Charge Quantization and the Standard Model from the $\mathbb{CP}^2$ and $\mathbb{CP}^3$ Nonlinear $\sigma$-Models

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We investigate charge quantization in the Standard Model (SM) through a $\mathbb{CP}^2$ nonlinear sigma model (NLSM), $SU(3)_C/(SU(2)_H \times U(1)_H)$, and a $\mathbb{CP}^3$ model, $SU(4)_C/(SU(3)_H \times U(1)_H)$. We also generalize to any $\mathbb{CP}^k$ model. Charge quantization follows from the consistency and dynamics of the NLSM, without a monopole or Grand Unified Theory, as shown in our earlier work on the $\mathbb{CP}^1$ model (arXiv:1309.0692). We find that representations of the matter fields under the unbroken non-abelian subgroup dictate their charge quantization under the $U(1)_H$ factor. In the $\mathbb{CP}^2$ model the unbroken group is identified with the weak and hypercharge groups of the SM, and the Nambu-Goldstone boson (NGB) has the quantum numbers of a SM Higgs. There is the intriguing possibility of a connection with the vanishing of the Higgs self-coupling at the Planck scale. Interestingly, with some minor assumptions (no vector-like matter and minimal representations) and starting with a single quark doublet, anomaly cancellation requires the matter structure of a generation in the SM. Similar analysis holds in the $\mathbb{CP}^3$ model, with the unbroken group identified with QCD and hypercharge, and the NGB having the up quark as a partner in a supersymmetric model. This can motivate solving the strong CP problem with a vanishing up quark mass.

I. INTRODUCTION

The quantization of electric charge was observed many decades ago, and remains an exquisitely confirmed aspect of nature today with no known exception. This experimental fact has inspired several endeavors to explain this mystery, the most well-studied and successful being the Dirac monopole [1] and Grand Unified Theories (GUTs) beginning with Georgi and Glashow [2]. Perhaps most particle physicists’ view is that the latter is mechanism is the more relevant one: at some energy scale much higher than the reach of current experiments, the gauge groups of the Standard Model (SM) are unified, and electromagnetic charge quantization follows from this unification into a single gauge group.

There are also several well-known drawbacks to GUTs and monopoles. Monopoles in these theories tend to be very heavy and cause cosmological problems, GUTs generically predict too fast a rate for proton decay, splitting the Higgs doublet and triplet masses is difficult, and so far no direct experimental evidence has been found. These are old problems which have a host of proposed solutions such as inflation, discrete symmetries, high mass scales, and so on. While GUTs remain relevant for model-building and phenomenology, it can be fruitful to think outside of the box (of GUTs).

In a previous work [3], we considered charge quantization in a $\mathbb{CP}^3$, or $SU(2)_C/U(1)_H$, nonlinear sigma model (NLSM). The subscripts $G$ and $H$ differentiate between an approximate global symmetry and an unbroken subgroup (which will be gauged and identified with some subset of the SM groups), respectively. We found that charge is quantized in half-integer units of the Nambu-Goldstone boson (NGB) charge. A key point is that the $SU(2)_C$ is never gauged — it is only an approximate, nonlinearly realized symmetry. The $U(1)_H$ is gauged and identified with the $U(1)_Y$ hypercharge of the SM.1 This model achieves charge quantization in the SM without a monopole or in the context a GUT, avoiding all of the associated problems. Furthermore, the NGB of this model is completely stable and fractionally charged, with a mass that can be light with intriguing phenomenological possibilities, such as applications in nuclear physics or as dark matter.

The derivation of charge quantization in [3] is reminiscent of the arguments given via monopoles (in the modern understanding due to Wu and Yang [5]), as we are requiring well-defined transformation laws for a matter field over a sphere ($\mathbb{CP}^1$ as a manifold), or GUTs, as it is the group structure which plays a critical role. However, our derivation is also rather distinct: there is no monopole and the $SU(2)_C$ is never gauged nor linearly realized. We work directly with the NLSM as we consider that this does not always imply the presence of a corresponding linear model. One can think of the quantization condition arising due to the compact origin of the unbroken $U(1)_H$, and thus topological in nature. In this spirit, it is natural to consider

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1 This gauging explicitly breaks the $SU(2)_C$, but this does not affect charge quantization. We require the presence of a consistent theory in the limit of Yukawa and gauge couplings vanishing. This leads to charge quantization, and as long as charge is conserved, these couplings cannot break charge quantization. This was addressed in [3], but we will comment more on this issue in subsequent work.

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2 There is also some similarity to earlier work in theories with Wess-Zumino terms [4].
other NLSMs which have a similar origin for a $U(1)_H$ subgroup: we will see that charge is quantized here as well.

A seemingly unrelated question turns out to be intimately linked to charge quantization in these models: why is the matter content of a generation in the SM what it is? There is no a priori reason for the structure of matter we observe, nor any relation between their quantum numbers. In a GUT, we have complete representations of the GUT group, and after breaking to the SM we have certain representations of the SM gauge groups. As we will see, this NLSM realization of charge quantization has a deep connection to a vanishing mass for the up quark and the strong CP problem. Finally, we discuss further extensions and related topics and give concluding remarks in Sec. IV.

II. CHARGE QUANTIZATION IN $\mathbb{CP}^k$ MODELS

A. Review of $\mathbb{CP}^1$

We start by briefly reviewing our earlier work on $\mathbb{CP}^1$ models\(^3\) (see [3] for the full derivation) which will be straightforwardly extended to the larger $\mathbb{CP}^k$ models. First, let us define our coordinates for $\mathbb{CP}^1$ as $\phi_{1,2}$ which satisfy the defining property of $\mathbb{CP}^1$, $(\phi_1, \phi_2) = (\lambda \phi_1, \lambda \phi_2)$. The ratio of these coordinates (automatically satisfying this property) are the affine coordinates $z_+ \equiv v \phi_1/\phi_2$ and $z_- \equiv v^2/z_+$, where $v$ is the symmetry breaking vev. In more physical terms (we are basically following the construction of [6]), the $z$-coordinates are the Nambu-Goldstone modes of the breaking from $SU(2)_G$ to $U(1)_H$, the NLSM description of $\mathbb{CP}^1$.

The $SU(2)_G$ is non-linearly realized, while the $U(1)_H$ is a good (linearly realized) symmetry. By explicitly considering consistent and well-defined transformation properties of a charged matter field, the complex scalar (for simplicity, or any other type) field $\chi$, over all of $\mathbb{CP}^1$ we are led to a charge quantization condition.\(^4\)

The infinitesimal generators\(^5\) of $SU(2)_G$ are labeled as $T_+, T_-$ and $T_0$. $\mathbb{CP}^1$, thought of as the manifold $S^2$, needs two coordinate patches, which we call the southern hemisphere ($z_- \neq 0$ everywhere, $z_+ = 0$ at the south pole) and the northern hemisphere ($z_+ \neq 0$ everywhere, $z_- = 0$ at the north pole).

Working first in the southern hemisphere with $z_+$ and $\chi$, the action on $z_+$ is

\[
\delta T_+ \circ z_+ = -\frac{1}{v} z^2_+ , \quad \delta T_- \circ z_+ = v , \quad \delta T_0 \circ z_+ = + z_+ .
\]

The $U(1)_H$ charge is defined as the eigenvalue under $T_0$, with the NGB $z_+$ having charge $+1$. $\chi$ has charge $\alpha$ and a nonlinear transformation under the other (broken) generators of $SU(2)_G$. After using the $SU(2)_G$ algebra and demanding the transformations are smooth at the south pole ($z_+ = 0$) the transformations on functions of $\chi$ and $z_+$ are determined to be

\[
\delta T_+ = -\frac{\alpha }{v} z_+ \partial_\chi - \frac{1}{v} z^2_+ \partial z_+ ,
\]

\[
\delta T_- = v \partial z_+ ,
\]

\[
\delta T_0 = \alpha \chi \partial_\chi + z_+ \partial z_+ .
\]

Switching to the northern hemisphere, we change coordinates to $z_-$. $\chi$ must also transform:

\[
\chi' \propto z_-^p \chi,
\]

with the form fixed by $\chi$ and $\chi'$ having definite eigenvalues under the same $U(1)_H$ — antipodal points are fixed by the same rotation generator, therefore the unbroken $U(1)_H$ at the two poles can be identified. Performing the coordinate and field transformations the full generators in the northern hemisphere are

\[
\delta T_0 = -z_- \partial z_- + (\alpha + p) \chi' \partial \chi' ,
\]

\[
\delta T_- = -\frac{z_-}{v} (z_- \partial z_- - p \chi' \partial \chi') ,
\]

\[
\delta T_+ = v \partial z_- - v (p + 2\alpha) z_+ \chi' \partial \chi' .
\]

4 If all of the fields in the NLSM form parts of complete linear multiplets of $G$, charge quantization follows trivially.

5 Here we will only work explicitly with the holomorphic generators and to linear order in the scalar field. For a complete discussion, see [3].

\(^3\) In our earlier work we discuss why we consider a supersymmetric model, despite not needing supersymmetry directly in our derivation. To summarize, supersymmetry ensures that the Kähler structure of the model is protected once matter is added. We will not comment further about supersymmetry in this work.
Therefore the charge $\alpha$ of the matter field $\chi$ is quantized in half-integer units of the NGB charge (+1 for $z_+$).

Another way to derive this charge quantization condition is to consider the kinetic terms for a charged field.\textsuperscript{6} The Kähler potential is fixed by enforcing that it is invariant under the full holomorphic plus antiholomorphic $SU(2)_G$ transformations, as well as being quadratic in $\chi, \chi^\dagger$, and invariant under phase rotations. It is given by

$$K_{\text{matter}} = \left(1 + \frac{|z_+|^2}{v^2}\right)^{-2\alpha} |\chi|^2$$ \hspace{1cm} (6)

in the southern hemisphere. Requiring that the kinetic terms have the same well-defined form in the northern hemisphere of $\mathbb{CP}^1$ leads to the same quantization condition given above.

\textbf{B. $\mathbb{CP}^k$ Models}

Although such an explicit derivation should be possible, in principle, for the larger $\mathbb{CP}^k$ models, it quickly becomes rather unwieldy. Now that we have a detailed understanding of the $\mathbb{CP}^1$ model, we can exploit it to understand the general $\mathbb{CP}^k$ models. We will do this by adding mass terms for the “extra” (beyond $\mathbb{CP}^1$) NGBs and flowing through renormalization to the $\mathbb{CP}^1$ model. In this way we will be able to derive a general charge quantization formula for any $\mathbb{CP}^k$ model.

Let us do this explicitly for the $\mathbb{CP}^2$ model and use this to generalize to larger $k$. We label the NGBs as $(z_1, z_2)$ from the group breaking of $G = SU(3)_G$ to $H = SU(2)_H \times U(1)_H$. The NGBs transform as fundamentals of the unbroken $SU(2)_H$. The generator of $U(1)_H$ is $q_H$ normalized by acting on the NGBs with charge +1. We will consider a matter field $\chi$ which is coupled in a $SU(3)_G$-symmetric way. $\chi$ is labeled by its representation/charge under the unbroken subgroup; the representations cannot necessarily be chosen arbitrarily.

We will label the eigenvalue of $\chi$ under $U(1)_H$ as $q_\chi$. The representation of $\chi$ under $SU(2)_H$ is given by the “2-ality” $T$ (i.e. “k-ality” with $k = 2$) of the representation: its eigenvalue, ±1, under the center of the group. With $J = T/2$ the isospin of the $SU(2)_H$ representation, we have $(-1)^J = (-1)^{2J}$. An important point is that the central element, $C$, can be generated by exponentiating any non-central element. If $K_2$ is any generator of $SU(2)_H$ normalized to have eigenvalues ±1/2 in the fundamental representation, then $C = \exp(2\pi i K_2)$. Comparing with $C = \exp(\pi i T)$, we see that $2K_2$ measures the “2-ality” $T$, defined modulo 2. For convenience, we will take $K_2$ to be the element that acts on the NGB doublet $(z_1, z_2)$ as the diagonal matrix with eigenvalues ±1/2.

Let us now add a mass term for one of the NGBs, $z_1$. We can do this while preserving an $SU(2)$ subgroup of $G$, which we call $G' \equiv SU(2)_{G'}$ (note: this is not the same subgroup as $SU(2)_H$). The theory then flows to the $\mathbb{CP}^1$ NLSM, where the unbroken subgroup is $U(1)_{H'}$.

After the renormalization group flow, the matter field $\chi$, which was coupled to the $\mathbb{CP}^2$ model in a $SU(3)_G$ symmetric way, is now coupled to the $\mathbb{CP}^1$ NLSM a way which preserves $SU(2)_{G'}$. This is by virtue of the fact that the $SU(2)_{G'}$ is a subgroup of $SU(3)_G$ that is preserved by the mass term. We can now apply the charge quantization condition we derived for the $\mathbb{CP}^1$ model: $q_{H'} = n/2$, where $n \in \mathbb{Z}$ and $q_{H'}$ is the eigenvalue for $\chi$ under $U(1)_{H'}$.

We now want to relate this charge to the eigenvalues for the matter field in the original $\mathbb{CP}^2$ NLSM. We exploit the fact that there must be a linear relation among $K_2, q_H$, and $q_{H'}$ as these are 3 commuting generators of $SU(3)_G$, which is only rank 2. With the $SU(3)_G$ generators as $t^A_{SU(3)_G}$, the Pauli matrices labeled $\sigma^a_{SU(2)}$, and writing the other group generators in (block) diagonal form as

$$SU(3)_G = t^A_{SU(3)_G}, \quad SU(2)_H = \text{diag}\{\sigma^a_{SU(2)}, 0\},$$

$$\tilde{K}_2 = \text{diag}\{+1/2, -1/2\}, \quad q_H = \text{diag}\{+1/3, +1/3, -2/3\},$$

$$SU(2)_{G'} = \text{diag}\{0, \sigma^a_{SU(2)}\}, \quad q_{H'} = \text{diag}\{0, +1/2, -1/2\},$$

the relation between the generators is

$$q_H = \frac{4}{3} q_{H'} + \frac{2}{3} \tilde{K}_2.$$ \hspace{1cm} (7)

Using the known charge quantization condition in $\mathbb{CP}^1$ and rewriting in terms of the “2-ality” of the $SU(2)_G$ representation of $\chi$, we have a charge quantization for a matter field in the $\mathbb{CP}^2$ model, with $n \in \mathbb{Z}$:

$$q_\chi = \frac{2n}{3} + \frac{1}{3} (\text{“2-ality” mod 2 of } \chi).$$ \hspace{1cm} (8)

More explicitly, the quantization condition can be written as

$$q_\chi = \begin{cases} \frac{2n}{3}, & \chi \text{ is a tensor representation of } SU(2)_H \\ \frac{2n+1}{3}, & \chi \text{ is a spinor representation of } SU(2)_H \end{cases}$$ \hspace{1cm} (9)

relative to the NGB charge (+1 in our conventions).

Having the charge quantization relation for both $\mathbb{CP}^1$ and $\mathbb{CP}^2$, we can now see quite easily how this will generalize for arbitrary $\mathbb{CP}^k$. We add mass terms for all but one of the NGBs, preserving an $SU(2)_{G'}$ subgroup and flowing through the renormalization group to the $\mathbb{CP}^1$ model. Let $T$ (mod $k$) represent the “k-ality” of the representation of $\chi$ under the $SU(k)_H$. For example, in the $\mathbb{CP}^3$ model $T = 0, -1, +1$ for a singlet, anti-fundamental, and fundamental representation, respectively. The general charge quantization condition is

$$q_\chi = \frac{kn + (T \text{ mod } k)}{k+1}$$ \hspace{1cm} (10)

relative to the NGB charge, which we always define as +1. The NGBs are always in the fundamental representation of the unbroken $SU(k)_H$. An interesting observation is that any non-singlet under $SU(k)_H$ must have nonzero $U(1)_H$ charge ($n \in \mathbb{Z}$ and $|T| < k$).
III. PHENOMENOLOGY

For phenomenology, and to successfully quantize electromagnetic charge, we must relate the NLSM $U(1)_H$ and the $U(1)_Y$ hypercharge of the SM. In order to fix the coefficient of proportionality between these generators, we will use a “minimality” condition: the smallest possible hypercharge should be the smallest hypercharge in the SM, 1/6.

In the $\mathbb{CP}^2$ model (see [3]) this led to hypercharges given by $q_Y = n/6$. The NGB has a fractional charge, is exactly stable, and has an electromagnetic mass from the gauging of $U(1)_Y$. This particle can have a collider accessible mass, and has implications for nuclear physics, especially nuclear fusion reactors. The NGB can be a component or possibly all of the dark matter, depending on the mass. We discussed the phenomenology of this model in more detail in [3].

For the $\mathbb{CP}^2$ model, the unbroken group is $SU(2)_H \times U(1)_H$. We want to identify this with the SM weak group, $SU(2)_L$ and the hypercharge group, $U(1)_Y$. The full SM is then $SU(3)_{\text{QCD}} \times \mathbb{CP}^2$. All of the charges in eq. (9) are given relative to the NGB charge, normalized to 1 in our conventions. In this model the NGB has the quantum numbers of the SM Higgs field. Thus, it is very interesting to identify the NGB with the SM Higgs boson, setting the coefficient of proportionality between hypercharge and $U(1)_H$ as 1/2. The SM matter hypercharges are

$$Q_Y = \begin{cases} \frac{n}{3}, & \text{for integer weak isospin} \\ \frac{2n+1}{6}, & \text{for half-integer weak isospin.} \end{cases}$$

For the $SU(2)_L$ quark and lepton doublets ($Q$ and $L$), we have $n = 0, -2$, respectively, to reproduce the correct hypercharges, while the $SU(2)_L$ singlet electrons, up, and down-type quarks (written as left-handed fields $\tilde{e}, \tilde{u}$, and $\tilde{d}$) have $n = 3, -2, 1$, respectively.

Once the $SU(2)_L \times U(1)_Y$ are gauged, the NGB is a pseudo-NGB, gaining a mass from gauge interactions, of order $\sqrt{\alpha_{\text{EW}} \Lambda}$, with $\Lambda$ a cutoff and $\alpha_{\text{EW}}$ the electroweak strength. We consider $\Lambda \sim M_p$, the Planck scale, and we need a fine tuning to explain a light Higgs mass at the electroweak scale.

A quartic self-coupling is generated via gauge and Yukawa interactions at the Planck scale which is one-loop suppressed and thus sufficiently small, $\mathcal{O}(10^{-3})$. There may be a connection with models (e.g., [7]) exploiting the possibility of the Higgs quartic coupling running to zero (within sizable errors, especially from the top mass) near the Planck scale (see [8] and references therein). A NGB hypothesis for the Higgs may be consistent if there is such a flat potential at the Planck scale, with appropriate assumptions on UV effects or boundary conditions. This would be a remarkable observation, connecting the 126 GeV Higgs mass [9] with physics at the Planck scale and charge quantization.

We also consider the phenomenology of the $\mathbb{CP}^3$ model, where the SM is now $\mathbb{CP}^3 \times U(1)_H$. The unbroken group from $\mathbb{CP}^3$ is $SU(3)_H \times U(1)_H$, which we take to be color and hypercharge, respectively. The proportionality constant between $U(1)_Y$ and $U(1)_H$ is $2/3$:

$$Q_Y = \frac{2}{3} q_H = \frac{3n + T}{6},$$

where $T$ is the “3-ality” of the $SU(3)_H$ representation, given by $T = 0, -1, +1$ for a singlet, anti-fundamental, and fundamental representation, respectively. This now corresponds to the QCD representation for the given matter field. The left-handed quark $SU(2)_L$ doublet $Q$ has $n = 0$ ($T = +1, Q_Y = 1/6$), a down-type $SU(2)_L$ singlet quark $\bar{d}$ has $n = +1$ ($T = -1, Q_Y = 1/3$), a lepton doublet $L$ has $n = -1$ ($T = 0, Q_Y = -1/2$), and so on.

In this model we consider the (conjugate) NGB in the anti-fundamental representation of $SU(3)_{\text{QCD}}$ with hypercharge $Q_Y = -2/3$: it has the quantum numbers of an $SU(2)_L$ singlet up squark. In fact, in a supersymmetric model the partner fermion $\tilde{u}$ can explain the smallness of the up quark mass in the SM. It can even be possible to have a massless up quark, avoiding the strong CP problem.

A. The SM generation content from $\mathbb{CP}^2$ and $\mathbb{CP}^3$

There is another intriguing phenomenological consequence of the $\mathbb{CP}^2$ and $\mathbb{CP}^3$ models: with some minor assumptions we can obtain the structure of the matter content of a generation in the SM. We impose the following restrictions:

- There is no vector-like matter (which would then have a natural mass scale of $M_p$).
- The theory is anomaly free for all gauge groups.\(^8\)
- The smallest representations and least amount of matter should be used.

We will start by looking at the $\mathbb{CP}^2$ model. We will add the color group, $SU(3)_{\text{QCD}}$, and one $SU(2)_L$ doublet quark (fundamental of $SU(3)_{\text{QCD}}$) with hypercharge

$$q_Q = Y \frac{2n_Q + 1}{3},$$

where $Y$ is the constant of proportionality between hypercharge and the NLSM $U(1)_H$ (or equivalently, the NGB charge which all charges are proportional to), which we will leave arbitrary in this analysis.

Let us first consider the $SU(3)_{\text{QCD}}$ anomaly. Since we have one quark doublet, based on our assumptions of no

\(^8\) This is reminiscent of an alternative observation of charge quantization by examining the SM anomalies (see [10]). Here, however, it is “opposite” in the sense that the charge quantization rule derived above leads to the SM matter content. For the relation of the matter representations and anomalies in the SM, see the earlier work of [11].
We have therefore "derived" the matter content of a generic model, with the Higgs $H$ realized as the Nambu-Goldstone boson. The normalization of the hypercharge is determined by the assigning $H$ to have $Y = 1/2$. $Q$ and $L$ are the $SU(2)_L$ quark and lepton doublets, while $\bar{u}, \bar{d}$, and $\bar{e}$ are the $SU(2)_L$ singlet up quark, down quark, and electron, respectively. All fermions are written as left-handed fields. The SM has $n_Q = 0, n_\bar{u} = -2, n_\bar{d} = 1, n_L = -2, n_\bar{e} = 3$.

vector-like matter and using minimal content, we must add two $SU(2)_L$ singlet anti-quarks, with charges $2Y n_\bar{u}, d/3$. The charges in the $SU(3)_{QCD}^2 U(1)_Y$ anomaly (all fields are left-handed) then require that

\[ 2n_Q + 1 = - (n_\bar{u} + n_\bar{d}). \tag{14} \]

Next, we have the $SU(2)_L^2 U(1)_Y$ anomaly. Only the quark doublet contributes, so we must add a lepton doublet. Writing its charge as $Y(2n_L + 1)/3$ the restriction on the charges from the anomaly is

\[ 2n_L + 1 = -3(2n_Q + 1) = 3(n_\bar{u} + n_\bar{d}). \tag{15} \]

We consider the (gravity)$^2 U(1)_Y$ anomaly. Again, we are required to add additional matter, which will be a singlet except for its $U(1)_Y$ charge: an $SU(2)_L$ singlet lepton with charge $2Y n_\bar{e}/3$. The anomaly constraint is

\[ n_\bar{e} = -3(n_\bar{u} + n_\bar{d}). \tag{16} \]

Finally, we have the $U(1)_Y^3$ anomaly. There is no extra matter that is required, if the integers giving the charges satisfy

\[ (n_\bar{u} + n_\bar{d})(2n_\bar{u} + n_\bar{d})(n_\bar{u} + 2n_\bar{d}) = 0. \tag{17} \]

Combined with the relation to $n_Q$ in eq. (14), and up to exchanging $n_\bar{u}, n_\bar{d}$, the unique solution is

\[ n_\bar{u} = -2n_\bar{d}. \tag{18} \]

We have therefore "derived" the matter content of a generation in the SM, with the final form of the $U(1)_Y$ charges given in Table I. We still need to fix the overall coefficient, $Y$, which we will set by taking the Higgs (NGB) hypercharge to be $1/2$. These charges are of course consistent with the usual SM charge assignments, as given previously. It seems rather unexpected, and remarkable, that such structure comes from the $\mathbb{CP}^2$ NLSM with rather minimal additional assumptions.

We can follow basically the same procedure in a supersymmetric $\mathbb{CP}^2$ model. In this case we add the weak group to complete the SM gauge groups. For this model we will work with an anti-fundamental NGB; it has $U(1)_H$ charge $-1$ and hypercharge $-Y$ (the normalization to hypercharge). With supersymmetry, the fermion partner to the NGB is an $SU(2)_L$ singlet quark. Whether it is an up or down quark depends on how we fix $Y$, which we again leave as a free parameter at this stage. We write this fermion as a left-handed field $\bar{u}$ which is an anti-fundamental of color with hypercharge $-Y$. With just a single colored fermion, we need to add matter to satisfy the $SU(3)_{QCD}^3$ anomaly. If we try adding another weak singlet, but with a different hypercharge to forbid a mass term, we then cannot satisfy the $SU(3)_{QCD}^2 U(1)_Y$ anomaly without additional colored matter (changing the $SU(3)_{QCD}^3$ anomaly). Instead, we add an $SU(2)_L$ quark doublet, with hypercharge $(3n_Q + 1)/Y$, and an $SU(2)_L$ singlet (anti-fundamental) quark, with hypercharge $(3n_Q - 1)/Y$. The $SU(3)_{QCD}^2 U(1)_Y$ anomaly relates their charges as

\[ 2n_Q + n_\bar{q} = 1. \tag{19} \]

The $SU(2)_L^2 U(1)_Y$ anomaly requires a (colorless) weak doublet: the lepton doublet with hypercharge $3Y n_L/4$. This charge is related to the quark doublet by

\[ n_L = -3(n_Q + 1). \tag{20} \]

Finally, we have the (gravity)$^2 U(1)_Y$ anomaly. The anomaly is zero only with additional matter (as $n_Q \in \mathbb{Z}$): an $SU(2)_L$ singlet lepton with hypercharge $3Y n_\bar{e}/4$. This charge is related to the quark charges by

\[ n_\bar{e} = 2(3n_Q + 1). \tag{21} \]

The final anomaly is the $U(1)_Y^3$ anomaly. For this anomaly to be satisfied we must satisfy the constraint

\[ n_Q(n_Q + 1)(3n_Q + 1) = 0. \tag{22} \]

There are two possible unique solutions for the integer $n_Q$ which specifies all of the matter hypercharges (the Higgs does not appear in the anomaly constraints). If $n_Q = -1$, then we can use one of the hypercharges of the SM to fix $Y = -1/3$. All of the hypercharges are specified and match their SM values, and the partner of the NGB is a down-type quark.

The more interesting possibility is if $n_Q = 0$. Fixing one of the hypercharges to the SM value requires $Y = 2/3$, which matches the "minimal" considerations we used in the previous section. In this case the NGB fermion partner is the up quark. This raises the possibility of connecting the $\mathbb{CP}^2$ model of charge quantization to the smallness of the up quark mass and the strong CP problem, which can be avoided with a massless up quark.

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| Field | $U(1)_Y$ Charge |
|-------|-----------------|
| $H$   | $Y \Rightarrow Y = \frac{2}{3}$ |
| $Q$   | $\frac{2n_Q + 1}{3}, 2n_Q + 1 = - (n_\bar{u} + n_\bar{d})$ |
| $\bar{u}$ | $\frac{n_\bar{u}}{3}, n_\bar{u} = -2n_\bar{d}$ |
| $\bar{d}$ | $\frac{n_\bar{d}}{3}, n_\bar{d} = 3(n_Q + n_\bar{d})$ |

\[ n_\bar{e} = -3(n_\bar{u} + n_\bar{d}). \]

\[ n_\bar{u} = -2n_\bar{d}. \]

\[ n_\bar{e} = 2(3n_Q + 1). \]

\[ n_Q(n_Q + 1)(3n_Q + 1) = 0. \]

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9 Remember that the $SU(2)^3_L$ anomaly, and anomalies with one $SU(3)_{QCD}$ or $SU(2)_L$ factor, are automatically satisfied due to the group structure.
In this work we have shown how to extend the earlier results for the $\mathbb{C}P^1$ model [3] to general $\mathbb{C}P^k$ models. The simplest way to do this is to add mass terms to additional NGBs and flow through renormalization to the $\mathbb{C}P^1$ model. We then arrive at a general charge quantization formula, which depends on a matter field’s representation under the unbroken group. We explored some of the phenomenological implications of these NLSMs with some part of the SM as the unbroken group.

The $\mathbb{C}P^2$ and $\mathbb{C}P^3$ models have very interesting phenomenology. The NGBs in the $\mathbb{C}P^2$ model have the quantum numbers of the SM Higgs boson, which presents some interesting model-building possibilities. In the $\mathbb{C}P^3$ model with supersymmetry, the fermion partner to the NGB is the up quark, connecting the model to the possibility of a vanishing up quark mass as a solution to the strong CP problem. Quite unexpectedly, both of these models, with some assumptions like chiral matter, lead to the structure of the SM matter generation content. This is due to the charge quantization formula enforcing that non-singlet fields have a nonzero hypercharge.

A logical continuation of this program would be to try to embed the entire SM as the unbroken group of a NLSM. This is currently under investigation, to appear in a future work. The charge quantization formula in this model can again be obtained by considering embedding the $\mathbb{C}P^1$ model, while the phenomenology is quite rich.

There are also several open questions related to these types of models which we are currently exploring. One question regards explicit breaking, beyond that of the gauging of the unbroken group. While the breaking due to gauging a symmetry is under control, determined by the (small) gauge coupling, what about other possible sources? Charge quantization in these models can be thought of as a topological effect (the structure and compactness of the group manifold), and thus may be robust against other breaking effects. Finally, there are also several interesting topics which are related to these types of theories which we are exploring. This includes anomaly considerations, beta functions, and more mathematical considerations. These NLSMs are proving to have quite a rich structure, probing deep questions in particle physics and the SM.

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