An Optimal Choice of Dirichlet Polynomials for the Nyman–Beurling Criterion

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In memory of Professor A. A. Karatsuba
on the 75th anniversary of his birth

1. INTRODUCTION

The Nyman–Beurling–Báez-Duarte approach to the Riemann hypothesis asserts that the Riemann hypothesis is true if and only if

$$\lim_{N \to \infty} d_N^2 = 0,$$

where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta A_N \left( \frac{1}{2} + it \right) \right|^2 dt \frac{1}{1/4 + t^2},$$

and the infimum is over all Dirichlet polynomials $A_N(s) = \sum_{n=1}^{N} a_n/n^s$ of length $N$ (see [1] for a nice account of this).

An open question is to determine what the rate of convergence of $d_n$ to zero is, assuming the Riemann hypothesis. Balazard and de Roton showed that, if the Riemann hypothesis is true, then

$$d_N^2 \ll \frac{(\log \log N)^{5/2+\varepsilon}}{\sqrt{\log N}},$$

for all $\varepsilon > 0$. On the other hand Báez-Duarte, Balazard, Landreau and Saias [2, 3] showed (unconditionally) that $d_N^2$ can not decay faster than a constant times $1/\log N$. More precisely, they showed that

$$\lim \inf_{N \to \infty} d_N^2 \log N \geq \sum_{\Re(\rho)=1/2} \frac{1}{|\rho|^2},$$

where the sum is restricted to distinct zeros of the Riemann zeta function on the critical line. The constant was later improved by Burnol [4] who showed

$$\lim \inf_{N \to \infty} d_N^2 \log N \geq \sum_{\Re(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2},$$

where $m(\rho)$ denotes the multiplicity of $\rho$. This lower bound is believed to be optimal and one expects that

$$d_N^2 \sim \frac{1}{\log N} \sum_{\Re(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2}.$$

(1)
Notice that under the Riemann hypothesis, one has
\[ \sum_{\text{Re}(\rho)=1/2} \frac{m(\rho)}{|\rho|^2} = 2 + \gamma - \log 4\pi \]
and in particular, if all the non-trivial zeros of \( \zeta(s) \) are simple, then (1) can be rewritten as
\[ d_N^2 \sim \frac{2 + \gamma - \log 4\pi}{\log N}. \]

It is the purpose of this note to prove (1) under the Riemann Hypothesis and assuming a mild condition on the growth of the mean value of \( 1/|\zeta'(\rho)|^2 \) over the non-trivial zeros \( |\rho| \leq T \) of \( \zeta(s) \). This will be achieved by using the Dirichlet polynomial \( V_N(s) := N \sum_{n=1}^{N} \left( 1 - \frac{\log n}{\log N} \right) \frac{\mu(n)}{n^s} \).

**Theorem 1.** If the Riemann hypothesis is true and if
\[ \sum_{|\text{Im}(\rho)| \leq T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{3/2-\delta} \]
for some \( \delta > 0 \), then
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta_N \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{1/4 + t^2} \sim \frac{2 + \gamma - \log 4\pi}{\log N}. \]

The condition (2) implicitly assumes that the zeros of the Riemann zeta function are all simple. Moreover, this upper bound is “mild” in the sense that a conjecture, due to Gonek and recovered by a different heuristic method of Hughes, Keating, and O’Connell [5], predicts that
\[ \sum_{|\rho| \leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{6}{\pi^3} T. \]

We remark that Theorem 1 is in contrast to what one might have expected after viewing the graphs of Landreau and Richards [6] which at first sight suggest that \( V_N \) is not optimal.

This behaviour of the Riemann zeta function resembles that of polynomials. In fact, Grenander and Rosenblatt [7] (see also [4, Theorem 2.1]) showed that for a polynomial \( P(z) \) one has that the zeros of \( P \) are all located outside or on the unit circle if and only if
\[ \lim_{N \to \infty} N \delta_N^2 = 0, \]
where
\[ \delta_N^2 = \frac{1}{2\pi} \inf_{Q_N} \int_{0}^{2\pi} |1 - P(z)Q_N(z)|^2 \, d\theta, \]
for \( z = e^{i\theta} \) and the infimum is over polynomials \( Q_N \) of degree at most \( N \). Moreover, if this happens, then
\[ \lim_{N \to \infty} N \delta_N^2 = \sum_{|\rho|=1} m(\rho)^2, \]
where the sum is restricted to the distinct zeros \( \rho \) of \( P(z) \) lying on the unit circle and \( m(\rho) \) is again the multiplicity of \( \rho \).

This analogy seems to apply also to the choices of optimal polynomials.

**Theorem 2.** Let \( P(z) \) be a polynomial whose zeros are all simple and lie outside or on the unit circle. Let
\[ W_N(z) := \sum_{n=0}^{N} \left( 1 - \frac{n}{N} \right) a_n z^n, \]
where
\[ \frac{1}{P(z)} = \sum_{n \geq 0} a_n z^n \]
is the Taylor expansion at \( z = 0 \) of the inverse of \( P(z) \) (i.e., it is the formal power series inverse of \( P(z) \)). Then
\[ \frac{1}{2\pi i} \int_0^{2\pi} |1 - P(z)W_N(z)|^2 \, d\theta \sim \frac{1}{N} \sum_{|\rho|=1} m(\rho)^2, \]
where \( z = e^{i\theta} \).

We remark that the proofs of Theorems 1 and 2 are very similar, the main difference being that the Riemann zeta function has infinitely many zeros. This generates some issues concerning the convergence of certain sums of \( \frac{1}{\zeta'(\rho)} \), which force us to assume condition (2).

2. POLYNOMIALS

Lemma 1. Let \( P(s) \) be a polynomial with \( P(0) \neq 0 \). We have
\[ W_N(s) = \frac{1}{P(s)} \left( 1 + \frac{s}{N} P'(s) \right) - \frac{s}{N} Y_N(s), \]
where \( W_N(s) \) is defined in (3),
\[ Y_N(s) := \sum_{\rho} \text{Res}_{z=\rho} \frac{s^N}{P(z)(z-s)^2 z^N}, \]
and the sum is over distinct zeros \( \rho \) of \( P(z) \).

Proof. Since \( P(0) \neq 0 \), we can take an \( \varepsilon > 0 \) such that all the zeros of \( P(z) \) lie outside of the circle \( |z| = \varepsilon \). Now, observe that we can assume \( 0 < |s| < \varepsilon \), since the result will then extend to all \( \mathbb{C} \) by analytic continuation. Denoting by \( C_y \) the circle of radius \( y > 0 \) (oriented in the positive direction), by the residue theorem we have that
\[ a_n = \frac{1}{2\pi i} \int_{C_y} \frac{1}{P(z)} \frac{dz}{z^{n+1}}, \]
therefore
\[ W_N(s) = \frac{1}{2\pi i} \int_{C_y} \frac{1}{P(z)} \sum_{n=0}^{N} \left( 1 - \frac{n}{N} \right) \left( \frac{s}{z} \right)^n \frac{dz}{z}. \]

Now,
\[ \sum_{n=0}^{N} \left( 1 - \frac{n}{N} \right) z^n = -\frac{1}{N} \frac{z - z^{N+1}}{(1-z)^2} + \frac{1}{1-z} \]
and thus
\[ W_N(s) = \frac{1}{2\pi i} \int_{C_y} \frac{1}{P(z)} \left( -\frac{1}{N} \frac{s z^{N} - s z^{N+1}}{(z-s)^2 z^N} + \frac{1}{z-s} \right) \frac{dz}{z}. \]

Now, by the residue theorem
\[ \frac{1}{2\pi i} \int_{C_y} \frac{1}{P(z)} \left( -\frac{1}{N} \frac{s}{(z-s)^2} + \frac{1}{z-s} \right) \frac{dz}{z} = \frac{1}{P(s)} \left( 1 + \frac{s}{N} P'(s) \right), \]
whereas, moving the line of integration to $C_y$ and letting $y$ tend to infinity, one has that
\[
\frac{1}{2\pi i N} \int_{C_y} \frac{1}{P(z)} \frac{s^{N+1}}{(z-s)^{2N}} \, dz = -\frac{s}{N} Y_N(s)
\]
and the Lemma follows.

**Proof of Theorem 2.** Let $\delta > 1$ be such that $P(s)$ does not have any zero on $1 < |s| \leq \delta$. We have
\[
\frac{1}{2\pi} \int_{0}^{2\pi} |1 - P(z)W_N(z)|^2 \, d\theta = \frac{1}{2\pi i} \int_{C_1} (1 - P(s)W_N(s)) \left( 1 - P \left( \frac{1}{s} \right) W_N \left( \frac{1}{s} \right) \right) \frac{ds}{s} 
\]
\[
= \frac{1}{2\pi i} \int_{C_3} (1 - P(s)W_N(s)) \left( 1 - P \left( \frac{1}{s} \right) W_N \left( \frac{1}{s} \right) \right) \frac{ds}{s}.
\]
Therefore, by Lemma 1, this is
\[
\frac{1}{2\pi i N^2} \int_{C_3} \left( \frac{P'}{P}(s) - P(s)Y_N(s) \right) \left( \frac{P'}{P} \left( \frac{1}{s} \right) - P \left( \frac{1}{s} \right) Y_N \left( \frac{1}{s} \right) \right) \frac{ds}{s} = O(1),
\]
therefore
\[
\frac{1}{2\pi i N^2} \int_{C_3} \left( \frac{P'}{P}(s) \frac{P'}{P} \left( \frac{1}{s} \right) + P(s)Y_N(s) \frac{P'}{P} \left( \frac{1}{s} \right) Y_N \left( \frac{1}{s} \right) \right) \frac{ds}{s} = O \left( \frac{1}{N^2} \right).
\]
Moreover for $s \in C_3$ one has that $Y_N(1/s) = O(\delta^{-N})$, thus
\[
-\frac{1}{2\pi i N^2} \int_{C_3} \left( \frac{P'}{P}(s) \frac{P'}{P} \left( \frac{1}{s} \right) Y_N \left( \frac{1}{s} \right) \right) \frac{ds}{s} = O \left( \frac{\delta^{-N}}{N^2} \right).
\]
Finally, by the residue theorem,
\[
-\frac{1}{2\pi i N^2} \int_{C_3} P(s)Y_N(s) \frac{P'}{P} \left( \frac{1}{s} \right) \frac{ds}{s}
\]
\[
= -\frac{1}{N^2} \sum_{|\rho| = 1} \text{Res} \ P(s)Y_N(s) \frac{P'}{P} \left( \frac{1}{s} \right) \frac{1}{s} - \frac{1}{2\pi i N^2} \int_{C_{1/\delta}} P(s)Y_N(s) \frac{P'}{P} \left( \frac{1}{s} \right) \frac{ds}{s}
\]
\[
= -\frac{1}{N^2} \sum_{|\rho| = 1} \text{Res} \ P(s)Y_N(s) \frac{P'}{P} \left( \frac{1}{s} \right) + O \left( \frac{\delta^{-N}}{N^2} \right).
\]
The Theorem then follows by observing that
\[
\text{Res} \ P(s)Y_N(s) \frac{P'}{P} \left( \frac{1}{s} \right) \frac{1}{s} = -N + O(1).
\]

3. THE RIEMANN ZETA-FUNCTION

We start with the following lemma, which is the analogue of Lemma 1. We remark that this lemma is unconditional.
Lemma 2. If $0 < \text{Re}(s) < 1$, then
\[
V_N(s) = \frac{1}{\zeta(s)} \left( 1 - \frac{1}{\log N} \zeta'(s) \right) + \frac{1}{\log N} \sum_{\rho} R_N(\rho, s) + \frac{1}{\log N} F_s \left( \frac{1}{N} \right),
\]
where the sum is over distinct non-trivial zeros $\rho$ of $\zeta(s)$ with
\[
R_N(\rho, s) = \text{Res}_{z=\rho} \frac{N^{z-s}}{\zeta(z)(z-s)^2},
\]
and where
\[
F_s(z) = \pi z^s \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n+1} z^{2n}}{(2n)! \zeta(2n+1)(2n+s)^2}
\]
is an entire function of $z$.

Proof. We have
\[
V_N(s) = \frac{1}{\log N} \frac{1}{2\pi i} \int_{(2)} \frac{N^w}{\zeta(s+w)} \frac{dw}{w^2},
\]
where we use the notation $\int_{(c)}$ to mean an integration up the vertical line from $c-i\infty$ to $c+i\infty$. Now we move the path of integration to $\text{Re}(w) = -\text{Re}(s) - 2M - 1$ for a large integer $M$. The residue at $w = \rho - s$ is $R_N(\rho, s)/\log N$. The residue at $s + w = -2n$ is
\[
\frac{N^{-2n-s}}{\zeta'(-2n)(2n+s)^2 \log N}
\]
and the integral on the new path is $\ll N^{-2M-1}$. Letting $M \to \infty$ and using
\[
\zeta'(-2n) = \frac{(-1)^n \pi(2n)!\zeta(2n+1)}{(2\pi)^{2n+1}}
\]
we obtain the result.

Lemma 3. Let $\varepsilon > 0$. Assume the Riemann hypothesis and that all the zeros of $\zeta(s)$ are simple. Then, if condition (2) holds, for $\text{Re}(s) = 1/2 \pm \varepsilon$ one has
\[
\sum_{\rho} R_N(\rho, s) \ll N^{\varepsilon} |s|^{3/4 - \delta/2 + \varepsilon}. \tag{4}
\]

Proof. Firstly observe that, by the Cauchy–Schwartz inequality, (2) implies
\[
\sum_{|\rho| \leq T} \frac{1}{|\zeta'(|\rho|)|} \ll \sqrt{N(T)} \sum_{|\rho| \leq T} \frac{1}{|\zeta'(|\rho|)|^2} \ll T^{5/4 - \delta/2} \sqrt{\log T},
\]
since
\[
N(T) := \frac{1}{2} \sum_{|\rho| \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).
\]
Therefore, by partial summation, we have that the series
\[
\sum_{\rho} \frac{1}{|\zeta'(|\rho|)|^\alpha}
\]
is convergent for any $\alpha > 5/4 - \delta/2$. Now, for a simple zero $\rho$, we have
\[
R_N(\rho, s) = \sum_{\rho} \frac{N^\rho - s}{\zeta'(|\rho|)(\rho - s)^2};
\]
Therefore
\[
N^{\pm s} \sum_{\rho} R_N(\rho, s) \ll \sum_{|\rho-s|<|\rho|/2} \frac{1}{|\zeta'(\rho)||\rho-s|^2} + \sum_{|\rho-s|>|\rho|/2} \frac{1}{|\zeta'(\rho)||\rho-s|^2} \\
\ll \sum_{|\rho-s|<|\rho|/2} \frac{1}{|\zeta'(\rho)||\rho-s|^2} + \sum_{|\rho-s|>|\rho|/2} \frac{1}{|\zeta'(\rho)||\rho|^2} \\
\ll \sum_{|\rho-s|<|\rho|/2} \frac{1}{|\zeta'(\rho)||\rho-s|^2} + 1.
\]

Now, by the Cauchy–Schwarz inequality,
\[
\sum_{|\rho-s|<|\rho|/2} \frac{1}{|\zeta'(\rho)||\rho-s|^2} \ll \sqrt{\left( \sum_{|\rho|<2|s|} \frac{1}{|\zeta'(\rho)|^2} \right) \left( \sum_{|\rho|<2|s|} \frac{1}{|\rho-s|^4} \right)} \ll |s|^{3/4-\delta/2+\varepsilon},
\]
since, by partial summation,
\[
\sum_{|\rho|<2|s|} \frac{1}{|\rho-s|^4} \ll \log(|s| + 2).
\]

This completes the proof of the lemma.

**Proof of Theorem 1.** We have
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta_N \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{1 + t^2} = \frac{1}{2\pi i} \int_{(1/2)} (1 - \zeta_N(s))(1 - \zeta_N(1-s)) \frac{ds}{s(1-s)} \\
= \frac{1}{2\pi i} \int_{(1/2-\varepsilon)} (1 - \zeta_N(s))(1 - \zeta_N(1-s)) \frac{ds}{s(1-s)}.
\]
By Lemma 2, this is
\[
\frac{1}{\log^2 N} \frac{1}{2\pi i} \int_{(1/2-\varepsilon)} \left( \frac{\zeta'}{\zeta^2}(s) - \sum_{\rho} R_N(\rho, s) - F_s \left( \frac{1}{N} \right) \right) \\
\times \left( \frac{\zeta'}{\zeta^2}(1-s) - \sum_{\rho} R_N(\rho, 1-s) - F_{1-s} \left( \frac{1}{N} \right) \right) \zeta(s)\zeta(1-s) \frac{ds}{s(1-s)}.
\]

Now, we have
\[
\frac{1}{\log^2 N} \frac{1}{2\pi i} \int_{(1/2-\varepsilon)} \sum_{\rho_1,\rho_2} R_N(\rho_1, s) R_N(\rho_2, 1-s) \frac{\zeta(s)\zeta(1-s)}{s(1-s)} ds \\
\ll \frac{1}{\log^2 N} \int_{(1/2-\varepsilon)} \sum_{|\rho-s|<|\rho|/2} \frac{|\zeta'(\rho)||\rho-s|^2 |\zeta(s)||s|^{-5/4+\delta/2-5\varepsilon}}{\log^2 N} + O\left( \frac{1}{\log^2 N} \right),
\]
where we used (4), (5) and the bound \( \zeta(1/2 \pm \varepsilon \pm it) \ll |t|^{\varepsilon} \) (which is a consequence of the Lindelöf hypothesis). Reversing the order of summation and integration, we have that this is bounded by
\[
\frac{1}{\log^2 N} \sum_{\rho} \frac{1}{|\zeta'(\rho)|} \int_{1/2-\varepsilon+i(\text{Im}(\rho)+|\rho|/2)}^{1/2-\varepsilon+i(\text{Im}(\rho)-|\rho|/2)} \frac{|ds|}{|\rho-s|^2 |s|^{5/4+\delta/2-5\varepsilon}} + O\left( \frac{1}{\log^2 N} \right)
\]
\[
\ll \frac{1}{\log^2 N} \sum_{\rho} \frac{1}{|\zeta'(\rho)| |\rho|^{5/4+\delta/2-5\varepsilon}} \ll \frac{1}{\log^2 N},
\]
if \( \varepsilon < \delta/10. \)
Now, by Lemma 3 and the trivial estimate $F_s(z) = O(N^{-5/2})$, all the other terms in (6) are trivially $O(1/\log^2 N)$ apart from
\[ -\frac{1}{\log^2 N} \frac{1}{2\pi i} \int_{(1/2-\varepsilon)} \frac{\zeta'}{\zeta} (1-s) \sum_{\rho} R_N(\rho, s) \frac{\zeta(s)}{s(1-s)} ds. \] (7)

The integrand has a double pole at every zero $\rho$ of residue
\[ \text{Res}_{s=\rho} \left( \frac{\zeta'}{\zeta} (1-s) \sum_{\rho} R_N(\rho, s) \frac{\zeta(s)}{s(1-s)} \right) = \left( \log N - \frac{1}{2} \frac{\zeta''(\rho)}{\zeta'(\rho)} + \frac{1}{\chi(\rho)} + \frac{1-2\rho}{|\rho|^2} \right) \frac{1}{|\rho|^2} \]
\[ = \frac{\log N}{|\rho|^2} + O\left( \frac{1}{|\rho|^{2-\varepsilon} |\zeta'(\rho)|} + \frac{1}{|\rho|^2} \right), \]
where we used the bound $\zeta''(1/2 + it) \ll |t|^\varepsilon$, which follows from the Lindelöf hypothesis and Cauchy’s estimate for the derivatives of a holomorphic function. It follows that moving the line of integration in (7) to $\text{Re}(s) = 1/2 + \varepsilon$ we get that the integral is equal to
\[ \frac{1}{\log^2 N} \sum_{\rho} \frac{1}{|\rho|^2} + O\left( \frac{1}{\log^2 N} \right), \]
and Theorem 1 then follows.

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