STATISTICAL MECHANICS OF THE COSMOLOGICAL MANY-BODY PROBLEM

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ABSTRACT

We derive analytical expressions for the grand canonical partition functions of point masses, and of extended masses (e.g., galaxies with halos), which cluster gravitationally in an expanding universe. From the partition functions, we obtain the system’s thermodynamic properties, distribution functions (including voids and counts in cells), and moments of distributions such as their skewness and kurtosis. This also provides an analytical calculation of the evolution of the distribution. Our results apply to both linear and nonlinear regimes of clustering. In the limit of point masses, these results reduce exactly to previous results derived from thermodynamics, thus providing a new, more fundamental foundation for the earlier results.

Subject headings: cosmology: theory — galaxies: clusters: general — gravitation — large-scale structure of universe — methods: analytical

1. INTRODUCTION

Galaxies cluster on very large scales under the influence of their mutual gravitation, and the characterization of this clustering is a problem of current interest. Different techniques such as percolation (Zeldovich, Eistenstok, & Shandarin 1982; Grimmet 1989), minimal spanning trees (Pearson & Coles 1995; Bhavsar & Splinter 1996), fractals (Mandelbrot 1982), correlation functions (Totsuji & Kihara 1969; Peebles 1980), and distribution functions (Saslaw & Hamilton 1984) have been introduced to understand the large-scale structure in the universe. Of all the description so far, correlations and distribution functions have been related most directly to physical theories of gravitational clustering. The consequences of these theories also agree well with observations.

Thus far, theories of the cosmological many-body galaxy distribution function have been developed mainly from a thermodynamic point of view. This starts from the first two laws of thermodynamics and, for quasi-equilibrium evolution, derives gravitational many-body equations of state in the context of the expanding universe. Application of the thermodynamic fluctuation theory to these equations of state, considering the galaxies as point gravitating masses, then gives their distribution function. Comparisons of gravitational thermodynamics to the cosmological many-body problem have been discussed on the basis of N-body computer simulation results (e.g., Saslaw et al. 1990; Itoh, Inagaki, & Saslaw 1988, 1993). Comparisons with the observed galaxy clustering (e.g., Sheth, Mo, & Saslaw 1994; Fang & Zou 1994; Raychaudhury & Saslaw 1996) along with other theoretical arguments (e.g., Zhan 1989; Zhan & Dyer 1989) support it further.

The applicability of thermodynamics to the cosmological many-body problem suggests that statistical mechanics should also apply. This close relation occurs because statistical mechanics is the microscopic (and therefore perhaps more fundamental) statistical description of particle (e.g., galaxy) positions and motions whose ensemble averages provide the macroscopic thermodynamic description of the system. Hitherto it has not been possible to develop a statistical mechanical theory of N-body galaxy clustering because the relevant partition function for the gravitational grand canonical ensemble could not be solved.

The general conditions under which statistical mechanics may describe the cosmological many-body problem are closely related to those for the applicability of thermodynamics, described in detail previously (Saslaw & Hamilton 1984; Saslaw & Fang 1996; Saslaw 2000), and we briefly discuss them in the present context. When the ensemble averaged thermodynamic quantities change more slowly than local dynamical crossing or clustering timescales, then the form of the statistical distribution functions remains essentially the same, and only their macroscopic variables evolve. In this quasi-equilibrium evolution, equilibrium statistical mechanics provides a good approximation to the distribution of particles and velocities at any given time with the values of the macroscopic variables at that time. In equilibrium, all permissible microstates of the systems in the ensemble have an equal a priori probability. This is the fundamental postulate of statistical mechanics. It implies that the approximate probability of finding a specified macrostate in the system is proportional to the number of permissible microstates having the macrostate’s properties.

Cosmological many-body systems generally satisfy the timescale criterion of quasi-equilibrium statistical mechan-
ics since macroscopic global variables such as average temperature, density, and the ratio of gravitational correlation energy to thermal energy change on timescales at least as long as the Hubble time, whereas local dynamical timescales in regions of clustering are shorter. The criterion of equal a priori probabilities for any microstate or configuration is less well understood, and its rigorous derivation remains an important unsolved problem even in classical statistical mechanics. It is closely related to statistical homogeneity and the absence of extensive very nonlinear structures over scales comparable with the system. $N$-body simulations in the previous references show that relaxation to the observed distributions (e.g., eq. [42] below) of quasi-equilibrium statistical mechanics occurs for initial power-law perturbation spectra with power-law indices between about $-1$ and $+1$. Systems with much stronger global initial correlations or anticorrelations relax only after many expansion timescales, or not at all. A more detailed analysis of the ranges of initial conditions that form the "basin of attraction" for equation (42) is an important future exploration.

In the present paper, we investigate the problem of nonlinear gravitational galaxy clustering from the point of view of statistical mechanics. The statistical mechanics of $N$-body systems is based on the $N$-body Hamiltonian. From this the partition function is formed as a function of an $N$-dimensional integral that incorporates the effects of interactions among all the particles. To evaluate such an integral analytically is generally very complicated. However, we will show how the partition function for the cosmological many-body problem can be evaluated analytically. This serves as a basic result for rigorously evaluating all the thermodynamic properties of the system, starting with the free energy. It is particularly interesting that the parameter $b$, which is the ratio of the gravitational potential correlation energy to twice the kinetic energy of peculiar motion, emerges directly in the partition function and in the equations of state. So we do not need to make any assumptions in the derivation of the functional form of $b(nT^{-3})$ as was done earlier (Saslaw & Hamilton 1984; Saslaw & Fang 1996). Once the many-body partition function is known, there is no difficulty in evaluating the grand canonical partition function, which represents the exchange both of particles and energy. From the grand canonical partition function, the distribution function of galaxies follows directly.

In addition to its generality and rigor, the main advantage of this statistical mechanical approach is that it can easily be extended to non–point-mass systems. Until now, all the thermodynamic techniques developed for the study of gravitational clustering have been applicable only to point mass galaxies. Actual galaxies have extended structures and haloes, and we shall see how this affects previous results. The introduction of a softening parameter enables us to include effects of large haloes of dark matter around galaxies. In § 2, we develop the analytical solution of the configuration integral for the cosmological gravitational systems. This integral may be applied to systems containing either point or extended masses. Then we analytically calculate the partition function and the free energy for systems composed of particles assumed to be point masses. From the free energy, all the thermodynamic functions follow directly, in § 3, including the equations of state. The proper thermodynamic dependence of the parameter $b$ emerges directly in the equations of state. Then we calculate the distribution function, $f(N)$, simply and directly. All these results agree exactly with earlier ones derived using thermodynamic arguments. These earlier results are then extended naturally to systems having non–point-mass particles. In § 4, the distribution function $f(N)$ is calculated more generally, taking the extended nature of the particles into account. Section 5 gives the moments, along with the skewness and kurtosis, for the distribution functions of extended galaxies that are clustering under their mutual gravitational. In § 6, we examine the evolution of $b$ for extended galaxies. Finally, our results are discussed in § 7.

2. THE COSMOLOGICAL MANY-BODY PARTITION FUNCTION

We start with the formalism of classical statistical mechanics for an ensemble of comoving cells containing gravitating particles (galaxies) in an expanding universe. If the size of the cells is smaller than the particle correlation length, then each member of this ensemble is correlated gravitationally with other cells. However, correlations within a cell are generally greater than correlations among cells, so that extensivity is a good approximation (e.g., Saslaw 2000; Sheth & Saslaw 2000). We consider a large system, which consists of an ensemble of cells, all of the same volume $V$, or radius $R$ (much smaller than the total volume) and average density $n$. Both the number of galaxies and their total energy will vary among these cells that, taken together, are represented by a grand canonical ensemble. In the system, galaxies have a gravitational pairwise interaction. We assume that their distribution is statistically homogenous over large regions.

Our starting point is the general partition function of a system of $N$ particles of mass $m$ interacting gravitationally with a potential energy $\phi$, having momenta $p_i$ and average temperature $T$:

$$Z_N(T, V) = \frac{1}{\Lambda^{3N}N!} \int \exp \left\{ -\sum_{i=1}^{N} \frac{p_i^2}{2m} + \phi(r_1, r_2, \ldots, r_N) \right\} T^{-1} \times d^{3N}p \ d^{3N}r.$$  

Here $N!$ takes the distinguishability of classical particles into account, and $\Lambda$ normalizes the phase space volume cell. Integration over momentum space yields

$$Z_N(T, V) = \frac{1}{N!} \left( \frac{2\pi m T}{\Lambda^2} \right)^{3N/2} Q_N(T, V)$$  

as usual, where

$$Q_N(T, V) = \int \cdots \int \exp[-\phi(r_1, r_2, \ldots, r_N)T^{-1}] \ d^{3N}r$$

is the configuration integral. In general, the gravitational potential energy $\phi(r_1, r_2, \ldots, r_N)$ is a function of the relative position vector $r_i = |r_i - r_j|$ and is the sum of the potential energies of all pairs. Evidently the evaluation of $Q_N(T, V)$ is the main task and involves a very complicated $3N$-fold integration, which usually cannot be carried out for general interparticle potentials. Often, as in the theory of imperfect gases or liquids, this configuration integral is approximated by a virial expansion in powers of the density, an approximation that becomes less accurate at higher densities. However, we show here that for the cosmological
gravitational many-body problem, it is not necessary to construct the usual virial expansion, and the partition function has a relatively simple exact solution.

In this gravitational system the potential energy $\phi(r_1, r_2, \ldots, r_N)$ is due to all pairs of particles (galaxies) composing the system, hence

$$\phi(r_1, r_2, \ldots, r_N) = \sum_{1 \leq i < j \leq N} \phi(r_{ij}) = \sum_{1 \leq i < j \leq N} \phi_{ij}(r).$$  \hspace{1cm} (4)

Thus, equation (3) becomes

$$Q_N(T, V) = \int \cdots \int \prod_{1 \leq i < j \leq N} \exp[-\phi_{ij}(r) T^{-1}] \, d^3N r. \hspace{1cm} (5)$$

Here $\phi_{ij}(r)$ is the potential energy of interaction between the $i$th and $j$th particle. To treat the problem we introduce the usual two-particle function defined by the relationship

$$f_{ij} = e^{-\phi_{ij}/T} - 1. \hspace{1cm} (6)$$

In the absence of interactions (ideal gas), the function $f_{ij}$ is identically zero, and in the presence of interactions it is non-zero. But at sufficiently high temperatures, it is quite small in comparison with unity. Substitution of equation (6) into equation (5) gives

$$Q_N(T, V) = \int \cdots \int \prod_{1 \leq i < j \leq N} (1 + f_{ij}) \, d^3r_1 d^3r_2 \ldots d^3r_N. \hspace{1cm} (7)$$

The usual virial low-density or high-temperature approximation to equation (7) results from dropping second and higher order product terms like $\sum f_{ij} f_{ij}$ and so forth. These higher order terms actually represent the interactions of more than two particles at once. In gravitating systems, however, all particles interact in pairs only. Hence, the product in the integral of equation (7) can be represented in the following way:

$$\prod_{1 \leq i < j \leq N} (1 + f_{ij}) = \prod_{1 \leq i < j \leq N} (1 + f_{ij}) \prod_{j=2, 3, 4, \ldots, N} (1 + f_{j1})(1 + f_{j2})(1 + f_{j3}) \ldots (1 + f_{jN}). \hspace{1cm} (8)$$

This sum excludes products involving self-energy terms of the form $f_{ij}$. Therefore, when $j = 2$, the above expression gives only one term $(1 + f_{12})$. For $j = 3$, we have just $(1 + f_{12})(1 + f_{23})$ terms and so on. Thus, we can write down the terms for other values of $j$ and equation (7) becomes

$$Q_N(T, V) = \int \cdots \int (1 + f_{12})(1 + f_{13})(1 + f_{23})(1 + f_{14}) \ldots \times (1 + f_{N-1, N}) d^3r_1 d^3r_2 \ldots d^3r_N. \hspace{1cm} (9)$$

We evaluate the above integral for different values of $N$ and then generalize it. For the point mass approximation, the partition function diverges when it includes energy states corresponding to $r_j = 0$, since the Hamiltonian diverges. However, this divergence can be removed by taking the extended nature of galaxies into account and introducing a softening parameter. Thus, each galaxy is considered to have a finite size that is incorporated in the potential represented by $(r^2 + \epsilon^2)^{-1/2}$, where $\epsilon$ is a constant average softening parameter representing approximately all galaxies (Ahmad 1987). The softening parameter $\epsilon$ describes the finite size of a galaxy in constant proper coordinates, and typically $0.01 \leq \epsilon \leq 0.05$ in units of the constant cell size. For a constant potential at small scales, the density would decrease as $r^{-2}$, which could approximately represent a spherical isothermal halo of dark matter around the galaxy. Thus, the interaction potential energy between two galaxies is represented by

$$\phi_{ij} = -\frac{Gm^2}{(r_{ij}^2 + \epsilon^2)^{1/2}}. \hspace{1cm} (10)$$

With the use of equation (10), equation (6) becomes

$$1 + f_{ij} = \exp \left[ \frac{Gm^2}{T(r_{ij}^2 + \epsilon^2)^{1/2}} \right]. \hspace{1cm} (11)$$

In the usual virial expansion, the ratio $Gm^2/rT \approx 3Gm^2/2rK$, where $K = m v^2/2$, averaged over the entire ensemble, represents the ratio of a typical interaction energy to a typical kinetic energy. (Here Boltzmann’s constant is taken to be unity). This ratio is usually considered to be small for either weak interactions, or high temperature, or low density (large average separation $<r_{ij}>$). In the cosmological many-body case, however, there is another physical interpretation of this ratio. It is equal to $3(W/K)/(W/K)_{vir}$. This is essentially the ensemble average of the ratio of gravitational potential energy to the kinetic energy of peculiar velocities but normalized to the value it would have if the system were virialized on all scales. Systems that are still clustering are, by definition, not virialized on all scales, and we refer to them as “undervirialized.” For such undervirialized systems, we can neglect higher powers of $Gm^2/rT$ in equation (11) so that $f_{ij}$ is given to a good approximation by

$$f_{ij} = \frac{Gm^2}{T(r_{ij}^2 + \epsilon^2)^{1/2}}. \hspace{1cm} (12)$$

Substituting equation (12) into equation (9) gives for $N = 1$,

$$Q_1(T, V) = V \hspace{1cm} (13)$$

and for $N = 2$

$$Q_2(T, V) = \int \int \left[ 1 + \frac{Gm^2}{T(r_{12}^2 + \epsilon^2)^{1/2}} \right] d^3r_1 d^3r_2. \hspace{1cm} (14)$$

There are two tricks that simplify the evaluation of the double integral in equation (14). The first is to fix the position of $r_1$ and carry out the integration over all the other particles. The second trick is to incorporate a feature that characterizes the cosmological many-body problem. This is that the expansion of the universe exactly cancels the effects of the long-range mean gravitational field on the particles' motions. This cancellation occurs for both Einstein-Friedmann models and for models incorporating a cosmological constant or quintessence (e.g., Saslaw 2001), leaving the dynamics to depend on local fluctuations of the potential. Thus, the integral in equation (14) is not over all space but only up to an average scale, $R_1$, beyond which density fluctuations do not depart significantly from the average density. This is where the expansion of the universe explicitly enters our solution. Since the function $f_{ij}$ extends over a rela-
tively small finite range of distances, this integration extends only over a limited region of space whose linear dimensions are of the order where \( f_x \) is appreciable. (Note that this might not apply to some fractal density distributions.) The integration over the coordinates \( r_i \) of the particles gives a straightforward factor of \( V \), while the second integral after transforming to spherical polar coordinates gives

\[
Q_2(T, V) = V^2 \left[ 1 + \frac{3Gm^2 \alpha(\epsilon/R_1)}{2T(\bar{n})^{-1/3}} \right], \tag{15}
\]

where

\[
\alpha\left(\frac{\epsilon}{R_1}\right) = \sqrt{1 + \left(\frac{\epsilon}{R_1}\right)^2} + \left(\frac{\epsilon}{R_1}\right) \ln \frac{(\epsilon/R_1)}{1 + \sqrt{1 + (\epsilon/R_1)^2}}. \tag{16}
\]

Here \( R_1 \) is the radius of the cell, and \( \bar{n} \) is the average number density per unit volume so that \( (\bar{n})^{-1/3} \approx r \).

Next consider the scale transformation for temperature \( T \) and number density \( \bar{n} \):

\[
T \rightarrow \lambda^{-1}T, \quad \bar{n} \rightarrow \lambda^{-3}\bar{n}. \tag{17}
\]

This leaves the factor \( 1/T(\bar{n})^{-1/3} \) in the second term of equation (15) invariant, i.e.,

\[
1/T(\bar{n})^{-1/3} \rightarrow T^{-3}. \tag{18}
\]

But the ratio \( Gm^2/T(\bar{n})^{-1/3} \) is dimensionless, and therefore the dimensionless scale invariance transformation for \( Gm^2/T(\bar{n})^{-1/3} \) gives

\[
Gm^2/T(\bar{n})^{-1/3} \rightarrow (Gm^2)^3\bar{n}T^{-3}. \tag{19}
\]

This is also clear from the scaling property of the partition function derived by Landau & Lifshitz (1980) and verified for the scaling property of the gravitational partition function (Saslaw & Fang 1996). Accordingly, the general solution of equation (1) has the form

\[
Z_N(T, V) = T^{3/2N}\bar{n}, \tag{20}
\]

where \( \bar{n} \) and \( T \) enter \( \eta \) as a function of just \( \bar{n}T^{-3} \). Consequently, any modifications of the thermodynamic functions owing to the potential energy term in equation (1) will depend on the intensive thermodynamic variables \( \bar{n} \) and \( T \) only in the combination \( \bar{n}T^{-3} \). Therefore, equation (15) is invariant under the transformation (17) and has the form

\[
Q_2(T, V) = V^2 \left[ 1 + \beta\bar{n}T^{-3}\alpha\left(\frac{\epsilon}{R_1}\right) \right], \tag{21}
\]

where \( \beta = (3/2)(Gm^2)^3 \) is a positive constant. Proceeding in the same way, we find \( Q_3(T, V), Q_4(T, V), \) and \( Q_N(T, V) \):

\[
Q_3(T, V) = \iiint (1+f_{12})(1+f_{23})d^3r_1d^3r_2d^3r_3 = V^3 \left[ 1 + \beta\bar{n}T^{-3}\alpha\left(\frac{\epsilon}{R_1}\right) \right]^2, \tag{22}
\]

\[
Q_4(T, V) = \iiint (1+f_{14})(1+f_{24})(1+f_{34})d^3r_1d^3r_2d^3r_3d^3r_4 = V^4 \left[ 1 + \beta\bar{n}T^{-3}\alpha\left(\frac{\epsilon}{R_1}\right) \right]^3, \tag{23}
\]

and, in general,

\[
Q_N(T, V) = V^N \left[ 1 + \beta\bar{n}T^{-3}\alpha\left(\frac{\epsilon}{R_1}\right) \right]^{N-1}. \tag{24}
\]

Finally, substituting equation (24) into equation (2) gives the partition function for gravitational clustering of galaxies incorporating their extended structures, such as spherical halos:

\[
Z_N(T, V) = \frac{1}{N!} \left( \frac{2\pi mT}{\Lambda^2} \right)^{3N/2} \times V^N \left[ 1 + \beta\bar{n}T^{-3}\alpha\left(\frac{\epsilon}{R_1}\right) \right]^{N-1} \tag{25}
\]

with \( \beta = (3/2)(Gm^2)^3 \) and \( \alpha(\epsilon/R_1) = [1 + (\epsilon/R_1)^2]^{1/2} + (\epsilon/R_1)^3\ln\{(\epsilon/R_1)/[1 + [1 + (\epsilon/R_1)^2]^{1/2}]\}. \) The effect of \( \epsilon/R_1 \) on the behavior of \( \alpha(\epsilon/R_1) \) is shown in Figure 1. For \( \epsilon/R_1 = 0, \) corresponding to point masses, \( \alpha(\epsilon/R_1) = 1. \) It decreases monotonically as \( \epsilon/R_1 \) increases up to its maximum realistic value less than 1.

3. THERMODYNAMIC FUNCTIONS FOR A SYSTEM OF EXTENDED GALAXIES

Once the partition function is known, it is straightforward to calculate the thermodynamic functions and the
equations of state. For example, the free energy is given by

\[ F = -T \ln Z_N(T, V) \]

\[ = NT \ln \left( \frac{N}{V} T^{-3/2} \right) - NT \ln \left[ 1 + \beta N T^{-3} \alpha \left( \frac{\epsilon}{R_1} \right) \right] \]

\[ - \frac{3}{2} NT - \frac{3}{2} NT \ln \left( \frac{2 \pi m}{\Lambda^2} \right). \]

(26)

In this expression we have approximated \( N - 1 \) by \( N \) for large \( N \). Note that \( n \) appears in the factor multiplying \( \alpha \) since this is an average over the system. From this free energy the other thermodynamic quantities such as entropy \( S \), internal energy \( U \), pressure \( P \), and chemical potential \( \mu \), follow directly:

\[ S = - \frac{\partial F}{\partial T} \]

\[ = - NT \ln \left( \frac{N}{V} T^{-3/2} \right) + N \ln \left( 1 + \beta N T^{-3} \alpha \left( \frac{\epsilon}{R_1} \right) \right) \]

\[ - \frac{3}{2} NB \frac{5}{2} N + \frac{3}{2} N \ln \left( \frac{2 \pi m}{\Lambda^2} \right), \]

(27)

\[ U = F + TS = \frac{3}{2} NT(1 - 2b_\epsilon), \]

(28)

\[ P = - \left( \frac{\partial F}{\partial V} \right)_{T, N} = \frac{NT}{V} (1 - 2b_\epsilon), \]

(29)

and

\[ \left( \frac{\mu}{T} \right) = \frac{1}{T} \left( \frac{\partial F}{\partial V} \right)_{T, V} \]

\[ = \ln \left( \frac{N}{V} T^{-3/2} \right) + \ln(1 - b_\epsilon) - b_\epsilon - \frac{3}{2} \ln \left( \frac{2 \pi m}{\Lambda^2} \right). \]

(30)

In these results, the quantity \( b_\epsilon \) appears naturally and is given by

\[ b_\epsilon = \frac{\beta N T^{-3} \alpha \left( \frac{\epsilon}{R_1} \right)}{1 + \beta N T^{-3} \alpha \left( \frac{\epsilon}{R_1} \right)} \].

Equation (31) for \( b_\epsilon \) reduces to the usual result for point masses for \( \epsilon = 0 \), implying \( \alpha = 1 \), from equation (16), and \( b_\epsilon \) is therefore related to \( b \) (point masses) by

\[ b_\epsilon = \frac{b \alpha \left( \frac{\epsilon}{R_1} \right)}{1 + b \alpha \left( \frac{\epsilon}{R_1} \right) - 1}. \]

(32)

Thus, for point masses (\( \epsilon = 0, \alpha = 1 \)), we get the same expressions originally derived by assuming the point mass form of equation (31) for \( b \) (Saslaw & Hamilton 1984) and later confirmed by various authors. More complicated relations for \( b \) (\( \epsilon T^{-3} \)) have been postulated from time to time but one can show (Saslaw & Fang 1996; F. Ahmad & S. Masood 2002, in preparation) that they lead to undesirable physical consequences such as distribution functions having negative probabilities, or violating the first or second laws of thermodynamics. However, in the present derivation, the functional form of \( b_\epsilon \) emerges directly from the partition function and its equations of state. This indicates a deep connection between the statistical mechanics and thermodynamics of gravitational galaxy clustering.

The overall clustering of galaxies is partially characterized by the full probability distribution function \( f(N) \). It contains the void distribution \( f_0(V) \) as well as the objective clustering statistic \( f_r(N) \) of counts of the number of galaxies in cells of a given size distributed throughout the system. We next use our result for the partition function to give a new much simpler derivation of the distribution function for the cosmological many-body system.

For a system of many particles (galaxies), the thermodynamic quantities obtained in \( \frac{1}{3} \) from the canonical partition function for the canonical ensemble are the same as in the grand canonical ensemble, since they are average relations for the entire system (e.g., Huang 1987). However, to
obtain the probability distribution function, we need to consider the departures from these averages, since the distribution functions describe the probabilities of finding fluctuations of all amplitudes. For the system of galaxies, where galaxies as well as energy can cross cell boundaries, we must use the grand canonical ensemble. The first step is thus to obtain the grand canonical partition function $Z_G$, which is a weighted sum of all the canonical partition functions defined by

$$Z_G(T, V, z) = \sum_{N=0}^{\infty} e^{\mu N/T} Z_N(T, V).$$

The probability of finding $N$-particles in a cell of the grand canonical ensemble is the sum over all of the energy states:

$$f(N) = \frac{\sum_{i=0}^{N} e^{\mu_i/T} e^{-U_i/T}}{Z_G(T, V, z)} = \frac{e^{\mu N/T} Z_N(T, V)}{Z_G(T, V, z)}.$$  \hspace{1cm} (34)

This is the basic equation for evaluating the distribution function. Recall that the weighting factor $z = e^{\mu N/T}$, the activity, is an average value over $N$, while the canonical partition function $Z_N(T, V)$ in equation (1) is a sum over all $N$-particles. For a system of point masses, $e^{\mu N/T}$ and $Z_N(T, V)$ are given by equations (30) and (25) with $\alpha = 1$ and therefore $\epsilon = 0$:

$$e^{\mu N/T} = \left(\frac{N}{V} T^{-3/2}\right)^N (1 - b)^N e^{-N h} \left(\frac{2\pi m}{\Lambda^2}\right)^{-3N/2}.$$ \hspace{1cm} (35)

and

$$Z_N(T, V) = \frac{1}{N!} \left(\frac{2\pi m}{\Lambda^2}\right)^{3N/2} \left(V T^{3/2}\right)^N \times \left[1 + \frac{Nb}{N(1 - b)}\right]^{N-1}.$$ \hspace{1cm} (36)

Also, the grand canonical partition function $Z_G$ is generally related to the thermodynamic potential $\Psi$ and thermodynamic variables by

$$\ln Z_G = \Psi = \frac{PV}{T} = N(1 - b).$$ \hspace{1cm} (37)

Substituting equations (35)–(37) into equation (34) gives the distribution function for point mass galaxies directly:

$$f(N) = \frac{\bar{N}(1 - b)}{N!} \left(\bar{N}(1 - b) + Nb\right)^{N-1} e^{-\bar{N}(1-b)-Nh}.$$ \hspace{1cm} (38)

This is the same distribution function for a system of point mass particles derived earlier by Saslaw & Hamilton (1984) and Saslaw & Fang (1996), although here we have derived it differently and quite simply, directly from statistical mechanics.

We proceed in a similar way to derive the distribution function for a cosmological system of galaxies or particles represented by non–point masses. For non–point masses, $e^{\mu N/T}$ and $Z_N(T, V)$ are obtained from equations (30), (25), and (31):

$$e^{\mu N/T} = \left(\frac{\bar{N}}{V} T^{-3/2}\right)^N (1 - b)^N e^{-Nb} \left(\frac{2\pi m}{\Lambda^2}\right)^{-3N/2}$$ \hspace{1cm} (39)

and

$$Z_N(T, V) = \frac{1}{N!} \left(\frac{2\pi m}{\Lambda^2}\right)^{3N/2} \times \left(V T^{3/2}\right)^N \left[1 + \frac{Nb}{N(1 - b)}\right]^{N-1}.$$ \hspace{1cm} (40)

Also, the grand partition function $Z_G$ for a non–point mass system is

$$\ln Z_G = \Psi = \frac{PV}{T} = \bar{N}(1 - b).$$ \hspace{1cm} (41)

Thus, substituting equations (39)–(41) into equation (34) gives the distribution function for non–point mass particles:

$$f(N, \epsilon) = \frac{\bar{N}(1 - b)}{N!} \left[\bar{N}(1 - b) + Nb\right]^{N-1} e^{-\bar{N}(1-b)-Nh}.$$ \hspace{1cm} (42)

As $\epsilon \to 0$ in $f(N, \epsilon)$, then this distribution tends to the distribution function of the point mass distribution.

We next compare the results of the distribution function for extended particles obtained in equation (42) with the earlier point mass distribution function of equation (38). Figure 3 shows two examples of the effects of finite galaxy size on the counts-in-cells distribution $f_T(N, \epsilon)$ for cells containing an average of $\bar{N} = \bar{n}V = 1$ galaxy (Fig. 3a) and $\bar{N} = 10$ galaxies (Fig. 3b). The solid lines plot the distributions for $\epsilon/R_1 = 0$ for comparison, showing how the distributions are modified for $b = 0.1, 0.6,$ and $0.9$ if $\epsilon/R_1 = 0.5$. These indicate that quite large values of $\epsilon/R_1$ are necessary to produce significant departures of the distributions from those of point masses.

5. MOMENTS FOR EXTENDED STRUCTURES

Sometimes it may not be necessary or practicable to know the exact shape of the distribution function. Then its general properties can be characterized by a sequence of moments of the distribution. Fluctuations may be related to variations of the numbers of particles, energy, etc., through moments. Higher order fluctuations are also informative and can be calculated straightforwardly as discussed by Callen (1985) and Saslaw (1969). For non–point masses, the fluctuation moments for the number of particles are

$$\langle(\Delta N)^2\rangle = \frac{\bar{N}}{(1 - b)}$$ \hspace{1cm} (43)

$$\langle(\Delta N)^3\rangle = \frac{\bar{N}}{(1 - b)^3}(1 + 2b)$$ \hspace{1cm} (44)

and

$$\langle(\Delta N)^4\rangle = \frac{\bar{N}}{(1 - b)^6}(1 + 8b + 6b^2) + 3\langle(\Delta N)^2\rangle^2.$$ \hspace{1cm} (45)
In calculating these moments, we have used the first-, second-, and third-order derivatives of equation (39) directly. In a similar way, we can evaluate fluctuations in energy and correlated energy-number fluctuations as

$$\langle (\Delta U)^2 \rangle = \frac{3NT^2}{4(1-b_\tau)^2}(5 - 20b_\tau + 34b_\tau^2 - 16b_\tau^3) , \quad (46)$$

$$\langle (\Delta N)(\Delta U) \rangle = \frac{3NT}{2(1-b_\tau)^2}(1 - 4b_\tau + 2b_\tau^2) , \quad (47)$$

and

$$\langle (\Delta N)^2(\Delta U) \rangle = \frac{3NT}{2(1-b_\tau)^2}(1 - 6b_\tau + 2b_\tau^2) . \quad (48)$$

Another aspect of these moments is their relation to correlation functions (e.g., Saslaw 2000). In addition, from the density moments we can measure the skewness and kurtosis of the distributions of extended galaxies clustering under gravitation. The skewness is given by

$$S = \bar{N} \left\langle \frac{(\Delta N)^3}{(\Delta N)^2} \right\rangle = 1 + 2b_\tau , \quad (49)$$

and the kurtosis is given by

$$K = \bar{N}^2 \left[ \left\langle \frac{(\Delta N)^4}{(\Delta N)^2} \right\rangle - 3 \left\langle (\Delta N)^2 \right\rangle^2 / (\left\langle (\Delta N)^2 \right\rangle)^2 \right] = 1 + 8b_\tau + 6b_\tau^2 . \quad (50)$$

These relations in equations (49) and (50) are especially important as they can easily be related to observed catalogs and to simulations.

6. EVOLUTION OF $b_t(t)$

As the universe expands, clustering generates entropy, which is reflected in the increase of $b_t$ in equation (27). For a statistically homogeneous universe, the local expansion in comoving coordinates would be adiabatic and satisfy the first law of thermodynamics $dQ = dU + PdV$ with no net heat flow into or out of a volume, which implies

$$dQ = 0 = dU + PdV \ . \quad (51)$$

Using the cosmic energy equation to calculate the adiabatic evolution for values of $b_t$ (for a point-mass system) over large scales as a function of expansion scale, $a$, gives (Saslaw 1992)

$$\frac{a}{a_\tau} = \frac{b_{1/8}}{(1-b_\tau)^{7/8}} , \quad (52)$$

where $a_\tau$ is a constant that determines the value of $b$ at some fiducial expansion state.

For extended masses, the time evolution of $b_t$ as a function of the expansion scale can be obtained from equation (51) by regarding $U$ and $b_t$ as functions of $P$ and $T$, since then

$$\frac{3N}{2}(1 - 2b_\tau)dT_p + \left( \frac{\partial U}{\partial b_\tau} \right)_T PdV = 0 . \quad (53)$$

To determine $b_t$ as a function of $P$ and $T$, we use equations (29) and (31) to find

$$a\beta PT^{-4} = b_\tau . \quad (54)$$

Equation (54) can be reexpressed after differentiation as

$$dT_p = -\frac{T}{4b_\tau} db_\tau . \quad (55)$$
Substituting $dT_p$ from equation (55) and $(\partial U/\partial b_*)_T$ from equation (28) into equation (53), and making use of $(dV/V) = (3d\alpha/\alpha)$, leads after integration to

$$\frac{\alpha}{\alpha_s} = \frac{b_1^{1/8}}{(1 - b_1)^7/8} = \frac{b_1^{1/8}}{(1 - b_1)^7/8} \left[ \alpha \left( \frac{\epsilon}{R_1} \right) \right]^{1/8} 	imes \left\{ 1 + b \left[ \alpha \left( \frac{\epsilon}{R_1} \right) - 1 \right] \right\}^{3/4}. \quad (56)$$

Therefore, the evolution of $b_1$ for extended galaxies has the same form as for point galaxies but with $b_1$ substituting for $b_1$.

7. DISCUSSION

It has long been an outstanding problem to relate the dynamical, thermodynamic, and statistical descriptions of gravitational many-body systems in various contexts. For canonical ensembles of finite systems, such as individual globular clusters or virialized clusters, which can exchange energy but not particles, these relations have been worked out in considerable detail (e.g., Saslaw 1987; de Vega & Sánchez 2001a, 2001b and references therein). For the cosmological many-body system, where linear and nonlinear clustering of galaxies and their haloes are present over many scales simultaneously, grand canonical ensembles are needed since cells can exchange galaxies as well as energy. This system has been described dynamically by $N$-body simulations and thermodynamically by quasi-equilibrium equations of state and fluctuations around them. But it has not hitherto had a statistical mechanical description.

There have been several perceived difficulties in providing a statistical mechanical description of the cosmological many-body system. One is the long-range nature of the gravitational field leading to divergences in integrals of potential. Another is the expansion of the universe. However, it has been realized (e.g., Saslaw & Fang 1996) that the mean field and the expansion essentially cancel each other in the Friedmann-Robertson-Walker universes, showing that the dynamical, thermodynamic, and statistical descriptions all depend only on the local departures from the mean field. This is the opposite situation from that prevailing in other many-body systems where the mean field dominates.

A third perceived difficulty with gravitational statistical mechanics are the divergences in the Hamiltonian, which result from two or more point particles approaching one another arbitrarily closely. Here we have resolved this problem by using a softened potential to eliminate the divergence, and also to represent galaxy haloes. We found that the statistical description continues to be valid even in the limit when the softening scale length vanishes. The reason for this, as in the thermodynamic description (Saslaw 2000), is that the probability of such point collisions in the expanding universe (although not always in finite globular clusters) is very small, and eliminating them does not significantly influence the thermodynamic state of the system, in agreement also with dynamical $N$-body simulations of point galaxy clustering.

Therefore, it has become reasonable to expect that a statistical mechanical description of the cosmological many-body system is possible, and we have found it by calculating the partition function analytically under these circumstances. The basic assumption necessary for this is that the universe is undervirialized, i.e., that it does not have a virialized hierarchy of structures on all scales, so $b = -W/2T < 1$ (with $W$ the average gravitational correlation energy in a cell). This is observed to be the case in our universe for which $b \approx 0.75$.

From the closed form of the resulting gravitational grand canonical partition function, we readily derive all the thermodynamic quantities and their equations of state. These give exactly the same previous results found directly from thermodynamic arguments in the limit of the zero-softening parameter and generalize them for a finite softening parameter $\epsilon$. Only for softening parameters that are a significant fraction, say $\epsilon \gtrsim 0.3$, of the cell size in physical coordinates, are the thermodynamics modified significantly. We calculate the modified galaxy distribution functions $f(N, \epsilon)$ and their moments, as well as their evolution.

Analyses of earlier galaxy surveys, mentioned in § 1 and having approximately $10^5$–$10^6$ galaxies, have determined values of $b_1$ within an uncertainty of about $\pm 0.05$. From Figure 3, this is only sufficient to provide an upper limit of about 0.1 to $\epsilon/R_1$. Much of the uncertainty comes from catalog incompleteness and resulting statistical fluctuations in the values of $f(N)$. Present ongoing catalogs, such as the 2dF survey, with about $10^5$–$10^6$ galaxies can decrease the uncertainty in $b_1$ as long as their selection criteria are homogeneous and their samples contiguous. Future catalogs with $10^6$–$10^7$ galaxies (e.g., the Sloan Survey) may reduce the uncertainty in $b_1$ by a factor of about 10 and provide a direct determination of $\epsilon/R_1$.

Another method for uncovering the effects of $\epsilon/R_1$ may be to examine $b_1$ over a range of redshifts and compare the results with equation (56). Both $\epsilon$ and $R_1$ will vary with time as galaxy haloes form and dissipate and as the scale length of galaxy clustering increases. Following these developments naturally requires detailed models, and their effects on $f(N)$ may be detectable and separable from other evolutionary properties in future extensive catalogs that observe out to $z \approx 3$. At these different redshifts, it may also be possible to determine how evolving haloes affect the value of $b_1$ as a function of scale, i.e., cell size.

In addition to its observational implications, the availability of the partition function in the form of equation (25) can provide new insights into the theory of cosmological many-body gravitational clustering, for example, into the nature of associated phase transitions.

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