Operator ordering and Classical soliton path
in Two-dimensional $N = 2$ supersymmetry
with Kähler potential

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Abstract

We investigate a two-dimensional $N = 2$ supersymmetric model which consists of $n$ chiral superfields with Kähler potential. When we define quantum observables, we are always plagued by operator ordering problem. Among various ways to fix the operator order, we rely upon the supersymmetry. We demonstrate that the correct operator order is given by requiring the super-Poincaré algebra by carrying out the canonical Dirac bracket quantization. This is shown to be also true when the supersymmetry algebra has a central extension by the presence of topological soliton. It is also shown that the path of soliton is a straight line in the complex plane of superpotential $W$ and triangular mass inequality holds. One half of supersymmetry is broken by the presence of soliton.

Keywords: Supersymmetry; operator ordering; soliton; Bogomol’nyi bound.
1 Introduction

In the process of quantization of field theory with curved target space, e.g. nonlinear
sigma model, we are usually plagued by the operator ordering in the definition of
various quantum observables. We must find out properly ordered quantum operator
\( \hat{F} \) from classical dynamical variable \( F(x, p) \). Various traditional ways are known for
this problem.

The first way of them comes from path integral. The midpoint prescription in
the path integral leads to the Weyl ordered operators:

\[
\int \frac{dp}{2\pi\hbar} e^{-ip(x-x')} H(x+x', p) = \langle x'| H(x, p) | x \rangle_{\text{Weyl}}
\]  

(1)

where \( H(x, p) \) is any classical dynamical variable [1]. However the Weyl ordering
does not always give sufficient answer, as the following examples indicate.

The second approach is connected with the self-adjoint extension of Hermitian
operators. To illustrate it, let us pick up a Hermitian operator

\[
F = x^3 p + px^3.
\]  

(2)

This has a pure imaginary eigenvalue

\[
F \psi = -i \psi
\]  

(3)

with eigenfunction

\[
\psi_{\pm}(x) = |x|^{-\frac{1}{2}} e^{-\frac{1}{4x^2}} \theta(\pm x)
\]  

(4)

where \( \theta(x) \) is a step function. On the other hand \(+i\) is not an eigenvalue. Therefore
the index of this operator is (0,2) which states that \( F \) does not have a self-adjoint
extension. Therefore an appropriate ordering of \( 2x^3 p \) should be sought out. However
the Weyl ordering makes no use here because \( F' = x^2 px + xpx^2 = F \). Therefore
some other prescription is needed to obtain the quantum counterpart of \( 2x^3 p \) [2].

The third approach is connected with symmetry: how symmetry dictates the
operator ordering. Mostafazadeh fixes the order of operators which appear in super-
charge \( Q \) by Peierls bracket quantization in supersymmetric system [3]. Pursuing
this approach, we consider a two-dimensional \( N = 2 \) Wess-Zumino type model which
consists of \( n \) chiral superfields with Kähler potential \( K(\phi, \phi^*) \). When this term is
flat, \( K(\phi, \phi^*) = \phi \phi^* \), there was no ordering problem because the target space is flat [4]. We shall admit that general Kähler potential raises the ordering problem. We
rely upon the supersymmetry algebra [5, 6] to fix the operator ordering: we will fix
the operator orders by requiring the super-Poincaré algebra.

When the kinetic term has a nonflat Kähler potential, \( K(\phi, \phi^*) \neq \phi \phi^* \), the
operator ordering problem appears. There are several nonequal operator orders in
the supercurrent and canonical momentum operator. To fix the operator orders
correctly, we require that each component fields \( \varphi \) satisfy the following relations:

\[
- i[\varphi, Q]_\pm - i[\varphi, \bar{Q}]_\pm = \delta \varphi
\]  

(5)
where \( Q \) and \( \bar{Q} \) are supercharges. When \( Q \) and \( \bar{Q} \) satisfy this relation we can obtain the correct supersymmetry algebra. Then we can take it as the correct operator order. This is also true when the supersymmetry algebra has a central extension by the presence of topological soliton.

In the supersymmetric field theory the properties of soliton solution is studied long ago. As Witten and Olive pointed out, in the two-dimensional \( N = 1 \) supersymmetric theory the supersymmetry algebra is modified to include central charges in the presence of topological soliton \([7]\). The phenomenon which breaks a part of supersymmetry is studied since 1980’s \([8]\). It also happen in our model in the presence of soliton which saturates the Bogomol’nyi mass bound. In the presence of soliton, there are central extensions in the two-dimensional \( N = 1 \) supersymmetry \([7]\) and the two-dimensional \( N = 2 \) supersymmetry with a flat metric \([4]\). We can obtain a central extension in the two-dimensional \( N = 2 \) supersymmetry with nonflat Kähler metric in the presence of soliton. In general soliton is a curve which connect each zero energy solution. On the other hand we can obtain a straight line in the complex plane of superpotential. Then we can obtain triangular inequality

\[
M_{IK} < M_{IJ} + M_{JK}
\]

among the classical mass of solitons. It means that there is attractive force between neighboring solitons. There isn’t marginal stability such as in refs. 9 and 10 because the equality does not hold. Because the classical mass of soliton saturates the Bogomol’nyi mass bound, it breaks a half of supersymmetry.

This paper is constructed as follows. In Sec. 2 we construct the two-dimensional \( N = 2 \) supersymmetry which consists of two-dimensional chiral superfield \( \phi \). We derive canonical quantization conditions through Dirac brackets. In Sec. 3 we fix the operator orders in \( j^\mu \) and \( \pi_{a^i} \). We obtain the correct operator order in \( j^\mu \) and \( \pi_{a^i} \) which satisfy the correct supersymmetry algebra. It contains a central charge. In Sec. 4 we apply Hamilton-Jacobi method of classical mechanics to bosonic Lagrangian. We obtain a straight line of soliton path in complex \( W \)-plane. Classical mass of soliton saturates Bogomol’nyi bound and it satisfy triangular mass inequality. In Sec. 5 we show the multiplet shortening of supersymmetry algebra. In Sec. 6 we check the “central charge” \( T \) commutes with other operators. Section 7 is devoted to conclusion of the work.

## 2 Noether current and Dirac bracket quantization

We consider the two-dimensional \( N = 2 \) supersymmetric theory. A chiral superfield is given by

\[
\phi = a(x_+) + \sqrt{2}\bar{\theta}^c \xi(x_+) + \bar{\theta}^c \theta f(x_+)
\]
where $x^\mu_+ = x^\mu + i \bar{\theta} \gamma^\mu \theta$ and $\xi$ and $\theta$ are two-dimensional Dirac spinors. We use 2-dimensional $\gamma$ matrices in the following representation:

$$\gamma^0 = \sigma_2, \quad \gamma^1 = -i \sigma_1, \quad \gamma_5 = \gamma^0 \gamma^1 = -\sigma_3.$$  \hspace{1cm} (8)

When the kinetic term has a nonflat Kähler potential, the Lagrangian of Wess-Zumino type model of $n$ chiral superfields $\phi^i$ is given as

$$\mathcal{L} = \int d^2 \theta d^2 \theta^* K(\phi^i, \phi^{*i}) + \int d^2 \theta W(\phi^i) + \int d^2 \theta^* W(\phi^{*i})$$

$$= \partial_{a^i} a^{*i} K_{i*} j \partial^a^i a^j + \frac{1}{2} i K_{i*} j \xi^i \gamma^\mu \partial_{\mu} \xi^j + K^{*i} j W_i^* W_j$$

$$+ \frac{1}{2} i K_{pqi} \xi^p \gamma^\mu \partial_{\mu} \xi^i - \frac{1}{2} i K_{i*} j k \xi^i \gamma^\mu \partial_{\mu} \xi^k - \frac{1}{2} i \left(W_{ij}^* - K_{i*} j k K^{*k} j W_i^* \right) \bar{\xi}^i \xi^j$$

$$+ \frac{1}{4} \left( K_{i*} j k \right. - K^{*m} n K_{i*} j m K^{*m} n \left. \right) \bar{\xi}^i \xi^j \bar{\xi}^k \xi^l - \frac{1}{4} \Box K(a^i, a^{*i}),$$  \hspace{1cm} (9)

where $K$ is a Kähler potential and the lower indices $i$ of $K$ and $W$ mean the derivatives by $a^i$ and $i^*$ means the derivatives by $a^{*i}$,

$$K_{i*} j = \frac{\partial^2 K(a^i, a^{*j})}{\partial a^{*i} \partial a^j},$$  \hspace{1cm} (10)

$$W_i = \frac{\partial W(a^i)}{\partial a^i}.$$  \hspace{1cm} (11)

$W_i^*$ means $(W_i)^*$ and the orders of indices are commutable each other. The matrix $K$ with upper indices of $K$ is the inverse matrix of $K$:

$$K^{*i} j K_{j*} k = \delta^i_k, \quad K_{i*} j K^{*k} j = \delta^*_k i.$$  \hspace{1cm} (12)

From this Lagrangian, canonical energy-momentum tensor $T^{\mu\nu}$ is given as follows:

$$T^{00} = \partial_{a^{*i}} K_{i*} j \partial_0 a^j + \partial_1 a^{*i} K_{i*} j \partial_1 a^j - \frac{1}{2} i K_{i*} j \left( \bar{\xi}^i \gamma^1 \partial_1 \xi^j - \partial_1 \bar{\xi}^i \gamma^1 \xi^j \right)$$

$$- \frac{1}{2} i K_{i*} j k \bar{\xi}^i \gamma^1 \xi^j \partial_1 a^k + \frac{1}{2} i K_{i*} j k \bar{\xi}^i \gamma^1 \xi^j \partial_1 a^{*k} + K^{*i} j W_i^* W_j$$

$$- \frac{1}{2} i \left(W_{ij} - K^{*m} n K_{i*} j m W_k \right) \bar{\xi}^i \xi^j + \frac{1}{2} i \left(W_{ij}^* - K^{*k} m K_{i*} j m W_k \right) \bar{\xi}^i \xi^j$$

$$- \frac{1}{4} \left( K_{i*} j k \right. - K^{*m} n K_{i*} j m K^{*m} n \left. \right) \bar{\xi}^i \xi^j \bar{\xi}^k \xi^l$$

$$T^{01} = -\partial_{a^{*i}} K_{i*} j \partial_0 a^j - \partial_1 a^{*i} K_{i*} j \partial_1 a^j - \frac{1}{2} i K_{i*} j \left( \bar{\xi}^i \gamma^0 \partial_1 \xi^j - \partial_1 \bar{\xi}^i \gamma^0 \xi^j \right)$$

$$- \frac{1}{2} i K_{i*} j k \bar{\xi}^i \gamma^0 \xi^j \partial_1 a^k + \frac{1}{2} i K_{i*} j k \bar{\xi}^i \gamma^0 \xi^k \partial_1 a^{*j}.$$  \hspace{1cm} (13)
The supercurrent $J^\mu$ is given by Noether procedure
\[
J^\mu = \bar{\eta}^c j^\mu + \eta^c \bar{j}^\mu,
\]
\[
j^\mu = \sqrt{2} \left( \partial_\nu a^i K_{i+j}^* \gamma^\mu \xi^j \right),
\]
\[
\bar{j}^\mu = \sqrt{2} \left( \bar{\xi}^i \gamma^\nu K_{i+j} \partial_\nu a^j - \bar{\bar{\xi}}^i \gamma^\mu W_i \right), \tag{14}
\]
where $\bar{\eta}^c$ and $\eta^c$ are the parameters of supersymmetry transformation.

The supercharge is defined by
\[
Q = \int_{-\infty}^{\infty} j^0(x) dx. \tag{15}
\]

In classical theory the orders of the operators appear in $j^\mu$ and $\bar{j}^\mu$ are freely changed. But we have to fix the order of the operators when we transfer from classical theory to quantum theory. Our basic recipe to fix operator order is that it gives the correct supersymmetry algebra.

Canonical momentum for $a$ and $\xi$ are given as follows:
\[
\pi_a^i = \partial^0 a^{*j} K_{j+i} + \frac{1}{2} i K_{ij} \xi^0 \xi^j,
\]
\[
\pi_{a^{*i}} = K_{i+j} \partial^0 a^j - \frac{1}{2} i K_{ij} \xi^0 \xi^j,
\]
\[
\pi_{\xi^i} = \frac{1}{2} i K_{j+i} \xi^j,
\]
\[
\pi_{\bar{\xi}^i} = \frac{1}{2} i K_{i+j} \bar{\xi}^j. \tag{16}
\]

When the kinetic term has a flat Kähler potential, $K(\phi, \phi^*) = \phi^* \phi$, the supercurrent and canonical momentum becomes as follows:
\[
j^\mu = \sqrt{2} \left( \partial_\nu a^i \gamma^\nu \gamma^\mu \xi - W^* \gamma^\mu \xi^c \right), \tag{17}
\]
\[
\pi_a = \partial^0 a^* , \tag{18}
\]
\[
\pi_\xi = \frac{1}{2} i \xi^\dagger. \tag{19}
\]

In this case, $K(\phi, \phi^*) = \phi^* \phi$, there is no ordering problem in $j^\mu$ because $\partial_0 a^*$ and $\xi$ commute. Then we can obtain the correct supersymmetry algebra.

On the other hand, when the kinetic term has a nonflat Kähler potential, $K(\phi, \phi^*) \neq \phi^* \phi$, $\partial_0 a^*$ and $\xi$ become noncommutable. Then we cannot change the order of bosons and fermions freely in $j^\mu$. In the same way two orders in $\pi_{a^i}$, $\partial^0 a^i K_{j+i}$ and $K_{j+i} \partial^0 a^j$, are not equal. So we have to use the correct operator order in $j^\mu$ and $\pi_{a^i}$ to obtain the correct supersymmetry algebra. As we will see later, the expressions (14) and (16) give the correct operator orders in $j^\mu$ and $\pi_{a^i}$, respectively.

On canonical quantization, canonical momenta for $\xi$ give primary constraints:
\[
\chi_{\xi^i} = \pi_{\xi^i} - \frac{1}{2} i K_{j+i} \xi^j = 0 ,
\]
\[
\chi_{\bar{\xi}^i} = \pi_{\bar{\xi}^i} - \frac{1}{2} i K_{i+j} \bar{\xi}^j = 0. \tag{20}
\]
These constraints are the second class constraints.

Canonical quantization condition is given through Dirac bracket. There are 17 nonzero Dirac brackets in 55 Dirac brackets. There are 10 independent Dirac brackets in these 17 nonzero Dirac brackets. On calculation we use the operator order in $\pi_a$ as:

\[
\{a^i, \pi_{a^j}\}_D = \delta^i_j,
\{a^{si}, \pi_{a^{sj}}\}_D = \delta^{sj}_i,
\{\xi^i, \xi^j\}_D = -iK^{ij},
\{\xi^i, \pi_{a^j}\}_D = -\frac{1}{2}K^{ilr}K_{l^*jm}\xi^m,
\{\xi^i, \pi_{a^{sj}}\}_D = -\frac{1}{2}K^{ilr}K_{l^*jm}\xi^m,
\{\xi^{si}, \pi_{a^{sj}}\}_D = -\frac{1}{2}K^{ilr}K_{m^*jl}\xi^m
\]

Replacing these Dirac brackets with (anti)commutator divided by $i$, we obtain the following canonical quantization conditions:

\[
[a^i(x, t), \partial_0 a^{s^j}(y, t)] = K^{i\gamma}(x, t) \cdot i\delta(x - y),
[a^{si}(x, t), \partial_0 a^j(y, t)] = K^{i\gamma}(x, t) \cdot i\delta(x - y),
\{\xi^i(x, t), \xi^{j}(y, t)\} = -iK^{ij}(x, t)I^l \cdot i\delta(x - y),
[\xi^i(x, t), \partial_0 a^{sj}(y, t)] = -\left(K^{imn}K^{j^n}K_{m^*nl}\xi^l\right)(x, t) \cdot i\delta(x - y),
[\xi^{i}(x, t), \partial_0 a^{j}(y, t)] = -\left(K^{i^m}K^{j^m}K_{mn*l}\xi^{l}\right)(x, t) \cdot i\delta(x - y),
[\partial_0 a^{si}(x, t), \partial_0 a^{sj}(y, t)] = \left(-K^{i^l}\partial_0 a^{m}K_{m^*l^*k}K^{ki^*} + K^{i^l}K^{ki^*}K_{m^*l^*k}\partial_0 a^m
+iK^{ij}K^{ki^*}K_{nkq}K_{m^*l^*p}\xi^{lm}\xi^q
-iK^{ij}K^{ki^*}K_{nm*l^*}\xi^{lm}\xi^q\right)(x, t) \cdot i\delta(x - y).
\]

3 Fixing of operator order and SUSY algebra

We have to fix the order of the operators appear in $j^\mu$ to transfer from classical theory to quantum theory. Our basis to obtain correct operator order is that it gives the correct supersymmetry algebra. We select the operator order which satisfy
the following relation:
\[-i[\varphi, Q]_+ - i[\varphi, \bar{Q}]_+ = \delta \varphi,\]  
(23)
where \(\varphi\) represents each component fields
\[
\delta a^i = \sqrt{2} \bar{\eta}^c \xi^i \\
\delta \xi^i = -\sqrt{2i} \partial_\mu a^i \gamma^\mu \eta^c + \sqrt{2} \eta f^i
\]  
(24)
and where \(f^i\) are auxiliary fields which satisfy
\[
f^i = \frac{1}{2} K^{im*} K_{m*kl} \bar{\xi}^e \xi^k - i K^{ij*} W^*_j.
\]  
(25)
The supercurrent which satisfies the above relation (23) gives the correct supersymmetry algebra.

With the supercurrent in the previous operator order (14), each component fields satisfy these relations (23). Then the supercurrent in this operator order gives a correct operator order because it gives the correct supersymmetry algebra as follows:
\[
\{Q, \bar{Q}\} = 2 \gamma^\mu P_\mu, \quad \{Q, \bar{Q}^c\} = T \gamma^5
\]  
(26)
provided that the operator order in \(\pi_{a^i}\) is given as (16). The other supercurrents which are equivalent to (14) also satisfy (23) and then they are also correct operator orders. But the supercurrents which are not equivalent to (14) do not satisfy (23) and then they do not give the correct supersymmetry algebra. So they are wrong operator orders. Then the relation (23) plays a role to select the correct operator orders from all possible orders.

In the above, we use the operator order (16) as \(\pi_{a^i}\). The other order of \(\pi_{a^i}, K_{j*} i \partial a^{*ij}\), does not give the supersymmetry algebra correctly. Then the operator order (16) gives the correct operator order of \(\pi_{a^i}\).

In the above supersymmetry algebra \(T\) is given as follows:
\[
T = -4 \int_{-\infty}^{\infty} \partial_t (W^*(a^*(x))) \ dx \\
= -4 \{W^*(a^*(x = \infty)) - W^*(a^*(x = -\infty))\} \\
\equiv -4 \Delta W^*.
\]  
(27)
If the superpotential \(W^*(a^*(x))\) have different values at \(x = \infty\) and \(x = -\infty\), solitonic configurations of \(a(x)\), \(T\) has a nonzero value. In this case it gives a central extension of supersymmetry algebra.

4 Classical Soliton path in complex \(W\)-plane

We assume that \(W(\phi^i)\) is a holomorphic function such that \(W_i = 0\) has \(N\) complex solutions
\[
a_{(I)} = \left(a_{(I)}^1, a_{(I)}^2, \cdots, a_{(I)}^N\right)
\]  
(28)
where $I$ runs from 1 to $N$.

There are not only $N$ classical vacuum configurations $a(t, x) = a(I)$ but also solitonic configurations. There are at most $N(N - 1)$ solitonic configurations characterized by

$$a(t, -\infty) = a(I), \quad a(t, \infty) = a(J)$$

which we call $(I, J)$-soliton.

In the presence of $(I, J)$-soliton, supersymmetry algebra has a central charge.

In the center of mass frame $(P^\mu) = (M_{IJ}, 0)$ it becomes

$$\{Q_\alpha, Q^\dagger_\beta\} = 2M_{IJ} \delta_{\alpha\beta}, \quad \{Q_\alpha, Q_\beta\} = -4i(\sigma_1)_{\alpha\beta} \Delta_{IJ} W^*$$

where $\Delta_{IJ} W = W(a(J)) - W(a(I))$ and $\alpha, \beta$ run 1 and 2. We put

$$A = Q_1 + ie^{i\theta} Q^\dagger_2, \quad B = Q_1 - ie^{i\theta} Q^\dagger_2$$

then we have Bogomol’nyi bound of $(I, J)$-soliton

$$M_{IJ} \geq 2|\Delta_{IJ} W|$$

from positivity of $A$

$$\{A, A^\dagger\} \geq 0.$$

Applying Hamilton-Jacobi method of classical mechanics, we can show that this Bogomol’nyi bound is saturated by classical solution. Dropping the fermion degrees of freedom from $T^0_0$ and considering the static configuration, and regarding $x$ as time, we have the following “Lagrangian”:

$$\mathcal{L}' = \partial_i a^{*i} K_{*j} \partial_j a^i + K^{ij} W_i^* W_j.$$  

(34)

From this “Lagrangian”, we have the following Hamiltonian:

$$\mathcal{H}' = K^{ki} (p_{a^k} - W_k W_i^*),$$

(35)

where $p_{a^k}$ is a conjugate momentum to $a^k$. This is a problem of a classical particle moving in the potential

$$U = -K^{ij} W_i^* W_j.$$  

(36)

We apply Hamilton-Jacobi method to this problem. The Hamilton-Jacobi equation for the action $S(a^i, a^{*i})$ is

$$K^{ij} \left( \frac{\partial S}{\partial a^i} \frac{\partial S}{\partial a^{*j}} - W_i W_j^* \right) = E.$$  

(37)
The following action solves the equation (37) for \( E = 0 \)

\[
S(a^i, a^{*i}, \alpha) = \alpha W(a^i) + \frac{1}{\alpha} W^*(a^{*i}),
\]

where \( \alpha = e^{i\omega} \) is a parameter. Except \( n = 1 \) case, this action is not a perfect solution because it has only one parameter. So it is a nonperfect solution of Hamilton-Jacobi equation. Then it does not fix a path which connect two classical vacua in complex \( n \)-dimensional \( a^i \)-space. When \( n = 1 \) case it becomes a perfect solution and then it describes a path which connects two classical vacua in complex one-dimensional \( a \)-plane.

But we can find a curve of soliton path which connects two classical vacua in complex \( W \)-plane as a projection of the soliton path in complex \( n \)-dimensional \( a^i \)-space. The action \( S \) satisfies the following condition:

\[
\frac{\partial S}{\partial \alpha} = \text{const.}
\]

Then we have

\[
\text{Im} \left( e^{i\omega} W(a^i) \right) = \text{const.}
\]

It means that we have a straight line of soliton path in complex \( W \)-plane although in general soliton path is a curve in complex \( n \)-dimensional \( a^i \)-space. When \( n = 1 \) we can find a path in complex 1-dimensional \( a \)-plane by solving (40) to \( a \).

Around \( a^i = a_{(I)}^i \) superpotential \( W(a) \) is expanded by power series of \( (a^i - a_{(I)}^i) \):

\[
W(a) = W(a_{(I)}) + \frac{1}{2} W_{ij}(a_{(I)}) (a^i(x) - a_{(I)}^i) (a^j(x) - a_{(I)}^j) + \frac{1}{3!} W_{ijk}(a_{(I)}) (a^i(x) - a_{(I)}^i) (a^j(x) - a_{(I)}^j) (a^k(x) - a_{(I)}^k) + \cdots.
\]

\( W(a) \) is not a single valued function around \( W(a_{(I)}) \). Then there is a branch cut extending from each \( W(a_{(I)}) \) to infinity. So only soliton paths which do not cross branch cuts can exist.

\( S \) also satisfies

\[
p_{a^i} = \frac{\partial S}{\partial a^i}.
\]

Then we have

\[
K_{i'^*j'} \partial_i a^{*i} \partial_{i'} a^{j'} = \left| \frac{dW}{dx} \right|
\]

because \( dW = |dW|\alpha^{-1} \).

Classical mass of \( (I, J) \)-soliton is given by

\[
M_{IJ} = \int_{-\infty}^{\infty} \mathcal{L}' dx = 2 \int_{-\infty}^{\infty} K_{i'^*j'} \partial_i a^{*i} \partial_{i'} a^{j'} dx
\]

\[
= 2 \int_{W(a_I)}^{W(a_J)} |dW| = 2 \left| \int_{W(a_I)}^{W(a_J)} dW \right| = 2 |\Delta_{IJ} W|
\]
since $W(a)$ is a straight line in the complex $W$-plane. This shows the Bogomol’nyi bound is saturated by classical solution. Classical mass of $(I,J)$-soliton is given by the length of its path in the complex $W$-plane. Then we have a inequality among the masses

$$M_{IK} < M_{IJ} + M_{JK}.$$  

(45)

The equality $M_{IK} = M_{IJ} + M_{JK}$ does not appear because in this case $(I,K)$-soliton path crosses $a(J)$ between $a(I)$ and $a(K)$ and it takes twice of infinite “time”. It means that there is attractive force between neighboring solitons.

5 Multiplet shortening of supersymmetry Algebra

In the presence of $(I,J)$-soliton and static case, each components of supercharge are given as

$$Q_1 = \sqrt{2} \int \left( \partial_1 a^{\ast i} K_{i,j} \xi_j^1 + i W_i^{\ast} \xi_i^1 \right) dx,$$

$$Q_2 = \sqrt{2} \int \left( -\partial_1 a^{\ast i} K_{i,j} \xi_j^2 - i W_i^{\ast} \xi_i^2 \right) dx.$$  

(46)

From these expressions $A$ and $B$ are given as

$$A = 0,$$

$$B = \sqrt{2} \int \left( e^{i\theta} W_i \xi_i^1 + i W_i^{\ast} \xi_i^2 \right) dx.$$  

(47)

Then $\{A, A^\dagger\} = 0$ means the Bogomol’nyi bound is saturated. And the anticommutator for $B$ becomes

$$\{B, B^\dagger\} = 8 M_{IJ}.$$  

(48)

So only $B$ and $B^\dagger$ can excite the fermionic modes. Then the number of bosonic and fermionic state is a half of that without $(I,J)$-soliton. If there is no $(I,J)$-soliton, then we have no central charge, so both $A$, $A^\dagger$ and $B$, $B^\dagger$ excite the fermionic modes. Then a half of supersymmetry is broken by the existence of $(I,J)$-soliton.

6 Commutation relations for $T$

Whether the operator $T$ commutes with the other operators or not is not trivial. We check that the operator $T$ commutes with the other operators.

Obviously $T$ commutes with itself,

$$[T, T] = 0.$$  

(49)
$T$ also commutes with $Q$ because $Q$ do not contain $\partial_0 a^i$, 

$$[T, Q] = 0.$$  \hspace{1cm} (50)

$T$ commutes with $\bar{Q}$, $P^0$ and $P^1$, 

$$[T, \bar{Q}] = -4\sqrt{2}i \int \partial_1 \left( \xi^i W_i^* \right) dx = 0,$$  \hspace{1cm} (51)

$$[T, P^0] = -4i \int \partial_1 \left( \partial_0 a^* W_i^* \right) dx = 0,$$  \hspace{1cm} (52)

$$[T, P^1] = -4i \int \partial_1 \left( \partial_1 a^* W_i^* \right) dx = 0.$$  \hspace{1cm} (53)

We calculate commutation relation for $T$ and the angular momentum operator. In 2-dimension the angular momentum operator $M^{01}$ is given as 

$$M^{01} = \int \left( T^{00} x - T^{01} t \right) dx.$$  \hspace{1cm} (54)

For the $T^{01}$ part

$$[T, \int T^{01}(y) t dy] = -4i \int \partial_1 \left( \partial_1 a^* W_i^* t \right) dx = 0.$$  \hspace{1cm} (55)

For the $T^{00}$ part

$$[T, \int T^{00}(y) y dy] = -4i \int \partial_1 \left( \partial_0 a^* W_i^* x \right) dx.$$  \hspace{1cm} (56)

If $W_i^*$ drops faster than $x^{-1}$ then this commutator vanishes. From (42) we have 

$$K_{ij} = W_j \alpha.$$  \hspace{1cm} (57)

$W_j(a)$ is expanded by power series around $a^i = a^i_{(I)}$: 

$$W_j(a) = W_{jI}(a_{(I)}) (a^i(x) - a^i_{(I)}) + \cdots.$$  \hspace{1cm} (58)

Then (57) becomes 

$$K_{ij}(a_{(I)}) \frac{\partial a^i}{\partial x} = \alpha W_{jI}(a_{(I)}) (a^i(x) - a^i_{(I)}).$$  \hspace{1cm} (59)

around $a^i = a^i_{(I)}$. In general a soliton path is a curve in the complex $a$-plane, but in the small area around $a^i = a^i_{(I)}$ the soliton path is approximated by a straight line. We rotate the real axis along this short line, we can take $a^i$ as real near $a^i = a^i_{(I)}$: 

$$K_{ij}(a_{(I)}) \frac{\partial a^i}{\partial x} = \alpha W_{jI}(a_{(I)}) (a^i(x) - a^i_{(I)}).$$  \hspace{1cm} (60)

Since $K_{ij}$ and $W_{ij}$ are complex symmetric matrices, we can diagonalize them by unitary matrix. Then $a(x)$ has a form 

$$K_{ii}(a_{(I)}) \frac{\partial a^i}{\partial x} = \alpha W_{ii}(a_{(I)}) (a^i(x) - a^i_{(I)}),$$  \hspace{1cm} (61)
where $K_{ij}$ and $\alpha W_{ij}$ must have same phase because $\partial a^i / \partial x$ and $a^i(x) - a^i(I)$ are real. We drop this phase, we have

$$ \frac{\partial a^i}{\partial x} = \text{const.} \times (a^i(x) - a^i(I)). \quad (62) $$

This equation has a solution

$$ a^i(x) - a^i(I) = (a^i(x_0) - a^i(I))e^{\text{const.} \times (x-x_0)}. \quad (63) $$

We can take the sign of the exponent properly. Then $a^i$ converges exponentially, $W_i(a)$ also converges exponentially, and $\mathcal{L}$ becomes zero. So the operator $T$ commutes with all the other operators, therefore we can call it as the central charge.

## 7 Conclusion

We investigated a two-dimensional $N = 2$ supersymmetric model which consists of $n$ chiral superfields with Kähler potential.

In the first half, we argued about the operator ordering problem. When the kinetic term has a flat Kähler potential, $K(\phi, \phi^*) = \phi \phi^*$, there is no ordering problem. On the other hand, when the kinetic term has a nonflat Kähler potential $K(\phi, \phi^*) \neq \phi \phi^*$, there arise ordering problem. We saw that general Kähler potential raises the ordering problem. When we transfer from classical theory to quantum theory, there are several ways to fix the operator orders. In the presence of supersymmetry it dictates the operator ordering: we can fix the operator orders by requiring the super-Poincaré algebra. It is also true when the supersymmetry algebra has a central extension by the presence of topological soliton.

In the latter half, we argued about some natures of soliton in this model. In general the path of soliton is a curve in complex $n$-dimensional $a^i$-space and it is difficult to find it except $n = 1$ case. But Hamilton-Jacobi method of classical mechanics leads the result that in complex $W$-plane we can find the path of the soliton and which is a projection of the path in complex $n$-dimensional $a^i$-space.

The path of the soliton is a straight line in complex $W$-plane. Classical mass of the soliton is given by the length of its path in complex $W$-plane. Then we obtain a triangular inequality among the masses. It means that there is attractive force between neighboring solitons. In the presence of soliton, a half of supersymmetry is broken because the Bogomol’nyi bound is saturated.

## Acknowledgments

The author (M. Y.) would like to thank K. Nishio and the author (N. M.) would like to thank S. Onizawa for useful discussions.
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