Quantum entanglement constitutes one of the most fundamental, complex and counter-intuitive aspects of quantum mechanics. It is an essential resource in quantum information theory \cite{1}, playing a key role in quantum teleportation \cite{2} and computation \cite{1, 3, 4}. Studies of finite chains, of most interest for quantum information applications, are presently also motivated by the possibility of their controllable simulation through quantum devices \cite{3, 10}.

A remarkable feature of interacting spin chains is the possibility of exhibiting exactly separable ground states (GS) for special values of the external magnetic field, first discovered in \cite{11, 12} in a 1D XYZ chain with first neighbor coupling. It was recently investigated in more general arrays under uniform fields \cite{13, 14, 15, 16, 17, 18, 19}, with a completely general method for determining separability introduced in \cite{18}. Another remarkable related aspect is the fact that in the immediate vicinity of these separability points (SP) the entanglement between two spins can reach infinite range \cite{13, 17}. In \cite{17} we have shown that the SP in finite cyclic spin 1/2 arrays in a transverse field corresponds actually to a GS transition between opposite parity states (the last level crossing for increasing field), with the entanglement between any two spins reaching there finite side limits irrespective of the coupling range. In a small chain, this SP plays then the role of a “quantum critical point”. In contrast, the entanglement range remains typically finite and low at the conventional phase transition \cite{16}.

The aim of this work is to generalize previous results to XYZ arrays of arbitrary spins and geometry in a general transverse field, not necessarily uniform. Moreover, we will also determine the exact side limits of the entanglement between any two subsystems (including those for the block entropy and those for any two spins or group of spins) at the SP analytically, for any spin value. A non-uniform field will be shown to allow exact separability with infinite entanglement range in its vicinity in quite diverse systems (such as open or non-uniform chains), including the possibility of field induced alternating separable solutions along separability curves, with controllable entanglement side limits. Illustrative results for the negativity between the first and the jth spin in an open spin s chain for different values of s and j are as well presented.

We consider n spins $s_i$ (which can be regarded as qubits of dimension $d_i=2s_i+1\geq 2$) not necessarily equal, interacting through XYZ couplings of arbitrary range in the presence of a transverse external field $b^z$, not necessarily uniform. The Hamiltonian reads

$$H = \sum_i b^z_i s_i^z - \frac{i}{4} \sum_{i,j} (v^x_{ij} s_i^x s_j^x + v^y_{ij} s_i^y s_j^y + v^z_{ij} s_i^z s_j^z),$$ \hspace{1cm} (1)

and commutes with the global $S_z$ parity or phase-flip $P_z = \exp[i\pi \sum_{i=1}^n (s^+_i s^z_i)]$ for any values of $b^z$, $v^x_{ij}$ or $s_i$. Self-energy terms ($i = j$), non-trivial for $s_i \geq 1$, are for instance present in recent coupled cavity based simulations of arbitrary spin XXZ models \cite{10} and will be allowed if $s_i \geq 1$.

We now seek the conditions for which such system will possess a separable parity breaking eigenstate of the form

$$|\Theta\rangle = \otimes_{i=1}^n \exp[i\theta_i s_i^y]|0_i\rangle$$ \hspace{1cm} (2)

$$= \otimes_{i=1}^n \sum_{k=0}^{2s_i} \sqrt{\binom{2s_i}{k} \cos^{2s_i-k} \frac{\theta_i}{2} \sin^{k} \frac{\theta_i}{2}} |k_i\rangle,$$ \hspace{1cm} (3)

where $s_i^z|k_i\rangle = (k - s_i)|k_i\rangle$ and $e^{i\theta_i s_i^y}|0_i\rangle$ is a rotated minimum spin state (coherent state \cite{20}). The choice of $y$ as rotation axis does not pose a loss of generality as
any state $e^{i\Phi_{\pm}}|0_i\rangle$ corresponds to a suitable complex character $\theta_i$ in (2). Replacing $s_{ij}^z$ in (11) by $e^{-i\theta_i}s_{ij}^x e^{i\theta_i} s_{ij}^x$, i.e., $s_{ij}^x \rightarrow s_{ij}^x \cos \theta_i \pm s_{ij}^y \sin \theta_i$, $s_{ij}^y \rightarrow s_{ij}^y$, the equation $H(\Theta) = E_0|\Theta\rangle$, i.e., $H_0|0\rangle = E_0|0\rangle$ with $|0\rangle = \otimes_{i=1}^n |0_i\rangle$ and $H_0 = -i\sum \theta_i s_{ij}^x H e^{i\sum \theta_i s_{ij}^x}$, leads to the equations

\[
v_{ij}^y = v_{ij}^x \cos \theta_i \cos \theta_j + v_{ij}^y \sin \theta_i \sin \theta_j, \\
b_i^x \sin \theta_i = \sum_j (s_j - \frac{1}{2} \delta_{ij})(v_{ij}^x \cos \theta_j \sin \theta_j - v_{ij}^y \sin \theta_j \cos \theta_j),
\]

which determine, for instance, the values of $v_{ij}^y$ and $b_i^x$ in terms of $v_{ij}^x, v_{ij}^y, s_i$ and $\theta_i$. The energy is then given by

\[
E_0 = -\sum_i b_i^x \cos \theta_i + \frac{1}{2} \sum_j (s_j - \frac{1}{2} \delta_{ij})(v_{ij}^x \cos \theta_j \sin \theta_j - v_{ij}^y \sin \theta_j \cos \theta_j) + v_{ij}^z \cos \theta_i \cos \theta_j + \frac{1}{2}(v_{ii}^x + v_{ii}^y + v_{ii}^z).
\]

For a 1D spin $s$ cyclic chain with first neighbor coupling ($v_{ij}^x = v_{ij}\delta_{i,j+1}$) in a uniform field ($b^x = b$) we recover the original GS separability conditions of ref. [12] for both the ferromagnetic ($v_i \geq 0, \theta_i = \theta$) and antiferromagnetic ($v_i \leq 0, \theta_i = (-1)^i \theta$) cases. Eqs. (4)–(6) are however completely general and actually hold also for complex values of $\theta_i$. $v_{ij}^x$ and $b_i^x$ can be satisfied by $\Theta$ and $b$, respectively: If $s_i \equiv 0$ for some $i$, this eigenvalue is degenerate: $|\Theta\rangle$ will break parity symmetry and therefore, the partner state $| - \Theta\rangle = P_z|\Theta\rangle = \otimes_{i=1}^n \exp[-i\Theta_i s_i^y]|0_i\rangle$.

\[
| - \Theta\rangle = P_z|\Theta\rangle = \otimes_{i=1}^n \exp[-i\Theta_i s_i^y]|0_i\rangle,
\]

will be an exact eigenstate of $H$ as well, with the same energy $|E_0|$. The points in parameter space where the states $| \pm \Theta\rangle$ become exact eigenstates correspond necessarily to the crossing of at least two opposite parity levels.

For real $\theta_i$, Eq. (5) is just the stationary condition for the energy $|E_0|$ at fixed $b^x, v_{ij}^x$. The state (2) can thus be regarded as a mean field trial state, with Eq. (5) the associated self-consistent equation. Eq. (4), which is spin independent (at fixed $v_{ij}^x$), ensures that it becomes an exact eigenstate by canceling the residual one and two-site matrix elements connecting $|\Theta\rangle$ with the remaining states. Moreover, if $\theta_i \in (0, \pi)$ for all $i$,

\[
|v_{ij}^y| \leq v_{ij}^y \forall i, j,
\]

we can ensure that $| \pm \Theta\rangle$ will be ground states of $H$: In the standard basis formed by the states $\{ \otimes_{i=1}^n |k_i\rangle\}$, the terms in $H$ depending on $s_i^y$ are diagonal whereas the rest lead to real non-negative off-diagonal matrix elements, as $\sum_{x,y,z} s_{ij}^y = \sum_{x,y,z} s_{ij}^y = \sum_{x=\pm} s_{ij}^{x+y} \left( s_{ij}^x \delta_{ij} - s_{ij}^y \right)$, where $s_{ij}^x = s_{ij}^x \pm i s_{ij}^y$ and $v_{ij}^y = \frac{1}{2}(v_{ij}^x + v_{ij}^z) \geq 0$ by Eq. (4). Hence, $\langle H \rangle$ can be minimized by a state of all coefficients real and of the same sign in this basis (different signs will not decrease $\langle H \rangle$), which then, cannot be orthogonal to all $|\Theta\rangle$ (Eq. (4)). With suitable phases for $\theta_i$, $| \pm \Theta\rangle$ can also be GS in other cases: A $\pi$ rotation around the $z$ axis at site $i$ leads to $\theta_i \rightarrow -\theta_i$ and $v_{ij}^x \rightarrow -v_{ij}^x, v_{ij}^y \rightarrow v_{ij}^y$ for $i \neq j$.

**Definite parity eigenstates of $H$ in the subspace generated by the states $| \pm \Theta\rangle$ can be constructed as**

\[
|\Theta\rangle = \frac{|\Theta\rangle \pm | - \Theta\rangle}{\sqrt{2(1 + O_\Theta)}},
\]

\[
O_\Theta = (-\Theta|\Theta\rangle = \prod_{i=1}^n \cos^{2s_i} \theta_i,
\]

which satisfy $P_\Theta|\Theta\rangle = \pm|\Theta\rangle, \langle \Theta'|\Theta\rangle = \delta_{\Theta'\Theta}$. Here we have set $\theta_i \in (0, \pi/2)$ (and hence $O_\Theta \geq 0$). (11) follows also from $|\Theta\rangle = \otimes_{i=1}^n \cos^{s_i} \theta_i$.

\[
\Delta M_i = \langle \Theta^- | s_i^z \Theta^- \rangle - \langle \Theta^+ | s_i^z \Theta^+ \rangle = 2s_i \sin^2 \theta_i O_\Theta \cos \theta_i (1 - O_\Theta^2).
\]

(In contrast, $\langle \Theta^- | s_i^2 \Theta^- \rangle = -s_i \cos \theta_i$, when $|\Theta^\pm\rangle$ are GS, a GS parity transition $\langle \Theta^- \rangle \rightarrow |\Theta^+\rangle$, characterized by a magnetization step $\Delta M_i = \sum_{i,j} \delta_{\Theta ij}$, will thus take place at the state $|\Theta^- \rangle$, where $s_i \equiv 0$ for all $i$, across the factorizing values $E_\Theta$. If $\Delta E$ or $\Delta M$ can be resolved or measured, the realization of the states $|\Theta^\pm\rangle$ is then ensured. Their magnitude is governed by the overlap $|\langle \Theta^- | s_i^2 \Theta^- \rangle|$, appreciable in small systems (if $|\theta_i \neq \pi/2$), but as well in finite systems with small angles $\theta_i^2 \approx \delta_i/n$, such that $O_\Theta \approx e^{-\sum_i \delta_i n}$. This implies $\langle \Theta^- \rangle$ systems close to the XXZ limit $v_{ij}^x = v_{ij}^y$. In this limit ($\theta_i \rightarrow 0$), $\Delta M_i \rightarrow 0$, with $|\Theta^+\rangle \rightarrow |0\rangle$ and $|\Theta^-\rangle \rightarrow \prod_i \sqrt{\theta_i} |1_i\rangle$ (weighted $W$-type state), where $|1_i\rangle = \otimes_{j \neq i} |j\rangle$.

**Entanglement of definite parity states.** In contrast with $|\Theta\rangle$, the states $|\Theta^\pm\rangle$ are entangled. If $s_i \equiv 0$ for all $i$, the Schmidt number for any global bipartition $(A, \bar{A})$ is 2 and the Schmidt decomposition is

\[
|\Theta^\pm\rangle = \sqrt{\rho^\pm_{A\bar{A}}} |\Theta^\pm_{A}| |\Theta^\pm_{\bar{A}}\rangle + \sqrt{\rho^\pm_{A}} |\Theta^\pm_{\bar{A}}| |\Theta^\pm_{A}\rangle,
\]

\[
p^\pm_{A\bar{A}} = \frac{1 + v_{A\bar{A}} (1 + v_{A\bar{A}})}{2(1 + O_\Theta)}, \quad O_\Theta = \langle -\Theta|\Theta\rangle ,
\]

where $|\Theta^\pm_{A}\rangle, |\Theta^\pm_{\bar{A}}\rangle$ denote the analogous normalized definite parity states for the subsystems $A, \bar{A}$, with $v = \pm \Theta_A$, $O_\Theta = O_{\bar{A}}$ and $p^+_A + p^-_{\bar{A}} = 1$. Hence, $|\Theta^\pm\rangle$ can be effectively considered as two qubit states with respect to any bipartition $(A, \bar{A})$, with $|\Theta^\pm_{A}\rangle, |\Theta^\pm_{\bar{A}}\rangle$, representing the orthogonal states of each qubit. Accordingly, the reduced density matrix $\rho^\pm_A$ of subsystem $A$ in the state $|\Theta^\pm\rangle$ is

\[
\rho^\pm_A = \rho^\pm_{A\bar{A}} |\Theta^\pm_{A}| |\Theta^\pm_{A}\rangle + \rho^-_{A\bar{A}} |\Theta^\pm_{\bar{A}}| |\Theta^\pm_{\bar{A}}\rangle.
\]

The entanglement between $A$ and its complement $\bar{A}$ can be measured through the global concurrence (square
root of the tangle \[23\] \( C_{AA} = \sqrt{2(1 - \text{tr}\rho_A^2)} \), which
for a rank 2 density is just an increasing function of the
entanglement entropy \( E_{A\bar{A}} = -\text{tr}\rho_A \log_2 \rho_A \), with
\( C_{AA} = E_{A\bar{A}} = 0 \) (1) for a separable (Bell) state. In the
states \([11]\) we then obtain
\[
C_{AA}^\pm = \sqrt{1 - 2\theta_0^2}(1 - 2\theta_0^2) / (1 - O_0^2)\,.
\]
(14)

These values represent the side limits of \( C_{AA} \) at the SP.
For \( O_\theta > 0 \), \( C_{AA}^- > C_{AA}^+ \), with \( C_{AA}^- = 1 \) if \( O_A = O_{\bar{A}} \).
Note that \( |\Theta^\pm\rangle\) are simultaneous Bell states for \((A, \bar{A}) \), only
if \( O_A = O_{\bar{A}} = 0 \) (GHZ limit of \( |\Theta^\pm\rangle\)). Increasing
overlaps will in general decrease the global entanglement.

At the SP, the entanglement entropy of a block of \( L \)
spins in a 1D first neighbor spin 1/2 XY chain in a
constant field was found in \(23\) to be \( S_L = -\text{tr}\rho_L \log \rho_L = \ln 2 \)
(i.e., \( C_{LL} = E_{LL} = 1 \)) in the thermodynamic limit, in
agreement with Eq. (14) for vanishing overlaps. Eq.
\([14]\) extends this result to general finite chains, leading
to a slightly smaller value: For small \( O_A \), \( O_{\bar{A}} \),
\( C_{AA}^\pm = 1 - \frac{1}{2}(O_A \pm O_{\bar{A}})^2 \) and \( S_L^\pm \approx \ln 2 - \frac{1}{2}(O_L \pm O_{\bar{L}})^2 \)
(with \( O_L = (\pm)^{1/2} \) in the \( s = 1/2 \) XY chain).

Pairwise and subsystem entanglement. On the other
hand, the entanglement of a subsystem is enabled by non-
zero overlaps. A remarkable feature of the states \([9]\)
is that any two spins or disjoint subsystems \( B, C \) will also
be entangled if the complementary overlap \( O_{BC} \) is non-
zero and \( O_C^2 < 1 \), \( O_B^2 < 1 \). Moreover, this entanglement
can be characterized by the concurrence
\[
C_{BC}^\pm = \frac{\sqrt{(1 - O_B^2)(1 - O_C^2)\Omega_{BC(+\bar{C})}}}{1 \pm O_\theta},
\]
or equivalently, the negativity \(\[24, 25\],\)
\[
N_{BC}^\pm = \frac{1}{2}\sqrt{(p_{A^+}^2) + (C_{BC}^2) / \Omega_{BC(+\bar{C})} - p_{A^+}^2},
\]
(15)
where \( A = B + C \). While the concurrence of an
arbitrary mixed state \( \rho_A \) (which can be defined through
the convex roof extension of the pure state definition
\([20]\) is not directly computable in general (the exception
being the case of two qubits \([27]\)), the negativity
\( N_{BC} = \frac{1}{2}|\text{Tr}\rho_A^B| - 1 \), where \( \rho_A^B \) denotes partial transp-
ose \([28]\), can always be calculated \([29]\), being just the
absolute value of the sum of the negative eigenvalues of
\( \rho_A^B \). Eq. (10) represents then the side-limits of \( N_{BC} \) at the SP.

**Proof:** For \( A = B + C \), we first note that if \( O_\theta = 0 \),
Eq. (13) becomes \( \rho_A^B = \frac{1}{2} (|\Theta_A\rangle\langle\Theta_A| + |\Theta_A\rangle\langle-\Theta_A|) \),
i.e., \( \rho_A^B \) coincident and separable (convex combination of
product densities \([30]\)). Entanglement between \( B \) and \( C \)
can then only arise if \( O_{BC} \neq 0 \). Next, using similar
Schmidt decompositions \([11]\) of the states \( |\Theta_A\rangle\), Eq. (13)
can also be considered as an *effective two-qubit mixed
state* with respect to any bipartition \((B, C)\) of \( A \): Its
support will lie in the subspace spanned by the four states
\(|\Theta_B^\nu\rangle\langle\Theta_C^\nu|, \nu, \nu' = \pm \), such that
\[
\rho_A^B = \begin{pmatrix}
(p_{A^+}^B + q_{BC}^+) & 0 & 0 & p_{A^+}^B \alpha_{BC}^+

0 & p_{A^+}^O \alpha_{BC}^- & p_{A^+}^B \alpha_{BC}^+ & 0

0 & 0 & p_{A^+}^O \alpha_{BC}^- & p_{A^+}^B \alpha_{BC}^+

(p_{A^+}^O \alpha_{BC}^-) & 0 & p_{A^+}^B \alpha_{BC}^- & 0
\end{pmatrix},
\]
where \( q_{BC}^+ = (1 + v_0 \sigma_0)(1 \pm o_{BC}) \), \( O_{BC} = \sqrt{2 \sigma_{BC}^+ q_{BC}^-} \)
and \( q_{BC}^+ + q_{BC}^- = 1 \). \( \rho_A^B \) will be entangled if its
partial transpose has a negative *eigenvalue* \([28]\), a condition
here equivalent to a positive mixed state concurrence
\([27]\) \( C_{BC}^2 = \max[C_{BC}^+ \sigma_{BC}^+, 0] \), where \( C_{BC}^+ = 2[p_{A^+}^B \alpha_{BC}^+ - p_{A^+}^B \alpha_{BC}^-] \) represent parallel \((\nu = +)\)
or antiparallel \((\nu = -)\) concurrences, i.e., driven by \( |\Theta_A^\nu\rangle\)
or \( |\Theta_{\bar{A}}^\nu\rangle\) in Eq. (13). This leads to Eq. (15), with \( C_{BC}^2 \) \( C_{BC}^2 \)
(parallel antiparallel). The ensuing negativity, given here
by minus the negative eigenvalue of the partial transpose
\( \rho_{A^+B} \), is then given by Eq. (16).

For \( B = A, C = \bar{A} \), \((O_{BC} = 1) \), Eq. (15) reduces
to (13), with \( N_{AA} = \frac{1}{4}C_{AA}^2 \). For a pair of spins \( i \neq j \),
\( O_B = \cos^2 \theta_i \), \( O_C = \cos^2 \theta_j \), and the result of (17)
is recovered from (15) if \( s_i = \frac{1}{2} \) and \( \theta_i = \theta \ \forall i \). We finally
note that if \( O_B = O_C \), \( N_{BC}^2 = C_{BC}^2 / 2 \), as in the case of
a global partition. In general, however, there is no
proportionality between \( N_{BC}^2 \) and \( C_{BC}^2 \).

The concurrences (15) fulfill the monogamy inequalities
\([31]\) \( C_{BC, C+D}^2 \geq C_{BC}^2 + C_{BD}^2 \) for any three disjoint
subsystems \( B, C, D \). We actually obtain here
\[
C_{BC}^2 + C_{BD}^2 = C_{BC, C+D}^2[1 - \frac{1 - (O_{BC}^2)(O_{BD}^2)}{1 - O_{BC}^2 - O_{BD}^2}],
\]
(17)

Let us also remark that subsystem entanglement persis-
ting, though attenuated, in the uniform mixture
\[
\rho^0 = \frac{1}{2}(|\Theta^+\rangle\langle\Theta^+| + |\Theta^-\rangle\langle\Theta^-|),
\]
(18)
which differs from \( \frac{1}{2}(|\Theta\rangle\langle\Theta|) \) if \( O_\theta = 0 \)
and represents the \( \tilde{T} \rightarrow 0^+ \) limit of the thermal state
\( \rho \propto \exp^{-3HT} \) at the SP when \( \pm \Theta \) are GS (the GS
degeneracy there is 2). Replacing \( p_{A^+}^B \) by \( \frac{1}{2}(p_{A^+}^B + p_{A^+}^-) \) in
(13), we find now antiparallel global and subsystem
concurrences, given for any disjoint subsystems \( B, C \) by
\[
C_{BC}^0 = \frac{1}{2}(C_{BC}^- - C_{BC}^+) = C_{BC}^2 O_\theta / (1 + O_\theta),
\]
(19)
i.e., half the parity splitting of \( C_{BC} \). Eq. (19) remains
valid for a global bipartition \((B = A, C = \bar{A}) \). The
ensuing negativity can be similarly calculated.

The order of magnitude of subsystem concurrences is
governed by the complementary overlap \( O_{BC} \). For small
subsystems (like a pair of spins) in a large system, \( C_{BC}^2 \)
will be appreciable just for sufficiently small angles in
the complementary system, i.e., \( \theta^2_\delta \approx \delta_i / n \), such that
\( O_{BC} \approx e^{-\delta_i / n} \) remains finite. This leads again
to systems with small XY anisotropy.
Uniform Solution. Let us now examine the possibility of a common angle \( \theta_i = \theta \forall i \). Eq. (4) leads then to
\[
v_{ij}^y - v_{ij}^z = (v_{ij}^y - v_{ij}^z) \cos^2 \theta,
\]
(20)
implying a fixed ratio \( \chi = (v_{ij}^y - v_{ij}^z)/(v_{ij}^y - v_{ij}^z) = \cos^2 \theta \) for all pairs with \( v_{ij}^y \neq v_{ij}^z \), and an isotropic coupling \( v_{ij}^y = v_{ij}^z \) if \( v_{ij}^x = v_{ij}^z \). A subset of isotropic couplings will not spoil this eigenstate \[32\]. Eq. (5) implies then \( b^i \) arbitrary if \( \theta = 0 \) or \( \pi \) \((XXZ case \( v_{ij}^x = v_{ij}^z \)) or otherwise
\[
b^i = \cos \theta \sum_j (v_{ij}^y - v_{ij}^z)(s_j - \frac{1}{2} \delta_{ij}).
\]
(21)
The energy \( \mathcal{E}_\theta \) becomes
\[
\mathcal{E}_\theta = -\frac{1}{2} \sum_{i,j} s_i (v_{ij}^x + v_{ij}^y + v_{ij}^z) + \delta_{ij} v_{ij}^z].
\]
(22)
A general field allows then a uniform separable eigenstate (a global coherent state) in cyclic as well as open chains with arbitrary spins \( s_i \) in any dimension if \( (20) \) holds \( \forall i,j \). For instance, in an open 1D spin \( s \) chain with first neighbor couplings \( v_{ij}^z = v_\mu \delta_{i,j+1} \), Eq. (21) yields \( b^i = b_\mu = 2s \sqrt{(v_x - v_z)(v_x - v_z)} \) at inner sites but \( b^1 = b^0 = \frac{1}{2} b_\mu \) at the borders.

Eqs. \((20)\)–\((22)\) are actually valid for general complex \( \theta \), but real fields imply \( \cos \theta \) real \((\chi \geq 0)\). The case \( \cos^2 \theta > 1 \) (imaginary \( \theta \)) corresponds to a rotation around the \( z \) axis but can be recast as a rotation around the \( y \) axis by a global rotation around the \( x \) axis. Hence, we may set \( \cos^2 \theta \in [0, 1], |\pm \Theta| \) will then be GS when Eq. \( [8] \) holds.

The concurrence \((15)\) becomes, setting \( \cos^2 \theta = \chi \),
\[
C_{BC}^+ = \sqrt{(1 - \chi^2 S)(1 - \chi^2 S)} \chi^{S-1} (S_B + S_C)
\]
(23)
where \( S_B = \sum_{i \in B} s_i \) is the subsystem total spin and \( S = \sum_i s_i \) the total spin. It is independent of separation and coupling range, depending solely on \( \chi^2 \) and the ratios \( S_B/S, S_C/S \). If \( \chi = 1 - \delta/2S \), with \( \delta > 0 \) and finite, \( \chi^2 \approx e^{-\delta/2} \) remains finite for large \( S \). Eq. \( (23) \) leads then to \( O(1/\sqrt{S}) \) and \( O(1/S) \) global and subsystems concurrences for small \( S_A, S_B \) and \( S_C \):
\[
C_{AA}^+ \approx \sqrt{\frac{S_A \delta \sqrt{1 - e^{-\delta}}}{S - 1} + e^{-\delta/2}},
\]
(24)
\[
C_{BC}^+ \approx \frac{\delta}{S} \sqrt{\frac{S_B S_C e^{-\delta/2}}{S - 1} + e^{-\delta/2}}.
\]
(25)

On the other hand, for \( S_A = \frac{1}{2} S \), \( C_{AA}^- = 1 \) whereas \( C_{AA}^+ = \tanh \frac{\delta}{2} \). Thus, while for large \( \delta \) both \( C_{AA}^+ \) rapidly approach 1 as \( S_A \) increases, for small \( \delta \) \((XXZ limit\) this occurs just for \( C_{AA}^- \) and \( S_A \) close to \( S/2 \) (here \( \Theta^+ \rightarrow 0 \)) but \( |\Theta^-| \) approaches the \( W \)-type state \( \propto \sum_i \sqrt{s_i} |1_i\rangle \).

Alternating solution and controllable entanglement at the SP. Among other possibilities allowed by Eqs. \( (3)\)–\((3)\), let us examine that of a field induced two-angle solution in a 1D chain (cyclic or open) of spin \( s \) with first neighbor XY couplings \( v_{ij}^y = \delta_{i,j+1} v_\mu \), with \( v_\mu = 0 \). We assume \( \chi = v_y/v_x \in [0, 1] \). A separable eigenstate with \( \theta_\mu = \theta \), \( \theta_{2i-1} = \theta_\mu \) is feasible if there is an alternating field \( b^1 = b_\mu \), \( b^{2i-1} = b_0 \) in inner sites satisfying (Eqs. \( (4)\)–\((5)\))
\[
b_\mu b_0 = (2s)^2 v_x v_y.
\]
(26)
This leads to a transverse separability curve. The ensuing satisfies \( \cos \theta_\mu \cos \Theta = v_y/v_x \) and are given by
\[
\cos^2 \theta_\mu = \frac{b_\mu^2 + (2sv_y v_x)^2}{b_0^2 + (2sv_x v_y)^2}, \quad \sigma = o, e,
\]
(27)
being field dependent. For \( b_\mu = b_0 \) we recover the previous uniform solution \( (b_\mu = 2s \sqrt{v_y v_x}) \). In an open chain we should just add, according to Eq. \( (5) \), the border corrections \( b^i = \frac{1}{2} b_\mu, b^0 = \frac{1}{2} b_\mu \). The states \( |\pm \Theta| \) will then be GS setting \( \theta_\mu > 0 > v_x > 0 > \theta_\mu > 0 > \theta_\mu > 0 \), etc., in the antiferromagnetic case \( v_x > 0 \) (for even \( n \) if chain is cyclic, to avoid frustration).

The definite parity states \( |\Theta^\pm \rangle \) will again lead to infinite entanglement range, but with three different field dependent (and hence controllable) pairwise concurrences between any two spins (Eq. \((13)\) for \( B = i, C = j \)); even-even, odd-odd and even-odd, satisfying \( C_{oo}^\pm \geq C_{ee}^\pm > C_{oe}^\pm \) if \( |b_\mu| < |b_0| \). Hence, \( C_{oo}^\pm \) can be made larger than \( C_{ee}^\pm \) despite the absence of odd-odd direct coupling. For sufficiently large \( b_\mu \), \( \cos \Theta \approx \chi \) just odd-odd pairs will be appreciably entangled in this limit at the SP.

Application. As illustration, we first depict in fig. 1 full exact results for the GS negativities \( N_{ij} \) between spins \( i \) and \( j \) in a small open chain of uniform spin \( s \) with first neighbor XY couplings in a uniform transverse field \( b^i = b \) for \( i = 2, \ldots, n - 1 \), with the border corrections \( b^1 = b^n = \frac{1}{2} b_\mu \). For \( \chi = v_y/v_x \in (0, 1) \) this chain will then exhibit an exact factorizing field \( b_\mu = 2sv_x \sqrt{\chi} \) where separable parity breaking states with uniform angle \( \cos \theta = \sqrt{\chi} \) will become exact GS if \( v_x > 0 \) (if \( v_x < 0 \), \( \theta_\mu = (-1)^i \theta \) instead in the GS). We have set \( \chi = 1 - \delta/(2ns) \), such that the side limits of the negativity at \( b_\mu \) are roughly independent of \( s \) and \( n \). It is first seen that the ensuing behavior of the \( N_{ij} \) in terms of the scaled field \( b/b_\mu \) is quite similar for the three spin values considered \((s = 1/2, 1, 3/2)\), the latter involving a diagonalization in 65536 states for \( n = 8 \).

The GS exhibits \( n s \) parity transitions as the field is increased from \( 0^+ \) to \( b_\mu \), with the last transition at \( b_\mu \). As the latter is approached, it is verified that the pairwise entanglement range increases, with all negativities approaching the common side limits \((16)\), distinct at each side, given here by \( N_{ij}^+ = \frac{1}{2} C_{ij}^+ \approx \frac{4s^3 e^{-\delta/2}}{2n(1+e^{-\delta/2})} \) (Eq. \((26)\)) and \( N_{ij}^- \approx \frac{(C_{ij}^-)^2 e^{-\delta/2}}{4p_{ij}^A} \approx \frac{4s^3 e^{-\delta/2}}{4n(1+e^{-\delta/2})} \). An interval of full range pairwise entanglement around \( b_\mu \) is then originated, which involves on the left side essentially the last state.
Fig. 1: (Color online) Negativities between the first and the \(j\)th spin in an open spin \(s\) chain with first neighbor XY coupling, as a function of the transverse field \(b\), with border corrections (see text), and three different values of \(s\). We have set an anisotropy \(v_y/v_x = 1 - \delta/(2n)\), with \(\delta = 2.5\) and \(n = 8\) spins. The factorizing field corresponds to the last parity transition, and is singled out as the field where all negativities merge, approaching common non-zero distinct side limits. The lowest line (in red) depicts the end-to-end negativity \((N_{1-n})\) before the last transition (roughly an \(W\)-state). \(b_s\) plays in this small chain the role of a quantum critical field.

The side limits at separability can actually be modified in this system by changing the even-odd field ratio \(\eta = b_e/b_o\), according to Eq. (26). Results for a fixed ratio \(\eta = 10\) (with pertinent border corrections) are shown in Fig. 3 in which case separability is exactly attained at an odd field \(b_{os} = b_s/\sqrt{\eta}\). We have again plotted just the negativities between the first and the \(j\) spin, which now approach two common side limits at each side, one for \(j\) even \((N_{oe}^\pm)\) and one for \(j\) odd \((N_{oo}^\pm)\). While the former become quite small, the latter become clearly appreciable, the final effect for such large ratios being essentially that just odd sites become uniformly entangled in the vicinity of \(b_{os}\). Even-even negativities \(N_{ee}^\pm\) (not shown) are of course also very small at \(b_{os}\). Notice finally that \(N_{13}\) can become much larger than \(N_{12}\) in the region around \(b_{os}\), despite the absence of second neighbor couplings.

In summary, we have first determined the conditions for the existence of separable parity breaking (and locally coherent) eigenstates in general \(X Y Z\) arrays of arbitrary spins in a general transverse field, showing in particular the possibility of exact separability in open as well as non-uniform chains through non-uniform transverse fields. We have also determined the entanglement properties of the associated definite parity states, through the evaluation of the concurrence and negativity for any pair of spins or subsystems, for any spin values. These states, which approach both GHZ and \(W\)-states in particular limits, exhibit full entanglement range when non-orthogonal, and can be seen as effective two qubit entangled states for any bipartition. Moreover, the same holds for their uniform mixture as well as for the reduced density of any subsystem. The finite entanglement limits at the SP become relevant in finite arrays close to the \(X X Z\) limit, where the separability field can be clearly identified with the last GS parity transition, as verified in the nu-
Numerical results presented, playing the role of a quantum critical field. The possibility of exact separability in an alternating field \( b_{\eta} = \eta b_0 \) for arbitrary even-odd ratios \( \eta \), leading to controllable entanglement side-limits, has also been disclosed. The present results provide a deeper understanding of the behavior of pairwise entanglement in finite XYZ spin arrays subject to transverse fields.

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