PERIODICITY OF \( p \)-ADIC EXPANSION OF RATIONAL NUMBER

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Abstract

In this paper we give an algorithm to calculate the coefficients of the \( p \)-adic expansion of a rational numbers, and we give a method to decide whether this expansion is periodic or ultimately periodic.

1 Introduction

It is known that in \( \mathbb{R} \), an element is rational if and only if its decimal expansion is ultimately periodic. An important analogous theorem for the \( p \)-adic expansion of rational number, is given by the following statement (see [1]):

**Theorem 1.1.** The number \( x \in \mathbb{Q}_p \) is rational if and only if the sequence of digits of its \( p \)-adic expansion is periodic or ultimately periodic.

For example, in \( \mathbb{Q}_3 \), the \( p \)-adic expansion of \( -\frac{1}{2} \) is \( 1 + 3 + 3^2 + 3^3 + ... = 111111111111 \), it is clear that this expansion is purely periodic. In the second example in \( \mathbb{Q}_3 \), the \( p \)-adic expansion of \( \frac{11}{5} \) is given by \( 1+1.3+1.3^2+2.3^3+1.3^4+0.3^5+... = 11121012101210121012101210... \). This expansion is ultimately periodic, with periodic block 1210. Another example in \( \mathbb{Q}_5 \), the \( p \)-adic expansion of \( \frac{243}{7} \) is given by \( 4+1.5+3.5^2+1.5^3+4.5^4+2.5^5+3.5^6+0.5^7+2.5^8+... = 413142302142302142302... \). This expansion is ultimately periodic, with periodic block 142302.

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Evertse in [3], gave an algorithm to calculate the coefficients of $p$-adic expansion of an element in $\mathbb{Z}_p$. We continue the study of the characterization of $p$-adic numbers (see [2]), we inspired by the works of Evertse, we propose the algorithm (2.1), to calculate a sequence of digits of a rational number $\frac{c}{d}$, then we prove that this sequence defines the $p$-adic expansion of $\frac{c}{d}$ (see lemma 2.2), and verified a relationship (2.2) (see lemma 2.3). Finally, in the main theorem, we demonstrate the periodicity of the $p$-adic expansion of $\frac{c}{d}$.

2 Definitions and properties

We will recall some definitions and basic facts from $p$-adic numbers (see [4]). Throughout this paper $p$ is a prime number, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{Q}^+$ is the field of nonnegative rational numbers and $\mathbb{R}$ is the field of real numbers. We use $|.|$ to denote the ordinary absolute value, $v_p$ the $p$-adic valuation and $|.|_p$ the $p$-adic absolute value. The field of $p$-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the $p$-adic absolute value. We denote the ring of $p$-adic integers by $\mathbb{Z}_p$. Every element of $\mathbb{Q}_p$ can be expressed uniquely by the $p$-adic expansion $\sum_{n=-j}^{+\infty} \alpha_n p^n$ with $\alpha_i \in \{0, 1, \ldots, p-1\}$ for $i \geq -j$. In $\mathbb{Z}_p$ we have simply $j = 0$.

Now, we give in the following definition the requested algorithm for a rational number

**Definition 2.1.** Let $\frac{c}{d} \in \mathbb{Q}^+ \cap \mathbb{Z}_p$, with $c \in \mathbb{N}$, $d \in \mathbb{N}^*$, and $(c, p) = 1$, $(d, p) = 1$, $(c, d) = 1$. We define the sequences $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ by

$$
\begin{align*}
\beta_0 &= c \\
\alpha_i &= \beta_i d^{-1} \text{mod} p, \forall i \geq 0 \\
\beta_{i+1} &= \frac{\beta_i - \alpha_i d}{p} \in \mathbb{Z}, \forall i \geq 0
\end{align*}
$$

(2.1)
Lemma 2.2. Under the hypothesis of the definition (2.1), the \( p \)-adic expansion of \( \frac{c}{d} \) is given by \( \sum_{i=0}^{+\infty} \alpha_i p^i \), with \( \alpha_i \in \{0, 1, \ldots, p - 1\} \), \( \forall i \geq 0 \). The opposite is true, i.e., if \( \frac{c}{d} = \sum_{i=0}^{+\infty} \alpha_i p^i \), then the sequences \( (\alpha_i)_{i\in\mathbb{N}} \) and \( (\beta_i)_{i\in\mathbb{N}} \) verified the algorithm (2.1).

Proof. Let \( (\alpha_i)_{i\in\mathbb{N}} \) and \( (\beta_i)_{i\in\mathbb{N}} \) as in the definition (2.1). We have

\[
\frac{c}{d} = \alpha_0 + \frac{\beta_1}{d} p
\]

\[
= \alpha_0 + \alpha_1 p + \frac{\beta_2}{d} p^2
\]

\[
\cdots
\]

\[
= \alpha_0 + \alpha_1 p + \ldots + \alpha_n p^n + \frac{\beta_{n+1}}{d} p^{n+1}
\]

So

\[
\left| \frac{c}{d} - \sum_{i=0}^{n} \alpha_i p^i \right|_p \leq \frac{1}{p^{n+1}}
\]

therefore \( \sum_{i=0}^{+\infty} \alpha_i p^i = \frac{c}{d} \).

For the second part, we suppose \( \frac{c}{d} = \sum_{i=0}^{+\infty} \alpha_i p^i \), and we prove by recursion that the sequences \( (\alpha_i)_{i\in\mathbb{N}} \) and \( (\beta_i)_{i\in\mathbb{N}} \) verified the algorithm (2.1). For \( i = 0 \), we have \( \frac{c}{d} = \alpha_0 \mod p \), then \( \alpha_0 = \frac{c d^{-1} \mod p}{d} = \beta_0 d^{-1} \mod p \). Now, suppose that \( \alpha_i = \beta_i d^{-1} \mod p \) and \( \beta_{i+1} = \frac{\beta_i - \alpha_i}{p} \), so we have

\[
\alpha_i = \beta_i d^{-1} \mod p \implies \alpha_{i+1} p + \alpha_i = \beta_i d^{-1} \mod p
\]

\[
\implies \alpha_{i+1} p = (\beta_i d^{-1} - \alpha_i) \mod p
\]

\[
\implies \alpha_{i+1} = \left(\frac{\beta_i - \alpha_i}{p}\right) d^{-1} \mod p = \beta_{i+1} d^{-1} \mod p
\]

therefore \( \forall i \geq 0 : \alpha_i = \beta_i d^{-1} \mod p \). \( \square \)
Lemma 2.3. Under the hypothesis of the definition (2.1), we have
\[ c = d \left( \sum_{n=0}^{i-1} \alpha_n p^n \right) + \beta_i p^i, \quad \forall i \in \mathbb{N}^* \] (2.2)

Proof. We prove this lemma, also, by induction. For \( i = 1 \), it’s obvious.
\[ d \left( \sum_{n=0}^{0} \alpha_n p^n \right) + \beta_1 p = d \alpha_0 + \left( \frac{c - \alpha_0 d}{p} \right) p = c \]

Suppose that, the relationship is true for \( i \). From (2.1), we have \( \beta_i = \alpha_i d + \beta_{i+1} p \). Then
\[ c = d \left( \sum_{n=0}^{i-1} \alpha_n p^n \right) + \beta_i p^i \]
\[ = d \left( \sum_{n=0}^{i-1} \alpha_n p^n \right) + (\beta_{i+1} p + \alpha_i d) p^i \]
\[ = d \left( \sum_{n=0}^{i} \alpha_n p^n \right) + \beta_{i+1} p^{i+1} \]

So, the relationship is true for all \( i \in \mathbb{N} \). \( \square \)

Remark 2.4. Let \( r = \frac{c'}{d'} \in \mathbb{Q}^+ \), but not in \( \mathbb{Z}_p \), i.e. the \( p \)-adic expansion of \( \frac{c'}{d'} \) is given by \( \sum_{n=-j}^{+\infty} \alpha_{n+j} p^n \), with \( j \neq 0 \) and \( \alpha_i \in \{0, 1, .., p-1\} \), \( \forall i \geq -j \). In this case, we can suppose \( c' = c \in \mathbb{N} \), \( d' = p^j d \in \mathbb{N}^* \), with \( (d, p) = 1 \), and \( (c, p) = 1 \). So, we have \( \frac{c}{d} = \sum_{n=0}^{+\infty} \alpha_n p^n \). We define a sequence \((\beta_i)_{i \in \mathbb{N}}\) by the same way
\[
\begin{align*}
\beta_0 &= c = c' \\
\beta_{i+1} &= \frac{\beta_i - \alpha_i d}{p} = \frac{\beta_i p^j - \alpha_i d'}{p^{j+1}} \in \mathbb{Z}
\end{align*}
\] (2.3)

3 Results and proof

To show that the algorithm (2.1) stops after a certain rank, it suffices to prove that the sequence \((|\beta_n|)_{n \in \mathbb{N}}\) is bounded or decreasing. This is the subject of the main theorem.
Main Theorem 3.1. The sequence \((\beta_i)_{i \in \mathbb{N}}\) given in (2.1) verified the following cases:

Case 1. If \(c < d\), then

\[0 \leq |\beta_i| < d, \quad \forall i \in \mathbb{N}\]

Case 2. If \(c > d\) and \(p \geq 3\), we have, also, two cases:

Case 2.1. If \(0 < \frac{c(p-1)}{2dp} < 1\), then for all \(i \in \mathbb{N}^*\), we have \(|\beta_i| < d\).

Case 2.2. If \(1 < \frac{c(p-1)}{2dp}\), then for a fixed integer

\[m = \left\lfloor \frac{\log \left(\frac{c(p-1)}{2dp}\right)}{\log p} \right\rfloor\]  \hspace{1cm} (3.1)

it comes that

\[
\begin{align*}
&\begin{cases}
   d < |\beta_i| < c & \text{for } 0 \leq i < m + 1 \\
   0 \leq |\beta_i| < d & \text{for } m + 1 < i \\
   0 \leq |\beta_i| < c & \text{for } m + 1 = i 
\end{cases}
\end{align*}
\]

Proof. We treat all cases:

Case 1. Let \(c < d\), we use the proof by induction. For \(i = 0\) is trivial. We suppose that in the rank \(n\) we have \(|\beta_i| < d\), and we prove the inequality \(|\beta_{i+1}| < d\). Indeed, we have

\[
|\beta_{i+1}| = \left| \frac{\beta_i - \alpha_i d}{p} \right| 
\]

\[
< \frac{1}{p} |\beta_i| + \frac{1}{p} |\alpha_i d| 
\]

\[
< \frac{1}{p} d + \frac{p-1}{p} d = d 
\]

Case 2. For \(c > d\) and \(p \geq 3\), we prove the two following cases:
Case 2.1. We suppose $0 < \frac{c(p-1)}{2dp} < 1$. Also, we prove by recurrence that $|\beta_i| < d$.

Starting with $i = 1$, we have

$$0 < \frac{c(p-1)}{2dp} < 1 \iff -\frac{\alpha_0 d}{p} < \frac{c}{p} - \frac{\alpha_0 d}{p} < \frac{2d}{p-1} - \frac{\alpha_0 d}{p}$$

So

$$-d < -\frac{\alpha_0 d}{p} < \beta_1 < d \left( \frac{2}{p-1} - \frac{\alpha_0}{p} \right) < d$$

Now, we assume that the property is true at rank $i$, and we show it at rank $i + 1$. Indeed, we have

$$-d < \beta_i < d \iff -d < -\frac{d(1 + \alpha_i)}{p} < \frac{\beta_i - \alpha_i d}{p} < \frac{d(1 - \alpha_i)}{p} < d$$

then $-d < \beta_{i+1} < d$. Which means that for every $i \in \mathbb{N}^*$, we have $|\beta_i| < d$.

Case 2.2. Let the integer $m$ given in (3.1), we suppose that $1 < \frac{c(p-1)}{2dp}$.

Firstly, we will prove that for all $0 \leq i \leq m$ the terms $\beta_i$ are strictly positive. Indeed, we assume that there is $k \in \{1, ..., m\}$, such that $\beta_k < 0$. From definition (2.1), we have

$$\frac{\beta_k - \alpha_k d}{p} < 0$$

which means $\beta_k < dp$. Multiplying both sides by $p^{k-1}$, and applying the lemma (2.3), it comes

$$c < d \left( \sum_{n=0}^{k-2} \alpha_n p^n \right) + dp^k$$

The coefficients $\alpha_n$ are strictly less than $p$, so

$$c < dp \left( \frac{p^{k-1} - 1}{p - 1} + p^{k-1} \right)$$

Then, after simplification

$$c < \frac{pd}{p-1} (p^k - 1) < \frac{2pd}{p-1} p^k$$
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Thus

$$\log \left( \frac{c(p - 1)}{2dp} \right) < k$$

however $m + 1 \leq k$. Where does the contradiction come from. Which means that for every $0 \leq i \leq m$, we have $\beta_k > 0$.

Now, we prove the inequalities $d \leq \beta_i \leq c$ for $i \in \{0, ..., m\}$.

The inequality in law is easily proved by recurrence for all $0 \leq i \leq m$. To prove the inequality in the left, we use the absurd. We assume that, there is a positive integer $k \in \{1, ..., m\}$ such that $0 < \beta_k < d$ (the condition $d < c$ implies that $k \neq 0$). By lemma (2.3) we obtain

$$\beta_k < d \iff c < d \left( \sum_{n=0}^{k-1} \alpha_n p^n \right) + dp^k$$

So

$$c < dp(1 + p + ... + p^{k-1} + p^{k-1})$$

Hence

$$c < \frac{dp}{p - 1} (2p^k - p^{k-1} - 1) \iff c < \frac{2pd}{p - 1} p^k$$

It comes that

$$\log \left( \frac{c(p - 1)}{2dp} \right) < k$$

However $m + 1 \leq k$, hence the contradiction. Which means that for all $0 \leq i \leq m$, we have $c \geq \beta_k \geq d$.

For the second part of this case, we suppose there is a positive integer $k > m + 1$ such that $|\beta_k| > d$, that is $\beta_k > d$ or $\beta_k < -d$. By lemma (2.3), we have

$$\beta_k > d \iff c > d \left( \sum_{n=0}^{k-1} \alpha_n p^n \right) + dp^k > dp^k$$
hence \( \frac{c(p - 1)}{2dp} > \left( \frac{p - 1}{2} \right)^{p^{k-1}} > p^{k-1} \), therefore

\[
\log \left( \frac{c(p - 1)}{2dp} \right) \log p > k - 1
\]

then

\[
m + 1 = \left\lceil \log \left( \frac{c(p - 1)}{2dp} \right) \log p \right\rceil + 1 > k
\]

Contradiction. For the second inequality, we have by the formula (2.1)

\[
\beta_k = \frac{\beta_k - \alpha_k d}{p} \leq -d
\]

then \( \beta_{k-1} \leq d(\alpha_k - p) \), however \( \alpha_k \leq p - 1 \), thus \( \beta_{k-1} \leq -d \). And so on, until \( \beta_0 = c \leq -d \), which is another contradiction. So, for all \( i \geq m + 2 \) we have \( |\beta_i| \leq d \). The last part is easily.

\( \square \)

**Example 3.2.** For \( p = 3 \), \( c = 7 \) and \( d = 11 \), the case 1 is verified (see table 1)

| \( k \) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \( \alpha_k \) | 2  | 2  | 0  | 0  | 1  | 1  | 2  | 0  | 0  | 1  | 1  | 2  | 0  | 0  | 1  | 1  |
| \( \beta_k \) | 7  | -5 | -9 | -3 | -1 | -4 | -5 | -9 | -3 | -1 | -4 | -5 | -9 | -3 | -1 | -4 |

For \( p = 3 \), \( c = 8 \) and \( d = 5 \), the case 2.1 is verified (see table 2)

| \( k \) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \( \alpha_k \) | 1  | 2  | 0  | 1  | 2  | 1  | 0  | 1  | 2  | 1  | 0  | 1  | 2  | 1  | 0  | 1  |
| \( \beta_k \) | 8  | 1  | -3 | -1 | -2 | -4 | -3 | -1 | -2 | -4 | -3 | -1 | -2 | -4 | -3 | -1 |

For \( p = 3 \), \( c = 17 \) and \( d = 5 \), we have \( m = 0 \) and the case 2.2 is verified (see table 3)
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Table 3: Case 2.2 for m=0

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $\alpha_k$ | 1 | 2 | 2 | 1 | 0 | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 0 | 1 | 2 | 1 |
| $\beta_k$ | 17 | 4 | -2 | -4 | -3 | -1 | -2 | -4 | -3 | -1 | -2 | -4 | -3 | -1 | -2 | -4 |

For $p = 3$, $c = 124$ and $d = 7$, we have $m = 1$ and the case 2.2 is verified (see table 4)

Table 4: Case 2.2 for m=1

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $\alpha_k$ | 1 | 0 | 1 | 2 | 2 | 0 | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 0 | 1 | 2 |
| $\beta_k$ | 124 | 39 | 2 | -6 | -2 | -3 | -1 | -5 | -4 | -6 | -2 | -3 | -1 | -5 | -4 | -6 |

For $p = 3$, $c = 247$ and $d = 7$, we have $m = 2$ and the case 2.2 is verified (see table 5)

Table 5: Case 2.2 for m=2

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $\alpha_k$ | 1 | 2 | 1 | 2 | 0 | 2 | 1 | 2 | 0 | 1 | 2 | 1 | 0 | 2 | 1 | 0 |
| $\beta_k$ | 247 | 80 | 22 | 5 | -3 | -1 | -5 | -4 | -6 | -2 | -3 | -1 | -5 | -4 | -6 | -2 |

In the following corollary, we give a particular case $p = 2$.

**Corollary 3.3.** For $p = 2$, The sequence $(\beta_i)_{i \in \mathbb{N}}$ given in (2.1) verified the same cases:

**Cas1.** If $c < d$, then

$$0 \leq |\beta_i| < d, \quad \forall i \in \mathbb{N}$$

**Cas2.** If $c > d$, we have also two cases:

**Cas2.1.** If $0 < \frac{c}{2d} < 1$, then for all $i \in \mathbb{N}^*$ we have $|\beta_i| < d$.

**Cas2.2.** If $1 < \frac{c}{2d}$, then for a fixed integer

$$m = \left\lfloor \frac{\log \left( \frac{c}{2d} \right)}{\log 2} \right\rfloor$$
it comes that

\[
\begin{cases}
  d \leq |\beta_i| \leq c & \text{for } 0 \leq i < m + 1 \\
  0 \leq |\beta_i| \leq d & \text{for } m + 1 \leq i \\
  0 \leq |\beta_i| < c & \text{for } m + 1 = i
\end{cases}
\]

**Proof.** The proof is similar to that of the main theorem.

**Example 3.4.** For \(p = 2, c = 5\) and \(d = 9\), the case 1 is verified (see table 6)

**Table 6: Case 1**

| \(k\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|---|----|---|----|---|----|---|
| \(\alpha_k\) | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| \(\beta_k\) | 5 | -2 | -1 | -5 | -7 | -8 | -4 | -2 | -1 | -5 | -7 | -8 | -4 | -2 | -1 | -5 |

For \(p = 2, c = 5\) and \(d = 3\), the case 2.1 is verified (see table 7)

**Table 7: Case 2.1**

| \(k\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|---|----|---|----|---|----|---|
| \(\alpha_k\) | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| \(\beta_k\) | 5 | 1 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -1 | -2 | -1 | -2 | -1 |

For \(p = 2, c = 7\) and \(d = 3\), we have \(m = 0\) and the case 2.2 is verified (see table 8)

**Table 8: Case 2.2 for \(m=0\)**

| \(k\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|---|----|---|----|---|----|---|
| \(\alpha_k\) | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| \(\beta_k\) | 7 | 2 | 1 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -2 | -1 |

For \(p = 2, c = 13\) and \(d = 3\), we have \(m = 1\) and the case 2.2 is verified (see table 9)

**Table 9: Case 2.2 for \(m=1\)**

| \(k\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|---|----|---|----|---|----|---|
| \(\alpha_k\) | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| \(\beta_k\) | 13 | 5 | 1 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -2 | -1 | -2 | -1 |
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For $p = 2$, $c = 25$ and $d = 3$, we have $m = 2$ and the case 2.2 is verified (see table 10)

| $k$  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $\alpha_k$ | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\beta_k$   | 25 | 11 | 4 | 2 | 1 | $-1$ | $-2$ | $-1$ | $-2$ | $-1$ | $-2$ | $-1$ | $-2$ | $-1$ | $-2$ | $-1$ |

Table 10: Case 2.2 for $m=2$

References

[1] G. Bachman, *Introduction to p-adic Numbers and Valuation Theory*, Academic press, New York and London. 1964.

[2] R. Belhadeef, H-A. Esbelin and T. Zerzaihi: *Transcendence of Thue-Morse $p$-adic Continued Fraction*, Mediterr. J. Math. 13(2016),1429-1434.

[3] J. H. Evertse, *p-adic Numbers*, Course Notes, 2011. [Online] Available: [http://www.math.leidenuniv.nl/~evertse/dio2011-padic.pdf](http://www.math.leidenuniv.nl/~evertse/dio2011-padic.pdf).

[4] F. Q. Gouvêa, *p-adic Numbers. An Introduction*, Springer-Verlag Berlin Heidelberg, New York,Second Edition, Universitext, 2000.

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