Error bounds of potential theoretic numerical integration formulas in weighted Hardy spaces

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Abstract
We give a mathematical analysis of potential-theoretical numerical integration formulas mainly proposed by Tanaka et al. over weighted Hardy spaces, which are spaces of analytic functions with a certain decay. We investigate the case of choosing sampling points by discrete energy minimization. In order to bound the convergence rate of the numerical integration error, we make use of the recent result of Hayakawa and Tanaka on the function approximation formula.

Keywords weighted Hardy space, numerical integration, potential theory

Research Activity Group Scientific Computation and Numerical Analysis

1. Introduction and background
In [1], the authors proposed numerical integration formulas of the form
\[ \int_1 F(t) dt = \int_\infty f(x) dx \]
for some bounded analytic function F over (-1,1). Here, \[ |F(\psi(x))\psi'(x)| \] decays at the same order as \[ |\psi'(x)| \] in the limit \[ x \to \pm \infty \]. Typical transformations are:
\[ \psi(x) = \text{tanh}(\frac{x}{2}) \quad \text{(TANH transformation [2,3])}, \]
\[ \psi(x) = \text{tanh}(\frac{x}{2} \sinh x) \quad \text{(DE transformation [4])}. \]

Given such a variable transformation, quadrature formulas with the trapezoidal rule
\[ \int_{-\infty}^{\infty} F(\psi(x))\psi'(x) dx \approx h \sum_{j=-N}^{N} F(\psi(jh))\psi'(jh) \]
for some small \( h > 0 \) can be considered.

Sugihara [5] showed near optimality of such formulas in the function space defined by
\[ \mathbb{H}^\infty(D_d, w) := \left\{ f : D_d \to \mathbb{C} \middle| f \text{ is analytic on } D_d, \|f\| := \sup_{z \in D_d} \frac{|f(z)|}{w(z)} < \infty \right\}, \]
where \( d > 0 \), \( D_d := \{z \in \mathbb{C} \mid |\text{Im} z| < d\} \), and \( w : D_d \to \mathbb{C} \) is a weight function with a certain asymptotic decay. The space \( \mathbb{H}^\infty(D_d, w) \) is considered to be the space of functions \( f = F(\psi(\cdot))\psi'(\cdot) \) after some variable transformation, where \( w \) represents their asymptotic decay. Note that the integrand \( f \) is supposed to be an analytic function over the real axis, whose domain can be analytically extended to a strip region about the real axis.

To show the near optimality in a fixed \( \mathbb{H}^\infty(D_d, w) \), Sugihara considered a criterion for the optimality of formulas of the form
\[ \int_{-\infty}^{\infty} f(x) dx \approx \sum_{p=1}^{\ell} \sum_{q=0}^{m_p-1} c_{p,q} f^{(q)}(a_p) \]
as follows:
\[ \varepsilon^\text{min}_N(\mathbb{H}^\infty(D_d, w)) := \inf_{\ell, c_{p,q}, m_p} \sup_{f \in \mathbb{H}^\infty(D_d, w)} \left| \int_{-\infty}^{\infty} f(x) dx - \sum_{p,q} c_{p,q} f^{(q)}(a_p) \right| \]
where \( c_{p,q} \in \mathbb{C} \) for each \( (p, q) \), \( 1 \leq \ell \leq 2N + 1 \) and \( m_1 + \cdots + m_\ell = 2N + 1 \) should hold. Note that (3) represents a general form of numerical integration formulas like (1) and (2).

In [1], the authors proposed new formulas based on the following expression of \( \varepsilon^\text{min}_N(\mathbb{H}^\infty(D_d, w)) \).

Theorem 1 ([1, Theorem 1]) It holds that
\[ \varepsilon^\text{min}_N(\mathbb{H}^\infty(D_d, w)) = \inf_{a_j \in \mathbb{R}} \int_{-\infty}^{\infty} w(x) \prod_{j=-N}^{N} T_d(x-a_j)^2 dx, \]
where \( T_d(x) = \frac{1}{\sqrt{2\pi}} \int_{-d}^{d} e^{-t^2/2} dt \) is a weight function with a certain asymptotic decay.
where
\[ T_d(x) := \tanh \left( \frac{\pi}{4d} \right). \]

For readers’ convenience, we describe a sketch of the proof of Theorem 1 in Section 5. Since (4) is given by a non-convex optimization problem, it is hard to get its minimizer \( a_{-N}, \ldots, a_N \). However, the problem to minimize

\[ \sup_{x \in \mathbb{R}} w(x) \prod_{j=-N}^N T_d(x - a_j)^2 \]  

(6)
can be approximately (but not exactly) solved by using potential-theoretical techniques [1, 6, 7]. In [1], the authors proposed using the point configuration \( a^* = (a_{-N}^*, \ldots, a_N^*) \in \mathbb{R}^{2N+1} \) approximately minimizing (6). They then suggested the numerical integration

\[ \int_{-\infty}^{\infty} f(x) \, dx \approx \int_{-\infty}^{\infty} \hat{f}_{2N+1}(a^*; x) \, dx, \]

(7)
where \( \hat{f}_{2N+1}(a; x) \) is the “generalized” Lagrange interpolation for \( a = (a_{-N}, \ldots, a_N) \in \mathbb{R}^{2N+1} \) given by

\[ \hat{f}_{2N+1}(a; x) := \frac{\sum_{j=-N}^N f(a_j) w(x) T'_d(x - a_j) - \pi/(4d) \prod_{i \neq j} T_d(x - a_i)}{w(a_j)} T_d(x - a_i). \]

(8)
This formula behaves well in numerical experiments, and shows a superiority to the trapezoidal rule for several weight functions [1].

In this paper, we mathematically analyze (7) and estimate the convergence rate of its error. To this end, we treat the following functions in addition to the weight function \( w : \mathcal{D}_d \to \mathbb{C} \) throughout the paper:

- \( Q : \mathbb{R} \to \mathbb{R} \) is defined as \( Q(x) := -1/(2) \log w(x) \),
- \( K : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) is defined as \( K(x) := -\log T_d(x) \).

We assume \( w \) and \( Q \) satisfy the following conditions.

(a) \( w \) is analytic and non-zero on \( \mathcal{D}_d \), and it holds that \( w(x) \in (0, 1] \) for every \( x \in \mathbb{R} \).

(b) \( \lim_{x \to \pm \infty} \int_{-\infty}^{d} w(x + iy) \, dy = 0 \)
and \( \lim_{y \to \pm \infty} \int_{-\infty}^{d} w(x + iy) \, dx < \infty \).

(c) \( Q \) is even and strictly convex on \( \mathbb{R} \).

2. Contribution of this study
Let \( a^* = (a_{-N}^*, \ldots, a_N^*) \) be the point configuration which approximately minimizes (6) (the definition of which is precisely given in the next section). Then, we have the following estimate for the minimum worst-case error. The proofs of theorems in this section are given in Section 3.

Assume \( \alpha_N > 0 \) satisfies

\[ \frac{2\alpha_N}{\pi \tanh(d)} \frac{Q(\alpha_N)^2 + Q'(\alpha_N)^2}{Q(\alpha_N)} \leq 2N + 1. \]

(9)

Theorem 2 It holds that
\[ c_N \min(\mathcal{H}^2(\mathcal{D}_d, w)) \leq 4e^3 \alpha_N \exp \left( -\frac{N}{2N+1} Q(\alpha_N) \right) + \exp(-2Q(\alpha_N)) \]
\[ \leq 2e^3 \exp \left( -\frac{N}{2N+1} Q(\alpha_N) \right). \]

Although the next evaluation is apparently loose, the following theorem is the first result which rigorously evaluates the convergence rate of (7).

Theorem 3 It holds that
\[ \sup_{\|f\| \leq 1} \left| \int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} \hat{f}_{2N+1}(a^*; x) \, dx \right| \]
\[ \leq \sqrt{8}e^3 \alpha_N \exp \left( -\frac{NQ(\alpha_N)}{2(2N+1)} \right) + \exp(-2Q(\alpha_N)). \]
Concrete applications of the above estimate are given in Section 4.

3. Further explanations and proofs
3.1 Definition and key estimate
We here define precisely \( a^* \) introduced in Section 2. Considering the logarithm, the minimization problem of (6) is equivalent to the following problem:

\[ \max \inf_{x \in \mathbb{R}} \left( \sum_{j=-N}^N K(x - a_j) + Q(x) \right) \]
\[ \text{s.t.} \quad a_{-N} < \cdots < a_N. \]
(10)

From a discrete analogue of potential-theoretical arguments, Tanaka & Sugihara [6] introduced the following problem:

\[ \min \sum_{i \neq j} K(a_i - a_j) + \frac{4N}{2N+1} \sum_{j=-N}^N Q(a_j) \]
\[ \text{s.t.} \quad a_{-N} < \cdots < a_N. \]
(11)

Since the problem (11) is a strictly convex optimization problem, there exists a unique optimal solution, which is denoted by \( a^* = (a_{-N}^*, \ldots, a_N^*) \) [6, Theorem 3.3]. Although (10) and (11) are different problems, \( a^* \) behaves well as an approximate solution of (10) in numerical experiments. It is also shown in [7] that \( a^* \) is “nearly optimal” solution of the problem (10).

The important estimate is the following theorem, which is an immediate consequence of [7, Section 2.3]:

Theorem 4 ([7]) It holds that

\[ \inf_{x \in \mathbb{R}} \left( \sum_{j=-N}^N K(x - a_j^*) + Q(x) \right) \]
\[ \geq \frac{3 + \log 2}{2(2N+1)} \]
where \( \alpha_N \) is the same as one defined in Section 2.

3.2 Proof of Theorem 2
From Theorem 4, the value (6) with \( a = a^* \) is upper-bounded as
\[ \sup_{x \in \mathbb{R}} w(x) \prod_{j=-N}^N T_d(x - a_j^*) \]
\[ = \exp \left( -2 \inf_{x \in \mathbb{R}} \left( \sum_{j=-N}^N K(x - a_j^*) + Q(x) \right) \right) \]
\[ \leq 2e^3 \exp \left( -\frac{N}{2N+1} Q(\alpha_N) \right). \]
Then, we have
\[
\int_{-\infty}^{\infty} w(x) \prod_{j=-N}^{N} T_d(x-a_j)^2 \, dx \\
\leq 2\alpha_N \sup_{x \in \mathbb{R}} w(x) \prod_{j=-N}^{N} T_d(x-a_j)^2 + 2 \int_{\alpha_N}^{\infty} w(x) \, dx \\
\leq 4e^3 \alpha_N \exp \left( -\frac{N}{2N+1} Q(\alpha_N) \right) \\
+ 2 \int_{\alpha_N}^{\infty} \exp(-2Q(x)) \, dx.
\]
Here, the second term of the right-hand side is estimated as
\[
2 \int_{\alpha_N}^{\infty} \exp(-2Q(x)) \, dx \\
= \int_{\alpha_N}^{\infty} \frac{2Q(\alpha_N)}{Q'(\alpha_N)} \exp(-2Q(x)) \, dx \\
\leq \frac{1}{Q'(\alpha_N)} \int_{\alpha_N}^{\infty} \left( -\frac{d}{dx} \exp(-2Q(x)) \right) \, dx \\
= \frac{\exp(-2Q(\alpha_N))}{Q'(\alpha_N)}.
\]
Thus, we have the desired estimate of \(\varepsilon_N^{\min}(\mathcal{H}^{\infty}(D_d, w))\).

3.3 Proof of Theorem 3
We first note a fact on the Lagrange interpolation formula and its error bound. In [6], the authors proposed the Lagrange interpolation formula \(L_{2N+1}\) (given in (8)) based on the function
\[
B_N(x;a,D_d) := \prod_{j=-N}^{N} T_d(x-a_j).
\]
They also estimated its error in [6, Proposition 2.3] as follows (originally obtained in [8, Lemma 4.3]):
\[
\sup_{||f|| \leq 1} \left| \int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} L_{2N+1}(a^*;x) \, dx \right| \\
\leq \int_{-\alpha_N}^{\alpha_N} \left| f(x) - L_{2N+1}(a^*;x) \right| \, dx + 2 \int_{\alpha_N}^{\infty} w(x) \, dx.
\]
Using this fact, we estimate the error by separating the difference of two integrals into two parts:
\[
\int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} L_{2N+1}(a^*;x) \, dx \\
\leq \int_{-\alpha_N}^{\alpha_N} \left| f(x) - L_{2N+1}(a^*;x) \right| \, dx + 2 \int_{\alpha_N}^{\infty} w(x) \, dx.
\]
In the second part, we have used (13). For the first part, we also use (13) to obtain
\[
\int_{-\alpha_N}^{\alpha_N} \left| f(x) - L_{2N+1}(a^*;x) \right| \, dx \\
\leq 2\alpha_N \sup_{x \in \mathbb{R}} \left| f(x) - L_{2N+1}(a^*;x) \right| \\
\leq 2\alpha_N \sup_{x \in \mathbb{R}} w(x) \prod_{j=-N}^{N} T_d(x-a_j)^2
\]
and so we have
\[
\frac{2\alpha_N}{\sup_{x \in \mathbb{R}} w(x)} \prod_{j=-N}^{N} T_d(x-a_j)^2
\subseteq 2\alpha_N \left[ \sup_{x \in \mathbb{R}} w(x) \prod_{j=-N}^{N} T_d(x-a_j)^2 \right].
\]
Then, by exploiting the evaluations in the previous section, we obtain the error estimate
\[
\int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} L_{2N+1}(a^*;x) \, dx \\
\leq \sqrt{8e^3} \alpha_N \exp \left( -\frac{NQ(\alpha_N)}{2(2N+1)} + \frac{-2Q(\alpha_N)}{Q'(\alpha_N)} \right),
\]
which does not depend on the choice of \(f\).

4. Examples
In this section, we give explicit asymptotic bounds of \(-\log \varepsilon_N^{\min}(\mathcal{H}^{\infty}(D_d, w))\) for some concrete asymptotic growth of \(Q(x)\) as \(x \to \pm \infty\). The analyses are based on [7, Section 5.2].

4.1 The case \(Q(x)\) has a polynomial growth
Consider the case \(Q(x) = (\beta|x|)^\rho\) holds for some \(\beta, \rho > 0\) and sufficiently large \(x\). This means that \(w\) is a single-exponential weight function. Then, we have from [7]
\[
2\alpha_N \frac{Q(\alpha)^2 + Q'(\alpha)^2}{\pi \tanh(d)} \leq 4\beta^\rho e^{\rho+1}.
\]
Therefore, \(\alpha_N\) in Theorem 2 can be taken as
\[
\alpha_N \leq \left[ \frac{\pi \tanh(d)}{4\beta^\rho (2N+1)} \right]^{1/\rho},
\]
and so we have
\[
Q(\alpha_N) \sim \beta^{1/\rho} (2N+1)^{1/\rho},
\]
as \(\log \alpha_N\) and \(\log Q'(\alpha_N)\) can be ignored when compared to \(Q(\alpha_N)\), we have the estimate
\[
-\log \varepsilon_N^{\min}(\mathcal{H}^{\infty}(D_d, w)) \gtrsim \beta^{1/\rho} (2N+1)^{1/\rho}.
\]

4.2 The case \(Q(x)\) has an exponential growth
Consider the case \(Q(x) = \beta \exp(\gamma |x|)\) holds for some \(\beta, \gamma > 0\) and sufficiently large \(x\). The paper [7] suggests that in this case we can take \(\alpha_N\) as
\[
\alpha_N \sim \frac{1}{\gamma} \log \left( \frac{\gamma (2N+1)}{\beta (1+\gamma^2)} \right),
\]
and so we have
\[
Q(\alpha_N) \sim \frac{\gamma}{1+\gamma^2} \frac{2N+1}{\log(2N+1)}.
\]
Therefore, we have the estimate
\[
-\log \varepsilon_N^{\min}(\mathcal{H}^{\infty}(D_d, w)) \gtrsim \frac{\gamma}{1+\gamma^2} \frac{2N+1}{\log(2N+1)}.
\]

5. Sketch of the proof of Theorem 1
We can prove Theorem 1 by showing the following two inequalities:
(i) \((\text{LHS of (2)}) \gtrsim (\text{RHS of (2)})\);
(ii) \((\text{LHS of (2)}) \leq (\text{RHS of (2)})\).
(i) We can show this inequality by using $B_N(x; a, D_d)$ introduced in (12),
\[
\varepsilon_N^{\min} (H^\infty (D_d, w)) \\
\geq \inf_{a \in 2N+1 \text{ points}} \sup_{\|f\|_{L^1} \leq 1} \left| \int_{-\infty}^{\infty} f(x) \, dx \right| \\
( : \text{ restricting } f \text{ at } a)
\]
\[
\geq \inf_{a \in 2N+1 \text{ points}} \left| \int_{-\infty}^{\infty} w(x) \, B_N(x; a, D_d) \, B_N(x; a, D_d) \, dx \right| \\
( : \text{ restricting } f \text{ again}).
\]

Note that the function $B_N(x; a, D_d)$ is analytic on $D_d$ because of the Schwarz reflection principle.

(ii) We can show this inequality by providing a numerical integration formula achieving $\varepsilon_N^{\min} (H^\infty (D_d, w))$. Such a formula can be given by integrating the Hermite interpolation $\tilde{f}_{2N+1}^H$ based on $B_n(x; a, D_d)$. $\tilde{f}_{2N+1}^H$ is explicitly written as
\[
\tilde{f}_{2N+1}^H(a; x) := \sum_{j=-N}^{N} f(a_j) u_j(a; x) + \sum_{j=-N}^{N} f'(a_j) v_j(a; x),
\]
where
\[
v_j(a; x) := \frac{1}{T_d(0)^2} w(x) \prod_{i \neq j} T_d(x - a_i) \prod_{i \neq j} T_d(a_j - a_i)^2 T_d(x - a_j)
\]
\[
u_j(a; x) := -\frac{\partial}{\partial a_j} \log \left( w(t) \prod_{i \neq j} T_d(t - a_i)^2 \right) \bigg|_{t=a_i} + \frac{w(x)}{w(a_j)} \prod_{i \neq j} T_d(x - a_i)^2 \prod_{i \neq j} T_d(a_j - a_i)^2 \cosh^4 \left( \frac{1}{T_d(x - a_j)} \right).
\]

In fact, the Hermite interpolation $\tilde{f}_{2N+1}^H$ is yielded by taking a limit $b_i \to a_i$ of the Lagrange interpolation with $2(2N+1)$ points
\[(a_1, \ldots, a_{2N+1}, b_1, \ldots, b_{2N+1}).\]

Therefore it follows from (13) that
\[
\inf_{\|f\|_{L^1} \leq 1} \left| \int_{-\infty}^{\infty} f(x) - \tilde{f}_{2N+1}^H(x) \, dx \right| \leq w(x) \|B_n(x; a, D_d)\|^2. \tag{14}
\]

Then we can obtain a $2(2N+1)$ point formula by integrating $\tilde{f}_{2N+1}^H$:
\[
\int_{-\infty}^{\infty} \tilde{f}_{2N+1}^H(x) \, dx = \sum_{j=1}^{2N+1} \left( \int_{-\infty}^{\infty} u_j(x) \, dx \right) f(a_j)
\]
\[
+ \sum_{j=1}^{2N+1} \left( \int_{-\infty}^{\infty} v_j(x) \, dx \right) f'(a_j),
\]
which does not seem to be suitable for a concrete formula achieving $\varepsilon_N^{\min} (H^\infty (D_d, w))$. However, it has been shown that the coefficients $\int_{-\infty}^{\infty} v_j(x) \, dx$ vanish for a minimizer $a^*$ of the RHS of (2). Then the above formula becomes an $(2N+1)$-point formula satisfying
\[
\left| \int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} \tilde{f}_{2N+1}^H(x) \, dx \right|
\leq \int_{-\infty}^{\infty} \left| f(x) - \tilde{f}_{2N+1}^H(x) \right| \, dx,
\]
which is the desired inequality.

We should note a little. We here use the Hermite interpolation for evaluating $\varepsilon_N^{\min} (H^\infty (D_d, w))$, but the actual numerical integration scheme we shall use is the Lagrange interpolation (as in Theorem 3). They are expected to coincide when the point configuration $a$ minimizes the value of (6), which was reported without proof in numerical experiments [1]. As the point configuration $a^*$ is an approximate minimizer of the problem (10), it seems better for us to use the Lagrange interpolation as the $(2N+1)$-point interpolation formula.

### 6. Concluding remarks

In this paper, we study the numerical integration formulas having proposed in [1], which exploited potential theory to get accurate formulas computing integrals of analytic functions contained in weighted Hardy spaces. The formula is determined by choosing sampling points as a solution of an optimization problem. The paper [7] gave an approximation error bound of the Lagrange interpolation. Then this paper exploits the bound to give a first theoretical bound of the numerical integration method (7). This paper also gives a theoretical bound of the worst-case error, which seems to be tight while the error bound of the numerical integration formula might be a little loose.

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