Cosmological $\Lambda$ converts to reheating energy and cold dark matter

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Suppose that the early Universe starts with a quantum spacetime originated cosmological $\Lambda$-term at the Planck scale $M_{pl}$. The cosmological energy density $\rho_{\Lambda}$ drives inflation and simultaneously reduces its value to create the pair-energy density $\rho_{M}$ via the continuous pair productions of massive fermions and antifermions. The decreasing $\rho_{\Lambda}$ and increasing $\rho_{M}$, in turn, slows down the inflation to its end when the pair production rate $\Gamma_{M}$ is larger than the Hubble rate $H$ of inflation. A large number of massive pairs is produced and reheating epoch starts. In addition to the Einstein equation and energy-conservation law, we introduce the Boltzmann-type rate equation describing the number of pairs produced from (annihilating to) the spacetime, and reheating equation describing massive unstable pairs decay to relativistic particles and thermodynamic laws. This forms a close set of four independent differential equations uniquely determining $H, \rho_{\Lambda}, \rho_{M}$ and radiation-energy density $\rho_{R}$, given the initial conditions at inflation end. Numerical solutions demonstrate three episodes of preheating, massive pairs dominate and genuine reheating. Results show that $\rho_{\Lambda}$ can efficiently convert to $\rho_{M}$ by producing massive pairs, whose decay accounts for reheating $\rho_{R}$, temperature and entropy of the Big-Bang Universe. The stable massive pairs instead account for cold dark matter. Using CMB and baryon number-to-entropy ratio measurements, we constrain the effective mass of pairs, Yukawa coupling and degeneracies of relativistic particles. As a result, the obtained inflation e-folding number, reheating scale, temperature and entropy are in terms of the tensor-to-scalar ratio in the theoretically predicated range $0.042 \lesssim r \lesssim 0.048$, consistently with current observations.

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I. INTRODUCTION

In the standard model of modern cosmology ($\Lambda$CDM), the cosmological constant, inflation, reheating, dark matter and coincidence problem have been long-standing basic issues for decades. The inflation [1] is a fundamental epoch and the reheating [2] is a critical mechanism, which transitions the Universe from the cold massive state left by inflation to the hot Big Bang [3]. The cosmic microwave background (CMB) observations have been attempting to determine a unique model of inflation and reheating. On the other hand, what is the crucial role that the cosmological $\Lambda$ term play in inflation and reheating, and what is the essential reason for the coincidence of dark-matter dominate matter density and the cosmological $\Lambda$ energy density. There are various models and many efforts, that have been made to approach these issues, and readers are referred to review articles and professional books, for example, see Refs. [4–23].

Suppose that the quantum gravity originates the cosmological term $\Lambda \sim M_{pl}^2$ at the Planck scale. The initial state of the Universe is an approximate de Sitter spacetime of the horizon $H_0 \approx (\Lambda/3)^{1/2}$ without any matter. The cosmological $\Lambda$ energy density drives the spacetime inflation with the scale factor $a(t) \approx e^{H_0 t}$. On the other hand, de Sitter spacetime is unstable against spontaneous particle creations [24, 25, 27–30]. By reducing its value, the cosmological $\Lambda$ term creates very massive pairs of fermions and antifermions $m \sim M_{pl}$ for matter content.

We adopt two independent equations of Friedmann and energy conservation law to completely determine the cosmological energy density $\rho_\Lambda$ and inflation rate $H$ [26],

\begin{align*}
H^2 &= \frac{8\pi G}{3} (\rho_\Lambda + \rho_M), \\
\dot{H} &= -\frac{8\pi G}{2} (1 + \omega_M)\rho_M,
\end{align*}

where the matter-energy density $\rho_M$ and its equation of state $\omega_M$ are calculated by using pair productions. The cosmological energy density $\rho_\Lambda$ governs the spacetime inflation rate $H$ and in the meantime produces the matter-energy density $\rho_M$, whose back reaction, in turn, slows down inflation to its end. The obtained results are in agreement with CMB observations. Suppose that after reheating the matter-energy density is much larger than the cosmological energy density, we show due to such back reaction that the cosmological term $\rho_\Lambda$ tracks down the matter term $\rho_M$ from the reheating end to the radiation-matter equilibrium, then it varies very slowly, $\rho_\Lambda \propto \text{constant}$, consistently leading to the cosmic coincidence in the present time. The detailed discussions and results of such scenario $\tilde{\Lambda}$CDM have been presented there.

We focus our attention on the detailed structure of reheating epoch to understand how the
cosmological energy density $\rho_\Lambda$ converts to the matter-energy density $\rho_M$ of massive pairs. Some of these massive pairs are unstable and decay to relativistic particles of radiation-energy density $\rho_R$. Others are stable, possibly accounting for the dark matter particles. For purpose of showing these, in addition to the aforementioned Einstein equation and conservation law, we study and solve other two independent equations of the Boltzmann-type: (i) the rate equation for massive particle-antiparticle pairs productions and annihilations; (ii) the reheating equation for massive pairs decay to relativistic particles, including the use of thermodynamic laws. These four independent equations allow us to completely determine the main properties in the reheating epoch. The obtained results are expressed in terms of the tensor-to-scalar ratio $0.042 \lesssim r \lesssim 0.048$, they are consistent with current observations.

We organise this article as follow. In Sec. II we briefly summarise the scenario $\Lambda$CDM and previous results of the pre-inflation and inflation epochs. In Sec. III we discuss the cosmic rate equation determining the relation between the matter density $\rho_M$ in the Einstein equations and the pair density $\rho_H^H(H)$ produced by the pair-production process. In Secs. IV and V we present the detailed discussions of three episodes of the reheating epoch, and calculations of relevant physical quantities, compared with observations. The studies of cold dark matter particles are arranged in Sec. VI. The article ends with a summary of the results and remarks.

In this article, $G = M_{pl}^{-2}$ is the the Newton constant, $M_{pl}$ is the Planck scale and reduced Planck scale $m_{pl} \equiv (8\pi)^{-1/2} M_{pl} = 2.43 \times 10^{18}$ GeV.

II. EINSTEIN EQUATION AND PAIR PRODUCTION

To proceed with more detailed discussions and calculations on the reheating epoch, for readers’ convenience, we present the brief and necessary summaries of the scenario $\Lambda$CDM and its applications to the pre-inflation and inflation epochs.

A. Generalized equation for the Friedmann Universe

The Universe evolution from the inflation to the reheating is rather complex. In order to study the different episodes of the reheating epoch after the inflation, we rewrite the generalised equations for the Friedmann Universe as,

$$H^2 = (3m_{pl}^2)^{-1}(\rho_\Lambda + \rho_M + \rho_R),$$

$$\dot{H} = -(3/2)(3m_{pl}^2)^{-1} [(1 + \omega_M)\rho_M + (1 + \omega_R)\rho_R].$$

(3)

(4)
FIG. 1: This figure is duplicated from Fig. 1 in Ref. [33]. Schematic evolution of the Hubble radius $H^{-1}$ and the physical length scale $\lambda(a)$, where physically interested scale $\lambda_0 = \lambda(a_0)$ at the present time $a_0 = 1$ crossed the Hubble horizon at the early time $a_4$. Setting the scale factor $a_4 = a_\ast$ at the inflation scale $H^{-1}_\ast$ fixed by the CMB pivot scale $\lambda_0 = \lambda_\ast = k^{-1}_\ast$. The pre-inflation $a > a_\ast$, the inflation $a_4 < a < a_3 = a_{\text{end}}$ and the inflation end $a_3 = a_{\text{end}}$, the reheating $a_3 < a < a_2 = a_R$, the genuine reheating at $a_R$ and the recombination at $a_1 = a_{\text{eq}}$.

where the Hubble rate $H \equiv \dot{a}/a$ of the scale factor $a(t)$ variation and $\dot{H} \equiv dH/dt$. The energy density $\rho_M$ is a specific notation for massive pairs of $\omega_M \approx 0$. The energy density $\rho_R$ a specific notation for relativistic particles of $\omega_R \approx 1/3$. Their Equations of States are defined as $\omega_{M,R} = p_{M,R}/\rho_{M,R}$ and $p_{M,R}$ is the pressure of massive pairs or relativistic particles respectively. The energy density $\rho_\Lambda \equiv \Lambda^2/(8\pi G)$ and $\omega_\Lambda \equiv -1$ are attributed to the cosmological $\Lambda$-term.

Moreover, we introduce the expanding time scale $\tau_H$ and the $\epsilon$-rate representing the rate of the $H$ variation in time:

$$\tau_H \equiv H^{-1}; \quad \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3\rho_M + (4/3)\rho_R}{2\rho_\Lambda + \rho_M + \rho_R},$$

(5)

whose values characterise different epochs in Universe evolution. Note that in the inflation epoch the $\epsilon$-rate is analogous to the so-called slow-roll parameter. As a convenient unit for calculations and expressions, we adopt the reduced Planck scale $m_{\text{pl}} \equiv (8\pi G)^{-1/2} = 1$, unless otherwise stated. In order to illustrate the pre-inflation, inflation and different episodes of the reheating epoch studied in this article, a schematic description of the Universe evolution for the standard cosmological scenario is presented in Fig. 1.
B. Particle-antiparticle pair production from spacetime

In this section, we briefly recall how to calculate the energy density $\rho^H_M$ of the matter produced from the spacetime horizon $H$ by the pair production of particles $F$ and antiparticles $\bar{F}$:

$$S \Rightarrow F + \bar{F},$$

(6)

where $S$ indicates the spacetime. Such pair production is considered to be a semi-classical process of producing particles and antiparticles in the slowly time-varying horizon $H$. In the physical regime, where produced $F$ and $\bar{F}$ particle masses $m$ are much larger than the horizon $H$ ($m/H \gg 1$), i.e., they are well inside the horizon, we approximately obtain the averaged number, mass densities and pressure of massive pairs produced from $t = 0$ to $t \approx 2\pi H^{-1/2}$:

$$\rho^H_M \approx 2\chi m^2 H^2 (1 + s),$$

(7)

$$p^H_M \approx (s/3)\rho^H_M, \quad s \approx 1/2(H/m)^2 \ll 1$$

(8)

$$\omega_M = p_M/\rho_M \approx s/3,$$

(9)

where the theoretical coefficient $\chi \approx 1.85 \times 10^{-3}$. These massive “non-reletivistic” pairs are produced in the thin shell with the thickness ($\chi m$)$^{-1}$ on the horizon. The pair-production rate $\Gamma_M$ is proportional to the $\epsilon$-rate of the Horizon variation in time.

In each epoch of the Universe evolution, we simply introduce the unique mass parameter $m$ to effectively characterize and describe the total contribution from all kinds of particle-antiparticle pairs (dark matter and/or usual matter) and their degeneracies to the produced pairs’ energy density (7) and pressure (8),

$$\rho^H_M \approx 2\chi H^2 m^2; \quad m^2 \equiv \sum_f g^f_d m^2_f,$$

(10)

where $g^f_d$ and $m_f$ are the degeneracy and mass of the particle of the flavor $f$, and the $\sum_f$ sums up all flavors produced. The pair-production energy density (7) and rate (9) show that the pair production process is in favor of massive pairs whose wavelengths are inside the Horizon $H^{-1}$. The inequality $m^2_f > H^2$ implies that the degeneracy $g^f_d$ should be small in the epoch of large $H^2$ value, whereas the such degeneracy $g^f_d$ should be large in the epoch of small $H^2$ value. Therefore the effective mass parameter $m$ in general depends on the epoch. The values of the mass parameter $m$ in different epochs are determined by observations.
At the end of this section, we present some discussions on the non-exponentially suppressed number and energy densities (7) of very massive particle-antiparticle pairs production and annihilation in cosmological evolution. They are approximately obtained in the static case “\(H = \text{const}\)” and leading orders of adiabatic approximation to smoothly varying background field. In the pioneering work [27], the authors show that the nonadiabaticity and discontinuous transition in the cosmic scale factor \(a(t)\) evolution result in the number and energy densities of superheavy particles produced fall off with a finite power of \(H/m\) for \(m \gg H\), see Eqs. (12,14,15) in their article. This situation is very different from the exponentially suppressed density \((\sim e^{-m/H})\) of superheavy particles produced in static or adiabatically evolutionary Universe. Besides, the authors show the superheavy particle production in cosmologically interesting quantities, such as dark matter relic abundance, which is plotted in terms of \(m/H\), see in their Fig. 2 the solid line for the inflationary epoch discontinuously into the matter dominate epoch. Moreover, their recent article [28] discusses the gravitational production of super-Hubble-mass particles from rapid background field oscillations and the exponential suppression can be avoided. Such situation can be realized in the parametric resonance [29] and the \(2 \rightarrow 2\) scattering case [30]. In our scenario under consideration, the nonadiabaticity is mainly due to the back and forth oscillating processes \(S \leftrightarrow \bar{F} + F\) of productions (6) and annihilation (20) of massive particle and antiparticle pairs. The oscillating frequency is proportional to pairs’ mass \(m \gg H\). The cosmological evolution and pair-production vacuum evolution are certainly non-adiabatic and discontinuous, since the cosmic scale factor \(a(t)\), the Hubble rate \(H(t)\) and their time derivatives are back reacted by producing massive pairs and these pairs annihilation and decay. It happens particularly in the transitions from one cosmic epoch to another, see Fig. 1. The detailed studies will be presented in a separate paper.

C. Naturally resultant inflation and CMB observations

In the pair-production process, the cosmological term \(\rho_\Lambda\) and the horizon \(H\) must decreases because the gravitational energy of the spacetime has to pay for the energy gain due to massive pair production and pairs’ kinetic energy. This back reaction of pair productions on the spacetime has been taken into account by the Einstein equation (3) and generalised conservation law (4). In the pre-inflation \((a < a_4)\) and inflation \((a_4 \leq a \leq a_3)\) epochs, as shown in Fig. 1, we approximately adopt \(\rho_M \approx \rho^H_M\), namely all produced pairs’ energy density \(\rho^H_M\) contribute to the matter-energy density \(\rho_M\) in the Einstein equations, neglecting the pair annihilation \(F + \bar{F} \Rightarrow S\). The reasons
are: (i) the pair energy density $\rho^H_M$ is small, particles and antiparticles are spaced rapidly far apart in the inflating spacetime $H \gtrsim \Gamma_M$; (ii) the pair annihilation is a self-quench process, as its rate is proportional to the pair density.

1. Inflation start scale determined by CMB measurements

We recall the results [26] of the inflation epoch ($a_4 \leq a \leq a_3$ in Fig. 1), when $H > \Gamma_M$, $\rho_\Lambda \gg \rho_M$, $\rho_R \approx 0$, and $\epsilon \approx (3/2)\rho_M/\rho_\Lambda \ll 1$. Using $\rho_M \approx \rho^H_M$ (7) and Eq. (4), we obtain the slowly decreasing

$$H = H_e^{-\chi^2 x} = H_e^{-\chi m^2 N}$$

in terms of the $e$-folding variable $x = \ln(a/a_*) = N$. The $m_*$ is the mass parameter in the inflation epoch. The initial scale $H_*$ corresponds to the interested mode of the pivot scale $k_*$ crossed the horizon ($k_* = H_*a_*$) for CMB power spectra observations. At this pivot scale, the scalar, tensor power spectra and their ratio are

$$\Delta^2_R = \frac{1}{8\pi^2 m^2_{pl} \epsilon c_s}; \quad \Delta^2_h = \frac{2}{\pi^2 m^2_{pl}}; \quad r \equiv \frac{\Delta^2_h}{\Delta^2_R} = 16 \epsilon c_s,$$

where $c_s < 1$ due to the Lorentz symmetry broken by the time dependence of the background [6].

Equation (12) relates to two CMB observational values: the spectral index $n_s \approx 0.965$ and the scalar amplitude $A_s = \Delta^2_R \approx 2.1 \times 10^{-9}$ at $k_* = 0.05$ (Mpc)$^{-1}$ [33]. As a result, we determine the mass parameter $m_*$ and $\epsilon$-rate value $\epsilon^* (n_s = 1 - 2\epsilon^*)$ at the inflation scale $H_*$

$$m_* \approx 3.08 m_{pl}, \quad \epsilon^* = \chi m^2_* \approx 1.75 \times 10^{-2},$$

$$H_* = 3.15 \times 10^{-5} (r/0.1)^{1/2} m_{pl},$$

where $r$ is the scalar-tensor-ratio and the dimensionless notation $\chi m^2_* \equiv \chi (m_*/m_{pl})^2$ for $m_{pl} = 1$ is used. The energy-density ratio of pairs and cosmological term is given by

$$\frac{\rho^*_M}{\rho_\Lambda} \approx \frac{2\chi(m_* H_*)^2}{3(m_{pl} H_*)^2} = \frac{2}{3} \chi m^2_* \approx 1.17 \times 10^{-2},$$

and $H^2_* \approx \rho^*_M/(3m^2_{pl})$. This shows that the pairs’ contribution $\rho_M$ to the Hubble horizon [3] is indeed negligible, compared with the cosmological term contribution $\rho_\Lambda$ in the inflation epoch.

Despite its smallness, the pairs’ energy density makes the Hubble rate $H$ slowly decrease.

2. Inflation end constraint on $n_s - r$ relation and reheating initial condition

The transition from the inflation epoch $H > \Gamma_M$ to the reheating epoch $\Gamma_M > H$ is physically continuous. We can estimate the inflation ending scale $H_{\text{end}}$ by using $H_{\text{end}} \lesssim \Gamma_M$. Approximately
using $2\epsilon \approx 2\chi m^2 \approx (1 - n_s)$ without any parameter, Equations (9) and (11) yield the inflation ending scale factor $a_{\text{end}}$ and e-folding number $N_{\text{end}}$,

$$N_{\text{end}} \gtrsim \ln \left( \frac{a_{\text{end}}}{a_*} \right) = \frac{2}{1 - n_s} \ln \left[ \frac{7.91 \times 10^{-4} (r/0.1)^{1/2}}{(1 - n_s)^{3/2} (\chi/2)^{1/2}} \right].$$

(16)

From Eq. (11), the inflation ending scale $H_{\text{end}}$ is

$$H_{\text{end}} = H_* e^{-\chi m^2 N_{\text{end}}}$$

$$\approx H_* e^{-(1 - n_s) N_{\text{end}}/2} \approx (0.42, 0.35) H_*,$$

(17)

for the e-folding number $N_{\text{end}} = (50, 60)$ and the tensor-to-scalar ratio $r = (0.037, 0.052)$. The numerical result $N_{\text{end}}$ depend on the CMB measurements of $r$ and $n_s$. The $r - n_s$ relation (16) in terms of $N_{\text{end}} = 50, 60$ is shown in Fig. 2 to compare with the constrains from CMB measurements.

In the inflation epoch, the scale factor increases from the beginning $a_*$ to the end $a_{\text{end}}$

$$\Delta_3 \equiv \frac{a_3}{a_4} = \frac{a_{\text{end}}}{a_*} = e^{N_{\text{end}}},$$

(18)

where $a_4 = a_*$, $a_3 = a_{\text{end}}$, the pre-inflation $a < a_4$, the inflation $a_* < a < a_{\text{end}}$, and the pivot length scale is the $\lambda_s = k_*^{-1} = \lambda_0 = k_0^{-1}$, as shown in Fig. 1 for the schematic evolution of the Hubble radius $H^{-1}$ and physically interested wavelength $\lambda(a)$ for the standard cosmological scenario.

Equations (14, 15, 17), $\chi \ll 1$ and $\epsilon^* \ll 1$ (13) show that the $H$-variation is very small in the inflation epoch, implying

$$H_{\text{end}}^2 = \frac{\rho_{\Lambda}^{\text{end}} + \rho_{M}^{\text{end}}}{3m_{\text{pl}}^2} \gtrsim \frac{\rho_{\Lambda}^{\text{end}}}{3m_{\text{pl}}^2} + \frac{\rho_{M}^{\text{end}}}{\rho_{\Lambda}^{\text{end}}},$$

(19)

namely, the cosmological term $\rho_{\Lambda}^{\text{end}} \approx 3m_{\text{pl}}^2 H_{\text{end}}^2$ is still dominant over the pair energy density $\rho_{M}^{\text{end}} \approx 2\chi m^2 H_{\text{end}}^2$ at the inflation end. We consider the ratio $\rho_{M}^{\text{end}} / \rho_{\Lambda}^{\text{end}}$ (15) and the scale $H_{\text{end}}$ (17) as initial conditions of the reheating epoch ($a_3 \leq a \leq a_2$ in Fig. 1) to be studied in Sec. IV.

### III. COSMIC RATE EQUATION FOR PAIRS AND SPACETIME

The inflation epoch ends and reheating epoch starts. The transitioning process from one to another cannot be instantaneous and must be very complex, due to the large density of particle-antiparticle pairs and the back reactions of microscopic and macroscopic processes. One of them is that pairs annihilate back to the spacetime, besides they are produced from the spacetime. This means that the produced pairs’ energy density $\rho_{M}^{H}$ is not the same as the matter energy density $\rho_{M}$.
FIG. 2: On the Figure 28 of the Planck 2018 results [34] for constraints on the tensor-to-scalar ratio $r$, the parameter-free $(n_s - r)$ relation [26] is plotted as two QFC curves respectively representing $N_{\text{end}} = 60$ and $N_{\text{end}} = 50$ are consistently inside the blue zone constrained by several observational data sets. The real values of $r$ ratio should be below the curves due to the nature of inequality (16).

in the Einstein equation $\rho_M \neq \rho_m$. They are related by the the Boltzmann type rate equation of pairs and spacetime, that is studied in this section and shown to be important for understanding the transition between the inflation end and reheating start in next Sec. [IV]

A. Particle-antiparticle pair annihilation and decay

We focus on the dynamics and kinematics of particle-antiparticle pairs after they are produced. From the microscopical points of view, the particle-antiparticle pairs can in turn annihilate to the spacetime, i.e., the inverse process of the production process (6)

$$F + \bar{F} \Rightarrow S.$$  \hspace{1cm} (20)

Such back and forth processes

$$S \leftrightarrow \bar{F}F,$$  \hspace{1cm} (21)

can be regarded as particle-antiparticle emissions and absorptions of the spacetime. As shown in Eqs. (5) and (9), the macroscopic time scale $\tau_H = H^{-1}$ of the spacetime expansion is much longer than the time scale $\Gamma_M^{-1}$ of microscopic pair productions and annihilations. Therefore, the CPT symmetry of local field theories should be held for microscopic processes, we consider that the
pair-annihilation rate is the same as the pair-production rate

$$\Gamma_{M}^{\text{Anni}} = \Gamma_{M}^{\text{Prod}} = \Gamma_{M},$$  \hspace{1cm} (22)

although the Universe expansion violates the $T$-symmetry of time translation and reflection.

Henceforth, we introduce the mass parameter $\hat{m}$ to represent not only the effective masses, but also the effective degeneracies of pairs produced in the reheating epoch. That is the substitution $m \rightarrow \hat{m}$ in Eqs. (7,8) and the rate (9) becomes

$$\Gamma_{M} \approx \left(\chi \hat{m}/4\pi\right)\epsilon. \hspace{1cm} (23)$$

These back and forth processes of massive pairs production and annihilation are due to purely gravitational interactions. Some of the massive pairs, however, can also carry the quantum numbers of gauge interactions. Therefore, in addition to their annihilation to the spacetime (20), these “unstable” massive pairs via gauge interactions decay to relativistic particles $\bar{\ell} \ell$, which are much lighter than massive pairs $\bar{F}F$ themselves,

$$\bar{F}F \Rightarrow \bar{\ell} \ell. \hspace{1cm} (24)$$

These relativistic particles $\bar{\ell} \ell$ represent elementary particles in the standard model (SM) of particle physics, and possible massless or massive sterile particles (neutrinos) for the dark matter, as well as composite particles [32] or particles of other theories beyond SM. In general, the decay rate of massive pairs is proportional to the pair mass $\hat{m}$ and can be parametrised as

$$\Gamma_{M}^{\text{de}} = g_{Y}^{2} \hat{m}, \hspace{1cm} (\bar{F}F \Rightarrow \bar{\ell} \ell) \hspace{1cm} (25)$$

where $g_{Y}$ is the Yukawa coupling between the massive pairs and relativistic particles. It is important to note that the decay rate $\Gamma_{M}^{\text{de}}$ depends not only on the Yukawa coupling $g_{Y}$, but also on the phase space of final particles, to which massive pairs decay. The some of spacetime produced massive pairs cannot decay to relativistic particles $\bar{\ell} \ell$, namely $\Gamma_{M}^{\text{de}} \approx 0$, and they would account for the cold massive dark matter.

Taking into account both pair-production and pair-annihilation processes, we study the semi-classical rate equation for the pair energy density $\rho_{M}$ based on the conservation of the total pair numbers in the Universe evolution obeying Einstein equations (3) and (4).

### B. Boltzmann rate equation for particle-antiparticle pairs and spacetime

We adopted the phase space density (the distribution function in phase space) is spatially homogenous and isotropic, and integrating over phase space, we have the pair number density
depending only on the time \( n_M(t) \), so that the Liouville operator in the phase space for the kinetic part is just \( d(a^3 n_M)/dt = a^3 n_M(t) + 3H a^3 n_M(t) \). Adopting the usual approach \cite{5,35} at the semi-classical level, we use the cosmic rate equation of the Boltzmann type for the pair number density \( n_M \),

\[
\frac{dn_M}{dt} + 3H n_M = \Gamma_M \left( n^H_M - n_M \right) - \Gamma^{de}_M n_M,
\]

\[
\frac{d\rho_M}{dt} + 3H \rho_M = \Gamma_M \left( \rho^H_M - \rho_M \right) - \Gamma^{de}_M \rho_M,
\]

(26)

where the second line is due to the massive pairs \( \rho_M \approx 2\hat{m} n_M \) and \( \rho^H_M \approx 2\hat{m} n^H_M \). These equations effectively describe the pair dynamics of the back and forth processes \( \chi \) and decay processes \( \delta \) in the Universe evolution. The term \( 3H n_M \) of the time scale \((3H)^{-1}\) represents the spacetime expanding effect on the pair density \( n_M \). It should be emphasized that \( \Gamma_M n^H_M \) is the source term of pair productions from the space time, and \( \Gamma_M n_M \) is the depletion term of pair annihilations into the space time. The spacetime horizon and particle-antiparticle pairs are coupled via the back and forth processes \( \chi \). The pair production and annihilation rates are assumed to be equal to \( \Gamma_M \) \( \chi \) in the detailed balance term

\[
\Gamma^{Prod}_M n^H_M - \Gamma^{Anni}_M n_M = \Gamma_M \left( n^H_M - n_M \right),
\]

(27)

in the RHS of the cosmic rate equation \( \chi \).

**C. A close set of fundamental equations in the reheating epoch**

In addition to the cosmic rate equation \( \chi \), there is another Boltzmann equation from the conservation of the radiation energy of relativistic particles from massive particle decays. Massive pairs decay \( \delta \) to relativistic particles \( \bar{\ell} \ell \), whose energy density is \( \rho_R \). The evolution of such radiation energy density \( \rho_R \) can be obtained by the energy conservation law, see for example Ref. \[5\],

\[
d(a^3 \rho_R) = -p_R d(a^3) - d(a^3 \rho_M) + \frac{\rho_R}{3} d(a^3) + (a^3 \rho_M) \Gamma^{de}_M dt,
\]

(28)

where \( d(a^3 \rho_M) = -(a^3 \rho_M) \Gamma^{de}_M dt \) is the massive pair energy that converts to the radiation energy of relativistic particles. This leads to the crucial reheating equation

\[
\dot{\rho}_R + 4H \rho_R = \Gamma^{de}_M \rho_M,
\]

(29)
in the reheating epoch.

As a consequence, we have a close set of four ordinary differential equations to uniquely determine the time evolutions of the Hubble rate $H$, the cosmological term $\rho_\Lambda$, massive pairs’ energy density $\rho_M$ and relativistic particles’ energy density $\rho_R$. They are the cosmic rate equation (26) for $\rho_M$, the reheating equation (29) for $\rho_R$, Einstein equations (3) and (4) for $H$ and $\rho_\Lambda$. In addition, there are three algebraic relations: the pair-production rate $\Gamma_M$ (23), the pair-decay rate $\Gamma^\text{de}_M$ (25) and the spacetime evolution $\epsilon$-rate (5). For further analysis, we recast these equations as the Einstein equations

$$\frac{h^2}{3} = \Omega_\Lambda + \Omega_M + \Omega_R,$$

$$\frac{dh^2}{dx} = -3\Omega_M - 4\Omega_R,$$

and the cosmic rate equations

$$\frac{d\Omega_M}{dx} + 3\Omega_M = \frac{\Gamma_M}{H} (\Omega_M^H - \Omega_M) - \frac{\Gamma^\text{de}_M}{H} \Omega_M,$$

$$\frac{d\Omega_R}{dx} + 4\Omega_R = \frac{\Gamma^\text{de}_M}{H} \Omega_M,$$

where, instead of the cosmic time $t$, we adopt the cosmic $e$-folding variable $x = \ln(a/a_{\text{end}})$ and $d(\cdots)/dx = d(\cdots)/(H dt)$ for the sake of simplicity and significance in physics.

In order to study the reheating epoch, we adopt in Eqs. (30-33) the scale factor $a_{\text{end}}$ at the end of the inflation, corresponding the normalisations $h \equiv H/H_{\text{end}}$, $\Omega_{\Lambda,M,R} \equiv \rho_{\Lambda,M,R}/\rho_c^{\text{end}}$ and

$$\Omega_M^H \equiv \frac{\rho_M^H}{\rho_c^{\text{end}}} = \frac{2}{3} \hat{m}^2 h^2, \quad \rho_c^{\text{end}} \equiv \frac{3H_{\text{end}}^2}{8\pi G},$$

in unit of the inflation ending scale $H_{\text{end}}$ and the corresponding characteristic density $\rho_c^{\text{end}}$. Using the rates $\Gamma_M$ (23) and $\Gamma^\text{de}_M$ (25), we write the ratios in Eqs. (32) and (33),

$$\frac{\Gamma_M}{H} = \left(\frac{\chi}{4\pi}\right) \left(\frac{\hat{m}}{H_{\text{end}}}ight) \epsilon \frac{1}{h}; \quad \frac{\Gamma^\text{de}_M}{H} = g_Y^2 \left(\frac{\hat{m}}{H_{\text{end}}}ight) \frac{1}{h},$$

which represent the rates $\Gamma_M$ and $\Gamma^\text{de}_M$ of the microscopic processes (21) and (25) compared with the Hubble rate $H$ of the macroscopic expansion of the spacetime. Moreover, we rewrite the $\epsilon$-rate (36) of time-varying horizon $H$ as

$$\epsilon \equiv -\frac{1}{H} \frac{dH}{dx} = \frac{3}{2} \frac{\Omega_M + (4/3)\Omega_R}{\Omega_\Lambda + \Omega_M + \Omega_R},$$

to characterise the different episodes of the reheating epoch.

Equations (30-36) can be numerically integrated, provided that the initial conditions (19), see Sec. II C, are given at the beginning of the reheating epoch. In order to understand the reheating
physics encoded, it is worthwhile to find the asymptotic solutions to physically characterise particular episodes in the entire reheating epoch. This epoch is represented by the scale factor changing from the inflation end \( a_3 = a_{\text{end}} \) to the genuine reheating \( a_2 = a_R \) in Fig. 1, the schematic diagram of the Universe evolution.

**IV. DIFFERENT EPISODES IN THE REHEATING EPOCH**

In the reheating epoch, general speaking, the horizon \( h \) and the cosmological term \( \Omega_\Lambda \) decreases, as the matter content \( \Omega_M \) or \( \Omega_R \) increases, meanwhile the ratio \( \Gamma_M/H \) and the \( \epsilon \)-rate increase. To gain the insight into the physics first, we use the \( \epsilon \)-rate values to characterize the different episodes in the reheating epoch. In each episode, the \( \epsilon \)-rate slowly varies in time, we approximately have the time scale of the spacetime expansion

\[
H^{-1} \approx \epsilon t. \tag{37}
\]

In the transition from one episode to another, the \( \epsilon \)-rate significantly changes its value. Using the characteristic \( \epsilon \) values \( \epsilon \ll 1, \epsilon \approx 3/2, \epsilon \approx 2 \), we identify the following three different episodes: \( \mathcal{P}\text{-episode} \), \( \mathcal{M}\text{-episode} \) and \( \mathcal{R}\text{-episode} \) in the reheating epoch. The \( \mathcal{P}\text{-episode} \) and the \( \mathcal{M}\text{-episode} \) have some similarities to the preheating phase in usual inflation models [2, 3].

**A. preheating episode: \( \mathcal{P}\text{-episode} \)**

The preheating \( \mathcal{P}\text{-episode} \) is a transition from the inflation end to the reheating start. In this episode, the pair production rate \( \Gamma_M \) is larger than the Hubble rate \( H \), that is still much larger than the pair decay rate \( \Gamma_{de}^M \),

\[
\Gamma_M > H \gg \Gamma_{de}^M, \quad \rho_\Lambda > \rho_M \gg \rho_R. \tag{38}
\]

The radiation energy density of relativistic particles is completely negligible \( \rho_R \approx 0 \), compared with the massive pairs’ energy density \( \rho_M \) and cosmological one \( \rho_\Lambda \). The studies presented in this and next sections are also relevant to the “stable” massive pairs which do not decay to relativistic particles \( \bar{\ell} \ell \), i.e. \( \Gamma_{de}^M = 0 \), via gauge interactions.
1. Numerical solutions of cosmological $\rho_\Lambda$ converted to matter $\rho_M$

After the inflation end $\Gamma_M \gtrsim H$, the pair-production rate $\Gamma_M$ increases as the $\epsilon$-rate

$$\epsilon \approx \frac{3}{2} \frac{\rho_M}{\rho_\Lambda + \rho_M} = \frac{3}{2} \frac{\Omega_M}{\Omega_\Lambda + \Omega_M}. \quad (39)$$

As a result, the massive pairs are significantly produced and pairs have the large density $n_M$ and rate $\Gamma_M n_M$ to annihilate into the spacetime. The spacetime horizon and pairs are thus coupled, and the back and forth processes have to be considered by the rate equation

$$\dot{\rho}_M + 3H\rho_M = \Gamma_M \left( \rho_M^H - \rho_M \right) \quad (40)$$

where the decay of massive pairs to relativistic particles is neglected. The reheating equation (33) is then not relevant, and the basic equations (30), (31) and (32) reduce to

$$h^2 = \Omega_\Lambda + \Omega_M, \quad (41)$$

$$\frac{dh^2}{dx} = -3\Omega_M, \quad (42)$$

$$d\Omega_M/dx + 3\Omega_M = (\Gamma_M/H) \left( \Omega_M^H - \Omega_M \right). \quad (43)$$

These equations uniquely determine the evolutions of the Hubble rate $H$, pairs’ energy densities $\rho_M$ and cosmological term $\rho_\Lambda$. The ratio $\Gamma_M/H$ of the pair-production rate and the Hubble rate has to be larger than one, $\Gamma_M/H > 1$ in Eq. (43).

We are in the position of solving these equations for the $\mathcal{P}$-episode. Using the follow values at the inflation end $H_{\text{end}}$ and energy density ratio $\rho_M^{\text{end}}/\rho_\Lambda^{\text{end}} \ll 1$, see Sec. II C

$$\Omega_M^{\text{end}} = 4.7 \times 10^{-3}, \text{ so that } (\Gamma_M/H)_{\text{end}} \approx 1 \quad (44)$$

as the initial conditions for starting the $\mathcal{P}$-episode, we numerically integrate Eqs. (30,31) and (32), by selecting values of the mass parameter $\hat{m}/m_{\text{pl}}$.

The numerical solutions are plotted in Figures 3 and 4 in terms of the $e$-folding variable $x = \ln(a/a_{\text{end}})$, where the scale factor $a_{\text{end}}$ correspondingly to the scale $H_{\text{end}}$ at the inflation end. These solutions show an important result that the cosmological energy density $\rho_\Lambda$ is significantly converted to the matter energy density $\rho_M$, as the pair-production rate $\Gamma_M$ increases and becomes larger than the Hubble rate $H$. In more details, we list that in the $\mathcal{P}$-episode the physical quantities vary in time as follow,

(i) the Hubble rate $h$ decreases rapidly in a few $e$-folding number, as the energy density $\rho_M$ of produced pairs becomes dominate over the energy density $\rho_\Lambda$ of the cosmological term, see Fig. 3(a);
FIG. 3: (Color Online). In a few $e$-folding number $x = \ln(a/a_{\text{end}})$, (a) the Hubble rate drops rapidly; (b) the pair energy density exceeds the energy density of the cosmological term; (c) the ratio $\Gamma_M/H > 1$ increases rapidly; (d) the $\epsilon$-rate of $H$ variation increases in the transition from $\epsilon \ll 1$ (inflation epoch) to the asymptotic value $\epsilon \sim O(1)$ ($\mathcal{M}$-episode, see Sec. IV B). These illustrations are plotted with the initial condition (44) and parameter $(\hat{m}/m_{\text{pl}}) = 27.7$.

(ii) the pair energy density $\rho_M$ increases at the expense of the energy density $\rho_\Lambda$ of the cosmological term, eventually $\rho_M$ exceeds and dominates $\rho_\Lambda$, see Fig. 3 (b);

(iii) the ratio $\Gamma_M/H > 1$ (38) of the pair-production rate $\Gamma_M$ and the Hubble rate $H$ increases and becomes much larger than unity ($\Gamma_M/H \gg 1$), see Fig. 3 (c);

(iv) the $H$ varying $\epsilon$-rate (39) increases from $\epsilon \ll 1$ to $\epsilon \sim O(1)$, indicating the transition from the inflation end to the preheating $\mathcal{P}$-episode, and it then approaches to an asymptotic value, see Fig. 3 (d).

In Figure 4 (left), we plot the energy densities $\Omega_\Lambda$ and $\Omega_M$ are plotted as functions of the horizon $h^2$, corresponding to Figures (a) and (b) in Fig. 3. It shows that two branches of asymptotic
solutions respectively for $h^2 > 0.9$, $h^2 < 0.9$

$$\Omega_\Lambda \sim \alpha_g h^2, \quad \Omega_\Lambda \sim \alpha_s h^2; \quad \alpha_g \gg \alpha_s,$$

(45)

$\Omega_M + \Omega_\Lambda = h^2$ and the turning point is about $h^2 \approx 0.98$ at which $\Omega_M$ exceed $\Omega_\Lambda$ and the rapid $\rho_\Lambda \gg \rho_m$ converting process takes place. The characteristic behaviour (45) in the reheating epoch is the same as $\Omega_\Lambda \propto h^2$ in the pre-inflation and inflation epoch, as well as the radiation and matter dominate epochs in the standard cosmology discussed in Ref. [26].

Figure 4 (right) shows that in the preheating $P$-episode, the ratio $\rho_M/\rho_\Lambda$ rapidly increases in a few e-folding numbers from $\rho_M/\rho_\Lambda \ll 1$ at the inflation end to a value $\rho_M/\rho_\Lambda \gtrsim O(1)$. As physically expected, these results show in the $P$-episode, the fast decrease of the Hubble rate $H$ that becomes much smaller than the pair-production rate $\Gamma_M$; the convention of the energy $\rho_\Lambda$ to the matter-energy $\rho_M$ that finally becomes dominant in the evolution. These properties of $P$-episode are very different from the properties of the inflation epoch. The $P$-episode end can be defined at

$$\rho_\Lambda \approx \rho_M, \quad a \gtrsim a_{\text{end}}$$

(46)

where $a \approx 1.11 a_{\text{end}}$, from Figs. 3 and 4 (right). This shows that the preheating $P$-episode is a very brief transition episode. The latter approached asymptotic values of $\epsilon$-rate and $\rho_M/\rho_\Lambda$, shown in Figs. 3 (d) and 4 (right), indicate another $M$-episode that will be discussed soon.
2. Threshold of produced pair mass and degeneracy for $\rho_M > \rho_\Lambda$

Moreover, another important result is that the numerical calculations shows how these solutions depend on the pair mass parameter $\hat{m}$ introduced for the reheating epoch:

(a) For large mass parameters $\hat{m}/m_{\text{pl}} > 20$, the pair energy density $\rho_M$ exceeds the energy density $\rho_\Lambda$ of the cosmological term and the asymptotic value of the ratio $\rho_M/\rho_\Lambda > 1$,

\[
\hat{m}/m_{\text{pl}} > 20, \quad \rho_M/\rho_\Lambda > 1.
\]

The physical explanation is that the effective degeneracy $g_d$ of massive pairs produced in the reheating epoch has to be large enough so that the production rate $\Gamma_M$ is much larger than the Hubble rate $\Gamma_M \gg H$ and the conversion from $\rho_\Lambda$ to $\rho_M$ is efficient. This case corresponds to the real physical situation that the most relevant matter of the Universe is generated in the reheating epoch, and afterwards the Universe expansion starts the radiation dominated epoch $\rho_R \gg \rho_M \gg \rho_\Lambda$ of the standard cosmology;

(b) For small mass parameters $\hat{m}/m_{\text{pl}} \lesssim 20$, the pair energy density $\rho_M$ never exceeds the energy density $\rho_\Lambda$ of the cosmological term, namely the ratio $\rho_M/\rho_\Lambda < 1$ is always smaller than one. This case should correspond to the unphysical situation that the Universe inflation would have never completely ended, i.e., $H^2$ is always dominated by the cosmological term $\rho_\Lambda$.

We will discuss the mass parameter $\hat{m}$ for the real physical situation (a) to see how its values relate to the CMB observations through the ratio $\rho_M/\rho_\Lambda > 1$ in Sec. 5.

Observe that the mass parameter $\hat{m}$ of the reheating epoch is larger than the mass parameter $m_*$ of the inflation epoch, i.e., $\hat{m} > m_*$. From the viewpoint of the pair production, the pair mass scale in the reheating epoch should be smaller than that in the inflation epoch, since the horizon $H$ of the reheating epoch is smaller than that of the inflation epoch. Therefore, this implies that the effective degeneracies $g_d$ of pairs produced in the reheating epoch $\Gamma_M/H > 1$ is much larger than the effective degeneracies of pairs produced in the inflation epoch $\Gamma_M/H < 1$.

Equation (39) shows that the asymptotic value of the Horizon variation $\epsilon$-rate relates to the ratio $\rho_M/\rho_\Lambda$ asymptotic value, see Figs. 3 (d) and 4 (right). For large mass parameter $\hat{m}/m_{\text{pl}} \gtrsim 27.7$, the asymptotic value of the ratio $\rho_M/\rho_\Lambda \gg 1$, the $\epsilon$-rate approaches to the constant $\epsilon \approx \epsilon_M = 3/2$. This shows the episode of massive pairs domination: $\mathcal{M}$-episode.
3. Minimal comoving radius \((Ha)^{-1}\) location

Before discussing the \(M\)-episode, we would like to mention the turning point at which the Universe acceleration vanishes \(\ddot{a} = 0\),

\[
2\rho_\Lambda = (1 + 3\omega_M)\rho_M - (1 + 3\omega_R)\rho_R,
\]

which is obtained from the 1–1 component of the Einstein equation

\[
\frac{2}{a} \frac{dH}{dt} + 2H^2 = \frac{2\dot{a}}{a} = \left[2\rho_\Lambda - (1 + 3\omega_M)\rho_M - (1 + 3\omega_R)\rho_R\right].
\]

At this turning point, the Universe stops acceleration \(\ddot{a} > 0\) and starts deceleration \(\ddot{a} < 0\). The turning point occurs at \(\rho_\Lambda = \rho_M/2\) for \(\omega_M \approx 0\) and \(\rho_R \approx 0\). This tells us the balance point of the competition between the cosmological term \(\rho_\Lambda\) and matter term \(\rho_M\) in the \(P\)-episode.

On the other hand, the minimal value of the comoving radius \((aH)^{-1}\) locates at

\[
d(aH)^{-1}/dt = 0 \Rightarrow \dot{H} + H^2 = 0.
\]

From Eqs. (3) and (4), we obtain

\[
\rho_\Lambda = \rho_M/2 + \rho_R \approx \rho_M/2, \quad \epsilon = \epsilon_\text{min} = 1,
\]

coinciding with the turning point \((48)\). Namely at the minimal comoving radius \((aH)^{-1}\), the Universe stops acceleration \(\ddot{a} > 0\) and starts deceleration \(\ddot{a} < 0\), beginning the reheating epoch and standard cosmology. This is indeed the case for large mass parameter \((\hat{m}/m_{\text{pl}}) > 20\) and the ratio \(\rho_M/\rho_\Lambda\) becomes larger than 2. The numerical results (Fig. 3) show that this turning/minimal point \(\epsilon_\text{min} = 1\) locates at \(x_{\text{min}} \approx 1.7 \times 10^{-2}\) and \(a_{\text{min}} \approx a_{\text{end}} \times \exp (1.7 \times 10^{-2}) = 1.02 a_{\text{end}}\).

While, the turning/minimal point \(\epsilon_\text{min} = 1\) is never reached, for the cases of the small mass parameter \((\hat{m}/m_{\text{pl}}) < 20\) and the ratio \(\rho_M/\rho_\Lambda\) is always smaller than 2, see Fig. 4 (right). The reason is that there is no enough matter of massive pairs produced to balance the cosmological term and slow down the Universe acceleration. As a consequence, the Universe keeps acceleration \(\ddot{a} > 0\) and does not run into the epoch of the standard cosmology. Therefore, the mass parameter range \((\hat{m}/m_{\text{pl}}) < 20\) should be excluded.

B. Massive pairs domination: \(M\)-episode

After the \(P\)-episode transition, the reheating epoch is in the \(M\)-episode of massive pair domination. The \(M\)-episode is characterised by

\[
\rho_M \gg \rho_\Lambda \gg \rho_R, \quad \Gamma_M > H > \Gamma_M^{\text{de}},
\]
so that the radiation energy density $\rho_R$ is negligible in the Einstein equations \([3,5]\) and the cosmic rate equation \([26]\). The $H$ variation $\epsilon$-rate $\epsilon_M$ is a constant, shown as an asymptotic value $\epsilon_M \approx 3/2$ in Fig. 3 (d) for $\rho_M/\rho_\Lambda \gg 1$, see Fig. 4 (right). In this episode, the Hubble rate $H$ and scale factor $a(t)$ vary as

$$H^{-1} \approx \epsilon_M t, \quad a(t) \sim t^{1/\epsilon_M},$$

and $h^2 \approx \Omega_M$, the pair energy density $\Omega_M \propto (a/a_{end})^{-2\epsilon_M}$ drops as in the matter dominated universe, analogously to the scenario \([36]\).

1. Spacetime and pairs are coupled in horizon $H$ evolution

In addition to the large pair energy density $\rho_M$, the pair production/annihilation rate $\Gamma_M$ is much larger than the Hubble rate $H$, i.e., $\Gamma_M/H \gg 1$, see Fig. 3 (c). The back and forth processes of pair production and annihilation $\bar{F}F \leftrightarrow S$ are important, as described by the cosmic rate equation \([40]\) with the detailed balance term $D_M$,

$$D_M \equiv \Gamma_M \left(\Omega_M^H - \Omega_M\right),$$

and the characteristic time scale $\tau_M$,

$$\tau_M^{-1} \equiv \frac{\Gamma_M}{\Omega_M} \left(\Omega_M^H - \Omega_M\right),$$

which is actually the time period of back and forth $\bar{F}F \leftrightarrow S$ oscillating processes. This macroscopic time scale is much smaller than the macroscopic expansion time scale $\tau_H = H^{-1}$, $\tau_M \ll \tau_H$. In this situation, the microscopic back and forth process $(S \leftrightarrow \bar{F}F)$ is much faster than the horizon expanding process, thus the space time and massive pairs are completely coupled each other via these back and forth processes.

Therefore, the back and forth oscillating process $\bar{F}F \leftrightarrow S$ can build a local chemical equilibrium of the quantum-number and energy equipartition

$$\rho_M \Leftrightarrow \rho_M^H; \quad \mu_F + \mu_{\bar{F}} = \mu_{\text{spacetime}}$$

between massive pairs and the space time. The chemical potential of particles is opposite to the chemical potential of antiparticles, i.e., $\mu_F = -\mu_{\bar{F}}$, so that particle and antiparticle pairs have zero chemical potential $\mu_{\text{pair}} = \mu_F + \mu_{\bar{F}} = 0$. The chemical equilibrium \([56]\) leads to the space time “chemical potential” is zero, i.e., $\mu_{\text{spacetime}} = 0$. In this case, the detailed balance term \([54]\)
for the oscillations $\rho_M \leftrightarrow \rho_M^H$ in the microscopic time scale $\tau_M$ should vanishes, in the sense of its time-averaged

$$\langle \rho_M - \rho_M^H \rangle = 0,$$

(57)

over the macroscopic time $\tau_H \gg \tau_M$. This means that $\rho_M \approx \rho_M^H$ in the macroscopic time scale $\tau_H$. The cosmic rate equation approximately becomes

$$\frac{d\rho_M}{dt} + 3H\rho_M \approx 0,$$

(58)

whose solution is $\rho_M \propto a^{-3}$. This is consistent with the solution to Eq. (3) for the massive pair domination $\rho_M \gg \rho_\Lambda$ and $\rho_M \gg \rho_R$, yielding $H^2 \sim \rho_M \propto a^{-3}$. This is also self-consistent with the pair-production formula (7) $\rho_M \approx \rho_H^M = \chi\hat{m}^2H^2 \propto a^{-3}$.

In order to verify these discussions, we check the solution (57) or (58) analytically and numerically. The approximately analytical solution $\rho_M \approx \rho_M^H = \chi\hat{m}^2H^2$ averaged over the time $\tau_H$ consistently obeys the cosmic rate equation (58),

$$\langle \dot{\rho}_M^H \rangle = \langle 2\chi\hat{m}^2HH \rangle = -\langle 2H\rho_M^H\epsilon \rangle \approx -3H\rho_M^H,$$

(59)

where $\langle \epsilon \rangle \approx \epsilon_M = 3/2$, and the detailed balance term (54) vanishes. Numerical results quantitatively show the same conclusion: $\rho_M$ approaches to $\rho_M^H$, and $n_M \approx \rho_M/2\hat{m}$ approaches to $n_M^H \approx \rho_M^H/2\hat{m}$

$$\rho_M \approx \rho_M^H = 2\chi\hat{m}^2H^2, \quad n_M \approx n_M^H = \chi\hat{m}H^2,$$

(60)

see Fig.5(a); correspondingly the detailed balance term (54) vanishes, see Fig.5(b). The detailed balance solution (60) is valid in the matter dominate evolution, and it is peculiar for stable massive pairs, which have no gauge interactions except gravitation one.

2. Preliminary discussions on CPT symmetry and increasing entropy

We observe that in the $\mathcal{M}$-episode, due to $\Gamma_M \gg H$ and $\tau_M \ll \tau_H$, the spacetime and pairs are tightly coupled each other and the local chemical equilibrium (56) can be established. This indicates that the CPT symmetry (22) is approximately preserved in the cosmic rate equation (26) involving the microscopical process $\mathcal{S} \leftrightarrow \bar{F}F$. On the other hand, the $T$-symmetry or time-reversal symmetry $t \rightarrow -t$ is macroscopically violated in the Universe expansion with increasing time and entropy.
This is not contradictory for the following reasons. The number and energy density of pair productions is slightly larger than that of pair annihilations, $\rho_{H}^{M} \gtrsim \rho_{M}^{H}$. The pair production $S \rightarrow \bar{F}F$ produce the entropy of pairs. Instead, the pair annihilation $\bar{F}F \rightarrow S$ eliminate the entropy of pairs. The net entropy of pairs produced is very small as pairs are very massive and the difference $\rho_{H}^{M} - \rho_{M}$ is very small. For this tiny production of pairs' entropy, the Universe consistently undergoes an entropically favourable expansion.

In the case that the number and energy densities of pair productions and annihilations are exactly the same $\rho_{M} \equiv \rho_{H}^{M}$, a local thermodynamic equilibrium (LTE) can be built between the spacetime and pairs, associating with the Hawking temperature $T_{H} = H/2\pi$ for a De Sitter spacetime of an exactly constant $H$. In this case, no pairs’ entropy is produced and the total entropy of pairs is conserved. Pairs’ temperature and entropy render the physical senses of the spacetime entropy and temperature, which is not in the scope of this article.

C. Relativistic particles domination: $R$-episode of the genuine reheating

After the $M$-episode of massive pairs domination, the massive pairs’ decay term $\Gamma_{M}^{\text{de}}\rho_{M}$ in the cosmic rate equation (40) starts to dominate, when the time $t \gtrsim \tau_{R}$, where $\tau_{R} = (\Gamma_{M}^{\text{de}})^{-1}$ is the characteristic time scale of massive pairs decay to relativistic particles, producing tremendous amounts of entropy. The reheating epoch starts its genuine reheating episode, i.e., $R$-episode. We discuss this episode following the line of Ref. 5.
FIG. 6: (Color Online). In a few $e$-folding number $x = \ln(a/a_{\text{end}})$, (a) the blue line $h^2$ and orange line $\Omega_R$, the Hubble rate drops more rapidly than the case neglecting $\Omega_R$ Fig. 3 (a); (b) $\Omega_R$, $\Omega_M$ and $\Omega_\Lambda$ are lines green, blue and orange; (c) the $e$-rate of $H$ variation increases in the transition from $e \ll 1$ ($\mathcal{P}$-episode) to the asymptotic value $e \sim \mathcal{O}(1)$ ($\mathcal{M}$-episode) and approaches to $e = 2$ of the radiation domination ($\mathcal{R}$-episode); (d) the ratios of $\Omega_M/\Omega_R$ (orange) and $\Omega_\Lambda/\Omega_R$ (blue), recalling $h^2 = \Omega_\Lambda + \Omega_M + \Omega_R$. These illustrations are plotted with the initial conditions (44) and $\Omega_{R_{\text{end}}} = 0$; the parameter $(\hat{m}/m_{\text{pl}}) = 27.7$ and $\sigma^2 = 10^{-9}$.

1. Massive pairs decay to relativistic particles

To study this $\mathcal{R}$-episode, we numerically solve the closed set of the basic equations (30-33) and the reheating equation (33) by taking into account relativistic particles of the radiation energy density $\Omega_R$ and the decay term,

$$ R_M = \Gamma_M^\text{de} \Omega_M, $$ (61)

so as to obtain the energy densities $\Omega_R$, $\Omega_M$, $\Omega_\Lambda$ and the horizon $H$ as functions of the $e$-folding number $x$ (time $t$). The initial condition of of the radiation energy density $\Omega_R = \Omega_{R_{\text{end}}} = 0$ is chosen at the inflation end $a_{\text{end}}$, in addition to the initial conditions (44). The numerical results are reported in Figures 6 which show that in the $H^2$ (30) the radiation energy density $\Omega_R$ of relativistic...
particles increases from the negligible contribution to the dominated contribution, compared with $\Omega_M$ and $\Omega_\Lambda$.

This phenomenon can be understood by comparing the decay term $R_M$ (61) with the detailed balance term $D_M$ (54) in the cosmic rate equation (32). The $\rho_R$ is negligible when $D_M > R_M$, while the $\rho_R$ is dominant when $R_M > D_M$, and the transition from the one to another occurs approximately at $R_M \gtrsim D_M$, where $\rho_R \lesssim h^2$, as shown in Figs. 7 (a) and (b).

More precisely, it is the comparison between the characteristic time scale $\tau_M$ (55) of the pair back and forth process $F \bar{F} \Leftrightarrow S$ (21) and the characteristic time scale $\tau_R$ of the pair decay process $F \bar{F} \Rightarrow \ell \bar{\ell}$ (25),

$$\tau_R^{-1} \equiv \frac{R_M}{\Omega_M} = \Gamma_{de} = g^2 \nu m. \quad (62)$$

When $\tau_M < \tau_R$, the process $F \bar{F} \Leftrightarrow S$ is faster thus dominates, whereas $\tau_R < \tau_M$, the process $F \bar{F} \Rightarrow \ell \bar{\ell}$ is faster thus dominates. In Figs. 7 (c) and (d), two time scales $\tau_M$ and $\tau_R$ are plotted as dimensionless quantities $\tau_R/\tau_H = (\Gamma_{de}^H)^{-1}$ and $\tau_M/\tau_H = (\Gamma_M/H)^{-1}$ to show two episodes:

(i) $\tau_R/\tau_H > 1$ and $\tau_M/\tau_H < 1$ ($D_M > R_M$), indicating the $M$-episode and the decay process $F \bar{F} \Rightarrow \ell \bar{\ell}$ (25) being irrelevant;

(ii) $\tau_R/\tau_H < 1$ and $\tau_M/\tau_H > 1$ ($D_M < R_M$); indicating the $R$-episode and the process $F \bar{F} \Leftrightarrow S$ (21) being irrelevant.

The separatrix of two episodes, i.e., the crossing point of two blue and orange lines in Figs. 7 roughly gives the scale factor $a_R$ at which the genuine reheating occurs, i.e., $R$-episode.

The reheating scale factor $a_R$ value depends on the Yukawa coupling $g_Y$, the larger $g_Y$ and the smaller $a_R$, as it should be. This is shown by the left column (a,c) and the right column (b,d) of Figs. 7. Around this point $a_R$, Figures 6 (b) and (d) show $\Omega_R \gg \Omega_M \gg \Omega_\Lambda$, and Fig. 6 (c) shows $\epsilon$-rate (36) approaches to two ($\epsilon \rightarrow 2$), indicating the genuine reheating occurrence. Note that at this point $a_R$ the numerical calculations of the basic equations (30-33) run into the stiffness system of step size being effective zero. However, in the case of genuine reheating starts, the analytical solution to these basic equations can be found and studied in the next section.

2. Energy densities of massive pairs and relativistic particles

After the $M$-episode, the decay term $R_M$ (61) prevails over the detailed balance term $D_M$ (54) in the cosmic rate equation (32), massive pairs undergo the process of decay to relativistic particles,
rather than the process of annihilating to the spacetime. The spacetime and pairs are decoupled in time evolution. It starts the $\mathcal{R}$-episode of the genuine reheating and radiation energy domination, which is characterised by

$$\rho_R \gg \rho_M \gg \rho_\Lambda, \quad \epsilon \to \epsilon_R \approx 2, \quad \text{(63)}$$

and $\Gamma_{M}^{\text{de}}/H > 1$, as shown in Figs. 6. As a result, Equations (30) and (31) or Eq. (36) gives

$$H^{-1} \approx \epsilon_R t, \quad a(t)/a_R \approx (t/\tau_R)^{1/\epsilon_R}, \quad \text{(64)}$$

where the period of massive particles decay $\tau_R = (\Gamma_{M}^{\text{de}})^{-1}$ is the reheating time scale and $a_R$ is the scale factor at the genuine reheating. Following the line presented in Ref. [5], we discuss how the annihilation/decay of $F\bar{F}$ pairs transfers their mass energy to relativistic particles, and calculate the radiation energy density $\rho_R$, entropy $S$ and temperature $T$ of relativistic particles.
Since massive pairs predominately decay to relativistic particles, the detailed balance term $D_M$ is negligible, and the cosmic rate equation (32) reduces to,

$$\frac{d(a^3\rho_M)}{dt} = a^3\frac{d\rho_M}{dt} + 3Ha^3\rho_M \approx -\tau_R^{-1}a^3\rho_M, \quad (65)$$

$$\Rightarrow \rho_M \approx \rho_M(a_R) \left(\frac{a}{a_R}\right)^{-3} \exp -t/\tau_R. \quad (66)$$

The reheating equation (33) becomes

$$\frac{d(a^4\rho_R)}{dt} = a^3\rho_M(a_R) \left(\frac{a}{a_R}\right)^{-3} \exp -t/\tau_R. \quad (67)$$

In theory it requires the the time integration from the initial time $t_i(a_i) \ll \tau_R$ when $\rho_R(a_i) = 0$ to the final time $t_f \gg \tau_R$ to obtain the radiation energy density $\rho_R$ of relativistic particles,

$$\rho_R = \left(\frac{a_R}{a}\right)^4 \frac{\rho_M(a_R)}{\tau_R} \int_{t_i}^{t_f} \left(\frac{t}{\tau_R}\right)^{1/\epsilon_R} e^{-t/\tau_R} dt \approx 0.89 \left(\frac{a_R}{a}\right)^4 \rho_M(a_R). \quad (68)$$

Through their gauge and/or other induced interactions, these relativistic particles $\bar{\ell}\ell$ including sterile particles and other particles beyond the SM, are quickly thermalised at a very high temperature $T_{RH}$, due to their high number and energy densities. The local thermalization time scale is very short than the expansion time scale $\tau_H = H^{-1}$, and thus the local thermal equilibrium (LTE) is built.

3. Temperature and entropy of reheating epoch

On the other hand, the second law of thermodynamics applied to a comoving volume element yields [5]

$$dS = \frac{dQ}{T} = -\frac{d(a^3\rho_M)}{T} \approx \frac{a^3\rho_M}{T} \tau_R^{-1} dt, \quad (69)$$

where $dQ$ is the pair mass energy and $dS$ is the entropy of relativistic particles produced from massive pairs decay. Therefore, in a comoving volume, the entropy and energy densities of relativistic particles at the thermal state of temperature $T$ are given by,

$$\rho_R = \frac{\pi^2}{30} g_* T^4, \quad S = \frac{2\pi^2}{45} g_* a^3 T^3, \quad \rho_R = \frac{3}{4} \left(\frac{45}{2\pi^2 g_*}\right)^{1/3} S^{4/3} a^{-4}, \quad (70)$$

where the appropriately time-averaged degeneracy (effective) $g_*$ over the decay period $\tau_R$ counts for the total number of effectively massless degrees of freedom, those species share common temperature $T$. Using the formula for the entropy (70), Eq. (69) can be written as,

$$S^{1/3} \dot{S} = \left(\frac{2\pi^2 g_*}{45}\right)^{1/3} a^4 \rho_M \tau_R^{-1}. \quad (71)$$
Integrating this equation over the $F\bar{F}$ pair decay interval from the initial scale factor $a_i$ to the reheating scaling factor $a_R > a_i$ leads to an approximate solution [5]

$$S_R^{4/3} = 1.09 \left( \frac{4}{3} \rho_M(a_i) a_R^{4/3} \right) \left( \frac{16\pi^3 g_* \rho_M(a_i)}{135 M_{pl}^2} \right)^{1/3} \tau_R^{2/3},$$

$$\Rightarrow S_R \approx 1.32 \left( \frac{16\pi^3 g_*}{135} \right)^{1/4} \rho_M(a_i) \left( \frac{a_i}{M_{pl}} \right)^{1/2} \left( \frac{\tau_R}{M_{pl}} \right)^{1/2},$$

(72)

where Eq. (66) is adopted and the initial entropy $S_i(a_i) \approx 0$, as massive pairs’ entropy is approximately zero. The integration is performed over the period of massive particles decay characterized by the time scale $\tau_R$. In theory it requires the integrating from the initial time $t_i(a_i) \ll \tau_R$ to the final time $t_i(a_R) \gg \tau_R$, when the entropy of relativistic particles is significantly increased. In practice, $a_i \lesssim a_R$ and $t_f \gtrsim \tau_R$ are approximately adopted in Eq. (72) for the reason that the entropy $S_R$ is mainly produced in the reheating time $\tau_R$ and around the scale factor $a_R(\tau_R)$.

At the reheating time scale $\tau_R$ and scale factor $a_R$, the reheating scale $H_{RH}$ can be obtained by either $\tau_R$ from the Friedmann equation (30) or the reheating temperature $T_{RH} \equiv T(t = \tau_R)$ from the thermalization (70) [5]:

$$H^2_{RH} \equiv H^2(t = \tau_R) \approx \frac{1}{4\tau_R^2},$$

(73)

$$H^2_{RH} \approx \frac{8\pi}{3M_{pl}^2} \rho_R \approx \frac{8\pi}{3M_{pl}^2} \left( \frac{\pi^2 g_* \tau^4}{30} \right),$$

(74)

leading to the reheating temperature

$$T_{RH} \approx 0.55 g_*^{-1/4} \left( \frac{M_{pl}}{\tau_R} \right)^{1/2} = 0.55 (g_*^{1/2} / g_Y^{1/2})^{1/2} (\dot{m}/M_{pl})^{1/2} M_{pl},$$

(75)

and the all-important entropy per comoving volume,

$$S_R \approx 1.32 \left( \frac{16\pi^3}{135} \right)^{1/4} \left( g_*^{1/2} / g_Y^{1/2} \right)^{1/2} (\dot{m}/M_{pl})^{1/2} a_R^3 \rho_M(a_i) / \dot{m},$$

(76)

and $s_R \equiv S_R / a_R^3$. Analytical Eqs. (73) and (74) physically mean that at the the genuine reheating (i) $H_{RH} \approx \Gamma_M^{de} / 2 = g_*^{1/2} \dot{m}/2$ the Hubble rate is the same order of the pair decay rate, (ii) $H^2_{RH} \approx \rho_R / 3m_{pl}^2$ the energy of relativistic particles is predominate. These results depend on the effective degeneracy $g_*$ (70) and the decay rate $\tau^{-1} = \Gamma_M^{de} = g_Y^{1/2} \dot{m}$ (25) of massive pairs $F\bar{F}$ to relativistic particles $\ell\ell$.

Indeed, our numerical calculations show the consistency of the approximation $a_i \lesssim a_R$ used in Eq. (72) and the agreement with the approximate analytical solutions (73) and (75). From Figs. 6 and 7 we find that the reheating predominately takes place around the reheating scale factor $a_R$, at
which $\tau_R \sim \tau_M$. At this reheating scale factor $a_R$, we indeed find the ratios $\tau_R/H_{RH} \approx \tau_M/H_{RH} \sim O(1)$ are about the order of unity. Moreover, from Fig. 6(a) we obtain the reheating scale

$$H_{RH} \approx 3.16 \times 10^{-4} H_{end} \approx 6.1 \times 10^9 \text{GeV}, \quad a_R \approx 20.1 a_{end}$$

(77)

for the case $g_Y^2 = 10^{-9}$ and $\hat{m} = 27.7 m_{pl}$, where $H_{end}$ (17) is used. While $H_{RH} \approx 10^{-1} H_{end} = 1.9 \times 10^{12} \text{GeV}$ (the plot is not present), and $a_R \approx 1.8 a_{end}$ for the case $g_Y^2 = 10^{-6}$. Note that the value $H_{end} \approx 7.95 \times 10^{-6} m_{pl}$ corresponds to $N_{end} = 60$ and $r = 0.052$ in Eq. (17).

4. Reheating scale factor and initial condition for standard cosmology

Our study shows that the reheating epoch composes the preheating $P$-episode, the $M$-episode of massive pairs domination, and the genuine reheating $R$-episode, and it lasts for the period from $a_{end} = a_3$ to $a_R = a_2$, see Fig. 1. From the numerical results $\epsilon$-rate of Figs. 6(c) and $\rho_M \gg \rho_\Lambda$ of Fig. 4 (right) for the mass parameter $\hat{m} > 20 m_{pl}$ (17), we observe that the $M$-episode $\epsilon \approx 3/2$ lasts much longer time than $P$-episode $\epsilon \ll 1$ and the $R$-episode $\epsilon \approx 2$. This implies that in the reheating epoch the $\rho_\Lambda \approx 3 m_{pl}^2 H_{end}^2$ energy density of the spacetime converts to the $\rho_M$ energy density of massive pairs, which then converts to the $\rho_R \approx 3 m_{pl}^2 H_{RH}^2$ energy density of relativistic particles at the the genuine reheating (69). This is the case that the reheating epoch ends at the scale factor $a_R$ with the condition $\rho_\Lambda \ll \rho_M \ll \rho_R$ to initiate the standard cosmological scenario.

In order to estimate the the scale factor change $a_R/a_{end}$ in the reheating epoch, we approximately use the conservation law (58) for the massive pair domination

$$\Delta_2 \equiv \frac{a_2}{a_3} = \frac{a_R}{a_{end}} \approx \left( \frac{\rho_M^i}{\rho_M^f} \right)^{1/3} \approx \frac{1}{\pi} \left( \frac{45}{4} \right)^{1/3} \left( \frac{H_{end}^2 M_{pl}^2}{g_y T_{RH}^4} \right)^{1/3},$$

(78)

assuming the initial pair energy density $\rho_M^i \approx \rho_{M_{end}}^i \approx 3 m_{pl}^2 H_{end}^2$ (19) at the beginning of the reheating epoch, and the final one $\rho_M^f \approx \rho_R \approx 3 m_{pl}^2 H_{RH}^2$ at the end of the reheating epoch, in virtue of Eqs. (70) and (74).

V. OBSERVATIONS TO FIX REHEATING TEMPERATURE AND MASS SCALE

Following the method proposed by Ref. [33, 37], we fix the reheating temperature by the CMB observations. Considering the cosmological evolution of the physical wavelength $\lambda(a)$ and wavenumber $k(a)$

$$\lambda(a) = \lambda_0 \frac{a_0}{a}, \quad k(a) = k_0 \frac{a_0}{a}, \quad \lambda(a) = 1/k(a),$$

(79)
FIG. 8: (Color Online). Fixed the observed spectral index $n_s = 0.965$, in terms of the tensor-to-scalar ratio $r$, we plot the inflation end $e$-folding number $N_{\text{end}}$ (16) and the reheating temperature $T_{\text{RH}}$ (84). These plots refer to their lower limits, due to the nature of inequality (16). The real values of $N_{\text{end}}$ and $T_{\text{RH}}$ should be slightly above the curves for a given $r$ value. The left plot $N_{\text{end}}$ is the supplement to Fig. 2, since the same equation (16) is used.

where the comoving wavenumber $k_0 = k(a_0)$ and wavelength $\lambda_0 = 1/k_0$ are constants in the cosmological evolution. The total increase of the scale factor from the horizon crossing $a_4$ to the present measurement $a_0 = 1$,

$$
\Delta_{\text{tot}} = \frac{a_0}{a_4} = \frac{\lambda_0}{\lambda(a_4)} = \frac{(H)_{\text{cross}}}{k_0},
$$

where $\lambda(a_4) = (H^{-1})_{\text{cross}}$, or $k_0 = a_4(H)_{\text{cross}}$ and $(H)_{\text{cross}}$ is the value of the Hubble parameter when the mode $\lambda(a) = 1/k(a)$ crossed the horizon at the scale factor $a_4$ during the inflation or pre-inflation epoch, see Fig. [1]. As an example, the pivot scale $k_0 = k_* = 0.05$ (Mpc)$^{-1}$ for the CMB observations [34], correspondingly to the scale factor $a_4 = a_*$ and $(H)_{\text{cross}} = H_*$ (14) at the inflation scale.

A. Reheating temperature and entropy vs tensor-to-scalar ratio $r \lesssim 0.048$

Using the scalar spectrum $\Delta^2_R$ (12) and the total scale factor $\Delta_{\text{tot}}$ (80) at the CMB pivot scale $k_*$, one arrives at

$$
\Delta_{\text{tot}} = \frac{M_{\text{pl}}}{k_*} \sqrt{\pi \epsilon_s A_s} = \frac{M_{\text{pl}}}{\sqrt{2} k_*} \sqrt{\pi (1 - n_s) A_s}.
$$

On the other hand, as illustrated in Fig. [1] $\Delta_{\text{tot}} = \Delta_3 \Delta_2 \Delta_1 \Delta_0$, where $\Delta_4 = (a_3/a_4)$ (18) and $\Delta_2 = (a_2/a_3)$ (78) are computed, whereas $\Delta_1 = (a_1/a_2) = (g_*/2)^{1/3} (T_{\text{RH}}/T_{\text{rec}})$ and $\Delta_0 = (a_0/a_1)$ =
1 + z_{rec} are given in terms of the temperature $T_{rec} = T_{CMB}(1 + z_{rec})$ and redshift $z_{rec}$ at the recombination [33],

$$\Delta_1 \Delta_0 = \frac{a_0}{a_2} = \frac{a_0}{a_R} \approx (g_s/2)^{1/3} (T_{RH}/T_{CMB}),$$  \hspace{1cm} (82)

as a result,

$$\Delta_{tot} = e^{N_{end}} \frac{1}{\pi} \left( \frac{45}{4} \right)^{1/3} \left( \frac{H_{end}^2 M_{pl}^2}{g_s T_{RH}^4} \right)^{1/3} T_{RH} \frac{T_{CMB}}{T_{CMB}} \left( \frac{g_s}{2} \right)^{1/3}.$$  \hspace{1cm} (83)

Equations (81) and (83) are independent of the effective reheating degeneracy $g_s$ and yield the reheating temperature

$$\frac{T_{RH}}{M_{pl}} = \left( \frac{45}{2^{3/2}} \right) \frac{e^{3 N_{end}}}{\pi^{9/2}} \left[ (1 - n_s) A_s \right]^{-3/2} \left( \frac{k_s}{T_{CMB}} \right)^{3} \left( \frac{H_{end}}{M_{pl}} \right)^{2}.$$  \hspace{1cm} (84)

in terms of the CMB observations $T_{CMB} = 2.725$ K = $2.348 \times 10^{-4}$ eV and $k_s = 0.05$Mpc$^{-1}$ (Mpc$^{-1}$ = $6.39 \times 10^{-30}$eV), as well as and $N_{end}$ [16] and $H_{end}$ [17], whose values depend on the CMB measurements $A_s$, $n_s$ and $r$, see Sec. II C.

Given the observed the scalar amplitude $A_s = 2.1 \times 10^{-9}$ and spectral index $n_s = 0.965$, the inflation ending e-folding number $N_{end}$ [16] and the reheating temperature $T_{RH}$ [84] are plotted in Fig. 8 as functions of the tensor-to-scalar ratio $r$ without any free parameter. Figure 8 shows that their values are $N_{end} \approx (50, 60)$ and $T_{RH}/M_{pl} \approx (5.5 \times 10^{-13}, 1.1 \times 10^9)$ in the range $r \approx (0.037, 0.052)$. This $r$ range is consistent with the observational constrain on the upper limit of the the tensor-to-scalar ratio $r < 0.1$ or $r < 0.065$ [34].

After obtaining the reheating temperature at the end $a_R$ of reheating, we calculate the entropy $S_{patch}$ produced within the physical patch of the volume $H_{RH}^{-3}$, which evolves from the initial patch of the volume $H_{s}^{-3}$ at the start $a_s = 1$ of inflation. The patch grows by a scale factor of Eqs. (18) and (78),

$$a_R^3 = (\Delta_3 \Delta_2)^3 \approx \frac{45 e^{3N_{end}}}{4\pi^3} \left( \frac{H_{end}^2 M_{pl}^2}{g_s T_{RH}^4} \right) = e^{3N_{end}} \left( \frac{H_{end}^2}{H_{RH}^2} \right).$$  \hspace{1cm} (85)

The entropy per comoving volume $S_R$ (76) at the end of reheating can be expressed as,

$$S_R \approx 2.2 \times 10^6 \left( \frac{16 \pi^3}{135} \right)^{1/4} \frac{3 a_R^3}{8 \pi} H_{end}^2 M_{pl}.$$  \hspace{1cm} (86)

where we use $\rho'_M(a_i) \approx \rho'_M(a_{end}) \approx 3 m_{pl}^2 H_{end}^2$ (78) and the constrain (89) below. The entropy $S_{patch}$ within the physical patch $H_{RH}^{-3}$ is given by,

$$S_{patch} = H_{RH}^{-3} S_R \approx 3.64 \times 10^5 e^{3N_{end}} \left( \frac{H_{end}^4 M_{pl}}{H_{RH}^2} \right).$$  \hspace{1cm} (87)
which is a function of \( r \) and \( g_* \). In Fig. 9 (left), we plot \( S_{\text{patch}} \) by using \( N_{\text{end}} \) \( (16) \), \( H_{\text{end}} \) \( (17) \) and \( H_{RH} \) \( (74) \). It shows that the calculated entropy accords with the observational vale \( S_{\text{patch}} \sim 10^{88} \) around \( r \sim 0.045 \). To have a better understanding how the physical patch horizon \( H_* > H_{\text{end}} > H_{RH} \) evolves, we plot in the same Fig. 9 (right) all characteristic Hubble scales from the inflation to the reheating: the inflation scale \( H_* \) \( (14) \), inflation end scale \( H_{\text{end}} \) \( (17) \), and reheating scale \( H_{RH} \) \( (74) \) in unit of the Planck scale \( M_{pl} \). It shows that the unphysical situation \( H_{RH} > H_{\text{end}} \) occurs when \( r > 0.047 \). Therefore the \( r > 0.047 \) range should be excluded and this theoretical upper limit is consistent with the observational one \( r < 0.065 \) \([34]\). Due to the dependence of \( H_{RH} \) on \( g_* \) and the approximations adopted in these preliminary calculations, we conservatively suggest a theoretical upper limit of the tensor-to-scalar ratio \( r \lesssim 0.048 \), which demands elaborated numerical calculations to precisely fix it.

These results show that the model \( \Lambda \)CDM we have been discussing so far is in accordance with observations. This implies that the precisely measuring \( r \)-value is uniquely not only to determine the e-folding number of the inflation, the reheating temperature, all characteristic Hubble scales and produced entropy, but also the tensor-to-scalar ratio upper and lower limits. The theoretical lower limit will be obtained in the next section.

\[ H_* \quad H_{\text{end}} \quad H_{RH} \]

\[ S_{\text{patch}} \]

\( \text{FIG. 9: (Color Online). Fixed the observed spectral index } n_s = 0.965, \text{ as functions of the tensor-to-scalar ratio } r, \text{ we plot (Left) the inflation scale } H_* \text{ (blue), inflation end scale } H_{\text{end}} \text{ (green), and reheating scale } H_{RH} \text{ (orange) in unit of the Planck scale } M_{pl}; \text{ (Right) the entropy } S_{\text{patch}} \text{ within the physical patch } H_{RH}^{-3} \text{ at the reheating end } a_R. \text{ Note that } H_{RH} \propto g_*^{1/2} \text{ and } S_{\text{patch}} \propto g_*^{-5/2} \text{ depend on the effective degeneracy } g_* \text{ of relativistic particles in reheating. Here we adopt } g_* \approx 10^2 \text{ for the standard model of particle physics, including sterile neutrinos.} \]
B. Reheating $\rho_R \gg \rho_\Lambda$ and lower limit of tensor-to-scalar ratio $r \gtrsim 0.042$

In the scenario $\Lambda$CDM, there are two parameters to describe the properties of the reheating epoch: (i) the effective mass parameter $\hat{m}/M_{pl}$ physically represents spacetime produced pairs’ masses and their degeneracies; (ii) the effective Yukawa coupling $(g_Y^2 / g^*_s^{1/2})$ represents pairs’ decay strength to relativistic particles of degeneracies $g_s$. We need at least two independent observations to see their possible values and to check the self-consistency and self-sufficiency of the scenario.

To determine these two parameters of the scenario $\Lambda$CDM, from Eqs. (75) and (84), we obtain one constrain,

$$\left( \frac{T_{RH}}{M_{pl}} \right)^2 = 0.3 \left( \frac{g_Y^2}{g^*_s^{1/2}} \right) \left( \frac{\hat{m}}{M_{pl}} \right).$$

(88)

Another constrain on these two parameters

$$(g_Y^2 / g^*_s^{1/2}) \left( \frac{M_{pl}}{\hat{m}} \right) \approx 3.6 \times 10^{-13},$$

(89)

comes from the combination of the reheating temperature $T_{RH}$ (75) and the ratio $T_{RH}/\hat{m} \approx 3.3 \times 10^{-7}$ obtained from the observed baryon number-to-entropy ratio $n_B/s$, which will be duly discussed in another article [40].

In Fig. 10 we numerically plot the constrains (88) and (89) on two parameters $\hat{m}/M_{pl}$ and $(g_Y^2 / g^*_s^{1/2})$ as a function of the tensor-to-scalar ratio $r$ in the range $(0.042, 0.048)$. However, on the other hand, we can constrain values of these two parameters from the theoretical point view, so as to provide an insight into the lower limit of values $r \neq 0$, a priori to observations. Recall that

![Graph](image-url)

FIG. 10: (Color Online). Using the observed spectral index $n_s = 0.965$, we plot the mass parameter $\hat{m}/M_{pl}$ (left) and the effective Yukawa coupling $(g_Y^2 / g^*_s^{1/2})$ (right), as functions of the tensor-to-scalar ratio $r$ in the same range $(0.042, 0.048)$ where the physically sensible values $N_{end}$ and $T_{RH}$ are also plotted in Fig. 8.
in Sec. IV A 2 we point out the theoretical threshold $\hat{m} > 20m_{pl} = 4M_{pl}$ (47) for $\rho_M \gg \rho_\Lambda$ that is necessary for the Universe evolution proceeding the standard cosmological scenario. Applying this theoretical threshold to the constrains of Eqs. (88) and (89) or numerical results Fig. 10 we definitely conclude that the tensor-to-scalar ratio $r \neq 0$ and conservatively suggest its lower limit should be around 0.042 in the present preliminary calculations. More elaborated numerical calculations are required to precisely determine the $r$ lower limit. The preliminarily theoretical estimate $r$-range $[0.042, 0.048]$ requires more elaborated numerical analyses. Nevertheless this $r$-range is relevant to the measurements by the next generation CMB observations, such as CMB-S4 which measures $r \gtrsim 10^{-3}$ [41]. In Ref. [31], the authors obtain the most recent constrain the tensor-to-scalar ratio $r < 0.044$ by using Planck data.

In the theoretically estimated $r$-range $[0.042, 0.048]$, it is shown that the inflation $e$-folding number $58 \gtrsim N_{end} \gtrsim 54$ and the reheating temperature $10^{-3} \gtrsim T_{RH}/M_{pl} \gtrsim 10^{-8}$ from the numerical results presented in Fig. 8 the inflation scale $H_s/M_{pl} \approx 4.0 \times 10^{-6}$, inflation end scale $H_{end}/M_{pl} \approx 1.5 \times 10^{-6}$, whereas $10^{-14} \gtrsim H_{RH}/M_{pl} \gtrsim 10^{-4}$ and the entropy $10^{120} \gtrsim S_{patch} \gtrsim 10^{76}$ from the numerical results presented in Fig. 9.

In the theoretically estimated $r$-range $[0.042, 0.048]$, the effective pair mass and degeneracy parameter $10^{-1} \lesssim \hat{m}/M_{pl} \lesssim 10^4$ and the effective Yukawa coupling $10^{-14} \lesssim (g^2/\xi^{1/2}) \lesssim 10^{-9}$. If $g_\pi \lesssim 10^2$ for the standard model of particle physics, including sterile neutrinos, $10^{-13} \lesssim g_Y^2 \lesssim 10^{-8}$. We check back the parameters used in Figs. 6 and 7 are $(\hat{m}/m_{pl}) = 27.7$ and $g_Y^2 = 10^{-9}$. This indicates that the model $\tilde{\Lambda}CDM$ we have been discussing so far is self consistent and self-contained as a theoretical framework.

VI. MASSIVE COLD DARK MATTER $\Omega_c^0$

Among massive fermion-antifermion pairs purely gravitationally generated from the spacetime, some of them possess no other or extremely weak gauge interactions, except gravitational ones. Henceforth, they can be stable against the decay to relativistic particles $\bar{\ell}\ell$ [24], and possibly have lifetimes longer than the Universe life. Thus these stable massive pairs can be candidates of massive cold dark matter in the $\Lambda$CDM model and contribute to the value $\Omega_c^0$ observed today. We consider that after produced at the reheating, stable massive pairs evolve in two different possibilities: (i) they decouple from the spacetime and evolve as a non-relativistic fluid; (ii) they couple with the spacetime via $S \leftrightarrow F\bar{F}$ from the reheating to the present time.
A. Massive cold dark matter produced in reheating

The gravitational productions of massive bosonic particles in the reheating can play an important role in explaining the massive cold dark matter \cite{25, 27, 30}. Following this line we consider the massive fermion pair energy density \(\rho^H_M\) produced in the reheating, that can be in general decomposed to the unstable and stable pair contributions:

\[
\rho^H_M \approx 2 \chi H^2 \hat{m}^2; \quad \hat{m}^2 \equiv \sum_f g_d^f m^2_f + \sum_{f'} g_d^{f'} m^2_{f'}. \tag{90}
\]

The contributions from the stable pairs of masses \(m_{f'}\) and degeneracies \(g_{d}^{f'}\) are

\[
\rho^{H\text{cold}}_M \equiv 2 \chi \hat{m}_{\text{cold}}^2 H^2 = (\hat{m}_{\text{cold}}^2/\hat{m}) \rho^H_M, \quad \hat{m}_{\text{cold}}^2 \equiv \sum_{f'} g_{d}^{f'} m^2_{f'}. \tag{91}
\]

where the effective mass and degeneracy parameter \(\hat{m}_{\text{cold}}\) is introduced for stable massive pairs and \(\hat{m}_{\text{cold}} < \hat{m}\). The stable pair number density \(n^{H\text{cold}}_M \approx \rho^{H\text{cold}}_M / 2 \hat{m}_{\text{cold}}\). Recall that these stable massive pairs follow the detailed balance solution \(\rho^H_M \approx \rho^{H\text{cold}}_M \approx \chi \hat{m}_{\text{cold}}^2 H^2RH\) at the reheating.

After these stable pairs produced, they couple with the spacetime in the \(M\)-episode, see Sec. \[IVB\] and then decouple in the \(R\)-episode of the reheating epoch. From the reheating epoch to the present time, they evolve as a non-relativistic fluid and conserve their numbers

\[
n^{H\text{cold}}_M(a_0) a^3_0 = n^{H\text{cold}}_M(a_R) a^3_R, \tag{92}
\]

per comoving volume. In connection with observations, the relic abundance of Eq. \(92\) can be defined as

\[
\Omega^0_c \equiv \frac{\rho^{H\text{cold}}_M(a_0) a^3_0}{\rho^0_c a^3_0} = \frac{\rho^{H\text{cold}}_M(a_0) a^3_0}{\rho^0_R a^3_R}, \tag{93}
\]

where \(\Omega^0_R = \rho^0_R / \rho^0_c\), the radiation energy density \(\rho^0_r\) and the critical density \(\rho^0_c = 3m_{pl}^2 H_0^2\) corresponding to the present horizon \(H_0\) and scale factor \(a_0\).

On the other hand, the entropy \(S(a) = s(a) a^3\) per comoving volume from the reheating epoch to the present time is conserved,

\[
S_R = S_0; \quad S_0 = \frac{2\pi^2}{45} g^*_0 a^3_0 T^3_0, \quad S_R = \frac{2\pi^2}{45} g^*_R a^3_R T^3_RH \tag{94}
\]

and the present temperature \(T_0 = T_{\text{CMB}} = 2.348 \times 10^{-4} \text{ eV}\), yielding

\[
(g^*_0)^{1/3} a_0 T_0 = (g^*_R)^{1/3} a_R T_{RH}, \tag{95}
\]
where \( g_0^* \) and \( g_R^* \equiv g_* \) are the effective degeneracies of relativistic particles respectively at the present time and reheating. Using the relationship (70) of radiation entropy and energy density, the conservations of particle number (92) and entropy (95), we obtain the relic abundance of the cold dark matter,

\[
\Omega_c^0 = \Omega_R^0 \left( \frac{4 \hat{m}_{\text{cold}}}{3 T_0} \right) \frac{n_{\text{M}}^0(a_R) a_R^3}{S_R(a_R)}. \tag{96}
\]

Moreover, using the entropy \( S_R \) (76) per comoving volume and the constrain (89) from the observed baryon number-to-entropy ratio, we have,

\[
\frac{n_M(a_R)a_R^3}{S_R(a_R)} \approx \frac{n_H(a_R)a_R^3}{S_R} \approx 4.55 \times 10^{-7} \left( \frac{135}{16\pi^3} \right)^{1/4}. \tag{97}
\]

As a result, the cold dark matter relic abundance is given by

\[
\Omega_c^0 = 4.4 \times 10^{-7} \Omega_R^0 \left( \frac{\hat{m}_{\text{cold}}}{T_0} \right). \tag{98}
\]

The observed values \( \Omega_c^0 h_0^2 \approx 0.12, \Omega_R^0 h_0^2 \approx 4.31 \times 10^{-5} \) and baryon component \( \Omega_b^0 h_0^2 \approx 2.24 \times 10^{-2} \) [34], the factor \( h^2 \) here comes from the convention \( H_0 = 100h_0 \text{km/sec/Mpc} \). The result (98) can possibly be of right order of magnitude, if the effective mass and degeneracy parameter of the stable massive pairs \( \hat{m}_{\text{cold}}/T_0 \sim O(10^{10}) \). It is not physically expected that the composition of stable pairs in total pairs produced is so small, \( \hat{m}_{\text{cold}} \ll \hat{m} \), where \( \hat{m}/M_{\text{pl}} \sim 10 \), see Sec. V B and Fig. 10 (left).

**B. Massive cold dark matter couples to the spacetime in its evolution**

We turn to another possibility that after produced at the reheating, the stable massive pairs always couple with the spacetime via the process \( S \leftrightarrow \bar{F}F \) (21), and follow the cosmic rate equation (43), rather than the free expansion of a non-relativistic fluid.

For the matter dominate case \( H^2 \approx (8\pi/3M_{\text{pl}})\rho_M \), we show in Sec. IV B 1 the detailed balance solution \( \rho_M^\text{cold} \approx 2\chi \hat{m}_{\text{cold}}^2 H^2 \) (60) to the cosmic rate equation (43). However, in the radiation dominate epoch after the reheating, one needs to find the numerical solution to the close set of Eqs. (30)-(33). We do not attempt to find the entire evolution history of the cold dark matter \( \rho_M^\text{cold} \) in this article. Instead, we focus on its evolution in the more recent epoch when the radiation component is negligible and the cosmological term starts to dominate over matter. In this case, Equations (30)-(33) can be approximately recasted as

\[
h^2 = \Omega_\Lambda + \Omega_M^\text{cold}, \tag{99}
\]
\[
\frac{d\Omega^\text{cold}_M}{dx} = -3\Omega^\text{cold}_M, \\
\frac{d\Omega^\text{cold}_M}{dx} + 3\Omega^\text{cold}_M = \frac{\Gamma_M}{H} \left( \Omega^\text{Hcold}_M - \Omega^\text{cold}_M \right),
\]

in unit of the critical density \( \rho^0_c = 3m^2_{\text{pl}}H^2_0 \) at the present horizon \( H_0 \) and scale factor \( a_0 \). Based on the fact that the cosmological term \( \Omega_\Lambda \) very slowly varies and \( \Omega_\Lambda \gtrsim \Omega_M \), the approximate solution to Eqs. (99-101) can be found at this preliminarily stage,

\[
\Omega^\text{cold}_M \approx \Omega^\text{Hcold}_M = \frac{16\pi}{3} \chi \left( \frac{\hat{m}}{M_{\text{pl}}} \right)^2 h^2, \quad h^2 = \left( \frac{H}{H_0} \right)^2,
\]

similarly to the detailed balance solution (60). As a result, the cold dark matter relic abundance at the present time is

\[
\Omega^0_c \approx \frac{16\pi}{3} \chi \left( \frac{\hat{m}_{\text{cold}}}{M_{\text{pl}}} \right)^2, \quad \chi \approx 1.85 \times 10^{-3}.
\]

Recall the discussions on the ratio \( \hat{m}/M_{\text{pl}} \approx 5 \sim 10 \) around the range of the tensor-to-scalar ratio \( r \approx 0.044 \), see Sec. V B and Fig. 10 (left). Thus we expect that the compositions of stable pairs and unstable pairs are comparable, i.e., \( \hat{m}_{\text{cold}} \lesssim \hat{m} \), and the cold dark matter relic abundance (102) can be of right order of magnitude, compared with observed value \( \Omega^0_c \approx 0.3 \).

We further numerically find the time-varying solution to Eqs. 99, 100 and 101 by considering slow \( \Omega_\Lambda \) variation in time from the radiation/matter dominate epoch \( \Omega_{R,M} \gg \Omega_\Lambda \) to the present epoch \( \Omega_M \sim \Omega_\Lambda \). The solution is consistent with the generalized Friedmann equation (104)

\[
h^2 \approx \Omega^0_M (1+z)^{3-\delta_M} + \Omega^0_\Lambda (1+z)^{\delta_\Lambda},
\]

where \( \Omega^0_M \) and \( \Omega^0_\Lambda \) are respectively present values of matter and cosmological terms. The effective parameters \( |\delta_M| \ll 1 \) and \( |\delta_\Lambda| \ll 1 \) (\( \delta_M, \delta_\Lambda > 0 \)) describe the tiny coupling between \( \Omega_\Lambda \) and \( \Omega_M \) via the \( \rho^\text{Hcold}_M \) and cosmic rate equation (101). Among the radiation and matter content \( \Omega_{R,M} \), the massive cold dark matter \( \Omega^\text{cold}_M \) plays the main role in interacting with \( \Omega_\Lambda \). The reason is that in the cosmic rate equation (101), the rate \( \Gamma_M \propto m \) and density \( \rho^\text{Hcold}_M \propto m^2_{\text{cold}} \) are proportional to the masses of superheavy cold dark matter particles. In the evolution epoch when \( \Omega_M > \Omega_\Lambda \), the cosmic rate equation (101) shows that \( \Omega_M \) decreases faster than \( (1+z)^3 \) (\( \delta_M < 0 \)) and \( \Omega_\Lambda \) increases (\( \delta_\Lambda < 0 \)) from large redshifts to low redshifts. This implies the cold dark matter annihilation to the spacetime (\( \Omega_\Lambda \)). The more details will be presented in a separated paper. On the other hand, the data analysis show the generalized Friedmann equation (104) has the better fit to observational data than the \( \Lambda \text{CDM} (\delta_M = \delta_\Lambda = 0) \) [38], and greatly relieved the \( \Lambda \text{CDM} \) \( H_0 \) tensions [39].
In this article, we leave these two possibilities of cold dark matter evolution as open questions for further discussions, since more theoretical studies are required and the contributions from bosonic candidates for cold dark matter particles have not been considered. We end this section by adding the following comments. Among relativistic particle $\bar{\ell}\ell$ from unstable massive pairs decay, there are massive and massless sterile particles of weak interacting with other SM particles. The smaller their interacting strengths are, the earlier they decouple from the thermal equilibrium bath of relativistic particles, and freely evolve as relativistic or non-relativistic fluids, accounting for the catalogue of warm dark matter in $\Lambda$CDM. We cannot determine their properties in this article.

VII. SUMMARY AND REMARK REHEATING EPOCH

We would like to make a summary to close this section. After the $\rho_\Lambda$-dominated inflation $H > \Gamma_M$, where the pair production density $\rho_M^H$ is negligible, the reheating epoch starts $\Gamma_M \gtrsim H$, $\rho_M^H$ becomes large and pair annihilation to spacetime and decay to relativistic particles are important. Therefore the cosmic rate equation (26) of the Boltzmann type governing the processes $\bar{F}F \leftrightarrow S$ and $\bar{F}F \rightarrow \bar{\ell}\ell$ are relevant. This is another dynamical equation in addition to two Einstein equations (3,4) and the reheating equation (29) from the energy conservation.

Described by the horizon $H$ and three cosmological constitutions $\rho_{\Lambda,M,R}$, the reheating epoch is determined by a close system of four dynamical equations (30-33) and pair production density $\rho_M^H$ (34), production rate $\Gamma_M$ and decay rate $\Gamma_{de,M}$ (35). Numerically solving this system, we find three characteristic episodes:

(i) the $P$-episode of the transition from the inflation end to the reheating start, the pair-production rate is much larger than the Hubble rate ($\Gamma_M \gg H$), the cosmological energy density $\rho_\Lambda$ quickly decreases and converts to the matter-energy density $\rho_M$ of massive pairs produced, as a consequence $\rho_\Lambda \ll \rho_M$;

(ii) the $M$-episode of massive pairs domination and their back and forth interaction to the space-time ($\bar{F}F \leftrightarrow S$), when the cosmic rate equation plays an essential role in build a local chemical equilibrium of the quantum-number and energy equipartition between massive pairs and spacetime;

(iii) the $R$-episode of the genuine reheating $\rho_R \gg \rho_M$ through massive pairs predominately decaying to relativistic particles quickly thermalised, the reheating temperature and entropy are
computed in terms of the CMB measured scalar amplitude $A_s$ and spectral index $n_s$, as well as the tensor-to-scalar ratio $r$.

As a result, using the CMB measurements at pivot scale $k_*=0.05\,(\text{Mpc})^{-1}$ and the Hubble scale $H_*$ at the beginning of inflation, we calculate the Hubble scale $H_{\text{end}}$ and energy densities $\rho_{\Lambda,M}^{\text{end}}$ at the end of inflation after $e$-folding number $N_{\text{end}}$. These are treated as initial conditions for studying the reheating epoch, and we calculate the Hubble scale $H_{\text{RH}}$, temperature $T_{\text{RH}}$ and entropy $S_{\text{patch}}$ at the genuine reheating episode. Based on the reheating temperature (84) determined by CMB measurements and the observed baryon number-to-entropy ratio, we consistently constrain the effective pair mass parameter ($\hat{m}/M_{\text{pl}}$) and the effective Yukawa coupling ($g_Y^2/g_s^{1/2}$) in the theoretical framework of $\tilde{\Lambda}$CDM. Thus no free parameter is adjustable.

We present these results as functions of the tensor-to-scalar ratio $r$, and show they are in accordance with observations. As a result, the final results are expressed as functions of the tensor-to-scalar ratio $r$ and their numerical values are in accordance with the CMB observations so far. Moreover, from purely theoretical points of views, we preliminarily limit the $r$ values in the range $0.042 \lesssim r \lesssim 0.048$. The upper limit is consistent with observations, while the lower limit shows $r \neq 0$ definitely, to be fixed by next-generation experiments, such as CMB-S4 [41].

There are two kinds of massive fermion-antifermion pairs gravitationally produced: (i) unstable pairs that couple and decay to relativistic particles; (ii) stable pairs that do not couple to other particles except gravity and possibly have a lifetime longer than the Universe. Considering stable massive pairs produced in the reheating and following the cosmic rate equation (43) for the back and forth processes $S \leftrightarrow \bar{F}F$, we discuss the sable massive pairs as cold dark matter candidates and calculate their relic abundance in connection with observations. We preliminarily find that in the matter dominate epoch the cold dark matter-energy density $\rho_M^{\text{cold}} \propto H^2$ in its evolution. More detailed analyses are required.

We would like to mention that at the reheating start $\rho_\Lambda \gg \rho_M \gg \rho_R$, the rapid converting process $\rho_\Lambda \Rightarrow \rho_M \Rightarrow \rho_R$ leads to $\rho_\Lambda \ll \rho_M \ll \rho_R$, and most relevant mass-energy and entropy of Universe had been produced by the end of reheating. This violent dynamical process could lead to the emission of primordial gravitational wave, see review [2, 42, 43] and references therein.

After the reheating epoch, the radiation dominated and entropy conserved the epoch of the standard cosmology starts. In standard cosmology epoch, the pairs production (7) and (8) become small and their contributions to the Hubble scale $H$ and its variation are strongly suppressed and negligible, compared with the matter produced in the reheating. However, the pair’s production
provides an indirect interaction between cosmological energy density $\rho_\Lambda$ and matter density $\rho_M$ in evolution, which can possibly account for the cosmological coincidence [26].

In this article, we have mainly addressed basic issues in the reheating epoch and cold dark matter candidates and presented some preliminary analysis and results. Further studies are still required and more elaborately numerical computations are very inviting, based on ongoing observations. Nevertheless, we expect that this theoretical scenario and present results provide some further understandings of the phenomena of the reheating and cold dark matter in the standard model $\Lambda$CDM for the modern cosmology.

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