Traveling wave solutions for delayed reaction-diffusion systems

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Abstract. This paper is concerned with the traveling waves of delayed reaction-diffusion systems where the reaction function possesses the mixed quasimonotonicity property. By the so-called monotone iteration scheme and Schauder's fixed point theorem, it is shown that if the system has a pair of coupled upper and lower solutions, then there exists at least a traveling wave solution. More precisely, we reduce the existence of traveling waves to the existence of an admissible pair of coupled quasi-upper and quasi-lower solutions which are easy to construct in practice.

Keywords: Traveling wave; Mixed quasimonotonicity; Upper and lower solutions.

MSC: Primary 35K10, 35K57; Secondary 35R20.

1 Introduction

Reaction diffusion system is used to model the spatial-temporal pattern. In the past decades, the traveling wave solutions of the reaction diffusion systems, which are studied as a paradigm for behavior, have been widely investigated due to significant applications in chemical engineering, population dynamics and biological models. Since the first instances in which traveling wave solutions

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were investigated were given in 1937 by Kolmogorov et al. [14] and Fisher [6].

Many methods have been used to study the traveling wave solutions of various parabolic equations and systems, for example, the phase plane technique in [4, 5, 11, 12, 27, 32, 33, 35], the degree theory method and the conley index method developed in [3, 7, 8, 34].

In many realistic models, the delays should be incorporated into the reaction diffusion system. Due to the presence of delays in the reaction diffusion system, the classical phase plane technique can not generate a monotone flow which ensures the existence of the traveling wave solution. Recently the classical monotone iteration technique was first used by Wu and Zou [36, 37] to establish the existence of traveling wave solution for delayed reaction diffusion system. They employed the idea of upper and lower solutions and an iteration scheme to construct a monotone sequence of upper solutions which was proved to converge to a solution of the corresponding wave equation of the reaction-diffusion system under consideration (see also [9, 10]). In fact, many researchers had used the monotone iteration technique to prove the existence of the reaction diffusion system in [11, 13, 15, 16, 17, 18, 25, 26, 28]. In [23, 24], Ma et al. proved some existence results for traveling wavefronts of reaction-diffusion systems by using Schauder’s fixed point theorem. One important feature of Ma’s method, which was different from the work of Zou and Wu [36], was that the upper solution of the wave equation was not necessary to converge to two distinct trivial solutions when \( t \to -\infty \) and \( t \to +\infty \) respectively. Li et al. developed a new cross iteration scheme and established the existence of the traveling wave for Lotka-Volterra competition system with delays [19, 20, 21].

More recently, Boumenir and Nguyen discussed in [2] a modified version of Perron Theorem for \( C^1 \)-solutions, and set up a rigorous framework for the monotone iteration method and then apply it to the predator-prey and Belousov-Zhabotinskii models with delays.

However, in the iteration process by the monotone iteration it is required that the nonlinear reaction function possesses a quasimonotone property in the sector between the upper and lower solutions [2, 23, 36]. This paper focus on the delayed reaction diffusion system without quasimonotonicity. Motivated by the above work and the upper and lower solution method developed by Pao [28, 29, 30, 31] and Li et al. [19] for reaction diffusion systems, we use the coupled upper and lower solutions to deal with the non-quasimonotonicity, which was first given out in [22]. Via the coupled upper and lower solutions, we construct an appropriate closed bounded convex set. By use of the Schauder’s fixed point
theorem in the convex set, we show the existence of the traveling wave solution. Moreover we reduce the existence of traveling wave solution to the existence of an admissible pair of quasi-upper and quasi-lower solutions which are easy to construct in practice.

This paper is organized as follows. In Section 2, we show the existence of the traveling wave solution by constructing the classical coupled upper and lower solutions. In Section 3 we relax the classical coupled upper and lower solutions to the \( C^1 \) smooth coupled quasi-upper and quasi-lower solutions. Section 4 deals with systems with quasimonotone nondecreasing functions, and the definition of ordered quasi-upper and quasi-lower solutions is introduced and an existence result of a traveling wavefront is given by the monotone iteration method. In Section 5 the main result is illustrated by and applied to a delayed Belousov-Zhabotinskii equation and a Mutualistic Lotka-Volterra model. This paper ends with a short discussion.

2 Coupled upper and lower solutions

In this paper, we will consider the following system of reaction-diffusion systems with time delays

\[
\frac{\partial}{\partial t} u(x, t) - D \frac{\partial^2}{\partial^2 x} u(x, t) = f(u, u_\tau),
\]

where \( x \in \mathbb{R}, t \in (0, \infty), u \equiv (u_1, \cdots, u_n) \in \mathbb{R}^n, u_\tau \equiv (u_1(x, t-\tau_1), \cdots, u_n(x, t-\tau_n)) \in \mathbb{R}^n \) for some positive constants \( \tau_1, \cdots, \tau_n \), which are so-called discrete delays and \( D = \text{diag}(d_1, \cdots, d_n) \) with \( d_i > 0 \), \( f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is Lipschitz continuous.

For convenience, we denote by \( C_b(\mathbb{R}, \mathbb{R}) \) the space of all bounded and continuous functions \( h: \mathbb{R} \rightarrow \mathbb{R} \) endowed with the super-norm. Moreover, for any \( k \in \mathbb{R}_+ \), we denote by \( C_b^k(\mathbb{R}, \mathbb{R}) \) the space of all continuous differentiable up to the \([k]\)-order functions \( h \) such that \( d^\gamma h \in C_b(\mathbb{R}, \mathbb{R}) \) for any \( |\gamma| \leq [k] \) ([k] denoting the integer part of \( k \)). The above spaces for vector-valued functions (with \( n \) components) are denoted by \( C_b(\mathbb{R}, \mathbb{R}^n) \) and \( C_b^k(\mathbb{R}, \mathbb{R}^n) \), respectively.

A traveling wave solution of (2.1) is a special translation invariant solution of the form \( u(x, t) = \varphi(t + x/c) \), where \( \varphi \in C_b^2(\mathbb{R}, \mathbb{R}^n) \) is the profile of the wave and \( c > 0 \) is a constant corresponding to the wave speed. The vibration at the space point \( x = 0 \) is \( u(t) = \varphi(t) \); The vibration \( u(t) \) propagating from the space value \( x = 0 \) to \( x \) costs the time \( x/c \), where \( c \) is wave velocity, in the case \( c > 0 \)
the traveling wave move to the left, in the case \( c < 0 \) the traveling wave move to the right. Our definition is more visual than \( \mathbf{u}(x, t) = \varphi(x + ct) \) in \([2, 23, 36]\).

Substituting \( \mathbf{u}(x, t) = \varphi(t + x/c) \) into (2.1) and letting \( s = t + x/c \), denoting also \( t \), we obtain the

\[
\varphi'(t) - \frac{D}{c^2} \varphi''(t) = f(\varphi(t), \varphi(t - \tau)), \quad t \in \mathbb{R}.
\]

(2.2)

corresponding wave equations If for some wave velocity \( c \), (2.2) has a solution \( \varphi \) defined on \( \mathbb{R} \) such that

\[
\lim_{t \to -\infty} \varphi(t) = \mathbf{u}_-, \quad \lim_{t \to +\infty} \varphi(t) = \mathbf{u}_+ \quad (2.3)
\]

exist, then \( \mathbf{u}(x, t) = \varphi(t + x/c) \) is called traveling wave with speed \( c \). Moreover, if \( \varphi \) is monotone in \( t \in \mathbb{R} \), then it is called a traveling wavefront.

Without loss of generality, we can assume \( \mathbf{u}_- = 0 \) and \( \mathbf{u}_+ = K > 0 \). Let

\[
C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) = \{ \varphi = (\varphi_1(t), \cdots, \varphi_n(t)) \in C_b(\mathbb{R}, \mathbb{R}^n) : 0 \leq \varphi_i(t) \leq K_i, t \in \mathbb{R} \}.
\]

Our aim is looking for a solution of (2.2) in \( C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \). Throughout this paper, the following hypothesis will be imposed on the reaction term \( f \):

\((H_1)\) \( f(0) = f(K) = 0 \).

Obviously, we should replace (2.3) with

\[
\lim_{t \to -\infty} \varphi(t) = 0, \quad \lim_{t \to +\infty} \varphi(t) = K \quad (2.4)
\]

In this paper, we explore the existence of the traveling wave solutions of (2.1) where the reaction term \( f \) is mixed quasimonotone.

\((H_2)\) The function \( f(\mathbf{u}, \mathbf{u}_\tau) = (f_1(\mathbf{u}, \mathbf{u}_\tau), \cdots, f_n(\mathbf{u}, \mathbf{u}_\tau)) \) is a \( C^1 \) function and possesses a mixed quasimonotone property in a subset \([0, K]\) of \( \mathbb{R}^n \).

The above hypothesis implies that there exist constants \( \beta_i \) such that \( f_i \) satisfies the Lipschitz condition

\[
|f_i(\mathbf{u}, \mathbf{u}_\tau) - f_i(\mathbf{v}, \mathbf{v}_\tau)| \leq \beta_i(||\mathbf{u} - \mathbf{v}|| + ||\mathbf{u}_\tau - \mathbf{v}_\tau||)
\]

for all \( \mathbf{u}, \mathbf{v}, \mathbf{u}_\tau \) and \( \mathbf{v}_\tau \) in \( C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \), \( i = 1, \cdots, n \), where \( || \cdot || \) and \( || \cdot || \) denote the super norm in \( \mathbb{R}^n \) and \( C(\mathbb{R}, \mathbb{R}^n) \), respectively.
Remark 2.1. If \( \phi \) for all \( i \), then the pair of vectors \( \bar{\phi} \) is monotone nondecreasing in \([u]_{a_i} \) and \([v]_{c_i} \) and is monotone nonincreasing in \([u]_{b_i} \) and \([v]_{d_i} \). If \( b_i = d_i = 0 \) for all \( i \) then \( f(u, v) \) is said to be quasimonotone nondecreasing.

The above general assumptions are used to establish the existence of a traveling wave solution to (2.2). Our approach to the problem is by the method of coupled upper and lower solutions which are defined as follows:

**Definition 2.1.** A pair of vectors \( \bar{\varphi} \equiv (\bar{\varphi}_1, \cdots, \bar{\varphi}_n) \), \( \hat{\varphi} \equiv (\hat{\varphi}_1, \cdots, \hat{\varphi}_n) \) in \( C^2_b(\mathbb{R}, \mathbb{R}^n) \) are called coupled upper and lower solutions of (2.2) if \( \bar{\varphi} \geq \hat{\varphi} \) and if

\[
\begin{align*}
\bar{\varphi}_i'(t) - \frac{d}{dt} \bar{\varphi}_i''(t) & \geq f_i(\bar{\varphi}_1, [\bar{\varphi}]_{a_1}, [\bar{\varphi}]_{b_1}, [\bar{\varphi}]_{c_1}, [\bar{\varphi}]_{d_1}), \\
\hat{\varphi}_i'(t) - \frac{d}{dt} \hat{\varphi}_i''(t) & \leq f_i(\hat{\varphi}_1, [\hat{\varphi}]_{a_1}, [\hat{\varphi}]_{b_1}, [\hat{\varphi}]_{c_1}, [\hat{\varphi}]_{d_1}) \quad (i = 1, \cdots, n),
\end{align*}
\]

where \( \varphi_r(t) = \varphi(t - \tau) \).

**Remark 2.1.** If \( f(\varphi, \varphi_r) \) is quasimonotone nondecreasing, that is, \( b_i = d_i = 0 \) for all \( i \), then the pair of vectors called ordered upper and lower solutions of (2.2)

Since that \( f \) satisfies \((H_2)\) and Lipschitz continuous, we have

\[
\begin{align*}
f_i(\varphi_1, [\varphi]_{a_1}, [\varphi]_{b_1}, [\varphi]_{c_1}, [\varphi]_{d_1}) - f_i(\bar{\varphi}_1, [\bar{\varphi}]_{a_1}, [\bar{\varphi}]_{b_1}, [\bar{\varphi}]_{c_1}, [\bar{\varphi}]_{d_1}) + \beta_i(\varphi_i - \bar{\varphi}_i) & \geq 0 \quad \text{for all} \quad 0 \leq \varphi_i \leq K_i, \quad i = 1, \cdots, n.
\end{align*}
\]

Next we define an operator \( H : C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \rightarrow C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \) by

\[
H(\varphi, \varphi_r)(t) = f(\varphi, \varphi_r) + \beta \varphi(t), \quad \varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n),
\]

where \( H = (H_1, \cdots, H_n) \), \( H_i(\varphi, \varphi_r)(t) = f_i(\varphi, \varphi_r) + \beta_i \varphi_i(t) \). Clearly, with the above notations, the system (2.2) is equivalent to the following system of ordinary differential equations

\[
\varphi'(t) - \frac{D}{c^2} \varphi''(t) + \beta \varphi(t) = H(\varphi, \varphi_r)(t), \quad t \in \mathbb{R}.
\]
Our first iteration involves the following linear system of ordinary differential equations
\begin{align}
    c\overline{x}_i^{(1)}' - \frac{d_i}{c^2}(\overline{x}_i^{(1)})'' + \beta_i \overline{x}_i^{(1)} &= \beta_i \hat{\varphi}_i + f_i(\hat{\varphi}_i, [\hat{\varphi}]_{a_i}, [\hat{\varphi}]_{b_i}, [\hat{\varphi}]_{c_i}, [\hat{\varphi}]_{d_i}), \\
    c\overline{\chi}_i^{(1)}' - \frac{d_i}{c^2}(\overline{\chi}_i^{(1)})'' + \beta_i \overline{\chi}_i^{(1)} &= \beta_i \hat{\varphi}_i + f_i(\hat{\varphi}_i, [\hat{\varphi}]_{a_i}, [\hat{\varphi}]_{b_i}, [\hat{\varphi}]_{c_i}, [\hat{\varphi}]_{d_i}).
\end{align}
\tag{2.9}

Note that
\begin{align*}
    \lambda_{1i} = \frac{c^2(1 - \sqrt{1 + 4\beta_i d_i/c^4})}{2d_i}, \quad \lambda_{2i} = \frac{c^2(1 + \sqrt{1 + 4\beta_i d_i/c^4})}{2d_i}
\end{align*}
are the negative and positive real roots of the equation
\begin{align}
    \frac{d_i}{c^2}\lambda^2 - \lambda - \beta_i = 0, \quad i = 1, 2, \ldots, n.
\end{align}

Using the Perron Theorem yields
\begin{align}
    \overline{x}_i^{(1)} &= \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}\left(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}(\beta_i \hat{\varphi}_i + f_i(\hat{\varphi}_i, [\hat{\varphi}]_{a_i}, [\hat{\varphi}]_{b_i}, [\hat{\varphi}]_{c_i}, [\hat{\varphi}]_{d_i}))ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}(\beta_i \hat{\varphi}_i + f_i(\hat{\varphi}_i, [\hat{\varphi}]_{a_i}, [\hat{\varphi}]_{b_i}, [\hat{\varphi}]_{c_i}, [\hat{\varphi}]_{d_i}))ds\right), \\
    \overline{\chi}_i^{(1)} &= \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}\left(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}(\beta_i \hat{\varphi}_i + f_i(\hat{\varphi}_i, [\hat{\varphi}]_{a_i}, [\hat{\varphi}]_{b_i}, [\hat{\varphi}]_{c_i}, [\hat{\varphi}]_{d_i}))ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}(\beta_i \hat{\varphi}_i + f_i(\hat{\varphi}_i, [\hat{\varphi}]_{a_i}, [\hat{\varphi}]_{b_i}, [\hat{\varphi}]_{c_i}, [\hat{\varphi}]_{d_i}))ds\right)
\end{align}
\tag{2.10}
for \(i = 1, \ldots, n\). Then by Lemma 2.1 of \[22\], \(\overline{x}_i^{(1)} \equiv (\overline{x}_1^{(1)}, \ldots, \overline{x}_n^{(1)})\) and \(\overline{\chi}_i^{(1)} \equiv (\overline{\chi}_1^{(1)}, \ldots, \overline{\chi}_n^{(1)})\) have the following properties.

**Lemma 2.1.** Let \(\overline{x}_i^{(1)}\) and \(\overline{\chi}_i^{(1)}\) be the solution of \(2.9\), then we have
\begin{enumerate}[(i)]
    \item \(\hat{\varphi} \leq \overline{x}_i^{(1)} \leq \overline{\chi}_i^{(1)} \leq \hat{\varphi}\).
    \item \(\overline{x}_i^{(1)}, \overline{\chi}_i^{(1)}\) are a pair of coupled upper and lower solutions of \(2.9\).
\end{enumerate}

Now in order to prove the existence of the traveling wave solution, we are in the position to apply the Schauder’s fixed point theorem. We define the operator
\[F = (F_1, \ldots, F_n) : C_{[0,\mathbf{K}]}(\mathbb{R}, \mathbb{R}^n) \rightarrow C_{[0,\mathbf{K}]}(\mathbb{R}, \mathbb{R}^n)\]
by

\[(F_i \varphi_i)(t) = \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}H_i(\varphi, \varphi_r)(s)ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}H_i(\varphi, \varphi_r)(s)ds)
\]

for \(i = 1, 2, \ldots, n\).

Let \(\rho > 0\) be such that \(\rho < \min\{-\lambda_{1i}, \lambda_{2i} : i = 1, 2, \ldots, n\}\), and let

\[B_\rho(\mathbb{R}, \mathbb{R}^n) = \{\varphi \in \mathcal{C}_{[0,K]}(\mathbb{R}, \mathbb{R}^n) : \sup_{t \in \mathbb{R}} |\varphi(t)|e^{-\rho|t|} < \infty\},\]

\[|\varphi|_\rho = \sup_{t \in \mathbb{R}} |\varphi(t)|e^{-\rho|t|};\]

Then it is easy to check that \(B_\rho(\mathbb{R}, \mathbb{R}^n), | \cdot |_\rho\) is a Banach space.

**Lemma 2.2.** Let the closed convex set \(\Gamma = \{\varphi \in \mathcal{C}_{[0,K]}(\mathbb{R}, \mathbb{R}^n) : \tilde{\varphi} \leq \varphi \leq \hat{\varphi}\}\), where \(\tilde{\varphi}\) and \(\hat{\varphi}\) are coupled upper and lower solutions of \((2.2)\), then \(F(\Gamma) \subseteq \Gamma\).

**Proof.** Since that the mixed quasimonotone property of the operator \(H\), \(\forall \varphi \in \Gamma\), we have

\[(F_i \varphi_i)(t) \leq \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}(\beta_i \tilde{\varphi}_i + f_i(\tilde{\varphi}_i, [\tilde{\varphi}]_{a_i}, [\tilde{\varphi}]_{b_i}, [\tilde{\varphi}]_{c_i}, [\tilde{\varphi}]_{d_i}))ds
+ \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}(\beta_i \tilde{\varphi}_i + f_i(\tilde{\varphi}_i, [\tilde{\varphi}]_{a_i}, [\tilde{\varphi}]_{b_i}, [\tilde{\varphi}]_{c_i}, [\tilde{\varphi}]_{d_i}))ds)\]

By the virtue of Lemma 2.1, we induce \((F_i \varphi_i)(t) \leq \tilde{\varphi}_i(t)\). Similarly, we have \((F_i \varphi_i)(t) \geq \hat{\varphi}_i(t)\) for all \(i = 1, \ldots, n\). Therefore \(F(\Gamma) \subseteq \Gamma\). \(\square\)

**Lemma 2.3.** If the hypothesis \((H_2)\) holds, then \(F : \mathcal{C}_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to \mathcal{C}_{[0,K]}(\mathbb{R}, \mathbb{R}^n)\) is continuous with respect to the norm \(\cdot \) in \(B_\rho(\mathbb{R}, \mathbb{R}^n)\).

**Proof.** \(\forall \varphi, \varphi' \in \mathcal{C}_{[0,K]}(\mathbb{R}, \mathbb{R}^n)\), in view of the definition of \(F\), we have

\[
F_i(\varphi, \varphi_r) - F_i(\varphi', \varphi_r') = \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}(\beta_i(\varphi_i - \varphi'_i) + f_i(\varphi, \varphi_r)
- f_i(\varphi', \varphi'_r))ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}(\beta_i(\varphi_i - \varphi'_i) + f_i(\varphi, \varphi_r) - f_i(\varphi', \varphi'_r))ds).
\]

(2.11)

Since that \(\varphi, \varphi' \leq K, (H_2)\) implies that \(f_i(\varphi, \varphi_r)\) is bounded for \(\varphi \in \mathcal{C}_{[0,K]}(\mathbb{R}, \mathbb{R}^n)\). Then the term \(\beta_i(\varphi_i - \varphi'_i) + f_i(\varphi, \varphi_r) - f_i(\varphi', \varphi'_r)\) is bounded, for convenience, we denote \(\beta_i(\varphi_i - \varphi'_i) + f_i(\varphi, \varphi_r) - f_i(\varphi', \varphi'_r) \leq M_i, (2.11)\) is transformed into

\[
F_i(\varphi, \varphi_r) - F_i(\varphi', \varphi'_r) \leq \frac{c^2M_i}{d_i(\lambda_{2i} - \lambda_{1i})}(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}ds)
= \frac{c^2M_i}{d_i(\lambda_{2i} - \lambda_{1i})}(\frac{1}{\lambda_{2i}} - \frac{1}{\lambda_{1i}}) = \frac{c^2M_i}{d_i\lambda_{1i}\lambda_{2i}}.
\]

(2.12)
It follows from (2.12) that
\[
|F_i(\phi, \varphi_\tau) - F_i(\phi', \varphi'_\tau)|e^{-\rho|t|} \leq \frac{c^2 M_i}{d_i \lambda_{11} \lambda_{2i}} e^{-\rho|t|} \leq \frac{c^2 M_i}{d_i \lambda_{11} \lambda_{2i}}.
\] (2.13)

Therefore \(|F_i(\phi, \varphi_\tau) - F_i(\phi', \varphi'_\tau)| \leq \frac{c^2 M_i}{d_i \lambda_{11} \lambda_{2i}} \) for all \(i = 1, \ldots, n\). That is, \(F : C_{\{0,K\}}(\mathbb{R}, \mathbb{R}^n) \to C_{\{0,K\}}(\mathbb{R}, \mathbb{R}^n)\) is continuous.

\[\Box\]

**Lemma 2.4.** If the hypothesis \((H_2)\) holds and \(\Gamma\) is defined in Lemma 2.2, then \(F : \Gamma \to \Gamma\) is compact.

**Proof.** First we compute \(\frac{dF_i}{dt}(\phi, \varphi_\tau)(t)\), for any \(\phi \in C_{\{0,K\}}(\mathbb{R}, \mathbb{R}^n)\), we have
\[
\frac{dF_i}{dt}(\phi, \varphi_\tau)(t) = \frac{c^2 \lambda_{1i}}{d_i(\lambda_{2i} - \lambda_{1i})} \int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} (\beta_i \phi_i + f_i(\phi, \varphi_\tau)) ds
+ \frac{c^2 \lambda_{2i}}{d_i(\lambda_{2i} - \lambda_{1i})} \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)} (\beta_i \phi_i + f_i(\phi, \varphi_\tau)) ds.
\] (2.14)

It follows from the similar argument of (2.12) that
\[
\frac{dF_i}{dt}(\phi, \varphi_\tau)(t) \leq \frac{c^2 \lambda_{1i} M_i}{d_i(\lambda_{2i} - \lambda_{1i}) \lambda_{1i}} + \frac{c^2 \lambda_{2i} M_i}{d_i(\lambda_{2i} - \lambda_{1i}) \lambda_{2i}} = \frac{2c^2 M_i}{d_i(\lambda_{2i} - \lambda_{1i})}.
\] (2.15)

It follows from (2.15) that
\[
\left| \frac{dF_i}{dt}(\phi, \varphi_\tau)(t) \right| e^{-\rho|t|} \leq \frac{2c^2 M_i}{d_i(\lambda_{2i} - \lambda_{1i})} e^{-\rho|t|} \leq \frac{2c^2 M_i}{d_i(\lambda_{2i} - \lambda_{1i})}.
\] (2.16)

Therefore \(\frac{dF_i}{dt}(\phi, \varphi_\tau)(t) \leq \frac{2c^2 M_i}{d_i(\lambda_{2i} - \lambda_{1i})}\) for all \(i = 1, \ldots, n\). Hence \(F\) is equicontinuous on \(C_{\{0,K\}}(\mathbb{R}, \mathbb{R}^n)\). In view of Lemma 2.2 \(F(\Gamma)\) is uniformly bounded.

Next we claim that \(F : \Gamma \to \Gamma\) is compact. Define the operator sequence \(\{F^{(n)}\}\), where \(F^{(n)} : C_{\{0,K\}}(\mathbb{R}, \mathbb{R}^n) \to C_{\{0,K\}}(\mathbb{R}, \mathbb{R}^n)\) by

\[
F^{(n)}(\phi, \varphi_\tau)(t) = \begin{cases} 
F(\phi, \varphi_\tau)(-n), & t \in (-\infty, -n), \\
F(\phi, \varphi_\tau)(t), & t \in [-n, n], \\
F(\phi, \varphi_\tau)(n), & t \in (n, +\infty). 
\end{cases}
\]

Hence the sequence \(\{F^{(n)}\}\) are uniformly bounded and equicontinuous. It follows from Arzela-Ascoli theorem that \(F^{(n)}\) is compact. Therefore we have
\[
|F_i^{(n)}(\phi, \varphi_\tau)(t) - F_i(\phi, \varphi_\tau)(t)| e^{-\rho|t|}
\]
\[
= \sup_{t \in (-\infty, -n) \cup (n, +\infty)} \left| F_i^{(n)}(\phi, \varphi_\tau)(t) - F_i(\phi, \varphi_\tau)(t) \right| e^{-\rho|t|}
\]
\[
\leq 2K_i e^{-\rho n} \to 0 \text{ as } t \to \infty,
\] (2.17)
where \( \varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \). By the virtue of proposition 2.1 in [39], the sequence \( \{ F^{(n)} \} \) converges to \( F \) in \( \Gamma \) with respect to the norm \( | \cdot |_\rho \). Therefore \( F : \Gamma \to \Gamma \) is compact.

\[ \square \]

**Theorem 2.1.** Assume that \((H_1)\) and \((H_2)\) hold. Suppose that \( \bar{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \), \( \hat{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \) be a pair of upper and lower solutions of \((2.2)\), and

\[
\lim_{t \to -\infty} \bar{\varphi}(t) = 0, \quad \lim_{t \to +\infty} \hat{\varphi}(t) = K, \tag{2.18}
\]

then \((2.2)\) and \((2.4)\) admit a solution. That is, the problem \((2.1)\) has a traveling wave solution.

**Proof.** First we define the following profile set such as in Lemma 2.2

\[ \Gamma = \{ \varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) : \hat{\varphi} \leq \varphi \leq \bar{\varphi} \}, \]

it is easy to show that \( \Gamma \) is a closed convex set.

Now we define the operator such as in Lemma 2.3

\[ F : \Gamma \to \Gamma, \]

in view of Lemma 2.2, Lemma 2.3, Lemma 2.4, \( F \) is continuous and compact with respect to \( | \cdot |_\rho \). By Schauder’s fixed point theorem, there exists a fixed point \( \varphi^* \in \Gamma \) such that \( F(\varphi^*, \varphi^*) = \varphi^* \). Using the Perron Theorem, \((2.8)\) has a solution \( \varphi^* \), that is \( \varphi^* \) is a solution of \((2.2)\).

Next we will show that \( \varphi^* \) satisfies the boundary condition \((2.4)\). In view of Lemma 2.2, \( \hat{\varphi} \leq \varphi^* \leq \bar{\varphi} \). It follows from \((2.18)\) that

\[
0 \leq \lim_{t \to -\infty} \varphi^*(t) \leq \lim_{t \to -\infty} \bar{\varphi} = 0, \quad K \geq \lim_{t \to +\infty} \varphi^*(t) \geq \lim_{t \to +\infty} \hat{\varphi} = K.
\]

Therefore \( \varphi^* \) is a traveling wave solution of the problem \((2.1)\). Thus the proof is completed. \[ \square \]

### 3 Coupled quasi-upper and quasi-lower solutions

In Theorem 2.1, we see that the smooth conditions on the coupled upper and lower solutions are too strong. In fact, it is very difficult to seek the \( C^2 \) smooth
coupled upper and lower solutions for a special model. We intend to relax the smoothness of the upper and lower solutions to $C^1$. Thus we should cite the modified Perron theorem in [2].

**Definition 3.1.** Considering the following scalar ordinary equation

$$u''(t) + \alpha u'(t) + \beta u(t) + f(t) = 0, \quad t \in \mathbb{R}, \ u(t) \in \mathbb{R},$$  

(3.1)

where $\beta < 0$, $f$ is a bounded and continuous function on $\mathbb{R} \setminus \{0\}$ and both $f(0^+)$ and $f(0^-)$ exist. Then, a function $u$ defined on $\mathbb{R}$ is said to be a generalized solution of (3.1) if

1. $u$ and $u'$ are bounded and continuous on $\mathbb{R}$.
2. $u''$ exists and is continuous on $\mathbb{R} \setminus \{0\}$, and both $u''(0^-)$ and $u''(0^+)$ exist.

**Lemma 3.1.** ([2]) Consider (3.1) with $\beta < 0$, and assume that

1. $f$ is a bounded and continuous function on $\mathbb{R} \setminus \{0\}$ and both $f(0^+)$ and $f(0^-)$ exist,
2. (3.1) holds in the classical sense for all $t$ except possibly at $t = 0$.

Then (3.1) has a unique generalized solution $u$ given by

$$u(t) = \frac{1}{\lambda_2 - \lambda_1} \left( \int_{-\infty}^{t} e^{\lambda_1(t-s)} f(s) ds + \int_{t}^{\infty} e^{\lambda_2(t-s)} f(s) ds \right),$$

(3.2)

where $\lambda_1$ and $\lambda_2$ are respectively the negative and positive roots of $\lambda^2 + \alpha \lambda + \beta = 0$.

Now we give the following definition of an admissible upper and lower solutions.

**Definition 3.2.** Assume that $\bar{\varphi}, \hat{\varphi} \in C^1_b(\mathbb{R}, \mathbb{R}^n)$, $\frac{d^2 \bar{\varphi}}{dt^2}(t)$ and $\frac{d^2 \hat{\varphi}}{dt^2}(t)$ exist and continuous on $\mathbb{R} \setminus \{0\}$, and

$$\sup_{t \to \mathbb{R} \setminus \{0\}} \left| \frac{d^2 \bar{\varphi}}{dt^2}(t) \right| < +\infty, \quad \text{and} \quad \lim_{t \to 0^-} \frac{d^2 \bar{\varphi}}{dt^2}(t), \lim_{t \to 0^+} \frac{d^2 \bar{\varphi}}{dt^2}(t) \text{ exist},$$

$$\sup_{t \to \mathbb{R} \setminus \{0\}} \left| \frac{d^2 \hat{\varphi}}{dt^2}(t) \right| < +\infty, \quad \text{and} \quad \lim_{t \to 0^-} \frac{d^2 \hat{\varphi}}{dt^2}(t), \lim_{t \to 0^+} \frac{d^2 \hat{\varphi}}{dt^2}(t) \text{ exist},$$

(3.3)

$\bar{\varphi}, \hat{\varphi}$ satisfy

$$\bar{\varphi}'(t) - \frac{d}{dt} \bar{\varphi}''(t) \geq f_i(\bar{\varphi},, [\bar{\varphi}]_a, \ [\bar{\varphi}]_b, \ [\bar{\varphi}]_c, \ [\bar{\varphi}]_d), \text{ for all } t \in \mathbb{R} \setminus \{0\},$$

$$c_1 \hat{\varphi}'(t) - \frac{d}{dt} \hat{\varphi}''(t) \leq f_i(\hat{\varphi},, [\hat{\varphi}]_a, \ [\hat{\varphi}]_b, \ [\hat{\varphi}]_c, \ [\hat{\varphi}]_d), \text{ for all } t \in \mathbb{R} \setminus \{0\},$$

(3.4)

for $i = 1, \cdots, n$. Then $\bar{\varphi}$ and $\hat{\varphi}$ are called coupled quasi-upper solution and quasi-lower solution of (2.2), respectively.
Lemma 3.2. If \( \tilde{\varphi}, \hat{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \) are a pair of quasi-upper and quasi-lower solutions of (2.2). Then \( \bar{X}^{(1)}(t) \) and \( \underline{X}^{(1)}(t) \) constructed by (2.10), where \( \tilde{\varphi} \) and \( \hat{\varphi} \) are replaced with these quasi-upper and quasi-lower solutions, are a pair of upper and lower solutions of (2.2), moreover

\[ \hat{\varphi} \leq \bar{X}^{(1)}(t) \leq \underline{X}^{(1)}(t) \leq \tilde{\varphi}. \]

Proof. Combining (3.4) and (2.10) yields

\[
(\bar{\pi}_1^{(1)})' - \frac{d_1}{c_2} (\bar{\pi}_1^{(1)})'' + \beta_1 \bar{\pi}_1^{(1)} = \beta_1 \bar{\varphi}_i + f_i(\bar{\varphi}_i, [\bar{\varphi}]_a, [\bar{\varphi}_\gamma]_c, [\bar{\varphi}_\tau]_{d_i}) \leq \bar{\varphi}_i - \frac{d_1}{c_2} \bar{\varphi}_i'' + \beta_1 \bar{\varphi}_i, \text{ for all } t \in \mathbb{R} \setminus \{0\}. \tag{3.5}
\]

In view of Lemma 3.1 we have

\[
\bar{X}_1^{(1)}(t) \leq \frac{1}{d_i(\lambda_2 - \lambda_1)} \left( \int_{-\infty}^{t} e^{\lambda_1(t-s)} (\bar{\varphi}_i'(s) - \frac{d_1}{c_2} \bar{\varphi}_i''(s) + \beta_1 \bar{\varphi}_i(s))ds \right)
+ \int_{t}^{+\infty} e^{\lambda_1(t-s)} (\bar{\varphi}_i'(s) - \frac{d_1}{c_2} \bar{\varphi}_i''(s) + \beta_1 \bar{\varphi}_i(s))ds \tag{3.5}
= \bar{\varphi}_i(t), \text{ for all } t \in \mathbb{R} \setminus \{0\}.
\]

In a similar way, we have \( \underline{X}_1^{(1)}(t) \geq \hat{\varphi}_i(t) \) for all \( t \in \mathbb{R} \setminus \{0\} \). Since that \( \bar{X}^{(1)}(t), \underline{X}^{(1)}(t), \bar{X}^{(1)}(t) \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \cap C^2(\mathbb{R}, \mathbb{R}^n) \) and (3.5) holds for all \( t \in \mathbb{R} \). A similar argument in Lemma 2.1 of [22] shows that \( \bar{X}^{(1)}, \underline{X}^{(1)} \) are a pair of coupled upper and lower solutions of (2.2).

Theorem 3.1. Assume that \((H_1)\) and \((H_2)\) hold. Suppose that \( \tilde{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n), \hat{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \) be a pair of coupled quasi-upper and quasi-lower solutions of (2.2), and

\[
\lim_{t \to -\infty} \tilde{\varphi}(t) = 0, \quad \lim_{t \to +\infty} \hat{\varphi}(t) = K.
\]

Then (2.2) and (2.4) admit a solution. That is, the problem (2.1) has a traveling wave solution.
Assume that Definition 4.2.

for $i\phi$ upper and lower solutions of (2.2). Replace Theorem 2.1 and obtain the same results directly.

Proof. Let

$$x^{(1)}_i = \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})} \left( \int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}(\beta_i \hat{\phi}_i + f_i(\hat{\phi}_i, [\hat{\phi}]_{a_i}, [\hat{\phi}]_{b_i}, [\hat{\phi}]_{e_i}, [\hat{\phi}]_{d_i}))ds \right. + \left. \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}(\beta_i \hat{\phi}_i + f_i(\hat{\phi}_i, [\hat{\phi}]_{a_i}, [\hat{\phi}]_{b_i}, [\hat{\phi}]_{e_i}, [\hat{\phi}]_{d_i}))ds \right),$$

for $i = 1, 2, \cdots, n$. Then by Lemma 3.2, $X^{(1)}$ and $X^{(1)}$ are a pair of coupled upper and lower solutions of (2.2). Replace $\bar{\phi}$, $\hat{\phi}$ with $\tilde{X}^{(1)}$, $\bar{X}^{(1)}$. We then use Theorem 2.1 and obtain the same results directly. 

\[\square\]

4 Ordered quasi-upper and quasi-lower solutions

If $f(\phi, \varphi_r)$ is quasimonotone nondecreasing, that is, $b_i = d_i = 0$ for all $i$, then the upper and lower solution are ordered. In this case [2, 23, 36] have showed the existence of the traveling wave solution.

Definition 4.1. A pair of vectors $\bar{\phi} \equiv (\bar{\phi}_1, \ldots, \bar{\phi}_n)$, $\hat{\phi} \equiv (\hat{\phi}_1, \ldots, \hat{\phi}_n)$ in $C^1_b(\mathbb{R}, \mathbb{R}^n)$ are called ordered upper and lower solutions of (2.2) if $\bar{\phi} \geq \hat{\phi}$ and if

$$\bar{\phi}'_i(t) - \frac{d}{dt} \bar{\phi}''_i(t) \geq f_i(\bar{\phi}, \hat{\phi}_r),$$

$$\hat{\phi}'_i(t) - \frac{d}{dt} \hat{\phi}''_i(t) \leq f_i(\bar{\phi}, \hat{\phi}_r), \quad (i = 1, \ldots, n),$$

(4.1)

where $\varphi_r(t) = \varphi(t - \tau)$.

Definition 4.2. Assume that $\bar{\phi}$, $\hat{\phi} \in C^1_b(\mathbb{R}, \mathbb{R}^n)$, $\frac{\partial^2 \bar{\phi}}{\partial t^2}(t)$ and $\frac{\partial^2 \hat{\phi}}{\partial t^2}(t)$ exist and continuous on $\mathbb{R} \setminus \{0\}$, and

$$\sup_{t \rightarrow R \setminus \{0\}} \left| \frac{d^2 \bar{\phi}}{dt^2}(t) \right| < +\infty, \quad \text{and} \quad \lim_{t \rightarrow 0^-} \frac{d^2 \bar{\phi}}{dt^2}(t), \lim_{t \rightarrow 0^+} \frac{d^2 \bar{\phi}}{dt^2}(t) \text{ exist},$$

$$\sup_{t \rightarrow R \setminus \{0\}} \left| \frac{d^2 \hat{\phi}}{dt^2}(t) \right| < +\infty, \quad \text{and} \quad \lim_{t \rightarrow 0^-} \frac{d^2 \hat{\phi}}{dt^2}(t), \lim_{t \rightarrow 0^+} \frac{d^2 \hat{\phi}}{dt^2}(t) \text{ exist},$$

(4.2)

$\bar{\phi}$, $\hat{\phi}$ satisfy

$$\bar{\phi}'_i(t) - \frac{d}{dt} \bar{\phi}''_i(t) \geq f_i(\bar{\phi}, \hat{\phi}_r), \text{ for all } t \in \mathbb{R} \setminus \{0\},$$

$$\hat{\phi}'_i(t) - \frac{d}{dt} \hat{\phi}''_i(t) \leq f_i(\bar{\phi}, \hat{\phi}_r), \text{ for all } t \in \mathbb{R} \setminus \{0\},$$

(4.3)
for \( i = 1, \cdots, n \). Then \( \bar{\varphi} \) and \( \breve{\varphi} \) are called ordered quasi-upper solution and quasi-lower solution of (2.2), respectively.

Next theorem shows that if the upper or quasi-upper solution \( \bar{\varphi}(t) \) is nondecreasing respect to \( t \), then the solution \( \varphi^*(t) \) is also nondecreasing. Using the similar argument in Lemma 4.1 of [22], we can induce the existence of the traveling wavefront of (2.2).

**Theorem 4.1.** Assume that \((H_1)\) and \((H_2)\) hold and \( f(\varphi, \varphi_\tau) \) is quasimonotone nondecreasing. Suppose that \( \bar{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \), \( \breve{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \) be a pair of ordered quasi-upper and quasi-lower solutions of (2.2), and \( \bar{\varphi}(t) \) is nondecreasing with respect to \( t \),

\[
\lim_{t \to -\infty} \bar{\varphi}(t) = 0, \quad \lim_{t \to +\infty} \breve{\varphi}(t) = K. \tag{4.4}
\]

Then the solution of (2.2) and (2.4) \( \varphi^* \) is nondecreasing with respect to \( t \). That is, the problem (2.1) at least has a traveling wavefront solution.

In Theorem 4.1, the condition (4.4) can be replaced by further restrictions on \( f \), that is

\[
(H_1^\ast) \quad f(u, u_\tau)|_{u=0} = f(u, u_\tau)|_{u=K} = 0 \text{ and } f(u, u_\tau)|_{u=L} \neq 0 \text{ for the constant-valued function } L \text{ with } 0 \leq L \leq K \text{ and } L \neq 0, L \neq K.
\]

As similar as the argument in Theorem 4.4 of [22], Theorem 4.1 can be transformed into

**Theorem 4.2.** Assume that \((H_1^\ast)\) and \((H_2)\) hold and \( f(\varphi, \varphi_\tau) \) is quasimonotone nondecreasing. Suppose that \( \bar{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \), \( \breve{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \) be a pair of ordered quasi-upper and quasi-lower solutions of (2.2), and \( \bar{\varphi}(t) \) is nondecreasing with respect to \( t \),

\[
0 \leq \breve{\varphi}(t) \leq \bar{\varphi}(t) \leq K, \quad \breve{\varphi}(t) \neq 0, \bar{\varphi}(t) \neq K \text{ in } \mathbb{R}, \tag{4.5}
\]

then the solution of (2.2) and (2.4) \( \varphi^* \) is nondecreasing with respect to \( t \). That is, the problem (2.1) at least has a traveling wavefront solution.

**Remark 4.1.** In Theorems 4.1 and 4.2, the ordered quasi-lower solution is not necessary nondecreasing. The results of Theorem 4.1 has been obtained by many authors, for example, Theorem 2.2 of [23] and Theorem 11 of [2], and Theorem 3.6 of [30].
In the similar way, Theorem 3.1 can be transformed into

**Theorem 4.3.** Assume that \((H_1^*)\) and \((H_2)\) hold. Suppose that \(\tilde{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)\), \(\hat{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)\) be a pair of coupled quasi-upper and quasi-lower solutions of \((2.2)\), and

\[
0 \leq \hat{\varphi}(t) \leq \tilde{\varphi}(t) \leq K, \quad \hat{\varphi}(t) \neq 0, \tilde{\varphi}(t) \neq K \text{ in } \mathbb{R},
\]

(4.6)

Then \((2.2)\) and \((2.4)\) admit a solution. That is, the problem \((2.1)\) has a traveling wave solution.

5 Application

In this section, we use our results obtained in previous sections to consider the delayed reaction diffusion models.

5.1 Belousov-Zhabotinskii equations

Consider the delayed Belousov-Zhabotinskii equations

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} &= u(x,t)(1 - u(x,t) - rv(x, t - \tau_2)), \\
\frac{\partial v(x,t)}{\partial t} - \frac{\partial^2 v(x,t)}{\partial x^2} &= -bu(x, t - \tau_1)v(x,t),
\end{align*}
\]

(5.1)

where \(u(x,t), v(x,t)\) are scalar functions, \(r > 0, b > 0\) are constants. In the biological sense, \(u\) and \(v\) represent the Bromic acid and bromide ion concentrations respectively (see more details in [27]). Without the delays, the existences of the traveling wave solutions were considered in [11] [12] [33] [38]. When \(\tau_1 = 0, \tau_2 \neq 0\), [2] [23] [36] studied the traveling wave solution by using of various methods to construct quasi-upper solution.

Now we seek the traveling wave solution, whose form is \(u(x,t) = \varphi_1(t + x/c), v(x,t) = \varphi_2(t + x/c)\). Clearly the wave equations corresponding to \((2.2)\) is the following form

\[
\begin{align*}
\varphi_1'(t) - \frac{1}{c^2} \varphi_1''(t) &= \varphi_1(t)(1 - \varphi_1(t) - r\varphi_2(t - \tau_2)), \\
\varphi_2'(t) - \frac{1}{c^2} \varphi_2''(t) &= -b\varphi_2(t)\varphi_1(t - \tau_1).
\end{align*}
\]

(5.2)

We seek a traveling wave solution of \((5.1)\) with the boundary conditions

\[
\begin{align*}
\lim_{t \to -\infty} \varphi_1(t) &= 0, \quad \lim_{t \to -\infty} \varphi_2(t) = 0, \\
\lim_{t \to +\infty} \varphi_1(t) &= 1, \quad \lim_{t \to +\infty} \varphi_2(t) = 1.
\end{align*}
\]
It is easy to check that \((H^*_2)\) and \((H_2)\) are satisfied. We only need seek the coupled quasi-upper and quasi-lower solutions of \([5.2]\).

If \(\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2)\) and \(\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2)\) are coupled quasi-upper and quasi-lower solutions of \([5.2]\), they must satisfy

\[
\begin{align*}
\bar{\varphi}'_1(t) - \frac{1}{c^2} \bar{\varphi}''_1(t) &\geq \bar{\varphi}_1(t) (1 - \bar{\varphi}_1(t) - r \hat{\varphi}_2(t - \tau_2)), \text{ for all } t \in \mathbb{R} \setminus \{0\}, \\
\bar{\varphi}'_2(t) - \frac{1}{c^2} \bar{\varphi}''_2(t) &\geq -b \hat{\varphi}_2(t) \bar{\varphi}_1(t - \tau_1), \text{ for all } t \in \mathbb{R} \setminus \{0\}, \\
\hat{\varphi}'_1(t) - \frac{1}{c^2} \hat{\varphi}''_1(t) &\leq \hat{\varphi}_1(t) (1 - \phi_1(t) - r \hat{\varphi}_2(t - \tau_2)), \text{ for all } t \in \mathbb{R} \setminus \{0\}, \\
\hat{\varphi}'_2(t) - \frac{1}{c^2} \hat{\varphi}''_2(t) &\leq -b \hat{\varphi}_2(t) \bar{\varphi}_1(t - \tau_1), \text{ for all } t \in \mathbb{R} \setminus \{0\}.
\end{align*}
\]

(5.3)

Assume that \(\bar{\varphi}\) and \(\hat{\varphi}\)

\[
\begin{align*}
\bar{\varphi}_1(t) &= \begin{cases} 
\frac{1}{2} e^{\lambda_1 t}, & t \leq 0, \\
\frac{1}{2} e^{-\lambda_1 t}, & t > 0,
\end{cases} \quad \text{and} \quad \bar{\varphi}_2(t) = \begin{cases} 
\frac{1}{2} e^{\lambda_2 t}, & t \leq 0, \\
\frac{1}{2} e^{-\lambda_2 t}, & t > 0,
\end{cases} \\
\hat{\varphi}_1(t) &= \begin{cases} 
\delta k e^{\lambda_3 t}, & t \leq 0, \\
\delta k e^{-\lambda_3 t}, & t > 0,
\end{cases} \quad \text{and} \quad \hat{\varphi}_2(t) = 0,
\end{align*}
\]

(5.4)

(5.5)

where \(\lambda_1, \lambda_2, \lambda_3, \delta, k\) are undetermined positive constants.

Direct calculations show that

\[
\begin{align*}
\bar{\varphi}'_1(t) &= \begin{cases} 
\frac{\lambda_1}{2} e^{\lambda_1 t}, & t \leq 0, \\
-\frac{\lambda_1}{2} e^{-\lambda_1 t}, & t > 0,
\end{cases} \quad \text{and} \quad \bar{\varphi}'_2(t) = \begin{cases} 
\frac{\lambda_2}{2} e^{\lambda_2 t}, & t \leq 0, \\
-\frac{\lambda_2}{2} e^{-\lambda_2 t}, & t > 0,
\end{cases} \\
\bar{\varphi}''_1(t) &= \begin{cases} 
\lambda_3 \delta k e^{\lambda_3 t}, & t \leq 0, \\
-\lambda_3 \delta k e^{-\lambda_3 t}, & t > 0,
\end{cases} \quad \text{and} \quad \bar{\varphi}''_2(t) = \begin{cases} 
\lambda_3 \delta k e^{\lambda_3 t}, & t \leq 0, \\
-\lambda_3 \delta k e^{-\lambda_3 t}, & t > 0,
\end{cases}
\end{align*}
\]

(5.6)

(5.7)

(5.8)

From \([5.6], [5.7]\) and \([5.8]\), we see that the first derivatives are continuous and the second derivatives exist and continuous on \(\mathbb{R} \setminus \{0\}\). Hence \(\bar{\varphi}\) and \(\hat{\varphi}\) satisfy \([4.12]\). Now we will choose proper \(\lambda_1, \lambda_2, \lambda_3, \delta, k\) such that \([5.3]\) holds.

Substituting \([5.4], [5.6]\) and \([5.7]\) into the first equation of \([5.3]\) yields

\[
\begin{align*}
\frac{\lambda_1}{2} e^{\lambda_1 t} - \frac{1}{c^2} \frac{\lambda_1^2}{2} e^{\lambda_1 t} &\geq \frac{\lambda_1}{2} e^{\lambda_1 t} (1 - \frac{1}{2} e^{\lambda_1 t}), \text{ for } t < 0, \\
\frac{\lambda_1}{2} e^{-\lambda_1 t} - \frac{1}{c^2} \frac{\lambda_1^2}{2} e^{-\lambda_1 t} &\geq (1 - \frac{1}{2} e^{-\lambda_1 t}) (1 - 1 - \frac{1}{2} e^{-\lambda_1 t}), \text{ for } t \geq 0.
\end{align*}
\]

(5.9)
In order to induce (5.9), the following is sufficient
\[ \lambda_1 - \frac{\lambda_1^2}{c^2} - 1 \geq 0. \] (5.10)

Hence we set
\[ \lambda_1 = \frac{c^2(1 - \sqrt{1 - \frac{1}{c^2}})}{2}. \] (5.11)

Substituting (5.4)-(5.8) into the second equation of (5.3) yields
\[
\begin{align*}
\frac{\lambda_2}{2} e^{\lambda_2 t} - \frac{1}{c^2} \frac{\lambda_2}{2} e^{-\lambda_2 t} & \geq -b \frac{\lambda_1}{2} e^{\lambda_3 (t - \tau_1)} & \text{for } t < 0, \\
\frac{\lambda_2}{2} e^{-\lambda_2 t} - \frac{1}{c^2} \left( \frac{\lambda_2}{2} e^{-\lambda_2 t} \right) & \geq -b(1 - \frac{1}{2} e^{-\lambda_2 t}) \delta k e^{\lambda_3 (t - \tau_1)} & \text{for } 0 \leq t \leq \tau_1, \\
\frac{\lambda_2}{2} e^{-\lambda_2 t} - \frac{1}{c^2} \left( \frac{\lambda_2}{2} e^{-\lambda_2 t} \right) & \geq -b(1 - \frac{1}{2} e^{-\lambda_2 t}) (k - \delta k e^{-\lambda_3 (t - \tau_1)}) & \text{for } t > \tau_1.
\end{align*}
\] (5.12)

In order to induce (5.12), the following is sufficient
\[ \lambda_2 - \frac{\lambda_2^2}{c^2} \geq 0. \] (5.13)

Hence we set
\[ \lambda_2 = \varepsilon_1, \text{ where } \varepsilon_1 \ll 1. \] (5.14)

Substituting (5.4)-(5.8) into the third equation of (5.3) yields
\[
\begin{align*}
\lambda_3 \delta k e^{\lambda_3 t} - \frac{1}{c^2} \lambda_3^2 \delta k e^{\lambda_3 t} & \leq \delta k e^{\lambda_3 t} (1 - \delta k e^{\lambda_3 t} - r \frac{1}{2} e^{\lambda_2 (t - \tau_2)}) & \text{for } t < 0, \\
\lambda_3 \delta k e^{-\lambda_3 t} + \frac{1}{c^2} \lambda_3^2 \delta k e^{-\lambda_3 t} & \leq (k - \delta k e^{-\lambda_3 t}) (1 - \delta k e^{-\lambda_3 t} - r \frac{1}{2} e^{\lambda_2 (t - \tau_2)}) & \text{for } 0 \leq t \leq \tau_2, \\
\delta k e^{-\lambda_3 t} \left( \lambda_3 + \frac{\lambda_3^2}{c^2} \right) & \leq (k - \delta k e^{-\lambda_3 t}) (1 - \delta k e^{-\lambda_3 t} - r (1 - \frac{1}{2} e^{-\lambda_2 t})) & \text{for } t > \tau_2.
\end{align*}
\] (5.15)

If we set \( \delta \ll 1, \lambda_2 \ll 1, \) in order to induce (5.15), the following is sufficient
\[
\begin{align*}
\lambda_3 - \frac{\lambda_3^2}{c^2} - 1 & < 0, \\
\delta k e^{-\lambda_3 t} \left( \lambda_3 + \frac{\lambda_3^2}{c^2} + 1 - k \right) & < k(1 - k), \\
\delta k e^{-\lambda_3 t} \left( \lambda_3 + \frac{\lambda_3^2}{c^2} + 1 - r - k \right) & < k(1 - r - k).
\end{align*}
\] (5.16)
To ensure (5.16), the parameter must satisfy

\[ r < 1 \]  

and we set

\[ \lambda_3 = \frac{c^2(1 - \sqrt{1 - \frac{4}{c^2}})}{2} - \varepsilon, \text{ where } \varepsilon_2 << 1, k << 1. \]  

(5.18)

It is easy to see that the fourth equation of (5.3) naturally holds. Therefore, we have proved

**Lemma 5.1.** Assume that the parameter of the problem (5.1) \( r < 1 \). Then there exists a constant \( c^* = 2 \) such that if \( c > c^* \), \( \tilde{\varphi} \) and \( \hat{\varphi} \), which are defined in (5.4) and (5.4'), are a pair of coupled quasi-upper and quasi-lower solutions of the problem (5.3).

Finally, by Theorem 4.3 we have

**Theorem 5.1.** Assume that the parameter of the problem (5.1) \( r < 1 \). Then there exists a constant \( c^* = 2 \) such that if \( c > c^* \), the problem (5.1) has a traveling wave solution

\[ u(x, t) = \varphi_1(t + x/c), \quad v(x, t) = \varphi_2(t + x/c), \]  

which connects \((0, 0)\) and \((1, 1)\).

**Remark 5.1.** In [2], the critical value of wave velocity \( c^* \) is dependent on the parameter \( b \). In our Theorem 5.1, we show that \( c^* \) may be a constant.

### 5.2 Mutualistic Lotka-Volterra model

The delayed mutualistic Lotka-Volterra model is as follows

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} - d_1 \frac{\partial^2 u(x, t)}{\partial x^2} &= ru(x, t)(1 - a_1 u(x, t) + b_1 v(x, t)), \\
\frac{\partial v(x, t)}{\partial t} - d_2 \frac{\partial^2 v(x, t)}{\partial x^2} &= v(x, t)(a_2 u(x, t - \tau) - b_2),
\end{align*}
\]  

(5.19)

where \( u(x, t), v(x, t) \) are scalar functions, and \( r, a_1, a_2, b_1, b_2 \) are all positive constants, \( d_1, d_2 \) are the positive diffusion coefficients. \( \tau \) represents the positive delay. For a detailed description of this model, we refer to [27]. The above model with \( \tau = 0, b_1 < 0 \) has been considered in [4, 5, 27]. The case with \( \tau = 0, b_1 < 0 \) was studied in [2, 23]. However, when \( b_1 < 0 \) the reaction term of (5.19) is not quasimonotone nondecreasing, there was not sufficient condition to prove Lemma 3.5 of [23] and Lemma 12 of [2].
If we assume that
\[ a_2 > a_1 b_2, \quad (5.20) \]
then the model \((5.19)\) has a unique positive equilibrium
\[ (u^*, v^*) = \left( \frac{b_2}{a_2}, \frac{1}{b_1} \left( \frac{a_1 b_2}{a_2} - 1 \right) \right). \quad (5.21) \]

The wave equations corresponding to \((2.2)\) is the following form
\[ \begin{array}{l}
\varphi_1'(t) - \frac{d_1}{a_2} \varphi_1''(t) = r_1 \varphi_1(t) \left( 1 - a_1 \varphi_1(t) + b_1 \varphi_2(t) \right), \\
\varphi_2'(t) - \frac{d_2}{a_2} \varphi_2''(t) = \varphi_2(t) \left( a_2 \varphi_1(t - \tau) - b_2 \right).
\end{array} \quad (5.22) \]

We will find the solution of the above wave equation such that
\[ \lim_{t \to -\infty} \varphi_1(t) = 0, \quad \lim_{t \to -\infty} \varphi_2(t) = 0, \]
\[ \lim_{t \to +\infty} \varphi_1(t) = u^*, \quad \lim_{t \to +\infty} \varphi_2(t) = v^*. \]

It is also easy to verify that \((H_1)\) and \((H_2)\) are satisfied. We only need seek the coupled quasi-upper and quasi-lower solutions of \((5.22)\).

If \( \tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2) \) and \( \hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2) \) are coupled quasi-upper and quasi-lower solutions of \((5.22)\), they must satisfy
\[ \begin{array}{l}
\tilde{\varphi}_1'(t) - \frac{d_1}{a_2} \tilde{\varphi}_1''(t) \geq r_1 \tilde{\varphi}_1(t) \left( 1 - a_1 \tilde{\varphi}_1(t) + b_1 \tilde{\varphi}_2(t) \right), \text{for all } t \in \mathbb{R} \setminus \{0\}, \\
\tilde{\varphi}_2'(t) - \frac{d_2}{a_2} \tilde{\varphi}_2''(t) \geq \tilde{\varphi}_2(t) \left( a_2 \tilde{\varphi}_1(t - \tau) - b_2 \right), \text{for all } t \in \mathbb{R} \setminus \{0\}, \\
\hat{\varphi}_1'(t) - \frac{d_1}{a_2} \hat{\varphi}_1''(t) \leq r_1 \hat{\varphi}_1(t) \left( 1 - a_1 \hat{\varphi}_1(t) - b_1 \hat{\varphi}_2(t) \right), \text{for all } t \in \mathbb{R} \setminus \{0\}, \\
\hat{\varphi}_2'(t) - \frac{d_2}{a_2} \hat{\varphi}_2''(t) \leq \hat{\varphi}_2(t) \left( a_2 \hat{\varphi}_1(t - \tau) - b_2 \right), \text{for all } t \in \mathbb{R} \setminus \{0\}. \quad (5.23)
\end{array} \]

Assume that \( \tilde{\varphi} \) and \( \hat{\varphi} \)
\[ \tilde{\varphi}_1(t) = \begin{cases} 
\frac{u^* e^{\lambda_1 t}}{2}, & t \leq 0, \\
u^* - \frac{u^*}{2} e^{-\lambda_1 t}, & t > 0,
\end{cases} \quad \text{and} \quad \hat{\varphi}_2(t) = \begin{cases} 
\frac{v^* e^{\lambda_2 t}}{2}, & t \leq 0, \\
v^* - \frac{v^*}{2} e^{-\lambda_2 t}, & t > 0.
\end{cases} \quad (5.24) \]

\[ \tilde{\varphi}_1(t) = \begin{cases} 
\delta k e^{\lambda_1 t}, & t \leq 0, \\
k - \delta k e^{-\lambda_1 t}, & t > 0,
\end{cases} \quad \text{and} \quad \hat{\varphi}_2(t) = 0, \quad (5.25) \]
where \( \lambda_1, \lambda_2, \lambda_3, \delta, k \) are undetermined positive constants.

As similar as the process in the above subsection, in order to \((5.23)\), the following is sufficient.
\[
\lambda_1 - \frac{d_1 \lambda_1^2}{c^2} - r \geq 0,
\]
\[
\lambda_1 + \frac{d_1 \lambda_1^2}{c^2} - b_1 v^* \geq 0,
\]
\[
\lambda_2 (1 - \frac{d_2 \lambda_2}{c^2}) \geq 0,
\]
\[
\lambda_3 - \frac{\lambda_3^2}{c^2} \leq r,
\]
\[
k \ll 1, \delta \ll 1.
\]

If
\[
c > \max\{2 \sqrt{rd_1}, \sqrt{d_1 (r + \frac{a_1 b_2}{a_2} - 1)}\},
\]
set
\[
\lambda_1 = \frac{c^2 (1 + \sqrt{1 - \frac{4rd - 1}{c^2}})}{2d_1}, \lambda_2 \ll 1, \lambda_3 \ll 1, k \ll 1, \delta \ll 1,
\]
then the sufficient conditions (5.26) hold. Therefore by Theorem 4.3, we have

**Theorem 5.2.** Assume that the parameter of the problem (5.19) satisfy (5.20). Then there exists a constant \(c^* = \max\{2 \sqrt{rd_1}, \sqrt{d_1 (r + \frac{a_1 b_2}{a_2} - 1)}\}\) such that if \(c > c^*\), the problem (5.1) has a traveling wave solution \(u(x, t) = \varphi_1(t + x/c)\), \(v(x, t) = \varphi_2(t + x/c)\), which connects \((0, 0)\) and \((\frac{b_2}{a_2}, \frac{1}{b_1} (\frac{a_1 b_2}{a_2} - 1))\).

## 6 Discussion

We aim to study the existence for the traveling wave solution of the discrete-delayed reaction diffusion systems, where the reaction term is mixed quasi-monotone. Our result is that the existence of the coupled quasi-upper and quasi-lower solutions ensure the traveling wave solution exists. In the equations of [2, 23, 36], the reaction term is quasimonotone nondecreasing. In fact the conditions of quasimonotone nondecreasing property is very strong. The predator-prey model which is studied in [2, 23] do not satisfy the quasimonotone nondecreasing property, and the existence theorem of [2, 23] is not suitable. Hence the application scope of Theorem 3.1 is wider than [2, 23, 36]. Moreover,
as a special case of Theorem 3.1, Theorem 4.1 contains the previous existence theorem for the traveling wave solution.

Our technique to deal with the mixed quasimonotony is constructing the coupled upper and lower solutions. Recently the method so-called cross iteration scheme was developed in [19] to deal with the traveling wave solution for the 2 dimensional competitive Lotka-Volterra model. Comparing with the model of [19], the systems in this paper are extensive to n dimension.

The classical coupled upper and lower solutions need the second order smoothness. It is very difficult to satisfy this condition in the real model. Thus it is necessary to relax the smoothness of the coupled upper and lower solutions to first order smoothness, which is called coupled quasi-upper and quasi-lower solutions. This paper apply the modified Perron theorem with the case $C^1$ smoothness, which is first proposed in [2]. Our existence theorem of the traveling wave solution is suitable to all 2 species or 3 species Lotka-Volterra systems.

References

[1] S. Ahmad, A. S. Vatsala, Comparison, results of reaction-diffusion equations with delay in abstract cones, *Rend. Sem. Mat. Univ. Padova* 65 (1981), 19-34.

[2] A. Boumenir, V. M. Nguyen, Perron theorem in the monotone iteration method for traveling waves in delayed reaction-diffusion equations, *J. Differential Equations* 244 (2008), 1551-1570.

[3] C. Conley, R. Gardner, An application of the generalized Morse index to traveling wave solutions of a competitive reaction-diffusion model, *Indiana Univ. Math. J.* 44 (1984), 319-343.

[4] S. R. Dunbar, Traveling wave solutions of diffusive Lotka-Volterra equations, *J. Math. Biol.* 17 (1983), 11-32.

[5] S. R. Dunbar, Traveling wave solutions of diffusive Lotka-Volterra equations: A hetero-clinic connection in $\mathbb{R}^4$, *Trans. Amer. Math. Soc.* 268 (1984), 557-594.

[6] R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* 7 (1937), 355-369.

[7] R. Gardner, Existence of traveling wave solutions of predator-prey systems via the connection index, *SIAM J. Appl. Math.* 44 (1984), 56-79.

[8] R. Gardner, Existence and stability of traveling wave solutions of competition models: A degree theoretic approach, *J. Differential Equations* 44 (1982), 343-364.
[9] J. Huang, G. Lu, S. G. Ruan, Existence of traveling wavefronts of delayed reaction-diffusion systems without monotonicity, *J. Math. Biol.* **46** (2003), 132-152.

[10] J. Huang, Z. F. Zou, Existence of traveling wave solutions in a diffusive predator-prey model *Discrete Contin. Dyn. Syst.* **9** (2003), 925-936.

[11] Ya. I. Kanel, Existence of a traveling-wave solution of the Belousov-Zhabotinskii system, *Differentsial’nye Uravneniya* **26** (1990), 652-660.

[12] A. Ya. Kapel, Existence of traveling-wave type solutions for the Belousov-Zhabotinskii system equations, *Sibirsk. Mat. Zh.* **32** (1991), 47-59.

[13] W. Kerscher, R. Nagel, Asymptotic behavior of one-parameter semigroups of positive operators, *Acta Appl. Math.* **2** (1984), 297-309.

[14] A. N. Kolomgorov, I. G. Petrovskii, N. S. Piskunov, Study of a diffusion equation that is related to the growth of a quality of matter, and its application to a biological problem, *Byul. Mosk. Gos. Univ. Ser. A Mat. Mekh.* **1** (1937), 1-26.

[15] K. Kunish, W. Schappacher, Order preserving evolution operators of functional differential equations, *Boll. Unione. Mate. Ital.* **16B** (1979), 480-500.

[16] G. ladas, V. Lakshmikantham, Differential equations in abstract spaces, Academic Press, New York, 1972.

[17] V. Lakshmikantham, S. Leela, Nonlinear differential equations in abstract spaces, Pergamon Press, Oxford, 1981.

[18] S. Leela, V. Moauro, Existence of solutions of delay differential equations on closed subsets of a Banach space, *Nonl. Anal. TMA* **2** (1978), 47-58.

[19] W. T. Li, G. Lin, S. G. Ruan, Existence of travelling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition system, *Nonlinearity* **19** (2006), 1253-1273.

[20] W. T. Li, S. G. Ruan, Z. C. Wang, On the diffusive Nicholson’s blowflies equation with nonlocal delays, *J Nonlinear Sci.* **17** (2007), 505-525.

[21] W. T. Li, Z. C. Wang, Traveling fronts in diffusive and cooperative Lotka-volterra system with ninlocal delays, *Z. Angew. Math. Phys.* **58** (2007), 571-591.

[22] Z. G. Lin, M. Pedersen, C. R. Tian, Traveling wave solutions for reaction-diffusion system, *Nonlinear Anal.*, 10.1016/j.na.2010.07.010

[23] S. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differential Equations* **171** (2001), 294-314.
[24] S. Ma, X. F. Zou, Existence, uniqueness and stability of traveling waves in a discrete reaction-diffusion monostable equation with delay, *J. Differential Equations* **217** (2005), 54-87.

[25] R. H. Martin, H. L. Smith, Abstract functional differential equations and reaction-diffusion systems, *Trans. Am. Math. Soc.* **321** (1990), 1-44.

[26] R. H. Martin, H. L. Smith, Reaction-diffusion systems with time delay: Monotonicity, invariance, comparison and convergence, *J. Reine. Angew. Math.* **414** (1991), 1-35.

[27] J.D. Murray, Mathematical Biology, 2nd corrected ed., Springer-Verlag, New York, 1993.

[28] C. V. Pao, Nonlinear parabolic and elliptic equations, Plenum Press, New York, 1992.

[29] C. V. Pao, Coupled nonlinear parabolic systems with time delays, *J. Math. Anal. Appl.* **196** (1995), 237-265.

[30] C. V. Pao, Dynamics of nonlinear parabolic systems with time delays, *J. Math. Anal. Appl.* **198** (1996), 751-779.

[31] C. V. Pao, Quasisolutions and global attractor of reaction-diffusion systems, *Nonlinear Anal.* **26** (1996), 1889-1903.

[32] M. M. Tang, P. Fife, Propagating fronts for competing species equations with diffusion, *Arch. Rational Mech. Anal.* **73** (1980), 69-77.

[33] W. C. Troy, The existence of traveling wavefront solutions of a model of the Belousov Zhabotinskii reaction, *J. Differential Equations* **36** (1980), 89-98.

[34] A. I. Volpert, V. A. Volpert, V. A. Volpert, Traveling wave solutions of parabolic systems, Translations of Mathematical Monographs, Vol. 140, Amer. Math. Soc., Providence 1994.

[35] J. H. van Vuuren, The existence of traveling plane waves in a general class of competition-diffusion systems, *IMA J. Appl. Math.* **55** (1995), 135-148.

[36] J. Wu, X. Zou, Traveling wave fronts of reaction-diffusion systems with delay, *J. Dynam. Differential Equations* **13** (2001), 651-687.

[37] X. Zou and J. Wu, Existence of raveling wavefronts in delayed reaction-diffusion system via monotone iteration method, *Proc. Amer. Math. Soc.* **125** (1997), 2589-2598.

[38] Q. Ye, M. Wang, Traveling wavefront solutions of Noyes-field system for Belousov-Zhabotinskii reaction, *Nonlinear Anal.* **11** (1987), 1289-1302.
[39] E. Zeidler, Nonlinear functional analysis and its applications: I, Fixed-Point Theorems, Springer, New York, 1986.