The Biedenharn Approach
to
Relativistic Coulomb-type Problems

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Abstract

The approach developped by Biedeharn in the sixties for the relativistic Coulomb problem is reviewed and applied to various physical problems.

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1 Introduction

In a paper anticipating supersymmetric quantum mechanics, [1], Biedenharn proposed a new approach to the Dirac-Coulomb problem. His idea has been to iterate the Dirac equation. The resulting quadratic equation, written in a non-relativistic Coulomb form, is readily solved using the ‘Biedenharn-Temple’ operator \( \Gamma \) analogous to the angular momentum operator (but with a fractional eigenvalues). Then the solutions of the first-order equation can be recovered from those of the second-order equation by projection.

In this review we apply the approach of Biedeharn to various physical problems.

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2 The Dirac approach

Let us first summarize the original approach of Dirac in his classic book [2]. He starts with the first-order Hamiltonian
\[
\mathcal{H} = -eA_0 + \rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3 m,
\] (2.1)

where the ‘Dirac’ matrices can be chosen as
\[
\rho_1 = \begin{pmatrix} 1 & 2 \\ 12 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1 & 2 \\ i12 & -i12 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 2 \\ 12 & -1 \end{pmatrix},
\] (2.2)

where $1_{2 \times 2}$ is the $2 \times 2$ unit matrix.

For a spherically symmetric potential, $A_0 = A_0(r)$, Dirac proposes the following solution. First, he proves the vector identity
\[
(\vec{\sigma} \cdot \vec{u})(\vec{\sigma} \cdot \vec{v}) = (\vec{u} \cdot \vec{v}) + i \vec{\sigma} \cdot (\vec{u} \times \vec{v}).
\] (2.3)

Then, applying to the orbital angular momentum and momentum, $\vec{u} = \vec{\ell} = \vec{x} \times \vec{p}$ and $\vec{v} = \vec{p}$, respectively, interchanging $\vec{u}$ and $\vec{v}$, he deduces that the two-component operator
\[
z = \vec{\sigma} \cdot \vec{\ell} + 1
\] (2.4)

anticommutes with $\vec{\sigma} \cdot \vec{p}$, $\{z, \vec{\sigma} \cdot \vec{p}\} = 0$. Therefore, the operator
\[
K = \rho_3 Z
\] where $Z = \vec{\Sigma} \cdot \vec{\ell} + 1$, $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} \\ \vec{\sigma} \end{pmatrix}$, (2.5)

commutes with all three terms in the Hamiltonian (2.1) and is hence a constant of the motion.

Next, applying to $\vec{u} = \vec{v} = \vec{\ell}$ allows him to infer, using the identity
\[
\vec{\ell} \times \vec{\ell} = i\vec{\ell},
\] (2.6)

that
\[
Z^2 = (\vec{\sigma} \cdot \vec{\ell} + 1)^2 = \vec{J}^2 + \frac{1}{4}, \quad \text{where} \quad \vec{J} \equiv \vec{\ell} + \frac{1}{2} \vec{\Sigma}.
\] (2.7)

$\vec{J}$ is here the total angular momentum operator. The eigenvalues of $K$ are therefore half-integers,
\[
\kappa = \pm (j + 1/2).
\] (2.8)

Further application of the identity (2.3) with $\vec{u} = \vec{x}$ and $\vec{v} = \vec{p}$ shows that
\[
(\vec{\sigma} \cdot \vec{x})(\vec{\sigma} \cdot \vec{p}) = r p_r + i(z - 1),
\] (2.9)

where $p_r = -i\partial_r$. Note that $[p_r, K] = 0$.

At this stage, Dirac introduces a second operator, namely
\[
\omega = \rho_1 W, \quad W = \begin{pmatrix} w \\ w \end{pmatrix}, \quad w = \vec{\sigma} \cdot \vec{x}/r.
\] (2.10)
which satisfies the relations
\[ \omega^2 = W^2 = w^2 = 1, \quad [\omega, \mathcal{K}] = 0. \] (2.11)

Finally, Dirac rewrites the Hamiltonian (2.1) in the form
\[ \mathcal{H} = -eA_0 + \omega (p_r + iZ - \frac{1}{r}) + \rho_m. \] (2.12)

In the Coulomb case, \( eA_0 = \alpha/r \), and the radial form (2.12) allows one to find the spectrum (3.15) of the relativistic hydrogen atom [2].

3 The Biedenharn approach to the Dirac-Coulomb problem.

Biedenharn [1] proposes instead to introduce the projection operators
\[ \mathcal{O}_\pm = i\rho_2 \vec{\sigma} \cdot \vec{p} \pm m - \rho_3 (E + \frac{\alpha}{r}), \] (3.1)
so that \( \mathcal{H} - E = \rho_3 \mathcal{O}_+ \), and observes that
\[ (\mathcal{H} - E)\psi = 0 \Rightarrow \mathcal{O}_- \mathcal{O}_+ \psi = 0, \quad \mathcal{O}_+ \phi = \mathcal{O}_+ \mathcal{O}_- \psi = \mathcal{O}_- \mathcal{O} - \psi = 0, \]
since the \( \mathcal{O}_\pm \) commute. The solutions of the first-order equation \( \mathcal{O}_+ \phi = 0 \) can be obtained, therefore, from those of the iterated equation by projection,
\[ \phi = \mathcal{O}_- \psi = 0. \] (3.2)

Then the ‘Biedenharn (Temple) operator’ is defined as
\[ \Gamma = -(Z + i\omega) \equiv - \left( \begin{array}{cc} z & i\omega \\ i\omega & z \end{array} \right). \] (3.3)

\( \Gamma \) is conserved for the iterated, but not for the first-order equation, and allows us to re-write \( \mathcal{O}_- \mathcal{O}_+ \psi = 0 \) in a form reminiscent of the non-relativistic Coulomb problem,
\[ \left[ - (\partial_r + \frac{1}{r})^2 + \frac{\Gamma(\Gamma + 1)}{r^2} + \frac{2\alpha E}{r} + m^2 - E^2 \right] \psi = 0. \] (3.4)

The operator \( \Gamma \) plays here a rôle of the angular momentum. However,
\[ \Gamma^2 = \mathcal{K}^2 - \alpha^2 = \mathcal{J}^2 + \frac{1}{4} - \alpha^2, \] (3.5)
so that the eigenvalues of \( \Gamma \) are
\[ \gamma = \pm \sqrt{\kappa^2 - \alpha^2} = \pm \sqrt{(j + 1/2)^2 - \alpha^2}, \quad \text{sign } \gamma = \text{sign } \kappa. \] (3.6)

For a \( \Gamma \)-eigenfunction,
\[ \Gamma(\Gamma + 1) = \ell(\gamma)(\ell(\gamma) + 1) \quad \text{with} \quad \ell(\gamma) = |\gamma| + \frac{1}{2}[\text{sign}(\gamma) - 1], \] (3.7)
i.e. the ‘angular momentum’ \( \ell(\gamma) \) is irrational. The operator \( \Gamma \) is hermitian as long as \( \alpha \leq 1 \), i.e., for nuclei with less than 137 protons.
To get explicit formulæ, remember [3] that the angular spinors
\[ \chi_\pm = \sqrt{\frac{\kappa + 1/2 \pm |\kappa|}{2 |\kappa| + 1}} \, Y_{j \pm 1/2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \mp \sqrt{\frac{\kappa + 1/2 \pm |\kappa|}{2 |\kappa| + 1}} \, Y_{j + 1/2} \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \] (3.8)
where the ± refers to the sign of \( \kappa \), and the \( Y \)'s are the spherical harmonics, are not only eigenfunctions of \( \vec{J}^2 \) and of \( J_3 \) with eigenvalues \( j(j+1) \), \( (j = 1/2, 3/2, \cdots) \) and \( \mu = -j, \ldots, j \) respectively, but also satisfy the crucial relations
\[ z \, \chi_\pm = \pm |\kappa| \, \chi_\pm \quad \text{and} \quad w \, \chi_\pm = \chi_\mp. \] (3.9)

Put
\[ \Xi_+ = \left( \begin{array}{c} \chi_\mu^+ \\ 0 \end{array} \right), \quad \Xi_- = \left( \begin{array}{c} 0 \\ \chi_\mu^- \end{array} \right), \quad \Upsilon_+ = \left( \begin{array}{c} 0 \\ \chi_\mu^+ \end{array} \right), \quad \Upsilon_- = \left( \begin{array}{c} \chi_\mu^- \\ 0 \end{array} \right). \] (3.10)

Then the
\[ \Phi_+ = -i\alpha \Xi_+ + (|\kappa| - |\gamma|)\Xi_-, \quad \Phi_- = (|\kappa| - |\gamma|)\Xi_+ + i\alpha \Xi_-, \] \[ \varphi_+ = -i\alpha \Upsilon_+ + (|\kappa| - |\gamma|)\Upsilon_-, \quad \varphi_- = (|\kappa| - |\gamma|)\Upsilon_+ + i\alpha \Upsilon_- \] (3.11)
are eigenfunctions of \( \Gamma \) with eigenvalues \( \pm |\gamma| \),
\[ \Gamma \Phi_\pm = \pm |\gamma| \, \Phi_\pm \quad \Gamma \varphi_\pm = \pm |\gamma| \, \varphi_\pm. \] (3.12)

Then, setting \( \psi_\pm = u_\pm \Phi_\pm \), the iterated equation takes indeed a non-relativistic Coulomb form with irrational angular momentum \( \ell(\gamma) \),
\[ \left[ - (\partial_r + \frac{1}{r})^2 + \frac{\ell(\gamma)(\ell(\gamma) + 1)}{r^2} + \frac{2\alpha E}{r} + m^2 - E^2 \right] u_\pm = 0, \] (3.13)
whose solutions are the well-known Coulomb eigenfunctions
\[ u_\pm(r) \propto r^{\ell(\gamma)} e^{ikr} F\left( \ell(\gamma) + 1 - i\alpha E/k, 2\ell(\gamma) + 2, -2ikr \right), \] (3.14)
where \( k = \sqrt{E^2 - m^2} \) and \( F \) denotes the confluent hypergeometric function. The energy levels are obtained from the poles of \( F \),
\[ \ell(\gamma) + 1 - i\alpha E/k = -n, \quad n = 0, 1, 2, \ldots, \]
yielding the familiar spectrum shown on FIG. II
\[ E_p = m \sqrt{1 - \frac{\alpha^2}{p^2 + \alpha^2}}, \quad p = \ell(\gamma) + 1 + n = |\gamma| + \frac{1}{2} \text{sign} \, \gamma + \frac{1}{2} + n, \quad n = 0, 1, \ldots \] (3.15)
Since \( \gamma \) and thus \( \ell \) are irrational, \( \ell + n = \ell' + n' \) is only possible for \( \gamma' = \pm \gamma \) so different \( j \)-sectors yield different \( E \)-values. For each fixed \( j \), the same energy is obtained in the \( \gamma > 0 \) sector for \( n - 1 \) as in the \( \gamma < 0 \) sector for \( n \). These energy levels are hence doubly degenerate. In the \( \gamma < 0 \) sector the \( n = 0 \) state is unpaired: each \( j \) sector admits a ground-state.
\[ u_0^j \propto r^{\ell(\gamma)-1} e^{-amr/(j+1/2)} \quad \text{with energy} \quad E_0^j = m \sqrt{1 - \frac{\alpha^2}{(j + 1/2)^2}}, \] (3.16)
Observe that eqn. (3.16) is consistent with (3.14) due to \( F(a, a, z) = e^z \).
Figure 1: The spectrum of a Dirac electron in the field of an H-atom. The ± signs refer to the sign of γ. In different j-sectors the energy levels are shifted by the fine structure.

4 Charged Dirac particle in a monopole field

A Dirac particle in the field of a Dirac monopole,

\[ \vec{B} = -g \frac{\vec{r}}{r^3}, \]

(4.1)
can be treated along the same lines [4]. The Hamiltonian is now

\[ \mathcal{H} = -\frac{\alpha}{r} + \rho_1 \vec{\sigma} \cdot \vec{\pi} + \rho_3 m, \quad \vec{\pi} = \vec{p} - e\vec{A}, \]

(4.2)

where \( \vec{A} \) is the vector potential of a Dirac monopole, \( \vec{\nabla} \times \vec{A} = \vec{B} \). Introducing again the projection operators

\[ \mathcal{O}_\pm = i\rho_2 \vec{\sigma} \cdot \vec{\pi} \pm m - \rho_3 (E + \frac{\alpha}{r}), \]

(4.3)

the solutions of the first-order equation can again be obtained from that of the iterated equation by projecting, cf. (3.2). Dirac’s operator,

\[ \mathcal{K} = -\rho_3 (\vec{\Sigma} \cdot \vec{\ell} + 1), \]

(4.4)
is formally the same as in (2.5), except for the replacements

\[ \vec{p} \rightarrow \vec{\pi}, \quad \Rightarrow \quad \vec{\ell} = \vec{r} \times \vec{\pi}. \]

(4.5)

Note that \( \vec{\ell} \) is now only part of the orbital angular momentum,

\[ \vec{L} = \vec{\ell} - q \frac{\vec{r}}{r}, \]

where \( q = eg \). The novelty is that, unlike in (2.8), the eigenvalues of \( \mathcal{K} \) became now irrational,

\[ \kappa = \sqrt{(j + 1/2)^2 - q^2}. \]

(4.6)

The iterated equation reads again as (3.4), with the Biedenharn operator \( \Gamma = -(Z + i\alpha \omega) \)

(3.3)

cf. (3.3). The square of \( \Gamma \) is now

\[ \Gamma^2 = \mathcal{K}^2 - \alpha^2 = \vec{J}^2 + \frac{1}{4} - q^2 - \alpha^2, \]

(4.7)
where
\[ \vec{J} = \vec{L} + \frac{1}{2} \vec{\Sigma} = \vec{L} - q\frac{\vec{r}}{r} + \frac{1}{2} \vec{\Sigma} \] (4.8)
is the total angular momentum. The eigenvalues of \( \Gamma \) are, therefore, ‘even more irrational’, since the monopole-charge term \( q^2 \) and the Coulomb-charge term \( \alpha^2 \) are both subtracted:
\[ \gamma = \pm \sqrt{\kappa^2 - \alpha^2} = \pm \sqrt{(j + 1/2)^2 - q^2 - \alpha^2}, \quad \text{sign } \gamma = \text{sign } \kappa. \] (4.9)

Observe that this yields now an imaginary \( \gamma \) for the lowest angular momentum \( j = q - 1/2 \) sector for any positive \( \alpha \), and the situation is worsened when \( \alpha \) is increased. These cases should be discarded.

Let us assume that \( \alpha \) is small, typically a few times 1/137 so that \( \gamma \) is real except for the lowest angular momentum sector. Assuming \( j \geq q + 1/2 \), consider those angular 2-spinors \( \chi_{\pm} \) in (3.8), i.e.
\[ \chi_{\pm}^\mu = \sqrt{\frac{|\kappa| + 1/2 \pm \mu}{2 |\kappa| + 1}} \ Y_{j\pm 1/2}^{\mu-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mp \sqrt{\frac{|\kappa| + 1/2 \mp \mu}{2 |\kappa| + 1}} \ Y_{j\pm 1/2}^{\mu+1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \] (4.10)
but with the \( Y \)’s being now replaced by the ‘Wu-Yang’ monopole harmonics [5]. These spinors are eigenfunctions of \( \vec{J}_2 \) and \( J_3 \) with eigenvalues \( j = q - 1/2, q + 1/2, \ldots \) and \( -j \leq \mu \leq j \), respectively. Then the \( \Phi_{\pm} \) and \( \varphi_{\pm} \) in Eq. (3.11) are eigenfunctions of \( \Gamma \) with eigenvalues \( \pm |\gamma| \), cf. (3.12). For the two signs
\[ \Gamma(\Gamma + 1) = \ell(\ell) + 1 \quad \text{with} \quad \ell(\gamma) = |\gamma| + \frac{1}{2} \text{[sign}(\gamma) - 1)] \] (4.11)
cf. (3.7). Setting \( \psi_{\pm} = u_{\pm} \Phi_{\pm} \) (and \( \psi_{\pm} = u_{\pm} \varphi_{\pm} \), respectively), the iterated Dirac equation \( O_- O_+ \) reduces to the non-relativistic Coulomb form \( \Box \) with solutions as in (3.13) and energy levels (3.15). The only difference is in the value of \( \gamma \).

The ground-states of the \( j = \text{const} \) sector are
\[ u_j^{(0)} \propto r|\gamma|-1 e^{-\alpha mr/\sqrt{\gamma^2 + \alpha^2}} \quad \text{with energy} \quad E_j^{(0)} = m \sqrt{1 - \frac{\alpha^2}{\gamma^2 + \alpha^2}}. \] (4.12)
The spectrum is shown on FIG 2.

For \( q = 0 \) (no monopole) we plainly recover Biedernharn’s results in [1] on the Dirac-Coulomb problem.

For \( \alpha = 0 \) (no Coulomb potential) one has a pure Dirac monopole [6]. The Biedernharn operator \( \Gamma \) reduces to \(-Z\). No further diagonalization in \( \rho \)-space is thus necessary. Since \([Z, \rho_3] = 0, \rho_3\) is now conserved for the iterated equation (but not for the first-order equation). The iterated equation splits therefore into two (identical) Pauli equations, and we can work with 2-spinors.

For \( j \geq q + 1/2 \), the angular eigenfunctions of \( \Gamma = -Z \) those \( \Xi \)’s in (3.10).\(^1\)

For \( j \geq q + 1/2 \) there are no bound states. The hypergeometric function reduces to a Bessel function and the radial eigenfunction becomes
\[ u_{\pm} \propto \frac{1}{\sqrt{k r}} J_{|\kappa| \pm 1/2}. \] (4.13)
\(^1\) the \( \Xi_{\pm} \)’s are proportional to those \( \xi^{(i)} \)’s \((i = 1, 2)\) in eqns. (11) and (19) of Kazama, Yang and Goldhaber [7]. Their \( \phi^{(i)} \)’s are just our \( \varphi_{\pm} \)’s in (3.11).
Figure 2: The bound-state spectrum of a Dirac particle in a charged monopole field \([q = 1/2]\). The ± refers to the sign of the Biedenharn operator \(\Gamma\). In each \(j = \text{const.}\) sector the energy levels are doubly degenerate except for a lowest-energy ground state, which occurs in the \(\gamma < 0\) sector. Different \(j\)-sectors are shifted by a modified fine structure. For \(j = 0\) there are no \(\gamma > 0\) states, and \(\Gamma\) is not hermitian. This critical case \(j = 0, \gamma < 0\) is not discussed here.

the same as eqn. # (37) in [11].

The \(j = q - 1/2\) case should not be discarded: the eigenvalue of \(Z \equiv \Gamma\) only vanishes, rather than becoming imaginary. The problem requires, nevertheless, special treatment. The Dirac Hamiltonian is indeed not self-adjoint [9] but admits a 1-parameter family of self-adjoint extensions, corresponding to different boundary conditions at \(r = 0\). These yield different physics. The one constructed by Callias [9] has further significance for the theta-angle in QCD. Kazama et al. [7, 8] suggest to cure the non-self-adjointness problem by adding an infinitesimal extra magnetic moment. For further discussion and details the reader is invited to consult the literature [9, 7, 8, 10].

5 Dyons

Let us consider a massless Dirac particle in the long-distance field of a (self-dual) Bogomolny-Prasad-Sommerfield monopole [13, 16, 17],

\[ e\vec{B} = -q\frac{\vec{r}}{r^3} \quad \text{and} \quad \Phi = q \left(1 - \frac{1}{r}\right). \]  

(5.1)

Identifying \(\Phi\) with the fourth component of a gauge field we get a static, self-dual Abelian gauge field in four euclidean dimensions

\[ A = qA_D, \quad A_4 = q \left(1 - \frac{1}{r}\right), \]  

(5.2)

where \(A_D\) denotes the vector potential of a Dirac monopole of unit strength. The associated Dirac Hamiltonian is therefore [13, 17]

\[ \Psi = \rho_1(\vec{\sigma} \cdot \vec{\pi}) - \rho_2\Phi = \begin{pmatrix} Q \end{pmatrix} \begin{pmatrix} Q^\dagger \\ \sigma \cdot \vec{\pi} + i\Phi \end{pmatrix} \begin{pmatrix} \sigma \cdot \vec{\pi} - i\Phi \end{pmatrix} = \begin{pmatrix} Q \end{pmatrix} Q^\dagger. \]  

(5.3)

In contrast to the Coulomb case, the scalar term \(\rho_2\Phi\) is now \emph{off-diagonal}, because it comes from the fourth, euclidean, direction, rather than from the time coordinate.
The total angular momentum, $\vec{J}$ in Eq. (4.8) is conserved. Using the notations and formulæ introduced for the charged monopole, we observe that the counterpart of Dirac’s operator (4.4),

$$K = -\rho Z = \begin{pmatrix} i & 1z \\ -i & 1z \end{pmatrix},$$

(5.4)

commutes with $\hat{h}$ and

$$K^2 = z^2 = \vec{J}^2 + \frac{1}{4} - q^2,$$

(5.5)

so that $z$ (and hence $Z$ and $K$) have irrational eigenvalues,

$$\kappa = \sqrt{(j + 1/2)^2 - q^2},$$

(5.6)

cf. Eq. (4.6). Since $j \geq q - 1/2$, $K$ is hermitian, but for $j = q - 1/2$ its eigenvalue $\kappa$ vanishes and thus $K$ is not invertible.

The Dirac operator (5.3) is, as in any even dimensional space, chiral-supersymmetric : $\{Q, Q^\dagger\}$ is a SUSY Hamiltonian and the SUSY sectors are the $\pm 1$ eigenspaces of the chirality operator $\rho_3$. The supercharges $Q$ and $Q^\dagger$ can be written as

$$Q = -iw(\partial_r + \frac{1}{r} - \frac{z + qw}{r} + qw) = -i(\partial_r + \frac{1}{r} + \frac{z - qw}{r} + qw)w,$$

(5.7)

$$Q^\dagger = iw(-\partial_r + \frac{1}{r} - \frac{z - qw}{r} + qw) = i(-\partial_r + \frac{1}{r}) - \frac{z + qw}{r} + qw)w.$$  

(5.8)

The square of (5.3) is

$$\hat{p}^2 = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} = \begin{pmatrix} Q^\dagger Q \\ QQ^\dagger \end{pmatrix},$$

(5.9)

where

$$H_0 = \left[ \pi^2 + q^2 \left( 1 - \frac{1}{r} \right)^2 \right]_2 \text{ and } H_1 = H_0 - 2\frac{\sigma \cdot \vec{r}}{r^3}.$$  

(5.10)

In the ‘lower’ (i.e. $\rho_3 = -1$) sector, the gyromagnetic ratio is $g = 0$, and $H_0$ can be viewed as describing two, uncoupled, spin 0 particles in the combined field of a Dirac monopole, of a Coulomb potential and of an inverse-square potential. This system has been solved many years ago; it has a Coulomb-type spectrum, whose degeneracy is explained by its ‘accidental’ $o(4)$ symmetry [15]. In the ‘upper” (i.e. $\rho_3 = 1$) sector $g = 4$; $H_1$ is the Hamiltonian of D’Hoker and Vinet in Ref. [14].

In terms of $Z$ and $w$, $\hat{p}^2$ is also

$$\hat{p}^2 = -(\partial_r + \frac{1}{r})^2 - \frac{2q^2}{r} + q^2 + \frac{Z^2 + q^2}{r^2} - \frac{1}{r^2} \begin{pmatrix} z + qw \\ z - qw \end{pmatrix}.$$  

(5.11)

The Biedenharn operator [17]

$$\Gamma = -(Z + q\rho_3 W) \equiv -\begin{pmatrix} z + qw \\ z - qw \end{pmatrix},$$

(5.12)

does not commute with $\hat{p}$, but it commutes with $\hat{p}^2$; it is thus conserved for the quadratic dynamics $H_0$ and $H_1$ [but not for the Dirac Hamiltonian $\hat{p}$]. In terms of $\Gamma$, $\hat{p}^2$ becomes

$$\hat{p}^2 = -(\partial_r + \frac{1}{r})^2 + \frac{\Gamma(\Gamma + 1)}{r^2} - \frac{2q^2}{r} + q^2.$$  

(5.13)
Now
\[ \Gamma^2 = z^2 + q^2 = \vec{J}^2 + \frac{1}{4}, \]  
(5.14)
because, unlike in (4.7), the \( q^2 \) comes with a positive sign. The eigenvalues of \( \Gamma \) are, therefore, (half)integers,
\[ \gamma = \pm (j + 1/2), \quad \text{sign} \, \gamma = \text{sign} \, \kappa. \]  
(5.15)
Hence, for a \( \Gamma \)-eigenfunction,
\[ \Gamma(\Gamma + 1) = L(\gamma)(L(\gamma) + 1) \quad \text{where} \quad L(\gamma) = j \pm \frac{1}{2}. \]  
(5.16)
(The sign is plus or minus depending on the sign of \( \gamma \)). \( L(\gamma) \) is now a (half)integer. Using the notations \( x = z - qw \) and \( y = z + qw \), the supercharges are written as
\[ Q = -iw\left(\partial_r + \frac{1}{r} - \frac{y}{r} + qw\right) = -i\left(\partial_r + \frac{1}{r} + \frac{x}{r} + qw\right)w, \]  
(5.17)
\[ Q^\dagger = iw\left(-\left(\partial_r + \frac{1}{r} + \frac{x}{r} + qw\right) = i\left(-\left(\partial_r + \frac{1}{r}\right) - \frac{y}{r}qw\right). \]  
(5.18)
Note that one can also write
\[ \Gamma = -\left(\vec{\sigma} \cdot \vec{L} + 1 + 2qw\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  
(5.19)
x and \( y \) are self-adjoint, \( x = x^\dagger, \ y = y^\dagger, \ w = w^\dagger \).

To find an explicit solution, we construct, cf. (4.10), angular 2-spinors \( \varphi^\mu_\pm \) and \( \Phi^\mu_\pm \), which are both eigenfunctions of \( \vec{J}^2 \) and \( J_3 \) with eigenvalues \( j(j + 1) \) and \( \mu \), and which diagonalize the operators \( x \) and \( y \):
\[ x\varphi^\mu_\pm = \mp | \gamma | \varphi^\mu_\pm \quad \text{and} \quad y\Phi^\mu_\pm = \mp | \gamma | \Phi^\mu_\pm. \]  
(5.20)
In the ‘lower’ sector, the coefficient of the \( r^{-2} \) term here is the square of the orbital angular momentum,
\[ x(x - 1) = \vec{L}^2 = L(\gamma)(L(\gamma) + 1), \]  
(5.21)
so that \( L(\gamma) \) is just the orbital angular quantum number. Due to the addition theorem of the angular momentum, if \( j \geq q + 1/2 \), \( L(\gamma) = j \pm 1/2 \), but for \( j = q - 1/2 \) the only allowed value of \( L(\gamma) \) is \( L(\gamma) = j + 1/2 \).

For \( j \geq q + 1/2 \) consider, therefore,
\[ \varphi^\mu_\pm = \sqrt{\frac{L(\gamma) + 1/2 + \mu}{2L(\gamma) + 1}} Y_{L(\gamma)}^{\mu - 1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \sqrt{\frac{L(\gamma) + 1/2 - \mu}{2L(\gamma) + 1}} Y_{L(\gamma)}^{\mu + 1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]  
(5.22)
where the \( Y \)'s are again the Wu-Yang [5] monopole harmonics, and the sign \( \pm \) refers to the sign of \( \gamma \). The \( \varphi \)'s satisfy\(^2\)
\[ \vec{J}^2 \varphi_\pm = j(j + 1) \varphi_\pm, \]  
(5.23)
\[ J_3 \varphi_\pm = \mu \varphi_\pm, \quad \mu = -j, \cdots, j, \]  
(5.24)
\[ \vec{L}^2 \varphi_\pm = L(\gamma)(L(\gamma) + 1) \varphi_\pm. \]  
(5.25)
\(^2\)the superscript \( \mu \) is dropped for the sake of simplicity.
Since $\vec{L} \cdot \vec{\sigma} = \vec{J}^2 - L^2 - 3/4$, we have

$$x \varphi_\pm = \left( \vec{L} \cdot \vec{\sigma} + 1 \right) \varphi_\pm = \mp \mid \gamma \mid \varphi_\pm,$$

as wanted.

For $j = q - 1/2$ no $\varphi_-$ (i.e. no $L(\gamma) = q - 1$) state is available, but eqn. 5.28 still yields $2(q - 1/2) + 1 = 2q$ $\varphi_0$'s with $L(\gamma) = q$, namely

$$\left( \varphi_0^0 \right)^\mu = \sqrt{q + 1/2 + \mu \over 2q + 1} Y_q^{\mu - 1/2} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \sqrt{q + 1/2 + \mu \over 2q + 1} Y_q^{\mu + 1/2} \left( \begin{array}{c} 0 \\ 1 \end{array} \right),$$

(5.27)

where $\mu = -(q - 1/2), \ldots, (q - 1/2)$. They are eigenstates of $x$ with eigenvalue $-q$.

The $y$-eigenspinors $\Phi$ of the ‘upper’ (i.e. $\rho_1 = 1$) sector are constructed indirectly. Assume first that one can find angular spinors $\chi_\pm$ which diagonalize $z = \vec{\sigma} \cdot \vec{L} + 1$,

$$z \chi_\pm = \pm \mid \kappa \mid \chi_\pm,$$

and also satisfy

$$\vec{J}^2 \chi_\pm^\mu = j(j + 1) \chi_\pm^\mu, \quad j = q - 1/2, q + 1/2, \ldots$$

(5.29)

$$J_3 \chi_\pm^\mu = \mu \chi_\pm^\mu, \quad \mu = -j, \ldots, j$$

(5.30)

and

$$w \chi_\pm^\mu = \chi_\pm^\mu.$$  

(5.31)

In the subspace spanned by the $\chi_\pm$'s, $x = z - qw$ and $y = z + qw$ have the remarkably symmetric matrix representations

$$[x] = \left( \begin{array}{cc} \mid \kappa \mid & -q \\ -q & \mid - \kappa \mid \end{array} \right) \quad \text{and} \quad [y] = \left( \begin{array}{cc} \mid \kappa \mid & q \\ q & \mid - \kappa \mid \end{array} \right).$$

(5.32)

The eigenvectors $\varphi_\pm$ and $\Phi_\pm$ of $x$ and $y$ with eigenvalues $\pm \mid \gamma \mid$ are thus

$$\varphi_+ = (\mid \kappa \mid + \mid \gamma \mid) \chi_+ - q \chi_-, \quad \varphi_- = q \chi_+ + (\mid \kappa \mid + \mid \gamma \mid) \chi_-,$$

$$\Phi_+ = (\mid \kappa \mid + \mid \gamma \mid) \chi_+ + q \chi_-, \quad \Phi_- = -q \chi_+ + (\mid \kappa \mid + \mid \gamma \mid) \chi_-.$$  

(5.33)

Expressing the $\chi$'s from the upper two equations in terms of the $x$-eigenspinors $\varphi$ yield the $z$-eigenspinors

$$\chi_+ = {1 \over 2 \mid \gamma \mid} \left( \varphi_+ + \frac{q}{\mid \gamma \mid + \mid \kappa \mid} \varphi_- \right), \quad \chi_- = -{1 \over 2 \mid \gamma \mid} \left( -\frac{q}{\mid \gamma \mid + \mid \kappa \mid} \varphi_+ + \varphi_- \right),$$

(5.34)

which do indeed satisfy (5.28). For $j = q - 1/2$, $\chi_-$ is missing and $\chi_+$ is proportional to the lowest $\varphi_0^0$ in (5.27).

Eliminating the $\chi$'s allows to deduce the $y$-eigenspinors $\Phi$ from the $x$-eigenspinors $\varphi$ according to

$$\Phi_+ = {1 \over \mid \gamma \mid} \left( \mid \kappa \mid \varphi_+ + q \varphi_- \right) \quad \text{and} \quad \Phi_- = {1 \over \mid \gamma \mid} \left( -q \varphi_+ + \mid \kappa \mid \varphi_- \right)$$

(5.35)

which, by construction, satisfy

$$\vec{J}^2 \Phi_\pm = j(j + 1) \Phi_\pm,$$

(5.36)

$$J_3 \Phi_\pm = \mu \Phi_\pm, \quad \mu = -j, \ldots, j.$$  

(5.37)

$$y \Phi_\pm = \mp \mid \gamma \mid \Phi_\pm.$$  

(5.38)
Finally, \( w = \vec{\sigma} \cdot \vec{r} / r \) interchanges the \( x \) and \( y \) eigenspinors,

\[
w \varphi_\pm^\mu = \Phi_\pm^\mu. \tag{5.39}\]

In contrast to what happens in the ‘lower’ (i.e. \( \rho_3 = -1 \)) sector, in the ‘upper’ (i.e. \( \rho_3 = 1 \)) sector

\[
y(y - 1) = \vec{L}^2 - 2\vec{\sigma} \cdot \vec{r} / r
\]

is not the square of an angular momentum and hence we do have \( L(\gamma) = q - 1 \) states: \( | \gamma | = q, \kappa = 0 \) for the lowest value of total angular momentum, \( j = q - 1/2 \), and for \( \gamma = -q \) eqn. (5.38) yields (5.27),

\[
\Phi_0 (= \Phi_-) = \varphi_0^0, \tag{5.40}
\]

while the entire \( \Phi_+ \) -tower is missing. This is a \((-1)\) eigenstate of \( w \),

\[
w \Phi_0 = -\Phi_0. \tag{5.41}\]

Since \( \varphi_0^0 \) is a \((-q)\) eigenstate of \( x \), \( \Phi_0 \) is an eigenstate of \( y = x + 2qw \) with eigenvalue \((+q)\).

Since

\[
\Gamma(\Gamma + 1) \Phi_\mu^\gamma = L(\gamma)(L(\gamma) + 1) \Phi_\gamma^\mu, \quad \Gamma(\Gamma + 1) \varphi_\gamma^\mu = L(\gamma)(L(\gamma) + 1) \varphi_\gamma^\mu, \tag{5.42}
\]

by construction, for \( j \geq q + 1/2 \) the eigenfunctions of \( \vec{D}^2 \) are found as

\[
\begin{align*}
\Psi_{\pm|\gamma|} &= u_\pm \begin{pmatrix} \Phi_\pm \\ 0 \end{pmatrix} \quad \text{for } a\rho_3 = 1, \\
\psi_{\pm|\gamma|} &= u_\pm \begin{pmatrix} 0 \\ \varphi_\pm \end{pmatrix} \quad \text{for } \rho_3 = -1
\end{align*}
\tag{5.43}
\]

where the radial functions \( u_\pm(r) \) solve the non-relativistic Coulomb-type equations

\[
\left[ -\left( \partial_r + \frac{1}{r} \right)^2 + \frac{L(\gamma)(L(\gamma) + 1)}{r^2} - \frac{2q^2}{r} + q^2 \right] u_\pm = E^2 u_\pm. \tag{5.44}
\]

By (5.16), these are just the upper (resp. lower) equations of

\[
- \left( \partial_r + \frac{1}{r} \right)^2 - \frac{2q^2}{r} + q^2 + \frac{1}{r^2} \left( (j - \frac{1}{2})(j + \frac{1}{2}) \right) (j + \frac{1}{2})(j + \frac{3}{2}) \tag{5.45}
\]

and hence

\[
u_\pm(r) \propto r^{L(\gamma)} e^{ikr} F \left( L(\gamma) + 1 - \frac{q^2}{k}; 2L(\gamma) + 2, -2ikr \right), \tag{5.46}\]

where \( k = \sqrt{E^2 - q^2} \).

For \( j = q - 1/2 \) we get the \((2q)\) spinors

\[
\psi_+ = u_+ \begin{pmatrix} 0 \\ \varphi_0^+ \end{pmatrix}, \quad \text{sign } \gamma = +1 \tag{5.47}
\]

in the \( \rho_3 = -1 \) sector with \( L(\gamma) = q^3 \), with \( u_+ \) still as in (5.36).
The energy levels are obtained from the poles of $F$,

$$L(\gamma) + 1 - i q^2 / k = -n, \ n = 0, 1, \ldots$$

Introducing the principal quantum number $p = L(\gamma) + 1 + n \geq q + 1$ we conclude that, in both $\rho_3$ sectors,

$$E_p = q^2 \left( 1 - \left( \frac{q}{p} \right)^2 \right), \quad p = q + 1, \ldots \quad (5.48)$$

The same energy is obtained if $L + n = L' + n'$. The degeneracy of a $p \geq q + 1$-level is hence $2(p^2 - q^2)$.

If $j = q - 1/2$, $(2q)$ extra states arise in the $\rho_3 = 1$ sector for $\gamma = -q$,

$$\Psi_0 = u_0 \begin{pmatrix} \Phi_0 \\ 0 \end{pmatrix} \quad \text{for } \rho_3 = 1 \quad \text{and} \quad \gamma = -q, \quad (5.49)$$

where $u_0$ solves (5.44) with $L(\gamma) = q - 1$. The principal quantum number is now $p = q$, yielding the $2q$-fold degenerate $0$-energy ground states. Since $F(0, a, z) = 1$, and the lowest $k$-value is $iq$, $u_0$ is simply

$$u_0 = r^{q-1} e^{-qr}, \quad (5.50)$$

cf. [13, 14]. The situation is shown in Figure 3-4:

Figure 3: The dyon spectrum in the $g = 0$ sector. The sign refers to that of $(-x)$. Each $j \geq q + 1/2$ sector is doubly degenerate. For $j = q - 1/2$ there are no $(-x) = -q$ states. The energy only depends on the principal quantum number $= L(\gamma) + 1 + n$.

### 6 Further applications

As yet another illustration, we consider a spin $\frac{1}{2}$ particle described by the four-component Hamiltonian

$$H = \begin{pmatrix} H_1 \\ H_0 \end{pmatrix} = \frac{1}{2} \left\{ q^2 - q \frac{\sigma \cdot \hat{r}}{r^2} + \lambda^2 \frac{\gamma^5 \sigma \cdot \hat{r}}{r^2} \right\} \quad (6.1)$$
Figure 4: The dyon spectrum in the $g = 4$ sector. The sign refers to that of $(-y)$. Each $j \geq q + 1/2$ sector is doubly degenerate. For $j = q - 1/2$ there are no $(-y) = +q$ states but $E = 0$ ground states arise for $(-y) = -q$.

where $\lambda$ is a real constant \[18\]. The Hamiltonian \[6.1\] can again be viewed as associated to a static gauge field on $\mathbb{R}^4$,

$$ A = qA_D, \quad A_4 = \lambda/r, \quad (6.2) $$

cf. \[5.2\]. The square of the associated Dirac operator

$$ \Psi = \begin{pmatrix} Q \\ Q^\dagger \end{pmatrix} = \begin{pmatrix} \sigma.\pi + i\frac{\lambda}{r} \\ \sigma.\pi - i\frac{\lambda}{r} \end{pmatrix}, \quad (6.3) $$

is precisely \[6.1\]. The partner hamiltonians of the chiral-supersymmetric Dirac operator have again the same spectra.

Much of the theory developed before in Sections \[4\] and \[5\] apply. The conserved total angular momentum is \[4.3\] and Dirac’s $K$ is again \[5.4\]. The supercharges $Q$ and $Q^\dagger$ can now be written as

$$ Q = -iw\left(\partial_r + \frac{1}{r} - \frac{y}{r}\right) = -i\left(\partial_r + \frac{1}{r} + \frac{x}{r}\right)w, \quad (6.4) $$

$$ Q^\dagger = -iw\left(\partial_r + \frac{1}{r} - \frac{x}{r}\right) = -i\left(\partial_r + \frac{1}{r} + \frac{y}{r}\right)w, \quad (6.4) $$

where

$$ x = z - \lambda w \quad \text{and} \quad y = z + \lambda w. \quad (6.5) $$

The Biedenharn operator, conserved for the quadratic dynamics, is now

$$ \Gamma = -(\sigma.\ell + 1 + \gamma^5 \lambda w) \quad \text{i.e.} \quad -(z + \gamma^5 \lambda w) \equiv -\begin{pmatrix} y \\ x \end{pmatrix}. \quad (6.6) $$

Since $\Gamma^2 = z^2 + \lambda^2 = J^2 + 1/4 + \lambda^2 - q^2$, the eigenvalues of $\Gamma$,

$$ \gamma = \pm \sqrt{(j + 1/2)^2 + \lambda^2 - q^2}, \quad \text{sign } \gamma = \text{sign } \kappa, \quad (6.7) $$
are in general again irrational. In terms of $\Gamma$, $\mathcal{D}^2$ is written
\[ \mathcal{D}^2 = \begin{pmatrix} Q^\dagger Q & \sqrt{Q} \end{pmatrix} = -(\partial_r + \frac{1}{r})^2 + \frac{\Gamma(\Gamma + 1)}{r^2}. \] (6.8)

The explicit solution.

The operator $\Gamma$ can be diagonalized as in Section 5, cf. [7, 8]. We get 2-spinors which diagonalize $x$ are
\[ \chi_+ = \frac{1}{2j+1} \begin{pmatrix} \varphi_+ \varphi_- \end{pmatrix} + \frac{q}{j+1/2|\kappa|} \] (6.9)
\[ \chi_- = \frac{1}{2j+1} \begin{pmatrix} -\varphi_+ \varphi_- \end{pmatrix} \]
where the $\phi_{\pm}$ are given in (5.22). Hence
\[ \phi_+ = (|\kappa| + j + \frac{1}{2}) \chi_+ - \lambda \chi_- \quad \phi_- = \lambda \chi_+ + (|\kappa| + j + \frac{1}{2}) \chi_- \] (6.10)
\[ \Phi_+ = (|\kappa| + j + \frac{1}{2}) \chi_+ + \lambda \chi_- \quad \Phi_- = -\lambda \chi_+ + (|\kappa| + j + \frac{1}{2}) \chi_- \]
diagonalize $\chi$ and $\psi$, hence
\[ x\phi_{\mu} = \mp |\gamma| \phi_{\mu} \quad \text{and} \quad y\Phi_{\mu} = \mp |\gamma| \Phi_{\mu}. \] (6.11)

The operator $w = \sigma \hat{r}$ interchanges the $x$ and $y$ eigenspinors,
\[ w\phi_{\mu} = \Phi_{\mu}. \] (6.12)

For $j = q - 1/2$, no $\phi_-$ is available and $\chi_-$ is hence missing. $\chi_+$ is proportional to the lowest $\varphi_+$ in (5.27). There are no $\phi_-$-states in the $\gamma_5 = -1$ sector and no $\Phi_+$ states in the $\gamma_5 = 1$ sector. However, in each $\gamma_5$ sector, (6.9) yields (2$\phi_0$) (+1)-eigenstates of $w$, namely
\[ (\phi_0^0)_{\mu} = (\Phi_0^0)_{\mu} \propto \sqrt{\frac{q + 1/2 + \mu}{2q + 1}} Y_{q}^{\mu-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{q + 1/2 + \mu}{2q + 1}} Y_{q}^{\mu+1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (6.13)

The eigenfunctions of $\mathcal{D}^2$ are then found as
\[ \begin{cases} \Psi_{\pm|\gamma|} = u_{\pm} \begin{pmatrix} \Phi_{\pm} \\ 0 \end{pmatrix} \quad \text{for} \quad \gamma_5 = 1 \\ \psi_{\pm|\gamma|} = u_{\pm} \begin{pmatrix} 0 \\ \phi_{\pm} \end{pmatrix} \quad \text{for} \quad \gamma_5 = -1 \end{cases} \]
for $j \geq q + 1/2$ (6.14)
\[ \begin{cases} \Psi_0 = u_0 \begin{pmatrix} \Phi_0 \\ 0 \end{pmatrix} \quad \text{for} \quad \gamma_5 = 1 \\ \psi_0 = u_0 \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} \quad \text{for} \quad \gamma_5 = -1 \end{cases} \]
for $j = q - 1/2$

Thus, the radial functions $u_{\pm}(r)$ solve
\[ \left[ -(\partial_r + \frac{1}{r})^2 + \frac{\gamma(\gamma + 1)}{r^2} - 2E \right] u_{\pm} = 0. \] (6.15)
This is the wave equation for a free particle except for the fractional ‘angular momentum’ \( \gamma \). Its solutions is hence given by the Bessel functions,

\[
 u_\pm(r) \propto r^{-1/2} J_{|\gamma|+\frac{1}{2}}(\sqrt{2E} \ r).
\]  

(6.16)

- For \( \lambda = 0 \) we recover the formulae in [19]. The well-known self-adjointness problem in the \( j = q - 1/2 \) sector shows up in that the eigenvalue \( \gamma \) vanishes in this case. (Self-adjointness of \( \hat{p}^2 \) requires in fact \( |\lambda| \geq 3/2 \) [18].)
- Another interesting particular value is \( \lambda = \pm q \), when the Biedenharn-Temple operator has half-integer eigenvalues,

\[
 \gamma = \pm (j + \frac{1}{2}).
\]  

(6.17)

In this case, \( \gamma(\gamma+1) \) is the same for \(-|\gamma|\) as for \(|\gamma|-1\), leading to identical solutions. Thus, the corresponding energy levels are two-fold degenerate. (This only happens for \(|\gamma| \geq |\gamma|_{\text{min}} + 1\) i.e. for \( j \geq q + 1/2 \). This can also be understood by noting that, for \( \lambda = \pm q \), the spin dependence drops out in one of the \( \gamma^5 \)-sectors. For \( \lambda = q \), e.g., the Hamiltonian (6.1) reduces to

\[
 H = \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \pi^2 + \frac{q^2}{r^2} - \frac{2q\sigma \cdot \vec{r}}{r^3} & \\ \frac{\pi^2 + q^2}{r^2} & \end{pmatrix},
\]  

(6.18)

i.e., \( H_0 \) describes a spin 0 particle, while \( H_1 = H_0 - 2q\sigma \cdot \vec{r}/r^3 \) corresponds to a particle with anomalous gyromagnetic ratio 4, cf. dyons in Section 5. The system admits hence an extra \( o(3) \) symmetry, generated by the spin vectors

\[
 S_0 = \frac{1}{2} \sigma \quad \text{for} \ H_0, \\
 S_1 = U^1 S_0 U \quad \text{for} \ H_1,
\]  

(6.19)

where \( U = Q/\sqrt{H_1} \) and \( U^{-1} = U^* = 1/\sqrt{H_1} \) are the unitary transformations which intertwine the non-zero-energy parts of the chiral sectors.

Each of the partner Hamiltonians \( H_1 \) and \( H_0 \) in (6.1) have a non-relativistic conformal \( o(2,1) \) symmetry [7] which combines, with \( \hat{p} \) and \(-i\gamma^5 \hat{p} \), into an \( osp(1/2) \) superalgebra [18].

The symmetries of the problem are studied in detail [18, 20].

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