Exact decoherence dynamics of 1/f noise

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Abstract

In this paper, we investigate the exact decoherence dynamics of a superconducting resonator coupled to an electromagnetic reservoir characterized by 1/f noise at finite temperature, where a full quantum description of the environment with 1/f^x noise (with x ≈ 1) is presented. The exact master equation and the associated non-equilibrium Green’s functions are solved exactly for such an open system. We show a clear signal of non-Markovian dynamics induced purely by 1/f noise. Our analysis is also applicable to other nano/micro mechanical oscillators. Finally, we demonstrate the non-Markovian decoherence dynamics of photon number superposition states using Wigner distribution that could be measured in experiments.

Keywords: 1/f noise, exact decoherence, non-Markovian dynamics

1. Introduction

The low-frequency noise spectrum $S(f) \sim 1/f$ was discovered in vacuum tubes and later observed in a variety of systems [1–4]. In electronics, this type of noise is commonly referred to as 1/f noise. Miniaturization of any material system in nanometer-scale devices can further increase 1/f noise levels and complicate practical applications [5, 6]. The sensitivity of amplifiers and transducers used in many types of sensors, particularly those that rely on an electrical response, is ultimately limited by the low-frequency noise level [7, 8]. The importance of 1/f noise in electronics has motivated various studies of its physical mechanisms, as well as the development of a variety of methods for its reduction [6]. Different physical origins of
fluctuation processes can be responsible for the occurrence of the 1/f noise in different materials and devices [9–11].

In almost all quantum computing nanodevices, the 1/f noise is found to be detrimental to the required maintenance of quantum coherent dynamics. In the last two decades, several experiments with single-electron transistors (SETs) [12–16] and superconducting circuits [17–23] at low temperatures have revealed low-frequency noise with a spectrum that scales inversely with frequency (1/f). The models and experimental observations [26–32] suggest that the proper scaling of this low-frequency noise should be 1/f^x where x may differ slightly from unity. We will investigate the exact decoherence dynamics of a superconducting resonator in the presence of this noise with varying values of x, including the situation under 1/f noise where x → 1. Our analysis is also applicable to other nanomechanical oscillators. Superconducting resonators and nano/micro mechanical oscillators have achieved sufficiently high frequencies (GHz range), and work at very low temperature (mK range), in which non-classical photon states can be rather easily generated [37–39]. Recently, 1/f noise has been measured both in superconducting resonators [40, 41], as well as in mechanical oscillators [42, 43].

Usually, in the literature on 1/f noise, the environment is treated classically as a random field describing a stochastic process [27–32]. We investigate in this paper the exact decoherence dynamics of a superconducting resonator coupled to an electromagnetic reservoir characterized by the 1/f frequency noise, where a full quantum description of the environment with 1/f noise is presented. We also describe the decoherence dynamics of the non-classical photon states of the resonator through the evolution of the Wigner distribution function. The paper is organized as follows. In section 2, we briefly discuss our model and the method we used. The noise power spectrum is explored in section 3 in the exact formalism of the non-equilibrium Green’s functions. In section 4, we present our numerical results on the exact decoherence dynamics of 1/f^x noise at finite temperature through the exact solution of the master equation associated with the non-equilibrium Green’s functions. Using Wigner distribution, we demonstrate in section 5 the exact decoherence dynamics of a superposition of photon number states under 1/f^x noise with different values of x. Finally, a conclusion is given in section 6.

### 2. The model and the exact master equation

If we consider a superconducting or a nanomechanical resonator coupled to an electromagnetic reservoir, the Hamiltonian of the total system can be written as

\[
H = \hbar \omega_0 a^\dagger a + \sum_k \hbar \omega_k b_k^\dagger b_k + \sum_k \hbar \left( V_k a^\dagger b_k + V_k^* a b_k^\dagger \right),
\]

where the first term is the Hamiltonian \( H_S \) of the single-mode resonator with frequency \( \omega_0 \), and \( a^\dagger \) and a are the creation and annihilation operators of the resonator, respectively; the second term is the Hamiltonian \( H_E \) of a general electromagnetic reservoir, as a collection of infinite photon or phonon modes, where \( b_k^\dagger \) and \( b_k \) are the corresponding creation and annihilation operators of the kth photon or phonon mode with frequency \( \omega_k \). The third term is the system–reservoir coupling \( H_I \), which characterizes photon scattering processes with the scattering amplitude \( V_k \) between the resonator and the kth reservoir mode. The non-linear photonic processes have been ignored in equation (1) because the 1/f noise spectrum only occurs in the very weak-coupling regime between the systems and its environment, as we will show in section 3.
We shall use the exact master equation method to describe the decoherence dynamics of the resonator under the influence of the $1/f$ noise. The master equation is given in terms of the reduced density operator, which is obtained from the density operator of the total system by tracing over the environmental degrees of freedom:

$$\rho = \text{Tr}_E[\rho_{\text{tot}}(t)]$$

The master equation is governed by the quantum evolution operator:

$$\rho(t) = e^{-\frac{i}{\hbar}Ht}\rho(0)e^{\frac{i}{\hbar}Ht}$$

As originally proposed by Feynman and Vernon [44–47], we take the initial state of the total system as a direct product of an arbitrary initial state of the system with the thermal state of the reservoir:

$$\rho(0) = \rho_0 \otimes \rho_E(0)$$

In this case, tracing over all the environmental degrees of freedom can be easily carried out using the Feynman–Vernon influence functional approach [44, 45] in the framework of coherent-state path-integral representations [48, 49]. The resulting master equation for the reduced density operator has the following form [49]:

$$\frac{d\rho(t)}{dt} = -i\omega'_0(t)\left[ a^\dagger a, \rho(t) \right] + \gamma(t)\left[ 2a\rho(t)a^\dagger - a^\dagger a\rho(t) - \rho(t)a^\dagger a \right] + \tilde{\gamma}(t)\left[ a\rho(t)a^\dagger + a^\dagger a\rho(t) - a^\dagger a\rho(t) - \rho(t)a\right]$$

(2)

where the time-dependent coefficient $\omega'_0(t)$ is the renormalized frequency of the resonator, while $\gamma(t)$ and $\tilde{\gamma}(t)$ describe the dissipation (damping) and fluctuation (noise) of the resonator due to its coupling to the reservoir. These coefficients can be exactly determined by the following relations [49–52]

$$\omega'_0(t) = -\text{Im}\left[ \frac{\dot{u}(t)}{u(t)} \right], \quad \gamma(t) = -\text{Re}\left[ \frac{\dot{u}(t)}{u(t)} \right]$$

$$\tilde{\gamma}(t) = \nu(t) - 2\nu(t)\text{Re}\left[ \frac{\dot{u}(t)}{u(t)} \right]$$

(3)

The function $u(t)$ is the non-equilibrium propagating (or spectral) Green’s function of the system, which satisfies the Dyson equation of motion

$$\dot{u}(t) + i\omega_0 u(t) + \int_{t_0}^t d\tau g(t - \tau)u(\tau) = 0$$

(4)

subject to the initial condition $u(t_0) = 1$. The non-equilibrium thermal fluctuation is characterized by the correlation function $v(t)$ through the non-equilibrium fluctuation–dissipation theorem [53], which is given explicitly by

$$v(t) = \int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' \tilde{g}(\tau - \tau')u^*(\tau)u(\tau')$$

(5)

The time correlation functions $g(t - \tau)$ and $\tilde{g}(\tau - \tau')$ in equations (4) and (5) are given by

$$g(t - \tau) = \int_0^\infty \frac{d\omega}{2\pi} f(\omega)e^{-i\omega(t-\tau)}$$

(6)

$$\tilde{g}(\tau - \tau') = \int_0^\infty \frac{d\omega}{2\pi} f(\omega)\tilde{n}(\omega, T)e^{-i\omega(\tau-\tau')}$$

(7)
which characterize all the non-Markovian back-action memory effects between the system and
the reservoir, where \( J(\omega) = 2\pi \sum_k |V_k|^2 \delta(\omega - \omega_k) \) is the spectral density, and \( V_k \) is the
coupling between the system and the reservoir. Furthermore, \( \bar{n}(\omega, T) = \frac{1}{\sum n_k \omega_k e^{\omega_k/k_B T}} \) is the particle
number distribution of the bosonic reservoir at the initial temperature \( T \). If the reservoir
spectrum is continuous, \( V_k \rightarrow V(\omega) \), we have \( J(\omega) = 2\pi q(\omega)|V(\omega)|^2 \), where \( q(\omega) \) is the
density of states of the reservoir.

3. Quantum description of 1/f noise spectrum

In the literature on 1/f noise, the environment is treated classically [27–32] with a classical
random field \( c(t) \) describing a stochastic process. The system–environment coupling is
described by a term \( -c \langle t \rangle A \), and \( A \) is a system operator. Then the system Hamiltonian becomes
stochastic due to the random nature of the classical random field \( c(t) \), which can take a different
form corresponding to different kinds of classical noise. A typical classical noise source is the
random telegraphic noise (RTN) which is used to classically model an environment for solid-
state devices [11, 31, 32] (and references therein), where the system is considered to be
interacting with a bistable fluctuator for which the time-dependent parameter is randomly
flipping between two values \( c(t) = \pm 1 \) with a switching rate \( \nu \). The Hamiltonian with such a
noise source describing a system subject to a random telegraph noise is usually used as a basic
building block to describe noises of the type 1/f\(^a\), and is characterized by an exponentially
decaying correlation function \( e^{-\nu |t|} \) of the fluctuating quantity \( c(t) \). Then, the noise spectrum is a
Lorentzian function

\[
S(\omega, \nu) = \int_{-\infty}^{+\infty} dt e^{i \omega t} c(t) \bar{n}(0) = \int_{-\infty}^{+\infty} dt e^{i \omega t} e^{-\nu |t|} = \frac{1}{\pi \frac{\nu}{\omega^2 + \nu^2}}.
\]

To reproduce the 1/f\(^a\) spectrum, the single RTN frequency power spectrum is then integrated
over the switching rate \( \nu \) with a suitable probability distribution:

\[
S_{1/f^a}(\omega) = \int_{\nu_1}^{\nu_2} S(\omega, \nu) p_\nu(\nu) d\nu,
\]

The integration is generally performed between a minimum and a maximum value of the
switching rates, \( \nu_1 \) and \( \nu_2 \), respectively. In order to simulate 1/f\(^a\) noise spectrum, the switching
rate distribution \( p_\nu(\nu) \) is considered to be proportional to \( 1/\nu^a \). When the integration in
equation (9) is performed, the spectrum shows 1/f\(^a\) behavior in a frequency interval, so that all
frequencies belonging to such an interval satisfy the condition \( \nu_1 \leq f \leq \nu_2 \). Although these low-
and high-frequency cutoff frequencies are artificially fixed in the literature [17–19, 21, 24], the
physical origin of the cutoff frequencies is debatable [25]. Another crucial point is that the
classical environment with the noise spectrum 1/f\(^a\) can be realized by a different configuration
of bistable fluctuators. The noise spectrum 1/f\(^a\) can be obtained by considering a single bistable
fluctuator whose switching rate is randomly chosen from a distribution \( p_\nu(\nu) \), as shown above.
The same 1/f\(^a\) spectrum can also be realized from the coupling of a system with a large number
of fluctuators, where the noise spectrum can be obtained as a result of a linear combination of
many Lorentzian functions, each characterized by a specific switching rate [11, 31, 32]. As
already pointed out in several papers [31, 33–36], different microscopic configurations of the
environment leading to the same spectra may correspond to different physical phenomena.
Thus, mere knowledge of the noise spectrum is not sufficient to describe the environmental
influence on the quantum dynamics of open systems and it is necessary to specify the model for the noise source in more detail.

In the previous section, we have presented a full quantum mechanical description of the system–environment coupling for a superconducting resonator coupled to an electromagnetic reservoir. Before we explore the decoherence dynamics of a superconducting resonator or a nanomechanical resonator, induced by the $1/f$ noise, it is important to justify the conditions for the occurrence of $1/f$ noise in a given electromagnetic reservoir characterized by the spectral density $J(\omega)$. In the literature, the spectral density $J(\omega)$ for an electromagnetic reservoir is found to be Ohmic-type [46], and is given by

$$J(\omega) = 2\pi \eta \omega \left(\frac{\omega}{\omega_c}\right)^{s-1} e^{-\omega/\omega_c},$$

with $0 < x = (1 - s) < 1$. It has been pointed out [56] that the $1/f$ noise spectrum corresponds to the special case of the spectral density equation (10) with $s \approx 0$ or $x \approx 1$. However, the exact connection of $J(\omega)$ with the noise spectrum $\omega S(\omega)$ has not been carried out so far in the literature. Here, within the exact master equation formalism, we find that the exact solutions of the non-equilibrium Green’s functions $u(t)$ and $v(t)$ carry all the information on the quantum nature of the noise spectrum $S(\omega)$.

Specifically, the noise spectrum of the system is quantum mechanically defined by the Fourier transform of the two-time particle correlation function:

$$S(\omega) = \lim_{t \to \infty} \int_{-\infty}^{\infty} e^{i\omega t} \langle a\dagger(t + \tau)\alpha(t)\rangle \, d\tau.$$  

As we have shown recently [49], the two-time particle correlation function $\langle a\dagger(t + \tau)\alpha(t)\rangle$ obeys the following relation in our exact master equation formalism,

$$\langle a\dagger(t + \tau)\alpha(t)\rangle = u^*(t + \tau)u(t)\langle a\dagger(t_0)\alpha(t_0)\rangle + v(t, t + \tau),$$

where

$$v(t, t + \tau) = \int_{t_0}^{t+\tau} \int_{t_0}^{t+\tau} \, d\tau_1 d\tau_2 u(t, \tau_1)\tilde{g}(\tau_1, \tau_2)u^*(t + \tau, \tau_2),$$

which is the more general form of equation (5). On the other hand, the exact analytic solution of the integro–differential equation (4) is also recently given in [53] as

$$u(t) = Z e^{-i\omega(t-t_0)} + \frac{1}{\pi} \int_0^\infty \frac{\gamma(\omega)e^{-i\omega(t-t_0)}}{[\omega - \omega_0 - \Delta(\omega)]^2 + \gamma^2(\omega)} \, d\omega,$$

where $\gamma(\omega) = J(\omega)/2$ and $\Delta(\omega) = P\int_0^\infty \frac{J(\omega')}{\omega - \omega'} \, d\omega'$, which are the real and imaginary parts of the self-energy correction, respectively, and $\Sigma(z)$, to the resonator, is induced by coupling between the resonator and the environment,

$$\Sigma(z) = \int_0^\infty \frac{J(\omega)}{z - \omega} \, d\omega.$$

The first term in equation (13) is the contribution of the dissipationless localized mode, where the localized mode frequency $\omega_b$ is determined by $\omega_b - \omega_0 - \Delta(\omega_b) = 0$, and $Z = [1 - \Sigma'(\omega_b)]^{-1}$ corresponds to the residue of $\Sigma(z)$ at the pole $z = \omega_b$, which gives the amplitude of the localized mode.
Using the exact solution of equation (13) and the relation (12), we find that the noise spectrum is given by

\[
\begin{align*}
S_1(\omega) &+ S_2(\omega),
\end{align*}
\]

where \(S_1(\omega)\) and \(S_2(\omega)\) are the contributions of the particle correlations from the system and the environment, respectively, to the noise spectrum, due to the coupling between them. Now, \(S_2(\omega)\) can have a power-series expansion with Legendre polynomials \(P_{\eta}(\xi)\) as

\[
S_2(\omega) = Z^2 J(\omega)\bar{n}(\omega, T) + J(\omega)\bar{n}(\omega, T) \left( \sum_{n=0}^{\eta} P_{\eta}(\xi)\xi^n \right)^2
\]

where \(\xi = \omega_0/\sqrt{\omega_0^2 + \gamma^2(\omega)}\), and \(\zeta = (\omega - \Delta(\omega))/\sqrt{\omega_0^2 + \gamma^2(\omega)}\). We should only take the low-frequency limits \(\hbar\omega/k_B T \ll 1\) and \(\omega \ll \omega_0\), in order to see the low-frequency behavior of the noise spectrum. Through a numerical check (see figure 1(a)), we find that for \(\eta \leq 10^{-3}\) and \(\omega/\omega_0 \leq 10^{-2}\), one has \(\xi, \zeta \ll 1\) and \(\omega_0^2 \gg \gamma^2(\omega)\). Meanwhile, when the coupling between the system and the environment \(\eta \leq 10^{-3}\), the localized mode amplitude \((Z)\) becomes negligibly small, and therefore it does not play any role in the low-frequency behavior of the noise spectrum. Thus, the noise spectrum is reduced to

\[
\text{Figure 1. (a) The first-order correction term } 2\xi\zeta \text{ in the expansion of the noise spectrum (16) for } \omega_c = \omega_0, \text{ which shows a very narrow range of } \eta \text{ and } \omega \text{ for the low frequency noise spectrum behaving as } 1/f^x \text{ power law. (b) The noise spectrum } S(\omega) \text{ is plotted for different values of } x \text{ as } x = 0.25 \text{ (green), } x = 0.5 \text{ (blue), } x = 0.75 \text{ (pink) and } x = 0.9999 \text{ (red) at } \eta = 10^{-3} \text{ with } \omega_c = \omega_0 = 5 \text{ GHz, and } T = 25 \text{ mK.}
\]
where $\eta = \eta_0^{1-s}/\hbar \omega_0^2$. For $s = 0$, equation (17) gives the exact $1/f$ noise spectrum. Generally, the low-frequency noise ($1/f^s$) dominates when $x > 0$ or $s < 1$ in the very weak coupling regime between the system and its reservoir. In figure 1(a), we show the range of $\eta$ and $\omega$ where the low-frequency noise spectrum behaves as a $1/f^s$ power law. For $x \approx 1$, the $1/f$ noise behavior shows up in the low-frequency domain where the coupling strength $\eta$ must be sufficiently weak. We also plot in figure 1(b) the noise spectrum $S(\omega)$ for $\eta = 10^{-3}$ with different values of $x$. Different $1/f^x$ behaviors are also displayed.

The above noise power spectrum, calculated analytically and exactly through the Fourier transform of the two-time particle correlation function (11), provides a fully quantum mechanical description of the $1/f^x$ noise. This quantum mechanical description of the $1/f$ noise shows that it is valid in a very narrow range of $\eta$ and $\omega$ when both the frequency and the coupling strengths are very small, in comparison with the energy scale of the system.

4. Decoherence dynamics under $1/f^x$ noise

Now we shall study the exact decoherence dynamics induced by $1/f$ noise under the condition $\hbar \omega/k_B T \ll 1$ and $\omega \ll \omega_0$, and when the dimensionless coupling strength $\eta$ is sufficiently weak. We start with the $1/f^x$ noise with $x = 0.25$, and gradually increase it to $x = 0.5$, 0.75, and 0.9999, to examine the change of dissipation and fluctuation dynamics of the resonator when the noise spectrum approaches to $1/f$. Dissipation and fluctuation dynamics through the exact solution of $u(t)$ and $v(t)$ are presented in figure 2 for $1/f^x$ noise (10) with $x = 0.25$, 0.5, 0.75, and 0.9999, respectively, corresponding to the four different curves in each graph. It shows how dissipation and fluctuation change as the reservoir spectra approach to low-frequency-dominated regime. One can see that for very weak coupling ($\eta = 10^{-3}$), the particle propagating function $u(t)$ has similar damping dynamics (monotinous decay) with different $x$ values (see figure 2(a)). We see similar decay dynamics for $u(t)$ (without having any non-Markovian effect) even if we increase the coupling strength to $\eta = 10^{-2}$ with changing $x$ values (see figure 2(b)).
Figures 2(c) and (d) show thermal fluctuations in terms of the correlation Green’s function $v(t)$. The correlation Green’s function $v(t)$ quantifies physically the thermal-fluctuation-induced average particle number inside the resonator, reflecting the noise effect of the decoherence dynamics. Figure 2(c) shows that the decoherence dynamics are significantly different for $f_1$ spectrum ($x = 0.9999$), as the oscillations of $v(t)$ is very strong, compared to other $f_1$ (with $x = 0.25$, 0.5, 0.75) at weak coupling ($\eta = 10^{-3}$). This is the signal of a non-Markovian decoherence dynamics purely induced by the noise effect. Physically, it can be seen from equation (7) that the initial particle distribution function $\pi(\omega, T)$ induces explicitly frequency dependence to the memory kernel $G(\tau - \tau')$, and in particular, this frequency dependence becomes stronger in the low-frequency regime $1/\omega$. This distinct oscillatory feature of $v(t)$ persists even at higher coupling strength ($\eta = 10^{-2}$), although it has a long time decay behavior (see figure 2(d)). In conclusion, we show here, for the first time, the physical mechanism of non-Markovian dynamics induced by $1/f$ noise, even though the system–environment coupling is very small.

With the above exact solution of the dissipation and fluctuation dynamics, we now present the dissipation and fluctuation coefficients in the master equation, $\gamma(t)$ and $\tilde{\gamma}(t)$, which manifests quantitatively the decoherence behavior of $1/f$ noise. Figures 3(a) and (b) show the time evolution of the dissipation coefficient $\gamma(t)$ for different $1/f^3$ noise with $x = 0.25$, 0.5, 0.75, and 0.9999 at two different values of the coupling strengths, $\eta = 10^{-3}$ and $10^{-2}$, partnered to the solutions in figures 2(a) and (b). In the very weak coupling region ($\eta = 10^{-3}$), including the $1/f$ noise, we see from figure 3(a) that the dissipation coefficient $\gamma(t) > 0$, which indicates that the corresponding dissipation solution is always Markovian [54, 55]. A similar behavior of $\gamma(t)$ with increased magnitude is seen when the coupling strength is increased ($\eta = 10^{-2}$ for figure 3(b)). However, the fluctuation coefficient $\tilde{\gamma}(t)$ is very sensitive to the noise spectrum and it behaves qualitatively differently from the dissipation coefficient $\gamma(t)$ in the weak-coupling regime, in particular, in the $1/f$ noise regime. As one can see from figure 3(c), $\tilde{\gamma}(t)$ oscillates between positive and negative values, resulting in a non-Markovian memory effect. This positive- and negative-bonded value is significantly larger in
the case of the $1/f$ noise spectrum ($x \sim 1$), compared to other spectra with smaller values of $x$. The distinct oscillatory feature of $\tilde{\gamma}(t)$ persists even at higher coupling strength ($\eta = 10^{-2}$), with a long time decay behavior (see figure 3(d)). The presence of this persistent oscillation between positive and negative values in fluctuation coefficient $\gamma(t)$, but not in the dissipation coefficient $\gamma(t)$, shows the evidence for strong non-Markovian dynamics associated with the $1/f$ noise, as we have just pointed out in the analysis of the correlation Green’s function $v(t)$. The decoherence dynamics of the resonator are very sensitive to the temperature for the $1/f$ spectrum. To show the effect of temperature dependence on the decoherence dynamics, we plot the non-equilibrium thermal fluctuation $v(t)$ and the fluctuation coefficient $\tilde{\gamma}(t)$ in figure 4 for $1/f^x$ spectrum with $x = 0.9999$ at various temperatures $T = 25$ mK, $T = 1.0$ K, and $T = 2.5$ K, respectively. It shows that the magnitude of both $v(t)$ and $\tilde{\gamma}(t)$ becomes significant with the rising temperature even if the system–reservoir coupling is very small ($\eta = 10^{-3}$). This temperature dependence comes through the particle number distribution $\tilde{n}(\omega, T)$ in equation (7), as purely a noise effect.

5. Decoherence dynamics of the resonator under $1/f^x$ noise using Wigner distribution

Next, we explore the decoherence dynamics of quantum photon states under the $1/f$ noise by examining the evolution of the corresponding Wigner function. With the help of the exact master equation (2), the exact Wigner function of an arbitrary quantum state at arbitrary time $t$ in the complex space $\{z\}$ is given by

$$W(z, t) = \int d\mu(\alpha_0) d\mu(\alpha_0^*) \langle \alpha_0 | \rho(t_0) | \alpha_0^* \rangle \mathcal{T}(z, t | \alpha_0, \alpha_0^*, t_0),$$

(18)

where $|\alpha\rangle = e^{z \alpha^*}|0\rangle$ is the coherent state, $d\mu(\alpha) = \frac{da^* da}{2\pi} e^{-|\alpha|^2}$ is the integral measure of the Bergman complex space, $\rho(t_0)$ is the reduced density matrix of the initial state, and the propagating function $\mathcal{T}(z, t | \alpha_0, \alpha_0^*, t_0)$ is given by [52, 57]
\[
    F(z, t | \alpha_0, \alpha^*_0, t_0) = W_0^0(z, t) \exp \left\{ \sum \Omega \left( t \right) u(t) \alpha_0 + \sum \Omega \left( t \right) u^*(t) \alpha^*_0 + \sum \Omega \left( t \right) \left[ 1 - |u(t)|^2 \right] \right\} \alpha_0, \quad (19)
\]

where
\[
    \Omega (t) = \frac{2}{1 + 2v(t)} \quad \text{and} \quad W_0^0(z, t) = \frac{2 \exp \left( -\sum \Omega (t) |z|^2 \right)}{\pi [1 + v(t)]}.
\]

We investigate the damping and decoherence dynamics of the resonator in the presence of low frequency \(1/f^x\) noise at a finite temperature. We see the effect of changing values of \(x\) on the dynamics of the resonator which is prepared initially in a superposition of Fock states. If the resonator is initially prepared in a photon number superposition state \(\rho (t_0) = |\psi_0 \rangle \langle \psi_0|\) with \(|\psi_0 \rangle = 1/\sqrt{2} \left( |0 \rangle + |1 \rangle \right)\). The time evolved Wigner function in this case is given by
\[
    W_n^0(z, t) = \frac{1}{2} \left[ W_0^0(z, t) + W_n^0(z, t) \right] + \frac{W_0^0(z, t)}{2 n!} \left[ \left( \sum \Omega (t) u^*(t) \right)^n + \left( \sum \Omega (t) u(t) \right)^n \right]. \quad (20)
\]

where \(W_0^0(z, t)\) is the time evolved Wigner function for the initial vacuum state, and
\[
    W_n^0(z, t) = W_0^0(z, t) \sum_{p=0}^{n} \frac{n!}{p!(n-p)!(n-p)!} \times \left[ \left( |u(t)|^2 \Omega (t) |z|^2 \right)^{n-p} \left( 1 - |u(t)|^2 \Omega (t) \right)^p \right]. \quad (21)
\]

The decoherence dynamics of the superposition state \(|\psi_0 \rangle = 1/\sqrt{2} \left( |n \rangle + |m \rangle \right)\) can be examined through the time evolution of its Wigner function, given by
\[
    W_n^m(z, t) = \frac{1}{2} \left[ W_n^m(z, t) + W_m^n(z, t) \right] + \frac{1}{2} W_0^0(z, t) \sum_{p=0}^{\min(n,m)} \frac{n! m!}{p!(n-p)!(m-p)!} \times \left\{ \left( \sum \Omega (t) u(t) \right)^{n-p} \left( \sum \Omega (t) u^*(t) \right)^{m-p} \left( 1 - |u(t)|^2 \Omega (t) \right)^p \right\}, \quad (22)
\]

where \(W_m^m(z, t)\) is the same as \(W_n^n(z, t)\), with \(n\) being replaced by \(m\). In figure 5, the time-dependent snapshots of the Wigner functions are shown at four different times: \(\omega_0 t = 0, 1, 1.5, \text{and} 2\). Figure 5(a) describes the decoherence dynamics of the resonator in the presence of the low frequency \(1/f^x\) noise with \(x = 0.25\) at a finite temperature \(T = 2.5\) K, and the resonator is initially prepared in a superposition state \(1/\sqrt{2} \left( |0 \rangle + |1 \rangle \right)\). The interference pattern consisting of three positive and three negative peaks is caused by the superposition between the states \(|0 \rangle\) and \(|3 \rangle\). As time evolves, the off-diagonal elements of the density matrix decays with time and the positive and negative peaks disappear. In figure 5(b), we plot the time evolution of the Wigner function for the same superposition state \(1/\sqrt{2} \left( |0 \rangle + |1 \rangle \right)\) in the presence of the \(1/f^x\) noise with \(x = 0.9999\). We see distinct decoherence dynamics for the
resonator initial state $\ket{\frac{1}{\sqrt{2}}(0 + 13)}$ under $1/f^x$ noise with different $x$ values, where the decoherence rate is much faster for $x = 0.9999$. Next, we plot (figures 5(c) and (d)) the decoherence dynamics of the resonator state $\ket{\frac{1}{\sqrt{2}}(2 + 13)}$, where the Wigner distribution functions show different decay dynamics of the interference fringe pattern due to the low frequency $1/f^x$ noise (with $x = 0.25$ and $x = 0.9999$) of the bosonic reservoir at a finite temperature $T = 2.5$ K. Faster decoherence dynamics is again observed as we increase the $x$ value of the noise spectrum.

6. Conclusion

In conclusion, the exact decoherence dynamics of a quantum resonator coupled to a low-frequency bosonic reservoir were explored. The noise power spectra were calculated analytically and exactly using the exact solutions of the non-equilibrium Green’s functions. It was found that the $1/f^x$ power law behavior of the noise spectrum is valid for a very narrow range of $\eta$ and $\omega$ when both the frequency and the coupling strengths are very small, in comparison with the energy scale of the system. The non-Markovian dynamics of the resonator in the weak coupling regime are produced by the noise effect. The correlation Green’s function $\nu(t)$ and hence the fluctuation coefficient $\tilde{\gamma}(t)$ shows a long-time non-Markovian oscillatory behavior which is qualitatively different from the Markovian dissipation dynamics described by the propagating Green’s function $u(t)$ and the dissipation coefficient $\gamma(t)$ in the ultra-weak coupling regime, and in particular, in the $1/f$ regime. We have shown through the exact master equation the evolution of a number of non-classical photon states of the resonator in the presence of $1/f$ noise, where the finite temperature effect of the bosonic reservoir is also examined. The faster decoherence behavior due to the $1/f^3$ noise was demonstrated by...
increasing the $x$ value. Our analysis is also applicable to other nano/micro mechanical oscillators, and we believe that the results presented here can enhance the understanding of non-Markovian decoherence dynamics for many solid-state quantum devices in the very weak system–reservoir coupling regime when the $1/f$ noise is dominated.

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