A Sequence Construction of Cyclic Codes over Finite Fields

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Abstract

Cyclic codes over finite fields are widely implemented in data storage systems, communication systems, and consumer electronics, as they have very efficient encoding and decoding algorithms. They are also important in theory, as they are closely connected to several areas in mathematics. There are a few fundamental ways of constructing all cyclic codes over finite fields, including the generator matrix approach, the generator polynomial approach, the generating idempotent approach, and the $q$-polynomial approach. Another one is a sequence approach, which has been intensively investigated in the past decade. The objective of this paper is to survey the progress in the past decade in this direction.

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1 Introduction

Let \( q \) be a power of a prime \( p \). An \([n, k, d]\) code over \( \mathbb{GF}(q) \) is a \( k \)-dimensional subspace of \( \mathbb{GF}(q)^n \) with minimum (Hamming) nonzero weight \( d \). Let \( A_i \) denote the number of codewords with Hamming weight \( i \) in a linear code \( C \) of length \( n \). The weight enumerator of \( C \) is defined by

\[
1 + A_1 z + A_2 z^2 + \cdots + A_n z^n.
\]

The weight distribution of \( C \) is the sequence \((1, A_1, \ldots, A_n)\).

An \([n, k, d]\) code over \( \mathbb{GF}(q) \) is called optimal if there is no \([n, k, d']\) code with \( d' > d \) or \([n, k', d]\) code with \( k' > k \) or \([n', k, d]\) code with \( n' < n \) over \( \mathbb{GF}(q) \) or the parameters \([n, k, d]\) meet a bound for linear codes over \( \mathbb{GF}(q) \). An \([n, k, d]\) code is said to be almost optimal if a linear code with parameters \([n, k + 1, d]\) or \([n, k, d + 1]\) or \([n - 1, k, d]\) is optimal.

A vector \((c_0, c_1, \ldots, c_{n-1}) \in \mathbb{GF}(q)^n\) is said to be even-like if \( \sum_{i=0}^{n-1} c_i = 0 \), and is odd-like otherwise. The even-like subcode of a linear code consists of all the even-like codewords of this linear code.

An \([n, k]\) code over a finite field is said to be cyclic if \((c_0, c_1, \ldots, c_{n-1}) \in C \) implies \((c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C \). Let \( \gcd(n, q) = 1 \). We identify a vector \((c_0, c_1, \ldots, c_{n-1}) \in \mathbb{GF}(q)^n \) with the polynomial

\[
c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \in \mathbb{GF}(q)[x]/(x^n - 1).
\]
Then a code $C$ of length $n$ over $\text{GF}(q)$ corresponds to a subset of the quotient ring $\text{GF}(q)[x]/(x^n - 1)$. The linear code $C$ is cyclic if and only if the corresponding subset in $\text{GF}(q)[x]/(x^n - 1)$ is an ideal of the ring $\text{GF}(q)[x]/(x^n - 1)$. It is well known that every ideal of $\text{GF}(q)[x]/(x^n - 1)$ is principal. Let $C = \langle g(x) \rangle$ be a cyclic code, where $g$ is monic and has the least degree. Then $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is referred to as the check polynomial of $C$. The dual code, denoted by $C^\perp$, of $C$ has generator polynomial $\overline{h}(x)$, which is the reciprocal of $h(x)$. The complement code, denoted by $C^c$, is generated by $h(x)$. It is known that $C^\perp$ and $C^c$ of $C$ have the same weight distribution.

The error correcting capability of cyclic codes may not be as good as some other linear codes in general. However, cyclic codes have wide applications in storage and communication systems because they have very efficient encoding and decoding algorithms [5, 21, 30].

Cyclic codes have been studied for decades and a lot of progress has been made [4, 24]. The total number of cyclic codes over $\text{GF}(q)$ and their constructions are closely related to cyclotomic cosets modulo $n$, and thus many areas of number theory. Cyclic codes are very important in practice, as some cyclic codes are widely implemented in data storage systems and communication systems.

A construction method is said to be fundamental, if every cyclic code over any finite field can be constructed with this method. Fundamental construction methods of cyclic codes include the following:

- The generator matrix (equivalently check matrix) approach.
- The generator polynomial (equivalently the check polynomial) approach.
- The generating idempotent approach.
- The $q$-polynomial approach [17].

Another way of constructing cyclic codes over $\text{GF}(q)$ with length $n$ is to use the generator polynomial

$$ \frac{x^n - 1}{\gcd(S(x), x^n - 1)} \tag{1} $$

where

$$ S(x) = \sum_{i=0}^{n-1} s_i x^i \in \text{GF}(q)[x] $$

and $s^\infty = (s_i)_{i=0}^\infty$ is a sequence of period $n$ over $\text{GF}(q)$. Throughout this paper, we call the cyclic code $C_s$ with the generator polynomial of (1) the code defined by the sequence $s^\infty$, and the sequence $s^\infty$ the defining sequence of the cyclic code $C_s$.

It is straightforward to see that every cyclic code of length $n$ over $\text{GF}(q)$ can be expressed as $C_s$ for some sequence $s^\infty$ of period $n$ over $\text{GF}(q)$. Because of this, this sequence construction of cyclic codes is fundamental. An impressive progress in the construction of cyclic codes with this approach has been made in the past decade [10, 11, 13, 14, 20, 32, 34].

The objective of this paper is to give a survey of recent development in this sequence construction of cyclic codes over finite fields. In view that this topic is huge,
we have to do a selective survey. Our idea is that this paper complements the monograph [14], so that the two references together could give a well rounded treatment of the sequence construction of cyclic codes over finite fields. It is hoped that this paper could stimulate further investigations into this sequence approach. This paper is an extended version of the survey paper [15].

2 Preliminaries

In this section, we present basic notation and results of \( q \)-cyclotomic cosets modulo \( n \), planar and almost perfect nonlinear functions, and sequences that will be employed in subsequent sections.

2.1 Some notation fixed throughout this paper

Throughout this paper, we adopt the following notation unless otherwise stated:

- \( p \) is a prime, \( q \) is a positive power of \( p \), and \( m \) is a positive integer.
- \( \mathbb{Z}_n = \{0, 1, \cdots, n-1\} \), the ring of integers modulo \( n \).
- \( N_q(x) \) is a function defined by \( N_q(x) = 0 \) if \( x \equiv 0 \) (mod \( q \)) and \( N_q(x) = 1 \) otherwise.
- \( \alpha \) is a generator of \( \text{GF}(q^m)^* \), the multiplicative group of \( \text{GF}(q^m) \).
- \( m_a(x) \) is the minimal polynomial of \( a \in \text{GF}(q^m) \) over \( \text{GF}(q) \).
- \( \text{Tr}(x) \) is the trace function from \( \text{GF}(q^m) \) to \( \text{GF}(q) \).
- \( \delta(x) \) is a function on \( \text{GF}(q^m) \) defined by \( \delta(x) = 0 \) if \( \text{Tr}(x) = 0 \) and \( \delta(x) = 1 \) otherwise.
- \( C_i \) denotes the \( q \)-cyclotomic coset modulo \( n \) containing \( i \).
- \( \Gamma \) is the set of all coset leaders of the \( q \)-cyclotomic cosets modulo \( n \).
- For any polynomial \( g(x) \in \text{GF}(q)[x] \) with \( g(0) \neq 0 \), \( \bar{g}(x) \) denotes the reciprocal of \( g(x) \).
- For a cyclic code \( C \) of length \( n \) over \( \text{GF}(q) \) with generator polynomial \( g(x) \), \( C^c \) denotes its complement code that is generated by \( h(x) := (x^n-1)/g(x) \), and \( C^\perp \) denotes its dual code with generator polynomial \( \bar{h}(x) \), i.e., the reciprocal of \( h(x) \).
- By the Database we mean the collection of tables of best linear codes known maintained by Markus Grassl at [http://www.codetables.de/](http://www.codetables.de/).
2.2 Planar and APN polynomials

A function \( f : \text{GF}(q^m) \rightarrow \text{GF}(q^m) \) is called almost perfect nonlinear (APN) if

\[
\max_{a \in \text{GF}(q^m)} \max_{b \in \text{GF}(q^m)} |\{x \in \text{GF}(q^m) : f(x+a) - f(x) = b\}| = 2,
\]

and is referred to as perfect nonlinear or planar if

\[
\max_{a \in \text{GF}(q^m)} \max_{b \in \text{GF}(q^m)} |\{x \in \text{GF}(q^m) : f(x+a) - f(x) = b\}| = 1.
\]

There is no perfect nonlinear (planar) function on \( \text{GF}(q^m) \) for even \( q \). But there are APN functions on \( \text{GF}(2^m) \). Both planar and APN functions over \( \text{GF}(q^m) \) for odd \( q \) exist. For example, \( x^2 \) is a planar function on \( \text{GF}(q^m) \) for odd \( q \), and \( x^{q^m-2} \) is an APN function over \( \text{GF}(q^m) \). Perfect nonlinear and APN functions were used to construct linear codes in different ways in the literature. Some planar and APN monomials will be employed to construct sequences first, and the obtained sequences will be used to construct cyclic codes in subsequent sections.

2.3 The \( q \)-cyclotomic cosets modulo \( n \)

Let \( \gcd(n,q) = 1 \). The \( q \)-cyclotomic coset containing \( j \) modulo \( n \) is defined by

\[
C_j = \{ j, qj, q^2 j, \ldots, q^{l_j-1} j \} \mod n \subset \mathbb{Z}_n
\]

where \( l_j \) is the smallest positive integer such that \( q^{l_j} j \equiv j \pmod{n} \), and is called the size of \( C_j \). It is known that \( l_j \) divides \( n \). The smallest integer in \( C_j \) is called the coset leader of \( C_j \). Let \( \Gamma \) denote the set of all coset leaders. By definition, we have

\[
\bigcup_{j \in \Gamma} C_j = \mathbb{Z}_n.
\]

It is well known that \( \prod_{j \in C_i} (x - \alpha^j) \) is an irreducible polynomial of degree \( l_i \) over \( \text{GF}(q) \) and is the minimal polynomial of \( \alpha^j \) over \( \text{GF}(q) \). Furthermore, the canonical factorization of \( x^n - 1 \) over \( \text{GF}(q) \) is given by

\[
x^n - 1 = \prod_{i \in \Gamma} \prod_{j \in C_i} (x - \alpha^j).
\]

2.4 The linear span and minimal polynomial of sequences

Let \( s^L = s_0 s_1 \cdots s_{L-1} \) be a sequence over \( \text{GF}(q) \). The linear span (also called linear complexity) of \( s^L \) is defined to be the smallest positive integer \( \ell \) such that there are constants \( c_0 = 1, c_1, \ldots, c_\ell \in \text{GF}(q) \) satisfying

\[-c_0 s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } \ell \leq i < L.\]

In engineering terms, such a polynomial \( c(x) = c_0 + c_1 x + \cdots + c_\ell x^\ell \) is called the feedback polynomial of a shortest linear feedback shift register (LFSR) that generates \( s^L \). Such an integer always exists for finite sequences \( s^L \). When \( L = \infty \), a sequence \( s^\infty \) is
called a semi-infinite sequence. If there is no such an integer for a semi-infinite sequence \( s^\infty \), its linear span is defined to be \( \infty \). The linear span of the zero sequence is defined to be zero. For ultimately periodic semi-infinite sequences such an \( \ell \) always exists.

Let \( s^\infty \) be a sequence of period \( L \) over \( \text{GF}(q) \). Any feedback polynomial of \( s^\infty \) is called a characteristic polynomial. The characteristic polynomial with the smallest degree is called the minimal polynomial of the periodic sequence \( s^\infty \). Since we require that the constant term of any characteristic polynomial be 1, the minimal polynomial of any periodic sequence \( s^\infty \) must be unique. In addition, any characteristic polynomial must be a multiple of the minimal polynomial.

For periodic sequences, there are a few ways to determine their linear span and minimal polynomials. One of them is given in the following lemma [18, p.87, Theorem 5.3].

**Lemma 1.** Let \( s^\infty \) be a sequence of period \( L \) over \( \text{GF}(q) \). Define \( S^L(x) = \sum_{i=0}^{L-1} s_i x^i \in \text{GF}(q)[x] \). Then the minimal polynomial \( M(x) \) of \( s^\infty \) is given by

\[
\frac{x^L - 1}{\gcd(x^L - 1, S^L(x))}
\]

and the linear span \( \mathbb{I}_s \) of \( s^\infty \) is given by \( L - \deg(\gcd(x^L - 1, S^L(x))) \).

The other one is given in the following lemma ([11 Theorem 3], [23]).

**Lemma 2.** Any sequence \( s^\infty \) over \( \text{GF}(q) \) of period \( q^m - 1 \) has a unique expansion of the form

\[
s_i = \sum_{t \geq 0} c_t \alpha^t \quad \text{for all } t \geq 0,
\]

where \( \alpha \) is a generator of \( \text{GF}(q^m) \) and \( c_t \in \text{GF}(q^m) \). Let the index set be \( I = \{ i \mid c_i \neq 0 \} \), then the minimal polynomial \( M_s(x) \) of \( s^\infty \) is \( M_s(x) = \prod_{i \in I} (1 - \alpha^i x) \), and the linear span of \( s^\infty \) is \( |I| \).

It should be noticed that in some references the reciprocal of \( M_s(x) \) is called the minimal polynomial of the sequence \( s^\infty \). So Lemma 2 is a modified version of the original one in [23].

### 3 Cyclic codes from combinatorial sequences

#### 3.1 The classical cyclic code \( \mathcal{C}_{\text{GF}(q)}(D) \) of a subset \( D \subseteq \mathbb{Z}_n \)

Let \( n \) be a positive integer, and let \( D \) be a subset of \( \mathbb{Z}_n \). Define \( B_i = D + i \) for all \( i \in \mathbb{Z}_n \). Then the pair \( (\mathbb{Z}_n, B) \) is called an incidence structure, where \( B = \{ B_0, B_1, \ldots, B_{n-1} \} \).

The incidence matrix \( M_D = (m_{ij}) \) of this incidence structure is an \( n \times n \) matrix, where \( m_{ij} = 1 \) if \( j \in B_i \) and \( m_{ij} = 0 \) otherwise. By definition, \( M_D = (m_{ij}) \) is a binary matrix. When \( M_D = (m_{ij}) \) is viewed as a matrix over \( \text{GF}(q) \), its row vectors span a cyclic code of length \( n \) over \( \text{GF}(q) \), which is denoted by \( \mathcal{C}_{\text{GF}(q)}(D) \) and called the classical code of \( D \). It is easily seen that the generator polynomial of \( \mathcal{C}_{\text{GF}(q)}(D) \) is given by

\[
\gcd \left( x^n - 1, \sum_{i \in D} x^i \right),
\]

(3)
where the greatest common divisor is computed over $\text{GF}(q)$[14][p. 66].

When $D$ has certain combinatorial structures, the cyclic code $C_{\text{GF}(q)}(D)$ has been well studied in the literature[2, 14]. This code is closely related to a code dealt with in the next section.

### 3.2 The cyclic code of the characteristic sequence of a subset $D \in \mathbb{Z}_n$

Let $D$ be a subset of $\mathbb{Z}_n$. The characteristic sequence $s(D)$ of $D$ is given by

$$s(D)_i = \begin{cases} 1 & \text{if } i \mod n \in D, \\ 0 & \text{otherwise}. \end{cases}$$

The binary sequence $s(D)$ can be viewed as a sequence of period $n$ over any field $\text{GF}(q)$, and can be employed to construct the code $C_s(D)$ over $\text{GF}(q)$. For any given pair of $n$ and $q$ with $\gcd(q, n) = 1$, the subset $D$ must be chosen properly, in order to construct a cyclic code $C_s(D)$ with desirable parameters. Intuitively, a good choice may be to select a subset $D$ of $\mathbb{Z}_n$ with certain combinatorial structures. It follows from the discussions in Section 3.1 that

$$C_s(D) = C_{\text{GF}(q)}(D)^c.$$  \hspace{1cm} (4)

Hence, in the case that the sequence $s^c$ over $\text{GF}(q)$ has only entries 0 and 1, the sequence code $C_s(D)$ is the complement code of the classical code of its support set. This is a major connection between the classical construction of cyclic codes with incidence structures and the sequence construction of this paper in the special case. However, the two approaches do not include each other.

Let $D$ be a $\kappa$-subset of $\mathbb{Z}_n$. The set $D$ is an $(n, \kappa, \lambda)$ difference set in $(\mathbb{Z}_n, +)$ if the multiset

$$\{x - y | x, y \in D\}$$

contains every nonzero element of $\mathbb{Z}_n$ exactly $\lambda$ times.

Let $D$ be a $\kappa$-subset of $\mathbb{Z}_n$. The set $D$ is an $(n, \kappa, \lambda, t)$ almost difference set (ADS) in $(\mathbb{Z}_n, +)$ if the multiset

$$\{x - y | x, y \in D\}$$

contains $t$ nonzero elements of $\mathbb{Z}_n$ exactly $\lambda$ times each and the remaining $n - 1 - t$ nonzero elements $\lambda + 1$ times each.

**Example 3.** The Singer difference set in $(\mathbb{Z}_n, +)$ is given by $D = \log_\alpha \{x \in \text{GF}(2^n) : \text{Tr}(x) = 1\} \subset \mathbb{Z}_n$, and has parameters $(2^m - 1, 2^{m-1}, 2^{m-2})$, where $\alpha$ is generator of $\text{GF}(2^n)$ and $n = 2^m - 1$.

Its characteristic sequence $(s_t)_{t \geq 0}^{\infty}$, where $s_t = \text{Tr}(\alpha^t)$ for any $t \geq 0$, a maximum-length sequence of period $2^m - 1$. The minimal polynomial of the Singer sequence is equal to the minimal polynomial $m_{\alpha^{-1}}(x)$ of $\alpha^{-1}$ over $\text{GF}(2)$, and its linear span is $m$. The cyclic code $C_s(D)$ defined by the characteristic sequence of the Singer sequence is equivalent to the Hamming code with parameters $[2^m - 1, 2^m - 1 - m, 3]$ and has generator polynomial $m_{\alpha^{-1}}(x)$. The code is optimal (perfect).

A proof of the following results can be found in[14][p. 193].
Example 4. Let $q = p^s$ be a prime power, where $p$ is a prime, and $s$ is a positive integer, and let $m \geq 3$ be a positive integer. Let $\alpha$ be a generator of $\text{GF}(q^m)^\ast$. Put $n = (q^m - 1)/(q - 1)$. Recall that

$$D = \{0 \leq i < n : \text{Tr}_{q^m/q}(\alpha^i) = 0\} \subset \mathbb{Z}_n$$

is the Singer difference set in $(\mathbb{Z}_n, +)$ with parameters

$$\left(\frac{q^m - 1}{q - 1}, \frac{q^{m-1} - 1}{q - 1}, \frac{q^{m-2} - 1}{q - 1}\right).$$

Let $s(D)^\infty$ be the characteristic sequence of $D$. Then the cyclic code $C_{s(D)}^c$ has parameters

$$\left[\frac{q^m - 1}{q - 1}, (p + m - 2)^s + 1, \frac{q^{m-1} - 1}{q - 1}\right].$$

(5)

Although the parameters of $C_{s(D)}^c$ are known, the following problem is still open.

Open Problem 5. Determine the minimum distance of the code $C_{s(D)}$ from the characteristic sequence of the Singer difference set $D$.

There are many families of difference sets and almost difference sets $D$ in $(\mathbb{Z}_n, +)$. In many cases, the dimension and generator polynomial of the classical code $C_{\text{GF}(q)}(D)$ or its complement $C_{\text{GF}(q)}(D)^c$ (hence, $C_{\text{GF}(q)}(D)^\perp$) are known. However, their minimum distances are open in general. Because of the relation in (4), the dimension and generator polynomial of the sequence code $C_{s(D)}$ are known in many cases, but its minimum distance is known only in some special cases. As this is a huge topic with a lot of results, it would be infeasible to survey the developments here. Thus, we refer the reader to the monograph [14] for detailed information.

Cyclotomic classes were employed to define binary sequences, which can be viewed as sequences over $\text{GF}(q)$ for any prime power $q$. Such sequences give cyclic codes $C_{s(D)}$ over $\text{GF}(q)$. The reader is referred to [10, 13, 14] for detailed information.

4 Cyclic codes from a construction of sequences from polynomials over $\text{GF}(q^m)$

Given a polynomial $f(x)$ on $\text{GF}(q^m)$, we define its associated sequence $s^\infty$ by

$$s_i = \text{Tr}(f(\alpha^i + 1))$$

(6)

for all $i \geq 0$, where $\alpha$ is a generator of $\text{GF}(q^m)^\ast$ and $\text{Tr}(x)$ denotes the trace function from $\text{GF}(q^m)$ to $\text{GF}(q)$. The code $C_s$ defined by the sequence $s^\infty$ in (6) is called the code from the polynomial $f(x)$ for simplicity.

It was demonstrated in [13, 20, 32] that the code $C_s$ may have interesting parameters if the polynomial $f$ is properly chosen. The objective of this section is to survey cyclic codes $C_s$ defined by special polynomials $f$ over $\text{GF}(q^m)$. 
4.1 Cyclic codes from special monomials

The following is a list of monomials over GF\(q^m\) with good nonlinearity (see Section 1.7 for definition and details).

- \(f(x) = x^{e^m-2}\) over GF\(q^m\) (APN).
- \(f(x) = x^{e^m+1}\) over GF\(q^m\), where \(m/\gcd(m,k)\) and \(q\) are odd (planar).
- \(f(x) = x^{(q^h-1)/(q-1)}\) over GF\(q^m\).
- \(f(x) = x^{(3^b+1)/2}\) over GF\(3^m\) (planar when \(\gcd(h,m) = 1\)).
- \(f(x) = x^{2^t+3}\) over GF\(2^m\) (APN).
- \(f(x) = x^e\) over GF\(2^m\), \(e = 2^{(m-1)/2} + 2^{(m-1)/4} - 1\) and \(m \equiv 1 \pmod{4}\) (APN).

When they are plugged into (6), sequences over GF\(q\) with certain properties are obtained. The corresponding sequence codes have interesting parameters. The objective of this section is to introduce the parameters of these cyclic codes.

Let \(t\) be a positive integer. We define \(T = 2^t - 1\). For any odd \(a \in \{1, 2, 3, \ldots, T\}\), define

\[
\epsilon_a^{(t)} = \begin{cases} 
1, & \text{if } a = 2^h - 1, \\
\left\lfloor \log_2 \frac{T}{a} \right\rfloor \mod 2, & \text{if } 1 \leq a < 2^h - 1 
\end{cases}
\]

and

\[
\kappa_a^{(t)} = \epsilon_a^{(t)} \mod 2. \tag{7}
\]

This function \(\kappa_a^{(t)}\) will be employed later.

4.1.1 Binary cyclic codes from \(f(x) = x^{2^t+3}\)

The monomial \(f(x) = x^{2^t+3}\) is APN over GF\(2^{2^t+1}\). Both the sequence in (6) defined by this monomial and the code \(C_s\) are interesting.

**Theorem 6.** [20] Let \(m = 2t + 1 \geq 7\). Let \(s^m\) be the sequence of (6), where \(f(x) = x^{2^{t+3}}\). Then the linear span \(L_s\) of \(s^m\) is equal to \(5m + 1\) and the minimal polynomial \(M_s(x)\) of \(s^m\) is given by

\[
M_s(x) = (x - 1)m_{\alpha - 1}(x)m_{\alpha - 3}(x)m_{\alpha - 5}(x)m_{\alpha - 7}(x)m_{\alpha - 9}(x). \tag{8}
\]

The binary code \(C_s\) has parameters \([2^m - 1, 2^m - 2 - 5m, d]\) and generator polynomial \(M_s(x)\) of (8), where \(d \geq 8\).

**Example 7.** Let \(m = 5\) and \(\alpha\) be a generator of GF\(2^m\) with \(\alpha^5 + \alpha + 1 = 0\). Then the generator polynomial of the code \(C_s\) is \(M_s(x) = x^{16} + x^{15} + x^{13} + x^{12} + x^8 + x^6 + x^3 + 1\), and \(C_s\) is a [31, 15, 8] binary cyclic code. Its dual is a [31, 16, 7] cyclic code. Both codes are optimal according to the Database.

**Example 8.** Let \(m = 7\) and \(\alpha\) be a generator of GF\(2^m\) with \(\alpha^7 + \alpha + 1 = 0\). Then the generator polynomial of the code \(C_s\) is \(M_s(x) = x^{36} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{21} + x^{12} + x^9 + x^7 + x^6 + x^5 + x^3 + x + 1\) and \(C_s\) is a [127, 91, 8] binary cyclic code.
It can be seen from Example 11 that the bound on the minimal distance of $C_s$ in Theorem 6 is tight in certain cases. The code $C_s$ in Theorem 6 could be optimal in some cases [20]. It would be interesting to settle the following problem.

**Open Problem 9.** Determine the minimum distance of the code $C_s$ in Theorem 6.

### 4.1.2 Binary cyclic codes from $f(x) = x^{2h-1}$

Consider the monomial $f(x) = x^{2h-1}$ over $\text{GF}(2^m)$, where $h$ is a positive integer with $1 \leq h \leq \left\lfloor \frac{m}{2} \right\rfloor$. As will be demonstrated below, it gives a binary sequence and binary code with special parameters.

**Theorem 10.** [20] Let $s^m$ be the sequence of (6), where $f(x) = x^{2h-1}$, $2 \leq h \leq \left\lfloor \frac{m}{2} \right\rfloor$. Then the linear span $\mathbb{L}_s$ of $s^m$ is given by

$$
\mathbb{L}_s = \left\{ \frac{m(2h+(-1)^{h-1})}{3}, \text{ if } m \text{ is even}, \quad \frac{m(2h+(-1)^{h-1})}{3}+3, \text{ if } m \text{ is odd.} \right\}
$$

(9)

The minimal polynomial

$$
\mathbb{M}_s(x) = (x-1)^{N_2(m)} \prod_{1 \leq j \leq \frac{m}{2}-1} m_{\kappa_j^{(h)-1}}(x),
$$

(10)

where $\kappa_j^{(h)}$ was defined in (7). $N_2(i) = 0$ if $i \equiv 0 \pmod{2}$ and $N_2(i) = 1$ otherwise.

Let $h \geq 2$. Then the binary code $C_s$ has parameters $[2^m-1, 2^{m-1}-1, \mathbb{L}_s, d]$ and generator polynomial $\mathbb{M}_s(x)$, where

$$
d \geq \begin{cases} 
2^{h-2} + 2 & \text{if } m \text{ is odd and } h > 2, \\
2^{h-2} + 1 & \text{otherwise.}
\end{cases}
$$

**Example 11.** Let $(m, h) = (7, 2)$ and $\alpha$ be a generator of $\text{GF}(2^m)^*$ with $\alpha^2 + \alpha + 1 = 0$. Then the generator polynomial of the code $C_s$ is $\mathbb{M}_s(x) = x^8 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, and $C_s$ is a [127, 119, 4] binary cyclic code and optimal according to the Database.

**Example 12.** Let $(m, h) = (7, 3)$ and $\alpha$ be a generator of $\text{GF}(2^m)^*$ with $\alpha^2 + \alpha + 1 = 0$. Then the generator polynomial of the code $C_s$ is $\mathbb{M}_s(x) = x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^7 + x^6 + x^5 + x^4 + 1$ and $C_s$ is a [127, 105, $d$] binary cyclic code, where $4 \leq d \leq 8$.

**Remark 13.** The code $C_s$ of Theorem 10 may be bad when $\gcd(h, m) \neq 1$. In this case the monomial $f(x) = x^{2h-1}$ is not a permutation of $\text{GF}(2^m)$. For example, when $(m, h) = (6, 3)$, $C_s$ is a [63, 45, 3] binary cyclic code, while the best known linear code in the Database has parameters [63, 45, 8]. Hence, we are interested in this code only for the case that $\gcd(h, m) = 1$, which guarantees that $f(x) = x^{2h-1}$ is a permutation of $\text{GF}(2^m)$.

The code $C_s$ in Theorem 10 could be optimal in some cases [20]. The lower bounds on $d$ given in Theorem 10 are quite tight. Nevertheless, it would be nice if the following problem could be solved.
Open Problem 14. Determine the minimum distance of the code $C_s$ in Theorem [70]

Let $f(x) = x^d - 1$ over $GF(q^m)$ and let $s^m$ be the sequence of [65]. Then the code $C_s$ was investigated in [26], where the results of this section were generalised. The reader is referred to [26] for detail.

4.1.3 Binary cyclic codes from $f(x) = x^e$, $e = 2^{(m-1)/2} + 2^{(m-1)/4} - 1$ and $m \equiv 1 \pmod{4}$

Let $f(x) = x^e$, where $e = 2^{(m-1)/2} + 2^{(m-1)/4} - 1$ and $m \equiv 1 \pmod{4}$. It is known that $f(x)$ is a permutation of $GF(2^m)$ and is APN. Properties of the binary sequence and binary code defined by $f(x) = x^e$ are documented in the following theorem.

**Theorem 15.** [20] Let $m \geq 9$ be odd. Let $s^m$ be the sequence of [6]. Then the linear span $\mathbb{L}_s$ of $s^m$ is given by

$$\mathbb{L}_s = \left\{ \begin{array}{ll} \frac{m(2^{(m+1)/2} + (-1)^{m-1})}{3} + 3, & \text{if } m \equiv 1 \pmod{8}, \\ \frac{m(2^{(m+1)/2} + (-1)^{m-1})}{3} + 3, & \text{if } m \equiv 5 \pmod{8}. \end{array} \right. \quad (11)$$

The minimal polynomial

$$M_s(x) = (x - 1) \prod_{i=0}^{2^{(m-1)/4} - 1} m_{\alpha^{-i-2} \cdot \alpha^{m-1}}(x) \prod_{\underset{1 \leq j < 2 \leq \frac{1}{2} \left( \frac{m-1}{4} \right)}{2 \leq j < 2 \leq \frac{m-1}{4}}} m_{\alpha^{-2j-1}}(x)$$

if $m \equiv 1 \pmod{8}$; and

$$M_s(x) = (x - 1) \prod_{i=1}^{2^{(m-1)/4} - 1} m_{\alpha^{-i-2} \cdot \alpha^{m-1}}(x) \prod_{\underset{1 \leq j < 2 \leq \frac{1}{2} \left( \frac{m-1}{4} \right)}{2 \leq j < 2 \leq \frac{m-1}{4}}} m_{\alpha^{-2j-1}}(x)$$

if $m \equiv 5 \pmod{8}$, where $\kappa_{2j+1}^{(h)}$ was defined in [7].

The binary code $C_s$ has parameters $[2^m - 1, 2^{m-1} - 1 - \mathbb{L}_s, d]$ and generator polynomial $M_s(x)$, and the minimum weight $d$ has the following bounds:

$$d \geq \left\{ \begin{array}{ll} 2^{(m-1)/4} + 2, & \text{if } m \equiv 1 \pmod{8}, \\ 2^{(m-1)/4}, & \text{if } m \equiv 5 \pmod{8}. \end{array} \right. \quad (12)$$

**Example 16.** Let $m = 5$ and $\alpha$ be a generator of $GF(2^m)^*$ with $\alpha^5 + \alpha^2 + 1 = 0$. Then the generator polynomial of the code $C_s$ is $M_s(x) = x^6 + x^2 + 1$, and $C_s$ is a $[31, 25, 4]$ binary cyclic code and optimal according to the Database.

**Example 17.** Let $m = 9$ and $\alpha$ be a generator of $GF(2^m)^*$ with $\alpha^9 + \alpha^4 + 1 = 0$. Then the generator polynomial of the code $C_s$ is $M_s(x) = x^{46} + x^{45} + x^{44} + x^{40} + x^{39} + x^{36} + x^{35} + x^{33} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{14} + x^{12} + x^7 + x^4 + x^2 + x + 1$ and $C_s$ is a $[511, 465, d]$ binary cyclic code, where $d \geq 6$. The actual minimum weight may be larger than 6.
The code $C_s$ in Theorem 15 could be optimal in some cases. The lower bounds on $d$ given in Theorem 10 are reasonably good. It would be interesting to work on the following problem.

**Open Problem 18.** Determine the minimum distance of the code $C_s$ in Theorem 15 or improve the lower bounds in (12).

### 4.1.4 Binary cyclic codes from $f(x) = x^{2^{2h} - 2^h + 1}$, where $\gcd(m, h) = 1$

Define $f(x) = x^{e}$, where $e = 2^{2h} - 2^h + 1$ and $\gcd(m, h) = 1$. It is known that $f$ is APN under these conditions. In this section, we restrict $h$ to the following range:

$$1 \leq h \leq \begin{cases} \frac{m - 1}{4} & \text{if } m \equiv 1 \pmod{4}, \\ \frac{m - 3}{4} & \text{if } m \equiv 3 \pmod{4}, \\ \frac{m - 4}{4} & \text{if } m \equiv 0 \pmod{4}, \\ \frac{m - 2}{4} & \text{if } m \equiv 2 \pmod{4}. \end{cases} \quad (13)$$

Some parameters of the binary sequence and the code defined by $f(x) = x^{e}$ are given in the following theorem.

**Theorem 19.** [20, 22] Let $h$ satisfy the conditions of (13). Let $s^m$ be the sequence of $f$. Then the linear span $\mathbb{L}_s$ of $s^m$ is given by

$$\mathbb{L}_s = \begin{cases} m(2^{h+2} + (-1)^{h-1} + 3N_x)(m) \pmod{4} & \text{if } h \text{ is even}, \\ m(2^{h+2} + (-1)^{h-1} - 6 + 3N_x)(m) \pmod{4} & \text{if } h \text{ is odd}. \end{cases} \quad (14)$$

The minimal polynomial

$$\mathbb{M}_s(x) = (x - 1)^{128}(m) \prod_{i=0}^{2^{h-1}-1} m_{x^{i+2^m - h}}(x) \prod_{1 \leq j \leq 2^{h-1}} m_{x^{2j-1}}(x)$$

if $h$ is even; and

$$\mathbb{M}_s(x) = (x - 1)^{128}(m) \prod_{i=1}^{2^{h-1}-1} m_{x^{i+2^m - h}}(x) \prod_{3 \leq j \leq 2^{h-1}} m_{x^{2j-1}}(x)$$

if $h$ is odd, where $x^{(h)}_{2j+1}$ was defined in 7.

The code $C_s$ has parameters $[2^m - 1, 2^m - 1 - \mathbb{L}_s, d]$ and generator polynomial $\mathbb{M}_s(x)$, and the minimum weight $d$ has the following bounds:

$$d \geq \begin{cases} 2^h + 2 & \text{if } h \text{ is even and } m \text{ is odd}, \\ 2^{h+1} & \text{if } h \text{ is even and } m \text{ is even}, \\ 2^h & \text{if } h \text{ is odd}. \end{cases} \quad (15)$$

**Example 20.** Let $(m, h) = (5, 2)$ and $\alpha$ be a generator of $\text{GF}(2^m)^+$. Then the generator polynomial of the code $C_s$ is $\mathbb{M}_s(x) = x^{16} + x^{14} + x^{10} + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ and $C_s$ is a $[31, 15, 8]$ binary cyclic code. Its dual is a $[31, 16, 7]$ cyclic code. Both codes are optimal according to the Database.
Example 21. Let \((m,h) = (7,2)\) and \(\alpha \) be a generator of \(\text{GF}(2^m)^*\) with \(\alpha^7 + \alpha + 1 = 0\). Then the generator polynomial of the code \(C_s\) is \(\mathbb{M}_s(x) = x^{36} + x^{28} + x^{27} + x^{23} + x^{21} + x^{20} + x^{18} + x^{13} + x^{12} + x^9 + x^7 + x^6 + x^3 + 1\) and \(C_s\) is a \([127, 91, 8]\) binary cyclic code.

The code \(C_s\) in Theorem [19] could be optimal in some cases [20]. It would be interesting to attack the following two problems.

Open Problem 22. Determine the minimum distance of the code \(C_s\) in Theorem [19].

The dimension and lower bounds on the minimum weight of the code \(C_s\) of this section were developed in [29] when \(h\) satisfies

\[
\begin{align*}
\frac{m-1}{2} &> h > \frac{m-1}{3} & \text{if } m \equiv 1 \pmod{4}, \\
\frac{m-2}{3} &> h > \frac{m-2}{3} & \text{if } m \equiv 3 \pmod{4}, \\
\frac{m-1}{4} &> h > \frac{m-1}{4} & \text{if } m \equiv 0 \pmod{4}, \\
\frac{m-2}{4} &> h > \frac{m-2}{4} & \text{if } m \equiv 2 \pmod{4}.
\end{align*}
\]

(16)

4.1.5 Binary cyclic codes from \(f(x) = x^{2^m-2}\) over \(\text{GF}(2^m)\)

Let \(\rho_i\) denote the total number of even integers in the 2-cyclotomic coset \(C_i\) modulo \(2^m - 1\). We then define

\[
v_i = \frac{m\rho_i}{\ell_i} \mod 2
\]

(17)

for each \(i \in \Gamma\), where \(\ell_i = |C_i|\) and \(\Gamma\) denotes the set of coset leaders modulo \(n = 2^m - 1\).

It is known that \(f(x) = x^{2^m-2}\) over \(\text{GF}(2^m)\) is APN. For the binary sequence and code defined by this monomial, we have the following.

Theorem 23. [17] Let \(s^\omega\) be the sequence of [5], where \(f(x) = x^{2^m-2}\). Then the linear span \(\mathbb{L}_s\) of \(s^\omega\) is equal to \((n+1)/2\) and the minimal polynomial \(\mathbb{M}_s(x)\) of \(s^\omega\) is given by

\[
\mathbb{M}_s(x) = \prod_{j \in \Gamma, v_j = 1} m_{\alpha^j}(x).
\]

(18)

The binary code \(C_s\) has parameters \([2^m - 1, 2^m-1 - 1, d]\) and generator polynomial \(\mathbb{M}_s(x)\). If \(m\) is odd, the minimum distance \(d\) of \(C_s\) is even and satisfies \(d^2 - d + 1 \geq n\), and the dual code \(C_s^\perp\) has parameters \([2^m - 1, 2^m-1, d^\perp]\), where \(d^\perp\) satisfies that \((d^\perp)^2 - d^\perp + 1 \geq n\).

Example 24. Let \(m = 5\) and \(\alpha\) be a generator of \(\text{GF}(2^m)^*\) with \(\alpha^5 + \alpha^2 + 1 = 0\). Then the generator polynomial \(\mathbb{M}_p(x)\) of the code \(C_s\) is

\[
(x + 1)m_{\alpha^3}(x)m_{\alpha^5}(x)m_{\alpha^{15}}(x)
= x^{16} + x^{14} + x^{13} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^2 + x + 1
\]

and \(C_s\) is a \([31, 15, 8]\) binary cyclic code. Its dual is a \([31, 16, 7]\) cyclic code. Both codes are optimal according to the Database.

When \(f(x) = x^{q^m-2}\) and \(q > 2\), the dimension of the code \(C_s\) over \(\text{GF}(q)\) was settled in [32]. But no lower bound on the minimum distance of \(C_s\) is developed.
4.1.6 Cyclic codes from \( f(x) = x^{q+1}, \) where \( m / \gcd(m, \kappa) \) and \( q \) are odd

Let \( f(x) = x^{q+1}, \) where \( m / \gcd(m, \kappa) \) and \( q \) are odd. It is known that \( f \) is planar. Properties of the sequence and code defined by this monomial are described below.

**Theorem 25.** [14] Let \( m \) be odd. Let \( s^* \) be the sequence of [2], where \( f(x) = x^{q+1}. \) Then the linear span \( L \) of \( s^* \) is equal to \( 2m + N_p(m) \) and the minimal polynomial \( M_s(x) \) of \( s^* \) is given by

\[
M_s(x) = (x-1)^{\text{gcd}(m)}m_{\alpha-1}(x)m_{\alpha^{-1}}(x),
\]

where \( N_p(i) = 0 \) if \( i \equiv 0 \text{ (mod } p) \) and \( N_p(i) = 1 \) otherwise.

The code \( C \) has parameters \([n, n-2m-N_p(m), d]\) and generator polynomial \( M_s(x), \) where

\[
\begin{cases}
  d = 4 & \text{if } q = 3 \text{ and } m \equiv 0 \text{ (mod } p), \\
  4 \leq d \leq 5 & \text{if } q = 3 \text{ and } m \not\equiv 0 \text{ (mod } p), \\
  d = 3 & \text{if } q > 3 \text{ and } m \equiv 0 \text{ (mod } p), \\
  3 \leq d \leq 4 & \text{if } q > 3 \text{ and } m \not\equiv 0 \text{ (mod } p).
\end{cases}
\]

**Example 26.** Let \((m, \kappa, q) = (3, 1, 3)\) and \( \alpha \) be a generator of \( GF(r)^* \) with \( \alpha^3 + 2\alpha + 1 = 0. \) Then \( C_s \) is a [26,20,4] ternary code with generator polynomial \( M_s(x) = x^6 + 2x^5 + 2x^4 + x^3 + x^2 + 2x + 1. \) This cyclic code is an optimal linear code according to the Database.

**Example 27.** Let \((m, \kappa, q) = (4, 4, 3)\) and \( \alpha \) be a generator of \( GF(r)^* \) with \( \alpha^4 + 2\alpha^3 + 2 = 0. \) Then \( C_s \) is a [80,71,5] ternary code with generator polynomial \( M_s(x) = x^9 + 2x^8 + x^7 + 2x^6 + x^4 + x^3 + 1. \) This cyclic code is an optimal linear code according to the Database.

Extending the work of [35], one can determine the weight distribution of \( C_s^1. \) With the MacWilliams identity, one can settle the minimum distance of the code \( C_s \) in Theorem 25.

4.1.7 Cyclic codes from \( f(x) = x^{(q^h-1)/(q-1)} \)

Let \( h \) be a positive integer satisfying the following condition:

\[
1 \leq h \leq \begin{cases} (m-1)/2 & \text{if } m \text{ is odd}, \\ m/2 & \text{if } m \text{ is even}. \end{cases}
\]

Let \( J \geq t \geq 2, \) and let \( N(J, t) \) denote the total number of vectors \((i_1, i_2, \cdots, i_{t-1})\) with \( 1 \leq i_1 < i_2 < \cdots < i_{t-1} < J. \) By definition, we have the following recursive formula:

\[
N(J, t) = \sum_{j=t-1}^{J-1} N(j, t-1). \tag{21}
\]

It is easily seen that

\[
N(J, 2) = J - 1 \text{ for all } J \geq 2 \tag{22}
\]
and
\[ N(J, 3) = \frac{(J - 1)(J - 2)}{2} \] for all \( J \geq 3 \). \hspace{1cm} (23)

It then follows from (21), (22) and (23) that
\[ N(J, 4) = \sum_{j=3}^{J-1} N(j, 3) = \sum_{j=3}^{J-1} \frac{(J - 1)(J - 2)}{2} = \frac{J^3 - 6J^2 + 11J - 6}{6}. \hspace{1cm} (24) \]

By definition, we have
\[ N(t, t) = 1 \] for all \( t \geq 2 \). \hspace{1cm} (25)

For convenience, we define \( N(J, 1) = 1 \) for all \( J \geq 1 \).

**Theorem 28.** [11] Let \( h \) satisfy the condition of (20). Let \( s^\alpha \) be the sequence of (6), where \( f(x) = x^{(q^3 - 1)/(q - 1)} \). Then the linear span \( \mathbb{L}_s \) and minimal polynomial \( \mathbb{M}_s(x) \) of \( s^\alpha \) are given by
\[
\mathbb{L}_s = \left( \mathbb{N}_p(h) + \sum_{i=1}^{h-1} \sum_{u=1}^{h-1} \mathbb{N}_p(h - u)\mathbb{N}(u, t) \right) m + \mathbb{N}_p(m)
\]

and
\[
\mathbb{M}_s(x) = (x - 1)^{\mathbb{N}_p(m)} \mathbb{m}_{\alpha^{-1}}(x)^{\mathbb{N}_p(h)} \prod_{1 \leq i \leq (q - 1)} \mathbb{m}_{\alpha^{-i}}(x) \times \prod_{2 \leq l \leq (q - 1)} \prod_{1 \leq i_1 < \cdots < i_{l-1} < u} m_{\alpha^{-i_1}}(x) \prod_{1 \leq j \leq (q - 1)} m_{\alpha^{-j}}(x).
\]

The code \( C_s \) has parameters \([n, n - \mathbb{L}_s, d]\) and generator polynomial \( \mathbb{M}_s(x) \).

**Open Problem 29.** Determine the minimum distance of the code \( C_s \) in Theorem 28 or develop a tight lower bound on it.

As a corollary of Theorem 28, we have the following.

**Corollary 30.** Let \( h = 3 \). The code \( C_s \) of Theorem 28 has parameters \([n, n - \mathbb{L}_s, d]\) and generator polynomial \( \mathbb{M}_s(x) \) given by
\[
\mathbb{M}_s(x) = (x - 1)^{\mathbb{N}_p(m)} \mathbb{m}_{\alpha^{-1}}(x) \mathbb{m}_{\alpha^{-1-q}}(x) \mathbb{m}_{\alpha^{-1-q^2}}(x) \mathbb{m}_{\alpha^{-1-q^2}}(x)
\]
if \( p \neq 3 \), and
\[
\mathbb{M}_s(x) = (x - 1)^{\mathbb{N}_p(m)} \mathbb{m}_{\alpha^{-1-q}}(x) \mathbb{m}_{\alpha^{-1-q}}(x) \mathbb{m}_{\alpha^{-1-q^2}}(x)
\]
if \( p = 3 \), where
\[
\mathbb{L}_s = \begin{cases} 
4m + \mathbb{N}_p(m) & \text{if } p \neq 3, \\
3m + \mathbb{N}_p(m) & \text{if } p = 3.
\end{cases} \hspace{1cm} (26)
\]

In addition,
\[
\begin{cases} 
3 \leq d \leq 8 & \text{if } p = 3 \text{ and } \mathbb{N}_p(m) = 1, \\
3 \leq d \leq 6 & \text{if } p = 3 \text{ and } \mathbb{N}_p(m) = 0, \\
3 \leq d \leq 8 & \text{if } p > 3.
\end{cases}
\]
Example 31. Let \((m, h, q) = (4, 3, 3)\) and \(\alpha\) be a generator of \(\text{GF}(r)^*\) with \(\alpha^3 + 2\alpha^2 + 2 = 0\). Then \(C_s\) is a \([80,69,5]\) ternary code with generator polynomial \(M_s(x) = x^{11} + 2x^8 + 2x^6 + 2x^5 + 2x^4 + x^3 + 2x^2 + x + 2\). This is an almost optimal linear code according to the Database. The known optimal linear code has parameters \([80,69,6]\) which is not cyclic. Notice that \(h > m/2\). Hence, the parameters of this code do not agree with those of the code in Corollary \[30\]. In this case \(f(x)\) is a permutation.

Example 32. Let \((m, h, q) = (5, 3, 3)\) and \(\alpha\) be a generator of \(\text{GF}(r)^*\) with \(\alpha^5 + 2\alpha + 1 = 0\). Then \(C_s\) is a \([242,226,5]\) ternary code with generator polynomial \(M_s(x) = x^{16} + 2x^{14} + 2x^{12} + 2x^{11} + x^{10} + x^9 + x^6 + x^3 + 2x^2 + 2\). Notice that \(h > m/2\). However, the parameters of this code do agree with those of the code in Corollary \[30\]. In this case \(f(x)\) is not a permutation. In fact, \(\gcd((q^h - 1)/(q - 1), q^n - 1) = 13\).

Example 33. Let \((m, h, q) = (6, 3, 3)\) and \(\alpha\) be a generator of \(\text{GF}(r)^*\) with \(\alpha^6 + 2\alpha^4 + \alpha^2 + 2\alpha + 2 = 0\). Then \(C_s\) is a \([728,710,6]\) ternary code with generator polynomial \(M_s(x) = x^{18} + 2x^{15} + 2x^{14} + 2x^{13} + 2x^{11} + x^{10} + 2x^9 + x^8 + x^6 + 2x^4 + x^3 + x^2 + 2\).

Notice that \(h = m/2\). Hence, the parameters of this code agree with those of the code in Corollary \[30\]. In this case \(f(x)\) is not a permutation. In fact, \(\gcd((q^h - 1)/(q - 1), q^n - 1) = 13\).

Open Problem 34. For the code \(C_s\) of Corollary \[30\], do the following lower bounds hold?

\[
d \geq \begin{cases} 
5 & \text{when } p = 3 \text{ and } \mathbb{N}_p(m) = 1, \\
4 & \text{when } p = 3 \text{ and } \mathbb{N}_p(m) = 0, \\
6 & \text{when } p > 3 \text{ and } \mathbb{N}_p(m) = 1, \\
5 & \text{when } p > 3 \text{ and } \mathbb{N}_p(m) = 0.
\end{cases}
\]

The work of this section was extended in \[31\], where the restriction in \[20\] was removed.

4.1.8 Cyclic codes from \(f(x) = x^{(3^h+1)/2}\)

Let \(h\) be a positive integer satisfying the following conditions:

\[
\begin{align*}
&h \text{ is odd,} \\
&\gcd(m, h) = 1, \\
&3 \leq h \leq \begin{cases} 
(m - 1)/2 & \text{if } m \text{ is odd,} \\
2m/2 & \text{if } m \text{ is even.}
\end{cases}
\end{align*}
\]

(27)

Theorem 35. \[11\] Let \(h\) satisfy the third condition of \[27\]. Let \(s^n\) be the sequence of \[32\], where \(f(x) = x^{(3^h+1)/2}\). Then the linear span \(L_s\) and minimal polynomial \(M_s(x)\) of \(s^n\) are given by

\[
L_s = \mathbb{N}_3(m) + \left( \sum_{i=0}^{h} \mathbb{N}_3(h - i + 1) \right) m + \\
\left( \sum_{i=2}^{h} \mathbb{N}(h,t) + \sum_{i=2}^{h-1} \mathbb{N}_3(h - i + 1) \mathbb{N}(i,t) \right) m 
\]

\[
16
\]
and

\[ M_s(x) = (x - 1)^{N_3(m)} m_{\alpha^{-1}}(x) N_3(h+1) m_{\alpha^{-2}}(x) \prod_{t=1}^{h-1} 1 \leq i_1 < \cdots < i_t \leq h-1 \ m_{\alpha^{-2} + \sum_{j=1}^{t} j}^t(x) \times \prod_{1 \leq i < h \atop N_3(h-i+1) = 1} m_{\alpha^{-1} + \sum_{j=1}^{i-1} j}^{i-1}(x), \]

where \( N_3(j) \) and \( N(j, t) \) were defined in Sections 2.7 and 4.1.7 respectively.

Furthermore, the code \( C_s \) has parameters \([n, n - L_s, d]\) and generator polynomial \( M_s(x) \).

As shown in Theorem 35, the linear span and the minimal polynomial of the sequence \( s^\infty \) have a complex formula. It looks difficult to discover further properties of the code in Theorem 35.

**Open Problem 36.** Determine the minimum distance of the code \( C_s \) in Theorem 35 or develop a tight lower bound on it.

As a corollary of Theorem 35 we have the following.

**Corollary 37.** Let \( h = 3 \). The code \( C_s \) of Theorem 35 has parameters \([n, n - L_s, d]\) and the generator polynomial \( M_s(x) \) given by

\[ M_s(x) = (x - 1)^{N_3(m)} m_{\alpha^{-1}}(x) m_{\alpha^{-2}}(x) m_{\alpha^{-5}}(x) m_{\alpha^{-10}}(x) m_{\alpha^{-11}}(x) m_{\alpha^{-13}}(x) m_{\alpha^{-14}}(x), \]

where \( L_s = 7m + N_3(m) \). In addition,

\[ \left\{ \begin{array}{ll} 5 \leq d \leq 16 & \text{if } N_3(m) = 1, \\ 4 \leq d \leq 16 & \text{if } N_3(m) = 0. \end{array} \right. \]

**Open Problem 38.** For the code \( C_s \) of Corollary 37 do the following lower bounds hold?

\[ d \geq \left\{ \begin{array}{ll} 9 & \text{when } N_p(m) = 1, \\ 8 & \text{when } N_p(m) = 0. \end{array} \right. \]

Let \( f(x) = x^{(p^h+1)/2} \) over \( \text{GF}(p^m) \) and let \( s^\infty \) be the sequence of \( \text{(6)} \). Then the code \( C_s \) was studied in [31], where the work of this section was generalised.

### 4.1.9 Cyclic codes from \( f(x) = x^{d^\ell+2} \) over \( \text{GF}(q^m) \)

In this section, we introduce cyclic codes defined by the monomial \( f(x) = x^{d^\ell+2} \) over \( \text{GF}(q^m) \), where \( m = 2\ell \).

**Lemma 39.** [26] Let \( m = 2\ell \) with \( \ell \) being a positive integer, and let \( s^\infty \) be the sequence of \( \text{(5)} \) defined by the monomial \( f(x) = x^{d^\ell+2} \) over \( \text{GF}(q^m) \).

When \( q \neq 2 \), the linear span \( L_s \) and minimal polynomial \( M_s(x) \) of \( s^\infty \) are given by

\[ L_s = 2m + N_p(3)m + N_p(4)\ell + N_p(m) \]
and
\[
\mathbb{M}_s(x) = \begin{cases} 
    m_{\alpha^{-1}}(x)m_{\alpha^{-2}}(x)m_{\alpha^{-(q^2+2)}}(x), & p = 2, \\
    (x-1)^{\mathbb{N}_p(m)}m_{\alpha^{-2}}(x)m_{\alpha^{-(q^2+1)}}(x)m_{\alpha^{-(q^2+2)}}(x), & p = 3, \\
    (x-1)^{\mathbb{N}_p(m)}m_{\alpha^{-1}}(x)m_{\alpha^{-2}}(x)m_{\alpha^{-(q^2+1)}}(x)m_{\alpha^{-(q^2+2)}}(x), & p > 3,
\end{cases}
\]

where \( \alpha \) is a generator of \( \text{GF}(q^m) \).

When \( q = 2 \), the linear span \( \mathbb{L}_s = m \) and minimal polynomial \( \mathbb{M}_s(x) \) of \( s^\infty \) is \( \mathbb{M}_s(x) = m_{\alpha^{-2(-1+1)}}(x) \).

**Theorem 40.** [26] Let \( C_s \) be the cyclic code defined by the sequence \( s^\infty \) in Lemma 39. Then we have the following.

- When \( q \neq 2 \), \( C_s \) has the generator polynomial \( \mathbb{M}_s(x) \) defined in Lemma 39 and parameters \([q^m - 1, q^m - 1 - (2m + \mathbb{N}_p(3)m + \mathbb{N}_p(4)\ell + \mathbb{N}_p(m)), d] \), where
  \[
  \begin{aligned}
  &\begin{cases} 
      3 \leq d \leq 6, & \text{if } p = 2 \text{ or } p = 3, \\
      3 \leq d \leq 8, & \text{if } p > 3.
    \end{cases}
  \end{aligned}
  \]

- When \( q = 2 \), \( C_s \) has the generator polynomial \( \mathbb{M}_s(x) \) defined in Lemma 39 and parameters \([2^m - 1, 2^m - 1 - m, d] \), where \( d = 2 \) if \( \ell \) is even and \( d = 3 \) otherwise.

## 4.2 Cyclic codes from Dickson polynomials

In this section, we survey known results on cyclic codes from Dickson polynomials over finite fields. All the results presented in this subsection come from [12].

### 4.2.1 Dickson polynomials over GF\( (q^m) \)

In 1896, Dickson introduced the following family of polynomials over \( \text{GF}(q^m) \) [7]:

\[
D_h(x,a) = \sum_{i=0}^{\lfloor \frac{h}{a} \rfloor} \frac{h}{a-i} \binom{h-i}{i} (-a)^i x^{h-2i},
\]

where \( a \in \text{GF}(q^m) \) and \( h \geq 0 \) is called the order of the polynomial. This family is referred to as the **Dickson polynomials of the first kind**.

Dickson polynomials of the second kind over \( \text{GF}(q^m) \) are defined by

\[
E_h(x,a) = \sum_{i=0}^{\lfloor \frac{h}{a} \rfloor} \binom{h}{h-i} (-a)^i x^{h-2i},
\]

where \( a \in \text{GF}(q^m) \) and \( h \geq 0 \) is called the order of the polynomial.

Dickson polynomials are an interesting topic of mathematics and engineering, and have many applications. For example, the Dickson polynomials \( D_s(x,a) = x^5 - ux - u^2x \) over \( \text{GF}(3^m) \) are employed to construct a family of planar functions [6] [19], and those planar functions give two families of commutative presemifields, planes, several classes of linear codes [3] [36], and two families of skew Hadamard difference sets [19].

The reader is referred to [27] for detailed information about Dickson polynomials. In subsequent subsections, we survey cyclic codes derived from Dickson polynomials.
4.2.2 Cyclic codes from the Dickson polynomial $D^p(x,a)$

Since $q$ is a power of $p$, it is known that $D_{hp}(x,a) = D_h(x,a)^p$ \[27\] Lemma 2.6]. It then follows that $D^p_{\omega}(x,a) = x^p$ for all $a \in \text{GF}(q)$.

The code $C_s$ over $\text{GF}(q)$ defined by the Dickson polynomial $f(x) = D^p_{\omega}(x,a) = x^p$ over $\text{GF}(q^m)$ has the following parameters.

**Theorem 41.** The code $C_s$ defined by the Dickson polynomial $D^p_{\omega}(x,a) = x^p$ has parameters $[n, n - m - \delta(1), d]$ and generator polynomial

$$M_s(x) = (x-1)^{\delta(1)} m_{\alpha-\delta}(x),$$

where

$$d = \begin{cases} 
4 & \text{if } q = 2 \text{ and } \delta(1) = 1, \\
3 & \text{if } q = 2 \text{ and } \delta(1) = 0, \\
3 & \text{if } q > 2 \text{ and } \delta(1) = 1, \\
2 & \text{if } q > 2 \text{ and } \delta(1) = 0,
\end{cases}$$

and the function $\delta(x)$ and the polynomial $m_{\alpha}(x)$ were defined in Section [2.1](#).

When $q = 2$, the code of Theorem 41 is equivalent to the binary Hamming weight or its even-weight subcode, and is thus optimal. The code is either optimal or almost optimal with respect to the Sphere Packing Bound.

4.2.3 Cyclic codes from $D_2(x,a) = x^2 - 2a$

In this section we consider the code $C_s$ defined by $f(x) = D_2(x,a) = x^2 - 2a$ over $\text{GF}(q^m)$. When $p = 2$, this code was treated in Section 4.2.2. When $p > 2$, the following theorem is a variant of Theorem 5.2 in [11], but has much stronger conclusions on the minimum distance of the code.

**Theorem 42.** Let $p > 2$ and $m \geq 3$. The code $C_s$ defined by $f(x) = D_2(x,a) = x^2 - 2a$ has parameters $[n, n - 2m - \delta(1 - 2a), d]$ and generator polynomial

$$M_s(x) = (x-1)^{\delta(1-2a)} m_{\alpha-1}(x)m_{\alpha-2}(x),$$

where

$$d = \begin{cases} 
4 & \text{if } q = 3 \text{ and } \delta(1-2a) = 0, \\
5 & \text{if } q = 3 \text{ and } \delta(1-2a) = 1, \\
3 & \text{if } q > 3 \text{ and } \delta(1-2a) = 0, \\
4 & \text{if } q > 3 \text{ and } \delta(1-2a) = 1,
\end{cases}$$

and the function $\delta(x)$ and the polynomial $M_{\alpha}(x)$ were defined in Section [2.1](#).

The code of Theorem 42 is either optimal or almost optimal for all $m \geq 2$.

4.2.4 Cyclic codes from $D_3(x,a) = x^3 - 3ax$

In this section we treat the code $C_s$ defined by the Dickson polynomial $D_3(x,a) = x^3 - 3ax$. We need to distinguish among the three cases: $p = 2$, $p = 3$ and $p \geq 5$. The case that $p = 3$ was covered in Section 4.2.2. So we need to consider only the two remaining cases.

We first handle the case $q = p = 2$ and state the following lemma.
Lemma 43. Let \( q = p = 2 \). Let \( s^\infty \) be the sequence of (6), where \( f(x) = D_3(x,a) = x^3 - 3ax = x^3 + ax \). Then the minimal polynomial \( M_s(x) \) of \( s^\infty \) is given by

\[
M_s(x) = \begin{cases} 
(x-1)^{\delta(1)}m_{\alpha^{-1}}(x) & \text{if } a = 0, \\
(x-1)^{\delta(1+a)}m_{\alpha^{-1}}(x)m_{\alpha^{-2}}(x) & \text{if } a \neq 0
\end{cases}
\]

where \( m_{\alpha^{-1}}(x) \) and the function \( \delta(x) \) were defined in Section 2.7, and the linear span \( \mathbb{L}_s \) of \( s^\infty \) is given by

\[
\mathbb{L}_s = \begin{cases} 
\delta(1) + m & \text{if } a = 0, \\
\delta(1+a) + 2m & \text{if } a \neq 0.
\end{cases}
\]

The following theorem gives information on the code \( C_s \).

**Theorem 44.** Let \( q = p = 2 \) and let \( m \geq 4 \). Then the binary code \( C_s \) defined by the sequence of Lemma 43 has parameters \([n, n - \mathbb{L}_s, d]\) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( \mathbb{L}_s \) are given in Lemma 43 and

\[
d = \begin{cases} 
2 & \text{if } a = 0 \text{ and } \delta(1) = 0, \\
4 & \text{if } a = 0 \text{ and } \delta(1) = 1, \\
5 & \text{if } a \neq 0 \text{ and } \delta(1+a) = 0, \\
6 & \text{if } a \neq 0 \text{ and } \delta(1+a) = 1.
\end{cases}
\]

**Remark 45.** When \( a = 0 \) and \( \delta(1) = 1 \), the code is equivalent to the even-weight sub-code of the Hamming code. We are mainly interested in the case that \( a \neq 0 \). When \( a = 1 \), the code \( C_s \) is a double-error correcting binary BCH code or its even-like sub-code. Theorem 44 shows that well-known classes of cyclic codes can be constructed with Dickson polynomials of order 3. The code is either optimal or almost optimal.

Now we consider the case \( q = p' \), where \( p \geq 5 \) or \( p = 2 \) and \( t \geq 2 \).

**Lemma 46.** Let \( q = p' \), where \( p \geq 5 \) or \( p = 2 \) and \( t \geq 2 \). Let \( s^\infty \) be the sequence of (6), where \( f(x) = D_3(x,a) = x^3 - 3ax \). Then the minimal polynomial \( M_s(x) \) of \( s^\infty \) is given by

\[
M_s(x) = \begin{cases} 
(x-1)^{\delta(-2)}m_{\alpha^{-3}}(x)m_{\alpha^{-2}}(x) & \text{if } a = 1, \\
(x-1)^{\delta(1-3a)}m_{\alpha^{-3}}(x)m_{\alpha^{-2}}(x)m_{\alpha^{-1}}(x) & \text{if } a \neq 1
\end{cases}
\]

where \( m_{\alpha^{-1}}(x) \) and the function \( \delta(x) \) were defined in Section 2.7, and the linear span \( \mathbb{L}_s \) of \( s^\infty \) is given by

\[
\mathbb{L}_s = \begin{cases} 
\delta(-2) + 2m & \text{if } a = 1, \\
\delta(1+a) + 3m & \text{if } a \neq 1.
\end{cases}
\]

The following theorem provides information on the code \( C_s \).

**Theorem 47.** Let \( q = p' \), where \( p \geq 5 \) or \( p = 2 \) and \( t \geq 2 \). Then the code \( C_s \) defined by the sequence of Lemma 46 has parameters \([n, n - \mathbb{L}_s, d]\) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( \mathbb{L}_s \) are given in Lemma 46 and

\[
d \geq 3 & \text{ if } a = 1, \\
d \geq 4 & \text{ if } a \neq 1 \text{ and } \delta(1-3a) = 0, \\
d \geq 5 & \text{ if } a \neq 1 \text{ and } \delta(1-3a) = 1, \\
d \geq 5 & \text{ if } a \neq 1 \text{ and } \delta(1-3a) = 0 \text{ and } q = 4, \\
d \geq 6 & \text{ if } a \neq 1 \text{ and } \delta(1-3a) = 1 \text{ and } q = 4.
\]
Remark 48. The code \( C_s \) of Theorem 47 is either a BCH code or the even-like subcode of a BCH code. One can similarly show that the code is either optimal or almost optimal.

When \( q = 4 \), \( a \neq 1 \), \( \delta(1 - 3a) = 1 \), and \( m \geq 3 \), the Sphere Packing Bound shows that \( d = 6 \). But the minimum distance is still open in other cases.

Open Problem 49. Determine the minimum distance \( d \) for the code \( C_s \) of Theorem 47.

4.2.5 Cyclic codes from \( D_4(x, a) = x^4 - 4ax^2 + 2a^2 \)

In this section we deal with the code \( C_s \) defined by the Dickson polynomial \( D_4(x, a) = x^4 - 4ax^2 + 2a^2 \). We have to distinguish among the three cases: \( p = 2 \), \( p = 3 \) and \( p \geq 5 \). The case \( p = 2 \) was covered in Section 4.2.2. So we need to consider only the two remaining cases.

We first take care of the case \( q = p = 3 \) and have the following lemma.

Lemma 50. Let \( q = p = 3 \) and \( m \geq 3 \). Let \( s^\infty \) be the sequence of (6), where \( f(x) = D_4(x, a) = x^4 - 4ax^2 + 2a^2 \). Then the minimal polynomial \( M_s(x) \) of \( s^\infty \) is given by

\[
M_s(x) = \begin{cases}
(x - 1)^{\delta(1)}m_{\alpha^{-4}}(x)m_{\alpha^{-1}}(x) & \text{if } a = 0, \\
(x - 1)^{\delta(1)}m_{\alpha^{-4}}(x)m_{\alpha^{-2}}(x) & \text{if } a = 1, \\
(x - 1)^{\delta(1-a-a^2)}m_{\alpha^{-1}}(x)m_{\alpha^{-2}}(x)m_{\alpha^{-1}}(x) & \text{otherwise},
\end{cases}
\]

where \( m_{\alpha^{-1}}(x) \) and the function \( \delta(x) \) were defined in Section 2.7 and the linear span \( L_s \) of \( s^\infty \) is given by

\[
L_s = \begin{cases}
\delta(1) + 2m & \text{if } a = 0, \\
\delta(1) + 2m & \text{if } a = 1, \\
\delta(1-a-a^2) + 3m & \text{otherwise}.
\end{cases}
\]

The following theorem gives information on the code \( C_s \).

Theorem 51. Let \( q = p = 3 \) and \( m \geq 3 \). Then the code \( C_s \) defined by the sequence of Lemma 50 has parameters \( [n, n - L_s, d] \) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( L_s \) are given in Lemma 50 and

\[
d = \begin{cases}
2 & \text{if } a = 1, \\
3 & \text{if } a = 0 \text{ and } m \equiv 0 \pmod{6}, \\
4 & \text{if } a = 0 \text{ and } m \not\equiv 0 \pmod{6}, \\
5 & \text{if } a^2 \neq a \text{ and } \delta(1-a-a^2) = 0, \\
6 & \text{if } a^2 \neq a \text{ and } \delta(1-a-a^2) = 1.
\end{cases}
\]

Remark 52. When \( a = 1 \), the code of Theorem 51 is neither optimal nor almost optimal. The code is either optimal or almost optimal in all other cases.

Open Problem 53. Determine the minimum distance \( d \) for the code \( C_s \) of Theorem 51.

Now we consider the case \( q = p^t \), where \( p \geq 5 \) or \( p = 3 \) and \( t \geq 2 \).
Lemma 54. Let \( m \geq 2 \) and \( q = p' \), where \( p \geq 5 \) or \( p = 3 \) and \( t \geq 2 \). Let \( s^\infty \) be the sequence of \( \mathcal{H} \), where \( f(x) = D_s(x,a) = x^5 - 4ax^2 + 2a^2 \). Then the minimal polynomial \( \mathcal{M}_s(x) \) of \( s^\infty \) is given by

\[
\mathcal{M}_s(x) = \begin{cases} 
(x - 1)^{\delta(1)}m_{\alpha^5}(x)m_{\alpha^3}(x) & \text{if } a = \frac{3}{2}, \\
(x - 1)^{\delta(1)}m_{\alpha^5}(x)m_{\alpha^3}(x) & \text{if } a = \frac{1}{2}, \\
(x - 1)^{\delta(1)-4a+2a^2}\prod_{i=1}^4 m_{\alpha^i}(x) & \text{if } a \notin \{\frac{3}{2}, \frac{1}{2}\},
\end{cases}
\]

where \( m_{\alpha^i}(x) \) and the function \( \delta(x) \) were defined in Section 2.7 and the linear span \( \mathcal{L}_s \) of \( s^\infty \) is given by

\[
\mathcal{L}_s = \begin{cases} 
\delta(1) + 3m & \text{if } a \in \{\frac{3}{2}, \frac{1}{2}\}, \\
\delta(1) - 4a + 2a^2 + 4m & \text{otherwise}.
\end{cases}
\]

The following theorem delivers to us information on the code \( C_s \).

Theorem 55. Let \( m \geq 2 \) and \( q = p' \), where \( p \geq 5 \) or \( p = 3 \) and \( t \geq 2 \). Then the code \( C_s \) defined by the sequence of Lemma 54 has parameters \([n,n-\mathcal{L}_s,d]\) and generator polynomial \( \mathcal{M}_s(x) \), where \( \mathcal{M}_s(x) \) and \( \mathcal{L}_s \) are given in Lemma 54 and

\[
d \geq 3 \quad \text{if } a = \frac{3}{2}, \quad d \geq 4 \quad \text{if } a = \frac{1}{2}, \\
d \geq 5 \quad \text{if } a \notin \{\frac{3}{2}, \frac{1}{2}\} \text{ and } \delta(1 - 4a + a^2) = 0, \\
d = 6 \quad \text{if } a \notin \{\frac{3}{2}, \frac{1}{2}\} \text{ and } \delta(1 - 4a + a^2) = 1.
\]

Remark 56. Except the cases that \( a \in \{\frac{3}{2}, \frac{1}{2}\} \), the code \( C_s \) of Theorem 55 is either optimal or almost optimal.

Open Problem 57. Determine the minimum distance \( d \) for the code \( C_s \) of Theorem 55

4.2.6 Cyclic codes from \( D_s(x,a) = x^5 - 5ax^3 + 5a^2x \)

In this section we deal with the code \( C_s \) defined by the Dickson polynomial \( D_s(x,a) = x^5 - 5ax^3 + 5a^2x \). We have to distinguish among the three cases: \( p = 2 \), \( p = 3 \) and \( p \geq 7 \). The case \( p = 5 \) was covered in Section 4.2.2. So we need to consider only the remaining cases.

We first consider the case \( q = p = 2 \) and have the following lemma.

Lemma 58. Let \( q = p = 2 \) and \( m \geq 5 \). Let \( s^\infty \) be the sequence of \( \mathcal{H} \), where \( f(x) = D_s(x,a) = x^5 - 5ax^3 + 5a^2x \). Then the minimal polynomial \( \mathcal{M}_s(x) \) of \( s^\infty \) is given by

\[
\mathcal{M}_s(x) = \begin{cases} 
(x - 1)^{\delta(1)}m_{\alpha^5}(x) & \text{if } a = 0, \\
(x - 1)^{\delta(1)}m_{\alpha^5}(x)m_{\alpha^3}(x) & \text{if } 1 + a + a^3 = 0, \\
(x - 1)^{\delta(1)}\prod_{i=0}^4 m_{\alpha^i}(x) & \text{if } a + a^2 + a^4 \neq 0
\end{cases}
\]

where \( m_{\alpha^i}(x) \) and the function \( \delta(x) \) were defined in Section 2.7 and the linear span \( \mathcal{L}_s \) of \( s^\infty \) is given by

\[
\mathcal{L}_s = \begin{cases} 
\delta(1) + m & \text{if } a = 0, \\
\delta(1) + 2m & \text{if } 1 + a + a^3 = 0, \\
\delta(1) + 3m & \text{if } a + a^2 + a^4 \neq 0.
\end{cases}
\]
The following theorem describes parameters of the code $C_s$.

**Theorem 59.** Let $q = p = 2$ and $m \geq 5$. Then the code $C_s$ defined by the sequence of Lemma 58 has parameters $[n, n - \mathbb{L}_s, d]$ and generator polynomial $\mathbb{M}_s(x)$, where $\mathbb{M}_s(x)$ and $\mathbb{L}_s$ are given in Lemma 58 and

\[
\begin{align*}
  d &= 2 & \text{if } a = 0 \text{ and } \delta(1) = 0 \text{ and } \gcd(5, n) = 5, \\
  d &= 3 & \text{if } a = 0 \text{ and } \delta(1) = 0 \text{ and } \gcd(5, n) = 1, \\
  d &= 4 & \text{if } a = 0 \text{ and } \delta(1) = 1, \\
  d &\geq 3 & \text{if } 1 + a + a^3 = 0 \text{ and } \delta(1) = 0, \\
  d &\geq 4 & \text{if } 1 + a + a^3 = 0 \text{ and } \delta(1) = 1, \\
  d &\geq 7 & \text{if } a + a^2 + a^4 \neq 0 \text{ and } \delta(1) = 0, \\
  d &= 8 & \text{if } a + a^2 + a^4 \neq 0 \text{ and } \delta(1) = 1.
\end{align*}
\]

**Remark 60.** The code of Theorem 59 is either optimal or almost optimal. The code is not a BCH code when $1 + a + a^3 = 0$, and a BCH code in the remaining cases.

**Open Problem 61.** Determine the minimum distance $d$ for the code $C_s$ of Theorem 59 for the three open cases.

We now consider the case $(p, q) = (2, 4)$ and have the following lemma.

**Lemma 62.** Let $(p, q) = (2, 4)$ and $m \geq 5$. Let $s^\omega$ be the sequence of (6), where $f(x) = D_s(x, a) = x^5 - 5ax^3 + 5a^2x$. Then the minimal polynomial $\mathbb{M}_s(x)$ of $s^\omega$ is given by

\[
\mathbb{M}_s(x) = \begin{cases} 
(x - 1)^{\delta(1)} m_{\alpha^{-1}}(x) & \text{if } a = 0, \\
(x - 1)^{\delta(1)} m_{\alpha^{-3}}(x)m_{\alpha^{-2}}(x) & \text{if } a = 1, \\
(x - 1)^{\delta(1) + a + a^2} m_{\alpha^{-3}}(x)m_{\alpha^{-2}}(x)m_{\alpha^{-1}}(x) & \text{if } a + a^2 \neq 0
\end{cases}
\]

where $m_{\alpha^{-i}}(x)$ and the function $\delta(x)$ were defined in Section 2.7 and the linear span $\mathbb{L}_s$ of $s^\omega$ is given by

\[
\mathbb{L}_s = \begin{cases} 
\delta(1) + m & \text{if } a = 0, \\
\delta(1) + 3m & \text{if } a = 1, \\
\delta(1) + 4m & \text{if } a + a^2 \neq 0.
\end{cases}
\]

The following theorem supplies information on the code $C_s$.

**Theorem 63.** Let $(p, q) = (2, 4)$ and $m \geq 5$. Then the code $C_s$ defined by the sequence of Lemma 62 has parameters $[n, n - \mathbb{L}_s, d]$ and generator polynomial $\mathbb{M}_s(x)$, where $\mathbb{M}_s(x)$ and $\mathbb{L}_s$ are given in Lemma 62 and

\[
\begin{align*}
  d &= 2 & \text{if } a = 0 \text{ and } \delta(1) = 0 \text{ and } \gcd(5, n) = 5, \\
  d &= 3 & \text{if } a = 0 \text{ and } \gcd(5, n) = 1, \\
  d &\geq 3 & \text{if } a = 1, \\
  d &\geq 6 & \text{if } a + a^2 \neq 0 \text{ and } \delta(1) = 0, \\
  d &\geq 7 & \text{if } a + a^2 \neq 0 \text{ and } \delta(1) = 1.
\end{align*}
\]

Examples of the code of Theorem 63 are documented in [arXiv:1206.4370](http://arxiv.org/abs/1206.4370) and many of them are optimal.
Open Problem 64. Determine the minimum distance $d$ of the code $C_s$ in Theorem 63.

We now consider the case $(p, q) = (2, 2^t)$, where $t \geq 3$, and state the following lemma.

Lemma 65. Let $(p, q) = (2, 2^t)$ and $m \geq 3$, where $t \geq 3$. Let $s^w$ be the sequence of (6), where $f(x) = D_5(x, a) = x^5 - 5ax + 5a^2x$. Then the minimal polynomial $M_s(x)$ of $s^w$ is given by

$$M_s(x) = \begin{cases} (x - 1)^{\delta(1)} m_{\alpha^{-5}}(x) m_{\alpha^{-1}}(x) & \text{if } a = 0, \\ \prod_{i=2}^{s} m_{\alpha^{-i}}(x) & \text{if } 1 + a + a^2 = 0, \\ (x - 1)^{\delta(1+a+a^2)} \prod_{i=1}^{5} m_{\alpha^{-i}}(x) & \text{if } a + a^2 + a^3 \neq 0, \end{cases}$$

where $m_{\alpha^{-i}}(x)$ and the function $\delta(x)$ were defined in Section 2.7 and the linear span $L_s$ of $s^w$ is given by

$$L_s = \begin{cases} \delta(1) + 3m & \text{if } a = 0, \\ \delta(1) + 4m & \text{if } 1 + a + a^2 = 0, \\ \delta(1) + 5m & \text{if } a + a^2 + a^3 \neq 0. \end{cases}$$

The following theorem provides information on the code $C_s$.

Theorem 66. Let $(p, q) = (2, 2^t)$, where $t \geq 3$. Then the code $C_s$ defined by the sequence of Lemma 65 has parameters $[n, n - L_s, d]$ and generator polynomial $M_s(x)$, where $M_s(x)$ and $L_s$ are given in Lemma 65 and

$$d \geq 3 \quad \text{if } a = 0 \text{ and } \delta(1) = 0, \\
\text{if } a = 0 \text{ and } \delta(1) = 1, \\
\text{if } a = 0 \text{ and } \delta(1) = 0, \\
\text{if } a = 0 \text{ and } \delta(1) = 1, \\
\text{if } a + a^2 + a^3 \neq 0 \text{ and } \delta(1) = 0, \\
\text{if } a + a^2 + a^3 \neq 0 \text{ and } \delta(1) = 1.$$

Open Problem 67. Determine the minimum distance $d$ of the code $C_s$ in Theorem 66.

Examples of the code of Theorem 66 can be found in [arXiv:1206.4370] and many of them are optimal. The code of Theorem 66 is not a BCH code when $a = 0$, and a BCH code otherwise.

We now consider the case $q = p = 3$ and state the following lemma and theorem.

Lemma 68. Let $q = p = 3$ and $m \geq 3$. Let $s^w$ be the sequence of (6), where $f(x) = D_5(x, a) = x^5 - 5ax + 5a^2x$. Then the minimal polynomial $M_s(x)$ of $s^w$ is given by

$$M_s(x) = \begin{cases} (x - 1)^{\delta(1+a+2a^2)} m_{\alpha^{-5}}(x) m_{\alpha^{-1}}(x) m_{\alpha^{-2}}(x) & \text{if } a - a^6 = 0, \\ (x - 1)^{\delta(1+a+2a^2)} \prod_{i=2}^{5} m_{\alpha^{-i}}(x) & \text{if } a - a^6 \neq 0, \end{cases}$$

where $m_{\alpha^{-i}}(x)$ and the function $\delta(x)$ were defined in Section 2.7 and the linear span $L_s$ of $s^w$ is given by

$$L_s = \begin{cases} \delta(1 + a + 2a^2) + 3m & \text{if } a - a^6 = 0, \\ \delta(1 + a + 2a^2) + 4m & \text{if } a - a^6 \neq 0. \end{cases}$$
The following theorem gives information on the code $C_s$.

**Theorem 69.** Let $q = p = 3$ and $m \geq 3$. Then the code $C_s$ defined by the sequence of Lemma 68 has parameters $[n, n - \mathbb{L}_s, d]$ and generator polynomial $\mathbb{M}_s(x)$, where $\mathbb{M}_s(x)$ and $\mathbb{L}_s$ are given in Lemma 68 and

\[
\begin{align*}
    d &\geq 4 \quad \text{if } a - a^6 = 0, \\
    d &\geq 7 \quad \text{if } a - a^6 \neq 0 \text{ and } \delta(1 + a + 2a^2) = 0, \\
    d &\geq 8 \quad \text{if } a - a^6 \neq 0 \text{ and } \delta(1 + a + 2a^2) = 1.
\end{align*}
\]

**Open Problem 70.** Determine the minimum distance $d$ of the code $C_s$ in Theorem 69 (our experimental data indicates that the lower bounds are the specific values of $d$).

Examples of the code of Theorem 69 are described in [arXiv:1206.4370](https://arxiv.org/abs/1206.4370) and some of them are optimal.

We now consider the case $(p, q) = (3, 3')$, where $t \geq 3$, and state the following lemma and theorem.

**Lemma 71.** Let $(p, q) = (3, 3')$ and $m \geq 2$, where $t \geq 2$. Let $s^\infty$ be the sequence of (6), where $f(x) = D_5(x, a) = a^5 - 5ax^3 + 5a^2x$. Then the minimal polynomial $\mathbb{M}_s(x)$ of $s^\infty$ is given by

\[
\mathbb{M}_s(x) = \begin{cases} 
(x - 1)^{\delta(1)} m_{\alpha - 5}(x)m_{\alpha - 4}(x)m_{\alpha - 3}(x) & \text{if } 1 + a = 0, \\
(x - 1)^{\delta(a - 1)} m_{\alpha - 5}(x)m_{\alpha - 4}(x)m_{\alpha - 3}(x)m_{\alpha - 2}(x) & \text{if } 1 + a^2 = 0, \\
(x - 1)^{\delta(1 + a + 2a^2)} \prod_{i=1}^{5} m_{\alpha - i}(x) & \text{if } (a + 1)(a^2 + 1) \neq 0,
\end{cases}
\]

where $m_{\alpha - i}(x)$ and the function $\delta(x)$ were defined in Section 2.7 and the linear span $\mathbb{L}_s$ of $s^\infty$ is given by

\[
\mathbb{L}_s = \begin{cases} 
\delta(1) + 4m & \text{if } a + 1 = 0, \\
\delta(a - 1) + 4m & \text{if } a^2 + 1 = 0, \\
\delta(1 + a + 2a^2) + 5m & \text{if } (a + 1)(a^2 + 1) \neq 0.
\end{cases}
\]

The following theorem supplies information on the code $C_s$.

**Theorem 72.** Let $(p, q) = (3, 3')$ and $m \geq 2$, where $t \geq 2$. Then the code $C_s$ defined by the sequence of Lemma 68 has parameters $[n, n - \mathbb{L}_s, d]$ and generator polynomial $\mathbb{M}_s(x)$, where $\mathbb{M}_s(x)$ and $\mathbb{L}_s$ are given in Lemma 71 and

\[
\begin{align*}
    d &\geq 3 \quad \text{if } a = -1 \text{ and } \delta(1) = 0, \\
    d &\geq 4 \quad \text{if } a = -1 \text{ and } \delta(1) = 1, \\
    d &\geq 5 \quad \text{if } a^2 = -1 \text{ and } \delta(a - 1) = 0, \\
    d &\geq 6 \quad \text{if } a^2 = -1 \text{ and } \delta(a - 1) = 1, \\
    d &\geq 6 \quad \text{if } (a + 1)(a^2 + 1) \neq 0 \text{ and } \delta(1 + a + 2a^2) = 0, \\
    d &\geq 7 \quad \text{if } (a + 1)(a^2 + 1) \neq 0 \text{ and } \delta(1 + a + 2a^2) = 1.
\end{align*}
\]

**Open Problem 73.** Determine the minimum distance $d$ of the code $C_s$ in Theorem 72.

Examples of the code of Theorem 72 are available in [arXiv:1206.4370](https://arxiv.org/abs/1206.4370) and some of them are optimal. The code is a BCH code, except in the case that $a = -1$.

We finally consider the case $p \geq 7$, and present the following lemma and theorem.
Lemma 74. Let \( p \geq 7 \) and \( m \geq 2 \). Let \( s^m \) be the sequence of \( \{a\} \), where \( f(x) = D_5(x, a) = x^5 - 5ax^3 + 5a^2x \). Then the minimal polynomial \( M_s(x) \) of \( s^m \) is given by

\[
M_s(x) = \begin{cases} 
(x - 1)^{\delta(1 - 5a + 5a^2)} m_{\alpha^{-5}}(x)m_{\alpha^{-4}}(x)m_{\alpha^{-3}}(x) & \text{if } a = 2, \\
(x - 1)^{\delta(1 - 5a + 5a^2)} m_{\alpha^{-5}}(x)m_{\alpha^{-4}}(x)m_{\alpha^{-3}}(x) & \text{if } a = \frac{2}{3}, \\
(x - 1)^{\delta(1 - 5a + 5a^2)} m_{\alpha^{-5}}(x)m_{\alpha^{-4}}(x)m_{\alpha^{-3}}(x) & \text{if } a^2 - 3a + 1 = 0, \\
(x - 1)^{\delta(1 - 5a + 5a^2)} \prod_{i=1}^{5} m_{\alpha^{-i}}(x) & \text{if } (a^2 - 3a + 1)(a - 2)(3a - 2) \neq 0,
\end{cases}
\]

where \( m_{\alpha^{-i}}(x) \) and the function \( \delta(x) \) were defined in Section 2.7 and the linear span \( L_s \) of \( s^m \) is given by

\[
L_s = \begin{cases} 
\delta(1 - 5a + 5a^2) + 4m, & \text{if } (a^2 - 3a + 1)(a - 2)(3a - 2) = 0, \\
\delta(1 - 5a + 5a^2) + 5m, & \text{otherwise}.
\end{cases}
\]

The following theorem provides information on the code \( C_s \).

Theorem 75. Let \( p \geq 7 \) and \( m \geq 2 \). Then the code \( C_s \) defined by the sequence of Lemma 74 has parameters \([n, n - L_s, d]\) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( L_s \) are given in Lemma 74 and

\[
\begin{align*}
\text{if } a & = 2 \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
\text{if } a & = 2 \text{ and } \delta(1 - 5a + 5a^2) = 1, \\
\text{if } a & = \frac{2}{3} \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
\text{if } a & = \frac{2}{3} \text{ and } \delta(1 - 5a + 5a^2) = 1, \\
\text{if } 1 - 3a + a^2 = 0 \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
\text{if } 1 - 3a + a^2 = 0 \text{ and } \delta(1 - 5a + 5a^2) = 1, \\
\text{if } (a^2 - 3a + 1)(a - 2)(3a - 2) \neq 0 \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
\text{if } (a^2 - 3a + 1)(a - 2)(3a - 2) \neq 0 \text{ and } \delta(1 - 5a + 5a^2) = 1.
\end{align*}
\]

Open Problem 76. Determine the minimum distance \( d \) of the code \( C_s \) in Theorem 75.

Examples of the code of Theorem 75 can be found in [arXiv:1206.4370] and some of them are optimal. The code is a BCH code, except in the cases \( a \in \{2, 2/3\} \).

4.2.7 Cyclic codes from other \( D_i(x, a) \) for \( i \geq 6 \)

Parameters of cyclic codes from \( D_i(x, a) \) for small \( i \) could be established in a very similar way. However, more cases are involved and the situation is getting more complicated when \( i \) gets bigger. Examples of the code \( C_s \) from \( D_7(x, a) \) and \( D_{11}(x, a) \) can be found in [arXiv:1206.4370].

4.2.8 Cyclic codes from Dickson polynomials of the second kind

Results on cyclic codes from Dickson polynomials of the second kind can be developed in a similar way. Experimental data indicates that the codes from the Dickson polynomials of the first kind are in general better than those from the Dickson polynomials of the second kind, though some cyclic codes from Dickson polynomials of the second kind could also be optimal or almost optimal.
4.2.9 Comments on the cyclic codes from Dickson polynomials

It is really amazing that in most cases the cyclic codes derived from the Dickson polynomials of small degrees within the framework of this paper are optimal or almost optimal (see arXiv:1206.4370 for examples of optimal codes).

We had to treat Dickson polynomials of small degrees case by case over finite fields with different characteristics as we do not see a way of treating them in a single strike. The generator polynomial and the dimension of the codes depend heavily on the degree of the Dickson polynomials and the characteristic of the base field.

5 Cyclic codes from the two-prime sequences

The objective of this section is to document a family of codes from the two-prime sequences. All the materials of this section come from [10].

5.1 The two-prime sequences

Throughout this section, let $n_1$ and $n_2$ be two distinct odd primes, define $n = n_1n_2$ and

$$N_1 = \{n_1, 2n_1, \cdots, (n_2 - 1)n_1\}, \quad N_2 = \{n_2, 2n_2, \cdots, (n_1 - 1)n_2\}.$$ 

The two-prime sequence, denoted by $\lambda^{\infty}$ throughout this section exclusively, is defined by

$$\lambda_i = \begin{cases} 
0, & \text{if } i \mod n \in \{0\} \cup N_2, \\
1, & \text{if } i \mod n \in N_1, \\
\left(1 - \left(\frac{i}{n_1}\right)\left(\frac{i}{n_2}\right)\right)/2, & \text{otherwise},
\end{cases}$$

(30)

where $\left(\frac{a}{n_1}\right)$ denotes the Legendre symbol. The autocorrelation values of this binary sequence $\lambda^{\infty}$ were determined in [8, 28]. Traditionally, these two-prime sequences are defined as binary sequences and a generalization of the binary twin-prime sequences [35]. Here in this section, we treat them as sequences over any finite field $\mathbb{GF}(q)$, where $\gcd(q, n) = 1$, and will use them to construct cyclic codes of length $n$ over $\mathbb{GF}(q)$.

We now give a cyclotomic description of the sequence $\lambda^{\infty}$. An integer $a$ is called a primitive root of (or modulo) $n$ if the multiplicative order of $a$ modulo $n$, denoted by $\text{ord}_n(a)$, is equal to $\phi(n)$, where $\phi(x)$ is the Euler function and $\gcd(a, n) = 1$.

Define $N = \gcd(n_1 - 1, n_2 - 1)$ and $e = (n_1 - 1)(n_2 - 1)/N$. It is well-known that any prime $n_1$ has $\phi(n_1 - 1)$ primitive roots. The Chinese Remainder Theorem guarantees that there are common primitive roots of both $n_1$ and $n_2$. Let $\pi$ be a fixed common primitive root of both $n_1$ and $n_2$, and $\rho$ be an integer satisfying

$$\rho \equiv \pi \pmod{n_1}, \quad \rho \equiv 1 \pmod{n_2}.$$ 

Whiteman proved that [35]

$$\mathbb{Z}_n^* = \{\pi^s\rho^i : s = 0, 1, \cdots, e - 1; \ i = 0, 1, \cdots, N - 1\},$$
where $\mathbb{Z}_n^*$ denotes the set of all invertible elements of the residue class ring $\mathbb{Z}_n$. The generalized cyclotomic classes $W_i$ of order $N$ with respect to $n_1$ and $n_2$ are defined by

$$W_i^{(N)} = \{\pi^s \rho^j : s = 0, 1, \ldots, e-1\}, \quad i = 0, 1, \ldots, N-1.$$ 

It was proved in [35] that

$$\mathbb{Z}_n^* = \bigcup_{i=0}^{N-1} W_i^{(N)}, \quad W_i^{(N)} \cap W_j^{(N)} = \emptyset \text{ for } i \neq j.$$ 

This generalized cyclotomy was introduced by Whiteman [35]. The motivation behind the investigation of the generalized cyclotomy with respect to two primes is the search for residue difference sets. The famous twin-prime difference sets are among such a class of difference sets.

Define

$$D_0^{(2)} = \bigcup_{i=0}^{(N-2)/2} W_i^{(N)}, \quad D_1^{(2)} = \bigcup_{i=0}^{(N-2)/2} W_i^{(N)}.$$ 

Clearly $D_0^{(2)}$ is a subgroup of $\mathbb{Z}_n^*$ and $D_1^{(2)} = \rho D_0^{(2)}$. The sets $D_0^{(2)}$ and $D_1^{(2)}$ are called the extended generalized cyclotomic classes of order two and are identical to Whiteman’s cyclotomic classes of order $N$ when and only when $N = 2$. In other words, the cyclotomy $\{D_0^{(2)}, D_1^{(2)}\}$ is different from Whiteman’s cyclotomy when $N > 2$.

Let

$$C_0 = \{0\} \cup N_2 \cup D_0^{(2)}, \quad C_1 = N_1 \cup D_1^{(2)}.$$ 

Then

$$C_0 \cup C_1 = \mathbb{Z}_n, \quad C_0 \cap C_1 = \emptyset.$$ 

It is not hard to verify that

$$\lambda_i = \begin{cases} 0, & \text{if } i \mod n \in C_0 \\ 1, & \text{if } i \mod n \in C_1 \end{cases}.$$ 

This is the cyclotomic description of the two-prime sequence $\lambda^\infty$ defined in [30].

In certain special cases, the two-prime sequence has optimal autocorrelation [9]. As will be seen later, the code $C_\lambda$ has also very good parameters in special cases.

### 5.2 The codes $C_\lambda$ defined by the two-prime sequences $\lambda^\infty$

After the preparations above, we are ready to present the linear span and minimal polynomial of the sequence $\lambda^\infty$ defined in [30]. To this end, we need to discuss the factorization of $x^n - 1$ over $\text{GF}(q)$.

Let $\theta$ be the same as before. Among the $n$th roots of unity $\theta^i$, where $0 \leq i \leq n-1$, the $n_2$ elements $\theta^i, i \in N_1 \cup \{0\}$, are $n_2$th roots of unity, the $n_1$ elements $\theta^i, i \in N_2 \cup \{0\}$, are $n_1$th roots of unity. Hence,

$$x^{n_1} - 1 = \prod_{i \in N_2 \cup \{0\}} (x - \theta^i), \quad x^{n_2} - 1 = \prod_{i \in N_1 \cup \{0\}} (x - \theta^i).$$

We define

$$d_i(x) = \prod_{i \in D_i^{(2)}} (x - \theta^i).$$
for \(i \in \{0,1\}\). If \(q \in D_0^{(2)}\), it is easily proved that \(d_i(x) \in \text{GF}(q)[x]\) for all \(i\).

Let \(d(x) = d_0(x) d_1(x) \in \text{GF}(q)[x]\). We have then

\[
x^n - 1 = \prod_{i=0}^{n-1} (x - \theta^i) = \frac{(x^{n_1} - 1)(x^{n_2} - 1)}{x-1} d(x).
\]

**Theorem 77.** Define for \(i \in \{1,2\}\)

\[
\Delta_i = \frac{n_i + (-1)^{i-1}}{2} \mod p, \, \Delta = \frac{(n_1 + 1)(n_2 - 1)}{2} \mod p.
\]

1. When \(n \equiv 3 \pmod{4}\) and \(\frac{n+1}{2} \mod p \neq 0\) or \(n \equiv 1 \pmod{4}\) and \(\frac{n-1}{2} \mod p \neq 0\), the minimal polynomial of the sequence \(\lambda^\infty\) is given by

\[
\mathcal{M}_{\lambda}(x) = \begin{cases} 
  x^n - 1, & \text{if } \Delta_1 \neq 0, \Delta_2 \neq 0, \Delta \neq 0 \\
  \frac{x^n - 1}{x^{n_1} - 1}, & \text{if } \Delta_1 \neq 0, \Delta_2 \neq 0, \Delta = 0 \\
  \frac{x^n - 1}{x^{n_2} - 1}, & \text{if } \Delta_1 = 0, \Delta_2 \neq 0, \Delta = 0 \\
  \frac{x^n - 1}{(x^{n_1} - 1)(x^{n_2} - 1)}, & \text{if } \Delta_1 = 0, \Delta_2 = 0 \\
\end{cases}
\]

(31)

The linear span of the sequence \(\lambda^\infty\) is equal to \(\deg(\mathcal{M}_{\lambda}(x))\). In this case, the cyclic code \(C_\lambda\) over \(\text{GF}(q)\) defined by the sequence \(\lambda^\infty\) has parameters \([n,k,d]\) and generator polynomial \(\mathcal{M}_{\lambda}(x)\) of (31), where the dimension \(k = n - \deg(\mathcal{M}_{\lambda}(x))\).

2. When \(n \equiv 3 \pmod{4}\) and \(\frac{n+1}{2} \mod p = 0\) or \(n \equiv 1 \pmod{4}\) and \(\frac{n-1}{2} \mod p = 0\), and \(d_i(x) \in \text{GF}(q)[x]\) for each \(i\). In this case, \(\Lambda(\theta) \in \{0,1\}\) and the minimal polynomial of the sequence \(\lambda^\infty\) is given by

\[
\mathcal{M}_{\lambda}(x) = \begin{cases} 
  \frac{x^n - 1}{d_0(x)}, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \Delta = 0, \Lambda(\theta) = 0 \\
  \frac{x^{n_1} - 1}{d_1(x)}, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \Delta = 0, \Lambda(\theta) = -1 \\
  \frac{x^{n_2} - 1}{(x^{n_1} - 1)d_0(x)}, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \Delta = 0, \Lambda(\theta) = 0 \\
  \frac{x^{n_2} - 1}{(x^{n_1} - 1)d_1(x)}, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \Delta = 0, \Lambda(\theta) = -1 \\
  \frac{x^n - 1}{(x^{n_1} - 1)(x^{n_2} - 1)}, & \text{if } \Delta_1 = 0, \Delta_2 = 0, \Lambda(\theta) = 0 \\
  \frac{x^{n_1} - 1}{(x^{n_2} - 1)d_0(x)}, & \text{if } \Delta_1 = 0, \Delta_2 = 0, \Lambda(\theta) = -1 \\
  \frac{x^{n_2} - 1}{(x^{n_1} - 1)d_1(x)}, & \text{if } \Delta_1 = 0, \Delta_2 = 0, \Lambda(\theta) = 0 \\
  \frac{x^n - 1}{(x^{n_1} - 1)(x^{n_2} - 1)}, & \text{if } \Delta_1 = 0, \Delta_2 = 0, \Lambda(\theta) = -1.
\end{cases}
\]

(32)

The linear span of the sequence \(\lambda^\infty\) is equal to \(\deg(\mathcal{M}_{\lambda}(x))\). In this case, the cyclic code \(C_\lambda\) over \(\text{GF}(q)\) defined by the sequence \(\lambda^\infty\) has parameters \([n,k,d]\) and generator polynomial \(\mathcal{M}_{\lambda}(x)\) of (32), where the dimension \(k = n - \deg(\mathcal{M}_{\lambda}(x))\).

### 5.2.1 The case \(q = 2\)

The following corollary follows directly from Theorem [77], and its conclusions on the linear span of the sequence \(\lambda^\infty\) is an extension of the work in [8].
Corollary 78. Let $q = 2$. We have the following conclusions:

1. If $n_1 \equiv 1 \pmod{8}$ and $n_2 \equiv 3 \pmod{8}$ or $n_1 \equiv -3 \pmod{8}$ and $n_2 \equiv -1 \pmod{8}$, we have

$$\mathbb{L}_\lambda = n - 1, \quad \mathbb{M}_\lambda(x) = \frac{x^n - 1}{x - 1}.$$ 

In this case, the cyclic code $C_\lambda$ over $\text{GF}(q)$ defined by the sequence $\lambda^\infty$ has generator polynomial $\mathbb{M}_\lambda(x)$ above and parameters $[n, 1, n - 1]$.

2. If $n_1 \equiv -1 \pmod{8}$ and $n_2 \equiv 3 \pmod{8}$ or $n_1 \equiv 3 \pmod{8}$ and $n_2 \equiv -1 \pmod{8}$, we have

$$\mathbb{L}_\lambda = n - n_2, \quad \mathbb{M}_\lambda(x) = \frac{x^n - 1}{x^{n_2} - 1}.$$ 

In this case, the cyclic code $C_\lambda$ over $\text{GF}(q)$ defined by the sequence $\lambda^\infty$ has generator polynomial $\mathbb{M}_\lambda(x)$ above and parameters $[n, n_2, n_1]$, where the minimum weight follows from Theorem 89.

3. If $n_1 \equiv -3 \pmod{8}$ and $n_2 \equiv 1 \pmod{8}$ or $n_1 \equiv 1 \pmod{8}$ and $n_2 \equiv -3 \pmod{8}$, we have

$$\mathbb{L}_\lambda = n - n_1, \quad \mathbb{M}_\lambda(x) = \frac{x^n - 1}{x^{n_1} - 1}.$$ 

In this case, the cyclic code $C_\lambda$ over $\text{GF}(q)$ defined by the sequence $\lambda^\infty$ has generator polynomial $\mathbb{M}_\lambda(x)$ above and parameters $[n, n_1, n_2]$, where the minimum weight follows from Theorem 89.

4. If $n_1 \equiv -1 \pmod{8}$ and $n_2 \equiv -3 \pmod{8}$ or $n_1 \equiv 3 \pmod{8}$ and $n_2 \equiv 1 \pmod{8}$, we have

$$\mathbb{L}_\lambda = n - (n_1 + n_2 - 1), \quad \mathbb{M}_\lambda(x) = \frac{(x^n - 1)(x - 1)}{(x^{n_2} - 1)(x^{n_1} - 1)}.$$ 

In this case, the cyclic code $C_\lambda$ over $\text{GF}(q)$ defined by the sequence $\lambda^\infty$ has generator polynomial $\mathbb{M}_\lambda(x)$ above and parameters $[n, n_1 + n_2 - 1, d]$, where $d$ has the lower bound of Theorem 97.

5. If $n_1 \equiv 1 \pmod{8}$ and $n_2 \equiv -1 \pmod{8}$ or $n_1 \equiv -3 \pmod{8}$ and $n_2 \equiv 3 \pmod{8}$, we have

$$\mathbb{L}_\lambda = n - \frac{(n_1 - 1)(n_2 - 1) + 2}{2}, \quad \mathbb{M}_\lambda(x) = \begin{cases} \frac{x^n - 1}{(x - 1)^{d_0(x)}}, & \text{if } \Lambda(\theta) = 0 \\ \frac{x^n - 1}{x^{d_1(x)}}, & \text{if } \Lambda(\theta) = 1. \end{cases}$$ 

In this case, the cyclic code $C_\lambda$ over $\text{GF}(q)$ defined by the sequence $\lambda^\infty$ has generator polynomial $\mathbb{M}_\lambda(x)$ above and parameters $[n, k, d]$, where $k = \frac{(n_1 - 1)(n_2 - 1) + 2}{2}$. 
6. If \( n_1 \equiv -1 \pmod{8} \) and \( n_2 \equiv 1 \pmod{8} \) or \( n_1 \equiv 3 \pmod{8} \) and \( n_2 \equiv -3 \pmod{8} \), we have

\[
\mathbb{L}_\lambda = n - \frac{(n_1 + 1)(n_2 + 1) - 2}{2}
\]

and

\[
\mathcal{M}_\lambda(x) = \begin{cases} 
\frac{(x^{e-1})(x-1)}{(x^{e-1})(x-1)} \text{ if } \Lambda(\theta) = 0 \\
\frac{(x^{e-1}-1)(x^{e-1}-d_1(x))}{(x^{e-1}-1)(x^{e-1}-d_0(x))} \text{ if } \Lambda(\theta) = 1.
\end{cases}
\]

In this case, the cyclic code \( C_{\lambda} \) over \( GF(q) \) defined by the sequence \( \lambda^\infty \) has generator polynomial \( \mathcal{M}_\lambda(x) \) above and parameters \( [n,k,d] \), where \( k = \frac{(n_1+1)(n_2+1)-2}{2} \) and \( d \) has the lower bound of Theorem 93.

7. If \( n_1 \equiv -1 \pmod{8} \) and \( n_2 \equiv -1 \pmod{8} \) or \( n_1 \equiv 3 \pmod{8} \) and \( n_2 \equiv 3 \pmod{8} \), we have

\[
\mathbb{L}_\lambda = n - \frac{(n_1 + 1)(n_2 - 1) + 2}{2}
\]

\[
\mathcal{M}_\lambda(x) = \begin{cases} 
\frac{x^{e-1}}{x^{e-1}} \text{ if } \Lambda(\theta) = 0 \\
\frac{(x^{e-1}-1)(x^{e-1}-d_0(x))}{(x^{e-1}-1)(x^{e-1}-d_0(x))} \text{ if } \Lambda(\theta) = 1.
\end{cases}
\]

In this case, the cyclic code \( C_{\lambda} \) over \( GF(q) \) defined by the sequence \( \lambda^\infty \) has generator polynomial \( \mathcal{M}_\lambda(x) \) above and parameters \( [n,k,d] \), where \( k = \frac{(n_1+1)(n_2-1)+2}{2} \) and \( d \) has the lower bound of Theorem 95.

8. If \( n_1 \equiv 1 \pmod{8} \) and \( n_2 \equiv 1 \pmod{8} \) or \( n_1 \equiv -3 \pmod{8} \) and \( n_2 \equiv -3 \pmod{8} \), we have

\[
\mathbb{L}_\lambda = n - \frac{(n_1 - 1)(n_2 + 1) + 2}{2}
\]

\[
\mathcal{M}_\lambda(x) = \begin{cases} 
\frac{x^{e-1}}{x^{e-1}} \text{ if } \Lambda(\theta) = 0 \\
\frac{(x^{e-1}-1)(x^{e-1}-d_0(x))}{(x^{e-1}-1)(x^{e-1}-d_0(x))} \text{ if } \Lambda(\theta) = 1.
\end{cases}
\]

In this case, the cyclic code \( C_{\lambda} \) over \( GF(q) \) defined by the sequence \( \lambda^\infty \) has generator polynomial \( \mathcal{M}_\lambda(x) \) above and parameters \( [n,k,d] \), where \( k = \frac{(n_1-1)(n_2+1)+2}{2} \) and \( d \) has the lower bound of Theorem 95.

**Example 79.** Let \( (p,m,n_1,n_2) = (2,1,3,5) \). Then \( q = 2, n = 15 \), and \( C_{\lambda} \) is a \([15,11,3]\) cyclic code over \( GF(q) \) with generator polynomial \( x^4+x^3+1 \). This code is optimal.

**Remark 80.** It was proved in [33] that \( C_1 = N_1 \cup D_1^2 \) is a difference set when \( n_2 = n_1 + 2 \) and \( n_1 \) are primes. Example 79 shows that a difference set may give an optimal cyclic code.

**Example 81.** Let \( (p,m,n_1,n_2) = (2,1,5,3) \). Then \( q = 2, n = 15 \), and \( C_{\lambda} \) is a \([15,5,7]\) cyclic code over \( GF(q) \) with generator polynomial \( x^{10}+x^9+x^8+x^6+x^5+x^2+1 \). This code is optimal.
Example 82. Let \((p, m, n_1, n_2) = (2, 1, 3, 7)\). Then \(q = 2, n = 21\), and \(C_\lambda\) is a \([21, 7, 3]\) cyclic code over \(\text{GF}(q)\) with generator polynomial \(x^{14} + x^7 + 1\). This is a bad cyclic code due to its poor minimum distance. The code in this case is bad because \(q \notin D_0^{(2)}\).

Remark 83. It was proved in [9, 28] that \(C_1 = \text{N}_1 \cup D^{(2)}_1\) is an almost difference set when \(n_2 = n_1 + 4\) and \(n_1\) both are primes. Example 82 shows that this difference set gives a bad cyclic code over \(\text{GF}(2)\). Later we will see that it may give an almost optimal cyclic code over other fields \(\text{GF}(q)\).

Example 84. Let \((p, m, n_1, n_2) = (2, 1, 7, 5)\). Then \(q = 2, n = 35\), and \(C_\lambda\) is a \([35, 11, 5]\) cyclic code over \(\text{GF}(q)\) with generator polynomial

\[
\begin{align*}
&x^{24} + x^{23} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + \\
&x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^6 + x^5 + x + 1.
\end{align*}
\]

5.2.2 The case \(q = 3\)

The following corollary follows directly from Theorem 77.

Corollary 85. Let \(q = 3\). We have the following conclusions:

1. If \(n_1 \equiv 1 \pmod{12}\) and \(n_2 \equiv -1 \pmod{12}\) or \(n_1 \equiv -5 \pmod{12}\) and \(n_2 \equiv 5 \pmod{12}\), we have

\[
\begin{align*}
&\mathbb{L}_\lambda = n - \frac{(n_1 - 1)(n_2 - 1)}{2}, \\
&M_{\lambda_\lambda}(x) = \left\{ \begin{array}{ll} \\
&\frac{x^{n_1 - 1}}{x_\lambda^{(n_1)}}, & \text{if } \Lambda(\theta) = 0 \\
&\frac{x^{n_2 - 1}}{x_\lambda^{(n_2)}}, & \text{if } \Lambda(\theta) = -1.
\end{array} \right.
\end{align*}
\]

In this case, the cyclic code \(C_\lambda\) over \(\text{GF}(q)\) defined by the sequence \(\lambda^\infty\) has generator polynomial \(M_{\lambda_\lambda}(x)\) above and parameters \([n, k, d]\), where \(k = \frac{(n_1 - 1)(n_2 - 1)}{2}\).

2. If \(n_1 \equiv -1 \pmod{12}\) and \(n_2 \equiv 5 \pmod{12}\) or \(n_1 \equiv 5 \pmod{12}\) and \(n_2 \equiv -1 \pmod{12}\), we have

\[
\begin{align*}
&\mathbb{L}_\lambda = n - n_2, \\
&M_{\lambda_\lambda}(x) = \frac{x^n - 1}{x^{n_2} - 1}.
\end{align*}
\]

In this case, the cyclic code \(C_\lambda\) over \(\text{GF}(q)\) defined by the sequence \(\lambda^\infty\) has generator polynomial \(M_{\lambda_\lambda}(x)\) above and parameters \([n, n_2, n_1]\), where the minimum weight follows from Theorem 89.

3. If \(n_1 \equiv -5 \pmod{12}\) and \(n_2 \equiv 1 \pmod{12}\) or \(n_1 \equiv 1 \pmod{12}\) and \(n_2 \equiv -5 \pmod{12}\), we have

\[
\begin{align*}
&\mathbb{L}_\lambda = n - n_1, \\
&M_{\lambda_\lambda}(x) = \frac{x^n - 1}{x^{n_1} - 1}.
\end{align*}
\]

In this case, the cyclic code \(C_\lambda\) over \(\text{GF}(q)\) defined by the sequence \(\lambda^\infty\) has generator polynomial \(M_{\lambda_\lambda}(x)\) above and parameters \([n, n_1, n_2]\), where the minimum weight follows from Theorem 89.
4. If \( n_1 \equiv -1 \pmod{12} \) and \( n_2 \equiv 1 \pmod{12} \) or \( n_1 \equiv 5 \pmod{12} \) and \( n_2 \equiv -5 \pmod{12} \), we have

\[
\mathbb{L}_\lambda = n - \frac{(n_1 + 1)(n_2 + 1) - 2}{2},
\]

\[
\mathbb{M}_\lambda(x) = \left\{ \begin{array}{ll}
\frac{(x^n - 1)(x - 1)}{(x^{n_1} - 1)(x^{n_2} - 1)d_0(x)}, & \text{if } \Lambda(\theta) = 0 \\
\frac{(x^n - 1)(x - 1)}{(x^{n_1} - 1)(x^{n_2} - 1)d_1(x)}, & \text{if } \Lambda(\theta) = -1.
\end{array} \right.
\]

In this case, the cyclic code \( C_\lambda \) over \( \operatorname{GF}(q) \) defined by the sequence \( \lambda^\infty \) has generator polynomial \( \mathbb{M}_\lambda(x) \) above and parameters \([n,k,d]\), where \( k = \frac{(n_1 + 1)(n_2 + 1) - 2}{2} \) and \( d \) has the lower bound of Theorem 93.

5. If \( n_1 \equiv 1 \pmod{12} \) and \( n_2 \equiv 5 \pmod{12} \) or \( n_1 \equiv -5 \pmod{12} \) and \( n_2 \equiv -1 \pmod{12} \), we have

\[
\mathbb{L}_\lambda = n, \quad \mathbb{M}_\lambda(x) = x^n - 1.
\]

In this case, the cyclic code \( C_\lambda \) over \( \operatorname{GF}(q) \) defined by the sequence \( \lambda^\infty \) has generator polynomial \( \mathbb{M}_\lambda(x) \) above and parameters \([n,0,0]\).

6. If \( n_1 \equiv -1 \pmod{12} \) and \( n_2 \equiv -5 \pmod{12} \) or \( n_1 \equiv 5 \pmod{12} \) and \( n_2 \equiv 1 \pmod{12} \), we have

\[
\mathbb{L}_\lambda = n - (n_1 + n_2 - 1), \quad \mathbb{M}_\lambda(x) = \frac{(x^n - 1)(x - 1)}{(x^{n_1} - 1)(x^{n_2} - 1)}.
\]

In this case, the cyclic code \( C_\lambda \) over \( \operatorname{GF}(q) \) defined by the sequence \( \lambda^\infty \) has generator polynomial \( \mathbb{M}_\lambda(x) \) above and parameters \([n,n_1 + n_2 - 1,d]\), where \( d \) has the lower bound of Theorem 97.

7. If \( n_1 \equiv 5 \pmod{12} \) and \( n_2 \equiv 5 \pmod{12} \) or \( n_1 \equiv -1 \pmod{12} \) and \( n_2 \equiv -1 \pmod{12} \), we have

\[
\mathbb{L}_\lambda = n - \frac{(n_1 + 1)(n_2 - 1) + 2}{2},
\]

\[
\mathbb{M}_\lambda(x) = \left\{ \begin{array}{ll}
\frac{x^n - 1}{(x^{n_2} - 1)d_0(x)}, & \text{if } \Lambda(\theta) = 0 \\
\frac{x^n - 1}{(x^{n_2} - 1)d_1(x)}, & \text{if } \Lambda(\theta) = -1.
\end{array} \right.
\]

In this case, the cyclic code \( C_\lambda \) over \( \operatorname{GF}(q) \) defined by the sequence \( \lambda^\infty \) has generator polynomial \( \mathbb{M}_\lambda(x) \) above and parameters \([n,k,d]\), where \( k = \frac{(n_1 + 1)(n_2 + 1) + 2}{2} \) and \( d \) has the lower bound of Theorem 95.

8. If \( n_1 \equiv 1 \pmod{12} \) and \( n_2 \equiv 1 \pmod{12} \) or \( n_1 \equiv -5 \pmod{12} \) and \( n_2 \equiv -5 \pmod{12} \), we have

\[
\mathbb{L}_\lambda = n - \frac{(n_1 - 1)(n_2 + 1) + 2}{2},
\]

\[
\mathbb{M}_\lambda(x) = \left\{ \begin{array}{ll}
\frac{x^n - 1}{(x^{n_2} - 1)d_0(x)}, & \text{if } \Lambda(\theta) = 0 \\
\frac{x^n - 1}{(x^{n_2} - 1)d_1(x)}, & \text{if } \Lambda(\theta) = -1.
\end{array} \right.
\]
In this case, the cyclic code $C_\lambda$ over $\text{GF}(q)$ defined by the sequence $\lambda^n$ has generator polynomial $M_\lambda(x)$ above and parameters $[n,k,d]$, where $k = \frac{(n-1)(n+1)+2}{2}$ and $d$ has the lower bound of Theorem 95.

Example 86. Let $(p,m,n_1,n_2) = (3,1,7,5)$. Then $q = 3$, $n = 35$, and $C_\lambda$ is a $[35,12,12]$ cyclic code over $\text{GF}(q)$ with generator polynomial
\[
x^{23} + 2x^{21} + 2x^{19} + 2x^{18} + x^{16} + 2x^{14} + x^{13} + 2x^{12} +
2x^{11} + 2x^{10} + x^8 + 2x^6 + 2x^5 + x^4 + x^3 + x^2 + 2x + 2.
\]
The best ternary linear code known of length 35 and dimension 12 has minimum distance 14.

Example 87. Let $(p,m,n_1,n_2) = (3,1,5,7)$. Then $q = 3$, $n = 35$, and $C_\lambda$ is a $[35,23,5]$ cyclic code over $\text{GF}(q)$ with generator polynomial $x^{12} + 2x^{11} + 2x^{10} + 2x^9 + 2x^7 + x^5 + 2x^4 + x^2 + 1$. The best ternary linear code known of length 35 and dimension 23 has minimum distance 6.

Remark 88. It was proved in [35] that $C_1 = N_1 \cup D_1^{(2)}$ is a difference set when $n_2 = n_1 + 2$ and $n_1$ are primes. Example 87 demonstrates that a difference set may give a very good cyclic code.

Finally, we present results on the minimum distance of some of the cyclic codes $C_\lambda$.

Theorem 89. The cyclic code over $\text{GF}(q)$ with the generator polynomial $g(x) = (x^n - 1)/(x^{ni} - 1)$ has parameters $[n,n_i,d_i]$, where
\[
d_i = n_{i-1}(-1)^i,
\]
where $i \in \{1,2\}$.

Example 90. Let $q = 2$ and $(n_1,n_2) = (7,5)$. Then the cyclic code over $\text{GF}(q)$ with the generator polynomial $g(x) = (x^n - 1)/(x^{n_1} - 1)$ has parameters $[35,7,5]$.

Theorem 91. Let $C_{(n_1,n_2,q)}$ denote the cyclic code over $\text{GF}(q)$ with the generator polynomial
\[
g(x) = (x-1)(x^n - 1)/(x^{n_1} - 1)(x^{n_2} - 1).
\]
Then the code $C_{(n_1,n_2,q)}$ has parameters $[n,n_1 + n_2 - 1,d_{(n_1,n_2)}]$, where
\[
d_{(n_1,n_2)} = \min(n_1,n_2).
\]

Example 92. Let $q = 2$ and $(n_1,n_2) = (7,5)$. The cyclic code over $\text{GF}(q)$ with the generator polynomial $g(x) = (x^n - 1)/(x^{n_1} - 1)(x^{n_2} - 1)$ has parameters $[35,11,5]$.

Theorem 93. Assume that $q \in D_0^{(2)}$. Let $C_{(i,q)}$ denote the cyclic code over $\text{GF}(q)$ with the generator polynomial $g(x) = (x^n - 1)/(x^{ni} - 1)(x^{n_2} - 1)$ has parameters $[n,(n_1 + 1)(n_2 + 1) - 2]/2,d_i]$, where
\[
d_i \geq \left\lceil \sqrt{\min(n_1,n_2)} \right\rceil.
\]
If $-1 \in D_1^{(2)}$, we have
\[
d_i^2 - d_i + 1 \geq \min(n_1,n_2).
\]
Example 94. Let \((p, m, n_1, n_2) = (3, 1, 5, 7)\). Then \(q = 3\), \(n = 35\), and \(C_{(0, q)}\) is a \([35, 23, 5]\) cyclic code over GF\((q)\) with generator polynomial \(x^{12} + 2x^{11} + 2x^{10} + x^9 + 2x^8 + x^7 + x^5 + 2x^4 + x^2 + 1\). In this case, \(-1 \in D_1^{(2)}\) and \(d_{(n_1, n_2)} = 5\). We have

\[
d_0^2 - d_0 + 1 \geq 5.
\]

Hence the lower bounds of (35) and (36) are 3. In this case, the two lower bounds are not met.

Theorem 95. Let \(q \in D_0^{(2)}\). Let \(C^{(i, j)}_{(n_1, n_2, q)}\) denote the cyclic code over GF\((q)\) with the generator polynomial

\[
g^{(i, j)}(x) = \frac{(x^{n_1} - 1)}{x - 1}d_j(x),
\]

and let \(d^{(i, j)}_{(n_1, n_2, q)}\) denote the minimum distance of this code, where \(i \in \{1, 2\}\) and \(j \in \{0, 1\}\). Then the code \(C^{(i, j)}_{(n_1, n_2, q)}\) has parameters

\[
\left[ n, \left(\frac{(n_1 - 1)(n_i - (-1)^i) + 1}{2}\right), d^{(i, j)}_{(n_1, n_2, q)} \right],
\]

where

\[
d^{(i, j)}_{(n_1, n_2, q)} \geq \lceil \sqrt{n_i} \rceil.
\]

If \(-1 \in D_1^{(2)}\), we have

\[
\left(\frac{d^{(i, j)}_{(n_1, n_2, q)}}{2}\right)^2 - d^{(i, j)}_{(n_1, n_2, q)} + 1 \geq n_i.
\]

Example 96. Let \((p, m, n_1, n_2) = (2, 1, 3, 11)\). Then \(q = 2\), \(n = 33\), and \(C_2\) is a \([33, 21, 3]\) cyclic code over GF\((q)\) with generator polynomial

\[
x^{12} + x^9 + x^7 + x^6 + x^5 + x^3 + 1 = \frac{x^{n_1} - 1}{x - 1}d_0(x).
\]

In this case, the lower bound of (37) is 2, while the actual minimum distance is 3.

6 Concluding remarks

In addition to the results presented in this paper, cyclic codes with interesting parameters were constructed with cyclotomic sequences of order 4 in [13] [33].

Recall that every cyclic code over any finite field could be expressed as a code \(C_s\) for a sequence \(s^\omega\). This approach can produce all cyclic codes over finite fields, including BCH codes. It is thus no surprise that some of the codes from Dickson polynomials are in fact BCH codes. Since the sequence approach is fundamental, it produces both good and bad cyclic codes. It is open what sequences over a finite field give cyclic codes with optimal parameters. There are many open problems in this direction.

Though a considerable amount of progress on this approach of constructing cyclic codes with sequences has been made, a lot of investigation should be further done, as there is a huge number of constructions of sequences in the literature. The reader is cordially invited to join the journey in this direction.
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