configuration is (anti-)self-dual. This means that from pure topological considerations the non-Abelian Born–Infeld action must satisfy this condition.

(ii) By construction our Lagrangian satisfies the above condition automatically, saturating Minkowski’s inequality (topological bound) when the (anti-)self-duality condition (10) is inserted in (3a). In this case, the NBI Lagrangian linearizes satisfying also the BPS-like conditions [28], becoming the Yang–Mills Lagrangian.

(iii) From point (ii) we can see that our action is, remaining minimally close to (below) the topological bound more than the other non-Abelian Lagrangians proposed.

These remarks become important when one studies instanton configurations in any background metric. The fact that any instanton configuration does not affect the background metric can be clearly seen in the context of the superstring theory. The energy–momentum tensor of the equations of motion of supergravity vanishes under a particular ansatz of the dilaton and R–R scalar field concerning the self-dual point [34, 35]. For instance, these D-instantons do not affect the Einstein equation. The inclusion of $B_{\mu\nu}$ was discussed in detail in [36].
5. Comparison with other prescriptions

Following the notation in [16], we have the following expression relating the determinant of a linear operator $A$ to traces:

$$\det(1 + A)^\beta = \exp[\beta \text{tr}[\log(1 + A)]]$$

$$= \sum_{n=0}^{\infty} \sum_{\alpha \in [S_n]} (-1)^n \prod_{p=1}^{n} \frac{1}{\alpha_p!} \left( -\frac{\beta}{p} \text{tr}(A^p) \right)^{\alpha_p}$$

where $\alpha \in [S_n]$ and $[S_n]$ is the set of equivalence classes of the permutation group of order $n$. The multi-index $\alpha$ is given by a Ferrer–Young diagram that equivalently satisfies the following relation:

$$\sum_{p=1}^{n} p\alpha_p = n \quad \alpha_p \geq 0$$

In order to analyse the structure of different Lagrangians up to any order in $F$ we can use this trace formula. We denote $\text{tr}_R \equiv \text{trace over representation indices}$, $\text{tr}_\otimes \equiv \text{trace over the tensor product}$ and $\text{tr}_M \equiv \text{the trace taken over spacetime indices}$. For our proposed Lagrangian (3a) we have

$$\det_M(1 + M)^{1/4} = \sum_{n=0}^{\infty} \sum_{\alpha \in [S_n]} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!} \left( -\frac{1}{4} \frac{\text{tr}_M(M^4)}{k} \right)^{\alpha_k}$$

$$= \sum_{n=0}^{\infty} \sum_{\alpha \in [S_n]} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!} \prod_{m=1}^{k} \left( -\frac{1}{4k} \text{tr}_M(M^m) \right)^{\alpha_k} \quad M_{\mu\nu} \equiv F_{\mu\alpha}^i F_{\nu\beta}^i \beta.$$ 

The expansion for the symmetrized trace prescription given by Tseytlin in [11] is given by

$$\frac{1}{dR} \text{tr}_R[\det_M(1 + iF^2 T_{\alpha})]^{1/2} = \frac{1}{dR} \text{tr}_R\sum_{n=0}^{\infty} \sum_{\alpha \in [S_n]} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!} \times$$

$$\times \left( -\frac{\text{tr}_M(F^{a_1} \cdots F^{a_{2k}})}{4k} T_{a_1} \cdots T_{a_{2k}} \right)^{\alpha_k} = \sum_{n=0}^{\infty} \sum_{\alpha \in [S_n]} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!} \prod_{m=1}^{k} \left( -\frac{\text{tr}_M(F^{a_1} \cdots F^{a_m})}{4k} \right) \frac{1}{dR} \text{tr}_R\left( \prod_{k=1}^{n} \prod_{m=1}^{k} T_{a_1} \cdots T_{a_m} \right)$$

and for the trace prescription inspired by non-commutative geometry given by Serie’ et al in [16] we have

$$\left[ \det_M(1 + F^2) \right]^{1/d_R} = \sum_{n=0}^{\infty} \sum_{\alpha \in [S_n]} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!} \left( -\frac{\text{tr}_\otimes(F^{2k})}{d_R k} \right)^{\alpha_k}$$

$$= \sum_{n=0}^{\infty} \sum_{\alpha \in [S_n]} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!} \prod_{m=1}^{k} \left( -\frac{\text{tr}_M(F^{a_1} \cdots F^{a_m})}{4k} \right) \frac{\text{tr}_R(T_{a_1} \cdots T_{a_m})}{d_R}.$$ 

It is very easy to see from the above expansions the following.

(i) In the three cases, the third-order and higher-order invariants of the non-Abelian electromagnetic field do not appear.
(ii) Clearly Serie’ et al and Tseytlin prescriptions have a strong dependence on the gauge group, which is not in our proposed non-Abelian Lagrangian because the basic block of the Lagrangian (3a) is 

\[ M_{\alpha\gamma} \equiv F_{\alpha\beta} F_{\gamma\beta}. \]

In this sense, the NBI Lagrangian (3a) has its gauge structure ‘hidden’ in the basic object \( M \). This is the reason why the Lagrangian (3a) remains closely below the topological bound given by Minkowski’s inequality more than the other non-Abelian proposals. And this is important from the BPS considerations and their relation to field configurations with minimal energy as in worldvolume theories of Dp-branes with specific values of \( p \) admitting ‘worldvolume (BPS) solitons’ [28]. It should be mentioned that the differences between our action and the symmetrized trace prescription for the usual gauge groups and for the leading terms in the expansions given above have no relevant importance in the context of the string theory; they can turn out to be very important in the brane theories. In particular, there are some discrepancies between the results arising from a symmetrized non-Abelian Born–Infeld theory and the spectrum to be expected in brane theories as pointed out in [37]. This fact will be tested with our proposed NABI generalization in the same brane context in order to see whether there exist similar discrepancies or not in future work [26]. It must be noted that this discussion should be viewed from within the context of a general analysis of possible NABI actions.

6. Topology of the gauge fields and spacetime: the reduced Lagrangian

Despite the apparent complexity of the non-Abelian Lagrangian (3a), there exists a possibility that it can be reduced to a square root form. This possibility is related to the choice of an ansatz where the gauge group and spacetime are highly identified (high symmetry). The requirement that the expression in the fourth root of the action (3a) becomes a perfect square is given when the following factorization property for the traceless \( \overline{M} \) holds:

\[ \overline{M}_{ab} \overline{M}^{\nu\rho} \approx \delta_a^\nu \gamma (F_{\mu\nu} F_{\rho}^{\mu\nu}) \]

where the scalar function \( \gamma \) depends obviously on the invariant \( F_{\mu\nu} F_{\mu\nu}^{\rho\rho} \). Several well-known ansätze reduce the form of the fourth-root Lagrangian (3a) to the square root form (e.g. Ogura and Hosoya [30], etc)

\[ S_{\text{NBI}} = \frac{b^2}{4\pi} \int \sqrt{-g} (1 - \mathcal{R}) \, dx^4 \]

where now

\[ \mathcal{R} = \sqrt{1 + \frac{1}{2b^2} (F_{\rho\lambda} F_{\lambda}^{\rho\lambda}) - \frac{1}{4b^4} (F_{\lambda\delta} F_{\rho}^{\lambda\delta}) (F_{\rho\lambda} F_{\delta}^{\rho\lambda}) + \frac{1}{8b^4} (F_{\rho\lambda} F_{\lambda}^{\rho\lambda})^2}. \]

Note that when the requirement (12) holds only the traces of even products of \( \overline{M} \), they will appear in the explicit computation of the determinant in the root of (3a) given the following result:

\[ \det (g_{\mu\nu} + \overline{F}_{\mu\lambda} \overline{F}^{\nu\lambda}) = g^2 \left[ 1 + \frac{1}{2b^2} (F_{\rho\lambda} F_{\lambda}^{\rho\lambda}) - \frac{1}{4b^4} (F_{\lambda\delta} F_{\rho}^{\lambda\delta}) (F_{\rho\lambda} F_{\delta}^{\rho\lambda}) + \frac{1}{8b^4} (F_{\rho\lambda} F_{\lambda}^{\rho\lambda})^2 \right]^2, \]

compare with expression (6). This property will be used when we solve and analyse different configurations with non-Abelian fields in the following sections.
7. Equations of motion for the non-Abelian Born–Infeld theory in curved spacetime

Now, we describe the dynamical equations for the non-Abelian electromagnetic fields in curved spacetime. Geometrically, introducing the generalizated exterior derivative ‘\( d \)’, the equations can be written as

\[
d F = 0; \quad d \tilde{F} = 0. \tag{15}
\]

The components of (15) in an orthonormal frame (tetrad) are

\[
D_B F^{A B a} = 0 \quad D_B \tilde{F}^{A B a} = \nabla_B F^{A B a} + \epsilon^{a b c} A_B^{a} \tilde{F}^{A B c} \tag{16}
\]

where

\[
(\nabla_B \tilde{F}^{A B})^a \equiv \frac{\delta \mathcal{L}}{\delta F^{A B a}} = (\epsilon_{B}^{\mu} \tilde{F}^{A B \mu} + \Gamma_{C D}^{A} \tilde{F}^{C D} + \Gamma_{D}^{A} \tilde{F}^{B} )^a \tag{17}
\]

and

\[
F^{a}_{b c} \equiv \frac{\partial L_{\text{NB}}}{\partial \dot{F}^{a}_{b c}} \tag{18}
\]

The Bianchi identity is automatically satisfied because, obviously, \( F^{a} = dA^{a} + A^{b} \wedge A^{c} \). \( \tilde{F}^{A B a} \) is the non-Abelian extension of the Born–Infeld dielectric displacement-like tensor in the Abelian Born–Infeld electrodynamics.

Trivial solutions to the dynamical equations of the non-Abelian electromagnetic fields are easily obtained from the action (3a). We will solve one of these cases as an example.

7.1. Spherically symmetric chromostatic solution

The equations that describe the dynamics of the non-Abelian electromagnetic fields of the Born–Infeld theory in curved spacetime are (16). Explicitly, and for the isotopic gauge [9], the equations are

\[
d F^{a} = d \left( F_{01}^{a} \omega^{0} \wedge \omega^{1} \right) = \partial_{\theta} \left( e^{\Lambda + \Phi} F_{01}^{a} \right) d\theta \wedge dt \wedge dr = 0 \tag{19}
\]

\[
d \tilde{F}^{a} = d \left( -F_{01}^{a} \omega^{3} \wedge \omega^{2} \right) = \partial_{r} \left( -e^{2G} F_{01}^{a} \right) dr \wedge d\phi \wedge d\theta = 0 \tag{20}
\]

where the 1-forms \( \omega^{\alpha} \) correspond to the spherically symmetric interval

\[
ds^{2} = -e^{2\Lambda} dr^{2} + e^{2\Phi} d\theta^{2} + e^{2F(r)} (d\theta^{2} + \sin^{2} \theta) d\phi^{2}. \tag{21}
\]

Explicitly,

\[
\omega^{0} = e^{\Lambda} dt \quad \Rightarrow \quad dt = e^{-\Lambda} \omega^{0}
\]

\[
\omega^{1} = e^{\Phi} dr \quad \Rightarrow \quad dr = e^{-\Phi} \omega^{1}
\]

\[
\omega^{2} = e^{2G} d\theta \quad \Rightarrow \quad d\theta = e^{-2G} \omega^{2}
\]

\[
\omega^{3} = e^{2F(r)} d\phi \quad \Rightarrow \quad d\phi = e^{-2F(r) (\sin \theta)^{-1}} \omega^{3}. \tag{22}
\]

From (19) and (20), we can see that

\[
F_{01}^{a} = f(r, a) \tag{23}
\]

and

\[
e^{2G} F_{01}^{a} = h(a). \tag{24}
\]

The simplest form for the fields and the potential that belongs to the above expression is

\[
F_{01}^{a} = f(r, a) = \delta^{a 0} \nabla_{r} \phi(r) \quad \text{and} \quad e^{2G} F_{01}^{a} = h(a) = \text{const.} \tag{25}
\]
On the other hand, from (18) we can see that
\[ F_{a01} = \frac{F_{a01}}{\sqrt{1 - (F_{01})^2}}, \quad (F_{01})^2 = \frac{F_{a01} F_{01}}{b^2}. \]  \( (26) \)

Then, inverting (26) together with (25)
\[ F_{a01} = \frac{b(a)}{\sqrt{1 + \left(\frac{h}{b^2}\right)^2}}. \]  \( (27) \)

In order to obtain the final expression for the non-Abelian electric field it is necessary to introduce (27) into the Einstein equations for the interval (21) (see the appendix) (for details of full computations in the Abelian case, see [13]). Having made it, the non-Abelian electromagnetic field for a static spherically symmetric configuration is
\[ F_{a01} = \frac{b}\delta az \sqrt{1 + \left[1 - \left(\frac{r_0}{a}\right)^2\right]^2 \left(\frac{r_0}{a}\right)^4} = -\delta az \nabla \phi(r) \]  \( (28) \)

where we have \( h = br_0^2 = Q [1, 3]\) and \( \lambda \) is a dimensionless parameter restricted to \( |\lambda| < 1 \) [13]. Integrating (28) we obtain the following expression for the potential \( \phi(r) \):
\[ \phi(r) = -(-1)^{1/4} br_0 F[\arcsin\left(-1\right)^{3/4} \overline{Y}(r), -1] \]  \( (29) \)

where
\[ \overline{Y}(r) = \left[1 - \left(\frac{r_0}{a}\right)^2\right] \frac{r}{r_0} \]

and \( F \) is the elliptic function of the first kind [12]. However, in nonlinear electrodynamics, the solution turns out to be of Coulomb type asymptotically (\( r \to \infty \)), but is strongly modified near the origin (\( r \to 0 \)) presenting regular behaviour without singularities [13]. Perhaps the isotopic ansatz is very ‘naive’ (embedded \( U(1) \) solution); the asymptotic behaviour of the solution is in complete agreement with the type of solutions given by Ikeda and Miyachi [9] for Yang–Mills in flat space.

7.2. About BPS and string theory considerations

In [29], the author finds an \( SO(4) \) invariant solution in four-dimensional Euclidean space adopting the ansatz of [33] into the following Lagrangian:
\[ S_{Park}[F, g] = \int R^4 \alpha \left( \det g_{\mu\nu} \otimes 1_{d_+} + \beta^{-1} F_{\mu
u}^a \otimes T_a \right)^{1/2} \]  \( (30) \)

obtaining, after the ansatz of [33] was introduced, that this action equation (32) does not become the Yang–Mills Lagrangian under the BPS condition. The main reason why this situation arises is that the expansion of the determinant into (32) contains odd powers of the fields and this fact, see [28] for details, makes it impossible for any proposed Lagrangian (independently of its nonlinearity or complexity) to be of the Yang–Mills form, when the BPS condition is introduced.

The non-Abelian Lagrangian proposed in our work straightforwardly linearizes when the ansatz of [33] is introduced because the determinant is reduced to a sum of squares. Note that this discussion should be viewed from the context of a general analysis of possible NBI. This means that the action relevant to the description of multiple D-branes has its origin in the superstring theory coupled to a non-Abelian gauge field, and it is well known that the effective action of the latter does not contain a term of the form \( F^3 \) [28]. We can see that our NBI action has the BPS properties discussed and from the point of view of string theory [11] the NBI action presented here is a strong candidate for the non-Abelian effective action.
8. Wormhole-instanton solution in NABI theory

The action is given by

\[ S = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R + \int d^4x \sqrt{g} \Lambda + \frac{1}{4\pi} \int d^4x \sqrt{g} L_{\text{NBI}} \]

\[ L_{\text{NBI}} = \left( \frac{b^2}{4\pi} \right) (1 - \Re) \]

\[ \Re = \sqrt{\gamma^2 - \frac{\gamma^2}{2} M^2 - \frac{\gamma^2}{3} \Lambda^3 + \frac{1}{8} (\Lambda^2)^2} - \frac{1}{4} \Lambda^4 \]

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (T_{\mu\nu} - \Lambda g_{\mu\nu}) \]

\[ T_{\mu\nu} = \frac{b^2}{4\pi} \left\{ \frac{g_{\mu\nu} - \sqrt{g} + M}{2\sqrt{g}} \right\} \]

\[ (31) \]

The scalar curvature \( R \) and the \( SU(2) \) Yang–Mills field strength \( F_{\mu\nu}^a \) are defined in terms of the affine connection \( \Gamma^i_{\mu\nu} \) and the \( SU(2) \) gauge connection \( A^a_{\mu} \) by

\[ R = g^{i\mu} R_{\mu\nu} \]

\[ R_{\mu\nu}^i = \partial_\mu \Gamma^i_{\nu\lambda} - \partial_\nu \Gamma^i_{\mu\lambda} + \cdots \]

\[ F_{\mu\nu}^a = \partial_\mu A^a_{\nu} - \partial_\nu A^a_{\mu} + \varepsilon^{abc}_{\mu} A^b_{\mu} A^c_{\nu} \]

\[ G \] and \( \Lambda \) are the Newton gravitational constant and the cosmological constant, respectively. As in the case of Einstein–Yang–Mills systems, our non-Abelian BI model can be interpreted as a prototype of gauge theories interacting with gravity (e.g. QCD, GUTs, etc). Upon varying the action (31), we obtain the Einstein equation

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (T_{\mu\nu} - \Lambda g_{\mu\nu}) \]

\[ (32) \]

and the Yang–Mills field equation in differential form

\[ d^* F_a + \frac{1}{2} \varepsilon_{abc} (A_b \wedge^* F_c - ^* F_b \wedge A_c) = 0 \]

\[ (34) \]

where we define as usual

\[ \frac{\partial L_{\text{NBI}}}{\partial F^{ab}_{\mu\nu}} = \frac{\partial L_{\text{NBI}}}{\partial F^{ab}_{\mu\nu}} \]

We are going to seek for a classical solution of equations (33) and (34) with the following spherically symmetric ansatz for the metric and gauge connection:

\[ ds^2 = d\tau^2 + a^2(\tau) \sigma^i \otimes \sigma^i \]

\[ (35) \]

where \( \tau \) is the Euclidean time and the dreibein is defined by \( e^i \equiv a^2(\tau) \sigma^i \). The gauge connection is

\[ A^a = A^a_{\mu} dx^\mu = h \sigma^a. \]

\[ (36) \]

The \( \sigma^i \) 1-form satisfies the \( SU(2) \) Maurer–Cartan structure equation:

\[ d\sigma^a + \varepsilon_{bce} \sigma^b \wedge \sigma^c = 0 \]

\[ (37) \]

Note that in the ansatz the frame and isospin indices are identified and the fourth-root NBI Lagrangian (3a) reduces to a square root expression, as we explained in section 6. The field strength 2-form

\[ F^\tau = \frac{1}{4} F^\tau_{\mu\nu} \ dx^\mu \wedge dx^\nu \]

becomes

\[ F^a = dA^a + \frac{1}{2} \varepsilon^a_{bce} A^b \wedge A^c \]

\[ \equiv (-h + \frac{1}{2} h^2) \varepsilon^a_{bce} \sigma^b \wedge \sigma^c. \]

\[ (39) \]
Inserting it in the Yang–Mills field equation (34) and into the NBI Lagrangian we obtain
\[ d^* F^a + \frac{1}{2} \epsilon^{abc} (A_b \wedge d^* F^c - d^* F_b \wedge A_c) = 0 \]
\[ = \epsilon^a_{hc} \, d\tau \wedge \sigma^b \wedge \sigma^c (-2h + h^2) (h - 1) (A_h/a) \]
where
\[ A_h \equiv \left[ 1 + 2 \left( \frac{r_0}{a} \right)^4 \left( -h + \frac{h^2}{2} \right) \right] \frac{\epsilon^a_{hc}}{R} \]
\[ \mathcal{R} = \sqrt{1 + 6 \left( -h + \frac{h^2}{2} \right) \left( \frac{r_0}{a} \right)^4 + 6 \left( -h + \frac{h^2}{2} \right) \left( \frac{r_0}{a} \right)^4}. \]
We can see that for \( h = 1 \) there exists a non-trivial solution
\[ F_{hc}^a = -\frac{\epsilon^a_{hc}}{a^2}, \quad F_{hc}^a = 0. \]
Namely, only the magnetic field is non-vanishing while the electric field vanishes. An analogous feature can be seen in the solution of Giddings and Strominger [31]. Substituting the expression for the field strength (39) into the Born–Infeld energy–momentum tensor, we reduce the Einstein equation (30) to an ordinary differential equation for the scale factor \( a \):
\[ 3 \left[ \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{a^2} \right] = 2G (b^2 - 4\pi \Lambda) - 2G b^2 \left[ 1 + 6 \left( \frac{r_0}{a} \right)^4 \left( 1 + \left( \frac{r_0}{a} \right)^4 \right) \right]^{1/2} \]
\[ \left[ 1 + 6 \left( -h + \frac{h^2}{2} \right) \left( \frac{r_0}{a} \right)^4 + 6 \left( -h + \frac{h^2}{2} \right) \left( \frac{r_0}{a} \right)^4 \right]. \]
where the relation with the \( H \) constant is given by \( H^2 = 8\pi G \Lambda \) and \( r_0 = \sqrt{\frac{Q}{G}} \). However, the above nonlinear differential equation has several integrability problems that can be avoided by taking \( 2G (b^2 - 4\pi \Lambda) / 3 = 0 \). The solution of (41) with \( 2G (b^2 - 4\pi \Lambda) / 3 = 0 \) is given by
\[ \pm \sqrt{-B} (\tau - \tau_0) = a^2 \left[ 12 + 6 \left( \frac{a}{r_0} \right)^4 + \left( \frac{a}{r_0} \right)^8 \right]^{1/4} \times F_1 \left[ \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{3}{2}, -\frac{1}{3}, \frac{1}{3} \right]. \]
where \( F_1 \) is the Appell hypergeometric function, \( B \equiv \frac{1}{2} G Q^2 \sqrt{6} \) and the relation between the charge \( Q \) and the Born–Infeld parameter \( b \) has been used. It is easy to note that we have the following relation between the cosmological constant and the absolute field of the Born–Infeld theory:
\[ \frac{b^2}{4\pi} = \Lambda \quad (G \neq 0). \]
The shape of the wormhole solution is given in figure 1. We now have the following different points of view for this result:
(i) for the magnetic field one can consider a imaginary magnetic charge producing it, \( \sqrt{-B} \) is a real value and the cosmological constant is for an anti-de Sitter spacetime. For this case, the solution is obviously Euclidean;
(ii) on the other hand, if \( b^2 \sim \Lambda \) is positive (\( \Lambda > 0 \)), from (42) we can see that the solution corresponds to a Lorenzian wormhole in de Sitter spacetime and electric charge. This is in some meaning a Wick rotation or, mathematically speaking, an analytical prolongation of the Euclidean interpretation given above. It is \( \tau \to \tau = i \), \( Q_M \to iQ_M = Q_E \), \( \Lambda < 0 \to \Lambda > 0 \) (\( ADS \to DS \)).
A more detailed analysis for this type of solution will be given elsewhere [26].
We show that the Lagrangian proposed as a candidate for the non-Abelian Born–Infeld theory gives a very wide spectrum of gravitational exact solutions, and also in the case of flat $O(4)$ configurations the structure of the proposed action fulfils the energy considerations and the topological bound (10). As was explained in the previous paragraphs, the action proposed reduces automatically to the Yang–Mills form under the BPS-like condition. This means that the Tseytlin prescription of symmetrized trace clearly is not the only one that takes a linear form under BPS considerations, as was claimed in [28]. We can see from the analysis of the previous sections that the Lagrangian presented here is consistent not only from the BPS point of view, but also from first principles: independent of the gauge group, conservation of its structure in all types of configurations remains close to (below) the topological bound (10) more than other proposals for a non-Abelian Born–Infeld action. In the next section, we will make the supersymmetric extension of our action, in order to complete the requirements clearly analysed in [28].

9. Supersymmetric extension

Having shown in the above sections several reasons to propose the action (3a) as a candidate for the non-Abelian Born–Infeld Lagrangian, we now discuss this problem from the point of view of supersymmetry. We have already seen in the previous sections that the very important property of our proposed non-Abelian action is its absolute independence of the gauge group and, for instance, also independence of any trace prescription (for a supersymmetric version of NBI with symmetric trace prescription, see [32]). This fundamental point makes the supersymmetric extension of our model not only possible but also simple. From the NABI Lagrangian (3a), we obtain after expansion of the determinant

$$S_{NBI} = \frac{b^2}{4\pi} \int \sqrt{-g} \, d^4x \left\{ 1 - \sqrt{\gamma^4 - \frac{\gamma^2 \bar{M}^2}{2} - \frac{\gamma \bar{M}^3}{3} + \frac{1}{8} \left( \bar{M}^2 \right)^2 - \frac{1}{4} \bar{M}^4} \right\}.$$ 

We write it in the form

$$L_{NBI} = \frac{b^2}{4\pi} \sum_{n=0}^{\infty} q_n (\Gamma + \Delta)^{n+1} \quad (44)$$
where we define
\[ \Gamma \equiv \gamma^4 - 1 \]  
(45)
\[ \Delta \equiv -\frac{\gamma^2 M^2}{2} - \frac{\gamma M^3}{3} + \frac{1}{8}(M^3)^2 - \frac{1}{4} M^4 \]  
(46)
and the coefficients \( q_n \) are given by the following formula:
\[ q_n = \begin{cases} 
\left( -\frac{1}{4} \right)^{n+1} \frac{1}{(n+1)!} \prod_{k=1}^{n} (4k - 1) & (n > 0) \\
-\frac{1}{4} & (n = 0).
\end{cases} \]  
(47)
\( L_{\text{NBI}} \) can be rewritten as
\[ L_{\text{NBI}} = b^2 \sum_{n=0}^{\infty} q_n \sum_{j=0}^{n+1} \binom{n+1}{j} \Gamma^j \Delta^{n+1-j}. \]  
(48)
The basic ingredient for the \( N = 1 \) supersymmetric extension of the non-Abelian Born–Infeld Lagrangian is the non-Abelian chiral superfield \( W_\alpha \):
\[ W_\alpha(y, \theta) = \frac{1}{8} D_\alpha D_\alpha e^{-2i/\Lambda} W_\alpha e^{2i/\Lambda}. \]  
(49)
Under a gauge transformation it transforms covariantly,
\[ W_\alpha \rightarrow e^{-2i/\Lambda} W_\alpha e^{2i/\Lambda}. \]
Its Hermitian conjugate transforms as
\[ W^\alpha \rightarrow e^{-2i/\Lambda^*} W^\alpha e^{2i/\Lambda^*}. \]
Written in components \( W_\alpha \) reads
\[ W_\alpha(y, \theta) = i\lambda_\alpha - \theta\alpha D_\alpha - \frac{1}{2} (\theta\sigma^\mu\sigma^\nu)_{\alpha} F_{\mu\nu} - \theta(\nabla\lambda)_{\alpha} \]
where a chiral variable was introduced,
\[ y^\mu = x^\mu + i\theta\sigma^\mu \bar{\theta} \]
with
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \]
and
\[ \nabla \lambda = (\sigma^\mu)_{\mu a} (\partial_\mu \lambda^a + i[A_\mu, \lambda^a]). \]
Similarly, as the SUSY extension of the \( N = 1 \) Yang–Mills theory can be constructed from \( W^2 \) and its Hermitian conjugate \( \bar{W}^2 \), we can construct the SUSY extension of the \( N = 1 \) NABI theory considering
\[ (W^\alpha W_\alpha)_{\mu\nu} = -\frac{\eta_{\mu\nu}}{4} [\lambda^2 + i(\theta\lambda D + D\theta\lambda) - \theta(\lambda\nabla\lambda + \bar{\lambda}\nabla\bar{\lambda} - D^2)] \]
\[ + \frac{1}{2} \left[ \theta(\sigma^\mu)_{\lambda} F_{\nu\lambda} + F_{\mu\rho}\theta(\sigma_\rho)_{\lambda} \lambda - \theta(\sigma^\mu F_{\nu\rho} + i\bar{F}_{\mu\rho} F_{\nu\rho}) \right] \]  
(50)
and analogically its Hermitian conjugate \( (\bar{W}^\alpha \bar{W}_\alpha)_{\mu\nu} \) where
\[ \bar{F}_{\mu\rho}^{a} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{a\nu\rho}. \]
Now we can define the object

\[ M_{\mu\nu} \equiv -\left[(W^{\alpha}W^\alpha)_{\mu\nu} + (\bar{W}_\alpha\bar{W}^{\alpha})_{\mu\nu}\right] \]  

with an on-shell purely bosonic part: \( \int (d^2\bar{\theta} + d^2\bar{\theta}) M_{\mu\nu} |_{\text{bos}} = F_{\mu\rho}^a F_{\nu\rho}^a. \)

Now, in order to construct higher powers of \( M_{\nu}^{\mu}(\bar{F}^2) \), necessary to obtain the non-Abelian Lagrangian, we define the superfield

\[ X_{\mu\nu} \equiv \frac{1}{4}[e^{2V}D^2(e^{-2V}(W^{\alpha}W^\alpha)_{\mu\nu})e^{2V} + e^{-2V}D^2(e^{2V}(W^{\alpha}W^\alpha)_{\mu\nu})e^{2V}] \]  

Their \( \theta = 0 \) component gives, as in the Abelian BI case, the obvious result

\[ X_{\mu\nu} \big|_{\theta = 0} = F_{\mu\rho}^a F_{\nu\rho}^a \]

and this field also transforms as under generalized gauge transformations \( X_{\mu\nu} \rightarrow = e^{-2i/\Lambda_1}X_{\mu\nu}e^{2i/\Lambda_1} \). With the supersymmetric gauge invariant objects, \( M_{\mu\nu} \) and \( X_{\mu\nu} \), the corresponding traceless objects \( \tilde{M}_{\mu\nu} \), \( \tilde{X}_{\mu\nu} \) and also \( \gamma \), can be easily constructed:

\[ \tilde{M}_{\mu\nu} \equiv M_{\mu\nu} - \frac{\eta_{\mu\nu}}{4}\tilde{X}_{\rho\lambda} \tilde{X}^{\rho\lambda} \]

From the above considerations, we propose the following supersymmetric Lagrangian for the non-Abelian generalization of the Born–Infeld theory:

\[ L_{NBI} = \sum_{n=0}^{\infty} C_{rst} \int (d^2\theta + d^2\bar{\theta})(\Gamma_2 + \Delta_2) \prod_{i=0}^{n} \Delta_{i} \]  

where

\[ \Delta_2 = -\frac{1}{2} \left[ \tilde{M}^\rho_\nu \tilde{X}_\rho_\mu \left( 1 + \frac{X_\rho_\mu}{4} \right)^2 \right] - \frac{1}{3} \left[ \tilde{M}^\rho_\nu \tilde{X}_\rho_\mu \tilde{X}_\sigma_\alpha \left( 1 + \frac{X_\rho_\mu}{4} \right) \right] \]

\[ + \frac{1}{8} \left[ (\tilde{X}_\rho_\mu \tilde{X}_\rho_\nu) \left( \tilde{X}_\rho_\mu \tilde{X}_\rho_\nu \right) \right] - \frac{1}{4} \left[ \tilde{M}^\rho_\nu \tilde{X}_\rho_\mu \tilde{X}_\rho_\mu \tilde{X}_\rho_\nu \right] \]

\[ \Delta_0 = -\frac{1}{2} \left[ \tilde{X}_\rho_\mu \tilde{X}_\rho_\nu \left( 1 + \frac{X_\rho_\mu}{4} \right)^2 \right] - \frac{1}{3} \left[ \tilde{X}_\rho_\mu \tilde{X}_\rho_\mu \tilde{X}_\rho_\mu \left( 1 + \frac{X_\rho_\mu}{4} \right) \right] \]

\[ + \frac{1}{8} \left[ (\tilde{X}_\rho_\mu \tilde{X}_\rho_\nu) \left( \tilde{X}_\rho_\mu \tilde{X}_\rho_\nu \right) \right] - \frac{1}{4} \left[ \tilde{X}_\rho_\mu \tilde{X}_\rho_\mu \tilde{X}_\rho_\mu \tilde{X}_\rho_\nu \right] \]

\[ \Gamma_2 \equiv M^\rho_\mu \left[ 1 + \frac{3}{8} (X_\rho_\mu) + \frac{1}{16} (X_\rho_\mu)^2 + \frac{1}{256} (X_\rho_\mu)^3 \right] \]

\[ \Gamma_0 \equiv X^\rho_\mu \left[ 1 + \frac{3}{8} (X_\rho_\mu) + \frac{1}{16} (X_\rho_\mu)^2 + \frac{1}{256} (X_\rho_\mu)^3 \right]. \]

Note that it remains to determine the arbitrary coefficients \( C_{rst} \) imposing the condition that the bosonic sector of the theory does coincide with the NABI Lagrangian. The particular choice for the coefficients \( C_{rst} \) that leads to the purely bosonic part of the NABI Lagrangian \((3a)\) is as follows:

\[ C_{0,s,t} = 0 \quad \text{for all} \ s, t \]

\[ C_{1,j,n-j} = \left\{ \begin{array}{ll} \left( \frac{b^2}{4\pi} \right) q_{n-j} \left( \begin{array}{c} n \\ j \end{array} \right) & \text{for } j \leq n \\ 0 & \text{for } j > n. \end{array} \right. \]  

(54)

With the knowledge of the coefficients \( C_{1,j,n-j} \) the supersymmetric Lagrangian can be easily written in the form \( L_{NBI|_{\text{susy}}} = L_{NBI} + L_{\text{fer}} + L_{\text{fb}} \), where \( L_{NBI} \) is the purely bosonic Lagrangian (putting fermions to zero), \( L_{\text{fb}} \) includes kinetic fermion and crossed boson–fermion terms and \( L_{\text{fer}} \) contains self-interacting fermion terms.
We have then been able to construct an $N = 1$ supersymmetric non-Abelian Lagrangian (53) in the context of superfields formulation, with a bosonic part expressed in terms of the fourth root of $|g| \det \left( g_{\mu \nu} + F_a^{\mu \alpha} F_a^{\nu \beta} \right)$. As usual, we have employed the natural curvature invariants as building blocks in the superfield construction arriving at a Lagrangian which, in its bosonic sector, depends only on the invariants given by expressions (3b), and it is expressed without any trace prescription. Odd powers of the field strength $F$ are absent because our starting point was the Abelian equivalent fourth-root version of the Born–Infeld Lagrangian (2) where the basic object of its structure is $M_{\mu \nu} = F_{\mu \sigma} F_{\sigma \nu}^\sigma$. Note also that there exists the technical impossibility of constructing a superfield functional of $W$ and $D W$ ($W$ and $D W$) containing $F^3$ in its higher $\theta$ component.

With the supersymmetric extension of our proposed non-Abelian Born–Infeld action, and taking account of all considerations and results from the previous sections in the different contexts: Born–Infeld theory itself, D-brane and superstring theory, we show that the action (3a) is a strong candidate for a concrete non-Abelian generalization of the Born–Infeld theory.

However, it will be interesting to analyse the non-Abelian Born–Infeld action from the point of view of nonlinear realizations [18, 19] as shown by Bagger and Galperin in [17] for the Abelian case. This will be our task in the near future [26].

10. Concluding remarks

In this work, a new non-Abelian generalization Born–Infeld action is proposed from a geometrical point of view. The advantage of this form of the non-Abelian Born–Infeld action over other attempts is based on several points:

1. the process of non-Abelianization of the new action is performed in a more natural form and is based on a geometrical property of the Abelian Born–Infeld Lagrangian in its determinantal form;
2. in the new action there are no trace prescriptions;
3. the new action has full independence of the gauge group;
4. by construction, our Lagrangian satisfies Minkowski’s inequality (topological bound), saturating the bound when the (anti-)self-duality condition (10) is inserted in (3a). In this case, the NBI Lagrangian linearizes satisfying also the BPS-like conditions [28], becoming the Yang–Mills Lagrangian;
5. from point (4) we can see that our action is minimal, remained close to (below) the topological bound more than the other non-Abelian Lagrangians proposed;
6. the supersymmetrization of the model is performed showing that the proposed action fulfills the requirements given by the BPS-SUSY relations;
7. by analogy with the Abelian case, the Lagrangian proposed satisfies all the following properties:
   (i) we find the ordinary Yang–Mills theory in the limit $b \to \infty$,
   (ii) the electric component $F_{\mu 0}^a$ non-Abelian electromagnetic tensor should be bounded for (i) when the magnetic components vanish,
   (iii) the action is invariant under diffeomorphisms and
   (iv) the action is real;
8. a static spherically symmetric regular solution of the Einstein–NABI system for an isotopic ansatz is obtained and the asymptotic behaviour of the solution is in agreement with the types of solutions given by Ikeda and Miyachi [9] for Yang–Mills in flat space, and a new instanton-wormhole solution in the non-Abelian Born–Infeld-Einstein theory is presented.
We show that the Lagrangian proposed as a candidate for the non-Abelian Born–Infeld theory gives gravitational exact solutions and that, in the case of the wormhole, one can see the following:

(i) there exists a link between the absolute field of Born and Infeld, $b$, and the cosmological constant $\Lambda$, and both ($\Lambda$ and $b$) can be identified;

(ii) the general shape of the wormhole and the tunnel radius are driven by the Born–Infeld theory itself without the necessity of introducing any additional field in particular.

This means that the NABI generalization presented here has a fundamental importance and physical meaning not only theoretically but also from the phenomenological point of view.

In the context of a general analysis of possible NABI actions, the NABI generalization presented in this work is a strong candidate for describing the low-energy dynamics of D-branes, the solitons in the non-perturbative spectrum of the (super)string theory. It must be noted that, as was suggested in [37], any candidate for the full NABI action should reproduce the correct spectra (spacing of energy levels) in a gauge theory configuration whose dual corresponds to 2-branes on $T^4$ where Tseytlin’s symmetrized trace prescription fails. We believe that our proposed action can resolve or improve this situation because of its good properties given above. We will reproduce with our action the explicit computations given in [37] in a forthcoming paper [26].

It is interesting to note that in our work we take the antisymmetric background tensor field $B_{\mu\nu} = 0$ for simplicity, but its importance is very well known from the point of view of the (super)string theories where it was first associated with massless modes [41] and, recently, the presence of such a background for the string dynamics has a very important implication in non-commutativity of the spacetime [42], and in many supergravity models [43]. Recently, a careful analysis of the $B_{\mu\nu}$ field in a five-dimensional brane-world scenario where it was associated with torsion was made [44]. The result of this analysis showed that it is possible to construct a topological configuration like a static cosmic string on the brane whose formation involves only the zero mode. We can expect that such types of configurations can also appear from our non-Abelian generalization of the Born–Infeld action in a similar five-dimensional scenario (i.e. Randall–Sundrum type). Also we expect that non-commutativity of the spacetime will be obtained as a result of switch-on the $B_{\mu\nu}$ field in the new NABI action proposed. These points will be discussed in more detail in the near future [26].

However, the geometry of the Lagrangian density is only a part of the whole picture. The full perspective about the advantages of the second-order actions with the different approaches in the treatment of the non-Abelian extensions to Born–Infeld Lagrangian will be given elsewhere [26], when we put the focus on D-brane actions, duality and quantization.

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Appendix

Explicitly, the components of the Einstein tensor for the line element (21) are

$$G^0_0 = e^{-2\Phi} \Psi - e^{-2\Phi}$$  \hspace{1cm} (A.1)
\[ \Psi \equiv [2\partial_\mu \partial_\nu F - 2\partial_\mu \Phi \partial_\nu F + 3(\partial_\mu F)^2] \quad G^1_1 = e^{-2\Phi} [2\partial_\mu \Lambda \partial_\nu F + (\partial_\mu F)^2] - e^{-2F} \tag{A.2} \]

\[ G^2_2 = G^3_3 = e^{-2\Phi} \left[ 2\partial_\mu \lambda \partial_\nu (F + \Lambda) - \Phi \partial_\mu \partial_\nu (F + \Lambda) + (\partial_\mu \Lambda)^2 + (\partial_\mu F)^2 + \partial_\mu F \partial_\nu \Lambda \right] \tag{A.3} \]

\[ G^1_3 = G^2_3 = G^0_2 = G^0_1 = G^1_2 = 0. \tag{A.4} \]

The components of the Born–Infeld energy–momentum tensor for the SU(2) gauge group in the tetrad (22) are

\[ -T_{00} = T_{11} = \frac{b^2}{4\pi} \left( 1 - \sqrt{\frac{F_0}{\epsilon G}} \right) \frac{4}{4 + 1} \tag{A.5} \]

\[ T_{22} = T_{33} = \frac{b^2}{4\pi} \left( 1 - \frac{1}{\sqrt{\frac{\lambda_0}{\epsilon^2}} + 1} \right). \tag{A.6} \]

In expressions (A.5)–(A.6) we utilized the isotopic ansatz and expression (27) for \( F_{01}^0 \).

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