The relativistic velocity addition law optimizes a forecast gambler’s profit

Edward W. Piotrowski\textsuperscript{a}, Jerzy Luczka\textsuperscript{b}

\textsuperscript{a}Institute of Mathematics, University of Bialystok, Lipowa 41, Pl-15424 Bialystok, Poland; ep@alpha.uwb.edu.pl

\textsuperscript{b}Institute of Physics, University of Silesia, Uniwersytecka 4, Pl-40007 Katowice, Poland; luczka@us.edu.pl

Abstract

We extend the projective covariant bookmaker’s bets model to the forecasting gamblers case. The probability of correctness of forecasts shifts probabilities of branching. The formula for the shift of probabilities leads to the velocity addition rule of the special theory of relativity. In the absence of information about bookmaker’s wagers the stochastic logarithmic rates completely determines the optimal stakes of forecast gambler.

Key words: gambling optimalization, gambling forecast, special relativity, thermodynamics, Kelly criterion

PACS: 89.65.Gh, 89.70.+c, 03.30.+p

1 Introduction

First, the true in Aristotle’s law of syllogism, the individuals’ preferences, the market prices are transitive \cite{1}. The transitivity is the fundamental property of effective markets. (However the different point of view was analysed e.g. in \cite{2,3}.) Second, in the asset closed games with incomplete information the player’s profit as the logarithmic rate fulfils the first law of thermodynamics \cite{5}. What are the connections between the first and the second observations? The considerations presented bellow try to answer this question. This problem concern the gamblers strategies based on Kelly criterion \cite{6,5}.
2 Quantitative transitivity property

Let us consider the transitivity property of the sequence $u : M \times M \to \mathbb{R}$ on quantities of goods. In the absence of transaction costs (or, dually, when we neglect market scale effects) the relative prices (or the growth ratios in time) $u_{XY}$ fulfil

$$u_{AB} \odot u_{BC} \odot u_{CA} = 1,$$

where $\odot$ is any multiplication, $u_{XY} \in \mathbb{R}_+$, $A, B, C, \ldots$ are the symbols of the units of goods, or the moments of the market time. Let us describe the above property (1) in the logarithmic quantities that fulfil the transitivity law in additive form that is convenient for statistical operations:

$$r_{AB} \oplus r_{BC} \oplus r_{CA} = 0,$$

where $r_{XY} := \ln u_{XY}$. Further on we will call “addition” $\oplus$ the r-addition.

In the ring of real numbers we can write Eq. (2) in the form of a condition, that the value of some function $f : \mathbb{R}^3 \to \mathbb{R}$ is zero:

$$f(r_{AB}, r_{BC}, r_{CA}) = 0.$$

We require that this equation has a unique solution, that is symmetric with respect to all permutations of indexes $A, B, C$, scale free and $r_{XX} = 0$. Then we can restrict generally this function $f$ to be of the following linear form (when we choose the natural units for $r_{XY}$):

$$r_{AB} + r_{BC} + r_{CA} = 0$$

for unbounded domain of $r_{XY}$ (that is well known case of logarithmic rate), or

$$r_{AB} + r_{BC} + r_{CA} + r_{AB}r_{BC}r_{CA} = 0 \text{ when } r_{AB} \in [-1, 1].$$

After ordering, we have:

$$r_{AC} = r_{AB} \oplus r_{BC} := \frac{r_{AB} + r_{BC}}{1 + r_{AB}r_{BC}}. \quad (3)$$

The above formula looks like the Einstein velocity addition law of the special relativity theory, expressed in the unit of the light velocity (the maximal velocity $c = 1$). Further considerations will be devoted to this very bounded case of “rate”. Substituting (probability Ansatz) $r_{XY} = p_{1XY} - p_{2XY}$, where $p_{jXY} \in [0, 1]$ and $p_{1XY} + p_{2XY} = 1$ we obtain the r-addition rule (3) in the simplest form:

$$\frac{p_{1AC}}{p_{2AC}} = \frac{p_{1AB}p_{1BC}}{p_{2AB}p_{2BC}}, \quad (4)$$

or (in the logarithms) in the form of ordinary addition on the group $(\mathbb{R}, +)$. For this reason this logarithmic “rate of rate” $\frac{1}{2} \ln \frac{p_{1XY}}{p_{2XY}}$ is convenient in calculations and is known as rapidity in the special theory of relativity. Eq. (4) is the
unique projective invariant known by ancient mathematicians as Menelaus’ theorem (see Fig. 2) [7].

Fig. 1. Mutually dual ancient nomographs (slide rules), respectively Menelaus’ and Ceva’s theorems: \( \frac{p_1}{p_2} = \frac{p_{1AB}}{p_{2AB}}, \) where: \( p_{1AB} = \lambda_{AB}|AL|, \) \( p_{2AB} = \lambda_{AB}|LB|, \) \( p_{1BC} = \lambda_{BC}|BK|, \) \( p_{2BC} = \lambda_{BC}|KC|, \) \( p_{1AC} = \lambda_{AC}|AM|, \) \( p_{2AC} = \lambda_{AC}|MC|, \) for \( \lambda_{AB}, \lambda_{BC}, \lambda_{AC} \in \mathbb{R}_+. \)

3 Canonical model: two-horse race

The bounded domain of the logarithmic rate \( r_{XY} \) appears not only in the special theory of relativity. We can observe this case in a lot of simple stochastic models. Let us assume that in the bookmaker game the gambler (say: the forecast-gambler, f-gambler) predicts future results of, for simplicity, two-horse races better than the average gambler. Let \( p_{1AB} \) be the probability of correctness of her/his forecast, and let \( p_{2AB} \) be the probability of her/his mistake. F-gambler’s knowledge \( r_{AB} \) decomposes two element space of events (the race results) in four element space, see Fig. 3.

Indexes 1, 2, 1(2)AC, 2BC, 1(2)AC\( |1 \), denote respectively the events: “the f-gambler places a bet on the white horse \( (W) \)”, “the f-gambler places a bet on the black horse \( (B) \)”, “the white horse win”, “the black horse win”, and “good (bad) guessing when the white horse win”. The indexes to the left of the vertical bar \( | \) represent the events and the expressions to the right of the vertical bar represent the conditions. \( r_{AB} = p_{1AB} - p_{2AB} \). We assume that events \( AB, BC \) are stochastically independent.
Let us describe (c.f. [5]) the sum of wagers from all the gamblers of the bet on event \( j \) by \( IN_j \), and the percentage share of f-gambler’s capital in four cases of bookmakers’ bets by \( l_{jk} \) (or equivalently by \( l_{jk} | k \)), \( L = (l_{kj}) \). F-gambler’s profit in a concrete horse race is equal to the projective invariant logarithmic rate [5]:

\[
z_{k|j} = z_{kj} := \ln(1 + \frac{IN_{3-k}}{IN_k} l_{kj} - l_{(3-k)j}) .
\]

And let us form the matrix \( P \) of probabilities of all four elementary events:

\[
P = (p_{kj}) := \begin{pmatrix}
  p_{1ABP1BC} & p_{2ABP1BC} \\
  p_{2ABP2BC} & p_{1ABP2BC}
\end{pmatrix}.
\]

Then the f-gambler expected profit is equal to:

\[
E_Z(L) := \sum_{k,j=1}^{2} p_{kj} z_{kj} = \sum_{k,j=1}^{2} p_{kj} \ln(1 + \frac{IN_{3-k}}{IN_k} l_{kj} - l_{(3-k)j}),
\]

where \( Z := (z_{kj}) \). The rational f-gambler bets the stakes \( \bar{L} = (\bar{l}_{kj}) \) such that her/his expected profit is the maximal one:

\[
E_Z(\bar{L}) := \max_{l_{11}, l_{12}, l_{21}, l_{22}} \{ E_Z(L) \}.
\]

Now, rationally, we find the global maximum of the function \( E_Z(L) \).

4 Maximizing profit strategy

After standard differential calculations we obtain that the f-gambler’s maximal profit is given by:

\[
E_Z(\bar{L}) = D_Z - S_{AC} .
\]

The first component of the maximal profit in [5] is the seer’s profit i.e. the profit on unpopularity of the winning bet:

\[
D_Z = - \sum_{j=1,2} p_{jBC} \ln \frac{IN_j}{IN_1 + IN_2},
\]

and the second denotes the (minus) Boltzmann/Shannon entropy \( -S_{AC} \) of the forecast:

\[
S_{AC} := -p_1 \sum_{j=1}^{2} p_{jAC|1} \ln p_{jAC|1} - p_2 \sum_{j=1}^{2} p_{jAC|2} \ln p_{jAC|2}.
\]

The profit difference \( -S_{AC} = \sum_j p_{jBC} \ln p_{jBC} \) between the rational forecast and the seer’s forecast \( (r_{AB} = 0) \) does not depend on the sums \( IN_j \) of wagers from all the gamblers of the bet.
The two families \((\bar{l}_{1j}, \bar{l}_{2j}) \in \mathbb{R}^2, j = 1, 2\), of the solutions–strategies of extremal problem are described by the following two straight line equations:

\[
(\bar{l}_{jj} - p_{1AC|j}) IN_{3-j} = (\bar{l}_{(3-j)j} - p_{2AC|j}) IN_j, \ j = 1, 2.
\]  

(6)

In the absence of short positions (a typical restriction on the bets \(\bar{l}_{1j}, \bar{l}_{2j} \geq 0\)) we assume that the rational f-gambler diversificates the risk in such a way that she/he bets only possible a minimal part of her/his resources. From all the strategies (6) we choose the optimal one:

\[
\text{if } p_{1AC|j} IN_{3-j} > p_{2AC|j} IN_j \\
\text{then } \bar{l}^*_{jj} = p_{1AC|j} - p_{2AC|j} \frac{IN_j}{IN_{3-j}}, \ \bar{l}^*_{(3-j)j} = 0 \ \\
\text{else } \bar{l}^*_{(3-j)j} = p_{2AC|j} - p_{1AC|j} \frac{IN_j}{IN_{3-j}}, \ \bar{l}^*_{jj} = 0.
\]

If we do not have the information about proportion of sums of wagers \(\frac{IN_j}{IN_{3-j}}\) then we use famous *Laplaces Principle of Indifference* \((IN_1 = IN_2)\), and the optimal stakes are (in the form of the step function with jump at zero, \([r]_+ := \max\{r, 0\}\)):

\[
L^* = \begin{pmatrix} [r_{AC|1}]_+ & [-r_{AC|2}]_+ \\
[-r_{AC|1}]_+ & [r_{AC|2}]_+ \end{pmatrix}.
\]

(7)

We have four cases of play: normal game (max-strategy is diagonal), misère game (antidiagonal) and two mixed normal-misère games (horizontal).

5 Conclusion

The formal description of the bookmaker bets with majority of branches of events might be created hierarchically as the binary tree with the leaves – elementary events, e.g. by analogy to the construction of tree-shaped key to compressing/decompressing Huffman code [8]. It follows that our binary bet is universal, i.e. many kinds of financial decisions we can describe as the systems based on a hierarchy of formal binary bets. The bet model presented in [5] and above, as a projective covariant counterpart for basic in financial economics instruments there are that Arrow-Debreu securities [9,10] has interesting econophysics interpretation in the fields of thermodynamics and special theory of relativity. In the absence of information about the wagers decomposition the pair \((r_{AC|1}, r_{AC|2})\) of “relativity-stochastic” rates in Eq. (7) fully determines the optimal stakes of f-gambler. Within presented motivation, in postmodern manner, such a rule we can call *the Menelaus-Kelly criterion* for bets [6].
References

[1] E. W. Piotrowski, J. Sładkowski, Geometry of Financial Markets – Towards Information Theory Model of Markets, Physica A, 382 (2007) 228-234.

[2] E. W. Piotrowski, M. Makowski, Cat’s Dillema – Transitivity vs. Intransitivity, Fluc. Noise Lett. 5 (2005) L85-L95.

[3] M. Makowski, E. W. Piotrowski, Quantum Cat’s Dilemma: an example of intransitivity in a quantum game, Phys. Lett. A, 355 (2006) 250-254.

[4] A. Szczypińska, E. W. Piotrowski, Projective Market Model Approach to AHP Decision–Making, APFA-6 Conference, Physica A, in press.

[5] E. W. Piotrowski, M. Schroeder, Kelly Criterion Revisited: Optimal Bets, Eur. Phys. J. B, 57 (2007) 201-203.

[6] J. L. Kelly, Jr., A New Interpretation of Information Rate, Bell System Technical Journal 6 (1956) 917-926; http://www.arbtrading.com/kelly.htm

[7] B. Grünbaum, G. C. Shephard, Ceva, Menelaus, and Selftransversality, Geometriae Dedicata, 65 (1997) 179-192.

[8] T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein. Introduction to Algorithms, MIT Press, Cambridge MA and McGraw–Hill, New York, 2001.

[9] K. J. Arrow, Essays in the Theory of Risk-Bearing, North-Holland Pub. Co., Amsterdam, 1971.

[10] L. Fortnow, J. Kilian, D. M. Pennock, M. P. Wellman, Betting Boolean-style: a framework for trading in securities based on logical formulas, Decision Support Systems 39(1) (2005) 87-104.