Weak turbulence of gravity waves.

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For the first time weak turbulent theory was demonstrated for the surface gravity waves. Direct numerical simulation of the dynamical equations shows Kolmogorov turbulent spectra as predicted by analytical analysis [1] from kinetic equation.

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In this Letter we study numerically the steady Kolmogorov spectra for spatially homogeneous gravity waves. According to the theory of weak turbulence the main physical process here is the stationary energy flow to the small scales, where the energy dissipates [1, 2]. This flow is described by kinetic equation which has power-like solutions – Kolmogorov spectra. This straightforward picture takes place experimentally and numerically for different physical situations. For capillary waves it was observed on the surface of liquid hydrogen [3], [4]. The numerical simulation of this process was performed in [5]. In nonlinear fiber optics these spectra were demonstrated in numerical simulation [6]. There are many other results [7, 8, 9, 10, 11]. One of the most interesting applications of the weak turbulence theory is the surface gravity waves. From the pioneering article by Toba [12] to the most recent observations [13] many experimentalists get the spectra predicted by the weak turbulence theory. But these experiments cannot be treated as a complete confirmation because the Zakharov-Filonenko spectrum is isotropic, while observed spectra are essentially anisotropic. It is worth to say that the wave kinetic equation, which is the keystone of this theory, was derived under several assumptions. Namely, it was assumed, that the phases of all interacting waves are random and are in state of chaotic motion. The validity of this proposition is not clear a priori. The direct numerical simulation of nonlinear dynamical equations can give us a confirmation this assumption valid or not. But for particular case of gravity surface waves the numerical confirmation was absent in spite of significant efforts were applied. The only successful attempt in this direction was the simulation of freely decaying waves [14]. The reason for that for our opinion was concerned with a choice of numerical scheme parameters.

Theoretical background. — Let us consider the potential flow of an ideal incompressible fluid of infinite depth and with a free surface. We use standard notations for velocity potential $\phi(r, z, t)$, $r = (x, y)$; $v = \nabla \phi$ and surface elevation $\eta(r, t)$. Fluid flow is irrotational $\Delta \phi = 0$. The total energy of the system can be represented in the following form

$$H = T + U,$$

where $g$ – is the gravity acceleration. It was shown [16] that under these assumptions the fluid is a Hamiltonian system

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta},$$

which is in the agreement with real experiments [12, 13].
where \( \psi = \phi(\mathbf{r}, \eta(\mathbf{r}, t), t) \) is a velocity potential on the surface of the fluid. In order to calculate the value of \( \psi \) we have to solve the Laplas equation in the domain with varying surface \( \eta \). This problem is difficult. One can simplify the situation, using the expansion of the Hamiltonian in powers of ”steepness”

\[
H = \frac{1}{2} \int \left( g\eta^2 + \psi \dot{\psi} \right) d^2r + \frac{1}{2} \int \eta \left[ (\nabla \psi)^2 - (k\psi)^2 \right] d^2r + \frac{1}{2} \int (k\psi)^2 \left[ \dot{\psi}_\alpha \eta \right] d^2r. \tag{5}
\]

For gravity waves it is enough to take into account terms up to the fourth order. Here \( \dot{\psi} \) is the linear operator corresponding to multiplying of Fourier harmonics by modulus of the wavenumber \( k \). In this case dynamical equations \((\ref{5})\) acquire the following form

\[
\begin{align*}
\dot{\psi} &= -g\eta - \frac{1}{2} \left[ (\nabla \psi)^2 - (k\psi)^2 \right] - [k\psi][k\psi \eta] \psi - \left[ k^2 \psi \eta \right] \psi \eta + D_r + F_r.
\end{align*}
\]

Here \( D_r \) is some artificial damping term used to provide dissipation at small scales; \( F_r \) is a pumping term corresponding to external force (having in mind wind blow, for example). Let us introduce Fourier transform

\[
\psi_k = \frac{1}{2\pi} \int \psi(r)e^{ikr}d^2r, \quad \eta_k = \frac{1}{2\pi} \int \eta(r)e^{ikr}d^2r.
\]

With these variables the Hamiltonian \((\ref{5})\) acquires the following form

\[
H = H_0 + H_1 + H_2 + \ldots,
\]

\[
H_0 = \frac{1}{2} \int \left( \frac{|k|}{|\eta_k|}^2 + g|\eta_k|^2 \right) dk,
\]

\[
H_1 = -\frac{1}{4\pi} \int L_k k_1 k_2 \psi_{k_1} \psi_{k_2} \eta_k \times \delta(k_1 + k_2 + k_3) dk_1 dk_2 dk_3,
\]

\[
H_2 = \frac{1}{16\pi} \int M_{k_1 k_2 k_3} \psi_{k_1} \psi_{k_2} \eta_k \times \delta(k_1 + k_2 + k_3 + k_4) dk_1 dk_2 dk_3 dk_4.
\]

Here

\[
L_{k_1 k_2} = (k_1 k_2) + |k_1||k_2|,
\]

\[
M_{k_1 k_2 k_3} = |k_1| |k_2| \left\{ \frac{1}{2} |k_1 + k_3|^2 + |k_1 + k_4|^2 + |k_2 + k_3|^2 + |k_2 - k_3|^2 \right\} - |k_1| - |k_2|.
\]

It is convenient to introduce the canonical variables \( a_k \) as shown below

\[
a_k = \sqrt{\frac{\omega_k}{2k}} \eta_k + i \sqrt{\frac{k}{2\omega_k}} \psi_k,
\]

where

\[
\omega_k = \sqrt{gk},
\]

this is the dispersion relation for the case of infinite depth. The similar formulas can be derived in the case of finite depth \([17]\). With these variables the equations \((\ref{4})\) take the following form

\[
\dot{a}_k = -i \frac{\delta H}{\delta a_k}. \tag{11}
\]

The dispersion relation \((\ref{10})\) is of the ”non-decay type” and the equations

\[
\omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \quad k_1 = k_2 + k_3 \tag{12}
\]

have no real solution. It means that in the limit of small nonlinearity, the cubic terms in the Hamiltonian can be excluded by a proper canonical transformation \( a(k, t) \rightarrow b(k, t) \) \([18]\). The formula of this transformation is rather bulky and well known \([17][18]\), so let us omit the details here.

For statistical description of a stochastic wave field one can use a pair correlation function

\[
< a_{k}a_{k'}^* >= n_k \delta(k - k'). \tag{13}
\]

The \( n_k \) is measurable quantity, connected directly with observable correlation functions. For instance, from \((\ref{9})\) one can get

\[
I_k = < |\eta_k|^2 > = \frac{1}{2} \frac{\omega_k}{g} (n_k + n_{-k}). \tag{14}
\]

In the case of gravity waves it is convenient to use another correlation function

\[
< b_{k}b_{k'}^* >= N_k \delta(k - k'). \tag{15}
\]

The function \( N_k \) cannot be measured directly. The relation connecting \( n_k \) and \( N_k \) is rather complex in the case of fluid of finite depth. But in the case of deep water it becomes very simple \([17]\]

\[
\frac{n_k - N_k}{n_k} \simeq \mu, \tag{16}
\]

where \( \mu = (ka)^2 \), here \( a \) is a characteristic elevation of the free surface. In the case of the weak turbulence \( \mu < < 1 \). The correlation function \( N_k \) obey the kinetic equation \([11]\)

\[
\frac{dN_k}{dt} = st(N, N, N) + f_p(k) - f_d(k). \tag{17}
\]

Here

\[
st(N, N, N) = 4\pi \int |T_{k_1 k_2 k_3}|^2 \times (N_{k_1}N_{k_2}N_{k_3} + N_{k_1}N_{k_2}N_{k_3} - N_{k_1}N_{k_1}N_{k_2} - N_{k_1}N_{k_1}N_{k_3}) \delta(k + k_1 - k_2 - k_3)dk_1dk_2dk_3. \tag{18}
\]
The complete form of matrix element \( T_{k_1,k_1,k_2,k_3} \) can be found in many sources [1, 2, 17]. Function \( f_\beta(k) \) in (16) corresponds to wave pumping due to wind blow for example. Usually it is located on long scales. Function \( f_\beta(k) \) represents the absorption of waves due to viscosity and wave-breaking. None of this functions are known to a sufficient degree.

Let us consider stationary solutions of the equation (17) assuming that

- The medium is isotropic with respect to rotations;
- Dispersion relation is a power-like function \( \omega = \alpha k^\beta \);
- \( T_{k_1,k_1,k_2,k_3} \) is a homogeneous function: \( T_{k_1,k_1,k_2,k_3} = e^{\beta k} T_{k_1,k_1,k_2,k_3} \).

Under this assumptions one can get Kolmogorov solutions [18]

\[
\begin{align*}
n_k^{(1)} & = C_1 P^{1/3} k^{-\frac{2\beta}{3} - d}, \\
n_k^{(2)} & = C_2 Q^{1/3} k^{-\frac{2\beta}{3} + \frac{1}{3} - d}.
\end{align*}
\]

Here \( d \) is a spatial dimension (\( d = 2 \) in our case). The first one is a Kolmogorov spectrum, corresponding to a constant flux of energy \( P \) to the region of small scales (direct cascade of energy). The second one is Kolmogorov spectrum, describing inverse cascade of wave action to large scales, and \( Q \) is a flux of action. In both cases \( C_1 \) and \( C_2 \) are dimensionless “Kolmogorov’s constants”.

In the case of deep water \( \omega = \sqrt{\gamma k} \) and, apparently, \( \beta = 3 \). It is known since [11] that on deep water

\[
n_k^{(1)} = C_1 P^{1/3} k^{-4}.
\]

In the same way [19] for second spectrum

\[
n_k^{(2)} = C_2 Q^{1/3} k^{-3/2}.
\]

In this Letter we will explore the first spectrum (energy cascade). Using [14] one can get

\[
I_k = \frac{C_1 g^{1/2} P^{1/3}}{k^{7/2}}.
\]

**Numerical Simulation** — Dynamical equations (6) are very hard for analytical analysis. One of the main obstacles is the \( \hat{k} \)-operator which is nonlocal. However, using Fourier technique practically makes no difference between derivative and \( \hat{k} \). The numerical simulation of the system is based upon consequent application of fast Fourier transform algorithm. The details of this numerical scheme will be published separately.

For numerical integration of (6) we used the functions \( F \) and \( D \) defined in Fourier space

\[
\begin{align*}
F_k & = f_k e^{i R_k(t)}, \\
f_k & = 4 F_0 \frac{(k - k_{p1})(k_{p2} - k)}{(k_{p2} - k_{p1})^2}; \\
D_k & = \gamma_k \psi_k, \\
\gamma_k & = -\gamma_1, k \leq k_{p1}, \\
\gamma_k & = -\gamma_2 (k - k_d)^2, k > k_d.
\end{align*}
\]

Here \( R_k(t) \) is the uniformly distributed random number in the interval \((0, 2\pi)\). We have solved system of equations (6) in the periodic domain \( 2\pi \times 2\pi \) (the wave-numbers \( k_x \) and \( k_y \) are integers in this case). The size of the grid was chosen \( 256 \times 256 \) points. Gravity acceleration \( g \) = 1. Parameters of the damping and pumping were the following: \( k_{p1} = 5, k_{p2} = 10, k_d = 64 \). Thus the inertial interval is about half of decade.

During the simulations we paid special attention to the problems which could “damage” the calculations. First of all, the “bottle neck” phenomenon at the boundary between inertial interval and dissipation region. This effect is very fast, but can be effectively suppressed by proper choice of damping value \( \gamma_2 \) in the case of moderate pumping values \( F_0 \). The second problem is the accumulation of “condensate” in low wave numbers. This mechanism for the case of capillary waves was examined in details in [15]. This obstacle can be overcome by simple adaptive damping scheme in the small wave numbers. After some time system reaches the stationary state, where the equilibrium between pumping and damping takes place. Important parameter in this state is the ratio of nonlinear energy to the linear one \( (H_1 + H_2)/H_0 \).

For example, in the case of \( F_0 = 2 \times 10^{-4}, \gamma_1 = 1 \times 10^{-3} \), \( \gamma_2 = 400 \) the level of nonlinearity was equal to \( (H_1 + H_2)/H_0 \approx 2 \times 10^{-3} \). The Hamiltonian as a function of time is shown in Fig. 11.

The surface elevation correlator function appears to be power-like in the essential part of inertial interval, where the influence of pumping and damping was small. The correlator is shown in Fig. 2.

One can try to estimate the exponent of the spectrum. It is worth to say that an alternative spectrum was proposed earlier by Phillips [20]. That power-like spectrum is due to wave breaking mechanism and gives us a surface elevation correlator as \( I_k \sim k^{-4} \). Compensated spectra are shown in the Fig. 8. It seems to be an evidence, that the Kolmogorov spectrum predicted by weak turbulence theory better fit the results of the numerical experiment.
The inertial interval was rather narrow (half a decade). But the obtained results allow us to conclude, that accuracy of experiment was good enough under the time constraints of simulation (we get the steady state after 20-30 h using available hardware, and we need several days to average $|\eta_k|^2$ function). The simulation on larger grid ($512 \times 512$, for example) can make the accuracy better. But even these results give us a clear qualitative picture.

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