Attractive Kane-Mele-Hubbard model at half filling: phase diagram and Cooperon condensation

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Recently, the attractive Kane-Mele-Hubbard (KMH) model on a honeycomb lattice at half filling has been studied in two papers: PRB 99, 184514 (2019) and PRB 94, 104508 (2016). The authors of the first one presented the phase diagram which interpolates the trivial and non-trivial topological states. However, the next-nearest-neighbor (NNN) hopping term has been neglected, although it is several orders of magnitude stronger than the internal spin-orbit coupling. We use the mean-field approximation to derive the phase diagram of the attractive KMH model with NNN hopping at half filling. The phase diagram without and the phase diagram with NNN hopping are significantly different in the non-trivial topological region.

The possibility to have superconducting instability in the attractive KMH model has been analyzed in the second paper within the T-matrix approximation. The question that naturally arises here is about the contributions due to the bubble diagrams, which are included in the Bethe-Salpeter (BS) equation, but neglected by the T-matrix approximation. To answer this question, we apply the BS formalism to calculate the slope of the Goldstone mode and the corresponding sound velocity. We found 4% difference between the values of the sound velocity provided by the T-matrix approximation and the BS equation. This small difference confirm previously reported result that close to the phase transition boundary the bubble-diagram contributions are not important.

PACS numbers: 71.10.Fd, 05.30.Rt, 37.10.Jk, 73.43.-f

I. INTRODUCTION

The present-day experiments with ultracold atoms in optical lattices allow us to simulate both the Haldane’s model [1] and the situation [2,3] consider by Kane and Mele (KM). The Haldane’s model [4,5] is a tight-binding representation of electron motion on a honeycomb lattice in the presence of a magnetic field, which vector potential has the full symmetry of the lattice and generates a magnetic field with zero total flux through the unit cell. It was pointed out by Haldane that in 2D honeycomb lattice the topological ordering requires time reversal symmetry breaking. Because of the zero magnetic flux through each unit cell, the phase accumulated through a nearest neighbor hopping vanishes, whereas the phase accumulated through nearest-neighbor (NNN) hopping is nonzero. This extra phase breaks the time-reversal symmetry. However, the electron spin is not included in the Haldane’s model.

It is known that the spin-orbit coupling preserves the time-reversal symmetry, but the spin-orbit effects can be used to get topological insulators. The first example was the KM Hamiltonian for the electrons in a graphene [6,7], which consists of two copies of the Haldane’s model, one for spin-up electrons and one for spin-down electrons. In the KM model each spin component breaks time-reversal symmetry, but the time reversal symmetry is restored when taking two copies with different signs for the spin together.

In a recent paper [8], the phase diagram of the attractive Kane-Mele-Hubbard (KMH) model at half filling has been obtained as a function of a tuning parameter \( x = 3\sqrt{3}\lambda/(m_{AB} + 3\sqrt{3}\lambda) \). Here \( \lambda \) is the strength of intrinsic spin-orbit (ISO) coupling, and \( m_{AB} \) is sublattice potential. However, the NNN hopping term has been neglected, although it is several orders of magnitude stronger than the ISO coupling.

In this paper, we have presented the phase diagram of the attractive KMH model with NNN hopping at half filling as a function of a tuning parameter \( x = 3\sqrt{3}\lambda/(3\sqrt{3}\lambda + 3t' + m_{AB}) \), where \( t' \) is the NNN hopping amplitude. It is shown that in the mean-field approximation we have two gap equations for \( \Delta_A \) and \( \Delta_B \), instead of a single gap \( \Delta \) when NNN hopping is neglected. We shall discuss the case of Fermi (spin-1/2) atoms loaded into honeycomb optical lattice, but the results are also valid for the tight-binding description of electrons in a graphene as well. Our tight-binding Hamiltonian \( H = H_{KM} + H_{NNN} + H_U \) includes the KM terms, as well as the NNN hopping and the onsite attractive Hubbard interaction, where

\[
H_{KM} = -t \sum_{<ij>} \left[ \Psi_{i\uparrow}^\dagger \Psi_{j\uparrow} + H.c. \right] + H_{ISO} - m_{AB} \sum_i \left( \Psi_{i\uparrow}^\dagger \Psi_{i\downarrow}^\dagger - \Psi_{i\downarrow}^\dagger \Psi_{i\uparrow}^\dagger \right),
\]

The first term in (1) takes into account the possibility for nearest-neighbor hopping. \( H_{ISO} \) represents the ISO interaction, which originates from the hybridization of the higher angular momentum orbit and it excerts opposite magnetic fields upon electrons with opposite spin polarizations. The third term in (1) describes the possible energy offset between sites of A and B sublattices. \( t \) is the nearest-neighbor hopping amplitudes, \( m_{AB} \) is the energy offset parameter, and \( \Psi_i = (\psi_{i\uparrow}^\dagger, \psi_{i\downarrow}^\dagger) \) is the creation operator for spin-up and
spin-down fermions at site \(i\). For hopping to the nearest neighbor sites the vectors \(\mathbf{r}_j\) in terms of lattice constant \(a = 1\) are: \(\delta_1 = (1/2, \sqrt{3}/2)\), \(\delta_2 = (1/2, -\sqrt{3}/2)\), \(\delta_3 = (-1, 0)\), and \(\delta_4 = (0, \sqrt{3})\). The attractive Hubbard interaction is described by \(H_U = -U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow}\), where \(U > 0\).

The KM Hamiltonian along with the NNN hopping term are represented by the following 4 \(\times\) 4 matrix in the momentum space on the basis of the four-component wave function \(\Psi = (\psi_A, \psi_B, \psi_A^\dagger, \psi_B^\dagger)^\dagger\):

\[
\hat{H}(k) = \begin{pmatrix}
Z(k) + h_1(k) - m_{AB} & 0 & h(k) & 0 \\
0 & -Z(k) + h_1(k) - m_{AB} & 0 & h(k) \\
h^*(k) & 0 & Z(k) + h_1(k) + m_{AB} & 0 \\
0 & h^*(k) & 0 & Z(k) + h_1(k) + m_{AB}
\end{pmatrix},
\]

where \(h(k) = -t \left[\cos(k_x) + 2 \cos\left(\frac{k_y}{2}\right) \cos\left(\frac{\sqrt{3}k_y}{2}\right)\right] + ut \left[\sin(k_x) - 2 \sin\left(\frac{k_y}{2}\right) \cos\left(\frac{\sqrt{3}k_y}{2}\right)\right], \quad h_1(k) = -2t' \left[\cos(\sqrt{3}k_y) + 2 \cos\left(\frac{k_y}{2}\right) \cos\left(\frac{\sqrt{3}k_y}{2}\right)\right], \quad Z(k) = -2\lambda \left[\sin(\sqrt{3}k_y) - 2 \cos\left(\frac{3k_y}{2}\right) \sin\left(\frac{\sqrt{3}k_y}{2}\right)\right], \quad \lambda = \text{strength of ISO interaction}.

The eigenvalues of equation (2) are \(\Omega_{1,2}(k) = h_1(k) \pm \sqrt{m_{AB} - Z(k)}^2 + |h(k)|^2\), and \(\Omega_{3,4}(k) = h_1(k) \pm \sqrt{m_{AB} + Z(k)}^2 + |h(k)|^2\).

The authors of Ref. [8] pointed out that depending on the value of the parameter \(0 \leq x \leq 1\), the system can be taken across the topological phase transition. For \(x < 1/2\) we have a topological trivial insulator, while for \(x > 1/2\) the system is in topological non-trivial insulator state. The phase diagram of the KMH model, reported in Ref. [9], was calculated within the self-consistent Bogoliubov-de Gennes theory using a supercell with six sites. The corresponding system parameters are \(\Lambda = (1 - x)E_g\), \(3\sqrt{3}\lambda = xE_g\), and \(E_g = t/2\).

In Sec. II, we apply the mean-field approximation to obtain the matrix elements of the single-particle Green’s functions of the KMH model with NNN hopping term at half filling. The above-mentioned supercell approach is more complicated than the mean-field approximation, but as can be seen from Fig. 2a in this paper, our approach reproduces the corresponding phase diagram (Fig. 2a in Ref. [3]). It turns out that the matrix elements of the mean-field single-particle Green’s function in the momentum space, proportional to \(\langle \Psi_{A,k_1}^\dagger \Psi_{A-k_1} \rangle\) and \(\langle \Psi_{B,k_1}^\dagger \Psi_{B-k_1} \rangle\), are different, and therefore, we have to introduce two gaps, \(\Delta_A\), and \(\Delta_B\). If the NNN hopping is neglected, we have \(\Delta_A = \Delta_B = \Delta\).

When the NNN hopping is included, we have \(m_{AB} = -3t' + (1 - x)E_g\), and \(3\sqrt{3}\lambda = xE_g\). At \(3t' + m_{AB} = 3\sqrt{3}\lambda\).
the the single-particle gap does close at $K = \left(\frac{2\pi}{3a}, \frac{2\pi}{3a}\right)$ while the mass of the other bands at $K' = \left(\frac{2\pi}{3a}, \frac{-2\pi}{3a}\right)$ remains constant throughout the transition for all values of $x$, and viceversa. As in Ref. 8, the ground state of the KM Hamiltonian with NNN hopping is topological nontrivial (or topological trivial), when the parameter $x = 3\sqrt{3}\Lambda/(3\sqrt{3}\Lambda + 3\hbar' + m_{AB})$ is $x > 1/2$ (or $x < 1/2$).

Our next goal is to examine the results about possible Cooperon condensation, discussed in Ref. 10, within the T-matrix approximation. The model Hamiltonian, used in Ref. 10, corresponds to $x = 1$ and $t' = 0$. Due to the attractive onsite interaction the system becomes unstable against the formation of a s-wave spin-singlet superfluid ground state. To the best of our knowledge, the superfluidity of fermion atoms in honeycomb optical lattice has been examined only in the above mentioned paper. According to the T-matrix approximation, the excitation spectrum of collective modes $\omega(Q)$ was derived by calculating the roots of the following secular 2 × 2 determinant:

$$\det \begin{pmatrix} U^{-1} - \Pi_{AA}(\omega, Q) & -\Pi_{AB}^{*}(\omega, Q) \\ -\Pi_{AB}(\omega, Q) & U^{-1} - \Pi_{BB}(\omega, Q) \end{pmatrix} = 0, \quad (3)$$

where

$$\Pi_{\xi\zeta} = \int \frac{d\Omega}{2\pi} \frac{d^{2}k}{(2\pi)^{2}} G_{\xi_i,\zeta_i}^{K_{M}}(k + Q, \Omega + \omega) G_{\zeta_i,\xi_i}^{K_{M}}(k, \Omega).$$

Here $\xi, \zeta$ are the sublattice indices, and $G_{\xi^\sigma,\zeta^\sigma}^{K_{M}}(k, \omega)$ is the Fourier transforms of the KM single-particle Green’s function $G_{\xi^\sigma,\zeta^\sigma}^{K_{M}}(t - t') = - \langle \hat{T}_{t}\{\psi_{\xi^\sigma}(t)\psi_{\zeta^\sigma}^{\dagger}(t')\} \rangle$. It is worth mentioning that the T-matrix approximation, also known as the ladder approximation in the Bethe-Salpeter (BS) equation, consists of the sum of ladder diagrams in the perturbation expansion in terms of $U$ where the corresponding single-particle KM Green’s functions are independent on the interaction. The question that naturally arises here is about the contributions due to the bubble diagrams, neglected by the T-matrix approximation.

To answer the above question, in Sec. III, we derive the BS equation by employing the Hubbard-Stratonovich transformation (HST). If no approximations were made in evaluating the corresponding functional integrals, it would not matter which of the possible HST is chosen. When approximations are taken, the final result depends on a particular form chosen. A possible approximation is to introduce the energy gap as an order parameter field, which allows us to integrate out the fermion fields and to arrive at an effective action. Next steps are to consider the state, which corresponds to the saddle point of the effective action, and to write the effective action as a series in powers of the fluctuations and their derivatives. The exact result can be obtained by explicitly calculating the terms up to second order in the fluctuations and their derivatives. This approximation, known as the Gaussian approximation, has been employed in the case of square geometry 12, but to the best of our knowledge, it has never been used in the case of honeycomb lattice. In our approach, the quartic terms are transformed to quadratic forms by introducing a boson field which mediates the interaction of fermions. This assumption is similar to the situation in quantum electrodynamics, where the photons mediate the interaction of electric charges, and it allows us to derive the Schwinger-Dyson (SD) equation for the poles of the single-particle Green’s function, as well as the BS equation in the generalized random phase approximation (GRPA) for the poles of the two-particle Green’s function. In the GRPA, the single particle excitations are replaced with those obtained by diagonalizing the Hartree-Fock (HF) mean-field Hamiltonian, while the collective modes are obtained by solving the BS equation in which the single-particle Green’s functions are calculated in HF mean-field approximation, and the BS kernel is obtained by summing ladder and bubble diagrams.

We have calculated the slope of the low-energy (Goldstone) mode and the corresponding sound velocity at half filling, using the same system parameters as in Ref. 10. We found that the T-matrix approximation is a good approximation because the sound velocity in the direction toward point $M$, calculated within the T-matrix approximation, is about 4% less than the result by employing the BS equation.

II. SINGLE-PARTICLE DISPERSION IN THE MEAN-FIELD APPROXIMATION

In the presence of an onsite attractive interaction between the fermions, the fermion atoms form bound (Cooper) pairs. As a result, the system becomes unstable against the formation of a s-wave spin-singlet superfluid ground state. At low energies the system admits an effective description in terms of massless Dirac fermions, therefore, in a honeycomb optical lattice we have a possibility to observe a superfluidity of Fermi atoms with the Dirac spectrum. We restricted our calculations to half-filling ($\mu = 0$), where the particle-hole symmetry takes place. We further assume that the BCS mean-field order parameters are real constants, i.e. $\Delta_{A(B)} = U < \psi_{A(B),-\downarrow}\psi_{A(B),\uparrow}>$. When the attractive Hubbard interaction is taken into account, the KM basis of the four-component wave function $\Psi^d = (\psi_{A,k,\uparrow}^{\downarrow}, \psi_{A,k,\downarrow}^{\uparrow}, \psi_{B,k,\downarrow}^{\uparrow}, \psi_{B,k,\uparrow}^{\downarrow})$ becomes a basis of the eight-component wave function $\Psi^f = (\psi_{A,k,\uparrow}^{\downarrow}, \psi_{B,k,\uparrow}^{\downarrow}, \psi_{B,-k,\downarrow}^{\downarrow}, \psi_{A,-k,\downarrow}^{\downarrow}, \psi_{A,k,\downarrow}^{\downarrow}, \psi_{B,k,\downarrow}^{\uparrow}, \psi_{B,-k,\uparrow}^{\downarrow}, \psi_{A,-k,\uparrow}^{\downarrow})$. Thus, the generalized Hamiltonian $H = H_{KM} + H_{NNN} + H_{U}$ in the mean-field approximation in the momentum space on the basis of the eight-component wave function is represented by the following $8 \times 8$ block-diagonal matrix:

$$\hat{H}_{MF}(k) = \begin{pmatrix} \hat{H}_1(k) & 0 \\ 0 & \hat{H}_2(k) \end{pmatrix}, \quad (4)$$
where the corresponding $4 \times 4$ blocks are defined by the following block-matrices:

$$
\tilde{H}_1(k) = \begin{pmatrix}
  h_1(k) + Z(k) - m_{AB} & h(k) & 0 & \Delta_A \\
  h^*(k) & h_1(k) - Z(k) + m_{AB} & 0 & 0 \\
  0 & \Delta_B & h_1(k) + Z(k) - m_{AB} & -h^*(k) \\
  \Delta_A & 0 & -h(k) & -h_1(k) - Z(k) + m_{AB}
\end{pmatrix}.
$$

The block $\tilde{H}_2(k)$ follows from $\tilde{H}_1(k)$ replacing $Z(k)$ and $\Delta_{A,B}$ by $-Z(k)$ and $-\Delta_{A,B}$.

The eigenvalues of the $8 \times 8$ Hamiltonian are as follows:

$$
\omega_1(k) = \sqrt{\Omega^2(k) + \omega^2(k)}, \quad \omega_2(k) = \sqrt{\Omega^2(k) - \omega^2(k)}, \quad \omega_3(k) = -\omega_1(k), \quad \omega_4(k) = -\omega_2(k),
$$

$$
\omega_5(k) = \sqrt{\Omega^2_+(k) + \omega^2_+(k)}, \quad \omega_6(k) = \sqrt{\Omega^2_-(k) - \omega^2_-(k)}, \quad \omega_7(k) = -\omega_5(k), \quad \omega_8(k) = -\omega_6(k),
$$

$$
\Omega_{\pm}(k) = \sqrt{(\Delta^2 + \delta^2)/4 + h_1^2(k) + [h(k)]^2 + (m_{AB} \pm Z(k))^2}, \quad \Delta = \Delta_A + \Delta_B, \quad \delta = \Delta_A - \Delta_B,
$$

$$
\omega_{\pm}^2(k) = \sqrt{4h_1^2(k) [h(k)]^2 + (m_{AB} \pm Z(k))^2 + \delta^2 [\Delta^2/4 + [h(k)]^2] - 2\delta \Delta h_1(k)(m_{AB} \pm Z(k)).}
$$

The single-particle excitations in the mean-field approximation manifest themselves as poles of the Matsubara single-particle Green’s function, defined as:

$$
\tilde{G}(k, \omega_m) = (\omega_m \hat{1} - \tilde{H}_{MF}(k))^{-1}
$$

$$
= \begin{pmatrix}
  G^{(1)}(k, \omega_m) & 0 \\
  0 & G^{(2)}(k, \omega_m)
\end{pmatrix}.
$$

Here, the Matsubara fermion energies are $\omega_m = (2\pi/\beta)(m + 1/2)$, $m = 0, 1, 2, ..., \beta = 1/(k_B T)$, and $k_B$ is the Boltzmann constant (throughout this paper we have assumed $\hbar = k_B = 1$). The corresponding zero-temperature Green’s function $\tilde{G}(k, \omega)$ is an $8 \times 8$ matrix with elements $G^{(1,2)}_{n_1,n_2}(k, \omega)$ \{n_1, n_2 = 1, 2, ... 8 written in the following form:

$$
G^{(1)}_{n_1,n_2}(k, \omega) = \frac{A^{(1)}_{n_1,n_2}(k)}{\omega - \omega_1(k) + i0^+} + \frac{B^{(1)}_{n_1,n_2}(k)}{\omega - \omega_2(k) - i0^+} + \frac{C^{(1)}_{n_1,n_2}(k)}{\omega - \omega_3(k) - i0^+} + \frac{D^{(1)}_{n_1,n_2}(k)}{\omega - \omega_4(k) + i0^+},
$$

$$
G^{(2)}_{n_1,n_2}(k, \omega) = \frac{A^{(2)}_{n_1,n_2}(k)}{\omega - \omega_1(k) + i0^+} + \frac{B^{(2)}_{n_1,n_2}(k)}{\omega - \omega_2(k) - i0^+} + \frac{C^{(2)}_{n_1,n_2}(k)}{\omega - \omega_3(k) - i0^+} + \frac{D^{(2)}_{n_1,n_2}(k)}{\omega - \omega_4(k) + i0^+}.
$$

The functions $A^{(1,2)}_{n_1,n_2}(k), B^{(1,2)}_{n_1,n_2}(k), C^{(1,2)}_{n_1,n_2}(k)$ and $D^{(1,2)}_{n_1,n_2}(k)$ can be numerically calculated by inverting the matrix \((\omega_m \hat{1} - \tilde{H}_{MF}(k))\).

The momentum distribution for the spin components $n^{(\uparrow,\downarrow)}_{\mu}(k)$ can be evaluated using the corresponding elements of the $8 \times 8$ Green’s function matrix:

$$
n^{(\uparrow)}_{\mu}(k) = \langle \psi_{A,k,\uparrow}^\dagger \psi_{A,k,\uparrow} \rangle + \langle \psi_{B,k,\uparrow}^\dagger \psi_{B,k,\uparrow} \rangle = \beta^{-1} \sum_{\omega_m} [G_{11}(k, \omega_m) + G_{22}(k, \omega_m)],
$$

$$
n^{(\downarrow)}_{\mu}(k) = \langle \psi_{A,k,\downarrow}^\dagger \psi_{A,k,\downarrow} \rangle + \langle \psi_{B,k,\downarrow}^\dagger \psi_{B,k,\downarrow} \rangle = \beta^{-1} \sum_{\omega_m} [G_{55}(k, \omega_m) + G_{66}(k, \omega_m)].
$$
Very similarly, one can derive a set of two gap equations $\Delta_A = U < \psi_A, -k, \psi_A, k > = U \beta^{-1} \sum_{\mathbf{k}, \mathbf{k}'} C^{(2)}_{67} (\mathbf{k}, \mathbf{k}')$, $\Delta_B = U < \psi_B, -k, \psi_B, k > = U \beta^{-1} \sum_{\mathbf{k}, \mathbf{k}'} C^{(2)}_{67} (\mathbf{k}, \mathbf{k}')$ which at a zero temperature assume the form:

$$\Delta_A = U \sum_{\mathbf{k} \in BZ} \left[ D^{(2)}_{67} (\mathbf{k}) + D^{(2)}_{58} (\mathbf{k}) \right], \quad \Delta_B = U \sum_{\mathbf{k} \in BZ} \left[ D^{(2)}_{67} (\mathbf{k}) + D^{(2)}_{58} (\mathbf{k}) \right].$$

(6)

Our next step is to solve the gap equations assuming $t' = -0.15t$ (the same value has been used previously in [11]). At $x = 1/2$, i.e. $3t' + m_{AB} = 3\sqrt{3} \lambda$, the band structure is at a topological phase transition because the Dirac bands at points $K$ or $K'$ being massless at $x = 1/2$, we have $\Omega_x (k_K) = 0$ and $\Omega_x (k_{K'}) = 0$ at points $K = (2\pi/3a, 2\pi/3\sqrt{3}a)$ and $K' = (2\pi/3a, -2\pi/3\sqrt{3}a)$. Let us assume that $\Delta_B = \alpha \Delta_A$, so we use Eqs. (6) to obtain an equation, $\alpha = f(\alpha, \Delta_A)$, for $\alpha$ and $\Delta_A$ with

$$f(\alpha, \Delta_A) = \frac{\sum_{\mathbf{k} \in BZ} \left[ D^{(2)}_{67} (\mathbf{k}) + D^{(2)}_{58} (\mathbf{k}) \right]}{\sum_{\mathbf{k} \in BZ} \left[ D^{(2)}_{67} (\mathbf{k}) + D^{(2)}_{58} (\mathbf{k}) \right]}$$

(7)

Next, we fixed the value of $\Delta_A$, and solve iteratively our equation for $\alpha$. Having $\alpha$ and $\Delta_A$, we used Eqs. (6) to obtain the corresponding value of $U$. The results of our numerical calculations for $x = 1/2$ are presented in Fig.

As can be seen, the minimum value of the Hubbard interaction that creates non-zero superfluid gaps is almost the same with and without the NNN hopping term. The difference between the two gaps becomes important for higher values of $U$.

III. GRPA FOR THE COLLECTIVE MODES

The Green’s functions in the functional-integral approach are defined by means of the so-called generating functional with sources for the boson and fermion fields. In our problem, the corresponding functional integrals cannot be evaluated exactly because the interaction part of the Hamiltonian is quartic in the Grassmann fermion fields. A possible way to deal with this problem is to transform the quartic terms to a quadratic forms by introducing a boson field which mediates the interaction of fermions. The boson field in the honeycomb lattice has to be an eight-component boson field $A_{\alpha}(z)$ ($\alpha = 1, 2, ..., 8$) interacting with eight-component fermion spinor fields $\hat{\Psi}(y) = (\psi^{\dagger}_{A, \uparrow}(y) \psi^{\dagger}_{A, \downarrow}(y) \psi_{A, \uparrow}(y) \psi_{A, \downarrow}(y))$, and $\hat{\Psi}(x) = \hat{\Psi}(y)$. Here, we have introduced composite variables, $z = (r_j, v) = (j, v)$, $y = \{r_j, u, z = (r_j, v) = (i, u) = \{r_j, u'\}$ are the lattice site vectors, and according to imaginary-time (Matsubara) formalism the variable $u, u'$ and $v$ range from 0 to $\beta$.

The action of this model system is assumed to be of the following form $S = S_0^{(F)} + S_0^{(B)} + S^{(F-B)}$, where:

$$S_0^{(F)} = \hat{\psi}(y) \tilde{G}^{(0)-1}(y; x) \hat{\psi}(x),$$

$$S_0^{(B)} = \frac{1}{2} A_{\alpha}(z) D^{(0)-1}(z, z') A_{\beta}(z'),$$

$$S^{(F-B)} = \hat{\psi}(y) \tilde{G}_\alpha^{(0)}(y, x | z) \hat{\psi}(x) A_{\alpha}(z).$$

Here we use the summation-integration convention: that repeated variables are summed up or integrated over. The action $S_0^{(F)}$ describes the fermion part of the system. The inverse Green’s function of free fermions $\tilde{G}^{(0)-1}(y; x)$ is given by the following matrix:
\[
\tilde{G}^{(0)-1}(y;x) = \sum_{k,\omega_m} \exp\left[i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - \omega_m(u - u')\right] G_{n_1,n_2}^{(0)-1}(k,\omega_m),
\]

where the non-interacting Green’s function is defined as \(\tilde{G}^{(0)-1}(k,\omega_m) = (\omega_m \mathbb{I} - \tilde{H}_0(k))\). The non-interacting Hamiltonian \(\tilde{H}_0(k)\) is obtained from \(\tilde{H}_{MF}(k)\) with \(\Delta_{A,B} = 0\).

The action \(S_0^B\) describes the boson field which mediates the fermion-fermion onsite interaction in the Hubbard Hamiltonian. The Fourier transform of the bare boson propagator \(\tilde{D}^{(0)}(\alpha\beta)(z,z')\) to find the first atom at the site \(i\)

\[
\tilde{D}^{(0)}_{\alpha\beta}(z,z') = \frac{1}{N} \sum_{k} \sum_{\omega_p} e \{ i [k \cdot (y_j - y_i) - \omega_p(u - v)] \} \tilde{G}_{\alpha\beta}^{(0)}(k), \quad \tilde{D}^{(0)}_{\alpha\beta}(k) = U (\delta_{1,\alpha}\delta_{2,\beta} + \delta_{2,\alpha}\delta_{1,\beta} + \delta_{5,\alpha}\delta_{6,\beta} + \delta_{6,\alpha}\delta_{5,\beta}).
\]

Here, the Matsubara boson energies are \(\omega_p = 2m\pi/\beta; p = 0, 1, 2, \ldots\).

The interaction between the fermion and the boson fields is described by the action \(S^{(F-B)}\). The bare vertex \(\tilde{\Gamma}^{(0)}_{\alpha}(y_1; x_2 | z) = \tilde{\Gamma}^{(0)}_{\alpha}(i_1, u_1; i_2, u_2 | j, v) = \delta(u_1 - u_2)\delta(u_1 - v)\delta_{i_1,i_2}\delta_{i,j}\) is a 8 \&times; 8 matrix \(\tilde{\Gamma}^{(0)}_{\alpha} = \begin{pmatrix} \tilde{\Gamma}^{(A)}_{\alpha} & 0 \\ 0 & \tilde{\Gamma}^{(B)}_{\alpha} \end{pmatrix}\),

where the 4 \&times; 4 blocks \(\tilde{\Gamma}^{(A/B)}_{\alpha}\) are defined in terms of the Dirac matrix \(\tilde{\gamma}_0\) and the matrices \(\tilde{\alpha}_i\) (\(\tilde{\alpha}_i\) matrices also appear in superconductivity [13]):

\[
\begin{align*}
\tilde{\Gamma}^{(A)}_{\alpha} &= \frac{1}{2} (\tilde{\gamma}_0 + \tilde{\alpha}_z) \delta_{\alpha 1} + \frac{1}{2} (\tilde{\gamma}_0 - \tilde{\alpha}_z) \delta_{\alpha 2} + \frac{1}{2} (\tilde{\gamma}_x + i \tilde{\gamma}_y) \delta_{\alpha 3} + \frac{1}{2} (\tilde{\gamma}_x - i \tilde{\gamma}_y) \delta_{\alpha 4}, \\
\tilde{\Gamma}^{(B)}_{\alpha} &= \frac{1}{2} (\tilde{\gamma}_0 + \tilde{\alpha}_z) \delta_{\alpha 5} + \frac{1}{2} (\tilde{\gamma}_0 - \tilde{\alpha}_z) \delta_{\alpha 6} + \frac{1}{2} (\tilde{\gamma}_x + i \tilde{\gamma}_y) \delta_{\alpha 7} + \frac{1}{2} (\tilde{\gamma}_x - i \tilde{\gamma}_y) \delta_{\alpha 8}, \\
\tilde{\gamma}_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \end{pmatrix}, \quad \tilde{\alpha}_i = \begin{pmatrix} \sigma_i & 0 \\
0 & \sigma_y \sigma_i \sigma_y \end{pmatrix}, \quad i = x, y, z.
\end{align*}
\]

The basic assumption in our BS formalism is that the bound states of two Fermi atoms in an optical lattice at zero temperature are described by the BS wave functions (BS amplitudes). The BS amplitude determines the probability amplitude to find the first atom at the site \(i\) at the moment \(t_1\) and the second atom at the site \(j\) at the moment \(t_2\). The BS amplitude depends on the relative internal time \(t_1 - t_2\) and on the "center-of-mass" time \((t_1 + t_2)/2\) [14]. Since the boson propagator \(\tilde{D}^{(0)}_{\alpha\beta}(k)\) is frequency independent, the spectrum of the collective modes will be obtained by solving the following BS equation for the equal-time BS amplitude \(\Psi^{Q}_{n_2,n_1}\):

\[
\Psi^{Q}_{n_2,n_1} = K^{(0)} \left[ \begin{array}{c} n_1 \\
n_2 \\
n_4 \end{array} \right] |\omega(Q)\rangle \left[ \begin{array}{c} I_d \left( \begin{array}{c} n_3 \\
n_5 \\
n_6 \end{array} \right) \\
I_{exc} \left( \begin{array}{c} n_3 \\
n_5 \\
n_6 \end{array} \right) \right] \Psi^{Q}_{n_2,n_1}.
\]

In the GRPA the two-particle propagator \(K^{(0)}_{\alpha\beta}\) is written in terms of the mean-field single-particle Green’s functions:

\[
K^{(0)}_{\alpha\beta} \left( \begin{array}{c} n_1 \\
n_2 \\
n_4 \end{array} |\omega(Q)\rangle \right) \equiv K^{(0)}_{n_1,n_3,n_4} = \int \frac{d\Omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} G_{n_3,n_4}(k + Q, \Omega + \omega(Q)) G_{n_1,n_2}(k, \Omega).
\]

The kernel of the BS equation is a sum of the direct \(I_d = \delta\Sigma^F/\delta G\) and exchange \(I_{exc} = \delta\Sigma^H/\delta G\) interactions, written as derivatives of the Fock and the Hartree parts of the self-energy. This means that the BS equation and the corresponding SD equation for the self-energy have to be solved self-consistently. In the Appendix A, we have presented an approximation which allows us to decouple the BS and SD equations, and to obtain the following expressions for the BS kernel:

\[
I_d \left( \begin{array}{c} n_1 \\
n_2 \\
n_4 \end{array} \right) = -\Gamma^{(0)}_{\alpha}(n_1,n_3) \Gamma_{\alpha\beta}^{(0)}(n_4,n_2), \quad I_{exc} \left( \begin{array}{c} n_1 \\
n_2 \\
n_4 \end{array} \right) = \frac{1}{2} \Gamma^{(0)}_{\alpha}(n_1,n_2) \Gamma_{\alpha\beta}^{(0)}(n_4,n_3).
\]

Here \(\Gamma^{(0)}_{\alpha}(n_1,n_2)\) is the corresponding matrix element of \(\tilde{\Gamma}^{(0)}_{\alpha}\). The BS equation, written in the matrix form, is \((\mathbb{I} + U \tilde{Z}) \Psi = 0\), where \(\mathbb{I}\) is the unit matrix, and the condition for the existence of non-trivial solution requires the
64 × 64 determinant \( \det \left| U^{-1} \mathbf{I} + \hat{Z} \right| = 0 \). By applying simple matrix algebra, the 64 × 64 determinant can be simplified to a 20 × 20 one of the following form

\[
\begin{vmatrix}
A_{4 \times 4}(\omega, \mathbf{Q}) & 0 & 0 \\
0 & B_{4 \times 4}(\omega, \mathbf{Q}) & 0 \\
0 & 0 & C_{12 \times 12}(\omega, \mathbf{Q})
\end{vmatrix} = 0.
\]

(13)

The elements of the above three blocks are given in the Appendix B. Blocks \( A_{4 \times 4} \) and \( B_{4 \times 4} \) have different elements, but \( \det |A_{4 \times 4}(\omega, \mathbf{Q})| = \det |B_{4 \times 4}(\omega, \mathbf{Q})| \). The above 20 × 20 determinant vanishing if \( \det |A_{4 \times 4}(\omega, \mathbf{Q})| = 0 \), or \( \det |C_{12 \times 12}(\omega, \mathbf{Q})| = 0 \). Our numerical calculations show that \( \det |A_{4 \times 4}(\omega = 0, \mathbf{Q} = 0) \neq 0 \). This means that the Goldstone mode dispersion within the BS formalism is provided by the secular determinant \( \det |C_{12 \times 12}(\omega, \mathbf{Q})| = 0 \).

To compare our numerical results with the T-matrix approximation, we assume the same system parameters as in Ref. [10]: \( m_{AB} = 0, t' = 0, \lambda = 0.1t, \) and \( U = 2.69t \). The gap equation provides \( \Delta = 0.151t \). Having the mean-field gap, we have calculated the sound velocity \( u = oat/\hbar \) at half filling in the direction of point \( \Gamma \) toward point \( M \). The slope \( \alpha \) of the linear part of the collective-mode dispersion has been calculated numerically by using three points \( \omega(Q_x, Q_y = 0) \) with \( Q_x = 0, aQ_x = 0, 0.0025\pi \) and 0.0050\( \pi \). The corresponding slope is \( \alpha = 1.45 \), and therefore, the sound velocity becomes \( u/v_F = 0.97 \), where we have introduced the Fermi velocity \( v_F = (3/2)ta/\hbar \) in a honeycomb lattice. For the similar system parameters, the slope, obtained from Fig. (5b) in Ref. [10], is \( \alpha = 1.34 \), that is about 4% difference.

IV. DISCUSSION

To summarize, we have numerically calculated the phase diagram of the attractive KMH model with NNN hoping at half filling within the mean-field approximation. It is shown that as soon as the NNN hoping is included, we have to solve two mean-field gap equations for \( \Delta_3 \) and \( \Delta_B \), instead of a single gap equation for \( \Delta \) in the case when NNN hopping is neglected. In the second part of this paper, we have calculated the slope of the low-energy (Goldstone) mode and the corresponding sound velocity in the direction toward point \( M \) within the BS formalism. We found that the T-matrix approximation provides the sound velocity which is about 4% less than the result obtained by employing the BS equation.

It is known that the Gaussian approximation also neglects the bubble diagrams, but in a square lattice the difference between the speeds of the sound calculated in the Gaussian and in the BS approximations is about 25% (see Fig 10 in Ref. [15]). To explain the small difference of 4% in our numerical calculations, we refer to the system parameters: \( x = 1, \lambda = 0.1t, U = 2.69t, \) and \( \Delta = 0.151t \). From the value of \( \lambda \) follows that \( E_g \approx 0.52t \), and therefore, the phase diagram is very close to that presented in Fig. (2). From another point of view, the value of \( U = 2.69t \) tells us that the system is very close to the topological phase transition line at \( x = 1 \). The fact that close to the phase transition boundary the speed of sound calculated in the Gaussian and the BS approaches is essentially the same has been previously found in a square lattice [14]. Thus, it is naturally to expect that away from the phase transition boundary contributions due to the bubble diagrams will be more important.

Appendix A

There is one-to-one correspondence between the KMH model and our model system, which is based on the following Hubbard-Stratonovich transformation for the fermion operators:

\[
\int \mu[A] \exp \left[ \hat{\psi}(y) \hat{\Gamma}_\alpha(y; z) \hat{\psi}(z) A_\alpha(z) \right] = \exp \left[ -\frac{1}{2} \hat{\psi}(y) \hat{\Gamma}_\alpha(y; z) \hat{\psi}(z) D_{\alpha\beta}(z, z') \hat{\psi}(y') \hat{\Gamma}_\beta(y'; z') \hat{\psi}(z') \right].
\]

The functional measure \( D\mu[A] \) is chosen to be:

\[
\mu[A] = DAe^{-\frac{1}{2} \int A_\alpha(z) D_{\alpha\beta}(z, z') A_{\beta}(z')} \int \mu[A] = 1.
\]

According to the field-theoretical approach, the expectation value of a general operator \( \hat{O}(u) \) can be expressed as a functional integral over the boson field \( A \) and the Grassmann fermion fields \( \hat{\psi} \) and \( \hat{\psi} \):

\[
< \hat{T}_u(\hat{O}(u)) > = \frac{1}{Z[J, M]} \int D\mu[\hat{\psi}, \hat{\psi}, A] \hat{O}(u) \exp \left[ J_\alpha(z)A_\alpha(z) - \hat{\psi} M \hat{\psi} \right] |_{J = M = 0},
\]
where the symbol $<...>$ means that the thermodynamic average is made. The functional $Z[J,M]$ is defined by

$$Z[J,M] = \int D\mu[\hat{\psi},\hat{\bar{\psi}},A] \exp \left[ J_\alpha(z) A_\alpha(z) - \hat{\bar{\psi}} M \hat{\psi} \right],$$

where the functional measure $D\mu[\hat{\psi},\hat{\bar{\psi}},A] = DAD\hat{\bar{\psi}}D\hat{\psi} \exp (S)$ satisfies the condition $\int D\mu[\hat{\psi},\hat{\bar{\psi}},A] = 1$. The quantity $J_\alpha(z)$ is the source of the boson field. The sources $M_{ij}(y;x)$ of the fermion fields are included in the $\hat{\Psi}(y)\hat{M}(y,x)\hat{\Psi}(x)$ term, where $\hat{M}(y,x)$ is an $8 \times 8$ matrix:

$$\hat{M}(y,x) = \begin{pmatrix} M_{11}(y;x) & \cdots & M_{18}(y;x) \\ \vdots & \ddots & \vdots \\ M_{81}(y;x) & \cdots & M_{88}(y;x) \end{pmatrix}. $$

In what follows, we introduce complex indices $1 = \{n_1,y_1\}$, and $2 = \{n_2,x_2\}$, so in short notations we have $M_{n_1,n_2}(y_1;x_2) = M(1;2)$.

By means of the definition of the thermodynamic average, one can express all Green’s functions in terms of the functional derivatives with respect to the corresponding sources of the generating functional of the connected Green’s functions $W[J,M] = \ln Z[J,M]$.

The boson Green’s function is $D_{\alpha\beta}(z,z')$ is a $8 \times 8$ matrix defined as

$$D_{\alpha\beta}(z,z') = -\frac{\delta^2 W}{\delta J_\alpha(z) \delta J_\beta(z')}.$$ 

The single-fermion Green’s function $\hat{G}(x,y) = -<\hat{T}_x \hat{\bar{\Psi}}(x) \otimes \hat{\Psi}(y)>$ includes all possible thermodynamic averages. Its matrix elements are $G_{n_1,n_2}(x_1;y_2) = -\delta W/\delta M_{n_2,n_1}(y_2;x_1)$. The Fourier transform of the single-particle Green’s function is given by

$$\hat{G}(1;2) = \frac{1}{N} \sum_k \sum_{\omega_m} \exp\{i \{ k \cdot (r_i - r_j) - \omega_m(u_i - u_j) \} \} \hat{G}(k,\omega_m).$$

The two-particle Green’s function $K\left(\begin{array}{cccc} n_1 & x_1 & n_3 & y_3 \\ n_2 & y_2 & n_4 & x_4 \end{array}\right)$ is defined as

$$K\left(\begin{array}{cccc} n_1 & x_1 & n_3 & y_3 \\ n_2 & y_2 & n_4 & x_4 \end{array}\right) = K\left(\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}\right) = \frac{\delta^2 W}{\delta M_{n_2,n_1}(y_2;x_1) \delta M_{n_3,n_4}(y_3;x_4)} = -\frac{\delta G_{n_1,n_2}(x_1;y_2)}{\delta M_{n_3,n_4}(y_3;x_4)}.$$ 

The vertex function $\hat{\Gamma}_\alpha(2;1|z)$ for a given $\alpha$ is a $8 \times 8$ matrix whose elements are:

$$\hat{\Gamma}_\alpha(i_2,u_2;i_1,u_1|\bar{\psi},j)_{n_2,n_1} = -\frac{\delta G_{n_2,n_1}^{-1}(i_2,u_2;i_1,u_1)}{\delta J_\beta(z')} D_{\beta\alpha}^{-1}(z',z).$$

Since the single-particle and the two-particle (collective) excitations manifest themselves as poles of the corresponding Green’s functions, our next step is to obtain equations of the boson and fermion Green’s functions. First, we shall obtain the SD equations, and they will be used to define the fermion self-energy (fermion mass operator) $\hat{\Sigma}(1;2)$. The simplest way to derive the SD equations is to use the fact that the measure $D\mu[\hat{\psi},\hat{\bar{\psi}},A]$ is invariant under the translations $\bar{\psi} \rightarrow \bar{\psi} + \delta \bar{\psi}$ and $A \rightarrow A + \delta A$:

$$D_{\alpha\beta}^{-1}(z,z') R_\alpha(z') + \frac{1}{2} Tr \left( \hat{G}(1;2) \hat{\Gamma}_\alpha(2;1|z) \right) + J_\alpha(z) = 0,$$

$$\hat{G}^{-1}(1;2) - \hat{G}^{-1}(0)(1;2) + \hat{\Sigma}(1;2) + \hat{M}(1;2) = 0,$$

where $R_\alpha(z) = \delta W/\delta J_\alpha(z)$ is the average boson field. The fermion self-energy $\hat{\Sigma}$, is a $8 \times 8$ matrix which can be written as a sum of Hartree $\hat{\Sigma}^H$ and Fock $\hat{\Sigma}^F$ parts. The Hartree part is a diagonal matrix whose elements are:

$$\Sigma^H(i_1,u_1;i_2,u_2)_{n_1,n_2} = \frac{1}{2} \hat{\Gamma}_\alpha(i_1,u_1;i_2,u_2|j,v)_{n_1,n_2} D_{\alpha\beta}^{(0)}(j,v;\bar{v},j') \hat{\Gamma}_\beta(i_3,u_3;i_4,u_4|\bar{v},j')_{n_3,n_4} G_{n_4,n_5}(i_4,u_4;i_3,u_3).$$
The Fock part of the fermion self-energy is given by:

\[ \Sigma^F (i_1, u_1; i_2, u_2)_{n_1n_2} = -\hat{\Gamma}_\alpha^0 (i_1, u_1; i_6, u_6 | j, v)_{n_1n_2} D_{\alpha\beta} (j, v; j', v') \hat{\Gamma}_\beta^0 (i_4, u_4; i_5, u_5 | j', v')_{n_4n_5} \times \]

\[ K \left( \begin{array}{c} n_5, i_5, u_5 \\ n_3, i_3, u_3 \\ n_4, i_4, u_4 \end{array} \right) G_{n_3n_2}^{-1} (i_3, u_3; i_2, u_2) \, G_{n_4n_2}^{-1} (i_4, u_4; i_2, u_2). \]

The Fock part of the fermion self-energy depends on the two-particle Green’s function \( K \); therefore the SD equations and the BS equation for \( K \) have to be solved self-consistently.

Our approach to the Hubbard model allows us to obtain exact equations of the Green’s functions by using the field-theoretical technique, in particular, the Legendre transforms. We can go over from the functional \( W[J, M] \) to a new functional \( V[R, G] = W[J,R,G] + S[J,R,M] + \text{Tr}(M[R,G]) \) such that the conjugate equations hold:

\[ \frac{\delta V}{\delta R_n} (z) = -J_n(z); \quad \frac{\delta V}{\delta G_{n_1n_2}(1; 2)} = M_{n_1n_2}(1; 2). \]

By means of the SD equations and the identity

\[ \delta (1 - 3) \delta (2 - 4) \delta_{n_1n_3} \delta_{n_2n_4} = \frac{\delta M_{n_1n_2}(1; 2)}{\delta M_{n_3n_4}(3; 4)} = \frac{\delta M_{n_1n_2}(1; 2)}{\delta R_n(z)} \frac{\delta R_n(z)}{\delta M_{n_3n_4}(3; 4)} + \frac{\delta M_{n_1n_2}(1; 2)}{\delta G_{n_5n_6}(5; 6)} \frac{\delta G_{n_5n_6}(5; 6)}{\delta M_{n_3n_4}(3; 4)}. \]

one sees that two-particle Green’s function satisfies the BS equation

\[ K^{-1} \left( \begin{array}{c} n_2, i_2, u_2 \\ n_1, i_1, u_1 \end{array} \right) = K^{(0)}^{-1} \left( \begin{array}{c} n_2, i_2, u_2 \\ n_1, i_1, u_1 \end{array} \right) - I \left( \begin{array}{c} n_2, i_2, u_2 \\ n_1, i_1, u_1 \end{array} \right).
\]

Here,

\[ K^{(0)} \left( \begin{array}{c} n_2, i_2, u_2 \\ n_1, i_1, u_1 \end{array} \right) = G_{n_2n_3} (i_2, u_3; i_2, u_2) G_{n_4n_1} (i_4, u_4; i_1, u_1) \]

is the two-particle free propagator constructed from a pair of fully dressed generalized single-particle Green’s functions.

The kernel \( I = \delta \Sigma/\delta G \) of the BS equation can be expressed as a functional derivative of the fermion self-energy \( \hat{\Sigma} \).

Since \( \hat{\Sigma} = \hat{\Sigma}^H + \hat{\Sigma}^F \), the BS kernel \( I = I_{exc} + I_d \) is a sum of functional derivatives of the Hartree \( \hat{\Sigma}^H \) and Fock \( \hat{\Sigma}^F \) contributions to the self-energy:

\[ I_{exc} \left( \begin{array}{c} n_2, i_2, u_2 \\ n_1, i_1, u_1 \end{array} \right) = \frac{\delta \Sigma^H (i_2, u_2; i_1, u_1)_{n_2n_1}}{\delta G_{n_3n_4}(i_3, u_3; i_4, u_4)}, \quad I_d \left( \begin{array}{c} n_2, i_2, u_2 \\ n_1, i_1, u_1 \end{array} \right) = \frac{\delta \Sigma^F (i_2, u_2; i_1, u_1)_{n_2n_1}}{\delta G_{n_3n_4}(i_3, u_3; i_4, u_4)}.
\]

The BS equation and the SD equations have to be solved self-consistently. In order to decouple them, we note that the identity

\[ K^{(0)} \left( \begin{array}{c} n_2, i_2, u_2 \\ n_1, i_1, u_1 \end{array} \right) = \frac{\delta \Sigma^H (i_2, u_2; i_1, u_1)_{n_2n_1}}{\delta G_{n_3n_4}(i_3, u_3; i_4, u_4)} \hat{\Gamma}_\beta (i_4, u_4; i_3, u_3 | z')_{n_3n_4} D_{\beta\alpha} (z', z).
\]

allows us to rewrite the Fock term as

\[ \Sigma^F (i_1, u_1; i_2, u_2)_{n_1n_2} = -\hat{\Gamma}_\alpha^0 (i_1, u_1; i_3, u_3 | j, v)_{n_1n_2} D_{\alpha\beta} (j, v; j', v') G_{n_3n_4} (i_3, u_3; i_4, u_4) \hat{\Gamma}_\beta (i_4, u_4; i_2, u_2 | j', v')_{n_4n_2}.
\]

To decouple SD and BS equations, we replace \( D \) and \( \hat{\Gamma} \) by the free boson propagator \( D^{(0)} \) and by the bare vertex \( \hat{\Gamma}^{(0)} \), respectively. In this approximation the Fock term assumes the form:

\[ \Sigma^F_0 (i_1, u_1; i_2, u_2)_{n_1n_2} = -\hat{\Gamma}_\alpha^0 (i_1, u_1; i_3, u_3 | j, v)_{n_1n_2} D_{\alpha\beta}^{(0)} (j, v; j', v') \hat{\Gamma}_\beta^{(0)} (i_4, u_4; i_2, u_2 | j', v')_{n_4n_2} G_{n_3n_4} (i_3, u_3; i_4, u_4).
\]

The total self-energy is \( \hat{\Sigma} (i_1, u_1; i_2, u_2) = \hat{\Sigma}^H (i_1, u_1; i_2, u_2) + \hat{\Sigma}^F (i_1, u_1; i_2, u_2) \). The Hartree part of the fermion self-energy is a diagonal 8 \times 8 matrix, but in the mean-field approximation, the elements on the major diagonal of \( \hat{\Sigma}^H (i_1, u_1; i_2, u_2) \) will be included into the chemical potential. To obtain an analytical expression for the single-particle Green’s function in the mean-field approximation, we neglect the frequency dependence of the Fourier transform.
of the Fock part of the fermion self-energy. In this approximation, the Fock term is an 8 × 8 matrix with non-zero elements \( \Sigma_{0}^{F}(i_{1}, u_{1}; i_{2}, u_{2}) = \delta(u_{1} - u_{2})\delta(r_{i_{1}} - r_{i_{2}})\Delta_{A/B} \) for \( \{n_{1}n_{2}\} = \{14\}, \{23\}, \{32\}, \{41\} \) for \( \Delta_{A} \) and \( \{n_{1}n_{2}\} = \{58\}, \{67\}, \{76\}, \{85\} \) for \( \Delta_{B} \).

**Appendix B**

The blocks in Eq. (13) are given by the following matrices:

\[
A_{4 \times 4}(\omega, Q) = 
\begin{pmatrix}
U^{-1} - K_{2211}^{(0)} & -K_{2141}^{(0)} & -K_{2141}^{(0)*} & -K_{2141}^{(0)*} \\
-K_{2141}^{(0)*} & U^{-1} - K_{6655}^{(0)} & -K_{6655}^{(0)*} & -K_{6655}^{(0)*} \\
-K_{2141}^{(0)*} & -K_{6655}^{(0)*} & U^{-1} - K_{3214}^{(0)} & -K_{3214}^{(0)*} \\
-K_{6655}^{(0)*} & -K_{3214}^{(0)*} & -K_{3214}^{(0)} & U^{-1} - K_{7218}^{(0)}
\end{pmatrix},
\]

\[
B_{4 \times 4}(\omega, Q) = 
\begin{pmatrix}
U^{-1} - K_{8433}^{(0)} & -K_{8433}^{(0)*} & -K_{8433}^{(0)*} & -K_{8433}^{(0)*} \\
-K_{8433}^{(0)*} & U^{-1} - K_{8877}^{(0)} & -K_{8877}^{(0)*} & -K_{8877}^{(0)*} \\
-K_{8433}^{(0)*} & -K_{8877}^{(0)*} & U^{-1} - K_{5436}^{(0)} & -K_{5436}^{(0)*} \\
-K_{8877}^{(0)*} & -K_{5436}^{(0)*} & -K_{5436}^{(0)} & U^{-1} - K_{4123}^{(0)}
\end{pmatrix},
\]

The elements of \( C_{12 \times 12}(\omega, Q) = 
\begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}
\) will be given by four 6 × 6 blocks:

\[
C_{11}(\omega, Q) = 
\begin{pmatrix}
U^{-1} + K_{1144}^{(0)} & K_{1144}^{(0)*} & K_{1144}^{(0)*} & K_{1144}^{(0)*} & 0 & 0 \\
K_{1144}^{(0)*} & U^{-1} + K_{1411}^{(0)} & K_{1411}^{(0)*} & K_{1411}^{(0)*} & 0 & 0 \\
K_{1144}^{(0)*} & K_{1411}^{(0)*} & U^{-1} + K_{8458}^{(0)} & K_{8458}^{(0)*} & 0 & 0 \\
K_{1411}^{(0)*} & K_{8458}^{(0)*} & K_{8458}^{(0)} & U^{-1} + K_{5588}^{(0)} & 0 & 0 \\
0 & 0 & 0 & 0 & U^{-1} + K_{2233}^{(0)} & K_{2233}^{(0)*} \\
0 & 0 & 0 & 0 & K_{2233}^{(0)*} & U^{-1} + K_{3322}^{(0)}
\end{pmatrix},
\]

\[
C_{21}(\omega, Q) = 
\begin{pmatrix}
0 & 0 & 0 & 0 & K_{7236}^{(0)} & K_{7236}^{(0)} \\
0 & 0 & 0 & 0 & K_{7236}^{(0)} & K_{7236}^{(0)} \\
K_{4414}^{(0)} & K_{4414}^{(0)*} & K_{4414}^{(0)*} & K_{4414}^{(0)*} & 0 & 0 \\
K_{8458}^{(0)} & K_{8458}^{(0)*} & K_{8458}^{(0)*} & K_{8458}^{(0)*} & 0 & 0 \\
-K_{1144}^{(0)} & -K_{1144}^{(0)*} & -K_{1144}^{(0)*} & -K_{1144}^{(0)*} & 0 & 0 \\
-K_{8458}^{(0)*} & -K_{8458}^{(0)*} & -K_{8458}^{(0)*} & -K_{8458}^{(0)*} & 0 & 0
\end{pmatrix}.
\]

\[
C_{22}(\omega, Q) = 
\begin{pmatrix}
U^{-1} + K_{7766}^{(0)} & K_{7766}^{(0)} & K_{7766}^{(0)} & K_{7766}^{(0)} & K_{7766}^{(0)} & K_{7766}^{(0)} \\
-K_{7766}^{(0)} & U^{-1} + K_{6677}^{(0)} & K_{6677}^{(0)} & K_{6677}^{(0)} & K_{6677}^{(0)} & K_{6677}^{(0)} \\
-K_{6266}^{(0)} & K_{6266}^{(0)*} & K_{6266}^{(0)*} & K_{6266}^{(0)*} & K_{6266}^{(0)*} & K_{6266}^{(0)*} \\
-K_{8448}^{(0)} & K_{8448}^{(0)*} & K_{8448}^{(0)*} & K_{8448}^{(0)*} & K_{8448}^{(0)*} & K_{8448}^{(0)*} \\
K_{2332}^{(0)} & K_{2332}^{(0)*} & K_{2332}^{(0)*} & K_{2332}^{(0)*} & K_{2332}^{(0)*} & K_{2332}^{(0)*} \\
K_{1111}^{(0)} & K_{1111}^{(0)*} & K_{1111}^{(0)*} & K_{1111}^{(0)*} & K_{1111}^{(0)*} & K_{1111}^{(0)*}
\end{pmatrix}.
\]

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1 G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, Experimental realization of the topological Haldane model with ultracold fermions, Nature (London) 515, 237 (2014).

2 M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B.
Paredes, and I. Bloch, Realization of the Hofstadter Hamiltonian with ultracold atoms in optical lattices, Phys. Rev. Lett. 111, 185301 (2013).

3 F. Grusdt, T. Li, I. Bloch, and E. Damler, Tunable spin-orbit coupling for ultracold atoms in two-dimensional optical lattices, Phys. Rev. A 95, 063617 (2017).

4 F. D. M. Haldane, Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the ”parity anomaly”, Phys. Rev. Lett. 61, 2015 (1988).

5 L. B. Shao, S. L. Zhu, L. Sheng, D. Y. Xing, and Z. D. Wang, Realizing and detecting the quantum Hall effect without Landau levels by using ultracold atoms, Phys. Rev. Lett. 101, 246810 (2008).

6 C. L. Kane and E. J. Mele, Z2 Topological order and the quantum spin Hall effect, Phys. Rev. Lett. 95, 146802 (2005).

7 C. L. Kane and E. J. Mele, Quantum spin Hall effect in graphene, Phys. Rev. Lett. 95, 226801 (2005).

8 S. Rachel and K. L. Hur, Topological insulators and Mott physics from the Hubbard interaction, Phys. Rev. B 82, 075106 (2010).

9 K. Lee, T. Hazra, M. Randeria, and N. Trivedi, Topological superconductivity in Dirac honeycomb systems, Phys. Rev. B 99, 184514 (2019).

10 S. Tsuchiya, J. Goryo, E. Arahata, and M. Sigrist, Cooperon condensation and intravalley pairing states in honeycomb Dirac systems, Phys. Rev. B 94, 104508 (2016).

11 E. Zhao, and A. Paramekanti, BCS-BEC crossover on the two-dimensional honeycomb lattice, Phys. Rev. Lett. 97, 230404 (2006).

12 J.R. Engelbrecht, M. Randeria, and C. A. R. Sá de Melo, BCS to Bose crossover: Broken-symmetry state, Phys. Rev. B 55, 15153 (1997).

13 K. Maki, p. 1035, in ”Superconductivity”, edited by R.D. Parks, Marcel Dekker, Inc., New York, (1969).

14 C. Itzykson and J. Zuber, Quantum Field Theory, McGraw-Hill, NY 1980.

15 Z. Koinov and S. Pahl, Spin-orbit-coupled atomic Fermi gases in two-dimensional optical lattice in the presence of a Zeeman field, Phys. Rev. A 95, 033634 (2017).

16 Z. Koinov and R. Mendoza, Rashba spin-orbit-coupled atomic Fermi gases in a two-dimensional optical lattice, J Low Temp. Phys., 181, 147, (2015).