Exact Models of Extremal Dyonic 4D Black Hole Solutions of Heterotic String Theory.

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Abstract

Families of exact $(0, 2)$ supersymmetric conformal field theories of magnetically and electrically charged extremal 4D black hole solutions of heterotic string theory are presented. They are constructed using a $(0, 1)$ supersymmetric $SL(2, \mathbb{R}) \times SU(2)$ Wess–Zumino–Witten model where anomalously embedded $U(1) \times U(1)$ subgroups are gauged. Crucial cancelations of the $U(1)$ anomalies coming from the supersymmetric fermions, the current algebra fermions and the gauging ensure that there is a consistency of these models at the quantum level. Various 2D models, which may be considered as building blocks for extremal 4D constructions, are presented. They generalise the class of 2D models which might be obtained from gauging $SL(2, \mathbb{R})$ and coincide with known heterotic string backgrounds. The exact conformal field theory presented by Giddings, Polchinski and Strominger describing the angular sector of the extremal magnetically charged black hole is a special case of this construction. An example where the radial and angular theories are mixed non–trivially is studied in detail, resulting in an extremal dilatonic Taub–NUT–like dyon.

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1. Introduction.

Since Witten’s observation[1] that an exact conformal field theory of a 2D black hole background for string theory may be obtained by an $SL(2, \mathbb{R})/U(1)$ gauged Wess–Zumino–Witten (WZW) model, there has been much activity in the field of string theory in non-compact curved spacetime backgrounds employing similar techniques.

Due to the fact that non-compact Lie Groups in general possess continuous spectra of unitary states, the conformal field theories derived from the WZW models based on these groups is poorly understood at present. So although it would be desirable to study the black hole and related conformal field theories in as much detail as the more familiar conformal field theories based on compact groups, little progress has been made beyond calculations which severely truncate the spectra\(^1\).

While waiting for the development of the tools needed to calculate extensively in these theories, researchers in the field have not been idle. There has been much energy directed towards the problem of finding more of this type of exact solution. At the level of a sigma model the solutions are the all orders in $\alpha'$ solution to the $\beta$–function equations of the background field problem in string theory[5]. Beyond the $\alpha'$ expansion, the non-perturbative information contained in these models describe aspects of quantum gravity which may not be contained in a metric theory of gravity.

Of particular interest is the problem of finding the conformal field theories corresponding to 4D black hole solutions of string theory. This is because of the expectation that string theory, as a possibly relevant quantum theory of gravity, might contribute significantly novel and useful features to the physics of these interesting objects. Two of the most pertinent problems here are the black hole information paradox and the nature of physics at curvature singularities. The exact conformal field theories corresponding to these backgrounds contain, in principle, the answers that string theory has to give about these problems.

One of the first lessons from string theory in the classroom of black hole physics is that the necessary presence of scalar fields significantly distinguishes the physics of

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\(^1\) See for example refs.[2][3][4] for some direct work on the black hole coset using conformal field theory techniques.
stringy black holes from that of their Einsteinian cousins. Consider the case when the black holes are charged. The relevant scalar fields are the dilaton and modulus fields arising from the compactification of the six internal dimensions of the heterotic string theory\(^2\). They combine to give an effective scalar–Maxwell coupling \(e^{-\alpha \phi F^2}\) in the effective 4D theory, where \(\alpha \geq 0\). In particular for \(\alpha\) non–zero, the Reissner–Nordstrøm solution, the prototype of the charged black hole in General Relativity, is inapplicable as a solution even in regions of spacetime where the curvature is mild. (This is in contrast to the Schwarzchild solution, which together with a constant dilaton is a stringy solution at one loop order in \(\alpha'\), and hence outside sufficiently large neutral black holes.)

In this paper only the pure dilaton solutions will be considered\(^3\), for which \(\alpha = 1\). For this case, the model charged solution is not Reissner–Nordstrom solution but instead the magnetically charged dilatonic black hole solution\(^8\) and its dyonic generalisations, some of which were found by exploiting an \(SL(2, \mathbb{R})\) duality symmetry of the equations of motion\(^9\). These types of solutions were only known to one loop order in \(\alpha'\) until relatively recently. An exact solution for the spherically symmetric magnetically charged 4D black hole solution to heterotic string theory was presented by Giddings, Polchinski and Strominger\(^10\) (GPS). This solution is the exact conformal field theory for which the dilatonic magnetic black hole of Gibbons and Maeda\(^8\) is the low energy limit. Stated more precisely, it describes the extremal black hole limit with an infinite throat. The conformal field theory takes the form of a product of the supersymmetric 2D linear dilaton black hole for the radial \((\sigma, t)\) sector and a monopole conformal field theory for the angular \((\theta, \phi)\) sector. The linear dilaton black hole is constructed using a supersymmetric gauged \(SL(2, \mathbb{R})/U(1)\) WZW model, while the monopole theory was identified as a \(Z_{2Q+2}\) orbifold of an \(SU(2)\) WZW model. This construction will be briefly reviewed in the next section.

\(^2\) The author is grateful to M J Duff for pointing out the role of the modulus field. See ref.\(^6\) for examples of solutions of heterotic string theory which include modulus fields arising in toroidal compactifications.

\(^3\) Ref.\(^7\) includes an extensive discussion of scalar–Maxwell black hole physics for more general values of \(\alpha\).
The significance of this factorisation into a product form is important. The physics of stringy black hole evaporation (and other processes) studied in 2D is quite relevant to 4D stringy black holes in the case of exact extremality. It is presumably possible to expand about this point in parameter space to study the physics of near extremal black holes. Generically the theory decomposes into a ‘black hole + wormhole’ situation, where the black hole is at the bottom of a semi–infinite wormhole. The wormhole has topology \( \mathbb{R}^2 \times S^2 \) in the infinite ‘throat’ region, and is asymptotic to Minkowski space after emerging from the ‘mouth’ region. In the throat region the two–sphere has a constant radius set by the charge of the black hole. The conformal field theory of GPS describes the black hole together with the semi–infinite throat. The same will be the case for the solutions described in this paper. Little is known about the conformal field theory describing the widening transition from the throat region to the mouth of the wormhole. However operators describing marginal perturbations of the conformal field theory, which represent clearing the throat region have been identified and discussed[11][10].

There have also been some conformal field theories of bosonic strings in axisymmetric 4D black hole backgrounds presented in refs.[12]. These solutions were obtained by gauging anomaly–free subgroups of \( SL(2, \mathbb{R}) \times SU(2) \times U(1) \) WZW models, or equivalently by doing some \( O(d, d) \) duality transformations on ungauged WZW models.

The group \( SL(2, \mathbb{R}) \times SU(2) \) has played a prominent role in many of the constructions generalising Witten’s work, starting with the rotating solutions of ref.[13] and including a cosmological solution[14], with numerous other constructions in the literature. This type of WZW model will also play a central role in this paper, with many novel features. A family of exact heterotic string solutions will be constructed as a \((0, 1)\) supersymmetric \(^5 SL(2, \mathbb{R}) \times SU(2)\) WZW model with \(U(1) \times U(1)\) subgroups gauged. The subgroup will be embedded in a way which will produce a classical anomaly in contrast to the non–anomalous embeddings more frequently used. This anomaly, together with the anomaly from the right–moving

\(^4\) The process of approaching extremality for the dilatonic magnetic black hole is described in ref.[10] and also discussed in section 9.

\(^5\) The final model later has \((0, 2)\) supersymmetry due to the nature of the coset. See later sections for a discussion of this.
supersymmetric fermions, will be cancelled against an anomaly coming from left–moving fermions which have been put in by hand to play the role of the current algebra fermions of the heterotic string theory. After integrating out the gauge fields, the resulting heterotic sigma model possesses a spacetime interpretation as extremal magnetically and electrically charged backgrounds with a non–trivial metric, dilaton, gauge and antisymmetric tensor fields. The metrics possesses asymptotic throat regions.

The model of GPS may be understood as a special case of the construction of this paper\(^6\), and although it was not presented in terms of a gauged WZW with fermions, this is a most natural framework within which to describe that model. This framework readily leads to fruitful generalisations, to which end this paper is a first step. The paper begins with a brief review in section 2 of the identification by GPS of their angular theory together with world sheet fermions as a WZW orbifold. Section 3 then recasts the theory as a gauged WZW model as a prototype illustration of the construction. It ends by discussing the non–trivial task of determining the correct spacetime metric from the final theory. This procedure differs from that which is involved in the more traditional gaugings of WZW models because of the non–negligible effect of the fermions on the spacetime. Section 4 constructs a large family of conformal field theories based on $SL(2,\mathbb{R})$ with heterotic fermions using the basic technique of this paper. The family includes the charged 2D black hole solutions already known to one loop as discussed in section 5. Extremal 4D solutions can then be constructed using products of the $SL(2,\mathbb{R})$ theories with the $SU(2)$ theories. The rest of the paper is devoted to constructing an extremal dyonic 4D solution as starting example of the rich class of heterotic string backgrounds which might be constructed in the manner outlined, by mixing the radial and angular theories non–trivially.

\(^6\) This construction was suggested to the author by E Witten.
2. The GPS Monopole theory.

The identification of the monopole theory à la Giddings, Polchinski and Strominger (GPS) begins with the $SU(2)$ WZW action at level $k$:

$$I_{WZW_k} = -\frac{k}{4\pi} \int_{\Sigma} d^2 z \ Tr(g^{-1} \partial_z g \cdot g^{-1} \partial_z g) - ik\Gamma(g) \tag{2.1}$$

where

$$\Gamma(g) = \int_B g^* \omega. \tag{2.2}$$

As is familiar this model is a theory of maps $g : \Sigma \to SU(2)$. $\Sigma$ is $(1 + 1$ dimensional) spacetime, the boundary of a ball $B$. Here $\omega$ is a $SU(2)_L \times SU(2)_R$ invariant 3–form on the manifold of $SU(2)$, normalised so that its integral over the whole manifold is $2\pi$.

It is convenient to use the following Euler parameterisation for $g \in SU(2)$:

$$g = e^{i\phi \sigma_3/2} e^{i\theta \sigma_2/2} e^{i\psi \sigma_3/2} = \begin{pmatrix} e^{\frac{i}{2} \phi + \cos \frac{\theta}{2}} & e^{\frac{i}{2} \phi - \sin \frac{\theta}{2}} \\ -e^{-\frac{i}{2} \phi - \sin \frac{\theta}{2}} & e^{-\frac{i}{2} \phi + \cos \frac{\theta}{2}} \end{pmatrix}, \tag{2.3}$$

where the $\sigma_i$ are the Pauli matrices and

$$\phi_{\pm} \equiv \phi \pm \psi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 4\pi. \tag{2.4}$$

This gives

$$I_{WZW_k} = \frac{k}{8\pi} \int d^2 z \ \left\{ G_{ab}^{S_3} \partial_z X^a \partial_z X^b \right\} - i\Gamma(g) =$$

$$= \frac{k}{8\pi} \int d^2 z \ \{ \partial_z \theta \partial_z \theta + \partial_z \phi \partial_z \phi + \partial_z \psi \partial_z \psi + \cos \theta (\partial_z \phi \partial_z \psi + \partial_z \psi \partial_z \phi) \}$$

$$+ \frac{k}{8\pi} \int d^2 z \ (\pm 1 - \cos \theta)(\partial_z \phi \partial_z \psi - \partial_z \psi \partial_z \phi). \tag{2.5}$$

The Wess–Zumino term is constructed using the fact that the unique (up to scalings) $G_L \times G_R$ invariant three–form on $SU(2)$ is the volume form $\omega_V$ of the three–sphere $S_3$, which is $\omega_V = |\det G_{ab}^{S_3}|^{1/2} d\theta \wedge d\phi \wedge d\psi$. The metric is

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos \theta \\ 0 & \cos \theta & 1 \end{pmatrix}, \tag{2.6}$$
and therefore (given that the volume of $S_3$ is $4\pi^2$)

$$\omega = \frac{1}{8\pi} \sin \theta d\theta \wedge d\phi \wedge d\psi. \quad (2.7)$$

The Wess–Zumino term is written as a two–dimensional integral by writing $\omega = d\lambda$ locally. This has solution $\lambda = \frac{1}{8\pi} (\pm 1 - \cos \theta) d\phi \wedge d\psi$ where the $\pm$ choice corresponds to the North and South poles $\theta = 0, \pi$ respectively. Using this together with $\epsilon^z_z = i = -\epsilon^z_z$ yields

$$\Gamma(g) = \int_B g^* \omega = \int \Sigma g^* \lambda = \int d^2 z \epsilon^{ab} \lambda_{ij} \partial_a \psi^i \partial_b \psi^j$$

$$= \frac{i}{8\pi} \int d^2 z \left( \pm 1 - \cos \theta \right) \left( \partial_z \phi \partial_z \psi - \partial_z \psi \partial_z \phi \right) \quad (2.8)$$

Now note that the $S^3$ metric may be decomposed as follows:

$$G_{\alpha\beta} \partial_z X^\alpha \partial_z X^\beta = (G_{\mu\nu}^{S_2} + 4 A^M \partial z X^\mu \partial z X^\nu - (1 \mp 2 \cos \theta) \partial_z \phi \partial_z \phi + \partial_z \psi \partial_z \psi + \cos \theta (\partial_z \phi \partial_z \psi + \partial_z \psi \partial_z \phi))$$

Here $X^1 = \theta$ and $X^2 = \phi$ and $A^M_{\phi}$ is the non–zero component of the field $A^M$ of a magnetic monopole:

$$A^M = \left( \begin{array}{c} 0 \\ \frac{1}{2} \mp \cos \theta \end{array} \right),\quad (2.9)$$

where the ‘$\pm$’ choice refers to the Northern or Southern hemisphere of $S^2$.

In [10] the identification of the heterotic sigma model is made with a Wess–Zumino–Witten model by identifying $\psi = \xi \mp \phi$ in the North (South) pole. Then with $\xi = X^3/Q_+$, and setting the level $k = 2Q_+Q_-$, the model is identical to that defined in equation (3.14) of Giddings, Polchinski and Strominger:

$$I_{GPS} = \frac{Q_+Q_-}{4\pi} \int d^2 z \left\{ (G_{\mu\nu}^{S_2} + 4 A^M \partial z X^\mu \partial z X^\nu + \frac{1}{Q_+^2} (\partial_z X^3 - 4 Q_+ A^M \partial z X^\mu) \partial z X^3) \right\}. \quad (2.10)$$

(Note that $Q_\pm \equiv Q \pm 1$)

In [10] the bosonic field $X^3$ arose in the sigma model by bosonising world sheet fermions and is $2\pi$ periodic. In order to complete the identification therefore,
given that $\xi$ is a $4\pi$ periodic field the further identification $\xi \sim \xi + 2\pi/Q_+$ must be made, and therefore the sigma model of [10] was identified as an $SU(2)/Z_{2Q_+}$ orbifold Wess–Zumino–Witten model at level $2Q_+Q_-$. This model is the angular sector of a 4D heterotic sigma model where the radial sector is a supersymmetric $SL(2, \mathbb{R})/U(1)$ coset. The product of these two models is a bosonised heterotic string theory whose background fields arise as the extremal limit of the magnetically charged dilaton black hole of [8][9]. This ends the review.

One of the purposes of this paper is to construct more general 4D heterotic string backgrounds by mixing the $(r, t)$ and $(\theta, \phi)$ sectors non–trivially. This is carried out in later sections. The next section will outline the central and motivating observation about the GPS model: It is equivalent to a gauged WZW model (with fermions) possessing many interesting properties.

3. The GPS Monopole theory as a gauged WZW model.

Now start again by gauging a right $U(1)$ subgroup of the $SU(2)$ Wess–Zumino–Witten model. The action of this subgroup (denoted $U(1)_R$) is:

$$g \rightarrow gh, \quad h = e^{i\sigma_3/2}$$

$$\psi \rightarrow \psi + \alpha.$$

This gauging procedure is anomalous. However an action for the gauged model may be written which expresses this (classical) anomaly in such a way that only depends upon the gauge field[15]:

$$I(g, A) = I_{WZW_k} + \frac{k}{2\pi} \int d^2 z \; \text{Tr}[A_z g^{-1} \partial_z g] - \frac{k}{4\pi} \int d^2 z \; \text{Tr}[A_z A_z]. \quad (3.1)$$

Under $\delta g = -gu$ and $\delta A_a = -\partial_a u - [A_a, u]$, the non–zero variation is

$$\delta I(g, A) = \frac{k}{4\pi} \int d^2 z \; \text{Tr}(u F_z) \equiv \frac{k}{4\pi} \int d^2 z \; \text{Tr}[u(\partial_z A_z - \partial_z A_z)]. \quad (3.2)$$

For the subgroup defined above, $u = -i\alpha \sigma_3/2$ (and $\delta A_a = \partial_a \alpha$) and the anomaly is

$$\delta I(g, A) = \frac{\alpha k}{8\pi} \int d^2 z \; F_z. \quad (3.3)$$
This may be checked directly in the Euler parameterisation of the action. (Note that the gauge fields have been written above as

\[ A_a = i A_a \sigma_3 / 2. \]

Also in the Euler parameterisation (2.3), \( \text{Tr}[\sigma_3 g^{-1} \partial_z g] = i [\partial_z \psi + \cos \theta \partial_z \phi] \). This action will be the first building block of the heterotic monopole model.

The rest of this heterotic sigma model is as follows. By asking for (0,1) supersymmetry there are two right–moving coset fermions, the supersymmetric partners of \( \theta \) and \( \phi \) in the coordinates of the previous sections. They are minimally coupled to the gauge fields with unit charge. They have an action

\[ I_{R}^F = \frac{i k}{4 \pi} \int d^2 z \ Tr(\Psi_R D\tau \Psi_R), \]  

where \( \Psi \) takes its values in the Lie algebra of \( SU(2)/U(1)_R \) and \( D\tau \) is the gauge covariant derivative. This action is classically gauge invariant under the above gauge transformations. However, there is the familiar quantum anomaly at one loop:

\[ \delta I_{R}^F = \frac{\alpha}{4 \pi} \int d^2 z \ F_{\tau\tau}. \]  

(3.4)

For the left–moving sector add two left–moving fermions which will play the role of the current algebra fermions in the heterotic string theory. They will be coupled to the two–dimensional gauge field with strength \( Q \) and will contribute an anomaly:

\[ \delta I_{L}^F = \frac{-Q^2 \alpha}{4 \pi} \int d^2 z \ F_{\tau\tau}. \]  

(3.5)

Notice that all of the anomalies discussed have the same structure. Therefore, an anomaly–free gauge invariant theory

\[ I_{\text{total}} = I(g, A, k) + I_{R}^F + I_{L}^F \]

may be constructed if the anomalies cancel, that is if

\[ k = 2Q^2 - 2 = 2Q_+ Q_. \]  

(3.6)

This is the same condition as in the model of GPS.
With the above ingredients it is difficult to carry out the remaining procedures involved in the construction of the coset theory. Although the model has been shown to be gauge invariant by considering it at one loop, the Lagrangian constructed out of the fermions and the WZW model is not classically gauge invariant, and in this form will not yield the correct coset theory, if (for example) the gauge fields were integrated out\textsuperscript{7}. One way to proceed is to bosonise the fermions. By doing this, the one–loop anomalies become classical anomalies, and thus $I_{total}$ can be written as a manifestly gauge invariant theory at tree level.

Now the fermions bosonise into a $2\pi$ periodic bosonic field $\Phi$ with the following action:

$$I_B = \frac{1}{4\pi} \int d^2 z \ (\partial_z \Phi - Q_+ A_z)(\partial_{\bar{z}} \Phi - Q_+ A_{\bar{z}}) - Q_\Phi F_{z\bar{z}}.$$  

(3.8)

The term proportional to $F_{z\bar{z}}$ will yield the anomaly of the fermionic theories, now made classical by the bosonisation procedure. The total action $I_{total}$ has the following $U(1)_R$ gauge invariance

$$\delta \psi = \alpha$$
$$\delta \Phi = Q_+ \alpha$$
$$\delta A_a = \partial_a \alpha.$$  

(3.9)

Next fix a gauge and integrate out the gauge fields. There are (at least) two interesting gauge choices:

1. $\Phi = 0$;
2. $\psi = \mp \phi$.

(3.10)

Condition (1) will return directly to the $SU(2)/Z_{2Q_+}$ theory of GPS. With $\Phi$ gauged to zero the earlier identification $\psi = \xi \mp \phi = X^3/Q_+ \mp \phi$ may be made. The extra identification $X^3 \sim X^3 + 2\pi$ corresponds to a residual discrete $Z_{2Q_+}$ subgroup of the original $U(1)_R$ gauge group: The gauge fixing (1) was incomplete.

\textsuperscript{7} Typically the quadratic terms in the gauge fields will not have the correct form.
In case (2) the following action results (renaming $\Phi$ as $X^3$):

$$I = I_{WZW} + \frac{k}{8\pi} \int d^2z \left\{ \frac{2}{k} \partial_z X^3 \partial_z X^3 + A_\tau \left( 4A_M^M \partial_z \phi - \frac{4}{k} \partial_z X^3 \right) 
+ A_z \left( -\frac{4Q}{k} \partial_z X^3 \right) \right\} \cdot$$

Finally, integrating out the gauge fields and recalling (3.7) leaves finally the GPS theory:

$$I_{GPS} = \frac{Q+Q-}{4\pi} \int d^2z \left\{ (G^{S\mu}_\nu + 4A_M^M A_\nu^M) \partial_z X^\mu \partial_z X^\nu + \frac{1}{Q^2+}(\partial_z X^3 - 4Q+ A_M^M \partial_z X^\mu) \partial_z X^3 \right\} \quad (3.11)$$

This completes the quantum construction of the ‘monopole’ sector of the heterotic sigma model of Giddings, Polchinski and Strominger, as a gauged Wess–Zumino–Witten model. Note that this model is actually has $(0,2)$ supersymmetry, as the $(0,1)$ supersymmetry is enhanced by that fact that $SU(2)/U(1)$ is a Kähler coset[16][17].

It is important to note here how to determine the 2D spacetime metric and gauge fields of the final heterotic string theory. This will be vitally important in more complicated models.

In more ‘traditional’ gauged WZW model constructions, there is no necessity for cancelation of anomalies coming from the spacetime sector with those of current algebra fermions. In these cases the spacetime metric would simply be read off a final action like (3.11) as $dS^2 = Q^2 \{d\theta^2 + (\sin^2 \theta + 4A_M^M A_\phi^M) d\phi^2 \}$, together with the $U(1)$ gauge field $QA_\phi^M$ coupling to the left movers. This naive procedure is incorrect here. The first sign that something is different here is the fact that the Ricci tensor $R_{\mu\nu}$ and the square of the electromagnetic field strength $F_{\mu\lambda}F_{\nu}^\lambda$ appear now at the same order in perturbation theory in the $\beta$–function equations. This is because perturbation theory in $\alpha' \sim 1/k$ has been replaced by perturbation theory in $1/Q$ due to the anomaly equation (3.7). Therefore the effects of the fermions on the metric usually negligible to leading order must be dealt with here.

Calculating this back–reaction of the fermions on the metric is again a non–trivial task which must be carried out at one loop. However, bosonisation is again the
key to carrying out the procedure efficiently. This is the more subtle outcome of the anomaly cancelling technique used to construct the models in this paper. The bosonisation, presented as a device to enable a classically gauge invariant Lagrangian to be written, also enables the correction to the naive metric to be calculated as follows. Looking at the bosonic heterotic sigma model (3.11), the conserved $U(1)$ current is readily identified as: $dX^3 - 2Q_+ A^M_\mu dX^\mu$, and factorising the action into the following form:

$$I = \frac{Q_+Q_-}{4\pi} \int d^2z \left\{ \partial_\theta \partial_\theta + \sin^2 \theta \partial_\phi \partial_\phi + \frac{1}{Q_+^2} \left( \partial_\theta X^3 - 2Q_+ A^M_\mu \partial_\theta X^\mu \right) \left( \partial_\phi X^3 - 2Q_+ A^M_\mu \partial_\phi X^\mu \right) - \frac{2A^M_\phi}{Q_+} \left( \partial_\theta X^3 \partial_\theta X^\mu - \partial_\phi X^3 \partial_\phi X^\mu \right) \right\}$$

which upon refermionisation is equivalent to:

$$I = \frac{Q_+Q_-}{4\pi} \int d^2z \left\{ \partial_\theta \partial_\theta + \sin^2 \theta \partial_\phi \partial_\phi + \frac{1}{2\pi} \int d^2z \left\{ \lambda_R \left( \partial_\theta - 2i A^M_\mu \partial_\theta X^\mu \right) \lambda_R + \lambda_L \left( \partial_\phi - 2i Q A^M_\mu \partial_\phi X^\mu \right) \lambda_L + \frac{1}{Q_+} \lambda_R \lambda_R \lambda_L \lambda_L \right\}.$$  

This clearly displays the heterotic sigma model with the promised bosonic and fermionic content. The metric is the round 2–sphere metric with radius $Q^2$ and the gauge field is a $U(1)$ monopole with charge $Q$. This was the starting point of GPS[10].

The rest of this paper deals with other constructions based on the ideas presented in this section. These constructions will yield many interesting models.

8 This form is not unrelated to the Kaluza–Klein form.

9 Crucially the refermionisation must be done taking into account the presence of the composite gauge field $A_a = -2Q_+ A^M_\mu \partial_a X^\mu$. See the next section for the more general case.
4. Some other 2D theories based on $SL(2, \mathbb{R})$.

The procedures used above to obtain the monopole theory as a gauged WZW model can be used in wider applications. An obvious one is the closely related problem of discovering new exact 4D black hole backgrounds which will be addressed later in this paper. First, another obvious avenue will be explored. What type of physics results from all of the various gaugings of $SL(2, \mathbb{R})$ beyond those which yield the 2D black hole\[1\] or Liouville theory–like models\[18\]? Much more freedom is allowed, typically the symmetry:

$$g \rightarrow e^{\epsilon \sigma L/2} g e^{\epsilon \sigma R/2},$$

(4.1)

where $g \in SL(2, \mathbb{R}); \{\sigma_R, \sigma_L\} \in \{\sigma_3, i\sigma_2, \sigma_1, \sigma_\pm = \sigma_1 \pm i\sigma_2\}$, might be gauged, producing an anomaly

$$A = \epsilon \text{Tr}[\sigma_L^2 - \sigma_R^2] \frac{k}{8\pi} \int d^2 z F_{zz}$$

(4.2)

for the appropriately chosen form of the gauged action as discussed in the previous section. Supersymmetric right moving fermions and some ‘current algebra’ left movers with a free charge $Q_L$ (or vice–versa) can be introduced as before to rescue the model from gauge non–invariance, if the total anomaly equation is satisfied$^{10}$:

$$-k \text{Tr}[\sigma_L^2 - \sigma_R^2] = 4(Q_L^2 - Q_R^2).$$

(4.3)

Here $Q_R$ is fixed by the choice of $\sigma_R$. For the normalisations chosen here $Q_R$ is positive and satisfies $Q_R^2 = |\text{Tr}\sigma_R^2/2|$.

The 2D models are (after bosonising the fermions to get the correct quadratic terms):

$$I = \frac{k}{4\pi} \int d^2 z \left\{ \text{Tr}(g^{-1} \partial_z g \cdot g^{-1} \partial_{\bar{z}} g) + ik \Gamma(g) \right\}$$

$$- \frac{k}{4\pi} \int d^2 z \left\{ A_{\bar{z}} \left( \text{Tr}[\sigma_R g^{-1} \partial_z g] + 4k Q_R \partial_z \Phi \right) + A_z \left( \text{Tr}[\sigma_L \partial_{\bar{z}} g g^{-1}] + 4k Q_L \partial_{\bar{z}} \Phi \right) \right.$$

$$- \frac{2}{k} \partial_z \Phi \partial_{\bar{z}} \Phi - \frac{1}{2} A_z A_{\bar{z}} \left( \text{Tr}[\sigma_L g \sigma_R g^{-1}] + \frac{Q_L \text{Tr}\sigma_R^2 - Q_R \text{Tr}\sigma_L^2}{Q_L - Q_R} \right) \right\}.$$  

(4.4)

$^{10}$ The sign of $k$ has been reversed to achieve a signature of $(− + +)$ for the $SL(2, \mathbb{R})$ manifold. This is desirable at least in the case of the black hole theory\[1\].
They are gauge invariant under (4.1) and
\[ \Phi \rightarrow \Phi + (Q_L + Q_R)\epsilon, \quad A_a \rightarrow A_a + \partial_a \epsilon \] (4.5)
and have \((0, 1)\) supersymmetry\(^{11}\) (after refermionisation via \(\partial_z \Phi \sim k\lambda^1_L\lambda^2_R, \partial_{\bar{z}} \Phi \sim k\lambda^1_R\lambda^2_L)\).

These models may be studied in their own right as 2D models, or used as building blocks for higher dimensional theories, as will be carried out later in the paper. To obtain a leading order approximation to the sigma–model geometry of these models, the gauge fields may be integrated out to give:

\[ I = I_{\text{WZW}} - \frac{k}{2\pi} \int d^2z \frac{1}{D} \left\{ \text{Tr}[\sigma_R g^{-1}\partial_z g]\text{Tr}[\sigma_L \partial_{\bar{z}} g^{-1}] \right\} \]
\[ + \frac{1}{2\pi} \int d^2z \frac{1}{D} \left\{ \left( D - \frac{8Q_L Q_R}{k} \right) \partial_z \Phi \partial_{\bar{z}} \Phi \right\} - 4 \left( Q_R \partial_z \Phi \text{Tr}[\sigma_L \partial_{\bar{z}} g^{-1}] + Q_L \partial_{\bar{z}} \Phi \text{Tr}[\sigma_R g^{-1}\partial_z g] \right), \]

where
\[ D = \text{Tr}[\sigma_L g\sigma_R g^{-1}] + \frac{Q_L \text{Tr}[\sigma_R^2 - Q_R \text{Tr}[\sigma_L^2]}{Q_L - Q_R} \] (4.6)

Considering the one–loop determinant arising from the integration measure\(^{19}\) will yield the dilaton:

\[ \hat{\Phi} = -\frac{1}{2} \log \left( \text{Tr}[\sigma_L g\sigma_R g^{-1}] + \frac{Q_L \text{Tr}[\sigma_R^2 - Q_R \text{Tr}[\sigma_L^2]}{Q_L - Q_R} \right) + \hat{\Phi}_0 \] (4.8)

where \(\hat{\Phi}_0\) is an arbitrary constant.

To proceed, it is natural to parameterise \(g\) as follows:
\[ g = e^{t_L \sigma_L/2} e^{t_R \sigma_R/2} \] (4.9)

such that the gauge transformations (4.1) act as shifts of \(t_L\) and \(t_R\) and \(G\) is a one–parameter \(SL(2, \mathbb{R})\) subgroup chosen to supply the (gauge invariant) third

\(^{11}\) As in the \(SU(2)\) case in the previous section, this is enhanced to \((0, 2)\) supersymmetry when the Kähler conditions for the coset are satisfied. This will be true for models based on gauging all of the subgroups except those generated by the strictly triangular \(\sigma^+\).
independent coordinate on the group manifold. \( \mathcal{G} \) should be chosen to be in a different conjugacy class from that of either of the neighbouring factors in order to define an independent third coordinate. This then corresponds to the parameterisation of the \( SL(2, \mathbb{R}) \) manifold by doing combinations of \( SO(1, 1) \) Lorentz boosts, \( SO(2) \) rotations and \( E(1) \) translations. The parameters will have ranges \([0, \infty)\) or \((-\infty, \infty)\) for the non–compact subgroups and \([0, 4\pi]\) for the rotations. Choosing \( \mathcal{G} = \exp(r \sigma_r / 2) \) where \( \sigma_r \in \{\sigma_3, i\sigma_2, \sigma_1, \sigma_\pm\} \) the coordinate invariant expressions reduce to:

\[
\text{Tr}[g^{-1} \partial_z g \cdot g^{-1} \partial_z g] = \frac{1}{4} \left( \partial_z r \partial_z r \text{Tr}\sigma_r^2 + \partial_z t_L \partial_z t_L \text{Tr}\sigma_L^2 + \partial_z t_R \partial_z t_R \text{Tr}\sigma_R^2 \right)
\]

\[
+ \left( \partial_z t_L \partial_z t_R + \partial_z t_R \partial_z t_L \right) \text{Tr}[\sigma_L G \sigma_R G^{-1}] \tag{4.10}
\]

and

\[
\text{Tr}[\sigma_R g^{-1} \partial_z g] = \frac{1}{2} \left( \partial_z t_R \text{Tr}\sigma_R^2 + \partial_z t_L \text{Tr}[\sigma_L G \sigma_R G^{-1}] \right)
\]

\[
\text{Tr}[\sigma_L \partial_z gg^{-1}] = \frac{1}{2} \left( \partial_z t_L \text{Tr}\sigma_L^2 + \partial_z t_R \text{Tr}[\sigma_L G \sigma_R G^{-1}] \right). \tag{4.11}
\]

As \( SL(2, \mathbb{R}) \) is three dimensional the unique choice (up to scalings) of the integrand of the 3D Wess–Zumino term is the volume form \( \omega_V \) obtained by simply reading off the metric \( M \) from (4.10) and forming \( \omega_V = |\text{det} M|^{1/2} dr \wedge dt_L \wedge dt_R \), where

\[
|\text{det} M| = \text{Tr}\sigma_r^2 \left( \text{Tr}\sigma_L^2 \text{Tr}\sigma_R^2 - \text{Tr}[\sigma_L G \sigma_R G^{-1}] \right) \tag{4.12}
\]

is clearly a function of \( r \) only. So it is possible to write the volume form as a closed form \( \omega_V = d\lambda \) and solve uniquely for the components of \( \lambda = \lambda_{RL}(r) dt_R \wedge dt_L \) in terms of the first \( r \)–integral of \( |\text{det} M|^{1/2} \). Depending upon the choices made for \( \mathcal{G} \), the solution will be able to be written globally or only locally. Either way, the torsion term of the sigma model can be written as a two–dimensional field theory due to the choices made above.

Given the choice (4.9) it is natural to work in terms of the gauge invariant combination \( t = \pm(t_L - t_R) \) by choosing a gauge in which either \( t_R \) or \( t_L \) (respectively) is
zero. In this gauge the Wess–Zumino term disappears, and the final theory is:

\[
I = \frac{k}{2\pi} \int d^2 z \left\{ \frac{\text{Tr} \sigma^2}{8} \partial_z r \partial_z t - \frac{\text{Tr} \sigma^2_R}{8} \partial_z t \partial_z t - \frac{\text{Tr}[\sigma_L G \sigma_R G^{-1}]}{\text{Tr}[\sigma_L G \sigma_R G^{-1}]} - \frac{\left( \frac{Q_L \text{Tr} \sigma^2_R - Q_R \text{Tr} \sigma^2_L}{Q_L - Q_R} \right)}{\left( \frac{Q_L \text{Tr} \sigma^2_R - Q_R \text{Tr} \sigma^2_L}{Q_L - Q_R} \right)} \right\} 
\]

\[
+ \left( \frac{1}{k} - \frac{8Q_LQ_R}{k^2 \left( \text{Tr}[\sigma_L G \sigma_R G^{-1}] + \left( \frac{Q_L \text{Tr} \sigma^2_R - Q_R \text{Tr} \sigma^2_L}{Q_L - Q_R} \right) \right)} \right) \partial_z \Phi \partial_z \Phi 
\]

\[
- \left( \frac{2Q_R \text{Tr}[\sigma_L G \sigma_R G^{-1}] - \left( \frac{Q_L \text{Tr} \sigma^2_R - Q_R \text{Tr} \sigma^2_L}{Q_L - Q_R} \right) \right) \partial_z \Phi \partial_z \Phi 
\]

\[
- \left( \frac{2Q_L \text{Tr} \sigma^2_R - \left( \frac{Q_L \text{Tr} \sigma^2_R - Q_R \text{Tr} \sigma^2_L}{Q_L - Q_R} \right) \right) \partial_z \Phi \partial_z \Phi \right) \right\} 
\]

(4.13)

Together with a dilaton given by (4.8). Also, when the WZW anomaly does not cancel, the level \( k \) is given by (4.3). Recall also that \( Q_R \) is fixed by supersymmetry and is given below equation (4.3).

When it comes to the determination of the metric of the underlying heterotic string theory the situation is the same as in the previous section. The \( \beta \)–function perturbation expansion parameter is \( 1/Q_L \). Generically, the metric is of order \( Q^2_L \) and as the gauge field coupling to the left mover \( \partial_z \Phi \) is of order \( Q_L \), the curvature, dilaton and gauge field strength terms are all of the same order. The naive metric read off from (4.13) has non–negligible corrections due to the back–reaction from the fermions. Again, the present bosonised form of the model allows an efficient determination of the correct spacetime metric, by writing it in a form which prepares it for re–fermionisation. The model is of the form:

\[
I = \frac{k}{2\pi} \int d^2 z \left\{ G_{rr} \partial_z r \partial_z r + G_{tt} \partial_z t \partial_z t + \frac{1}{k} \left( (1 + F(r)) \partial_z \Phi \partial_z \Phi + 2A_L \partial_z t \partial_z \Phi + 2A_R \partial_z t \partial_z \Phi \right) \right\} 
\]

(4.14)
which should be written as:

\[
I = \frac{k}{2\pi} \int d^2z \left\{ G^0_{rr} \partial_z r \partial_{\bar{r}}r + \left( G^0_{tt} - \frac{1}{k}[A_L + A_R]^2 \right) \partial_z t \partial_{\bar{t}}t + \frac{1}{k} F(r) \partial_z \Phi \partial_{\bar{z}}\bar{\Phi} \right.
\]
\[
+ \frac{1}{k} \left( \partial_z \Phi + [A_L + A_R] \partial_z t \right) \left( \partial_{\bar{z}} \bar{\Phi} + [A_L + A_R] \partial_{\bar{z}}t \right)
\]
\[
\left. + \left[ A_L - A_R \right] \left( \partial_z t \partial_{\bar{z}}\Phi - \partial_{\bar{z}}t \partial_z \Phi \right) \right\}
\]

By studying the 2D boson–fermion relations in the presence of vector and axial couplings to a \(U(1)\) gauge field \(A_\alpha\) it is easy to read off the \((0,1)\) heterotic sigma model as:

\[
I = \frac{k}{2\pi} \int d^2z \left( G_{rr} \partial_z r \partial_{\bar{r}}r + G_{tt} \partial_z t \partial_{\bar{t}}t \right)
\]
\[
+ \frac{1}{2\pi} \int d^2z \left\{ \lambda_R \left( \partial_{\bar{z}} - 2i \Omega_t \partial_z t \right) \lambda_R + \lambda_L \left( \partial_z - 2i Q_L A_t \partial_z t \right) \lambda_L 
\]
\[
\left. + \frac{1}{2} F_{rt} \psi_R^r \psi_R^t \lambda_L \lambda_L \right\},
\]

where \(G^0_{rr}, G^0_{tt}, F(r), A_L\) and \(A_R\) are all read off from (4.16) using (4.14) and the metric, gauge field, tangent space connection and four–fermi term are given by:

\[
G_{rr} = G^0_{rr}
\]
\[
G_{tt} = G^0_{tt} - \frac{1}{k}[A_L + A_R]^2
\]
\[
A_t = A_L
\]
\[
\Omega_t = A_R
\]
\[
F_{rt} \psi_R^r \psi_R^t = F(r) \lambda_R \lambda_R;
\]

As a quick check, it is useful to recover a familiar case. When \(\sigma_R = \sigma_L = \sigma_3\) then \(G\) should be chosen as \(\exp r\sigma_1/2\) and \(Q_R = 1\). Then the WZW model is anomaly free for arbitrary \(k\) without recourse to the fermions which must now be anomaly–free on their own. This amounts to the choice \(Q_L^2 = 1\). Notice that for this case (and any case where the WZW anomaly is zero) the spacetime metric is simply the naive metric which would be deduced prior to re–fermionisation, as the contribution to

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the $\beta$–function from the gauge field is suppressed by $1/k$. Choosing $Q_L = -1$ yields the solution:

$$\begin{align*}
    dS^2 &= \frac{k}{4} \left( dr^2 - \tanh^2 \left( \frac{r}{2} \right) dt^2 \right), \\
    \hat{\Phi} &= - \log \cosh \left( \frac{r}{2} \right) + \hat{\Phi}_0, \\
    A_t &= \frac{2}{\cosh(r) + 1},
\end{align*}$$

which is the Lorentzian 2D black hole[1] with a fixed charge $U(1)$ gauge field (assuming that $k$ has been fixed to yield the desired value of the central charge). The choice $\sigma_R = \sigma_L = i\sigma_2$ reverses the sign on the timelike component of the metric displayed above, $t$ is now a compact coordinate and the cigar shaped metric of the Euclidean black hole is recovered. Those two exact charged solutions were first displayed by Ishibashi, Li and Steif[20] as a bosonic solution\(^{12}\). The $U(1)$ boson coupled to the theory there to carry the background gauge field had a free $U(1)$ charge which gave the black hole an arbitrary charge. Here the right movers’ charge has been chosen in order to yield $(0,1)$ supersymmetry, and because the WZW anomaly is zero this fixes the left movers’ charge also. This is of course not a necessary requirement here. However, including supersymmetry will be essential later for constructing 4D heterotic string solutions.

Now it is interesting to move slightly away from the familiar case. One way to introduce an arbitrary charge on the 2D black hole is to use a one–parameter deformation of the charged black hole theories above which may be obtained by simply taking $\sigma_R = \delta \sigma_L = \delta \sigma_3$. The above choice for $G$ is still appropriate here.

\(^{12}\) The attentive reader will note that their solution is in a different gauge arising from their different gauge fixing choice. Their solution is related to the one here by simply adding a constant to $A_t$. 

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This yields the solution (for large $Q$ and rescaling $\delta t \to t$):

$$d\tau^2 = \frac{k}{4} \left[ dr^2 - \frac{\cosh^2(r) - 1}{(\cosh(r) + \delta)^2} dt^2 \right]$$

$$\hat{\Phi} = -\frac{1}{2} \log(\cosh(r) + \delta) + \hat{\Phi}_0,$$

$$A_t = -\frac{2Q}{\cosh(r) + \delta},$$

$$k = 2 \frac{Q^2}{(\delta^2 - 1)}.$$  \hfill(4.19)

This family of Lorentz signature solutions\textsuperscript{13} has an arbitrary $U(1)$ charge due to the inclusion of the free parameter $\delta$. In two dimensions this is calculated by simply taking the large $r$ limit of the scalar

$$\tilde{F} = e^{2(\hat{\Phi} - \hat{\Phi}_0)} \epsilon^{\alpha\beta} F_{\alpha\beta},$$

which yields

$$Q_E = Q.$$  

The solutions also possess interesting physical behaviour at various radii, for generic values of $Q_E$: The curvature scalar, given by $R = G''_{tt}/G_{tt} - (G'_{tt})^2/2$ shows that there is a curvature singularity on the surface given by $\cosh(r) = -\delta$. There is also a horizon at the radius given by $\cosh^2 r = 1$. This is in direct analogy with the original 2D black hole as described by Witten\textsuperscript{[1]}. There $G_{tt} = \tanh^2 r/2 = (\cosh r - 1)/(\cosh r + 1)$ and so there is a horizon at $\cosh r = 1$ and a singularity at the (analytically continued) position $\cosh r = -1$. In the present solution, to leading order in $Q$, the position of the singularity is a function of $\delta$.

Notice that when $\delta = -1$, $k$ is free again, the expression for the metric (4.19) is invalid (due to the use of (4.3) in its derivation) and the familiar trumpet shaped solution dual\textsuperscript{14} to (4.18) is obtained. Of course, the same construction could have been carried out with the Euclidean theory $\sigma_R = \delta \sigma_L = i \delta \sigma_2$.

\textsuperscript{13} These solutions may be embedded in a larger conformal field theory in order to allow a sensible ($U(1)$ charge independent) central charge to be chosen.

\textsuperscript{14} Here duality refers to the metric of the bosonic theory alone\textsuperscript{[1][2]}. For all of the theories of this paper the full conformal field theory including the fermions will be dual to a much richer class of theories than those obtained by WZW models with no additional fermions. It would be interesting to study them.
Now it is interesting to turn to the unusual case where the left and right generators of the $U(1)$ action on the group manifold are in different conjugacy classes. In close analogy with the previous case taking $\sigma_L = \sigma_3, \sigma_R = i\delta\sigma_2, \sigma_r = \sigma_1$ yields the solution:

$$dS^2 = \frac{k}{4} \left[ dr^2 + \frac{\sinh^2(r) + 1}{(\sinh(r) + \delta)^2} dt^2 \right],$$

$$\hat{\Phi} = -\frac{1}{2} \log(\sinh(r) + \delta) + \hat{\Phi}_0,$$

$$A_t = \frac{2Q}{\sinh(r) + \delta},$$

$$k = -\frac{2Q^2}{(\delta^2 + 1)}. \quad (4.20)$$

The coordinate $t$ is now compact with period $4\pi$. This solution has a number of interesting properties. The most obvious one is the singularity at the radius given by $\sinh r = -\delta$ and a horizon at $\sinh r = 1$. Also note that as $r \to \infty$ the metric approaches that of a cylinder. Analytically continuing to non–compact geometry, this means that the solution is asymptotically flat.

Notice that the cases where both $\sigma_L$ and $\sigma_R$ are either of $\sigma_{\pm}$ there is no supersymmetry, as the right–moving fermions decouple. The WZW anomaly is already zero, so all of the fermions may be neglected. As is familiar now[2][18][21], the resulting theory is a 1D model of the Liouville form, with cosmological coupling $\mu$, which in the straightforward case here is zero, but may be made non–zero by modifying the constraints on the $\sigma_{\pm}$ generators.

By putting $\sigma_L = \sigma_{\pm}$ but leaving $\sigma_R$ as one of the other generators, the right moving supersymmetry is retained and more non–trivial 2D geometries can be obtained. For example $\sigma_L = \sigma_+, \sigma_R = \delta\sigma_3$ and $\sigma_r = \sigma_1$ results in

$$dS^2 = \frac{k}{4} \left[ dr^2 - \frac{\sinh^2(2r) + 1}{(\sinh(2r) + 2\delta)^2} dt^2 \right],$$

$$\hat{\Phi} = -\frac{1}{2} \log(\sinh(2r) + 2\delta) + \hat{\Phi}_0,$$

$$A_t = -\frac{2Q}{\sinh(2r) + 2\delta},$$

$$k = \frac{2Q^2}{(\delta^2 - 1)}. \quad (4.21)$$
Here $-\infty < t < \infty$. The solution is asymptotically flat at $r = \infty$ and as before for generic $Q$ there is a horizon and a singularity. This case is similar to the previous one and may be thought of as the Lorentzian version of it, after a trivial rescaling of $r$ and $\delta$.

Again, these models can in principle be arranged as part of a larger conformal field theory where the central charge can be set so as to fix the space time dimension and leave $Q_E$ free.

5. Charged two–dimensional black holes in string theory.

The previous section described a family of conformal field theories which may be interpreted as low dimensional charged black hole solutions of string theory, to one loop in the $\alpha'$ expansion. This means that they are solutions to the $\beta$–function equations:

\begin{align*}
R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \hat{\Phi} - 2 F_{\mu\lambda} F^\lambda_\nu &= 0 \\
\nabla^\mu (e^{-2\hat{\Phi}} F_{\mu\nu}) &= 0 \\
4 \nabla^2 \hat{\Phi} - 4 (\nabla \hat{\Phi})^2 + \Lambda + R - F^2 &= 0.
\end{align*} (5.1)

The black hole solutions to these equations (for $F_{\mu\nu} = 0$) have been studied directly by Mandal, Sengupta and Wadia[22]. Witten’s exact conformal field theory derived from the $SL(2, \mathbb{R})/U(1)$ coset coincided with this solution at one loop, as indeed it should. McGuigan, Nappi and Yost[23] generalised the solution of ref.[22] to the case where gauge fields are present, introduced either by coupling to an open string sector or by heterotic compactification of closed strings. This latter case should be of concern here in order to interpret the results of the previous section.

In general, the solution takes the following form:

\begin{align*}
\frac{ds^2}{2} &\sim (1 - 2me^{-Qr} + q^2 e^{-2Qr})^{-1} dt^2 - (1 - 2me^{-Qr} + q^2 e^{-2Qr}) dr^2 \\
A_t &\sim - e^{-Qr},
\end{align*} (5.2)

with linear dilaton

\[ \hat{\Phi} - \hat{\Phi}_0 = - \frac{Qr}{2}. \] (5.3)
Here, \( Q \) is a constant, \( q \) sets the charge of the black hole and \( m \) is related to the physical mass. The curvature singularity is located at \( r = -\infty \), the solution is asymptotically flat for \( r \to \infty \) and there are horizons at the zeros of \( G_{tt} \). For \( q = 0 \) (the uncharged case) the coordinate transformation

\[
e^{Qr} = m(\cosh \sigma + 1)
\]

yields (after absorbing \( Q \) into \( \sigma \))

\[
dS^2 \sim d\sigma^2 - \tanh^2 \frac{\sigma}{2} dt^2
\]

\[
\hat{\Phi} - \hat{\Phi}_0 = -\frac{1}{2} \log(\cosh^2 \sigma),
\]

which is the familiar form of the 2D black hole.

Similarly, the transformation

\[
e^{Qr} = \frac{m}{\delta}(\cosh \sigma + \delta)
\]

produces (for \( q^2 = m^2(\delta^2 - 1)/\delta^2 \))

\[
dS^2 \sim d\sigma^2 - \cosh^2 \frac{\sigma - 1}{(\cosh \sigma + \delta)^2} dt^2
\]

\[
\hat{\Phi} - \hat{\Phi}_0 = -\frac{1}{2} \log(\cosh \sigma + \delta),
\]

\[
A_t \sim -\frac{1}{(\cosh \sigma + \delta)}
\]

which is the solution (4.19) of the previous section. Furthermore, the solutions (4.20) and (4.21) may be obtained by similar transformations, replacing \( \cosh \sigma \) by \( \sinh \sigma \), etc. This explains why those solutions all appear to be simply analytic continuations of each other.

The conclusion here is that the interpretation of the conformal field theories explicitly examined in the previous section is indeed the 2D heterotic string theory in a black hole background with \( U(1) \) gauge field. In the leading order approximation, the different solutions found are different coordinate transformations of the same basic solution presented in ref.[23]. The mass and charge of the 2D black hole,
which are independent in general, have been set proportional to one another\textsuperscript{15}. Note however that this is not analogous to the familiar 4D extremality condition as the singularity and horizon are still distinct. The details of the spacetime structure of the charged 2D black holes was analysed in ref.[23].

As mentioned in the introduction, extremal 4D black holes factorise into a 2D radial theory and a 2D angular theory. Section 3 discussed how to realise a magnetic monopole angular theory background of heterotic string theory as a gauged WZW model with fermions. Section 4 described radial conformal field theories of electrically charged linear dilaton black holes by the same construction. Arbitrary products of these two types of theories can be made to realise extremal 4D black holes with $U(1) \times U(1)$ gauge group where one $U(1)$ factor carries the electric charge and the other the magnetic.

It would be interesting however to construct less trivial combinations of the radial and angular degrees of freedom, using the same techniques. This would yield exact conformal field theories of more interesting 4D backgrounds. This is the subject of the rest of this paper. The result will be a $U(1) \times U(1)$ dyon, where both $U(1)$ factors contain electric and magnetic components. The metric will not be a product metric.

6. A gauged $SL(2, \mathbb{R}) \times SU(2)$ Wess–Zumino–Witten model.

The construction starts by gauging certain $U(1)$ subgroups of an $SL(2, \mathbb{R}) \times SU(2)$ Wess–Zumino–Witten model:

\[
I_{WZW} = \frac{k_1}{4\pi} \int_\Sigma d^2z \ Tr(g_1^{-1}\partial_z g_1 \cdot g_1^{-1}\partial_\Sigma g_1) - \frac{k_2}{4\pi} \int_\Sigma d^2z \ Tr(g_2^{-1}\partial_z g_2 \cdot g_2^{-1}\partial_\Sigma g_2) + ik_1\Gamma(g_1) - ik_2\Gamma(g_2) \tag{6.1}
\]

where

\[
\Gamma(g) = \int_B g^* \omega. \tag{6.2}
\]

Here $g_1 \in SL(2, \mathbb{R})$ and $g_2 \in SU(2)$. This model is a theory of maps from $\Sigma$ to $G = SL(2, \mathbb{R}) \times SU(2)$. $\Sigma$ is $(1 + 1$ dimensional) spacetime, the boundary of an

\textsuperscript{15}Generically there is only one free parameter left in the theories of the previous section.
auxiliary space $B$. Here $\omega$ is a $G_L \times G_R$ invariant 3–form on $G$:
\[
\omega = \frac{1}{12\pi} \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)
\]
(6.3)

The first subgroup of interest is
\[
U(1)_A \times U(1)_B : \begin{cases}
g_1 \to e^{\epsilon_A \sigma_3/2} g_1 e^{(\delta \epsilon_A + \lambda \epsilon_B)\sigma_3/2} \\
g_2 \to g_2 e^{i\epsilon_B \sigma_3/2}
\end{cases}
\]
(6.4)

(Notice that the limit $\lambda \to 0, \delta \to 1$ will yield the gauging for the monopole theory of the section 3.)

As in the previous construction gauging this subgroup will result in anomalies. It is not possible to construct a gauge invariant extension of the Wess–Zumino term $\Gamma(g)$. It is possible however to construct an extension $\Gamma(g, A^A, A^B)$ for which the terms violating gauge invariance do not depend upon the group element[15]:
\[
\Gamma(g, A^A, A^B) = \Gamma(g) - \frac{1}{4\pi} \int_{\Sigma} d^2z \ A^a \wedge \text{Tr}(t_{a,L}dgg^{-1} + t_{a,R}g^{-1}dg) \\
- \frac{1}{8\pi} \int_{\Sigma} d^2z \ A^a \wedge A^b \text{Tr}(t_{a,R}g^{-1}t_{b,L}g - t_{b,R}g^{-1}t_{a,L}g),
\]
(6.5)

where the indices $a, b$ can take the values $A, B$. Also $A^A$ and $A^B$ are gauge fields for $U(1)_A$ and $U(1)_B$ respectively, and $t_{a,L(R)}$ are the generators for the left(right) action of the groups. Under gauge transformations this action will have anomalies
\[
A_{ab} = \frac{1}{4\pi} \text{Tr}(t_{a,L}t_{b,L} - t_{a,R}t_{b,R}) \int_{\Sigma} d^2z \ F^b_{2\pi}
\]
where $F^b_{2\pi} = \partial_z A^b_z - \partial^2 A^b_z$ and there is no sum intended on the index $b$ in the formula immediately above. The metric part of the Wess–Zumino–Witten model is coupled invariantly to gauge fields by simply replacing the derivative with its gauge–covariant extension:
\[
\partial_{\mu} g \to D_{\mu} g = \partial_{\mu} g + A_{\mu}^a(t_{a,L}g - gt_{a,R}).
\]

Inserting the explicit expressions for the generators:
\[
t_{A,R}^{(1)} = \delta \frac{\sigma_3}{2}; \quad t_{A,L}^{(1)} = \sigma_3; \quad t_{B,R}^{(1)} = \lambda \frac{\sigma_3}{2}; \quad t_{B,R}^{(2)} = i \frac{\sigma_3}{2};
\]

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yields the following total action for the gauged Wess–Zumino–Witten model:

\[ I = I_{WZW} + \frac{k_1}{8\pi} \int d^2z \left\{ -2 \left( \delta A_+^A + \lambda A_+^B \right) \Tr[\sigma_3 g_1^{-1} \partial_z g_1] - 2A_+^A \Tr[\sigma_3 \partial_z g_1 g_1^{-1}] \\
+ A_+^A A_+^A \left( 1 + \delta^2 + \delta \Tr[\sigma_3 g_1 \sigma_3 g_1^{-1}] \right) + \lambda^2 A_+^B A_+^B \\
+ \delta \lambda A_+^A A_+^B + A_+^B A_+^A \left( \delta \lambda + \lambda \Tr[\sigma_3 g_1 \sigma_3 g_1^{-1}] \right) \right\} \\
+ \frac{k_2}{8\pi} \int d^2z \left\{ 2i A_+^B \Tr[\sigma_3 g_2^{-1} \partial_z g_2] + A_+^B A_+^B \right\} \]

(6.6)

with anomalies

\[ A_{AA} = - \frac{k_1 (1 - \delta^2) \epsilon_A}{8\pi} \int d^2z F_{zz}^A \]
\[ A_{BB} = (k_2 + k_1 \lambda^2) \frac{\epsilon_B}{8\pi} \int d^2z F_{zz}^B \]
\[ A_{AB} = k_1 \delta \lambda \frac{\epsilon_A}{8\pi} \int d^2z F_{zz}^B \]
\[ A_{BA} = k_1 \delta \lambda \frac{\epsilon_B}{8\pi} \int d^2z F_{zz}^A. \]

(6.7)

Turning to the fermionic content, consider the four (0,1) right-moving supersymmetric coset fermions. They are minimally coupled to the above gauge theory:

\[ I^F_R = \frac{i}{4\pi} \int d^2z \Tr(\Psi_R D_z \Psi_R). \]

(6.8)

Here \( \Psi_R \) takes values in the orthogonal complement of the Lie algebra of \( U(1)_A \times U(1)_B \) and the covariant derivative is

\[ D_z \Psi_R = \partial_z \Psi_R - \sum_a A^A_{zz} [t_{a,R}, \Psi_R] \]

(6.9)

where the \( t_{a,R} \) are the generators of the subgroup acting on the right. As the generators of the \( U(1) \) subgroups of \( SL(2, \mathbb{R}) \) and \( SU(2) \) are represented as diagonal antihermitian \( 2 \times 2 \) matrices, the fermions are of the form:

\[ \Psi_{R,1} = \begin{pmatrix} 0 & \lambda_1^R \\ \lambda_2^R & 0 \end{pmatrix}, \]
\[ \Psi_{R,2} = \begin{pmatrix} 0 & \lambda_3^R \\ \lambda_4^R & 0 \end{pmatrix} \]

(6.10)
with ‘1’ and ‘2’ denoting $SL(2, \mathbb{R})$ and $SU(2)$ respectively. The $\text{Tr}$ in (6.8) decomposes as $-k_1 \text{Tr}_1 + k_2 \text{Tr}_2$. The resulting action is therefore:

$$I_R^F = -\frac{ik_1}{4\pi} \int d^2 z \left\{ 2\lambda_R^1 \partial_{\overline{z}} \lambda_R^2 + \lambda_R^1 [\delta A_\overline{z}^A + \lambda A_\overline{z}^B, \lambda_R^2] \right\} + \frac{ik_2}{4\pi} \int d^2 z \left\{ 2\lambda_R^3 \partial_{\overline{z}} \lambda_R^4 + \lambda_R^3 \lambda_R^4 \right\}. \tag{6.11}$$

The model now has invariance under the $(0, 1)$ supersymmetry

$$\delta g_1 = i \epsilon g_1 \Psi_{R,1};$$
$$\delta g_2 = i \epsilon g_2 \Psi_{R,2};$$
$$\delta \Psi_{R,1} = \epsilon (g_1^{-1} \partial_z g_1 + \frac{A_A^A}{2} g_1^{-1} \sigma_3 g_1);$$
$$\delta \Psi_{R,2} = \epsilon (g_2^{-1} \partial_z g_2)$$
$$\delta A_i^a = 0,$$

(modulo the equations of motion $\partial_z \Psi_{R,1} = 0, \partial_z \Psi_{R,2} - A_z^A [\sigma_3, \Psi_{R,2}]$, which may be verified by direct calculation. In general this symmetry is enhanced to $(0, 2)$ supersymmetry in a general $G/H$ coset model under certain circumstances[17]: The complexification of the space $\text{Lie}(G/H)$ denoted $T_C$ can be decomposed into the form $T_C = T \oplus \overline{T}$. Here $T$ and $\overline{T}$ are represented in this case by strictly upper and strictly lower triangular $2 \times 2$ matrices respectively. They therefore automatically satisfy the required conditions

$$[T, T] \in T, \quad [T, \overline{T}] \in \overline{T} \tag{6.13}$$

and

$$\text{Tr}(t_1, t_2) = 0 \quad \text{for} \quad t_1, t_2 \in T \text{ or } \overline{T}. \tag{6.14}$$

These are precisely the algebraic conditions for a coset space to be Kähler, which is the requirement for enhancing $N = 1$ to $N = 2$ supersymmetry as pointed out by Kazama and Suzuki[16]. The fermions $\Psi_R$, which take their values in $T_C$, have been decomposed in this case as given by equation (6.10). By inspection of the action (6.11) it can be seen that it is possible to assign a classical$^{16}$ $R$–symmetry to

$^{16}$To identify the full $U(1)$ $R$–symmetry of these cosets the classical and quantum anomalies coming from the bosonic and fermionic sectors respectively have to be treated correctly. See for example ref[24].
the system under which \( g_1, g_2 \) and the gauge fields have charge zero, fields valued in \( T \) have charge 1 and fields valued in \( \overline{T} \) have charge \(-1\). This symmetry does not commute with the transformations (6.12) and therefore a second supersymmetry action may be extracted. The explicit \((0, 2)\) supersymmetry transformations for the fields in this model may be found by using the decomposition (6.10) (together with the parameterisations of \( g_1 \) and \( g_2 \) and to be discussed later) in the transformations (6.12).

Next add four left–moving fermions \( \Psi_L = (\lambda^1_L, \lambda^2_L, \lambda^3_L, \lambda^4_L) \). These carry a global \( SO(4)_L \) symmetry, the maximal torus of which will be identified with the gauged \( U(1)_A \times U(1)_B \). The generators of this gauge symmetry will be chosen as:

\[
\hat{Q}_A = \begin{pmatrix} 0 & Q_A & 0 & 0 \\ -Q_A & 0 & 0 & 0 \\ 0 & 0 & 0 & P_A \\ 0 & 0 & -P_A & 0 \end{pmatrix}, \quad \hat{Q}_B = \begin{pmatrix} 0 & Q_B & 0 & 0 \\ -Q_B & 0 & 0 & 0 \\ 0 & 0 & 0 & P_B \\ 0 & 0 & -P_B & 0 \end{pmatrix} \tag{6.15}
\]

giving the action:

\[
I_F^L = \frac{-i k_1}{4\pi} \int d^2 z \left\{ \lambda^1_L [\partial z + Q_A A^A_z + Q_B A^B_z] \lambda^2_L \right\} + \frac{i k_2}{4\pi} \int d^2 z \left\{ \lambda^2_L [\partial z + P_A A^A_z + P_B A^B_z] \lambda^1_L \right\}. \tag{6.16}
\]

(Here the couplings \( k_1 \) and \( k_2 \) have explicitly been chosen.) Note that to connect smoothly to the pure monopole theory, the charges \( P_A \) and \( Q_B \) should be sent to zero as \( \delta \to 1 \) and \( \lambda \to 0 \). (This is necessary to ensure that the remaining mixed anomaly from the left moving fermions cancel in this limit. \( Q_A \) remains non–zero to cancel the anomaly from the \( SL(2, \mathbb{R}) \) right–moving fermions.)

At one–loop the fermions will produce anomalies of the same form as above when coupled to gauge fields. They are:

\[
\begin{align*}
A^F_{AA} &= -2(Q_A^2 + P_A^2 - \delta^2) \frac{\epsilon_A}{8\pi} \int d^2 z \, F^A_{z\overline{z}} \\
A^F_{BB} &= -2(Q_B^2 + P_B^2 - (1 + \lambda^2 \lambda^2)) \frac{\epsilon_B}{8\pi} \int d^2 z \, F^B_{z\overline{z}} \\
A^F_{AB} &= -2(Q_A Q_B + P_A P_B - \lambda \delta) \frac{\epsilon_A}{8\pi} \int d^2 z \, F^B_{z\overline{z}} \\
A^F_{BA} &= -2(Q_A Q_B + P_A P_B - \lambda \delta) \frac{\epsilon_B}{8\pi} \int d^2 z \, F^A_{z\overline{z}}.
\end{align*} \tag{6.17}
\]
Adding these anomalous fermionic actions to the anomalously gauged Wess-Zumino-Witten model will yield a gauge invariant theory if:

\[-k_1(1 - \delta^2) = 2(Q_A^2 + P_A^2 - \delta^2);\]
\[k_2 + k_1 \lambda^2 = 2(Q_B^2 + P_B^2 - (1 + \lambda^2));\]
\[k_1 \delta \lambda = 2(Q_A Q_B + P_A P_B - \lambda \delta).\] \hspace{1cm} (6.18)

Also, as this is a four-dimensional heterotic string solution, the resulting central charge of this conformal field theory\(^\text{17}\) should be \(c = 6\). This means that

\[c = \frac{3k_1}{k_1 - 2} + \frac{3k_2}{k_2 + 2} = 6,\] \hspace{1cm} (6.19)

(the \(-2\) from gauging is cancelled by the \(+2\) from the four fermions) which gives \(k_1 = k_2 + 4\).

All these fermions are equivalent to two periodic bosons \(\Phi_1\) and \(\Phi_2\) which display the anomalies (6.17) classically:

\[I_B = \frac{1}{4\pi} \int d^2 z \left\{ (\partial_z \Phi_2 - P_A A_z^A - (P_B + 1) A_z^B)^2 \right. \]
\[+ (\partial_z \Phi_1 - (Q_B + \lambda) A_z^B - (Q_A + \delta) A_z^A)^2 \]
\[- \Phi_1 \left[ (Q_B - \lambda) F_z^B + (Q_A - \delta) F_z^A \right] \]
\[- \Phi_2 \left[ (P_B - 1) F_z^B + P_A F_z^A \right] \]
\[+ \left[ A_z^A A_z^B - A_z^A A_z^B \right] \left[ Q_B - Q_A \lambda - P_A \right].\] \hspace{1cm} (6.20)

The \(U(1)_A \times U(1)_B\) action on the bosons is:

\[\delta \Phi_1 = (Q_A + \delta) \epsilon_A + (Q_B + \lambda) \epsilon_B \quad \delta \Phi_2 = P_A \epsilon_A + (P_B + 1) \epsilon_B;\]
\[\delta A_a^A = \partial_a \epsilon_A \quad \delta A_a^B = \partial_a \epsilon_B,\] \hspace{1cm} (6.21)

under which it may be verified that the action (6.20) yields precisely the anomalies displayed in (6.17).

\(^{17}\) The rest of the central charge needed to construct a consistent string theory will be supplied by a \(c = 9\) internal conformal field theory in the usual way. This will not be of concern here.
7. The complete theory.

The complete theory is now (after using the anomalies to simplify the quadratic terms in the gauge fields):

\[
I_{\text{total}} = I_{WZW} + \frac{1}{4\pi} \int d^2 z \left\{ \partial_z \Phi_1 \partial_z \Phi_1 + \partial_z \Phi_2 \partial_z \Phi_2 \right\} \\
- \frac{k_1}{8\pi} \int d^2 z \left\{ A_z^A \left( 2 \text{Tr}[\sigma_3 g_1^{-1}] + \frac{4}{k_1} (Q_A \partial_z \Phi_1 + P_A \partial_z \Phi_2) \right) \\
+ A_z^A \left( 2 \delta \text{Tr}[\sigma_3 g_1^{-1}] + \frac{4}{k_1} \delta \partial_z \Phi_1 \right) \\
+ A_z^B \left( \frac{4}{k_1} (Q_B \partial_z \Phi_1 + P_B \partial_z \Phi_2) \right) \\
+ A_z^B \left( 2 \lambda \text{Tr}[\sigma_3 g_1^{-1}] - 2 \frac{k_2}{k_1} \text{Tr}[\sigma_3 g_2^{-1}] \right) \\
+ \frac{4}{k_1} (\partial_z \Phi_2 + \lambda \partial_z \Phi_1) \right\} \\
- A_z^A A_z^B \left( \lambda \text{Tr}[\sigma_3 g_1^{-1}] + \frac{4}{k_1} [P_A (P_B + 1) + Q_A (Q_B + \lambda)] \right) \\
- A_z^B A_z^A \left( \frac{4}{k_1} [P_A P_B + Q_B (Q_A + \delta)] \right) \\
- A_z^A A_z^A \left( 2 + \delta \text{Tr}[\sigma_3 g_1^{-1}] + \frac{4}{k_1} [P_A^2 + Q_A (Q_A + \delta)] \right) \\
- A_z^B A_z^B \left( \frac{4}{k_1} [P_B (P_B + 1) + Q_B (Q_B + \lambda)] \right) \right\}
\]

Due to the anomaly cancelation (6.18) it is gauge invariant and hence conformally invariant, as can be be verified explicitly.

As the gauge fields are non–dynamical and appear quadratically in the action they
may be integrated out with the following result:

\[ I = I_{WZW} + \]

\[ \frac{k_1}{8\pi} \int d^2z \, \frac{1}{D} \left\{ \frac{2D}{k_1} \left[ \partial_z \Phi_2 \partial_z \Phi_2 + \partial_z \Phi_1 \partial_z \Phi_1 \right] \right. \]

\[ -4Tr_a \left[ (\lambda F_B - \delta G_B)(Tr_b + 2 \frac{2}{k_1} \partial_z \Phi_1) + F_B(Tr_c + 2 \frac{2}{k_1} \partial_z \Phi_2) \right] \]

\[ + (Q_A \partial_z \Phi_1 + P_A \partial_z \Phi_2) \left[ 8(\lambda F_B - \delta G_B)(Tr_b + 2 \frac{2}{k_1} \partial_z \Phi_1) + 8F_B(Tr_c + 2 \frac{2}{k_1} \partial_z \Phi_2) \right] \]

\[ - (Q_B \partial_z \Phi_1 + P_B \partial_z \Phi_2) \left[ 8(\lambda F_A - \delta G_A + \lambda \frac{k_1}{2})(Tr_b + 2 \frac{2}{k_1} \partial_z \Phi_1) \right. \]

\[ \left. + (8F_A - 2\delta Tr_d + 4k_1)(Tr_c + 2 \frac{2}{k_1} \partial_z \Phi_2) \right\} \]

where

\[ D \equiv Tr_d(\lambda F_B - \delta G_B) - 2G_B + \frac{4(F_B G_A - G_B F_A)}{k_1} \quad (7.3) \]

with

\[ F_A = P_A^2 + Q_A(Q_A + \delta); \]
\[ F_B = P_A P_B + Q_B(Q_A + \delta); \]
\[ G_A = P_A(P_B + 1) + Q_A(Q_B + \lambda); \]
\[ G_B = P_B(P_B + 1) + Q_B(Q_B + \lambda); \]
\[ Tr_a = Tr[\sigma_3 \partial_z g_1 g_1^{-1}], \quad Tr_b = Tr[\sigma_3 g_1^{-1} \partial_z g_1], \]
\[ Tr_c = -i \frac{k_2}{k_1} Tr[\sigma_3 g_2^{-1} \partial_z g_2], \quad Tr_d = Tr[\sigma_3 g_1 \sigma_3 g_1^{-1}]. \]

Due to the unusual construction of this model, using bosonisation to arrive at the desired quadratic terms for the gauge fields to achieve gauge invariance, it is worth checking that gauge invariance is present after the integration process. Using the gauge transformations (6.4) and (6.21), together with repeated use of the anomaly equations (6.18), gauge invariance is indeed verified.

Naively there is no reason to expect that the integration procedure is correct beyond the leading order in \( k_1 \), so the terms appearing in (7.2) are the leading order terms in a large \( k_1 \) expansion. Also \( k_1/k_2 \rightarrow 1 \) in this limit, following from equation (6.19). In what follows \( k_1 \) and \( k_2 \) will be denoted \( k \) and in this limit the anomaly equations...
simplify:
\[ k = 2 \frac{Q_A^2 + P_A^2}{\delta^2 - 1} = 2 \frac{Q_B^2 + P_B^2}{\lambda^2 + 1} - 2 \frac{Q_A Q_B + P_A P_B}{\delta \lambda}. \] (7.4)

It is also extremely useful that
\[ F_A = \frac{k}{2} (\delta^2 - 1), \]
\[ F_B = G_A = \frac{k}{2} \lambda \delta, \]
\[ G_B = \frac{k}{2} (\lambda^2 + 1). \] (7.5)

It is comforting to note that the potentially clumsy expressions resulting from (7.2) and (7.3) simplify enormously in this large radius limit using these expressions, as will be shown in the next section once the model has been endowed with some coordinates.

8. The 4D solutions

The next task is to choose a parameterisation for \(g_1\) and \(g_2\) and appropriate gauge conditions. The choice of parameterisation for \(SL(2, \mathbb{R})\) shall be the Euler angles:
\[ g_1 = e^{t_L \sigma_3/2} e^{\sigma_1/2} e^{t_R \sigma_3/2} = \begin{pmatrix} e^{t_+ \cosh \frac{\sigma}{2}} & e^{t_- \sinh \frac{\sigma}{2}} \\ e^{-t_- \sinh \frac{\sigma}{2}} & e^{-t_+ \cosh \frac{\sigma}{2}} \end{pmatrix}, \] (8.1)

where the \(\sigma_i\) are the Pauli matrices and
\[ t_\pm \equiv t_L \pm t_R, \quad 0 \leq \sigma \leq \infty, \quad -\infty \leq t_L \leq \infty, \quad -\infty \leq t_R \leq \infty, \] (8.2)

and Euler angles \((\phi, \theta, \psi)\) for \(SU(2)\) are chosen as in (2.3) and (2.4). For these choices the traces are:
\[ \text{Tr}[\sigma_3 \partial_\sigma g_1^{-1}] = \partial_\psi t_L + \partial_\pi t_R \cosh \sigma, \]
\[ \text{Tr}[\sigma_3 g_1^{-1} \partial_\sigma g_1] = \partial_\sigma t_R + \partial_\sigma t_L \cosh \sigma, \]
\[ \text{Tr}[\sigma_3 g_1 \sigma_3 g_1^{-1}] = 2 \cosh \sigma, \]
\[ \text{Tr}[\sigma_3 g_2^{-1} \partial_\sigma g_2] = i(\partial_\psi \phi + \partial_\pi \phi \cos \theta). \] (8.3)
The gauge transformations in these coordinates are:

\[ t_L \rightarrow t_L + \epsilon_A; \]
\[ t_R \rightarrow t_R + \delta \epsilon_A + \lambda \epsilon_B; \]
\[ \psi \rightarrow \psi + \epsilon_B; \]
\[ \Phi_1 \rightarrow \Phi_1 + (Q_A + \delta)\epsilon_A + (Q_B + \lambda)\epsilon_B; \]
\[ \Phi_2 \rightarrow \Phi_2 = P_A\epsilon_A + (P_B + 1)\epsilon_B. \]

There are three gauge choices of interest which may be chosen for this model:

1. \( \Phi_1 = \Phi_2 = 0; \)
2. \( t_L = 0, \psi = \mp \phi; \) \hspace{1cm} (8.5)
3. \( t_L = t_R = 0. \)

Choices (1) and (2) are equivalent (up to a simple change of variables) for all values of the parameters, in analogy with the two gauge choices discussed for the monopole theory in section 3. With gauge (1), \( t_R \) and \( \psi \) can be identified as the bosonised world sheet fermions, after a discrete identification to obtain \( 2\pi \) periodicity. This modding is the analogue of that in the GPS monopole theory.

The gauges (3) and (2) correspond respectively to whether the gauge conditions are applied entirely to the \( SL(2, \mathbb{R}) \) coordinates or shared between the \( SL(2, \mathbb{R}) \) and \( SU(2) \) coordinates. These gauges are also equivalent up to simple coordinate transformations away from \( \lambda = 0 \), where gauge (3) cannot be implemented. However when \( \lambda = 0 \) the angular and radial sectors of the theory decouple; the angular sector is simply the monopole theory of section 3 while the radial sector is the 2D charged black hole model of sections 4 and 5.

Choosing gauge (2) (as it is the most transparent and generic) and examining the gauge transformations (8.4) the remaining fields and coordinates are defined in terms of the gauge invariant combinations of the original fields:

\[ \sigma, \theta, \phi, \]
\[ t' = t_R - \delta t_L - \lambda(\psi \pm \phi), \]
\[ \Phi'_1 = \Phi_1 - (Q_A + \delta)\epsilon_A - (Q_B + \lambda)(\psi \pm \phi), \]
\[ \Phi'_2 = \Phi_2 - P_A\epsilon_A - (P_B + 1)(\psi \pm \phi). \]
Having chosen the $U(1)_B$ gauge, some care must be exercised to trace the global $SU(2)_L$ invariance in the now gauge-fixed coordinates. An $SU(2)_L$ rotation preserves the two–sphere defined by $(\theta, \phi)$ and the $U(1)$ monopole field undergoes a gauge transformation:

$$A^M = \frac{(\pm 1 - \cos \theta)}{2} d\phi \to A^M + d\Lambda. \quad (8.7)$$

$SU(2)_L$ invariance is preserved by the fact that the forms $dt', d\Phi'_1, d\Phi'_2$ all shift by amounts proportional to $d\Lambda$. This structure is guaranteed to preserve $SU(2)_L$ invariance as it is inherited directly from the world sheet $U(1)_B$ gauge theory.

Dropping the primes the model may be written as follows:

$$I = \frac{k_1}{8\pi} \int d^2 z \left\{ G_{\sigma\sigma} \partial_z \sigma \partial_z \sigma + G_{t\phi}^{0} \partial_z t \partial_z t + G_{t\phi}^{0} (\partial_z t \partial_z \phi + \partial_z t \partial_z \phi) + G_{\theta\theta} \partial_z \theta \partial_z \theta + G_{\phi\phi}^{0} \partial_z \phi \partial_z \phi + + B_{t\phi} (\partial_z t \partial_z \phi - \partial_z t \partial_z \phi) \right.$$  

$$+ \frac{1}{k} \left[ (2 + F_{11}) \partial_z \Phi_1 \partial_z \Phi_1 + (2 + F_{22}) \partial_z \Phi_2 \partial_z \Phi_2 \right.$$  

$$\left. + F_{12} \partial_z \Phi_1 \partial_z \Phi_2 + F_{21} \partial_z \Phi_2 \partial_z \Phi_1 \right.$$  

$$+ (A_1 \partial_z \phi + A_t \partial_z t) \partial_z \Phi_1 + (A_2 \partial_z \phi + A^2_t \partial_z t) \partial_z \Phi_2$$  

$$+ \tilde{A}_1 \partial_z t \partial_z \Phi_1 + \tilde{A}_t \partial_z \Phi_2 \right\}, \quad (8.8)$$

where (for large charge):

$$G_{\sigma\sigma} = 1, \quad G_{\theta\theta} = 1, \quad G_{\phi\phi}^{0} = \sin^2 \theta + 4A^M_{\phi} A^M_{\phi};$$

$$G_{t\phi}^{0} = -\cosh \sigma - \delta \over \cosh \sigma + \delta, \quad G_{t\phi}^{0} = -2\lambda A^M_{\phi} \cosh \sigma \over \cosh \sigma + \delta, \quad B_{t\phi} = -B_{\phi t} = 2\lambda A^M_{\phi} \cosh \sigma \over \cosh \sigma + \delta;$$

$$A_1^t = -4Q_A \over k(\cosh \sigma + \delta), \quad A_2^t = -4P_A \over k(\cosh \sigma + \delta);$$

$$A_1^\phi = -8A^M_{\phi} \lambda Q_A - Q_B (\cosh \sigma + \delta) \over \cosh \sigma + \delta, \quad A_2^\phi = -8A^M_{\phi} \lambda P_A - P_B (\cosh \sigma + \delta) \over \cosh \sigma + \delta; \quad (8.9)$$

$$\tilde{A}_1^t = -4 \cosh \sigma \over k(\cosh \sigma + \delta), \quad \tilde{A}_2^t = 4\lambda \cosh \sigma \over k(\cosh \sigma + \delta);$$

$$F_{11} = -8Q_A \over k(\cosh \sigma + \delta), \quad F_{12} = -8P_A \over k(\cosh \sigma + \delta);$$

$$F_{21} = 8Q_A - Q_B (\cosh \sigma + \delta) \over k(\cosh \sigma + \delta), \quad F_{22} = 8\lambda P_A - P_B (\cosh \sigma + \delta) \over k(\cosh \sigma + \delta).$$
Notice that the functions $F_{ij}$ and $\tilde{A}^k_t$ appear at subleading order. After re-fermionisation they become the four–fermi interactions and tangent space connections for the right movers, respectively. There is also the dilaton, which may be calculated by various standard methods (which all amount to the evaluation of the one–loop determinant arising from the integration over the gauge fields[5][19]) to give the action

$$I_{\text{dilaton}} = \frac{1}{16\pi} \int d^2 z \ R^{(2)} \hat{\Phi}$$

where $R^{(2)}$ is the world sheet two–dimensional curvature scalar and

$$\hat{\Phi} = -\frac{1}{2} \log[\cosh \sigma + \delta] + \hat{\Phi}_0$$

where $\hat{\Phi}_0$ is a constant.

In the above, the superscript ‘0’ on some of the metric components denotes that they are distinct from those which will appear in the heterotic sigma model after re–fermionisation, due to large corrections from the $A_\phi, A_t$ gauge field interactions with the fermions. As in the 2D cases of preceding sections, the corrections to the metric components are computed easily in this bosonic form by simply symmetrising and antisymmetrising the leading $\Phi_1, \Phi_2$ interaction terms, yielding (after repeated use of the three anomaly equations to simplify expressions):

$$G_{tt} = G^0_{tt} - \frac{1}{4k} \left[ (A_1^t)^2 + (A_2^t)^2 \right] = -\frac{\cosh^2 \sigma - 1}{(\cosh \sigma + \delta)^2}$$

$$G_{t\phi} = G^0_{t\phi} - \frac{1}{4k} \left[ A_1^t A_1^\phi + A_2^t A_2^\phi \right] = 2\lambda A^M_\phi G_{tt}$$

$$G_{\phi\phi} = G^0_{\phi\phi} - \frac{1}{4k} \left[ (A_1^\phi)^2 + (A_2^\phi)^2 \right] = 4\lambda^2 A^M_\phi A^M_\phi G_{tt}.$$  

(8.12)

After re–fermionising, the whole model may be written in the standard fashion as a heterotic sigma model:

$$I = \frac{k}{8\pi} \int d^2 z \left\{ G_{\mu\nu} + B_{\mu\nu} \right\} \partial_z X^\mu \partial_{\bar{z}} X^\nu + \frac{k}{8\pi} \int d^2 z \left\{ i\lambda_R^a (\partial_z - \Omega_{\mu ab} \partial_{\bar{z}} X^\mu) \lambda_R^b 

+ i\lambda_L^a (\partial_z - A_{\mu ab} \partial_{\bar{z}} X^\mu) \lambda_L^b + \frac{1}{2} F_{\mu\nu\alpha\beta} \Psi_R^\mu \Psi_R^\nu \lambda_L^\alpha \lambda_L^\beta \right\}$$

(8.13)

where the $(a, b)$ and $(\alpha, \beta)$ indices are tangent space and $U(1) \times U(1)$ current algebra indices respectively.
9. Some 4D spacetime physics.

The spacetime metric derived from the exact theory (7.1) is:

\[
\frac{k}{4} \left\{ d\sigma^2 - \frac{\cosh^2 \sigma - 1}{(\cosh \sigma + \delta)^2} \left( dt + 2\lambda A^M_\phi d\phi \right)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right\},
\]

(9.1)

together with the dilaton (8.11) and the antisymmetric tensor \( B_{t\phi} \) and gauge fields \( A^i_t, A^i_\phi \) given in (8.9). This metric is invariant under spacetime rotations providing that \( t \) transforms to cancel the \( U(1) \) gauge variation produced by the monopole. This requires \( t \) to have periodicity as \( 2\lambda \phi \), which is \( 4\lambda \pi \). Examining the gauging (6.4), where \( U(1)_B \) mixes compact and non-compact variables, this modding of \( t \) can be anticipated. Another way to see this is to note that there is a ‘Dirac string’ coordinate singularity in the spacetime which runs along the \( z \)-axis (\( \theta = 0, \pi \)). In the Northern hemisphere, \( 2A^M_\phi = 1 - \cos \theta \). The metric is well-behaved everywhere but in the Southern hemisphere, small loops about the \( z \)-axis do not shrink to zero length at the South pole (\( \theta = \pi \)). So the metric is singular along this line. This warrants the use of \( 2A^M_\phi = -1 - \cos \theta \) when down South, moving the string to the North Pole (\( \theta = 0 \)). This change of coordinate patch (which is the \( U(1) \) gauge transformation \( A^M \to A^M - 1 \)) is equivalent to changing variables from \( t \) to \( t - 2\lambda \phi \). Given that \( \phi \) has periodicity \( 2\pi \), \( t \) has periodicity \( 4\lambda \pi \). The Dirac string also introduces a quantisation condition on the charges of the model in the standard way.

This metric is reminiscent of a very special form of the Taub–NUT metric[25][26], and deserves further comment. As a solution to the empty space Einstein equations, the full Taub–NUT metric is:

\[
\frac{k}{4} \left\{ d\sigma^2 - \frac{\cosh^2 \sigma - 1}{(\cosh \sigma + \delta)^2} \left( dt + 2\lambda A^M_\phi d\phi \right)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right\},
\]

(9.2)

where

\[
f^2 = 1 - 2\frac{Mr + \lambda^2}{r^2 + \lambda^2}.
\]

(9.3)

It reduces to the Schwarzchild metric as \( \lambda \to 0 \). Together with a constant dilaton, this solution will satisfy the one loop \( \beta \)-function equations as a low-energy solution.
to string theory, as does the Schwarzchild metric. Given the simple relation between
the dilatonic magnetically charged black hole[8]

\[ dS^2 = \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 - \left( 1 - \frac{2M}{r} \right) dt^2 - r^2 \left( 1 - \frac{Q^2}{Mr} \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + r^2 \left( \frac{1 - Q^2}{Mr} \right) (d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ \hat{\Phi} - \hat{\Phi}_0 = -\frac{1}{2} \log \left( 1 - \frac{Q^2}{Mr} \right), \] (9.4)

and the Schwarzchild solution, it is tempting to conjecture that there exists a
similarly simple modification of the Taub–NUT metric, such that when supple-
mented with an antisymmetric tensor with non–zero components \( B_{\tau \phi} = -B_{\phi \tau}, \) and
gauge fields \( A_\phi, A_\tau, \) a dilatonic Taub–NUT solution to the one loop \( \beta \)-function
equations is constructed. The solution (9.4) reduces to the product form at
extremality as follows: First forming the string metric by multiplying the metric
(9.4) by \( \exp 2(\hat{\Phi} - \hat{\Phi}_0), \) perform the change of variables \( r = 2M + \Delta \sinh^2 \sigma \) where
\( \Delta = 2M - Q^2/(2M) \to 0 \) at extremality. Taking the limit \( \Delta \to 0 \) while holding
the dilaton finite in the region of interest by the absorption of an infinite additive
constant into it via \( \exp(-2\hat{\Phi}_0') = \exp(-2\hat{\Phi}_0) \Delta, \) yields: \( (\tau = t/(2Q)) \)

\[ dS^2 = 4Q^2 \left( d\sigma^2 - \tanh^2 \sigma d\tau^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

\[ \hat{\Phi} - \hat{\Phi}_0' = -\frac{1}{2} \log \cosh^2 \sigma. \] (9.5)

This is the low energy solution for which the product of the monopole conformal
field theory of GPS and the supersymmetric linear dilaton black hole conformal field
theory of Witten is the full heterotic string theory\(^\text{18}\). It is likely that there exists
an analogous transformation to scaled variables for the above conjectured dilatonic
Taub–NUT dyon solution which would yield the low energy solution (8.9) (with the
extra background fields (9.1)) as its extremal limit. The exact string theory is given
by the heterotic coset (7.1).

This extremal dyonic theory is a natural generalisation of the simple product model
(9.5) which is the familiar extremal magnetically charged black hole of Gibbons and

\(^\text{18}\) In ref.[10] it is described how different choices for how the dilaton behaves at
extremality will yield alternatively either the asymptotically flat limit or the mouth
region. The black hole + throat region is the most interesting limit for the purposes
of this paper.

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Maeda[8]. It is smoothly connected to it in the limit $\lambda \to 0, \delta \to 1$. The gauge group of this model is $U(1) \times U(1)$, as it is necessary to have $P_A$ and $Q_B$ distinct from $Q_A$ and $P_B$ (respectively), in order to preserve the smooth limit to the $U(1)$ magnetic case as $\lambda \to 0$ while cancelling all of the anomalies to all orders, as discussed in section 6. Sending $P_A$ and $Q_B$ to zero in this limit recovers the $U(1)$ gauge group.

Because of the lack of a $\sigma$–dependent factor multiplying the round 2–sphere metric, the solution does not have any behaviour resembling the approach to an asymptotically flat limit\(^{19}\). This and all of the solutions studied in this paper as simple products of the angular and radial theories of sections (3) and (4) become throat or wormhole–like in the large $\sigma$ limit. This is the familiar throat geometry typical of extremal black hole solutions. The radius of the throat is set by the charges of the model as can be seen from (9.1) and (7.4). It is a simple matter to evaluate the electric and magnetic charges on this extremal black hole solution. They are given by integrating the electromagnetic field strengths $F^{i}$ and their duals $\star F^{i}$ over the asymptotic 2-sphere at infinity:

$$Q^i_E = \frac{1}{4\pi} \int_{S^2} \star F^i_{\theta\phi}(\sigma \to \infty)$$

$$Q^i_M = \frac{1}{4\pi} \int_{S^2} F^i_{\theta\phi}(\sigma \to \infty). \quad (9.6)$$

where a dilaton factor is necessary to take the dual\(^{20}\). This yields

$$Q^1_E = 4Q_A, \quad Q^2_E = 4P_A, \quad Q^1_M = 4Q_B, \quad Q^2_M = 4P_B. \quad (9.7)$$

These four charges are completely independent as the construction began with eight free parameters, from which four degrees of freedom were removed by the three anomaly equations (6.18) together with the condition on the central charge of the theory (6.19). There is also a dilaton charge and two type of axion charge on the theory. These however are determined in terms of $Q^i_E$ and $Q^i_M$, given that there were only four free parameters left over from the construction of the model. As

\(^{19}\) The full Taub–NUT is asymptotically flat, although it is not possible to write the flat limit in global Cartesian form $G_{\mu\nu} \sim \eta_{\mu\nu} + O(1/\sigma)$.

\(^{20}\) This may be deduced by studying the action for the effective field theory in the standard way[5].
there is a Dirac string, the usual quantisation conditions for $U(1)$ magnetic and electric charges apply, requiring the product of the electric and magnetic charges to be integer (in units of $\hbar c/2$).

10. Conclusions and Outlook.

This paper has presented methods for constructing a rich class of exact heterotic string backgrounds using what might be called a ‘heterotic coset’ technique$^{21}$. This technique combines a gauged WZW model with supersymmetric right–moving fermions and current–algebra left–moving fermions in a way which produces a conformally invariant theory. The novelty here is that none of the three ingredients above need be conformally invariant on its own. This introduces a close relationship between the charges of the fermions and the level of the WZW model. The role of bosonisation and subsequent re–fermionisation is emphasised as crucial in determining the correct gauge invariant lagrangian and in the later determination of the correct spacetime metric of the heterotic string background. These features will be common to any model constructed using this type of mixing of fermions and gauged WZW models$^{22}$.

The application considered in this paper was the construction of dyonic extremal charged black holes. The prototype of the angular theory of the models is the ‘monopole’ conformal field theory of Giddings, Polchinski and Strominger$^{[10]}$, which

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$^{21}$ This is in contrast to earlier constructions$^{[27]}$ of heterotic string backgrounds using ungauged (0, 1) supersymmetric WZW models via superfield constructions.

$^{22}$ In ref.$^{[28]}$, the use of supersymmetric gauged WZW models truncated to (0, 1) supersymmetry was considered as a means of constructing heterotic string backgrounds. It was noted there that in order to cancel the 2d gauge anomaly, the addition of internal fermions was an option. They were coupled to an external (spacetime) gauge field. However, the only external gauge fields which were considered in that paper were such that the theory was again effectively (1, 1) supersymmetric. This is in contrast to the more general couplings considered in the present paper, allowing truly (0, 1) (or (0, 2)) supersymmetric models to be defined. See also ref.$^{[29]}$ for a discussion of heterotic string backgrounds using ‘chiral gauged’ WZW models. The author is grateful to A A Tseytlin for directing his attention to refs.$^{[28][29]}$. 

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was presented by them as an orbifold of an $SU(2)$ WZW model. Here it is shown to be equivalent to a heterotic coset of the type presented in this paper. Similar constructions applied to $SL(2, \mathbb{R})$ yields a family of 2D charged black hole solutions. These are exact conformal field theories which reduce to the one–loop solutions of McGuigan, Nappi and Yost[23], and a special case of the bosonic solution of Ishibashi, Li and Steif[20].

Products of these with the monopole theory will give 4D models which may be interpreted as 4D dyonic extremal black holes. Also considered was a model where the radial and angular sectors were mixed by embedding $U(1) \times U(1)$ non–trivially into $SL(2, \mathbb{R}) \times SU(2)$. This yielded a 4D dyonic model with a metric resembling what would result from extremising a conjectured dilatonic Taub–NUT metric. Such a dilatonic Taub–NUT (with torsion) would be interesting for many reasons. The Taub–NUT solution has interesting cosmological interpretations[26] and the existence of a stringy Taub–NUT would be interesting in that context. Also, (Euclidean) Taub–NUT is a self–dual gravitational instanton[30], and so a string theoretic solution in such a background is certainly interesting and deserves further study. An axionic, dyonic and extremal stringy Taub–NUT with wormhole, found here as an exact conformal field theory may be a useful solution for studying string instantons representing spacetime topology change[31]. The use of Euclidean Taub–NUT in constructing monopole solutions to Kaluza–Klein theories as first presented by Sorkin[32], and Gross and Perry[33] may also be relevant. The presence of such solutions in string theory is important[34]. It would be easy to construct that type of solution (at least in some limit) using these methods. The compact time in the solution of this paper would become another internal coordinate while the role of time would be introduced by tensoring the heterotic coset with a free field.

The theories constructed in this paper all have $(0, 2)$ world sheet supersymmetry. The consequences of this were not pursued in this paper, although the pure magnetic theory was shown in ref.[10] to not satisfy the conditions for $N = 1$ spacetime supersymmetry. Leaving behind the black–hole applications, the construction of this paper can certainly be applied to the study of $(0, 2)$ string compactification. These are less well–studied than their more specialised $(2, 2)$ cousins, but are extremely important for realisations of string vacua which may be relevant to the
world in which we live.

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Note Added.
After this paper was submitted for publication, the paper of ref.[35] appeared, in which 4D dyonic black hole solutions and Bertotti–Robinson Spacetimes were constructed as conformal fields theories, using analogous techniques to those employed in ref.[10]. In particular, an $SL(2, \mathbb{R})/Z$ orbifold is central to the construction of those solutions. Those models are a special case of the models described in section 4. This is easily seen by a choice of gauge analogous to that used in section 3 to show the equivalence between the GPS $SU(2)$ orbifold and the gauged WZW model described there.
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