Rigidity for nearly umbilical hypersurfaces in space forms

Xu Cheng and Detang Zhou*

Abstract

In [P], Perez proved some $L^2$ inequalities for closed convex hypersurfaces immersed in the Euclidean space $\mathbb{R}^{n+1}$, more generally, for closed hypersurfaces with non-negative Ricci curvature, immersed in an Einstein manifold. In this paper, we discuss the rigidity of these inequalities when the ambient manifold is $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$, or the closed hemisphere $\mathbb{S}^{n+1}_+$. We also obtain a generalization of the Perez’s theorem to the hypersurfaces without the hypothesis of non-negative Ricci curvature.

1 Introduction

In this paper, we suppose $\Sigma$ is a smooth connected oriented closed (i.e. compact and without boundary) hypersurface immersed in an $(n+1)$-dimensional Riemannian manifold $(M, \tilde{g})$ with induced metric $g$. Recall that $\Sigma$ is called totally umbilical if its second fundamental form $A$ is multiple of its metric $g$ at every point of $\Sigma$, that is, $A = \frac{H}{n} g$ on $\Sigma$. Here, $A$ is defined by $A(X,Y) = - \langle \tilde{\nabla}_X Y, \nu \rangle$, where $\nu$ denotes the outward unit normal to $\Sigma$, $X, Y \in T\Sigma$, $\tilde{\nabla}$ denote the Levi-Civita connection of $(M, \tilde{g})$. $H = \text{tr} A$ denotes the mean curvature of $\Sigma$, which is the trace of $A$. A classical theorem in differential geometry states that a closed totally umbilical surface in the Euclidean space $\mathbb{R}^3$ must be a round sphere $\mathbb{S}^2$ and its second fundamental form $A$ is a constant multiple of its metric $g$. This theorem also holds for

*Both authors are partially supported by CNPq and Faperj of Brazil.
higher dimensional cases. There are various generalizations of this theorem (for instance, cf. a survey [R]).

In 2005, De Lellis and Müller [dLT] considered a stability of the above theorem and proved that if \( \Sigma \subset \mathbb{R}^3 \) is a closed connected surface with normalized area \( 4\pi \), then

\[
||A - \text{Id}||_{L^2(\Sigma)} \leq C||A - \frac{\text{tr} A}{2}\text{Id}||_{L^2(\Sigma)},
\]

where \( C \) is a universal constant.

Recently, D. Perez [P] generalized the inequality of De Lellis and Müller to convex hypersurfaces. He proved that

**Theorem 1.1.** ([P]) Let \( \Sigma \) be a smooth, closed and connected hypersurface in \( \mathbb{R}^{n+1}, n \geq 2 \), with induced Riemannian \( g \) and non-negative Ricci curvature, then

\[
\int_{\Sigma} |A - \frac{1}{n} g| \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|,
\]

and equivalently

\[
\int_{\Sigma} (H - \overline{H})^2 \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2,
\]

where \( \overline{H} = \frac{1}{\text{Vol}(\Sigma)} \int_{\Sigma} H \). In particular, the above estimate holds for smooth, closed hypersurfaces which are the boundary of a convex set in \( \mathbb{R}^{n+1} \).

For a closed hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \), it is known that \( \text{Ric} \geq 0 \) is equivalent to \( A \geq 0 \) (that is, \( \Sigma \) is convex) (see its proof, for instance, in [P], Pg.48). So this implies that Theorem 1.1 holds for closed convex hypersurfaces. In [P], the author also showed that the constants in inequalities (1.1) and (1.2) are sharp, and that without assumption on non-negativity of the Ricci curvature, the inequality cannot hold with a universal constant. Moreover, as pointed out by De Lellis and Topping [dLT], Perez’s theorem holds even for the closed hypersurfaces with non-negative Ricci curvature when the ambient space is Einstein. Indeed this can be seen by an observation on the Codazzi’s equation satisfied by hypersurfaces in an Einstein manifold (more precisely, see equation (2.1) of this paper). Hence a slight modification of the proof of Perez [P] gives...
**Theorem 1.2.** Let \((M^{n+1}, \tilde{g}), n \geq 2,\) be an Einstein manifold. Let \(\Sigma\) be a smooth, closed and connected hypersurface immersed in \(M\) with non-negative Ricci curvature, then

\[
\int_{\Sigma} |A - \frac{1}{n} \overline{H} g|^2 \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2, \tag{1.3}
\]

and equivalently

\[
\int_{\Sigma} (H - \overline{H})^2 \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2, \tag{1.4}
\]

where \(\overline{H} = \frac{1}{\text{Vol}_n(\Sigma)} \int_{\Sigma} H.\)

In this paper, we will study the rigidity of Perez’s inequalities (Theorem 1.1 and Theorem 1.2). We study what happens if (1.1) and (1.2), or (1.3) and (1.4) hold as an equality. When the ambient spaces are \(\mathbb{R}^{n+1}, \mathbb{H}^{n+1}\) and the closed hemisphere \(S_{+}^{n+1},\) we prove that

**Theorem 1.3.** Assume \(M^{n+1}(c), c = 0, -1, 1,\) are the Euclidean space \(\mathbb{R}^{n+1},\) the hyperbolic space \(\mathbb{H}^{n+1}(-1),\) and the closed hemisphere \(S_{+}^{n+1}(1),\) respectively. Let \(\Sigma\) be a smooth, closed and connected hypersurface immersed in \(M\) with non-negative Ricci curvature. Then

\[
\int_{\Sigma} |A - \frac{1}{n} \overline{H} g|^2 = \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2, \tag{1.5}
\]

and equivalently

\[
\int_{\Sigma} (H - \overline{H})^2 = \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2, \tag{1.6}
\]

hold if and only if \(\Sigma\) is a totally umbilical hypersurface, where \(\overline{H} = \frac{1}{\text{Vol}_n(\Sigma)} \int_{\Sigma} H,\) that is, \(\Sigma\) is a distance sphere \(S^n\) in \(M^{n+1}(c).\)

Here a distance sphere \(S^n\) in a complete Riemannian manifold \(M^{n+1}\) is defined as the set of points in \(M\) which have the same distance from a fixed point in \((M, \tilde{g}).\) It is known that a closed totally umbilical hypersurface in a space form is a distance sphere (especially, a distance sphere in \(\mathbb{R}^{n+1}\) is a round sphere) and its second fundamental form \(A\) is a constant multiple of its metric. Theorem 1.3 discuss rigidity of the above known fact.
If the ambient manifold $M$ is any Einstein manifold, we cannot expect the equalities like (1.5) and (1.6) hold for a closed totally umbilical hypersurface in $M$. For instance, the complex projection space $\mathbb{CP}^{n+1}$ has no closed totally umbilical hypersurfaces. On the other hand, the constants in Theorem 1.2 are also sharp when the ambient Einstein manifold is a space form besides $\mathbb{R}^{n+1}$, which was proved by A. Juárez [J].

We further study the general case in which the hypersurfaces has no assumption on its Ricci curvature. In this case, although an inequality with universal constant cannot hold as demonstrated in [P], we may still obtain quantitative $L^2$ inequalities and discuss the rigidity of these inequalities. Precisely, we prove that

**Theorem 1.4.** Let $(M^{n+1}, \bar{g})$, $n \geq 2$, be an Einstein manifold. Let $\Sigma$ be a smooth, connected and closed hypersurface immersed in $M$ with induced metric $g$. Then

$$\int_{\Sigma} |A - \frac{\mathcal{R}}{n} g|^2 \leq \frac{n}{n-1} (1 + \frac{K}{\eta_1}) \int_{\Sigma} |A - \frac{H}{n} g|^2,$$

(1.7)

and equivalently

$$\int_{\Sigma} (H - \frac{\mathcal{R}}{n})^2 \leq \frac{n}{n-1} (1 + \frac{nK}{\eta_1}) \int_{\Sigma} |A - \frac{H}{n} g|^2,$$

(1.8)

where $\eta_1$ is the first nonzero eigenvalue of the Laplacian operator on $\Sigma$, $K \geq 0$ is a nonnegative constant such that the Ricci curvature of $\Sigma$ satisfies $\text{Ric} \geq -K$.

Moreover, when $M^{n+1}$ is the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}(-1)$, or the closed hemisphere $S^{n+1}_+(1)$, the equality in (1.7) (or equivalently (1.8)) holds if and only if $\Sigma$ is a totally umbilical hypersurface, that is, $\Sigma$ is a distance sphere $S^n$ in $M^{n+1}$.

Note that take $K = 0$ in Theorem 1.4, we obtain Theorem 1.2 and Theorem 1.3. Hence Theorem 1.4 generalizes Theorem 1.2 and Theorem 1.3. We take the method used in [P] to prove inequalities (1.7) and (1.8) in Theorem 1.4. On proof of Theorem 1.3 and the case of equalities in Theorem 1.4, the methods used in [dLT] and [GWX] could not be applied. However we can use the submanifold theory to prove them.

It is worth mentioning that there is a similar phenomenon in Riemannian geometry. The classical Schur lemma states that the scalar curvature of an Einstein manifold of dimension $n \geq 3$ must be constant. In [dLT], De

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4
Lellis and Topping discussed the stability and rigidity of this assertion and proved an almost-Schur lemma. Recently, the first author of the present paper [C1] generalized the almost-Schur lemma to the case in which the Ricci curvature has no non-negative lower bound and in [C2] obtained a generalization of almost-Schur lemma for symmetric $(2,0)$-tensors and made some applications.

2 Proof of theorems

Throughout the paper, we use the same notation to denote a symmetric $(2,0)$-tensor and its corresponding $(1,1)$-tensor. For example, Ric denotes the symmetric $(2,0)$ Ricci tensor of $\Sigma$ and its corresponding $(1,1)$ Ricci tensor.

Proof of Theorem 1.4. Take a local orthonormal frame $\{e_i\}, 1 \leq i \leq n$, on $\Sigma$, which can be extended to a local orthonormal frame $\{e_i\}, 1 \leq i \leq n+1$ on $M$, where $e_{n+1} = \nu$. $h_{ij} = A(e_i, e_j), 1 \leq i, j \leq n$. The Codazzi equations are

$$h_{ij,k} - h_{ik,j} = \tilde{R}_{n+1ijk}, 1 \leq i, j, k \leq n$$

where $\tilde{R}$ denotes the Riemannian curvature tensor on $(M, \tilde{g})$.

Take $j = i$ in the Codazzi equations and take the sum of the index $i$ from 1 to $n$. Then

$$H_k = \sum_{i=1}^{n} h_{ii,k} = \sum_{i=1}^{n} h_{ik,i} + \sum_{i=1}^{n} \tilde{R}_{n+1iik} = \sum_{i=1}^{n} h_{ik,i} + \tilde{R}_{n+1k}.$$ 

Since $M$ is Einstein, $\tilde{R}_{n+1k} = \frac{R}{n} \tilde{g}_{n+1k} = 0$. We have

$$\nabla H = \text{div} A,$$ 

(2.1)

where \(\text{div} A = \sum_{i=1}^{n} (\nabla e_i A)(e_i, e_k) = \sum_{i=1}^{n} h_{ik,i} e_k\).

Denote by $\tilde{A} = A - \frac{H}{n} g$ the traceless tensor of $A$. Then

$$\text{div} \tilde{A} = \text{div} A - \text{div} \left( \frac{H}{n} g \right) = \text{div} A - \frac{\nabla H}{n}.$$ 

So by (2.1),

$$\nabla H = \frac{n}{n-1} \text{div} \tilde{A}. \quad \text{(2.2)}$$

5
Let $\phi$ be the unique solution of the Poisson equation on $\Sigma$:

$$
\Delta \phi = H - \overline{H}, \quad \int_\Sigma \phi = 0.
$$

(2.3)

If $\phi \equiv \text{Constant on } \Sigma$, inequality (1.8) obviously holds. So now we suppose $\phi$ is not identically zero on $\Sigma$. We have

$$
\int_\Sigma |H - \overline{H}|^2 = \int_\Sigma (H - \overline{H}) \Delta \phi = - \int_\Sigma \langle \nabla H, \nabla \phi \rangle
$$

$$
= - \frac{n}{n-1} \int_\Sigma \langle \text{div} \mathring{A}, \nabla \phi \rangle
$$

$$
= \frac{n}{n-1} \int_\Sigma \langle \mathring{A}, \nabla^2 \phi \rangle
$$

$$
= \frac{n}{n-1} \int_\Sigma \langle \mathring{A}, \nabla^2 \phi - \frac{1}{n} (\Delta \phi) g \rangle
$$

$$
\leq \frac{n}{n-1} \left( \int_\Sigma |\mathring{A}|^2 \right)^{\frac{1}{2}} \left[ \int_\Sigma |\nabla^2 \phi - \frac{1}{n} (\Delta \phi) g|^2 \right]^{\frac{1}{2}}
$$

$$
= \frac{n}{n-1} \left( \int_\Sigma |\mathring{A}|^2 \right)^{\frac{1}{2}} \left[ \int_\Sigma |\nabla^2 \phi|^2 - \frac{1}{n} \int_\Sigma (\Delta \phi)^2 \right]^{\frac{1}{2}}
$$

(2.4)

Applying the Bochner formula to $\phi$, integrating, and applying the Stokes’ formula, we have

$$
\int_\Sigma |\nabla^2 \phi|^2 = \int_\Sigma (\Delta \phi)^2 - \int_\Sigma \text{Ric}(\nabla \phi, \nabla \phi).
$$

(2.5)

By (2.4) and (2.5), we have

$$
\int_\Sigma |H - \overline{H}|^2 \leq \frac{n}{n-1} \left( \int_\Sigma |\mathring{A}|^2 \right)^{\frac{1}{2}} \left[ \int_\Sigma (\Delta \phi)^2 - \frac{n-1}{n} \int_\Sigma \text{Ric}(\nabla \phi, \nabla \phi) \right]^{\frac{1}{2}}
$$

(2.6)

Since $\text{Ric} \geq -(n-1)K, K \geq 0$,

$$
\int_\Sigma \text{Ric}(\nabla \phi, \nabla \phi) \geq -(n-1)K \int_\Sigma |\nabla \phi|^2.
$$

Thus,

$$
\int_\Sigma |H - \overline{H}|^2 \leq \frac{n}{n-1} \left( \int_\Sigma |\mathring{A}|^2 \right)^{\frac{1}{2}} \left[ \frac{n-1}{n} \int_\Sigma (\Delta \phi)^2 + (n-1)K|\nabla \phi|^2 \right]^{\frac{1}{2}}
$$

$$
= \sqrt{\frac{n}{n-1}} \left( \int_\Sigma |\mathring{A}|^2 \right)^{\frac{1}{2}} \left[ \int_\Sigma (\Delta \phi)^2 + nK|\nabla \phi|^2 \right]^{\frac{1}{2}}
$$

(2.7)
Let $\eta_1$ denote the first nonzero eigenvalue of the Laplace operator on $\Sigma$, i.e.,

$$\eta_1 = \inf \left\{ \frac{\int_M |\nabla \varphi|^2}{\int_M \varphi^2} : \varphi \in C^\infty(M) \text{ is not identically zero and } \int_M \varphi = 0 \right\}.$$  

We have

$$\int_\Sigma |\nabla \phi|^2 = -\int_\Sigma \phi \Delta \phi = -\int_\Sigma \phi (H - \overline{H})$$

$$\leq \left( \int_\Sigma \phi^2 \right)^{\frac{1}{2}} \left[ \int_\Sigma (H - \overline{H})^2 \right]^{\frac{1}{2}}$$

$$\leq \left( \frac{\int |\nabla \phi|^2}{\eta_1} \right)^{\frac{1}{2}} \left[ \int (H - \overline{H})^2 \right]^{\frac{1}{2}}$$

(2.8)

So

$$\int_\Sigma |\nabla \phi|^2 \leq \frac{1}{\eta_1} \int (H - \overline{H})^2.$$  

(2.9)

Substitute (2.9) and (2.3) into (2.7). We obtain

$$\int_\Sigma |H - \overline{H}|^2$$

$$\leq \frac{n}{n-1} \left( \int_\Sigma |\overline{A}|^2 \right)^{\frac{1}{2}} \left[ \frac{n-1}{n} \int_\Sigma (H - \overline{H})^2 + \left( \frac{(n-1)K}{\eta_1} \right) \int_\Sigma (H - \overline{H})^2 \right]^{\frac{1}{2}}$$

$$= \sqrt{\frac{n}{n-1}} \left( \int_\Sigma |\overline{A}|^2 \right)^{\frac{1}{2}} \left[ (1 + \frac{nK}{\eta_1}) \int (H - \overline{H})^2 \right]^{\frac{1}{2}}$$

(2.10)

So we obtain inequality (1.8):

$$\int_\Sigma |H - \overline{H}|^2 \leq \frac{n}{n-1} \left( 1 + \frac{nK}{\eta_1} \right) \int_\Sigma |\overline{A}|^2.$$  

By (1.8) and the identity: $|A - \frac{H}{n} g|^2 = |A - \overline{g}|^2 - \frac{1}{n} (H - \overline{H})^2$, we obtain (1.7):

$$\int_\Sigma |A - \frac{H}{n} g|^2 \leq \frac{n}{n-1} \left( 1 + \frac{K}{\eta_1} \right) \int_\Sigma |A - \frac{H}{n} g|^2.$$  

Now we prove the conclusion about the equalities in (1.7) and (1.8).
Assume the ambient manifold $M$ is the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}(-1)$, or the closed hemisphere $S^{n+1}_+(1)$. Firstly, suppose $\Sigma$ is totally umbilical, that is, $A = \frac{H}{n}g$. Then the right side of (1.7) vanishes and the equality holds. Secondly, suppose the equality in (1.7) holds. Thus the equality in (1.8) also holds. We discuss two cases: constant $K = 0$ and $K > 0$ separately.

Case 1. When $K = 0$, by the proof of (1.7), it holds that, on $\Sigma$,

(i) $\text{Ric} (\nabla \phi, \nabla \phi) = 0$ and

(ii) $\hat{A}$ and $\nabla^2 \phi - \frac{1}{n} (\Delta \phi) g$ are linearly dependent.

If $\hat{A} \equiv 0$ on $\Sigma$, $\Sigma$ is totally umbilical. If $\nabla^2 \phi - \frac{1}{n} (\Delta \phi) g \equiv 0$ on $\Sigma$, by (2.4), $H = \mathcal{H}$ on $\Sigma$. Since the equality in (1.8) holds, $A = \frac{H}{n}g$ on $\Sigma$. So $\Sigma$ is totally umbilical. If both $\hat{A}$ and $\nabla^2 \phi - \frac{1}{n} (\Delta \phi) g$ are not identically zero on $\Sigma$, then by (ii), there exists a nonzero constant $\mu \neq 0$ such that, on $\Sigma$,

$$\mu \hat{A} = \nabla^2 \phi - \frac{1}{n} (\Delta \phi) g. \tag{2.11}$$

Fix a point $o \in M$. Since $\Sigma$ is closed, there is a point $p \in \Sigma$ such that $p$ realizes the maximum $d$ of the extrinsic distances between a point in $\Sigma$ and $o$ in the metric $\tilde{g}$ of $M$. Let $B_d(o)$ denote the closed geodesic ball of $M$ with the radius $d$ centered at $o$. Then $\Sigma$ is contained in $B_d(o)$. Since $M^{n+1}(c) = \mathbb{R}^{n+1}, \mathbb{H}^{n+1}, S^{n+1}_+(1)$, the distance sphere $S_d(o) = \partial B_d(o)$ is a smooth closed hypersurface and tangent to $\Sigma$ at $p$. By the Gauss equation, we have

$$\text{Ric} = (n - 1)cg + HA - A^2. \tag{2.12}$$

We may choose $e_i, 1 \leq i \leq n$, such that they are the orthonormal eigenvectors of $A$ at $p$ with $Ae_i = \lambda_i e_i, 1 \leq i \leq n$. Here we still use $A$ to denote the shape operator of $\Sigma$: $T\Sigma \to T\Sigma$, defined by $\langle A(X), Y \rangle = A(X, Y)$. Then

$$\text{Ric}(e_i) = c(n - 1)e_i + \left( \sum_{j \neq i} \lambda_j \right) \lambda_i e_i = [c(n - 1) + \left( \sum_{j \neq i} \lambda_j \right) \lambda_i] e_i = \tau_i e_i,$$

where $\tau_i = c(n - 1) + \left( \sum_{j \neq i} \lambda_j \right) \lambda_i, 1 \leq i \leq n$. This says at $p$, $e_i$ are the orthonormal eigenvectors of Ricci tensor, corresponding to the eigenvalues $\tau_i$. Now we claim that the Ricci tensor $\text{Ric}$ of $\Sigma$ is positive definite at $p$.

Recall that the principle curvatures of a hypersurface are the eigenvalues of its shape operator $A$. It is known that under the above notations, at $p$,
the principal curvatures $\lambda_i$ of $\Sigma$ are no less than the principal curvatures $\eta$ of $S_d(o)$ when $M$ is as above. For completeness of proof, we give the proof of this conclusion here.

Fix $i$, $1 \leq i \leq n$. Let $\gamma(s) : (-\epsilon, \epsilon) \to \Sigma$ be a smooth curve with the arc-length parameter, satisfying $\gamma(0) = p$, $\gamma'(0) = e_i$. Take $h(s) = r(\gamma(s))$, where $r$ denotes the extrinsic distance function from the point $o$. We have $h(0) = r(p) = \max_{s \in (-\epsilon, \epsilon)} h(s)$. Hence $h'(0) = 0$, $h''(0) \leq 0$. Note

$$h'(s) = \langle \tilde{\nabla} r, \gamma'(s) \rangle(s),$$

$$h''(0) = \langle \tilde{\nabla} \gamma'(0), \tilde{\nabla} r, \gamma'(0) \rangle(0) + \langle \tilde{\nabla} r, \tilde{\nabla} \gamma'(0) \rangle(0)$$

$$= \text{Hess}(\gamma'(0), \gamma'(0)) + \langle \nu, \nabla \gamma'(0) \rangle(0)$$

$$= \text{Hess}(e_i, e_i) - A(e_i, e_i) = \eta - \lambda_i \leq 0.$$ 

Thus we have proved the conclusion mentioned above.

If $M = \mathbb{R}^{n+1}$, then $\eta = \frac{1}{d}$. So $\lambda_i \geq \eta > 0$ and hence $\tau_i \geq (n - 1) \frac{1}{d^2} > 0$ at $p$.

If $M = \mathbb{H}^{n+1}(-1)$, it is known the principle curvature $\eta = \coth d > 1$. Then at $p$, $\lambda_i \geq \eta > 1$ and

$$\tau_i = -(n - 1) + \sum_{j \neq i} \lambda_j \lambda_i > 0.$$ 

If $M = \mathbb{S}^{n+1}(1)$, the principle curvature $\eta = \cot d \geq 0$. Then at $p$, $\lambda_i \geq \eta \geq 0$ and

$$\tau_i = (n - 1) + \sum_{j \neq i} \lambda_j \lambda_i > 0.$$ 

This means that the Ricci tensor of $\Sigma$ is positive definite at $p$. We have proved the claim.

Then there exists a neighborhood $N_{\epsilon}(p)$ of $p$ in $\Sigma$ such that the Ricci tensor of $\Sigma$ is positive-definite on $N_{\epsilon}(p)$. Note (i) $\text{Ric}(\nabla \phi, \nabla \phi) = 0$. It must hold that $\nabla \phi = 0$ on $N_{\epsilon}(p)$. By this and the definition of $\phi$, we have $\phi \equiv C$ and $H = \overline{H}$ on $N_{\epsilon}(p)$. By (2.11), $A = \frac{H}{n} g = \overline{H} \frac{n}{n} g$ on $N_{\epsilon}(p)$. By continuity, on the closed neighborhood $\overline{N}_{\epsilon}(p)$, $A = \overline{H} n g$. Hence by (2.12), $\text{Ric} = (n - 1)cg + \frac{H^2}{n} g - \frac{H^2}{n^2} g^2$ on $\overline{N}_{\epsilon}(p)$, which says that on $\overline{N}_{\epsilon}(p)$, the Ricci tensor is a constant tensor and thus is positive definite.
By the same argument as above, we may have a neighborhood of every point on the boundary \( \partial N_\epsilon(p) \) of \( N_\epsilon(p) \) such that \( A = \frac{\mathcal{H}}{n} g \). Let \( D \) be the connected subset of \( \Sigma \) such that \( p \in D \) and \( A = \frac{\mathcal{H}}{n} g \). From the above argument, \( D \) is both open and closed. Since \( \Sigma \) is connected, \( D = \Sigma \) and thus \( \Sigma \) is totally umbilical.

Case 2. When \( K > 0 \), we have that, on \( \Sigma \),

(I) \((\text{Ric} + (n - 1)Kg) (\nabla \phi, \nabla \phi) = 0\),

(II) \( \mathcal{H} \) and \( \nabla^2 \phi - \frac{1}{n}(\Delta \phi)g \) are linearly dependent,

(III) \( \phi \) and \( H - \mathcal{H} \) are linearly dependent, and

(IV) \( \int_\Sigma (|\nabla \phi|^2 - \eta_1 \phi^2) = 0 \).

Like the proof in Case 1, \( M \) is obviously totally umbilical if \( \mathcal{H} \equiv 0 \) or \( \nabla^2 \phi - \frac{1}{n}(\Delta \phi)g \equiv 0 \). We now consider that both \( \mathcal{H} \) and \( \nabla^2 \phi - \frac{1}{n}(\Delta \phi)g \) are not identically zero on \( \Sigma \), then by (II), there exists a nonzero constant \( \mu \neq 0 \) such that

\[
\mu \mathcal{H} = \nabla^2 \phi - \frac{1}{n}(\Delta \phi)g. \tag{2.13}
\]

Take points \( o \) and \( p \) as in Case 1. Observe that the positivity of Ricci curvature on the neighborhood \( N_\epsilon(p) \) of \( p \) and \( K > 0 \) still implies \( \nabla \phi = 0 \) on \( N_\epsilon(p) \). Similar to case 1, we may prove that \( \Sigma \) is totally umbilical.

We complete the proof of the theorem. \( \square \)

Theorem 1.3 is obtained by taking \( K = 0 \) in Theorem 1.4.

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10
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Xu Cheng
Instituto de Matemática
Universidade Federal Fluminense - UFF
Centro, Niterói, RJ 24020-140 Brazil
e-mail:xcheng@impa.br

Detang Zhou
Instituto de Matemática
Universidade Federal Fluminense - UFF
Centro, Niterói, RJ 24020-140 Brazil
e-mail: zhou@impa.br