A tomographic description for classical and quantum cosmological perturbations

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Abstract
Classical and quantum perturbations can be described in terms of marginal distribution functions in the framework of tomographic cosmology. In particular, the so-called Radon transformation and the mode-parametric quantum oscillator description can give rise to links between quantum and classical regimes. The approach results in a natural scheme to discuss the transition from quantum to classical perturbations and then it could be a workable scheme to connect primordial fluctuations with the large-scale structure observed today.

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1. Introduction

Addressing the problem of cosmic evolution starting from the inflationary mechanism [1, 2] up to the observed scenario as well as the problem of producing classical perturbations starting from the quantum ones have been the subject of several studies during the last decades [3–5]. In particular, the Friedmann–Lemaître–Robertson–Walker (FLRW) homogeneous and isotropic metric, described by the expansion factor \( a(t) \), is usually considered the standard background on which one can formulate the classical and quantum theory of cosmological perturbations. The goal is to provide a suitable description which, starting from the quantum regime, can lead to the observed large-scale structures such as galaxies, galaxy clusters, super-clusters and voids.

On the other hand, the task of quantum cosmology is to achieve a quantum description of the cosmic initial state, which by the following evolution could determine, in principle, all the features of the latest classical epochs that we observe today by astrophysical measurements. The initial quantum state is usually associated with the so-called ‘wave function of the universe’, in analogy with standard quantum mechanics [6], or with its density matrix [7, 8]. A useful representation to consider the quantum-to-classical transition for the quantum state of the universe is the Wigner function \( W(X,\mu,\nu) \) (see e.g. [5, 9]).

Such a Wigner function is the analogue of the classical probability distributions on phase-space \( f(p, q) \) which are, in general, non-negatively defined. On the other hand, due to uncertainty relations, the Wigner function can assume negative values in some domain of phase-space. In view of this fact, it is not a fair probability distribution and is called a quasi-distribution.

Besides, quantum cosmology in its development has always adopted new results from quantum mechanics and quantum field theory. Recently, a new tomographic probability representation of quantum states has been found [10]. In this picture, the fair non-negative probability distribution, containing complete information on the states, is used instead of the wave function or the density matrix. Although the new probability representation is essentially equivalent to all the other available representations adopted in quantum mechanics [11], it has its own merits. An important property of this representation is that the quantum state is associated with the same tomographic probability density \( \mathcal{W}(X,\mu,\nu) \), connected with the Wigner function by the standard integral Radon transform [12]. Besides, the classical state is described using the Radon component of the standard classical probability density \( f(p, q) \) [13]. Thus, in tomographic probability representation, both classical and quantum states can be described by the same non-negative probability densities \( \mathcal{W}(X,\mu,\nu) \); this fact makes it easier to consider the classical-to-quantum mutual relations and transitions.

Some aspects of quantum cosmology have already been studied in the framework of the tomographic probability...
is devoted to a detailed description of the universe that can be suitably defined by assigning a set of cosmological observables. As shown in [3], it is possible to find the same equations for perturbations in the case of hydrodynamic matter, of scalar fields and of alternative theories of gravity as $f(R)$. Perturbation evolution can be described by the Hamiltonian formalism corresponding to small vibrations on a classical background. Field theory can be considered in the tomographic picture as shown in [20]. The Hamiltonian for small vibrations corresponds to interacting oscillators with time-dependent frequencies and mutual coupling constants. In quantum mechanics, the one-dimensional parametric oscillator was studied by Husimi [21] and new integrals of motion, linear in field quadratures of the oscillator, were found in [22, 23]. A comprehensive approach to the system of interacting parametric oscillators has been presented in [24]. The theory of such oscillators with dissipation can be developed in connection with the non-stationary Casimir effect studied in [29].

The advantage of the tomographic approach for this particular Hamiltonian is related to the fact that, in both quantum and classical domains, propagators providing the tomograms of states at time $t$, in terms of the state tomograms at some earlier time, are identical. For these particular propagators, the difference between the quantum and classical universe descriptions is connected to the choice of the initial state tomogram only.

This paper is organized as follows. In section 2, the main points of the field theory for cosmological perturbations are sketched. The probability representation of cosmological perturbations is discussed in section 3. A general scheme for approaching classical and quantum perturbations is given in section 4. Section 5 is devoted to a detailed description of the one-mode parametric quantum oscillator, while the multi-mode case is considered in section 6 in view of the classical-to-quantum relations. Conclusions are drawn in section 7.

2. The field theory of perturbations

In [3], a complete gauge invariant theory for cosmological perturbations is discussed. The cases of hydrodynamic and scalar fields are studied and the formulation in terms of theories alternative to general relativity is considered.

Gauge invariance is a crucial requirement for quantization, because it allows one to properly quantize the physical degrees of freedom.

According to the inflationary paradigm, the origin of cosmological perturbations, which lead to the hierarchy of structures that form the observed universe, can be retraced back to the quantum fluctuations of the very early universe. The quantization of cosmological perturbations allows one in principle, to study the initial state of the universe, at least from a theoretical point of view. In order to relate the early quantum states of the universe to cosmological observations, it is necessary to develop a model for the evolution of quantum fluctuations into classical perturbations.

A unifying picture of these different models for perturbations has been proposed in order to quantize them. To this aim, the total action for gravity and matter is

$$S = \frac{1}{16\pi G} \int R \sqrt{-g} \text{d}^4x + \int \mathcal{L}_m(g)\sqrt{-g} \text{d}^4x$$  \hspace{1cm} (1)

if we are considering general relativity. On the other hand, actions of the forms

$$S = \int f(R) \sqrt{-g} \text{d}^4x + \int \mathcal{L}_m(g)\sqrt{-g} \text{d}^4x,$$ \hspace{1cm} (2)

$$S = \int \sqrt{-g} \text{d}^4x \left[ F(\Phi) + \frac{1}{2} g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} - W(\Phi) \right]$$

$$+ \int \mathcal{L}_m(g)\sqrt{-g} \text{d}^4x,$$ \hspace{1cm} (3)

where $f(R)$, $F(\Phi)$, $W(\Phi)$ are generic functions of the Ricci scalar $R$ and a scalar field $\Phi$, respectively, can be adopted in view of facing several cosmological problems ranging from inflation to dark energy (see e.g. [25–28] for reviews).

Considering the simplest case (general relativity), the action for the perturbations can be derived from (1) by writing first the metric (in conformal time $\eta$) in Arnowitt–Deser–Moser (ADM) formulation [30]

$$\text{d}s^2 = \left( N^2 - N_iN^i \right) \text{d}\eta^2 - 2N_i \text{d}\eta \text{d}x^i - \gamma_{ij} \text{d}x^i \text{d}x^j,$$ \hspace{1cm} (4)

where the lapse function is

$$N = a(\eta) \left( 1 + \phi - \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} B_{ij} B^{ij} \right),$$ \hspace{1cm} (5)

the shift functions are

$$N_i = a^2(\eta) B_{ij},$$ \hspace{1cm} (6)

the spatial metric is

$$\gamma_{ij} = a^2 (1 - 2 \psi) \delta_{ij} + 2a^2 E_{ij},$$ \hspace{1cm} (7)

and its inverse is

$$\gamma^{ij} = a^{-2} (\delta_{ij} + 2\psi \delta_{ij} - 2E_{ij} + 4E_{il}E_{lj} - 8E_{ij}\psi),$$ \hspace{1cm} (8)

where the convention of summation has been adopted on the repeated lower indices.

The fields $\phi$, $\psi$, $E$ and $B$, introduced in the previous equations, form the tensor $\delta_{ij}$, which represents scalar perturbations of the FLRW spacetime metric. Each of these perturbation terms is gauge dependent, but as shown in [3], linear combinations of these functions and of their derivatives can be taken into account to construct gauge invariant objects.

Writing the gravitational part of the action in ADM form and expressing it in terms of the perturbed metric, we arrive at the equation for a gauge invariant combination of the fields $\phi$, $\psi$, $E$ and $B$. The perturbations have been studied in three cases, i.e. hydrodynamical, scalar field and $f(R)$-gravity.
In all these three cases, the action for the perturbations, described in terms of gauge invariant fields (omitting total derivatives), takes the form

$$ S_{\text{perturb.}} = \frac{1}{2} \int \left( v'^2 - c_s^2 \gamma v^i v^i + \frac{\dot{z}''}{z} v^2 \right) \sqrt{T} \, d^4 x, \quad (9) $$

where $z$ is a time-dependent function. Equation (9) is the action of a scalar field $v$ with a time-dependent mass $m^2(v) = \frac{\dot{z}''}{z}$.

Turning to the Hamiltonian formulation, the conjugate momentum for the field $v$ is

$$ \pi(\eta, x) = \frac{\delta L}{\delta \dot{v}} = v'(\eta, x), \quad (10) $$

and the resulting Hamiltonian is

$$ \mathcal{H} = \frac{1}{2} \int \left( \pi^2 + c_s^2 \gamma v^i v^i - \frac{\dot{z}''}{z} v^2 \right) \sqrt{T} \, d^4 x, \quad (11) $$

where $c_s$ is the speed of sound and

$$ z = \frac{a(K + \mathcal{H}^2 - \mathcal{H}')}{{\mathcal{H} c_s}}. \quad (12) $$

The transition to the quantum formulation is obtained once the variables $v$ and $\pi$ are replaced with the operators $\hat{v}$ and $\hat{\pi}$, satisfying the following commutation relations:

$$ [\hat{v}(\eta, x), \hat{\pi}(\eta, x')] = [\hat{\pi}(\eta, x), \hat{v}(\eta, x')] = 0, \quad (13) $$

$$ [\hat{v}(\eta, x), \hat{\pi}(\eta, x')] = i\delta(x-x'), $$

where the delta function $\delta(x-x')$ is normalized by requiring

$$ \int \sqrt{T} \delta(x-x') \, d^4 x = 1. \quad (14) $$

3. The probability representation for cosmological perturbations

Instead of going into the usual procedure of canonical quantization of the above system, we want to indicate a different approach to quantize the perturbation field in terms of probability distribution functions in the way proposed in [10].

In this formulation of quantum mechanics, marginal distribution functions with classical-like evolution replace the wave functions. The main advantage of this formulation is that a quantum theory can be entirely expressed in terms of observable functions, which are comparable with their classical counterparts.

This approach has already been applied in quantum cosmology with the purpose of studying the evolution of a quantum universe into a classical one and to obtain all the information of the initial quantum cosmological stages from the observations today.

Recently this approach has been extended to quantum field theory [30]. We shall briefly recall the results here.

Let us consider the quantum Hamiltonian for a scalar field in a $d+1$ spacetime

$$ \hat{H} = \int \left[ \frac{1}{2} \hat{\pi}^2 + \frac{1}{2} \sum_{D=1}^{d} (\delta_D \hat{\varphi}(x))^2 + U(\hat{\varphi}(x)) \right] \, d^4 x, \quad (15) $$

and the combination

$$ \hat{\Phi}(x) = \mu(x) \hat{\varphi}(x) + v(x) \hat{\pi}(x). \quad (16) $$

Introducing the quantum characteristic function

$$ \chi(k(x)) = \exp \left( i \int d^4 x k(x) \hat{\Phi}(x) \right), \quad (17) $$

we can define the marginal distribution functional

$$ \mathcal{W}(\hat{\Phi}(x), \mu(x), v(x), t) = \int Dk e^{-i \int d^4 x k(x) \hat{\Phi}(x)} \chi(k), \quad (18) $$

which satisfies the following evolution equation [20]:

$$ \dot{\mathcal{W}}(\hat{\Phi}(x), \mu(x), v(x), t) = \left\{ \int d^4 x \left[ \mu(x) \frac{\delta}{\delta v(x)} + 2v(x) \frac{\delta}{\delta \phi(x)} \right] \Delta \left[ \left( \frac{\delta}{\delta \phi(x)} \right)^{-1} \frac{\delta}{\delta \mu(x)} \right] - \frac{i}{\hbar} \left[ U \left( -\frac{\delta}{\delta \phi(x)} \right)^{-1} \frac{\delta}{\delta \mu(x)} + \frac{i v(x) \hbar}{2} \frac{\delta}{\delta \phi(x)} \right] \right\} \mathcal{W}(\hat{\Phi}(x), \mu(x), v(x), t), \quad (19) $$

where $\Delta f(x) = f(x + \Delta x) - f(x)$ and the operator $(-\delta/\delta \phi(x))^{-1}$ is defined by

$$ \left( \frac{-\delta}{\delta \phi(x)} \right)^{-1} \int Dk e^{-i \int d^4 x k(x) \hat{\Phi}(x)} \right) \hat{\Phi}(x) = \int Dk \left\{ \frac{i}{k(x)} e^{-i \int d^4 x k(x) \hat{\Phi}(x)} \hat{\Phi}(x) \right\}, $$

and the dot represents the time derivative. The above formulas allow one to develop a theory of cosmological perturbations at classical and quantum levels.

4. Classical and quantum cosmological perturbations

In view of Hamiltonian (11) and equation (19), the evolution of cosmological quantum perturbations can be described in terms of a marginal distribution function that satisfies the equation

$$ \dot{\mathcal{W}}(v(x), \mu(x), v(x), t) = \left\{ \int d^4 x \left[ \mu(x) \frac{\delta}{\delta v(x)} + 2v(x) \frac{\delta}{\delta \phi(x)} \right] \Delta \left[ \left( \frac{\delta}{\delta \phi(x)} \right)^{-1} \frac{\delta}{\delta \mu(x)} \right] - \frac{\delta}{\delta \phi(x)} \left( \frac{-\delta}{\delta \phi(x)} \right)^{-1} \frac{\delta}{\delta \mu(x)} \right\} \mathcal{W}(v(x), \mu(x), v(x), t). \quad (20) $$

Because the potential is quadratic, classical perturbations follow a similar evolution equation. The only difference is in the initial conditions, which are restricted by the Heisenberg uncertainty principle for quantum perturbations. A relevant

4 Note that the meaning of the function $\Phi(x)$ is different with respect to the scalar field mentioned in action (3).
expression is also the evolution equation for the Fourier transform of the tomogram
\[ \chi(k, \mu, v, t) = \int dX e^{i kX} V(X, \mu, v, t). \]  
\( \text{(21)} \)

It is
\[ \dot{\chi}(k(x), \mu(x), v(x), t) = \left\{ \int d^4x \left[ \frac{1}{m(x)} \frac{\partial}{\partial v(x)} - 2i\hbar \frac{\delta}{\delta v(x)} \right] \right\} \chi(k(x), \mu(x), v(x), t). \]
\[ \text{(22)} \]

5. The one-mode-parametric quantum oscillator

In the previous section, the Hamiltonian describing perturbations has been constructed and presented in the form of a sum of Hamiltonians of oscillators with time-dependent frequencies. In this section, we will consider in detail the one-mode evolution for the quantum parametric oscillator both in the Schrödinger representation and in the tomographic probability representation. Let us use, for the one-dimensional parametric oscillator, dimensionless units, i.e. with the Planck constant \( \hbar = 1 \), the ‘mass’ of the oscillator \( m = 1 \) and the time-dependent frequency, at the characteristic initial time \( t_0 \), equal to unity, i.e. \( \omega(t_0) = 1 \). The Hamiltonian of the parametric oscillator in these units reads as
\[ \hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2(t) \hat{q}^2}{2}. \]
\( \text{(23)} \)

This system has two integrals of motion, linear in position and momentum \([24, 31]\),
\[ \hat{A}(t) = \frac{i}{\sqrt{2}} (\varepsilon(t) \hat{p} - \hat{\varepsilon}(t) \hat{q}), \]
\( \text{(24)} \)
\[ \hat{A}^\dagger(t) = -\frac{i}{\sqrt{2}} (\varepsilon^*(t) \hat{p} - \hat{\varepsilon}^*(t) \hat{q}). \]
\( \text{(25)} \)

In equation (24), the complex function of time \( \varepsilon(t) \) (where \( t \) can be the conformal time) obeys the classical equation of motion for the oscillator:
\[ \dot{\varepsilon}(t) + \omega^2(t)\varepsilon(t) = 0. \]
\( \text{(26)} \)

The initial conditions
\[ \varepsilon(t_0) = 1, \quad \dot{\varepsilon}(t_0) = i \]
\( \text{(27)} \)

provide the commutation relations of the integrals of motion (24) and (25):
\[ [\hat{A}(t), \hat{A}^\dagger(t)] = 1. \]
\( \text{(28)} \)

For the initial time \( t_0 \) (we shall use the initial time \( t_0 = 0 \)) the integrals of motion coincide with the standard creation and annihilation operators
\[ \hat{A}(t_0) = a, \quad \hat{A}^\dagger(t_0) = a^\dagger, \]
\( \text{(29)} \)

where
\[ \hat{a} = \frac{1}{\sqrt{2}} (\hat{q} + i \hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{q} - i \hat{p}). \]
\( \text{(30)} \)

Here \( \hat{q} \) and \( \hat{p} \) are the position and momentum operators, respectively. The Schrödinger equation
\[ i\dot{\psi}(x, t) = -\frac{1}{2} \frac{\partial^2 \psi(x, t)}{\partial x^2} + \frac{\omega^2(t)x^2}{2} \psi(x, t), \]
\( \text{(31)} \)

has the solution \( \psi_0(x, t) \) that corresponds to the initial vacuum state, obeying the equation
\[ \hat{A}(t) \psi_0(x, t) = 0, \]
\( \text{(32)} \)

with the initial condition
\[ \psi_0(x, t_0) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}, \]
\( \text{(33)} \)

which is the standard ground state of the oscillator obeying the vacuum condition
\[ \hat{a} \psi_0(x, t_0) = 0. \]
\( \text{(34)} \)

The solution \( \psi_0(x, t) \) has the form of a Gaussian wave packet:
\[ \psi_0(x, t) = \frac{1}{\sqrt{\pi t}} e^{(\epsilon(t)x^2)/2}. \]
\( \text{(35)} \)

The Fock states \( \psi_n(x, t) \), which are solutions of the Schrödinger equation (31), are constructed by starting from the vacuum state (35) by standard algebraic formulas using the integrals of motion (24) and (25):
\[ \psi_n(x, t) = \frac{1}{\sqrt{n!}} (\hat{A}^\dagger(t))^n \psi_0(x, t). \]
\( \text{(36)} \)

Solutions (36) have the explicit form
\[ \psi_n(x, t) = \psi_0(x, t) \frac{1}{\sqrt{2^n n!}} \left( \frac{\epsilon(t)}{\epsilon(t)} \right)^{n/2} H_n \left( \frac{x}{|\epsilon(t)|} \right). \]
\( \text{(37)} \)

Here, \( H_n \) are Hermite’s polynomials. There exist Gaussian packets that are squeezed coherent states, which are obtained by means of the Weyl displacement operator acting on the vacuum state, i.e.
\[ \psi_\alpha(x, t) = \hat{D}(\alpha) \psi_0(x, t), \]
\( \text{(38)} \)

where the Weyl system reads as
\[ \hat{D}(\alpha) = \exp(\alpha \hat{A}^\dagger(t) - \alpha^* \hat{A}(t)). \]
\( \text{(39)} \)

Here \( \alpha \) are complex numbers \( \alpha = \alpha_1 + i \alpha_2 \) and the squeezed coherent states have the properties
\[ \int \psi_\alpha^*(x, t) \psi_\beta(x, t) \, dx = \exp \left( -|\alpha|^2 - |\beta|^2 - \frac{1}{2} |\alpha \beta|^2 + \alpha^* \beta \right) \]
\( \text{(40)} \)

and
\[ \frac{1}{\pi} \int \int \psi_\alpha^*(x, t) \psi_\alpha(x', t) \, dx' \, dx = \delta(x - x'). \]
\( \text{(41)} \)
Let us now consider the tomographic probability distribution of the quantum parametric oscillator. The symplectic tomogram of the oscillator quantum vacuum state $\psi_0(x, t)$ is expressed in terms of the wave function

$$W_0(x, \mu, v, t) = \frac{1}{2\pi|v|} \left| \int \psi_0(y, t) e^{i\mu y + i\frac{v}{2} y^2} dy \right|^2. \quad (42)$$

Using the explicit expression of the wave function, we obtain a Gaussian tomographic probability distribution of the form

$$W_0(x, \mu, v, t) = \frac{1}{\sqrt{2\pi \sigma_{\mu v}^2}} \exp \left( -\frac{X^2}{2\sigma_{\mu v}^2} \right), \quad (43)$$

where the dispersion of the random position $X$ depends on the parameters $\mu$ and $v$ and the function $\varepsilon(t)$ as follows:

$$\sigma_{\mu v}^2 = \mu^2 \sigma_{qq} + v^2 \sigma_{pp} + 2\mu v \sigma_{pq}, \quad (44)$$

where

$$\sigma_{qq} = \frac{|\varepsilon(t)|^2}{2}, \quad \sigma_{pp} = \frac{\varepsilon'(t)^2}{2}, \quad \sigma_{pq}^2 = \frac{1}{4} (|\varepsilon(t)\varepsilon'(t)|^2 - 1). \quad (45)$$

The state corresponding to tomogram (43) is the squeezed vacuum state. Depending on $\varepsilon(t)$ and $\varepsilon'(t)$ the fluctuations of position or momentum can be either smaller than 1/2 or larger than 1/2. The state has the position–momentum correlation $\sigma_{pq} \neq 0$ and this correlation satisfies the minimization of the Schrödinger–Robertson uncertainty relation [33, 34]

$$\sigma_{pp} \sigma_{qq} - \sigma_{pq}^2 \geq \frac{1}{4}. \quad (46)$$

The coherent state $\psi_q(x, t)$ also has a Gaussian tomogram

$$W_q(X, \mu, v, t) = \frac{1}{\sqrt{2\pi \sigma_{\mu v}^2}} \exp \left[ -\frac{(X - \bar{X})^2}{2\sigma_{\mu v}^2} \right], \quad (47)$$

where the dispersion $\sigma_{\mu v}^2$ is given by equations (44) and (45) and the mean value reads

$$\bar{X} = \mu \langle \hat{q} \rangle_a + v \langle \hat{p} \rangle_a, \quad (48)$$

where

$$\langle \hat{q} \rangle_a = \sqrt{2} \text{Re} \alpha(t), \quad \langle \hat{p} \rangle_a = \sqrt{2} \text{Im} \alpha(t). \quad (49)$$

Here

$$\alpha(t) = \frac{i}{\sqrt{2}} (\varepsilon(t) \bar{p} - \varepsilon'(t) \bar{q}), \quad (50)$$

$$\alpha^*(t) = -\frac{i}{\sqrt{2}} (\varepsilon(t) \bar{p} - \varepsilon'(t) \bar{q}),$$

and

$$\bar{q} = \sqrt{2} \text{Re} \alpha, \quad \bar{p} = \sqrt{2} \text{Im} \alpha, \quad (51)$$

where $\alpha = \alpha_1 + i\alpha_2$, i.e. $\alpha$ is a constant complex number labeling coherent states $\psi_\alpha(x, t) = \psi_\alpha^*(t)\psi_\alpha(x, t)$.

The probability distributions $W_0(X, \mu, v, t)$ and $W_q(X, \mu, v, t)$ satisfy the kinetic equation

$$\dot{W}(X, \mu, v, t) = -\mu \frac{\partial}{\partial \mu} W(X, \mu, v, t) + \dot{\alpha}^2(t) v \frac{\partial}{\partial \mu} W(X, \mu, v, t) = 0. \quad (52)$$

The kinetic equation corresponds to the classical Liouville equation in classical mechanics, that is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial \mathcal{H}}{\partial q} = 0, \quad (53)$$

where the potential energy $\mathcal{V}(q, t)$ is the energy for the parametric oscillator:

$$\mathcal{V}(q, t) = \frac{1}{2} \dot{q}^2(t) q^2. \quad (54)$$

However, the same equation (52) corresponds to the quantum von Neumann equation for the density operator of the parametric oscillator

$$\frac{1}{i\hbar} \frac{\partial}{\partial t} \rho(t) = [\hat{H}(t), \rho(t)], \quad (55)$$

with $\hbar = 1$, where

$$\hat{H}(t) = \frac{p^2}{2} + \omega_0^2(t) \hat{q}^2. \quad (56)$$

In the tomographic probability representation equations (53) and (55) coincide. The evolution of the tomogram $W(X, \mu, v, t)$ can be expressed in terms of the propagator

$$W(X, \mu, v, t) = \int \Pi(X, \mu, v, X', \mu', v', t) \times W(X', \mu', v', 0) dX' d\mu' dv'. \quad (57)$$

The propagator $\Pi(X, \mu, v, X', \mu', v', t)$ in this integral relation is identical for both classical and quantum evolutions of the one-mode parametric oscillator with a time-dependent frequency. The difference appears in the initial conditions. However, classical tomograms do not have to respect the uncertainty relation

$$\left\{ \int X^2 W(X, \mu, v, t) dX \right|_{\mu=1, v=0} \geq \frac{1}{4}, \quad (58)$$

while the tomogram of a quantum oscillator must satisfy these inequalities. This result means that since the evolution is governed by the same equations, both situations, classical and quantum, are represented. Since the field is the collection of modes, the arguments presented above are applicable to all the fields. This means that for the field tomogram, which is the product of tomograms of all modes, the evolution result is classical. Quantumness is hidden in the initial state, but summing up all states gives, as a result, the classical field. The perturbations described by the Hamiltonian, in the tomographic picture can be discussed using the quantum or the classical propagator of the field since both coincide.

It is straightforward that the Hamiltonian describing the cosmological perturbations is the Hamiltonian of field
harmonic vibrations. It is the natural Hamiltonian of a system whose properties fluctuate around the background state (the FLRW background) and where the fluctuations are small (small vibrations). Deviations from the ‘equilibrium’ background states obey the Hooke law. Only for large deviations from the equilibrium state is the Hooke law violated and strong anharmonicity appears. The ‘spring’ providing the harmonic vibrations of a system with a very large amplitude of vibrations can even break, possibly allowing the system to come out far away. This simple mechanism in the classical description of small vibrations and in the quantum domain can give different results. In fact the ‘ground’ state of the field is the squeezed state of harmonic linear oscillators with time-dependent frequencies. Fluctuations of these vibrations in the classical domain, at zero temperature, are also equal to zero. The small vibrations cannot provide, in this case, large deviations from the equilibrium state even if one takes into account the possible appearance of Hooke’s law violation for large amplitudes of the oscillations. For a quantum domain, even these small vibrations have quantum fluctuations. Thus deviations of the oscillation position \( \hat{\epsilon}(t) \) and of the momentum \( \hat{\epsilon}(t) \) can become large in comparison with the standard vacuum ones and be equal to 1/2, due to the influence of the possible large values of the contributions of \( |\epsilon(t)\rangle \) (or \( |\hat{\epsilon}(t)\rangle \)). This mechanism of quantum fluctuations in the presence of time-dependent frequencies can, for some parts of the system, create large amplitudes that violate the Hooke law. In such a case, the system can give rise to domains of larger densities and non-uniformities, which result as inhomogeneities and anisotropies with respect to the background. This could be a coherent scheme to match quantum microscopic primordial perturbations with the large-scale structure observed today.

6. The multi-mode parametric small perturbations

Let us now consider the most general multi-mode Hamiltonian. This situation is more realistic in order to take into account the problem of cosmological perturbations. As in [5], we consider periodic conditions. Due to these, the Hamiltonian can be taken in the following form:

\[
\mathcal{H}(t) = \frac{1}{2} \vec{Q} B(t) \vec{Q} + \vec{C}(t) \vec{Q}. \tag{59}
\]

Here, we consider \( N \) modes (\( N \) can be equal to \( \infty \)) and the vector operator

\[
\vec{Q} = (\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_N; \hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N). \tag{60}
\]

The system properties are coded by the interaction \( 2N \times 2N \) matrix \( B(t) \), which depends on time \( t \), that is

\[
B(t) = \begin{pmatrix} B_1(t) & B_2(t) \\ B_3(t) & B_4(t) \end{pmatrix}. \tag{61}
\]

The \( N \times N \) block matrices \( B_k(t) \), \( k = 1, 2, 3, 4 \), correspond to quadrature interactions, namely \( B_1(t) \) describes \( \hat{P}_k \)-quadrature interactions and matrices \( B_2(t) \) and \( B_3(t) \) correspond to interaction terms due to \( \hat{q}_k \) and \( \hat{P}_k \) couplings. The \( 2N \)-vector \( \vec{C}(t) \) provides the interaction terms corresponding to homogeneous ‘electric-like’ fields acting on charged particles.

The classical counterpart of this Hamiltonian, in the same quadratic form, is given by a vector (60) composed of classical momenta and positions with standard Poisson brackets. The solution of the Schrödinger equation with first integrals of motion, in position and momenta, determines an inhomogeneous symplectic group element parameterized by \( 2N \times 2N \) matrix \( \Lambda(t) \) and a \( 2N \)-vector \( \Delta(t) \). One has the \( 2N \)-vector \( \vec{I}(t) \) whose components are integrals of motion. The vector reads

\[
\vec{I}(t) = \Lambda(t) \vec{Q} + \Delta(t). \tag{62}
\]

The symplectic matrix \( \Lambda(t) \) satisfies the equation

\[
\dot{\Lambda}(t) = \Lambda(t) \Sigma B(t), \quad \Sigma = \begin{pmatrix} 0 & -1_N \\ 1_N & 0 \end{pmatrix} \tag{63}
\]

and the initial condition

\[
\Lambda(0) = 1_{2N}. \tag{64}
\]

The vector \( \Delta(t) \) satisfies the evolution equation

\[
\dot{\Delta} = \Lambda(t) \Sigma \vec{C}(t) \tag{65}
\]

with initial value \( \Delta(0) = 0 \).

The evolution of the system of field modes with Hamiltonian (59) is described by the Green function of the Schrödinger non-stationary equation determined in terms of matrix \( \Lambda(t) \), vector \( \Delta \) and the Gaussian Green function [24]. It is

\[
G(\vec{x}, \vec{x}', t) = \frac{1}{\sqrt{\text{det}[\Lambda]}} \exp \left[ -\frac{1}{2} \left( \vec{x} \Lambda^{-1} \vec{x} \right) \right. \\
\left. - 2 \vec{x}_1 \Lambda^{-1} \vec{x}_1' \right] + \vec{x}_1 \Lambda^{-1} \vec{x}_1' + 2 \vec{x}_2 \Lambda^{-1} \vec{x}_2' \\
+ \vec{x}_3 \Lambda^{-1} \vec{x}_3' + \vec{x}_4 \Lambda^{-1} \vec{x}_4' \\
\left. - 2 \frac{\dot{\delta}_1(t)}{\dot{\delta}_2(t)} \delta_1(t) \delta_2(t) \right]. \tag{66}
\]

Here, the symplectic matrix \( \Lambda(t) \) as well as the vector \( \Delta \) are in block form:

\[
\Lambda(t) = \begin{pmatrix} \lambda_1(t) & \lambda_2(t) \\ \lambda_3(t) & \lambda_4(t) \end{pmatrix}, \quad \Delta(t) = \begin{pmatrix} \delta_1(t) \\ \delta_2(t) \end{pmatrix}. \tag{67}
\]

This means that, given the initial quantum state at \( t = 0 \), \( \psi(\vec{x}, 0) \) where \( \vec{x} = (x_1, x_2, \ldots, x_N) \), the state at time \( t \) reads

\[
\psi(\vec{x}, t) = \int G(\vec{x}, \vec{x}', t) \psi(\vec{x}', 0) \, d\vec{x}'. \tag{68}
\]

If the initial state is given by a density matrix \( \rho(\vec{x}, \vec{x}', 0) \), the density matrix at time \( t \) reads

\[
\rho(\vec{x}, \vec{x}', t) = \int G(\vec{x}, \vec{y}, t) \rho(\vec{y}, \vec{y}', 0) G^*(\vec{x}', \vec{y}', t) \, d\vec{y} \, d\vec{y}'. \tag{69}
\]

In [4], some initial states are studied. The first is that the initial state is the vacuum state with a Gaussian wave function corresponding to a set of oscillators with different frequencies.
Another possibility is that the initial state is chosen as a thermal density matrix of the set of oscillators at temperature $T$. The density matrix is also Gaussian and the state at $t = 0$ can be chosen as a pure squeezed state with a large, but finite, number of particles with wave functions expressed in terms of Hermite polynomials. The true vacuum state and the thermal state can be considered in both classical and quantum universe evolution pictures. The squeezed Fock states of the evolving universe correspond to the quantum domain only because the states cannot be realized as an initial state in the classical domain. Below we focus on the vacuum initial state to show, in tomographic probability representation, its evolution both in classical and quantum pictures.

The quantum tomogram $\mathcal{W}(\vec{X}, \vec{\mu}, \vec{v}, t)$, where $\vec{X} = (X_1, X_2, \ldots, X_N)$, $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$, $\vec{v} = (v_1, v_2, \ldots, v_N)$, which is a joint probability density of random variables $X_k, k = 1, 2, \ldots, N$, is expressed in terms of the wave function

$$\mathcal{W}(\vec{X}, \vec{\mu}, \vec{v}, t) = \frac{1}{\prod_{k=1}^{N} 2\pi |v_k|} \left| \int \psi(\vec{y}, t) \right|^2 \times \exp \left\{ i \sum_{k=1}^{N} \left( \frac{\mu_k}{2v_k} \frac{X_k}{v_k} - \frac{X_k}{v_k} \right) \right\} \, d\vec{y}. \tag{70}$$

The Wigner function of the state is expressed as the wave function

$$W(\vec{q}, \vec{p}, t) = \int \psi^*(\vec{q} + \frac{\vec{p}}{2}, t) \psi(\vec{q} - \frac{\vec{p}}{2}, t) e^{-i\vec{p} \cdot \vec{q}} \, d\vec{q}. \tag{71}$$

Tomogram (70) is connected with the above Wigner function by the Radon transform

$$\mathcal{W}(\vec{X}, \vec{\mu}, \vec{v}, t) = \int W(\vec{q}, \vec{p}, t) \times \prod_{k=1}^{N} \left[ \delta \left( X_k - \mu_k q_k - v_k p_k \right) \frac{d q_k d p_k}{2\pi} \right]. \tag{72}$$

The density matrix in position representation satisfies the von Neumann equation

$$i \frac{\partial \rho(\vec{x}, \vec{x}', t)}{\partial t} = \frac{1}{2} \left[ \hat{Q}_x B(t) \hat{Q}_x^\dagger \rho(\vec{x}, \vec{x}', t) + \hat{C}(t) \hat{Q}_x \rho(\vec{x}, \vec{x}', t) \right] - \frac{1}{2} \left[ \hat{Q}_x B(t) \hat{Q}_x^\dagger \rho(\vec{x}, \vec{x}', t) - \hat{C}(t) \hat{Q}_x \rho(\vec{x}, \vec{x}', t) \right]. \tag{73}$$

Here the operators $\hat{Q}_x$ and $\hat{Q}_x^\dagger$ are given by (60) where $\hat{P}_{\vec{k}} \rightarrow -i \frac{\partial}{\partial k}$, $\hat{q}_{\vec{k}} \rightarrow x_k$ and $\hat{P}_{\vec{k}} \rightarrow -i \frac{\partial}{\partial k}$, $\hat{q}_{\vec{k}} \rightarrow x'_k$ respectively. The equation for the Wigner function (the Moyal equation, see [32]) is given by (73) with the replacements

$$\begin{align*}
\frac{\partial}{\partial x_k} &\rightarrow \frac{1}{2} \frac{\partial}{\partial q_k} + ip_k, \quad \frac{\partial}{\partial x'_k} \rightarrow \frac{1}{2} \frac{\partial}{\partial q_k} - ip_k \\
x_k &\rightarrow q_k + \frac{i}{2} \frac{\partial}{\partial p_k}, \quad x'_k \rightarrow q_k - \frac{i}{2} \frac{\partial}{\partial p_k}.
\end{align*} \tag{74}$$

The evolution equation for tomogram (70) or (72) is obtained from the equation for the Wigner function by the replacements

$$W(\vec{q}, \vec{p}, t) \rightarrow \mathcal{W}(\vec{X}, \vec{\mu}, \vec{v}, t), \quad \frac{\partial}{\partial q_k} \rightarrow \mu_k \frac{\partial}{\partial X_k}, \quad \frac{\partial}{\partial P_k} \rightarrow \frac{\partial}{\partial X_k}, \quad q_k \rightarrow -\frac{\partial}{\partial \mu_k} \left( \frac{\partial}{\partial X_k} \right)^{-1}, \quad P_k \rightarrow -\frac{\partial}{\partial \nu_k} \left( \frac{\partial}{\partial X_k} \right)^{-1}. \tag{75}$$

It is important to point out that the equation for the Wigner function and for the tomogram will coincide for both domains, quantum and classical, with the equations for probability densities $f(\vec{q}, \vec{p}, t)$ satisfying the Liouville equation and for the quantum and the classical tomogram

$$\mathcal{W}(\vec{X}, \vec{\mu}, \vec{v}, t) = \int f(\vec{q}, \vec{p}, t) \times \prod_{k=1}^{N} \left[ \delta \left( X_k - \mu_k q_k - v_k p_k \right) \frac{d q_k d p_k}{2\pi} \right]. \tag{76}$$

respectively.

The tomogram is the probability density. Hence one can introduce the Shannon entropy [16] associated with the probability density. The formula for the entropy reads as

$$H_1(\vec{\theta}) = -\int \mathcal{W}(\vec{X}, \vec{\mu}, \vec{v}) \ln \mathcal{W}(\vec{X}, \vec{\mu}, \vec{v}) \, d\vec{X}, \tag{77}$$

$$\mu_k = x_k \cos \theta_k, \quad v_k = x_k^{-1} \sin \theta_k. \tag{79}$$

This means that for evolution of the Wigner function $W(\vec{q}, \vec{p}, t)$, in the quantum domain, and the probability distribution $f(\vec{q}, \vec{p}, t)$, in the classical domain, the propagators are identical. It is a property of systems with Hamiltonians of the form (59) and corresponds to the Ehrenfest theorem.

Thus, given the evolution of the tomogram in two states, the evolution of entropy can be defined as

$$H(\vec{\theta}, t) = H_1(\vec{\theta}, t) + H_2(\vec{\theta}, t) \tag{78}$$

where

$$\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_N) \tag{79}$$

and

$$H_2(\vec{\theta}) = H_1 \left( \vec{\theta} + \frac{\vec{\theta}}{2} \right), \quad \vec{\theta} + \frac{\vec{\theta}}{2} = \left\{ \theta_k + \frac{\pi}{2} \right\}. \tag{80}$$

One can introduce the sum entropy

$$H(\vec{\theta}, t) = H(\vec{\theta}, t) + H \left( \vec{\theta} + \frac{\vec{\theta}}{2} \right), \tag{81}$$

where

$$\vec{\theta} + \frac{\vec{\theta}}{2} = \left\{ \theta_k + \frac{\pi}{2} \right\}. \tag{82}$$

The entropy $H(\vec{\theta}, t)$ satisfies the inequality, see e.g. [16],

$$H(\vec{\theta}, t) \geq N \ln \pi e. \tag{83}$$

For the initial ground state of the universe, one has the saturation of the above inequality, i.e.

$$H(\vec{\theta}) = N \ln \pi e. \tag{84}$$


In the quantum domain, the entropy $\hat{H}(\tilde{\theta})$ cannot be less than the value $N \ln \pi e$. In the classical domain, the entropy $\hat{H}(\tilde{\theta})$ characterizes the order or disorder in the field state. In the process of evolution of the universe, this entropy is changing. The tomograms for classical states are probability distributions evolving for the quadratic Hamiltonian (59) identically. If one measures the tomogram at the late time $t$, corresponding to the classical epoch, one can learn what was the initial state, since at this period of time the tomogram corresponding to the classical state provides the possibility of calculating the probability density $f(\vec{q}, \vec{p}, t)$, using the inverse Radon transform

$$f(\vec{q}, \vec{p}, t) = \frac{1}{(2\pi)^N} \int W(\vec{X}, \vec{\mu}, \vec{\nu}, t)e^{\sum_{k=1}^{N} \left( X_k - \mu_k q_k - \nu_k p_k \right)} d\vec{X} \prod_{k=1}^{N} d\mu_k d\nu_k.$$  

(85)

In principle, this density can be used to describe the distribution of matter in galaxies [35] and in clusters of galaxies [36]. Besides, one can introduce another characterization of the universe state in terms of the tomogram.

7. Conclusions

To conclude, we point out the main results of the paper. We have considered classical and quantum perturbations under the same tomographic standard. The small vibration Hamiltonian with time-dependent parameters can give an account of the basic model that can be adopted in the quantum and in the classical regime. Specifically, the classical and quantum descriptions can be associated with the tomographic probability distributions. The classical initial states (classical tomograms) and quantum initial states (quantum tomograms) have different properties because the classical tomographic entropies can violate the quantum bound of the tomographic entropy. The tomographic entropy can be estimated at the present epoch by measuring the space and momentum distributions of the matter in the universe. Since the propagator for both classical and quantum tomograms of the universe states is the same, one can trace back, in principle, the entropy data to the initial state of the universe. The observed data, extrapolated to the initial state by means of the classical tomogram, can be less than the quantum bound. The existence of bound property can be used to discriminate between cosmological quantum and classical behavior.

The physical meaning of the entropy which we have considered is that it represents the sum of two entropies. One corresponds to the spatial probability density of matter. Then, we have to assume $\dot{\theta} = 0$. The other corresponds to the momentum distribution. For this case, we have to assume the value $\tilde{\theta} = \frac{1}{T}$. By measuring these distributions at the present epoch, one can trace back the cosmological evolution to entropy initial values corresponding to small vibrations. In other words, the tomographic description of the universe provides the possibility of correlating classical and quantum cosmological perturbations in the same unitary scheme.

The final goal of this picture is to find a dynamical and self-consistent approach capable of connecting the primordial quantum perturbations to the large-scale structure observed today [36]. In a forthcoming paper, we will discuss the above results considering data from the observations and their relations with initial values of cosmological parameters.

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