Charge conjugation from space-time inversion in QED: discrete and continuous groups

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Abstract
We show that the CPT groups of QED emerge naturally from the $\mathcal{PT}$ and $\mathcal{P}$ (or $\mathcal{T}$) subgroups of the Lorentz group. We also find relationships between these discrete groups and continuous groups, like the connected Lorentz and Poincaré groups and their universal coverings.

Keywords: CPT groups; space-time inversion; Lorentz and Poincaré groups

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1 Introduction

It was shown in [1] that the CPT group, $G_{\hat{\theta}}(\hat{\psi})$ ($\hat{\theta} = \hat{C} \ast \hat{P} \ast \hat{T}$), of the Dirac quantum field is a non abelian group with sixteen elements isomorphic to the direct product of the quaternion group, $Q$, and the cyclic group, $\mathbb{Z}_2$:

$$G_{\hat{\theta}}(\hat{\psi}) \cong Q \times \mathbb{Z}_2.$$ (1)

Unlike $G_{\hat{\theta}}(\hat{\psi})$ [1, 2, 3], the CPT group, $G_{\hat{\theta}}(\hat{A}_\mu)$, of the electromagnetic field is an abelian group of eight elements with three generators [2]:

$$G_{\hat{\theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3.$$ (2)

As the CPT transformation properties of the interacting $\hat{\psi} - \hat{A}_\mu$ fields are the same as for the free fields [4], the complete CPT group for QED, $G_{\hat{\Theta}}(QED)$, is the direct product of the above mentioned two groups, $G_{\hat{\Theta}}(\hat{\psi})$ and $G_{\hat{\Theta}}(\hat{A}_\mu)$, i.e.,

$$G_{\hat{\Theta}}(QED) = G_{\hat{\Theta}}(\hat{\psi}) \times G_{\hat{\Theta}}(\hat{A}_\mu) \cong (Q \times \mathbb{Z}_2) \times \mathbb{Z}_2^3.$$ (3)

2 C from $\mathcal{PT}$

It was shown in [3] that $Q$ becomes isomorphic to a subgroup $H$ of $SU(2)$, being $\lambda$ the isomorphism:

$$Q \xrightarrow{\lambda} H < SU(2),$$

$1 \mapsto I, \quad \iota \mapsto -i\sigma_1, \quad \gamma \mapsto -i\sigma_2, \quad \kappa \mapsto -i\sigma_3,$ (4)

where $\iota$, $\gamma$, $\kappa$ are the three imaginary units of the quaternion group and $\sigma_k$ ($k = 1, 2, 3$) are the Pauli matrices; and taking also into account that $\mathbb{Z}_2$ is isomorphic to the center of $SU(2)$: $\{I, -I\}$, then:

$$G_{\hat{\theta}}(\hat{\psi}) \cong H \times (\text{center of } SU(2)).$$ (5)

Since $SU(2)$ is the universal covering group of $SO(3)$:

$$SU(2) \xrightarrow{\Phi} SO(3),$$ (6)
then $\Phi(H)$ has 4 elements and, for that reason, the unique candidates are groups isomorphic to $C_4$ and $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein group. A simple application of $\Phi$ to the elements of $H$ led to:

$$\Phi(H) = \{I, R_x(\pi), R_y(\pi), R_z(\pi)\},$$

(7)

with $R_x(\pi), R_y(\pi), R_z(\pi)$ the rotations in $\pi$ around the axes $x, y$ and $z$, respectively, and $I$, the unit matrix in $SO(3)$. It was then immediately verified that the multiplication table of $\Phi(H) < SO(3)$ is the same as for $D_2$.

Then, we have:

$$G_\theta(\hat{\psi}) \cong \Phi^{-1}(D_2) \times \mathbb{Z}_2.$$

(8)

Within the Lorentz group $O(3,1)$, the transformations of parity $\mathcal{P}$ and time reversal $\mathcal{T}$, together with their product $\mathcal{P}\mathcal{T}$ and the $4 \times 4$ unit matrix $E$, lead to the subgroup of the Lorentz group, called the $\mathcal{P}\mathcal{T}$-group, which is also isomorphic to $D_2$.

On the other hand, $\mathcal{P}$ or $\mathcal{T}$ separately, together with the unit $4 \times 4$ matrix $E$, give rise to the group $\mathbb{Z}_2$. Then, we obtain the desired result for the Dirac field:

$$G_\theta(\hat{\psi}) \cong \Phi^{-1}(\langle \{ \mathcal{P}, \mathcal{T} \} \rangle \times \langle \{ \mathcal{P} \} \rangle)$$

(9)

or

$$G_\theta(\hat{\psi}) \cong \Phi^{-1}(\langle \{ \mathcal{P}, \mathcal{T} \} \rangle \times \langle \{ \mathcal{T} \} \rangle);$$

(10)

while, for the electromagnetic field, we have:

$$G_\theta(\hat{\mathcal{A}}_{\mu}) \cong \langle \{ \mathcal{P}, \mathcal{T} \} \rangle \times \langle \{ \mathcal{P} \} \rangle$$

(11)

or

$$G_\theta(\hat{\mathcal{A}}_{\mu}) \cong \langle \{ \mathcal{P}, \mathcal{T} \} \rangle \times \langle \{ \mathcal{T} \} \rangle.$$

(12)

The above result suggests that the Minkowskian space-time structure of special relativity, in particular the unconnected component of its symmetry group, the real Lorentz group $O(3,1)$, implies the existence of the CPT group as a whole, and therefore the existence of the charge conjugation transformation, and thus the proper existence of antiparticles.
3 Discrete and continuous groups

The relationships between the discrete groups: $Q, G_{PT} = \{P, T\}$, $G_{\hat{\phi}(\hat{\psi})}$ and $G_{\hat{\phi}(\hat{A}_\mu)}$ and continuous groups, like the Lorentz group and its universal covering group, can be summarized in the following diagram:

\[
\begin{array}{cccccc}
Z_2 & \downarrow & Z_2 & \downarrow & Z_2 & \downarrow \\
G_{\hat{\phi}(\hat{\psi})} \cong Q \times Z_2 & \xleftarrow{\psi} & Q & \xrightarrow{\mu} & SU(2) & \xrightarrow{\beta} SL_2(\mathbb{C}) & \xrightarrow{\gamma} \mathbb{R}^4 \times SL_2(\mathbb{C}) \\
G_{\hat{\phi}(\hat{A}_\mu)} \cong \mathbb{Z}_2^3 & \xleftarrow{\bar{\alpha}} & G_{PT} \cong \mathbb{Z}_2^2 & \xrightarrow{\bar{\nu}} SO(3) & \xrightarrow{\bar{\beta}} SO^c(3,1) & \xrightarrow{\bar{\gamma}} \mathbb{R}^4 \times SO^c(3,1) \\
\end{array}
\]

The homomorphism $\mu$ is defined by $\mu(q) = \lambda(q)$ (see (14)) and the homomorphism $\Phi$ was described in (6); $\tilde{\Phi}$ and $\bar{\Phi}$ are the homomorphisms between the connected Lorentz ($SO^c(3,1)$) and Poincaré ($\mathbb{R}^4 \times SO^c(3,1) \equiv \mathcal{P}_4^c$) groups, respectively, and their universal coverings ($SL_2(\mathbb{C})$ and $\mathbb{R}^4 \times SL_2(\mathbb{C}) \equiv \bar{\mathcal{P}}_4^c$); while $\rho$, $\psi$, $\bar{\rho}$, $\bar{\alpha}$, $\beta$, $\bar{\beta}$, $\gamma$ and $\bar{\gamma}$ are given by:

\[
Q \xrightarrow{\rho} \frac{Q}{\mathbb{Z}_2} \cong G_{PT}, \quad q \mapsto [q], \quad (14)
\]

\[
G_{\hat{\phi}(\hat{\psi})} \xrightarrow{\psi} \frac{Q \times \mathbb{Z}_2}{\mathbb{Z}_2} \cong G_{\hat{\phi}(\hat{A}_\mu)}, \quad (q,1) \mapsto [(q,1)], \quad (q,-1) \mapsto [(q,1)], \quad (15)
\]

\[
G_{PT} \xrightarrow{\bar{\nu}} SO(3), \quad [q] \mapsto \Phi(h), \quad h = \lambda(q), \quad (16)
\]

\[
Q \xrightarrow{\alpha} G_{\hat{\phi}(\hat{\psi})}, \quad q \mapsto (q,1), \quad (17)
\]

\[
G_{PT} \xrightarrow{\bar{\alpha}} G_{\hat{\phi}(\hat{A}_\mu)}, \quad [q] \mapsto [(q,1)], \quad (18)
\]

\[
SU(2) \xrightarrow{\beta} SL_2(\mathbb{C}), \quad A \mapsto A, \quad (19)
\]

\[
SO(3) \xrightarrow{\bar{\beta}} SO^c(3,1), \quad R \mapsto \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad (20)
\]
\[ SL_2(\mathbb{C}) \xrightarrow{\gamma} \mathbb{R}^4 \rtimes SL_2(\mathbb{C}), \quad B \mapsto (0, B), \quad (21) \]

\[ SO^c(3, 1) \xrightarrow{\bar{\gamma}} \mathbb{R}^4 \rtimes SO^c(3, 1), \quad \Lambda \mapsto (0, \Lambda). \quad (22) \]

Let \( \nu \) the function which goes from \( Q \times \mathbb{Z}_2 \) to \( SU(2) \):

\[ Q \times \mathbb{Z}_2 \xrightarrow{\nu} SU(2), \quad (q, g) \mapsto \nu(q, g) := sg(g)\lambda(q), \quad (23) \]

where \( sg(g) = 1 \) if \( g = 1 \) and \( sg(g) = -1 \) if \( g = -1 \).

Then, it holds:

- \( \nu \) is an homomorphism.
  
  Proof:

\[ \nu((q', g')(q, g)) = \nu(q'q, g'g) = sg(g')\lambda(q'q) = sg(g')sg(g)\lambda(q')\lambda(q) \]

\[ = (sg(g')\lambda(q'))(sg(g)\lambda(q)) = \nu(q', g')\nu(q, g). \quad (24) \]

- \( \nu \) is 2 to 1.
  
  Proof:

\[ \nu(q, -1) = \nu(-q, 1). \quad (25) \]

\( \bar{\nu} \) is determined by \( \nu \) due to the commutative diagram:

\[ G_{\hat{\theta}}(\hat{\psi}) \xrightarrow{\nu} SU(2) \]

\[ \downarrow \psi \quad \downarrow \Phi \]

\[ G_{\hat{\theta}}(\hat{A}_\mu) \xrightarrow{\bar{\nu}} SO(3) \quad (26) \]

and is also a 2 to 1 homomorphism. If \( x \in \mathbb{Z}_2^3 \) then \( \psi^{-1}\{x\} = \{y_1, y_2\} \subset Q \times \mathbb{Z}_2 \). Hence:

\[ \bar{\nu}(x) = \bar{\nu}(\psi(y_k)) = \bar{\nu} \circ \psi(y_k) = \Phi \circ \nu(y_k) \]

\[ = \Phi(\nu(y_k)) = \Phi(\nu(q_k, g_k)), \quad (27) \]

with \( k = 1 \) or 2. Then:

- \( \bar{\nu} \) is an homomorphism.
  
  Proof:

\[ \bar{\nu}(x'x) = \Phi(\nu((q'_k, g'_k)(q_l, g_l))) = \Phi(\nu(q_k, g_k))\Phi(\nu(q_l, g_l)) \]

\[ = \Phi \circ \nu(q'_k, g'_k)\Phi \circ \nu(q_l, g_l) = \bar{\nu} \circ \psi(q'_k, g'_k)\bar{\nu} \circ \psi(q_l, g_l) \]

\[ = \bar{\nu}(x')\bar{\nu}(x), \quad (28) \]

with \( l = 1 \) or 2.
4 Discussion

• $\bar{\nu}$ is 2 to 1.

Proof: From $\Phi \circ \nu = \bar{\nu} \circ \psi$ and the fact that $\Phi, \nu$ and $\psi$ are 2 to 1, it follows that $\bar{\nu}$ is also 2 to 1.

Taking into account diagrams (13) and (26), the group homomorphisms:

$$\varphi = \gamma \circ \beta \circ \nu$$

and

$$\bar{\varphi} = \bar{\gamma} \circ \bar{\beta} \circ \bar{\nu},$$

make commutative the following diagram:

$$\begin{array}{c}
G_\psi(\hat{\psi}) \xrightarrow{\varphi} \mathcal{P}_4^c \\
\downarrow \psi \quad \downarrow \bar{\psi} \\
G_\phi(\hat{A}_\mu) \xrightarrow{\bar{\varphi}} \mathcal{P}_4^c;
\end{array}$$

making explicit the close and possibly deep relationship between these discrete and continuous groups.

4 Discussion

In summary, we have that $G_\psi(\hat{\psi})$ and $G_\psi(\hat{A}_\mu)$, which are groups acting at the quantum field level that include the charge conjugation operator, emerge in a natural way from the $\mathcal{PT}$-group and its $\mathcal{P}$ (or $\mathcal{T}$) subgroups. That is, from matrices acting on Minkowski classical space-time.

It is important to note that $G_{\mathcal{PT}}$ generates $G_\phi(\hat{A}_\mu)$, the CPT group of the electromagnetic field, without passing through $SU(2)$. That is, without the need of using spinors; while the group $SU(2)$ is needed in order to generate $G_\psi(\hat{\psi})$, the CPT group of the Dirac field.

Finally, another important thing that we found is the relationship between discrete groups, like $G_\phi(\hat{A}_\mu)$ and $G_\psi(\hat{\psi})$, and continuous groups, like the connected Poincaré group ($\mathcal{P}_4^c$) and its universal covering ($\bar{\mathcal{P}}_4^c$). This is shown in diagram (31).

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References

[1] M. Socolovsky, *The CPT group of the Dirac Field*, Int J Theor Phys, **43** (2004), pp. 1941-1967; arXiv: math-ph/0404038.

[2] B. Carballo Pérez and M. Socolovsky, *Irreducible representations of the CPT groups in QED*, IJPAM (in press) (2010); arXiv: math-ph/0906.2381v3.

[3] B. Carballo Pérez and M. Socolovsky, *Charge conjugation from spacetime inversion*, Int J Theor Phys, **48** (2009), pp. 1712-1716; arXiv: hep-th/0811.0842v1.

[4] J. A. de Azcárraga, *P, C, T, θ in Quantum Field Theory*, GIFT 7/75 (1975), 69.