Axisymmetric equilibria of a gravitating plasma with incompressible flows

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Abstract

It is found that the ideal magnetohydrodynamic equilibrium of an axisymmet-
ric gravitating magnetically confined plasma with incompressible flows is governed
by a second-order elliptic differential equation for the poloidal magnetic flux func-
tion containing five flux functions coupled with a Poisson equation for the gravi-
tation potential, and an algebraic relation for the pressure. This set of equations
is amenable to analytic solutions. As an application, the magnetic-dipole static
axisymmetric equilibria with vanishing poloidal plasma currents derived recently
by Krasheninnikov, Catto, and Hazeltine [Phys. Rev. Lett. 82, 2689 (1999)] are
extended to plasmas with finite poloidal currents, subject to gravitating forces
from a massive body (a star or black hole) and inertial forces due to incompress-
ible sheared flows. Explicit solutions are obtained in two regimes: (a) in the
low-energy regime $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1$, where $\beta_0$, $\gamma_0$, $\delta_0$, and $\epsilon_0$ are related
to the thermal, poloidal-current, flow and gravitating energies normalized to the
poloidal-magnetic-field energy, respectively, and (b) in the high-energy regime
$\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$. It turns out that in the high-energy regime all four
forces, pressure-gradient, toroidal-magnetic-field, inertial, and gravitating con-
tribute equally to the formation of magnetic surfaces very extended and localized
about the symmetry plane such that the resulting equilibria resemble the accre-
tion disks in astrophysics.

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1. Introduction

The difficult problem of equilibrium with flow has been the subject of an increasing number of investigations (e.g., in relation to the present work see Refs. [1]-[14], [16], [20]) on both laboratory and astrophysical plasmas. Even for vanishing gravity the ideal magnetohydrodynamic (MHD) equilibrium of a symmetric (two dimensional) plasma with arbitrary compressible flows associated, e.g. with isothermal magnetic surfaces, satisfies a second-order partial differential equation for the poloidal magnetic flux function $\psi$ coupled with an algebraic Bernoulli equation for the density [1]. Depending on the value of the poloidal Mach number $M^2$ (defined in Sec. 2), the above mentioned equation can be either elliptic or hyperbolic. Experimental evidence [11, 12, 13, 14], however, confirms that (a) both density and temperature are to a very good approximation flux functions, i.e. functions constant on magnetic surfaces, and (b) the poloidal flows (e.g. involved in the transition from the low to the high confinement regime in tokamaks) lie within the first elliptic region. Similar conditions may also prevail in astrophysical plasmas at least at distances not close to centers of gravity. For magnetically confined plasmas the equation is expected elliptic because hyperbolicity, associated with shock waves, would imply open magnetic surfaces and thereby abrupt confinement degradation. In this respect, equilibria with incompressible flows for which the differential equation becomes always elliptic has been of particular interest for both laboratory [2, 3, 4, 5, 6] and astrophysical [7, 8, 9, 10] plasmas.

In previous studies [5, 6] we considered ideal MHD equilibria with incompressible flows in cylindrical and toroidal geometries. For a toroidal plasma we found that the differential equation decouples from the pressure relation, thus making the problem analytically solvable. Several classes of analytic solutions of linearized versions of the above mentioned differential equation associated with sheared flows were also obtained. For vanishing flows this equation reduces to the Grad-Schlüter-Shafranov equation. Analytic solutions of the nonlinear Grad-Schlüter-Shafranov equation for a plasma with vanishing poloidal current at either low or high pressure confined by a dipolar magnetic field were obtained recently in Ref. [15]. These studies were then extended to equilibria with purely toroidal flow [16] (being inherently incompressible because of axisymmetry), to gravitating magnetic dipolar plasmas without flow [17], and to plasmas with anisotropic pressure [18].
The purpose of the present work is twofold: (a) to extend our equilibrium equations [6] to the case of a gravitating plasma with incompressible flows, and (b) with employment of the separable eigenvalue technique introduced in Refs [15]-[18] to derive analytic magnetic dipolar equilibria for a plasma at finite pressure and poloidal current with incompressible sheared flows having non-vanishing toroidal and poloidal components, under the exertion of gravitational forces from a massive body (a star or a black hole).

In Section 2 the equilibrium equations for an axisymmetric gravitating magnetically confined plasma with incompressible flows are derived. These equations are then reduced further for a plasma confined by the magnetic field of a point dipole in Section 3. Analytic magnetic dipolar solutions are constructed in Sections 4 and 5 in the following regimes: (a) in the low-energy regime \( \beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1 \), where \( \beta_0, \gamma, \delta_0, \) and \( \epsilon_0 \) are related to the thermal, poloidal-current, flow, and gravitating energies normalized to the poloidal-magnetic-field energy, respectively, and (b) in the high-energy regime \( \beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1 \). Finally, the conclusions are summarized in Sec. 6.

2. Stationary equilibrium equations for a gravitating plasma

The ideal MHD equilibrium states of a gravitating plasma with flow are governed by the following set of equations, written in convenient units:

\[
\nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}
\]

\[
\rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla P - \rho \nabla \Omega \tag{2}
\]

\[
\nabla^2 \Omega = G_0 \rho_t \tag{3}
\]

\[
\nabla \times \mathbf{E} = 0 \tag{4}
\]

\[
\nabla \times \mathbf{B} = \mathbf{j} \tag{5}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{6}
\]

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \tag{7}
\]

Here, \( \Omega, 4\pi G_0, \) and \( \rho_t \) are, respectively, the gravitation potential, the constant of gravity and the total density including contributions from the plasma itself.
and from external mass sources. Standard notations are used in the rest of Eqs. (1)-(7).

For an axially symmetric plasma the divergence-free fields can be expressed in terms of the functions $I(R,z)$, $F(R,z)$ and $\Theta(R,z)$ as

\begin{align*}
B &= I \nabla \phi + \nabla \phi \times \nabla \psi, \quad (8) \\
\mathbf{j} &= \Delta^* \psi \nabla \phi - \nabla \phi \times \nabla I, \quad (9)
\end{align*}

and

$$\rho \mathbf{v} = \Theta \nabla \phi + \nabla \phi \times \nabla F. \quad (10)$$

Here, $R,\phi,z$ are cylindrical coordinates with $z$ corresponding to the axis of symmetry, constant $\psi$ surfaces are the magnetic surfaces, and $\Delta^* \equiv R^2 \nabla \cdot (\nabla / R^2)$.

Eqs. (1)-(7) can be reduced by means of certain integrals of the system, which are shown to be flux functions. To identify three of these quantities, the time independent electric field is expressed by $E = -\nabla \Phi$ and the Ohm’s law (7) is projected along $\nabla \phi$, $B$ and $\nabla \psi$, respectively, yielding $F = F(\psi)$, $\Phi = \Phi(\psi)$, and

$$\frac{1}{\rho R^2} (IF' - \Theta) = \Phi'. \quad (11)$$

A fourth flux function is derived from the component of the force-balance equation (2) along $\nabla \phi$:

$$I \left( 1 - \frac{(F')^2}{\rho} \right) + R^2 F' \Phi' \equiv X(\psi), \quad (12)$$

the flux function $X(\psi)$ being related to the toroidal magnetic field (see also Eq. (21) in Sec. 3). Note that the toroidal quantities $I(\psi, R)$ and $\Theta(\psi, R)$ are not flux functions. With the aid of the flux functions identified the components of Eq. (2) along $\mathbf{B}$ and perpendicular to a magnetic surface are put in the respective forms

$$\mathbf{B} \cdot \left[ \nabla \left( \frac{v^2}{2} + \frac{\Theta}{\rho} \Phi' \right) + \frac{\nabla P}{\rho} + \nabla \Omega \right] = 0 \quad (13)$$

and

\begin{align*}
\left\{ \nabla \cdot \left[ \left( 1 - \frac{(F')^2}{\rho} \right) \nabla \psi \right] + \frac{F'' F'}{\rho} \frac{\left| \nabla \psi \right|^2}{R^2} \right\} \left| \nabla \psi \right|^2 \\
+ \left[ \frac{\rho}{2} \left( \nabla v^2 - \frac{\nabla (\Theta/\rho)^2}{R^2} \right) + \frac{\nabla (I^2)}{2 R^2} + \nabla P + \rho \nabla \Omega \right] \cdot \nabla \psi &= 0 \quad (14)
\end{align*}
It is pointed out that Eqs. (13) and (14) hold for any equation of state.

We now consider incompressible flows, \( \nabla \cdot \mathbf{v} = 0 \), which on account of Eq. (1) implies that the density is a flux function. With the aid of \( \rho = \rho(\psi) \), Eq. (13) can be integrated to yield an expression for the pressure, i.e.

\[
P = P_s(\psi) - \rho \left( \frac{v^2}{2} + \Omega - \frac{R^2(\Phi')^2}{1 - (F')^2/\rho} \right),
\]

where \( P_s(\psi) \) is a flux function which for vanishing flow and gravity (\( \mathbf{v} = \Omega = 0 \)) represents the static pressure. Defining the Alfvén velocity associated with the poloidal magnetic field, \( v_{Ap}^2 \equiv |\nabla \psi|^2/\rho \), and the Mach number \( M^2 \equiv v_p^2/v_{Ap}^2 = (F')^2/\rho \) and inserting the expression (15) into Eq. (14), we arrive at the elliptic differential equation

\[
(1 - M^2) \Delta \psi - \frac{1}{2} (M^2)' |\nabla \psi|^2 + \frac{1}{2} \left( \frac{X^2}{1 - M^2} \right)' + R^2 (P_s' - \Omega') + \frac{R^4}{2} \left( \frac{\rho(\Phi')^2}{1 - M^2} \right)' = 0.
\]

Eq. (16) contains the arbitrary flux functions \( F(\psi), \Phi(\psi), X(\psi), \rho(\psi) \) and \( P_s(\psi) \) which must be found from other physical considerations. For \( \Omega \equiv 0 \) Eqs. (15) and (16) constitute concise forms of the equilibrium equations we derived formerly [6] [Eqs. (19) and (22) therein]. We note here that the physically plausible class of equilibria with isothermal magnetic surfaces, \( T = T(\psi) \), was examined in our previous studies [5, 6] for vanishing gravity. Incompressible flows and \( T = T(\psi) \) imply that the pressure becomes a flux function. For toroidal plasmas it was shown that equilibria of this kind are possible. In addition, steady states with incompressible flows and isobaric magnetic surfaces were proposed in Ref. [3] as approximate equilibria for the Joint-European-Torus tokamak. It is also noted that for non-ideal plasmas with arbitrary flows, i.e. when a finite resistivity is introduced in Eq. (7), some of the integrals found in this section in the form of flux functions are destroyed and the tractability of an extension of the present investigation becomes questionable.

Recapitulating, the axisymmetric ideal MHD equilibria of a gravitating plasma with incompressible flows are governed by the set of Eqs. (8) and (16) for \( \psi \) and \( \Omega \), and relation (15) for the pressure.
Under the transformation \( [19, 20] \)

\[
U(\psi) = \int_0^\psi [1 - M^2(\psi')]^{1/2} d\psi', \quad M^2 < 1,
\]

(17)

Eq. (16) reduces (after dividing by \((1 - M^2)^{1/2}\)) to

\[
\Delta \ast U + \frac{1}{2} \frac{d}{dU} \left( \frac{X^2}{1 - M^2} \right) + R^2 \left( \frac{dP_s}{dU} - \Omega \frac{d\rho}{dU} \right)
+ R^4 \frac{d}{dU} \left[ \rho \left( \frac{d\Phi}{dU} \right)^2 \right] = 0.
\]

(18)

Eq. (18) is free of the nonlinear term \(1/2(M^2' \mid \nabla \psi)^2\) and, therefore, for \(M^2 < 1\) the equilibrium can be determined from the more tractable set of Eqs. (3), (15) and (18).

### 3. Magnetic-dipole equilibrium equations

The equilibrium of a plasma confined by the magnetic field of a current ring lying on the symmetry plane and centered at the origin of the system of coordinates is now considered. Employing spherical coordinates \(r, \theta\) and \(\phi\) with \(\mu = \cos \theta\) and \(R = r \sin \theta\), we seek separable solutions of Eq. (18) of the form

\[
U(r, \mu) = U_0 H(\mu) \left( \frac{r_0}{r} \right)^\alpha.
\]

(19)

Here, \(H\) is an unknown function of \(\mu\) alone such that \(H(0) = 1\), and \(U_0\) and \(r_0\) are normalization constants specifying a reference flux surface location. The parameter \(\alpha\) plays the role of an eigenvalue of equations (3) and (15). It equals unity or \(-2\) in the vacuum limit to recover the dipolar solutions \(\psi_{\text{vac}} \propto (1 - \mu^2)/r\) and \(\psi_{\text{vac}} \propto (1 - \mu^2)r^2\) describing, respectively, the flux surfaces far away from and close to the origin. We are interested in configurations symmetric with respect to the symmetry plane. Accordingly, the boundary conditions

\[
H(\mu \rightarrow 1) \propto 1 - \mu, \quad \frac{dH}{d\mu} \bigg|_{\mu=0} = 0
\]

(20)

are chosen to keep the magnetic field finite at \(\theta = 0\) and parallel to the axis of symmetry at \(\theta = \pi/2\), respectively. Using Eqs. (8), (11), (12), (17), and (20) we
find for the magnetic field associated with Eq. (19)

\[
B = \frac{1}{r(1 - \mu^2)^{1/2}} \left[ \frac{X}{1 - M^2} - r^2(1 - \mu^2) \frac{dF}{dU} \frac{d\Phi}{dU} \right] e_\phi \\
+ B_0 \left( \frac{r_0}{r} \right)^{2+\alpha} (1 - M^2)^{-1/2} \left[ (1 - \mu^2)^{-1/2} H(\mu)e_\theta + \frac{1}{\alpha} \frac{dH(\mu)}{d\mu} e_r \right],
\]

(21)

where \( B_0 = \alpha U_0/r_0^2 \). Note the finite toroidal magnetic field induced by the flow and the poloidal currents as can also be seen from Eq. (12).

We now consider a plasma subject only to gravitating forces from a star or black hole of mass \( M_s \) placed at \( r = 0 \); the plasma self gravity is neglected. Consequently, Eq. (3) decouples from (18) and has the solution

\[
\Omega = -\frac{G_0 M_s}{4\pi r}.
\]

(22)

Inspection of Eq. (18) with the gravitation potential (22) implies that the separable solution (19) is only possible provided

\[
P_s = P_{s0} \left( \frac{U}{U_0} \right)^{2+4/\alpha},
\]

(23)

\[
\frac{X^2}{1 - M^2} = X_{0}^2 \left( \frac{U}{U_0} \right)^{2+2/\alpha},
\]

(24)

\[
\rho = \rho_0 \left( \frac{U}{U_0} \right)^{2+3/\alpha},
\]

(25)

and

\[
\rho \left( \frac{d\Phi}{dU} \right)^2 = \rho_0 \left( \frac{\Phi_0}{U_0} \right)^2 \left( \frac{U}{U_0} \right)^{2+6/\alpha},
\]

(26)

where \( P_{s0}, X_0, \rho_0, \) and \( \Phi_0 \) are normalization constants associated with the reference flux surface \( U_0 \). Inserting Eqs. (19) and (22)-(26) into Eq. (18), we obtain

\[
\frac{d^2H}{d\mu^2} + \frac{\alpha(\alpha + 1)}{1 - \mu^2} H = -\beta_0 \alpha(2 + \alpha) H^{1+4/\alpha} - \gamma_0 \alpha(1 + \alpha)(1 - \mu^2)^{-1} H^{1+2/\alpha} \\
- \delta_0 \alpha(3 + \alpha)(1 - \mu^2) H^{1+6/\alpha} - \epsilon_0 \alpha(3/2 + \alpha) H^{1+3/\alpha}.
\]

(27)
Here, \( \beta_0, \gamma_0, \delta_0, \) and \( \epsilon_0 \) are related, respectively, to the static thermal, poloidal-current, flow and gravitating energies normalized to the poloidal-magnetic-field energy on the reference surface:

\[
\beta_0 \equiv \frac{P_1}{B_0^2/2}, \quad \gamma_0 \equiv \frac{(X_0/r_0)^2/2}{B_0^2/2}, \quad \delta_0 \equiv \frac{\rho_0(\Phi_0 r_0/U_0)^2/2}{B_0^2/2}, \quad \epsilon_0 \equiv \frac{\rho_0 G_0 M_s/(4 \pi r_0)}{B_0^2/2}.
\]

Solutions of Eq. (27) will be constructed in the low-energy regime, \( \beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1 \), and in the high-energy regime, \( \beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1 \).

4. Solution in the low-energy regime

\( (\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1) \)

For this case it is convenient to put Eq. (27) in the form

\[
\frac{d}{d\mu} \left[(1 - \mu^2)^2 \frac{d}{d\mu} \left( \frac{H}{1 - \mu^2} \right) \right] - (1 - \alpha)(2 + \alpha)H = -\beta_0 \alpha(2 + \alpha)(1 - \mu^2) H^{1+4/\alpha} - \gamma_0 \alpha(1 + \alpha)H^{1+2/\alpha} - \delta_0 \alpha(3 + \alpha)(1 - \mu^2)^2 H^{1+6/\alpha} - \epsilon_0(\frac{3}{2} + \alpha)(1 - \mu^2) H^{1+3/\alpha}
\]

where \( H \to 1 - \mu^2 \) as \( \beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \to 0 \). With the use of the boundary conditions (20), integration of Eq. (29) from \( \mu = 0 \) to \( \mu = 1 \) yields

\[
(2 + \alpha) [(1 - \alpha)P_1 - \alpha \beta_0 P_2] - \alpha \left[ \gamma_0 (1 + \alpha)P_3 + \delta_0 (3 + \alpha)P_4 + \epsilon_0 \left( \frac{3}{2} + \alpha \right)P_5 \right] = 0
\]

with

\[
P_1 = \int_0^1 H d\mu, \quad P_2 = \int_0^1 (1 - \mu^2) H^{1+4/\alpha} d\mu, \quad P_3 = \int_0^1 H^{1+2/\alpha} d\mu,
\]

\[
P_4 = \int_0^1 (1 - \mu^2)^2 H^{1+6/\alpha} d\mu, \quad P_5 = \int_0^1 (1 - \mu^2) H^{1+3/\alpha} d\mu.
\]

To appreciate the impact of finite pressure, finite poloidal current, flow, and gravity on the vacuum equilibrium, the relations \( H = 1 - \mu^2 \) and \( \alpha \to 1 \) are employed into Eq. (30) except for the term \( 1 - \alpha \). The departure of \( \alpha \) from unity is then found

\[
1 - \alpha = \frac{512}{1001} \beta_0 + \frac{16}{33} \gamma_0 + \frac{131072}{230945} \delta_0 + \frac{320}{693} \epsilon_0.
\]
Therefore, the modifications of the vacuum equilibrium from the finite pressure, poloidal current, flow, and gravity are of the order of magnitude of \( \beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \). As a result, an analytic solution of Eq. (29) can be derived by using the vacuum solution \( H = 1 - \mu^2 \) in the five terms in which \( H \) appears undifferentiated.

Using the boundary condition (20) at \( \mu = 1 \) and introducing \( t = 1 - \mu^2 = \sin^2 \theta \), integration of Eq. (29) from 1 to \( \mu \) yields

\[
\frac{d}{dt}\left( \frac{H}{t} \right) = \frac{1}{4t^2(1-t)^{1/2}} \left[ 3(1-\alpha) \int_0^t \frac{x \, dx}{(1-x)^{1/2}} - 3\beta_0 \int_0^t \frac{x^6 \, dx}{(1-x)^{1/2}} 
- 2\gamma_0 \int_0^t \frac{x^3 \, dx}{(1-x)^{1/2}} - 4\delta_0 \int_0^t \frac{x^9 \, dx}{(1-x)^{1/2}} - \frac{5}{2}\epsilon_0 \int_0^t \frac{x^5 \, dx}{(1-x)^{1/2}} \right].
\]

(33)

Evaluating the integrals in Eq. (33) and integrating again, using \( H(\mu = 0) = 1 \), we obtain the following low-energy solution valid at all distances from a point dipole:

\[
\frac{H}{1-\mu^2} = 1 - \left[ \frac{192}{1001}(1-t) + \frac{160}{2002}(1-t^2) 
+ \frac{20}{429}(1-t^3) + \frac{18}{572}(1-t^4) + \frac{3}{130}(1-t^5) \right] \beta_0 
- \left[ \frac{6}{35}(1-t) + \frac{5}{70}(1-t^2) \right] \gamma_0 
- \left[ \frac{49152}{230945}(1-t) + \frac{8192}{92378}(1-t^2) + \frac{7168}{138567}(1-t^3) + \frac{42256}{923780}(1-t^4) 
+ \frac{2688}{104975}(1-t^5) + \frac{192}{9690}(1-t^6) + \frac{36}{2261}(1-t^7) + \frac{2}{152}(1-t^8) \right] \delta_0 
- \left[ \frac{40}{231}(1-t) + \frac{100}{1386}(1-t^2) + \frac{25}{594}(1-t^3) + \frac{5}{176}(1-t^4) \right] \epsilon_0.
\]

(34)

For vanishing poloidal current, flow and gravity (\( \gamma_0 = \delta_0 = \epsilon_0 = 0 \)) Eq. (34) reduces to the low-pressure static equilibrium solution of Ref. [15] [Eq. (12) therein].

5. Solution in the high energy regime

\( (\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1) \)
From the low-energy solution we anticipate that for $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$ it holds that $|\alpha| \ll 1$ and, consequently, assume that

$$\frac{1}{\beta_0} \ll |\alpha| \ll 1,$$

(35)

to be verified a posteriori. The term $\alpha(1 + \alpha)H/(1 - \mu^2)$ in Eq. (27) is small everywhere, via relations (20) and (35), and can be neglected.

We first search for solutions in the region $0 \leq \mu \leq \mu_c \equiv 1/\beta_0 \ll 1$. In this region the approximations $1 - \mu^2 \approx 1$, $H \approx 1$ can be made and, therefore, Eq. (27) can be written in the form

$$\frac{d^2 H}{d\mu^2} = -\alpha \left(2\beta_0 + \gamma_0 + 3\delta_0 + \frac{3}{2}\epsilon_0\right).$$

(36)

Integrating twice Eq. (36) with $dH/d\mu|_{\mu=0} = 0$ and $H(\mu = 0) = 1$ we obtain the solution

$$H = 1 + \alpha \left(2\beta_0 + \gamma_0 + 3\delta_0 + \frac{3}{2}\epsilon_0\right) \mu \left(1 - \frac{\mu}{2}\right).$$

(37)

Evaluation of Eq. (37) at $\mu = \mu_c$ yields

$$|\alpha| = O(1 - H(\mu_c)) \ll 1,$$

(38)

consistent with assumption (35).

We now consider the whole regime of variation of $\mu$. For $0 \leq \mu \leq 1$ the terms on the RHS of Eq. (27) are large at $\mu = 0$ and rapidly decrease to zero as $H$ decreases from $H(\mu = 0) = 1$ toward $H(\mu = 1) = 0$ since $|\alpha| \ll 1$. In particular, the variations of the terms $\gamma_0(1 - \mu^2)^{-1}\alpha(1 + \alpha)H^{1+2/\alpha}$ and $\delta_0(1 - \mu^2)\alpha(3 + \alpha)H^{1+6/\alpha}$ from $1 - \mu^2$ are much weaker than those from $H^{1+2/\alpha}$ and $H^{1+6/\alpha}$, respectively. Therefore, in the above terms, $1 - \mu^2$ can be approximated by unity and, consequently, Eq. (27) can be written in the form

$$\frac{d^2 H}{d\mu^2} = -\alpha \left(2\beta_0 H^{1+4/\alpha} + \gamma_0 H^{1+2/\alpha} + 3\delta_0 H^{1+6/\alpha} + \frac{3}{2}\epsilon_0 H^{1+3/\alpha}\right).$$

(39)

Multiplying Eq. (39) by $dH/d\mu$ and integrating from $\mu = 0$ where $dH/d\mu = 0$, we find

$$\frac{dH}{d\mu} = -|\alpha| \left[\beta_0 \left(1 - H^{2+4/\alpha}\right) + \gamma_0 \left(1 - H^{2+2/\alpha}\right) + \delta_0 \left(1 - H^{2+6/\alpha}\right) + \epsilon_0 \left(1 - H^{2+3/\alpha}\right)\right]^{1/2} + O(\sqrt{|\alpha|}).$$

(40)
Integration again from $\mu = 0$, where $H(\mu = 0) = 1$, to $\mu$ yields

$$
|\alpha|\mu = \int_{H}^{1} \left[ \beta_0 \left( 1 - x^{2+4/\alpha} \right) + \gamma_0 \left( 1 - x^{2+2/\alpha} \right) \\
+ \delta_0 \left( 1 - x^{2+6/\alpha} \right) + \epsilon_0 \left( 1 - x^{2+3/\alpha} \right) \right]^{-1/2} \, dx \\
\mu \rightarrow 1 \rightarrow (1 - H)(\beta_0 + \gamma_0 + \delta_0 + \epsilon_0)^{-1/2}.
$$

(41)

To satisfy $H(\mu = 1) = 0$, Eq. (41) requires

$$
|\alpha| = (\beta_0 + \gamma_0 + \delta_0 + \epsilon_0)^{-1/2} + \mathcal{O}(1/(\beta_0 + \gamma_0 + \delta_0 + \epsilon_0)).
$$

(42)

Note that for $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$, $H = 1 - \mu$ holds everywhere except in a small region $0 \leq \mu \leq (\beta_0 + \gamma_0 + \delta_0 + \epsilon_0)^{-1/2} \ll 1$, where $H$ remains close to unity, but with a large second derivative of the order of $(\beta_0 + \gamma_0 + \delta_0 + \epsilon_0)^{1/2}$ [see Eq. (39)]. Eq. (42) implies that the distance between adjacent flux surfaces at the symmetry plane $\mu = 0$ increases as either of $\beta_0$, $\gamma_0$, $\delta_0$, and $\epsilon_0$ increases. Indeed, as $\alpha$ decreases the spacing must adjust to keep $U \approx (r_0/r)^\alpha$ fixed and, therefore, the magnetic surfaces become more extended and localized about the symmetry plane. The resulting equilibria resemble the accretion disks in astrophysics. It is noted that Krasheninnikov and Catto [16, 17] came to the conclusion that in the strong gravity limit ($\epsilon_0 \gg 1$), gravity and flow affect the flux surfaces but not the eigenvalue $\alpha$. This does not contradict our conclusion because it concerns different density, current and flow regimes, i.e. arbitrary density profiles, vanishing poloidal currents, and purely toroidal flows were considered in Refs. [16, 17] while the present study concerns finite poloidal currents and magnetic surfaces of constant density associated with incompressible flows with non-vanishing toroidal and poloidal components.

6. Conclusions

It has been shown that the equilibrium of a gravitating axisymmetric magnetically confined plasma with incompressible flows is governed by a second-order elliptic differential equation for the poloidal magnetic flux function $\psi$ [Eq. (16)] containing five flux functions coupled with a Poisson equation for the gravitational potential, and an algebraic relation for the pressure [Eq. (13)]. The above mentioned elliptic equation can be transformed to one [Eq. (18)] possessing a
differential part identical to that of the Grad-Schlüter-Shafranov equation, which permits the derivation of analytic solutions.

Analytic solutions for a plasma confined by a dipolar magnetic field and subject to gravitating forces from a massive body have been obtained in two energy regimes: (a) in the low-energy regime $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1$, where $\beta_0$, $\gamma_0$, $\delta_0$, and $\epsilon_0$ are related to the thermal, poloidal-current, flow and gravitating energies normalized to the poloidal-magnetic-field energy, respectively, and (b) in the high-energy regime $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$. These solutions generalize the static magnetic-dipole equilibria with vanishing poloidal currents obtained in Ref. [15]. It turns out that in the high-energy regime all four forces, pressure-gradient, toroidal-magnetic-field, inertial and gravitating, contribute equally to the formation of magnetic surfaces very extended and localized about the symmetry plane such that the resulting equilibria resemble the accretion disks in astrophysics.

Finally, it may be noted that, in addition to their astrophysical concern, the equilibrium investigations of Refs. [15, 16, 17] and of the present work may help in developing possible novel magnetic confinement devices. In this view further studies on the impact of compressible flows or/and self gravity on the equilibrium properties of plasmas confined in dipolar magnetic fields are of particular interest.

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