A Polya operator for automorphic L-functions

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Introduction

1 Schwartz-Bruhat functions and Haar measures

Let $k$ be a number field and let $M$ be a simple $k$-algebra with center $k$ of rank $n$, i.e. $\dim_k(M) = n^2$. Fix a $k$-involution $x \mapsto x^\tau$ on $M$, i.e. $(xy)^\tau = y^\tau x^\tau$ as well as $(x^\tau)^\tau = x$ and $x^\tau = x$ when $x \in k$. We will assume that $M$ and $\tau$ are defined over the ring of integers $\mathcal{O}$ of $k$. The standard example will be the algebra of $n \times n$ matrices $\text{Mat}_n(k)$ and $x^\tau = x^t$ the transposed matrix.

For any ring $R$ over $k$ let $M(R) = M \otimes_k R$ and let $G$ be the $k$-groupscheme of invertible elements, i.e. $G(R) = M(R)^\times$. For almost all places $v$ of $k$ we have $M(k_v) \cong \text{Mat}_n(k_v)$. In this case we say that $M$ splits at $v$. At any splitting place $v$ we assume a $\mathcal{O}_v$-isomorphism $M(k_v) \to \text{Mat}_n(k_v)$ fixed, where $\mathcal{O}_v$ is the local ring of integers.

Let $\mathbb{A} = \mathbb{A}_f \times \mathbb{A}_\infty$ be the ring of adeles of $k$ where $\mathbb{A}_f$ is the finite and $\mathbb{A}_\infty$ the infinite adeles. Let $dx$ denote the additive Haar-measure on $\mathbb{A}$ given
by \(dx = \otimes_v dx_v\), the product being extended over all places of \(k\), where at a finite place \(v\) we have for the valuation ring \(O_v \subset k_v\) that \(dx_v(O_v) = 1\). At the infinite places we will normalize the measures in a way such that the lattice \(k \subset \mathbb{A}\) has covolume 1, i.e. \(\text{vol}(\mathbb{A}/k) = 1\). We also write \(dx = \otimes_v dx_v\) for an additive Haar-measure on \(M(\mathbb{A})\), where \(dx_v\) is given componentwise by the above at each splitting place \(v\).

For \(a \in \mathbb{A}\) let \(|a|\) be its modulus, so \(dx(aA) = |a|dx(A)\). For \(x \in M(\mathbb{A})\) let \(|x| = |\text{det}(x)|\), where \(\text{det} : M \to k\) is the reduced norm. Note that \(\text{det}\) equals the determinant over each field splitting \(M\) and that \(\text{det}(x^\tau) = \text{det}(x)\) for any \(x\). Let \(dx^\times = \otimes_v dx_v^\times\) be the Haar measure on \(G(\mathbb{A})\) given by \(dx_v^\times(G(O_v)) = 1\) for \(v\) finite and \(dx_v^\times = \frac{dx_v}{|x_v|}\) at the infinite places. The measure

\[
|x|^n dx^\times = \bigotimes_{v < \infty} |x_v|^n dx_v^\times \otimes \bigotimes_{v | \infty} dx_v
\]

is translation invariant on \(M(\mathbb{A})\) but it is not a Haar measure since it is infinity on any open set.

Let \(N(\mathbb{A})\) be the set of all \(m \in M(\mathbb{A})\) with \(\text{det}(m) = 0\). Then for \(y \in M(k)\) and \(m \notin N(\mathbb{A})\) we have \(ym \neq m\). Let \(\mathcal{S}(M(\mathbb{A}))\) be the space of Schwartz-Bruhat functions on \(M(\mathbb{A})\), that is, any \(f \in \mathcal{S}(N(\mathbb{A}))\) is a finite sum of functions of the form \(f = \otimes_v f_v\), where \(f_v\) is the characteristic function of the set \(O_v\) for almost all \(v\) and \(f_v \in \mathcal{S}(M(k_v))\) at all places, where \(\mathcal{S}(M(k_v))\) is the usual Schwartz-Bruhat space if \(v\) is infinite and is the space of locally constant functions of compact support if \(v\) is finite.

To define Fourier transforms we will fix an additive character \(\psi\) as follows. At first assume \(k = \mathbb{Q}\), then \(\psi = (\prod_p \psi_p)\psi_\infty\) with \(\psi_p(\mathbb{Z}_p) = 1\), \(\psi_p(p^{-n}) = e^{2\pi i / p^n}\), and \(\psi_\infty(x) = e^{2\pi i x}\). For general \(k\) note that the trace map \(Tr_{k/\mathbb{Q}} : k \to \mathbb{Q}\) induces a trace \(Tr : \mathbb{A}_k \to \mathbb{A}_\mathbb{Q}\) and let \(\psi_k := \psi_\mathbb{Q} \circ Tr\). The character \(\psi\) identifies \(\mathbb{A}\) with its dual via the pairing \(\langle x, y \rangle = \psi(xy)\). Note that \(\psi\) is chosen in a way that the lattice \(k \subset \mathbb{A}\) is its own dual, i.e.

\[
\langle x, y \rangle = 1 \quad \forall y \in k \quad \iff \quad x \in k.
\]

For \(f \in \mathcal{S}(\mathbb{A})\) its Fourier transform is defined by

\[
\hat{f}(x) = \int_{\mathbb{A}} f(y)\psi(xy)dy.
\]

We lift these notions to \(M(\mathbb{A})\). Let \(\psi_M : M(\mathbb{A}) \to \mathbb{C}\) be defined by \(\psi_M(x) = \psi(\text{Tr}_{M/k}(x))\). Then \(\psi_M\) sets \(M(\mathbb{A})\) in self duality and \(M(k)\) is a self dual
lattice. The Fourier transform for \( f \in \mathcal{S}(M(A)) \) is

\[
\hat{f} = \int_{M(A)} f(y) \psi_M(xy) dy.
\]

Let \( \mathcal{S}_0 = \mathcal{S}(M(A))_0 \) be the space of all \( f \in \mathcal{S}(M(A)) \) such that \( f \) and \( \hat{f} \) send \( N(A) \) to zero.

2 Local factors

Let \( K \) be a nonarchimedean local field with ring of integers \( \mathcal{O} \). We assume \( K \supset k \) and \( M \) splits over \( K \). So in this section read \( M(K) = \text{Mat}_n(K) \) and \( G(K) = \text{GL}_n(K) \). Let \( \varpi \) be a uniformizing element and let \( |\cdot|: K \to \mathbb{R}_{\geq 0} \) be the absolute value normalized so that \( |\varpi| = q^{-1} \), where \( q \) is the number of elements of the residue class field \( \mathcal{O}/\varpi \mathcal{O} \).

Let \( \pi \) be an irreducible admissible Hilbert representation of \( G(K) \), that is, \( \pi \) is a continuous representation of \( G(K) \) on a Hilbert space \( V_\pi \) with \( \dim V_\pi < \infty \) for any open subgroup \( U \) of \( G(K) \), where \( V_\pi \) is the set of vectors fixed by any element of \( U \).

Suppose \( \pi \) is a class one representation, i.e. \( \pi \) has nonzero fixed vectors under the maximal compact subgroup \( G(\mathcal{O}) \) of \( G(K) \). The Hecke algebra \( \mathcal{H} = \mathcal{H}(G(K), G(\mathcal{O})) \) is the convolution algebra of all compactly supported functions \( f: G(K) \to \mathbb{C} \) which are bi-invariant under \( G(\mathcal{O}) \), that is \( f(xk) = f(kx) = f(x) \) for all \( k \in G(\mathcal{O}) \).

Let \( A \subset G(K) \) be the subgroup of diagonal elements and \( N \subset G(K) \) be the subgroup of upper triangular matrices with ones on the diagonal. Then \( B = AN \) is the Borel subgroup of upper triangular matrices. For \( a = \text{diag}(a_1, \ldots, a_n) \in A \) let

\[
\delta(a) := |\text{det}(a|\text{Lie } N)| = |a_1|^{n-1} |a_2|^{n-3} \cdots |a_n|^{-(n-1)}
\]

the modulus of \( B \). Let \( dn \) be the Haar measure on \( N \) normalized so that \( \text{vol}(N \cap G(\mathcal{O})) = 1 \). The Satake transform (see [1], p. 146)

\[
Sf(a) := \delta(a)^{1/2} \int_N f(an)dn
\]

gives an isomorphism \( \mathcal{H} \to C[\lambda]^W \), where \( \lambda = A/A \cap G(\mathcal{O}) \) and \( C[\lambda] \) is the convolution algebra (group algebra) of all finitely supported functions on \( \lambda \). Finally \( W \) is the Weyl group of all permutations of the diagonal entries of
A. The set of characters \( \text{Hom}(\lambda, \mathbb{C}^*) \) forms the complex points of a torus \( T \subset \text{GL}_n(\mathbb{C}) \). The Hecke algebra \( \mathcal{H} \) acts on the one dimensional space \( V_{\pi}^{G(\mathcal{O})} \) through a character \( \chi_\pi \) which by the Satake isomorphism is given by an element of \( T \). The \textbf{local L-factor} of \( \pi \) is defined by
\[
L(\pi) = \det(1 - \chi_\pi)^{-1}.
\]
We will need to make this more explicit. Let \( \omega_j = \text{diag}(1, \ldots, \omega, \ldots, 1) \) (the \( \omega \) on the \( j \)-th position. The Satake isomorphism gives a bijection
\[
\text{Hom}_{\text{alg}}(\mathcal{H}, \mathbb{C}) \rightarrow \text{Hom}(\lambda, \mathbb{C}^*)/W
\]
and using these terms the local factor is given by
\[
L(\pi)^{-1} = \det(1 - \chi_\pi) = \prod_j (1 - \chi_\pi(\omega_j)).
\]
Let \( e \in \mathbb{C}[\lambda]^W \) be given by \( e = \prod_j (1 - \omega_j) \). Then there exists a unique \( f \in \mathcal{H} \) with \( Sf = e \) and for this \( f \) we have
\[
\text{tr}\pi(f) = L(\pi)^{-1}.
\]
But also \( L(\pi) \) itself rather than its inverse can sometimes be given as a trace. Let \( |\pi| := \max_j(\chi_\pi(\omega_j)) \). The characteristic function \( 1_{M(\mathcal{O})} \) restricted to \( G(K) \) does not lie in \( \mathcal{H} \) since it is not compactly supported but it can be written as an infinite sum of elements of \( \mathcal{H} \). The same applies to the function \( 1_{M(\mathcal{O})}|x|^{-\frac{n-1}{2}} \).

**Proposition 2.1** If \( |\pi| < 1 \) then \( \pi(1_{M(\mathcal{O})}|x|^{-\frac{n-1}{2}}) \) exists and is of trace class with
\[
\text{tr}\pi(1_{M(\mathcal{O})}|x|^{-\frac{n-1}{2}}) = L(\pi).
\]

**Proof:** We compute the Satake transform of \( f(x) = 1_{M(\mathcal{O})}(x)|x|^{-\frac{n-1}{2}} \).

At first note that \( f(an) \neq 0 \) implies \( a \in A \cap M(\mathcal{O}) \), that is \( |a_j| \leq 1 \) for \( j = 1, \ldots, n \). For such an \( a \) we substitute \( y_k = a_jn_{j,k} \), where \( n_{j,k} \) is the corresponding entry of \( n \). Since \( \int_N f(n)dn = 1 \) this gives
\[
\int_N f(an)dn = |a|^{\frac{n-1}{2}}|a_1|^{-(n-1)}|a_2|^{-(n-2)} \cdots |a_{n-1}|^{-1}.
\]
And so
\[
Sf(a) = \delta(a)^{\frac{1}{2}} \int_N f(an)dn
= |a|^{-\frac{n}{2}} |a_1a_2...a_n|^{-\frac{n-1}{2}}
= 1.
\]

So that \( Sf = 1_{A \cap M(O)} \).

The condition \( |\pi| < 1 \) implies that
\[
L(\pi) = \det(1 - \chi_{\pi})^{-1}
= \prod_j (1 - \chi_{\pi}(\omega_j))^{-1}
= \prod_j \sum_{k=0}^{\infty} \chi_{\pi}(\omega_j)^k
= \chi_{\pi} \left( \prod_j \sum_{k=0}^{\infty} \omega_j^k \right)
= \chi_{\pi} \left( 1_{A \cap M(O)} \right).
\]

The unramified character \( g \mapsto |g|^s \) for some \( s \in \mathbb{C} \) is a class one admissible representation of \( G \) and so is \( \pi_s = |\cdot|^s \pi : g \mapsto |g|^s \pi(g) \). We compute
\[
\chi_{\pi_s} = q^{-s} \chi_{\pi}
\]
and
\[
|\pi_s| = q^{-\text{Re}(s)}|\pi|,
\]
so that we have the

**Corollary 2.2** For any irreducible admissible \( \pi \) and \( s \in \mathbb{C} \) with \( |\pi| < q^\text{Re}(s) \) it holds
\[
L(\pi, s) := L(\pi_s)
= \text{tr} |\cdot|^s \pi \left( 1_{M(O)}(x)|x|^{\frac{n-1}{2}} \right)
= \text{tr} \pi \left( 1_{M(O)}(x)|x|^{s+\frac{n-1}{2}} \right).
\]

\( \square \)
3 Global L-functions

For \( f \in S(M(\mathbb{A}))_0 \) let \( f^\tau(x) := f(x^\tau) \). Let then \( E(f) : G(\mathbb{A}) \to \mathbb{C} \) be defined by

\[
E(f) = |x|^{\frac{n}{2}} \sum_{\gamma \in G(k)} f(\gamma x) = |x|^{\frac{n}{2}} \sum_{\gamma \in M(k)} f(\gamma x).
\]

Note that \( f \in S(M(\mathbb{A}))_0 \) implies that \( f \) vanishes on the set \((M(k) - G(k))M(\mathbb{A})\). Therefore it doesn’t matter whether the sum is extended over \( G(k) \) or \( M(k) \).

**Lemma 3.1** For any \( f \in S(M(\mathbb{A}))_0 \) the sum \( E(f)(x) \) converges locally uniformly in \( x \) with all derivatives. For \( x \in G(\mathbb{A}) \) it holds

\[
E(f)(x) = E(\hat{f}^\tau)(x^{-\tau}),
\]

where \( x^{-\tau} = (x^{-1})^\tau = (x^\tau)^{-1} \). Finally for any \( n \in \mathbb{N} \) there is a \( C > 0 \) such that

\[
|E(f)(x)| \leq C \min(|x|, \frac{1}{|x|})^n.
\]

**Proof:** With \( f \) any derivative of \( f \) is again in \( S(M(\mathbb{A})) \) so we only need to check convergence for \( f \) itself. Now \( M(k) \) is a lattice in \( M(\mathbb{A}) \) and so is \( M(k)x \) for any \( x \in G(\mathbb{A}) \). This lattice further depends continuously on \( x \), which gives the claim.

For any lattice \( \Gamma \subset M(\mathbb{A}) \) and any \( f \in S(M(\mathbb{A})) \) the Poisson summation formula says

\[
\text{covol}(\Gamma) \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{\eta \in \Gamma^\perp} \hat{f}(\eta),
\]

where \( \Gamma^\perp \) is the dual lattice. Let \( x \in G(\mathbb{A}) \), then \( M(k)x \) still is a lattice in \( M(\mathbb{A}) \) and its dual is \( x^{-1}M(k) \). Further, the covolume of \( M(k)x \) is \( |x|^n \) so that

\[
|x|^n \sum_{\gamma \in M(k)} f(\gamma x) = \sum_{\Gamma \in M(k)} \hat{f}(x^{-1}\gamma)
\]

\[
= \sum_{\Gamma \in M(k)} \hat{f}^\tau(\gamma x^{-\tau}).
\]
Now it remains to control the growth of $E(f)$ and by the last assertion it suffices to control the growth when $|x|$ is large. We may assume that $f$ is of the form $f = f_{\text{fin}} \otimes f_\infty$ with $f_{\text{fin}} \in \mathcal{S}(M(\mathbb{A}_{\text{fin}}))$ and $f_\infty \in \mathcal{S}(M(\mathbb{A}_\infty))$. The function $f_{\text{fin}}$ has compact support, say $C \subset M(\mathbb{A}_{\text{fin}})$. Shifting factors we may assume that $|f_{\text{fin}}| \leq 1_C$ and so we can estimate

$$|E(f)(x)| \leq |x|^{\frac{n}{2}} \sum_{\gamma \in G(k)} |f(\gamma x)|$$

$$\leq |x|^{\frac{n}{2}} \sum_{\gamma \in G(k) \cap C x_{\text{fin}}^{-1}} |f_\infty(\gamma x_\infty)|.$$
Let $G(\mathbb{A})^1$ be the kernel of the map $g \mapsto |g|$. The exact sequence

$$1 \to G(\mathbb{A})^1 \to G(\mathbb{A}) \to \mathbb{R}_+^\times \to 1$$

splits. Moreover, there is splittings which make $G(\mathbb{A})$ a direct product of $G(\mathbb{A})^1$ and $\mathbb{R}_+^\times$. For example fix a place $v_0|\infty$ and let $s : \mathbb{R}_+^\times \to G(\mathbb{A})$ be given by $s(t)_v = 1$ if $v \neq v_0$ and $s(t)_{v_0} = t^{1/2} 1$ then the map $(i, s) : G(\mathbb{A})^1 \times \mathbb{R}_+^\times \to G(\mathbb{A})$ is an isomorphism. We will fix such a splitting from now on. For $x \in G(\mathbb{A})$ we will write $x = (x^1, |x|)$ in these coordinates.

The group $G(k)$ is a lattice in $G(\mathbb{A})^1$. A representation $\pi$ of $G(\mathbb{A})^1$ is called automorphic if it occurs as a subrepresentation in $L^2(G(k) \backslash G(\mathbb{A})^1)$. With respect to our fixed splitting we are able to lift $\pi$ to a representation of $G(\mathbb{A})$ by $\pi(x^1, |x|) := \pi(x^1)$ and we will thus consider it as a representation of $G(\mathbb{A})$. Then $\pi$ decomposes into an infinite tensor product $\pi = \otimes_v \pi_v$. We will write $Z$ for the image of the splitting map $s$ and we will thus identify $L^2(G(k) \backslash G(\mathbb{A})^1)$ to $L^2(ZG(k) \backslash G(\mathbb{A}))$. Write $R$ for the representation of $G(\mathbb{A})$ on the latter. That is, for $\varphi \in L^2(ZG(k) \backslash G(\mathbb{A}))$ and $y \in G(\mathbb{A})$ we write $R(y)\varphi(x) = \varphi(xy)$.

Now assume $\pi$ is automorphic and fix a $G(\mathbb{A})^1$-homomorphism $V_\pi \to L^2(G(k) \backslash G(\mathbb{A})^1)$. Let $\varphi \in L^2(ZG(k) \backslash G(\mathbb{A}))$ be the image of a vector $\alpha = \otimes_v \alpha_v \in V_\pi = \otimes_v V_{\pi,v}$ such that $\alpha_v$ is a normalized class one vector at almost all places. Further assume that $\varphi$ is smooth and $\varphi(1) \neq 0$. The latter can be achieved by replacing $\varphi(x)$ with $\varphi(xy)$ for a suitable $y$ if necessary.

For any set of places $S$ let $\mathbb{A}_S$ be the restricted product of $k_v, v \in S$ and let $G_S = G(\mathbb{A}_S)$. We consider $G_S$ as a subgroup of $G(\mathbb{A})$.

**Lemma 3.3** Let $f \in S(M(\mathbb{A}))_0$ be of the form $f_S \otimes f^S$ for a finite set of places $S$, where $f_S \in S(M(\mathbb{A}_S))$ and $f^S = \prod_{v \notin S} 1_{M(\mathbb{Q}_v)}$. Take $S$ so large that $\varphi$ is of class one outside $S$, then for $\text{Re}(s) > 0$ we have

$$\int_{G(k) \backslash G(\mathbb{A})} E(f)(x)\varphi(x)|x|^s d^* x = L_S(\pi, s + \frac{1}{2}) \int_{G_S} f(x)\varphi(x)|x|^{s+n/2} d^* x.$$

The left hand side is entire in $s$. The second factor on the right hand side is entire if we assume that $f_S$ has compact support in $G_S$. The $L$-function is entire $\Xi$], hence the identity holds for all $s \in \mathbb{C}$.

**Remark.** The holomorphicity of the $L$-function is in $\Xi]$ only given for cuspidal ones. The formula of the Lemma gives it for all automorphic $L$-functions.
Proof: For \( \text{Re}(s) >> 0 \) we compute that

\[
\int_{G(k) \backslash G(\mathbb{A})} E(f)(x) \varphi(x) |x|^s d^* x
\]
equals

\[
\int_{G(\mathbb{A})} f(x) |x|^{\frac{n}{2} + s} \varphi(x) d^* x = \int_{G(\mathbb{A})} f(x) |x|^{\frac{n}{2} + s} R(x) \varphi(1) d^* x
\]

\[
= \prod_{v \notin S} L(\pi_v, s + \frac{1}{2}) \int_{G_S} f(x) |x|^{\frac{n}{2} + s} \varphi(x) d^* x.
\]

For the justification of this computation note that for \( \text{Re}(s) >> 0 \) the integral over \( G(\mathbb{A}) \) converges absolutely.

In the notations of the lemma let

\[
\Delta_{S, \varphi, s}(f) := \int_{G_S} f(x) \varphi(x) |x|^{s + \frac{n}{2}} d^* x.
\]

For \( \delta > 0 \) let \( L^2_\delta(G(k) \backslash G(\mathbb{A})) \) denote the space of measurable functions \( f \) on \( G(k) \backslash G(\mathbb{A}) \) with

\[
\int_{G(k) \backslash G(\mathbb{A})} |f(x)|^2 \left( 1 + (\log |x|)^2 \right)^{\delta/2} d^* x < \infty
\]
modulo nullfunctions. The sum \( E \) defines a linear map \( E : S(M(\mathbb{A}))_0 \rightarrow L^2_\delta(G(k) \backslash G(\mathbb{A})) \). The group \( G(\mathbb{A}) \) acts on both sides by right translation \( R \) and it is easy to see that for any \( y \in G(\mathbb{A}) \)

\[
ER(y) = |y|^{-n/2} R(y) E,
\]

so that the image of \( E \) is an invariant subspace.

The pairing

\[
(f, g) = \int_{G(k) \backslash G(\mathbb{A})} f(x) g(x) d^* x
\]
identifies \( L^2_\delta(G(k) \backslash G(\mathbb{A})) \) to the dual space of \( L_\delta(G(k) \backslash G(\mathbb{A})) \). The space \( L^2_\delta(G(k) \backslash G(\mathbb{A})) \) can be viewed as Hilbert space tensor product

\[
L^2(G(k) \backslash G(\mathbb{A}))^1 \otimes L^2_\delta(\mathbb{R}),
\]
where \( L^2_\delta(\mathbb{R}) \) is the Fourier transform of the \( \delta \)-Sobolev space, i.e. the space of all functions \( f \) on \( \mathbb{R} \) with \( \int_\mathbb{R} |f(x)|^2 (1 + x^2)^{\delta/2} dx < \infty \).
For an irreducible unitary representation $\pi$ of $G(\mathbb{A})^1$ let

$$L^2(G(k)\backslash G(\mathbb{A}))(\pi)$$

denote the isotypical component in $L^2(G(k)\backslash G(\mathbb{A}))(\pi)$ and let

$$L^2_G(G(k)\backslash G(\mathbb{A}))(\pi) = L^2(G(k)\backslash G(\mathbb{A}))(\pi) \otimes L^2_0(\mathbb{R})$$

be its isotype in $L^2(G(k)\backslash G(\mathbb{A}))$. Let $\tilde{\pi}$ be the dual representation then the space $L^2(G(k)\backslash G(\mathbb{A}))(\tilde{\pi})$ is dual to $L^2_G(G(k)\backslash G(\mathbb{A}))(\pi)$. Let $\tilde{\varphi} = \varphi \otimes \psi \in L^2_\delta(G(k)\backslash G(\mathbb{A}))$, then $\tilde{\varphi}$ is orthogonal to the image of $E$ if and only if

$$\int_{G(k)\backslash G(\mathbb{A})} E(f)(x) \varphi(x) \psi(\log |x|) d^* x = 0.$$ 

We consider $\psi$ as a distribution and write formally $\psi(\log |x|) = \int_{\mathbb{R}} \hat{\psi}(t) |x|^{it} dt$. The above becomes

$$\int_{G(k)\backslash G(\mathbb{A})} \int_{\mathbb{R}} E(f)(x) \varphi(x) \hat{\psi}(t) |x|^{it} dtd^* x$$

which equals

$$\int_{\mathbb{R}} L_S(\pi, \frac{1}{2} + it) \hat{\psi}(t) \Delta_{S, \varphi, it}(f) dt$$

if we plug in functions $f$ as in Lemma 3.3. To justify this computation note that, since we are interested in the orthogonal space of an invariant space we can assume $R(h)\varphi = \varphi$ for some $h$ such that $\hat{h}$ has compact support. Then $\hat{\psi}$ has compact support in $\mathbb{R}$.

At the finite places in $S$ let $f_v(x) = 1_{G(O_v)} \varphi_v(x)$. Then there is a constant $c > 0$ such that

$$\Delta_{S, \varphi, it}(f) = c \int_{G(\mathbb{A}_\infty)} f_\infty(x) \varphi(x) |x|^{it + \frac{n}{2}} d^* x.$$ 

We can choose $\varphi$ so that there is a finite place $v$ with $R(G(O_v))\varphi = 0$ which implies that $f_\infty$ can be chosen arbitrarily and still $f \in S(M(\mathbb{A})).$ If we let run $f_\infty$ through an approximate identity at the unit element we get that the distribution $L_S(\pi, \frac{1}{2} + it) \hat{\psi}(t)$ on $\mathbb{R}$ is zero, i.e.

$$L_S(\pi, \frac{1}{2} + it) \hat{\psi}(t) = 0. \quad (1)$$
Now $\psi \in L^2_{\delta}$ implies that its Fourier transform is a distribution of order $< \delta - 1$. For this recall that for $k \in \mathbb{N}$ the $k$-th derivative of the Dirac distribution $\delta^{(k)}(h) = h^{(k)}(0)$ has Fourier transform $(ix)^k$ which lies in $L^2_{\delta - \delta - 1}(\mathbb{R})$ if and only if $k < \delta - 1$. Therefore equation (I) is satisfied precisely for $\psi$ being a linear combination of $\delta^{(k)}(t)$ where $k < \delta - 1$ and $\frac{1}{2} + it$ is a zero of $L(\pi)$ of order $> k$. We will interprete this as a spectral decomposition of the $t$-multiplication. More precisely let $H_\pi$ be the Hausdorff cokernel of the composite map

$$ S(M(\mathbb{A}))_0 \to L^2_{\delta}(G(k) \backslash G(\mathbb{A})) \to L^2_{\delta}(G(k) \backslash G(\mathbb{A}))(\pi). $$

Then $H_\pi$ is isomorphic to the orthogonal space of the image of $E$ in the dual space. On $H_\pi$ we have the operator $D_\pi$ given by

$$ D_\pi \xi = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} (R(s(e^{\epsilon}')) - 1) \xi, $$

where $s : \mathbb{R}^\times_+ \to G(\mathbb{A})$ is our fixed section. The domain $\text{dom } D_\pi$ is the set of $\xi$ for which the limit exists. A computation shows that

$$ \| R(g) \|_{\delta} \leq 2^{\delta/4} (1 + (\log |g^{-1}|)^2)^{\delta/4}, $$

so that $D_\pi$, which is the infinitesimal generator of $R(s(e^t))$, has purely imaginary spectrum. Its resolvent $R_k = (D_\pi - k)^{-1}$ is given for $\text{Re}(k) > 0$ by

$$ R_k = -\int_0^\infty R_\pi(s(e^t))e^{-kt}dt $$

and for $\text{Re}(k) < 0$ by

$$ R_k = -\int_0^\infty R_\pi(s(e^{-t}))e^{kt}dt. $$

The operator $D_\pi$ is closed since $v_n \to v$ and $D_\pi v_n \to y$ implies $v = R_k y - kR_k v$ lies in $\text{im } R_k = \text{dom } D_\pi$. The operator $D_\pi$ commutes with the action of $G(\mathbb{A})^1$. Therefore its eigenspaces are $G(\mathbb{A})^1$-modules, in fact, are multiples of $\pi$. Let $m(\pi)$ be the multiplicity of $\pi$ in $L^2(G(k) \backslash G(\mathbb{A}))$. Now recall that there is a definition [3] for $L(\pi, s)$ in the global case by attaching factors at the ramified places in a way that the zero or poles of $L(\pi, s)$ along the line $\text{Re}(s) = \frac{1}{2}$ coincide with those of $L_S(\pi, s)$. We have proven
Theorem 3.4 For $\delta > 1$ the operator $D_\pi$ has discrete spectrum in $i\mathbb{R}$ consisting of all $ip \in i\mathbb{R}$ such that $\frac{1}{2} + ip$ is a zero of $L(\pi, \cdot)$. The eigenspace at $\rho$ is as a $G(\mathbb{A}^1)$-module a multiple of $\pi$ of multiplicity $m(\pi)n(\rho)$, where $n(\rho)$ is the largest integer $n < \delta - 1$ such that $n < \text{multiplicity of the zero } L(\pi, \frac{1}{2} + ip)$.

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