HITCHIN COMPONENTS FOR ORBIFOLDS

DANIELE ALESSANDRINI, GYE-SEON LEE, AND FLORENT SCHAFFHAUSER

Abstract. We extend the notion of Hitchin component from surface groups to orbifold groups and prove that this gives new examples of Higher Teichmüller spaces. We show that the Hitchin component of an orbifold group is homeomorphic to an open ball, and we compute its dimension explicitly. For example, the Hitchin component of the right-angled hyperbolic \(\ell\)-polygon reflection group into \(\text{PGL}(2m,\mathbb{R})\), resp. \(\text{PGL}(2m+1,\mathbb{R})\), is homeomorphic to an open ball of dimension \((\ell-4)m^2+1\), resp. \((\ell-4)(m^2+m)\).

We also give applications to the study of the pressure metric and the deformation theory of real projective structures on 3-manifolds.

Contents

1. Introduction 1
2. Hitchin representations for orbifolds 4
3. Hitchin’s equations in an equivariant setting 12
4. Parametrization of Hitchin components 23
5. Invariant differentials 27
6. Applications 32
Appendix A. Expected dimensions of Hitchin components 39
References 40

1. Introduction

1.1. Hitchin components. The Teichmüller space parametrizes the space of hyperbolic structures on a closed orientable surface \(X\) of genus \(g \geq 2\). It can also be seen as a connected component of the representation space

\[
\text{Rep}(\pi_1 X, \text{PGL}(2, \mathbb{R})) := \text{Hom}(\pi_1 X; \text{PGL}(2, \mathbb{R}))/\text{PGL}(2, \mathbb{R})
\]

consisting of the conjugacy classes of discrete and faithful representations of \(\pi_1 X\) into \(\text{PGL}(2, \mathbb{R})\). It is well-known that the Teichmüller space is homeomorphic to an open ball of dimension \((6g-6)\). In 1992, N. J. Hitchin [Hit92] replaced the group \(\text{PGL}(2, \mathbb{R})\) by the split real form \(G\) of a complex simple Lie group, and found a special component of \(\text{Rep}(\pi_1 X, G) := \text{Hom}(\pi_1 X; G)/G\) homeomorphic to an open ball of dimension \((2g-2)\dim G\), which is now called the Hitchin component of the surface group \(\pi_1 X\) into \(G\). The geometry of the representations in the Hitchin components was studied by Choi-Goldman [CG93] for \(G = \text{PGL}(3, \mathbb{R})\), Labourie [Lab00] and Guichard [Gui08] for \(G = \text{PGL}(n, \mathbb{R})\), Guichard-Wienhard [GW08] for \(G = \text{PGL}(4, \mathbb{R})\) and [GW12] in higher generality. From these works, we see that the Hitchin components share many properties with Teichmüller spaces, and they are part of an interesting family of spaces called Higher Teichmüller spaces (see Wienhard [Wie18] for a survey of this theory).

In this paper, we generalize the notion of Hitchin components of surface groups to a more general family of groups, namely the fundamental groups of closed 2-dimensional orbifolds with negative orbifold Euler characteristic. This family includes all the groups isomorphic to a uniform lattice in \(\text{PGL}(2, \mathbb{R})\). This is a large family of groups including, for example, the fundamental groups of closed non-orientable surfaces and surfaces with boundary, and all the 2-dimensional hyperbolic Coxeter groups.
The first instance of Hitchin components for these groups was studied by Thurston [Thu79] who showed that the Teichmüller space (i.e. the space of hyperbolic structures on a closed orbifold $Y$) is parametrized by the subspace of $\text{Rep}(\pi_1 Y, \text{PGL}(2,\mathbb{R}))$ consisting of the conjugacy classes of discrete and faithful representations of the orbifold fundamental group $\pi_1 Y$ into $\text{PGL}(2,\mathbb{R})$, and he described its topology. Then Choi-Goldman [CG05] studied the Hitchin component for $\pi_1 Y$ in $\text{PGL}(3,\mathbb{R})$, describing its topology and showing that it parametrizes the deformation space of convex real projective structures on $Y$. Finally,Labourie and McShane introduced $\text{PGL}(n,\mathbb{R})$ Hitchin components for orientable surfaces with boundary, in order to generalize McShane-Mirzakhani identities from hyperbolic geometry to arbitrary cross ratios ([LM09]). In this paper, we study the general case. We fix $\mathfrak{g}$ to be a split real form of a complex simple Lie algebra, and let $G$ be the group of real points of $\text{Int}(\mathfrak{g} \otimes \mathbb{C})$. For the classical Lie algebras, $G$ is one of the groups $\text{PGL}(n,\mathbb{R})$, $\text{PSp}^\pm(2m,\mathbb{R})$, $\text{PO}(m,m+1)$, $\text{PO}^\pm(m,m)$, while for the exceptional Lie algebras we do not have specific names for these groups. A representation of $\pi_1 Y$ in $G$ is said to be Fuchsian if it is the composition of a discrete and faithful representation of $\pi_1 Y$ in $\text{PGL}(2,\mathbb{R})$ with the principal representation $\kappa: \text{PGL}(2,\mathbb{R}) \to G$. We define the Hitchin component $\text{Hit}(\pi_1 Y, G)$ of $\pi_1 Y$ into $G$ as the connected component of $\text{Rep}(\pi_1 Y, G)$ which contains the Fuchsian representations.

1.2. Results in the orbifold case. We first verify that, also in the case of an orbifold $Y$, every Hitchin representation $\rho: \pi_1 Y \to \text{PGL}(n,\mathbb{R})$ shares some of the nice properties of the Fuchsian representations into $\text{PGL}(2,\mathbb{R})$.

**Theorem A** (Section 2.5). Every Hitchin representation $\rho: \pi_1 Y \to \text{PGL}(n,\mathbb{R})$ is $B$-Anosov, where $B$ is any Borel subgroup of $\text{PGL}(n,\mathbb{R})$, and it is discrete and faithful. Moreover, for all $\gamma$ of infinite order in $\pi_1 Y$, the element $\rho(\gamma)$ is diagonalizable with distinct real eigenvalues. A representation of $\pi_1 Y$ in $\text{PGL}(n,\mathbb{R})$ is Hitchin if and only if it is hyperconvex.

The proof of Theorem A uses the analogous results for surfaces, as proved by Labourie [Lab06], Guichard [Gu08], and Guichard and Wienhard [GW12]. This theorem implies that the Hitchin components for orbifold groups form a new family of higher Teichmüller spaces according to the definition given by Wienhard [Wie18]. Having established these geometric properties, we determine the topology of Hitchin components for orbifold groups. This is the main theorem of this paper:

**Theorem B** (Theorem 5.6). Let $Y$ be a closed 2-orbifold of negative orbifold Euler characteristic, with $k$ cone points of orders $m_1, \ldots, m_k$ and with $\ell$ corner reflectors of orders $n_1, \ldots, n_\ell$. Let $G$ be the group of real points of $\text{Int}(\mathfrak{g} \otimes \mathbb{C})$, where $\mathfrak{g}$ is a split real form of a complex simple Lie algebra of rank $r$. If $|Y|$ is the underlying topological surface of $Y$, and $d_1, \ldots, d_r$ are the exponents of $\mathfrak{g}$, then the Hitchin component $\text{Hit}(\pi_1 Y, G)$ is homeomorphic to an open ball of dimension

$$-\chi(|Y|) \dim G + \sum_{a=1}^r \left( 2 \sum_{i=1}^k O(d_a + 1, m_i) + \sum_{j=1}^\ell O(d_a + 1, n_j) \right)$$

where $O(d,m) = \left\lfloor d - \frac{d}{m} \right\rfloor$ denotes the biggest integer not greater than $(d - \frac{d}{m})$.

As a corollary, we obtain that the Hitchin component of Labourie and McShane is homeomorphic to an open ball, whose dimension is still given, in this case, by Thurston’s formula (Corollary 5.7 and Remark 5.8).

**Remark 1.1.** In the special case where $Y$ is a sphere with 3 cone points and $G = \text{PGL}(n,\mathbb{R})$ (resp. $G = \text{PSp}^+(2m,\mathbb{R})$ or $\text{PO}(m,m+1)$), Long and Thistlethwaite [LT18] (resp. Weir [Wei18]) computed the dimension of the Hitchin component of $\pi_1 Y$ into $G$. Our result confirms the formulae they found in this case, and in addition shows that those Hitchin components are homeomorphic to open balls.

1.3. Idea of the proof. The key idea of the proof of Theorem B is that the Hitchin components of orbifold groups can be identified with particular subsets of the Hitchin components of surface subgroups in the following way: Every orbifold $Y$ is covered by some orientable surface $X$ of genus $g \geq 2$, in other words, the orbifold $Y$ is a quotient of the surface $X$ by an action of a finite group $\Sigma$. The restriction of a representation of $\pi_1 Y$ to the subgroup $\pi_1 X$ gives a map $j: \text{Hit}(\pi_1 Y, G) \to \text{Hit}(\pi_1 X, G)$, which is in fact injective (Proposition 2.11). Moreover the image is exactly the set $\text{Fix}_\Sigma \text{Hit}(\pi_1 X, G)$ of the Hitchin representations for $X$ that are fixed by the action of $\Sigma$ (see Theorem 2.13). In order to prove Theorem 2.13 i.e. that the map $j$
is also surjective onto $\text{Fix}_{\Sigma} \text{Hit}(\pi_1 X, G)$, we develop a $\Sigma$-equivariant version of the Non-Abelian Hodge Correspondence between the representation space of completely reducible representations of $\pi_1 Y$ into $G$ and the moduli space of isomorphism classes of $\Sigma$-polystable equivariant $G$-Higgs bundles on $X$ (see Section 3.4) and we show that a section of the Hitchin fibration with Hitchin base $B_X(g)$ is $\Sigma$-equivariant, which induces a homeomorphism $\text{Fix}_{\Sigma}(B_X(g)) \simeq \text{Fix}_{\Sigma}(\text{Hit}(\pi_1 X, G))$ (see Lemma 4.3). Finally, we study the spaces of regular differentials on orbifolds, computing their dimension, and we define the Hitchin base of genus $0$ and components can be understood completely. In this paper we classify all Hitchin components of dimensions large, as their dimension grows quadratically with the rank of the group. The Hitchin components for orbifolds of genus zero can be small, hence they can be used as toy models where the geometry of the Hitchin components can be understood completely. In this paper we classify all Hitchin components of dimensions 0 and 1 and, for orientable orbifolds, also 2 (Theorems 6.2, 6.12 and Remark 6.13).

The zero-dimensional Hitchin components give us many examples of rigidity phenomena, where a representation cannot be deformed. In the case of $G = \text{PGL}(2, \mathbb{R})$, rigid cases were already classified by Thurston [Thi79], and in $G = \text{PGL}(3, \mathbb{R})$ they were classified by Choi and Goldman [CG05]. We extend this classification to the general case. For example, for the family $\text{PGL}(n, \mathbb{R})$, we find rigid orbifolds for $n \leq 5$, and we prove that this does not happen any more when $n \geq 6$ (see Theorem 6.2). Moreover, we encounter another type of rigidity phenomenon: some Hitchin components contain no Zariski dense representations (all such Hitchin components are classified in Section 6). The existence of Hitchin components which do not contain any Zariski dense representation is surprising and it is in contrast with what happens for surface groups: in the latter case, the subset of Zariski dense representations is always dense in the character variety ([KP14]).

1.4. Applications. An interesting feature of orbifold groups is that they give examples of small Hitchin components. For orientable surfaces, the smallest Hitchin component is the Teichmüller space of the surface of genus 2, which has real dimension 6. For target groups of higher rank, Hitchin components soon become large, as their dimension grows quadratically with the rank of the group. The Hitchin components for orbifolds of genus zero can be small, hence they can be used as toy models where the geometry of the Hitchin components can be understood completely. In this paper we classify all Hitchin components of dimensions 0 and 1 and, for orientable orbifolds, also 2 (Theorems 6.2, 6.12 and Remark 6.13).

The Hitchin components of dimension 1 are particularly remarkable, because they are geodesics for these metrics, they are the first geodesics that can be constructed explicitly. This is interesting, since almost nothing is known about geodesics for such metrics. In Section 6 we classify all the Hitchin components of dimension 1. For the family $\text{PGL}(n, \mathbb{R})$, we find one-dimensional Hitchin components for $n \leq 11$, and we prove that they do not exist any more when $n \geq 12$ (see Theorem 6.12 and the Remark 6.13).

Recall that Hitchin representation for surface groups are parametrized by the Hitchin base, a direct sum of spaces of holomorphic differentials. It is possible to define special loci in the Hitchin components corresponding to the vanishing of some of these differentials. In some special cases, such loci have a geometric interpretation, but in general we do not understand their geometric meaning. We will show how, in some especially interesting cases, the Hitchin representations for orbifold groups allow us to geometrically construct some Hitchin representations of surface groups lying in these loci where no geometric interpretation is known. Further study of these examples might help to understand the meaning of such loci. Moreover, the same orbifolds give us many examples of Hitchin components for orbifold groups where all the corresponding Higgs bundles are cyclic or $(n-1)$-cyclic. This is particularly interesting since many properties of these components can be understood using analytic techniques.

We also have applications to the study of the deformation spaces of real projective structures on Seifert fibered 3-manifolds (see Section 6.5). Guichard and Wienhard [GW08] showed that the Hitchin component of a surface group in $\text{PGL}(4, \mathbb{R})$ parametrizes the deformation space of convex foliated projective structures on the unit tangent bundle of the surface. In a similar way, the Hitchin component on an orientable orbifold in $\text{PGL}(4, \mathbb{R})$ parametrizes a connected component of the deformation space of real projective structures on some Seifert fibered 3-manifolds which have $Y$ as Seifert base (see Theorem 6.15). Now Theorem 6.15 allows us to determine the topology of this connected component of the deformation space. When this is put together with our classification of the rigid Hitchin components in $\text{PGL}(4, \mathbb{R})$, we can produce many examples of closed 3-manifolds with a rigid real projective structure.
1.5. **Organization of the paper.** In Section 2, we define the Hitchin component for an orbifold $Y$ and the restriction map $j : \text{Hit}(\pi_1 Y, G) \to \text{Fix}_X \text{Hit}(\pi_1 X, G)$, where $Y$ is a quotient of the surface $X$ by an action of a finite group $\Sigma$, and prove that the map $j$ is injective. In addition, we prove Theorem 3. Section 3 describes the $\Sigma$-equivariant non-Abelian Hodge correspondence via three bijective maps: (i) between the representations of $\pi_1 Y$ into $G$ and $\Sigma$-equivariant flat $G$-bundles on $X$, (ii) between $\Sigma$-equivariant flat $G$-bundles on $X$ and $\Sigma$-equivariant harmonic bundles, and (iii) between $\Sigma$-equivariant harmonic bundles and $\Sigma$-equivariant Higgs bundles. In Section 4, we show that a section of the Hitchin fibration with the Hitchin base $B_X(g)$ is $\Sigma$-equivariant, and we show how this implies that the map $j$ is surjective. In Section 5, we describe the space of differentials for orbifolds, thus computing the dimension of the Hitchin components. This completes the proof of Theorem 3. In Section 6, we classify the Hitchin components of dimensions 0 and 1, and we present the applications to the study of the pressure metric on the Hitchin components for surface groups, to the Higgs bundles corresponding to a single differential and to the deformation theory of real projective structures on Seifert fibered 3-manifolds. Finally, in the Appendix we show that the dimensions of Hitchin components $\text{Hit}(\pi_1 Y, \text{PGL}(n, \mathbb{R}))$ are equal to the dimensions one can guess from examining a presentation of $\pi_1 Y$.

**Acknowledgments.** We thank Olivier Guichard, Qiongling Li, Anna Wienhard and Tengren Zhang for helpful conversations.

D. Alessandrini was supported by the DFG grant AL 1978/1-1 within the Priority Programme SPP 2026 “Geometry at Infinity”. G.-S. Lee was supported by the DFG research grant “Higher Teichmüller Theory”, by the European Research Council under ERC-Consolidator Grant 614733, and by the DFG grant LE 3901/1-1 within the Priority Programme SPP 2026 “Geometry at Infinity”. F. Schaffhauser was supported by *Convocatoria 2018-2019 de la Facultad de Ciencias (Uniandes), Programa de investigación “Geometría y Topología de los Espacios de Módulos”, the European Union’s Horizon 2020 research and innovation programme under grant agreement No 795222 and the University of Strasbourg Institute of Advanced Study (USIAS).

The authors acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 “RNMS: Geometric structures And Representation varieties” (the GEAR Network).

2. **Hitchin representations for orbifolds**

2.1. **Hyperbolic 2-orbifolds.** For background on orbifolds, we refer for instance to [Thu79, Sco83, CHK00, CG05]. Let $Y$ be a closed connected smooth orbifold of dimension 2. Recall that a singularity of $Y$ is a point $y \in Y$ of one of the following three types:

1. A cone point of order $m$ if $y$ has a neighborhood isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \backslash \mathbb{R}^2$, where $\mathbb{Z}/m\mathbb{Z}$ acts on $\mathbb{R}^2$ via a rotation of angle $\frac{2\pi}{m}$;
2. A mirror point if $y$ has a neighborhood isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \backslash \mathbb{R}^2$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ via a reflection though a line;
3. A corner reflector (or dihedral point) of order $n$ if $y$ has a neighborhood isomorphic to $D_n \backslash \mathbb{R}^2$, where the action of the dihedral group $D_n \simeq (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ on $\mathbb{R}^2$ is generated by the reflections through two lines with angle $\frac{\pi}{n}$ between them.

In the rest of the paper, for an orbifold $Y$, we will denote by $k$ the number of cone points (of respective orders $m_1, \ldots, m_k$) and by $\ell$ the number of corner reflectors (of respective orders $n_1, \ldots, n_\ell$). We will denote by $\tilde{Y}$ the orbifold universal cover of $Y$; recall that $\tilde{Y}$ is necessarily simply connected as a topological space but, in general, it may have non-trivial orbifold structure. We will denote by $\pi_1 Y$ the orbifold fundamental group of $Y$, defined as the group of deck transformation of $\tilde{Y}$:

$$\pi_1 Y := \text{Aut}_Y(\tilde{Y}).$$

The underlying topological space $|Y|$ of a 2-orbifold $Y$ is always homeomorphic to a compact surface, which has boundary if and only if $Y$ has mirror points, in which case $\partial |Y|$ is the set of mirror points and corner reflectors of $Y$. A 2-orbifold $Y$ is called orientable if $|Y|$ is orientable and $Y$ has only cone points as singularities. For instance, the universal orbifold covering $\tilde{Y}$ is always orientable as an orbifold. Recall that $Y$ may be non-orientable as an orbifold even though $|Y|$ is an orientable surface (this happens if and only if $|Y|$ is an orientable topological surface with non-empty boundary). Note that the setting that we have just described includes the case where $|Y|$ is non-orientable as a topological surface (possibly with boundary).
In particular, our results will hold for non-orientable surfaces with trivial orbifold structure (or with only mirror points as orbifold singularities).

![Fig. 1](image_url)

**Figure 1.** Some closed 2-dimensional orbifolds.

We shall assume throughout that $Y$ has negative (orbifold) Euler characteristic, i.e. the rational number $\chi(Y)$ defined below is strictly negative:

\begin{equation}
\chi(Y) := \chi(Y) - \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) - \frac{1}{2} \sum_{j=1}^{\ell} \left(1 - \frac{1}{n_j}\right) < 0.
\end{equation}

Every orbifold of negative Euler characteristic can always be seen as a quotient of a closed orientable surface. This is a classical result, but, for completeness, we provide a proof here, because this fact is a crucial technical tool in this paper.

**Definition 2.1.** A presentation of a closed connected orbifold $Y$ is a triple $(X, \Sigma, \varphi)$, where $X$ is a closed connected orientable surface, $\Sigma$ a finite subgroup of Diff$(X)$, and $\varphi$ an orbifold isomorphism:

$$\varphi : Y \longrightarrow \Sigma \backslash X.$$ 

In other words, a presentation is a finite, Galois, orbifold cover $p : X \longrightarrow Y$ of $Y$ by a closed, connected, orientable surface $X$. In the following, to keep the notation more compact, we will denote a presentation $(X, \Sigma, \varphi)$ simply by $Y \cong [\Sigma \backslash X]$, leaving $\varphi$ implicit.

**Proposition 2.2.** Let $Y$ be a closed connected 2-orbifold of negative Euler characteristic. Then $Y$ admits a presentation $Y \cong [\Sigma \backslash X]$. Moreover, we can always assume that $X$ is a Riemann surface and that $\Sigma$ acts on $X$ by holomorphic or anti-holomorphic maps.

**Proof.** By a theorem due to Thurston [Thu79], if $Y$ is a closed 2-orbifold of negative Euler characteristic, it admits at least one hyperbolic structure. We fix one of these hyperbolic structures on $Y$. This implies that the orbifold universal cover $\tilde{Y}$ is orbifold isomorphic to $H^2_\mathbb{H}$ (which has trivial orbifold structure), and that $\pi_1 Y$ acts on $H^2_\mathbb{H}$ via hyperbolic isometries. This gives a representation $\pi_1 Y \rightarrow PGL(2, \mathbb{H})$.

It follows from the orbifold version of the Seifert–van Kampen theorem ([Sco83]) that $\pi_1 Y$ is finitely generated for $Y$ closed. So Selberg’s lemma (see for instance [Rat94]) ensures the existence of a torsion-free normal subgroup of finite index $\Gamma' \leq \pi_1 Y$. Let then $\Gamma' \rightarrow Y$ be the orientation cover of $Y$ (in particular, $\pi_1 Y' \rightarrow PSL(2, \mathbb{R})$) and consider $\Gamma := \Gamma' \cap \pi_1 Y'$. As $\pi_1 Y'$ is of index at most 2 in $\pi_1 Y$, the group $\Gamma$ is still a torsion-free normal subgroup of finite index of $\pi_1 Y$ and the connected surface $X := \Gamma \backslash H^2_\mathbb{H}$ satisfies all the requirements of the proposition.

Moreover, $X$ inherits a hyperbolic structure from $H^2_\mathbb{R}$. This gives a Riemann surface structure on $X$ with the required properties. \[\square\]

The proposition implies the existence of a short exact sequence:

\begin{equation}
1 \longrightarrow \pi_1 X \longrightarrow \pi_1 Y \longrightarrow \Sigma \longrightarrow 1.
\end{equation}

The group homomorphism

$$\Sigma \subset Diff(X) \longrightarrow MCG(X) \simeq \text{Out}(\pi_1 X)$$

taking a transformation $\sigma : X \rightarrow X$ to its (extended) mapping class coincides, through the Dehn-Nielsen-Baer theorem, with the canonical group homomorphism $\Sigma \rightarrow \text{Out}(\pi_1 X)$ induced by the short exact sequence (2.2). In general, it does not lift to a group homomorphism $\Sigma \rightarrow Aut(\pi_1 X)$ (it does, though, if $\Sigma$ happens to have a global fixed point in $X$, in which case the short exact sequence (2.2) splits and $\pi_1 Y$ is isomorphic, non-canonically in general, to the semi-direct product $\pi_1 X \rtimes \Sigma$; this fact will be used in Remark 5.8).
2.2. Principal representation. Let \( g \) be a (real or complex) semisimple Lie algebra. Recall that the adjoint representation \( \text{ad} : g \rightarrow \text{End}(g) \) is faithful. Its image \( \text{ad}(g) \) is a subalgebra of \( \text{End}(g) \) isomorphic to \( g \). The adjoint group of \( g \), denoted by \( \text{Int}(g) \), is defined as the connected Lie subgroup of \( \text{GL}(g) \) whose Lie algebra is \( \text{ad}(g) \), a centerless group. In the rest of the paper, when \( g_C \) is a complex semisimple Lie algebra, we will denote its adjoint group by \( G_C := \text{Int}(g_C) \). A real form of \( g_C \) is a real Lie subalgebra \( g \subset g_C \) which is the set of fixed points of a real involution \( \tau : g_C \rightarrow g_C \). The involution \( \tau \) also induces an involution on \( \text{Int}(g_C) \). We will denote by \( G \) the group \( G = \text{Int}(g_C)^\tau \subset G_C \) consisting of all the inner automorphisms of \( g_C \) that commute with \( \tau \). The group \( G \) is a real semisimple Lie group with Lie algebra \( g \). It has trivial center, but it is not connected in general. Its identity component is the group \( \text{Int}(g) \).

Example 2.3. Here are some examples:

- If \( g = \mathfrak{sl}(n, \mathbb{R}) \), then \( G_C \simeq \text{PSL}(n, \mathbb{C}) \) and \( G \simeq \text{PGL}(n, \mathbb{R}) \simeq \text{PSL}^\pm(n, \mathbb{R}) \), which is connected if and only if \( n \) is odd, and \( \text{Int}(g) \simeq \text{PSL}(n, \mathbb{R}) \) for all \( n \). Here, for each subgroup \( H \) of \( \text{GL}(n, \mathbb{K}) \), we denote by \( \text{PGL}(H) \) the projectivization of \( H \), i.e., \( \text{PGL}(H) = H/(H \cap \mathbb{C}) \) with \( C \) the center of \( \text{GL}(n, \mathbb{K}) \), and \( \text{SL}^\pm(n, \mathbb{R}) = \{ A \in \text{GL}(n, \mathbb{R}) \mid \det(A) = \pm 1 \} \).
- If \( g = \mathfrak{sp}(2m, \mathbb{R}) \), then \( G_C \simeq \text{PSp}(2m, \mathbb{C}) \) and \( G \simeq \text{PSp}^\pm(2m, \mathbb{R}) \), which has two connected components. We recall that given a symplectic form \( \omega \) on \( \mathbb{R}^{2m} \), we have:
  \[
  \text{Sp}^\pm(2m, \mathbb{R}) = \{ A \in \text{GL}(2m, \mathbb{R}) \mid A^T\omega A = \pm \omega \}.
  \]
- If \( g = \mathfrak{so}(p, q) \), then \( G_C \simeq \text{PO}(p + q, \mathbb{C}) \) and \( G \simeq \text{PO}^\pm(p, q) \), which is always disconnected. Again recall that given a non-degenerate bilinear form \( J \) of signature \( (p, q) \), we have:
  \[
  \mathbb{O}^\pm(p, q) = \{ A \in \text{GL}(p + q, \mathbb{R}) \mid A^TJA = \pm J \}.
  \]

Let us assume, from now on, that \( g \) is the split real form of a complex simple Lie algebra \( g_C \), defined by an involution \( \tau \). As in [Hit92], we can choose a principal 3-dimensional subalgebra \( \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow g_C \) such that \( \mathfrak{sl}(2, \mathbb{C}) \) is \( \tau \)-invariant and induces a subalgebra \( \mathfrak{sl}(2, \mathbb{R}) \hookrightarrow g \). Denote by

\[
(2.3) \quad \kappa_C : \text{PGL}(2, \mathbb{C}) \simeq \text{Int}(\mathfrak{sl}(2, \mathbb{C})) \rightarrow G_C
\]

the induced group homomorphism, and let

\[
(2.4) \quad \kappa : \text{PGL}(2, \mathbb{R}) \rightarrow G
\]

be its restriction to the subgroup \( \text{PGL}(2, \mathbb{R}) < \text{PGL}(2, \mathbb{C}) \). We will call \( \kappa \) the principal representation of \( \text{PGL}(2, \mathbb{R}) \) in \( G \). In this paper we use that the representation \( \kappa \) is defined on the whole group \( \text{PGL}(2, \mathbb{R}) \).

In the examples discussed above (Example 2.3), the principal representation \( \kappa \) can be described explicitly. Consider the \( n \)-dimensional vector space \( H_{n-1} \) of homogeneous polynomials of degree \( n-1 \) in two variables \( X \), \( Y \). A matrix \( A \in \text{GL}(2, \mathbb{R}) \) induces a linear map \( \tilde{\kappa}(A) \) that sends a polynomial \( P(X, Y) \in H_{n-1} \) to the polynomial \( P(A^{-1} \cdot (X, Y)) \). This gives an explicit irreducible representation \( \tilde{\kappa} : \text{GL}(2, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R}) \) whose projectivization is conjugate to the principal representation \( \kappa : \text{PGL}(2, \mathbb{R}) \rightarrow \text{PGL}(n, \mathbb{R}) \). In this way, we can see that \( \kappa \) makes the Veronese embedding

\[
\mathbb{RP}^1 \rightarrow \mathbb{P}(H_{n-1}) \quad \left[ a:b \right] \mapsto \left[(aX - bY)^{n-1}\right]
\]

\( \text{PGL}(2, \mathbb{R}) \)-equivariant. If \( n = 2m \) is even, the image of \( \tilde{\kappa} \) is contained in \( \text{Sp}^\pm(2m, \mathbb{R}) \), and if \( n = 2m + 1 \) is odd, it is contained in \( \text{O}(m, m + 1) \). If \( n = 7 \), then the projective image of \( \tilde{\kappa} \) is contained in \( G_2 \). So the projectivization of \( \tilde{\kappa} \) is an explicit model for the principal representation in \( \text{PGL}(n, \mathbb{R}), \text{PSp}^\pm(2m, \mathbb{R}), \text{PO}(m, m + 1) \) and \( G_2 \). The principal representation in \( \text{PO}^\pm(m, m) \) is given by the composition of \( \kappa : \text{PGL}(2, \mathbb{R}) \rightarrow \text{PO}(m - 1, m) \) with the block embedding \( \text{PO}(m - 1, m) \hookrightarrow \text{PO}^\pm(m, m) \).

\[\text{6}\]
2.3. Hitchin representations. Thurston ([Thu79]) studied the space of isotopy classes of hyperbolic structures on a closed 2-orbifold of negative Euler characteristic, called the Teichmüller space of $Y$ and denoted by $T(Y)$. The map sending a hyperbolic structure to its holonomy representation induces a homeomorphism between $T(Y)$ and a connected component of the representation space:

$$\text{Rep}(\pi_1Y, \text{PGL}(2, \mathbb{R})) := \text{Hom}(\pi_1Y, \text{PGL}(2, \mathbb{R}))/\text{PGL}(2, \mathbb{R})$$

This connected component consists exactly of the $\text{PGL}(2, \mathbb{R})$-conjugacy classes of discrete and faithful representations from $\pi_1Y$ to $\text{PGL}(2, \mathbb{R}) \simeq \text{Isom}(\mathbb{H}^2)$. Such representations are usually called Fuchsian representations and, in what follows, we will constantly identify $T(Y)$ with the space of the conjugacy classes of Fuchsian representations. Thurston proved that $T(Y)$ is homeomorphic to an open ball of dimension:

$$-\chi(|Y|) \dim \text{PGL}(2, \mathbb{R}) + 2k + \ell = -3\chi(|Y|) + 2k + \ell.$$ 

Let $g$ be the split real form of a complex simple Lie algebra $g_C$ and let $G$ be defined as above. In this paper we will study the Hitchin component, a connected component of the representation space

$$\text{Rep}(\pi_1Y, G) := \text{Hom}(\pi_1Y; G)/G$$

that generalizes the Teichmüller space. The first step is to use the principal representation to define Fuchsian representations taking values in $G$.

**Definition 2.4** (Fuchsian representations). A group homomorphism $\varrho : \pi_1Y \to G$ is called a Fuchsian representation if there is a discrete, faithful representation $h : \pi_1Y \to \text{PGL}(2, \mathbb{R})$ such that $\kappa \circ h = \varrho$, where $\kappa$ is the principal representation from (2.4).

Definition 2.4 says that a representation $\varrho : \pi_1Y \to G$ is Fuchsian if and only if there exists a hyperbolic structure on $Y$ whose holonomy representation $h$ makes the following diagram commute.

\[
\begin{array}{ccc}
\text{PGL}(2, \mathbb{R}) & \xrightarrow{\kappa} & G \\
\pi_1Y & \xrightarrow{\kappa} & \text{Hom}(\pi_1Y; G) \\
& \searrow & \downarrow \kappa \\
& h & \varrho
\end{array}
\]

In particular, as $\chi(Y) < 0$, there exist Fuchsian representations of $\pi_1Y$. The set of $G$-conjugacy classes of Fuchsian representations is called the Fuchsian locus of $\text{Rep}(\pi_1Y; G)$. This defines a continuous map (which is actually injective, see Corollary 2.12)

$$T(Y) \to \text{Rep}(\pi_1Y, G)$$

(2.5)

from the Teichmüller space onto the Fuchsian locus of the representation space. Since $T(Y)$ is connected, the Fuchsian locus is contained in a well-defined connected component of $\text{Rep}(\pi_1Y; G)$ called the Hitchin component and denoted by $\text{Hit}(\pi_1Y, G)$. For instance, $\text{Hit}(\pi_1Y, \text{PGL}(2, \mathbb{R})) \simeq T(Y)$. As any two principal $3$-dimensional subalgebras $\mathfrak{sl}(2, \mathbb{C}) \subset G_C$ are related by an interior automorphism of $G_C$ (see [Kos59]), the map (2.5) does not depend on that particular choice in the construction.

**Definition 2.5** (Hitchin representation). A group homomorphism $\varrho : \pi_1Y \to G$ is called a Hitchin representation if its $G$-conjugacy class is an element of the Hitchin component $\text{Hit}(\pi_1Y, G)$.

**Remark 2.6.** It follows from the definition of a Fuchsian representation that if $Y$ is orientable (for instance, if $Y = X$ is a closed orientable surface), then any Fuchsian representation of $\pi_1Y$ in $G$ is in fact contained in $\text{Hom}(\pi_1Y; G_0)$, where $G_0$ is the identity component of $G$ (because the holonomy representation of any hyperbolic structure on an orientable orbifold is contained in $\text{PSL}(2, \mathbb{R})$). If we consider such representations up to $G_0$-conjugacy, it may happen that there are two connected components of $\text{Hom}(\pi_1Y; G_0)/G_0$ containing conjugacy classes of Fuchsian representations, but these are related by an interior automorphism of $G$. This happens for instance if $g = \mathfrak{sl}(2, \mathbb{R})$, in which case $G \simeq \text{PGL}(2, \mathbb{R})$ and $G_0 \simeq \text{PSL}(2, \mathbb{R})$. If we want to have potentially more than one Hitchin component in $\text{Hom}(\pi_1Y; G)/G$, we can replace the Teichmüller space $T(Y) \simeq \text{Hom}^{\text{d.f.}}(\pi_1Y; \text{PGL}(2, \mathbb{R}))/\text{PGL}(2, \mathbb{R})$ by the representation space with two connected components $\text{Hom}^{\text{d.f.}}(\pi_1Y; \text{PGL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R})$. Here, we denote by $\text{Hom}^{\text{d.f.}}(\pi_1Y; \text{PGL}(2, \mathbb{R}))$ the subspace of $\text{Hom}(\pi_1Y; \text{PGL}(2, \mathbb{R}))$ consisting of discrete and faithful representations.
Remark 2.7. As the morphism $\kappa : \text{PGL}(2, \mathbb{R}) \to \text{PGL}(n, \mathbb{R})$ has image contained in $\text{PSp}^\pm(2m, \mathbb{R})$ if $n = 2m$, $\text{PO}(m, m + 1)$ if $n = 2m + 1$ and $G_2$ if $n = 7$, for any orbifold $Y$ we have maps:

$$\text{Hit}(\pi_Y, \text{PSp}^\pm(2m, \mathbb{R})) \to \text{Hit}(\pi_Y, \text{PGL}(2m, \mathbb{R}))$$

$$\text{Hit}(\pi_Y, \text{PO}(m, m + 1)) \to \text{Hit}(\pi_Y, \text{PGL}(2m + 1, \mathbb{R}))$$

$$\text{Hit}(\pi_Y, G_2) \to \text{Hit}(\pi_Y, \text{PO}(3, 4)) \to \text{Hit}(\pi_Y, \text{PGL}(7, \mathbb{R}))$$

If $Y = X$ is a closed orientable surface, it is a consequence of Hitchin’s parametrization \([\text{Hit92}]\) recalled in Section 5 that these maps are injective. For the same reason, $\mathcal{T}(X) \simeq \text{Hit}(\pi_X, \text{PGL}(2, \mathbb{R}))$ embeds into each $\text{Hit}(\pi_X, X, G)$.

2.4. Restriction of Hitchin representations. Assume now that $Y \simeq [\Sigma \backslash X]$ is a presentation of $Y$ (which always exists by Proposition 2.2). In particular, $\pi_X < \pi_Y$ with finite index and $\Sigma \simeq \pi_Y / \pi_X$. The restriction of a representation to a subgroup gives a map:

$$j : \text{Rep}(\pi_Y, X, G) \to \text{Rep}(\pi_X, X, G)$$

$$[\varrho] \mapsto [\varrho|_{\pi_X}] .$$

Recall that there is a canonical group homomorphism $\Sigma \to \text{Out}(\pi_X)$ and that $\text{Out}(\pi_X)$ acts on $\text{Rep}(\pi_X, X, G)$. We will denote by $\text{Fix}_\Sigma(\text{Rep}(\pi_X, X, G))$ the fixed locus of this action.

**Lemma 2.8.** The image of the map $j$ is contained in $\text{Fix}_\Sigma(\text{Rep}(\pi_X, X, G))$.

*Proof.* Take $\sigma \in \Sigma$ and choose a lift $\gamma \in \pi_Y$. If $\varrho : \pi_Y \to G$ is a representation, then $\sigma \cdot [\varrho|_{\pi_X}]$ is, by definition, the $G$-conjugacy class of the representation:

$$\sigma \cdot [\varrho]|_{\pi_X} : \pi_X \to G$$

$$\delta \mapsto \varrho|_{\pi_X}(\gamma^{-1} \delta \gamma).$$

As $\varrho|_{\pi_X}(\gamma^{-1} \delta \gamma) = \varrho(\gamma)^{-1} \varrho|_{\pi_X}(\delta) \varrho(\gamma)$ with $\varrho(\gamma) \in G$, we have indeed that $\sigma \cdot [\varrho]|_{\pi_X}$ lies in the $G$-conjugacy class of $[\varrho]|_{\pi_X}$.

Note that the formula $(\gamma \cdot \varrho)(\delta) := \varrho(\gamma)^{-1} \varrho(\delta) \varrho(\gamma)$ indeed defines a left action of $\pi_Y$ on $\text{Hom}(\pi_Y; G)$ because

$$(\gamma_1 \cdot (\gamma_2 \cdot \varrho))(\delta) = (\gamma_2 \cdot \varrho)(\gamma_1)^{-1} (\gamma_2 \cdot \varrho)(\delta) (\gamma_2 \cdot \varrho)(\gamma_1).$$

In general, the map

$$(2.6) \quad j : \text{Rep}(\pi_Y, X, G) \to \text{Fix}_\Sigma(\text{Rep}(\pi_X, X, G))$$

defined by means of Lemma 2.8 is neither injective nor surjective. A crucial observation of the present paper is that if we restrict to Hitchin components, $j$ induces a bijective map.

**Lemma 2.9.** If $\varrho : \pi_Y \to G$ is a Hitchin representation and $Y' \to Y$ is a finite orbifold cover, then $\varrho|_{\pi_Y} : \pi_Y \to G$ is a Hitchin representation.

*Proof.* If $\varrho : \pi_Y \to G$ is a Fuchsian representation, then, for every finite orbifold cover $Y' \to Y$, the representation $\varrho|_{\pi_Y}$ is also Fuchsian. As Hitchin components are connected, this implies the statement. \(\square\)

**Lemma 2.9** implies that $j(\text{Hit}(\pi_Y, X, G)) \subset \text{Hit}(\pi_X, X, G)$. Moreover, the group $\text{Out}(\pi_X, X, G)$ preserves the Fuchsian locus, therefore it also preserves the Hitchin component. We denote the fixed locus of this action by $\text{Fix}_\Sigma(\text{Hit}(\pi_X, X, G))$. Hence we have a map:

$$j : \text{Hit}(\pi_Y, X, G) \to \text{Fix}_\Sigma(\text{Hit}(\pi_X, X, G)).$$

To prove that the map $j$ is injective, we will need the following lemma.

**Lemma 2.10.** Let $\varrho : \pi_Y \to G$ be a Hitchin representation. Then $\varrho$ is $G_\Sigma$-strongly irreducible, meaning that its restriction to every finite index subgroup is $G_\Sigma$-irreducible. Moreover, $\varrho$ has trivial centralizer in $G$ and in $G_\Sigma$, i.e., if an element $g \in G_\Sigma$ satisfies $gg(\gamma) = g(\gamma)g$ for every $\gamma \in \pi_Y$ then $g$ is the identity.

Recall that for a (real or complex) reductive Lie group $H$, a representation is $H$-irreducible if its image is not contained in any parabolic subgroup of $H$. When $G = \text{PGL}(n, \mathbb{R})$ or $\text{PGL}(n, \mathbb{C})$, this is equivalent to the well-known definition. As expected, being $G_\Sigma$-irreducible implies being $G$-irreducible.
Proof of Lemma 2.10. Choose a presentation $Y \simeq [\Sigma \backslash X]$, and consider $\varrho|_{\pi_1 X}$. Hitchin proved in [Hit92] Section 5 that the Higgs bundles in the Hitchin components are smooth points of the moduli space of $G_C$-Higgs bundles, and hence these Higgs bundles are $G_C$-stable and simple. By the non-abelian Hodge correspondence, this implies that the representation $\varrho|_{\pi_1 X}$ is $G_C$-irreducible and has trivial centralizer in $G_C$. The same properties therefore hold for $\varrho$. Moreover, if $\Gamma' < \pi_1 Y$ is a finite index subgroup, then $\Gamma'$ is the orbifold fundamental group of a finite orbifold covering $Y'$, and by Lemma 2.9 we see that the restriction to $\Gamma'$ is still $G_C$-irreducible. □

Proposition 2.11. The map $j : \text{Hit}(\pi_1 Y, G) \rightarrow \text{Fix}_G(\text{Hit}(\pi_1 X, G))$ is injective.

Proof. Let $\varrho_1, \varrho_2$ be two Hitchin representations of $\pi_1 Y$ into $G$ such that $\varrho_1|_{\pi_1 Y}$ and $\varrho_2|_{\pi_1 Y}$ are $G$-conjugate. Replacing $\varrho_2$ by $\text{Int}_g \circ \varrho_2$ for some $g \in G$ if necessary, we may assume that $\varrho_1|_{\pi_1 X}$ and $\varrho_2|_{\pi_1 X}$ are equal. The abstract situation (compare [LR99, Lemma 3.1]) is then as follows: we have a normal subgroup $N < \Gamma$ and two representations $\varrho_1, \varrho_2 : \Gamma \rightarrow G$ such that $\varrho_1|_N = \varrho_2|_N =: \varrho$ has trivial centralizer in $G$. For a fixed $\gamma \in \Gamma$, consider the representation $N \rightarrow G$ defined, for all $n \in N$, by

$$n \mapsto \varrho_1(\gamma^{-1}) \varrho_2(\gamma) \varrho(n) \varrho_2(\gamma^{-1}) \varrho_1(\gamma).$$

This is equal to

$$\varrho_1(\gamma^{-1}) \varrho_2(\gamma \varrho \gamma^{-1}) \varrho_1(\gamma) = \varrho(n)$$

because $\gamma \varrho \gamma^{-1} \in N < \Gamma$. Hence $\varrho_1(\gamma^{-1}) \varrho_2(\gamma)$ centralizes the Hitchin representation $\varrho$, and by Lemma 2.10 it is the identity. Thus, $\varrho_1(\gamma) = \varrho_2(\gamma)$ for all $\gamma \in \pi_1 Y$.

Corollary 2.12. Let $\varrho_1, \varrho_2 : \pi_1 Y \rightarrow G$ be two Fuchsian representations: $\varrho_1 = \kappa \circ h_1$ and $\varrho_2 = \kappa \circ h_2$ where $h_1, h_2 : \pi_1 Y \rightarrow \text{PGL}(2, \mathbb{R})$ are discrete and faithful representations. If $\varrho_1$ and $\varrho_2$ are $G$-conjugate, then $h_1$ and $h_2$ are $\text{PGL}(2, \mathbb{R})$-conjugate. Equivalently, the Teichmüller space $\mathcal{T}(Y)$ of the orbifold $Y$ embeds onto the Fuchsian locus of $\text{Rep}(\pi_1 Y, G)$ through the map (2.5).

Proof. Let $Y \simeq [\Sigma \backslash X]$. Recall that $\mathcal{T}(Y) \simeq \text{Hit}(\pi_1 Y, \text{PGL}(2, \mathbb{R}))$, and similarly for $X$. The map $h \mapsto h|_{\pi_1 X}$ then induces a commutative diagram

$$\begin{array}{ccc}
\mathcal{T}(Y) & \longrightarrow & \text{Fix}_G\mathcal{T}(X) \\
\downarrow & & \downarrow \\
\text{Hit}(\pi_1 Y, G) & \longrightarrow & \text{Fix}_G\text{Hit}(\pi_1 X, G)
\end{array}$$

whose vertical arrows are induced by composition by the principal representation $\kappa : \text{PGL}(2, \mathbb{R}) \rightarrow G$ and whose horizontal arrows are injective, by Proposition 2.11 (as a matter of fact, we only need the injectivity of the top one). Since the vertical arrow $\mathcal{T}(X) \rightarrow \text{Hit}(\pi_1 X, G)$ is injective (see [Hit92] and Remark 2.7), it follows that so is the vertical arrow $\mathcal{T}(Y) \rightarrow \text{Hit}(\pi_1 Y, G)$. □

Theorem 2.13. Let $Y$ be a closed connected 2-orbifold of negative Euler characteristic. Let $\mathfrak{g}$ be the split real form of a complex simple Lie algebra and let $G$ be the group of real points of $\text{Int}(\mathfrak{g} \otimes \mathbb{C})$. Given a presentation $Y \simeq [\Sigma \backslash X]$, the map $\varrho \mapsto \varrho|_{\pi_1 X}$ induces a homeomorphism

$$j : \text{Hit}(\pi_1 Y, G) \rightarrow \text{Fix}_G(\text{Hit}(\pi_1 X, G))$$

between the Hitchin component of $\text{Rep}(\pi_1 Y, G)$ and the $\Sigma$-fixed locus in $\text{Hit}(\pi_1 X, G)$.

The injectivity was proved in Proposition 2.11. We postpone the proof of surjectivity to Section 4.2.

Corollary 2.14. Let $Y' \rightarrow Y$ be a finite Galois cover of $Y$ and let $\Sigma' := \pi_1 Y/\pi_1 Y'$. Then the map $\varrho \mapsto \varrho|_{\pi_1 Y'}$ induces a homeomorphism:

$$\text{Hit}(\pi_1 Y, G) \simeq \text{Fix}_G\text{Hit}(\pi_1 Y', G).$$
Proof. Let $X$ be a finite Galois cover of $Y$ by a closed orientable surface. By pulling back this cover to $Y'$ if necessary, we can assume that $X$ is a (finite and Galois) cover of $Y'$. Then, by Theorem 2.13 one has:

\[ \text{Hit}(\pi_1 Y, G) \simeq \text{Fix}_{\pi_1 Y/\pi_1 X} \text{Hit}(\pi_1 X, G) \simeq \text{Fix}_{\pi_1 Y'/\pi_1 Y} \left( \text{Fix}_{\pi_1 Y/\pi_1 X} \text{Hit}(\pi_1 X, G) \right) \]

which is homeomorphic to $\text{Fix}_{\pi_1 Y'} \text{Hit}(\pi_1 Y'; G)$, again by Theorem 2.13. \qed

**Corollary 2.15.** Let $\varphi : \pi_1 Y \rightarrow G$ be a representation and let $Y' \rightarrow Y$ be a finite cover of $Y$, not necessarily Galois. Then $\varphi$ is Hitchin if and only if $\varphi|_{\pi_1 Y'}$ is Hitchin.

**Proof.** The obvious direction of the corollary is given by Lemma 2.9. Conversely, assume that $\varphi : \pi_1 Y \rightarrow G$ satisfies that $\varphi|_{\pi_1 Y'}$ is Hitchin. Let $Y''$ be a finite Galois cover of $Y$ that covers $Y'$ (again, this may be obtained by pulling back of a finite Galois cover of $Y$). By Lemma 2.9 the representation $\varphi|_{\pi_1 Y''}$ is Hitchin. And by 2.6, $\varphi|_{\pi_1 Y''}$ lies in the fixed-point set of $\pi_1 Y/\pi_1 Y''$ in $\text{Rep}(\pi_1 Y''; G)$. Therefore, Corollary 2.14 shows that $\varphi$ is Hitchin. \qed

2.5. Properties of Hitchin representations for orbifolds. We end Section 2 with a series of properties satisfied by Hitchin representations of orbifold fundamental groups in $\text{PGL}(n, \mathbb{R})$ that directly generalize known ones for fundamental groups of closed orientable surfaces (strong irreducibility, discreteness, faithfulness, hyperconvexity). The first three are all simple consequences of the fact that $\pi_1 Y$ contains the fundamental group of a closed orientable surface $X$ as a normal subgroup of finite index (Proposition 2.2). For the remaining one, we apply Theorem 2.13. We shall assume that $G \simeq \text{PGL}(n, \mathbb{R})$ until the end of this section.

**Remark 2.16.** By Remark 2.7, the results of this subsection also apply to Hitchin representations in the groups $\text{PSp}^\pm(2m, \mathbb{R})$, $\text{PO}(m, m + 1)$ and $G_2$, since they are special cases of Hitchin representations in $\text{PGL}(n, \mathbb{R})$.

We refer to [GW12, Definition 2.10] for the definition of Anosov representations.

**Proposition 2.17.** Let $B$ be a Borel subgroup of $\text{PGL}(n, \mathbb{R})$. Then every Hitchin representation $\varphi : \pi_1 Y \rightarrow \text{PGL}(n, \mathbb{R})$ is $B$-Anosov.

**Proof.** Choose a presentation $Y \simeq [\Sigma \setminus X]$. Labourie [Lab06] proved that $\varphi|_{\pi_1 X}$ is $B$-Anosov. Now [GW12, Corollary 3.4] implies that $\varphi$ is also $B$-Anosov since $\pi_1 X$ is a finite index subgroup of $\pi_1 Y$. \qed

**Corollary 2.18.** Every Hitchin representation $\varphi : \pi_1 Y \rightarrow \text{PGL}(n, \mathbb{R})$ has discrete image.

**Proof.** By [GW12, Theorem 5.3], all Anosov representations have this property. \qed

Moreover, the theory of domains of discontinuity, developed by Guichard and Wienhard [GW12] and by Kapovich, Leeb and Porti [KLP18], can be applied to Hitchin representations of orbifold groups. An important special case is studied in detail in Section 6.5.

**Remark 2.19.** We can also prove Corollary 2.18 as follows. Let $\varphi$ be a Hitchin representation of $\pi_1 Y$. By [Lab06, Lemma 10.4], $\varphi|_{\pi_1 X}$ has discrete image in $G$. Being a subgroup of $G$, $\text{Im} \varphi|_{\pi_1 X}$ is therefore closed in $G$. Since $\pi_1 X$ has finite index in $\pi_1 Y$, the group $\pi_1 Y$ is the union of finitely many cosets of $\pi_1 X$, so the image of $\pi_1 Y$ under $\varphi$ is a finite union of closed discrete subsets of the Lie group $G$; therefore, $\text{Im} \varphi$ is discrete.

**Proposition 2.20.** Every Hitchin representation $\varphi : \pi_1 Y \rightarrow \text{PGL}(n, \mathbb{R})$ is faithful.

**Proof.** Let $\varphi$ be a Hitchin representation of $\pi_1 Y$. By [Lab06, Proposition 3.4], elements of $(\text{Im} \varphi|_{\pi_1 X}) \setminus \{1\}$ in $\text{PGL}(n, \mathbb{R})$ are diagonalizable with distinct, real eigenvalues. In particular, $\varphi|_{\pi_1 X}$ is faithful. Consider now $\gamma \in \pi_1 Y$. Since $\pi_1 Y/\pi_1 X$ is a finite group, there exists a minimal integer $q$ such that $\gamma^q \in \pi_1 X$. If $\gamma^q \neq 1_{\pi_1 X}$, then by Labourie’s result, $\varphi(\gamma)^q = \varphi(\gamma^q) \neq 1$ in $\text{PGL}(n, \mathbb{R})$. In particular, $\varphi(\gamma) \neq 1$. If $\gamma^q = 1_{\pi_1 X}$, then (as $\varphi(\gamma)^q = \varphi(\gamma^q) = 1$) the eigenvalues of $\varphi(\gamma)$, as an endomorphism of $\mathfrak{g}$, are all $q$-th roots of unity. Since those form a finite (in particular, discrete) subset, the latter is invariant by continuous deformation of $\varphi$. Let us then consider a Fuchsian representation $\varphi' : \pi_1 Y \rightarrow \text{PGL}(n, \mathbb{R})$. By what we have just said, $\varphi'(\gamma)$ and $\varphi(\gamma)$ have the same eigenvalues. Since $\gamma$ is not trivial in $\pi_1 Y$, the element $\varphi'(\gamma)$ is not trivial in $\text{PGL}(n, \mathbb{R})$, so it has an eigenvalue that is not equal to 1. Therefore $\varphi(\gamma)$ also has an eigenvalue that is not equal to 1. In particular, $\varphi(\gamma) \neq 1$. \qed
Remark 2.21. Wienhard [Wie18] gave a definition of Higher Teichmüller spaces as a union of connected components of the representation space $\text{Rep}(\pi_1(X); G)$ where each representation is discrete and faithful. In her definition, the group $\pi_1(X)$ is the fundamental group of an orientable surface. If we want to generalize her definition to orbifold groups, then Corollary 2.18 and Proposition 2.20 say that the Hitchin components for orbifold groups give new examples of Higher Teichmüller spaces.

Proposition 2.22. If $\varrho : \pi_1 Y \longrightarrow \text{PGL}(n, \mathbb{R})$ is a Hitchin representation, then for all $\gamma$ of infinite order in $\pi_1 Y$, the element $\varrho(\gamma)$ of $\text{PGL}(n, \mathbb{R})$ is purely loxodromic (i.e. diagonalizable with distinct real eigenvalues).

Proof. In the case of a closed orientable surface $X$, all elements of $\pi_1 X$ are of infinite order and Labourie has shown in [Lab06] that their image under a Hitchin representation is purely loxodromic. If $Y = [\Sigma \setminus X]$ with $\Sigma$ finite, then for all $\gamma \in \pi_1 Y$ there exists $q \geq 1$ such that $\gamma^q \in \pi_1 X$ and if $\gamma$ is of infinite order, then $\gamma^q \neq 1$ in $\pi_1 X$. So $\varrho(\gamma)^q = \varrho(\gamma^q)$ is purely loxodromic. Therefore, so is $\varrho(\gamma)$.

However, if $\gamma$ is of finite order in $\pi_1 Y$, then $\varrho(\gamma) \in G$ may have non-distinct eigenvalues, as we can already see from the case $G = \text{PGL}(3, \mathbb{R})$, where $\kappa$ takes the rotation matrix of angle $\tfrac{\pi}{2}$, say, to a matrix conjugate to the diagonal matrix diag($-1, 1, -1$). This happens for instance if $Y$ has a cone point of angle $\pi$ (i.e. of order $2$): a small loop around that cone point will map, under the holonomy representation of a hyperbolic structure on $Y$, to an element of $\text{PGL}(2, \mathbb{R})$ conjugate to diag($i, -i$) in $\text{PGL}(2, \mathbb{C})$ and this indeed maps to diag($-1, 1, -1$) under $\kappa_C$.

Finally, by applying Theorem 2.13 we can extend the Labourie–Guichard characterization of Hitchin representations into $G = \text{PGL}(n, \mathbb{R})$ as hyperconvex representations [Lab06, Gui08] to the orbifold case. Following Labourie [Lab06], a $\text{PGL}(n, \mathbb{R})$-representation $\varrho$ of $\pi_1 Y$ is called hyperconvex if there exists a continuous map $\xi : \partial_{\infty} \pi_1 Y \longrightarrow \mathbb{RP}^{n-1} = \mathbb{P}(\mathbb{R}^n)$ that is $\pi_1 Y$-equivariant with respect to $\varrho$ and hyperconvex in the sense that for all $n$-tuples of pairwise distinct points $(x_1, \ldots, x_n)$ in $\partial_{\infty} \pi_1 Y \simeq \partial H^n \simeq S^n$, we have:

$$\xi(x_1) + \cdots + \xi(x_n) = \mathbb{R}^n.$$  

Lemma 2.23. If $X \longrightarrow Y$ is a finite cover, then there is a canonical homeomorphism $\partial_{\infty} \pi_1 X \simeq \partial_{\infty} \pi_1 Y$, which is $\pi_1 X$-equivariant with respect to the inclusion $\pi_1 X \hookrightarrow \pi_1 Y$.

Proof. This is proved for example in [GdlH90, KB02].

Theorem 2.24. A representation of $\pi_1 Y$ in $\text{PGL}(n, \mathbb{R})$ is Hitchin if and only if it is hyperconvex.

Proof. Let $Y = [\Sigma \setminus X]$ be a presentation of $Y$ (see Proposition 2.22) and let $\varrho$ be a Hitchin representation of $\pi_1 Y$. Then $\varrho|_{\pi_1 X}$ is Hitchin by Lemma 2.9. By [Lab06] Theorem 1.4, there exists a $\pi_1 X$-equivariant, hyperconvex curve $\xi : \partial_{\infty} \pi_1 X \longrightarrow \mathbb{RP}^{n-1}$. Given an element $\gamma \in \pi_1 Y$, let us consider the map

$$\varrho(\gamma) \circ \xi \circ \gamma^{-1} : (\partial_{\infty} \pi_1 Y = \partial_{\infty} \pi_1 X) \longrightarrow \mathbb{RP}^{n-1}$$

identifying $\partial_{\infty} \pi_1 X$ with $\partial_{\infty} \pi_1 Y$ via the $\pi_1 X$-equivariant homeomorphism in Lemma 2.23. It is straightforward to check that this map is hyperconvex. Moreover, it is $\pi_1 X$-equivariant: if $\delta \in \pi_1 X$, we have, as $\pi_1 X$ is normal in $\pi_1 Y$, that:

$$(\varrho(\gamma) \circ \xi \circ \gamma^{-1}) \circ \delta = \varrho(\gamma) \circ (\varrho(\gamma^{-1} \delta \gamma) \circ \xi \circ \gamma^{-1}) = \varrho(\delta) \circ (\varrho(\gamma) \circ \xi \circ \gamma).$$

So, by uniqueness of such a map [Gui08 Proposition 16]), we have $\varrho(\gamma) \circ \xi \circ \gamma^{-1} = \xi$. As this holds for all $\gamma \in \pi_1 Y$, we have that $\xi$ is $\pi_1 Y$-equivariant.

Conversely, assume that $\varrho$ is hyperconvex and let $\xi : \partial_{\infty} \pi_1 Y \longrightarrow \mathbb{RP}^{n-1}$ be the associated $\pi_1 Y$-equivariant hyperconvex curve. Since $\pi_1 X \hookrightarrow \pi_1 Y$, the curve $\xi$ is also $\pi_1 X$-equivariant. So, by [Gui08 Théorème 1], $\varrho|_{\pi_1 X}$ is a Hitchin representation. It then follows from Corollary 2.15 that $\varrho$ is a Hitchin representation of $\pi_1 Y$.

As a last remark, the hyperconvex curve, if it exists, is unique.

Proposition 2.25. Let $\varrho : \pi_1 Y \longrightarrow \text{PGL}(n, \mathbb{R})$ be a hyperconvex representation of $\pi_1 Y$, then the $\pi_1 Y$-equivariant hyperconvex curve $\xi : \partial_{\infty} \pi_1 Y \longrightarrow \mathbb{RP}^{n-1}$ is unique.
Proof. Notice that the case where \( Y \) is a closed surface is proved in [Gui08 Proposition 16]. Here we only need to show that this implies the general case. By definition of hyperconvexity, the assumption on \( \varrho \) is that a \( \pi_1 Y \)-equivariant hyperconvex curve \( \xi \) exists. Assume there is another such curve \( \xi' \), and let \( Y = [\Sigma \setminus X] \) be a presentation (see Proposition 2.2). Then \( \xi \) and \( \xi' \) are \( \pi_1 X \)-equivariant. Hence [Gui08 Proposition 16] implies that \( \xi = \xi' \).

3. Hitchin’s equations in an equivariant setting

In this section, we give a short presentation of the results of equivariant non-Abelian Hodge theory that we need for our purposes, using previous work of C. Simpson ([Sim88, Sim92]), N. K. Ho, G. Wilkin and S. Wu ([HWW13]) and O. García-Prada and G. Wilkin ([GPW16]). Since we are only interested in certain particular groups of adjoint type in this paper, we can afford to work with Lie algebra bundles and avoid the formalism of principal bundles (except perhaps in Lemma 3.11).

3.1. From orbifold representations to equivariant flat bundles. Let us fix a presentation \( Y \simeq [\Sigma \setminus X] \) as in Proposition 2.2. In particular, there is a short exact sequence \( 1 \rightarrow \pi_1 X \rightarrow \pi_1 Y \rightarrow \Sigma \rightarrow 1 \) and the universal covers of \( X \) and \( Y \) are \( \pi_1 X \)-equivariantly isomorphic: \( \tilde{X} \simeq \tilde{Y} \). It is well-known that, if \( G \) is a Lie group of adjoint type with Lie algebra \( \mathfrak{g} \), and \( \varphi : \pi_1 X \rightarrow G \) is a representation of \( \pi_1 X \) in \( G \), there is, associated to \( \varphi \), a flat Lie algebra \( G \)-bundle \( \mathcal{E}_\varphi := \pi_1 X \backslash (\tilde{X} \times \mathfrak{g}) \) on \( X \). We now recall that, if \( \varphi : \pi_1 X \rightarrow G \) is the restriction to \( \pi_1 X \) of a representation of \( \pi_1 Y \) into \( G \), then the action of \( \Sigma \) on \( X \) lifts to \( \mathcal{E}_\varphi \), giving it a structure of \( \Sigma \)-equivariant bundle in the following sense.

**Definition 3.1** (Equivariant bundle). A \( \Sigma \)-equivariant Lie algebra \( G \)-bundle over \((X, \Sigma)\) is a pair \((E, \tau)\) consisting of a smooth Lie algebra \( G \)-bundle \( E \) and a family \( \tau = (\tau_\sigma)_{\sigma \in \Sigma} \) of bundle homomorphisms

\[
\begin{array}{ccc}
E & \xrightarrow{\tau_\sigma} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & X
\end{array}
\]

satisfying \( \tau_{1_\Sigma} = \text{Id}_E \) and, for all \( \sigma_1, \sigma_2 \in \Sigma \), \( \tau_{\sigma_1 \sigma_2} = \tau_{\sigma_1} \tau_{\sigma_2} \). A homomorphism of \( \Sigma \)-equivariant bundles over \( X \) is a bundle homomorphism (over \( \text{Id}_X \)) that commutes to the \( \Sigma \)-equivariant structures. A \( \Sigma \)-sub-bundle of \((E, \tau)\) is a sub-bundle \( F \subset E \) such that, for all \( \sigma \in \Sigma \), \( \tau_\sigma(F) \subset F \). In particular, \((F, \tau|_F)\) is itself a \( \Sigma \)-equivariant bundle on \( X \).

When we say Lie algebra \( G \)-bundle, we mean a locally trivial \( G \)-bundle whose fibers are modeled on a Lie algebra equipped with an effective action of \( G \) by Lie algebra isomorphisms. The most important case for us is when the Lie algebra is \( \mathfrak{g} = \text{Lie}(G) \), and \( G \) acts on it by the adjoint representation (in this case, \( G \) has trivial center). Homomorphisms of such bundles are understood to be Lie algebra homomorphisms fiberwise.

A definition similar to Definition 3.1 of course holds for usual vector bundles, as well as for principal bundles. If \((E, \tau)\) is a \( \Sigma \)-equivariant bundle on \( X \), there are canonical isomorphisms \( \varphi_\sigma : E \xrightarrow{\sim} \sigma^* E \), satisfying \( \varphi_{1_\Sigma} = \text{Id}_E \) and \( \varphi_{\sigma_1 \sigma_2} = (\sigma_2^* \varphi_{\sigma_1}) \varphi_{\sigma_2} \) for all \( \sigma_1, \sigma_2 \in \Sigma \). Conversely, such a family \((\varphi_\sigma)_{\sigma \in \Sigma}\) defines a \( \Sigma \)-equivariant structure \( \tau \) on \( E \), the relation between the two notions being given by the following commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\tau_\sigma} & E \\
\downarrow & & \downarrow \\
\sigma^* E & \xrightarrow{\tilde{\varphi}_E} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & X
\end{array}
\]

so that \( \tau_\sigma = \tilde{\varphi}_E \circ \varphi_\sigma \), where \( \tilde{\varphi}_E \) is the canonical map \( \sigma^* E \rightarrow E \) over \( \sigma : X \rightarrow X \), satisfying \( \tilde{\varphi}_1^* = \tilde{\varphi}_1 \tilde{\varphi}_2^* \) and \( \varphi_1 \varphi_2 = (\tilde{\varphi}_2 \tilde{\varphi}_1)^{-1} \varphi_1 \tilde{\varphi}_2 \). In what follows, given a \( \Sigma \)-equivariant bundle \((E, \tau)\), we will
always identify $E$ with $\sigma^*E$ using $\varphi_\sigma$. In particular, there is an induced action of $\Sigma$ on the space of $G$-connections on $E$, as we shall see momentarily. But first, recall that a $G$-connection on a Lie algebra $G$-bundle $E$ is a linear map $\nabla : \Omega^0(X;E) \rightarrow \Omega^1(X;E)$ that satisfies the Leibniz identity $\nabla(f \cdot s) = (df) \cdot s + f \cdot \nabla s$ and is compatible with the Lie bracket of sections in the sense that:

$$\nabla[s_1,s_2] = [\nabla s_1,s_2] + [s_1,\nabla s_2]$$

It induces linear maps $\nabla^{(k)} : \Omega^k(X;E) \rightarrow \Omega^{k+1}(X;E)$ by imposing the conditions:

1. $\forall \alpha \in \Omega^k(X), \forall s \in \Omega^{k}(X;E)$, $\nabla^{(k+\ell)}(\alpha \wedge s) = (d\alpha) \wedge s + (-1)^k \alpha \wedge \nabla^{(\ell)}s$,

2. $\forall (s_1,s_2) \in \Omega^k(X;E) \times \Omega^{k}(X;E)$, $\nabla^{(k)}[s_1,s_2] = [\nabla^{(k)} s_1,s_2] + (-1)^k [s_1,\nabla^{(k)} s_2]$.

In particular, the curvature $F_\nabla := \nabla^{(1)} \circ \nabla$ of $\nabla$ may be seen as an $E$-valued 2-form on $X$, where $E \rightarrow \text{End}(E)$ via the fiberwise adjoint action (defined by the Lie bracket on $E$). The space $A_E$ of all $G$-connections on $E$ is an affine space whose group of translations is $\Omega^1(X;E)$. If $\tau$ is a $\Sigma$-equivariant structure on $E$, then $\Sigma \subset \text{Diff}(X)$ acts on $A_E$ in the following way: if $\nabla$ is a $G$-connection on $E$ and $\sigma \in \Sigma$, then $\nabla^\sigma := \sigma^* \nabla$ is a connection on $\sigma^* E$, which has been canonically identified with $E$ via $\varphi_\sigma$. This is a right action of $\Sigma$ on $A_E$, which may, equivalently, be defined by noting that $\sigma$ acts on $\Omega^\ell(X;E) = \Gamma((k\ell)T^*X \otimes E)$, by $\sigma \cdot \omega := (\sigma \otimes \tau_\sigma) \circ \omega \circ \sigma^{-1}$ (see also (3.7)), and setting $\nabla^\sigma := \sigma^{-1} \nabla \sigma$ (see also Proposition 3.24). The group $\Sigma$ also acts on the gauge group $\mathcal{G}_E$ of $E$ via $u^\sigma := \sigma^* u$ (or, equivalently, $u^\sigma = \tau^{-1}_\sigma \circ u \circ \tau_\sigma$). The $\Sigma$-action on $A_E$ is then compatible with the gauge action on that space, in the sense that $(u^{-1} \nabla u)^\sigma = (u^\sigma)^{-1} \nabla^\sigma u^\sigma$.

In particular, $\text{Fix}_\Sigma(\mathcal{G}_E)$ acts on $\text{Fix}_\Sigma(A_E)$. We also observe that $F_\nabla^\sigma = \sigma^* F_\nabla =: F_\nabla^\sigma$ in $\Omega^2(X;E)$. In particular, $\Sigma$ acts on the set $F^{-1}(0)$ of flat connections on $E$.

It remains to see that, if $\varrho : \pi_1Y \rightarrow G$ is a group homomorphism, then there is indeed a canonical $\Sigma$-equivariant structure on the Lie algebra bundle $\mathcal{E}_\varrho := \pi_1 X \setminus (\widetilde{X} \times \mathfrak{g})$ over $X$.

**Proposition 3.2.** Given $\gamma \in \pi_1 Y$, the map

$$\bar{\tau}_\gamma : \widetilde{X} \times \mathfrak{g} \rightarrow \widetilde{X} \times \mathfrak{g} \quad \begin{array}{ccc} (\eta,v) & \rightarrow & (\gamma \cdot \eta, \varrho(\gamma) \cdot v) \end{array}$$

descends to a map $\tau_\gamma$ on $\mathcal{E}_\varrho$ that only depends on the class $\sigma$ of $\gamma$ in $\Sigma = \pi_1 Y / \pi_1 X$. The collection $\tau : = (\tau_\sigma)_{\sigma \in \Sigma}$ of these maps defines a $\Sigma$-equivariant structure on $\mathcal{E}_\varrho$. Moreover, the canonical flat connection on $\mathcal{E}_\varrho$, induced by the trivial connection on $\widetilde{X} \times \mathfrak{g}$, is $\Sigma$-invariant with respect to the action of $\Sigma$ on the space of connections on $\mathcal{E}_\varrho$ associated to $\tau$.

**Proof.** Let us first check that $\bar{\tau}_\gamma$ descends to $\mathcal{E}_\varrho$: if $\delta \in \pi_1 X$, then

$$\bar{\tau}_\gamma(\delta \cdot \eta, \varrho(\delta) \cdot v) = (\gamma \delta \cdot \eta, \varrho(\gamma \delta) \cdot v) \sim_{\gamma \delta \gamma^{-1}} (\gamma \cdot \eta, \varrho(\gamma) \cdot v) = \bar{\tau}_\gamma(\eta, v),$$

where $\gamma \delta \gamma^{-1} \in \pi_1 X$. A similar computation shows that the induced transformation of $\mathcal{E}_\varrho$ indeed only depends on the class of $\gamma$ modulo $\pi_1 X$. The connection on $\mathcal{E}_\varrho$ induced by the trivial connection on $\widetilde{X} \times \mathfrak{g}$ is $\Sigma$-invariant because the trivial connection on $\widetilde{X} \times \mathfrak{g}$ is $\pi_1 Y$-invariant with respect to the $\pi_1 Y$-equivariant structure $(\bar{\tau}_\gamma)_{\gamma \in \pi_1 Y}$ on that bundle (note that $\widetilde{X} \times \mathfrak{g}$ does not admit a $\Sigma$-equivariant structure in general).

Therefore, given a presentation $Y \simeq [\Sigma \cdot X]$, we have set up a map

$$\text{Hom}(\pi_1 Y; G)/G \rightarrow \{ \Sigma\text{-equivariant flat } G\text{-bundles on } X \}/\text{isomorphism}$$

where a $\Sigma$-equivariant flat bundle is defined as follows.

**Definition 3.3** (Equivariant flat bundle). A $\Sigma$-equivariant flat bundle on $(X, \Sigma)$ is a triple $(E, \nabla, \tau)$ where $(E, \nabla)$ is a flat bundle on $X$ and $\tau$ is a $\Sigma$-equivariant structure on $E$ that leaves the connection $\nabla$ invariant. A homomorphism of $\Sigma$-equivariant flat bundles is a homomorphism of flat bundles that commutes with the $\Sigma$-equivariant structures.
An inverse map to (3.2) is provided by the holonomy of \( \Sigma \)-invariant flat connections. More precisely, as in \cite{HI09, Sch17}, we will have one such holonomy map for each isomorphism type of \( \Sigma \)-equivariant bundles. To prove this, we first need a description of \( \pi_1 Y \) in terms of paths in \( X \). Let us choose a point \( x \in X \) and consider the set \( P_x \), consisting of pairs \( ([c], \sigma) \) where \( \sigma \in \Sigma \) and \([c]\) is the homotopy class of a path \( c : [0; 1] \to X \) satisfying \( c(0) = x \) and \( c(1) = \sigma(x) \), equipped with the group law

\[
([c_1], \sigma_1) \cdot ([c_2], \sigma_2) = ([c_1 \sigma_1 \circ c_2], \sigma_2).
\]

Our convention for concatenating paths is from left to right, so the above group law is well-defined. Note that, if \( \Sigma \) has fixed points in \( X \) and \( x \in \text{Fix}_\Sigma(X) \), then \( P_x \cong \pi_1 X \rtimes \Sigma \) for the natural left action of \( \Sigma \) on \( \pi_1 X \). In what follows, we denote by \( \tilde{x} \) the base point of \( \tilde{X} \) corresponding to the homotopy class of the constant path at \( x \) in \( X \). Recall that \( \pi_1 Y = \text{Aut}_Y(\tilde{Y}) \).

**Lemma 3.4.** The map \( \pi_1 Y \to P_x \) sending \( \gamma \in \pi_1 Y \) to \( ([c], \sigma) \), where \( c : [0; 1] \to X \) is the projection to \( X \) of an arbitrary path from \( \tilde{x} \) to \( \gamma(\tilde{x}) \) in \( \tilde{X} \) and \( \sigma \) is the class of \( \gamma \) in \( \Sigma = \pi_1 Y/\pi_1 X \), is a group isomorphism.

**Proof.** Note that \([c]_1 \) is well-defined because \( \tilde{X} \) is simply connected. Moreover, the map \( \gamma \mapsto ([c], \sigma) \) is a group homomorphism. To see that it is injective, assume that \( ([c_1], \sigma) = (\tilde{x}, 1_\Sigma) \). As \( \sigma = 1_\Sigma \), we have that \( \gamma \in \pi_1 X \). And since \( c_1 \) is homotopic to the constant path at \( x \) in \( X \), the path it lifts to in \( \tilde{X} \) goes from \( \tilde{x} \) to \( \tilde{x} \). In particular, \( \gamma(\tilde{x}) = \tilde{x} \). And since \( \gamma \in \pi_1 X \) and \( \pi_1 X \) acts freely on \( \tilde{X} \), this implies that \( \gamma = 1_{\pi_1 X} = 1_{\pi_1 Y} \). To see that our map \( \pi_1 Y \to P_x \) is surjective, take \( ([c], \sigma) \in P_x \) and let us denote by \( q \) the universal covering map \( q : \tilde{X} \to X \). The path \( c \) goes from \( x \) to \( \sigma(x) \) in \( X \) and it lifts to a path \( \tilde{c} \) from \( \tilde{x} \) to a point \( \eta \in \tilde{X} \), that lies in the fiber of \( q \) over \( \sigma(x) \). Since \( \sigma \circ q : \tilde{X} \to X \) is also a universal covering map, there exists a unique continuous map \( \gamma : \tilde{X} \to \tilde{X} \) such that \( q \circ \gamma = \sigma \circ q \) and \( \gamma(\tilde{x}) = \eta \). Since \( \gamma : \tilde{X} \to \tilde{X} \) lifts over \( \sigma : X \to X \), we have that \( \gamma \) maps to \( \sigma \) in \( \pi_1 Y/\pi_1 X \) and, since \( \sigma : X \to X \) lies over \( \text{Id}_Y \), we also have that \( \gamma \in \text{Aut}_Y(\tilde{Y}) = \pi_1 Y \). Finally, by definition of \((c_\cdot)\), we have that \([c_\cdot]_1 = [c]_1 \) for that \( \gamma \).

For all \((c_\cdot), \sigma) \in P_x \), we consider the map \( \tau_\sigma^{-1} \circ T^\nabla_\cdot \circ \tau_\sigma^\nabla : E_x \to E_x \) obtained by composing the parallel transport operator along the path \( c \) with respect to \( \nabla \) by the bundle map \( \tau_\sigma^{-1} \). Because of our convention on concatenation of paths, this will be a group anti-homomorphism from \( P_x \) to \( \text{Aut}(E_x) \), as we now show.

**Theorem 3.5.** Given a presentation \( Y \cong [\Sigma \setminus X] \) and a \( \Sigma \)-equivariant flat \( G \)-bundle \((E, \nabla, \tau)\) over \( X \), there is a group homomorphism

\[
\tilde{\varphi}^\nabla : \pi_1 Y \cong P_x \to G
\]

obtained by taking \((c_\cdot), \sigma) \in P_x \) to \( \tau_\sigma^{-1} \circ T^\nabla_\cdot \in \text{Aut}(E_x) \). Moreover, the restriction of \( \tilde{\varphi}^\nabla \) to \( \pi_1 X \subset \pi_1 Y \) is the holonomy representation \( \varphi^\nabla : \pi_1 X \to G \).

Two gauge-equivalent connections induce conjugate representations and we obtain a continuous map

\[
\text{Fix}_\Sigma(F^{-1}(0))/(\text{Fix}_\Sigma(G_E)) \to \text{Hom}(\pi_1 Y; G)/G
\]

from gauge orbits of \( \Sigma \)-invariant, flat connections on \((E, \tau)\) to \( \text{Rep}(\pi_1 Y, G) \) which, composed with the map \( (3.2) \), is the identity map of \( \text{Fix}_\Sigma(F^{-1}(0))/(\text{Fix}_\Sigma(G_E)) \).

**Proof.** The statement follows from the definition of \( P_x \) and the properties of parallel transport operators, namely that, if \( T^\nabla_\cdot \) is the parallel transport operator along the path \( c \) with respect to a connection \( \nabla \), there is a commutative diagram

\[
\begin{array}{ccc}
E_{c(0)} & \xrightarrow{T^\nabla_\cdot} & E_{c(1)} \\
\downarrow{\tau_\cdot} & & \downarrow{\tau_\cdot} \\
E_{\sigma(c(0))} & \xrightarrow{T^\nabla_{\sigma(c(0))}} & E_{\sigma(c(1))}
\end{array}
\]

where, as earlier, \( \nabla^\sigma = \sigma^* \nabla \) (by definition). For a detailed proof of the above, we refer for instance to \cite[Section 4.1]{Sch17}. In particular, if \( \nabla^\sigma = \nabla \), then \( T^\nabla_{\sigma(c)} = \tau_\sigma T^\nabla_\cdot \tau_\sigma^{-1} \), which readily implies that the map \((c, \sigma) \mapsto \tau_\sigma^{-1} \circ T^\nabla_\cdot \) is a group anti-homomorphism from \( \pi_1 Y \) to \( \text{Aut}(E_x) \) (since \( T^\nabla_{\sigma(c)} \sigma(c_\cdot) = T^\nabla_\cdot \sigma(c_\cdot) \circ T^\nabla_\cdot \), due to our convention on concatenation of paths). The rest of the theorem is proved as in the case where \( \Sigma \) is trivial. \( \square \)
Corollary 3.6. Under the assumptions of Theorem 3.3, there is a homeomorphism
\[ \text{Hom}(\pi_1 Y; G)/G \simeq \bigsqcup_{[E, \tau] \in \mathcal{P}_G} \text{Fix}_G(F^{-1}(0))/\text{Fix}_G(G_E), \]
where \( \mathcal{P}_G \) is the set of isomorphism classes of \( \Sigma \)-equivariant smooth Lie algebra \( G \)-bundles with fiber \( g = \text{Lie}(G) \) on \( X \).

3.2. From equivariant flat bundles to equivariant harmonic bundles. Let now \( g \) be a real semisimple Lie algebra and let \( G \) be the group of real points of \( \text{Int}(g \otimes \mathbb{R}) \mathbb{C} \). Let \( (E, \nabla) \) be a flat Lie algebra bundle over \( X \), with typical fiber \( g \) and structure group \( G \). Choose a Cartan involution \( \theta : G \rightarrow G \) and denote by \( K := \text{Fix}(\theta) \subset G \) the associated maximal compact subgroup of \( G \). The induced Lie algebra automorphism will also be denoted by \( \theta \). Let \( \varphi : \pi_1 X \rightarrow G \) be the holonomy representation associated to the flat connection \( \nabla \). For a flat \( G \)-bundle, a reduction of structure group from \( G \) to \( K = \text{Fix}(\theta) \) (also called a \( K \)-reduction) can be defined as \( \pi_1 X \)-equivariant map \( f : \tilde{X} \rightarrow G/K \), where \( \pi_1 X \) acts on \( G/K \) via \( \varphi \) and left translations by elements of \( G \). Indeed, if such a map \( f \) is given, then \( E = \pi_1 X \times (\tilde{X} \times g) \) inherits an involutive automorphism \( \theta_f \), induced by the map
\[ \tilde{\theta}_f : \tilde{X} \times g \rightarrow (\tilde{X} \times g) \quad (\eta, v) \mapsto (\eta, \text{Ad}_{f(\eta)}(\theta_f(v))^{-1} \theta(v)) \]
(by a computation similar to the one used in Proposition 3.10, it is immediate to check that \( \tilde{\theta}_f \) indeed extends to the bundle \( \pi_1 X \times (\tilde{X} \times g) \) and it can be observed that this uses the fact that \( G \) acts on \( g \) by the adjoint action). The map \( \theta_f \) induces in turn a direct sum decomposition \( E \simeq E_K \oplus P \) where \( E_K := \text{Fix}(\theta_f) \) is a Lie algebra \( K \)-bundle with typical fiber \( \mathfrak{k} := \text{Lie}(K) \) and \( P := \{ v \in E \mid \theta_f(v) = -v \} \) is a vector \( K \)-bundle with typical fiber \( \mathfrak{p} := \{ v \in g \mid \theta(v) = -v \} \simeq T_K(G/K) \), satisfying \( [E_K, E_K] \subset P \) and \([P, P] \subset E_K \) with respect to the fiberwise Lie bracket. The Killing form \( \kappa \) of \( g \) induces a bilinear form \( B \) on the fibers of \( E \) and, since \( K \) is a maximal compact subgroup of \( G \), the associated \( \mathbb{R} \)-bilinear form \( B_{\theta_f}(v_1, v_2) := -B(\theta_f(v_1), v_2) \) is a positive definite metric on \( E \). The decomposition \( E = E_K \oplus P \) will be called a Cartan decomposition and the involutions \( \theta_f \) a Cartan involution of \( E \).

Example 3.7. If \( G \simeq \text{PGL}(n, \mathbb{R}) \) then \( K \simeq \text{PO}(n) \) and a \( K \)-reduction of \( E \) is induced by the choice of a Riemannian metric on \( E \) that is fiberwise conjugate to \( (v_1, v_2) \mapsto \text{tr}(v_1 v_2) \), where \( \theta : v \mapsto -v \) is “the” Cartan involution of \( g = \mathfrak{sl}(n, \mathbb{R}) \) and \( b(v_1, v_2) = \text{tr}(v_1 v_2) \) is the Killing form. Similar considerations apply for \( G \simeq \text{PGL}(n, \mathbb{C}) \) and \( K \simeq \text{PU}(n) \) when \( \theta(v) = -iv \) on \( g = \mathfrak{sl}(n, \mathbb{C}) \), the latter being viewed as a real Lie algebra.

Note that we can view \( \theta_f \) as a gauge transformation of \( E \). In particular, there is a well-defined connection \( \theta_f^{-1} \nabla \theta_f := \theta_f \nabla \theta_f^{-1} \). Therefore, once a \( K \)-reduction \( f \) of \( E \) has been chosen, the connection \( \nabla \) decomposes uniquely into \( \nabla = A_f + \psi_f \), where \( A_f = \frac{1}{2}(\nabla + \theta_f^{-1} \nabla \theta_f) \) is a \( G \)-connection on \( E \) whose restriction to \( E_K \) is a \( K \)-connection, and \( \psi_f = \nabla - A_f \in \Omega^1(X; P) \subset \Omega^1(X; E) \) acts on sections of \( E \) via the Lie bracket (thus preserving \( E_K \)). It follows that we can identify \( A_E \) to \( A_{E_K} \otimes \Omega^1(X; P) \). We recall that \( \nabla \) being flat is then equivalent to the following two conditions:
\[ F_{A_f} + \frac{1}{2} [\psi_f, \psi_f] = 0 \quad \text{and} \quad d_{A_f} \psi_f = 0, \]
where \( F_{A_f} \in \Omega^2(X; E_K) \subset \Omega^2(X; E) \) is the curvature of \( A_f |_{E_K} \) and \( d_{A_f} \psi_f \in \Omega^2(X; P) \). If we now fix an orientation and a Riemannian metric \( g \) on \( X \), then the choice of a maximal compact subgroup \( K = \text{Fix}(\theta) \) and a \( K \)-reduction \( f : \tilde{X} \rightarrow G/K \) of \( E \) induces a positive definite scalar product on the space of \( E \)-valued \( k \)-forms on \( X \), called the \( L^2 \)-metric (depending on \( g \) and \( f \)) and defined by
\[ (s_1 \wedge s_2)_{\Omega^k(X; E)} := \int_X B_{\theta_f}(s_1 \wedge \ast_g s_2) \]
where \( \ast_g \) is the Hodge star operator associated to \( g \) and the orientation of \( X \) (here, \( s_1 \) and \( s_2 \) are \( E \)-valued \( k \)-forms, \( \ast_g s_2 \) is an \( E \)-valued \((n-k)\)-form, \( s_1 \wedge \ast_g s_2 \) is an \((E \otimes E)\)-valued \( n \)-form and \( B_{\theta_f}(s_1 \wedge \ast_g s_2) \) is a real-valued \( n \)-form on \( X \)). The covariant derivative \( d_{A_f} : \Omega^k(X; P) \rightarrow \Omega^{k+1}(X; P) \) then has an adjoint operator \( d_{A_f}^* : \Omega^{k+1}(X; P) \rightarrow \Omega^k(X; P) \) (depending on \( g \) and \( f \)) and the \( K \)-reduction \( f \) of the flat bundle
$(E, \nabla)$ that induces the decomposition $\nabla = A_f + \psi_f$ is called harmonic if $d_{A_f}^* \psi_f = 0$ in $\Omega^0(X; P)$. We recall that this is the case if and only if the reduction $f : \tilde{X} \to G/K$ is a harmonic map in the usual sense, or equivalently, if it minimizes the energy functional

$$E_\nabla(f) := \frac{1}{2} \int_X \|\psi_f\|^2_{\nabla} d\text{vol}_g,$$

where $\theta_f$ and $\psi_f$ both depend on $f$ (see for instance [Gui10, Lemma 9.11]). A theorem of Corlette (and Donaldson for the case $G \simeq \text{PGL}(2, \mathbb{C})$, [Don87]) gives a necessary and sufficient condition on $\nabla$ for a harmonic reduction to exist, namely that $\nabla$ be completely reducible. We state it below in a form adapted to our Lie algebra bundle setting (Theorem 3.9) but first we recall the definition of the relevant stability condition in the context of flat bundles.

**Definition 3.8 (Stability condition).** [Cor88] Definition 3.1] A flat Lie algebra $G$-bundle $(E, \nabla)$ on $X$ is called:

- **irreducible** (or **stable**), if it contains no non-trivial $\nabla$-invariant sub-bundle (or equivalently, if the holonomy representation $\varrho_\nabla : \pi_1 X \to G \subset \text{Aut}(g)$ turns $g$ into an irreducible $\pi_1 X$-module).
- **completely reducible** (or **polystable**) if $(E, \nabla)$ is isomorphic to a direct sum $\bigoplus_{1 \leq i \leq r} (E_i, \nabla_i)$ of irreducible flat bundles (or equivalently, if $g$ is isomorphic, as a $\pi_1 X$-module, to a direct sum $\bigoplus_{1 \leq i \leq r} g_i$ of irreducible $\pi_1 X$-modules).

By definition here, a $\pi_1 X$-module is a pair $(g, q)$ consisting of a Lie algebra $g$ and a homomorphism $q : \pi_1 X \to G$ to the group of real points of $\text{Int}(g \otimes_{\mathbb{R}} \mathbb{C})$. Another possible characterization of complete reducibility is to say that every $\nabla$-invariant sub-bundle $F$ of the flat bundle $(E, \nabla)$ has a $\nabla$-invariant complement (or equivalently, that any sub-$\pi_1 X$-module of $g$ has a $\pi_1 X$-invariant complement).

**Theorem 3.9.** [Cor88] Theorem 3.4.4] Let $g$ be a real semisimple Lie algebra and let $G$ be the group of real points of $\text{Int}(g \otimes_{\mathbb{R}} \mathbb{C})$. Let $K \subset G$ be a maximal compact subgroup and let $(E, \nabla)$ be a flat Lie algebra $G$-bundle with fiber $g$ over $X$. Then $(E, \nabla)$ admits a harmonic $K$-reduction $f : \tilde{X} \to G/K$ if and only if it is polystable.

Thus, a polystable flat bundle $(E, \nabla)$ admits a harmonic $K$-reduction $f : \tilde{X} \to G/K$, which gives rise to a solution $(A_f, \psi_f) \in \mathcal{A}_{E_K} \times \Omega^1(X; P)$ of the Hitchin equations

$$F_{A_f} + \frac{1}{2} [\psi_f \wedge \psi_f] = 0, \quad d_{A_f} \psi_f = 0 \quad \text{and} \quad d_{A_f}^* \psi_f = 0,$$

where $E_K \oplus P$ is the Cartan decomposition of $E$ associated to the Cartan involution $\theta_f$ associated to the $K$-reduction $f$. The triple $(E, \nabla, f)$ is called a harmonic bundle.

We now want to provide an analogue of Theorem 3.9 in the $\Sigma$-equivariant setting. So let $(E, \nabla, \tau)$ be a $\Sigma$-equivariant flat bundle on $X$, in the sense of Definition 3.3. By Theorem 3.5 the holonomy representation $\varrho_\nabla : \pi_1 X \to G$ extends to a group homomorphism $\varrho_\nabla : \pi_1 Y \to G$. We will be interested in $\pi_1 X$-equivariant maps $f : \tilde{X} \to G/K$ that are in fact $\pi_1 Y$-equivariant (with respect to $\varrho_\nabla$).

**Proposition 3.10.** Let $(E, \nabla, \tau)$ be a $\Sigma$-equivariant flat bundle on $X$ and let $Y$ be the orbifold $[\Sigma \backslash X]$. We shall denote by $\varepsilon : \pi_1 Y \to \pi_1 X$ the canonical morphism $\varepsilon : \pi_1 Y \to \pi_1 X \simeq \Sigma$. Then the following properties hold:

1. The group $\pi_1 Y$ acts on the set of $\pi_1 X$-equivariant maps $f : \tilde{X} \to G/K$ by $f^\gamma := \varrho_\nabla(\gamma^{-1}) (f \circ \gamma)$.
2. If $\theta_f$ is the Cartan involution of $E$ associated to $f$, then, for all $\gamma \in \pi_1 Y$, we have $\theta_f^\gamma = \varrho_\nabla(\gamma^{-1} \theta_f \gamma)$, where $\varepsilon(\gamma) \in \Sigma$ acts on $\theta_f$ via the $\Sigma$-action on gauge transformations of $(E, \tau) : \theta_f^\gamma = \tau_{\sigma}^{-1} \theta_f \tau_{\sigma}$ for all $\sigma \in \Sigma$.
3. Let $E \simeq E_K \oplus P$ be the Cartan decomposition of $E$ associated to $f$ and let $(A_f, \psi_f) \in \mathcal{A}_{E_K} \times \Omega^1(X; P)$ be the associated decomposition of $\nabla$ into a $K$-connection and a $P$-valued 1-form, i.e. $\nabla = A_f + \psi_f$. Then, for all $\gamma \in \pi_1 Y$, we have $A_f^\gamma = A_f^\varepsilon(\gamma)$ and $\psi_f^\gamma = \psi_f^\varepsilon(\gamma)$.
4. If the map $f : \tilde{X} \to G/K$ is $\pi_1 Y$-equivariant, then $\theta_f$ commutes to the $\Sigma$-equivariant structure $\tau$ on $E$. In particular, the restriction of $\tau$ induces $\Sigma$-equivariant structures on the $K$-bundles $E_K$ and $P$, therefore also a $\Sigma$-action on $\mathcal{A}_{E_K}$ and $\Omega^1(X; P)$ and the pair $(A_f, \psi_f) \in \mathcal{A}_{E_K} \times \Omega^1(X; P)$ is $\Sigma$-invariant.
Proof. Let us show the listed properties.

(1) We check that the map \( \tilde{\varphi}(\gamma^{-1})(f \circ \gamma) \) from \( \tilde{X} \) to \( G/K \) is \( \pi_1 X \)-equivariant. For all \( \eta \in \tilde{X} \) and all \( \delta \in \pi_1 X \), we have

\[
\tilde{\varphi}(\gamma^{-1})(f \circ \gamma)(\delta \cdot \eta) = \tilde{\varphi}(\gamma^{-1})(f((\gamma \delta \gamma^{-1}) \cdot (\gamma \cdot \eta))) = \tilde{\varphi}(\gamma^{-1})\varphi(\gamma \delta \gamma^{-1})f(\gamma \cdot \eta) = \varphi(\delta)(\tilde{\varphi}(\gamma^{-1})(f \circ \gamma)(\eta))
\]

since \( \gamma \delta \gamma^{-1} \in \pi_1 X \cap \pi_1 Y \).

(2) It suffices to show that the map \( \tilde{\theta}_f \) defined in \( \text{[3.3]} \) satisfies \( \tilde{\theta}_f \tilde{\varphi} = \tilde{\varphi}^{-1} \tilde{\theta}_f \tilde{\varphi} \), where \( \tilde{\varphi} \) is the map defined in \( \text{[3.1]} \). For all \( (\eta, v) \in \tilde{X} \times g \), one has, on the one hand (writing simply \( \rho \) for \( \tilde{\varphi} \)),

\[
\tilde{\varphi}^{-1}\tilde{\theta}_f \tilde{\varphi}(\eta, v) = \tilde{\varphi}^{-1}\tilde{\theta}_f (\gamma \cdot \eta, \rho(\gamma) \cdot v) = \tilde{\varphi}^{-1}(\gamma \cdot \eta, \text{Ad}_f(\gamma \cdot \eta)\theta(\rho(\gamma) \cdot v)) = (\eta, \theta(\gamma^{-1}) \cdot \text{Ad}_f(\gamma \cdot \eta)\theta(\rho(\gamma) \cdot v))
\]

and, on the other hand,

\[
\tilde{\theta}_f \tilde{\varphi}(\eta, v) = (\eta, \text{Ad}_f(\gamma \cdot \eta)\theta(\rho(\gamma) \cdot v)) = (\eta, \text{Ad}_f(\gamma^{-1})\theta(\rho(\gamma) \cdot v)) = (\eta, \theta(\gamma^{-1}) \cdot \text{Ad}_f(\gamma \cdot \eta)\theta(\rho(\gamma) \cdot v))
\]

where in the last equation we use that \( G \) acts on \( g \) by the adjoint action.

(3) This is a computation similar to the one above, using the explicit definition of \( A_f \) and \( \psi_f \) in terms of \( \nabla \) and \( \theta_f \), and the \( \Sigma \)-invariance of \( \nabla \). One has:

\[
\theta_f^{-1}\nabla \theta_f = (\theta_f^{(\gamma)})^{-1}\nabla^{(\gamma)} \theta_f^{(\gamma)} = (\theta_f^{-1}\nabla \theta_f)^{(\gamma)}
\]

so

\[
A_f = \frac{1}{2}(\nabla + \theta_f^{-1}\nabla \theta_f) = \frac{1}{2}(\nabla^{(\gamma)} + (\theta_f^{-1}\nabla \theta_f)^{(\gamma)}) = \frac{1}{2}(\nabla + \theta_f^{-1}\nabla \theta_f)^{(\gamma)} = A_f^{(\gamma)}
\]

and \( \psi_f = \nabla - A_f = \nabla^{(\gamma)} - A_f^{(\gamma)} = (\nabla - A_f)^{(\gamma)} = \psi_f^{(\gamma)} \).

(4) This follows immediately from the previous three properties. \( \square \)

Of course, for Proposition \( \text{[3.10]} \) to be useful, we need to make sure that \( \pi_1 Y \)-equivariant maps \( f : \tilde{X} \rightarrow G/K \) indeed exist. One way to do this is as follows. Let \( P_E \) be the principal \( G \)-bundle associated to \( E \) and let \( P_E(G/K) := P_E \times_K (G/K) \) be the bundle whose sections are \( K \)-reductions of \( P_E \). The \( \Sigma \)-equivariant structure \( \tau \) on \( E \) induces a \( \Sigma \)-equivariant structure on \( P_E \), that we shall still denote by \( \tau \). Note that this is an \( \Sigma \)-equivariant structure in the principal bundle sense, so we have the compatibility relation \( \tau_\sigma(p \cdot g) = \tau_\sigma(p) \cdot g \) between the action of \( G \) on \( P_E \) and and the \( \Sigma \)-equivariant structure \( \tau = (\tau_\sigma)_{\sigma \in \Sigma} \). Then \( P_E(G/K) \) also has a \( \Sigma \)-equivariant structure, given by

\[
P_E \times_G (G/K) \rightarrow P_E \times_K (G/K) \mid (p, gK) \mapsto (\tau_\sigma(p), gK)
\]

which is indeed well-defined by the previous remark. In particular, it makes sense to speak of \( \Sigma \)-equivariant sections of \( P_E(G/K) \), and these do exist as we can average an arbitrary section of \( P_E(G/K) \) over the finite group \( \Sigma \), since there is a notion of center of mass in the complete Riemannian manifold of non-positive curvature \( G/K \) (the fiber of \( P_E(G/K) \)). Denote now by \( q : \tilde{X} \rightarrow X \) the universal covering map. As \( E \) is flat, so are \( P_E \) and \( P_E(G/K) \). Therefore \( q^*P_E(G/K) \) is a flat bundle on the simply connected manifold \( \tilde{X} \), which implies that it can be \( \pi_1 X \)-equivariantly identified with \( \tilde{X} \times G/K \), using parallel transport on \( \tilde{X} \). Note that \( \pi_1 X \backslash (\tilde{X} \times G/K) \simeq P_E(G/K) \) as bundles over \( X \).
Lemma 3.11. The $\pi_1X$-equivariant isomorphism $q^*P_E(G/K) \simeq \tilde{X} \times G/K$ induces a bijection between $\Sigma$-equivariant sections of $P_E(G/K) \to X$ and $\pi_1Y$-equivariant maps $f : \tilde{X} \to G/K$. In particular, the latter do exist.

Proof. Recall that there is a short exact sequence $1 \to \pi_1X \to \pi_1Y \to \Sigma \to 1$. In particular, for all $\eta \in \tilde{Y} \simeq \tilde{X}$ and all $\gamma \in \pi_1Y$, we have $q(\gamma \cdot \eta) = \varepsilon(\gamma) \cdot q(\eta)$ in $X$. Moreover, $\pi_1Y$ acts on $q^*P_E(G/K)$ via

$$(\eta, [p, gK]) \mapsto (\gamma \cdot \eta, \varepsilon(\gamma) \cdot [p, gK]),$$

which is well-defined because $\tau_\sigma(p \cdot g) = \tau_\sigma(p) \cdot g$ for all $\sigma \in \Sigma, p \in P$ and $g \in G$. So we have a commutative diagram

$$
\begin{array}{ccc}
q^*P_E(G/K) & \longrightarrow & P_E(G/K) \\
\downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & X
\end{array}
$$

where all maps are $\pi_1Y$-equivariant (on the right-hand-side, $\pi_1Y$ acts through the group homomorphism $\varepsilon : \pi_1Y \to \Sigma$). As the flat connection $\nabla_{\tilde{x}/K}$ induced by $\nabla$ on $P_E(G/K)$ is $\Sigma$-invariant, the $\pi_1X$-equivariant isomorphism $q^*P_E(G/K) \simeq \tilde{X} \times G/K$ provided by parallel transport is actually $\pi_1Y$-equivariant with respect to the extended holonomy representation $\tilde{\gamma} : \pi_1Y \to G$, whose existence is guaranteed by Theorem 3.5. Therefore, $\Sigma$-equivariant sections of $P_E(G/K) \to X$ are in bijective correspondence with $\pi_1Y$-equivariant sections of $q^*P_E(G/K) \to \tilde{X}$, which themselves are in bijective correspondence with $\pi_1Y$-equivariant sections of $\tilde{X} \times G/K \to \tilde{X}$. \qed

Note that, in general, it does not make sense to speak of a $\Sigma$-equivariant map $f : \tilde{X} \to G/K$, because $\tilde{X}$ does not act on $\tilde{X}$ in any interesting way. We will however, slightly abusively, speak of $\Sigma$-invariant reductions in the following sense.

Definition 3.12 (Invariant $K$-reduction). Let $(E, \nabla, \tau)$ be a $\Sigma$-equivariant flat bundle on $X$. A $\pi_1Y$-equivariant map $f : \tilde{X} \to G/K$ will be called a $\Sigma$-invariant $K$-reduction of $(E, \nabla, \tau)$.

As Lemma 3.11 shows, $\pi_1Y$-equivariant maps $f : \tilde{X} \to G/K$ exist, and as Proposition 3.10 shows, the Cartan decomposition $E \simeq E_k \oplus P$ associated to such an $f$ is compatible with the $\Sigma$-action in the sense that $\tau_\sigma(E_k) \subset E_k$ and $\tau_\sigma(P) \subset P$ for all $\sigma \in \Sigma$. In the context of $\Sigma$-equivariant flat bundles, Definition 3.8 then extends as follows, which will eventually lead to a generalization of Theorem 3.9.

Definition 3.13 (Stability condition for equivariant flat bundles). A $\Sigma$-equivariant flat Lie algebra $G$-bundle $(E, \nabla, \tau)$ on $X$ is called:

- $\Sigma$-irreducible (or $\Sigma$-stable) if it contains no non-trivial $\nabla$-invariant $\Sigma$-sub-bundle (or equivalently, if the extended holonomy representation $\tilde{\gamma} : \pi_1Y \to G \subset \text{Aut}(g)$ of Theorem 3.5 turns $g$ into an irreducible $\pi_1Y$-module).
- $\Sigma$-completely reducible (or $\Sigma$-polystable) if $(E, \nabla, \tau)$ is isomorphic to a direct sum $\bigoplus_{1 \leq i \leq k} (E_i, \nabla_i, \tau_i)$ of irreducible $\Sigma$-equivariant flat bundles (or equivalently, if $g$ is isomorphic, as a $\pi_1Y$-module, to a direct sum $\bigoplus_{1 \leq i \leq k} g_i$ of irreducible $\pi_1Y$-modules).

Again here, a $\pi_1Y$-module is a pair $(g, \rho)$ consisting of a Lie algebra $g$ and a homomorphism $\rho : \pi_1Y \to G$ to the group of real points of $\text{Int}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$, and another possible characterization of complete reducibility is to say that every $\nabla$-invariant $\Sigma$-sub-bundle $F$ of the flat bundle $(E, \nabla)$ has a complement that is both $\Sigma$-invariant and $\nabla$-invariant (or equivalently, that any sub-$\pi_1Y$-module of $g$ has a $\pi_1Y$-invariant complement). Evidently, if a flat bundle $(E, \nabla)$ is stable, then, for any $\Sigma$-equivariant structure $\tau$ leaving $\nabla$ invariant, the equivariant flat bundle $(E, \nabla, \tau)$ is $\Sigma$-stable. But $\Sigma$-stability of $(E, \nabla, \tau)$ only implies polystability of $(E, \nabla)$ in general. As a matter of fact, $(E, \nabla, \tau)$ is $\Sigma$-polystable if and only if $(E, \nabla)$ is polystable, as follows from the following result.

Proposition 3.14. Let $g$ be a real semisimple Lie algebra and let $G$ be the group of real points of $\text{Int}(g \otimes_{\mathbb{R}} \mathbb{C})$. Let $[\Sigma \setminus X] \simeq Y$ be a presentation of the orbifold $Y$ and let $\rho : \pi_1Y \to G$ be a representation of the orbifold
fundamental group of $Y$ in $G$. Then $\mathfrak{g}$ is completely reducible as a $\pi_1 Y$-module if and only if it is completely reducible as a $\pi_1 X$-module.

Proof. Since $\pi_1 X$ is a normal subgroup of finite index of $\pi_1 Y$, the result follows for instance from [Ser94, Lemma 5].

The next result lays the groundwork for the first half of the non-Abelian Hodge correspondence for $\Sigma$-equivariant bundles: if the $\Sigma$-equivariant flat bundle $(E, \nabla, \tau)$ is $\Sigma$-polystable, it admits a $\Sigma$-invariant harmonic $K$-reduction $f$ (in the sense of Definition 3.12), which defines a $\Sigma$-equivariant harmonic bundle $(E, \nabla, f, \tau)$, i.e. a harmonic bundle $(E, \nabla, f)$ endowed with a $\Sigma$-equivariant structure $\tau$ that leaves the connection $\nabla$, the harmonic $K$-reduction $f$, the connection $A_f$ and the 1-form $\psi_f$ all invariant.

**Theorem 3.15** (Invariant harmonic reductions of equivariant bundles). [HWW13, Theorem 2.2] Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $G$ be the group of real points of $\text{Int}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. Let $K \subset G$ be a maximal compact subgroup and let $(E, \nabla, \tau)$ be a $\Sigma$-equivariant flat Lie algebra $G$-bundle with fiber $\mathfrak{g}$ over $X$. Then $(E, \nabla, \tau)$ admits a $\Sigma$-equivariant harmonic $K$-reduction $f : \tilde{X} \rightarrow G/K$ if and only if it is $\Sigma$-polystable.

**Remark 3.16.** In [HWW13, Theorem 3.15] is proved in the special case where $\Sigma \simeq \mathbb{Z}/2\mathbb{Z}$, but their techniques extend to the case where $\Sigma$ is any finite group. Note that, in [HWW13], $X$ is of arbitrary dimension.

3.3. **From equivariant Higgs bundles to equivariant harmonic bundles.** Let $(X, \Sigma)$ be a closed orientable surface equipped with an action of a finite group $\Sigma$. We fix an orientation and a $\Sigma$-invariant Riemannian metric $g$ on $X$, and denote by $J$ the associated complex structure. Then a transformation $\sigma \in \Sigma$ is holomorphic with respect to $J$ if it preserves the orientation of $X$; otherwise, it is anti-holomorphic (note that, here, $\Sigma$ is a subgroup of $\text{Diff}(X)$, not $\text{MCG}(X) = \pi_0(\text{Diff}(X))$, so finding a complex structure $J$ on $X$ such that $\Sigma \subset \text{Aut}_{\mathbb{C}}(X, J)$ is elementary). A $\Sigma$-equivariant structure $\tau$ on a holomorphic vector bundle $\mathcal{E} \rightarrow X$ is a family $\tau = (\tau_\sigma)_{\sigma \in \Sigma}$ of either holomorphic or anti-holomorphic transformations of $\mathcal{E}$ satisfying:

1. For all $\sigma \in \Sigma$, the diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\tau_\sigma} & \mathcal{E} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & X
\end{array}
$$

is commutative,

2. The bundle map $\tau_\sigma$ is fiberwise $\mathbb{C}$-linear if $\sigma : X \rightarrow X$ is holomorphic and fiberwise $\mathbb{C}$-antilinear if $\sigma : X \rightarrow X$ is anti-holomorphic,

3. One has $\tau_{\Sigma} = \text{Id}_{\mathcal{E}}$ and, for all $\sigma_1, \sigma_2 \in \Sigma$, $\tau_{\sigma_1 \sigma_2} = \tau_{\sigma_1} \tau_{\sigma_2}$.

For instance, the canonical bundle $K_X$ of $X$ has a $\Sigma$-equivariant structure induced by the $\Sigma$-action on $X$. Moreover, if $(\mathcal{E}, \tau)$ is a $\Sigma$-equivariant holomorphic vector bundle on $X$, then all associated bundles inherit a $\Sigma$-equivariant structures. For instance, $\text{End}(\mathcal{E}) \simeq \mathcal{E}^* \otimes \mathcal{E}$ has the induced $\Sigma$-equivariant structure $\xi \otimes v \mapsto (\xi \circ \tau_\sigma^{-1}) \otimes \tau_\sigma(v)$, which shall simply denote by $\tau$. As a consequence, $K_X \otimes \text{End}(\mathcal{E})$ also has an induced $\Sigma$-equivariant structure, that we denote by $(\sigma \otimes \tau_\sigma)_{\sigma \in \Sigma}$, and the spaces of sections of $K_X \otimes \text{End}(\mathcal{E})$, just as the space of sections of any equivariant bundle, inherit a $\Sigma$-action respectively defined, for all $\sigma \in \Sigma$, by

$$(3.7) \quad \sigma(\varphi) := (\sigma \otimes \tau_\sigma) \circ \varphi \circ \sigma^{-1}.$$ 

Whenever the holomorphic vector bundle $\mathcal{E}$ has an extra structure (for instance, a holomorphic Lie bracket), we will assume, in the definition of a $\Sigma$-equivariant structure $\tau$, that the bundle maps $\tau_\sigma : \mathcal{E} \rightarrow \mathcal{E}$ are compatible with that structure.

In this paper, we shall consider $G$-Higgs bundles on $X$ for $G$ a real form of a connected semisimple complex Lie group of adjoint type $G_{\mathbb{C}}$. We denote by $\mathfrak{g}$ the Lie algebra of $G$. If $\theta : G \rightarrow G$ is a Cartan involution, $K := \text{Fix}(\theta) \subset G$ is the associated maximal compact subgroup and $K \rightarrow \text{GL}(\mathfrak{p})$ is the isotropy (adjoint) representation of $K$ on the $(-1)$-eigenspace of $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, then, by definition, a $G$-Higgs bundle on $X$ is a pair $(\mathcal{P}, \varphi)$ consisting of a holomorphic principal $K_{\mathbb{C}}$-bundle $\mathcal{P}$, where $K_{\mathbb{C}}$ is the complexification of $K$, and a holomorphic section $\varphi \in H^0(X; K_X \otimes \mathcal{P}(\mathfrak{p}_{\mathbb{C}}))$, where $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} \otimes \mathbb{C}$ and $\mathcal{P}(\mathfrak{p}_{\mathbb{C}}) := \mathcal{P} \times_{K_{\mathbb{C}}} \mathfrak{p}_{\mathbb{C}}$. 

19
Let now us now specialize this definition to the case where $G$ is the group of real points of $G_C := \text{Int}(\mathfrak{g}_C)$, where $\mathfrak{g}$ is a real semisimple Lie algebra and $\mathfrak{g}_C := \mathfrak{g} \otimes \mathbb{C}$. We let $\mathfrak{t}$ be a maximal compact Lie subalgebra of $\mathfrak{g}$, with respect to the Killing form $\kappa$, and we denote by $K_C \subset G_C$ be the connected subgroup corresponding to the Lie algebra $\mathfrak{k}_C := \mathfrak{t} \otimes \mathbb{C}$ (i.e. here, $K_C = \text{Int}(\mathfrak{t} \otimes \mathbb{C})$). We denote by $\theta_C$ (resp. $\kappa_C$) the $\mathbb{C}$-linear extension to $\mathfrak{g}_C$ of the Cartan involution $\theta$ (resp. Killing form $\kappa$) of $\mathfrak{g}$. Then $K_C = \text{Fix}(\theta_C)$ in $G_C$ and we set $K := \text{Fix}(\theta)$ in $G$. Moreover, the positive definite quadratic form $B_\theta(x,y) := -\kappa(\theta(x),y)$ on $\mathfrak{g}$ induces a non-degenerate $\mathbb{C}$-valued quadratic form $B_{\theta_C}$ on $\mathfrak{g}_C$, whose group of isometries contains $K_C$ and whose space of symmetric endomorphisms contains the space of adjoint transformations of the form $\text{ad}_x = [x, \cdot]$ for $x \in \mathfrak{p}_C := \mathfrak{p} \otimes \mathbb{C}$. Using the faithful representations $K_C \hookrightarrow \text{O}(\mathfrak{g}_C, B_{\theta_C})$ and $\mathfrak{p}_C \hookrightarrow \text{Sym}(\mathfrak{g}_C, B_{\theta_C})$, we can now give the following definition of a $G$-Higgs bundle for $G$ as above.

**Definition 3.17** (Higgs bundles for real forms of connected semisimple Lie groups of adjoint type). Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $G$ be the group of real points of $\text{Int}(\mathfrak{g} \otimes \mathbb{C})$. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with Cartan involution $\theta$ and set $K_C := \text{Int}(\mathfrak{t} \otimes \mathbb{C})$. By a $G$-Higgs bundle on the Riemann surface $X$, we shall mean a pair $(\mathcal{E}, \varphi)$ consisting of:

- a holomorphic Lie algebra bundle $\mathcal{E}$ with typical fiber $\mathfrak{g}_C := \mathfrak{g} \otimes \mathbb{C}$ and structure group $K_C$, and
- a holomorphic 1-form $\varphi \in H^0(X; K_X \otimes \text{ad}_{\mathfrak{p}_C}(\mathcal{E}))$ called the Higgs field,

where by $\text{ad}_{\mathfrak{p}_C}(\mathcal{E})$ we mean the bundle of symmetric adjoint endomorphisms of $\mathcal{E}$, i.e. endomorphisms of $\mathcal{E}$ that are given, locally, by a transformation of the form

$$\text{ad}_\xi = [\xi, \cdot] : \mathfrak{g}_C \rightarrow \mathfrak{g}_C,$$

for some $\xi \in \mathfrak{p}_C := \mathfrak{p} \otimes \mathbb{C}$. This notion is indeed independent of the choice of local trivialization because the adjoint action of $K_C$ preserves $\mathfrak{p}_C$.

**Remark 3.18.** Giving a $G$-Higgs bundle in the sense of Definition 3.17 is equivalent to giving a triple $(\mathcal{E}, \beta, \varphi)$ where:

- $\mathcal{E}$ is a holomorphic Lie algebra bundle $\mathcal{E}$ with typical fiber $\mathfrak{g}_C := \mathfrak{g} \otimes \mathbb{C}$ and structure group $G_C$,
- $\beta \in H^0(X; S^2\mathcal{E}*)$ is a non-degenerate quadratic form on $\mathcal{E}$ which is compatible with the Lie bracket in the sense that $\beta([v_1, v_2], v_3) = \beta(v_1, [v_2, v_3])$, and
- $\varphi \in H^0(X; K_X \otimes \text{ad}(\mathcal{E}))$ is symmetric with respect to $\beta$.

Indeed, $\beta$ will be fiberwise of the form $B_{\theta_C}$ for $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ a (fixed) Cartan involution, thus inducing a reduction of structure group from $G_C$ to $K_C$, so the Higgs field $\varphi$ is symmetric with respect to $\beta$ if and only if it is $\text{ad}_{\mathfrak{p}_C}(\mathcal{E})$-valued.

It will be convenient, at times, to see a holomorphic vector bundle $\mathcal{E}$ as a pair $(E, \nabla_E)$ consisting of a smooth complex vector bundle $E$ on $X$ and a Dolbeault operator $\nabla_E : \Omega^0(X; E) \rightarrow \Omega^{0,1}(X; E)$.

As an example of $G$-Higgs bundle for $G$ as above, consider the case where $\mathfrak{g} = \mathfrak{h}_C$ is already a complex semisimple Lie algebra. Then $K_C \simeq H_C$, $\mathfrak{p}_C \simeq \mathfrak{h}_C$ and $\mathcal{P}(\mathfrak{p}_C) \simeq \mathcal{P} \times H_C \mathfrak{h}_C \simeq \text{ad}(\mathcal{P})$. So, when $G = H_C$ is a connected complex semisimple Lie group of adjoint type, an $H_C$-Higgs bundle can be thought of as a holomorphic Lie algebra vector bundle $\mathcal{E}$, with typical fiber $\mathfrak{h}_C$ and structure group $H_C \simeq \text{Int}(\mathfrak{h}_C)$, equipped with a holomorphic 1-form $\varphi$ with values in adjoint endomorphisms of $\mathcal{E}$. Another elementary example is given by the case where $\mathfrak{g} = \mathfrak{t}$ is a compact semisimple Lie algebra. Then $G = K$, so $\mathfrak{p} = 0$, and a $K$-Higgs bundle is a pair $(\mathcal{E}, \varphi) = (E, 0)$ consisting of a holomorphic Lie algebra vector bundle $E$, with typical fiber $\mathfrak{t}_C$ and structure group $K_C \simeq \text{Int}(\mathfrak{t}_C)$. Note that when $\mathfrak{t} = \mathfrak{u}(n)$, then $K \simeq \text{PU}(n)$. A more elaborate example is given as follows: given a real semisimple Lie algebra $\mathfrak{g}$ and $G$ the group of real points of $\text{Int}(\mathfrak{g}_C)$, if $f : \widetilde{X} \rightarrow G/K$ is a harmonic $K$-reduction of a polystable flat Lie algebra $G$-bundle $(E, \nabla)$, with associated Cartan decomposition $E = E_K \oplus \mathcal{P}$ and $\nabla = A_f + \psi_f$, then $(E \otimes \mathbb{C}, d_A^{0,1}, \psi_f^{0,1})$ is a $G$-Higgs bundle. In this last example, the vector bundle $\mathcal{E}$ in particular has vanishing Chern classes.

In our context, the following definition is then natural (and is a special case of the notion of pseudo-equivariant $G$-Higgs bundle developed for an arbitrary semisimple Lie group $G$ in [GPW16, HS18]).

**Definition 3.19** (Equivariant Higgs bundles). A $\Sigma$-equivariant $G$-Higgs bundle on $(X, \Sigma)$ is a triple $(\mathcal{E}, \varphi, \tau)$ consisting of a $G$-Higgs bundle $(\mathcal{E}, \varphi)$ and a $\Sigma$-equivariant structure $\tau = (\tau_x)_{x \in \Sigma}$ leaving the Higgs field $\varphi$ invariant, i.e. such that, for all $\sigma \in \Sigma$, one has $\sigma(\varphi) = \varphi$ for the action of $\Sigma$ on $H^0(X; K_X \otimes \text{ad}_{\mathfrak{p}_C}(\mathcal{E}))$ defined
in (3.7), respectively. A homomorphism of $\Sigma$-equivariant $G$-Higgs bundles is a homomorphism of $G$-Higgs bundles that commutes to the $\Sigma$-equivariant structures.

The $\Sigma$-invariance condition on the Higgs field $\varphi$ can also be phrased in the following way: for all $\sigma \in \Sigma$, the following diagram, where by $\tau$ we mean the $\Sigma$-equivariant structure of $\text{ad}_{P_c}(E) \subset \text{End}(E)$ induced by that of $E$, is commutative.

\[
\begin{array}{ccc}
\text{ad}_{P_c}(E) & \xrightarrow{\varphi} & K_X \otimes \text{ad}_{P_c}(E) \\
\downarrow \tau_\sigma & & \downarrow \sigma \otimes \tau_\sigma \\
\text{ad}_{P_c}(E) & \xrightarrow{\varphi} & K_X \otimes \text{ad}_{P_c}(E)
\end{array}
\]

We will now further restrict ourselves to $G$-Higgs bundles that have vanishing Chern classes, because, in that case, we can take semistability of a principal $G$-Higgs bundle $(P, \varphi)$ to mean that the vector $G$-Higgs bundle $(\mathcal{P}(V_C), \varphi_{V_C})$ associated to $(P, \varphi)$ via a faithful representation $K \rightarrow \text{GL}(V_C)$ is semistable ([Sim92, p.86]). In the $\Sigma$-equivariant setting, we then have the following definition, which will be sufficient for our purposes.

**Definition 3.20** (Stability condition for equivariant Higgs bundles). Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $G$ be the group of real points of $\text{Int}(\mathfrak{g} \otimes \mathbb{C})$. A $\Sigma$-equivariant $G$-Higgs bundle $(E, \varphi, \tau)$ with vanishing first Chern class on $X$ is called:

- $\Sigma$-semistable if, for all non-trivial sub-bundle $F \subset E$ such that $\varphi(F) \subset K_X \otimes F$ and $\tau_\sigma(F) \subset F$ for all $\sigma \in \Sigma$, the degree of $F$ is non-positive, i.e. $\deg(F) \leq 0$.
- $\Sigma$-stable if the above inequality is strict.
- $\Sigma$-polystable if it is isomorphic to a direct sum of $\Sigma$-stable equivariant Higgs bundles of degree 0.

The point of this definition is that any $\Sigma$-semistable equivariant $G$-Higgs bundle has an associated $\Sigma$-polystable equivariant Higgs bundle (the graded object associated to any choice of a Jordan-Hölder filtration of the initial bundle, defined up to isomorphism) and that such objects admit a characterization in terms of special metrics, namely Hermitian-Yang-Mills metrics (Theorem 3.22 which is due to Simpson in [Sim88, Sim92]). An example of such a $\Sigma$-polystable equivariant $G$-Higgs bundle is provided by the equivariant $G$-Higgs bundle $(E \otimes \mathbb{C}, d_A^{0,1}, \psi^{1,0}, \tau)$ associated to a $\Sigma$-invariant $K$-reduction $f : \tilde{X} \rightarrow G/K$ of a $\Sigma$-polystable equivariant flat bundle $(E, \nabla, \tau)$. Note that such a map $f$ exists by Theorem 3.15 and that the existence and uniqueness of a maximally destabilizing sub-bundle $(F, \varphi |_F)$ of $(E, \varphi)$ has the following standard consequence (that we do not need for our purposes).

**Lemma 3.21.** A $\Sigma$-equivariant $G$-Higgs bundle $(E, \varphi, \tau)$ is $\Sigma$-semistable if and only if the $G$-Higgs bundle $(E, \varphi)$ is semistable.

It is well-known that Lemma 3.21 no longer holds if one replaces semistability with stability: $\Sigma$-stable equivariant objects are not stable in general. We will see, however, that they are polystable (Corollary 3.23). In some cases (see for instance Sch12 for the case where $\Sigma \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $X$ by an anti-holomorphic involution and $G \simeq \mathbb{P}U(n)$ is itself endowed with a non-trivial action of $\mathbb{Z}/2\mathbb{Z}$), this can be seen directly, by characterizing explicitly all $\Sigma$-stable objects. As a matter of fact, based on that characterization, it is a simple matter to prove a stronger result: that $(E, \varphi, \tau)$ is $\Sigma$-polystable if and only if $(E, \varphi)$ is polystable.

Here, we take a different and in a way more fundamental approach, based on Simpson’s characterization of polystable objects in terms of special Hermitian metrics, the point being that Theorem 1 of [Sim88] is already stated in a $\Sigma$-equivariant setting, for $\Sigma$ a finite group of holomorphic automorphisms of $X$. The extension to the case where $\Sigma$ is allowed to contain anti-holomorphic transformations of $X$ is not difficult, once one realizes that such a group $\Sigma$ still acts on the space of smooth Hermitian metrics on a holomorphic vector bundle $E$, by setting, for all $x \in X$ and all $v_1, v_2$ in $T_x X$,

\[
h_\Sigma^x(v_1, v_2) = \begin{cases} 
\frac{h_{\sigma(x)}(\tau_\sigma(v_1), \tau_\sigma(v_2))}{h_{\sigma(x)}(\tau_\sigma(v_1), \tau_\sigma(v_2))} & \text{if } \sigma \text{ is holomorphic on } X, \\
\frac{h_{\sigma(x)}(\tau_\sigma(v_1), \tau_\sigma(v_2))}{h_{\sigma(x)}(\tau_\sigma(v_1), \tau_\sigma(v_2))} & \text{if } \sigma \text{ is antiholomorphic on } X.
\end{cases}
\]

We can therefore use Simpson’s theorem ([Sim88, Theorem 1]). Note that Simpson’s version actually has one extra degree of generality, namely the Higgs field $\varphi$ is not assumed to be preserved by the $\Sigma$-action, instead
it suffices that there exist a character \( \chi : \Sigma \to \mathbb{C}^* \) such that, for all \( \sigma \in \Sigma \), \( \sigma(\phi) = \chi(\sigma)\phi \); when \( \Sigma \) contains anti-holomorphic transformations, the group homomorphism \( \chi : \Sigma \to \mathbb{C}^* \) could be replaced by a crossed homomorphism, with respect to the action of \( \Sigma \) on \( \mathbb{C}^* \) defined by the canonical morphism \( \Sigma \to \mathbb{Z}/2\mathbb{Z} \) followed by complex conjugation on \( \mathbb{C}^* \), but in any case this is not necessary for us, as we only consider Lie groups with trivial center. As a matter of fact, we also need Simpson’s extension of his result to \( G \)-Higgs bundles with \( G \) a real form of a complex semisimple Lie group ([Sim92 Corollary 6.16]). A different approach to Theorem 3.22 below and its generalization to pseudo-equivariant \( G \)-Higgs bundles can be found in [GPW16 Theorem 4.4].

Given a \( G \)-Higgs bundle \((\mathcal{E}, \varphi)\) (with vanishing first Chern class) equipped with a Hermitian metric \( h \), we denote by \( \varphi^* \) the fiberwise adjoint of the Higgs field \( \varphi \) with respect to \( h \), and by \( A_h \) the Chern connection associated to \( h \). Recall that \( h \) is called a Hermitian-Yang-Mills metric on \((\mathcal{E}, \varphi)\) if the Chern connection \( A_h \) satisfies the self-duality equation \( F_{A_h} + [\varphi, \varphi^*] = 0 \). In such a case, the triple \((\mathcal{E}, \varphi, h)\) defines a harmonic bundle in the sense of Section 3.2 and the next, fundamental, result of Simpson’s says that all harmonic bundles arise in this way from polystable Higgs bundles with vanishing first Chern class.

**Theorem 3.22.** Sim88 Sim92 Let \( g \) be a real semisimple Lie algebra and let \( G \) be the group of real points of \( \text{Int}(g \otimes \mathbb{R} \subset \mathbb{C}) \). Let \((\mathcal{E}, \varphi, \tau)\) be \( \Sigma \)-equivariant \( G \)-Higgs bundle with vanishing first Chern class on \( X \). Then there exists a \( \Sigma \)-invariant Hermitian-Yang-Mills metric \( h \) on the holomorphic vector bundle \( \mathcal{E} \) if and only if \((\mathcal{E}, \varphi, \tau)\) is \( \Sigma \)-polystable.

**Corollary 3.23.** Let \((\mathcal{E}, \varphi, \tau)\) be \( \Sigma \)-equivariant \( G \)-Higgs bundle on \( X \). If \((\mathcal{E}, \varphi, \tau)\) is \( \Sigma \)-polystable as an equivariant \( G \)-Higgs bundle, then \((\mathcal{E}, \varphi)\) is polystable as a \( G \)-Higgs bundle.

**Proof.** Assume that \((\mathcal{E}, \varphi, \tau)\) is \( \Sigma \)-polystable. Then, by Theorem 3.22 it admits a \( \Sigma \)-invariant Yang-Mills metric \( h \). Such a metric is in particular Hermitian-Yang-Mills, so \((\mathcal{E}, \varphi)\) is polystable as a \( G \)-Higgs bundle, again by Theorem 3.22. \( \square \)

Our next goal is to show that the Chern connection of a \( \Sigma \)-invariant metric is necessarily \( \Sigma \)-invariant. This will follow from a very elementary observation (Proposition 3.24). Let \( \mathcal{E} \) be a holomorphic vector bundle on \( X \) and think of it as a smooth vector bundle \( E \) equipped with a Dolbeault operator \( \overline{\partial}_E : \Omega^0(X; E) \to \Omega^{0,1}(X; E) \). Saying that \( \tau = (\tau_\sigma)_{\sigma \in \Sigma} \) is a \( \Sigma \)-equivariant structure in the holomorphic sense on \( \mathcal{E} \) is equivalent to saying that \( \tau \) is a \( \Sigma \)-equivariant structure in the smooth sense on \( E \) such that, additionally, \( \sigma \overline{\partial}_E \sigma^{-1} = \overline{\partial}_E \) for all \( \sigma \in \Sigma \), i.e. the Dolbeault operator \( \overline{\partial}_E \) is equivariant with respect to the \( \Sigma \)-actions induced by \( \tau \) on \( \Omega^0(X; E) \) and \( \Omega^{0,1}(X; E) \). Indeed, that equivariance condition implies that each \( \tau_\sigma \) preserves the space ker\( \overline{\partial}_E \) of \( \Sigma \)-invariant holomorphic sections of \( \mathcal{E} \), therefore is either holomorphic or anti-holomorphic with respect to \( \overline{\partial}_E \).

**Proposition 3.24.** Let \((E, \overline{\partial}_E, \tau)\) be a \( \Sigma \)-equivariant holomorphic vector bundle on \( X \). Then there is a right action \( D \mapsto D^\sigma \) of the group \( \Sigma \) on the space of linear connections \( D : \Omega^0(X; E) \to \Omega^{1}(X; E) \) satisfying \( D^{0,1} = \overline{\partial}_E \). Moreover, if \( h \) is a Hermitian metric on \((E, \overline{\partial}_E)\) and \( A_h \) is the Chern connection associated to \( h \), then one has, for all \( \sigma \in \Sigma \), \( A^\sigma_h = A_{h^\sigma} \) with respect to the action of \( \Sigma \) on the space of metrics defined in (3.10). In particular, if \( h \) is \( \Sigma \)-invariant, then so is the Chern connection \( A_h \).

**Proof.** First, we define an action of \( \Sigma \) on the space of linear connections on \( E \) that are compatible with the holomorphic structure \( \overline{\partial}_E \). Set \( D^\sigma := \sigma^{-1}D\sigma \), where \( \sigma \) acts on \( \Omega^k(X; E) \) in the usual way (see for instance (3.7)). It is clear that this action preserves the subspaces of \((1, 0)\) and \((0, 1)\) pseudo-connexions, as conjugation by \( \sigma \) will always be \( \mathbb{C} \)-linear. Then \( (\sigma^{-1}D\sigma)^{0,1} = \sigma^{-1}D^{0,1}\sigma = h^{-1}\overline{\partial}_E h = \overline{\partial}_E \), so \( D^\sigma \) is indeed compatible with \( \overline{\partial}_E \).

Next, we prove that \( A^\sigma_h = A_{h^\sigma} \). Recall that the Chern connection \( A_h \) associated to the metric \( h \) and the holomorphic structure \( \overline{\partial}_E \) is the linear connection \( A_h := D_h + \overline{\partial}_E \) where \( D_h \) is the operator of type \( (1, 0) \) uniquely determined by the condition \( \overline{\partial}_j(h(s_1, s_2)) = h(D_h s_1, s_2) + h(s_1, \overline{\partial}_E s_2) \) for all smooth sections \( s_1, s_2 \) of \( E \) (and where \( \overline{\partial}_j \) is the Cauchy-Riemann operator associated to \( J \) on \( X \)). Then, one has, for all \( \sigma \in \Sigma \):

\[
\overline{\partial}_j(h^\sigma(s_1, s_2)) = \overline{\partial}_j\left(h(\sigma(s_1), \sigma(s_2))\right) = h(D_h \sigma(s_1), \sigma(s_2)) + h(\sigma(s_1), \overline{\partial}_E \sigma(s_2))
= h(\sigma^{-1}D_h \sigma(s_1), \sigma(s_2)) + h(\sigma(s_1), \sigma^{-1}\overline{\partial}_E \sigma(s_2))
= h(\sigma^{-1}D_h \sigma(s_1, s_2)) + h(\sigma(s_1, \overline{\partial}_E s_2)) = 0
\]
so \( D_{h^*} = \sigma^{-1} D_h \sigma = D_h^* \) and \( A_{h^*} = D_{h^*} + \bar{\nabla}_E = \sigma^{-1} D_h \sigma + \sigma^{-1} \bar{\nabla}_E \sigma = \sigma^{-1} A_h \sigma = A_h^* \).

Combining Proposition 3.24 with Simpson’s Theorem 3.22, we obtain the main result of this section, which is the second half of the non-Abelian Hodge correspondence for \( \Sigma \)-equivariant bundles: if the \( \Sigma \)-equivariant \( G \)-Higgs bundle \((E, \varphi, \tau) = (E, \bar{\nabla}_E, \varphi, \tau)\) is \( \Sigma \)-polystable, it admits a \( \Sigma \)-invariant Hermitian-Yang-Mills metric \( h \), which defines a \( \Sigma \)-equivariant harmonic bundle \((E, \bar{\nabla}_E, \varphi, h, \tau) = (E, \nabla_h, h, \tau)\), with \( \Sigma \)-invariant flat connection \( \nabla_h := A_h + \psi_h \) where \( \psi_h = \varphi + \varphi^h \).

**Theorem 3.25.** Let \( \mathfrak{g} \) be a real semisimple Lie algebra and let \( G \) be the group of real points of \( \text{Int}(\mathfrak{g} \otimes \mathbb{C}) \). Let \((E, \varphi, \tau)\) be a \( \Sigma \)-equivariant \( G \)-Higgs bundle on \( X \). Then \((E, \varphi, \tau)\) is \( \Sigma \)-polystable if and only if there exists a \( \Sigma \)-invariant Hermitian metric \( h \) on \( E \) such that the associated Chern connection \( A_h \) is a \( \Sigma \)-invariant solution of the self-duality equation, namely:

1. \( F_{A_h} + [\varphi, \varphi^h] = 0 \),
2. For all \( \sigma \in \Sigma \), \( A_h^\sigma = A_h \).

**Proof.** Assume that \((E, \varphi, \tau)\) is \( \Sigma \)-polystable. The existence of a \( \Sigma \)-invariant metric \( h \) such that \( A_h \) satisfies \( F_{A_h} + [\varphi, \varphi^h] = 0 \) is provided by [Sim88] and [Sim92], as recalled in Theorem 3.22. The \( \Sigma \)-invariance of the associated Chern connection then comes from Proposition 3.24. \( \square \)

### 3.4. Non-Abelian Hodge correspondence

Putting together the results of Section 3, we obtain, given a real semisimple Lie algebra \( \mathfrak{g} \), a hyperbolic 2-orbifold \( Y \) and a presentation \( Y \simeq [\Sigma \setminus X] \) of that orbifold as a quotient of a closed orientable hyperbolic surface \( X \) by the action of a finite group of isometries \( \Sigma \), a homeomorphism

\[
\text{Hom}^{c.r.}(\pi_1 Y; G)/G \simeq \mathcal{M}(X, \Sigma)(G)
\]

between the representation space \( \text{Hom}^{c.r.}(\pi_1 Y; G)/G \) of completely reducible representations of the orbifold fundamental group \( \pi_1 Y \) into the group of real points of \( \text{Int}(\mathfrak{g} \otimes \mathbb{C}) \) and the moduli space

\[
\mathcal{M}(X, \Sigma)(G) := \left\{ \text{\( \Sigma \)-polystable equivariant } G \text{-Higgs bundles } (E, \varphi, \tau) \text{ with vanishing first Chern class on } X \right\} / \text{isomorphism}
\]

of isomorphism classes of \( \Sigma \)-polystable equivariant \( G \)-Higgs bundles on \( X \). We shall refer to that homeomorphism as a \textbf{non-Abelian Hodge correspondence for orbifolds}, depending on the presentation \( Y \simeq [\Sigma \setminus X] \), and we now proceed to analyzing the Hitchin component of \( \text{Hom}(\pi_1 Y; G)/G \) in terms of that correspondence.

### 4. Parametrization of Hitchin components

Throughout this section, we fix a presentation \( Y \simeq [\Sigma \setminus X] \) of the orbifold \( Y \), where \( X \) is assumed to be a Riemann surface and \( \Sigma \) acts on \( X \) by transformations that are either holomorphic or anti-holomorphic (see Lemma 2.10), and we let \( G \) be the group of real points of \( \text{Int}(\mathfrak{g} \otimes \mathbb{C}) \), where \( \mathfrak{g} \) is now a \textit{split} real form of a complex \textit{simple} Lie algebra. By Lemma 2.10, if \( \varphi : \pi_1 Y \longrightarrow G \subset \text{GL}(\mathfrak{g}) \) is a Hitchin representation, then \( \mathfrak{g} \) is an irreducible \( \pi_1 Y \)-module under \( \varphi \). So, using the non-Abelian Hodge correspondence for orbifolds recalled in Section 3.4, we can think of \( \text{Hit}(\pi_1 Y, G)/G \) as a connected component of

\[
\text{Hom}^{c.r.}(\pi_1 Y; G)/G \simeq \mathcal{M}(X, \Sigma)(G).
\]

By Corollary 3.23 there is a well-defined map

\[
J : \mathcal{M}(X, \Sigma)(G) \longrightarrow \mathcal{M}_X(G)
\]

forgetting the \( \Sigma \)-equivariant structure \( \tau \). The group \( \Sigma \) acts on \( \mathcal{M}_X(G) \) (by pullback of bundles and Higgs fields) and, as in Lemma 2.8, the image of \( J \) is contained in \( \text{Fix}_\Sigma(\mathcal{M}_X(G)) \) but the resulting map

\[
J : \mathcal{M}(X, \Sigma)(G) \longrightarrow \text{Fix}_\Sigma(\mathcal{M}_X(G))
\]

is again neither injective nor surjective in general (see [Sch19] for an analysis of this map in the vector bundle case). In this section, we will show that, if we restrict it to \( \text{Hit}(\pi_1 Y, G) \subset \mathcal{M}(X, \Sigma)(G) \), then the map \( J \) induces a homeomorphism

\[
\text{Hit}(\pi_1 Y, G) \simeq \text{Fix}_\Sigma(\text{Hit}(\pi_1 X, G)).
\]
4.1. Equivariance of the Hitchin fibration. Recall that $\Sigma$ acts on $X$ by transformations that are either holomorphic or anti-holomorphic. The induced action on the canonical bundle $K_X$ defines a $\Sigma$-equivariant structure $(\tau_\sigma)_{\sigma \in \Sigma}$ in the holomorphic sense on $K_X$. As seen in Section 3.3 this in turn induces an action of $\Sigma$ on all tensor powers $K_X^d$ of the canonical bundle, and on sections of such bundles: if $s \in H^0(X; K_X^d)$ and $\sigma \in \Sigma$, we set $\sigma(s) := \tau_\sigma \circ s \circ \sigma^{-1}$.

\[
\begin{array}{ccc}
K_X^d & \xrightarrow{\tau_\sigma} & K_X^d \\
\downarrow{s} & & \downarrow{\sigma(s)} \\
X & \xrightarrow{\sigma} & X \\
\end{array}
\]

Since $\sigma$ and $\tau_\sigma$ are either simultaneously holomorphic or simultaneously anti-holomorphic, $\sigma(s)$ is indeed a holomorphic section of $K_X^d$. Explicitly for $d = 1$, as $\tau_\sigma : K_X \rightarrow K_X$ is just the transpose of the tangent map $T\sigma^{-1}$, we have
\[
\sigma(s) = \left\{ \begin{array}{ll}
(\sigma^{-1})^*s & \text{if } \sigma \text{ is holomorphic}, \\
(\sigma^{-1})^*s & \text{if } \sigma \text{ is anti-holomorphic},
\end{array} \right.
\]
where, by definition, $(\sigma^{-1})^*s$ sends $v \in T_zX$ to
\[
(s(\sigma^{-1}(z)) \circ T_z\sigma^{-1}) \cdot v \in \mathbb{C}.
\]

And finally, if $X$ is an open set in $\mathbb{C}$ with an action of $\Sigma$ and $s(z) = f(z) dz$, then
\[
\sigma(s) = \left\{ \begin{array}{ll}
(f \circ \sigma^{-1}) (\partial \sigma) dz & \text{if } \sigma \text{ is holomorphic}, \\
(f \circ \sigma^{-1}) (\partial \sigma) dz & \text{if } \sigma \text{ is anti-holomorphic},
\end{array} \right.
\]
where by $\partial \sigma$ we denote the $\mathbb{C}$-linear part of the differential $\partial \sigma$ of the $\mathbb{R}$-differentiable map $\sigma : \mathbb{C} \rightarrow \mathbb{C}$.

We now recall the definition of the Hitchin fibration $F : \mathcal{M}_X(G) \rightarrow \mathcal{B}_X(\mathfrak{g})$, where the Hitchin base $\mathcal{B}_X(\mathfrak{g})$ will be defined in (4.4).

**Remark 4.1.** This fibration was introduced by N. Hitchin for simple complex Lie groups $G_C$ in \cite{Hit92}. For a real Lie group $G$ like ours (=split real form of a connected simple complex Lie group of adjoint type), there are two possibilities to define the Hitchin fibration. Either, as in \cite{Hit92}, by composing the original Hitchin fibration $F_C : \mathcal{M}_X(G_C) \rightarrow \mathcal{B}_X(\mathfrak{g}_C)$ with the canonical map $\mathcal{M}_X(G) \rightarrow \mathcal{M}_X(G_C)$, or, as in \cite{GPPNR18}, by a direct definition generalizing the one in \cite{Hit87}. The latter is perhaps preferable from our point of view, because it avoids the injectivity defect of the canonical map $\mathcal{M}_X(G) \rightarrow \mathcal{M}_X(G_C)$. For the two approaches to actually coincide, one needs in particular to have $\mathcal{B}_X(\mathfrak{g}) = \mathcal{B}_X(\mathfrak{g}_C)$, which is true by the assumption that $\mathfrak{g}$ is a split real form of $\mathfrak{g}_C$ (see \cite{GPPNR18}).

Let $\mathfrak{g}$ be the split real form of a simple complex Lie algebra $\mathfrak{g}_C$ and let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ (with respect to the Killing form). Let $\mathfrak{t}_C := \mathfrak{t} \otimes \mathbb{C}$ and $\mathfrak{p}_C := \mathfrak{p} \otimes \mathbb{C}$ be the complexification of the subalgebra $\mathfrak{t}$ and the subspace $\mathfrak{p}$, respectively. As usual, we set $G_C := \text{Int}(\mathfrak{g}_C)$ and $K_C := \text{Int}(\mathfrak{t}_C)$, and we let $G$ (resp. $K$) denote the real form of $G_C$ (resp. $K_C$) with respect to the canonical real structure of $G_C = \text{Int}(\mathfrak{g} \otimes \mathbb{C})$. Then $G$ (resp. $K$) is a Lie group (not connected in general) with Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{t}$). In particular, $K$ is compact. The adjoint action of $K \subset G$ on $\mathfrak{g}$ preserves $\mathfrak{p}$, and the induced action of $K$ on $\mathfrak{p}$ is compatible with the canonical real structures of these spaces, in the sense that $A_k \xi = \text{Ad}_x \xi$ for all $k \in K_C$ and all $x \in \mathfrak{p}_C$. Let $r := \text{rk}(\mathfrak{g})$ denote the real rank of $\mathfrak{g}$. Note that since $\mathfrak{g}$ is split by assumption, this is equal to the rank of $\mathfrak{g}_C$. By a theorem due to Kostant and Rallis (\cite{KR71}), the $\mathbb{R}$-algebra of $K$-invariant regular functions on $\mathfrak{p}$ is generated by exactly $r$ homogeneous polynomials $(P_1, \ldots, P_r)$:

\[
\mathbb{R}[\mathfrak{p}]^K = \mathbb{R}[P_1, \ldots, P_r].
\]

We set $d_\alpha := \text{deg } P_\alpha - 1$ for all $\alpha \in \{1, \ldots, r\}$. The $(d_\alpha)_{1 \leq \alpha \leq r}$ depend only on the real Lie algebra $\mathfrak{g}$ and are called the exponents of $\mathfrak{g}$. Following \cite{Hit87} and \cite{GPPNR18}, every such family defines a fibration

\[
F : \mathcal{M}_X(G) \rightarrow \mathcal{B}_X(\mathfrak{g}) := \bigoplus_{\alpha=1}^r H^0(X; K_X^{d_\alpha+1})
\]

\[
(\mathcal{E}, \varphi) \mapsto (P_1(\varphi), \ldots, P_r(\varphi)).
\]
By Definition 3.17, the Higgs field $\varphi \in H^0(X; K_X \otimes \text{ad}_{p_\xi}(\mathcal{E}))$ of a $G$-Higgs bundle $(\mathcal{E}, \varphi)$ is a holomorphic 1-form on $X$ with values in the bundle of symmetric adjoint endomorphisms of $\mathcal{E}$, i.e. endomorphisms that are locally of the form $\text{ad}_g$ for some $\xi \in p_\xi$. Since $\text{ad}_{p_\xi}(\mathcal{E})$ has fiber $p_\xi$ and structure group $K_\xi$, and each $P_\alpha \in \mathbb{R}[p]^K$ defines a $K_\xi$-invariant $\mathbb{C}$-valued polynomial function on $p_\xi$, we have that $P_\alpha(\varphi)$ is indeed a (homogeneous) holomorphic differential of degree equal to $\deg P_\alpha = d_\alpha + 1$ on $X$.

We shall now see that the Hitchin fibration (4.4) is $\Sigma$-equivariant. Recall first (see 4.2) that the finite group $\Sigma$, consisting of transformations of $X$ that are either holomorphic or anti-holomorphic, acts on each complex vector space $H^0(X; K_X^{d_\alpha+1})$. Moreover, if $(\mathcal{E}, \varphi)$ is a $G$-Higgs bundle on $X$ and $\sigma \in \Sigma$, then there is a $G$-Higgs bundle

\begin{equation}
(\sigma(\mathcal{E}), \sigma(\varphi)) = \begin{cases}
((\sigma^{-1})^*\mathcal{E}, (\sigma^{-1})^*\varphi) & \text{if } \sigma \text{ is holomorphic}, \\
((\sigma^{-1})^*\mathcal{E}, (\sigma^*)^*\varphi) & \text{if } \sigma \text{ is anti-holomorphic},
\end{cases}
\end{equation}

where $\sigma(\varphi) \in H^0(X; \sigma(K_X) \otimes \text{ad}_{p_\xi}(\sigma(\mathcal{E}))) = H^0(X; K_X \otimes \text{ad}_{p_\xi}(\sigma(\mathcal{E})))$, since $K_X$ has a canonical $\Sigma$-equivariant structure (therefore is canonically isomorphic to $K_X$). The point is that $\sigma(\varphi)$ is indeed a Higgs field on the $K_\xi$-bundle $\sigma(\mathcal{E})$. Note that if, additionally, a $\Sigma$-equivariant structure $\tau$ on $\mathcal{E}$ has been given, then there is a canonical isomorphism $\sigma(\mathcal{E}) \simeq \mathcal{E}$, and $\sigma(\varphi)$ may therefore be viewed as a Higgs field on the original holomorphic bundle $\mathcal{E}$: we recover in this way the canonical $\Sigma$-action $\varphi \mapsto \sigma(\varphi)$ on sections of the $\Sigma$-equivariant bundle $K_X \otimes \text{ad}_{p_\xi}(\mathcal{E})$, as defined in (3.7). We now want to compare $F(\sigma(\mathcal{E}), \sigma(\varphi))$ and $\sigma(F(\mathcal{E}, \varphi))$, where $F$ is the Hitchin fibration (4.4) and $\sigma \in \Sigma$.

**Proposition 4.2.** Let $\mathfrak{g}$ be the split real form of a simple complex Lie algebra $\mathfrak{g}_C$ and let $G$ be the associated real form of the simple complex Lie group of adjoint type $G_C := \text{Int}(\mathfrak{g}_C)$. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and let $K$ be the compact real form of $\text{Int}(\mathfrak{t} \otimes \mathbb{C})$. Let $X$ be a compact connected Riemann surface of genus $g \geq 2$ and let $\Sigma$ be a finite group acting effectively on $X$ by transformations that are either holomorphic or anti-holomorphic. For any choice of generators $(P_1, \ldots, P_r)$ of the $\mathbb{R}$-algebra $\mathbb{R}[\mathfrak{p}]^K$, we denote by $F$ the associated Hitchin fibration

$$F : \mathcal{M}_X(G) \longrightarrow \mathcal{B}_X(\mathfrak{g}) := \bigoplus_{\alpha=1}^r H^0(X; K_X^{d_\alpha+1})$$

$$\mathcal{E}, \varphi \longmapsto (P_1(\varphi), \ldots, P_r(\varphi)).$$

where $d_\alpha := \deg P_\alpha - 1$ and $r := \text{rk}(\mathfrak{g})$.

Then $F : \mathcal{M}_X(G) \longrightarrow \mathcal{B}_X(\mathfrak{g})$ is $\Sigma$-equivariant with respect to the $\Sigma$-action on $\mathcal{M}_X(G)$ defined in (4.5) and the $\Sigma$-action on $\mathcal{B}_X(\mathfrak{g})$ defined by means of (4.2).

**Proof.** Let us write simply $P_\alpha(\varphi)$ and $P_\alpha(\sigma(\varphi))$ for $P_\alpha(\mathcal{E}, \varphi)$ and $P_\alpha(\sigma(\mathcal{E}), \sigma(\varphi))$, respectively.

The Higgs field $\varphi$ is a section of $\text{ad}_{p_\xi}(\mathcal{E}) \otimes K_X$, so it is locally of the form $\xi \otimes dz$, where $\xi$ is a $p_\xi$-valued holomorphic function. So, on the one hand, for all $\alpha \in \{1, \ldots, r\}$, the holomorphic differential $P_\alpha(\varphi)$ is locally of the form $(P_\alpha \circ \xi)(dz)^{d_\alpha+1}$. And on the other hand, for all $\sigma \in \Sigma$, the Higgs field $\sigma(\varphi)$ (on $\sigma(\mathcal{E})$) is locally of the form $\sigma(\xi) \otimes \sigma(dz)$, where $\sigma(dz) = (\sigma^{-1})^*dz$ and $\sigma(\xi) = \xi \circ \sigma^{-1}$ if $\sigma$ is holomorphic on $X$, and $\sigma(dz) = (\sigma^{-1})^*dz$ and $\sigma(\xi) = \xi \circ \sigma^{-1}$ if $\sigma$ is anti-holomorphic on $X$ (in the latter expression, complex conjugation in $p_\xi$ is taken with respect to the real form $\mathfrak{p}$).

To compute $P_\alpha(\sigma(\varphi))$, let us recall that $(P_1, \ldots, P_r)$ are generators of the $\mathbb{R}$-algebra $\mathbb{R}[\mathfrak{p}]^K$. Equivalently, they are generators of the $\mathbb{C}$-algebra $\mathbb{C}[[\mathfrak{p}]]^K \simeq \mathbb{R}[\mathfrak{p}]^K \otimes \mathbb{C}$ that, in addition, are fixed points of the canonical real structure of $\mathbb{R}[\mathfrak{p}]^K \otimes \mathbb{C}$. Therefore, if $\sigma$ is anti-holomorphic, we have $P_\alpha(\xi \circ \sigma^{-1}) = P_\alpha(\xi \circ \sigma^{-1})$, where complex conjugation on the right-hand side is the usual one on $\mathbb{C}$, i.e. the function $P_\alpha : p_\xi \longrightarrow \mathbb{C}$ is a real function in the sense that it commutes to the given real structures of $p_\xi$ and $\mathbb{C}$ (which is indeed the case for polynomial functions with real coefficients in a real basis of $p_\xi$). We note that the $\mathbb{C}$-valued function $\xi$ depends on the choice of a local trivialization of $\text{ad}_{p_\xi}(\mathcal{E})$, but $P_\alpha \circ \xi$ is independent of such a choice because the function $P_\alpha : p_\xi \longrightarrow \mathbb{C}$ is $K_\xi$-invariant and $K_\xi$ is the structure group of $\text{ad}_{p_\xi}(\mathcal{E})$.

Thus, we have shown that, if $\varphi$ is locally of the form $dz \otimes \xi$, then $P_\alpha(\varphi)$ is locally of the form $(P_\alpha \circ \xi)(dz)^{d_\alpha+1}$ and $P_\alpha(\sigma(\varphi))$ is locally of the form

$$\begin{cases}
((P_\alpha \circ \xi) \circ \sigma^{-1}) \otimes ((\sigma^{-1})^*dz)^{d_\alpha+1} & \text{if } \sigma \text{ is holomorphic}, \\
((P_\alpha \circ \xi) \circ \sigma^{-1}) \otimes ((\sigma^{-1})^*dz)^{d_\alpha+1} & \text{if } \sigma \text{ is anti-holomorphic}.
\end{cases}$$
Comparing this with the definition of the $\Sigma$-action on $H^0(X; K^d_X)$ given in (4.3), we have indeed that $P_0(\sigma(\varphi)) = \sigma(P_0(\varphi))$. 

4.2. Invariant Hitchin representations. In [Hit92], Hitchin constructed a section $s : B_X(\mathfrak{g}) \to M_X(G)$ of the Hitchin fibration $F : M_X(G) \to B_X(G)$ whose image is exactly the Hitchin component $\text{Hit}(\pi_1, X, G)$, and we will now check that this section is $\Sigma$-equivariant in our context. This will enable us to prove Theorem 21.13.

Let us first briefly recall Hitchin’s construction of his section, which uses Lie-theoretic results of Kostant ([Kos63]). One starts with a split real form $\mathfrak{g}$ of a complex simple Lie algebra $\mathfrak{g}_C$ and a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then one chooses a regular nilpotent element $e \in \mathfrak{p}$ (i.e. $\text{ad}_e$ is a nilpotent endomorphism of $\mathfrak{g}$ whose centralizer is of the smallest possible dimension, equal to the rank of $\mathfrak{g}$). By the strong Jacobson-Morozov Lemma ([KR71 Proposition 4]), $e$ can be embedded in a copy of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, i.e. one can find $x \in \mathfrak{k}$ semisimple and $\tilde{e} \in \mathfrak{p}$ nilpotent such that

$$[x, e] = e, \quad [x, \tilde{e}] = -\tilde{e} \quad \text{and} \quad [e, \tilde{e}] = x.$$ 

We henceforth fix such a triple $(x, e, \tilde{e})$ and we let

$$\mathfrak{g}_C = \bigoplus_{\alpha=1}^{r} V_\alpha$$

be the decomposition of the $\mathfrak{sl}(2, \mathbb{C})$-module $\mathfrak{g}_C$ into $r = \text{rk}(\mathfrak{g}_C)$ irreducible representations ([Kos59]): each $V_\alpha$ is of odd dimension $2d_\alpha + 1$ where the $(d_\alpha)_{1 \leq \alpha \leq r}$ are the exponents of $\mathfrak{g}_C$ (or equivalently, of $\mathfrak{g}$, since we are assuming that $\mathfrak{g}$ is split), and the eigenvalues of the restriction of $\text{ad}_x$ to $V_\alpha$ are the integers in the interval $[-d_\alpha, d_\alpha]$. For all $\alpha \in \{1, \ldots, r\}$ and all $d \in [-d_\alpha, d_\alpha] \cap \mathbb{Z}$, let us denote by $\mathfrak{g}_C^{(d)}$ the subspace of $\mathfrak{g}_C$ on which $\text{ad}_x$ acts with eigenvalue $d$. Then

$$\mathfrak{g}_C = \bigoplus_{d=-M}^{M} \mathfrak{g}_C^{(d)},$$

where $M = \max_{1 \leq \alpha \leq r} d_\alpha$. Note that the eigenvalues of $\text{ad}_x$ are all real and that $\mathfrak{g}_C^{(d)} = \mathfrak{g}_C^{(\overline{d})} \otimes \mathbb{C}$ has a canonical real structure (likewise $V_\alpha$ has a canonical real structure, induced by that of $\mathfrak{g}_C$, since the latter is a real $\mathfrak{sl}(2, \mathbb{C})$-module with respect to the real form $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{sl}(2, \mathbb{C})$). Let us now consider the Lie algebra bundle

$$(4.7) \quad \mathcal{E}_\text{can} := \bigoplus_{d=-M}^{M} K_X^d \otimes \mathfrak{g}_C^{(d)}$$

with fiber $\mathfrak{g}_C$ and structure group $K_C := \text{Int}(\mathfrak{k} \otimes \mathbb{C})$. This is the bundle introduced by Hitchin in [Hit92 Section 5]. It is endowed with the canonical Higgs field $\varphi_0 := \tilde{e}$, where the latter element is seen as a section of $K_X \otimes (K_X^{-1} \otimes \mathfrak{g}_C^{(-1)}) \simeq \mathfrak{g}^{(-1)}$: indeed, $\tilde{e} \in \mathfrak{g}_C^{(-1)}$ because $[x, \tilde{e}] = -\tilde{e}$ by construction of the triple $(x, e, \tilde{e})$. We note that $\mathcal{E}_\text{can}$ has a canonical $\Sigma$-equivariant structure, induced by the canonical $\Sigma$-equivariant structure of $K_X$ and the canonical real structure of $\mathfrak{g}_C^{(d)} = \mathfrak{g} \otimes \mathbb{C}$: if $\sigma \in \Sigma$, then $\sigma$ acts on $K_X^d \otimes \mathfrak{g}_C^{(d)}$ via $\tau^d_\sigma \otimes \varepsilon_\sigma$, where $\tau_\sigma$ is the transformation of $K_X$ induced by $\sigma$ and $\varepsilon_\sigma : \mathfrak{g}_C^{(d)} \to \mathfrak{g}_C^{(\overline{d})}$ is the identity map if $\sigma$ is holomorphic on $X$ and complex conjugation with respect to $\mathfrak{g}_C^{(d)}$ if $\sigma$ is anti-holomorphic on $X$.

The Hitchin section is then defined as follows. Given $p = (p_1, \ldots, p_r) \in B_X(\mathfrak{g}) = \bigoplus_{\alpha=1}^{r} H^0(X; K_{X}^{d_\alpha+1})$, one sets

$$\varphi(p) := \tilde{e} + \sum_{\alpha=1}^{r} p_\alpha \otimes e_\alpha,$$

where $e_1, \ldots, e_r$ are the highest weight vectors of the $\mathfrak{sl}(2, \mathbb{C})$-module $\mathfrak{g}_C$ (with respect to the choice of the Lie sub-algebra of $\mathfrak{g}_C$ generated by the $\mathfrak{sl}(2, \mathbb{R})$-triple $(x, e, \tilde{e})$, i.e. $e_\alpha \in V_\alpha \cap \mathfrak{g}$ and $\text{ad}_x e_\alpha = d_\alpha e_\alpha$). Since $p_\alpha$ is a section of $K_{X}^{d_\alpha+1}$ and $\tilde{e}$ and all the $e_\alpha$ lie in $\mathfrak{p}$, one has $\varphi(p) \in H^0(X; K_X \otimes \text{ad}_{p_\alpha}(\mathcal{E}_\text{can}))$, so $\varphi(p)$ is indeed a Higgs field on $\mathcal{E}_\text{can}$. Hitchin proved in [Hit92] that the map $p \mapsto (\mathcal{E}_\text{can}, \varphi(p))$ is a section of the Hitchin fibration (4.4), whose image is exactly $\text{Hit}(\pi_1 X, G)$.
Lemma 4.3. The Hitchin section

\[ s : \mathcal{B}_X(\mathfrak{g}) \rightarrow \text{Hit}(\pi_1 X, G) \]

is Σ-equivariant. In particular, it induces a homeomorphism \( \text{Fix}_\Sigma(\mathcal{B}_X(\mathfrak{g})) \simeq \text{Fix}_\Sigma(\text{Hit}(\pi_1 X, G)) \).

Proof. As \( \mathcal{E}_\text{can} \) is Σ-equivariant, there are canonical identifications \( \sigma(\mathcal{E}_\text{can}) \simeq \mathcal{E}_\text{can} \) for all \( \sigma \in \Sigma \) and we can think of \( \sigma(\varphi(p)) \) as a Higgs field on \( \mathcal{E}_\text{can} \) itself. Recall that, by definition, \( \varphi(p) = \tilde{e} + \sum_{a=1}^{r} p_a \otimes e_a \). Since \( \tilde{e} \) and all the \( e_a \) are real with respect to the canonical real structure of \( \mathfrak{g}_C = \mathfrak{g} \otimes \mathbb{C} \), the Σ-equivariance of \( s \) follows immediately from the definition of the Σ-action on the Hitchin base \( \mathcal{B}_X(\mathfrak{g}) \) and the Σ-action on the set of Higgs fields on a fixed Σ-equivariant bundle:

\[ \varphi(\sigma(p)) = \tilde{e} + \sum_{a=1}^{r} \sigma(p_a) \otimes e_a = \sigma(\tilde{e}) + \sum_{a=1}^{r} \sigma(p_a) \otimes \sigma(e_a) = \sigma(\tilde{e} + \sum_{a=1}^{r} p_a \otimes e_a) = \sigma(\varphi(p)) \]

where \( \tilde{e} \) and all the \( e_a \) are indeed Σ-equivariant when seen as sections of the Σ-equivariant bundles \( X \times \mathfrak{g}_C^{(d)} \) because they are real elements of \( \mathfrak{g}_C^{(d)} = \mathfrak{g}^{(d)} \otimes \mathbb{C} \).

We can now prove Theorem 2.13:

Proof of Theorem 2.13 We already know, by Proposition 2.11, that, given a presentation \( Y \simeq [\Sigma \backslash X] \), the map

\[ j : \text{Hit}(\pi_1 Y, G) \rightarrow \text{Hit}(\pi_1 X, G) \]

is injective. To prove that it is surjective, let us consider an element \([\varrho] \in \text{Fix}_\Sigma \text{Hit}(\pi_1 X, G)\) and let us fix a hyperbolic structure on \( Y \) (or, equivalently, on \( X \), with \( \Sigma \) acting by transformations that are either holomorphic or anti-holomorphic). By Lemma 4.3 there is a unique \( p \in \text{Fix}_\Sigma(\mathcal{B}_X(\mathfrak{g})) \) such that \( s(p) = (\mathcal{E}_\text{can}, \varphi(p)) \) is the \( G \)-Higgs bundle corresponding to \( \varrho \). Since \( \mathcal{E}_\text{can} \) is Σ-equivariant and \( \varphi(p) \) is Σ-invariant, the Non-Abelian Hodge Correspondence of Section 3.4 shows that there is an Σ-equivariant flat \( \mathfrak{g} \)-bundle \( (\mathcal{E}_\text{can}, \nabla_p) \) associated to the Σ-equivariant \( G \)-Higgs bundle \( (\mathcal{E}_\text{can}, \varphi(p)) \). In particular, the flat connection \( \nabla_p \) is Σ-invariant, so, by Theorem 3.5, the associated holonomy representation \( \varrho_{\nabla_p} = \varrho \), from \( \pi_1 X \) to \( G \), extends to a representation \( \varrho_{\nabla_p} : \pi_1 Y \rightarrow G \).

It remains to prove that \( \varrho_{\nabla_p} \) is indeed a Hitchin representation of \( \pi_1 Y \). This follows from the connectedness of the real vector space \( \text{Fix}_\Sigma(\mathcal{B}_X(\mathfrak{g})) \) and the fact that the representation \( \varrho_{\nabla_p} : \pi_1 Y \rightarrow G \) associated to the origin \( p = 0 \) via the construction above is precisely the Fuchsian representation associated to the fixed hyperbolic structure on \( Y \) (in particular, \( \varrho_0 \) is a Hitchin representation).

Corollary 4.4. The Hitchin component \( \text{Hit}(\pi_1 Y, G) \) is homeomorphic to the real vector space \( \text{Fix}_\Sigma(\mathcal{B}_X(\mathfrak{g})) \).

In particular, it is a contractible space.

5. Invariant differentials

5.1. Spaces of differentials for orientable orbifolds. Assume, for this subsection, that \( Y \) is orientable, i.e., its underlying topological space \( |Y| \) is a closed orientable surface, and the only singularities are cone points. We denote by \( x_1, \ldots, x_k \) the cone points of order, respectively, \( m_1, \ldots, m_k \geq 2 \). As usual, we assume that \( \chi(Y) < 0 \). We fix, for the rest of this subsection, an orbifold complex analytic structure on \( Y \). This structure induces a complex structure on the underlying surface \( |Y| \): on the open subset \( U = Y \setminus \{x_1, \ldots, x_k\} \), the complex structure on \( |Y| \) is identical to the orbifold analytic structure on \( Y \); the complex structure of the open subset \( U \) can be completed to a complex structure on the closed surface \( |Y| \) in a unique way. With this complex structure, \( |Y| \) becomes a Riemann surface, and we denote by \( K|Y| \) the canonical bundle of \( |Y| \). Define the number

\[ O(d, m) := \left\lfloor \frac{d - \frac{d}{m}}{m} \right\rfloor, \]

where the brackets \( \lfloor \ldots \rfloor \) stand for integer part, the biggest integer not greater than the argument. For every point \( x \in |Y| \), we denote by \( L_x \) the point line bundle supported on \( x \), this is characterized as the unique line
bundle on $|Y|$ admitting a holomorphic section with a zero of order one at $x$ and no other zeros. For every integer $d \geq 2$, we consider the following line bundle

$$K(Y,d) := K^{d}_{|Y|} \otimes \bigotimes_{i=1}^{k} L^{O(d,m_{i})}_{x_{i}}.$$

Holomorphic sections of $K(Y,d)$ can be seen as meromorphic $d$-forms (sections of $K^{d}_{|Y|}$) which are allowed to have a pole of order at most $O(d,m_{i})$ at the point $x_{i}$, and no other pole. The vector space of all such sections is, as usual, denoted by $H^{0}(Y,K(Y,d))$. We will say that a holomorphic section of $K(Y,d)$ is a regular $d$-differential on $Y$. Such differentials are allowed to have poles of controlled order at the singular points of $Y$.

**Proposition 5.1.** Let $d \geq 2$ be an integer. Then we have the formula:

$$\dim_{\mathbb{C}}(H^{0}(Y,K(Y,d))) = -\frac{1}{2} \chi(|Y|)(2d-1) + \sum_{i=1}^{k} O(d,m_{i}).$$

**Proof.** If $g$ is the genus of $|Y|$, the degree of $K(Y,d)$ is

$$\deg(K(Y,d)) = \deg(K^{d}_{|Y|}) + \sum_{i} O(d,m_{i}) = 2d(g-1) + \sum_{i} O(d,m_{i}).$$

If $M$ is a line bundle with $\deg(M) > 2g - 2$, we have by the Riemann-Roch theorem: $\dim_{\mathbb{C}}(H^{0}(Y,M)) = \deg(M) + 1 - g$. So we get the formula. \qed

Let us now choose a presentation $Y \simeq [\Sigma \setminus X]$, and we denote the projection by $\pi : X \rightarrow Y$. We consider on $X$ the pull back of the orbifold complex analytic structure on $Y$. Since $X$ has trivial orbifold structure, this is a complex structure in the usual sense. The map $\pi : X \rightarrow |Y|$ is then holomorphic. We will now describe a natural identification between the space $H^{0}(Y,K(Y,d))$ of regular $d$-differentials on $Y$ and the space of $d$-differentials on $X$ which are invariant by the action of $\Sigma$ defined in Section 4.2.

**Proposition 5.2.** For every meromorphic $d$-form $q$ on $|Y|$, the pull-back $\pi^{*}q$ is a $\Sigma$-invariant meromorphic $d$-form on $X$. If $q \in H^{0}(Y,K(Y,d))$, then the pull-back $\pi^{*}q$ is holomorphic, giving a map

$$\pi^{*} : H^{0}(Y,K(Y,d)) \rightarrow \text{Fix}_{\Sigma}(H^{0}(X,K^{d}_{X})).$$

Moreover, this map is an isomorphism of $\mathbb{C}$-vector spaces.

With this, Proposition 5.1 gives us the dimension of $\text{Fix}_{\Sigma}(H^{0}(X,K^{d}_{X}))$. We note that the dimension of the space $\text{Fix}_{\Sigma}(H^{0}(X,K^{d}_{X}))$ was also computed by J. Lewittes as part of his PhD thesis ([Lew63] ). The proof depends on the following:

**Lemma 5.3.** Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function from an open neighborhood $U$ of $0 \in \mathbb{C}$. Assume that $f$ has a zero of order $m$. Let $q$ be a meromorphic differential of degree $d$ on a neighborhood of $0 \in \mathbb{C}$, with a pole of order $s$. Then the pull-back $f^{*}q$ is holomorphic if and only if $s \leq O(d,m)$.

**Proof.** We can assume $f(z) = z^{m}$ and $q(z) = \frac{1}{2\pi i}(dz)^{d}$. So $(f^{*}q)(z) = \frac{1}{2\pi i}(dz^{m})^{d} = m^{d}z^{d(m-1)-ms}(dz)^{d}$ and this is holomorphic if and only if $d(m-1) - ms \geq 0$, i.e. $s \leq O(d,m)$. \qed

**Proof of Proposition 5.2.** If $q$ is a meromorphic differential of degree $d$ on $Y$, with poles only at the points $x_{i}$ of order not greater than $O(d,m_{i})$, then, by Lemma 5.3, the pull-back of $q$ by the holomorphic map $\pi$ is a holomorphic differential on $X$ that is invariant by $\Sigma$. \qed

5.2. **Spaces of differentials for non-orientable orbifolds.** Assume, for this subsection, that $Y$ is not orientable. We denote by $x_{1}, \ldots, x_{k}$ its cone points of order, respectively, $m_{1}, \ldots, m_{k} \geq 2$, and by $y_{1}, \ldots, y_{\ell}$ its corner reflectors of order, respectively, $n_{1}, \ldots, n_{\ell} \geq 2$. Denote by $Y^{+}$ its orientation double cover, an orientable orbifold with a $(2 : 1)$-orbifold covering $\eta : Y^{+} \rightarrow Y$ and a $\mathbb{Z}/2\mathbb{Z}$-action such that $(\mathbb{Z}/2\mathbb{Z}) \backslash Y^{+} = Y$. We denote by $u_{i}, v_{i}$ the two cone points of $Y^{+}$ in $\eta^{-1}(x_{i})$, each of order $m_{i}$, and $w_{j}$ the cone point of $Y^{+}$ in $\eta^{-1}(y_{j})$ of order $n_{j}$. The $\mathbb{Z}/2\mathbb{Z}$-action sends $u_{i}$ to $v_{i}$ for all $i$, and fixes $w_{j}$, for all $j$. We have that $\chi(Y^{+}) = 2\chi(Y)$ and $\chi(|Y^{+}|) = 2\chi(|Y|)$. An orbifold dianalytic structure on $Y$ can be defined as an orbifold complex analytic structure on $Y^{+}$ such that $\mathbb{Z}/2\mathbb{Z}$ acts with an anti-holomorphic map. We will fix an orbifold
Proposition 5.5. The group $\mathbb{Z}/2\mathbb{Z}$-action is a real structure on $|Y^+|$, which lifts in a natural way to a real structure $\tau$ on the line bundle $K(Y^+, d)$. We will denote by $K(Y, d)$ the $\mathbb{Z}/2\mathbb{Z}$-equivariant line bundle $(K(Y^+, d), \tau)$. The sections of $K(Y, d)$ are defined as the $\mathbb{Z}/2\mathbb{Z}$-invariant sections of $K(Y^+, d)$ (also called its real sections). We will write

$$H^0(Y, K(Y, d)) := \text{Fix}_\tau(H^0(Y^+, K(Y^+, d))).$$

Notice that $\mathbb{Z}/2\mathbb{Z}$ acts on $H^0(Y^+, K(Y^+, d))$ by a $\mathbb{C}$-anti-linear involution, hence $H^0(Y, K(Y, d))$ is only a real vector subspace. We define the regular $d$-differentials on $Y$ as the elements of $H^0(Y; K(Y, d))$.

**Proposition 5.4.** Let $d \geq 2$ be an integer. Then we have the formula:

$$\dim \mathbb{R} H^0(Y, K(Y, d)) = \dim \mathbb{C} H^0(Y^+, K(Y^+, d)) = -\chi(|Y|)(2d - 1) + 2 \sum_{i=1}^k O(d, m_i) + \sum_{j=1}^\ell O(d, n_j).$$

**Proof.** The fixed point set of a $\mathbb{C}$-anti linear involution is always a real vector space of real dimension equal to the complex dimension of the ambient space. The formula comes from Proposition 5.1.

Let us now choose a presentation $Y \simeq [\Sigma \setminus X]$, and we denote the projection by $\pi : X \to Y$. Since $X$ is orientable, the map $\pi$ factors through a map $\pi^+ : X \to Y^+$. Let $\Sigma^+ \subset \Sigma$ be the subgroup consisting of orientation-preserving maps. Then $[\Sigma^+ \setminus X]$ is a presentation of $Y^+$. We consider on $X$ the pull back of the orbifold complex analytic structure on $Y^+$. We will now describe a natural identification between the space $H^0(Y, K(Y, d))$ of regular $d$-differentials on $Y$ and the space of $d$-differentials on $X$ which are invariant by the action of $\Sigma$ defined in Section 4.2.

**Proposition 5.5.** The $\mathbb{C}$-linear isomorphism $\pi^+ : H^0(Y^+, K(Y^+, d)) \to \text{Fix}_{\Sigma^+}(H^0(Y, K(X^+, d)))$ defined in Proposition 5.2 restricts to an isomorphism of $\mathbb{R}$-vector spaces

$$\pi^* : H^0(Y, K(Y, d)) = \text{Fix}_{\Sigma^+}(H^0(Y^+, K(Y^+, d))) \to \text{Fix}_{\Sigma^+}(H^0(Y, K(X^+, d))).$$

**Proof.** The group $\Sigma/\Sigma^+ \simeq \mathbb{Z}/2\mathbb{Z}$ is acting on $\text{Fix}_{\Sigma^+}(H^0(Y, K(X^+, d)))$ via a $\mathbb{C}$-anti linear involution. The $\mathbb{C}$-linear isomorphism $\pi^+$ conjugates this action with the $\mathbb{Z}/2\mathbb{Z}$-action on $H^0(Y^+, K(Y^+, d))$.

Therefore, Proposition 5.4 gives us the dimension of $\text{Fix}_{\Sigma^+}(H^0(Y, K(X^+, d)))$ in the general case.

5.3. **Topology of the Hitchin components.** Let $G$ be the group of real points of $\text{Int}(\mathfrak{g} \otimes \mathbb{C})$, where $\mathfrak{g}$ is a split real form of a complex simple Lie algebra of rank $r$. Let $d_1, \ldots, d_r$ be the exponents of $\mathfrak{g}$, as in Section 4. Let $Y$ be an orbifold with negative orbifold Euler characteristic, and we choose an orbifold complex analytic structure (or dianalytic structure, if $Y$ is non-orientable) on $Y$. We define the *Hitchin base* of $Y$ as

$$B_Y(\mathfrak{g}) = \bigoplus_{\alpha=1}^r H^0(Y, K(Y, d_\alpha + 1)),
$$

where $K(Y, d)$ is the holomorphic line bundle defined in (5.1). We can now state the main result of this paper.

**Theorem 5.6.** The Hitchin component $\text{Hit}(\pi_1 Y, G)$ is homeomorphic to $B_Y(\mathfrak{g})$. In particular, it is homeomorphic to a real vector space of dimension

$$-\chi(|Y|) \dim G + \sum_{\alpha=1}^r \left( 2 \sum_{i=1}^k O(d_\alpha + 1, m_i) + \sum_{j=1}^\ell O(d_\alpha + 1, n_j) \right).$$

**Proof.** The first statement is given by Corollary 4.4 and Proposition 5.5. The dimensions are computed using Proposition 5.4 and the following formula, a consequence of 4.6:

$$\sum_{\alpha=1}^r (2d_\alpha + 1) = \dim G.$$

This becomes more explicit using the classification of the simple complex Lie algebras. We can list the dimensions of the groups and the exponents (see for example [Dam14, Table 2]).
5.4. Approximation formula. We have the following corollary of Theorem 5.6.

**Corollary 5.7.** If \( Y \) is such that \( k = \ell = 0 \), we have

\[
\dim \text{Hit}(\pi_1 Y, G) = -\chi(Y) \dim G = -\chi(|Y|) \dim G.
\]

When \( Y \) is an orientable surface, this formula was found by Hitchin [Hitchin92]. Our corollary proves that the same formula is still valid when \( Y \) is a non-orientable surface, or an orbifold having only mirror points as singularities.

**Remark 5.8.** In [LM09], F. Labourie and G. McShane introduced a notion of Hitchin component for the \( \text{PGL}(n, \mathbb{R}) \)-representation variety of the topological fundamental group of a compact orientable surface with boundary \( S \). The boundary condition that they impose is that a simple loop around a boundary component should go to a purely loxodromic element of \( \text{PGL}(n, \mathbb{R}) \), thus generalizing the classical Teichmüller space of hyperbolic structures with totally geodesic boundary on \( S \). Under this assumption, they show in [LM09, Theorem 9.2.2.2] that \( \varrho : \pi_1^{\text{top}}(S) \to \text{PSL}(n, \mathbb{R}) \) is a Hitchin representation in their sense if and only if it extends to a Hitchin representation \( \hat{\varrho} : \pi_1^{\text{top}}(\hat{S}) \to \text{PSL}(n, \mathbb{R}) \), where \( \hat{S} \) is the doubled surface, such that \( \hat{\varrho} \) is \( \mathbb{Z}/2\mathbb{Z} \)-equivariant with respect to the natural involution of \( \hat{S} \) and the involution of \( \text{PSL}(n, \mathbb{R}) \) given by conjugation by \( J_n := \text{diag}(1, -1, 1, -1, \ldots) \in \text{PGL}(n, \mathbb{R}) \). Equivalently, \( \hat{\varrho} \) is a representation

\[
\pi_1^{\text{top}}(\hat{S}) \times (\mathbb{Z}/2\mathbb{Z}) \to \text{PSL}(n, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z}) \subset \text{PGL}(n, \mathbb{R}).
\]

Since \( \pi_1^{\text{top}}(\hat{S}) \times (\mathbb{Z}/2\mathbb{Z}) \) is isomorphic to the orbifold fundamental group of the orbifold \( Y_\hat{S} \) with underlying space \( S \) obtained by decreeing that all boundary points of \( S \) are mirror points, the Hitchin component of \( S \) in the sense of Labourie and McShane is homeomorphic to the Hitchin component of \( Y_\hat{S} \) in the sense of Definition 2.5. In particular, Theorem 5.6 and Corollary 5.7 show that that space is homeomorphic to an open ball of dimension \( -\chi(S) \dim G = -\chi(|Y|) \dim G \).

The formula in Corollary 5.7 does not extend immediately to the case of orbifolds, because their Euler characteristic is a rational number in general. However, it is interesting to remark that, in general, this formula gives a good approximation for the dimension of the Hitchin components.

**Proposition 5.9.** Let \( r(Y, G) := -\chi(Y) \dim G - \dim \text{Hit}(\pi_1 Y, G) \). Then the following estimate holds:

\[
-rk(G) \left( k + \frac{\ell}{2} \right) < r(Y, G) < \frac{3}{2} rk(G) \left( k + \frac{\ell}{2} \right).
\]

More precisely,

\[
-rk(G) \left( \sum_{i=1}^{k} \left( 1 - \frac{1}{m_i} \right) + \frac{\ell}{2} \sum_{j=1}^{\ell} \left( 1 - \frac{1}{n_j} \right) \right) \leq r(Y, G) \leq rk(G) \left( \sum_{i=1}^{k} \left( 1 + \frac{1}{m_i} \right) + \frac{\ell}{2} \sum_{j=1}^{\ell} \left( 1 + \frac{1}{n_j} \right) \right).
\]

**Proof.** Write the quantity \( r(Y, G) \) using Theorem 5.6 (2.1) and (5.3). Then use the inequality

\[
0 \leq (d + 1) \left( 1 - \frac{1}{m} \right) - \left( (d + 1) \left( 1 - \frac{1}{m} \right) \right) < 1.
\]

It is worth noting that in the families of classical Lie groups, the dimension of the group grows quadratically with the rank, so the estimate is asymptotically good.
Remark 5.10. When $H$ is a split simple real algebraic group and $Y$ is orientable, Larsen and Lubotzky [LL13] gave an asymptotic estimate of the dimension of $\text{Hom}^\text{epi}(\pi_1 Y; H)$, which is the Zariski-closure in $\text{Hom}(\pi_1 Y; H)$ of the set of Zariski-dense representations $\pi_1 Y \rightarrow H(\mathbb{R})$. More precisely,

$$\dim \text{Hom}^\text{epi}(\pi_1 Y; H)/H = -\chi(Y) \dim(H) + O(\text{rk}(H)).$$

The spaces that they study are not always comparable with the Hitchin components, see Section 6.1 for examples of Hitchin components containing no Zariski dense representations.

5.5. Formulae for the dimension. In the next section, we need some explicit computations for the dimension of the Hitchin components. We will give here some basic results that will be needed later. To keep the notation simpler, we will only consider the case of an orientable orbifold $Y$. This suffices, because we proved in Proposition 5.5 that

$$\dim \text{Hit}(\pi_1 Y, G) = \frac{1}{2} \dim \text{Hit}(\pi_1 Y^+, G)$$

We will denote by $g$ the genus of the surface $[Y]$. We use the symbol $O(d, m)$ only for integers $d \geq 2$ and integers $m \geq 2$. For a fixed $d$, the function $O(d, m)$ has minimum value $\lfloor \frac{d}{2} \rfloor \geq 1$ when $m = 2$, and it then increases until its maximum value $d - 1$, which it achieves exactly when $m \geq d$.

Corollary 5.11. The dimension of the spaces of regular $d$-differentials on $Y$ can be written as:

$$\dim \mathcal{H}^0(Y, K(Y, d)) = (2d - 1)(g - 1) + (d - 1)k - c(d, m_1, \ldots, m_k)$$

where $c(d, m_1, \ldots, m_k) \geq 0$ is a correction term that can be computed explicitly for every value of $d$. We introduce the notation $k_m = \# \{ i : m_i = m \}$. Then we have:

$$\begin{align*}
c(2, m_1, \ldots, m_k) &= 0 \\
c(3, m_1, \ldots, m_k) &= k_2 \\
c(4, m_1, \ldots, m_k) &= k_2 + k_3 \\
c(5, m_1, \ldots, m_k) &= 2k_2 + k_3 + k_4 \\
c(6, m_1, \ldots, m_k) &= 2k_2 + k_3 + k_4 + k_5 \\
c(7, m_1, \ldots, m_k) &= 3k_2 + 2k_3 + k_4 + k_5 + k_6 \\
c(8, m_1, \ldots, m_k) &= 3k_2 + 2k_3 + k_4 + k_5 + k_6 + k_7 \\
c(9, m_1, \ldots, m_k) &= 4k_2 + 2k_3 + 2k_4 + k_5 + k_6 + k_7 + k_8 \\
c(10, m_1, \ldots, m_k) &= 4k_2 + 3k_3 + 2k_4 + k_5 + k_6 + k_7 + k_8 + k_9
\end{align*}$$

These formulae give us formulae for the dimension of the Hitchin components that are similar to the formulae for $\text{PGL}(2, \mathbb{R})$ and $\text{PGL}(3, \mathbb{R})$ given by Thurston [Thu79] and Choi-Goldman [CG05].

Corollary 5.12. In the notation of Corollary 5.11 one has:

$$\begin{align*}
\dim \mathcal{H}^0(\pi_1 Y, \text{PGL}(2, \mathbb{R})) &= 6(g - 1) + 2k \\
\dim \mathcal{H}^0(\pi_1 Y, \text{PGL}(3, \mathbb{R})) &= 16(g - 1) + 6k - 2k_2 \\
\dim \mathcal{H}^0(\pi_1 Y, \text{PSp}^\pm(4, \mathbb{R})) &= 20(g - 1) + 8k - 2k_2 - 2k_3 \\
\dim \mathcal{H}^0(\pi_1 Y, \text{PSp}^\pm(6, \mathbb{R})) &= 42(g - 1) + 18k - 6k_2 - 4k_3 - 2k_4 - 2k_5 \\
\dim \mathcal{H}^0(\pi_1 Y, \text{PO}(4, \mathbb{R})) &= 30(g - 1) + 12k - 4k_2 - 2k_3 \\
\dim \mathcal{H}^0(\pi_1 Y, \text{PO}(6, \mathbb{R})) &= 48(g - 1) + 20k - 8k_2 - 4k_3 - 2k_4 \\
\dim \mathcal{H}^0(\pi_1 Y, \text{PO}(4, \mathbb{R})) &= 56(g - 1) + 24k - 8k_2 - 6k_3 - 2k_4 - 2k_5 \\
\dim \mathcal{H}^0(\pi_1 Y, \text{PO}(6, \mathbb{R})) &= 70(g - 1) + 30k - 12k_2 - 6k_3 - 4k_4 - 2k_5
\end{align*}$$

Lemma 5.13. If $\dim \mathcal{H}^0(Y, K(Y, d)) = 0$, then $g = 0$. 

31
Proof. This follows from the formula in Proposition 5.1 since with $g \geq 1$, all terms are non-negative and $O(d,m) \geq 1$.

Lemma 5.14. For $g = 0$, we have: $\dim C H^0(Y,K(Y,d)) \geq 1 - 2d + (d - 1 - \left\lfloor \frac{d-1}{2} \right\rfloor) k$.

Proof. This follows from Proposition 5.1 using the inequality $O(d,m) \geq O(d,2)$.

Lemma 5.15. If $\dim C H^0(Y,K(Y,d)) = 0$, then

1. If $d = 3$, we have $g = 0$ and $k \leq 5$. Moreover, if $k = 5$, then $\forall i, m_i = 2$.
2. If $d \geq 5$ and $d$ odd, we have $g = 0$ and $k \leq 4$.
3. If $d$ is even, we have $g = 0$ and $k = 3$.

Proof. This follows from the inequality in Lemma 5.14.

Lemma 5.16. If $\dim C H^0(Y,K(Y,d)) = 0$, $g = 0$, $k = 5$, then $d = 3$ and $\forall i, m_i = 2$. Instead, if $\dim C H^0(Y,K(Y,d)) = 0$, $g = 0$, $k = 4$, then $d = 3, 5$ or 7. In this case, $m_1 = m_2 = m_3 = 2$. Moreover, if $d = 7$, $m_4 = 3$, if $d = 5$, $m_4 = 3$ or 4.

Proof. If $k = 5$, we know from Lemma 5.15 that $d = 3$. In this case, Corollary 5.11 says that all the 5 cone points have order 2. If $k = 4$, $d$ must be odd, say $d = 2\delta + 1$. Then we have

$$\dim C H^0(Y,K(Y,d)) \geq -(2d-1) + 3\delta + \left\lfloor \frac{4\delta + 2}{3} \right\rfloor$$

because, by Euler characteristic, at least one of the $m_i$ is $\geq 3$.

6. Applications

Theorem 5.6 about the topology of the Hitchin components for orbifold groups, reveals new phenomena that cannot be observed with ordinary surface groups: we will discuss here some new and interesting rigidity phenomena and we will classify the Hitchin components of dimension $0$, $1$ and $2$. Moreover, our result also has applications to the study of the Hitchin components of ordinary surface groups, we will present here some applications to the theory of Higgs bundles, and to the study of the pressure metric on the Hitchin components for surface groups. We will also describe some connected components of the deformation space of real projective structures on Seifert fibered 3-manifolds.

6.1. Rigidity phenomena. An interesting feature of representations of orbifold groups is that they show us many examples of rigidity phenomena. By rigidity we mean representations that cannot be deformed. For this subsection, $Y$ will always be an orientable orbifold of negative Euler characteristic.

Remark 6.1. Rigidity of non-orientable orbifolds can be deduced immediately from the orientable case discussed here using (5.6).

The first type of rigidity is given by the zero-dimensional Hitchin components. For the target group $\text{PGL}(2,\mathbb{R})$, they were classified by Thurston [Thu79], who saw that $\dim \text{Hit}(\pi_1 Y, \text{PGL}(2,\mathbb{R})) = 0$ if and only if $Y$ has genus $0$ and $3$ cone points. For the target group $\text{PGL}(3,\mathbb{R})$, they were classified by Choi and Goldman [CG05], who saw that, $\dim \text{Hit}(\pi_1 Y, \text{PGL}(3,\mathbb{R})) = 0$ if and only if $Y$ has genus $0$, one cone point of order $2$ and $2$ other cone points. Both Thurston’s and Choi-Goldman’s classification can be recovered using our formulae for the dimension and we can complete the classification of the Hitchin components of dimension $0$ for all groups $G$.

Theorem 6.2. For an orientable orbifold $Y$ of negative Euler characteristic, if $\dim \text{Hit}(\pi_1 Y, G) = 0$, then $Y$ has genus $0$ and $3$ cone points. As usual, we denote the orders of the cone points by $m_1 \leq m_2 \leq m_3$.

1. $\dim \text{Hit}(\pi_1 Y, \text{PGL}(4,\mathbb{R})) = 0$ if and only if $m_1 = 2, m_2 = 3$.
2. $\dim \text{Hit}(\pi_1 Y, \text{PGL}(5,\mathbb{R})) = 0$ if and only if $m_1 = 2, m_2 = 3$.
3. For $n \geq 6$, $\dim \text{Hit}(\pi_1 Y, \text{PGL}(n,\mathbb{R})) > 0$.
4. $\dim \text{Hit}(\pi_1 Y, \text{PSp}^{\pm}(4,\mathbb{R})) = 0$ if and only if $m_1 = 2, m_2 = 3$ or $m_1 = 3, m_2 = 3$.
5. For $m \geq 3$, $\dim \text{Hit}(\pi_1 Y, \text{PSp}^{\pm}(2m,\mathbb{R})) > 0$.
6. For $m \geq 3$, $\dim \text{Hit}(\pi_1 Y, \text{PO}(m, m + 1)) > 0$.
7. For $m \geq 4$, $\dim \text{Hit}(\pi_1 Y, \text{PO}^{\pm}(m, m)) > 0$. 

(8) \( \dim \Hit(\pi_1 Y, G_2) = 0 \) if and only if \( m_1 = 2, m_2, m_3 \leq 5 \).

(9) If \( G \) is exceptional and not \( G_2 \), \( \dim \Hit(\pi_1 Y, G) > 0 \).

Proof. If \( \dim \Hit(\pi_1 Y, G) = 0 \), then \( \dim \Hit(\pi_1 Y, PGL_2(\mathbb{R})) = 0 \), hence \( Y \) has genus 0 and 3 cone points. The various statements are an immediate consequence of Corollary 5.11.

Orbifold groups also give us examples of a second type of rigidity which happens when representations in a certain target group can never be Zariski dense, but are forced to take image in a certain subgroup. This contrasts with what happens for surface groups, for which the subset of Zariski dense representations is always a certain target group can never be Zariski dense, but are forced to take image in a certain subgroup. This statement is actually an inclusion

\[ \kappa : PGL(2, \mathbb{R}) \to G, \]

and we are in case 1. By Lemmas 5.15 and 5.16, we see that the only other cases are the ones described in the Hitchin base of \( G \). If an inclusion is surjective, then \( H \) is not conjugate to a representation in \( H \) which is not conjugate to a representation in \( H \).

Theorem 6.4. The inclusion \( \Hit(\pi_1 Y, G) \to PGL_2(\mathbb{R}) \) is surjective if and only if

(1) both spaces have dimension 0 (see Theorem 6.2), or

(2) \( G = PGL(3, \mathbb{R}) \), \( g = 0, k = 4 \) and \( m_1, m_2, m_3 = 2 \), or \( k = 5 \) and all \( m_i = 2 \).

Proof. By Lemma 6.3, if the inclusion is surjective, then \( g = 0 \). If \( k = 3 \), then \( \dim \Hit(\pi_1 Y, PGL_2(\mathbb{R})) = 0 \), and we are in case 1. By Lemmas 5.15 and 5.16, we see that the only other cases are the ones described in case 2.

Theorem 6.5. The inclusions \( \Hit(\pi_1 Y, PSp^+(2n, \mathbb{R})) \to \Hit(\pi_1 Y, PGL_2(\mathbb{R})) \) and \( \Hit(\pi_1 Y, PO(n - 1, n)) \to \Hit(\pi_1 Y, PGL_2(2n - 1, \mathbb{R})) \) are surjective if and only if:

(1) \( g = 0, k = 5 \), all \( m_i = 2 \), \( n = 2 \).

(2) \( g = 0, k = 4 \), \( m_1 = m_2 = m_3 = 2 \), \( n \leq 4 \). If \( n = 4 \), then \( m_4 = 3 \). If \( n = 3 \), then \( m_4 = 3 \) or 4. If \( n = 2 \), then \( m_4 \) can assume every value.

(3) \( g = 0, k = 3 \), \( m_1 = 2 \). If \( m_2 = 4, m_3 = 5 \) for \( n \leq 7 \), if \( m_2 = 4, m_3 = 6 \) for \( n \leq 5 \). If \( m_2 = 3, m_3 = 7 \) for \( n \leq 10 \), if \( m_2 = 3, m_3 = 8 \) for \( n \leq 7 \). If \( m_2 = 3, m_3 \geq 9 \) for \( n \leq 4 \). If \( m_2 = 4, m_3 \geq 7 \) for \( n \leq 3 \).

Proof. From the formulae in Corollary 5.11, we can classify all the cases where the spaces of odd degree differentials of degree \( \leq 2n \) have dimension zero.

Theorem 6.6. The inclusions

\[ \Hit(\pi_1 Y, G_2) \to \Hit(\pi_1 Y, PO(3, 4)) \]

and \( \Hit(\pi_1 Y, G_2) \to \Hit(\pi_1 Y, PGL_2(7, \mathbb{R})) \)

are surjective if and only if \( g = 0, k = 3, m_1 = 2, m_2 = 3 \).

Proof. This follows from the formulae in Corollary 5.11.

Theorem 6.7. The inclusion \( \Hit(\pi_1 Y, PO(n - 1, n)) \to \Hit(\pi_1 Y, PO^+(n, n)) \) is surjective if and only if \( \dim(H^0(Y, K(Y, n))) = 0 \). This dimension is given by Proposition 5.7. Depending on \( Y \), this might happen for some values \( n \leq 43 \).

Proof. For \( g = 0, k = 3, m_1 = 2, m_2 = 3, m_3 = 7 \), we have \( \dim(H^0(Y, K(Y, 43))) = 0 \), and in higher degree it is always non-zero.
6.2. Higgs bundles with a single differential. Let $X$ be a Riemann surface. The Hitchin component $\Hit(\pi_1 X, G)$ is parametrized by the Hitchin base $B_X(g)$, which is a direct sum of spaces of differentials. It is possible to define special loci inside $\Hit(\pi_1 X, G)$ given by the vanishing of some of the differentials. In a few cases, such loci have a clear geometric interpretation, but in general they are very mysterious. For example, the locus $\{(q_2, \ldots, q_n) \in \Hit(\pi_1 X, \PGL(n, \mathbb{R})) \mid q_2 = 0\}$ corresponds to those representations in $\Hit(\pi_1 X, \PGL(n, \mathbb{R}))$ admitting an equivariant minimal surface in the symmetric space inducing the conformal structure $X$ on the surface. Similarly, the locus $\{(q_2, \ldots, q_n) \in \Hit(\pi_1 X, \PGL(n, \mathbb{R})) \mid \forall i < \frac{n}{2}, q_{2i+1} = 0\}$ corresponds to those representations which are conjugate to representations in the symplectic group or in the split orthogonal group. At any rate, other loci are rather mysterious. For example, no known geometric interpretation exists for the following loci:

$$\{(q_2, q_3, q_4) \in \Hit(\pi_1 X, \PGL(4, \mathbb{R})) \mid q_4 = q_2 = 0\}$$

$$\{(q_2, q_4, q_6) \in \Hit(\pi_1 X, \PSp^+ (6, \mathbb{R})) \mid q_6 = q_2 = 0\}$$

$$\{(q_2, q_4, q_6) \in \Hit(\pi_1 X, \PSp^+ (6, \mathbb{R})) \mid q_4 = q_2 = 0\}$$

These loci are nonetheless interesting, since they describe the image of the following maps:

**Proposition 6.8.** There are mapping class group equivariant maps

$$\Psi_3 : \Hit(\pi_1 X, \PGL(3, \mathbb{R})) \rightarrow \Hit(\pi_1 X, \PGL(4, \mathbb{R})),$$

$$\Psi_4 : \Hit(\pi_1 X, \PSp^+ (4, \mathbb{R})) \rightarrow \Hit(\pi_1 X, \PSp^+ (6, \mathbb{R})),$$

$$\Psi_6 : \Hit(\pi_1 X, G_2) \rightarrow \Hit(\pi_1 X, \PSp^+ (6, \mathbb{R}))$$

**Proof.** This is a consequence ofLabourie’s conjecture, proved byLabourie [Lab17] for groups of rank 2. In the case of $\PGL(3, \mathbb{R})$, the map is defined in the following way: byLabourie’s conjecture, a representation $\rho$ in $\Hit(\pi_1 X, \PGL(3, \mathbb{R}))$ admits a unique complex structure $X(\rho)$ on the surface such that the equivariant harmonic map is a minimal immersion. The Higgs bundle corresponding to $\rho$ for the complex structure $X(\rho)$ is parametrized by differentials $(0, q_3)$. With the data of $X(\rho)$ and $q_3$, it is possible to construct a Higgs bundle for $\PGL(4, \mathbb{R})$ with Riemann surface $X(\rho)$ and differentials $(0, q_3, 0)$. This defines the map to $\Hit(\pi_1 X, \PGL(4, \mathbb{R}))$. A similar construction works for the groups $\PSp^+ (4, \mathbb{R})$ and $G_2$. 

The images of these maps are interesting mapping class group invariant submanifolds of the Hitchin components $\Hit(\pi_1 X, \PGL(4, \mathbb{R}))$ and $\Hit(\pi_1 X, \PSp^+ (6, \mathbb{R}))$, for which no geometric interpretation is known. Orbifold groups and our results about the dimensions of the spaces of invariant differentials allow us to construct the first geometric examples of Hitchin representations of surface groups lying in the image of the maps $\Psi_3, \Psi_4, \Psi_6$ described in Proposition 6.8.

**Theorem 6.9.** Let $Y$ be an orbifold with $g = 0$, $k = 3$, and choose a presentation $Y \simeq [\Sigma, X]$, where $X$ is endowed with the unique $\Sigma$-invariant complex structure. Recall that $\pi_1(X) < \pi_1(Y)$.

1. Assume that $m_1 = 3, m_2 = 3$. Then, every Hitchin representation $\rho : \pi_1 Y \rightarrow \PGL(4, \mathbb{R})$ restricts to a Hitchin representation $\rho|_{\pi_1 X} : \pi_1 X \rightarrow \PGL(4, \mathbb{R})$ parametrized by $(0, q_3, 0)$, for some cubic differential $q_3$. In particular, $\rho|_{\pi_1 X}$ is in the image of the map $\Psi_3$.

2. Assume that $m_1 = 3, m_2 = 3, m_3 = 4$. Then, every Hitchin representation $\rho : \pi_1 Y \rightarrow \PGL(5, \mathbb{R})$ restricts to a Hitchin representation $\rho|_{\pi_1 X} : \pi_1 X \rightarrow \PGL(5, \mathbb{R})$ parametrized by $(0, q_3, 0, 0)$, for some cubic differential $q_3$.

3. Assume that $m_1 = 2, m_2 = 4, m_3 = 5$. Then, every Hitchin representation $\rho : \pi_1 Y \rightarrow \PGL(6, \mathbb{R})$ or $\PGL(7, \mathbb{R})$ restricts to a Hitchin representation $\rho|_{\pi_1 X} : \pi_1 X \rightarrow \PGL(6, \mathbb{R})$ or $\PGL(7, \mathbb{R})$ parametrized by $(0, q_4, 0, 0)$ or $(0, q_4, 0, 0, 0)$, for some quartic differential $q_4$. In particular, such representations in $\PGL(6, \mathbb{R})$ are automatically in $\PSp^+ (6, \mathbb{R})$, and they are in the image of the map $\Psi_4$.

4. Assume that $m_1 = 2, m_2 = 3$. Then, every Hitchin representation $\rho : \pi_1 Y \rightarrow \PGL(6, \mathbb{R})$ restricts to a Hitchin representation $\rho|_{\pi_1 X} : \pi_1 X \rightarrow \PGL(6, \mathbb{R})$ parametrized by $(0, 0, 0, 0, q_6)$, for some differential $q_6$ of degree 6. In particular, these representations are in $\PSp^+ (6, \mathbb{R})$, and they are in the image of the map $\Psi_6$.
(5) Assume that $m_1 = 2, m_2 = 3, m_3 = 7$. Then, every Hitchin representation $\varphi : \pi_1 Y \to \text{PGL}(n, \mathbb{R})$ with $n \leq 11$ restricts to a Hitchin representation $\varphi|_{\pi_1 X} : \pi_1 X \to \text{PGL}(n, \mathbb{R})$ such that the only differential appearing in the parametrization of $\varphi|_{\pi_1 X}$ is the differential of degree 6.

**Proof.** This follows from the formula for the dimension of differentials (Proposition 5.1). \qed

A Higgs bundle in the Hitchin component of $\text{PGL}(n, \mathbb{R})$ is said to be cyclic if it is parametrized by $(0, \ldots, 0, q_n)$ (see Baraglia [Bar15]), and $(n - 1)$-cyclic if it is parametrized by $(0, \ldots, 0, q_{n-1}, 0)$ (see Collier Col16). The Hitchin’s equations on Higgs bundles of these types have a form that is especially simple, and many analytical properties can be studied only for these types of Higgs bundles, see for example [Bar15], Col16, CL17, DL16, DL17. The orbifold groups listed in Theorem 6.9 interesting also because they give many examples of Hitchin components that consists entirely of cyclic or $(n - 1)$-cyclic Higgs bundles. For these Hitchin components, the results about cyclic or $(n - 1)$-cyclic Higgs bundles in the papers cited above are valid for all the points of the Hitchin components. This never happens for a surface group. For example, the description of the asymptotic behavior of families of Higgs bundles going at infinity given by Collier and Li [CL17] gives a good description of the behavior at infinity of these special Hitchin representations.

### 6.3. Geodesics for the pressure metric

When $X$ is an orientable surface, there are two definitions of $\text{Out}(\pi_1 X)$-invariant Riemannian metrics on the Hitchin components $\text{Hit}(\pi_1 X, \text{PGL}(n, \mathbb{R}))$, the **Pressure metric** [BCLS15] and the **Liouville pressure metric** [BCLS17]. By restriction, they give $\text{Out}(\pi_1 X)$-invariant Riemannian metrics also on $\text{Hit}(\pi_1 X, \text{PSp}^6(2n, \mathbb{R}))$, $\text{Hit}(\pi_1 X, \text{PO}(n, n+1))$ and $\text{Hit}(\pi_1 X, \text{G}_2)$. In the special case of $\text{Hit}(\pi_1 X, \text{PGL}(3, \mathbb{R}))$, there is also another invariant Riemannian metric, the **Li metric** [Li10]. All these Riemannian metrics restrict to the Weil-Petersson metric on Teichmüller space. Very little is known about their geometric properties, for example almost nothing is known about their geodesics.

If $Y$ is a closed orbifold and $Y \simeq [\Sigma \backslash X]$ is a presentation, our inclusion

$$\text{Hit}(\pi_1 Y, G) \simeq \text{Fix}_\Sigma(\text{Hit}(\pi_1 X, G)) \subset \text{Hit}(\pi_1 X, G)$$

gives, by pullback, Riemannian metrics on $\text{Hit}(\pi_1 Y, G)$. Here we want to underline the fact that the knowledge of the spaces $\text{Hit}(\pi_1 Y, G)$ gives important information about the geometric properties of the Riemannian metrics on $\text{Hit}(\pi_1 X, G)$.

**Proposition 6.10.** The submanifold $\text{Fix}_\Sigma(\text{Hit}(\pi_1 X, G))$ is a totally geodesic submanifold in the Hitchin component $\text{Hit}(\pi_1 X, G)$ for all the $\text{Out}(\pi_1 X)$-invariant Riemannian metrics.

**Proof.** The group $\Sigma$ acts on $\text{Hit}(\pi_1 X, G)$ as a subgroup of $\text{Out}(\pi_1 X)$, which acts on $\text{Hit}(\pi_1 X, G)$ by isometries. It is a basic fact of Riemannian geometry that a fixed point set of a subgroup of isometries is always totally geodesic. \qed

Hence our description of the Hitchin components for orbifolds gives many examples of totally geodesic submanifolds of the Hitchin components for ordinary surface groups. These submanifolds are explicit both in geometric terms (because they correspond to representations with very high symmetry) and in analytic terms (because their parametrization in terms of the holomorphic differentials can be understood). The most interesting case is when $\text{Hit}(\pi_1 Y, G)$ has real dimension one: in that case we have many examples of explicit geodesics for the invariant Riemannian metrics in the Hitchin components for surfaces. We can find these kind of geodesics in all Hitchin components for surfaces into $\text{PGL}(n, \mathbb{R})$ with $n \leq 11$. To make this more explicit, in the next subsection, we will classify all the one-dimensional Hitchin components.

### 6.4. Small-dimensional Hitchin components

An interesting feature of orbifold groups is that they give examples of very small Hitchin components. For orientable surfaces, the smallest Hitchin component is the Teichmüller space of the surface of genus 2, which has real dimension 6. For target groups of higher rank they soon became huge, since the dimensions grow quadratically with the rank of the group. The Hitchin components for orbifolds of genus zero can be very small, hence they can be used as toy models where the geometry of the Hitchin components can be understood completely. For example, we saw in Section 6.3 that the Hitchin components for orbifold groups inherit Riemannian metrics from the Hitchin components of surfaces; an interesting question is to describe these Riemannian metrics. For one-dimensional Hitchin components this amounts to compute their length: do they have infinite or finite length? This question is non-trivial, because these Riemannian metrics are, in general non-complete. For two-dimensional Hitchin
components, an interesting question is whether the Riemannian metrics have negative curvature. Moreover, we explained in Section 6.3 that one-dimensional Hitchin components give examples of explicit geodesics in the Hitchin components for surfaces. The one-dimensional Hitchin components are also the simplest case where one can study the asymptotic behavior of a sequence of representations going to infinity. For these reasons, it is interesting to classify the low-dimensional Hitchin components. From Thurston’s and Choo-Goldman’s formulae for the dimension of Hitchin components for \( \text{PGL}(2, \mathbb{R}) \) and \( \text{PGL}(3, \mathbb{R}) \), we can see the following.

**Proposition 6.11.** Let \( Y \) be an orientable orbifold of negative Euler characteristic, genus \( g \) and \( k \) cone points. Then:

1. \( \dim \text{Hit}(\pi_1 Y, \text{PGL}(2, \mathbb{R})) = 2 \) if and only if \( m_1 = 2, m_2 \geq 4 \) or \( m_1 = m_2 = 3 \).
2. \( \dim \text{Hit}(\pi_1 Y, \text{PGL}(3, \mathbb{R})) = 2 \) if and only if \( m_1 = m_2 = 3, m_3 = 4 \) or \( m_1 = 2, m_2 = 4 \).
3. For \( n = 6, 7 \), \( \dim \text{Hit}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) = 2 \) if and only if \( m_1 = 2, m_2 = 3 \) or \( m_1 = 2, m_2 = 4, m_3 \leq 5 \).
4. For \( n = 8, 9, 10, 11 \), \( \dim \text{Hit}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) = 2 \) if and only if \( m_1 = 2, m_2 = 3, m_3 = 7 \).
5. For \( n \geq 12 \), \( \dim \text{Hit}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) > 2 \).
6. \( \dim \text{Hit}(\pi_1 Y, \text{PSp}^\pm(4, \mathbb{R})) = 2 \) if and only if \( m_1 \leq 3, m_2 \geq 4 \).
7. \( \dim \text{Hit}(\pi_1 Y, \text{PSp}^\pm(6, \mathbb{R})) = 2 \) if and only if \( m_1 = 2, m_2 = 3 \) or \( m_1 = 2, m_2 = 4, m_3 \leq 5 \) or \( m_1 = m_2 = 3, m_3 \leq 5 \).
8. For \( n = 4, 5 \), \( \dim \text{Hit}(\pi_1 Y, \text{PSp}^\pm(2n, \mathbb{R})) = 2 \) if and only if \( m_1 = 2, m_2 = 3, m_3 = 7 \).
9. For \( n \geq 6 \), \( \dim \text{Hit}(\pi_1 Y, \text{PSp}^\pm(2n, \mathbb{R})) > 2 \).
10. \( \dim \text{Hit}(\pi_1 Y, \text{PO}(n, n + 1)) = \dim \text{Hit}(\pi_1 Y, \text{PSp}^\pm(2n, \mathbb{R})) \).
11. \( \dim \text{Hit}(\pi_1 Y, \text{PO}(4, 4)) = 2 \) if and only if \( m_1 = 2, m_2 = 3 \) or \( m_1 = 2, m_3 = 7 \).
12. \( \dim \text{Hit}(\pi_1 Y, \text{PO}(5, 5)) = 2 \) if and only if \( m_1 = 2, m_2 = 3, m_3 = 7 \).
13. For \( n \geq 6 \), \( \dim \text{Hit}(\pi_1 Y, \text{PO}(n, n)) > 2 \).
14. \( \dim \text{Hit}(\pi_1 Y, G_2) = 2 \) if and only if \( m_1 = 2, m_2 \leq 5, m_3 \geq 6 \) or \( m_1 > 2, m_1, m_2, m_3 \leq 5 \).
15. If \( G \) is exceptional and not \( G_2 \), \( \dim \text{Hit}(\pi_1 Y, G) > 2 \).

**Proof.** If \( g \geq 1 \), or \( g = 0 \) and \( k \geq 4 \), we have that \( \dim \text{Hit}(\pi_1 Y, \text{PGL}_2(\mathbb{R})) = 2 \), and the space of quartic differentials has complex dimension at least 1. Hence we must have \( g = 0 \) and \( k = 3 \). The various statements are a consequence of corollary 6.11.

**Remark 6.13.** Using Theorem 6.12, we can also classify all the one-dimensional Hitchin components. A Hitchin component can have odd dimension only if the orbifold is non-orientable, then we can use 6.6. The one-dimensional Hitchin components are especially interesting, because they give geodesics for the \( \text{Out}(\pi_1 X) \)-invariant Riemannian metrics on the Hitchin components of surfaces.

### 6.5. Projective structures on Seifert-fibered 3-manifolds

In this subsection, we describe an application of our results to the study of the deformation spaces of geometric structures on closed 3-manifolds. A **geometry** is a pair \( (X, G) \), where \( G \) is a Lie group and \( X \) is a manifold endowed with a transitive and effective action of \( G \). When the group \( G \) acting on \( X \) is clear, we will denote the geometry simply by \( X \). Given a manifold \( M \) of the same dimension as \( X \), an **X-structure** on \( M \) is an atlas of charts taking values in \( X \), whose transition functions are locally restrictions of elements of \( G \). An **X-isomorphism** between two \( X \)-structures on \( M \) is a self-homeomorphism of \( M \) which, when expressed in charts for the two structures, is locally restriction of an element of \( G \). The deformation space of \( X \)-structures on \( M \), denoted here by \( \mathcal{D}_X(M) \), is the set of all \( X \)-structures on \( M \) up to \( X \)-isomorphisms isotopic to the identity. For an introduction to geometric structures on manifolds and their parameter spaces, the reader is referred to Gol88, Thu97.
In the field of geometric topology in dimension 3, the most important geometry is the hyperbolic geometry \( \mathbb{H}^3 = (\mathbb{H}^3, \text{PO}(1,3)) \), one of Thurston’s eight geometries featured in the geometrization theorem. Here we will consider the 3-dimensional geometry of the Lie group \( \text{PSL}(2, \mathbb{R}) \), acting on itself by left translations: 

\[
\text{PSL}(2, \mathbb{R}) = (\text{PSL}(2, \mathbb{R}), \text{PSL}(2, \mathbb{R}))
\]

This geometry can be considered as a subgeometry of another one of Thurston’s eight geometries, the geometry of the Lie group \( \text{SL}(2, \mathbb{R}) \): \( (\text{SL}(2, \mathbb{R}), \text{Isom}(\text{SL}(2, \mathbb{R}))) \). The latter geometry has a bigger symmetry group, of dimension 4. We will consider also two other important geometries in dimension 3, the projective geometry \( \mathbb{RP}^3 = (\mathbb{RP}^3, \text{PGL}(4, \mathbb{R})) \) and the contact projective geometry \( \mathbb{RP}^3_\omega = (\mathbb{RP}^3, \text{PSp}^3_\omega(4, \mathbb{R})) \), the latter having this name because the group \( \text{PSp}^3_\omega(4, \mathbb{R}) \) acts on \( \mathbb{RP}^3 \) preserving the contact form induced by the standard symplectic form \( \omega \) on \( \mathbb{R}^4 \). The hyperbolic geometry has a projective model, the Klein model: the hyperbolic isometries act on the ellipsoid as projective transformations given by the standard embedding \( \text{PO}(1,3) < \text{PGL}(4, \mathbb{R}) \). This means that every hyperbolic structure on \( M \) induces a projective structure and this gives an embedding of the deformation spaces \( D_{\mathbb{H}^3}(M) \subset D_{\mathbb{RP}^3}(M) \).

When \( M \) is a closed manifold admitting a hyperbolic structure, \( D_{\mathbb{H}^3}(M) \) has only one point by Mostow rigidity, but the space \( D_{\mathbb{RP}^3}(M) \) might be bigger. The connected component of \( D_{\mathbb{RP}^3}(M) \) containing the hyperbolic structure is made of special projective structures called convex projective structures. The study of the deformation space of convex projective structures on a closed 3-manifold is an area of active research. For some \( M \), it is just one point (projectively rigid hyperbolic manifolds), for other \( M \) it is actually possible to deform the hyperbolic structure to some other convex projective structures (see for example [CLT07], [HPT1], [Mar10], [CL15]).

As an application of our results, we will here describe a similar picture for manifolds admitting a \( \text{PSL}(2, \mathbb{R}) \)-structure. This geometry also has a projective model: we use the principal representation \( \kappa : \text{PGL}(2, \mathbb{R}) \rightarrow \text{PGL}(4, \mathbb{R}) \). The group \( \kappa(\text{PSL}(2, \mathbb{R})) \) acts on \( \mathbb{RP}^3 \) with two open orbits \( \Omega^+, \Omega^- \subset \mathbb{RP}^3 \), and on one of them, say \( \Omega^+ \), the action is simply transitive (see [GW08] for details).

The action of \( \kappa(\text{PSL}(2, \mathbb{R})) \) on \( \Omega^+ \) can be seen as a projective model for the \( \text{PSL}(2, \mathbb{R}) \)-geometry. Moreover, since the image of \( \kappa \) is contained in \( \text{PSp}^3_\omega(4, \mathbb{R}) \), this model also has an invariant contact form. This gives maps between deformation spaces of \( \text{PSL}(2, \mathbb{R}) \)-structures in the deformation spaces of projective structures, but this map is not injective, it is 2 : 1, because of the action of the disconnected group \( \kappa(\text{PGL}(2, \mathbb{R})) \), which still preserves \( \Omega^+ \). We will denote the quotient by this action by \( D_{\text{PSL}(2, \mathbb{R})}(M) \). We have embeddings

\[
D_{\text{PSL}(2, \mathbb{R})}(M) \subset D_{\mathbb{RP}^3}(M) \subset D_{\mathbb{RP}^3}(M)
\]

Here we want to describe these deformation spaces for every closed 3-manifold \( M \) admitting a \( \text{PSL}(2, \mathbb{R}) \)-structure. Such manifolds are a special type of Seifert fibered 3-manifolds which are closely related with closed orientable 2-orbifolds. Recall that \( \pi_1 \text{PSL}(2, \mathbb{R}) = \mathbb{Z} \), hence, for every \( d \), this group has a unique (\( d : 1 \))-covering group that we will denote by \( \text{PSL}(2, \mathbb{R})^d \).

**Proposition 6.14.** Assume that \( M \) is a closed 3-manifold admitting at least one \( \text{PSL}(2, \mathbb{R}) \)-structure. Then there exists a natural number \( d \), a closed orientable 2-orbifold \( Y \) and a representation \( \varrho \in \text{Hit}(\pi_1 Y, \text{PSL}(2, \mathbb{R})) \) such that

1. \( \varrho \) can be lifted to a representation \( \varrho^d : \pi_1 Y \rightarrow \text{PSL}^d(2, \mathbb{R}) \),
2. \( M \) is homeomorphic to \( \text{PSL}^d(2, \mathbb{R})/\varrho^d(\pi_1 Y) \).

From this, we see that \( M \) is a Seifert-fibered space with Seifert base equal to the orbifold \( Y \), and that \( Y \) and \( d \) are uniquely determined by \( M \).

Moreover, for such \( Y \) and \( d \), every representation \( \varrho \in \text{Hit}(\pi_1 Y, \text{PGL}(2, \mathbb{R})) \) can be lifted to a representation \( \varrho^d : \pi_1 Y \rightarrow \text{PSL}^d(2, \mathbb{R}) \), and \( M \) is always homeomorphic to the quotient space \( \text{PSL}^d(2, \mathbb{R})/\varrho^d(\pi_1 Y) \). This quotient carries a natural \( \text{PSL}(2, \mathbb{R}) \)-structure, which gives a homeomorphism

\[
\text{Hit}(\pi_1 Y, \text{PGL}(2, \mathbb{R})) \ni \varrho \mapsto \text{PSL}^d(2, \mathbb{R})/\varrho^d(\pi_1 Y) \in D_{\text{PSL}(2, \mathbb{R})}(M) \]

**Proof.** We can fix on \( \text{PSL}(2, \mathbb{R}) \) a left-invariant Riemannian metric. Hence, all \( \text{PSL}(2, \mathbb{R}) \)-structures on closed manifolds are complete (see [Thu97] Prop. 3.4.10]). Fix a \( \text{PSL}(2, \mathbb{R}) \)-structure on a closed manifold \( M \), and denote by \( D : \tilde{M} \rightarrow \text{PSL}(2, \mathbb{R}) \) the developing map and by \( h : \pi_1 M \rightarrow \text{PSL}(2, \mathbb{R}) \) the holonomy representation (see [Gol88], [Thu97]). Completeness of the structure implies that \( D \) is a covering map, and actually it is the universal covering of \( \text{PSL}(2, \mathbb{R}) \), i.e. \( \tilde{M} \simeq \text{SL}(2, \mathbb{R}) \). The action of \( \pi_1 M \) on \( \tilde{M} \) by deck transformations is isometric, giving \( M \) also a \( (\text{SL}(2, \mathbb{R}), \text{Isom}(\text{SL}(2, \mathbb{R}))) \)-structure. Now [Thu97] Cor. 4.7.3 says that \( h \) has discrete image and infinite kernel. Let us denote by \( \Gamma \) the quotient group \( \pi_1 M / \ker(h) \), and
notice that $h$ factors through a discrete and faithful representation $h' : \Gamma \to \text{PSL}(2, \mathbb{R})$. The developing map $D$ factors through a map $D' : \tilde{M}/\ker(h) \to \text{PSL}(2, \mathbb{R})$. From the classification of the coverings of the circle, we see that $D'$ is a finite covering, say a $(d : 1)$-covering, hence $\tilde{M}/\ker(h) \simeq \text{PSL}^d(2, \mathbb{R})$.

There is a unique group structure on such a covering that makes $D$ a group homomorphism, such group is usually denoted by $\text{PSL}^d(2, \mathbb{R})$. The action of the group $\Gamma = \pi_1\tilde{M}/\ker(h)$ on $\tilde{M}/\ker(h)$ is a representation $h^d : \Gamma \to \text{PSL}^d(2, \mathbb{R})$ that lifts the representation $h'$. The manifold $M$ is homeomorphic to the quotient $\text{PSL}^d(2, \mathbb{R})/h^d(\Gamma)$. This implies that $h'(\Gamma)$ is a cocompact discrete subgroup of $\text{PSL}(2, \mathbb{R})$, in particular, $\Gamma$ is isomorphic to $\pi_1Y$, for some orientable orbifold $Y$, and $h'$ is a representation in $\text{Hit}(\pi_1Y, \text{PGL}(2, \mathbb{R}))$.

This proves the first statement. The fact that $Y$ and $d$ are unique follows from the classification of Seifert fibered 3-manifolds. The possibility of lifting a representation from $\text{PSL}(2, \mathbb{R})$ to $\text{PSL}^d(2, \mathbb{R})$ only depends on the connected component of the character variety where the representation lies. Hence if we can lift one Hitchin representation, we can lift all of them.

To see the topology of $M$ better, consider that $\text{PSL}(2, \mathbb{R})$ can be identified with the unit tangent bundle $T^1\mathbb{H}$ of the hyperbolic space, and similarly, $\text{PSL}^d(2, \mathbb{R})$ can be identified with the $(d : 1)$-covering of $T^1\mathbb{H}$. All these spaces are circle bundles over $\mathbb{H}^2$, and the manifold $M$ is an orbifold circle bundle over $Y$ (this is an equivalent definition of a Seifert fiber space).

In the special case when $M = T^1X$, the unit tangent bundle of a closed orientable surface, this theorem has been proved in [GuW08], here we generalized their result to all the closed 3-manifolds admitting a $\text{PSL}(2, \mathbb{R})$-structure. We can use Theorem 5.6 to completely understand the deformations of these geometric structures.

\[\begin{align*}
\Psi : \text{Hit}(\pi_1Y, \text{PGL}(4, \mathbb{R})) & \ni \varphi \mapsto \varphi^+ \in \text{Hom}(\pi_1M, \text{PGL}(4, \mathbb{R})) / \text{PGL}(4, \mathbb{R}), \\
\forall \varphi^+ & \in \text{Hom}(\pi_1M, \text{PGL}(4, \mathbb{R})) / \text{PGL}(4, \mathbb{R}),
\end{align*}\]
answering the rigidity questions which are so hard for the case of the $\mathbb{H}^3$-geometry. Theorem 6.2 shows us the cases of rigidity, when the spaces have dimension zero. Theorem 6.4 tells us that if there are non-trivial deformations inside the $\text{PSL}(2,\mathbb{R})$-structures, then they can also be deformed to some other real projective structures. From Theorem 6.5 we can see cases when all real projective structures are contact, while from Theorem 6.2 we can see cases when the spaces of $\text{PSL}(2,\mathbb{R})$-structures and contact structures are just one point, but we can deform to other non-contact projective structures. To conclude the proof of Theorem 7.6 we need the following lemma, which can be of independent interest. Again, in the special case when $M = T^1X$, the unit tangent bundle of a closed orientable surface, this was proved in GW08.

**Lemma 6.16.** Let $M$ be a closed Seifert fibered 3-manifold, whose Seifert base is a closed 2-orbifold $Y$ with $\chi(Y) < 0$. Let $G$ be one of the groups $\text{PGL}(n,\mathbb{R}), \text{PSp}^+/(2m,\mathbb{R}), \text{PO}(m,m+1)$ or $G_2$. Denote by $\varphi: \pi_1M \to \pi_1Y$ the projection to the fundamental group of the Seifert base. Then the map $\varphi^*: \text{Hom}(\pi_1Y,G) \to \text{Hom}(\pi_1M,G)/G$ restricts to a homeomorphism from $\text{Hit}(\pi_1Y,G)$ to a connected component of $\text{Hom}(\pi_1M,G)/G$.

In particular, $\text{Hom}(\pi_1M,G)/G$ has a connected component homeomorphic to a ball.

**Proof.** The homomorphism $\varphi$ is surjective, hence the map $\varphi^*$ is injective and its image is the set of all representations of $\pi_1M$ having kernel which includes $\ker \varphi$. This is a closed condition, hence the map $\varphi^*$ is a closed map. Let us now restrict our attention to the case when $G = \text{PGL}(n,\mathbb{R})$. We claim that the map $\varphi^*$, when restricted to the Hitchin component, is an open map. Consider a set of generators $\gamma_1, \ldots, \gamma_s$ of $\pi_1Y$, if $Y$ is orientable, and of $\pi_1Y^+ < \pi_1Y$ if $Y$ is not orientable. We can lift them to elements $\bar{\gamma}_1, \ldots, \bar{\gamma}_s$ of $\pi_1M$. If $g \in \text{Hom}(\pi_1Y,\text{PGL}(n,\mathbb{R}))$, it is strongly irreducible by Lemma 2.10 hence all representations in a neighborhood $U$ of $\varphi^*(g)$ send the elements $\bar{\gamma}_1, \ldots, \bar{\gamma}_s$ to elements of $\text{PGL}(n,\mathbb{R})$ generating an irreducible subgroup. The subgroup $\ker \varphi$ is central in $\pi_1M$ if $Y$ is orientable and its centralizer is $\varphi^{-1}(\pi_1Y^+)$ if $Y$ is non-orientable (see [Bri93, Lemma 2.4.15]), in either cases it commutes with all the elements $\bar{\gamma}_1, \ldots, \bar{\gamma}_s$. This implies that every representation in $U$ sends $\ker \varphi$ to the identity element, because in $\text{PGL}(n,\mathbb{R})$ only the identity commutes with an irreducible subgroup. Hence all the elements of $U$ are in the image of the map $\varphi^*$, and this implies our claim. When $G$ is another group in the given list, by Remark 2.7 all Hitchin representation in $G$ are also Hitchin representations in $\text{PGL}(n,\mathbb{R})$, hence strongly irreducible as representations in $\text{PGL}(n,\mathbb{R})$, and we can use the same argument to prove openness of the map on the Hitchin component. □

**Appendix A. Expected dimensions of Hitchin components**

In this appendix, we compare the dimension of the Hitchin component $\text{Hit}(\pi_1Y,\text{PGL}(n,\mathbb{R}))$, which was determined in Theorem 5.6, with the dimension that it is possible to guess by examining a presentation of the group $\pi_1Y$. We will call the latter dimension the *expected* dimension of the Hitchin component, and we will show here that the two dimensions agree. For simplicity, we will restrict our attention to orientable orbifolds $Y$ and to the target group $\text{PGL}(n,\mathbb{R})$. So let $Y$ be a closed orientable 2-orbifold with $k$ cone points of orders $m_1, \ldots, m_k$, with underlying space $|Y|$ a surface of genus $g$. Then $\pi_1Y$ has a presentation of the standard form

$$\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_k | [a_1, b_1] \cdots [a_g, b_g] x_1 \cdots x_k = 1 = x_1^{m_1} = \cdots = x_k^{m_k} \rangle.\tag{A.1}$$

We can define the *expected dimension* $\dim_\text{e} \text{Hit}(\pi_1Y,\text{PGL}(n,\mathbb{R}))$ of the Hitchin component considering that for each $i = 1, \ldots, g$, the generators $a_i$ and $b_i$ can be mapped to elements of $\text{PGL}(n,\mathbb{R})$ that form an open subset (see Proposition 2.22), so we count a dimension $\dim \text{PGL}(n,\mathbb{R})$ for each one of them. For each $j = 1, \ldots, k$, instead, we have seen in the proof of Proposition 2.20 that the generator $x_j$ can be mapped to an element of $\text{PGL}(n,\mathbb{R})$ which is conjugate to $\kappa(\tau_m)$, where $\kappa$ is the principal representation, as in (2.4), and $\tau_m$ is the matrix

$$\begin{bmatrix}
\cos \frac{\pi}{m} & -\sin \frac{\pi}{m} \\
\sin \frac{\pi}{m} & \cos \frac{\pi}{m}
\end{bmatrix} \in \text{PGL}(2,\mathbb{R}).$$

We denote by $D_n(\mathbb{Z}/m\mathbb{Z})$ the component of the representation variety $\text{Hom}(\mathbb{Z}/m\mathbb{Z},\text{PGL}(n,\mathbb{R}))$ containing the representation $\varphi$ given by $\varphi(1) = \kappa(\tau_m)$. For every $x_j$, we count a dimension $\dim D_n(\mathbb{Z}/m_j\mathbb{Z})$. We also need to consider the relation $[a_1, b_1] \cdots [a_g, b_g] x_1 \cdots x_k = 1$, so we subtract a term equal to $\dim \text{PGL}(n,\mathbb{R})$. Finally,
since the Hitchin component is the space of conjugacy classes of Hitchin representations, we subtract again \( \dim \text{PGL}(n, \mathbb{R}) \). We obtain in this way the expected dimension of the Hitchin components:

\[
\dim_{\text{e}} \text{Hit}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) := (2g - 2) \dim \text{PGL}(n, \mathbb{R}) + \sum_{i=1}^{k} \dim D_n(\mathbb{R}/m, z).
\]

**Remark A.1.** The arguments used to define the expected dimension are just a way to guess the dimension, but they do not constitute a proof that the actual dimension is the same. We will now prove that this expected dimension agree with the actual dimension, this is interesting because it tells us that those arguments work in this case. More importantly, Theorem 5.6 determines not only the dimension, but also the topology of the Hitchin component.

**Remark A.2.** We may similarly define the expected dimensions of the other components of the representation space \( \text{Rep}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) \), however as in the previous remark, it is a different story to give a proof that the expected dimension is in fact equal to the actual dimension. So in this paper, we did not mention the expected dimensions of components of \( \text{Rep}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) \) other than the Hitchin component.

**Proposition A.3.** One has \( \dim_{\text{e}} \text{Hit}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) = \text{dim}_{\text{e}} \text{Hit}(\pi_1 Y, \text{PGL}(n, \mathbb{R})) \).

**Proof.** Since we know from Theorem 5.6 that the Hitchin component is of dimension \( (2g - 2) \dim \text{PGL}(n, \mathbb{R}) + 2 \sum_{d=2}^{n} \sum_{i=1}^{k} O(d, m_i) \), the only thing we need to show is that \( \dim D_n(\mathbb{R}/m, z) = 2 \sum_{d=2}^{n} O(d, m) \). In [LT18], Long and Thistlethwaite introduced an arithmetic function of two variables \( \sigma(n, m) \) for \( n, m \geq 2 \), and showed that \( D_n(\mathbb{R}/m, z) \) is of dimension \( n^2 - \sigma(n, m) \). Here, if \( q \) and \( r \) are the quotient and remainder on dividing \( n \) by \( m \), respectively, i.e. \( n = mq + r \) with \( 0 \leq r < m \), then \( \sigma(n, m) = (n + r)q + r \). Then the lemma is a consequence of the following simple computation:

\[
\sum_{d=2}^{n} O(d, m) = 2 \sum_{d=2}^{n} \left( \left\lfloor \frac{d}{m} \right\rfloor + \frac{d - \left\lfloor \frac{d}{m} \right\rfloor}{m} \right) = 2 \sum_{d=2}^{n} \left( d - 2 \sum_{i=1}^{\sigma(n, m)} \left\lfloor \frac{d}{m} \right\rfloor \right) = 2 \cdot \frac{n(n + 1)}{2} - 2 \left( m \cdot \frac{q(q + 1)}{2} + r(q + 1) \right) = n^2 - \sigma(n, m).
\]

\[\Box\]

**References**

[Bar15] D. Baraglia. Cyclic Higgs bundles and the affine Toda equations. *Geom. Dedicata*, 174:25–42, 2015.

[BCLS15] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino. The pressure metric for Anosov representations. *Geom. Funct. Anal.*, 25(4):1089–1179, 2015.

[BCLS17] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino. Simple root flows for Hitchin representations. arXiv:1708.01675, 2017.

[Bri93] M. G. Brin. *Seifert Fibered Spaces. Notes for a course given in the Spring of 1993*. arXiv:0711.1346v2, 1993.

[CG93] S. Choi and W. M. Goldman. Convex real projective structures on closed surfaces are closed. *Proc. Amer. Math. Soc.*, 112(2):657–661, 1993.

[CG05] S. Choi and W. M. Goldman. The deformation spaces of convex \( \mathbb{R}P^2 \)-structures on 2-orbifolds. *Amer. J. Math.*, 127(5):1019–1102, 2005.

[CHK00] D. Cooper, C. D. Hodgson, and S. P. Kerckhoff. *Three-dimensional orbifolds and cone-manifolds*, volume 5 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2000. With a postface by S. Kojima.

[CL15] S. Choi and G.-S. Lee. Projective deformations of weakly orderable hyperbolic Coxeter orbifolds. *Geom. Topol.*, 19(4):1777–1828, 2015.

[CL17] B. Collier and Q. Li. Asymptotics of Higgs bundles in the Hitchin component. *Adv. Math.*, 307:488–558, 2017.

[CLT06] D. Cooper, D. D. Long, and M. B. Thistlethwaite. Computing varieties of representations of hyperbolic 3-manifolds into SL(4, R). *Experiment. Math.*, 15(3):291–305, 2006.

[CLT07] D. Cooper, D. D. Long, and M. B. Thistlethwaite. Flexing closed hyperbolic manifolds. *Geom. Topol.*, 11:2413–2440, 2007.

[Col16] B. Collier. *Finite order automorphisms of Higgs Bundles: Theory and application*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)—University of Illinois at Urbana-Champaign.

[Cor88] K. Corlette. Flat \( G \)-bundles with canonical metrics. *J. Differential Geom.*, 28(3):361–382, 1988.

[Dam14] P. A. Damianou. A beautiful sine formula. *Amer. Math. Monthly*, 121(2):120–135, 2014.

[DL16] S. Dai and Q. Li. Minimal surfaces for Hitchin representations. arXiv:1605.09590, 2016.

[DL17] S. Dai and Q. Li. On cyclic Higgs bundles. arXiv:1710.10725, 2017.

[Don87] S. K. Donaldson. Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc.*, 55(1):127–131, 1987.
[Sim88] C. T. Simpson. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Amer. Math. Soc., 1(4):867–918, 1988.

[Sim92] C. T. Simpson. Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math., 75:5–95, 1992.

[Thu79] W. P. Thurston. Geometry and topology of 3-manifolds. Lecture notes, Princeton University, 1979.

[Thu97] W. P. Thurston. Three-dimensional geometry and topology. Vol. 1, volume 35 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[Wei18] E. A. Weir. The dimension of the restricted Hitchin component for triangle groups, 2018. Thesis (Ph.D.)–The University of Tennessee.

[Wie18] A. Wienhard. An invitation Higher Teichmüller Theory. In Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. 1, pages 1007–1034. World Scientific, 2018.

Mathematisches Institut, Universität Heidelberg, INF 205, Heidelberg, Germany.
E-mail address: daniele.alessandrini@gmail.com

Mathematisches Institut, Universität Heidelberg, INF 205, Heidelberg, Germany.
E-mail address: lee@mathi.uni-heidelberg.de

Departamento de Matemáticas, Universidad de Los Andes, Bogotá, Colombia & IRMA, Université de Strasbourg, France.
E-mail address: schaffhauser@math.unistra.fr