Constraints For Topological Strings In $D \geq 1$

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Abstract

New relations of correlation functions are found in topological string theory; one for each second cohomology class of the target space. They are close cousins of the Deligne-Dijkgraaf-Witten’s puncture and dilaton equations. When combined with the dilaton equation and the ghost number conservation, the equation for the first chern class of the target space gives a constraint on the topological sum (over genera and (multi-)degrees) of partition functions. For the CP$^1$ model, it coincides with the dilatation constraint which is derivable in the matrix model recently introduced by Eguchi and Yang.

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1. Introduction

The solution of two dimensional quantum gravity [1] was a striking event toward an understanding of the nature of quantum geometry and also of non-perturbative string theory. Apart from the interest in its own right, it provides a ground to test new methods and techniques and it may become a source of inspiration. Namely, it promotes further investigations.

Use of matrix integral to count the number of triagulated surfaces was essential in the treatment. This integral makes possible to see that the scaling fields generate the flows of the KdV hierarchy [2]. Moreover, the Schwinger-Dyson equations lead to a system of constraints on the partition function — the Virasoro constraints [3]. However, in gravity or string theory in physically interesting dimensions it is difficult to gain some definite results by a direct generalization of the method. It seems that we must take other approaches.

Soon after the solution of [1], it is pointed out [4, 5] that topological gravity with perturbation gives results identical to those of the matrix model. This was a surprise since topological gravity counts numbers of interest in intersection theory on the moduli space of Riemann surfaces. Amazingly, relations of intersection numbers derived by a purely geometric argument coincides with the first two of the Virasoro constraints; the puncture equation [6] gives the string equation $L_{-1}Z = 0$ and the dilaton equation [7] together with dimensional consideration gives the dilatation constraint $L_0Z = 0$. The higher constraints $L_nZ = 0$ could be derived by a rather involved treatment of quantum field theory [8]. It is also notable that a certain matrix integral [9] makes transparent the relation of the intersection theory and the KdV hierarchy. In addition, there exists considerable evidence [10, 11] that the twisted $N = 2$ minimal superconformal models coupled to topological gravity (which we refer to as the minimal models) have similar structures and are equivalent to the physical models of matrix chains. These successful observations invite us to take topological field theory as an alternative method in gravity or string theory.

Topological strings are the coupled systems of topological gravity and topological sigma models [12]. A model is determined for each choice of compact complex or almost complex manifold as a target space. The $k$-th minimal model can formally be considered as a topological string theory with the target space of dimension $\frac{k}{k+2}$ [14]. It is an interesting problem to see to what extent the structures found in the minimal models — such as integrable flows or as Virasoro constraints — can be generalized to topological string theories in dimension $\geq 1$ [1].

Recently, Eguchi and Yang have proposed a matrix model for the topological string theory with $\mathbb{C}P^1$ target (the $\mathbb{C}P^1$ model) [13]. The Schwinger-Dyson equations would impose several constraints on the partition function. A natural question to ask is whether these can be derived by a purely geometric argument. The string equation [13] is derivable by the puncture equation that holds in any topological string theory [9]. However, the

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1In this paper, by dimension of a space, we always mean its complex dimension.
dilatation constraint (associated with rescaling of matrices) is not derivable only from the dimensional consideration and the dilaton equation. This is because the dimension of the moduli space depends on the degree of the maps and the full partition function is the sum of contributions from all genera and all degrees. Therefore, we need other equations with degree dependence. In this paper, we find such equations in general models. In the CP\(^1\) model, the constraint derived from these equations agrees with the dilatation constraint for the matrix integral.

The rest of the paper is organized as follows. In section 2, we derive the new relations of correlation functions. It turns out that they give rise to constraints on the topological sum generalizing the first two of the Virasoro constraints. In section 3, we compare our constraints for the CP\(^1\) model with the results obtained by matrix model of [13]. Section 4 provides a field theoretical background for the description of the correlation functions used in section 2.

2. New Equations Associated With 2nd Cohomology Classes

This section contains the main result of the paper. We refer the reader to section 4 for some field theoretical origin of the geometric description of the amplitudes.

The bosonic elementary fields of topological string theory consists of a Riemannian metric on a compact oriented surface and a map of the surface to the target space \(M\). The topological type of such fields is classified by the genus of the surface and the degree of the map. The degree \(d\) of a map \(f: \Sigma \to M\) is defined here as the homology class \(d = f_*[\Sigma] \in H_2(M; \mathbb{Z})\) of the image. If \(H_2(M; \mathbb{Z}) \cong \mathbb{Z}\), by choosing a generator \(\omega \in H^2(M; \mathbb{Z})\), it is specified by an integer

\[
\int d \omega = \int_{\Sigma} f^* \omega. \tag{2.1}
\]

If \(H_2(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}\) (\(r\)-components), choosing a base \(\omega_1, \cdots, \omega_r\) of \(H^2(M; \mathbb{Z})\), it is specified by a multiple of integers \((d_1, \cdots, d_r)\) defined by \(d_i = \int_{\Sigma} \omega_i\).

Physical fields in the theory are classified into primary fields and their gravitational descendants. Primary fields are in one to one correspondence with the de Rham cohomology classes of the target space. We denote by \(\sigma_n(\Omega)\) the \(n\)-th descendant of the primary field (= \(\sigma_0(\Omega)\)) corresponding to \(\Omega \in H^*(M; \mathbb{C})\).

2.1 Topological String Amplitudes

The instanton calculus reduces the functional integration for a physical amplitude to an integration over a finite dimensional moduli space of Riemann surfaces with holomorphic maps. It seems that the moduli space of stable maps introduced in [13, 16] is the natural one to appear here. We recall the definition:

\[
\mathcal{M}_{g,a}(M, d) = \left\{ (\Sigma, x_1, \cdots, x_a, f) \right\}/ \cong, \tag{2.2}
\]
A representative of an element consists of a Riemann surface $\Sigma$ of genus $g$ (possibly with ordinary double points), $s$-distinct marked points in $\Sigma$ and a holomorphic map $f : \Sigma \to M$ such that every genus 0 (resp. 1) component which maps to a point contains at least 3 (resp. 1) marked or singular points. For each $i$, we have the evaluation map

$$\phi_i : \mathcal{M}_{g,s}(M,d) \to M, \quad [\Sigma, x_1, \ldots, x_s; f] \mapsto f(x_i). \quad (2.3)$$

Also, the cotangent space $T^*_\Sigma$ varies as the fibre of a complex line bundle $L_{(i)}$ over $\mathcal{M}_{g,s}(M,d)$. At the point $[\Sigma, x_1, \ldots, x_s; f]$ with smooth $\Sigma$, denoting by $T$ the tangent space to $\mathcal{M}_{g,s}(M,d)$, we have the exact sequence

$$0 \to H^0(f^*T_M) \to T \to H^1(T^*_\Sigma) \overset{j^*}{\to} H^1(f^*T_M), \quad (2.4)$$

where $T_\Sigma$ and $T_M$ are the tangent bundles of $\Sigma$ and $M(\mathbb{P})$ and the last map is induced by the homomorphism $T_\Sigma \to f^*T_M$; $v^\flat \mapsto v^\flat \partial_2 f^\flat$. Hence, the dimension of the tangent space $T$ is the sum of the dimension $T$ of the cokernel of $f_*$ and the following virtual dimension (the ghost number anomaly):

$$\int_d c_1(M) + (3 - \dim M)(g - 1) + s. \quad (2.5)$$

Under these definitions, the degree $d$ contribution to the $g$-loop $s$-point physical amplitude is given as follows (See §4). Let us take $\Omega_i \in H^{2g_i}(M)$ ($i = 1, \ldots, s$).

(I) If $\mathcal{M}_{g,s}(M,d)$ is empty or if the ghost number $\sum (n_i + q_i)$ does not match the virtual dimension $\langle 2.3 \rangle$, we have

$$\langle \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d} = 0. \quad (2.6)$$

(II) If the ghost number match the virtual dimension, we have

$$\langle \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d} = \prod_{i=1}^s n_i! \int_{\mathcal{M}_{g,s}(M,d)} c_T(V_{g,s,d}) \prod_{i=1}^s c_1(L_{(i)})^{n_i} \phi_i^* \Omega_i, \quad (2.7)$$

where $V_{g,s,d}$ is the rank $T$ vector bundle described as follows: At $X = [\Sigma, x_1, \ldots, x_s; f]$ with smooth $\Sigma$, the fibre $V|_X$ is the cokernel of the map $f_*$ in $(2.4)$. The dual is the space of anti-ghost zero modes, i.e. the space of holomorphic differentials with values in $f^*T_M$ that are annihilated by $df : \rho_{zi} \mapsto \rho_{zi} \partial_z f^i$. Namely, $V^*|_X = H^0(\Sigma, F)$ where $F$ is the kernel of the sheaf map $df$:

$$0 \to F \to K_2 \otimes f^*T^*_M \overset{df}{\to} K^2_2, \quad (2.8)$$

in which $K_2$ is the sheaf holomorphic differentials on $\Sigma$ (or the cotangent bundle of $\Sigma$). This definition of $V^*|_X$ is applicable to the case in which $\Sigma$ is singular, by considering $K_2$ as the dualizing sheaf $\mathcal{O}(\mathbb{P})$.

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2 In this paper, in order to simplify the notation, we denote the sheaf of germs of holomorphic sections of a holomorphic vector bundle $E$ by the letter $E$ itself.

3 Let $U$ be a neighborhood of a double point $x_0$ and $U_1, U_2 \subset U$ be the two branches with coordinates $z, w$ ($zw = 0$). Denoting $f|_{U_1}$ by $f_1$, one can define $df_1 = z\psi \partial_z f_1 \frac{dz}{\partial z}$ and $df_2 = -w\psi \partial_w f_2 \frac{dw}{\partial w}$ where $\psi = \frac{dz}{z} = -\frac{dw}{w}$. Then, $df$ is defined on $U$ as $df_1 + df_2$ and $(2.8)$ makes sense.
Examples

(i) \( \overline{\mathcal{M}}_{g,s}(M,d) \) is empty when \( g = 0, s \leq 2 \) and \( d = 0 \) or \((g, s, d) = (1, 0, 0)\).

(ii) If \( \Sigma \) is smooth, the differential \( df \) determines a one (or zero) dimensional subbundle of \( K_\Sigma \otimes f^*T_M \) with the section \( df \) and hence a subbundle \( \ell_f \) of \( f^*T_M \). The latter defines the normal bundle \( N_f = f^*T_M / \ell_f \) and the conormal bundle \( N_f^\perp = (\ell_f)^\perp \subset f^*T_M \). Then, we have \( \mathcal{F} = K_\Sigma \otimes N_f^* \) and \( \mathcal{V}|_X = H^1(\Sigma, N_f) \).

(iii) The degree zero moduli space factorizes as

\[
\overline{\mathcal{M}}_{g,s}(M,0) = \overline{\mathcal{M}}_{g,s} \times M ,
\]

where \( \overline{\mathcal{M}}_{g,s} \) is the moduli space of stable Riemann surfaces with marked points. Since \( df = 0 \) for a degree zero map, \( N_f \) is the trivial bundle \( \Sigma \times T_x M; x = f(\Sigma) \), and hence

\[
\mathcal{V}|_{[\Sigma, x_1, \ldots, x_s, f]} = H^0(\Sigma, K_\Sigma)^* \otimes T_x M.
\]

As is well known, \( H^0(\Sigma, K_\Sigma) \) has dimension \( g \) if \( \Sigma \) is smooth. If \( \Sigma \) has a double point \( x_+ \), the holomorphic section of \( K_\Sigma \) is admitted to have a simple pole at \( x_+ \) with opposite residues on the two branches. Calculation using the Riemann-Roch formula gives again \( \dim H^0(\Sigma, K_\Sigma) = g \). Anyway, \( \mathcal{V}_{g,s,0} \) is a vector bundle whose rank is \( g \dim M \).
Remarks

(i) One crucial remark is that in general the moduli space $M_{g,s}(M,d)$ is not smooth and it is not obvious whether (2.7) makes sense. However, when $M$ satisfy a certain condition (called convexity [15]), the smoothness (as an orbifold) for $g = 0$ is proved [16] and hence (2.7) makes sense. Modification will be required in some other cases but we proceed expecting that it would not affect our main conclusion given below for a certain class of target spaces.

(ii) If a generic element of $M_{g,s}(M,d)$ has non-trivial automorphisms, the right hand side of (2.7) should be divided by the order of the automorphism group. In particular, since a generic elliptic curve with one point has order two automorphism, degree zero contribution is given by

$$\langle \sigma_n \phi \rangle_{1,0} = \frac{1}{2} n! \int c_1(L)^n \phi^* \Omega.$$  

(iii) There may be a "pathological" case in which (a component of) the moduli space has dimension less than the dimension $\int c_1(M) + (3 - \dim M)(g - 1) + T$ of each tangent space $\mathcal{T}$. This happens for example when $M = \mathbb{CP}^3$, $g = 24$ and $d = 14$ [17]. In such a case, the number of ghost zero modes is greater than the number of moduli. Not having a good remedy at present, we simply put zero the contribution of such component. This automatically follows from the expression (2.7).

2.2 The New Equation

We derive a new relation of correlation functions (i.e. string amplitudes) associated with the primary field $\sigma_0(\omega)$ for each 2nd cohomology class $\omega \in H^2(M)$. It expresses the $s + 1$ point function

$$\langle \sigma_0(\omega) \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d}$$  

by the sum of certain $s$ point functions. As in the derivation of the puncture- and the dilaton equations [6, 7], use of the forgetful map

$$\pi : \mathcal{M}_{g,s+1}(M,d) \longrightarrow \mathcal{M}_{g,s}(M,d),$$  

is essential. We assume that the space $\mathcal{M}_{g,s}(M,d)$ is non-empty for a while and shall deal with the empty case separately.

We denote the image of $[\Sigma, x_0, x_1, \cdots, x_s; f]$ under $\pi$ by $[\Sigma', x_1, \cdots, x_s; f']$: “0” is the mark for the point to be forgotten. $\Sigma'$ is obtained by contracting (if exists) the genus 0 and degree 0 component of $\Sigma$ containing $x_0$ and only two other marked or singular points. We denote by $L_{(i)}$ (resp. $L'_{(i)}$) the line bundle over $\mathcal{M}_{g,s+1}(M,d)$ (resp. $\mathcal{M}_{g,s}(M,d)$) associated with the $i$-th point and by $\phi_i$ (resp. $\phi'_i$) the evaluation map of $\mathcal{M}_{g,s+1}(M,d)$ (resp. $\mathcal{M}_{g,s}(M,d)$) at the $i$-th point. As is explained in [6], $c_1(L_{(i)})$ for $i \neq 0$ is different from the pull back $\pi^*c_1(L'_{(i)})$ but is given by

$$c_1(L_{(i)}) = \pi^*c_1(L'_{(i)}) + [D_i].$$  

Here, $D_i$ is the divisor in $\mathcal{M}_{g,s+1}(M,d)$ consisting of those configurations which include a genus 0 and degree 0 component containing only the 0-th point, the $i$-th point and
one node. Also it is shown that $c_1(L_{(i)})[D_i] = 0$. Since $\pi$ sends the evaluation maps; $\phi_i = \phi_i' \circ \pi$, we have

$$\phi_i^* \Omega_i = \pi^* \phi_i'^* \Omega_i, \quad i = 1, \ldots, s.$$  \hfill (2.14)

Combining these observations, we have

$$c_1(L_{(1)})^{-n_1} \phi_1^* \Omega_1 \cdots c_1(L_{(s)})^{-n_s} \phi_s^* \Omega_s = \pi^* \left\{ c_1(L'_{(1)})^{-n_1} \phi_1'^* \Omega_1 \cdots c_1(L'_{(s)})^{-n_s} \phi_s'^* \Omega_s \right\}$$

$$+ \sum_{i=1}^{s} [D_i] \pi^* \left\{ \cdots c_1(L'_{(i)})^{-n_{i-1}} \phi_i'^* \Omega_i \cdots \right\}. \hfill (2.15)$$

One can also show that $V_{g,s+1,d}$ is the pull back of $V_{g,s,d}$ and hence

$$c_T(V_{g,s+1,d}) = \pi^* c_T(V_{g,s,d}). \hfill (2.16)$$

This is essentially because the construction of $V_{g,s,d}$ does not refer to the marked points. However, one should be careful if $\Sigma \neq \Sigma'$. This is the case when (A) $[\Sigma, x_0, \ldots, x_s; f] \in [D_i]$ for some $i = 1, \ldots, s$ or when (B) $\Sigma$ includes a genus 0 degree 0 component containing only $x_0$ and two nodes, say $x_0$ and $y_s$. In each case, $H^0(\Sigma, F)$ is canonically isomorphic to $H^0(\Sigma', F')$ since $H^0(CP^1_0, K) = H^0(CP^1_0, K \otimes O(1)) = 0$ for (A) and $H^0(CP^1_0, K \otimes O(x_s) \otimes O(y_s)) = C dz/z$ for (B) where $CP^1_0$ is the degree 0 component containing $x_0$ and $z$ is the coordinate of $CP^1_0$ with $z(x_*) = 0$ and $z(y_*) = \infty$.

The identities (2.15) and (2.16) lead to the puncture and the dilaton equations:

$$\langle P \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d} = \sum_{i=1}^{s} n_i \langle \sigma_{n_i-1}(\Omega_i) \prod_{j \neq i} \sigma_{n_j}(\Omega_j) \rangle_{g,d}, \hfill (2.17)$$

$$\langle \sigma_1(P) \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d} = (2g - 2 + s) \cdot \langle \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d}, \hfill (2.18)$$

where we denote $\sigma_0(1) = P$ and $\sigma_n(1) = \sigma_n(P)$. The latter equation is due to $L_{(0)}[\text{fibre of } \pi] = K \otimes \prod_{i=1}^{s} O(x_i)$ and $c_1(L_{(0)})[D_i] = 0$ [7]. Finally, we perform the integration along the fibre of $\pi$ with $\phi_0^* \omega$-insertion. Integration of the sole form $\phi_0^* \omega$ gives the topological number $\int_d \omega$. One can easily see

$$\phi_0^* \omega [D_i] = [D_i] \pi^* \phi_i'^* \omega. \hfill (2.19)$$

Hence, we have the equation

$$\langle \sigma_0(\omega) \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d} = \int_d \omega \cdot \langle \sigma_{n_1}(\Omega_1) \cdots \sigma_{n_s}(\Omega_s) \rangle_{g,d}$$

$$+ \sum_{i=1}^{s} n_i \langle \sigma_{n_i-1}(\omega \wedge \Omega_i) \prod_{j \neq i} \sigma_{n_j}(\Omega_j) \rangle_{g,d}. \hfill (2.20)$$

Here, $\omega \wedge \Omega_i$ is the product in the classical cohomology ring $H^*(M; \mathbb{C})$. Note that $\sigma_0(\omega)$ counts the degree of the maps (the first term) as the dilaton field counts the Euler number of the surfaces, and it also has contact interactions with other fields (the succeeding terms) as the puncture has.
Remark. Actually, for \((g, s, d) = (1, 1, 0)\) the relation (2.13) does not hold but \([D_i]\) must be replaced by \(\frac{1}{2}[D_i]\) due to the \(\mathbb{Z}_2\)-symmetry. However, on account of the definition of one point functions (remark (ii) in §2.1), the equations (2.17), (2.18) and (2.20) hold without modification.

The Exceptional Cases

We consider here, as we have promised, the case in which \(\mathcal{M}_{g,s}(M,d)\) is empty but \(\mathcal{M}_{g,s+1}(M,d)\) is not: a) \((g, s, d) = (0, 2, 0)\) and b) \((g, s, d) = (1, 0, 0)\).

a) The moduli space is just \(\mathcal{M}_{0,3}(M,0) = M\). Hence, the gravitational descendants \(\sigma_n(\Omega), n > 0\) decouples and we have

\[
\langle P \sigma_0(\Omega_1) \sigma_0(\Omega_2) \rangle_{0,0} = \int_M \Omega_1 \wedge \Omega_2,
\]

\[
\langle \sigma_0(\omega) \sigma_0(\Omega_1) \sigma_0(\Omega_2) \rangle_{0,0} = \int_M \omega \wedge \Omega_1 \wedge \Omega_2.
\]

b) The moduli space is \(\mathcal{M}_{1,1}(M,0) = \mathcal{M}_{1,1} \times M\). Recall the example (iii) of §2.1. On a smooth torus, a cotangent vector at any point determines a holomorphic differential on the whole surface. Even when there is a double point, this is also the case if the chosen point is non-singular. Thus, \(H^0(\Sigma, K)\) is isomorphic to \(T^*_x \Sigma\) for every \([\Sigma, x_0] \in \mathcal{M}_{1,1}\) and hence we have \(\mathcal{V}_{1,1,0} \cong \mathcal{L}^{-1}_{(0)} \otimes T_M\). So, the Euler class is expressed as

\[
c_T(\mathcal{V}_{1,1,0}) = c_{\dim M}(M) - c_1(\mathcal{L}_{(0)}) c_{\dim M-1}(M) + \cdots.
\]

Since the one point function \(\frac{1}{2} \int_{\mathcal{M}_{1,1}} c_1(\mathcal{L}_{(0)})\) of topological gravity is \(\frac{1}{24}\), we have

\[
\langle \sigma_1(\Omega) \rangle_{1,0} = \frac{1}{24} \chi(M),
\]

\[
\langle \sigma_0(\omega) \rangle_{1,0} = -\frac{1}{24} \int_M \omega \wedge c_{\dim M-1}(M).
\]

Note that \(\langle P \rangle_{1,0} = 0\) on dimensional ground.

2.3 The Constraints On The Topological Sum

Consider the theory perturbed by the physical fields. We choose a base \(\{\Omega_\alpha\}_\alpha\) of the cohomology group \(H^*(M; \mathbb{C})\) and denote by \(t^n_\alpha\) the coupling constant for the field \(\sigma_n(\Omega_\alpha)\). The genus \(g\) degree \(d\) free energy is defined as the formal series

\[
F_{g,d}(t) = \left\langle \exp \left( \sum_{n,\alpha} t^n_\alpha \sigma_n(\Omega_\alpha) \right) \right\rangle_{g,d} = \sum_{\{m_n^\alpha\}} \prod_{n,\alpha} \frac{(t^n_\alpha)^{m_n^\alpha}}{m_n^\alpha!} \left\langle \prod_{n,\alpha} \sigma_n(\Omega_\alpha)^{m_n^\alpha} \right\rangle_{g,d}.
\]

We wish to derive constraints on the topological sum

\[
F = F(t; \lambda, \Theta) = \sum_{g,d} \lambda^{2g-2} e^{\int_{\mathbb{S}^d} \Theta} F_{g,d}(t),
\]
where $\lambda$ is the string coupling constant and $\Theta$ is valued in some subset of $H^2(M; \mathbb{C})$. The puncture equations (2.17), the dilaton equations (2.18), and the new equations (2.20) with various insertions are compiled into the following three respectively:

$$
\frac{\partial F_{g,d}}{\partial t_0} = \sum_{n,\alpha} n t_n^\alpha \frac{\partial F_{g,d}}{\partial t_n^\alpha} + \frac{1}{2} \sum_{\alpha,\beta} \eta_{\alpha\beta} t_0^{\alpha} t_0^{\beta} \delta_{0,0},
$$

$$
\frac{\partial F_{g,d}}{\partial t_1} = (2g - 2) F_{g,d} + \sum_{n,\alpha} t_n^\alpha \frac{\partial F_{g,d}}{\partial t_n^\alpha} + \frac{\chi(M)}{24} \delta_{1,0},
$$

$$
\omega^\alpha \frac{\partial F_{g,d}}{\partial t_0^\alpha} = \int_d \omega F_{g,d} + \sum_{n,\alpha,\beta} n C_{\alpha \beta} t_n^{\alpha} \frac{\partial F_{g,d}}{\partial t_n^{\beta}} + \frac{\delta_{0,0}}{2} \sum_{\alpha,\beta} C_{\alpha \beta} t_0^{\alpha} t_0^{\beta} - \frac{\delta_{1,0}}{24} \int_M \omega c_{\dim M - 1}(M)
$$

where $\eta_{\alpha\beta} = \int_M \Omega_{\alpha} \Omega_{\beta}$, $\omega \Omega_{\alpha} = \sum \Omega_{\beta} C_{\alpha \beta}^\beta$ and $C_{\alpha \beta} = \int_M \omega \Omega_{\alpha} \Omega_{\beta} = \eta_{\beta\gamma} C_{\alpha \gamma}$. The inhomogeneous terms have come from the exceptional contributions (2.21), (2.22), (2.24) and (2.25). Note also that the dimensional consideration ($\S$2.1 (I)) gives the selection rule

$$
\sum_{n,\alpha} (n + q_\alpha - 1) t_n^{\alpha} \frac{\partial F_{g,d}}{\partial t_n^{\alpha}} = \left( \int_d c_1(M) + (3 - \dim M)(g - 1) \right) F_{g,d},
$$

where $2q_\alpha$ denotes the dimension of the base element $\Omega_{\alpha}$.

Now we can write down two constraints on the topological sum $F$ which do not involve derivative with respect to $\lambda$ nor $\Theta$. The puncture equation (2.28) directly gives

$$
\sum_{n,\alpha} n t_n^{\alpha} \frac{\partial F}{\partial t_n^{\alpha}} + \frac{\lambda^2}{2} \sum_{\alpha,\beta} \eta_{\alpha\beta} t_0^{\alpha} t_0^{\beta} = 0.
$$

The dilaton equation (2.29) and the selection rule (2.31) together with the new equation (2.30) for $\omega = c_1(M)$ give

$$
\sum_{n,\alpha} (n + q_\alpha + \frac{1 - \dim M}{2} t_n^{\alpha} \frac{\partial F}{\partial t_n^{\alpha}} + \sum_{n,\alpha,\beta} n C_{M\alpha}^{\beta} t_n^{\alpha} \frac{\partial F}{\partial t_n^{\beta}}
$$

$$
+ \frac{\lambda^2}{2} \sum_{\alpha,\beta} C_{M\alpha}^{\beta} t_0^{\alpha} t_0^{\beta} + \frac{1}{24} \left( \frac{3 - \dim M}{2} \chi(M) - \int_M c_1(M)c_{\dim M - 1}(M) \right) = 0.
$$

In the above expressions, we have used the parameters $\tilde{t}_n^{\alpha}$ defined by

$$
\tilde{t}_n^{\alpha} = \begin{cases} t_1^{P} - 1, & \text{if } (n, \alpha) = (1, P) \\ t_n^{\alpha}, & \text{otherwise}. \end{cases}
$$

Also, we have denoted $C_{c_1(M)\alpha}^{\beta}$ etc. by $C_{M\alpha}^{\beta}$ etc. for brevity.

At a glance, we see that (2.32) and (2.33) are expressed as $L_{-1} e^F = 0$ and $L_0 e^F = 0$ respectively, using first order differential operators $L_0$ and $L_{-1}$ on the space of coupling constants $\{t_n^{\alpha}\}$. We can check that the operators satisfy

$$[L_0, L_{-1}] = L_{-1}.$$
Consider a hypothetical space $M_k$ such that $\dim M_k = \frac{k}{k+2}$, $\chi(M_k) = k + 1$, $c_1(M_k) = 0$ and $H^*(M_k)$ is a $k + 1$ dimensional space generated by $\{\Omega_\alpha\}_{\alpha=0,1,...,k}$ with $q_\alpha = \frac{\alpha}{k+1}$ and $\eta_{\alpha\beta} = \delta_{\alpha+\beta,k}$. Then we see that (2.32) and (2.33) are precisely the first two of the Virasoro constraints for the $k$-th minimal model (see [10, 18]). In this sense, we may consider (2.32) and (2.33) as the proper generalization of the string equation and the dilatation constraint to topological strings in dimension $\geq 1$.

3. The CP\textsuperscript{1} Model

Eguchi and Yang have recently proposed a matrix model for the CP\textsuperscript{1} model [13]. In this section, we derive some Schwinger-Dyson equations for the matrix integral and compare them with the constraints obtained in the preceding section.\footnote{This section is totally based on a discussion with T. Eguchi.}

### 3.1 The Eguchi-Yang Model

Let $N \in \mathbb{R}$ and $\nu \in \mathbb{N}$ be large numbers of the same order. We consider the integration over $\nu \times \nu$ ‘hermitian’ matrices

$$Z(\nu; N) = \int d^\nu M e^{N\text{tr}V(M)}, \quad (3.1)$$

where

$$V(M) = -2M(\log M - 1) + \sum_{n=1}^{\infty} 2t_n^p M^n(\log M - a_n) + \sum_{n=1}^{\infty} \frac{1}{n^q} M^n; \quad (3.2)$$

with $a_n = \sum_{j=1}^{n} \frac{1}{j}$. Though the logarithm is not defined for a general hermitian matrix, we could give a definition to the integral (3.1) by taking the integration region away from the set of hermitian matrices. Namely, we replace (3.1) by

$$Z(\nu; N) = \frac{1}{\nu!} \int_{\mathbb{C}^\nu} \prod_{i=1}^{\nu} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{N\sum_{i=1}^{\nu} V(\lambda_i)}, \quad (3.3)$$

in which, instead of the real line, the contour $\mathcal{C}$ is chosen so that the integrand and the integral make sense. (Here we have changed the normalization by multiplying a $\nu$-dependent factor.) However, not having a particularly good choice, we do not specify the contour $\mathcal{C}$ and, in what follows, we develop a formal argument by assuming that the integration by parts does not pick up the boundary contribution.

The orthogonal polynomials are defined by

$$\psi_n(\lambda) = \lambda^n + \text{lower order terms}, \quad (3.4)$$

$$\int_{\mathbb{C}} d\lambda e^{N\text{tr}V(\lambda)} \psi_n(\lambda)\psi_m(\lambda) = \delta_{n,m}h_n. \quad (3.5)$$
Denoting the norm squared $h_n$ by $e^{N \phi_n}$, the partition function is written as

$$Z(n; N) = h_0 \cdots h_{n-1} = h_0^n \exp \left\{ \sum_{n=1}^{\nu-1} \left( \frac{\nu}{N} - \frac{n}{N} \right) N^2 (\phi_n - \phi_{n-1}) \right\}.$$  \hspace{1cm} (3.6)

We denote by $u(n; N)$ the derivative $\phi_n$ of $\phi_n$ with respect to $\frac{N}{N}$, in which $\phi_n$ is considered as a function of $\frac{N}{N}$ and $N$. We also denote $\int d\lambda e^{N\nu \lambda} (\psi_n(\lambda))^2 = v(n; N)$. The basic ansatz of \cite{13} is that, under the identification $\frac{N}{N} \Rightarrow t_0^P$, $u$ and $v$ are the two point functions of the CP$^1$ model at some value of $\Theta$

$$u = \langle PP \rangle,$$

$$v = \langle PQ \rangle,$$  \hspace{1cm} (3.7)

with $N^{-1}$ being the string coupling constant. Here $Q$ is the primary field corresponding to the Kähler form of CP$^1$ of volume 1. In particular $N^2 u$ and $N^2 v$ are the derivatives of the free energy $F$ that is expanded with respect to the genus

$$F\left( \frac{n}{N} \right) = \sum_{g=0}^{\infty} N^{2-2g} F_g \left( \frac{n}{N} \right).$$  \hspace{1cm} (3.8)

Starting with this ansatz, Eguchi and Yang derived for small $g$ ($g = 0, 1, 2, 3, 4$) the flow equations that can be anticipated by a general argument \cite{19} (for $g = 0, 1$ they coincides with the topological recursion relations of \cite{3}). Also, just as in the matrix model for 2D gravities, they derived the string equation — the equation obtained by differentiating our string equation (2.32) with respect to $t_0^P$ and $t_0^Q$.

### 3.2 The Constraints

In this subsection, we give further confirmation of (3.7) by deriving from the matrix model (certain derivatives of) the dilatation constraint (2.33) and the dilaton equation (2.29) for the CP$^1$ model. As a preparation, we express the ansatz in terms of the free energies of the models. We introduce a function $f(x; N)$ such that $N^2(\phi_n - \phi_{n-1}) = \frac{1}{N} f''(\frac{n}{N}; N)$. Using the Euler-Maclaurin formula, we have the following expression for the free energy $F^M = \log Z$ of the matrix model:

$$F^M(n; N) = \nu \log h_0 + \frac{1}{N} \sum_{n=1}^{\nu-1} \left( \frac{\nu}{N} - \frac{n}{N} \right) f''(\frac{n}{N})$$

$$= \nu \log h_0 + \int_0^{\frac{\nu}{N}} \left( \frac{\nu}{N} - y \right) f''(y) dy - \frac{1}{2N} \left. \frac{\nu}{N} f''(0) \right|_0^{\frac{\nu}{N}}$$

$$+ \sum_{r \geq 1} \left( -1 \right)^{r+1} B_r \frac{1}{2N} \left( \left( \frac{\nu}{N} - y \right) f''(y) \right) (2r-1) \left|_0^{\frac{\nu}{N}} \right.$$

$$= N x \log h_0 + f(x) - f(0) - x f'(0) - \frac{1}{2N} x f''(0)$$

$$+ \sum_{r \geq 1} \left( -1 \right)^{r+1} B_r \frac{1}{2N} \left[ (2r-1) \left( f^{(2r)}(x) - f^{(2r)}(0) \right) + x f^{(2r+1)}(0) \right]$$  \hspace{1cm} (3.10)
where \( B_r \) are the Bernoulli numbers and \( x = \frac{\nu}{N} \). On the other hand, the ansatz (3.7) enables us to put

\[
f(x; N) = F(x; N) - \frac{1}{2N} F'(x; N) + \cdots + \frac{(-1)^r}{(r+1)! N^r} F^{(r)}(x; N) + \cdots,
\]

where \( F \) is the free energy of the CP\(^1\) model.

**The Virasoro Constraints**

Consider the differential operators

\[
\mathcal{L}_n = \sum_{i,j=1}^{\nu} \frac{\partial}{\partial M_{ij}} (M^{n+1})_{ij}, \quad n \geq 1,
\]

acting on functions of the matrix \( M \), where \((M^{n+1})_{ij}\) stands for the \((i,j)\)-th component of the \((n+1)\)-th power of \( M \). They satisfy the commutation relations \([\mathcal{L}_n, \mathcal{L}_m] = (m - n)\mathcal{L}_n\).

When \( \mathcal{L}_n \) for \( n \geq 0 \) are applied to the weight \( e^{\nu t^V(M)} \), the results can be expressed as the responses to differential operators on the space of coupling constants \( t^P_{n+1}, t^Q_n \):

\[
\mathcal{L}_n e^{\nu t^V(M)} = \mathcal{L}_n e^{\nu t^V(M)}, \quad n \geq 0,
\]

where

\[
\mathcal{L}_n = \frac{1}{N^2} \sum_{l=1}^{n-1} l(n-l) \frac{\partial}{\partial t^P_{l-1}} \frac{\partial}{\partial t^Q_{n-l-1}} + \sum_{m=0}^{\infty} \left( m \tilde{t}^P_m \frac{\partial}{\partial t^P_{m+n}} + (m + n + 1) t^Q_m \frac{\partial}{\partial t^Q_{m+n}} \right)
\]

\[
+ 2 \sum_{m=0}^{\infty} (m + n)(1 - m(a_{m+n} - a_m)) \tilde{t}^P_m \frac{\partial}{\partial t^Q_{m+n-1}} \quad n \geq 1,
\]

and

\[
\mathcal{L}_0 = \sum_{m=0}^{\infty} \left( m \tilde{t}^P_m \frac{\partial}{\partial t^P_m} + (m + 1) t^Q_m \frac{\partial}{\partial t^Q_m} \right) + 2 \sum_{m=1}^{\infty} m \tilde{t}^P_m \frac{\partial}{\partial t^Q_{m-1}} + N^2 x^2,
\]

in which \( x = t^P_0 = \frac{\nu}{N} \) and \( \tilde{t}^P_m \) is defined as in (2.34). Due to the translational invariance of the measure \( d^2 M \) (or \( \prod d\lambda_i \)), the relation (3.13) leads to the following constraints on the partition function:

\[
\mathcal{L}_n Z = 0, \quad n \geq 0.
\]

As a consequence of the commutation relations of \( \mathcal{L}_n \), we find

\[
[\mathcal{L}_n, \mathcal{L}_m] = (n - m)\mathcal{L}_{n+m}, \quad n, m \geq 0.
\]

Hence, (3.16) may be called the Virasoro constraints.

**The Rescaling Constraint**

Rescaling \( N \) with \( \nu = Nx \) being fixed, we obtain the equation

\[
\left( N \frac{\partial}{\partial N} - x \frac{\partial}{\partial x} \right) Z = \sum_{n=0}^{\infty} \left( \tilde{t}^P_{n+1} \frac{\partial}{\partial t^P_{n+1}} + t^Q_n \frac{\partial}{\partial t^Q_n} \right) Z.
\]
The right hand side comes out since the action $NtrV(M)$ depends linearly on the coupling constants $t_n^{P+1}$ and $t_n^Q$ ($n \geq 0$). We call this the rescaling constraint. Note that it takes the same form as the dilaton equation (2.23) except for the inhomogeneous term $\chi^2/4$.

Comparison With The Constraints For The $\mathbb{C}P^1$ Model

We are now in a position to compare the constraints for the matrix model obtained above with the constraints for the $\mathbb{C}P^1$ model. In particular, we compare the dilatation constraint $L_0Z = 0$ (the first of the Virasoro constraints (3.16)) with our new constraint (2.33) applied to $M = \mathbb{C}P^1$

$$\left(\tilde{L}_0 + N^2(t_0^P)^2\right)e^F = 0,$$

(3.19)

where $\tilde{L}_0 = \sum_{m=0}^{\infty}(mt_m^P \partial_{t_m^P} + (m + 1)t_m^Q \partial_{t_m^Q}) + 2\sum_{m=1}^{\infty} mt_m^P \partial_{t_m^P - \partial_{t_m^Q}}$. We also compare the rescaling constraint (3.18) with the dilaton equation (2.29)

$$\left(D - N \frac{\partial}{\partial N} + \frac{1}{12}\right)e^F = 0,$$

(3.20)

where $D = \sum_{m=0}^{\infty}(mt_m^P \partial_{t_m^P} + t_m^Q \partial_{t_m^Q})$. Though they have the same (similar) forms, the comparison is a non-trivial task since $F^M$ and $F$ do not coincide but are related by (3.10) with (3.11) under the ansatz (3.7). For instance, we will need the following equations:

$$\left(\tilde{L}_0 + 1\right)h_0 = 0,$$

(3.21)

$$\left(D - N \frac{\partial}{\partial N}\right)h_0 = 0,$$

(3.22)

which are the constraints $L_0Z = 0$ and (3.18) for the special case $\nu = 1$.

We first show that the constraints (3.19) and (3.20) for the $\mathbb{C}P^1$ model lead through (3.10) with (3.11) to the dilatation and the rescaling constraints on $Z = e^{F^M}$. Using (3.19) and its derivatives with respect to $x = t_0^P$, we find

$$\tilde{L}_0 f(x) = -N^2 x^2 + Nx - \frac{1}{3}.$$

(3.23)

This together with its derivatives shows that

$$\tilde{L}_0 F^M = Nx \tilde{L}_0 \log h_0 - N^2 x^2 + Nx.$$

(3.24)

If we use (3.21), we obtain the dilatation constraint $L_0 e^{F^M} = 0$. On the other hand, from (3.20) we find

$$\left(D - N \frac{\partial}{\partial N}\right)\frac{1}{N^m} F^{(m)} = 0 \quad m \geq 1,$$

(3.25)

which leads to $(D - N \frac{\partial}{\partial N}) f(x) = -\frac{1}{12}$. It then follows that

$$\left(D - N \frac{\partial}{\partial N}\right)\frac{x}{N^{m-1}} f^{(m)}(0) = 0 \quad m \geq 1.$$

(3.26)
Now, using (3.22), we obtain the rescaling constraint (3.18).

Conversely, if we start with $\tilde{L}_0 F^M = -\nu^2$ and $(D - N \frac{\partial}{\partial N}) F^M = 0$, by differetiating them by $x = \frac{\nu}{N}$, we obtain recursively (namely, order by order with respect to $N^{-1}$) the second derivatives of our equations $\tilde{L}_0 F = -N^2 x^2$ and $(D - N \frac{\partial}{\partial N}) F = -\frac{1}{12}$.

We have thus observed that some of the matrix model results coincides with the ones in the $\mathbb{C}P^1$ model obtained by geometric arguments. It seems an interesting problem to find further constraints for the $\mathbb{C}P^1$ model which correspond to the higher Virasoro constraints $L_n Z = 0$, $n \geq 1$. In particular, we wish to know whether there are Virasoro constraints $L_n e^F = 0$, $n \geq -1$ for the $\mathbb{C}P^1$ model. (We have already found $L_{-1}$ and $L_0$ for general models.) A more challenging problem is to find matrix models for other models of topological strings so as to derive the constraints obtained in §2. For this purpose, we give in Appendix the explicit form of the constraint for the case in which the target space is the projective space or the grassmannian.

4. Field Theoretical Background

This section provides a field theoretical background for the description of string amplitudes given in §2, especially for the emergence of the top chern class of the vector bundle $\mathcal{V}_{g,s,d}$. Though the argument is rather standard [22, 11, 21], we present it in some detail because it exhibits that the general principle also works in a model in which gravity and matter are coupled in non-trivial way (5).

We start with describing briefly the elementary fields, the classical action and the symmetries. Bosonic elementary variables consist of a metric $g$ on the world sheet $\Sigma$ and a map $\phi$ of $\Sigma$ to the Kähler manifold $M$. The BRST transformation introduces their superpartners — the ghosts (6):

\[
\begin{align*}
\delta g_{\mu\nu} &= -2\chi_{\mu\nu} + \cdots, \quad \delta \chi_{\mu\nu} = \cdots, \\
\delta \phi^I &= \chi^I, \quad \delta \chi^I = 0.
\end{align*}
\]

In the above expression, $\cdots$ denotes the contribution of diffeomorphism ghost and the secondary ghost in the topological gravity multiplet (see e.g. [22]) which will not be used in our discussion. Also, we will eventually fix the local scale and restrict our attention to the complex structure $J_{\mu}^\nu$ induced by the metric $g_{\mu\nu}$. Hence, we shall neglect the trace part $g^{\mu\nu} \chi_{\mu\nu}$ from the start. The classical action of the system is the sum

\[
I = I_g + I_m.
\]
of the gravity part \( I_g \) (e.g. eqn (5.9) in [22]) and the sigma model part

\[
I_m = \frac{1}{2\pi} \int_{\Sigma} d^2z \left\{ g_{ij} \partial_z \phi^i \partial_{\bar{z}} \phi^j + i \rho_{zi} (D_z \chi^i + \chi^i \partial_z \phi^i) + i \tilde{\rho}_{zi} (D_{\bar{z}} \chi^i + \chi^i \partial_{\bar{z}} \phi^i) + \chi^i \chi^j R_{kl} \tilde{\rho}^i_{z_k} \rho^i_{z_l} - \chi^i \chi^j \rho^i_{\bar{z}_k} \rho^j_{\bar{z}_l} \right\},
\]

where \( d^2z = idz d\bar{z} \) and \( D_\mu \chi^i = (\partial_\mu \delta^i_j + \partial_\mu f^{K\mu} K^I_{KJ}) \chi^j \) in which \( D_K v^I = (\partial_K \delta^I_J + \Gamma^I_{KJ}) v^J \) is the covariant derivative with respect to the Kähler metric \( g \). The metric is used also to raise and lower the indices. We follow the convention \( D \) is the covariant derivative with respect to the Kähler metric \( g \) with values in \( K_\Sigma \otimes f^* T_M^* \) (resp. \( \overline{K}_\Sigma \otimes f^* \overline{T}_M^* \)).

This action is invariant under the BRST transformation

\[
\delta J^z = -2i \chi^z, \quad \delta \chi^z = 0, \quad \delta \phi^z = \chi^z, \quad \delta \chi^z = 0, \quad \delta \phi^z = \chi^z, \quad \delta \chi^z = 0,
\]

\[
\delta \rho^z_i = i \partial_z \phi^i - \Gamma^z_{jk} \chi^z \rho^k_j, \quad \delta \rho^z_i = i \partial_{\bar{z}} \phi^i - \Gamma^i_{jk} \chi^z \rho^z_j.
\]

The action (4.3) and the transformation rule above is obtained by neglecting the secondary ghosts etc. in the corresponding expressions given in [12].

**The Space Of Instantons**

The most important fact in the quantum theory is that the WKB approximation is exact. This is because the action is obtained by eliminating some auxiliary fields from a BRST exact functional. Thus, we are interested in instantons — configurations that minimize the action.

Since the local scale degrees of freedom will eventually be fixed, we look for instantons in the space \( C_\Sigma \times \text{Map}(\Sigma, M) \) of pairs of complex structures on \( \Sigma \) and maps of \( \Sigma \) to \( M \). Looking at the action (4.3), we see that the space of instantons is given by

\[
\mathcal{H}\sigma = \left\{ (J, f) : \Sigma_J \to M \text{ is holomorphic.} \right\},
\]

where \( \Sigma_J \) stands for the Riemann surface determined by the complex structure \( J \). The quotient of this space by a certain group of diffeomorphisms is an open subset of the moduli space of stable maps (2.2). With respect to local coordinates, the instanton condition is written as

\[
\frac{1}{2} dx^\mu (\delta^\mu_\mu + i J^\mu_\mu) \partial_\mu f^i = 0.
\]

Taking the first order variation, we find that the tangent space to \( \mathcal{H}\sigma \) at \( (J, f) \) is spanned by such \( (\delta J, \delta f) \) that

\[
\partial_\bar{z} \delta f^i + \frac{i}{2} \delta J^z \partial_\bar{z} f^i = 0.
\]

We can (and we do) consider the space \( C_\Sigma \times \text{Map}(\Sigma, M) \) as a complex manifold such that \( \delta^0.1 J^z = 0 \) and \( \delta^0.1 f^i = 0 \). Since the tangential condition (4.7) is holomorphic, each tangent
space is a complex subspace of the tangent space of \( \mathcal{C}_\Sigma \times \text{Map}(\Sigma, M) \). As in §2, we assume that the instanton space \( \mathcal{H}\mathcal{O}\ell \subset \mathcal{C}_\Sigma \times \text{Map}(\Sigma, M) \) is a smooth complex submanifold.

Note that the above description naturally leads to the description \( \text{[24]} \) of the tangent space to the moduli space of stable maps. Note also that a ghost zero mode, i.e. \( \chi^s_\Sigma, \chi^i \) with \( \partial_z \chi^i + \chi^s \partial_z f^i = 0 \), gives a \((1,0)\)-tangent vector of \( \mathcal{H}\mathcal{O}\ell \).

### 4.1 The Instanton Calculus

We proceed to the Gaussian approximation around an instanton. We first note that, after certain partial quantization of topological gravity, we are left with the integration over the moduli space \( \overline{\mathcal{M}}_{g,s} \) of Riemann surfaces with the measure provided by the integration over the fermionic fields \( \chi^s_\Sigma, \chi^i \). Therefore, we assume in what follows that the complex structure \( J \) varies in a representative family \( \{J_m\} \) parametrized by the finite moduli \( m^1, \ldots, m^{3g-3+s} \) and that \( \chi^s_\Sigma \) takes values in the tangent space of the moduli space: \( \chi^s_\Sigma = \frac{i}{2} \sum \hat{m}_a \frac{\partial}{\partial m_a} J^s_\Sigma \). Also, since the position of the puncture does not affect our argument, we put \( s = 0 \) from the start.

Now take an instanton \( (J, f) \in \mathcal{H}\mathcal{O}\ell \) and consider the first order variation

\[
    f \rightarrow f + \delta f, \\
    J \rightarrow J + \delta J,
\]

in the direction transversal to \( \mathcal{H}\mathcal{O}\ell \). This is generated by a smooth section \( v \) of the bundle \( f^*T_M \); \( \delta f^i = v^i \) and a Beltrami differential \( \omega \) representing a tangent vector \( [\omega] \in H^{0,1}(T_{\Sigma}) \cong H^1(T_{\Sigma}) \) of the ordinary moduli space; \( \delta J^s_\Sigma = -2i \omega^s_\Sigma \). The transversality condition is realized by the requirement

\[
    v \in (H^0(f^*T_M))^\perp \subset \Omega^{0,0}(f^*T_M), \\
    \omega \in (\text{Ker } f^*)^\perp \subset (\partial \Omega^{0,0}(T_{\Sigma}))^\perp. 
\]

The expressions here are explained as follows: For a vector bundle \( E \) over \( \Sigma \), we denote by \( \Omega^{p,q}(E) \) the space of smooth \((p, q)\)-forms with values in \( E \). For \( E = T_{\Sigma} \) or \( f^*T_M \), we use the metrics \( g_{zz} \) and \( g_{ij} \) to define a hermitian inner product on the space \( \Omega^{p,q}(E) \). The space \( H^0(f^*T_M) \) of holomorphic sections is considered as a subspace of \( \Omega^{0,0}(f^*T_M) \) and \( (H^0(f^*T_M))^\perp \) is its orthocomplement. The cokernel \( H^{0,1}(E) \) of \( \bar{\partial} : \Omega^{0,0}(E) \rightarrow \Omega^{0,1}(E) \) for \( E = T_{\Sigma} \) or \( f^*T_M \) is identified with the orthocomplement of the image. As in §2, we have the map \( f_* : H^{0,1}(T_{\Sigma}) \rightarrow H^{0,1}(f^*T_M) \) that comes from the cochain map \( f_*^* : \Omega^{p,q}(T_{\Sigma}) \rightarrow \Omega^{p,q}(f^*T_M) \) induced by \( T_{\Sigma} \rightarrow f^*T_M; v^z \mapsto v^i \partial_z f^i \).

Before starting the calculation, we remark on the complex conjugates \( \bar{v} \) and \( \bar{\omega} \) of \( v \) and \( \omega \). Note first that the metrics \( g_{zz} \) and \( g_{ij} \) give the identification of bundles

\[
    T_{\Sigma} \cong T_{\Sigma}^* = K_{\Sigma}, \quad f^*T_M \cong f^*T_M^*. 
\]

The transversality conditions \( \text{[4.10]} \) and \( \text{[4.11]} \) can then be stated as

\[
    \bar{v} \in (H^0(f^*T_M^*))^\perp \subset \Omega^{0,0}(f^*T_M^*), \\
    \bar{\omega} \in \text{Im } df \subset H^0(K_{\Sigma}^{\otimes 2}), 
\]
where \( df \) is the map \( H^0(K^*_V \otimes f^* T^*_M) \rightarrow H^0(K^*_S \otimes f^* T^*_M) \) induced by the map (2.8): \( \rho_{zi} \rightarrow \rho_{zi} \partial_z f^i \).

The equivalence of (4.11) and (4.14) can be seen as follows: Under the identification \( \Omega^{0,1}(T_S) \cong \Omega^{0,0}(K^*_S) \), \( (\partial \Omega^{0,0}(T_S))^\perp \) is the null space of \( \partial \Omega^{0,0}(T_S) \) and hence coincides with \( H^0(K^*_S \otimes f^* T^*_M) \). It then suffices to note that \( df \) is the dual map of \( f_* \).

The Calculation

We expand the action \( I_m \) up to quadratic terms in \( v, \bar{v}, \omega \) and \( \omega \). Under the variation of \( J \) and \( \phi \), the spaces of fields \( \chi^i \) and \( \rho_{zi} \) are varied and we use the metric connection on \( M \) to transport them:

\[
\begin{align*}
\chi^i \frac{\partial}{\partial z^i} |_f & \rightarrow \left( \chi^i - v^k \Gamma^i_{kj} \chi^j \right) \frac{\partial}{\partial z^i} |_{f + \delta f}, \\
\rho_{zi} dz \otimes dz^i |_f & \rightarrow \left( \rho_{zi} dz + v^k \Gamma^i_{kj} \rho_{zj} dz + \omega^z \rho_{zi} dz \right) \otimes dz^i |_{f + \delta f}.
\end{align*}
\] (4.15) (4.16)

In the above expressions, \( \frac{\partial}{\partial z^i} |_f \) (resp. \( dz^i |_f \)) stand for the local sections of \( f^* T_M \) (resp. \( f^* T^*_M \)) associated with local coordinates \( z^i \) of \( M \). By a direct calculation, we obtain the quadratic approximation \( I^{(2)}_m = \frac{1}{2\pi} I_t + \frac{1}{2\pi} I_b \) where

\[
I_t = \int_{\Sigma} d^2 z \left\{ i\rho_{zi}(\partial_z \chi^i + \chi^i \partial_z f^i) + i\rho_{zi}(\partial_z \chi^i + \chi^i \partial_z f^i) + \chi^i R_{klj} \rho_{zi}^j - \chi^i \chi^i \rho_{zi} \right\},
\] (4.17)

and

\[
I_b = \int_{\Sigma} d^2 z \left\{ -g^{ij} v_i D_z D_z \bar{v}_j + D_z v_i \omega_z \partial_z f^i + \omega_z \partial_z f^i D_z \bar{v}_i + g_{ij} \omega_z \partial_z f^i \bar{\omega}_z \partial_z f^j \right. \\
- i v^i \phi_{zzi} + i \bar{\phi}_{zzi} \bar{v}^i - i \omega_z D_z \chi^i \rho_{zi} + i \rho_{zi} D_z \chi^i \bar{\omega}^z \right\} + \cdots,
\] (4.18)

in which

\[
\phi_{zzi} = \chi^k \partial_z f^k R_{klij} \rho_{zi}, \quad \bar{\phi}_{zzi} = \rho_{zj} \chi^k \partial_z f^k R_{klij} \bar{\rho}_{zi}.
\] (4.19) (4.20)

and \( \cdots \) are terms that do not contribute to the functional integral.

We decompose the anti-ghost \( \rho \) as \( \rho^{(0)} + \rho^{(\perp)} + \rho^{(+)} \) according to the orthogonal decomposition

\[
\Omega^{1,0}(f^* T^*_M) = H^0(K^*_V \otimes N^*_f) \oplus H^0_\perp \oplus D\Omega^{0,0}(f^* T^*_M),
\] (4.21)

where \( N^*_f \) is the conormal bundle (see example (ii) in \( \S 2.1 \)) and \( H^0_\perp \) is the orthocomplement of \( H^0(K^*_V \otimes N^*_f) \) in \( H^0(K^*_V \otimes f^* T^*_M) \). The ghosts are also decomposed as \( \chi^i = \chi^{i(0)} + \chi^{i(\perp)} \) and \( \chi^i = \chi^{i(0)} + \chi^{i(+)} \) etc. where

\[
\chi^{i(\perp)} \in (\text{Ker} f_*)^\perp, \quad \chi^{i(+)} \in (H^0(f^* T_M))^\perp,
\] (4.22) (4.23)

With a shift of the variables \( \chi^{i(+)}, \bar{\chi}^{i(+)} \) and dropping terms that are irrelevant upon functional integration, \( I_t \) can be rewritten as follows:

\[
I_t = \int_{\Sigma} d^2 z \left\{ i \rho^{(0)} \bar{D} \chi^{i(0)} + i \rho^{(\perp)} f_* \chi^{i(\perp)} + i \rho^{(0)} D \bar{\chi}^{i(\perp)} + i \rho^{(\perp)} f_* \bar{\chi}^{i(\perp)} + i \rho^{(0)} R_{\rho^{(0)}} - \chi^{i(0)} \bar{\chi}^{i(0)} \right\}.
\] (4.24)
Similarly, the fermionic fields $\chi, \bar{\chi}, \rho, \bar{\rho}$ in $I_b$ can be replaced by the zero mode components $\chi(0), \bar{\chi}(0), \rho(0), \bar{\rho}(0)$.

We introduce the Green’s operator $G$ for the Laplacian $\bar{D}D$:

$$G : \Omega^{1,1}(f^*T_M^*) \xrightarrow{\pi_{\text{im},D}} \text{Im} \bar{D} \xrightarrow{(\bar{D}D)^{-1}} (\text{Ker}D)^\perp \hookrightarrow \Omega^{0,0}(f^*T_M^*),$$

where $\pi_{\text{im},D}$ is the orthogonal projection and $\bar{D}D$ is the restriction of $\bar{D}D$ to $(\text{Ker}D)^\perp$. As a consequence of the replacement $\chi \to \chi(0), \rho \to \rho(0)$, the two form $\varphi$ defined in (4.19) is in the image of $D$. This can be seen by showing that the pairing of $\varphi$ and any holomorphic section of $f^*T_M$ is zero. Now we see that $I_b$ is rewritten as

$$I_b = \int_\Sigma d^2z \left\{ -(v + \cdots) \bar{D}D(\bar{v} + \cdots) + \varphi G(\varphi) + f_z \omega(1 - DG\bar{D})\bar{f}_z \omega - if_z \omega DG(\varphi) - i\omega(D\bar{\chi})\rho - i\varphi G(\bar{D}\bar{f}_z \omega) + i\rho(\bar{D}\bar{\chi})\omega \right\},$$

where $(\bar{f}_z \omega)\bar{z}_i = g_{ij} g^{zz} \partial_z \bar{f}_z \bar{\omega}_{zz}$. Here and in what follows, we drop the sign ‘(0)’ for the fermion zero modes. We note that $1 - DG\bar{D}$ is the orthogonal projection to the kernel of $\bar{D}$. It is easy to see that the map $\pi_{\text{H}} \circ \bar{f}_z : H^0(K_\Sigma^2) \to H^0(K_\Sigma \otimes f^*T_M)$ corresponds to $\bar{f}_z : H^{0,1}(\Sigma) \to H^{0,1}(f^*T_M)$ under the identification $K_\Sigma \cong T_\Sigma$ and $f^*T_M \cong f^*T_M$. Since $(K \Sigma \otimes f^*T_M)^\perp$ corresponds to $\text{Im} df$ and the restriction of $\bar{f}_z$ to $H^0(K_\Sigma \otimes f^*T_M)$ is the dual $\bar{f}_z$ of $f_z$, we see that $\bar{f}_z \pi_{\text{H}} \bar{f}_z$ is identified with $f_z \bar{f}_z$ and gives, by restriction, an invertible operator on $\text{Im} df$. Using the zero mode condition for the fermions, one can see that $D_z \chi^i \rho_{zi} + \partial_z f^i D_z G_i(\varphi)$ is a holomorphic quadratic differential. Moreover, it seems that it is also in the image of $df$ (see the next subsection). Now $I_b$ can be written as

$$I_b = \int_\Sigma d^2z \left\{ -(v + \cdots) \bar{D}D(\bar{v} + \cdots) + (\omega + \cdots) f_z \pi_{\text{H}} \bar{f}_z + \varphi G(\varphi) - (\bar{\rho}(\bar{D}\bar{\chi}) - \varphi G\bar{D}\bar{f}_z)\left( f_z \pi_{\text{H}} \bar{f}_z \right)^{-1}((D\chi)\rho + f_z DG(\varphi)) \right\}.$$ 

When the transversal modes $v, \omega, \rho(\cdot)^{\pm}\chi(\cdot)^{\pm}, \rho(\cdot)^{\perp}\chi(\cdot)^{\perp}$ (and their conjugates) are integrated out, the bosonic and fermionic determinants cancel with each other, and we are left with the following effective action:

$$I_{\text{eff}} = \frac{1}{2\pi} \int_\Sigma d^2z \left\{ \chi^k \bar{R}_{kl} \bar{\rho}_l z_i - \chi^k \chi^\xi \partial_z \bar{f}_i z_i + \varphi_{zz} \bar{G}_i(\varphi) \right.$$

$$\left. - (\bar{\rho}(\bar{D}\bar{\chi}) - \varphi G\bar{D}\bar{f}_z)_{zz} \left( f_z \pi_{\text{H}} \bar{f}_z \right)^{-1}((D\chi)\rho + f_z DG(\varphi)) \right\}.$$ 

We note again that the fermions in the above expression are the zero mode components: $\partial_z \chi^i + \chi^\xi \partial_z \bar{f}_i = 0, \partial_z \rho_{zi} = 0$ and $\rho_{zi} \partial_z \bar{f}_i = 0$.

**4.2 Connection And Curvature Of $\mathcal{V}$**
If we integrate $e^{-\beta H}$ over the $\rho$-zero modes, we obtain a differential form on the space $\mathcal{H}\sigma\ell$ of instantons. In this subsection, we show that this is the top chern class of the dual $\mathcal{V}$ of the bundle $\mathcal{V}^*$ of $\rho$-zero modes $[\mathcal{I}]$. To be more precise, we introduce a hermitian connection of the bundle $\mathcal{V}^*$, calculate the curvature, and then compare it with the expression $[4.29]$.

**The Hermitian Vector Bundle $\mathcal{V}^*$**

As the instanton $(J, f)$ varies, the first cohomology group $H^1(N_\ell)$ of the normal bundle $N_\ell$ (see example (ii) of §2.1) varies as the fibre of a bundle $\mathcal{V}$ over $\mathcal{H}\sigma\ell$. Its dual $\mathcal{V}^*$ is the bundle of $\rho$-zero modes $[\mathcal{I}]$:

$$\partial_z \rho_{zi} = 0, \quad \rho_{zi} \partial_z f^i = 0.$$  \hfill (4.30)

Since each fibre $H^0(K_\Sigma \otimes f^* T^*_M)$ is in the space $\Omega^{1,0}(f^* T^*_M)$ with a hermitian product (induced by the Kähler metric of $M$), we can consider $\mathcal{V}^*$ as a hermitian vector bundle.

**The Hermitian Connection Of $\mathcal{V}^*$**

We shall introduce a connection of $\mathcal{V}^*$ by finding a method of parallel transport and then show that it is hermitian. Consider the variation $(J, f) \to (J + \delta J, f + \delta f)$ in $\mathcal{H}\sigma\ell$:

$$\partial_z \delta f^i + \frac{i}{2} \delta J^z \partial_z f^i = 0.$$  \hfill (4.31)

We determine $\eta_{zi}$ by requiring also that $\tilde{\rho}$ is holomorphic with respect to $J + \delta J$ and conormal with respect to $f + \delta f$. The requirements are written down as

$$\partial_z \eta_{zi} = -\frac{i}{2} \partial_z (\delta J^z \rho_{zi}),$$  \hfill (4.32)

$$\eta_{zi} \partial_z f^i = -\rho_{zi} \partial_z \delta f^i.$$  \hfill (4.33)

In order to express these in covariant forms, we consider the parallel transport $\tau \tilde{\rho} \in \mathcal{V}^*|_{(J + \delta J, f + \delta f)}$ induced by the metric $g_{ij}$ of $M$:

$$\tau \tilde{\rho} = \left( dz \rho_{zi} - \frac{i}{2} d\bar{z} \delta J^z \rho_{zi} + dz \eta_{zi} \right) \otimes d\bar{z} |_{f + \delta f},$$  \hfill (4.34)

where $\Delta \rho_{zi} = \eta_{zi} - \delta f^k \Gamma^{ij}_{kl} \rho_{zj}$. Then, the requirements $[4.32]$ and $[4.33]$ are written as

$$D_z \Delta \rho_{zi} = \varphi_{zzi},$$  \hfill (4.35)

$$\Delta \rho_{zi} \partial_z f^i = -\rho_{zi} D_z \delta f^i,$$  \hfill (4.36)

in which $\varphi_{zzi} = \delta f^k \partial_z f^i \bar{R}^{j}_{kl} \rho_{zj} - \frac{i}{2} D_z (\delta J^z \rho_z)$.  \hfill (4.37)

---

7 As explained in [11], whether you take $c_T(\mathcal{V}^*)$ or $c_T(\mathcal{V})$ is a matter of convention. We take the latter so that the resulting constraint $[2.33]$ recover the dilatation constraint for the minimal model.

8 In this subsection, we consider $\rho_{zi}$ as a bosonic field.
As noted in §4.1, if ρ is a zero mode (i.e. satisfies (1.30)) and if (δJ, δf) is a variation in \( H\mathfrak{o}\ell \) (a ghost zero mode), one can show that \( \varphi \) is in the image of \( \tilde{D} \). Therefore, (1.35) can be solved by using the Green’s operator \( G \) for \( \tilde{D} \), defined in (4.23), as
\[
\Delta \rho_{zi} = D_z G_i(\varphi) - \xi_{zi} \quad ; \quad \partial_z \xi_{zi} = 0.
\] (4.37)

We shall determine \( \xi_{zi} \) so that this satisfies the second requirement (1.37):
\[
\xi_{zi} \partial_z f^i = \rho_{zi} D_z \delta f^i + D_z G_i(\varphi) \partial_z f^i.
\] (4.38)

Using again the zero mode condition of \( \rho \) and (δJ, δf), one can show that the right hand side is a holomorphic quadratic differential. Though we do not have an explicit proof, we expect that it is in the image of \( df : H^0(K_\Sigma \otimes f^* T^*_M) \to H^0(K_\Sigma^{\otimes 2}) \). This is based on the fact that there exists a solution to our problem, namely, there exists a connection of \( \mathcal{V}^* \).

Let us introduce the ‘Green’s operator’ for \( df \):
\[
\Gamma : H^0(K_\Sigma^{\otimes 2}) \xrightarrow{\pi_I} \text{Im} df \xrightarrow{(df)^{-1}} (\text{Ker} df)^\perp \to H^0(K_\Sigma \otimes f^* T^*_M),
\] (4.39)

where the projection \( \pi_I \) and the orthocomplement are determined with respect to the inner products on \( H^0(K^2) \) and \( H^0(K \otimes f^* T^*_M) \) inherited from \( \Omega^{1,0}(K) \) and \( \Omega^{1,0}(f^* T^*_M) \). A solution of (4.38) is given by
\[
\xi_{zi} = \Gamma_{zi}((D \delta^{1,0} f) \rho + \bar{t}_z D G(\varphi)),
\] (4.40)

where \( \delta^{1,0} f \) is the variation of in the holomorphic direction. Thus, we have determined a method of transportation \( \rho \mapsto \bar{\rho} \).

We can now define the covariant derivative using this transport:
\[
\nabla \rho = \tau \left( (\rho_i + \delta \rho_i) \otimes dz^i f + \delta f - \rho \right)
\]
\[
= \left( \delta \rho_i - \delta f^k \bar{\Gamma}_{ki}^j \rho_j + \frac{i}{2} dz \delta J_{zi} \rho_{zi} \right.
\]
\[
- dz D_z G_i(\varphi) + dz \Gamma_{zi}((D \delta^{1,0} f) \rho + \bar{t}_z D G(\varphi)) \bigg) \otimes dz^i f.
\] (4.41)

An important property of this connection is that it preserves the hermitian structure of \( \mathcal{V}^* \). This follows in particular from the choice (4.40) of the solution of (4.38) because the image of \( \Gamma \) is orthogonal to the fibre Ker \( df = \tilde{H}^0(K_\Sigma \otimes N^*_f) \). This fact implies that only the (1, 1)-form part of the curvature \( F \) is non-vanishing and the calculation becomes relatively easy.

**The Curvature**

Now we calculate the curvature:
\[
F \rho_i = F^{(1,1)} \rho_i = \{ \nabla^{0,1}, \nabla^{1,0} \} \rho_i
\]
\[
= \delta f^k \delta f^j \bar{\partial}_k \bar{\Gamma}_{ki}^j \rho_j - \frac{1}{4} dz \delta J^z \delta J_{zi} \rho_{zi}
\]
\[
- (\delta^{0,1} D) G_i(\varphi) + (\delta^{0,1} \Gamma_i)((D \delta^{1,0} f) \rho + \bar{t}_z D G(\varphi)) + D(\cdots)_i + \Gamma_i(\cdots).
\] (4.42)
Since we will eventually take the inner product with an element of $H^0(K_{\Sigma} \otimes N_f^*)$, we can forget the terms $D(\cdots)$ and $\Gamma_i(\cdots)$.

The $(0,1)$-variation of the covariant derivative $D = dzD_z$ is directly calculated:

\[
(\delta^{0,1}D)G_i(\varphi) = \delta^{0,1}\left(\frac{dx^\mu}{2}\frac{\delta \varphi}{\partial \tau^\mu} - iJ^\nu(\partial_\nu \delta_i^f - \partial_\nu f^k\Gamma_k^i)\right)G_j(\varphi)
\]
\[
= -dz\partial_z f^k\delta_l^j R_{ik}^j G_j(\varphi) - \frac{i}{2}dz\delta J_z^z D_z G_i(\varphi).
\]

(4.43)

Taking the inner product with $\rho' \in H^0(K_{\Sigma} \otimes N_f^*)$, we have

\[
(\rho', (\delta^{0,1}D)G(\varphi)) = \int_\Sigma d^2z \varphi_{zz}f G_j(\varphi),
\]

(4.44)

where $\varphi_{zz} = \rho_{zz}'\delta f\partial_z f R_{ik}^j G_j(\varphi) + \frac{i}{2}dz(\rho_z^i\delta J_z^z)i$.

We have to determine the $(0,1)$-variation of the Green’s operator $\Gamma$ for $df$. For this, we use the variational formula elaborated in [23], section 3b:

\[
\delta \Gamma = \Gamma \delta(df)\Gamma + \pi_K \delta(df)^\Gamma\Gamma + \Gamma \Gamma^\dagger \delta(df)\Gamma^\dagger \pi_{f^\perp},
\]

(4.45)

where $\pi_K$ and $\pi_{f^\perp}$ is the orthogonal projection to the kernel and the orthocomplement of the image of $df$. Since $\Gamma$ is valued in $(\text{Ker} df)^\perp$, we can neglect the first and the third terms. To evaluate the second term, we look at the hermitian conjugate of $df$:

\[
(\eta, (df)^\dagger \beta) = (df(\eta), \beta) = \int_\Sigma d^2z \overline{\varphi}_{zz}g^{ij}(g_{ij}g^{zz}\partial_z f^i\beta_{zz}).
\]

(4.46)

Namely, we have

\[
(df)^\dagger = \pi_\mathcal{H} \circ \tilde{f}_z,
\]

(4.47)

where we recall that $\tilde{f}_z$ is the map $\Omega^{1,0}(T_{\Sigma}) \to \Omega^{1,0}(f^*T_M^*)$; $\beta_{zz} \mapsto g_{ij}g^{zz}\partial_z f^i\beta_{zz}$ and $\pi_\mathcal{H}$ is the projection to $H^0(K_{\Sigma} \otimes f^*T_M^*)$. Since $df$ is the dual of $f_*$, we have $(df)(df)^\dagger = (f_\pi \pi_\mathcal{H} \tilde{f}_z$, and hence its Green’s operator $\Gamma^\dagger \Gamma$ has the restriction

\[
\Gamma^\dagger \Gamma = (f_\pi \pi_\mathcal{H} \tilde{f}_z)^{-1} \text{ on } \text{Im } df.
\]

(4.48)

The $(0,1)$-variation of the map $\tilde{f}_z$ is directly calculated:

\[
(\delta^{0,1} \tilde{f}_z)\beta = \delta^{0,1}(dx^\mu g_{ij}g^{\nu\sigma}\partial_\sigma f^i)|\beta_{\mu\nu}dz^i|f
\]
\[
= dz g_{ij}g^{zz}D_z \delta^{0,1}f^i\beta_{zz}dz^i|f.
\]

(4.49)

As for the variation of $\pi_\mathcal{H}$, since it is the orthogonal projection to the kernel of the operator $\tilde{D} : \Omega^{1,0}(f^*T_M^*) \to \Omega^{1,1}(f^*T_M^*)$, we can again use the variational formula of [23]:

\[
\delta^{0,1} \pi_\mathcal{H} = -\tilde{D}^{-1}(\delta^{0,1}\tilde{D})\pi_\mathcal{H} - \pi_\mathcal{H}(\delta^{0,1}\tilde{D})^\dagger(\tilde{D}^{-1})^\dagger,
\]

(4.50)
where \(D^{-1}\) is the Green’s operator for \(D\). Since \(D^{-1}\) is valued in \((H^0(K_{\Sigma} \otimes f^*T^*_M))^\perp\), we can neglect the first term in the right hand side. Using \(\tilde{D}^\dagger = -D^*\) where \(* : \Omega^{1,1} \to \Omega^{0,0}\) is the Hodge operator, we obtain

\[
\delta^{0,1} \pi_H = -\pi_H(\delta^{0,1} D) - D^{-1} + \cdots = -\pi_H(\delta^{0,1} D) G \tilde{D} + \cdots,
\]

in which \(\delta^{0,1} D\) is calculated in (4.43). Now, the inner product of \(\rho' \in H^0(K_{\Sigma} \otimes N^*_f)\) and \(\delta^{0,1} (df)^\dagger \beta\) is expressed as follows:

\[
\left( \rho', \delta^{0,1} (df)^\dagger \beta \right) = \left( \rho', \pi_H(\delta^{0,1} f)' \beta + (\delta^{0,1} \pi_H) f \beta \right) = \int_{\Sigma} d^2z \{ \tilde{\rho}' \delta g z \tilde{z} D \delta^{0,1} f \beta \cdot \cdot \cdot - \varphi' \delta z \tilde{z} G_1 (\tilde{D} \tilde{f} \beta) \}.
\]

Using the variational formulae (4.43), (4.52) and the equality (4.48), we have

\[
\left( \rho', (\delta^{0,1} \Gamma)((D^{\delta^{1,0} f}) \rho + t f \delta G(\varphi)) \right) = \int_{\Sigma} d^2z \{ \tilde{\rho}' \delta g z \tilde{z} D \delta^{0,1} f \cdot \cdot \cdot - \varphi' \delta z \tilde{z} G_1 (\tilde{D} \tilde{f} \beta) \}.
\]

Gathering all we have got, we obtain the following expression of the curvature:

\[
\left( \rho', F \rho \right) = \int_{\Sigma} d^2z \{ \frac{1}{4} \tilde{\rho}' \delta J^z \bar{z} J^z \rho_z - \tilde{\rho}' \delta f \delta \bar{f} R_{k\bar{l}i} \rho_{zi} - \varphi' \rho_{zi} - \varphi' \delta z \tilde{z} G_1 (\varphi) \}
\]

\[
+ \left( \rho' \delta f \cdot \cdot \cdot - \varphi' \delta G \cdot \cdot \cdot - \varphi' \delta z \tilde{z} G_1 (\bar{f} \beta) \right)^{-1}((D^{\delta^{1,0} f}) \rho + t f \delta G(\varphi)) \}.
\]

We see that this coincides with \(-2\pi I_{\text{eff}}\) under the substitution \(\delta f i \to \chi_i, \frac{i}{2} \delta J^z \to \chi^z_\bar{z}, \tilde{\rho}_z \to \rho_{z \bar{z}}\) and \(\rho_{zi} \to \rho_{zi}\). This establishes our claim.

5. Concluding Remarks

We finish this paper by commenting on the physical significance of our new constraint (2.33). In the two-dimensional gravity theory, the dilatation constraint \(L_0 Z = 0\) together with the dilatation equation determines the scaling behaviour of the system. In addition, it can be used to search for multi-critical points [24]. To see whether this holds in general topological string theory, following [24] we take the combination \((\frac{1}{2} \sum_{d} \frac{\dim M \cdot F_{g,d}}{(2.29)} + \frac{1}{\delta} + 2)\) of the constraints on \(F_g = \sum_d e^{\theta} F_{g,d}\) where \(\delta\) is a positive number. This leads to

\[
\frac{x \partial F_g}{\partial x} = \sum_{n} \left( \frac{n + q_n}{\delta} - 1 \right) \bar{r}_{n} \frac{\partial F_g}{\partial \bar{r}_n} + \sum_{n, \alpha, \beta} \frac{n \cdot C_{\alpha \beta}}{\delta} \bar{r}_{n} \frac{\partial F_g}{\partial \bar{r}_n^\alpha} + \left( 2 - \frac{\dim M - 1}{\delta} \right) (1 - g) F_g + \cdots,
\]
where \( x = t_0^P \) and \( \sum' \) is the sum over all \((n, \alpha)\) except \((0, P)\). The term \( \cdots \) is the exceptional one present only for \( g = 0 \) or \( 1 \).

When \( c_1(M) = 0 \), we can use this to find multi-critical points: If we tune the coupling constant as \( t_1^P = 1, t_{n_1}^\alpha \neq 0 \) and other \( t_n^\alpha = 0 \), the system is on the multi-critical point with the following string susceptibility and scaling dimension:

\[
\gamma_{\text{string}} = \frac{\dim M - 1}{\delta_1}, \quad \gamma_{n,\alpha} = \frac{n + q_\alpha}{\delta_1},
\]

(5.2)

where \( \delta_1 = n_1 + q_{\alpha_1} \). Applying this to the hypothetical space \( M_k \) (see the end of \( \S 2 \)), we recover the KPZ formula \([25]\) for the Virasoro minimal models coupled to gravity.

For a target space with \( c_1(M) \neq 0 \), however, we cannot expect such multi-critical behaviour because of the ‘off-diagonal’ term \( \sum \sigma M a \partial_{n_1} \sigma_{n_1} \). (One cannot ‘diagonalize the equation’ by whatever means since the operation \( \sigma_n(\Omega) \mapsto \sigma_{n-1}(c_1(M)\wedge \Omega), \sigma_0(\Omega) \mapsto 0 \) is nilpotent.) Hence, the constraint \((2.33)\) exhibits in a precise way how the first chern class \( c_1(M) \) obstructs the scaling behaviour.

Although we cannot expect critical behaviour for the full system, it seems possible to find a subsystem that admits multi-criticality. Namely, we restrict our attention to those observables that decouple from the primary field \( \sigma_0(c_1(M)) \):

\[
\sigma_0(\Omega) \quad \text{and} \quad \sigma_n(\Omega') \text{ with } c_1(M) \wedge \Omega' = 0.
\]

(5.3)

Then, we can again find multi-critical points, possibly with logarithmic scaling violation due to the exceptional terms on the sphere and the torus. Though it is not at all obvious what this restriction means physically, this does not seem a bad thing to do since the known relation (such as the string equation and topological recursion relations) remain closed.

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Appendix

In this appendix, we give an explicit expression of the new constraint (2.33):

\[ L_0 e^F = 0, \quad (A.1) \]

for the case in which the target space is the complex projective space \( \mathbb{C}P^n \) or the complex grassmann manifold \( G(k, N) \). Though \( \mathbb{C}P^n = G(1, n + 1) \), we treat them separately.

**The \( \mathbb{C}P^n \) Model**

The cohomology ring \( H^*(\mathbb{C}P^n) \) of \( M = \mathbb{C}P^n \) is generated by the Kähler class \( \omega \in H^2(\mathbb{C}P^n) \) such that \( \omega^n \) has volume one. In particular, as a vector space, \( H^*(\mathbb{C}P^n) \) has the base \( 1, \omega, \ldots, \omega^n \). We denote by \( t^a_m (a = 0, \ldots, n) \) the coupling constant for the observable \( \sigma^m(\omega^a) \). Using the Bott residue formula, we can see

\[ c_r(\mathbb{C}P^n) = (n+1)^r \omega^r. \]

In particular we have

\[ \chi(M) = n + 1, \]

\[ \int_M c_1(M) \omega^a \omega^b = (n + 1) \delta_{b,n-1-a}, \quad (A.2) \]

\[ \int_M c_1(M) c_n-1(M) = \frac{(n+1)^2}{2}. \]

These are enough to see

\[ L_0 = \sum_{m=0}^{\infty} \sum_{a=0}^{n} \left( m + a + \frac{1-n}{2} \right) \tilde{t}_m^a \frac{\partial}{\partial \tilde{t}_m^a} + (n + 1) \sum_{m=1}^{\infty} \sum_{a=0}^{n-1} m \tilde{t}_m^a \frac{\partial}{\partial t_{m-1}^a} \]

\[ + \frac{n + 1}{2 \lambda^2} \sum_{a=0}^{n-1} \tilde{t}_0^a t_0^{n-1-a} - \frac{1}{48} (n^2 - 1)(n + 3), \quad (A.3) \]

where \( \lambda \) is the string coupling constant.

**The Grassmannian Model**

The complexgrassmann manifold \( M = G(k, N) \) is the space of \( k \)-dimensional subspaces of \( \mathbb{C}^N \). The cohomology ring of this space is conveniently described in terms of the Schubert cycles. For each sequence \( a = a_1, \ldots, a_k \) of integers with \( N - k \geq a_1 \geq \cdots \geq a_k \geq 0 \), we take a subspace \( W_a \subset \mathbb{C}^N \) with the base \( e_{N-k-a_1+1}, e_{N-k-a_2+2}, \ldots, e_{N-a_k} \) where \( \{e_i\}_{i=1}^N \) is the standard base of \( \mathbb{C}^N \). The orbit through \( W_a \) of the group of \( N \times N \) upper triangular regular matrices is a cell \( C_a \subset M \) whose closure \( \overline{C_a} \) is a submanifold of codimension \( |a| = a_1 + \cdots + a_k \) (the Schubert cycle). Let \( \Omega_a \in H^{2|a|}(M) \) be the Poincaré dual of \( \overline{C_a} \). It is known that \( \{\Omega_a\}_a \) is a base of the vector space \( H^*(M) \). Hence,

\[ \chi(M) = \binom{N}{k}. \quad (A.4) \]

The ring structure is also well known (the Schubert calculus [20]). In particular, we have

\[ \int_M \Omega_a \Omega_b = \delta_{a_1}^{N-k-b_1} \cdots \delta_{a_k}^{N-k-b_k}. \quad (A.5) \]
In addition, the generator \( \omega = \Omega_{1,0,\ldots,0} \) of \( H^2(M) \) satisfies

\[
\omega \wedge \Omega_{a_1,\ldots,a_k} = \sum_{i=1}^{k} \Omega_{a_1,\ldots,a_i+1,\ldots,a_k}, \tag{A.6}
\]

in which we put \( \Omega_{a_1,\ldots,a_k} = 0 \) unless \( a_{i-1} \geq a_i + 1 \geq a_{i+1} \). Since \( \omega \) is the first Chern class of “the tautological quotient bundle” (whose fibre at \( W \in M \) is the quotient space \( \mathbb{C}^N/W \)), we can easily see \( c_1(M) = N \omega. \tag{A.7} \)

Using the Bott residue formula, we find that

\[
\int_M c_1(M)c_{\dim M-1}(M) = \frac{k}{2}(N^2 - N + 2 - 2k) \binom{N}{k}. \tag{A.8}
\]

Denoting by \( t^a_n \) the coupling constant for \( \sigma_n(\Omega_a) \), we obtain the expression

\[
L_0 = \sum_{n=0}^{\infty} \sum_a \left( n + |a| + \frac{1-k(N-k)}{2} \right) \tilde{t}^a_n \frac{\partial}{\partial t^a_n} + N \sum_{n=1}^{\infty} \sum_{i=1}^{k} \sum_a n \tilde{t}^{a_1,\ldots,a_k}_n \frac{\partial}{\partial \tilde{t}^{a_1,\ldots,a_{i+1},\ldots,a_k}_{n-1}} \tag{A.9}
\]

\[
+ \frac{N}{2\lambda^2} \sum_{i=1}^{k} \sum_a \tilde{t}^{a_1,\ldots,a_{i-1},\ldots,a_k}_n \tilde{t}^{N-k-a_1,\ldots,N-k-a_k}_0 - \frac{1}{48}(kN^2 - 3k^2 + 2k - 3) \binom{N}{k},
\]

where each sum \( \sum_a \) is over the sequences for which the summands are defined.

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