Truncated Milstein method for non-autonomous stochastic differential equations and its modification

Juan Liao  Wei Liu*  Xiaoyan Wang
Department of Mathematics
Shanghai Normal University, Shanghai, 200234, China

Abstract

The truncated Milstein method, which was initial proposed in (Guo, Liu, Mao and Yue 2018), is extended to the non-autonomous stochastic differential equations with the super-linear state variable and the Hölder continuous time variable. The convergence rate is proved. Compared with the initial work, the requirements on the step-size is significantly released. In addition, the technique of the randomized step-size is employed to raise the convergence rate of the truncated Milstein method.

Key words: non-autonomous stochastic differential equations, truncated Milstein method, randomized step-size, super-linear state variable, Hölder continuous time variable.

1 Introduction

Numerical methods for stochastic differential equations (SDEs) with super-linear coefficients have been attracting lots of attention in recent years. Due to the classical Euler-Maruyama (EM) method fails to converge for those types of SDEs [15], different new methods have been proposed.

Implicit methods are natural alternatives, since they have been successful in handling the stiffness in ordinary differential equations (ODEs) [11]. The implicit methods of the Euler’s type for SDEs were studied in [2] [14] [27] [32] [33]. The Milstein-type implicit methods for SDEs were discussed in [13] [19] [29] [37]. The multi-stage implicit methods were investigated in [13] [33]. We just mention some

*Corresponding author, Email: weiliu@shnu.edu.cn; lwbvb@hotmail.com
of the works on implicit methods here and refer the readers to the references therein.

Compared with implicit methods, explicit methods still have their advantages, such as simple structures, easy to implement, cheap to simulate large numbers of paths [12]. Recently, many different explicit methods have been proposed to approximate SDEs with super-linear coefficients. The tamed Euler method was initially proposed in [16]. By using the idea of taming the coefficients, different types of tamed methods have been proposed [7, 8, 28, 30, 31, 34, 36]. The truncated EM method is another modification of the classical EM, which was initialized in [25, 26]. Afterwards, the truncating technique have been employed to develop different kinds of truncated methods [6, 9, 17, 21, 22, 35].

Most of the works mentioned above dealt with autonomous SDEs, where the time variable does not appear explicitly. Meanwhile, it is well-known that the non-smoothness of the time variable in non-autonomous SDEs leads to significant difference in the convergence rate of numerical methods for both ODEs [4, 18] and SDEs [20, 23].

Motivated by all the issues mentioned above, we investigate the truncated Milstein method for non-autonomous SDEs with the time variable satisfying Hölder’s continuity and the state variable containing super-linear terms, and prove the finite time convergence rate. This result could be regarded as an extension of [10], where autonomous SDEs were considered. We also propose the randomized truncated Milstein method to overcome the low convergence rate due to the Hölder continuous time variable.

The main contribution of this paper are twofold.

- Compared with the existing work [10], our results cover the non-autonomous case and release the requirements on the step-size significantly.

- The randomized truncated Milstein method is proposed. By using numerical simulations, this new method is demonstrated to outperform the truncated Milstein method for non-autonomous SDEs.

This paper is constructed in the following way. Notations, assumptions and some necessary lemmas are presented in Section 2. Section 3 contains the main results and their proofs. The randomized truncated Milstein method is proposed in Section 4. Numerical simulations are conducted in Section 5. Section 6 sees the conclusion and some discussions on future research.

2 Mathematical preliminary

This section is divided into two parts. Notations, assumptions and the truncated Milstein method for non-autonomous SDEs are introduced in the first part. Some
useful lemmas are presented in the second part.

2.1 Notation, assumptions and the truncated Milstein method

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all p-null sets). Let $B(t)$ be an one-dimensional Brownian motion defined in the probability space and is $\mathcal{F}_t$-adapted. And let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace norm in $\mathbb{R}^{n \times m}$; Moreover, for two real numbers $a$ and $b$, we use $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$. For a given set $G$, its indicator function is denoted by $I_G$, namely $I_G(x) = 1$ if $x \in G$ and 0 otherwise.

We are concerned with the $d$-dimension SDEs

$$dy(t) = \mu(t, y(t))dt + \sigma(t, y(t))dB(t), \quad t \geq t_0, \quad \text{with} \quad y(t_0) = y_0 \in \mathbb{R}^d, \quad (2.1)$$

where the drift coefficient function $\mu : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient function $\sigma : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$. and $y(t) = (y^1(t), y^2(t), ..., y^d(t))^T$.

We define:

$$L\sigma(t, y) = \sum_{l=1}^{d} \sigma^l(t, y) \frac{\partial \sigma(t, y)}{\partial y^l},$$

Where $\sigma = (\sigma^1, \sigma^2, ..., \sigma^d)^T$, $\sigma^l : \mathbb{R}^d \to \mathbb{R}$.

And define the derivative of vector $\sigma(t, y)$ with respect to $y^l$ by

$$G^l(t, y) := \left( \frac{\partial \sigma^1(t, y)}{\partial y^l}, \frac{\partial \sigma^2(t, y)}{\partial y^l}, ..., \frac{\partial \sigma^d(t, y)}{\partial y^l} \right).$$

Moreover, we assume that both $\sigma$, $\mu$ have second-order derivative, and we make the following assumptions.

**Assumption 2.1.** There exist constants $C_1 > 0$ and $\beta > 0$ such that

$$|\mu(t, x) - \mu(t, y)| \lor |\sigma(t, x) - \sigma(t, y)| \lor |L\sigma(t, x) - L\sigma(t, y)| \leq C_1(1 + |x|^\beta + |y|^\beta)|x - y|,$$

for all $x, y \in \mathbb{R}^d$.

**Assumption 2.2.** There exist constants $q \geq 2$ and $C_2 > 0$ such that

$$\langle x - y, \mu(t, x) - \mu(t, y) \rangle + \frac{q-1}{2} |\sigma(t, x) - \sigma(t, y)|^2 \leq C_2|x - y|^2,$$

for all $x, y \in \mathbb{R}^d$. 
Assumption 2.3. There exist constants $p > 2$ and $C_3 > 0$ such that
\[ \langle y, \mu(t, y) \rangle + (p - 1) |\sigma(t, y)|^2 \leq C_3(1 + |y|^2), \]
where $C_3$ depends on $C_2$ and $\sup_{t_0 \leq t \leq T} (|\mu(t, 0)| + |\sigma(t, 0)|)$.

Assumption 2.4. There exist constants $\beta > 0$, $C_4 > 0$ and $\alpha \in (0, 1]$ such that
\[ |\mu(t_1, y) - \mu(t_2, y)| \lor |\sigma(t_1, y) - \sigma(t_2, y)| \leq C_4(1 + |y|^{\beta + 1})|t_1 - t_2|^\alpha, \]
for all $y \in \mathbb{R}^d$, any $t \in [t_0, T]$, where the $\beta$ is which in the Assumption 2.1.

In addition, we can derive the boundness of the moment of the analytically solution from Assumption 2.2, that is, there exists a constant $M_1$, which is dependent on $t$ and $q$, such that
\[ E|y(t)|^q \leq M_1(1 + |y(0)|^q). \quad (2.2) \]
And it can be observed from Assumption 2.1 that all $y \in \mathbb{R}^d$ and $t \in [t_0, T]$
\[ |\mu(t, y)| \lor |\sigma(t, y)| \lor |L\sigma(t, y)| \leq M_2(1 + |y|^\beta + 1) \]
where $C$ depends on $C_1$ and $\sup_{t_0 \leq t \leq T} (|\mu(t, 0)| + |\sigma(t, 0)|)$.

We further assume that there exists a positive constant $M_3$ such that
\[ \left| \frac{\partial \mu(t, y)}{\partial y} \right| \lor \left| \frac{\partial^2 \mu(t, y)}{\partial y^2} \right| \lor \left| \frac{\partial \sigma(t, y)}{\partial y} \right| \lor \left| \frac{\partial^2 \sigma(t, y)}{\partial y^2} \right| \leq M_3(1 + |y|^{\beta + 1}). \quad (2.4) \]

To make the paper self-contained, let us revisit the truncated Milstein method. Firstly, we choose a strictly increasing function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $f(u) \rightarrow \infty$ as $u \rightarrow \infty$ and
\[ \sup_{t \in [t_0, T]} \sup_{|y| \leq u} (|\mu(t, y)| \lor |\sigma(t, y)| \lor |L\sigma(t, y)|) \leq f(u), \quad \forall u \geq 1 \quad (2.5) \]
Then we use $f^{-1}$ denote the inverse function of $f$. we can easily observe that $f^{-1}$ is also a strictly increasing continuous function from $[f(0), \infty)$ to $\mathbb{R}^+$. We choose a constant $\hat{h} \geq 1 \lor |f(0)|$ and a strictly decreasing function $h : (0, 1] \rightarrow [[f(0)], \infty)$ such that
\[ \lim_{\Delta \rightarrow 0} h(\Delta) = \infty \text{ and } \Delta^\dagger h(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0, 1]. \quad (2.6) \]
For a given stepsize $\Delta \in (0, 1]$, and any $y \in \mathbb{R}^d$, define the truncated functions
by

\[
\sigma_{\Delta}(t, y) = \sigma \left( t, (|y| \wedge f^{-1}(h(\Delta))) \frac{y}{|y|} \right),
\]

(2.7)

\[
\mu_{\Delta}(t, y) = \mu \left( t, (|y| \wedge f^{-1}(h(\Delta))) \frac{y}{|y|} \right),
\]

(2.8)

\[
G_{\Delta}(t, y) = G \left( t, (|y| \wedge f^{-1}(h(\Delta))) \frac{y}{|y|} \right),
\]

(2.9)

where we set \( y/|y| = 0 \) if \( y = 0 \). It is clear that

\[
|\sigma_{\Delta}(t, y)| \vee |\mu_{\Delta}(t, y)| \vee |G_{\Delta}(t, y)| \leq f \left( f^{-1}(h(\Delta)) \right) = h(\Delta).
\]

(2.10)

We can also obtain the fact that there exists a positive constant \( M \) such that

\[
\left| \frac{\partial \mu_{\Delta}(t, y)}{\partial y} \right| \vee \left| \frac{\partial^2 \mu_{\Delta}(t, y)}{\partial y^2} \right| \vee \left| \frac{\partial \sigma_{\Delta}(t, y)}{\partial y} \right| \vee \left| \frac{\partial^2 \sigma_{\Delta}(t, y)}{\partial y^2} \right| \leq M,
\]

(2.11)

for any \( t \in [t_0, T] \) and \( y \in \mathbb{R}^d \).

The discrete-time truncated milstein numerical solution \( X_i \), to approxiamate \( y(t_i) \) for \( t_i = i\Delta + t_0 \), are formed by setting \( X_0 = y_0 \) and computing

\[
X_{i+1} = X_i + \mu_{\Delta}(t, X_i)\Delta + \sigma_{\Delta}(t, X_i)\Delta B_i + \frac{1}{2} \sum_{l=1}^{d} \sigma^l_{\Delta}(t, X_i)G^l_{\Delta}(t, X_i)(\Delta B_i^2 - \Delta),
\]

(2.12)

for \( i = 0, 1, \ldots, N \), where \( N \) is the integer part of \( T/\Delta \) and let \( t_{N+1} = T \) while \( \Delta B_i = B(t_{i+1}) - B(t_i) \).

To simplify the notation, we set

\[
L\sigma_{\Delta}(t, X_i) := \sum_{l=1}^{d} \sigma^l_{\Delta}(t, X_i)G^l_{\Delta}(t, X_i).
\]

The continous version of the truncated milstein method is defined by

\[
X(t) = \bar{X}(t) + \int_{t_k}^{t} \sigma_{\Delta} \left( \kappa(s), \bar{X}(s) \right) ds + \int_{t_k}^{t} \sigma_{\Delta} \left( \kappa(s), \bar{X}(s) \right) dB(s)
\]

\[
+ \int_{t_k}^{t} L\sigma_{\Delta} \left( \kappa(s), \bar{X}(s) \right) \Delta B(s)dB(s),
\]

(2.13)

where \( \bar{X}(t) = X_i \) and \( \kappa(t) = t_i \) for \( t_i \leq t < t_{i+1} \).
We need the following version of Taylor expansion.

If a function \( \phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \) is twice differentiable, by Taylor formula we have:

\[
\phi(\kappa(t), x) - \phi(\kappa(t), x^*) = \frac{\partial \phi}{\partial x}(\kappa(t), x) \bigg|_{x=x^*} (x - x^*) + R_1(\phi),
\]

where \( R_1(\phi) = \int_0^1 (1 - \tau) \frac{\partial^2 \phi}{\partial x^2}(\kappa(t), x) \bigg|_{x=x^*}(x - x^*, x - x^*)d\tau \), for any fixed \( t \in [t_0, T] \)

For any \( y, j_1, j_2 \in \mathbb{R}^d \), the expressions of the derivatives are as follows:

\[
\frac{\partial \phi}{\partial y}(\kappa(t), y)(j_1) = \sum_{i=1}^d \frac{\partial \phi}{\partial y^i} j_1^i, \quad \frac{\partial \phi}{\partial y}(\kappa(t), y)(j_1, j_2) = \sum_{i,j=1}^d \frac{\partial^2 \phi}{\partial y^i \partial y^j} j_1^i j_2^j,
\]

where \( \frac{\partial \phi}{\partial y} = \left( \frac{\partial \phi_1}{\partial y^1}, \ldots, \frac{\partial \phi_d}{\partial y^d} \right) \), \( \phi = (\phi_1, \phi_2, \ldots, \phi_d) \).

If we replace \( x \) and \( x^* \) by \( X(t) \) and \( \bar{X}(t) \) respectively from (2.14), we have:

\[
\phi(\kappa(t), X(t)) - \phi(\kappa(t), \bar{X}(t)) = \frac{\partial \phi}{\partial X}(\kappa(t), X) \bigg|_{X(t)} \int_{t_i}^t \sigma(\kappa(s), \bar{X}(s))dB(s) + \bar{R}_1(\phi).
\]

(2.15)

Here,

\[
\bar{R}_1(\phi) = \frac{\partial \phi}{\partial X}(\kappa(t), X) \bigg|_{X(t)} \left( \int_{t_i}^t \mu(\kappa(s), X(s))ds + \int_{t_i}^t L\sigma(\kappa(s), X(s))\Delta B(s)dB(s) \right) + R_1(\phi).
\]

(2.16)

Thus, replacing \( \phi \) by \( \sigma_\Delta \) from (2.15), we obtain

\[
\bar{R}_1(\sigma_\Delta) = \sigma_\Delta(\kappa(t), X(t)) - \sigma_\Delta(\kappa(t), \bar{X}(t)) + L\sigma_\Delta(\kappa(t), \bar{X}(t))\Delta B(t),
\]

(2.17)

for \( t_i \leq t < t_{i+1} \), where \( \Delta B(t) = B(t) - B(t_i) \).

### 2.2 Some useful lemmas

**Lemma 2.5.** Assume that Assumption 2.3 holds, then for all \( \Delta \in (0, 1] \), we have

\[
\langle y, \mu_\Delta(t, y) \rangle + \frac{p-1}{2} |\sigma_\Delta(t, y)|^2 \leq C_5 (1 + |y|^2), \quad \forall y \in \mathbb{R}^d,
\]

where \( C_5 = C_3 \left( 1 \lor \left[ 1/f^{-1}(h(1)) \right] \right) \).

**Lemma 2.6.** For any \( \Delta \in (0, 1] \), \( t \in [t_0, T] \) and all \( p \geq 2 \)

\[
\mathbb{E}|X(t) - \bar{X}(t)|^p \leq C\Delta^\frac{p}{2}(h(\Delta))^p,
\]

where \( C \) is a constant independent of \( \Delta \), consequently,

\[
\lim_{\Delta \to 0} \mathbb{E}|X(t) - \bar{X}(t)|^p = 0, \quad \forall t \in [t_0, T].
\]
Applying the Itô formula, we obtain Lemma 2.7.

Proof. Fix the step size $\Delta \in (0,1]$ arbitrarily. For any $t \geq 0$, there exist a constant $i \geq 0$ such that $t_i \leq t < t_{i+1}$. We derive from (2.13) that

\[
\mathbb{E}|X(t) - \bar{X}(t)|^p \leq C \mathbb{E} \left( \left| \int_{t_i}^{t} \mu_\Delta (\kappa(s), \bar{X}(s)) \, ds \right|^p + \left| \int_{t_i}^{t} \sigma_\Delta (\kappa(s), \bar{X}(s)) \, dB(s) \right|^p \right.
\]

\[
+ \left. \left| \int_{t_i}^{t} L\sigma_\Delta (\kappa(s), \bar{X}(s)) \Delta B(s) \, dB(s) \right|^p \right).
\]

Where the elementary inequality $|\sum_{i=1}^{m} a_i|^p \leq m^{p-1} \sum_{i=1}^{m} |a_i|^p$ have been used, and $C$ is a positive constant independent of $\Delta$ that may change from line to line. We derive from the elementary inequality, the Hölder inequality and Theorem 7.1 in [24] that

\[
\mathbb{E}|X(t) - X(t)|^p \leq C \left( \Delta^{p-1} \mathbb{E} \int_{t_i}^{t} |\mu_\Delta (\kappa(s), X(s))|^p \, ds + \Delta^{\frac{p-2}{2}} \mathbb{E} \int_{t_i}^{t} |\sigma_\Delta (\kappa(s), X(s))|^p \, ds \right.
\]

\[
+ \left. \Delta^{\frac{p-2}{2}} \mathbb{E} \int_{t_i}^{t} |L\sigma_\Delta (\kappa(s), X(s)) \Delta B(s)|^p \, ds \right).
\]

By using (2.10) and the inequality of $\mathbb{E}|\Delta B(s)|^p \leq C \Delta^{p/2}$ for $s \in [t_i, t_{i+1})$, we get

\[
\mathbb{E}|X(t) - \bar{X}(t)|^p \leq C \left( \Delta^p h(\Delta)^p + \Delta^p h(\Delta)^p + \Delta^p h(\Delta)^{2p} \right).
\]

Applying (2.8), the desired assertion holds. \[\square\]

**Lemma 2.7.** Let Assumption 2.3 holds. Then, for any $\Delta \in (0,1]$ and any $T > 0$

\[
\sup_{0 < \Delta \leq 1} \sup_{t_0 \leq t \leq T} \mathbb{E}|X(t)|^p \leq K(1 + \mathbb{E}|X(0)|^p), \tag{2.18}
\]

where $K$ is a positive constant dependent on $T$ but independent of $\Delta$.

Proof. It follows from (2.13) that

\[
X(t) = X(t_0) + \int_{t_0}^{t} \mu_\Delta (\kappa(s), \bar{X}(s)) \, ds + \int_{t_0}^{t} \sigma_\Delta (\kappa(s), \bar{X}(s)) \, dB(s) + \int_{t_0}^{t} L\sigma_\Delta (\kappa(s), \bar{X}(s)) \Delta B(s) \, dB(s).
\]

Applying the Itô formula, we obtain

\[
\mathbb{E}|X(t)|^p \leq \mathbb{E}|X(t_0)|^p + p\mathbb{E} \int_{t_0}^{t} |X(s)|^{p-2} \langle X(s), \mu_\Delta (\kappa(s), \bar{X}(s)) \rangle \, ds
\]

\[
+ \frac{p(p-1)}{2} \mathbb{E} \int_{t_0}^{t} |X(s)|^{p-2} \left| \sigma_\Delta (\kappa(s), \bar{X}(s)) + L\sigma_\Delta (\kappa(s), \bar{X}(s)) \Delta B(s) \right|^2 \, ds,
\]

7
with the fact that $p|X(s)|^{p-2}\langle X(s), \sigma_\Delta (\kappa(s), \bar{X}(s)) + L\sigma_\Delta (\kappa(s), \bar{X}(s)) \Delta B(s) \rangle$ is $\mathcal{F}_s$ measurable and

$$
\mathbb{E} \left( \int_{t_0}^t p|X(s)|^{p-2} \langle X(s), \sigma_\Delta (\kappa(s), \bar{X}(s)) + L\sigma_\Delta (\kappa(s), \bar{X}(s)) \Delta B(s) \rangle \, dB(s) \right) = 0
$$

are used. We rewrite the inequality as

$$
\mathbb{E}|X(t)|^p \leq \mathbb{E}|X(t_0)|^p + p\mathbb{E} \int_{t_0}^t |X(s)|^{p-2} \left( \langle X(s), \mu_\Delta (\kappa(s), \bar{X}(s)) \rangle + (p-1)|\sigma_\Delta (\kappa(s), \bar{X}(s))|^2 \right) \, ds
$$

$$
+ p(p-1)\mathbb{E} \int_{t_0}^t |X(s)|^{p-2} \left| L\sigma_\Delta (\kappa(s), \bar{X}(s)) \right|^2 \, ds
$$

$$
+ p\mathbb{E} \int_{t_0}^t |X(s)|^{p-2} \langle X(s) - \bar{X}(s), \mu_\Delta (\kappa(s), \bar{X}(s)) \rangle \, ds.
$$

By Lemma 2.5 and (2.10), we get

$$
\mathbb{E}|X(t)|^p \leq \mathbb{E}|X(t_0)|^p + K\mathbb{E} \int_{t_0}^t |X(s)|^{p-2} \left( 1 + |\bar{X}(s)|^2 \right) \, ds
$$

$$
+ K\mathbb{E} \int_{t_0}^t |X(s)|^{p-2} |h(\Delta)|^4 \, ds + p\mathbb{E} \int_{t_0}^t |X(s)|^{p-2} \langle X(s) - \bar{X}(s), \mu_\Delta (\kappa(s), \bar{X}(s)) \rangle \, ds.
$$

where K is a constant independent of $\Delta$ that may change from line to line. By using the Young inequality

$$
a^{p-2}b \leq \frac{p-2}{p}a^p + \frac{2}{p}b^\frac{p}{2},
$$

we have

$$
\mathbb{E}|X(t)|^p \leq \mathbb{E}|X(t_0)|^p + K\mathbb{E} \int_{t_0}^t |X(s)|^p \, ds + K\mathbb{E} \int_{t_0}^t |\bar{X}(s)|^p \, ds
$$

$$
+ K\mathbb{E} \int_{t_0}^t |h(\Delta)|^{2p-\frac{p}{2}} \, ds + K\mathbb{E} \int_{t_0}^t |X(s) - \bar{X}(s)|^{p-2} \left| \mu_\Delta (\kappa(s), \bar{X}(s)) \right|^\frac{p}{2} \, ds.
$$

By (2.6), (2.10) and Lemma 2.6, we obtain

$$
\mathbb{E} \int_{t_0}^t |X(s) - \bar{X}(s)|^{\frac{p}{2}} \left| \mu_\Delta (s, \bar{X}(s)) \right|^{\frac{p}{2}} \, ds \leq C \int_{t_0}^t |h(\Delta)|^{p-\frac{p}{2}} \, ds \leq C(t-t_0). \quad (2.20)
$$

Substituting (2.20) into (2.19), by using (2.6) we see that

$$
\mathbb{E}|X(t)|^p \leq \mathbb{E}|X(t_0)|^p + K(t-t_0) + KC(t-t_0) + K \int_{t_0}^t \left( \sup_{t_0 \leq u \leq s} \mathbb{E}|X(u)|^p \right) \, ds.
$$
Under the fact that the sum of the right-hand-side in the above inequality is a increasing function of $t$, we obtain

$$\sup_{t_0 \leq s \leq t} \mathbb{E}|X(s)|^p \leq \mathbb{E}|X(0)|^p + K(t - t_0) + K \int_{t_0}^{t} \left( \sup_{t_0 \leq u \leq s} \mathbb{E}|X(u)|^p \right) ds.$$ 

Applying the Gronwall inequality, the desired assertion holds. \hfill \Box

### 3 Main results

This part is divided into three subsections. The main theorem of this paper and the comparison with the existing result are presented in Section 3.1. Some important lemmas are presented and proved in Section 3.2. The proof of the main theorem is put in Section 3.3.

#### 3.1 Main theorem

**Theorem 3.1.** Let Assumptions 2.1, 2.2, 2.3 and 2.4 hold and assume that $p > 2(1 + \beta)q$, then, for any $\bar{q} \in [2, q)$ and $\Delta \in (0, 1]$ 

$$\mathbb{E}|y(T) - X(T)|^{\bar{q}} \leq H \left( \Delta^{\alpha \bar{q}} + \Delta^{\bar{q}} (h(\Delta))^{2\bar{q}} + \Delta^{\bar{q}} + \left( f^{-1}(h(\Delta)) \right)^{(\beta + 1)\bar{q} - p} \right)$$

and

$$\mathbb{E}|y(T) - \bar{X}(T)|^{\bar{q}} \leq H \left( \Delta^{\alpha \bar{q}} + \Delta^{\bar{q}} (h(\Delta))^{2\bar{q}} + \Delta^{\bar{q}} + \left( f^{-1}(h(\Delta)) \right)^{(\beta + 1)\bar{q} - p} \right).$$

**Theorem 3.2.** Let Assumptions 2.1, 2.2 and 2.4 hold and Assumption 2.3 hold for any $p > 2$. Then for any $\bar{q} > 2$, $\varepsilon \in (0, 1/4)$ and $\Delta \in (0, 1]$, 

$$\mathbb{E}|y(T) - X(T)|^{\bar{q}} \leq H \left( \Delta^{\min(1-2\varepsilon, \alpha)\bar{q}} \right)$$

and

$$\mathbb{E}|y(T) - \bar{X}(T)|^{\bar{q}} \leq H \left( \Delta^{\min(1-2\varepsilon, \alpha)\bar{q}} \right).$$

**Proof.** First we define $f(u) = H_4 u^{\beta+2}$, $\forall u \geq 1$. It is easy to get 

$$f^{-1}(u) = \left( \frac{u}{H_4} \right)^{\frac{1}{\beta+2}}.$$ 

Then, let 

$$h(\Delta) = \Delta^{-\varepsilon}.$$
for some $\varepsilon \in (0, 1/4)$ and $\hat{h} > (1 \lor |f(0)|)$.
Applying Theorem 3.1 we can see that
\[
E\left|y(T) - X(T)\right|^q \leq H \left(\Delta_{\min}\left(z^{(p-(\beta+1)q)}_{\beta+2}, aq_1(1-2\varepsilon)q\right)\right),
\]
and
\[
E\left|y(T) - \bar{X}(T)\right|^q \leq H \left(\Delta_{\min}\left(z^{(p-(\beta+1)q)}_{\beta+2}, aq_1(1-2\varepsilon)q\right)\right).
\]
We can get the desired assertions easily by choosing a sufficiently large $p$.

To explain the improvement of the main theorem of this paper, we recall the main theorem in [10] as follows.

**Theorem 3.3.** Let Assumptions 2.1, 2.2 and (2.6) hold. Furthermore, assume that for any given $p \geq 1$, there exists a $q \in (p, \infty)$ and $\Delta^*$ satisfying (2.8). In addition, if
\[
h(\Delta) \geq \mu \left((\Delta^p (h(\Delta))^{2p})^{-1/(q-p)}\right)
\]
holds for all sufficiently small $\Delta \in (0, \Delta^*)$, then for any fixed $T = N\Delta > 0$ and sufficiently small $\Delta \in (0, \Delta^*)$,
\[
E|X(T) - Y_N|^{2p} \leq K\Delta^{2p} (h(\Delta))^{4p}
\]
holds, where $K$ is a positive constant independent of $\Delta$.

**Remark 3.4.** Consider the scalar SDE
\[
dX(t) = (X(t) - 2X^5(t)) \, dt + X^2(t)dB(t), \quad t \geq t_0.
\]
with $t_0 = 0$ and the initial value $X(t_0) = 1$.
Due to the fact that
\[
\sup_{|x| \leq u} (|\mu(x)| \lor |\sigma(x)| \lor |L\sigma(x)|) \leq 3u^5, \quad \forall u \geq 1,
\]
so we choose $f(u) = 3u^5$ and define $h(u) = u^{-\varepsilon}$ for $\varepsilon \in (0, 1/4]$.
Choose $\varepsilon = 1/4$, to satisfy condition (3.1) for $p = 1$ and $q = 12$, $\Delta \leq 10^{-21}$ must be true. Let $\varepsilon = 1/4$ and $\Delta \leq 10^{-21}$ and by Theorem 3.3 we can conclude that
\[
E|X(T) - Y_N|^{2p} \leq K\Delta^p
\]
This is to say that the order of $L^{2p}$-convergence of the truncated Milstein method is $1/2$.
However, take $\varepsilon$ to be $1/4$ and choose $p$ sufficiently large, it can be derived from Theorem 3.1 that
\[
E|y(T) - X(T)|^q \leq K\Delta^{q/4}
\]
where the influence of $\alpha$ are ignored. It can be shown that the convergence rate of this method is $1/2$.
Comparing with Theorem 3.3, $\Delta \leq 10^{-21}$ is not required, and we get the same conclusion. This means that we do relax the step size a little bit.
\textbf{3.2 Important lemmas}

\textbf{Lemma 3.6.} If Assumptions 2.1, 2.2 and (2.4) hold, then for all \( t \in [t_0, T] \),
\[
\sup_{0 \leq \Delta \leq 1} \sup_{0 \leq \tau \leq T} \left[ \mathbb{E}\left| \mu(t, X(t)) \right|^2 \mathbb{E}\left| \sigma(t, X(t)) \right|^2 \mathbb{E}\left| \frac{\partial \mu}{\partial X} (t, X(t)) \right|^2 \mathbb{E}\left| \frac{\partial \sigma}{\partial X} (t, X(t)) \right|^2 \right] < \infty.
\]

\textbf{Lemma 3.7.} If Assumptions 2.1, 2.3 and (2.4) hold and assume that \( p \geq 2(1 + \beta)q \), then for any \( \bar{q} \in (2, q) \) and \( \Delta \in (0, 1] \),
\[
\mathbb{E}|\tilde{R}_1(\mu)|^{\bar{q}} \vee \mathbb{E}|\tilde{R}_1(\sigma)|^{\bar{q}} \vee \mathbb{E}|\tilde{R}_1(\sigma_\Delta)|^{\bar{q}} \leq C \Delta^{\bar{q}} (h(\Delta))^{2\bar{q}},
\]
where \( C \) is a constant independent of \( \Delta \).

\textbf{Proof.} Firstly, we give an estimate on \( |\tilde{R}_1(\mu_\Delta)|^{\bar{q}} \), by Lemmas 2.6 and 2.7 we obtain a constant \( C \) such that
\[
\mathbb{E}|\tilde{R}_1(\mu)|^{\bar{q}} \leq \int_0^1 (1 - \tau)^{\bar{q}} \mathbb{E}\left| \frac{\partial \mu}{\partial X} (\kappa(t), X) \right|_{X(t) + \tau (X(t) - X_0)} (X(t) - X(t), X(t) - X(t)) d\tau
\]
\[
\leq \int_0^1 \mathbb{E}\left| \frac{\partial \mu}{\partial X} (\kappa(t), X) \right|_{X(t) + \tau (X(t) - X_0)} \left( \mathbb{E}\left| X(t) - \tilde{X}(t) \right|^{4q} \right)^{\frac{1}{2}} \left( \mathbb{E}\left| X(t) - \tilde{X}(t) \right|^{4q} \right)^{\frac{1}{2}} d\tau
\]
\[
\leq C \left( 1 + \mathbb{E}\left| X(t) \right|^{2(1 + \beta)q} + \mathbb{E}\left| \tilde{X}(t) \right|^{2(1 + \beta)q} \right)^{\frac{1}{2}} \left( \mathbb{E}\left| X(t) - \tilde{X}(t) \right|^{4q} \right)^{\frac{1}{2}}
\]
\[
\leq C \Delta^{\bar{q}} h(\Delta)^{2\bar{q}} \tag{3.4}
\]
where the H"older inequality and the Jensen’s inequality are used.

Then we can observe from (2.16) that
\[
\mathbb{E}|\tilde{R}_1(\mu)|^{\bar{q}} \leq C \left[ \Delta^{\bar{q}} \mathbb{E}\left| \frac{\partial \mu}{\partial X} (\kappa(t), X) \right|_{X(t)}^{\mu_\Delta (\kappa(t), \tilde{X}(t))} \right]^{\bar{q}}
\]
\[
+ \frac{1}{2} \mathbb{E}\left| \frac{\partial \mu}{\partial X} (\kappa(t), X) \right|_{X(t)}^{L \sigma_\Delta (\kappa(t), \tilde{X}(t)) (\Delta B(t)^2 - \Delta)}^{\bar{q}} + \mathbb{E}|R_1(\mu)|^{\bar{q}}, \tag{3.5}
\]
for \( t_i \leq t < t_{i+1} \).

We can derive from the Hölder inequality

\[
\mathbb{E} \left| \Delta B(t)^2 - \Delta \right|^q \leq 2^{q-1} \left( \mathbb{E} \left| \Delta B(t) \right|^{2q} + \Delta^q \right) \leq 2^{q-1} (\Delta^q + \Delta^q) \leq 2^q \Delta^q. \tag{3.6}
\]

By using Lemma 3.6, (2.10) and the Hölder inequality, we can see that for \( t_0 \leq t \leq T \),

\[
\mathbb{E} \left| \frac{\partial \mu}{\partial X} (\kappa(t), X) \right|_{\mathbb{E}} \mu_\Delta (\kappa(t), \bar{X}(t)) \right|^q \leq \left[ \mathbb{E} \left| \frac{\partial \mu}{\partial X} (\kappa(t), X) \right|_{\mathbb{E}} \mu_\Delta (\kappa(t), \bar{X}(t)) \right]^{2q} \leq C(\Delta)^q \tag{3.7}
\]

Substituting (3.4), (3.6) and (3.7) into (3.5) and using the independence between \( \bar{X}(t) \) and \( \Delta B(t) \), we have

\[
\mathbb{E} \left| \Delta \hat{R}_1(\mu) \right|^q \leq C \Delta^q (h(\Delta))^{2q}
\]

We obtain the desired result.

Similarly, we can show

\[
\mathbb{E} \left| \Delta \hat{R}_1(\sigma) \right|^q \leq C \Delta^q (h(\Delta))^{2q}.
\]

The proof is complete.

### 3.3 Proof of Theorem 3.1

**Proof.** Fix \( \tilde{q} \in [2, q) \) and \( \Delta \in (0, 1] \) arbitrarily, let \( e(t) = y(t) - X(t) \) for \( t > t_0 \), we define the stopping time for each integer \( n > |X_0| \)

\[
\theta_n = \inf \left\{ t \geq t_0 : |X(t)| \vee |y(t)| \geq n \right\}.
\]

We can derive from Itô formula that for any \( t_0 \leq t \leq T \),

\[
\mathbb{E} \left| e(t \wedge \theta_n) \right|^q \leq \tilde{q} \mathbb{E} \int_{t_0}^{t \wedge \theta_n} |e(s)|^{q-2} \left( y(s) - X(s), \mu(s, y(s)) - \mu_\Delta (\kappa(s), \bar{X}(s)) \right) ds
\]

\[
+ \frac{q}{2} \mathbb{E} \int_{t_0}^{t \wedge \theta_n} |e(s)|^{q-2} \left( \sigma(s, y(s)) - \sigma_\Delta (\kappa(s), \bar{X}(s)) \right)
\]

\[- L \sigma_\Delta (\kappa(s), \bar{X}(s)) \Delta B(s) \right|^2 ds. \tag{3.8}
\]
Substituting (2.17) into (3.8), we have

\[
\mathbb{E}|e(t \wedge \theta_n)|^q \leq \tilde{q} \mathbb{E} \int_{t_0}^{t \wedge \theta_n} |e(s)|^{q-2} \left\langle e(s), \mu(s, y(s)) - \mu_{\Delta}(\kappa(s), X(s)) \right\rangle ds \\
+ \frac{\tilde{q}}{2} \mathbb{E} \int_{t_0}^{t \wedge \theta_n} |e(s)|^{q-2} \left| \sigma(s, y(s)) - \sigma_{\Delta}(\kappa(s), X(s)) + \tilde{R}_1(\sigma_{\Delta}) \right|^2 ds.
\]

By the Young inequality \(2ab \leq \varepsilon a^2 + b^2/\varepsilon\) for any \(a, b \geq 0\) and \(\varepsilon\) arbitrarily, we choose \(\varepsilon = \frac{q-2q+1}{2(q-1)}\) here.

\[
(q-1) \left| \sigma(s, y(s)) - \sigma_{\Delta}(\kappa(s), X(s)) \right|^2
\]

\[
= (q-1) \left[ (1 + \frac{q-2q+1}{2(q-1)}) \left| \sigma(s, y(s)) - \sigma(s, X(s)) \right|^2 \\
+ \left( 1 + \frac{2(q-1)}{q-2q+1} \right) \left| \sigma(s, X(s)) - \sigma_{\Delta}(\kappa(s), X(s)) \right|^2 \right]
\]

\[
= \frac{q-1}{2} \left| \sigma(s, y(s)) - \sigma(s, X(s)) \right|^2 + \frac{(q-1)(q-1)}{q-2q+1} \left| \sigma(s, X(s)) - \sigma_{\Delta}(\kappa(s), X(s)) \right|^2.
\]

Then

\[
\mathbb{E}|e(t \wedge \theta_n)|^q = J_1 + J_2 + J_3,
\]

where

\[
J_1 = \mathbb{E} \int_{t_0}^{t \wedge \theta_n} |e(s)|^{q-2} \left\langle e(s), \mu(s, y(s)) - \mu(s, X(s)) \right\rangle + \frac{q-1}{2} \left| \sigma(s, y(s)) - \sigma(s, X(s)) \right|^2 ds,
\]

\[
J_2 = \mathbb{E} \int_{t_0}^{t \wedge \theta_n} |e(s)|^{q-2} \left\langle e(s), \mu(s, X(s)) - \mu_{\Delta}(\kappa(s), X(s)) \right\rangle \\
+ \frac{q(q-1)}{q-2q+1} \left| \sigma(s, X(s)) - \sigma_{\Delta}(\kappa(s), X(s)) \right|^2 ds,
\]

\[
J_3 \leq \mathbb{E} \int_{t_0}^{t \wedge \theta_n} \tilde{q}(q-1)|e(s)|^{q-2}|\tilde{R}_1(\sigma_{\Delta})|^2 ds.
\]

By Assumption 2.2, we have

\[
J_1 \leq H_1 \int_{t_0}^{t \wedge \theta_n} |e(s)|^q ds.
\]
Rearranging $J_2$, we get

\[
J_2 \leq \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \left( \left< e(s), \mu(s, X(s)) - \mu(\kappa(s), X(s)) \right> \right) ds \\
+ \frac{2\bar{q}(\bar{q} - 1)(q - 1)}{q - 2\bar{q} + 1} \left| \sigma(s, X(s)) - \sigma(\kappa(s), X(s)) \right|^2 ds \\
+ \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \left( \left< e(s), \mu(\kappa(s), X(s)) - \mu_\Delta(\kappa(s), \bar{X}(s)) \right> \right) ds \\
+ \frac{2\bar{q}(\bar{q} - 1)(q - 1)}{q - 2\bar{q} + 1} \left| \sigma(s, X(s)) - \sigma_\Delta(\kappa(s), X(s)) \right|^2 ds
\]

(3.11)

We estimate $J_{21}$ first. Applying the Young inequality $a^{p-2}b^2 \leq (p-2)a^p/p + 2b^{p/2}/p$ for any $a, b \geq 0$ and $t_0 \leq t \land \theta_n \leq t \leq T$, we obtain

\[
J_{21} \leq \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \left( \frac{1}{2} |e(s)|^2 + \frac{1}{2} |\mu(s, X(s)) - \mu(\kappa(s), X(s))|^2 \right) ds \\
+ \frac{2\bar{q}(\bar{q} - 1)(q - 1)}{q - 2\bar{q} + 1} \left| \sigma(s, X(s)) - \sigma(\kappa(s), X(s)) \right|^2 ds \\
\leq H_2 \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |e(s)|^q ds + \mathbb{E} \int_{t_0}^{t \land \theta_n} |\mu(s, X(s)) - \mu(\kappa(s), X(s))|^q ds \right) \\
+ \mathbb{E} \int_{t_0}^{t \land \theta_n} \left| \sigma(s, X(s)) - \sigma(\kappa(s), X(s)) \right|^q ds
\]

(3.12)

\[
\leq H_2 \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |e(s)|^q ds + 2C_4 \mathbb{E} \int_{t_0}^{t \land \theta_n} \left( 1 + |X(s)|^{(1+\beta)q} \right) \Delta^{\alpha q} ds \right) \\
\leq H_2 \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |e(s)|^q ds + \Delta^{\alpha q} \right),
\]

Where the Assumption 2.4 and Lemma 2.7 are used. And by rearranging $J_{22}$ show that

\[
J_{22} \leq \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \left( \left< e(s), \mu(\kappa(s), X(s)) - \mu(\kappa(s), \bar{X}(s)) \right> \right) ds \\
+ \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \left( \left< e(s), \mu(\kappa(s), X(s)) - \mu_\Delta(\kappa(s), \bar{X}(s)) \right> \right) ds \\
+ \frac{2\bar{q}(\bar{q} - 1)(q - 1)}{q - 2\bar{q} + 1} \left| \sigma(\kappa(s), X(s)) - \sigma_\Delta(\kappa(s), X(s)) \right|^2 ds
\]

(3.13)

\[
= I_1 + I_2.
\]
where
\[
I_1 = \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \langle e(s), \mu(\kappa(s), X(s)) - \mu(\kappa(s), \bar{X}(s)) \rangle ds,
\]
\[
I_2 = \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \left( \langle e(s), \mu(\kappa(s), X(s)) - \mu(\kappa(s), \bar{X}(s)) \rangle + \frac{2\bar{q}(q - 1)}{q - 2\bar{q} + 1} \left| \sigma(\kappa(s), X(s)) - \sigma(\kappa(s), \bar{X}(s)) \right|^2 \right) ds.
\]

We can derive from the Young inequality, and (2.4) that
\[
I_1 \leq \mathbb{E} \int_{t_0}^{t \land \theta_n} \bar{q} |e(s)|^{q-2} \langle e(s), \frac{\partial \mu}{\partial X} (\kappa(s), X) \rangle_{\bar{X}(s)} \int_{t_0}^{s} \sigma(\kappa(s_1), \bar{X}(s)) dB(s_1) + \tilde{R}_1(\mu) \rangle ds
\]
\[
\leq H_{21} \mathbb{E} \int_{t_0}^{t \land \theta_n} \left( |e(s)|^q + |e(s)|^T \frac{\partial \mu}{\partial X} (\kappa(s), X) \right)_{\bar{X}(s)} \int_{t_0}^{s} \sigma(\kappa(s_1), \bar{X}(s)) dB(s_1) \right|^q ds
\]
\[
\leq H_{21} \mathbb{E} \int_{t_0}^{t \land \theta_n} \left( |e(s)|^q + \left| \tilde{R}_1(\mu) \right|^q + I_{11} \right) ds,
\]

where
\[
I_{11} := \mathbb{E} \int_{t_0}^{t \land \theta_n} \left| e(s) \frac{\partial \mu}{\partial X} (\kappa(s), X) \right)_{\bar{X}(s)} \int_{t_0}^{s} \sigma(\kappa(s_1), \bar{X}(s)) dB(s_1) \right|^q ds.
\]

Following a very similar approach used for (3.35) in [somewhere], we get
\[
I_{11} \leq H_{21} \Delta^q.
\]

Combining (3.14), (3.15) and Lemma 3.7 we obtain
\[
I_1 \leq H_{21} \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |e(s)|^q ds + \Delta^q (h(\Delta))^{2\bar{q}} + \Delta^q \right).
\]
And applying the Young inequality and Assumption 2.1, we can show that

\[ I_2 \leq H_2 \left( \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} |c(s)|^{\frac{q}{p}} ds + \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} \left( \left| \mu_\Delta (\kappa(s), X(s)) - \mu_\Delta (\kappa(s), \bar{X}(s)) \right|^q + \left| \sigma_\Delta (\kappa(s), X(s)) - \sigma_\Delta (\kappa(s), \bar{X}(s)) \right|^q \right) ds \right) \]

\[ \leq H_2 \left( \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} |c(s)|^{\frac{q}{p}} ds + \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} \left( 1 + \left| \bar{X}(s) \right|^{\beta q} + \left| \kappa(s) \wedge f^{-1}(h(\Delta)) \right|^{\beta q} \right) \left| X(s) - \left( \left| X(s) \right| \wedge f^{-1}(h(\Delta)) \right) \frac{X(s)}{|X(s)|} \right|^q ds \right) \]

\[ \leq H_2 \left( \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} |c(s)|^{\frac{q}{p}} ds + \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} \left( \mathbb{E} \left[ 1 + \left| \bar{X}(s) \right|^{p} + \left| \kappa(s) \wedge f^{-1}(h(\Delta)) \right|^{p} \right]^{\frac{p}{p-\beta q}} \right)^{\frac{\beta q}{p}} \right) \left( \mathbb{E} \left[ 1 + \left| \kappa(s) \wedge f^{-1}(h(\Delta)) \right|^{p} \right]^{\frac{p}{p-\beta q}} \right) ds \]

\[ \leq H_2 \left( \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} |c(s)|^{\frac{q}{p}} ds + \mathbb{E} \int_{t_0}^{t_\wedge \theta_n} \left( \mathbb{E} I \{ \bar{X}(s) > f^{-1}(h(\Delta)) \} \left| \bar{X}(s) \right|^{\frac{\beta q}{p-\beta q}} \right) ds \right) \]
where the Hölder inequality and Lemma 2.7 are used above, and using the Cheby-
shev inequality yields

\[
I_2 \leq H_{22} \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |c(s)|^q \, ds + \int_{t_0}^{t \land \theta_n} \left( \left[ P \left\{ |X(s)| > f^{-1}(h(\Delta)) \right\} \right] \mathbb{E} |X(s)|^p \right)^{\frac{p-\beta q-q}{p-q}} \, ds \right) \\
+ \int_{t_0}^{t \land \theta_n} \left( \left[ P \left\{ |X(s)| > f^{-1}(h(\Delta)) \right\} \right] \mathbb{E} |X(s)|^p \right)^{\frac{p-\beta q-q}{p-q}} \, ds \\
\leq H_{22} \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |c(s)|^q \, ds + \int_{t_0}^{t \land \theta_n} \left( \mathbb{E} |X(s)|^p \right)^{\frac{p-\beta q-q}{p-q}} \, ds \right) \\
\leq H_{22} \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |c(s)|^q \, ds + \left( f^{-1}(h(\Delta)) \right)^{(\beta+1)q-p} \right).
\] (3.17)

Substituting (3.16) and (3.17) into (3.13) gives

\[
J_{22} \leq H_{2} \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |c(s)|^q \, ds + \left( f^{-1}(h(\Delta)) \right)^{(\beta+1)q-p} + \Delta \left( h(\Delta) \right)^q + \Delta \right). \tag{3.18}
\]

Due to the Young inequality and Lemma 3.7, we derive that

\[
J_3 \leq H_{3} \mathbb{E} \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |c(s)|^q + |\tilde{R}_1(\sigma_\Delta)|^q \right) ds \leq H_{3} \mathbb{E} \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |c(s)|^q + \Delta \left( h(\Delta) \right)^{2q} \right) ds,
\] (3.19)

where \(H_{21}, H_{22}, H_{3}\) and the following \(H\), etc. are generic constants independent
of \(\Delta\) that may change from line to line.

Combining (3.9), (3.10), (3.11), (3.12), (3.18) and (3.19) together, we can see that

\[
\mathbb{E}|c(t \land \theta_n)|^q \leq H \left( \mathbb{E} \int_{t_0}^{t \land \theta_n} |c(s)|^q \, ds + \Delta^{\alpha q} + \Delta \left( h(\Delta) \right)^{2q} + \Delta \left( f^{-1}(h(\Delta)) \right)^{(\beta+1)q-p} \right).
\]

An application of the Gronwall inequality yields that

\[
\mathbb{E}|c(T \land \theta_n)|^q \leq H \left( \Delta^{\alpha q} + \Delta \left( h(\Delta) \right)^{2q} + \Delta \left( f^{-1}(h(\Delta)) \right)^{(\beta+1)q-p} \right).
\]

By using the Fatou Lemma, let \(n \to \infty\), and Lemma 2.6 to get the desired
assertion. \qed

4 Randomized Truncated Milstein method

As already observed in Remark 3.3, the convergence rate of the truncated Milstein
method is dominated by the Hölder index \(\alpha\). The purpose of this section is to
propose some new method to improve the convergence rate.
Inspired by [20], we embed the randomized time step into (2.12) and propose the following randomized truncated Milstein method.

Define a uniform mesh $T^N : t_0 = t_1 \leq \ldots \leq t_N = T$ with $t_i = i\Delta$ where $\Delta = T/N$; Then the randomized truncated Milstein method on the mesh $T^N$ is given by the recursion

$$
X_{i+1}^T = X_i + \tau_i \Delta \mu(t_i, X_i) + \sigma(t_i, X_i) \int_{t_i}^{t_i+\tau_i \Delta} dB(s),
$$

$$
X_{i+1} = X_i + \Delta \mu(t_i, X_i, X_{i+1}^T) + \sigma(t_i, X_i) \Delta B_i + \frac{1}{2} \sum_{l=1}^{d} \sigma_l(t_i, X_i) G^l(t_i, X_i)(\Delta B^2_i - \Delta),
$$

for all $i \in \{0, 1, \ldots, N\}$ and the initial value $X_0 = y_0$, where

$$
\int_{t_i}^{t_i+\tau_i \Delta} dB(s)
$$

and $(\tau_i)_{i \in \mathbb{N}}$ be an i.i.d family of $U(0, 1)$-distributed randomized variables.

Based on [20], we have the following conjecture on the convergence rate. Briefly speaking, with the employment of the randomized technique the convergence is improved from $\min(1 - 2\varepsilon, \alpha)$ to $\min(1 - 2\varepsilon, \alpha + 1/2)$.

Since we have still been working on the proof of it, we will demonstrate this conjecture by using numerical simulation in the next section.

**Conjecture 4.1.** Suppose Assumptions 2.1, 2.2, 2.3 and 2.4 hold for any $p > 2$, then for any $\tilde{q} > 0$, $\varepsilon \in (0, 1/4)$ and $\Delta \in (0, 1]$, the randomized truncated Milstein method satisfies

$$
\mathbb{E}|y(T) - X(T)|^\tilde{q} \leq H \left(\Delta^{\min(1 - 2\varepsilon, \alpha + \frac{1}{2})}\right),
$$

and

$$
\mathbb{E}|y(T) - \bar{X}(T)|^\tilde{q} \leq H \left(\Delta^{\min(1 - 2\varepsilon, \alpha + \frac{1}{2})}\right).
$$

5 Numerical examples

The purpose of the example discussed in this section is twofold. On one side, it is used to illustrate Theorem 3.2. On the other side, it demonstrates that the convergence rate in Conjecture 4.1 is promising.

**Example 5.1.** Consider the scaler SDE

$$
\begin{cases}
    dx(t) = \left(\left[ t (1 - t) \right]^\frac{1}{2} X^2(t) - X^5(t) \right) dt + \left[ t (1 - t) \right]^\frac{3}{2} X(t) dB(t), \\
    X(t_0) = 2,
\end{cases}
$$

where $t_0 = 0$, $T = 1$ and $B(t)$ is a scalar Brownian motion.
For any $q > 2$, $t \in [0, 1]$ we can see that

$$(x - y)^T (\mu (t, x) - \mu (t, y)) + \frac{q - 1}{2} |\sigma (t, x) - \sigma (t, y)|^2$$

$$\leq (x - y)^2 \left( [t (1 - t)]^\frac{1}{4} (x + y) - (x^4 + x^3y + x^2y^2 + xy^3 + y^4) + \frac{q - 1}{2} [t (1 - t)]^\frac{3}{2} \right).$$

But

$$-(x^3y + xy^3) = -xy(x^2 + y^2) \leq 0.5(x^2 + y^2)^2 = 0.5(x^4 + y^4) + x^2y^2.$$

Hence

$$(x - y)^T (\mu (t, x) - \mu (t, y)) + \frac{q - 1}{2} |\sigma (t, x) - \sigma (t, y)|^2$$

$$\leq (x - y)^2 \left( [t (1 - t)]^\frac{1}{4} (x + y) - 0.5 (x^4 + y^4) + \frac{q - 1}{2} [t (1 - t)]^\frac{3}{2} \right)$$

$$\leq C(x - y)^2.$$

Under the fact that polynomials with negative coefficient for the highest order term can always be bounded, we can obtain the assertion above. It means that Assumption 2.2 is satisfied.

Similarly, for any $p > 2$ and any $t \in [0, 1]$, we have

$$x^T \mu (t, x) + (p - 1)\sigma (t, x)|^2 = [t (1 - t)]^\frac{1}{4} x^3 - x^6 + (p - 1) [t (1 - t)]^\frac{3}{2} x^2 \leq C (1 + |x|^2),$$

this indicates that Assumption 2.3 holds.

Applying the mean value theorem for the temporal variable, Assumption 2.1 and 2.4 are satisfied with $\alpha = 1/4$ and $\beta = 3$. Due to the fact that

$$\sup_{0 \leq t \leq 1} (|\mu (t, x)| \lor |\sigma (t, x)| \lor |L\sigma (t, x)|) \leq 2u^5, \quad \forall u \geq 1,$$

Let $f(u) = 2u^5$ and $h(\Delta) = \Delta^{-\varepsilon}$, for any $\varepsilon \in (0, 1/4)$. Choose $p$ sufficiently large, we can derive from Theorem 3.2 that

$$\sup_{0 \leq t \leq 1} E |y(T) - X(T)|^q \leq H \Delta^\frac{1}{4},$$

and

$$\sup_{0 \leq t \leq 1} E |y(T) - \bar{X}(T)|^q \leq H \Delta^\frac{1}{4}.$$
This indicates that the convergence rate of truncated Milstein method for the SDE (5.1) is $1/4$. To approximate the mean square error, we run $M = 1000$ independent trajectories for 5 different step sizes. And we regard the numerical solution with the step size $10^{-6}$ as the true solution for the SDE. By numerical simulation we can see in Figure 1 that the slope of the error against the step sizes is approximately $0.2527$.

Let us turn to the discussion on the randomized truncated Milstein method. By Conjecture 4.1 we would expect

$$\sup_{0 < t \leq 1} \mathbb{E}|y(t) - X(t)|^q \leq H \Delta^{\frac{3}{4}q},$$

and

$$\sup_{0 < t \leq 1} \mathbb{E}|\bar{y}(t) - \bar{X}(t)|^q \leq H \Delta^{\frac{3}{4}q}.$$
Now, it is clearly seen in Figure 2 that the convergence rate of the RTM method is indeed improved to be 0.7548. This shows that Conjecture 4.1 is reasonable.

6 Conclusion and future research

This paper revisited the truncated Milstein method and proved the strong convergence of the method for non-autonomous SDEs, which extended and improved the existing result.

With the observation that the convergence rate could be very low due to the Hölder continuous time variable, the randomized truncated Milstein method was proposed. The conjecture on the improvement of the convergence rate is reported. Numerical simulations demonstrate the conjecture is promising.

One of the main future works is to prove the conjecture. In addition, we are working on the stability of the truncated Milstein method and the randomized truncated Milstein method in different senses.
References

[1] S. Amiri and S.M. Hosseini, Stochastic Runge–Kutta Rosenbrock type methods for SDE systems, Applied Numerical Mathematics, Volume 115, 1 May 2017, Pages 1-15.

[2] W.-J. Beyn and R. Kruse, Two-sided error estimates for the stochastic theta method, Discrete and Continuous Dynamical Systems - Series B, Volume 14, Issue 2, September 2010, Pages 389-407.

[3] K. Burrage and T. Tian, Implicit stochastic Runge-Kutta methods for stochastic differential equations, BIT Numerical Mathematics, Volume 44, Issue 1, 2004, Pages 21-39.

[4] T. Daun, On the randomized solution of initial value problems, Journal of Complexity, Volume 27, Issue 3-4, June-August 2011, Pages 300-311.

[5] K. Debrabant and A. Rößler, Diagonally drift-implicit Runge-Kutta methods of weak order one and two for Itô SDEs and stability analysis, Applied Numerical Mathematics, Volume 59, Issue 3-4, March 2009, Pages 595-607.

[6] S. Deng, W. Fei, W. Liu and X. Mao, The truncated EM method for stochastic differential equations with Poisson jumps, Journal of Computational and Applied Mathematics, Volume 355, 1 August 2019, Pages 232-257.

[7] K. Dareiotis, C. Kumar and S. Sabanis, On tamed Euler approximations of SDEs driven by Lévy noise with applications to delay equations, SIAM Journal on Numerical Analysis, Volume 54, Issue 3, 2016, Pages 1840-1872.

[8] S. Gan, Y. He and X. Wang, Tamed Runge-Kutta methods for SDEs with super-linearly growing drift and diffusion coefficients, Applied Numerical Mathematics, Volume 152, June 2020, Pages 379-402.

[9] Q. Guo, W. Liu, X. Mao and R. Yue, The partially truncated Euler–Maruyama method and its stability and boundedness, Applied Numerical Mathematics, Volume 115, 1 May 2017, Pages 235-251.

[10] Q. Guo, W. Liu, X. Mao and R. Yue, The truncated Milstein method for stochastic differential equations with commutative noise, Journal of Computational and Applied Mathematics, Volume 338, 15 August 2018, Pages 298-310.

[11] E. Hairer and G. Wanner, Solving ordinary differential equations. II. Stiff and differential-algebraic problems. Second revised edition. Springer Series in Computational Mathematics, 14. Springer-Verlag, Berlin, 2010.
[12] D.J. Higham, Stochastic ordinary differential equations in applied and computational mathematics, IMA Journal of Applied Mathematics, Volume 76, Issue 3, June 2011, Pages 449-474.

[13] D.J. Higham, X. Mao and L. Szpruch, Convergence, non-negativity and stability of a new Milstein scheme with applications to finance, Discrete and Continuous Dynamical Systems - Series B, Volume 18, Issue 8, October 2013, Pages 2083-2100.

[14] Y. Hu, Semi-implicit Euler-Maruyama scheme for stiff stochastic equations. Stochastic analysis and related topics, V (Silivri, 1994), 183–202, Progr. Probab., 38, Birkhäuser Boston, Boston, MA, 1996.

[15] M. Hutzenthaler, A. Jentzen and P.E. Kloeden, Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, Volume 467, Issue 2130, 8 June 2011, Pages 1563-1576.

[16] M. Hutzenthaler, A. Jentzen and P.E. Kloeden, Strong convergence of an explicit numerical method for sdes with nonglobally lipschitz continuous coefficients, Annals of Applied Probability, Volume 22, Issue 4, August 2012, Pages 1611-1641.

[17] Y. Jiang, Z. Huang and W. Liu, Equivalence of the mean square stability between the partially truncated Euler–Maruyama method and stochastic differential equations with super-linear growing coefficients, Advances in Difference Equations Volume 2018, Issue 1, 1 December 2018, Article No. 355.

[18] B.Z. Kacewicz, Optimal solution of ordinary differential equations, Journal of Complexity, Volume 3, Issue 4, December 1987, Pages 451-465.

[19] C. Kahl and H. Schurz, Balanced Milstein methods for ordinary SDEs, Monte Carlo Methods and Applications, Volume 12, Issue 2, April 2006, Pages 143-170.

[20] R. Kruse and Y. Wu, A randomized milstein method for stochastic differential equations with non-differentiable drift coefficients, Discrete and Continuous Dynamical Systems - Series B, Volume 24, Issue 8, August 2019, Pages 3475-3502.

[21] G. Lan and F. Xia, Strong convergence rates of modified truncated EM method for stochastic differential equations, Journal of Computational and Applied Mathematics, Volume 334, 15 May 2018, Pages 1-17.
[22] X. Li, X. Mao and G. Yin, Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: Truncation methods, convergence in pth moment and stability, IMA Journal of Numerical Analysis, Volume 39, Issue 2, 2019, Pages 847-892.

[23] W. Liu, X. Mao, J. Tang and Y. Wu, Truncated Euler-Maruyama method for classical and time-changed non-autonomous stochastic differential equations, Applied Numerical Mathematics, Volume 153, July 2020, Pages 66-81.

[24] X. Mao, Stochastic Differential Equations and Applications, 2nd Edition, Horwood Publishing, Chichester, 2007.

[25] X. Mao, The truncated Euler-Maruyama method for stochastic differential equations, Journal of Computational and Applied Mathematics, Volume 290, 22 June 2015, Pages 370-384.

[26] X. Mao, Convergence rates of the truncated Euler-Maruyama method for stochastic differential equations, Journal of Computational and Applied Mathematics, Volume 296, 1 April 2016, Pages 362-375.

[27] X. Mao and L. Szpruch, Strong convergence rates for backward Euler-Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients, Stochastics, Volume 85, Issue 1, February 2013, Pages 144-171.

[28] H.L. Ngo and D.T. Luong, Tamed Euler–Maruyama approximation for stochastic differential equations with locally Hölder continuous diffusion coefficients, Statistics and Probability Letters, Volume 145, February 2019, Pages 133-140.

[29] V. Reshniak, A.Q.M. Khaliq, D.A. Voss and G. Zhang, Split-step Milstein methods for multi-channel stiff stochastic differential systems, Applied Numerical Mathematics, Volume 89, January 2015, Pages 1-23.

[30] M. Song, Y. Lu and M. Liu, Convergence of the Tamed Euler Method for Stochastic Differential Equations with Piecewise Continuous Arguments Under Non-global Lipschitz Continuous Coefficients, Numerical Functional Analysis and Optimization, Volume 39, Issue 5, 4 April 2018, Pages 517-536.

[31] X. Wang and S. Gan, The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients, Journal of Difference Equations and Applications, Volume 19, Issue 3, March 2013, Pages 466-490.

[32] X. Wang, J. Wu and B. Dong, Mean-square convergence rates of stochastic theta methods for SDEs under a coupled monotonicity condition, BIT Numerical Mathematics, Volume 60, Issue 3, 1 September 2020, Pages 759-790.
[33] Z. Yan, A. Xiao and X. Tang, Strong convergence of the split-step theta method for neutral stochastic delay differential equations, Applied Numerical Mathematics, Volume 120, October 2017, Pages 215-232.

[34] Z. Zhang and H. Ma, Order-preserving strong schemes for SDEs with locally Lipschitz coefficients, Applied Numerical Mathematics, Volume 112, 1 February 2017, Pages 1-16.

[35] W. Zhang, M. Song and M. Liu, Strong convergence of the partially truncated Euler–Maruyama method for a class of stochastic differential delay equations, Journal of Computational and Applied Mathematics, Volume 335, June 2018, Pages 114-128.

[36] X. Zong, F. Wu and C. Huang, Convergence and stability of the semi-tamed Euler scheme for stochastic differential equations with non-Lipschitz continuous coefficients, Applied Mathematics and Computation, Volume 228, 1 February 2014, Pages 240-250.

[37] X. Zong, F. Wu and G. Xu, Convergence and stability of two classes of theta-Milstein schemes for stochastic differential equations, Journal of Computational and Applied Mathematics, Volume 336, July 2018, Pages 8-29.