Hey Bill, what’s the deal with Threshold Progressions in a Variety of Covering and Packing Contexts?

Many combinatorial objects can be viewed as covering other combinatorial objects. For example, the set:

\[ \{1, 4, 7\} \]

is “covered” by the set

\[ \{1, 2, 4, 7, 9\} \]

because it is a subset. Similarly, the permutation

\[ 132 \]

is covered by the permutation

\[ 3174256 \]

because it exists as a subpermutation (i.e., if we delete all the non-bold symbols and pretend like we have a permutation on 3 elements, it’s a 132 permutation). In order to motivate the problems addressed in this paper, let’s focus on the first example—subsets of the numbers \( \{1, 2, 3, \ldots, n\} := [n] \).

Let’s say that \( C \) is a collection of subsets of \([n]\) which each have cardinality 5. We wish to introduce the primary notions investigated in this project with the specific example of size 3 subsets of \([n]\):

- If every subset which has size 3 is covered by at least one element of \( C \), we say that \( C \) covers the size 3 subsets of \([n]\). If every subset which has size 3 is covered by at least \( \lambda \) elements of \( C \), we say \( C \) is a \( \lambda \)-covering of the size 3 subsets of \([n]\).
- If every subset which has size 3 is covered by at most one element of \( C \), we say that the size 3 subsets of \([n]\) pack into \( C \). If every subset which has size 3 is covered by at most \( \lambda \) elements of \( C \), we say that the size 3 subsets of \([n]\) \( \lambda \)-pack into \( C \).

We can discuss \( \lambda \)-covering and \( \lambda \)-packing for any objects which have some kind of subobject relation. We now need a few notions from probability theory.

We say that a family of events \( E_n \) in a probability space happens with high probability (abbreviated whp) if \( P(E_n) \to 1 \) as \( n \to \infty \). Similarly, we say that \( E_n \) happens with low probability (wlp) if \( P(E_n) \to 0 \) as \( n \to \infty \). We say that a value \( p_0 = p_0(n) \) is a threshold for a property \( C \) if for all \( p = p(n) \) where \( p < p_0 \) the property \( C \) occurs wlp, while for all \( p \gg p_0 \) the property \( C \) occurs whp. A simple example would be to look at \( G(n, p) \), the graph on \( n \) vertices where we include each potential edge with probability \( p \), and exclude it with probability \( (1 - p) \). Note that for each \( p \), \( G(n, p) \) is a probability distribution on the collection of all graphs on \( n \) vertices. It can be shown that if \( p < 1/n \), then \( P(G(n, p) \text{ has a cycle}) \to 0 \), while if \( p \gg 1/n \), then \( P(G(n, p) \text{ has a cycle}) \to 1 \), and hence \( 1/n \) serves as a threshold for \( G(n, p) \) containing a cycle.

Given some combinatorial objects \( A \), and we select some random collection \( C \) of (larger) objects with probability \( p \). We can ask for which values of \( p \) is \( C \) a \( \lambda \)-cover or when \( A \) \( \lambda \)-packs into \( C \) with high probability. We establish probabilistic thresholds for the following:

- Let \( k < t \), and let \( A = \binom{[n]}{k} \) and \( C \subseteq \binom{[n]}{t} \) be chosen randomly with each member selected with probability \( p \). We establish a threshold for both \( \lambda \) coverings and \( \lambda \) packings, where \( \lambda \geq 2 \).
- Let \( A \) be all permutations on \( n \) symbols and \( C \) a random subset of the permutations on \( n+1 \) symbols, where the subobject relation is “order isomorphic subpattern”. In [?] Allison, Godbole, Hawley, and K. showed the covering and packing thresholds for \( \lambda = 1 \), and in this paper we establish them for all \( \lambda \geq 2 \).
- Suppose a set \( A \subseteq [n] \) satisfies the property that for all \( k \in [h, nh] \) there are fewer than \( \lambda \) solutions to the equation:

\[
\sum_{i=1}^{h} a_i = k
\]

where \( a_1 < \ldots < a_h \subseteq A \). We call such a set \( \lambda \)-Sidon. In some sense, we can think about the collection of sums with \( h \) terms from the set \( A \) as \( \lambda \)-packing into the set of potential sums from \( A \). We establish the packing threshold for all \( \lambda \).

This is joint work with Anant Godbole, Thomas Grubb, and Kyutae (Paul) Han.