Equation of State of the Transplanckian Dark Energy and the Coincidence Problem

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Observational evidence suggests that our universe is presently dominated by a dark energy component and undergoing accelerated expansion. We recently introduced a model, motivated by string theory for short-distance physics, for explaining dark energy without appealing to any fine-tuning. The idea of the transplanckian dark energy (TDE) was based on the freeze-out mechanism of the ultralow frequency modes, \(\omega(k)\) of very short distances, by the expansion of the background universe, \(\omega(k) \leq H\). In this paper we address the issue of the stress-energy tensor for the nonlinear short-distance physics and explain the need to modify Einstein equations in this regime. From the modified Einstein equations we then derive the equation of state for the TDE model, which has the distinctive feature of being continually time-dependent. The explanation of the coincidence puzzle relies entirely on the intrinsic time-evolution of the TDE equation of state.

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1 Introduction

Cosmological observations of large scale structure, SN1a, age of the universe and cosmic microwave background (CMB) data, strongly indicate that the universe is dominated by a dark energy component with negative pressure [1]. Besides the difficulty of coming up with a natural explanation for the smallness of the observed dark energy, an equal challenge is the "cosmic coincidence" problem.

Recently we proposed a model [2] for explaining the observed dark energy without appealing to fine-tuning or anthropic arguments. This model is based on the nonlinear behavior of transplanckian metric perturbation modes which was motivated by closed string theory [3, 4] and quantum gravity [5]. The transplanckian dark energy (TDE) model was based on the freeze-out mechanism of the short-distance modes with ultralow energy, by the expansion of the background universe, \( H \), and it naturally explained the smallness of the observed dark energy.

In this paper we study the stress-energy tensor of the TDE model in order to calculate the equation of state for these short-distance stringy modes. As we will show, the frozen tail modes start having a negative pressure of the same order as their positive energy density soon after the matter domination era. Thus it is only at low redshifts that they become important for driving the universe into an accelerated expansion and dominate the Hubble expansion rate \( H \). A distinctive feature of the TDE model is that its equation of state, \( w_H \), is always strongly time-dependent at any epoch in the evolution of the universe, (e.g. \( w_H = -1/3 \) during radiation dominated era but it becomes \( w_H = -1/2 \) at matter domination). It becomes more and more negative at late times until it approaches the limiting value \( w_H = -1 \), after the matter domination time, \( t_{eq} \).

The calculation of the components of the stress-energy tensor, \( T_{\mu\nu} \), namely the pressure and energy density, is given in Section 2. Due to the nonlinearity of short distance physics, Bianchi identity is generically violated for all these models. Therefore one needs to modify the Einstein equations, \( (T_{\mu\nu}) \), such that the modified ones satisfy Bianchi identity.

From physical considerations, the need for modifying Einstein equations in the nonlinear regime of short distance physics is to be expected, due to nonequilibrium dynamics of the short distance modes. In practical terms this is not an easy fair to carry out in an unambiguous way, for a simple reason: we do not have a unique effective theory valid at transplanckian energies or a lagrangian description of the theory in this regime [6]. The only information available to most transplanckian models [4, 2, 3, 8, 11, 12, 13] is the field equation of motion (with a few exceptions like [14]). Nevertheless all these models do violate Bianchi identity and the energy conservation law, if \( T_{\mu\nu} \) is not modified accordingly.

Based on the equation of motion as our sole information for short-distance physics, we therefore use a kinetic theory approach for modifying Einstein equations in the absence of an effective lagrangian description. The assumption made is that a kinetic theory description of the cosmological fluid is valid even in the transplanckian regime. Despite its nonlinear behavior at short distances, this imperfect fluid shares the same symmetries, namely homogeneity and isotropy, as the background Friedman-Robertson-Walker (FRW) universe. Then the corrections \( \tau_{\mu\nu} \) to the stress energy tensor \( T_{\mu\nu} \) will also be of a diagonal
In Section 3 we explore the observational consequences of the model with the puzzle of 'cosmic coincidence' in mind. A summary is given in Section 4. A discussion of the nonequilibrium dynamics and distribution function for the transplanckian modes, as well as details of averaging of their energy and pressure, are attached in the Appendix.

2 The Equation of State from the Modified Einstein Equations

2.1 Analytical expression for \( T_{\mu\nu} \)

Transplanckian models that investigate the sensitivity of the CMB spectrum or Hawking radiation to short-distance physics, all introduce a nonlinear, time-dependent frequency for the very short wavelength modes: \( \omega[p] = f[p] = f[k/a] \).

The physical momentum \( p \) is related to the comoving wavenumber \( k \) by \( p = k/a \), with \( a \) the scale factor. Most of these models lack a lagrangian description and, all the information they propose about short-distance physics is contained in the mode equation of motion:

\[
[\Box + \omega(k, a)^2] \phi_k = 0 .
\]  

The expectation that Einstein equations will not hold unless they are modified in the nonlinear regime of short distance physics is fully reasonable and it is based on the fact that Bianchi identity and energy conservation law will be violated due to the nonlinear time dependence of \( \omega \). In terms of kinetic theory, the time-dependence of the group velocity \( v_g \) indicates departure from equilibrium (see Appendix). Here we study the modifications of \( T_{\mu\nu} \) for a specific class, the TDE model \(^1\). Our approach is based on kinetic theory and the pressure modifications are obtained through balance equations.

In the TDE model we are considering, the dispersed frequency for short-distance metric perturbation modes is:

\[
\begin{align*}
\omega^2[p] &= p^2 \mathcal{E}[p/p_c] \\
&= p^2 \left[ \frac{\epsilon_1}{1 + u} + \frac{\epsilon_3 u}{(1 + u)^2} \right] , \tag{4}
\end{align*}
\]

\[
\begin{align*}
u &= \exp[2p/p_c] , \tag{5}
\end{align*}
\]

\(^1\)The \( \omega^2 \) term collectively denotes the generalized frequency that appears as a mass squared term in the equation. Depending in the particular problem studied it may also include other terms to it, like for example, the coupling of the modes to the curvature of the universe, \( a''/a \), if the equation under consideration is that of metric perturbations.
where $p_c$ is of order the Planck mass or string scale $M$, $p$ is the physical momentum, $\epsilon_i$ arbitrary constants. The maximum of $\omega[p]$ is around $p \approx p_c$. The frequency function behaves as

$$\omega^2[p] \approx p^2 \left( 1 + O(p^2/M^2) \right) \quad p \ll M$$

for the modes in the sub-planckian regime, and like

$$\omega^2[p] \approx \sqrt{\epsilon_1 + \epsilon_3 p^2 \exp[-2p/M]} \quad p \gg M.$$  

for those modes in the TP regime. The nonlinear exact function Eq (4) for the frequency can be fitted to a good accuracy to $\omega[p]^2 \sim \frac{p^2}{\cosh[p/p_c]-1}$. 

Let us refer to the wavepackets of the modes centered around a momentum $p_i$ as particles. Then, their group velocity $v_g = d\omega/dp$ is time dependent through its nonlinear $p$-dependence, and is different from the phase velocity, $v_c = \omega/p$.

Lorentz invariance is broken due to the nonlinearity at short distances. Therefore, the fixed cuttoff scale $p_c = M$, together with all the transplanckian modes pick a preferred frame, the CMB frame. This frame is freely falling along the comoving geodesics, with respect to the physical FRW Universe. Sometimes we will refer to transplanckian modes as the modes inside a small box with fixed Planck size, $l_p = 1/M$, in the preferred frame, since their wavelength $\lambda_{TP} < l_p$ is smaller than the "size of the box", $p > M$. In this picture, Lorentz invariance is broken in the small box but restored in the large box with size $L = a/M$, i.e the Universe. Thus the physical momenta modes for the "small box" bound observers in the preferred frame are the comoving wavenumber modes for the "outside" observers, in the Lorentz invariant FRW Universe, that "see" the preferred frame in a free fall.

Let us address below the issue of how the energy density components behave with time, prior to the pressure modifications. The short distance modes are out of thermal equilibrium, due to their nonlinear frequency and group velocity, $v_g \neq 1$. Meanwhile a thermal state is restored at large scales, ($\lambda \gg l_p$), where the frequency is nearly linear and thus $v_g \approx 1$. Thus we need to average the contribution of the short distance modes to the energy and pressure in the Universe, over many of their wavelengths, in order to obtain an effective large scale thermal state. That is why in obtaining the equation of state, $< w_i >$ prior to the pressure modifications $\Pi_i$, for the transplanckian modes, the averaging is done in time-scales of cosmological order. Details of averaging are provided in the Appendix, 5.2. The equation of state $< w_i >$ prior to viscous pressure modifications, is obtained from the expression $< w_i >= \frac{< \tilde{p}_i >}{< \rho_i >}$ with $< \tilde{p}_i >= - < \rho_i > - < (a/3)d\rho_i/da >$ and $a$ the scale factor. Based on the behavior of $v_g$ with $p$, we divide the dispersion function into 4 regions (see Fig.1)

- **Region 0**: Linear regime, up to $p \approx p_c$, such that $\omega[p] \simeq p$. These modes behave as radiation, Eq.(4), with the averaged pressure expression being $\bar{p}_0 = \rho_0/3$.

- **Region I**: Around the maximum of the dispersion function, up to some value $p_B > M$ in physical momentum, where $\omega$ can be expanded in a polynomial series, and the leading

\[ \text{See [15] for a very nice treatment of issues related to a fixed physical cuttoff in a preferred frame.} \]
order terms in Eq. (3) are the first ones. Region I is dual to region 0.

\[ \omega[p] \approx p \left( a_0 + a_1 \frac{p}{M} + a_2 \frac{(p/M)^2 + \cdots} \right) \]  

where \( a_i \) are constants, \( (a_2 < 0) \). We use Eq. (3) to estimate the energy density

\[ \rho_I \approx \frac{C}{a^4} \int_M^{k_B} dk k^3 \left( a_0 + a_1 \frac{k}{aM} + a_2 \left( \frac{k}{aM} \right)^2 + \cdots \right) \]  

\[ \approx \frac{CM^4}{a^4} \left( \frac{a_0}{4} (x_B^4 - 1) + \frac{a_1}{5a} (x_B^5 - 1) + \cdots \right) \]  

\[ \propto M^4 \frac{a_4}{a^4}, \]  

where \( x_B = k_B/M > 1, \langle x_B \rangle = O(1) \). The constant \( C = |\beta_p|^2 \) denotes the Bogolubov coefficient squared, which in our model does not depend on the momentum \( p [2] \). Therefore, \( \rho_I \) behaves like radiation plus \( O(1/a^2) \) corrections in its averaged equation of state, \( \langle \tilde{p}_I \rangle = (1/3 - A/a^2 + \ldots) \langle \rho_I \rangle \). Regions I and 0 contribute to the radiation energy component in the Universe.
**Region II**: From some mode, \( p_B \gg M \), onwards defined such that its frequencies can be best fitted to an exponential dependence on \( p \), \( \omega[p] \approx p \exp[-p/M] \). The energy for this region is

\[
\rho_{II} \simeq \frac{C}{a^4} \int_{k_B}^{k_H} dk k^3 \exp[-2k/aM] \tag{13}
\]

\[
= \frac{CM^4}{a^3} (F[x_B] - F[x_H]) \tag{14}
\]

where,

\[
F[x_i] = \left( \frac{x_i^3}{2} + \frac{3x_i^2}{4} + \frac{3x_i}{4} + \frac{3}{8} \right) \exp[-2x_i/a] \tag{15}
\]

and \( x_i = k_i/M \). Since \( x_B > 1 \) then \( F[x_B] \simeq \frac{1}{2} x_B^3 \exp[-2x_B/a] \). Thus \( \rho_{II} \) behaves as matter when averaged over many oscillations and its averaged pressure is \( \langle \bar{p}_{II} \rangle \simeq (B/a) \langle \rho_{II} \rangle \simeq 0 \). Also since \( x_H > x_B > 1 \) then \( F[x_B] \gg F[x_H] \).

**Region “H”**: This is our “tail”\(^2\), defined as the part of the graph for which the frequency of the modes is smaller than the Hubble parameter, \( H \). The functional behavior of the frequency with \( p \) is the same as in region II, therefore the averaged pressure expression for this region is the same as that of Region II, that is \( \langle \bar{p}_H \rangle \simeq 0 \). But the lower limit of integration \( k_H \) (or \( p_H \)) is given by the physical condition of the freeze-out of the modes by the expansion of the background universe,

\[
\omega_H[p_H] = H. \tag{16}
\]

This region includes the modes from \( p_H \) to \( \infty \) in the range where \( \omega \) is exponentially suppressed. The energy density of the tail is

\[
\rho_H \simeq \frac{C}{a^3} \frac{k_H^3}{2M^3} \exp[-2k_H/aM] \tag{17}
\]

Notice that due to the freeze-out, the evolution of the \( k_H \) mode is highly nontrivial and thus corrections to the averaged pressure term \( \langle \bar{p}_H \rangle \simeq 0 \) will be important.

Modes in the tail, between \( p_H \) to \( \infty \), behave differently from the other modes, since their time-dependence is controlled by the Hubble expansion, Eq. \( (16) \). On the other hand, all modes with momenta \( p \leq p_H \) redshift in the same way with the scale factor, towards decreasing values, i.e the linear regime\(^4\). Nevertheless, these regions, \( (0, I, II) \), also receive small modifications to their pressure term from the deep transplanckian regime. We show below that the modifications due to the \( p_H \)-defrosting effect, are non negligible and important only in the highly nonlinear regime, around \( p_H \).

Now, we would like to estimate the corrections to pressure, \( \Pi_i \), for all these regions, with the notation, \( P_i \) for the effective modified pressure

\[
P_i \to \langle \bar{p}_i \rangle + \Pi_i, \tag{18}
\]

\(^3\)See Appendix for details of averaging.

\(^4\)Modes in the linear regime are referred to as "normal modes".
where the index runs to \( i = 0, I, II, H \). The averaged unmodified pressure expressions, \( \langle \bar{p}_i \rangle \), are

\[
\langle \bar{p}_{0,I} \rangle \simeq \left( \frac{1}{3} - \frac{A}{a^2} + \cdots \right) \langle \rho_{0,I} \rangle ,
\]

\[
\langle \bar{p}_{II,H} \rangle \simeq \left( -\frac{B}{a} + \cdots \right) \langle \rho_{II,H} \rangle .
\]

(19)

(20)

The terms that go as inverse powers of \( a \) can be neglected and \( A, B \) are numerical constants related to the averaging (see Appendix for details). We refer to these expressions as "bare pressures" in order to distinguish them from the viscous pressure modifications terms (defined below).

In a similar manner to particle creation cases \([16]\) in imperfect fluids \([17, 18]\), the highly nontrivial time-dependence of the mode \( p_H \) and the transfer of energy between regions, due to the defrosting of this mode across the boundary \( p_H \), gives rise to pressure corrections in the fluid energy conservation law. The defrosting of the modes results in a time-dependent "particle number" for regions near \( p_H \). From kinetic theory we know that this "particle creation", (the defrosting of the modes), gives rise to effective viscous pressure modifications \([17, 18]\). The term \( \Pi_i \) denotes the effective viscous pressure modification to the "bare" pressure, \( \langle \bar{p}_i \rangle \).

The criteria we will use for modifying \( T_{\mu\nu} \) is that Bianchi identity must be satisfied \([19]\) with the new expressions for pressure\( \bar{P}_i \),

\[
\Sigma_i [\dot{\rho}_i + 3H(\rho_i + \bar{p}_i + \Pi_i)] = \Sigma_i [\dot{\rho}_i + 3H(\rho_i + \bar{P}_i)] = 0 ,
\]

(21)

with \( i = 0, I, II, H \). Let us write this expression explicitly in terms of its energy and bare pressure components, and collect the contributions of regions 0, I into one combined radiation energy, \( \rho_R = \rho_0 + \rho_I \):

\[
\dot{\rho}_{II} + 3H(\rho_{II} + \bar{p}_{II}) + \dot{\rho}_R + 3H(\rho_R + \bar{p}_R) = -3H\Pi_{II} ,
\]

(22)

\[
\dot{\rho}_{H} + 3H(\rho_{H} + \bar{p}_{H}) = -3H\Pi_{H} ,
\]

(23)

where \( \bar{p}_{II,H} \simeq 0, \bar{p}_R \simeq 1/3 \). So, we have imperfect fluids in regions II and "H", and eventually their energy is transferred, due to the redshifting effect, to regions 0 and I, which is why these regions also receive pressure modifications. Nevertheless, the viscous pressure corrections to the "radiation" modes are very small since the energy and the volume of phase space occupied by them is very large (\( a^3 \) times larger the Planck size volume). These regions are in a nearly equilibrium situations, (see Appendix). All pressure corrections are estimated below.

Let us find out \( \Pi_i \), in order to solve Eqs. (22) and (23). As explained, the presence of \( \Pi_i \) is due to the exchange of energy between the two regions, from the defrosting of the modes \( p_H \) at the boundary. This is directly related to the time dependence of the boundary \( p_H \), which in turn is going to be controlled by the Hubble parameter \( H \). In essence, there is an exchange of modes between region \( (R + II) \) and "H". Although the specific number

\[\text{From here on we drop the } \langle \cdot \cdot \rangle \text{ notation and denote the averaged "bare" pressure by simply } \bar{p}_i \text{ instead of } \langle \bar{p}_i \rangle.\]
of particles in each of these regions, \( N_{\Pi} \) and \( N_{H} \), is not conserved, their rate of change, in the physical FRW Universe, is related through the conservation of the total number of particles which contains both of these components

\[
\dot{N}_T = 0, \tag{24}
\]

Each component satisfies

\[
\dot{N}_{\Pi} = \Gamma_{\Pi}N_{\Pi}, \tag{25}
\]
\[
\dot{N}_{H} = (3H - \Gamma_{H})N_{H}, \tag{26}
\]

where the ”decay rates” of the regions \( \Gamma_i \) account for the rate of change in the number of their “particles” (modes), due to the defrosting effect.

The system is not yet in equilibrium. The change in the number of ”particles” is giving rise to the effective viscous pressure, \( \Pi_i \). Even prior to the freeze-out effects, that is, even for \( \Gamma_{H,\Pi} = 0 \), the short distance modes in region \( \Pi \) and III were out of thermal equilibrium, due to their nonlinear frequency and group velocity, \( v_g \neq 1 \).

The contribution terms to pressure, \( \Pi_i \), are related to \( \Gamma_i \) through \[17, 18\]

\[
3H\Pi_{H} = -\left(\rho_{H} + \bar{p}_{H}\right)\Gamma_{H}, \tag{27}
\]
\[
3H\Pi_{\Pi} = -\left[(\rho_{\Pi} + \bar{p}_{\Pi}) + (\rho_{R} + \bar{p}_{R})\right]\Gamma_{\Pi}. \tag{28}
\]

Therefore, Eq. (23) reduces to

\[
\dot{\rho}_{H} + (3H - \Gamma_{H})(\rho_{H} + \bar{p}_{H}) = 0, \tag{29}
\]

which can be also recast as:

\[
\dot{\rho}_{H} = \frac{\dot{n}_{H}}{n_{H}}\left(\rho_{H} + \bar{p}_{H}\right), \tag{30}
\]

where \( n_{H} = \frac{N_{H}}{a^3} \) is the “particle” number density for the region of modes from \( \rho_{H} \) to infinity. The flow of particles is described by \( n_{H} = n_{H}u_{a} \), with \( u_{a} \) the unit 4-velocity vector of the fluid. Notice that since the group velocity in the H-region is negative, particles in this region flow in a direction \( v_g \) which opposite to the direction of their momenta, \( k \). The rate \( \Gamma_{H} \), calculated below, is positive, and the increase in the number of particles \( N_{H} \) as given by \( \Gamma_{H} \) does not allow the energy density of the ”tail” to redshift as fast as matter. This indicates that although small, \( \rho_{H} \) eventually will will come to dominate the total energy density.

---

6We are loosely using the term particle here to refer to the wavepackets of the transplanckian modes, centered around a momenta \( p_i \)

7Vector objects related to the flow direction of the fluid are denoted in bold letters, e.g. \( N_i = \dot{N}_iu_a \) with \( u_a \) being the unit 4-velocity of the fluid and the corresponding modulus of this vector being \( N_i \). Notice that the factor \((3H_{N_{H}})\) in Eq. \( \#3 \) is related to the fact that the preferred frame for the tail modes is falling along comoving geodesics in the Universe.
The number of "particles" $N_H$ contained in the tail regime, in its preferred frame, is given by

$$N_H \simeq C_H \int_{k_H}^{\infty} dk k^2 \exp[-k/aM]$$

$$\simeq C_H (aM) k_H^2 \exp[-k_H/aM].$$  \hspace{1cm} (31)

where $C_H$ is the constant proportional to the Bogolubov coefficient $\beta_p$. We can now calculate the energy transfer, due to the defrosting of the modes $k_H$, between the tail region and region $II$ from the balance equation for $N_H$, Eq. (26), where $\dot{N}_H$ is:

$$\dot{N}_H \simeq C_H \int_{k_H}^{\infty} dk k^2 \left( \frac{k}{aM} \right) \exp[-k/aM] - C_H k_H^2 \exp[-k_H/aM] \dot{k}_H$$

$$\simeq 3HN_H - C_H k_H^2 \exp[-k_H/aM] (\dot{k}_H - Hk_H)$$

$$\simeq N_H \left( 3H + \frac{k_H/aM}{k_H/aM - 1} \left( \frac{\dot{H}}{H} \right) \right).$$  \hspace{1cm} (32)

In the last line we have used the approximation in Eq. (31), and:

$$\frac{\dot{p}_H}{p_H} = \frac{p_H/M}{1 - p_H/M} \left( \frac{\dot{H}}{H} \right),$$  \hspace{1cm} (33)

derived from Eq. (16). When $p_H \gg M$ (which always holds), we have for $\Gamma_H$:

$$\Gamma_H \simeq 3H + \frac{\dot{H}}{H} \simeq 3H \left( \frac{1 - w_{total}}{2} \right),$$  \hspace{1cm} (34)

where $w_{total} = \rho_{total}/\rho_{total}$. Therefore, when $w_{total} \to -1$ then $\Gamma_H$ reaches its limit, $\Gamma_H \to 3H$. $\Gamma_H$ can not change anymore once this limit is reached because the Hubble constant and the mode $p_H$ freeze to a time-independent value. Notice that $\Gamma_H$ is positive for all equations of state $w_{total} \leq 1$ and thus it slows down the dilution of the tail with the scale factor.

We can repeat the same procedure for the modes in region 0, I, II in order to obtain a closed equation for $\dot{\rho}_{R,II}$, similar to Eq. (29), i.e. that is given entirely in terms of $\rho_{R,II}$ and $\bar{p}_{R,II}$

$$\dot{\rho}_{II,R} + (3H - \Gamma_{II})(\rho_{II,R} + \bar{p}_{II,R}) = 0,$$  \hspace{1cm} (35)

where we have used Eq. (28).

Let us now try to relate $\Gamma_H$ to $\Gamma_{II}$. The total number of not frozen particles, $N_{II}$, in the region from zero to $k_H$ is given by

$$N_{II} \simeq C_H [4M^2 + Mp_H \omega H].$$  \hspace{1cm} (36)

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In our model $|\beta_p|^2$ calculated in \[3\] resulted in a scale invariant spectrum. It does not depend on the wavenumber $p$, the reason why it can be pulled out of the integral and factored into the coefficient $C_H = N|\beta_p|^2$, with $N$ an overall normalization constant we are keeping for the sake of generality.
From the total balance equation for the particle number between the two regions, \((R + II)\) and region “H” in the comoving volume, we have \(\dot{N}_{\text{total}} = 0\), where \(\dot{N}_{II} = \Gamma_{II} N_{II}\) and \(N_H = M p_H^2 \exp(-p_H/M) u = M p_H H u\). Thus

\[
\Gamma_{II} = \frac{M p_H H}{4 M^3 + M p_H H} (\Gamma_H - 3H),
\]

and \(N_T = N_{II} - C_H (M p_H \omega_H) = 4 C_H M^3\). In obtaining the scalar quantity for the number of particles \(N_T\) from their flow \(N_T\), the negative sign picked up in the second term in \(N_T\) is related to the fact that the flow of the tail’s defrosted modes is in the opposite direction to their momenta, due to their negative group velocity. Therefore, by plugging in the expression of \(N_{II}\) from Eq. (36), we get that in the limit \(N_{II} \gg M p_H H\), \(\Gamma_{II}\) is smaller than \(\Gamma_H\) and negative, given by the expression:

\[
\Gamma_{II} = -(3H - \Gamma_H) \frac{p_H H}{4M M} .
\]

Since \(p_H H^2 < M^2 H\) then \(y = \frac{p_H H}{M^2} \simeq O(H/M)\) is going to be much less than 1 for as long as the expansion is not dominated by the tail. From the condition \(\omega_H[p_H] = H\), and the time evolution of the physical momentum \(p_H\) in Eq. (33), we have that \(\dot{p}_H/p_H \to -\dot{H}/H\), when \(p_H \gg M\). The exact value of \(y\) does not matter and it is small. The tail domination case, when \(\rho_H\) becomes comparable to \(\rho_{II}\), should be treated separately since \(\Gamma_{II} \to 0\). The pressure modification \(\Pi_{II}\) increases the dilution of \(\rho_{II}\) as determined by the equation for \(\dot{\rho}_{II}\). This equation shows that due to the modified pressure effects, \(\rho_{II}\) goes to zero faster than a matter energy density component.

### 2.2 Equation of State

In this part we calculate the effective equation of state for all the regions, from the pressure expressions, \(\bar{p}_i, \Pi_i\), that were obtained in the previous section. Starting with region “H”, we have

\[
\frac{\dot{\rho}_H}{\rho_H} = (\Gamma_H - 3H)(1 + w_H) = -\frac{3H}{2}(1 + w_{total})(1 + w_H)
\]

where we have defined

\[
\frac{\dot{H}}{H} = -\frac{3}{2}(1 + w_{total}),
\]

with \(w_{total}\) referring to the effective equation of state for the total energy density. The “effective” equation of state for the tail can be read from this expression to be

\[
1 + \bar{w}_H = \frac{1}{2}(1 + w_{total})(1 + w_H),
\]

with \(w_H \simeq 0\). The time evolution for \(\rho_H\) is

\[
\rho_H = \rho_H(0) \exp[-\frac{3}{2} \int (1 + w_{total})(1 + w_H) d \ln a]
\]

\[
= \rho_H(0) \exp[-3 \int (1 + \bar{w}_H) d \ln a].
\]
During radiation domination, \( w_{\text{total}} = 1/3 \), then the effective equation of state for the tail is \( \tilde{w}_H = -1/3 \); for matter domination, \( w_{\text{total}} = 0 \), then \( \tilde{w}_H = -1/2 \); at the start of the accelerated expansion, \( q = 0 \), \( \rho_{II} = \rho_H \), we have \( w_{\text{total}} = -1/3 \), \( \tilde{w}_H = -2/3 \); and finally, if the tail dominates, then the only solution to the Friedman equation is given by \( w_{\text{total}} \simeq \tilde{w}_H \), with \( \tilde{w}_H = -1 \). Therefore, the tail starts behaving as dark energy only recently, when its equation of state \( \tilde{w}_H \) becomes close to the limiting value, \( \tilde{w}_H = -1 \).

By the same procedure, we can now estimate the effective equation of state for \( \rho_{II} \) in terms of \( \Gamma_{II} \):

\[
\frac{\dot{\rho}_{II}}{\rho_{II}} = \Gamma_{II} - 3H = -3H[1 + \left(\frac{1 - w_{\text{total}}}{2}\right)y]
\]

(44)

Thus the effective equation of state for region II, obtained from Eq. (44) is:

\[
1 + \tilde{w}_{II} = (1 + y\frac{1 - w_{\text{total}}}{2})(1 + w_{II}),
\]

(45)

where \( y = \left(\frac{\rho_{II}}{\rho_{M}}\right)(\frac{H}{M}) = O(H/M) \) and \( w_{II} \simeq 0 \).

The time evolution of the \( \rho_{II} \) energy density component is

\[
\rho_{II} = \rho_{II}(0)\exp[-3 \int (1 + \tilde{w}_{II})d\ln a],
\]

(46)

In a radiation dominated universe, \( H = M/a^2 \) therefore \( (H/M) = (1/a^2) \), and matter dominated, \( H = M/a^{3/2} \), so then \( (H/M) = (1/a^{3/2}) \). The point is that the correction \( y \) to the matter equation of state for region II is really small for up to the equality time. During most of the history of the Universe, \( y \) goes as an inverse power of the scale factor \( a \). So region II behaves pretty much like matter. The special era when the tail eventually dominates the expansion and \( H \) becomes a constant, \( (at p_H = \text{constant}, \tilde{w}_H = w_{\text{total}} = -1) \), is discussed below in Sect.3.

Similarly, by repeating the same steps, from Eq. (22), it can be shown that the modifications to pressure for \( \rho_R \), regions 0,1, are very small indeed. Thus their effective equation of state remains very nearly that of radiation, \( \tilde{w}_R = 1/3 \). In order to avoid repetition, we will not carry out the calculation for the effective equation of state of \( \rho_R \), as the procedure is essentially the same as for \( \rho_{II} \), and it results in an effective radiation equation of state

\[
1 + \tilde{w}_R = (1 + y\frac{1 - w_{\text{total}}}{2})(1 + w_R),
\]

(47)

3 The Issue of Coincidence and Comparison to Observation

From the computation of the pressures \( \bar{p}_H \) and \( \Pi_i \), it is clear that the initial radiation is redshifted faster than the other components of the total energy density. We can ask the question at what time, \( t_{eq} \), the \( \rho_{II} \) components of matter becomes comparable to radiation

\[
\rho_R \simeq \rho_{II} \simeq \frac{\rho_{\text{total}}}{2} = \frac{3}{2}H^2_{eq}M^2,
\]

(48)
\[ \rho_{II} \simeq C p_B^3 \exp[-2p_B/M] \simeq \frac{3}{2} H_{eq}^2 M^2. \]  

(49)

From the above equation we obtain \( p_B \) at \( t_{eq} \):

\[ p_B \simeq \frac{M}{2} \ln \left( \frac{2Cp_B^3}{3H_{eq}^2 M^2} \right). \]  

(50)

On the other hand, from Eq. (16) we have:

\[ p_H \simeq M \ln \left( \frac{p_H}{H} \right), \]  

(51)

and comparing Eqs. (50) and (51), it is clear that \( p_B(t_{eq}) < p_H(t_{eq}) \), and therefore

\[ \rho_H(t_{eq}) < \rho_{II}(t_{eq}). \]  

(52)

So matter-radiation equality takes place well before the eventual “tail” domination. From the equations of state, \( \tilde{w}_H \) and \( \tilde{w}_{II} \), Eqs. (11) and (15), we have that \( \rho_{II} \) always dilutes faster than \( \rho_H \). Thus the inequality in Eq. (60) holds true not only at \( t = t_{eq} \) but at all earlier times before \( t_{eq} \). Generally, there may be other sources of matter and radiation in the Universe, besides the contribution from the transplanckian modes. Although these components would not be affected by the viscous pressure corrections, \( \Pi_i \), their contribution should be included in the Friedman equation when determining the equality time, \( a(t_{eq}) = a_{eq} \). Since their effect to the expansion is well studied and known, here we chose to focus our attention only on the role of the transplanckian modes.

Let us now estimate the time at which the tail takes over to dominate the expansion and address the issue of the cosmic coincidence. As we will see below, the effective equation of state \( \tilde{w}_{II} \) for \( \rho_{II} \) changes from \( \tilde{w}_{II} \simeq 0 \) to \( \tilde{w}_{II} \geq 0 \). Let us start by asking at what time, \( a_{DE} \), we have

\[ \rho_H = \rho_{II} = \rho_{total}/2 \]  

(53)

or in terms of the density parameters \( \Omega_H = \Omega_{II} \). From the Friedmann Eq. for the expansion and the relation of \( \tilde{w}_H \) to \( w_{total} \) it is straightforward to find out that at \( a = a_{DE} \) we have:

\[ \tilde{w}_H(a_{DE}) = -\frac{2}{3}, \quad w_{total}(a_{DE}) = -\frac{1}{3}, \]  

(54)

and therefore

\[ a_{DE} = \left( \frac{\rho_H(0)}{\rho_{II}(0)} \right)^{2/(w_{total}-1)} = \left( \frac{\rho_H(0)}{\rho_{II}(0)} \right)^{1/2} = \left( \frac{\Omega_H(0)}{\Omega_{II}(0)} \right)^{1/2} \left( \frac{3}{a^3_{eq}} \right)^{1/2}. \]  

(55)

with \( \rho_i(0) \) being the value of the i-th component at equality time, \( a_{eq} \). It is interesting to notice that \( w_{total} = -1/3 \) corresponds to the transition time where the deceleration parameter,

\[ q \simeq \frac{1}{2}(3w_{total} + 1), \]  

(56)
changes sign and goes through zero. This means that acceleration starts at the same time \( a_q \) as the dominance of the tail, \( a_{DE} \), i.e., \( a_q = a_{DE} \). Using the Friedman expansion law, we can find the solution for the scale factor \( a \), after the time \( a_q \):

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\rho_I^{(0)}}{3M^2} + \frac{\rho_H^{(0)}}{3M^2} = \frac{H^2(a_q)}{2} \left[ \left( \frac{a_q}{a} \right)^2 + \frac{a_q}{a} \right].
\]  

Therefore,

\[
\int_1^{a/a_q} \frac{a/a_q}{(a/a_q)^2 + 1} d(a/a_q) = \frac{H(a_q)}{\sqrt{2}} (t - t_q).
\]  

This integral can be done exactly and it is messy. The important point about it is that it gives a power law accelerated expansion, \( \frac{a}{a_q} \sim t^n \) with \( n \geq 2 \). Clearly the tail is behaving as dark energy and it is dominating the expansion soon after \( a_q \).

Let us understand physically what is happening around the time \( a = a_q \), and why \( a_q = a_{DE} \). As showed in Sect. 2, due to the strong coupling of the tail evolution to the Hubble constant, \( H \), and therefore a coupling to \( \rho_{total} \), \( \bar{w}_H \) becomes more and more negative with decreasing values of \( w_{total} \), soon after \( a_{eq} \). Thus the tail starts behaving as dark energy, dominates the expansion, and approaches its limiting value \( \bar{w}_H = -1 \) only at late times, when other energy contributions to \( \rho_{total} \) become negligible. From the Eqs. (57) and (58), when \( \rho_{II} \simeq \rho_H \) we obtain \( \Gamma_{II} \simeq -2y\Gamma_H \). When the tail comes to dominate, the only solution to the Friedman equation is \( \Gamma_H \rightarrow 3H \), and then \( w_{total} = \bar{w}_H \simeq -1 \). This means that the tail dominates the expansion \( (w_{total} = -1) \) very fast and, around that time \( \rho_{II} \) has become nearly zero due to the \( \Gamma_{II} \) viscous pressure corrections. Recall, that in this estimation we assumed that around the time \( a_q \), \( \rho_{II} \) is the only source of matter. The fast dilution of \( \rho_{II} \) as compared to \( \rho_H \), due to the viscous pressure effects of \( \Gamma_{II} \), is the reason why \( q = 0 \) occurs at the same time as the tail dominance, \( a_{DE} \). With no other sources of matter, \( w_{total} \) almost immediately goes from \( w_{total} = -1/3 \) to \( w_{total} = -1 \). Thus \( a_{DE} \) has occurred very recently indeed. If we consider other matter contributions in the Friedman equation, that are independent of \( \rho_{II} \), \( \Gamma_i \), then the time the expansion takes between the start of the accelerated expansion \( a_q \), and the time of tail dominance time over \( \rho_{total} \), (i.e. when \( w_{total} = w_H \simeq -1 \)) becomes a bit longer. This time interval from \( a_q \) to present is also the time interval when the tail has acquired a dark energy equation of state, until it reaches its limiting value, \( \bar{w}_H = -1 \).

Therefore, cosmic coincidence is explained naturally by the intrinsic time evolution of the effective equation of state for the tail, \( \bar{w}_H \).

4 Summary

Models of nonlinear short distance physics discussed recently in literature [8, 9, 11, 12, 13, 15], usually introduce a time dependent frequency, at the level of the equation of motion for the field. As a result Bianchi identity is generically violated which indicates that Einstein equations need to be modified in the high energy regime. It is difficult to
do so without an effective lagrangian description of the theory in the transplanckian (TP) regime.

We therefore used a kinetic theory approach, in order to estimate the short distance modification to the cosmic fluid stress energy tensor for the model of [2]. It is not clear to us whether this procedure determines the modifications in a unique unambiguous way, or whether fluid idealization and the assumption that kinetic theory remains valid at such high wavenumber modes, is a good approximation. Nevertheless, we believe that without an effective lagrangian, kinetic theory is the only available tool to get some sensible results for the contribution that TP modes have in the long wavelength regime.

In a previous paper [2], we showed that the energy contribution of the tail modes is comparable in magnitude to the observed dark energy in the universe. In this work we calculated the effective equation of state, \( \tilde{w}_H \), for these tail modes in order to address the cosmic coincidence issue and showed that the tail modes behave as dark energy only at late times.

The tail has an exponentially suppressed frequency and all the modes with \( \omega < H \) are frozen out by the expansion of the background universe. However, due to the short distance pressure modification, the tail does not always behave as dark energy. The highly nontrivial time-dependence of tail’s dominant mode \( p_H \), tracks the evolution of the total energy density \( \rho_{\text{total}} \) through its strong dependence on the Hubble constant, \( H \). The dependence of \( p_H \) on \( H \) is given by the freeze-out condition.

As a result of the coupling of the \( p_H \) mode to \( \rho_{\text{total}} \), the tail equation of state \( \tilde{w}_H \) follows the evolution of \( w_{\text{total}} \), such that \( \tilde{w}_H \propto \frac{(w_{\text{total}} - 1)}{2} \). For this reason, \( \tilde{w}_H \) acquires more and more negative values as \( w_{\text{total}} \) decreases, from radiation to matter. The tail has a slower dilution with the scale factor compared to the other components and it starts dominating the expansion and behaving as dark energy only recently, from the time when the deceleration parameter \( q \) changes sign at time \( a_q \). From this point, \( a_q \) (with \( w_{\text{total}} = -1/3, \tilde{w}_H = -2/3 \)) and onwards, the tail drives the Universe into an accelerated expansion, and soon reaches its limiting value of \( w_{\text{total}} \approx \tilde{w}_H \approx -1 \) with \( \rho_{\text{total}} \approx \rho_H \). Therefore, cosmic coincidence in this model is explained naturally from the time evolution of the tail’s \( \tilde{w}_H = f[w_{\text{total}}] \). This is the most important result of this paper.

The TDE model was motivated by closed string theory of Brandenberger-Vafa model [3]. Therefore its observational implications may be explored as indirect string signatures. Some of the distinctive features of the TDE model are the predictions that [4]: the change in sign of the deceleration parameter, \( q = 0 \), occurs at the same time as the start of tail dominance, ie. the time when the tail energy is half of the total, \( a_q = a_{DE} \); this accelerated expansion occurs very recently; due to the viscous pressure effects, the matter contribution from the TP modes goes through a change of its equation of state such that it starts decaying faster than normal matter. The latter effect shortens the time the Universe takes to change the parameters from \( w_{\text{total}} \approx -1/3, q = 0 \) to the time when \( w_{\text{total}} \approx -1, \rho_{\text{total}} \approx \rho_H \). To get these numbers, we ignored other matter sources and also the short distance corrections to the equations of state \( \langle w_i \rangle \) that go as inverse powers of the scale factor, \( O(1/a^n) \). Perhaps these corrections may be important recently, in delaying the time it takes the tail to behave

\[ \text{[These predictions are in the absence of other matter sources.]} \]
as dark energy, $\bar{w}_H \simeq -1$. These features can be scrutinized with respect to observation [20].

*Note Added*: Results obtained in this paper for the tail’s equation of state, $\bar{w}_H$, differ from those reported in Lemoine et al. [21].

The fact that we include the curvature term $a''/a$ under the definition of the generalized frequency $\omega^2$, while they keep the two contributions separate, is not the only source of discrepancy. There are fundamental differences between the two studies. Authors of [21] ignore the crucial effects of the out-of-equilibrium dynamics and the breaking of Lorentz invariance, by the nonlinear short distance modes, when carrying out their calculation for $\rho_H$ and $p_H$. In their approach these effects would be contained in the dynamics of the vector field, $u_\mu$ described by a lagrangian $\mathcal{L}_u$ for this field. Obviously, the choice of $u_\mu$ field dynamics and its Lagrangian $\mathcal{L}_u$ *strongly depend on the details of the short distance nonlinear model considered*. The expression for $\mathcal{L}_u$ these authors borrow from the Corley-Jacobson (CJ) model [7] is *consistent only with the dispersion function of the CJ model for stationary backgrounds*, since both $\mathcal{L}_{cor}$ and the terms given by $\mathcal{L}_u$ contain up to 4-th order derivatives, together with the antisymmetric tensor $F_{\mu\nu}$. Therefore, it should come as no surprise that $\mathcal{L}_u$ gives zero corrections to $\rho_H$, $p_H$ when applied to the FLRW Universe (where clearly the antisymmetric tensor vanishes identically $F_{\mu\nu}$) and hence, higher order counterterms $(u_\mu)^n$ in $\mathcal{L}_u$ are ignored, while at the same time in $\mathcal{L}_{cor}$ they have a series of all higher order derivative terms, up to $n \to \infty$. It is easy to check they break energy conservation law and Bianchi identity, by plugging in the energy conservation equation their expression for $\rho$ and $p$ in their Eqs.(27) and (28)

$$\langle \dot{\rho} \rangle + 3H\langle \rho + p \rangle \neq 0 \quad (59)$$

which we suspect is due to the abovementioned reasons.

Perhaps they could reconstruct their formalism, either by identifying the correct $\mathcal{L}_u$ which would be appropriate for the model given by their $\mathcal{L}_{cor}$, or by defining an inner product (following the construction in [13], $(\phi_{in},\phi_{in}) = \int dx \phi F(k)\phi$) for the fields in such a way that it accounts for the *nonlinear dynamics and Lorentz noninvariance* of short distance modes with frequency $F(k)$, in an *expanding* Universe. However these issues extend beyond the scope of this paper and we do not intend to elaborate further.

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Appendix

5.1 Distribution function for the Linear, Crossover and Transplanckian regime.

We have a time translation Killing vector for future infinity that determines our outgoing positive frequency modes. Let us consider our Universe as an expanding box, with size $L = a/M$, filled with modes. Inside this box we have a smaller box with fixed size $l_p \simeq 1/M$ that determines the range for the transplanckian (TP) modes. There is a preferred frame attached to the small box (due to the breaking of Lorentz invariance by the short-distance modes) but Lorentz invariance is restored in the big box, the expanding Universe. Following along the arguments and derivation in [15] for the ‘particle’ distribution function, it is straightforward to apply their expression to our model. Below we consider three regimes depending on the value of the momentum $p$ with respect to the cuttoff scale $M$

(a) $p/M \ll 1$, the “normal regime”. In this regime the frequency of the modes becomes linear,

$$\omega[p] \simeq p e^{-p/M} \rightarrow p/M \ll 1 \quad p.$$  \hspace{1cm} (60)

The wavelength of these modes is then $\lambda \simeq O(L) \gg l_p$.

(b) $p/M \simeq O(1)$, the crossover or intermediate regime during which the TP modes go from the “TP box” with fixed size $l_p = 1/M$ into the “big box” with size $L = a/M$. This process occurs due to the redshifting of the modes, $p_i = k_i/a$. Each mode $p_i$ will crossover and become a “normal mode” at some time $a_i = k_i/M$.

(c) $p/M \gg 1$, the TP regime, such that $\lambda \simeq 1/k \ll l_p = 1/M$.

We do not report the derivation of the distribution function since the reader can find it in detail in [15]. In what follows we apply it to our case, to lend support to our assumption.
that the TP modes, *modes in the small box with fixed size* \( l_p = 1/M \) *with respect to the preferred frame*, are not in thermal equilibrium, while modes in the range of the linear regime, *(modes in the big box with size* \( L=a/M \), *where Lorentz invariance is restored, the FRW Universe)* are in a nearly thermal equilibrium situation.

\[
\mathcal{N}(\omega) = \frac{\left[(\omega_{in})v_g(p_{in})\right]}{\left[(\omega_{out})v_g(p_{out})\right]} \left| \frac{\beta_p}{\alpha_p} \right|^2.
\]

(61)

The index *in (out)* refers to the incoming (outcoming) modes as defined in [2]; \( v_g \) is the group velocity,

\[
v_g(p) = \frac{d\omega}{dp},
\]

(62)

and \( \beta_p, \alpha_p \) are the Bogoliubov coefficients, which in our model do not depend on the momentum \( p \) [2]. The dispersed frequency is given in Eq. (5), and thus

\[
|v_g(p_{in})| = \frac{\omega_{in}}{p_{in}} \left| 1 - \frac{p_{in}}{M} \right|,
\]

(63)

and

\[
v_g(p_{out}) \approx 1,
\]

(64)

since \( \omega(p_{out}) \approx p_{out} \). The phase velocity is defined as \( v_c = \omega/p \). Therefore

\[
\mathcal{N}(\omega) = \left| \frac{\beta_p}{\alpha_p} \right|^2 e^{-p/M_P} \left( e^{-p/M_P} \left( 1 - \frac{p}{M_P} \right) \right).
\]

(65)

The thermal distribution is immediately recovered in the limit \( p/M \ll 1 \), i.e., \( v_g \to 1 \) and \( \omega(p) \to p \).

Now we can evaluate the distribution function for the different regimes above:

(a) In this case

\[
\mathcal{N}_a(\omega) \approx \left| \frac{\beta_p}{\alpha_p} \right|^2,
\]

(66)

as it should be since we recover in the *out region* the normal linear frequency for the modes, \( p/M \ll 1 \) and \( v_g \approx 1 \).

(b) In the intermediate crossover regime, \( p/M \approx O(1) \), we have

\[
\mathcal{N}_b(\omega) \approx \frac{C}{2} \frac{e^{-2}}{e^{\beta \omega} - 1},
\]

(67)

where we have identified the inverse of temperature \( \beta \approx a \) and the linear term in \( p/M \) with

\[
\beta \omega \approx \frac{1}{1 - p/M} = a \frac{M}{aM - k}, \quad \beta \approx a.
\]

(68)

Clearly in this regime \( |v_g| \) goes to zero, and \( \beta \omega \) goes to infinity. The spectrum is nearly thermal, however \( \mathcal{N}(\omega) \) goes to zero, since as seen from the TP box the group velocity of
these modes as they approach \( p \approx M \) is becoming zero; or as seen from the normal particles in the big box, these modes have a high frequency \( (\omega \approx M) \) thus they do not contribute very much to the energy of modes in (a).

(c) However in the TP regime the distribution function strongly deviates from that of thermal equilibrium, since in this case \( v_g(n) \neq 1 \), and the frequency is highly nonlinear:

\[
\mathcal{N}_c(\omega) \approx \frac{C}{2} \left| e^{-p/M} \right| e^{-p/M} \left| 1 - \frac{p}{M} \right|, \tag{69}
\]

with \( p/M \gg 1 \). Nevertheless \( \mathcal{N}_c(\omega) \to 0 \) when \( p/M \to \infty \), thus their contribution to the energy is suppressed. The suppression comes directly from the frequency \( \omega(p) \approx pe^{-p/M} \) in this case.

The volume element in momentum space, \( dV_P \), for the dispersed ‘particles’, whose world-line intersect an hypersurface element \( d\Sigma \) around \( x \), having momenta in the range \((p, p+dp)\) is

\[
dV_P = 2\delta(p_\mu p^\mu) dp^4, \tag{70}
\]

which is consistent with the quantum field theory expressions of currents and ‘particle’ number densities, given in terms of creation and annihilation operators.

The distribution function, Eq. (69), justifies our assumptions of deviation from thermal equilibrium in the TP regime.

5.2 Averaging of the TP energies, pressure and equation of state.

We have shown that modes in the TP regime of such very short wavelength \( \lambda_{TP} \ll l_P \) are out of thermal equilibrium. Thus we need to average their effective contribution to the energy and pressure over many wavelengths, in order to obtain a nearly thermal, large scale state. This averaging is done over many wavelengths since clearly scales of cosmological order that are of interest to us, are much much longer than any TP wavelengths. In what follows, we are interested to find the averaged bare quantities \( \langle \rho_i \rangle \) and \( \langle \bar{p}_i \rangle \) before including any modification \( \Pi_i \). The viscous pressure modifications, \( \Pi_i \), occur due to the freeze-out and the change in the number of particles and are accounted for separately in Sect.2. Therefore region II will be grouped together with region H, since both have a highly dispersed TP frequency and, for the moment, we are ignoring the freeze-out corrections. Approximately we can write the energy of regions \((0+I)\) and \((II+H)\) in one compact form to avoid repetition:

\[
\rho = p^4 \Theta(M - p) + \frac{M}{2} p^3 e^{-(p/M)\Theta(p-M)}, \tag{71}
\]

where \( \Theta(p-M) \) is the unit step function that takes the value \( \Theta(p-M) = 1 \) for \( p > M \) and zero otherwise. Clearly, for modes with \( p/M \gg 1 \) we get \( \rho_{II} (\rho_H) \). And for modes \( M > p \) we get the radiation energy density of the (nearly) linear modes.
Using the energy conservation law (while ignoring the freeze-out effects):

\[ \rho + \bar{p} = \frac{p}{3} \frac{dp}{dp} = \frac{p}{3} \left( \frac{d\rho_1}{dp} + \frac{d\rho_2}{dp} \right) = (\rho_1 + \bar{p}_1) + (\rho_2 + \bar{p}_2), \]  

(72)

with

\[ \rho_1 = \frac{M}{2} p^3 e^{-p/M}, \quad p > M, \]  

(73)

\[ \rho_2 = p^4, \quad p < M. \]  

(74)

And from Eq. (72) we find:

\[ w_1 = \frac{\bar{p}_1}{\rho_1} = -\frac{k_B}{3aM} \Theta(k_B - aM), \]  

(75)

\[ w_2 = \frac{\bar{p}_2}{\rho_2} = \frac{1}{3}. \]  

(76)

It is clear from the previous section that since the (nearly) linear modes, \( \rho_2 \) are nearly in thermal equilibrium then we do not need to bother with the averaging procedure for them. (One can however check to verify the result \( \langle w_2 \rangle = 1/3 \)). This is not the case for the modes with \( p/M > 1 \), since these short wavelengths are out of equilibrium. Let us calculate \( \langle \bar{p}_1 \rangle, \langle \rho_1 \rangle, \) and \( \langle w_1 \rangle \):

\[ \langle \rho_1 \rangle = \frac{\int_0^a \rho_1 a^2 da}{\int_0^a a^2 da} = 3 \frac{\int_0^{a^*} a^2 da}{\int_0^{a^*} a^2 da} \rho_1 a^2 da \approx \frac{14M^4a^*}{16a^3}, \]  

(77)

Similarly,

\[ \langle \bar{p}_1 \rangle = \frac{\int_0^a \bar{p} a^2 da}{\int_0^a a^2 da} = 3 \frac{\int_0^{a^*} a^2 da}{\int_0^{a^*} a^2 da} \bar{p} a^2 da = -\frac{14a^*}{20a} \langle \rho_1 \rangle. \]  

(78)

and

\[ \langle w_1 \rangle = \frac{\langle \bar{p}_1 \rangle}{\langle \rho_1 \rangle} = 0 - \frac{14a^*}{20a} \approx -\frac{B}{a}, \]  

(79)

where in Planck units, \( a^* = l_p \) and we have used the normalization that Planck length \( l_p = 1/M = a^* = a(t_P) = 1 \). The time \( a^* \) corresponds to the crossover time when the wavelength of the TP modes is on average of order the size of the small box, \( p(a^*) = M \). This time scale of order the Planck box is much smaller than scales of cosmological interest, \( L = a/M \). External observers bound to the large scale, Lorentz invariant Universe with size \( L = a/M \) and in thermal equilibrium, 'feel' the energy and pressure contributions from the TP modes given by Eq. (69).

Of course the real equation of state for these modes is given by their effective equation of state, \( \tilde{w}_i \), Sect.2, that takes into account the relativistic kinetic theory modifications due to the freeze-out.

In a similar manner, one can obtain \( \langle w_2 \rangle \) and in particular the numerical coefficient \( A \) for \( \langle w_1 \rangle, \bar{p}_1, \rho_1 \).