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Convolutions and Applications for the Offset Linear Canonical Transform Via Hermite Weights

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Abstract. The main purpose of this paper is to present three new convolutions for the offset linear canonical transform, with the Hermite weights, and to illustrate their potential applications. In view of this, new factorization theorems are obtained and new Young’s convolution inequalities will be introduced. Within the more applied side, the way to design filters (including multiplicative filters in the time domain) is also discussed in the last section.

INTRODUCTION

The offset linear canonical transform (OLCT) (see [7]) of a signal \(f(t)\) with real parameters \(A = (a, b, c, d, u_0, \omega_0)\), (satisfying \(ad - bc = 1\)) is defined as

\[
F_A(u) := \mathcal{O}_A(f(t))(u) := \begin{cases}
\int_{\mathbb{R}} f(t)\mathcal{K}_A(u, t)dt, & b \neq 0 \\
\sqrt{d} e^{j\frac{d}{2}(u-u_0)^2} f(d(u-u_0)), & b = 0,
\end{cases}
\]

where \(\mathcal{K}_A(u, t) := K_A e^{j\left(\frac{d}{2}u^2 - \frac{b}{2}u + \frac{a}{2}t^2 + \frac{ic}{b}(u-u_0)t\right)}\), and \(K_A = \frac{\sqrt{ad}}{\sqrt{2\pi bj}}\). The inverse of the OLCT is given by

\[
f(t) = \mathcal{O}_{A^{-1}}\{F_A(u)\}(t) = C \int_{\mathbb{R}} F_A(u)\mathcal{K}_{A^{-1}}(u, t)du,
\]

where \(A^{-1} = (d, -b, -c, a, b\omega_0 - du_0, cu_0 - a\omega_0)\), and \(C = e^{j\left(\frac{1}{4}cd^2 - \frac{1}{2}adu_0 + abu_0\right)}\). In this paper, we will always consider \(b \neq 0\) since the OLCT becomes a chirp multiplication operation otherwise. We recall that the Fourier transform and its inverse are defined by \(\Psi_{FT}(f(t))(u) = \int_{\mathbb{R}} f(t)e^{-jut}dt\) and \(f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_{FT}(f(t))(u)e^{jut}du\), respectively. If \(f, h \in L^1(\mathbb{R})\), then the classic (Fourier) convolution in the time domain is expressed as

\[
(f * h)(t) := \int_{\mathbb{R}} f(\tau)h(t - \tau)d\tau,
\]

and the factorization property as follows

\[
\Psi_{FT}\{f * h\}(t) = \Psi_{FT}\{f(t)\}(u) \cdot \Psi_{FT}\{h(t)\}(u).
\]
For any real number \( \lambda \neq 0 \), we have
\[
(f * h)(\lambda t) = \lambda f(\lambda t) * h(\lambda t).
\] (5)

We also have the Young’s inequality (see [2]). If \( f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R}) \), and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} + 1 \) (with \( p, q \geq 1 \)). Then, the following inequality holds
\[
\|f * h\|_r \leq C_1 \|f\|_p \cdot \|h\|_q, \text{ for some } C_1 > 0.
\] (6)

We notice that when \( u_0 = \omega_0 = 0 \), \( \Omega_A \) is the well-known linear canonical transform (LCT) (see [4]). Remind that if \( A = (a, b, c, d, 0, 0) \), \( |a + d| < 2 \), and \( \phi_n(t) \), \( \mu_n \) are the eigenfunctions and the eigenvalues of the OLCT (or the LCT) (see [6]), then we have
\[
\mu_n \phi_n(u) = \Omega_A[\phi_n(t)](u),
\] (7)

where
\[
\phi_n(t) := \frac{1}{\sqrt{\beta 2^n n! \sqrt{\pi}}} e^{-\frac{(b-t)^2}{2\beta}} H_n\left(\frac{t}{\beta}\right), \quad \mu_n := e^{-\frac{j}{4} n^2} \quad (n \in \mathbb{N}),
\]

and \( H_n \) is the \( n \)-th Hermite polynomial. The constants \( \alpha, \beta, \theta \) can be taken from
\[
\alpha := \frac{sgn(b) \cdot (a - d)}{\sqrt{4 - (a + d)^2}}, \quad \beta := \frac{2|b|}{\sqrt{4 - (a + d)^2}}, \quad \theta := \cos^{-1}\left(\frac{a + d}{2}\right).
\] (8)

Throughout this paper, for convenience, we denote
\[
E_A(t) := e^{i\frac{\theta}{2} t^2 + \frac{\alpha t}{2}}, \quad \bar{f}(t) := E_A(t)f(t), \quad \mathcal{E}_A^m(t) := e^{im\theta} E_A(t) \quad (m \in \mathbb{R}).
\]

The identity (1) becomes
\[
F_A(u) = \Omega_A(f)(u) = K_A e^{i\frac{\theta}{2} u^2 + \frac{\alpha u}{2}} \int_{\mathbb{R}} \bar{f}(t) e^{-\frac{\theta}{2} t^2} dt.
\] (9)

This paper is divided into four sections and organized as follows. In Section 2, we introduce the relationship between the Hermite functions and the OLCT, which are displayed in Theorem 1 and Theorem 2. Three new convolutions for the OLCT with the Hermite weights and their product theorems are studied in the Section 3. Some special cases of these convolutions are also obtained. In the last section, we propose some applications of these convolutions as well as new Young’s convolution inequalities and designing multiplicative filter in the time domain.

**HEMITE FUNCTIONS AND THE OLCT**

For \( \lambda \neq 0 \), it is easy to realize that \( 1 = ad - bc = (a \lambda)(\frac{d}{2}) - (\frac{c}{2})(c \lambda) \). Let \( \lambda \neq 0 \), and the parameters \( A_\lambda := (a \lambda, \frac{b}{2}, c \lambda, \frac{d}{2}, 0, 0) \) satisfy
\[
|a \lambda + d| < 2. \quad (10)
\]

Under the condition (10), let \( \phi_n^\lambda(t) \), \( \mu_n^\lambda \) be the eigenfunctions and the corresponding eigenvalues of the OLCT with parameters \( A_\lambda = (a \lambda, \frac{b}{2}, c \lambda, \frac{d}{2}, 0, 0) \). We then have \( \mu_n^\lambda \cdot \phi_n^\lambda(u) = \Omega_{A_\lambda} \phi_n^\lambda(t)](u) \). The eigenfunctions \( \phi_n^\lambda(t) \) and the eigenvalues \( \mu_n^\lambda \) corresponding parameters \( A_\lambda \) can be calculated as in (8).

**Theorem 1** Let the parameters \( A_1 = (a, b, c, d, 0, 0) \), and one of the following conditions is satisfied:

(i) \( |a + d| < 2 \); (ii) \( |a + d| \geq 2 \) and \( 1 - ad > 0 \).

Then, there exists a constant \( \lambda > 0 \) such that the following relation holds
\[
\phi_n^\lambda(u) = \frac{1}{\mu_n^\lambda} \Omega_{A_1} \phi_n^\lambda(t)](u).
\] (11)
Proof. If $|u + d| < 2$ then from relation (7) we choose $\lambda = 1$. Thus, (11) is fulfilled.
If $|u + d| \geq 2$ and $1 - ad > 0$. By changing the variable $t = \lambda r$ ($\lambda > 0$), we get

$$\mathcal{K}_A(u, t) = K_A e^{A(u^2 - \frac{j}{4})} = K_A e^{\left(\frac{j}{2}\right)u^2 - \frac{j}{4} + \frac{1}{\lambda}(1 - ad)x^2} = \frac{1}{\sqrt{\lambda}} \mathcal{K}_A(u, \tau).$$

It follows

$$\mathcal{K}_A(u, \tau) = \sqrt{\lambda} \mathcal{K}_A(u, t).$$

Assume that the condition (10) is satisfied. We then have

$$(a\lambda^2 - 2\lambda + d)(a\lambda^2 + 2\lambda + d) < 0.$$ \hspace{1cm} (13)

If $a = 0$, then from (13) we derive $|\alpha| > \frac{|d|}{2}$. Thus, there exists $\lambda > 0$ such that (10) is satisfied. If $a \neq 0$, since $\delta = 1 - ad > 0$ then we denote $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be four solutions of the following equation $(a\lambda^2 - 2\lambda + d)(a\lambda^2 + 2\lambda + d) = 0$. Solving this equation, we receive

$$\lambda_i \in \left\{-1 \pm \frac{\sqrt{5}}{a}, \frac{1 \pm \sqrt{5}}{a}\right\}, i \in \{1, 2, 3, 4\}.$$

From (13), we deduce $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \lambda_4)$, and $\lambda_4 = \frac{1 + \sqrt{5}}{a} > 0$. Hence, there exists $\lambda > 0$ such that (10) is fulfilled. Therefore, the OLCT with parameters $A, \lambda$ has the eigenfunctions $\phi_n^A(t)$ and the eigenvalues $\mu_n^A$:

$$\mu_n^A \cdot \phi_n^A(u) = \int_{\mathbb{R}} \mathcal{K}_A(u, t) \cdot \phi_n^A(t) dt.$$ \hspace{1cm} (14)

Substituting the relation (12) into (14) results in

$$\mu_n^A \cdot \phi_n^A(u) = \sqrt{\lambda} \int_{\mathbb{R}} \mathcal{K}_A(u, t) \cdot \phi_n^A(t) dt.$$ \hspace{1cm} (15)

That implies $\phi_n^A(u) = \frac{1}{\mu_n^A} \int_{\mathbb{R}} \mathcal{K}_A(u, t) \cdot \phi_n^A(t) dt$. Hence $\phi_n^A(u) = \frac{1}{\mu_n^A} \mathcal{K}_A (\frac{t}{\lambda}) (u)$. The proof is completed.

**Theorem 2** Let $A = (a, b, c, d, u_0, \omega_0)$, and one of the following conditions be fulfilled: (i) $|u + d| < 2$; (ii) $|u + d| \geq 2, a + d \neq 2$ and $1 - ad > 0$. Then, there exists a positive constant $\lambda$, such that the following relation holds

$$e^{im_1 u} \cdot \phi_n^A(u - m_2) = \frac{e^{i\left(m_1 - \frac{d\omega_0}{\omega}\right)t}}{\mu_n^A} \mathcal{K}_A \left\{ e^{im_1 t} \cdot \phi_n^A\left(\frac{t - m_2}{\lambda}\right)\right\} (u),$$ \hspace{1cm} (16)

provided

$$\begin{aligned}
   m_1 &= \frac{(u_0 - d\omega_0) + m_1 (1 - d)}{2d(u + d - 2)} \\
   m_2 &= \frac{(b\omega_0 - d\omega_0 + u_0 \cdot \omega)}{2(b\omega_0 - d\omega_0 + u_0)} \\
   m_3 &= \frac{(b\omega_0 - d\omega_0 + m_1 \cdot \omega)(2b\omega_0 + m_1 + \omega)}{2(b\omega_0 - d\omega_0 + u_0 \cdot \omega)}. \hspace{1cm} (17)
\end{aligned}$$

Proof. We realize that

$$\int_{\mathbb{R}} \mathcal{K}_A(u, t) \cdot e^{i(m_1 + m_3) + e^{-im_1 u} \cdot \phi_n^A\left(\frac{t - m_2}{\lambda}\right)} dt = K_A \int_{\mathbb{R}} e^{\left(\frac{1}{2} u^2 - \frac{j}{4} + \frac{1}{\lambda}(1 - ad)x^2\right)} e^{i(m_1 + m_3) + e^{-im_1 u} \cdot \phi_n^A\left(\frac{t - m_2}{\lambda}\right)} dt$$

$$= K_A \int_{\mathbb{R}} e^{\left(\frac{1}{2} u^2 - \frac{j}{4} + \frac{1}{\lambda}(1 - ad)x^2\right)} e^{i(m_1 + m_3) + e^{-im_1 u} \cdot \phi_n^A\left(\frac{t - m_2}{\lambda}\right)} dt.$$

Let

$$\frac{d}{dt} u^2 = \frac{1}{b} tu + \frac{a}{2b} t^2 + \left(\frac{b\omega_0 - d\omega_0}{b} - m_1\right) u + \left(\frac{u_0}{b} + m_1\right) t + m_3 = \frac{d}{dt} \left(u^2 - m_2^2\right) - \frac{1}{b} (t - m_2)(u - m_2) + \frac{a}{2b} (t - m_2)^2. \hspace{1cm} (18)$$
then
\[
\begin{align*}
\begin{cases}
m_1 - \frac{d - 1}{b} m_2 &= \frac{b u_0 - d u_0}{b} \\
m_1 + \frac{a - 1}{b} m_2 &= -\frac{u_0}{b} \\
m_3 &= m_2 \left( \frac{d f}{b} - \frac{1}{b} + \frac{a}{2b} \right).
\end{cases}
\end{align*}
\]
Remind that \(a + d \neq 2\). Then, the solution of this system equations is given as
\[
\begin{align*}
\begin{cases}
m_1 &= \frac{(a - 1)(b u_0 - d u_0) + a (1 - d)}{(b a + d - 2)} \\
m_2 &= -\frac{b u_0 - d u_0}{a b + d - 2} \\
m_3 &= \frac{(b u_0 - d u_0)^2}{2 b (a b + d - 2)}.
\end{cases}
\end{align*}
\]
Substituting equation (18) into equation (17), we obtain
\[
\int_{\mathbb{R}} \mathcal{K}_A(u,t) e^{\im i m_1 + m_3} e^{-\im j m_1 u} \phi_n^j \left( \frac{t - m_2}{\lambda} \right) dt = e^{\frac{\im \lambda u}{2}} \int_{\mathbb{R}} \mathcal{K}_A(u,m_2,t-m_2) \phi_n^j \left( \frac{t - m_2}{\lambda} \right) dt = e^{\frac{\im \lambda u}{2}} \mathbb{O}_A \left( \hat{\phi}_n^j \left( \frac{t}{\lambda} \right) \right) (u-m_2).
\]
Thanks to equation (11), we derive \(\int_{\mathbb{R}} \mathcal{K}_A(u,t) \cdot e^{\im i m_1 + m_3} e^{-\im j m_1 u} \cdot \phi_n^j (t-m_2) dt = e^{\frac{\im \lambda u}{2}} \cdot \mu_n \cdot \sqrt{A} \cdot \phi_n^j (u-m_2)\), which implies that \(e^{\im j m_1} \cdot \frac{\im \lambda u}{2} \cdot \sqrt{A} \cdot \phi_n^j (u-m_2) = \int_{\mathbb{R}} \mathcal{K}_A(u,t) \cdot e^{\im i m_1} e^{-\im j m_1 u} \cdot \phi_n^j \left( \frac{t - m_2}{\lambda} \right) dt\). This means
\[
e^{\im j m_1} \phi_n^j (u-m_2) = e^{\im i m_1} \cdot \mu_n \cdot \sqrt{A} \mathbb{O}_A \left( e^{\im i m_1} \cdot \phi_n^j \left( \frac{t - m_2}{\lambda} \right) \right) (u).
\]
The theorem is achieved.

**Remark 1** If \(a + d = 2\), and \(b u_0 - (1 - a) u_0 = 0\), then the relation (15) holds for the following conditions \(m_1 = \frac{(a - 1)(b u_0 - d u_0)}{b (a b + d - 2)} - \frac{u_0}{b}, m_2 \in \mathbb{R}\) and \(m_3 = 0\).

**Example 1** Consider the case \(A = (-\frac{2}{3}, \frac{1}{3}, -9, 3, 1, 3), A_1 = (-\frac{2}{3}, \frac{1}{3}, -9, 3, 0, 0)\). Since \(|a + d| = \frac{7}{3} > 2\), then it is easily seen that
\[
\lambda \in \left( \frac{3 \sqrt{3} - 3}{2}, \frac{3 - 3 \sqrt{3}}{2} \right) \cup \left( \frac{3 \sqrt{3} + 3}{2}, \frac{3 + 3 \sqrt{3}}{2} \right).
\]
If we choose \(\lambda = \frac{3}{2}\), then \(A_1 = (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, 2, 0, 0)\). The Hermite functions and the values \(\mu_n\) can be expressed as
\[
\phi_n^j (t) = \frac{3}{\sqrt{2 \pi} n!} e^{-\frac{3}{2} t^2} H_n \left( \frac{9 \sqrt{3} t}{4} \right), \quad \mu_n = e^{-\frac{t^2}{4}} (n \in \mathbb{N}).
\]
Therefore, the relation (11) becomes \(\phi_n^j (t) = \sqrt{\frac{2}{\pi}} e^{\frac{t^2}{4}} (n \in \mathbb{N})\). From (16), we obtain \(m_1 = 12, m_2 = 3, m_3 = \frac{9}{2}\). The relation (15) gives \(e^{\im j m_1} \phi_n^j (t-3) = \sqrt{\frac{2}{\pi}} e^{\frac{t^2}{4}} (n \in \mathbb{N})\). From (16), we obtain \(m_1 = 12, m_2 = 3, m_3 = \frac{9}{2}\). The relation (15) gives \(e^{\im j m_1} \phi_n^j (t-3) = \sqrt{\frac{2}{\pi}} e^{\frac{t^2}{4}} (n \in \mathbb{N})\).

**CONVOLUTIONS FOR THE OFFSET LINEAR CANONICAL TRANSFORM WITH HERMITE WEIGHTS**

In this section, the space \(L^p(\mathbb{R})\) will be endowed with the norm \(\| \cdot \|_p\) defined by \(\| f \|_p := \left( \int_{\mathbb{R}} |f|^p dt \right)^{\frac{1}{p}}\), \(p \geq 1\). Assume that the conditions of Theorem 1 and Theorem 2 are satisfied. The convolution for the OLCT of two signals \(f(t)\) and \(h(t)\) is defined by
\[
(f \otimes h)(t) := \frac{K_A^2 e^{\im i (m_1 - n \tau)} (E_A(t))^{-1}}{\mu_n \sqrt{\lambda}} \int_{\mathbb{R}^2} f(\tau) \bar{h}(v) \cdot E_A^{m_1} (t - \tau - v) \phi_n^j \left( \frac{t - \tau - v - m_2}{\lambda} \right) d\tau dv,
\]
provided that the integral in (19) is well-defined. Moreover, if \(f, h \in L^1(\mathbb{R})\), then the function defined in (19) belongs to \(L^1(\mathbb{R})\), and \(\| f \otimes h \|_1 \leq C_2 \| f \|_1 \cdot \| h \|_1\), where \(C_2\) is a positive constant. We have the following product theorem.
Theorem 3  
Assume that $f, h \in L^1(\mathbb{R})$, $F_A$ and $H_A$ denote the OLCT of the signals $f(t)$ and $h(t)$ with parameters $A$, respectively. We have

$$
\mathcal{O}_A \left\{ f \odot h \right\}(t) = e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) \cdot F_A(u) \cdot H_A(u).
$$

Moreover, if $e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) \cdot F_A(u) \cdot H_A(u) \in \mathcal{O}_A(L^1(\mathbb{R}))$, then

$$
(f \odot h)(t) = \mathcal{O}_A^{-1} \left\{ e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) \cdot F_A(u) \cdot H_A(u) \right\}(t).
$$

Proof.  
Using the identity (9) and Theorem 2, we realize that

$$
e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) \cdot F_A(u) \cdot H_A(u)$$

$$= e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) K_A e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \int_{\mathbb{R}^2} f(\tau) \overline{h}(v) e^{-im \tau} e^{-im \omega} d\tau dv$$

$$= e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} K_A e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \int_{\mathbb{R}^2} e^{\frac{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}}{A}} f(\tau) \overline{h}(v) e^{-im \tau} e^{-im \omega} d\tau dv$$

$$= e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} K_A e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \int_{\mathbb{R}^2} e^{\frac{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}}{A}} f(\tau) \overline{h}(v) \cdot \mathcal{E}_A^{m}(t) \cdot \phi^A_n(t) \cdot \phi^A_n(t) dt dv.$$

By making $\tau = \tau, \nu = \nu$ and $s = t + \tau + \tau$, we obtain

$$
e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) \cdot F_A(u) \cdot H_A(u)$$

$$= K_A e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \int_{\mathbb{R}} e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \mathcal{E}_A^{m}(s) \int_{\mathbb{R}^2} f(\tau) \overline{h}(v) \cdot \mathcal{E}_A^{m}(s - \tau - \nu) \cdot \phi^A_n(t) \cdot \phi^A_n(t) \cdot \phi^A_n(t) ds dv$$

$$= \int_{\mathbb{R}} \mathcal{K}_A(u, s) \left\{ e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \mathcal{E}_A^{m}(s) \int_{\mathbb{R}^2} f(\tau) \overline{h}(v) \cdot \mathcal{E}_A^{m}(s - \tau - \nu) \cdot \phi^A_n(t) \cdot \phi^A_n(t) \cdot \phi^A_n(t) ds dv \right\} ds$$

$$= \int_{\mathbb{R}} \mathcal{K}_A(u, s) \cdot \left( f \odot h \right)(s) ds = \mathcal{O}_A \left\{ \left( f \odot h \right) \right\}(u).$$

The proof is concluded.

By using the same method as in Theorem 3, we derive the next result.

Theorem 4  
Let $f, h \in L^1(\mathbb{R})$, $F_A$ and $H_A$ denote the OLCT of the signals $f(t)$ and $h(t)$ with parameters $A$, respectively. The transform

$$
(f \odot h)(t) := e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) \cdot F_A(u) \cdot H_A(u).
$$

defines a convolution belonging $L^1(\mathbb{R})$, and turns possible the following factorization identity

$$
\mathcal{O}_A \left\{ f \odot h \right\}(t) = e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \cdot \phi^A_n(u - m_2) \cdot F_A(u) \cdot H_A(u).
$$

The convolution for the OLCT of two signals $f(t)$ and $h(t)$ associated with the Hermite functions $\phi^A_n(\sqrt{3} \tau - m_2)$ scaled by the chirp $e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}}$, is defined as

$$
(f \odot h)(t) := \frac{\sqrt{3} K_A e^{im \omega \cdot \hat{\tau} - \hat{v}^2 + \frac{2m \cdot d(u)}{A^2}} \mathcal{E}_A^{m}(t)}{\mu_\lambda^3 \cdot \sqrt{A}} \int_{\mathbb{R}^2} f(\tau) \overline{h}(v) \cdot \mathcal{E}_A^{m}(t) \cdot \frac{\sqrt{3} \tau - \tau - v + \kappa}{A} \cdot \phi^A_n(t) \cdot \phi^A_n(t) \cdot \phi^A_n(t) dv,$$

where $\kappa = (3 - \sqrt{3})(b\omega_0 - d_0)$ (as long as the integral in (22) is well-defined). Moreover, if $f, h \in L^1(\mathbb{R})$ then $(f \odot h)(t) \in L^1(\mathbb{R})$ since $\|f \odot h\|_1 \leq C_3 \|f\|_1 \cdot \|h\|_1$ for some $C_3 > 0$. 

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Theorem 5  Let \( f, h \in L^1(\mathbb{R}) \), \( F_A \) and \( H_A \) denote the OLCT of the signals \( f(t) \) and \( h(t) \), with parameter \( A \), respectively. The following factorization identity holds
\[
\mathcal{O}_A \left\{ \left( f \otimes h \right)(t) \right\}(u) = e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( \frac{m}{\sqrt{3}} - m_2 \right) \cdot F_A \left( \frac{m}{\sqrt{3}} \right) \cdot H_A \left( \frac{m}{\sqrt{3}} \right).
\]
Moreover, if \( e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( \frac{m}{\sqrt{3}} - m_2 \right) \cdot F_A \left( \frac{m}{\sqrt{3}} \right) \cdot H_A \left( \frac{m}{\sqrt{3}} \right) \in \mathcal{O}_A(L^1(\mathbb{R})) \), then
\[
\left( f \otimes h \right)(t) = \mathcal{O}_A \left\{ e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( \frac{m}{\sqrt{3}} - m_2 \right) \cdot F_A \left( \frac{m}{\sqrt{3}} \right) \cdot H_A \left( \frac{m}{\sqrt{3}} \right) \right\}(t).
\] (23)

Proof. Based on (9) and (16), we have
\[
e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( u - m_2 \right) \cdot F_A \left( u \right) \cdot H_A \left( u \right) = e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( u - m_2 \right) \cdot K_A^2 e^{\left( \frac{2}{\sqrt{3}} + \frac{m}{\mu} \frac{v}{\sqrt{3}} \right) u} \int_{\mathbb{R}^2} \hat{f}(\tau) \overline{h}(v) e^{-\frac{\mu \tau}{\sqrt{3}}} e^{-\frac{\mu v}{\sqrt{3}}} d\tau dv
\]
\[
e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( u - m_2 \right) \cdot F_A \left( u \right) \cdot H_A \left( u \right) = e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( u - m_2 \right) \cdot K_A^2 e^{\left( \frac{2}{\sqrt{3}} + \frac{m}{\mu} \frac{v}{\sqrt{3}} \right) u} \int_{\mathbb{R}^2} \hat{f}(\tau) \overline{h}(v) e^{-\frac{\mu \tau}{\sqrt{3}}} e^{-\frac{\mu v}{\sqrt{3}}} d\tau dv
\]
Performing the change of variables \( \tau = \tau, v = v \) and \( s = t + \tau + v - \kappa \), we achieve
\[
e^{\frac{m \mu}{\sqrt{3}}} \cdot \phi_n^A \left( u - m_2 \right) \cdot F_A \left( u \right) \cdot H_A \left( u \right) = K_A e^{\left( \frac{2}{\sqrt{3}} + \frac{m}{\mu} \frac{v}{\sqrt{3}} \right) u} \int_{\mathbb{R}^2} e^{-\frac{\mu \tau}{\sqrt{3}}} \hat{E}_A \left( \frac{s - \tau - v + \kappa}{\sqrt{A}} \right) \cdot \hat{E}_A \left( \frac{s - \tau - v + \kappa}{\sqrt{A}} \right) d\tau dv \times
\]
\[
\int_{\mathbb{R}} \hat{J}(\tau) \overline{h}(v) \cdot \hat{E}_A \left( s - \tau - v + \kappa \right) \cdot \phi_n^A \left( \frac{s - \tau - v + \kappa}{\sqrt{A}} \right) d\tau dv \times
\]
\[
\int_{\mathbb{R}} K_A \left( u \sqrt{3}, \frac{v}{\sqrt{3}} \right) \left( \frac{\kappa}{\mu \sqrt{A}} \right) e^{\left( \frac{2}{\sqrt{3}} + \frac{m}{\mu} \frac{v}{\sqrt{3}} \right) u} \int_{\mathbb{R}^2} \hat{J}(\tau) \overline{h}(v) \cdot \hat{E}_A \left( s - \tau - v + \kappa \right) \cdot \phi_n^A \left( \frac{s - \tau - v + \kappa}{\sqrt{A}} \right) d\tau dv ds
\]
\[
\int_{\mathbb{R}} K_A \left( u \sqrt{3}, \frac{v}{\sqrt{3}} \right) \left( \frac{\kappa}{\mu \sqrt{A}} \right) e^{\left( \frac{2}{\sqrt{3}} + \frac{m}{\mu} \frac{v}{\sqrt{3}} \right) u} \int_{\mathbb{R}^2} \hat{J}(\tau) \overline{h}(v) \cdot \hat{E}_A \left( s - \tau - v + \kappa \right) \cdot \phi_n^A \left( \frac{s - \tau - v + \kappa}{\sqrt{A}} \right) d\tau dv ds
\]
which proves the theorem.

Corollary 1  Let \( f, h \in L^1(\mathbb{R}), k \in \{1, 2\} \), \( F_A \) and \( H_A \) denote the LCT of the signals \( f(t) \) and \( h(t) \) with parameters \( A_1 \), respectively. The convolution of two signals \( f(t), h(t) \) for the LCT is defined as follows
\[
(f \otimes h)(t) = \frac{K_A^2 \left( E_{A_1}(t) \right)^{-1}}{\mu_n^A \cdot \sqrt{A}} \int_{\mathbb{R}^2} \hat{J}(\tau) \overline{h}(v) \cdot \phi_n^A \left( -\tau - v \right) d\tau dv.
\] (24)

and we have
\[
\mathcal{O}_A \left\{ (f \otimes h)(t) \right\}(u) = \phi_n^A \left( u \right) \cdot e^{\frac{m \mu}{\sqrt{3}}} \cdot F_A \left( u \right) \cdot H_A \left( u \right).
\] (25)

Corollary 2  Let \( f, h \in L^1(\mathbb{R}), F_A \) and \( H_A \) are the LCT of the signals \( f(t) \) and \( h(t) \) with parameters \( A_1 \). The convolution of two signals \( f(t), h(t) \) for the LCT with the Hermite weights \( \phi_n^A \left( \frac{m}{\sqrt{3}} \right) \) is defined by
\[
(f \otimes h)(t) = \frac{K_A^2 \left( E_{A_1}(t) \right)^{-1}}{\mu_n^A \cdot \sqrt{A}} \int_{\mathbb{R}^2} \hat{J}(\tau) \overline{h}(v) \cdot \phi_n^A \left( \frac{\sqrt{3}t - \tau - v}{\lambda} \right) d\tau dv.
\] (26)

and the following relation holds
\[
\mathcal{O}_A \left\{ (f \otimes h)(t) \right\}(u) = \phi_n^A \left( \frac{u}{\sqrt{3}} \right) \cdot F_A \left( \frac{u}{\sqrt{3}} \right) \cdot H_A \left( \frac{u}{\sqrt{3}} \right).
\] (27)
APPLICATIONS

Young’s Convolution Inequalities

Note that \(1 = |e^ith| = |E_A(t)| = |E_A^n(t)| = |\mu_1^n|\), and \(|f(t)| = |\overline{f(t)}|\). We derive the following theorem (see [1, 3]).

**Theorem 6**  Suppose that \(p, q, r, s \geq 1\), and \(\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, k \in \{1, 2, 3\} \). Then,

(i) \(\|f \otimes h\|_r \leq C_4\|\phi_h\|_s \cdot \|f\|_r \cdot \|h\|_l\), for any \(f, h \in L^1(\mathbb{R})\).

(ii) \(\|f \otimes h\|_r \leq C_5\|\phi_h\|_1 \cdot \|f\|_p \cdot \|h\|_q\), for any \(f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})\), where \(C_1, C_2, C_3, C_4\) are some positive constants.

**Proof.** We will present the proof for the case \(k = 3\). The cases \(k \in \{1, 2\}\) will be omitted because the proofs are analogous. Remind that \(\phi_h(t)\) are rapidly decreasing functions. By applying the Minkowski’s integral inequality and

changing variable we obtain

\[
\left[ \int_{\mathbb{R}} \left| f \otimes h(t) \right|^s dt \right]^{1/s} = \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(t) \overline{h}(v) \cdot \phi_h^{1/3} t \frac{1}{\sqrt{v^2 t^2}} \right) drdv \right]^{1/s}
\]

\[
= \left[ \int_{\mathbb{R}} \left| f(t) \overline{h}(v) \cdot \phi_h^{1/3} t \frac{1}{\sqrt{v^2 t^2}} \right|^s dt \right]^{1/s} \leq C_4 \|\phi_h\|_s \|f\|_{L^p} \|h\|_{L^q},
\]

where \(C_4\) is a constant. Thus, we obtain (i).

Now, we turn to the proof of (ii). Due to the formula (5) the convolution (22) can be also expressed as

\[
(f \otimes h)(t) = 3K_A(E_A(t))^{-1} \cdot \left( \overline{f(\sqrt{3}t)} \ast \overline{h(\sqrt{3}t)} \ast G_A(\sqrt{3}t) \right),
\]

where \(G_A(t) := \sqrt{3K_A} \cdot \frac{(m_1^1 + \xi)}{\sqrt{3}} \cdot E_A(t) + \kappa \cdot \phi_h^{1/3} t \frac{1}{\sqrt{v^2 t^2}} \). Remind that \(f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})\). By performing a change of variable, we realize that \(f(\sqrt{3}t) \in L^p(\mathbb{R}), h(\sqrt{3}t) \in L^q(\mathbb{R})\). Applying the Young’s inequality (6) for the case \(\frac{1}{r} + 1 = \frac{1}{r} + 1\), we have \(f(\sqrt{3}t) \ast h(\sqrt{3}t) \in L^r(\mathbb{R})\). Since the Hermite functions \(\phi_h(t)\) are rapidly decreasing functions then, applying the Young’s inequality (6) for the case \(\frac{1}{r} + \frac{1}{r} = \frac{1}{r} + 1\), we get \(f(\sqrt{3}t) \ast h(\sqrt{3}t) \ast G_A(\sqrt{3}t) \in L^r(\mathbb{R})\). Moreover, we also achieve

\[
\|f \otimes h\|_r \leq C_5 \|\phi_h\|_1 \cdot \|f\|_p \cdot \|h\|_q,
\]

where \(C_5\) is a positive constant. The proof is completed.

The Multiplicative Filter in the OLCT Domain

In this subsection, we will discuss an application of the new convolution to the design of multiplicative filters in the OLCT domain (see [7]). We only consider the convolution (19) when \(n = 0\). The Hermite function \(\phi_h(t)\) and the value \(\mu_0\) are given by

\[
\phi_h(t) = \frac{1}{\sqrt{\beta}} e^{-\frac{(t^2 + \mu_0^2)}{2p^2}}, \quad \mu_0 = e^{\frac{\beta}{2}},
\]

where \(\alpha, \beta, \theta\) can be taken from (8). We shall denote by \(r_{in}(t)\) and \(r_{out}(t)\) the input signal and output signal, respectively. From (20), the output signal can be expressed as

\[
r_{out}(t) = \bigcup_{A} \left\{ \sum_{n} \{ r_{in}(t) \} (u) \cdot e^{j\pi n u} e^{-\frac{\beta}{2} u^2 + 2^{(b_0 + d_0)} u} \cdot \phi_h^{1/3} u \ast H_A(u) \right\} (t).
\]
Let us now denote
\[
H_A(u) = e^{jm_2u}e^{-j\frac{\phi_0^A(u-m_2)}{\mu_0}u} \cdot \phi_0^A(u-m_2) \cdot H_A(u).
\] (30)

Then, it follows \(H_A(u) = e^{-jm_2u}e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} \cdot (\phi_0^A(u-m_2))^{-1} \cdot \overline{H_A(u)}\).

Based on different transforms \(\overline{H}(u)\), there are many ways to design a multiplicative filter. For instance, we can choose the function \(h(t)\) such that \(\overline{H}_A(u)\) is constant over \([-\Omega, \Omega]\), and zero or with rapid decay outside that region. Let \(T\) be a constant and

\[
\overline{H}_A(u) = \begin{cases} T, & u \in [-\Omega, \Omega] \\ 0, & u \not\in [-\Omega, \Omega]. \end{cases}
\] (31)

Thus, we obtain

\[
\text{output signal} = T \cdot \mathbb{R}^{-1}\left\{\mathbb{R}\{\text{input signal}\} \cdot \mathbb{R}\{\text{output signal}\}\right\}(t).
\]

From (3), the output signal \(\text{output signal}\) can be rewritten as

\[
\text{output signal} = K_A(\mathbb{E}_A{t})^{-1} \left(\mathbb{R}_t * \ell(t)\right),
\]

where the convolution function \(\ell(t)\) is given by

\[
\ell(t) = \frac{K_Ae^{j(m_2 - \frac{\phi_0^A}{\mu_0})u}}{2\pi} \cdot \int_{\mathbb{R}} \overline{h}(\nu) \cdot \mathbb{E}_A^{m_2} \cdot \phi_0^A \left(\frac{t - \nu - m_2}{A}\right) d\nu.
\] (33)

This shows that we can achieve the multiplicative filter through the classic Fourier convolution of \(\text{input signal}\) and \(\ell(t)\) in the time domain. A realization of the method is displayed in Figure 1 (see also [7]).

Using the expression (2), we obtain

\[
\begin{align*}
h(t) &= CK_A^{-1} \int_{\mathbb{R}} H_A(u)e^{-j\frac{(\mu_0u^2 + \frac{\phi_0^A(u-m_2)}{\mu_0}u)}{\mu_0^2}du} \cdot \overline{H}_A(u) \cdot \phi_0^A(u-m_2) \cdot H_A(u) e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} du \\
&= CK_A^{-1} \int_{\mathbb{R}} e^{-jm_2u}e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} \cdot (\phi_0^A(u-m_2))^{-1} \cdot \overline{H}_A(u) \cdot e^{-j\frac{\phi_0^A(u-m_2)}{\mu_0}u} \cdot \phi_0^A(u-m_2) \cdot \overline{H}_A(u) e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} du \\
&= CK_A^{-1}(\mathbb{E}_A{t})^{-1} \int_{\mathbb{R}} e^{-jm_2u}e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} \cdot (\phi_0^A(u-m_2))^{-1} \cdot \overline{H}_A(u) e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} du.
\end{align*}
\]

Then,

\[
\overline{h}(t) = CK_A^{-1} \int_{\mathbb{R}} e^{-jm_2u}e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} \cdot (\phi_0^A(u-m_2))^{-1} \cdot \overline{H}_A(u) e^{j\frac{\phi_0^A(u-m_2)}{\mu_0}u} du.
\] (34)
Substituting (34) into (33), gives rise to

\[ \ell(t) = \frac{K_A e^{i\frac{m_2}{\lambda}}}{\nu_0^A \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \bar{h}(v) \cdot \mathcal{E}_A^m(t - v) \cdot \phi_0^\dagger(t - v - m_2)dv \]

\[ = \frac{K_A e^{i\frac{m_2}{\lambda}}}{\nu_0^A \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \left( CK_{A^{-1}} \int_{\mathbb{R}} e^{-jm_1u} e^{i\frac{m_1}{\lambda} u + \frac{i(m_1 - m_2)}{\lambda} v} \cdot (\phi_0^\dagger(u - m_2))^{-1} \cdot \bar{H}_A(u) e^{i\frac{m_1}{\lambda} u} \right) \mathcal{E}_A^m(t - v) \cdot \phi_0^\dagger(t - v - m_2)dv \]

\[ = \frac{CK_{A^{-1}} e^{i\frac{m_2}{\lambda}}}{\nu_0^A \cdot \sqrt{\lambda}} \int_{\mathbb{R}} \left( e^{-jm_1u} (\phi_0^\dagger(u - m_2))^{-1} \cdot \bar{H}_A(u) \right) \mathcal{E}_A^m(t - v) \cdot \phi_0^\dagger(t - v - m_2)dvdu \]

By taking \( s = t - v \), it shows that

\[ K_A e^{i\frac{m_2}{\lambda} u} \int_{\mathbb{R}} \bar{H}_A(u) e^{i\frac{m_1}{\lambda} u} dv = e^{i\frac{m_1}{\lambda} u} K_A e^{i\frac{m_1}{\lambda} u + \frac{i(m_1 - m_2)}{\lambda} u} \int_{\mathbb{R}} e^{i\frac{m_1}{\lambda} u} \mathcal{E}_A^m(s) \left( e^{i\frac{m_1}{\lambda} u} \phi_0^\dagger(s - m_2) \right) ds \]

\[ = e^{i\frac{m_1}{\lambda} u} \mathcal{E}_A^m(\phi_0^\dagger(s - m_2)) \] (35)

Manipulating (15), we have

\[ K_A e^{i\frac{m_2}{\lambda} u} \int_{\mathbb{R}} e^{i\frac{m_1}{\lambda} u} \mathcal{E}_A^m(t - v) \cdot \phi_0^\dagger(t - v - m_2)dv = \nu_0^A \cdot \sqrt{\lambda} e^{-i\frac{m_2}{\lambda} u} e^{i\frac{m_1}{\lambda} u} \phi_0^\dagger(u - m_2) \] (36)

Therefore, \( \ell(t) = CK_{A^{-1}} \int_{\mathbb{R}} \bar{H}_A(u) e^{i\frac{m_1}{\lambda} u} du \). Based on the relation (31), we derive

\[ \ell(t) = CK_{A^{-1}} \int_{-\Omega}^{\Omega} Te^{i\frac{m_1}{\lambda} u} du = 2bCK_{A^{-1}} T \cdot \sin \left( \frac{\Omega}{t} \right) \]

Hence \( \ell(t) = \frac{T}{\pi} \cdot \sin \left( \frac{\Omega}{t} \right)\).

In the following example, we shall use the proposed multiplicative filter to restore an observed signal \( r_{in}(t) = X(t) + N(t) \), where \( X(t), N(t) \) denote the desired signal and the additive noise, respectively.
Example 2. We use \( r_i(t) = e^{-t^2} \cdot \sin(1.5t) + e^{t(t+10)^2} \), \( X(t) = e^{-t^2} \cdot \sin(1.5t) \), and \( N(t) = e^{t(t+10)^2} \). For convenience, let \( u_0 = \omega_0 = 0 \). The Wigner distributions of \( X(t) \) and \( r_i(t) \) are shown in Figure 2. Thus, we can choose (see [5]) \( a = -\frac{2}{3}, b = \frac{1}{3}, \Omega = 2 \). The transfer function reads

\[
\widehat{H}_A(u) = \begin{cases} 
1, & u \in [-2, 2] \\
0, & u \notin [-2, 2],
\end{cases}
\]

and \( \ell(t) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin \delta t}{\delta t} \). The output signal can be expressed as \( r_{out}(t) = \sqrt{\frac{3}{2\pi}} e^{it^2} \cdot (\widehat{r}_i * \ell)(t) \). The consequent result of the multiplicative filter is displayed in Figure 3.

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