Bounds on Entanglement Catalysts

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Given a finite dimensional pure state transformation restricted by entanglement assisted local operations and classical communication (ELOCC), we derive minimum and maximum bounds on the entanglement of an ancillary catalyst that allows that transformation. These bounds are non-trivial even when the Schmidt number of both the original and ancillary states becomes large. Along with these bounds, we present further constraints on ELOCC transformations by identifying restrictions on the Schmidt coefficients of the target state. In addition, an example showing the existence of qubit ELOCC transformations with multiple ranges of potential ancillary states is provided. This example reveals some additional difficulty in finding strict bounds on ELOCC transformations, even in the qubit case. Finally, a comparison of the bounds in this paper with previously discovered bounds is presented.

I. INTRODUCTION

In recent years, entanglement has been identified as a valuable resource that has proven to be integral for use in multi-party quantum information protocols. Formally, entanglement can be defined as the resource which allows parties to overcome the limitations imposed by local operations and classical communication (LOCC) [1, 2]. Often entanglement is not available in its most pure form. Rather, a quantum system may be partially entangled and given in a form that is mixed with noise. Consequently, such systems may not be optimal for certain quantum information tasks. One can conclude that certain quantum states are more desirable than others depending on the objective at hand. For example, one may desire a Bell state for the most efficient use of a quantum teleportation protocol. One important effect of LOCC is the conversion from one bipartite pure state to another. Such transformations involve the consumption of shared entanglement between parties such that the output system is less entangled than the input system. The precise characterization of these types of transformations is stated in Nielsen’s Theorem [3].

Nielsen’s Theorem is deeply rooted in the theory of majorization. Suppose that $p$ and $q$ are two vectors in $\mathbb{R}^d$. Then $p$ is said to be majorized by $q$, denoted $p \prec q$, if:

$$\sum_{i=1}^{l} p_i^k \leq \sum_{i=1}^{l} q_i^k \quad \forall l \in \{1, 2, \ldots, d\} \quad (1)$$

with equality when $l = d$. Here, $p_i^k$ is the $i$-th element of the vector $p$, and $q_i^k$ is the $i$-th element of the vector $q$. Majorization can be thought of as a heat engine undergoing a cyclic process that is returned to its original form when the thermal process has concluded [7]. Yet, these conditions only work for a subset of catalytic examples. It is desirable to obtain a thorough understanding of ELOCC transformations as they play a vital role in quantum thermodynamics where a catalyst can be thought of as a heat engine undergoing a cyclic process that is returned to its original form when the thermal process has concluded [8]. In the case of thermal majorization [9], it is thermodynamically superior to be an entanglement catalyst rather than entanglement, however the underlying principles governing thermal state transformations remain consistent with ELOCC transformations.

The trouble with ELOCC transformations is that there is still no way to fully classify the properties of ancillary entangled states. Necessary and sufficient conditions have recently been outlined on the existence of a catalyst state for any given transformation using the Renyi entropies and power means [5, 6]. However, these results did not present any conditions on the catalyst state itself, only on the existence of such a state. Furthermore, conditions found on catalyst states including the catalyst dimension have been presented in [7]. It is desirable to obtain a thorough understanding of ELOCC transformations as they play a vital role in quantum thermodynamics where a catalyst can be thought of as a heat engine undergoing a cyclic process that is returned to its original form when the thermal process has concluded [5]. In the case of thermal majorization [9], it is thermodynamically superior to be an entanglement catalyst rather than entanglement, however the underlying principles governing thermal state transformations remain consistent with ELOCC transformations.

It has also been shown that any entangled target state can be embezzled (up to a small amount of error) using only a family of bipartite catalysts [10]. Unfortu-
nately, embezzling with small error requires the catalyst Schmidt number to diverge to infinity which limits its utility. This paper will focus on finite dimensional ELOCC transformations in which embezzling is not possible with a high degree of accuracy. Specifically, bounds limiting the amount of entanglement a potential catalyst state can contain for any pure state transformation will be shown. These bounds restrict the values of specific Schmidt coefficients of the catalyst state and depend on Schmidt coefficients of the initial and target states. The bounds on catalyst entanglement are supplemented with additional conditions on the target state that must be satisfied for an ELOCC transformation to be possible. In addition, the existence of transformations with multiple regions of qubit catalysis will be shown, which impose challenges for finding further bounds on ELOCC transformations, even in the qubit case. Finally, a brief comparison between the new bounds proved in this paper with the bounds presented in [2] will be shown.

II. BOUNDS ON ENTANGLEMENT CATALYSTS

There are three important properties of ELOCC transformations that will be used extensively in this paper. All three properties were originally proved by [2]. For the remainder of the paper, we assume that $p, q \in \mathbb{R}^d$ and $r \in \mathbb{R}^k$ are all arranged in non-increasing order.

Property 1: For two incomparable Schmidt vectors $p, q \in \mathbb{R}^d$ and some catalyst vector $r \in \mathbb{R}^k$ such that $p \otimes r \preceq q \otimes r$, the largest element of $p$ is always smaller than the largest element of $q$ ($p_1 \leq q_1$) and the smallest element of $p$ is always larger than the smallest element of $q$ ($p_d \geq q_d$). In addition,

$$\sum_{i=1}^{d-1} p_i \leq \sum_{i=1}^{d-1} q_i$$

(2)

Property 2: When $d = 2$, either $p \preceq q$ or $q \preceq p$, which makes borrowing a catalyst irrelevant. If $p, q \in \mathbb{R}^d$ are incomparable Schmidt vectors and $d = 3$, then no catalyst vector $r$ exists such that $p \otimes r \preceq q \otimes r$.

Property 3: No transformation can be catalysed by the maximally entangled state, $r = (1/k, 1/k, \ldots, 1/k)$.

A. Bounds on Minimum and Maximum Entanglement of Catalyst States

This section outlines bounds that quantify the amount of entanglement a catalyst state must contain in order to catalyse an incomparable state transformation.

Let $p$ and $q$ be incomparable Schmidt vectors. Then there exists at least one $l \in \{1, 2, \ldots, d\}$ such that $\sum_{i=1}^{l-1} p_i > \sum_{i=1}^{l-1} q_i$, thus violating the majorization criterion \(\mathbb{1}\). Denote $m$ as the smallest $l$ such that $\sum_{i=1}^{l} p_i > \sum_{i=1}^{l} q_i$. That is, define the set of all $l$ values such that $\sum_{i=1}^{l} p_i > \sum_{i=1}^{l} q_i$ to be $\mathcal{L}$:

$$\mathcal{L} = \left\{ l \in \{1, 2 \cdot \cdot \cdot d\} \mid \sum_{i=1}^{l} p_i - q_i > 0 \right\}$$

(3)

Then $m \equiv \min(\mathcal{L})$. Due to property 1, $m \neq 1, d - 1$. Additionally, $m \neq d$ since $\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} q_i = 1$. Since $p_1 \leq q_1$, $m$ can be thought of as the minimum $l$ causing $p$ and $q$ to be incomparable.

Suppose that the Schmidt vector $r$ is the product state $(1, 0, \ldots, 0)$. Then $r$ cannot catalyse the transformation because the non-zero elements of $p \otimes r$, and $q \otimes r$ are identical to $p$ and $q$ respectively. Similarly, if $r$ is the maximally entangled state $(1/k, 1/k, \ldots, 1/k)$, then by property 3, $r$ cannot catalyse the transformation. The question that arises is: How far can one deviate $r$ from the product state or the maximally entangled state before an ELOCC transformation becomes possible? This question is resolved in Theorem 1.

Theorem 1: For any incomparable Schmidt vectors $p, q \in \mathbb{R}^d$ and for any Schmidt vector $r \in \mathbb{R}^k$, if $p \otimes r \preceq q \otimes r$, then $r$ must satisfy:

$$\frac{r_1}{r_2} < \frac{q_1}{q_m}$$

(4)

$$\frac{r_1}{r_k} > \frac{p_m}{q_m}$$

(5)

Remark 1: To see the symmetry between bounds (4) and (6), (4) can be written as:

$$\frac{r_1}{r_2} < \max\left(\frac{p_1}{p_m}, \frac{q_1}{q_m}\right)$$

(6)

It will be proven that $\frac{q_2}{q_m} > \frac{p_1}{p_m}$ which reduces (6) to (4). There is no such simplification for (5) except for when $d = 4$ as seen in the corollary of the theorem.

Remark 2: It appears that bound (4) is only determined by the Schmidt vector $q$, however $q_m$ is identified using both the Schmidt vectors $p$ and $q$ through the definition of $m$. Thus, (4) is not independent of $p$.

Proof: Let $p$ and $q$ be incomparable Schmidt vectors and let $p \otimes r \preceq q \otimes r$ for some catalyst Schmidt vector $r$. To begin, we prove that:

$$\frac{p_1}{p_m} < \frac{q_1}{q_m}$$

(7)

From the definition of $m$, we get:

$$\sum_{i=1}^{m} p_i > \sum_{i=1}^{m} q_i$$

and

$$\sum_{i=1}^{m-1} p_i \leq \sum_{i=1}^{m-1} q_i$$

(8)
Therefore:
\[ \sum_{i=1}^{m} p_i = \sum_{i=1}^{m-1} p_i + p_m > \sum_{i=1}^{m-1} q_i + q_m = \sum_{i=1}^{m} q_i \]  
(9)

This implies \( p_m > q_m \). Because \( p \otimes r < q \otimes r \), it holds that \( p_1 \leq q_1 \) by property 1. Combining \( p_1 \leq q_1 \) and \( p_m > q_m \), we get:
\[ \frac{p_1}{p_m} < \frac{q_1}{q_m} \]

Thus, we have proved the claim stated in Remark 1. We are now ready to prove bound [4]. We use proof by contradiction. Let:
\[ r_1 \geq \frac{q_1}{q_m} \]
(10)

Thus, \( q_m r_1 \geq q_1 r_2 \). This implies that \( q_2 r_1 \geq q_1 r_2 \) \( \forall x \leq m \). The vector \( q \otimes r \) can therefore be partially ordered as follows:
\[ q_1 r_1 \geq q_2 r_1 \geq \cdots \geq q_m r_1 \geq q_1 r_2 \]
(11)

From (7) and (10), we get:
\[ \frac{r_1}{r_2} \geq \frac{q_1}{q_m} > \frac{p_1}{p_m} \]
(12)

Thus, \( p_m r_1 \geq p_1 r_2 \). This implies that \( p_1 r_1 \geq p_1 r_2 \) \( \forall x \leq m \). The vector \( p \otimes r \) can therefore be partially ordered as follows:
\[ p_1 r_1 \geq p_2 r_1 \geq \cdots \geq p_m r_1 \geq p_1 r_2 \]
(13)

The elements \( p_1 r_1 \) and \( q_1 r_1 \) are the largest in the ordered vectors \( (p \otimes r)^j \) and \( (q \otimes r)^j \) respectively. Thus, the first \( m \) elements of \( (p \otimes r)^j \) and \( (q \otimes r)^j \) are identical to the case when \( r \) is the product state. From (11) and (13), we have:
\[ \sum_{j=1}^{m} (p \otimes r)^j = \sum_{j=1}^{m} p_j > \sum_{j=1}^{m} q_j = \sum_{j=1}^{m} (q \otimes r)^j \]
(14)

Where we used the fact that \( \sum_{j=1}^{m} p_j > \sum_{j=1}^{m} q_j \) from the definition of \( m \). From (14) and because \( p_1 \leq q_1 \) implies \( p_1 r_1 \leq q_1 r_1 \), we conclude \( p \otimes r \not\preceq q \otimes r \), which contradicts the assumption that \( p \otimes r \prec q \otimes r \). This completes the derivation of bound [4].

We will now prove bound [3]. Again, we use proof by contradiction. Let:
\[ \frac{r_1}{r_k} \leq \min \left( \frac{p_m}{p_{m+1}}, \frac{q_m}{q_{m+1}} \right) \]
(15)

**Case 1:** Let:
\[ \frac{r_1}{r_k} \leq \frac{p_m}{p_{m+1}} \leq \frac{q_m}{q_{m+1}} \]
(16)

Therefore, \( p_{m+1} r_1 \leq p_m r_k \) which implies that the first \( mk \) largest elements of \( p \otimes r \) are given by \( \{p_x r_y\} \) with \( 1 \leq x \leq m \) and \( 1 \leq y \leq k \). Similarly, from (16) we get that \( q_{m+1} r_1 \leq q_m r_k \) which implies that the first \( mk \) largest elements of \( q \otimes r \) are given by \( \{q_x r_y\} \) with \( 1 \leq x \leq m \) and \( 1 \leq y \leq k \). The result is that the first \( km \) elements of \( (p \otimes r)^j \) and \( (q \otimes r)^j \) are identical to the case when \( r \) is the maximally entangled state. It must follow that:
\[ \sum_{i=1}^{km} (p \otimes r)^j_i = \sum_{j=1}^{m} (\sum_{j=1}^{k} p_j) (\sum_{h=1}^{m} r_h) = \sum_{j=1}^{m} p_j \]
\[ > \sum_{j=1}^{m} q_j = \sum_{j=1}^{m} (\sum_{j=1}^{k} q_j) (\sum_{h=1}^{m} r_h) \]
(17)

\[ = \sum_{i=1}^{km} (q \otimes r)^j_i \]

Where we used the fact that \( \sum_{h=1}^{m} r_h = 1 \) and that \( \sum_{j=1}^{m} q_j > \sum_{j=1}^{m} p_j \) from the definition of \( m \). From (17) and because \( p_1 \leq q_1 \) implies \( p_1 r_1 \leq q_1 r_1 \), we can conclude \( p \otimes r \not\preceq q \otimes r \), which contradicts the assumption that \( p \otimes r \prec q \otimes r \). Therefore, we have derived \( \frac{r_1}{r_k} > \frac{p_m}{p_{m+1}} \) by contradiction.

**Case 2:** Let:
\[ \frac{r_1}{r_k} \leq \frac{q_m}{q_{m+1}} < \frac{p_m}{p_{m+1}} \]
(18)

Analogously to case 1, we can again state that the first \( mk \) largest elements of \( p \otimes r \) are given by \( \{p_x r_y\} \) and the first \( mk \) largest elements of \( q \otimes r \) are given by \( \{q_x r_y\} \) with \( 1 \leq x \leq m \) and \( 1 \leq y \leq k \). Therefore, (17) is again derived and we conclude that \( \frac{r_1}{r_k} > \frac{q_m}{q_{m+1}} \) by contradiction.

From both case 1 and case 2, we have determined that:
\[ \frac{r_1}{r_k} > \frac{p_m}{p_{m+1}} \quad \text{and} \quad \frac{r_1}{r_k} > \frac{q_m}{q_{m+1}} \]
(19)

This is equivalent to:
\[ \frac{r_1}{r_k} > \min \left( \frac{p_m}{p_{m+1}}, \frac{q_m}{q_{m+1}} \right) \]
(20)

Thus, we have derived bound [3]. This completes the proof.

Note that if \( q_1 = q_m \), then there is no Schmidt vector \( r \) that catalyses the transformation. This follows since \( r_1 \geq r_2 \) and \( q_1 \geq q_m \) by definition. If \( q_1 = q_m \), then bound (4) states that \( \frac{r_1}{r_k} < 1 \) implying \( r_1 < r_2 \). This contradicts \( r_1 \geq r_2 \). Thus, \( q_1 \neq q_m \) if a catalyst is to exist for the transformation. This condition is strongest when \( d = 4 \) because \( m \neq 1, d - 1, d \) which implies that \( m = 2 \). Specifically, it states that if \( d = 4 \) and \( q_1 = q_2 \), then \( r \) cannot catalyse the transformation.

**Corollary:** If \( d = 4 \), then (5) reduces to:
\[ \frac{r_1}{r_k} > \frac{q_2}{q_3} \]
(21)
Proof: Let \( d = 4 \). Then it must be that \( m = 2 \) since \( m \neq 1, d-1, d \). We have proved that \( p_m > q_m \) which means \( p_2 > q_2 \). It remains to prove that \( q_3 > p_3 \). We use proof by contradiction. Let \( q_3 \leq p_3 \), then:

\[
q_3 = \sum_{i=1}^{3} q_i - \sum_{i=1}^{2} p_i \leq \sum_{i=1}^{3} p_i - \sum_{i=1}^{2} p_i = p_3
\]

\[
\Rightarrow \sum_{i=1}^{3} (q_i - p_i) \leq \sum_{i=1}^{3} (q_i - p_i)
\]

Since \( \sum_{i=1}^{3} (q_i - p_i) \geq 0 \) by property 1 and \( \sum_{i=1}^{2} (q_i - p_i) < 0 \) by the definition of \( m \), (22) is contradicted. Thus, \( q_3 > p_3 \). Because \( p_2 > q_2 \) and \( q_3 > p_3 \), it must hold that \( p_2 q_3 > q_2 p_3 \) which implies \( l_p > l_q \). Thus, (5) simplifies to \( \frac{r_k}{r_k} > \frac{p_2}{q_3} \). This completes the proof.

Bound (4) limits how similar to the product state the probability distribution of \( r \) can be. If the ratio between the first and second element of \( r \) is too large, then \( r \) is not entangled enough to catalyse the transformation. In other words, (4) represents the minimum entanglement \( r \) must contain in order to facilitate the transformation. Bound (5) limits how flat (how close to the maximally entangled state) the probability distribution of \( r \) can be. It states that if the ratio between the first element and last element of \( r \) is too close to one, then \( r \) is too entangled to catalyse the transformation. Thus, this bound represents the maximum entanglement a catalyst vector can contain and still potentially catalyse the transformation.

III. EXAMPLES

In this section, the maximum probability of a state transformation and the majorization distance will be harnessed to observe specific examples of ELOCC transformations that emphasize the utility of bounds (4) and (5). Specifically, we begin with a simple example in the qubit catalyst case that shows bounds (4) and (5) limiting the set of potential catalysts. We then prove the existence of ELOCC state transformations with more than one distinct region of potential qubit catalysts facilitating them. This indicates a difficulty of deriving further bounds on ELOCC transformations. Finally, a higher dimensional (non-qubit) transformation will be presented providing a comparison of the bounds shown in (7) with the bounds on minimum and maximum entanglement derived in this paper.

A. Maximum Probability of Transformation and Majorization Distance

For any two incomparable bipartite states \(|\psi\rangle\) and \(|\phi\rangle\), there exists a maximum probability (< 1) that the transformation \(|\psi\rangle \rightarrow |\phi\rangle\) is achieved. The maximum probability of a pure bipartite state transformation was first proved by [11]. We wish to observe how the maximum probability of an incomparable transformation varies when a qubit Schmidt vector \( r = (1 - t, t) \), \( t \in [0, 1/2] \) is borrowed to catalyse the transformation. The modified maximum probability of the product state transformation is:

\[
P_{max}(|\psi\rangle \otimes |\chi\rangle) = \min_{t' \in \{1, 2, \ldots, 2d\}} \frac{E_{l'}(|\psi\rangle \otimes |\chi\rangle)}{E_{l'}(|\phi\rangle \otimes |\chi\rangle)}
\]

where \( E_{l'}(|\psi\rangle \otimes |\chi\rangle) = 1 - \sum_{i=1}^{l'-1} (p \otimes r)^i_1 \). When \( t' = 1 \), both sums in (23) are zero, making \( P_{max}(t) = 1 \). Thus, the maximum probability of a transformation never exceeds one.

In addition to the maximum probability, the majorization distance provides a measure of how close a vector \( q \) is to majorizing another vector \( p \) and was first defined by [12] in the context of approximate majorization. If \( r \) is a qubit Schmidt vector, then analogously to the maximum probability, we can observe how the majorization distance varies when \( r \) is borrowed to catalyse the transformation. The modified majorization distance of the product state transformation is:

\[
\delta(t) = 2 \max_{t' \in \{1, \ldots, 2d\}} \sum_{i=1}^{l'} ((p \otimes r)^i_1 - (q \otimes r)^i_1)
\]

Since both (23) and (24) are functions of \( t \), we can plot \( P_{max}(t) \) and \( \delta(t) \) for \( t \in [0, 1/2] \). Both the maximum probability and the majorization distance reveal which values of \( t \) make the Schmidt vector \( r \) an effective catalyst for the transformation. Specifically, when \( P_{max}(t) = 1 \), or alternatively, when \( \delta(t) = 0 \), \( r \) is an effective catalyst for the transformation. Both of these functions will be used to observe specific examples in subsequent sections.

B. A Simple Qubit Example

In the qubit catalyst case, bounds (4) and (5) produce direct upper and lower bounds on potential ancillary states because \( r_1/r_2 = r_1/r_k \). Specifically, if it is assumed that \( r = (1 - t, t) \), where \( t \in [0, 1/2] \), then:

\[
\frac{1}{q_m} + 1 < t < \frac{1}{\min \left( \frac{p_m}{p_m+1}, \frac{q_m}{q_m+1} \right)} + 1
\]

For example, consider the vectors:

\[
p = (0.450775, 0.347115, 0.123023, 0.079087)
q = (0.558785, 0.212395, 0.170259, 0.058561)
\]

Clearly these vectors are incomparable since

\[
0.450775 < 0.558785 \text{ but } 0.450775 + 0.347115 > 0.558785 + 0.212395. \text{ From (25), we see that}
0.275416 < t < 0.444942. \text{ This region of potential catalysis can be visualized by plotting the maximum}
\]
FIG. 1. (color online) The maximum probability of transformation (top) and the majorization distance (bottom) plotted against $t$. The region in which $P_{\text{max}}(t) = 1$ and $\delta(t) = 0$ is the region of catalysis. The vertical solid green lines bound the exact region of qubit catalysis for this transformation. The left and right vertical red dashed lines represent the bounds on minimum and maximum catalyst entanglement, respectively.

C. Multiple Regions of Catalysis

In this section, the existence of a new class of catalytic state transformations is proven. Namely, the existence of incomparable vectors with multiple distinct ranges of potential qubit catalysts is shown. Consider the intervals $I_1 = [a, b]$, $I_2 = (b, c)$ and $I_3 = [c, d]$, where $0 \leq a < b < c < d \leq 1/2$. It will be shown that there exists $a$, $b$, $c$, $d$ and incomparable Schmidt vectors $p$ and $q$ such that when $t \in I_1$ or $t \in I_3$, $r = (1 - t, t)$ is an effective catalyst for the state transformation, while when $t \in I_2$, $r$ is not an effective catalyst for the transformation. That is, there are two distinct regions of qubit catalysts that will do the job.

For example, let

$$p = (0.487318, 0.304082, 0.129458, 0.0647176, 0.0014244)$$

$$q = (0.558879, 0.250609, 0.100547, 0.0888719, 0.0010931)$$

The bounds on qubit catalyst entanglement (25) state that $0.152477 < t < 0.469182$. FIG. 2 shows the maximum probability of the transformation and the majorization distance as functions of $t$ for this example. It is remarkable to see that when $t = 0.2$ and when $t = 0.4$, $r$ is an effective catalyst for the transformation, while when $t = 0.3$ it does not!

The discovery of this class of incomparable state transformations has a few important implications. Namely, the existence of multiple regions of catalysis makes it much harder to find exact bounds on qubit ELOCC transformations. No matter how close the
bounds given by (25) are to the minimal or maximal effective catalyst, it is not guaranteed that all \( r \) that lie within the bounded region catalyse the transformation. There may be additional gaps (such as in FIG. 2) where catalysis does not occur. In order to fully characterise ELOCC transformations even in the qubit case, one would have to find conditions on \( p \) and \( q \) to which multiple regions of catalysis exist. More specifically, one would have to find bounds that accurately disregard incomparable regions surrounded by regions in which the catalyst is effective. Nevertheless, the bounds presented in this paper substantially restrict the domain of catalysts that need to be considered for any given transformation.

In general, it appears that there are no incomparable four-dimensional vectors \( p \) and \( q \) that have multiple regions of qubit catalysis like the example shown in FIG. 2. Moreover, state transformations are not limited to only two regions of catalysis. Examples have been found where there are three or more regions of potential catalysts. It is conjectured that there is no limit on the maximum number of distinct regions of potential qubit catalysts allowing a particular ELOCC transformation. The likelihood of multiple regions appears to rise as the dimension of the Schmidt vectors \( p \) and \( q \) becomes large. Thus, there is still more work to be done to fully understand ELOCC transformations of this kind.

D. A Higher Dimensional Example

We will now show a higher dimensional (non-qubit) catalytic example demonstrating that (4) and (5) are able to impose conditions on incomparable state transformations that previous bounds, first discovered by [7], could not. This example will demonstrate the utility of the bounds (4) and (5), even when the minimum catalyst dimension is unknown.

The bounds in [7] were discovered using the class of Schur-convex/concave functions. Any real valued function \( f \) is said to be Schur-convex if \( p < q \) implies \( f(p) \leq f(q) \). The function \( f \) is said to be Schur-concave if the inequality is flipped.

By exploiting the Schur convexity/concavity of the elementary and power sum symmetric polynomials using Newton’s identities [13], a bound on the minimum dimension \( r \) must be in order to catalyse a particular incomparable transformation was presented in [7]. Namely it was found that:

\[
k \geq \frac{\log_2(e_d-1(q)) - \log_2(e_d-1(p))}{\log_2(e_d(p)) - \log_2(e_d(q))} + 1 \quad (28)
\]

Where \( e_j(p), j \in \{0, 1, \ldots, d\} \) is the \( j \)th elementary symmetric polynomial of \( p \). (28) is only non-trivial when the logarithmic term is larger than one (when \( k \geq 2 \)). In addition to (28), the following bound was also presented in [7]:

\[
\mathcal{R}(r) \geq \frac{-e_3(p) - e_3(q)}{e_2(p) - e_2(q)} \quad (29)
\]

Where

\[
\mathcal{R}(r) = \frac{e_2(r) - 2e_3(r)}{1 - 2e_2(r) + 3e_3(r)} \geq 0 \quad (30)
\]

Note that \( e_2(p) - e_2(q) > 0 \) is a monotone under ELOCC, however \( e_3(p) - e_3(q) \) is not [7]. Therefore, in order for (30) to be non-trivial, \( e_3(p) - e_3(q) < 0 \). Because both (28) and (30) must satisfy conditions to be non-trivial, they only provide information on the catalyst Schmidt vector \( r \) for a subset of all catalytic examples.

To demonstrate this, consider the vectors:

\[
p = (0.470974, 0.367948, 0.138787, 0.022291)
\]
\[
q = (0.527578, 0.309212, 0.146736, 0.016474)
\]

Bound (28) states that \( k \geq 0.795029 \) which rounds to \( k \geq 1 \). Furthermore, bound (30) states that \( \mathcal{R}(r) \geq -0.198101 \). Because at minimum, \( k \geq 2 \) and \( \mathcal{R}(r) \geq 0 \forall r \), these two bounds are trivial for this example.

On the contrary, bound (4) states that \( \rho_{11} / \rho_{22} < 1.7062 \) and bound (5) states that \( \rho_{11} / \rho_{22} > 2.10727 \). Thus, the bounds on minimum and maximum entanglement produce non-trivial results for this incomparable transformation. This shows that the new bounds presented in this paper provide conditions on the subset of incomparable state transformations that could not be classified with the bounds presented in [7] alone. Furthermore, this shows that bounds (4) and (5) remain non-trivial, even when the minimum allowable catalyst dimension is unknown.

IV. CONCLUSIONS

In this paper, we address the following problem. Consider a state transformation \( |\psi\rangle \rightarrow |\phi\rangle \) that cannot be achieved with LOCC alone. What catalyst \( |\chi\rangle \) changes the process into the ELOCC transformation \( |\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\chi\rangle \)? We provide a partial answer to this question by considering the minimum deviation a potential catalyst must have from both the product state and the maximally entangled state for an ELOCC transformation to become possible. In particular, we have shown that for any incomparable Schmidt vectors \( p \) and \( q \), if the Schmidt vector \( r \) is a catalyst for the transformation, then it must have enough entanglement to satisfy (4), but not so much entanglement that it violates (5).

The solution in this paper is only partial as bounds (4) and (5) are not tight bounds excluding all catalysts that are not effective. However, these bounds substantially restrict the set of potential catalysts for any particular state transformation, and as such, provide restrictions on the amount of entanglement a catalyst state
may contain. We have shown an example demonstrating that bounds [4] and [5] provide conditions on the set of catalytic transformations which the previous bounds in [7] could not. Of course, in an ideal setting, both the bounds in [7] and the ones presented in this paper will be used in conjunction to provide the best possible restrictions on potential catalysts.

Additionally, we have shown the existence of qubit ELOCC transformations that have multiple distinct regions of effective qubit catalysts allowing them. The existence of these examples demonstrates how difficult exact bounds on ELOCC transformations would be to achieve, even in the qubit case. Thus, to proceed from this work to a more general description of ELOCC transformations, one would have to determine the conditions for when an incomparable state transformation has more than one continuous region of catalysis and furthermore would have to bound these additional regions precisely. It is vitally important to fully understand transformations that require the use of ancillary systems as they provide additional conversion power with little to no increase in resource cost. This conversion power is valuable as most quantum information processes require a precise state for optimal efficiency. This work provides a step in the right direction to a full understanding of ELOCC state transformations.

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