On the Lefschetz property for quotients by monomial ideals containing squares of variables

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ABSTRACT
Let \( \Delta \) be an (abstract) simplicial complex on \( n \) vertices. One can define the Artinian monomial algebra \( A(\Delta) = k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2, I_\Delta) \), where \( k \) is a field of characteristic 0 and \( I_\Delta \) is the Stanley-Reisner ideal associated to \( \Delta \). In this paper, we aim to characterize the Weak Lefschetz Property (WLP) of \( A(\Delta) \) in terms of the simplicial complex \( \Delta \). We are able to completely analyze when the WLP holds in degree 1, complementing work by Migliore, Nagel and Schenck on the WLP for quotients by quadratic monomials. We give a complete characterization of all 2-dimensional pseudomanifolds \( \Delta \) such that \( A(\Delta) \) satisfies the WLP. We also construct Artinian Gorenstein algebras that fail the WLP by combining our results and the standard technique of Nagata idealization.

ARTICLE HISTORY
Received 29 September 2022
Revised 24 July 2023
Communicated by Daniel Erman

KEYWORDS
Artinian algebra; bipartite graph; face 2-colorable; Gorenstein ring; Grünbaum coloring; Lefschetz properties; monomial ideal; pseudomanifolds; simplicial complex; Stanley-Reisner ring; triangulation

2020 MATHEMATICS SUBJECT CLASSIFICATION
05C25; 05E40; 13E10; 13F55; 13H10

1. Introduction

Let \( A \) be an Artinian standard algebra over a field \( k \) of characteristic 0. \( A \) is said to satisfy the Weak Lefschetz Property (WLP for short) if the map between graded pieces \( A_i \rightarrow A_{i+1} \) of \( A \) induced by multiplying by a general linear form has full rank for all \( i \). This condition mimics the property of the cohomology ring of a projective variety where the Lefschetz’s hyperplane Theorem holds, hence the name.

The last few decades have seen tremendous growth in the study of the WLP, due to its many fascinating connections to other areas of mathematics: Dilworth number of posets, Schur-Weyl duality, the \( g \)-conjecture and its generalizations, syzygy bundles, Laplace equations and Togliatti systems, just to name a few. (See [1, 4, 10, 12, 18–20, 25].)

Even in the case where \( A \) is defined by a \textit{monomial ideal}, the literature is quite extensive, see for instance [2, 5, 10, 17–20, 22] and the references therein. In this work, we focus on a special class of Artinian monomial ideal. Let \( J \subseteq S = k[x_1, \ldots, x_n] \) be a monomial ideal containing squares of all the variables. Clearly, \( J = \langle x_1^2, x_2^2, \ldots, x_n^2, I \rangle \) where \( I \) is a \textit{square-free} monomial ideal. Thus, we can write \( I \) as the Stanley-Reisner ideal of a simplicial complex \( \Delta \). Because of this correspondence, we shall write \( J_\Delta \) for \( J \) and \( A(\Delta) \) for the Artinian quotient \( S/J_\Delta \).

Information about \( I \) and \( J_\Delta \) such as Hilbert functions, Betti numbers, type, regularity, can then be conveniently studied via combinatorial and topological properties of \( \Delta \), see [17, Theorem 2.1], [22, Lemma 2.1] and Proposition 2.3. We note also that the polarization of \( J_\Delta \) has been utilized recently in [6] to study the Koszul properties of quadratic monomial ideals.
In this paper, we study the WLP of \( A(\Delta) \) in degree 1 and \( d = \dim \Delta \), when \( k \) is a field of characteristic 0 (our results also hold if the characteristic of \( k \) is large enough, see [21, Proposition 7.2]). Our first main result completely characterizes the WLP in degree 1, complementing the work in [22].

**Theorem 1.1 (Theorem 3.3).** Let \( \Delta \) be a simplicial complex, \( G(\Delta) \) the 1-skeleton of \( \Delta \), and \( A = A(\Delta) \) the Artinian ring defined by the Stanley-Reisner ideal of \( \Delta \) plus the squares of all variables.

(i) If \( \dim_k A_2 \geq \dim_k A_1 \), then the WLP holds in degree 1 if and only if \( G(\Delta) \) has no bipartite components.

(ii) If \( \dim_k A_2 < \dim_k A_1 \), then the WLP holds in degree 1 if and only if each bipartite component of \( G(\Delta) \) (if it exists) is a tree and each non-bipartite component satisfies the property that the number of edges in the component is equal to the number of vertices in the component. Further, in this case, the WLP holds in degree 1 implies the WLP holds in all degrees.

As an application, we show that if \( J \) is a monomial ideal containing the squares of variables with Artinian quotient \( A \), and the multiplication map by a general linear form \( A_1 \to A_2 \) is surjective, then the regularity of \( J \) is at most 4 (See Corollary 3.7). This is related to the work by Eisenbud-Huneke-Ulrich on regularity of \( \text{Tor} \) in [7] and the \( p \)-basepoint freeness condition and Hankel index studied by Blekherman-Sinn-Velasco [3]. See [3, Definition 2.3, Theorem 2.4 in Section 2.2].

Finally, we use similar ideas to establish the WLP in degree \( d = \dim \Delta \) using the dual graph of \( \Delta \) (see Corollary 4.6). In this direction, we have (see Theorems 4.3 and 4.5):

**Theorem 1.2.** For \( A = A(\Delta) \) corresponding to a \( d \)-dimensional pseudomanifold \( \Delta \), the WLP holds in degree \( d \) if and only if:

(i) \( \Delta \) has boundary or

(ii) \( \Delta \) has no boundary and the dual graph of \( \Delta \) is not bipartite.

The above results combine to give the following corollary: when \( \Delta \) is a two-dimensional pseudomanifold, the WLP fails for \( A(\Delta) \) if and only \( \Delta \) has no boundary and the dual graph of \( \Delta \) is bipartite (Corollary 4.6). Furthermore, when \( \Delta \) is a planar triangulation without boundary, the dual graph is not bipartite if and only if the graph of \( \Delta \) is not Eulerian. (Corollary 4.12). Moreover, the first barycentric subdivisions of triangulations of a two-dimensional pseudomanifold without boundary always give \( A(\Delta) \) that fail the WLP (Corollary 4.14).

In Section 5, we construct Artinian Gorenstein algebras that fail the WLP using our previous results and the technique of Nagata idealization inspired by [16, 23] (see Example 5.3). In particular, we construct Artinian Gorenstein algebra in \( 2n \) variables that fail the WLP in degree 1 and having unimodal Hilbert series, for each \( n \geq 4 \).

### 2. Preliminary definitions

Let \( V \) be a finite non-empty set of \( n \) vertices. An \textit{(abstract) simplicial complex} \( \Delta \) is a non-empty subset of the power set of \( V \) such that \( F \subseteq E, E \in \Delta \implies F \in \Delta \). The elements of \( \Delta \) consisting of \( k + 1 \) vertices are called \textit{k-faces} or \textit{k-simplices} (dimension of the face is \( k \)). The faces which are maximal with respect to inclusion are called \textit{facets}. The dimension of \( \Delta \), \( \dim(\Delta) = \max(\dim(F) : F \text{ is a facet of } \Delta) \), \( \Delta \) is said to be \textit{pure} if every facet has the same dimension. The \textit{d-skeleton} of \( \Delta \), denoted by \( \Delta_d \), is the set of all faces of dimension at most \( d \). The \textit{f-vector} of \( \Delta \), \( f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{\dim(\Delta)}(\Delta)) \) is defined as \( f_i(\Delta) = \text{number of } i\text{-faces of } \Delta \), with \( f_{-1}(\Delta) = 1 (\emptyset \in \Delta) \). We define \( \Delta \) by defining the facets.

Let \( S = \mathbb{k}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over a field \( \mathbb{k} \) of characteristic 0. For a given \( \Delta \) such that every vertex of \( V \) is a 0-face, define the ideal \( I_\Delta = \langle x_{i_1} \ldots x_{i_m} : \{i_1 \ldots i_m\} \notin \Delta \rangle \). Then, \( \mathbb{k}[\Delta] = S/I_\Delta \) is the Stanley-Reisner ring (or face ring) of \( \Delta \). We further define the ideal \( J_\Delta = \langle x_1^2, x_2^2, \ldots, x_n^2, I_\Delta \rangle \) and consider the quotient ring \( A(\Delta) = S/J_\Delta \).
Example 2.1. $\Delta = \langle 123, 134, 45 \rangle$. The non-empty faces of $\Delta$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{12\}$, $\{13\}$, $\{23\}$, $\{14\}$, $\{34\}$, $\{45\}$, $\{123\}$, $\{134\}$.

Definition 2.2. For a standard graded Artinian ring $A$, socle of $A$ is given by $(0 : A_{n \geq 1})$. We call any highest degree monomial of the ring a top socle, and the degree of this monomial is the (top) socle degree, denoted by $\text{socdeg}(A)$. If $(0 : A_{n \geq 1}) = A_{\text{socdeg}(A)}$, then $A$ is a level $k$-algebra and, if $\dim_k(0 : A_{n \geq 1}) = 1$, $A$ is Gorenstein.

By labeling the vertices of $V$ by $x_i$, we see that each face $F = \{i_1 i_2 \ldots i_k\}$ of $\Delta$ represents a monomial $x_F = x_{i_1} x_{i_2} \ldots x_{i_k}$ in $A(\Delta)$, i.e., a monomial that does not belong to $J_\Delta$. This gives us the following facts.

Proposition 2.3.

(i) The monomials in $A(\Delta)$ are in one-one correspondence with the faces of $\Delta$. For $i > 0$, $A(\Delta)_i$ is a $k$-vector space with a basis given by $\{x_F : F \text{ is an } (i-1)\text{-face of } \Delta\}$.

(ii) The Hilbert Series of $A(\Delta)$ is given by $\text{Hilb}_{A(\Delta)}(t) = \sum_{i \geq 0} f_{i-1}(\Delta) t^i$.

(iii) $A(\Delta) = \bigoplus_{i \leq \dim(\Delta) + 1} (A(\Delta))_i$.

(iv) A top socle of $A(\Delta)$ is given by $x_F$, where $F$ is any facet of $\Delta$ of dimension $\dim(\Delta)$. The top socle degree, $\text{socdeg}(A(\Delta)) = \dim(\Delta) + 1$. $A(\Delta)$ being a finite length graded module over $S$,

$$\text{reg } A(\Delta) := \max\{i : A(\Delta)_i \neq 0\} = \text{socdeg}(A(\Delta))$$

where $\text{reg } A(\Delta)$ is the (Castelnuovo-Mumford) regularity of $A(\Delta)$.

The following definition is from [21].

Definition 2.4. Let $A$ be a graded Artinian algebra and $\ell$ be a general linear form.

$A$ has the Weak Lefschetz Property (the WLP) if the homomorphism induced by multiplication by $\ell$, $\mu_i : A_i \rightarrow A_{i+1}$, has maximal rank for all $i$ (i.e., is injective or surjective).

Such an $\ell$ is called a Lefschetz element.

$A$ is said to have the Strong Lefschetz Property (SLP) if the homomorphism induced by multiplication by $\ell^j$, $\mu_{i}^j : A_i \rightarrow A_{i+j}$, has maximal rank for all $i$ and $j$.

Note that unless specified otherwise, $\mu_i = \mu_1^i$.

The motivation for the study of Lefschetz properties comes from a result first proved by R. P. Stanley in 1980, which states that SLP holds for $S/I$, where $S = \mathbb{k}[x_1, \ldots, x_n]$ with char $\mathbb{k} = 0$ and $I$ is an Artinian monomial complete intersection [24].
Proposition 2.5. [20, Proposition 2.2] Let \( I \subseteq S \) be an Artinian monomial ideal and assume that the field \( \kappa \) is infinite. Then, \( S/I \) has the WLP if and only if \( \ell := x_1 + \cdots + x_n \) is a Lefschetz element for \( S/I \).

Remark 2.6. By part (iii) of Proposition 2.3, to check for the WLP in degree \( i \), i.e., \( A(\Delta)_{i} \rightarrow A(\Delta)_{i+1} \), we need to consider only \( A(\Delta)_{i} \).

Henceforth, we use \( \mu_i : A(\Delta)_i \rightarrow A(\Delta)_{i+1} \) to denote the multiplication map by the linear form \( \ell := x_1 + \cdots + x_n \). We consider the \( \kappa \)-bases of \( A(\Delta)_i \) and \( A(\Delta)_{i+1} \), consisting of their monomials in lex order \( (x_1 > x_2 > \cdots > x_n) \), to get the matrix representing the map \( \mu_i \), and this matrix shall be denoted by \([\mu_i]\). In terms of the simplicial complex \( \Delta \), by part (i) of Proposition 2.3, for any \( 1 \leq i \leq \dim(\Delta) \) and an \((i-1)\)-face \( F \),

\[
\mu_i(x_F) = \sum_{x_G \in \Delta} x_G
\]

where \( x_F \) is the monomial in \( A(\Delta) \) that corresponds to face \( F \in \Delta \).

Example 2.7. Let \( I_\Delta \subseteq S = \mathbb{k}[x_1, \ldots, x_7] \) be the edge ideal of path on 7 vertices and \( J_\Delta = (x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_7^2, I_\Delta) \). Note that \( \Delta \) here is the independence complex of the path on 7-vertices. We wish to examine the WLP in degree 1 for \( A(\Delta) = S/J_\Delta \). Taking \( \ell = x_1 + \cdots + x_7 \), the associated matrix is given by

\[
\begin{bmatrix}
    x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
    x_1x_3 & 0 & 1 & 0 & 0 & 0 & 0 \\
    x_1x_4 & 0 & 0 & 1 & 0 & 0 & 0 \\
    x_1x_5 & 0 & 0 & 0 & 1 & 0 & 0 \\
    x_1x_6 & 0 & 0 & 0 & 0 & 1 & 0 \\
    x_1x_7 & 0 & 0 & 0 & 0 & 0 & 1 \\
    x_2x_4 & 0 & 1 & 0 & 1 & 0 & 0 \\
    x_2x_5 & 0 & 1 & 0 & 0 & 1 & 0 \\
    x_2x_6 & 0 & 1 & 0 & 0 & 0 & 1 \\
    x_2x_7 & 0 & 1 & 0 & 0 & 0 & 1 \\
    x_3x_5 & 0 & 1 & 0 & 0 & 0 & 1 \\
    x_3x_6 & 0 & 0 & 1 & 0 & 1 & 0 \\
    x_3x_7 & 0 & 0 & 1 & 0 & 0 & 1 \\
    x_4x_6 & 0 & 0 & 0 & 1 & 0 & 1 \\
    x_4x_7 & 0 & 0 & 0 & 0 & 1 & 0 \\
    x_5x_7 & 0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix}
\]

The above matrix has full rank. Hence, the WLP holds in degree 1 for \( A(\Delta) \). In general, examining the WLP by checking the rank of a matrix can be cumbersome! In Example 3.4, we revisit the WLP in degree 1 for \( A(\Delta) \) where \( \Delta \) is the independence complex of \( n \)-path.

The following result is a restatement of [20, Proposition 2.1 (a)], in the context of the ring \( A(\Delta) \).

Proposition 2.8. Given a simplicial complex \( \Delta \) of dimension \( d \) and the corresponding \( A(\Delta) \), if \( \dim_k A(\Delta)_i > \dim_k A(\Delta)_{i+1} \) for some \( i \geq 1 \) and if the WLP holds in degree \( i \), then, the WLP holds in all degrees greater than \( i \).

Proof. When \( \dim_k A(\Delta)_i > \dim_k A(\Delta)_{i+1} \) for some \( i \geq 1 \), if the WLP holds in degree \( i \) then, \( \dim_k \left( \frac{A(\Delta)}{\ell A(\Delta)} \right)_{i+1} = 0 \) by surjectivity. So, every monomial in \( A(\Delta) \) of degree \( i + 1 \) is in the ideal \( \ell A(\Delta) \). By Proposition 2.3 (i), for any \( j > i \) and monomial \( x_E \) of degree \( j + 1 \), the corresponding \( j \)-face \( E \) has some
i-face $F$, i.e., $x_F \in \langle x_F \rangle \subseteq \ell A(\Delta)$. Thus, $\dim_k \left( \frac{A(\Delta)}{\ell A(\Delta)} \right)_j = 0$ for all $j > i$. Hence, $\mu_j : A(\Delta)_j \to A(\Delta)_{j+1}$ is surjective for all $j \geq i$. □

3. Weak Lefschetz Property in degree 1

In this section, we characterize the WLP in degree 1 for $A(\Delta)$ in terms of the 1-skeleton of $\Delta$. We denote the 1-skeleton of $\Delta$ by $G = G(\Delta)$, where $V = \{v_1, \ldots, v_n\}$ is the vertex set and $E$ is the edge set consisting of monomial quadrics in $A(\Delta)$. Let $I(G)$ be the incidence matrix of $G(\Delta)$, i.e., a matrix having dimension $|V| \times |E|$, with $(i,j)$th-entry equal to 1 if vertex $v_i$ is incident with edge $e_j$ (i.e., vertex $v_i$ divides monomial $e_j$), and zero otherwise. We take the vertices and edges in lex order for writing the incidence matrix.

The following result [8, Theorem 8.2.1] is instrumental in proving the results of this paper. For a graph $G$, we define the notation $b_G$ as the number of connected bipartite components.

Proposition 3.1. Given a graph $G$ with $n$ vertices and $b_G$ bipartite connected components, the rank of the incidence matrix of $G$ over a field of characteristic 0 is given by $n - b_G$.

Let $\mu_1 : A(\Delta)_1 \to A(\Delta)_2$ be the multiplication map by the linear form $\ell = x_1 + x_2 + \cdots + x_n$ and $[\mu_1]$ be the matrix representing this map with the monomial $k$-bases of $A(\Delta)_1$, $A(\Delta)_2$ taken in lex order.

Lemma 3.2. The matrix $[\mu_1]$ is the transpose of the incidence matrix $I(G)$ of the 1-skeleton $G(\Delta)$ of $\Delta$.

Proof. Given $\Delta$, by part (i) of Proposition 2.3, the monomials in $A(\Delta)_2$ are in one-one correspondence with the edges of $G(\Delta)$. For a vertex $v_i$,

$$\mu_1(v_i) = x_i \cdot \ell = \sum_{\{x_i, x_j\} \in \Delta} x_ix_j$$

In the matrix $[\mu_1]$, the column corresponding to $v_i$ will have 1 in the rows corresponding to quadrics in $A(\Delta)$ of the form $x_ix_j$ and zero otherwise. Hence, this is the transpose of $I(G)$, where the rows are labeled by the vertices and columns by the edges. □

Theorem 3.3.

(i) If $\dim_k A(\Delta)_2 \geq \dim_k A(\Delta)_1$, then, the WLP holds in degree 1 if and only if the 1-skeleton $G(\Delta)$ of $\Delta$ has no bipartite components.

(ii) If $\dim_k A(\Delta)_2 < \dim_k A(\Delta)_1$, then, the WLP holds in degree 1 if and only if each bipartite component of $G(\Delta)$ (if it exists) is a tree and each non-bipartite component satisfies the property that the number of edges in the component is equal to the number of vertices in the component. Further, in this case, the WLP holds in degree 1 implies the WLP holds in all degrees.

Proof. By Lemma 3.2, the WLP holds in degree 1 if and only if $[\mu_1]$ and hence, the incidence matrix $I(G)$ of $G(\Delta)$ has maximal rank. Note that $n = |V| = \dim_k A(\Delta)_1$ and $|E| = \dim_k A(\Delta)_2$. Hence, the WLP holds in degree 1 if and only if

$$\text{rank}(I(G)) = \min\{\dim_k A(\Delta)_1, \dim_k A(\Delta)_2\}$$

Also, from Proposition 3.1, $\text{rank}(I(G)) = n - b_G$ where $b_G$ is the number of connected bipartite components of $G(\Delta)$.

We consider the following two cases -

(i) $\dim_k A(\Delta)_2 \geq \dim_k A(\Delta)_1$: Hence, $\text{rank}(I(G)) = \dim_k A(\Delta)_1$, i.e., $n - b_G = n$, that is to say, $G$ has no bipartite component.
(ii) \( \dim_k A(\Delta)_2 < \dim_k A(\Delta)_1 \) : Hence, \( \text{rank}(I(G)) = \dim_k A(\Delta)_2 \), i.e., \( |V| - b_G = |E| \iff |V| - |E| = b_G \), i.e.,

\[
\sum \text{C: bipartite component} \quad (|V_C| - |E_C| - 1) + \sum \text{C: nonbipartite component} \quad (|V_C| - |E_C|) = 0
\]

where \( |V_C| \) and \( |E_C| \) are the number of vertices and edges respectively in the connected component \( C \) of the graph \( G(\Delta) \).

For a connected bipartite component \( C \), \( |E_C| \geq |V_C| - 1 \). For each nonbipartite connected component \( C \), since it cannot be a tree (as a tree is bipartite), \( |E_C| \geq |V_C| \). Since each term in the above sum is non-positive, they become equal to zero. Thus, each bipartite component of \( G(\Delta) \) has \( |E_C| = |V_C| - 1 \) and so, has to be a tree. Each nonbipartite component of \( G(\Delta) \) satisfies \( |E_C| = |V_C| \).

Further, in this case, by Proposition 2.8, if the WLP holds in degree 1, then the WLP holds in all degrees.

\[ \square \]

**Example 3.4.** Let \( I_\Delta \) be the edge ideal of a path on \( n \) vertices and \( A(\Delta)_1 = S/\langle x_1^2, \ldots, x_n^2, I_\Delta \rangle \). \( \Delta \) is the independence complex of the path on \( n \)-vertices. Let \( G(\Delta) \) be the 1-skeleton of \( \Delta \). For small values of \( n \), it is easy to observe that the WLP holds in degree 1 for \( A(\Delta) \). For \( n \geq 5 \), we have \( \dim_k A(\Delta)_2 \geq \dim_k A(\Delta)_1 \) (since \( |E| \geq |V| \)) and \( G(\Delta) \) is connected and admits a triangle (an odd cycle), thus, having no bipartite components. Hence, the WLP holds in degree 1 for all \( n \).

Given below is \( G(\Delta) \) for \( n = 5 \):

We look at an interesting consequence of Proposition 3.3.

**Definition 3.5.** Given a graph \( G(V, E) \), any \( C \subseteq V \) is a clique of \( G \) if every pair of vertices in \( C \) are adjacent and \( \max\{|C| : C \subseteq V \text{ is a clique} \} \) is called the clique number of \( G \), denoted by \( \omega(G) \).

**Lemma 3.6.** \( \text{socdeg}(A(\Delta)) \leq \omega(G(\Delta)) \).

**Proof.** Suppose \( x_F = x_{i_1}x_{i_2} \ldots x_{i_j} \) is a top socle of \( A(\Delta) \). By part (i) of Proposition 2.3, \( F \) is a face of \( \Delta \) of dimension \( (j - 1) \). For every quadric \( x_{i_k}x_{i_j} \) that divides \( x_F \), there is a corresponding edge in \( G(\Delta) \). Thus, \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_j}\} \) is a clique of \( G(\Delta) \) and hence, \( \text{socdeg}(A(\Delta)) = \deg(x_F) \leq \omega(G(\Delta)) \). \[ \square \]

**Corollary 3.7.** When \( A(\Delta) \) has the WLP in degree 1, if \( \dim_k A(\Delta)_2 \leq \dim_k A(\Delta)_1 \), then \( \text{socdeg}(A(\Delta)) \) is at most 3.

**Proof.** When the WLP holds in degree 1, if \( \dim_k A(\Delta)_2 \leq \dim_k A(\Delta)_1 \), from part (ii) of Theorem 3.3, we see that \( G(\Delta) \) may have bipartite components, in which case the maximum possible clique size is 2, or every connected nonbipartite component has the property that the number of edges in the component is equal to the number of vertices in the component, allowing clique size at most 3 since for any clique of size greater than 3, number of edges is greater than the number of vertices. Hence, the clique number of \( G(\Delta) \) and by Lemma 3.6, \( \text{socdeg}(A(\Delta)) \) is at most 3. \[ \square \]
4. Weak Lefschetz property in degree \(d\) for pseudomanifolds

In this section, we look at the Weak Lefschetz property in degree \(d\) of \(A(\Delta)\) corresponding to the \(d\)-dimensional pseudomanifold \(\Delta\).

**Definition 4.1.** A combinatorial \(d\)-dimensional pseudomanifold is a simplicial complex such that:

(i) each facet is a \(d\)-simplex.

(ii) every \((d - 1)\)-simplex is a face of at most two \(d\)-simplices for \(d > 1\).

(iii) given any two \(d\)-simplices \(F_1, F_k\), there exists a chain of \(d\)-simplices \(F_1, F_2, F_3, \ldots, F_k\) such that \(F_i \cap F_{i+1}\) is a \((d - 1)\)-simplex for \(1 \leq i \leq k - 1\).

We refer to a \(d\)-dimensional pseudomanifold to be one with boundary if there exists at least one boundary \((d - 1)\)-simplex, i.e., a \((d - 1)\)-simplex that is a face of exactly one \(d\)-simplex. We refer to a \(d\)-simplex having a boundary \((d - 1)\)-simplex as a boundary \(d\)-simplex. A \(d\)-dimensional pseudomanifold having no boundary \(d\)-simplex is called a pseudomanifold without boundary.

Given a pure \(d\)-dimensional complex \(\Delta\), any face of dimension \(d - 1\) is called a ridge. The dual graph \(G^*(\Delta)\) of a pure complex \(\Delta\) is the graph in which every facet of \(\Delta\) becomes a vertex and two vertices in \(G^*(\Delta)\) are adjacent (i.e., are connected by an edge) if and only if the corresponding facets share a ridge.

**Remark 4.2.** For \(A(\Delta)\) corresponding to a \(d\)-dimensional pseudomanifold \(\Delta\), \(\dim_k A(\Delta)_d = \) number of ridges and \(\dim_k A(\Delta)_{d+1} = \) number of \(d\)-simplices. Further, \(A(\Delta)\) is a level algebra.

4.1. Pseudomanifold without boundary

**Theorem 4.3.** For \(d \geq 1\), \(A(\Delta)\) corresponding to a \(d\)-dimensional pseudomanifold without boundary has the WLP in degree \(d\) if and only if the dual graph is not bipartite.

**Proof.** For \(d \geq 1\), let \(\Delta\) be a \(d\)-dimensional pseudomanifold without boundary, i.e., every ridge is the face of exactly two \(d\)-simplices and every \(d\)-simplex contains \((d + 1)\) many ridges. Then,

\[
\dim_k A(\Delta)_d = \left(\frac{d + 1}{2}\right) \cdot \dim_k A(\Delta)_{d+1} \tag{4.1}
\]

Hence, \(\dim_k A(\Delta)_d \geq \dim_k A(\Delta)_{d+1}\).

Now, consider the dual graph \(G^*(\Delta)\) of \(\Delta\). The number of vertices in \(G^*(\Delta)\), \(|V_{G^*(\Delta)}|\), equals the number of \(d\)-simplices of \(\Delta\), and the number of edges in \(G^*(\Delta)\), \(|E_{G^*(\Delta)}|\), equals the number of ridges of \(\Delta\). The map \(\mu_d : A(\Delta)_d \to A(\Delta)_{d+1}\) then gives the transpose of the map \(\mu_1 : V_{G^*(\Delta)} \to E_{G^*(\Delta)}\) and hence, by Lemma 3.2, the corresponding matrix is the incidence matrix \(I(G^*(\Delta))\). So, \(\mu_d\) has full rank if and only if \(I(G^*(\Delta))\) has full rank. Note that here, by Eq. (4.1),

\[
|E_{G^*(\Delta)}| \geq |V_{G^*(\Delta)}|
\]

Hence, by part (i) of Theorem 3.3, we see that \(\mu_d\) has full rank if and only if the connected graph \(G^*(\Delta)\) is not bipartite.

**Example 4.4.** Let \(\Delta\) be the octahedron (2-dimensional pseudomanifold without boundary). For the corresponding \(A(\Delta)\), the WLP fails in degree 2.
This can be observed from the fact that the dual graph of the octahedron is the 1-skeleton (edge graph) of a cube, and is bipartite.

4.2. Pseudomanifold with boundary

For $d \geq 1$, let $\Delta$ be a $d$-dimensional pseudomanifold with boundary, i.e., there exists at least one boundary $(d - 1)$-simplex.

**Theorem 4.5.** The WLP in degree $d$ always holds for $A(\Delta)$ corresponding to a $d$-dimensional pseudomanifold $\Delta$ with boundary.

**Proof.** We use the definition of the WLP to prove this theorem by showing that $\mu_d : A(\Delta)_d \to A(\Delta)_{d+1}$ is surjective. We use the notation $x_F$ for the monomial in $A(\Delta)$ corresponding to the face $F \in \Delta$.

For a boundary $d$-simplex $F'$, any boundary ridge $B'$ such that $B' \subseteq F'$ gives $\mu_d(x_{F'}) = x_{F'}$. Now we consider a $d$-simplex $F$ which is not a boundary facet. By part (iii) in Definition 4.1, we get some boundary $d$-simplex $F_k$, and a chain of $d$-simplices $F = F_1, F_2, F_3, \ldots, F_k$ such that $F_i \cap F_{i+1} = B_i$ is a ridge for $1 \leq i \leq k - 1$. Let $B_k$ be a boundary ridge of $F_k$.

We have $\mu_d(x_{B_k}) = x_{F_k}$ and $\mu_d(x_{B_i}) = x_{F_i} + x_{F_{i+1}}$ for $1 \leq i \leq k - 1$.

Now, define

$$x_{M_1} := x_{B_k}, 
\quad x_{M_j} := x_{B_{k-j+1}} - x_{M_{j-1}}$$

for $2 \leq j \leq k$.

Then, $\mu_d(x_{M_1}) = x_{F_k}, \mu_d(x_{M_2}) = x_{F_{k-1}}$ and continuing this way, $\mu_d(x_{M_k}) = x_{F_1} = x_F$. Hence, $\mu_d$ is surjective. \(\square\)

We now look at some interesting corollaries of the above results.

**Corollary 4.6.** Let $\Delta$ be a 2-dimensional pseudomanifold. $A(\Delta)$ has the WLP if and only if $\Delta$ has boundary or if $\Delta$ has no boundary and the dual graph $G^*(\Delta)$ is not bipartite.

**Proof.** Let $\Delta$ be a 2-dimensional pseudomanifold. Note that here we only need to check for the WLP in degrees 1 and 2. Firstly, for degree 1, the 1-skeleton $G(\Delta)$ has only one connected component. Further, each vertex is incident with at least two edges and each edge is incident with two vertices. Thus,

$$\left(\frac{k}{2}\right) \cdot \dim_k A(\Delta)_1 = \dim_k A(\Delta)_2$$

where $k \geq 2$. Hence, $\dim_k A(\Delta)_2 \geq \dim_k A(\Delta)_1$. Since a 2-dimensional pseudomanifold has triangular facets making $G(\Delta)$ nonbipartite, by part (i) of Theorem 3.3, the WLP holds in degree 1 for $A(\Delta)$.\(\square\)
Next, in degree 2, we see that the WLP holds for $A(\Delta)$ either if $\Delta$ has boundary (Theorem 4.5) or, if $\Delta$ has no boundary, the dual graph $G^*(\Delta)$ is not bipartite (Theorem 4.3).

**Definition 4.7.** A simplicial complex $\Delta$ is a *triangulation* if there exists a homeomorphism between the geometric realization of $\Delta$ and a topological manifold. In the specific case when $\Delta$ is a 2-dimensional pseudomanifold, the facets of $\Delta$ are triangles (2-simplices) and the triangulation is said to be *face 2-colorable* if it is possible to color all the triangles using 2-colors such that no two adjacent triangles have the same color.

**Theorem 4.8.** Let $\Delta$ be a 2-dimensional pseudomanifold without boundary that is a triangulation. Then, the WLP holds in degree 2 for the corresponding $A(\Delta)$, if and only if the triangulation is not face 2-colorable.

**Proof.** When $\Delta$ is a 2-dimensional pseudomanifold without boundary, by Theorem 4.3, $A(\Delta)$ has the WLP in degree 2 if and only if the dual graph has no bipartite components. Since the vertices in the dual graph correspond to triangles in the triangulation, this implies that the WLP holds in degree 2 if and only if the triangulation is not face 2-colorable.

**Example 4.9.** A tetrahedron is a triangulation without boundary of the sphere $S^2$. It is clearly not 2-colorable, hence, the corresponding $A(\Delta)$ satisfies the WLP in degree 2.

**Remark 4.10.** Face 2-colorability has some interesting connections with the concept of Grünbaum coloring [14]. In particular, from [14, Theorem 1], we see that if the given triangulation of a closed manifold is not Grünbaum colorable, then, it is not face 2-colorable and hence, the WLP holds in degree 2.

**Definition 4.11.** A graph is said to be *Eulerian* if and only if it has no vertex of odd degree. A *planar triangulation* is a triangulation that can be embedded in the plane.

By [15, Theorem 1.4(b)], [11, Proposition 2], a planar triangulation of a graph $G$ is face 2-colorable if and only if the graph $G$ is Eulerian. This gives the following corollary.

**Corollary 4.12.** Let $\Delta$ be a 2-dimensional pseudomanifold without boundary that is a planar triangulation. Then, the WLP holds in degree 2 for the corresponding $A(\Delta)$, if and only if the 1-skeleton $G(\Delta)$ is not Eulerian.

**Definition 4.13.** The *first barycentric subdivision* of a triangulation is obtained by introducing 4 new vertices and 6 new edges in every triangle as follows—the midpoints of the edges and the centroid of the triangle are the new vertices, and the new edges are created by joining the centroid with the midpoints of the three edges and the vertices of the triangle.

**Corollary 4.14.** Let $\Delta$ be a 2-dimensional pseudomanifold without boundary that is the first barycentric subdivision of a triangulation. Then, the WLP in degree 2 fails for $A(\Delta)$.

**Proof.** When $\Delta$ is a 2-dimensional pseudomanifold without boundary that is the first barycentric subdivision of a triangulation, by [14, Corollary 2], we see that the complex is now face 2-colorable and hence the WLP fails in degree 2.

**Example 4.15.** For the first barycentric subdivision of a tetrahedron, the WLP fails in degree 2 for $A(\Delta)$.

We note that [13] has more results on Lefschetz property of barycentric subdivisions.
5. Examples of Artinian Gorenstein algebras that fail the WLP

In this section we construct Artinian Gorenstein algebras that fail the WLP by combining our results and the standard technique of Nagata idealization. We briefly recall the basic set-up that’s relevant for our purpose. Let $k$ be a field of characteristic 0. Let $R$ be a standard graded level Artinian $k$-algebra with socle degree $d$, that is, the socle of $R$ is $R_d$ (see Definition 2.2). Let $ω$ be the graded canonical module of $R$, and let $\tilde{R} = R \otimes ω(−d−1)$ be the Nagata idealization of $ω$. Then $\tilde{R}$ is a standard graded Artinian Gorenstein algebra with socle degree $d+1$, and $\dim_k \tilde{R}_i = \dim_k R_i + \dim_k R_{d+1−i}$. For more details on this construction we refer to [16].

**Proposition 5.1.** Assume the set-up above with $d ≥ 2$. Further assume that $\dim_k R_2 + \dim_k R_{d−1} ≥ \dim_k R_1 + \dim_k R_d$. If $R$ fails surjectivity at degree $d−1$, then $\tilde{R}$ fails surjectivity, and hence fails the WLP in degree $d−1$.

**Proof.** Let $l ∈ \tilde{R}_1$ be a general linear form. If $l$ induces a surjective map $\tilde{R}_{d−1} → \tilde{R}_d$, then the restriction of $l$ to $R_1$ must induce a surjective map from $R_{d−1} → R_d$. But indeed no linear form can give surjective map $R_{d−1} → R_d$ (as general linear forms have maximal possible rank). Finally, to show that failing surjectivity implies failing the WLP, we need to show that $\dim_k \tilde{R}_{d−1} ≥ \dim_k \tilde{R}_d$, which follows from the assumption on $R$. □

**Corollary 5.2.** Let $d ≥ 2$ and $Δ$ be a $(d−1)$-dimensional pseudomanifold without boundary such that the dual graph of $Δ$ is bipartite. Let $R = A(Δ)$. The idealization $\tilde{R}$ of $R$ is an Artinian Gorenstein ring that fails the WLP in degree $d−1$.

**Proof.** $R$ is a level algebra (Remark 4.2) that fails surjectivity at degree $d−1$ by Theorem 4.3. Thus, by applying Proposition 5.1, we just need to show that $\dim_k R_2 + \dim_k R_{d−1} ≥ \dim_k R_1 + \dim_k R_d$. Note that $\dim_k R_1 = f_{d−1}(Δ)$. Clearly $f_1(Δ) ≥ f_0(Δ)$: the 1-skeleton of $Δ$ is connected and hence $f_1(Δ) ≥ f_0(Δ)−1$ with equality if and only if it is a tree, which is impossible as $Δ$ is a pseudo-manifold. Equally clearly, $f_{d−2}(Δ) = (d/2)f_{d−1}(Δ)$: each $(d−2)$-face is in exactly 2 facets, and each facet has exactly $d$ many $(d−2)$-faces. Thus the desired inequality follows. □

**Example 5.3.** Let $Δ = {12, 23, 34, 14}$, a 4-cycle which is certainly a 1-dimensional pseudomanifold without boundary. $R = A(Δ) = k[x_1, x_2, x_3, x_4]/I_Δ$ where $I_Δ = (x_1^2, x_2^2, x_3^2, x_4^2)$, is a graded Artinian level $k$-algebra with socdeg($A(Δ)$) = 2. By Theorem 3.3 (i), since $G(Δ)$ is bipartite, the WLP fails in degree 1 (this can also be observed from Theorem 4.3, since the dual graph of $Δ$ is also a 4-cycle which is bipartite).

We now refer to [16, Lemma 3.3] to compute $\tilde{R}$, which is a quotient of the polynomial ring $T = k[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$, where the variables $y_1, y_2, y_3, y_4$ correspond to the generators of $ω_R$. From the presentation of $ω_R$, we get the ideal defining the relations involving generators of $ω_R$, $I = ⟨−x_1y_1, −x_2y_2, −x_3y_3, −x_4y_4, x_1y_1 − x_3y_2, −x_2y_2 + x_4y_4, x_1y_1 − x_3y_2, −x_2y_2 + x_4y_4, x_1y_1 − x_3y_2, −x_2y_2 + x_4y_4, x_1y_1 − x_3y_2, −x_2y_2 + x_4y_4⟩$. We see that the idealization $\tilde{R}$ is given by $T/J'$ where $J' = I_Δ + ⟨y_1, y_2, y_3, y_4⟩^2 + I$. It can be checked that the WLP fails in the degree $d−1 = 1$, as predicted by Corollary 5.2. We note that the Hilbert function of $\tilde{R}$ is $(1, 8, 8, 1)$. Further, it can be checked using Macaulay2 that $J'$ has a Gröbner basis consisting of quadrics, and thus, $\tilde{R}$ is Koszul (see [6, Proposition 2.12]).

**Example 5.3** can be generalized as follows: Consider $Δ$ to be any even cycle on $n = 2a$ vertices for $a ≥ 2$ and $R = A(Δ) = k[x_1, . . . , x_n]/I_Δ$ (this ensures that $G(Δ)$ is bipartite and hence, $R$ fails the WLP in degree 1). In the polynomial ring $T = k[x_1, . . . , x_n, y_1, . . . , y_n]$, where $y_1, . . . , y_n$ correspond to the generators of $ω_R$, define ideal $I ⊆ T$ using the relations involving $⟨y_1, . . . , y_n⟩$. To get a Gorenstein algebra $\tilde{R}$ in $2n$ variables, let $\tilde{R} = T/J'$ where $J' = I_Δ + ⟨y_1, . . . , y_n⟩^2 + I$. The Hilbert function of $\tilde{R}$ is $(1, 2n, 2n, 1)$. This $\tilde{R}$ fails the WLP in degree 1. This way, we get a family of Artinian Gorenstein
algebras in $2n$ variables that fail the WLP in degree 1 and having unimodal Hilbert function $(1, 2n, 2n, 1)$, for $n \geq 4$.

We note that the resulting Gorenstein algebra constructed in this section bear resemblance to the ones in [9], albeit from a rather different point of view. See also [6, Section 8] for a detailed discussion of and some corrections to the results of [9].

Acknowledgments

We thank Aaron Dall, Alessio D’Alì, Serge Lawrencenko, Hal Schenck, Alexandra Seceleanu, and an anonymous referee for some helpful correspondence on the topics of this work. We also thank Daniel Erman and an anonymous referee for their valuable comments.

References

[1] Adiprasito, K., Papadakis, S. A., Petrotou, V. (2021). Anisotropy, biased pairings, and the Lefschetz property for pseudomanifolds and cycles. Preprint.
[2] Altafi, N., Boij, M. (2020). The weak Lefschetz property of equigenerated monomial ideals. *J. Algebra* 556:136–168.
[3] Blekherman, G., Sinn, R., Velasco, M. (2017). Do sums of squares dream of free resolutions?. *SIAM J. Appl. Algebra Geom.* 1:175–199.
[4] Brenner, H., Kaid, A. (2007). Syzygy bundles on $\mathbb{P}^2$ and the weak Lefschetz property. *Illinois J. Math.* 51:1299–1308.
[5] Cook, D., Nagel, U. (2016). The weak Lefschetz property for monomial ideals of small type. *J. Algebra* 462:285–319.
[6] D’Alì, A., Venturello, L. (2023). Koszul Gorenstein algebras from Cohen-Macaulay simplicial complexes. *Int. Math. Res. Not.* 2023(6):4998–5045.
[7] Eisenbud, D., Huneke, C., Ulrich, B. (2006). The regularity of Tor and graded Betti numbers. *Amer. J. Math.* 128:573–605.
[8] Godsil, C., Royle, G. (2001). *Algebraic Graph Theory.* New York: Springer-Verlag, p. 166.
[9] Gondim, R., Zappalà, G. (2018). Lefschetz properties for Artinian Gorenstein algebras presented by quadrics. *Proc. Amer. Math. Soc.* 146(3):993–1003.
[10] Harima, T., Maeno, T., Morita, H., Numata, Y., Wachi, A., Watanabe, J. (2013). *The Lefschetz Properties.* Springer Lecture Notes in Mathematics, Vol. 2080. Heidelberg: Springer.
[11] Hertrich, C., Schröder, F., Steiner, R. (2020). Coloring drawings of graphs. Preprint.
[12] Iarrobino, A., McDaniel, C., Seceleanu, A. (2022). Connected sums of graded Artinian Gorenstein algebras and Lefschetz properties. *J. Pure Appl. Algebra* 226(1):106787.
[13] Kubitzke, M., Nevo, E. (2009). The Lefschetz property for barycentric subdivisions of shellable complexes. *Trans. Amer. Math. Soc.* 361:6151–6163.
[14] Lawrencenko, S., Vyalii, M. N., Zgonnik, L. V. (2017). Grünbaum coloring and its generalization to arbitrary dimension. *Australas. J. Combin.* 67(2):119–130.
[15] Liu, W., Lawrencenko, S., Chen, B., Ellingham, M. N., Hartsfield, N., Yang, H., Ye, D., Zha, X. (2019). Quadrangular embeddings of complete graphs and the Even Map Color theorem. *J. Combin. Theory Ser. B* 139(1):1–26.
[16] Mastroeni, M., Schenck, H., Stillman, M. (2021). Quadratic Gorenstein rings and the Koszul property I. *Trans. Amer. Math. Soc.* 374(2):1077–1093.
[17] Mermin, J., Peeva, I., Stillman, M. (2008). Ideals containing the squares of the variables. *Adv. Math.* 217(5):2206–2230.
[18] Mezzetti, E., Miró-Roig, R., Ottaviani, G. (2013). Laplace equations and the weak Lefschetz property. *Can. J. Math.* 65(3):634–654.
[19] Michałek, M., Miró-Roig, R. (2016). Smooth monomial Togliatti systems of cubics. *J. Combin. Theory Ser. A* 143:66–87.
[20] Migliore, J., Miró-Roig, R., Nagel, U. (2011). Monomial ideals, almost complete intersections and the Weak Lefschetz property. *Trans. Amer. Math. Soc.* 363:229–257.
[21] Migliore, J., Nagel, U. (2013). A tour Of the weak and strong Lefschetz properties. *J. Commut. Algebra* 5:329–358.
[22] Migliore, J., Nagel, U., Schenck, H. (2020). The weak Lefschetz property for quotients by quadratic monomials. *Math. Scand.* 126:41–60.
[23] McCullough, J., Seceleanu, A. (2020). Quadratic Gorenstein algebras with many surprising properties. *Arch. Math. (Basel)* 115(5):509–521.
[24] Stanley, R. P. (1980). Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods* 1:168–184.
[25] Stanley, R. P. (1983). Combinatorial applications of the Hard Lefschetz theorem. In: *Proceedings of the International Congress of Mathematicians Warsaw (1983)*, pp. 447–453.