Branching problem on winding subalgebras of affine Kac-Moody algebras $A^{(1)}_1$ and $A^{(2)}_2$

Duc Khanh Nguyen

Abstract

Consider an affine Kac-Moody algebra $g$ with Cartan subalgebra $h$. Given $\Lambda$ in the set $P^+$ of dominant integral weights of $g$, we denote by $L(\Lambda)$ the integrable highest weight $g$-module with highest weight $\Lambda$. For $\mu \in h^*$, we denote by $L(\Lambda)_\mu$ the corresponding weight space. Consider the support $\Gamma(g,h)$ of the decompositions of the $L(\Lambda)$ as a $h$-module:

$$\Gamma(g,h) = \{(\Lambda,\mu) : L(\Lambda)_\mu \neq \{0\}\}.$$  

Consider now the winding subalgebra $g[u]$ (for some positive integer $u$). The winding subalgebra $g[u]$ is isomorphic to $g$ but with a nontrivial embedding in $g$ depending on the parameter $u$. Given $\lambda$ in the set $\check{P}^+$ of dominant integral weights of $g[u]$, we denote by $\check{L}(\lambda)$ the integrable highest weight $g[u]$-module with highest weight $\lambda$. Then the $g$-module $L(\Lambda)$ decomposes as a direct sums of simple $g[u]$-modules $\check{L}(\lambda)$ with finite multiplicities. In this paper, we are interested in the supports of this decomposition, i.e., the set of pairs $(\Lambda,\lambda)$ in $P^+ \times \check{P}^+$ such that the integrable highest weight $g[u]$-modules $\check{L}(\lambda)$ is a submodule of $L(\Lambda)$. We show that both $\Gamma(g,h)$ and $\Gamma(g[g[u]])$ are semigroups. Moreover, for the cases $A^{(1)}_1$ and $A^{(2)}_2$, we determine explicitly $\Gamma(g[h])$.

Finally, we describe explicit subsets of $P^+ \times \check{P}^+$ where the two semigroups coincide.

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1 Introduction

One of the most important question in representation theory is how an irreducible module of a Lie algebra $g$ can be decomposed when we consider it as a representation of some given Lie subalgebra $\hat{g}$. Assume first that $g$ and $\hat{g}$ are finite dimensional and semi-simple. Then, the irreducible $g$-modules (resp. $\hat{g}$-modules) are parametrized by the semi-group $P_+$ (resp. $\hat{P}_+$) of dominant integral weights. Given $\Lambda \in P_+$, under the action of $\hat{g}$, the irreducible $g$-module $L(\Lambda)$ of highest weight $\Lambda$ decomposes as

$$L(\Lambda) = \bigoplus_{\lambda \in \hat{P}_+} \hat{L}(\lambda)^{mult_{\Lambda,\hat{g}}(\lambda)},$$

where $mult_{\Lambda,\hat{g}}(\lambda)$ is the multiplicity of $\hat{L}(\lambda)$ in $L(\Lambda)$. Understanding the number $mult_{\Lambda,\hat{g}}(\lambda)$ is referred as the branching problem. For example, for $\hat{g}$ diagonally embedded in $g = \hat{g} \times \hat{g}$, the coefficients $mult_{\Lambda,\hat{g}}(\lambda)$ are the multiplicities of the tensor product decomposition of two irreducible representations of $\hat{g}$. If $\hat{g} = gl_n(\mathbb{C})$ then $\hat{P}_+$ identifies with the set of non-increasing sequences $\nu = (\nu_1 \geq \cdots \geq \nu_n)$ of $n$ integers and the coefficients are the
Littlewood-Richardson coefficients $e_{\lambda u}^\kappa$. If $\hat{\mathfrak{g}}$ is a Cartan subalgebra of $\mathfrak{g}$, the multiplicities $\text{mult}_{\Lambda, \hat{\mathfrak{g}}}(\lambda)$ are the Kostka coefficients. The support

$$\Gamma(\mathfrak{g}, \hat{\mathfrak{g}}) = \{ (\Lambda, \lambda) \in P_+ \times \hat{P}_+ : \text{mult}_{\Lambda, \hat{\mathfrak{g}}}(\lambda) \neq 0 \}$$

of these multiplicities is also a fascinating object. Actually, it is a finitely generated semigroup that generates a polyhedral convex cone. For $\hat{\mathfrak{g}}$ diagonally embedded in $\mathfrak{g} = \hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$ this cone is the famous Horn cone. Its description has a very long and rich story (see [3], [2], [8], [1], [13]).

In this paper, we are interested in similar questions for affine Kac-Moody algebras. Assume now that $\mathfrak{g}$ is an affine Kac-Moody algebra and consider integrable highest weight $\mathfrak{g}$-modules $L(\Lambda)$ as module over some subalgebra $\hat{\mathfrak{g}}$. In the following three cases, we have decompositions similar to (1) with finite multiplicities:

1. $\hat{\mathfrak{g}} = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$;
2. $\hat{\mathfrak{g}}$ diagonally embedded in $\mathfrak{g} = \hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$; case of tensor product decomposition.
3. $\hat{\mathfrak{g}}$ is a winding subalgebra of $\mathfrak{g}$ introduced by V. G. Kac and M. Wakimoto in [6].

Recently, several authors studied $\Gamma(\mathfrak{g}, \hat{\mathfrak{g}})$ in the case of the tensor product decomposition of affine (or symmetrizable) Kac-Moody Lie algebras (see [12], [10], [11], [9]).

Here, we start a study of $\Gamma(\mathfrak{g}, \hat{\mathfrak{g}})$ in the winding case. Fix a winding subalgebra $\hat{\mathfrak{g}} = \mathfrak{g}[u]$ of $\mathfrak{g}$ for some given positive integer $u$. It is a subalgebra of $\hat{\mathfrak{g}}$ isomorphic to $\mathfrak{g}$ but nontrivially embedded (the embedding depends on $u$) (see [6] or subsection 4.1 for details).

Let $\delta$ be the basic imaginary root and $d$ be the scaling element, for any set $S \subset \mathfrak{h}^*$, we denote by $\overline{S}$ the subset of all $\lambda \in S + \mathbb{C}\delta$ such that $\lambda(d) = 0$. Let $k \in \mathbb{Z}_{\geq 0}$, we denote by $S^k$ the subset of $S$ of weights of level $k$.

Let $P$ (resp. $\hat{P}$) be the set of all integral weights of $\mathfrak{g}$ (resp. $\mathfrak{g}[u]$). The subset of dominant integral weights of $P$ is denoted by $P_+$. For $\Lambda \in P_+$ (resp. $\lambda \in \hat{P}_+$), let $L(\Lambda)$ (resp. $\hat{L}(\lambda)$) be the integrable irreducible highest weight $\mathfrak{g}$-module (resp. $\mathfrak{g}[u]$-module) with highest weight $\Lambda$ (resp. $\lambda$).

Let $Q$ be the root lattice of $\mathfrak{g}$. Fix $\Lambda \in P_+$, then for each $\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P}$, there exists an unique number $b_{\Lambda, \lambda} \in \mathbb{C}$ such that $\lambda + (b_{\Lambda, \lambda} + n)\delta$ is a weight of $L(\Lambda)$ only for $n \in \mathbb{Z}_{\leq 0}$. Also for each $\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \overline{\hat{P}}$, there exists an unique number $b_{\Lambda, \lambda, u} \in \mathbb{C}$ such that $\hat{L}(\lambda + (b_{\Lambda, \lambda, u} + n)\delta) \subset L(\Lambda)$ only for $n \in \mathbb{Z}_{\leq 0}$.

The two semigroups can be described as

$$\Gamma(\mathfrak{g}, \hat{\mathfrak{g}}) = \bigcup_{\Lambda \in P_+} \bigcup_{\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P}} (\Lambda, \lambda + (b_{\Lambda, \lambda} - \mathbb{Z}_{\geq 0})\delta), \quad (2)$$

$$\Gamma(\mathfrak{g}, \mathfrak{g}[u]) = \bigcup_{\Lambda \in P_+} \bigcup_{\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \overline{\hat{P}}} (\Lambda, \lambda + (b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0})\delta). \quad (3)$$

Our first main result is the following theorem.
Theorem 1.1. As a subset of $\mathfrak{h}^* \times \mathfrak{h}^*$, the set $\Gamma(g, g[u])$ is a semigroup. Moreover, we have
\[ \Gamma(g, g[u]) \subset \Gamma(g, \mathfrak{h}) \cap (P_+ \times \bar{P}_+). \] (4)

In particular, $b_{\Lambda, \lambda, u} \leq b_{\Lambda, \lambda}$ for any $\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \bar{P}_+$.

For the cases $g$ is of type $A_1^{(1)}$ and $A_2^{(2)}$, we can compute explicitly the number $b_{\Lambda, \lambda}$. Let $\Lambda_0$ be the first fundamental weight and $\alpha$ be the second simple root. Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} + b\delta \in P^m$, $(m \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}_{\geq 0}, b \in \mathbb{C})$. We then describe a subset $A_u(\Lambda)$ of the set of all $\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \bar{P}_+$ such that $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$. Namely, for the case $A_1^{(1)}$, we set
\[ A_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \bigg| j' \in [0, um] \cap (j + 2\mathbb{Z}) \cap [j - 1, um - j] \right\} \] if $u$ is even,
\[ A_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \bigg| j' \in [0, um] \cap (j + 2\mathbb{Z}) \cap [j - 1, m(u - 1) + j + 1] \right\} \] if $u$ is odd.

For the case $A_2^{(2)}$, we set
\[ A_u(\Lambda_0 + b\delta) = \left\{ \Lambda_0 + \frac{j\alpha}{2} \bigg| j' \in [0, u/2] \cap \mathbb{Z} \right\}, \] (7)
\[ A_u(2\Lambda_0 + b\delta) = \left\{ 2\Lambda_0 + \frac{j\alpha}{2} \bigg| j' \in [0, u] \cap \mathbb{Z} \right\}, \] (8)
\[ A_u(2\Lambda_0 + \frac{\alpha}{2} + b\delta) = \left\{ 2\Lambda_0 + \frac{j\alpha}{2} \bigg| j' \in [0, u] \cap \mathbb{Z} \right\}, \] (9)
\[ A_u(\Lambda) = A_u^{(1)}(\Lambda) \sqcup A_u^{(2)}(\Lambda) \text{ if } m > 2, \] (10)

where
\[ A_u^{(1)}(\Lambda) = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \bigg| j' \in [j, \frac{um}{2}] \cap \mathbb{Z} \cap \left( \frac{m(u - 1)}{2} - j + \mathbb{Z}_{\geq 0} \right) \right\}, \] (11)
\[ A_u^{(2)}(\Lambda) = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \bigg| j' \in [j, \frac{um}{2}] \cap \mathbb{Z} \cap \left( \frac{m(u - 1)}{2} - j + \mathbb{Z}_{< 0} \right) \right\}. \] (12)

For $A_1^{(1)}$ and $A_2^{(2)}$, we define a subset $A_u$ of $P_+ \times \bar{P}_+$ by
\[ A_u = \bigcup_{\Lambda \in P_+} \bigcup_{\lambda \in A_u(\Lambda)} (\Lambda, \lambda + (b_{\Lambda, \lambda} - \mathbb{Z}_{\geq 0}) \delta). \] (13)

For the case $A_2^{(2)}$, we define smaller subsets of $A_u$ by
\[ A_u^{(1)} = \bigcup_{\Lambda \in P^m, m > 2} \bigcup_{\lambda \in A_u^{(1)}(\Lambda)} (\Lambda, \lambda + (b_{\Lambda, \lambda} - \mathbb{Z}_{\geq 0}) \delta), \] (14)
\[ A_u^{(2)} = \bigcup_{\Lambda \in P^m, m > 2} \bigcup_{\lambda \in A_u^{(2)}(\Lambda)} (\Lambda, \lambda + (b_{\Lambda, \lambda} - \mathbb{Z}_{\geq 0}) \delta). \] (15)
Theorem 1.2. Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $A_1^{(1)}$ or $A_2^{(2)}$. Let $\Lambda \in P_+$. Then for any $\lambda \in \mathcal{A}_u(\Lambda)$ we have
\[ b_{\Lambda,\lambda,u} = b_{\Lambda,\lambda}. \]  
Hence
\[ \mathcal{A}_u = \mathcal{A}_u \cap \Gamma(\mathfrak{g}, \mathfrak{g}[u]) = \mathcal{A}_u \cap \Gamma(\mathfrak{g}, \mathfrak{h}). \]  
Moreover, $\mathcal{A}_u$ is a semigroup only for the case $A_1^{(1)}$. In the case $A_2^{(2)}$, the restriction of $\mathcal{A}_u$ on the pairs of weights of level greater than 2 is a disjoint union of two semigroups $\mathcal{A}_1^{(1)}$ and $\mathcal{A}_2^{(2)}$.

This article is organized as follows. In Section 2, we prepare fundamental knowledge of affine Kac-Moody algebras. In Section 3, we present results around branching on Cartan subalgebras. We prove that the set $\Gamma(\mathfrak{g}, \mathfrak{h})$ is a semigroup. We also compute the number $b_{\Lambda,\lambda}$ for the cases $A_1^{(1)}$ and $A_2^{(2)}$. In Section 4, we introduce the notion of winding subalgebras. The main results of this article, Theorem 1.1 and Theorem 1.2 above, will be presented in Section 4.

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2 Preliminaries

In this section, we recall basic fact about affine Kac-Moody algebras in [4], [5], [6].
2.1 Generalized Cartan matrix of affine type

Set $I = \{0, \ldots, l\}$. Let $A = (a_{ij})_{i, j \in I}$ be a generalized Cartan matrix of affine type, i.e., $A$ is indecomposable of corank 1, $a_{ii} = 2$, $-a_{ij} \in \mathbb{Z}_{\geq 0}$ for $i \neq j$, $a_{ij} = 0$ iff $a_{ji} = 0$ and there exists a column vector $u$ with positive integer entries such that $Au = 0$.

Let $a = t(a_0, \ldots, a_l)$ and $c = (c_0, \ldots, c_l)$ be the vectors of relatively prime integers such that $a_i, c_i > 0$ and $Aa = cA = 0$. The Coxeter number and dual Coxeter number of $A$ are defined by $h = \sum_{i \in I} a_i$ and $h^\vee = \sum_{i \in I} c_i$.

2.2 Realization of a generalized Cartan matrix

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $A$ where $\mathfrak{h}$ is a $\mathbb{C}$-vector space of dimension $l + 2$, $\Pi^\vee = \{h_0, \ldots, h_l\}$ is a linearly independent subset in $\mathfrak{h}$ and $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ is a linearly independent subset in $\mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) such that $\alpha_i(h_j) = \delta_{ij}$.

Let $K = \sum_{i \in I} c_i h_i$ be the canonical central element and $\delta = \sum_{i \in I} a_i \alpha_i$ be the basic imaginary root. Let $d \in \mathfrak{h}$ be the scaling element, i.e., $\alpha_0(d) = 1, \alpha_i(d) = 0$ for $i > 0$. Let $\Lambda_i (i \in I)$ be the fundamental weights, i.e., $\Lambda_i(h_j) = \delta_{ij}, \Lambda_i(d) = 0$ for all $j \in I$. Set $\rho = \sum_{i \in I} \Lambda_i$. Then $\{\alpha_0, \ldots, \alpha_l, \Lambda_0\}$ is a basis of $\mathfrak{h}^*$ and $\{h_0, \ldots, h_l, d\}$ is a basis of $\mathfrak{h}$.

2.3 Affine Kac-Moody algebras

Let $\mathfrak{g}(A)$ be the affine Kac-Moody algebra corresponding to the matrix $A$. We call $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}(A)$ and $\Pi, \Pi^\vee$ the set of simple roots, coroots of $\mathfrak{g}(A)$, respectively. We have a triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_-$ is the negative subalgebra of $\mathfrak{g}(A)$ and $\mathfrak{n}_+$ is the positive subalgebra of $\mathfrak{g}(A)$.

An affine Kac-Moody algebra has type $X^{(r)}_N$ with $r = 1, 2, 3$ (here we use the standard notation in [5] to label the type of affine Kac-Moody algebras). In particular, the untwisted affine Kac-Moody of type $A^{(1)}_1$ is defined by the generalized Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

and the twisted affine Kac-Moody algebra of type $A^{(2)}_2$ is defined by the generalized Cartan matrix

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$
2.4 Weyl group

Let $Q = \mathbb{Z}\Pi$ be the root lattice and let $Q_+ = \mathbb{Z}_{\geq 0}\Pi$. We define an order on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. For each $i \in I$, set $\alpha_i^\vee = \frac{2}{\alpha_i} \alpha_i$. We define a sublattice $M$ of $Q$ by

$$M = \bigoplus_{i=0}^{l} \mathbb{Z}\alpha_i^\vee$$

if $r^\vee = 1$,

$$M = \bigoplus_{i=0}^{l} \mathbb{Z}\alpha_i^\vee$$

if $r^\vee > 1$.

Let $(\cdot \mid \cdot)$ be the standard invariant form on $\mathfrak{h}^*$ which is defined by

$$(\alpha_i \mid \alpha_j) = \frac{c_{ij}}{a_i} a_{ji}, \quad (\alpha_i \mid \Lambda_0) = \frac{1}{a_0} \alpha_i(d), \quad (\Lambda_0 \mid \Lambda_0) = 0 \quad \forall i,j \in I.$$ (22)

For each $\alpha \in \mathfrak{h}^*$, we define $t_\alpha \in GL(\mathfrak{h}^*)$ by

$$t_\alpha(\lambda) = \lambda + \lambda(K)\alpha - \left( \lambda + \frac{\lambda(K)\alpha}{\tau} \right) \delta.$$ (23)

Let $W$ be the Weyl group of $\mathfrak{g}(A)$ the group generated by fundamental reflections $\{ s_0, \ldots, s_l \}$ where $s_i \in GL(\mathfrak{h}^*)$ is defined by $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. Then we have

$$W \cong t_M \times \bar{W}$$ (24)

where $t_M = \{ t_\alpha \mid \alpha \in M \}$ and $\bar{W}$ is the subgroup of $W$ generated by $\{ s_1, \ldots, s_l \}$.

2.5 Realization of affine Kac-Moody algebras

Let $B = (b_{ij})_{i,j \in \{1,\ldots,l\}}$ be a Cartan matrix of finite type. Let $\mathfrak{g}(B)$ be the simple Lie algebra associated to $B$ with Lie bracket $[,]_0$, the standard invariant form $(\cdot \mid \cdot)_0$ (Here we use the notion of standard invariant form for a simple Lie algebra in the book of Carter [4]). We extend $\mathfrak{g}(B)$ to a new Lie algebra

$$\tilde{\mathfrak{g}}(B) = C[t, t^{-1}] \otimes \mathfrak{g}(B) \oplus CK \oplus Cd$$ (25)

with new Lie bracket

$$[t^i \otimes x + \lambda K + \mu d, t^j \otimes y + \lambda' K + \mu' d]$$

$$= t^{i+j} \otimes [x, y]_0 + \mu j t^{i} \otimes y - \mu' t^{j} \otimes x + i \delta_{i+j,0}(x|y)_0 K.$$ (26)

for all $i,j \in \mathbb{Z}$, $x,y \in \mathfrak{g}(B)$ and $\lambda, \lambda', \mu, \mu' \in \mathbb{C}$.

Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\tilde{\mathfrak{g}}(B)$ and let $\tilde{\Phi}$ be the root system of $\tilde{\mathfrak{g}}(B)$. For each $\alpha \in \tilde{\Phi}$, let

$$\tilde{\mathfrak{g}}(B)_\alpha = \{ x \in \tilde{\mathfrak{g}}(B) \text{ such that } [h, x]_0 = \alpha(h)x \text{ for all } h \in \tilde{\mathfrak{h}} \}.$$ (27)

Let $\tilde{\Pi} = \{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_l \}$ be the set of simple roots and $\tilde{\Pi}^\vee = \{ \tilde{h}_1, \ldots, \tilde{h}_l \}$ be the set of coroots of $\tilde{\mathfrak{g}}(B)$. It is known that the dimension of $\mathfrak{g}(B)_\alpha$ is one for each $\alpha \in \tilde{\Phi}$. For each $i \in \{1, \ldots, l\}$, let $\tilde{e}_i$ be a basis vector of $\tilde{\mathfrak{g}}(B)_{\tilde{\alpha}_i}$ and $\tilde{f}_i$ be a basis vector of $\tilde{\mathfrak{g}}(B)_{-\tilde{\alpha}_i}$. Then $\mathfrak{g}(B)$ is a Lie algebra with generators $\{ \tilde{h}_1, \ldots, \tilde{h}_l, \tilde{e}_1, \ldots, \tilde{e}_l, \tilde{f}_1, \ldots, \tilde{f}_l \}$.

For any $\sigma \in S_l$ such that $b_{ij} = b_{\sigma(i)\sigma(j)}$ for all $i,j \in \{1, \ldots, l\}$, we can consider it as an automorphism of $\tilde{\mathfrak{g}}(B)$ by sending

$$\tilde{e}_i \mapsto \tilde{e}_{\sigma(i)}, \quad \tilde{f}_i \mapsto \tilde{f}_{\sigma(i)}, \quad \tilde{h}_i \mapsto \tilde{h}_{\sigma(i)}.$$ (28)
Let $m$ be the order of $\sigma$ and $\eta = e^{2\pi i/m}$. We define the automorphism $\tau$ of $\hat{g}(B)$ by

$$
\tau(t^i \otimes x) = \eta^{-i}t^i \otimes \sigma(x), \tau(K) = K, \tau(d) = d
$$

for all $i \in \mathbb{Z}, x \in g(B)$. This map is called a twisted automorphism of $\hat{g}(B)$.

Now, let $A$ be an affine Cartan matrix of type $X_{rN}^{(r)}$. Let $\bar{A}$ be the finite Cartan matrix of type $X_{N}^{(r)}$. If $A$ is an untwisted affine Cartan matrix, i.e., type $X_{N}^{(1)}$, then we have

$$
g(A) \simeq \hat{g}(\bar{A}).
$$

If $A$ is a twisted affine Cartan matrix, i.e., type $X_{N}^{(r)}$ for $r = 2, 3$, we have

$$
g(A) \simeq \hat{g}(\bar{A})^{(r)}.
$$

The simple coroots $h_1, \ldots, h_l$ of $g(A)$ have property

$$
h_i \in 1 \otimes \bar{h}
$$

for each $i \in \{1, \ldots, l\}$, where $\bar{h}$ is the Cartan subalgebra of $\hat{g}(\bar{A})$. For the details, we refer the reader to the proofs of Theorem 18.5, Theorem 18.9 and Theorem 18.14 in [4].

### 2.6 Dominant integral weights and integrable irreducible modules

Let

$$
P_{+} = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i + \mathbb{C}\delta
$$

be the set of dominant integral weights. For any set $S \subset h^*$, we denote by $\overline{S}$ the subset of all $\lambda \in S + \mathbb{C}\delta$ such that $\lambda(d) = 0$. We have

$$
\overline{P}_{+} = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i.
$$

For each $\lambda \in P_{+}$, the number $\lambda(K)$ is a non-negative integer and we call it the level of $\lambda$. For each $k \in \mathbb{Z}_{\geq 0}$, we denote by $P_{+}^k$ the set of all dominant integral weights of level $k$. Then we have

$$
P_{+}^k = \left\{ \sum_{i \in I} m_i \Lambda_i \left| \sum_{i \in I} m_i c_i = k, m_i \in \mathbb{Z}_{\geq 0} \right. \right\} + \mathbb{C}\delta.
$$

To forget the dominant property, we use the notations $P, \overline{P}, P^k$.

For each $\lambda \in P_{+}$, let $L(\lambda)$ be the integrable irreducible highest weight $g(A)$-module with highest weight $\lambda$.

### 3 Branching on Cartan subalgebras

In this section, we recall some facts about branching on Cartan subalgebras of affine Kac-Moody algebras.

Let $\Lambda \in P_{+}$ be a dominant integral weight of $g(A)$, $\mathfrak{h}$ be a Cartan subalgebra of $g(A)$. Regarding $g(A)$-module $L(\Lambda)$ as an $\mathfrak{h}$-module, it can be decomposed into direct sum of weights spaces

$$
L(\Lambda) = \bigoplus_{\lambda \in h^*} L(\Lambda)^{mult_{A,\mathfrak{h}}}(\lambda),
$$

(36)
where
\[ L(\Lambda) = \{ v \in L(\Lambda) \mid hv = \lambda(h)v \ \forall h \in \mathfrak{h} \}. \] (37)
The decomposition (36) is encoded by a formal series \( ch_{\Lambda} \) on \( \mathfrak{h}^* \) as follows
\[ ch_{\Lambda} = \sum_{\lambda \in \mathfrak{h}^*} \text{mult}_{\Lambda, \mathfrak{h}}(\lambda) e^{\lambda}, \] (38)
where \( e^{\lambda}(\mu) = \delta_{\lambda, \mu} \) for \( \mu \in \mathfrak{h}^* \). We call \( ch_{\Lambda} \) the character of \( L(\Lambda) \). The set of weights of \( L(\Lambda) \) is defined by
\[ P(\Lambda) = \{ \lambda \in \mathfrak{h}^* \mid \text{mult}_{\Lambda, \mathfrak{h}}(\lambda) \neq 0 \}. \] (39)
For each \( \lambda \in \mathfrak{h}^* \), we say \( \lambda \) a maximal weight of \( L(\Lambda) \) if \( \lambda \in P(\Lambda) \) but \( \lambda + n\delta \notin P(\Lambda) \) for any \( n > 0 \). Let \( \text{max}(\Lambda) \) be the set of all maximal weights of \( L(\Lambda) \). We have
\[ \text{max}(\Lambda) = W((\Lambda - Q_+) \cap P_+) \] (40)
and
\[ P(\Lambda) = W((\Lambda - Q_+) \cap P_+) = \text{max}(\Lambda) - Z_{\geq 0}\delta. \] (41)

3.1 About the character \( ch_{\Lambda} \)
We now recall some facts about the character \( ch_{\Lambda} \) for any affine Kac-Moody algebra.

Denote \( e^{-\delta} \) by \( q \). For each \( \lambda \in \mathfrak{h}^* \), we define the string function \( c^A_{\lambda} \in \mathbb{C}(q) \) of \( L(\Lambda) \) associated to \( \lambda \) by
\[ c^A_{\lambda} = \sum_{n \in \mathbb{Z}} \text{mult}_{\Lambda, \mathfrak{h}}(\lambda - n\delta)q^n. \] (42)
Then for any \( w \in W \) we have
\[ c^A_{\lambda} = c^A_{w\lambda}. \] (43)
and
\[ ch_{\Lambda} = \sum_{\lambda \in \text{max}(\Lambda)} c^A_{\lambda} e^{\lambda}. \] (44)
The character \( ch_{\Lambda} \) can be written in terms of Weyl group, what we call the Weyl-Kac character formula (see Corollary 19.18 in [4]):
\[ ch_{\Lambda} = \frac{\sum_{w \in W} \epsilon(w)e^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w)e^{w(\rho)}}. \] (45)

3.2 About the set of weights \( P(\Lambda) \)
In this subsection, we show some facts about the set of weights \( P(\Lambda) \) for any affine Kac-Moody algebra and in particular for the cases \( A_1^{(1)} \) and \( A_2^{(2)} \).

The formula (41) says that for each \( \lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \mathcal{P} \), there exists uniquely a number \( b_{\Lambda, \lambda} \in \mathbb{C} \) such that \( \lambda + b_{\Lambda, \lambda}\delta \in \text{max}(\Lambda) \). Hence,
\[ P(\Lambda) = \bigcup_{\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \mathcal{P}} (\lambda + (b_{\Lambda, \lambda} - Z_{\geq 0})\delta). \] (46)
Since $\lambda + b\delta \in P(\Lambda + b\delta)$ if and only if $\lambda \in P(\Lambda)$ for all $b \in \mathbb{C}$, we have
\[
b_{\Lambda+b\delta,\lambda} = b_{\Lambda,\lambda} + b. \tag{47}\]

We may assume that $\Lambda \in \overline{P}_+$. With this assumption, in particular, for the cases $A_1^{(1)}$ and $A_2^{(2)}$, we can compute explicitly the set $\text{max}(\Lambda)$, hence the number $b_{\Lambda,\lambda}$ and the set $P(\Lambda)$. The idea of computations bases on the work on S. Kumar and M. Brown in [9].

### 3.2.1 Semigroup structure

In this part, we study the set $\Gamma(g, h) \subset \mathfrak{h}^* \times \mathfrak{h}^*$.

**Theorem 3.1.** As a subset of $\mathfrak{h}^* \times \mathfrak{h}^*$, the set $\Gamma(g, h)$ is a semigroup.

**Proof.** Let $(\Lambda, \lambda)$ and $(\bar{\Lambda}, \bar{\lambda})$ be elements in the set $\Gamma(g, h)$. We will show that $(\Lambda + \bar{\Lambda}, \lambda + \bar{\lambda})$ is still in this set. Indeed, $\lambda + \bar{\lambda}$ is a weight of $L(\Lambda) \otimes L(\bar{\Lambda})$. Hence $\lambda + \bar{\lambda}$ is a weight of $L(\Lambda'')$ for some $\Lambda'' \in ((\Lambda + \bar{\Lambda}) - Q_+) \cap \overline{P}_+$. By (41), we have
\[
P(\Lambda'') \subset P(\Lambda + \bar{\Lambda}). \tag{48}\]

It means $\lambda + \bar{\lambda} \in P(\Lambda + \bar{\Lambda})$ and then $\Gamma(g, h)$ is a semigroup. \qed

**Remark 3.2.** We can prove that $\Gamma(g, h)$ is a semigroup for any symmetrizable Kac-Moody algebra $\mathfrak{g}$ by the same argument.

### 3.2.2 Computation for the case $A_1^{(1)}$

Let $A$ be the affine Cartan matrix of type $A_1^{(1)}$. Fix $m \in \mathbb{Z}_{\geq 0}$. Let $\alpha$ be the second simple root of $g(A)$. We have
\[
\overline{P}^m_+ = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \right\} \quad j \in [0, m] \cap \mathbb{Z}. \tag{49}\]

We can describe explicitly the set $\text{max}(\Lambda)$ and number $b_{\Lambda,\lambda}$ for the case $A_1^{(1)}$ as follows.

**Proposition 3.3.** With the setting for $A_1^{(1)}$, let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \overline{P}^m_+$. For each $k \in \mathbb{Z}$, let $n_k$ be a number which is uniquely determined by $k, m, j$ as follows

1. Write $k = mq + r$ for some $q \in \mathbb{Z}, r \in [0, m]$.

2. Set
\[
n_k = -q(k + r + j) + \begin{cases} -r & \text{if } r \in [0, m - j], \\ -m - j - 2r & \text{if } r \in [m - j, m]. \end{cases} \tag{50}\]

Then we have
\[
\text{max}(\Lambda) = \{ \Lambda + k\alpha + n_k\delta \mid k \in \mathbb{Z} \}. \tag{51}\]

Or, equivalently, for each $\lambda = m\Lambda_0 + \frac{j'\alpha}{2} \in \text{max}(\Lambda)$, we have
\[
b_{\Lambda,\lambda} = n_{j' + 2r}. \tag{53}\]
To prove above proposition, we need the following lemma.

**Lemma 3.4.** For any affine Kac-Moody algebra, let \( \Lambda \in P_+ \), then

\[
\max(\Lambda) \cap P_+ = \left\{ \Lambda - \sum_{i \in I} m_i \alpha_i \mid m_i \in \mathbb{Z}_{\geq 0} \text{ for all } i, m_i < a_i \text{ for some } i \in I \right\} \cap P_+. \tag{54}
\]

**Proof.** Since \( P(\Lambda) = W((\Lambda - Q_+) \cap P_+) \) and if \( \mu \in P(\Lambda) \), then \( \mu \leq \Lambda \), we have

\[
\max(\Lambda) \cap P_+ = \{ \mu \in P(\Lambda) \cap P_+, \mu + \delta \notin P(\Lambda) \}
= \{ \mu \in P(\Lambda) \cap P_+, \mu + \delta \notin \Lambda \}
= \left\{ \mu \in P(\Lambda) \cap P_+, \mu = \Lambda - \sum_{i \in I} m_i \alpha_i \mid m_i < a_i \text{ for some } i \in I \right\}
= \left\{ \Lambda - \sum_{i \in I} m_i \alpha_i \mid m_i \in \mathbb{Z}_+ \text{ for all } i, m_i < a_i \text{ for some } i \in I \right\} \cap P_+.
\]

So, we get the conclusion. \( \square \)

With the aid of Lemma 3.4, we can prove Proposition 3.3 as follows.

**Proof.** We have \( \max(\Lambda) = W(\max(\Lambda) \cap P_+) \). By Lemma 3.4,

\[
\max(\Lambda) \cap P_+ = \left\{ \Lambda - m_0(\delta - \alpha), \Lambda - m_1 \alpha \mid m_i \in \mathbb{Z}_{\geq 0}, m_0 \leq \frac{m-j}{2}, m_1 \leq \frac{j}{2} \right\}. \tag{55}
\]

Recall that \( W = \{ t_{n\alpha}, t_{n\alpha}s_1, \ldots \mid n \in \mathbb{Z} \} \). We have

\[
t_{n\alpha}(\Lambda - m_0(\delta - \alpha)) = \Lambda + (m_0 + mn)\alpha - ((j + 2m_0 + mn)n + m_0)\delta, \tag{56}
\]

\[
t_{n\alpha}s_1(\Lambda - m_0(\delta - \alpha)) = \Lambda + (-j - m_0 + mn)\alpha - ((-j - 2m_0 + mn)n + m_0)\delta, \tag{57}
\]

\[
t_{n\alpha}(\Lambda - m_1 \alpha) = \Lambda + (-m_1 + mn)\alpha - (j - 2m_1 + mn)n\delta, \tag{58}
\]

\[
t_{n\alpha}s_1(\Lambda - m_1 \alpha) = \Lambda + (-j + m_1 + mn)\alpha - (-j + 2m_1 + mn)n\delta. \tag{59}
\]

So an element in \( \max(\Lambda) \) has form \( \Lambda + k\alpha + n_k' \delta \) for some \( n_k' \), \( k \in \mathbb{Z} \). Suppose that \( \Lambda + r\alpha + n_r' \delta \in \max(\Lambda) \), then

\[
t_{qm}(\Lambda + r\alpha + n_r' \delta) = \Lambda + (mq + r)\alpha + (n_r' - (j + 2r + mq)q)\delta \tag{60}
\]

is still in \( \max(\Lambda) \). So, if we write \( k = mq + r \) for some \( q \in \mathbb{Z} \), then

\[
n_r' = n_r' - q(k + r + j). \tag{61}
\]

Let \( 0 \leq r < m \), then the expression \( k = mq + r \) is unique. By (56), (57), (58), (59) we get

\[
n_r' = \begin{cases} -r & \text{if } r \in [0, m-j], \\ m - j - 2r & \text{if } r \in [m-j, m]. \tag{62} \end{cases}
\]

Hence we obtain \( n_k' = n_k \) given by (50). It means

\[
\max(\Lambda) = \{ \Lambda + k\alpha + n_k \delta \mid k \in \mathbb{Z} \}. \tag{63}
\]

Since \( \lambda = \Lambda + \frac{i\alpha}{2} \) then \( \lambda + n_{\frac{i\alpha}{2}} \delta \in \max(\Lambda) \). Hence \( b_{\Lambda, \lambda} = n_{\frac{i\alpha}{2}} \) by definition. \( \square \)
3.2.3 Computation for the case $A_2^{(2)}$

For the case $A_2^{(2)}$, the computation is similar. Namely, let $A$ be the affine Cartan matrix of type $A_2^{(2)}$. Fix $m \in \mathbb{Z}_{\geq 0}$. Let $\alpha$ be the second simple root of $\mathfrak{g}(A)$. We have

$$\mathcal{P}_+^m = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \mid j \in \left[0, \frac{m}{2}\right] \cap \mathbb{Z} \right\}. \quad (64)$$

We can describe explicitly the set $\max(\Lambda)$ and number $b_{\Lambda, \lambda}$ for the case $A_2^{(2)}$ as follows.

**Proposition 3.5.** With the setting for $A_2^{(2)}$, let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \mathcal{P}_+^m$. For each $k \in \frac{1}{2}\mathbb{Z}$, let $n_k$ be a number which is uniquely determined by $k, m, j$ as follows.

1. Write $k = \frac{m}{2}q + r$ for some $q \in \mathbb{Z}, r \in [0, \frac{m}{2})$.
2. Set

$$n_k = -q(k + r + j) + \begin{cases} -r & \text{if } r \in [0, \frac{m}{2} - j], \\ \frac{m}{2} - j - 2r & \text{if } r \in \left(\frac{m}{2} - j, \frac{m}{2}\right) \cap \left(\frac{m}{2} + \mathbb{Z}\right) \\ \frac{m-1}{2} - j - 2r & \text{if } r \in \left(\frac{m-1}{2} - j, \frac{m}{2}\right) \cap \left(\frac{m-1}{2} + \mathbb{Z}\right). \end{cases} \quad (65)$$

Then we have

$$\max(\Lambda) = \left\{ \Lambda + k\alpha + n_k\delta \mid k \in \frac{1}{2}\mathbb{Z} \right\}. \quad (66)$$

Or, equivalently, for each $\lambda = m\Lambda_0 + \frac{j\alpha}{2}$ in the set

$$(\Lambda - Q + \mathbb{C}\delta) \cap \overline{\mathcal{P}} = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in \mathbb{Z} \right\}, \quad (67)$$

we have

$$b_{\Lambda, \lambda} = n_{j' - j}. \quad (68)$$

**Proof.** We have $\max(\Lambda) = W(\max(\Lambda) \cap P_+)$. By Lemma 3.4, $\max(\Lambda) \cap P_+$ contains exactly elements

$$\Lambda - m_0\alpha_0, \Lambda - m_1\alpha, \Lambda - \alpha_0 - m_2\alpha \quad (69)$$

such that

$$m_0, m_1, m_2 \in \mathbb{Z}_{\geq 0}, m_0 \leq \frac{m}{2} - j, m_1 \leq \frac{j}{2} \leq \frac{j+1}{2} \leq m_2 \leq \frac{j+1}{2}. \quad (70)$$

Recall that $W = \left\{ t_{\frac{m}{2}}, t_{\frac{m}{2}}s_1 \mid n \in \mathbb{Z} \right\}$. We have

$$t_{\frac{m}{2}}(\Lambda - m_0\alpha_0) = \Lambda + \frac{mn + m_0}{2}\alpha - \frac{(mn + 2j + 2m_0)n + m_0\delta}{2}, \quad (71)$$

$$t_{\frac{m}{2}}s_1(\Lambda - m_0\alpha_0) = \Lambda + \frac{mn - 2j - m_0}{2}\alpha - \frac{(mn - 2j - 2m_0)n + m_0\delta}{2}, \quad (72)$$

$$t_{\frac{m}{2}}(\Lambda - m_1\alpha) = \Lambda + \frac{mn - 2m_1}{2}\alpha - \frac{(mn + 2j - 4m_1)n + m_0\delta}{2}, \quad (73)$$

$$t_{\frac{m}{2}}s_1(\Lambda - m_1\alpha) = \Lambda + \frac{mn - 2j + 2m_1}{2}\alpha - \frac{(mn - 2j + 4m_1)n + m_0\delta}{2}, \quad (74)$$

$$t_{\frac{m}{2}}(\Lambda - \alpha_0 - m_2\alpha) = \Lambda + \frac{mn + 1 - 2m_2}{2}\alpha - \frac{(mn + 2j + 2 - 4m_2)n + 1}{2}\delta, \quad (75)$$

$$t_{\frac{m}{2}}s_1(\Lambda - \alpha_0 - m_2\alpha) = \Lambda + \frac{mn - 1 - 2j + 2m_2}{2}\alpha - \frac{(mn - 2j - 2 + 4m_2)n + 1}{2}\delta. \quad (76)$$

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So an element in $\text{max}(\Lambda)$ has form $\Lambda + k\alpha + n_k'\delta$ for some $n_k'$, $k \in \frac{1}{2}\mathbb{Z}$. Suppose that $\Lambda + r\alpha + n_k'\delta \in \text{max}(\Lambda)$, then
\[
\text{max}(\Lambda) = \Lambda + \left(\frac{m}{2}q + r\right)\alpha + \left(n_k' - \left(j + 2r + \frac{m}{2}q\right)q\right)\delta \quad (77)
\]
is still in $\text{max}(\Lambda)$. So, if we write $k = \frac{m}{2}q + r$ for some $q \in \mathbb{Z}$, then
\[
n_k' = n_k' - q(k + r + j). \quad (78)
\]
Let $0 \leq r < \frac{m}{2}$, then the expression $k = \frac{m}{2}q + r$ is unique. By (71), (72), (73), (74), (75), (76) we get
\[
n_k' = \left\{ \begin{array}{ll}
-r & \text{if } r \in [0, \frac{m}{2} - j], \\
\frac{m}{2} - j - 2r & \text{if } r \in \left[\frac{m}{2} - j, \frac{m}{2}\right) \cap (\frac{m}{2} + \mathbb{Z}), \\
\frac{m}{2} - j - 2r & \text{if } r \in \left[\frac{m}{2} - j, \frac{m}{2}\right) \cap (\frac{m}{2} - 1 + \mathbb{Z}).
\end{array} \right. \quad (79)
\]
Hence, we obtain $n_k' = n_k$ given by (65). It means
\[
\text{max}(\Lambda) = \left\{ \Lambda + k\alpha + n_k\delta \mid k \in \frac{1}{2}\mathbb{Z} \right\}. \quad (80)
\]
Since $\lambda = \Lambda + \frac{\delta}{\alpha + \delta$, then $\lambda + n\frac{\alpha}{\alpha + \delta} \in \text{max}(\Lambda)$. Hence, $b_{\lambda, \lambda} = n\frac{\alpha}{\alpha + \delta}$ by definition. \(\square\)

4 Branching on winding subalgebras

In this section, we study the branching problem on winding subalgebras.

4.1 Winding subalgebras of an affine Kac-Moody algebra

In this subsection, we recall the notation of winding subalgebras of an affine Kac-Moody algebra in [6].

Let $g(A)$ with $A$ of type $X_N^{(r)}$ be an affine Kac-Moody algebra which is defined by (30), (31). Fix $u \in \mathbb{Z}_{>0}$ relatively prime with $r$. We define the Lie homomorphism $\psi_u : g(A) \to g(A)$ by
\[
t^i \otimes x \mapsto t^{u_i} \otimes x, K \mapsto uK, d \mapsto \frac{d}{u} \quad (81)
\]
where $i \in \mathbb{Z}, x \in g(\bar{A})$. It is easy to check that $\psi_u$ is an injective Lie homomorphism. Let $g(A)[u]$ be the image of this map. Then $g(A)[u]$ is a subalgebra of $g(A)$ and isomorphic to $g(A)$. We call $g(A)[u]$ the winding subalgebra of $g(A)$ associated to $u$.

Set $\bar{K} = \psi_u(K) = uK$. Let $\tilde{\psi}_u : \mathfrak{h} \to \mathfrak{h}$ be the restriction of $\psi_u$ over the Cartan subalgebra $\mathfrak{h}$ of $g(A)$. For each $i \in I$, set $\hat{h}_i = \tilde{\psi}_u(h_i)$. Then by (32), we see that
\[
\hat{h}_i = h_i \text{ for all } i > 0 \text{ and } \hat{h}_0 = \frac{u - 1}{c_0}K + h_0. \quad (82)
\]
Let $t\psi_u : \mathfrak{h}^* \to \mathfrak{h}^*$ the dual map of $\tilde{\psi}_u$. Namely, for each $\lambda \in \mathfrak{h}^*$ we define $t\psi_u$ by
\[
t\psi_u(\lambda)(h) = \lambda(\tilde{\psi}_u(h)) \quad (83)
\]
for all $h \in \mathfrak{h}$. For each $i \in I$, set $\hat{\alpha}_i = t\psi_u(\alpha_i)$. Then by (82), (83) we have
\[
\hat{\alpha}_i = \alpha_i \text{ for all } i > 0 \text{ and } \hat{\alpha}_0 = \frac{u - 1}{a_0}\delta + \alpha_0. \quad (84)
\]
For each $i \in I$, set $\hat{\Lambda}_i = i\hat{\psi}_u(\Lambda_i)$ and $\hat{\rho} = i\hat{\psi}_u(\rho)$. By (82), (83) we have

$$\hat{\Lambda}_i = \Lambda_i + \left(\frac{1}{u} - 1\right) \frac{c_i}{c_0} \Lambda_0 \quad (85)$$

$$\hat{\rho} = \sum_i \hat{\Lambda}_i = \rho + \left(\frac{1}{u} - 1\right) h^\vee \Lambda_0. \quad (86)$$

The map $i\hat{\psi}_u$ induces simple reflections $\hat{s}_i \in Aut(\mathfrak{b}^*)$, which are defined by

$$\hat{s}_i(\lambda) = \lambda - \lambda(h_i)\hat{\alpha}_i. \quad (87)$$

The Weyl group $\hat{W}$ of $\mathfrak{g}(A)[u]$ is generated by simple reflections $\hat{s}_i (i \in I)$ turns out to be

$$\hat{W} \cong t_{u,M} \times \hat{W}. \quad (88)$$

Let

$$\hat{P}_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \hat{\Lambda}_i + \mathbb{C}\delta \quad (89)$$

be the set of dominant integral weights of $\mathfrak{g}(A)[u]$. For each $k \in \mathbb{Z}_{\geq 0}$, let $\hat{P}_+^k$ be the set of dominant integral weights of $\mathfrak{g}(A)[u]$ of level $k$, i.e.,

$$\hat{P}_+^k = \left\{ \sum_{i \in I} m_i \hat{\Lambda}_i \mid \sum_{i \in I} m_i c_i = k, m_i \in \mathbb{Z}_{\geq 0} \right\} + \mathbb{C}\delta. \quad (90)$$

Let $\lambda \in \hat{P}_+$, we denote the irreducible integrable $\mathfrak{g}(A)[u]$-module of highest weight $\lambda$ by $\hat{L}(\lambda)$. The winding subalgebra $\mathfrak{g}(A)[u]$ has a triangular decomposition

$$\mathfrak{g}(A)[u] = \hat{n}_- \oplus \mathfrak{h} \oplus \hat{n}_+, \quad (91)$$

where $\hat{n}_-$ is the negative subalgebra of $\mathfrak{g}(A)[u]$ and $\hat{n}_+$ is the positive subalgebra of $\mathfrak{g}(A)[u]$.

### 4.2 The set of weights $P_{A,u}(\Lambda)$

For each $\Lambda \in P_+^k (k \in \mathbb{Z}_{\geq 0})$, the $\mathfrak{g}(A)$-module $L(\Lambda)$ can be regarded as a $\mathfrak{g}(A)[u]$-module of level $uk$. Then it can be decomposed into direct sum of integrable irreducible $\mathfrak{g}(A)[u]$-module of level $uk$

$$L(\Lambda) = \bigoplus_{\lambda \in \hat{P}_+^k} \hat{L}(\lambda)^{mult_{A,\hat{\psi}_u}(\lambda)}. \quad (92)$$

Set

$$P_{A,u}(\Lambda) = \{ \lambda \in \hat{P}_+ \mid mult_{A,\hat{\psi}_u}(\lambda) \neq 0 \}. \quad (93)$$

For each $\lambda \in \hat{P}_+$, the $\mathfrak{g}(A)[u]$-module $\hat{L}(\lambda)$ is said to be maximal if $\lambda \in P_{A,u}(\Lambda)$ but $\lambda + n\delta \not\in P_{A,u}(\Lambda)$ for any $n > 0$. Let $max_{A,u}(\Lambda)$ be the set of all weight $\lambda \in \hat{P}_+$ such that $\hat{L}(\lambda)$ is a maximal $\mathfrak{g}(A)[u]$-submodule of $L(\Lambda)$. Similarly to Cartan subalgebras, we have

$$P_{A,u}(\Lambda) = max_{A,u}(\Lambda) - \mathbb{Z}_{\geq 0}\delta. \quad (94)$$

The formula (94) says that for each $\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \hat{P}_+$, there exists an unique number $b_{\Lambda,\lambda,u} \in \mathbb{C}$ such that $\lambda + b_{\Lambda,\lambda,u}\delta \in max_{A,u}(\Lambda)$. Hence

$$P_{A,u}(\Lambda) = \bigcup_{\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \hat{P}_+} (\lambda + (b_{\Lambda,\lambda,u} - \mathbb{Z}_{\geq 0})\delta). \quad (95)$$

Since $\hat{L}(\lambda + b\delta) \subset L(\Lambda + b\delta)$ if and only if $\hat{L}(\lambda) \subset L(\Lambda)$ for any $b \in \mathbb{C}$, we have

$$b_{\Lambda + b\delta,\lambda,u} = b_{\Lambda,\lambda,u} + b. \quad (96)$$
4.3 An identity of characters

Let $\Lambda \in P^k_+ (k \in \mathbb{Z}_{\geq 0})$, by (40), (43), (44) we have

$$
\left(\sum_{w \in W} \epsilon(w) e^{w(\rho)}\right) c_{\Lambda}^A = \sum_{\lambda \in \max(\Lambda)} \left(\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}\right) c_{\lambda}^A.
$$

(97)

We can suppose that $\lambda + \rho$ in the above equality is regular with respect to $W$. In this case, there exists unique $\sigma \in W$ and $\lambda' \in \hat{P}_+$ such that $\sigma(\lambda + \rho) = \lambda' + \hat{\rho}$. Let $p(\lambda)$ and $\{\lambda\}$ be $\epsilon(\sigma)$ and $\lambda'$ in this case, respectively. In the case $\lambda + \hat{\rho}$ is nonregular, set $p(\lambda)$ and $\{\lambda\}$ be 0. Since

$$
\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)} = p(\lambda) \sum_{w \in W} \epsilon(w) e^{w\{\lambda\} + \hat{\rho}},
$$

(98)

it follows from the identities (45), (97) that:

**Proposition 4.1.**

$$
ch_{\Lambda} = \sum_{\lambda \in \max(\Lambda)} p(\lambda) ch_{\{\lambda\}} c_{\lambda}^A.
$$

(99)

4.4 Semigroup structure

We state our first result about the set $\Gamma(g, g[u])$.

**Theorem 4.2.** As a subset of $\mathfrak{h}^* \times \mathfrak{h}^*$, the set $\Gamma(g, g[u])$ is a semigroup. Moreover, we have

$$
\Gamma(g, g[u]) \subset \Gamma(g, \mathfrak{h}) \cap (P_+ \times \hat{P}_+).
$$

(100)

In particular, $b_{\Lambda, \lambda, \mu} \leq b_{\Lambda, \lambda}$ for any $\lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \bar{P}_+$.

**Proof.**

A. Let $(\Lambda, \lambda)$ and $(\bar{\Lambda}, \bar{\lambda})$ be elements in the set $\Gamma(g, g[u])$. We need to show that $(\Lambda + \bar{\Lambda}, \lambda + \bar{\lambda}) \in \Gamma(g, g[u])$. The pair $(\Lambda, \lambda)$ is an element of $\Gamma(g, g[u])$ if and only if $\hat{L}(\lambda) \subset L(\Lambda)$. The condition is equivalent to the existence of a nonzero vector $v \in L(\Lambda)$ such that

$$
g(v) = 0, \forall g \in \hat{\mathfrak{n}}_+ \quad \text{and} \quad h(v) = \lambda(h)v, \forall h \in \mathfrak{h}.
$$

(101)

Let $\tilde{v}$ be a nonzero vector in $L(\bar{\Lambda})$ satisfying the same conditions but for the pair $(\bar{\Lambda}, \bar{\lambda})$. To show the semi-group structure of $\Gamma(g, g[u])$ we just need to show the existence of a nonzero vector $\tilde{v}$ in $L(\Lambda + \bar{\Lambda})$ which satisfies the conditions (101) but for the pair $(\Lambda + \bar{\Lambda}, \lambda + \bar{\lambda})$. We make the details in the two following steps.

**Step 1. Construction of the vector $\tilde{v}$.** By the fact that $L(\Lambda + \bar{\Lambda})$ is a $g$-submodule of $L(\Lambda) \otimes L(\bar{\Lambda})$ of multiplicity one, there exists an unique $g$-stable complementary subspace, which we denote by $S$ such that

$$
L(\Lambda) \otimes L(\bar{\Lambda}) = L(\Lambda + \bar{\Lambda}) \oplus S.
$$

(102)

Let $\pi : L(\Lambda) \otimes L(\bar{\Lambda}) \to L(\Lambda + \bar{\Lambda})$ be the projection with kernel $S$. Set $\tilde{v} = \pi(v \otimes \bar{v})$. We will show that $\tilde{v} \neq 0$ and satisfies the conditions (101) in the next steps.
Step 2. \( \tilde{v} \) is nonzero. Let

\[
L(\Lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} L(\Lambda)_\mu
\]

(103)

be the weight spaces decomposition of \( L(\Lambda) \). We define

\[
L(\Lambda)^\vee = \bigoplus_{\mu \in \mathfrak{h}^*} (L(\Lambda)_\mu)^*.
\]

(104)

There exists a nonzero vector \( \psi \in L(\Lambda)^\vee \) such that

\[
g(\psi) = 0, \forall g \in \mathfrak{n}_- \quad \text{and} \quad h(\psi) = -\Lambda(h)\psi, \forall h \in \mathfrak{h}.
\]

(105)

Let \( G \) be the minimal Kac-Moody group corresponding to the Kac-Moody algebra \( \mathfrak{g} \) (see [7]). To the vector \( v \in L(\Lambda) \) defined above, we associate a function \( f_v : G \to \mathbb{C}, g \mapsto \psi(g^{-1}(v)) \). Since \( L(\Lambda) \) is irreducible, the function \( f_v \) is nonzero (\( f_v = 0 \) implies \( Gv \subset \text{ker} \psi \)). Let \( B^- \) be the negative Borel subgroup of \( G \). We have

\[
(1, b).f_v = \Lambda(b)^{-1}f_v \quad \text{for all} \quad b \in B^-.
\]

(106)

Indeed,

\[
(1, b).f_v(g) = f_v((1, b)^{-1}.g)
\]

\[
= f_v(gb)
\]

\[
= \psi((b^{-1}g^{-1})(v))
\]

\[
= b(\psi)(g^{-1}(v))
\]

\[
= \Lambda(b)^{-1}\psi(g^{-1}(v))
\]

\[
= \Lambda(b)^{-1}f_v(g).
\]

Similarly, for \( L(\tilde{\Lambda}) \), we define \( \tilde{\psi} \in L(\tilde{\Lambda})^\vee \) and \( f_{\tilde{v}} : G \to \mathbb{C}, g \mapsto \tilde{\psi}(g^{-1}(\tilde{v})) \). Then \( f_{\tilde{v}} \) is nonzero and

\[
(1, b).f_{\tilde{v}} = \tilde{\Lambda}(b)^{-1}f_{\tilde{v}} \quad \text{for all} \quad b \in B^-.
\]

(107)

Set \( f = f_v.f_{\tilde{v}} \). Since \( G \) is irreducible as an indvariety, the function \( f \) is a well-defined nonzero function on \( G \). And of course,

\[
(1, b).f = (\Lambda + \tilde{\Lambda})(b)^{-1}f \quad \text{for all} \quad b \in B^-.
\]

(108)

Moreover, we have

\[
f(g) = (\psi \otimes \tilde{\psi})(g^{-1}(v \otimes \tilde{v})).
\]

(109)

Indeed, by definition

\[
f(g) = f_v(g)f_{\tilde{v}}(g)
\]

\[
= \psi(g^{-1}(v))\tilde{\psi}(g^{-1}(\tilde{v}))
\]

\[
= (\psi \otimes \tilde{\psi})(g^{-1}(v) \otimes g^{-1}(\tilde{v}))
\]

\[
= (\psi \otimes \tilde{\psi})(g^{-1}(v \otimes \tilde{v})).
\]

Now, \( \psi \otimes \tilde{\psi} \) is an element of

\[
L(\Lambda)^\vee \otimes L(\tilde{\Lambda})^\vee = (L(\Lambda) \otimes L(\tilde{\Lambda}))^\vee = L(\Lambda + \tilde{\Lambda})^\vee \oplus S^\vee.
\]

(110)

By (108), (109) we have

\[
\psi \otimes \tilde{\psi} \in L(\Lambda + \tilde{\Lambda})^\vee.
\]

(111)
It implies that
\[ \ker(\psi \otimes \bar{\psi}) \supset S. \quad \tag{112} \]

Rewrite \( v \otimes \bar{v} = \pi(v \otimes \bar{v}) + s \) for some \( s \in S \). Then we have
\[
(\psi \otimes \bar{\psi})(g^{-1}(v \otimes \bar{v})) = (\psi \otimes \bar{\psi})(g^{-1}(\pi(v \otimes \bar{v}) + s)) = (\psi \otimes \bar{\psi})(g^{-1}(\pi(v \otimes \bar{v}))) + (\psi \otimes \bar{\psi})(g^{-1}(s)) = (\psi \otimes \bar{\psi})(g^{-1}(\pi(v \otimes \bar{v}))).
\]

It means \( f(g) = (\psi \otimes \bar{\psi})(g^{-1}(\pi(v \otimes \bar{v}))) \). Since \( f \neq 0 \), we have \( \bar{v} = \pi(v \otimes \bar{v}) \neq 0 \).

**Step 3.** \( \bar{v} \) satisfies the conditions (101). For any \( g \in \mathfrak{n}_+ \) and \( h \in \mathfrak{h} \), we have:
\[
g(\pi(v \otimes \bar{v})) = \pi(g(v \otimes \bar{v})) = \pi(g(v) \otimes \bar{v} + v \otimes g(\bar{v})) = \pi(0) = 0,
\]
\[
h(\pi(v \otimes \bar{v})) = \pi(h(v \otimes \bar{v})) = \pi(h(v) \otimes \bar{v} + v \otimes h(\bar{v})) = \pi((\lambda + \bar{\lambda})(h)(v \otimes \bar{v})) = (\lambda + \bar{\lambda})h(\pi(v \otimes \bar{v})).
\]

We conclude that the set \( \Gamma(\mathfrak{g},\mathfrak{g}[u]) \) is a semigroup.

B. The inclusion (100) comes from the fact that if \( \mu \in (\Lambda - Q + \mathbb{C}\delta) \cap \hat{P}_+ \) and \( \hat{L}(\mu) \subset L(\Lambda) \) then \( \mu \in P(\Lambda) \). Consequently, \( \lambda + b_{\Lambda,\lambda,u} \delta \in P(\Lambda) \). It implies that \( b_{\Lambda,\lambda,u} \leq b_{\Lambda,\lambda} \).

\[ \square \]

**Remark 4.3.** By the same argument, we can prove that the set \( \Gamma(\mathfrak{g},\hat{\mathfrak{g}}) \) is a semigroup for any symmetrizable Kac-Moody algebra \( \mathfrak{g} \) and \( \hat{\mathfrak{g}} \) is a subalgebra of \( \mathfrak{g} \) such that \( \hat{\mathfrak{g}} = (\hat{\mathfrak{g}} \cap \mathfrak{n}_-) \oplus (\hat{\mathfrak{g}} \cap \mathfrak{h}) \oplus (\hat{\mathfrak{g}} \cap \mathfrak{n}_+) \).

### 4.5 About the cases \( A_1^{(1)} \) and \( A_2^{(2)} \)

In this subsection, we describe a subset \( \mathcal{A}_u(\Lambda) \) of the set of all \( \lambda \in (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P_+} \) such that \( b_{\Lambda,\lambda,u} = b_{\Lambda,\lambda} \). We obtain a set \( \mathcal{A}_u \) when the two sets \( \Gamma(\mathfrak{g},\mathfrak{g}[u]) \) and \( \Gamma(\mathfrak{g},\hat{\mathfrak{g}}) \) coincide.

**The case \( A_1^{(1)} \).** Let \( \Lambda_0 \) be the first fundamental weight and \( \alpha \) be the second simple root of \( \mathfrak{g}(A) \). Fix \( m \in \mathbb{Z}_{\geq 0} \), \( u \in \mathbb{Z}_{>0} \). Let \( \Lambda = m\Lambda_0 + \frac{j\alpha}{2} + b\delta \in P_+^m \) for some \( j \in [0,m] \cap \mathbb{Z}. \)

We define a subset \( \mathcal{A}_u(\Lambda) \) of \( (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P_+^m} \) as follows.

1. If \( u \) is even, set
\[
\mathcal{A}_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in [0,um] \cap (j + 2\mathbb{Z}) \cap [j - 1, um - j] \right\}. \quad (113)
\]

2. If \( u \) is odd, set
\[
\mathcal{A}_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in [0,um] \cap (j + 2\mathbb{Z}) \cap [j - 1, m(u - 1) + j + 1] \right\}. \quad (114)
\]
The case $A_2^{(2)}$. Let $A$ be the affine Cartan matrix of type $A_2^{(2)}$. Let $\Lambda_0$ be the first fundamental weight and $\alpha$ be the second simple root of $g(A)$. Fix $m \in \mathbb{Z}_{>0}, u \in \mathbb{Z}_{>1}$ such that $(u, 2) = 1$. Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} + b\delta \in P_+^m$ for some $j \in [0, \frac{m}{2}] \cap \mathbb{Z}$. We define a subset $A_u(\Lambda)$ of $(\Lambda - Q + \mathbb{C}\delta) \cap P_+^{um}$ as follows.

1. If $m = 1$, then $j = 0$ and set
   
   \[ A_u(\Lambda_0 + b\delta) = \left\{ \Lambda_0 + \frac{j'\alpha}{2} \mid j' \in \left[0, \frac{u}{2}\right] \cap \mathbb{Z} \right\}. \tag{115} \]

2. If $m = 2$, then $j = 0$ or $j = 1$ and set
   
   \[ A_u(2\Lambda_0 + b\delta) = \left\{ 2\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in [0, u] \cap \mathbb{Z} \right\}, \tag{116} \]
   \[ A_u(2\Lambda_0 + \frac{\alpha}{2} + b\delta) = \left\{ 2\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in [0, u] \setminus \{u - 1\} \cap \mathbb{Z} \right\}. \tag{117} \]

3. If $m > 2$, set
   
   \[ A_u(\Lambda) = A_u^{(1)}(\Lambda) \cup A_u^{(2)}(\Lambda), \tag{118} \]
   
   where
   
   \[ A_u^{(1)}(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in \left[j, \frac{um}{2}\right] \cap \mathbb{Z} \cap \left(\frac{m(u-1)}{2} - j + 2\mathbb{Z}_{\geq 0}\right) \right\}, \tag{119} \]
   \[ A_u^{(2)}(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in \left[j, \frac{um}{2}\right] \cap \mathbb{Z} \cap \left(\frac{m(u-1)}{2} - j + \mathbb{Z}_{< 0}\right) \right\}. \tag{120} \]

For $A_1^{(1)}$ and $A_2^{(2)}$, we define a subset $A_u$ of $P_+ \times \hat{P}_+$ by

\[ A_u = \bigcup_{\Lambda \in P_+} \bigcup_{\lambda \in A_u(\Lambda)} (\Lambda, \lambda + (b_{\Lambda,\lambda} - \mathbb{Z}_{\geq 0})\delta). \tag{121} \]

For the case $A_2^{(2)}$, we define smaller subsets of $A_u$ by

\[ A_u^{(1)} = \bigcup_{\Lambda \in P_+^m, m > 2} \bigcup_{\lambda \in A_u^{(1)}(\Lambda)} (\Lambda, \lambda + (b_{\Lambda,\lambda} - \mathbb{Z}_{\geq 0})\delta), \tag{122} \]
\[ A_u^{(2)} = \bigcup_{\Lambda \in P_+^m, m > 2} \bigcup_{\lambda \in A_u^{(2)}(\Lambda)} (\Lambda, \lambda + (b_{\Lambda,\lambda} - \mathbb{Z}_{\geq 0})\delta). \tag{123} \]

Our second main result for the cases $A_1^{(1)}$ and $A_2^{(2)}$ can be stated as follows.

**Theorem 4.4.** Let $g$ be an affine Kac-Moody algebra of type $A_1^{(1)}$ or $A_2^{(2)}$. Let $\Lambda \in P_+$. Then for any $\lambda \in A_u(\Lambda)$ we have

\[ b_{\Lambda,\lambda,u} = b_{\Lambda,\lambda}. \tag{124} \]

Hence

\[ A_u = A_u \cap \Gamma(g, g[\lambda]) = A_u \cap \Gamma(g, \delta). \tag{125} \]

Moreover, $A_u$ is a semigroup only for the case $A_1^{(1)}$. In the case $A_2^{(2)}$, the restriction of $A_u$ on the pairs of weights of level greater than 2 is a disjoint union of two semigroups $A_u^{(1)}$ and $A_u^{(2)}$. 
**Proof.** The (non)semigroup property of the sets are trivial by definition and the fact that \( \Gamma(\mathfrak{g}, \mathfrak{h}) \) is a semigroup in Theorem 3.1. We just need to prove that \( b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda} \) for all \( \lambda \in \mathcal{A}_{\mathfrak{g}}(\Lambda) \). By equalities (47), (96), we can suppose that \( \Lambda \in \overline{P}^+ \). The details are given as follows.

**The case \( A_1^{(1)} \).** The first step is writing explicitly \( ch_{\Lambda} \) in Proposition 4.1. It can be done as follows:

1. We use the result of Proposition 3.3 that

   \[ max(\Lambda) = \{ \Lambda + k \alpha + n_k \delta \mid k \in \mathbb{Z} \} \]

   where \( n_k \) is given by (50). Take \( \lambda_k = \Lambda + k \alpha + n_k \delta \in max(\Lambda) \), by using the following data

   \[ |\alpha|^2 = 2, \quad \hat{\rho} = \frac{2}{\mathfrak{u}} \Lambda_0 + \frac{1}{2} \alpha, \quad \hat{W} = \{ t_{\alpha n}, t_{\alpha n} s_1 \mid n \in \mathbb{Z} \} \]

   we can easily compute \( p(\lambda_k) \) and \( \{ \lambda_k \} \) for \( \lambda_k + \hat{\rho} \) regular with respect to \( \hat{W} \). As in Proposition 4.7 below, the only possible values of \( k \) are

   \[ k = \frac{j' - j}{2} - n(um + 2) \text{ and } k = -\frac{j' + j}{2} - 1 + n(um + 2) \]

   where \( j' \in [0, um] \cap (j + 2\mathbb{Z}) \) and \( n \in \mathbb{Z} \). Set

   \[ N_k = -n_k + un(j' + 1 - num - 2n). \]  

   If \( k = \frac{j' - j}{2} - n(um + 2) \), then \( p(\lambda_k) = 1 \) and \( \{ \lambda_k \} = m\Lambda_0 + \frac{j' \alpha}{2} - N_k \delta \).

   If \( k = -\frac{j' + j}{2} - 1 + n(um + 2) \), then \( p(\lambda_k) = -1 \) and \( \{ \lambda_k \} = m\Lambda_0 + \frac{j' \alpha}{2} - N_k \delta \).

2. Substitute values of \( p(\lambda_k) \) and \( \{ \lambda_k \} \) into the formula of Proposition 4.1, we can rewrite \( ch_{\Lambda} \) as follows

   \[ \sum_{j' \in [0, um] \cap (j + 2\mathbb{Z})} \hat{c}_h_{m\Lambda_0 + \frac{j' \alpha}{2}} \left( \sum_{n \in \mathbb{Z}} q^{N_k c_{\lambda_k}^{A}} - \sum_{n \in \mathbb{Z}} q^{N_k c_{\lambda_k}^{A}} \right) \]

   The coefficients of \( c_{\lambda_k}^{A} \) in the formula (130) are always positive integers since \( \lambda_k \in max(\Lambda) \). Proposition 4.7 below says that \( N_k \) depends on \( n \in \mathbb{Z} \) for each case of \( k \). It attains minimums at \( n = 0 \) for the first case and at \( n = 0 \) or \( n = 1 \) for the second case. The corresponding minimums of \( N_k \) are

   \[ -n \frac{j' - j}{2} \text{ and } \min \left( -n \frac{j' - j}{2} - 1, u - n \frac{j' + j}{2} + 1 \right) \]

   Proposition 4.8 below says that

   \[ -n \frac{j' - j}{2} \leq \min \left( -n \frac{j' - j}{2} - 1, u - n \frac{j' + j}{2} + 1 \right) \]

   Moreover, the equality

   \[ -n \frac{j' - j}{2} = \min \left( -n \frac{j' + j}{2} - 1, u - n \frac{j' + j}{2} + 1 \right) \]

   happens for \( j' \in [0, um] \cap (j + 2\mathbb{Z}) \) if and only if one of the next three conditions follows is satisfied:
1. \( m > 1 \) and \( j' \leq j - 2 \).

2. \( m > 1 \), \( u \) is even, \( j' \geq um - j + 1 \).

3. \( m > 1 \), \( u \) is odd, \( j' \geq m(u - 1) + j + 2 \).

Or equivalently, the equality (133) happens if and only if

\[
 m\Lambda_0 + \frac{j'\alpha}{2} \in (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P_{um}^+} \setminus \mathcal{A}_u(\Lambda). \tag{134}
\]

So, for any \( \lambda = m\Lambda_0 + \frac{j'\alpha}{2} \in \mathcal{A}_u(\Lambda) \), we have

\[
 - n_{\frac{j'-j}{2}} < \min \left( -n_{\frac{j'+j}{2}-1}, u - n_{\frac{j'+j}{2}+1} \right). \tag{135}
\]

By (130), in this case we have \( \lambda + n_{\frac{j'-j}{2}} \delta \in \text{max}_{\mathcal{A}_u}(\Lambda) \). Hence

\[
 b_{\lambda,\mu,\nu} = n_{\frac{j'-j}{2}} = b_{\lambda,\mu}. \tag{136}
\]

**The case** \( A_2^{(2)} \). The strategy is the same as in the case \( A_1^{(1)} \). The first step is writing explicitly \( ch_{\Lambda} \) in Proposition 4.1. It can be done as follows:

1. We use the result of Proposition 3.5 that

\[
 \text{max}(\Lambda) = \left\{ \Lambda + k\alpha + n_k\delta \middle| k \in \frac{1}{2}\mathbb{Z} \right\} \tag{137}
\]

where \( n_k \) is given by (65). Take \( \lambda_k = \Lambda + k\alpha + n_k\delta \in \text{max}(\Lambda) \), by using the following data

\[
 |\alpha|^2 = 4, \quad \hat{\rho} = \frac{3}{u}\Lambda_0 + \frac{1}{2}\alpha, \quad \hat{W} = \left\{ \frac{t_{\mu_0} s}{\alpha}, \frac{t_{\mu_0} s_1}{\alpha} \middle| n \in \mathbb{Z} \right\} \tag{138}
\]

we can easily compute \( p(\lambda_k) \) and \( \{\lambda_k\} \) for \( \lambda_k + \hat{\rho} \) regular with respect to \( \hat{W} \). As in Proposition 4.9 below, the only possible values of \( k \) are

\[
 k = \frac{j' - j}{2} - n \frac{um + 3}{2} \quad \text{and} \quad k = -\frac{j' + j}{2} - 1 + n \frac{um + 3}{2} \tag{139}
\]

where \( j' \in [0, \frac{um}{2}] \cap \mathbb{Z} \) and \( n \in \mathbb{Z} \). Set

\[
 N_k = -n_k + un \left( j' + 1 - n \frac{um + 3}{2} \right). \tag{140}
\]

If \( k = \frac{j'-j}{2} - n \frac{um+3}{2} \), then \( p(\lambda_k) = 1 \) and \( \{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} - N_k\delta \).

If \( k = -\frac{j'+j}{2} - 1 + n \frac{um+3}{2} \), then \( p(\lambda_k) = -1 \) and \( \{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} - N_k\delta \).

2. Substitute values of \( p(\lambda_k) \) and \( \{\lambda_k\} \) into the formula of Proposition 4.1 we can rewrite \( ch_{\Lambda} \) as follows

\[
 \sum_{j' \in [0, \frac{um}{2}] \cap \mathbb{Z}} ch_{m\Lambda_0 + \frac{j'\alpha}{2}} \left( \sum_{k = \frac{j'-j}{2} - n \frac{um+3}{2}} q^{N_k} c_{\lambda_k}^{\Lambda} - \sum_{k = -\frac{j'+j}{2} - 1 + n \frac{um+3}{2}} q^{N_k} c_{\lambda_k}^{\Lambda} \right). \tag{141}
\]

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The coefficients of $c_{\lambda_k}^\Lambda$ in the formula (141) are always positive integers since $\lambda_k \in max(\Lambda)$. Proposition 4.9 below says that the number $N_k$ depends on $n \in \mathbb{Z}$ for each case of $k$. It attains minimums at $n = 0$ for the first case and at $n = 0$ or $n = 1$ for the second case. The corresponding minimums of $N_k$ are

$$- n_{\alpha_{k+1}^j} \quad \text{and} \quad \min \left( - n_{\alpha_{k+1}^j-1} \frac{u}{2} - n_{\alpha_{k+1}^j+\frac{1}{2}} \right).$$

(142)

Proposition 4.10 below says that

$$- n_{\alpha_{k+1}^j} \leq \min \left( - n_{\alpha_{k+1}^j-1} \frac{u}{2} - n_{\alpha_{k+1}^j+\frac{1}{2}} \right)$$

Moreover, the equality

$$- n_{\alpha_{k+1}^j} = \min \left( - n_{\alpha_{k+1}^j-1} \frac{u}{2} - n_{\alpha_{k+1}^j+\frac{1}{2}} \right)$$

(144)

happens for $j' \in [0, \frac{um}{2}] \cap \mathbb{Z}$ if and only if one of the next two conditions is satisfied:

1. $m > 2$ and $j' \leq j - 1$.
2. $m \geq 2, j' \in \frac{m(u-1)}{2} + 1 - j + 2\mathbb{Z}_{\geq 0}$.

Or equivalently, the equality (144) happens if and only if

$$m\Lambda_0 + \frac{j'\alpha}{2} \in (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P_{u,R}^{\mathbb{Z}}} \setminus \mathcal{A}_u(\Lambda).$$

(145)

So, for any $\lambda = m\Lambda_0 + \frac{j'\alpha}{2} \in \mathcal{A}_u(\Lambda)$, we have

$$- n_{\alpha_{k+1}^j} < \min \left( - n_{\alpha_{k+1}^j-1} \frac{u}{2} - n_{\alpha_{k+1}^j+\frac{1}{2}} \right).$$

(146)

By (141), in this case we have $\lambda + n_{\alpha_{k+1}^j} \delta \in max_{\Lambda,u}(\Lambda)$. Hence

$$b_{\Lambda,\lambda,u} = n_{\alpha_{k+1}^j} = b_{\Lambda,\lambda}.$$  

(147)

We have proven the theorem. □

**Corollary 4.5.** We have

$$P_{A,u}(\Lambda) = \bigcup_{\lambda \in \mathcal{A}_u(\Lambda)} \lambda + (b_{\Lambda,\lambda} - \mathbb{Z}_{\geq 0}) \delta$$

(148)

if $A$ is of type $A_1^{(1)}$ and one of the next three conditions is satisfied:

1. $m = 1, u$ is arbitrary, $j \in \{0, 1\}$.
2. $m > 1, u$ is even, $j \in \{0, 1\}$.
3. $m = 2, u$ is odd, $j = 1$.

or $A$ is of type $A_2^{(2)}$ and $(u, 2) = 1, m = 1, j = 0$.

**Proof.** We just need to check that if one of the conditions is satisfied then

$$\mathcal{A}_u(\Lambda) = (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P_{u,R}^{\mathbb{Z}}}.$$  

The conclusion therefore comes from Theorems 4.2 and 4.4. □

Here is an example of Corollary 4.5.
Example 4.6. Consider the case A is of type $A_1^{(1)}$ and $\Lambda = 9\Lambda_0 + \frac{9}{2}$, $u = 4$. Then

$$A_0(\Lambda) = (\Lambda - Q + \mathbb{C}\delta) \cap \overline{P_+} = \left\{ 9\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in \{1, 3, 5, \ldots, 35\} \right\}. \quad (149)$$

Hence the set $\text{max}_{A_4}(9\Lambda_0 + \frac{9}{2})$ contains exactly 18 elements given by

$$9\Lambda_0 + \frac{\alpha}{2}, \quad 9\Lambda_0 + \frac{3\alpha}{2} - \delta, \quad 9\Lambda_0 + \frac{5\alpha}{2} - 2\delta,$$
$$9\Lambda_0 + \frac{7\alpha}{2} - 3\delta, \quad 9\Lambda_0 + \frac{9\alpha}{2} - 4\delta, \quad 9\Lambda_0 + \frac{11\alpha}{2} - 5\delta,$$
$$9\Lambda_0 + \frac{13\alpha}{2} - 6\delta, \quad 9\Lambda_0 + \frac{15\alpha}{2} - 7\delta, \quad 9\Lambda_0 + \frac{17\alpha}{2} - 8\delta,$$
$$9\Lambda_0 + \frac{19\alpha}{2} - 10\delta, \quad 9\Lambda_0 + \frac{21\alpha}{2} - 13\delta, \quad 9\Lambda_0 + \frac{23\alpha}{2} - 16\delta,$$
$$9\Lambda_0 + \frac{25\alpha}{2} - 19\delta, \quad 9\Lambda_0 + \frac{27\alpha}{2} - 22\delta, \quad 9\Lambda_0 + \frac{29\alpha}{2} - 25\delta,$$
$$9\Lambda_0 + \frac{31\alpha}{2} - 28\delta, \quad 9\Lambda_0 + \frac{33\alpha}{2} - 31\delta, \quad 9\Lambda_0 + \frac{35\alpha}{2} - 34\delta.$$

Then $P_{A_4}(9\Lambda_0 + \frac{9}{2}) = \text{max}_{A_4}(9\Lambda_0 + \frac{9}{2}) - \mathbb{Z}_{\geq 0}\delta$.

The next one is a proposition we used in the proof of Theorem 4.4 for the case $A_1^{(1)}$.

Proposition 4.7. Let A be the affine Cartan matrix of type $A_1^{(1)}$. Fix $m \in \mathbb{Z}_{>0}$, $u \in \mathbb{Z}_{>0}$. Let $\Lambda = m\Lambda_0 + \frac{i\alpha}{2} \in \overline{P_+}$.

1. We parametrize $\lambda \in \text{max}(\Lambda)$ such that $\lambda + \dot{\rho}$ is regular with respect to $\dot{W}$ by $\lambda_k = \Lambda + k\alpha + n_k\delta$. Then the only possible values of $k$ are

$$k = \frac{j' - j}{2} - n(um + 2) \quad \text{and} \quad k = -\frac{j' + j}{2} - 1 + n(um + 2) \quad (150)$$

where $j' \in [0, um] \cap (j + 2\mathbb{Z})$ and $n \in \mathbb{Z}$.

2. Let

$$N_k = -n_k + un(j' + 1 - un - 2n). \quad (151)$$

Then

i. If $k = \frac{j' - j}{2} - n(um + 2)$, then $p(\lambda_k) = 1$ and $\{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} - N_k\delta$.

ii. If $k = -\frac{j' + j}{2} - 1 + n(um + 2)$, then $p(\lambda_k) = -1$ and $\{\lambda_k\} = m\Lambda_0 + \frac{j\alpha}{2} - N_k\delta$.

iii. The function $N_k$ is considered as a function on $n$ and it attains the minimum at $n = 0$ in the first case and at $n = 0$ or $n = 1$ in the second case.

Proof. Since $\lambda_k + \dot{\rho}$ is regular with respect to $\dot{W}$, there exists unique $\sigma \in \dot{W}$ and $\mu = m\Lambda_0 + \frac{i\alpha}{2} + b\delta \in \dot{P}^\text{num}$ such that $\sigma(\lambda_k + \dot{\rho}) = \mu + \dot{\rho}$.

a. (Proof of 1. and 2.i.) If $\sigma = t_{una}$ for some $n \in \mathbb{Z}$, then $\sigma(\lambda_k + \dot{\rho}) - \dot{\rho}$ equals

$$m\Lambda_0 + \left( \text{num} + 2n + k + \frac{j}{2} \right) \alpha + (n_k - un(2k + j + 1 + num + 2n))\delta. \quad (152)$$

Hence

$$j' \in [0, um] \cap (j + 2\mathbb{Z}) \quad \text{and} \quad k = \frac{j' - j}{2} - n(um + 2). \quad (153)$$

In this case, we have

$$p(\lambda_k) = 1 \quad \text{and} \quad \{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} + (n_k - un(j' + 1 - num - 2n))\delta. \quad (154)$$
b. (Proof of 1. and 2.ii.) If \( \sigma = t_{\alpha} s_1 \) for some \( n \in \mathbb{Z} \), then \( \sigma (\lambda_k + \dot{\rho}) - \dot{\rho} \) equals
\[
m \Lambda_0 + \left( \text{num} + 2n - k - \frac{j}{2} - 1 \right) \alpha + (n_k - \text{un}(-2k - j - 1 + \text{num} + 2n)) \delta. \tag{155}
\]
Hence
\[
j' \in [0, \text{um}] \cap (j + 2\mathbb{Z}) \text{ and } k = -\frac{j' + j}{2} - 1 + n(\text{um} + 2). \tag{156}
\]
In this case, we have
\[
p(\lambda_k) = -1 \text{ and } \{ \lambda_k \} = m \Lambda_0 + \frac{j' \alpha}{2} + (n_k - \text{un}(j' + 1 - \text{num} - 2n)) \delta. \tag{157}
\]

c. (Proof of 2.iii.) Put \( M = \text{um} + 2 \). We consider the first case when \( k = \frac{j' + j}{2} + nM \).

Write \( k = qm + r \) for some \( q \in \mathbb{Z}, 0 \leq r < m \), then
\[
-N_k = n_r - q(k + r + j) + un(j' + 1 + nM) = n_r - \frac{(j' - j + nM - r)(j' + j + nM + r)}{m} + un(j' + 1 + nM) \tag{158}
\]
\[
= n^2 M \left( u - \frac{M}{m} \right) + n \left( u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{4m} + \left( \frac{r^2}{m} + \frac{rj}{m} + n_r \right). \tag{159}
\]
We have
\[
\frac{r^2}{m} + \frac{rj}{m} + n_r = \begin{cases} \frac{1}{m} r(r + j - m) & \text{if } 0 \leq r \leq m - j, \\ \frac{1}{m} (r - m)(r + j - m) & \text{if } m - j \leq r < m. \end{cases} \tag{160}
\]
The condition \( 0 \leq r \leq m - j \) can be rewritten as
\[
\frac{j - m}{2} \leq nM + \frac{j' - j}{2} - m \left( q + \frac{1}{2} \right) \leq \frac{m - j}{2} \tag{162}
\]
and \( m - j \leq r < m \) can be rewritten as
\[
-\frac{j}{2} \leq nM + \frac{j' - j}{2} - m(q + 1) < \frac{j}{2} \tag{163}
\]
It implies that \( \frac{r^2}{m} + \frac{rj}{m} + n_r \) equals
\[
\begin{cases} \frac{|nM + j' - \frac{mp}{2} - (m-j)^2|}{m} & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that } \frac{nM + j' - \frac{mp}{2}}{2} \leq \frac{m-j}{2}, \\ \frac{|nM + j' - \frac{mp}{2}|^2 - \frac{j^2}{4}}{m} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that } \frac{|nM + j' - \frac{mp}{2}|}{2} \leq \frac{1}{2}. \end{cases} \tag{164}
\]

So \( -N_k = F_{j,j'}(n) \) where \( F_{j,j'} : \mathbb{Z} \to \mathbb{R} \) is defined by
\[
F_{j,j'}(t) = t^2 M \left( u - \frac{M}{m} \right) + t \left( u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{4m} + P_{j,j'}(t) \tag{165}
\]
with \( P_{j,j'}(t) \) is given by

\[
\begin{cases}
\frac{|tM + \frac{j'}{2} - \frac{p}{m}|^2 - (m-j)^2}{m} & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that } |tM + \frac{j'}{2} - \frac{p}{m}| \leq \frac{m-j}{2}, \\
\frac{|tM + \frac{j'}{2} - \frac{p}{m}|^2 - \frac{1}{2}}{m} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that } |tM + \frac{j'}{2} - \frac{p}{m}| \leq \frac{1}{2}.
\end{cases}
\]

We will show that the maximum of \( F_{j,j'}(n) \) appears when \( n = 0 \), i.e., \( k = \frac{j+j'}{2} \). To do that, we consider the function \( F : \mathbb{R} \times [0,m] \times [0,um] \to \mathbb{R} \) given by \( F(t,j,j') = F_{j,j'}(t) \) (we also define \( P(t,j,j') \) from \( P_{j,j'}(t) \)). Let \( \Delta(t,j,j') = F(t+1,j,j') - F(t,j,j') \), then it is nonincreasing in \( t \) and \( \Delta(-1,j,j') > 0 > \Delta(0,j,j') \). It implies that \( F(0,j,j') > F(t,j,j') \) for any \( t \in \mathbb{Z}, t \neq 0 \), i.e., \( F_{i,j}(n) \) attains its maximum when \( n = 0 \). Indeed:

\[
\Delta(t,j,j') = 2tM \left( u - \frac{M}{m} \right) + (M+j') \left( u - \frac{M}{m} \right) + u + P(t+1,j,j') - P(t,j,j').
\]

We denote the numbers \( p \) defined on \( P_{j,j'}(t+1) \) and \( P_{j,j'}(t) \) by \( p_1, p_0 \), respectively. Use definition, we have \( \frac{p_1 - p_0}{2} \geq u \). Hence

\[
\partial_t \Delta(t,j,j') = 2M \left( u - \frac{p_1 - p_0}{2} \right) \leq 0, \text{ i.e., } \Delta \text{ is nonincreasing in } t, \tag{168}
\]

\[
\partial_{j'} \Delta(t,j,j') = u - \frac{p_1 - p_0}{2} \leq 0, \text{ i.e., } \Delta \text{ is nonincreasing in } j'. \tag{169}
\]

So \( \Delta(0,j,j') \leq \Delta(0,j,0) \) and \( \Delta(-1,j,um) \leq \Delta(-1,j,j') \). We can easily check that \( \Delta(0,j,0) < 0 < \Delta(-1,j,um) \). Hence, in the case \( k = \frac{j+j'}{2} + nM \), the minimum of \( N_k \) occurs when \( n = 0 \).

For the case \( k = -\frac{j+j'}{2} - 1 + nM \). Since \( k = \frac{j+j'}{2} + \left( n - \frac{j+1}{M} \right) M \), we have

\[
-N_k = F \left( n - \frac{j+1}{M}, j, j' \right). \tag{170}
\]

Then \( N_k \) attains its minimum when \( n = 0 \) or \( 1 \).

\[\square\]

Here is the next proposition we used in the proof of Theorem 4.4 for the case \( A_1^{(1)} \).

**Proposition 4.8.** With \( n_k \) is defined as in (50). For each \( j \in [0,m] \) and \( j' \in [0,um] \cap (j + 2\mathbb{Z}) \), we always have

\[
-n_{\frac{j'}{2}} \leq \min \left( -n_{\frac{j+1}{2} - 1}, u - n_{\frac{j+1}{2} + 1} \right). \tag{171}
\]

Moreover, the equality

\[
-n_{\frac{j'}{2}} = \min \left( -n_{\frac{j+1}{2} - 1}, u - n_{\frac{j+1}{2} + 1} \right) \tag{172}
\]

happens for \( j' \in [0,um] \cap (j + 2\mathbb{Z}) \) if and only if one of the next three conditions follows is satisfied:
1. \( m > 1 \) and \( j' \leq j - 2 \).

2. \( m > 1, u \) is even, \( j' \geq um - j + 1 \).

3. \( m > 1, u \) is odd, \( j' \geq m(u - 1) + j + 2 \).

Proof. Then inequality comes from (53), (130) and Theorem 4.2. To study the equality, we use a fact that

\[
 n_{-(j+k)} = n_k. \tag{173}
\]

Indeed, if \( \Lambda = m\Lambda_0 + \frac{i\alpha}{2} + b\delta \in P^m_+ \) and \( \lambda = \Lambda + k\alpha + n_k\delta \in \max(\Lambda) \), then \( s_1(\lambda) = \Lambda - (j + k)\alpha + n_k\delta \in \max(\Lambda) \). We use the equality (173) to rewrite

\[
 n_{-\frac{j' + j}{2} + 1} = n_{\frac{j' - j}{2}}. \tag{174}
\]

as \( n_x = n_{x+1} \), where \( x = -\frac{j' + j}{2} - 1 \). Use definition (50) for \( n_x \) we check that it happens if and only if

\[
 m > 1 \text{ and } j' \leq j - 2. \tag{175}
\]

We use the equality (173) to rewrite

\[
 -u + n_{\frac{j' + j}{2} + 1} \leq n_{\frac{j' - j}{2}} \tag{176}
\]

as \( n_{x+1} - u \leq n_x \), where \( x = -\frac{j' + j}{2} \). Use definition (50) for \( n_x \) we can check that it happens if and only if

\[
 m > 1, u \text{ is even, } j' \geq um - j + 1, \tag{177}
\]

or

\[
 m > 1, u \text{ is odd, } j' \geq m(u - 1) + j + 2. \tag{178}
\]

Thus we have proven the proposition. □

The next one is a proposition we used in the proof of Theorem 4.4 for the case \( A_2^{(2)} \).

**Proposition 4.9.** Let \( A \) be the affine Cartan matrix of type \( A_2^{(2)} \). Fix \( m \in \mathbb{Z}_{\geq 0}, u \in \mathbb{Z}_{> 0} \) such that \((u, 2) = 1\). Let \( \Lambda = m\Lambda_0 + \frac{i\alpha}{2} \in \mathcal{T}^m_+ \).

1. We parametrize \( \lambda \in \max(\Lambda) \) such that \( \lambda + \dot{\rho} \) is regular with respect to \( \dot{W} \) by \( \lambda_k = \Lambda + k\alpha + n_k\delta \). Then the only possible values of \( k \) are

\[
 k = \frac{j' - j}{2} - n \frac{um + 3}{2} \text{ and } k = -\frac{j' + j}{2} - 1 + n \frac{um + 3}{2} \tag{179}
\]

where \( j' \in [0, \frac{um}{2}] \cap \mathbb{Z} \) and \( n \in \mathbb{Z} \).

2. Let

\[
 N_k = -n_k + un \left( j' + 1 - n \frac{um + 3}{2} \right). \tag{180}
\]

Then

i. If \( k = \frac{j' - j}{2} - n \frac{um + 3}{2} \), then \( p(\lambda_k) = 1 \) and \( \{\lambda_k\} = m\Lambda_0 + \frac{i\alpha}{2} - N_k\delta \).

ii. If \( k = -\frac{j' + j}{2} - 1 + n \frac{um + 3}{2} \), then \( p(\lambda_k) = -1 \) and \( \{\lambda_k\} = m\Lambda_0 + \frac{i\alpha}{2} - N_k\delta \).

iii. The function \( N_k \) is considered as a function on \( n \) and it attains the minimum at \( n = 0 \) in the first case and at \( n = 0 \) or \( n = 1 \) in the second case.
Proof. Since \( \lambda_k + \hat{\rho} \) is regular with respect to \( \hat{W} \), there exists unique \( \sigma \in \hat{W} \) and \( \mu = m\Lambda_0 + \frac{1}{2}\hat{\rho} + b\delta \in \hat{P}_+ \) such that \( \sigma(\lambda_k + \hat{\rho}) = \mu + \hat{\rho} \).

a. (Proof of 1. and 2.i.) If \( \sigma = \frac{t \alpha_0}{2} \) for some \( n \in \mathbb{Z} \), then \( \sigma(\lambda_k + \hat{\rho}) - \hat{\rho} \) equals

\[
m\Lambda_0 + \left( k + \frac{j}{2} + n \frac{um + 3}{2} \right) \alpha + \left( n_k - un \left( 2k + j + 1 + n \frac{um + 3}{2} \right) \right) \delta.
\]

Hence

\[
j' \in \left[ 0, \frac{um}{2} \right] \cap \mathbb{Z} \quad \text{and} \quad k = \frac{j' - j}{2} - n \frac{um + 3}{2}.
\]

In this case, we have

\[
p(\lambda_k) = 1 \quad \text{and} \quad \{\lambda_k\} = m\Lambda_0 + \frac{j' \alpha}{2} + \left( n_k - un \left( j' + 1 - n \frac{um + 3}{2} \right) \right) \delta.
\]

b. (Proof of 1. and 2.ii.) If \( \sigma = \frac{t \alpha_0}{2} \) for some \( n \in \mathbb{Z} \), then \( \sigma(\lambda_k + \hat{\rho}) - \hat{\rho} \) equals

\[
m\Lambda_0 + \left( -k - \frac{j}{2} - 1 + n \frac{um + 3}{2} \right) \alpha + \left( n_k - un \left( -2k - j - 1 + n \frac{um + 3}{2} \right) \right) \delta.
\]

Hence

\[
j' \in \left[ 0, \frac{um}{2} \right] \cap \mathbb{Z} \quad \text{and} \quad k = \frac{j' + j}{2} - 1 + n \frac{um + 3}{2}.
\]

In this case, we have

\[
p(\lambda_k) = -1 \quad \text{and} \quad \{\lambda_k\} = m\Lambda_0 + \frac{j' \alpha}{2} + \left( n_k - un \left( j' + 1 - n \frac{um + 3}{2} \right) \right) \delta.
\]

c. (Proof of 2.iii.) Put \( M = um + 3 \). We consider the first case \( k = \frac{j' - j}{2} + \frac{nM}{2} \). Write \( k = \frac{4r}{2} q + r \) for some \( q \in \mathbb{Z}, 0 \leq r < \frac{m}{2} \), then

\[
-N_k = n_r - q(k + r + j) + un \left( j' + 1 + n \frac{M}{2} \right)
\]

\[
= n_r - 2 \left( \frac{j'}{2} + \frac{M}{2} - r \right) \left( \frac{j}{2} + \frac{nM}{2} + r \right) + un \left( j' + 1 + \frac{nM}{2} \right)
\]

\[
= n^2 M \left( u - \frac{M}{m} \right) + n \left( u + u j' \frac{M}{m} \right) + \frac{j^2 - j'^2}{2m} + \left( \frac{2r^2}{m} \frac{2rj}{m} + n_r \right).
\]

We have \( \frac{2r^2}{m} + \frac{2rj}{m} + n_r \) equals

\[
\begin{cases} 
\frac{2}{m} r (r + j - \frac{m}{2}) & \text{if } 0 \leq r \leq \frac{m}{2} - j, \\
\frac{2}{m} (r - \frac{m}{2}) (r + j - \frac{m}{2}) & \text{if } \frac{m}{2} - j \leq r < \frac{m}{2}, r \in \frac{m}{2} + \mathbb{Z}, \\
\frac{2}{m} (r - \frac{m}{2}) (r + j - \frac{m}{2}) - \frac{1}{2} & \text{if } \frac{m}{2} - 1 j \leq r < \frac{m}{2}, r \in \frac{m}{2} + \mathbb{Z}.
\end{cases}
\]

The condition \( 0 \leq r \leq \frac{m}{2} - j \) can be rewritten as

\[
\frac{2j - m}{4} \leq \frac{nM + j'}{2} - \frac{m}{2} \left( q + \frac{1}{2} \right) \leq \frac{m - 2j}{4}.
\]
and \( \frac{m}{2} - j \leq r < \frac{m}{2} \) can be rewritten as
\[
-\frac{j}{2} \leq \frac{nM + j'}{2} - \frac{m}{2}(q + 1) < \frac{j}{2}.
\] (192)

It implies that \( \frac{2r^2}{m} + \frac{2r}{m} + n_r \) equals
\[
\begin{cases}
\frac{2}{m} | \frac{nM + j'}{2} - \frac{m}{4} p |^2 - | \frac{m-2j}{4} |^2 & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that } \frac{nM + j'}{2} - \frac{m}{4} p \leq \frac{m-2j}{4}, \\
\frac{2}{m} | \frac{nM + j'}{2} - \frac{m}{4} p |^2 - \frac{j^2}{4} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that } \frac{nM + j'}{2} - \frac{m}{4} p \leq \frac{j}{2}, \\
\frac{2}{m} | \frac{nM + j'}{2} - \frac{m}{4} p |^2 - \frac{j^2}{4} - \frac{1}{2} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that } \frac{nM + j'}{2} - \frac{m}{4} p \in \left[ -\frac{1+j}{2}, \frac{-j}{2} \right],
\end{cases}
\] (193)

So \(-N_k = F_{j,j'}(n)\) where \( F_{j,j'} : \mathbb{Z} \to \mathbb{R} \) is defined by
\[
F_{j,j'}(t) = t^2 \frac{M}{2} \left( u - \frac{M}{m} \right) + t \left( u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{2m} + P_{j,j'}(t)
\] (194)

with \( P_{j,j'}(t) \) is given by
\[
\begin{cases}
\frac{2}{m} | \frac{|M+1|}{2} - \frac{m}{4} p |^2 - | \frac{m-2j}{4} |^2 & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that } \frac{|M+1|}{2} - \frac{m}{4} p \leq \frac{m-2j}{4}, \\
\frac{2}{m} | \frac{|M+1|}{2} - \frac{m}{4} p |^2 - \frac{j^2}{4} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that } \frac{|M+1|}{2} - \frac{m}{4} p \leq \frac{j}{2}, \\
\frac{2}{m} | \frac{|M+1|}{2} - \frac{m}{4} p |^2 - \frac{j^2}{4} - \frac{1}{2} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that } \frac{|M+1|}{2} - \frac{m}{4} p \in \left[ -\frac{1+j}{2}, \frac{-j}{2} \right],
\end{cases}
\] (195)

We will show that the maximum of \( F_{j,j'}(n) \) appears when \( n = 0 \), i.e., \( k = \frac{-j-j'}{2} \). To do that, we show that the upper bound function \( F^+ : \mathbb{R} \times \left[ 0, \frac{m}{2} \right] \times \left[ 0, \frac{um}{m} \right] \to \mathbb{R} \) of \( F_{j,j'}(t) \) and lower bound function \( F^- : \mathbb{R} \times \left[ 0, \frac{m}{2} \right] \times \left[ 0, \frac{um}{m} \right] \to \mathbb{R} \) of \( F_{j,j'}(t) \) given below attains their maximum along \( t \in \mathbb{Z} \) when \( t = 0 \).

\[
F^+(t, j, j') = t^2 \frac{M}{2} \left( u - \frac{M}{m} \right) + t \left( u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{2m} + P^+(t, j, j'), \quad (196)
\]
\[
F^-(t, j, j') = t^2 \frac{M}{2} \left( u - \frac{M}{m} \right) + t \left( u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{2m} + P^-(t, j, j'), \quad (197)
\]

where \( P^+(t, j, j') \) is
\[
\begin{cases}
\frac{2}{m} | \frac{|M+1|}{2} - \frac{m}{4} p |^2 - | \frac{m-2j}{4} |^2 & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that } \frac{|M+1|}{2} - \frac{m}{4} p \leq \frac{m-2j}{4}, \\
\frac{2}{m} | \frac{|M+1|}{2} - \frac{m}{4} p |^2 - \frac{j^2}{4} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that } \frac{|M+1|}{2} - \frac{m}{4} p \leq \frac{j}{2},
\end{cases}
\] (198)
and $P^-(t, j, j')$ is
\[
\begin{align*}
\frac{2}{m}((\frac{t+1}{2} - \frac{m}{4} p)^2 - \frac{m - 2j}{4}) & \quad \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that } \\
\frac{2}{m}((\frac{t+1}{2} - \frac{m}{4} p)^2 - \frac{t}{4}) - \frac{1}{2} & \quad \text{if } \exists p \in 2\mathbb{Z} \text{ such that } \frac{t}{4} \notin \mathbb{Z}.
\end{align*}
\]
\hspace{1cm} (199)

We just consider the function $F^+$ and apply similar arguments for $F^-$. Let $\Delta^+(t, j, j') = F^+(t+1, j, j') - F^+(t, j, j')$, then it is nonincreasing in $t$ and $\Delta^+(-1, j, j') > 0 > \Delta^+(0, j, j')$. It implies that $F^+(0, j, j') > F^+(t, j, j')$ for any $t \in \mathbb{Z}, t \neq 0$. The same results is true for $F^-$. Hence $F_t(n)$ attains its maximum when $n = 0$. Indeed: $\Delta^+(t, j, j')$ equals
\[
tM \left(u - \frac{M}{m} \right) + \left( j' + \frac{M}{2} \right) \left(u - \frac{M}{m} \right) + u + P^+(t+1, j, j') - P^+(t, j, j').
\]  
\hspace{1cm} (200)

We denote the numbers $p$ defining $P^+(t+1, j, j')$ and $P^+(t, j, j')$ by $p_1, p_0$, respectively. Use definition, we have $\frac{p_1 - p_0}{m} \geq u$. Hence
\[
\partial_t \Delta^+(t, j, j') = M \left(u - \frac{p_1 - p_0}{2} \right) \leq 0, \text{ i.e., } \Delta^+ \text{ is nonincreasing in } t,
\]  
\hspace{1cm} (201)
\[
\partial_{j'} \Delta^+(t, j, j') = u - \frac{p_1 - p_0}{2} \leq 0, \text{ i.e., } \Delta^+ \text{ is nonincreasing in } j'.
\]  
\hspace{1cm} (202)

So $\Delta^+(0, j, j') \leq \Delta^+(0, j, 0)$ and $\Delta^+(-1, j, \frac{um}{2}) \leq \Delta^+(-1, j, j')$. We can check that $\Delta^+(0, j, 0) < 0 < \Delta^+(-1, j, \frac{um}{2})$. Hence, in the case $k = \frac{j' - j}{2} + \frac{4m}{2}$, the minimum of $N_k$ occurs when $n = 0$.

For the case $k = -\frac{j' + j}{2} - 1 + \frac{4M}{2}$. Since $k = \frac{j' - j}{2} + \left(n - 2\frac{j' + 1}{M}\right) \frac{M}{2}$, we have
\[
- N_k = F \left(n - 2\frac{j' + 1}{M}, j, j' \right).
\]  
\hspace{1cm} (203)

Then $N_k$ attains its minimum when $n = 0$ or 1.

Here is the next proposition we used in the proof of Theorem 4.4 for the case $A_2^{(2)}$.

**Proposition 4.10.** With $n_k$ is defined as in (65). For each $j \in \left[0, \frac{m}{2}\right]$ and $j' \in \left[0, \frac{um}{2}\right] \cap \mathbb{Z}$, we always have
\[
- n' \frac{u}{2} \leq \min \left(-n - \frac{j' + 1}{2} - 1, \frac{u}{2} - n - \frac{j' + 1}{2} + \frac{1}{2}\right).
\]  
\hspace{1cm} (204)

Moreover, the equality
\[
- n \frac{u}{2} = \min \left(-n - \frac{j' + 1}{2} - 1, \frac{u}{2} - n - \frac{j' + 1}{2} + \frac{1}{2}\right)
\]  
\hspace{1cm} (205)

happens for $j' \in \left[0, \frac{um}{2}\right] \cap \mathbb{Z}$ if and only if one of the next two conditions is satisfied:

1. $m > 2$ and $j' \leq j - 1$.
2. $m \geq 2, j' \in \frac{m(u-1)}{2} + 1 - j + 2\mathbb{Z}_{\geq 0}$.
Proof. Then inequality comes from (68), (141) and Theorem 4.2. We again use the equality (173) to rewrite
\[ n_{-\frac{j+j}{2}} - 1 = n_{\frac{j}{2}} \quad \text{and} \quad -\frac{u}{2} + n_{-\frac{j+j}{2} + \frac{1}{2}} = n_{\frac{j}{2}} \quad (206) \]
as \( n_x = n_{x-1} \) and \( n_x = -\frac{u}{2} + n_{x+\frac{1}{2}} \), where \( x = -\frac{j+j}{2} \). Use definition (65), we can check that it happens for \( j' \in [0, \frac{um}{2}] \cap \mathbb{Z} \) if and only if one of two conditions follows is satisfied:

1. \( m > 2 \) and \( j' \leq j - 1 \).

2. \( m \geq 2 \), \( j' \in \frac{m(u-1)}{2} + 1 - j + 2\mathbb{Z}_{\geq 0} \).

We have proven the theorem. \( \square \)

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Universite Lyon, Universit Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 Boulevard du 11 Novembre 1918, F-69622 Villeurbanne cedex, France
E-mail: khanh.mathematic@gmail.com.