Universality of correction to Lüscher term in Polchinski-Strominger effective string theories

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We show, by explicit calculation, that the next correction to the universal Lüscher term in the effective string theories of Polchinski and Strominger is also universal. We find that to this order in inverse string-length, the ground-state energy as well as the excited-state energies are the same as those given by the Nambu-Goto string theory, the difference being that while the Nambu-Goto theory is inconsistent outside the critical dimension, the Polchinski-Strominger theory is by construction consistent for any space-time dimension. Our calculation explicitly avoids the use of any field redefinitions as they bring in many other issues that are likely to obscure the main points.

I. INTRODUCTION

Fundamental string theories can only be consistently quantised in the so-called critical dimension which is $D = 26$ for bosonic and $D = 10$ for supersymmetric theories. On the other hand string-like defects or solitons occur in a wide variety of physical circumstances, the most well-known being vortices in superfluids, the Nielsen-Olesen vortices of quantum field theories, vortices in Bose-Einstein condensates and QCD strings. These objects do clearly exist in dimensions other than the previously mentioned critical dimensions. The challenge then is to find means of consistently quantising such effective string theories without restriction on the dimension.

Polchinski and Strominger (PS)\textsuperscript{1} indeed showed how to do this. Their proposal is in spirit very close to that of chiral perturbation theory\textsuperscript{2}, which is an effective theory of chiral perturbation theory\textsuperscript{2}, which is an effective action, they are to be used in a long-string vacuum, for which the dominant term in the action is the usual quadratic action. This allows perturbation in the small parameter $R^{-1}$ where $2\pi R$ is the length of the (closed) string. In addition, PS dropped terms in the action which are proportional to equations of motion and constraints, to appropriate orders in $1/R$.

The plan of the paper is as follows. In the next section we briefly review the PS scheme to order $R^{-2}$ terms. We then prove, in very general terms, the absence of additional terms in the action which are of order $R^{-3}$. This is crucial in establishing the results of this paper. Using this we carry out the analysis of the spectrum to higher orders, where we show the absence of order-$R^{-2}$ terms as well absence of corrections to order-$R^{-3}$ terms.

In our analysis, we have carefully avoided the use of any additional ingredients such as field redefinitions. Field redefinitions bring with them a number of new issues like associated changes in measures and intrinsic arbitrariness. While we have nothing against these per se, we wish to present an analysis that is not obscured by them.

II. LEADING-ORDER ANALYSIS

Here, we review the analysis given by Polchinski and Strominger\textsuperscript{1}. They begin with the action

$$S = \frac{1}{4\pi} \int d\tau^+ d\tau^- \left\{ \frac{1}{a^2} \partial_+ X^\mu \partial_- X_\mu + \beta \frac{\partial_+^2 X \cdot \partial_- X \cdot \partial_+^2 X}{(\partial_+ X \cdot \partial_- X)^2} + O(R^{-3}) \right\}. \quad (1)$$

This action is invariant under the modified conformal transformations

$$\delta_- X^\mu = e^- (\tau^-) \partial_- X^\mu \mp \frac{\beta a^2}{2} \partial_\nu e^- (\tau^-) \frac{\partial_+ X^\mu}{\partial_+ \cdot \partial_- X} \partial_\nu X. \quad (2)$$

(and another; $\delta_+ X$ with + and − interchanged) leading to the energy momentum tensor (which agrees with eqn(11) of [1] to the relevant order)

$$T_{PS} = -\frac{1}{2a^2} \partial_+ X \cdot \partial_- X + \frac{\beta}{2L^2} (L \partial_\mu L - (\partial_\mu L)^2$$

$$+ \partial_+ X \cdot \partial_- X \partial_\mu^2 X \cdot \partial_\nu^2 X - \partial_+ L \partial_- X \cdot \partial_\nu^2 X) \quad (3)$$

where we have omitted terms proportional to the leading-order equation of motion, $\partial_+ \partial_- X^\mu = 0$ which has the solution $X^\mu_{cl} = e^\mu_+ R \tau^+ + e^\mu_- R \tau^-; \text{ here } e^\mu_+ = e^\mu_- = 0 \text{ and } e^\mu_+ \cdot e^- = -1/2$. Fluctuations around the classical solution are denoted by $Y^\mu$, so that $X^\mu = X^\mu_{cl} + Y^\mu$. The energy-momentum tensor in terms of the fluctuation field is then

$$T_{--} = -\frac{R}{a^2} e_- \partial_+ Y - \frac{1}{2a^2} \partial_\mu Y \cdot \partial_\nu Y \frac{\beta}{R} \delta^\mu_+ \partial_\nu^3 Y + \ldots \quad (4)$$

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with the OPE of $T_{-\tau}(\tau^-)T_{-\tau}(0)$ given by

$$\frac{D + 12\beta}{2(\tau^-)^4} + \frac{2}{(\tau^-)^2}T_{-\tau} + \frac{1}{\tau^-} \partial_\tau T_{-\tau} + O(R^{-1}).$$

It should be noted that due to the $-\frac{R}{a^2} e_- \cdot \partial Y$ term in $T_{-\tau}$, in principle the order-$R^{-2}$ term in the $Y \cdot Y$ propagator could contribute. It turns out that for the PS field definition it does not. The full equation of motion is $E^\mu = 0$;

$$E^\mu = -\frac{1}{2\pi a^2} \partial_+ X^\mu + \frac{\beta}{4\pi} \left\{ \partial_+ X^\mu \left( \frac{\partial^2 X \cdot \partial_+ X}{L^2} \right) \right\} + 2\partial_+ \left\{ \frac{\partial_+ X^\mu (\partial_+ X \cdot \partial X)}{L^2} \right\} - \partial_- \left\{ \frac{\partial_+ X^\mu (\partial_+ X \cdot \partial_+ X)}{L^2} \right\} + \{ + \leftrightarrow - \}. $$

where we have used the notation $L = \partial_+ X \cdot \partial X$. When this equation is restricted to terms linear in $Y^\mu$ we get an equation from which the two-point function can be computed:

$$\langle Y^\mu Y^{\nu'} \rangle = -a^2 \log(\tau^+ \tau^-) \eta^{\mu\nu} + \frac{2\beta a^4}{R^2} \epsilon^{(\mu \nu \nu')} \delta^3(\tau)$$

Consequently the potential contribution to the central charge $\frac{2\beta}{a^2} e_- e_- \langle Y^\mu Y^{\nu'} \rangle$ vanishes, as $e_- \cdot e_- = 0$. This is not always true as can be checked by redefining the $X^\mu$ field. Of course the total central charge does not change. One must add the contribution $-26$ from the ghosts, leading to the total central charge $D + 12\beta - 26$. Vanishing of the conformal anomaly thus requires

$$\beta_c = \frac{-D - 26}{12},$$

valid for any dimension $D$.

Using standard techniques the spectrum of this effective theory can be worked out. PS have shown how to do this at the leading order. We briefly reproduce their results here in order to set the stage for the rest of the paper. The Virasoro generators operate on the Fock space basis provided by $\partial_- Y^\mu = a \sum_{m = -\infty}^{\infty} a_n^\mu e^{-im\tau}$ and are given by

$$L_n = \frac{R}{a} e_- \cdot \alpha_n + \frac{1}{2} \sum_{m = -\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m :$$

$$+ \frac{\beta_c}{2} \delta_n - \frac{a\beta_c n^2}{R} e_+ \cdot \alpha_n + O(R^{-2}).$$

The quantum ground state is $|k, k; 0\rangle$ which is also an eigenstate of $\alpha_0^\mu$ and $\alpha_0^\mu$ with common eigenvalue $ak^\mu$. This state is annihilated by all $\alpha_n^\mu$ for positive-definite $n$. The ground state momentum is $p^\mu_{\text{gnd}} = \frac{2\pi}{2a^2} (e_+ + e_-) + k^\mu$ while the total rest energy is

$$(-p^2)^{1/2} = \sqrt{\left( \frac{R}{2a^2} \right)^2 - k^2 - \frac{R}{a^2} (e_+ + e_-) \cdot k}.$$ 

The physical state conditions $L_0 = \tilde{L}_0 = 1$ fix $k$, so that

$$k^1 = 0, \quad k^2 + \frac{R}{a^2} (e_+ + e_-) \cdot k = \frac{2 - \beta_c}{a^2}. $$

The first follows from the periodic boundary condition for the closed string which gives $e_+^\mu - e_-^\mu = \delta_0^\mu$. Substituting the critical value $\beta_c = (26 - D)/12$ one arrives at

$$(-p^2)^{1/2} = \frac{R}{2a^2} \sqrt{1 - \frac{D - 2}{12} \left( \frac{2a}{R} \right)^2},$$

which is the precise analog of the result obtained by Arvis for open strings [3]. Expanding this and keeping only the first correction, one obtains for the static potential

$$V(r) = \frac{R}{2a^2} - \frac{D - 2}{12} \frac{1}{R} + \cdots.$$ 

### III. ABSENCE OF ADDITIONAL TERMS AT ORDER $R^{-3}$

It is of crucial importance for the arguments of this paper that the next possible candidate term in the action is not $R^{-3}$ order. PS have stated without proof in [1] that the next such term is actually of order $R^{-4}$. However, as this is such a vital point we give here the most general proof for it. We follow PS and construct actions that are $(1, 1)$ in the naive sense: that is, the net number of $(+, -)$ indices is $(1, 1)$. We include no terms proportional to the leading order constraints $\partial_+ X \cdot \partial_+ X$ or to the leading order equations of motion $\partial_+ X^\mu$; otherwise they can be of arbitrary form. Clearly such actions can be constructed out of skeletal forms of the type

$$X_1^{\mu_1} X_2^{\mu_2} \cdots X_{MN}^{\mu_N} L^M$$

by contracting the Lorentz indices $\mu_1, \mu_2, \ldots, \mu_N$ with the help of invariant tensors, that is, with either $\eta_{\mu\nu}$ or $\epsilon_{\mu_1 \mu_2 \ldots \mu_p}$. Let us consider the potentially parity-violating terms involving the Levi-Civita symbols later. Here $X_{\mu_i}^{\nu_i}$ stands for $m$ derivatives of type $s = \pm$ acting on $X^\mu$. The numbers $\{m_i\}, M$ are adjusted to achieve the (naive) $(1, 1)$ nature.

The PS lagrangian is not strictly a $(1, 1)$ form as can be checked explicitly. However the PS action, to the desired accuracy, is invariant under the transformation laws of eqn.(2). It is $(1, 1)$ only in the naive sense mentioned above. The naive criterion is necessary but not sufficient, thus it suffices to prove the absence of action terms that are $R^{-3}$ using this criterion. In fact, it is desirable to have a formulation that is manifestly covariant. This will be presented elsewhere [3].

Only powers of $L$ have been used in the denominator to get a $(1, 1)$ form. It may appear that any scalar in target space would have sufficed. However, the action should not become singular on any fluctuation. Thus a scalar,
say, of the type $\partial^2 X \cdot \partial^2 X$ would not be permissible as it vanishes with $Y$. Whatever is in the denominator must be of the form $\partial^2 X \cdot \partial^2 X \cdots$; this can always be expanded around the dominant $L$ term to produce forms as in eqn. (14). A covariant formulation [4] gives a natural explanation for this as well as for the forms considered in eqn. (14).

All those cases where the Lorentz contractions produce additional factors of $L$ can be reduced to forms with lower $N$; we therefore need not consider cases where the number of factors with higher derivatives ($m \geq 2$) is smaller than the number with only single derivatives. On the other hand cases with more higher-derivative factors than single-derivative factors are less dominant. Thus for the even-$N$ case considered first (taken as $2N$ from now onwards) we need to consider the maximal case of exactly $N$ single-derivative terms and $N$ terms with all possible higher derivatives.

Among the single-derivative terms, let $n_+$ be the number with $+-$derivatives; then there are $N - n_+$ single derivative terms with $-$derivatives. Among the higher derivative terms let $p_+$ be the number of terms with only $+-$derivatives, and likewise $p_-$. Let $m_+$ be the total number of higher $+-$derivatives and $m_-$ the corresponding number of higher $-+$ derivatives. As $p_+ + p_- = N$, $m_+ \geq 2p_+$, $m_- \geq 2p_-$, it follows that $m_+ + m_- \geq 2N$, $m_+ + n_+ = m_- + N - n_+$, $M = m_+ + n_+ - 1$, and subsequently that $2m_+ \geq 3N - 2n_+$.

Now the leading-order behaviour of such a term is $R^{N-2(2n_+ + n_- - 1)}$. On noting that $N = 2 - 2n_+ - 2m_- \leq 2 - 2N$ we see that for $N \geq 3$ the leading behaviour of the action is at most $R^{-4}$. The case $N = 2$ is precisely the PS action with $R^{-2}$ behaviour. The dominant case among the subdominant class for $N = 2$ (four factors) is where there are three factors with only higher derivatives and one with a single derivative which we can take to be of $+-$type without loss of generality. If $l_-$ denotes the total number of $+-$derivatives among higher derivatives and likewise $l_+$, we must have $l_- - l_+ = l$. As before, if $p_-$ denotes the number of terms with only $+-$derivatives and likewise $p_+$, we have $p_+ + p_- = 3$ and then $l_- \geq 2p_+$, $l_+ \geq 2p_-$ and $l_+ + l_- \geq 6$. These lead to $l_-^{\text{min}} = 4$, $l_+^{\text{min}} = 3$, giving $M = 3$ and the leading-order behaviour is then of order $R^{-5}$. For $N = 1$ (two factors) we can only have higher-derivative terms and it is easy to see that the dominant term is $\partial^2 X \cdot \partial^2 X / L$, which in the context of this analysis is equivalent to the PS action.

Finally we turn to parity-violating cases and first to the case where there is an odd number of $X$ fields present. This can only happen when $D$ is odd. The contraction must be between $\varepsilon_{\mu_1 \cdots \mu_{2n+1}}$ and an expression of the form

$$\partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial_+ X^{\mu_3} \partial_- X^{\mu_4} \cdots \partial_+ X^{\mu_n} \partial_- X^{\mu_{n+1}}.$$ \hspace{0.5cm} (15)

The total number of $+-$derivatives is $n(n+1)/2 + n + 1$, while the total number of $-$derivatives is $n(n+1)/2$. The above expression multiplied by $\partial_+ X^{\mu_1} \cdot \partial_- X$ balances the $+, -$ derivatives (terms with $+$ and $-$ interchanged are also allowed). This has to be divided by $(\partial_+ X \cdot \partial_- X)^{n(n+1)/2+n+1}$, producing a leading behaviour of $R^{3-n-2n-2} \text{ or } R^{-(n+3-n-1)}$. This has the potential $R^{-3}$ behaviour in $D = 3$ and less dominant behaviour for higher $D$. In $D = 3$ this behaviour is $R^{-3} \varepsilon_{\mu_1 \mu_2 \mu_3} e_\mu e_\nu \partial_+ Y^{\mu_1} \partial_- Y^{\mu_2} \partial_+ Y^{\mu_3}$ which can be rewritten by partial integration as $-R^{-3} \varepsilon_{\mu_1 \mu_2 \mu_3} e_\mu \partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial_+ X^{\mu_3}$ and can therefore be dropped as it is proportional to the leading-order equation of motion.

In even dimensions a similar analysis shows that the leading behaviour is $R^{3-D(D+2)/4}$ which need not be considered for $D \geq 6$. For $D = 4$ this is superficially $R^{-2}$ but again both $R^{-2}$ and $R^{-3}$ terms are proportional to $\partial_+ Y$.

### IV. Higher Corrections to Ground-State Energy

From the expression for the ground-state momentum, it is clear that all higher corrections are determined by $k^2 + \frac{R}{4}(e_+ + e_-) \cdot k$ (eqn. (11)) which was only calculated to leading order in eqn. (11). Thus an order-$R^{-n}$ correction to this would result in order-$R^{-n-1}$ and higher corrections to the spectrum; here we need to investigate both $R^{-1}$ and $R^{-2}$ corrections. As this quantity is just a sum of the $L_0$ and $\tilde{L}_0$ conditions, we need to calculate up to order-$R^{-2}$ corrections to $L_0$ and $\tilde{L}_0$, or equivalently to $T_{-\cdot}$. As the transformation laws (2) have a leading part linear in $R$, additional terms in the action at order $R^{-3}$ would in principle have induced $R^{-2}$ corrections to $T_{-\cdot}$. That would in turn have changed the $R^{-3}$ terms in the ground-state energy. This is the reason why the absence of such terms in the action needs to be established so carefully. Absence of such terms also means that the expression for $T_{-\cdot}$ in eqn. (16) can be consistently expanded to keep order-$R^{-2}$ terms. We give here the on-shell expression to the desired order;

$$T_{-\cdot} = \frac{R}{a^2} \cdot \partial_- Y - \frac{1}{2a^2 \partial_- Y \cdot \partial_+ Y} - \frac{1}{R \partial_+ Y} \cdot \partial_+ Y \cdot \partial_+ Y \cdot \partial_- Y \cdot \partial_- Y \cdot \partial_+ Y - \frac{1}{R \partial_+ Y} \cdot \partial_+ Y \cdot \partial_+ Y \cdot \partial_- Y \cdot \partial_- Y \cdot \partial_+ Y.$$ \hspace{0.5cm} (16)

We see hence that $L_0$ and $\tilde{L}_0$ do not receive any order-$R^{-1}$ correction. At this point, the $T_{-\cdot}$ of eqn. (16) does not seem holonomic as there are $+-$derivatives terms occurring in $T_{-\cdot}$, while the Noether procedure necessarily gives a $T_{-\cdot}$ which satisfies $\partial_{\tau} T_{-\cdot} = 0$. The resolution of this apparent contradiction lies in the fact that the solution of the full equation of motion (16) can no longer be split into a sum of holonomic and anti-holonomic pieces.

Because of the absence of additional terms in the action with the leading $R^{-3}$ behaviour, the equation of motion (16) is sensible inclusive of $R^{-3}$ terms. Now we expand
this expression and retain terms up to order $R^{-3}$:
\[
\frac{2}{a^2} \partial_+ Y^\mu = -4 \beta R^2 \partial_+ \partial^2 Y^\mu \\
-4 \frac{\beta}{R^2} \left[ \partial_+ \{ \partial_2 Y^\mu (e_+ \cdot \partial_+ Y + e_- \cdot \partial_- Y) \} \\
+ \partial_2 Y^\mu (e_+ \cdot \partial_+ Y + e_- \cdot \partial_- Y) \} \\
+ 4e_0^\mu \partial_- (\partial_+ \cdot \partial^2 Y) + e_0^\mu \partial_+ (\partial_2 Y \cdot \partial_2 Y) \right].
\]

We can solve this equation iteratively by writing $Y^\mu = Y_0^\mu + Y_1^\mu$, where $Y_0^\mu$ is a solution of the leading order equation of motion. Keeping terms only up to order $R^{-3}$ we obtain an expression which can be readily integrated to yield
\[
\frac{2}{a^2} \partial_- Y_1^\mu = 4 \frac{\beta}{R^3} (e_+ \partial_+ Y_0 \cdot \partial_2 Y_0 + e_- \partial_- Y_0 \cdot \partial_2 Y_0 \\
- \partial_2 Y_0^\mu (e_+ \cdot \partial_2 Y_0 - \partial_2 Y_0^\mu e_+ \cdot \partial_2 Y_0),
\]

Exaining eqn (17) one sees that to order $R^{-2}$ only the first term linear in $R$ contributes additional non-holonomic terms which exactly cancel the remaining non-holonomic pieces. This immediately leads to the manifestly holonomic representation of $T_{--}$ to order $R^{-2}$,
\[
T_{--} = -\frac{R}{a^2} e_- \cdot \partial_- Y_0 \frac{1}{2a^2} \partial_- \partial_2 Y_0 \cdot \partial_- Y_0 - \frac{\beta}{R} e_+ \cdot \partial_2 Y_0 \\
- 2 \frac{\beta}{R^2} e_+ \cdot \partial_2 Y_0 e_+ \cdot \partial_- Y_0 - 2 \frac{\beta}{R^2} (e_+ \cdot \partial_2 Y_0)^2,
\]

whence we obtain the Virasoro generators with higher-order corrections,
\[
L_n = \frac{R}{a} e_- \cdot \alpha_n + \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m} \alpha_m : + \frac{\beta}{R} e_0^\mu \partial_2 Y_0 \\
- \frac{a \beta n^2}{R} e_n + \frac{\beta}{R} \frac{a^2 n^2}{R^2} \sum_{m=-\infty}^{\infty} : e_{n-m} e_m :,
\]

where $e_n \equiv e_+ \cdot \alpha_n$.

Thus we have established that $L_0$ and $\tilde{L}_0$ have no corrections at either $R^{-1}$ or $R^{-2}$ order. As mentioned earlier, this means all the terms in the ground state energy and the excited state energies, inclusive of the order-$R^{-3}$ term, are identical to those in the Nambu-Goto theory.

\section{Conclusions}

Not only is the Polchinski-Strominger action \cite{11} the unique effective first-order action for a consistent conformal theory of long strings, but as we have carefully shown it is unique up to and including terms of third order in the inverse string length. The only remaining freedom in the action is the substitution of the other equivalent form of the PS term, which we mentioned in sec. III and which does not alter our results.

Furthermore, the spectrum is found to coincide with that of the Nambu-Goto theory, including third order terms. This universality explains why comparisons between potentials and excited state energies in lattice computations \cite{10,2,3,4,5} and Nambu-Goto theory have been favourable in the past even beyond the universal Lüscher term \cite{10} (in the case of the ground state energy), despite the inconsistency of the Nambu-Goto string outside the critical dimension.

\textbf{Note Added:} The preprint \cite{11} came to our attention during the preparation of this manuscript. Although this work claims to derive eqn (20) it is marred by errors and missing proofs. Subtleties regarding the field redefinition in eqns (2.17-2.19) and how this leads to a non-trivial energy momentum tensor are ignored. In fact, as mentioned previously, field redefinitions must be handled with great care. We have depended on no such field redefinitions in this paper.

A serious deficiency of \cite{11} is the assertion that eqns (2.7-2.10) give the next leading corrections to the PS action. Not only is this incorrect, as new terms do appear at order $R^{-4}$, but the faulty analysis thus does not provide convincing evidence that there are no new terms at order $R^{-3}$. The issue of at which orders correction terms exist should be taken seriously. In particular, the absence of order-$R^{-3}$ terms is absolutely essential, as can be seen from several stages of sec. IV above; the explicit proof thereof is a main result of the present work.

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