Q-operator and $T - Q$ relation from the fusion hierarchy

Wen-Li Yang, $^{a,b}$ Rafael I. Nepomechie$^c$ and Yao-Zhong Zhang$^b$

$^a$ Institute of Modern Physics, Northwest University, Xian 710069, P.R. China
$^b$ Department of Mathematics, University of Queensland, Brisbane, QLD 4072, Australia
$^c$ Physics Department, P.O. Box 248046, University of Miami, Coral Gables, FL 33124, USA

Abstract

We propose that the Baxter $Q$-operator for the spin-1/2 XXZ quantum spin chain is given by the $j \to \infty$ limit of the transfer matrix with spin-$j$ (i.e., $(2j + 1)$-dimensional) auxiliary space. Applying this observation to the open chain with general (nondiagonal) integrable boundary terms, we obtain from the fusion hierarchy the $T-Q$ relation for generic values (i.e. not roots of unity) of the bulk anisotropy parameter. We use this relation to determine the Bethe Ansatz solution of the eigenvalues of the fundamental transfer matrix. This approach is complementary to the one used recently to solve the same model for the roots of unity case.

PACS: 03.65.Fd; 04.20.Jb; 05.30.-d; 75.10.Jm
Keywords: Spin chain; reflection equation; Bethe Ansatz; Q-operator; fusion hierarchy
1 Introduction

The Baxter $Q$-operator is a fundamental object in the theory of exactly solvable models [1]. Nevertheless, it has been an enigma. Indeed, while the transfer matrix has a systematic construction in terms of solutions of the Yang-Baxter equation, the $Q$-operator’s original construction – its brilliance notwithstanding – was ad hoc. In particular, the $Q$-operator seemed to be absent from the quantum inverse scattering method (QISM). It was later understood [2, 3] that the $Q$-operator could be realized by a transfer matrix whose associated auxiliary space is infinite dimensional. However, its relation to the QISM remained unclear.

Motivated in part by [2, 3], we propose here that the $Q$-operator $\bar{Q}(u)$ for a spin-$1/2$ XXZ quantum spin chain is given by the $j \to \infty$ limit of the transfer matrix $t^{(j)}(u)$ with spin-$j$ (i.e., $(2j + 1)$-dimensional) auxiliary space,

$$\bar{Q}(u) = \lim_{j \to \infty} t^{(j)}(u - 2j\eta), \quad (1.1)$$

where $\eta$ is the anisotropy parameter. This relation makes it clear that the $Q$-operator does in fact fit naturally within the QISM. Moreover, this relation together with the fusion hierarchy for the closed-chain transfer matrix [4, 5, 6, 7]

$$t^{(j)}(u) \bar{Q}(u) = \sinh^N(u + \eta) \sinh^N(u - \eta) \bar{Q}(u - \eta) + \bar{Q}(u + \eta), \quad (1.3)$$

immediately leads to the Baxter $T-Q$ relation

$$t^{(1/2)}(u) \bar{Q}(u) = \sinh^N(u + \eta) \sinh^N(u - \eta) \bar{Q}(u - \eta) + \bar{Q}(u + \eta), \quad (1.3)$$

from which it is possible to derive the well-known expression for the eigenvalues of the fundamental transfer matrix $t^{(1/2)}(u)$ and the associated Bethe Ansatz equations. However, we emphasize that the above argument is formal: we assume without proof that the limit in (1.1) exists, and we do not evaluate the right-hand-side explicitly.

It is interesting to apply this observation to the open spin-$1/2$ XXZ quantum spin chain with general integrable boundary terms [8, 9]. Indeed, this model remains unsolved, although the special case of diagonal boundary terms was solved long ago [10, 11, 12]. Significant progress has been made recently for the case of nondiagonal boundary terms where the boundary parameters obey some constraints. One approach [13] (see also [14]) is based on
the generalized algebraic Bethe Ansatz [15, 16]. A second approach, which was developed in [17], exploits functional relations obeyed by the transfer matrix at roots of unity to obtain the eigenvalues of the transfer matrix. These functional relations are a consequence of the truncation of the fusion hierarchy of the transfer matrix at roots of unity [19, 20].

In this paper, we develop a third approach, which is complementary to the second one [17]. Indeed, as in [17], we make use of the fusion hierarchy for the open XXZ chain [21, 22]. However, here we consider instead generic values (i.e. not roots of unity) of the bulk anisotropy parameter, for which the fusion hierarchy does not truncate. We instead use the relation (1.1) to obtain the T-Q relation. We then use the latter relation, together with some additional properties of the transfer matrix, to determine the eigenvalues of the transfer matrix and the associated Bethe Ansatz equations. The expressions for the eigenvalues are generalizations of those found in [17], and this new derivation explains their validity for generic anisotropy values [18].

The paper is organized as follows. In Section 2, we introduce our notation and some basic ingredients. In Section 3, we derive the T-Q relation from (1.1) and the fusion hierarchy of the open XXZ chain. Through that relation, we determine the eigenvalues of the transfer matrix and the associated Bethe Ansatz equations in Section 4. Finally, we summarize our conclusions and mention some interesting open problems in Section 5.

2 Transfer matrix

Throughout, let us fix a generic complex number \( \eta \), and let \( \sigma^x, \sigma^y, \sigma^z \) be the usual Pauli matrices. The well-known six-vertex model R-matrix \( R(u) \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \) is given by

\[
R(u) = \begin{pmatrix}
\sinh(u + \eta) & \sinh(u) & \sinh(\eta) \\
\sinh(u) & \sinh(\eta) & \sinh(u) \\
\sinh(\eta) & \sinh(u) & \sinh(u + \eta)
\end{pmatrix}.
\] (2.1)

Here \( u \) is the spectral parameter and \( \eta \) is the so-called bulk anisotropy parameter. The R-matrix satisfies the quantum Yang-Baxter equation and the properties,

\[
\text{Unitarity relation : } R_{1,2}(u)R_{2,1}(-u) = -\xi(u) \text{id}, \quad \xi(u) = \sinh(u + \eta) \sinh(u - \eta), \] (2.2)

\footnote{It is not yet clear whether this approach can give all the eigenvalues of the transfer matrix. Indeed, in order to obtain all the levels, two sets of Bethe Ansatz equations are required [18], and therefore, presumably two pseudovacua. However, it is not yet clear how to construct the second pseudovacuum.}

\footnote{Additional solutions have recently been found at roots of unity which are not valid for generic anisotropy values [23].}
Crossing relation : \( R_{1,2}(u) = V_1 R_{1,2}^\ell (\alpha - \beta) V_1, \quad V = -i \sigma_y, \) \quad (2.3)

Periodicity property : \( R_{1,2}(u + i\pi) = -\sigma_z^z R_{1,2}(u) \sigma_z^z. \) \quad (2.4)

Here \( R_{1,2}(u) = P_{12} R_{1,2}(u) P_{12} \) with \( P_{12} \) being the usual permutation operator and \( t_i \) denotes transposition in the \( i \)-th space. Here and below we adopt the standard notations: for any matrix \( A \in \text{End}(\mathbb{C}^2), A_j \) is an embedding operator in the tensor space \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots, \) which acts as \( A \) on the \( j \)-th space and as identity on the other factor spaces; \( R_{i,j}(u) \) is an embedding operator of \( R \)-matrix in the tensor space, which acts as identity on the factor spaces except for the \( i \)-th and \( j \)-th ones.

The transfer matrix \( t(u) \) of the open XXZ chain with general integrable boundary terms is given by \cite{12}

\[
t(u) = tr_0 \left( K_0^+(u) T_0(u) K_0^-(u) \hat{T}_0(u) \right),
\]
where \( T_0(u) \) and \( \hat{T}_0(u) \) are the monodromy matrices

\[
T_0(u) = R_{0,N}(u) \cdots R_{0,1}(u), \quad \hat{T}_0(u) = R_{1,0}(u) \cdots R_{N,0}(u),
\]
and \( tr_0 \) denotes trace over the “auxiliary space” 0. We consider the most general solutions \( K^\mp(u) \) \cite{8 9} to the reflection equation and its dual \cite{12 24}. The matrix elements are given respectively by

\[
K_{11}^+(u) = 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) + \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)),
\]
\[
K_{22}^+(u) = 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) - \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)),
\]
\[
K_{12}^+(u) = e^{\theta_-} \sinh(2u), \quad K_{21}^+(u) = e^{-\theta_-} \sinh(2u),
\]
\quad (2.7)

and \( K^+(u) = K^-(\alpha - \beta)(\alpha_-, \beta_-) \rightarrow (-\alpha_+ - \beta_+, \theta_+). \) Here \( \alpha_+, \beta_+, \theta_+ \) are the boundary parameters which are associated with boundary interaction terms. The K-matrices have the periodicity property: \( K^+(u + i\pi) = -\sigma_z^z K^+(u) \sigma_z^z. \) Sklyanin has shown that the transfer matrices with different spectral parameters commute with each other: \([t(u), t(v)] = 0. \) This ensures the integrability of the open XXZ chain. Furthermore, one can show that the transfer matrix also has

\[ i\pi \text{-periodicity : } t(u + i\pi) = t(u), \] \quad (2.8)

Crossing symmetry : \( t(-u - \eta) = t(u), \) \quad (2.9)

Initial condition : \( t(0) = -2^3 \sinh^{2N}(\eta) \sinh(\alpha_-) \cosh(\beta_-) \sinh(\alpha_+) \cosh(\beta_+) \text{id}, \) \quad (2.10)

Asymptotic behavior : \( t(u) \sim \frac{\cosh(\theta_- - \theta_+) e^{\pm(2N+4)u+(N+2)\eta}}{22^N+1} \text{id} + \ldots, \) for \( u \to \pm\infty. \) \quad (2.11)
3 \ T-Q relation

We shall use the fusion procedure, which was first developed for R-matrices \[4, 5\] and then later generalized for K-matrices \[21, 22\], to obtain the Baxter T-Q relation. The fused spin-\((j, \frac{1}{2})\) R-matrix \((j = \frac{1}{2}, 1, \frac{3}{2}, \ldots)\) is given by \[5\]

\[
R_{(1\ldots 2j),2j+1}(u) = P_{1\ldots 2j}^{(+)} R_{1,2j+1}(u) R_{2,2j+1}(u + \eta) \cdots R_{2j,2j+1}(u + (2j - 1)\eta) P_{1\ldots 2j}^{(+)},
\]

where \(P_{1\ldots 2j}^{(+)}\) is the completely symmetric projector. Following \[21, 22\], the fused spin-\(j\) K-matrix \(K_{(1\ldots 2j)}^{-}(u)\) is given by

\[
K_{(1\ldots 2j)}^{-}(u) = P_{1\ldots 2j}^{(+)} \left\{ K_{2j}(u) R_{2j,2j-1}(2u + \eta) K_{2j-1}^{-}(u + \eta) \right. \\
\times R_{2j-2}(2u + 2\eta) R_{2j-1,2j-2}(2u + 3\eta) K_{2j-2}^{-}(u + 2\eta) \\
\times \cdots R_{2,1}(2u + (4j - 3)\eta) K_{1}^{-}(u + (2j - 1)\eta) \left. \right\} P_{1\ldots 2j}^{(+)}. \tag{3.2}
\]

The fused spin-\(j\) K-matrix \(K_{(1\ldots 2j)}^{+}(u)\) is given by

\[
K_{(1\ldots 2j)}^{+}(u) = F(u|2j) K_{(1\ldots 2j)}^{-}(-u - 2j\eta) \bigg|_{(\alpha_{-},\beta_{-},\theta_{-}) \rightarrow (-\alpha_{+},-\beta_{+},\theta_{+})}, \tag{3.3}
\]

where the scalar functions \(F(u|q)\) are given by \(F(u|q) = 1/\left(\prod_{l=1}^{q-1} \prod_{k=1}^{l} \xi(2u + l\eta + k\eta)\right)\), for \(q = 1, 2, \ldots\). The fused transfer matrix \(t^{(j)}(u)\) constructed with a spin-\(j\) auxiliary space is given by

\[
t^{(j)}(u) = tr_{1\ldots 2j} \left( K_{(1\ldots 2j)}^{+}(u) T_{(1\ldots 2j)}(u) K_{(1\ldots 2j)}^{-}(u) \hat{T}_{(1\ldots 2j)}(u + (2j - 1)\eta) \right), \tag{3.4}
\]

where

\[
T_{(1\ldots 2j)}(u) = R_{(1\ldots 2j),N}(u) \cdots R_{(1\ldots 2j),1}(u),
\]

\[
\hat{T}_{(1\ldots 2j)}(u + (2j - 1)\eta) = R_{(1\ldots 2j),1}(u) \cdots R_{(1\ldots 2j),N}(u). \tag{3.5}
\]

The transfer matrix \(\hat{T}_{(1\ldots 2j)}(u)\) corresponds to the fundamental case \(j = \frac{1}{2}\), i.e., \(t(u) = t^{(\frac{1}{2})}(u)\). The fused transfer matrices constitute commutative families, namely,

\[
[t^{(j)}(u), t^{(k)}(v)] = 0. \tag{3.6}
\]

They satisfy a so-called fusion hierarchy \[21, 22\]

\[
t^{(j)}(u - (2j - 1)\eta) = t^{(j - \frac{1}{2})}(u - (2j - 1)\eta) t(u) - \frac{\Delta(u - \eta)}{\xi(2u)} t^{(j-1)}(u - (2j - 1)\eta), \tag{3.7}
\]
where \( \xi(u) \) is given by (2.2); and the coefficient function \( \frac{\Delta(u-\eta)}{\xi(2u)} \), which we now denote by \( \delta(u) \), is given by \(^3\)

\[
\delta(u) = \frac{\Delta(u-\eta)}{\xi(2u)} = 2^4 \sinh^2N(u-\eta) \sinh^2N(u+\eta) \frac{\sinh(2u-2\eta) \sinh(2u+2\eta)}{\sinh(2u-\eta) \sinh(2u+\eta)} \times \sinh(u+\alpha_-) \sinh(u-\alpha_-) \cosh(u+\beta_-) \cosh(u-\beta_-) \times \sinh(u+\alpha_+) \sinh(u-\alpha_+) \cosh(u+\beta_+) \cosh(u-\beta_+). 
\]

(3.8)

In the above hierarchy, \( t^{(0)}(u) = \text{id} \). For generic \( \eta \), the fusion hierarchy does not truncate (c.f. the roots of unity case [17]). Hence \( \{t^{(j)}(u)\} \) constitute an infinite hierarchy, namely, \( j \) taking values \( \frac{1}{2}, 1, \frac{3}{2}, \ldots \).

The commutativity (3.6) of the fused transfer matrices \( \{t^{(j)}(u)\} \) and the fusion relation (3.7) imply that the corresponding eigenvalue of the transfer matrix \( t^{(j)}(u) \), denoted by \( \Lambda^{(j)}(u) \), satisfies the following hierarchy

\[
\Lambda^{(j)}(u+\eta-2j\eta) = \Lambda^{(j-\frac{1}{2})}(u-2(j-\frac{1}{2})\eta) \Lambda(u) - \delta(u) \Lambda^{(j-1)}(u-\eta-2(j-1)\eta). 
\]

(3.9)

Here we have used the convention \( \Lambda(u) = \Lambda^{(\frac{1}{2})}(u) \) and \( \Lambda^{(0)}(u) = 1 \). Dividing both sides of (3.9) by \( \Lambda^{(j-\frac{1}{2})}(u-2(j-\frac{1}{2})\eta) \), we have

\[
\Lambda(u) = \frac{\Lambda^{(j)}(u+\eta-2j\eta)}{\Lambda^{(j-\frac{1}{2})}(u-2(j-\frac{1}{2})\eta)} + \delta(u) \frac{\Lambda^{(j-1)}(u-\eta-2(j-1)\eta)}{\Lambda^{(j-\frac{1}{2})}(u-2(j-\frac{1}{2})\eta)}. 
\]

(3.10)

We now consider the limit \( j \to \infty \). We make the fundamental assumption (1.1) (in particular, that the limit exists), which implies for the corresponding eigenvalues

\[
\bar{Q}(u) = \lim_{j \to +\infty} \Lambda^{(j)}(u-2j\eta). 
\]

(3.11)

It follows from (3.10) that

\[
\Lambda(u) = \frac{\bar{Q}(u+\eta)}{Q(u)} + \delta(u) \frac{\bar{Q}(u-\eta)}{Q(u)}. 
\]

(3.12)

Assuming the function \( Q(u) \) has the decomposition \( Q(u) = f(u)Q(u) \) with

\[
Q(u) = \prod_{j=1}^{M} \sinh(u-u_j) \sinh(u+u_j+\eta), 
\]

(3.13)

\(^3\)In [17, 23], the function \( \frac{\Delta(u-\eta)}{\xi(2u)} \) is denoted instead by \( \delta(u-\eta) \).
$M$ being an integer such that $M \geq 0$, Eq. (3.12) becomes
\[
\Lambda(u) = H_1(u) \frac{Q(u + \eta)}{Q(u)} + H_2(u) \frac{Q(u - \eta)}{Q(u)}.
\]  
(3.14)

Here $H_1(u) = \frac{f(u+\eta)}{f(u)}$ and $H_2(u) = \delta(u) \frac{f(u-\eta)}{f(u)}$. It is easy to see that the functions \( \{H_i(u)\,|\,i = 1, 2\} \) satisfy the relation
\[
H_1(u - \eta)H_2(u) = \delta(u),
\]
(3.15)

where the function $\delta(u)$ is given by (3.8).

In summary, the eigenvalue $\Lambda(u)$ of the fundamental transfer matrix $t(u)$ (2.5) has the decomposition form (3.14), where the coefficient functions \( \{H_i(u)\} \) satisfy the constraint (3.15). In the next section, we use the analytic property of the eigenvalue $\Lambda(u)$ and the other properties derived from the transfer matrix to determine the functions \( \{H_i(u)\} \) and therefore the eigenvalue $\Lambda(u)$.

## 4 Eigenvalues and Bethe Ansatz equations

It follows from (2.8)-(2.11) that the eigenvalue $\Lambda(u)$, as a function of $u$, has the following properties,

- **Periodicity**: $\Lambda(u + i\pi) = \Lambda(u)$,  
(4.1)

- **Crossing symmetry**: $\Lambda(-u - \eta) = \Lambda(u)$,  
(4.2)

- **Initial condition**: $\Lambda(0) = -2^3 \sinh^{2N}(\eta) \cosh(\eta) \sinh(\alpha_-) \cosh(\beta_-) \sinh(\alpha_+) \cosh(\beta_+)$,  
(4.3)

- **Asymptotic behavior**: $\Lambda(u) \sim -\cosh(\theta_- - \theta_+) e^{\pm i[2N+4]u+(N+2)\eta]} + \ldots$, for $u \to \pm \infty$.  
(4.4)

The commutativity of the transfer matrix $t(u)$ and the analyticity of the R-matrix and K-matrices imply that $\Lambda(u)$ further obeys the property

- **Analyticity**: $\Lambda(u)$ is an analytic function of $u$ at finite $u$.  
(4.5)

Moreover the semiclassical property of the R-matrix, $R(u)|_{\eta=0} = \sinh(u) \text{id}$, leads to the following property of $\Lambda(u)$,

\[
\Lambda(u)|_{\eta=0} = 2^3 \sinh^{2N}(u) \left\{ -\sinh(\alpha_-) \cosh(\beta_-) \sinh(\alpha_+) \cosh(\beta_+) \cosh^2(u) \\
+ \cosh(\alpha_-) \sinh(\beta_-) \cosh(\alpha_+) \sinh(\beta_+) \sinh^2(u) \\
- \cosh(\theta_- - \theta_+) \sinh^2(u) \cosh^2(u) \right\}.
\]
(4.6)
The $T$-$Q$ relations (3.14) and (3.15), together with the above properties (4.11)-(4.5), can be used to determine the eigenvalues of the transfer matrix. For $\{\epsilon_i = \pm 1 \mid i = 0, 1, 2, 3\}$, let us introduce

$$
\begin{align*}
H_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) &= -2^2 \epsilon_2 \sinh^{2N}(u) \frac{\sinh(2u)}{\sinh(2u + \eta)} \sinh(u \pm \alpha_- + \eta) \cosh(u \pm \epsilon_1 \beta_- + \eta) \\
&\quad \times \sinh(u \pm \epsilon_2 \alpha_+ + \eta) \cosh(u \pm \epsilon_3 \beta_+ + \eta), \\
H_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) &= -2^2 \epsilon_2 \sinh^{2N}(u + \eta) \frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)} \sinh(u \mp \alpha_-) \cosh(u \mp \epsilon_1 \beta_-) \\
&\quad \times \sinh(u \mp \epsilon_2 \alpha_+) \cosh(u \mp \epsilon_3 \beta_+).
\end{align*}
$$

(4.7)

(4.8)

One may readily check that the functions $H_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3)$ and $H_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3)$ indeed satisfy (3.16), namely,

$$
H_1^{(\pm)}(u - \eta|\epsilon_1, \epsilon_2, \epsilon_3)H_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) = \delta(u).
$$

(4.9)

The general solution to (3.15) can be written as follows:

$$
H_1(u) = H_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) g_1(u), \quad H_2(u) = H_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) g_2(u),
$$

(4.10)

where $\{g_i(u)\}$ satisfy the following relations,

$$
\begin{align*}
g_1(u - \eta)g_2(u) &= 1, \\
g_1(u + i\pi) &= g_1(u), \\
g_2(u + i\pi) &= g_2(u).
\end{align*}
$$

(4.11)

The solutions to (4.11) have the following form,

$$
\begin{align*}
g_1(u) &= a \prod_{j=1}^{N_1} \frac{\sinh(u - u_j^+)}{\sinh(u - u_j^-)}, \\
g_2(u) &= \frac{1}{a} \prod_{j=1}^{N_2} \frac{\sinh(u - u_j^- - \eta)}{\sinh(u - u_j^+ - \eta)},
\end{align*}
$$

(4.12)

where $N_1$ and $N_2$ are integers such that $N_1, N_2 \geq 0$, and $a$ is a non-zero constant. In the above equation, we assume that $u_j^+ \neq \pm \alpha_- - \eta, \pm \epsilon_1 \beta_- - \eta - i\pi/2, \pm \epsilon_2 \alpha_+ - \eta, \pm \epsilon_3 \beta_+ - \eta - i\pi/2$ and $u_j^+ \neq \epsilon_0 \alpha_-, \epsilon_1' \beta_- - i\pi/2, \epsilon_2 \alpha_+, \epsilon_3' \beta_+ - i\pi/2$, otherwise the corresponding factors in $g_1(u)$ make transitions among $\{H_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3)\}$ $\{H_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3)\}$ respectively. Then the analyticity of $\Lambda(u)$ (4.6) requires that $g_1(u)$ and $g_2(u)$ have common poles; i.e., $N_1 = N_2$, and $u_j^- = u_j^+ + \eta$. This means

$$
\begin{align*}
g_1(u) &= a \prod_{j=1}^{N_1} \frac{\sinh(u - u_j^- + \eta)}{\sinh(u - u_j^-)}, \\
g_2(u) &= \frac{1}{a} \prod_{j=1}^{N_1} \frac{\sinh(u - u_j^- - \eta)}{\sinh(u - u_j^-)}.
\end{align*}
$$

(4.13)
Since $H_2^{(±)}(u|\epsilon_1, \epsilon_2, \epsilon_3) = H_1^{(±)}(-u-\eta|\epsilon_1, \epsilon_2, \epsilon_3)$, the crossing symmetry of $\Lambda(u)$ implies that $H_2(u) = H_1(-u-\eta)$. Hence, $N_1$ is even and

$$g_1(u) = a \prod_{j=1}^{N_1} \frac{\sinh(u - u_j^- + \eta) \sinh(u + u_j^- + 2\eta)}{\sinh(u - u_j^-) \sinh(u + u_j^- + \eta)},$$

$$g_2(u) = a \prod_{j=1}^{N_1} \frac{\sinh(u - u_j^+ - \eta) \sinh(u + u_j^+)}{\sinh(u - u_j^+) \sinh(u + u_j^+ + \eta)}, \quad a = \pm 1.$$  \hfill (4.14) (4.15)

This is equivalent to having additional Bethe roots; and the corresponding factors, except $a$, can be absorbed into those of $Q(u)$ \hfill (3.13). Therefore the eigenvalues of the transfer matrix can be uniquely expressed in the following form:

$$\Lambda(u) = aH_1^{(±)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u + \eta)}{Q(u)} + aH_2^{(±)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u - \eta)}{Q(u)}, \quad a = \pm 1.$$  \hfill (4.16)

The initial condition \hfill (4.3) implies $a = +1$. The asymptotic behavior \hfill (4.4), together with \hfill (4.16), requires that the boundary parameters should obey a constraint among the boundary parameters. The resulting constraint and the semiclassical property of the eigenvalues \hfill (4.6) finally give rise to a further constraint among the discrete parameters $\{\epsilon_i\}$. Indeed, if the boundary parameters satisfy any of the following constraints:

$$\alpha_- + \epsilon_1 \beta_- + \epsilon_2 \alpha_+ + \epsilon_3 \beta_+ = \epsilon_0 (\theta_- - \theta_+) + \eta k + \frac{1 - \epsilon_2}{2} i\pi \text{ mod } (2i\pi), \quad \epsilon_1 \epsilon_2 \epsilon_3 = +1,$$ \hfill (4.17)

where $k$ is an integer such that

$$|k| \leq N - 1, \quad \text{and } N - 1 + k \text{ is even},$$ \hfill (4.18)

then the eigenvalues of the corresponding transfer matrix are

$$\Lambda^{(±)}(u) = H_1^{(±)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q^{(±)}(u + \eta)}{Q^{(±)}(u)} + H_2^{(±)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q^{(±)}(u - \eta)}{Q^{(±)}(u)}.$$ \hfill (4.19)

Here

$$Q^{(±)}(u) = \prod_{j=1}^{M^{(±)}} \sinh(u - v_j^{(±)}) \sinh(u + v_j^{(±)} + \eta), \quad M^{(±)} = \frac{1}{2} (N - 1 + k),$$ \hfill (4.20)

and the parameters $\{v_j^{(±)}\}$ satisfy the associated Bethe Ansatz equations respectively,

$$\frac{H_2^{(±)}(v_j^{(±)}|\epsilon_1, \epsilon_2, \epsilon_3)}{H_2^{(±)}(-v_j^{(±)} - \eta|\epsilon_1, \epsilon_2, \epsilon_3)} = -\frac{Q^{(±)}(v_j^{(±)} + \eta)}{Q^{(±)}(v_j^{(±)} - \eta)}, \quad j = 1, \ldots, M^{(±)}.$$ \hfill (4.21)
One can verify that, for generic values of $\eta$, both $\Lambda^{(\pm)}(u)$ have the same desirable asymptotic behavior \((4.4)\) and semiclassical property \((4.6)\) provided the constraint \((4.17)\) is satisfied. We note that Refs. \([17, 18]\) treat explicitly only the case $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. One can check numerically along the lines of \([18]\) that for a given set of bulk and boundary parameters satisfying \((4.17)\), the eigenvalues $\Lambda^{(-)}(u)$ and $\Lambda^{(+)}(u)$ together give the complete set of eigenvalues of the transfer matrix $t(u)$.

5 Conclusions

We have argued that the Baxter $Q$-operator for the spin-1/2 XXZ chain is given by the $j \to \infty$ limit of the transfer matrix with spin-$j$ auxiliary space \((1.1)\). Indeed, this relation together with the fusion hierarchy lead to the Baxter $T$-$Q$ relation for both the closed \((1.3)\) and open \((3.14), (3.15)\) integrable chains. Since the (fused) transfer matrices are standard objects in the QISM, the relation \((1.1)\) shows that the $Q$-operator also fits naturally in the QISM. In contrast to the approach \([17]\), here it is presumably essential that the bulk anisotropy parameter $\eta$ have generic values, for which case there exist $U_q(sl_2)$ representations of arbitrary spin. For the open chain, we have shown in detail how the $T$-$Q$ relation, together with some additional properties, determine the eigenvalues \((4.19)\) of the transfer matrix and the associated Bethe Ansatz equations \((4.21)\). For a given set of bulk and boundary parameters satisfying the constraint \((4.17)\), the eigenvalues $\Lambda^{(-)}(u)$ and $\Lambda^{(+)}(u)$ together are expected to constitute the complete set of eigenvalues of the transfer matrix $t(u)$ \([18]\). Our results complement and also generalize those obtained in \([17]\) at roots of unity.

It would be interesting to determine the conditions for which the limit \((1.1)\) exists. We have seen that, for the open chain with generic values of $\eta$, \((4.19), (4.20)\) is a solution if the constraint \((4.17)\) is satisfied. This suggests that, for generic values of $\eta$, the constraint \((4.17)\) may be a necessary condition for the existence of the limit \((1.1)\). This, in turn, suggests that (again, for generic values of $\eta$), if the constraint \((4.17)\) is not satisfied, then there may not be a $T$-$Q$ relation – a most unusual situation for an integrable model, which merits further investigation. It would also be interesting to explicitly evaluate the $Q$-operator directly from Eq. \((1.1)\).
Note added: Some special cases of the constraint equation (4.17), corresponding to particular values of the discrete parameters $\epsilon_i$, were noted previously in [13, 25, 26]. It was noted in [26] that the constraint equation corresponds to points where the representation theory of the two-boundary Temperley-Lieb algebra is non-semisimple, giving rise to indecomposable representations.

Acknowledgments

Financial support from the Australian Research Council (WLY and YZZ) and the National Science Foundation under Grant PHY-0244261 (RIN) is gratefully acknowledged.

References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).

[2] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, *Commun. Math. Phys.* **190** (1997), 247 [hep-th/9604044].

[3] A. Antonov and B. Feigin, *Phys. Lett.* B **392** (1997), 115 [hep-th/9603105]; M. Rossi and R. Weston, *J. Phys.* A **35** (2002), 10015.

[4] M. Karowski, *Nucl. Phys.* B **153** (1979), 244.

[5] P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, *Lett. Math. Phys.* 5 (1981), 393.

[6] P.P. Kulish and E.K. Sklyanin, *Lecture Notes in Physics*, Vol. 151, (Springer, 1982) 61.

[7] A.N. Kirillov and N.Yu. Reshetikhin, *J. Sov. Math.* 35 (1986), 2627; *J. Phys.* A20 (1987), 1565.

[8] H.J. de Vega and A. González-Ruiz, *J. Phys.* A **26** (1993), L519 [hep-th/9211114].

[9] S. Ghoshal and A.B. Zamolodchikov, *Int. J. Mod. Phys.* A **9** (1994), 3841 [hep-th/9306002].

[10] M. Gaudin, *Phys. Rev.* A **4** (1971), 386.

[11] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, *J. Phys.* A **20** (1987), 6397.
[12] E. K. Sklyanin, *J. Phys.* A 21 (1988), 2375.

[13] J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, *Nucl. Phys.* B 663 (2003), 487.

[14] W.-L. Yang, Y.-Z. Zhang and M. Gould, *Nucl. Phys.* B 698 (2004), 503 [hep-th/0411048].

[15] R. J. Baxter, *Ann. Phys.* (NY) 76 (1973), 1.

[16] L. D. Faddeev and L. A. Takhtajan, *Russ. Math. Surv.* 34 (1979), 11.

[17] R. I. Nepomechie, *J. Phys.* A 34 (2001), 9993 [hep-th/0110081]; *Nucl. Phys.* B 622 (2002), 615 [hep-th/0110116]; *J. Stat. Phys.* 111 (2003), 1363 [hep-th/0211001]; *J. Phys.* A 37 (2004), 433 [hep-th/0304092].

[18] R. I. Nepomechie and F. Ravanini, *J. Phys.* A 36 (2003), 11391; Addendum, *J. Phys.* A 37 (2004), 1945 [hep-th/0307095].

[19] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Commun. Math. Phys.* 177 (1996), 381 [hep-th/9412229]; *Commun. Math. Phys.* 200 (1999), 297 [hep-th/9805008].

[20] A. Kuniba, K. Sakai and J. Suzuki, *Nucl. Phys.* B 525 (1998), 597.

[21] L. Mezincescu and R. I. Nepomechie, *J. Phys.* A 25 (1992), 2533.

[22] Y. K. Zhou, *Nucl. Phys.* B 458 (1996), 504 [hep-th/9510095].

[23] R. Murgan and R. I. Nepomechie, *J. Stat. Mech.* P05007 (2005); Addendum, *J. Stat. Mech.* P11004 (2005) [hep-th/0504124]; *J. Stat. Mech.* P08002 (2005) [hep-th/0507139].

[24] I. V. Cherednik, *Theor. Math. Fiz* 61 (1984), 35.

[25] J. de Gier and P. Pyatov, *J. Stat. Mech.* P03002 (2004) [hep-th/0312235].

[26] J. de Gier, A. Nichols, P. Pyatov and V. Rittenberg, *Nucl. Phys.* B 729 (2005), 387 [hep-th/0505062].