Microscopic Foundation of Stochastic Game Dynamical Equations

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Abstract

The game dynamical equations are derived from Boltzmann-like equations for individual pair interactions by assuming a certain kind of imitation behavior, the so-called proportional imitation rule. They can be extended to a stochastic formulation of evolutionary game theory which allows the derivation of approximate and corrected mean value and covariance equations. It is shown that, in the case of phase transitions (i.e. multi-modal probability distributions), the mean value equations do not agree with the game dynamical equations. Therefore, their exact meaning is carefully discussed. Finally, some generalizations of the behavioral model are presented, including effects of expectations, other kinds of interactions, several subpopulations, or memory effects.

1 Introduction

Since von Neumann and Morgenstern have initiated the field of game theory, it has often proved of great value for the quantitative description and understanding of the competition and co-operation between individuals. Game theory focusses on two questions: 1. Which is the optimal strategy in a given situation? 2. What is the dynamics of strategy choices in cases of repeatedly interacting individuals? In this connection game dynamical equations find a steadily increasing interest. Although they agree with the replicator equations of evolution theory (cf. Sec. 2), they cannot be substantiated in the same way. Therefore, we will be looking for a foundation of the game dynamical equations which bases on individual actions and decisions (cf. Sec. 4).

In addition, we will formulate a stochastic version of evolutionary game theory (cf. Sec. 3). This allows to investigate the effects of fluctuations on the dynamics of social systems.

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1 J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*. Princeton: Princeton University Press 1944.

2 P. Taylor and L. Jonker, “Evolutionarily Stable Strategies and Game Dynamics”, in: *Math. Biosciences* 40, 1978, pp. 145–156.

J. Hofbauer and K. Sigmund, *Evolutionstheorie und dynamische Systeme*. Berlin: Parey 1984.
In order to illustrate the essential ideas, a concrete model for the self-organization of behavioral conventions is presented (cf. Sec. 5). We will see that the game dynamical equations describe the average evolution of social systems only for a certain time period. Therefore, a criterium for their validity will be developed (cf. Sec. 6). Finally, we will present possible extensions to more general behavioral models and discuss the actual meaning of the game dynamical equations (cf. Sec. 7).

2 The Game Dynamical Equations

Let \( p_x(t) \) with
\[
0 \leq p_x(t) \leq 1 \quad \text{and} \quad \sum_x p_x(t) = 1
\]
denote the proportion of individuals pursuing the behavioral strategy \( x \in S \) at time \( t \). We assume the considered strategies to be mutually exclusive. The set \( S \) of strategies may be discrete or continuous, finite or infinite. The only difference will be that sums over \( x \) are to be replaced by integrals in cases of continuous sets. By \( A_{xy} \) we will denote the possibly time-dependent payoff of an individual using strategy \( x \) when confronted with an individual pursuing strategy \( y \). Hence, his/her expected success \( \langle E_x \rangle_t \) will be given by the weighted mean value
\[
\langle E_x \rangle_t = \sum_y A_{xy} p_y(t) ,
\]
(2)
since \( p_y \) is the probability that the interaction partner uses strategy \( y \). In addition, the average expected success will be
\[
\langle E \rangle_t = \sum_x p_x(t) \langle E_x \rangle_t = \sum_x \sum_y p_x(t) A_{xy} p_y(t) .
\]
(3)

Assuming that the relative temporal increase \( (dp_x/dt)/p_x \) of the proportion \( p_x \) of individuals pursuing strategy \( x \) is proportional to the difference between the expected success \( \langle E_x \rangle_t \) and the average expected success \( \langle E \rangle_t \), we obtain the game dynamical equations
\[
\frac{dp_x(t)}{dt} = \nu p_x(t) \left[ \langle E_x \rangle_t - \langle E \rangle_t \right]
\]
(4)
where the possibly time-dependent proportionality factor \( \nu \) is a measure for the interaction rate with other individuals. According to (4), the proportions of strategies with an above-average success \( \langle E_x \rangle_t > \langle E \rangle_t \) increase, whereas the other strategies will be diminished. Note, that the proportion of a strategy does not necessarily increase or decrease monotonically. Certain payoffs are related with an oscillatory or even chaotic dynamics.

Equations (4) are identical with the replicator equations from evolutionary biology. They can be extended to the selection-mutation equations
\[
\frac{dp_x(t)}{dt} = \nu p_x(t) \left[ \langle E_x \rangle_t - \sum_y p_y(t) \langle E_y \rangle_t \right]
\]
\[+ \sum_y \left[ p_y(t) w(y \to x) - p_x(t) w(x \to y) \right] .
\]
(5)

³W. Schnabl, P. F. Stadler, C. Forst, and P. Schuster, “Full Characterization of a Strange Attractor. Chaotic Dynamics in Low-Dimensional Replicator Systems”, in: Physica D 48, 1991, pp. 65–90.
The terms which agree with (4) describe a selection of superior strategies. The new terms correspond to the effect of mutations, i.e. to spontaneous changes from strategy $x$ to other strategies $y$ with possibly time-dependent transition rates $w_1(x \rightarrow y)$ (last term) and the inverse transitions. They allow to describe trial and error behavior or behavioral fluctuations.

3 Stochastic Dynamics: The Master Equation

Let us consider a social system consisting of a constant number

$$N = \sum_x n_x(t)$$

(6)

of individuals. Herein, $n_x(t)$ denotes the number of individuals who pursue strategy $x$ at time $t$. Hence, the time-dependent vector

$$\vec{n} = (n_1, n_2, \ldots, n_x, \ldots, n_y, \ldots)$$

(7)

reflects the strategy distribution in the social system and is called the socioconfiguration. If the individual strategy changes are subject to random fluctuations (e.g. due to trial and error behavior or decisions under uncertainty), we will have a stochastic dynamics. Therefore, given a certain socioconfiguration $\vec{n}_0$ at time $t_0$, for the occurrence of the strategy distribution $\vec{n}$ at a time $t > t_0$ we can only calculate a certain probability $P(\vec{n}, t)$. Its temporal change $dP/dt$ is governed by the so-called master equation

$$\frac{dP(\vec{n}, t)}{dt} = \sum_{\vec{n}'} [P(\vec{n}', t)W(\vec{n}' \rightarrow \vec{n}) - P(\vec{n}, t)W(\vec{n} \rightarrow \vec{n}')]$$

(8)

The sum over $\vec{n}'$ extends over all socioconfigurations fulfilling $n_x \in \{0, 1, 2, \ldots\}$ and $\vec{n}$. According to equation (8), an increase of the probability $P(\vec{n}, t)$ of having socioconfiguration $\vec{n}$ is caused by transitions from other socioconfigurations $\vec{n}'$ to $\vec{n}$. While a decrease of $P(\vec{n}, t)$ is related to changes from $\vec{n}$ to other socioconfigurations $\vec{n}'$. The corresponding changing rates are proportional to the configurational transition rates $W(\vec{n} \rightarrow \vec{n}')$ of changes to socioconfigurations $\vec{n}'$ given the socioconfiguration $\vec{n}$ and to the probability $P(\vec{n}, t)$ of having socioconfiguration $\vec{n}$ at time $t$.

The configurational transition rates $W$ have the meaning of transition probabilities per time unit and must be non-negative quantities. Frequently, the individuals can be assumed to change their strategies independently of each other. Then, the configurational transition rates have the form

$$W(\vec{n} \rightarrow \vec{n}') = \begin{cases} n_x w(x \rightarrow y; \vec{n}) & \text{if } \vec{n}' = \vec{n}_{xy} \\ 0 & \text{otherwise} \end{cases}$$

(9)

i.e. they are proportional to the number $n_x$ of individuals who may change their strategy from $x$ to another strategy $y$ with an individual transition rate $w(x \rightarrow y; \vec{n}) \geq 0$. In relation (9), the abbreviation

$$\vec{n}_{xy} = (n_1, n_2, \ldots, n_x - 1, \ldots, n_y + 1, \ldots)$$

(10)

\footnote{W. Weidlich and G. Haag, Concepts and Models of a Quantitative Sociology. The Dynamics of Interacting Populations. Berlin: Springer, 1983.}
means the socioconfiguration which results after an individual has changed his/her strategy from \( x \) to \( y \).

It can be shown that the master equation has the properties
\[
P(\vec{n}, t) \geq 0 \quad \text{and} \quad \sum_{\vec{n}} P(\vec{n}, t) = 1
\]
for all times \( t \), if they are fulfilled at some initial time \( t_0 \). Therefore, the master equation actually describes the temporal evolution of a probability distribution.

4 Approximate Mean Value Equations

In order to connect the stochastic model to the game dynamical equations, we must specify the individual transition rates \( w \) in a suitable way. Therefore, we derive the mean value equations related to the master equation (8) and compare them to the selection-mutation equations (5).

The proportion \( p_x \) is defined as the mean value
\[
\langle f \rangle_t = \sum_{\vec{n}} f(\vec{n}, t)P(\vec{n}, t)
\]
of the number \( f(\vec{n}, t) = n_x \) of individuals pursuing strategy \( x \), divided by the total number \( N \) of considered individuals:
\[
p_x(t) = \frac{\langle n_x \rangle_t}{N} = 1 \frac{1}{N} \sum_{\vec{n}} n_x P(\vec{n}, t) .
\]

Taking the time derivative of \( \langle n_x \rangle_t \) and inserting the master equation gives
\[
\frac{d\langle n_x \rangle_t}{dt} = \sum_{\vec{n}} n_x [P(\vec{n}', t)W(\vec{n}' \rightarrow \vec{n}) - P(\vec{n}, t)W(\vec{n} \rightarrow \vec{n}')] = \sum_{\vec{n}} (n'_{x} - n_x)W(\vec{n} \rightarrow \vec{n}')P(\vec{n}, t) ,
\]
where we have interchanged \( \vec{n} \) and \( \vec{n}' \) in the first term on the right hand side. Taking into account relation (8), we get
\[
\frac{d\langle n_x \rangle_t}{dt} = \sum_{y} n_y w(y \rightarrow x; \vec{n})P(\vec{n}, t) - \sum_{\vec{n}} n_x w(x \rightarrow y; \vec{n})P(\vec{n}, t) = \sum_{y} [n_y w(y \rightarrow x; \vec{n}) - n_x w(x \rightarrow y; \vec{n})]P(\vec{n}, t) .
\]

With (13) this finally leads to the approximate mean value equations
\[
\frac{dp_x(t)}{dt} = \sum_{y} [p_y(t)w(y \rightarrow x; \langle \vec{n} \rangle_t) - p_x(t)w(x \rightarrow y; \langle \vec{n} \rangle_t)]
\]
However, these are only exact, if the individual transition rates \( w \) are independent of the socioconfiguration \( \vec{n} \). Anyhow, they are approximately valid as long as the probability distribution \( P(\vec{n}, t) \) is narrow, so that the mean value \( \langle f(\vec{n}, t) \rangle_t \) of a function \( f(\vec{n}, t) \) can
be replaced by the function $f(\langle \vec{n} \rangle_t, t)$ of the mean value. This problem will be discussed in detail later on.

Comparing the rate equations (16) with the selection-mutation equations (5), we find a complete correspondence for the case

$$w(y \to x; \vec{n}) = w_1(y \to x) + w_2(y \to x) n_x$$

(17)

with

$$w_2(y \to x) = \frac{\nu}{N} \max(E_x - E_y, 0)$$

(18)

and the success

$$E_x = \sum_y A_{xy} \frac{n_y}{N},$$

(19)

since

$$\max(\langle E_x \rangle_t - \langle E_y \rangle_t, 0) - \max(\langle E_y \rangle_t - \langle E_x \rangle_t, 0) = \langle E_x \rangle_t - \langle E_y \rangle_t.$$  

(20)

Whereas $w_1$ is again the mutation rate (i.e. the rate of spontaneous transitions), the additional term in (17) describes imitation processes, where individuals take over the strategy $x$ of their respective interaction partner. Imitation processes correspond to pair interactions of the form

$$y + x \to x + x.$$  

(21)

Their frequency is proportional to the number $n_x$ of interaction partners who may convince an individual of strategy $x$. The proportionality factor $w_2$ is the imitation rate.

Relation (18) is called the proportional imitation rule and can be shown to be the best learning rule. It was discovered in 1992 and says that an imitation behavior only takes place, if the strategy $x$ of the interaction partner turns out to have a greater success $E_x$ than the own strategy $y$. In such cases, the imitation rate is proportional to the difference $(E_x - E_y)$ between the success' of the alternative $x$ and the previous strategy $y$, i.e. strategy changes occur more often the greater the advantage of the new strategy $x$ would be.

All specifications of the type

$$w_2(y \to x) = C + \frac{\nu}{N} [\lambda E_x - (1 - \lambda) E_y]$$

(22)

with an arbitrary parameter $\lambda$ also lead to the game dynamical equations. However, individuals would then, with a certain rate, take over the strategy $x$ of the interaction partner, even if its success $E_x$ is smaller than that of the previously used strategy $y$. Moreover, if $C$ is not chosen sufficiently large, the individual transition rates $w \geq 0$ can become negative.

In summary, we have found a microscopic foundation of evolutionary game theory which bases on four plausible assumptions: 1. Individuals evaluate the success of a strategy as its average payoff in interactions with other individuals (cf. (19)). 2. They compare the success of their strategy with that of the respective interaction partner, basing on observations or an exchange of experiences. 3. Individuals imitate each others behavior. 4. In doing so, they apply the proportional imitation rule (18) [or (22)].

5K. H. Schlag, “Why Imitate, and if so, How? A Bounded Rational Approach to Multi-Armed Bandits”, Discussion Paper No. B-361, Department of Economics, University of Bonn.

6D. Helbing, “Interrelations between Stochastic Equations for Systems with Pair Interactions”, in Physica A 181, 1992, pp. 29–52.


5 Self-Organization of Behavioral Conventions

For illustrative reasons, we will now discuss an example which allows to understand how social conventions emerge. We consider the simple case of two alternative strategies $x \in \{1, 2\}$ and assume them to be equivalent so that the payoff matrix is symmetrical:

$$
(A_{xy}) = \begin{pmatrix} A + B & B \\ B & A + B \end{pmatrix}
$$

(23)

If $A > 0$, the additional payoff $A$ reflects the advantage of using the same strategy like the respective interaction partner. This situation is, for example, given in cases of network externalities like in the historical rivalry between the video systems VHS and BETA MAX\(^7\). Finally, the mutation rates are taken constant, i.e. $w_1(x \to y) = W_1$.

The resulting game dynamical equations are

$$
\frac{dp_x(t)}{dt} = -2\left[p_x(t) - \frac{1}{2}\right]\left[W_1 + \nu A p_x(t)[p_x(t) - 1]\right].
$$

(24)

Obviously, they have only one stable stationary solution if the (control) parameter

$$
\kappa = 1 - \frac{4W_1}{\nu A}
$$

(25)
is smaller than zero. However, for $\kappa > 0$ equation (24) can be rewritten in the form

$$
\frac{dp_x(t)}{dt} = -2\nu A \left[p_x(t) - \frac{1}{2}\right] \left[p_x(t) - \frac{1 + \sqrt{\kappa}}{2}\right] \left[p_x(t) - \frac{1 - \sqrt{\kappa}}{2}\right].
$$

(26)

The stationary solution $p_x = 1/2$ is unstable, then, but we have two new stable stationary solutions $p_x = (1/2 \pm \sqrt{\kappa}/2)$. That is, dependent on the detailed initial condition, one strategy will win the majority of users although both strategies are completely equivalent. This phenomenon is called symmetry breaking. It will be suppressed, if the mutation rate $W_1$ is larger than the advantage effect $\nu A/4$.

The above model allows to understand how behavioral conventions come about. Examples are the pedestrians’ preference for the right-hand side (in Europe), the revolution direction of clock hands, the direction of writing, or the already mentioned triumph of the video system VHS over BETA MAX.

It is very interesting how the above mentioned symmetry breaking affects the probability distribution $P(\vec{n}, t) = P(n_1, n_2, t) = P(n_1, N - n_1, t)$ of the related stochastic model (cf. Fig. [ II]). For $\kappa < 0$ the probability distribution is located around $n_1 = N/2 = n_2$ and stays small so that the approximate mean value equations are applicable. At the so-called critical point $\kappa = 0$, a phase transition to a qualitative different system behavior occurs and the probability distribution becomes very broad. As a consequence, the game dynamical equations do not correctly describe the temporal evolution of the mean strategy distribution anymore.

\(^7\)W. B. Arthur, “Competing Technologies, Increasing Returns, and Lock-In by Historical Events”, in The Economic Journal 99, 1989, pp. 116–131.

\(^8\)For illustrative reasons, a small number of individuals ($N = 40$) and a broad initial probability distribution have been chosen. In each picture, the box is twice as high as the maximal occurring value of the probability.
For $\kappa > 0$, a bimodal and symmetrical probability distribution evolves. That is, the likelihood that one of the two equivalent strategies will win through is much larger than the likelihood to find approximately equal proportions of both strategies. At the beginning, the initial state or maybe some random fluctuation determines, which strategy has better chances to win. However, in the long run both strategies have exactly the same chance. It is clear, that in such cases the game dynamical equations fail to describe the mean system behavior (cf. Fig. 3), which would correspond to the average temporal evolution of an ensemble of identical social systems. In cases of oscillatory or chaotic solutions of the game dynamical equations the situation is even worse.

6 Exact, Approximate, and Corrected Mean Values and Variances

In the last section we have seen that the approximate mean value equations

$$\frac{d\langle n_x \rangle_t}{dt} = M_x(\langle \vec{n} \rangle_t)$$

(27)

with the so-called first jump moments

$$M_x(\vec{n}) = \sum_{\vec{n}'} (n_x' - n_x)W(\vec{n} \rightarrow \vec{n}')$$

(28)

(cf. (14)) are not sufficient. This calls for corrected mean value equations and a criterium for the time period of their validity. If the individual transition rates $w(x \rightarrow y; \vec{n})$ depend on the socioconfiguration, the exact mean value can only be evaluated via formula (13). This requires the calculation of the probability distribution $P(\vec{n}, t)$ and, therefore, the numerical solution of the respective master equation (8). Since the number of possible socioconfigurations is normally very large, an extreme amount of computer time would be necessary for this.

Luckily, it is possible to derive from (14) the corrected mean value equations

$$\frac{\partial \langle n_x \rangle_t}{\partial t} = M_x(\langle \vec{n} \rangle_t) + \frac{1}{2} \sum_y \sum_{y'} \sigma_{yy'}(t) \frac{\partial^2 M_x(\langle \vec{n} \rangle_t)}{\partial \langle n_y \rangle_t \partial \langle n_{y'} \rangle_t}$$

(29)

by means of a suitable Taylor approximation. This equation depends on the covariances

$$\sigma_{xy}(t) = \langle (n_x - \langle n_x \rangle_t)(n_y - \langle n_y \rangle_t) \rangle_t = \sum_{\vec{n}} (n_x - \langle n_x \rangle_t)(n_y - \langle n_y \rangle_t)P(\vec{n}, t)$$

(30)

which can be determined by means of the covariance equations

$$\frac{\partial \sigma_{xx'}(t)}{\partial t} = M_{xx'}(\langle \vec{n} \rangle_t) + \frac{1}{2} \sum_y \sum_{y'} \sigma_{yy'}(t) \frac{\partial^2 M_{xx'}(\langle \vec{n} \rangle_t)}{\partial \langle n_{y} \rangle_t \partial \langle n_{y'} \rangle_t} + \sum_y \left[ \sigma_{xy}(t) \frac{\partial M_{x'}(\langle \vec{n} \rangle_t)}{\partial \langle n_y \rangle_t} + \sigma_{x'y}(t) \frac{\partial M_{x}(\langle \vec{n} \rangle_t)}{\partial \langle n_y \rangle_t} \right].$$

(31)

The functions

$$M_{xy}(\vec{n}) = \sum_{\vec{n}'} (n_x' - n_x)(n_y' - n_y)W(\vec{n} \rightarrow \vec{n}')$$

(32)
are called the second jump moments. Equations (29) and (31) build a closed system of equations, but still no exact one, since this would depend on higher moments of the form $\langle n_x n_y n_z \cdots \rangle_t$. Nevertheless, according to Figure 2 the corrected mean value equations yield significantly better results than the approximate ones. As a consequence, they are valid for a much longer time period. Suitable validity criteria are the relative variances

$$V_x(t) := \frac{\sigma_{xx}(t)}{\langle n_x \rangle_t^2},$$

(33)

since these are a measure for the relative width of the probability distribution $P(\bar{n}, t)$. It can be shown that the covariances and all higher moments are small, if only $V_x(t)$ is much smaller than 1 for every $x$. Numerical investigations indicate that the approximate mean value equations begin to separate from the exact ones as soon as one of the relative variances $V_x(t)$ becomes greater than 0.04. The corrected mean value equations and covariances remain reliable as long as $V_x(t)$ is smaller than 0.12 for all $x$ (cf. Fig. 2).

A more detailed discussion of the above matter is presented elsewhere.

7 Diverse Generalizations

The above discussed behavioral model can be generalized in different respects.

**Modified transition rates:** The strange cusp at $n_1 = N/2$ in Figure 1, which comes from the discontinuous derivative of $w_2(x \rightarrow y)$ at $E_x = E_y$, can be avoided by the modified imitation rates

$$w_2(y \rightarrow x) = \frac{\nu}{N} \frac{\exp(E_x - E_y)}{D_{xy}} \text{ with } D_{xy} = D_{yx} = 2.$$ 

(34)

This ansatz agrees with relation (22) in linear approximation for $C = \nu/(2N)$ and $\lambda = 1/2$, but it always yields non-negative imitation rates. Similar to (18) it guarantees two essential things: 1. The imitation rate grows with an increasing gain $(E_x - E_y)$ of success. 2. If the alternative strategy $x$ is inferior, the imitation rate is very small (but, due to uncertainty, not negligible). The results of the corresponding stochastic behavioral model are presented in Figure 3. They show the usual flatness of the probability distribution $P(n_1, N - n_1, t)$ at the critical point $\kappa = 0$, where again a phase transition occurs.

**Dynamics with expectations:** The decisions of individuals are often influenced by their expectations $\langle E_x \rangle_{t'}$ about the success of a strategy $x$ at future times $t' > t$. These will base on some kind of extrapolation of past experiences with the success of $x$. If expected payoffs at future times $t'$ are weighted exponentially with their distance $(t' - t)$ from the present time $t$, one would set

$$\langle E_x \rangle_t = \frac{1}{T} \int_t^\infty dt' \langle E_x \rangle_{t'} \exp \left( \frac{t' - t}{T} \right).$$

(35)

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9D. Helbing, “A Stochastic Behavioral Model and a ‘Microscopic’ Foundation of Evolutionary Game Theory”, in Theory and Decision 40, 1996, pp. 149–179.

10N. S. Glance and B. A. Huberman, “Dynamics with Expectations”, in Physics Letters A 165, 1992, pp. 432–440.
Other kinds of pair interactions: Apart from imitative behavior, individuals also sometimes show an avoidance behavior

\[ x + x \rightarrow y + x , \]  

especially if they dislike their interaction partner (so-called ‘snob effect’). This can be taken into account by an additional contribution to the individual interaction rates:

\[ w(y \rightarrow x; \vec{n}) = w_1(y \rightarrow x) + w_2(y \rightarrow x) n_x + w_3(y \rightarrow x) n_y . \]  

(37)

\( w_3 \) denotes the avoidance rate.

Several subpopulations: Sometimes one has to distinguish different subpopulations \( a \), i.e. different kinds of individuals. This is necessary, if not all individuals have the same set \( S \) of strategies.\[ ^{11} \] A similar thing holds, if the considered social system consists of competing groups, where only individuals of the same group behave cooperatively. The generalized behavioral equations are

\[ \frac{dp^a_x(t)}{dt} = \sum_{y} [p^a_y(t) w^a(y \rightarrow x; \langle \vec{n} \rangle_t) - p^a_x(t) w^a(x \rightarrow y; \langle \vec{n} \rangle_t) ] \]  

(38)

with individual interaction rates of the form

\[ w^a(y \rightarrow x; \vec{n}) = w^a_1(y \rightarrow x) + \sum_b [w^a_{2b}(y \rightarrow x) n^b_x + w^a_{3b}(y \rightarrow x) n^b_y] . \]  

(39)

Inclusion of memory effects: If the strategy distribution at past times \( t' < t \) influences present decisions in a non-Markovian way, the approximate mean value equations have the form

\[ \frac{dp^a_x(t)}{dt} = \sum_{y} \int_{-\infty}^{t} dt' [p^a_y(t') w^a_{-t'}(y \rightarrow x; \langle \vec{n} \rangle_{t'}) - p^a_x(t') w^a_{-t'}(x \rightarrow y; \langle \vec{n} \rangle_{t'}) ] . \]  

(40)

For example, in cases of an exponentially decaying memory one would have

\[ w^a_{-t'}(x \rightarrow y; \langle \vec{n} \rangle_{t'}) = w^a(x \rightarrow y; \langle \vec{n} \rangle_{t'}) \frac{1}{\tau} \exp \left( \frac{t - t'}{\tau} \right) . \]  

(41)

8 Summary and Conclusions

We have found a microscopic foundation of the game dynamical equations, basing on a certain kind of imitative behavior. Moreover, a stochastic version of evolutionary game theory has been formulated. It allowed to understand the self-organization of social conventions as a phase transition which is related with symmetry breaking. Moreover, we have

\[ ^{11} \] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems*. Cambridge: Cambridge University Press 1988.

\[ ^{12} \] D. Helbing, *Quantitative Sociodynamics. Stochastic Methods and Models of Social Interaction Processes*. Dordrecht: Kluwer Academic 1995.
seen that the game dynamical equations correspond to approximate mean value equations. Normally, they agree with the mean value equations of stochastic game theory for a certain time period only, which can be determined by calculating the relative variances. For an improved description of the average system behavior we have derived corrected mean value equations which require the solution of additional covariance equations.

The interpretation of the game dynamical equations follows by reformulating these in terms of a social force model\(^1\) assuming a continuous strategy set:

\[
\frac{dx_\alpha(t)}{dt} = f_1(x_\alpha) + \sum_{\beta(\neq \alpha)} f_2(x_\alpha, x_\beta) + \text{fluctuations}.
\] (42)

The force term

\[
f_1(x_\alpha) = \int dx (x - x_\alpha) w_1(x_\alpha \rightarrow x)
\] (43)

delineates spontaneous strategy changes by individual \(\alpha\), whereas

\[
f_2(x_\alpha, x_\beta) = (x_\beta - x_\alpha) w_2(x_\alpha \rightarrow x_\beta) + \int dx (x - x_\alpha) w_3(x_\alpha \rightarrow x) \delta(x_\alpha - x_\beta)
\] (44)

is the interaction force which originates from individual \(\beta\) and influences individual \(\alpha\). Here, \(\delta(x - y)\) denotes Dirac’s delta function (which yields a contribution for \(x = y\) only).

According to \(^{12}\), the game dynamical equations describe the most probable strategy changes rather than the average (representative) evolution of a social system. Therefore, they neglect the effects of fluctuations on the system behavior.

A more detailed discussion of the results presented in this paper is available elsewhere\(^{12,13}\).

\(^{12}\)D. Helbing, Stochastische Methoden, nichtlineare Dynamik und quantitative Modelle sozialer Prozesse, 2nd edition. Aachen: Shaker 1996.
Figure 1: Probability distribution $P(\vec{n}, t) = P(n_1, N - n_1; t)$ of the socioconfiguration $\vec{n}$ for varying values of the control parameter $\kappa$ according to the stochastic version of the game dynamical equations.
Figure 2: The numerical solutions of the approximate mean value equations (⋯) agree with those of the exact mean value equations (—) only for a short time interval. The corrected mean value equations (– –) yield much better results, although they also deviate from the exact curves when the relative variances (⋯) become too large. Nevertheless, they describe the average long-term behavior properly.
Figure 3: Probability distribution $P(\vec{n}, t) = P(n_1, N - n_1; t)$ of the socioconfiguration $\vec{n}$ according to the modified stochastic game dynamical equations.