Study of the Spectrum of Inflaton Perturbations

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We examine the spectrum of inflaton fluctuations resulting from any given long period of exponential inflation. Infrared and ultraviolet divergences in the inflaton dispersion summed over all modes do not appear in our approach. We show how the scale-invariance of the perturbation spectrum arises. We also examine the spectrum of scalar perturbations of the metric that are created by the inflaton fluctuations that have left the Hubble sphere during inflation and the spectrum of density perturbations that they produce at reentry after inflation has ended. When the inflaton dispersion spectrum is renormalized during the expansion, we show (for the case of the quadratic inflaton potential) that the density perturbation spectrum approaches a mass-independent limit as the inflaton mass approaches zero, and remains near that limiting value for masses less than about 1/4 of the inflationary Hubble constant. We show that this limiting behavior does not occur if one only makes the Minkowski space subtraction, without the further adiabatic subtractions that involve time derivatives of the expansion scale factor \( a(t) \). We also find a parametrized expression for the energy density produced by the change in \( a(t) \) as inflation ends. If the end of inflation were sufficiently abrupt, then the temperature corresponding to this energy density could be very significant. We also show that fluctuations of the inflaton field that are present before inflation starts are not dissipated during inflation, and could have a significant observational effect today. The mechanism for this is caused by the initial fluctuations through stimulated emission from the vacuum.

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I. INTRODUCTION

Cosmological inflation predicts an amplification of quantum fluctuations that perturbs the background homogeneity of the universe and leads to regions of spacetime of nonuniform density [1]. The dispersion of the quantized inflaton fluctuation field, \( \delta \phi \), is taken as a measure of the inhomogeneity of the field during inflation. Inhomogeneity perturbs the gravitational field during inflation, and its perturbations set the initial conditions for the acoustic oscillations of the plasma and matter that are present after the inflationary era.

The dispersion of \( \delta \phi \) has an ultraviolet divergence that results from integrating over all modes at a given time during inflation. As the highest frequency modes correspond to wavelengths that never exit the de Sitter horizon \( H^{-1} \) during inflation, one may think that such modes can be ignored or dealt with by means of a cut-off or by standard curved-spacetime regularization and renormalization methods without observable consequences. However, [2] showed that when the method of adiabatic regularization is used to renormalize the dispersion of \( \delta \phi \), the dispersion spectrum at wavelengths that have left the de Sitter horizon is significantly affected. Adiabatic regularization in Robertson-Walker universes has been shown [3] to give the same result as other forms of regularization and renormalization, including point-splitting regularization; and the adiabatic condition has been shown to be closely related to the Hadamard condition in curved spacetime. Because of the time-translation properties of the universe during inflation, renormalization of the spectrum of inflaton fluctuations does not alter the near scale-invariance of the spectrum of perturbations as they enter the Hubble horizon of the post-inflationary universe. However, renormalization may alter the relation between the magnitude of the inflaton perturbation spectrum and the implied value of \( H \) during inflation. This would be of importance in testing theories that predict the value of \( H \) directly. The effects studied in [2] have been further elucidated in [4, 5].

Is there a way to obtain the dispersion spectrum of inflaton fluctuations by using only the well-tested techniques of quantum field theory in Minkowski space without appealing to renormalization in curved spacetime? Such a method was developed by one of the authors (LP) in the early 1960’s in the first part of his Ph. D. thesis [6]. First we explain how and why the method works. Then we trace the evolution of the quantized inflaton fluctuation field on the background metric, without considering the metric perturbations and the density perturbations that they lead to after reheating. In that case, we can use the method to obtain the dispersion spectrum and to examine the scale invariance of the spectrum resulting from a long period of inflation. We also demonstrate how the method avoids both infrared and ultraviolet divergences in the dispersion of the inflaton fluctuation field.

Then we turn to processes involving the production of scalar perturbations of the metric by the modes of the inflaton fluctuation field after they exit the inflationary Hubble sphere; and to the spectrum of initial density perturbation they create upon reentry into the Hubble...
sphere after the end of inflation. For these considerations we consider the consequences of using only the Minkowski space regularization, as compared with regularizing infinities of the inflaton dispersion that involve derivatives of $a(t)$ during the expansion, and using the regularized inflaton dispersion at the time when they induce significant metric perturbation after leaving the inflating Hubble horizon. We find that there are significant differences in the magnitudes of the density perturbations and their dependence on the effective mass of the inflaton fluctuation field. In particular, using the regularized dispersion leads to initial density perturbations that approach a nonzero value in the limit of zero inflaton mass, and that are almost independent of the inflaton mass if that mass is less than a significant fraction of the inflationary Hubble constant.

In [6], LP showed that an expanding universe creates elementary particles. He assumed that the quantized field is evolved by the generally covariant field equation in the expanding universe. If the quantized particle field was expanded in terms of mode function solutions of the field equation and the creation and annihilation operators were defined as usual in terms of the coefficients of the mode functions, then he found that the particle number density created in a given mode was finite, but when summed over all modes the particle number density had an ultraviolet divergence. This raised the questions, (1) should the particle number operator that yields the particle number per unit physical volume be renormalized during the expansion of the universe, and (2) is it possible to prove, using only the known and tested quantum field theory in Minkowski space, that the density of particles created during the expansion of the universe, as predicted by quantum field theory, is actually finite?

To answer the latter question, he considered a general expansion$^1$ of the universe that started smoothly from Minkowski space in the distant past and after expanding in an arbitrary smooth manner approached a Minkowski space again in the distant future. It is reasonable to assume that the density of real particles created during the expansion is not greatly disturbed by the gradual and smooth joining to the early- and late-time Minkowski spaces. He showed that for such asymptotically Minkowskian expansions of the universe, the number density of particles created by the expansion of the universe was finite. No renormalization of the particle number in the curved spacetime during the expansion of the universe was used to obtain this result. Only the standard definition of particle number in the initial and final Minkowski spaces was used. He employed general mathematical theorems to prove that the particle number in a given co-expanding volume is an adiabatic invariant, and that the total number of particles created in the co-expanding volume by the expansion of the universe from the initial to the final Minkowski space is finite when summed over all modes. This was proved by considering the Bogoliubov transformation, $a_k = \alpha_k A_k + \beta_k A_k^\dagger$, that relates the annihilation operator $a_k$ for particles in mode $k$ at early times to the annihilation and creation operators at late times. He showed that the quantity $|\beta_k|^2$ that determines the average number of particles, created in mode $k$ from the early-time vacuum by the expansion of the universe, vanishes faster than any inverse power of $k$, as $k \to \infty$. Consequently, the average number of created particles, summed over all modes, is finite as measured in the late-time Minkowski space. This result is independent of renormalization in curved spacetime.

He addressed the former question (about renormalization of the particle number operator during the expansion) by considering the properties of a device that measures the number of particles per unit physical volume. As shown in detail in [6] and summarized in [7], the natural assumption that the measuring instrument is not able to measure a particle number that has very fast small oscillations, together with the requirement that the number measured in a given physical volume should be an integer, leads to a renormalized definition of the Hermitian number operator corresponding to the quantity actually measured by such an instrument. This physically relevant renormalized number operator in an expanding universe is the “bare” number operator obtained from the mode function expansion of the field, renormalized by making the adiabatic subtractions to second adiabatic order. The method of adiabatic regularization was further developed and applied to the energy-momentum tensor with Fulling and Hu in [8].

In order to obtain unambiguous results, independent of regularization and renormalization in curved spacetime, for the dispersion of the inflaton field resulting from a long period of inflation, we will consider a model of the inflationary universe that starts smoothly from Minkowski space in the distant past and, after undergoing any given number of e-foldings of inflation, approaches a Minkowski space again in the distant future. (We do not take up perturbations of the background metric that are produced by the created inflaton fluctuations until later in this paper.) We first take the initial state of the quantized inflaton perturbation field to be the Minkowski vacuum. The asymptotic Minkowski space in the distant past may be regarded as a convenient way of specifying unambiguous initial conditions that can be interpreted without reference to curved spacetime. The asymptotic Minkowski space at late times plays a similar role in permitting us to unambiguously interpret the spectrum of inflaton perturbations at late times after any given number of e-foldings of inflation. As explained above in our discussion of [6, 7], we must do the joining at early and late times as smoothly as possible to obtain results that do not have an ultraviolet divergence when summed over all modes. Our present method, in which $a(t)$ is asymptoti-

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$^1$ The word “expansion” is used in a general sense, to include the possibility of contraction.
cally flat at early times, also avoids infrared divergences, such as those that were found in the treatment of graviton production from inflation by [9]. The infrared and ultraviolet divergences found by [10] are also absent.

In Section III, we specify the inflationary spacetime by joining together a composite scale factor built of segments for which the solution to the evolution equation of the inflaton fluctuation field is known analytically. Because observation [11, 12] favors a period of nearly exponential inflation with a slowly changing inflationary Hubble constant, $H_{\text{infl}}$, we include as the middle part of our composite scale factor a region of exponential inflation. We take $H_{\text{infl}}$ as a constant, but we can incorporate a slow change by using the usual adiabatically adjusted solution in which the mode-function solutions are adjusted by neglecting the time-derivatives, but including the slow change of $H_{\text{infl}}$. The initial and final asymptotically Minkowskian segments have a number of adjustable parameters that allow us to choose any value of $H_{\text{infl}}$ and any number of e-folds of inflation, while joining the scale factor $a(t)$ continuously and with continuous first and second derivatives. That is, $a(t)$ will be a $C^2$ function. This is the minimum degree of continuity of $a(t)$ that, in general, will give a finite energy density in the late-time Minkowski space. We can also adjust parameters that determine how quickly inflation ends. In Section III, we obtain the asymptotic conditions on the modes of the inflaton perturbation field.

In Section IV, we discuss the average number of quanta of the inflaton perturbation field for a pure state (the early-time vacuum) and for a statistical mixture of states having different numbers of particles present at early times. We discuss a surprising feature that results from the stimulated creation of particles if there are particles present at early times. This effect remains significant at late times, particularly at scales that may be relevant to the large scale structure of the universe. Without amplification by stimulated emission, the initial inflaton fluctuations (i.e., particles or quanta of the inflaton fluctuation field) would be dispersed by the inflationary expansion and would have negligible effect. We will take this up further in a later paper. In standard treatments of inflation, it is assumed that the initial conditions have no significant effect after a sufficient number of e-foldings. This statement, although it seems intuitively obvious, is not correct unless the initial state of the inflaton field is the vacuum state.

In Section V, we solve the evolution equation for quantized fluctuations of the inflaton field for the above class of scale factors $a(t)$. We do this by matching the analytic solutions for the modes of the inflaton field perturbations, $\delta \phi$, and their time-derivatives at the joining points where we spliced together the different segments of the scale factor. We introduce the general evolution equation for $\delta \phi$ with a constant effective mass and focus initially on the exact solutions of the massless case. In Section VI, we find the number of particles created in each mode for inflaton fluctuations of 0 effective mass, and we discuss the effect of discontinuities in $a(t)$ and its derivatives on the average number of inflaton perturbation quanta created in each mode.

In Section VII, we obtain the dispersion spectrum of the massless inflaton fluctuation field. As mentioned above, there are no infrared or ultraviolet divergences in these quantities. We also find effects that are evident in our dispersion spectrum and depend on the phases of the inflaton perturbation modes at the time that inflation begins in our model.

In Sections VIII and IX, we find the corresponding results for quantized inflaton fluctuations of non-zero effective mass. In Section X, we discuss the dependence of the spectral index on the effective mass of $\delta \phi$. We also investigate the effect of the total duration of inflation on the scale-invariance of very long wavelength perturbation modes. In Section XI, we discuss the initial density perturbations we would expect to be produced after reheating, and we show that there is a significant difference when we employ renormalization of the inflaton dispersion during the expansion, as compared with employing only the Minkowski space renormalization. In Section XII, we use our parametrized scale factor to calculate the contribution of the changing gravitational field to reheating (see also [13]).

II. COMPOSITE SCALE FACTOR

We consider the metric

$$ds^2 = dt^2 - a^2(t)((dx)^2 + (dy)^2 + (dz)^2).$$

The time $t$ will run continuously from $-\infty$ to $\infty$. The scale factor $a(t)$ will be composed of three segments. Our scale factor will generally be $C^2$, i.e., a continuous function with continuous first and second derivatives everywhere, including at the joining points between segments. The initial and final segments are asymptotically Minkowskian in the distant past and future, respectively. The middle segment is an exponential expansion. We choose specific forms for $a(t)$ in these segments that have exact solutions of the evolution equations for inflaton quantum fluctuations of zero effective mass.

We emphasize that the initial and final asymptotically flat regions permit us to unambiguously interpret our results for free fields without having to perform any renormalization in curved spacetime. The final asymptotically flat region will not significantly affect the result obtained for the spectrum of inflaton perturbations created by the inflationary segment of the expansion. The initial asymptotically flat region should have a negligible effect on the

2 We also briefly consider scale factors that are only $C^1$ or $C^0$ at the joining points.
spectrum resulting from a long period of inflation. If there are cases in which no inflaton perturbations are created by the period of exponential inflation, then the inflaton perturbations would result from the initial asymptotically flat region, possibly amplified by the long period of inflation.

![Image of a graph showing the scale factor, $a(t(\tau))$, and dimensionless Hubble parameter, $sH(t(\tau)) = sa^{-1}da/dt = sa^{-4}da/d\tau$, of Eq. (2) with $a_1 = 1$, $a_2 = 2$, $b = 0$, and $s = 1$. Note in the graph that the maximum of $H$ occurs at a value of $a(t(\tau))$ closer to $a_1$ than to $a_2$. In both the case where $a_2 \gg a_1$ and the case where $a_2 \approx a_1$, $H_{\text{max}}$ occurs at a value of the scale factor where $a(t(\tau)) \approx a_1$.](image)

**FIG. 1:** Scale factor, $a(t(\tau))$, and dimensionless Hubble parameter, $sH(t(\tau)) = sa^{-1}da/dt = sa^{-4}da/d\tau$, of Eq. (2) with $a_1 = 1$, $a_2 = 2$, $b = 0$, and $s = 1$. Note in the graph that the maximum of $H$ occurs at a value of $a(t(\tau))$ closer to $a_1$ than to $a_2$. In both the case where $a_2 \gg a_1$ and the case where $a_2 \approx a_1$, $H_{\text{max}}$ occurs at a value of the scale factor where $a(t(\tau)) \approx a_1$. The relationship between proper time and $\tau$-time is given by Eq. (3).

We base each asymptotic segment on a scale factor of the form,

$$a(t(\tau)) = \left\{a_1^4 + e^\tau/[a_2^4 - a_1^4](e^{\tau/s} + 1) + b(e^{\tau/s} + 1)^{-2}\right\}^{1/4}$$

where $\tau$ is related to the proper time $t$, as defined through Eq. (1), by

$$d\tau \equiv a(t)^{-3}dt.$$  

(3)

The form of the scale factor in Eq. (2) is based on the form of the index of refraction used by Epstein to model the scattering of radio waves in the upper atmosphere and by Eckart to model the potential energy in one-dimensional scattering in quantum mechanics. It was first used in the cosmological context by Parker to model $a(t)$. As can be seen from Fig. 1 this scale factor approaches the constant $a_1$ at early times and the constant $a_2$ at late times, and the constant $s$ determines roughly the interval of $\tau$-time for $a(t)$ to go from $a_1$ to $a_2$. (A sufficiently large magnitude of $b$ would produce a bump or valley in $a(t)$.) The parameters $a_1$, $a_2$, $b$, and $s$ are different in the initial and final asymptotically flat segments. Where confusion would arise we will include subscripts $i$ in the initial set of parameters and $f$ in the final set of parameters. The equation for the middle (inflationary) segment of our composite scale factor is given in terms of proper time by

$$a(t) = a(t_1)e^{H_{\text{ini}}(t-t_1)},$$  

(4)

where $H_{\text{ini}}$ is the constant value of $H(t) \equiv a^{-1}da/dt$ during the exponential expansion of the middle segment.

We define the quantity $N_c \equiv \ln (a_{2f}/a_{1i})$, when there is a long period of exponential growth, $N_c$ is essentially the number of $e$-folds of inflation. Typically, $N_c$ will be about 60. Within the final asymptotically flat scale factor, the ratio of $a_{2f}$ to $a_{1f}$ determines how gradually the exponential expansion transitions to the asymptotically flat late-time region. (For example, this ratio might be 1 $e$-fold, which we would consider to be relatively gradual, or it might be 1.0001, which we would consider to be relatively abrupt.)

With our choices of $a(t)$ in the three segments, we are able to join them so that $a(t)$ and its first and second derivatives with respect to time are continuous everywhere. This requires that we join the exponentially expanding segment, in which $H(t)$ has the constant value

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3 We do find remnants of the early initial conditions in the late-time inflaton dispersion spectrum, which we discuss later.
\[ H_{\text{inf}} \], to the initial and final segments at the times when \( H(t) \) is a maximum. This maximum value must equal \( H_{\text{inf}} \). A simple power law form of the scale factor, such as that of a radiation-dominated universe, could not be used to simultaneously maintain the continuity of the scale factor and its first and second derivatives when matched directly to the inflationary segment of exponential expansion.

With \( b_1 = 0 \) and \( b_f = 0 \), we then find the following expressions. The time \( \tau_i \) at which the first segment joins to the exponential segment is

\[
\tau_i = s_l \ln \left( \frac{3a_{i1}^4 - 3a_{2i}^4 + C_i}{8a_{2i}} \right). \tag{5}
\]

The constant \( a(t_1) \) in Eq. (1) is

\[
a(t_1) = \left( \frac{-3a_{i1}^4 - 3a_{2i}^4 + C_i}{2} \right)^{1/4}. \tag{6}
\]

Because the maximum value of \( H(t) \) in the first segment must equal \( H_{\text{inf}} \), we find that

\[
H_{\text{inf}} = \left[ \frac{2^{3/4} (-a_{i1}^4 + a_{2i}^4)}{a_{2i}^4 \left( 11a_{i1}^4 - 3a_{2i}^4 + C_f \right)^2 s_l} \times \left( -3a_{i1}^4 - 3a_{2i}^4 + C_i \right)^{1/4} \times \left( 3a_{i1}^4 - 3a_{2i}^4 + C_i \right) \right], \tag{7}
\]

where

\[
C_i \equiv \sqrt{9a_{i1}^8 + 46a_{i1}^4 a_{2i}^4 + 9a_{2i}^8}. \tag{8}
\]

Once we choose values for \( a_{i1} \) and \( a_{2i} \), the remaining constants are determined to have the following values:

\[
s_f = \left[ \frac{2^{3/4} (-a_{i1}^4 + a_{2f}^4)}{a_{2f}^4 \left( 11a_{i1}^4 - 3a_{2f}^4 + C_f \right)^2 H_{\text{inf}}} \times \left( -3a_{i1}^4 - 3a_{2f}^4 + C_f \right)^{1/4} \times \left( 3a_{i1}^4 - 3a_{2f}^4 + C_f \right) \right]. \tag{9}
\]

Denote the parameter \( \tau \) of Eq. (2) as \( \tau^\prime \) in the final segment. At the time \( \tau^\prime_f \) when the exponential segment joins to the final segment, we find that

\[
\tau^\prime_f = s_f \ln \left( \frac{3a_{i1}^4 - 3a_{2f}^4 + C_f}{8a_{2f}} \right). \tag{10}
\]

The corresponding proper time \( t \) at which the exponential segment joins to the final segment is

\[
t_2 = \frac{1}{4H_{\text{inf}}} \ln \left( \frac{-3a_{i1}^4 - 3a_{2f}^4 + C_f}{-3a_{i1}^4 - 3a_{2i}^4 + C_i} \right) + t_1, \tag{11}
\]

where

\[
C_f \equiv \sqrt{9a_{i1}^8 + 46a_{i1}^4 a_{2f}^4 + 9a_{2f}^8}. \tag{12}
\]

See Fig. 2 for a schematic diagram of how we match our segments of the scale factor together.

### III. Early and Late Asymptotic Conditions on \( \delta \phi \)

Consider an inflaton field composed of a spatially homogeneous term plus a first order perturbation,

\[
\phi(x, t) = \phi(0)(t) + \delta \phi(x, t). \tag{13}
\]

We investigate, in units of \( \hbar = c = 1 \), a minimally coupled scalar field that obeys the evolution equation:

\[
\partial_t^2 \delta \phi + 3H \partial_t \delta \phi - a^{-2}(t) \sum_{i=1}^{3} \partial_i^2 \delta \phi + m(\phi(0))^2 \delta \phi = 0. \tag{14}
\]

The mass term is related to the inflationary potential by

\[
m(\phi(0))^2 = \frac{d^2V}{d(\phi(0))^2}. \tag{15}
\]

For simplicity, we take \( m(\phi(0))^2 \) as a constant, \( m^2 \).

The quantized field \( \delta \phi \) can be written in terms of the early-time creation and annihilation operators, \( A_k^\dagger \) and \( A_k^\dagger \), as

\[
\delta \phi = \sum_k \left(A_k^\dagger f_k + A_k^\dagger f_k^* \right) \equiv \sum_k \delta \phi_k^*, \tag{16}
\]

where\(^4\)

\[
f_k^* = V^{-1/4} e^{i k \cdot \vec{x}} \psi_k(t(\tau)). \tag{17}
\]

The function \( \psi_k(t) \) satisfies

\[
\partial_t^2 \psi_k(t) + 3H \partial_t \psi_k(t) + \frac{k^2}{a^2(t)} \psi_k(t) + m^2 \psi_k(t) = 0, \tag{18}
\]

where \( k = 2\pi n/L \), with \( n \) an integer. Because the creation and annihilation operators in Eq. (16) correspond to particles at early times, we require that \( \psi_k \) satisfies the early-time positive frequency condition

\[
\lim_{\tau \to -\infty} \psi_k(t(\tau)) \sim \frac{1}{\sqrt{2a_{i1}^2 \omega_{i1}(k)}} e^{-i a_{i1}^2 \omega_{i1}(k) \tau}, \tag{19}
\]

where \( \omega_{i1}(k) \equiv \sqrt{(k/a_{i1})^2 + m^2} \).

At late times, this solution will have the asymptotic form

\[
\lim_{\tau \to \infty} \psi_k(t(\tau)) \sim \frac{1}{\sqrt{2a_{i1}^2 \omega_{i1}(k)}} \left[ \alpha_k e^{-i a_{i1}^2 \omega_{i1}(k) \tau} + \beta_k e^{i a_{i1}^2 \omega_{i1}(k) \tau} \right]. \tag{20}
\]

\(^4\) For simplicity, we are imposing periodic boundary conditions upon a cubic coordinate volume, \( V = L^3 \). In the continuum limit \( L \) would go to infinity.
where \( \omega_2(k) \equiv \sqrt{(k/a_2)^2 + m^2} \). Here, one has
\[
|\alpha_k|^2 - |\beta_k|^2 = 1, \quad (21)
\]
from the conserved Wronskian of Eq. (18). The quantity \(|\beta_k|^2\) is the average number of particles in mode \( \vec{k} \) created by the expansion of the scale factor from a state that initially has no particles \([6, 7]\). Because our scale factor is asymptotically Minkowskian, the meaning of particles at early and late times has no ambiguity. The late-time creation and annihilation operators, \( a_{\vec{k}} \) and \( a_{\vec{k}}^\dagger \), are related to the early-time creation and annihilation operators through a Bogoliubov transformation:
\[
a_{\vec{k}} = \alpha_k A_k + \beta_k A_{-\vec{k}}^\dagger. \quad (22)
\]

IV. PARTICLE NUMBER IN PURE AND MIXED STATES

As noted above, if no particles are present at early times, then the average particle number at late times in mode \( \vec{k} \) is
\[
\langle N_{\vec{k}} \rangle_{\tau \to \infty} = \langle 0| a_{\vec{k}}^\dagger a_{\vec{k}} |0 \rangle = |\beta_k|^2, \quad (23)
\]
where \(|0\rangle\) is the state annihilated by the early-time annihilation operators \( A_{\vec{k}} \).

Let us define \( \left| \delta \phi_k^{(\text{un})} \right|^2 \equiv \langle 0| \left( \delta \phi_k \right)^2 |0 \rangle_{\text{un}} = |f_{\vec{k}}|^2 \). Here, “un” refers to unrenormalized values. In the continuum limit, this reduces to \( \left| \delta \phi_k^{(\text{un})} \right|^2 = (2\pi)^{-3} |\psi_k|^2 \).

Later, we will find that renormalization is necessary and may have a significant effect on the magnitude of the dispersion, even for modes that leave imprints on the CMB and large-scale structure that can be observed in the present universe. Let \( \left| \delta \phi_k^{(\text{re})} \right|^2 \equiv \langle 0| \left( \delta \phi_k \right)^2 |0 \rangle_{\text{re}} \), where
\[
\sum_{\vec{k}} \langle 0| \left( \delta \phi_k \right)^2 |0 \rangle_{\text{re}} \equiv \langle 0| \langle \delta \phi(x) \rangle^2 |0 \rangle_{\text{re}}. \quad (24)
\]
The value of \( \langle 0| \langle \delta \phi(x) \rangle^2 |0 \rangle_{\text{re}} \) would diverge without renormalization. In Minkowski space, the renormalization would be equivalent to subtracting the vacuum zero-point contributions. In the expanding universe there are also divergent contributions coming from the time-dependence of the scale factor \( a(t) \), and to take those into account we must use a curved spacetime renormalization method such as adiabatic regularization or point-splitting Hadamard regularization. We will take this up again later in Section V.

If the initial state of the universe were not the vacuum, but instead were a statistical mixture of pure states, each of which contains a definite number of particles at early times, then we would find the analog of stimulated emission, where the initial presence of scalar particles tends to increase the number of scalar particles created by the expansion of the universe \([6, 7]\):
\[
\langle N_{\vec{k}} \rangle_{\tau \to \infty} = \langle N_{\vec{k}}^0 \rangle + |\beta_k|^2 \left( 1 + 2 \langle N_{\vec{k}}^0 \rangle \right). \quad (25)
\]
Here, \( \langle N_{\vec{k}} \rangle_{\tau \to \infty} \) is the average particle number in mode \( \vec{k} \) at late times, and \( \langle N_{\vec{k}}^0 \rangle \) is the average number of particles in mode \( \vec{k} \) at early times. Because of this stimulated-emission effect, the initial presence of particles could lead to larger inflaton perturbations than would be the case with an initial Minkowski vacuum state. Furthermore, it could affect the scale invariance of the inflaton perturbation spectrum at late times. The particles that were present at early times disperse so that their relic density becomes negligible as a result of inflation. This is evident from the first term on the right of Eq. (25). However, the density of quantized inflaton perturbations created as a result of the inflationary expansion remains significant and is determined by the second term. In that term, the factor of \(|\beta_k|^2\) multiplying \( \langle N_{\vec{k}}^0 \rangle \) makes the effect of the initial number of particles present in the coordinate volume \( L^3 \) significant if \( \langle N_{\vec{k}}^0 \rangle \) is larger than or of order 1. This effect could conceivably lead to observable consequences for the large scale structure of the universe, and will be considered in later work. In the rest of this paper, we assume our initial state is asymptotically a Minkowski vacuum.

V. JOINING CONDITIONS FOR \( \psi_k \)

Consider a spacetime composed of three segments of the scale factor, \( a(t) \), in a homogeneous background metric given by Eq. (1). For an example, see Figs. 2 and 3. The first and second segments are joined at the time \( t_1 \), and the second and third segments are joined at the time \( t_2 \).

We have two linearly independent solutions to the evolution equation in both the second segment, with solutions \( h_1(t) \) and \( h_2(t) \); and the third segment, with solutions \( g_1(t) \) and \( g_2(t) \); for a total of four separate functions. These functions are multiplied by constant coefficients that we must determine. During the second segment, from \( t_1 \) to \( t_2 \), we have:
\[
\psi_k(t) = Ah_1(t) + Bh_2(t), \quad (26)
\]
\[
\psi_k'(t) = Ah_1'(t) + Bh_2'(t). \quad (27)
\]
For \( t > t_2 \), we have:
\[
\psi_k(t) = Cg_1(t) + Dg_2(t), \quad (28)
\]
\[
\psi_k'(t) = Cg_1'(t) + Dg_2'(t). \quad (29)
\]
If we require that \( \psi_k(t) \) and \( \psi_k'(t) \) be continuous at \( t_1 \) and \( t_2 \). This imposes 4 matching conditions:
\[
Ah_1(t_1) + Bh_2(t_1) = \psi_k(t_1), \quad (28)
\]
\[
Ah_1'(t_1) + Bh_2'(t_1) = \psi_k'(t_1),
\]
\[
Cg_1(t_2) + Dg_2(t_2) = Ah_1(t_2) + Bh_2(t_2),
\]
\[
Cg_1'(t_2) + Dg_2'(t_2) = Ah_1'(t_2) + Bh_2'(t_2).
\]
The calculation given in Appendix A then shows us that

\[ C = \frac{1}{(g'_1 g_2 - g_1 g'_2)_{t=t_2}} \times \left\{ \frac{\psi'_1 h_2 - \psi_1 h'_2}{h'_1 h_2 - h_1 h'_2} (h'_1 g_2 - h_1 g'_2)_{t=t_2} + \frac{\psi'_1 h_1 - \psi_1 h'_1}{h'_2 h_1 - h_2 h'_1} (h'_2 g_2 - h_2 g'_2)_{t=t_2} \right\}, \]

\[ D = \frac{1}{(g'_2 g_1 - g_2 g'_1)_{t=t_2}} \times \left\{ \frac{\psi'_1 h_2 - \psi_1 h'_2}{h'_1 h_2 - h_1 h'_2} (h'_1 g_1 - h_1 g'_1)_{t=t_2} + \frac{\psi'_1 h_1 - \psi_1 h'_1}{h'_2 h_1 - h_2 h'_1} (h'_2 g_1 - h_2 g'_1)_{t=t_2} \right\}, \]

(29)

(30)

where \( \psi_{k1} \equiv \psi_k(t_1) \) and \( \psi'_{k1} \equiv \psi'_k(t_1) \). We find \( \psi_{k1} \) and \( \psi'_{k1} \) from the solution to the evolution equation in the initial asymptotically flat segment of the scale factor. In the massless case, this solution is given by Eq. (22) below. The functions \( h_1(t) \) and \( h_2(t) \) are to be related to the evolution equation solutions in the inflationary middle segment of the scale factor, which are given in Eqs. (33) and (34) below. In terms of those solutions, one finds that \( A \) and \( B \) in Eq. (27) are given by \( A = E(k) \) and \( B = F(k) \). Similarly, the functions \( g_1(t) \) and \( g_2(t) \) are related to the solution of the evolution equation in the final asymptotically flat segment of the scale factor. The latter solution is given in Eq. (34), which allows us to specify that \( C = N_1(k) \) and \( D = N_2(k) \).

VI. MASSLESS PARTICLE PRODUCTION

We will first consider the case, \( m = 0 \). Rewriting the evolution equation, Eq. (18), in terms of \( \tau \) instead of \( t \) leads to

\[ \frac{d^2 \psi_k}{d \tau^2} = -k^2 a^4 \psi_k. \]

(31)

For the first segment of our composite scale factor, the solution of (31) having positive frequency form \( 19 \) at early times is the hypergeometric function \( 15 \)

\[ \psi_k(t(\tau)) = \frac{1}{\sqrt{2k_{a_1} a_2}} e^{-ika_{12}^2 \tau} F(-ika_{12}^2 s_i + ika_{22}^2 s_i, -ika_{12}^2 s_i - ika_{22}^2 s_i; 1 - 2ika_{12}^2 s_i; e^{-\tau}), \]

(32)

where \( F(a; b; c; d) \) is the hypergeometric function as defined in \( 17 \) see 15.1.1.

For the exponentially expanding segment of the scale factor in the massless case

\[ \psi_k(t) = -a(t)^{\frac{a}{2}} \sqrt{\pi} H_{\text{inf}} \left( E(k) H_{\text{inf}}^{(1)} \left( -\frac{k}{a(t) H_{\text{inf}}} \right) + F(k) H_{\text{inf}}^{(2)} \left( -\frac{k}{a(t) H_{\text{inf}}} \right) \right), \]

(33)

where \( H_{\text{inf}}^{(1)} \) and \( H_{\text{inf}}^{(2)} \) are the Hankel functions of the first and second kind. The variables \( t \) and \( \tau \) are related by Eq. (5). The coefficients \( E(k) \) and \( F(k) \) are determined by the matching conditions of the first joining point at \( t = t_1 \). We note that the finite period of exponential inflation lacks the full symmetries of a de Sitter universe. In the pure de Sitter case, as shown in \( 18 \), the \( k = 0 \) mode has to be chosen in a special way to avoid infrared divergences. For our \( a(t) \), infrared divergences do not arise (see Sec. VI).

For the final segment of our composite scale factor, the solution of the evolution equation (31) is a linear combination of hypergeometric functions \( 15 \)

\[ \psi_k(t(\tau)) = N_1(k) e^{-ika_{12}^2 \tau} F(-ika_{12}^2 s_f + ika_{22}^2 s_f, -ika_{12}^2 s_f - ika_{22}^2 s_f; 1 - 2ika_{12}^2 s_f; e^{-\tau}) + N_2(k) e^{ika_{12}^2 \tau} F(ika_{12}^2 s_f + ika_{22}^2 s_f, ika_{12}^2 s_f - ika_{22}^2 s_f; 1 + 2ika_{12}^2 s_f; e^{-\tau}), \]

(34)

where the coefficients \( N_1(k) \) and \( N_2(k) \) are determined by the matching conditions of the second joining point at \( t = t_2 \).

An example of the evolution for a particular mode is plotted for a specific choice of parameters using our composite scale factor in Fig. 3. We emphasize that our considerations below apply to any length of the middle exponentially inflating segment of the expansion of the universe, not just to the one used in Figs. 2 and 3.

For our choice of the final asymptotically flat segment given by Eq. (2), where we use Eq. (34) to define our functions \( g_1(t) \) and \( g_2(t) \) in terms of the relationship \( \psi_k(t) = N_1(g_1(t(\tau)) + N_2 g_2(t(\tau)) \), we find the coefficients \( \alpha_k \) and \( \beta_k \) of Eq. (20) from the large argument asymptotic forms \( 15 \)

\[ \alpha_k = \sqrt{2k a_2} \left[ \frac{C \Gamma(1 - 2c_1) \Gamma(-2c_2)}{\Gamma(1 - c_1 - c_2) \Gamma(1 + c_1 - c_2)} \right], \]

(35)

\[ + \frac{D \Gamma(1 + 2c_1) \Gamma(-2c_2)}{\Gamma(1 + c_1 - c_2) \Gamma(1 + c_2 - c_2)}, \]

and

\[ \beta_k = \sqrt{2k a_2} \left[ \frac{C \Gamma(1 - 2c_1) \Gamma(2c_2)}{\Gamma(1 - c_1 + c_2) \Gamma(1 + c_1 + c_2)} \right], \]

(36)

\[ + \frac{D \Gamma(1 + 2c_1) \Gamma(2c_2)}{\Gamma(1 + c_1 + c_2) \Gamma(1 + c_1 + c_2) \Gamma(1 + c_2 - c_2)}. \]
Recall that $C$ and $D$ and the functions $g_1(t)$ and $g_2(t)$ were defined in Sec. V. As the Wronskian of the evolution equation is conserved with our joining conditions, Eq. (21) should hold to arbitrary accuracy. As a useful check on our calculations, we verified that in all the cases considered in this paper, the relation $|\alpha_k|^2 - |\beta_k|^2 = 1$ is satisfied to at least 500 significant figures.

The quantity $|\beta_k|^2$ is the average number of particles present in mode $k$ at late times. These particles are created by the expansion of the universe through $N_e$ e-folds from a state having no particles present at early times.

We use the dimensionless variable

$$q_2 \equiv \frac{k}{a_{2f}H_{\text{inf}}}.$$  \hspace{1cm} (37)

where $k$ is the wave number, $a_{2f}$ is the constant scale factor that is approached at very late times, and $H_{\text{inf}}$ is the constant value of $(\dot{a}(t)/a(t))$ during the exponential expansion of the middle segment. It is convenient to express our results in terms of the dimensionless quantity $q_2$. For example, for $N_e = 60$, as in Fig. 4, the graph is the same if we change the range of $k$, and the values of $a_{2f}$ and $H_{\text{inf}}$, while keeping the range of $q_2$ unchanged. We will often write $|\beta_{q_2}|^2$ for the quantity $|\beta_k|^2$ with $k = q_2 a_{2f} H_{\text{inf}}$.

We define three regions of $q_2$. Values of $q_2 \lesssim \exp(-N_e)$ are in the “small-$q_2$ region.” Values of $\exp(-N_e) \lesssim q_2 \lesssim 1$ are in the “intermediate-$q_2$ region.” Values of $1 \lesssim q_2$ are in the “large-$q_2$ region.”

When $a(t)$ is at least $C^1$, i.e. when $H_{\text{inf}}$ is continuous, we find numerically that the particle production per mode in the small-$q_2$ region, $(q_2 \lesssim e^{-N_e})$, is to good approximation given by

$$\beta_{q_2} = \sinh[N_e].$$  \hspace{1cm} (38)

We also find this to be the case, analytically, by taking the limit $k \to 0$. For at least a moderate number of e-folds, this simplifies to

$$|\beta_{q_2}|^2 \approx \frac{1}{4} e^{2N_e}.$$  \hspace{1cm} (39)

The dependence in the intermediate-$q_2$ region ($e^{-N_e} \lesssim q_2 \lesssim 1$) for the $C^2$ or $C^1$ massless case is

$$|\beta_{q_2}|^2 \approx \frac{1}{4} q_2^{-2}.$$  \hspace{1cm} (40)

When $N_e$ is finite, with our composite scale factor there are no infrared divergences. For infinite inflation, where $N_e \to \infty$, we find the infrared divergences of a de Sitter universe. This problem is resolved for a true de Sitter universe in [13]. Our composite scale factor is different from a purely de Sitter universe in that our initial conditions are specified by our initial asymptotically flat region of the scale factor.

Discontinuities in the derivatives of the scale factor at the matching points cause increased particle production for large-$q_2$, (i.e., $q_2 \gtrsim 1$). This is evident in Fig. 4. As shown by Parker in [12,13], the particle number created by the expansion of the universe is related to an adiabatic invariant of the harmonic oscillator, and for such an oscillator the continuity of the frequency and its derivatives is related to the change in the adiabatic invariant [13].
The increased particle production associated with discontinuities in the derivatives of the scale factor $a(t)$ is a consequence of this relation. This result is also seen in \[20\].

For the $C^1$ case, where the scale factor and $H = \dot{a}(t)/a(t)$ are both continuous, the large-$q_2$ region goes like

$$|\beta_{q_2}|^2 = n_4 q_2^{-4}. \tag{41}$$

For the $C^2$ case, where the scale factor and $H = \dot{a}(t)/a(t)$ and $H(t)$ are all continuous, the large-$q_2$ region goes like

$$|\beta_{q_2}|^2 = n_6 q_2^{-6}. \tag{42}$$

Here $n_4$ and $n_6$ are constant coefficients, with $n_4 \approx n_6 \approx O(1)$ for a gradual end to inflation. For a sufficiently abrupt end to inflation, $n_4$ and $n_6$ can be made to be arbitrarily large. See Sec. XV.

When $H(t)$ is not continuous, we find quite a different behavior in the $C^0$ case. The evolution equation, Eq. (15), may be written \[3\]

$$\frac{d^2 \psi_k(t)}{dt^2} + \left[ \frac{k^2}{a(t)^2} + m^2 - \frac{3}{4} \left( \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{3}{2} \frac{\dot{a}(t)}{a(t)} \right] \psi_k(t) = 0. \tag{43}$$

At the discontinuity in $\dot{a}(t)$ if we express the jump as a step function, then the form of $\dot{a}(t)$ picks up a delta-function contribution. Thus, there is a finite jump in $dv_k(t)/dt$ across the discontinuity. In the $C^0$ case, $|\beta_{q_2}|^2$ is proportional to $q_2^{-2}$ in the small- and large-$q_2$ regions, and it is proportional to $q_2^{-4}$ in the intermediate-$q_2$ region. A $C^0$ scenario would suffer from these problems in addition to the divergences mentioned earlier, hence we will not consider it further.

For a non-composite scale factor composed of one asymptotically flat scale factor defined by Eq. \[2\], for large values of $q_2$, the value of $|\beta_{q_2}|^2$ falls off faster than any power of $q_2$, and in terms of $k$ we have: \[15, 16\]

$$|\beta_k|^2 = \frac{\sin^2 \left( \frac{1}{2} \sqrt{1 + \sqrt{1 + 4k^2 s^4 b}} \right) + \sin^2 \left[ \pi k s (a_1^2 - a_2^2) \right]}{\sin^2 \left[ \pi k s (a_1^2 + a_2^2) \right] - \sin^2 \left[ \pi k s (a_1^2 - a_2^2) \right]} \tag{44}$$

In the limit that $k \to 0$ for the case of the scale factor of Eq. \[2\], which is asymptotically flat at early and late times and has no exponential segment, we find that $\lim_{k \to 0} |\beta_k|^2 = \sinh^2[N_c]$, where in this case $N_c$ is $\ln (a_2/a_1)$. This is the same small-$q_2$ limit for the average number of particles created per mode as we found above in Eq. (38).

**VII. DISPERSION SPECTRUM**

The unrenormalized dispersion spectrum is \[16, 21\]

$$\langle 0 | \delta \phi^2 | 0 \rangle_{\text{un}} = \frac{1}{2(a_2 f L)^2} \sum_k \frac{1 + 2|\beta_k|^2}{\sqrt{(k/a_2)^2 + m^2}}, \tag{45}$$

where the expectation value is with respect to the state $|0\rangle$ having no particles at early times.

![FIG. 5: (color online). Dispersion $\langle |\delta \phi^2| \rangle_{q_2}/H_{\text{inf}}^2$ given by Eq. (50) for our composite scale factor continuous in $a(t)$, $\dot{a}(t)$, and $H(t)$ over an expansion of 60 e-folds. The y-axis, $\langle |\delta \phi^2| \rangle_{q_2}/H_{\text{inf}}^2$, is shown multiplied by a factor of $e^{-N_c}$; and the x-axis, $q_2$, is shown multiplied by a factor of $e^{N_c}$. When using this scaling, for a given set of values of $a_{1i}$ and $a_{2i}$ the region plotted in this graph would look identical for any number of e-folds larger than about 10. In the case of $a_{2i} = a_{1i} + da_{1i}$, where $da_{1i} \equiv 10^{-26} a_{1i}$, we see marked peaks in the dispersion spectrum. When we change the parameters in the initial asymptotically flat region to $a_{2i} = 10 a_{1i}$, these peaks are somewhat damped, as shown. The ending conditions of the final asymptotically flat segment do not affect these peaks.

We will first consider the massless case, $m = 0$. See Sec. IX for the massive case. If $|\beta_k|^2$ were 0 in Eq. (15), then one would be left with the contribution of the late-time Minkowski space vacuum. As is standard in flat spacetime quantum field theory, we subtract off this contribution of the Minkowski vacuum, leaving the physically relevant, renormalized dispersion,

$$\langle 0 | \delta \phi^2 | 0 \rangle_{\text{re}} = \frac{1}{2(a_2 f L)^2} \sum_k \frac{2|\beta_k|^2}{\sqrt{(k/a_2)^2 + m^2}} \tag{46}$$

where the subscript “re” refers to renormalized. In the continuum limit, this becomes

$$\langle 0 | \delta \phi^2 | 0 \rangle_{\text{re}} = \frac{1}{2\pi^2 a_2 f} \int_0^\infty \frac{|\beta_k|^2 |k|}{k} dk, \tag{47}$$

or with spherical symmetry,

$$\langle 0 | \delta \phi^2 | 0 \rangle_{\text{re}} = \frac{1}{2\pi^2 a_2 f} \int_0^\infty \frac{|k|\beta_k|^2}{k} dk. \tag{48}$$

With $k = q_2 a_2 f H_{\text{inf}}$ and $dk = dq_2 a_2 f H_{\text{inf}}$, we have

$$\langle 0 | \delta \phi^2 | 0 \rangle_{\text{re}} = \frac{H_{\text{inf}}^2}{2\pi^2} \int_0^\infty q_2 |\beta_{q_2}|^2 dq_2. \tag{49}$$
The $q_2$-component of the dispersion in the massless case is thus,

$$\langle |\delta \phi^2| \rangle_{q_2} = \frac{q_2^2 |\beta_{q_2}|^2 H_{\text{infl}}^2}{2 \pi^2}. \quad (50)$$

In the late-time Minkowski space, this subtraction is all that is necessary, but during the time when the universe is expanding this subtraction alone would give an infinite dispersion when summed over all modes. During the expansion, there are additional subtractions necessary. Adiabatic regularization \cite{2, 5} includes those and reduces to this standard Minkowski space subtraction in flat spacetime. In Sec. \textbf{VIII}, we consider the scalar perturbations of the metric that are created by the inflaton fluctuations. Those metric perturbations for a given mode are formed during inflation shortly after the inflaton has exited the Hubble sphere. As explained in \cite{2, 3}, we regard the metric perturbations as classical at the time when they are produced and assume that they respond to the renormalized dispersion of the inflaton fluctuation field. This implies that the relevant adiabatic subtractions that influence the metric perturbation for a given mode are those evaluated near the time that the quantized inflaton fluctuations exit the Hubble sphere.

The definition of the spectrum of inflaton perturbations, $P_{\delta \phi}$, given in \cite{23} is

$$\langle 0| \delta \phi^2 |0 \rangle_{\text{te}} = \int_{0}^{\infty} P_{\delta \phi} \frac{dq_2}{q_2}. \quad (51)$$

Therefore, we obtain the spectrum of inflaton fluctuations as

$$P_{\delta \phi} = \frac{q_2^2 |\beta_{q_2}|^2 H_{\text{infl}}^2}{2 \pi^2}. \quad (52)$$

We plot $\langle |\delta \phi^2| \rangle_{q_2}/H_{\text{infl}}^2$ in Fig. \textbf{5} and we plot $P_{\delta \phi}/H_{\text{infl}}^2$ in Fig. \textbf{6}. The peaks in these figures do not depend on the number of e-folds and are the result of the state $|0\rangle$ having no particles in the asymptotically flat part of the expansion at early times. In Fig. \textbf{7} we plot $\ln(P_{\delta \phi}/H_{\text{infl}}^2)$ versus $\ln(q_2 \exp(60))$ for an expansion of 60 e-folds. In this figure, we have plotted the full range of the inflationary part of the expansion. We see that in both of the cases plotted, where $a(t)$, $\dot{a}(t)$, and $\ddot{a}(t)$ are all continuous; and the case where only $a(t)$ and $\dot{a}(t)$ are continuous; $\langle 0| \delta \phi^2 |0 \rangle_{\text{te}}$ is finite without the need for any renormalization beyond subtracting off the Minkowski vacuum contribution at late times. If the first derivative of the scale factor were not continuous, then the integrated dispersion would diverge.

**VIII. MASSIVE PARTICLE PRODUCTION**

In the case of a massive scalar field, the evolution equation \cite{14} with $a(t)$ given by the inflationary exponential of Eq. \textbf{4}, has the exact solution

$$\psi_k(t) = -a(t)^{-\frac{3}{2}} \frac{i}{2} \sqrt{\frac{\pi}{H_{\text{infl}}}} \left[ E(k) \frac{H_{(1)}^{(1)}}{\sqrt{\frac{3}{2} - m_{\psi}^2}} \left( \frac{k}{a(t) H_{\text{infl}}} \right) \right. \left. + F(k) \frac{H_{(2)}^{(2)}}{\sqrt{\frac{3}{2} - m_{\psi}^2}} \left( \frac{k}{a(t) H_{\text{infl}}} \right) \right], \quad (53)$$

where

$$m_{\psi} = \frac{m}{H_{\text{infl}}}. \quad (54)$$
The evolution equation for arbitrary \( a(t) \) can be written in terms of \( \tau \) (defined in Eq. (3)) as

\[
\frac{d^2 \psi_k}{d\tau^2} = -(k^2 a^4 + m^2 a^6) \psi_k. \tag{55}
\]

In the initial and final asymptotically flat segments of \( a(t) \) that are joined to the inflationary segment at early and late times, we do not have an exact solution of this equation for nonzero mass. Therefore, we used approximations to obtain the plots in Fig. 8 and Fig. 9. In these graphs, we see that at low momentum there are many more particles with \( m_H = 0.1 \) than with \( m_H = 1 \) present at late times.

**FIG. 8:** (color online). Log-Log plot of the average number of created particles versus \( q_2 \). The effect of three different masses is shown for an expansion of 60 e-folds. The beginning and end segments of \( a(t) \) are rather abrupt, with \( a_{2f} = a_{1f}(1 + 10^{-26}) \), \( a_{2f} = a_{1f} e^{50} \), and \( a_{1f} = 0.9999a_{2f} \). Comparing the graph plotted here for \( m_H = 0 \) with the corresponding graph plotted for a gradual end to inflation in Fig. 4, one sees that at low momentum there are many more particles of low momentum than does the massless case. See also Fig. 9.

**FIG. 9:** (color online). The dependence of particle production (\( \langle |\beta_{q_2}|^2 \rangle \)) on mass is shown for an expansion of 60 e-folds. This graph is different from Fig. 8 in that the transition from exponential expansion to the final asymptotic segment of the scale factor is more gradual, happening over about an e-fold. For values of \( q_2 \lesssim 1 \), this graph is identical to that of Fig. 8.

**FIG. 10:** (color online). Comparison of dispersion spectrum, \( \langle |\delta \phi^2| \rangle_{q_2}/H_{\text{infl}}^2 \), given by Eq. (56) and normalized to 1, for our composite scale factor continuous in \( a(t) \), \( \dot{a}(t) \), and \( \ddot{a}(t) \) over an expansion of 60 e-folds for various masses. The values of \( \langle |\delta \phi^2| \rangle_{q_2}/H_{\text{infl}}^2 \) were divided by the maximum value of the primary peak located at \( q_2 \approx 10^{-N_\beta} \) for each. To normalize these peaks, \( \langle |\delta \phi^2| \rangle_{q_2}/H_{\text{infl}}^2 \) was divided by the following factors: 1.3 \( \times 10^{22} \) for the massless case, 2.3 \( \times 10^{22} \) for \( m_H = 0.1 \), and 2.2 \( \times 10^{21} \) for \( m_H = 1 \).

The dispersion spectrum is plotted for three different cases of \( m_H \) in Fig. 10. We have found that, even in the massive case, the observed humps are dependent only upon the initial conditions. The shape of the curves is fixed above a moderate number of e-folds. We define the variable \( J \) such that the maximum value of \( \langle |\delta \phi^2| \rangle_{q_2}/H_{\text{infl}}^2 \) for the major peak, which is the peak located nearest to \( q_2 = e^{-N_\beta} \), is \( J e^{(P-1)N_\beta} \) in the mass-
less case and is \( J e^{(P-2)N_e} \) in the massive case. Then, the normalization factor scales like \( e^{(P-1)N_e} \) in the massless case, as can be seen from Eq. (50); and the normalization factor scales like \( e^{(P-2)N_e} \) in the massive case, as can be seen from Eq. (54), where we define the exponent \( P \) in the following way:

\[
|\beta_{q_2}|^2 \approx \frac{1}{4} q_2^{-P}
\]

in the region of intermediate-\( q_2 \), \( e^{-9N_e} \lesssim q_2 \lesssim 1 \), and

\[
|\beta_{q_2}|^2 \approx \frac{1}{4} e^{PN_e}
\]

in the small-\( q_2 \) region, \( q_2 \lesssim e^{-N_e} \). The exponent \( P \) is well described by a \( q_2 \)-independent value in the case of \( m = 0 \) and in the case of \( 0.01 \lesssim m_H^2 \lesssim 9/4 \).

In the massless case, \( P = 2 \), as can be seen by Eqs. (39) and (40). The height of the major peak in the graph of the massless case in Fig. 10 grows with an increasing number of e-folds as \( e^{N_e} \), while the widths of the peaks narrow with an increasing number of e-folds as \( e^{-N_e} \). The area under an individual peak in the massless graph therefore does not change appreciably when changing the number of e-folds of expansion, provided there are at least a few e-folds of inflation. For the massive cases, we see that \( P = 2.93358 \) when \( m_H^2 = 0.1 \), and that \( P = 2.23607 \) when \( m_H = 1 \). See Fig. 12.

We wish now to approximate the dependence of \( P \) on the number of e-folds. This dispersion is obtained from Eq. (45) and its continuum limit. It is proportional to the area under the curves of \( \langle |\delta\phi|^2 \rangle_{q_2} / H_{\text{inf}}^2 \) in Fig. 10 for the particular values of the mass shown. The main contribution to the area under each curve comes from the intermediate values of \( q_2 \) (i.e., \( \exp(-N_e) < q_2 < 1 \)). For

\[
\langle |\delta\phi|^2 \rangle_{q_2} \propto m_H^2 \frac{m_H}{4} \int_{q_2}^{1} \frac{m_H}{4} \frac{dN_e}{H_{\text{inf}}^2}
\]

FIG. 11: (color online). Dispersion spectrum \( \mathcal{P}_{\delta\phi}/H_{\text{inf}}^2 \) given by Eq. (62) for our composite scale factor continuous in \( a(t) \), \( \dot{a}(t) \), and \( \ddot{a}(t) \) over an expansion of 60 e-folds. The masses considered are the same as in Figs. 8 and 10 where the average number of created particles was plotted.

FIG. 12: The dependence of the variable \( P \), as defined in Eq. (58), upon \( m_H = m/H_{\text{inf}} \). The calculated data points shown lie on the curve \( P = \sqrt{9 - 4m_H^2} \). Outside of the region plotted, however, \( P \) does not have a constant, \( q_2 \)-independent value. For \( m_H > 1.5 \), the argument, \( \sqrt{9/4 - m_H^2} \), of the Hankel functions becomes imaginary, and \( |\beta_{q_2}|^2 \) oscillates with changing \( q_2 \). For an example of a non-zero mass much smaller than \( H_{\text{inf}} \), see Fig. 13.

FIG. 13: Particle production as a function of \( q_2 \) is plotted for 60 e-folds for both the massless case and the case of \( m = 10^{-10}H_{\text{inf}} \), labeled as “\( m << H \).” As the mass decreases, the region of overlap of the solid curve with the dotted \( (m = 0) \) curve becomes larger and approaches the dotted curve in the limit as \( m \to 0 \). The solid curve breaks away from the dotted curve when \( q_2 \lesssim m_H \). The solid curve takes the constant value, \( (1/4)q_2^{3N_e} \), when \( q_2 < m_H \exp(-N_e) \). In the region of \( m_H \exp(-N_e) < q_2 < m_H \), we see \( (k/a(t))^2 \gg m^2 \) in the initial asymptotically flat region and \( (k/a(t))^2 \lesssim m^2 \) in the final asymptotically flat region. Between \( q_2 \simeq m_H \exp(-N_e) \) and \( q_2 \simeq \exp(-N_e) \), we see \( |\beta_{q_2}|^2 \approx q_2^{-1} \); and between \( q_2 \simeq \exp(-N_e) \) and \( q_2 \simeq m_H \), we see \( |\beta_{q_2}|^2 \propto q_2^{-3} \). In light of these characteristics, a comparison of Eqs. (50) and (56) can be made with consideration to where \( (k/a(t))^2 \gg m^2 \) and to where \( (k/a(t))^2 \ll m^2 \). Such an analysis shows that in the tiny mass limit of \( m_H \ll 1 \), the dispersion spectrum reduces to the massless dispersion spectrum.
TABLE I: Configuration Space Dispersion

| $m_H$ | $\langle |\delta\phi|^2|q_2\rangle/H^2$ | $H_{\text{inf}}^2$ |
|-------|-----------------------------------|------------------|
| $0.01 \lesssim m_H \lesssim 1.49$ | $\frac{1}{3}P - \frac{1}{3P - \delta + N_e}J$ | $\frac{1}{2} + N_e$ |
| $m_H = 0$ | $\frac{1}{3}P + \frac{1}{3P - \delta + N_e}e^{(P - 3)N_e}J$ |

this range of $q_2$, the value of $\langle |\delta\phi|^2|q_2\rangle/H^2$ decreases as $(q_2e^{N_e})^2-P$ in the massive case—or as $(q_2e^{N_e})^1-P$ in the massless case—until the onset of large-$q_2$ behavior at $q_2 = 1$. The values of the exponents containing $P$ can be seen from Eqs. (50), (56), and (58). (Recall that $P = 2$ in the massless case, and the range of $P$ for the massive case is shown in Fig. 12.) The value $q_2 = 1$ effectively serves as a cut-off because of the rapid fall-off in the massless case, namely $\langle |\delta\phi|^2|q_2\rangle/H^2$ as $q_2$ in the large $q_2$ region. We find that the actual height of the major peak corresponding to the one normalized to unity in Fig. 11 can be approximated as $J e^{(P - 2)N_e}$ in the massive case, and as $J e^{(P - 1)N_e}$ in the massless case, with $J$ having the same value, $J \simeq 0.01$. Then we find that

\[
0.01 \lesssim m_H \lesssim 1.49 : \quad \frac{\langle |\delta\phi|^2|0\rangle}{H_{\text{inf}}^2} = \left(\frac{1}{2} - P\right) \frac{1}{J} e^{(P - 3)N_e} + \int_{-N_e}^{1} dq_2 \, J \, q_2^{2-P},
\]

\[
m_H = 0 : \quad \frac{\langle |\delta\phi|^2|0\rangle}{H_{\text{inf}}^2} = \left(\frac{1}{2} - P\right) \frac{1}{J} \int_{-N_e}^{1} dq_2 \, J \, q_2^{2-P}. \tag{60}
\]

The resulting configuration space dispersion is given in TABLE 1. From this, one can deduce that the small mass limit, for which $P \to 3$, gives the same result for $|\delta\phi|$, as one has in the massless case, namely

\[
|\delta\phi| = \sqrt{\frac{\langle |\delta\phi|^2|0\rangle}{10} \, H_{\text{inf}}^2} \sim 1 \sqrt{N_e} + \frac{1}{2}. \tag{61}
\]

From the behavior discussed in the caption to Fig. 13 one can show that this continuity also holds for each mode separately.

X. SPECTRAL INDEX

We say that a given mode $k$ of the perturbation field $\delta \Phi$ is crossing the Hubble radius at the time when $k/(a(t)H(t)) = 1$. For larger values of $k/(a(t)H(t))$ (shorter wavelengths) the mode is said to be inside the Hubble radius, and for smaller values it is said to be outside the Hubble radius. Modes in the intermediate-$q_2$ range, as defined after Eq. (77), exit during inflation to eventually re-enter the Hubble radius at some time after inflation has ended. Using our composite scale factor, we note that after a few e-folds of inflation, the quantum perturbations that are exiting the Hubble radius are found numerically to satisfy—

\[
|\psi_k|^2 = \frac{H_{\text{inf}}^2}{k^3 D(m_H)}, \tag{62}
\]

where it will be recalled that $|\delta\phi_k^{(un)}|^2 = (2\pi)^{-3} |\psi_k|^2$, as given by Eqs. (10) and (17). The variable $D(m_H)$ is a constant of order 1 that we have evaluated numerically to be—

\[
D(m_H = 0) = 1.00, \quad D(m_H = \sqrt{10}) = 1.04, \quad D(m_H = 1) = 1.45. \tag{63}
\]

One finds approximately that $D(m_H) \simeq (1 + \frac{1}{2}m_H^2)^2$. A few Hubble times after the mode has exited the Hubble horizon, the value of $|\psi_k|^2$ approaches a constant value of 1/2 that given in Eq. (62) when $m_H = 0$. In the case when $m_H \ll 1$, the value of $|\psi_k|^2$ approaches about 1/2 the value in Eq. (62) after a few Hubble times, but then decreases very slowly over many Hubble times.

The scalar spectral index $n_s$ is defined by

\[
n_s = 1 + \frac{d \ln P_{\delta \phi}}{d \ln k}. \tag{64}
\]

At the time of exit from the horizon, or at several Hubble times after exit, we see that the spectrum defined by Eq. (61) satisfies the proportionality

\[
P_{\delta \phi} \propto k^{3} \left|\delta\phi_k^{(re)}\right|^2, \quad \tag{65}
\]

with $\left|\delta\phi_k^{(re)}\right|^2 \propto k^{-3}$. Therefore the spectral index $n_s$ is 1 for constant $H$ as measured at any given number of Hubble times after horizon exit during inflation. This is true for the range of values of $m_H$ considered in Eq. (63).

The modes that exit the Hubble radius at early times before, or shortly after, the scale factor begins growing exponentially are not described by Eq. (62). These modes, which are primarily in the small-$q_2$ region, are not scale-invariant. The gravitational perturbations induced by these modes could re-enter the Hubble radius during the matter-dominated or dark-energy dominated stages of expansion, and would appear as long wavelength modes having a spectrum that is not scale invariant. If the total number of e-folds of inflation is sufficiently small, it would be possible to observe a breaking of the scale-invariance of the large-scale structure of the universe at sufficiently large scales. See Figs. 4, 8, and 9 in which the breaking of scale-invariance is seen in the small-$q_2$ region. The curves of $\log(\langle |\beta_k|^2\rangle)$ have a slope of $-2$ in the scale-invariant region of intermediate values of $q_2$ and a slope of 0 in the region of small $q_2$ values.

Because the small-$q_2$ modes of large enough wavelength exit the Hubble radius before evolving away from the early-time conditions specified by Eq. (19), we would expect a massless inflaton to generate a spectral index of $n_s = 3$ in the small-$q_2$ region. (This follows from the fact that the square of the amplitude in Eq. (19) goes as $k^{-1}$ for 0 mass.) Similarly, we would expect a massive inflaton to generate a spectral index of $n_s = 4$ in the small-$q_2$
region, if \( m_H \) satisfies \( q_2 \ll \exp(-N_c) m_H \) in that region. If scale-invariance continued indefinitely for large wavelength modes, the dispersion would be infrared divergent, so we expect an end to scale-invariance to occur at very large length scales for any reasonable spectrum of inflaton perturbations. Let us examine this breaking of scale invariance for the \( a(t) \) we have been considering. The modes responsible for galaxy-size structures today, left the Hubble radius approximately 45 e-folds before the end of inflation [28, pp. 285], so if \( N_c \) were not too much larger than 45, we would expect it to be possible to observe the breaking of scale invariance at sufficiently large scales within our observable universe.

It is of interest to understand the behavior of the coefficients \( E(k) \) and \( F(k) \) in the expression, given in Eq. (53), of the mode functions as a superposition of Hankel functions. These coefficients determine the quantum state of the inflaton perturbation field \( \delta \phi \). Because the modes evolve separately in this model, the state vectors can be considered for each mode separately. For the modes that exhibit a scale-invariant spectrum, we will see that the quantum state is the Bunch-Davies vacuum [24] in de Sitter spacetime. Since \( \beta_k \) is essentially 1 and the value of \( |\delta_k|^2 \) is reasonable that the initial Minkowski vacuum goes over into what is essentially the Bunch-Davies vacuum for values of \( k \) that satisfy this condition at the time inflation starts in our model.

The large-\( q_2 \) modes do not exit the Hubble horizon before inflation ends with the joining to the late-time asymptotically Minkowskian region at \( t_2 \) in our model. Hence, for those modes one still has at late times that \( k/a(t_2) > H_{\text{infl}} \), so in those modes there are no created particles of the field \( \delta \phi \), i.e., no perturbations of the inflaton field are created. This behavior can be seen in Figs. 4, 8, and 9 where a rapid fall off in \( |H_{\text{infl}}|^2 \) is seen for the large-\( q_2 \) region. By a similar argument to that in the previous paragraph, we would expect the number of particles of \( \delta \phi \) to negligible in the final Minkowski region for these modes \( k \) that did not exit the Hubble horizon during inflation. For these modes, the Bunch-Davies vacuum goes over into the Minkowski vacuum at late times.

On the other hand, for intermediate-\( q_2 \) modes \( a(t_2)/a(t_1) \) is sufficiently large that \( k/(a(t_2)H_{\text{infl}}) \lesssim 1 \). Therefore, we would expect the intermediate-\( q_2 \) modes to contain created particles of the \( \delta \phi \) field in the final Minkowski space. This can be confirmed from Figs. 4, 8 and 9.

For these intermediate values of \( q_2 \), the value of the argument \( z \) of the Hankel functions is small in the late stage of inflation. Therefore, one can use the asymptotic form,

\[
|H_v^{(1)}(z)|^2 \simeq |H_v^{(2)}(z)|^2 \simeq \left( \frac{\Gamma(v)}{\pi} \right)^2 \left( \frac{1}{2z} \right)^{-2v},
\]

in Eq. (63) to find that \( |\psi_i|^2 \simeq a^{-3} |H^{(1)}(z)|^2 \propto a^{-3}z^{-2v} \), where \( z = k/(a(t)H_{\text{infl}}) \) and \( v = \sqrt{(9/4) - m_H^{-2}} \). From Eq. (63), we see that \( |\psi_i|^2 \) is constant when \( m_H = 0 \) and decreases with time as \( a(t)((9/4) - m_H^{-2})^{-3/2} \).

Using this small argument approximation with

\footnote{For a discussion of the Bunch-Davies vacuum in connection with the energy-momentum tensor see [23].}
Eq. (65) leads to
\[ \frac{d \ln P_{\delta \phi}}{d \ln k} = 3 - \sqrt{9 - 4m_H^2}, \] (67)

Then Eq. (64) gives a spectral index of
\[ n_s = 4 - \sqrt{9 - 4m_H^2}. \] (68)

When \( m_H = 0 \), this gives the scale-invariant spectrum of inflaton fluctuations corresponding to \( n_s = 1 \). The slow-roll approximation will reduce the value of \( n_s \) because \( H_{\text{inf}} \) decreases with time. This easily could be incorporated into our model, but will not be pursued further here.

For \( 3/2 > m_H^2 > 0 \), Eq. (68) would seem to give \( n_s > 1 \), but this is because in obtaining that equation we have used the same given \( t \) for all modes, which means that \( z \) is proportional to \( k \) for each mode. However, the perturbation spectrum of \( \delta \phi \) is believed to induce the spectrum of scalar perturbations of the metric when the modes are at a given number of wavelengths outside the inflationary Hubble horizon \[ 23, 26. \] This corresponds to the same value of \( z \), not \( t \), for each mode. When that is taken into account, the quantity \( d \ln P / d \ln k = 0 \), thus giving the scale-invariant value \( n_s = 1 \) for any value of \( m_H \).

Because the gravitational perturbations are effectively massless, if one were to include them and propagate them to late-times using the scale-factor \( a(t) \) in our model, one would find the same scale-invariant spectrum as we obtained for an inflaton perturbation field with \( m_H \ll 1 \).

The graph of the inflaton spectra at late times for intermediate \( q_2 \) in Fig. 11 has the spectral index of Eq. (68) because the joining to the asymptotically flat late time region occurs at a given time \( t_2 \). For massless inflatons, as for massless metric perturbations, the fact that inflation ends approximately at a given time does not affect the scale-invariance of the spectrum.

### XI. DENSITY PERTURBATIONS

In this section, we assume as usual that the inflaton perturbation field \( \delta \phi_k^{(\text{re})} \) sets up scalar perturbations of the metric shortly after their wavelengths exit the Hubble horizon. We will consider two scenarios.

In the first scenario, the modes of the quantized inflaton field directly give rise to the scalar metric perturbations, without taking into account any regularization or renormalization of the ultraviolet divergences of the dispersion (or variance) of the quantized inflaton perturbation field, beyond the Minkowski vacuum energy subtraction. If there are no further subtractions during the time when the scalar metric perturbations are induced, then the dispersion of the quantized inflaton field has an ultraviolet divergence. It is only in the late-time Minkowski space that this subtraction would give a finite dispersion for the quantized inflaton field.\(^7\) Thus UV divergences are ignored in this first scenario, presumably because the modes relevant to observable scales today are assumed to be unaffected by renormalization of the ultraviolet divergences in the dispersion. However, it is not at all obvious that this is correct. This will become evident when we consider the second scenario.

In the second scenario, the modes of the quantized inflaton field will again induce scalar metric perturbations, but we will regularize or renormalize the dispersion of the ultraviolet divergences of the quantized inflaton field, including the renormalization terms that are required for the dispersion to remain finite during the expansion of the universe, and not just the terms required in the late time Minkowski limit. We assume that at a time shortly after exit from the Hubble horizon, the scalar perturbations of the metric are induced and may be treated as essentially classical after that time \[ 8, 27. \] It is most straightforward to use adiabatic regularization \[ 8 \] to find the dispersion because it gives the result explicitly for each mode of the quantized inflaton perturbations. It should not make a significant difference to our result if one were to use, for example, Hadamard point-splitting regularization to obtain the dispersion \[ 8. \] One way to understand the rational for adiabatic regularization is analogous to that originally given in \[ 8, 7 \] in which a measuring instrument was considered. In the present case, one may suppose that the classical gravitational metric acts like a classical measuring instrument that cannot support the ultraviolet divergences that arise from the short-wavelength modes of the quantized inflaton field. The natural subtraction procedure that arises from this assumption is embodied by adiabatic regularization.

We now evaluate the expected dispersion of the scalar metric perturbations by both methods for the simple inflaton potential quadratic in the inflaton field. We work with the case when \( H_{\text{inf}} \) is constant. As in \[ 2, 4 \], we find that when regularization is taken into account, the relevant metric perturbations induced by a given value of \( H_{\text{inf}} \) are considerably smaller than they would be if no regularization of the quantized inflaton field were taken into account. (By “relevant modes,” we mean those responsible for temperature variations in the CMB and large scale structure that we observe today.) Put another way, when regularization is taken into account, the value of \( H_{\text{inf}} \) responsible for the these observed variations must be larger than it would be without regularization.

The basic reason for the smaller dispersion found in

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6 The scale-invariance of the initial spectrum of scalar perturbations of the metric when inflation ends, thus places an upper limit on the graviton mass, at least in principle.

7 This can be seen, for example, by looking at the terms in the adiabatic subtraction that vanish when \( \dot{a} \) and \( \ddot{a} \) are 0 – those terms are divergent when integrated over all modes.
the second scenario for the scalar metric perturbations is that the metric perturbations are induced, and acquire classical properties, within about a Hubble time after the inflaton perturbations have left the horizon. The dispersion of the classical scalar metric perturbations depends on the value of the dispersion of the inflaton field at the time when the perturbations are induced. Because that time is soon after the inflaton perturbations have exited the Hubble horizon, the adiabatic subtractions are quite significant. After that time the induced scalar metric perturbations are treated as classical, requiring no further subtractions.

If the scalar metric perturbations were induced only at the end of the inflationary era, then the relevant modes of the inflaton perturbation field would have left the Hubble horizon a large number of Hubble times earlier and the adiabatic subtractions would be small. In that case, the adiabatic subtractions would give only a small difference from the first scenario. However, in the second scenario we are assuming the metric perturbations are induced soon after horizon exit.

In the case when \( m_H^{-2} \ll 1 \), we find numerically that the duration and manner in which inflation ends does not have much affect on the value of \( \beta_{\phi_k}^{(m)} \), after a given mode in the relevant range of wavelengths has exited the Hubble radius. Thus, the value of \( \beta_{\phi_k}^{(m)} \) at late times is approximately equal to the value it has soon after leaving the Hubble horizon. At late times, Eqs. (17), (20), and (22) show that the expectation value is

\[
\left| \beta_{\phi_k}^{(m)} \right|^2 = \frac{1}{2L^3a_{2f}^3\omega_{2f}} \left( |\alpha_k|^2 + |\beta_k|^2 \right) = \frac{1}{2L^3a_{2f}^3\omega_{2f}} \left( 1 + 2 |\beta_k|^2 \right). \tag{69}
\]

The value of \( \beta_{\phi_k}^{(m)} \) obtained from Eqs. (20), (17), and (22), however, is un-renormalized. In the late time Minkowski spacetime, the renormalization consists of subtracting the vacuum contribution corresponding to the late time Minkowski spacetime for which the scale factor has the value \( a_{2f} \). This is equivalent to replacing \( (1 + 2 |\beta_{\phi_k}|^2) \) by \( 2 |\beta_{Q_1}|^2 \) in Eq. (69), as was done to obtain Eq. (21). Thus, at late times

\[
\left| \beta_{\phi_k}^{(re)} \right|^2 = \frac{1}{2L^3a_{2f}^3\omega_{2f}} |\beta_k|^2. \tag{70}
\]

For intermediate values of \( q_2 \) in the massless case, it follows from Eq. (40) that at late times

\[
|\beta_{Q_2}|^2 \simeq \frac{1}{4} q_2^{-2}. \tag{71}
\]

We write the renormalized spectrum of \( \delta \phi(x) \) in terms of the modes of the renormalized expectation value of the dispersion (or variance), \( \langle 0 \mid (\delta \phi(x))^2 \mid 0 \rangle_{re} \), which are defined in Eq. (24). This gives the spectrum as

\[
P_{\delta \phi^{(re)}} = \left( \frac{L}{2\pi} \right)^3 4\pi k^3 \langle 0 \mid (\delta \phi_k^{(re)})^2 \mid 0 \rangle_{re} \]

\[
= \left( \frac{L}{2\pi} \right)^3 4\pi k^3 |\beta_{\phi_k}^{(re)}|^2. \tag{72}
\]

From Eqs. (70) and (71), we obtain, in the continuum limit, the late-time renormalized inflaton perturbation spectrum

\[
P_{\delta \phi^{(re)}} = \frac{H_{\text{inf}}^2}{8\pi^2}. \tag{73}
\]

From our discussion of Fig. 13 this result also holds in the massive case when \( m_H^{-2} \ll 1 \). Here, in the range where \( m^2 \gg k^2/a_{2f}^2 \), we have \( |\beta_{Q_2}|^2 \propto q_2^{-3} \). As can be seen in Fig. 13 this region is continuous with the massless case given by Eq. (71) for \( k^2/a_{2f}^2 \gg m^2 \). This continuity tells us that the region of \( |\beta_{Q_2}|^2 \) proportional to \( q_2^{-3} \) must follow

\[
|\beta_{Q_2}|^2 \simeq \frac{m_H}{4} q_2^{-3}, \tag{74}
\]

and thus

\[
P_{\delta \phi^{(re)}} = \frac{H_{\text{inf}}^2}{8\pi^2}, \tag{75}
\]

for both the massless case and the case where \( m_H^{-2} \ll 1 \). This is the result for the spectrum in the first scenario, in which only the late-time Minkowski vacuum subtractions are considered and the renormalization of the inflaton dispersion during the expansion of the universe is ignored.

However, by late-times it is usually assumed that the inflaton perturbations have already induced the scalar metric perturbations that will give rise to the initial scale-invariant plasma oscillations in the reheated universe. Because the universe is still expanding rapidly at the time that the relevant scalar metric perturbations are induced, one should consider the effect of renormalization of the inflaton perturbations in the rapidly expanding universe on the amplitude of the metric perturbations that they induce. Following [2, 4, 5], we take the scalar metric perturbations to be classical in nature shortly after they are induced. Therefore, in our second scenario, it is the renormalized spectrum of the quantized inflaton perturbation field shortly after their wavelengths exit the Hubble horizon during inflation that we use to determine the magnitude of the spectrum of the induced scalar metric perturbations. The latter propagate as classical quantities during the long remaining duration of inflation, and therefore their amplitudes remain almost constant up to the time that they induce acoustic plasma oscillations after reheating. The adiabatic subtractions coming from renormalization are significant at the time that the scalar perturbations of the metric are induced, and hence have an important influence on the magnitude of the metric
p perturbations and consequently on the CMB anisotropies and the large-scale structure observed today. Slow-roll inflation will alter the scale-invariant spectrum, as considered with the effect of renormalization in [3]. The effect of renormalization is found there to be important for interpreting the existing measurements of the CMB anisotropies.

Next, we consider the effect of renormalization on the amplitude of the spectrum of scalar metric perturbations induced by the quantized inflaton field during inflation. We give the results for both scenarios; the first, in which only the Minkowski subtractions are made and the late-time dispersion in the final Minkowski space is used; and the second, in which the renormalization is carried out near the time when the wavelength of the mode first exceeds the Hubble radius during inflation.

The curvature perturbations (or scalar perturbations) of the metric [23] are defined by

\[ R_k = -\frac{H}{\dot{\phi}} \left| \delta \phi_k^{(\text{ren})} \right|, \tag{76} \]

where \( \dot{\phi} \) represents \( \dot{\phi}^{(0)} \) in the present section. In [23], the possible effects of renormalization by adiabatic regularization or point-splitting were not considered, and the unrenormalized value \( \left| \delta \phi_k^{(\text{un})} \right| \) was used. However, here we will use the renormalized value, as it is clear that at late times one must subtract the Minkowski vacuum contribution. This subtraction is already included in adiabatic regularization (and in point-splitting) when applied to Minkowski space. However, when \( a \) or \( \dot{a} \) are nonzero, the simple Minkowski space subtraction is not enough to yield a finite value for the dispersion of the inflaton fluctuation field in the expanding universe. As noted in [2, 4, 8], the effect of renormalization in the expanding universe may have a significant effect on the magnitude of the inflaton dispersion spectrum. As discussed in the previous paragraph, we take the relevant time at which the magnitude of the inflaton perturbation field induces scalar metric perturbations to be shortly after horizon crossing. Therefore, in the second scenario we use the horizon crossing time as a characteristic time at which to evaluate the renormalized spectrum in Eq. (76). One will obtain a similar result as long as the characteristic time is taken within a few Hubble times after horizon crossing.

The spectrum, \( P_R = (L/2\pi)^3 4\pi k^3 R_k^2 \), of scalar metric perturbations in the continuum limit now takes the form

\[ P_R = \frac{H^2}{\phi^2} P_{\delta \phi^{(\text{ren})}}. \tag{77} \]

The difference between the two scenarios boils down to the time at which the renormalized dispersion is calculated.

So far, our parameterized scale factor has not been linked to any particular potential or model of inflation. In what comes next, we choose a simple potential that was found to be in good agreement with the 3-year WMAP data [11]. For our example, we use the quadratic chaotic-inflation potential [29]

\[ V = \frac{1}{2} m^2 \phi^2. \tag{78} \]

The two slow roll conditions are [1]

\[ H^2 \approx \frac{8\pi G}{3} V, \tag{79} \]

and

\[ \dot{\phi} \approx -\frac{dV/d\phi}{3H}. \tag{80} \]

We combine these two slow roll equations with the present quadratic potential to find

\[ \dot{\phi} \approx -m \sqrt{\frac{2}{3}} \frac{1}{\sqrt{8\pi G}}. \tag{81} \]

Then with the quadratic potential, in the first scenario, we find from Eqs. (75) and (77) with only the late-time Minkowski subtraction, that

\[ P_R = \frac{3}{16\pi^2 m_H^2} \left( \frac{H_{\text{infl}}}{2.436 \times 10^{18} \text{GeV}} \right)^2. \tag{82} \]

In the second scenario, we take account of the further subtractions necessary to renormalize the inflaton dispersion when the time-derivatives of \( a(t) \) cannot be neglected and obtain for the inflaton dispersion spectrum. We use the result given in [2] for the renormalized spectrum, \( P_{\delta \phi^{(\text{ren})}} \). As noted before, this should be evaluated shortly after the mode has exited the Hubble radius. For simplicity, we use the time of exit to characterize the order of magnitude one will obtain. This gives

\[ P_{\delta \phi^{(\text{ren})}} \approx \frac{H_{\text{infl}}^2}{32 \pi^2} \left( \frac{4\pi}{m_H^6 + 33 m_H^4 + 46 m_H^2 + 16} \right) \left( \frac{m_H^2 + 1}{17/2} \right). \tag{83} \]

from which one obtains,

\[ P_R \approx \frac{3}{64\pi^2 m_H^2} \left( \frac{H_{\text{infl}}}{2.436 \times 10^{18} \text{GeV}} \right)^2 \times \left( \frac{4\pi}{m_H^6 + 33 m_H^4 + 46 m_H^2 + 16} \right) \left( \frac{m_H^2 + 1}{17/2} \right). \tag{85} \]
After inflation has ended, at the time when once again \( k/(a(t)H(t)) = 1 \), the curvature perturbations reenter the Hubble sphere. At that reentry time, their amplitudes are related to the amplitudes, \( \delta p_k \), of the density perturbations by relations of the form \( \delta p_k = \delta \rho_k = \chi R_k \),

\[
\delta \rho_k = \frac{\chi}{\rho} = \delta_k = \chi R_k,
\]

where \( \chi \) is 2/5 if reentry occurs during the matter-dominated stage, and is 4/9 if reentry occurs during the radiation-dominated stage. We use the value 2/5 in the rest of this section. Then, in the first scenario, the spectrum of density perturbations created by the scalar metric perturbations at the time of reentry is

\[
\mathcal{P}_\delta = \left(\frac{2}{5}\right)^2 \mathcal{P}_R = \frac{3}{100 \pi^2 m_H^2} \left(\frac{H_{\text{inf}}}{2.436 \times 10^{18} \text{GeV}}\right)^2.
\]

In the second scenario, assuming that the magnitude of the scalar metric perturbations does not change much from the value it has at the time that the metric perturbations are induced to the time of reentry, we have

\[
\mathcal{P}_\delta \approx \frac{3}{400 \pi^2 m_H^2} \left(\frac{H_{\text{inf}}}{2.436 \times 10^{18} \text{GeV}}\right)^2 \times \left(4\pi H^{(1)}_{\text{infl}} \left(\frac{4}{q} - m_H^2\right) \right)^2 - \frac{8 m_H^2 + 33 m_H^4 + 46 m_H^2 + 16}{(m_H^2 + 1)^{7/2}}.
\]

See TABLE III for sample values of the density contrast, \( \mathcal{P}_\delta \), given by Eq. (87) in the first scenario, and see TABLE IV for sample values of the density contrast given by Eq. (88) in the second scenario. In both cases, the potential is the quadratic one of Eq. (78). The quantity \( \mathcal{P}_\delta \) in our table is related to the measured quantity denoted by \( \Delta^2_R \) in the WMAP five-year results [31] see their Table 2]. Their measurements give for the most likely value, \( \Delta^2_R = 2.41 \times 10^{-9} \). In our present paper, \( \Delta^2_R \) is denoted by \( \mathcal{P}_R \), and from the first line of Eq. (87), we obtain its observed value: \( \mathcal{P}_\delta = (2/5)^2 \mathcal{P}_R \approx 3.86 \times 10^{-10} \). We have chosen \( H \) to give values of \( \mathcal{P}_\delta \) in the first row of each table that are near the observed value of \( \mathcal{P}_\delta \).

A comparison of TABLE IV with TABLE III reveals that adiabatic regularization leads to a constant non-zero value of \( \mathcal{P}_\delta \) for small values of \( m_H \). Thus, for the quadratic potential, all the values of \( m_H \) shown for adiabatic regularization in TABLE IV give a value of \( \mathcal{P}_\delta \) that agrees with the observed value when \( H \approx 10^{15} \) GeV. In TABLE III we see that \( \mathcal{P}_\delta \) grows to as \( m_H \) decreases. For the quadratic potential it approaches infinity in the limit as \( m_H \to 0 \) without adiabatic regularization, but approaches a constant non-zero value when the full adiabatic regularization is used during the expansion. This can be confirmed by expanding the right-hand-side of Eq. (88) in powers of \( m_H \) and taking the limit as \( m_H \to 0 \). The fact that the value of \( \mathcal{P}_\delta \) is almost constant in TABLE III for all the values of \( m_H \leq 0.25 \) in each column, implies that one could predict the value of \( H_{\text{inf}} \) for this, and perhaps also for other inflaton potentials, when adiabatic regularization is taken into account.

### XII. REHEATING

When we maintain continuity of \( a(t), \dot{a}(t), \) and \( \ddot{a}(t) \), the particle number per

| \( H = 10^{16} \) GeV | \( H = 10^{17} \) GeV | \( H = 10^{18} \) GeV |
|------------------|------------------|------------------|
| \( m_H = 0.0001 \) | \( 4.60 \times 10^{-12} \) | \( 4.60 \times 10^{-10} \) | \( 4.60 \times 10^{-8} \) |
| \( m_H = 0.01 \) | \( 4.60 \times 10^{-12} \) | \( 4.60 \times 10^{-10} \) | \( 4.60 \times 10^{-8} \) |
| \( m_H = 0.1 \) | \( 4.64 \times 10^{-12} \) | \( 4.64 \times 10^{-10} \) | \( 4.64 \times 10^{-8} \) |
| \( m_H = 0.25 \) | \( 4.79 \times 10^{-12} \) | \( 4.79 \times 10^{-10} \) | \( 4.79 \times 10^{-8} \) |

When the end of inflation is gradual, we found (as seen in Fig. 3) that beyond \( q_2 \approx 1 \), the quantity \( |\beta_{q_2}|^2 \) falls off as \( q_2^{-6} \). For small \( m_H \), using the behavior of \( |\beta_{q_2}|^2 \) summarized in Eqs. (99), (100), and (102) for the case of a gradual end to inflation with \( a_{2f}/a_{1f} < 2 \), we find (by blue shifting the temperature at late times back in time) that at the time when the inflationary expansion is joined to the final asymptotically flat segment of the expansion, the energy-density has an effective temperature that is slightly larger than the Gibbons-Hawking temperature \( H_{\text{inf}}/2\pi \). Particle creation by the expanding universe was also investigated in [13] as a possible cause of reheating, with a similar result to the present one we find for a gradual end to inflation.

We next use our results to investigate the effective temperature if there is a fairly abrupt end to inflation. With a fairly abrupt end to inflation, for which \( a_{2f}/a_{1f} \approx 1 \), the range in \( q_2 \) where \( |\beta_{q_2}|^2 \propto q_2^{-6} \) only begins until a large value of \( q_2 \), which we denote by \( q_{2\text{cut-off}} \). From our numerical results, we find that \( q_{2\text{cut-off}} \approx a_{2f}/(a_{2f} - a_{1f}) \). We define the region between \( 1 \leq q_2 \leq q_{2\text{cut-off}} \) as the “stretched” region. In this “stretched” region, \( |\beta_{q_2}|^2 \propto q_2^{-2} \). As an example, see Fig. 5, where \( q_{2\text{cut-off}} \approx 10^4 \) and \( a_{2f}/(a_{2f} - a_{1f}) \approx 10^4 \). The “stretched” region is a result of particle production caused by the rapid fall-off in \( H(t) \) as inflation ends.

With sufficient stretching, the particle number per
mode, $|β_{q_2}|^2$, in the “stretched” region is proportional to $q_2^{-2}$ for any value of $m_H < 3/2$. In the intermediate-$q_2$ region, the behavior of $|β_{q_2}|^2$ is governed by the power $P$ defined in Eq. (58) and shown in Fig. 12. As is evident from Fig. 8 the stretched region extends over a much larger range of values of $q_2$ than do the intermediate- and small-$q_2$ regions. Because the $a(t)$ we defined is only $C^2$ across the joining points, $|β_{q_2}|^2$ falls off as $q_2^{-6}$ for $q_2 > q_{2\text{cut-off}}$. For an $a(t)$ that is $C^∞$ for all $t$, the rate of fall off for $q_2 > q_{2\text{cut-off}}$ would be much faster. Therefore, in our calculation below of the total energy density, we neglect the contribution to the energy density from this UV-range of $q_2$. (However, in the $C^2$ case, one can show that the contribution of this UV-range to the total energy-density is of the same magnitude as the contribution of the stretched region; so in that case the energy density would be twice the value we obtain below.)

Therefore, for a fairly abrupt transition with $a_{2f}/a_{1f} ≃ 1$, the total energy density is

$$\langle \frac{E}{V} \rangle \simeq \frac{1}{(2πa_{2f})^3} \int_{q_{2\text{cut-off}}}^{q_{2}} \frac{k}{a_{2f}} |β_k|^2 d^3k = \int_{1}^{q_{2\text{cut-off}}} \frac{q_2^3 |β_{q_2}|^2 H_{\text{inf}}^4}{2π^2} dq_2 = \int_{1}^{q_{2\text{cut-off}}} \frac{q_3 H_{\text{inf}}^4}{8π^2} dq_2.$$  

(89)

In the present case, $q_{2\text{cut-off}} ≃ (a_{2f}/[a_{2f} - a_{1f}]) \gg 1$, so we find

$$\langle \frac{E}{V} \rangle \simeq \frac{H_{\text{inf}}^4 (a_{2f}/a_{2f} - a_{1f})^2}{16π^2}.  \tag{90}$$

It follows that a fairly abrupt end to inflation can lead to energy densities that correspond to an effective temperature large with respect Gibbons-Hawking temperature. This would certainly be large enough to reheat the universe if some process caused a fairly rapid end to inflation. A more gradual end to inflation could also produce a temperature sufficient for reheating if the value of $H_{\text{inf}}$ were large enough, as pointed out in [13].

**XIII. CONCLUSIONS**

We have set up an exactly solvable model of exponential inflation joined to initial and final asymptotically flat expansions of the universe. There are eight adjustable parameters, three of which pertain to the initial asymptotically flat segment, three to the final asymptotically flat segment, and two to the inflationary stage of the expansion. In the case when the scale factor $a(t)$ is required to be continuous with continuous first and second derivatives at the joining times, there are three independent conditions on the parameters, leaving five adjustable parameters. Among these are the rate of inflation, $H_{\text{inf}}$, the number of e-foldings of inflation, $N_e$, as well as three parameters characterizing the asymptotic regions. The parameter $H_{\text{inf}}$ is constant in this model, and $N_e$ can be arbitrarily large. Our principal focus in this work has been to use this exactly solvable model to look for generic behaviors that have not been previously studied. In this way, we have found a number of new effects that may be relevant to the inflationary universe. We summarize these as follows.

We find that if the quantized inflaton fluctuation field is in its vacuum state in the early time asymptotic Minkowski space, then the inflaton fluctuations (which correspond to particles of a quantized scalar field) are created by the expanding universe in the manner thoroughly analyzed in [6, 7]. We show that the creation of these particles, or inflaton fluctuations, leads to a scale-invariant spectrum of created particles, or fluctuations, for the range of momenta or wavelengths that are relevant to the present universe when $N_e$ is sufficiently large, for example, larger than about 60. For an initial Minkowski vacuum, we find that for modes having wavelengths that are not so large that they leave the Hubble sphere within the first few e-foldings of inflation, the Minkowski vacuum state evolves into the Bunch-Davies de Sitter vacuum after a few e-folds of inflation. So the results for such modes of the inflaton fluctuation field should agree with those obtained by assuming the de Sitter vacuum, as is usually done in treatments of inflation.

If the total number of e-foldings of inflation prior to the exit of all the observationally relevant modes is not sufficiently long, then our results would imply that the longest wavelength perturbations that may come within the range of observation should have a spectrum that deviates in a certain way from scale invariance. This deviation is a result of the fact that the Minkowski vacuum has not had time to evolve to the de Sitter vacuum before the modes exit from the Hubble sphere.

We find that if there were perturbations (i.e., particles) present at early times, then their effects can be propagated all the way to the end of inflation, by means of “stimulated emission” from the vacuum. These perturbations present prior to the start of inflation would then have observable effects. This is an important new effect that we are considering in a paper [32] in preparation.

We investigated the spectrum of scalar perturbations of the metric that would be induced by the inflaton fluctuations, and we found the spectrum of initial density perturbation that these would produce after the end of inflation. We found a significant difference between the properties of the density perturbations that one would obtain by taking account (a) only of the vacuum subtraction in the final Minkowski space, as compared with (b) subtracting the full set of adiabatic regularization terms at a time within a few e-foldings after a mode ex-

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8 We are not incorporating the slow-roll parameters, which of course would modify the scale-invariance.
its the Hubble sphere during inflation, as in §3. A new result we find is that in case (b), the density perturbations have a non-zero value that is nearly independent of the inflaton mass for masses $m_H \lesssim 0.25$. This type of behavior may permit one to use our knowledge of the initial density perturbations to make a prediction about the value of $H$ during inflation. This mass-independence does not appear if one only uses the Minkowski space vacuum subtraction.

We also considered the energy density that would be created by the change of the scale factor $a(t)$ at the end of inflation. We found agreement with earlier treatments for a gradual end to inflation. We also were able to estimate the energy density created if there were a fairly abrupt, but smooth, end to inflation. The effective reheating temperature produced could be very high. We give the value of the created energy density as a function of a parameter characterizing the abruptness of the end to inflation.

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**APPENDIX A: BOUNDARY MATCHING CALCULATION**

Given the values of $\psi_{k1}$ and $\psi'_{k1}$, and the matching conditions

$$A h_1(t_1) + B h_2(t_1) = \psi_k(t_1) = \psi_{k1},$$

$$A h'_1(t_1) + B h'_2(t_1) = \psi'_k(t_1) = \psi'_{k1},$$

$$C g_1(t_2) + D g_2(t_2) = A h_1(t_2) + B h_2(t_2),$$

$$C g'_1(t_2) + D g'_2(t_2) = A h'_1(t_2) + B h'_2(t_2),$$

we wish to calculate the constant coefficients $C$ and $D$ in terms of the functions $h_1(t), h_2(t), g_1(t)$, and $g_2(t)$; and the values of $\psi_{k1}$, $\psi'_{k1}$, $t_1$, and $t_2$. (Here prime denotes derivative with respect to $t$.) Rearranging the first two matching conditions leads to

$$B = \left[ \frac{\psi_{k1} - A h_1}{h_2} \right]_{t=t_1}, \quad (A2)$$

$$A = \left[ \frac{\psi'_{k1} - B h'_2}{h'_1} \right]_{t=t_1}. \quad (A3)$$

Combining these two equations leads to

$$A = \left[ \frac{\psi'_{k1} h_2 - \psi_{k1} h'_2}{h'_1 h_2 - h_1 h'_2} \right]_{t=t_1},$$

$$B = \left[ \frac{\psi'_{k1} h_1 - \psi_{k1} h'_1}{h'_2 h_1 - h_2 h'_1} \right]_{t=t_1}. \quad (A3)$$

At the time, $t_2$, we have:

$$\psi_k(t_2) = A h_1(t_2) + B h_2(t_2)$$

$$= \left\{ \left[ \psi'_{k1} h_2 - \psi_{k1} h'_2 \right] h_1(t_2) \right. + \left[ \psi'_{k1} h_1 - \psi_{k1} h'_1 \right] h_2(t_2) \right\}, \quad (A4)$$

and

$$\psi'_k(t_2) = A h'_1(t_2) + B h'_2(t_2)$$

$$= \left\{ \left[ \psi'_{k1} h_2 - \psi_{k1} h'_2 \right] h'_1(t_2) \right. + \left[ \psi'_{k1} h_1 - \psi_{k1} h'_1 \right] h'_2(t_2) \right\}. \quad (A5)$$

Let us also define $\psi_{k2} \equiv \psi_k(t_2)$ and $\psi'_{k2} \equiv \psi'_k(t_2)$. In terms of $\psi_{k2}$ and $\psi'_{k2}$ the last two boundary conditions in Eq. (A1) become

$$C = \left. \frac{\psi'_{k2} g_2 - \psi_{k2} g'_2}{g_1 g_2 - g_2 g'_1} \right\}_{t=t_2}, \quad (A6)$$

$$D = \left. \frac{\psi'_{k2} g_2 - \psi_{k2} g'_2}{g_2 g_1 - g_2 g'_1} \right\}_{t=t_2}. \quad (A7)$$

Substituting for $\psi_{k2}$ and $\psi'_{k2}$ yields

$$C = \left. \frac{[A h'_1 + B h'_2] g_2 - [A h_1 + B h_2] g'_2}{g_1 g_2 - g_2 g'_1} \right\}_{t=t_2}, \quad (A8)$$

Finally, expressing $A$ and $B$ in terms of the given values of $\psi_{k1}$ and $\psi'_{k1}$ specified at $t_1$ leads to

$$C = \left. \frac{1}{(g'_2 g_2 - g_2 g'_1)} \right\}_{t=t_2} \times \left\{ \left[ \frac{\psi'_{k1} h_2 - \psi_{k1} h'_2}{h'_1 h_2 - h_1 h'_2} \right]_{t=t_1} (h'_4 g_2 - h_1 g'_2)_{t=t_2} \right. + \left[ \frac{\psi'_{k1} h_1 - \psi_{k1} h'_1}{h'_2 h_1 - h_2 h'_1} \right]_{t=t_1} (h'_2 g_2 - h_2 g'_2)_{t=t_2} \right\}, \quad (A9)$$

and

$$D = \left. \frac{1}{(g'_2 g_2 - g_2 g'_1)} \right\}_{t=t_2}. \quad (A10)$$

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9 At least for the quadratic inflaton potential that we considered, but probably also more generally
\[
\times \left\{ \begin{array}{c}
\left[ \frac{\psi'_k h_2 - \psi_k h'_2}{h'_2 h_2 - h_1 h'_1} \right]_{t=t_1} (h'_1 g_1 - h_1 g'_1)_{t=t_2} \\
\left[ \frac{\psi'_k h_1 - \psi_k h'_1}{h'_2 h_1 - h_2 h'_1} \right]_{t=t_1} (h'_2 g_1 - h_2 g'_1)_{t=t_2}
\end{array} \right\},
\]
(A9)

which are the combined joining conditions for \( \psi_k \) and \( \psi'_k \).

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