Spin current swapping in the two dimensional electron gas

Ka Shen,1 R. Raimondi,2 and G. Vignale1

1Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA
2CNISM and Dipartimento di Matematica e Fisica, Università Roma Tre, Via della Vasca Navale 84, 00146 Rome, Italy

(Dated: February 5, 2015)

We analyze the effect known as “spin current swapping” due to electron-impurity scattering in a two-dimensional electron gas. In this effect a primary spin current $J_i$ (lower index for spatial direction, upper index for spin direction) generates a secondary spin current $J_j$ with $j \neq i$ if $i = a$. By employing microscopic diagrammatic calculations, as well as spin-dependent drift-diffusion equations, we show that, contrary to naive expectation, the homogeneous spin current associated with the uniform drift of the spin polarization in the presence of an electric field does not act as a source of spin current swapping. On the other hand, the inhomogeneous spin current associated with spin diffusion is a legitimate source of spin current swapping and does generate a transverse spin current. An experimental setup for the observation of the effect is therefore proposed.

PACS numbers: 72.25.-b, 71.70.Ej, 72.20.Dp, 85.75.-d

I. INTRODUCTION

The generation, manipulation and detection of spin currents are central issues in realizing spintronic devices.1,2 Recently, Lifshits and D’yakonov3 described an interesting and potentially important “spin current swapping” (SCS) effect: a primary spin current, $[J_i^x]^{(0)}$ flowing along $a$-th direction with spin polarization along $i$-th direction, generates a transverse spin current, which can be expressed in a general form as

$$[J_i^x]^{SCS} = \kappa \left( [J_i^z]^{(0)} - \delta_{i\alpha} \sum_J [J_J^l]^{(0)} \right), \quad \text{Eq. (1)}$$

with the generation efficiency parameter $\kappa = \lambda^2 k_F^2$ proportional to the square of the effective Compton wavelength $\lambda$ (which controls the strength of the spin-orbit coupling) and the square of the Fermi wave vector $k_F$. Here the lower index, $i$, denotes the spatial direction of flow of the spin current, while the upper index $a$ denotes the orientation of the spin. As shown by Lifshits and D’yakonov in Ref.3 the spin current swapping effect originates from the spin precession of the propagating electrons under the impurity-generated spin-orbit field. In a classical picture, when an electron passes near an impurity, not only its momentum changes, but also its spin undergoes a rotation around the effective magnetic field associated with the impurity potential. This effective field is normal to the plane defined by the electron momentum and the gradient of the local electric potential, and its sign depends on whether the electron passes on the left or on the right side of the impurity. The correlation between the scattering direction and the sign of the spin precession is the essence of the spin-current swapping effect, as it causes, for example, spins initially oriented in the $+y$ direction and propagating along $+x$ (primary spin current $J_x^y$) to acquire a positive $x$ spin component when they are deflected in the positive $y$ direction, and a negative $x$ component when they are de-

flected in the negative $y$ direction: this results in a secondary spin current $J_y^x$ [see Fig.1(a) and its caption].

At first sight the detection of the spin current swapping effect seems quite straightforward. Consider, for example, a setup consisting of a two dimensional electron gas confined to the $x$-$y$ plane, with a spontaneous or externally induced in-plane spin polarization described by average homogeneous spin densities $S^x$ and $S^y$. An electric field applied in the $-x$ direction will produce primary spin currents $[J_x^y]^{(0)}$ and $[J_y^y]^{(0)}$ proportional to the charge current $J_x^y$ and to the spin densities $S^x$ and $S^y$, respectively. The effective magnetic field created by the spin-orbit interaction with the impurities is perpendicular to the plane, i.e., along the $z$ direction. Then, according to Eq. (1), the spin currents generated by the spin current swapping effect are

$$[J_y^y]^{SCS} = -\kappa [J_x^y]^{(0)}, \quad \text{Eq. (2)}$$

$$[J_y^x]^{SCS} = \kappa [J_y^y]^{(0)}, \quad \text{Eq. (3)}$$

and it might seem a relatively easy matter to detect the spin accumulations associated with one or the other component of the spin current. Notice that both $[J_y^y]^{SCS}$ and $[J_y^x]^{SCS}$ are transverse with respect to the direction of flow of the primary current. However, at variance with the well-known transverse spin current induced by spin Hall effect $J_y^x$ (Refs.4,7), here only the in-plane spin components are relevant.

Unfortunately, things are not so simple. As explained above, Eq. (2) takes into account only the effect of the out-of-plane magnetic field from impurity spin-orbit coupling. The in-plane external electric field that drives the primary spin current — a plain drift current — will also contribute to the SCS, because it generates, via spin-orbit coupling, an effective magnetic field that lies exactly in the opposite direction as the impurity-induced one [see Fig.1(b)]. Therefore, the total transverse spin current will be the sum of two contributions, one from the impurities and the other from the electric field, and
These two contributions cancel each other. This issue was raised in our recent paper where we demonstrated the exact cancellation (in a homogeneous system) from spin-dependent drift-diffusion equations. The cancellation can be understood as a consequence of the force balance between external electric field and impurities at the steady state, i.e., $\langle \nabla V_{\text{imp}} \rangle = \bar{E}$, with $\langle \ldots \rangle$ representing the average over the electron density distribution.

While these findings essentially rule out the possibility of observing the SCS in the naive homogeneous setup described above, it leaves well open the possibility of observing the effect in inhomogeneous settings, such as the one described in Ref. [3] where the spin current was injected from ferromagnetic leads into a nonmagnetic conductor. In this case the spin current is a diffusion current, driven by a spin density gradient rather than by an electric field, and our arguments leading to the cancellation of the SCS for drift currents do not apply.

In the present paper, we reexamine the problem by employing microscopic diagrammatic calculations of spin dependent conductivity. We demonstrate that, consistent with our earlier findings, a primary spin current driven entirely by an electric field (drift current) indeed produces no detectable swapped spin current. Moreover, we show that the spin current swapping effect is present and should be observable when the primary spin current has a diffusive component, even in the presence of the intrinsic spin-orbit coupling.

II. DIAGRAMMATIC THEORY

We start from a homogeneous two dimensional electron gas with a finite homogeneous spin polarization along $x$ direction, which is maintained by an external magnetic field or by an internal exchange field with, say, $d$-electrons. A longitudinal electric field $E_x$ produces a drift current of charge ($J_x$) and spin ($J^z_x$). We use the standard Kubo formula to calculate the transverse spin swapping current ($J^y_x$) in the presence of spin-orbit coupling with extrinsic impurities, and show that it is zero. Our model Hamiltonian can be expressed as

$$H = \frac{\hbar^2}{2m} + V(\mathbf{r}) - (\Delta/2)\hat{\sigma}^x - \lambda^2 \hat{\sigma} \cdot \nabla V(\mathbf{r}) \cdot \hat{p}$$

with $\hat{p} = -i\nabla$ and $V(\mathbf{r})$ representing a short-range impurity potential with zero average and Gaussian distribution given by $\langle V(\mathbf{r})V(\mathbf{r}') \rangle = v_0^2 \delta(\mathbf{r} - \mathbf{r}')$. Notice that we have set $\hbar = 1$. Here $\Delta$ is the difference of the Fermi energies, $E_+$ and $E_-$ of the two spin bands with $\sigma^z = \pm 1$: $\Delta = E_+ - E_-$. Within the self-consistent Born approximation, the retarded and advanced Green’s functions have the form

$$\tilde{G}^{R/A}_{\mathbf{k}}(\epsilon) = \hat{\sigma}^0 G^{R/A}_{0\mathbf{k}}(\epsilon) + \hat{\sigma}^x G^{R/A}_{1\mathbf{k}}(\epsilon),$$

where

$$G^{R/A}_{0\mathbf{k}}(\epsilon) = \frac{1}{2}(G^{R/A}_{+\mathbf{k}} + G^{R/A}_{-\mathbf{k}}),$$

$$G^{R/A}_{1\mathbf{k}}(\epsilon) = \frac{1}{2}(G^{R/A}_{+\mathbf{k}} - G^{R/A}_{-\mathbf{k}}),$$

with $G^{R/A}_{\pm\mathbf{k}}(\epsilon) = (\epsilon - \xi_{\mathbf{k}} \pm \Delta/2 + i/2\tau)^{-1}$ and $G^{A}_{\pm\mathbf{k}} = (G^{R}_{\pm\mathbf{k}})^*$. Here, $\xi_{\mathbf{k}} = k^2/(2m) - E_F$ with the average Fermi energy $E_F = (E_+ + E_-)/2$, and $\hat{\sigma}^i$ (with $i = 0, x, y, z$) are the usual Pauli matrices. The scattering time has the standard expression $\tau^{-1} = 2\pi n_i N_0 v_0^2$, with $N_0 = m/2\pi$ and $n_i$ being the density of states and impurity concentration in two dimensions, respectively. Notice that, in using the self-consistent Born approximation, we have absorbed the $\hat{\sigma}^0$ and $\hat{\sigma}^x$ components of the real part of the Green function self-energy into the renormalization of the chemical potential and Zeeman energy $\Delta$, respectively (See Ref. [9] for details).

According to the linear response theory, the longitudinal and transverse spin currents arising from the application of an electric field along the $x$ axis are

$$J^x_x = \sigma^x_{xx} E_x,$$

and

$$J^y_x = \sigma^y_{xx} E_x,$$

where $\sigma^x_{xx}$ and $\sigma^y_{xx}$ are the longitudinal and transverse spin conductivities, respectively. The longitudinal spin conductivity is given by

$$\sigma^x_{xx} = \lim_{\omega \to 0} \frac{\langle \langle \hat{J}^x_x ; \hat{J}_x \rangle \rangle_{\omega}}{i\omega},$$

where the double bracket denotes the Kubo product $\langle \langle A ; B \rangle \rangle_{\omega} = -\frac{1}{\pi} \int_0^\infty \langle \langle \hat{A}(t) ; \hat{B}(0) \rangle \rangle e^{i\omega t} dt$. The zero-th diagram, shown in Fig. 2(a), gives

$$\sigma^x_{xx} = \frac{1}{2\pi} \sum_{\mathbf{k}} \text{Tr} \left( \hat{j}_x^z \tilde{G}^R \hat{j}_x \tilde{G}^A \right),$$

FIG. 1: (a) Impurity-induced spin current swapping for primary spin current $J^y_x$. The electric force induced by the impurity potential ($-\nabla V_{\text{imp}}$ shown as the blue arrows) gives rise to an effective magnetic field $\vec{B}_{\text{imp}} \approx \lambda^2 m \vec{E}_{\text{imp}} \times \vec{v}$, which, so as to the spin precession, is in opposite directions for the two trajectories, leading to a transverse spin current $J^y_x$. (b) The spin current swapping effect due to the external electric field. Here, the spin precession is caused by the effective magnetic field $\vec{B}_E \approx -\lambda^2 m e \vec{E} \times \vec{v}$. Observe how the “swapped” spin current $J^y_x$ produced by the electric field in (b) is opposite to the one produced by the impurities in (a).
where we have introduced charge-current and spin-current vertices as

\[ \hat{J}_x = (-e)\hat{v}_x \]

with velocity operator \( \hat{v}_x = k_x/m \). Note that \( e \) is the positive unit charge and we assign to electrons a charge \(-e\). By performing the integral over momentum, we get

\[ \sigma_{xx}^x = (-e) \frac{N_0 D}{2} \frac{2\Delta}{mv_F^2}, \]

where \( D = v_F^2 \tau/2 \) is the diffusion coefficient with \( v_F = \sqrt{2E_F/m} \) being the Fermi velocity. By noting that the difference between the squares of the Fermi momenta of the two Fermi surfaces is \( k_{F+}^2 - k_{F-}^2 = 2m\Delta \), Eq. (14) can also be written as

\[ \sigma_{xx}^x = (-e) \frac{1}{2} (N_0 D_+ - N_0 D_-) = \frac{(-e)}{4\pi} \tau \Delta, \]

with \( D_\pm = k_{F\pm}^2 \tau/(2m^2) \). One sees that the longitudinal spin conductivity is simply equal to the difference between the Drude conductivities of the two spin channels and vanishes in the absence of uniform spin polarization at \( \Delta \to 0 \).

By replacing the spin current vertex \( \hat{J}_x^x \) in Eq. (11) by \( \hat{J}_y^y \), one can calculate the transverse spin conductivity \( \sigma_{yx}^y \) from the same diagram as

\[ \sigma_{yx}^y = \frac{1}{2\pi} \sum_k \text{Tr} \left( \hat{J}_y^y \hat{G}^R J_x \hat{G}^A \right), \]

where the spin current vertex is given by

\[ \hat{J}_y = \frac{1}{2} k_y \hat{\sigma}^y. \]

Unfortunately, we find that \( \sigma_{yx}^y \) from Eq. (16) vanishes after the trace over the Pauli matrices. This forces us to go beyond the zero-th order approximation and consider the velocity correction arising from the spin-orbit coupling with impurities [diagrams in Fig. 2(b) and (c)] as well as vertex corrections [diagrams in Fig. 2(d) and (e)]. Explicit expressions for these diagrams are given in Appendix A. Specifically, the last term of our Hamiltonian in Eq. (4) gives rise to an anomalous velocity operator

\[ \delta \hat{v}_x \equiv \delta \hat{v}_{x,k,k'} = i\lambda^2 (k_y - k'_y) \hat{\sigma}^z v_0. \]

The presence of \( \hat{\sigma}^z \) together with the matrix structure of the Green function allows to get an effective vertex which behaves as \( \hat{\sigma}^y \) and then survives when traced with \( J_y^y \). The diagrams in Figs. 2(b) and (c) then give a non-vanishing contribution

\[ \sigma_{yx}^y (b + c) = (-e) \frac{N_0 D}{2} \frac{2m\lambda^2\Delta}{1 + \Delta^2 \tau^2}, \]

leading to the following ratio between transverse and longitudinal spin currents

\[ \frac{J_y^y (b + c)}{J_x^x} = \lambda^2 k_F^2. \]

at \( \Delta \tau \ll 1 \). Note that the impurity-induced correction at the spin current vertex \( J_y^y \) is irrelevant because of the vanishing anti-commutator between \( \hat{\sigma}^y \) and \( \hat{\sigma}^z \).

Further contributions arising from vertex corrections are shown in Figs. 2(d) and (e), where the impurity line connects a simple impurity potential insertion (full dot) with the spin-orbit field due to the impurity (cross). The right part of those diagrams (including the impurity line) can be seen as a correction of the charge current vertex

\[ \delta \hat{J}_{x,VC} = -2m\lambda^2 \Delta \hat{J}_y^y, \]

where the superscript “VC” stands for vertex corrections. Evaluating the diagrams of Figs. 2(d) and (e) according to the formulas given in Appendix A yields

\[ \sigma_{yx}^y (d + e) = e \frac{N_0 D}{2} \frac{2m\lambda^2\Delta}{1 + \Delta^2 \tau^2}, \]

which exactly cancels the contribution from Figs. 2(b) and (c) given by Eq. (19), revealing the vanishing of transverse spin current \( J_y^y \) hence of the entire spin current swapping. This result is perfectly consistent with our earlier findings from the drift-diffusion equations.

One may notice that the external electric field, which played a crucial role in our physical explanation of the vanishing of the SCS, does not show up explicitly in the above calculation. This is because of the particular gauge of the electric field in Kubo formula. In the Kubo formula, the spin-orbit coupling to the electric field is replaced by that due to impurity under steady-state condition, which implies, via the force balance equation, a kind of interchangeability between \( \vec{E} \) and \( \nabla \vec{V} \).
III. DRIFT-DIFFUSION EQUATIONS

In order to show the role of the electric-field-induced spin-orbit coupling in a more apparent way, we now turn to the spin-dependent drift-diffusion equations. Beyond the simple model used above, in this section, we extend our discussion into more general cases by taking into account (i) the inhomogeneity of electronic spin density and (ii) spin-orbit coupling of “intrinsic” origin, i.e., not related to the impurities. The Hamiltonian can now be written in the SU(2) form as

\[ H = \frac{\hat{p}^2}{2m} + \frac{1}{m}\hat{p}_iA'_i\hat{\sigma}^i + e\mathbf{E}\cdot\mathbf{r} + V(r) - \lambda^2\hat{\sigma}\times \nabla V(r) \cdot \hat{p}, \]  

where the SU(2) gauge field \( A'_i \) includes not only the intrinsic spin-orbit coupling, but also the one due to external electric field. For example, in a (001) two dimensional quantum well we have \( A'_x = m(\alpha + \beta) \) and \( A'_y = m(\beta - \alpha) \) with \( \alpha \) and \( \beta \) corresponding to the coefficients of Rashba and Dresselhaus spin-orbit couplings separately. In addition, the in-plane electric field gives \( A'_z = \lambda^2 emE_y \) and \( A''_y = -\lambda^2 emE_x \).

The conventional SU(2) drift-diffusion equation for the spin current (defined as \( J'_a = \{\tilde{v}_i, \hat{\sigma}^a\}/2 \)) read \[ J'_a = -[(v_i + D\nabla_i)S]^a - \theta_{SH}\epsilon^{ija}J_j, \]  

where the last term on the right-hand side describes the spin-Hall term with \( \epsilon^{ija} \) being the Levi-Civita antisymmetric tensor. Here, \( v_i = e\tau E_i/m \) represents the drift velocity due to the external electric field. The covariant derivative \( (\nabla_i O)^a = \partial_i O^a - 2\epsilon^{abc}A^b_i O^c \). However, as we noticed in Ref.[8] the spin precession due to spin-orbit coupling with impurities is not included in Eq. (24). This effect can be carried out from the collision integral

\[ J'_k(t) = -\left( \int_c dt'[\Sigma_k(t,t')G_k(t',t) - G_k(t,t')\Sigma_k(t',t')] \right) <, \]

with the second-order self-energy

\[ \Sigma_k = -i\lambda^2 \epsilon^{ija} \sum_{k'}(\hat{\sigma} \cdot \mathbf{k} \times \mathbf{k'}) \rho_{k'} G_{k'}. \]

Here, \( G_{k}(r,t,t') \) and \( \Sigma_{k}(r,t,t') \) stand for the contour-ordered Green’s function and the self-energy, respectively. The superscript “<” denotes the lesser component of the contour integral. Since the detailed technique to calculate Eq. (25) has been presented in Ref.[8] here we jump to the result

\[ I^{\text{SCS}}_k = -i\lambda^2(2\pi)^{-1} \int_{0}^{2\pi} d\theta_{k'} [\hat{\sigma} \cdot \mathbf{k} \times \mathbf{k'}] \]

where \( \theta_{k'} \) is the angle between \( \mathbf{k} \) and \( \mathbf{k'} \), and \( \rho_{k} = \sum g_{i} \hat{\sigma}^i \) is the spin-dependent density matrix at momentum \( \mathbf{k'} \). In the steady state, Eq. (27) leads to the following correction to the spin-dependent density matrix:

\[ \delta g^i_{k} = (2\lambda^2 m/N_0) \sum_{lmn} \epsilon^{ijl} \epsilon^{lmn} k_{m} J^l_{n}, \]

where the spin currents on the right-hand side are the “unperturbed” ones: \( J^l_{n} = \sum_{k} (k_{l}/m)g_{k}. \) Then the additional contribution in spin current due to \( \delta g^i_{k} \) can be evaluated via

\[ [J^i_{k} ]^{\text{SCS}} \simeq \sum_{k}(k_{l}/m)\delta g^i_{k} = \kappa \left( J^i_{k} - \delta_{ij} J^j_{k} \right), \]

which describes the role of the electric-field-induced “side-jump” contribution to the spin Hall conductivity, where the kinetic equation approach requires the inclusion of the spin-orbit coupling due to the electric field while the Kubo formula does not.\[ ^3\text{[15]} \]

One notices that \( J^x \) and \( J^y \) are coupled with \( J^z \) separately, while the spin Hall term does not contribute to the expressions for these two components of the spin current:

\[ J^x = -(v_x + D\partial_x)S^x + 2D\epsilon^{xbc}A^b_x S^c - \kappa J^y, \]

\[ J^y = -(v_y + D\partial_y)S^y + 2D\epsilon^{abc}A^b_y S^c - \kappa J^x. \]

The first two terms on the right-hand side in each equation can be recognized as primary spin currents. Naturally, we can define the drift part of the spin currents as a product of the drift velocity and spin density, i.e.,

\[ (J^x)^{\text{drift}} = -v_x S^x, \]

\[ (J^y)^{\text{drift}} = -v_y S^y. \]

The other part resulting from the diffusion effect can be written as

\[ (J^x)^{\text{diff}} = -D\partial_x S^x + 2Dm(\alpha + \beta) S^x, \]

\[ (J^y)^{\text{diff}} = -D\partial_y S^y + 2Dm(\alpha - \beta) S^y. \]

One can see that in addition to the spatial inhomogeneity of the in-plane spin polarization, the out-of-plane spin polarization also contributes to the spin currents. This contribution comes from the spin precession under the intrinsic spin-orbit effective magnetic field. Then, Eqs. (31) and (32) can be rewritten as

\[ J^x = (J^x)^{\text{drift}} + (J^x)^{\text{diff}} + \kappa(J^y)^{\text{drift}} - \kappa J^y, \]

\[ J^y = (J^y)^{\text{drift}} + (J^y)^{\text{diff}} + \kappa(J^x)^{\text{drift}} - \kappa J^x. \]

The third term on the right-hand side in each equation is obtained by substituting the vector potential \( A^b_x, y \) into Eqs. (31) and (32), i.e., by taking into account the spin-orbit coupling due to the electric field. The appearance
of this term reduces the efficiency of the spin current swapping effect. Indeed, the equations show clearly that only the diffusion part of the primary spin current is a source of SCS. By solving these equations, we obtain

\[
J_x^y = \left( J_x^y \right)^{\text{drift}} + \frac{1}{(1 - \kappa^2)} \left[ (J_x^y)^{\text{diff}} - \kappa (J_y^x)^{\text{diff}} \right],
\]

from which we see that (i) the drift component of the primary spin current \(J_x^y\) does not generate SCS; (ii) a transverse spin current, \(J_y^y\), is generated from the diffusive component of \(J_x^y\) via SCS. The final expressions for the remaining spin currents, \(J_x^x\) and \(J_y^y\), can be obtained by simply replacing \(J_x^x\), \(J_y^y\) and \(\kappa\) by \(J_y^x\), \(J_x^y\) and \((-\kappa)\), respectively.

The spin current swapping effect should be observable in an experiment such as the one described in Fig. 3. The idea is to inject a pure spin current \(J_y^y\) from a ferromagnetic contact into the longitudinal (x) arm of a cross-shaped device. Spin current swapping then injects a spin current \(J_x^y\) into the transverse (y) arm of the cross resulting in opposite spin accumulations at the ends of the transverse arm. These spin accumulations could in principle be detected by Faraday rotation spectroscopy (if the cross is made of a semiconductor material) or by inverse spin Hall effect (for metals). In addition to the spin swapping current there is also a spin current \(J_y^y\) flowing along the cross arm, originating from the diffusion of y-oriented spins from the center of the cross into the transverse arm. This diffusion current produces equal spin accumulations on the two ends of the transverse arm and therefore does not contribute to the asymmetry. We also notice that the inverse spin Hall effect associated with the primary spin current does not generate a potential difference between the ends of the transverse arm.

IV. ACKNOWLEDGEMENTS

We gratefully acknowledge support from NSF Grant No. DMR-1406568.

Appendix A: EXPLICIT EXPRESSIONS OF TRANSVERSE SPIN CONDUCTIVITY

Up to the first order in \(\lambda^2\), the transverse spin conductivity \(\sigma_{yx}^y\), from side-jump-like diagrams, corresponding to Figs. 2(b) and (e), is given by

\[
\sigma_{yx}^y (b + c) = -i \lambda^2 n_i v_0^2 \frac{1}{2\pi} \sum_{kk'} \frac{k_y k_y'}{2m} (k_y - k_y') \times \text{Tr}[\sigma^y G_k^R (G_k^R \sigma_z - \sigma_z G_k^A) G_k^A], (A1)
\]

The impurity-vertex-correction diagrams, i.e., Figs. 2(d) and (e), lead to

\[
\sigma_{yx}^y (d + e) = -i \lambda^2 n_i v_0^2 \frac{1}{2\pi} \sum_{kk'} \frac{k_y k_y'}{2m} (k_y - k_y') \times \text{Tr}[\sigma^y G_k^R [\sigma_z, G_k^R \sigma_z, G_k^A [\sigma^z, G_k^A]]]. (A2)
\]
8 K. Shen, R. Raimondi, and G. Vignale, Phys. Rev. B 90, 245302 (2014).
9 P. Schwab and R. Raimondi, Eur. Phys. J. B 25, 483 (2002).
10 E. M. Hankiewicz and G. Vignale, J. Phys.: Condens. Matter 21, 253202 (2009).
11 D. Culcer, E. M. Hankiewicz, G. Vignale, and R. Winkler, Phys. Rev. B 81, 125332 (2010).
12 X. Bi, P. He, E. M. Hankiewicz, R. Winkler, G. Vignale, and D. Culcer, Phys. Rev. B 88, 035316 (2013).
13 W.-K. Tse and S. Das Sarma, Phys. Rev. Lett. 96, 056601 (2006).
14 W.-K. Tse and S. Das Sarma, Phys. Rev. B 74, 245309 (2006).
15 E. M. Hankiewicz, G. Vignale, and M. E. Flatté, Phys. Rev. Lett. 97, 266601 (2006).
16 C. Gorini, P. Schwab, R. Raimondi, and A. L. Shelankov, Phys. Rev. B 82, 195316 (2010).
17 R. Raimondi, P. Schwab, C. Gorini, and G. Vignale, Ann. Phys. 524, 153 (2012).