LOCALLY SEMI-SIMPLE REPRESENTATIONS OF QUIVERS.

D.A. SHMELKIN

Abstract. We suggest a geometrical approach to the semi-invariants of quivers based on Luna’s slice theorem and the Luna-Richardson theorem. The locally semi-simple representations are defined in this spirit but turn out to be connected with stable representations in the sense of GIT, Schofield’s perpendicular categories, and Ringel’s regular representations. As an application of this method we obtain an independent short proof of the theorem of Skowronsky and Weyman about semi-invariants of the tame quivers.

1. Introduction

Let $Q$ be a finite quiver, i.e., an oriented graph. We fix the notation as follows: denote by $Q_0$ and $Q_1$ the sets of the vertices and the arrows of $Q$, respectively. For any arrow $\varphi \in Q_1$ denote by $t\varphi$ and $h\varphi$ its tail and its head, respectively. A representation $V$ of $Q$ over an algebraically closed field $k$, $\text{char } k = 0$, consists in defining a vector space $V(i)$ over $k$ for any $i \in Q_0$ and a $k$-linear map $V(\varphi) : V(t\varphi) \to V(h\varphi)$ for any $\varphi \in Q_1$. The dimension vector $\text{dim } V$ is the collection of $\text{dim } V(i)$, $i \in Q_0$. For a fixed dimension $\alpha$ we may set $V(i) = k^{|\alpha| i}$. Then the set $R(Q, \alpha)$ of the representations of dimension $\alpha$ is converted into the vector space

$$(1) \quad R(Q, \alpha) = \bigoplus_{\varphi \in Q_1} \text{Hom}(k^{|\alpha| t\varphi}, k^{|\alpha| h\varphi}).$$

A homomorphism $H$ of a representation $U$ of $Q$ to another representation, $V$ is a collection of linear maps $H(i), U(i) \to V(i) \in Q_0$ such that for any $\varphi \in Q_1$ holds $V(\varphi)H(t\varphi) = H(h\varphi)U(\varphi)$. The endomorphisms, automorphisms, and isomorphisms are defined naturally. Hence, the isomorphism classes of representations of $Q$ are the orbits of a reductive group $GL(\alpha) = \prod_{i \in Q_0} GL(|\alpha| i)$ acting naturally on $R(Q, \alpha)$: $(g(V))(\varphi) = g(h\varphi)V(\varphi)(g(t\varphi))^{-1}$. Set $SL(\alpha) = \prod_{i \in Q_0} SL(|\alpha| i) \subseteq GL(\alpha)$.

Assume that $Q$ has no oriented cycles. Then for any dimension $\alpha$ the algebra $k[R(Q, \alpha)]^{GL(\alpha)}$ of $GL(\alpha)$-invariant regular functions is trivial and the unique $GL(\alpha)$-closed orbit is the origin of $R(Q, \alpha)$. It is however interesting to study the $SL(\alpha)$-invariant functions or the semi-invariants of $GL(\alpha)$. The aim of this paper is to suggest a geometrical approach to this study in the spirit of Luna’s papers [Lu1], [Lu2]. For this we need to describe the closed orbits of $SL(\alpha)$. Consider a more general setting of a connected reductive group $G$ acting on an affine variety $X$, $x \in X$, $G' \subseteq G$ is the commutant. In [2.1] we prove that $G'x$ is closed in $X$ if and only if $Gx$ is closed in an open affine neighborhood $X_f \subseteq X$, where $f$ is semi-invariant. We call such $x$ a locally semi-simple point. In [2.5] we prove that there exists a generic stabilizer of locally semi-simple points and in [2.9] we obtain a special version of the Luna-Richardson theorem (see [1.12]) that can be called the Luna-Richardson theorem about semi-invariants.
Locally semi-simple representations of quivers turn out to be closely connected with the stable representations in the sense of GIT (see [Kac, MR]) and the perpendicular categories introduced in [Sch]. Namely we prove in [E2] that a representation is locally semi-simple if and only if it is a sum of simple objects in a perpendicular category; actually this is just a more strong version of [Kac, Proposition 3.2].

Recall ([Kac]) that a decomposition \( \alpha = \sum_{i=1}^{t} \beta_i \) with \( \alpha, \beta_1, \cdots, \beta_t \in \mathbb{Z}_{\geq 0}^k \) is called canonical if \( \beta_1, \cdots, \beta_t \) are Schur roots and the set of representations \( R_1 + \cdots + R_t \) such that \( R_i \) is indecomposable and \( \dim R_i = \beta_i \) contains an open dense subset in \( R(Q, \alpha) \). On the other hand, Ringel applied in [Ri] the term “canonical decomposition” for a different notion and we need that notion too. We therefore call the decomposition introduced by Kac generic (as e.g., in [SKW]).

It is well-known that the generic decomposition corresponds to the generic stabilizer in the sense that the torus \( T \subseteq GL(\alpha) \) of rank \( t \) naturally corresponding to this decomposition is a maximal torus in a generic stabilizer for the action of \( GL(\alpha) \). Analogically, the maximal torus of the generic stabilizer of locally semi-simple points yields another decomposition of \( \alpha \) that we call generic locally semi-simple. Given a non-trivial locally semi-simple point \( V \in R(Q, \alpha) \), we show in [E3] how to describe both decompositions in terms of those for a quiver \( \Sigma_V \) and a dimension vector \( \gamma \). The corresponding linear group \( (GL(\gamma), R(\Sigma_V, \gamma)) \) is nothing else but the image of \( Aut(V) \) under the slice representation. Note that for \( V \) semi-simple such a form of the slice representation is known after [LBP]. Applying [AD] for the quiver setting, we get a useful general description of \( k[R(Q, \alpha)]^{SL(\alpha)} \) of the module of \( \chi \)-eigenvectors of \( G \) acted upon by \( G \).

We apply the above methods to the case when \( Q \) is a tame quiver. Ringel introduced in [Ri] the category \( R \) of regular representations. For the tame quivers he described explicitly the simple objects of \( R \). We prove in [E4] that \( R \) is the union of the perpendicular categories \( \perp S \) with \( S \) running over the homogeneous simple regular representations. For a dimension vector \( \alpha \) such that \( R(Q, \alpha) \) contains a regular representation Ringel introduced a canonical decomposition of \( \alpha \) into a sum of the dimensions of the simple regular representations. An important observation is that the regular representations \( V \in R(Q, \alpha) \) corresponding to this decomposition are locally semi-simple. Moreover, the linear group \( (GL(\gamma), R(\Sigma_V, \gamma)) \) corresponds to a quiver being a disjoint union of equioriented \( A_n \)-type quivers. For the latter quivers we can easily describe the generic and the generic locally semi-simple decomposition. Together with [E3] this yields a way to determine both decompositions for \( Q \) and \( \alpha \) (in what concerns the generic decomposition we recover [Ri] Theorem 3.5).

The algebras of semi-invariants of tame quivers \( Q \) have been studied in several papers including [Ri], [HH], [SchW]. In [SKW] Skowronsky and Weyman proved that \( k[R(Q, \alpha)]^{SL(\alpha)} \) is a complete intersection for any \( \alpha \). Using [DW] and [E3] we give an independent short proof of this result in [E5].

2. Locally semi-simple points.

Throughout this section let \( G \) denote a connected reductive group, \( G' \) stands for its commutant, \( T = G/G' \) is a torus; let \( X \) be an irreducible affine variety acted upon by \( G \). For a \( T \)-character \( \chi \in \Xi(T) \) denote by \( k[X]^G_{\chi} \) the module of eigenvectors of \( G \) with weight \( \chi \), denote by \( G_{\chi} \) the kernel of \( \chi \). Clearly, the algebra \( k[X]^{G'} \) of \( G' \)-invariant functions on \( X \) is the direct sum: \( k[X]^{G'} = \oplus_{\chi \in \Xi(T)} k[X]^{G}_{\chi} \).
Theorem 2.1. The following properties of \( x \in X \) are equivalent:

(i) for a semi-invariant \( f \in k[X]_{\chi}^{(G)} \), \( f(x) \neq 0 \) and \( Gx \) is closed in \( X_f \)

(ii) for a character \( \chi \in \Xi(T) \), the orbit \( G_\chi x \) is closed in \( X \)

(iii) the orbit \( G'x \) is closed in \( X \)

(iv) the closure of the orbit \( G'x \) in \( X \) is contained in \( Gx \).

Proof. First observe that if \( f \in k[X]_{\chi}^{(G)} \), \( f(x) \neq 0 \), and \( g \in G \), then \( g(G_\chi x) = Gx \cap \{ y \in X | f(y) = f(gx) \} \). This yields the implication \( (i) \Rightarrow (ii) \). Also we note that \( Gx \) is a disjoint union of \( t(G_\chi x) \) with \( t \) running over the 1-dimensional coset space \( G/G_\chi \). The implication \( (ii) \Rightarrow (iii) \) follows from the fact that the subgroup \( G' \) is normal in \( G_\chi \).

Let us prove the implication \( (iii) \Rightarrow (i) \). Let \( \pi_{G'} : X \to X/G' \) denote the quotient map. The torus \( T \) acts on the quotient \( X/G' \); consider a \( T \)-equivariant embedding of \( X/G' \) to a \( T \)-module \( W \). Clearly, if \( y = \pi_{G'}(x) \) belongs to \( W^T \), then \( Ty = y \) implies \( Gx = G'x \) and \( (i) \) holds with \( f \) being a constant function. Otherwise, \( y \) is a sum of non-zero \( T \)-eigenvectors; let \( f \) be the product of the corresponding linear \( T \)-eigenfunctions. Then \( f \) is a \( T \)-semi-invariant function on \( W \) with respect to a character \( \chi \in \Xi(T) \), and \( f(y) \neq 0 \). Moreover, the orbit of \( y \) with respect to the kernel \( T_\chi \) of \( \chi \) is closed, because \( y \) is a sum of \( T_\chi \)-eigenvectors such that the sum of their characters (with respect to \( T_\chi \)) is zero. Consequently, the orbit \( G_\chi x = T_\chi G'x \) is closed, because is equal to the intersection of the closed pull-back \( \pi_{G'}^{-1}(T_\chi y) \) with the closed subset \( X_{\dim G'x} = \{ z \in X | \dim G'z \leq \dim G'x \} \). So we got \( (ii) \) and besides, \( f(x) \neq 0 \) for the above \( f \) thought of as a \( G \)-semi-invariant function on \( X \). Set \( m = \dim G_\chi x \). Then the closure \( \overline{Gx} \) of \( Gx \) in \( X \) is an irreducible variety of dimension \( m + 1 \) and for any \( t \in G/G_\chi \) the set \( \{ y \in Gx | f(y) = f(tx) \} \) is an equidimensional closed subvariety in \( X \) of dimension \( m \) with one of irreducible components being equal to \( t(G_\chi x) \). Since \( G/G_\chi \) acts transitively on the fibers of \( f \), we get \( \overline{Gx} = \overline{Gx} \), that is, \( Gx \) is closed in \( X_f \).

The implication \( (iii) \Rightarrow (iv) \) is obvious. If \( G'x \) is not closed, then there is \( z \in \overline{Gx} \) such that \( \dim G'z < \dim G'x \). However, for any \( z \in Gx \) we have \( \dim G'z = \dim G'x \). Therefore \( (iv) \) implies \( (iii) \).

The points \( x \in X \) such that \( Gx \) is closed in \( X \) are called semi-simple in \( [GV] \).

If \( X \) is a variety of representations of associative algebras, then this is not just a definition, since it is proved that the module corresponding to \( x \) is semi-simple if and only if \( Gx \) is closed (see e.g. [M]). This motivates

Definition 2.2. We call \( x \in X \) locally semi-simple if \( x \) fulfills the equivalent conditions of the above theorem.

Remark 2.1. The property of local semi-simplicity is intermediate between those of stability and semi-stability introduced by Mumford ([MF]). Recall that in our context \( x \) is called \( \chi \)-semistable if \( f(x) \neq 0 \) for a non-constant semi-invariant \( f \in k[X]_{\chi}^{(G)} \), and \( x \) is called \( \chi \)-stable if \( x \) is \( \chi \)-semistable, the stabilizer of \( x \) is equal to the kernel of the action \( G : X \), and the orbit \( Gx \) is closed in \( X_f \). So locally semi-simple points meeting condition \( (i) \) of [2.1] are \( \chi \)-semistable and \( x \) is \( \chi \)-stable if and only if \( x \) is locally semi-simple with trivial stabilizer.
For a reductive group $M$ acting on an affine variety $Y$ D.Luna introduced in
[Lu1, Lu2] the concept of étale slice at a semi-simple point $y \in Y$. First of all by
Matsushima’s criterion [Ma], the stabilizer $M_y$ is reductive. The Luna slice theorem
([Lu1]) states that there exists an étale slice $M$ at $y$ such that $S \supseteq y$ is affine,
locally closed, $M_y$-stable, and the natural map $\varphi_y : M \ast M_y S \rightarrow Y, [m, y] \mapsto my$ is
efficient (see precise definition in [Lu1]), in particular the image of $\varphi_y$ is affine and
the restriction of $\varphi_y$ to any fiber of the $M$-quotient map is an isomorphism. Further,
assume that $Y = V$ is a vector space and $M$ acts on $V$ by a linear representation,
v = $y$; choose a $M_v$-stable complementary subspace $N$ to $T_v M v$ in $V$. Then as
$S$ we can take $S = v + N_0$ for an open affine subset $N_0 \subseteq N$ containing 0. The
representation $\sigma_v : M_v \rightarrow GL(N)$ is called in this case the slice representation of $v$
and can be calculated by the formula (Ad stands for the adjoint representation):

$$\sigma_v = (V + Ad M_v) / (Ad M)_{|M_v}.$$  

Let $\pi_{G'} : X \rightarrow X / G' = Spec k[X]^{G'}$ denote the quotient map. For $\xi \in X / G'$
de note by $O_\xi$ the unique $G'$-closed orbit in $\pi_{G'}^{-1}(\xi)$. The quotient $X / G'$ carries the
Luna stratification ([Lu1]) by the disjoint locally closed subvarieties $(X / G')_{|L} = \{\xi \in X / G' | O_\xi \cong G' / L\}$, where $L$ is a subgroup in $G'$. We consider a similar
stratification with respect to the action of $G$, as follows. For a (reductive) subgroup
$M \subseteq G$ denote by $(X / G')_{|M}$ the set of all $\xi \in X / G'$ such that $G_{z,}\xi$ is $G$-conjugate
to $M$ for $z \in O_\xi$. Clearly, if the subgroups $M_1$ and $M_2$ are $G$-conjugate, then
$M_1 \cap G'$ and $M_2 \cap G'$ are $G'$-conjugate, hence each Luna stratum $(X / G')_{|L}$ is a
union of $(X / G')_{|M}$ with $M \cap G'$ being $G'$-conjugate to $L$.

**Proposition 2.3.** $(X / G')_{|M}$ is locally closed.

*Proof.* Apply the slice theorem for $z \in O_\xi^M$ and $G'$. Since $z$ is $M$-invariant and
$M$ normalizes $G'_z$, the slice $S$ can be chosen to be $M$-stable. Then the map $\varphi_z$
is $M$-equivariant. Denote by $\varphi_z / G' : S / G'_z \rightarrow X / G'$ the étale covering of a
neighborhood of $\xi$ in $X / G'$ given by the slice theorem. Then the $M$-stratum is
covered by $S^M$, hence is locally closed. \hfill \Box

**Proposition 2.4.** The stratification $X / G' = \sqcup M (X / G')_{|M}$ is finite.

*Proof.* Since the Luna stratification is finite, it is sufficient to show that each Luna
stratum $(X / G')_{|L}$ is decomposed into finitely many strata $(X / G')_{|M}$. Clearly,
we may assume $M \cap G' = L$. Take $\xi \in (X / G')_{|L}$; then $\xi \in (X / G')_{|M}$ with
$M / L \cong (G / G')_\xi$ so $M$ is an extension of $L$ by a diagonalizable subgroup in the
normalizer $N_G(L)$. Choose a maximal torus $A \subseteq N_G(L)$. Then each point $z$
with $G'_z = L$ is $N_G(L)$-conjugate to a point $w$ such that the identity component of $G_w$
is contained in $LA$. On the other hand, $X^L$ can be divided into finitely many subsets with constant stabilizer with respect to $A$. So there are finitely many
$N_G(L)$-conjugacy classes of stabilizers in $G$ of points $z$ with $G'_z = L$. \hfill \Box

Recall that a subgroup $H_0 \subseteq G'$ is called principal isotropy group if $(X / G')_{|H_0}$
is the unique open and dense Luna stratum. By Propositions 2.3 and 2.4 we have:

**Definition-Proposition 2.5.** A subgroup $H \subseteq G$ is called generic stabilizer of a
locally semi-simple point if $(X / G')_{|H}$ is the unique open and dense $G$-stratum of
$X / G'$. The intersection $H \cap G'$ is a principal isotropy group and the image of $H$
in $G / G'$ is the kernel of the action $G / G' : X / G'$.
We now want to describe locally semi-simple points and their stabilizers in terms of the Luna slice theorem with respect to the group $G$. Indeed, if $x$ is locally semi-simple and $Gx$ is closed in $X_f$ for a semi-invariant $f$, then there is an étale slice $S$ at $x$ with respect to $X_f$, and if $X = V$ is a vector space, then an étale slice of type $x + N_0$ exists.

**Proposition 2.6.** If $G : V$ is a linear representation, $v \in V$ is a locally semi-simple point, then for any $n \in N_0$, $v + n$ is locally semi-simple with respect to $G$ if and only if $n \in N$ is locally semi-simple with respect to $G_v$.

**Proof.** Assume that $G(v + n)$ is closed in $V_f$ for a $G$-semi-invariant $f$. Then $\varphi_v^{-1}G(v + n)$ is closed in $\varphi_v^{-1}V_f = G \ast_G, (v + (N_0)_{f'})$, where $f' \in k[N]$ is defined as $f'(n') = f(v + n'), n' \in N$ so that $f'$ is $G_v$-semi-invariant. Moreover, $\varphi_v$ is excellent implies that $\varphi_v^{-1}G(v + n)$ is a union of finitely many orbits, hence $G[e, v + n] = G \ast_G, (v + G_0n)$ is also closed in $G \ast_G, (v + (N_0)_{f'})$, equivalently $G_vn$ is closed in $(N_0)_{f'}$. Since $N_0$ is affine and $N$ is a vector space, $N_0 = N_d$ for some $d \in k[N]$ and $N_0$ is $G_v$-stable implies that $d$ is $G_v$-semi-invariant. So $G_vn$ is closed in $N_{f'}$ and we proved the "only if" part.

Assume that $G_vn$ is closed in $N_{f'}$. As above, we may additionally assume that $N_{f'}$ is contained in $N_0$. Then by the properties of an excellent map we have that $G(v + n)$ is closed in an open subset $V_1$ in the image $V_0$ of $\varphi_v$, such that $V_0 \setminus V_1$ is an equidimensional subvariety of codimension 1 in $V_0$. Since $V_0$ is affine and $V$ is a vector space, we get $V_1 = V_f$ for some $f \in k[V]$, and $V_0$ is $G$-stable implies $f$ is $G$-semi-invariant. \hfill \qed

It is well-known that the isotropy group $G_v$ for a semi-simple $v \in V$ is principal if and only if the only semi-simple point in $(G_v, N/N^{G_v})$ is 0.

**Corollary 2.7.** Let $N = N^{G_v} \oplus N_+$ be a $G_v$-stable decomposition. Then $v$ is generic if and only if the only $G_v$-locally semi-simple point in $N_+$ is 0.

**Proof.** If $N_+ \setminus \{0\}$ contains a $G_v$-locally semi-simple point, then $N$ contains a $G_v$-locally semi-simple point $n$ with a proper isotropy subgroup $(G_v)_n \subsetneq G_v$. Multiplying $n$ by a scalar, we may assume $n \in N_0$, hence by the Proposition, $v + n$ is $G$-locally semi-simple with stabilizer $G_{v+n} = (G_v)_n$. Then the closure of $(X/G')_{v+n}^G$ contains $(X/G')_{(G_v)n}^G$ so the closure of the latter can not be equal to $X/G'$ and $v$ is not generic. Conversely, if the only $G_v$-locally semi-simple point in $N_+$ is 0, then $v$ is a generic locally semi-simple point in the image $V_0$ of $\varphi_v$. Let $V_{lss} \subset V$ be the subset of $G$-locally semi-simple points. By Theorem 2.2, $V_{lss}$ is also the union of $G'$-closed orbits; since $V/G'$ is irreducible, the closure $V_{lss}$ also is. Therefore $V_{lss}$ is the closure of its intersection with $V_0$ and $v$ is generic in $V$. \hfill \qed

**Corollary 2.8.** If $n \in N_0$ is a generic locally semi-simple point for the action of $G_v$, then $v + n$ is generic for $G$ acting on $V$.

**Proof.** By the slice theorem $G_{v+n} = (G_v)_n$; by the Proposition, $v + n$ is locally semi-simple. Applying formula 2, we get: $\sigma_{v+n} = \sigma_n$. Applying Corollary 2.7 we conclude the proof. \hfill \qed

The notion of the locally semi-simple point can be used in order to describe the semi-invariants of $G$. Recall that the Luna-Richardson theorem [Lu2] says that if $H_1$ is a principal isotropy group for the action $G : X$, then the embedding $X^{H_1} \subset X$
Proposition 3.2. Assume that $G$ is a generic stabilizer of a locally semi-simple point for the action $G : X$. Then the embedding $X^H \subseteq X$ and the group homomorphism $\theta_H : N_G(H)/N_{G'}(H) \to G/G'$ give rise to an isomorphism:

$$k[X]^G \cong k[X]^H_{N_G(H)}$$

of a $\Xi(G/G')$-graded algebra onto a $\Xi(N_G(H)/N_{G'}(H))$-graded algebra. Moreover generic orbit of $N_{G'}(H)$ is closed in $X^H$.

Proof. First prove that we have an isomorphism of algebras. Since $H$ normalizes $G'$, we have $HG'$ is a reductive subgroup in $G$. Moreover $H$ is a principal isotropy group for $HG'$ acting on $X$ so $k[X]^{HG'}$ restricts isomorphically onto $k[X]^H_{N_{HG'}(H)} = k[X]^H_{N_{G'}(H)} = k[X]^H_{N_{G'}(H)}$. Also we have: $k[X]^{G'} = k[X]^{HG'}$. Indeed for any $h \in H, f \in k[X]^G$ we have $hf = f$ on $X^H$, hence on $G'X^H$. But $X^H$ intersects all closed $HG'$-orbits, so any $HG'$-invariant function is completely defined by its restriction to $G'X^H = G'X^H$. Thus we got that restricting $G'$-invariant functions to $X^H$ we have an isomorphism: $k[X]^{G'} \cong k[X]^H_{N_{G'}(H)}$. Clearly a $G'$-semi-invariant function of weight $\chi \in \Xi(G/G')$ restricts to a $N_{G'}(H)$-semi-invariant function of weight $\theta_H(\chi)$. Finally observe that generic point $x \in X^H$ has a closed $HG'$-orbit, because $H$ is a principal isotropy group of $HG'$. By [Lu2], $N_{HG'}(H)x = N_{G'}(H)x$ is also closed in $X^H$. □

Remark 2.2. The description of $k[X]^{G'}$ given by this theorem is in general different from the given by the Luna-Richardson theorem for $G'$. For instance, take as $G'$ the group $SL_2$ acting naturally on $k^2 \oplus k^2$; as $G$ take the extension of $G'$ by $T = \{ diag(t, t, s, s) | t, s \in k^* \}$. Then principal isotropy group of $G'$ is trivial but $H = \{ diag(u, 1, 1, u^{-1}) | u \in k^* \}$.

3. Locally semi-simple representations of quivers.

Definition 3.1. A representation $V$ of a quiver $Q$ is called locally semi-simple if $V$ is a locally semi-simple point of $R(Q, \text{dim} V)$ with respect to $GL(\text{dim} V)$.

We start with observations as follows:

Proposition 3.2. Assume that $V$ is a locally semi-simple representation.
1. If $V = V_1 + V_2$, then both $V_1$ and $V_2$ are locally semi-simple.
2. If $V$ is indecomposable, then $\text{Aut}(V) = k^*$.
3. If $V = V_1 + V_2$, both $V_1$ and $V_2$ are indecomposable, and $V_1 \not\cong V_2$, then we have: $\text{Hom}(V_1, V_2) = 0, \text{Hom}(V_2, V_1) = 0$.

Proof. The assertions 1-3 follow from Theorem 1.2 below. We give however an independent proof. Set $\alpha = \text{dim} V$.

1. Set $\beta = \text{dim} V_1$. Note that $SL(\alpha)$ contains a subgroup naturally isomorphic to $SL(\beta)$ and $SL(\beta) V_1 + V_2$ is contained in $SL(\alpha)V$. Assuming that $V_1$ is not locally semi-simple, we get by Theorem 2.1 that $SL(\beta) V_1$ contains a representation non-isomorphic to $V_1$, hence $SL(\alpha) V$ contains a representation non-isomorphic to $V$ and $V$ is not locally semi-simple.

2. By Fitting’s Lemma $\text{End}(V)$ is local. By Matsushima’s criterion $\Xi$ the stabilizer $SL(\alpha) V$ is reductive; since $\text{Aut} V = GL(\alpha) V$ and $GL(\alpha) V / SL(\alpha) V$ is a subgroup in the center of $GL(\alpha)$, $\text{Aut} V$ is reductive, hence $\text{Aut} V = k^*$. 

3. By 2 we know $\text{End}(V_1) = k$, $\text{End}(V_2) = k$, and by Matsushima’s criterion $\text{End}(V)$ is reductive. The decomposition $\text{End}(V) = \bigoplus_{i,j=1,2} \text{Hom}(V_i, V_j)$ implies that $\text{End}(V)$ is either $k \oplus k$ or $\text{End}(k^2)$. In the latter case let $H_{12}$ and $H_{21}$ be the generators of $\text{Hom}(V_1, V_2)$ and $\text{Hom}(V_2, V_1)$, respectively. The isomorphism $\text{End}(V) \cong \text{End}(k^2)$ implies $H_{21}H_{12} \in \text{End}(V_1)$ does not vanish, hence is a scalar operator on $V_1$. So we have $V_1 \cong V_2$, a contradiction. □

A representation $V$ such that $\text{Aut}(V) = k^*$ is called Schurian. The converse to Proposition 3.4 is not true, i.e., not any Schurian representation is locally semi-simple:

Example 3.3. Let $Q$ be the quiver with one vertex and two attached loops. Let $V$ be a 2-dimensional representation of $Q$ such that the corresponding pair of endomorphisms of $k^2$ generates the algebra $B$ of the upper triangular matrices, in a basis. Then $\text{End}(V)$ is the centralizer of $B$ in $\text{End}(k^2)$, so $V$ is Schurian. Since the center of $GL(\alpha)$ acts trivially on representations, the locally semi-simple representations are in this case just the semi-simple representations. A semi-simple Schurian representation must be simple, but $V$ has a 1-dimensional subrepresentation, so $V$ is not semi-simple.

Recall that each quiver $Q$ determines two forms on $Z^{Q_0}$, the Tits quadratic form $q_Q(\alpha) = \sum_{i \in Q_0} \alpha_i^2 - \sum_{\varphi \in Q_1} \alpha_{t \varphi} \alpha_{h \varphi}$, and the Euler bilinear form:

$$
\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{\varphi \in Q_1} \alpha_{t \varphi} \beta_{h \varphi}.
$$

Note also that the Euler form is not symmetric and $\langle \alpha, \alpha \rangle = q_Q(\alpha)$.

Proposition 3.4. If $V$ is a Schurian representation and $q_Q(\dim V) = 1$ (in other words, $\alpha = \dim V$ is a real Schur root), then $V$ is locally semi-simple.

Proof. The hypothesis implies that the $GL(\alpha)$-orbit of $V$ is dense in $R(Q, \alpha)$, so the generic stabilizer for the action of $SL(\alpha)$ is trivial. By [Po], generic $SL(\alpha)$-orbit are closed. Hence, $SL(\alpha)V$ is closed. □

4. Semi-invariants of quivers and perpendicular categories.

The character group of $GL(\alpha)$ is generated by the determinants of the $GL(\alpha_a)$-factors, $\alpha \in Q_0$ so is isomorphic to $Z^{Q_0}$ such that $\chi \in Z^{Q_0}$ gives rise to the character $\chi = \prod_{\alpha > 0} \det_{\alpha}^\chi$. We also can think of $\chi$ as of an integer function on the dimensions of representations such that $\chi(\alpha) = \sum_{\alpha \in Q_0} \chi_{\alpha} \alpha_{\alpha}$. We have:

$$
\kappa[R(Q, \alpha)]_{SL(\alpha)}^{GL(\alpha)} = \bigoplus_{\chi \in Z^{Q_0}} \kappa[R(Q, \alpha)]_{\chi}^{GL(\alpha)},
$$

where $\kappa[R(Q, \alpha)]_{\chi}^{GL(\alpha)} = \{ f \in \kappa[R(Q, \alpha)] | gf = \chi(g)f, \forall g \in GL(\alpha) \}$.

Recall that for dimension vectors $\alpha, \beta \in Z^{Q_0}$ such that $\langle \alpha, \beta \rangle = 0$ Schofield introduced in [Sch] a function $c$ on $R(Q, \alpha) \times R(Q, \beta)$ such that:

(i) $c(V, W) \neq 0$ if and only if $\text{Hom}(V, W) = 0$

(ii) $c$ is $GL(\alpha) \times GL(\beta)$-semi-invariant; if $c(V, \cdot) \neq 0$, then its character is equal $\langle \alpha, \cdot \rangle$; if $c(\cdot, W) \neq 0$, then its character is equal $-\langle \cdot, \beta \rangle$.

Derksen and Weyman proved in [DWW] that each vector space $\kappa[R(Q, \alpha)]_{\sigma}^{GL(\alpha)}$ is generated by the functions $c_W = c(\cdot, W)$ such that for any $\alpha \in Z^{Q_0}$, $-\langle \alpha, \dim W \rangle = \sigma(\alpha)$. Recall the Ringel formula ([Ri]):

$$
\dim \text{Ext}(V, W) = \dim \text{Hom}(V, W) - \langle \dim V, \dim W \rangle.
$$
This formula and the above properties imply that for a given \( V \in R(Q, \alpha) \) the semi-invariants \( c_W \) such that \( c_W(V) \neq 0 \) correspond to the representations \( W \) such that \( \text{Hom}(V, W) = 0, \text{Ext}(V, W) = 0 \). Schofield called in [Sch] the set of such representations, \( V^\perp \), the right perpendicular category of \( V \). The left perpendicular category, \( \perp V \), is defined similarly. Note that \( S \in V^\perp \) is equivalent to \( c(V, S) \neq 0 \) and the same for the left category. Schofield proved that the perpendicular categories are Abelian subcategories. In particular, simple objects in \( V^\perp \) are Schurian representations, homomorphisms between non-isomorphic simple objects are trivial, any representation has a unique Jordan-Hölder decomposition.

On the other hand, we have the notion of \( \chi \)-stability (see Remark 2.1) and for representations of quivers King proved in [Ki, Proposition 3.1] that \( V \) is \( \chi \)-stable if and only if \( \chi(\dim V) = 0 \) and \( \chi(\dim V') < 0 \) for any subrepresentation \( V' \subseteq V \) different from 0 and \( V \).

**Proposition 4.1.** \( S \in \perp W \) is a simple object if and only if \( S \) is \( -\langle \cdot, \dim W \rangle \)-stable.

**Proof.** \( S \) is a simple object means that there are no proper subrepresentations \( S' \subseteq S \) such that \( \text{Hom}(S', W) = 0 \) and \( \langle \dim S', \dim W \rangle = 0 \). So the "if" part follows. Now assume \( S \) is simple in \( \perp W \) and \( S' \subseteq S \) is a proper subrepresentation. This means that \( c(S, W) \neq 0 \) and \( c(S', W) = 0 \). By [DW, Lemma 1], \( \langle \dim S', \dim W \rangle < 0 \) would contradict our condition \( c(S, W) \neq 0 \) and \( \langle \dim S', \dim W \rangle = 0 \) would imply that \( c(S', W) \neq 0 \). Thus we have: \( \langle \dim S', \dim W \rangle > 0 \). \( \square \)

Now we give a criterion for a representation to be locally semi-simple. The above discussion shows that it is sufficient to find out for representations \( V, W \) such that \( \text{Hom}(V, W) = 0, \text{Ext}(V, W) = 0 \) whether the \( GL(\dim V) \)-orbit of \( V \) is closed in \( R(Q, \dim V)_{c_W} \).

**Theorem 4.2.** The orbit of \( V \) is closed in \( R(Q, \dim V)_{c_W} \) if and only if \( V = p_1 S_1 + \cdots + p_i S_i \), where \( S_1, \ldots, S_i \) are simple objects in \( \perp W \).

**Remark 4.1.** Taking into account Proposition 1 one can see that this Theorem is similar to [Ki, Proposition 3.2]. However, the property of the orbit to be closed in \( R(Q, \dim V)_{c_W} \) is more strong than that to be closed in the open set of semi-stable points as in [Ki].

**Proof.** Clearly the sum of Jordan-Hölder factors of \( V \) in \( \perp W \) belongs to the closure of \( V \)-orbit and to \( R(Q, \dim V)_{c_W} \). So if the orbit is closed, then \( V \) is isomorphic to the sum of simple objects. Conversely, assume that \( V \) is a sum of simple objects and let \( U \) be the closed orbit in the closure of the orbit of \( V \) in \( R(Q, \dim V)_{c_W} \). Then there is a 1-parameter subgroup \( g(t) \in GL(\dim V), t \in \mathbb{K}^* \) such that \( \lim_{t \to 0} g(t)V \subseteq U \). Considering the \( g(t) \)-eigenspace decomposition of \( V(i), i \in Q_0 \) one easily sees that the limit exists means that these eigenspaces yield a filtration of \( V \) in \( \perp W \) such that \( U \) is the associated graded of \( V \). Hence, the Jordan-Hölder factors for \( U \) are the same as for \( V \), so \( V \) belongs to the closure of the orbit of \( U \), in other words, the orbits are equal. \( \square \)

**Remark 4.2.** Clearly, the Theorem implies Proposition 5 above.

**Example 4.3.** Let \( Q = A_n; \circ \to \circ \to \cdots \circ \to \circ \), where \( n \) stands for the number of vertices. Let \( \varepsilon_1, \ldots, \varepsilon_n \) denote the standard basis of \( \mathbb{Z}^n \). It is well-known that the indecomposable representations of \( A_n \) are the representations \( S_{ij}, 1 \leq i \leq j \leq n \).
such that \( \dim S_{ij} = \varepsilon_i + \cdots + \varepsilon_j \) and these representations are Schurian. It can be directly verified (and follows e.g. from \(|\text{Sh}|\) Th. 10))

\[
\text{Hom}(S_{kl}, S_{ij}) \neq 0 \iff i \leq k \leq j \leq l.
\]

Hence, the condition \( \text{Hom}(S_{ij}, S_{kl}) = 0 = \text{Hom}(S_{kl}, S_{ij}) \) is equivalent to:

\[
j < k, \text{ or } l < i, \text{ or } i < k \leq l < j, \text{ or } k < i < j < l,
\]

in other words the segments \([i,j]\) and \([k,l]\) either are disjoint sets or one of them contains another in the interior.

**Proposition 4.4.** Let \( V = \bigoplus_{p,q} m_{pq} S_{pq} \) and set \( I = \{(p,q) \mid m_{pq} > 0\} \). Then \( V \) is locally semi-simple if and only if any pair \((i,j), (k,l) \in I \) meets condition \(|\text{S}|\).

**Proof.** The "only if" part follows from \(|\text{S}|\). To prove the "if" part we first observe that the condition \( \langle \dim S_{kl}, \dim S_{ij} \rangle = 0 \) implies \( \text{Hom}(S_{kl}, S_{ij}) = 0 \), because of \(|\text{S}|\). Using condition \(|\text{S}|\), one can show that the dimensions \( \alpha_1, \cdots, \alpha_t \) of the representations \( S_{pq}, (p,q) \in I \) are linear independent and moreover, the sublattice

\[
\langle \alpha_1, \cdots, \alpha_t \rangle = \{ \beta \in \mathbb{Z}^n \mid \langle \alpha_i, \beta \rangle = 0, i = 1, \cdots, t \}
\]

is generated by \( n - t \) linear independent roots. By the above observation the corresponding \( n - t \) indecomposable representations belong to \( V^\perp \); let \( W = R_1 + \cdots + R_k \) be the sum of all simple factors of these. Clearly, \( \dim R_1, \cdots, \dim R_k \) also generate \( \langle \alpha_1, \cdots, \alpha_t \rangle \); since homomorphisms between simple objects \( R_i, R_j \) are trivial for \( i \neq j \), dim \( R_1, \cdots, \dim R_k \) are linearly independent, so \( k = n - t \) and \( \alpha_1, \cdots, \alpha_t \) is a basis of the sublattice \( \langle \dim R_1, \cdots, \dim R_{n-i} \rangle \). By construction, \( V \in W \). We claim that \( S_{pq}, (p,q) \in I \) are simple objects in \( W \) and this implies the assertion, thanks to Theorem \(|\text{GL}|\). Indeed, assume that a summand, \( S_{ij} \) is not simple, i.e., a proper subrepresentation \( S' \subseteq S_{ij} \) belongs to \( W \). Any proper subrepresentation of \( S_{ij} \) is isomorphic to \( S_{kl} \) with \( i < k \leq j \), so \( \dim S_{kl} \) is a linear combination of \( \alpha_1, \cdots, \alpha_t \). Using condition \(|\text{S}|\), one can easily see this is false. \( \Box \)

5. **DECOMPOSITIONS AND SLICES.**

Let \( V \) be a locally semi-simple representation of \( Q \), \( \dim V = \alpha \); by Proposition \(|\text{S}|\) we know: \( V = \bigoplus_{i=1}^t m_i S_i \), where \( S_i \) are pairwise non-isomorphic Schurian representations with trivial homomorphism spaces between them. Hence, \( \text{Aut}(V) \cong \prod_{i=1}^t GL(m_i) \). Note that the group \( \text{Aut}(V) \) and its embedding to \( GL(\alpha) \) are completely determined by the decomposition \( \alpha = m_1 \dim S_1 + \cdots + m_t \dim S_t \).

**Definition 5.1.** A decomposition \( \alpha = \sum_{i=1}^t m_i \beta_i \) with \( \alpha, \beta_1, \cdots, \beta_t \in \mathbb{Z}^Q_+ \), \( m_i \in \mathbb{N} \) is called **locally semi-simple** if for each \( i \) there exists a representation \( S_i \) such that \( \dim S_i = \beta_i \), \( \dim \text{Hom}(S_i, S_j) = \delta_{ij} \), and \( V = \bigoplus_{i=1}^t m_i S_i \) is a locally semi-simple representation. If moreover \( V \) is generic, then we call this decomposition generic.

Note that a locally semi-simple decomposition determines the isomorphism class of the representation if and only if all the components are real Schur roots. Note also that there can be equal summands \( \beta_i = \beta_j = \beta \) in such a decomposition; the condition \( \dim S_i, \dim S_j = 0 \) implies that \( \beta \) is an imaginary root.

Assume that \( V = \bigoplus_{i=1}^t m_i S_i \) is locally semi-simple. By Ringel’s formula \(|\text{BL}|\) we have: \( \delta_{ij} = \dim S_i, \dim S_j = \dim \text{Ext}(S_i, S_j) \geq 0 \). Le Bruyn and Procesi showed in \(|\text{LBP}|\) that the slice representations for \( V \) semi-simple can be expressed in terms of quivers. Following \(|\text{LBP}|\), we introduce a quiver \( \Sigma_V \) with vertices \( a_1, \cdots, a_t \).
corresponding to the summands $S_1, \ldots, S_t$ and $\delta_{ij} - \langle \dim S_i, \dim S_j \rangle$ arrows from $a_i$ to $a_j$; set $\gamma = (m_1, \ldots, m_t) \in \mathbb{Z}^{S_2}$. It is known (see e.g. [Kac]) that for any representation $W$ the normal space to the isomorphism class of $W$ at $W$, $R(Q, W)/TWGL(\dim W)W$, is isomorphic to $Ext(W, W)$. Hence, we get a helpful form of the slice representation $\sigma_V$ of $V$ (the same as in [LBP] for the semi-simple case): $(GL(\alpha_V, \sigma_V) = (Aut(V), Ext(V, V)) =
abla \mathbb{GL}(m_i), \bigoplus_{i,j=1}^t Ext(S_i, S_j) \otimes \text{Hom}(k^{m_i}, k^{m_j})) = (GL(\gamma), R(\Sigma_V, \gamma)).$

Let $D_V : \mathbb{Z}^{S_2} \to \mathbb{Z}^{Q_0}$ denote the linear map taking $i$-th basis vector of $\mathbb{Z}^{S_2}$ to $\dim S_i, i = 1, \ldots, t$. Note that $D_V(\gamma) = \alpha$. The definition of the Euler form yields:

**Proposition 5.2.** $D_V$ preserves the Euler form: $\langle D_V(\gamma_1), D_V(\gamma_2) \rangle = \langle \gamma_1, \gamma_2 \rangle$.

**Proposition 5.3.** Consider a decomposition $\gamma = \sum_{j=1}^s \rho_j p_j$ and the corresponding decomposition $\alpha = \sum_{j=1}^s \rho_j D_V(p_j)$ of $\alpha$.

1. If the decomposition of $\gamma$ is generic, then that of $\alpha$ is.
2. If the decomposition of $\gamma$ is locally semi-simple, then that of $\alpha$ is.
3. If the decomposition of $\alpha$ is generic locally semi-simple, then that of $\alpha$ is.

**Proof.** A general remark is that by Luna’s slice theorem for any representation $W \in R(\Sigma_V, \gamma)$ there exists a representation $V_1 \in R(Q, \alpha)$ with $Aut(V_1) = Aut(W)$, where $Aut(W) \subseteq GL(\gamma)$ is embedded to $GL(\alpha)$ via the embedding $GL(\gamma) = Aut(V) \subseteq GL(\alpha)$. Therefore if the maximal torus of $Aut(W)$ corresponds to the given decomposition of $\gamma$, then the maximal torus of $Aut(V_1)$ corresponds to the given decomposition of $\alpha$. Now 1 follows from the fact that the generic decompositions are determined by generic stabilizers and by Luna’s slice theorem if $W$ is generic for $(GL(\alpha_V, \sigma_V))$, then $V_1$ is generic for $(GL(\alpha), R(Q, \alpha))$. By Proposition 5.3 $W$ is locally semi-simple implies $V_1$ is locally semi-simple, so we proved 2. Applying Corollary 2.8 we also get 3. \hfill \Box

**Proposition 5.4.** 1. If $\gamma_1 \in \mathbb{Z}_+^{S_2}$ is a (real) Schur root, then $D_V(\gamma_1)$ is.

2. If $\gamma_1, \gamma_2 \in \mathbb{Z}_+^{S_2}$ are real Serre roots, $V_1 \in R(S_V, \gamma_1), W_2 \in R(S_V, \gamma_2)$ and $V_1 \in R(Q, D_V(\gamma_1)), V_2 \in R(Q, D_V(\gamma_2))$ are Schurian representations, then $Ext(V_1, W_2) = 0 = Ext(W_1, V_2)$ implies $Ext(V_1, V_2) = 0 = Ext(W_2, V_1)$ and $Hom(W_1, W_2) = 0 = Hom(V_1, V_2)$ implies $Hom(V_1, V_2) = 0 = Hom(W_2, V_1)$.

**Proof.** Note that the map $D_V$ depends on the indecomposable summands of $V$ not of $V$ itself. So in 1 we may assume that $\dim V = D_V(\gamma_1)$. Then by 5.1 $\dim V = \dim V$ is the generic decomposition, that is $D_V(\gamma_1)$ is a Schur root. By 5.2 $D_v(D_V(\gamma_1)) = q_2(\gamma_1)$. So $\gamma_1$ is real implies $D_V(\gamma_1)$ is. In 2 we assume $\dim V = \alpha = D_V(\gamma_1) + D_V(\gamma_2)$. Then by Kac the condition $Ext(W_1, W_2) = 0 = Ext(W_2, W_1)$ implies that $\gamma = \gamma_1 + \gamma_2$ is the generic decomposition. Then by 5.31, $\alpha = D_V(\gamma_1) + D_V(\gamma_2)$ is the generic decomposition so again applying Kac we get $Ext(W_1, V_2) = 0 = Ext(V_2, W_1)$. The condition $Hom(W_1, W_2) = 0 = Hom(W_2, V_1)$ yields $Aut(W_1 + W_2)$ is the corresponding embedding of $(k^*)^2$ to $GL(\gamma)$. By Luna’s slice theorem there exists a representation $V' \in R(Q, \alpha)$ with $Aut(V')$ being the image of $Aut(W_1 + W_2)$ under the embedding $GL(\gamma) \subseteq GL(\alpha)$. This means that $V' = V'_1 + V'_2$ with $V'_1, V'_2$ indecomposable of dimensions $D_V(\gamma_1), D_V(\gamma_2)$ and $Hom(V'_1, V'_2) = 0 = Hom(V'_2, V'_1)$. Clearly, this is equivalent to what we assert. \hfill \Box
Now apply the Luna-Richardson theorem to this situation. Assume that \( \alpha = \sum_{i=1}^{r} m_i \beta_i \) is a generic locally semi-simple decomposition. For a locally semi-simple decomposition we observed above that equal summands \( \beta_i = \beta_j \) must be imaginary roots. Besides in generic case the multiplicity \( m_j \) of any imaginary root \( \beta_j \) is equal 1. Indeed, if we have a locally semi-simple representation \( V = \oplus_{i=1}^{r} m_i S_i \) then by Theorem 4.2, \( S_1, \ldots, S_t \) are simple objects in the category \( \perp W \) for some \( W \). Clearly, being perpendicular to \( W \) and being simple object in \( \perp W \) are open conditions on \( R(Q, \beta_j) \). Since \( R(Q, \beta_j) \) contains infinitely many isomorphism classes of indecomposable representations, we could replace \( m_j S_j \) with \( m_j > 1 \) by a sum of \( m_j \) generic representations of dimension \( \beta_j \) to get a locally semi-simple representation with a smaller automorphism group.

**Theorem 5.5.** 1. A generic locally semi-simple decomposition has the form:

\[
\alpha = \delta_1 + \cdots + \delta_1 + \cdots + \delta_r + \cdots + \delta_r + m_1 \beta_1 + \cdots + m_s \beta_s,
\]

where \( \delta_1, \ldots, \delta_r \) are pairwise non-equal imaginary Shur roots and \( \beta_1, \ldots, \beta_s \) are pairwise non-equal real Shur roots.

2. Generic stabilizer \( H \) of a locally semi-simple point is isomorphic to \( \prod_{i=1}^{r} (k^*)^p_i \times \prod_{j=1}^{s} GL(m_j) \). The linear group \( \prod_{i=1}^{r} (k^*)^p_i \times \prod_{j=1}^{s} GL(m_j) \) is isomorphic to

\[
\bigoplus_{i=1}^{r} ((GL(\delta_i), R(Q, \delta_i)) \oplus \cdots \oplus ((GL(\delta_i), R(Q, \delta_i)) \oplus \bigoplus_{j=1}^{s} (GL(\beta_j), R(Q, \beta_j))).
\]

3. \( k[R(Q, \alpha)]^{SL(\alpha)} \cong k[R(Q, p_1 \delta_1) \oplus \cdots R(Q, p_r \delta_r) \oplus R(Q, \beta_1) \oplus \cdots R(Q, \beta_s)]^{G} \), where \( G \subseteq GL[p_1 \delta_1 \times \cdots \times GL[p_r \delta_r \times GL(\beta_1) \times \cdots \times GL(\beta_s)] \) consists of the elements such that for each vertex the product of determinants is 1. Moreover, generic \( G \)-orbit in \( R(Q, p_1 \delta_1) \oplus \cdots R(Q, p_r \delta_r) \oplus R(Q, \beta_1) \oplus \cdots R(Q, \beta_s) \) is closed.

4. \( k[R(Q, \alpha)]^{SL(\alpha)} \cong \bigoplus_{\chi \in \Lambda} k[R(Q, \beta_j)]^{(GL(\beta_j))} \neq 0, j = 1, \ldots, s \).

**Proof.** The assertion 1 is showed above. The form of \( H \) in 2 follows from 1. For any vertex \( \alpha \in Q_0 \) each summand \( \rho = \delta_i \) or \( \beta_j \) yields an isotypical component of the \( H \)-module \( k^a \) being the sum of \( \rho_a \) irreducible factors of type \( GL(m, k^m) \), where \( m = 1 \) for \( \delta_i \) and \( m = m_j \) for \( \beta_j \). These isotypical components are stable with respect to the centralizer \( Z_{GL(\alpha)}(H) \) of \( H \) and each of them yields a factor \( (GL(\rho), R(Q, \rho)) \) of \( (Z_{GL(\alpha)}(H), R(Q, \alpha))^{H} \). Elements of \( N_{GL(\alpha)}(H) \) act on \( Z_{GL(\alpha)}(H) \) induce an outer automorphism of the group \( H \) and the corresponding permutation of the isotypical components in each space \( k^a \). Clearly, the isotypical components corresponding to non-equal summands can not be permuted, so \( N_{GL(\alpha)}(H) \) is contained in the extension of \( Z_{GL(\alpha)}(H) \) by the groups \( S_p, i = 1, \ldots, r \), and the latter extension does normalize \( H \) so 2 is proved.

By Proposition 2.1, \( k[R(Q, \alpha)]^{SL(\alpha)} \cong k[R(Q, \alpha)]^{H} \). Consider a subgroup \( N \triangleleft N_{GL(\alpha)}(H) \) consisting of the elements such that the restrictions to the irreducible \( N_{GL(\alpha)}(H) \)-submodule of \( k^a \) are unimodular for any \( \alpha \in Q_0 \). By 1 \( N \) acts independently on the summands of \( R(Q, \alpha)^H \) and we have: \( k[R(Q, \alpha)^H] \cong (\bigotimes_{i=1}^{r} k[R(Q, \delta_i) \oplus \cdots \oplus R(Q, \delta_i)]^N) \otimes (\bigotimes_{j=1}^{s} k[R(Q, \beta_j)]^N) \). Note that \( p_i \delta_i = \delta_i + \cdots + \delta_i \) is the generic locally semi-simple decomposition of \( p_i \delta_i \). Note also that \( N_{GL(\alpha)}(H) \) acts on \( R(Q, \delta_i) \oplus \cdots \oplus R(Q, \delta_i) \) as \( N \) extended by the center of \( GL(p_i \delta_i) \).
Therefore by \( \Xi(\mathcal{G}(p, \delta_i)) \oplus \cdots \oplus R(Q, \delta_i) \) and \( k[R(Q, p, \delta_i)]^{SL(p, \delta_i)} \) are isomorphic as \( \Xi(\mathcal{G}(p, \delta_i)) \)-graded algebras.

Next, fix \( j \in \{1, \cdots, s\} \), set \( m = m_j, \beta = \beta_j \) and consider the direct summand of \( R(Q, \alpha)^H \) corresponding to \( m \beta \). The determinant \( \det_\alpha \) restricts to \( GL(\beta) \subseteq N_{GL(\alpha)}(H) \) as the \( m \)-th power of the corresponding determinant on \( GL(\beta)_a \). Therefore \( N \) acts on \( R(Q, \beta) \) as the group \( SL(\beta)_{\Gamma_m} \), where \( \Gamma_m = \{ g \in GL(\beta) | g_\alpha = c_\alpha \text{Id}, c_\alpha \in \sqrt{V} \} \) is a finite group. Since \( \beta \) is a real Schur root, \( k[R(Q, \beta)]^{SL(\beta)} \) is generated by semi-invariants with linear independent weights such that their common kernel is the group \( k^* \) of scalar operators. One can deduce from this that for each weight \( \chi \) there is an element \( w \in \Gamma_m \) such that \( \chi(w) \) is a prime unity root of order \( m \) and all other weights take \( w \) to 1. Consequently, \( k[R(Q, \beta)]^N \) is generated by the \( m \)-th powers of the generators of \( k[R(Q, \beta)]^{SL(\beta)} \). So we have an isomorphism of \( k[R(Q, \beta)]^{SL(\beta)} \) onto \( k[R(Q, \beta)]^N \) taking \( GL(\beta) \)-eigenvectors of weight \( \chi \) to \( N_{GL(\alpha)(H)}/N \)-eigenvectors of weight \( m \chi \).

Thus we proved an isomorphism

\[
(12) \quad \expval{Q}{\alpha}^H \cong \bigotimes_{j=1}^r \bigotimes_{i_1=1}^s \bigotimes_{i_2=1}^s k[R(Q, p, \delta_i)]^{SL(p, \delta_i)} \otimes \bigotimes_{j=1}^r k[R(Q, \beta_j)]^{SL(\beta_j)},
\]

where a subspace of weight \( \sigma_1 + \cdots + \sigma_r + \chi_1 + \cdots + \chi_s \) in the right hand side algebra corresponds to that of weight \( \sigma_1 + \cdots + \sigma_r + m \chi_1 + \cdots + m \chi_s \) in \( k[R(Q, \alpha)^H]^N \). Consequently, the \( GL(p_i, \delta_i) \times \cdots \times GL(\beta)_s \)-weights vanishing on \( G \) correspond to the \( N_{GL(\alpha)(H)}^N \)-weights vanishing on \( N_{SL(\alpha)(H)} \) so the subalgebra of \( N_{SL(\alpha)(H)} \)-invariants corresponds under the isomorphism to that of \( G \)-invariants. By \( 2.19 \) generic \( N_{SL(\alpha)(H)} \)-orbits are closed in \( R(Q, \alpha)^H \); clearly, this implies the same for the \( G \)-orbits, and \( 3 \) is proved.

Elements of \( \expval{R(Q, p, \delta_i)}_{\gamma_i}^{GL(p, \delta_i)} \oplus \expval{R(Q, \beta_j)}_{\chi_j}^{GL(\beta)} \) are \( G \)-invariant if and only if the corresponding character \( \sigma_1 + \cdots + \sigma_r + \chi_1 + \cdots + \chi_s \) is a linear combination of the sums of determinants for each vertex, that is, \( \sigma_1 = \sigma_2 = \cdots = \sigma_r = \chi_1 = \cdots = \chi_s \) as elements of \( \mathbb{Z}Q_\alpha \). So the algebra of \( G \)-invariants is equal to \( \bigoplus_{\chi \in \Lambda} \expval{R(Q, p, \delta_i)}^{GL(p, \delta_i)}_{\chi} \oplus \expval{R(Q, \beta_j)}^{GL(\beta)}_{\chi} \). On the other hand, \( \beta_j \) are real Schur roots, hence, \( \dim k[R(Q, \beta_j)]^{GL(\beta)}_{\chi} = 1 \) for any \( j \) and \( \chi \in \Lambda \). So restricting \( G \)-invariants to \( \bigoplus_{i=1}^r R(Q, p, \delta_i) \), we get an isomorphism of \( k[(\bigoplus_{i=1}^r R(Q, p, \delta_i)) \oplus (\bigoplus_{j=1}^r R(Q, \beta_j))]^G \) onto \( \bigoplus_{\chi \in \Lambda} \bigotimes_{i=1}^r k[R(Q, p, \delta_i)]^{GL(p, \delta_i)}_{\chi} \).

### 6. Decompositions for \( A_n \) quiver.

In this section we describe generic and generic locally semi-simple decompositions for \( Q \) being the equioriented \( A_n \)-quiver that we considered in Example 4.3. Since \( Q \) is finite, there is a dense isomorphism class in \( R(Q, \alpha) \) for all \( \alpha \) so \( V \) is generic is equivalent to \( V \) having the dense orbit or \( \text{Ext}(V, V) = 0 \). So we are looking for a sum of \( S_{ij} \) with trivial \( \text{Ext} \)-spaces for summands. Using Ringel formula 4.3 and 4.7, we see that the condition \( \text{Ext}(S_{ij}, S_{kl}) = 0 = \text{Ext}(S_{kl}, S_{ij}) \) is equivalent to:

\[
(13) \quad j < k - 1, \text{ or } l < i - 1, \text{ or } i \leq k \leq l \leq j, \text{ or } k \leq i \leq j \leq l.
\]

so either the distance between the segments \([i, j] \) and \([k, l] \) is at least 2, or one of them contains another. This property yields a simple algorithm for calculating generic decomposition, exactly the same as in 4.3 Lemma 3.3:
Algorithm 6.1. For \( \alpha \in \mathbb{Z}_{+}^{Q_0} \) set \( m = \min\{\alpha_a | a \in Q_0\} \). If \( m > 0 \), then \( \alpha = m(1, \cdots, 1) + \pi, \pi \in \mathbb{Z}_{+}^{Q_0} \) and the generic decomposition of \( \alpha \) is \( \alpha = m(1, \cdots, 1) + \pi \) where \( \pi \) is the terms of the generic decomposition of \( m \). Otherwise, if \( \alpha_t = 0 \), then \( \alpha = \pi + \sigma, \pi = (\alpha_1, \cdots, \alpha_{t-1}, 0, \cdots, 0), \sigma = (0, \cdots, 0, \alpha_{t+1}, \cdots, \alpha_n) \) and the generic decomposition of \( \alpha \) is that of \( \pi + \sigma \) for the appropriate proper subquivers.

Now we consider locally semi-simple decompositions. The following observation follows from Proposition 4.4.

Proposition 6.2. If \( 0 < m = \alpha_t < \alpha_i \) for any \( i \neq t \), then any locally semi-simple representation \( V \) of dimension \( \alpha \) decomposes as \( V = mS_{\pi t} + \) other summands.

Algorithm 6.3. For \( \alpha \in \mathbb{Z}_{+}^{Q_0} \) set \( m = \min\{\alpha_a | a \in Q_0\}, t = \min\{a \in Q_0 | \alpha_a = m\} \), \( s = \max\{a \in Q_0 | \alpha_a = m\} \). If \( m > 0 \), then \( \alpha = \pi + \sigma + m \dim S_{ts}, \pi = (\alpha_1, \cdots, \alpha_{t-1}, 0, \cdots, 0), \sigma = (0, \cdots, 0, \alpha_{s+1}, \cdots, \alpha_s) \) and the generic locally semi-simple decomposition of \( \alpha \) is \( \alpha = m \dim S_{ts} + \) the terms of the decompositions of \( \pi, \sigma, \rho \) for the appropriate proper subquivers. Otherwise, if \( \alpha_t = 0 \), then \( \alpha = \pi + \sigma, \pi = (\alpha_1, \cdots, \alpha_{t-1}, 0, \cdots, 0), \sigma = (0, \cdots, 0, \alpha_{t+1}, \cdots, \alpha_n) \) and the generic locally semi-simple decomposition of \( \alpha \) is that of \( \pi + \sigma \) for the appropriate proper subquivers.

Proof. The second case \( m = 0 \) is obvious. In the first case set \( \mu = \alpha - m \dim S_{ts} \) and take a representation \( V = mS_{ts} + \mu S_{11} + \cdots + \mu \dim S_{nn}, \dim V = \alpha \). Since \( \mu_S = 0 \), \( V \) is locally semi-simple. The Ext-spaces for the summands of \( V \) are non-zero and one-dimensional only for \( \text{Ext}(S_{ti}, S_{t+1}, \cdots, S_{s-1}) \) and \( \text{Ext}(S_{ts}, S_{s+1}) \). So the graph \( \Sigma_V \) is the disjoint union of \( A_{n-s+t} \) on the vertices corresponding to \( S_{11}, \cdots, S_{t-1}, S_{ts}, S_{s+1} \) and \( A_{s-t} \) (if \( s - t \geq 2 \)) on the vertices corresponding to \( S_{t+1}, \cdots, S_{s-1} \). The induced dimension \( \gamma \) is \( (\pi, m, \sigma) \) on \( A_{n-s+t} \) and \( \rho \) on \( A_{s-t-1} \). By Proposition 4.3, the generic locally semi-simple decomposition for \( \alpha \) is the sum of that for \( \rho \) and that for \( (\pi, m, \sigma) \). Applying the map \( D_V \) to the summands of the decomposition, we conclude the proof. \( \square \)

7. Regular representations of tame quivers.

The tame quivers can be described by several equivalent conditions; in particular, these are the quivers with the underlying graph being an extended Dynkin diagram of type \( A_n, D_n, E_6, E_7, E_8 \) (the number of vertices is in all cases the subscript + 1). So let \( Q \) be a tame quiver and assume additionally that \( Q \) does not have oriented cycles (this is a restriction only for the underlying graph being \( A_n \)).

For quivers without oriented cycles Bernstein, Gelfand, and Ponomarev introduced in [BGP] Coxeter functors \( C^+ \) and \( C^- \) (defined not uniquely) acting on representations of \( Q \). The corresponding linear Coxeter transformation \( c \) is defined by the rule \( c(\dim V) = \dim C^+ V \) for a representation \( V \) of dimension \( \alpha \); note that \( \dim C^-(V) = c^{-1} \dim V \). Indecomposable representations \( V \) such that \( C^n V = 0 \) for natural \( n \) are called preprojective, the preinjective representations being defined symmetrically. Representation having neither preprojective nor preinjective direct summands are called regular.

For tame quivers regular indecomposable representations \( V \) can be described in term of a certain defect function \( \sigma \) such that \( V \) is regular if and only if \( \sigma(\dim V) = 0 \).
This $\sigma$ is presented explicitly in [Ri] (for special orientations), and one can easily check in all cases:
\[ \sigma(\alpha) = (\alpha, \delta), \]
where $\delta$ is the non-divisible imaginary root such that $\langle \delta, \delta \rangle = 0$.

In [Ri] Ringel proved that the regular representations form an Abelian subcategory $\mathcal{R}$ closed under direct sums, direct summands, homomorphisms, extensions etc. Note that by definition and [14], the simple regular objects are the $\delta$-stable representations. These simple objects are as follows. In dimension $\delta$ there is a 1-parameter family of simple regular objects; these representations are called homogeneous. We follow [Ri] and denote by $I$ the set of the dimensions of regular simple objects different from $\delta$ and by $e_i$ the dimension corresponding to $i \in I$.

It is known that the set $I$ consists of regular elements such that there is a unique simple representation $E_i$ of dimension $e_i$, up to isomorphism. Furthermore, the set $I$ is finite and stable with respect to the Coxeter transformation $c$; moreover, $c$ has at most 3 orbits in $I$. The sum of dimensions over a $c$-orbit is equal $\delta$.

The category $\mathcal{R}$ is connected with perpendicular ones:

**Proposition 7.1.** Let $S \in \mathcal{R}$ be a homogeneous simple object and let $V \in \mathcal{R}$ be an indecomposable representation. If not all Jordan-Hölder factors of $V$ are isomorphic to $S$, then $V \in ^+ S$ and $V \in S^\perp$.

**Proof.** By definition and [14], $\langle \dim V, \dim S \rangle = \langle \dim V, \delta \rangle = \sigma(\dim V) = 0$. On the other hand, for any $\alpha$, $\langle \alpha, \delta \rangle + \langle \delta, \alpha \rangle = q_\delta(\alpha + \delta) - q_\delta(\alpha) - q_\delta(\delta) = 0$, because $\delta$ is in the kernel of $q_\delta$. So we need to check: $\text{Hom}(V, S) = 0 = \text{Hom}(S, V)$ or (by the Ringel formula) $\text{Ext}(V, S) = 0 = \text{Ext}(S, V)$. Assume the converse and apply induction on $\dim V$. If $\text{Hom}(V, S) \neq 0$, then we get an exact sequence $0 \to V' \to V \to S \to 0$, because $S$ is simple. So $V'$ also has a Jordan-Hölder factor different from $S$, since $V$ has. Consequently, at least one of the direct summands $V'_i$ of $V'$ meets this condition and $\text{Ext}(S, V'_i) = 0$ by induction. Decompose $V'$ as $V' = V'_1 + V'_2$. Applying the definition of the Ext-functor, one can find a subrepresentation $V'_3 \subseteq V$ containing $V'_2$ such that $V = V'_1 + V'_3$. This is a contradiction, because $V$ is indecomposable. Analogously, $\text{Hom}(S, V) \neq 0$ and the induction imply $\text{Ext}(V'_i, S) = 0$ for an indecomposable summand $V'_i \subseteq V/S$ and this also contradicts to $V$ being indecomposable.

Denote by $D_r$ the dimensions of regular representations. If $\alpha \notin D_r$, then by [Ri] Theorem 3.2] $R(Q, \alpha)$ contains a dense orbit. Otherwise, if $\alpha \in D_r$, then $\alpha$ decomposes as $\alpha = p\delta + \sum_{i \in I} p_i e_i$ and there is a unique decomposition of such a type with an additional condition that for every $c$-orbit there is an element $j$ such that $p_j = 0$. Ringel called this decomposition canonical. This decomposition yields locally semi-simple representations:

**Proposition 7.2.** Let $\alpha = p\delta + \sum_{i \in I} p_i e_i$ be the canonical decomposition of $\alpha \in D_r$. Consider a representation $V = S_1 + \cdots + S_p + \sum_{i \in I} p_i E_i$, where $S_1, \ldots, S_p$ are homogeneous representations. Then $V$ is a locally semi-simple representation.

**Proof.** Take $S$ to be a homogeneous simple object non-isomorphic to $S_1, \ldots, S_p$. By Proposition 7.1 $S_1, \ldots, S_p$ and $E_i$ for all $i \in I$ belong to $^+ S$. Since all these are $\delta$-stable, these are also simple in $^+ S$, by Proposition 4.1. So the assertion follows from Theorem 4.2.
In [9] we described the slice at a locally semi-simple point \( V \) in terms of the quiver \( \Sigma_V \) with dimension \( \gamma \). For \( V \) being as in [9] \( \Sigma_V \) has a simple structure. Denote by \( E(Q) \) the quiver with \( E(Q)_0 = I \) and an arrow from \( i \) to \( j \) for each pair \((i, j)\) such that \( c(e_i) = e_j \). Note that \( E(Q) \) is a disjoint union of circular quivers.

**Proposition 7.3.** Let \( \alpha \) and \( V \) be as in Proposition [7.2] such that \( S_1, \ldots, S_p \) are pairwise non-isomorphic. Then \( (\Sigma_V, \gamma) \) is a disjoint union of 1-dimensional representations of the loops sitting at the vertices corresponding to \( S_1, \ldots, S_p \) and \( (E(Q), p_i, i \in I) \).

**Proof.** The quiver \( \Sigma_V \) is defined in terms of the Euler form or the Ext-spaces for the summands of \( V \). Since \( \langle \delta, \delta \rangle = 0 \) and \( \langle \delta, e_i \rangle = 0 \), each of the vertices corresponding to \( S_1, \ldots, S_p \) is incident to the unique arrow-loop and the dimension sitting there is 1. It remains to describe \( \text{Ext}(E_i, E_j) \). Applying the formula:

\[
\dim \text{Ext}(U, W) = \dim \text{Hom}(W, C^+U).
\]

(see e.g. [Ri, p.219]) we get: \( \dim \text{Ext}(E_i, E_j) = \dim \text{Hom}(E_j, C^+E_i) \). Since \( \dim \text{Hom}(E_i, E_j) = \delta_{ij} \), \( \dim \text{Ext}(E_i, E_j) \) is either 0 or 1, the latter being equivalent to \( c(e_i) = e_j \).

Now we have 3 ingredients that allow to calculate the generic and the generic locally semi-simple decompositions for \( \alpha \in D_\gamma \). First, given the canonical decomposition of \( \alpha \), we have a locally semi-simple representation \( V \) and the description of \( \Sigma_V \) in Proposition [7.2]. Thanks to the condition that \( p_i = 0 \) for at least one \( i \) in each \( c \)-orbit, the group \((GL(\gamma), R(\Sigma_V, \gamma))\) is isomorphic, up to a \( p \)-dimensional invariant subspace to a direct sum of groups \((GL(\gamma_i), R(A_n, \gamma_i))\). Secondly, Proposition [7.3] reduces both decompositions to the same for the quivers \( A_n \). Thirdly, Algorithms 6.1 and 6.3 yield both decompositions for \( A_n \).

In what concerns the generic decomposition our algorithm recovers that by Ringel from [Ri, Theorem 3.5]. It should be noted, however, that Ringel used an equivalence of categories instead of the slice theorem.

**Example 7.4.** Consider the quiver \( \tilde{E}_6 \) (over each vertex we placed the index):

\[
\begin{align*}
\tilde{E}_6: & 1 \quad \rightarrow \quad 2 \quad \rightarrow \quad 7 \quad \leftarrow \quad 4 \quad \leftarrow \quad 3 \\
& 5 \quad \rightarrow \quad 6 \quad \leftarrow \quad 3 \\
& 5 \quad \rightarrow \quad 6 \quad \leftarrow \quad 3 
\end{align*}
\]

We have \( \delta = (1, 2, 1, 2, 1, 2, 3) \) so that \( \sigma(\alpha) = 3\alpha_7 - \alpha_1 - \cdots - \alpha_6 \). The sequence of the vertices in the order defined by the indices is admissible in the sense of [BGP], i.e., for any arrow \( \varphi \) holds \( h\varphi > t\varphi \). So the composition \( C^+ = R_1^+ R_2^+ \cdots R_7^+ \) of the reflection functors at sinks is well-defined. Hence we have \( c = r_1 r_2 \cdots r_7 \) where \( r_i \) is the reflection at the vertex \( i \). There are 3 \( c \)-orbits of dimensions of simple regular representations: \( e_3 \rightarrow e_2 \rightarrow e_1 \rightarrow e_3, e_6 \rightarrow e_5 \rightarrow e_4 \rightarrow e_6, e_8 \rightarrow e_7 \rightarrow e_8 \):

\[
(16) \quad e_1 = (1, 1, 0, 1, 0, 0, 1); e_2 = (0, 0, 1, 0, 1, 1, 1); e_3 = (0, 1, 0, 0, 1, 0, 1); e_6 = (0, 0, 0, 1, 1, 1, 1); e_8 = (0, 1, 1, 1, 0, 0, 1);
\]

\[
(17) \quad e_4 = (1, 1, 0, 0, 0, 1, 1); e_5 = (0, 0, 0, 0, 1, 1, 1); e_8 = (0, 1, 1, 1, 0, 0, 1);
\]

\[
(18) \quad e_7 = (0, 1, 0, 1, 0, 1, 1); e_8 = (1, 1, 1, 1, 1, 1, 2).
\]
For example take $\alpha = (6, 10, 7, 14, 5, 9, 17)$. The canonical decomposition of $\alpha$ is: $\alpha = 2\delta + 3e_1 + 2e_2 + 2e_5 + 2e_6 + e_8$. So $(E(\delta), \gamma)$ is the direct sum of $(A_2, (2, 3)), (A_2, (2, 2)), \text{and } (A_1, (1))$. Applying Algorithm [6.3] we get the generic decomposition for $(E(\delta), \gamma)$: $(A_2, 2(1, 1) + (0, 1)), (A_2, 2(1, 1)), \text{and } (A_1, (1))$. So by Proposition [5.3.3], the generic decomposition of $\alpha$ is $\alpha = 2\delta + 2(e_1 + e_2) + e_1 + 2(e_5 + e_6) + 2e_8$, where $e_1 + e_2$ and $e_5 + e_6$ are real Schur roots. Next, applying Algorithm [6.3], we get the generic locally semi-simple decomposition for $(E(\delta), \gamma)$: $(A_2, 2(1, 0) + 3(0, 1)), (A_2, 2(1, 1)), \text{and } (A_1, (1))$. So by Proposition [5.3.3], the generic locally semi-simple decomposition of $\alpha$ is $\alpha = 2\delta + 3e_1 + 2e_2 + 2(e_5 + e_6) + e_8$.

8. Semi-invariants of tame quivers.

The algebras of semi-invariants of tame quivers $Q$ have been studied in several papers including [Ri, HI, SchW]. In [SkW] Skowronsky and Weyman proved that $k[R(Q, \alpha)]^{SL(\alpha)}$ is a complete intersection for any $\alpha$; moreover in most cases $k[R(Q, \alpha)]^{SL(\alpha)}$ is a polynomial algebra and in all other cases is a hypersurface.

Note that after [Kac], it is known that the reflection functors give rise to so called castling transforms of semi-invariants, so given a description of semi-invariants for $Q$ and $\alpha$, one can describe the semi-invariants for any quiver and dimension obtained by reflection functors. In particular, one may fix a convenient orientation for $Q$ (in the case of $\tilde{A}_n$, one of the convenient orientations). If $\alpha \notin D_r$, then by [Ri, Theorem 3.2], $R(Q, \alpha)$ contains a dense orbit, hence $k[R(Q, \alpha)]^{SL(\alpha)}$ is a polynomial algebra by the theorem of Sato-Kimura ([SK]). Moreover, one can always apply one of the Coxeter functors $C^+$ or $C^-$ and describe the semi-invariants of $Q$ in dimension $\alpha$ in terms of the castling transforms of those in dimension $\beta = c(\alpha)$ or $c^{-1}(\alpha)$, respectively. It is well known that for $\alpha \notin D_r$ this process is not cyclic and in the end we reduce the question to $\beta$ being the dimension of a representation of a projective or an injective module where the semi-invariants are obvious (see an example of such an approach in [SchW] for $\tilde{D}_4$ quiver). That is why we may and will assume from now on: $\alpha = p\delta + \sum_{i \in I} p_i e_i \in D_r$.

Ringel described the field $k[R(Q, \alpha)]^{GL(\alpha)}$ of invariants. Namely, he constructed semi-invariants $f_0, \cdots, f_p$ of weight $\sigma$ and proved in [Ri, Theorem 4.1] that the fractions $f_0/L_0, \cdots, f_p/L_p$ generate $k[R(Q, p\delta)]^{GL(p\delta)}$. Moreover, it is stated on [Ri, p.237] that $f_0, \cdots, f_p$ form a basis of $k[R(Q, p\delta)]^{GL(p\delta)}$ and one can actually deduce this from the proof of [Ri, Theorem 4.1].

First consider the homogeneous case $\alpha = p\delta$. The generators of $k[R(Q, \alpha)]^{SL(\alpha)}$ can be obtained using the following Corollary of the results from [DW].

**Proposition 8.1.** If $W \in V^+$ and $m_1S_1 + m_2S_2 + \cdots + m_tS_t$ is the sum of Jordan-Hölder factors of $W$ in $V^+$, then $c_W = c_{S_1}^{m_1} c_{S_2}^{m_2} \cdots c_{S_t}^{m_t}$.

**Proof.** We have a filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_d = W$ such that $W_j \in V^+$, $W_j/W_{j-1} \cong S_p$, $j = 1, \cdots, d$. Applying [DW, Lemma 1], we decompose $c_W$. $\square$

Denote by $n_0$ the number of $c$-orbits in $I$; then $n_0 = 2$ if $\Gamma = \tilde{A}_n$ and $n_0 = 3$, otherwise. For each orbit $O_i, i = 1, \cdots, n_0$, denote by $P_i$ the product of $c_{E_i}$ over the orbit; clearly this semi-invariant is of weight $\sigma$.

**Theorem 8.2.**

1. The algebra $k[R(Q, p\delta)]^{SL(p\delta)}$ is generated by $c_{E_i}, i \in I$ and $f_0, \cdots, f_p$. 


2. A minimal system of generators of \( k[R(Q, p\delta)]^{SL(p\delta)} \) consists of \( c_{E_i}, i \in I \) and \( \max(p+1-n_o, 0) \) elements from \( f_0, \ldots, f_p \). If \( p+1 \geq n_o \), then these generators are algebraically independent; otherwise, if \( p = 1, n_o = 3 \), then the generators fulfill a syzygy \( c_1P_1 + c_2P_2 + c_3P_3 = 0, c_1, c_2, c_3 \in k^* \) and the ideal of syzygies is generated by this one.

Remark 8.1. This statement is the same as [SW Theorem 2.4].

Proof. Denote by \( E_\lambda, \lambda \in \Lambda \subseteq k \) a 1-parameter family of pairwise non-isomorphic simple homogeneous regular representations of \( Q \) of dimension \( \delta \). A generic representation of dimension \( p\delta \) is locally semi-simple and is isomorphic to \( E_{\lambda_1} + \cdots + E_{\lambda_p} \).

By [DW], \( k[R(Q, \alpha)]^{SL(\alpha)} \) is generated by the semi-invariants \( c_W \) such that \( W \in \langle E_{\lambda_1} + \cdots + E_{\lambda_p} \rangle \) for some collection \( \lambda_1, \ldots, \lambda_p \); moreover, by Proposition 8.1 we can assume \( W \) to be a simple object of this category, hence, a \( \sigma \)-stable representation, by Proposition 4.1. The \( \sigma \)-stable representations are the simple regular ones. Note also that for \( W \) being homogeneous simple, \( c_W \subseteq k[R(Q, p\delta)]^{SL(p\delta)} \) is proved. So the assertion 1 is proved.

The generic stabilizer of \( GL(p\delta) \) is \( (k^*)^p \), hence the generic stabilizer of \( SL(p\delta) \) is \( (k^*)^{p-1} \). So \( \dim k[R(Q, p\delta)]^{SL(p\delta)} = \dim R(Q, p\delta) - \dim SL(p\delta) + (p-1) = q_Q(p\delta) + |Q_0| + (p-1) = n + p \), where \( n = |Q_0| - 1 \).

Consider the semi-invariants \( P_j, j = 1, \ldots, n_o \). Set \( W_j = \sum_{i \in \Omega_j} E_i \). Since \( \Hom(W_j, E_i) \) is non-trivial if and only if \( i \in \Omega_j \), \( P_j \) vanishes on \( W_k \) if and only if \( k = j \). Hence, \( P_j \) and \( P_j \) are non-proportional for \( i \neq j \). Moreover, if \( n_O = 3 \) and \( p \geq 2 \), then \( P_1, P_2, P_3 \) are linearly independent because of the values of these on representations \( W_1 + W_2, W_1 + W_3, W_2 + W_3 \).

Therefore, if \( p+1 \geq n_o \), then the semi-invariants \( P_1, \ldots, P_{n_o} \) are linear independent and \( k[R(Q, p\delta)]^{SL(p\delta)} \) is generated by \( c_{E_i}, i \in I \) and \( p+1-n_o \) elements of \( f_0, \ldots, f_p \). One can see that \( I \) consists of \( n + n_O - 1 \) elements. So this system of generators consists of \( (n + n_O - 1) + (p+1-n_O) = n + p \) elements, hence, this is a minimal system of algebraically independent generators.

Finally, if \( p = 1, n_O = 3 \), then \( P_1, P_2, \) and \( P_3 \) are non-proportional elements of two-dimensional vector space \( k[R(Q, p\delta)]^{SL(p\delta)} \), hence we get a syzygy as in 2. Since the number of generators is \( n+2 = n+p+1 \), and because our syzygy is of degree 1 by each of the generators, the assertion 2 is proved. \( \square \)

Now consider the general case: \( \alpha = p\delta + \sum_{i \in I} p_i e_i \). The quiver \( E_Q \) is the union of 2 or 3 circular quivers; for \( i \in I \) define by \( n(i) \) and \( p(i) \) the next and the previous vertex of \( E_Q \) so that \( c(e_i) = e_{n(i)}, n(p(i)) = i \). A subset \( [k, l] = \{k, n(k), \ldots, l\} \subseteq I \) will be called an arc. By Proposition 4.1, each arc \([k, l] \) with \( k \neq l \) yields a real Schur root \( c_{k,l} = e_k + \cdots + e_l \); pick a Schurian representation \( E_{k,l} \in R(Q, e_{k,l}) \).

By Proposition 5.8 and Algorithm 6.3, the generic locally semi-simple decomposition of \( \alpha \) is \( \alpha = p\delta + \sum_{[k, l] \in \Omega} m_{k,l} e_{k,l} \), where \( \Omega \) is a set of arcs such that for different arcs \([k_1, l_1], [k_2, l_2] \in \Omega \):
- either \([k_1, l_1] \cap [k_2, l_2] = \emptyset \) or \([k_1, l_1] \subseteq [n(k_2), p(l_2)] \) or else \([k_2, l_2] \subseteq [n(k_1), p(l_1)] \)
- if \( m_{k_1, l_1} = m_{k_2, l_2} \), then \( p_{k_1} \neq p_{k_2} \).

Proposition 8.3. \( \dim k[R(Q, \alpha)]^{SL(\alpha)} = \dim k[R(Q, p\delta)]^{SL(p\delta)} - |\Omega| \).

Proof. By Theorem 5.8, \( k[R(Q, \alpha)]^{SL(\alpha)} \cong k[R(Q, p\delta)] \oplus \bigoplus_{[k, l] \in \Omega} R(Q, e_{k,l})^G \), where \( G \subseteq GL(p\delta) \times \prod_{[k, l] \in \Omega} GL(e_{k,l}) \) consists of the elements with the product...
of determinants at any vertex being 1; moreover, generic $G$-orbit is closed. Next, since $e_{k,l}$ are real Shur roots, $G$ acts on $\bigoplus_{[k,l] \in \Omega} R(Q, e_{k,l})$ with an open orbit $G/K$, where $K$ is the kernel of that action. Consequently, generic $K$-orbit is closed in $R(Q, p\delta)$ and $\dim k[R(Q,\alpha)]_{\Lambda}^{SL(\alpha)} = \dim k[R(Q, p\delta)]_{\Lambda}^{K}$. Observe that $K$ acts on $R(Q, p\delta)$ as a subgroup $T \mathcal{S}L(p\delta) \subseteq GL(p\delta)$, where $T$ is a central torus in $GL(p\delta)$ of dimension $|\Omega|$. One can easily check that $T$ acts effectively on $R(Q, p\delta)/\mathcal{S}L(p\delta)$, hence $\dim k[R(Q, p\delta)]_{\Lambda}^{K} = \dim k[R(Q, p\delta)/\mathcal{S}L(p\delta)] = \dim k[R(Q, p\delta)]_{\Lambda}^{[SL(p\delta)]} - |\Omega|$. 

By Theorem 8.4, $k[R(Q, \alpha)]_{\Lambda}^{[GL(\alpha)]}$ is isomorphic to $\bigoplus_{\chi \in \Lambda} k[R(Q, p\delta)]_{\chi}^{[GL(\alpha)]}$, where $\Lambda \subseteq \mathbb{Z}^{Q_\delta}$ consists of weights such that $k[R(Q, p\delta)]_{\chi}^{[GL(\alpha)]} \neq 0$ and for each $[k, l] \in \Omega$, $k[R(Q, e_{k,l})]_{\chi}^{[GL(e_{k,l})]} \neq 0$. By Theorem 8.4, $k[R(Q, p\delta)]_{\chi}^{[GL(\alpha)]} \neq 0$ implies $\chi = -\langle , e \rangle$, where $e \in \langle e_i, i \in I \rangle_{\mathbb{Z}_+}$. So in order to determine $\Lambda$, we need to find the dimensions $e \in \langle e_i, i \in I \rangle_{\mathbb{Z}_+}$ such that $E_{k,l}^{+} \cap R(Q, e) \neq 0$ for any $[k, l] \in \Omega$. 

**Proposition 8.4.** $E_{k,l}^{+} \cap R(Q, e) \neq 0$ iff $e \in \langle e_{k,n(l)}, e_i | i \in I, i \neq k, n(l) \rangle_{\mathbb{Z}_+}$.

**Proof.** Clearly, a necessary condition for $E_{k,l}^{+} \cap R(Q, e) \neq 0$ is $\langle e_{k,l}, e \rangle = 0$. By Proposition 8.2 we have:

\[
\langle e_{k,l}, \sum_{i \in I} q_i e_i \rangle = q_k - q_{n(l)}.
\]

Hence, the semi-group $\{e = \sum_{i \in I} q_i e_i, q_i \in \mathbb{Z}_+ | \langle e_{k,l}, e \rangle = 0\}$ is generated by dimensions $e_{k,n(l)}, e_i \in I \setminus \{k, n(l)\}$. So it is sufficient to check either of the equivalent conditions $\text{Hom}(E_{k,l}, E) = 0$ or $\text{Ext}(E_{k,l}, E) = 0$ for $E = E_{k,n(l)}, E_i, i \neq k, n(l)$. For all $E$ with the exception of $E_{k,n(l)}, E_i$ we have by 8.3 and Proposition 8.4.2: $\text{Hom}(E_{k,l}, E) = 0 = \text{Hom}(E, E_{k,l})$. On the other hand, for $E = E_{k,n(l)}, E_i, i \neq k, n(l)$ we have: $p(k) \neq l, p(n(l)) \neq k$. 

Let $J \subseteq I$ consist of elements being $k$ or $n(l)$ for an arc $[k, l] \in \Omega$. It can happen that $J$ consists of less than $2|\Omega|$ elements because there can be arcs like $[k, l]$ and $[n(l), m]$ in $\Omega$ such that their union is again an arc. So we can introduce a new set $\Delta$ of arcs such that each arc from $\Delta$ is a disjoint union of arcs from $\Omega$, each arc from $\Omega$ is contained in an arc from $\Delta$, and for any $[k_1, l_1], [k_2, l_2] \in \Delta$ we have: $p(k_1) \neq l_2, p(n(l_1)) \neq k_2$.

**Proposition 8.5.** $\Lambda$ is generated by $|I| - |\Omega|$ elements $\chi = -\langle , e \rangle$, where $e \in \{e_i, e_{k,n(l)} | i \in I \setminus J, [k, l] \in \Delta\}$.

**Proof.** By formula 8.3 a necessary condition for a character $\chi = -\langle , \sum_{i \in I} q_i e_i \rangle$ to be in $\Lambda$ is $q_{n(l)} = q_k$ for any arc $[k, l] \in \Omega$. Hence, the semigroup of dimension vectors meeting this condition is generated by $e_i, i \in I \setminus J$ and $e_{k,n(l)}, [k, l] \in \Delta$. On the other hand, by Proposition 8.4.3 for each $e$ of this generators and for each arc $[k, l] \in \Omega$ there is a representation of dimension $e$ perpendicular to $E_{k,l}$. It remains to note: $|I| = |\Delta| + |\Omega|$. 

**Theorem 8.6.** Let $\alpha = p\delta + \sum_{i \in I} p_i e_i, p > 0$. If $p = 1$, $I$ consists of 3 orbits, and for each orbit at least two coefficients $p_i$ vanish, then $k[R(Q, \alpha)]_{\Lambda}^{[SL(\alpha)]}$ is a hypersurface; in all other cases it is a polynomial algebra.
Proof. Clearly, $A = \bigoplus_{\chi \in \Lambda} k[R(Q, p\delta)](GL(p\delta))_\chi$ is generated as an algebra by the subspaces $k[R(Q, p\delta)](GL(p\delta))_\chi$, where $\chi$ is $\sigma$ or a generator of $\Lambda$ from $\text{Sec}$ Moreover, for each orbit $O_j \subseteq I$, $\delta$ can be obtained as a non-negative linear combination of the generators of $\Lambda$ corresponding to $O_j$, hence $P_j$ is a corresponding product of $\delta$ generators of $A$. So $A$ is generated by $c_{E_i}, i \subseteq I \setminus J, c_{E_k} \cdots c_{E_m(\delta)}, [k, l] \in \Delta$, and $\max(p + 1 - n_\sigma, 0)$ elements from $f_0, \ldots, f_p$. The number of these generators of $A$ is less than the number of generators of $k[R(Q, p\delta)]^{SL(p\delta)}$ by $|\Omega|$, hence by Proposition $\text{Sec}$ if $k[R(Q, p\delta)]^{SL(p\delta)}$ is polynomial algebra, then $A$ is and if $k[R(Q, p\delta)]^{SL(p\delta)}$ is a hypersurface, then $A$ is generated by $\dim A + 1$ elements. Moreover, in the latter case the unique relation between the generators is $c_1 P_1 + c_2 P_2 + c_3 P_3 = 0$. If for some $O_j$, $P_j = c_{E_k} \cdots c_{E_m(\delta)} [k, l] \in \Delta$, then the relation says that this generator is redundant, so $A$ is in fact a polynomial algebra. This happens precisely when $n(n(l)) = k$ or equivalently, $p_n(l) = 0$ and $p_l \neq 0$ for all other $i \in O_j$. This completes the proof. □

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