Theory of tangential idealizers
and
tangentially free ideals

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Abstract. We generalize the theory of logarithmic derivations through a self-contained
study of modules here dubbed tangential idealizers. We establish reflexiveness criteria for
such modules, provided the ring is a factorial domain. As a main consequence, necessary
and sufficient conditions for their freeness are derived and the class of tangentially free ideals
is introduced, thus extending (algebraically) the theory of free divisors proposed by K. Saito
around 30 years ago.

1 Introduction

Derivations of commutative noetherian rings constitute a classical branch of research into
both commutative algebra and algebraic geometry. Modern papers reveal a renewed interest
on the theme as well as on various related topics, based upon foundational works of authors
like J.-P. Jouanolou, J. Lipman, Y. Nakai, K. Saito, A. Seidenberg and O. Zariski (cf. [15],
[17], [19], [21], [24], [23], [34]). The fundamental case of finite type algebras over a field
deserves special attention, as it relates closely to the setting of tangent vector fields (hence
of foliations, typically in the complex context) defined on algebraic varieties.

Loosely speaking, the general purpose of this paper is to contribute to the subject by
developing a self-contained study on distinguished submodules of whole derivation modules.
Such submodules, here called tangential idealizers, are naturally attached to ideals of the
(same) base ring — thus, heuristically, one might expect to catch their essence after some
type of ideal-theoretic inheritance. As it will be detailed, the main goal of such a study is
to extend and generalize Kyoji Saito’s theory of logarithmic vector fields and free divisors
([21]), which has deserved substantial attention since it was proposed around 30 years ago.

Before the first technicalities, let us say a few words on the influence of Saito’s fruitful
theory. Its legacy is present in lots of works published on the theme, where contributions
were given with diverse focuses, for instance on discriminants of mappings, versal unfoldings,
bifurcation varieties and arrangements of hyperplanes; central references are [2], [5], [9], [26],
[27], [28], [30] and [33]. Locally quasi-homogeneous free divisors are considered by Calderón-
Moreno and Narváez-Macarro in [7]. Koszul free divisors are treated in [6], where a study
is made on the logarithmic de Rham complex associated to a free divisor as well as the role
played by its cohomology (see also [7]). The concept of almost free divisor is defined by J.
Damon and applied to the study of topological properties of certain singularities (see, e.g.,
\[\text{[1]}\]

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We emphasize that the characterization features an *tangential idealizer* be a *free* which we denote by $Y$. They are tangent along into the new class of free varieties. Moreover, as the ambient variety the research that has been done on the case of free divisors should be somehow adapted in our sense, recover the concept of free divisor. Then, one might hope that a portion of Denote by $T$ zero (assumed algebraically closed for geometric simplicity), and let $T$ fully described. As it will be clear, the project landed on searching first the reflexiveness of $d$: smooth, our factorial setting generalized. Of more recent vintage is the notion of *linear free divisor* developed in [5] by R.-O. Buchweitz and D. Mond. From an algebraic viewpoint, Saito’s theory relies primevaly on studying modules formed with derivations preserving principal ideals in regular local rings, and mainly their freeness. Here, in essence, we ask about the same property for analogous modules defined from ideals (in more general rings) that are not necessarily principal. As it turned out, the elaboration of the theory we present here emerged step-by-step in order to solve the problem. It allows us to extend and generalize Saito’s theory, through the description of the tangentially free ideals — free ideals, for short — in suitable factorial domains.

Geometrically, one may speak about free varieties. To a glimpse at how they come around, fix an arithmetically factorial algebraic variety $X$ over a field $k$ of characteristic zero (assumed algebraically closed for geometric simplicity), and let $Y \subset X$ be a subvariety. Denote by $T_{X/k}(Y)$ the set of tangent vector fields defined on $X$ satisfying the property that they are tangent along $Y$ (at its smooth part, a natural necessity that we will always leave implicit). Then, equivalent conditions for the (geometric) tangential idealizer $T_{X/k}(Y)$ to be free as a module over the coordinate ring of $X$ will be clarified. In this case, we shall baptize $Y$ a free variety (in $X$).

Such setting extends Saito’s, as $Y$ is not necessarily a divisor in $X$. Free hypersurfaces, in our sense, recover the concept of free divisor. Then, one might hope that a portion of the research that has been done on the case of free divisors should be somehow adapted into the new class of free varieties. Moreover, as the ambient variety $X$ is not necessarily smooth, our factorial setting generalizes Saito’s, even though clearly in a different context, since he worked within the local complex analytic ambient. Further, the preparatory results we shall obtain — which may presumably raise independent interest — do not require the base ring to be factorial, which in turn, in several cases, is not even assumed to contain a field.

With the exception of a few geometric comments, our approach here is purely algebraic, starting from the main definition itself. Consider a commutative ring $R$ with identity and an $R$-module $M$. For an ideal $a \subset R$, a subring $k \subset R$ and an $R$-submodule $N \subset M$, we define the tangential idealizer (over $k$) of $a$ with respect to $N$ to be the set of all $k$-derivations $d: R \rightarrow M$ satisfying $d(a) \subset N$, that is, $d(x) \in N$ for every $x \in a$. This is an $R$-module, which we denote by $T_{R/k}^M(a, N)$, or simply $T_k(a, N)$ if no confusion arises.

The main situation arises when $M = R$ is the ring itself and $N = b$ is an $R$-ideal, yielding the module $T_k(a, b)$ consisting of the $k$-derivations conducting $a$ into $b$. In the collapsing case $a = b$, it is denoted $T_k(a)$, the (absolute) tangential idealizer of $a$. A direct explanation for our choice of such terminology comes from the geometric context where $a = a_X$ is the ideal of vanishing functions on a complex reduced algebraic variety $X$, for instance in a fixed affine space $A^n$. In this case, the module $T_C(a_X)$ collects the globally defined vector fields that are tangent along $X$, that is, $T_C(a_X) = T_{A^n/C}(X)$.

Back to our algebraic setting, consider a factorial domain $R$ that is of finite type or essentially of finite type over a field $k$ of characteristic zero, and let $a \subset R$ be an ideal with tangential idealizer $T_{R/k}(a)$ as defined above. Then, equivalent conditions for $T_{R/k}(a)$ to be a free $R$-module — in which case we say that $a$ is a (tangentially) free ideal — will be fully described. As it will be clear, the project landed on searching first the reflexivity of $T_{R/k}(a)$, that is, when the canonical map from $T_{R/k}(a)$ into its $R$-bidual is an isomorphism. We emphasize that the characterization features an effective side, as it will be possible to
handle mainly one of its conditions.

As it’s well-known, the idea of considering derivations preserving ideals is not new and has received different treatments from both commutative and non-commutative algebra, as well as notably from complex analytic geometry, with a view kept on Lie-theoretic properties of (modular) Lie algebras of vector fields. For instance, we point out the important role played by the so-called tangent algebra, which is the Lie algebra formed with the derivations fixing the defining ideal of an algebraic variety; it determines the isomorphism type of the variety (see [12] and references therein).

On the other hand, apart from focusing on the Lie algebra side, the topic has not been treated widely after J. Wahl’s major progress ([33]) on the case of quasi-homogeneous complete intersections with isolated singularities (see also [1]). The monomial case was studied in [3] (generalized in [29] for rings of differential operators), with an angle to algebraic combinatorics. In [25], an interpretation is given in the particular setup of polynomial principal ideals.

We now pass to a simplified covering of each section of the paper.

In Section 2, we give the general definition and start a systematic study on tangential idealizers. Initial module-properties (e.g., on the graded case), a basic exact sequence and a relative general notion of differential ideal (Definition 2.12) are dealt with. Various comparison results between tangential idealizers are investigated — some of them will turn out to be crucial for later results; some instances are Proposition 2.20 and Corollary 2.23. In Theorem 2.24 we obtain a primary decomposition for the tangential idealizer of an ideal without embedded component (the necessity of this hypothesis is shown in Example 2.25), and a consequence is noticed for whole derivation modules of algebras (essentially) of finite type over a field. A study is also made on tangential idealizers of ordinary and symbolic powers of an ideal, and of its radical as well (see, e.g., Proposition 2.33 and Remark 2.34(ii)).

Our main results are established in Section 3, which we divided into two parts.

In the first part 3.1 — where, for instance, results of K. Saito and H. Terao are improved — we exploit the module of abstract logarithmic derivations, which corresponds to the generalized principal ideal case. We explicit generators (Proposition 3.3) and establish that, quite generally, such modules are reflexive (Proposition 3.6). The so-called abstract free divisors (Definition 3.8) are characterized in Propositions 3.10 and 3.11 (the latter, in terms of Cohen-Macaulayness of abstract jacobian ideals; see Definition 3.1). Examples are given, a main one being Example 3.15, where a free divisor in the projective twisted cubic curve is exhibited, and where we illustrate that abstract jacobian ideals do not yield “true” jacobian ideals in general.

In the second part 3.2, we present the promised extension of K. Saito’s theory of free divisors. Our main result (Theorem 3.17) furnishes reflexiveness criteria for the tangential idealizer, provided the ring is a factorial domain (essentially) of finite type over a field containing the rationals. Freeness criteria will follow automatically (Theorem 3.22), so that we shall be in position to introduce the above mentioned class of free ideals (Definition 3.23), with respect to which the class of free divisors will turn out to be, in fact, a subclass — a crucial one, let us emphasize, in virtue of the role it plays into the structure result given in Theorem 3.22. A free ideal in the factorial ring of the hypersurface $x^2 + y^3 + z^5 = 0$ is exhibited in Example 3.25. A general family of free ideals is provided by Proposition 3.26.
Finally, in Section 4, a few simple geometric comments within the language of vector fields are made. In particular, the geometry of free varieties (algebraic varieties arisen from free ideals) is briefly explained.

Throughout this paper, all rings are tacitly assumed to be commutative with identity.

2 Tangential idealizers

We start with the general definition and a systematic study of the main object of interest. Several of its facets will be presented, to wit, basic module-properties, various comparison results and primary decomposition. A natural generalization of the well-known notion of differential ideal will be given conveniently, as it relates rather closely to our subject.

2.1 Definition and first properties

Let $R$ be a ring and $M$ be an $R$-module. Recall that a derivation of $R$ with values in $M$ is an additive map $d: R \to M$ with the Leibniz’s rule: $d(xy) = xd(y) + yd(x)$, for all $x, y \in R$, where, on the right side, the structural operation of $M$ as an $R$-module is meant. The set of all such derivations is usually denoted $\mathcal{D}er(R, M)$, which is clearly an $R$-module. If $k \subset R$ is a subring, we may consider the $R$-submodule $\mathcal{D}er_k(R, M) \subset \mathcal{D}er(R, M)$ consisting of the $k$-derivations of $R$ into $M$, that is, derivations $R \to M$ vanishing on $k$. In case $M = R$, the notation is simplified to $\mathcal{D}er_k(R)$. In general, in certain situations, one is concerned with the structure of $\mathcal{D}er_k(R)$ as a Lie algebra (with the usual Lie bracket: $[d_1, d_2] = d_1 d_2 - d_2 d_1$, for $d_1, d_2 \in \mathcal{D}er_k(R)$), but here we focus on its aspects as an $R$-module.

Definition 2.1 Let $a \subset R$ be an ideal and $N \subset M$ be an $R$-submodule. The tangential idealizer (over $k$) of $a$ with respect to $N$ is the set of all $k$-derivations $d: R \to M$ satisfying $d(a) \subset N$, that is, $d(x) \in N$ for every $x \in a$. We should denote it, in full notation, by $\mathcal{T}_k^M(a, N)$, but one may simply write $\mathcal{T}_k(a, N)$ whenever there is no risk of confusion — that is, once $R$-modules $N \subset M$ are fixed in the context, $\mathcal{T}_k(a, N)$ must not be regarded as a subfunctor of $\mathcal{D}er_k(R, N)$, unless one takes $N = M$, in which case, trivially, $\mathcal{T}_k(a, M) = \mathcal{D}er_k(R, M)$. We say that each of its elements conducts $a$ into $N$. If $M = R$ and $N = b$ is also an $R$-ideal, we write $\mathcal{T}_k(a, b)$, and if $a = b$, the notation is reduced to $\mathcal{T}_k(a)$, the (absolute) tangential idealizer of $a$. For any $d \in \mathcal{T}_k(a)$, we say that $a$ is $d$-invariant or that $d$ preserves $a$.

It’s clear that $\mathcal{T}_k(a, N)$ is an $R$-submodule of $\mathcal{D}er_k(R, M)$. Now, assume that $a \subset N: M = \{x \in R \mid xM \subset N\}$ (the annihilator of $M/N$ in $R$). Then, for any given $d \in \mathcal{D}er_k(R, M)$, the condition $d \in \mathcal{T}_k(a, N)$ is easily seen to be equivalent to $d(x) \in N$, for every element $x \in a$ in some (hence any, and not necessarily finite) set of generators of $a$. In fact, let $\{x_\alpha\}_{\alpha \in A}$ be a generating set of $a$ and let $x \in a$ be an arbitrary element. Since, in general, a (possibly infinite) sum of modules consists, by definition, of finite sums of elements, one gets an expression $x = \sum_{\alpha \in F} y_\alpha x_\alpha$, with $F \subset A$ a finite subset and $y_\alpha \in R$. Applying $d$ to this sum we obtain

$$d(x) = \sum_{\alpha \in F} y_\alpha d(x_\alpha) + \sum_{\alpha \in F} x_\alpha d(y_\alpha) \in N + aM = N,$$

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thus showing that $d \in \mathcal{T}_k(a, N)$. Notice that, in particular, there’s no dependence with respect to choice of generators of $a$. In the peculiar case when $a$ is a principal ideal, the corresponding tangential idealizer — which we may call \textit{module of abstract logarithmic derivations} (see \cite{3.1}) — will be written $\mathcal{T}_k(x)$, where $x$ is any single generator of $a$.

\textit{Torsion-freeness and rank.} First we investigate torsion-freeness of tangential idealizers as well as their rank as a module. For a ring extension $k \subset R$, the $R$-module of the $k$-derivations of $R$ with values in an $R$-module $M$ is easily seen to be torsion-free whenever $M$ is torsion-free (in particular, $\text{Der}_k(R)$ is torsion-free). Hence, as a submodule of $\text{Der}_k(R, M)$, the tangential idealizer $\mathcal{T}_k(a, N)$ is torsion-free for every ideal $a \subset R$ and every $R$-submodule $N$ of the torsion-free module $M$. As to the rank, we give a proposition below. Recall that a finitely generated module $M$ over a noetherian ring $R$ has (generic, constant) rank $r$ if there’s an isomorphism of $R_\wp$-modules $M_\wp \simeq R_\wp^r$ for each associated prime $\wp$ of $R$. Notation: $\text{rk}_R M = r$. Note that, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finitely generated $R$-modules such that two of them have a rank, then so has the remaining one, and $\text{rk}_R M_2 = \text{rk}_R M_1 + \text{rk}_R M_3$.

\textbf{Proposition 2.2} Let $k \subset R$ be a noetherian ring extension and $M$ be an $R$-module such that $\text{Der}_k(R, M)$ is finitely generated over $R$. Then, for any $R$-submodule $N \subset M$ and any $R$-ideal $a \subset N$: $M$ containing a regular element, the $R$-module $\mathcal{T}_k(a, N)$ has a well-defined rank if and only if $\text{Der}_k(R, M)$ has too. In this case, the ranks are equal.

\textbf{Proof.} Since $aM \subset N$, direct inspection shows that the cokernel $C$ of the inclusion $\mathcal{T}_k(a, N) \subset \text{Der}_k(R, M)$ is annihilated by $a$. As this ideal has positive grade, $C$ has rank 0 and the assertions follow. \hfill \Box

\textbf{Remark 2.3} Let $k$ be a field of characteristic zero and let $R$ be a finitely generated $k$-algebra. In this case, the $R$-module $\Omega_{R/k}$ of \textit{Kähler differentials} of $R$ over $k$ is presented by the (transposed) jacobian matrix, with entries seen in $R$, of any generating set of an ideal defining $R$. One knows that $\text{Der}_k(R) \simeq \text{Hom}_R(\Omega_{R/k}, R)$. It follows, in particular, that $\text{Der}_k(R)$ is finitely generated (see also \cite[Theorem 30.7]{18}). Now let $d > 0$ be the Krull dimension of $R$ and let $\mathfrak{d}_R \subset R$ be its \textit{jacobian ideal}, that is, the $d$-th \textit{Fitting ideal} of $\Omega_{R/k}$. One sees that $\text{Der}_k(R)$ has a well-defined rank, necessarily equal to $d$, if and only if $\mathfrak{d}_R$ has positive grade. Thus, applying the proposition above to such context, one concludes that the tangential idealizer $\mathcal{T}_k(a)$ of any ideal $a \subset R$ of positive grade has a well-defined rank if and only if $\mathfrak{d}_R$ contains a regular element. In particular, in case $R$ is a domain (\textit{e.g.}, a polynomial ring) and $a \subset R$ is any ideal, one has $\text{rk}_R \mathcal{T}_k(a) = d$ (clearly, the zero ideal also verifies the formula). For a simple illustration of the opposite situation, consider the 1-dimensional ring $R = k[s, t, u]$, with $st = su = u^2 = 0$. Its jacobian ideal $\mathfrak{d}_R = (s^2, tu)$ has grade zero. Hence, the $R$-module $\text{Der}_k(R)$ does not have a well-defined rank and consequently the tangential idealizers of its regular ideals do not have too.

Also notice that, in the setting of Proposition 2.2, if an ideal $a \subset N$: $M$ has grade at least 2, and if $T^*$ denotes the $R$-dual $\text{Hom}_R(T, R)$ of an $R$-module $T$, then there’s an isomorphism $\mathcal{T}_k(a, N)^* \simeq \text{Der}_k(R, M)^*$ (when $M = R$, this is the module of \textit{Zariski differentials of $R$ over $k$}; cf. \cite{34}).

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Graded tangential idealizers. Recall that, if \( R = \bigoplus_{i \in \mathbb{Z}} R_i \) is a \( \mathbb{Z} \)-graded ring and \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) is a \( \mathbb{Z} \)-graded \( R \)-module (one could also consider the context of gradings over general abelian semigroups with identity), then \( R_0 \subset R \) is a subring and each graded piece \( M_i \) is an \( R_0 \)-module. We want to verify that tangential idealizers inherit a grading from their ambient derivation modules. Note that a natural way to make \( \text{Der}_{R_0}(R, M) \) into a graded \( R \)-module is the following: for \( s \in \mathbb{Z} \), an \( R_0 \)-derivation \( d: R \to M \) is homogeneous of degree \( s \) if, for any \( i, d(x) \in M_{i+s} \) whenever \( x \in R_i \). Hence, the set \( \text{Der}_{R_0}(R, M)_s \) of all such derivations is an \( R_0 \)-module.

Proposition 2.4 Let \( R \) and \( M \) be as above. Let \( \mathfrak{a} \subset R \) be an homogeneous ideal and \( N \subset M \) be a graded \( R \)-submodule. Then

\[
\mathcal{T}^M_{R/R_0}(\mathfrak{a}, N) = \bigoplus_{s \in \mathbb{Z}} (\mathcal{T}^M_{R/R_0}(\mathfrak{a}, N) \cap \text{Der}_{R_0}(R, M)_s),
\]

that is, the tangential idealizer \( \mathcal{T}^M_{R/R_0}(\mathfrak{a}, N) \) is a graded \( R \)-submodule of \( \text{Der}_{R_0}(R, M) \).

Proof. Let \( d = \sum_{j=t}^r d_j, \ t \leq r, \) be an element of \( \text{Der}_{R_0}(R, M) \), with \( d_j \)'s homogeneous of degree \( j \). Assume that \( d \in \mathcal{T}_{R_0}(\mathfrak{a}, N) \). We need to verify that \( d_j \in \mathcal{T}_{R_0}(\mathfrak{a}, N) \) for each \( j \).

Pick \( x \in \mathfrak{a} \). Since \( \mathfrak{a} \) is homogeneous, one may assume that \( x \) is homogeneous, of degree, say, \( n \). One has \( d(x) \in N \), that is, \( \sum_{j=t}^r d_j(x) \in N \). As the \( d_j(x) \)'s are the homogeneous terms of \( d(x) \) (since \( d_j(x) \in M_{j+n} \)) and \( N \) is graded, it follows that \( d_j(x) \in N \) for each \( j \), as wanted.

Remark 2.5 A special situation comes when \( R \) is a polynomial ring in variables \( x_1, \ldots, x_n \) over a field \( k \). In this case, the \( R \)-module \( \text{Der}_k(R) \) is free, a basis being \( \{ \frac{\partial}{\partial x_i} \}_{i=1}^n \). One considers the standard grading \( R = \bigoplus_{i \geq 0} R_i \), where \( R_0 = k \). For \( s \in \mathbb{Z} \), an element of \( \text{Der}_k(R)_s \) is of the form \( \sum_{j=1}^n h_j \frac{\partial}{\partial x_j} \), with \( h_j \in R_{s+1} \) for all \( j \). Hence \( \text{Der}_k(R)_s = 0 \) for \( s < -1 \) (for much more on the (non)-existence of derivations of negative degree — or weight — over more general rings, see [32]). Now, Proposition 2.4 above guarantees that the tangential idealizer of any homogeneous ideal \( \mathfrak{a} \subset R \) may be generated by homogeneous derivations. Thus,

\[
\mathcal{T}_k(\mathfrak{a}) = \mathcal{T}_k(\mathfrak{a})_{-1} \oplus \mathcal{T}_k(\mathfrak{a})_0 \oplus \mathcal{T}_k(\mathfrak{a})_1 \oplus \cdots
\]

Quite often, the \( k \)-vector subspace \( \mathcal{T}_k(\mathfrak{a})_{-1} \subset \bigoplus_i k \frac{\partial}{\partial x_i} \) vanishes, but this is not always true. For instance, pick the homogeneous polynomial \( f = x^d y + x^d z \) in \( k[x, y, z] \), where \( d \) is any positive integer. Then, the algebraic vector field \( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \) belongs to \( \mathcal{T}_k(f)_{-1} \). A distinguished element of \( \text{Der}_k(R)_0 \) is the Euler (or radial) derivation \( \epsilon = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \). As \( \epsilon(f) = sf \) whenever \( f \in R_n \), it follows that \( \epsilon \in \mathcal{T}_k(\mathfrak{a})_0 \) for any homogeneous ideal \( \mathfrak{a} \). Such polynomial context is closely related to the theory of holomorphic foliations on complex projective spaces (some references will be suggested in the last section).

A basic exact sequence. Although possibly not so tight in general, the link from tangential idealizers to whole derivation modules goes in a very natural way:
Proposition 2.6 Let $k \subset R$ be a ring extension and $N \subset M$ be $R$-modules. Then, for any $R$-ideal $a \subset N$: $M$, one has an exact sequence of $R$-modules

$$0 \longrightarrow \text{Der}_k(R, N) \longrightarrow \mathcal{M}_{R/k}^M(a, N) \longrightarrow \text{Der}_k\left(\frac{R}{a}, \frac{M}{N}\right)$$

Proof. Set $M' = M/N$ and $R' = R/a$. The condition $a \subset N$: $M = 0$: $M'$ guarantees that $M'$ is an $R'$-module. Consider the surjection $\rho: M \twoheadrightarrow M'$. For each $d \in \mathcal{M}_{R/k}^M(a, N)$ we form the composite $\rho d: R \twoheadrightarrow M'$, which clearly belongs to $\mathcal{D}_k(R, M')$. Since $d(a) \subset N$, one has a well-defined induced application $d': R' \twoheadrightarrow M'$, given by $d'(x') = (\rho d)(x)$, with $x \in R$ and $x'$ denoting its image in $R'$. Moreover, $d'$ is easily seen to be a $k$-derivation. This process defines a natural map

$$\tau_{a, N}^M: \mathcal{M}_{R/k}^M(a, N) \longrightarrow \text{Der}_k(R', M')$$

which is $R$-linear and has kernel $\mathcal{D}_k(R, N)$. \hfill \Box

We furnish a simple consequence of the proposition above. For a ring $R$ and an $R$-module $M$, the set of associated primes of $M$ is denoted $\text{Ass}_R M$, and its support is $\text{Supp}_R M$. Just for the sake of completeness, recall the general definition of the module $\Omega_{R/k}$ of Kähler differentials of $R$ over a subring $k$, as being the conormal module $D/D^2$ of the diagonal ideal $D = \text{ker } \mu$, where $\mu: R \otimes_k R \to R$ is the multiplication map. Note that the ring isomorphism $(R \otimes_k R)/D \to R$ induced by $\mu$ makes $\Omega_{R/k}$ into an $R$-module. For any $R$-module $M$, one may identify $\text{Hom}_R(\Omega_{R/k}, M)$ with $\mathcal{D}_k(R, M)$, via the isomorphism $\phi \mapsto \phi d_{R/k}$ (for $R$-linear $\phi$: $\Omega_{R/k} \to M$), where $d_{R/k}: R \to \Omega_{R/k}$ is the universal derivation with which $\Omega_{R/k}$ comes equipped.

Corollary 2.7 In the general setting of Proposition 2.6 (and its proof, with the same notation), assume at least one of the following situations:

(i) $k$ is a field and $R'$ is a finite direct product of separable algebraic field extensions of $k$;

(ii) $k$ is noetherian, $R'$ is a finitely generated $k$-algebra and $(\Omega_{R'/k})_{\mathfrak{p}'} = 0$, for every $\mathfrak{p}' \in \text{Ass}_{R'} M'$.

Then, it follows an equality $\mathcal{M}_k(a, N) = \mathcal{D}_k(R, N)$, that is, any $k$-derivation $R \to M$ conducting $a$ into $N$ must send the whole $R$ into $N$.

Proof. (i) Writing $R' = \prod_{i=1}^s k_i$, one may use that $\Omega_{R'/k} = \prod_{i=1}^s \Omega_{k_i/k}$ and the fact that $\Omega_{k_i/k} = 0$ for each $i$, since $k_i$ is separable and algebraic over $k$ (for details, see [10]). Hence $\Omega_{R'/k} = 0$. Then, being isomorphic to $\text{Hom}_{R'}(\Omega_{R'/k}, M')$, the module $\mathcal{D}_k(R', M')$ vanishes, and the desired follows from the exact sequence given in Proposition 2.6.

(ii) In this situation, one sees that $R'$ is noetherian and that $\Omega_{R'/k}$ is a finitely generated $R'$-module. Hence, $\text{Ass}_{R'} \text{Hom}_{R'}(\Omega_{R'/k}, M') = \text{Supp}_{R'} \Omega_{R'/k} \cap \text{Ass}_{R'} M'$. The hypothesis on the module of differentials means that this intersection is empty. Then $\mathcal{D}_k(R', M') = 0$ and one again uses Proposition 2.6. \hfill \Box

A couple of comments on Proposition 2.6 is in order.
Remarks 2.8 (i) If \( a \subseteq R \) is any ideal annihilating \( M \), then every \( k \)-derivation \( R \to M \) vanishing on \( a \) gives rise, uniquely, to a \( k \)-derivation \( R/a \to M \). This is simply expressed in terms of the injective map

\[
\tau^M_{a,0} : \mathcal{T}^M_{R/k}(a,0) \hookrightarrow \mathcal{D}er_k(R/a, M)
\]

to which Proposition 2.6 is reduced if one takes \( N = 0 \). Although this seems to provide a natural shortcut, it would not be possible to write tangential idealizers in full generality; for instance, if moreover \( M \) is faithful (that is, \( 0: M = (0) \)), there would be nothing to see but the trivial fact that \( \tau^M_{(0),0} \) is the identity map \( \mathcal{D}er_k(R, M) \to \mathcal{D}er_k(R, M) \).

(ii) In the fundamental case of \( R \)-ideals \( a \subseteq b \), one gets an exact sequence

\[
0 \to \mathcal{D}er_k(R, b) \to \mathcal{T}_k(a, b) \xrightarrow{\tau^R_{a,b}} \mathcal{D}er_k \left( \frac{R}{a} : \frac{R}{b} \right).
\]

Denoting by \( \rho_a \) and \( \rho_b \) the projections \( R \to R/a \) and \( R \to R/b \), respectively, one is led to ask whether any given \( k \)-derivation \( \vartheta : R/a \to R/b \) satisfies \( \vartheta \rho_a = \rho_b d_\vartheta \), for some \( d_\vartheta \in \mathcal{T}_k(a, b) \) depending on \( \vartheta \). In other words, is the map \( \tau^R_{a,b} \) surjective? In general, when is \( \tau^M_{a,N} \) surjective?

We are now going to verify that the question raised in (ii) above is fulfilled when \( k \) is a field and \( R \) is a polynomial ring over \( k \) or a localization thereof — that is, residue class rings of \( R \) are algebras of finite type or essentially of finite type over \( k \). Moreover, it will be clear that, in the polynomial case, the operation of taking tangential idealizers commutes with formation of fractions. If \( S \) is a ring and \( M \) is a \( S \)-module, write \( M_U \) for the module of fractions of \( M \) with respect to a multiplicative set \( U \subseteq S \). Note that each \( \delta \in \mathcal{D}er(S, M) \) induces a derivation \( \delta_U \in \mathcal{D}er(S_U, M_U) \), given by \( \delta_U(\frac{s}{u}) = \frac{1}{u^2}(u\delta(s) - s\delta(u)) \), for \( s \in S \) and \( u \in U \).

Proposition 2.9 Let \( k \) be a field and \( R \) be a polynomial ring \( S = k[x_1, \ldots, x_n] \) or a ring of fractions of \( S \) with respect to a multiplicative set \( U \subseteq S \).

(i) Given ideals \( a \subseteq b \subseteq R \), one has an isomorphism of \( R/b \)-modules

\[
\mathcal{D}er_k \left( \frac{R}{a} : \frac{R}{b} \right) \cong \bigoplus_{i=1}^{n} \frac{b}{b} \frac{\partial}{\partial x_i} \subset \bigoplus_{i=1}^{n} \left( \frac{R}{b} \right) \frac{\partial}{\partial x_i}
\]

(ii) Given ideals \( a \subseteq b \subseteq S \), one has an isomorphism of \( S_U \)-modules

\[
\mathcal{T}_{S/k}(a, b)_U \cong \mathcal{T}_{S_U/k}(aS_U, bS_U).
\]

Proof. The proposed isomorphism of \( R/b \)-modules is well-known in case \( R = S \) and \( a = b \) (see, e.g., [3]). For ideals \( a \subseteq b \) the proof is essentially the same, which we present here for completeness. Write \( a = (f_1, \ldots, f_m) \) and recall that the module of differentials of \( S/a \) over \( k \) may be presented, in canonical bases, by the (transposed) jacobian matrix of the \( f_i \)'s, with entries seen in \( S/a \). After applying the functor \( \text{Hom}_{S/a}(-, S/b) \) to such presentation, one can identify the \( S/b \)-module of derivations \( \mathcal{D}er_k(S/a, S/b) \) with the kernel
of the induced map $(S/b)^n \to (S/b)^m$ whose matrix is the referred jacobian matrix, but now with entries in $S/b$. Then, denoting by $h'$ the image of any $h \in S$ in the residue class ring $S/b$, each $\vartheta \in \text{Der}_k(S/a, S/b)$ may be explicitly written as $\vartheta = \sum_j h'_j \frac{\partial}{\partial x_j}$ — meaning that its effect on any given $f \text{ mod } a$ is $\sum_j h'_j \frac{\partial f}{\partial x_j} = (\sum_j h_j \frac{\partial f}{\partial x_j})'$ — with the $h_j'$'s satisfying $\sum_j h'_j \frac{\partial f}{\partial x_j} = 0'$ for every $i$, that is, $\sum_j h_j \frac{\partial f}{\partial x_j} \in b$. Now, putting $d_\vartheta = \sum_j h_j \frac{\partial}{\partial x_j}$, one has $d_\vartheta \in \text{Der}_k(a, b)$; in other words, letting $\tau = \tau_{ab}$ be the homomorphism considered previously, one may write $\vartheta \tau = \tau(d_\vartheta)$ and then $\tau$ is surjective. As to the kernel of $\tau$, recall that $\ker \tau = \text{Der}_k(S, B)$. If $D_\vartheta = \sum_j g_j \frac{\partial}{\partial x_j} \in \text{Der}_k(S, b)$ then, for each $j$, one has $g_j = (\sum_i g_i \frac{\partial}{\partial x_i})(x_j) \in b$, showing that $\text{Der}_k(S, b) \subset b\text{Der}_k(S)$; the opposite inclusion being obvious, one gets $\ker \tau = b\text{Der}_k(S) = \oplus_{i=1}^n b \frac{\partial}{\partial x_j} \simeq b^{\oplus n}$. Thus, in case $R = S$, the desired isomorphism follows.

We now treat the case $R = S_U$. Note that $\{\left( \frac{\partial}{\partial x_i} \right) u \}_{i=1}^n$ is a basis for the free $S_U$-module $\text{Der}_k(S_U)$. If $f U \in \mathcal{T}_{S/k}(a, b) U \subset \text{Der}_k(S) U$, for some $d = \sum_j h_j \frac{\partial f}{\partial x_j} \in \mathcal{T}_k(a, b)$, one considers the derivation $d_U = \sum_j \left( \frac{h_j}{f} \right) \left( \frac{\partial f}{\partial x_j} \right) u \in \text{Der}_k(S_U)$. Set $a_U = a S_U$, $b_U = b S_U$. For $f \in a$ and $u \in U$, one has

$$
d_U \left( \frac{f}{u} \right) = \frac{1}{u} d(f) - \left( \frac{d(u)}{u^2} \right) f \in b_U + a_U = b_U,
$$

thus showing that $d_U$ acts $a_U$ into $b_U$. It follows an application $\kappa: \mathcal{T}_{S/k}(a, b) U \to \mathcal{T}_{S_U/k}(a_U, b_U)$ given naturally by $\kappa(v^{-1} d) = v^{-1} d_U$, for $d \in \mathcal{T}_k(a, b)$, $v \in U$, which is clearly a well-defined $S_U$-linear map that fits into a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & (b^{\oplus n})_U \\
\| & \downarrow \iota & \| \downarrow \kappa \\
0 & \longrightarrow & \mathcal{T}_{S/k}(a, b) U \\
\| & \downarrow \tau_{a, b} & \| \downarrow \iota \\
0 & \longrightarrow & \mathcal{T}_{S_U/k}(a_U, b_U) \\
\| & \downarrow \tau_{a_U, b_U} & \| \downarrow \kappa \\
0 & \longrightarrow & \text{Der}_k(S/a, S/b)_U \\
\| & \downarrow \iota & \| \downarrow \kappa \\
0 & \longrightarrow & \text{Der}_k(S/a_U, S/b_U)
\end{array}
$$

where $\iota$ and $\kappa$ are the natural isomorphisms. In the same way $\tau_{a, b}$ has kernel $b\text{Der}_k(S) \simeq b^{\oplus n}$, one also sees that $\ker \tau_{a_U, b_U} \simeq (b_U)^{\oplus n}$. We then conclude that $\kappa$ is an isomorphism (for instance, by the Snake Lemma) and hence that $\tau_{a_U, b_U}$ is surjective.

**Example 2.10** Let $C \subset \mathbb{P}^3$ be the twisted cubic in projective 3-space (over the complex number field $\mathbb{C}$), that is, the smooth curve given by the intersection of the quadrics defined by $f = y^2 - xz$, $g = yz - xw$ and $h = z^2 - yw$, where $x, y, z, w$ are homogeneous coordinates in $\mathbb{P}^3$. Write $\varphi = (f, g, h)$, the (prime) ideal of $C$ in the standard-graded polynomial ring $S = \mathbb{C}[x, y, z, w]$. The $S$-module $\mathcal{T}_C(\varphi) \subset S_{x2} \oplus S_{y2} \oplus S_{z2} \oplus S_{xw} \simeq S^4$ is minimally generated by the column-vectors of the matrix

$$
\mathfrak{A}_C = \begin{pmatrix}
0 & 0 & 3y & x & 3z^2 & 0 & 0 & 0 & 0 & 0 \\
x & y & 2z & y & 2zw & z^2 & 0 & 0 & 0 & 0 \\
2y & 2z & w & z & w^2 & 2zw & h & 0 & 0 & 0 \\
3z & 3w & 0 & w & 0 & 3w^2 & 0 & h & g & f
\end{pmatrix},
$$

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which thus define ambient vector fields — hence foliations on \( \mathbf{P}^3 \) — under which \( C \) is invariant. As previously guaranteed by Proposition 2.9, such generators are homogeneous, and by Proposition 2.10 they are representatives (liftings) for the generators of the \( S/\wp \)-module \( \text{Der}_C(S/\wp) \) of the tangent vector fields on \( C \). Hence, eliminating redundancy and setting \( R = S/\wp \), one sees that, minimally,

\[
\text{Der}_C(R) = R\overline{d}_1 + R\overline{d}_2 + R\overline{d}_3 + R\overline{\epsilon}
\]

where \( \overline{d}_j \) denotes residue class with respect to \( \wp^{\otimes 4} \), \( \epsilon \) is the Euler vector field on \( S \), and \( d_1, d_2, d_3 \) are given respectively by the first 3 column-vectors of \( \mathfrak{A}_C \) (notice that they all have degree 0 as elements of \( \text{Der}_C(R) \)). See Example 3.15, where these generators are used in order to exhibit a free divisor in \( C \).

**Example 2.11** Let \( S = \mathbb{C}[x,y,z] \) be a standard-graded polynomial ring, and consider the complete intersection ring \( R = S/\wp = S/(f,g) \), where \( f = x^2y - z^3 \) and \( g = y^3 - x^2z^2 \). The \( S \)-module \( \mathcal{I}_C(\wp) \subset S \frac{\partial}{\partial x} \oplus S \frac{\partial}{\partial y} \oplus S \frac{\partial}{\partial z} \) \( \simeq S^3 \) is minimally generated by the column-vectors of the matrix

\[
\mathfrak{A}_\wp = \begin{pmatrix}
y^3z & x & z^3 & y^4 & 0 & 0 & 0 \\
xz^3 & y & x^2y & xyz & f & 0 & g \\
xz^3 & z & xyz & xz^3 & f & 0 & g
\end{pmatrix}.
\]

Reducing modulo \( \wp \) and applying Proposition 2.9 one sees that \( \text{Der}_C(R) = R\overline{d} + R\overline{d} \), where \( \overline{d} \) denotes residue class with respect to \( \wp^{\otimes 3} \) and \( d \) is given by the first column-vector of \( \mathfrak{A}_\wp \) (note that it has degree 3).

Relative differential ideals. Let \( k \subset R \) be a ring extension. An ideal \( \mathfrak{a} \subset R \) is said to be \((k-)differential\) if \( d(\mathfrak{a}) \subset \mathfrak{a} \), for every \( d \in \text{Der}_k(R) \). The classical reference, in case \( R \) is an algebra of finite type containing the field of rational numbers, is A. Seidenberg’s paper [24].

As the differentiability of \( \mathfrak{a} \) can be expressed by an equality \( \mathcal{I}_k(\mathfrak{a}) = \text{Der}_k(R) \), the theory of differential ideals is naturally related to the concept of tangential idealizer. Then, it does not seem senseless to develop a piece of investigation inside our theory. We first propose, in generality, a naive relative version of this well-known notion.

**Definition 2.12** Let \( k \subset R \) be a ring extension, \( M \) an \( R \)-module and \( \mathfrak{a} \subset R \) an ideal. For a subset \( \sigma \subset M \), we say that \( \mathfrak{a} \) is a \((k-)differential ideal with respect to \( \sigma \) if every \( k \)-derivation \( \sigma \to M \) conducts \( \mathfrak{a} \) into \( \sigma \). When \( \sigma = N \) is an \( R \)-submodule, this means that \( \mathcal{I}_k(\mathfrak{a},N) = \text{Der}_k(R,M) \). In the fundamental particular case \( M = R \), any ideal which is differential with respect to itself will be simply called differential, in accordance with the traditional terminology.

**Examples 2.13** Let us see some initial examples (in case \( M = R \)). Clearly, the trivial ideals \( (0) \) and \( (1) \) are differential. If \( \{d_{\alpha}\}_{\alpha \in \mathcal{A}} \) is a generating set for the module of \( k \)-derivations of \( R \) and if \( x \in R \) is any element, then the principal ideal \( (x) \) is differential with respect to the corresponding gradient ideal \( (d_{\alpha}(x))_{\alpha \in \mathcal{A}} \subset R \). For any ideal \( \mathfrak{a} \subset R \) and any integer \( r \geq 1 \), direct use of Leibniz’s rule yields that the \( r \)-th power \( \mathfrak{a}^r \) of \( \mathfrak{a} \) is differential with respect to its \( (r-1) \)-th power \( \mathfrak{a}^{r-1} \) (this simple fact shows that every derivation of \( R \) is \( \mathfrak{a} \)-adically continuous, and hence induces a derivation of the completion of \( R \) with respect to
the $\mathfrak{a}$-adic topology). If $R$ is a finitely generated reduced algebra over a field of characteristic zero, and has finite integral closure $\overline{R}$, then a classical result of Seidenberg (cf. [23]) states that the conductor ideal $R: \overline{R} = \{ x \in R \mid x \overline{R} \subset R \}$ is differential (for a relative situation involving integral closure of ideals, see Example 2.16). Several other instances will be given later in 2.2.

**Remark 2.14** Obviously, the ideal $(0)$ is differential with respect to any submodule $N \subset M$. Then, by the exact sequence of Proposition 2.6 we simply recover the rather tautological short exact sequence

$$0 \longrightarrow \text{Der}_k(R, N) \longrightarrow \text{Der}_k(R, M) \longrightarrow D^\rho_k(R) \longrightarrow 0$$

associated to the projection $\rho: M \rightarrow M' = M/N$, where $D^\rho_k(R) \subset \text{Der}_k(R, M')$ stands for the submodule consisting of the $k$-derivations $d': R \rightarrow M'$ such that $d'$ can be factored as $d' = \rho d$, for some $d \in \text{Der}_k(R, M)$ (it would be of interest to investigate when every derivation with values in $M'$ can be factored in this way, that is, $D^\rho_k(R) = \text{Der}_k(R, M')$).

As one expects, the roles played by the submodules $T_{M/k}^M(a, 0) \subset \text{Der}_k(R, M')$ and $T_{R/k}^M(a, N) \subset \text{Der}_k(R, M)$ are similar, since they “differ” only by the projection $\rho$. We now study the relationship between them by means of giving a description of the cokernel $\mathcal{C}_{a,N}$ of the latter inclusion, which thus measures how far the ideal $a$ is from being differential with respect to $N$ (see Proposition 3.5 for the simple case where $M = R$ and $a$ is a principal ideal).

**Proposition 2.15** Keep the notation above.

(i) One has a short exact sequence of $R$-modules

$$0 \longrightarrow \text{Der}_k(R, N) \longrightarrow \mathcal{T}_{R/k}^M(a, N) \longrightarrow \mathcal{T}_{R/k}^{M'}(a, 0) \cap D^\rho_k(R) \longrightarrow 0$$

(ii) There’s an isomorphism of $R$-modules

$$\mathcal{C}_{a,N} \cong \frac{\mathcal{T}_{R/k}^{M'}(a, 0) + D^\rho_k(R)}{\mathcal{T}_{R/k}^M(a, 0)}$$

(iii) The ideal $a$ is differential with respect to $N$ if and only if $D^\rho_k(R) \subset \mathcal{T}_{R/k}^{M'}(a, 0)$, that is, every factored derivation with values in $M'$ vanishes on $a$.

(iv) If $a$ is differential with respect to $0 \subset M'$ then $a$ is differential with respect to $N$. The converse holds if $D^\rho_k(R) = \text{Der}_k(R, M')$ (the condition mentioned at the end of Remark 2.14).

**Proof.** (i) Notice that $\rho$ induces an $R$-linear application $\mathcal{T}_{R/k}^M(a, N) \rightarrow \mathcal{T}_{R/k}^{M'}(a, 0)$. It has kernel $\text{Der}_k(R, N)$ and was implicitly used in Proposition 2.6 as a component of the map $\mathcal{T}_{a,N}^M$, since it is clearly seen to be the restriction to $\mathcal{T}_{R/k}^M(a, N)$ of the natural surjection $\text{Der}_k(R, M) \rightarrow D^\rho_k(R)$. The desired follows.
(ii) One easily compares the short exact sequence obtained in (i) above with the one mentioned in Remark 2.14. By the Snake Lemma, one gets

$$\mathcal{C}_{a,N} \cong \frac{\mathcal{D}_k^\rho(R)}{\mathcal{D}_k^\rho(R) \cap \mathcal{T}_{R/k}(a,0)}$$

and the wanted follows.

(iii) Since $a$ is differential with respect to $N$ if and only if $\mathcal{C}_{a,N} = 0$, the assertion follows directly from the isomorphism of (ii).

(iv) It follows, from (ii), an injection of cokernels

$$\mathcal{C}_{a,N} \hookrightarrow \frac{\mathcal{D}_k(R, M')}{\mathcal{T}_{R/k}(a,0)}$$

which implies that, if $a$ is differential with respect to $0 \subset M'$, then $\mathcal{C}_{a,N} = 0$. For the converse, if $\mathcal{C}_{a,N} = 0$ and $\mathcal{D}_k(R) = \mathcal{D}_k(R, M')$, then again by (ii) (or directly by (iii)) one gets that $\mathcal{T}_{R/k}(a,0) = \mathcal{D}_k(R, M')$, as wanted. \(\square\)

Example 2.16 Let $S$ be a noetherian domain, with finite integral closure, containing a field $k$ of characteristic zero. For any ideal $a \subset S$ and any integer $r \geq 0$, write $a^r$ for the integral closure of the $r$-th power of $a$ (recall that the integral closure of an ideal $b$ is the ideal $\overline{b}$ formed with the elements $z$ such that $z^n + \sum_{i=1}^{n} a_i z^{n-i} = 0$, for some $n \geq 1$ and elements $a_i$'s such that $a_i \in b^i$). A result of B. Ulrich and W. Vasconcelos (see [31, Theorem 7.14]) asserts that, for every $k$-derivation $d: S \to S$, one has $d(a^{r+1}) \subset a^r$. Hence $\mathcal{T}_k(a^{r+1}, a^r) = \mathcal{D}_k(S)$, that is, the ideal $a^{r+1}$ is differential with respect to $a^r$. Then, if for any given $r$ one denotes by $A_r$ the ring $S/a^r$, which is a $S$-module as well as an $A_{r+1}$-module, one concludes that every $k$-derivation $\overline{d}: S \to A_r$ that may be factored as $S \xrightarrow{d} S \xrightarrow{\rho} A_r$ (where $d \in \mathcal{D}_k(S)$ and $\rho$ is the projection) must necessarily vanish on $a^{r+1}$. This fact illustrates Proposition 2.15(iii). Also notice that, if one takes $S$ to be a polynomial ring in indeterminates $x_1, \ldots, x_n$ over $k$, Proposition 2.9 gives $\mathcal{D}_k(A_{r+1}, A_r) = \bigoplus_{i=1}^{n} A_r \frac{\partial}{\partial x_i}$, a free $A_r$-module of rank $n$.

2.2 Comparing tangential idealizers

Our objective now is to compare tangential idealizers of ideals that are related somehow, with an angle kept on the theory of differential ideals. Given a ring extension $k \subset R$ and ideals $a \subset b$, we begin asking whether $\mathcal{T}_k(a) \subset \mathcal{T}_k(b)$ or $\mathcal{T}_k(b) \subset \mathcal{T}_k(a)$, that is, whether the operation of taking tangential idealizers preserves or reverts inclusion. As it will be clear, there is not a general rule in any direction, but not so rarely one can detect well-behaved situations. For instance, we shall see that primary decompositions of ideals without embedded components are preserved.

First, as a matter of preliminary illustration, we put some trivial facts into terms of tangential idealizers.

Proposition 2.17 Let $k \subset R$ be a ring extension and let $\{a_\alpha\}_\alpha$ be a family of $R$-ideals. Then:
(i) $\bigcap \mathcal{T}_k(a_\alpha) \subset \mathcal{T}_k(\bigcap a_\alpha)$.

(ii) $\bigcap \mathcal{T}_k(a_\alpha) \subset \mathcal{T}_k(\bigcap a_{\alpha_1}, \ldots, \bigcap a_{\alpha_n})$; in particular, one has $\mathcal{T}_k(a_\alpha) + \mathcal{T}_k(a_\beta) \subset \mathcal{T}_k(a_\alpha, a_\beta)$ whenever $a_\alpha \subset a_\beta$.

(iii) $\mathcal{T}_k(\bigcap a_\alpha) = \bigcap \mathcal{T}_k(a_\alpha, a_\beta)$; in particular, if $\{x_\alpha\}_\alpha$ is a generating set for an ideal $a \subset R$, one has $\mathcal{T}_k(a) = \bigcap \mathcal{T}_k(x_\alpha, a)$.

**Proof.** Direct verification. $\square$

As an immediate consequence, an arbitrary intersection of differential ideals is also differential. Moreover, an arbitrary sum $a$ of ideals $a_\alpha$’s is differential if and only if each $a_\alpha$ is differential with respect to $a$.

The fact below was first noticed by I. Kaplansky (cf. [16]) and we state it here in virtue of its usefulness.

**Proposition 2.18** Let $R$ be a noetherian ring containing a field $k$ of characteristic zero, and let $a \subset R$ be an ideal. Then,

$$\mathcal{T}_k(a) \subset \mathcal{T}_k(\sqrt{a}).$$

In particular, the ni-radical $\sqrt{(0)}$ of $R$ is a differential ideal.

**Proof.** Take $d \in \mathcal{T}_k(a)$ and set $z = d(x)$, for a given $x \in \sqrt{a}$. We must show that $z \in \sqrt{a}$. From $x^n \in a$ (for some positive integer $n$) it follows that $d(x^n) = nx^{n-1}z \in a$. Since $d(d(x^n)) \in a$, one has $n(n-1)x^{n-2}z^2 + nx^{n-1}d(z) \in a$, that is, $x^{n-2}z^2 + x^{n-1}y_1 \in a$, where $y_1 = \frac{d(z)}{n-1}$. Applying $d$ again (if necessary) yields $x^{n-3}z^3 + x^{n-2}y_2 \in a$, where $y_2 = \frac{1}{n-2}(3zd(z) + \frac{d(d(z))}{n-1})$. Continuing in this way, one finds $z^n + xy_{n-1} \in a$, for some $y_{n-1} \in R$. But then, as $x \in \sqrt{a}$, the desired follows. $\square$

Later on, in Proposition 2.33 we shall give a large class of ideals $a$ for which $\mathcal{T}_k(a)$ equals $\mathcal{T}_k(\sqrt{a})$. As one knows, this is not true in general, as simple examples show.

**Example 2.19** Pick the polynomial ideal $a = (x^2, xy, yz^2) \subset k[x, y, z]$ ($k$ is an arbitrary field). The $k$-derivation $d = yz \frac{\partial}{\partial x}$ preserves $\sqrt{a} = (x, yz)$, but $d(xy) = y^2z \notin a$.

**Basic comparisons.** In this part, a ring extension $k \subset R$ is fixed. Given ideals $a \subset b \subset R$, we consider the situation where $R/b$ has positive depth with respect to the colon ideal $a/b = \{x \in R \mid xb \subset a\}$, the annihilator of $b/a$ in $R$. In the next step we will look at product ideals.

**Proposition 2.20** Let $a \subset b \subset R$ be ideals such that $a/b$ contains an $R/b$-regular element. Then

$$\mathcal{T}_k(a, b) = \mathcal{T}_k(b).$$

In particular, $\mathcal{T}_k(a) \subset \mathcal{T}_k(b)$; moreover, $b$ is differential if and only if $a$ is differential with respect to $b$. 

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Proof. Since \( a \subset b \), the inclusion \( \mathcal{I}_k(b) \subset \mathcal{I}_k(a,b) \) is obvious. Now pick any \( d \in \mathcal{I}_k(a,b) \) and \( y \in b \). We want to show that \( d(y) \in b \). By hypothesis there exists \( x \in a: b \) which is regular modulo \( b \). Since \( xy \in a \) we have \( d(xy) \in b \), which means that \( yd(x) + xd(y) \in b \), hence \( xd(y) \in b \) and necessarily \( d(y) \in b \), since \( x \) does not divide zero in \( R/b \). This shows the proposed equality. In particular we have \( \mathcal{I}_k(a) \subset \mathcal{I}_k(b) \), since clearly \( \mathcal{I}_k(a) \subset \mathcal{I}_k(a,b) \).

Remark 2.21 A typical situation where one can use Proposition 2.20 is when \( R \) is a noetherian ring, \( a \subset R \) is an ideal without embedded primary components, and \( b = \wp \) is one of the (minimal) associated prime ideals of \( a \). Indeed, in this case one has \( a: b = \wp \). A consequence is the (known) fact that every minimal prime \( \wp/a \subset R/a \) is differential. See also Theorem 2.24.

Proposition 2.22 Let \( a = bc \subset R \) be a product of ideals \( b, c \subset R \) such that \( a : b \) contains an \( R/b \)-regular element. Then

\[
\mathcal{I}_k(a) = \mathcal{I}_k(b) \cap \mathcal{I}_k(c, a : b).
\]

In particular, \( a \) is differential if and only if both \( b \) is differential and \( c \) is differential with respect to \( a : b \).

Proof. Take a derivation \( d \in \mathcal{I}_k(a) \). Applying Proposition 2.20 we get \( \mathcal{I}_k(a) \subset \mathcal{I}_k(b) \). Thus, \( d(b) \subset b \). For any \( y \in b \) and \( z \in c \), we have \( yz \in bc = a \), hence \( d(yz) \in a \), so that \( yd(z) + zd(y) \in a \). But \( zd(y) \in a \) since \( d(y) \in b \). We conclude that \( yd(z) \in a \), which shows the inclusion \( \mathcal{I}_k(a) \subset \mathcal{I}_k(b) \cap \mathcal{I}_k(c, a : b) \). Conversely, pick \( d \in \text{Der}_k(R) \) satisfying \( d(b) \subset b \) and \( b d(c) \subset a \). As any element \( x \in a \) may be written as a finite sum of terms of the form \( y_i z_i \in bc \), we may suppose, by additivity, that \( x = yz \), with \( y \in b \) and \( z \in c \); hence \( d(x) = yd(z) + d(y)z \in a + bc = a \), as wanted.

The consequence below will play a crucial role later into the proof of Theorem 3.17.

Corollary 2.23 Let \( x \in R \) be a non-zero-divisor and let \( c \subset R \) be an ideal containing an \( R/(x) \)-regular element. Then

\[
\mathcal{I}_k(xc) = \mathcal{I}_k(x) \cap \mathcal{I}_k(c).
\]

Proof. One applies the proposition above, noting that \( c = xc : (x) \), as \( x \) is \( R \)-regular.

Primary decomposition. We now obtain a primary decomposition of the tangential idealizer of an ideal \( a \) in a noetherian ring \( R \), starting from a primary decomposition of \( a \), provided it does not have embedded components.

Recall that, if \( N \subset M \) are \( R \)-modules and \( N = \bigcap_{i=1}^t N_i \) is a primary decomposition of \( N \) in \( M \), in the usual sense that the radical of each \( N_i : M \) is a prime ideal \( \wp_i \) (the unique associated prime of the \( R \)-module \( M/N_i \)), then the decomposition is said to be minimal if \( \wp_i \neq \wp_j \) whenever \( i \neq j \). In particular, a primary decomposition \( a = \bigcap_{i=1}^t a_i \) of an ideal \( a \subset R \) is minimal if \( \sqrt{a_i} \neq \sqrt{a_j} \) for \( i \neq j \).

Also, recall that an ideal \( a \subset R \) is \( k \)-differential if \( \mathcal{I}_k(a) = \text{Der}_k(R) \).
Theorem 2.24 Let $k \subseteq R$ be a noetherian ring extension and let $a \subseteq R$ be an ideal with minimal primary decomposition $a = \cap_{i=1}^s q_i$, without embedded components. Then $\mathcal{I}_k(a) = \cap_{i=1}^s \mathcal{I}_k(q_i)$. Moreover, setting $I = \{i \mid q_i \text{ is not } k\text{-differential}\}$, one has

$$\mathcal{I}_k(a) = \bigcap_{i \in I} \mathcal{I}_k(q_i),$$

and this is a minimal primary decomposition of $\mathcal{I}_k(a)$ in $\text{Der}_k(R)$. In particular, $a$ is differential if and only if its primary components $q_i$'s are differential.

Proof. Set $\varphi_i = \sqrt{q_i}$. Since the given primary decomposition of $a$ is minimal and $a$ has only minimal associated primes, one has $a_{\varphi_i} = (q_i)_{\varphi_i}$ for each $i \in \{1, \ldots, s\}$, that is, $a \cap q_i \not\subseteq \varphi_i$. This means that $a \cap q_i$ contains an $R/q_i$-regular element. Applying Proposition 2.20 one gets $\mathcal{I}_k(a) \subseteq \mathcal{I}_k(q_i)$, hence $\mathcal{I}_k(a) \subseteq \cap_j \mathcal{I}_k(q_j)$. The opposite inclusion is immediate (Proposition 2.17(i)), and the proposed equality follows. Since $\mathcal{I}_k(q_j) = \text{Der}_k(R)$ for $j \not\in I$, we may write $\mathcal{I}_k(a) = \cap_{i \in I} \mathcal{I}_k(q_i)$. Let us show that such decomposition is primary. Fixed $i \in I$, set $q_i = q$ and $\varphi = \sqrt{q}$ for simplicity. If $P \in \text{Ass} \text{Der}_k(R)/\mathcal{I}_k(q)$, one has $P = \mathcal{I}_k(q) : d$, for some $d \in \text{Der}_k(R) \setminus \mathcal{I}_k(q)$. We claim that $P = \mathcal{I}_k(q)$. If $x \in \varphi$, there is a positive integer $r$ such that $x^r \in q$. Hence $x^r d \in q \text{Der}_k(R) \subseteq \mathcal{I}_k(q)$, from which it follows $x^r \in P$, and then $x \in P$. Conversely, pick $y \in P$, that is, $(yd)(q) \subseteq q$. Since $d \notin \mathcal{I}_k(q)$, there exists $z \in q$ with $d(z) \notin q$. On the other hand, $yd(z) \in q$ and hence, necessarily, a power of $y$ lies in $q$, yielding $y \in \varphi$. Thus, we have shown that $\text{Ass} \text{Der}_k(R)/\mathcal{I}_k(q_i) = \{\varphi_i\}$ for each $i \in I$, as desired. Equivalently, the radical of the annihilator of the $R$-module $\text{Der}_k(R)/\mathcal{I}_k(q_i)$ is exactly $\varphi_i$, from which one notes the asserted minimality, as the given decomposition of $a$ has this property.

Example 2.25 We want to illustrate the necessity of the absence of embedded components, in the theorem above. Pick, in the polynomial ring $S = k[x, y, z]$ ($k$ is an arbitrary field), the ideal $a = (xz, yz, x^2, y^2)$, which has minimal primary decomposition $a = q_1 \cap q_2$, with $q_1 = (x, y)$ and $q_2 = (x^2, y^2, z)$. Then, the intersection $\mathcal{I}_k(q_1) \cap \mathcal{I}_k(q_2)$ is strictly contained in $\mathcal{I}_k(a)$. In fact, setting $d = xy \frac{\partial}{\partial z}$, one has $d(a) \subseteq a$, but $d(q_2) \not\subseteq q_2$ since $d(z) = xy \notin q_2$.

Remark 2.26 Theorem 2.24 complements [13, Lemme 1(d)] substantially, first as to the generality of the context (concerning both $R$ and the subring $k$, which is not assumed to be a field), and mainly because here we show that the obtained intersection of the idealizers of the (non-differential) primary components of $a$ yields a primary decomposition of $\mathcal{I}_k(a)$; on the other hand, we point out that the focus of [13] lies on the Lie algebra aspects of the idealizer — called tangent algebra therein — instead of purely on its module-theoretic side. See also [24, Theorem 1].

A consequence of Theorem 2.24 is that one can find a primary decomposition of the derivation module of an algebra $R$ of finite type, or essentially of finite type, over a field $k$. Note that, expressing $R$ as a quotient of a polynomial ring $S$ (or a localization thereof) in indeterminates $x_1, \ldots, x_n$ over $k$, one has an embedding $\text{Der}_k(R) \subseteq \bigoplus_{j=1}^n R \frac{\partial}{\partial x_j}$, where $\bigoplus_{j=1}^n R \frac{\partial}{\partial x_j}$ is the free $R$-module $\text{Der}_k(S) \otimes_S R \simeq R^n$. 

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Corollary 2.27 Let $S$ be as above, and let $a \subset S$ be an ideal with minimal primary decomposition $a = \cap_{i=1}^s q_i$ without embedded components. Writing $\mathcal{T}_a = \oplus_{j=1}^n a_{\partial \over \partial x_j}$ (the submodule of “trivial” vector fields) and $R = S/a$, one has

$$\text{Der}_k(R) = \bigcap_{i=1}^s \frac{\mathcal{T}_k(q_i)}{\mathcal{T}_a},$$

and this is a minimal primary decomposition of the $R$-module $\text{Der}_k(R)$ in $\oplus_{j=1}^n R_{\partial \over \partial x_j}$.

**Proof.** Proposition 2.24 yields $\text{Der}_k(R) = \mathcal{T}_k(a)/\mathcal{T}_a$. On the other hand, by Theorem 2.22 the tangential idealizer of $a$ has a minimal primary decomposition $\mathcal{T}_k(a) = \cap_{i=1}^s \mathcal{T}_k(q_i)$ in $\text{Der}_k(S)$ (notice that $I = \{1, \ldots, s\}$, as non-zero ideals in polynomial rings are known to be non-differential). Since $(\cap_{i=1}^s \mathcal{T}_k(q_i))/\mathcal{T}_a = (\cap_{i=1}^s (\mathcal{T}_k(q_i))/\mathcal{T}_a)$, the proposed equality follows. Now, for each $i$, the cokernel of the inclusion $\mathcal{T}_k(q_i)/\mathcal{T}_a \subset \oplus_{j=1}^n R_{\partial \over \partial x_j}$ may be identified with the non-zero $R$-module $\text{Der}_k(S)/\mathcal{T}_k(q_i)$, which has a single associated prime $\sqrt{q_i}$. We then conclude that the decomposition of $\text{Der}_k(R)$ just obtained is primary and minimal as well. □

Ordinary and symbolic powers. We wish to compare the tangential idealizer of an ideal with the idealizers of its ordinary and symbolic powers. A noetherian ring extension $k \subset R$ is fixed.

We notice an elementary property that will be useful to our next result.

**Lemma 2.28** Given an integer $n \geq 2$ and elements $x_1, \ldots, x_n \in R$, write $z_i = \prod_{j \neq i} x_j$ for each $i$. Then, for any derivation $d: R \to R$, one has an expression

$$(n-1) \cdot d(x_1 \cdots x_n) = \sum_{i=1}^n x_i d(z_i).$$

**Proof.** Set $z = x_1 \cdots x_n$. For any $i$, one may write $z = x_i z_i$ and hence $d(z) = x_i d(z_i) + z_i d(x_i)$. Summing over $i \in \{2, \ldots, n\}$, one gets $(n-1)d(z) = \sum_{i \geq 2} x_i d(z_i) + \sum_{i \geq 2} z_i d(x_i)$. On the other hand, iterated use of Leibniz’s rule gives $d(z_1) = x_3 \cdots x_n d(x_2) + \ldots + x_2 \cdots x_{n-1} d(x_n)$, thus yielding $x_1 d(z_1) = \sum_{i \geq 2} z_i d(x_i)$, as needed. □

**Proposition 2.29** Given an ideal $a \subset R$ and a positive integer $r$, one has $\mathcal{T}_k(a) \subset \mathcal{T}_k(a^r)$ and $r \mathcal{T}_k(a^r) \subset \mathcal{T}_k(a^{r+1})$. Moreover, if $k$ is a field of characteristic zero, the descending chain of the powers of $a$ induces an ascending chain of submodules of $\text{Der}_k(R)$,

$$\mathcal{T}_k(a) \subset \mathcal{T}_k(a^2) \subset \mathcal{T}_k(a^3) \subset \cdots$$

In particular, the ordinary powers of a differential ideal are also differential.

**Proof.** The inclusion $\mathcal{T}_k(a) \subset \mathcal{T}_k(a^r)$ follows easily from iterated use of Leibniz’s rule. Now let $d \in \mathcal{T}_k(a^r)$ and pick a typical generator $x = x_1 \cdots x_{r+1}$ of $a^{r+1}$, $x_i \in a$. Applying Lemma 2.22 with $n = r + 1$, we get $(rd)(x) = rd(x) = \sum_{i=1}^{r+1} x_i d(z_i)$, with $z_i = \prod_{j \neq i} x_j \in a^r$. Thus $d(z_i) \in a^r$, and $x_i d(z_i) \in a^{r+1}$, hence $(rd)(x) \in a^{r+1}$, as we want. If $k$ contains the field of rationals, the proposed ascending chain follows. □

In the radical case (even in more generality; see Remark 2.34(ii)), each inclusion in the chain detected above is in fact an equality.
Corollary 2.30  If \( k \) is a field of characteristic zero and \( a \subset R \) is a radical ideal, then, for any positive integer \( r \), one has
\[
\mathcal{T}_k(a) = \mathcal{T}_k(a^r).
\]

Proof. In virtue of Proposition 2.29, it suffices to show that \( \mathcal{T}_k(a^r) \subset \mathcal{T}_k(a) \). But then, as in the present case \( \sqrt{a^r} = a \), one may apply Proposition 2.18. \( \square \)

Proposition 2.31  Assume that \( R \) is local, and that \( 2 \) is invertible. For an ideal \( a \subset R \) generated by an \( R \)-sequence, one has
\[
\mathcal{T}_k(a) = \mathcal{T}_k(a^2).
\]

Proof. It’s enough to show that \( \mathcal{T}_k(a^2) \subset \mathcal{T}_k(a) \). Let \( \{x_j\}_{j=1}^m \) be an \( R \)-sequence generating \( a \), and for each \( i \) denote by \( a_i \) the subideal generated by the set \( \{x_j\}_{j=1}^m \setminus \{x_i\} \). Since \( R \) is a noetherian local ring, any permutation of the \( x_i \)’s is also an \( R \)-sequence, hence for every \( i \) one has \( a_i : (x_i) = a_i \). For any \( d \in \mathcal{T}_k(a^2) \) one may write \( 2x_id(x_i) = d(a_i^2) \in a^2 \). Therefore, there exist \( z_{i1}, \ldots, z_{im} \in a \) such that \( x_id(x_i) = \sum_{i=1}^m z_{ii}x_i \), which implies \( z_{i1} - d(x_i) \in a_i : (x_i) = a_i \) and hence \( d(x_i) \in a_i \). In the same way, one shows that \( d(x_2), \ldots, d(x_m) \in a \). \( \square \)

For an ideal \( a \subset R \) and an integer \( r \geq 2 \), one may look at the \( r \)-th symbolic power \( a^{(r)} \) of \( a \). This is the ideal formed with the \( x \in a \) such that \( yx \in a^r \) for some \( R/a \)-regular element \( y \). Notice that \( a \) and \( a^{(r)} \) have the same radical, since clearly \( a^r \subset a^{(r)} \subset a \).

Proposition 2.32  If \( a \subset R \) is an ideal and \( r \) is a positive integer, then \( \mathcal{T}_k(a^r) \subset \mathcal{T}_k(a^{(r)}) \). If \( k \) is a field of characteristic zero, one has \( \mathcal{T}_k(a) \subset \mathcal{T}_k(a^{(r)}) \), and if further \( a \) is radical, equality holds:
\[
\mathcal{T}_k(a) = \mathcal{T}_k(a^{(r)}).
\]

In particular, the symbolic powers of a (not necessarily radical) differential ideal are also differential.

Proof. Pick \( d \in \mathcal{T}_k(a^r) \) and \( x \in a^{(r)} \). There exists a \( R/a \)-regular element \( y \) such that \( yx \in a^r \). Then \( xd(y) + yd(x) \in a^r \). Multiplying by \( y \), we get \( 2yd(x) = d(x^r) \), hence \( d(x) \in a^{(r)} \), as wanted. If \( k \) is a field containing the rationals, Proposition 2.29 may be applied and the inclusion \( \mathcal{T}_k(a) \subset \mathcal{T}_k(a^{(r)}) \) follows. Finally, as \( \sqrt{a^{(r)}} = a \) in the radical case, one uses Proposition 2.18 and the desired equality holds. \( \square \)

Radical. Quite generally, the tangential idealizer of the radical of an ideal contains the idealizer of the ideal itself (Proposition 2.18). Now we search sufficient conditions for equality, as a generalization of one direction of a study made by H. Hauser and J.-J. Risler (see [13]). They show that, if \( \mathcal{O} \) is the ring of germs of real analytic functions and \( I \subset \mathcal{O} \) is an ideal without embedded components, then the module consisting of the derivations preserving \( I \), denoted \( \mathcal{D}_I \), therein, coincides with \( \mathcal{D}_{\sqrt{I}} \) if and only if \( I \) can be expressed as a finite intersection \( I = \cap_{i=1}^s \mathcal{O}_k^{(k_i)} \), for positive integers \( k_i \)’s and prime ideals \( \mathcal{O}_k \)’s with \( k_i \)-th symbolic power \( \mathcal{O}_k^{(k_i)} \). It was also pointed out that the same is true in the algebraic context, that is, after replacing \( \mathcal{O} \) by a polynomial ring over the field of real numbers.

Our result is as follows:
Proposition 2.33  Let $R$ be a noetherian ring containing a field $k$ of characteristic zero, and let $a \subset R$ be an ideal having minimal primary decomposition, without embedded components, of the form

$$a = \bigcap_{i \in I_1} \mathfrak{p}_i^{n_i} \cap \bigcap_{i \in I_2} \mathfrak{p}_i^{(n_i)}$$

where $\mathfrak{p}_i$’s are prime ideals, $n_i$’s are positive integers, and $I_1$, $I_2$ are finite (possibly empty) index sets. Then, it follows an equality

$$T_k(a) = T_k(\sqrt{a}).$$

In particular, $a$ is differential if (and only if) $\sqrt{a}$ is differential.

Proof.  Theorem 2.24 yields

$$T_k(a) = \bigcap_{i \in I_1} T_k(\mathfrak{p}_i^{n_i}) \cap \bigcap_{i \in I_2} T_k(\mathfrak{p}_i^{(n_i)}).$$

Similarly, one may write

$$T_k(\sqrt{a}) = \bigcap_{i \in I_1} T_k(\mathfrak{p}_i) \cap \bigcap_{i \in I_2} T_k(\mathfrak{p}_i^{(n_i)}).$$

Since $T_k(\mathfrak{p}_i^{n_i}) = T_k(\mathfrak{p}_i^{(n_i)})$ for any $i \in I_1$ (Corollary 2.30), and $T_k(\mathfrak{p}_i) = T_k(\mathfrak{p}_i^{(n_i)})$ for any $i \in I_2$ (Proposition 2.32), one gets the desired equality.  \[\square\]

Remarks 2.34  (i) Notice, from the proof above, that no condition on the subring $k$ is required in case the $n_i$’s (in the primary decomposition of $a$) are only 1 or 2, since, in general, any derivation preserving an ideal must also preserve its ordinary and symbolic squares.

(ii) The equalities proved in Corollary 2.30 and Proposition 2.32 are now seen to be valid for the broader class of ideals satisfying Proposition 2.33.

3  Tangentially free ideals: extending Saito’s theory

This is the main section of the paper. Our main goal will be to present concrete criteria for the tangential idealizer of an ideal (that is not necessarily principal) to be free provided the base ring is a factorial domain of finite type or essentially of finite type over a field containing the rationals. As it turned out, the project landed first on an investigation of reflexivity, to such an extent that necessary and sufficient conditions for freeness will follow as an easy consequence. We shall then be in position to introduce the class of tangentially free ideals, thus furnishing an extension of K. Saito’s theory of free divisors.

As it will be clarified, the principal ideal case figures as a fundamental piece into our main results. It then seems natural to treat it first, separately, as it represents a whole subject of interest; this was essentially done by K. Saito and H. Terao in the local complex analytic setup — notably, a more restrictive setup than the one we adopt herein.

3.1  Abstract logarithmic derivations and free divisors

Fixed a ring extension $k \subset R$ and an element $x \in R$, one may consider the module of abstract logarithmic derivations of $x$ (over $k$), that is, the tangential idealizer

$$T_k(x) = \{ \chi \in \text{Der}_k(R) \mid \chi(x) \in (x) \}$$

of the principal ideal $(x)$ (or of $x$, simply). It could also be referred to as a Saito module, paying tribute to K. Saito and his article [21], where in particular he studies logarithmic
vector fields and free divisors. We pass, by completeness and for the reader’s convenience, to a glimpse on these concepts in their original local analytic setup.

Let $X$ be an $n$-dimensional complex manifold with holomorphic function sheaf $\mathcal{O}_X$, and let $D \subset X$ be a divisor. Consider the $\mathcal{O}_X$-module sheaf $\mathcal{D}(\log D)$ — also denoted $\mathcal{D}(\mathcal{O}_X, \log D)$ — whose stalk $\mathcal{D}(\mathcal{O}_X, \log D)_p$ at a point $p \in D$ is the so-called module of logarithmic derivations of $D$ at $p$, formed with the (germs of) $\mathcal{C}$-derivations $\delta_p : \mathcal{O}_X \rightarrow \mathcal{O}_X$ such that $\delta_p(f) \in (f)_p \cdot \mathcal{O}_X$, where $f_p$ is a local reduced equation for $D$ at $p$. The divisor $D$ is said to be free (at $p$) if the $\mathcal{O}_X$-module $\mathcal{D}(\mathcal{O}_X, \log D)_p$ is free (necessarily of rank $n$); in this case, $f_p$ is also dubbed free divisor. By choosing local coordinates $z_1, \ldots, z_n$ in an open neighborhood of a smooth point $p \in D$, each logarithmic derivation may be interpreted as a vector field $\sum_{i=1}^n h_i \frac{\partial}{\partial z_i}$ (with $h_i$’s in $\mathcal{O}_{X,p}$) tangent to $D$ at $p$. Finally, we quote Saito’s freeness criterion (see [21]): $D$ is free at a point $p \in D$ if and only if there exist $\delta_1, \ldots, \delta_n \in \mathcal{D}(\mathcal{O}_X)$, written $\delta_i = \sum_{j=1}^n \alpha_{ij} \frac{\partial}{\partial z_j} \in \mathcal{O}_{X,p}$, such that the determinant of the matrix $(\alpha_{ij})_{i,j}$ is a unit multiple of a local reduced equation for $D$ at $p$.

One then sees that, setting $R = \mathcal{O}_{X,p}$, the $R$-module $\mathcal{D}(\mathcal{O}_X, \log D)_p$ of logarithmic vector fields around $p$ is the tangential idealizer $\mathcal{I}_{R/C}(f_p)$ of $f_p$. Although clearly just a matter of symbology, the study of this basic situation (in terms of tangential idealizers and their first properties) was the startpoint of our research. We also point out that, under a rather schematic point of view, we allow free divisors to be non-reduced.

We shall be focused, in this part, on the following tasks: (i) To explicit a generating set for $\mathcal{I}_k(x)$ as an $R$-module; (ii) To establish the reflexiveness of the Saito module $\mathcal{I}_k(x)$ provided $x \in R$ is a non-zero-divisor, and (iii) To study a natural “abstract” version of the notion of free divisor, mainly finding an effective criterion for a regular divisor to be free, at least when the ambient derivation module is projective.

First, we introduce the notion of abstract jacobian ideal (of a divisor), which will be useful in the sequel.

**Definition 3.1** Let $k \subset R$ be a ring extension. For any element $x \in R$, the gradient ideal of $x$ (over $k$), denoted $\mathcal{G}_x$ herein, is the image of the natural $R$-linear map $\mathcal{D}(k)(R) \rightarrow R$ given by evaluation at $x$. Thus, one may write $\mathcal{G}_x = \langle d_x \rangle$, for any given set of generators $\{d_x\}_{x \in A}$ of $\mathcal{D}(k)(R)$. We define $\mathcal{J}_x = \langle \mathcal{G}_x, x \rangle$, the abstract jacobian ideal of $x$ (over $k$). For instance, if $\mathcal{D}(k)(R)$ admits a finite generating set $\{d_i\}_{i=1}^n$ as an $R$-module, then $\mathcal{J}_x = \langle d_1(x), \ldots, d_n(x), x \rangle$. Of course, one has $\mathcal{J}_x = \mathcal{G}_x$ whenever $x \in \mathcal{G}_x$ (a concrete typical situation is that of quasi-homogeneous polynomials over a field).

**Remark 3.2** Let $R$ be an algebra of finite type over a field $k$, and let $x \in R$ be a non-invertible divisor with abstract jacobian ideal $\mathcal{J}_x$ as defined above. The ring $R/(x)$ has a presentation $S/\mathfrak{a}$, for some ideal $\mathfrak{a}$ in a polynomial ring $S$ over $k$. Then, one may speak about the “true” jacobian ideal $\text{Jac}(R/(x))$ of $R/(x)$. We warn that the ideal $\mathcal{J}_x/(x) \subset R/(x)$ — which may be seen in $S/\mathfrak{a}$ too — does not coincide with $\text{Jac}(R/(x))$ in general; this will be illustrated at the end of Example 3.15.

**Proposition 3.3** Let $k \subset R$ be a noetherian ring extension such that $\mathcal{D}(k)(R)$ is finitely generated as an $R$-module, and let $\{d_i\}_{i=1}^n$ be a finite set of generators. For an element
$x \in R$, fix the (ordered, signed) generating set \{$d_1(x), \ldots, d_n(x), x$\} of its abstract jacobian ideal $\mathfrak{j}_x$, together with a free presentation

$$R^n \xrightarrow{\Phi_x} R^{n+1} \longrightarrow \mathfrak{j}_x \longrightarrow 0 \quad \Phi_x = (h_{ij})_{i,j} \quad i = 1, \ldots, n+1 \quad j = 1, \ldots, s$$

with respect to the canonical bases of $R^n$ and $R^{n+1}$. Then, the $R$-module $\mathfrak{I}_k(x)$ of abstract logarithmic derivations of $x$ is generated by the $k$-derivations

$$\chi_j = \sum_{i=1}^n h_{ij} d_i, \quad j = 1, \ldots, s$$

obtained from the column-vectors of the matrix $\Phi_x$ after deletion of its last row.

**Proof.** Pick $\chi \in \mathfrak{I}_k(x)$. There exists $z \in R$ such that $\chi(x) + zx = 0$, that is, writing $\chi = \sum_{i=1}^n z_i d_i$, one has $z_1 d_1(x) + \ldots + z_n d_n(x) + zx = 0$. This means that $\mathfrak{j} = (z_1, \ldots, z_n, z) \in R^{n+1}$ is a relation of the ideal $\mathfrak{j}_x$. But the module of first-order syzygies is generated by the column-vectors of $\Phi_x$, hence there exist $q_1, \ldots, q_s \in R$ such that $\mathfrak{j} = \sum_{j=1}^s q_j \mathfrak{d}_j$, where $\mathfrak{d}_j = (h_{ij}, \ldots, h_{nj}, h_{n+1,j})$, which yields

$$\chi = \sum_{i=1}^n z_i d_i = \sum_{i=1}^n \left( \sum_{j=1}^s q_j h_{ij} \right) d_i = \sum_{j=1}^s q_j \left( \sum_{i=1}^n h_{ij} d_i \right) = \sum_{j=1}^s q_j \chi_j,$$

thus showing that $\mathfrak{I}_k(x) \subset \sum_{j=1}^s R \chi_j$. To get the equality, it suffices to check that each $\chi_j$ preserves $(x)$. This is clear, since each relation

$$\sum_{i=1}^n h_{ij} d_i(x) + h_{n+1,j} x = 0, \quad j = 1, \ldots, s,$$

may be rewritten as $\chi_j(x) = -h_{n+1,j} x$. \hfill $\square$

**Example 3.4** Consider the cubic $f = x^2 y - z^3 \in S = C[x, y, z]$. In this case, $\mathfrak{j}_f = (2xy, x^2, -3z^2, f)$. The matrix

$$\Phi_f = \begin{pmatrix} x & 0 & 3z^2 & x \\ -2y & 3z^2 & 0 & y \\ 0 & x^2 & 2xy & z \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

presents $\mathfrak{j}_f$, hence Proposition \ref{3.3} yields $\mathfrak{I}_C(f) = S \chi_1 + S \chi_2 + S \chi_3 + S \epsilon$, where $\epsilon$ is the Euler derivation, $\chi_1 = x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y}$, $\chi_2 = 3z^2 \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z}$, $\chi_3 = 3z^2 \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial z}$. This is easily seen to be a minimal generating set.

Knowing a set of generators \{$\chi_j$\}_{j=1}^s of the module $\mathfrak{I}_k(x)$ will be fundamental — beyond the detection of *abstract free divisors* (see Definition \ref{3.8}) — later into Theorems \ref{3.17} and \ref{3.22} as consequently they will be given an *effective* character.

The following simple description of the cokernel $\mathfrak{C}_x = \text{Der}_k(R)/\mathfrak{I}_k(x)$ (a particular case of the module $\mathfrak{C}_{a,N}$ treated in Proposition \ref{2.15}) will be very useful.
Proposition 3.5 For any \( x \in R \), there’s an isomorphism of \( R \)-modules \( \mathfrak{C}_x \cong \mathfrak{J}_x/(x) \).

Proof. By definition, the gradient ideal \( \mathfrak{J}_x \) of \( x \) fits into a surjective homomorphism \( \mathcal{D}er_k(R) \to \mathfrak{J}_x \) (evaluation at \( x \)). By composition with the projection \( \mathfrak{J}_x \to (\mathfrak{J}_x, x) / (x) \), one gets a surjective \( R \)-linear map \( \mathcal{D}er_k(R) \to \mathfrak{J}_x/(x) \) whose kernel is easily seen to be \( \mathfrak{T}_k(x) \).

We must now recall some notions and facts. If \( R \) is a noetherian ring, then a finitely generated \( R \)-module \( M \) satisfies the “Serre type” \( \tilde{S}_n \) condition, for a given positive integer \( n \), if depth \( M_\varphi \geq \min\{n, \text{depth } R_\varphi\} \) for every prime ideal \( \varphi \subset R \) (here, for any finitely generated \( R \)-module \( N \), one denotes by depth \( N_\varphi \) the maximal length of an \( N_\varphi \)-regular sequence contained in the maximal ideal \( \varphi R_\varphi \)). Moreover, \( M \) is reflexive if the canonical map from \( M \) into its bidual \( M^{**} = \text{Hom}_R(\text{Hom}_R(M, R), R) \) is an isomorphism. The ring \( R \) is Cohen-Macaulay if, locally, its depth equals its Krull dimension. One knows that, if \( R \) is Gorenstein locally in depth 1 (that is, the local ring \( R_\varphi \) is Cohen-Macaulay and has a canonical module isomorphic to \( R_\varphi \) itself, for every prime ideal \( \varphi \subset R \) such that depth \( R_\varphi \leq 1 \)), then a finitely generated \( R \)-module \( M \) is reflexive if and only if it has the \( \tilde{S}_2 \) condition; if \( M \) is torsion-free, it suffices to check that depth \( M_\varphi \geq 2 \) whenever depth \( R_\varphi \geq 2 \). For details, see [1] Propositions 16.31, 16.33, and Remark 16.35.

The result below makes use of the observations above, and widely extends the first part of [21] Corollary 1.7.

Proposition 3.6 Let \( k \subset R \) be a noetherian ring extension such that \( \mathcal{D}er_k(R) \) is a finitely generated \( R \)-module with the \( \tilde{S}_2 \) condition. Assume that \( R \) is a Gorenstein ring locally in depth 1. If \( x \in R \) is a non-zero-divisor, then \( \mathfrak{T}_k(x) \) (and hence its \( R \)-dual \( \mathfrak{T}_k(x)^* \)) is a reflexive \( R \)-module.

Proof. One may assume that \( R \) is local, with depth \( R \geq 2 \). As in the present case the \( R \)-module \( D = \mathcal{D}er_k(R) \) is reflexive, one may suppose that the principal ideal \( (x) \subset R \) is not differential, that is, \( \mathfrak{J}_x \neq (x) \); in particular, \( x \) must belong to the maximal ideal of \( R \). In order for the (torsion-free) module \( \mathfrak{T}_k(x) \) to be reflexive, we need to show that its depth is at least 2. We shall proceed by standard depth-chase along the short exact sequence of \( R \)-modules
\[
0 \longrightarrow \mathfrak{T}_k(x) \longrightarrow D \longrightarrow \mathfrak{J}_x \longrightarrow 0
\]
derived from Proposition [3.3] where we set \( \mathfrak{J}_x = \mathfrak{J}_x/(x) \). If depth \( \mathfrak{J}_x \geq \text{depth } D \), then depth \( \mathfrak{T}_k(x) \geq \text{depth } D \), whence depth \( \mathfrak{T}_k(x) \geq 2 \) in virtue of the \( \tilde{S}_2 \) condition of \( D \). Thus we may assume that depth \( \mathfrak{J}_x < \text{depth } D \), in which case depth \( \mathfrak{T}_k(x) = \text{depth } \mathfrak{J}_x + 1 \). It now suffices to check that depth \( \mathfrak{J}_x > 0 \). Assume the contrary. Then certainly depth \( \mathfrak{J}_x > \text{depth } \mathfrak{J}_x \), since \( x \) is a non-zero-divisor. As \( \mathfrak{J}_x \) is the cokernel of the “multiplication by \( x \)” injection \( R \hookrightarrow \mathfrak{J}_x \), one would get depth \( R = 1 \), a contradiction.

Remark 3.7 The proposition above may be applied into the setting of algebras of finite type or essentially of finite type over a field, that are, say, complete intersections locally in depth 1. In fact, in this situation the module of derivations is (finitely generated and) reflexive, as it satisfies the \( \tilde{S}_2 \) property, being a module of second-order syzygies (the dual of the module of Kähler differentials).
Now, following Saito’s original definition of free divisor, it seems natural to propose the following abstract version:

**Definition 3.8** With respect to a fixed ring extension \( k \subset R \), an element \( x \in R \) is said to be an abstract free divisor — free divisor, for short — if its tangential idealizer \( \mathcal{T}_k(x) \) is a locally free \( R \)-module, that is, \( \mathcal{T}_k(x)_\wp \) is a free \( R_\wp \)-module for every prime ideal \( \wp \subset R \).

Clearly, for a ring extension \( k \subset R \), an element \( x \in R \) for which the principal ideal \((x)\) is differential (e.g., \( x = 1 \)) is a free divisor if and only if the \( R \)-module \( \mathcal{D} \text{er}_k(R) \) is locally free.

It turns out that Proposition 3.6 recovers also the second part of [21, Corollary 1.7], which is a well-known result of Saito stating that divisors in complex smooth surfaces are free.

**Corollary 3.9** Let \( R \) be a regular local ring of dimension 2 that is essentially of finite type over a perfect field. Then, any \( x \in R \) is a free divisor.

**Proof.** In this situation, \( \mathcal{D} \text{er}_k(R) \) is a free \( R \)-module (in particular, there is nothing to show if \( x = 0 \)). Moreover, \( \mathcal{T}_k(x) \) must have finite homological dimension, and its depth is 2 by Proposition 3.6. Thus, by the well-known Auslander-Buchsbaum formula, one concludes that \( \mathcal{T}_k(x) \) is free. \( \square \)

For the next result, the proof we present is similar to the one used in Proposition 3.6. In our statement, the base ring is not even required to contain a field. If \( k \subset R \) is an extension of noetherian local rings, we shall say, for brevity, that a given \( x \in R \) is quasi-smooth (over \( k \)) if \( \mathcal{J}_x \) is free as an \( R \)-module, that is, if the ideal \( \mathcal{J}_x \) is principal generated by a non-zero-divisor — possibly, the element \( x \) itself, which means that \((x)\) is a differential ideal. For instance, any smooth \( x \in R \), in the sense that \( \mathcal{J}_x = R \), is automatically quasi-smooth. If \( M \) is a finitely generated \( R \)-module, one denotes by \( \text{hd}_R M \) the homological dimension of \( M \) over \( R \). Our goal is to show that, in a suitable setting, a regular divisor \( x \in R \) is free if and only if \( \text{hd}_R \mathcal{J}_x \leq 1 \).

**Proposition 3.10** Let \( k \subset R \) be an extension of noetherian local rings such that \( \mathcal{D} \text{er}_k(R) \) is a free \( R \)-module of finite rank. Let \( x \in R \) be a non-zero-divisor. Then, \( x \) is a free divisor if and only if either \( x \) is quasi-smooth or its abstract jacobian ideal \( \mathcal{J}_x \) has a minimal free resolution of the form

\[
0 \longrightarrow R^n \longrightarrow R^{n+1} \longrightarrow \mathcal{J}_x \longrightarrow 0
\]

for some integer \( n \geq 1 \).

**Proof.** Write \( F = \mathcal{D} \text{er}_k(R) \) and \( \mathcal{J}_x = \mathcal{J}_x/(x) \), and assume that \( x \) is a free divisor that is not quasi-smooth. In particular, \( \mathcal{J}_x \) is non-zero. From the short exact sequence (Proposition 3.5)

\[
0 \longrightarrow \mathcal{T}_k(x) \longrightarrow F \longrightarrow \mathcal{J}_x \longrightarrow 0
\]

it follows that \( \text{hd}_R \mathcal{J}_x \leq 1 \) and hence necessarily \( \text{hd}_R \mathcal{J}_x = 1 \), as \( x \) is regular and annihilates \( \mathcal{J}_x \). But then, by chasing homological dimension along the exact sequence

\[
0 \longrightarrow R \longrightarrow \mathcal{J}_x \longrightarrow \mathcal{J}_x \longrightarrow 0
\]
one concludes that $\text{hd}_R \mathcal{J}_x \leq 1$, which must be exactly 1, as $x$ is not quasi-smooth. For the converse, one may assume that $\mathcal{J}_x \neq 0$. If $\text{hd}_R \mathcal{J}_x \leq 1$ then, by the latter exact sequence, one again must have $\text{hd}_R \mathcal{J}_x = 1$, and hence, by the former, $\mathcal{T}_R(x)$ must be free. \hfill \Box

The result below gives a version of the proposition above and extends results of H. Terao (cf. [26, Proposition 2.4], [28, Proposition 3]), who looked at the question of characterizing freeness by means of Cohen-Macaulayness of gradient ideals. The height of a proper ideal $\mathfrak{a} \subset R$ is denoted $\text{ht} \mathfrak{a}$, and by a widely accepted abuse of terminology, we say that $\mathfrak{a}$ is Cohen-Macaulay if $R/\mathfrak{a}$ is a Cohen-Macaulay ring.

**Proposition 3.11** Let $k \subset R$ be an extension of noetherian local rings, with $R$ Cohen-Macaulay, such that $\text{Der}_k(R)$ is a free $R$-module of finite rank. Let $x \in R$ be a regular non-smooth divisor such that $\text{ht} \mathcal{J}_x \geq 2$. Then, $x$ is a free divisor if and only if $\mathcal{J}_x$ is a Cohen-Macaulay ideal with $\text{hd}_R \mathcal{J}_x < \infty$ and height exactly 2.

**Proof.** If $x$ is free, Proposition 3.10 gives $\text{hd}_R \mathcal{J}_x \leq 1 < \infty$ and then $\text{ht} \mathcal{J}_x \leq \text{hd}_R R/\mathcal{J}_x \leq 2$, hence $\text{ht} \mathcal{J}_x = 2$, and $\mathcal{J}_x$ must be Cohen-Macaulay by the Auslander-Buchsbaum formula. Conversely, again by Proposition 3.10 all we have to notice is that if $\mathcal{J}_x$ is Cohen-Macaulay, with $\text{hd}_R \mathcal{J}_x < \infty$ and $\text{ht} \mathcal{J}_x = 2$, then $\text{hd}_R \mathcal{J}_x = 1$. This in fact holds, as in this case $\mathcal{J}_x$ is a perfect ideal of height 2, which means that $\mathcal{J}_x$ has the desired homological dimension. \hfill \Box

**Remarks 3.12** (i) It should be noticed, by the well-known Hilbert-Burch theorem, that a resolution map $\phi_x: R^n \to R^{n+1}$ associated to a free divisor $x$, in the setting above, recaptures the ideal $\mathcal{J}_x$ in the sense that $\mathcal{J}_x$ equals the image of the induced application $\wedge^{n+1} \phi_x: \wedge^{n+1} R^n \to R$. In other words, the maximal subdeterminants of $\phi_x$ (now seen as a matrix, taken with respect to canonical bases) form a generating set for $\mathcal{J}_x$.

(ii) A few comments about the condition of freeness of $\text{Der}_k(R)$, required in propositions 3.10 and 3.11. Freeness of derivation modules is the crucial point in the long standing Zariski-Lipman Conjecture. Let $R$ be a local ring which is an algebra essentially of finite type over a field $k$ of characteristic zero. Thus, the conjecture predicts that $R$ is regular if the $R$-module $\text{Der}_k(R)$ is free. In the presence of this hypothesis, it’s known that $R$ must be at least a normal domain (cf. [17]). The problem has been settled positively in fundamental situations, but it remains open in the concrete, critical setting where $R$ is the localization of a 2-dimensional affine complete intersection ring (even in 4 variables). If one drops the characteristic zero hypothesis, the regularity condition on $R$ is, in general, stronger than the freeness condition on $\text{Der}_k(R)$. This can be easily observed in a simple example: the local ring of the surface $z^p = x y$ at the origin, over a field $k$ with prime characteristic $p$, is not regular but its derivation module is free. A nice reference for an overview on this and other conjectures is J. Herzog’s survey [14].

(iii) Clearly, the hypotheses of freeness of $\text{Der}_k(R)$ and finiteness of $\text{hd}_R \mathcal{J}_x$ (the latter holds if for instance $\mathcal{J}_x$ is generated by a regular sequence; see Example 3.14) are both fulfilled if $k$ is a perfect field and $R$ is a regular local ring essentially of finite type over $k$. Also, propositions 3.10 and 3.11 may be applied within the context of graded polynomial rings, where, for instance, the ideal $\mathcal{J}_f$ coincides with the lifted jacobian ideal of $f$, for any given polynomial $f$. Thus, in this special situation, Proposition 3.11 recovers the well-known fact that non-smooth reduced free divisors have a Cohen-Macaulay singular locus of high dimension (equal to $n - 2$, where $n$ is the dimension of the ambient space).
Examples 3.13 (i) The \( n \)-sphere \( \sum_{i=1}^{n} x_i^2 = 1 \), in real affine \( n \)-space, is smooth and hence a free divisor.

(ii) The normal crossing divisor \( f = x_1x_2 \cdots x_n \) in the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \) is a free divisor, since \( J_f \) is Cohen-Macaulay of codimension 2. A free basis for \( J_C(f) \) is \( \{ \epsilon, \chi_1, \ldots, \chi_{n-1} \} \), where \( \epsilon \) is the Euler derivation and \( \chi_i = x_i \frac{\partial}{\partial x_i} - x_n \frac{\partial}{\partial x_n} \), for \( i = 1, \ldots, n-1 \). Another basis is \( \{ x_i \frac{\partial}{\partial x_i} \}_{i=1}^{n} \).

(iii) The homogeneous polynomial \( f = x^2y + xyz + z^3 \) is a free divisor in the polynomial ring \( S = k[x, y, z] \), where \( k \) is a field of characteristic 3. This follows from Proposition 3.10
since \( J_f \) has a minimal free resolution of the form
\[
0 \rightarrow S^2 \rightarrow S^3 \rightarrow (yz - xy, \ x^2 + xz, \ xy) \rightarrow 0
\]

(iv) The polynomial \( f = 256x^3 - 128x^2z^2 + 16x^4 + 144y^2z - 4x^3y^2 - 27y^4 \) is an irreducible quasi-homogeneous free divisor in \( C[x, y, z] \), a basis for \( J_C(f) \) being \( \{ \chi_1, \chi_2, \chi_3 \} \), where \( \chi_1 = 6y \frac{\partial}{\partial x} + (8x - 2x^2) \frac{\partial}{\partial y} - x y \frac{\partial}{\partial z} \), \( \chi_2 = (4x^2 - 48z) \frac{\partial}{\partial x} + 12xy \frac{\partial}{\partial y} + (9y^2 - 16xz) \frac{\partial}{\partial z} \), and \( \chi_3 = 2x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 4z \frac{\partial}{\partial z} \). This example is due to K. Saito (cf. [20]).

(v) As one expects, product of free divisors is not necessarily a free divisor. For instance, in the polynomial ring \( C[x, y, z] \), pick \( f = xyz \) (which is free, by the example (ii) above) and \( g = x + y + z \) (free, by smoothness). The divisor \( fg = x^2yz + xy^2z + xyz^2 \) is not free, since \( J_{fg} \) is not Cohen-Macaulay.

We now want to illustrate Proposition 3.11 in the non-polynomial case.

Example 3.14 Let \( k \) be a field of characteristic 3 and let \( R \) be the local ring of the surface \( z^2 = y(y^2 + x) \) at the origin. In this case, the \( R \)-module \( \mbox{Der}_k(R) \) is free, a basis being \( \{ \partial_1, \partial_2 \} \), where \( \partial_1, \partial_2 \) are, respectively, the images in \( \mbox{Der}_k(R) \) of \( d_1 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \ d_2 = \frac{\partial}{\partial z} \). Let \( \eta \) be the image of \( x + z^2 \) in \( R \). One gets \( J_\eta = (x, z)R \), which is a height 2 complete intersection. Thus, by Proposition 3.11, \( \eta \) must be a free divisor in \( R \).

For a \( d \)-dimensional algebra \( R \) essentially of finite type (or standard graded of finite type) over a perfect field \( k \), with \( \mbox{Der}_k(R) \) not necessarily free, and such that its jacobian ideal has positive grade (in which case there’s a well-defined rank, equal to \( d \); see Remark 2.3), one may check whether a given algebraic divisor \( x \in R \) is free by resorting to Proposition 3.11 and extracting a minimal generating set of \( J_k(x) \); if \( d \) elements suffice, then \( x \) is free.

Example 3.15 Let \( R = S/\mathfrak{g} = \mathbb{C}[x, y, z, w]/(y^2 - xz, yz - xw, z^2 - yw) \) be the homogeneous coordinate ring of the twisted cubic curve in complex projective 3-space. Example 2.10 gives an explicit minimal generating set \( \{ \partial_1, \partial_2, \partial_3, \partial_4 \} \) for the (non-free) \( R \)-module \( \mbox{Der}_C(R) \). Denote by \( \overline{f} \) the image in \( R \) of any given \( f \in S \). We claim that \( \overline{f} \) is a free divisor in \( R \). Computing a presentation matrix of the abstract jacobian ideal \( J_{\overline{f}} \) with respect to its generating set \( \{ \partial_1(\overline{f}), \partial_2(\overline{f}), \partial_3(\overline{f}), \partial_4(\overline{f}) \} = \{ \overline{f}, \overline{y}, \overline{x}, \overline{z} \} \), and applying Proposition 3.2, one gets that \( \{ \overline{f}, \overline{x}, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8 \} \) is a set of generators for \( J_C(\overline{f}) \), where \( \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8 \) are, respectively, the images in \( \mbox{Der}_C(R) \) of the global vector fields (tangent to the twisted cubic) given by \( xzd - 2xzd, \ yd_3 - 2zd, \ zdc - 2wd, \ xdc - yd_1, \ yd_2 - zd_1, \ zd_2 - wd_1 \). Using the defining equations of \( R \), one may write \( \chi_3 = 3\overline{f}(\chi - \partial_2), \chi_4 = \ldots \)
\[ 3 \psi (\tau - d_2), \quad \chi_5 = 3 \psi (\tau - d_2) \]. Moreover, the derivations \( \chi_6, \chi_7, \chi_8 \) vanish identically on \( R \).

This shows that the set \( \{ \tau, d_2 \} \) generates \( \mathcal{T}_C(\psi) \). As \( R \) is 2-dimensional, the freeness of \( \psi \) follows. Explicitly:

\[
\mathcal{T}_C(\psi) = R \tau \oplus R \overline{d}_2 \cong R^2
\]

In order to dispel the hope that linear forms in \( R \) should be free divisors, note that \( \{ \tau, d_1, d_2 \} \subset \mathcal{T}_C(\psi) \) and hence \( \tau \) cannot be free. Finally, we want to use the present example to justify the warning made in Remark 3.2. In the present case, the ideal \( \mathfrak{p} = \mathcal{T}_C(\psi) \subset R/\psi \) may be identified with \( (x, y, z)/a \subset S/a \), where \( a = (y, \psi) = (y, xz, xw, z^2) \). On the other hand, a direct calculation gives that the “true” jacobian ideal \( \text{Jac}(R/\psi) \), seen in \( S/a \), equals \( (x^2, zw, a)/a \). Therefore, \( \mathfrak{p}/(\psi) \neq \text{Jac}(R/\psi) \).

**Remark 3.16** Let \( R = \bigoplus_{i \geq 0} R_i \) be a standard graded algebra of finite type over a field \( R_0 = k \), and let \( \ell \in R_1 \) be a linear form that is a free divisor. Example 3.15 above illustrates a situation where a basis for \( \mathcal{T}_k(\ell) \), with \( \ell = \psi \), is a subset of a set of generators of \( \text{Der}_k(R) \). However, this is far from being true in general — easy examples are linear forms in polynomial rings. For a non-polynomial instance, consider the graded \( k \)-algebra \( R = k[x, y, z]/(xy + z^2) \), where \( k \) is a field of characteristic 2. One has \( \text{Der}_k(R) = R \overline{d}_1 \oplus R \overline{d}_2 \), where \( \overline{d}_1, \overline{d}_2 \) are represented, respectively, by the polynomial derivations \( d_1 = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} \), \( d_2 = \frac{\partial}{\partial z} \). Then, the image \( \ell = \overline{\psi} \) of the variable \( z \) in \( R \) is a (smooth) free divisor, and a basis for its idealizer is \( \{ \overline{d}_1, \ell \overline{d}_2 \} \).

### 3.2 Criteria for reflexive and free tangential idealizers

In the previous part we showed that, in a suitable setting, modules of abstract logarithmic derivations of regular divisors are reflexive, and proved, independently, criteria for a divisor to be free. Now, we first exhibit necessary and sufficient conditions for the reflexiveness of the tangential idealizer (of a not necessarily principal ideal), provided the ring is a “geometric” factorial domain. As it will turn out, freeness criteria may be derived easily, giving rise to a distinguished class of ideals here dubbed *tangentially free*. Such a class provides, thus, an extension of Saito’s theory of free divisors. By this reason, this part contains the core of our theory.

**Reflexive tangential idealizers.** Before proving reflexiveness criteria, let us recall a very useful fact: if \( R \) is a noetherian normal domain and \( M \subset N \) are finitely generated \( R \)-modules, with \( M \) reflexive and \( N \) torsion-free, such that the height of the ideal \( M : N \subset R \) is at least 2 (that is, \( M \) and \( N \) coincide locally in height 1), then \( M = N \). This follows easily from a classical result of P. Samuel (see [22] Proposition 1], which guarantees that the reflexive module \( M \) may be expressed as the intersection (taken in the vector space \( M \otimes_R L \), where \( L \) is the fraction field of \( R \)) of the localizations of \( M \) at the height 1 prime ideals of \( R \). It can be shown that this holds in more generality, but, as we shall need the factorial hypothesis, the present case suffices for our purposes.

**Theorem 3.17** Let \( R \) be a factorial domain that is of finite type or essentially of finite type over a field \( k \) of characteristic zero. For an ideal \( a \subset R \), the following assertions are equivalent:

(i) The \( R \)-module \( \mathcal{T}_k(a) \) is reflexive.
For each Lie bracket \([d_i, d_j]\) vanishes at \(x\), for some finite generating set \(\{d_i\}_{i=1}^n\) of \(\text{Der}_k(R)\).

Then

\[ \mathcal{T}_k(x) \subset \mathcal{T}_k(\mathcal{J}_x) \]
As we recall the fact that any derivation preserving an ideal Remark 3.19 which concludes the proof. □

\[
(\chi \in \mathcal{J}_x = (d_1(x), \ldots, d_n(x), x)). \text{ Set } \chi_1 = \sum_{j=1}^{n} x_i d_i(j(x)), \text{ which by Leibniz's rule may be written as }
\]
\[
z_1 = d_1(\chi(x)) - \sum_{j=1}^{n} d_1(x_i) d_i(x)
\]
As \(\chi(x) = zx\), for some \(z \in R\), one has \(d_1(\chi(x)) \in \mathcal{J}_x\). Hence \(z_1 \in \mathcal{J}_x\). On the other hand, by the condition on the Lie brackets, one may write \(z_1 = \sum_{j=1}^{n} x_i d_i(d_1(x)) = \chi(d_1(x))\), which concludes the proof.

**Remark 3.19** We recall the fact that any derivation preserving an ideal \(a \in \text{ a polynomial ring } S\) over a field \(k\) of characteristic zero, must also preserve its lifted jacobian ideal \((a, \mathcal{J}_a) \subset S\), where \(\mathcal{J}_a\) is the ideal generated by the \(c \times c\) subdeterminants \(c\) is the codimension of \(a\) of the jacobian matrix of a generating set of \(a\) (see, e.g., [12, Proposition 5.1(a)] and comments therein). In essence, this was first detected by A. Seidenberg (cf. [24]), who noticed that the ideal defining the singular locus of an algebraic variety over \(k\) is differential. Here, Proposition 3.18 treats the principal ideal case in more generality.

**Corollary 3.20** In the setting of Theorem 3.17, let \(x \in R\) be such that each Lie bracket \([d_i, d_j]\) vanishes at \(x\), for some generating set \(\{d_i\}_{i=1}^{n}\) of \(\text{Der}_k(R)\). Then \(\mathcal{T}_k(x, \mathcal{J}_x) = \mathcal{T}_k(x)\), a reflexive \(R\)-module.

**Proof.** It follows immediately from Theorem 3.17 and Proposition 3.18. □

**Corollary 3.21** Let \(S = k[x_1, \ldots, x_n]\) be a polynomial ring over a field \(k\) of characteristic zero. Then, for any \(f \in S\), the tangential idealizer of the ideal \((f \frac{\partial f}{\partial x_1}, \ldots, f \frac{\partial f}{\partial x_n}, f^2)\) equals the reflexive module \(\mathcal{T}_k(f)\).

**Proof.** Apply Corollary 3.20 with \(d_i = \frac{\partial}{\partial x_i}\), for each \(i\). □

**Tangentially free ideals.** We are going to write down the promised characterization of ideals with free tangential idealizer. The result is stated here as a separated theorem, even though it follows immediately from Theorem 3.17.

**Theorem 3.22** Let \(R\) be a factorial domain that is of finite type or essentially of finite type over a field \(k\) of characteristic zero. For an ideal \(a \subset R\), the following assertions are equivalent:

(i) The \(R\)-module \(\mathcal{T}_k(a)\) is locally free.

(ii) Either \(\text{Der}_k(R)\) is locally free and \(a\) is differential, or \(a \subset (x)\) for some non-zero abstract free divisor \(x \in R\) such that \(\mathcal{T}_k(a) = \mathcal{T}_k(x)\).

(iii) Either \(\text{Der}_k(R)\) is locally free and \(a\) is differential, or \(a = xa'\) for some ideal \(a' \subset R\) and some non-zero abstract free divisor \(x \in R\) such that \(\mathcal{T}_k(x) \subset \mathcal{T}_k(a')\).
We are now in position to introduce a class of ideals that extends, algebraically, the well-known class of free divisors, as now the ideals are not required to be principal anymore. In spite of the hypothesis of factoriality imposed above, we propose a definition in full generality.

**Definition 3.23** An ideal \( a \) of a ring \( R \) is said to be a **tangentially free ideal** (in \( R \), over \( k \)), or simply free ideal, if the \( R \)-module \( J_k(a) \) is locally free (that is, free locally at every prime ideal of \( R \)).

**Remark 3.24** In the setting of Theorem 3.22, if \( a = xa' \subset R \) is a free ideal, then it follows from item (ii) that

\[
J_k(a) = \bigoplus_{j=1}^s R \chi_j \cong R^s,
\]

for any free basis \( \{\chi_j\}_{j=1}^s \) of \( J_k(x) \) (see Proposition 3.3). Clearly, the effective side of the theorem is provided by item (iii), since it suffices to verify whether \( a' = \chi_j \)-invariant, for each \( j \).

**Example 3.25** Take \( f = x^2 + y^3 + z^5 \) in the polynomial ring \( S = \mathbb{C}[x, y, z] \), and set \( R = S/(f) \), which is a well-known factorial domain — a distinguished one, as its completion, with respect to the maximal ideal of the origin, has a special uniqueness feature. We intend to produce a free ideal in \( R \). The (non-free) \( R \)-module \( \text{Der}_R(R) \), seen as a submodule of \( R^3 \), is generated by the images \( \overrightarrow{d}_1, \overrightarrow{d}_2, \overrightarrow{d}_3, \overrightarrow{d}_4 \) of the vectors \( d_1 = (15x, 10y, 6z), \ d_2 = (3y^2, -2x, 0), \ d_3 = (5z^4, 0, -2x), \ d_4 = (0, 5z^4, -3y^2) \). For \( g \in S \), denote by \( \overline{g} \) its image in \( R \). We first claim that \( \overline{\tau} \) is a free divisor. Write \( \overline{\tau} = (6\overline{x}, \overline{y}, -2\overline{x}, -3\overline{y}^2, \overline{z}) \) and apply the recipe described in Proposition 3.3. One gets that \( \{\overrightarrow{d}_1, \overrightarrow{d}_2, \chi_3, \chi_4, \chi_5, \chi_6\} \) is a set of generators for \( J_C(\overline{\tau}) \), where \( \chi_3, \chi_4, \chi_5, \chi_6 \) are, respectively, the images in \( \text{Der}_C(R) \) of the derivations of \( S \) given by \( xd_1 + 3zd_3, y^2d_1 + 2zd_4, 3y^2d_3 - 2xd_4, z^5d_1 - 3zd_3 - 2yd_4 \). But \( \chi_3 = -5\overline{y}\overline{d}_2 \), \( \chi_4 = 5\overline{z}\overline{d}_2 \), \( \chi_5 = 5\overline{z}\overline{d}_2 \), and \( \chi_6 \) vanishes identically on \( R \), so that \( \overline{\tau} \) is a free divisor, with \( J_C(\overline{\tau}) = R\overrightarrow{d}_1 \oplus R\overrightarrow{d}_2 \). Since clearly \( \overline{\tau} \) is invariant under both \( \overrightarrow{d}_1 \) and \( \overrightarrow{d}_2 \), we obtain an inclusion \( J_C(\overline{\tau}) \subset J_C(\overline{\tau}) \). Thus, we may apply Theorem 3.22(iii) and affirm that the ideal \( \overline{\tau} \overline{J}_x = (\overline{\tau}, \overline{y}^2\overline{z}, \overline{x}^2) \subset R \) is tangentially free. A basis of \( J_C(\overline{\tau}) \) is \( \{\overrightarrow{d}_1, \overrightarrow{d}_2\} \).

Naturally, one may consider ideals \( b \subset R \) that are “pseudo-free”, in the sense that there exists a free divisor \( x \in R \) with \( xb \) a free ideal. The next result shows that, quite generally (but still within the factorial context), abstract jacobian ideals are of this type.

**Proposition 3.26** In the setting of Theorem 3.22, let \( x \in R \) be an abstract free divisor such that each Lie bracket \( [d_i, d_j] \) vanishes at \( x \), for some finite generating set \( \{d_i\}_{i=1}^n \) of \( \text{Der}_k(R) \). Then \( x \overline{J}_x \) is a tangentially free ideal.

**Proof.** Apply Corollary 3.20 with \( x \) an abstract free divisor. \( \square \)

It follows immediately:

**Corollary 3.27** Let \( S = k[x_1, \ldots, x_n] \) be a polynomial ring over a field \( k \) of characteristic zero. Then, for any free divisor \( f \in S \), the ideal \( (f \partial_{x_1}, \ldots, f \partial_{x_n}, f^2) \) is tangentially free.
We point out that free ideals are not necessarily of the form \( x \partial x \), as shown in the example below.

**Example 3.28** Let \( S = \mathbb{C}[x, y, z] \) be a polynomial ring. Pick the free divisor \( f = xyz \), a basis of \( \mathcal{J}_C(f) \) being \( \{ \epsilon, x \frac{\partial}{\partial x} - z \frac{\partial}{\partial z}, y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \} \). Consider the ideals \( \mathfrak{a} = (xz, yz) \) and \( \varphi = (x, y) \). Clearly, \( \mathfrak{a} \subseteq \mathcal{J}_f \subseteq \varphi \), and one easily checks that each element of the basis preserves both \( \mathfrak{a} \) and \( \varphi \). Hence, by Theorem 3.22, \( f \mathfrak{a} \) and \( f\varphi \) are tangentially free ideals (with the same tangential idealizer).

### 4 Geometric comments

We conclude the paper with a simple short section on the geometric side of some of the results and notions treated. Having studied tangential idealizers, the natural geometric language seems to be that of vector fields, which is very closely related to the beautiful theory of holomorphic foliations, not considered in a direct way herein (we refer the interested reader to the classical work [15], as well as the papers [8], [11] and their recommended references).

For convenience, we shall adopt the field \( \mathbb{C} \) of complex numbers as our ground field, and algebraic varieties will be assumed to be reduced — otherwise, an approach by means of scheme theory might be plausible; moreover, we point out that such an assumption is not so restrictive to our purposes in virtue of Proposition 2.33, that is, we could simply require that algebraic varieties must not have embedded components.

*Tangency of vector fields.* Let \( S = \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial ring, seen as the coordinate ring of the complex affine \( n \)-space \( \mathbb{A}^n \). First, as usual, we interpret a \( \mathbb{C} \)-derivation \( \delta : S \to S \) as a polynomial vector field on \( \mathbb{A}^n \), as one can write \( \delta = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i} \), for polynomial functions \( h_i : \mathbb{A}^n \to \mathbb{C} \), \( i = 1, \ldots, n \). In other words, to any point \( p \in \mathbb{A}^n \) one associates the vector \( \delta(p) \) with complex coordinates \( h_1(p), \ldots, h_n(p) \), with respect to the basis \( \{ (\frac{\partial}{\partial x_i})_{p} \}_{i=1}^n \) of the tangent \( \mathbb{C} \)-vector space \( \mathcal{T}_p \mathbb{A}^n \) of \( \mathbb{A}^n \) at \( p \).

Now let \( X \subset \mathbb{A}^n \) be an algebraic variety defined by an ideal \( \mathfrak{a}_X \subset S \). Then, as one sees easily (and almost tautologically, by interpreting the role played by the jacobian matrix of a set of generators of \( \mathfrak{a}_X \) within the definition of tangential idealizer), the geometric meaning of its tangential idealizer \( \mathcal{T}_C(X) = \mathcal{T}_C(\mathfrak{a}_X) \) is that it collects the ambient vector fields — that is, vector fields defined globally on \( \mathbb{A}^n \) — that are tangent to \( X \). It should not be confused with the module \( \text{Der}_C(S/\mathfrak{a}_X) \) consisting of the tangent vector fields defined on \( X \).

Any vector field \( \delta = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i} \in \mathcal{T}_C(X) \) yields an algebraic tangent vector field \( \bar{\delta} \) on \( X \) given by \( \bar{\delta}(f \text{ mod } \mathfrak{a}_X) = \delta(f) \text{ mod } \mathfrak{a}_X \), for \( f \in S \), which is clearly well-defined. Proposition 2.9 says that, conversely, any \( \delta \in \text{Der}_C(S/\mathfrak{a}_X) \) may be obtained in this way, that is, \( \delta \) lifts to an element of \( \mathcal{T}_C(X) \).

A word on relative tangential idealizers: for a subvariety \( Y \subset X \) with ideal \( \mathfrak{a}_Y \subset S \), an element of the module \( \mathcal{T}_C(\mathfrak{a}_X, \mathfrak{a}_Y) \) — formed with the \( C \)-derivations sending \( \mathfrak{a}_X \) into \( \mathfrak{a}_Y \) — may be simply interpreted as a restriction, to \( Y \), of a global vector field tangent along \( X \). Of course, such a restriction does not have to be tangent along \( Y \) in general. There is, however, a simple instance where tangency is kept: let \( X = \bigcup_i X_i \) be an algebraic variety and let \( Y = \bigcup_j Y_j \subset X \) be a subvariety, both decomposed in terms of their irreducible...
components. If, for every \( j \),
\[
Y_j \not\subseteq \bigcup_{X_i \not\subseteq Y} X_i
\]
then, for any vector field \( \vartheta \) tangent along \( X \), its restriction \( \vartheta|_Y \) to \( Y \) must be tangent along \( Y \). Note that this is the geometric content of Proposition 2.20 for radical ideals (standard prime avoidance is being used here). In case \( X \) is already irreducible, the above condition is fulfilled if and only if \( Y = X \).

**Free varieties.** We define a *(tangentially) free variety* as being an algebraic variety whose defining ideal is tangentially free in our sense. As we suggested the notion of abstract free divisor in a purely algebraic fashion, let us reserve the denomination *free hypersurface* for geometric free divisors. Thus, a geometric interpretation of a special case of Theorem 3.22 follows at once: a non-empty proper subvariety \( Y \) of a smooth (hence, arithmetically factorial) variety \( X \) is a free variety if and only if it may be written \( Y = Z \cup Y' \), where \( Z \subset X \) is a free hypersurface and \( Y' \subset X \) is a subvariety with the property that every tangent vector field on \( X \) that is tangent along \( Z \) must also be tangent along \( Y' \). Note that free varieties are non-reduced, in general. Smoothness of the ambient \( X \) here is being imposed in order to avoid “differential” subvarieties, as regular rings are differentially simple. This can be applied more concretely in case \( X = \mathbb{A}^n \).

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