Symmetric Numerical Integration Techniques for Singular Integrals in the Method-of-Moments Implementation of the Electric-Field Integral Equation

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Abstract

In this paper, we present two approaches for designing geometrically symmetric quadrature rules to address the logarithmic singularities arising in the method of moments from the Green’s function in integrals over the test domain. These rules exhibit better convergence properties than quadrature rules for polynomials and, in general, lead to better accuracy with a lower number of quadrature points. We demonstrate their effectiveness for several examples encountered in both the scalar and vector potentials of the electric-field integral equation (singular, near-singular, and far interactions) as compared to the commonly employed polynomial scheme and the double Ma–Rokhlin–Wandzura (DMRW) rules, whose sample points are located asymmetrically within triangles.

Keywords: method of moments, singular integrals, geometrically symmetric quadrature rules

1. Introduction

The method of moments (MoM) is a useful technique in computational electromagnetics for solving the electric-field integral equation (EFIE), the magnetic-field integral equation (MFIE), and the combined-field integral equation (CFIE), upon discretizing surfaces using planar or curvilinear mesh elements. Through this approach, four-dimensional integrals are evaluated, which integrate over source and test elements. However, the presence of a Green’s function in these equations yields scalar and vector potential terms with singularities (in their higher-order derivatives) when the test and source elements share one or more edges or vertices and near-singularities when they are otherwise close.

Many approaches have been developed to address the singularity and near-singularity for the inner, source-element integral. While, originally, singularity subtraction schemes were proposed [1, 2, 3], more recent approaches use singularity cancellation schemes [4, 5, 6, 7, 8, 9], through which the Jacobian from a variable transformation cancels the (near-)singularity, permitting the use of Gauss–Legendre quadrature rules. More recently, a hybrid scheme that combines these two methods has been proposed [10].

Approaches have also been developed to address the singularity in the outer, test-element integral. In [11], the authors use the outer product of one-dimensional rules from [12], which they map to the triangular test element through a Duffy transformation [13]. In [14], the authors use a series of variable transformations and integration reordering to integrate the four-dimensional integrals. In [15], the authors present an approach for coplanar source and test elements, extended in [16] to general element orientations. In [17], for the CFIE, the authors avoid the singularity in the test-element integral by modifying the integrand. In [18], for the MFIE, the authors use a singularity-extraction method for the test-element integral, which they use in [19] to implement the MFIE in a manner that eliminates some of the restrictions due to the singularity. In [20], the authors use double-exponential quadrature integration schemes. In [21], the authors expand the integrand in a (truncated) power series and analytically integrate term by term. This approach, however, cannot be applied to integrals that do not involve a homogeneous-medium Green’s function.

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In this paper, we derive geometrically symmetric quadrature rules better suited for evaluating the logarithmic singularities in the test integral, and we compare with a standard polynomial scheme [22] and with the asymmetric double Ma–Rokhlin–Wandzura (DMRW) rules [11, 12]. In particular, we present two approaches here: (a) Approach 1, suitable for a moderate number of points, with comparable efficiency to polynomial quadrature rules (leading to about 6 or 7 digits in accuracy), and (b) Approach 2, suitable for a large number of points but less efficient (leading to machine accuracy). Symmetric rules that can efficiently handle singularities are desirable because their mapping to the integration domain is straightforward and points are not heavily concentrated near some vertices. Asymmetric rules, on the other hand, generally employed to integrate singularities, require the determination of vertex mapping, and points may be concentrated nonuniformly at the vertices.

This paper is organized as follows. In Sec. 2, we characterize the singularities in the test-element integrand. In Sec. 3, we use the characterizations from Sec. 2 to construct appropriate geometrically symmetric quadrature rules. In Sec. 4, we demonstrate the effectiveness of these rules and compare them to a standard polynomial scheme and the DMRW rules. In Sec. 5, we provide concluding remarks.

2. Logarithmic Singularities in the MoM Test Integrand for the EFIE

Singularities will appear in the source potential when it becomes the integrand of a test integral. Under the \(e^{j\omega t}\) time-harmonic convention, the singular integrals for the EFIE that occur when using the MoM take the forms

\[
I_s = \int_{A_T} \nabla \cdot \mathbf{A}_T^i (x_T) \int_{A_S} \frac{e^{-jkR(x_S,x_T)}}{R(x_S,x_T)} \nabla \cdot \mathbf{A}_S^i (x_S) dA_S dA_T \tag{1}
\]

and

\[
I_v = \int_{A_T} \mathbf{A}_T^i (x_T) \cdot \int_{A_S} \frac{e^{-jkR(x_S,x_T)}}{R(x_S,x_T)} \mathbf{A}_S^i (x_S) dA_S dA_T, \tag{2}
\]

where \(I_s\) in (1) appears in the electric scalar potential and \(I_v\) in (2) appears in the electric vector potential. \(x_S\) and \(x_T\) are the source and test points, respectively; \(A_S\) and \(A_T\) are the source and test elements surfaces, respectively; \(R(x_S,x_T) = \|x_S - x_T\|_2\), \(\mathbf{A}_T^i\) is the test basis function associated with edge \(j\); and \(\mathbf{A}_S^i\) is the source basis function associated with edge \(i\). In (1) and (2), \(k = k_0\sqrt{\epsilon_r \mu_r}\), where \(k_0 = 2\pi/\lambda\) is the free-space wavenumber, \(\lambda\) is the wavelength, and \(\epsilon_r\) and \(\mu_r\) are the relative permittivity and permeability of the medium, respectively.

When \(\mathbf{A}_T^i\) and \(\mathbf{A}_S^i\) are linear, as in the Rao–Wilton–Glisson (RWG) basis functions [3], \(\nabla \cdot \mathbf{A}_T^i (x_T)\) and \(\nabla \cdot \mathbf{A}_S^i (x_S)\) are constants, such that (1) becomes

\[
I_s = C_1 \int_{A_T} \int_{A_S} \frac{e^{-jkR(x_S,x_T)}}{R(x_S,x_T)} dA_S dA_T. \tag{3}
\]

Upon performing a Taylor-series expansion of the exponential factor about \(R\), the test integrand in (3) can be expressed as

\[
f(x_T) = \sum_{p=0}^{\infty} \frac{(-jk)^p}{p!} \int_{A_S} R(x_S,x_T)^{p-1} dA_S. \tag{4}
\]

Even \(p\) terms in (4), which raise \(R\) to odd powers, yield terms with unbounded derivatives near the boundaries of \(A_S\). Odd \(p\) terms in (4) yield even powers of \(R\), which remain smooth and integrable.

In (2), when \(\mathbf{A}_T^i\) and \(\mathbf{A}_S^i\) are unnormalized RWG basis functions [3], \(\mathbf{A}_T^i (x_T) = x_T - x_j\) and \(\mathbf{A}_S^i (x_S) = x_S - x_i\), where \(x_j\) is the vertex of the test element opposite edge \(j\) and \(x_i\) is the vertex of the source element opposite edge \(i\), then

\[
\mathbf{A}_T^i \cdot \mathbf{A}_S^i = (x_T - x_j) \cdot (x_S - x_i) = \left( \hat{x} + \frac{x_T - x_S}{2} - x_j \right) \cdot \left( \hat{x} - \frac{x_T - x_S}{2} - x_i \right) = D_0 + D_1 R + D_2 R^2, \tag{5}
\]
where

\[ D_0(x_S, x_T) = \| \tilde{x} \|_2^2 - (x_i + x_j) \cdot \tilde{x} + x_i \cdot x_j, \]

\[ D_1(x_S, x_T) = \frac{\| x_j - x_i \|_2^2 \cos \phi}{2}, \]

\[ D_2 = -1/4, \]

\[ \tilde{x}(x_S, x_T) = (x_S + x_T)/2, \]

and \( \phi(x_S, x_T) \) is the angle between \( (x_T - x_S) \) and \( (x_j - x_i) \). Using (5), (2) becomes

\[ I_v = \int_{A_T} \int_{A_S} D_0(x_S, x_T) e^{-jkR(x_S, x_T)} dA_S dA_T + \int_{A_T} \int_{A_S} D_1(x_S, x_T) e^{-jkR(x_S, x_T)} dA_S dA_T \]

\[ + D_2 \int_{A_T} \int_{A_S} e^{-jkR(x_S, x_T)} R(x_S, x_T) dA_S dA_T. \]  

Performing a Taylor series expansion of the exponential factor in (9) leads to integer powers of \( R \). Once more, odd powers of \( R \) yield singularities, whereas even powers remain smooth and integrable.

We describe the singularities arising from the odd powers of \( R \) in

\[ \int_{A_S} R(x_S, x_T)^q dA_S, \quad \text{for} \ q = -1, 0, 1, \ldots \]  

for two cases: (1) when \( A_S \) and \( A_T \) are coplanar (\( \theta = 0^\circ \) or \( \theta = 180^\circ \) and \( \Delta z = 0 \) in Fig. 1) and (2) when \( A_S \) and \( A_T \) are perpendicular and share an edge (\( \theta = \pm 90^\circ \) and \( \Delta y = \Delta z = 0 \) in Fig. 1). Note that, although (10) is useful to understand and discuss singularities, integrations will not be carried out using (10); rather, the more general integrals in (1) and (2) will be used in later sections. Though the primary focus of this paper is on triangular elements, for simplicity, we use here rectangular domains to describe the singularities so that the equations are more tractable. However, we have also investigated these cases for the triangular elements and, though the expressions are more complicated, they retain the same singularities as the rectangular elements shown below.

![Figure 1: Relative positions of \( A_S \) and \( A_T \). The \( y' \)-axis is on the \( x = 0 \) plane, parallel to the \( y \)-axis, offset by \( \Delta z \).](image)

### 2.1. Coplanar Domains

In this subsection, we analyze the kind of singularities exhibited for coplanar domains. We demonstrate this behavior by letting \( A_S \) be the rectangle \( x_S \in [a, b] \times y_S \in [c, d] \) and \( A_T \) be coplanar with \( A_S \). For \( q = -1 \)
in (10), \( \int_{A_s} R(x_s, x_T)^{-1} dA_S \) is
\[
\int_{c}^{d} \int_{a}^{b} \frac{1}{\sqrt{(x_T - x_S)^2 + (y_T - y_S)^2}} dx_S dy_S = \frac{4}{\alpha} \left\{ \alpha_i \ln \left[ \beta_i + \sqrt{\alpha_i^2 + \beta_i^2} \right] - \alpha_i \ln \left[ \gamma_i + \sqrt{\alpha_i^2 + \gamma_i^2} \right] \right\},
\]
where
\[
\alpha = \{ y_T - c, y_T - d, x_T - a, x_T - b \},
\]
\[
\beta = \{ x_T - a, x_T - b, y_T - c, y_T - d \},
\]
\[
\gamma = \{ x_T - b, x_T - a, y_T - d, y_T - c \}.
\]

In (11), each pair of terms in the summation yields singularities in derivatives with respect to the test coordinates along \( \alpha_i = 0 \). The vertices have the following singularities: as \( \beta_i \to 0 \), \( \alpha_i \ln \left[ \beta_i + \sqrt{\alpha_i^2 + \beta_i^2} \right] \) becomes \( \alpha_i \ln |\alpha_i| \), and as \( \gamma_i \to 0 \), \( \alpha_i \ln \left[ \gamma_i + \sqrt{\alpha_i^2 + \gamma_i^2} \right] \) becomes \( \alpha_i \ln |\alpha_i| \). Additionally, the pair of terms can be written as \( \alpha_i \ln \left[ \frac{\beta_i + \sqrt{\alpha_i^2 + \beta_i^2}}{\gamma_i + \sqrt{\alpha_i^2 + \gamma_i^2}} \right] \). On an edge of the rectangle, \( \beta_i \) and \( \gamma_i \) have opposite signs and, as \( \alpha_i \to 0 \), the argument of the logarithm is dominated by \( -\frac{4\beta_i \gamma_i}{\alpha_i^2} \) or \( -\frac{\alpha_i^2}{4\beta_i \gamma_i} \), depending on whether \( \beta_i > \gamma_i \) or \( \beta_i < \gamma_i \). This yields a term containing \( \alpha_i \ln |\alpha_i| \), indicating the edge has a singularity as well. Along the edge of the rectangle, a series expansion of (11) as \( \alpha_i \to 0 \) yields the following terms:
\[
1, \alpha_i, \alpha_i \ln |\alpha_i|, \alpha_i^2, \alpha_i^3, \alpha_i^4, \alpha_i^5, \ldots
\]

For \( q = 1 \) in (10), \( \int_{A_s} R(x_s, x_T) dA_S \) yields additional terms, however, similar analysis yields similar observations. The terms
\[
\alpha_i^3 \ln \left[ \beta_i + \sqrt{\alpha_i^2 + \beta_i^2} \right], \quad \alpha_i^2 \ln \left[ \gamma_i + \sqrt{\alpha_i^2 + \gamma_i^2} \right]
\]
yield singularities of the form \( \alpha_i^q \ln |\alpha_i| \) along the edges and vertices and, along the edge, a series expansion of \( \int_{A_s} R(x_s, x_T) dA_S \) as \( \alpha_i \to 0 \) yields the following terms:
\[
1, \alpha_i, \alpha_i^2, \alpha_i^3, \alpha_i^4, \alpha_i^5, \ldots
\]

The trend continues for odd powers of \( R \), with one-dimensional characterizations \( \alpha_i^{q+2} \ln |\alpha_i| \) and 2D characterizations
\[
\alpha_i^{q+2} \ln \left[ \beta_i + \sqrt{\alpha_i^2 + \beta_i^2} \right], \quad \alpha_i^{q+2} \ln \left[ \gamma_i + \sqrt{\alpha_i^2 + \gamma_i^2} \right].
\]

For even powers of \( R \), the test integrand consists of a linear combination of monomials \( x_T^s y_T^t \), where \( 0 \leq s \leq q \), \( 0 \leq t \leq q \), and \( 0 \leq s + t \leq q \).

These derivations indicate that, when \( A_T = A_S \), the entire boundary of \( A_T \) has singularities. When \( A_T \) and \( A_S \) are co-planar and share an edge, the shared edge and its vertices have singularities. The numerical procedure here developed is applicable to both self and touching elements.

### 2.2. Perpendicular Domains

In this subsection, we analyze the kind of singularities exhibited in perpendicular domains. We demonstrate this behavior by letting \( A_S \) be the rectangle \( y_S \in [c, d] \times z_S \in [0, a] \) on the \( x = 0 \) plane and \( A_T \) be the rectangle \( x_T \in [0, b] \times y_T \in [c, d] \) on the \( z = 0 \) plane (note the shared edge is along the \( y \)-axis, where \( x_T = 0 \) and \( z_S = 0 \)). For \( q = -1 \) in (10), \( \int_{A_s} R(x_s, x_T)^{-1} dA_S \) is
\[
\int_{c}^{d} \int_{a}^{b} \frac{1}{\sqrt{x_T^2 + (y_T - y_S)^2 + z_S^2}} dy_S dz_S =
\]
derivations indicate that, when terms:

\[ x^T \arctan \frac{a(c - y_T)}{x_T \sqrt{x_T^2 + a^2 + (c - y_T)^2}} + x_T \arctan \frac{a(y_T - d)}{x_T \sqrt{x_T^2 + a^2 + (y_T - d)^2}} \]

\[ + \frac{c - y_T}{2} \ln \left[ x_T^2 + (c - y_T)^2 \right] + \frac{y_T - d}{2} \ln \left[ x_T^2 + (y_T - d)^2 \right] \]

\[- (c - y_T) \ln \left[ a + \sqrt{x_T^2 + a^2 + (c - y_T)^2} \right] - (y_T - d) \ln \left[ a + \sqrt{x_T^2 + a^2 + (y_T - d)^2} \right] \]

\[- a \ln \left[ c - y_T + \sqrt{x_T^2 + a^2 + (c - y_T)^2} \right] + a \ln \left[ d - y_T + \sqrt{x_T^2 + a^2 + (y_T - d)^2} \right]. \quad (19) \]

In (19), there are unbounded derivatives at the shared vertices \((0, c, 0)\) and \((0, d, 0)\) that respectively arise from \(-\frac{c - y_T}{2} \ln \left[ x_T^2 + (c - y_T)^2 \right] \) and \(-\frac{y_T - d}{2} \ln \left[ x_T^2 + (y_T - d)^2 \right] \). As \(x_T \to 0\), these become \(\alpha_i \ln |\alpha_i|\), where \(\alpha = \{c - y_T, y_T - d\}\). A series expansion of \(\int_{A_S} R(x_S, x_T)^{-1} dA_S\) from the interior of \(A_T\) to a shared vertex yields the following terms:

\[ 1, \alpha_i, \alpha_i \ln |\alpha_i|, \alpha_i^2, \alpha_i^3, \alpha_i^4, \alpha_i^5, \ldots. \quad (20) \]

As with the coplanar case in Sec. 2.1, for even powers of \(R\), the test integrand consists of a linear combination of monomials \(x_T^2 y_T\), where \(0 \leq s \leq q, 0 \leq t \leq q\), and \(0 \leq s + t \leq q\).

For \(q = 1\) in (10), \(\int_{A_S} R(x_S, x_T) dA_S\) yields additional terms, however, similar analysis yields similar observations. The terms

\[ \frac{(3x_T^2 + (c - y_T)^2)(c - y_T)}{12} \ln[x_T^2 + (c - y_T)^2], \quad (21) \]

\[ \frac{(3x_T^2 + (y_T - d)^2)(y_T - d)}{12} \ln[x_T^2 + (y_T - d)^2] \quad (22) \]

yield singularities of the form \(\alpha_i^3 \ln |\alpha_i|\) as \(x_T \to 0\). A series expansion of \(\int_{A_S} R(x_S, x_T) dA_S\) from the interior of \(A_T\) to a shared vertex yields the following terms:

\[ 1, \alpha_i, \alpha_i^2, \alpha_i^3, \alpha_i^4, \alpha_i^5, \ldots. \quad (23) \]

The trend continues for the odd powers of \(R\), with one-dimensional characterizations \(\alpha_i^{q+2} \ln |\alpha_i|\). These derivations indicate that, when \(A_S\) and \(A_T\) are perpendicular with a shared edge, the shared vertices have singularities.

3. Geometrically Symmetric Quadrature Rules for Logarithmic Singularities

Due to the edge and vertex singularities described in Sec. 2, quadrature rules for polynomials are not well suited for integrating the test integrand. Therefore, we construct two types of symmetric quadrature rules for triangles, using Approaches 1 and 2 of [23].

For Approach 1, which provides sufficient accuracy with the least number of quadrature points, we construct a two-dimensional function sequence that consists of polynomials and the two-dimensional characterizations from Sec. 2.1:

\[ 1, x, x \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}), x \ln(y + \sqrt{x^2 + y^2}), x^2, x y, x^3, x^2 y, x^3 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}), \]

\[ x^3 \ln(y + \sqrt{x^2 + y^2}), x^4, x^3 y, x^2 y^2, x^5, x^4 y, x^3 y^2, x^5 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}), x^5 \ln(y + \sqrt{x^2 + y^2}), \]

\[ x^6, x^5 y, x^4 y^2, x^3 y^3, x^7, x^6 y, x^5 y^2, x^4 y^3, x^7 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}), x^7 \ln(y + \sqrt{x^2 + y^2}), \]

\[ x^8, x^7 y, x^6 y^2, x^5 y^3, x^4 y^4, x^9, x^8 y, x^7 y^2, x^6 y^3, x^5 y^4, x^9 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}), x^9 \ln(y + \sqrt{x^2 + y^2}), \]

\[ x^{10}, x^9 y, x^8 y^2, x^7 y^3, x^6 y^4, x^5 y^5, \ldots. \quad (24) \]

The points and weights that integrate the function sequence (24) are obtained by solving a multidimensional unconstrained optimization problem, as described in [23]. For each of these functions, we map \(x \to \alpha\) and
\[ y \rightarrow \beta, \text{ where } \alpha \text{ and } \beta \text{ are the barycentric coordinates of a triangle (see Appendix A). Because the rules are geometrically symmetric, these quadrature rules are able to account for the singularities at each edge and vertex. In this paper, we consider only the two-dimensional characterizations from Sec. 2.1 because these are more severe than those in Sec. 2.2. The points and weights are listed in Appendix A, together with a pictorial representation of the proposed approach.}

For Approach 2, which provides monotonic improvement in accuracy as the number of quadrature points is increased, we construct a one-dimensional function sequence

\[ 1, x, x \ln x, x^2, x^3 \ln x, x^4, x^5 \ln x, x^6, \ldots, \]

which applies to the singularities in both Sec. 2.1 and 2.2.

The one-dimensional points and weights are listed in Appendix B, for \( x \in [0, 1] \), together with a pictorial representation of the proposed approach.

Letting \( x' = 1 - x \), Approach 2 can be directly applied to quadrilateral elements by taking the outer product of the one-dimensional rules that exactly integrate

\[ 1, x, x \ln x, x^2, x^3 \ln x, x^4, x^5 \ln x, x^6 \ln x, x^7, \ldots, \]

Although, in this context, we use quadrature rules for logarithmic functions per the singularities observed in Sec. 2, Approaches 1 and 2 are well suited to work with the logarithmic singularity in (24) and (25) replaced by other integrable singular functions.

4. Numerical Experiments for Singular, Near-Singular, and Far interactions

To assess the effectiveness of the quadrature rules described in Sec. 3, we consider multiple configurations for \( A_S \) and \( A_T \). For \( A_S \), we consider the triangular element with vertices \((0 \text{ m}, 0 \text{ m}), (1/20 \text{ m}, 1/20 \text{ m}), \) and \((-1/20 \text{ m}, 1/20 \text{ m})\), and we use a triangular element with the same shape for \( A_T \). We parameterize these configurations by defining an angle \( \theta \) between the planes of \( A_S \) and \( A_T \). Additionally, we consider displacements \( \Delta y \) and \( \Delta z \). These parameters are depicted in Fig. 1, and are listed in Table 1 for the cases considered in this paper, with \( \delta_y = (6\sqrt{2} - 1)/60 \text{ m} \) and \( \delta_z = 1/2000 \text{ m} \).

| Case | \( \theta \) | \( \Delta y \) | \( \Delta z \) | Interaction | Potential |
|------|---------|--------|--------|-------------|-----------|
| 1    | 0°      | 0      | 0      | Singular    | Scalar    |
| 2    | 45°     | 0      | 0      | Singular    | Scalar    |
| 3    | 90°     | 0      | 0      | Singular    | Scalar    |
| 4    | 180°    | 0      | 0      | Singular    | Scalar    |
| 5    | 180°    | 0      | \( \delta_z \) | Near-singular | Scalar |
| 6    | 0°      | \( \delta_y \) | 0      | Far         | Scalar    |
| 7    | 90°     | 0      | 0      | Singular    | Vector    |
| 8    | 180°    | 0      | 0      | Singular    | Vector    |

Table 1: Parameters describing the configurations of \( A_S \) and \( A_T \) to analyze singular, near-singular, and far interactions for scalar and vector potentials. Note that Cases 4 and 8 are cases where the source and test triangles coincide.

We consider the integrals for a free-space medium

\[ I_{s,c} = \int_{A_T} \int_{A_S} \frac{\cos(2\pi R)}{R} dA_S dA_T, \]

\[ I_{s,s} = \int_{A_T} \int_{A_S} \frac{\sin(2\pi R)}{R} dA_S dA_T, \]

\[ I_{v,c} = \int_{A_T} (x_T - x_j) \cdot \int_{A_S} \frac{\cos(2\pi R)}{R} (x_S - x_i) dA_S dA_T, \]

\[ I_{v,s} = \int_{A_T} (x_T - x_j) \cdot \int_{A_S} \frac{\sin(2\pi R)}{R} (x_S - x_i) dA_S dA_T, \]
where \( \mathbf{x}_j = (1/20 \text{ m}, 1/20 \text{ m}) \) and \( \mathbf{x}_i = (-1/20 \text{ m}, 1/20 \text{ m}) \). When \( \lambda = 1 \text{ m} \), (27) and (28) respectively correspond to the even- and odd-term components of (3), and (29) and (30) respectively correspond to the even- and odd-term components of (9). Additionally the maximum edge lengths of \( A_S \) and \( A_T \) are \( 1/10 \lambda \).

The integrals \( I_{s,c} \) and \( I_{v,c} \) can be (nearly-)singular, depending on the distance between \( A_S \) and \( A_T \). On the other hand, \( I_{s,s} \) and \( I_{v,s} \) are nonsingular. Cases 1–4 and 7–8 in Table 1, with \( \Delta y = \Delta z = 0 \), are singular, and Case 5, with \( \Delta z = \delta z \), is nearly singular. In Case 6, \( \Delta y = \delta y \) is large enough that the integrands of \( I_{s,c} \) and \( I_{v,c} \) are smooth.

We compute reference solutions using Mathematica [24] with 34 digits of working precision and precision and accuracy goals of 17 digits. To compute \( I_{s,c} \) and \( I_{v,c} \), we use the radial–angular transformation presented in [6]. To verify the implementation of the radial–angular transformation, we compute \( I_{s,s} \) and \( I_{v,s} \), which are nonsingular, with and without the transformation to confirm both reference solutions are the same. The amount of time required to compute each reference solution \( I_{s,c} \) and \( I_{v,c} \) is hours, compared to the fraction of a second required for the quadrature integration.

To perform the quadrature integration, which we denote by \( \bar{I} \), we use polynomial and polynomial-root Gauss–Legendre rules with the radial–angular transformation [6] for the integral over \( A_S \). For the one-dimensional polynomial rules, we use 100 points and, for the one-dimensional polynomial-root rules, we use 9 points. For the integral over \( A_T \), we use two-dimensional polynomial rules [25, 22, 26, 27], the DMRW rules [11, 12], and the rules from Approaches 1 and 2 from Sec. 3. Because the DMRW rules are asymmetric, we compare the average error of these three DMRW choices.

Fig. 2 shows the relative errors \( \varepsilon = \left| (\bar{I} - I) / I \right| \) for \( I_{s,s} \) for Case 1, with \( \theta = 0^\circ \), \( \Delta y = 0 \), and \( \Delta z = 0 \). Because \( I_{s,s} \) is not singular, the polynomial rules perform the most efficiently; however, Approach 1 achieves similar efficiency. The averaged DMRW rules are the least efficient. This trend also occurs for the other values of \( \theta \); therefore, we omit those plots. Furthermore, \( I_{v,s} \) is also smooth and nonsingular, and shows similar properties to \( I_{s,s} \).

Fig. 3 shows the relative errors for Cases 1–4, with \( I_{s,c} \) for \( \Delta y = 0 \) and \( \Delta z = 0 \). For the coplanar cases (\( \theta = 0^\circ \) in Fig. 3a) and (\( \theta = 180^\circ \) in Fig. 3d), both approaches generally outperform the polynomial quadrature rules and the averaged DMRW rules, and Approach 1 often outperforms the polynomial quadrature rules by orders of magnitude. In Fig. 3d, for example, Approach 1 outperforms the polynomial rules by at least two orders of magnitude for \( n_T = 27 \). For the noncoplanar cases (\( \theta = 45^\circ \) in Fig. 3b) and (\( \theta = 90^\circ \) in Fig. 3c), both approaches often outperform the polynomial quadrature rules and the averaged DMRW rules. Approach 1, though designed for coplanar elements, generally outperforms the polynomial rules and the averaged DMRW rules. Approach 2, though less efficient for small \( n_T \), yields a monotonically decreasing relative error as the number of quadrature points is increased. We believe that the singularities derived for the two special cases in Sec. 2.1 and 2.2 are more generally applicable, as the good convergence properties observed in Fig. 3 suggest. The techniques shown in [2] and [28] may allow one to derive the singularities.
Figure 3: Relative error for $I_{s,c}$, corresponding to Cases 1–4, with $\Delta y = 0$ and $\Delta z = 0$. Note that $\theta = 135^\circ$ has similar features to $\theta = 45^\circ$; thus, we omit that result.

Figure 4: Relative error for $I_{s,c}$, corresponding to Case 5, with $\theta = 180^\circ$, $\Delta y = 0$, and $\Delta z = \delta_z$. 
Figure 5: Relative error for $I_{s,c}$, corresponding to Case 6, with $\theta = 0^\circ$, $\Delta y = \delta_y$, and $\Delta z = 0$.

Figure 6: Relative error for $I_{v,c}$, corresponding to Cases 7–8, with $\Delta y = 0$ and $\Delta z = 0$.

for completely general source and test triangle configurations.

Fig. 4 shows the relative errors for $I_{s,c}$ for Case 5, with $\theta = 180^\circ$, $\Delta y = 0$, and $\Delta z = \delta_z$. This case is nearly singular. Approaches 1 and 2 are not designed for this case; nonetheless, Approach 1 often outperforms the polynomial rules and the averaged DMRW rules, and Approach 2 monotonically converges.

Fig. 5 shows the relative errors for $I_{s,c}$ for Case 6 with $\theta = 0^\circ$, $\Delta y = \delta_y$, and $\Delta z = 0$. This case is not singular, but Approach 1 converges nearly as rapidly as the polynomial rules. Approach 2 converges faster than the averaged DMRW rules.

Finally, Fig. 6 shows the relative errors for Cases 7–8, with $I_{v,c}$ for $\Delta y = 0$ and $\Delta z = 0$. For the coplanar case ($\theta = 180^\circ$ in Fig. 6b), both approaches generally outperform the polynomial quadrature rules and the averaged DMRW rules for larger numbers of points. For the noncoplanar case ($\theta = 90^\circ$ in Fig. 6a), Approach 2 and the averaged DMRW rules yield monotonically decreasing relative errors as the number of quadrature points is increased, and Approach 1 fluctuates less than the polynomial quadrature rules.

5. Conclusions

In this paper, we presented two symmetric quadrature rule approaches to address some of the singularities encountered in the method of moments for the EFIE. Our first approach was often able to outperform polynomial rules by several orders of magnitude for singular cases and exhibited similar convergence properties for nonsingular cases. Though not as efficient as Approach 1 for singular integrals or the polynomial rules
for nonsingular integrals, the relative error arising from Approach 2 decreases monotonically with respect to
the number of integration points, a feature that is not observed with the polynomial scheme when applied
to integrands with logarithmic singularities. Additionally, for large numbers of integration points, the points
arising from Approach 2 take less time to compute than those from Approach 1 since they are computed
from one-dimensional rules. Generally, Approaches 1 and 2, which have the favorable property of geometric
symmetry, outperform the averaged DMRW rules.

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Appendix A. Quadrature Points and Weights for Approach 1

Table A.1 provides the points and weights for Approach 1, which have been ordered similarly to those
in [22] to facilitate comparison. Figure A.1 shows a pictorial representation of Approach 1 for a (1,2,1) rule,
illustrating \( w, \alpha, \beta, \) and \( \gamma \) (more details in [23]).

![Figure A.1: Pictorial representation of Approach 1 for a (1,2,1) rule. Note each set of colored points is associated with a single orbit and is therefore symmetric.](image-url)
| u   | w   | α   | β   | γ   |
|-----|-----|-----|-----|-----|
| 1   | 1.000000000000000 | 0.333333333333333 | 0.333333333333333 | 0.333333333333333 |
| 3   | 0.333333333333333 | 0.695787795710222 | 0.152510612144899 | 0.152510612144899 |
| 4   | -0.714428571428571 | 0.695787795710222 | 0.690421702429153 | 0.352891128754099 |
| 6   | 0.257014576980601 | 0.987813809527068 | 0.451093280597361 | 0.451093280597361 |
| 7   | 0.859150927451144 | 0.690551440177062 | 0.497427429999135 | 0.497427429999135 |
| 12  | 0.149283457065075 | 0.5166984894836 | 0.241695675425822 | 0.241695675425822 |
| 16  | 0.118243901597933 | 0.333333333333333 | 0.333333333333333 | 0.333333333333333 |
| 25  | 0.105991450511953 | 0.333333333333333 | 0.333333333333333 | 0.333333333333333 |
| 27  | 0.019225313240704 | 0.0012565568381 | 0.499937172180911 | 0.499937172180911 |
| 33  | 0.010658306417199 | 0.0012565568381 | 0.499937172180911 | 0.499937172180911 |
| 42  | 0.021915502462300 | 0.0012565568381 | 0.499937172180911 | 0.499937172180911 |
| 50  | 0.032896741607277 | 0.0012565568381 | 0.499937172180911 | 0.499937172180911 |

**Table A.1:** Quadrature points and weights for Approach 1.
Appendix B. One-Dimensional Quadrature Points and Weights for Approach 2

Table B.1 provides the one-dimensional points and weights for Approach 2. Figure B.1 shows a pictorial representation of Approach 2 to subdivide a triangle into three quadrilaterals (more details in [23]).

| n' | w' | ξ |
|----|----|----|
| 1  | 1.000000000000000 | 0.500000000000000 |
| 2  | 0.416874772299575 | 0.15853759535360 |
|    | 0.583121522770053 | 0.74481339598618 |
| 3  | 0.189997117971354 | 0.60827366949223 |
|    | 0.469254348225795 | 0.48489594837560 |
|    | 0.349747472060753 | 0.854959900000000 |
| 4  | 0.106882575161292 | 0.03428786806880 |
|    | 0.395788771158858 | 0.234171483281889 |
|    | 0.377297249510236 | 0.58759535849673 |
|    | 0.226031404163614 | 0.908595817252184 |
| 5  | 0.057875593510600 | 0.020052668459088 |
|    | 0.188381905418622 | 0.140436646107533 |
|    | 0.297422543453810 | 0.380919795252978 |
|    | 0.296295612221600 | 0.698863855001447 |
|    | 0.158803685502429 | 0.936974233334311 |
| 6  | 0.036467705409931 | 0.012544980340007 |
|    | 0.125413284365877 | 0.090649292816740 |
|    | 0.220345408168324 | 0.26588636715017 |
|    | 0.267514284590912 | 0.51519742477348 |
|    | 0.236265294911663 | 0.772638772660649 |
|    | 0.117635891713573 | 0.953295979568199 |

Table B.1: 1D quadrature points and weights for Approach 2.

Figure B.1: Pictorial representation of Approach 2 to map one of the three quadrilaterals that form the original triangle.

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