Nonasymptotic effects in critical sound propagation associated with spin-lattice relaxation

A. Pawlak
Institute of Physics, A. Mickiewicz University, Poznań, Poland

The nonasymptotic critical behavior of sound attenuation coefficient has been studied in an elastically isotropic Ising system above the critical point on the basis of a complete stochastic model including both spin-energy and lattice-energy modes linearly coupled to the longitudinal sound wave. The effect of spin-lattice relaxation on the ultrasonic attenuation is investigated. The crossover between weak-singularity behavior \( t^{-2\alpha} \) and strong-singularity behavior \( t^{-(2+\alpha)} \) is studied as dependent on the values of ultrasonic frequency, reduced temperature, relaxation times etc.. A new high-frequency regime with a singularity of the type \( t^{-4+\alpha} \) is discovered in the magnetic systems. This new regime corresponds to an adiabatic sound propagation and is very similar to the ones in binary mixture and liquid helium. The scaling functions are given in all regimes to the first order in \( \epsilon \). A new frequency-dependent specific-heat being the harmonic average of the bare lattice and critical spin specific-heats is introduced. It was shown that such specific-heat describes the process of equilibration between spin and lattice subsystems and includes the most important features of critical sound attenuation. In some regions of coupling constants the acoustic self-energy can be very well approximated solely by this quantity.

05.70.Jk, 62.65.+k, 64.60.Ht, 64.60.Ak

I. INTRODUCTION

A strong anomaly of sound attenuation measured in Ising like magnetic metals \cite{1,2} is well described by the theories \cite{3,4,5,6} in which the sound wave is assumed to couple to two critical spin fluctuations. The sound attenuation coefficient is found then as the imaginary part of the four-spin response function. On the contrary to metals, in magnetic insulators we observe a weak anomaly in sound attenuation \cite{7,8}, which has been phenomenologically explained \cite{9} by postulating a linear coupling of the sound mode to the spin-energy with the spin-lattice (or more correctly spin-energy-lattice-energy) relaxation time playing the essential role. Assuming only the specific-heat-like singularity of the spin-lattice relaxation time and simple separability of contributions coming from both types of coupling, Kawasaki \cite{7} was able to obtain weak divergence of the sound attenuation coefficient proportional to the square of the specific-heat for the magnets where the first coupling is negligible. However, as it was shown by the renormalization-group analysis of the model of coupled spin and energy fields \cite{10} the first assumption is not correct sufficiently close to the critical point where the energy relaxes with the same characteristic exponent as the spin fluctuations, \( z_E = z \), because the singular part of the energy response function is dominated by the four-spin response function. Also the second assumption concerning the separability fails near \( T_C \) as was recently shown \cite{8}. The objective of this paper is to use a complete stochastic model which includes interactions between the order parameter (spin), the acoustic phonon, spin-energy and lattice-energy modes in order to find the asymptotic as well as nonasymptotic behavior of the sound attenuation coefficient. We shall restrict our discussion to the disordered phase. We consider an elastically isotropic Ising model, where above the critical temperature a phonon (longitudinal) mode \( Q \) is coupled to a scalar order parameter \( S \) as well as to the fluctuations of spin- and lattice-energy densities \( e_S \) and \( e_L \). Introduction of the second energy density associated with the lattice (or conduction electrons in some metals) allows us to consider much richer dynamics than that of the two limiting cases: pure relaxation to a reservoir with infinite thermal conductivity or pure diffusion of energy. Our model permits also a study of the intermediate dynamics between these limiting cases. We have obtained an general expression for the sound attenuation coefficient properly describing the ultrasound attenuation in very broad temperature and frequency range. The purpose of this paper is to demonstrate that the strong sound attenuation anomaly (sometimes called Murata-Iro-Schwabl behavior) as well as the weak anomaly (Kawasaki behavior) can be obtained within our stochastic model depending on the relative size of the reduced temperature, frequency and other parameters of the model. The nonuniversal amplitudes for both types of behavior are modified considerably by the presence of the second energy field in comparison with one-energy-field model. In the high-frequency regime we obtain a new kind of critical behavior, which we shall call the “adiabatic” behavior, analogous to the critical attenuation behavior in \(^4\)He and in binary mixtures. Contrary to the Murata-Iro-Schwabl and Kawasaki behavior the amplitude for the “adiabatic” limit is determined completely only by one coupling constant, the one describing the interaction of the sound mode with two fluctuations of the order parameter. The “adiabatic” limit does not change by the inclusion of the field \( e_L \). We shall also show that in
some regions of parameters the acoustic self-energy is simply proportional to a frequency-dependent specific-heat $C_-$ which is the harmonic average of the bare lattice specific-heat and the specific-heat of an idealized spin system (model A in terminology of [10]). $C_-$ has a simple interpretation: it equals to the ratio of the heat transferred from one sub-system to the other, to the induced temperature difference between subsystems. The important point is that such specific-heat shows three main types of singularities characterizing the sound attenuation coefficient as the critical temperature is approached for low or high frequencies.

The paper is organized as follows. In Sec.II we introduce the model and then perform some dynamical decoupling transformations. In Sec.III we obtain a general expression for the acoustic self-energy taking into account the reducibility of the latter with respect to energy propagators. In Sec.IV many different regimes for sound-attenuation are discussed as a function of frequency, temperature and relaxation times. The scaling functions are determined within the one-loop approximation. We then show that for some regions of parameters the ultrasonic attenuation can be written as the imaginary part of $C_-(\omega)$. The relevance of this specific-heat to the description of nonasymptotic sound attenuation is shown. In Sec.V we summarize the results.

II. MODEL AND BASIC FORMULAS

A. Equations of motion

The entropy of the system [11] can be written as

$$H = \frac{1}{2} \int d^4x \left\{ rS^2 + (\nabla S)^2 + \frac{u}{2} S^4 + C_{12} \left( \sum_\alpha e_{\alpha\alpha} \right)^2 + 2C_{44} \sum_{\alpha,\beta} e_{\alpha\beta}^2 
+ 2g \sum_\alpha e_{\alpha\alpha} S^2 + 2fe_S S^2 + 2w(\epsilon_S + \epsilon_L) \sum_\alpha e_{\alpha\alpha} + \frac{\epsilon_S^2}{C_S} + \frac{\epsilon_L^2}{C_L} \right\}, \tag{2.1}$$

where $e_{\alpha\beta}(x)$ are components of the strain tensor and the symbols $C_{\alpha\beta}$ stand for the bare elastic constants and $C_S$ and $C_L$ are the spin and the lattice specific-heat, respectively. The unitary mass density and $k_B T_C = 1$ have been assumed. The first three terms in the total Hamiltonian describing the static behavior of the system, make the Ginzburg-Landau functional for the spin variable. The elastic energy is given by the 4th and 5th term in Eq. (2.1). Here, we have made use of the relation $C_{11} - C_{12} = 2C_{44}$ applicable to the isotropic cubic systems. The last two terms results from lowest order expansion of the entropy functional with respect to energy fields. The other terms in the Hamiltonian describe interactions. The constant $g$ denotes the coupling of the (longitudinal) sound mode to two spin fluctuations. The coupling of the sound to the energy fields is characterized by the constants $w$ and $aw$ where the parameter $a$ is the ratio of the couplings of the phonon mode to the lattice-energy $\epsilon_L$ and spin-energy density $\epsilon_S$, respectively, $f$ is the coupling constant between the order parameter and the spin energy generating the divergence of the specific-heat.

After introducing the normal mode expansion of the strain tensor, the dynamics of the system can be described by the coupled Langevin equations [41]

$$\dot{S}_k = -\Gamma \frac{\delta H}{\delta S_{-k}} + \xi_k, \tag{2.2a}$$

$$\dot{Q}_k = -\frac{\delta H}{\delta Q_{-k}} - \theta k^2 \dot{Q}_k + \eta_k, \tag{2.2b}$$

$$\dot{\varphi}_k = -\left( \gamma_S + \lambda_S k^2 \right) \frac{\delta H}{\delta \varphi_{-k}} + \frac{\delta H}{\delta \psi_{-k}} + \varphi_k, \tag{2.2c}$$

$$\dot{\psi}_k = -\left( \gamma_L + \lambda_L k^2 \right) \frac{\delta H}{\delta \psi_{-k}} + \frac{\delta H}{\delta \varphi_{-k}} + \psi_k, \tag{2.2d}$$

where $Q_k$ is the longitudinal phonon normal coordinate and $\xi_k, \eta_k, \varphi_k$ and $\psi_k$ are the Fourier components of Gaussian white noises with variances related to the bare damping terms $\Gamma, \theta k^2, (\gamma_S + \lambda_S k^2)$ and $(\gamma_L + \lambda_L k^2)$ through the Einstein relations.
The first two of these equations have been commonly used in investigation of critical sound propagation \[14\,15\]. The other two describe the energy flows between the sub-systems. In the absence of the nonlinear terms and noises they transform into the equations describing the decay of total energy \( e = e_S + e_L \) and equalization of temperatures of both sub-systems

\[
\dot{e} = -\frac{\Lambda_+}{C_+} e - \Lambda_{\Delta e} C_{-} \Delta
\]  
(2.3)

\[
\dot{\Delta} = -\frac{\Lambda_{\Delta e}}{C_+} e - \Lambda C_{-} \Delta
\]  
(2.4)

where \( \Delta = \delta T_L - \delta T_S = e_L/C_L - e_S/C_S \) is the temperature difference between both sub-systems, \( \Lambda_+ = \gamma_S + \gamma_L - 2\gamma + \lambda_+ k^2 \), \( \Lambda_{\Delta e} = \frac{2\gamma - \gamma_S + \lambda_L k^2}{C_L} \) is the temperature difference between both sub-systems, \( \Lambda_+ = \gamma_S + \gamma_L - 2\gamma + \lambda_+ k^2 \), \( \Lambda_{\Delta e} = \frac{2\gamma - \gamma_S + \lambda_L k^2}{C_L} \), \( \lambda_+ = \lambda_S + \lambda_L, C_+ = C_S + C_L \), and \( C_- = (1/C_S + 1/C_L)^{-1} \). If \( \gamma_S = \gamma_L = \gamma \), then the total energy is conserved (for \( k \to 0 \)) and \( \frac{\Delta}{C_+} = \frac{\gamma}{C_+} k^2 \) is the rate of the thermal conduction with \( \lambda_S \) and \( \lambda_L \) being the thermal conductivity for spin and lattice sub-systems, respectively. For this case, the temperature difference \( \Delta \) decays at the rate \( \Lambda C_- \approx \gamma/C_- \). Therefore, \( \Lambda C_- \) can be interpreted as the frequency of the spin-lattice relaxation. It reduces to \( \gamma/C_+ \) for the lattice of infinite specific heat. For non-conserved total energy, as e.g. for \( \gamma_L > \gamma_S = \gamma \) (we assume that the spin-energy can change only through interactions with lattice degrees of freedom i.e. \( \gamma_S = \gamma \)),

\[
\dot{e} = -\frac{\Lambda_+}{C_+} e - \frac{\Lambda_+ C_{-}}{C_L} \Delta
\]  
(2.5)

\[
\dot{\Delta} = -\frac{\Lambda_{\Delta e}}{C_+} e - \Lambda C_{-} \Delta
\]  
(2.6)

with \( \Lambda_+ \approx \gamma_L - \gamma \). The total energy, in the absence of the temperature difference \( \Delta \), decays at the frequency \((\gamma_L - \gamma)/C_+ \) and the temperature difference relaxes (in the absence of total energy fluctuations) at the rate \( \Lambda C_- = \gamma/C_- + \frac{\Lambda_{\Delta e} C_S}{C_L} \).

In a typical case \( e_L \) is the sum of energies of all phonon branches except for the longitudinal acoustic phonons with wave-vectors restricted to the sphere \( 0 < |k| < 1 \). These phonons are taken explicitly into account in the model as the quantity of our primary interest in this paper is the phonon response function. In some metals, where the lattice can be treated fully adiabatically (\( C_L \to \infty \)), \( e_L \) may be understood as the energy density of conduction electrons. In that case, the total energy in the system composed of the localized electrons and the conduction electrons sub-systems is no longer conserved so we have admitted the possibility of nonconserved total energy \( e_L + e_S \) (\( \gamma_L \neq \gamma \)) in the equations of motion.

**B. Lagrangian**

Instead of using Eq. (2.2), it is convenient to represent the model in terms of the equivalent functional form \[14\,15\] with a Lagrangian given by

\[
L = \int \sum_k \left\{ \Gamma \tilde{S}_{\kappa,\omega} \tilde{S}_{-\kappa,\omega} + \Theta k^2 \tilde{Q}_{\kappa,\omega} \tilde{Q}_{-\kappa,\omega} + \Gamma_S(k) \tilde{e}_{\kappa,\omega}^S \tilde{e}_{-\kappa,\omega} + \Gamma_L(k) \tilde{e}_{\kappa,\omega}^L \tilde{e}_{-\kappa,\omega} - 2\gamma \tilde{e}_{\kappa,\omega}^S \tilde{e}_{-\kappa,\omega} \right\}
\]  
(2.7)

\[
- \dot{Q}_{\kappa,\omega} \left[ (\omega^2 + i\theta k^2 \omega) Q_{-\kappa,\omega} + \frac{\partial H}{\partial Q_{\kappa,\omega}} \right] - \tilde{S}_{\kappa,\omega} \left[ i\omega S_{-\kappa,\omega} + \Gamma \frac{\partial H}{\partial S_{\kappa,\omega}} \right]
\]

\[
- \tilde{e}^S_{\kappa,\omega} \left[ i\omega e_{-\kappa,\omega} + \Gamma_S(k) \frac{\partial H}{\partial e_{\kappa,\omega}^S} - \gamma \frac{\partial H}{\partial e_{\kappa,\omega}^S} \right] - \tilde{e}^L_{\kappa,\omega} \left[ i\omega e_{-\kappa,\omega} + \Gamma_L(k) \frac{\partial H}{\partial e_{\kappa,\omega}^L} - \gamma \frac{\partial H}{\partial e_{\kappa,\omega}^L} \right]
\]

where \( \tilde{S}, \tilde{Q}, \tilde{e}^S \) and \( \tilde{e}^L \) are auxiliary “response” fields and \( \Gamma_i(k) = \gamma_i + \lambda_i k^2 \) for \( i = S, L \). The Lagrangian does not contain the Jacobian because we have assumed the Heaviside step function to be zero for time \( t = 0 \). This assumption excludes the accusal terms in the perturbation theory \[14\,15\]. In this formalism the correlation and response functions are given by path integrals weighted with a density \( exp(L) \). For instance if the phonon variable is coupled to an external

3
field $h$ in the entropy functional (2.1), an additional contribution $h_{k,\omega} \tilde{Q}_{-k,-\omega}$ appears in the stochastic functional $L$. Then the phonon response function can be written in the form

$$G(k, \omega) = \langle \tilde{Q}_{-k,-\omega} Q_{k,\omega} \rangle = \frac{1}{Z} \int \mathcal{D}[S] \mathcal{D}[i\tilde{S}] \mathcal{D}[Q] \mathcal{D}[i\tilde{Q}] \mathcal{D}[\tilde{e}_S] \mathcal{D}[\tilde{e}_L] \mathcal{D}[i\tilde{e}_L] \tilde{Q}_{-k,-\omega} Q_{k,\omega} \exp(L) ,$$

(2.8)

where $\mathcal{D}[\cdot]$ denotes a suitable integration measure and $Z$ is a normalization factor. The bilinear terms in the Lagrangian determine the free response and correlation propagators. Usually, we include the terms proportional to the coupling $w$ as well as the term $\gamma \tilde{e}_S \tilde{e}_L$ into the interaction part rather than into the free part of Lagrangian. Then, the free propagators will be given by the expressions presented in Table 8, where $c = C_1^{1/2}$. Note that in order to obtain the free response functions for the spin, spin-energy and lattice-energy fields we have to multiply the response propagator by the corresponding damping coefficient $|c|$. Once the propagators are defined in principle a perturbation expansion for any quantity can be performed. Note also that because of the term $\gamma \tilde{e}_S \tilde{e}_L$, the spin-energy densities $\tilde{e}_S$, $e_S$ and the lattice-energy densities $\tilde{e}_L$, $e_L$ are dynamically coupled, even if $w = 0$ in contrast to the static case where they are decoupled from each other in the functional $H$ for $w = 0$.

### C. Decoupling transformations

We have found it more convenient first to decouple $\tilde{e}_S, e_S$ from $\tilde{e}_L, e_L$ by the Gaussian transformation:

$$e^L_{k,\omega} \rightarrow e^L_{k,\omega} + \frac{\gamma}{C_S \Gamma_L} D_{L0}(k, \omega) e^S_{k,\omega} + A(k, \omega) \tilde{e}^S_{k,\omega} ,$$

$$\tilde{e}^L_{k,\omega} \rightarrow \tilde{e}^L_{k,\omega} + \frac{\gamma}{C_L \Gamma_L} D_{L0}(k, -\omega) e^S_{k,\omega} .$$

(2.9)

Here $A(k, \omega) = -2 \left[ \frac{\gamma}{C_L \Gamma_L} D_{L0}(k, \omega) - \frac{\gamma}{C_S \Gamma_S} |D_{L0}(k, \omega)|^2 \right]$ is a coefficient. For the case of the total energy conserved, this transformation can be also looked at as a substraction of the slow component associated with thermal conduction mode from $e_L$, so the new variable $e^L_{n,\omega}$ contains only the fast mode of equalization of temperatures: $e^L_{n,\omega} = C_L \left( \frac{\partial H}{\partial e^L_{n,\omega}} - \frac{\partial H}{\partial e_S} \right)$ for $\omega = 0$. The Lagrangian expressed in new variables takes the form

$$L = \int \frac{\omega}{2} \sum_k \left\{ \Gamma S^2_{k,\omega} \langle S_{-k,-\omega} \rangle + \theta k^2 \langle Q_{k,\omega} \tilde{Q}_{-k,-\omega} \rangle + \Gamma S^2_{k,\omega} \langle e^S_{k,\omega} e^S_{-k,-\omega} \rangle + \Gamma L(k, \omega) \langle \tilde{e}^L_{k,\omega} e^S_{-k,-\omega} + \tilde{e}^L_{-k,-\omega} e^S_{k,\omega} \rangle - \tilde{Q}_{k,\omega} S_{-k,-\omega} \tilde{Q}_{-k,-\omega} - \frac{\gamma}{C_S \Gamma_L} \sum_k S^2_{k,\omega} e^S_{k,\omega} e^S_{-k,-\omega} - \frac{\gamma}{C_L \Gamma_L} \sum_k e^S_{k,\omega} e^S_{-k,-\omega} - \tilde{e}^L_{k,\omega} D_{L0}^{-1}(k, \omega) e^L_{-k,-\omega} - \tilde{e}^L_{-k,-\omega} D_{L0}^{-1}(k, -\omega) e^L_{k,\omega} - \frac{\gamma}{C_L \Gamma_L} \sum_k e^S_{k,\omega} e^S_{-k,-\omega} \right\} ,$$

(2.10)

where $S^2_{k,\omega} = (\tilde{S} S)_{k,\omega}$, $S^2_{k,\omega}$ is dynamically coupled, even if $w = 0$ in contrast to the static case where they are decoupled from each other in the functional $H$ for $w = 0$.
where \( \omega = \omega_S/\omega_L \) and \( \bar{\omega} = \omega/\omega_L \) is the frequency reduced with respect to the bare spin-energy \( \omega_S = \Gamma_S/C_S \) and lattice-energy \( \omega_L = \Gamma_L/C_L \) relaxation rate, respectively. It is easy to see that for very slow lattice-energy relaxation \( \bar{\omega} \gg 1 \), which takes place for \( \gamma_L = \gamma \rightarrow 0 \) or \( C_L \rightarrow \infty \), the dressed spin-energy propagator reduces to the bare one and the model with two energies reduces to the model without the lattice-energy field, in which only the equation

\[
\dot{\varepsilon}_S = -\Gamma_S(k)\frac{\delta H}{\delta \varepsilon_S} + \varphi
\]

matters in our considerations. Depending on the ratio of the bare relaxation rate of the energy diffusion mode \( \frac{\lambda \bar{k}^2}{\gamma} \) to the bare spin-lattice relaxation frequency \( \gamma/\gamma_L \), the spin system can be then regarded either as thermally isolated from the lattice \( \left( \frac{\lambda \bar{k}^2}{\gamma} \gg 1 \right) \) or as freely relaxing to an infinite heat bath at each lattice site, or to a bath with an infinite thermal conductivity \( \left( \frac{\lambda \bar{k}^2}{\gamma} \ll 1 \right) \). The same happens for the nonconserved energy case. If \( \gamma_L \rightarrow \infty \) the lattice energy always takes its equilibrium value. Then \( \bar{\omega} \ll 1, \frac{\gamma^2}{\Gamma(\Gamma)} \rightarrow 0 \) and again \( \dot{D}_S \rightarrow D_S \).

The one energy field model we recover also for \( \gamma \rightarrow \infty \) or \( C_L \rightarrow 0 \) in the conserved total energy case. Then \( \bar{\omega} \ll 1 \) and expanding Eq. (2.11) we get \( \dot{D}_S(k,\omega) = C_S \Gamma^{-1}_S m^{-1}_e \left( 1 - i \frac{\omega C_S}{\lambda \bar{k}^2} \right)^{-1} \) where \( m(k) \equiv 1 - \frac{\gamma^2}{\Gamma_S(k)\Gamma_L(k)} \frac{\lambda \bar{k}^2}{\gamma} \) with \( \lambda^+ = \lambda_S + \lambda_L \) and \( C^+ = C_S + C_L \). being the thermal conductivity and specific-heat of the total \( S + L \) system. At the same time the dressed damping coefficient \( \Gamma_S(k,\omega) \rightarrow \Gamma_S(k) m \) so the dressed spin-energy response function \( \Gamma_S(k,\omega) \dot{D}_S(k,\omega) \rightarrow C_S \left( 1 - i \frac{\omega C_S}{\lambda \bar{k}^2} \right)^{-1} \). It can be said that on very short time scales, of an order of \( C_L/\gamma \), the lattice comes to equilibrium with the spin system and then for much longer times, of an order of the inverse of ultrasonic frequency, only the diffusion of the total energy \( \varepsilon_S + \varepsilon_L \) matters and instead of Eqs. (2.2a,2.2b) only the equation

\[
\dot{\varepsilon} = -\lambda^+ \bar{k}^2 \frac{\delta H}{\delta \varepsilon} + \phi
\]

may be used. This limiting case for dynamics was considered by Drossel and Schwabl \[17\]. The reason for introducing the second energy field is now becoming clear. We expect that the model with only one energy density correctly describes the dynamics only in the two limiting cases: for pure diffusion of the energy field and for pure relaxation to an infinite heat bath. As we want to be able also to describe the dynamics which is intermediate between these two cases we have included \( \varepsilon_L \) as a separate variable.

Next, we decouple the sound mode from the spin and energy fluctuations by the transformation

\[
Q_{k,\omega} \rightarrow Q_{k,\omega} - g k G_0(k,\omega) S_{k,\omega} + w k J(k,\omega) G_0(k,\omega) \varepsilon_{k,\omega}^S - a w k G_0(k,\omega) \varepsilon_{k,\omega}^L + B(k,\omega) S_{k,\omega}^2 + E(k,\omega) \varepsilon_{k,\omega}^S + F(k,\omega) \varepsilon_{k,\omega}^L ,
\]

\[
\dot{Q}_{k,\omega} \rightarrow \dot{Q}_{k,\omega} - 2 g k G_0(k,-\omega) \Gamma S^2_{k,\omega} - w k K(k,-\omega) \Gamma S G_0(k,-\omega) \dot{\varepsilon}_{k,\omega}^S - w k(a \Gamma_L - \gamma) G_0(k,-\omega) \dot{\varepsilon}_{k,\omega}^L . \tag{2.12}
\]

Here the abbreviations

\[
B(k,\omega) = -4 \Gamma g k |G_0|^2 \bar{k}^2 ,
\]

\[
E(k,\omega) = -2 w k K(k,-\omega) \Gamma |G_0|^2 \bar{k}^2 - a w k A(k,\omega) G_0(k,\omega) ,
\]

\[
F(k,\omega) = -2 w k(a \Gamma_L - \gamma) |G_0|^2 \bar{k}^2 ,
\]

have been introduced. Finally, we apply the transformations:

\[
e_{L_{k,\omega}} \rightarrow e_{L_{k,\omega}} + R(k,\omega) S_{k,\omega}^2 + T(k,\omega) \varepsilon_{k,\omega}^S + U(k,\omega) S_{k,\omega}^2 + V(k,\omega) \varepsilon_{k,\omega}^S ,
\]

\[
\dot{e}_{L_{k,\omega}} \rightarrow \dot{e}_{L_{k,\omega}} + \dot{D}_{L,0}(k,-\omega) \left[ 2 a w g k G_0(k,-\omega) \Gamma S^2_{k,\omega} + a w \bar{k}^2 K(k,-\omega) G_0(k,-\omega) \Gamma S \varepsilon_{k,\omega}^S \right] , \tag{2.13}
\]

and
\[
\hat{\epsilon}^S_{k,\omega} \rightarrow \hat{\epsilon}^S_{k,\omega} + \tilde{D}_{S0}(k,\omega) \Gamma_S \left[ w g k^2 K(k,\omega) G_0(k,\omega) - f N(k,\omega) \right] S^2_{k,\omega} + W(k,\omega) \tilde{S}^2_{k,\omega},
\]
\[
\hat{\epsilon}^L_{k,\omega} \rightarrow \hat{\epsilon}^L_{k,\omega} - 2 \tilde{D}_{S0}(k,-\omega) \left[ f - w g k^2 J(k,-\omega) G_0(k,-\omega) \right] \Gamma \tilde{S}^2_{k,\omega},
\]
\[\text{(2.14)}\]

decoupling \(e_S\) and \(e_L\) from the spin fluctuations, where

\[
\tilde{D}_{L0}^{-1}(k,\omega) = D_{L0}^{-1}(k,\omega) - (a \Gamma_L - \gamma) a w^2 k^2 G_0(k,\omega),
\]
\[
\tilde{D}_{S0}^{-1}(k,\omega) = \tilde{D}_{S0}^{-1}(k,\omega) - w^2 g^2 k^2 J(k,\omega) K(k,\omega) G_0(k,\omega),
\]

are “dressed” by transformations \[\text{(2.9)}\] and \[\text{(2.14)}\] energy-density propagators. Here the coefficients are given by

\[
R(k,\omega) = [\gamma f + (a \Gamma_L - \gamma) w g k^2 G_0(k,\omega)] \tilde{D}_{L0}(k,\omega),
\]
\[
T(k,\omega) = w^2 g^2 J(k,\omega) (a \Gamma_L - \gamma) G_0(k,\omega) \tilde{D}_{L0}(k,\omega),
\]
\[
U(k,\omega) = 4 \Gamma w g k^2 (a \Gamma_L - \gamma) |G_0|^2 \theta k^2 \tilde{D}_{L0}(k,\omega) + 4 a g w k^2 \Gamma \Gamma_L G_0(k,-\omega) \tilde{D}_{L0} |^2,
\]
\[
V(k,\omega) = \omega k \left\{ 2 a w k K(k,-\omega) |1 + (a \Gamma_L - \gamma) a w^2 k^2 |G_0|^2 \theta k^2 G_0(k,-\omega)| \tilde{D}_{L0} |^2 \Gamma \Gamma_L - (a \Gamma_L - \gamma) E(k,-\omega) \tilde{D}_{L0}(k,\omega) \right\},
\]
\[
W(k,\omega) = 2 w g k^2 \left\{ a G_0(k,-\omega) A(k,\omega) + 2 K(k,\omega) \Gamma S |G_0|^2 \theta k^2 \right\} \Gamma \tilde{D}_{S0}(k,\omega) - 4 \left[ f - w g k^2 J(k,-\omega) G_0(k,-\omega) \right] \left( 1 - \frac{\gamma^2 \Gamma L |\tilde{D}_{L0}|^2}{\Gamma S \tilde{S}^2_{k,\omega} L_{en}} \right) \Gamma \Gamma_S \tilde{D}_{S0} |^2.
\]

### III. ACOUSTIC SELF-ENERGY

Having applied the transformations \[\text{(2.9)}, \text{(2.12)}, \text{(2.14)}\] the acoustic phonon response function, to the leading order in coupling constants \(g\) and \(w\), can be written as

\[
\langle \tilde{Q}_{-k,-\omega} Q_{k,\omega} \rangle = G_0(k,\omega) + G_0^2(k,\omega) \Sigma^{red}(k,\omega),
\]
\[\text{(3.1)}\]

where

\[
\Sigma^{red}(k,\omega) = k^2 \left\{ X(k,\omega) \tilde{D}_{L0}(k,\omega) + Y(k,\omega) \tilde{D}_{S0}(k,\omega) + 2 g_1(k,\omega) g_2(k,\omega) (\Gamma \tilde{S}^2_{-k,-\omega} S^2_{k,\omega}) L_{en} \right\},
\]
\[\text{(3.2)}\]

and

\[
X(k,\omega) = a (a \Gamma_L - \gamma) w^2, \quad Y(k,\omega) = w^2 \Gamma S K(k,\omega) J(k,\omega),
\]
\[
g_1(k,\omega) = g - w f_1 \Gamma S K(k,\omega) \tilde{D}_{S0}(k,\omega) + a w f \Gamma L \tilde{D}_{L0}(k,\omega),
\]
\[
g_2(k,\omega) = g - w f_2 J(k,\omega) \Gamma S \tilde{D}_{S0}(k,\omega),
\]

with \(f = \gamma f + (a \Gamma_L - \gamma) w g k^2 G_0(k,\omega), f_1 = f N(k,\omega) - w g k^2 K(k,\omega) G_0(k,\omega)\) and \(f_2 = f - w g k^2 J(k,\omega) G_0(k,\omega)\). The crucial point in Eq. \[\text{(3.2)}\] is that the four-spin response function is calculated with the Lagrangian which depends solely on the spin variables. As the result of coupling of the spin variable to the energy densities and phonons, the effective spin Lagrangian \(L_{eff}\) obtained by transformations \[\text{(2.9)}, \text{(2.12)}, \text{(2.14)}\] contains phonon- as well as energy-density-mediated four-spin non-local interactions \(u(k,\omega) S^2_{-k,-\omega} S^2_{k,\omega}\) with
\[ u(k, \omega) = u - 2g^2k^2G_0(k, \omega) - 2awgk^2\Gamma_L G_0(k, \omega)\tilde{D}_L 0(k, \omega) - 2\tilde{f}_1\tilde{f}_2\Gamma_S \tilde{D}_S 0(k, \omega) \,. \]

A general expression for the interacting phonon response function is given by
\[ G^{-1}(k, \omega) = G_0^{-1}(k, \omega) - \Sigma(k, \omega), \] where the phonon self-energy \( \Sigma(k, \omega) \) is the irreducible, with respect to phonon lines, part of \( \Sigma^{\text{red}}(k, \omega) \). From Eqs. (3.1) and (3.4) the irreducible and reducible parts are related through
\[ \Sigma(k, \omega) = \frac{\Sigma^{\text{red}}(k, \omega)}{1 + G_0(k, \omega)\Sigma^{\text{red}}(k, \omega)}. \] Next we may eliminate the dangerous resonances \( [12, 9] \) by replacing \( G_0(k, \omega) \) with \( G_0(0, \omega) \) in \( \Sigma(k, \omega) \) i.e. we set the strongly irrelevant parameters - the coefficients at \( \omega k^2 \) and \( \omega^2 \) - equal to zero. In this paper we will not discuss the role of the denominator in Eq. (3.5) assuming the elastic couplings \( g \) and \( w \) to be very small. This assumption gives \( \Sigma(k, \omega) = \Sigma^{\text{red}}(k, \omega). \) It is consistent with neglecting the higher order (in \( g \) and \( w \)) terms in \( \Sigma^{\text{red}}(k, \omega). \) Then also \( \tilde{D}_L 0(k, \omega) = D_L 0(k, \omega) \) and \( \tilde{D}_S 0(k, \omega) = D_S 0(k, \omega) \) together with small \( g \) and \( w \) let us also neglect the macroscopic instability \( [13, 14] \), which is believed to take place in compressible spin systems with positive specific-heat exponent near the transition temperature. The weak first-order transition in such systems is a result of a non-analytical character of the coupling constant \( u(k, \omega) \) at \( k = 0 \). For small \( g \) and \( w \) the first-order regime can be probably observed only extremely close to \( T_C \). We could in principle put other irrelevant parameter in Eq. (3.5) equal to zero as the coefficient at \( \omega \) in the energy-density propagators for non-conserved systems, but then we would obtain only the asymptotic behavior of the sound attenuation coefficient. As we are also interested in the nonasymptotic behavior associated with the coupling to the energy-density we will proceed in another way, preserving the nonasymptotic effects. First, note, that because for small \( g \) and \( w \)
\[ u(k, \omega) = u - 2f^2N(k, \omega)\Gamma_S \tilde{D}_S 0(k, \omega), \]

thus \( \langle \tilde{S}_2^2 \rangle_{\omega} \) is reducible with respect to (new) spin-energy propagators and so is \( \Sigma \). We assume that the coupling constant \( f \) need not to be very small but the positivity of the four-spin coupling must be guaranteed \( [9] \). The perturbation expansion with respect to \( u(k, \omega) \) let us express \( \langle \tilde{S}_2^2 \rangle_{\omega} \), the reducible with respect to \( \tilde{D}_S \) four-spin response function, in terms of the irreducible one
\[ \langle \tilde{S}_2^2 \rangle_{\omega} = \frac{\langle \tilde{S}_2^2 \rangle_{\omega}^{\text{irr}}}{1 - 2f^2\Gamma_S N(k, \omega)\tilde{D}_S 0(k, \omega)(\langle \tilde{S}_2^2 \rangle_{\omega}^{\text{irr}})}, \]

Eq. (3.6) is equivalent to
\[ \langle \tilde{S}_2^2 \rangle_{\omega}^{\text{irr}} = \frac{\langle \tilde{S}_2^2 \rangle_{\omega}^{\text{eff}}}{1 + 2f^2\Gamma_L N(k, \omega)\tilde{D}_L 0(k, \omega)(\langle \tilde{S}_2^2 \rangle_{\omega}^{\text{eff}})}. \]

It is easy to note that the coefficient preceding \( \langle \tilde{S}_2^2 \rangle_{\omega} \) in the denominator of Eq. (3.7) can be written as \( 2f^2\tilde{D}_S(k, \omega) \) with \( \tilde{D}_S(k, \omega) = C_S (1 - i\omega b(q, \omega))^{-1} \) where \( b(q, \omega) = \frac{1 - i\omega}{m - \omega}. \) In the \( \omega = 0 \) limit it is equal to \( v = 2f^2C_S. \) As \( \langle \tilde{S}_2^2 \rangle_{\omega}^{\text{irr}} \) is a vertex function, containing no resonances, we may again replace strongly irrelevant parameters in Eq. (3.7) like the coefficient at \( \omega \) in \( \tilde{D}_S 0, \) in the case of nonconserved systems, by zero. We then obtain
\[ \langle \tilde{S}_2^2 \rangle_{\omega}^{\text{irr}} = \frac{\langle \tilde{S}_2^2 \rangle_{\omega}^{\text{eff}}}{1 + v(\langle \tilde{S}_2^2 \rangle_{\omega}^{\text{eff}})}, \]

where \( L_A \) is the action of the model A of Halperin, Hohenberg and Ma \([10]\), with \( u_A = u - v. \) In the case of a system with the conserved total energy we cannot proceed in this way as \( D_S 0 \) contains nonstatic relevant terms and \( \langle \tilde{S}_2^2 \rangle_{\omega}^{\text{irr}} \) should be calculated using energy generated dynamic interactions in \( L_{\text{eff}} \). The asymptotic behavior of the ultrasonic attenuation coefficient in a model with conserved energy field was investigated by Drossel and Schwabl \([17]\). They showed that the difference of the asymptotic scaling functions (compared to the model A)
originates mainly from different exponent $z$ for model C. In the present paper we will restrict our discussion to the nonconserved systems and a detailed analysis of the conserved systems will be presented elsewhere.

Now we return to Eq. (3.8) which inserted into Eq. (3.6) yields

$$
\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{Latt}} = \frac{\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}}}{1 + 2f^2(C_S - \bar{D}_S)\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}}}.
$$

(3.9)

Next from Eqs. (3.2), (3.5) and (3.9) we obtain

$$
\Sigma(k, \omega) = k^2 \left\{ X(k, \omega) D_{L0}(k, \omega) + Y(k, \omega) \bar{D}_{S0}(k, \omega) + \frac{2g_1(k, \omega)g_2(k, \omega)\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}}}{1 + 2f^2(C_S - \bar{D}_S)\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}}} \right\}.
$$

(3.10)

It is not difficult to note that

$$
g_1(k, \omega) = g_2(k, \omega) = \dot{g}(k, \omega) = g - wfP(k, \omega)\bar{D}_S(k, \omega),
$$

with

$$
P(k, \omega) = \frac{m - i\omega(1 - \frac{a^2}{m})}{m - i\omega}.
$$

Now let us split $AD_{L0} + B\bar{D}_{S0}$ into the following terms

$$
a^2w^2k^2mC_L \frac{m - i\omega}{m - i\omega} + w^2k^2P^2\bar{D}_S(k, \omega)
$$

and reduce the second term to the common denominator with the third term in Eq. (3.11). This results in the following expression for $\Sigma$

$$
\frac{\Sigma(k, \omega)}{k^2} = \frac{a^2w^2C_L m}{m - i\omega} + \frac{2w^2f^2P^2C_S \bar{D}_S(k, \omega) - 4wfgP \bar{D}_S(k, \omega) + 2g^2\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}} + w^2P^2}{1 + 2f^2(C_S - \bar{D}_S)\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}}}.
$$

(3.11)

Remembering that $C_S - \bar{D}_S(k, \omega) = -i\bar{\omega}b(k, \omega)\bar{D}_S(k, \omega)$ and ignoring the first noncritical term in Eq. (3.11) the acoustic self-energy can be written as

$$
\frac{\Sigma(k, \omega)}{k^2} = \frac{2[g^2(k, \omega) - i\bar{\omega}b(k, \omega)g^2]\langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}} + w^2P^2C_S}{1 - i\bar{\omega}b(k, \omega)(1 + v(\Gamma S^2_{-k,-\omega} S^2_{k,\omega}))_{\text{L}}}
$$

(3.12)

with $g(k, \omega) = g - wfC_S P(k, \omega)$.

IV. DISCUSSION

A. General expressions

The critical contribution to the coefficient of attenuation is determined by the imaginary part of $\Sigma$ and Eq. (3.12) implies

$$
\alpha(\omega, t) = \left(\frac{\omega}{2e^3}\right)^2 W_1|\Psi|^2 + W_2\text{Im}\Psi + W_3\text{Re}\Psi
$$

(4.1)

where $\Psi = \langle \Gamma S^2_{-k,-\omega} S^2_{k,\omega} \rangle_{\text{L}}$ and $W_1 = 2v\bar{\omega}\text{Re}[\bar{b}g^2(k, \omega)]$,

$$
W_2 = 2\text{Re}g^2(k, \omega) + 2\bar{\omega}|b|^2g^2 + \bar{\omega}\text{Im}[v\bar{b}w^2C_S P^2 + 2bg^2 - 2\bar{b}g^2(k, \omega)],
$$

$$
W_3 = 2\bar{\omega}\text{Re}[\bar{b}g^2(k, \omega) - bg^2 + \bar{b}w^2f^2C_S^2 P^2] + 2\text{Im}g^2(k, \omega).
$$
In the limit $\Gamma_L \to \infty$ these coefficients take the corresponding values from the model without lattice-energy fields \[ \mathrm{I} \] i.e. $W_1 = 2e\tilde{\omega}\tilde{g}^2$, $W_2 = 2(\tilde{g}^2 + \tilde{\omega}^2\tilde{g}^2)$ and $W_3 = -4\omega f CS\tilde{\omega}\tilde{g}$ with $\tilde{g} = g - wf CS$. The four-spin response function usually is evaluated in the limit $k = 0$ since in the ultrasonic experiments the wavelength is much longer than the correlation length ($k\xi \ll 1$) while the ultrasonic frequency can be comparable to the characteristic frequency of the spin fluctuations. Then the singular part of the four-spin response function $\Psi$ obeys the scaling relation
\[
\Psi = t^{-\alpha}\Phi(y) \quad ,
\]
with $y = \omega t^{-\nu}/\Gamma$ as the reduced frequency and $t$ proportional to the reduced temperature. Here $\alpha$, $\nu$ and $z$ are usual critical exponents. The scaling function $\Phi$ is known to the leading order in $\epsilon = 4 - d$ \[ \mathrm{II} \] : 
\[
\Phi(y) = \Theta^{-\alpha/\nu}\left\{\frac{\nu}{\alpha} + \frac{i}{y}\left[i(1 - iy/2)\arctan(y/2) - \frac{1}{2}\ln(1 + (y/2)^2)\right]\right\}K_4,
\]
where $\Theta = [1 + (y/2)^2]^{-1/4}$ and $K_4 = (8\pi^2)^{-1}$. With the aid of Eq. (4.1) the attenuation coefficient can be written as
\[
\alpha(\omega, t) = \frac{\omega^2}{2e^3}\left[\frac{CS}{\Gamma S}\tilde{W}_1 t^{-2\alpha} F_1(y; \omega, t) + \frac{1}{\Gamma}\tilde{W}_2 t^{-(\alpha + \nu)} F_2(y; \omega, t) + \tilde{W}_3 t^{-\alpha} F_3(y; \omega, t)\right], \tag{4.2}
\]
where $\tilde{W}_1 = W_1 K_4/\tilde{\omega}$, $\tilde{W}_2 = W_2 K_4$, $\tilde{W}_3 = W_3 K_4/\omega$ and $F_1 = |\Phi/M|^2$, $F_2 = \text{Im}(\Phi)y^{-1}|M|^2$, $F_3 = \text{Re}(\Phi)|M|^2$ with the denominator of Eq. (4.1) denoted by
\[
M(t, \omega) = 1 - i\tilde{\omega}b(1 + vt^{-\alpha}\Phi).
\]

B. Regime I

Eq. (4.2) shows many different regimes. First let us consider the regime I: $m \gg \tilde{\omega}$ ($\omega \ll (g - \gamma CS)\Delta_{L}/CL$) for which $P(\omega) \to 1$. It seems that the regime I may be adequate for some metals, where $c_L$ should be understood as the spin energy of the conduction electrons and also infinite specific-heat associated with phonons can be assumed.

1. Low-frequency region

In the low-frequency region, $\tilde{\omega}|e| \simeq \frac{\hbar}{\Lambda} \ll 1$, the denominator in Eq. (4.1) does not play any role and the second term in the numerator with strong singularity dominates
\[
\alpha(\omega, t) \simeq \frac{\tilde{g}^2}{C^2}\omega^{-\nu}\frac{\text{Im}\Phi(y)}{y}. \tag{4.3}
\]
This type of singularity in the ultrasonic attenuation, which we shall also call Murata-Iro-Schwabl behavior, has been so far obtained by neglecting the energy density fields \[ \mathrm{II} \] and it is believed to take place in magnetic metals \[ \mathrm{II} \].

Note, however that the coefficient $W_2$ in this limit is equal $2(g - wf CS)^2$ to be compared with $2g^2$ in the model without coupling to the energy fields \[ \mathrm{I} \].

When $\tilde{\omega}/m \ll 1$ i.e. for $M(t, \omega) \simeq 1$ while $t$ is not extremely small there may be a competition between the first and the second term in Eq. (4.2). For $t > t_{\text{cross}}$ with $t_{\text{cross}} = \left(\frac{\alpha^2 m^2 \gamma S}{4 KL (1 + \epsilon S \Gamma)}\right)^{\frac{\nu-\alpha}{\nu}}$ the weak-singularity term dominates and
\[
\alpha(\omega, t) \simeq \frac{\tilde{g}^2 wf CS}{C^3 m^2 \gamma S} \omega^2 t^{-2\alpha}|\Phi(y)|^2. \tag{4.4}
\]
Such behavior was first obtained by Kawasaki \[ \mathrm{I} \]. He postulated phonon-spin-energy coupling in order to explain extremely small sound attenuation exponents observed in magnetic insulators. Note however the difference between the corresponding scaling functions. What also arises from our analysis, is that such behavior is generated even if there is no direct coupling of the sound mode to the energy-density fields, and that it cannot be truly asymptotic as both terms (4.3) and (4.4) are proportional (in the $\omega \to 0$ limit) to the square of the same coupling constant $g$ so even if $g = 0$ the strong singularity described by Eq. (4.3) is still present and become dominant for sufficiently small $t$. The dominance of a given type of behavior does not depend on the relative strength of the coupling constants $g$.
and $\nu$ but rather on the ratio of the bare relaxation times $\frac{2m}{v(m)}$ and also on the ratio of $\nu$ to its fixed point value $v^* \simeq (\alpha/v)K_4^{-1}$. Slow spin-lattice relaxation (and/or fast decay of spin fluctuations) together with strong coupling $v$ favors the weak-singularity type behavior.

It should be noted that the amplitudes of the Murata-Iro-Schwabl as well as the Kawasaki term are proportional to the same coupling constant $\tilde{g}^2$ only in the limit $\tilde{\omega} = \tilde{\omega} = 0$. For finite $\tilde{\omega}$ or $\tilde{\omega}$ both amplitudes begin to differ from each other and become frequency dependent as can be seen from the expressions for the coefficients $\tilde{W}_1$ and $\tilde{W}_2$. For example $\tilde{W}_1$ is not longer equal to $\tilde{W}_2 - 2\tilde{\omega}^2|b|^2g^2$. To be exact this frequency-dependence (except $2\tilde{\omega}^2|b|^2g^2$ term in $\tilde{W}_2$) of the coefficients $\tilde{W}_i$ results from the inclusion of the lattice-energy field to the model in which spins, phonons and spin-energy modes interact. The nonuniversal frequency-dependent coefficients may be useful in making comparisons with experimental results.

2. High-frequency region

As the sound frequency increases, $\tilde{\omega}/m$ approaches unity and the denominator in Eq. (4.1) as well as the second term in the expression for $\tilde{W}_2$ become important. With a further increase of the sound frequency, two scenarios, can be realized. In the first one, $\tilde{W}_2 \ll \tilde{W}_1$ i.e. the sound frequency is not high enough and/or $g$ is much smaller then $\tilde{g}$. Then, the situation resembles the high frequency regime for Kawasaki classical relaxation function [24]. The sound attenuation coefficient saturates and $\alpha(\omega, t) \propto \text{const}$. The second scenario happens when $\tilde{W}_2$ in which the term $2g^2\tilde{\omega}^2|b|^2$ dominates, is much greater then $\tilde{W}_1$ and then:

(i) for $vt^{-\alpha} \Phi \gg 1$ i.e. with noticeable singularity in the specific-heat a new type of behavior can be observed

$$\alpha(\omega, t) \propto g^2\tilde{\omega}^2 t^{-(\nu - \alpha)} \frac{\text{Im}\Phi(y)}{y}, \quad (4.5)$$

which can be obtained from the one described by Eq. (4.3) by a simple replacement $\alpha \rightarrow -\alpha, \Phi \rightarrow \Phi^{-1}$ and $\tilde{g} \rightarrow g$ giving a bit weaker than Murata-Iro-Schwabl singularity but much stronger than the anomaly in the energy-relaxation-dominated region described by Eq.(4.3). The difference in exponents $\nu + \alpha$ and $\nu - \alpha$ is very small and it may be difficult to distinguish both regimes in experiment. This new type of behavior of ultrasonic attenuation in magnets is very interesting. In this regime the frequency of the sound mode is much higher than the (modified by the lattice) spin-energy relaxation frequency so contrary to the low-frequency region described by Eqs. (1.3) and (1.4) now the “lattice” and the spin systems no longer can be treated as remaining at a local equilibrium. The local temperature of the spin system is not able to follow alternate hot and cold temperature variations produced by the ultrasonic wave. One can say that we have here some kind of “adiabatic” sound propagation although the “lattice” (whatever it is in the spin system is not able to follow alternate hot and cold temperature variations produced by the ultrasonic wave.

The behavior described by Eq. (1.3) can be regarded as dominant for the whole asymptotic region $t, \omega \rightarrow 0$ except only for a very narrow area $vt^{-\alpha}\tilde{\omega}/m \gg 1$ where the “adiabatic” behavior (1.3) is dominant. The latter area, however, can be experimentally inaccessible as for sufficiently low sound-frequency the reduced temperature needed to make $vt^{-\alpha}\tilde{\omega}/m$ big enough may be too small, as a consequence of smallness of the exponent $\alpha$, to be approached in real experiment. For this reason we shall sometimes call the Murata-Iro-Schwabl behavior also the asymptotic behavior keeping in mind that it concerns only the experimentally accessible part of the asymptotic limit. On the other hand, in many ultrasonic experiments the measured frequency range is limited only to one or two decades so for the magnets with slow-spin-lattice relaxation this asymptotic region can not be approached at all. Because the “adiabatic” behavior comes from the $2\tilde{\omega}^2|b|^2g^2$ term in the coefficient $\tilde{W}_2$ we can also say that as the reduced frequency $\tilde{\omega}$ increases, the Murata-Iro-Schwabl behavior turns into the “adiabatic” one.

(ii) For $vt^{-\alpha} \Phi \ll 1$ we recover the asymptotic behavior (1.3) but with $\tilde{g}^2$ replaced by $g^2$. What is rather unexpected we observe the asymptotic behavior in quite a nonasymptotic region because here $\tilde{\omega} \gg 1$ and $t$ is also large. Moreover, the proportionality coefficient $g^2$ is exactly equal to that from the theory which neglects all effects of the energy-density fields. However this region is not fully critical because as a result of smallness of $vt^{-\alpha} \Phi$ also the critical contribution to the specific-heat is very small so the specific-heat does not show any noticeable divergence in this region.

The last statement is easily seen in the static case, where the temperature dependent specific-heat is given by $C_S(t) = C_S(1 + \frac{T}{2}(S^2_{S^2_{S^2_{S}}}H))$, remembering that the four-spin response function $2(\Gamma S^2_{S^2_{S^2_{S}}})^{LA}$ transforms into $\langle S^2_{S^2_{S^2_{S}}}H \rangle$ for $\omega = 0$. Here the static average (...)$_{\omega}$ was calculated with the weight $\exp(-H)$.

In the above we have assumed that despite of $\tilde{\omega} \gg 1$ the frequency $\tilde{\omega}$ is still small. This assumption not always seems to be correct and if so the reduced frequency $\tilde{\omega}$ may become comparable with 1. Then the coefficients $\tilde{W}_i$ and $b(\omega)$ began to evolve, however the “adiabatic” limit ($\tilde{\omega} \rightarrow \infty$) does not change as the term $2\tilde{\omega}^2|b|^2g^2$ in $\tilde{W}_2$ is reduced with the dominant term in the denominator of Eq.(4.1).
3. Comparison of the “adiabatic” limit with critical ultrasonic attenuation in binary mixtures and liquid $^4$He

The analytic relation (4.5) resembles the corresponding asymptotic formulas for sound attenuation in binary mixtures [23,24,25,26,27], where the sound attenuation exponent is also equal to $\nu - \alpha$. Looking at Eq.(4.3) it is easy to see that it can also be transformed into the form

$$\alpha(\omega, t) \propto g^2 \omega \text{Im} (1 + v (\Gamma_{S2}^{-2})_{-k,-\omega} S^2_{k,\omega}^{2})^{-1},$$

(4.6)

in this region, which is also identical in form to the expression used by Ferrel and Bhattacharjee [22] for the ultrasonic attenuation near the $\lambda$-transition in liquid helium (although the specific-heat exponent is very close to zero for such systems):

$$\alpha(\omega, t) \propto -\omega \text{Im} C_{FB}(t, \omega)^{-1},$$

(4.7)

where $C_{FB}(t, \omega)$ is a phenomenological frequency-dependent specific-heat. Comparing Eqs.(4.6) and (4.7) we get the correspondence between these two equations by interpreting

$$C^A_S(t, \omega) = C_S[1 + v (\Gamma_{S2}^{-2})_{-k,-\omega} S^2_{k,\omega}^{2}]^{-1}$$

(4.8)

as a frequency-dependent specific-heat in the Ferrel-Bhattacharjee sense, where as usual we put $k = 0$. A very similar identification, in the case of $^4$He, was made by Pankert and Dohm [23] who gave the statistical meaning to $C_{FB}(t, \omega)$. Note however that in spite of the fact that for $\omega \to 0$ Eq.(4.8) transforms (up to bilinear terms in small couplings $w$ and $g$) into the exact relation known from the statics

$$C_S(t, k) = C_S + (f - w g^2 c_0^2) C^A_S(S^2_{-k,S} S^2_{-k}) e_h,$$

(4.9)

where $C_S(t, k) \equiv \langle \epsilon^2_k \widetilde{e}^S_k \rangle e_h$ is the static temperature and wave vector dependent specific-heat, with $C^A_S = C^A_S - w^2 c_0^{-2}$ and $c_0^{-2} = C^A_S - a^2 w^2 C_L$, the interpretation of Eq.(4.6) in terms of true frequency-dependent specific-heat is inaccurate as in dynamics the latter is given by the product of the spin-energy propagator $D_S(k, \omega)$ and the frequency-dependent damping coefficient $\Gamma_S(k, \omega)$, which can be found as the coefficient staying in front of the $\epsilon_S \widetilde{e} \epsilon$ term in the Lagrangian after transformation (2.14). From the transformations (2.9,2.13,2.14) we find

$$D_S(k, \omega) = \tilde{D}_{S0}(k, \omega) + 2 \tilde{D}_{S0}(k, \omega) \tilde{f}_1(\omega) \tilde{f}_2(\omega) (\Gamma_{S2}^{-2})_{-k,-\omega} S^2_{k,\omega}^{2} L_{eff}$$

(4.10)

and $\Gamma_S(k, \omega) \equiv \Gamma_{S0}(k, \omega)$ if we drop the terms of order of $w^2$. The determined in this way frequency-dependent specific-heat, although giving the same static limit (4.9) for $\omega \to 0$ differs significantly from $C^A_S(t, \omega)$ for high frequencies ($\omega \to \infty$) contrary to $\tilde{C}_S$ which is a constant. Also the four-spin response functions in Eqs.(4.8) and (4.10) show different behavior as the coupling constant $u$ is frequency independent in the idealized Lagrangian $L_A$ in contrast to $u(k, \omega)$ in $L_{eff}$ (Eq.(2.3)). The advantage of the function $C^A_S(t, \omega)$ is that it is much simpler in calculations, as $L_A$ does not contain the details of the dynamics. It is worth noting that $C^A_S(t, \omega)$ is defined with the help of the constant $v = 2 f^2 C_S$ which is assumed to be finite even if the coupling constants $g$ and $w$ are very small i.e. also in the so-called “weak-coupling” limit (2.13).

The “adiabatic” formula (4.6) shows a close analogy to the critical sound attenuation in two quite different systems belonging to different universality classes and in the case of $^4$He even the order parameter dimensionality is different, $n = 2$ and $\alpha \simeq 0$. It may suggest that as concerns the adiabatic sound propagation this kind of singularity may be quite common in the critical systems where the order parameter is coupled to the energy and sound mode by two different coupling constants. The exception is the gas-liquid system but there, as was noted by Pankert and Dohm [23], the order-parameter itself is proportional to the energy and there is only one static coupling between the order-parameter and the sound variable.

C. Regime II

Now let us consider a regime which is more appropriate for description of the common lattice consisting mostly of phonons. The new regime, opposite to regime I, is defined by the inequality: $m \ll \tilde{\omega}$ ($\omega \gg \gamma + \lambda + \kappa^2 / C_L$). Now the function $b(\omega) \to m + \tilde{\omega}^2 + i \tilde{\omega} (1 - m)$, and $m \ll \tilde{\omega}^2$ then $b(\omega) \simeq 1 + \tilde{\omega}$. 
frequency contributions, the Murata-Iro-Schwabl and Kawasaki term. Note only, the different amplitude, proportional slightly singular and we have a kind of competition, with frequency-dependent weights, between two kinds of low-
systems stay at equilibrium with each other. For very high $\omega$, term proportional to $e^{\frac{vC_S(t,\omega)}{2}}$.

Thus

$$\tilde{\omega} \approx \omega$$

and

$$\delta T_{\omega} \approx \tilde{\omega}$$

when they were at equilibrium with each other ($\delta T = \tilde{\omega}$). 1

The total volume, in both subsystems. The total temperature difference $\delta E$ is the fluctuation only contribution to $\Delta C_S(t,\omega) = C_S(t,\omega) - C_S$ is the fluctuation only contribution to $C_S(t,\omega)$.

1. Low-frequency and high-frequency regions

It is seen from Eq. (4.14) that as long as $|C_L(\omega)| \gg |C_S(t,\omega)|$ i.e. when $\tilde{\omega}$ is not too high the denominator is only slightly singular and we have a kind of competition, with frequency-dependent weights, between two kinds of low-frequency contributions, the Murata-Iro-Schwabl and Kawasaki term. Note only, the different amplitude, proportional to $g^2$ in the limit $\tilde{\omega} = 0$, instead of $g^2$ in Eqs. (4.3) and (4.4). In this low-frequency region again the spin and lattice systems stay at equilibrium with each other. For very high $\tilde{\omega}$:

$$\omega \approx \tilde{\omega}$$

in Eqs. (4.3) and (4.4). In this low-frequency region again the spin and lattice systems stay at equilibrium with each other. For very high $\tilde{\omega}$: $|C_L(\omega)| \approx \tilde{\omega} \ll C_S$, which is equivalent to the condition $\tilde{\omega} \gg 1$ the specific-heat $C_S(t,\omega)$ begins to outweigh in the denominator of Eq.(4.11) and the “adiabatic” term proportional to

$$g^2 \Delta C_S(t,\omega) = \text{const} - g^2 C_S(t,\omega)$$

dominates the acoustic self-energy, because the spin and lattice subsystems are no longer in equilibrium.

2. Relevant frequency-dependent specific-heats

Introducing $C_A(t,\omega) = \frac{C_L(\omega)C_S(t,\omega)}{C_L(\omega) + C_S(t,\omega)}$ and $C_A(t,\omega) \equiv C_L(\omega) + C_A(t,\omega)$, Eq.(4.11) can also be written as

$$\frac{vC_S(S(k,\omega))}{k^2} = 2g^2(t,\omega)C_A(t,\omega) + 2g^2C_S(t,\omega) + [w^2v(1-a)^2C_S - 2g^2]C_S(t,\omega),$$

where $C_S(t,\omega) = \frac{C_L}{1 - \frac{1}{1 - \frac{1}{1}}}$ is the free, frequency-dependent, lattice specific-heat, $\tilde{g} = g - w(f(1-a)C_S$ and $\Delta C_S(t,\omega) = C_S(t,\omega) - C_S$ is the fluctuation only contribution to $C_S(t,\omega)$.

$C_S(t,\omega)$ is a kind of specific heat which by analogy with the static, $\omega = 0$, case can be given the following interpretation. The energy contribution to the functional $H$, describing the statics, can be rewritten as

$$\frac{\epsilon_S^2}{C_S} + \frac{\epsilon_L^2}{C_L} = \frac{\epsilon_A^2}{C_A} + \frac{\epsilon_A^2}{C_A}$$

where $1/C_+ = 1/C_S + 1/C_L$ and $\epsilon_A = C_A(\frac{\epsilon_S}{C_S} - \frac{\epsilon_L}{C_L})$. If the lattice and the spin systems are at equilibrium with each other, then the local fluctuations of temperature $\delta T_S = \frac{\delta T_S}{C_S}$ and $\delta T_L = \frac{\delta T_L}{C_L}$ are equal, and consequently $\Delta = \epsilon_A = 0$. Thus $\epsilon_A$ can be interpreted as excess energy in one system (and deficiency in the other) in comparison with the case when they were at equilibrium with each other ($\delta T_S = \delta T_L$), so if we shift the energy $\delta E_S$ from the spin system to the lattice system ($\delta E_L = -\delta E_S$) it results in the shifts of temperatures $\delta T_S = \frac{\delta E_S}{C_SV}$ and $\delta T_L = \frac{\delta E_L}{C_LV} = -\frac{\delta E_S}{C_LV}$, where $V$ is the total volume, in both subsystems. The total temperature difference

$$T_S - T_L = \delta T_S - \delta T_L = \frac{\delta E_S}{C_SV} + \frac{\delta E_S}{C_LV} = \frac{\delta E_S}{C_V}$$

is proportional to the energy transported from one system to the other and $C_V = \frac{\delta E_S}{\delta T_L}$ can be understood as the net heat capacity for the process of differentiation (or equalization) of temperatures between both subsystems, whereas the total specific heat $C_A$ characterizes the thermal properties of the system as the whole (after equilibrization).

Coming back to Eq.(4.14) note that it can also be rewritten as

$$\frac{vC_S(S(k,\omega))}{k^2} = 2g^2(t,\omega)C_A(t,\omega) + 2\frac{g^2 - \tilde{g}^2}{C_S(t,\omega) + w^2v(1-a)^2C_S(t,\omega)}$$

(4.14)
where $\tilde{C}^A(t, \omega) = \frac{(C_{L0}(\omega) + C_S)\Delta C^A_S(t, \omega)}{C_{L0}(\omega) + C_S + \Delta C^A_S(t, \omega)}$ may be looked upon again as the net specific-heat for the process of equilibration for two sub-systems with the only difference that this time the background term $C_S$ has been “moved” from the spin to the lattice sub-system. Now the lattice degrees of freedom plus short-wavelength spin fluctuations consist together the one (noncritical) sub-system and the long-wavelength (critical) spin fluctuations the other. In Eqs. (4.12) and (4.14) the second terms vanish for $g = 0$ and $w = 0$, respectively. We shall call them the mixing terms because in the case when one of the coupling constants $g$ or $wf(1-a)C_S$ dominates $g$, the acoustic self-energy is composed of the dominant term proportional to the corresponding specific-heat $\tilde{C}^A(t, \omega)$ or $C^A_S(t, \omega)$ and a small correction resulting from a slightly different ways the constants $g$ and $w$ couple the sound to the spin system [23]. The former one couples the sound mode only to the part of the spin system, the long wavelength critical spin fluctuations, in contrast to the latter coupling which couples effectively the whole spin system (including the noncritical background) to the sound. Looking at $\tilde{C}^A(t, \omega)$ it is easy to see that for $C_L \gg C_S$ it contains both low-frequency terms: the Murata-Iro-Schwabl and Kawasaki terms, as well as the “adiabatic” limit. Namely, the first term in

$$\text{Im}\tilde{C}^A(t, \omega) = \frac{|\Delta C^A_S(t, \omega)|^2 \text{Im}\tilde{C}_{LO}(\omega) + |C_{LO}(\omega) + C_S|^2 \text{Im}\Delta C^A_S(t, \omega)}{|C_{LO}(\omega) + C^A_S(t, \omega)|^2}$$

(4.15)

can be identified as the Kawasaki term for $\tilde{\omega} \ll 1$ ($C_{LO}(\omega) \gg C^A_S(t, \omega)$) and the second as the Murata-Iro-Schwabl one. The crossover temperature $t_{\text{cross}} = \left(\frac{\alpha^2}{4K_L^2\rho v^2\epsilon^2}\right)^\frac{1}{\nu}$ depends mostly on the ratio of bare relaxation frequencies $\omega_S/\Gamma$. In the opposite case $\tilde{\omega} \gg 1$, $C_{LO}(\omega) \ll C^A_S(t, \omega)$, and $\text{Im}\tilde{C}^A_S(t, \omega) \simeq \frac{C^A_S |\Delta C^A_S(t, \omega)|^2}{|C^A_S(t, \omega)|^2}$ i.e. the “adiabatic” limit is recovered. The situation described above may be also superimposed by the effects associated with the crossover in the specific-heat $C^A_S(t, \omega)$, because the expression for $C^A_S(t, \omega) = C_S(1 + v/\omega)\Phi)$ may not display the required $t^{-\alpha}$ singularity in the physical region of interest (if $v$ is very small). Then, for example, the measured effective sound attenuation exponent in the Kawasaki region will be much smaller than $2\gamma$, sometimes close to zero.

In closing we should mention that we have not discussed yet the third term in the numerator of Eq. (4.2) as well as its analogue in Eq.(4.11). It gives the weakest singularity $\propto t^{-\alpha}$ which is usually smaller then the first two terms. However, as it is proportional, in the $\omega = 0$ limit, to $w/fC_S \hat{g}$ i.e. to the first power of $\hat{g}$, whereas $W_1$ and $W_2$ are proportional to the second power, then the $t^{-\alpha}$ term may dominate the ultrasonic attenuation in the degenerated case of $\hat{g} \to 0$.

V. SUMMARY

We have performed a detailed analysis of the critical sound attenuation in magnets. The relaxation of spin-energy to the lattice has been fully taken into account in the prediction of the temperature and frequency dependencies of the acoustic self-energy. The important point here is that the acoustic self-energy is reducible with respect to the energy propagators. We were able to express $\Sigma(k, \omega)$ in terms of the four-spin response function $\langle S^a_{k,-\omega}S^a_{k,\omega} \rangle^L_A$ of the idealized, phonon- and energy-free model A. The last quantity can be relatively easy calculated by the renormalization group method. In the low-frequency region, two kinds of critical behavior described by Eqs. (4.3) and (4.4) compete with each other and the relative weights are determined by the coupling constants $g$ and $w$ as well as the frequencies $\omega, \omega_S, \omega_L$ and the other parameters. The high-frequency or “adiabatic” limit is determined completely only by one coupling constant $g$ - describing the interaction of sound mode with two spin fluctuations. The dynamic scaling functions for these three regimes are calculated within the one-loop approximation.

Another point of interest is the possibility of expressing the acoustic self-energy by a suitable frequency-dependent specific-heat. We have shown that in the case of one coupling $g$ or $wfC_S(1-a)$ being much stronger then the other, the acoustic self-energy can be very well approximated by the specific-heat $C^A(t, \omega)$ or $\tilde{C}^A_S(t, \omega)$, the net specific-heats for the process of equilibration between the lattice and spin sub-systems or in the latter case between the noncritical background composed of the lattice degrees of freedom plus the noncritical short-wavelength spin fluctuations and the critical subsystem composed of the critical long-wavelength spin fluctuations. The difference in definitions of $C^A(t, \omega)$ and $\tilde{C}^A_S(t, \omega)$ comes from a slightly different ways the couplings $g$ and $wfC_S(1-a)$ couple the acoustic phonon to the fluctuations of the order parameter. The virtue of Eqs. (4.12) and (4.14) is that (depending on the relative strength of the coupling constants) in some regions of parameters the acoustic self-energy can be expressed entirely by the frequency dependent specific-heat $\tilde{C}^A_S(t, \omega)$ (or $\tilde{C}^A_S(t, \omega)$), governing the equalization of temperatures between the lattice and spin systems. $\tilde{C}^A_S(t, \omega)$ shows its relevance in the ultrasonic propagation in magnetic systems due to the fact that depending on the frequency, the reduced temperature and the ratio $\omega_S/\Gamma$ it can reveal the most important types of singularities in the critical sound attenuation.
A similar analysis is also possible for the magnets with the order parameter dimensionality greater than one. For example, the Eq. (4.12) is still valid there. However, the specific-heat exponent $\alpha$ is negative in X-Y and Heisenberg systems, therefore, the scaling behavior of the Kawasaki as well as the “adiabatic” limit will change.

[1] B. Lüthi, in Dynamical Properties of Solids, edited by G. K. Horton and A. A. Marudin, ( North-Holland, Amsterdam 1980), Vol. 3, p. 245.
[2] K. Kawasaki, in Phase Transitions and Critical Phenomena, edited by C. Domb and M. S. Green, (Academic Press, New York 1976), Vol.5a, p. 165.
[3] J. Pankert and V. Dohm, Phys. Rev. B 40
[4] R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. Lett. 44, 1
[5] K. Kawasaki, Intern. J. Magnetism
[6] The mixing term in Eq.(4.12) participates also in forming the correct “adiabatic” limit which always is proportional to
[7] D. M. Kroll and J. M. Ruhland, Phys. Rev. A 23
[8] R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. A 31, 1
[9] R. Folk and G. Moser, Phys. Rev. E 58, 6246 (1998).
[10] B. I. Halperin, P. C. Hohenberg and S. Ma, Phys. Rev. B 13, 279 (1990); Phys. Rev. B 44, 5296 (1991).
[11] H. K. Janssen, Z. Phys. B 23, 377 (1976).
[12] A. Pawlak, Eur. Phys. J. B 4, 179 (1998).
[13] B. I. Halperin, P. C. Hohenberg and S. Ma, Phys. Rev. B 10, 139 (1974).
[14] A. Pawlak, Acta Phys. Pol. A 92, 449 (1997).
[15] R. Dengler and F. Schwabl, Z. Phys. B 69, 327 (1987).
[16] A. Pawlak, Z. Phys. B 79, 279 (1990); Phys. Rev. B 44, 5296 (1991).
[17] B. Drossel and F. Schwabl, Z. Phys. B 91, 93 (1993).
[18] D. M. Kroll and J. M. Ruhland, Phys. Rev. A 23, 371 (1981).
[19] R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. A 31, 1788 (1985).
[20] R. Folk and G. Moser, Phys. Rev. E 58, 6246 (1998).
[21] R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. Lett. 44, 403 (1980); Phys. Rev. B 23, 2434 (1981).
[22] J. Pankert and V. Dohm, Phys. Rev. B 40, 10842, 10856 (1989).
[23] K. Kawasaki, Intern. J. Magnetism 1, 171 (1971).
[24] The mixing term in Eq.(4.12) participates also in forming the correct “adiabatic” limit which always is proportional to $g^2$.

TABLE I. Propagators of the model

| Propagator                  | Analytic expression                                                                 | Free Lagrangian average |
|-----------------------------|-------------------------------------------------------------------------------------|-------------------------|
| Spin response-              | $H_0(k, \omega) = -\omega + \frac{1}{2} k^2$                                       | $(S_{k,\omega} S_{-k,-\omega})_0$ |
| Spin correlation-           | $K_{S0}(k, \omega) = \frac{1}{2\Gamma_{s}(k)}$                                     | $(S_{k,\omega} S_{-k,-\omega})_0$ |
| Phonon response-            | $G_0(k, \omega) = \frac{1}{c^2 k^2 + \omega^2}$                                    | $(Q_{k,\omega} Q_{-k,-\omega})_0$ |
| Phonon correlation-         | $K_{Q0}(k, \omega) = \frac{1}{c^2 k^2 + \omega^2}$                                 | $(Q_{k,\omega} Q_{-k,-\omega})_0$ |
| Spin-energy response-       | $D_{S0}(k, \omega) = \frac{1}{2\Gamma_{s}(k)c^2}$                                 | $(\epsilon_{k,\omega} e_{S_{k,-\omega}}^S)_0$ |
| Spin-energy correlation-    | $K_{s,0}(k, \omega) = \frac{1}{2\Gamma_{s}(k)c^2}$                               | $(\epsilon_{k,\omega} e_{S_{k,-\omega}}^S)_0$ |
| Lattice-energy response-    | $D_{L0}(k, \omega) = \frac{1}{\omega^2 + 2\Gamma_{L}(k)c^2}$                     | $(\epsilon_{k,\omega} e_{L_{k,-\omega}}^L)_0$ |
| Lattice-energy correlation- | $K_{s,0}(k, \omega) = \frac{1}{2\Gamma_{s}(k)c^2}$                               | $(\epsilon_{k,\omega} e_{L_{k,-\omega}}^L)_0$ |