Translatable radii of an operator in the direction of another operator II

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Abstract

One of the couple of translatable radii of an operator in the direction of another operator introduced in earlier work[13] is studied in details. A necessary and sufficient condition for a unit vector $f$ to be a stationary vector of the generalized eigenvalue problem $Tf = \lambda Af$ is obtained. Finally a theorem of Williams[16] is generalized to obtain a translatable radius of an operator in the direction of another operator.

1 Introduction.

Let $T$ and $A$ be two bounded linear operators on a complex Hilbert space $H$ with inner product $(,)$ and norm $\|\|$. Consider the generalized eigenvalue problem $Tf = \lambda Af$ where $f \in H$ and $\lambda \in \mathbb{C}$, $\lambda$ is called the eigenvalue of the above equation and $f$ the corresponding eigenvector. The non-negative functional

$$M_T(f) = \|Tf - \frac{(Tf, Af)}{(Af, Af)} Af\|, \text{ provided } \|Af\| \neq 0,$$

gives the deviation of a unit vector $f$ from being an eigenvector and

$$M_T(A) = \sup_{\|f\|=1} \{\|Tf - \frac{(Tf, Af)}{(Af, Af)} Af\|\}, \text{ provided } 0 \notin \sigma_{\text{app}} A,$$

gives the supremum of all those deviations, where $\sigma_{\text{app}} A$ is the set of approximate eigenvalues of $A$.

Geometrically $Tf - \frac{(Tf, Af)}{(Af, Af)} Af$ is the component of $Tf$ perpendicular to $Af$. For $A = I$ problems related to the concepts considered here have been studied by Bjorck and Thomee[2], Garske[8], Prasanna[14], Fujii and Prasanna[6], Furuta et al[7], Fujii and Nakamoto[5], Isumino[9], Nakamoto and Sheth[11], Mustafaev and Shulman[10] and many others.

Keywords: Stationary distance vectors, Translatable radii.
Bjorck and Thomee[2] have shown that for a normal operator $T$, 

$$M_T = \sup_{\|f\| = 1} \{\|Tf - (Tf, f)f\| = R_T, $$

where $R_T$ is the radius of the smallest circle containing the spectrum. Garske[8] improved on the result to prove that for any bounded linear operator $T$, 

$$M_T = \sup_{\|f\| = 1} \{\|Tf - (Tf, f)f\| \geq R_T. $$

Stampfli[15] proved that for a bounded linear operator $T \exists$ a unique complex scalar $c_T$, defined as the center of mass of $T$ such that 

$$\|T - c_T I\|^2 + |\lambda|^2 \leq \|T - c_T I + \lambda I\|^2, \quad \forall \lambda \in C.$$

With the help of Stampfli’s result Prasanna[14] proved that $M_T = \|T - c_T I\|$. Later Fujii and Prasanna[6] improved on the inequality of Garske to show that $M_T \geq w_T$ where $w_T$ is the radius of the smallest circle containing the numerical range.

In [12] we proved that for any two bounded linear operators $T$ and $A$ if $0 \notin \sigma_{app} A$ then there exists a unique complex scalar $\lambda_0$ such that $\|T - \lambda_0 A\| \leq \|T - \lambda A\| \forall \lambda \in C$. We defined $T - \lambda_0 A$ as the minimal-norm translation of $T$ in the direction of $A$. The equality of $\inf_\lambda \|T - \lambda A\| = M_T(A)$ was also studied by E.Asplund and V.Pták[1]

Then in [13] we introduced a couple of translatable radii of an operator $T$ in the direction of another operator $A$ as follows:

If $0$ does not belong to the approximate point spectrum of $A$ let

$$M_T(A) = \sup_{\|f\| = 1} \{\|Tf - \frac{(Tf, Af)}{(Af, Af)} Af\|\}$$

i.e., $M_T(A) = \sup_{\|f\| = 1} \{\|Tf\|^2 - \frac{|(Tf, Af)|^2}{(Af, Af)}\}^{1/2}$

and if $0 \notin \overline{W(A)}$, where $\overline{W(A)}$ stands for the closure of the numerical range of $A$, let

$$\tilde{M}_T(A) = \sup_{\|f\| = 1} \{\|Tf - \frac{(Tf, f)}{(Af, f)} Af\|\}.$$ 

We defined $M_T(A)$ and $\tilde{M}_T(A)$ as translatable radii of the operator $T$ in the direction of $A$ and proved in [13] that if $0 \notin \overline{W(A)}$ then

$$\tilde{M}_T(A) \geq M_T(A) \geq m_T(A)/\|A^{-1}\|,$$
where $m_T(A)$ is the radius of the smallest circle containing the set $W_T(A) = \{ (Tf, Af)/(Af, Af) : \|f\| = 1 \}$.

Das[4] introduced the concept of stationary distance vectors while studying the eigenvalue problem $Tf = \lambda f$. Following the ideas of Das we here use the concept of stationary distance vectors to study the generalized eigenvalue problem $Tf = \lambda Af$ and the translatable radius $M_T(A)$. We investigate the structure of the vectors for which the translatable radius $M_T(A)$ is attained and prove that if $M_T(A)$is attained at a vector $f$ then $M_T(A^*)$ is attained at the vector $h/\|h\|$, where $h = Tf - (Tf, Af)/(Af, Af) Af$. We also show that if $g$ is a state (normalized positive functional) on the Banach algebra $B(H,H)$ of all bounded linear operators on $H$ then

$$M_T(A) = \sup\{ g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : g \text{ is a state and } g(A^*A) \neq 0 \}.$$

The last result mentioned here is a generalization of a theorem of Williams [16].

2 Stationary distance vectors of the generalized eigenvalue problem $Tf = \lambda Af$

In this section we study the following:

“For any two bounded linear operators $T$ and $A$ what are the vectors that are nearest to or farthest from being eigenvectors of the equation $Tf = \lambda Af$ in the sense that $\|Tf - (Tf, Af)/(Af, Af) Af\|$ with unit $f$ is minimum or maximum?”

We give a necessary and sufficient condition that a unit vector $f$ is at a stationary distance from being an eigenvector. We call such $f$’s the stationary distance vectors and the corresponding $\lambda = (Tf, Af)/(Af, Af)$ the stationary distance value of the eigenvalue problem $Tf = \lambda Af$. We use the concept of stationary vectors the definition of which is given below:

Definition 1 Stationary vector.

Let $\varphi$ be a functional defined on the unit sphere of $H$. Then a unit vector $f$ is said to be a stationary vector and $\varphi$ is said to have a stationary value at $f$ of $\varphi$ iff the function $w_g(t)$ of a real variable $t$, defined as

$$w(t) = \varphi\left(\frac{f + tg}{\|f + tg\|}\right)$$

has a stationary value at $t=0$ i.e., $w'_g(0) = 0$ for any arbitrary but fixed vector $g \in H$. e.g., If $\varphi(f) = \|Tf - (Tf, Af)/(Af, Af) Af\|^2$ then a stationary vector $f$ of functional $\varphi$ is called the stationary distance vector of the eigenvalue problem $Tf = \lambda Af$. 

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We assume that 0 does not belong to the approximate point spectrum of A and prove the following theorem:

**Theorem 1.** The necessary and sufficient condition for a unit vector \( f \) to be a stationary distance vector of the generalized eigenvalue problem \( T f = \lambda A f \) is that it satisfies the following

\[
(T^* - \bar{\lambda} A^*)(T - \lambda A) f = \|h\|^2 f
\]

where \( h = T f - \lambda A f \) and \( \lambda = \frac{(T f, A f)}{(A f, A f)} \).

**Proof.** Consider \( M_T(f) = \|T f - (T f, A f)/(A f, A f) A f\| \). Define the function \( w_g(t) \) of a real variable \( t \) as follows

\[
w_g(t) = M_T^2 \left( \frac{f + tg}{\|f + tg\|} \right) = \frac{\|T(f + tg)\|^2}{\|f + tg\|^2} - \frac{|(T(f + tg), A(f + tg))|^2}{(A(f + tg), A(f + tg)) \|f + tg\|^2}
\]

where \( g \) is arbitrary but fixed vector in \( \mathbb{H} \).

At a stationary vector \( f \) we have \( w_g'(0) = 0 \) and so

\[
2 \Re (T^* T f, g) - \|T f\|^2 2 \Re (f, g) - \frac{\|A f\|^2}{\|A f\|^4} \left[ (T f, A f) \right.
\]

\[
\left\{ (T f, A g) + (T g, A f) \right\} + (T f, A f) \left\{ (T f, A g) + (T g, A f) \right\}
\]

\[
+ \frac{|(T f, A f)|^2}{\|A f\|^4} \left\{ \|A f\|^2 2 \Re (f, g) + 2 \Re (A^* A f, g) \right\} = 0 .
\]

Since \( g \) is arbitrary we get,

\[
T^* T f - \|T f\|^2 f - \lambda T^* A f - \bar{\lambda} A^* T f + \|A f\|^2 \lambda^2 f + \lambda^2 A^* A f = 0 ,
\]

where \( \lambda = (T f, A f)/(A f, A f) \).

Let \( h = T f - \lambda A f \), then \((h, A f) = 0 \) and \( \|h\|^2 = \|T f\|^2 - |(T f, A f)|^2/(A f, A f) \).

So we get

\[
(T^* - \bar{\lambda} A^*)(T - \lambda A) f = \|h\|^2 f .
\]

Thus the theorem is proved.

We now prove the following corollary:

**Corollary 1.** If \( M_T(A) \) is attained at \( f \) then \( M_T(A^*) \) is also attained at \( h/\|h\| \) where \( h = T f - (T f, A f)/(A f, A f) A f \).
Proof. Suppose \( M_T(A) \) is attained at a vector \( f \) and \( \lambda = \frac{(Tf, Af)}{(Af, Af)} \). Then \( f \) is a stationary distance vector and so we get
\[
(T^* - \bar{\lambda} A^*)(T - \lambda A)f = \|h\|^2 f
\]
\[
\Rightarrow (T^* - \bar{\lambda} A^*)h = \|h\|^2 f
\]
\[
\Rightarrow (T^*h, A^*h) = \bar{\lambda}(A^*h, A^*h)
\]
\[
\Rightarrow \bar{\lambda} = \frac{(T^*h, A^*h)}{(A^*h, A^*h)}
\]

Now \( T^*h = \bar{\lambda} A^*h + \|h\|^2 f \)
\[
\Rightarrow \|T^*h\|^2 = |\bar{\lambda}|^2 \|A^*h\|^2 + \|h\|^4
\]
\[
\Rightarrow \|T^*h\|^2 = \|h\|^2 \left\{ \|Tf\|^2 - \frac{|(Tf, Af)|^2}{(Af, Af)} \right\} + \frac{|(Tf, Af)|^2}{(Af, Af)} \cdot \|A^*h\|^2
\]

If the minimal-norm translation of \( T \) in the direction of \( A \) is \( T \) itself then the minimal-norm translation of \( T^* \) in the direction of \( A^* \) is also \( T^* \). So if \( M_T(A) = \|T\| \) then \( M_{T^*}(A^*) = \|T^*\| \).

Let \( M_T(A) = \|T\| = \|Tf\|, \quad (Tf, Af)/(Af, Af) = 0 \).

Then \( M_{T^*}(A^*) = \|T^*\| = \|T\| = \|T^*h\|/\|h\|, \quad \text{since} \quad (Tf, Af)/(Af, Af) = 0 \).

This completes the proof.

Next we prove the following theorem:

**Theorem 2.** Suppose \( T \) and \( A \) are two selfadjoint operators and \( f \) be a unit stationary distance vector such that \((Tf, Af)\) is real, then \( f \) can be expressed as the linear combination of two eigenvectors of the problem \( T f = \lambda A f \).

**Proof.** As both \( T \) and \( A \) are selfadjoint and \( f \) is a stationary distance vector with \((Tf, Af)\) real we get from the last theorem
\[
(T - \lambda A)^2 f = \|h\|^2 f.
\]

So we get
\[
\Rightarrow (T - \lambda A)^2 f \pm \|h\|f = \|h\|^2 f \pm \|h\|f
\]
\[
\Rightarrow T(T f - \lambda A f \pm \|h\|f) = (\lambda A \pm \|h\|)(T f - \lambda A f \pm \|h\|f)
\]

Let \( g_1 = T f - \lambda A f + \|h\|f \)
and \( g_2 = T f - \lambda A f - \|h\|f \).

Then we get
\[
T g_1 = (\lambda A + \|h\|)g_1 \quad \text{and} \quad T g_2 = (\lambda A - \|h\|)g_2
\]
so that
\[(T - \lambda A)g_1 = \|h\|g_1 \text{ and } (T - \lambda A)g_2 = -\|h\|g_2.\]
Thus \(f = (g_1 - g_2)/(2\|h\|)\) completes the proof.

3 On the attainment of \(M_T(A)\)

Suppose \(\{f_n\}\) be a sequence of unit vectors such that
\[
\|T f_n\|^2 - \frac{|(T f_n, A f_n)|^2}{(A f_n, A f_n)} \to M_T(A)^2.
\]

As the unit sphere in \(H\) is weakly compact without loss of generality we may assume that \(\{f_n\}\) converges weakly to \(f\), i.e., \(f_n \rightharpoonup f\).

We now prove the following theorem:

**Theorem 3.** Suppose \(\{f_n\}\) be a weakly convergent sequence of unit vectors such that
\[
\|T f_n\|^2 - \frac{|(T f_n, A f_n)|^2}{(A f_n, A f_n)} \to M_T(A)^2.
\]
If the weak limit \(f\) is non-zero then \(M_T(A)\) is attained for the vector \(f/\|f\|\). If the supremum is not attained then all such sequences must tend weakly to zero.

**Proof.** Since \(M_T(A)\) is translation invariant in the direction of \(A\) so without any loss of generality we may assume that the minimal-norm translation of \(T\) in the direction of \(A\) is \(T\) itself, i.e., \(M_T(A) = \|T\|\).

So there exists a sequence \(\{f_n\}, f_n \in H, \|f_n\| = 1\) such that \(\|T f_n\| \to \|T\|\) and \((T f_n, A f_n) \to 0\). Considering the positive operator \(\|T\|^2 I - T^* T\) we have
\[
\frac{(\|T\|^2 f_n - T^* T f_n, f_n)}{(A f_n, A f_n)} \to 0
\]
\[
\implies \frac{\|T\|^2 f_n - T^* T f_n}{(A f_n, A f_n)} \to 0, \text{ by property of positive operators.}
\]
If \(f \neq 0\) we have
\[
\|T\|^2(f_n, f) - (T^* T f_n, f) \to 0.
\]
Since \(f_n \rightharpoonup f\) and weak limit \(f\) is unique we get
\[
\|T\|^2 = \frac{\|T f\|^2}{\|f\|^2}.
\]
The result that “if \(f_n \rightharpoonup f, \|T f_n\| \to \|T\|\) and \(f \neq 0\) then \(\|T\|\) is attained at \(f/\|f\|\)” follows directly from the corollary 1 of Das[3].
As \(M_T(T) = \|A\|\) the theorem is proved.
4 On generalization of a Theorem of Williams

Let $B$ denote the set of all normalized positive linear functionals (states) on $B(H, H)$ i.e.,

$$B = \{ g : g \in L(B(H, H), C) \text{ and } g(I) = 1 = \|g\| \}$$

Clearly $B$ is weak$^*$ compact. Let $P = \{ g : g \in B \text{ and } g(A^*A) \neq 0 \}$.

Williams[16] proved that for any bounded linear operator $T$, $\|T\| \leq \|T - \lambda I\| \forall \lambda \in C$ iff there exists a state $f$ such that $f(T^*T) = \|T^*T\|$ and $f(T) = 0$. We here show that if for two bounded linear operators $T$ and $A$, $\|T\| \leq \|T - \lambda A\| \forall \lambda \in C$ then $\|T\|^2 = \sup g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : g \text{ is a state and } g(A^*A) \neq 0$.

We now prove the following theorem:

**Theorem 4.** $[M_T(A)]^2 = \sup g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : g \text{ is a state and } g(A^*A) \neq 0$.

**Proof.** Let $[S_T(A)]^2 = \sup g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : g \text{ is a state and } g(A^*A) \neq 0$.

Clearly $S_{T+\lambda A}(A) = S_T(A)$ and $M_{T+\lambda A}(A) = M_T(A)$ so that both are translation invariant in the direction of $A$. Without loss of generality we assume that $M_T(A) = \|T\|$.

Now for each $x \in H, \|x\| = 1$, let $g_x : B(H, H) \rightarrow C$ be defined as $g_x(U) = (Ux, x) \forall U \in B(H, H)$.

Then $g_x$ is a state and $g_x(A^*A) \neq 0$.

So

$$\|T\| = \sup g_x(T^*T) - \frac{|g_x(A^*T)|^2}{g_x(A^*A)}^{1/2}$$

$$\leq \sup g_x \{g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)}\}^{1/2}$$

$$\leq \sup g_x \{g(T^*T)\}^{1/2}$$

$$= \|T\|^2.$$ 

This completes the proof.

**Note.** For $A=I$ the result of Williams follows easily from Theorem 4.

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References

[1] E.Asplund and V.Pták, A minimax inequality for operators and a related numerical range, *Acta Mathematica*, 126 (1971), 53-62.

[2] G.Bjorck and V.Thomee, A property of bounded normal operators in Hilbert Space, *Arkiv for Math.*, 4 (1963), 551-555.

[3] K.C.Das, Extrema of the Rayleigh Quotient and normal Behavior of an operator, *Journal of Mathematical Analysis and Applications*, Vol.41 No.3 (1973) 765-774.

[4] K.C.Das, Stationary distance vectors and their relation with eigenvectors, *Science Academy Medals for Young Scientists-Lectures*, (1978) 44-52.

[5] M.Fujii and R. Nakamoto, An estimation of the transcendental radius of an operator, *Math. Japonica*, 27 (1982), 637-638.

[6] M.Fujii and S.Prasanna, Translatable radii for operators, *Mathematica Japonica*, 26 (1981) 653-657.

[7] T.Furuta, S.Izumino and S.Prasanna, A characterisation of centroid operators, *Math. Japonica*, 27 (1982) 105-106.

[8] G.Garske, An equality concerning the smallest disc that contains the spectrum of an operator, *Proc. Amer. Math. Soc.*, 78 (1980), 529-532.

[9] S.Izumino, An estimation of the transcendental radius of an operator, *Math. Japonica*, 27 No.5 (1982), 645-646.

[10] G.S.Mustafaev and V.S.Shulman, An estimate of the norms of inner derivation in some operator algebras. *Math. Notes(English. Russian original)* 45, No.4 (1989) 337-341; translation from Mat. Zametki 45, No.4, 105-110 (1989).

[11] R. Nakamoto and I.H.Sheth, On centroid operators. *Math. Japonica*, 29, No.2 (1984) 287-289.

[12] K.Paul, Sk.M.Hossein and K.C.Das, Orthogonality on B(H,H) and minimal-norm operator, *Journal of Analysis and Applications*, Vol. 6, No. 3 (2008) 169-178.

[13] K.Paul, Translatable radii of an operator in the direction of another operator, *Scientiae Mathematicae*, Vol.2 No.1 (1999) 119-122.

[14] S.Prasanna, The norm of a derivation and the Bjorck-Thomee-Istratescu theorem, *Mathematica Japonica*, 26 (1981), 585-588.
[15] G. Stampfli, The norm of a derivation, Pacific J. math., 33 (1970) 737-747.

[16] J.P. Williams, Finite operators, Proc. Amer. Math. Soc. Vol.26 (1970) 129-136.

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