On exponential functionals, harmonic potential measures and
undershoots of subordinators

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Abstract

We establish a link between the distribution of an exponential functional \( I \) and the undershoots of a subordinator, which is given in terms of the associated harmonic potential measure. This allows us to give a necessary and sufficient condition in terms of the Lévy measure for the exponential functional to be multiplicative infinitely divisible. We then provide a formula for the moment generating function of an exponential functional \( I \) and the so called remainder random variable \( R \) associated to it. We provide a realization of the remainder random variable \( R \) as an infinite product involving independent last position random variables of the subordinator. Some properties of harmonic measures are obtained and some examples are provided.

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1 Introduction and main results

Let \( \xi \) be a (possibly killed) subordinator; that is a \([0, \infty)\]-valued Lévy process with non-decreasing paths, absorbing state \( \Delta := \{+\infty\} \) and lifetime \( \zeta := \inf\{s > 0; \xi_s = \Delta\} \). Thus, either \( \zeta = +\infty \), in which case we say that \( \xi \) is immortal, or \( \zeta \) is exponentially distributed with parameter \( q \), for some \( q > 0 \), in which case \( \xi \) jumps to the cemetery state \( \Delta \) at time \( \zeta \) and stays there forever. In any case, \( \xi \) can always be seen as an immortal subordinator killed at an independent exponential time with parameter \( q \geq 0 \), and the case \( q = 0 \) is included to permit \( \zeta = \infty \) a.s. We denote by \( \phi: \mathbb{R}^+ \to \mathbb{R}^+ \) its Laplace exponent, which is defined by

\[
-\log \left\{ \mathbb{E}[e^{-\lambda \xi}] \right\} := \phi(\lambda), \quad \lambda \geq 0.
\]

By the Lévy Khintchine formula, we know that

\[
\phi(\lambda) = q + \lambda a + \int_{(0,\infty)} (1 - e^{-\lambda x})\Pi(dx), \quad \lambda \geq 0,
\]

where \( a, q \geq 0 \) and \( \Pi \) is a measure on \((0, \infty)\) such that \( \int_{(0,\infty)} (x \wedge 1)\Pi(dx) < \infty \). Equivalently, \( \phi \) is a Bernstein function. So, associated to \( \xi \) there is a Bernstein function. Conversely, given a Bernstein function \( \phi \) it is well known that the function \( \exp\{-\phi(\lambda)\} \) is the Laplace transform of an infinitely

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divisible probability measure concentrated on $\mathbb{R}^+$, and thus associated to it there is a subordinator. For background on subordinators see [6] and on Bernstein functions see [30].

We denote by $V(dx)$ the potential measure (p.m. for short) of $\xi$, that is

$$V(dx) = \int_{\mathbb{R}^+} dt P(\xi_t \in dx), \quad x \geq 0.$$ 

Observe that the Laplace transform of $V$ is

$$\int_{\mathbb{R}^+} V(dx) e^{-\lambda x} = 1/\phi(\lambda), \quad \lambda \geq 0.$$  

Thus, both $(q, a, \Pi)$ and $V$ characterize the law of the subordinator $\xi$. For $\alpha \geq 0$, the $\alpha$-resolvent of $\xi$, say $V_\alpha$, is the measure given by

$$V_\alpha(dx) = \int_0^\infty dt e^{-\alpha t} P(\xi_t \in dx), \quad x \geq 0.$$ 

Observe that $V_0$ equals $V$ so hereafter the latter and former quantities will refer to the same object.

One of the objectives of this work is to emphasize the importance of the so-called harmonic potential measure (h.p.m. for short) of a subordinator, in particular within the characterization of Lévy measures of random variables related to exponential functionals of subordinators and undershoots. The h.p.m. restricted to $(0, \infty)$ is defined by

$$H(dx) = \int_0^\infty dt P(\xi_t \in dx), \quad x > 0.$$ 

As we will see later in the examples, there are several subordinators for which the h.p.m. $H$ can be obtained explicitly. More can be extracted from references such as [23] and [30].

We start by establishing an identity relating the h.p.m. to the Lévy measure of the last position of $\xi$ at a random barrier. For that end, let $L_t = \inf \{ s > 0 : \xi_s > t \}$ be the first passage time above the level $t$, for $\xi$, and $G_t = \xi_{L_t-}$ be the last position below the level $t$. We define the Lévy tail $\overline{\Pi} : \mathbb{R}^+ \to \mathbb{R}$ by setting $\overline{\Pi}(x) = q + \Pi(x, \infty)$, for $x > 0$, and $\overline{\Pi}(+\infty) = q$.

**Lemma 1.** Assuming that $e_\alpha$ is an exponential random variable with parameter $\alpha > 0$, which is independent of $\xi$, the following assertions hold true.

1) The last position below the level $e_\alpha$ and the undershoot, $G_{e_\alpha}$ and $e_\alpha - G_{e_\alpha}$ respectively, are independent r.v.’s and their Laplace transforms are given, for every $\lambda \geq 0$, by

$$E[\exp{-\lambda G_{e_\alpha}}] = \frac{\phi(\alpha)}{\phi(\alpha + \lambda)},$$  

$$E[\exp{-\lambda (e_\alpha - G_{e_\alpha})}] = \frac{\alpha}{\phi(\alpha)} \frac{\phi(\alpha + \lambda)}{\alpha + \lambda}.$$ 

2) $G_{e_\alpha}$ is a positive infinitely divisible random variable having the probability distribution

$$P(G_{e_\alpha} \in dx) = \phi(\alpha)e^{-\alpha x} V(dx), \quad x \geq 0$$ 

and the Bernstein function $\lambda \mapsto \log \frac{\phi(\lambda + \alpha)}{\phi(\alpha)} = \int_{(0,\infty)} (1 - e^{-\lambda x})e^{-\alpha x} H(dx), \lambda \geq 0$. 

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3) $e_\alpha - G_{e_\alpha}$ has the distribution

$$P(e_\alpha - G_{e_\alpha} \in dx) = \frac{\alpha}{\phi(\alpha)} \left( a\delta_0(dx) + e^{-\alpha x} \Pi(x) dx \right), \quad x \geq 0.$$  

Relationship (3) can be found in Bertoin [6] Lemma 1.11, we also refer to Winkel [32] for the study of some related distributions. For the sake of completeness, a proof will be given below in section 2.1.

Remark 1. In the case where $x \mapsto \Pi(x)$ is log-convex, which is equivalent to say that the function $x \mapsto a e^{-\alpha x} \Pi(x)/\phi(\alpha)$ bears the same property for some (and then for every) $\alpha > 0$, Steutel’s theorem (Theorem 10.2 in Chapter III in Steutel and van Harn [28]) insures then that the overshoot $e_\alpha - G_{e_\alpha}$ is infinitely divisible.

We will now relate the h.p.m. to the Lévy measure of a remainder of an exponential functional, in the sense of Hirsch and Yor [23]. Associated to $\xi$ we define the so called exponential functional $I$ given by

$$I = \int_0^\infty \exp\{-\xi_s\} \, ds.$$

For background on exponential functionals and related objects see the thorough review by Bertoin and Yor [10]. The random variable $I$ is finite almost surely because either the life-time is finite a.s or the strong law of large numbers for subordinators ensures that they grow at least linearly. Following [10], the distribution of $I$ is determined by its integer moments which, as proved by Carmona et al. [12], satisfy

$$E[I^n] = \prod_{i=1}^n \frac{i}{\phi(i)}, \quad n \geq 1.$$

We also refer to Gnedin et al. [15] for combinatorial derivations of the moments formula. Bertoin and Yor [9] proved the existence of a unique random variable $R$, the these days called remainder random variable associated to $I$, whose entire moments are given by

$$E[R^n] = \prod_{i=1}^n \phi(i), \quad n \geq 1,$$

and such that if $R$ is independent of $I$ then

$$IR \overset{\text{law}}{=} e$$

where $e$ is a standard exponential random variable. The focus in the following Theorem is on a characterization of the infinite divisibility of the r.v.’s $e_\alpha - G_{e_\alpha}$ and log $I$ in terms of $\phi$ and $H$.

Theorem 1. The following assertions are equivalent:

(i) $e_\alpha - G_{e_\alpha}$ is infinitely divisible for some, and hence for all, $\alpha > 0$;

(ii) the probability measure

$$\frac{\alpha}{\phi(\alpha)} \left( a\delta_0(dx) + e^{-\alpha x} \Pi(x) dx \right), \quad x \geq 0,$$

is the law of an infinitely divisible r.v. for some, and hence for all, $\alpha > 0$;

(iii) log $I$ is infinitely divisible.
(iv) $\lambda \mapsto \frac{1}{\lambda} - \frac{\phi'(\lambda)}{\phi(\lambda)}$ is completely monotone (c.m. for short);

(v) The measure $dx - xH(dx)$, $x > 0$, is nonnegative.

(vi) The h.p.m. has a density $\rho$ with respect to the measure $dx/x$ such that $\rho(x) \leq 1$ for every $x > 0$.

We define the exponential functional stopped at time $t > 0$ by

$$I_t = \int_0^t \exp\{-\xi_s\} \, ds.$$  

Before establishing a connection between the h.p.m. of $\xi$ and the exponential functional and the remainder, we give a formula which describes the moments of $I$ and $R$ in terms of the so called generalized gamma function. It is worth mentioning that a closely related result appeared recently in the paper by Patie and Savov [26].

We say that a function $g : (0, \infty) \to (0, \infty)$ satisfies Webster’s conditions if

$$g \text{ is log-concave } \quad \text{and} \quad \lim_{s \to \infty} \frac{g(s+c)}{g(s)} = 1, \quad \text{for every} \quad c > 0$$  

Following Webster [31], we set

$$a_n = \frac{(g'_-(n) + g'_+(n))}{2g(n)}, \quad n \geq 1,$$

and define the generalized gamma function associated to $g$ by

$$\Gamma_g(s) := e^{-\gamma_g s} \prod_{n=1}^\infty \frac{g(n)}{g(n+s)} e^{a_n s}, \quad s > 0.$$  

Theorem 2. The following assertions hold true.

(1) The functions $\lambda \mapsto \phi(\lambda)$ and $\lambda \mapsto \phi^*(\lambda) := \lambda/\phi(\lambda)$ satisfy (6) and the moment generating functions of $R$ and $I$ are given by

$$E[R^s] = \Gamma_{\phi}(s+1) \quad \text{and} \quad E[I^s] = \Gamma_{\phi^*}(s+1) = \frac{\Gamma(s+1)}{\phi(s+1)}, \quad s > -1.$$  

(2) We assume here that $q = \phi(0) = 0$. Let $\alpha > 0$, $e_\alpha$ be an exponentially distributed random variable with parameter $\lambda$ independent of $\xi$ and denote $\phi_{c,\alpha}(\cdot) = \phi(\cdot + c) + \alpha$, $c \geq 0$. Then, the joint distribution of $(I_{e_\alpha}, \xi_{e_\alpha})$ is characterized by

$$E[(I_{e_\alpha})^se^{-\mu\xi_{e_\alpha}}] = \frac{\alpha}{\alpha + \phi(\mu)} \frac{\Gamma(s+1)}{\phi_{\mu,\alpha}(s+1)}, \quad \mu \geq 0, \quad s > -1.$$  

It is well-known that $\log R$ is infinitely divisible, see Berg [4]. In the next theorem, we write its Lévy measure in terms of the h.p.m. $H$ and, more interestingly, we provide an identity in law describing $R$ as an infinite product of independent last position of $\xi$ below random barriers. This result and the Corollary [1] can be seen as a generalisation of Gordon’s representation [16] of a log-gamma random variable involving a sequence of independent standard exponential random variables. See the Example [1] below for further details.
Theorem 3. The Laplace exponent of $\log R$ defined by the function $\lambda \mapsto \log \mathbb{E}[R^\lambda]$ is given by the expression

$$\log \Gamma_\phi(\lambda + 1) = -\lambda \gamma_\phi + \int_{(0,\infty)} \left( e^{-\lambda x} - 1 + \lambda x \right) \frac{e^{-x}}{1 - e^{-x}} H(dx). \quad (10)$$

Its Lévy measure is the image by the map $x \mapsto -x$ of the measure $1_{x>0}(e^x - 1)^{-1} H(dx)$. Furthermore, we have the following equality in law

$$\log R \overset{\text{Law}}{=\gamma_\phi + \sum_{k=1}^{\infty} \left( \mathbb{E}[G^{(k)}] - G^{(k)} \right) \phi_k(1) - G^{(k)} \right). \quad (12)$$

where the random variables $(G^{(k)}, k \geq 1)$ are assumed to be independent, $G^{(k)}$ being a copy of $G_{\phi_k}$ for $k = 1, 2, \ldots$ and $\mathbb{E}[G^{(k)}] = \frac{\varphi_k(1)}{\phi(k)}$. Equivalently, we have

$$\log R \overset{\text{Law}}{=} \log \phi(1) + \sum_{k=1}^{\infty} \left( \varphi_k(1) - G^{(k)} \right). \quad (12)$$

and $\varphi_k(1) = -\log \mathbb{E} \left[ \exp(-G^{(k)}) \right] = \int_{(0,\infty)} (1 - e^{-x}) e^{-kx} H(dx) = \log \frac{\phi(k+1)}{\phi(k)}$.

Corollary 1. With the same notation as in the latter theorem we have the representation

$$\log R \overset{\text{Law}}{=\gamma_\phi + \sum_{k=1}^{n} \left( \mathbb{E}[G^{(k)}] - G^{(k)} \right) \phi(n+1) + B_n,}$$

where $R_{(n)}$ is the remainder random variable associated to the subordinator with Laplace exponent $\lambda \mapsto \phi(\lambda + n)$, $\lambda \geq 0$, it is independent of $(G^{(k)}, 1 \leq k \leq n)$, and it is such that $\log \frac{R_{(n)}}{\phi(n+1)} \to 0$ in probability; and finally

$$B_n = \int_{(0,\infty)} H(dx) \frac{e^{-x}}{(1 - e^{-x})} (e^{-x} - 1 + x) e^{-nx} \to 0 \quad as \quad n \to \infty.$$

As a consequence of the first assertion in the above theorem and properties of the h.p.m. we will establish the following corollaries.

Corollary 2. If the subordinator $\xi$ has a strictly positive drift then the random variable $\log I$ is not infinitely divisible.

Corollary 3. If the Lévy tail $\Pi$ is log-convex, then the random variable $\log R$ belongs to class of self-decomposable distributions; if furthermore $\Pi$ is c.m., then $\log R$ belongs to the class of extended generalized gamma convolutions.

See the books of Bondesson [11] and Sato [27] for background on self-decomposable and extended generalized convolutions distributions.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminary results and proofs. To be more precise, we start with Subsection 2.1 where we recall some basic facts on subordinators and state Lemma 1. Theorem 2 is then proved in Subsection 2.2 where also some related results are recalled. In Section 3 we obtain some basic results about harmonic potential measures and provide a proof of Theorem 1 and of Corollaries 2 and 3. Section 4 is devoted to the study of the distribution of the remainder random variable. In particular, Theorem 3 is established therein. To finish, in Section 5 some examples are studied.
2 Preliminaries and proofs

2.1 Proof of Lemma 1

As for the p.m., the h.p.m. is related to $\phi$ via Laplace transforms, but in a more involved way as it is shown in the following identities. Indeed, by Frullani’s integral, we have the following identities: for $\lambda \geq 0$, $\alpha > 0$,

$$
\frac{\phi(\alpha)}{\phi(\alpha + \lambda)} = \exp \left\{ \log \left( \frac{\phi(\alpha)}{\phi(\alpha + \lambda)} \right) \right\} \\
= \exp \left\{ \int_0^\infty \frac{dt}{t} \left( e^{-t\phi(\alpha + \lambda)} - e^{-t\phi(\alpha)} \right) \right\} \\
= \exp \left\{ \int_0^\infty \frac{dt}{t} \left( E[e^{-(\lambda+\alpha)\xi_t}] - E[e^{-\alpha \xi_t}] \right) \right\} \\
= \exp \left\{ -\int_0^\infty \frac{dt}{t} \int_{\mathbb{R}^+} P(\xi_t \in dx) e^{-\alpha x} (1 - e^{-\lambda x}) \right\} \\
= \exp \left\{ -\int_{\mathbb{R}^+} H(dx) e^{-\alpha x} (1 - e^{-\lambda x}) \right\}.
$$

(13)

Now, let us observe the representation for the Bernstein function $\phi$,

$$
\phi(\lambda) = q + \lambda a + \int_{\mathbb{R}^+} (1 - e^{-\lambda x}) \Pi(dx) = \lambda a + \int_{[0, \infty]} (1 - e^{-\lambda x}) (\Pi(dx) + q \delta_{\infty}(dx)), \quad \lambda \geq 0.
$$

This allows us to extend the Lévy-Itô representation of $\xi$ as

$$
\xi_t = at + \sum_{s \leq t} \Delta_s, \quad t \geq 0,
$$

where $((t, \Delta_t), t \geq 0)$ forms a Poisson point process on $(0, \infty) \times (0, \infty]$ with intensity measure $ds \otimes (\Pi(dx) + q \delta_{\infty}(dx))$. Said otherwise, the killing of the subordinator arises when there appears a jump of infinite size. A different way to see a subordinator $\xi$ is by means of an immortal subordinator, that is one with infinite lifetime, whose characteristics are $(0, a, \Pi)$, say $\xi^\dagger$, killed at a time $e_q$, where $e_q$ is an independent exponential random variable with parameter $q > 0$; that is to say by killing $\xi^\dagger$ at rate $q$. More precisely, the process defined by

$$
\xi_{t,q}^\dagger := \begin{cases} 
\xi_t^\dagger, & \text{if } t < e_q; \\
\infty, & \text{otherwise},
\end{cases}
$$

is a subordinator with characteristics $(q, a, \Pi)$, and hence with the same law as $\xi$. This makes that the transition probabilities of $\xi^\dagger$ are related to those of $\xi$ by an exponential factor viz.

$$
P(\xi_t \in dx) = e^{-qt} P(\xi_{t,q}^\dagger \in dx), \quad \text{on } [0, \infty), \text{ for all } t \geq 0.
$$

It is also easily seen that if we kill at a rate $\alpha$ a killed subordinator with characteristics $(q, a, \Pi)$ then we obtain again a killed subordinator with characteristics $(q + \alpha, a, \Pi)$. This elementary observations have as a consequence that the $(\alpha+q)$-resolvent of $\xi^\dagger$ equals the potential of the subordinator with characteristics $(\alpha + q, a, \Pi)$ or still the $\alpha$-resolvent of the subordinator with characteristics $(q, a, \Pi)$. These remarks will be useful in the following proof and in section 3.

We have now all the elements to give a proof to Lemma 1.
**Proof of Lemma** The proof of this result follows from the identities

\[ P(G_t \in dz, G_t < t) = V(dz)\Pi(t - z), \quad 0 \leq z < t, \]  

(15)

and

\[ P(G_t = t) = av(t), \quad t \geq 0, \]  

(16)

where \( v \) stands for the density of the p.m. \( V \), which we know exists whenever the drift \( a > 0 \). The latter identities can essentially be read in Bertoin ([5], Proposition III.2) where the result is proved for immortal subordinators but the same argument hold for killed subordinators. Indeed, arguing as in the proof of the aforementioned Proposition of [5] it is proved that

\[ P(G_t \in dz, G_t < t) = V_q^\dagger(dz) (\Pi(t - z, \infty) + q), \]

where \( V_q^\dagger \) denotes the \( q \)-potential of \( \xi^\dagger \), and the term \( qV_q^\dagger(dz) \) comes from the possibility that the process passes above the level by jumping to its cemetery state \( \Delta \), that is \( L_t = \zeta \). Furthermore, to verify that (10) holds true, we use the formulae

\[
\begin{align*}
P(G_t = t) &= P(G_t^\dagger = t, L_t < e_q) \\
&= \mathbb{E} \left[ 1_{\{\xi_{L_t}^\dagger = t\}} e^{-qL_t} \right] \\
&= \mathbb{E}[e^{-qT_{\{t\}^\dagger}} 1_{\{T_{\{t\}^\dagger < \infty\}}] \\
&= av_q^\dagger(t), \quad t > 0,
\end{align*}
\]

where \( T_{\{t\}^\dagger} = \inf\{s > 0: \xi_{s}^\dagger = t\} \) and \( v_q^\dagger \) denotes the density of the measure \( V_q^\dagger \), which is known to exists whenever the drift is strictly positive, and the last equality follows from the equality in the bottom of page 80 in [5]. From the discussion preceding this proof, we have that \( v = v_q^\dagger \). In the remainder of this proof, we will use the formula

\[
\frac{\phi(\lambda)}{\lambda} = a + \int_0^\infty e^{-\lambda x} \Pi(x) \, dx. \tag{17}
\]

which is obtained by integrating by parts (11).

1) Taking joint Laplace transforms, we get for every \( \lambda, \mu \geq 0, \)

\[
\Psi(\lambda, \mu) = \mathbb{E} \left[ \exp\{-\lambda G_{e_\alpha} - \mu(e_\alpha - G_{e_\alpha})\} \right]
\]

\[
= \alpha \int_0^\infty dt e^{-(\mu+\alpha)t} \left( \mathbb{E}[G_t \in dz] e^{(\mu-\lambda)z} \right)
\]

\[
= \alpha \int_0^\infty dt e^{-(\mu+\alpha)t} \left( V(dz) \Pi(t - z) e^{(\mu-\lambda)z} + \int_0^\infty dt e^{-(\alpha+\lambda)t} av(t) \right)
\]

\[
= \alpha \left( \mathbb{E}[V(dz) e^{(\mu-\lambda)z}] e^{-(\mu+\alpha)z} \int_z^\infty dt e^{-(\mu+\alpha)(t-z)} \Pi(t - z) + \frac{\alpha a}{\phi(\alpha + \lambda)} \right)
\]

By making use of (17), we obtain

\[
\Psi(\lambda, u) = \frac{\alpha}{\phi(\alpha + \lambda)} \left( a + \phi(\alpha + \mu) - a(\alpha + \mu) \right)
\]

\[
= \frac{\phi(\alpha)}{\phi(\alpha + \lambda)} \times \frac{\alpha \phi(\alpha + \mu)}{\phi(\alpha) (\alpha + \mu)}
\]
and one concludes on the independence.

2) The infinite divisibility property of $G_{e_n}$ follows from (13) while the expression giving the probability distribution $P(G_{e_n} \in dx)$ is obtained from (2).

3) From the above identities, it is immediate that we have
\[
\mathbb{E}[\exp\{-\lambda (e_n - G_{e_n})\}] = \frac{\alpha}{\phi(\alpha)} \phi(\alpha + \lambda) = \frac{\alpha}{\phi(\alpha)} \int_0^\infty e^{-\lambda x} [e^{-a_0} \Pi(x) dx + a\delta_0(dx)]
\]
where we used (17) to get the second equality.

\[\square\]

2.2 Proof of Theorem 2

Following [13], we start by observing that the following functional equation for the moments of $I$ holds
\[
\mathbb{E}[I^s] = \frac{s}{\phi(s)} \mathbb{E}[I^{s-1}] \quad \text{for} \quad s > 0 \quad \text{and} \quad \mathbb{E}[I^0] = 1.
\]
By using (5), we get also a functional equation for the moments of $R$
\[
\mathbb{E}[R^s] = \phi(s) \mathbb{E}[R^{s-1}] \quad \text{for} \quad s > 0 \quad \text{and} \quad \mathbb{E}[R^0] = 1.
\]
Given a log-concave function $g: \mathbb{R}_+ \to \mathbb{R}_+$, we will denote by $g'_+$ and $g'_-$ the right and left derivatives of $g$, respectively. Consider the more general functional equation
\[
f(s+1) = g(s)f(s) \quad \text{for} \quad s > 0 \quad \text{and} \quad f(1) = 1.
\]
This equation was studied by Webster in [31] where he was motivated by investigations of generalized gamma functions and their characterizations by a Bohr-Mollerup-Artin type theorem.

**Theorem 4.** (Webster [31]) Assume that $g$ is log-concave and $\lim_{s \to \infty} g(s+c)/g(s) = 1$ for all $c > 0$. Then there exists a unique log-convex solution $f: \mathbb{R}_+ \to \mathbb{R}_+$ to the functional equation (21) satisfying $f(1) = 1$ which is given by the generalized gamma function
\[
\Gamma_g(s) := \frac{e^{-\gamma_g s}}{g(s)} \prod_{n=1}^{\infty} \frac{g(n)}{g(n+s)} e^{a_n s}, \quad s > 0,
\]
where $a_n = (g'_+(n) + g'_-(n))/2g(n)$, $n \geq 1$ and $\gamma_g = \lim_{n \to \infty} (\sum_1^n a_j - \log g(n))$. Furthermore, if $g(s) \to 1$, as $s \to \infty$, then we have
\[
\Gamma_g(s) = \frac{1}{g(s)} \prod_{n=1}^{\infty} \frac{g(n)}{g(n+s)}.
\]

**Theorem 4** is obtained from a combination of Theorem 4.1 of [31] and the following discussion in Section 7 for the case when $g$ (resp. $f$) is log-concave (resp. log-convex) on $\mathbb{R}_+$. Note that when $g(x) = x$ in formula (5) gives the well-known infinite product representation of the Gamma function
\[
\Gamma(s) = \frac{1}{s} e^{-\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n} \quad \text{for} \quad s > 0
\]
where $\gamma$ stands for the Euler-Mascheroni constant. A combination of the aforementioned results and some arguments taken from the proof of Theorem 2.2 in [24] leads to the first assertion of Theorem 2.
Proof of Theorem 2. (1) To prove the first assertion of (1), we observe that the Frullani integral allows to write
\[
\log(\phi(\lambda)/\phi(1)) = \int_{(0,\infty)\times(0,\infty)} (e^{-x} - e^{-\lambda x}) P(\xi_t \in dx) \frac{dt}{t}, \quad \lambda > 0,
\]
which implies the log-concave property for \( \phi \). The function \( \phi^* \) is log-concave since by (17), \( 1/\phi^* \) is c.m. and then log-convex. The slowly varying property, which means that
\[
\lim_{s \to \infty} \phi(s+c)/\phi(s) = 1 \quad \text{for every} \quad c \geq 0,
\]
is seen from (3) and the fact that \( e_\alpha \) converges a.s. to 0 as \( \alpha \to \infty \). Theorem 4 implies that (21), with \( g = \phi \), has the unique log-convex solution given by \( \Gamma_\phi(s) \). From Guillemin et al. [19] and Maulik and Zwart [24], we know that the function \( s \to \mathbb{E}[R^s] \) is log-convex. Indeed, since \( R \) has some exponential moments, differentiating twice under the expectation symbol and using Cauchy-Schwarz inequality, we conclude that \( \frac{d^2}{ds^2} \log \mathbb{E}[R^s] \) is nonpositive. Hence, \( \mathbb{E}[R^s] = \Gamma_\phi(s + 1) \). Now, the function \( h : s \to \mathbb{E}[R^s] \mathbb{E}[I^s] \) satisfies (21) with \( g(s) = s \), and \( h(1) = 1 \). Hence, by Bohr-Mollerup-Artin theorem for the gamma function, we conclude that \( h(s) = \Gamma(s) \). For the second statement, on the one hand, we have \( \mathbb{E}[e^{I_s}] = \Gamma(s+1) \). On the other, using the factorization (5), we can write
\[
\mathbb{E}[e^{I_s}] = \mathbb{E}[I^s] \mathbb{E}[R^s] = \mathbb{E}[I^s] \Gamma_\phi(s + 1).
\]
Equalizing the latter equations, we obtain the result.

(2) Assuming that \( \xi \) is immortal, let \( Y \) be \( \xi \) killed at rate \( \alpha \). Plainly, \( Y \) is a subordinator with Laplace exponent \( \phi(\cdot) + \alpha \). Thus, we have
\[
I_{e^\alpha} = \int_0^\infty e^{-Y_s} \, ds.
\]
For convenience, in the remainder of this proof, we replace the notation \( \Gamma_\phi(s) \) by \( \Gamma(\phi(\cdot); s) \). The first assertion of Theorem 2 when applied to \( Y \), implies that
\[
\mathbb{E}[I_{e^\alpha}] = \frac{\Gamma(s+1)}{\Gamma(\phi(\cdot) + \alpha; s + 1)}, \quad s > 0.
\]
Next, let us introduce a new probability measure by setting
\[
d P^{(\mu)} = e^{-\mu \xi_t + \phi(\mu) t} d P_{\xi_t}, \quad t \geq 0,
\]
and denote by \( \mathbb{E}^{(\mu)} \) the expectation under \( P^{(\mu)} \). Clearly, under \( P^{(\mu)} \), \( \xi \) is a subordinator with Laplace exponent \( \phi(\cdot + \mu) - \phi(\mu) \). Thus, applying the first assertion, we obtain
\[
\mathbb{E}^{(\mu)}[I_{e^\alpha}] = \frac{\Gamma(s+1)}{\Gamma(\phi(\cdot + \mu) - \phi(\mu) + \alpha; s + 1)}.
\]
Now, by performing a change of measure, we can write
\[
\mathbb{E}^{(\mu)}[I_{e^\alpha}] = E[I_{e^\alpha}^s e^{-\mu \xi_t + \phi(\mu) e^\alpha}] = \alpha \int_0^\infty e^{-(\alpha - \phi(\mu)) t} \mathbb{E}[(I_t)^s e^{-\mu \xi_t}] \, dt = \frac{\Gamma(s+1)}{\Gamma(\phi(\cdot + \mu) - \phi(\mu) + \alpha; s + 1)}.
\]
Since \( \alpha \) can be taken to be arbitrary, replacing \( \alpha \) by \( \alpha + \phi(\mu) \) in the second and last equalities and rearranging terms, we get the second statement. \( \square \)
3 Further properties of harmonic potentials of subordinators

Before proving Theorem 3 in the next section, we make here a relatively long digression to establish some identities and properties for harmonic potentials. Those desiring to read the details of the proof of Theorem 3 just need to know the result in the first Lemma of this section.

We start by defining \( \kappa(dx) \), on \([0, \infty)\), as the unique measure whose Laplace transform is given by the expression

\[
\int_{(0,\infty)} \kappa(dx) e^{-\lambda x} = \frac{\phi'(\lambda)}{\phi(\lambda)}, \quad \lambda > 0.
\]  

(27)

This measure plays a crucial role in the identification of the Lévy measure of the random variable \( R \) or remainder of an exponential functional of subordinator, see [4] and [23]. The following result relates the measure \( \kappa \) with the h.p.m. of \( \xi \).

Lemma 2. We have the following identity,

\[
\int_{(0,\infty)\times(0,\infty)} ye^{-\lambda y} P(\xi_t \in dy) \frac{dt}{t} = \frac{\phi'(\lambda)}{\phi(\lambda)}, \quad \lambda > 0.
\]

As a consequence, we have that

\[
\kappa(dx) = xH(dx), \quad x > 0.
\]

Furthermore, \( H(dx) \) is the unique measure such that

\[
\log \frac{\phi(\lambda)}{\phi(1)} = \int_{(0,\infty)} (e^{-x} - e^{-\lambda x}) H(dx), \quad \lambda > 0.
\]

As a consequence, for any fixed \( c > 0 \), the measure \( e^{-cx} H(dx) \) is the h.p.m. of the subordinator with Laplace exponent \( \phi(\cdot + c) \) and the Lévy measure of the r.v. \( G_c \).

Proof. Observe that the left hand side in the statement in Lemma 2 equals

\[
\int_0^\infty \frac{dt}{t} \mathbb{E}[\xi_t e^{-\lambda \xi_t}] = \int_0^\infty \frac{dt}{t} t \phi'(\lambda) e^{-t\phi(\lambda)} = \frac{\phi'(\lambda)}{\phi(\lambda)}
\]

since

\[
\mathbb{E}[\xi_t e^{-\lambda \xi_t}] = - \left( \mathbb{E}[e^{-\lambda \xi_t}] \right)' = t\phi'(\lambda) e^{-t\phi(\lambda)}, \quad \lambda, t > 0.
\]

The identity relating \( \kappa \) and the h.p.m. of \( \xi \) follows by inverting Laplace transforms. The last formula is obtained by integrating (27) over \([1, \lambda]\). □

We have now all the elements to prove the Theorem 1.

Proof of Theorem 1. The equivalence between (iii) and (iv) is proved in Theorem 1.9 in Berg [3], with \( \alpha = 1 \) therein. The equivalence between (iv) and (v) is obtained from Theorem 3.3 in [23] and the identification of the measure \( \kappa \) in the Lemma 2. The equivalence between (v) and (vi) is straightforward. Finally, the equivalence between the assertions (v) and (i) is deduced from the identity

\[
\mathbb{E}[\exp\{-\lambda (e_\alpha - G_{e_\alpha})\}] = \exp \left\{ - \int_0^\infty \left( 1 - e^{-\lambda x} \right) e^{-\alpha x} \left( \frac{dx}{x} - H(dx) \right) \right\}, \quad \lambda \geq 0,
\]

which follows from [4], [13] and a further application of Frullani’s integral. That (i) and (ii) are equivalent follows from the third assertion in Lemma 1. Finally, the fact that the statements in (i) and (ii) hold for all \( \alpha > 0 \) when they hold for some \( \alpha_0 > 0 \) follows from the fact that if the measure \( e^{-\alpha x}(x^{-1}dx - H(dx)) \) is nonnegative for \( \alpha = \alpha_0 \) then it is also nonnegative for all \( \alpha > 0 \). □
The h.p.m. is related to the p.m. as follows.

**Lemma 3.** We have the equality of measures on \( \mathbb{R}^+ \)

\[
xH(dx) = \begin{cases} 
\int_{[0,x]} \Pi(dy)yV(dx-y) & \text{if } a = 0; \\
av(x)dx + \int_{[0,x]} \Pi(dy)yv(x-y)dx & \text{if } a > 0,
\end{cases}
\]

where \( v \) denotes the density of the p.m. \( V \) which we know exists when \( a > 0 \).

**Proof.** We have

\[
\int_{\mathbb{R}^+} e^{-\lambda x} \left( av(x)dx + \int_{[0,x]} \Pi(dy)yv(x-y)dx \right) = a \frac{\phi'(\lambda) - a}{\phi(\lambda)} + \int_{\mathbb{R}^+} xe^{-\lambda x} \Pi(dx), \quad \forall \lambda > 0,
\]

where we used \( \phi'(\lambda) = a + \int_{\mathbb{R}^+} xe^{-\lambda x} \Pi(dx) \) and formulae (2) and (27). The result follows from the injectivity of the Laplace transform.

Let \( S^{(\gamma)} \) be a \( \gamma \)-stable subordinator, that is it has no killing term, no drift and Lévy measure given by \( cx^{-1-\gamma}dx, x > 0, \) with \( c \in \mathbb{R}^+ \) a constant. We assume that \( S^{(\gamma)} \) is independent of \( \xi \) and define a new subordinator \( \xi^{(\gamma)} \) by subordinating \( \xi \) to \( S^{(\gamma)} \), that is

\[
\xi^{(\gamma)}_t = \xi_{S^{(\gamma)}_t}, \quad t \geq 0.
\]

We denote by \( H^{(\gamma)} \) the h.p.m. of \( \xi^{(\gamma)} \). We have the following simple but important result relating \( H^{(\gamma)} \) and \( H \).

**Lemma 4.** We have that

\[
H^{(\gamma)}(dx) = \gamma H(dx), \quad x > 0.
\]

**Proof.** First of all we prove that the h.p.m. associated \( S^{(\gamma)} \) is given by \( c x^{-1-\gamma}dx, x > 0 \). This follows easily from the following identities valid for any \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) measurable test function

\[
\int_{(0,\infty)} \frac{dt}{t} E[f(S^{(\gamma)}_t)] = \gamma \int_{(0,\infty)} \frac{du}{u} f(u),
\]

where we used Fubini’s theorem and we have performed the change of variable \( s = t^{1/\gamma}S^{(\gamma)}_1 \). Using the independence between \( \xi \) and \( S^{(\gamma)} \) and the former formula for the h.p.m. of \( S^{(\gamma)} \) we obtain the equalities

\[
\int_{(0,\infty)} H^{(\gamma)}(dx)f(x) := \int_{(0,\infty)} \frac{dt}{t} E[f(\xi^{(\gamma)}_t)] = \gamma \int_{\mathbb{R}^+(0,\infty)} \frac{dx}{x} E[f(\xi_x)] = \gamma \int_{\mathbb{R}^+(0,\infty)} H(dy)f(y).
\]
A natural question is to know how the p.m. and the h.p.m. are related. This question will be answered below and will lead to interesting consequences about the harmonic potential measure.

Lemma 5. We have the equality of measures

\[ H(dx) = \int_0^\infty V_\alpha(dx) \, d\alpha, \quad x \geq 0. \]

Proof. This follows from the elementary identity \( t^{-1} = \int_0^\infty e^{-\alpha t} \, d\alpha \) and Fubini’s Theorem. \( \square \)

An interesting consequence of the Lemma 5 is the following formula for the h.p.m., which is reminiscent of Kingman’s well known result that the p.m. has a bounded density that it is proportional to the creeping probability.

Corollary 4. For \( x > 0 \), denote the first hitting time of \( x \) by \( T_{\{x\}} = \inf\{t > 0 : \xi_t = x\} \). If \( \xi \) has a strictly positive drift \( a \) then the h.p.m. has a density \( h \) which is given by

\[ h(x) = E \left[ \frac{1}{aT_{\{x\}}} 1_{\{T_{\{x\}} < \infty\}} \right], \quad x > 0. \]

Proof. According to the discussion in (5, pp. 80-81), for each \( \alpha > 0 \) the \( \alpha \)-potential of \( \xi \) has a density, say \( v_\alpha \), and is related to the law of the hitting time of singletons, via the formula

\[ E \left[ \exp\{-\alpha T_{\{x\}}\} 1_{\{T_{\{x\}} < \infty\}} \right] = av_\alpha(x), \quad x > 0. \]

It follows that

\[ H(dx) = dx \int_0^\infty v_\alpha(x) \, d\alpha = \frac{1}{a} \int_0^\infty E \left[ \exp\{-\alpha T_{\{x\}}\} 1_{\{T_{\{x\}} < \infty\}} \right] \, d\alpha = \frac{dx}{a} E \left[ \frac{1}{T_{\{x\}}} 1_{\{T_{\{x\}} < \infty\}} \right], \]

for \( x > 0 \). \( \square \)

By an application of Lemma 5 together with the results in [23] we obtain the result in Corollary 2.

Proof of Corollary 2. Observe that if the drift \( a \) of \( \xi \) is strictly positive then necessarily

\[ T_x \leq x/a \quad \text{on} \quad \{T_{\{x\}} < \infty\}. \]

Hence, the h.p.m. has a density such that \( xh(x) \geq 1 \) for all \( x > 0 \). This has as a consequence that the random variable \( \log I \) is not infinitely divisible because according to [23] a necessary and sufficient condition for this to hold is that \( \kappa(dx) \leq dx \), but, in this setting, we have \( \kappa(dx) = xh(x)dx \geq dx \) for all \( x > 0 \). \( \square \)

A combination of Lemma 5 and the results of Gripenberg [17, 18], Friedman [20], Hawkes [21] and Hirsch [22] allows us to establish without much effort the following Proposition.

Proposition 1. If the Lévy tail \( \Pi \) is log-convex then the h.p.m. \( H \) has a density on \((0, \infty)\) which is non-increasing; if furthermore the former function is c.m. then the density of \( H \) on \((0, \infty)\) is also c.m.
Remark 2. In fact, more can be said in the latter case. It can be verified that \( x \mapsto \Pi(x) \) is c.m. if and only if \( \Pi \) has a completely monotone density, which is equivalent to require that \( \phi \) is a complete Bernstein function, which in turn is a necessary and sufficient condition for the density of \( H \) on \((0,\infty)\) to be the Laplace transform of some function \( \eta \) on \((0,\infty)\), taking values in \([0,1]\). For further details about these statements see the monograph of Schilling et al. ([29], Theorem 6.10, p. 58).

Proof. Let \( \alpha > 0 \) fixed. We will prove that under the assumptions of the proposition, the \( \alpha \)-potential of \( \xi \), \( V_\alpha \), has a non-increasing (c.m.) density. The result will follow from Lemma 5. Let us recall from the discussion in subsection 2.1 that the \( \alpha \)-potential of the subordinator \( \xi \), whose characteristics are \((q,a,\Pi)\), correspond to the 0-potential of a subordinator with characteristics \((q+\alpha,a,\Pi)\). With this remark at hand, the arguments in the proof of Lemma 1 allow us to ensure that, by integrating (15) over \([0,t]\), we obtain

\[
1 = av_\alpha(t) + \int_0^t (\Pi(t-s,\infty) + q + \alpha) V_\alpha(ds), \quad t > 0.
\]

According to [17, 18, 20, 21, 22], it follows that \( V_\alpha \) has a non-increasing (c.m.) density in \((0,\infty)\) whenever \( q + \alpha + \Pi(x,\infty) \) is a log-convex (c.m.) function. But this is a straightforward consequence of the assumption that \( \Pi(\cdot) \) bears this property.

Taking for granted the first claim in Theorem 3 and applying Proposition 1 we can prove Corollary 3.

Proof of Corollary 3. According to the latter Lemma we just need to notice that \( x \mapsto e^{-x}/(1-e^{-x}) \), \( x > 0 \), is a c.m. function. This is true because, on the one hand, the product of two c.m. functions is a c.m. function. On the other hand, \( e^{-x} \) is the Laplace transform of the Dirac’s delta measure concentrated at 1 while \((1-e^{-x})^{-1} \) is the Laplace transform of the p.m. of a Poisson process with rate one. We conclude by applying Theorem 15.10 in [27] and Theorem 3.1.1 in [11], respectively.

4 Remainder of exponential functionals

It has been proved by Berg [4], Theorem 2.2, that the random variable \( \log R \) is a spectrally negative infinitely divisible one and that its characteristic exponent, which is defined by \( \mathbb{E}\{\exp\{i\lambda \log R\}\} = \exp\{-\Psi(\lambda)\} \), is specified by

\[
\Psi(\lambda) = i\lambda \log \phi(1) + \int_{(0,\infty)} \left( e^{-i\lambda x} - 1 + i\lambda(1-e^{-x}) \right) \frac{e^{-x}}{1-e^{-x}} \kappa(dx)
\]

for \( \lambda \in \mathbb{R} \), and \( \kappa(dx) \) is the measure characterized by (27). The Lévy measure of the distribution of \( \log R \) is the image by the map \( x \mapsto -x \) of the measure given on the positive half-line by \( x^{-1}(e^x - 1)^{-1}\kappa(dx) \), see [23] for more details. An application of Lemma 2 and the identity (32) below gives the proof of the first claim in Theorem 3.

An analytic extension argument, consisting in replacing \( i\lambda \) by a positive real \( s \) in the expression in the first claim in Theorem 3 allows to get a moment generating transform version of Berg’s result. We state this version and however prove it using Theorem 4. Note that this appeared, while we were preparing this paper, in [23].

Proposition 2. We have the integral representation

\[
\log \Gamma_\phi(s+1) = s \log \phi(1) + \int_{(0,\infty)} \left( s - \frac{1 - e^{-sx}}{1-e^{-x}} \right) e^{-x} H(dx), \quad s > 0.
\]
Proof. Let us call \( F(s+1) \) the right hand side of (29). Clearly, \( F : \mathbb{R}_+ \to \mathbb{R}_+ \) is a log-convex function on \( \mathbb{R}_+ \) satisfying \( F(1) = 1 \). It also satisfies (19) because we have

\[
\log \Gamma \phi(s+1) = \log \phi(1) + \int_{(0,\infty)} \left( s - 1 - \frac{1 - e^{-sx}}{1 - e^{-x}} \right) e^{-x} H(dx)
\]

\[
= \log \phi(1) + \int_{(0,\infty)} \left( e^{-x} - e^{-sx} \right) H(dx)
\]

\[
= \log \phi(1) - \log \phi(1) = \log \phi(s).
\]

We conclude that \( \Gamma \phi(s) = F(s) \) for all \( s > 0 \) by using Theorem 4.

We have now all the elements to conclude the Proof to Theorem 3.

Proof of Theorem 3. Let us prove (11). Notice that we have

\[
E \left[ \exp \left( \gamma \phi(k) \right) \right] = \phi(k)
\]

For all \( s > 0 \), we can write

\[
E \left[ e^{-sG(k)} \right] = e^{-\gamma \phi(s)} \prod_{k=1}^{N} e^{\phi(k)} E \left[ e^{-sG(k)} \right]
\]

\[
= e^{-\gamma \phi(s)} \prod_{k=1}^{N} e^{\phi(k)} \phi(s+k) \to E \left[ R^s \right] \quad \text{as} \quad N \to \infty,
\]

where for the last equation we used Theorem 2 and the injectivity of Mellin transform. Let us now prove (12) using the aforementioned Berg’s result. We have the identity

\[
E \left[ \exp \left( \sum_{k=1}^{N} \left( \varphi_k(1) - G(k) \right) \right) \right] = \prod_{k=1}^{N} E \left[ \exp \left( \varphi_k(1) - G(k) \right) \right]
\]

\[
= \prod_{k=1}^{N} \exp - \int_{\mathbb{R}^+} H(dx) e^{-\lambda x} \left( 1 - e^{-\lambda x} - \lambda(1 - e^{-x}) \right)
\]

\[
= \exp - \int_{\mathbb{R}^+} H(dx) e^{-\lambda x} \left( 1 - e^{-N \lambda x} - \lambda(1 - e^{-x}) \right)
\]

\[
\to \exp - \int_{\mathbb{R}^+} H(dx) e^{-\lambda x} \left( 1 - e^{-\lambda x} - \lambda(1 - e^{-x}) \right) \quad \text{as} \quad N \to \infty.
\]

where the last equality is obtained by using the dominated convergence theorem. We conclude using Proposition 2 and the injectivity of the moment generating transform. Finally let us check that (11)
and (12) are equivalent. For that, we have

\[ -\gamma \phi + \sum_{k=1}^{N} \left( E[G^{(k)}] - G^{(k)} \right) - \log \phi(1) - \sum_{k=1}^{N} \left( \varphi_k(1) - G^{(k)} \right) \]

\[ = -\gamma \phi - \log \phi(1) + \sum_{k=1}^{N} \left( E[G^{(k)}] - \varphi_k(1) \right) \]

\[ = -\gamma \phi - \log \phi(1) + \sum_{k=1}^{N} \left( \frac{\phi'(k)}{\phi(k)} + \log E[e^{-G^{(k)}}] \right) \]

\[ = -\gamma \phi - \log \phi(1) + \sum_{k=1}^{N} \left( \frac{\phi(k)}{\phi(k)} + \log \frac{\phi'(k)}{\phi(k+1)} \right) \]

\[ = -\gamma \phi + \sum_{k=1}^{N} \frac{\phi'(k)}{\phi(k)} - \log \phi(N + 1) \to 0 \quad \text{as} \quad N \to \infty. \]

This shows that the right hand sides (11) and (12) are actually equal.

\[ \square \]

**Proof of Corollary [1]** Formula (11) can be written as

\[ \log R \overset{\text{Law}}{=} -\gamma \phi + \sum_{k=1}^{n} \left( E[G^{(k)}] - G^{(k)} \right) + \sum_{k=n+1}^{\infty} \left( E[G^{(k)}] - G^{(k)} \right) \]

\[ \overset{\text{Law}}{=} -\gamma \phi + \sum_{k=1}^{n} \left( E[G^{(k)}] - G^{(k)} \right) + \sum_{k=1}^{\infty} \left( E[G^{(k+n)}] - G^{(k+n)} \right) \]

\[ \overset{\text{Law}}{=} -\gamma \phi + \sum_{k=1}^{n} \left( E[G^{(k)}] - G^{(k)} \right) + \log R(n) + \gamma_{\phi(n+n)} \]

where \( R(n) \), which is assumed to be independent of the r.v.’s in the r.h.s. of the latter formula, is the analogue of \( R \) when we work with a subordinator having the Laplace exponent \( \phi(\cdot + n) \) instead of \( \phi(\cdot) \).

Next, we have

\[ \gamma_{\phi(n+n)} = \lim_{m \to \infty} \left\{ \sum_{k=1}^{m} \frac{\phi'(k+n)}{\phi(k+n)} - \log \phi(m+n) \right\} \]

\[ = \lim_{m \to \infty} \left\{ \sum_{k=1}^{m+n} \frac{\phi'(k)}{\phi(k)} - \log \phi(m+n) \right\} - \sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} \]

\[ = \gamma \phi - \sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)}. \]

It follows that

\[ \log R \overset{\text{Law}}{=} \sum_{k=1}^{n} \left( E[G^{(k)}] - G^{(k)} \right) + \log \frac{R(n)}{\phi(n+1)} + d_n \]

where

\[ d_n := -\sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} + \log \phi(n + 1). \]
Now, we can write

\[
d_n = \log \phi(1) - \sum_{k=1}^{n} \left[ \frac{\phi'(k)}{\phi(k)} - \log \frac{\phi(k+1)}{\phi(k)} \right]
\]

\[
= \log \phi(1) - \sum_{k=1}^{n} \int_{0}^{\infty} e^{-(k-1)x} e^{-x}(x - (1 - e^{-x}))H(dx)
\]

\[
= \log \phi(1) - \int_{(0,\infty)} H(dx) \frac{e^{-x}}{1 - e^{-x}} (e^{-x} - 1 + x) (1 - e^{-nx}),
\]

where to obtain the second equality we used Lemma \ref{lem:gamma}. Moreover, the above calculations show that the constant \(\gamma_\phi\) defined in (11) admits the representation

\[
- d_n \xrightarrow{n \to \infty} \gamma_\phi = - \log \phi(1) + \int_{(0,\infty)} (e^{-x} - 1 + x) \frac{e^{-x}}{1 - e^{-x}} H(dx). \tag{32}
\]

The representation of the error term \(B_n\) follows immediately from this expression. Recall that \(H^{(n)}(dx) = e^{-nx}H(dx)\) is the h.p.m. associated to \(\phi(\cdot + n)\). Thus, by applying Proposition \ref{prop:gamma} we can write for all \(s > 0\)

\[
\mathbb{E} \left( \frac{\left(R_n/\phi(n+1)\right)^s}{\phi(1+n)^s} \right) = \frac{\Gamma(\phi(n) + s)}{\phi(n+1)^s}
\]

\[
= \exp \int_{\mathbb{R}^+} \left( s - 1 - \frac{1 - e^{-sx}}{1 - e^{-x}} \right) e^{-x} H^{(n)}(dx)
\]

\[
= \exp \int_{\mathbb{R}^+} \left( s - 1 - \frac{1 - e^{-sx}}{1 - e^{-x}} \right) e^{-(n+1)x} H(dx) \to 1 \quad \text{as} \quad n \to \infty,
\]

by the dominated convergence theorem. This concludes the proof of the Corollary. \(\square\)

5 Examples

Example 1. Let \(K > 0\) and \(\xi\) be the deterministic subordinator \(\xi_t = Kt, \ t \geq 0\) killed at rate \(q \geq 0\). Then, the corresponding Bernstein function is \(\phi(\lambda) = q + K\lambda\) and the associated p.m. \(V\) and harmonic potential measures \(H\) are

\[
V(dx) = K^{-1} e^{-qx/K} dx, \quad \text{and} \quad H(dx) = e^{-qx/K} x^{-1} dx, \quad x > 0.
\]

In this case \(I\) has the same law as \(K^{-1} \mathcal{B}_{1,q/K}\) where the r.v. \(\mathcal{B}_{1,q/K}\) follows a Beta\((1,q/K)\)-law. This subordinator is special and its conjugated pair is \(\phi^*(\lambda) = \lambda/(\lambda K + q)\), see example \ref{ex:beta} below for the meaning of this terminology. The latter is the Laplace exponent of a compound Poisson process with jumps of exponential size of parameter \(q/K\) and arrival rate 1. Its h.p.m. is hence given by

\[
H^*(dx) = (1 - e^{-qx/K}) x^{-1} dx, \quad x > 0.
\]

From standard properties of the Beta and Gamma distributions and some factorizations of the exponential distribution, see e.g. \cite{Feller}, it is easy to see that the remainder random variable \(R\) has the same law as \(K \gamma_{1+q/K}\) where the latter Gamma variable has the p.d.f. \(\Gamma(1+q/K) x^{q/K} e^{-x}, \ x > 0\). This assertion can also be easily verified by calculating the expression of the moments of \(R\). Taking \(K = 1\) in this setting, with the identity in law (11) we recover Gordon’s representation \cite{Gordon} of a log-gamma
r.v. with shape parameter \( t = 1 + q \). Indeed, let \((e^{(i)}, i \geq 1)\) be an i.i.d. sequence of r.v.’s following a standard exponential distribution. For any \( k \geq 1\) the random variables \( G_{ek} \) have an exponential distribution with parameter \( q + k \), and therefore \( \mathbf{E}(G_{ek}) - G_{ek} \overset{\text{law}}{=} \frac{1}{q+k} - \frac{\mathbf{e}(k)}{q+k} \). Identity (33) becomes

\[
\log \gamma_{1+q} \overset{\text{law}}{=} \log R = -\gamma_{\phi} + \sum_{j=0}^{\infty} \frac{1}{(q+1)+j} - \frac{\mathbf{e}(k)}{q+1} + j,
\]

and it is readily verified that the constant \( \gamma_{\phi} + \sum_{k=1}^{\infty} \frac{q}{k(q+k)} \) equals Euler constant. The above representation of a Gamma r.v. with shape parameter \( 1 + q \) is the one obtained by Gordon in his Theorem 2, formula (3).

Finally, the measures

\[
H_{\alpha}(dx) = \alpha e^{-q x} x^{-1} dx \quad \text{and} \quad H_{\alpha}^{*}(dx) = \alpha (1 - e^{-q x}) x^{-1} dx, \quad x > 0,
\]

are the h.p.m.‘s of subordinators whose Laplace exponents are given by the Complete Bernstein functions \( \lambda \mapsto (\lambda + q)\alpha \) and \( \lambda\alpha / (\lambda + q)\alpha \), respectively. These naturally arise when subordinating, with an \( \alpha \)-stable subordinator, the above described subordinators.

**Example 2.** Let \( S \) be a \( \alpha \)-stable subordinator, that is it has no killing term, no drift and Lévy measure given by \( \Pi(dx) = c x^{-(1+\alpha)} dx_+ \), with \( c \in \mathbb{R}^+ \) and \( \alpha \in (0, 1) \). Recall the so-called Mittag-Leffler function is defined by

\[
E_{\alpha}(z) = \sum_{k \geq 0} \frac{z^k}{\Gamma(k\alpha + 1)} = \mathbf{E} \left[ e^{z S_{1}^{-\alpha}} \right], \quad z \in \mathbb{C}.
\]

**Proposition 3.** The h.p.m. \( H_{\alpha}^{*}(dx) \) associated to the subordinator obtained by killing \( S \) at rate \( q \geq 0 \), that we will denote by \( X = (X_s, s \geq 0) \), is

\[
H_{\alpha}(dx) = \alpha E_{\alpha}(-qx^{\alpha}) x^{-1} dx, \quad x > 0.
\]

**Proof.** Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be any measurable function. Using the scaling property, Fubini’s theorem and a change of variables \( s = t^{1/\alpha} S_1 \), we obtain the following formula for the h.p.m. of \( X \)

\[
\int_{(0,\infty)} H_{\alpha}^{*}(dx) f(x) := \int_0^\infty \frac{dt}{t} \mathbf{E}[f(X_t)]
= \int_0^\infty \frac{dt}{t} e^{-qt} \mathbf{E} \left[ f(t^{1/\alpha} S_1) \right]
= \int_0^\infty \frac{ds}{s} e^{-qs} \mathbf{E} \left[ \exp(-qs S_{1}^{-\alpha}) \right] f(s).
\]

The result follows. \( \square \)

As a consequence of this fact we get that the r.v. \( \log I \), with \( I := \int_0^\infty e^{-X_s} ds \) belongs to the \( B(\mathbb{R}) \)-class, see Barndorff-Nielsen et al. [2], and its Lévy measure has the density

\[
\frac{e^x}{|x|(1-e^x)} \left( (1-\alpha) + \alpha \mathbf{E} \left[ 1 - e^{-q(-x)^\alpha S_{1}^{-\alpha}} \right] \right), \quad x < 0.
\]
Similarly, the random variable \( \log R \) associated to \( X \), belongs to the class of Extended Generalized Gamma Convolutions.

The above remark has as a consequence that \( \log R \) is a random variable whose Lévy measure is absolutely continuous with respect to the probability distribution of a one sided Linnik r.v. We think that this link deserves to be studied in further detail.

**Example 3.** The following Lemma is useful to construct explicit examples of the Lévy measure of the random variable \( R \). Its proof is elementary and it is based on elementary properties of Poisson processes, so we omit the details.

**Lemma 6.** Assume \( \xi \) is a subordinator with drift \( \alpha \geq 0 \), no killing term and finite Lévy measure \( \Pi \). Let \( (N_t, t \geq 0) \) be a Poisson process with intensity \( \Pi(0, \infty) =: \alpha \), \( (Y_i)_{i \geq 1} \) i.i.d. random variables which are independent of \( N \), with common distribution \( \alpha^{-1} \Pi(dx) \), and \( S_n = \sum_{i=1}^n Y_i \), for \( t \geq 0 \). When the drift \( \alpha \) is strictly positive, we have the equality of measures

\[
\int_0^\infty \frac{dt}{t} P(\xi_t \in dx) = e^{-\frac{\alpha}{\alpha+1} x} \frac{dx}{x} + \left( \sum_{n=1}^\infty \frac{\alpha}{n!} E \left[ \left( \frac{\alpha}{\alpha}(x-S_n) \right)^{n-1} \exp \left\{ -\frac{\alpha}{\alpha}(x-S_n) \right\} 1_{\{S_n \leq x\}} \right] \right) \frac{dx}{x},
\]
on \((0, \infty)\). While, when the drift \( \alpha \) is zero, we have

\[
\int_0^\infty \frac{dt}{t} P(\xi_t \in dx) = \sum_{n \geq 1} \frac{1}{n} P(S_n \in dx)
\]
on \((0, \infty)\); that is the h.p.m. of \( \xi \) coincides with the h.p.m. of the random walk \( (S_n, n \geq 1) \).

Assume for instance that \( \xi \) be a subordinator with no killing, drift 1, and Lévy measure

\[
\Pi(dx) = \frac{\alpha}{\Gamma(\beta)} x^{\beta-1} e^{-x} dx, \quad x > 0,
\]

where \( \beta > 0 \). Equivalently, the Laplace exponent is given by \( \lambda \mapsto \lambda + \alpha (1 - \frac{1}{1+\lambda^\beta}) \) It follows from Lemma 6 and elementary calculations that the h.p.m. of \( \xi \) is given by

\[
\left( 1 + \sum_{n=1}^\infty \frac{\alpha^n x^{n(1+\beta)} \Gamma(n\beta)}{n! \Gamma(n(1+\beta))} \right) e^{-\alpha x} x^{-1} dx, \quad x > 0.
\]

Furthermore, assuming the same jump structure, but zero drift, that is the Laplace exponent is given by \( \lambda \mapsto \alpha (1 - \frac{1}{1+\lambda^\beta}) \), we find the harmonic measure

\[
\left( \sum_{n=1}^\infty \frac{x^n \beta}{\Gamma(n\beta+1)} \right) e^{-x} x^{-1} dx, \quad x > 0.
\]

**Example 4.** Complete Bernstein functions, which are necessarily special, are those Bernstein functions for which the Lévy measure of the corresponding subordinator is completely monotonic, for example see [29] Chapter 6. The latter have the following representation

\[
\frac{\phi(\lambda)}{\phi(1)} = \exp \int_0^\infty \left( \frac{1}{1+t} - \frac{1}{\lambda+t} \right) \eta(t) dt = \exp \int_0^\infty \frac{\lambda-1}{\lambda+t} \frac{\eta(t)}{1+t} dt
\]

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for all \( \lambda \geq 0 \), were \( \eta : \mathbb{R}_+ \to [0, 1] \) is a measurable function. The density of the h.p.m., in this case, is the Laplace transform of \( \eta \) i.e. \( \rho(x) = \int_0^\infty e^{-xt} \eta(t) dt \). Marchal [25] considered this class of Bernstein functions using the representation

\[
\phi^{(\alpha)}(\lambda) = \exp \int_0^1 \frac{\lambda - 1}{1 + (\lambda - 1)x} \alpha(x) dx.
\]

Making the substitution \( x = 1/(1 + t) \), we obtain that

\[
\phi^{(\alpha)}(\lambda) = \exp \int_0^1 \frac{\lambda - 1}{\lambda + t} \frac{\alpha(1/(1 + t))}{1 + t} dt
\]

which is indeed of the form \([36]\) with \( \eta(t) = \alpha(1/(1 + t)) \). Furthermore, in Section 16.11 in [29] we can find several interesting families of subordinators where the harmonic potential density is known explicitly.

**Example 5.** Assume that \( \xi \) is the ascending ladder time, with Laplace exponent \( \phi \), of a general real valued Lévy process \( (X_t, t \geq 0) \). It follows from the Lemma 1 and the identity (5) in page 166 in [5], which is a consequence of Fristedt’s formula, that the h.p.m. of \( \xi \) is given by

\[
H(dx) = x^{-1} P(X_x \geq 0) dx, \quad x > 0.
\]

We further note that, in the stable case we have by the self-similarity of \( X \) that \( P(X_t \geq 0) = \beta \) for all \( t \), and some \( \beta \in (0, 1) \), called the positivity index of \( X \). Recall that \( E[e^s] = \Gamma(s + 1) \), where \( e \) is a standard exponential random variable, and we have the integral representation of the gamma function

\[
\log \Gamma(s + 1) = \int_0^\infty \left( s - \frac{1 - e^{-st}}{1 - e^{-t}} \right) e^{-t} dt
\]

found for example in Theorem 1.6.2 in P.28 of [1]. It follows from Proposition 2 that \( \Gamma_\phi(s) = (\Gamma(s))^{\beta} \), \( \Gamma^*(\phi(\cdot); s) = \Gamma^{\phi^*}(s) = (\Gamma(s))^{1-\beta} \), and an expression for the characteristics of \( \log R \) and \( \log I \) is easily deduced from the above expression.

**Example 6.** The processes of examples [1] and [5] are particular cases of special subordinators. We remind that \( \phi \) is said to be a special Bernstein function if the function \( \phi^* \) defined by

\[
\phi^*(\lambda) = \lambda \frac{1}{\phi(\lambda)}, \quad \lambda > 0,
\]

is Bernstein function. In this case we say that \((\phi, \phi^*)\) is a pair of conjugated Bernstein functions and we denote by \((q^*, a^*, \Pi^*)\) the characteristics of \( \phi^* \), that is

\[
\phi^*(\lambda) = q^* + \lambda a^* + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Pi^*(dx), \quad \lambda \geq 0.
\]

It is well known that a necessary and sufficient condition for \( \phi \) to be special is for the p.m. of \( \phi \) to be such that \( V(dx) = v(x) dx \), where \( v : \mathbb{R}_+ \to \mathbb{R}_+ \) is a non-increasing function. Sufficient conditions on the Lévy measure are also known see e.g. Hirsch [22] and Song and Vondracek [30]. It is known, and easy to see that for a special subordinator with Laplace exponent \( \phi \), and its conjugate \( \phi^* \), we have that

\[
qq^* = 0, \quad aa^* = 0, \quad \Pi^*(x, \infty) + q^* = v(x)
\]
where \( G \) see the well-known fact that the equation (13) applied to \( \varphi \) and \( G \) By Lemma 1, we already know that obtain information about the exponential functional \( I \) notations, and therefore all the results obtained for the remain variable \( \xi \), \( \xi \) to write the analogue of formulae (11) and (12) of Theorem 3 for it and so on. To prove the claimed to determine when it has a self-decomposable or extended generalized gamma convolution distribution, \( H \) recall that it follows from the fact that \( \alpha > 0 \), \( \lambda \geq 0 \). Let \((\xi, \xi^*)\) be a a conjugated pair of subordinators associated to the pair \((\phi, \phi^*)\). For convenience, all objects associated to \( \xi^* \), which we defined for \( \xi \), are denoted with the superscript \(*\). We immediately see the well-known fact that \( e_\alpha - G_{e_\alpha} \overset{\text{Law}}{=} G_{e_\alpha}^* \) and \( G_{e_\alpha} + G_{e_\alpha}^* \overset{\text{Law}}{=} e_\alpha \), where \( G_{e_\alpha} \) and \( G_{e_\alpha}^* \) are independent. Furthermore, there exists a function \( \rho : (0, \infty) \to [0, 1] \) such that \( H(dx) = \rho(x) \frac{dx}{x} \) and \( H^*(dx) = (1 - \rho(x)) \frac{dx}{x}, \quad x > 0, \) and \( \rho \) is related to \( \phi \), \( \Pi \) and \( v \) as follows \[ \int_{(0, \infty)} dy \rho(y) e^{-\lambda y} = \frac{\phi'(\lambda)}{\phi(\lambda)}, \quad \lambda > 0, \] and \[ \rho(x) = av(x) + \int_{[0, x]} \Pi(dy) yv(x-y), \quad x > 0. \] It is worth pointing out that in this case we have the identities \( I \overset{\text{Law}}{=} R^* \) and \( I^* \overset{\text{Law}}{=} R \), with obvious notations, and therefore all the results obtained for the remainder random variable \( R \) can be applied to obtain information about the exponential functional \( I \). For instance, this can be used for the r.v. \( \log I \) to determine when it has a self-decomposable or extended generalized gamma convolution distribution, to write the analogue of formulae (11) and (12) of Theorem 3 for it and so on. To prove the claimed facts about \( H \), we recall that it follows from the fact that \( \phi, \phi^* \) are conjugated Bernstein functions and the equation (13) applied to \( \phi \) and \( \phi^* \) that for \( \alpha > 0, \lambda \geq 0 \), \begin{equation} \begin{aligned} \exp \left\{ - \int_{(0, \infty)} \frac{dx}{x} e^{-\alpha x} (1 - e^{-\lambda x}) \right\} &= \frac{\alpha}{\alpha + \lambda} \\ &= \frac{\phi(\alpha)}{\phi(\alpha + \lambda)} \frac{\phi^*(\alpha)}{\phi^*(\alpha + \lambda)} \\ &= \exp \left\{ - \int_{(0, \infty)} H(dx) e^{-\alpha x} (1 - e^{-\lambda x}) \right\} \exp \left\{ - \int_{(0, \infty)} H^*(dx) e^{-\alpha x} (1 - e^{-\lambda x}) \right\} \end{aligned} \end{equation} For \( \alpha > 0 \), we deduce the equality of Lévy measures \[ e^{-\alpha x} H(dx) + e^{-\alpha x} H^*(dx) = e^{-\alpha x} x^{-1} dx, \quad x > 0. \] Hence \( H(dx) + H^*(dx) = x^{-1} dx \), and that \( H(dx) \) and \( H^*(dx) \) are absolutely continuous with respect to \( x^{-1} dx \) and thus there exists a density function \( \rho \) such that \( \rho(x)x^{-1} dx = H(dx) \) and \( 0 \leq \rho(x) \leq 1 \). Analogously we have that \( H^*(dx) = \rho^*(x)x^{-1} dx \) with \( 0 \leq \rho^*(x) \leq 1 \). Furthermore, as \( H(dx) + H^*(dx) = x^{-1} dx \), it follows that \( \rho(x) + \rho^*(x) = 1 \) for a.e. all \( x > 0 \). The rest follows from the results in Section 3.
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