Corrections to scaling for percolative conduction: anomalous behavior at small $L$

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Recently Grassberger has shown that the correction to scaling for the conductance of a bond percolation network on a square lattice is a nonmonotonic function of the linear lattice dimension with a minimum at $L = 10$, while this anomalous behavior is not present in the site percolation networks. We perform a high precision numerical study of the bond percolation random resistor networks on the square, triangular and honeycomb lattices to further examine this result. We use the arithmetic, geometric and harmonic means to obtain the conductance and find that the qualitative behavior does not change: it is not related to the shape of the conductance distribution for small system sizes. We show that the anomaly at small $L$ is absent on the triangular and honeycomb networks. We suggest that the nonmonotonic behavior is an artifact of approximating the continuous system for which the theory is formulated by a discrete one which can be simulated on a computer. We show that by slightly changing the definition of the linear lattice size we can eliminate the minimum at small $L$ without significantly affecting the large $L$ limit.

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I. INTRODUCTION

According to finite size scaling theory, the conductance of a finite random resistor network is expected to vary with the system size:

$$<\sigma> = L^{-t}[a + bf(L)]$$

(1)

Here $t \approx 0.982$ is the conductivity exponent and $f(L)$ is the correction to scaling term which vanishes as the size of the system becomes infinite. In this work we study the behavior of the correction-to-scaling for finite systems that can be simulated on a computer.

Recently Grassberger numerically studied the conductance of random resistor networks on the square lattice by numerical simulations for $2 \leq L \leq 4096$. He found that for bond percolation networks the corrections-to-scaling are nonmonotonic: there is a dip in the corrections-to-scaling present at small lattice sizes. Even though this effect is rather weak, it can be clearly distinguished by high precision numerical simulations. In the site percolation networks this behavior is absent. Grassberger showed that the bond percolation data could be fitted to the log-periodic form $<\sigma> \sim \sin(\log(L))$ and pointed out that only the comparison with site percolation resistor networks enables us to reject this. Because this behavior is not understood we wanted to investigate it further by studying systems defined on different lattices. One question we ask is whether this behavior is the result of the strong shape dependence of the probability distribution of conductances for small system sizes. To answer this we simulated square lattice percolating networks and calculated the arithmetic, geometric and harmonic means (Grassberger used the arithmetic mean to obtain the conductance). Another question is whether the oscillations are universal or particular to the square lattice used in Grassberger’s work. To test the universality we also simulated bond percolation resistor networks on triangular and honeycomb lattices.

Our paper is organized as follows: in Section 2 we give the details of our numerical work and present our results, Section 3 contains the discussion and conclusions.

II. NUMERICAL SIMULATIONS

The systems we study are random resistor networks that are connected to perfectly conducting busbars at the two opposite edges of the array, while the transverse boundaries are free. To simulate a random resistor network we start with a bond percolation network and assign unit conductance to the bonds that are present while a missing bond gives zero conductance. The calculation of the network conductance for a given configuration of conductances is done by the Lobb-Frank algorithm, which uses a succession of star-triangle (ST) and triangle-star (TS) transformations to reduce the network to an equivalent chain of resistors.

Our square lattice simulations were performed on rectangles $(0 \leq x \leq L, 1 \leq y \leq L)$, with the perfectly conducting busbars placed at $x = 0$ and $x = L$. This means that the lattice (busbars excluded) has dimensions $(L-1,L)$. The reason for this choice is to ensure that the bond system is self-dual (see Ref. 3 and references therein). To simulate systems on a triangular lattice we add a diagonal resistor to each unit cell of the square lattice without changing the position of the busbars. This way of representing the triangular network changes the geometry of the system - effectively the shape of the boundary becomes a rhombus. Simulations on the honeycomb lattice are done by starting with a honeycomb resistor network and replacing the leftmost column of resistors with perfect conductors and attaching the rightmost column of nodes to a perfectly conducting busbar. The pictures of these networks can be found in Ref. 4. The systems were taken to be at the percolation threshold, where $p_c = 0.5$ for square lattice,
$p_c = 2 \sin(\pi/18) = 0.34729$ for the triangular lattice and $p_c = 1 - 2 \sin(\pi/18) = 0.65271$ for the honeycomb lattice \cite{26}. For the square lattice the conductance was obtained by exact enumeration for $L \leq 4$, while for the triangular lattice exact enumeration was used to obtain the result for $L = 2$. For other system sizes the number of random configurations generated was at least $10^7$.

Our results for the square lattice are shown in Fig. 1, for the triangular lattice in Fig. 2 and for the honeycomb lattice in Fig. 3. We plot the conductance multiplied by $L$ raised to the conductivity exponent $t = 0.982$ as a function of $L$ on a semilogarithmic scale.

**FIG. 1.** Average conductance on square lattices of size $L \times L$, multiplied by $L^{0.982}$. The upper set of data is obtained by using the arithmetic mean, the middle set is obtained from the geometric mean and the lower set is the harmonic mean.

**FIG. 2.** Average conductance on triangular lattices of size $L \times L$, multiplied by $L^{0.982}$. The symbols have the same meaning as in Fig. 1.
FIG. 3. Average conductance on honeycomb lattices of size $L \times L$, multiplied by $L^{1.982}$. The notation is the same as in Fig. 1 and Fig. 2.

In these figures full points (circles) are obtained by using the arithmetic mean, stars are obtained from the geometric mean and crosses correspond to the harmonic mean (where the averages are computed using only the samples with nonzero conductance). The errors are smaller than the size of the points on the graphs. The points in Fig. 1 obtained by the arithmetic mean match the data obtained by Grassberger in [1].

III. DISCUSSION AND CONCLUSIONS

Let us first consider our results for the networks defined on a square lattice.

The data obtained by using the arithmetic mean start at a low value for the lowest lattice size ($L = 2$) and then there is a jump to a high value at lattice size $L = 3$ followed by the monotonic decrease up to $L \approx 10$. If we use the harmonic or geometric mean the jump is not present - $< \sigma > L^{i}$ is a monotonically decreasing function until it reaches a minimum. Thus the behavior at $L = 2$ is affected by the peculiar (and very discrete) distribution of conductances.

In all three cases a minimum is present at low $L$. Our numerical investigation of conduction on a square lattice shows that the dip in the correction-to-scaling term for small systems does not get averaged out by using different means to obtain the conductance. We did not find the anomalous behavior for the systems defined on triangular and honeycomb lattices. The dip at small $L$ is not present and the behavior is quite similar to those obtained in Ref. 1 for the case of site percolation on a square lattice.

The anomaly present in the case of square lattice is rather weak - only high precision simulations are able to reveal it. We notice that changing $L$ by a fraction of the lattice spacing will strongly affect the behavior of $< \sigma > L^{i}$ for small $L$ while the large $L$ results will be practically unaffected. Replacing $L$ by $L - \Delta$ as the independent variable gives a monotonic function for $\Delta = 0.05$; for $\Delta = 0.1$ the corrections-to-scaling behave qualitatively in the same way as the results obtained for the other systems mentioned above (Ziff [7] used a similar constant in connection with crossing probability problems). This is illustrated in Fig. 4.

FIG. 4. Average conductance on square lattices of size $L \times L$, multiplied by $(L - \Delta)^{0.982}$. The upper set of data is obtained by using the arithmetic mean (this set is represented in Fig. 1 by the same symbol), the middle set is obtained by correcting $L$ by 0.05 and the lower set is obtained with a correction of 0.1.

In the case of triangular and honeycomb systems the rescaling of $L$ does not qualitatively change the behavior of $< \sigma > L^{i}$ - in particular, the oscillation at small $L$ does not occur. This is shown in Fig. 5 and Fig. 6 for triangular and honeycomb lattices, respectively.
FIG. 5. Average conductance on triangular lattices of size $L \times L$, multiplied by $(L - \Delta)^{0.982}$. The middle set of data is obtained by using the arithmetic mean (this set is represented in Fig. 2. by the same symbol), the lower set is obtained by correcting $L$ by 0.05 and the upper set is obtained with a correction of -0.05.

FIG. 6. Average conductance on honeycomb lattices of size $L \times L$, multiplied by $(L - \Delta)^{0.982}$. The middle set of data is obtained by using the arithmetic mean (this set is represented in Fig. 3. by the same symbol), the lower set is obtained by correcting $L$ by 0.05 and the upper set is obtained with a correction of -0.05.

The justification for this adjustment of $L$ is that the theory is formulated in the continuum limit while the discrete systems we simulated have the smallest scale equal to the lattice spacing. While for the large systems this should not matter, the small system results might be affected by this if we are doing very precise simulations.

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