On the Spontaneous Breaking of $U(N)$ symmetry in invariant Matrix Models

Fabio Franchini

SISSA and I.N.F.N, Sezione di Trieste, Via Bonomea 265, 34136, Trieste, Italy and
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Matrix Models have a strong history of success in describing a variety of situations, from nuclei spectra to conduction in mesoscopic systems, from strongly interacting systems to various aspects of mathematical physics. Traditionally, the requirement of base invariance has lead to a factorization of the eigenvalue and eigenvector distribution and, in turn, to the conclusion that invariant models describe extended systems. Moreover, Wigner-Dyson statistics for the eigenvalues is a hallmark of eigenvector delocalization. We show that deviations of the eigenvalue statistics from the Wigner-Dyson universality (in the form of a gap) reflects itself on the eigenvector distribution and that the phase transition observed when the eigenvalue density become disconnected corresponds to a breaking of the $U(N)$ symmetry to a smaller one. This spontaneous symmetry breaking means that the system looses ergodicity, with implications on localization problems, as well as for fundamental theories.
Much of our understanding of Physics in the last century has been built on the progressive appreciation of the role of symmetries, from the classification of different phases of matter, to the description of the fundamental (and effective) forces of nature. A special role in this journey has been played by the realization that a system can spontaneously break the symmetry that characterizes it. The prototypical example of this phenomenon is ferromagnetism: although the system is invariant under rotations, at sufficiently low temperatures the majority of the microscopic magnetic moments align along a particular direction. In gauge theories, the spontaneous breaking of a symmetry (SSB) is accompanied by the by Higgs mechanism, which gives masses to the mediator of the broken symmetries.

SSB mechanisms have been studied in virtually all branches of physics, but, so far, not in a matrix model formulation. Matrix models are well suited to describe systems with many interacting degrees of freedom. They were originally introduced by Wigner in the fifties to describe the spectra of heavy nuclei, under the hypothesis that the Hamiltonian governing the interactions could be well approximated by a randomly generated matrix $\mathcal{H}$. The success of this hypothesis has led over the years to the application of matrix models to virtually every branch of physics and beyond: from nuclear physics to string theory, from 2-D quantum gravity to condensed matter physics, from statistical physics to econo-physics, from neuroscience to chaos theory, from number theory to integrable systems, and so on on [2–5]. In the different contexts the matrices can correspond to very different physical quantities: in some cases the matrices are taken as dynamical fields with many internal degrees of freedom; in some other instances they are interpreted as links between points in a network (or a discretized manifold); in other situations they are matrix representation of the operators of a many-body quantum theory; and so on.

Despite the differences in physical interpretation of these models, they all share the same mathematical structure, which has offered the opportunity for cross-fertilization between different areas of physics and mathematics over the years. The central object defining the model is the probability distribution function (PDF) $P(M)$, which provides the weight for the different elements of the $N \times N$ matrix $M$. In the following, we will assume $M$ to be hermitian, although other symmetry classes can be considered as well. In most application one assumes basis invariance, so that the PDF can be written as

$$P(M) \propto e^{-N \text{Tr} V(M)},$$

and is specified by the potential $V(x)$. One important reason for working with this type of PDF is that invariant models enjoy powerful mathematical approaches which, over the years, have granted important and general results.

A key property of invariant models is the possibility of factorizing separately the eigenvalues from the rotational degrees of freedom. That is, writing $M = U^\dagger A U$, where $A$ is diagonal and $U$ is the unitary matrix encoding the eigenvectors of $M$, the PDF does not depend on $U$. This independence has been interpreted to imply that $U$ is uniformly distributed over an $N$-dimensional sphere. Since each eigenvector can extend over the whole Hilbert space of the system, invariant matrix models are supposed to describe extended (i.e. conducting) phases. For a matrix $U$ belonging to the unitary $U(N)$ group, the distribution of its entries is the Porter-Thomas distribution [6]

$$\mathcal{P} \left( |U_{ij}|^2 \right) = \chi \exp \left[ -\chi |U_{ij}|^2 \right],$$

with $\chi = N$, which means that the mean amplitude of each eigenvector entry is $1/N$ and it is equal to its standard deviation.

Very much like for the ferromagnet, we want to show that even in invariant models some rotational degrees of freedom of the eigenvectors can be frozen by the eigenvalue distribution. In particular, a gap between two eigenvalue sets prevents the mixing between the corresponding eigenvectors. This coupling between the eigenvector and eigenvalue distributions represent a novel SSB mechanism, for which the $U(N)$ symmetry is spontaneously broken into the product of smaller unitary groups $U(n)$s. The critical and universal behavior at the threshold when a gap in the eigenvalue density is about to open has been observed in the literature [7, 8], but now our work characterizes it in terms of symmetries being broken. This realization can unlock the possibility of study new non-trivial properties of invariant matrix models and to apply them to new phenomena, such as partially localized systems. Moreover, matrix models arise naturally in the description of strongly interacting theories, such as string theories and holographic models. Thus, the discovery of their SSB mechanism can be applied to the description of nature at the fundamental level.

I. GENERAL CONSIDERATIONS

The SSB mechanism in matrix models is easily understood by noticing that the de Haar measure of integration for matrices is flat in matrix elements space, but curved in the eigenvalue/eigenvectors representation, with a curvature
induced just by the eigenvalues \[2\]. The line element is given by

\[ ds^2 = \text{tr} \left( dM \right)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2 \sum_{j>l}^{N} (\lambda_j - \lambda_l)^2 |dA_{jl}|^2 . \]

(3)

where \( dA \equiv U^t dU \). Keeping the eigenvalues \( \lambda_j \) fixed, the angular degree of freedom \( dA_{jl} \) lives on a sphere of radius \( r_{jl} = |\lambda_j - \lambda_l| \). If the two eigenvalues are very distant \( (r_{jl} \gg 1) \), even a small angular change \( dA_{jl} \) can move to a very distant point, producing a large \( ds \).

This geometric argument is at the heart of the SSB mechanism, as can be understood by introducing a fictitious dynamics in the model, in the form of the Brownian motion originally proposed by Dyson \[9\]. In this picture, one introduces a random noise and let the eigenvalue and eigenvectors evolve to the equilibrium distribution. The stochastic differential system for this Brownian motion is \[10\]

\[ d\lambda_j = -\frac{dV(\lambda_j)}{d\lambda_j}dt + \frac{\beta}{2N} \sum_{l \neq j} \frac{dt}{\lambda_j - \lambda_l} + \frac{1}{\sqrt{N}} dB_j(t) , \]

\[ d\vec{U}_j(t) = -\frac{1}{2N} \sum_{l \neq j} \frac{dt}{(\lambda_j - \lambda_l)^2} \vec{U}_j + \frac{1}{\sqrt{N}} \sum_{l \neq j} \frac{dW_{jl}(t)}{\lambda_j - \lambda_l} \vec{U}_l , \]

(4)

where \( dB_j \) and \( dW_{jl} = dW_{lj}^* \) are delta-correlated, independent, stochastic sources, and \( \vec{U}_j \) denotes the \( j \)-th column of \( U \). We notice that, while the eigenvalues dynamics is self-consistent, the motion of the eigenvectors depends on eigenvalue distribution through their distances. This effect is the source of the coupling between the eigenvectors and the eigenvalues which is not apparent in the equilibrium measure. We remind that eigenvalues are spread on two characteristic scales: while neighboring ones are separated by level repulsion at a distance of the order of \( 1/N \), the whole distribution spans a length of order \( 1 \). Thus, we see that if two sets of eigenvalues are separated by a gap of the order of unity, eq. \[11\] indicates that the evolution of the eigenvectors toward the subspace spanned by eigenvectors belonging to the distant eigenvalues is suppressed. In this case, the eigenvectors cannot spread ergodically over the \( N \)-sphere, indicating a spontaneous breaking of \( U(N) \) symmetry.

II. SSB IN A DOUBLE WELL MODEL

Equipped with these general geometric considerations, we are now going to provide a more concrete construction. In doing so, we will consider the specific example of a double well matrix model, that is, a potential in \[11\] of the form

\[ V_{2W}(x) = \frac{1}{4} x^4 - \frac{t}{2} x^2 , \]

(5)

which is known to have a disjoint support for the eigenvalues for \( t > 2 \) \[11, 12\]. That is, for sufficiently deep and separated wells, half of the eigenvalues accumulate around one minimum and the other half around the other minimum\[ ? \]. We want to show that the gap separating the negative eigenvalues from the positive ones induces a breaking of \( U(N) \) symmetry into \( U(N/2) \times U(N/2) \).

One of the most direct way to detect a SSB is to introduce an explicit symmetry breaking term and to remove it only after the thermodynamic limit. The most natural choice for this term in matrix models would be \( tr \left( [\tilde{M}, S] \right)^2 \), since it would favor the alignment of the eigenvectors of \( M \) along those of the Hermitian matrix \( S \). Unfortunately, this choice turns out too complicated to handle.

Thus, we will consider the following perturbation of the PDF

\[ P(M) \propto e^{-N \text{Tr} \left[ V_{2W}(M) + J | \hat{A} \hat{T} - M \hat{S} \right] } , \]

(6)

where the symmetry breaking term serves the same purpose of favoring the alignment of the eigenvectors. Here, \( J \) acts as a source term and \( T \) is a diagonal matrix with the eigenvalues of \( S = V^t T V \). The absolute value in \[9\] can be removed by ordering the entries in the diagonal matrices \( A \) and \( T \) in increasing order, since this insures the positiveness of the trace. In order to induce the correct symmetry breaking, it is convenient to take \( N/2 \) eigenvalues of \( S \) to be equal to \( t \) and the remaining \( N/2 \) equal to \( -t \) (without lose of generality, we can further set \( t = 1 \)). This choices preserve \( U(N/2) \) rotational freedom for the eigenvectors corresponding to eigenvalues within one well, while it penalizes rotations in the remaining directions\[ ? \].
In the Methods section, we show how to compute the generating function

\[ W(J) = \ln \int dM e^{-N \text{Tr} \left[ V_2W(M) + J |\Lambda - M S| \right]} , \]  

whose derivative with the respect to the coupling \( J \) provides the expectation value of the order parameter

\[ \frac{dW(J)}{dJ} \bigg|_{J=0} = \langle \Lambda - M S \rangle . \]  

This order parameter measures the misalignment between the eigenvectors of \( M \) and \( S \). Note that, contrary to the usual case, with our definition the breaking of the symmetry is signaled by a vanishing order parameter, while a non-vanishing one indicates a symmetry restoration. We find that, in the \( J \to 0 \) limit the order parameter remains finite as long as \( N \) is finite. However, taking the \( J \to 0 \) limit after the thermodynamic limit \( N \to \infty \) yields a vanishing order parameter, indicating that the \( U(N) \) symmetry has been spontaneously broken into \( U(N/2) \times U(N/2) \).

Moreover, the calculation shows that, at finite sizes, corrections to the symmetry broken configuration appear in the form \( \exp \left[ -N J \delta \lambda_{j,l} \right] \), where \( \delta \lambda_{j,l} \equiv \left| \lambda_j^{(1)} - \lambda_l^{(2)} \right| \) is the distance between two eigenvalues in different wells. These
contributions have a natural interpretation in terms of instantons which exchange the two eigenvalues between the wells and progressively restore the broken symmetries. The $J \to 0$ limit renders these instantons “massless” and thus restores the symmetry to the full $U(N)$. In contrast, the $N \to \infty$ limit suppresses these contributions completely and induces the SSB. Note that $\delta \lambda_{j,l}$ is of the order of unity because the eigenvalues belong to different well. If we apply the same symmetry breaking term to a single well case, $\delta \lambda_{j,l}$ would be of the order of $1/N$ and thus these contributions would not disappear in the thermodynamic limit and there would be no SSB.

Having established the existence of a SSB mechanism in matrix models, we are now going to provide a numerical procedure to detect it. In the spirit of [4], we ask how do the eigenvectors respond to a perturbation. In a Gaussian matrix models, they would rotate over the $N$-dimensional sphere and the new eigenvectors would quickly acquire a non-vanishing overlap with virtually all the original, non-perturbed, eigenvectors. In a double well model, instead, the SSB would confine each set of eigenvectors to rotate in the $N/2$-dimensional sphere spanned by the eigenvectors of each well, with only suppressed overspill in the remaining space, vanishing in the thermodynamic limit. Thus, starting from a matrix $M$ which is diagonal. In Fig. 1 we plot the squared absolute value of the entries of the matrix in fig. 1. We see that they both fit a Porther-Thomas distribution like (2), corresponding to eigenvalues belonging to the same well. In Fig. 2 we plot the distributions of the diagonal and off-diagonal block entries of the matrix in fig. 1. We see that they both fit a Porther-Thomas distribution like (2), but with

$$
\chi_D = \frac{N}{2} \quad \text{and} \quad \chi_{OD} = \frac{2N^2}{n}
$$

for the diagonal and off-diagonal elements, respectively. The former indicates that the eigenvectors are (approximately) uniformly distributed over a $N/2$-dimensional sphere, while the latter is compatible with the expectation from a simple perturbation theory analysis. Another important quantity in the context of localization is the overlap between eigenstates, defined as $O_{jl} = \sum_m |\tilde{U}_{mj}|^2 |\tilde{U}_{ml}|^2$. In extended phases, the overlap scales like $1/N$: a faster decay indicates a suppression of the overlap and hence the onset of localization. From the above matrix element distributions, we expect the overlap between eigenstates belonging to the same well (diagonal blocks of $O$) and of different wells (off-diagonal blocks of $O$) to be

$$
\langle |O_{jl}| \rangle_D = \langle |U_{jl}| \rangle_D = \langle |\Delta U_{jl}| \rangle_D = \frac{1}{\chi_D},
$$

$$
\langle |O_{jl}| \rangle_{OD} = 2\langle |U_{jl}| \rangle_{OD} = 2\langle |\Delta U_{jl}| \rangle_{OD} = \frac{2}{\chi_{OD}}.
$$

We check these predictions against the numerical results obtained over several realizations of the applied perturbations in fig. 3, concentrating in particular on the finite-size behavior, finding a remarkable agreement for all the quantities considered.

These results show that the off-diagonal blocks of the unitary matrix are suppressed by a power of $N$ compared to diagonal ones, indicating that in the planar limit (i.e., in the leading order in $N$) the full rotational symmetry is broken into a $U(N/2) \times U(N/2)$.

III. DISCUSSION & CONCLUSIONS

In this work, we have proven, both analytically and through a numerical experiment, that the rotational invariance of matrix models can be spontaneously broken, when the distribution of the eigenvalues has a disconnected support (the so-called “multi-cuts solutions” [14, 15]). Although the mathematical literature has already discussed extensively the critical behavior of the eigenvalue distribution at the point where a gap in the distribution is about to open, our results characterizes it as a phase transition to a phase where the $U(N)$ symmetry has been broken to a smaller one. This consideration, as well as the calculation shown in the method section, establishes once more the strict connection between the eigenvalue and eigenvector statistics.
We discussed extensively the double well case, for which the eigenvalues distribute over two intervals, but all our analysis can be extended straightforwardly to any number of intervals. In the double well case, the eigenvectors get localized to a \( N/2 \)-dimensional sphere, with spillage outside it suppressed and vanishing in the thermodynamic limit. The location of this sphere in the whole Hilbert space is random and uniform, similarly to the magnetization of a ferromagnet. But once this direction is chosen, the system shows a certain rigidity against changing it, which gets completely frozen in the thermodynamic limit.

The problem of eigenvector localization in random matrices has already been discussed, by considering non-invariant PDFs, (most noticeably, the so-called Banded Random Matrices \[17\]), which introduce an explicit basis dependence in the definition of the model. Unfortunately, these models are hardly treatable in an analytical way (with the exception of a few perturbative approaches, valid only in certain regimes \[18\]–\[21\]) and one has to rely mainly on numerics for the analysis. Our approach disclose the full power of the techniques available for invariant matrix models to the study of broken symmetries.

Two important applications that come to mind are the description of the Anderson Metal-Insulator transition (which still lacks a description amenable to analytical treatment) and the study of SSB in string theories models, such as ABJM, which admit a localization limit (that is, a description in terms of matrix models). These models are relevant both as fundamental theoretical models, as well as for their holographic description of strongly interacting field theories in condensed matter systems or QCD. Additionally, our results indicate that the calculation of overlap (and inverse participation ratios) alone are not sufficient to detect a localized phase. This observation can be relevant in the context of many-body localization and it implies that, in addition to the usual quench disorder, one should also consider correlated disorder, generated by adding small perturbations to a given disorder realization, and compare the localization properties of the perturbed eigenstates.

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Appendix A: Methods

1. Methods: Symmetry Breaking Term

To evaluate the effect of (7) we use the Itzykson-Zuber formula for the integration over the unitary group \[22\]

\[
\int dU e^{i Tr A U B U^+} \propto \frac{\det \left[ e^{a_j b_l} \right]}{\Delta(\{a\}) \Delta(\{b\})}, \tag{A1}
\]

where \( \Delta(\{a\}) = \prod_{j<l}(a_j - a_l) \) is the Van der Monde determinant constructed with the eigenvalues \(\{a\}\) of \(A\). For its application to (7) we have \(A = M\) and \(B = NJS\). The L.H.S. of eq. (A1) is singular when two or more eigenvalues of either \(A\) or \(B\) coincide: both the determinant at the numerator and at the denominator vanish, because two or more rows (or columns) become identical. Since both singularities are of the same order, taking the limit where the eigenvalues become progressively close (point-splitting method), one achieves a finite expression. If \(B\) has two sets of \(N/2\) degenerate eigenvalues equal to \(\pm b\) we have

\[
\int dU e^{i Tr A U B U^+} \propto \frac{1}{\Delta(\{a\})} \det \left( \frac{a_j^{-1} e^{-b a_i}}{a_j^{-1} e^{+b a_i}} \right)_{j=1...N/2, l=1...N} \tag{A2}
\]

The determinant at the numerator can be further evaluated as

\[
\det \left( \frac{a_j^{-1} e^{-b a_i}}{a_j^{-1} e^{+b a_i}} \right) = \sum_{\{a\} \cup \{a'\} = \{a\}^c} e^{-b \sum_{j} (a_j - a'_j)} \Delta(\{a\}) \Delta(\{a'\}) \tag{A3}
\]
where the sum is over all the partitions of the eigenvalues of $A$ into two sets of $N/2$ elements. We see that the degeneracies of the eigenvalues of $B$ induces a suppression of the interaction between the $A$ eigenvalues, in the form of a reduced Van der Monde determinant.

Combining these formulas into (7), we see that the effect of the symmetry breaking term is to isolate one such partition (given by $\Lambda_T$), which corresponds to assigning each eigenvalue to a given well, and to suppress the interaction between eigenvalues of different wells. This interaction is progressively restored by the other partitions, which represent configurations in which two eigenvalues have switched wells. However, the latter contributions have a weight in the partition function given by $\exp\left(-b|\alpha_j - \alpha'_j|\right)$ for each pair of eigenvalues switched compared to the leading configuration. These contributions are suppressed for large $b = NJ\tau$ or large eigenvalue distance and disappear in the thermodynamic limit.

2. Methods: Effect of a perturbation

In the numerical experiment, we solve the plasma equation for the eigenvalues, to determine their equilibrium distribution under the effect of the potential (5). Having determined $A$, we obtain a typical matrix $M = U^\dagger A U$ of the double well matrix ensemble by generating a unitary matrix $U$ with the algorithm in [23]. As a perturbation, we generate $\Delta M$, a sparse, Hermitian random Gaussian matrix, with $n \times N$ non-zero elements. We determine the unitary matrix $U'$ which diagonalizes $M + \Delta M$ and study it in the basis where $M$ is diagonal: that is, we are looking at $\tilde{U} = U' U^\dagger$. For Fig. 3, we generated the equilibrium distribution of eigenvalues for $t = 4, 8$ and consider perturbations with $n = 150, 300$. Furthermore, each data point is the result of averaging over several realizations (we took $30$ realizations for $N < 700$, $20$ for $N < 1200$, and $10$ otherwise, noticing that each realization is already quite accurately self-averaging).

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Figure 2. Distribution of the squared absolute value entries of the diagonal (Top) and off-diagonal (Bottom) blocks of the matrix in Fig. 1. In red the numerics, while the blue lines are the Porther-Thomas distribution (2), but with the effective rank $\chi$ given by (1).
Figure 3. Log-log plot of the means and standard deviations of the diagonal and off-diagonal blocks of the eigenvectors amplitudes matrices and of the means of the blocks of the overlap matrices. These results obtained by averaging over several realizations are in remarkable agreement with the analytical expectations (10 11), also plotted as continuous lines.
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