Central Submonads and Notions of Computation

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Overview

- For any monoid $M$, its centre $Z(M)$ is a commutative submonoid;
- For any semiring $R$, its centre $Z(R)$ is a commutative subsemiring.
- For any group $G$, its centre $Z(G)$ is a commutative subgroup (aka abelian subgroup);
- What about monads?
  - Context:
    - A symmetric monoidal category $(C, I)$,
    - A strong monad $(T, \eta, \mu, \tau)$.
  - We wonder:
    - Is there a commutative submonad of $T$ which is its centre? When does it exist?
    - Is there an appropriate computational interpretation?
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- a strong monad $(\mathcal{T}, \eta, \mu, \tau)$.

We wonder:
- Is there a commutative submonad of $\mathcal{T}$ which is its centre? When does it exist?
- Is there an appropriate computational interpretation?
Background
Given a monoid $M$, its centre is defined as

$$Z(M) \overset{\text{def}}{=} \{ x \in M \mid \forall y \in M. \ x \cdot y = y \cdot x \}.$$ 

Notice there is an implicit swap in the arguments.

**But**, the definition of a monad is independent of any monoidal structure on the base category.

Unclear how to define a suitable notion of centre for such monads.
The Strength of a Monad

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- Notice there is an implicit swap in the arguments.

- *But*, the definition of a monad is independent of any monoidal structure on the base category.

- Unclear how to define a suitable notion of centre for such monads.

- Instead, we introduce the centre for *strong* monads acting on symmetric monoidal categories.

- The monadic strength is a natural transformation

$$\tau_{X,Y} : X \otimes T Y \to T (X \otimes Y)$$

that satisfies some coherence conditions w.r.t. monoidal structure.

- The monadic left strength is a natural transformation

$$\tau'_{X,Y} : T X \otimes Y \to T (X \otimes Y)$$

that may be defined via $\tau$ and the monoidal symmetry.
Commutative Monads

**Definition (Commutative Monad)**

A strong monad $\mathcal{T}$ is said to be *commutative* if the following diagram:

\[
\begin{align*}
\mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau_{\mathcal{T}X, \mathcal{T}Y}} \mathcal{T}(\mathcal{T}X \otimes \mathcal{Y}) & \xrightarrow{\mathcal{T}\tau'_{X, Y}} \mathcal{T}^2(X \otimes Y) \\
\mathcal{T}(X \otimes \mathcal{T}Y) & \xrightarrow{\mathcal{J}_{\mathcal{T}X, \mathcal{T}Y}} \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} \mathcal{T}(X \otimes Y)
\end{align*}
\]

commutes for every choice of objects $X$ and $Y$. 
The Centre of a Monad on Set
The first example

Given a monoid \((M, e, m)\), the writer monad: \((M \times -) : \textbf{Set} \to \textbf{Set}\) has the following monad structure:

- \(\eta_X : X \to M \times X :: x \mapsto (e, x)\);
- \(\mu_X : M \times (M \times X) \to M \times X :: (z, (z', x)) \mapsto (m(z, z'), x)\),
- \(\tau_{X,Y} : X \times (M \times Y) \to M \times (X \times Y) :: (x, (z, y)) \mapsto (z, (x, y))\).

What should be the centre? What about \(Z(M) \times -\)? Indeed, it is a commutative submonad of \((M \times -)\).
$\mathcal{T} : \textbf{Set} \rightarrow \textbf{Set}$ is said to be \textit{commutative} if the following diagram:

\[
\begin{array}{ccc}
\mathcal{T} X \times \mathcal{T} Y & \xrightarrow{\tau_{\mathcal{T} X, \mathcal{T} Y}} & \mathcal{T} (\mathcal{T} X \times Y) \\
\downarrow{\tau'_{X, \mathcal{T} Y}} & & \downarrow{\mathcal{T}\mu_{X \times Y}} \\
\mathcal{T} (X \times \mathcal{T} Y) & \xrightarrow{\mathcal{T}\tau_{X, Y}} & \mathcal{T}^2 (X \times Y) \\
\mu_{X \times Y} & & \mu_{X \times Y}
\end{array}
\]

commutes for every choice of sets $X$ and $Y$. 
\( \mathcal{T} : \text{Set} \rightarrow \text{Set} \) is said to be \textit{commutative} if the following diagram:

\[
\begin{array}{ccc}
\mathcal{T} X \times \mathcal{T} Y & \xrightarrow{\tau_{\mathcal{T} X, \mathcal{T} Y}} & \mathcal{T}(\mathcal{T} X \times Y) & \xrightarrow{\mathcal{T} \tau_{\mathcal{T} X, Y}'} & \mathcal{T}^2(X \times Y) \\
\downarrow{\tau'_{\mathcal{T} X, \mathcal{T} Y}} & & \downarrow{\mathcal{T} \tau_{\mathcal{T} X, Y}'} & & \downarrow{\mu_{X \times Y}} \\
\mathcal{T}(X \times \mathcal{T} Y) & \xrightarrow{\mathcal{T} \tau_{\mathcal{T} X, Y}} & \mathcal{T}^2(X \times Y) & \xrightarrow{\mu_{X \times Y}} & \mathcal{T}(X \times Y)
\end{array}
\]

commutes for every choice of sets \( X \) and \( Y \).

How would you define a \textit{central submonad} \( \mathcal{Z} \) of \( \mathcal{T} \)?
The trick is to consider all the monadic elements of $TX$ that make the previous diagram commute.

**Definition (Centre)**

Given a set $X$, the *centre* of $\mathcal{T}$ at $X$, written $\mathcal{Z}X$, is defined to be the set

$$\mathcal{Z}X \overset{\text{def}}{=} \{ t \in TX \mid \forall Y \in \text{Ob}(\textbf{Set}). \forall s \in \mathcal{T}Y. \mu(\mathcal{T}\tau'(\tau(t, s))) = \mu(\mathcal{T}\tau(\tau'(t, s))) \}.$$ 

We write $\iota_X : \mathcal{Z}X \subseteq TX$ for the indicated subset inclusion.
The Centre

- **Lemma**: The assignment $\mathcal{Z}(-)$ extends to a functor $\mathcal{Z} : \text{Set} \to \text{Set}$ when we define
  \[ \mathcal{Z}f \overset{\text{def}}{=} T f|_{\mathcal{Z}X} : \mathcal{Z}X \to \mathcal{Z}Y, \]
  for any function $f : X \to Y$.

- **Lemma**: For any two sets $X$ and $Y$, the monadic unit $\eta_X : X \to TX$, the monadic multiplication $\mu_X : T^2X \to TX$, and the monadic strength $\tau_{X,Y} : X \times TY \to T(X \times Y)$ (co)restrict respectively to functions $\eta^\mathcal{Z}_X : X \to \mathcal{Z}X$, $\mu^\mathcal{Z}_X : \mathcal{Z}^2X \to \mathcal{Z}X$ and $\tau^\mathcal{Z}_{X,Y} : X \times \mathcal{Z}Y \to \mathcal{Z}(X \times Y)$.

- **Theorem**: The assignment $\mathcal{Z}(-)$ extends to a commutative submonad $(\mathcal{Z}, \eta^\mathcal{Z}, \mu^\mathcal{Z}, \tau^\mathcal{Z})$ of $T$ with $\iota_X : \mathcal{Z}X \subseteq TX$ the submonad morphism. Furthermore, there exists a canonical\(^1\) isomorphism $\text{Set}_\mathcal{Z} \cong \mathcal{Z}(\text{Set}_T)$.

\[\text{\(^1\)Details later.}\]
Examples

- Continuation monad: \( \mathcal{T} = [[-, S], S] : \text{Set} \to \text{Set}. \)
Examples

- Continuation monad: \( T = [\_, S], S : \text{Set} \to \text{Set} \).
  - \( \mathcal{Z}X = \eta_X(X) \cong X \),
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  - \( c \mathcal{S} X = \eta_X(X) \cong X \),
  - The image of the monadic unit is always in the centre.
Examples

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  - $\mathcal{Z}X = \eta_X(X) \cong X$,
  - The image of the monadic unit is always in the centre.
  - The centre is naturally isomorphic to the identity monad; therefore the centre is trivial.
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- If \( \mathcal{T} \) is commutative, its centre is itself.
Examples

- Continuation monad: $\mathcal{T} = [[\_ , S], S] : \textbf{Set} \rightarrow \textbf{Set}$.
  - $\mathcal{Z}X = \eta_X(X) \cong X$,
  - The image of the monadic unit is always in the centre.
  - The centre is naturally isomorphic to the *identity monad*; therefore the centre is trivial.
- If $\mathcal{T}$ is commutative, its centre is itself.
- The centre of $(M \times \_)$ is indeed $(Z(M) \times \_)$. 
Link with Lawvere theories

- In a Lawvere theory $T$, we say that $f: A^n \to A^{n'}$ and $g: A^m \to A^{m'}$ commute if and only if $f^{n'} \circ g^n$ (also written $f \star g$) and $g^{n'} \circ f^m$ (also written $g \star f$) are equal, up to isomorphism.

- If $S$ is a subcategory of $T$, the commutant of $S$ in $T$ is a subcategory of $T$ whose morphisms commute with the morphisms of $S$. This commutant is written $S^\perp$, and is also a Lawvere subtheory of $T$.

- Considering this, $T^\perp$ is seen as the centre of the Lawvere theory $T$.

- From $T$ arises a finitely strong monad $\mathcal{T}$ on $\text{Set}$, and its centre $\mathcal{Z}$ is the monad of $T^\perp$. 
Central Submonads in Symmetric Monoidal Categories
A *central cone* of $\mathcal{T}$ at $X$ is given by a pair $(Z, \iota)$, an object $Z$ and a morphism $\iota : Z \to \mathcal{T}X$, such that the diagram:

\[
\begin{array}{cccccc}
Z \otimes \mathcal{T}Y & \xrightarrow{\iota \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \longrightarrow & \mathcal{T}(X \otimes \mathcal{T}Y) \\
\downarrow \iota \otimes \mathcal{T}Y & & \downarrow \mathcal{T}\tau_{X,Y} & & \\
\mathcal{T}X \otimes \mathcal{T}Y & & \mathcal{T}^2(X \otimes Y) & & \text{commutes.}
\end{array}
\]
Definition (Central Submonad)

Given a strong monad \((S, \eta^S, \mu^S, \tau^S)\) which is a submonad of \(T\) with monad monomorphism \(\iota\), we say that \(S\) is a central submonad of \(T\) if for any object \(X\), \((SX, \iota_X)\) is a central cone for \(T\) at \(X\). Besides, this last condition implies that \(S\) is commutative.
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- There always is at least one central submonad for \(T\): the identity functor is one;
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- There always is at least one central submonad for \(T\): the identity functor is one;
- They form a category with strong monad morphisms. If the category has a terminal object, the latter is the centre of \(T\).
Centralisable Monads in Symmetric Monoidal Categories
Centralisable Monad

If \((Z, \iota)\) and \((Z', \iota')\) are two central cones of \(T\) at \(X\), then a *morphism of central cones* \(\varphi : (Z', \iota') \to (Z, \iota)\) is a morphism \(\varphi : Z' \to Z\), such that \(\iota \circ \varphi = \iota'\).
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A terminal central cone is a terminal object in the category of central cones. Its morphism component always is a monomorphism.
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**Definition**

We say that the monad \(T\) is *centralisable* if for any object \(X\), a terminal central cone of \(T\) at \(X\) exists. We write \((\mathcal{E}X, \iota_X)\) for the terminal central cone of \(T\) at \(X\).
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We say that the monad \(\mathcal{T}\) is *centralisable* if for any object \(X\), a terminal central cone of \(\mathcal{T}\) at \(X\) exists. We write \((Z^X, \iota^X)\) for the terminal central cone of \(\mathcal{T}\) at \(X\).

**Theorem**

The assignment \(Z(\_\_)\) extends to a commutative submonad \((Z, \eta^Z, \mu^Z, \tau^Z)\) of \(\mathcal{T}\) with \(\iota : Z \Rightarrow \mathcal{T}\) the submonad monomorphism.

Note that a submonad morphism induces a canonical embedding \(\mathcal{I} : \mathcal{C}_Z \rightarrow \mathcal{C}_\mathcal{T}\).
Kleisli Categories and Premonoidal Categories
Premonoidal category

- If $\mathbf{C}$ is symmetric monoidal and $\mathcal{T}: \mathbf{C} \to \mathbf{C}$ a strong monad;
- then $\mathbf{C}_{\mathcal{T}}$ does not necessarily have a monoidal structure,

Definition (Central morphism \[Power and Robinson, 1997\])

A morphism $f: X \to Y$ in $\mathbf{C}_{\mathcal{T}}$ is central if for any morphism $f': X' \to Y'$ in $\mathbf{C}_{\mathcal{T}}$, the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

commutes in $\mathbf{C}_{\mathcal{T}}$.

Central cones and central morphisms are actually equivalent notions!
Premonoidal category

- If $\mathbf{C}$ is symmetric monoidal and $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ a strong monad;
- then $\mathbf{C}_\mathcal{T}$ does not necessarily have a monoidal structure,
- $\mathbf{C}_\mathcal{T}$ has a premonoidal structure [Power and Robinson, 1997].
If $\mathbf{C}$ is symmetric monoidal and $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ a strong monad;
then $\mathbf{C}_\mathcal{T}$ does not necessarily have a monoidal structure,
$\mathbf{C}_\mathcal{T}$ has a \textit{premonoidal structure} [Power and Robinson, 1997].
there are two families of functors $(- \otimes_l X')$ and $(X \otimes_r -)$ on $\mathbf{C}_\mathcal{T}$. 

\textbf{Central morphism} [Power and Robinson, 1997]

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$\mathbf{C}_\mathcal{T}$ has a *premonoidal structure* \cite{power1997premonoidal}.
there are two families of functors $(- \boxtimes_l X')$ and $(X \boxtimes_r -)$ on $\mathbf{C}_\mathcal{T}$.

**Definition (Central morphism \cite{power1997premonoidal})**

A morphism $f: X \to Y$ in $\mathbf{C}_\mathcal{T}$ is *central* if for any morphism $f': X' \to Y'$

\[
\begin{array}{ccc}
X \otimes X' & \xrightarrow{f \otimes_l X'} & Y \otimes X' \\
\downarrow & & \downarrow \\
X \otimes Y' & \xrightarrow{f \otimes_l Y'} & Y \otimes Y'
\end{array}
\]

in $\mathbf{C}_\mathcal{T}$, the following diagram:

commutes in $\mathbf{C}_\mathcal{T}$. 
Premonoidal category

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\[
\begin{array}{ccc}
X \otimes X' & \xrightarrow{f \otimes_l X'} & Y \otimes X' \\
\downarrow X \otimes_r f' & & \downarrow Y \otimes_r f' \\
X \otimes Y' & \xrightarrow{f \otimes_l Y'} & Y \otimes Y'
\end{array}
\]

commutes in $\mathbf{C}_\mathcal{T}$.

Central **cones** and central **morphisms** are actually equivalent notions!
- $Z(\mathbf{C}_\tau)$: the wide subcategory of $\mathbf{C}_\tau$ with central morphisms.
- It is symmetric monoidal [Power and Robinson, 1997].
- \( Z(\mathbf{C}_\mathcal{T}) \): the wide subcategory of \( \mathbf{C}_\mathcal{T} \) with central morphisms.
- It is symmetric monoidal [Power and Robinson, 1997].

**Proposition**

*If the strong monad \( \mathcal{T} \) is centralisable, then the canonical embedding \( \mathcal{I} : \mathbf{C}_\mathcal{Z} \to \mathbf{C}_\mathcal{T} \) corestricts to an isomorphism of categories \( \hat{\mathcal{I}} : \mathbf{C}_\mathcal{Z} \to Z(\mathbf{C}_\mathcal{T}) \).*
Premonoidal Centre

- $Z(C_T)$: the wide subcategory of $C_T$ with central morphisms.
- It is symmetric monoidal [Power and Robinson, 1997].

**Proposition**

*If the strong monad $T$ is centralisable, then the canonical embedding $I : C_Z \to C_T$ corestricts to an isomorphism of categories $\hat{I} : C_Z \to Z(C_T)$.***

This is why we call $Z$ the central submonad of $T$. 
Premonoidal adjunction
In the Kleisli adjunction between $\mathbf{C}$ and $\mathbf{C}_T$, the left adjoint, $\mathcal{J}: \mathbf{C} \to \mathbf{C}_T$ always corestricts to $\hat{\mathcal{J}}: \mathbf{C} \to Z(\mathbf{C}_T)$.
In the Kleisli adjunction between $\mathbf{C}$ and $\mathbf{C}_T$, the left adjoint, $\mathcal{J} : \mathbf{C} \to \mathbf{C}_T$ always corestricts to $\hat{\mathcal{J}} : \mathbf{C} \to Z(\mathbf{C}_T)$.

**Proposition**

*If the strong monad $\mathcal{T}$ is centralisable, then $\hat{\mathcal{J}}$ is also a left adjoint and the adjunction induces the central submonad $\mathcal{Z}$.*
Characterisation
The Main Theorem

Theorem (Centralisability)

Let $\mathbf{C}$ be a symmetric monoidal category and $\mathcal{T}$ a strong monad on it. The following are equivalent:

1. For any object $X$ of $\mathbf{C}$, $\mathcal{T}$ admits a terminal central cone at $X$;

2. There exists a commutative submonad $\mathcal{Z}$ of $\mathcal{T}$ such that the canonical embedding functor $\mathcal{I}: \mathbf{C}_\mathcal{Z} \to \mathbf{C}_\mathcal{T}$ corestricts to an isomorphism of categories $\mathbf{C}_\mathcal{Z} \cong Z(\mathbf{C}_\mathcal{T})$;

3. The corestriction of the Kleisli left adjoint $\mathcal{J}: \mathbf{C} \to \mathbf{C}_\mathcal{T}$ to the premonoidal centre $\hat{\mathcal{J}}: \mathbf{C} \to Z(\mathbf{C}_\mathcal{T})$ also is a left adjoint.
Some Centralisable Monads and a non Centralisable one

- Using the main theorem, it follows every strong monad on many categories of interest (e.g., \textbf{Set}, \textbf{DCPO}, \textbf{Meas}, \textbf{Top}, \textbf{Hilb}, \textbf{Vect}) is centralisable.
- If \( \mathbf{C} \) is a symmetric monoidal closed category that is total, then every strong monad on it is centralisable.
- If \( \mathcal{T} \) is a commutative monad, then \( \mathcal{T} \) is centralisable and its centre coincides with itself.

Is every strong monad centralisable?
Some Centralisable Monads and a non Centralisable one

- Using the main theorem, it follows every strong monad on many categories of interest (e.g., Set, DCPO, Meas, Top, Hilb, Vect) is centralisable.

- If $\mathcal{C}$ is a symmetric monoidal closed category that is total, then every strong monad on it is centralisable.

- If $\mathcal{T}$ is a commutative monad, then $\mathcal{T}$ is centralisable and its centre coincides with itself.

Is every strong monad centralisable? No!
Example built with a full subcategory $\mathcal{C}$ of Set where not all subsets of $\mathcal{T}X$ are objects of $\mathcal{C}$. 
Example

The valuation monad $\mathcal{V}: \text{DCPO} \rightarrow \text{DCPO}$ is strong, but its commutativity is an open problem [Jones, 1990]. The central submonad of $\mathcal{V}$ is precisely the "central valuations monad" described in [Jia et al., 2021].
Computational interpretation
A meta language

Refinement of Moggi’s metalanguage;
A meta language

Refinement of Moggi’s metalanguage;

\[ A, B ::= 1 | A \times B | A \rightarrow B | \exists A | T A \]
A meta language

Refinement of Moggi’s metalanguage;

\[ A, B ::= 1 \mid A \times B \mid A \rightarrow B \mid ZA \mid TA \]

\[
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A. M : A \rightarrow B} \quad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}
\]

\[
\frac{\Gamma \vdash M : A}{\Gamma \vdash \text{ret}_Z M : ZA} \quad \frac{\Gamma \vdash M : ZA \quad \Gamma, x : A \vdash N : ZB}{\Gamma \vdash \text{do } x \leftarrow_Z M ; N : ZB}
\]

\[
\frac{\Gamma \vdash M : ZA}{\Gamma \vdash \mu M : TA} \quad \frac{\Gamma \vdash M : TA \quad \Gamma, x : A \vdash N : TB}{\Gamma \vdash \text{do } x \leftarrow_T M ; N : TB}
\]
Computational use case for the centre of a monad

If at least one of `op1` or `op2` is central, then the two programs are contextually equivalent!

```
do
  x <- op1
  y <- op2
  f x y

  y <- op2
  x <- op1
  f x y
```
Ongoing and Future Work
- Completeness and internal language result for the computational interpretation;
- Notion of Commutant for monads in general;
- Link with Garner’s results on commutativity.
Thank you!
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