Adiabatic evolution and shape resonances

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Abstract

Motivated by a problem of one mode approximation for a nonlinear evolution with charge accumulation in potential wells, we consider a general linear adiabatic evolution problem for a semi-classical Schrödinger operator with a time dependent potential with a well in an island. In particular, we show that we can choose the adiabatic parameter $\varepsilon$ with $\ln \varepsilon \asymp -1/h$, where $h$ denotes the semi-classical parameter, and get adiabatic approximations of exact solutions over a time interval of length $\varepsilon^{-N}$ with an error $O(\varepsilon^N)$. Here $N > 0$ is arbitrary.\footnote{While deciding the general strategy through joint discussions, the coauthors have invested various amounts of time in the actual elaboration. The main authors of the}
Résumé
Motivés par un problème d’approximation à un mode pour une évolution avec accumulation de charge dans des puits de potentiel, nous considérons un problème d’évolution linéaire pour un opérateur de Schrödinger avec un potentiel dépendant du temps avec un puits dans une île. En particulier, nous montrons que nous pouvons choisir le paramètre adiabatique \( \varepsilon \) avec \( \ln \varepsilon \asymp -1/h \), où \( h \) désigne paramètre semi-classique, et obtenir des approximations adiabatiques de solutions exactes sur des intervalles de temps de longueur \( \varepsilon^{-N} \) avec une erreur \( O(\varepsilon^N) \). Ici \( N > 0 \) est arbitraire.

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different sections are indicated by their initials as follows:
Section 1 MH, AM, JS,
Sections 2, 3 AM, JS,
Sections 4, 5, 6, 7 JS,
Section 8 MH, JS,
Section 9 MH, AM, JS.
1 Introduction and main results

Our work is connected with the modelling of the axial transport through resonant tunneling structures like highly doped p-n semiconductor heterojunctions (Esaki diodes), multiple barriers or quantum wells diodes. The scattering of charge carriers in such devices has been described using non-linear Schrödinger-Poisson Hamiltonians with quantum wells in a 1D semiclassical island (see [19]). The quantum wells regime is defined as a perturbation of the semiclassical Laplacian $-\hbar^2 \partial_x^2$ by the superposition of a potential barrier plus an attractive term, with support of size $h$, modelling one or more quantum wells. In the simplest setting of a single well separating two linear barriers, the linear part of the potential has the shape in Figure 1. In connection with the modelling of a mesoscopic semi-conductor device, this scheme represents “metallic conductors” at $[-\infty, a]$ and $[d, +\infty]$ while the double barrier describes the interaction of charge carriers in a semiconductor junction. Here $a < b < d$ are fixed, $c = b + h$ with $h \to 0$, while $V_1$
defines an exterior voltage applied between the two infinite conductors. In this framework, \( h \) corresponds to a rescaled Fermi length fixing the quantum scale of the system (see for instance [5]) and, coherently with the features of the physical model, is assumed to be small. The shape resonances (i.e. those with energies below \( V_0 \)) define the Fermi levels of the junction and the corresponding resonant states describe (in the one-particle approximation) the concentration of charges in the depletion region. In particular, the exterior potential bias \(-V_1\) is introduced in order to select only incoming waves with positive momentum as contributions to the charging process (about this point the reader may refer to the analysis developed in [6]).

In linear models, the small-\( h \) asymptotic behaviour of the shape resonances generated by quantum wells has been understood in the work of B. Helffer and J. Sjöstrand [16]; for operators of the form

\[
H^h := -h^2 \partial_x^2 + V^h
\]

with \( V^h \) fulfilling the scaling of Figure 1 (i.e.: a semiclassical barrier supported on \([a, d]\) plus quantum wells) and suitable regularity assumptions, the approach of [16] allows to localize the shape resonances w.r.t. the spectrum of a corresponding Dirichlet operator

\[
H^h_D := -h^2 \Delta^D_{(a,d)} + 1_{(a,b)} V^h,
\]

where \( \Delta^D_{(a,d)} \) denotes the Dirichlet Laplacian on the barrier interval. In particular, very accurate Agmon-type estimates show that to each \( \lambda \in \sigma_p (H^h_D) \cap (0, V_0) \) corresponds a unique resonance of \( H^h \), \( E^h_{\text{res}} = E^h - i \Gamma^h \), and the estimates

\[
|\lambda - E^h| + \Gamma^h \lesssim e^{-S_0/h},
\]

hold with \( \Gamma^h > 0 \) fixing the imaginary part of \( E^h_{\text{res}} \) and \( S_0 > 0 \) depending on the Lithner-Agmon distance separating the well from the boundary of the barrier. In the time evolution problems, the imaginary part of resonances fixes the lifetime of the corresponding quasiresonant states, which are \( L^2 \)-functions defined by a cut-off of the resonant states outside the interaction region (see the definition in [13]). This general idea has been investigated in [35] in the framework of 3D Schrödinger operators with exponentially decaying potentials (see also [37] and [25]); for operators exhibiting the scaling introduced above, a precise exponential decay estimate has been provided in [13, Th. 4.3]. Let \( u_{E^h_{\text{res}}} \) denote the resonant state of \( H^h \) for the resonance \( E^h_{\text{res}} \) (i.e.: a solution of \((H^h - E^h_{\text{res}}) u_{E^h_{\text{res}}} = 0\)); under the assumption \( \Gamma^h \gtrsim e^{-2S_0/h} \) (which holds for a large class of models including the case of sharp barriers (see [7])) we have

\[
e^{-itH^h} 1_{(a,d)} u_{E^h_{\text{res}}} = e^{-itE^h_{\text{res}}} 1_{(a,d)} u_{E^h_{\text{res}}} + R^h (t),
\]
where $R^h(t) = O(e^{-S_0/h})$, in the $L^2$-norm sense, on the time scale: $t \lesssim 1/\Gamma^h$. Then the estimate (1.3) implies
\[ |e^{-itH^h}1_{(a,d)}u_{E^h_{\text{res}}}| \approx e^{-t\Gamma^h} |1_{(a,d)}u_{E^h_{\text{res}}}|. \] (1.5)

The comparison between the shape resonance problem for $H^h$ and the eigenvalue equation for the corresponding truncated Dirichlet model $H^h_D$ also shows that the quasiresonant states are mainly supported near the wells (see [10]). Hence, according to the above relation, the time evolution preserves this concentration of $L^2$-mass on the time scale $1/\Gamma^h$ which is exponentially large w.r.t. $h$.

In the non-linear modelling, the repulsive effect due to the concentration of charges in the depletion region is taken into account by a Poisson potential term depending on the charge density. The corresponding non-linear steady state problem
\[ (H^h + V^h_{NL} - E) u = 0, \quad \partial_x^2 V^h_{NL} = |u|^2, \] (1.6)
has been investigated in [6]-[7] under far-from-equilibrium assumptions; in this case, following the scaling introduced above, the underlying linear model $H^h$ is defined by using an array of quantum wells of the form
\[ W^h = - \sum_{n=1}^N w_n ((x - x_n) 1/h), \quad w_n \in C^0(\mathbb{R},\mathbb{R}^+), \quad \text{supp } w_n = [-d,d]. \] (1.7)

An accurate microlocal analysis of the tunnel effect as $h \to 0$ then shows that the estimates (1.3) still hold in the stationary nonlinear framework and determine the limit occupation number of resonant states. This analysis leads to a simplified equation for the Poisson problem where the limit charge density is described by a superposition of delta-shaped distributions centered in the points $\{x_n\}$. In the time-dependent case, the non-linear evolution equation reads as
\[ i\partial_t u = (H^h + V^h_{NL}) u, \quad \partial_x^2 V^h_{NL} = |u|^2. \] (1.8)

When the initial state is formed by a superposition of incoming waves with energies close to the Fermi level ($E_F$ the resonant energy), this interacts with resonant states which, as the estimates (1.5) suggest, are expected to evolve in time according to a quasi-stationary dynamics. In this picture, $u$ behave as a metastable state and the charge density $|u|^2$ remains concentrated in a neighbourhood of the wells for a large range of time fixed by the imaginary
part of the (nonlinear) resonances. Depending on the position of the wells, this possibly induces a local charging process; then, the nonlinear coupling in (1.8) generates a positive response (depending on the charge in the wells) which modifies the potential profile and reduces the tunnelling rate.

The above scheme outlines the behaviour of the nonlinear dynamics under non-equilibrium initial conditions and assuming $\hbar$ small. In particular, $V_{NL}^h$ is expected to define an adiabatic process with variations in time of size $\varepsilon = \Gamma^h$. The relevance of adiabatic approximations in the small-$\hbar$ asymptotic analysis of the nonlinear quantum transport was pointed out in ([19], [29], [30]) where this dynamics was considered within a simplified framework.

1.1 The works of C. Presilla and J. Sjöstrand

Following the work [19] (with G. Jona-Lasinio), in [29], [30] C. Presilla and J. Sjöstrand considered a non-linear evolution problem for a mesoscopic semiconductor device and did some heuristic work. It is assumed that the incoming charged particles (entering from the left) have energies $E \geq 0$ distributed according to the density $g(E)dE$ supported on $[0, E_F]$, where $E_F < V_0$ is the Fermi level. Moreover, these particles interact only inside the device (i.e. in the region $[a,d]$ in Figure 1) through a modification of the common potential due to charge accumulation there. After a rescaling, the model is described
by a nonlinear Hamiltonian
\[ H_{NL}^h = H^h + s(u^h(t, \cdot)) W_0(x), \]
where the linear part \( H^h \) is defined as in (1.1) and Figure 1, while the Poisson potential is replaced by an affine function \( W_0(x) \), with a fixed “profile” and support in \([a, d]\), multiplied by the charge accumulated inside the device. This is defined in terms of the nonlinear evolution of generalized eigenfunctions \( u(t, x, E) \) according to
\[
s(u^h(t, \cdot)) := \int \int \frac{(c+d)/2}{(a+b)/2} \left| u^h(t, x, E) \right|^2 g(E) \, dx \, dE, \tag{1.9}\]
and the corresponding nonlinear evolution problem is
\[
\begin{align}
\begin{cases}
    i \partial_t u^h(t, x, E) = (H^h + s(u^h(t, \cdot)) W_0(x)) u^h(t, x, E), \\
    (H^h + s(u^h(0, \cdot)) W_0(x) - E) u^h(0, x, E) = 0.
\end{cases}
\end{align} \tag{1.10}
\]
The heuristic analysis in [29] was based on the 1-mode approximation
\[
u^h(t, x, E) \approx \mu(t, x, E) + e^{-iEt/h} z^h(t, E) e^h(x, s(u^h(t, \cdot))), \tag{1.11}\]
where
\[\cdot\mu(t, x, E) \text{ is the solution of a linear evolution problem, obtained by “filling” the potential well } [b, c], \]
\[e^h = e^h(x, s(u^h(t, \cdot))) \text{ is a resonant state } (\notin L^2) \text{ corresponding to a resonance } \lambda^h(s(u^h(t, \cdot))) \text{ in the lower half-plane.} \]
From this the authors derived a simpler evolution equation
\[
h \partial_t z^h(t, E) = i(E - \lambda^h(s)) z^h(t, E) + B^h(t, s, E), \tag{1.12}\]
where \( B^h(t, x, E) \) is a “driving term” derived from \( \mu \) and \( e^h \). Then, using an adiabatic approximation for the evolution of the nonlinear resonant state in term of instantaneous resonances and WKB expansions, an even further simplified differential equation for \( s(u^h(t, \cdot)) \) was obtained in the limit \( h \to 0 \) (eq. 9.7 in [29]). From this one could describe fixed points of the vector field in (1.12), and hysteresis phenomena under slow variations of the exterior bias \( V_1 = V_1(t) \).
1.2 Adiabatic evolution of resonant states

A rigorous study of the model (1.8) in the small $h$ regime is a very vast program. In this connection, we remark that the lack of an error bound in the adiabatic formulas used in [29] prevents to control the possible remainder terms in the asymptotic limit. Hence, a strong adiabatic theorem for resonant states, with adiabatic parameter $\varepsilon$ satisfying

$$\ln \varepsilon \asymp -1/h,$$

seems to be a key point of this program. The adiabatic problem for resonances can be considered following different approaches. One consists in using the unitary propagator associated to the physical (selfadjoint) Hamiltonian and study the adiabatic evolution of $L^2$ states spectrally localized near the resonant energy (or $L^2$ functions obtained by truncating resonant states). This point of view was adopted in [28] where the condition (1.13), connecting the adiabatic parameter to the resonance lifetime, was also taken into account; in the case of a single time-dependent resonance $E_{\text{res}}(t)$ with:

$$\text{Im} E_{\text{res}}(t) \sim \text{Im} E_{\text{res}}(0) = e^{-\frac{c}{h}}$$

an adiabatic formula was obtained on the specific time range $(\text{Im} E_{\text{res}}(0))^{-1}$.

A different approach consists in using a complex deformation to define resonances as eigenvalues of a deformed (non-selfadjoint) Hamiltonian; in this framework, the resonant states identify with $L^2$ eigenvectors and the corresponding evolution problem is naturally formulated in terms of the deformed operator. Then, an adiabatic approximation can be studied by adapting the standard adiabatic theorem with gap condition to the non-selfadjoint case. (a similar strategy was implemented in [1]). It is worthwhile to remark that this requires uniform-in-time bounds for the deformed dynamical system (we refer to [27]): the lack of this condition, due to the complex deformation, is the main obstruction to implement such an approach in some relevant physical models (including those we are considering here). In [20], the time-adiabatic evolution in Banach spaces has been considered, under a fixed gap condition, for semigroups exhibiting an exponential growth in time. In this framework, which could be adapted to the case of resonant states, the exponential growth of the dynamical system is compensated by the small error of the adiabatic approximation under analyticity-in-time assumptions and an adiabatic formula for the evolution of spectral projectors is provided with an error which is small on a suitable range of time.

More recently, an alternative approach to the adiabatic evolution of shape resonances has been proposed [10]. For a 1D Schrödinger operator describing the regime of quantum well in a semiclassical island, artificial (non-selfadjoint) interface conditions are added at the boundary of the potential’s
support. Depending on the deformation parameter \( \theta \), these may be chosen in such a way that the corresponding modified and complex deformed operator is maximal accretive: hence, the deformed dynamical system allows uniform-in-time estimates and the adiabatic theory for isolated spectral sets can be developed in the deformed setting. In particular, the authors show that artificial interface conditions introduce small perturbations on shape resonances, preserving the relevant physical quantities (the exponentially small scales). Using an exterior complex deformation polynomially small w.r.t. the quantum scale \( h \), an adiabatic theorem for the resonant states associated to shape resonances is provided. In this framework, the adiabatic parameter \( \varepsilon = e^{-c/h} \) can be fixed with any \( c > 0 \) independently of the resonance lifetime (see [10, Th. 7.1] for the precise statement), while the loss due to the small spectral gap (induced by the \( h \)-dependent deformation) is compensated by the small error of the adiabatic expansion which is now given by \( \varepsilon^{1-\delta} \) where \( \delta \in (0, 1) \).

We study the adiabatic evolution problem for resonant states in connection with models of mesoscopic transport. Our aim is to avoid the unphysical modification of the selfadjoint operator introduced in [10]. Our approach consists in using adapted Hilbert spaces that contain the relevant resonant states, with the goal of having an adiabatic approximation to all orders in \( \varepsilon \), over time intervals of length \( \varepsilon^{-N} \) for any fixed \( N \geq 0 \). In our framework, the evolution is no longer unitary and our first result says that we can arrange so that the generator of our evolution has an imaginary part which is \( \leq \varepsilon^N \) for any \( N \). A second result (for the moment limited to the case of one space dimension), gives appropriate control on the resolvent in the same spaces, and we get adiabatic approximation over long time intervals for exact solutions of linear adiabatic evolution equations. (The multidimensional case will be attacked in a future work.)

1.3 Aims and ideas

The aim of the present paper is to establish asymptotic approximations for solutions of adiabatic evolution equations of the form

\[
(\varepsilon D_t + P(t))u(t) = 0
\]

that are valid over time intervals of length \( \varepsilon^{-N} \), with errors \( O(\varepsilon^N) \) for arbitrary \( N > 0 \). Here \( P(t) = -\hbar^2 \Delta + V(t, x) \) is a self-adjoint Schrödinger operator with a single well in a potential island which is assumed to generate a shape resonance with real energy \( E = E(t) \). Typically, \( \varepsilon \) should be comparable to the tunneling relaxation time for \( P(t) \) on a logarithmic scale. More precisely, we should have \( \ln(1/\varepsilon) \asymp 1/\hbar \). The approximations should be of
the form
\[ u_{\text{ad}}(t) \sim (\nu_0(t) + \varepsilon \nu_1(t) + ...) e^{-\frac{i}{\hbar} \int \lambda(x, \varepsilon) \, ds} \]  
(1.15)
where \( \nu_j(t) \) are well-behaved smooth functions of \( t \) with values in the (by assumption) common domain of the \( P(t) \), and
\[ \lambda(t, \varepsilon) \sim \lambda_0(t) + \varepsilon \lambda_1(t) + ..., \]  
(1.16)
where \( \lambda_0(t) \) is a shape resonance of \( P(t) \), satisfying
\[ 0 \leq -\Im \lambda_0(t) \leq e^{-2(1+o(1)) S(t)/\hbar}, \]  
(1.17)
and \( \nu_0(t) \) is a corresponding resonant state; \( (P(t) - \lambda_0(t))\nu_0(t) = 0 \). Here \( S(t) > 0 \) is the Lithner-Agmon distance for \( P(t) - E(t) \) from the potential well to the sea surrounding the potential island.

Since the non-linearity is concentrated to a bounded region, we can choose an ambient Hilbert function space \( \mathcal{H} \) quite freely such that the space of restrictions of its elements to some neighborhood \( \Omega \) of the island is equal to \( L^2(\Omega) \).

One such choice is \( \mathcal{H} = L^2(\mathbb{R}^n) \). A nice feature with this choice is that the evolution (1.14) is norm preserving: \( \| u(t) \|_{\mathcal{H}} = \| u(s) \|_{\mathcal{H}} \). A difficulty with this choice is that the resonant state \( \nu_0(t) \) does not belong to \( L^2(\mathbb{R}^n) \) (and \( \lambda_0(t) \) does not belong to the \( L^2 \)-spectrum of \( P(t) \)), so the adiabatic expansion (1.15) can hold only locally in \( \mathbb{R}^n \).

Rather, we choose \( \mathcal{H} \) to be a Hilbert space that contains the resonant state \( \nu_0(t) \) and such that \( \lambda_0(t) \) belongs to the \( \mathcal{H} \)-spectrum of \( P(t) \). When replacing \( L^2 \) with some other Hilbert space we lose (in general) the self-adjointness of the operators \( P(t) \) and the corresponding unitarity of the evolution (1.14).

The original non-linear problem is not time reversible, so we only wish to have a good control of the solutions in the forward time direction. If we can choose \( \mathcal{H} \), depending on \( \varepsilon \) but not on \( t \), so that
\[ \Im P(t) \leq \tau(\varepsilon), \]  
(1.18)
for \( P(t) \) as an unbounded operator \( \mathcal{H} \rightarrow \mathcal{H} \), where \( \tau(\varepsilon) \geq 0 \), then if \( u(t) \) solves (1.14) for \( t \) in some interval, we would have
\[ \| u(t) \|_{\mathcal{H}} \leq e^{\tau(\varepsilon)(t-s)} \| u(s) \|_{\mathcal{H}}, \]  
for \( t \geq s \),  
(1.19)
so the solution will grow at most exponentially with rate \( \tau(\varepsilon) \) in the direction of increasing time. Correspondingly, we can expect well-posedness for solutions of the initial value problem,
\[ \begin{cases} 
(\varepsilon D_t + P(t))u(t) = 0, & 0 \leq t \leq T, \\
u(0) = u_0, 
\end{cases} \]  
(1.20)
assuming, to fix the ideas, that $P(t)$ is defined for $t \in [0, T], T > 0$. Then

$$\|u(t)\|_\mathcal{H} \leq e^{\tau(\varepsilon)T} \|u_0\|_\mathcal{H}$$

and in order to avoid exponential growth of the upper bound we require

$$\tau(\varepsilon)T \leq \mathcal{O}(1).$$

With $T = \varepsilon^{-N_0}$ for some fixed $N_0 > 0$ we then need $\tau(\varepsilon) \leq \mathcal{O}(\varepsilon^{N_0})$. Assume that we can perform the adiabatic constructions in (1.15)–(1.17) with $\exists \lambda(t, \varepsilon) \leq \varepsilon^{N+1}, N \geq N_0$, and that we have (1.18) with $\tau(\varepsilon) \leq \varepsilon^{N+1}$ for $0 \leq t \leq T, T \leq \varepsilon^{-N_0}$. Then taking a suitable realization of $u_{ad}$ we expect to have

$$(\varepsilon D_t + P(t))u_{ad} = \mathcal{O}(\varepsilon^{N+1})$$

in $\mathcal{H}$, $\|u(t)\|_\mathcal{H} = \mathcal{O}(1)$, for $0 \leq t \leq T$ and using (1.19) we would be able to solve

$$(\varepsilon D_t + P(t))v = -(\varepsilon D_t + P(t))u_{ad}, \quad \|v(0)\|_\mathcal{H} = 0,$$

with $\|v\| = \mathcal{O}(\varepsilon^{N-N_0})$. Then $u(t) = u_{ad}(t) + v(t)$ is an exact solution of $(\varepsilon D_t + P(t))u = 0$ on $[0, T]$ with $u - u_{ad} = \mathcal{O}(\varepsilon^{N-N_0})$ in $\mathcal{H}$.

The easiest choice of $\mathcal{H}$, at first sight, would be to follow the method of exterior complex distortions \[2, 3, 32, 17\] in the spirit of \[35\] so that $\mathcal{H} = L^2(\Gamma)$, where $\Gamma \subset \mathbb{C}^n$ is a totally real manifold of real dimension $n$, obtained as a deformation of $\mathbb{R}^n$, coinciding with $\mathbb{R}^n$ along the island. We did not succeed with this particular choice however (see further comments below). A. Faraj, A. Mantile and F. Nier [10] followed this path. They defined an operator $\tilde{P}(t)$ from $P(t)$ living on a distorted contour having an artificial interface condition between real part of the contour near the island and the complex distorted part. This way the discrete spectrum of $P(t)$ needs no longer to consist of resonances of $P(t)$, so the exact link with the original evolution problem is disrupted.

In this paper we use the spaces of \[16\]. Such spaces $\mathcal{H} = H(\Lambda_{vG}), 0 \leq v \ll 1$ are defined with the help of a suitable real symbol $G(x, \xi)$ vanishing for large $|\xi|$, and very roughly $H(\Lambda_{vG})$ is the space of functions $u(x)$ on $\mathbb{R}^n$, such that $\bar{u}(x, \xi)e^{-\varepsilon G(x, \xi)/\hbar} \in L^2(\mathbb{R}^{2n})$, where $\bar{u}$ denotes a suitable FBI transform. The associated “1-Lagrangian” manifold $\Lambda_{vG}$ is given by $\mathcal{H}(x, \xi) = \nu H_G(\Re(x, \xi))$, where $H_G = \partial_x G \cdot \partial_x - \partial_x G \cdot \partial_\xi \simeq (\partial_\xi G, -\partial_\xi G)$ is the Hamilton field of $G$. $\Lambda_{vG}$ is then the natural classical phase space associated with $H(\Lambda_{vG})$. Thus if we consider a semi-classical Schrödinger operator $P = -h^2 \Delta + V(x)$ with leading symbol $p(x, \xi) = \xi^2 + V(x)$ where $V$ extends far enough in the complex domain, the natural leading symbol of the unbounded operator $P : \mathcal{H} \rightarrow \mathcal{H}$ is $p|_{\Lambda_{vG}}$. Here, by Taylor expansion we have

$$p(x, \xi) = p(\Re(x, \xi)) + i\nu H_G(p)(\Re(x, \xi)) + \mathcal{O}(v^2) = p(\Re(x, \xi)) - i\nu H_p(G)(\Re(x, \xi)) + \mathcal{O}(v^2),$$

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locally uniformly (and also globally after putting the right order function into the remainder). If $H_p G \geq 0$, then (at least up to $O(v^2)$) we have $\Im p|_{\Lambda \psi G} \leq 0$ and this is a first step towards having (1.18) for $P$. In order to define a resonance $\lambda_0$ for $P$ we need to choose $G$ and $\psi$ so that near infinity $\Im p|_{\Lambda \psi G} < \Im \lambda_0$ when $\Re p|_{\Lambda \psi G}$ belongs to a neighborhood of $\Re \lambda_0$. This can be obtained by choosing $G$ to be an escape function, meaning roughly that $H_p G > 1/O(1)$ near infinity on $p^{-1}(E_0)$ where $E_0$ is some fixed real energy and we assume that $\Re \lambda_0 \approx E_0$.

The method of (small) contour distortions follows this scheme with the symbol $G(x, \xi)$ chosen to be linear or possibly affine linear in $\xi$. With such restrictions it is harder (maybe impossible) to find $G$ so that $H_p G \geq 0$ everywhere and $H_p G > 0$ near infinity on $p^{-1}(E_0)$.

For the construction of formal adiabatic solutions we will also need a good control over $(P(t) - z)^{-1}$ for $z$ on some small closed contour enclosing $\lambda_0(t)$.

1.4 The main results

The results of this paper concern the linear adiabatic theory for time dependent potentials with a well in an island, in a fairly general setting. We hope that they will be useful for non-linear problems of the type described above and also that they are interesting in their own right. We study

1) Semi-boundedness as in (1.18)

2) Resolvent estimates in $H(\Lambda \psi G)$-spaces

3) Adiabatic approximations over long time intervals.

Here 3) will be a fairly direct consequence of 1) and 2), using general arguments from adiabatic constructions, that we shall review in Section 2, see also Section 3.

In Sections 4 5 we review some of the theory in [16]. Let $r(x), R(x)$ be positive smooth functions on $\mathbb{R}^n$ satisfying (4.1): $r \geq 1$, $rR \geq 1$.

Define $\tilde{r}(x, \xi) = (r(x)^2 + \xi^2)^{1/2}$ as in (4.2) and the symbol spaces $S(m)$ as in (4.3). We assume (4.4):

$$m \in S(m), \ r \in S(r), \ R \in S(R).$$

Let $1 \leq m_0 \in S(m_0)$. We consider the formally self-adjoint semi-classical differential operator in (4.6):

$$P = \sum_{|\alpha| \leq N_0} a_\alpha(x; h)(hD)^\alpha, \ a_\alpha \in S(m_0r^{-|\alpha|}),$$
where \( a_\alpha \) is a finite sum in powers of \( h \) as in (4.7) with leading term \( a_{\alpha,0}(x) \). The full symbol of \( P \) for the standard left quantization will also be denoted by \( P \) (see (4.8)) and the semi-classical principal symbol will be denoted by \( p(x,\xi) \) (4.13):

\[
p(x,\xi) = \sum_{|\alpha| \leq N_0} a_{\alpha,0}(x)\xi^\alpha \in S(m)
\]

where \( m = m_0(x)(\tilde{r}(x,\xi)/r(x))^{N_0} \). We also have \( P(x,\xi;h) \in S(m) \) (uniformly with respect to \( h \)). We make the ellipticity assumption (4.14), where \( p_{\text{class}} \) is the classical (PDE) principal symbol in (4.12). Then for every fixed real \( E_0 \) the energy surface \( \Sigma_{E_0} = p^{-1}(E_0) \) has the property that

\[
|\xi| \leq \mathcal{O}(r(x)), \text{ for } (x,\xi) \in \Sigma_{E_0}.
\]

We say (see Definition 4.1) that the real-valued function \( G \in S(\tilde{r}R) \) is an escape function if

\[
H_p G(\rho) \geq \frac{m(\rho)}{\mathcal{O}(1)} \text{ on } \Sigma_{E_0} \setminus K,
\]

for some compact set \( K \). We make the technical assumption (4.22) (there stated with \( E_0 \) replaced with 0, a reduction obtained by replacing \( p \) with \( p - E_0 \)):

For every \( r_0 > 0 \), there exists \( \epsilon_0 > 0 \), such that

\[
|p - E_0| \geq \epsilon_0 m \text{ on } \mathbb{R}^{2n} \setminus \bigcup_{\rho \in \Sigma_{E_0}} B_{g(\rho)}(\rho, r_0).
\]

Here \( g \) is the natural metric associated to the scales \( \tilde{r}, R \), see (4.5).

Proposition 4.2 (where again we took the case \( E_0 = 0 \)) states that if we have an escape function for a given energy \( E_0 \), then we can modify it on a bounded set to get an escape function \( G \) which vanishes on any given compact set, such that \( H_p G \geq 0 \).

The main example we have in mind is that of the Schrödinger operator

\[
P = -h^2 \Delta + V(x),
\]

where \( V \) is real-valued and

\[
\partial_x^\alpha V = o(\langle x \rangle^{-|\alpha|}), \text{ } |x| \to \infty.
\]

Then we take \( r(x) = 1, R(x) = \langle x \rangle, m = \xi^2 \) and \( P(x,\xi;h) = p(x,\xi) = \xi^2 + V(x) \). When \( E_0 > 0 \), we have the escape function \( x \cdot \xi \) and after multiplication with a cutoff \( \chi(p(x,\xi) - E_0) \) we can also assume that \( G \) has compact support in \( \xi \). In this case (1.22) holds automatically.
Let $G = G(x, \xi)$ be real-valued and sufficiently small in $S(\tilde{r}R)$. Let $\Lambda_G$ be the corresponding I-Lagrangian manifold $\mathcal{I}(x, \xi) = H_G(\mathcal{R}(x, \xi))$, given in (5.1), which is also symplectic for $\Re \sigma$, where $\sigma = \sum d\xi_j \wedge dx_j$ is the complex symplectic form on $\mathbb{C}^{2n}$. We assume that $|\xi| \leq O(r(x))$ on the support of $G$ and define the weight function $H$ on $\Lambda_G$ by (5.2). It is also of class $S(\tilde{r}R)$ when using the natural parametrization of $\Lambda_G$ in (5.1).

Let $T$ be an FBI-transform defined as in (5.3)–(5.6) so that $T : \mathcal{E}'(\mathbb{R}^n) \to C^\infty(\Lambda_G; \mathbb{C}^{n+1})$. If $m$ is an order function on $\Lambda_G$ ($m \in S(m)$), we define the Hilbert spaces $H(\Lambda_G, m)$ as in Definition 5.2 and put $H(\Lambda_G) = H(\Lambda_G, 1)$. When $G = 0$ this gives $L^2(\mathbb{R}^n)$ with equivalence of norms. In Section 5.4 we review pseudodifferential operators, Fourier integral operators and Toeplitz operators, acting on these spaces.

Let $r, R, m(x, \xi) = m_0(x)(\tilde{r}(x, \xi)/r(x))^{N_0}$ be as above and let $P$ be a formally self-adjoint $h$-differential operator as in (4.6), (4.7), fulfilling (4.14) as well as the technical assumption (4.22). We also make the exterior analyticity assumption (4.11). If $G \in S(\tilde{r}R)$ with $|\xi| \leq O(r(x))$ on supp $G$, then (cf. (5.36)) we can view $P$ as a bounded operator

$$P : H(\Lambda_{vG}, m) \to H(\Lambda_{vG}),$$

for $0 \leq v \ll 1$, provided that the coefficients $a_{a,k}$ of $P$ are analytic in a neighborhood of the $x$-space projection of supp $G$. In Section 6 we prove a first semiboundedness result:

**Theorem 1.1** Under the above assumptions, assume in addition that $P$ has an escape function $G_0$ at energy $E_0 \in \mathbb{R}$. Let $K \subset \mathbb{R}^n$ be compact, containing the analytic singular support of $P$, i.e. the smallest closed set $\tilde{K}$ such that the coefficients of $P$ (more precisely all the $a_{a,k}$ in (4.7)) are analytic in $\mathbb{R}^n \setminus \tilde{K}$. Then we can find an escape function at energy $E_0$;

$$G(x, \xi; h) \sim G^0 + hG^1 + \ldots \text{ in } S(\tilde{r}R),$$

supported in $|\xi| \leq O(r(x))$, where $G^0 = G_0$ near infinity on $\Sigma_{E_0}$, $\pi_x \text{supp } G^j$, $\pi_x \text{supp } G^j$ are disjoint from a fixed neighborhood of $K$, such that for $P$ as a closed unbounded operator: $H(\Lambda_{vG}) \to H(\Lambda_{vG})$, we have

$$\mathbb{S}(P u | u)_{H(\Lambda_{vG})} \leq v O(h^\infty) \| u \|^2_{H(\Lambda_{vG}, m^{1/2})}, \quad (1.25)$$

for $v \geq 0$ and $h > 0$ small enough. In the Schrödinger case ($m = \langle \xi \rangle^2$), we can replace $H(\Lambda_{vG}, m^{1/2})$ with $H(\Lambda_{vG})$.

As we shall see, we can arrange so that

$$\| u \|_{H(\Lambda_{vG})} = \| u \|_{L^2}, \text{ when } u \in L^2_{\text{comp}}(K).$$

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The half-estimate (1.25) is of independent interest, when compared with various versions of Gårding’s inequality, but for adiabatic theory in connection with a potential well in an island, we would like to replace $O(h^N)$ for every $N > 0$ with $O(e^{-N/h})$ for every $N > 0$. This improvement will be obtained with a scaling argument.

Keeping the above assumptions, we also assume (7.1), (7.2):

$$r = 1, \quad R(x) = \langle x \rangle, \quad m_0(x) = 1, \quad m = \langle \xi \rangle^{N_0},$$

as well as (7.3) which states that $p(x, \xi)$ converges to a limiting polynomial $p_{\infty}(\xi)$ when $x \rightarrow \infty$ in the natural sense for the semi-norms of $S(m)$. The main fact that we exploit is now that in a region where $x = \mu \tilde{x}$, $|\tilde{x}| \approx 1$, $P(x,hD_x;h)$ can be viewed as an $\tilde{h}$-differential operator $P(\mu \tilde{x}, \tilde{h}D_{\tilde{x}};h)$ with $\tilde{h} = h/\mu$. Combination of this observation with Theorem 1.1 leads to (cf. (7.30)):

**Theorem 1.2** We make the assumptions of Theorem 1.1 and the two additional assumptions above. Let $\pi_x : \mathbb{R}^n \times \mathbb{R}^n \ni (x,\xi) \mapsto x \in \mathbb{R}^n$. Then uniformly for $\mu \in [1, +\infty[$, we can find an escape function

$$G(x,\xi,\mu;h) \sim G^0 + hG^1 + ... \quad \text{in} \quad S(\tilde{r}R),$$

with support in $\pi_x^{-1}((\mathbb{R}^n \setminus B(0,\mu)) \cap (\mathbb{R}^n_x \times B(0,r_0)))$ for some fixed $r_0 > 0$ and with $G^0 = G_0$ on $\Sigma_{E_0} \cap \pi_x^{-1}(\mathbb{R}^n \setminus B(0,2\mu))$, independent of $\mu$ for $|x| \geq 2\mu$, such that for $P$ as an unbounded closed operator $H(\Lambda_{\nu G}) \rightarrow H(\Lambda_{\nu G})$, we have

$$\exists (Pu|u)_{H(\Lambda_{\nu G})} \leq vO((h/\mu)^\infty)\|u\|^2_{H(\Lambda_{\nu G},m^{1/2})}, \quad 0 \leq v \ll 1. \quad (1.26)$$

In the Schrödinger operator case, we can replace $H(\Lambda_{\nu G}, m^{1/2})$ in (1.26) with $H(\Lambda_{\nu G})$.

In this result we use a decoupling property which can be obtained with a suitable choice of norm in $H(\Lambda_{\nu G})$, namely that

$$\|u\|_{H(\Lambda_{\nu G},1)} = \|u\|_{L^2}, \quad \text{when} \quad \text{supp} \quad u \subset B(0,\mu/2).$$

We next consider resolvent estimates. For simplicity we assume right away that $P$ is a semi-classical Schrödinger operator as in (1.23), (1.24). We will also assume that we are in the 1-dimensional case; $n = 1$, even though we now think that the higher dimensional case is within reach. With the higher dimensional case in mind we will formulate certain statements as if we were in that general case, even though (for the moment) $n = 1$. In order to fit with (1.24), we take

$$r(x) = 1, \quad R(x) = \langle x \rangle, \quad m = \langle \xi \rangle^2,$$
and note that $P(x,\xi;h) = p(x,\xi) = \xi^2 + V(x)$. We keep the exterior analyticity assumption (4.11) which takes the form (8.3). Let $E > 0$. Let us first consider the non-trapping case (cf. Proposition 8.3).

**Theorem 1.3** Assume that the $H_p$-flow on $\Sigma_E = p^{-1}(E)$ is non-trapping (in the sense that no maximal $H_p$ trajectory in $p^{-1}(E)$ is contained in a compact set). Let $E > 0$. Let us first consider the non-trapping case (cf. Proposition 8.3).

Let $G$ be as in Theorem 1.2, where we choose $\mu = h/\epsilon$ where $0 < \epsilon \ll h$ is a small parameter. Let $\vartheta > 0$ be small and fix $\upsilon > 0$ sufficiently small. If $\delta_0 > 0$, $C > 0$ are respectively small and large enough, then for $z$ in the range (8.45):

$$\Re z \in [E - \delta_0/2, E + \delta_0/2], \quad -\epsilon_\vartheta/C \leq \Im z \leq 1/C,$$

we have that $P - z : H(\Lambda_{\upsilon \epsilon G}, \langle \xi \rangle^2) \to H(\Lambda_{\upsilon \epsilon G})$ is bijective and

$$m_\epsilon(x;h)^{\frac{1}{2}}(z-P)^{-1}m_\epsilon(x;h)^{\frac{1}{2}} = \mathcal{O}(1) : H(\Lambda_{\upsilon \epsilon G}) \to H(\Lambda_{\upsilon \epsilon G}, \langle \xi \rangle^2).$$

Here we have put

$$m_\epsilon(x,\xi) := \frac{h}{\langle x \rangle^{1+\vartheta}} + \epsilon_\vartheta, \quad \epsilon_\vartheta = \left(\frac{\epsilon}{h}\right)^\vartheta \epsilon.$$

We next consider a trapping case, namely that of a potential well in an island. Let $E > 0$ and keep the assumptions above, except the one about non-trapping. Let $\bar{O} \subset \mathbb{R}^n$ be open (still with $n = 1$) and let $U_0 \subset \bar{O}$ be a compact subset. Assume (8.51), (8.52):

$$V - E < 0 \text{ in } \mathbb{R}^n \setminus \bar{O}, \quad V - E > 0 \text{ in } \bar{O} \setminus U_0, \quad V - E \leq 0 \text{ in } U_0,$$

$$\text{diam}_d U_0 = 0,$$

where $d$ is the Lighthill-Agmon distance associated to the metric

$$(V - E)_+ dx^2, \quad (V - E)_+ = \max(V - E, 0).$$

Assume (8.53):

The $H_p$-flow has no trapped trajectories in $p^{-1}(E)|_{\mathbb{R}^n \setminus \bar{O}}$.

Let $M_0 \subset \bar{O}$ be a connected compact set with smooth boundary (i.e. a compact interval in the present 1D case) such that (8.54) holds:

$$M_0 \supset \{ x \in \bar{O}; d(x, \partial \bar{O}) \geq \epsilon_0 \},$$

for some small $\epsilon_0 > 0$. Let $P_0$ denote the Dirichlet realization of $P$ on $M_0$. 

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Let $J(h) \subset \mathbb{R}$ be an interval tending to $\{E\}$ as $h \to 0$. Assume (8.67):

$$P_0 \text{ has no spectrum in } \partial J(h) + [-\delta(h), \delta(h)],$$

where the parameter $\delta(h)$ is small but not exponentially small;

$$\ln \delta(h) \geq -o(1)/h.$$  

$\sigma(P_0) \cap J(h)$ is a discrete set of the form $\{\mu_1(h), ..., \mu_m(h)\}$ where $m = m(h) = O(h^{-n})$ and we repeat the eigenvalues according to their multiplicity. Let $\Gamma(h)$ denote the set of resonances of $P$ in $J(h) - i[0, \epsilon_\vartheta/C]$, $C \gg 1$, also repeated according to their (algebraic) multiplicity. Assume (8.68):

$$\epsilon \geq e^{-1/(Ch)},$$

where $S_0 > 0$ denotes the Lithner-Agmon distance from $U_0$ to $\partial \tilde{O}$. In Proposition 8.6 we recall a result from [16] when $V$ is analytic everywhere and due to Fujiié, Lahmar-Benbernou, Martinez [11], for potentials that are merely smooth on a bounded set, stating that there is a bijection $b: \{\mu_1, ..., \mu_m\} \to \Gamma(h)$ such that

$$b(\mu) - \mu = \tilde{O}(e^{-2S_0/h}) := O(e^{\omega-2S_0/h}), \ \omega = \omega(\epsilon_0) \to 0, \ \epsilon_0 \to 0.$$  

We give a proof in Section 8.

**Theorem 1.4** For $C \gg 1$ sufficiently large, let

$$z \in \{z \in J(h) + i] - \epsilon_\vartheta/C, 1/C] ; \ \text{dist} (z, \sigma(P_0) \cap J(h)) = \text{dist} (z, \sigma(P_0))\}.$$  

(1.27)

Assume either that $m$ (the number of elements in $\Gamma(h)$) is equal to 1, or that $	ext{dist} (z, \sigma(P_0)) \geq \tilde{O}(e^{-2S_0/h})$. Then we have (8.106):

$$(z - P)^{-1} = O(h/\delta) + O(1) + O(h/\text{dist} (z, \Gamma)) : m^{\frac{1}{2}}\mathcal{H}_{\text{sbd}} \to m^{-\frac{1}{2}}\mathcal{D}_{\text{sbd}},$$

where the first two terms are holomorphic in the interior of the set (1.27). Here

$$\mathcal{H}_{\text{sbd}} = H(\Lambda_{\text{rotG}}), \ \mathcal{D}_{\text{sbd}} = H(\Lambda_{\text{rotG}}, \langle \xi \rangle^2)$$

and $\nu, G$ are as in Theorem 1.2.

We next turn to adiabatic expansions. Let $I \subset \mathbb{R}$ be an interval and let

$$V_t = V(t, x) \in C^{\infty}_b (I \times \mathbb{R}^n; \mathbb{R}).$$  

(1.28)
Here $C^\infty_b(\Omega)$ denotes the space of smooth functions on $\Omega$ that are bounded with all their derivatives. We assume (cf. (8.3))

$$V_t \text{ has a holomorphic extension (also denoted } V_t \text{) to }$$

$$\{ x \in \mathbb{C}^n; |\Re x| > C, |\Im x| < |\Re x|/C \}$$

such that $V_t(x) = o(1)$, $x \to \infty$. 

(1.29)

$$\partial_t V_t(x) = 0 \text{ for } |x| \geq C, \text{ for some constant } C > 0$$

(1.30)

It is tacitly assumed that $V(t, x)$ does not depend on $h$. However, when considering a narrow potential well in an island, of diameter $\propto h$, we will have to make an exception and allow such an $h$-dependence in a small neighborhood of the well.

Let $0 < E_- < E'_- < E'_+ < E_+ < \infty$ and let

$$E_0(t) \in C^\infty_b(I; [E'_-, E'_+]).$$

(1.31)

We assume that $V_t - E_0(t)$ has a potential well in an island as above.

Let $\bar{O} = \bar{O}(t) \subset \mathbb{R}^n$ be a connected open set and let $U_0(t) \subset \bar{O}(t)$ be compact. Assume (cf. (8.51)), still with $n = 1$,

$$V_t - E_0(t) \begin{cases} < 0 \text{ in } \mathbb{R}^n \setminus \bar{O}(t), \\ > 0 \text{ in } \bar{O}(t) \setminus U_0(t), \\ \leq 0 \text{ in } U_0(t), \end{cases}$$

(1.32)

$$\text{diam}_{d_t}(U_0(t)) = 0.$$ 

(1.33)

Here $d_t$ is the Lithner-Agmon distance on $\bar{O}(t)$, given by the metric $(V_t - E_0(t))_+ dx^2$.

Also assume that with $p_t = \xi^2 + V_t(x)$,

$$\text{the } H_{p_t}\text{-flow has no trapped trajectories in } p_t^{-1}(E_0(t))_{\mathbb{R}^n \setminus \bar{O}(t)}.$$ 

(1.34)

It follows that

$$d_x V_t \neq 0 \text{ on } \partial \bar{O}(t),$$

(1.35)

so $\partial \bar{O}(t)$ is smooth and depends smoothly on $t$. Thus $\bar{O}(t)$ is a manifold with smooth boundary, depending smoothly on $t$. Further, $U_0(t)$ depends continuously on $t$.

For $\epsilon_0 > 0$ small, we define

$$M_0(t) = \{ x \in \bar{O}(t); d_t(x, \partial \bar{O}(t)) \geq \epsilon_0 \},$$

(1.36)

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so $M_0(t) \Subset \tilde{O}(t)$ is a compact set with smooth boundary, depending smoothly on $t$. (Here we use the structure of $d_t(x, \tilde{O}(t))$ that follows from [135], see Sections 9, 10 of [16].) More precisely, for every fixed $t$ (consequently suppressed from the notation) the function $\phi(x) = d(x, \partial \tilde{O})$ in $\tilde{O}$ is analytic in the interior, continuous up to the boundary, where it vanishes, and solves the eikonal equation $|\nabla \phi|^2 = V(x)$. Over a neighborhood of the boundary, we have the Lagrangian manifold $\Lambda$, defined as the flow out of the eikonal equation $|\nabla \phi|^2 = V(x)$. The manifold $\Lambda$ has a simple fold over the boundary and $\Lambda_\phi : \xi = \phi(x)$ describes “one of the two covering halves” of $\Lambda$. It is quite well known then that if we choose analytic local coordinates $y = (y_1, \ldots, y_{n-1}, V) = (y', V)$ near a boundary point, then $\phi$ is an analytic function of $y', V^{1/2}$ and has a convergent expansion $\phi = a_3(y)V^{3/2} + a_4(y)V^2 + \cdots$ with $a_k$ analytic and $a_3 > 0$.

Since we would like to allow $I$ in (1.28) to be a very long interval, we introduce the following uniformity assumption:

$$(V_t, E_0(t)) \in \mathcal{K}, \forall t \in I,$$ where $\mathcal{K}$ is a compact subset of

$${V} \subset C_0^\infty (\mathbb{R}^n; \mathbb{R})\quad V \text{ satisfies (1.29) with a fixed constant } C \times [E'_-, E'_+]$$

such that $(V, E)$ satisfies the assumptions (1.30), with a fixed $C$ as well as (1.32), (1.33), (1.34).

Let $P_0(t)$ denote the Dirichlet realization of $P(t) = -h^2 \Delta + V_t(x)$ on $M_0(t)$. If we enumerate the eigenvalues of $P_0(t)$ in $]E_-, E_+[ in increasing order (repeated with multiplicities) we know (as a general fact for 1-parameter families of self-adjoint operators), that they are uniformly Lipschitz functions of $t$. Let $\mu_0(t) = \mu_0(t; h)$ be such an eigenvalue and assume,

$$\mu_0(t; h) = E_0(t) + o(1), \quad h \to 0, \text{ uniformly in } t.$$ (1.38)

$\mu_0(t; h)$ is a simple eigenvalue and

$$\sigma(P_0(t)) \cap [E_0(t) - \delta(h), E_0(t) + \delta(h)] = \{\mu_0(t; h)\}.$$ (1.39)

Here, as above, $\delta(h) > 0$ is small but not exponentially small,

$$\ln \delta(h) \geq -o(1)/h, \quad h \to 0.$$ (1.40)

In addition to (8.68), we assume (9.57), so we have (9.114):

$$e^{-1/(C_1 h)} \leq \epsilon \leq \min(h/C_2, \delta), \quad C_1, C_2 \gg 1.$$ (1.41)

Let $\lambda_0(t)$ be the unique resonance of $P(t)$ in $D(\mu_0(t), \delta(h))$ (the open disc in $\mathbb{C}$ with center $\mu_0(t)$ and radius $\delta(h)$), so that $\lambda_0(t) - \mu_0(t) = \tilde{O}(e^{-2S(t)/h})$. 19
As we shall see in (9.60):

\[ \partial_t^k \lambda_0(t) = O(\delta(h)^{-k}), \ k \geq 1. \]

Let \( e^0(t) \) be the corresponding resonant state, \( (P(t) - \lambda_0(t))e^0(t) = 0 \), uniquely determined up to a factor \( \pm 1 \) (that we take independent of \( t \)) by the condition

\[ \int_{\mathbb{R}^n} e^0(t, x)^2 \, dx = 1. \]

As we shall see in Section 9, we can find an escape function \( G \) as in Theorem 1.2 which applies simultaneously to all \( (P(t), E(t)) \). Moreover, we can choose \( G \) such that \( G(x, -\xi) = -G(x, \xi) \) and with this choice the bilinear scalar product

\[ \langle u | v \rangle = \int u(x)v(x) \, dx \]

is well-defined and bounded on \( H(\Lambda_{veG}) \times H(\Lambda_{veG}) \). Then \( e^0 = O(1) \) in \( H(\Lambda_{veG}) \). Recall that \( H_{\text{sbd}} = H(\Lambda_{veG}), \ D_{\text{sbd}} = H(\Lambda_{veG}, \langle \xi \rangle^2) \) where \( \nu > 0 \) is small and fixed. Then we have (9.52):

\[ \partial_t^k e^0(t) = O(1) \left( \frac{h}{\hat{\epsilon}^2} \right)^k \text{ in } D_{\text{sbd}} \text{ for } k \geq 0. \]

Using the resolvent estimates in Theorem 1.4 we will establish (as Proposition 9.2) the following result:

**Theorem 1.5** Under the assumptions above there exist two formal asymptotic series,

\[ \nu(t, \epsilon) \sim \nu_0(t) + \epsilon \nu_1(t) + \epsilon^2 \nu_2(t) + \ldots \text{ in } C^\infty(I; D_{\text{sbd}}), \]  

\[ \lambda(t, \epsilon) \sim \lambda_0(t) + \epsilon \lambda_1(t) + \epsilon^2 \lambda_2(t) + \ldots \text{ in } C^\infty(I), \]

such that

\[ (\epsilon D_t + P(t) - \lambda(t, \epsilon))\nu(t, \epsilon) \sim 0 \]

as a formal asymptotic series in \( C^\infty(I; \mathcal{H}_{\text{sbd}}) \). Here,

\[ \partial_t^k \nu_j = O(1)(h/\hat{\epsilon}^2)^{2j+k} \text{ in } D_{\text{sbd}}, \ j \geq 0, \ k \geq 0, \]  

\[ \partial_t^k \lambda_j = O(1)(h/\hat{\epsilon}^2)^{2j-1+k}, \ j \geq 1, k \geq 0, \]

where

\[ \hat{\epsilon} := \min(\epsilon_{\theta}, (\epsilon_{\theta} \delta h)^{1/2}). \]
We continue the discussion under the assumptions of Theorem 1.5. Put for $N \geq 1$

$$\nu^{(N)} = \nu_0 + \varepsilon \nu_1 + ... + \varepsilon^N \nu_N,$$  \hspace{1cm} (1.47)

$$\lambda^{(N)} = \lambda_0 + \varepsilon \lambda_1 + ... + \varepsilon^N \lambda_N, \hspace{1cm} N \geq 1.$$ \hspace{1cm} (1.48)

Then by the proof of Theorem 1.5 (cf. (2.4)),

$$\left( \varepsilon D_t + P(t) - \lambda^{(N)} \right) \nu^{(N)} = r^{(N+1)},$$ \hspace{1cm} (1.49)

where

$$r^{(N+1)} = \varepsilon^{N+1} D_t \nu_N - \sum_{j+k \leq N, j+k \geq N+1} \varepsilon^{j+k} \lambda_j \nu_k.$$ \hspace{1cm} (1.50)

In the following, we assume that

$$\frac{\varepsilon^{1/2} h}{\varepsilon_0^2} \ll 1.$$ \hspace{1cm} (1.51)

Recall from (1.40) that $\delta = \delta(h)$ is small, but not exponentially small and that $\epsilon_\theta = (\epsilon/h)^\theta \epsilon$. Then (1.51) holds if we assume that $\varepsilon$ is exponentially small:

$$0 < \varepsilon \leq O(1) \exp\left(-1/(Ch)\right), \hspace{1cm} \text{for some } C > 0,$$ \hspace{1cm} (1.52)

and choose

$$\varepsilon \geq \varepsilon^{\frac{1}{2(1+\theta)\alpha}}.$$ \hspace{1cm} (1.53)

for some $\alpha \in [0,1/(4(1+\theta))]$. We also assume (9.95), stating that $\varepsilon \leq O(\varepsilon^{1/N_0})$ for some $N_0 > 0$.

Having assumed (1.51) we get, as we shall see:

$$\partial_t r^{(N+1)} = O(1) \varepsilon^{\frac{1}{2}} \left( \frac{\varepsilon^{1/2} h}{\varepsilon_0^2} \right)^{2N+1} \left( \frac{h}{\varepsilon_0^2} \right)^k \text{ in } D_{sbd}.$$ \hspace{1cm} (1.54)

Also,

$$\|\nu^{(N)}(t)\|_{H_{sbd}} = (1 + O(\varepsilon h^2/\varepsilon_0^4)) \|\nu_0(t)\|_{H_{sbd}} \ll 1.$$ \hspace{1cm} (1.55)

From (1.25) with $\mu$ in (8.26), $\mu = h/\epsilon$, we get:

$$\Im(P(t)u|u)_{H_{sbd}} \leq O(\epsilon^\infty) ||u||^2_{H_{sbd}}.$$ \hspace{1cm} (1.56)

Let $I \ni t \mapsto u(t) \in H(\Lambda_{ueG}, \langle \xi \rangle)$ be continuous such that $\partial_t u$ is continuous with values in $H(\Lambda_{ueG}, \langle \xi \rangle^{-1})$. Assume that $u$ is a solution of

$$\left( \varepsilon D_t + P(t) \right) u(t) = 0.$$
It then follows from (1.56) that
\[ \| u(t) \|_{\mathcal{H}_{sbd}} \leq e^{O(\epsilon^\infty)(t-s)} \| u(s) \|_{\mathcal{H}_{sbd}}, \quad t \geq s. \] (1.57)

From (1.56) we will derive a resolvent bound which leads to the fact that for every \( u_0 \in \mathcal{D}_{sbd} \) and every \( s \in I \), there exist \( u_0 \in C(I \cap [s, \infty[; \mathcal{D}_{sbd}) \cap C^1(I \cap [s, \infty[; \mathcal{H}_{sbd}) \) such that
\[ (\epsilon D_t + P(t))u(t) = 0 \text{ for } s \leq t \in I, \quad u(s) = u_0. \] (1.58)
Again the solution satisfies (1.57). When \( P(t) = P(t_0) \) is independent of \( t \), this follows from the Hille–Yosida theorem. In the general case we can use [22, Theorem 6.1 and Remark 6.2].

This allows us to define the forward fundamental matrix \( E(t, s), I \ni t \geq s \) of \( \epsilon D_t + P(t) \):
\[
\begin{cases}
(\epsilon D_t + P(t))E(t, s) = 0, & t \geq s, \\
E(t, t) = 1
\end{cases}
\]
and we have
\[ \| E(t, s) \|_{\mathcal{L}(\mathcal{H}_{sbd}, \mathcal{H}_{sbd})} \leq \exp((t-s)O(\epsilon^\infty)), \quad t \geq s, \quad t, s \in I. \] (1.59)
If \( v \in C(I; \mathcal{H}_{sbd}) \) vanishes for \( t \) near \( \inf I \), we can solve \((\epsilon D_t + P(t))u = v\) on \( I \) by
\[ u(t) = \frac{i}{\epsilon} \int_{\inf I}^t E(t, s)v(s)ds. \]

Now return to (1.47)–(1.49) with \( \lambda_j, \nu_j \) as in Theorem 1.5 and \( \rho^{(N+1)} \) satisfying (1.54). We notice that
\[ \lambda^{(N)} = \lambda_0 + O(1)\epsilon^{\frac{1}{2}} \left( \frac{\epsilon^2 h}{\epsilon^\alpha} \right)^\frac{1}{2}. \] (1.60)
We can choose \( \nu_0(t) = e^0(t) \) implying that \( \lambda_1 = 0 \) and (1.60) improves to
\[ \lambda^{(N)} = \lambda_0 + O(1)\epsilon^{\frac{1}{2}} \left( \frac{\epsilon^2 h}{\epsilon^\alpha} \right)^\frac{3}{2}. \] (1.61)
See Remark 9.3. We choose \( \nu_0 = e^0 \) in the remainder of this introduction.

Assume, to fix the ideas, that \( 0 \in I \), and restrict the attention to \( I_+ = \{ t \in I; \ t \geq 0 \} \). From (1.49), we get
\[ (\epsilon D_t + P(t))u^{(N)} = \rho^{(N+1)}, \quad t \in I_+, \] (1.62)
where
\[ u^{(N)} = e^{-i \int_0^t \lambda^{(N)} ds / \epsilon} \nu^{(N)}, \quad \rho^{(N+1)} = e^{-i \int_0^t \lambda^{(N)} ds / \epsilon} r^{(N+1)}. \] (1.63)

By (1.55), (1.54), we have
\[ \| \rho^{(N+1)} \|_{H_{\text{sbd}}} = O(1) \epsilon^{-\frac{1}{2}} \left( \frac{\epsilon \frac{1}{2} h}{\epsilon_0^2} \right)^{2 N+1} \| u^{(N)} \|_{H_{\text{sbd}}}. \] (1.64)

Using again (1.56), we get as in Section 9,
\[ \| u^{(N)} (t) \| \leq e^{O(1) t \epsilon^{-1/2} (\epsilon \epsilon_0^{1/2} h)^{2 N+1}} \| u^{(N)} (0) \|, \quad t \in I_+. \] (1.65)

Assume (9.108):
\[ (\sup_I) \epsilon^{-\frac{1}{2}} \left( \frac{\epsilon \frac{1}{2} h}{\epsilon_0^2} \right)^{2 N+1} \leq O(1). \] (1.66)

Then, for \( t \in I_+ \),
\[ \| u^{(N)} (t) \|_{H_{\text{sbd}}} \leq O(1), \quad \| \rho^{(N+1)} (t) \|_{H_{\text{sbd}}} \leq O(1) \epsilon^{-\frac{1}{2}} \left( \frac{\epsilon \frac{1}{2} h}{\epsilon_0^2} \right)^{2 N+1}. \] (1.67)

Using the fundamental matrix \( E \) to correct the error \( \rho^{(N+1)} \) we have the exact solution
\[ u = u^{(N)} = u^{(N)}(t) - \frac{i}{\epsilon} \int_0^t E(t, s) \rho^{(N+1)}(s) ds \] (1.68)

of the equation
\[ (\epsilon D_t + P(t)) u = 0 \text{ on } I_+. \]

From (1.66) we get
\[ \sup_I \leq \epsilon^{-N_0}, \] (1.69)

for some fixed finite \( N_0 \). Then by (1.59)
\[ \| E(t, s) \|_{L(H_{\text{sbd}}, H_{\text{sbd}})} \leq e^{O(\epsilon \infty)} = 1 + O(\epsilon \infty), \] (1.70)

and using this and (1.67) in (1.68), we get
\[ \| u - u^{(N)} \|_{H_{\text{sbd}}} \leq O(1) \epsilon^{-1} (\sup_I) \epsilon^{-\frac{1}{2}} \left( \frac{\epsilon \frac{1}{2} h}{\epsilon_0^2} \right)^{2 N+1}. \] (1.71)

This estimate is the main result of the present work. Let us recollect the assumptions and the general context in the following theorem (same as Theorem 9.5 below).
Theorem 1.6 Let $V_t = V(t, x) \in C_0^\infty(I \times \mathbb{R}^n; \mathbb{R})$, where $n = 1$, $0 < E_- < E'_- < E'_+ < E_+ < \infty$, $E_0(t) \in C^\infty(I; [E'_-, E'_+])$, $\hat{O}(t) \in \mathbb{R}^n$, $U_0(t) \subset \hat{O}(t)$ be as in the discussion around and including (1.28), (1.35), (1.37). Let $\mu_0(t)$ be a Dirichlet eigenvalue of $-P_0$ as in (1.47), (1.48). Let $\lambda_0(t)$ be the solution of (9.27): $
abla(t) := \{z \in D(\mu_0(t); h); \delta(h)/2; \Im z \geq -\epsilon_0/C\}$.

It is simple and satisfies (9.28):

$$\lambda_0(t) - \mu_0(t) = \widetilde{O}(\epsilon^{-2S_t}/h), \quad S_t := d_t(U_0(t), \partial \hat{O}(t)).$$

Here $\epsilon_0 = (\epsilon/h)^{\bar{\nu}}, \nu \in [e^{-1/(Ch)}, h/C]$ for some sufficiently large constant $C > 0$ and $\bar{\nu} > 0$ is a fixed small constant. Assume (9.57), (8.68):

$$e^{-1/(C(h))} \leq \epsilon \leq \min(h/C_2, \delta), \quad C_1, C_2 \gg 1,$$

(1.72)
as well as (9.95): $\epsilon \leq O(e^{1/(N_0)})$ for some $N_0 > 0$.

Define the spaces $\mathcal{H}_{sbd} = H(\Lambda_{\text{ext}} G), D_{sbd} = H(\Lambda_{\text{ext}} G; \xi^2)$ as earlier in this section, so that $\lambda_0(t)$ is the unique eigenvalue in $\nabla(t)$ of $P(t): \mathcal{H}_{sbd} \to \mathcal{H}_{sbd}$ with domain $D_{sbd}$.

Then we have the formal asymptotic series $\nu(t, \varepsilon), \lambda(t, \varepsilon)$ in Theorem 1.5, where we choose $\nu_0(t) = \epsilon^0(t)$. For $N \geq 1$, define the partial sums $\nu^{(N)}(t), \lambda^{(N)}(t)$ as in (1.47), (1.48). Let $\varepsilon$ be small enough so that (1.51) holds and notice that this would follow from (1.52), (1.53). Assume (to fix the ideas) that $0 \in I$, and assume (1.60) so that sup $I \leq \varepsilon^{-N_0}$ for some constant $N_0 > 0$ and put $I_+ = I \cap [0, +\infty]$. Let $u(t) \in C^1(I_+; \mathcal{H}_{sbd}) \cap C^0(I_+; D_{sbd})$ be the solution of

$$\epsilon D_t + P(t)u = 0 \text{ on } I_+, \quad u(0) = u^{(N)}(0),$$

(1.73)

where $u^{(N)}$ is defined in (1.63). Then (1.71) holds uniformly for $t \in I_+$.

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2 Formal adiabatic solutions for an isolated eigenvalue

Let $\mathcal{H}$ be a separable complex Hilbert space, let $I \subset \mathbb{R}$ be a compact interval and let $P = P(t): \mathcal{H} \to \mathcal{H}$ be a closed densely defined operator, depending
on $t \in I$ such that 

(H1) The domain $\mathcal{D} = \mathcal{D}(t)$ is independent of $t \in I$ and the domain norms $\|u\|_{\mathcal{D}(t)}$ are uniformly equivalent to each other in the sense that there exists a constant $C \geq 1$ such that $C^{-1}\|u\|_{\mathcal{D}(s)} \leq \|u\|_{\mathcal{D}(t)} \leq C\|u\|_{\mathcal{D}(s)}$, $s, t \in I$, $u \in \mathcal{D}$

(H2) $P(t) \in C^\infty(I; \mathcal{L}(\mathcal{D}, \mathcal{H}))$ in the natural sense: the successive derivatives are uniformly bounded $\mathcal{D} \rightarrow \mathcal{H}$ and again differentiable in the uniform sense.

In this section we shall also assume

(H3) $P(t)$ has a simple eigenvalue $\lambda_0(t)$ which depends continuously on $t$ and is isolated from the rest of the spectrum:

$$\text{dist} \ (\lambda_0(t), \sigma(P(t)) \setminus \{\lambda_0(t)\}) \geq 1/C,$$

where $C > 0$ is independent of $t$.

It follows from these assumptions that $\lambda_0(t)$ is a smooth function of $t$. In the following result we review the formal adiabatic construction.

**Proposition 2.1** There exist two asymptotic series

$$\nu(t, \varepsilon) \sim \nu_0(t) + \varepsilon \nu_1(t) + \varepsilon^2 \nu_2(t) + \ldots \text{ in } C^\infty(I; \mathcal{D}), \quad (2.1)$$

$$\lambda(t, \varepsilon) = \lambda_0(t) + \varepsilon \lambda_1(t) + \ldots \text{ in } C^\infty(I), \quad (2.2)$$

where $\nu_0(t)$ is non-vanishing, such that

$$(\varepsilon D_t + P(t) - \lambda(t, \varepsilon))\nu(t, \varepsilon) \sim 0, \quad (2.3)$$

as an asymptotic series in $C^\infty(I; \mathcal{H})$.

**Proof.** We insert the developments for $\nu$ and $\lambda$ into (2.3) and try to cancel the successive powers of $\varepsilon$:

$$(\varepsilon D_t + P(t) - \lambda(t, \varepsilon))\nu(t, \varepsilon) =$$

$$(P(t) - \lambda_0(t))\nu_0(t) + \varepsilon ((D_t - \lambda_1(t))\nu_0(t) + (P - \lambda_0)\nu_1(t))$$

$$+ \ldots$$

$$+ \varepsilon^k ((P(t) - \lambda_0(t))\nu_k + (D_t - \lambda_1)\nu_{k-1} - \lambda_2\nu_{k-2} - \ldots - \lambda_k\nu_0)$$

$$+ \ldots \quad (2.4)$$
In order to annihilate the $\varepsilon^0$ term it is necessary and sufficient to let $\nu_0(t)$ be a non-vanishing eigenvector associated to $\lambda_0(t)$, which depends smoothly on $t$ and we fix such a choice.

By assumption, $\nu_0(t)$ is unique up to a smooth non-vanishing scalar factor. The corresponding spectral projection (independent of the scalar factor) is

$$\pi_0(t) = \frac{1}{2\pi i} \int_{\gamma(t)} (z - P(t))^{-1} dz,$$

(2.5)

where $\gamma(t)$ is the oriented boundary of the disc $D(\lambda_0(t), r)$ and $r > 0$ is a sufficiently small constant. We know that $\pi_0(t)$ is a projection of rank 1 and hence of the form

$$\pi_0(t) u = (u|\delta_0(t))\nu_0(t).$$

(2.6)

This projection and its adjoint, $\pi_0(t)^* v = (v|\nu_0(t))\delta_0(t)$ depend smoothly on $t$ and we deduce that $\delta_0(t)$ (like $\nu_0(t)$) depends smoothly on $t$. Since $\pi_0(t)(P(t) - \lambda_0(t)) = 0$, we have $((P(t) - \lambda_0(t))u|\delta_0(t)) = 0$ for all $u \in \mathcal{D}$ and it follows that $\delta_0(t) \in D(P_0(t)^*)$ and that $(P(t)^* - \lambda_0(t))\delta_0(t) = 0$. From $\pi_0(t)^2 = \pi_0(t)$ it follows that $(\nu_0(t)|\delta_0(t)) = 1$.

By holomorphic functional calculus, we know that

$$\mathcal{H} = \mathcal{R}(1 - \pi_0(t)) \oplus \mathcal{R}(\pi_0(t)) = \mathcal{N}(\pi_0(t)) \oplus \mathcal{R}(\pi_0(t)) = \delta_0(t)^{\perp} \oplus \mathbb{C}\nu_0(t).$$

(2.7)

Further, $P(t) : \delta_0(t)^{\perp} \to \delta_0(t)^{\perp}$ is a closed densely defined operator with spectrum equal to $\sigma(P(t)) \setminus \{\lambda_0(t)\}$. From this we conclude that the equation

$$(P - \lambda_0(t))u = v$$

(2.8)

has a solution precisely when $v \perp \delta_0(t)$ and when this condition is fulfilled the general solution is of the form $u = \tilde{u} + z\nu_0(t)$, where $\tilde{u}$ is the unique solution in $\delta_0(t)^{\perp}$ and $z \in \mathbb{C}$ is arbitrary.

In order to annihilate the $\varepsilon^1$ term in (2.4) we need to find $\nu_1(t) \in \mathcal{D}$ depending smoothly on $t$ such that

$$(P(t) - \lambda_0(t))\nu_1(t) = \lambda_1(t)\nu_0(t) - D_t\nu_0(t).$$

(2.9)

As we have just seen, this equation can be solved precisely when

$$0 = (\lambda_1(t)\nu_0(t) - D_t\nu_0(t)|\delta_0(t)) = \lambda_1(t) - (D_t\nu_0|\nu_0),$$

(2.10)

so we choose $\lambda_1 = (D_t\nu_0|\nu_0)$. Choose a smooth solution $\nu_1(t)$ of (2.9) (which is unique up to a term $z(t)\nu_0(t)$ where $z(t)$ is a smooth scalar function). Then, to annihilate the $\varepsilon^2$ term, we have the equation,

$$(P(t) - \lambda_0(t))\nu_2(t) + (D_t - \lambda_1(t))\nu_1(t) - \lambda_2\nu_0 = 0,$$

and we see that the solvability with respect to $\nu_2(t)$ imposes a unique choice of $\lambda_2$. By iterating this argument we get the proposition. \hfill \Box
Remark 2.2 (a) Let $\lambda, \nu$ be as in the proposition and let
$$\theta(t, \varepsilon) \sim \theta_0(t) + \varepsilon\theta_1(t) + \ldots \text{ in } C^\infty(I).$$
Then,
$$\tilde{\lambda} = \lambda + \varepsilon \partial_t \theta, \quad \tilde{\nu} = e^{i\theta} \nu$$
is another pair as in the Proposition. Indeed,
$$0 \sim e^{i\theta} (\varepsilon D_t + P - \lambda) e^{-i\theta} \tilde{\nu} = (\varepsilon D_t + P - \tilde{\lambda}) \tilde{\nu}.$$
Now, any function $\tilde{\lambda} \sim \lambda_0(t) + \varepsilon \lambda_1(t) + \ldots \text{ in } C^\infty(I)$ is of the form $\tilde{\lambda} = \lambda + \varepsilon \partial_t \theta$ for a suitable $\theta$ as above (which is unique up to a constant $C(\varepsilon) \sim C_0 + \varepsilon C_1 + \ldots$) and we conclude that $\lambda(t, \varepsilon)$ in the proposition can be any asymptotic series as in (2.2), with leading term $\lambda_0$ given in (H3).
(b) If $(\lambda, \nu)$ and $(\lambda, \tilde{\nu})$ are two pairs as in the proposition, then there exists $C(\varepsilon) \sim C_0 + \varepsilon C_1 + \ldots$, such that $\tilde{\nu} = C(\varepsilon) \nu$. In fact, writing $\tilde{\nu}_0 = C_0(t) \nu$ and the corresponding equation for $\tilde{\nu}_1$;
$$(P - \lambda_0) \tilde{\nu}_1 = C_0(\lambda_1 - D_t) \nu_0 - (D_t C_0) \nu_0,$$
we see that $D_t C_0$ has to vanish, so that $C_0$ is constant. Repeat the argument for
$$(\varepsilon D_t + P - \lambda) \left( \frac{\tilde{\nu} - C_0 \nu}{\varepsilon} \right) = 0,$$
to see that $\tilde{\nu} = C_0 \nu + \varepsilon C_1 \nu_0 + O(\varepsilon^2)$, where $C_1$ is a constant. By iteration we get the statement.

3 Some further adiabatic results

3.1 Formal adiabatic solutions for an isolated group of eigenvalues

Let $P(t)$ satisfy the assumptions of Section 2 except for the assumption (H3) that we generalize to

(H4) For some integer $N_0 \geq 1$, $P(t)$ has a group of $N_0$ eigenvalues $\lambda_1(t), \ldots, \lambda_{N_0}(t)$ counted with their multiplicities, depending continuously on $t$ and isolated from the rest of the spectrum:
$$\text{dist} \left( \{\lambda_j(t)\}, \sigma(P(t)) \setminus \{\lambda_j(t)\} \right) \geq 1/C,$$
where $C > 0$ is independent of $t$. Here $\{\lambda_j(t)\} = \{\lambda_j(t); 1 \leq j \leq N_0\}$.
This assumption can be reformulated as follows:

- There exists a simple closed $C^1$ loop, $\gamma = \gamma_t : S^1 \to \mathbb{C}$, enclosing some non-empty part of the spectrum, such that $\text{dist}(\gamma_t(S^1), \sigma(P(t))) \geq 1/C$ and such that the spectral projection

$$
\pi_0(t) = \frac{1}{2\pi i} \int_{\gamma_t} (z - P(t))^{-1} \, dz
$$

(3.1)

has finite rank, necessarily equal to some constant $N_0 \geq 1$.

The formal adiabatic problem is now to construct

$$
U(t, \varepsilon) \sim U_0(t) + \varepsilon U_1(t) + \ldots \in C^\infty(I; \mathcal{L}(C^{N_0}, \mathcal{D}))
$$

(3.2)

and

$$
\Lambda(t, \varepsilon) \sim \Lambda_0(t) + \varepsilon \Lambda_1(t) + \ldots \in C^\infty(I; \mathcal{L}(C^{N_0}, C^{N_0})),
$$

(3.3)

such that

$$
(\varepsilon D_t + P(t)) U(t, \varepsilon) - U(t, \varepsilon) \Lambda(t, \varepsilon) = 0
$$

(3.4)

as a formal powerseries with values in $\mathcal{L}(C^{N_0}, \mathcal{H})$,

$$
\sigma(\Lambda_0(t)) = \{\lambda(t)\},
$$

(3.5)

$$
U_0(t) \text{ is injective and } U_0(t) = \pi_0(t)U_0(t).
$$

(3.6)

As in the case of a single eigenvalue ($N_0 = 1$) we substitute (3.2), (3.3) into (3.4) and try to annihilate the successive powers of $\varepsilon$. This leads to the equations,

$$
PU_0 - U_0 \Lambda_0 = 0,
$$

(3.7)

$$
PU_1 - U_1 \Lambda_0 + D_t U_0 - U_0 \Lambda_1 = 0,
$$

(3.8)

$$
PU_k - U_k \Lambda_0 + D_t U_{k-1} - U_{k-1} \Lambda_1 - U_{k-2} \Lambda_2 - \ldots - U_0 \Lambda_k = 0,
$$

(3.9)

As for (3.7), we let

$$
U_0(t) : C^{N_0} \to \mathcal{R}(\pi_0(t))
$$

(3.10)

be any injective map which depends smothly on $t$ and then take $\Lambda_0(t) = U_0(t)^{-1}P(t)U_0(t)$, where $U_0^{-1}$ denotes the inverse of (3.10). Then

$$
\sigma(\Lambda_0(t)) = \sigma\left(P(t)_{|\mathcal{R}(\pi_0(t))}\right),
$$
so (3.5) is fulfilled.

In order to solve (3.8) we first choose $\Lambda^1(t)$ so that

$$\pi_0(D_tU_0 - U_0\Lambda^1) = 0,$$

(3.11)
i.e. $\Lambda^1 = U_0^{-1}\pi_0 D_t U_0$, where $U_0^{-1}$ denotes the inverse of the operator in (3.10).

Then we look for $U_1$ with

$$(1 - \pi_0)U_1 = U_1,$$

(3.12)
i.e. $U_1 : C^{N_0} \to \mathcal{R}(1 - \pi_0) \cap \mathcal{D}$, and it suffices to find such a (smooth family of) map(s) such that

$$PU_1 - U_1\Lambda_0 + (1 - \pi_0)D_t U_0 = 0.$$

(3.13)

Here $P$ is identified with $P|_{\mathcal{R}(1 - \pi_0)} : \mathcal{R}(1 - \pi_0) \to \mathcal{R}(1 - \pi_0)$, which is closed, densely defined and satisfies

$$\sigma (P|_{\mathcal{R}(1 - \pi_0)}) \cap \sigma (\Lambda_0) = \emptyset.$$

(3.14)

**Lemma 3.1** Let $\mathcal{H}$ be a complex separable Hilbert space and let $A : \mathcal{H} \to \mathcal{H}$ be closed and densely defined. Let $B \in \mathcal{L}(C^{N_0}, C^{N_0})$ and assume that

$$\sigma(A) \cap \sigma(B) = \emptyset.$$  

(3.15)

Then for every $V \in \mathcal{L}(C^{N_0}, \mathcal{H})$, there is a unique $U \in \mathcal{L}(C^{N_0}, D(A))$ such that

$$AU - UB = V.$$

(3.16)

**Proof.** Decompose

$$C^{N_0} = \bigoplus_{\lambda \in \sigma(B)} E_\lambda,$$

where $E_\lambda$ is the spectral subspace, so that $B : E_\lambda \to E_\lambda$ and $B|_{E_\lambda} = \lambda + N$, where $N = N_\lambda$ is nilpotent. It suffices to find, for every $\lambda \in \sigma(B)$, a unique linear operator $U = U_\lambda : E_\lambda \to D(A)$ such that

$$AU - U(\lambda + N) = V_\lambda,$$

where $V_\lambda = V|_{E_\lambda}$.

We write this as

$$(A - \lambda)U - UN = V_\lambda$$

(3.17)
and notice that when $N = 0$, the unique solution is $U = (A - \lambda)^{-1}V_\lambda$.
In the general case we look for $U$ of the form $U = (A - \lambda)^{-1}\tilde{U}$, $\tilde{U} \in \mathcal{L}(E_{\lambda}, \mathcal{H})$ and (3.17) becomes

$$\tilde{U} - \tilde{N}(\tilde{U}) = V_{\lambda},$$

(3.18)

where $$\tilde{N}(\tilde{U}) := (A - \lambda)^{-1}\tilde{U}N.$$ It then suffices to observe that $\tilde{N}$ is nilpotent, so that (3.18) has a unique solution.

By a simple Neumann series argument, if $A = A(t)$, $B = B(t)$, $V = V(t)$ depend smoothly on a real parameter and $\mathcal{D}(A(t))$ is independent of $t$, then the same holds for $U(t)$.

Applying Lemma 3.1 and the above observation to (3.13), we get a unique solution $U_1(t) : C^0 \rightarrow \mathcal{D} \cap \mathcal{R}(1 - \pi_0)$ which is smooth in $t$. Thus, there is a unique solution $(U_1, \Lambda_1)$ to (3.11)–(3.13).

Assuming that $U_0, ..., U_{k-1}, \Lambda_1, ..., \Lambda_{k-1}$ have been constructed, we solve (3.9) in the same way: First, make the unique choice of $\Lambda_k$ for which

$$\pi_0(D_tU_{k-1} - U_{k-1}\Lambda_1 - U_{k-2}\Lambda_2 - ... - U_0\Lambda_k) = 0.$$  

(3.19)

Then, let $U_k$ be the unique map: $C^0 \rightarrow \mathcal{R}(1 - \pi_0) \cap \mathcal{D}$, such that

$$PU_k - U_k\Lambda_0 + (1 - \pi_0)(D_tU_{k-1} - \Lambda_1U_{k-1} - ... - \Lambda_kU_0) = 0.$$

Summing up the discussion, we have

**Proposition 3.2** The problem (3.2)–(3.6) has a solution with

$$U_k \in C^\infty(I; \mathcal{L}(C^0, \mathcal{D})), \quad \Lambda_k \in C^\infty(I; \mathcal{L}(C^0, C^0)).$$

The solution is unique if we first choose $U_0, \Lambda_0$ as in (3.7) and then require that $(1 - \pi_0(t))U_k(t) = U_k(t)$ for $k \geq 1$.

### 3.2 Adiabatic projections

We keep the assumption of the preceding section. Recall the notion of adiabatic spectral projections, [26, 27] in the presentation of [34]. Consider

$$\mathcal{P}(t, \varepsilon D_t; \varepsilon) = \varepsilon D_t + P(t)$$

(3.20)

as a vector valued $\varepsilon$-pseudodifferential operator (see e.g. [9]) with symbol

$$\mathcal{P}(t, \tau; \varepsilon) = \tau + P(t).$$

(3.21)
Then for $z \in \text{neigh}(\gamma_t)$, where $\gamma_t$ is as in (3.1), we define the formal resolvent
\[(z - \mathcal{P})^{-1} = S(t, \varepsilon D_t, z; \varepsilon), \quad (3.22)\]
as a formal $\varepsilon$-pseudodifferential operator with symbol
\[
S(t, \tau, z; \varepsilon) \sim S_0(t, \tau, z) + \varepsilon S_1(t, \tau, z) + ..., \quad (3.23)
\]
defined for $t \in I$, $z - \tau \in \text{neigh}(\gamma_t)$ and obtained by the standard elliptic parametrix construction, so that $S_0(t, \tau, z) = (z - \tau - P(t))^{-1}$. The corresponding adiabatic projection is the formal $\varepsilon$-pseudodifferential operator,
\[
\pi(t, \varepsilon D_t; \varepsilon) = \frac{1}{2\pi i} \int_{\gamma_t} (z - \mathcal{P})^{-1} dz, \quad (3.24)
\]
defined on the symbol level, for $\tau \in \text{neigh}(0, C)$.

Using the property, $S(t, \varepsilon D_t; \varepsilon) = S(t, \varepsilon D_t; \varepsilon)$, it is shown in [34] that the symbol $\pi(t, \tau; \varepsilon)$ is independent of $\tau$, so that
\[
\pi(t, \varepsilon D_t; \varepsilon)u = \pi(t, \varepsilon)u(t), \quad (3.25)
\]
where
\[
\pi(t; \varepsilon) = \pi_0(t) + \varepsilon \pi_1(t) + ... \in C^\infty(I; \mathcal{L}(\mathcal{H}, \mathcal{D})). \quad (3.26)
\]

$\pi_0(t)$ is the spectral projection for $P(t)$ in (3.1). Moreover (cf. (16), (17) in [34]),
\[
[\varepsilon D_t + P(t), \pi(t; \varepsilon)] = 0. \quad (3.27)
\]

**Proposition 3.3** Let $U \sim U_0(t) + \varepsilon U_1(t) + ..., \Lambda \sim \Lambda_0(t) + \varepsilon \Lambda_1(t) + ...$ be a solution of the problem (3.2)–(3.4). Put $\tilde{U} = \pi(t, \varepsilon)U$. Then $\tilde{U} \sim U_0 + \varepsilon U_1 + ...$ (with $\tilde{U}_0 = \pi_0 U_0$), and $(\tilde{U}, \tilde{\Lambda}) := (\tilde{U}, \Lambda)$ is a solution of (3.2)–(3.4) and we have
\[
\pi(t, \varepsilon)\tilde{U}(t, \varepsilon) = \tilde{U}(t, \varepsilon). \quad (3.29)
\]

In view of (3.29) and the fact that $\pi$ is asymptotically a projection, we shall say that $\mathcal{R}(U) \subset \mathcal{R}(\pi)$ (pointwise in $t$).
Proof. By \((3.4), (3.28)\), we get
\[
(\varepsilon D_t + P(t))\tilde{U} - \tilde{U}\Lambda = \pi((\varepsilon D_t + P(t)U - U\Lambda) = 0
\]
\[
\square
\]

**Proposition 3.4** Let \((U, \Lambda), (\tilde{U}, \tilde{\Lambda})\) be two solutions of the problem \((3.2)-(3.4)\) with \(\mathcal{R}(U), \mathcal{R}(\tilde{U}) \subset \mathcal{R}(\pi)\) in the sense of \((3.29)\). Assume that \(U\) satisfies \((3.6)\). Then \(\exists!\)
\[
M(t; \varepsilon) \sim M_0(t) + \varepsilon M_1(t) + ... \in C^\infty(I; \mathcal{L}(\mathbb{C}^{N_0}, \mathbb{C}^{N_0}))
\]
such that
\[
\tilde{U} = UM.
\]
Conversely, if \(U\) solves \((3.2)-(3.6)\) and \(M\) is of the form \((3.30)\) with \(M_0(t)\) invertible, then \((\tilde{U}, \tilde{\Lambda})\) solves \((3.2)-(3.4)\) where
\[
\tilde{U} := UM, \tilde{\Lambda} := M^{-1}\varepsilon D_t(M) + M^{-1}\Lambda M.
\]

**Proof.** We first prove the converse part by direct calculation
\[
(\varepsilon D_t + P(t))\tilde{U} = ((\varepsilon D_t + P(t)U)M + U\varepsilon D_t(M)
\]
\[
= U(\Lambda M + \varepsilon D_t(M)
\]
\[
= UM(M^{-1}\Lambda M + M^{-1}\varepsilon D_t(M))
\]
\[
= \tilde{U}\tilde{\Lambda}.
\]

We next prove the direct part. Let \((U, \Lambda), (\tilde{U}, \tilde{\Lambda})\) be as in the beginning of the proposition, both solving \((3.2)-(3.4)\) with \(\pi U = \pi \tilde{U} = \tilde{U}\) and such that \(U\) satisfies \((3.6)\).

Writing \(\tilde{U} = \tilde{U}_0 + \varepsilon \tilde{U}_1 + ...\), we conclude that \(\tilde{U}_0\) maps \(\mathbb{C}^{N_0} \to \mathcal{R}(\pi_0)\) pointwise in \(t\). \(\tilde{U}_0(t) : \mathbb{C}^{N_0} \to \mathcal{R}(\pi_0(t))\) has the same property and is bijective. Hence there is a unique \(M_0(t) : \mathbb{C}^{N_0} \to \mathbb{C}^{N_0},\) smooth in \(t\), such that \(\tilde{U}_0(t) = U_0(t)M_0(t)\). From the proof of the “converse” part, we see that
\[
V_1(t) := \tilde{U}_0(t) - U(t)M_0(t) \sim: \varepsilon V_{1,1}(t) + \varepsilon^2 V_{2,1}(t) + ...
\]
solves \((3.2)-(3.4)\) with \(\Lambda\) replaced by a new matrix \(\varepsilon \Lambda_1(t; \varepsilon)\). We also have \(\pi V_1 = 0\), so \(\pi_0(t)V_{1,1}(t) = 0\) and hence \(\exists M_1(t) : \mathbb{C}^{N_0} \to \mathbb{C}^{N_0}\), such that \(V_{1,1}(t) = U_0(t)M_1(t)\). Then
\[
V_2(t) := \tilde{U}_0(t) - U(t)(M_0(t) + \varepsilon M_1(t)) \sim: \varepsilon^2 V_{2,2}(t) + \varepsilon^3 V_{2,3}(t) + ...
\]
and \(\pi(t)V_2(t) = 0\), so \(\pi_0(t)V_{2,2}(t) = 0\). Iterating this procedure we get \(M(t) \sim M_0(t) + \varepsilon M_1(t) + ...\) with the required properties. \(\square\)
### 3.3 Extension to the case of variable $\varepsilon$

Consider the evolution equation
\[(D_s + P(s))\nu(s) = 0 \quad (3.33)\]
on some large interval $I$, where $P(s)$ are closed densely defined operators with common domain $\mathcal{D}$. Assume that
\[
\partial_s^k P(s) = \mathcal{O}(\varepsilon(s)^k), \quad k = 0, 1, 2, \ldots, \quad (3.34)
\]
as bounded operators from $\mathcal{D}$ to $\mathcal{H}$. Here the function $\varepsilon(s)$ is assumed to satisfy
\[
\varepsilon > 0, \quad \partial_s^k \varepsilon = \mathcal{O}(\varepsilon^{k+1}), \quad k \geq 0. \quad (3.35)
\]

**Remark 3.5** Under the same assumptions, if $\tilde{\varepsilon} \geq \varepsilon$ is a second function on $I$ which satisfies (3.35), then (3.34) holds with $\varepsilon$ replaced by $\tilde{\varepsilon}$.

Let $f(s)$ be the strictly increasing function, uniquely determined up to a constant, by
\[
f'(s) = \varepsilon(s). \quad (3.36)
\]
Then, if $t = f(s)$, we have $D_s = f'(s)D_t = \varepsilon(s)D_t$ and (3.33) takes the form,
\[
(\varepsilon(g(t))D_t + P(g(t)))u(t) = 0, \quad u(t) = \nu(g(t)). \quad (3.37)
\]
Here, $g := f^{-1}$.

Differentiating $f(g(t)) = t$, we first get $f'(g(t))g'(t) = 1$, so
\[
g'(t) = \frac{1}{\varepsilon(g(t))}. \quad (3.38)
\]
Differentiating $m$ times, where $m \geq 2$, we get
\[
f'(g(t))\partial_t^m g(t) + \sum_{k=2}^{m} \sum_{m_j \geq 1, m_1 + \ldots + m_k = m} C_{m_1, \ldots, m_k} f^{(k)}(g(t))\partial_t^{m_1} g(t) \cdot \ldots \cdot \partial_t^{m_k} g(t) = 0.
\]
Assuming by induction, that
\[
\partial_t^{\tilde{m}} g = \mathcal{O}(1/\varepsilon(g(t))), \quad \tilde{m} < m,
\]
we get
\[
\varepsilon(g(t))\partial_t^m g + \sum_{k=2}^{m} \sum_{m_1 + \ldots + m_k = m} \mathcal{O}(1)\varepsilon(g(t))^k \varepsilon(g(t))^{-k} = 0,
\]

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so we get
\[ \partial_t^m g = O(1/\varepsilon(g(t))), \quad m \geq 1. \] (3.39)

Now,
\[ \partial_t^m (P(g(t))) = \sum_{k=1}^{m} C_{m_1,\ldots,m_k} P^{(k)}(g(t)) \partial_{t_1}^{m_1} \ldots \partial_{t_k}^{m_k} g = \sum O(1) \varepsilon^{k-k} = O(1). \] (3.40)

Similarly,
\[ \partial_t^m \varepsilon(g(t)) = \sum O(1) \varepsilon^{(k)} \partial_{t_1}^{m_1} \ldots \partial_{t_k}^{m_k} g = O(\varepsilon(g(t))). \] (3.41)

This shows that (3.37) is a very nice semi-classical equation.

4 General facts about operators and escape functions

This section is a review of some material from [16, 14] and we add some remarks for later use. We adopt the framework of [16]: Choose two positive smooth scale functions \( r(x), R(x) \) on \( \mathbb{R}^n \) with
\[ r \geq 1, \quad rR \geq 1, \]
(4.1)

Let
\[ \tilde{r}(x, \xi) = (r(x)^2 + \xi^2)^{\frac{1}{2}} \in C^\infty(\mathbb{R}^{2n}). \]
(4.2)

If \( 0 < m \in C^\infty(\mathbb{R}^{2n}) \) we say that \( a \in C^\infty(\mathbb{R}^{2n}) \) belongs to the space \( S(m) \), if
\[ |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta} m(x, \xi) R(x)^{-|\alpha|} \tilde{r}(x, \xi)^{-|\beta|}. \]
(4.3)

We will always assume that the weight \( m \) and the scale functions belong to their own symbol classes:
\[ m \in S(m), \quad r \in S(r), \quad R \in S(R). \]
(4.4)

It follows that \( \tilde{r} \in S(\tilde{r}) \). The naturally associated metric on \( \mathbb{R}^{2n} \) in the spirit of Hörmander’s Weyl calculus of pseudodifferential operators [18] is given by
\[ g = \left( \frac{d\xi}{r} \right)^2 + \left( \frac{dx}{R} \right)^2. \]
(4.5)

It is slowly varying, but another important assumption of that calculus will not be satisfied in our case however, namely the \( \sigma \)-temperance.
Often, \(a\) and even \(m\) will depend on the semi-classical parameter \(h\), and it will then be implicitly assumed that all estimates involved in the statements \(a \in S(m)\) and \(m \in S(m)\) are uniform with respect to \(h\) (and possibly other parameters as well). We define \(h^k S(m) := S(h^k m)\). When \(a : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}\) is a smooth map and \(g\) is a smooth metric on \(\mathbb{R}^{2n}\), we say that \(a\) is of class \(S(m)\) for the metric \(g\), if \(g_a(x,\xi) (\partial_x^\alpha \partial_\xi^\beta a(x,\xi))\) satisfy the estimates in (4.3) uniformly.

Let \(1 \leq m_0(x) \in S(m_0)\) and let \(P = P(x,hD;h)\) be a semi-classical differential operator on \(\mathbb{R}^n\) of the form

\[
P = \sum_{|\alpha| \leq N_0} a_\alpha(x;h)(hD_x)^\alpha,
\]

(4.6)

where \(a_\alpha(x;h) \in S(m_0(x)r^{-|\alpha|})\) and

\[
a_\alpha(x;h) = \sum_{k=0}^{N_0-|\alpha|} h^k a_{\alpha,k}(x), \quad a_{\alpha,k} \in S(m_0r^{-|\alpha|(rR)^{-k}}).
\]

(4.7)

Such operators form an algebra in the natural way. Then

\[
h^k a_{\alpha,k} \xi^\alpha \in S\left(m_0(\tilde{r}/r)^{N_0}(h/(\tilde{r}R))^k\right)
\]

and we have the full semi-classical symbol for the standard left quantization

\[
P(x,\xi;h) = \sum_{|\alpha| \leq N_0} a_\alpha(x;h)\xi^\alpha \in S(m), \quad \text{where} \quad m(x,\xi) = m_0(x)(\tilde{r}/r)^{N_0}
\]

(4.8)

We can write

\[
P(x,\xi;h) = p(x,\xi) + hp_1(x,\xi) + h^2 p_2(x,\xi) + \ldots + h^{N_0} p_{N_0}(x,\xi),
\]

where

\[
p_j \in S(m(\tilde{r}R)^{-j}),
\]

(4.9)

(4.10)

We also assume analyticity in \(x\) near infinity:

\[
\exists C > 0 \text{ such that } P \text{ extends to a holomorphic function}
\]

in \(\{ x \in \mathbb{C}^n; |\Re x| > C, |\Im x| \leq R(\Re x)/C \}\) and the

symbol properties above extend in the natural sense.

This could be formulated more directly in terms of the coefficients \(a_\alpha\) in (4.6).

We assume that \(P\) is formally self-adjoint, so that the classical and the semi-classical principal symbols, given respectively by

\[
p_{\text{class}}(x,\xi) = \sum_{|\alpha|=N_0} a_{\alpha,0}(x)\xi^\alpha
\]

(4.12)
and
\[ p(x, \xi) = \sum_{|\alpha| \leq N_0} a_{\alpha,0}(x)\xi^\alpha \] (4.13)
are real-valued. We make the ellipticity assumption in the classical PDE sense,
\[ p_{\text{class}}(x, \xi) \geq m_0(x)(|\xi|/r)^{N_0}, \quad (x, \xi) \in \mathbb{R}^{2n}. \] (4.14)
This implies for the zero energy surface,
\[ \Sigma_0 = \{(x, \xi) \in \mathbb{R}^{2n}; \, p(x, \xi) = 0\}, \] (4.15)
that
\[ |\xi| \leq \text{Const.} \, r(x) \text{ on } \Sigma_0. \] (4.16)

The same holds on \( \Sigma_E := p^{-1}(E) \) for every fixed \( E \), but we shall mainly concentrate on the case \( E = 0 \) to simplify the notation (observing that after replacing \( p \) with \( p - E \), we are reduced to that case).

We are particularly interested in the following situation:
\[ P = -h^2\Delta + V(x) - 1 \] (4.17)
with symbol
\[ P(x, \xi; h) = p(x, \xi) = \xi^2 + V(x) - 1, \] (4.18)
We will assume that \( V \) is smooth, real-valued and extends holomorphically to the set in (4.11) and tends to 0 when \( x \to \infty \) in that set. This enters into the general framework with
\[ r = 1, \quad R = \langle x \rangle, \quad m_0(x) = 1, \quad m = \langle \xi \rangle^2. \] (4.19)

We next discuss escape functions. If \( a_j \in S(m_j), \, j = 1, 2, \) then \( a_1a_2 \in S(m_1m_2) \) and
\[ H_{a_1a_2} = \{a_1, a_2\} \in S(\frac{m_1m_2}{rR}). \] (4.20)

(4.20) remains valid if we weaken the assumption on \( a_1, a_2 \) to \( a_j \in \hat{S}(m_j) \) for \( j = 1, 2 \) where we let \( \hat{S}(m) \) denote the space of smooth functions \( a \) on \( \mathbb{R}^{2n} \) which satisfy the estimates (4.3) for all non-vanishing \((\alpha, \beta) \in \mathbb{N}^{2n}\).

**Definition 4.1** A real-valued function \( G \in \hat{S}(rR) \) is called an escape function if there exists a constant \( C_0 \) and a compact set \( K \subset \mathbb{R}^{2n} \) such that
\[ H_pG(\rho) \geq \frac{m(\rho)}{C_0}, \text{ for all } \rho \in \Sigma_0 \setminus K. \] (4.21)
When specifying the energy level we say that $G$ in the definition above is an escape function at energy 0. More generally we can define escape functions for $p$ at a real energy $E$, replacing $\Sigma_0$ by $\Sigma_E$.

As we have already noticed, $|\xi| \leq r(x)$ on $\Sigma_0$, so $m \asymp m_0$ there. Also, $H_pG \in S(m)$ for all $G \in \hat{S}(\tilde{r}R)$, so when $G$ is an escape function, we have $H_pG \asymp m$ on $\Sigma_0$ near infinity.

We will also need to know that $|p|$ cannot be very small away from $\Sigma_0$ and therefore make the following assumption:

For every $r_0 > 0$, there exists $\epsilon_0 > 0$, such that

$$|p| \geq \epsilon_0 m \text{ on } \mathbb{R}^{2n} \setminus \bigcup_{\rho \in \Sigma_0} B_{g(\rho)}(\rho, r_0).$$

(4.22)

Here, $g$ is the metric in (4.5) and $B_{g(\rho)}(\rho, r_0)$ denotes the open ball with center $\rho$ and radius $r_0$ for the constant metric $g(\rho)$.

For $\epsilon_0 > 0$ sufficiently small, we introduce the energy shell

$$\Sigma_{[-\epsilon_0, \epsilon_0]} = \{ \rho \in \mathbb{R}^{2n}; |p(\rho)| \leq \epsilon_0 \}.$$  

(4.23)

The assumption (4.22) implies that for every $r_0 > 0$, there exists $\epsilon_0 > 0$ such that

$$\Sigma_{[-\epsilon_0, \epsilon_0]} \subset \bigcup_{\rho \in \Sigma_0} B_{g(\rho)}(\rho, r_0).$$

(4.24)

From (4.21), (4.24) and the fact that $H_pG \in S(m)$ it follows that there exist $C_0, \epsilon_0 > 0$ and a compact set $K \subset \mathbb{R}^{2n}$ such that

$$H_pG(\rho) \geq \frac{m(\rho)}{C_0}, \text{ for all } \rho \in \Sigma_{[-\epsilon_0, \epsilon_0]} \setminus K.$$  

(4.25)

For the Hamilton field,

$$H_G = \partial_\xi G(x, \xi) \cdot \partial_x - \partial_x G(x, \xi) \cdot \partial_\xi,$$

we get when $G \in \hat{S}(\tilde{r}R)$,

$$\|H_G\|_g \asymp \frac{|\partial_\xi G|}{R} + \frac{|\partial_x G|}{\tilde{r}} = \mathcal{O}(1).$$

(4.26)

Using that $p \in S(m)$, we also have

$$H_pG = \tilde{r} p'_\xi \cdot \frac{G'_\xi}{\tilde{r}} - R p'_x \cdot \frac{G'_x}{R} = \mathcal{O}(m) \|H_G\|_g.$$  

Thus, if $G$ is an escape function we get with $\epsilon_0, K$ as in (4.25),

$$\|H_G\|_g \asymp 1 \text{ in } \Sigma_{[-\epsilon_0, \epsilon_0]} \setminus K$$

(4.27)
and here, $H_pG \geq (m/C)\|H_G\|_g$. Also notice that

$$\|H_p\|_g = \mathcal{O}\left(\frac{m}{\tau R}\right).$$

(4.28)

Until further notice we restrict the attention to $\Sigma_{[\epsilon_0,\epsilon_0]}$. We next review the appendix in [14] and especially how to improve the escape function $G$ by modifying it on a bounded set. Let

$$| - \tau_-(\rho), \tau_+(\rho)| \ni t \mapsto \exp tH_p(\rho)$$

be the maximal $H_p$-integral curve through the point $\rho \in \Sigma_{[\epsilon_0,\epsilon_0]}$, where $0 < \tau_{\pm}(\rho) \leq +\infty$ are lower semi-continuous.

If $K \subset \Sigma_{[\epsilon_0,\epsilon_0]}$ is a compact subset as in (4.25), then there exists a finite number $T = T(K) > 0$ such that

$$- T(K) < G < T(K) \text{ on } K.$$

(4.29)

The set $\{\rho \in \Sigma_{[\epsilon_0,\epsilon_0]} ; G(\rho) \geq T(K)\}$ is invariant under the $H_p$-flow in the positive time direction:

$$G(\rho) \geq T(K) \implies G(\exp tH_p(\rho)) \geq T(K), 0 \leq t < \tau_+(\rho).$$

If $G(\rho) \geq T(K)$, $\epsilon_0 > 0$ we have $\exp tH_p(\rho) \in B_g(\rho, \epsilon_0)$ for $0 \leq t \leq t_0(\epsilon_0)\tau R/m$, when $t_0(\epsilon_0) > 0$ is sufficiently small and $G(\exp tH_p(\rho))$ will increase by $\sim \tau R(\rho) \geq 1$ during such a time interval. Then repeat the same consideration with $\rho$ replaced by $\exp t_0(\epsilon_0)H_p(\rho)$ and so on. The trajectory will have to go through infinitely many balls as above and we conclude that $G(\exp tH_p(\rho)) \to +\infty$ when $t \to \tau_+(\rho)$, for every $\rho \in \Sigma_{[\epsilon_0,\epsilon_0]} \cap G^{-1}([-T(K), +\infty[)$. Similarly, $G(\exp tH_p(\rho)) \to -\infty$ when $0 \geq t \to -\tau_-(\rho)$ for every $\rho \in G^{-1}([-\infty, -T(K)])$.

By a similar argument, if $K_1 \subset \Sigma_{[\epsilon_0,\epsilon_0]}$ is a sufficiently large compact set containing $K$, then for every $\rho \in G^{-1}([-T(K), T(K)]) \setminus K_1$, we have $\exp tH_p(\rho) \notin K$, $t \in \mathbb{R}$, and $G(\exp tH_p(\rho)) \to \pm \infty$ when $t \to \pm \tau_\pm(\rho)$.

Define the outgoing and incoming tails $\Gamma_+, \Gamma_- \subset \Sigma_{[\epsilon_0,\epsilon_0]}$ respectively, by

$$\Gamma_\pm = \{\rho \in \Sigma_{[\epsilon_0,\epsilon_0]} ; \exp tH_p(\rho) \not\to \infty, t \to \mp \tau_\pm(\rho)\}. \quad (4.30)$$

In [14] it was shown that $\Gamma_\pm$ are closed sets,

$$\Gamma_+ \subset G^{-1}([-T(K), +\infty[), \quad \Gamma_- \subset G^{-1}([-\infty, T(K)]),$$

and that

$$\Gamma_+ \cap G^{-1}([-\infty, T]), \quad \Gamma_- \cap G^{-1}([-T, +\infty[)$$
are compact for every $T \in \mathbb{R}$. In particular the trapped set $\Gamma_+ \cap \Gamma_-$ is a compact subset of $G^{-1}([- T(K), T(K)])$ and

$$\Gamma_\pm = \{ \rho \in \Sigma_{[-\epsilon_0, \epsilon_0]}; \exp tH_p(\rho) \to \Gamma_+ \cap \Gamma_-, \; t \to +\tau_+(\rho) \}. \quad (4.31)$$

Having fixed $T = T(K)$ above, let $\tilde{K} \subset G^{-1}([- T(K), T(K)])$ be a compact set containing the trapped set $\Gamma_+ \cap \Gamma_-$. For $\rho \in G^{-1}(T(K))$, define

$$\sigma_+(\rho) = \sup\{t \in [0, \tau_-(\rho)]; \exp(-[0, t]H_p)(\rho) \subset G^{-1}([- T(K), T(K)]) \setminus \tilde{K}\}. \quad (4.32)$$

When $\rho$ is outside the set $K_1$ above, assumed to be large enough, the $(-H_p)$-trajectory through $\rho$ will hit $G^{-1}(-T(K))$ without reaching $\tilde{K}$ or get trapped and $\sigma_+(\rho)$ is the corresponding hitting time which depends locally smoothly on $\rho$. For $\rho \in K_1 \cap G^{-1}(T(K))$ it may also happen that the trajectory hits $\tilde{K}$ at the finite time $\sigma_+(\rho)$ or converges to $\Gamma_+ \cap \Gamma_-$ without hitting $\tilde{K}$, in which case $\sigma_+(\rho) = \tau_-(\rho) = +\infty$.

Notice that $\sigma_+$ is a lower semi-continuous function. Define the open subset $\Omega_+$ of $G^{-1}(T(K)) \times [0, +\infty[$ by

$$\Omega_+ := \{(\rho, t) \in G^{-1}(T(K)) \times [0, +\infty[; 0 \leq t < \sigma_+(\rho)\}. \quad (4.33)$$

Then

$$\tilde{\Omega}_+ = \{\exp(-tH_p)(\rho); (\rho, t) \in \Omega_+\}$$

is an open subset of $G^{-1}([- T(K), T(K)])$ and the map

$$\Omega_+ \ni (\rho, t) \mapsto \exp(-tH_p)(\rho) \in \tilde{\Omega}_+$$

is a diffeomorphism. We have

$$\tilde{\Omega}_+ = G^{-1}([- T(K), T(K)]) \setminus \Gamma_-(\tilde{K}), \quad (4.35)$$

where $\Gamma_-(\tilde{K})$ denotes the incoming $\tilde{K}$-tail, defined as

$$\Gamma_-(\tilde{K}) = \Gamma_- \cup \{\exp(-tH_p)(\rho); \rho \in \tilde{K}, \; 0 \leq t < \tau_-(\rho)\}. \quad (4.36)$$

The intersection of $\Gamma_-(\tilde{K})$ with $G^{-1}([- T, +\infty[)$ is compact for every $T \in \mathbb{R}$.

Let $f_+ \in C^\infty(\tilde{\Omega}_+; ]0, +\infty[)$ be equal to $H_pG$ near $G^{-1}(T(K))$ and outside some bounded set. Define $G_+ \in C^\infty(\tilde{\Omega}_+)$ by

$$H_pG_+ = f_+, \; G_+ = G \text{ on } G^{-1}(T(K)) \quad (4.37)$$

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and observe first that $G_+ = G$ near $G^{-1}(T(K))$. By choosing $f_+$ large enough we may arrange so that

$$\limsup_{\tilde{\Omega}_+ \ni \nu \to \partial \tilde{\Omega}_+} G_+(\nu) \leq -T(K) - \frac{1}{C}, \quad C > 0,$$

(4.38)

$$\limsup_{\tilde{\Omega}_+ \ni \nu \to G^{-1}(-T(K))} G_+(\nu) \leq -T(K),$$

where $\partial \tilde{\Omega}_+$ denotes the boundary of $\tilde{\Omega}_+$ as a subset $G^{-1}(\{ -T(K), T(K) \})$, so that $\partial \tilde{\Omega}_+ \subset \Gamma_-(\tilde{K})$. Since $G_+ = G$ near $G^{-1}(T(K))$, we can extend $G_+$ by $G$ to a smooth function $G_+ \in C^\infty(\tilde{\Omega}_+ \cup G^{-1}(\{ T(K), \infty \})).$

By construction, if $\chi \in C^\infty(\mathbb{R})$ and supp $\chi \subset [-T(K), +\infty[$, then $\chi \circ G_+$ is well-defined in $C^\infty(\mathbb{R}^{2n})$.

Next we briefly introduce the analogous quantities, $\Omega_-, G_-$. For $\rho \in G^{-1}(-T(K))$, define

$$\sigma_-(\rho) = \sup\{t \in [0, \tau_+(\rho)]; \exp([0, t]H_p)(\rho) \subset G^{-1}(\{ -T(K), T(K) \}) \setminus \tilde{K} \}.$$  

(4.39)

$\sigma_-$ is a lower semi-continuous function. Let us also define the open subset $\Omega_-$ of $G^{-1}(-T(K)) \times [0, +\infty[$ by

$$\Omega_- := \{(\rho, t) \in G^{-1}(-T(K)) \times [0, +\infty[; 0 \leq t < \sigma_-(\rho)\}.$$  

(4.40)

Then

$$\tilde{\Omega}_- := \{\exp(tH_p)(\rho); (\rho, t) \in \Omega_-\}$$

is an open subset of $G^{-1}([-T(K), T(K)])$ and the map

$$\Omega_- \ni (\rho, t) \mapsto \exp(tH_p)(\rho) \in \tilde{\Omega}_-$$

(4.41)

is a diffeomorphism. We have

$$\tilde{\Omega}_- = G^{-1}([-T(K), T(K)]) \setminus \Gamma_+(\tilde{K}),$$

(4.42)

where $\Gamma_+(\tilde{K})$ denotes the outgoing $\tilde{K}$-tail, defined as

$$\Gamma_+(\tilde{K}) = \Gamma_+ \cup \{\exp(tH_p)(\rho); \rho \in \tilde{K}, 0 \leq t < \tau_+(\rho)\}. $$

(4.43)

The intersection of $\Gamma_+(\tilde{K})$ with $G^{-1}(\{ -\infty, T \})$ is compact for every $T \in \mathbb{R}$.

Let $f_- \in C^\infty(\tilde{\Omega}_+; [0, +\infty[)$ be equal to $H_pG$ near $G^{-1}(-T(K))$ and outside some bounded set. Define $G_- \in C^\infty(\tilde{\Omega}_-)$ by

$$H_pG_- = f_-, \quad G_- = G \text{ on } G^{-1}(-T(K))$$

(4.44)
and observe that $G_\pm = G$ near $G^{-1}(-T(K))$. By choosing $f_-$ large enough we may arrange so that

$$\liminf_{\tilde{\Omega}_- \ni \nu \to \partial \tilde{\Omega}_-} G_-(\nu) \geq T(K) + \frac{1}{C}, \quad C > 0,$$

$$\liminf_{\tilde{\Omega}_- \ni \nu \to g^{-1}(T(K))} G_-(\nu) \geq T(K),$$

where $\partial \tilde{\Omega}_-$ denotes the boundary of $\tilde{\Omega}_+$ as a subset $G \cap \tilde{\Omega}_-$, so that $\partial \tilde{\Omega}_- \subset \Gamma_+(\tilde{K})$.

Since $G_- = G$ near $G^{-1}(-T(K))$, we can extend $G_-$ by $G$ to a smooth function $G_- \in C^\infty(\tilde{\Omega}_- \cup G^{-1}(-\infty, -T(K)])$. By construction, if $\chi \in C^\infty(R^n)$ and $\text{supp} \chi \subset [-\infty, T(K)]$, then $\chi \circ G_-$ is well-defined in $C^\infty(R^{2n})$. With $T = T(\tilde{K})$, we define

$$\tilde{G} = \chi_+ \circ G_+ + \chi_- \circ G_- \in C^\infty(\Sigma_{[-\epsilon_0, \epsilon_0]}),$$

where

- $\chi_\pm \in C^\infty(R; R)$, $\pm \chi_\pm \geq 0$,
- $\chi_+(t) + \chi_-(t) = t$,
- $\chi_+'> 0$ on $]-T, +\infty[,$
- $\chi_-'> 0$ on $[-\infty, T[,$
- $\text{supp} \chi_+ = [-T, +\infty[,$ $\text{supp} \chi_- = ]-\infty, T[,$

We notice that $\tilde{G} = G$ outside a bounded subset of $G^{-1}([-T(K), T(K)])$ and that

$$\tilde{G}^{-1}(0) \supset \{\rho; G_+(\rho) \leq -T(K), \ G_-(\rho) \geq T(K)\} \supset \Gamma_+(\tilde{K}) \cap \Gamma_-(\tilde{K}).$$

It is also clear that $\tilde{G}^{-1}(0) \subset G^{-1}(]-T, T[)$. Moreover, we can choose $f_+, f_-$ so that

the set $G_+(\rho) \leq -T(K)$ is contained in an arbitrarily small neighborhood of $G^{-1}(-T(K)) \cup \Gamma_-(\tilde{K})$,

the set $G_-(\rho) \geq T(K)$ is contained in an arbitrarily small neighborhood of $G^{-1}(T(K)) \cup \Gamma_+(\tilde{K})$.  

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Outside a bounded set, we have \( \tilde{G} = G \), \( \|H_G\|_g \asymp 1 \) and \( H_pG \geq m/O(1) \) by (4.27), (4.21). In a bounded set, we use (4.46) and get

\[
H_p\tilde{G} = (\chi'_+ \circ G_+)f_+ + (\chi'\circ G_-)f_- \asymp \chi'_+ \circ G_+ + \chi'_\circ G_-,
\]

(4.48)

\[
\|H_G\|_g = (\chi'_+ \circ G_+)\|H_{G_+}\|_g + (\chi'\circ G_-)\|H_{G_-}\|_g \asymp \chi'_+ \circ G_+ + \chi'_\circ G_-,
\]

(4.49)

so

\[
H_p\tilde{G} \asymp m\|H_G\|_g \text{ uniformly on } \Sigma_{[-\epsilon_0,\epsilon_0]},
\]

(4.50)
in addition to the fact that \( H_p\tilde{G} \in S(m) \) and \( \|H_G\|_g \asymp 1 \) away from a bounded set.

We can arrange so that \( H_p\tilde{G} > 0 \) outside an arbitrarily small neighborhood of \( \Gamma_+(\tilde{K}) \cap \Gamma_-(\tilde{K}) \).

We now strengthen the assumption \( G \in \dot{S}(\tilde{r}R) \) to

\[
G \in S(\tilde{r}R).
\]

(4.51)

Then \( \tilde{G} \in S(\tilde{r}R) \) and outside a bounded set we have

\[
\tilde{G} = O(\tilde{r}R)\|H_G\|_g.
\]

Choose \( \chi_{\pm} \) so that

\[
\chi_{\pm} = O(\chi_{\pm}') \text{ uniformly on any bounded set}.
\]

Then from (4.46), (4.49), we conclude that

\[
\tilde{G} = O(\tilde{r}R)\|H_G\|_g \text{ uniformly on } \Sigma_{[-\epsilon_0,\epsilon_0]}.
\]

(4.52)

Assume

\[
m \asymp 1 \text{ on } \Sigma_{[-\epsilon_0,\epsilon_0]}.
\]

(4.53)

Let \( \chi \in C_0^\infty([-\epsilon_0,\epsilon_0];[0,1]) \) be equal to 1 on \([-\epsilon_0/2,\epsilon_0/2]\) and define globally,

\[
G^0(\rho) = \chi(p(\rho))\tilde{G}(\rho) \in C^\infty(\mathbb{R}^{2n}),
\]

(4.54)

with the convention that \( G^0 = 0 \) outside \( \Sigma_{[-\epsilon_0,\epsilon_0]} \). By (4.53) we have \( \chi(p(\rho)) \in S(1) \) and hence \( G^0 \in S(\tilde{r}R) \) by (4.51) and the subsequent observation. Then

\[
H_pG^0 = \chi(p)H_p\tilde{G},
\]

(4.55)

\[
G^0 = \chi(p)\tilde{G} + \tilde{G}\chi'(p)H_p.
\]

(4.56)
From the last equation, (4.28) and (4.53), we get

\[ \|H G^0\|_g \leq \chi(p)\|H \tilde{G}\|_g + O(1)\frac{\chi'(p)\|\tilde{G}\|}{R} \leq \chi(p)\|H \tilde{G}\|_g + O(1)\frac{\chi'(p)\|\tilde{G}\|}{R} = O(1), \]

leading to

\[ \|H G^0\|_g^2 \leq O(1) \left( \chi(p)^2\|H \tilde{G}\|_g^2 + \frac{\chi(p)^2\|\tilde{G}\|_g^2}{R^2} \right), \]

in view of the standard estimate, \( \chi' = O(\chi^{1/2}) \) for non-negative smooth functions. Now apply (4.52) to get

\[ \|H G^0\|_g^2 \leq O(1)\chi(p)\|H \tilde{G}\|_g^2 \leq O(1)\chi(p)\|H \tilde{G}\|_g, \quad (4.57) \]

where the last inequality follows from (4.26) which also holds for \( \tilde{G} \). Combining this with (4.50), (4.55), we get

\[ H_p G^0 \geq m \frac{\chi(p)}{O(1)}\|H G^0\|_g^2, \quad (4.58) \]

We sum up the constructions in

**Proposition 4.2** Let \( r, R, \tilde{r} \) be as in (4.1)–(4.4), define the metric \( g \) by (4.5). Let \( P, p, m \) be as in (4.6)–(4.10), where \( 1 \leq m_0 \in S(m_0) \). Assume (4.14) with \( p_{class} \) as in (4.12). Define the energy slice \( \Sigma_{[-\epsilon_0, \epsilon_0]} \) by (4.23) for some \( \epsilon_0 > 0 \) and let \( G \in S(\tilde{r} R) \) be an escape function in the sense of Definition 4.1 and assume (4.22), so that (4.25) holds if \( \epsilon_0 > 0 \) is small enough, and fix such a choice of \( \epsilon_0 \). Let \( \tilde{K} \subset \Sigma_{[-\epsilon_0, \epsilon_0]} \) be a compact set which contains the trapped set \( \Gamma_+ \cap \Gamma_- \) (cf. (4.30)). Define the outgoing and incoming \( \tilde{r} \)-tails \( \Gamma_+ (\tilde{K}), \Gamma_- (\tilde{K}) \) by (4.43), (4.36), so that \( \tilde{K} := \Gamma_+ (\tilde{K}) \cap \Gamma_- (\tilde{K}) \subset \Sigma_{[-\epsilon_0, \epsilon_0]} \) is a compact set; “the \( H_p \)-convex hull” of \( K \).

Then, after modifying \( G \) on a bounded set to a new function \( \tilde{G} \), we can achieve that

- \( H_p \tilde{G} \asymp m \|H G\|_g \) uniformly on \( \Sigma_{\epsilon_0} \),
- \( H_p \tilde{G} > 0 \) outside any fixed given neighborhood of \( \tilde{K} \),
- \( \tilde{G} = 0 \) in a neighborhood of \( \tilde{K} \).

If we also assume (4.53) and define \( G^0 \in S(\tilde{r} R) \) as in (4.54), then we have (4.58).
5 Microlocal approach to resonances ([16])

In this section we review some basic notions developed in the first half of [16].

5.1 I-Lagrangian manifolds

Let \( G \in \dot{S}(\tilde{r}R) \) be real-valued. Then the manifold

\[
\Lambda_G = \{ (x, \xi) \in \mathbb{C}^{2n}; \Im(x, \xi) = H_G(\Re(x, \xi)) \}
\]  

is I-Lagrangian, i.e. Lagrangian in \( \mathbb{C}^{2n} \) for the real symplectic form \(-\Im\sigma\), where \( \sigma = \sum d\xi_j \wedge dx_j \) is the complex symplectic form. Since \( \Lambda_G \) is I-Lagrangian, \( d(-\Im(\xi \cdot dx)|_{\Lambda_G}) = -\Im\sigma|_{\Lambda_G} = 0 \) and since \( \Lambda_G \) is topologically trivial, we know that \(-\Im(\xi \cdot dx)|_{\Lambda_G}\) is exact and hence \( = dH \) for some smooth function \( H \in C^\infty(\Lambda_G) \). The primitive \( H \) is unique up to a constant and we can choose

\[
H = -\Re\xi \cdot \Im x + G(\Re(x, \xi)) = G(\Re(x, \xi)) - \Re\xi \cdot G'(\Re(x, \xi)).
\]  

(5.2)

If we also assume that \( G \) is small in \( \dot{S}(\tilde{r}R) \), then \( \Lambda_G \) is \( \mathbb{R} \)-symplectic, i.e. a symplectic submanifold of \( \mathbb{C}^{2n} \), equipped with the symplectic form \( \Re\sigma \). In other words, \( \sigma|_{\Lambda_G} \) is a (real) symplectic form on \( \Lambda_G \) and we have the volume element

\[
d\alpha = \frac{1}{n!}\sigma^n|_{\Lambda_G}.
\]

5.2 FBI-transforms

(5.1) gives a parametrization \( \mathbb{R}^{2n} \ni \rho \mapsto \rho + iH_G(\rho) \) of \( \Lambda_G \) and we can then define symbol spaces \( S(m) = S(m, \Lambda_G) \) of functions on \( \Lambda_G \) by pulling back functions and weights to \( \mathbb{R}^{2n} \). In particular, we define the scales \( \tilde{r} \) and \( R \) by this pull back. Let \( \lambda = \lambda(\alpha) \in S(\tilde{r}R^{-1}, \Lambda_G) \) be positive, elliptic (in the sense that \( \lambda \) is non-vanishing and \( 1/\lambda \in S((\tilde{r}R^{-1})^{-1}, \Lambda_G) \)) and put

\[
\phi(\alpha, y) = (\alpha_x - y)\alpha_\xi + i\lambda(\alpha)(\alpha_x - y)^2, \quad \alpha = (\alpha_x, \alpha_\xi) \in \Lambda_G, \quad y \in \mathbb{C}^n.
\]  

(5.3)

This will be the phase in our FBI-transform.

The amplitude will be a \( \mathbb{C}^{n+1} \)-valued smooth function \( t(\alpha, y; h) \) on \( \Lambda_G \times \mathbb{C}^n \) which is affine linear in \( y \). When discussing symbol properties of such functions we restrict the attention to a region

\[
|y - \alpha_x| < O(1)R(\alpha_x),
\]  

(5.4)
and with this convention, we require that $t \in h^{-3n/4} S(\tilde{r}^{n/4} R^{-n/4})$ and that $t, \partial_{y_1}t, ..., \partial_{y_n}t$ are maximally linearly independent in the sense that with $t$ treated as a column vector,

$$|\det \left( \begin{array}{cccc} t & \partial_y t & \ldots & \partial_{y_n} t \end{array} \right)| \asymp R^{-n} \left( h^{-\frac{3n}{r^2} - \frac{n}{4}} \right)^{n+1}. \quad (5.5)$$

Notice that the determinant is independent of $y$. If $e_0, e_1, ..., e_n$ is the canonical basis in $\mathbb{C}^{n+1}$, we can choose $t(\alpha; y) = t_0(\alpha; h) + \sum_{j=1}^{n} (\alpha_{x_j} - y_j)t_j(\alpha; h)$,

where,

$$t_j = t_je_j, \text{ and } t_j(\alpha; h) = \frac{t_0(\alpha; h)}{R} \text{ for } j > 0$$

and $t_0 \in h^{-3n/4} S(\tilde{r}^{n/4} R^{-n/4})$ is elliptic.

**Remark 5.1** If $s(\alpha; y; h)$ is a second amplitude with the same properties as $t$, then it is not hard to show that there exists $U(\alpha; h) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1},$ independent of $y$ and invertible, such that

$$U, U^{-1} \in S(1) \text{ and } s(\alpha, y; h) = U(\alpha; h)t(\alpha, y; h).$$

Let $\chi \in C_0^\infty(B(0, 1/C))$ be equal to one in $B(0, 1/(2C))$, where $C > 0$ is large enough. We define the FBI-transform $T : \mathcal{D}'(\mathbb{R}^n) \rightarrow C^\infty(\Lambda_G; \mathbb{C}^{n+1})$ by

$$Tu(\alpha; h) = \int e^{\frac{i}{h}\phi(\alpha, y)} t(\alpha, y; h)\chi_{\alpha}(y)u(y)dy, \quad (5.6)$$

where $\chi_{\alpha}(y) = \chi((y - R\alpha_x)/R(\Re\alpha_x)).$ Here the domain of integration is equal to $\mathbb{R}^n$ and the integral is defined as the bilinear scalar product of $u \in \mathcal{D}'(\mathbb{R}^n)$ and a test function in $C_0^\infty(\mathbb{R}^n)$.

We assume from now on that $G$ belongs to $S(\tilde{r}R)$. We also assume:

$$\exists g_0 = g_0(x) \in S(rR), \text{ such that } G(x, \xi) - g_0(x)$$

has its support in a region where $|\xi| \leq O(r(x)) \quad (5.7)$

and $G(x, \xi) - g_0(x)$ is sufficiently small in $S(rR)$.

We will also consider the more special situation, when $G \in S(m_G)$:

$$\exists g = g(x) \in S(m_G^0), \text{ such that } G(x, \xi) - g(x)$$

has its support in a region where $|\xi| \leq O(r(x)). \quad (5.8)$

and $G(x, \xi) - g_0(x)$ is sufficiently small in $S(m_G)$.
Here \( m_G \leq \tilde{r}R \) is an order function, and we have put \( m^0_G(x) = m_G(x, 0) \).

Let \( H \) be given in (5.2). Then \( H \in S(\tilde{r}R) \). Under the more restrictive assumption (5.8), we have

\[
H \in S(m_G).
\]

(5.9)

Using \( T \) we shall define the function spaces \( H(\Lambda_G, m) \), essentially by requiring that

\[
Tu \in L^2(\Lambda_G, m^2 e^{-2H/h} d\alpha) =: L^2(\Lambda_G, m).
\]

Here, \( m \) is an order function: \( 0 < m \in S(m) \). An intuitive reason for the appearance of \( H \) here is the following: The function

\[
f(y, \theta) = -\Im y \cdot \theta + G(\Re y, \theta), \quad \theta \in \mathbb{R}^n
\]

(5.10)

is a nondegenerate phase function on \( \mathbb{C}^n \times \mathbb{R}^n \) in the sense of Hörmander’s theory of Fourier integral operators (apart from a homogeneity condition) with \( \theta \) as the fiber variables. The corresponding critical manifold \( C_f \) is given by

\[
f_y'(y, \theta) = 0 : \Im y = G_y' (\Re y, \theta)
\]

and the associated I-Lagrangian manifold is

\[
\{ (y, \partial_y f(y, \theta)) ; (y, \theta) \in C_f \} = \Lambda_G.
\]

We are beyond the scope of Hörmander’s theory, but from this it is natural to define the space \( H(\Lambda_G, m) \) by saying that a distribution \( u \) should belong to it when \( Tu \in L^2(\Lambda_G, m^2 e^{-2H/h} d\alpha) \), where

\[
\tilde{H}(\alpha) = v.c.((y, \theta)) (-\Im \phi(\alpha, y) + f(y, \theta)).
\]

Here \( v.c.((y, \theta)) \) indicates that we take the critical value with respect to the variables \((y, \theta)\). The critical point is nondegenerate and given by \((y, \theta) = (\alpha_x, \Re \alpha_\xi)\) and we get

\[
\tilde{H}(\alpha) = H(\alpha).
\]

Letting \( \Lambda_0 = \mathbb{R}^{2n} \), we can find an FBI-transform

\[
T_{\Phi} : \mathcal{D}'(\mathbb{R}^n) \to C^\infty(\Lambda_0; \mathbb{C}^{n+1})
\]

given by

\[
T_{\Phi}u(\beta; h) = \int e^{\frac{i}{h} \phi(\beta, y)} s(\beta, y; h) \chi(\beta) u(y) dy,
\]

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which is equivalent to $T$ in the sense of (5.13) below, provided that
\[
\phi_0(\beta, y) = (\beta_x - y) \cdot \beta_x + i\lambda_0(\beta)(\beta_x - y)^2
\]
and $\mathbf{s}$, $\chi$ are chosen suitably. First we need a bijection $\Lambda_G \ni \alpha \mapsto \beta \in \Lambda_0$ and we define $\beta = \beta(\alpha)$ by imposing the condition
\[
\{\beta\} = \Lambda_0 \cap \{(y, -\partial_y \phi(\alpha, y)); y \in \mathbb{C}^n\},
\]
which gives
\[
\beta = (\beta_x, \beta_\xi) = \left(\Re \alpha_x + \frac{3\alpha_\xi}{2\lambda(\alpha)}, \Re \alpha_\xi - 2\lambda(\alpha)\Im \alpha_x\right).
\]
This gives a bijection $\Lambda_G \rightarrow \Lambda_0$ with inverse $\beta \rightarrow \alpha(\beta)$, both having the natural symbol properties. We define the elliptic element $0 < \lambda_0 \in S(\tilde{r}R^{-1})$ by
\[
\lambda_0(\beta) = \lambda(\alpha(\beta)).
\]
By construction the two quadratic polynomials $\phi(\alpha, \cdot)$ and $\phi_0(\beta, \cdot)$ have the same gradients and Hessians at the point $y = \beta_x$, so they differ by a constant (independent of $y$). More explicitly,
\[
\phi(\alpha, y) = \phi(\alpha, \beta_x) + \phi_0(\beta, y).
\]
Finally, choose
\[
\mathbf{s}(\beta, y) = \mathbf{t}(\alpha, y), \quad \chi_\beta(y) = \chi_\alpha(y).
\]
Then
\[
Tu(\alpha; h) = e^{i\tilde{\phi}(\alpha, \beta_x)}T_0u(\beta; h),
\]
which expresses the equivalence of $T$ and $T_0$.

It follows that if we identify order functions on $\Lambda_G$ and on $\Lambda_0$ in a natural way, then we have the equivalence
\[
Tu \in L^2(\Lambda_G, m^2 e^{-2H/h} d\alpha) \iff T_0u \in L^2(\Lambda_0, m^2 e^{-2F/h} d\beta),
\]
where
\[
F = H + \Im \phi(\alpha, \beta_x) = v.c., y, \theta \left(-\Im \phi_0(\beta, y) + f(y, \theta)\right).
\]
Let $G_1, G_2 \in S(\tilde{r}R)$ be as above and let $f_1$, $f_2$ and $F_1$, $F_2$ be the corresponding functions. In [16] it was shown, using (5.14) and a corresponding inverse “Legendre” formula, that we have the equivalence,
\[
G_1 \leq G_2 \iff F_1 \leq F_2.
\]
From this and the description with the help of $T_0$ it will follow that we have the inclusion $H(\Lambda_{G_1}, m) \subset H(\Lambda_{G_2}, m)$, when $G_1 \leq G_2$. 

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5.3 Sobolev spaces with exponential phase space weights

Let $G$ satisfy (5.7) and be sufficiently small in $S(\tilde{r}R)$. Define $H$ as in (5.2), let $m$ be an order function on $\Lambda G$ and let $T$ be an associated FBI-transform as in (5.6). In [16] it is shown that $T$ is injective on $C_0^\infty(\mathbb{R}^n)$ and also on more general Sobolev spaces with exponential weights, by the construction of an approximate left inverse of $T$ which works with exponentially small errors.

**Definition 5.2** $H(\Lambda_G, m)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ for the norm

$$
\|u\|_{H(\Lambda_G, m)} = \|Tu\|_{L^2(\Lambda_G, m^2e^{-2g(x)/h}d\alpha)}.
$$

(5.16)

The following facts were established in [16]:

- $H(\Lambda_G, m)$ is a Hilbert space
- If we modify the choice of $\lambda$ and $t$ in the definition of $T$, we get the same space $H(\Lambda_G, m)$ and the new norm is uniformly equivalent to the earlier one, when $h \to 0$.
- When $G = g(x)$ is independent of $\xi$ and $m = m_0(x)$, we get

$$
H(\Lambda_G, m) = L^2(\mathbb{R}^n; m_0^2e^{-2g(x)/h}dx)
$$

with uniform equivalence of norms. More generally, when $m(x, \xi) = m_0(x)(\tilde{r}(x, \xi)/r(x))^{N_0}$, $N_0 \in \mathbb{R}$, then $H(\Lambda_G, m)$ is the naturally defined exponentially weighted Sobolev space.

**Remark 5.3** From the last point, we know that $L^2(\mathbb{R}^n) = H(\Lambda_0, 1)$ (when $G = 0$) with uniformly equivalent norms. As in [16], this can be improved:

There exists a positive weight $1 \times M_0(\alpha; h) \in S(1)$ such that if $L_0(d\alpha) = M_0(\alpha; h)^2L(d\alpha)$ ($L$ being the Lebesgue measure), then

$$
(u|v)_{L^2(\mathbb{R}^n)} = \int_{\Lambda_0} TuTvL_0(d\alpha) + (Ku|v)_{L^2(\mathbb{R}^n)},
$$

(5.17)

where $K$ is negligible of order 1 (as defined in the beginning of Subsection 5.4) so that for every $N \in \mathbb{N}$,

$$
K = O(1): H(\Lambda_0, (\tilde{r}R/h)^{-N}) \to H(\Lambda_0, (\tilde{r}R/h)^N).
$$

Notice that the weight $H$ is zero when $G = 0$. 

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5.4 Pseudodifferential- and Fourier integral operators

Such operators can be defined directly (cf. (6.3), (7.8) in [16]). We will only need their descriptions on the FBI-side, somewhat in the spirit of Toeplitz operators.

Let \( m \) be an order function on \( \Lambda_G \). We say that \( R : H(\Lambda_G, m) \to H(\Lambda_G, 1) \) is negligible of order \( m \) if for every order function \( \tilde{m} \) and every \( N_0 \in \mathbb{N} \), \( R \) is a well defined operator \( H(\Lambda_G, \tilde{m} R/h^{N_0}) \to H(\Lambda_G, \tilde{m}/h) \) which is uniformly bounded in the limit \( h \to 0 \). ("Well defined" here refers to the existence of a unique extension from the dense subspace \( C^\infty_0(R^n) \).

We have a completely analogous notion of negligible operators of order \( m \): \( L^2(\Lambda_G, m) \to L^2(\Lambda_G, 1) \). Here, we write \( L^2(\Lambda_G, m) = L^2(\Lambda_G, 2 e^{-2H/h} d\alpha) \) for short. We will use the abbreviation

\[ \text{nop} = \text{negligible operator}, \]
\[ \text{pop} = \text{pseudodifferential operator}, \]
\[ \text{top} = \text{Toeplitz operator}. \]

Let \( \Pi \) be the orthogonal projection \( L^2(\Lambda_G, m) \to TH(\Lambda_G, m) \). Then (see [16], (7.24) and the adjacent discussion)

\[ \Pi = \tilde{\Pi} + \Pi_{-\infty}, \quad \Pi_{-\infty} \text{ is } L^2\text{-negligible of order } 1, \]
\[ \tilde{\Pi} u(\alpha) = \int p(\alpha, \beta; h) e^{i \psi(\alpha, \beta)} u(\beta) m(\beta)^2 e^{-2H(\beta)/h} d\beta, \tag{5.18} \]

where \( \psi \) is independent of \( m \) and of class \( S(\tilde{r} R) \) in a region \( \{ (\alpha, \beta); d_\gamma(\alpha, \beta) \leq 1/\mathcal{O}(1) \} \) and satisfies,

\[ -\Im \psi(\alpha, \beta) - H(\alpha) - H(\beta) \asymp - \left( \frac{\tilde{r}}{R} |\alpha_x - \beta_x|^2 + \frac{R}{\tilde{r}} |\alpha_\xi - \beta_\xi|^2 \right). \tag{5.19} \]

Moreover,

\[ p \in S(m^{-2}h^{-n}) \text{ is supported in a region } d_\gamma(\alpha, \beta) \leq 1/\mathcal{O}(1) \tag{5.20} \]

and we have

\[ \overline{\psi(\alpha, \beta)} = -\psi(\beta, \alpha), \quad p(\alpha, \beta; h) = p(\beta, \alpha; h). \tag{5.21} \]

We refrain from recalling the characterization of \( TH(\Lambda_G, m) \) as the approximate null space of a left ideal of pseudodifferential operators.

We also have a class of pseudodifferential operators of order \( m \) ([16]) \( A : H(\Lambda_G, \tilde{m}) \to H(\Lambda_G, \tilde{m}/m), \forall \tilde{m} \). Such an operator has an associated
principal symbol $\sigma(A) \in \mathcal{S}(m, \Lambda_G)/\mathcal{S}(mh/\tilde{r}R, \Lambda_G)$ which determines the operator $A$ up to an operator of order $mh/\tilde{r}R$ and the principal symbol map is a bijection between the corresponding quotient spaces of operators and of symbols. We also have the usual result for the composition modulo negligible operators.

When $P$ is an $h$-differential operator as in (4.6)–(4.13) with coefficients that are holomorphic near $\pi_{\text{supp}} G$, then $P$ is an $h$-pseudodifferential operator of order $m = m_0(x)(\tilde{r}(x, \xi)/r(x))^{N_0}$, associated to $\Lambda_G$ and the corresponding principal symbol is

$$p_{\Lambda_G}. \quad (5.22)$$

According to Proposition 7.3 in [16] the classes $\{\Pi b \Pi; b \in \mathcal{S}(m)\}$ and $\{TAT^{-1}\Pi; A$ is an $h$-pseudodifferential operator of order $m$ associated to $\Lambda_G\}$ coincide modulo negligible operators of order $m$. Moreover, $b$ and $A$ are related by

$$b \equiv \sigma_A \mod \mathcal{S}(\frac{mh}{\tilde{r}R}). \quad (5.23)$$

Now, let $G_0, G_1$ be two functions with the properties of $G$ above. Then (see the beginning of Chapter 7 in [16]) there exists a smooth real bijective canonical transformation $\kappa : \Lambda_{G_0} \to \Lambda_{G_1}$ such that, writing $(x, \xi) = \kappa(y, \eta)$, we have

$$x - y \in \mathcal{S}(R), \quad \xi - \eta \in \mathcal{S}(\tilde{r}),$$

either as functions of $(y, \eta) \in \Lambda_{G_0}$ or of $(x, \xi) \in \Lambda_{G_1}$. We can then define Fourier integral operators of order $m$, associated to $\kappa$; $A = \mathcal{O}(1): \mathcal{H}(\Lambda_{G_0}, \tilde{m}) \to \mathcal{H}(\Lambda_{G_1}, \tilde{m}/m)$, $\forall \tilde{m}$. Such operators have the usual composition result up to negligible operators. Moreover, we have the usual notion of elliptic operators: If $U : \mathcal{H}(\Lambda_{G_0}, m) \to \mathcal{H}(\Lambda_{G_1}, 1)$ is an elliptic Fourier integral operator of order $m$, then (for $h$ small enough) $U$ is bijective and the inverse is an elliptic Fourier integral operator of order $m^{-1}$ associated to $\kappa^{-1}$ up to a negligible operator of order $m^{-1}$. We also have a corresponding Egorov’s theorem: With $U$ as above, let $A$ be a pseudodifferential operator of order $\hat{m}$ associated to $\Lambda_{G_1}$. Then $B = U^{-1}AU$ is a pseudodifferential operator of order $\hat{m}$, associated to $\Lambda_{G_0}$ (up to a negligible operator of the same order), and the principal symbols are related by

$$\sigma_B = \sigma_A \circ \kappa. \quad (5.24)$$

We now specify the above in the case when

$$G_0 = 0, \quad G_1 = G \quad (5.25)$$
and in doing so we go slightly beyond §16. Since there will be several different
symplectic frameworks, let us denote the standard real Hamilton field of $G$
on $\mathbb{R}^{2n}$, by $H^G_{\mathbb{R}^{2n}}$. Recall that
\[ \Lambda_{vG} = \{ \rho \in \mathbb{C}^{2n}; \Im \rho = vH^G_{\mathbb{R}^{2n}}(\Re \rho) \}. \quad (5.26) \]
We let $\sigma = \sum_{1}^{n} d\xi_j \wedge dx_j$ denote the complex symplectic form on $\mathbb{C}^n_C \times \mathbb{C}^n_C$. The real and imaginary parts $\Re \sigma$ and $\Im \sigma$ are real symplectic forms. When $f$ is a real $C^1$ function on some open subset of $\mathbb{C}^{2n}$, we let $H^R_{\mathbb{R}^{2n}}$ and $H^I_{\mathbb{R}^{2n}}$ denote the corresponding Hamilton fields. In general, if $r = p + iq$ is differentiable with complex-linear differential at some point, then at that point (cf. §33, (11.5), (11.6)),
\[ \hat{H}_r = H^I_{\mathbb{R}^{2n}} \text{ and } \hat{J} \hat{H}_r = H^I_{\mathbb{R}^{2n}}. \quad (5.27) \]
Here, $J$ = multiplication of tangent vectors with $i$, $H_r$ denotes the complex Hamilton field for $\sigma$ (of type 1,0) and the hat indicates that we take the corresponding real vector field; $\hat{H}_r = H_r + \overline{\Pi}_r$, $H_r = r' \cdot \partial_z - r' \cdot \partial_{\overline{z}}$.

Returning to (5.26), if $G(\rho) = G(\Re \rho)$ is considered as a function on $\mathbb{C}^{2n}$, we have
\[ H^3_{\mathbb{R}^{2n}} = JH^R_{\mathbb{R}^{2n}}. \]
Then we can view the family $\Lambda_{vG}$ as obtained from $\Lambda_0$ by deformation with the field
\[ \nu_v = H^3_{\mathbb{R}^{2n}}|_{\Lambda_{vG}}. \]
Since $\Lambda_{vG}$ is $J$-Lagrangian and we get the same deformation is we modify $\nu_v$ by adding a field tangent to $\Lambda_{vG}$, we can replace $\nu_v$ with $\tilde{\nu}_v = H^3_{\mathbb{R}^{2n}}|_{\Lambda_{vG}}$, if $F_v$ is real, smooth and $F_v = G$ on $\Lambda_{vG}$.

Let $\tilde{G}_v$ be an almost holomorphic extension from $\Lambda_{vG}$ of $G|_{\Lambda_{vG}}$. Then at $\Lambda_{vG}$,
\[ J\hat{H}_{\tilde{G}_v} = H^3_{\mathbb{R}^{2n}}|_{\Lambda_{vG}} \equiv H^3_{\mathbb{R}^{2n}} \mod TA_{vG}, \]
by (5.27), so $J\hat{H}_{\tilde{G}_v}$ generates the family $\Lambda_{vG}$ by deformation from $\Lambda_0$.

$G_v$ can be constructed in the following way: Consider the map
\[ \theta = \theta_v : \mathbb{R}^{2n} \ni \rho \mapsto \rho + ivH^R_{\mathbb{R}^{2n}}(\rho) =: \rho + i\gamma_v(\rho) \in \mathbb{C}^{2n}. \]
For $k \in \mathbb{N}$, $\partial^k_{\rho} \gamma_v$ is of class $S(1)$ for the metric $g$. (Here we use that $H_G$ is of class $S(1)$ for the metric $g$.) Thus $\partial^k_{\rho} \theta_v$ is of class $S(1)$ when $k = 0$ and of class $S(1)$ when $k \geq 1$.

Let
\[ \tilde{\theta}_v : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \]
be an almost holomorphic extension of $\theta_v$ with the same symbol properties and let $\tilde{G} \in S(\tilde{r}R)$ be an almost holomorphic extension of $G$. We notice that $\tilde{\theta}_v$ is a local diffeomorphism and that $\tilde{\theta}_v^{-1} : \text{neig}(\Lambda_{vG}, C^{2n}) \to C^{2n}$ is almost holomorphic at $\Lambda_{vG}$ with the same symbol properties. Then $\tilde{G}_v := \tilde{G} \circ \tilde{\theta}_v^{-1}$ has the required properties. One can also see that it can be defined in a $1/\mathcal{O}(1)$-neighborhood of $\Lambda_{vG}$ for $g$, and be of class $S(\tilde{r}R)$ there with all its $t$-derivatives. Using $\tilde{G}_v$, we get a smooth family of canonical transformations $\kappa_v : \Lambda_0 \to \Lambda_{vG}$ by integration of

$$\dot{\kappa}_v(\rho) = H_{\tilde{G}_v}(\kappa_v(\rho)), \quad \rho \in \Lambda_0$$

(identifying $H_{\tilde{G}_v} \simeq J\hat{H}_{\tilde{G}_v}$).

In this way $\kappa_v$ is defined in a $1/\mathcal{O}(1)$-neighborhood of $\Lambda_0$ for $g$ and almost holomorphic at $\Lambda_0$. $\kappa_v \in \tilde{S}(1), \partial^k_v \kappa_v \in S(1)$ for $k \geq 1$.

Write $G^\ell_v = \tilde{G}_v$ and let $G^r_v$ be the almost holomorphic function at $\Lambda_0$ which is given by

$$G^\ell_v \circ \kappa_v = G^r_v. \quad (5.28)$$

We have

$$\partial^k_v G^\ell_v, \partial^k_v G^r_v \in S(\tilde{r}R) \text{ for } k \geq 0.$$

Then on $\Lambda_{vG}$:

$$H_{G^\ell_v} = (\kappa_v)_* H_{G^r_v},$$

where $(\kappa_v)_*$ denotes the operation of push forward of vector fields.

Let $G^r_v, G^\ell_v$ be pseudodifferential operators of order $\tilde{r}R$ associated to $\Lambda_0, \Lambda_{vG}$ with principal symbols $G^r_v, G^\ell_v$ respectively. We can also assume that $\partial^k_v G^r_v$ is a pseudodifferential operator of order $\tilde{r}R$ for all $k$. Then we have elliptic Fourier integral operators $U_v, \tilde{U}_v$ of order 1 associated to $\kappa_v$, such that

$$hD_v U_v + iU_v G^r_v = K^r_v, \quad U_0 = 1, \quad (5.29)$$

$$hD_v \tilde{U}_v + i\tilde{U}_v G^r_v = K^\ell_v, \quad \tilde{U}_0 = 1, \quad (5.30)$$

where $K^r_v, \partial^k_v K^r_v, K^\ell_v, \partial^k_v K^\ell_v$ are negligible operators of order $\tilde{r}R$. This is a straightforward WKB-solution of Cauchy problems within the framework of [16]. Now replace $G^r_v$ with $G^r_v + iU_v^{-1} K^r_v$ and $G^\ell_v$ with $G^\ell_v + iK^\ell_v \tilde{U}_v^{-1}$ and notice that $U_v^{-1} K^r_v$ and $K^\ell_v \tilde{U}_v^{-1}$ are negligible of order 1 with all their $v$-derivatives. Then we get

$$hD_v U_v + iU_v G^r_v = 0, \quad U_0 = 1, \quad (5.31)$$

$$hD_v \tilde{U}_v + i\tilde{U}_v G^\ell_v = 0, \quad \tilde{U}_0 = 1. \quad (5.32)$$

If we choose first $G^r_v, U_v$ in (5.31) and then determine $G^\ell_v$ by

$$G^\ell_v U_v = U_v G^r_v \quad (5.33)$$
(in formal agreement with Egorov’s theorem and (5.28)), we get \( \tilde{U}_v = U_v \) in (5.32):

\[
 hD_v U_v + iG^G_v U_v = 0, \quad U_0 = 1.
\]

(5.34)

Using also (5.42) below, we get

\[
 (hD_v)^k G^G_v = U_v (hD_v - i\text{ad}_{G^G_v})^k (G^G_v)^U_0^{-1},
\]

(5.35)

which shows that for every \( k \geq 0 \), \( \partial^k G^G_v \) is the sum of a pseudodifferential operator and a negligible operator of order \( \tilde{r}R \). Here \( \text{ad}_A(B) \) denotes the commutator \([A,B]\).

Let \( P \) be an \( h \)-differential operator of order \( m = m_0(x)(\tilde{r}/r)^{N_0} \) as in (4.6)–(4.13), so that \( P \) is also an \( h \)-pseudodifferential operator

\[
 P : H(\Lambda_{vG}, m) \rightarrow H(\Lambda_{vG}, 1)
\]

(5.36)

with principal symbol \( p_{\Lambda_{vG}} \) as in (5.22). Here we also assume that the coefficients of \( P \) are analytic in a neighborhood of the \( x \)-space projection of \( \text{supp}G \). The study of \( P \) in (5.36) is equivalent to that of

\[
 V_vPU_v =: P_v : H(\Lambda_0, m) \rightarrow H(\Lambda_0, 1), \quad \text{where} \quad V_v = U_0^{-1}.
\]

(5.37)

We will often write \( H(\Lambda_{vG}) = H(\Lambda_{vG}, 1) \). Notice that

\[
 (Pu|v)_{H(\Lambda_{vG})} = (P_vV_vu|V_vv),
\]

if we define the norm on \( H(\Lambda_{vG}) \) by

\[
 \|v\|_{H(\Lambda_{vG})} = \|V_vv\|_{L^2},
\]

(5.38)

making the operators \( U_v : L^2 \rightarrow H(\Lambda_{vG}) \) and \( V_v : H(\Lambda_{vG}) \rightarrow L^2 \) unitary. This norm is uniformly equivalent to the one in (5.16).

**Remark 5.4** Let \( \Omega \Subset \mathbb{R}^n \) be open and assume that \( G(x,\xi) = 0 \) whenever \( x \in \tilde{\Omega} \), where \( \tilde{\Omega} \) is a neighborhood of \( \Omega \). We can choose first the formal pseudodifferential operator part of \( G^G_v \) with symbol equal to zero over \( \tilde{\Omega} \). Then formally, \( U_v \) is a Fourier integral operator equal to 1 on \( L^2(\tilde{\Omega}) \). It follows from the way Fourier integral operators are defined in [16], that we can choose a realization of \( U_v \) (that we denote with the same symbol) such that

\[
 U_vu = u, \quad \text{when} \quad u \in C_0^\infty(\Omega).
\]

(5.39)

As before, let \( V_v = U_0^{-1} \). Applying \( V_v \) to (5.39), we get

\[
 V_vu = u, \quad \text{when} \quad u \in C_0^\infty(\Omega).
\]

(5.40)

After that we modify \( G^G_v \) with negligible terms as above, so that (5.31), (5.34) hold. From (5.38), we now get

\[
 \|v\|_{H(\Lambda_{vG})} = \|v\|_{L^2}, \quad v \in C_0^\infty(\Omega).
\]

(5.41)
From (5.31), we first notice that
\[ hD_vV_v - iG'_vV_v = 0, \]
(5.42)
and then that
\[ h\partial_vP_v = [P_v, G'_v]. \]
(5.43)

We already know that
\[ P_v = P'_v + N_v \]
where \( P'_v, N_v \) are continuous in \( v \) with values in the pseudodifferential and negligible operators respectively, of order \( m \). See the statements 1–3 after Theorem 7.2 in [16]. Write (5.43) as
\[ \left( \partial_v + \frac{1}{h} \text{ad}_{G'_v} \right) P_v = 0, \]
which implies,
\[ \left( \partial_v + \frac{1}{h} \text{ad}_{G'_v} \right)^k P_v = 0, \quad k = 1, 2, ... \]

From this we deduce that \( \partial_v^k P_v \) has the same structure. From Taylor’s formula with integral remainder, we get
\[ P_v = P_{v,k} + N_{v,k} \]
for every \( k \in \mathbb{N} \), where \( v \mapsto P_{v,k}, v \mapsto N_{v,k} \) are of class \( C^k \) with values in the pops of order \( m \) and nops of order \( m \) respectively.

On the other hand, since the machineries are based on the (complex) method of stationary phase, we also know that the Weyl symbols of \( P_v \) and \( P_{v,k} \) are of the form
\[ \sim \sum_0^\infty h^j p_j(v, x, \xi), \]
(5.44)
where \( p_j \in S(m/(\tilde{r}R)^j) \) are independent of \( k \) and therefore smooth in \( v \). We conclude that \( P_v = P'_v + N_v \) where \( P'_v, N_v \) are smooth in \( v \) with values in the pops and nops respectively, of order \( m \).

The equation for \( p_0(v, x, \xi) = p_v(x, \xi) \) is
\[ \partial_v p_v = iH_{G'_v}p_v, \quad p_v=0 = \text{the principal symbol of } P. \]

We recover the fact (already known by Egorov’s theorem) that
\[ p_v(\rho) = p(\kappa_v(\rho)) =: \tilde{p}_v. \]
(5.45)
Indeed, the two symbols are equal when \( v = 0 \) and
\[ \partial_v \tilde{p}_v(\rho) = \langle \dot{\kappa}_v(\rho), dp(\kappa_v(\rho)) \rangle = i \langle (\kappa_v)_*H_{G'_v}, dp(\kappa_v(\rho)) \rangle \]
\[ = \langle H_{G'_v}, \kappa_v^*(dp(\kappa_v(\rho))) \rangle = i \langle H_{G'_v}, d(p \circ \kappa_v(\rho)) \rangle = iH_{G'_v} \tilde{p}_v. \]
From the construction of $\kappa_\upsilon$ prior to (5.28), we see that
\[
\kappa_\upsilon(\rho) = \rho + ivH_G(\rho) + \mathcal{O}(v^2),
\] (5.46)
where the remainder is $\mathcal{O}(v^2)$ as a smooth function of $\upsilon$ with values in $S(1)$ (with respect to the metric $g$). Using this in (5.45), we get
\[
p_\upsilon(\rho) = p(\rho) - ivH_\rho G + \mathcal{O}(v^2 m)
\] (5.47)
in the sense of smooth functions neigh $(0, \mathbb{R}) \ni \upsilon \mapsto S(1)$.

From (5.46) we get
\[
\tilde{\rho} := \Re \kappa_\upsilon(\rho) = \rho + \mathcal{O}(v^2),
\]
so that by (5.26) $\kappa_\upsilon(\rho) = \tilde{\rho} + ivH_G(\tilde{\rho})$ (5.48)
and hence,
\[
p_\upsilon(\rho) = p(\tilde{\rho} + ivH_G(\tilde{\rho})).
\] (5.49)

If $G = G_\upsilon \in S(\tilde{\upsilon}\mathbb{R})$ is real and depends smoothly on $s \in \text{neigh } (0, \mathbb{R})$, then the smooth dependence on $s$ diffuses into the whole construction above and we get (with the obvious notation) that $P_{\upsilon,s} := V_{\upsilon,s}P_{\upsilon,s}$ in (5.37) is a smooth function of $(\upsilon, s)$ with values in the pops+nops of order $m$. (Recall that we sometimes abbreviate: pop=pseudodifferential operator, nop=negligible operator.)

Let $\Pi$ be the orthogonal projection $L^2(\Lambda_0, M_0) \rightarrow TL^2(\mathbb{R}^n)$ (cf. Remark 5.3) whose properties were recalled in (5.18)–(5.21). Combining the above properties of $P_{\upsilon,s}$ with Proposition 7.3 in [16], we get
\[
TP_{\upsilon,s}T^{-1}\Pi = \Pi P_{\upsilon,s}^{\text{top}} \Pi + N_{\upsilon,s}
\] (5.50)
where $N_{\upsilon,s}$ is smooth in $(\upsilon, s)$ with values in the nops of order $m$
and
\[
P_{\upsilon,s}^{\text{top}}(\rho; h) \sim \sum_{0}^{\infty} h^k p_{\upsilon,s}^{k}(\rho), \ \rho \in \Lambda_0,
\] (5.51)
in $C^\infty(\text{neigh } (0, 0), S(m))$ and with the general term in the sum belonging to
$C^\infty(\text{neigh } (0, 0), h^k S(m/(\tilde{\upsilon}\mathbb{R})^k))$. Here, as already recalled in (5.23),
\[
p_{\upsilon,s}^{0} = p_{\upsilon,s}
\] (5.52)
is the principal symbol of $P_{\upsilon,s}$.

From (5.50) we infer that
\[
(\text{PU}^{\upsilon,s}u|\upsilon v)_{H(\Lambda_0,G_\upsilon)} = (P_{\upsilon,s}u|\upsilon v)
\]
\[
= \int_{\Lambda_0} P_{\upsilon,s}^{\text{top}}(\rho; h) Tu(\rho) \cdot \overline{Tv(\rho)} L_0(d\rho) + (N_{\upsilon,s}u|\upsilon v),
\] (5.53)
for \( u, v \in H(\Lambda_0, m^{1/2}) \) (cf. Remark 5.3 and (5.38)). This can be expressed in the coordinates \( \tilde{\rho} \) in (5.48). Here the scalar product in the middle is the one of \( L^2(\mathbb{R}^n) \). The Jacobian satisfies

\[
J_{v,s}(\tilde{\rho}) := \frac{d\rho}{d\tilde{\rho}} = 1 + \mathcal{O}(\nu^2) \quad \text{in } S(1)
\] (5.54)

and is a smooth function of \( v, s \). We can write

\[
P_{v,s}^{\text{top}}(\rho; h) = \tilde{P}_{v,s}^{\text{top}}(\tilde{\rho}; h),
\] (5.55)

so (5.53) becomes

\[
(P_{v,s}u|v) = \int_{\Lambda_0} \tilde{P}_{v,s}^{\text{top}}(\tilde{\rho}; h) \tilde{T}u(\tilde{\rho}) \cdot \tilde{T}v(\tilde{\rho}) J_{v,s}(\tilde{\rho}) L_0(d\tilde{\rho}) + (N_{v,s}u|v),
\] (5.56)

where \( \tilde{T}u(\tilde{\rho}) := Tu(\rho) \). \( \tilde{P}_{v,s}^{\text{top}}(\tilde{\rho}; h) \) has an asymptotic expansion as in (5.51) with \( \tilde{p}_{v,s}(\tilde{\rho}) = p_{v,s}(\rho) \) and the advantage with (5.56) is that \( \tilde{p}_{v,s} = p_{v,s} \) satisfies

\[
\tilde{p}_{v,s}(\tilde{\rho}) = p(\tilde{\rho} + ivH_{G,s}(\tilde{\rho})).
\] (5.57)

All this remains valid if we replace the single parameter \( s \) by \( s = (s_1, \ldots, s_k) \in \text{neigh}(0, \mathbb{R}^k) \).

If \( p \) is real-valued on \( \Lambda_0 \), we get

\[
\Im p_{v,0}(\tilde{\rho}) = \nu H_{G,s}(p) + \mathcal{O}(\nu^3 \|H_{G,s}\|_2^3) = -\nu H_p(G_0) + \mathcal{O}(\nu^3 \|H_{G,s}\|_2^3).
\] (5.58)

We summarize the results in this section.

**Proposition 5.5** Let \( P \) be an \( h \)-differential operator of order \( m(x, \xi) = m_0(x)(\tilde{r}/r)^N_0 \) as in (4.6)–(4.13). Let \( G \in S(\tilde{r}R) \) satisfy \( (5.7) \) and assume that the coefficients of \( P \) are analytic in a neighborhood of the \( x \)-space projection of \( \text{supp} (G - g_0) \). Then for \( 0 \leq v \leq 1 \), \( P : H(\Lambda_{vG}, m) \to H(\Lambda_{vG}, 1) \) is the sum of an \( h \)-pop and a \( \text{nop} \) both of order \( m \), depending smoothly on \( v \).

The principal symbol is equal to \( p_{\Lambda_{vG}} \).

We can find a canonical transformation \( \kappa_v : \Lambda_0 \to \Lambda_{vG} \) of class \( \dot{S}(1) \) for the metric \( g \), depending smoothly on \( v \in [0, 1] \) in that class, satisfying (5.46) and an operator \( U_v : H(\Lambda_0, 1) \to H(\Lambda_{vG}, 1) \) of the form \( U'_v + N_v \), where \( U'_v \) is an elliptic Fourier integral operator of order 1 associated to \( \kappa_v \) and \( N_v \) is a \( \text{nop} \) of order 1, with \( U'_0 = \text{id} \), such that \( U'^{-1}_v = V_v = V'_v + M_v \) has the analogous properties (with \( \kappa_v \) replaced with \( \kappa_v^{-1} \), such that \( P_v := V_vPU_v \) has the following properties:
• $P_v$ is the sum of a pop and a nop of order $m$, both depending smoothly on $v$ in the corresponding spaces of operators.

• The principal symbol of $P_v$ is given by (5.49), (5.48).

• Writing the Weyl symbol of $P_v$ as \( \sim \sum_0^\infty h^j p_j(v, x, \xi) \), we have

\[
\text{supp} \left( p_j(v, \cdot) - p_j(0, \cdot) \right) \subset \text{supp} G, \quad j \geq 0
\]

• We have the Toeplitz representation (5.50), (5.51), (5.53) (without the parameters $s$ for the moment), where the leading symbol in (5.51) is equal to the one of $P_v$ as a pseudodifferential operator, i.e. $p_0(v, x, \xi)$.

When $G$ depends smoothly on additional parameters $s \in \text{neigh}(0, \mathbb{R}^k)$ we have the corresponding smooth dependence of all terms above.

When $G \in S(m_G)$ satisfies the more special condition (5.8), we can choose $U_v$ so that $P_v - P$ is of order $mm_G/(\tilde{r}R)$. More precisely, $\partial_v P_v \sim \sum_0^\infty h^j \partial_v p_j$ in $S(mm_G/(\tilde{r}R))$, $\partial_v p_j \in S(mm_G/(\tilde{r}R)^{j+1})$. A similar statement holds for $P_{v,s}^{\text{top}}$, $\tilde{P}_{v,s}^{\text{top}}$ and we here retain that $\partial_t (P_{v,s}^{\text{top}} - p_{v,s}) \in S(hmm_G/(\tilde{r}R)^2)$.

The extension in the last paragraph of the proposition follows from an inspection of the proofs.

6 Semi-boundedness in $H(\Lambda_{vG})$ spaces

We continue the discussion from the preceding section and work with the representations (5.56), (5.57), where we drop the tildes until further notice. From (5.57), we get

\[
\partial_s p_{v,s}(\rho) = iv \langle H_{\partial_s G_s}(\rho), d\rho(\rho + ivH_{G_s}(\rho)) \rangle,
\]

which can also be written

\[
\partial_s p_{v,s}(\rho) = -iv \langle H_p(\rho), d\partial_s G_s(\rho) \rangle, \quad (6.1)
\]

If $p \in S(m)$ is real-valued on $\Lambda_0$, we get by Taylor expansion “in the $g$-metric”,

\[
\partial_s \Im p_{v,s}(\rho) = -v \langle H_p(\rho), d\partial_s G_s(\rho) \rangle + O \left( \frac{m}{\tilde{r}R} \|H_{G_s}\|_g^2 \|d\partial_s G_s\|_{g^*} \right). \quad (6.2)
\]

Here the factor $m/(\tilde{r}R)$ corresponds to the estimate (4.28) and $g^*$ is the dual metric to $g$:

\[
g^* = (\tilde{r}d\xi)^2 + (Rdx)^2, \quad (6.3)
\]
so
\[\|df\|_{g^s}^2 = \overline{\tau^2}|\partial_s f|^2 + R^2|\partial_x f|^2 = O(\overline{\tau^2}R^2),\]
when \(f \in \hat{S}R\). Hence, if we assume a uniform bound on \(\partial_s G_s\) in \(S(\overline{\tau}R)\), (6.2) simplifies to
\[\partial_s \mathcal{A}p_v,\rho = -vH_p(\partial_s G_s)(\rho) + O(mv^3\|H_{G_s}\|_{g^s}^2).\]
(6.4)

In this formula we can take \(s = (s_1, \ldots, s_k)\) close to 0 in \(\mathbb{R}^k\) and replace \(\partial_s\) with \(\partial_{s_k}\):
\[\partial_{s_k} \mathcal{A}p_v,\rho = -vH_p(\partial_{s_k} G_s)(\rho) + O(mv^3\|H_{G_s}\|_{g^s}^2).\]
Taylor expansion at \(s = 0\) gives,
\[\partial_{s_k} \mathcal{A}p_v,\rho = -vH_p(\partial_{s_k} G_s)_{s_k=0}(\rho) + O(mv^3s_k) + O(mv^3\|H_{G_0}\|_{g}^2).\]
Writing \(s = (s', s_k)\) and integrating (6.5) from 0 to \(s_k\), gives
\[-\mathcal{A}p_v,\rho + \mathcal{A}p_v(s', 0)(\rho) = \nu_k H_p(\partial_{s_k})_{s_k=0}G_s + O(mv^3s_k) + O(mv^3\|s\|s_k) + O(mv^3s_k\|H_{G_0}(\rho)\|_{g}^2).\]
(6.6)

We now consider the situation in Proposition 4.2. Let
\[K_0 \supset K_1 \supset K_2 \supset \ldots\]
be a sequence of compact \(H_p\)-convex sets in \(\Sigma_{[-\epsilon_0, \epsilon_0]}\) that contain the trapped set. We choose \(K_j\) so that \(K_{j+1}\) is contained in the interior of \(K_j\). For \(j \in \mathbb{N}\), let \(\chi_j \in C_0^\infty([-\epsilon_0, \epsilon_0]; [0, 1])\) be equal to 1 on \([-\epsilon_0/2, \epsilon_0/2]\) and such that \(\chi_{j+1} = 1\) on \(\text{supp} \chi_j\).

Let \(G_0\) be a modification of \(G\) on a bounded subset of \(\Sigma_{[-\epsilon_0, \epsilon_0]}\) as "\(\overline{G}\)" in Proposition 4.2 with \(\overline{K}\) there equal to \(K_0\). Let \(G_j\) be constructed similarly with \(\overline{K}\) equal to \(K_j\) in such a way that \(H_pG_{j+1} > 0\) in \(\Sigma_{[-\epsilon_0, \epsilon_0]} \setminus K_j\). This implies that \(H_pG_{j+1} > 0\) on \(\text{supp} G_j\).

Let \(G_j^0 = \chi_j(p)G_j\). Since \(H_pG_j^0 = \chi_j(p)H_pG_j\), we get
\[H_pG_{j+1}^0 \geq \frac{m}{O(1)} \quad \text{on} \quad \text{supp} G_j^0.\]
(6.7)

Let
\[G^{(N)} = G_0^0 + hG_1^0 + \ldots + h^NG_N^0\]
and notice that this enters into the framework of Section 5:
\[G^{(N)} = G_s = G_0^0 + s_1G_1^0 + \ldots + s_NG_N^0, \quad s = (h, h^2, \ldots, h^N).\]
We apply (5.56) (with the tildes dropped since the beginning of this section) with $\nu > 0$ small, and recall that we have (5.51), (5.57) (with the tildes dropped). When going from $G^{(k-1)}$ to $G^{(k)}$, the leading term $p_{\nu,s}$ changes according to (6.6). Writing $p_{\nu}^{(k)} = p_{\nu,(h,...,h^k,0,...,0)}$, we get

$$-\Im P_{\nu}^{(k)} + \Im P_{\nu}^{(k-1)} = \nu h^k H_p(G_0^0) + O(m\nu h^{2k}) + O(m\nu^3 h^{k+1}) + O(m\nu^3 h^k \|H_{G_0}^0\|^2)$$

$$= \nu h^k H_p(G_0^0) + O(m\nu h^k) + O(m\nu^3 h^k \|H_{G_0}^0\|^2).$$

(6.8)

Also, by (5.58),

$$-\Im p_{\nu,0} = \nu H_p(G_0^0) + O(m\nu^3 \|H_{G_0}^0\|^3)$$

$$= \nu(1 + O(v^2)) H_p(G_0^0),$$

(6.9)

where we also used (4.58) for $G_0 = G_0^0$. Using that estimate also for the last term in (6.8) and summing over $k$, we get

$$-\Im p_{\nu}^{(N)} = \nu \left(1 + O^{(0)}(v^2) + O^{(1)}(v^2 h) + ... + O^{(N)}(v^2 h^N)\right) H_p(G_0^0)$$

$$+ \nu \left(h H_p G_1^0 + h^2 H_p G_2^0 + ... + h^N H_p G_N^0\right)$$

$$+ (m\nu O^{(1)}(h^2) + m\nu O^{(2)}(h^3) + ... + m\nu O^{(N)}(h^{N+1})).$$

(6.10)

Here $O^{(k)}(\cdot)$ denotes a term which depends on $G_0^0,...,G_k^0$ but not on $G_{k+1}^0,...$ and whose support is contained in that of $G_k^0$. We see that after successive replacements, $G_j^0 \mapsto \alpha_j G_j^0$ with $\alpha_j > 0$ large enough, we can achieve that

$$h^2 H_p G_2^0 + m\nu O^{(1)}(h^2) \geq 0,$$

$$h^3 H_p G_3^0 + m\nu O^{(2)}(h^3) \geq 0,$$

$$...$$

$$-\Im p_{\nu}^{(N)} \geq \nu(1 + O(v^2)) H_p(G_0^0) - O(m h^{N+1}).$$

(6.11)

Now recall (5.51) (after adding and removing the tildes), where $p_{\nu,s}^k$ are real for $\nu = 0$. Taylor expand each $p_{\nu,s}$ to sufficiently high order at $s = 0$ and take $s = (h,h^2,...,h^N)$. Then we get with $P_{\nu}^{top(N)} = P_{\nu,(h,...,h^N)}^{top}$,

$$-\Im P_{\nu}^{top(N)}(\rho;h) =$$

$$-\Im p_{\nu}^{(N)} + hO^{(0)}(m\nu) + h^2 O^{(1)}(m\nu) + ... + h^{N+1} O^{(N)}(m\nu).$$

(6.12)

Here the factors $O^{(j)}$ belong to $S(m\nu)$. They are independent of $h$ for $j \leq N - 1$. By successive replacements $G_j^0 \mapsto \alpha_j G_j^0$, we can achieve, using (6.10), that

$$-\Im P_{\nu}^{top(N)} \geq \nu(1 + O(v^2)) H_p(G_0^0) - O(m h^{N+1}).$$

(6.13)
Hence, by (5.56),

\[-\Im(P_v(N)u|u) \geq \int_{\Lambda_0} v(1 + O(v^2))H_p(G_0^0)\|Tu(\rho)\|^2J_v(h, \ldots, h^N, L_0(d\rho) - O(v)h^{N+1}\|u\|^2_{H(\Lambda_0, m^{1/2})}.
\]

(6.14)

The replacement \(G_j^0 \mapsto \alpha_jG_j^0\) does not depend on the value of \(N \geq j\), so we get a full sequence \(G_0^0, G_1^0, \ldots\). Consider an asymptotic sum \(G^0 \sim \sum_0^\infty G^0_j h^j\) in \(S(\tilde{r}R)\).

(6.15)

Then for every \(N \geq 1\), \(G^0 = G^{(N)} + h^{N+1}\tilde{G}^0_{N+1}, \tilde{G}^0_{N+1} \in S(\tilde{r}R)\) and if \(P_v = V_v PU_v\) (with the natural definitions of \(U_v\) and \(V_v = U_v^{-1}\)) we have the analogue of (5.56) (now with the tildes dropped),

\[(P_v u|v) = \int_{\Lambda_0} P_v^{\text{top}}(\rho; h)Tu(\rho; h) \cdot \overline{Tv(\rho; h)J_v(\rho; h)L_0(d\rho)} + (N_v u|v).\]

(6.16)

Here \(N_v\) is negligible of order \(m\), \(\Im P^\text{top}_v\) and \(\Im N_v\) vanish for \(v = 0\). We can replace \(P^{(N)}_v\) with \(P_v\) in (6.14) and the discussion leading to that estimate shows that

\[0 < J_v(\rho; h) = 1 + O(v^2), \quad -\Im P_v^{\text{top}} \geq v(1 + O(v^2))H_pG_0^0 - O(mvh^\infty).\]

(6.17)

In particular,

\[-\Im(P_v u|u) \geq -vO(h^\infty)\|u\|^2_{H(\Lambda_0, m^{1/2})}.\]

(6.18)

Recall that \(P_v\) is just a reduction to \(H(\Lambda_0)\) of the restriction to \(H(\Lambda_vG^0, m)\) of \(P\), so with the norm and scalar product on \(H(\Lambda_vG^0)\) induced by \(U_v\), we get

\[-\Im(Pu|u)_{H(\Lambda_vG^0, 1)} \geq -vO(h^\infty)\|u\|^2_{H(\Lambda_vG^0, m^{1/2})},\]

(6.19)

for \(u \in H(\Lambda_vG^0, m)\).

**Remark 6.1** Only \(m_{\Sigma[-\epsilon_0,\epsilon_0]}\) matters in the calculations. Especially, in the Schrödinger case (discussed in Section [ ]), we have \(m = (\xi)^2\), so we can replace \(m\) with 1.

We end this section with an observation about decoupling of the exterior and the interior part in certain situations. Let \(G^0\) be as in (6.15). Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set such that

\[\overline{\Omega} \cap \pi_x(\text{supp } G^0) = \emptyset, \quad \pi_x(x, \xi) = x.\]

(6.20)
We have seen in Remark 5.4 that we can choose the Fourier integral operators $U_\upsilon, V_\upsilon$ in Section 5 so that (5.39)–(5.41) hold for $G = G^0$:

$$\|v\|_{H(\Lambda \upsilon G^0)} = \|v\|_{L^2}, \quad v \in C_0^\infty(\Omega). \quad (6.21)$$

It follows that

$$-\Im(\mathcal{H} u | u)_{H(\Lambda \upsilon G^0)} = -\Im(\tilde{\mathcal{H}} u | u)_{H(\Lambda \upsilon G^0)}$$

if $\tilde{\mathcal{H}}$ is a new formally self-adjoint operator (with respect to $L^2(\mathbb{R}^n)$) such that $\text{supp} (\tilde{\mathcal{H}} - \mathcal{H}) \subset \Omega$, where $\text{supp} (\tilde{\mathcal{H}} - \mathcal{H})$ is defined to be the union of the supports of the coefficients of $\tilde{\mathcal{H}} - \mathcal{H}$. In particular, we may then replace $\mathcal{H}$ with $\tilde{\mathcal{H}}$ in (6.19).

7 Far away improvement

In this section we discuss improvements in the semi-bound estimates, when the escape function is supported far away. We let $\mathcal{H}, m, r, \tilde{r}, R$ be as in Section 4 with the following special choices,

$$r = 1, \quad R(x) = \langle x \rangle, \quad m_0(x) = 1, \quad (7.1)$$

implying

$$m(x, \xi) = \langle \xi \rangle^{N_0}, \quad \tilde{r}(x, \xi) = \langle \xi \rangle. \quad (7.2)$$

With $p(x, \xi)$ still denoting the semi-classical principal symbol, we assume that $p(x, \xi) \to p_\infty(\xi) \in S(m)$ as $x \to \infty$ in the following sense: For all $\alpha, \beta \in \mathbb{N}_n$,

$$\partial_\alpha x \partial_\beta \xi (p(x, \xi) - p_\infty(\xi)) = o(1) m(\xi) R(x)^{-|\alpha|} \tilde{r}(x, \xi)^{-|\beta|}, \quad x \to \infty, \quad (7.3)$$

uniformly with respect to $\xi$.

If we assume the existence of an escape function in $\Sigma_{[-\epsilon_0, \epsilon_0]}$ for $\epsilon_0 > 0$ small enough, then

$$p_\infty(\xi) = 0 \Rightarrow \partial_\xi p_\infty(\xi) \neq 0. \quad (7.4)$$

From the ellipticity assumption (4.14) we know in addition to (4.15), (4.16) that $p_\infty^{-1}(0)$ is bounded. Conversely, if we assume (7.4), then

$$G(x, \xi) = x \cdot \frac{\partial_\xi p_\infty(\xi)}{\langle \xi \rangle^{N_0-2}} \in S(R\tilde{r})$$

has the required properties. Indeed, when $x \to \infty$,

$$H_p G(x, \xi) \to H_{p,\infty} G = \frac{(\partial_\xi p_\infty)^2}{\langle \xi \rangle^{N_0-2}} < m \text{ on } p_\infty^{-1}(0).$$
In the classical Schrödinger operator case, this gives \( G(x, \xi) = 2x \cdot \xi \), which up to the factor 2 is the escape function appearing in standard complex scaling. We study the situation in a domain \(|x| > \mu/\mathcal{O}(1)\), for \( \mu \gg 1 \), and eventually we will choose our escape function \( G_0 \) with its support contained in such domains. It is natural to make the change of variables, \( x = \mu \tilde{x} \), so that \(|\tilde{x}| > 1/\mathcal{O}(1)\).

Consider first the principal symbol. Put

\[
p_\mu(\tilde{x}, \tilde{\xi}) = p(\mu \tilde{x}, \tilde{\xi}) = p \circ \kappa_\mu(\tilde{x}, \tilde{\xi}),
\]

(7.5)

where

\[
\kappa_\mu(\tilde{x}, \tilde{\xi}) = (\mu \tilde{x}, \tilde{\xi}),
\]

(7.6)

\[
\kappa_\mu^* \sigma = \mu \sigma.
\]

(7.7)

Then in any region, \(|\tilde{x}| > 1/\mathcal{O}(1)\) we have

\[
\partial_\alpha \partial_\beta p_\mu(\tilde{x}, \tilde{\xi}) = \mathcal{O}(1)m(\tilde{\xi})\tilde{r}^{-|\beta|} \hat{R}(\tilde{x})^{-|\alpha|},
\]

(7.8)

\[
\partial_\alpha \partial_\beta (p_\mu(\tilde{x}, \tilde{\xi}) - p_\infty(\tilde{\xi})) = o(1)m(\tilde{\xi})\tilde{r}^{-|\beta|} \hat{R}(\tilde{x})^{-|\alpha|}, \mu \to \infty
\]

(7.9)

Here \( \hat{R} = \hat{R}_\mu \) is given by

\[
\hat{R}(\tilde{x}) = \frac{R(\mu \tilde{x})}{\mu},
\]

(7.10)

so that

\[
\hat{R}(\tilde{x}) \asymp R(\tilde{x}), \ |\tilde{x}| > 1/\mathcal{O}(1).
\]

We restrict the attention to a region,

\[
\Sigma_{\mu, \epsilon_0} = p_\mu^{-1}([-\epsilon_0, \epsilon_0]).
\]

(7.11)

**Proposition 7.1** The “balls” \( \pi_{\tilde{x}}^{-1}B(0, r_0) \cap \Sigma_{\mu, \epsilon_0} \) are \( H_{p_\mu} \)-convex for \( r_0 \geq 1/\mathcal{O}(1) \) when \( \mu \) is large enough. More precisely, every \( H_{p_\mu} \)-trajectory in \( \Sigma_{\mu, \epsilon_0} \) can visit such a ball only during at most one time interval which can be finite or infinite.

**Proof.** It suffices to check that

\[
H_{p_\mu}^2(\tilde{x}^2/2) > 0, \ (\tilde{x}, \tilde{\xi}) \in \Sigma_{\mu, \epsilon_0}, \ |\tilde{x}| \geq 1/\mathcal{O}(1).
\]
With a somewhat simplified notation, we get
\[
H^2_{p_\mu} \left( \frac{\tilde{x}^2}{2} \right) = \left( \frac{\partial p_\mu}{\partial \xi} \cdot \frac{\partial}{\partial \tilde{x}} - \frac{\partial p_\mu}{\partial \tilde{x}} \cdot \frac{\partial}{\partial \xi} \right) \left( \frac{\partial p_\mu}{\partial \xi} \cdot \tilde{x} \right)
\]
\[
= \left( \frac{\partial p_\mu}{\partial \xi} \right)^2 + \frac{\partial^2 p_\mu}{\partial \tilde{x} \partial \xi} \cdot \tilde{x} - \frac{\partial p_\mu}{\partial \tilde{x}} \cdot \frac{\partial^2 p_\mu}{\partial \xi^2} \cdot \tilde{x}
\]
\[
\rightarrow \left( \frac{\partial p_\infty}{\partial \xi} \right)^2 > 0, \quad \mu \to \infty.
\]

From this proposition and (7.20) below, it will follow that the “balls” \( \pi^{-1}_x(B(0, \mu)) \cap \Sigma_{\epsilon_0} \) are \( H_p \) convex for \( \mu \) large enough.

We next apply the change of variables \( x = \mu \tilde{x} \) to the operator \( P \) in (4.6). We get,
\[
P(x, hD_x; h) = P(\mu \tilde{x}, \tilde{h}D_{\tilde{x}}; \tilde{h}) =: P_\mu(\tilde{x}, \tilde{h}D_{\tilde{x}}; \tilde{h}), \quad \tilde{h} = \frac{h}{\mu},
\]
(7.12)

More explicitly, in view of (4.6), (4.7):
\[
P_\mu(\tilde{x}, \tilde{h}D_{\tilde{x}}; \tilde{h}) = \sum_{|\alpha| \leq N_0} a_\alpha^\mu(\tilde{x}; \tilde{h})(\tilde{h}D_{\tilde{x}})^\alpha,
\]
(7.13)

where,
\[
a_\alpha^\mu(\tilde{x}; \tilde{h}) = a_\alpha(\mu \tilde{x}; h) = \sum_{k=0}^{N_0-|\alpha|} \tilde{h}^k a_{\alpha,k}(\mu \tilde{x}) = \sum_{k=0}^{N_0-|\alpha|} \tilde{h}^k a_{\alpha,k}^\mu(\tilde{x}).
\]
(7.14)

Here
\[
a_{\alpha,k}^\mu = \mu^k a_{\alpha,k}(\mu \tilde{x}) \in S(\tilde{R}^{-k}), \quad |\tilde{x}| \geq 1/\mathcal{O}(1),
\]
(7.15)

and \( \tilde{R} \approx R(\tilde{x}) \) as in (7.10). This means that \( P_\mu \) satisfies the general assumptions for \( P \) in the region, \( |\tilde{x}| \geq 1/\mathcal{O}(1) \) and we have the analogue of (4.9):
\[
P_\mu(\tilde{x}, \tilde{\xi}; \tilde{h}) = p_{0,\mu}(\tilde{x}, \tilde{\xi}) + \tilde{h} p_{1,\mu}(\tilde{x}, \tilde{\xi}) + \ldots + \tilde{h}^{N_0} p_{N_0,\mu}(\tilde{x}, \tilde{\xi}),
\]
(7.16)

\[
p_{0,\mu}(\tilde{x}, \tilde{\xi}) = p_{\mu}(\tilde{x}, \tilde{\xi}),
\]
\[
p_{j,\mu}(\tilde{x}, \tilde{\xi}) \in S(m(\tilde{R})^{-j}), \quad |\tilde{x}| \geq 1/\mathcal{O}(1),
\]
(7.17)
We next check that $\Lambda_{\nu G}$ scales naturally when $G$ is an escape function. We expect the scaled weight $G_\mu$ to satisfy,

$$e^{\nu G(x,\xi)/h} = e^{\nu G_\mu(\tilde{x},\tilde{\xi})/\tilde{h}}, \quad (x,\xi) = \kappa_\mu(\tilde{x},\tilde{\xi}),$$

i.e. $G_\mu(\tilde{x},\tilde{\xi}) = G(x,\xi)/\mu$, so we define:

$$G_\mu(\tilde{x},\tilde{\xi}) = \frac{1}{\mu} G(\mu\tilde{x},\tilde{\xi}) = \frac{1}{\mu} (G \circ \kappa_\mu)(\tilde{x},\tilde{\xi}). \quad (7.18)$$

We have,

$$\Lambda_{\nu G} = \kappa_\mu^{-1}(\Lambda_{\nu G}). \quad (7.19)$$

Indeed, for $(\tilde{x},\tilde{\xi}) \in \Lambda_{\nu G}$, we have

$$\mu \Im \tilde{x} = \mu \nu \partial_\xi G_\mu(\Re \tilde{x},\Re \tilde{\xi}) = \nu \partial_\xi G(\mu \Re \tilde{x},\Re \tilde{\xi}),$$

$$\Im \tilde{\xi} = -\nu \partial_x G_\mu(\Re \tilde{x},\Re \tilde{\xi}) = -\nu \partial_x G(\mu \Re \tilde{x},\Re \tilde{\xi}),$$

which shows that $(x,\xi) \in \Lambda_{\nu G}$ if $(x,\xi) = \kappa_\mu(\tilde{x},\tilde{\xi})$.

In the same spirit, we observe that

$$(\kappa_\mu)_* H_p = \mu H_p. \quad (7.20)$$

We finally apply the natural scaling

$$\Lambda_{\nu G} \ni \alpha \mapsto \tilde{\alpha} = \kappa_\mu^{-1}(\alpha) \in \Lambda_{\nu G},$$

to $Tu$ in (5.5), (5.6). Starting from (5.6), we put $u(y) = \tilde{u}(\tilde{y})$, where $y = \mu \tilde{y}$. Again, with $h = h/\mu$, we get from (5.3):

$$\frac{1}{\tilde{h}} \tilde{\phi}(\alpha,\tilde{y}) = \frac{1}{h} \phi(\alpha,\tilde{y}), \quad \text{where}$$

$$\tilde{\phi}(\tilde{\alpha},\tilde{y}) = (\tilde{\alpha}_x - \tilde{y}) \cdot \tilde{\alpha}_\xi + i \lambda_\mu(\tilde{\alpha})(\tilde{\alpha}_x - \tilde{y})^2 \quad (7.21)$$

and

$$\lambda_\mu(\tilde{\alpha}) = \mu \lambda(\mu \tilde{\alpha}_x,\tilde{\alpha}_\xi) \in S(\tilde{r}(\tilde{\alpha}) R(\tilde{\alpha}_x)^{-1}) \quad (7.22)$$

for $|\tilde{\alpha}_x| \geq 1/O(1)$, where we also used that $\mu/R(\mu \tilde{\alpha}_x) \sim 1/R(\tilde{\alpha}_x)$. With the same changes of variables in (5.6), we get

$$Tu(\alpha; h) = \int e^{\frac{i}{\tilde{h}} \tilde{\phi}(\tilde{\alpha},\tilde{y})} \mu^n t(\mu \tilde{\alpha}_x,\tilde{\alpha}_\xi,\mu \tilde{y}; \mu \tilde{h}) \chi \left( \frac{\tilde{y} - \Re \tilde{\alpha}_x}{R(\tilde{\alpha}_x)} \right) \tilde{u}(\tilde{y}) d\tilde{y},$$

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still with $\hat{R} \simeq R$ as in (7.10), so the cutoff is the naturally scaled one. The new amplitude
\[
\tilde{t}(\tilde{\alpha}, \tilde{y}; \tilde{h}) = \mu^n t(\mu \tilde{\alpha}_x, \tilde{\alpha}_x, \mu \tilde{y}; \mu \tilde{h}),
\]
belongs to
\[
S \left( \mu^n h^{\frac{3n}{4}} \hat{r}(\tilde{\alpha}_x)^{\frac{n}{2}} R(\mu \tilde{\alpha}_x)^{-\frac{n}{2}} \right) = S \left( \tilde{h}^{\frac{3n}{4}} \hat{r}(\tilde{\alpha}_x)^{\frac{n}{2}} \hat{R}(\tilde{\alpha}_x)^{-\frac{n}{2}} \right),
\]
which is the right symbol class (working still in $|\tilde{\alpha}_x| \geq 1/O(1)$).

Furthermore,
\[
|\det \begin{pmatrix} t & \partial_{y_1} t & \ldots & \partial_{y_n} t \end{pmatrix}| = \mu^{(n+1)n+n} |\det \begin{pmatrix} t & \partial_{y_1} t & \ldots & \partial_{y_n} t \end{pmatrix}| \lesssim \hat{R}^{-n} \left( \tilde{h}^{\frac{3n}{4}} \hat{r}^{\frac{n}{2}} \hat{R}^{-\frac{n}{2}} \right)^{n+1},
\]
which is analogous to (5.5).

In conclusion, for $|\alpha_x| \geq \mu/O(1)$, we have
\[
Tu(\alpha; h) = \tilde{T} \tilde{u}(\tilde{\alpha}; \tilde{h}), \Lambda_{\nu G} \ni \alpha = \kappa_\mu(\tilde{\alpha}), \tilde{\alpha} \in \Lambda_{\nu \mu},
\]
\[
\tilde{u}(\tilde{y}) = u(y), \ y = \mu \tilde{y}, \ \tilde{h} = h/\mu,
\]
where $\tilde{T}$ has all the general properties of an FBI-transform in any fixed region $|\tilde{\alpha}_x| \geq 1/O(1)$.

If the two order functions $m$ and $\tilde{m}$ are related by
\[
\tilde{m} = m \circ \kappa_\mu,
\]
which is fulfilled under the assumptions (7.1), (7.2), when $\tilde{m} = \langle \tilde{\xi} \rangle^{N_0}$, then we can define the Sobolev spaces $H(\Lambda_{\nu G}, m)$, $H(\Lambda_{\nu \mu}, \tilde{m})$ as in Section 5.3 by (5.16), with $G$ replaced by $\nu G$, and its analogue,
\[
\|\tilde{u}\|_{H(\Lambda_{\nu \mu}, \tilde{m})} = \|\tilde{T} \tilde{u}\|_{L^2(\Lambda_{\nu \mu}, \tilde{m}^{2e^{-2\nu \mu} / \tilde{h}})},
\]
(7.25)

Here we use $H$ in (5.16) adapted to $G$ (cf. (5.2)) so that $\tilde{v} H$ is adapted to $\nu G$, define $H_\nu$ by the analogous relation and notice that $\tilde{H}_\mu(\tilde{x}, \tilde{\xi}) = \mu^{-1} H(\mu \tilde{x}, \tilde{\xi})$. ($G$ and $\tilde{H}$ also depend on $\mu$ and we use the subscript $\mu$ to indicate when we work in the scaled variables $(\tilde{x}, \tilde{\xi})$.) If we extend the definition of $\kappa_\mu$ to maps: $\mathbb{R}^n \to \mathbb{R}^n$ by putting $\kappa_\mu(\tilde{y}) = \mu \tilde{y}$, and let $\kappa^*_\mu$ denote right composition with $\kappa_\mu$ in the usual way, then (7.23) tells us that
\[
\kappa^*_\mu \circ T = \tilde{T} \circ \kappa^*_\mu.
\]
Moreover, \( d\alpha = \mu^n d\tilde{\alpha}, \) \( dy = \mu^n d\tilde{y}, \) so
\[
\|u\|_{L^2}^2 = \mu^n \|\tilde{u}\|_{L^2}^2,
\]
\[
\|Tu\|_{L^2(\Lambda_{\nu G, m^2 e^{-2\nu H/\mu d\alpha}})}^2 = \mu^n \|\tilde{T}\tilde{u}\|_{L^2(\Lambda_{\nu G, \tilde{m}^2 e^{-2\nu H/\mu d\tilde{\alpha}}})}^2.
\]

(7.26)

Thus,
\[
\|u\|_{H(\Lambda_{\nu G, m})} = \mu^\frac{1}{2} \|\tilde{u}\|_{H(\Lambda_{\nu G, \tilde{m}})}.
\]

(7.27)

This applies in particular to the spaces \( H(\Lambda_{\nu G}), H(\Lambda_{\nu G, \nu}) \).

Now we apply the discussion at the end of Section 6 to the operator \( P_\mu \), whose symbol properties we have verified in any region \( |\tilde{x}| \geq 1/\mathcal{O}(1) \). In view of Proposition 7.1 we have the strictly decreasing sequence of \( H^p \)-convex sets in \( \Sigma_{\mu, r_0} \):
\[
\tilde{K}_j = \pi^{-1}_x(B(0, r_j)) \cap \Sigma_{\mu, r_0}, \quad j = 0, 1, 2, ...
\]

(7.28)

where \( 3/2 > r_0 > r_1 > r_2, ... \), \( j \to \infty \). This gives rise to a weight \( G^0_\mu \), vanishing over a neighborhood of \( B(0, 1/2) \), for which we would have
\[
-\Im(P_\mu \tilde{u}|\tilde{u})_{H(\Lambda_{\nu G, \nu})} \geq -v\mathcal{O}(\tilde{h}^\infty) \|\tilde{u}\|_{H(\Lambda_{\nu G, \nu}, \tilde{m}^1/2)}^2.
\]

(7.29)

had it been true that \( P_\mu \) is a differential operator of the right symbol class also inside a region \( |\tilde{x}| \leq 1/\mathcal{O}(1) \). However, this symbol property is guaranteed only outside such balls, but according to the observation at the end of Section 6, we can choose the \( H(\Lambda_{\nu G, \nu}) \) norm, so that
\[
-\Im(P_\mu u|u)_{H(\Lambda_{\nu G, \nu})} = -\Im(\tilde{P}_\mu u|u)_{H(\Lambda_{\nu G, \nu})},
\]

whenever \( \text{supp} (\tilde{P}_\mu - P_\mu) \) is contained in some small fixed neighborhood of \( B(0, 1/2) \). Moreover we can find such a \( \tilde{P}_\mu \) satisfying (7.29). Hence (7.29) holds for \( P_\mu \). (It suffices to take \( \tilde{P}_\mu = (1 - \chi) P_\mu (1 - \chi) \), where \( \chi \in C^\infty_0(B(0, 1/2); \mathbb{R}) \) is equal to 1 on \( B(0, 1/3) \).)

In order to get the corresponding estimates for \( P \), we put
\[
G^0 = \mu G^0_\mu \circ \kappa^{-1}_\mu : \quad G^0(x, \xi) = \mu G^0_\mu(x/\mu, \xi).
\]

Then (7.29) gives,
\[
-\Im(Pu|u)_{H(\Lambda_{\nu G, 0})} \geq -v\mathcal{O}((h/\mu)^\infty) \|u\|_{H(\Lambda_{\nu G, 0}, m^{1/2})}^2.
\]

(7.30)

By Remark 6.1 we can replace \( m \) by 1 in the Schrödinger case. In (7.29), the semi-classical parameter is \( \tilde{h} = h/\mu \) while in (7.30) we are back to using \( h \).
To fix the ideas, we assume right away that $n = 1$ and that

$$P = -h^2 \partial_x^2 + V(x), \quad V \in C^\infty(\mathbb{R}; \mathbb{R}).$$

(8.1)

We adopt the general assumptions of Section 7. More precisely, we assume

(7.1): $r = 1$, $R(x) = \langle x \rangle$, $m_0(x) = 1$ and (7.2) with $N_0 = 2$ so that $m(x, \xi) = \langle \xi \rangle^2$. Recall that $\tilde{r}(x, \xi) = \langle \xi \rangle$. We also assume (7.3) with $p(x, \xi) = \xi^2 + V(x)$, $p_\infty(\xi) = \xi^2$, which amounts to

$$\partial_\alpha x V(x) = o(1) \langle x \rangle^{-|\alpha|}, \quad x \to \infty.$$

(8.2)

We also assume dilation analyticity near $\infty$:

$V$ has a holomorphic extension to $\{x \in \mathbb{C}; \mathbb{R} x > C, \ |\Im x| < |\mathbb{R} x|/C\}$

and denoting the extension also by $V$, we have $V(x) = o(1)$,

(8.3)

when $x \to \infty$ in the truncated sector above.

The earlier discussion was focused on the energy level $E = 0$. Here we will apply it with $P$ replaced by $P - E$ for $E \in [E_-, E_+]$ for $0 < E_- < E_+ < +\infty$. In other terms we will mainly work in

$$p^{-1}([E_-, E_+]), \quad 0 < E_- < E_+ < +\infty,$$

(8.4)

and the slight difference with the earlier discussion is that we now take a wider energy range $[E_-, E_+]$ instead of $[-\epsilon_0, \epsilon_0]$.

Assume that for a choice $E \in ]E_-, E_+]$,

$$\text{the } H_p\text{-flow is non trapping in every unbounded connected component of } p^{-1}(E).$$

(8.5)

Later, we shall strengthen this assumption to non-trapping in $p^{-1}(E)$. Using the special structure of the symbol $p(x, \xi) = \xi^2 + V(x)$, we see that every unbounded connected component $\Sigma_E'$ of $p^{-1}(E)$ is a simple smooth integral curve $\gamma : \mathbb{R} \ni t \mapsto \gamma(t) = (x(t), \xi(t))$ of $H_p$ with one of the following 4 properties:

1) $t \xi(t) > 0$, $x(t) \to +\infty$, when $|t| \to \infty$,

2) $t \xi(t) < 0$, $x(t) \to -\infty$, when $|t| \to \infty$,

3) $\xi(t) > 0$, $x(t) \to \pm \infty$, when $t \to \pm \infty$,
4) \( \xi(t) < 0, x(t) \to \mp \infty \), when \( t \to \pm \infty \).

Moreover, the union \( \Sigma_E \) of all unbounded components of \( p^{-1}(E) \) is the union of two different components as above where either

I) One is of type 1) and the other is of type 2),

or

II) one is of type 3) and the other is of type 4)

Using a sequence of cutoffs, \( \chi_j(p), \chi_j \in C_0^\infty([E_-, E_+]) \), where

\[
1_{[E_- + \delta, E_+ - \delta]} < \chi_0 < \chi_1 < \ldots,
\]
a corresponding sequence of escape functions \( G_0, G_1, G_2, \ldots \) with \( G_0 < G_1 < G_2 < \ldots \) and a dilation \( \tilde{x} \mapsto \mu \tilde{x}, \mu \geq 1 \), we obtain as in Sections 6, 7 a function \( G^0 = G^0_\mu \) of class \( S(\tilde{r}R) \), uniformly with respect to \( \mu \), with support in \( \Sigma_{[E_-, E_+]} \cap \{ (x, \xi); |x| \geq \mu/(2C) \} \) such that we have the semi-boundedness property (7.30) for \( 0 \leq \nu \ll 1, \mu \geq 1 \) and

\[
H_p G^0 \geq \chi_0(p)/C, \text{ on } \{|x| \geq \mu/C\}. \tag{8.6}
\]

(The only difference with Sections 6, 7 is that we have replaced \([-\epsilon_0, \epsilon_0]\) with \([E_-, E_+]\) which is quite straightforward in the Schrödinger case.) Since we shall next turn to resolvent estimates with more escape functions, it is convenient to rename \( G^0 \):

\[
G^\mu_{\text{sbd}} = G_{\text{sbd}} := G^0. \tag{8.7}
\]

For the resolvent estimates, we need to supply a suitable escape function in the set \( \Sigma_E \cap \{|x| \leq \mu/C\} \) and to merge it to \( G^\mu_{\text{sbd}} \). Here we assume that \( E \in [E_+ - \delta, E_+ + \delta] \).

First we can find an escape function

\[
G \in C^\infty(\Sigma_E; \mathbb{R}) \tag{8.8}
\]
of class \( S(\tilde{r}R) \) such that

\[
H_p G > 0, \tag{8.9}
\]

\[
G(x, -\xi) = -G(x, \xi), \tag{8.10}
\]

\[
G(x, \xi) = x \cdot \xi, \ |x| \gg 1. \tag{8.11}
\]

Observe that, since we are in the 1D case,

\[
\langle G(x, \xi) \rangle \asymp \langle x \rangle, \tag{8.12}
\]

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so $\langle G \rangle$ and $\langle x \rangle$ are equivalent weights.

Let $0 \leq \vartheta \ll 1$, to be fixed small enough. Let $f = f_{\vartheta} \in C^\infty(\mathbb{R}; \mathbb{R})$ be given by

$$f(0) = 0, \quad f'(t) = \frac{h}{\langle t \rangle^{1+\vartheta}}.$$  \hfill (8.13)

Then $f$ is odd, and when $\vartheta > 0$, we have

$$f(t) = h(\pm C_{\vartheta} + \mathcal{O}(\langle t \rangle^{-\vartheta})), \quad t \to \pm \infty.$$  \hfill (8.14)

Here,

$$C_{\vartheta} = \int_{0}^{+\infty} \frac{1}{\langle t \rangle^{1+\vartheta}} dt.$$  

When $\vartheta = 0$, we get

$$f(t) = h(\ln t + \mathcal{O}(1)), \quad t \to +\infty.$$ \hfill (8.15)

Because of the unboundedness in this case, we assume from now on that $\vartheta > 0$.

Define the (new) function $G^0$ by

$$G^0 = f(G).$$ \hfill (8.16)

Then,

$$H_p G^0 = f'(G)H_p G \asymp \frac{h}{\langle G \rangle^{1+\vartheta}} \asymp \frac{h}{\langle x \rangle^{1+\vartheta}}.$$ \hfill (8.17)

Also, $H_G G^0 = f'(G)H_G = \frac{h}{\langle G \rangle^{1+\vartheta}} H_G$ and recalling that $\|H_G\|_g = \mathcal{O}(1)$, we get

$$\|H_G G^0\|_g = \mathcal{O}(h) \langle x \rangle^{1+\vartheta}.$$ \hfill (8.18)

This also follows from (8.19) below.

**Proposition 8.1** We have

$$G^0 \in \dot{S} \left( \frac{h}{\langle x \rangle^{\vartheta}} \right),$$ \hfill (8.19)

$$G^0 \in S(h).$$ \hfill (8.20)

**Proof.** For $k \geq 1$, we have

$$f^{(k)}(t) = \mathcal{O}(h) \langle t \rangle^{-\vartheta-k}.$$  

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and for \((\alpha, \beta) \in \mathbb{N}^2 \setminus 0\), we can write \(\partial_x^\alpha \partial_\xi^\beta G^0\) as a finite linear combination of terms
\[
\tag{8.21}
f^{(k)}(G) \left( \partial_x^{\alpha_1} \partial_\xi^{\beta_1} G \right) \cdots \left( \partial_x^{\alpha_k} \partial_\xi^{\beta_k} G \right),
\]
where \(k \geq 1\), \((\alpha_j, \beta_j) \neq (0, 0)\), \(\alpha_1 + \cdots + \alpha_k = \alpha\), \(\beta_1 + \cdots + \beta_k = \beta\). Since \(\langle G \rangle \approx \langle x \rangle\), it follows that the term in (8.21) is
\[
\mathcal{O}(1) \frac{h}{\langle x \rangle^{\vartheta+k}} \langle x \rangle^{1-\alpha_1} \cdots \langle x \rangle^{1-\alpha_k} = \mathcal{O}(h) \langle x \rangle^{\vartheta+\alpha}
\]
and (8.19) follows. Now (8.20) follows from (8.19) and the fact that \(G^0 = \mathcal{O}(h)\) by (8.14). \qed

Until further notice, we assume that
\[
\tag{8.22}
\text{The } H_p \text{ flow is non-trapping on } p^{-1}(E).
\]
In other words, \(\Sigma_E\) is equal to all of \(p^{-1}(E)\). Let \(\chi^0 \in C_0^\infty(\text{neigh } (E, \mathbb{R}); [0, 1])\) be equal to 1 near \(E\) and with its support contained in \(\chi_0^{-1}(1)\), where \(\chi_0, \chi_1, \ldots\) are the cutoffs used before. Put
\[
\tag{8.23}
G_{\text{lap}} = \chi^0(p) G^0 \in S(h).
\]
Here “lap” stands for “limiting absorption principle”, because \(G_{\text{lap}}\) can be used to give a quick proof of the semi-classical limiting absorption principle of Robert–Tamura [31], also proved by Gérard–Martinez [12]. This is also related to Martinez’ result [23] on the absence of resonances for non-trapping potentials that are merely smooth on some bounded set. We put
\[
\tag{8.24}
G_{\epsilon} = G_{\text{lap}} + \epsilon G_{\text{sbd}}, \quad 0 < \epsilon \ll h,
\]
where we recall that \(G_{\text{sbd}}\) depends on a large parameter \(\mu\). We next choose \(\mu\) as a function of \(\epsilon\), and to do so we notice that when the support of \(\chi^0\) is narrow enough,
\[
\tag{8.25}
H_p G_{\text{lap}} \asymp \chi^0(p) \frac{h}{\langle G \rangle^{1+\vartheta}} \asymp \chi^0(p) \frac{h}{\langle x \rangle^{1+\vartheta}},
\]
which is \(\asymp h/\langle x \rangle^{1+\vartheta}\) where \(\chi_0(p) = 1\) and in particular in \(p^{-1}([E-\delta_0, E+\delta_0])\) when \(\delta_0 > 0\) is small enough. On the other hand \(H_p(\epsilon G_{\text{sbd}})\) is \(\mathcal{O}(\epsilon)\) and of order of magnitude \(\epsilon\) in \(\{\chi_0(p) = 1\} \cap \{|x| \geq \mu/C\}\), by (8.6). Accepting a loss due to the positivity of \(\vartheta\), we choose \(\mu\) so that
\[
\frac{h}{\mu} = \epsilon,
\]

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\[ \mu = \frac{h}{\epsilon} \gg 1. \]  
(8.26)

Then,
\[ G_\epsilon \in S(h + \epsilon(x)), \]  
(8.27)
and in particular,
\[ \|H_{G_\epsilon}\|_g = \mathcal{O}\left( \frac{h}{\langle x \rangle} + \epsilon \right). \]  
(8.28)

In addition to (8.25), (8.6), we know that
\[ H_pG_{\text{sbd}} \geq 0. \]  
(8.29)

Since \( \chi^0 \prec \chi_0 \), it follows that
\[ H_pG_\epsilon \gtrsim \chi^0(p) \left( \frac{h}{\langle x \rangle^{1+\delta}} + \epsilon 1_{\{|x| \geq \mu/C\}} \right) \]  
(8.30)
to be compared with the upper bound, that follows from (8.27) and the fact that \( \text{supp } G_\epsilon \subset p^{-1}(E_-, E_+) \):
\[ H_pG_\epsilon = \mathcal{O}(1) \left( \frac{h}{\langle x \rangle} + \epsilon \right), \]  
(8.31)

where
\[ \frac{h}{\langle x \rangle} + \epsilon \asymp \frac{h}{\langle x \rangle} + \epsilon 1_{\{|x| \geq \mu/C\}}. \]  
(8.32)

**Remark 8.2** We define the spaces \( H(\Lambda_{\nu G_{\text{sbd}}}) \), \( H(\Lambda_{\nu G_\epsilon}) \) as in (5.16), with \( G \) replaced by \( \nu G_{\text{sbd}} \) and \( \nu G_\epsilon \) respectively. Since \( G_{\text{lap}} \in S(h) \) by (8.23), we see that \( H^{\nu G_\epsilon} - H^{\nu G_{\text{sbd}}} = \mathcal{O}(\nu h) \), where \( H^{\nu G_\epsilon}, H^{\nu G_{\text{sbd}}} \) are defined in (5.2), with \( G \) replaced by \( \nu G_\epsilon \) and \( \nu G_{\text{sbd}} \) respectively. We conclude that
\[ \|u\|_{H(\Lambda_{\nu G_\epsilon})} \asymp \|u\|_{H(\Lambda_{\nu G_{\text{sbd}}})}, \]  
(8.33)
uniformly with respect to \( \nu, h, u \), when \( 0 \leq \nu \leq \nu_0 \ll 1, \ h \leq h_0 \ll 1 \).

We now apply Proposition 5.5 and get
\[ \Im(Pu|u)_{H(\Lambda_{\nu G_\epsilon})} = \int_{\Lambda_{\nu G_\epsilon}} P_{\nu}^{\text{top}} Tu \cdot T\nu^{-2\nu H_\epsilon/h} d\alpha + (N_{\nu}u|u)_{H(\Lambda_{\nu G_\epsilon})}, \]  
(8.34)
where we have preferred the more invariant integration on \( \Lambda_\nu G \), rather than
to reduce everything to \( \Lambda_0 \). \( H_\epsilon \) is defined as in (5.2) with \( G \) replaced by \( G_\epsilon \).

Here (cf. (8.27))

\[
P^{\text{top}}_\nu = p^{\text{top}}_\nu + \mathcal{O}(1) \frac{vh(h + \epsilon \langle x \rangle)}{\langle x \rangle^2} = p^{\text{top}}_\nu + \mathcal{O}(1) \frac{vh}{\langle x \rangle} \left( \frac{h}{\langle x \rangle} + \epsilon \right), \quad (8.35)
\]

\[
p^{\text{top}}_\nu(\partial) = \Im(p(\Re \partial + i v H_\epsilon(\Re \partial))) = -v H_\rho G_\epsilon(\Re \partial) + \mathcal{O}(v^3) \| H_\epsilon \|_g^3 \quad (8.36)
\]

Further, \( N_\nu \) is negligible of order \( \nu \).

Here we notice that by (8.32)

\[
\frac{h}{\langle x \rangle^{1+\vartheta}} + \epsilon 1_{\{ |x| \geq \mu/C \}} \gtrsim \left( \frac{h}{\langle x \rangle^{1+\vartheta}} + \epsilon_\vartheta \right) =: m_\epsilon(x; h), \quad \epsilon_\vartheta := \left( \frac{\epsilon}{K} \right)^{\vartheta} \epsilon. \quad (8.37)
\]

(In the limiting case, \( \epsilon = h/C \), where \( C > 0 \) is a large constant, \( \mu \) is of the
order of a large constant and \( \epsilon_\vartheta \asymp h \asymp m_\epsilon \).

Thus if we fix \( \vartheta > 0 \) small enough, we have

\[
\frac{vh}{\langle x \rangle} \left( \frac{h}{\langle x \rangle} + \epsilon \right), \quad v^3 \left( \frac{h}{\langle x \rangle} + \epsilon \right)^3 \ll v \left( \frac{h}{\langle x \rangle^{1+\vartheta}} + \epsilon 1_{\{ |x| \geq \mu/C \}} \right)
\]

It then follows from (8.30), (8.35), (8.36) that

\[
-\Im P^{\text{top}}_\nu(\rho) \geq \frac{v}{C} \chi^0(\rho) \left( \frac{h}{\langle x \rangle^{1+\vartheta}} + \epsilon 1_{\{ |x| \geq \mu/C \}} \right) - v \tilde{k}_\nu, \quad (8.38)
\]

where the right hand side is evaluated at the point \( \Re \rho \),

\[
\tilde{k}_\nu = \mathcal{O}(1) \left( \frac{h}{\langle x \rangle} + \epsilon \right), \quad \text{supp} \tilde{k}_\nu \subset p^{-1}([E_-, E_+] \setminus [E - \delta_0, E + \delta_0]). \quad (8.39)
\]

We retain from this and (8.37), that

\[
-\Im P^{\text{top}}_\nu \geq \frac{v}{C} m_\epsilon(x; h) - v k_\nu, \quad (8.40)
\]

where

\[
k_\nu = \mathcal{O} \left( \frac{h}{\langle x \rangle} + \epsilon \right), \quad \text{supp} k_\nu \cap p^{-1}([E - \delta_0, E + \delta_0]) = \emptyset. \quad (8.41)
\]
Using again the identification of $h$-pops and $h$-tops, we see that

$$
\int_{\Lambda_{v,G_\alpha}} u_k \mathcal{L}_u \cdot \mathcal{M} u e^{-\nu H_v/h} d\alpha \leq v \| R_v u \|_{H(\Lambda_{v,G_\alpha})}^2 + (N_v u | u), \quad (8.42)
$$

where $N_v$ is negligible of order $v$ and $R_v$ is an $h$-pop whose symbol is $O(1)$ in $S((h/\langle x \rangle + \epsilon)^{1/2})$ and with support disjoint from $p^{-1}(E - \delta_0, E + \delta_0)$. Thus with a new negligible operator of order $v$,

$$
- \Im (P u | u)_{H(\Lambda_{v,G_\alpha})} \geq \frac{v}{C} \| m_{1/2}^v u \|_{H(\Lambda_{v,G_\alpha})}^2 - v \| R_v u \|_{H(\Lambda_{v,G_\alpha})}^2 - (N_v u | u). \quad (8.43)
$$

Assume from now on that $v > 0$ is fixed and sufficiently small. \quad (8.44)

We next remark that the arguments work virtually without any changes if we replace $P$ by $P - z$ for $z \in \mathbb{C}$, satisfying,

$$
\Re z \in [E - \delta_0/2, E + \delta_0/2], \quad - \frac{1}{C} \| z \| \leq \Im z \leq \frac{1}{C}, \quad (8.45)
$$

for $C$ sufficiently large. Also, since the support of the symbol of $R_v$ is contained in a region where $|p - z| \geq 1/\mathcal{O}(1)$, we have for every fixed $N > 0$,

$$
\| R_v u \|_{H(\Lambda_{v,G_\alpha})} \leq \mathcal{O}(1) \| (P - z) u \|_{H(\Lambda_{v,G_\alpha})} + \mathcal{O}(1) \| (h/\langle x \rangle)^N (h/\langle x \rangle + \epsilon)^{1/2} u \|_{H(\Lambda_{v,G_\alpha})}. \quad (8.46)
$$

Here (8.46) follows by the calculus of $h$-pseudodifferential operators associated to $\Lambda_{v,G_\alpha}$, see Section 6 of \cite{16}. Using this in (8.43) with $P$ there replaced by $P - z$, we get,

$$
- \Im ((P - z) u | u)_{H(\Lambda_{v,G_\alpha})} \geq \frac{1}{C} \| m_{1/2}^v u \|_{H(\Lambda_{v,G_\alpha})}^2 - C \| (P - z) u \|_{H(\Lambda_{v,G_\alpha})}^2. \quad (8.47)
$$

Here, we have for every $\alpha > 0$,

$$
- \Im ((P - z) u | u)_{H(\Lambda_{v,G_\alpha})} \\
\leq \| m_{\epsilon} (x; h)^{-1/2} (P - z) u \|_{H(\Lambda_{v,G_\alpha})} \| m_{\epsilon} (x; h)^{1/2} u \|_{H(\Lambda_{v,G_\alpha})} \\
\leq \frac{\alpha}{2} \| m_{\epsilon} (x; h)^{-1/2} (P - z) u \|_{H(\Lambda_{v,G_\alpha})}^2 + \frac{1}{2\alpha} \| m_{\epsilon} (x; h)^{1/2} u \|_{H(\Lambda_{v,G_\alpha})}^2.
$$

Use this in (8.47) for a fixed large enough $\alpha$ together with the observation

$$
\| (P - z) u \|_{H(\Lambda_{v,G_\alpha})} \lesssim \| m_{\epsilon} (x; h)^{-1/2} (P - z) u \|_{H(\Lambda_{v,G_\alpha})},
$$

to conclude that

$$
\| m_{\epsilon} (x; h)^{1/2} u \|_{H(\Lambda_{v,G_\alpha})}^2 \leq \mathcal{O}(1) \| m_{\epsilon} (x; h)^{-1/2} (P - z) u \|_{H(\Lambda_{v,G_\alpha})}^2. \quad (8.48)
$$

Summing up, we have:

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Proposition 8.3 Under the assumptions above, in particular about non trapping, we fix $\delta_0 > 0$ small and then $\nu > 0$ small enough. Then for $z$ in the region (8.45), where $C > 0$ is large enough,

$$P - z : H(\Lambda_{\nu G_e}, \langle \xi \rangle^2) \rightarrow H(\Lambda_{\nu G_i})$$

is bijective and

$$m_\nu(x; h)^{\frac{1}{2}}(z - P)^{-1}m_\nu(x; h)^{\frac{1}{2}} = O(1) : H(\Lambda_{\nu G_e}) \rightarrow H(\Lambda_{\nu G_i}), \quad (8.49)$$

where $m_\nu$ is defined in (8.37).

Remark 8.4 By Remark 8.2, we can replace $G_e$ with $\epsilon G_{sbd}$ in (8.49). Now $G_{sbd}$ vanishes for $\langle x \rangle \leq \mu/(2C)$ and it follows that $\|u\|_{H(\Lambda_{\nu G_{sbd}})} \leq \|u\|_{L^2}$ for $u$ with support in a fixed compact set. Moreover $m_\nu(x; h) \sim h$ for $x$ in any fixed compact set. Hence from (8.49), we deduce that

$$\chi(z - P)^{-1}\chi = O(1/h) : L^2 \rightarrow L^2,$$

for every fixed $\chi \in C_0^\infty(\mathbb{R}^n)$.

In order to shorten the notation, we will often write

$$\mathcal{H}_{sbd} = H(\Lambda_{\nu G_{sbd}}), \quad D_{sbd} = H(\Lambda_{\nu G_{sbd}}, \langle \xi \rangle^2), \quad (8.50)$$

where $\nu > 0$ is small and fixed, as above.

We finally treat a trapping case, namely that of a potential well in an island, generating shape resonances. As before, let $E \in ]E_+ - \delta, E_+ + \delta[$ be a fixed energy level. (We can also allow it to vary, as we shall do in the next section, but then some geometric quantities will also vary.) Let $\bar{\Omega} \subset \mathbb{R}^n$ be a connected open set (still assuming $n = 1$ but trying to keep the discussion as general as possible). Let $U_0 \subset \bar{\Omega}$ be a compact subset. Assume:

$$V - E < 0 \text{ in } \mathbb{R}^n \setminus \bar{\Omega}, \quad V - E > 0 \text{ in } \bar{\Omega} \setminus U_0, \quad V - E \leq 0 \text{ in } U_0, \quad (8.51)$$

$$\text{diam}_d U_0 = 0, \quad (8.52)$$

where $d$ is the Lithner-Agmon distance given by the metric $(V - E)_+(x)dx^2$,

$$(V - E)_+ = \max(V - E, 0),$$

The $H_p$-flow has no trapped trajectories in $p^{-1}(E)_{\mathbb{R}^n \setminus \bar{\Omega}}$. \quad (8.53)

Let $M_0 \subset \bar{\Omega}$ be a connected compact set with smooth boundary such that

$$M_0 \supset \{x \in \bar{\Omega}; d(x, \partial \bar{\Omega}) \geq \epsilon_0 \}, \quad (8.54)$$
for some small $\epsilon_0 > 0$. Let $P_0$ denote the Dirichlet realization of $P$ in $M_0$, equipped with the domain $\mathcal{D}(P_0) = H^2(M_0) \cap H^1_0(\tilde{M}_0)$. (The right hand side in (8.54) has smooth boundary, as we recalled after (1.36).)

From Agmon estimates we have the well-known fact that if $\tilde{M}_0 \subset \tilde{O}$ has the same properties as $M_0$ with the same value of $\epsilon_0$, then in any $\alpha(1)$-neighborhood of $E$, the eigenvalues of $P_0$ and $\tilde{P}_0$ differ by $O(1) \exp 2(\alpha + \epsilon_0 - d(U_0, \partial \tilde{O})/h)$ for every $\alpha > 0$. (Cf. [15])

Let $K(h) \subset \mathbb{C}$ converge to \{E\}, when $h \to 0$ such that uniformly for all $z \in K(h)$,

\begin{align}
\text{dist} (z, \sigma(P_0)) & \geq \lambda(h), \\
\lambda(h) & > 0
\end{align}

where $\lambda(h) > 0$ and

\begin{align}
\ln \lambda(h) & \geq -o(1)/h, \ h \to 0. \\
(8.57)
\end{align}

Let $\tilde{V} = V + W$, where $0 \leq W \in C_0^\infty(\text{neigh}(U_0))$ has its support in a small neighborhood of $U_0$ and $V + W - E > 0$ in $\tilde{O}$. Let $\tilde{P} := -h^2 \Delta + V + W$, $\tilde{p}(x, \xi) = \xi^2 + V + W$. Then $\tilde{p}$ satisfies (8.22) if $E_- < E < E_+$ and $E_+ - E_-$ are small enough. Then the resolvent estimate (8.49) applies to $\tilde{P}$ for $z \in K(h)$ and we recall Remark 8.4. We can then apply Agmon estimates inside $\tilde{O}$, as explained in Section 6 of [9], and we get

\begin{align}
(\tilde{P} - z)^{-1}(x,y) = \mathcal{O}(\epsilon^{-d(x,y)/h}), \ x, y \in \tilde{O}, \ z \in K(h), \\
(8.58)
\end{align}

where the notation $\mathcal{O}$ is explained in Proposition 9.3 in [16]. Under the assumptions (8.56), (8.57) we get the same estimate for $P_0$, i.e. we can replace $(\tilde{P}, \tilde{O})$ by $(P_0, M_0)$ in (8.58).

Recall the elementary telescopic formula, for the moment under the a priori assumption that $(P - z)^{-1}$ exists for $z \in K(h)$ (which will follow from the discussion):

\begin{align}
(P - z)^{-1} = (\tilde{P} - z)^{-1} + (\tilde{P} - z)^{-1}W(\tilde{P} - z)^{-1} + (\tilde{P} - z)^{-1}W(P - z)^{-1}W(\tilde{P} - z)^{-1}. \\
(8.59)
\end{align}

It reduces the study of $(P - z)^{-1}$ to that of $W(P - z)^{-1}W$ and we shall make a perturbation series approach. Let $\chi \in C_0^\infty(\tilde{O})$ be equal to 1 on $\{x \in \tilde{O}; \ d(x, \partial \tilde{O}) \geq 2\epsilon_0\}$ and let $\chi_0 \in C_0^\infty(\text{neigh}(\text{supp} W))$ be equal to one near supp $W$. Take the neighborhood small enough so that \text{supp} $\chi_0 \cap \text{supp} (1 - \chi) = \emptyset$. Let $\chi_1 \in C_0^\infty(\tilde{O})$ satisfy $1_{B_d(U_0, 2\epsilon_0)} < \chi_1 < 1_{B_d(U_0, 2\epsilon_0)}$, where
\( B_d(U_0, r) \) denotes the open ball of center \( U_0 \) and radius \( r \) for the Lithner-Agmon distance and \( S_0 := d(U_0, \partial \tilde{\mathcal{O}}) \). As a first approximation to \((P - z)^{-1}\), we take

\[
E = \chi(P_0 - z)^{-1} \chi_1 + (\tilde{P} - z)^{-1}(1 - \chi_1) =: E_0 + \tilde{E}. \tag{8.60}
\]

Then,

\[
(P - z)E = 1 + [P, \chi](P_0 - z)^{-1} \chi_1 - W(\tilde{P} - z)^{-1}(1 - \chi_1) =: 1 - K. \tag{8.61}
\]

From (8.58) and the corresponding estimate for \( P_0 \), we see that

\[
\|K\|_{\mathcal{L}(m_{1/2}^1 H_{sbd}, L^2(\tilde{\mathcal{O}}))} = \tilde{O}(1) \exp \left( -\frac{S_0}{2h} \right), \tag{8.62}
\]

where \( \tilde{O}(1) \) indicates a quantity which is \( \mathcal{O}(e^{\alpha/h}) \) for some \( \alpha > 0 \) which tends to 0 when \( \epsilon_0 \) and \( d(\supp \chi_0, U_0) \) tend to 0. Thus for \( h > 0 \) small enough, the Neumann series,

\[
1 + K + K^2 + K^3 + \ldots \tag{8.63}
\]

converges to \((1 - K)^{-1} = 1 + \tilde{O}(1) \exp \left( -\frac{S_0}{2h} \right) \) in \( \mathcal{L}(m_{1/2}^1 H_{sbd}, L^2(\tilde{\mathcal{O}})) \). It follows that \( E(1 - K) \) is a right inverse of \( P - z : D_{sbd} \to H_{sbd} \) and since the latter is of index 0 by the general theory of resonances ([16]) it is a two-sided inverse.

**Proposition 8.5** Let \( C \supset K(h) \to \{E\}, h \to 0 \) and assume (8.55) – (8.57). Then for \( h > 0 \) small enough and for \( z \in K(h) \), \( P - z : D_{sbd} \to H_{sbd} \) is bijective with inverse

\[
(P - z)^{-1} = \chi(P_0 - z)^{-1} \chi_1 (1 - K)^{-1} + (\tilde{P} - z)^{-1}(1 - \chi_1)(1 - K)^{-1}. \tag{8.64}
\]

Here,

\[
\|\chi(P_0 - z)^{-1} \chi_1 (1 - K)^{-1}\|_{\mathcal{L}(H_{sbd}, H_{sbd})} = \frac{\mathcal{O}(1)}{\dist(z, \sigma(P_0))} \tag{8.65}
\]

and by Proposition 8.3

\[
m_\epsilon(x; h)^{1/2}(\tilde{P} - z)^{-1}(1 - \chi_1)(1 - K)^{-1}m_\epsilon(x; h)^{1/2} = \mathcal{O}(1) : H_{sbd} \to H_{sbd}. \tag{8.66}
\]

We next study the situation when \( z \) gets closer to \( \sigma(P_0) \). Let \( J(h) \subset \mathbb{R} \) be an interval tending to \( \{E\} \) as \( h \to 0 \). Assume that

\[
P_0 \text{ has no spectrum in } \partial J(h) + [-\delta(h), \delta(h)] \tag{8.67}
\]
where the parameter $\delta(h)$ is small but not exponentially small;

$$\ln \delta(h) \geq -o(1)/h.$$  

$\sigma(P_0) \cap J(h)$ is a discrete set of the form $\{\mu_1(h), ..., \mu_m(h)\}$ where $m = m(h) = \mathcal{O}(h^{-n})$ and we repeat the eigenvalues according to their multiplicity. Let $\Gamma(h)$ denote the set of resonances of $P$ in $J(h) - i[0, \epsilon_\phi/C], C \gg 1$, also repeated according to their (algebraic) multiplicity. Assume that

$$\epsilon \geq e^{-1/(Ch)},$$  

for some $C \gg 1$, so that

$$\epsilon \left(\frac{\epsilon}{h}\right) \geq e^{-\frac{1}{\epsilon_\phi} - \frac{2S_0}{h}}.$$  

Then we have,

**Proposition 8.6** There is a bijection $b : \{\mu_1, ..., \mu_m\} \to \Gamma(h)$, such that

$$b(\mu) - \mu = \tilde{O}(e^{-2S_0/h}),$$

where $S_0 = d(U_0, \partial\tilde{O})$ and the tilde indicates that the right hand side is $\mathcal{O}(e^{(\omega - 2S_0)/h})$, where $\omega = \omega(\epsilon_0) > 0$ and $\omega(\epsilon_0) \to 0$, when $\epsilon_0 \to 0$.

We shall prove the proposition and also get precise information about the resolvent by studying an associated Grushin problem. Let $e_0^0(\epsilon_0)$ be an orthonormal system of eigenfunctions of $P_0$ associated to the eigenvalues $\mu_1(h), ..., \mu_m(h)$. Then we know from Chapter 6 of [9] that

$$e_j^0 = \tilde{O}(e^{-d(U_0, x)/h}).$$  

A first trivial Grushin problem for $P_0$ is defined by the matrix

$$P_0 = \begin{pmatrix} P_0 - z & R_0^0 \\ R_0^+ & 0 \end{pmatrix} : \mathcal{D}(P_0) \times \mathbb{C}^m \to L^2(M_0) \times \mathbb{C}^m,$$

where

$$R_0^+ u(j) = (u|e_j^0), \quad R_0^- = (R_0^+)^*.$$  

Let

$$\tilde{J}(h) = J(h) + i[\epsilon_\phi/C, 1/C]$$

with $C > 0$ sufficiently large (cf. (8.45)). Then it follows from Section 9 of [10] that for $z \in \tilde{J}(h)$, the operator $P_0(z)$ is bijective with inverse

$$E_0 = \begin{pmatrix} E_0^0(z) & E_0^0(z) \\ E_0^+(z) & E_0^-(z) \end{pmatrix} : L^2(M_0) \times \mathbb{C}^m \to \mathcal{D}(P_0) \times \mathbb{C}^m.$$
where, with $\Pi_0$ denoting the spectral projection onto the space spanned by $\epsilon_0, \ldots, \epsilon_m$,

$$E^0_0(z) = (P_0 - z)^{-1} (1 - \Pi_0) = \mathcal{O}(1/\delta(h)) : L^2 \rightarrow \mathcal{D}(P_0),$$  \hspace{1cm} (8.74)

$$E^0_+ (z) v_+ = \sum v_+ (j) e^0_j, \| E^0_+ \|_{L^\infty(C_m, \mathcal{D}(P_0))} \leq \mathcal{O}(1),$$  \hspace{1cm} (8.75)

$$E^0_-(z) u(j) = (u|e^0_j), \| E^0_\|_{L_2(C_m)} \leq \mathcal{O}(1),$$  \hspace{1cm} (8.76)

$$E^0_{-+}(z) = \text{diag} (z - \mu_j).$$  \hspace{1cm} (8.77)

Choose $\chi \in C^\infty_0(M_0)$ as after (8.59) and put

$$R_+ = R_\chi^\circ : \mathcal{H}_{sbd} \rightarrow C^m,$$  \hspace{1cm} (8.78)

$$R_- = \chi R_\circ : C^m \rightarrow \mathcal{H}_{sbd}.$$  \hspace{1cm} (8.79)

Define,

$$\mathcal{P}(z) = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}_{sbd} \times C^m \rightarrow \mathcal{H}_{sbd} \times C^m.$$  \hspace{1cm} (8.80)

This is a Fredholm operator of index 0, so to show that it is bijective it suffices to construct a right inverse.

Let $\chi_0, \chi_1, \chi$ be as in the construction of $(P - z)^{-1}$ in Proposition 8.5 (where the assumptions on $z$ were different). Following the same path as there, we put

$$\tilde{E} = \begin{pmatrix} \chi E^0_0 \chi & \chi E^0_+ \\ E^0_- \chi & E^0_+ \end{pmatrix} + \begin{pmatrix} (P - z)^{-1} (1 - \chi_1) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{E} & \tilde{E}_+ \\ \tilde{E}_- & \tilde{E}_{-+} \end{pmatrix},$$  \hspace{1cm} (8.81)

$$\tilde{E} = \mathcal{O}(1/\delta(h)) + \underbrace{\mathcal{O}(1)}_{\mathcal{H}_{sbd} \rightarrow \mathcal{D}_{sbd}} \underbrace{m_2^{1/2} \mathcal{D}_{sbd} \rightarrow m_2^{1/2} \mathcal{D}_{sbd}}_{\mathcal{H}_{sbd} \rightarrow \mathcal{D}_{sbd}} = \mathcal{O}(1/\epsilon_0) : \mathcal{H}_{sbd} \rightarrow \mathcal{D}_{sbd}.$$  \hspace{1cm} (8.82)

A straightforward calculation gives

$$\mathcal{P}(z) \tilde{E}(z) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$  \hspace{1cm} (8.83)

where

$$A_{11} = (P - z) \chi E^0_0 \chi + \chi R_\circ^0 E^0_+ \chi + 1 - \chi_1 - W(P - z)^{-1} (1 - \chi_1),$$

$$A_{12} = (P - z) \chi E^0_+ + \chi R_\circ^0 E^0_+,$$

$$A_{21} = R_\circ^0 \chi^2 E^0_0 \chi + R_\circ^0 \chi (P - z)^{-1} (1 - \chi_1)$$

$$A_{22} = R_\circ^0 \chi^2 E^0_+.$$
Here, using standard Lithner-Agmon estimates for $E^0$, $(\tilde{P} - z)^{-1}$, together with the fact that

$$E^0_\chi = E^0_\chi + \tilde{O}(e^{-S_0/(2h)})$$

we get

$$(P - z)E^0_\chi + \chi R^0_\chi E^0_\chi = \chi ((P - z)E^0 + R^0 E^0)\chi + \tilde{O}(e^{-S_0/(2h)})$$

$$= \chi_1 + \tilde{O}(e^{-S_0/(2h)})$$

and

$$W(\tilde{P} - z)^{-1}(1 - \chi_1) = \tilde{O}(e^{-S_0/(2h)})$$.

Hence

$$A_{11} = 1 + \tilde{O}(e^{-S_0/(2h)})$$ \quad (8.84)

Similarly,

$$A_{12} = \chi ((P - z)E^0_{+^0} + R^0_{+^0}E^0_{-^0}) + [P, \chi]E^0_{+^0} = \tilde{O}(e^{-S_0/h})$$ \quad (8.85)

$$A_{21} =
\begin{align*}
R^0_\chi \chi_1 + R^0_\chi (\chi^2 - 1_{M_0})E^0_\chi = \tilde{O}(e^{-S_0/2h}) \\
\tilde{O}(e^{-S_0/(2h)})
\end{align*}$$

$$A_{22} =
\begin{align*}
R^0_\chi E^0_+ + R^0_\chi (\chi^2 - 1_{M_0})E^0_+ = 1 + \tilde{O}(e^{-2S_0/h}) \quad \text{(8.86)}
\end{align*}$$

A first conclusion is that

$$P(z)\tilde{E}(z) = 1 + \tilde{O}(e^{-S_0/(2h)}): \mathcal{H}_{ab} \times \mathbb{C}^m \rightarrow \mathcal{H}_{ab} \times \mathbb{C}^m,$$

where the remainder has entries with distribution kernels supported in $M_0 \times M_0$, $M_0 \times \mathbb{C}^m$, $\mathbb{C}^m \times M_0$, $\emptyset$ respectively. so $P(z)$ is bijective with inverse

$$E(z) = \tilde{E}(z)(1 + \tilde{O}(e^{-S_0/(2h)})) = \tilde{E}(z) + \tilde{O}(e^{-S_0/(2h)}): m^{1/2}_{\epsilon} \mathcal{H}_{ab} \times \mathbb{C}^m \rightarrow m^{1/2}_{\epsilon} \mathcal{D}_{ab} \times \mathbb{C}^m \quad (8.88)$$

In particular, if we write

$$E(z) = \begin{pmatrix}
E(z) & E_+(z) \\
E_-(z) & E_-(z)
\end{pmatrix}, \quad (8.89)$$

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we get
\[ E_{-+} - E_{++}^0 = \tilde{O}(e^{-S_0/(2h)}) \]
and \( E \) satisfies the estimate in (8.82).

We shall improve this estimate by working with exponential weights in \( \tilde{O} \). For \( \phi \in C^\infty_0(M_0) \) real-valued, we put
\[ \mathcal{H}_{\phi} = e^{\phi/h}\mathcal{H}_{sbd} \]
equipped with the norm \( \| e^{-\phi/h}u \|_{\mathcal{H}_{sbd}} \). As a vector space it is equal to \( \mathcal{H}_{sbd} \).
The constructions above work without any great changes if we assume that
\[ \text{supp} \nabla \phi \cap U_0 = \emptyset, \quad (\nabla \phi)^2 \leq V - E - \epsilon_0. \]

By varying \( \phi \) we see that \( m^{\frac{1}{2}} E (m^{\frac{1}{2}} (x,y)) = \tilde{O}(e^{-d(x,y)/h}), \quad (m^{\frac{1}{2}} E_\pm)(x) = \tilde{O}(e^{-d(x,U_0)/h}), \quad (E m^{\frac{1}{2}})(y) = \tilde{O}(e^{-d(U_0,y)/h}), \quad x,y \in B_d(U_0,S_0), \) \( (8.90) \)

where we use the same symbols to denote the distribution kernels of \( E, E_\pm \).

For \( v_+ \in C^m \), the solution \((u,u_-)\) of the problem
\[{\begin{cases} (P - z)u + R_- u_- = 0, \\ R_+ u = v_+ \end{cases}} \tag{8.91} \]
is given by \( u = E_+ v_+, \quad u_- = E_- v_- \). As an approximate solution to (8.91), we take \( u^0 = \chi E_+^0 v_+, \quad u_-^0 = E_- v_- \). Then
\[{\begin{cases} (P - z)u^0 + R_- u_-^0 = [P,\chi] E_+^0 v_+, \\ R_+ u^0 = v_+ + R_+^0 (\chi^2 - 1_M) E_+^0 v_+ \end{cases}} \]
so we get the solution to (8.90) in the form
\[{\begin{cases} u = u^0 - E [P,\chi] E_+^0 v_+ - E_+ R_+^0 (\chi^2 - 1_M) E_+^0 v_+, \\ u_- = u_-^0 - E_- [P,\chi] E_+^0 v_+ - E_+ R_+^0 (\chi^2 - 1_M) E_+^0 v_+ \end{cases}} \tag{8.92} \]

Now it follows from (8.90) and the corresponding estimates for \( E^0, E_\pm^0 \), that
\[ |u_- - u_-^0| = \tilde{O}(e^{-2S_0/h}) |v_+| , \]
which means that
\[ E_{-+} - E_{-+}^0 = \tilde{O}(e^{-2S_0/h}). \tag{8.93} \]
Proof of Proposition 8.6. By (8.77), (8.93) reads
\[ E_{-+} - \text{diag}(z - \mu_j) = \tilde{O}(e^{-2S_0/h}). \] (8.94)

Now the resonances of \( P \) in \( \tilde{J}(h) \) are the zeros of \( \det(E_{-+}) \) and we get the proposition by means of elementary arguments for zeros of holomorphic functions of one variable. \( \square \)

The first equation in (8.92) can be written
\[ E_+v_+ - \chi E_0^0 v_+ = -\left( E[P, \chi]E_0^0 + E_+ R_0^0 (\chi^2 - 1) E_0^0 \right) v_+ \]
and it follows that
\[ \| m_{\tilde{\epsilon}}^\frac{1}{2} (E_+v_+ - \chi E_0^0 v_+) \|_{\mathcal{H}_{sbd}} = \tilde{O}(e^{-S_0/h})|v_+|, \]
i.e.
\[ m_{\tilde{\epsilon}}^\frac{1}{2} (E_+ - \chi E_0^0) = \tilde{O}(e^{-S_0/h}) : \mathbb{C}^m \rightarrow \mathcal{H}_{sbd}. \] (8.95)

Taking the adjoints with respect to the scalar product on \( L^2(\mathbb{R}^n) \times \mathbb{C}^m \), we have
\[ \mathcal{P}(z)^* \mathcal{E}(z)^* = 1, \]
where
\[ \mathcal{P}(z)^* = \begin{pmatrix} P^* - z & R_+^* \\ R_-^* & 0 \end{pmatrix}, \quad \mathcal{E}(z)^* = \begin{pmatrix} E(z)^* & E_+(z)^* \\ E_+(z)^* & E_-(z)^* \end{pmatrix}, \]
and hence \( \mathcal{E}(z)^* \) can be constructed by starting with
\[ \hat{\mathcal{E}}(z) = \begin{pmatrix} \chi E_{0+}^0 & \chi E_{-+}^0 \\ E_0^0 & \chi \end{pmatrix} + \begin{pmatrix} (\bar{P}^* - \bar{z})^{-1}(1 - \chi_1) & 0 \\ 0 & 0 \end{pmatrix}. \]
In analogy with (8.95) we get
\[ m_{\tilde{\epsilon}}^\frac{1}{2} (E_+^* - \chi E_0^0) = \tilde{O}(e^{-S_0/h}) : \mathbb{C}^m \rightarrow \mathcal{H}_{sbd}^*, \]
where we notice that \( \mathcal{H}_{sbd}^* = H(\Lambda_{\mathbb{R}G_{sbd}}) \), in view of Proposition 8.8 of [1].

By duality,
\[ (E_- - E_0^0 \chi) m_{\tilde{\epsilon}}^\frac{1}{2} = \tilde{O}(e^{-S_0/h}) : \mathcal{H}_{sbd} \rightarrow \mathbb{C}^m. \] (8.96)

Recall the standard formula for Grushin problems:
\[ (z - P)^{-1} = -E(z) + E_+(z)E_-(z)^{-1}E_-(z), \quad z \in \tilde{J} \setminus \Gamma(h). \] (8.97)
Here, $E(z)$ is holomorphic and by (8.89), (8.81), (8.88),
\[
E(z) = \chi E^0 \chi_1 + (\tilde{P} - z)^{-1} (1 - \chi_1) + \mathcal{O}(e^{-S_0/(2h)}) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to m^\frac{1}{2}_e \mathcal{D}_{sbd}.
\]
(8.98)
\[
E(z) = \mathcal{O}(h/\delta) + \mathcal{O}(1) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to m^\frac{1}{2}_e \mathcal{D}_{sbd}.
\]
(8.99)
Here the term $\mathcal{O}(h/\delta)$ represents the term $\chi E^0 \chi_1$ which is $\mathcal{O}(1/\delta)$ as an operator on $L^2(\mathbb{R}^n)$.

When either $m = 1$ or $\text{dist}(z, \sigma(P_0)) \geq \tilde{\mathcal{O}}(e^{-2S_0/h})$, it follows from (8.93) that
\[
E(z) = O(h/\delta) + O(1) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to m^\frac{1}{2}_e \mathcal{D}_{sbd}.
\]
(8.100)
Here we also assumed for simplicity that $\text{dist}(z, \sigma(P_0) \cap J(h)) = \text{dist}(z, \sigma(P_0))$, which can be achieved by a slight shrinking of the interval $J(h)$. Thus, when (8.100) holds, we get
\[
E(z) = O(h/\delta) + O(1) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to m^\frac{1}{2}_e \mathcal{D}_{sbd},
\]
(8.101)
where we also used that by (8.95), (8.96),
\[
E_+ = O(1) : \mathcal{C}^m \to m^\frac{1}{2}_e \mathcal{D}_{sbd};
\]
(8.102)
\[
E_- = O(1) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to \mathcal{C}^m.
\]
(8.103)
Now (8.101) implies that $\psi E_+ E_+^{-1} E_- \psi$ is $\mathcal{O}(1/(h \text{dist}(z, \Gamma))) : L^2 \to L^2$ for every $\psi \in \mathcal{C}^\infty_c$. Using (8.95), (8.96) more directly, we get
\[
E_+ E_+^{-1} E_- - \chi E_+ E_+^{-1} E_- \chi = \frac{1}{\text{dist}(z, \Gamma)} \tilde{\mathcal{O}}(e^{-S_0/h}) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to m^\frac{1}{2}_e \mathcal{D}_{sbd}
\]
(8.104)
and here
\[
\chi E_+ E_+^{-1} E_- \chi = \mathcal{O}(h/\text{dist}(z, \Gamma)) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to m^\frac{1}{2}_e \mathcal{D}_{sbd}.
\]
(8.105)
From (8.97), (8.98), (8.99), (8.100), (8.102), (8.103), we get

**Proposition 8.7** We let $z$ vary in the set $\tilde{J}(h)$ in (8.72). Assume that $m = 1$ or $\text{dist}(z, \sigma(P_0)) \geq \tilde{\mathcal{O}}(e^{-2S_0/h})$, and also that $\text{dist}(z, \sigma(P_0) \cap J(h)) = \text{dist}(z, \sigma(P_0))$. Then we have,
\[
(z - P)^{-1} = \mathcal{O}(h/\delta) + \mathcal{O}(1) + \mathcal{O}(h/\text{dist}(z, \Gamma)) : m^\frac{1}{2}_e \mathcal{H}_{sbd} \to m^\frac{1}{2}_e \mathcal{D}_{sbd},
\]
(8.106)
where the first two terms to the right are holomorphic in $z$. 
9 Back to adiabatics

Let \( I \subset \mathbb{R} \) be an interval and let

\[
V_t = V(t, x) \in C^\infty_b(I \times \mathbb{R}^n, \mathbb{R}).
\]

(9.1)

We assume that (cf. (8.3))

\[
V_t \text{ has a holomorphic extension (also denoted } V_t \text{) to } \{ x \in \mathbb{C}^n; |\Re x| > C, |\Im x| < |\Re x|/C \} \text{ such that } V_t(x) = o(1), x \to \infty.
\]

(9.2)

\[
\partial_t V_t(x) = 0 \text{ for } |x| \geq C, \text{ for some constant } C > 0
\]

(9.3)

It is tacitly assumed that \( V(t, x) \) does not depend on \( h \). However, when considering a narrow potential wells in an island, of diameter \( \asymp h \), we will have to make an exception and allow such an \( h \)-dependence in a small neighborhood of the well.

Let \( 0 < E_- < E'_- < E'_+ < E_+ < \infty \) and let

\[
E_0(t) \in C^\infty_b(I; [E'_-, E'_+]).
\]

(9.4)

We assume that \( V_t - E_0(t) \) has a potential well in an island as in Section 8.

Let \( \tilde{O} = \tilde{O}(t) \subset \mathbb{R}^n \) be a connected open set and let \( U_0(t) \subset \tilde{O}(t) \) be compact. Assume (cf. (8.51))

\[
V_t - E_0(t) \begin{cases} < 0 \text{ in } \mathbb{R}^n \setminus \tilde{O}(t), \\ > 0 \text{ in } \tilde{O}(t) \setminus U_0(t), \\ \leq 0 \text{ in } U_0(t), \end{cases}
\]

(9.5)

\[
d_t d(U_0(t)) = 0.
\]

(9.6)

Here \( d_t \) is the Lithner-Agmon distance on \( \tilde{O}(t) \), given by the metric \( (V_t - E_0(t))_+ dx^2 \).

Also assume that with \( p_t = \xi^2 + V_t(x) \),

the \( H_{p_t} \)-flow has no trapped trajectories in \( p_t^{-1}(E_0(t))|_{\mathbb{R}^n \setminus \tilde{O}(t)} \).

(9.7)

It follows that

\[
d_x V_t \neq 0 \text{ on } \partial \tilde{O}(t),
\]

(9.8)

so \( \partial \tilde{O}(t) \) is smooth and depends smoothly on \( t \). Thus \( \tilde{O}(t) \) is a manifold with smooth boundary, depending smoothly on \( t \). Further, \( U_0(t) \) depends
continuously on $t$. (This will still be true when we allow $h$-dependence near $U_0(t).$)

For $\epsilon_0 > 0$ small, we define

$$M_0(t) = \{x \in \tilde{O}(t); d_t(x, \partial \tilde{O}(t)) \geq \epsilon_0\}, \quad (9.9)$$

so $M_0(t) \Subset \tilde{O}(t)$ is a compact set with smooth boundary, depending smoothly on $t$. (Here we use the structure of $d_t(x, \tilde{O}(t))$ that follows from (9.8), see \[10\].)

When $I$ is a fixed compact interval, the assumptions above are fulfilled uniformly in $t$. Since we also want to allow $I$ to be a very long interval, we add the following compactness assumption:

$$(V_t, E_0(t)) \in \mathcal{K}, \ \forall t \in I, \text{ where } \mathcal{K} \text{ is a compact subset of } \{V \in C^\infty_b(\mathbb{R}^n; \mathbb{R}); V \text{ satisfies } (9.2) \text{ with a fixed constant } C\} \times [E'_-, E'_+]$$

such that $(V, E)$ satisfies the assumptions \[9.3\], with a fixed $C$ as well as \[9.5\], \[9.6\], \[9.7\].

$$(9.10)$$

Let $P_0(t)$ denote the Dirichlet realization of $P(t) = -h^2 \Delta + V_t(x)$ on $M_0(t)$. If we enumerate the eigenvalues of $P_0(t)$ in $]E_-, E_+[$ in increasing order (repeated with multiplicities) we know (as a general fact for 1-parameter families of self-adjoint operators), that they are uniformly Lipschitz functions of $t$. Let $\mu_0(t) = \mu_0(t; h)$ be such an eigenvalue and assume (cf. (8.67)),

$$\mu_0(t; h) = E_0(t) + o(1), \ h \to 0, \text{ uniformly in } t. \quad (9.11)$$

$$\mu_0(t; h) \text{ is a simple eigenvalue and } \quad (9.12)$$

$$\sigma(P_0(t)) \cap [E_0(t) - \delta(h), E_0(t) + \delta(h)] = \{\mu_0(t; h)\}. \quad (9.12)$$

Here, as in Section 8, $\delta(h) > 0$ is small but not exponentially small,

$$\ln \delta(h) \geq -o(1)/h, \ h \to 0. \quad (9.13)$$

We restrict the spectral parameter $z$ to $D(\mu_0(t), \delta(h)/2)$. In this region we have,

$$(z - P_0(t))^{-1} = \mathcal{O} \left( \frac{1}{|z - \mu_0(t)|} \right): L^2 \to \mathcal{D}(P_0(t)), \quad (9.14)$$

and more generally,

$$\partial_t^k(z - P_0(t))^{-1} = \mathcal{O} \left( \frac{1}{|z - \mu_0(t)|^{1+k}} \right): L^2 \to \mathcal{D}(P_0(t)). \quad (9.15)$$
Strictly speaking, we work on sufficiently small time intervals, where we can replace $P_0(t)$ with the unitarily equivalent operator $U(t)^{-1}P_0(t)U(t)$, where $U(t): L^2(M_0(t)) \to L^2(M_0(t_0))$ is induced by a diffeomorphism $\kappa_t: M_0(t_0) \to M_0(t)$, depending smoothly on $t$. The spectral projection $\Pi_0(t)$, associated to $(P_0(t), \mu_0(t))$ is given by

$$\Pi_0(t) = \frac{1}{2\pi i} \int_{\partial D(\mu_0(t), r)} (z - P_0(t))^{-1} dz, \quad 0 < r \leq \delta(h)/2,$$

and choosing $r = \delta(h)/2$, we see that

$$\partial_t^k \Pi_0(t) = O(\delta(h)^{-k}) : L^2 \to D(P_0(t)).$$

It follows that we can choose a normalized eigenfunction $e_0(t)$:

$$P_0(t)e_0(t) = \mu_0(t)e_0(t), \quad \|e_0\|_{L^2} = 1,$$

such that

$$\partial_t^k e_0(t) = O(\delta(h)^{-k}) \text{ in } D(P_0(t)), \quad k = 1, 2, \ldots.$$

Now it is classical that

$$\mu_0(t) = (P_0(t)e_0(t)|e_0(t)),$$

$$\partial_t \mu_0(t) = (\partial_t P_0 e_0|e_0) + (P_0 \partial_t e_0|e_0) + (P_0 e_0|\partial_t e_0),$$

where the sum of the last two terms is equal to 0:

$$(\partial_t e_0|P_0 e_0) + (P_0 e_0|\partial_t e_0) = \mu_0(t) ((\partial_t e_0|e_0) + (e_0|\partial_t e_0)) = \mu_0(t) \partial_t (e_0|e_0) = 0.$$

Thus,

$$\partial_t \mu_0(t) = (\partial_t P_0 e_0|e_0) = O(1),$$

and after differentiating in $t$:

$$\partial_t^k \mu_0(t) = O(\delta(h)^{-k+1}), \quad k = 1, 2, \ldots.$$ 

For our purposes, it will be enough to work with the weaker estimate

$$\partial_t^k \mu_0(t) = O(\delta(h)^{-k}), \quad k = 1, 2, \ldots$$

Before discussing shape resonances, it will be convenient to discuss some simple symmetry properties. In [16], (7.17) it was shown that

$$(u|v)_{H(\Lambda_G)} = (Bu|v)_{L^2(\mathbb{R^n})}, \quad u, v \in H(\Lambda_G),$$
where $B : H(\Lambda_G) \to H(\Lambda_{-G})$ is the sum of an elliptic Fourier integral operator and a nop of order 1. (Here $G$ denotes the function “$\nu G$” in Section 8, where the parameter “$\nu$” is fixed according to (8.44).) Taking complex conjugates and exchanging $u$ and $v$, we get

$$(u|v)_{H(\Lambda_G)} = (u|Bv)_{L^2(\mathbb{R}^n)}, \quad u, v \in H(\Lambda_G).$$

Write

$$\langle u|v \rangle = \int_{\mathbb{R}^n} uv dx$$

for the bilinear scalar product on $L^2$, so that

$$\langle u|v \rangle = \langle u|\Gamma v \rangle_{L^2}, \quad \text{where } \Gamma v := \overline{v}.$$ 

**Proposition 9.1** We have $\Gamma = \mathcal{O}(1) : H(\Lambda_G) \to H(\Lambda_{\tilde{G}})$, where $\tilde{G}(x, \xi) = G(x, -\xi)$.

**Proof.** This follows from 3 easily checked facts, where $\tilde{G}(x, \xi) = G(x, -\xi)$:

1) $(x, \xi) \in \Lambda_{\tilde{G}} \iff (\overline{x}, -\overline{\xi}) \in \Lambda_G$.

2) If $T$ is an FBI-transformation adapted to $\Lambda_G$, then for $u \in H(\Lambda_G)$,

$$T \Gamma u(\alpha) = \overline{T u(\overline{x}, -\overline{\xi})}, \quad \alpha \in \Lambda_{\tilde{G}},$$

where $\overline{T}$ is an FBI-transformation adapted to $\Lambda_G$.

3) Let $H$ be the function on $\Lambda_G$, defined in (5.2) and let $\tilde{H}$ be the corresponding function on $\Lambda_{\tilde{G}}$. Then

$$\tilde{H}(x, \xi) = H(\overline{x}, -\overline{\xi}).$$

Then,

$$\langle u|\Gamma B v \rangle = (u|v)_{H(\Lambda_G)}, \quad u, v \in H(\Lambda_G),$$

and here $\Gamma B$ is an antilinear bijection $H(\Lambda_G) \to H(\Lambda_{-G})$ with $\Gamma B$ and $(\Gamma B)^{-1}$ uniformly bounded.

Since $P = P^t$ is symmetric (with “$t$” indicating transpose for the bilinear scalar product),

$$\langle Pu|v \rangle = \langle u|Pv \rangle,$$
and hence $p(x, -\xi) = p(x, \xi)$ (also clear from the explicit formula $p(x, \xi) = \xi^2 + V(x)$), we see that $-\check{G}$ is also an escape function and hence also $\frac{1}{2}G - \frac{1}{2}\check{G}$. Replacing $G$ with the latter we get a new escape function $\check{G}$ satisfying

$$ -\check{G} = G. \quad (9.25) $$

Now $\Gamma B$ becomes an antilinear bijection: $H(\Lambda_G) \to H(\Lambda_G)$, uniformly bounded with its inverse. Replacing $v$ with $(\Gamma B)^{-1}v$ in (9.24), we get

$$ \langle u|v \rangle = (u|(\Gamma B)^{-1}v)_{H(\Lambda_G)}, \; u, v \in H(\Lambda_G). \quad (9.26) $$

Then $\langle u|v \rangle$ is a bilinear nondegenerate scalar product on $H(\Lambda_G)$ (In the case of ordinary complex scaling, this is seen more directly by a shift of contour in (9.23).)

We resume the earlier discussion with $G = G_{sbd}$ (assuming for simplicity that the parameter “$\upsilon$” in Section 8 is equal to 1) and apply Propositions 8.6, 8.7 with $J = [\mu_0(t; h) - \delta(h)/2, \mu_0(t; h) + \delta(h)/2]$. Let

$$ \Omega(t) := \left\{ z \in D(\mu_0(t; h), \delta(h)/2); \Im z \geq -\epsilon_\theta/C \right\}, \quad (9.27) $$

where we recall that $\epsilon_\theta = (\epsilon/h)^\theta \epsilon$. Then $P(t)$ has a unique resonance $\lambda_0(t) = \lambda_0(t; h)$ in $\Omega(t)$. It is simple and

$$ \lambda_0(t) - \mu_0(t) = \tilde{O}(e^{-2S_1/h}), \; S_1 := d_t(U_0(t), \partial\tilde{O}(t)). \quad (9.28) $$

(8.106) gives

$$ (z - P(t))^{-1} = O\left(\frac{h}{\delta(h)}\right) + O(1) + O\left(\frac{h}{z - \lambda_0(t)}\right): m_\epsilon^{\frac{1}{2}}H_{sbd} \to m_\epsilon^{-\frac{1}{2}}D_{sbd}, \quad (9.29) $$

where the first two terms in the right hand side are holomorphic in $z$. In addition to (8.24), (8.68) and the assumption $\ln \delta(h) \geq -o(1)/h$, we assume from now on that

$$ \epsilon \leq \delta(h), \quad (9.30) $$

so that the first two terms in (9.29) drop out when $|z - \lambda_0(t)| \leq \epsilon_\theta$.

We have the spectral projection

$$ \pi_0(t) = \frac{1}{2\pi i} \int_{\partial D(\lambda_0(t), r)} (z - P(t))^{-1}dz, \quad (9.31) $$

where $0 < r \leq \epsilon_\theta/(2C)$ and choosing the maximal value for $r$, we get from (9.29),

$$ \pi_0(t) = O(h): m_\epsilon^{\frac{1}{2}}H_{sbd} \to m_\epsilon^{-\frac{1}{2}}D_{sbd}. \quad (9.32) $$
For the higher $t$ derivatives, we write

$$
\partial^k_t \pi_0(t) = \frac{1}{2\pi i} \int_{\partial D(\lambda_0(t), r)} \partial^k_t (z - P(t))^{-1} dz,
$$

(9.33)

where the integrand is a linear combination of terms,

$$(z - P(t))^{-1}(\partial^{k_1}_t P)(z - P(t))^{-1}(\partial^{k_2}_t P)\ldots(\partial^{k_\ell}_t P)(z - P(t))^{-1},
$$

(9.34)

with $k_j \geq 1$, $k_1 + k_2 + \ldots + k_\ell = k$. In view of (8.37), we have $m_\epsilon \asymp h$ on every fixed compact set and since $P$ is independent of $t$ outside such a set, we conclude that

$$
\partial^k_t P(t) = O(1/h) : m_\epsilon^{-\frac{1}{2}} \mathcal{H}_{sbd} \rightarrow m_\epsilon^{\frac{1}{2}} \mathcal{H}_{sbd}.
$$

Also, for $z \in \Omega(t)$, we have

$$(z - P(t))^{-1} = \frac{O(h)}{z - \lambda_0(t)} : m_\epsilon^{\frac{1}{2}} \mathcal{H}_{sbd} \rightarrow m_\epsilon^{-\frac{1}{2}} \mathcal{H}_{sbd}, \text{ when } |z - \lambda_0(t)| \lesssim \epsilon_0.
$$

Hence the term (9.34) is

$$
\frac{O(h)}{(z - \lambda_0(t))^{k+1}} : m_\epsilon^{\frac{1}{2}} \mathcal{H}_{sbd} \rightarrow m_\epsilon^{-\frac{1}{2}} \mathcal{D}_{sbd},
$$

so the integrand in (9.33) is

$$
\frac{O(h)}{(z - \lambda_0(t))^{k+1}} : m_\epsilon^{\frac{1}{2}} \mathcal{H}_{sbd} \rightarrow m_\epsilon^{-\frac{1}{2}} \mathcal{D}_{sbd},
$$

and we conclude that

$$
\partial^k_t \pi_0(t) = \frac{O(h)}{\epsilon_0^k} : m_\epsilon^{\frac{1}{2}} \mathcal{H}_{sbd} \rightarrow m_\epsilon^{-\frac{1}{2}} \mathcal{D}_{sbd}.
$$

(9.35)

Let us fix $t \in I$ for a while and write $P = P(t)$. $P$ is symmetric for the bilinear scalar product (9.26) and so is the Grushin operator $P$ in (8.80) ($m = 1$) if we use

$$
\left\langle \begin{pmatrix} u \\ u_- \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{u}_- \end{pmatrix} \right\rangle = \langle u|\tilde{u} \rangle + u_- \tilde{u}_-,
$$

and take care to use real eigenfunctions of $P_0$, when defining $R_{\pm}$. Then the inverse $\mathcal{E} : \mathcal{H}_{sbd} \times \mathbb{C} \rightarrow \mathcal{D}_{sbd} \times \mathbb{C}$ is symmetric:

$$
E^t = E, \quad E^t_+ = E_-.
$$

(9.36)
Using that

\[(z - P)^{-1} = -E(z) + E_+(z)E_{-+}(z)^{-1}E_-(z)\]

in (9.31), we get (with \(\lambda_0 = \lambda_0(t)\) etc.),

\[\pi_0 = \frac{1}{E'_{-+}(\lambda_0)}E_+(\lambda_0)E_-(\lambda_0).\]  

(9.37)

Here, by (8.94) and the Cauchy inequality,

\[E'_{-+} = 1 + \tilde{O}(1)e^{-2S_t/h}\epsilon/\vartheta.\]  

(9.38)

From (cf. (8.68)), we get

\[\epsilon/\vartheta \geq e^{-S_t/h}.\]  

(9.39)

Now, by (8.95), (9.36) we have

\[E_+(\lambda_0)v_+ = v_+e_+,\ E_-(\lambda_0)v = \langle v|e_+\rangle,\]  

(9.40)

\[m_0^\frac{1}{2}(e_+ - \chi e_0) = \tilde{O}(e^{-S_t/h}) \text{ in } D_{sbd}.\]  

(9.41)

Since \(m_0 \geq \epsilon_\phi\), this implies that,

\[e_+ - \chi e_0 = \tilde{O}(e^{-S_t/h}\epsilon_\phi^{1/2}) \text{ in } D_{sbd}.\]  

(9.42)

\[m_0^{-\frac{1}{2}}(e_+ - \chi e_0) = \tilde{O}(e^{-S_t/h}/\epsilon_\phi) \text{ in } D_{sbd}.\]  

(9.43)

From (9.37), (9.40), we get

\[\pi_0 u = \frac{1}{E'_{-+}(\lambda_0)}\langle u|e_+\rangle e_+,\]  

(9.44)

and in particular that \(e_+\) is a resonant state. The reproducing property \(\pi_0^2 = \pi_0\) means that \(\pi_0 e_+ = e_+\), which by (9.44) is equivalent to

\[\frac{1}{E'_{-+}(\lambda_0)}\langle e_+|e_+\rangle = 1.\]  

(9.45)

Put

\[e^0 = (E'_{-+}(\lambda_0))^{-\frac{1}{2}}e_+ = (1 + \tilde{O}(e^{-2S_t/h}/\epsilon_\phi))e_+.\]  

(9.46)

Then

\[\pi_0 u = \langle u|e^0\rangle e^0, \quad \langle e^0|e^0\rangle = 1.\]  

(9.47)
We next estimate the $t$-derivatives of $e^0 = e^0(t)$. We work in a small neighborhood of a variable point $t_0 \in I$. Then

$$f(t) = \pi_0(t)e^0(t_0)$$

is collinear to $e^0(t)$ and we recover $e^0(t)$ from the formula

$$e^0(t) = \langle f(t) | f(t) \rangle^{-\frac{1}{2}} f(t). \tag{9.48}$$

By (9.35), we have

$$\| m_\epsilon^{1/2} \partial_t^k f(t) \|_{\mathcal{D}_{sbd}} = \mathcal{O}(h \epsilon^{-k}) \| m_\epsilon^{-1/2} e^0(t_0) \|_{\mathcal{H}_{sbd}}. \tag{9.49}$$

Using that $m_\epsilon \geq \epsilon_\theta$, we conclude that

$$\| \partial_t^k f(t) \|_{\mathcal{D}_{sbd}} = \mathcal{O}(1) \left( \frac{h}{\epsilon_\theta} \right)^k, \quad k \geq 0. \tag{9.50}$$

for $k \geq 1$. For $k = 0$ we have $\| f \|_{\mathcal{D}_{sbd}} = 1$ so we have the simpler but weaker estimate,

$$\| \partial_t^k f(t) \|_{\mathcal{D}_{sbd}} = \mathcal{O}(1) \left( \frac{h}{\epsilon_\theta} \right)^k, \quad k \geq 0. \tag{9.51}$$

From (9.48) it then follows that

$$\| \partial_t^k e^0(t) \|_{\mathcal{D}_{sbd}} = \mathcal{O}(1) \left( \frac{h}{\epsilon_\theta} \right)^k, \quad k \geq 0, \tag{9.52}$$

and this implies that

$$\partial_t^k \pi_0(t) = \mathcal{O}(1) \left( \frac{h}{\epsilon_\theta} \right)^k : \mathcal{H}_{sbd} \to \mathcal{D}_{sbd}, \quad k \geq 0. \tag{9.53}$$

Next, we estimate $\partial^k \lambda_0(t)$, $k = 1, 2, \ldots$. We start with

$$\lambda_0(t) = \langle P(t)e^0(t) | e^0(t) \rangle. \tag{9.54}$$

By the symmetry of $P(t)$ and the fact that $\langle e^0(t) | e^0(t) \rangle = 1$, we get

$$\partial_t \lambda_0(t) = \langle (\partial_t P(t))e^0(t) | e^0(t) \rangle = \mathcal{O}(1). \tag{9.55}$$

It follows from (9.52) that

$$\partial_t^{k+1} \lambda_0 = \mathcal{O}(1) \left( \frac{h}{\epsilon_\theta} \right)^k, \quad k \geq 0,$$
and hence,
\[ \partial_t^k \lambda_0 = \mathcal{O}(1) \left( \frac{h}{\epsilon_\vartheta} \right)^{(k-1)+}, \quad k \geq 0. \tag{9.56} \]

Recall from (8.24), (9.30), that
\[ \epsilon \leq \min(\frac{h}{C}, \delta), \quad C \gg 0. \tag{9.57} \]

\( \lambda_0 \) does not depend on the choice of \( \epsilon \) and if we make the maximal choice in
(9.57), we get \( \epsilon \approx \min(1, \delta/h)^{\vartheta} \min(h, \delta) \) and (9.56) gives
\[ \partial_t^k \lambda_0 = \mathcal{O}(1) \left( h \min(1, \delta/h)^{2+2\vartheta} \right)^{-(k-1)+} \tag{9.58} \]

Then (9.21) implies similar subexponential estimates for \( \partial_t^k (\lambda_0 - \mu_0) \), \( k \geq 1 \). Combining this with (9.28) and elementary interpolation estimates, we get
\[ \partial_t^k (\lambda_0 - \mu_0) = \tilde{\mathcal{O}}(e^{-2S_t/h}), \quad k \geq 0, \tag{9.59} \]

and we can then use (9.21) again, to get
\[ \partial_t^k \lambda_0 = \mathcal{O}(\delta(h)^{-k}), \quad k \geq 1. \tag{9.60} \]

We next study \( (\lambda_0(t) - P(t))^{-1}(1 - \pi_0(t)) \) and its derivatives. In the discussion leading to (9.35) we have seen that
\[ m_t^4 \partial_t^k (z - P(t))^{-1} m_t^4 = \frac{\mathcal{O}(h)}{(z - \lambda_0)^{k+1}} : \mathcal{H}_{ab} \to \mathcal{D}_{ab}, \quad |z - \lambda_0| \leq \epsilon_\vartheta/C \]
and hence
\[ \partial_t^k (z - P(t))^{-1} = \frac{\mathcal{O}(h)}{\epsilon_\vartheta(z - \lambda_0)^{k+1}} : \mathcal{H}_{ab} \to \mathcal{D}_{ab}, \quad |z - \lambda_0| \leq \epsilon_\vartheta/C. \]

Combining this with (9.53), we get
\[ \partial_t^k ((z - P(t))^{-1}(1 - \pi_0(t))) = \mathcal{O}(1) \sum_{k_1 + k_2 = k} \frac{h}{\epsilon_\vartheta(z - \lambda_0)^{k_1+1}} \left( \frac{h}{\epsilon_\vartheta} \right)^{k_2} : \mathcal{H}_{ab} \to \mathcal{D}_{ab}. \]

When \( |z - \lambda_0| \approx \epsilon_\vartheta \), the majorant is
\[ \leq \mathcal{O}(1) \sum_{k_1 + k_2 = k} \frac{h}{\epsilon_\vartheta^2 \epsilon_\vartheta^{k_1}} \left( \frac{h}{\epsilon_\vartheta^2} \right)^{k_2} \]
\[ \leq \mathcal{O}(1) \sum_{k_1 + k_2 = k} \frac{h}{\epsilon_\vartheta} \left( \frac{h}{\epsilon_\vartheta} \right)^{k_1} \left( \frac{h}{\epsilon_\vartheta} \right)^{k_2} \leq \mathcal{O}(1) \left( \frac{h}{\epsilon_\vartheta} \right)^{k+1}. \]
Since \((z - P(t))^{-1}(1 - \pi_0(t))\) and its \(t\)-derivatives are holomorphic near \(z = \lambda_0(t)\) the maximum principle gives for \(|z - \lambda_0(t)| \leq \epsilon_\delta / C:\)

\[
\partial^k_t ((z - P(t))^{-1}(1 - \pi_0(t))) = \mathcal{O}(1) \left( \frac{h}{\epsilon_\delta} \right)^{k+1} : \mathcal{H}_{sbd} \to \mathcal{D}_{sbd}. \tag{9.61}
\]

With the Cauchy inequalities, this extends to

\[
\partial^k_z \partial^k_t ((z - P(t))^{-1}(1 - \pi_0(t))) = \mathcal{O}(1) \epsilon_\delta^{-\ell} \left( \frac{h}{\epsilon_\delta} \right)^{k+1} = \mathcal{O}(1) \left( \frac{h}{\epsilon_\delta} \right)^{k+\ell+1} : \mathcal{H}_{sbd} \to \mathcal{D}_{sbd}. \tag{9.62}
\]

Finally we put \(z = \lambda_0(t)\) and get with the natural meaning of “lincomb”

\[
\partial^k_t \left( (\lambda_0(t) - P_0(t))^{-1}(1 - \pi_0(t)) \right) = \text{lincomb}_{\ell_1 + \cdots + \ell_\lambda = k} \partial^m_t \partial^{\lambda}_z \left( (z - P_0(t))^{-1}(1 - \pi_0(t)) \right)_{z = \lambda_0(t)}^\prime (\partial^\ell_1 \lambda_0) \cdots (\partial^\ell_\lambda \lambda_0).
\]

Using \((9.62), (9.60)\), we see that the \(\mathcal{L}(\mathcal{H}_{sbd}, \mathcal{D}_{sbd})\)-norm of the general term is

\[
\mathcal{O}(1) \epsilon_\delta^{-\lambda} \left( \frac{h}{\epsilon_\delta} \right)^{m+1} \delta^{-(\ell_1 + \cdots + \ell_\lambda)} \leq \mathcal{O}(1) \left( \frac{1}{\delta \epsilon_\delta} \right)^{k-m} \left( \frac{h}{\epsilon_\delta} \right)^{m+1} \leq \mathcal{O}(1) \frac{h}{\epsilon_\delta} \left( \frac{\max(h, \epsilon_\delta / \delta)}{\epsilon_\delta^2} \right)^k,
\]

where we used that \(\lambda \leq \ell_1 + \cdots + \ell_\lambda = k - m\). Thus for every \(k \in \mathbb{N}\),

\[
\partial^k_t \left( (\lambda_0(t) - P_0(t))^{-1}(1 - \pi_0(t)) \right) = \mathcal{O}(1) \frac{h}{\epsilon_\delta} \left( \frac{\max(h, \epsilon_\delta / \delta)}{\epsilon_\delta^2} \right)^k : \mathcal{H}_{sbd} \to \mathcal{D}_{sbd}. \tag{9.63}
\]

When \(\epsilon\) is exponentially small, \(\epsilon = \exp(-1/\mathcal{O}(h))\), or more generally when \(\epsilon_\delta \leq \delta h\), the estimate simplifies to

\[
\partial^k_t \left( (\lambda_0(t) - P_0(t))^{-1}(1 - \pi_0(t)) \right) = \mathcal{O}(1) \left( \frac{h}{\epsilon_\delta} \right)^{k+1} : \mathcal{H}_{sbd} \to \mathcal{D}_{sbd}, \quad k \geq 0. \tag{9.64}
\]

We next consider formal adiabatic solutions in the spirit of Proposition 2.1. For the moment, we let \(\epsilon, \varepsilon\) be independent parameters.
Proposition 9.2  Under the assumptions above, there exist two formal asymptotic series,

\[ \nu(t, \varepsilon) \sim \nu_0(t) + \varepsilon \nu_1(t) + \varepsilon^2 \nu_2(t) + \ldots \text{ in } C^\infty(I; D_{sbd}), \]

\[ \lambda(t, \varepsilon) \sim \lambda_0(t) + \varepsilon \lambda_1(t) + \varepsilon^2 \lambda_2(t) + \ldots \text{ in } C^\infty(I), \]

such that

\[ (\varepsilon D_t + P(t) - \lambda(t, \varepsilon))\nu(t, \varepsilon) \sim 0 \]

as a formal asymptotic series in \( C^\infty(I; \mathcal{H}_{sbd}) \). Here,

\[ \partial_k t \nu_j = O(1)(h/\hat{\epsilon}^2 \vartheta)^{2j+k} \text{ in } D_{sbd}, \quad j \geq 0, \quad k \geq 0, \]

\[ \partial_k t \lambda_j = O(1)(h/\hat{\epsilon}^2 \vartheta)^{2j-1+k}, \quad j \geq 1, \quad k \geq 0. \]

Here, \( \hat{\epsilon} := \max(1, \epsilon_0/(\delta h))^{1/2} = \min(\epsilon_0, (\epsilon_0 \delta h)^{1/2}). \)

Proof. We sacrifice optimal sharpness for simplicity and work with the weaker form of (9.63):

\[ \partial_k t (\lambda_0(t) - P_0(t))^{-1}(1 - \pi_0(t)) = O(1)(h/\hat{\epsilon}^2 \vartheta)^{1+k}; \]

which is equivalent to (9.64) in the most interesting case when \( \epsilon_0 \leq \delta h \). We shall use (9.60): \( \partial_k t \lambda_0 = O(\delta^{-k}) \) and the following weakened form of (9.52):

\[ \partial_k t e^0(t) = O(1)(h/\hat{\epsilon}^2 \vartheta)^k \text{ in } D_{sbd}. \]

We follow the proof of Proposition 2.1 and annihilate successively the powers of \( \varepsilon \) in the right hand side of (2.4). The first equation is then

\[ (P(t) - \lambda_0(t))\nu_0(t) = 0, \]

so we choose

\[ \nu_0(t) = \theta_0(t)e^0(t) \]

with the condition

\[ \theta_0(t) \asymp 1, \quad \partial_k t \theta_0 = O(1)(h/\hat{\epsilon}^2 \vartheta)^k, \]

so that \( \nu_0 \) satisfies (9.68). Then the \( e^0 \) term in (2.4) vanishes.

To annihilate the \( \varepsilon^i \)-term, we need to solve (2.9) which is solvable precisely when (cf. (2.10))

\[ 0 = \langle \lambda_1(t)\nu_0(t) - D_t\nu_0(t)|e^0(t) \rangle = \theta_0(t)\lambda_1(t) - \langle D_t\nu_0(t)|e^0(t) \rangle. \]
Here,
\[ \langle D_t \nu_0(t) | e^0(t) \rangle = D_t \theta_0(t) + \theta_0(t) \langle D_t e^0(t) | e^0(t) \rangle = D_t \theta_0(t), \]
since
\[ \langle D_t e^0(t) | e^0(t) \rangle = \frac{1}{2} D_t \langle e^0 | e^0 \rangle = 0, \]
recalling that \( \langle e^0 | e^0 \rangle = 1 \).
Thus, \( \lambda_1 \) should satisfy \( \theta_0(t) \lambda_1(t) - D_t \theta_0(t) = 0, \)
\[ \lambda_1(t) = \frac{D_t \theta_0}{\theta_0}, \tag{9.76} \]
and in particular, \( \partial^k \lambda_1(t) = \mathcal{O}(1)(h/\hat{\epsilon}_0^2)^{1+k} \), so \( \lambda_1 \) satisfies (9.69).

**Remark 9.3** A natural choice of \( \theta_0 \) is \( \theta_0 = 1 \). Then we get \( \lambda_1 = 0 \) in (9.76).

With this unique choice of \( \lambda_1 \), we can solve (2.9) and the general solution is
\[ \nu_1(t) = (P(t) - \lambda_0(t))^{-1}(1 - \pi_0(t))(\lambda_1(t)\nu_0 - D_t\nu_0(t)) + z(t)e^0(t), \tag{9.77} \]
where we are free to choose \( z(t) \), and we will take \( z(t) = 0 \) for simplicity. From (9.71), the estimate (9.69) for \( \lambda_1 \) and (9.68) for \( \nu_0 \), we get
\[ \partial^k \nu_1 = \mathcal{O}(1)(h/\hat{\epsilon}_0^2)^{2+k} \text{ in } D_{sbd}, \; k \geq 0, \tag{9.78} \]
i.e. \( \nu_1 \) satisfies (9.68).

The equation for annihilating the \( \varepsilon_j \)-term in (2.4) is
\[ (P(t) - \lambda_0(t))\nu_j = (\lambda_1 - D_t)\nu_{j-1} + \lambda_2\nu_{j-2} + \ldots + \lambda_j\nu_1 + \lambda_j\nu_0. \tag{9.79} \]
Let \( N \geq 2 \) and assume that we have already constructed \( \nu_j, \lambda_j \) for \( j \leq N-1 \), satisfying (9.68), (9.69), (9.79) for \( j \leq N-1 \). Consider (9.79) for \( j = N \). The condition for finding a solution \( \nu_N \) is that the right hand side is orthogonal (for \( \langle \cdot | \cdot \rangle \) to \( e^0 \) and since \( \langle \nu_0 | e^0 \rangle = \theta_0(t) \), we get
\[ \lambda_N = \theta_0^{-1}((D_t - \lambda_1)\nu_{N-1} + \lambda_2\nu_{N-2} + \ldots + \lambda_{N-1}\nu_1 | e^0). \tag{9.80} \]

Here
\[ \partial^k D_t \nu_{N-1} = \mathcal{O}(1)(h/\hat{\epsilon}_0^2)^{2(N-1)+k+1} = \mathcal{O}(1)(h/\hat{\epsilon}_0^2)^{2N-1+k} \tag{9.81} \]
and for \( 1 \leq \ell \leq N-1 \):
\[ \partial^k (\lambda_0 \nu_{N-\ell}) = \mathcal{O}(1)(h/\hat{\epsilon}_0^2)^{2\ell-1+2(N-\ell)+k} = \mathcal{O}(1)(h/\hat{\epsilon}_0^2)^{2N-1+k}. \tag{9.82} \]
Using also (9.75), we see that \( \lambda_N \) satisfies (9.69).
We can now solve for $\nu_N$ in (9.79):

$$
\nu_N = (P(t) - \lambda_0(t))^{-1}(1 - \pi_0(t)) \\
((\lambda_1 - D_t)\nu_{N-1} + \lambda_2\nu_{N-2} + \ldots + \lambda_{N-1}\nu_1 + \lambda_N\nu_0) + z(t)e^0(t).
$$

(9.83)

Again we take $z = 0$ for simplicity and get, using (9.71), (9.81), (9.82):

$$
\partial_t^k \nu_N = O(1)(h/\hat{\epsilon}_\phi)^{1+2N-1+k} = O(1)(h/\hat{\epsilon}_\phi)^{2N+k},
$$
so $\nu_N$ satisfies (9.68) and this finishes the inductive proof. \(\square\)

**Remark 9.4** The construction of $\nu_j, \lambda_j$ is independent of the choice of ambient spaces and if we choose $\epsilon$ maximal in (9.57) we see as after that inequality that (9.69) becomes

$$
\partial_t^k \lambda_j = O(1) \left( \min(1, \delta/h)^{\vartheta} \min(\delta, \min(1, \delta/h)^{1+\vartheta}) \right)^{-(2j+k-1)}, \quad j \geq 1, \quad k \geq 0.
$$

(9.84)

When $\delta \leq h$ this simplifies to

$$
\partial_t^k \lambda_j = O(1) \left( \left( \frac{h}{\delta} \right)^{1+\vartheta} \frac{1}{\delta} \right)^{2j-1+k}.
$$

This can probably be improved as in the proof of (9.60).

We continue the discussion under the assumptions of Proposition 9.2. Put for $N \geq 1$

$$
\nu^{(N)} = \nu_0 + \epsilon \nu_1 + \ldots + \epsilon^N \nu_N, \quad (9.85)
$$

$$
\lambda^{(N)} = \lambda_0 + \epsilon \lambda_1 + \ldots + \epsilon^N \lambda_N, \quad N \geq 1. \quad (9.86)
$$

Then by construction (cf. (2.4)),

$$
(\epsilon D_t + P(t) - \lambda^{(N)})\nu^{(N)} = r^{(N+1)},
$$

(9.87)

where

$$
r^{(N+1)} = \epsilon^{N+1} D_t \nu_N - \sum_{\substack{j,k \leq N \ \ j+k \geq N+1}} \epsilon^{j+k} \lambda_j \nu_k. \quad (9.88)
$$

From the estimates in Proposition 9.2 we get

$$
r^{(N+1)} = O(1) \left( \epsilon^{N+1} (h/\hat{\epsilon}_\phi)^{2(N+1)} + \sum_{\substack{j,k \leq N \ \ j+k \geq N+1}} \epsilon^{j+k} (h/\hat{\epsilon}_\phi)^{2(j+k)-1} \right) \text{ in } D_{sbd}.
$$
In the following, we assume that
\[
\frac{\varepsilon^{1/2}h}{\varepsilon_0^2} \ll 1. \tag{9.89}
\]
Recall from (9.13) that \(\delta = \delta(h)\) is small, but not exponentially small and that \(\epsilon_\delta = (\epsilon/h)^u \varepsilon\). Then (9.89) holds if we assume that \(\varepsilon\) is exponentially small:
\[
0 < \varepsilon \leq O(1) \exp\left(-1/(Ch)\right), \quad \text{for some } C > 0, \tag{9.90}
\]
and choose
\[
\varepsilon \geq \varepsilon^{u/(u+\theta) - \alpha}, \tag{9.91}
\]
for some \(\alpha \in ]0, 1/(4(1 + \theta))].\)

Having assumed (9.89) we get
\[
\partial_t r^{(N+1)} = O(1) \varepsilon^{1/2} \left(\frac{\varepsilon h}{\varepsilon_0^2}\right)^{2N+1} \text{ in } D_{\text{sbd}} \text{ and more generally,}
\]
\[
\partial_t^k r^{(N+1)} = O(1) \varepsilon^{1/2} \left(\frac{\varepsilon h}{\varepsilon_0^2}\right)^{2N+1} \left(\frac{h}{\varepsilon_0^2}\right)^k \text{ in } D_{\text{sbd}}. \tag{9.92}
\]
Also, since \(\|\nu_0(t)\|_{H_{\text{sbd}}} = \|e^0(t)\|_{H_{\text{sbd}}},\) we get
\[
\|\nu^{(N)}(t)\|_{H_{\text{sbd}}} = (1 + O(\varepsilon h^2/\varepsilon_0^4))\|\nu_0(t)\|_{H_{\text{sbd}}} \asymp 1. \tag{9.93}
\]
Recall (7.30) with the subsequent observation and the choice of \(\mu\) in (8.26):
\[
-\Im(P(t)u|u)_{H_{\text{sbd}}} \geq -O(\varepsilon^\infty)\|u\|_{H_{\text{sbd}}}^2. \tag{9.94}
\]
Let \(I \ni t \mapsto u(t) \in H(\Lambda_\varepsilon G, \{\xi\})\) be continuous such that \(\partial_t u\) is continuous with values in \(H(\Lambda_\varepsilon G, \{\xi\}^{-1})\), \(G = G_{\text{sbd}}\). Assume that \(u\) is a solution of
\[
(\varepsilon D_t + P(t))u(t) = 0. \tag{9.95}
\]
Then,
\[
\varepsilon \partial_t \|u(t)\|_{H_{\text{sbd}}}^2 = 2\Im(P(t)u|u) \leq O(\varepsilon^\infty)\|u\|_{H_{\text{sbd}}}^2,
\]
implying
\[
\|u(t)\|_{H_{\text{sbd}}} \leq e^{O(\varepsilon^\infty)(t-s)/\varepsilon}\|u(s)\|_{H_{\text{sbd}}}, \quad t \geq s.
\]
Assume
\[
\varepsilon \leq O(\varepsilon^{1/N_0}), \quad \text{for some fixed } N_0 > 0. \tag{9.95}
\]
Then,
\[
\|u(t)\|_{H_{\text{sbd}}} \leq e^{O(\varepsilon^\infty)(t-s)}\|u(s)\|_{H_{\text{sbd}}}, \quad t \geq s. \tag{9.96}
\]

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From (9.94) and the fact that $P(t_0) - z : D_{sbd} \to H_{sbd}$ is Fredholm of index 0, when $\Re z > 0$, we see that

$$\| (P(t_0) - z)^{-1} \|_{L(H_{sbd}, H_{sbd})} \leq \frac{1}{\Re z - y}, \quad \text{for } \Re z > y,$$

(9.97)

where $y = O(\epsilon^\infty)$ can be chosen independent of $t_0 \in I$. By the Hille-Yosida theorem, $-iP(t_0)$ generates a strongly continuous semi-group leading to: If $u_0 \in D_{sbd}$, then $\exists! u \in C([0, +\infty]; D_{sbd}) \cap C^1([0, +\infty]; H_{sbd})$ such that

$$\epsilon D_t + P(t_0) u(t) = 0 \quad \text{for } t \geq 0, \quad u(0) = u_0,$$

(9.98)

where we entered the parameter $\epsilon > 0$ to conform to the general discussion.

Now $P(t) - P(t_0)$ is a smooth function of $t$ with values in $L(H_{sbd}, H_{sbd})$ and an application of [22, Theorem 6.1 and Remark 6.2], allows us to conclude that for every $u_0 \in D_{sbd}$ and every $s \in I$, there exist $u_0 \in C(I \cap [s, \infty]; D_{sbd}) \cap C^1(I \cap [s, \infty]; H_{sbd})$ such that

$$\epsilon D_t + P(t) u(t) = 0 \quad \text{for } s \leq t \in I, \quad u(s) = u_0.$$

(9.99)

Again the solution satisfies (9.96).

This allows us to define the forward fundamental matrix $E(t, s)$, $I \ni t \geq s \in I$ of $\epsilon D_t + P(t)$:

$$
\begin{cases}
(\epsilon D_t + P(t))E(t, s) = 0, & t \geq s, \\
E(t, t) = 1
\end{cases}
$$

and from [22, Theorem 6.1 and Remark 6.2] we infer, in particular, that $E(t, s)$ is strongly continuous in the $H_{sbd}$-norm both in $t$ and $s$, such that

$$\| E(t, s) \|_{L(H_{sbd}, H_{sbd})} \leq \exp((t - s)O(\epsilon^\infty / \epsilon)), \quad t \geq s, \quad t, s \in I.
$$

(9.100)

If $v \in C(I; H_{sbd})$ vanishes for $t$ near $\inf I$, we can solve $(\epsilon D_t + P(t)) u = v$ on $I$ by

$$u(t) = \frac{i}{\epsilon} \int_{\inf I}^t E(t, s)v(s)ds.$$

Now return to (9.85)–(9.87) with $\lambda_j, \nu_j$ as in Proposition 9.2 and $r^{(N+1)}$ satisfying (9.92). We notice that

$$\lambda^{(N)} = \lambda_0 + O(1)\epsilon^{\frac{1}{2}} h^{\frac{1}{2}},$$

(9.101)
and that this improves to
\[ \lambda^{(N)} = \lambda_0 + \mathcal{O}(1) \varepsilon^{\frac{1}{2}} \left( \frac{\varepsilon^{\frac{1}{2}} h}{\varepsilon_0^{\frac{1}{2}}} \right)^3, \quad (9.102) \]
if we take
\[ \theta_0 = 1, \lambda_1 = 0 \quad (9.103) \]
as in Remark 9.3. We assume (9.103) in the following.

Assume, to fix the ideas, that \(0 \in I\), and restrict the attention to \(I_+ = \{t \in I; t \geq 0\}\). From (9.87), we get
\[ (\varepsilon D_t + P(t))u^{(N)} = \rho^{(N+1)}, \quad t \in I_+, \quad (9.104) \]
where
\[ u^{(N)} = e^{-i \int_0^t \lambda^{(N)} ds / \varepsilon \nu^{(N)}}, \quad \rho^{(N+1)} = e^{-i \int_0^t \lambda^{(N)} ds / \varepsilon \tau^{(N+1)}}. \quad (9.105) \]
By (9.93), (9.92), we have
\[ \|\rho^{(N+1)}\|_{\mathcal{H}_{ab}} = \mathcal{O}(1) \varepsilon^{\frac{1}{2}} \left( \frac{\varepsilon^{\frac{1}{2}} h}{\varepsilon_0^{\frac{1}{2}}} \right)^{2N+1} \|u^{(N)}\|_{\mathcal{H}_{ab}}. \quad (9.106) \]
Taking the imaginary part of the scalar product in \(\mathcal{H}_{ab}\) with \(u^{(N)}\), we get with norms and scalar products in \(\mathcal{H}_{ab}\):
\[ -\frac{1}{2} \varepsilon \partial_t \|u^{(N)}\|^2 + (\Re P u^{(N)} | u^{(N)}) = \Re(\rho^{(N+1)} | u^{(N)}), \]
\[ \varepsilon \partial_t \|u^{(N)}\|^2 = 2(\Re P u^{(N)} | u^{(N)}) - 2 \Re(\rho^{(N+1)} | u^{(N)}) \]
\[ \leq \mathcal{O}(\varepsilon^{\infty}) \|u^{(N)}\|^2 + 2 \|\rho^{(N+1)}\| \|u^{(N)}\|. \]
Hence, by (9.106) and the assumption (9.95),
\[ \varepsilon \partial_t \|u^{(N)}\|^2 \leq \mathcal{O}(1) \varepsilon^{\frac{1}{2}} \left( \frac{\varepsilon^{\frac{1}{2}} h}{\varepsilon_0^{\frac{1}{2}}} \right)^{2N+1} \|u^{(N)}\|^2, \]
leading to
\[ \|u^{(N)}(t)\| \leq e^{\mathcal{O}(1) t \varepsilon^{-1/2} (\varepsilon^{1/2} h / \varepsilon_0^{1/2})^{2N+1}} \|u^{(N)}(0)\|, \quad 0 \leq t \in I. \quad (9.107) \]
Assume,
\[ (\sup I) \varepsilon^{-\frac{1}{4}} \left( \frac{\varepsilon^{\frac{1}{2}} h}{\varepsilon_0^{\frac{1}{2}}} \right)^{2N+1} \leq \mathcal{O}(1). \quad (9.108) \]
Then, for $0 \leq t \in I$,

$$
\|u^{(N)}(t)\|_{H_{sbd}} \leq O(1)\|u^{(N)}(0)\|_{H_{sbd}},
$$

$$
\|\rho^{(N+1)}(t)\|_{H_{sbd}} \leq O(1)\varepsilon^{\frac{1}{2}} \left( \frac{\varepsilon^2 h}{\varepsilon^2} \right)^{2N+1} \|u^{(N)}(0)\|_{H_{sbd}}.
$$

(9.109)

Using the fundamental matrix $E(t, s)$ to correct the error $\rho^{(N+1)}$ we have the exact solution $u = u^{(N)}_{\text{exact}},$

$$
u = u^{(N)}(N) + \int_0^t E(t, s)\rho^{(N+1)}(s)ds
$$

(9.110)

of the equation $(\varepsilon D_t + P(t))u = 0$ on $I_+.$

From (9.108) we get

$$
sup I \leq \varepsilon^{-N_0},
$$

(9.111)

for some fixed finite $N_0.$ Then by (9.100), (9.95),

$$
\|E(t, s)\|_{L(H_{sbd}, H_{sbd})} \leq e^{O(\varepsilon^\infty)} = 1 + O(\varepsilon^\infty),
$$

(9.112)

and using this and (9.109) in (9.110), we get

$$
\|u - u^{(N)}\|_{H_{sbd}} \leq O(1)\varepsilon^{-1}(sup I)\varepsilon^{\frac{1}{2}} \left( \frac{\varepsilon^2 h}{\varepsilon^2} \right)^{2N+1} \|u^{(N)}(0)\|_{H_{sbd}}.
$$

(9.113)

This estimate is the main result of the present work. Let us recollect the assumptions and the general context in the following theorem.

**Theorem 9.5** Let $V_t = V(t, x) \in C^\infty_b(I \times \mathbb{R}^n; \mathbb{R})$, where $n = 1, 0 < E_-, E'_- < E_+ < E'_+ < E_+ < \infty,$ $E_0(t) \in C^\infty(I; [E'_-, E'_+]),$ $\bar{O}(t) \in \mathbb{R}^n,$ $U_0(t) \subset \bar{O}(t)$ be as in the discussion around and including (9.1)–(9.8), (9.9)–(9.13). Let $\mu_0(t)$ be a Dirichlet eigenvalue of $-h^2\Delta + V(t, \cdot)$ on $M_0$ as in (9.9)–(9.13). The operator $P(t)$ has a unique resonance $\lambda_0(t)$ in the set $\Omega(t)$ in (9.27). It is simple and satisfies (9.28). Here $\epsilon_\theta = (\epsilon/h)^\theta \epsilon,$ for some $\epsilon \in [e^{-1/(\epsilon h)}, \epsilon/C]$ for some sufficiently large constant $C > 0$ and $\theta > 0$ is a fixed small constant. Assume (9.57), (8.68):

$$
e^{-1/(C_1 h)} \leq \epsilon \leq \min(h/C_2, \delta), \quad C_1, C_2 \gg 1.
$$

(9.114)

Define the spaces $H_{sbd} = H(\Lambda_{G_{sbd}}),$ $\mathcal{D}_{sbd} = H(\Lambda_{G_{sbd}}; \langle \xi \rangle^2)$ as earlier in this section, so that $\lambda_0(t)$ is the unique eigenvalue in $\Omega(t)$ of $P(t) : H_{sbd} \rightarrow H_{sbd}$ with domain $\mathcal{D}_{sbd}.$
Then we have the formal asymptotic series $\nu(t, \varepsilon)$, $\lambda(t, \varepsilon)$ in Proposition 9.2, where we choose $\nu_0(t)$ in (9.74) with $\theta_0(t) = 1$, so that $\lambda_1(t) = 0$. For $N \geq 1$, define the partial sums $\nu^{(N)}$, $\lambda^{(N)}$ as in (9.85), (9.86). Let $\varepsilon$ be small enough so that (9.89) holds (and notice that this would follow from (9.90), (9.91)) and bounded from below by some positive power of $\varepsilon$ as in (9.95). Assume (to fix the ideas) that $0 \in I$, and assume (9.108) so that $\sup I \leq \varepsilon^{-N_0}$ for some constant $N_0 > 0$ and put $I_+ = I \cap [0, +\infty[$. Let $u(t) \in C^4(I_+; \mathcal{H}_{abd}) \cap C^0(I_+; \mathcal{D}_{abd})$ be the solution of

$$
(\varepsilon D_t + P(t))u = 0 \text{ on } I_+, \ u(0) = u^{(N)}(0),
$$

where $u^{(N)}$ is defined in (9.105). Then (9.113) holds uniformly for $t \in I_+$.

We shall finally describe a situation appearing in the mesoscopic problems studied in [19], [29], where the potential well is of diameter $\sim h$. The potential will be result of drilling a well of width $\sim h$ in a “filled potential”. Let us first describe the filled potential $\tilde{V}_t = \tilde{V}(t, x)$. Assume,

$$
\tilde{V}(t, x) \in C^\infty_0(I \times \mathbb{R}^n; \mathbb{R})
$$

still with $n = 1$ for the moment.

$\tilde{V}_t$ has a holomorphic extension (also denoted $\tilde{V}_t$) to

$$
\{ x \in \mathbb{C}^n; \ |\Re x| > C, \ |\Im x| < |\Re x|/C \}
$$

such that $\tilde{V}_t(x) = o(1)$, $x \to \infty$ for some constant $C > 0$.

We next define $V_t(x)$ by drilling a thin well of $t$-dependent depth and of diameter $2h$. Fix a point $x_0 \in \mathbb{R}^n$ and assume,

$$
\tilde{V}_t(x_0) \geq 1/C, \ t \in I.
$$

Let $I \ni t \mapsto \alpha^t \in [1/C, C]$ (where $C > 0$ is a new constant) be a smooth function with

$$
\partial_k^t \alpha^t = O_k(1), \ k \in \mathbb{N}
$$

and put

$$
V_t(x) = \tilde{V}_t(x) - \alpha^t 1_{U_0}(x), \ U_0 = B(x_0, h).
$$

We need a first reference operator. Choose $\tilde{V}_t(x) \in C^\infty_0(\mathbb{R}^n; \mathbb{R})$ such that

$$
\tilde{V}_t(x) = \tilde{V}_t(x) \text{ in a neighborhood of } x_0, \text{ independent of } t, h,
$$

$$
\tilde{V}_t \geq \tilde{V}_t(x_0) - \delta_0,
$$

$$
100
$$
for some small fixed constant $\delta_0 > 0$. Put $\hat{V}_t = \tilde{V}_t - \alpha^t 1_{\tilde{U}_0}$. Then $\hat{P}_t := -h^2 \Delta + \hat{V}_t$ is a self-adjoint operator (defined by means of Friedrichs extension) with purely discrete spectrum in $]-\infty, V(x_0) - \delta_0[$, bounded from below by $\min(\hat{V}_t(x_0) - \delta_0, \hat{V}_t(x_0) - \alpha^t - O(h))$.

The eigenvalues in the interval $]-\infty, V(x_0) - 2\delta_0[$ can be obtained by scaling and simple semi-classical analysis: Let

\[ e_0(\alpha) < e_1(\alpha) < ... < e_{k(\alpha)}(\alpha) < 0 \]

be the negative eigenvalues of $-\partial^2 - \alpha_1_{-1,1}(x)$ on $\mathbb{R}$. Then for $h < 0$ small enough, the eigenvalues of $\hat{P}_t$ in $]-\infty, V(x_0) - 2\delta_0[$ are of the form

\[ \hat{E}_k(t; h) \sim E_{k,0}(t) + hE_{k,1}(t) + ... \]

where

\[ E_{k,0}(t) = \tilde{V}_t(x_0) + e_k(\alpha^t) \]

belongs to $]-\infty, \tilde{V}_t(x_0) - 3\delta_0/2[$ (in the limit of small $h$) and we get all such eigenvalues this way (one for each $k$) when $h > 0$ is small enough.

Now fix a $k \in \mathbb{N}$ and assume that we have the well-defined eigenvalue $\hat{E}_k(t; h) = \hat{E}(t; h)$ of $\hat{P}_t$ in $]\delta_0, \tilde{V}_t(x_0) - 2\delta_0[$ for all $t \in I$ for $0 < h \ll 1$. (The positivity is required since we look for shape resonances of $P_t$.) Define the $t$-dependent potential island

\[ \bar{O}(t) = \{ x \in \mathbb{R}^n; \tilde{V}_t(x) > E_0(t) \}, \quad E_0(t) := E_{k,0}(t). \]

Let $\tilde{\rho}_t = \xi^2 + \tilde{V}_t(x)$ be the semi-classical principal symbol of $\hat{P}_t$. Assume that

\[ \text{The } H_{\tilde{\rho}_t}\text{-flow is non-trapping on } (\tilde{\rho}_t)^{-1}(E_0(t))_{R^n \setminus \bar{O}}. \]

In $\bar{O}(t)$ we have the Lithner-Agmon distance $d_t$, associated to the metric $(\tilde{V}_t(x) - E_0)dx^2$. Let $S_t := d_t(x_0, \partial \bar{O}(t)) > 0,$

\[ M_0(t) := \{ x \in \bar{O}(t); \tilde{V}_t(x) > E_0(t) + \delta \}, \]

where $\delta > 0$ is a small constant. Notice that $M_0(t)$ has smooth boundary and depends smoothly on $t$. Let $P_0^t$ be the Dirichlet realization of $P_t$ in $M_0(t)$. Then $P_0^t$ has a unique eigenvalue $E(t; h)$ such that $E(t; h) = \hat{E}(t; h) = o(1), h \to 0$ and the two eigenvalues are exponentially close:

\[ E(t; h) = \hat{E}(t; h) + O(e^{-1/(Ch)}). \]
As in the beginning of this section, we know that $P_t$ has a unique resonance with
\[ \Re \lambda_0(t) - E(t; h) = o(1), \quad \Im \lambda_0(t) \geq -C h \ln(1/h). \]
Moreover, we have
\[ \lambda_0(t; h) = E(t; h) + \tilde{O}(e^{-2S_t/h}). \quad (9.128) \]
This means that (apart from the fact that our potential is $h$-dependent near $U_0$) we can apply Theorem 9.5 with $\delta(h) \asymp 1$.

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