EXPONENTIAL STABILITY FOR THE COUPLED KLEIN-GORDON-SCHRÖDINGER EQUATIONS WITH LOCALLY DISTRIBUTED DAMPING

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Abstract. The following coupled damped Klein-Gordon-Schrödinger equations are considered
\[ i\psi_t + \Delta \psi + iab(x)(|\psi|^2 + 1)\psi = \phi \psi \chi_\omega \text{ in } \Omega \times (0, \infty), \quad (\alpha > 0) \]
\[ \phi_{tt} - \Delta \phi + a(x)\phi_t = |\psi|^2 \chi_\omega \text{ in } \Omega \times (0, \infty), \]
where \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \), with smooth boundary \( \Gamma \) and \( \omega \) is a neighbourhood of \( \partial \Omega \) satisfying the geometric control condition. Here \( \chi_\omega \) represents the characteristic function of \( \omega \). Assuming that \( a, b \in L^\infty(\Omega) \) are nonnegative functions such that \( a(x) \geq a_0 > 0 \) in \( \omega \) and \( b(x) \geq b_0 > 0 \) in \( \omega \), the exponential decay rate is proved for every regular solution of the above system. Our result generalizes substantially the previous ones given by Cavalcanti et. al in the reference [9] and [1].

1. Introduction. We consider the following model of Klein-Gordon-Schrödinger equations with locally distributed damping
\[
\begin{aligned}
&i\psi_t + \Delta \psi + iab(x)(|\psi|^2 + 1)\psi = \phi \psi \chi_\omega \text{ in } \Omega \times (0, \infty), \\
&\phi_{tt} - \Delta \phi + a(x)\phi_t = |\psi|^2 \chi_\omega \text{ in } \Omega \times (0, \infty), \\
&\psi = \phi = \phi_t = 0 \text{ on } \Gamma \times (0, \infty), \\
&\psi(0) = \psi_0 \in H^1_0(\Omega) \cap H^2(\Omega), \\
&\phi(0) = \phi_0 \in H^1_0(\Omega) \cap H^2(\Omega), \\
&\phi_t(0) = \phi_1 \in H^1_0(\Omega),
\end{aligned}
\]
where \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \), with smooth boundary \( \Gamma \) and \( \omega \) is an open subset of \( \Omega \) such that \( \text{meas}(\omega) > 0 \) and satisfying the geometric control condition. In what follows, \( \alpha \) is a positive constant and \( \chi_\omega \) represents the characteristic function, that is, \( \chi = 1 \) in \( \omega \) and \( \chi = 0 \) in \( \Omega \setminus \omega \). We consider \( a, b \in L^\infty(\Omega) \) nonnegative functions such that
\[ a(x) \geq a_0 > 0 \text{ in } \omega, \quad \text{and} \quad b(x) \geq b_0 > 0 \text{ in } \omega, \]

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so that the nonlinearity $\psi$ exists where the damping terms
\[ a(x)\phi_t \quad \text{and} \quad i\alpha b(x)(|\psi|^2 + 1)\psi \]
are, in fact, effective and reciprocally. If the damping is effective in whole domain, i.e., $a(x) \geq a_0 > 0$ in $\Omega$ and $b(x) \geq b_0 > 0$ in $\Omega$ we can consider $\chi_\omega \equiv 1$ in $\Omega$. This is required in order to turn the system dissipative. Indeed, the presence of the damping terms given in (2) is not necessary, by itself, to guarantee that the energy $E(t)$ associated to problem (1) (see the definition of $E(t)$ in (9)) is a non-increasing function on the parameter $t$. This will clarified in section 4. Uniform decay rate estimates to problem (1) has been considered in the previous results due to Cavalcanti et al [11], [9]. While in [11] a full damping was in place in both equations, in contrast, in [9] a full damping has been considered in the Schrödinger equation but just a localized damping has been considered for the wave equation. It is worth mentioning that in the recent the article [1] the authors generalize both previous results just considering two localized dampings in both equations, namely:

\[ i\dot{\psi} + \Delta \psi + i\alpha b(x)(-\Delta)^{\frac{1}{2}} b(x)\psi = \phi \psi \chi_\omega \quad \text{in} \quad \Omega \times (0, \infty), \]  
\[ \phi_{tt} - \Delta \phi + a(x)\phi_t = |\psi|^2 \chi_\omega \quad \text{in} \quad \Omega \times (0, \infty), \]

making use of the multipliers method combined with integral inequalities of energy and a regularizing effect due to Aloui [2], [3].

The main purpose of the present article is to generalize substantially all the previous results above mentioned by considering the weaker damped structure $i\alpha b(x)(|\psi|^2 + 1)\psi$ instead of $i\alpha b(x)(-\Delta)^{\frac{1}{2}} b(x)\psi$ assumed in [1]. For this purpose, we make use of the observability inequality of both, the wave and the Schrödinger linear equations (see [7] and [16], [32] respectively), combined with other tools in order to prove the exponential decay rate as has been considered previously in [15] for the wave equation. This will be clarified during the proof. It is important to be mentioned that the use of the observability inequality associated to the linear problems of the wave and Schrödinger equations instead of the multiplier technique allows us to consider sharp regions $\omega$ satisfying the geometric control condition. Indeed, the inequalities given in (7) and (8) are proved by means of microlocal analysis and produce sharp regions when compared with the multiplier method. As a consequence, the present paper generalizes substantially our previous results not only with concerns to consider a weaker damping but also regarding the sharpness of the region $\omega$ (less damping as possible) where the damping acts.

Problem (1) (see [47]) has its origin in the canonical model of the Yukawa interaction of conserved complex nucleon field $\psi$ with neutral real meson field $\phi$ given by

\[
\begin{cases}
    i\psi_t + \Delta \psi = \phi \psi \quad \text{in} \quad \Omega \times (0, \infty), \\
    \phi_{tt} - \Delta \phi + \mu^2 \phi = |\psi|^2 \quad \text{in} \quad \Omega \times (0, \infty), \\
    \psi = \phi = 0 \quad \text{on} \quad \Gamma \times (0, \infty), \\
    \psi(0) = \psi_0, \phi(0) = \phi_0, \phi_t(0) = \phi_1.
\end{cases}
\]

Here, $\psi$ is a complex scalar nucleon field while $\phi$ is a real scalar meson one and the positive constant $\mu$ represents the mass of a meson. Since we are considering a bounded domain, the term $\mu^2 \phi$ does not affect our arguments in the proof of the asymptotic stability. So, for simplicity this term will be omitted.

It is important to note that problem (3) is not naturally dissipative. So, the introduction of the dissipative mechanisms given by the terms in (2) are necessary to force the energy to decay to zero when $t$ goes to infinity. In fact, the dissipative
KGS equation has been widely studied, see for example the following references: [22], [24], [28], [29], [21], [14] and references therein. The majority of works in the literature deal with linear dissipative terms acting in both equations, except for the works [23] and [11].

We would like to mention other papers in connection with problem (3), namely: Fukuda and Tsutsumi [17], [18], [19], [20], Bachelot and Chadam [5] and Hayashi and W. Von Wahl [27]. In the above articles the unique global existence to problem (3) is established and some conservation laws are verified. We also would like to quote some nice important papers in connection with Klein-Gordon-Schrödinger (KGS) system from various points of view as: [4], [6], [8], [10], [13], [14], [25], [27], [26], [28], [29], [37], [21], [38], [40], [33], [43], [44], [42], [41], [45], [46], [47], [39] and references therein.

Our paper is organized as follows. In section 2 we give the precise assumptions and state our main result, in section 3 we give an idea of the proof of existence and in section 4 we give the proof of the main theorem.

2. Main result. In what follows let us consider the Hilbert space $L^2(\Omega)$ of complex valued functions on $\Omega$ endowed with the inner product $$(u, v) = \int_{\Omega} u(x) \overline{v(x)} \, dx,$$
and the corresponding norm $$||u||_2^2 = (u, u).$$

We also consider the Sobolev space $H^1(\Omega)$ endowed with the scalar product $$(u, v)_{H^1(\Omega)} = (u, v) + (\nabla u, \nabla v).$$

We define the subspace of $H^1(\Omega)$, denoted by $H^1_0(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in the strong topology of $H^1(\Omega)$. This space endowed with the norm induced by the scalar product $$(u, v)_{H^1_0(\Omega)} = (\nabla u, \nabla v)$$ is, thanks to the Poincaré’s inequality $||u||_2 \leq \lambda ||\nabla u||_2$, for all $u \in H^1_0(\Omega)$:

$$||u||_2 \leq \lambda ||\nabla u||_2, \quad \text{for all } u \in H^1_0(\Omega); \tag{4}$$

a Hilbert space. We set the norms $||u||_p^p = \int_\Omega |u(x)|^p \, dx$, $||u||_{\Gamma,p}^p = \int_{\Gamma} |u(x)|^p \, d\Gamma$, $||u||_\infty = \text{ess sup}_{x \in \Omega} |u(x)|$.

In the particular case when $n = 2$ we have the Gagliardo-Nirenberg inequality,$$
||u||_4 \leq c ||u||_{\frac{n}{2}} \frac{||\nabla u||_2^2}{2}, \quad \text{for all } u \in H^1_0(\Omega). \tag{5}
$$

We denote by $\omega$ the intersection of $\Omega$ with a neighborhood of $\partial \Omega$ in $\mathbb{R}^2$ and we will call it a neighborhood of the boundary of $\Omega$.

The following assumptions are made:

Conjecture 1. We assume that $a, b \in L^\infty(\Omega)$ are nonnegative functions such that

$$a(x) \geq a_0 > 0, \quad \text{in } \omega,$$
$$b(x) \geq b_0 > 0, \quad \text{in } \omega.$$

In addition,

If $a(x) \geq a_0 > 0$ in $\Omega$, then we consider $\chi_\omega \equiv 1$ in $\Omega$,

If $b(x) \geq b_0 > 0$ in $\Omega$, then we consider $\chi_\omega \equiv 1$ in $\Omega$. 

**Definition 2.1.** (Geometric Control Condition): $\omega$ geometrically controls $\Omega$, i.e. there exists $T_0 > 0$, such that every geodesic of $\Omega$ travelling with speed 1 and issued at $t = 0$, enters the set $\omega$ in a time $t < T_0$.

So, the couple $(\omega, T_0)$ satisfies the geometric control condition (GCC, in short) if every geodesic of $\Omega$, traveling with speed 1 and issued at $t = 0$ enters the open set $\omega$ before the time $T_0$.

**Conjecture 2.** We assume that $\omega$ satisfies the geometric control condition (mentioned above). The standard example is when $\omega$ is a neighbourhood of $\Gamma(x_0)$ where

$$\Gamma(x_0) := \{ x \in \Gamma; (x - x_0) \cdot \nu(x) > 0 \}$$

and $\nu(x)$ is the unit outward normal at $x \in \Gamma$.

As an example of a domain $\Omega$ satisfying the above assumption let us consider the figure 1 (See [35]), although there exist a wide assortment of much more interesting examples as those ones considered in Bardos, Lebeau and Rauch [7].

As a consequence of assumption (2) it follows that there exists a couple $(\omega, T_0)$, with $T_0 > 0$, such that the following observability inequalities occur:

$$||\psi_0||^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |\psi(x,t)|^2 \, dx \, dt,$$

concerning problem

$$\begin{cases} \quad \psi_t + \Delta \psi = 0 \text{ in } \Omega \times (0,T), \\ \psi = 0 \text{ on } \Gamma \times (0,T), \\ \psi(0) = \psi_0 \in L^2(\Omega), \end{cases}$$

and

$$||\phi_1||^2_{L^2(\Omega)} + ||\nabla \phi_0||^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |\phi_t(x,t)|^2 \, dx \, dt,$$

regarding problem

$$\begin{cases} \quad \phi_{tt} - \Delta \phi = 0 \text{ in } \Omega \times (0,T), \\ \phi = 0 \text{ on } \Gamma \times (0,T), \\ \phi(0) = \phi_0 \in H^1_0(\Omega), \\ \phi_t(0) = \phi_1 \in L^2(\Omega), \end{cases}$$

for some positive constant $C = C(\omega, T_0)$ and for all $T > T_0$. The proof of (7) can be found in [36] and [32] while the proof of (8) is established in [35] and [7].
The energy associated to problem (1) is defined by

\[ E(t) := \frac{1}{2} \int_{\Omega} \left( |\psi(x,t)|^2 + |\nabla \phi(x,t)|^2 + |\phi(x,t)|^2 \right) dx. \]  

(9)

Now, we are in position to state our main results.

**Theorem 2.2.** Given \( \{\psi_0, \phi_0, \phi_1\} \in \{H^1_0(\Omega) \cap H^2(\Omega)\}^2 \times H^1_0(\Omega) \) and assuming that assumption 1 holds and that \( \alpha \geq \frac{5}{2a_0b_0} \), then, there exists a unique regular solution to problem (1) such that

\[
\psi \in L^\infty(0, \infty; H^1_0(\Omega) \cap H^2(\Omega)), \quad \psi' \in L^\infty(0, \infty; L^2(\Omega)), \\
\phi \in L^\infty(0, \infty; H^1_0(\Omega) \cap H^2(\Omega)), \quad \phi' \in L^\infty(0, \infty; H^1_0(\Omega)), \\
and \phi'' \in L^\infty(0, \infty; L^2(\Omega)).
\]

Setting \( \mathcal{H} := \{H^1_0(\Omega) \cap H^2(\Omega)\}^2 \times H^1_0(\Omega) \), in the next theorem, below, we provide a local uniform decay of the energy. Indeed, we shall consider the initial data taken in bounded sets of \( \mathcal{H} \), namely, \( \|\{\psi_0, \phi_0, \phi_1\}\|_H \leq L \), where \( L \) is a positive constant. This is strongly necessary due to the non linear character of system (1) and since the energy \( E(t) \) is not naturally a non increasing function of the parameter \( t \). Thus, the constants, \( C \) and \( \gamma \) which appear in (10) will depend on \( L > 0 \). Now, we are in position to state our stabilization result.

**Theorem 2.3.** Assume that the assumption of Theorem 2.2 are in place and the assumption (2) hold. Then, there exist \( C, \gamma \) positive constants such that the following decay rate holds

\[ E(t) \leq Ce^{-\gamma t} E(0), \quad \text{for all } t \geq 0. \]  

(10)

for every regular solution of problem (1) in the class given in previous theorem, provided the initial data are taken in bounded sets of \( \mathcal{H} \).

**Remark 1.** The assumption \( \alpha \geq \frac{5}{2a_0b_0} \) is required in the proof of the existence. On the other hand, take \( \alpha \) sufficiently large is natural to guarantee the dissipativity of the system.

3. **Existence and uniqueness.** In this section we derive a priori estimates for the solutions of the Klein-Gordon-Schrödinger system (1). In what follows, for simplicity, we will denote \( u_t = u' \). Let us represent by \( \{\omega_i\} \) a basis in \( H^1_0(\Omega) \cap H^2(\Omega) \) formed by the eigenfunctions of \(-\Delta\), by \( V_m \), the subspace of \( H^1_0(\Omega) \cap H^2(\Omega) \) generated by the first \( m \) vectors and by

\[
\psi_m(t) = \sum_{i=1}^{m} g_{im}(t) \omega_i, \quad \phi_m(t) = \sum_{i=1}^{m} h_{im}(t) \omega_i,
\]

where \( \{\psi_m(t), \phi_m(t)\} \) is the solution to the following Cauchy problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\psi''_m(t, v) + i\nabla \psi_m(t), \nabla u \right) + \alpha(b(x)|\psi_m(t)|^2 \psi_m(t), u \right) + \alpha(b(x)\psi_m(t), u) \\
= -i(\phi_m(t) \psi_m(t) \chi_m, v), \quad \forall u \in V_m,
\end{array}
\right.
\]

(11)

\[
\left\{ \begin{array}{l}
\phi''_m(t) + (\nabla \phi_m(t), \nabla v) + (a(x)\phi_m(t), v) = (|\psi_m(t)|^2 \chi_m, v), \forall v \in V_m, \\
\psi_m(0) = \psi_0m \rightarrow \psi_0, \quad \phi_m(0) = \phi_0m \rightarrow \phi_0 \quad \text{in } H^1_0(\Omega) \cap H^2(\Omega),
\end{array}
\right.
\]

for every \( \{\psi_0m, \phi_0m\} \rightarrow \psi_0, \phi_0 \in H^1_0(\Omega) \cap H^2(\Omega) \).
The approximate system (11) is a finite system of ordinary differential equations which has a solution in $[0, t_m]$. The extension of the solution to the whole interval $[0, T]$, for all $T > 0$, is a consequence of the first estimate we are going to obtain below.

**A priori estimates**

*The First Estimate:* Considering $u = \bar{\psi}_m$ in the first equation of (11) and taking the real part, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\psi_m(t)\|_2^2 + \alpha \int_\Omega b(x)|\psi_m(x, t)|^4 \, dx + \alpha \int_\Omega b(x)|\psi_m(x, t)|^2 \, dx = 0. \tag{12}
$$

So, since $b(x) > b_0$ almost everywhere in $\omega$, we have

$$
\frac{1}{2} \frac{d}{dt} \|\psi_m(t)\|_2^2 + \alpha b_0 \|\psi_m(t)\|_{L^4(\omega)}^4 + \alpha b_0 \|\psi_m(t)\|_{L^2(\omega)}^2 \leq 0. \tag{13}
$$

Multiplying (13) by 2, integrating over $(0, t)$, $t \in [0, t_m)$, we obtain

$$
\|\psi_m(t)\|_2^2 + 2\alpha b_0 \int_0^t \|\psi_m(s)\|_{L^4(\omega)}^4 \, ds + 2\alpha b_0 \int_0^t \|\psi_m(s)\|_{L^2(\omega)}^2 \, ds \leq \|\psi_{m0}\|_2^2. \tag{14}
$$

Then, from convergence $\psi_m(0) = \psi_{0m} \to \psi_0$ in $H^1_0(\Omega) \cap H^2(\Omega)$, we have

$$
(\psi_m) \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)), \tag{15}
$$

and there exist $C_1, C_2 > 0$, such that

$$
\int_0^\infty \|\psi_m(s)\|_{L^4(\omega)}^4 \, ds \leq C_1 = C(\|\psi_0\|_2), \tag{16}
$$

$$
\int_0^\infty \|\psi_m(s)\|_{L^2(\omega)}^2 \, ds \leq C_2 = C(\|\psi_0\|_2). \tag{17}
$$

Now, considering $v = \phi'_m$ in the second equation of (11) and making use of Hölder and Young inequalities, we deduce

$$
\frac{1}{2} \frac{d}{dt} \|\phi'_m(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 + \int_\Omega a(x)|\phi'_m(x, t)|^2 \, dx \leq \frac{1}{2a_0} \|\phi'_m(t)\|_{L^4(\omega)}^4 + \frac{1}{2} \int_\Omega a(x)|\phi'_m(x, t)|^2 \, dx. \tag{18}
$$

Thus,

$$
\frac{1}{2} \frac{d}{dt} \|\phi'_m(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 + \frac{1}{2} \int_\Omega a(x)|\phi'(x, t)|^2 \, dx \leq \frac{1}{2a_0} \|\phi'_m(t)\|_{L^4(\omega)}^4. \tag{19}
$$

So,

$$
\frac{1}{2} \frac{d}{dt} \|\phi'_m(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 \leq \frac{1}{2a_0} \|\phi'_m(t)\|_{L^4(\omega)}^4. \tag{20}
$$

since $\frac{1}{2} \int_\Omega a(x)|\phi'_m(x, t)|^2 \, dx \geq 0$.

Multiplying (20) by 2, integrating over $(0, t)$, $t \in [0, t_m)$, observing that $\phi_m(0) = \phi_{0m} \to \phi_0$ in $H^1_0(\Omega) \cap H^2(\Omega)$, $\phi'_m(0) = \phi_{1m} \to \phi_1$ in $H^1_0(\Omega)$, we obtain

$$
\|\phi'_m(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 \leq C + \frac{1}{2a_0} \int_0^t \|\psi_m(s)\|_{L^4(\omega)}^4 \, ds, \tag{21}
$$

where $C := C(\|\phi_1\|_2, \|\nabla \phi_0\|_2)$. 
So, taking (16) into account, yields,
\[ \|\psi_m'(t)\|_2^2 + \|\nabla\psi_m(t)\|_2^2 \leq C \]
(22)
where \( C := C (\|\psi_0\|, \|\phi_1\|_2, \|\nabla\phi_0\|_2) \).
Thus,
\[
(\phi_m) \text{ is bounded in } L^\infty(0, \infty; H_0^1(\Omega)),
\]
(23)
\[
(\phi_m') \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)).
\]
(24)

The Second Estimate: Taking the derivative on time of the first equation in (11), considering \( u = \psi_m \), we have
\[
(\psi_m''(t), \psi_m'(t)) + i(\nabla\psi_m'(t), \nabla\psi_m'(t)) + \alpha([b(x)\psi_m(t)]^2\psi_m'(t), \psi_m'(t)) + \alpha([b(x)\psi_m(t)]', \psi_m'(t)) = -i[\phi_m(t)\psi_m(t)\chi_{\omega}]', \psi_m'(t)).
\]
(25)
Considering \( I_1 := ([b(x)\psi_m(t)]^2\psi_m'(t), \psi_m'(t)) \), we infer
\[
I_1 := (2b(x)|\psi_m(t)|^2\psi_m'(t) + b(x)\psi_m(t)^2\psi_m'(t), \psi_m'(t))
\]
(26)
\[
= 2 \int_\Omega b(x)|\psi_m(x,t)|^2|\psi_m'(x,t)|^2 \, dx + \int_\Omega b(x)|\psi_m(x,t)|^2 \, dx.
\]
(27)
Observing that
\[ 2[\text{Re}(z_1\overline{z_2})]^2 = |z_1|^2|z_2|^2 + \text{Re}(z_1\overline{z_2})]^2, \forall z_1, z_2 \in \mathbb{C}, \]
and thus taking the part real of (26) and since \( b(x) \geq b_0 > 0 \) a.e in \( \omega \), we deduce
\[
b_0 \int_\omega |\psi_m(x,t)|^2|\psi_m'(x,t)|^2 \, dx + 2b_0 \int_\omega [\text{Re}(\psi_m(x,t)\overline{\psi_m'(x,t)})]^2 \, dx
\]
\[
\leq \int_\Omega b(x)|\psi_m(x,t)|^2|\psi_m'(x,t)|^2 \, dx + 2 \int_\Omega b(x)[\text{Re}(\psi_m(x,t)\overline{\psi_m'(x,t)})]^2 \, dx.
\]
(28)
It follows from (25), (26) and (27) that
\[
\text{Re} \, [(\psi_m''(t), \psi_m'(t)) + i(\nabla\psi_m'(t), \nabla\psi_m'(t))] + \alpha b_0 \int_\omega |\psi_m(x,t)|^2|\psi_m'(x,t)|^2 \, dx
\]
\[
+ 2\alpha b_0 \int_\omega [\text{Re}(\psi_m(x,t)\overline{\psi_m'(x,t)})]^2 \, dx + \alpha([b(x)\psi_m(t)]', \psi_m'(t))
\]
\[
\leq \text{Re}[-i[\phi_m(t)\psi_m(t)\chi_{\omega}]', \psi_m'(t))]
\]
that is,
\[
\frac{1}{2} \frac{d}{dt} \|\psi_m'(t)\|_2^2 + \alpha b_0 \int_\omega |\psi_m(x,t)|^2|\psi_m'(x,t)|^2 \, dx
\]
\[
+ 2\alpha b_0 \int_\omega [\text{Re}(\psi_m(x,t)\overline{\psi_m'(x,t)})]^2 \, dx + \alpha b_0 \|\psi_m'(t)\|_{L^2(\omega)}^2
\]
\[
\leq \int_\omega |\phi_m'(x,t)||\psi_m(x,t)||\psi_m'(x,t)| \, dx.
\]
(29)
Thus, using the inequalities of Hölder and Young to estimate the term on the right hand side of (29), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\psi_m'(t)\|^2 + \left(\alpha b_0 - \frac{1}{2a_0} \right) \int_\Omega |\psi_m(x, t)|^2 |\psi_m'(x, t)|^2 \, dx \\
+ 2\alpha b_0 \int_\Omega |\text{Re}(\psi_m(x, t)\psi_m'(x, t))|^2 \, dx + \alpha b_0 \|\psi_m'(t)\|_{L^2(\Omega)}^2 \\
\leq \frac{1}{2} \int_\Omega a(x) |\phi_m'(x, t)|^2 \, dx.
\]

(30)

Taking the derivative in \( t \) of the second equation in (11), considering \( v = \phi_m'' \) and taking the real parte, we have

\[
\frac{1}{2} \frac{d}{dt} \|\phi_m''(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \phi_m'(t)\|^2 + \int_\Omega a(x) |\phi_m''(x, t)|^2 \, dx \\
\leq 2 \int_\Omega |\psi_m(x, t)||\psi_m'(x, t)||\phi_m''(x, t)| \, dx.
\]

(31)

So, making use of the Hölder’s generalized inequality combined with the inequality of Young to estimate the term on the right hand side of (31), we get

\[
\frac{1}{2} \frac{d}{dt} \|\phi_m''(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \phi_m'(t)\|^2 + \frac{1}{2} \int_\Omega a(x) |\phi_m''(x, t)|^2 \, dx \\
\leq \frac{2}{a_0} \int_\Omega |\psi_m(x, t)||\psi_m'(x, t)|^2 \, dx.
\]

(32)

Adding (30), (32), and (19) we have

\[
\frac{1}{2} \frac{d}{dt} [\|\psi_m'(t)\|^2 + \|\phi_m'(t)\|^2 + \|\nabla \phi_m(t)\|^2 + \|\phi_m''(t)\|^2 + \|\nabla \phi_m'(t)\|^2] \\
\left(\alpha b_0 - \frac{1}{2a_0} - \frac{2}{a_0} \right) \int_\Omega \psi_m(x, t)^2 \psi_m'(x, t)^2 \, dx + \frac{1}{2} \int_\Omega a(x) \phi_m''(x, t)^2 \, dx \\
\alpha b_0 \|\psi_m'(t)\|_{L^2(\Omega)}^2 + 2\alpha b_0 \int_\Omega |\text{Re}(\psi_m(x, t)\psi_m'(x, t))|^2 \, dx \leq \frac{1}{2a_0} \|\psi_m(t)\|_{L^4(\Omega)}^4.
\]

(33)

Thus, considering that \( \alpha \geq \frac{5}{2a_0b_0} \), we have

\[
\left(\alpha b_0 - \frac{1}{2a_0} - \frac{2}{a_0} \right) \int_\Omega \psi_m(x, t)^2 \psi_m'(x, t)^2 \, dx \geq 0,
\]

and we also have to

\[
2\alpha b_0 \int_\Omega |\text{Re}(\psi_m(x, t)\psi_m'(x, t))|^2 \, dx \geq 0, \quad \frac{1}{2} \int_\Omega a(x) \phi_m''(x, t)^2 \, dx \geq 0.
\]

So,

\[
\frac{1}{2} \frac{d}{dt} [\|\psi_m'(t)\|^2 + \|\phi_m'(t)\|^2 + \|\nabla \phi_m(t)\|^2 + \|\phi_m''(t)\|^2 + \|\nabla \phi_m'(t)\|^2] \\
\alpha b_0 \|\psi_m'(t)\|_{L^2(\Omega)}^2 \leq \frac{1}{2a_0} \|\psi_m(t)\|_{L^4(\Omega)}^4.
\]

(34)
Multiplying (34) by 2, integrating over \((0, t), t \in [0, T]\), we infer
\[
\|\psi_m'(t)\|_2^2 + \|\phi_m'(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 + \|\phi_m''(t)\|_2^2 + \|\nabla \phi_m'(t)\|_2^2 + \|\nabla \phi_m''(t)\|_2^2 \\
+ 2\alpha_0 \int_0^t \|\psi_m'(s)\|_{L^2(\omega)}^2 \, ds \\
\leq \|\psi_m'(0)\|_2^2 + \|\phi_m'(0)\|_2^2 + \|\nabla \phi_m(0)\|_2^2 + \|\phi_m''(0)\|_2^2 + \|\nabla \phi_m'(0)\|_2^2 + \|\nabla \phi_m''(0)\|_2^2 \\
\frac{1}{a_0} \int_0^t \|\psi_m(s)\|_{L^4(\omega)}^4 \, ds.
\]

Making \(u = \psi_m'(0)\) in the first equation of (11) and \(v = \phi_m''(0)\) in the second equation and observing the convergences in (11), it is verified that there is \(C > 0\) such that \(\|\psi_m'(0)\|^2 + \|\phi_m''(0)\|^2 < C\). Then, from (16) we obtain
\[
\|\psi_m'(t)\|_2^2 + \|\phi_m'(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 \leq C,
\]
where \(C := C(\|\psi_0\|_2, \|\nabla \psi_0\|_2, \|\Delta \phi_0\|_2, \|\nabla \phi_0\|_2)\). Thus,
\[
(\psi_m') \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)),
(\phi_m') \text{ is bounded in } L^\infty(0, \infty; H_0^1(\Omega)),
(\phi_m'') \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)).
\]

**The Third Estimate:** In (11) considering \(u = \overline{\psi_m}\) and taking the imaginary part, we obtain
\[
\|\nabla \psi_m(t)\|_2^2 = -\text{Im}(\psi_m'(t), \psi_m(t)) - \int_\omega \phi_m(x, t)|\psi_m(x, t)|^2 \, dx.
\]

Noting that \(H_0^1(\Omega) \hookrightarrow L^2(\Omega)\) and making use of the Hölder’s inequality and taking inequality (5) into consideration, we deduce
\[
|\text{Im}(\psi_m'(t), \psi_m(t))| \leq C \|\psi_m'(t)\|_2 \|\nabla \psi_m(t)\|_2,
\]
\[
- \int_\omega \phi_m(x, t)|\psi_m(x, t)|^2 \, dx \leq C \|\nabla \phi_m(t)\|_2 \|\psi_m(t)\|_2 \|\nabla \psi_m(t)\|_2.
\]

So, from (24), (36), (41) and (42) we conclude
\[
\|\nabla \psi_m(t)\|_2 \leq C, \text{ for all } t > 0,
\]
where \(C = C(\|\nabla \psi_0\|_2, \|\Delta \phi_0\|_2, \|\nabla \phi_0\|_2)\).

Thus,
\[
\psi_m \text{ is bounded in } L^\infty(0, \infty; H_0^1(\Omega)).
\]

Taking \(v = \Delta \phi_m\) in the second equation of (11), we have
\[
\|\Delta \phi_m(t)\|_2^2 \leq \int_\omega |\psi_m(x, t)|^2 |\Delta \phi(x, t)| \, dx + \|\phi_m''(t)\|_2 \|\nabla \phi_m(t)\|_2 \\
+ \|a\|_\infty \|\phi_m'(t)\|_2 \|\Delta \phi_m(t)\|_2.
\]

We estimate the first term on the right hand side of (44) using the Hölder inequality and (5). Then,
\[
\|\Delta \phi_m(t)\|_2 \leq C, \text{ for all } t > 0,
\]
where \(C = C(\|\nabla \psi_0\|_2, \|\Delta \phi_0\|_2, \|\nabla \phi_0\|_2)\).

Therefore,
\[
(\phi_m) \text{ is bounded in } L^\infty(0, \infty; H_0^1(\Omega) \cap H_0^2(\Omega)).
\]
Taking \( u = \Delta \psi_m \) in the first equation of (11), we have
\[
\| \Delta \psi_m(t) \|_2^2 \leq \| \psi_m(t) \|_2 \| \Delta \psi_m(t) \|_2 + \alpha \| b \|_\infty \| \nabla \psi_m(t) \|_2 \| \Delta \psi_m(t) \|_2 \tag{47}
+ \alpha \| b \|_\infty \| \psi_m(t) \|_2 \| \Delta \psi_m(t) \|_2 + \| \nabla \psi_m(t) \|_2 \| \Delta \psi_m(t) \|_2.
\]

Then,
\[
\| \Delta \psi_m(t) \|_2 \leq C, \text{ for all } t > 0,
\tag{48}
\]
where \( C = C(\| \psi_0 \|_2, \| \phi_1 \|_2, \| \nabla \psi_0 \|_2, \| \Delta \phi_0 \|_2, \| \nabla \phi_0 \|_2) \).

Therefore,
\[
(\psi_m) \text{ is bounded in } L^\infty(0, \infty; H^1_0(\Omega) \cap H^2_0(\Omega)).
\tag{49}
\]

The rest of the proof follows the same basic steps as the one of [11], Theorem 2.1.

**Uniqueness:** Let \( \{ \psi_1, \phi_1 \} \) and \( \{ \psi_2, \phi_2 \} \) solutions do problem (1). Then, the uniqueness follow defining \( z = \psi_1 - \psi_2 \) and \( w = \phi_1 - \phi_2 \) and repeating verbatim the same arguments already used in the first estimate.

4. **Uniform decay rates.** In this section we work it regular solutions \( \{ \psi(t), \phi(t), \phi_\alpha(t) \} \) to problem (1), that is, those ones that lie in \( H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \) and taking \( \{ \psi_0, \phi_0, \phi_\alpha \} \in H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) := \mathcal{H} \), such that \( \| \{ \psi_0, \phi_0, \phi_\alpha \} \|_\mathcal{H} \leq L \), where \( L > 0 \). So, from first equation of (1), we have
\[
\psi' - i \Delta \psi + \alpha b(x)(|\psi|^2 + 1) \psi = -i \phi \chi_\omega.
\tag{50}
\]

Multiplying (50) by \( \overline{\psi} \) and integrating over \( \Omega \), we have
\[
\int_\Omega \psi' \overline{\psi} \, dx + i \int_\Omega \nabla |\psi|^2 \, dx + \alpha \int_\Omega b(x)(|\psi|^4 + |\psi|^2) \, dx = -i \int_\Omega \phi |\psi|^2 \, dx.
\tag{51}
\]

Taking the real part in (51) we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \psi(t) \|_2^2 + \alpha \int_\Omega b(x)(|\psi|^4 + |\psi|^2) \, dx = 0.
\tag{52}
\]

Multiplying the second equation by \( \phi' \), integrating over \( \Omega \) and making use of Green formula, we deduce that
\[
\frac{1}{2} \frac{d}{dt} (\| \phi' \|_2^2 + \| \nabla \phi \|_2^2) \, dx + \int_\Omega a(x)|\phi'|^2 \, dx = \int_\Omega |\psi|^2 \phi' \, dx.
\tag{53}
\]

Adding (52) and (53) we obtain
\[
E'(t) + \alpha \int_\Omega b(x)(|\psi|^4 + |\psi|^2) \, dx + \int_\Omega a(x)|\phi'|^2 \, dx = \int_\Omega |\psi|^2 \phi' \, dx.
\tag{54}
\]

Next, we will analyze the last term on the RHS of (54). We have, from Assumption 1 and making use of the Cauchy-Schwarz inequality that
\[
\left| \int_\Omega |\psi|^2 \phi' \, dx \right| \leq \left( \int_\Omega |\psi|^4 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega |\phi'|^2 \, dx \right)^{\frac{1}{2}}
\leq b_0^{-\frac{1}{2}} \left( \int_\Omega b_0 |\psi|^4 \, dx \right)^{\frac{1}{2}} a_0^{-\frac{1}{2}} \left( \int_\Omega a_0 |\phi'|^2 \, dx \right)^{\frac{1}{2}}
\leq b_0^{-\frac{1}{2}} \left( \int_\Omega b(x) |\psi|^4 \, dx \right)^{\frac{1}{2}} a_0^{-\frac{1}{2}} \left( \int_\Omega a(x) |\phi'|^2 \, dx \right)^{\frac{1}{2}}
\leq \frac{1}{2a_0 b_0} \int_\Omega b(x) |\psi|^4 \, dx + \frac{1}{2} \int_\Omega a(x) |\phi'|^2 \, dx.
\tag{55}
\]
Combining (54), (55) and since \( \alpha \geq \frac{5}{2a_0 b_0} \) follow that \( \beta := \alpha - \frac{1}{2a_0 b_0} > 0 \) and it holds that
\[
E'(t) \leq -\beta \int_{\Omega} b(x)|\psi|^4 \, dx - \alpha \int_{\Omega} b(x)|\psi|^2 \, dx - \frac{1}{2} \int_{\Omega} a(x)|\phi'|^2 \, dx
\]
\[
\leq -k \left[ \int_{\Omega} b(x)(|\psi|^4 + |\psi|^2) \, dx + \int_{\Omega} a(x)|\phi'|^2 \, dx \right] dt,
\]
where \( k = \min\{\frac{1}{\beta}, \beta\} \).

**Remark 2.** From (56) we deduce two facts: (i) the map \( t \in (0, \infty) \mapsto E(t) \) is non-increasing, and, in addition, (ii) we have the following inequality of the energy
\[
E(t_2) - E(t_1) \leq -k \int_{t_1}^{t_2} \left[ \int_{\Omega} a(x)|\phi'|^2 \, dx + \int_{\Omega} b(x)(|\psi|^4 + |\psi|^2) \, dx \right] dt,
\]
for \( 0 \leq t_1 \leq t_2 < +\infty \), which will be crucial in the proof.

Let \( T_0 > 0 \) considered sufficiently large for our purpose. We will prove the following lemma:

**Lemma 4.1.** For all \( T > T_0 \) there exists a positive constant \( C = C(T) \) such that if \((\psi, \phi)\) is the regular solution of (1) with initial data \( \{\psi_0, \phi_0, \phi_1\} \) taken in limited of \( \mathcal{H} \), we have
\[
E(0) \leq C \int_0^T \left[ \int_{\Omega} b(x)(|\psi|^4 + |\psi|^2) \, dx + \int_{\Omega} a(x)|\phi'|^2 \, dx \right] dt.
\]

**Proof.** We argue by contradiction. Let us suppose that (57) is not verified and let \( \{\psi_k(0), \phi_k(0), \phi_k'(0)\} \) be a sequence of initial data where the corresponding solutions \( \{\psi_k, \phi_k, \phi_k'\} \) with \( E_k(0) \) uniformly bounded in \( k \), verifies
\[
\lim_{k \to +\infty} \frac{E_k(0)}{\int_0^T \int_{\Omega} b(x)(|\psi_k|^4 + |\psi_k|^2) \, dx \, dt + \int_0^T \int_{\Omega} a(x)|\phi_k'|^2 \, dx \, dt} = +\infty,
\]
or,
\[
\lim_{k \to +\infty} \frac{\int_0^T \int_{\Omega} b(x)(|\psi_k|^4 + |\psi_k|^2) \, dx \, dt + \int_0^T \int_{\Omega} a(x)|\phi_k'|^2 \, dx \, dt}{E_k(0)} = 0.
\]

Since \( E_k(t) \) is non-increasing and \( E_k(0) \) remains bounded then, we obtain a subsequence, still denoted by \( \{\psi_k, \phi_k\} \) which verifies
\[
\psi_k \rightharpoonup \psi \text{ weak star in } L^\infty(0, T; L^2(\Omega)),
\]
\[
\phi_k \rightharpoonup \phi \text{ weak star in } L^\infty(0, T; H^1(\Omega)),
\]
\[
\phi_k' \rightharpoonup \phi' \text{ weak star in } L^\infty(0, T; L^2(\Omega)).
\]

We also have, employing compactness results (see Theorem 5.1 in Lions [34]) that
\[
\phi_k \to \phi \text{ strongly in } L^2(0, T; L^2(\Omega)).
\]

Now, from (58), (59) and (60) we deduce that
\[
\lim_{k \to +\infty} \int_0^T \int_{\Omega} a(x)|\phi_k'|^2 \, dx \, dt = 0,
\]
\[
\lim_{k \to +\infty} \int_0^T \int_{\Omega} b(x)(|\psi_k|^4 + |\psi_k|^2) \, dx \, dt = 0.
\]
Consider the sequence of problems
\[
\begin{aligned}
&i\psi_k' + \Delta \psi_k + ib(x)(|\psi_k|^2\psi_k + \psi_k) = \phi_k\psi_k \chi_\omega, \\
&\phi_k'' - \Delta \phi_k + a(x)\phi_k = |\psi_k|^2\chi_\omega, \\
&\psi_k = 0 \text{ on } \Gamma \times (0, T), \\
&\phi_k = 0 \text{ on } \Gamma \times (0, T), \\
&\phi_k' \to 0 \text{ in } L^2(0, T; L^2(\omega)).
\end{aligned}
\]

Defining
\[
c_k := [E_k(0)]^{1/2}, \\
\hat{\phi}_k = \frac{1}{c_k}\phi_k, \quad \hat{\psi}_k = \frac{1}{c_k}\psi_k,
\]
we infer
\[
\hat{E}_k(t) = \frac{1}{2} \left[ \int_\Omega |\hat{\psi}_k|^2 dx + \int_\Omega |\hat{\phi}_k'|^2 dx + \int_\Omega |\nabla \hat{\phi}_k|^2 dx \right] = \frac{E_k(t)}{c_k^2}.
\]

In particular,
\[
\hat{E}_k(0) = 1.
\]

Now, consider the sequence of problems
\[
\begin{aligned}
i\hat{\psi}_k' + \Delta \hat{\psi}_k + ib(x)(|\psi_k|^2\hat{\psi}_k + \hat{\psi}_k) = \hat{\phi}_k\psi_k \chi_\omega, \\
\hat{\phi}_k'' - \Delta \hat{\phi}_k + a(x)\hat{\phi}_k = |\psi_k|^2\chi_\omega, \\
\hat{\psi}_k = 0 \text{ on } \Gamma \times (0, T), \\
\hat{\phi}_k = 0 \text{ on } \Gamma \times (0, T), \\
\hat{\phi}_k' \to 0 \text{ in } L^2(0, T; L^2(\omega)),
\end{aligned}
\]
and, from (68), for a subsequence \{\hat{\psi}_k, \hat{\phi}_k\}, we obtain
\[
\hat{\psi}_k \rightharpoonup \hat{\psi} \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \\
\hat{\phi}_k \rightharpoonup \hat{\phi} \text{ weak star in } L^\infty(0, T; H^1_0(\Omega)), \\
\hat{\phi}_k' \rightharpoonup \hat{\phi}' \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \\
\hat{\phi}_k \to \hat{\phi} \text{ strongly in } L^2(0, T; L^2(\Omega)).
\]

Consequently, from (63) and (64) and since \(E_k(0) \leq L\),
\[
\lim_{k \to +\infty} \int_0^T \int_\Omega a(x)|\hat{\phi}_k'|^2 dx dt = 0,
\]
\[
\lim_{k \to +\infty} \int_0^T \int_\Omega b(x) \left( \frac{|\psi_k|^4}{c_k^2} + |\psi_k|^2 \right) dx dt = 0.
\]

Considering the immersion \(H^2(\Omega) \hookrightarrow L^\infty(\Omega)\) for \(n = 2\) and (64) yields
\[
\lim_{k \to \infty} \int_0^T \int_\Omega |ib(x)|\psi_k|^2 \hat{\psi}_k|^2 dx dt = 0,
\]
and from (64) we also have
\[
\lim_{k \to \infty} \int_0^T \int_\omega |\hat{\phi}_k\psi_k|^2 dx dt = 0.
\]
Passing to the limit when \( k \to \infty \) in (69) and taking the convergences above and (63), (64), (76) and (77) into account, we get
\[
\begin{cases}
\hat{\phi}'' - \Delta \hat{\phi} = 0 \text{ in } \Omega \times (0, T), \\
\hat{\phi} = 0 \text{ on } \Gamma \times (0, T), \\
\hat{\phi}' = 0 \text{ a.e. in } \omega \times (0, T).
\end{cases}
\] (78)

and
\[
\begin{cases}
i\hat{\psi}'' + \Delta \hat{\psi} = 0 \text{ in } \Omega \times (0, T), \\
\hat{\psi} = 0 \text{ on } \Gamma \times (0, T).
\end{cases}
\] (79)

Thus for \( \hat{\phi}' = v \) in (78), we obtain, in the distributional sense that
\[
\begin{cases}
v'' - \Delta v = 0 \text{ in } D'(\Omega \times (0, T)), \\
v = 0 \text{ on } \Gamma \times (0, T) \text{ (in } H^{-1}(0, T; H^{1/2}(\Gamma))), \\
v = 0 \text{ a.e. in } \omega \times (0, T).
\end{cases}
\] (80)

From Holmgren’s uniqueness theorem for the wave equation we conclude that \( v \equiv 0 \), that is, \( \phi' \equiv 0 \). Returning to (78) we obtain the following elliptic equation for a.e. \( t \in (0, T) \):
\[
\begin{cases}
-\Delta \hat{\phi} = 0 \text{ in } \Omega, \\
\hat{\phi} = 0 \text{ on } \Gamma.
\end{cases}
\] (81)

thus, \( \hat{\phi} = 0 \). Analogously, from Holmgren’s uniqueness theorem we conclude that \( \hat{\psi} = 0 \) a.e in \( \Omega \). Therefore we have \( \phi = \hat{\psi} = 0 \) and thus the convergences given in (70) - (73) are valid with \( \hat{\psi} = \phi = \hat{\phi}' = 0 \).

In order to achieve a contradiction it is enought to prove that \( \hat{E}_k(0) \to 0 \) where \( k \to \infty \). Indeed, one has
\[
\hat{E}_k(0) = \frac{1}{2} \int_{\Omega} \left[ |\hat{\psi}_k(x, 0)|^2 + |\hat{\phi}_k(x, 0)|^2 + |\nabla \hat{\phi}_k(x, 0)|^2 \right] \, dx \, dt.
\] (82)

We deduce
\[
\hat{E}_k(0) = E_{\hat{\psi}_k}(0) + E_{\hat{\phi}_k}(0).
\] (83)

We shall prove that \( \hat{E}_k(0) \to 0 \) when \( k \to \infty \) proving, separately, that \( E_{\hat{\psi}_k}(0) \to 0 \) and \( E_{\hat{\phi}_k}(0) \to 0 \) and using the observability results for the linear Schrödinger and wave equations.

Consider the following problems
\[
\begin{cases}
i\hat{\psi}_k' + \Delta \hat{\psi}_k + ib(x)(|\psi_k|^2 \hat{\psi}_k + \hat{\psi}_k) = \hat{\phi}_k \psi_k \chi_\omega \text{ in } \Omega \times (0, T), \\
\hat{\psi}_k = 0 \text{ on } \Gamma \times (0, T), \\
\hat{\psi}_k(0) = \psi_{0k} \text{ in } \Omega.
\end{cases}
\] (84)

and
\[
\begin{cases}
i\hat{\omega}_k' + \Delta \hat{\omega}_k = -ib(x)(|\psi_k|^2 \hat{\psi}_k + \hat{\psi}_k) + \hat{\phi}_k \psi_k \chi_\omega \text{ in } \Omega \times (0, T), \\
\hat{\omega}_k = 0 \text{ on } \Gamma \times (0, T), \\
\hat{\omega}_k(0) = 0 \text{ in } \Omega.
\end{cases}
\] (85)

Note that \( \hat{\psi}_k = v_k + \omega_k \) is the solution of (84), where \( v_k \) comes from (85) and \( \omega_k \) is the solution of (86). In addition, \( \hat{\psi}_k = v_k + \omega_k = 0 \in \Gamma \times (0, \infty) \) and
\[ \hat{\psi}_k(0) = v_k(0) + \omega_k(0) = \psi_{0k}. \] Thus, it follows that
\[ E_{\psi_k}(0) = E_{v_k}(0) \leq c_1 \int_0^T \int_\omega |v_k|^2 \, dx \, dt \tag{87} \]
\[ \leq c_1 \int_0^T \int_\omega |\hat{\psi}_k|^2 + |\omega_k|^2 \, dx \, dt \]
\[ \leq \hat{c}_1 \left( \int_0^T \int_\Omega b(x)|\hat{\psi}_k|^2 \, dx \, dt + \int_0^T \int_\omega |\omega_k|^2 \, dx \, dt \right). \]

Let us define the following linear and continuous form
\[ T : L^2(\Omega) \times L^1(0, T, L^2(\Omega)) \to L^\infty(0, T, L^2(\Omega)) \]
\[ (z_0, f) \mapsto T(z_0, f) = z \]
where \( z \) is the solution of the problem
\[ \begin{cases} iz' + \Delta z = f & \text{in } \Omega \times (0, T), \\ z = 0 & \text{on } \Gamma \times (0, T), \\ z(0) = z_0 & \in L^2(\Omega). \end{cases} \tag{88} \]

Clearly \( T \) is linear. We will prove that \( T \) is continuous. In fact, \( z \) is the solution of (88), so \( z \) satisfies the integral equation
\[ z(t) = S(t)z_0 + \int_0^t S(t-s)f(s) \, ds \]
where \( S(t) \) is the semigroup generated by the monotonous maximal operator \( A \), and
\[ A : D(A) = H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega) \]
\[ y \mapsto Ay = -i\Delta y. \]

Thus, taking into account that \( \|S(t)\|_{L(L^2(\Omega))} \leq M \), we arrive at the following estimate
\[ \|z(t)\|_2^2 \leq c_3 \|z_0\|_2^2 + c_4 \left( \int_0^t \|f(s)\|_2 \, ds \right)^2 \]
\[ \leq C(\|z_0\|_2^2 + \|f\|_{L^2(0, T, L^2(\Omega))}^2) = C(\|z_0\|_{L^2(\Omega)}^2 \times \|f\|_{L^1(0, T, L^2(\Omega))}^2) \]
which shows that \( T \) is continuous.

In this way, since \( L^\infty(0, T, L^2(\Omega)) \to L^2(0, T, L^2(\Omega)) \) and denoting \( f(t) = -ib(x)(|\psi_k(t)|^2 \psi_k(t) + \psi_k(t)) + \hat{\phi}_k(t)\psi_k(t)\chi_\omega, z_0 = 0 \), follows from the Holder inequality and the continuity of \( T \)
\[ I := \int_0^T \int_\omega |\omega_k|^2 \, dx \, dt \]
\[ \leq \|\omega_k\|_{L^\infty(0, T, L^2(\Omega))}^2 \]
\[ \leq C\|f\|_{L^1(0, T, L^2(\Omega))}^2 \]
\[ \leq C(T) \left( \int_0^T \int_\Omega (\hat{\phi}_k \psi \psi_k \chi_\omega - ib(x)(|\psi_k|^2 \hat{\psi}_k + \hat{\psi}_k)^2) \, dx \, dt \right). \tag{89} \]

Therefore, from (87) and (89) we have
\[ E_{\psi_k}(0) \leq C(T) \left( \int_0^T \int_\Omega b(x)|\hat{\psi}_k|^2 \, dx \, dt \right) \]
When \( n \to \infty \). Analogously, consider the following problems

\[
\begin{aligned}
\phi_k'' - \Delta \phi_k + a(x)\phi_k' &= \frac{|\psi_k|^2}{c_k} \chi_\omega \quad \text{in } \Omega \times (0, T), \\
\phi_k &= 0 \quad \text{on } \Gamma \times (0, T), \\
\phi_k(0) &= \phi_{0k} \quad \text{in } \Omega, \\
\phi_k(0) &= \phi_{1k} \quad \text{in } \Omega;
\end{aligned}
\]

(91)

and

\[
\begin{aligned}
\omega_k'' + \Delta \omega_k &= -a(x)\phi_k' + \frac{|\psi_k|^2}{c_k} \chi_\omega \quad \text{in } \Omega \times (0, T) \\
\omega_k &= 0 \quad \text{on } \Gamma \times (0, T), \\
\omega_k(0) &= 0 \quad \text{in } \Omega, \\
\omega_k'(0) &= 0 \quad \text{in } \Omega.
\end{aligned}
\]

(93)

Note that \( \phi_k = v_k + \omega_k \) is the solution of (91), where \( v_k \) comes from (92) and \( \omega_k \) is the solution of (93). In addition, \( \phi_k = v_k + \omega_k = 0 \) in \( \Gamma \times (0, \infty) \) and \( \phi_k'(0) = v_k'(0) + \omega_k'(0) = \phi_{1k} \). Thus, it follows that

\[
E_{\phi_k}(0) = E_{v_k}(0) \leq c_1 \int_0^T \int_\omega |v_k'|^2 \, dx \, dt
\]

\[
\leq c_1 \int_0^T \int_\omega |\hat{\phi}_k'|^2 + |\omega_k'|^2 \, dx \, dt
\]

\[
\leq \hat{c}_1 \left( \int_0^T \int_\Omega a(x)|\hat{\phi}_k'|^2 \, dx \, dt + \int_0^T \int_\omega |\omega_k'|^2 \, dx \, dt \right).
\]

Let us define the following linear and continuous form

\[
T : \mathcal{R} \times L^1(0, T, L^2(\Omega)) \to L^\infty(0, T, L^2(\Omega))
\]

\[
(z_0, z_1, f) \mapsto T(z_0, z_1, f) = z'
\]

where \( \mathcal{R} = H^1_0(\Omega) \times L^2(\Omega) \) and \( z \) is the solution of the problem

\[
\begin{aligned}
z'' - \Delta z &= f \quad \text{in } \Omega \times (0, T) \\
z &= 0 \quad \text{on } \Gamma \times (0, T) \\
z(0) &= z_0 \in H^1_0(\Omega) \\
z'(0) &= z_1 \in L^2(\Omega)
\end{aligned}
\]

(94)
Clearly $T$ is linear. We will prove that $T$ is continuous. In fact, since $Z = \begin{pmatrix} z \\ z' \end{pmatrix}$ is the solution of the problem

$$
\begin{cases}
Z' + AZ = F \\
Z(0) = Z_0
\end{cases}
$$

(95)

where $A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 \\ f \end{pmatrix}$, we have $Z$ satisfies the integral equation

$$Z(t) = S(t)Z_0 + \int_0^t S(t-s)F(s) \, ds$$

where $S(t)$ is the semigroup generated by the monotonous maximal operator $A$. Thus, taking into account that $\|S(t)\|_{L(L^2(\Omega))} \leq M$, we arrive at the following estimate

$$\|z'(t)\|_2^2 \leq \|Z(t)\|_R^2 \leq c_1 \|(z_0, z_1)\|_R^2 + c_2 \left( \int_0^t \left\| \begin{pmatrix} 0 \\ f \end{pmatrix} \right\|_R \, ds \right)^2 \leq c_3 \|(z_0, z_1)\|_R^2 + c_3 \|f\|_{L^1(0, T; L^2(\Omega))}^2 \leq c_4 \|(z_0, z_1, f)\|_{R \times L^1(0, T; L^2(\Omega))}^2.$$ 

Therefore, $\|z'(t)\|_\infty^2 \leq c_4 \|(z_0, z_1, f)\|_{R \times L^1(0, T; L^2(\Omega))}^2$, so that $c_4 = \max\{c_1, c_3\}$ which shows that $T$ is continuous.

In this way, since $L^\infty(0, T, L^2(\Omega)) \hookrightarrow L^2(0, T, L^2(\Omega))$ and denoting $f(t) = -a(x)\hat{\phi}_k'(t) + \hat{\psi}_k^2(t)\chi_\omega$ and $z_0 = z_1 = 0$, follows from the Hölder inequality and the continuity of $T$

$$I := \int_0^T \int_\omega |\omega_k'|^2 \, dx \, dt \leq \|\omega_k\|_{L^\infty(0, T, L^2(\Omega))} \leq C\|f\|_{L^1(0, T; L^2(\Omega))}^2 \leq C(T) \left( \int_0^T \int_\Omega \left| \frac{\hat{\psi}_k}{c_k} \chi_\omega - a(x)\hat{\phi}_k' \right|^2 \, dx \, dt \right).$$

(96)

Therefore, from (94) and (96) we have

$$E_{\hat{\phi}_k}(0) \leq C(T) \left( \int_0^T \int_\Omega \left| \frac{\hat{\psi}_k}{c_k} \chi_\omega - a(x)\hat{\phi}_k' \right|^2 \, dx \, dt \right).$$

Thus, from (74) and (75) we obtain

$$E_{\hat{\phi}_k}(0) \rightarrow 0,$$

(97)

when $n \rightarrow \infty$.

Therefore,

$$E_k(0) = E_{\hat{\psi}_k}(0) + E_{\hat{\phi}_k}(0) \rightarrow 0$$

when $n \rightarrow +\infty$ which is absurd in view of (68).
Take $T_0$ large enough. From the inequality of energy we deduce

$$E(T_0) - E(0) \leq -k \int_0^{T_0} \left( \int_{\Omega} a(x)|\phi'|^2 \, dx + \int_{\Omega} b(x)(|\psi|^4 + |\psi|^2) \, dx \right) \, dt,$$

(98)

and from Lemma (4.1) we have

$$E(0) \leq C \int_0^{T_0} D(t) \, dt.$$  

(99)

of (98) follows that

$$C \int_0^{T_0} D(t) \, dt \leq -\frac{C}{k} E(T_0) + \frac{C}{k} E(0).$$

(100)

Combining (99) and (100) it holds that

$$E(T_0) \leq E(0) \leq C \int_0^{T_0} D(t) \, dt \leq -\frac{C}{k} E(T_0) + \frac{C}{k} E(0).$$

So,

$$\left(1 + \frac{C}{k}\right) E(T_0) \leq \frac{C}{k} E(0).$$

Thus,

$$E(T_0) \leq \frac{C}{k + C} E(0),$$

that is,

$$E(T_0) \leq \alpha E(0),$$

where $0 < \alpha < 1$.

So, taking $T_0$ large enough for $T > T_0$ we obtain

$$E(T) \leq E(T_0) \leq \alpha E(0).$$

Hence,

$$E(T) \leq \alpha E(0), \quad \text{for all } T > T_0,$$

where, $\alpha < 1$.

Proceeding in a similar way we have done previously from $T$ to $2T$ and we deduce, as before, $E(2T) \leq \alpha E(T)$, for all $T > T_0$, and, consequently,

$$E(2T) \leq \alpha^2 E(0), \quad \text{for all } T > T_0.$$

In general,

$$E(nT) \leq \alpha^n E(0), \quad \text{for all } T > T_0.$$

Let us consider, now, $t > T_0$, then $t = nT_0 + r$ for $0 \leq r < T_0$. Thus,

$$E(t) \leq E(t - r) = E(nT_0) \leq \alpha^n E(0) = \frac{\ln \alpha}{\ln \alpha} E(0) = e^\frac{\ln \alpha}{\ln \alpha} E(0),$$

which implies the exponential stability. The arguments used above were inspired by [12], [30], [31] among others.
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