1. Introduction and results

Fix a nonabelian free group $\mathbb{F}$ of finite rank and let $H$ be a finitely generated (or f.g. for short) group with a f.g. subgroup $P$. In his work on the Tarski problem, Zlil Sela considers the following question. In how many ways can a given homomorphism $P \to \mathbb{F}$ be extended to $H$? Of course without further restrictions the answer is often infinitely many. He goes on to define a natural equivalence relation on the set of extensions (described below in our setting) and obtains the remarkable result:

**Theorem 1.1** (Sela [7]). Suppose that $H$ is freely indecomposable rel $P$. There is a number $N = N(H,P)$ and a finite set $\mathcal{F} = \{q_i : H \to L_i\}$ of proper quotients so that each homomorphism $P \to \mathbb{F}$ has at most $N$ equivalence classes of extensions to $H$ with the property that no element of the equivalence class factors through an element of $\mathcal{F}$.

The set $\mathcal{F}$ is a factor set for $(H,P)$. A homomorphism from $H$ to $\mathbb{F}$ with the property that no equivalent element factors through an element of $\mathcal{F}$ is solid with respect to $\mathcal{F}$. Not much was known about $N(H,P)$. For example, Sela asked whether there was a sequence of examples $(H_i, P_i)$ with $\lim N(H_i, P_i) = \infty$. Our main result is that

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there is such a sequence. In fact, in our sequence $H_i$ will be the fundamental group of an orientable surface of genus $i$ with $P_i$ representing its boundary and we show that $N(H_i,P_i) \geq 2^i$.

We now describe our results in more detail. Identify $H_g = \langle a_1, b_1, \ldots, a_g, b_g \rangle$, a free group of rank $2g$, with the fundamental group of a surface $S_g$ of genus $g$ and one boundary component and set $\partial_g = [a_1, b_1] \cdots [a_g, b_g]$ so that $\partial_g$ is represented by the boundary of $S_g$. For $x \in F$, a genus $g$ representation of $x$ is a homomorphism $h \in \text{Hom}(H_g,F)$ such that $h(\partial_g) = x$. Set $P_g = \langle \partial_g \rangle$.

**Definition 1.2.** Two genus $g$ representations $h$ and $h'$ of $x$ are related by a fractional Dehn twist if one of the following holds:

- $H_g = A \ast_C B$ with $C$ cyclic, $\partial_g \in A$, and there is $z \in F$ centralizing $h(C)$ such that $h' = h \ast (i_z \circ h)$ (by which we mean $h'|A = h|A$ and $h'|B = i_z \circ (h|B)$ where $i_z$ denotes conjugation by $z$).
- $H_g = A \ast \phi = \langle A, t \mid t^{-1}ct = \phi(c), c \in C \rangle$ where $\phi : C \to C'$ is the bonding isomorphism, $C$ is cyclic, $\partial_g \in A$, and there is $z \in F$ centralizing $h(C)$ such that $h'|A = h|A$ and $h'(t) = zh(t)$.

The equivalence relation “$\sim$” on representations of $x$ is generated by $h \sim h'$ if $h$ and $h'$ are related by a fractional Dehn twist.

**Remark 1.3.**

1. It is a result of Stallings that splittings of $H_g$ as in Definition 1.2 are all induced by some simple closed curve $\sigma$ in $S_g$. The different items correspond to whether or not $\sigma$ is separating.

2. If $z \in h(C)$ then there is an automorphism $\tau$ of $H_g$ such that $h' = h \circ \tau$. By [1], $\tau$ is a (classical) Dehn twist.

3. Using the trivial splitting $H_g = H_g \ast_{P_g} P_g$, we see that $i_z \circ h \sim h$ where $z$ is a root of $x$ in $F$.

4. The group of outer automorphisms of $H_g$ preserving the conjugacy class of $\partial_g$ may be identified with the modular group $\text{Mod}(S_g)$, see [9]. $\text{Mod}(S_g)$ is generated by Dehn twists [2, 3]. It follows that if $\phi$ is an automorphism of $H_g$ fixing $\partial_g$ then $h \circ \phi \sim h$.

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1To prove [1], first resolve the given splitting to find a collection of pairwise disjoint simple closed curves that induces a splitting of $H_g$ that can be folded to the given splitting. Then, show that only in trivial situations is a fold possible; see [1].
Example 1.4. Let $F = \langle u, v \rangle$, $H_1 = \langle a_1, b_1 \rangle$, and $x_m = [u^m, v]$. For $m, n \in \mathbb{Z}$, the homomorphism $h_{m,n} : H_1 \to F$ given by $a_1 \mapsto u^m$ and $b_1 \mapsto vu^n$ is a genus 1 representation for $x_m$. The homomorphisms $h_{m,n}$ and $h_{m,n'}$ are related by fractional Dehn twist whereas they differ by a Dehn twist iff $n \equiv n' \mod m$. In particular, Theorem 1.1 is false if the equivalence relation is defined only using Dehn twists.

Definition 1.5. For $x \in F$, a genus $g$ representation is $\sim$-injective if all equivalent genus $g$ representations are also injective. Define

$$\text{num}_g x$$

to be the number of equivalence classes of $\sim$-injective genus $g$ representations of $x$ in $\text{Hom}(H_g, F)$. Finally, define

$$f_{\mathcal{F}}(g) = \sup \{ \text{num}_g x \mid x \in F \}.$$

If $h \in \text{Hom}(H_g, F)$ is $\sim$-injective, then $h$ is solid with respect to any factor set, and in particular

$$f_{\mathcal{F}}(g) \leq N(H_g, P_g).$$

That $f_{\mathcal{F}}(g)$ is finite is a consequence of Theorem 1.1. In Corollary 4.6 we show that $f_{\mathcal{F}}$ is independent of $F$. It is not hard to see that if $x \in F$ is a “generic” element with a genus 1 representation, then $\text{num}_g x = 1$. However, it should also be reasonable to expect that $f_{\mathcal{F}}(1) > 1$ – take a “generic” map from the genus 2 surface to a graph representing $F$, then the element $x \in F$ represented by the image of the waist curve is written as $[p, q]$ in two inequivalent ways, giving two representations. It takes a little bit of work to show that these representations really are inequivalent. This is the content of Section 3 and reproduces a result of Lyndon and Wicks [4].

For higher genera this conceptual argument fails to show $f_{\mathcal{F}}(g) > 2$. The reason is that we do not know explicitly the MR-diagram$^3$ for the group obtained by gluing say 3 surfaces with boundary along their boundaries. The only “obvious” quotients are obtained by identifying two of the surfaces or killing the common boundary. To find interesting examples one would have to show that there are other maximal limit group quotients of this group, see Remark 4.7.

However, in Section 4 we will argue that $f_{\mathcal{F}}(g) \geq 2^g$. For example, to see $f_{\mathcal{F}}(2) \geq 4$ we form the “boundary connected sum” of genus 1

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$^2$Thanks to Leo Comerford for pointing us to this article.

$^3$Some comments are meant for those familiar with Sela’s work on the Tarski problems. The theorems and proofs in this paper do not depend on such a familiarity.
examples. Each piece bounds in two ways, so we expect the sum to bound in 4 ways.

In order to deal with fractional Dehn twists it is convenient to consider more restrictive representations.

**Definition 1.6.** An injective representation \( h \in \text{Hom}(H_g, \mathbb{F}) \) of \( x \in \mathbb{F} \) is admissible if \( \text{Im} h \) is a primitive\(^4\) subgroup of \( \mathbb{F} \).

**Proposition 1.7.** Suppose \( h \in \text{Hom}(H_g, \mathbb{F}) \) is an admissible representation for \( x \in \mathbb{F} \). If \( h' \sim h \), then \( h' \) is also admissible and \( \text{Im} h' = \text{Im} h \).

In particular, \( h \) is \( \sim \)-injective.

**Proof.** Simple closed curves represent indivisible\(^5\) elements of \( H_g \), and hence (in the presence of admissibility) fractional Dehn twists are Dehn twists, see Remark 1.3(2). It follows that there is an automorphism \( \tau \) of \( H_g \) such that \( h' = h \circ \tau \) and \( h'(H_g) = h \circ \tau (H_g) = h(H_g) \).

**Definition 1.8.** For \( x \in \mathbb{F} \) define \( \text{num}'_g x \) to be

\[
\left| \{ \text{Im} h : h \in \text{Hom}(H_g, \mathbb{F}) \text{ is an admissible representation of } x \} \right|
\]

and

\[
f'_g(g) = \sup \{ \text{num}'_g x \mid x \in \mathbb{F} \}.
\]

We then have

\[
N(H_g, P_g) \geq f_g(g) \geq f'_g(g).
\]

We will see that \( f'_g \) is also independent of \( \mathbb{F} \). Our main results are:

**Theorem 1.9.** \( f'_g(1) \geq 2 \) and \( f'_g(m + n) \geq f'_g(m) f'_g(n) \).

**Corollary 1.10.** \( N(H_g, P_g) \geq f_g(g) \geq f'_g(g) \geq 2^g \).

We go on to consider a class of representations called minimal.

**Definition 1.11.** For \( x \) in the commutator subgroup \([\mathbb{F}, \mathbb{F}]\) of \( \mathbb{F} \) the algebraic genus of \( x \), denoted \( a\text{-genus } x \), is the smallest \( g \geq 0 \) such that there is \( h \in \text{Hom}(H_g, \mathbb{F}) \) with \( h(\partial_g) = x \). A genus \( g \) representation of \( x \) is minimal if \( g = a\text{-genus } x \). We now make the same definitions as before, but restrict ourselves to minimal representations. Define

\[
\text{num } x \text{ to be the number of equivalence classes of minimal } \sim \text{-injective representations of } x,
\]

\[
\text{num}'_x = \left| \{ \text{Im} h : h \text{ is a minimal admissible representation of } x \} \right|,
\]

\[
f'_g(g) = \sup \{ \text{num } x \mid a\text{-genus } x = g \}, \text{ and}
\]

\(^4\) closed under taking roots, root-closed in \[4\]

\(^5\) not a proper power
\[ \hat{f}'_{\mathcal{F}}(g) = \sup \{ \text{num}' x \mid \text{a-genus} x = g \}. \]

Again we have
\[ N(H_g, P_g) \geq \hat{f}'_{\mathcal{F}}(g) \geq \hat{f}'_{\mathcal{F}}(g). \]

We will see that \( \hat{f}'_{\mathcal{F}} \) and \( \hat{f}'_{\mathcal{F}} \) are independent of \( \mathcal{F} \). Also,

**Theorem 1.12.** \( \hat{f}'_{\mathcal{F}}(1) \geq 2 \) and \( \hat{f}'_{\mathcal{F}}(m + n) \geq \hat{f}'_{\mathcal{F}}(m) \hat{f}'_{\mathcal{F}}(n) \).

**Corollary 1.13.** \( N(H_g, P_g) \geq \hat{f}'_{\mathcal{F}}(g) \geq \hat{f}'_{\mathcal{F}}(g) \geq 2^g. \)

2. Labeled graphs

A reference for this section is [8]. \( \mathcal{F} \) is a non-abelian free group with fixed finite basis \( \mathcal{B} \). The cyclic word obtained by cyclically reducing the \( \mathcal{B} \)-word \( w \) is denoted \( [\![w]\!] \). There is a 1-1 correspondence between cyclically reduced cyclic \( \mathcal{B} \)-words and conjugacy classes of elements of \( \mathcal{F} \). If \( x \in \mathcal{F} \), then \( [\![x]\!] \) denotes its conjugacy class. We will sometimes blur the distinction between \( \mathcal{B} \)-words (or cyclic \( \mathcal{B} \)-words) and the elements (or conjugacy classes) that they represent.

Let \( R_{\mathcal{B}} \) denote the wedge of \( |\mathcal{B}| \) oriented circles with fundamental group identified with \( \mathcal{F} \). \( R_{\mathcal{B}} \) is an example of a labeled graph. More generally, a labeled graph is a connected non-empty finite graph \( \Gamma \) together with a combinatorial\(^7\) map \( l : \Gamma \to R_{\mathcal{B}} \) called a labeling. We consider two labelings \( l \) and \( l' \) to be the same if, for each edge \( e \), the paths \( l|e \) and \( l'|e \) are homotopic rel endpoints. Thus, a labeling is equivalent to a choice of \( u(e) \in \mathcal{B}^{\pm 1} := \mathcal{B} \sqcup \mathcal{B}^{-1} \) for each oriented edge \( e \) of \( \Gamma \) such that \( u(e^{-1}) = u(e)^{-1} \) where \( e^{-1} \) is the edge opposite to \( e \). A labeling also induces labelings of edge paths in \( \Gamma \).

If \( l : \Gamma \to R_{\mathcal{B}} \) is an immersion and if \( \Gamma \) has no valence 1 vertices then we say that \( l \) or \( \Gamma \) is tight. A morphism of labeled graphs \( l_1 : \Gamma_1 \to R_{\mathcal{B}} \) and \( l_2 : \Gamma_2 \to R_{\mathcal{B}} \) is a combinatorial map \( f : \Gamma_1 \to \Gamma_2 \) that preserves labels, i.e. \( l_1 = l_2 \circ f \). An injective homomorphism \( \phi : \mathcal{F}_1 \to \mathcal{F}_2 \) induces a cellular map \( R_{\mathcal{B}_1} \to R_{\mathcal{B}_2} \) that immerses each edge. A morphism is obtained by subdividing edges of \( R_{\mathcal{B}_1} \). If \( l : \Gamma \to R_{\mathcal{B}_1} \) is a labeling then \( \phi(l) : \phi(\Gamma) \to R_{\mathcal{B}_2} \) is the induced labeled graph

\[ \Gamma \xrightarrow{l} R_{\mathcal{B}_1} \to R_{\mathcal{B}_2}. \]

Similarly, if \( f : \Gamma_1 \to \Gamma_2 \) is a morphism then there is an induced morphism \( \phi(f) : \phi(\Gamma_1) \to \phi(\Gamma_2) \).

For a labeling \( l : \Gamma \to R_{\mathcal{B}}, \text{Im} \pi_1(l) \) is a well-defined conjugacy class \( \mathcal{H} \) of a subgroup of \( \mathcal{F} \) and we say that \( l \) is a labeling for \( \mathcal{H} \) or that \( l \)

\(^6\)1-dimensional CW-complex

\(^7\)cellular taking open edges homeomorphically to open edges
represents $\mathcal{H}$. There is a 1-1 correspondence between tight labelings of finite graphs and conjugacy classes of f.g. subgroups of $\mathbb{F}$. A labeling $l: \Gamma \to R_B$ of a finite graph can always be folded until it is an immersion, see $\S$. Valence one vertices can then be iteratively pruned until it is tight. Let $\tau(l): \tau(\Gamma) \to R_B$ denote the resulting tight labeling. This tight labeling is unique unless $\Gamma$ is contractible in which case $\tau(\Gamma)$ will consist of a single vertex.

Based labeled graphs, i.e. labeled graphs with a base point, are also useful. The definitions in Section 2 have analogues if we allow base points. The base point of the $R_B$ is its unique vertex. Of course, labelings automatically take base points to base points. We require that morphisms do the same. A labeling of a based labeled graph is tight if it is an immersion and the only valence one vertex, if any, is the base point. A based labeling $l: (\Gamma, \ast) \to (R_B, \ast)$ represents the subgroup $S$ of $F_B$ that is identified with $\text{Im} l_\ast \subset \pi_1(R_B, \ast)$. Without the base point $l$ represents the conjugacy class in $F_B$ of $S$. If $\Gamma$ is an oriented circle with base point, then we also say that $l$ represents the element $x \in F_B$ identified with $l_\ast([\Gamma])$ where $[\Gamma] \in \pi_1(\Gamma, \ast) \cong \mathbb{Z}$ is the generator determined by the orientation. Without the base point, $l$ represents the conjugacy class $[[x]]$ of $x$ in $F_B$. There is a 1-to-1 correspondence between tight based labeled graphs and f.g. subgroups of $F_B$. As mentioned above, there is a 1-to-1 correspondence between tight labeled graphs and conjugacy classes of f.g. subgroups of $F_B$.

3. Genus 1

Here $B = \{u, v\}$ and so $\mathbb{F}$ is a free group of rank 2. We use the convention that if $w$ is a $B$-word then $W$ denotes its inverse.

**Proposition 3.1** (Lyndon-Wicks). $f'_w(1) \geq 2$. Specifically, if $h_1$ is the representation given by

$$u \mapsto uvuvv, v \mapsto UUVU$$

and if $h_2$ is given by

$$u \mapsto vuvv, v \mapsto UUVUV$$

then $h_1$ and $i_u \circ h_2$ are inequivalent admissible representations for

$$uvuvvUUVUVV = [h_1(u), h_1(v)] = i_u([h_2(u), h_2(v)]).$$

The proof of Proposition 3.1 will rely on two lemmas.

**Lemma 3.2.** $\text{Im} h_1$ and $\text{Im} h_2$ are not conjugate.
Proof. The tight labelings representing the conjugacy classes of $\text{Im} h_1$ and $\text{Im} h_2$ are pictured in Figure 1. Since these labelings are not isomorphic, $\text{Im} h_1$ and $\text{Im} h_2$ are not conjugate.

**Lemma 3.3.** $\text{Im} h_1$ and $\text{Im} h_2$ are primitive.

**Proof.** If $\phi \in \text{Aut}(F)$ interchanges $u$ and $v$ then $\phi(\text{Im} h_1) = \text{Im} h_2$. So, it is enough to argue that $\text{Im} h_1$ is primitive. We will show that $\text{Im} h_1$ is malnormal in $F$, i.e. that if $w \in F$ satisfies $i_w(\text{Im} h_1) \cap \text{Im} h_1 \neq \{1\}$ then $w \in \text{Im} h_1$. This clearly implies that $\text{Im} h_1$ is primitive. The pullback of two copies of the tight labeling for $\text{Im} h_1$ has only one component that is not contractible—that of the “diagonal”. From [8], it follows that $\text{Im} h_1$ is malnormal in $F$. \hfill \square

Proposition 3.1 is proved. \hfill \square

The homomorphisms $h_1$ and $h_2$ in Proposition 3.1 were found by a computer search. The original homomorphisms found by Lyndon and Wicks were $h'_1$ given by

$$u \mapsto uvuvUvuvu, v \mapsto vuvuvUvuvUvuvuv$$

and $h'_2$ given by

$$u \mapsto vuvUvuvuvuvuvuvuvuvuvuvuv, v \mapsto UvuvuvU.$$  

It is easy to check that $[h'_1(u), h'_1(v)]$ and $[h'_2(u), h'_2(v)]$ are conjugate. They argue that $\text{Im} h'_1$ and $\text{Im} h'_2$ are primitive and point out that the abelianizations of $h'_1$ and $h'_2$ are not in the same $SL_2\mathbb{Z}$-orbit. Hence $h'_1$ and $h'_2$ are not equivalent.

**4. Higher genus**

Here we prove:

**Proposition 4.1.**

$$f'_x(m + n) \geq f'_x(m)f'_x(n)$$
Definition 4.2. Let $F_1$ and $F_2$ be two nonabelian free groups with fixed finite bases $B_1$ and $B_2$. For a homomorphism $\phi : F_1 \to F_2$, set $m(\phi) = \min \{ \text{length } \phi(u) \mid u \in B_1 \}$ where length is with respect to $B_2$. We say that $\phi$ is an $\alpha$-map (for some $\alpha > 0$) if

- for all $u \in B_1$, a subword of $\phi(u)$ of length $\geq \alpha m(\phi)$ appears exactly once as a subword of $\phi(u)$, and
- for $u, v \in B_1^{\pm 1}$, if $\phi(u)$ and $\phi(v)$ have subwords of length $\geq \alpha m(\phi)$ that are isomorphic preserving orientation, then $u = v$.

Remark 4.3. An equivalent definition is that $\phi$ is an $\alpha$-map if, for any reduced $B_2$-word $w$ of length $\geq \alpha m(\phi)$, $w$ appears at most once as a subword in the sequence $\{ \phi(u) \mid u \in B_1^{\pm 1} \}$. For this reason, it is often convenient not to distinguish between a subword and its inverse. For example, we will say that two $B_2$-words $p$ and $q$ share a subword $w$ if $w$ or $W$ appears in $p$ and $w$ or $W$ appears in $q$.

The idea of $\alpha$-maps goes back to Sacerdote [6].

Example 4.4. Say $F_1 = F_2 = \langle u, v \rangle$. Let

$$
\phi(u) = uv^2vu^3v \cdots u^nv
$$

and

$$
\phi(v) = uv^2u^2v^2u^3v^2 \cdots u^n v^2
$$

As $n \to \infty$, this is an $\alpha$-map for $\alpha \to 0$.

While working with an $\alpha$-map $\phi : F_1 \to F_2$ the natural unit of length is $\alpha m(\phi)$. We say that an edge path in a $B_2$-labeled graph or a $B_2$-word is $n$-long if it has length at least $n \alpha m(\phi)$. Otherwise it is $n$-short.

Lemma 4.5. Set $\alpha = 1/4$. For all $\alpha$-maps $\phi : F_1 \to F_2$, the following holds.

1. $\phi$ is injective.
2. For all $x, x' \in F_1$, $x$ and $x'$ are $F_1$-conjugate if and only if $\phi(x)$ and $\phi(x')$ are $F_2$-conjugate.
3. For all $x \in F_1$ and subgroups $S$ of $F_1$, $x$ is conjugate into $S$ if and only if $\phi(x)$ is conjugate into $\phi(S)$.
4. For all f.g. subgroups $S$ and $S'$ of $F_1$, $S$ is $F_1$-conjugate to $S'$ if and only if $\phi(S)$ is $F_2$-conjugate to $\phi(S')$.
5. For all $x \in F_1$, $x$ is indivisible in $F_1$ if and only if $\phi(x)$ is indivisible in $F_2$.

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8 Recall the convention that an element of $F_2$ is identified with the reduced $B_2$-word representing it.

9 Recall the convention that corresponding small and capital letters are mutually inverse.
(6) For all subgroups $S$ of $\mathbb{F}_1$, $S$ is primitive in $\mathbb{F}_1$ if and only if $\phi(S)$ is primitive in $\mathbb{F}_2$.

Proof. (1): Here $\alpha < 1/2$ works. Let $l : C \to R_{B_1}$ represent the cyclically reduced $B_1$-word $x = u_1 \ldots u_N$ where $C$ is a circle. The labeling $\phi(l)$ represents the cyclic $B_2$-word $\phi(x) = \phi(u_1) \cdot \ldots \cdot \phi(u_N)$ and $\phi(l)$ is nearly tight in that folds can only occur in $\alpha \eta(\phi)$-neighborhoods of the initial vertices of the $\phi(u_i)$'s. Since $\alpha < 1/2$, for each $i$, not all of $\phi(u_i)$ is involved in a fold and so $\phi(x)$ is not trivial.

(2): The “$\Rightarrow$” direction is obvious and holds for any homomorphism $\mathbb{F}_1 \to \mathbb{F}_2$. For the other direction, assume $[[\phi(x)]] = [[\phi(x')]]$. Let $l : C \to R_{B_1}$ be a labeling representing $x = u_1 \ldots u_N$ and let $l' : C' \to R_{B_1}$ represent $x' = u'_1 \ldots u'_{N'}$ as cyclically reduced cyclic $B_1$-words. The labelings $\phi(l) : \phi(C) \to R_{B_2}$ and $\phi(l') : \phi(C') \to R_{B_2}$ represent respectively the cyclic $B_2$-words $\phi(u_1) \cdot \phi(u_2) \cdot \ldots \cdot \phi(u_N)$ and $\phi(u'_1) \cdot \phi(u'_2) \cdot \ldots \cdot \phi(u'_{N'})$. As in the proof of (1), the labelings $\phi(l)$ and $\phi(l')$ are nearly tight. Since $\alpha = 1/4$, there are 2-long subwords $p_i$ of $\phi(u_i)$ and $p'_j$ of $\phi(u'_j)$ that survive the folding with $\tau(\phi(l))$ and $\tau(\phi(l'))$ representing the same cyclic words $p_1 \ldots p_N = p'_1 \ldots p'_{N'}$.

Claim: If $p_i$ and $p'_j$ share a 1-long subword $p$ then $p_i = p'_j$.

Before proving the claim, we show that it implies (2). The $p_i$'s and $p'_j$'s are 2-long and so some $p_i$ shares a 1-long subword with some $p'_j$. By the claim, $p_i = p'_j$. Up to a cyclic permutation, we may assume that $i = j = 1$. Then $p_2$ and $p'_2$ share a 1-long subword and $p_2 = p'_2$, etc.

We now prove the claim. We may assume that $p$ is chosen to be maximal, i.e. $p$ is contained in no longer shared subword. We will show that $p_i = p = p'_j$. Set $\phi(u_i) = spt$ and $\phi(u'_j) = s'pt'$. Since $p$ is 1-long, Definition 1.2 gives $u_i = u'_j$, $s = s'$, and $t = t'$. Now, $p_i = spt_i$ (so $s_i$ is the subword of $s$ that survives cancellation). Similarly, $p'_j = s'_jpt'_j$. The claim is that $s_i$, $s_j$, $t_i$, and $t'_j$ are all trivial. Since $p$ is maximal one of $s_i$ and $s'_j$, say $s_i$, is the empty word. If $s'_j$ is not also empty then the terminal letter of $s'_j$ and the terminal letter of $s$ are the same letter $b$ and $\phi(u_{i-1})$ contains the subword $bB$, contradiction. See Figure 2. That $t_i$ and $t'_j$ are trivial is similar.

(3) is a direct consequence of (2). Indeed, if $\phi(x)$ is conjugate into $\phi(S)$ then, for some $s \in S$, $\phi(x)$ is conjugate to $\phi(s)$. By (2) $x$ is conjugate to $s$.

(1): Suppose that $S$ and $S'$ are f.g. subgroups of $\mathbb{F}_1$ such that $\phi(S)$ and $\phi(S')$ are conjugate in $\mathbb{F}_2$. Let $l : \Gamma \to R_{B_1}$ and $l' : \Gamma' \to R_{B_1}$
be tight labelings representing the conjugacy classes of $S$ and $S'$ respectively. \( \text{(2)} \) is a special case with $\Gamma = C$ and $\Gamma' = C'$. So, we may assume that $S$ and $S'$ are not cyclic.

Consider a natural edge $e$ of $\Gamma$ viewed as a labeled edge path representing the word $u_1 \ldots u_n$. The edge path $\phi(e)$ is a natural edge of the graph $\phi(\Gamma)$ representing $\phi(u_1) \ldots \phi(u_N)$. The edge path $\tau(\phi(e))$ nearly represents a natural edge of $\tau(\phi(\Gamma))$. That is, there are 2-long subwords $p_i$ of $\phi(u_i)$ so that $\tau(\phi(e))$ agreeing with $\tau(\phi(e))$ except perhaps in 1-short initial and terminal subwords. It follows exactly as in (2) that there is a corresponding natural edge of $\phi(\Gamma')$ representing $\phi(u_1) \ldots \phi(u_N)$ and (4) follows.

(5): The “$\implies$” direction is obvious. For the other direction, let $l : C \to R_{B_2}$ represent the non-trivial indivisible cyclic word $x = u_1 \ldots u_N$ which we may assume is cyclically reduced. Suppose that $\tau(\phi(l)) : \tau(\phi(C)) \to R_{B_2}$ represents $\lbrack \phi(x) \rbrack = y^n$ with $n > 1$ maximal and $y$ cyclically reduced. Rotation by $2\pi/n$ induces a (label preserving) isomorphism $\rho : \tau(\phi(C)) \to \tau(\phi(C))$. As in (2), $y^n = p_1 \cdots p_N$ where $p_i$ is the 2-long subword of $\phi(u_i)$ that survives cancellation. If we set $p'_i = \rho(p_i)$ then $p_i$ shares a 1-long subword with some $p'_j$. Exactly as in (2), $p_i = p'_j$. It follows that $\rho$ leaves the set of $p_i$’s invariant and that $x$ is not indivisible, contradiction.

(6) follows directly from (5).

**Corollary 4.6.** Set $\alpha = 1/4$. If $x \in F_1$ and if $\phi : F_1 \to F_2$ is an $\alpha$-map, then

- $\text{num}_g^{' \phi} \phi(x) \geq \text{num}_g^{' \phi} x$; and
- $\text{num}_g \phi(x) \geq \text{num}_g x$.

In particular, $f^g_2(g)$ and $f^g_1(g)$ do not depend on $F$.

**Proof.** If $h$ is an admissible representation of $x$ then $\phi \circ h$ is an admissible representation of $\phi(x)$ by Items \( \boxed{1} \) and \( \boxed{6} \) of Lemma \( \boxed{4.5} \). Again by Lemma \( \boxed{4.5} \), $\phi$ induces an injective map from the set of subgroups of $F_1$ to the set of subgroups of $F_2$. Hence, $\text{num}_g^{' \phi} \phi(x) \geq \text{num}_g^{' \phi} x$. 

**Figure 2.** Adjacent parallel segments should be viewed as overlapping.
For the second item, let \( h \) and \( h' \) be \( \sim \)-injective representations of \( x \). To show that \( \text{num}_g \phi(x) \geq \text{num}_g x \), we must show two things:

1. \( \phi \circ h \) and \( \phi \circ h' \) are also \( \sim \)-injective.
2. If \( \phi \circ h \sim \phi \circ h' \) then \( h \sim h' \).

First (1). It is enough to show that \( \phi \circ h \) is \( \sim \)-injective. Suppose \( h'' \) and \( \phi \circ h \) are related by a fractional Dehn twist. There are two cases corresponding to the two bullets in Definition 1.2. Let \( z \) be an indivisible element of \( \mathbb{F}_1 \) that commutes with \( h_i(C) \). By Lemma 4.5, \( \phi(z) \) is an indivisible element of \( \mathbb{F}_2 \) that commutes with \( \phi \circ h_i(C) \). It follows that for some \( k \)

\[
h'' = (\phi \circ h) \ast (i_{\phi(z)}^k \circ (\phi \circ h)) = \phi \circ (h \ast (i_{z_k} \circ h)).
\]

The representation \( h \ast (i_{z_k} \circ h) \) is injective because it is equivalent to \( h \). Since \( \phi \) is injective \( h'' \) is also injective. The case corresponding to the second bullet in Definition 1.2 is similar and left to the reader.

Continuing with (2), there is a sequence

\[
h_0, h_1, \ldots, h_k
\]

of representations of \( \phi(x) \) where \( h_0 = \phi \circ h, h_k = \phi \circ h', \) and \( h_i \) and \( h_{i+1} \) are related by a fractional Dehn twist. Suppose by induction that \( h_i = \phi \circ h_i' \) for some \( h_i' \sim h \). Again there are two cases corresponding to the bullets in Definition 1.2. Suppose first that the fractional Dehn twist relating \( h_i \) and \( h_{i+1} \) results from a splitting \( H_g = A \ast_C B \) as in the first bullet of Definition 1.2. Let \( z \) be an indivisible element of \( \mathbb{F}_1 \) that commutes with \( h_i(C) \). As in the proof of (1) above, \( \phi(z) \) is an indivisible element of \( \mathbb{F}_2 \) that commutes with \( \phi \circ h_i(C) \) and for some \( k \)

\[
h_{i+1} = h_i \ast (i_{\phi(z)}^k \circ h_i) = (\phi \circ h_i') \ast (i_{\phi(z)}^k \circ \phi \circ h_i') = \phi \circ (h_i' \ast (i_{z_k} \circ h_i')).
\]

If we set \( h_{i+1}' = h_i' \ast (i_{z_k} \circ h_i') \) then \( h_{i+1} = \phi \circ h_{i+1}' \) and \( h_{i+1}' \sim h \). At the end of the induction, we get \( \phi \circ h' = h_k = \phi \circ h_k'. \) Since \( \phi \) is injective, \( h' = h_k' \sim h \). Again, the second case is similar and is left to the reader.

To prove the final statement of this corollary, let \( x \in \mathbb{F}_1 \) also satisfy \( f_{\mathbb{F}}(g) = \text{num}_g x \) then

\[
f_{\mathbb{F}_1}(g) = \text{num}_g x \leq \text{num}_g \phi(x) \leq f_{\mathbb{F}_2}(g).
\]

Since \( \mathbb{F}_1 \) and \( \mathbb{F}_2 \) were arbitrary, \( f_{\mathbb{F}_1}(g) = f_{\mathbb{F}_2}(g) \). The case of \( f'_{\mathbb{F}} \) is similar.

We are now ready for the proof of our main proposition.

**Proof of Proposition 4.4.** Let \( x \in \mathbb{F} \) and \( y \in \mathbb{F} \) satisfy \( \text{num}'_m x = f'_{\mathbb{F}}(m) \) and \( \text{num}'_n y = f'_{\mathbb{F}}(n) \). Since \( \text{num}'_m x \) depends only on the conjugacy
class of \( x \), we may take \( x \) and \( y \) to be cyclically reduced. Consider 
\( z = xy \in \mathbb{F} \ast \mathbb{F} \). It follows from the next sublemma that 
\( \text{num}_{m+n} z \geq \text{num}_{m} x \cdot \text{num}_{n} y \).

**Sublemma.** Suppose that \( h_x \) and \( h_x' \) are admissible representations of 
\( x \) and suppose that \( h_y \) and \( h_y' \) are admissible representations of \( y \).

1. \( \text{Im} h_x \ast h_y = \text{Im} h_x \ast \text{Im} h_y \) and \( \text{Im} h_x \ast h_y = \text{Im} h_x' \ast \text{Im} h_y' \).
2. \( h_x \ast h_y : H_{m+n} = H_{m} \ast H_{n} \to \mathbb{F} \ast \mathbb{F} \) and \( h_x \ast h_y' \) are admissible representations of \( z \).
3. If \( \text{Im} h_x \neq \text{Im} h_x' \) or if \( \text{Im} h_y \neq \text{Im} h_y' \) then \( \text{Im} h_x \ast h_y \neq \text{Im} h_x' \ast h_y' \).

**Proof.** (1) follows from the uniqueness of normal forms in a free product; see [5] for example.

(2): We must show that \( h_x \ast h_y \) is injective and has primitive image. Set \( A = \text{Im} h_x \) and \( B = \text{Im} h_y \). Since the rank of \( A \ast B \) is the sum of the 
ranks of \( A \) and \( B \) and since free groups are Hopfian \( h_x \ast h_y \) is injective. 
Again, [5] is a reference.

By uniqueness of normal forms, an element of \( \mathbb{F} \ast \mathbb{F} \) in normal form 
(with respect to \( \mathbb{F} \ast \mathbb{F} \)) is in \( A \ast B \) if and only if it is in normal form 
with respect to \( A \ast B \). So:

- an element of \( \mathbb{F} \ast \mathbb{F} \) in normal form is in \( A \ast B \) if and only if 
each of its factors is either in \( A \) or \( B \).

Now suppose \( t^n \in A \ast B \) with \( t \in \mathbb{F} \ast \mathbb{F} \) and \( n > 0 \). We want to show 
that \( t \in A \ast B \). To distinguish between the factors of \( \mathbb{F} \ast \mathbb{F} \), let \( \mathbb{F}_1 \) 
denote the first factor and \( \mathbb{F}_2 \) the second. We want to show \( t \in A \ast B \).

Write \( t = t_1 t_0^n t_1^{-1} \) as a reduced word in \( \mathbb{F} \ast \mathbb{F} \) with \( t_0 \) cyclically reduced.
Then, \( t^n = t_1 t_0^n t_1^{-1} \) is also reduced. As a first case, suppose that the 
normal form for \( t_0 \) has more than one factor. Let \( a \) be the first factor 
in the normal form for \( t_0 \). We may assume that \( a \in \mathbb{F}_1 \). So, \( t_0 = aw \) 
where \( w \) is reduced and has first factor in \( \mathbb{F}_2 \) and \( t^n = t_1 (aw)^n t_1^{-1} \). The 
occurances of \( wa \) in this expression for \( t^n \) are product of factors in the 
normal form. By the bulleted fact above, we may remove these factors 
from \( t^n \) and the result is still in \( A \ast B \). Hence, \( t = t_1 (aw)t_1^{-1} \) is in 
\( A \ast B \) and we are done in this case. The other case is that \( t_0 \in \mathbb{F}_1 \cup \mathbb{F}_2 \).

We may assume that \( t_0 \in \mathbb{F}_1 \) and write \( t_1 = wa \) where \( a \) is the last 
factor of \( t_1 \) if it is in \( \mathbb{F}_1 \) and trivial if the last factor is in \( \mathbb{F}_2 \). Using 
the bulleted fact again, since \( at_0^na^{-1} = (at_0a^{-1})^n \) and \( w \) are factors of 
\( t^n \), we conclude that \( at_0^na^{-1} \in A \) and \( w \in A \ast B \). Since \( A \) is primitive, 
\( at_0^na^{-1} \in A \) and so \( t = wat_0a^{-1}w^{-1} \in A \ast B \).

(3) follows immediately from (1). This finishes the proof of the sub-
lemma. \( \square \)
We now finish the proof of Proposition 4.1. We have established that 
\[ \text{num}_{m+n} x \cdot \text{num}_{n} y = f'_{\mathbb{F}}(m) \cdot f'_n(n). \]
Also, according to Corollary 4.6, for an \( \alpha \)-map \( \phi : \mathbb{F} \ast \mathbb{F} \to \mathbb{F} \) with \( \alpha = 1/4 \), we have
\[ \text{num}_{m+n} \phi(z) \geq f'_{\mathbb{F}}(m) \cdot f'_{\mathbb{F}}(n). \]
Hence,
\[ f'_{\mathbb{F}}(m + n) \geq \text{num}_{m+n} \phi(z) \geq f'_{\mathbb{F}}(m) \cdot f'_{\mathbb{F}}(n). \]
\[ \square \]

**Remark 4.7.** We discovered a new limit group quotient that does not factor through any of the obvious quotients. For example, take \( H \) to be the union of 4 genus 2 surfaces with one boundary component glued along their boundaries. Take \( L \) to be the wedge of two genus two surfaces. Map \( H \to L \) by sending the common boundary to the product of the two waist curves, and sending each genus two membrane to the “boundary connected sum” of two halves (there are 4 possible combinations – use all 4).

### 5. More labeled graphs–boundings

The rest of the paper is devoted to reproving the main results of this paper in the context of minimal representations; see Definition 1.11.

We now consider the problem of extending a labeling \( l : C \to R_B \) of an oriented circle \( C \) to a surface. Suppose that we have a way of pairing up edges of \( C \) so that paired oriented edges have the same label in \( B \) and are inconsistently oriented with respect to the orientation induced by \( C \). There is a labeling induced on the quotient graph \( \Gamma \) obtained from \( C \) by gluing paired edges and there is an induced morphism \( b : C \to \Gamma \).

The morphism \( b \) has two key properties:

1. \( b \) is generically 2-to-1 and generically locally of degree 0, i.e. the \( b \)-preimage of an open edge consists of two inconsistently oriented open edges in \( C \); and
2. the *Whitehead graphs* of vertices of \( \Gamma \) are connected.

By (2), we mean the following. The link \( Lk_{\Gamma(b)}(v) \) of a vertex \( v \) of \( \Gamma(b) \) is a union of points, one for each oriented edge with initial endpoint \( v \). For each point \( \hat{v} \) in the \( b \)-preimage of \( v \) there is an induced map \( Lk_C(\hat{v}) \to Lk_{\Gamma(b)}(v) \). The *Whitehead graph of \( v \) has vertex set \( Lk_{\Gamma(b)}(v) \) and an edge connecting the vertices in the image of \( Lk_C(\hat{v}) \to Lk_{\Gamma(b)}(v) \) for each \( \hat{v} \in b^{-1}(v) \). For any \( b \) satisfying (1), the Whitehead graph of a vertex of \( \Gamma \) is a disjoint union of circles. So, to require that Whitehead graphs are connected is equivalent to requiring them to be circles. There is a 1-to-1 correspondence between pairings of edges of \( C \) as above and morphisms \( b \) satisfying (1) and (2).
Definition 5.1. Suppose that \( l : C \to R_B \) is a labeling where \( C \) is an oriented circle. A \textit{bounding} of \( l \) is a morphism \( b \) satisfying (1) and (2) from \( l \) to a labeling \( l(b) : \Gamma(b) \to R_B \).

We say that two closed edges of \( C \) with the same \( b \)-image are \textit{\( b \)-paired}. The mapping cylinder \( S \) of \( b \) is a surface with boundary \( C \). Let \( \mathcal{N}V(b) \) denote the set of natural vertices of \( \Gamma(b) \), i.e. the set of vertices of valence other than 2 and let \( \mathcal{N}E(b) \) denote the set of natural edges of \( \Gamma(b) \), i.e. the closures of components of \( \Gamma(b) \backslash \mathcal{N}V(b) \). Set \( v(b) = |\mathcal{N}V(b)| \) and \( e(b) = |\mathcal{N}E(b)| \). The \textit{geometric genus} of \( b \) is defined to be

\[
g\text{-genus } b = \frac{1}{2} \cdot (1 - v(b) + e(b))
\]

and equals the genus of \( S \). If \( l \) represents a cyclic \( B \)-word \( w \) then we also say that \( b \) is a \textit{bounding} of \( w \) (or of the conjugacy class \([w]\)). The \textit{geometric genus} of the conjugacy class \( \omega \) of an element in \([F, F]\) is

\[
g\text{-genus } \omega := \min\{g\text{-genus } b \mid b \text{ is a bounding of } \omega\}
\]

If \( C \) is the concatenation of edge paths \( p_1 \cdots p_{4g} \) and if the induced edge paths \( b_*(p_j) \) and \( b_*(p_{j+2}^{-1}) \) coincide for \( j \equiv 1 \text{ or } 2 \mod 4 \), then \( b \) is a \textit{standard bounding}.

Of course, there is a close relationship between boundings and representations. Choose a base point \( * \in C \) and suppose that \( l \) represents \( x \in F_B \). Also, choose an isomorphism \( H_g \to \pi_1(\Gamma(b), b(*)) \) so that \( \partial_g \) maps to the generator of \( \mathrm{Im}b_* \) determined by the orientation for \( C \). Here \( g = g\text{-genus } b \). Since we have identified the fundamental group of \( R_B \) with \( F_B \), a genus \( g \) representation of \( x \) is given by:

\[
H_g \to \pi_1(\Gamma(b), b(*)) \overset{l(b)}{\to} \pi_1(R_B, *)
\]

The choices here were the base point of \( C \) and the isomorphism \( H_g \to \pi_1(\Gamma(b), b(*)) \). It follows that if \( h_1 \) and \( h_2 \) are two representations obtained from \( b \) in this manner, then there is \( y \in F_B \) and a representation \( h \) that is equivalent to \( h_1 \) such that \( h_2 = i_y \circ h \); see Remark 1.3. We may say that \( b \) determines a representation of the conjugacy class of \( x \) that is well-defined up to equivalence.

Example 5.2. In Figure 3 there are three related boundings. The first \( b \) is a standard bounding of \( l : C \to R_B \) where \( B = \{u, v, w\} \) and \( l : C \to R_B \) represents the cyclic word \([uv, wU]\). The labeling \( \hat{b} \) is a labeling of \( \tau(l) \) and represents the cyclic word \( wuwUVW \). One way to create new boundings from \( b \) is to collapse two edges that are \( b \)-paired (and then “pull apart” any vertices that may have disconnected
Figure 3. A bounding of a labeled graph, a bounding of its tightening, and a bounding obtained by collapsing $b$-paired edges.

Whitehead graph). The bounding $b'$ is obtained by collapsing the two thicker edges and is a bounding for $vwUW$. Note that $[uv, wU]$ and $uvwUVW$ represent the same conjugacy class, but $vwUVuW$ represents a different conjugacy class.

The next lemma and corollary are classical. The lemma can be proved, for example, using cut-and-paste surface techniques and folding.

**Lemma 5.3.** Let $b : C \to \Gamma(b)$ be a bounding for the labeling $l : C \to R_B$ representing the cyclic $B$-word $w$. 

**Corollary.**
Recall that $\tau(l) : \tau(C) \to R_\Sigma$ is the labeling obtained by tightening $l$. There is a bounding denoted $\hat{b} : \tau(C) \to \Gamma(\hat{b})$ for $\tau(l)$ with $g$-genus $\hat{b} \leq g$-genus $b$.

There is a labeled graph $l' : C' \to R_\Sigma$ representing the conjugacy class $[[w]]$ with a standard bounding $b' : C' \to \Gamma(b')$ such that $g$-genus $b' \leq g$-genus $b$. □

See Figure 3.

**Corollary 5.4.** For $x \in [F, F]$, $a$-genus $x = g$-genus $x$. □

**Definition 5.5.** For $x \in [F, F]$, the genus of $x$, denoted genus $x$, is the number $a$-genus $x = g$-genus $x$. Similarly genus $[[x]] := a$-genus $[[x]] = g$-genus $[[x]]$.

**Warning 5.6.** The labeled graph $\Gamma(\hat{b})$ in Lemma 5.3(1) need not be tight. Even though $\tau(C)$ is tight and therefore $\hat{b}$ is an immersion, it is possible that, after a fold of $\Gamma(\hat{b})$, the induced map from $\tau(C)$ is no longer generically 2-to-1 and therefore not a bounding. Folding at a bad vertex (see Figure 5) would be an example. Note however that no folding is possible at a valence two vertex of $\Gamma(\hat{b})$.

We record the next easy lemma for later use.

**Lemma 5.7.** Let $b : C \to \Gamma(b)$ be a bounding for the labeling $l : C \to R_\Sigma$.

1. Suppose $b'$ is the new bounding for a new labeling obtained by first collapsing an edge of $\Gamma(b)$ and its $b$-preimage and then “pulling apart” any vertex with disconnected Whitehead graph. Then, $g$-genus $b' \leq g$-genus $b$.

2. If $l$ represents a cyclically reduced word then $\Gamma(b)$ has no valence one vertices. In particular, $v(b) \leq 4 \cdot g$-genus $b - 2$ and $e(b) \leq 6 \cdot g$-genus $b - 3$. □

The inequalities in 2 follow from $2 \cdot (g$-genus $b) = 1 - v(b) + e(b)$ and $3v(b) \leq 2e(b)$.

**Remark 5.8.** The bounding $b'$ in Lemma 5.7(1) is usually a bounding of a different conjugacy class than the bounding $b$. For example, see Figure 8.

**Remark 5.9.** It is sometimes convenient to view a labeling $l : \Gamma \to R_\Sigma$ as a morphism and this can lead to some confusion because the $\phi$-image of $l$ as a labeling is not usually the same as the $\phi$-image of $l$ as a morphism. To avoid this confusion, we let $l_\#$ denote the morphism induced by $l$. See Figure 8.
Figure 4. An example of $\phi(l)$ and $\phi(l_{\#})$ where $\phi : \langle u, v \rangle \rightarrow \langle w, x \rangle$ is given by $u \mapsto wxw$ and $v \mapsto xw$.

### 6. More $\alpha$-maps–preserving genus

**Lemma 6.1.** Let $x \in \mathbb{F}_1$ have genus $g$. There is $\alpha > 0$ such that, for all $\alpha$-maps $\phi : \mathbb{F}_1 \rightarrow \mathbb{F}_2$, $\phi(x)$ has genus $g$.

**Proof.** Suppose $x = u_1 \cdots u_M \in \mathbb{F}_1$ is cyclically reduced and has genus $g$. Represent $u_1 \cdots u_M$ by a tight labeling $l : C \rightarrow R_{B_1}$ (so $C$ has $M$ edges). Choose $\alpha < [4M(48g - 24 + M)]^{-1}$. This reason for this choice will become clear later. Let $\phi : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ be an $\alpha$-map and set $m := m(\phi)$. Consider the induced labeling $\phi(l) : \phi(C) \rightarrow R_{B_2}$ (so $\phi(C)$ has $|\phi(u_1)| + \cdots + |\phi(u_M)|$ edges). We can identify subwords of $\phi(u_i)$ in $\phi(u_1) \cdots \phi(u_M)$ with certain edge paths in $\phi(C)$. If $w_i$ is a subword of some $\phi(u_i)$ and if $u_i$ equals $u_j$ or $U_j$, then there is a corresponding subword $w_j$ of $\phi(u_j)$ or $\phi(U_j)$. More formally, if $w_i$ (respectively $w_j$) is represented by the edge path $p_i : I \rightarrow \phi(C)$ (respectively $p_j$) then $w_i$ and $w_j$ correspond if the edge paths $\phi(l_{\#}) \circ p_i$ and $\phi(l_{\#}) \circ p_j$ in $\phi(R_{B_1})$.
are equal, see Remark 5.4. The two edge paths in the lower left circle of Figure 4 indicated by the thicker lines correspond.

As in Lemma 4.5, $\phi(l)$ is almost tight and $\tau(\phi(l))$ is obtained by folding $\phi(l)$ in 1-short neighborhoods of at most $M$ of the vertices of $\phi(C)$. Suppose that $\tau(\phi(l))$ represents the cyclically reduced word $v_1 \cdots v_M$ where each $v_i$ is the surviving subword of $\phi(u_i)$ (so $\tau(\phi(C))$ has $|v_1| + \cdots + |v_M|$ edges). Since $\alpha < 1/4$, the length of each $v_i$ is at least $m/2$. In order to obtain a contradiction, assume that $\tau(\phi(l))$ has a bounding $b_{\tau(\phi(l))}$ with geometric genus $g_{\tau(\phi(l))}$ less than $g$ (see Lemma 5.3(1)). Our ultimate goal is to obtain a bounding for $x$ of geometric genus $\leq g_{\tau(\phi(l))}$. By Lemma 5.7(2), $\Gamma(b_{\tau(\phi(l))})$ has no valence 1 vertices, $v(b_{\tau(\phi(l))}) < 4g - 2$, and $c(b_{\tau(\phi(l))}) < 6g - 3$. The natural edges of $\Gamma(b_{\tau(\phi(l))})$ are labeled with $B_2$-subwords of $v_1 \cdots v_M$ and, as above, we can talk of their lengths. We may also identify the $v_i$’s with edge subpaths of $\phi(C)$ via the labeling $\phi(l)$. The proof of this lemma will be more involved than that of Lemma 4.5 primarily because some of these natural edges may be 1-short and because $\Gamma(b_{\tau(\phi(l))})$ need not be tight (see Warning 5.6).

The edges labeled $u_i$ in $C$ map to edge paths labeled $\phi(u_i)$ in $\phi(C)$. Therefore, a bounding of $\phi(l)$ that pairs $\phi(u_i)$’s with $\phi(U_j)$’s can be pulled back to give a bounding of $l$ with the same geometric genus (forget the $\phi$’s). Call such a bounding of $\phi(l)$ good. A weaker condition on a bounding of $\phi(l)$ is that it be saturated, i.e. paired edges correspond. It is easy to see that a saturated bounding $b_{\phi(l)}$ of $\phi(l)$ is good if it has the additional property:

- for each natural vertex $v$ of $\Gamma(b_{\phi(l)})$, $b_{\phi(l)}^{-1}(v)$ consists only of initial and terminal vertices of $\phi(u_i)$’s, i.e. $b_{\phi(l)}^{-1}(v)$ contains no vertices that are interior to a $\phi(u_i)$.

Indeed, if this is the case then each natural edge of $\Gamma(b_{\phi(l)})$ is a union of $\phi(u_i)$’s and, because $b_{\phi(l)}$ is saturated $\phi(u_i)$’s will be paired with $\phi(U_j)$’s.

So, our proof can be completed in two steps. In the first step, we find a saturated bounding for $\phi(l)$ of geometric genus at most $g_{\tau(\phi(l))}$. In the second step, we adjust the bounding discovered in the first step without increasing geometric genus until it satisfies the bulleted property above and so is good.

**Step 1.** (Find a saturated bounding $b_{\phi(l)}$ of $\phi(l)$ with geometric genus at most $g_{\tau(\phi(l))}$.) Consider a point $y$ in a natural edge $e$ of $\Gamma(b_{\tau(\phi(l))})$ whose distance from $\mathcal{N}Y(b_{\tau(\phi(l))})$ is at least $4\alpha m$. Since the length of each $v_i$ is more than $m/2$ and $\alpha < 1/8$, the $b_{\tau(\phi(l))}$-image of some
$v_j$ meets $e$ in a 2-long maximal subpath $p$ containing $y$, i.e. if we view $v_j$ as a path in $\Gamma(b_{r(\phi(l))})$ then $p$ is the maximal common subpath of $v_j$ and $e$ containing $y$. Further, the $b_{r(\phi(l))}$-image of some $V_k$, $k \neq j$ shares a maximal 1-long subpath $q$ with $p$. Arguing exactly as in Lemma \ref{lem:5.2}, $p = q$ and the maximal common subpaths of $v_j$ and $V_k$ (again viewed as paths in $\Gamma(b_{r(\phi(l))})$) in $e$ and containing $p$ (equivalently $y$) correspond. We conclude that an edge of $\tau(\phi(C))$ whose $b_{r(\phi(l))}$-image contains a point outside the $4am$-neighborhood of $\mathcal{N}(b_{r(\phi(l))})$ corresponds with its $b_{r(\phi(l))}$-paired edge. The number of edges of $\Gamma(b_{r(\phi(l))})$ in the $4am$-neighborhood of $\mathcal{N}(b_{r(\phi(l))})$ is at most $4am$ times the number of directions at vertices in $\mathcal{N}(b_{r(\phi(l))})$ which in turn is at most $4am \cdot 2e(b_{r(\phi(l))}) \leq 8am(6g-3)$ by Lemma \ref{lem:5.2}. Since boundings are generically 2-to-1, the number of edges of $\tau(\phi(C))$ not corresponding with their $b_{r(\phi(l))}$-paired edge is at most $16am(6g-3)$.

The difference in the number of edges of $\phi(C)$ and $\tau(\phi(C))$ is at most $2amM$. Viewing the edges of $\tau(\phi(C))$ as edges of $\phi(C)$, we have a paired off corresponding edges of $\phi(C)$ except for at most $16am(6g-3) + 2amM = 2am(48g-24+M)$ edges. So, at this point we have a partial bounding $\mathcal{P}$ of edges of $\phi(C)$. The bounding is partial in that not all edges of $\phi(C)$ are $\mathcal{P}$-paired with another edge, such edges are $\mathcal{P}$-unpaired. If an edge is $\mathcal{P}$-paired with some other edge, then we say that $\mathcal{P}$ is defined on that edge. A partial bounding that is defined on all edges determines a bounding.

There are at most $2am(48g-24+M)$ $\mathcal{P}$-unpaired edges in $\phi(C)$ and two edges that are $\mathcal{P}$-paired correspond. From $\mathcal{P}$ we want to construct a saturated partial bounding where by saturated here we mean a partial bounding that in addition to the property that $\mathcal{P}$-paired edges correspond also has the property that if an edge is $\mathcal{P}$-unpaired then all corresponding edges are also $\mathcal{P}$-unpaired. This can be achieved by starting with $\mathcal{P}$ and forgetting $\mathcal{P}$-pairings of all edges that correspond to a $\mathcal{P}$-unpaired edge. Since an edge has at most $M$ corresponding edges, we now have a saturated partial bounding, still called $\mathcal{P}$, of edges of $\phi(C)$ that is defined on all but at most $2amM(48g-24+M) < m/2$ edges. Since $|v_i| \geq m/2$, in each $v_i$ and hence in each $\phi(u_i)$, there is at least one edge on which $\mathcal{P}$ is defined. This explains our choice of $\alpha$.

Consider the bounding $b'$ induced from $\mathcal{P}$ by collapsing to a point each $\mathcal{P}$-unpaired edge of $\phi(C)$ as in Lemma \ref{lem:5.4}. By construction, two edges that are $\mathcal{P}$-paired are $b_{r(\phi(l))}$-paired. So, $b'$ can also be obtained by first collapsing to a point each edge of $\phi(C)$ that is not in some $v_i$ (giving $\tau(\phi(C))$) and then iteratively collapsing to points two $b_{r(\phi(l))}$-paired edges that are $\mathcal{P}$-unpaired. By Lemma \ref{lem:5.1}, $g$-genus $b' \leq g_{r(\phi(l))}$. As noted in Remark \ref{rem:5.8}, $b'$ is probably not a
bounding for $\phi(l)$, but nonetheless we will use $b'$ and the fact that $P$ is saturated to complete Step 1 by extending $P$ to the sought-after bounding $b_{\phi(l)}$ of $\phi(l)$ with $g$-genus $b_{\phi(l)} = g$-genus $b' \leq g_{r_i(\phi(l))}$.

Recall that $\phi(l)$ represents $\phi(u_1) \ldots \phi(u_M)$ and we may view the $\phi(u_i)$'s as edge paths in $\phi(C)$. Suppose that $p$ is a non-trivial maximal subpath of some $\phi(u_i)$ consisting of $P$-unpaired edges. Since $\phi(u_i)$ contains an edge on which $P$ is defined, an edge $w$ of $p$ shares an endpoint with an edge $q$ of $\phi(u_i)$ on which $P$ is defined. Since $P$ is saturated, it is defined on all edges of $\phi(C)$ corresponding to $q$ and determines a pairing of edges corresponding to $w$ as follows. If $q_1$ and $q_2$ are $P$-paired edges corresponding to $q$ and if $w_k$ corresponds to $w$ and shares an endpoint with $q_k$, $k = 1, 2$, then pair $w_1$ with $w_2$. In this way, we extend $P$. The extended partial bounding is still saturated and has fewer unpaired edges. Further, if we now collapse edges that are unpaired with respect to the extended partial pairing then we get a bounding $b''$ such that $\Gamma(b')$ is obtained from $\Gamma(b'')$ by collapsing disjoint partial natural edges. In particular, $g$-genus $b'' = g$-genus $b'$. Continue until there are no unpaired edges. This completes Step 1.

**Step 2.** (Find a good bounding of $\phi(l)$ of geometric genus less than $g$. ) We start with $b_{\phi(l)}$ found in Step 1. Recall that $b_{\phi(l)}$ is saturated in that $b_{\phi(l)}$-paired edges correspond. As previously mentioned, if, for each natural vertex $v$ of $\Gamma(b_{\phi(l)})$, $b_{\phi(l)}^{-1}(v)$ consists only of initial and terminal vertices of $\phi(u_i)$'s then $b_{\phi(l)}$ would be the desired bounding. A natural vertex $v$ not having this property is bad. Let $N(v)$ be the closed neighborhood of $v$ consisting of the union of all closed edges incident to $v$. We now examine the structure of $N(v)$ for bad $v$.

Suppose $v$ is bad. Give each of the edge paths $\phi(u_i)$ in $\phi(C)$ an orientation so that corresponding $\phi(u_i)$'s have the same orientation. Since $v$ is bad, each point of $b_{\phi(l)}^{-1}(v)$ is an interior vertex of some $\phi(u_i)$. Indeed, because $v$ is bad some element of $b_{\phi(l)}^{-1}(v)$ is an interior vertex and because $b_{\phi(l)}$ is saturated all elements are interior vertices. It follows that $b_{\phi(l)}^{-1}(N(v))$ consists of the vertices $\tilde{v}$ in $b_{\phi(l)}^{-1}(v)$ together with, for each $\tilde{v}$, the pair of edges $I_{\tilde{v}}$ and $O_{\tilde{v}}$ incident to $\tilde{v}$. We choose the notation so that, with respect to the orientation on the $\phi(u_i)$'s, $I_{\tilde{v}}$ has initial vertex $\tilde{v}$ (and so is incoming) and $O_{\tilde{v}}$ has terminal vertex $\tilde{v}$ (and so is outgoing). Finally, since $b_{h(l)}$ is saturated and Whitehead graphs are connected, all $I_{\tilde{v}}$'s correspond and all $O_{\tilde{v}}$'s correspond.

We now introduce a move that produces from $b_{\phi(l)}$ a new saturated bounding for $\phi(l)$ with no greater geometric genus. We will then show that we get a good bounding after iterating this move finitely many
times. Intuitively, we “push the problem forward”. Since the oriented edges $I_v$ correspond for $\tilde{v} \in b_{\phi(l)}^{-1}(v)$, they are all labeled with the same element $i_v$ of $B_2$. Similarly all $O_v$’s are labeled with the same element $o_v$ of $B_2$. The new bounding is obtained by collapsing the $O_v$’s, relabeling the $I_v$’s with $i_vo_v$, and pulling apart any vertices with disconnected Whitehead graph. See Figure 5. It is clear that the new bounding has the advertised properties.

For a vertex $\tilde{v}$ in $\Gamma(C)$ that is interior to some $\phi(u_i)$, define $|\tilde{v}|$ to be the distance from $\tilde{v}$ to the terminal endpoint of $\phi(u_i)$ (remember our orientation on the $\phi(u_i)$’s). It is easy to check that the following number decreases upon each iteration:

$$\sum\{|\tilde{v}| : b_{\phi(l)}(\tilde{v}) \text{ is a bad vertex of } \Gamma(b_{\phi(l)})\}.$$ 

This completes Step 2 and the proof of the lemma.

7. Proof of main results for minimal representations

**Corollary 7.1.** Given $x \in F_1$ there is $\alpha > 0$ such that, for any $\alpha$-map

- $\text{num}' \phi(x) \geq \text{num}' x$, and
- $\text{num} \phi(x) \geq \text{num} x$.

In particular, $\hat{f}_{\phi}(g) \text{ and } \hat{f}_{\phi}(g)$ do not depend on $F$.

**Proof.** The proof is the same as that of Corollary 4.6 as long as we choose $\alpha < 1/4$ and such that $\text{genus} \phi(x) = \text{genus} x$.\qed
Lemma 7.2. Suppose that $x$ (respectively $y$) is a cyclically reduced $B_1$-word (respectively $B_2$-word). Let $z = x * y \in F_{B_1} * F_{B_2}$.

- If $b_z : C \to R_{B_1 \sqcup B_2}$ be a bounding for $z$ and let $b_x$ (respectively $b_y$) be the bounding for $x$ (respectively $y$) obtained by collapsing to points the edges of $C$ labeled with elements of $B_2$ (respectively $B_1$). Then,
  \[ g \text{-genus } b_z + g \text{-genus } b_y = g \text{-genus } b_z. \]

- genus $x + \text{genus } y = \text{genus } z$

Proof. There are two special vertices $c$ and $c'$ in $C$ where $x$ and $y$ meet. Note that $b_z(c) = b_z(c')$. In fact, this is a consequence of the restriction of $b_z$ to the edge path in $C$ labeled $x$. To see this, complete the edge path labeled $x$ to another circle $C'$ by adding an unlabeled edge (connecting $c$ and $c'$). If we glue together $b_z$-paired edges of $C'$, the quotient is a surface with boundary and the boundary is the image of the unlabeled edge. The image of the unlabeled edge is a circle and hence $b_z(c) = b_z(c')$. It now also follows that Whitehead graphs of vertices in $\Gamma(b_x)$ and $\Gamma(b_y)$ are connected and so $b_x$ and $b_y$ are indeed boundings.

There is a 1-to-1 correspondence between natural vertices in $\Gamma(b_z)$ other than $b_z(c)$ and natural vertices of $\Gamma(b_x) \sqcup \Gamma(b_y)$ other than $b_x(c)$ and $b_y(c)$. Similarly, there is a 1-to-1 correspondence between natural edges in $\Gamma(b_z)$ not containing $b_z(c)$ and natural edges of $\Gamma(b_x)$ and natural edges of $\Gamma(b_z) \sqcup \Gamma(b_y)$ not containing $b_x(c)$ or $b_y(c)$. Since the four labels of edges incident to $c$ and $c'$ all have different labels ($x$ and $y$ are cyclically reduced), $b_z(c)$ is a natural vertex of $\Gamma(b_z)$. In case $b_z(c)$ is a natural vertex of $\Gamma(b_x)$ and $b_y(c)$ is a natural vertex of $\Gamma(b_y)$ then there is a 1-to-1 correspondence between natural edges of $\Gamma(b_z)$ and natural edges of $\Gamma(b_x) \sqcup \Gamma(b_y)$. In this case,
\[ e(b_z) - v(e_z) = e(b_x) - v(e_x) + e(b_y) - v(e_y) - 1 \]
where the 1 arises because $b_z(c)$ corresponds to the two vertices $b_x(c)$ and $b_y(c)$. Hence $g \text{-genus } b_z = g \text{-genus } b_x + g \text{-genus } b_y$. Since the labels in $x$ of the edge incident to and the edge incident to $c'$ are different, $b_x(c)$ has valence at least two in $\Gamma(b_x)$. The same holds for $b_y(c)$. Including vertices of valence two in our definition of geometric genus produces the same number. We conclude that in any case $g \text{-genus } b_z = g \text{-genus } b_x + g \text{-genus } b_y$.

The second item follows from the first item and the observation that if $b_x$ is a bounding for $x$ and if $b_y$ is a bounding for $y$, then a bounding $b_z$ of $z$ is induced by “concatenating” $b_x$ and $b_y$. \qed
We can now prove Theorem 1.12.

Proof of Theorem 1.12. The representations found in Proposition 3.1 have minimal genus and hence $\hat{f}_F'(1) \geq 2$. By the second item of Lemma 7.2, the representations found in the proof of Proposition 4.1 are also minimal and so $\hat{f}_F'(m + n) \geq \hat{f}_F'(m) \cdot \hat{f}_F'(n)$.

□

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