RESTRICTION OF SECTIONS OF ABELIAN SCHEMES

NAJMUDDIN FAKHRUDDIN

1. Introduction

1.1. Let $A \to B$ be an abelian scheme. For a subvariety $C$ of $B$, we shall denote by $A(C)$ the group of sections of the abelian scheme $A \times_B C \to C$. We shall prove the following:

**Theorem 1.1.** Let $B$ be a smooth, irreducible, quasi-projective variety over the complex numbers and assume that $B$ has a projective compactification $\bar{B}$ such that $\bar{B} - B$ is of codimension at least two in $\bar{B}$. Then there exists a family of smooth irreducible curves $\{C_q\}_{q \in Q}$ in $B$ parametrised by an irreducible variety $Q$ such that if $p : A \to B$ is an abelian scheme and $q \in Q$ is a generic point, then the restriction map on sections $A(B) \to A(C_q)$ is an isomorphism.

This answers, in a special case, a question of Graber, Harris, Mazur and Starr [5, Question 4].

1.2. Our method of proof is briefly as follows: we first prove the theorem for isotrivial abelian schemes and then reduce the general case to a cohomological statement using the cycle class map. The Lefschetz hyperplane section theorem allows us to further reduce to the case that $B$ is a surface. This is then handled by a monodromy argument involving the cohomology of Lefschetz pencils with coefficients in a local system.

**Remark 1.2.** It seems possible that our method can be extended to any smooth, quasi-projective base $B$. However, the monodromy computations become much more difficult in this generality.

1.3. Conventions. All our varieties will be over the field of complex numbers $\mathbb{C}$. By a generic point of such a variety we shall mean a closed point lying outside a countable union of proper closed subvarieties while by a general point we shall mean a closed point lying in some Zariski open subset.

2. Preliminary reductions

2.1. In this section $A \to B$, $A' \to B$ will always be abelian schemes with $B$ a smooth, irreducible quasi-projective variety of dimension $\geq 1$. Unless stated otherwise, $C$ will be a smooth, irreducible curve in $B$ such that the map $\pi_1(C) \to \pi_1(B)$ is surjective.

**Lemma 2.1.** Suppose the map $A(B) \otimes \mathbb{Q} \to A(C) \otimes \mathbb{Q}$ is an isomorphism. Then the map $A(B) \to A(C)$ is also an isomorphism.

**Proof.** Since all elements of the kernel of the restriction map are torsion, the kernel must be zero since a non-zero torsion point always specializes to a non-zero torsion point. For any $\sigma \in A(C)$ there exists $\tau \in A(B)$ and $n > 0$ such that the restriction of $n\tau$ to $C$ is $\sigma$. Let $Z(\tau, n) = [n]^{-1}(\tau(B))$, where $[n] : A \to A$ is the multiplication by $n$ map. $Z(\tau, n)$ is finite étale over $B$, so the surjectivity assumption on fundamental groups implies that
the restriction map induces a bijection from the set of components of $Z(\tau, n)$ and those of $Z(\tau, n) \times_B C$. Since $\sigma$ corresponds to a component of $Z(\tau, n) \times_B C$ which is of degree 1 over $C$, it follows that it must be the restriction of an element of $A(B)$.

**Lemma 2.2.** Suppose the map $A(B) \to A(C)$ is an isomorphism and let $A \to A'$ be an isogeny of abelian schemes over $B$. Then the map $A'(B) \to A'(C)$ is also an isomorphism.

**Proof.** This follows from Lemma 2.1 since $A(B) \otimes \mathbb{Q} \cong A'(B) \otimes \mathbb{Q}$. □

**Lemma 2.3.** Suppose both the maps $A(B) \to A(C)$, and $A'(B) \to A'(C)$ are isomorphisms. Then the map $A \times_B A'(B) \to A \times_B A'(C)$ is an isomorphism.

**Proof.** This is clear since $A \times_B A'(B) = A(B) \times A'(B)$. □

**Lemma 2.4.** Suppose $A = A_0 \times B$ where $A_0$ is an abelian variety i.e. $A$ is a constant abelian scheme with fibre $A_0$. Let $\bar{B}$ be a normal projective compactification of $B$ and embed $\bar{B}$ in $\mathbb{P}^n$ for some $n$. Then for a generic complete intersection curve $C$ in $B$ of large degree, the map $A(B) \to A(C)$ is an isomorphism.

**Proof.** Sections of a constant abelian scheme correspond to maps from the base to the fibre. For a smooth, irreducible, quasi-projective variety $X$ we denote by $Alb(X)$ the Albanese variety of any smooth projective compactification of $X$. This is universal for morphisms of $X$ to abelian varieties and is determined by the mixed Hodge structure on $H_1(X, \mathbb{Z})$ [3].

Suppose $\dim(B) > 2$. Then by the theorem of Goresky and MacPherson [1, p. 150], it follows that if $B'$ is a general hyperplane section of $B$ then the map $H_1(B', \mathbb{Z}) \to H_1(B, \mathbb{Z})$ is an isomorphism. It follows that $Alb(B') \to Alb(B)$ is also an isomorphism hence

$$A(B) = Mor(Alb(B), A_0) \to Mor(Alb(B'), A_0) = A(B)$$

is an isomorphism.

We may thus assume that $\dim(B) = 2$. Let $\bar{C}$ be a general hypersurface section of $\bar{B}$ and let $C = \bar{C} \cap B$. It follows from loc. cit. that the map $H_1(\bar{C}, \mathbb{Z}) \to H_1(\bar{B}, \mathbb{Z})$ is a surjection, where $\bar{B}$ is a resolution of singularities of $\bar{B}$. Since $Alb(C) = Alb(\bar{C})$ and $Alb(B) = Alb(\bar{B})$, we get an exact sequence of abelian varieties

$$0 \to K \to Alb(C) \to Alb(B) \to 0.$$

If $\bar{B}$ is smooth, the usual theory of Lefschetz pencils [1, Exposé XVIII] implies that if $C$ is generic then $K$ is a simple abelian variety which moreover varies in moduli as $C$ varies, hence $Hom(K, A_0) = 0$. If $B$ is only normal, using Lemma 3.2 one sees that if the degree of hypersurface is sufficiently large then the usual arguments show that $K$ has no “fixed part” as $C$ varies so we still have $Hom(K, A_0) = 0$ for generic $C$. Therefore $Hom(Alb(B), A_0) = Hom(Alb(C), A_0)$, and so

$$A(B) = Mor(Alb(B), A_0) \to Mor(Alb(C), A_0) = A(C)$$

is an isomorphism. □

**Lemma 2.5.** Suppose $A \to B$ is isotrivial (i.e. there exists a finite étale cover $B' \to B$ such that the abelian scheme $A \times_B B' \to B'$ is constant) and let $C$ be a generic complete intersection curve of large degree. Then the map $A(B) \to A(C)$ is an isomorphism.
Proof. We first note that an isotrivial abelian scheme is determined by its monodromy representation \( \pi_1(B) \to Aut(A_0) \), where \( A_0 \) is the fibre over the basepoint.

Suppose \( A(B) \) is non-torsion. Let \( A'' \) be the connected component containing the zero section of the Zariski closure of the union of the images of the elements of \( A(B) \). This is a (non-trivial) abelian subscheme of \( A \) and is moreover constant since the monodromy acts trivially on it. Then \( A \) is isogenous to \( A'' \times_B (A/A'') \), with \( A/A'' \) also isotrivial. The lemma follows in this case from Lemmas 2.4 and 2.3 along with induction on the relative dimension.

If \( A(B) \) is torsion, then \( A(C) \) must also be torsion. For otherwise \( A \times_B C \) would contain a non-trivial constant abelian subscheme (by the argument above) which is not possible because of the surjectivity of \( \pi_1(C) \to \pi_1(B) \). So the lemma follows from Lemma 2.1. \( \square \)

2.2. Let \( p : A \to B \) be an abelian scheme of relative dimension \( n \) and consider the local system \( \mathbb{V} = R^{2n-1}p_! \mathbb{Q} \) on \( B \). The monodromy representation corresponding to this local system is semisimple, so breaks up as a direct sum of irreducible representations. Let \( \mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2 \), where \( \mathbb{V}_1 \) is the local system corresponding to the direct sum of all the irreducible summands with finite image and \( \mathbb{V}_2 \) corresponds to the direct sum of those with infinite image. Then there exist abelian schemes \( p_i : A_i \to B, i = 1, 2 \) such that \( A \) is isogenous to \( A_1 \times_B A_2 \) and \( R^{2n-1}p_i_! \mathbb{Q} \cong \mathbb{V}_i \). \( A_1 \) is isotrivial and \( A_2 \) contains no isotrivial abelian subschemes. By Lemmas 2.2 and 2.5 the proof of Theorem 1.1 reduces to the case that \( p : A \to B \) is an abelian scheme which has no isotrivial abelian subschemes.

Now suppose that \( p : A \to B \) is as above with \( \mathbb{V}_1 = 0 \). Recall from [7] (or see [2]) that the vector space \( H^1(B, \mathbb{V}) \) carries a natural mixed Hodge structure of weights \( \geq 2n \).

Lemma 2.6. \( A(B) \) is a finitely generated abelian group and the cycle class map identifies\(^1\) \( A(B) \otimes \mathbb{Q} \) (via the Leray spectral sequence) with the group of Hodge classes of type \((n, n)\) of \( H^1(B, \mathbb{V}) \).

Proof. The finite generation follows from the theorem of Lang and Néron [6].

To prove the injectivity of the map we may assume that \( B \) is a curve. The cycle class map referred to above (after tensoring with \( \mathbb{Q} \)) can be viewed as being induced by the boundary map

\[
H^0(B, A) = A(B)_{an} \xrightarrow{\partial} H^1(B, \mathbb{V}_Z)
\]

coming from the long exact cohomology sequence corresponding to the short exact sequence of sheaves on \( B \) in the analytic topology

\[
0 \to \mathbb{V}_Z \to \text{Lie}(A/B) \to A \to 0.
\]

Here \( A(B)_{an} \) denotes the abelian group of complex analytic sections, \( \mathbb{V}_Z = R^{2n-1}p_* \mathbb{Z} \) and \( \text{Lie}(A/B) \), the relative Lie algebra of \( A \) over \( B \), is the locally free sheaf associated to the complex local system \( \mathbb{V}_Z \otimes \mathbb{C} \).

Let \( \bar{B} \) be the smooth compactification of \( B \) and let \( \bar{p} : \bar{A} \to \bar{B} \) be the Néron model of \( A \). If \( \partial(\sigma) = 0 \) for some \( 0 \neq \sigma \in A(B) \), then it lifts to a non-zero element \( \bar{\sigma} \) of \( H^0(\bar{B}, \text{Lie}(\bar{B}/A)) \). Since this is a complex vector space, the images of the elements \( t\bar{\sigma}, t \in \mathbb{C}^* \) give a 1-parameter family of elements of \( A(B)_{an} \). By the property of Neron models, \( \sigma \) extends to an element \( \bar{\sigma} \in \bar{A}(\bar{B}) \), hence by continuity any element of \( A(B)_{an} \) close to \( \sigma \) also extends to an element of \( \bar{A}(\bar{B})_{an} = \bar{A}(\bar{B}) \). This implies that \( A \) must contain a non-trivial constant abelian subscheme, which contradicts the hypotheses.

\(^1\)For the proof of Theorem 1.1, the surjectivity of the map is not essential.
The surjectivity follows from Lefschetz’s theorem on $1,1$ classes since cup product with $\theta^{n-1}$, where $\theta \in H^0(B, R^2 p_* Q)$ is the class of a polarisation, induces an isomorphism of mixed Hodge structures

$$H^1(B, R^1 p_* Q) \to H^1(B, R^{2n-1} p_* Q) \otimes Q(n - 1).$$

□

Remark 2.7. If $C$ is a generic complete intersection curve in $B$ then the restriction map $H^1(B, \mathcal{V}) \to H^1(C, \mathcal{V}|_C)$ is always an injection; the difficulty lies in showing that all Hodge classes of type $(n,n)$ lie in the image.

3. LEFSCHETZ PENCILS WITH COEFFICIENTS

3.1. In this section we state mild generalisations of a couple of the results of the theory of Lefschetz pencils.

Lemma 3.1. Let $D$ be the open unit disc in $\mathbb{C}$, $X$ a connected, two dimensional, complex manifold and $\pi : X \to D$ a proper analytic map whose differential is non-zero except at a single point $x_0 \in X$ above $0 \in D$ where it has a non-degenerate critical point. Let $\mathcal{V}$ be a local system of $\mathbb{Q}$-vector spaces on $X$. Then the monodromy of $R^1 \pi_* \mathcal{V}$ restricted to $D^* = D - \{0\}$ is unipotent.

Proof. This follows from the results in SGA7 II [11 Exposés XIII & XIV]: we only indicate the slight changes that need to be made. Let $y \in D^*$ be a basepoint and $\sigma : H^1(X_y, \mathcal{V}) \to H^1(X_y, \mathcal{V})$ be the monodromy automorphism. Then $\sigma - 1$ is the composite of the following sequence of maps

$$H^1(X_y, \mathcal{V}) \xrightarrow{j^*} H^1(V, \mathcal{V}) \xrightarrow{\text{Var}} H^1(V^0, \mathcal{V}) \xrightarrow{\partial_*} H^1(X_y, \mathcal{V})$$

where $V$ is defined on [11 p. 136], $j : V \to X_y$, $j^0 : V^0 \to X_y$, are the inclusions and, following [11 p. 151], the map $\text{Var}$ is as follows: since $\mathcal{V}$ is constant on $V$ we may write $\mathcal{V}|_V \cong \oplus_{i=1}^r \mathcal{Q} \cdot \nu_i$, where $\nu_i$, $i = 1,2,\ldots,r = \text{rank}(\mathcal{V})$, is a local basis of sections. This gives rise to elements $\delta_i \in H^1_c(V^0, \mathcal{V})$ which are well defined upto sign. The basis also gives an isomorphism $\mathcal{V} \to \tilde{\mathcal{V}}$ restricted to $V$, where $\tilde{\mathcal{V}}$ is the dual local system of $\mathcal{V}$, and hence induces an isomorphism $H^1_c(V^0, \mathcal{V}) \cong H^1_c(V^0, \tilde{\mathcal{V}})$. We let $\tilde{\delta}_i \in H^1_c(V^0, \tilde{\mathcal{V}})$, $i = 1,2,\ldots,r$ be the elements corresponding to the $\delta_i$’s under this isomorphism. Then for any $x \in H^1(V, \mathcal{V})$

$$\text{Var}(x) = - \sum_{i=1}^r (x \cdot \tilde{\delta}_i) \delta_i,$$

where the pairing $(x \cdot \tilde{\delta}_i)$ is the natural duality pairing.

It is clear that $(j^*(j^0(\delta_k)) \cdot \tilde{\delta}_l) = 0$ for all $k,l = 1,2,\ldots,r$, hence $(\sigma - 1)^2 = 1$. Thus $\sigma$ is unipotent.

Lemma 3.2. Let $X$ be a normal projective surface, $S$ a finite subset of $X$ including all its singular points and $Y = X - S$. There exists a pencil of very ample curves on $X$, $\{C_p\}_{p \in \mathbb{P}^1}$, with the following properties:

1. A general element of the pencil is smooth and all $C_p$, $p \neq \infty$ are irreducible with at most a single ordinary double point.
2. $S \subset C_\infty$
(3) $C_p$ and $C_q$ for $p \neq q$ meet transversally (at smooth points of $X$).

**Proof.** Let $\mathcal{L}$ be a very ample line bundle on $X$ and choose a trivialisation of $\mathcal{L}$ restricted to $S$ i.e. an isomorphism $\mathcal{L}|_S \cong \mathcal{O}_S$ (which induces a similar isomorphism for all tensor powers of $\mathcal{L}$). For $n > 0$, let $V_n$ be the subspace of $H^0(X, \mathcal{L}^\otimes n)$ consisting of all sections whose restriction to $S$ is a constant section. If $n$ is sufficiently large, this linear system is base point free and induces a morphism $\phi_n : X \to \mathbb{P}(V_n)$ which is an embedding on $X - S$ and maps $S$ to a single point.

Fix $n$ as above and let $X_n = \phi_n(X) \subset \mathbb{P}(V_n)$. Let $\tilde{X}_n \subset \tilde{\mathbb{P}}(V_n)$ be the dual variety of $X_n$. It consists of two irreducible components, the general point of one corresponding to hyperplanes in $\mathbb{P}(V_n)$ tangent to a smooth point of $X_n$ and the points of the other corresponding to hyperplanes containing $\phi_n(S)$.

The proof of the existence of Lefschetz pencils in [1] Exposé XVII goes through without any changes to show that a general pencil in $\tilde{\mathbb{P}}(V_n)$ gives rise to a pencil of hyperplane sections of $X_n$, which when pulled back to $X$ satisfies all the conditions of the lemma. \qed

### 4. Proof of the theorem

4.1. Let $p : A \to B$ be an abelian scheme of relative dimension $n$ with $B$ a smooth connected surface and let $\mathbb{V} = R^{2n-1}p_*\mathbb{Q}$. Let $Y = B$ and let $X$ be a normal projective compactification of $Y$ with $S = X - Y$ a finite set. Assume that $\mathbb{V}$ does not contain any non-trivial sub-local systems with finite monodromy.

**Proposition 4.1.** Let $\mathcal{L}$ be an ample line bundle on $X$. Then for $n$ sufficiently large (depending only on $X$) and $C$ a generic element of $[H^0(X, \mathcal{L}^\otimes n)]$, the restriction map $A(B) \to A(C)$ is an isomorphism.

**Proof.** We have seen in Section 2.2 that it suffices to prove that the restriction map $H^1(Y, \mathbb{V}) \to H^1(C, \mathbb{V}|_C)$ induces a surjection on Hodge classes of type $(n, n)$.

Let $f : \tilde{X} \to \mathbb{P}^1$ be obtained by blowing up the basepoints of a pencil on $X$ obtained by applying Lemma 3.2. Let $\tilde{Y}$ be the inverse image of $Y$ in $\tilde{X}$ and let $\tilde{\mathbb{V}}$ be the pullback of $\mathbb{V}$ to $\tilde{Y}$. Let $U$ be the open subset of $\mathbb{P}^1$ over which $f$ is smooth, let $Y' = f^{-1}(U)$, $f' = f|_{Y'} : Y' \to U$, $\mathbb{V}' = \tilde{\mathbb{V}}|_{Y'}$ and let $\mathbb{W} = R^1f'_*\mathbb{V}'$. Our hypothesis on the monodromy of $\mathbb{V}$ implies that $f'|_U = 0$, hence from the Leray spectral sequence for $f'$ it follows that $H^0(U, \mathbb{W}) = H^1(Y', \mathbb{V}')$.

Let $u$ be a generic element of $U$ and let $C = C_u = f^{-1}(u)$. Let $\alpha$ be a Hodge class of type $(n, n)$ in $H^1(C, \mathbb{V}|_C)$. Since $u$ is generic and $\alpha$ is an algebraic class, it follows that there exist an étale morphism $g : T \to U$ with $T$ a smooth connected curve, $t \in T$ such that $g(t) = u$, and an element $\beta \in H^0(T, g^*\mathbb{W})$ such that $t^*(\beta) = \alpha$. It follows from Lemma 3.1 that the local monodromies of $\mathbb{W}$ around all the points of $\mathbb{P}^1 - (U \cup \{\infty\})$ are unipotent. Since the image of $\pi_1(T)$ in $\pi_1(U)$ is of finite index and $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ is simply connected, it follows that $H^0(U, \mathbb{W}) = H^0(T, g^*\mathbb{W})$. So $\beta$ is already defined over $U$, hence comes from an element of $H^1(Y', \mathbb{V}')$.

The proof is completed by observing that the pullback map $H^1(Y, \mathbb{V}) \to H^1(\tilde{Y}, \tilde{\mathbb{V}})$ is an isomorphism and that the restriction map $H^1(\tilde{Y}, \tilde{\mathbb{V}}) \to H^1(Y', \mathbb{V}')$ induces a surjection (in fact an isomorphism) on Hodge classes of type $(n, n)^2$. \qed

This is elementary in our situation since we know that such classes are algebraic.
4.2. We now complete the proof of the main result of the paper:

Proof of Theorem 1.1. Choose a normal, projective compactification $\tilde{B}$ of $B$ with $\tilde{B} - B$ of codimension at least two in $\tilde{B}$ and embed $\tilde{B}$ in $\mathbb{P}^n$ for some $n$. If $B'$ is a general hypersurface section of $B$ a theorem of Goresky and MacPherson [4, p. 150] implies that if $\dim(B') > 1$ then $\pi_1(B') \to \pi_1(B)$ is an isomorphism, and so also the restriction map

$$H^1(B, R^{2n-1}p_*\mathbb{Q}) \to H^1(B', R^{2n-1}p_*\mathbb{Q}).$$

Since a complete intersection curve in $B'$ is also a complete intersection curve in $B$, it follows from the above and the reductions in Section 2 that it suffices to prove the theorem in the case $B$ is a surface. This follows from the reductions in Section 2 and Proposition 4.1. We see that the family of curves $\{C_q\}_{q \in Q}$ can be chosen to be all smooth complete intersection curves in $B$ of a fixed large multidegree. 

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: naf@math.tifr.res.in