Current Renormalisation in NRQCD for Semi-leptonic $B \to D$ Decays

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We present a calculation of the renormalisation constants for the temporal vector current, $Z_{V_0}$ and spatial axial current, $Z_{A_0}$, to $O(M^4)$ for $B \to D$ transitions using the $O(M^4)$ NRQCD action for both $b$ and $c$ quarks evaluated for a large range of mass parameters. Considerations for the renormalisation of the spatial vector current and the temporal axial current are discussed and initial results for a mixed lattice current are presented for the spatial vector current.

1. Introduction

The semi-leptonic $B \to D, D^*$ decay is phenomenologically interesting since its Feynman amplitude involves the poorly known CKM matrix element $V_{cb}$

$$M(B \to Xl\bar{\nu}) = \frac{i}{2} V_{cb} \bar{u} \gamma^\mu (1 - \gamma_5) v_L H_\mu$$

(1)

$$H_\mu = (X(p')|\bar{c} \gamma_\mu (1 - \gamma_5) b |B(p))$$

$X = D, D^*$

where the hadronic tensor $H_\mu$, and its Lorentz decomposition into form factors is calculable using lattice QCD.

We perform the matching of the Lattice to continuum $M_{\text{MS}}$ currents to 1 loop using the on-shell scheme in Feynman gauge with vanishing external spatial momenta. In the on-shell scheme the wavefunction renormalisation and vertex corrections individually contain (cancelling) infra-red divergences which we control using a fictitious gluon mass, $\lambda$. We use dimensional regularisation to control the ultra-violet behaviour in the continuum calculation.

2. Continuum Calculation

Expanding the 1-loop correction to the currents to $O(M^4)$, and writing $\epsilon = \frac{M^2}{M_{\text{MS}}^2}$, we obtain

$$\delta V_0/\frac{\alpha_s}{3\pi} = \gamma_0 \left[ 3 \frac{\epsilon + 1}{\epsilon - 1} \log \epsilon - 4 \right] - 2 + O(1/M^2)$$

(3)

$$\delta V_k/\frac{\alpha_s}{3\pi} = \gamma_k \left[ 3 \frac{\epsilon + 1}{\epsilon - 1} \log \epsilon - 4 \right] + \frac{\epsilon + 1}{4 M_{\text{MS}}} \left[ \frac{2 \log \epsilon}{\epsilon + 1} - \frac{2 \log(\epsilon^2 - 1)}{(\epsilon - 1)^2} \right] + O(1/M^2)$$

(4)

$$\delta A_k/\frac{\alpha_s}{3\pi} = \gamma_k \gamma_5 \left[ 3 \frac{\epsilon + 1}{\epsilon - 1} \log \epsilon - 8 \right] + O(1/M^2)$$

(5)

$$\delta A_0/\frac{\alpha_s}{3\pi} = \gamma_0 \gamma_5 \left[ 3 \frac{\epsilon + 1}{\epsilon - 1} \log \epsilon - 8 \right] + \gamma_5 \left[ 6 \frac{\epsilon + 1}{\epsilon - 1} \right] + O(1/M^2)$$

(6)

Here the currents $V_0$ and $A_k$ can be renormalised to $O(M^4)$ in the usual manner, however the currents $V_k$ and $A_0$ will involve operator mixing at this order.

3. 1-loop Lattice Correction

We use the $O(M^4)$ lattice NRQCD action [1] for both $b$ and $c$ quarks,

$$aL_{\text{NRQCD}} = \psi^\dagger (x) \psi (x)$$

$$- \psi^\dagger (x + \hat{t}) \left( 1 - \frac{a \delta H}{2} \right) \left( 1 - \frac{a H_0}{2n_0} \right) \frac{U_j (x)}{n_0}$$

(7)

$$H_0 = \frac{\Delta}{2 M_0}$$

$$\delta H = - C_B \frac{g_\sigma \cdot B}{2 M_0}$$

(8)

for which the Feynman rules may be found in reference [1].

The Foldy-Wouthysen transformation of the continuum current corrections and dimensional arguments for which currents can contribute at $O(M^4)$ suggest we take the bases of operators in Table 1 for the lattice currents.

For those currents that are unmixed, the calculation is similar to [1], and we write the renormalisation constant $Z_X = 1 + \alpha_s Z_X^{[1]} = 1 + \delta Z_X^{\text{lat}} - \delta Z_X^{\text{lat}}$, where the lattice integrals for $\delta Z_X^{\text{lat}}$ were
3.2. V

stronger since the line for rent is in Figure 2. The mass dependence is parameter \( n \) is no correction on the line the continuum current is conserved, so that there is the conserved current of the lattice action, and manner such that \( M \) values of \( \epsilon \) \( Z \) constant the current \( \bar{\psi} \) derivatives of the diagrams with respect to the ex- nal axial currents it is necessary to evaluate the Lattice Current Bases

| \( V_0 \) | \( V_k \) | \( A_0 \) | \( A_k \) |
|----------|----------|----------|----------|
| \( J_{1V}^0 = -\sigma_k \frac{\sigma}{2M_0} \) | \( J_{1V}^k = \sigma_k \frac{\sigma}{2M_k} \) | \( J_{1A}^0 = -\sigma_k \frac{\sigma}{2M_0} \) | \( J_{1A}^k = \sigma_k \frac{\sigma}{2M_k} \) |
| \( J_{2V}^0 = \frac{\sigma}{2M_0} \sigma_k \) | \( J_{2V}^k = \frac{\sigma}{2M_k} \sigma_k \) | \( J_{2A}^0 = \frac{\sigma}{2M_0} \sigma_k \) | \( J_{2A}^k = \frac{\sigma}{2M_k} \sigma_k \) |
| \( J_{3V}^0 = -\frac{\sigma}{2M_0} \omega \) | \( J_{3V}^k = -\frac{\sigma}{2M_k} \omega \) | \( J_{3A}^0 = \frac{\sigma}{2M_0} \omega \) | \( J_{3A}^k = \frac{\sigma}{2M_k} \omega \) |
| \( J_{4V}^0 = \frac{\sigma}{2M_0} \) | \( J_{4V}^k = \frac{\sigma}{2M_k} \) | \( J_{4A}^0 = \frac{\sigma}{2M_0} \) | \( J_{4A}^k = \frac{\sigma}{2M_k} \) |

computed numerically using VEGAS \( \int \), performing the temporal loop momentum integra- tion analytically for those contributions that were infra-red divergent.

In the case of the spatial vector and temporal axial currents it is necessary to evaluate the derivatives of the diagrams with respect to the external momentum to match with the continuum. This is done by numerically evaluating the integral of the analytically taken derivative, resolving the pieces of the Pauli structure by taking different derivative directions.

In the continuum, both the vertex correction to the current \( \bar{\psi} \Gamma b \), and the wavefunction renormalisation contained a logarithmic divergence proportional to \( \frac{1}{\alpha} \log \frac{M}{M_0} \Gamma \) where the sign is positive in the vertex correction. The same infra-red diver- gences were found in the lattice vertex correction and wavefunction renormalisation so that as one would expect the infra-red behaviour of the theory is identical to that in the continuum.

3.1. \( V_0 \) and \( A_k \) Currents

The 1-loop contribution to the renormalisation constant \( V_{V_0} \) is plotted in Figure 1 for various values of \( \epsilon = \frac{M}{M_0} \) and \( M_0 \). Here the stabilisation parameter \( n \) has been chosen in a mass dependent manner such that \( M_0 \times n \geq 3 \). The lattice current is the conserved current of the lattice action, and the continuum current is conserved, so that there is no correction on the line \( \epsilon = 1 \).

The 1-loop correction to the spatial axial current is in Figure 2. The mass dependence is stronger since the line for \( \epsilon = 1 \) is not protected.

3.2. \( V_k \) Current

We define the 1-loop lattice mixing matrix \( Z_{ij} \),

\[
\langle c(p') | J_i | b(p) \rangle = \sum_j Z_{ij} \Omega_{ij}^E \tag{9}
\]

![Spatial Vector Current](image)

Figure 1. \( z_{V_0}^{[1]} \) as a function of \( \frac{1}{M} \) and \( \frac{M}{M_0} \). The lines correspond to \( \frac{M}{M_0} = 0.1, \ldots, 1.0 \) in increments of 0.1, with the \( \frac{M}{M_0} = 0.1 \) curve topmost.

where the \( \Omega_{ij}^E \) are the continuum analogs of the lattice operators given in Table 1. We define the contribution to the mixing matrix arising from the lattice vertex correction, \( \xi_{ij} \), via

\[
Z_{ij} = \delta_{ij} + \alpha_s \left[ \frac{1}{2} \left( \tilde{Z}_{\psi \psi} + \tilde{Z}_{\bar{\psi} \bar{\psi}} \right) \delta_{ij} + \tilde{\xi}_{ij} \right] \tag{10}
\]

We then invert the mixing matrix and match to the continuum. Here the coefficients \( B_i \) may be inferred from equation 11 and the \( Z_m \) factors arise from matching the bare mass in lattice currents to the pole mass in the tree level continuum current.

\[
V_{k_{\overline{MS}}} = 1 + \alpha_s \left[ B_1 - \frac{1}{2} \left( \tilde{Z}_{\psi \bar{\psi}} + \tilde{Z}_{\bar{\psi} \psi} \right) - \xi_{11} + \tilde{\xi}_{21} - \tilde{\xi}_{31} \right] J_1 + \alpha_s \left[ B_2 - \frac{1}{2} \left( \tilde{Z}_{\psi \bar{\psi}} + \tilde{Z}_{\bar{\psi} \psi} \right) - \xi_{12} + \tilde{\xi}_{22} - \tilde{\xi}_{32} \right] J_2 + \alpha_s \left[ B_3 - \tilde{\xi}_{13} - \tilde{\xi}_{23} \right] J_3 + \alpha_s \left[ B_4 - \tilde{\xi}_{14} - \tilde{\xi}_{24} \right] J_4 \tag{11}
\]

The diagonal elements \( \xi_{jj} \) contain infra-red divergences which cancel with those in the lattice wavefunction renormalisations. Those currents appearing at tree level also carry tadpole im- provement counter terms. We therefore rewrite
Figure 2. $Z_A^{[1]}$ as a function of $\frac{1}{M_b}$ and $\frac{M_c}{M_b}$. The lines correspond to $\frac{M_c}{M_b} = 0.1, \ldots, 1.0$ in increments of 0.1, with the $\frac{M_c}{M_b} = 0.1$ curve topmost.

Equation 11 in terms of IR divergence and tadpole counter term free quantities, as follows:

$$V^{\text{MS}}_k = \begin{pmatrix} 1 + \alpha_s \left( \frac{1}{3} \left( Z_{\chi}^{\text{lat}} + Z_{\psi}^{\text{lat}} \right) \right) \end{pmatrix} J_1 + \alpha_s \left( \frac{1}{3} \left( Z_{\chi}^{\text{lat}} - Z_{\psi}^{\text{lat}} \right) \right) J_2 + \alpha_s (B_3 - \xi_{13} - \xi_{23}) J_3 + \alpha_s (B_4 - \xi_{14} - \xi_{24}) J_4$$

(12)

Preliminary calculations of the mixed lattice have been performed on a few mass values for degenerate quarks. The degenerate case may in fact be of use in certain lattice simulations to obtain the Isgur-Wise function, however the results should really be considered illustrative of the method, and the calculation over a similar parameter regime to our previous results for the other currents will be performed. In this case the coefficients of $J_3$ and $J_4$, and of $J_3$ and $J_4$ are identical, and we only present the coefficients for $J_1$ and $J_3$ in Table 2. We evaluate the tadpole improvement counter terms using $u_0^{[1]}(\text{plaq}) = \frac{\pi}{4}$, and take values for $Z_{\chi}$ from [2]. Also, for the degenerate case $B_1 = B_2 = -\frac{1}{3}\xi_{11}$ and $B_3 = B_4 = -\frac{1}{3}\xi_{11}$, so that

| $M_{Q, n}$ | $C_1$ | $C_2$ |
|------------|-------|-------|
| 2.2        | -0.776(4) | 0.145(1) |
| 4.2        | -0.571(4) | 0.098(1) |
| 10.1       | -0.247(4) | -0.010(1) |

Table 2
One Loop Correction Coefficients

Table 3
One Loop Correction Coefficients

when the mixed current is written as

$$V^{\text{MS}}_k = (1 + \alpha_s C_1) (J_1 + J_2) + \alpha_s C_2 (J_3 + J_4),$$

(13)

we obtain the coefficients in Table 3.

4. Future Direction

We are nearing completion of the calculation of the mixed current for $V_k$, while there is a smaller basis of operators for $A_0$, so that the corresponding calculation should be somewhat easier. Thereafter we shall perform a similar calculation for the $b \to c$ transition using the NRQCD action for the $b$ quark, and the O(a) improved Wilson action for the $c$ quark. The extension of the calculation to $O(\frac{1}{M^2})$ would be of interest.

5. Acknowledgements

We would like to thank the Physics Department and the ITP, UCSB for their hospitality while this work was carried out. PB is funded by PPARC grant PP/CBA/62, CD thanks the Fulbright commission and the Leverhulme trust.

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