Class of axially symmetric solutions of vacuum Einstein field equations including the Papapetrou solution as particular case has been found. It has been shown that the derived solution describes the external axial symmetric gravitational field of the source with nonvanishing mass. The general solution is obtained for this class of functions. As an example of physical application, the spacetime metric outside a line gravitomagnetic monopole has been obtained from Papapetrou solution of vacuum equations of gravitational field.

**Keywords:** Einstein equations; exact vacuum solutions; axial symmetry

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1. Introduction

A wide range of massive objects from the compact relativistic stars, as black holes and neutron stars, up to modern cosmological models, including Early Universe and its evolution \(^1\) where the solutions of Einstein equations are applicable and important is currently known. But the logic of scientific development and practical experience requires an introduction into the consideration of principally new objects, like cosmic strings, gravitomagnetic monopoles, quasiclosed worlds etc. The extension of objects under research leads to the additional analysis and extension of well known existing solutions of fundamental field equations. Therefore reconsideration of old and finding new exact exterior solutions of Einstein field equations describing space-time around these objects is one of the interesting and important branches of general relativity and theoretical physics. There exist a lot of inter-
testing problems, especially in quantum gravity, because it is still not clear how to build quantum gravity (see, for example, \(^2\)) and in this level of our understanding it is very useful to investigate objects which are on the boundary between classical and quantum gravity. Of special importance also are external gravitational fields of isolated stars, as today is recognized they are at the core of most intriguing astrophysical phenomena. In spite of the fact that many exact solutions of Einstein equations can be found in classical monographs as \(^3\) it is still interesting to study new possible solutions of this fundamental equation.

In our present work we concentrate our efforts in investigation and analysis of some exact solutions of Einstein equations \(^3\), namely, on the exterior axisymmetric fields in the vacuum, arising around rotating gravitational objects. In our consideration we look for some mathematical features and structure of Einstein equations, which could determine their physically acceptable solutions.

In fact, our motivation is, that the Papapetrou class of axially-symmetric solutions \(^4,5\) of vacuum Einstein equations does not contain physically interesting one, because in the classical Newtonian limit there is no term being proportional to \(r^{-1}\), what means that it describes rotation field for the bounded systems with zero mass \(^5\). However it should be noted, that one of the solutions of Einstein equations with the similar asymptotic behavior has been considered and interpreted as gravitational field of massless quadrupole \(^6\). From geometrical point of view, the gravitational source is characterized by a topological closure of internal three dimensional space with zero integral contribution according to its zero mass.

In the Papapetrou case the solutions are given in the class of pure harmonic functions \(\zeta\) and our assumption is to investigate this solution in the more extended functional space of functions \(\zeta\).

Here we follow to the logic of investigation, brilliantly reflected in \(^5\). As we will show in the Section 2, small difference, introduced by us into the approach described in \(^5\), gives us a possibility to obtain solution of Einstein equations for more general class, which includes Papapetrou one as particular case.

Throughout, we use a space-like signature \((+,-,-,-)\) and a system of units in which \(G = 1 = c\). Greek indices are taken to run from 0 to 3 and Latin indices from 1 to 3; partial derivatives are denoted with a comma.

### 2. Extension of Papapetrou class of solutions

We consider a stationary axially symmetric object in cylindrical coordinates \(x^\alpha = (t, \rho, \phi, z)\) and assume that there is no any gravitating matter outside of it. For the rotating fields in GR, the most general form of the metric of space-time of axially symmetric objects can be written as Weyl - Lewis - Papapetrou one, presented in the form \(^5\)

\[
ds^2 = f(dt - \omega d\phi)^2 - \rho^2 \cdot f^{-1} d\phi^2 - e^\mu (d\rho^2 + dz^2),
\]

where \(fl + k^2 = \rho^2\), \(f, \omega, \mu, k, l\) are unknown metric functions may depend on the cylindrical coordinates \(\rho\) and \(z\).
The external gravitational field is described by the symmetric Ricci tensor \( R_{\alpha\beta} \) which obeys to the exterior Einstein vacuum equations \( R_{\alpha\beta} = 0 \). The nonvanishing \( R_{00} \) and \( R_{03} \) components of Einstein equations can be written as

\[
\begin{align*}
&f(f_{,\rho\rho} + f_{,zz} + \rho^{-1} f_{,\rho}) - f_{,\rho}^2 - f_{,zz}^2 + \rho^{-2} f^4 (\omega_{,\rho}\rho^2 + \omega_{,z}^2) = 0 , \\
&f(\omega_{,\rho\rho} + \omega_{,zz} - \rho^{-1} \omega_{,\rho}) + 2 f_{,\rho} \omega_{,\rho} + 2 f_{,z} \omega_{,z} = 0 .
\end{align*}
\]

Following to the approach presented in \(^5\) one can select the function \( \omega \) satisfying to equation

\[
\omega_{,\rho\rho} + \omega_{,zz} - \rho^{-1} \omega_{,\rho} = 0 ,
\]

as \( \omega = C \rho \zeta_{,\rho} \), where \( C \) is a constant. Mathematical properties of function \( \zeta \) will be defined below. It is easy to see, that after substituting \( \omega \) in (4) and using simple relations

\[
\begin{align*}
\omega_{,\rho} &= C (\zeta_{,\rho} + \rho \zeta_{,\rho\rho}) , \\
\omega_{,\rho\rho} &= C (2 \zeta_{,\rho\rho} + \rho \zeta_{,\rho\rho\rho}) , \\
\omega_{,z} &= C \rho \zeta_{,z} \rho , \\
\omega_{,zz} &= C \rho \zeta_{,z\rho} ,
\end{align*}
\]

one can write

\[
\zeta_{,\rho\rho} + \zeta_{,zz} + (\rho^{-1} \zeta_{,\rho})_{,\rho} = 0 .
\]

Integration of the equation (6) gives an expression for \( \zeta \) :

\[
\nabla^2 \zeta = \zeta_{,\rho\rho} + \zeta_{,zz} + \rho^{-1} \zeta_{,\rho} = \eta(z) .
\]

Therefore we introduced an arbitrary function \( \eta = \eta(z) \). Introduction of this function is very important, because it could generate different kind of solutions for set of differential equations (2) and (3). Hereafter we have essential difference with compare to \(^5\) and our aim is to solve Einstein equations (2) and (3) in terms of more wide class of functions \( \zeta \), which include harmonic functions as a particular case.

In the case \( \eta(z) = 0 \) equation (7) becomes Laplace one, which is similar to the expression given in \(^5\). However in the general case when \( \eta(z) \neq 0 \), we have Poisson like equation (7) for function \( \zeta \) instead of Laplace one. Using the relations (5) for the function \( \omega \) and condition (4) one can rewrite field equation (3) as

\[
-f_{,\rho} (\zeta_{,zz} - \eta) + f_{,z} \zeta_{,\rho}\rho = 0 .
\]

It follows from the equations (7) and (8) the general solution for the metric function \( f \) takes the following form

\[
f = f \left( \zeta_{,z} - \int \eta(z) dz \right) .
\]
After introducing new variable $\tilde{\zeta}, z$ and derivative $f'$

$$\tilde{\zeta}, z = \zeta, z - \int \eta(z) dz, \quad f' = \frac{\partial f}{\partial \tilde{\zeta}, z}$$

(10)

one can rewrite the derivatives of the metric function $f$ in the following way

$$f, \rho = f' \tilde{\zeta}, z\rho, \quad f, \rho\rho = f'' \tilde{\zeta}, z\rho + f' \tilde{\zeta}, z\rho\rho,$$

$$f, z = f' [\zeta, z - \eta(z)], \quad f, zz = f'' (\zeta, zz - \eta)\eta + f' (\zeta, zzz - \eta, z).$$

(11)

Using the equations (7) and (11) one can get

$$f, \rho\rho + f, zz + \rho^{-1} f, \rho = f'' [\tilde{\zeta}, z\rho + (\zeta, zz - \eta)^2],$$

(12)

$$f^2, \rho + f^2, z = f' [\tilde{\zeta}, z\rho + (\zeta, zz - \eta)^2],$$

(13)

$$\omega^2, \rho + \omega^2, z = C^2 \rho^2 [\tilde{\zeta}, z\rho + (\zeta, zz - \eta)^2].$$

(14)

Finally, after inserting the derived expressions into the equation (2) we obtain equation

$$f f'' - f'^2 + C^2 f^4 = 0$$

(15)

which looks like an extension of Papapetrou class of solutions of gravitational field equations:

$$f(\rho, z) = \left\{ \alpha \cdot \cosh \left[ \tilde{\zeta}, z - \int \eta(z) dz \right] - \beta \sinh \left[ \tilde{\zeta}, z - \int \eta(z) dz \right] \right\}^{-1},$$

(16)

where the functions $\eta(z)$ and $\zeta(\rho, z)$ satisfy to the equation (7) and $C^2 = \alpha^2 - \beta^2$. Here parameters $\alpha$ and $\beta$ have the same meaning as in the Papapetrou solutions. As a particular case, in the limit $\eta(z) = 0$, the solution (16) reduces to the Papapetrou one.

3. Asymptotic properties

As a rule, one of the main difficulties, after finding an exact solution of Einstein equations is a physical interpretation of it. In the case of the Papapetrou solutions in the Newtonian limit that is at large arguments $r = \sqrt{\rho^2 + z^2} \to \infty$, the asymptotics looks like

$$f(\rho, z) = \alpha^{-1} \left[ 1 + \frac{\beta z}{\alpha r^3} + O(r^{-2}) \right]$$

(17)

and does not contain a term being proportional to $r^{-1}$. From comparison of expression (17) with the metric given in

$$ds^2 = \left[ 1 - \frac{2M}{r} + O(r^{-2}) \right] dt^2 - \left[ 4\epsilon_{ijk} \frac{S^i_2 j k}{r^3} + O(r^{-3}) \right] dt \cdot dx^i -$$

7
one can conclude that for any given harmonic function $\zeta$, the metric (16) with $\eta(z) = 0$ does not contain solution with nonzero mass of the rotating axially symmetric source. Here $M$ is the mass of the source and the three dimensional spatial vector $S_j$ is the total angular momentum.

We expect that in general case when $\eta \neq 0$ it would be possible to construct a solution, which gives flat asymptotics at infinity for massive sources. As an example, one of the possible solutions could be obtained in the following way. Assume that the function $\zeta(\rho, z) = f_1(\rho)f_2(z)$ could be taken as a product of two functions $f_1$ and $f_2$. Then from equation (7) one can get

$$f_1_{,\rho\rho}f_2 + f_1f_2_{,zz} + \rho^{-1}f_1_{,\rho}f_2 = \eta(z).$$

Taking solution for the function $f_2$ in the form $f_2 = f_{2,zz} = \eta(z)$, with $\eta(z) = C_0(e^z - e^{-z})$, we have for the function $f_1$ the following equation

$$g_1_{,\rho\rho} + \rho^{-1}g_1_{,\rho} + g_1 = 0.$$

Here function $f_1 = g_1 + 1$ and $C_0$ is constant. The last equation is the Bessel one and $f_1 = 1 + J_0(\rho)$, where $J_0(\rho)$ is zero rank Bessel function. Finally

$$\zeta(\rho, z) = C_0[1 + J_0(\rho)](e^z - e^{-z}),$$

and according to (10) it gives an expression for the function $\tilde{\zeta}_z$

$$\tilde{\zeta}_z = C_0J_0(\rho)(e^z + e^{-z}),$$

which takes the simple form $\tilde{\zeta}_{z,|z=0} = 2C_0J_0(\rho)$ at the plane $z = 0$. For the large arguments $\rho \to \infty$, zero rank Bessel function has the following asymptotic behavior

$$\lim_{\rho \to \infty} J_0(\rho) = \frac{2}{\rho} \cos(\rho - \frac{\pi}{4})$$

and consequently

$$\tilde{\zeta}_{z,|z=0} = 2C_0\frac{2}{\rho} \cos(\rho - \frac{\pi}{4}) = \frac{1}{\sqrt{\rho}} \cos(\rho - \frac{\pi}{4}) \equiv x,$$

where we put, for the simplification $C_0 = \frac{1}{2} \sqrt{2}$. Using asymptotic properties of hyperbolic functions one could write

$$\cosh(x) = 1 + \frac{1}{2} \rho^{-1} \cos^2(\rho - \frac{\pi}{4}) + \frac{1}{24} \rho^{-2} \cos^4(\rho - \frac{\pi}{4}) + O(\rho^{-2}),$$

$$\sinh(x) = \rho^{-\frac{1}{2}} \cos(\rho - \frac{\pi}{4}) + \frac{1}{6} \rho^{-\frac{3}{2}} \cos^3(\rho - \frac{\pi}{4}) + O(\rho^{-2}).$$
Substituting obtained asymptotic expressions (25) and (26) in solution (16), we can write the asymptotics of function \( f(\rho, z) \), at the plane \( z = 0 \) when \( \rho \to \infty \), in the following form

\[
\lim_{\rho \to \infty} f_{z=0} = \alpha^{-1} \left[ 1 - \frac{1}{4} \rho^{-1} - \frac{1}{4} \rho^{-1} \sin(2\rho) - \frac{1}{24} \rho^{-2} \cos^4(\rho - \frac{\pi}{4}) + \frac{\beta}{\alpha} \rho^{-\frac{5}{4}} \cos(\rho - \frac{\pi}{4}) + \frac{\beta}{6\alpha} \rho^{-\frac{7}{4}} \cos^3(\rho - \frac{\pi}{4}) \right] + O(\rho^{-2}) .
\] (27)

We can see from the asymptotics (27) that in contrast to the Papapetrou one (17) there is a term being proportional to \( \rho^{-1} \) and at the same time we have asymptotically flat solution at \( z = 0 \) plane.

4. General solution

The general solution for function \( \zeta(\rho, z) \) can be derived by the simple integration of right hand side of the Poisson like equation (7):

\[
\zeta(\rho, z) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\eta(z')dz'}{\sqrt{\rho^2 + (z-z')^2}} .
\] (28)

Thus in the general case the solution (16) for function \( f \) contains function \( \zeta(\rho, z) \) which satisfy to the integral (28) and as result function \( \eta(z) \) generates new solutions for the function \( f(\rho, z) \). At the end we would like to underline that for this class of functions \( \zeta \), satisfying to the Poisson like equation (7), this is most general solution.

Using nonvanishing components of Einstein vacuum equations

\[
R_{11} = R_{12} = R_{22} = 0
\] (29)

and following to the book 5 we can easily obtain the connection of function \( \mu(\rho, z) \) with functions \( \zeta(\rho, z) \), \( \omega(\rho, z) = C\rho \zeta,\rho \) and \( f(\rho, z) \):

\[
\mu,\rho = -\frac{f,\rho}{f} + \frac{\rho}{2f^2}(f^2 - f,^2_{,z}) - \frac{f^2}{2\rho}(\omega^2_{,\rho} - \omega^2_{,z}) ,
\] (30)

\[
\mu,\omega = -\frac{f,\omega}{f} + \frac{\rho}{2f^2}f,\rho f,\omega - \frac{f^2}{\rho} \omega_{,\rho}\omega_{,z} ,
\] (31)

and therefore the metric, presented by expression (1) is defined completely.

One can see form equation (16) that the general solution is invariant under the transformation

\[
\zeta \to \zeta + h(z)
\]

in the functional space of function \( \zeta \), where function \( h(z) \) satisfies to the condition

\[
h(z),zz = \eta(z).
\]

This is gauge condition directly connected with the source of gravitational field defined by the metric function \( \eta(z) \). From our point of view this statement is analogous to the Lorentz gauge condition for the Maxwell equations in electrodynamics.
5. Spacetime of line gravitomagnetic monopole

It is obviously, that any solution of Einstein equations becomes important if it has any physical application and identified with the source of gravitational field. Here we demonstrate, that the obtained extension of Papapetrou solution contains, for example, solution for the line gravitomagnetic monopole and can be useful in the interpretation of the connection of the metric source with the gravitomagnetic monopole momentum $L$.

If we take $\zeta(\rho, z)$ in the form

$$\zeta(\rho, z) = z \ln \left( \frac{\rho}{\rho_0} \right)^{2m} + \int F(z) dz + \eta_0 ,$$

(32)

where $m$, $\rho_0$, $\eta_0$ are some constants and $F(z) = \int \eta(z) dz$, then as it follows from the expression (32), so defined function $\zeta(\rho, z)$ satisfies to the equation (7) for an arbitrary function $\eta(z)$. Choosing in our solution (16) $\beta = 0$ and $\alpha = L/2m$, for the simplicity, we can obtain

$$f(\rho, z) = \frac{2m}{L} \cdot \frac{1}{\cosh \left( 2m \ln \frac{\rho}{\rho_0} \right)} .$$

(33)

The function (33) coincides with the metric function $g_{00}(\rho)$ from the paper 9 and therefore presents the space time around this source, reproducing cylindrical analogue of NUT space, named for Newman, Tamburino and Unti 10.

From this simple example one can see using (32) that the consideration of the line gravitomagnetic monopole metric in the Weyl - Lewis - Papapetrou form leads to

$$\omega = \alpha \rho \zeta_{,\rho} = 2m \alpha \cdot z = L \cdot z ,$$

(34)

from where one can conclude, that gravitomagnetic monopole momentum $L = 2m \alpha$ in accordance with definition of parameter $\alpha$. Further investigation of the effect of gravitational field of a line gravitomagnetic monopole on test electromagnetic fields has been considered in our accompanying paper 11.

6. Conclusion

We conclude, that obtained solution (16) is an extension of the well known Papapetrou class of solutions, which can be applied to the physical systems, presenting rotating bounded nonzero masses. We have constructed the solution for the particular function $\eta(z)$, which in some sense, demonstrates the existence of $\rho^{-1}$ term in asymptotics. In addition using properties of Poisson like equation for $\zeta$ function, we have found the general solution (28) for any given function $\eta(z)$, which generates various metric functions $f(\rho, z)$. The connection of above obtained solution with metric of a line gravitomagnetic monopole was considered as a particular case. This is preliminary result and the physical interpretation of the obtained solution is outside the framework of this paper. More detailed investigation and application...
to gravitational sources is the scope of our future research. In this respect as the next step we want to investigate the regularity conditions at the axis $\rho = 0$ in dependence from the choice of function $\eta(z)$.

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