On integrable discretization of the inhomogeneous Ablowitz-Ladik model.

V. V. Konotop
Department of Physics and Center of Mathematical Sciences,
University of Madeira, Praça do Município, Funchal, P-9000, Portugal.

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Abstract
An integrable discretization of the inhomogeneous Ablowitz-Ladik model with a linear force is introduced. Conditions on parameters of the discretization which are necessary for reproducing Bloch oscillations are obtained. In particular, it is shown that the step of the discretization must be comensurable with the period of oscillations imposed by the inhomogeneous force. By proper choice of the step of the discretization the period of oscillations of a soliton in the discrete model can be made equal to an integer number of periods of oscillations in the underline continuous-time lattice.

1 Introduction
During the last few years a great deal of attention has been paid to integrable discretizations of nonlinear evolution equations. The interest is naturally justified by needs of the computational physics [1, 2]. One of the purposes of integrable discretization is the construction of a discrete analogue of a continuum model which preserves main features of the last one. This point becomes especially important when one deals with inhomogeneous models. In that case even "the first step" of the discretization of a one-dimensional nonlinear evolution equation, i.e. discretization with respect to the spatial coordinate, may introduce qualitatively new features into the dynamics. So, for instance, in the case of the inhomogeneous nonlinear Schrödinger equation a constant force, which linearly depends on the spatial coordinate, results only in the renormalization of the phase and velocity of the one-soliton solution [3] while the same force leads to oscillations of solitons of the inhomogeneous Ablowitz-Ladik (AL) [4] model:

\begin{align}
  i\dot{q}_n + (1 - q_n r_n)(q_n - 1 + q_{n+1}) + 2\chi n q_n &= 0 \\
  - i\dot{r}_n + (1 - q_n r_n)(r_{n-1} + r_{n+1}) + 2\chi n r_n &= 0
\end{align}

(here \( r_n = \pm \bar{q}_n \), \( \chi \) is a real constant, which from the physical point of view determines the strength of the linear force, a dot stands for the derivative with respect to time, and a bar stands for the complex conjugation) [5, 6, 7]. Periodic dependence on time is a property of any solution of (1), (2) and it is caused by the discreteness. In the case of the one-soliton solution which reads

\begin{align}
  q_n^{(s)} &= -\bar{r}_n^{(s)} = \frac{\sinh(2w)}{\cosh(2w - X(t) - X_0)} e^{i\Phi(t) - 2n\chi(t-t_0)}
\end{align}

where

\begin{align}
  \Phi(t) &= \frac{1}{\chi} \cosh(2w) \sin[2\chi(t-t_0)]
\end{align}
\[ X(t) = \frac{1}{\chi} \sinh(2w) \cos[2\chi(t - t_0)] \]  

(5)

\( w, \ t_0, \) and \( X_0 \) are real constants, the soliton dynamics has deep analogy with the well known Bloch oscillations of an electron in a lattice potential affected by a constant electric field (due to this reason such behaviour is referred to as Bloch oscillations \( \text{[3]} \)). As it follows from \( \text{(3)} \) the period of oscillations is given by \( \tau_0 = \pi/\chi \).

The phenomenon of Bloch oscillations becomes especially interesting if one looks for the possibility of integrable discretization of the inhomogeneous AL model with respect to time. Indeed, Bloch oscillations are characterised by the additional temporal scale, \( \tau_0 \). This scale is determined by the strength of the force and must lead to some constrains on the step of discretization. Thus the purpose of the present communication is to introduce integrable discretization of the model \( \text{(1)}, \text{(2)} \) and to obtain conditions on parameters of the discretization which preserve the effect of Bloch oscillations in the discrete scheme.

2 **Integrable discretization**

At \( \chi = 0 \) system \( \text{(1)}, \text{(2)} \) transforms to the conventional AL model which discretization is well known \( \text{[2, 8]} \). In particular, it can be achieved by using the discrete analogue of the zero-curvature condition

\[ U(n, t + h)V(n, t) = V(n + 1, t)U(n, t). \]  

(6)

In the case \( \chi \neq 0 \) the same condition involving \( U \)-matrix as follows

\[ U(n, t) = \begin{pmatrix} \lambda e^{-i\chi t} & q(n, t) \\ r(n, t) & e^{i\chi t}/\lambda \end{pmatrix} \]  

(7)

and \( V \)-matrix having the elements

\[ V_{11} = i - h\alpha_0 + \delta_1 \left( \lambda^2 e^{-i\chi(2t+nh)} - A(n, t) \right) + h \left( \frac{\alpha_2}{\lambda^2} e^{i\chi(2t-nh)} - \delta_2 q(n, t + h)r(n - 1, t) \right) \Lambda(n, t) \]

\[ V_{12} = h \left( \alpha_1 \lambda e^{-i\chi(t+nh)} q(n, t) - \delta_1 e^{i\chi n h} q(n - 1, t + h) \right) + h \left( \delta_2 e^{-i\chi(t-nh)} q(n, t + h) - \frac{\alpha_2}{\lambda} e^{i\chi(n-h)} q(n - 1, t) \right) \Lambda(n, t) \]

\[ V_{21} = h \left( \alpha_1 \lambda e^{-i\chi(t+nh)} r(n - 1, t + h) - \frac{\delta_1}{\lambda} e^{i\chi n h} r(n, t) \right) + h \left( \delta_2 e^{-i\chi(t-nh)} r(n - 1, t) - \frac{\alpha_2}{\lambda} e^{i\chi(n-h)} r(n, t + h) \right) \Lambda(n, t) \]

\[ V_{22} = i + h\delta_0 - h\delta_1 \left( e^{i\chi(2t+nh)}/\lambda^2 - D(n, t) \right) - h \left( \delta_2 \lambda^2 e^{-i\chi(2t-nh)} - \alpha_2 q(n - 1, t)r(n, t + h) \right) \Lambda(n, t) \]

where \( \lambda \) is a spectral parameter, \( \alpha_j \) and \( \delta_j \) are parameters and \( h \) \( (h > 0) \) is a step of the discretization, results in the system

\[ ih^{-1} \left[ q(n, t + h) - q(n, t) \right] = \delta_1 q(n, t + h) e^{i\chi(n+h)} - \delta_0 q(n, t + h) - \alpha_0 q(n, t) + \alpha_1 q(n + 1, t) e^{-i\chi(n+1)} - \delta_1 q(n, t + h) D(n, t) - \alpha_1 q(n, t) A(n + 1, t) \]

\[ \alpha_2 q(n - 1, t) e^{i\chi(n-1)} - q(n, t + h) r(n, t + h) \Lambda(n, t) + +\delta_2 q(n + 1, t + h) e^{i\chi(n+2)} - q(n, t) r(n, t) \Lambda(n + 1, t), \]  

(8)
respectively we have to require which allow existence of "bright" solitons. Hence it will be assumed that

\[ \delta_0 r(n,t) + \delta_1 r(n+1,t) e^{ihx(n+1)} - \delta_1 r(n+1,t) A(n,t) - \delta_1 r(n+1,t) D(n+1,t) + \delta_2 r(n-1,t) [e^{ihx(n+1)} - q(n,t+h)] r(n,t+1) + \delta_2 r(n-1,t) [e^{-ihx(n+2)} - q(n,t)] \Lambda(n,t) + \alpha_2 r(n+1,t+1) [e^{-ihx(n+2)} - q(n,t)] \Lambda(n+1,t) \] (9)

\[ \alpha_1 [A(n+1,t) - A(n,t) e^{-ihx}] - h^{-1}(1 - e^{-ihx}) = \] (10)

\[ \delta_1 [D(n+1,t) - D(n,t) e^{ihx}] + h^{-1}(1 + h\delta_0) (1 - e^{ihx}) = \] (11)

\[ \Lambda(n,t) [1 - q(n,t+h)] r(n,t+1) \Lambda(n+1,t) = \Lambda(n+1,t) [1 - q(n,t)] r(n,t) \] (12)

Then the discrete analogue of the AL model (1), (2) is obtained from (8)-(12) by means of the

\[ r(n,t) = \pm \bar{q}(n,t) \] (13)

which requires the following relation among the parameters: \( \alpha_i = \bar{\delta}_j \).

In order to define solutions of (8)-(12) one has to fix boundary conditions for \( r(n,t), q(n,t), A(n,t), \) and \( D(n,t) \). In what follows we deal only with the case of zero boundary conditions

\[ \lim_{n \to \pm \infty} q(n,t) = \lim_{n \to \pm \infty} r(n,t) = 0 \] (14)

which allow existence of "bright" solitons. Hence it will be assumed that \( r(n,t) = -\bar{q}(n,t) \).

respectively we have to require

\[ \lim_{n \to \pm \infty} A(n,t) = A_0(n,t) = \frac{-i + h\delta_0}{h\alpha_1} (e^{-ih\chi n} - 1), \] (15)

\[ \lim_{n \to \pm \infty} D(n,t) = D_0(n,t) = \frac{i + h\delta_0}{h\delta_1} (e^{ih\chi n} - 1). \] (16)

Notice that (13), (14) transform to the zero boundary conditions in the case of the homogeneous

AL model \( \chi = 0 \). Eqs. (10), (11) subject to (13), (14) allow one to express \( A(n,t) \) and \( D(n,t) \) through \( q(n,t) \) and \( r(n,t) \):

\[ A(n,t) = A_0(n,t) + \alpha_1^{-1} \sum_{k=1}^{\infty} f_A(n-k,t) e^{-i\chi h k} \] (17)

\[ D(n,t) = D_0(n,t) + \delta_1^{-1} \sum_{k=1}^{\infty} f_D(n-k,t) e^{i\chi h k} \] (18)

Here \( f_A(n,t) \) and \( f_D(n,t) \) stand for the right hand sides of the equations (14) and (15), correspondingly. It is to emphasised that formulae (14) and (15) do not give yet explicit solutions for
As it has been shown in [6, 9] a convenient approach to treat inhomogeneous discrete models is the use of the gauge transformation which allows one to restrict the study only to the temporal behaviour of the scattering data. The gauge transformation in the discrete case is given by

\[ \tilde{U}(n,t) = G(n+1,t)U(n,t)G^{-1}(n,t) \]  
\[ \tilde{V}(n,t) = G(n,t+h)V(n,t)G^{-1}(n,t) \]  

By choosing \( G(n,t) = \exp\{i\chi n t \sigma_3\} \), where \( \sigma_3 \) is the Pauli matrix one reduces \( \tilde{U}(n,t) \) to the form which corresponds to the \( U \)-matrix of the underline homogeneous model (i.e. to the form which does not have explicit dependence on time and can be obtained from (11) by the replacement \( \exp(i\chi t) \rightarrow 1 \)). Then the dependence of the transfer matrix \( T(t) \), associated with \( \tilde{U}(n,t) \), on the discrete time is governed by the equation

\[ T(t+h) = V_h T(t) V_h^{-1} \]  

where \( V_h \) is a diagonal matrix, \( V_h = \text{diag}(\theta_1(\lambda,t),\theta_2(\lambda,t)) \), with the elements

\[ \theta_1(\lambda,t) = i - h\alpha_0 + h\lambda^2 \alpha_1 e^{-2i\chi t} + h\lambda^{-2} \alpha_2 e^{2i\chi t}. \]  
\[ \theta_2(\lambda,t) = i + h\delta_0 - h\lambda^2 \delta_2 e^{-2i\chi t} - h\lambda^{-2} \delta_1 e^{2i\chi t}. \]  

Let us now assume that \( t = mh \) and \( m = 0,1,... \) (it is straightforward to generalise the results to the case \( t = mh + t_0 \) where \( t_0 \) is an arbitrary real constant playing the role of initial moment of time). Then the element \( T^{(11)}(t) \) of the matrix \( T(t) \) does not depend on \( m \) (or \( t \)) while for \( T^{(12)}(t) \equiv b_m \) one obtains

\[ b_{m+1} = b_0 \prod_{n=0}^{m} \mu_n(\lambda) \]  

where

\[ \mu_m(\lambda) = \frac{\theta_1(\lambda,mh)}{\theta_2(\lambda,mh)}. \]  

In the case of solitonic solutions (24) formally solves the discrete Cauchy problem since it defines dependence of the scattering data on time and the solution of the eigenvalue problem for the matrix \( U(n,0) \) is well known [4]. Below we concentrate on some ”physical” consequences of that result.

### 3 Bloch oscillations in the discrete model

As it has been mentioned in the Introduction each step of discretization can introduce new features into the dynamics (even in cases of integrable models). One of such features is the oscillatory behaviour of the solutions of the AL model affected by the linear force (Bloch oscillations). Now we address to the question whether it is possible to preserve such evolution subject to the discretization with respect to time.

To this end we take into account that periodic behaviour means that there exists a positive integer \( M \) such that

\[ \prod_{n=m+1}^{m+M} \mu_n(\lambda) = 1 \]  

for any \( \lambda \) (which can be considered, say, inside the unit circle on the complex plane) and any \( m \). The period \( \tau \) of oscillations is then given by \( \tau = Mh \) (evidently \( M \) is considered to be the smallest possible integer).
The discretization of the homogeneous AL model is a three parametric one (this, in particular, allows one to represent it in a form of a product of the local maps \( \mathcal{A} \)). In the inhomogeneous case the imposed conditions lead to constraints on the parameters. To find them we first consider the limit \( |\lambda| \to 0 \) (or \( |\lambda| \to \infty \)). Then from (24), (25), (26), and (27) one finds that there must exist relations

\[
 a_1 = -a_2 \quad \phi_1^{(l)} + \phi_2^{(l)} = 2\pi \frac{l}{M}, \quad l = 0, 1, ..., M - 1
\]

(27)

where \( a_{1,2} \) and \( \phi_{1,2} \) are real parameters connected to \( \alpha_{1,2} \), \( \alpha_{1,2} = a_{1,2} \exp(\phi_{1,2}) \), and the upper index has been attributed to the “quantized” phases. Next, the independence of (26) on \( m \) implies \( \mu_m = \mu_{m+M} \) which means that

\[
 \chi h_1 M = \pi \tilde{l}
\]

(28)

where \( \tilde{l} \) is a positive integer. In other words the step of the discretization is not arbitrary (the subindex \( \tilde{l} \) is introduced to label different discrete values of \( h \)). Physical sense of the last requirement is quite transparent. Recalling that the period of Bloch oscillations in the continuous-time model is given by \( \tau_0 = \pi / \chi \), one concludes that (23) means that the period of the Bloch oscillations in the discretized model, \( \tau_1 \), is \( \tilde{l} \) times bigger than the period \( \tau_0 \): \( \tau_1 = \tilde{l} \tau_0 \) and the number \( \tilde{l} \) is related to the chosen step of discretization \( h \). On the other hand rewriting (28) as \( h_1 = l \tau_1 / M \) one can interpret it as a condition for the discretization step to be commensurable with the period of oscillations. As it is evident, for the direct coincidence of the result obtained on the discrete lattice with its continuum counterpart one must let \( \tilde{l} = 1 \). Below we concentrate on this case. Then \( \mu_n(\lambda) \) takes the form

\[
 \mu_n(\lambda) = \frac{1 + a \exp(i(\Gamma_1 + \gamma_{l,n})) \lambda^2 - a \exp(i(\Gamma_1 - \gamma_{l,n})) \lambda^{-2}}{1 - a \exp(-i(\Gamma_1 + \gamma_{l,n})) \lambda^{-2} + a \exp(-i(\Gamma_1 - \gamma_{l,n})) \lambda^2} e^{2i\phi_0}
\]

(29)

where

\[
 \Gamma_1 = \frac{l}{M} \pi - \phi_0, \quad \gamma_{l,n} = \frac{1}{2} \left( \phi_1^{(l)} - \phi_2^{(l)} \right) - \frac{2\pi n}{M}
\]

(30)

\[ a = |h\alpha_1/(i - h\alpha_0)| \text{ and } \phi_0 = \arg(i - h\alpha_0). \]

Now we consider the unit circle where \( \lambda^2 = \exp(i\psi) \) (\( \psi \) being real). Then one can find two possibilities to satisfy the requirement (26). One simplest solution corresponds to \( M \) even and \( \phi_0 = \pi / M + \pi / 2 + p\pi \) (\( p \) is an integer). Then \( \mu_n(\lambda) = \exp\{2\pi i(l/M - 1/2)\} \). By the direct algebra one ensures that this is the degenerated case, when the limiting transition \( h \to 0 \) results in a trivial linear equation instead of the AL model. A nontrivial and physically relevant solution corresponds to the case when \( M = 4N \) (\( N \) is an integer) and \( \phi_0 = \phi_{l,p} = \pi l / M + p\pi \) (in that case \( \mu_n \mu_{n+2N} \mu_{n+2N} = 1 \)). Then \( \mu_n(\lambda) \) which determines evolution of the one-soliton solution associated with the eigenvalue \( \lambda_1 = \exp(-w + i\theta) \) is given by

\[
 \mu_n(\lambda_1) = \frac{1 - 2(-1)^p a \sinh[2w - i(\gamma_{l,n} + 2\theta)]}{1 + 2(-1)^p a \sinh[2w - i(\gamma_{l,n} + 2\theta)]} \exp\left(2\pi i \frac{l}{M}\right)
\]

(31)

Let us illustrate the discrete-time dynamics on example of the one-soliton solution. We assume that (27) holds. For the sake of simplicity we let \( \phi_1 = -\phi_2 = \pi / 2 \) and \( \phi_0 = 0 \). Then the one-soliton solution of (8), (9) can be written down in the form (recall \( t = mh \))

\[
 q^{(s)}(n,m) = -\bar{r}^{(s)}(n,m) = \frac{\sinh(2w)}{\cosh(2nw - X_m)} e^{i\Phi(n,m)}
\]

(32)

where

\[
 \Phi(n,m) = \sum_{k=0}^{m-1} \arctan \left[ \frac{4a \cos(2\chi kh - 2\theta) \cosh(2w)}{1 + 2a^2 \cos(4\chi kh - 4\theta) + 4a^2 \cosh(4w)} \right] - 2i\chi nh
\]
\[ X_m = \frac{1}{2} \sum_{k=0}^{m-1} \ln \left[ \frac{2a^2 (\cosh(4w) + \cos(4\chi kh - 4\theta)) + 1 - 4a \sinh(2w) \sin(2\chi kh - 2\theta)}{2a^2 (\cosh(4w) + \cos(4\chi kh - 4\theta)) + 1 + 4a \sinh(2w) \sin(2\chi kh - 2\theta)} \right] \] (33)

Comparing these formulae with (4), (5) one can see that choosing \( a = h/2 \) the last ones are obtained by the limiting transition \( h \to 0 \).

To be more specific we concentrate on \( X_m \) which describes evolution of the centre of the soliton. To this end we represent \( h = \pi \xi / (M \chi) \) (the so chosen step of the discretization satisfies (28) when \( \xi \) is integer). The results are summarized in Fig. 1 for three values of the parameter \( \xi \) displaying different situations: (a) When \( \xi = 1 \) the discrete model exactly reproduces Bloch oscillations of the continuous-time model. (b) At \( \xi = \sqrt{3} \) the evolution of the discrete model is not periodic (notice that lines in the figure are used for the sake of convenience of presentation: the truth trajectories are sets of points). However, by considering analytic continuation of the solution the periodicity can be considered between discrete time steps \([2]\). (c) At \( \xi = 2 \) the period of soliton oscillations is two times more than the period of Bloch oscillation of the AL model which is obtained by the limiting transition \( h \to 0, M \to \infty \) with \( hM = \pi / \chi \) (it is to be mentioned that the minima of the curve (c) about \( h = 48 \) and \( h = 145 \) are not numerical zeros).

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Figure caption

Trajectory of the soliton centre corresponding to different steps of discrete time (a) \( h = 1/97 \), (b) \( h = \sqrt{3}/97 \), and (c) \( h = 2/97 \). Other parameters are as follows: \( M = 97, w = 0.5, \chi = \pi, a = 0.1 \).
position of the center

\begin{align*}
\text{time} & \quad 0 & 20 & 40 & 60 & 80 & 100 & 120 & 140 & 160 & 180 \\
\text{position of the center} & \quad 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18
\end{align*}