Non-uniform dependence for a generalized Degasperis-Procesi equation

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Abstract

In the paper, we consider the Cauchy problem for a generalized Degasperis-Procesi equation. We prove that the data-to-solution map is not uniformly continuous.

Key Words: A generalized Degasperis-Procesi equation; non-uniform dependence.

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1 Introduction

In this paper, we study a generalized Degasperis-Procesi equation introduced by Novikov in [38]:

\[(1 - \partial_x^2)u_t = \partial_x(2 - \partial_x)(1 + \partial_x)u^2.\] (1.1)

It was showed in [38] that Eqs. (1.1) possesses a hierarchy of local higher symmetries and the first non-trivial one is \(u_\tau = \partial_x[(1 - \partial_x u)]^{-1}\).

Eqs. (1.1) belongs to the following class [38]:

\[(1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}),\] (1.2)

which has attracted much attention on the possible integrable members of (1.2).

The first well-known integrable member of (1.2) is the Camassa-Holm (CH) equation [5]:

\[(1 - \partial_x^2)u_t = -(3u u_x - 2u_x u_{xx} - u u_{xxx}).\]
The CH equation can be regarded as a shallow water wave equation [5, 6, 16]. It is completely integrable [5, 8], has a bi-Hamiltonian structure [7, 27], and admits exact peaked solitons of the form $ce^{-|x-ct|}$, $c > 0$, which are orbitally stable [18]. It is worth mentioning that the peaked solitons present the characteristic for the traveling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [10, 14, 15, 40]. The local well-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was discussed in [11, 12, 19, 21, 39, 32]. It was shown that there exist global strong solutions to the CH equation [9, 11, 12] and finite time blow-up strong solutions to the CH equation [9, 11, 12, 13]. The existence and uniqueness of global weak solutions to the CH equation were proved in [17, 45]. The global conservative and dissipative solutions of CH equation were discussed in [3, 4].

The second well-known integrable member of (1.2) is the Degasperis-Procesi (DP) equation [23]:

$$(1 - \partial_x^2) u_t = -(4uu_x - 3u_xu_{xx} - uu_{xxx}).$$

The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as the CH shallow water equation [24]. The DP equation is integrable and has a bi-Hamiltonian structure [22]. An inverse scattering approach for computing n-peakon solutions to the DP equation was presented in [37]. Its traveling wave solutions were investigated in [31, 41]. The local well-posedness of the Cauchy problem of the DP equation in Sobolev spaces and Besov spaces was established in [28, 29, 48]. Similar to the CH equation, the DP equation has also global strong solutions [34, 49, 51] and finite time blow-up solutions [25, 26, 34, 35, 48, 49, 50, 51]. In addition, it has global weak solutions [2, 25, 50, 51]. Although the DP equation is similar to the CH equation in several aspects, these two equations are truly different. One of the novel features of the DP equation different from the CH equation is that it has not only peakon solutions [22] and periodic peakon solutions [50], but also shock peakons [36] and the periodic shock waves [26].

The third well-known integrable member of (1.2) is the Novikov equation [38]

$$(1 - \partial_x^2) u_t = 3uu_xu_{xx} + u^2u_{xxx} - 4u^2u_x.$$ 

The most difference between the Novikov equation and the CH and DP equations is that the former one has cubic nonlinearity and the latter ones have quadratic nonlinearity. It was showed that the Novikov equation is integrable, possesses a bi-Hamiltonian structure, and admits exact peakon solutions $u(t, x) = \pm \sqrt{c}e^{\pm |x-ct|}$, $c > 0$ [30]. The local well-posedness for the Novikov equation in Sobolev spaces and Besov spaces was studied in [43, 44, 46, 47, 44]. The global existence of strong solutions was established in [43] under some sign conditions and the blow-up phenomena of
the strong solutions was shown in \cite{47}. The global weak solutions for the Novikov equation were discussed in \cite{12}.

The local well-posedness and global existence of strong solutions for the generalized Degasperis-Procesi were studied in \cite{33}.

To our best knowledge, Eqs. \eqref{1.1} can transform into the following equivalent form:

\[
\begin{aligned}
  &u_t - 2uu_x = \partial_x(1 - \partial_x^2)^{-1}(u^2 + (u^2)_x), \quad t > 0, \ x \in \mathbb{R}, \\
  &u(0, x) = u_0(x), \ x \in \mathbb{R}.
\end{aligned}
\tag{1.3}
\]

Using this result and the method of approximate solutions, we prove the following nonuniform dependence result.

\textbf{Theorem 1.1.} If \(s > \frac{3}{2}\), then the data-to-solution map for the generalized Degasperis-Procesi equation defined by the Cauchy problem \eqref{1.3} is not uniformly continuous from any bounded subset in \(H^s\) into \(C([0, T]; H^s)\).

\textbf{Notations.} Since all spaces of functions are over \(\mathbb{R}\), for simplicity, we drop \(\mathbb{R}\) in our notations of function spaces if there is no ambiguity.

\section{Proof of the main theorem}

In this section, we will give the proof of the main theorem. Motivated by \cite{29}, we first construct a sequence approximate solutions. Lately, we will show that the distance between approximate solutions and actual solutions is decaying. Finally, we can conclude that the Cauchy problem \eqref{1.3} is not uniformly continuous.

\textbf{Lemma 2.1.} \textit{(\cite{1})} For any \(s > 0\), there exists a positive constant \(c = c(s)\) such that

\[ ||fg||_{H^s} \leq c(||f||_{H^s}||g||_{L^\infty} + ||g||_{H^s}||f||_{L^\infty}). \]

For any \(s > \frac{1}{2}\), there exists a positive constant \(c\) such that

\[ ||f||_{L^\infty} \leq c||f||_{H^s}. \]

\textbf{Lemma 2.2.} \textit{(\cite{1})} Let \(s > \frac{1}{2}\). Assume that \(f_0 \in H^s\), \(F \in L^1_T(H^s)\) and \(\partial_x v \in L^1_T(H^{s-1})\). If \(f \in C([0, T]; H^s)\) solves the following 1-D linear linear transport equation:

\[
\begin{aligned}
  &\partial_t f + v\partial_x f = F, \\
  &f(0, x) = f_0,
\end{aligned}
\]

then there exists a positive constant \(C = C(s)\) such that

\[ ||f||_{H^s} \leq e^{CV(t)}(||f_0||_{H^s} + \int_0^t e^{-CV(\tau)}||F(\tau)||_{H^s}d\tau), \]

where

\[
V(t) = \begin{cases} 
  \int_0^t ||\partial_x v||_{H^{s-1}}d\tau, & s > \frac{3}{2}, \\
  \int_0^t ||\partial_x v||_{H^s}d\tau, & \frac{1}{2} < s \leq \frac{3}{2}.
\end{cases}
\]
Lemma 2.3. (29) Let $\phi \in S(\mathbb{R})$, $\delta > 0$ and $\alpha \in \mathbb{R}$. Then for any $s \geq 0$ we have that

$$\lim_{n \to \infty} n^{-\frac{1}{2} \delta - s} ||\phi(\frac{x}{n^\delta})\cos(nx - \alpha)||_{H^s} = \frac{1}{\sqrt{2}} ||\phi||_{L^2},$$

$$\lim_{n \to \infty} n^{-\frac{1}{2} \delta - s} ||\phi(\frac{x}{n^\delta})\sin(nx - \alpha)||_{H^s} = \frac{1}{\sqrt{2}} ||\phi||_{L^2}.$$  

Lemma 2.4. (33) Let $s > \frac{3}{2}$ and $u_0 \in H^s$. There exists a positive time $T = T(||u_0||_{H^s})$ such that (1.2) has a solution $u \in C([0,T]; H^s)$. Moreover, there exists a constant $C = C(s) > 0$ such that

$$||u||_{L^\infty(\mathbb{R})} \leq C||u_0||_{H^r}, \quad \forall \ r \geq s.$$

Proof of the theorem: Let $\omega \in \{0, 1\}$ and $\varphi$ be a $C_0(\mathbb{R})$ such that

$$\varphi(x) = \begin{cases} 
1, & |x| \leq 1, \\
0, & |x| \geq 2.
\end{cases}$$

Let $\phi$ be a $C_0(\mathbb{R})$ such that $\phi(x)\varphi(x) = \varphi(x)$. We introduce the following sequence of high frequency approximate solutions:

$$u_{\omega}^{h,n} = n^{-\frac{s}{2} - s}(\frac{x}{n^\delta})\cos(nx + 2\omega t).$$

We also define the solution $u_{\omega}^{\ell,n}$ which satisfies the following equation with the low frequency initial data:

$$\begin{cases}
\partial_t u_{\omega}^{\ell,n} - 2u_{\omega}^{\ell,n} \partial_x u_{\omega}^{\ell,n} = \partial_x(1 - \partial_x^2)^{-1}[(u_{\omega}^{\ell,n})^2 + \partial_x(u_{\omega}^{\ell,n})^2], \\
u_{\omega}^{\ell,n}(0, x) = \omega n^{-1}\phi(\frac{x}{n^\delta}).
\end{cases}$$

Since

$$\partial_t u_{\omega}^{h,n} = -2\omega n^{-\frac{s}{2} - s}(\frac{x}{n^\delta})\sin(nx + 2\omega t) = -2nu_{\omega}^{\ell,n}(0, x) n^{-\frac{s}{2} - s}(\frac{x}{n^\delta})\sin(nx + 2\omega t),$$

$$2u_{\omega}^{\ell,n} \partial_x u_{\omega}^{h,n} = -2nu_{\omega}^{\ell,n} n^{-\frac{s}{2} - s}(\frac{x}{n^\delta})\sin(nx + 2\omega t) + 2u_{\omega}^{\ell,n} n^{-\frac{s}{2} - s} \partial_x \phi(\frac{x}{n^\delta}) \cos(nx + 2\omega t),$$

then we can find that

$$\partial_t u_{\omega}^{h,n} - 2u_{\omega}^{h,n} \partial_x u_{\omega}^{h,n} = 2n[u_{\omega}^{\ell,n}(t, x) - u_{\omega}^{\ell,n}(0, x)] n^{-\frac{s}{2} - s}(\frac{x}{n^\delta})\sin(nx + 2\omega t) - 2u_{\omega}^{\ell,n}(t, x) n^{-\frac{s}{2} - s} \partial_x \phi(\frac{x}{n^\delta}) \cos(nx + 2\omega t).$$

4
Letting $U^n_\omega = u^{h,n}_\omega + u^{\ell,n}_\omega$, we obtain $U^n_\omega$ satisfies the following equation:

$$
\partial_t U^n_\omega - 2U^n_\omega \partial_x U^n_\omega = -2u^{h,n}_\omega \partial_x u^{\ell,n}_\omega - 2u^{h,n}_\omega \partial_x u^{h,n}_\omega \\
+ 2n[u^{\ell,n}_\omega(t, x) - u^{\ell,n}_\omega(0, x)]n^{-\frac{\delta}{2}}\varphi\left(\frac{x}{n^\delta}\right) \sin(nx + 2\omega t) \\
- 2u^{\ell,n}_\omega(t, x)n^{-\frac{\delta}{2}}\partial_x \varphi\left(\frac{x}{n^\delta}\right) \cos(nx + 2\omega t) \\
+ \partial_x (1 - \partial_x^2)^{-1}[(u^{\ell,n}_\omega)^2 + \partial_x (u^{\ell,n}_\omega)^2].
$$

Now, letting $V^n_\omega$ be the solution of the Cauchy problem for the equation:

$$
\begin{align*}
\partial_t V^n_\omega - 2V^n_\omega \partial_x V^n_\omega &= \partial_x (1 - \partial_x^2)^{-1}[(V^n_\omega)^2 + \partial_x (V^n_\omega)^2], \\
V^n_\omega(0, x) &= n^{-\frac{\delta}{2}}\varphi\left(\frac{x}{n^\delta}\right) \cos(nx) + \omega n^{-1} \varphi\left(\frac{x}{n^\delta}\right).
\end{align*}
$$

Denoting $W^n_\omega = U^n_\omega - V^n_\omega$, it easy to show that

$$
\begin{align*}
\partial_t W^n_\omega - 2U^n_\omega \partial_x W^n_\omega - 2W^n_\omega \partial_x V^n_\omega \\
= -2u^{h,n}_\omega \partial_x u^{\ell,n}_\omega - 2u^{h,n}_\omega \partial_x u^{h,n}_\omega \\
+ 2n[u^{\ell,n}_\omega(t, x) - u^{\ell,n}_\omega(0, x)]n^{-\frac{\delta}{2}}\varphi\left(\frac{x}{n^\delta}\right) \sin(nx + 2\omega t) \\
- 2u^{\ell,n}_\omega(t, x)n^{-\frac{\delta}{2}}\partial_x \varphi\left(\frac{x}{n^\delta}\right) \cos(nx + 2\omega t) \\
+ \partial_x (1 - \partial_x^2)^{-1}[(u^{\ell,n}_\omega)^2 + \partial_x (u^{\ell,n}_\omega)^2] \\
- \partial_x (1 - \partial_x^2)^{-1}[(V^n_\omega)^2 + \partial_x (V^n_\omega)^2] := \sum_{i=1}^5 I_i.
\end{align*}
$$

According to Lemmas 2.3 and 2.4, we have for $\sigma \geq 0$

$$
||u^{h,n}_\omega(t)||_{H^\sigma} \leq Cn^{\sigma - s}, \quad ||u^{\ell,n}_\omega(t)||_{H^\sigma} \leq C||u^{\ell,n}_\omega(0)||_{H^\sigma} \leq Cn^{\frac{1}{2} - \frac{s}{2}}.
$$

Choosing $\delta > 0$ such that $s - 1 - \delta > \frac{1}{2}$, we have $||f||_{L^\infty} \leq ||f||_{H^{s-1-\delta}}$. Therefore, by Lemma 2.1 and (2.1), we have

$$
||I_1||_{H^{s-1}} \leq C(||u^{h,n}_\omega||_{H^{s-1}}||\partial_x u^{\ell,n}_\omega||_{H^{s-1}} + ||u^{h,n}_\omega||_{H^{s-1-\delta}}||\partial_x u^{h,n}_\omega||_{H^{s-1}} \\
+ ||u^{h,n}_\omega||_{H^{s-1}}||\partial_x u^{h,n}_\omega||_{H^{s-1-\delta}}) \\
\leq Cn^{\frac{1}{2} - \delta - 2} + Cn^{-\delta},
$$

$$
||I_2||_{H^{s-1}} \leq Cn \int_0^t ||\partial_x u^{\ell,n}_\omega(\tau)||_{H^{s-1}} d\tau \cdot ||n^{-\frac{\delta}{2}}\varphi\left(\frac{x}{n^\delta}\right) \sin(nx - 2\omega t)||_{H^{s-1}} \\
\leq C(||u^{\ell,n}_\omega||_{H^{s}}||\partial_x u^{\ell,n}_\omega||_{H^{s}} + ||\partial_x (1 - \partial_x^2)^{-1}[(u^{\ell,n}_\omega)^2 + \partial_x (u^{\ell,n}_\omega)^2]||_{H^{s-1}}) \\
\leq C||u^{\ell,n}_\omega||_{H^{s}} + ||\partial_x u^{\ell,n}_\omega||_{H^{s}} \leq Cn^{\delta - 2},
$$

5
\[ ||I_3||_{H^{s+1}} \leq C||n^{-\frac{3}{2}} - s \partial_x \varphi (\frac{x}{n^\delta}) \cos(nx - 2\omega t)||_{H^{s-1}} \leq C n^{-1-\delta}, \]

\[ ||I_4||_{H^{s-1}} \leq C||\partial_x (1 - \partial_x^2)^{s_1} [(u_{\omega}^e)^2 + \partial_x(u_{\omega}^e)^2]||_{H^{s-1}} \leq C n^{\delta - 2} \]

\[ ||I_5||_{H^{s-1}} \leq ||\partial_x (1 - \partial_x^2)^{-s_1} [(V_n^e)^2 + \partial_x(V_n^e)^2]||_{H^{s-1}} \leq C n^{\delta - 2}. \]

Now, setting \( \delta < \min\{s - \frac{1}{2}, 1\} \), it follows from Lemma \( \ref{lem:2.2} \) that

\[ ||W_n^e||_{H^{s-1}} \leq C n^{-1}(n^{-\delta} + n^{\delta - 1}). \]

According to Lemmas \( \ref{lem:2.4} \), we have

\[ ||W_n^e(t)||_{H^{s+1}} \leq ||U_n^e(t)||_{H^{s+1}} + ||V_n^e(t)||_{H^{s+1}} \]

\[ \leq ||u_n^h(t)||_{H^{s+1}} + C(||u_n^f(0)||_{H^{s+1}} + ||V_n^e(0)||_{H^{s+1}}) \]

\[ \leq C n, \]

which implies

\[ ||W_n^e||_{H^s} \leq C||W_n^e||_{H^{s-1}}^{\frac{1}{2}} ||W_n^e||_{H^{s+1}}^{\frac{1}{2}} \leq C (n^{-\delta} + n^{\delta - 1})^{\frac{1}{2}}. \]  

(2.2)

Then, combining (2.1) and (2.2), we have

\[ ||V_1^n(t) - V_0^n(t)||_{H^s} \geq ||U_1^n(t) - U_0^n(t)||_{H^s} - C \varepsilon_n \]

\[ \geq ||u_1^h(t) - u_0^h(t)||_{H^s} - C \varepsilon'_n \]  

(2.3)

\[ \geq 2|\sin t| \cdot ||n^{-\frac{3}{2}} - s \varphi (\frac{x}{n^\delta}) \sin(nx + t)||_{H^s} - C \varepsilon'_n, \]

where

\[ \varepsilon_n = (n^{-\delta} + n^{\delta - 1})^{\frac{1}{2}}, \quad \varepsilon'_n = (n^{-\delta} + n^{\delta - 1})^{\frac{1}{2}} + n^{\frac{1}{2} \delta - 1}. \]

Letting \( n \) go to \( \infty \) and using Lemma \( \ref{lem:2.3} \), \( \ref{lem:2.2} - \ref{lem:2.3} \), it follows that

\[ \lim_{n \to \infty} ||V_1^n(t) - V_0^n(t)||_{H^s} \geq \sqrt{2} ||\varphi||_{L^2} |\sin t|. \]  

(2.4)

Noticing that \( |\sin t| = \sin t \) when \( t \in [0, \pi] \), then \( \sin t > 0 \) in an interval \((0, t_0)\) for some \( 0 < t_0 < \pi \). This together with the fact that

\[ \lim_{n \to \infty} ||V_1^n(0) - V_0^n(0)||_{H^s} \leq ||n^{-1} \phi (\frac{x}{n^\delta})||_{H^s} \leq n^{\delta - 1} \to 0, \quad \text{as} \quad n \to \infty, \]

complete the proof of Theorem \( \ref{thm:1.1} \).

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References

[1] H. Bahouri, J. Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften, vol. 343, Springer-Verlag, Berlin, Heidelberg, 2011.
[2] G. M. Coclite and K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, J. Funct. Anal., 233 (2006), 60-91.
[3] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, Arch. Ration. Mech. Anal., 183 (2007), 215-239.
[4] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, Anal. Appl., 5 (2007), 1-27.
[5] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett., 71 (1993), 1661-1664.
[6] R. Camassa, D. Holm and J. Hyman, A new integrable shallow water equation, Adv. Appl. Mech., 31 (1994), 1-33.
[7] A. Constantin, The Hamiltonian structure of the Camassa-Holm equation, Exposition. Math., 15 (1997), 53-85.
[8] A. Constantin, On the scattering problem for the Camassa-Holm equation, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457 (2001), 953-970.
[9] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, Ann. Inst. Fourier (Grenoble), 50 (2000), 321-362.
[10] A. Constantin, The trajectories of particles in Stokes waves, Invent. Math., 166 (2006), 523-535.
[11] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), 303-328.
[12] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, Comm. Pure Appl. Math., 51 (1998), 475-504.
[13] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math., 181 (1998), 229-243.
[14] A. Constantin and J. Escher, Particle trajectories in solitary water waves, Bull. Amer. Math. Soc., 44 (2007), 423-431.
[15] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, Ann. of Math., 173 (2011), 559-568.
[16] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, Arch. Ration. Mech. Anal., 192 (2009), 165-186.
[17] A. Constantin and L. Molinet Global weak solutions for a shallow water equation, Comm. Math. Phys., 211 (2000), 45-61.
[18] A. Constantin and W. A. Strauss, Stability of peakons, Comm. Pure Appl. Math., 53 (2000), 603-610.
[19] R. Danchin, A few remarks on the Camassa-Holm equation. Differential Integral Equations, 14 (2001), 953-988.
[20] R. Danchin, Fourier Analysis Methods for PDEs, in Lecture Notes, 2005, 14 November.
[21] R. Danchin, A note on well-posedness for Camassa-Holm equation, J. Differential Equations, 192 (2003), 429-444.
[22] A. Degasperis, D. D. Holm, and A. N. W. Hone, A new integrable equation with peakon solutions, Theoret. and Math. Phys., 133 (2002), 1463-1474.
[23] A. Degasperis, M. Procesi, *Asymptotic integrability*, in Symmetry and Perturbation Theory (Rome, 1998), page 23C37, World Sci. Publ., River Edge, NJ, 1999.

[24] H. R. Dullin, G. A. Gottwald, and D. D. Holm, *On asymptotically equivalent shallow water wave equations*, Phys. D, **190** (2004), 1-14.

[25] J. Escher, Y. Liu and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*, J. Funct. Anal., **241** (2006), 457-485.

[26] J. Escher, Y. Liu and Z. Yin, *Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation*, Indiana Univ. Math. J., **56** (2007), 87-177.

[27] A. Fokas and B. Fuchssteiner, *Symplectic structures, their B"acklund transformation and hereditary symmetries*, Phys. D, **4** (1981/82), 47-66.

[28] G. Gui and Y. Liu, *On the Cauchy problem for the Degasperis-Procesi equation*, Quart. Appl. Math., **69** (2011), 445-464.

[29] A. A. Himonas and C. Holliman, *The Cauchy problem for the Novikov equation*, Nonlinearity, **25** (2012), 449-479.

[30] A. N. W. Hone and J. Wang, *Integrable peakon equations with cubic nonlinearity*, J. Phys. A, **41** (2008), 372002, 10 pp.

[31] J. Lenells, *Traveling wave solutions of the Degasperis-Procesi equation*, J. Math. Anal. Appl., **306** (2005), 72-82.

[32] J. Li and Z. Yin, *Remarks on the well-posedness of Camassa-Holm type equations in Besov spaces*, J. Differential Equations, **261** (2016), 6125-6143.

[33] J. Li and Z. Yin, *Well-posedness and global existence for a generalized Degasperis-Procesi equation*, Nonlinear Anal. RWA., **28** (2016), 72-90.

[34] Y. Liu and Z. Yin, *Global Existence and Blow-up Phenomena for the Degasperis-Procesi Equation*, Comm. Math. Phys., **267** (2006), 801-820.

[35] Y. Liu and Z. Yin, *On the blow-up phenomena for the Degasperis-Procesi equation*, Int. Math. Res. Not. IMRN, **23** (2007), rnm117, 22 pp.

[36] H. Lundmark, *Formation and dynamics of shock waves in the Degasperis-Procesi equation*, J. Nonlinear Sci., **17** (2007), 169-198.

[37] H. Lundmark and J. Szmigielski, *Multi-peakon solutions of the Degasperis-Procesi equation*, Inverse Problems, **19** (2003), 1241-1245.

[38] V. Novikov, *Generalization of the Camassa-Holm equation*, J. Phys. A, **42** (2009), 342002, 14 pp.

[39] G. Rodríguez-Blanco, *On the Cauchy problem for the Camassa-Holm equation*, Nonlinear Anal., **46** (2001), 309-327.

[40] J. F. Toland, *Stokes waves*, Topol. Methods Nonlinear Anal., **7** (1996), 1-48.

[41] V. O. Vakhnenko and E. J. Parkes, *Periodic and solitary-wave solutions of the Degasperis-Procesi equation*, Chaos Solitons Fractals, **20** (2004), 1059-1073.

[42] X. Wu and Z. Yin, *Global weak solutions for the Novikov equation*, J. Phys. A, **44** (2011), 055202, 17pp.

[43] X. Wu and Z. Yin, *Well-posedness and global existence for the Novikov equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **11** (2012), 707-727.

[44] X. Wu and Z. Yin, *A note on the Cauchy problem of the Novikov equation*, Appl. Anal., **92** (2013), 1116-1137.

[45] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Comm. Pure Appl. Math., **53** (2000), 1411-1433.

[46] W. Yan, Y. Li and Y. Zhang, *The Cauchy problem for the integrable Novikov equation*, J. Differential Equations, **253** (2012), 298-318.
[47] W. Yan, Y. Li and Y. Zhang, The Cauchy problem for the Novikov equation, NoDEA Nonlinear Differential Equations Appl., 20 (2013), 1157-1169.

[48] Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, Illinois J. Math., 47 (2003), 649-666.

[49] Z. Yin, Global existence for a new periodic integrable equation, J. Math. Anal. Appl., 283 (2003), 129-139.

[50] Z. Yin, Global weak solutions for a new periodic integrable equation with peakon solutions, J. Funct. Anal., 212 (2004), 182-194.

[51] Z. Yin, Global solutions to a new integrable equation with peakons, Indiana Univ. Math. J., 53 (2004), 1189-1210.