Distance Enumerators for Number-Theoretic Codes

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Abstract—The number-theoretic codes are a class of codes defined by single or multiple congruences and are mainly used for correcting insertion and deletion errors. Since the number-theoretic codes are generally non-linear, the analysis method for such codes is not established enough. The distance enumerator of a code is a unary polynomial whose $i$th coefficient gives the number of the pairs of codewords with distance $i$. The distance enumerator gives the maximum likelihood decoding error probability of the code. This paper presents an identity of the distance enumerators for the number-theoretic codes. Moreover, as an example, we derive the Hamming distance enumerator for the Varshamov-Tenengolts (VT) codes.

I. INTRODUCTION

The number-theoretic codes [1] are a class of codes defined by single or multiple congruences. These codes are mainly used for correcting insertion and deletion errors [2–6] or for correcting asymmetric errors [7], [8]. In general, the number-theoretic codes are non-linear. Unfortunately, analysis methods for the number-theoretic codes have not been established enough compared with linear codes.

The distance enumerators for codes characterize the error correcting capability. In particular, the Hamming distance enumerators [9], [10] are used for analyzing the error probability of the maximum likelihood decoder for the codes through symbol error channels. The distance enumerator of a code is a unary polynomial whose $i$th coefficient gives the number of pairs of codewords with distance $i$. For linear codes, we can easily derive the Hamming distance enumerator from the Hamming weight enumerator. Thus, many works have investigated the Hamming distance enumerators for the linear codes.

On the other hand, for non-linear codes, the Hamming weight enumerators have been much researched. Delsarte [9] defined the Hamming distance enumerator for linear/non-linear codes and derived an upper bound for the cardinality of code with designed Hamming distance. Kalai and Linial [10] analyzed the asymptotic behavior of growth rate for the Hamming distance enumerator. Mimura [11] analyzed the asymptotic growth rate for the Hamming distance enumerator for a class of non-linear code ensemble.

We introduced the simultaneous congruence (SC) code, a general class of the number-theoretic code, in previous work [6]. Moreover, we presented an identity for the Hamming weight enumerators for the SC codes and derived their cardinalities. In this paper, we provide an identity for the distance enumerators for the SC codes. This identity gives the distance enumerator related to not only Hamming distance but also other distances, e.g., Levenshtein distance [3] and Lee distance [12]. This identity can be derived as a natural extension of the previous work. This paper also derives the Hamming distance enumerators for Varshamov-Tenengolts (VT) codes as an example.

The rest of the paper is organized as follows: Section II defines notations and definitions used throughout the paper. Section III presents an identity for the distance enumerators for the SC codes. Section IV derives the Hamming distance enumerator for the VT codes and shows a numerical example. Section V concludes the paper.

II. DEFINITIONS AND NOTATIONS

This section gives the definitions and notations used throughout the paper. Moreover, we define several codes and the distance enumerators.

A. Notations

Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}_{\geq 0}$, and $\mathbb{C}$ be the set of all integers, positive integers, non-negative real numbers, and complex numbers, respectively. For $a, b \in \mathbb{Z}$, denote the integers between $a$ and $b$, by $[a, b]$, i.e., $[a, b] := \{i \in \mathbb{Z} | a \leq i \leq b\}$. In particular we denote $[r] := [0, r - 1]$. Let $\{P\}$ be the indicator function, which equals 1 if the proposition $P$ is true and equals 0 otherwise. Denote the cardinality of a set $T$, by $|T|$. Denote the vector of length $n$, by $x = (x_1, x_2, \ldots, x_n)$.

For $a, b \in \mathbb{Z}$, we write $a \mid b$ if $a$ divides $b$. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, denote $a \equiv b \pmod{n}$ if $(a - b) \mid n$. For $a, b \in \mathbb{Z}$, let $\gcd(a, b)$ be the greatest common divisor of $a, b$. Let $i$ be the imaginary unit. Define $e(x) := \exp(2\pi i x)$.

B. Number-Theoretic Codes

Bibak and Milenkovic [4] defined the binary linear congruence (BLC) codes as follows:

Definition 1 ([4]): Denote the code length, by $n \in \mathbb{Z}^+$. Let $m \in \mathbb{Z}^+$, $h = (h_1, h_2, \ldots, h_n) \in \mathbb{Z}^n$, and $a \in [m]$. Then, the BLC code of length $n$ with parameters $m, a, h$ is defined by

$$\text{BLC}_a(n, m, h) := \{(x_1, x_2, \ldots, x_n) | x_i \in \{0, 1\}^n | \sum_{i=1}^n h_ix_i \equiv a \pmod{m}\}.$$ 

We [6] defined the SC codes by extending the definition of the BLC codes as follows:

Definition 2 ([4, Def. 2]): Denote the code length, by $n \in \mathbb{Z}^+$. Let $r, s \in \mathbb{Z}^+$, $m := (m_1, m_2, \ldots, m_s) \in (\mathbb{Z}^+)^s$,
and \(a := (a_1, a_2, \ldots, a_s) \in [m_1] \times [m_2] \times \cdots \times [m_s]\). For all \(i \in \{1, s\}\), let \(\rho_i : [r]^n \rightarrow \mathbb{Z}\) and denote \(\rho := (\rho_1, \rho_2, \ldots, \rho_s)\). Then, the \(r\)-ary SC code of length \(n\) with parameters \(a, \rho, a, m\) is

\[
C_{\rho, a, m}(n, r, s) := \{x \in [r]^n \mid \forall i \in \{1, s\}, \rho_i(x) \equiv a_i \pmod{m_i}\}.
\]

There are examples of the number-theoretic codes included in [2], which is known as a single insertion/deletion correcting \(s\), \(n\sum_{i=1}^{s} \rho_i(x) \equiv n\) and \(a\). The distance on \([r]^n\) is a function \(d : [r]^n \times [r]^n \rightarrow \mathbb{R}_\geq 0\), satisfying the following conditions:

1) For all \(x, y \in [r]^n\), \(d(x, y) \geq 0\). Moreover, \(d(x, y) = 0\) iff \(x = y\).
2) For all \(x, y \in [r]^n\), \(d(x, y) = d(y, x)\).
3) For all \(x, y, z \in [r]^n\), \(d(x, y) \leq d(x, z) + d(z, y)\).

The following distances are often used in the coding theory.

1) For all \(x, y \in [r]^n\), the Hamming distance enumerator related to a \(T\) is defined by the \(n, r, s\). The cardinality of VT codes [13], [14] satisfies \(|VT_r(n)| = |VT_a(n)| (a \in \{0, n\})\).

C. Distance and Distance Enumerator

A distance on \([r]^n\) is a function \(d : [r]^n \times [r]^n \rightarrow \mathbb{R}_\geq 0\), satisfying the following conditions:

1) For all \(x, y \in [r]^n\), the Hamming distance \(d_H(x, y)\) is the minimum number of substitutions required to change \(x\) to \(y\), i.e.,

\[
d_H(x, y) := \{i \in \{1, n\} \mid x_i \neq y_i\}\.
\]

2) The longest common subsequence (or indel) distance \(d_I(x, y)\) is the minimum number of insertions and deletions required to change \(x\) to \(y\).

3) The Levenshtein distance \(d_{lev}(x, y)\) [3] is the minimum number of insertions, deletions, and substitutions required to change \(x\) to \(y\).

4) The Lee distance \(d_L(x, y)\) [12] is defined by

\[
d_L(x, y) := \sum_{i=1}^{n} |x_i - y_i|,
\]

\[
d_L(x, y) := \min\{|x - y|, r - |x - y|\}.
\]

For a code \(T \subseteq [r]^n\), the distance enumerator related to a distance \(d\) is defined by

\[
D(T; z) := \sum_{x \in \{z^d(x, y) = i\} \subseteq T} z^i.
\]

Example 1: To simplify the notation, we denote the binary vector \((x_1, x_2, \ldots, x_n) \in [2]^n\) by \(x_1 x_2 \ldots x_n\). Consider \(VT_0(5) = \{00000, 10001, 01010, 00111, 11100, 11011\}\). The Hamming distance enumerator for this code is

\[
\mathcal{D}_H(VT_0(5); z) = 6z^0 + 8z^2 + 16z^3 + 6z^4.
\]

Remark 1: By normalizing the Hamming distance enumerator by the cardinality \(|T|\) of the code, we get the average Hamming distance enumerator [9], [10].

\[
\mathcal{D}_H(T; z) := \frac{\mathcal{D}_H(T; z)}{|T|} = \frac{\mathcal{D}_H(T; z)}{\mathcal{D}_H(T; 0)}.
\]

In words, the average Hamming distance enumerator \(\mathcal{D}_H(T; z)\) is derived from Hamming distance enumerator \(\mathcal{D}_H(T; z)\).

This paper investigates the extended distance enumerator, which is a generalization of the distance enumerator.

Definition 3: Let \(n, r, s \in \mathbb{Z}_+.\) Let \(\rho_i : [r]^n \rightarrow \mathbb{Z}\) for \(i \in \{1, s\}\). Denote \(\rho = (\rho_1, \rho_2, \ldots, \rho_s)\), \(u = (u_1, u_2, \ldots, u_s)\), \(v = (v_1, v_2, \ldots, v_s)\). We define the extended distance enumerator parameterized by \(\rho\) for a code \(T \subseteq [r]^n\) as

\[
\mathcal{E}(T, \rho; z, u, v) = \sum_{x \in T} \sum_{y \in T} \prod_{i=1}^{n} u_i^{\rho_i(x)} v_i^{\rho_i(y)}.
\]

In particular, we denote the extended Hamming distance enumerator, by \(\mathcal{E}_H(T, \rho; z, u, v)\).

Example 2: We continue from Example 1. The extended Hamming distance enumerator for \(VT_0(5)\) is

\[
\mathcal{E}_H(VT_0(5); z, u, v) = (1 + u^6 + v^6)(1 + 2u^6v^6) + 2(u^6 + v^6)(1 + u^6v^6)z^2
\]

\[
+ (1 + u^6)(1 + u^6)(u^6 + v^6 + 2u^6v^6)z^3
\]

\[
+ (u^6 + v^6)(u^6 + u^6 + u^6v^6)z^4.
\]

Remark 2: Define \(\mathbf{1} := (1, 1, \ldots, 1)\). Then, the distance enumerator \(\mathcal{D}(T; z)\) is derived from the extended distance enumerator \(\mathcal{E}(T, \rho; z, u, v)\) as follows:

\[
\mathcal{E}(T, \rho; z, 1, 1) = \mathcal{D}(T; z).
\]

III. EXTENDED DISTANCE ENUMERATORS FOR SC CODES

A. Main Result and Corollary

The following theorem presents an important formula to derive the extended distance enumerator.

Theorem 1: Define the SC codes (resp. extended weight enumerator) as in Definition 2 (resp. 3). Define \(\omega(e) := \)
\[
(1) = \left( \frac{1}{n^2} \right) \sum_{j,k} a_j^2 e^{-\frac{a(j+k)}{m}} \]
\[
(2) = \sum_{j,k} \cdots \sum_{j,k} \left[ \prod_{i=1}^n \left( 1 + e \left( \frac{a_i(j_i + k_i)}{m_i} \right) \right) \right] 
\times E([r]^n; \rho; z, u e(j/m), u e(k/m)).
\]

2) Proof of Corollary \[7\]
The following holds:
\[
E_H ([2]^n, \ell_h; z, u, v) = \prod_{i=1}^n \left( 1 + u^{h_i} z + v^{h_i} z + (uv)^{h_i} \right). 
\]
Combining this identity and \(1\), we get
\[
E_H (BLC_a(n, m, h), \ell_h; z, u, v) = \sum_{j,k} \left[ \prod_{i=1}^n \left( 1 + e \left( \frac{a_i(j_i + k_i)}{m_i} \right) \right) \right] 
\times E([r]^n; \rho; z, u e(j/m), u e(k/m)).
\]

From this equation and Remark \[2\] we have
\[
D_H (BLC_a(n, m, h), \ell_h; z, u, v) = \sum_{j,k} \left[ \prod_{i=1}^n \left( 1 + e \left( \frac{a_i(j_i + k_i)}{m_i} \right) \right) \right] 
\times \prod_{i=1}^n \left( 1 + e \left( \frac{h_i j_i}{m_i} \right) z + e \left( \frac{h_i k_i}{m_i} \right) z + e \left( \frac{h_i(j_i + k_i)}{m_i} \right) \right).
\]

IV. HAMMING DISTANCE ENUMERATOR FOR VT CODES

This section derives the Hamming distance enumerator for the VT codes. Section \[IV-A\] gives some properties which are useful to calculate the Hamming distance enumerator. Section \[IV-B\] shows algorithms to calculate the Hamming distance enumerator efficiently. Section \[IV-C\] gives a numerical example.

A. Properties

To simplify the notation, we denote \( m := n + 1 \). Corollary \[1\] leads the Hamming distance enumerators for the VT codes:
\[
D_H (VT_a(n); z) = \frac{1}{m^2} \sum_{j,k} e \left( \frac{a(j+k)}{m} \right) A_{m,j,k}(z),
\]
\[
A_{m,j,k}(z) := \prod_{i=1}^{m-1} \left( 1 + e \left( \frac{i(j+k)}{m} \right) z + e \left( \frac{ik}{m} \right) z \right).
\]

Define polynomial \( B_{m,j,k}(z) \) as
\[
B_{m,j,k}(z) := (2 + 2z) A_{m,j,k}(z) 
\]
\[
= \prod_{i=1}^{m-1} \left( 1 + e \left( \frac{i(j+k)}{m} \right) z + e \left( \frac{ik}{m} \right) z \right). 
\]
From this definition, \( B_{m,j+k+im}(z) = B_{m,j,k}(z) \) holds for all \( s, t \in \mathbb{Z} \). Hence, if we get \( B_{m,j,k}(z) \) for all \( j, k \in [m] \), we have the Hamming distance enumerators as follows:

\[
D_H(VT_a(n); z) = \frac{1}{m^2} \sum_{j \in [m]} \epsilon\left(-\frac{aj}{m}\right) F_{m,j}(z),
\]

\[
F_{m,j}(z) := \frac{1}{2z + 2} \sum_{k \in [m]} B_{m,k,j-k}(z).
\]

This section gives some properties of \( B_{m,j,k}(z) \) to calculate the Hamming distance enumerators. All the proofs are in Appendix.

**Lemma 2:** For all \( m \in \mathbb{Z}^+ \), \( j, k \in [m] \), the following hold

\[
B_{m,j,k}(z) = B_{m,k,j}(z),
\]

\[
B_{m,j,k}(z) = (-1)^{j(m+1)} z^m B_{m,m-j,k}(z^{-1}),
\]

\[
B_{m,j,k}(z) = (-1)^{(j+k+1)m} z^m B_{m,j,m-k}(z^{-1}).
\]

Note that for an \( m \)th degree polynomial \( f(z) = \sum_{i=0}^{m} a_i z^i \), its reciprocal polynomial is written by \( z^m f(z^{-1}) = \sum_{i=0}^{m} a_{m-i} z^i \). Denote the floor function for \( z \in \mathbb{R} \), by \( \lfloor z \rfloor \), i.e., \( \lfloor z \rfloor = \max\{i \in \mathbb{N} \mid i \leq z \} \). By this lemma, we need to derive \( B_{m,j,k}(z) \) for only \( j, k \in [0, \lfloor \frac{m-1}{2} \rfloor] \). Moreover, the following lemma allows us to reuse the calculation results.

**Lemma 3:** Suppose integers \( t, m \) are coprime. Then,

\[
B_{m,j,t,k}(z) = B_{m,j,k}(z).
\]

Denote the Chebyshev polynomials of second and third kind, by \( U_n(z) \) and \( V_n(z) \), respectively. The explicit formulas for \( U_n(x) \) and \( V_n(z) \) are known as

\[
U_n(x) = \sum_{k=0}^{|n/2|} (-1)^k \binom{n-k}{k} (2x)^n - 2k,
\]

\[
V_n(z) = \sum_{k=0}^n (-1)^k \binom{2n-k}{k} 2^{n-k}(x-1)^{n-k}.
\]

For some pairs \( (j, k) \), we have the explicit formulas of \( B_{m,j,k}(z) \).

**Lemma 4:** Define \( d := \gcd(m,j) \) and denote \( m' := m/d \). Denote \( \bar{m} := |\frac{m-1}{2}| \). For all \( m \in \mathbb{Z} \) and all \( j \in [m] \), the following hold:

\[
B_{m,j,0}(z) = B_{m,0,j}(z) = 2^d (1 + z)^m [m' : \text{odd}],
\]

\[
B_{m,j,1}(z) = \begin{cases}
(-1)^j 2^{2d} (z^2 - 1)^d U_{\bar{m}}(z) 2^d, & (m' : \text{even}), \\
2^d (z + 1)^d V_{\bar{m}}(z) 2^d, & (m' : \text{odd}),
\end{cases}
\]

\[
B_{m,j,m-k}(z) = \begin{cases}
2^{2d} (z^2 - 1)^d \bar{m} U_{\bar{m}}(z^{-1}) 2^d, & (m' : \text{even}), \\
2^d (z + 1)^d \bar{m} V_{\bar{m}}(z^{-1}) 2^d, & (m' : \text{odd}).
\end{cases}
\]

Now, consider the general case of \( (j, k) \).

**Lemma 5:** Denote \( d := \gcd(m,j,k) \). For all \( m \in \mathbb{Z}^+ \) and \( j, k \in [m] \), we have

\[
B_{m,j,k}(z) = \left( B_{\frac{m}{d}, \frac{j}{d}, \frac{k}{d}}(z) \right)^d.
\]

**Algorithm 1** Brute-force algorithm for calculating \( D_H(VT_a(n); z) \)

**Require:** Code length \( n \) and an integer \( a \in [0, n] 

**Ensure:** Hamming distance distribution \( D_H(VT_a(n); z) = \sum_{i=0}^{n} D_i z^i 

1: Initialize \( D_i \leftarrow 0 \) for all \( i \in [0, n] 
2: Enumerate codewords \( c_1, c_2, \ldots, c_u \) in \( VT_a(n) 
3: for j = 1, 2, \ldots, u do 
4: for k = 1, 2, \ldots, u do 
5: \( D_{di}(c_j, c_k) \leftarrow D_{di}(c_j, c_k) + 1 
6: end for 
7: end for 
8: Output \( \sum_{i=0}^{n} D_i z^i 

This lemma implies that \( B_{m,j,k}(z) \) is derived from \( B_{\frac{m}{d}, \frac{j}{d}, \frac{k}{d}}(z) \), where \( \gcd\left(\frac{m}{d}, \frac{j}{d}, \frac{k}{d}\right) = 1 \). Hence, the following two lemmas suppose the case of \( \gcd(m, j, k) = 1 \).

**Lemma 6:** Suppose \( m \) is an odd integer. For \( j, k \in [m] \) such that \( \gcd(m, j, k) = 1 \), we get

\[
B_{m,j,k}(z) = 2^m (1 + z) \prod_{i=1}^{\frac{m-1}{2}} \left( \cos\left(\frac{\pi ij + k}{m}\right) + z \cos\left(\frac{\pi ij - k}{m}\right) \right)^2.
\]

**Lemma 7:** Suppose \( m \) is an even integer. For \( j, k \in [m] \) such that \( \gcd(m, j, k) = 1 \), we get

\[
B_{m,j,k}(z) = 2^m (1 - z^2) \prod_{i=1}^{\frac{m-1}{2}} \left( \cos\left(\frac{\pi ij + k}{m}\right) + z \cos\left(\frac{\pi ij - k}{m}\right) \right)^2.
\]

**B. Algorithms**

This section shows some algorithms to calculate the Hamming distance enumerators for VT codes. Firstly, we give a brute-force algorithm (Algorithm 1) and evaluate its complexity. Next, we give an efficient algorithm (Algorithm 2) based on the previous section results.

At first, let us consider a brute-force algorithm. In this algorithm, we enumerate all the codewords in \( VT_a(n) \) and evaluate the Hamming distance between all the pairs of codewords. Algorithm 1 gives this brute-force algorithm. Here, \( a \leftarrow b \) represents substituting \( b \) for \( a \). This algorithm’s complexity is \( O(u^2) \), where \( u \) represents the number of codewords in the VT code. Since the cardinality \( u \) of an VT code is approximated by \( 2^n/(n+1) \), this algorithm’s complexity is \( O(2^{2n}/n^2) \). In other words, this brute-force algorithm is exponential time.

Next, let us consider an efficient algorithm based on Theorem 1 and lemmas given in the previous section. This algorithm calculates \( D_H(VT_a(n); z) \) by deriving \( B_{m,j,k}(z) \). Algorithm 2 shows the details of this algorithm. For all \( j, k \in [0, n] \), to derive \( B_{m,j,k}(z) \) with Lemmas 5 and 7 we need \( O(n^2) \) times multiplication in the real number.
Algorithm 2 Algorithm for calculating $D_H(VT_a(n); z)$ based on Theorem 1

Require: Code length $n$ and an integer $a \in [0, n]$

Ensure: Hamming distance distribution $D_H(VT_a(n); z)$

1: Calculate $B_{m,j,k}(z)$ for $j, k \in [0, n]$
2: for $j = 0, 1, 2, \ldots, n$ do
3: Calculate $F_{m,j}(z)$ by Eq. (5)
4: end for
5: Calculate $D_H(VT_a(n); z)$ by Eq. (4)
6: Output $D_H(VT_a(n); z)$

Algorithm 3 Calculation of $B_{m,j,k}(z)$

Require: Integer $m \in \mathbb{Z}^+$

Ensure: Polynomials $B_{m,j,k}(z)$ for $j, k \in [m]$

1: Calculate $B_{m,0,k}(z)$ by Eq. (10) for $j \in [m]$
2: for $j = 1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor$ do
3: $d' \leftarrow \gcd(m, j), d'' \leftarrow \gcd(m, j/d')$
4: if $d' \neq 1$ then
5: $B_{m,j,k}(z) \leftarrow B_{m,k,j}(z)$ for $k = 0, 1, \ldots, j - 1$
6: Calculate $B_{m,j,k}(z)$ by Eq. (11)
7: Calculate $B_{m,j,k}(z)$ by Lemmas 5, 6, 7 for $k = j + 1, \ldots, \lfloor \frac{m-1}{2} \rfloor$
8: Set $B_{m,j,k}(z)$ by Eq. (7) for $k = \lfloor \frac{m-1}{2} \rfloor + 1, \ldots, m - 1$
9: else
10: $B_{m,j,k/d}(z) \leftarrow B_{m,d,k}(z)$ for $k \in [m]$
11: end if
12: end for
13: for $j = \lfloor \frac{m-1}{2} \rfloor + 1, \ldots, m - 1$ do
14: Set $B_{m,j,k}(z)$ by Eq. (8) for all $k \in [m]$
15: end for
16: Output $B_{m,j,k}(z)$ for $j, k \in [m]$

C. Numerical Example

This section gives a numerical example of Hamming distance enumerator for VT codes.

Table II displays the Hamming distance enumerator for VT codes $VT_a(15)$ with code length $n = 15$. Note that $D_H(VT_a(n); z)$ depends on $d := \gcd(a, m)$. In other words, the column labeled with $d = 16$ gives the case of $a = 0$, and the column labeled with $d = 1$ gives the cases of $a = 1, 3, 5, 7, 9, 11, 13, 15$.

From this table, we see that $D_1 = 0$ for all $d$. This result is easily checked since the VT codes correct single insertion/deletion. This table also shows that $D_0$ has the same value. This value coincides the cardinalities of VT codes [13].

Moreover, by comparing $D_2/D_0$, we see that the case of $d = 4$ has the smallest value. Hence, in the case of $n = 15$, the VT codes with $a = 4, 12$ are the best codes from the Hamming distance perspective. On the other hand, in general, the VT code with $a = 0$ is the best code from the perspective of the cardinality of the code. Summarizing above, there are cases that the VT code with $a = 0$ is not the best in term of the Hamming distance.

V. CONCLUSION AND FUTURE WORKS

This paper has presented the identity of the distance enumerators for the SC codes. Using this result, we have shown an efficient algorithm to calculate the Hamming distance enumerators for the VT codes. Moreover, there are cases that $VT_0(n)$ is not the best in term of the Hamming distance enumerator.

As future work, we derive the distance enumerator for other SC codes and other distances.

APPENDIX

Proof of Lemma 2. From (3), we get (6). Equation (3) leads

$$z^m B_{m,m-j,k}(z^{-1})$$

$$= \prod_{i=1}^{m} \left( z + e \left( \frac{i(j+k)}{m} \right) z + e \left( \frac{-ij}{m} \right) + e \left( \frac{ik}{m} \right) \right)$$

$$= \prod_{i=1}^{m} e \left( \frac{-ij}{m} \right)$$

$$\times \prod_{i=1}^{m} e \left( \frac{ij}{m} \right) + z + e \left( \frac{ik}{m} \right) z + 1 + e \left( \frac{i(j+k)}{m} \right)$$

$$= (-1)^{j(m+1)} B_{m,j,k}(z).$$

Hence, we have (7). Similarly, we get (8).
Proof of Lemma 3. For $a \in \mathbb{Z}$, $m \in \mathbb{Z}^+$, denote the remainder in the division of $a$ by $m$, by $\langle a \rangle_m$. Then, since $t$ satisfies $\gcd(t, m) = 1$, we have $\{\langle it \rangle_m | i \in [m]\} = [m]$. Combining this and 3, we get 3.

Proof of Lemma 4. The first equality of (10) follows (6). From (3), we get

$$B_{m,j,0}(z) = (1 + z)^m \prod_{i=1}^m \left(1 + e^{i j / m}\right).$$

Note that

$$\prod_{i=1}^m \left(1 + e^{i j / m}\right) = (1 - (-u)^m / m)^d.$$ Substituting $u = 1$, we get

$$\prod_{i=1}^m \left(1 + e^{i j / m}\right) = 2^d \lfloor m / \text{odd} \rfloor.$$ Combining (15) and (16), we get the second equality of (10).

From (3), we have

$$B_{m,j,k}(z) = \prod_{i=1}^m \left(1 + e^{2 i j / m} + 2 e^{i j / m} z\right)$$

$$= \prod_{i=1}^m \left(2 e^{i j / m}\right) \prod_{i=1}^m \left(\cos\left(2\pi i j / m\right) + z\right)$$

$$= 2^m (-1)^{j(m+1)} \prod_{i=1}^m \left(\cos\left(2\pi i j / m\right) / m\right)^d$$

$$= (-1)^j(m+1) \left\{ \prod_{i=1}^m \left(\cos\left(2\pi i / m'\right)\right) \right\}^d,$$

where the fourth equality follows from $\gcd(j/d, m') = 1$. The Chebyshev polynomials of second and third kind (e.g., see (14)) are expressed as

$$U_n(z) = 2^n \prod_{i=1}^n \left(z - \cos\left(\pi i / n + 1\right)\right)$$

$$= 2^n \prod_{i=1}^n \left(z + \cos\left(\pi i / n + 1\right)\right),$$

$$V_n(z) = 2^n \prod_{i=1}^n \left(z - \cos\left(\pi 2i / 2n + 1\right)\right)$$

$$= 2^n \prod_{i=1}^n \left(z + \cos\left(\pi 2i / 2n + 1\right)\right).$$

Consider even $m'$, i.e., $m' = 2m + 2$. Then,

$$G_{m'}(z) = 4(z^2 - 1) \left\{ \prod_{i=1}^m \left(z + \cos\left(\pi i / m + 1\right)\right) \right\}^2$$

$$= 4(z^2 - 1) \{U_m(z)\}^2.$$

Here, $\prod_{i=m+2}^{2m+1} \left(z + \cos\left(\pi i / m + 1\right)\right)$ leads the first equality. Consider odd $m'$, i.e., $m' = 2m + 1$. Then,

$$G_{m'}(z) = 2(z + 1) \left\{ \prod_{i=1}^{m} \left(z + \cos\left(\pi 2i / 2m + 1\right)\right) \right\}^2$$

$$= 2(z + 1) \{V_m(z)\}^2.$$

Based on this equation, we will derive Lemmas 5 and 7. Proof of Lemma 6. From (20), we have

$$B_{m,j,k}(z)$$

$$= \prod_{i=1}^m \left(1 + e^{i j / m} + 2 e^{i j / m} z\right)$$

$$= \prod_{i=1}^m \left(2 e^{i j / m}\right) \prod_{i=1}^m \left(\cos\left(\pi i j / m\right) + z\cos\left(\pi i j / m\right)\right).$$

This concludes the proof.
Here, the first equality follows from the fact that \((j + k)\) is even iff \((j - k)\) is even; The second equality follows from even \(m\) and \(\gcd(m, j, k) = 1\). Combining this equation and (20) leads

\[
B_{m,j,k}(z) = 2^m (-1)^{(j+k)/2} (1 + z) \\
\quad \times \mathbb{I}[j : \text{odd}] \mathbb{I}[k : \text{odd}] (-1)^{(j+k)/2} (1 - z) \\
\quad \times \prod_{i=1}^{m-1} \left( \cos \left( \pi i \frac{j + k}{m} \right) + z \cos \left( \pi i \frac{j - k}{m} \right) \right) \\
\quad \times \prod_{i=\frac{m}{2}+1}^{m-1} \left( \cos \left( \pi i \frac{j + k}{m} \right) + z \cos \left( \pi i \frac{j - k}{m} \right) \right) \\
= 2^m (1 - z^2) \mathbb{I}[j : \text{odd}] \mathbb{I}[k : \text{odd}] \\
\quad \times \prod_{i=1}^{m-1} \left( \cos \left( \pi i \frac{j + k}{m} \right) + z \cos \left( \pi i \frac{j - k}{m} \right) \right)^2.
\]

This concludes the proof.

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