Summation formulas of q-hyperharmonic numbers

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Abstract
In this paper, several weighted summation formulas of q-hyperharmonic numbers are derived. As special cases, several formulas of hyperharmonic numbers of type \( \sum_{\ell=1}^{n} \ell^{p} H_{\ell}^{(r)} \) and \( \sum_{\ell=0}^{n} \ell^{p} H_{n-\ell}^{(r)} \) are obtained.

Keywords Hyperharmonic numbers · Stirling numbers · q-generalizations

1 Introduction

Spieß [15] gives some identities including the types of \( \sum_{\ell=1}^{n} \ell^{k} H_{\ell}, \sum_{\ell=1}^{n} \ell^{k} H_{n-\ell} \) and \( \sum_{\ell=1}^{n} \ell^{k} H_{\ell} H_{n-\ell} \). In particular, explicit forms for \( r = 0, 1, 2, 3 \) are given. In this paper, several identities including \( \sum_{\ell=1}^{n} \ell^{k} H_{\ell}^{(r)} \) and \( \sum_{\ell=1}^{n} \ell^{k} H_{n-\ell}^{(r)} \) are shown as special cases of more general results, where \( H_{\ell}^{(r)} \) denotes hyperharmonic numbers defined in (4). When \( r = 1, H_{n} = H_{n}^{(1)} \) is the original harmonic number defined by \( H_{n} = \sum_{j=1}^{n} 1/j \). This paper is also motivated from the summation \( \sum_{\ell=1}^{n} \ell^{k} \), which is related to Bernoulli numbers. In [1], Stirling numbers are represented via harmonic numbers and hypergeometric functions related to Euler sums. In this paper, the sums involving harmonic numbers and their q-generalizations are expressed by using Stirling numbers and their q-generalizations.

There are many generalizations of harmonic numbers. Furthermore, some q-generalizations of hyperharmonic numbers have been proposed. In this paper, based upon a certain type of q-harmonic numbers \( H_{n}^{(r)} (q) \) defined in (3), several formulas of q-hyperharmonic numbers are also derived as q-generalizations. These results are also motivated from the q-analogues of the sums of consecutive integers [9,14,16].

In order to consider the weighted summations, we are motivated by the fact that the sum of powers of consecutive integers \( 1^{k} + 2^{k} + \cdots + n^{k} \) can be explicitly expressed in terms of

\[ \sum_{\ell=0}^{n} \ell^{p} H_{n-\ell}^{(r)} \]
Bernoulli numbers or Bernoulli polynomials. After seeing the sums of powers for small $k$:
\[
\sum_{\ell=1}^{n} \ell = \frac{n(n+1)}{2}, \quad \sum_{\ell=1}^{n} \ell^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{\ell=1}^{n} \ell^3 = \left(\frac{n(n+1)}{2}\right)^2, \ldots ,
\]
the formula can be written as
\[
\sum_{\ell=1}^{n} \ell^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} \tag{1}
\]
\[
= \frac{1}{k+1} (B_{k+1}(n+1) - B_{k+1}(1)) \quad [6] ,
\tag{2}
\]
where Bernoulli numbers $B_n$ are determined by the recurrence formula
\[
\sum_{j=0}^{k} \binom{k+1}{j} B_j = k + 1 \quad (k \geq 0)
\]
or by the generating function
\[
\frac{t}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} ,
\]
and Bernoulli polynomials $B_n(x)$ are defined by the following generating function
\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} .
\]
If Bernoulli numbers $\mathfrak{B}_n$ are defined by
\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \mathfrak{B}_n \frac{t^n}{n!} ,
\]
we can see that $B_n = (-1)^n \mathfrak{B}_n$. Then
\[
\sum_{\ell=1}^{n} \ell^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} (-1)^j \mathfrak{B}_j n^{k+1-j} .
\]
We recall the well-known Abel’s identity, which is frequently used in the present paper.

**Lemma 1** (Abel’s identity) For any positive integer $n$,
\[
\sum_{\ell=1}^{n} a_{\ell} b_{\ell} = s_n b_n + \sum_{\ell=1}^{n-1} s_{\ell} (b_{\ell} - b_{\ell+1}) .
\]
where
\[
s_n = \sum_{\ell=1}^{n} a_{\ell} .
\]

In the weight of harmonic numbers $H_n$, we have the following formulas.
Proposition 1  For $n, k \geq 1$,
\[
\sum_{\ell=1}^{n} \ell^k H_{\ell} = \frac{H_n}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} - \sum_{\ell=1}^{n-1} (H_n - H_{\ell}) \ell^k .
\]

Proof  Set $a_{\ell} = \ell^k$ and $b_{\ell} = H_{\ell}$ in Lemma 3. With
\[
\sum_{\ell=1}^{n-1} s_{\ell} (H_{\ell} - H_{\ell+1}) = s_1 (H_1 - H_2) + \cdots + s_{n-1} (H_{n-1} - H_n)
\]
\[
= 1^k H_1 + \cdots + (n-1)^k H_{n-1} - s_{n-1} H_n
\]
\[
= - \sum_{\ell=1}^{n-1} (H_n - H_{\ell}) \ell^k ,
\]
formula (1) gives the result.  \(\square\)

Proposition 2  For $n, k \geq 1$,
\[
\sum_{\ell=1}^{n} \ell^k H_{\ell} = \frac{H_n}{k+1} (B_{k+1} (n+1) - B_{k+1} (1)) - \sum_{\ell=1}^{n-1} \frac{B_{\ell+1} (\ell+1) - B_{\ell+1} (1)}{(k+1)(\ell+1)} .
\]

Proof  Set $a_{\ell} = \ell^k$ and $b_{\ell} = H_{\ell}$ in Lemma 3. Formula (2) gives the result.  \(\square\)

2 Weighted summations of $q$-hyperharmonic numbers

Many types of $q$-generalizations have been studied for harmonic numbers (e.g., [11,17]). In this paper, a $q$-hyperharmonic number $H_n^{(r)}(q)$ (see [12]) is defined by
\[
H_n^{(r)}(q) = \sum_{j=1}^{n} q^j H_j^{(r-1)}(q) \quad (r, n \geq 1)
\]
with
\[
H_n^{(0)}(q) = \frac{1}{q[n]_q}
\]
and
\[
[n]_q = \frac{1 - q^n}{1 - q} .
\]
Note that
\[
\lim_{q \to 1} [n]_q = n .
\]
In this $q$-generalization,
\[
H_n(q) = H_n^{(1)}(q) = \sum_{j=1}^{n} q^{j-1} [j]_q
\]
is a \(q\)-harmonic number. When \(q \to 1\), \(H_n = \lim_{q \to 1} H_n(q)\) is the original harmonic number and \(H^{(r)}_n = \lim_{q \to 1} H^{(r)}_n(q)\) is the \(r\)-th order hyperharmonic number, defined by

\[
H^{(r)}_n = \sum_{\ell=1}^n H^{(r-1)}_\ell \quad \text{with} \quad H^{(1)}_n = H_n. \tag{4}
\]

Mansour and Shattuck [12, Identity 4.1, Proposition 3.1] give the following identities

\[
H^{(r)}_n(q) = \left(\frac{n + r - 1}{r - 1}\right)_q (H_{n+r-1}(q) - H_{r-1}(q)) \tag{5}
\]

\[
= \sum_{j=1}^n \left(\frac{n + r - j - 1}{r - 1}\right)_q \frac{q^{r-1}}{[j]_q}, \tag{6}
\]

where

\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}
\]

is a \(q\)-binomial coefficient with \(q\)-factorials \([n]_q! = [n]_q [n-1]_q \cdots [1]_q\). Note that the identities (5) and (6) are \(q\)-generalization of the identities (7) and (8), respectively.

\[
H^{(r)}_n = \left(\frac{n + r - 1}{r - 1}\right)(H_{n+r-1} - H_{r-1}) \tag{7}
\]

\[
= \sum_{j=1}^n \left(\frac{n + r - j - 1}{r - 1}\right) \frac{1}{j} \tag{2}.
\]

So, we can see the recurrence relation for \(r \geq 1\)

\[
H^{(r+1)}_n = \frac{n + r}{r} H^{(r)}_n - \frac{1}{r} \left(\frac{n + r - 1}{r}\right).
\]

The generating function of this type of \(q\)-hyperharmonic numbers is given by

\[
\sum_{n=1}^\infty H^{(r)}_n(q) z^n = \frac{-\log_q (1 - q^r z)}{q(z; q)_r} \quad (r \geq 0) \tag{9}
\]

([12, Theorem 3.2]), where

\[
- \log_q (1 - t) = \sum_{m=1}^\infty \frac{t^m}{[m]_q}
\]

is the \(q\)-logarithm function and

\[
(z; q)_k := \prod_{j=0}^{k-1} (1 - zq^j)
\]

is the \(q\)-Pochhammer symbol. When \(q \to 1\), (9) is reduced to the generating function of hyperharmonic numbers:

\[
\sum_{n=1}^\infty H^{(r)}_n z^n = \frac{-\log(1 - z)}{(1 - z)^r} \quad (r \geq 0).
\]
In fact, the same form is given by Knuth [10] as
\[ \sum_{n=r-1}^{\infty} \binom{n}{r-1} (H_n - H_{r-1}) z^{n-r+1} = \frac{-\log(1-z)}{(1-z)^r} \quad (r \geq 0). \]

By (5), we have
\[ H_n^{(r+1)}(q) - \frac{[n+r]_q}{[r]_q} H_n^{(r)}(q) = -\frac{q^{r-1}}{[r]_q} \binom{n+r-1}{r}_q. \]

Hence,
\[ H_n^{(r+1)}(q) = \frac{[n+r]_q}{[r]_q} H_n^{(r)}(q) - \frac{q^{r-1}}{[r]_q} \binom{n+r-1}{r}_q. \quad (10) \]

By replacing \( n \) by \( n+1 \) and \( r \) by \( r-1 \) in (10), together with the definition in (3), we have
\[ [n+r]_q H_n^{(r)}(q) = [n+1]_q H_{n+1}^{(r)}(q) - q^{n+r-1} \binom{n+r-1}{r}_q. \quad (11) \]

Mansour and Shattuck [12, Theorem 3.3] also give the following formula,
\[ H_n^{(r)}(q) = \sum_{j=1}^{n} q^{j(r-m)} \binom{n+r-m-j-1}{r-m-1}_q H_j^{(m)}(q). \quad (12) \]

When \( q \to 1 \), (12) is reduced to
\[ H_n^{(r)} = \sum_{j=1}^{n} \binom{n+r-m-j-1}{r-m-1}_q H_j^{(m)}. \]

(see also [2], [3, 2.4.Theorem]). When \( m = 0 \), (12) is reduced to (6).

We prove a more general result of (5).

**Theorem 1** For nonnegative integers \( n \) and \( k \) and a positive integer \( r \), we have
\[ \binom{k+r-1}{k}_q H_n^{(k+r)}(q) = \binom{n+k}{n}_q H_n^{(r)}(q) - \binom{n+k+r-1}{n}_q H_k^{(r)}(q). \]

**Remark** If \( r = 1 \) and \( k \) is replaced by \( r - 1 \) in Theorem 1, we have the identity (5). If \( q \to 1 \) in Theorem 1, we have the version of the original hyperharmonic numbers in [13, Theorem 1].

**Proof of Theorem 1** The proof is done by induction on \( k \). When \( k = 0 \), the identity is clear since both sides are equal to \( H_n^{(r)}(q) \). Assume, then, that the identity has been proved for \( 0, 1, \ldots, k \). We give some explanations for the following calculation. Firstly, by replacing \( r \) by \( k + r \) in (10), we get the first identity. Secondly, by using the inductive assumption, we get the second identity. Thirdly, by replacing \( n \) by \( n + k \) and \( n \) by \( k \) respectively in (11), we
get the third identity. Then, we have

$$
\binom{k+r}{k+1} q H_{n}^{(k+r+1)}(q)
$$

$$
= \binom{k+r}{k+1} q \frac{n+k+r}{k+r} q H_{n}^{(k+r)}(q) - \binom{k+r}{k+1} q \frac{n+k+r-1}{k+r} q H_{n}^{(k+r)}(q)
$$

$$
= \frac{[n+k+r]}{[k+1]q} \binom{n+k}{n} q H_{n+k}^{(r)}(q) - \frac{[n+k+r]}{[k+1]q} \binom{n+k+r-1}{n} q H_{n+k}^{(r)}(q)
$$

$$
\quad - \frac{q^{k+r-1}}{[k+r]q} \binom{k+r}{k+1} q (n+k+r-1)
$$

$$
= \frac{[n+k+1]}{[k+1]q} \binom{n+k}{n} q H_{n+k+1}^{(r)}(q) - \frac{[n+k+1]}{[k+1]q} \binom{n+k-r}{n} q H_{k+1}^{(r)}(q)
$$

$$
\quad - \frac{q^{k+r-1}}{[k+r]q} \binom{k+r}{k+1} q (n+k+r-1)
$$

$$
= \binom{n+k+1}{n} q H_{n+k+1}^{(r)}(q) - \binom{n+k+r}{n} q H_{k+1}^{(r)}(q)
.$$
from (5),

\[
\frac{H_{n+1}^{(n+1)}(q)}{H_n^{(n)}(q)} = \frac{(2n+1)_q}{(2n-1)_q} \frac{H_{2n+1}(q) - H_n(q)}{H_{2n-1}(q) - H_{n-1}(q)}
\]

\[\rightarrow 1 \cdot q = q.\]

\[\square\]

**Theorem 2**  For positive integers \(n\) and \(r\),

\[
\sum_{\ell=1}^{n} q^{\ell-1} [\ell]_q H_{\ell}^{(r)}(q) = \frac{[n]_q [n+r]_q}{[r+1]_q} H_n^{(r)}(q) - \frac{q^r [n-1]_q [n]_q}{(r+1)_q^2} \binom{n+r-1}{r-1}_q
\]

\[= \frac{[n]_q [r]_q}{[r+1]_q} H_{n+r}^{(r+1)}(q) + \frac{q^{r-1}}{[r+1]_q} \binom{n+r}{r+1}_q. \quad (13)\]

**Proof**  Set \(a_\ell = q^{\ell-1} \left( \frac{\ell+r}{r} \right)_q\) and \(b_\ell = H_{\ell+r-1}(q)\). By using Lemma 3, we have

\[\sum_{\ell=1}^{n} q^{\ell-1} \left( \frac{\ell+r-1}{r} \right)_q H_{\ell+r-1}(q)\]

\[= \sum_{\ell=1}^{n} q^{\ell-1} \left( \frac{\ell+r-1}{r} \right)_q H_{n+r-1}(q) - \sum_{\ell=1}^{n-1} q^{\ell+r-1} \left( \frac{\ell+r}{r+1} \right)_q\]

\[= \binom{n+r}{r+1}_q H_{n+r-1}(q) - \frac{q^r}{[r+1]_q} \binom{n+r-1}{r+1}_q. \quad (14)\]

Hence,

\[\sum_{\ell=1}^{n} q^{\ell-1} [\ell]_q H_{\ell}^{(r)}(q)\]

\[= \sum_{\ell=1}^{n} q^{\ell-1} [\ell]_q \left( \frac{\ell+r-1}{r-1} \right)_q (H_{\ell+r-1}(q) - H_{r-1}(q))\]

\[= [r]_q \sum_{\ell=1}^{n} q^{\ell-1} \left( \frac{\ell+r-1}{r} \right)_q (H_{\ell+r-1}(q) - H_{r-1}(q))\]

\[= [r]_q \sum_{\ell=1}^{n} \left( \frac{\ell+r-1}{r} \right)_q H_{\ell+r-1}(q) - [r]_q H_{r-1}(q) \binom{n+r}{r+1}_q. \quad (15)\]

With the help of (5), (14) and (15), we get the desired result. \[\square\]

When \(q \rightarrow 1\), Theorem 2 is reduced to the following.

**Corollary 1**  For \(n, r \geq 1\),

\[
\sum_{\ell=1}^{n} \ell H_{\ell}^{(r)} = \frac{n(n+r)}{r+1} H_n^{(r)} - \frac{(n-1)^{(r+1)}}{(r-1)!(r+1)^2} \]

\[= \frac{nr}{r+1} H_n^{(r+1)} + \frac{1}{r+1} \binom{n+r}{r+1}_q.
\]

where \((x)^{(n)} = x(x+1) \cdots (x+n-1) (n \geq 1)\) denotes the rising factorial with \((x)^{(0)} = 1\).
In order to establish similarly structured theorems of \( q \)-hyperharmonic numbers, we recall the \( q \)-Stirling numbers of the second kind, denoted by \( S_q(n, m) \), defined by Carlitz (see e.g. [4]) as

\[
([x]_q)^n = \sum_{m=0}^{n} q^{\binom{m}{2}} S_q(n, m) ([x]_q)_{(m)}, \quad (n \in \mathbb{N}),
\]

(16)

where \( ([x]_q)_{(m)} = [x]_q[x-1]_q \cdots [x-m+1]_q \) denotes the \( q \)-falling factorial with \( ([x]_q)_0 = 1 \). The \( q \)-Stirling numbers of the second kind \( S_q(n, m) \) satisfy the recurrence relation

\[
S_q(n + 1, m) = S_q(n, m - 1) + [m]_q \cdot S_q(n, m)
\]

with boundary values

\[
S_q(n, 0) = S_q(0, n) = \delta_{n0}, \quad (n \geq 0)
\]

[8].

We need a \( q \)-version of the relation by Spieß [15], which is essential in the proof of the following structured theorem of \( q \)-hyperharmonic numbers of type \( \sum_{\ell=0}^{n} q^{\ell-1}([\ell]_q)^p H^{(r)}_\ell(q) \).

Lemma 2 Given summation formulas \( \sum_{\ell=0}^{n} q^{\ell-1}([\ell]_q)^p [c\ell]_q = F_q(n, j) \) for \( n, j \in \mathbb{N}, \) one has

\[
\sum_{\ell=0}^{n} q^{\ell-1}([\ell]_q)^p [c\ell]_q = \sum_{\ell=0}^{p} q^{\binom{\ell}{j}} S_q(p, \ell) \cdot ([\ell]_q)! \cdot F_q(n, \ell).
\]

where \( S_q(p, \ell) \) denote the \( q \)-Stirling numbers of the second kind.

Proof Using (16), we have

\[
\sum_{\ell=0}^{n} q^{\ell-1}([\ell]_q)^p [c\ell]_q = \sum_{\ell=0}^{p} q^{\binom{\ell}{j}} S_q(p, j) \cdot ([\ell]_q)! \cdot [c\ell]_q
\]

\[
= \sum_{j=0}^{p} q^{\binom{\ell}{j}} S_q(p, j) [j]_q! \cdot \sum_{\ell=0}^{n} q^{\ell-1}([\ell]_q)^p [c\ell]_q
\]

\[
= \sum_{j=0}^{p} q^{\binom{\ell}{j}} S_q(p, j) \cdot [j]_q! \cdot F_q(n, j).
\]

\[\square\]

We introduce some notations. For \( n, r, p \in \mathbb{N} \), set

\[
\sum_{\ell=0}^{n} q^{\ell-1}([\ell]_q)^p H^{(r)}_\ell(q) = A_q(p, r, n) H^{(r)}_n(q) - B_q(p, r, n).
\]

From (10), for \( p = 0 \), \( A_q(0, r, n) = \frac{[n+r]_q}{[r]_q} \), \( B_q(0, r, n) = \frac{q^{r-1} [n+r-1]_q}{[r]_q} \). From Theorem 2, for \( p = 1 \), we know that

\[
A_q(1, r, n) = \frac{[n]_q[n+r]_q}{[r+1]_q},
\]

\[
B_q(1, r, n) = \frac{q^r [n-1]_q[n]_q}{([r+1]_q)^2} \frac{(n+r-1)}{q^r-1}.
\]
Theorem 3 For $n, r, p \geq 1$,

$$\sum_{\ell=0}^{n} q^{\ell-1}[\ell]_q p H^{(r)}_{\ell}(q) = A_q(p, r, n)H^{(r)}_{n}(q) - B_q(p, r, n),$$

where

$$A_q(p, r, n) = \sum_{\ell=0}^{p} q^{(\ell)+p-1} S_q(p, \ell)[\ell]_q! \left( \binom{n+r-1}{r-1} q^{r+\ell-1} q^{r+\ell} \right),$$

$$B_q(p, r, n) = \sum_{\ell=0}^{p} q^{(\ell)+r+2p-2} \frac{r+\ell-1}{[r+\ell]_q} S_q(p, \ell)[\ell]_q! \left( \binom{r+n-1}{r-1} q^{r+\ell} q^{r+\ell} \right).$$

Proof Set $[c_\ell]_q = H^{(r)}_{\ell}(q)$ in Lemma 17. Then by using Lemma 3, we have

$$F_q(n, p) = \sum_{\ell=0}^{n} q^{\ell-1}\binom{\ell}{p}_q H^{(r)}_{\ell}(q)$$

$$= \sum_{\ell=1}^{n} q^{\ell-1}\binom{\ell}{p}_q \left( \binom{\ell+r-1}{r-1} H_{\ell+r-1}(q) - H_{r-1}(q) \right)$$

$$= \sum_{\ell=1}^{n} q^{\ell-1}\binom{r+p-1}{p}_q \left( \binom{\ell+r-1}{r-1} H_{\ell+r-1}(q) - H_{r-1}(q) \right)$$

$$= q^{p-1}\binom{r+p-1}{p}_q \left( \binom{r+n}{r+p} H_{n+r-1}(q) - q^{r+p-1}\binom{r+p-1}{p}_q \sum_{\ell=1}^{n-1} q^{\ell-1}\binom{\ell+r}{r+p}_q \right)$$

$$- q^{r+p-1}\binom{r+p-1}{p}_q \left( \binom{r+n}{r+p} H_{r-1}(q) \right)$$

$$= q^{p-1}\binom{r+p-1}{p}_q \left( \binom{r+n}{r+p} H_{n+r-1}(q) - q^{r+p-1}\binom{r+p-1}{p}_q \right)$$

$$- q^{r+p-1}\binom{r+p-1}{p}_q \left( \binom{r+n}{r+p} \right) q^{r+n-1} q^{r+\ell}.$$ (17)

With the help of (5) and (17), Lemma 17 gives the result.

When $q \to 1$, Theorem 3 is reduced to the following.

Corollary 2 For $n, r, p \geq 1$,

$$\sum_{\ell=0}^{n} \ell^p H_{\ell}^{(r)} = A(p, r, n)H_{n}^{(r)} - B(p, r, n),$$
where

\[
A(p, r, n) = \sum_{\ell=0}^{p} S(p, \ell) q^{\ell} \binom{n+r-1}{r-1}^{-1} \binom{r+n}{\ell} \binom{r+\ell}{r+\ell},
\]

\[
B(p, r, n) = \sum_{\ell=0}^{p} \frac{1}{r+\ell} S(p, \ell) q^{\ell} \binom{r+\ell-1}{r-1} \binom{r+n}{\ell} \binom{r+\ell}{r+\ell}.
\]

Example 1 \( p = 2 \) gives

\[
\sum_{\ell=1}^{n} q^{\ell-1} [\ell]_q [\ell+1]_q H_{\ell}^{(r)}(q) = \frac{[n]_q [n+r]_q (1 + q[r+1]_q [n]_q)}{[r+1]_q [r+2]_q} H_{\ell}^{(r)}(q)
\]

\[- q^r [n-1]_q [n]_q \binom{n+r-1}{r-1} \binom{2}{} [q[r+1]_q [n]_q - q^2 [r+2]_q [2]_q} {[r+1]_q [r+2]_q} \cdot \binom{2}{} .
\]

Note that \([\ell+1]_q = 1 + q \cdot [\ell]_q \) and \([\ell+2]_q = [2]_q + q^2 \cdot [\ell]_q \). With the help of Theorem 3 and identities (13) and (18), we have the following identities. For positive integers \( n \) and \( r \),

\[
\sum_{\ell=1}^{n} q^{\ell-1} [\ell]_q [\ell+1]_q [\ell+2]_q H_{\ell}^{(r)}(q) = \frac{[n]_q [n+r]_q [2]_q [n+2]_q + q^3 [r-1]_q [n+1]_q}{[r+1]_q [r+2]_q} H_{\ell}^{(r)}(q)
\]

\[- q^r [n-1]_q [n]_q \binom{n+r-1}{r-1} \binom{2}{} [q[r+1]_q [n+2]_q + q^4 [r+1]_q [n-2]_q} {[r+1]_q [r+2]_q} \cdot \binom{2}{} .
\]

To give a more general result, we need the \( q \)-unsigned Stirling numbers of the first kind \( s_{uq}(n, k) \) defined by

\[
[\ell]_q^{(n)} = [\ell]_q [\ell+1]_q \cdots [\ell+n-1]_q = \sum_{k=0}^{n} s_{uq}(n, k) [\ell]_q^{k}, \quad (n \in \mathbb{N}).
\]

The \( q \)-unsigned Stirling numbers of the first kind \( s_{uq}(n, k) \) are well defined since \([\ell+m]_q = [m]_q + q^m \cdot [\ell]_q \).
Theorem 4 For positive integers \( n, p \) and \( r \),
\[
\sum_{\ell=1}^{n} q^{-1}[\ell]_{q}^{(p)} H_{\ell}^{(r)}(q) = A_{1q}(p, r, n)H_{n}^{(r)} - B_{1q}(p, r, n),
\]
where
\[
\begin{align*}
A_{1q}(p, r, n) &= \sum_{m=0}^{p} s_{aq}(p, m)A_{q}(m, r, n), \\
B_{1q}(p, r, n) &= \sum_{m=0}^{p} s_{aq}(p, m)B_{q}(m, r, n).
\end{align*}
\]

Proof
\[
\begin{align*}
\sum_{\ell=1}^{n} q^{-1}[\ell]_{q}^{(p)} H_{\ell}^{(r)}(q) &= \sum_{\ell=1}^{n} q^{-1} \sum_{m=0}^{p} s_{aq}(p, m)[\ell]_{q}^{m} H_{\ell}^{(r)}(q) \\
&= \sum_{m=0}^{p} s_{aq}(p, m) \sum_{\ell=1}^{n} q^{-1}[\ell]_{q}^{m} H_{\ell}^{(r)}(q) \\
&= \sum_{m=0}^{p} s_{aq}(p, m) (A_{q}(m, r, n)H_{n}^{(r)}(q) - B_{q}(m, r, n)) \\
&= \left( \sum_{m=0}^{p} s_{aq}(p, m)A_{q}(m, r, n) \right) H_{n}^{(r)}(q) - \left( \sum_{m=0}^{p} s_{aq}(p, m)B_{q}(m, r, n) \right).
\end{align*}
\]
\( \square \)

When \( q \to 1 \), Theorem 4 is reduced to the following.
Corollary 3 For positive integers \( n, p \) and \( r \),
\[
\sum_{\ell=1}^{n} (\ell)^{(p)} H_{\ell}^{(r)} = A_{1}(p, r, n)H_{n}^{(r)} - B_{1}(p, r, n),
\]
where
\[
\begin{align*}
A_{1}(p, r, n) &= \sum_{m=0}^{p} (-1)^{p+m}s(p, m)A(m, r, n), \\
B_{1}(p, r, n) &= \sum_{m=0}^{p} (-1)^{p+m}s(p, m)B(m, r, n).
\end{align*}
\]
3 Backward summations

Now we consider backward summations of $q$-hyperharmonic numbers.

**Theorem 5** For positive integers $n$ and $r$,

\[
\sum_{\ell=1}^{n} q^{2n-2\ell} [\ell]_q H_{n-\ell}^{(r)}(q) = [n]_q [n+r]_q [r]_q [r+1]_q H_n^{(r)}(q) - (n+r) \left( \frac{q^{r-1}}{r} + \frac{q^r}{r+1} - \frac{q^{n+r-1}}{[n+r]_q} \right).
\]

**Proof** Set $a_\ell = q^{n-\ell} H_{n-\ell}^{(r)}(q)$, and $b_\ell = q^{n-\ell} [\ell]_q$. By using Lemma 3 and $[\ell+1]_q - q[\ell]_q = 1$, we have

\[
\sum_{\ell=1}^{n} q^{2n-2\ell} [\ell]_q H_{n-\ell}^{(r)}(q) = [n]_q \cdot H_{n-1}^{(r+1)}(q) + \sum_{\ell=1}^{n-1} (H_n^{(r+1)}(q) - H_{n-\ell-1}^{(r+1)}(q)) (q^{n-\ell} [\ell]_q - q^{n-\ell-1} [\ell+1]_q)
\]

\[
= [n]_q \cdot H_{n-1}^{(r+1)}(q) + \sum_{\ell=1}^{n-1} H_n^{(r+1)}(q) (q^{n-\ell} [\ell]_q - q^{n-\ell-1} [\ell+1]_q)
\]

\[
+ \sum_{\ell=1}^{n-1} H_{n-\ell-1}^{(r+1)}(q) (-q^{n-\ell} [\ell]_q + q^{n-\ell-1} [\ell+1]_q)
\]

\[
= q^{n-1} H_{n-1}^{(r+1)}(q) + \sum_{\ell=1}^{n-1} q^{n-\ell-1} H_{n-\ell-1}^{(r+1)}(q)
\]

\[
= H_{n-1}^{(r+2)}(q).
\]

With the help of (5), we get the desired result. \qed

When $q \to 1$, Theorem 5 is reduced to the following.

**Corollary 4** For positive integers $n$ and $r$,

\[
\sum_{\ell=1}^{n} \ell H_{n-\ell}^{(r)} = \frac{n(n+r)}{r(r+1)} H_n^{(r)} - \frac{(n)^{(r)} ((2r+1)n + r^2)}{(r-1)r^2(r+1)^2}.
\]

It is more complicated to get a summation formula for the backward summations of higher power. In the case where $q \to 1$, we have more relations, including the following.

**Theorem 6** For positive integers $n$, $p$ and $r$,

\[
\sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(r)} = A_2(p, r, n) H_n^{(r)} - B_2(p, r, n).
\]
where \(A_2(p, r, n)\) and \(B_2(p, r, n)\) satisfy the following relations:

\[
A_2(p, r, n) = A_2(0, r, n) \left( 1 + \sum_{j=0}^{p-1} \binom{p}{j} A_2(j, r + 1, n - 1) \right),
\]

\[
B_2(p, r, n) = B_2(0, r, n) \left( 1 + \sum_{j=0}^{p-1} \binom{p}{j} A_2(j, r + 1, n - 1) \right) + \sum_{j=0}^{p-1} \binom{p}{j} B_2(j, r + 1, n - 1),
\]

with the initial values \(A_2(0, r, n) = \frac{n}{r}\) and \(B_2(0, r, n) = \frac{1}{r} \binom{n+r-1}{r}\).

Nevertheless, we can have a different backward summation formula without weights.

**Theorem 7** For positive integers \(n, p\) and \(r\),

\[
\sum_{\ell=1}^{n} q^{p(n-\ell)} H_{n-\ell}^{(r)}(q) = C_q(p, r, n)H_n^{(r)}(q) - D_q(p, r, n),
\]

where \(C_q(p, r, n)\) and \(D_q(p, r, n)\) satisfy the following recurrence relation.

\[
C_q(p, r, n) = \frac{[n]_q}{[r]_q} \left( q^{(p-1)(n-1)} + (1 - q^{p-1})C_q(p-1, r+1, n-1) \right)
\]

\[
D_q(p, r, n) = \frac{q^{-1}[n]_q}{([r]_q)^2} \left( q^{p(n-1)} + (1 - q^{p-1})C_q(p-1, r+1, n-1) \right)
\]

\[+ (1 - q^{p-1})D_q(p-1, r+1, n-1).\]

**Proof** Set \(a_\ell = q^{n-\ell} H_{n-\ell}^{(r)}(q)\) and \(b_\ell = q^{(p-1)(n-\ell)}\). By using Lemma 3 and \([\ell + 1]_q - q[\ell]_q = 1\), we have

\[
\sum_{\ell=1}^{n} q^{p(n-\ell)} H_{n-\ell}^{(r)}(q)
\]

\[= H_{n-1}^{(r+1)}(q) + \sum_{\ell=1}^{n-1} (H_{n-\ell}^{(r+1)}(q) - H_{n-\ell-1}^{(r+1)}(q)) (q^{(p-1)(n-\ell)} - q^{(p-1)(n-\ell-1)})
\]

\[= H_{n-1}^{(r+1)}(q) + \sum_{\ell=1}^{n-1} H_{n-\ell}^{(r+1)}(q) (q^{(p-1)(n-\ell)} - q^{(p-1)(n-\ell-1)})
\]

\[+ \sum_{\ell=1}^{n-1} H_{n-\ell-1}^{(r+1)}(q) (-q^{(p-1)(n-\ell)} + q^{(p-1)(n-\ell-1)})
\]

\[= q^{(p-1)(n-1)} H_{n-1}^{(r+1)}(q) + (1 - q^{p-1}) \sum_{\ell=1}^{n-1} q^{(p-1)(n-\ell-1)} H_{n-\ell-1}^{(r+1)}(q)
\]

\[= q^{(p-1)(n-1)} H_{n-1}^{(r+1)}(q)
\]

\[+ (1 - q^{p-1}) \left( C_q(p-1, r+1, n-1) H_{n-1}^{(r+1)}(q) - D_q(p-1, r+1, n-1) \right).\]

With the help of (5), we get the desired result. \(\square\)
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