Moufang symmetry VII.
Moufang transformations

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Abstract
Concept of a birepresentation for the Moufang loops is elaborated.

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1 Introduction

Groups are often said to be an algebraic abstraction of the notion of symmetry. As a slight generalization of this one can introduce the notion of the Moufang symmetry. The latter can be defined as a hypothetic kind of symmetry associated with the Moufang loops. By paraphrasing these words, we can also say that the Moufang loops are an algebraic abstraction of the Moufang symmetry.

By introducing such a notion of symmetry, one finds oneself confronted with a question about its real meaning. If one feels like looking at world affairs from viewpoint of the Moufang symmetry, one needs a suitable mathematical machinery for identification of this symmetry. As in case of groups, one really has to elaborate representation theory of the Moufang loops, and this is the logical way to get an answer to the question.

In the present paper we elaborate a concept of a birepresentation for the Moufang loops. Throughout the paper, ideas presented in [3] are very useful.

2 Moufang loops

A Moufang loop [4] (see also [2] [3] [7]) is a set $G$ with a binary operation (multiplication) $\cdot : G \times G \to G$, denoted also by juxtaposition, so that the following three axioms are satisfied:

1) in equation $gh = k$, the knowledge of any two of $g, h, k \in G$ specifies the third one uniquely,
2) there is a distinguished element $e \in G$ with the property $eg = ge = g$ for all $g \in G$,
3) the Moufang identity

\[(gh)(kg) = g(hk)g\]  \hspace{1cm} (2.1)

hold in $G$.

Recall that a set with a binary operation is called a groupoid. A groupoid $G$ with axiom 1) is called a quasigroup. If axioms 1) and 2) are satisfied, the groupoid (quasigroup) $G$ is called a loop. The element $e$ in axiom 2) is called the unit (element) of the (Moufang) loop $G$.

In a (Moufang) loop, multiplication need not be neither associative nor commutative. Associative (Moufang) loops are well known and called groups. The associativity and commutativity laws read, respectively,

\[g \cdot hk = gh \cdot k, \quad gh = hg, \quad \forall g, h, k \in G\]
The most familiar kind of loops are those with the *associative* law, and these are called *groups*. A (Moufang) loop $G$ is called *commutative* if the commutativity law holds in $G$, and (only) the commutative associative (Moufang) loops are said to be *Abelian*.

The most remarkable property of the Moufang loops is their *diassociativity*; in a Moufang loop $G$ every two elements generate an associative subloop (group) [4]. In particular, from this it follows that

\[ g \cdot gh = g^2h, \quad hg \cdot g = hg^2, \quad gh \cdot g = g \cdot hg, \quad \forall g, h \in G \tag{2.2} \]

The first and second identities in (2.2) are called the left and right *alternativity*, respectively, and the third one is said to be *flexibility*.

The unique solution of equation $xg = e \ (gx = e)$ is called the left (right) *inverse* element of $g \in G$ and is denoted as $g_L^{-1}$ ($g_R^{-1}$). It follows from diassociativity of the Moufang loop that

\[
\begin{align*}
g_R^{-1} & = g_L^{-1} = g^{-1} \quad \tag{2.3a} \\
g^{-1} \cdot gh & = hg \cdot g^{-1} \quad \tag{2.3b} \\
(g^{-1})^{-1} & = g \quad \tag{2.3c} \\
(gh)^{-1} & = h^{-1}g^{-1}, \quad \forall g, h \in G \quad \tag{2.3d}
\end{align*}
\]

## 3 Moufang transformations

Let $X$ be a set and let $\mathcal{F}(X)$ denote the transformation group of $X$. Elements of $X$ are called *transformations* of $X$. Multiplication in $\mathcal{F}(X)$ is defined as composition of transformations, and unit element of $\mathcal{F}(X)$ coincides with the identity transformation $id$ of $X$.

Let $G$ be a Moufang loop with the unit element $e \in G$ and let $(S, T)$ denote a pair of maps $S, T : G \to \mathcal{F}(X)$.

**Definition 3.1** (birepresentation). The pair $(S, T)$ is said to be an *action* of $G$ on $X$ if

\[
\begin{align*}
S_e & = T_e = id \quad \tag{3.1a} \\
S_gT_gS_h & = S_{gh}T_g \quad \tag{3.1b} \\
S_gT_gT_h & = T_{hg}S_g \quad \tag{3.1c}
\end{align*}
\]

hold for all $g, h \in G$. The pair $(S, T)$ is called also a *birepresentation* of $G$ (in $\mathcal{F}(X)$). Transformations $S_g, T_g \in \mathcal{F}(X) \ (g \in G)$ are called $G$-transformations or the *Moufang transformations* of $X$. The set of all Moufang transformations is denoted as $\mathcal{E}_G(S, T)$.

**Example 3.2.** Define the left $(L)$ and right $(R)$ translations of $G$ by $gh = L_g h = R_h g$. Then it follows from the (2.1) that the pair $(L, R)$ of maps $L_g, R_g : G \to \mathcal{F}(G)$ is a birepresentation of $G$ in $\mathcal{F}(G)$.

The Moufang transformations need not close, but generate a subgroup of $\mathcal{F}(X)$. This subgroup is called an *enveloping group* of $\mathcal{E}_G(S, T)$ and is denoted as $\mathcal{E}_G(S, T)$. In other words, the Moufang transformations are generators of group $\mathcal{E}_G(S, T)$ – the enveloping group of birepresentation $(S, T)$ of $G$. The defining relations of $\mathcal{E}_G(S, T)$ are (3.1a–c). The enveloping group $\mathcal{E}_G(S, T)$ can be called the *multiplication group* of the birepresentation $(S, T)$ as well.

**Definition 3.3** (kernel). The set

\[ K = \text{Ker}(S, T) = \{ g \in G \mid S_g = T_g = \text{id} \} \]

is called the *kernel* of birepresentation $(S, T)$. If $K = \{ e \}$, then birepresentation $(S, T)$ is called *faithful* and action of the Moufang loop $G$ on $X$ is called *effective*.

**Example 3.4.** Birepresentation $(L, R)$ is exact.
4 Properties of Moufang transformations

Proposition 4.1. We have

\[ S_gT_g = T_gS_g, \quad \forall g \in G \] (4.1)

Proof. Set \( h = u \) in (3.1c) \( \Box \)

Proposition 4.2. We have

\[ S^{-1}_g = S^{-1}_g, \quad T^{-1}_g = T^{-1}_g, \quad \forall g \in G \] (4.2)

Proof. In (3.1b,c) first set \( h = g^{-1} \):

\[ S_gS_g^{-1} = T_gT_g^{-1} = \text{id} \]

Analogously, setting in (3.1b,c) \( g = h^{-1} \), we have

\[ S_{h^{-1}}S_h = T_{h^{-1}}T_h = \text{id} \]

Thus

\[ S_gS_g^{-1} = S^{-1}_gS_g = \text{id}, \quad T_gT_g^{-1} = T_gT_g = \text{id}, \quad \forall g \in G \] \( \Box \)

Lemma 4.3. The defining relations of the Moufang transformations can equivalently be written as follows:

\[ S_e = T_e = \text{id} \] (4.3a)
\[ S_hT_gS_g = T_gS_{hg} \] (4.3b)
\[ T_hT_gS_g = S_gT_{gh} \] (4.3c)

for all \( g, h \) in \( G \).

Proof. It follows from (3.1b) and (4.2) that

\[ S^{-1}_hT^{-1}_gS^{-1}_g = T^{-1}_gS^{-1}_g \]

which implies

\[ S_{h^{-1}}T_{g^{-1}}S_{g^{-1}} = T_{g^{-1}}S_{(gh)^{-1}} = T_{g^{-1}}S_{h^{-1}}g^{-1} \]

Thus, replacing \( g^{-1} \to g \) and \( h^{-1} \to h \) we obtain (4.3b). Analogously (4.3c) can be checked. \( \Box \)

Theorem 4.4. The Moufang transformations satisfy the following relation:

\[ S_gS_hT_hT_g = T_hT_gS_gS_h, \quad \forall g, h \in G \] (4.4)

Proof. In (3.1b) interchange \( g \) and \( h \) to obtain

\[ S_hT_gS_g = T_gS_{hg} \]

and comparing the resulting formula with (3.1b) we get

\[ S_gT_gS_{h^{-1}} = T_{h^{-1}}S_gT_hS_h \]

which implies the desired relation. \( \Box \)
Lemma 4.5. The defining relations of the Moufang transformations satisfy the following relations:

\[
\begin{align*}
S_{g^{-1}h} &= T_g^{-1}S_g^{-1}S_hT_g \\
T_{g^{-1}h} &= S_gT_hT_g^{-1}S_g^{-1} \\
S_{hg^{-1}} &= T_g^{-1}S_g^{-1}S_g^{-1}T_g \\
S_{hg^{-1}} &= S_g^{-1}T_g^{-1}T_hS_g
\end{align*}
\]

for all \(g, h\) in \(G\).

Theorem 4.6. We have:

1) \(\text{Ker}(S, T)\) is a subloop of the Moufang loop \(G\),
2) \(S_g = S_h (T_g = T_h)\) iff and only if \(S_{g^{-1}h} = \text{id} (T_{g^{-1}h} = \text{id})\),
3) birepresentation \((S, T)\) is faithful if and only if from \(S_g = S_h\) and \(T_g = T_h\) follows \(g = h\).

Proof. Use Lemma 4.3.

5 Triality

Define the quadratic Moufang transformations as

\[ P_g = S_g^{-1}T_g^{-1} \in \mathfrak{E}_G(S, T), \quad g \in G \quad (5.1) \]

Note that \(P_g\) commutes both with \(S_g\) and \(T_g\). Thus we can equivalently define \(P_g\) by the symmetric relation

\[ S_gT_gP_g = \text{id}, \quad g \in G \quad (5.2) \]

Proposition 5.1. We have

\[ P_e = \text{id} \]

\[ P_g^{-1}P_g = P_gP_g^{-1} = \text{id}, \quad \forall g \in G \]

Corollary 5.2. We have

\[ P^{-1}_g = P_g^{-1}, \quad \forall g \in G \]

Denote by \((S, T, P)\) the triple of maps \(S, T, P : g \to \Sigma(\mathfrak{X})\).

Theorem 5.3. Let \((S, T)\) be a birepresentation of the Moufang loop \(G\). Then the following pairs are birepresentations of \(G\) as well:

\[
\begin{align*}
(T^{-1}, S^{-1}) : & \quad g \to T_g^{-1}, \quad S_g^{-1} \\
(T, P) : & \quad g \to T_g, \quad g \to P_g \\
(P^{-1}, P^{-1}) : & \quad g \to P_g^{-1}, \quad T_g^{-1} \\
(P, S) : & \quad g \to P_g, \quad g \to S_g \\
(S^{-1}, P^{-1}) : & \quad g \to S_g^{-1}, \quad P_g^{-1}
\end{align*}
\]

Proof. As an example, check the defining relations for pair \((T, P)\):

\[
\begin{align*}
T_gP_gT_h &= T_{gh}P_g \\
T_gP_gP_{gh} &= P_{hg}T_g
\end{align*}
\]
To get the first relation, express $P_g$ from (5.2) and replace into this relation, the resulting relation is equivalent to (3.1b). The second relation can equivalently be written as

$$P_{h^{-1}g} = S_g P_h T_g$$

Now calculate:

$$P_{h^{-1}g} (5.3) = S_{h^{-1}g}^{-1} T_{h^{-1}g}^{-1}$$

$$P_{h^{-1}g} (5.2), (2.3) = S_{gh^{-1}} T_{gh^{-1}}$$

$$S_{h^{-1}g} (\text{from } (5.2) \text{ and replace into this relations, the resulting relation is equivalent to (3.1b)})$$

$$S_{h^{-1}g} (\text{from } (5.2) \text{ and replace into this relations, the resulting relation is equivalent to (3.1b)})$$

The defining relations for other pairs can be checked analogously.

**Corollary 5.4.** The defining relations of the birepresentations from triple $(S, T, P)$ can be collected to the following table:

| $(S, T)$ | $(T^{-1}, S^{-1})$ | $(T, P)$ | $(P^{-1}, T^{-1})$ | $(P, S)$ | $(S^{-1}, P^{-1})$ |
|----------|-------------------|---------|-------------------|---------|-------------------|
| $(5.3a)$ | $(5.3b)$          | $(5.3b)$| $(5.3c)$          | $(5.3a)$| $(5.3a)$          |
| $(5.4a)$ | $(5.4b)$          | $(5.4b)$| $(5.4c)$          | $(5.3a)$| $(5.3a)$          |

where

$$S_{g^{-1}h} = P_x S_h T_g \quad T_{g^{-1}h} = S_x T_h P_g \quad P_{g^{-1}h} = T_x P_h S_g$$

$$S_{h^{-1}g} = T_g S_h P_g \quad T_{h^{-1}g} = P_g T_h S_g \quad P_{h^{-1}g} = S_g P_h T_g$$

**Corollary 5.5.** It follows from (5.3a–c) and (5.4a–c) that

$$P_x S^{-1}_h T_g = T_{g^{-1}h} P_g, \quad S_x T^{-1}_h P_g = P_g T^{-1}_h S_g, \quad T_x P^{-1}_h S_g = S_g P^{-1}_h T_g$$

The latter are equivalent to (4.7) and to

$$T_g T_h P_g = P_h P_g T_g T_h, \quad T_g T_h P_g = P_h P_g T_g T_h$$

Collecting above properties of birepresentations we can propose

**Theorem 5.6** (principle of triality). The defining relations of the Moufang transformations are invariant under the triality substitutions

$$\text{id} = (S \to S)(T \to T)(P \to P)$$

$$\tau = (S \to T^{-1} \to S)(P \to P^{-1})$$

$$\rho = (S \to T \to P \to S)$$

$$\rho^2 = (S \to P \to T \to S)$$

$$\rho \circ \tau = (S \to P^{-1} \to S)(T \to T^{-1})$$

$$\rho^2 \circ \tau = (T \to P^{-1} \to P)(S \to S^{-1})$$

*Hence all algebraic consequences of the defining relations must be triality invariant as well.*
6 Reconstruction Theorem

It turns out that the triality symmetry is a characteristic property of the Moufang transformations.

Theorem 6.1 (reconstruction). Let $G$ be a groupoid and $(S, T, P)$ a triple of maps $S, T, P : G \to \mathcal{T}(X)$ such that:

1) $S_g T_g P_g = \text{id}$ for all $g \in G$,

2) for every $g \in G$ there exists $\overline{g} \in G$ such that $S_\overline{g}^{-1} = S_g$ and $T_\overline{g}^{-1} = T_g$,

3) for all $g, h \in G$ relations

\[
S_{gh} = P_g S_h T_g, \quad T_{gh} = S_x T_h P_g, \quad P_{gh} = T_x P_h S_g
\]

are satisfied in $\mathcal{T}(X)$,

4) from $S_g = S_h$ and $T_g = T_h$ it follows that $g = h$.

Then $G$ is a Moufang loop. The unit element of $G$ is $\overline{e} = e$, where the latter does not depend on the choice of $g$ in $G$, and the inverse element of $g$ is $\overline{g}$.

Proof. The detailed proof is presented in [6].

7 Triple closure

Theorem 7.1. The Moufang transformations satisfy the triple closure relations:

\[
S_g S_h S_g \overset{(a)}{=} S_{gh}, \quad T_g T_h T_g \overset{(b)}{=} T_{gh}, \quad P_g P_h P_g \overset{(c)}{=} P_{gh}, \quad \forall g \in G
\]  

(7.1)

Proof. Calculate:

\[
S_{gh} \overset{(5.3)}{=} P^{-1} g S_h T_g^{-1}
\]

\[
\overset{(5.4)}{=} P^{-1} g T^{-1}_g S_h P^{-1} g T^{-1}_g
\]

\[
\overset{(5.2)}{=} S^{-1} g S_h S^{-1} g
\]

\[
\overset{(4.2)}{=} S_g S_h S_g
\]

The remaining relations $(7.1b, c)$ can be checked analogously.

Remark 7.2. It follows from Theorem 7.1 that the Moufang transformations realize the triple family of transformations $\mathcal{T}(X)$ of $X$.

8 Minimality conditions

We call birepresentation $(S, T)$ associative if the Moufang transformations satisfy the closure relations

\[
S_g S_h \overset{(a)}{=} S_{gh}, \quad T_g T_h \overset{(b)}{=} T_{gh}, \quad S_g T_h \overset{(c)}{=} T_h S_g, \quad \forall g, h \in G
\]  

(8.1)

It follows from (8.1) that these conditions are equivalent.
It has to be noted that the non-associative Moufang loops do not have faithful associative birepresentations. Really, for the associative birepresentation we have

\[ S_g S_h S_k = S_{ghk}, \quad T_g T_h T_k = S_{ghk} = T_{ghk} \]

from which it follows that \((gh \cdot k)^{-1}(g \cdot hk) \in \text{Ker}(S, T)\). But for the faithful birepresentation \(\text{Ker}(S, T) = \{e\}\), hence \(g \cdot hk = g \cdot hk\).

Denote the commutator of transformations \(A, B\) by \([A, B] = ABA^{-1}B^{-1}\). Equivalence of the associativity constraints \([8.1]\) can be also seen from

**Theorem 8.1** (minimality conditions). The Moufang transformations satisfy relations

\[
[T_h, S_{h}^{-1}] \equiv S_{gh}^{-1}S_gS_{h} \equiv T_{gh}T_{g}^{-1}T_{h}^{-1} \equiv [S_{g}, T_{h}] \equiv S_{g}^{-1}S_{h}^{-1}S_{hg} \equiv T_{g}T_{h}T_{hg}^{-1} \quad (8.2)
\]

**Proof.** It is easy to check that \((8.2a) \equiv (5.3a)\), \((8.2b) \equiv (5.3b)\), \((8.2c) \equiv (5.3c)\), \((8.2d) \equiv (5.3d)\) and \((8.2e) \equiv (5.3e)\). Note that other possible equalities from \((8.2a-e)\) give rise also \((5.3a-c)\), \((5.4a-c)\) or the triple closure relations \((7.1a-c)\).

**Definition 8.2** (associators). Let \((S, T)\) be a birepresentation of the Moufang loop \(G\). Elements from group \(\mathcal{E}_G(S, T)\) of form

\[
S(g; h) = S_{gh}^{-1}S_gS_h, \\
T(g; h) = T_{gh}T_{g}^{-1}T_{h}^{-1}, \\
[T_g, S_{h}^{-1}] = T_{g}S_{h}^{-1}T_{g}^{-1}S_h, \\
[S_{g}^{-1}, T_h] = S_{g}^{-1}T_hS_gT_{h}^{-1}
\]

are called associators of birepresentation \((S, T)\).

It is easy to see from Theorem \([8.2]\)

**Corollary 8.3** (minimality conditions). Associate of a birepresentation \((S, T)\) satisfy the minimality conditions

\[
[T_g, S_{h}^{-1}] = S(g; h) = T(g; h) = [S_{g}^{-1}, T_h] = S^{-1}(h; g) = T^{-1}(h; g) \quad (8.3)
\]

**Remark 8.4.** For associative Moufang transformations we have

\[
[T_g, S_{h}^{-1}] = S(g; h) = T(g; h) = [S_{g}^{-1}, T_h] = S^{-1}(h; g) = T^{-1}(h; g) = \text{id} \quad (8.4)
\]

Comparing \((8.3)\) and \((8.4)\) one can say that the Moufang transformations have the property that their associativity is spoiled in the minimal way. Constraints \((8.2)\) and \((8.3)\) are hence called the minimality conditions.

By trial we can propose

**Theorem 8.5** (triality and minimality). The Moufang transformations satisfy the minimality conditions:

\[
[P_h, T_{g}^{-1}] \equiv T_{gh}^{-1}T_{g}T_{h} \equiv P_{gh}P_{g}^{-1}P_{h}^{-1} \equiv [T^{-1}_{g}, T_h] \equiv T_{g}^{-1}T_{h}^{-1}T_{hg} \equiv P_{g}P_{h}P_{hg}^{-1} \quad (8.5)
\]

\[
[S_{h}, P_{g}^{-1}] \equiv P_{gh}^{-1}P_{g}P_{h} \equiv S_{gh}S_{g}^{-1}S_{h}^{-1} \equiv [P_{g}^{-1}, P_h] \equiv P_{g}^{-1}P_{h}^{-1}P_{hg} \equiv S_{g}S_{h}S_{hg}^{-1} \quad (8.6)
\]

**Proof.** Constraints \((8.5a-e)\) and \((8.6a-e)\) hold because \((T, P)\) and \((P, S)\) are birepresentations of \(G\). 

\[\square\]
9 Theorem on kernel of birepresentation

Definition 9.1 (normal divisor [2]). A subloop $N$ of the Moufang loop $G$ is called a normal divisor of $G$ if it is invariant with respect to the following transformations of $X$ from the group $E_G(S, T)$:

$$L(g; h) = L_g^{-1}L_g L_h, \quad M_g = R_g L_g^{-1}$$

If $L(g; h) = id$ for all $g, h$ in $G$, the Moufang loop $G$ is a group and then every $M_g^+$ ($g \in G$) is an inner automorphism of $G$.

Theorem 9.2. The kernel Ker$(S, T)$ of a birepresentation $(S, T)$ of the Moufang loop $G$ is a normal divisor of $G$.

Proof. We know from Theorem 4.6 Ker$(S, T)$ is a subloop of $G$, thus it is sufficient to check that for all $g, h$ in $G$ and $k$ in Ker$(S, T)$ we have

$$S_{M_g^+ k}^{(a)} = id, \quad T_{M_g^+ k}^{(a)} = id \quad (9.1)$$

First calculate

$$S_{M_g^+ k} = S_{R_g L_g^{-1} k} = T_g^{-1} S_{L_g^{-1} k} P_g^{-1} = T_g^{-1} P_g S_k T_g P_g^{-1} = T_g^{-1} P_g T_g^{-1} P_g^{-1} = T_g^{-1} S_g^{-1} P_g^{-1} = (P_g S_g T_g)^{-1} = id$$

Condition (9.1b) can be checked analogously. Next calculate

$$S_{L(g; h) k} = S_{L_{gh}^{-1} L_g L_h k} = P_{gh} S_{L_h L_h k} T_{gh} = P_{gh} P_g^{-1} S_{L_h T_h^{-1} T_{gh}} = P_{gh} P_g^{-1} P_h^{-1} S_h T_h^{-1} T_h^{-1} T_{gh} = P_{gh} P_g^{-1} P_h^{-1} T_h^{-1} T_h^{-1} T_{gh} = P_{gh} P_g^{-1} P_h^{-1} T_h^{-1} T_h^{-1} T_{gh} = (P_h P_g^{-1} T_h^{-1} T_h^{-1} T_{gh}) = id$$

Condition (9.2b) can be checked analogously.

10 Birepresentation of quotient loop $G/\text{Ker}(S, T)$

Recall some basic facts [2] from theory of the Moufang loops.

Let $N$ be a normal divisor of $(S, T)$. The we can define on the Moufang loop $G$ the left (right) equivalence: for $g, h$ in $G$ we set $g \overset{L}{\sim} h$ ($g \overset{R}{\sim} h$) if $g^{-1} h \in N$ ($h g^{-1} \in N$). The resulting
equivalence classes are called the left (right) cosets with respect to the normal divisor \( N \). It turns out \([2]\) that the left and right cosets can be presented as \( gN \) and \( Ng \), respectively, and coincide: \( gN = Ng \). On the set of cosets of \( G \) with respect to \( N \) we can define multiplication:

\[
(gN)(hN) \doteq (gh)N
\]

which satisfy all the Moufang loop axioms. The resulting Moufang loop is called the quotient loop with respect to \( N \) and is denoted by \( G/N \). The unit element of \( G/N \) is \( K \).

The normal divisors coincide \([2]\) with kernels of homomorphisms.

**Theorem 10.1.** Let \((S, T)\) be a birepresentation of the Moufang loop \( G \) and \( K \doteq \text{Ker}(S, T) \) be kernel of the birepresentation \((S, T)\). Then the pair of maps \( gK \to S_g \), \( gK \to T_g \) is a faithful birepresentation of the quotient Moufang loop \( G/K \).

**Proof.** A pair \((S', T')\) of maps \( gK \to S'_gK \), \( gK \to T'_gK \) is a birepresentation of \( G \) if the following conditions are satisfied:

\[
S'_K = T'_K = \text{id} \quad (10.1a)
\]

\[
S'_{gK} T'_{gK} S'_{hK} = S'_{(gK)(hK)} T'_{gK} = S'_{(gh)K} T'_{gK} \quad (10.1b)
\]

\[
S'_{gK} T'_{gK} T'_{hk} = T'_{(hK)(gK)} S'_{gK} = T'_{(hg)K} S'_{gK} \quad (10.1c)
\]

Define \( S'_{gK} \) and \( S'_{gK} \) by the following simple formulae:

\[
S'_{gK} = S_g, \quad T'_{gK} = T_g, \quad \forall gK \in G/K
\]

First of all, note that the definition of \( S'_{gK} \) and \( S'_{gK} \) does depend on the choice of representatives in coset \( gK \). Really, if \( k \in gK \), then \( k = gn \), where \( n \in K \). Then we have

\[
S'_{gK} = S_k = S_{gn} = P^{-1}_g S_n T^{-1}_g = P^{-1}_g T^{-1}_g = S_g
\]

\[
T'_{gK} = T_k = T_{gn} = T^{-1}_g S_n P^{-1}_g = T^{-1}_g P^{-1}_g = T_g
\]

Thus the maps \( gK \to S'_{gK} \), \( gK \to T'_{gK} \) are defined uniquely. The defining relations of \((10.1a), (b), (c)\) follow from \((3.1a–c)\) and the above definition of \( S'_{gK} \) and \( S'_{gK} \). This means that the pair of maps \( gK \to S'_{gK} \), \( gK \to T'_{gK} \) is a birepresentation of \( G/K \). The set

\[
\text{Ker}(S', T') \doteq \{gK \in G/K | S'_{gK} = T'_{gK} = \text{id}\}
\]

is the kernel of birepresentation \((S', T')\). Evidently, \( K \in \text{Ker}(S', T') \). If \( gK \in \text{Ker}(S', T') \), then it follows from

\[
S'_{gK} = S_g = \text{id}, \quad T'_{gK} = T_g = \text{id}
\]

that \( g \in K \). Thus, \( \text{Ker}(S', T') = \{K\} \), from which it follows that birepresentation \( gK \to S'_{gK} \doteq S_g \), \( gK \to T'_{gK} \doteq T_g \) is faithful.

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