Cosmic String Solution in a Born-Infeld Type Theory of Gravity

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In this work we derive an exact solution for the exterior metric of a local cosmic string in an effective theory of gravitation, the so-called NDL theory, which is inspired in the Born-Infeld theory. The solution is given by a family of parameters which presents quite different features from that of the General Relativity theory. The differences come from the specific choice of the gravitation Lagrangian which is based on a spin-1 construction of the gravitation theory.

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I. INTRODUCTION

Advances in the formal structure of string theory point to the emergence, and necessity, of a scalar-tensorial theory of gravity. It seems that, at least at high energy scales, the Einstein’s theory is not enough to explain the gravitational phenomena [1]. In other words, the existence of a scalar (gravitational) field acting as a mediator of the gravitational interaction together with the usual purely rank-2 tensorial field is, indeed, a natural prediction of unification models as supergravity, superstrings and M-theory [2]. This type of modified gravitation was first introduced in a different context in the 60’s in order to incorporate the Mach’s principle into relativity [3], but nowadays it acquired different sense in cosmology and gravity theories.

Although such unification theories are the most acceptable, they all exist in higher dimensional spaces. The compactification from these higher dimensions to the 4-dimensional physics is not unique and there exist many effective theories of gravity which come from the unification process. Each of them must, of course, satisfy some predictions. Here, in this paper, we will deal with one of them. The so-called NDL theory [5].

One important assumption in General Relativity is that all fields interact in an universal way with gravity. This is the so called Strong Equivalence Principle (SEP). It is well known, with good accuracy, that this is true when it concerns to matter-gravity interaction, i.e, the Weak Equivalence Principle(WEP). But, until now, there is no direct observational confirmation of this assumption to what concerns the gravity-gravity interaction. In [5], an extension of the field theoretical approach of General Relativity built by [6, 7] proposes an alternative field theory of gravity. In this theory, gravitons propagate in a different spacetime of the matter fields. The velocity of propagation of the gravitational waves does not coincide with the General Relativity predictions because of the violation of the SEP and the self-interaction graviton-graviton predicts a massive graviton in this theory.

In this paper, our main purpose is to investigate the properties of a straight cosmic string in the NDL theory. To do that, this manuscript is organized as follows. In the section 2 we give a brief presentation of the NDL theory based on the original paper [5]. In the section 3, we write down the detailed calculations to find the exterior metric of a local cosmic string and finally in the Section 4 we summarize and discuss our results.

II. THE NDL THEORY

In this section we summarize the NDL theory following [4, 5]. The more detailed presentation is presented in the ref. [5]. To start with, the main lines of the NDL theory are:

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The gravitational interaction is represented by a symmetric tensor $\varphi_{\mu\nu}$ that obeys a nonlinear equation of motion.

The matter (but not gravity) couples to gravity through the metric $g_{\mu\nu} = \gamma_{\mu\nu} + \varphi_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the flat background metric.

The self interaction of the gravitational field breaks the universal modification of the space time geometry, i.e., the gravity couples to gravity in a special way distinct from all different forms of energy.

We begin defining the tensor $F_{\alpha\beta\mu}$, which is anti-symmetric in the two first indices, called the gravitational field:

$$F_{\alpha\beta\mu} := \frac{1}{2} (\varphi_{\mu[\alpha;\beta]} + F_{[\alpha} \gamma_{\beta]\mu}) ,$$

where $[x, y] = xy - yx$ and the covariant derivative is constructed with the background metric. Indices are raised and lowered with that metric also, and

$$F_{\alpha} = F_{\alpha\mu\nu} \gamma_{\mu\nu} = \varphi_{,\alpha} - \varphi_{\alpha\mu;\nu} \gamma_{\mu\nu}$$

In order to have a nonlinear theory of the gravitational field $F_{\alpha\beta\mu}$ with the correct weak field limit, we assume that the interaction of gravity with itself is described by a functional of $A$ and $B$ which are invariants built from the gravitational field $F_{\alpha\beta\mu}$:

$$A = F_{\alpha\beta\mu} F^{\alpha\beta\mu} \quad \text{and} \quad B = F_{\alpha} F^{\alpha}.$$  

In our case we will use the Born-Infeld Lagrangian:

$$\mathcal{L} = \frac{b^2}{k} \sqrt{1 - \frac{A - B}{b^2}} - 1,$$

where $k$ is the Einstein’s constant. Thus, the gravitational action will be:

$$S = \int d^4x \sqrt{\gamma} \mathcal{L},$$

where $\gamma$ is the determinant of the Minkowski metric in an arbitrary coordinate system. Taking the variation of the action (3) with respect to the potential $\varphi_{\mu\nu}$, we obtain the following equations of motion:

$$\left( \mathcal{L}_U F^{\lambda(\mu\nu)} \right)_{;\lambda} = -\frac{1}{2} T^{\mu\nu}$$

where $(x, y) = xy + yx, U = A - B$ and $\mathcal{L}_U = \frac{d\mathcal{L}}{dU}$.

III. COSMIC STRING SOLUTION IN NDL THEORY

We consider here the exterior region of a local cosmic string in the NDL theory. In this case our energy momentum tensor will be identically zero in the region outside the string and our source has a cylindrical symmetry. Thus, we begin with the following ansätze:

- The background metric, i.e. $\gamma_{\mu\nu}$, will be Minkowski in cylindrical coordinates, following Vilenkin in [8]:

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - dz^2.$$  

- The potential $\varphi_{\mu\nu}$ will be:

$$\varphi_{11} = \alpha(r), \quad \varphi_{33} = -c(r) \quad \text{and} \quad \varphi_{44} = -\beta(r).$$  

(5)
In a way that our cylindrical metric for the straight cosmic string in the NDL theory is:

\[ ds^2 = (1 + \alpha(r))dt^2 - dr^2 - (r^2 + c(r))d\theta^2 - (1 + \beta(r))dz^2. \]  

(6)

In order to obtain the equations of motion, first we need to compute some elements. The trace and the covariant derivatives of the potential are, respectively:

\[ \varphi = \varphi_{11}\gamma^{11} + \varphi_{33}\gamma^{33} + \varphi_{44}\gamma^{44} = \alpha + \beta + \frac{c}{r^2}, \]
\[ \varphi_{11:2} = \varphi_{11,2} = \alpha'(r), \]
\[ \varphi_{33:2} = \varphi_{33,2} - 2\Gamma^3_{23} = -c' + 2\frac{c}{r}, \]
\[ \varphi_{44:2} = \varphi_{44,2} = -\beta', \]
\[ \varphi_{32:3} = \Gamma^3_{32} \varphi_{33} = \frac{c}{r}. \]

The only non-vanishing component of \( F_{\alpha} \) is \( F_2 \):

\[ F_2 = \varphi_{,2} - \varphi_{23;3}\gamma^{33} = \alpha' + \beta' + \frac{c'}{r^2} - 2\frac{c}{r^3} - \frac{c}{r} \left( -\frac{1}{r^2} \right) = \alpha' + \beta' - \frac{c'}{r^3}. \]

With these, we can start calculating the \( F^{\mu\nu\lambda} \) components:

\[ F^{211} = -F_{211} = -\frac{1}{2} \left( \beta' + \frac{c'}{r^2} - \frac{c}{r^3} \right) \]
\[ F^{233} = -F_{233} = \frac{1}{2} \left( \alpha' + \frac{\beta'}{r^2} \right) \]
\[ F^{244} = -F_{244} = \frac{1}{2} \left( \alpha' + \frac{c'}{r^2} - \frac{c}{r^3} \right). \]

To write explicitly the Lagrangian we must find the invariants \( A \) and \( B \) for our particular metric:

\[ A = -\beta'^2 - \alpha'^2 - \frac{c'^2}{r^4} - \frac{c^2}{r^6} - \alpha' \beta' - \frac{\alpha' c'}{r^2} + \frac{\alpha' c'}{r^3} + \beta' \frac{c}{r^3} + 2\frac{c c}{r^5}, \]
\[ B = -\left( \alpha'^2 + 2\alpha' \beta' + \beta'^2 + \frac{c'^2}{r^4} - 2\frac{c c}{r^5} + \frac{c^2}{r^6} + 2\frac{\alpha' c'}{r^2} + 2\frac{\beta' c}{r^3} - 2\frac{\beta' c}{r^2} - 2\frac{2\beta' c}{r^3} \right), \]
\[ U = A - B = \alpha' \beta' + \frac{\alpha' c'}{r^2} + \frac{\beta' c}{r^3} - \frac{\beta' c}{r^3} - \frac{\alpha' c}{r^3}. \]

And, finally, the covariant derivatives of the gravitational tensor \( F^{\lambda\mu\nu} \):

\[ F^{211}_{2} = \frac{1}{2} \left( \beta'' + \frac{c''}{r^2} - 3\frac{c'}{r^3} + 3\frac{c}{r^4} \right), \]
\[ F^{244}_{2} = \frac{1}{2} \left( \alpha'' + \frac{c''}{r^2} - 3\frac{c'}{r^3} + 3\frac{c}{r^4} \right), \]
\[ F^{344}_{3} = \frac{1}{2r} \left( \alpha' + \frac{c'}{r^2} - \frac{c}{r^3} \right), \]
\[ F^{311}_{3} = \frac{1}{2} \left( \frac{c}{r^4} - \frac{c'}{r^3} - \frac{\beta'}{r} \right), \]
\[ F^{322}_{3} = \frac{1}{2r} \left( \alpha' + \beta' \right), \]
\[ F^{233}_{2} = \frac{1}{2} \left( \frac{\alpha''}{r^2} + \frac{\beta''}{r^2} \right). \]
The non vanishing equations of motion are:

\[ L_U F^{211} + L_U F^{212} + L_U F^{311}_{:3} = 0, \]  

\[ L_U F^{322}_{:3} = 0, \]  

\[ L_U F^{233} + L_U F^{233}_{:2} = 0, \]  

\[ L_U F^{244} + L_U F^{244}_{:2} + L_U F^{344}_{:3} = 0. \]  

Knowing that,

\[ L_U = \frac{-1}{2} \frac{1}{k^2 \sqrt{1 - \frac{v}{c}^2}}, \]

\[ L_U' = L_U \frac{U'}{2b^2} \frac{1}{1 - \frac{v}{c}}, \]

where prime means \( \frac{\partial}{\partial r} \), these equations can be written by:

\[ L_U \frac{U'}{2b^2} \left( 1 - \frac{U}{b^2} \right)^{-1} \left( -\frac{1}{2} \right) \left( \beta' + \frac{c'}{r^2} - \frac{c}{r^3} \right) + L_U \left( -\frac{1}{2} \right) \left( \beta'' + \frac{c''}{r^2} - \frac{2c'}{r^3} + \frac{2c}{r^4} + \frac{\beta'}{r} \right) = 0, \]  

\[ L_U \frac{U'}{2b^2} \left( 1 - \frac{U}{b^2} \right)^{-1} \left( \frac{1}{2} \right) \left( \frac{\alpha' + \beta'}{r^2} \right) + L_U \left( \frac{1}{2} \right) \left( \frac{\alpha'' + \beta''}{r^2} \right) = 0, \]  

\[ L_U \frac{U'}{2b^2} \left( 1 - \frac{U}{b^2} \right)^{-1} \left( \frac{1}{2} \right) \left( \alpha' + \frac{c'}{r^2} - \frac{c}{r^3} \right) + L_U \frac{1}{2} \left( \frac{\alpha'' + \beta''}{r^2} - \frac{2c'}{r^3} + \frac{2c}{r^4} + \frac{\alpha'}{r} \right) = 0. \]

From (12) we have \( \alpha' = -\beta' \) and equation (13) is identically satisfied.

If we add equation (11) to the equation (14) and use \( \beta' = -\alpha' \) we have:

\[ L_U' \alpha' + L_U \left( \alpha'' + \frac{\alpha'}{r} \right) = 0. \]  

Since

\[ L_U = L_U \frac{U'}{2b^2} \frac{1}{1 - \frac{v}{c}} = -L_U \frac{\alpha' \alpha''}{b^2} \frac{1}{1 + \frac{\alpha'}{b^2 r}} \]

we finally have:

\[ \frac{\alpha'}{r} + \alpha'' + \frac{\alpha'^3}{b^2 r} = 0. \]  

The solution of this equation is \( \alpha(r) = \pm b \sqrt{k_1} \ln \frac{k_1 r + \sqrt{\left(-r^2 + k_1 k_1 \sqrt{k_2}\right) + k_2}}{\sqrt{k_1}} \), where \( k_1 \) is a positive constant and \( k_2 \) is an arbitrary real constant. We have to find the equation for \( c(r) \). Making now the difference (14)-(11) we have:

\[-\frac{\alpha' \alpha''}{b^2} \left( 1 + \left( \frac{\alpha'}{b^2} \right)^2 \right)^{-1} \left( \frac{c'}{r^2} - \frac{c}{r^3} \right) + \left( \frac{c''}{r^2} - \frac{2c'}{r^3} + \frac{2c}{r^4} \right) = 0. \]

Using the fact that \( \alpha'' = -\left( \frac{2c'}{r^2} + \frac{2c}{r^4} \right) \) we have:

\[ c' r^4 - 2c' r^3 + 2c r^2 - k_1 c'' r^2 + 3k_1 c' r - 3k_1 c = 0, \]

which give the solution:

\[ c(r) = k_3 r + k_4 r \sqrt{r^2 - k_1}. \]
We know that $\beta' = -\alpha'$. Thus:

$$\beta(r) = -\alpha(r) + k_5,$$

where $k_3$, $k_4$ and $k_5$ are constants to be determined. $k_5$ can be absorbed into a new coordinate $z'$ in the metric (6) by a straightforward reparametrization of the coordinate $z$. Then,

$$\beta(r) = -\alpha(r),$$

which indicates that our local string is invariant Lorentz boost. However, the other integration constants can be determined only after relating them with the internal structure of the cosmic string and making a proper match between the internal and the exterior solutions of the metric.

### IV. SUMMARY AND ANALYSIS OF THE COSMIC STRING SOLUTION

The main goal of this paper was to consider a straight and neutral cosmic string in the NDL theory. This result has been presented in the section III. Interesting enough, although the nature of this string is neutral (e.g., without internal electric or magnetic current), the solution found for the function $\alpha(r)$ (and, consequently, $\beta(r)$) here resembles pretty much to the superconducting string either in the General Relativity theory\[9\] as well as in the Scalar-Tensor theory\[10\]. In ref. [11], one of us investigated the internal nature of ordinary cosmic vortices in some scalar-tensor extensions of gravity. Solutions were found for which the dilaton field condenses inside the vortex core. These solutions could be interpreted as raising the degeneracy between the eigenvalues of the effective stress-energy tensor, namely the energy per unit length $U$ and the tension $T$, by picking a privileged spacelike or timelike coordinate direction; in the latter case, a *phase frequency threshold* occurred that is similar to what is found in ordinary neutral current-carrying cosmic strings. It has been found that the dilaton contribution for the equation of state, once averaged along the string worldsheet, vanishes, leading to an effective Nambu-Goto behavior of such a string network in cosmology, *i.e.* on very large scales. It has been shown also that, on small scales, the energy per unit length and tension depend on the string internal coordinates in such a way as to permit the existence of centrifugally supported equilibrium configuration, also known as *vortons*, whose stability, depending on the very short distance (unknown) physics, can lead to catastrophic consequences on the evolution of the Universe.

Here, the reason for this "apparent" superconducting properties of the cosmic string is the non-linear nature of the gravitation Lagrangian which has been built through a spin-1 field, just like in the electromagnetic theory. Therefore, it is the nature of the background gravity which entitles the string with a "current" flow property.

The other remarkable feature of this local string is that the "deficit angle" given by the function $c(r)$ is not constant, but depends on the position $r$. Since we are violating the SEP, this means that observers located at different distances from the string "see" the string with different values of its deficit angle. This is the main difference from the NDL cosmic string and the GR or the scalar-tensor cosmic string solutions.

Finally, we must stress that, looking at the expressions for the solutions of $\alpha(r)$ and $c(r)$, we conclude that $r$ must satisfy $r > k_1$, otherwise we obtain imaginary values for the position $r$. We conjecture that it might be possible to separate the exterior metric into two regions, the $r < k_1$ and the $r > k_1$ in an analogous way as in the Rindler space. We plan to return to this point in a forthcoming paper.

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