GROWTH AND SPECTRUM OF DIFFUSIONS

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INTRODUCTION

Let $X$ be a locally compact, second countable Hausdorff space $X$ and let $dx$ be a Radon measure on $X$, which sometimes we shall call the volume of $X$.

Let $H \geq 0$ be a self-adjoint operator associated with a diffusion $(a, D[a])$, that is, a strongly local, regular Dirichlet form on $L^2 = L^2(\Omega, dx)$ [4]. In this paper we examine the relation between the growth (of the volume) of $X$ and the bottom of the spectrum of the operator $H$. More precisely, we introduce a (pseudo-)distance $d(x, y)$ on $X$ through $(a, D[a])$ and assume that the metric topology is equivalent to the original topology of $X$; we fix a point $o \in X$ and, denoting by $V(r)$ the measure of the ball $B(o, r)$, $r > 0$, introduce the growth of $X$ by means of the quantity

$$
\mu := \limsup_{r \to +\infty} \frac{\ln V(r)}{r};
$$

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finally we also introduce

\[ \lambda_e := \sup \lambda_o(X \setminus K), \]  

where the supremum is taken on the family of compact subsets of \( X \), and

\[ \lambda_o(X \setminus K) = \inf \left\{ \frac{a[u,u]}{\int_X u^2 \, dx} : u \neq 0, \ u \in D[a] \cap C_c(X \setminus K) \right\} \]

is the bottom of the spectrum of \( H \) on \( L^2(X \setminus K) \) with Dirichlet boundary condition on \( \partial K \).

The formula (0.1) above can be seen as a generalization of the well-known formula [2], [3] that holds when \( H \) is the Laplace-Beltrami operator on a Riemannian manifold \( X \) (\( dx \) is then the Riemannian volume) and \( \lambda_e \) is the bottom of the essential spectrum of \( H \). The number \( \lambda_e \) may be infinite, which is the case when \( H \) has discrete spectrum and the essential spectrum is empty.

The main result of this paper is an extension of a result by R. Brooks [1, Theorem 1] and goes as follows.

**Theorem 1.** If \( X \) is non-compact, the volume of \( X \) is infinite and the metric space \((X, d)\) is complete, then

\[ \lambda_e \leq \frac{\mu^2}{4}. \]

In particular, if the measure \( dx \) has polynomial growth (e.g., if \( dx \) satisfies a doubling condition), then \( \mu = 0 \), hence the
bottom of the essential spectrum is $\lambda_e = 0$. See Corollary [4]. On the other hand, if the growth is exponential (as in the case, e.g., of $H^{n+1}$, the $n + 1$-dimensional hyperbolic space), then we recover a well-known result on the bottom of the Laplace-Beltrami operator on $H^{n+1}$; cf., e.g., the result by H.P. McKean [7].

The above result sharpens a similar one for $\lambda_o(X)$ (with $K = \emptyset$) by K.-T. Sturm [10, Theorem 5].

Theorem [1] is sharp in the following sense. It is known that non-complete metric spaces may have discrete spectra; cf. e.g. results for the Laplace operator in “quasibounded domains” of $\mathbb{R}^n$ by D. Hewgill [5], [6] (see also the references cited in these two papers). Moreover, compact Riemannian manifolds and some Riemannian manifolds with finite volume are known to have discrete spectrum: the former is classical, while instances of the latter have been constructed by H. Donnelly & P. Li in [3].

On the other hand, it must be remarked that there are examples where the upper bound on $\lambda_e$ in term of the growth of $X$ is not sharp in that $\lambda_e = 0$ and $\mu > 0$; cf. the discussion in the Introduction of [1].

The organization of the paper is as follows: in the first section we fix the notation, introduce the relevant concepts and prove the preliminary results that will be needed in the second section,
which is devoted to the proof of the main result. The proof of our main result follows the lines of the proof of Theorem 1 in \[1\] and is based on Theorem \[2\], which is a result that may have some interest in its own.

1. Preliminaries

Let $X$ be a locally compact, second countable Hausdorff space and let $dx$ be a Radon measure on it.

Dirichlet forms \[3\]. We let $(a,D[a])$ denote the (Dirichlet) form associated with the self-adjoint operator $H$ on $L^2 = L^2(X,dx)$, so that

$$\langle Hf, g \rangle = a[f,g], \quad f \in D(H), \quad g \in D[a],$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2$.

We shall say that $u \in D_{loc}[a]$ if for every compact set $C \subset X$ there exists $\overline{u} \in D[a]$ such that $u = \overline{u}, \text{ } dx\text{-a.e. on } C$.

We shall assume in the sequel that $(a,D[a])$ is a diffusion, \textit{i.e.}, $(a,D[a])$ satisfies the following property:

$$a[u,v] = 0, \quad u,v \in D[a],$$

whenever $u = \text{const. on supp } v$.

Then it is standard that we can write the form as

$$a[u,u] = \int_{\Omega} d\Gamma[u,u],$$
where the map \((u, v) \mapsto d\Gamma[u, v]\), defined on \(D[a] \times D[a]\) with values in the space of signed Radon measures, is a non-negative symmetric bilinear form (the energy measure associated with the form \((a, D[a])\)). The energy measure can be defined according to the formula
\[
\int_X \phi d\Gamma[u, u] = a[u, \phi u] - \frac{1}{2} a[u^2, \phi],
\]
for every \(u \in D[a] \cap L^\infty(X, dx)\) and every \(\phi \in D[a] \cap C^c_c(X)\).

**Proposition 1.2.** The energy measure satisfies the following properties:

(D_1) The Leibnitz rule. For every \(u, v \in D_{loc}[a] \cap L^\infty(X, dx)\) and \(w \in D_{loc}[a]\)
\[
d\Gamma[uv, w] = v(x) d\Gamma[u, w] + u(x) d\Gamma[v, w]
\]
in the sense of measures.

(D_2) The Schwarz rule. If \(u, v \in D[a], f \in L^2(\Omega, d\Gamma[u, u]), g \in L^2(\Omega, d\Gamma[v, v])\), then \(fg\) is integrable w.r.t. the absolute variation \(|d\Gamma[u, v]|\) of \(d\Gamma[u, v]\), and
\[
|fg| |d\Gamma[u, v]| \leq \frac{\eta}{2} |f|^2 d\Gamma[u, u] + \frac{1}{2\eta} |g|^2 d\Gamma[v, v],
\]
for \(\eta > 0\);

(D_3) The chain rule. Let \(\eta \in C^1(\mathbb{R})\) with bounded derivative. Then \(u \in D_{loc}[a]\) implies \(\eta(u) \in D[a]\) and
\[
d\Gamma[\eta(u), v] = \eta'(u) d\Gamma[u, v],
\]
for every \( v \in D[a] \cap L^\infty(X, dx) \).

(D_4) The truncation property. ([8, Lemma 3-(o)] For every \( u \in D_{loc}[a] \))

\[
0 \leq d \Gamma[u_+, v] := 1_{\{u > 0\}}d \Gamma[u, v],
\]

where \( u_+(x) := \max\{u(x), 0\}, x \in X \).

(D_5) Locality. If \( A \) is an open set and \( u_1 = u_2 \) dx-a.e. on \( A \), \( u_1, u_2 \in D[a] \), then

\[
1_A(x)d \Gamma[u_1, u_1] = 1_A(x)d \Gamma[u_2, u_2]
\]

on \( X \); moreover

\[
1_A(x)d \Gamma[u, v] = 0,
\]

on \( X \), whenever \( u \in D[a] \) is constant on \( A \), for arbitrary \( v \in D[a] \).

**Definition 1.3.** For \( x, y \in \Omega \) let

\[
d(x, y) := \sup\{\psi(x) - \psi(y) : \psi \in C, \ d \Gamma[\psi, \psi] \leq dx\}.
\]

Then it is not difficult to prove that \( d(\cdot, \cdot) \) is a (pseudo-)distance on \( \Omega \), which we call the Carnot-Carathéodory distance (associated with the form \((a, D[a])\)); cf. [11, § III-4].

**Remark 1.4.** Notice that \( d(x, y) = 0 \) not necessarily implies \( x = y \). Moreover \( d(x, y) \) may be equal to 0 or \( \infty \), for some \( x \neq y \).
Warning. In the rest of the paper we shall make the assumption that the topology induced by the metric is equivalent to the original topology of $X$ and that $(X, d(\cdot, \cdot))$ is a complete metric space.

This in turn is equivalent \cite{9} (cf. also \cite{10, §4}) to the fact that all balls are relatively compact in $X$.

Remark 1.5. Notice that in the case of the Dirichlet integral on a bounded open set of $\mathbb{R}^n$ the energy measure $d\Gamma[u, u]$ is absolutely continuous with respect to the Lebesgue measure; the Radon-Nikodym derivative is equal to $|\nabla u|^2$ so that

$$d(x, y) = \sup\{\psi(x) - \psi(y) : \psi \in C, \ |\nabla \psi| \leq 1 \ a.e.\}.$$ Notice that $d(x, y)$ is locally equivalent to the standard euclidean one, that is, for any $x \in \Omega$ and any neighborhood $U$ of $x$, there exists a constant $c > 0$ such that

$$c^{-1}d(x, y) \leq |x - y| \leq c d(x, y),$$

for $y \in U$.

We shall need the following technical result to localize functions in $D[a]$.

Lemma 1.6. For each compact set $K \subset X$ there exists a function $\chi$ such that $\chi_K \in D_{loc} \cap C(X)$, $\chi_K(x) = 1$ on $K$, the support of $\chi_K$ is contained in a neighborhood $B$ of $K$ and $d\Gamma[\chi_K, \chi_K] \leq 16(diamB)^{-2}dx$. 
Proof. As $K$ is compact, then $K \subset B(x_o, R)$, for some $x_o \in X$, $R > \text{diam } K \geq 0$, and $B = B(x_o, 2R)$. Consider the function $\eta \in C^1(R)$ such that $\eta(t) = 1$, for $t \in (-\infty, 1)$, $\eta(t) = 0$, for $t \in [2, +\infty)$ and $|\eta'| \leq 1$; let

$$\chi_K(x) := \eta \left( \frac{d(x, x_o)}{R} \right).$$

The function $\rho_{x_o}(x) := d(x, x_o)$ is in $D_{\text{loc}}[a] \cap C(X)$ and $d\Gamma[\rho_{x_o}, \rho_{x_o}] \leq dx$ [10] § 4, Lemma A', thus by the chain rule $\chi_K \in D_{\text{loc}} \cap C(X)$ and $d\Gamma[\chi_K, \chi_K] \leq R^{-2}dx$; moreover, by definition, $\chi_K = 1$ on $(B(x_o, R)$, hence on) $K$ and $\chi_K = 0$ outside $B = B(x_o, 2R)$, so the proof is completed.

**Proposition 1.7.** Let $A$ be any subset of $X$; then

$$\phi(x) := \text{dist}(x, A) = \inf \{d(x, y) : y \in A\}$$

is in $D_{\text{loc}}[a] \cap C(X)$, with $d\Gamma[\phi, \phi] \leq dx$.

Proof. We first prove the continuity of the map $\phi$, by showing

that $\phi(x_n) \to \phi(x)$ whenever $d(x, x_n) \to 0$. By definition

$$\phi(x_n) \leq d(x_n, y), \ y \in A,$$

thus

$$\limsup_{n \to +\infty} \phi(x_n) \leq \limsup_{n \to +\infty} d(x_n, y) = d(x, y),$$

hence, taking the supremum over all $y \in A$,

$$\limsup_{n \to +\infty} \phi(x_n) \leq \inf \{d(x, y) : y \in A\} = \phi(x).$$
On the other hand for any $\varepsilon > 0$ there is $y_\varepsilon \in A$ such that
\[
d(x_n, y_\varepsilon) \leq \phi(x_n) + \varepsilon,
\]
thus, choosing $\varepsilon = 1/n$ and using the triangle inequality $d(x, y_\varepsilon) \leq d(x, x_n) + d(x_n, y_\varepsilon)$, we have
\[
\phi(x) \leq d(x, x_n) + \phi(x_n) + 1/n;
\]
hence
\[
\phi(x) \leq \liminf_{n \to +\infty} \left[ d(x, x_n) + \phi(x_n) + 1/n \right]
\]
\[
= \liminf_{n \to +\infty} \left[ (d(x, x_n) + 1/n) + \phi(x_n) \right]
\]
\[
\leq \limsup_{n \to +\infty} \left[ d(x, x_n) + 1/n \right] + \liminf_{n \to +\infty} \phi(x_n)
\]
\[
\leq \liminf_{n \to +\infty} \phi(x_n)
\]
thus
\[
\phi(x) \leq \liminf_{n \to +\infty} \phi(x_n).
\]
Therefore
\[
\limsup_{n \to +\infty} \phi(x_n) \leq \phi(x) \leq \liminf_{n \to +\infty} \phi(x_n),
\]
which implies that the map $\phi$ is continuous on $X$.

Now let us prove that $\phi \in D_{loc}[a]$, with $d\Gamma[\phi, \phi] \leq dx$. Again by definition there exists a sequence $(y_n)$ of points in $A$ such that
\[
\phi(x) = \lim_{n \to +\infty} d(x, y_n).
\]
Consider the map \( \phi_n(x) := d(x, y_n), \ n = 1, 2, \ldots \), so that \( \phi_n(x) \to \phi(x), \ x \in X \); in fact, the convergence is uniform on any relatively compact open subset \( Y \subset X \). Indeed, it is easy to see that each \( \phi_n \) is Lipschitz continuous with Lipschitz constant less than or equal to 1 so that the sequence is pre-compact in \( C(Y) \), according to the Ascoli-Arzelá criterion. Moreover and without loss of generality, we can also assume that \( \phi_n \leq \phi_{n+1}, \ n = 1, 2, \ldots \) (consider otherwise \( \sup_{1 \leq i \leq n} \phi_n \)). We notice [II, § 4, Lemma A] that \( \phi_n \in D_{\text{loc}}[a] \cap C(X) \) and \( d\Gamma[\phi_n, \phi_n] \leq dx, \ n = 1, 2, \ldots \). By localization on \( Y \), we also have that
\[
\int_Y d\Gamma[\phi_n, \phi_n] + \int_Y \phi_n^2 \, dx \leq \text{const}(Y).
\]
Therefore the family
\[
\{ \phi_n : \phi_n \in D_{\text{loc}}[a] \cap C(X), \ d\Gamma[\phi_n, \phi_n] \leq dx, \ n \in N \}
\]
is convex and uniformly bounded in \( D[a] \). By the Banach-Saks theorem, there exists (possibly a subsequence of) \( (\phi_n) \) that converges strongly (in \( D[a] \), hence strongly in) \( L^2 \) to some \( \bar{\phi} \in D_{\text{loc}}[a] \); from the strong convergence in \( D[a] \) we also have \( d\Gamma[\bar{\phi}, \phi] \leq dx \) as measures. On the other hand the whole sequence converge uniformly on \( Y \) to \( \phi \), therefore \( \phi(x) = \bar{\phi}(x) \) for q.e. \( x \in Y \), and so we have \( \phi \in D_{\text{loc}}[a] \) and also \( d\Gamma[\phi, \phi] \leq dx \).
2. THE MAIN RESULT

Denote by \( \rho(x) := d(x, o) \), the function “distance from a given point \( o \in X \)”.

As in Brooks’s paper [1], our Theorem 1 is a consequence of the following result.

**Theorem 2.** Let \( K \subset X \) be a compact set (possibly empty), let

\[
\lambda_o(X \setminus K) = \inf \left\{ \frac{a[u, u]}{\int_{X \setminus K} u^2 \, dx} : 0 \neq u \in D[a] \cap C_c(X \setminus K) \right\}.
\]

If

\[
\int_{X \setminus K} \exp(-2\alpha \rho(x)) \, dx < +\infty,
\]

for some \( \alpha \in (0, \sqrt{\lambda_o(X \setminus K)}) \), then

\[
\int_{X \setminus K} \exp(2\alpha \rho(x)) \, dx < +\infty.
\]

Theorem 3 implies Theorem 4. Recall that \( V(r) \) stands for the volume of \( B(o, r) \), \( r > 0 \). If \( 2\alpha < \mu \) then

\[
\int_{X \setminus K} \exp(-2\alpha \rho(x)) \, dx \leq \sum_{r=1}^{+\infty} \left[ V(r) - V(r - 1) \right] e^{-2\alpha(r-1)}
\]

\[
= \sum_{r=1}^{+\infty} V(r) e^{-2\alpha r} \left[ e^{2\alpha} - 1 \right]
\]

and the latter sum is finite, as it follows by comparing it with a geometric series and by the fact that \( 2\alpha > \mu \). Thus if \( 2\alpha > \mu \).
and $\alpha < \sqrt{\lambda_o(X \setminus K)}$ then by Theorem 2 it follows that
\[
\int_{X \setminus K} \exp(2\alpha \rho(x)) \, dx < +\infty,
\]
but this inequality cannot be true, as the volume of $(X$, hence the volume of) $X \setminus K$ is infinite. (Recall that $dx$ is a Radon measure, hence the volume of any compact set is finite.) Therefore $2\alpha \leq \mu$, and letting $\alpha$ approach $\sqrt{\lambda_o(X \setminus K)}$, we have
\[
\lambda_o(X \setminus K) \leq \frac{\mu^2}{4},
\]
and the right-hand side does not depend on $K$. Taking the supremum over $K$, we have $\lambda_e \leq \mu^2/4$, which proves Theorem 4.

\[\square\]

Proof of Theorem 3: Without loss of generality we can assume that $\lambda_o(X \setminus K) \neq 0$ and hence, by a possible rescaling of the measure $dx$, that $\lambda_o(M \setminus K) = 1$. Let us consider the function $f(x) = e^{h(x)}\chi(x)$, where $\chi \in D[a] \cap C_c(X \setminus K)$ and $h$ is a bounded function in $D[a]$. (We’ll make a choice of these two functions later on.) Let us compute $d\Gamma[f, f] = d\Gamma[\chi e^h, \chi e^h]$. We have
\[
d\Gamma[f, f] = d\Gamma[\chi e^h, \chi e^h]
\]
(Leibniz rule)
\[
= e^{2h}d\Gamma[\chi, \chi] + \chi^2 d\Gamma[e^h, e^h] + 2\chi e^h d\Gamma[e^h, \chi]
\]
(chain rule)
\[
= e^{2h}d\Gamma[\chi, \chi] + e^{2h}\chi^2 d\Gamma[h, h] + 2\chi e^{2h} d\Gamma[h, \chi]
\]
\[
= e^{2h}d\Gamma[\chi, \chi] + f^2 d\Gamma[h, h] + 2\chi e^{2h} d\Gamma[h, \chi].
\]
As
\[
1 = \lambda_0(X \setminus K) \leq \frac{\int_{X \setminus K} d\Gamma[f, f]}{\int_{X \setminus K} f^2 \, dx},
\]
we have
\[
\int_{X \setminus K} f^2 \, dx \leq \int_{X \setminus K} \left[ e^{2h}d\Gamma[\chi, \chi] + f^2d\Gamma[h, h] + 2\chi e^{2h}d\Gamma[h, \chi] \right],
\]
that is,
\[
(2.1)
\]
\[
\int_{X \setminus K} f^2 \, dx - \int_{X \setminus K} f^2d\Gamma[h, h] \leq \int_{X \setminus K} \left[ e^{2h}d\Gamma[\chi, \chi] + 2\chi e^{2h}d\Gamma[h, \chi] \right].
\]
Moreover, by the Schwarz rule and taking into account that \( \chi \) is a bounded function,
\[
(2.2)
\]
\[
\int_{X \setminus K} 2\chi e^{2h}d\Gamma[h, \chi] \leq C \left( \int_{X \setminus K} e^{2h}d\Gamma[\chi, \chi] \right)^{1/2} \times \left( \int_{X \setminus K} e^{2h}d\Gamma[h, h] \right)^{1/2},
\]
for some constant \( C > 0 \). Now we turn to the choice of the functions \( h, \chi \). Let \((K_i)\) be a sequence of compact sets in \( X \setminus K \).
which increases to \( X \setminus K \). For \( i = 1, 2, \ldots \), let

\[
\chi_i(x) := \begin{cases} 
\frac{1}{\delta} \text{dist}(x, X \setminus K_i), & \text{if } 0 \leq \text{dist}(x, X \setminus K_i) \leq \delta \\
1, & \text{if } \text{dist}(x, X \setminus K_i) > \delta.
\end{cases}
\]

By Proposition 1.7 \( \chi_i \in D_{loc} \cap C(X) \), \( d\Gamma[\chi_i, \chi_i] \leq \delta^{-2} dx \) and by definition the support of \( \chi_i \) is compact and contained in \( X \setminus K \); moreover, by the locality property of the energy measure, the support of \( d\Gamma[\chi_i, \chi_i] \) is contained in a neighborhood \( B_\delta(\partial K_i) \) of \( K_i \). Furthermore, let us choose the function \( h \) such that \( d\Gamma[h, h] \leq \alpha dx \), for \( \alpha \in (0, 1) \). Then from (2.1), (2.2) we get

\[
\int_{X \setminus K} f^2 (1 - \alpha^2) dx \leq C \left( \frac{2}{\delta} + \frac{1}{\delta^2} \right) \int_{B_\delta(\partial K_i)} e^{2h} dx.
\]

Now the proof follows the proof given by R. Brooks in [1, Theorem 2]: consider

\[
h_n(x) := \min \{ \alpha \rho(x), -\alpha \rho(x) + n \}, \ n = 1, 2, \ldots ;
\]

notice that \( h_n \in D_{loc}[a] \cap L^\infty(X, dx), \ n = 1, 2, \ldots; \ d\Gamma[h_n, h_n] \leq \alpha dx \) and, under the assumption that \( \exp(-\alpha \rho(x)) \) is integrable on \( X \setminus K \), \( h_n \) is integrable for all \( n \), so \( h_n \) is an admissible function in the definition of \( f \). Notice also that \( h_n(x) \leq h_{n+1}(x) \to h = \alpha \rho(x), \ x \in X \setminus K \), as \( n \to +\infty \). Thus for \( n \) sufficiently large,

\[
\int_{X \setminus K} e^{2h_n} (1 - \alpha^2) dx \leq C \left( \frac{2}{\delta} + \frac{1}{\delta^2} \right) \int_{B_\delta(\partial K)} e^{2\alpha \rho(x)} dx,
\]
that is,
\[ \int_{X\setminus K} e^{2h_\rho} dx \leq \text{const} < +\infty, \]
where the constant at the right-hand side of the above inequality does not depend on \( n \). Taking the limit, as \( n \to +\infty \), we have
\[ \int_{X\setminus K} \exp(2\alpha\rho(x)) dx \leq \text{const} < +\infty, \]
which concludes the proof of the theorem. \( \square \)

**Corollary 1.** If \( X \) has sub-exponential growth, then \( \lambda_e = 0 \).

The condition that \( X \) has sub-exponential growth is precisely the fact that \( \mu = 0 \).

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