THE NP-HARD PROBLEM OF COMPUTING THE MAXIMAL SAMPLE VARIANCE OVER INTERVAL DATA IS SOLVABLE IN ALMOST LINEAR TIME WITH HIGH PROBABILITY

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Abstract. We consider the algorithm by Ferson et al. (Reliable computing 11(3), p. 207–233, 2005) designed for solving the NP-hard problem of computing the maximal sample variance over interval data, motivated by robust statistics (in fact, the formulation can be written as a nonconvex quadratic program with a specific structure). First, we propose a new version of the algorithm improving its original time bound $O(n^22^\omega)$ to $O(n \log n + n \cdot 2^\omega)$, where $n$ is number of input data and $\omega$ is the clique number in a certain intersection graph. Then we treat input data as random variables (as it is usual in statistics) and introduce a natural probabilistic data generating model. We get $2^\omega = O(n^{1/\log \log n})$ and $\omega = O(\log n / \log \log n)$ on average. This results in average computing time $O(n^{1+\epsilon})$ for $\epsilon > 0$ arbitrarily small, which may be considered as “surprisingly good” average time complexity for solving an NP-hard problem. Moreover, we prove the following tail bound on the distribution of computation time: hard instances, forcing the algorithm to compute in time $2^{\Omega(n)}$, occur rarely, with probability tending to zero at the rate $e^{-n \log \log n}$.

Keywords. Nonconvex quadratic programming, average complexity, tail probability

Subject classification. Quadratic programming, Geometric probability and stochastic geometry
1. Introduction and motivation

1.1. Problem formulation. Ferson et al. (2005) studied the pair of optimization problems

\[
\begin{align*}
\text{(1.1)} & & \min_{x \in \mathbb{R}^n} V(x) & \text{ s.t. } \underline{x} \leq x \leq \overline{x}, \\
\text{(1.2)} & & \max_{x \in \mathbb{R}^n} V(x) & \text{ s.t. } \underline{x} \leq x \leq \overline{x},
\end{align*}
\]

where and \( \underline{x} \leq \overline{x} \in \mathbb{Q}^n \) are given input data and

\[
V(x) := \frac{1}{n} \sum_{i=1}^{n} \left( x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right)^2.
\]

It is obvious that (1.1) is a convex quadratic program (CQP) solvable in polynomial time, while (1.2) is easily proven to be NP-hard. It is worth noting that a general CQP solver yields a weakly polynomial algorithm for (1.1), but Ferson et al. (2005) introduced a strongly polynomial method.

They also introduced a method for solving (1.2) which works in exponential time in the worst case (not surprisingly). The method will be described in Section 2. Abbreviating the names of all authors (Ferson, Ginzburg, Kreinovich, Longpré and Aviles), we will refer to their method as FGKLA algorithm.

1.2. Summary of results. In this text we focus on the NP-hard case (1.2) and the FGKLA algorithm. Our contribution is twofold.

Improving the worst-case complexity of the FGKLA algorithm. Firstly, we show that there exists an implementation of the FGKLA algorithm working in time

\[
O(n \log n + n \cdot 2^\omega),
\]

where \( \omega \) is the size of the largest clique in a certain intersection graph. The graph will be introduced in Definition 2.2. This improves the bound \( O(n^2 \cdot 2^\omega) \) from the original paper. For further discussion see Remark 2.7.
Proving a “good” behavior in a probabilistic setting. Secondly, we treat the input data \( \vec{x}, \bar{x} \) as random variables. We introduce a natural and fairly general probabilistic model (details are in Section 3), under which we show that

(i) on average, the algorithm works in time

\[
O(n^{1+\epsilon}) \quad \text{for all } \epsilon > 0,
\]

which is surprisingly good considering the problem is NP-hard,

(ii) the probability that the algorithm computes in time \( 2^{\Omega(n)} \) tends to zero faster than exponentially with \( n \to \infty \). In other words, we show that “hard” instances occur indeed rarely.

More specifically: (i) we prove that under the probabilistic model it holds

\[
E2^\omega = O(n^{\frac{1}{\log \log n}}),
\]

where \( E[\cdot] \) stands for the expected value of \( [\cdot] \). Combination of (1.3) with (1.5) yields (1.4) as \( n \log n = O(n^{1+\epsilon}) \) and \( n^{\frac{1}{\log \log n}} = O(n^\epsilon) \) for any \( \epsilon > 0 \). In the entire text, “log” stands for the natural logarithm.

To achieve (ii) from (1.3) it follows that the computing time is exponential when \( \omega \geq \delta n \) with \( \delta > 0 \). We prove that

\[
\Pr[\omega \geq \delta n] \leq e^{-n \log \log n}
\]

for every \( \delta > 0 \) and a sufficiently large \( n \).

1.3. Motivation from statistics. Problems (1.1) and (1.2) are studied in statistics; see e.g. Antoch et al. (2010) and references therein, and a pseudopolynomial method in Černý & Hladík (2014). The statistical motivation is as follows: we are interested in sample variance \( V(x) \) of a dataset \( x = (x_1, \ldots, x_n)^T \). However, the data \( x \) is not observable. What is available instead is a collection of intervals \( \vec{x}_i := [\underline{x}_i, \bar{x}_i], i = 1, \ldots, n \), such that \( \underline{x}_i \leq x_i \leq \bar{x}_i \) (for example, instead of the exact values \( x \) we have rounded versions only). Then, \( V(x) \) cannot be computed exactly, but we can get
tight bounds for $V(x)$ in the form (1.1) and (1.2). In econometrics, this phenomenon is sometimes called \textit{partial identification}, see Manski (2003).

The problem is more general and is studied for various statistics in place of $V(x)$ in (1.1) and (1.2) see the reference books by Kreinovich \textit{et al.} (1998) and Nguyen \textit{et al.} (2012).

1.4. Related work. In general, this paper contributes to the analysis of complexity of optimization problems and algorithms when input data can be assumed to be random, drawn from a particular distribution or a class of distributions. As a prominent example recall the famous average-time analysis of the Simplex Algorithm by Borgwardt (1982) and Spielman & Teng (2004), where the phenomenon “exponential in the worst case but fast on average” has been studied since 1980’s.

The phenomenon is particularly interesting for NP-hard problems since the worst-case exponential time seems to be unavoidable. In the area of quadratic optimization, the simplex-constrained case has been studied by several authors (see e.g. Bomze \textit{et al.} (2017) and references therein). It turns out that a quadratic form with entries randomly generated from “natural” distributions attains, with a high probability, its global maximum in a face of a small dimension. This property implies that the problem can be solved efficiently by enumerating faces.

Another nice example is the analysis of the average-case complexity of an NP-hard variant of the open shop scheduling problem by Lu & Posner (1993). Their setup is similar to ours: they assume that input data (the job processing times) are generated from a certain class of probabilistic distributions and prove that the average complexity is polynomial in the number of jobs.

Finally, we mention the average-case complexity analysis of the NP-hard $k$-CLIQUE problem. Rossman (2014) derived bounds on average-case complexity on monotone circuits. Fountoulakis \textit{et al.} (2015) then extended the analysis into a probabilistic setting, showing whether the “hard” instances occur frequently or rarely. Their results are, in a sense, analogous to ours: if the edges are sampled from a “natural” distribution, then $k$-CLIQUE can be solved in polynomial time with probability tending to one faster than
polynomially.

Aside from the above mentioned general average-case results, the *weak average-case complexity* paradigm by Amelunxen & Lotz (2017) is worth mentioning; here, the word “weak” refers to omitting a “small” subset of hard instances from the set of all possible instances and performing the average-case complexity analysis on such a reduced class of instances. It turns out that for many interesting problems the removal of a subset with exponentially small measure is sufficient to making hard problems tractable.

2. FGKLA Algorithm

Recall that the input instance is given by the pair \( \mathbf{x} = (x_1, \ldots, x_n)^T \) and \( \overline{\mathbf{x}} = (x_1, \ldots, x_n)^T \). Compact intervals will be denoted in boldface, e.g. \( \mathbf{x}_i = [\overline{x}_i, \underline{x}_i] \). For \( i = 1, \ldots, n \) define

\[
\begin{align*}
x_i^* &:= \frac{1}{2}(x_i + \overline{x}_i), \quad x_i^\Delta := \frac{1}{2}(\overline{x}_i - x_i), \\
x_i^{1/n} &:= [x_i^* - \frac{1}{n} x_i^\Delta, x_i^* + \frac{1}{n} x_i^\Delta].
\end{align*}
\]

The numbers \( x_i^*, x_i^\Delta \) are referred to as *center* and *radius* of \( \mathbf{x}_i \), respectively, and \( \mathbf{x}_i^{1/n} \) is called a *narrowed interval* (i.e., \( \mathbf{x}_i \) shrunk by factor \( n \) around its center). For \( x \in \mathbb{R}^n \) we define \( \mu[x] := \frac{1}{n} \sum_{i=1}^{n} x_i \) (the *mean* of \( x \)).

Our version of the FGKLA algorithm is summarized as Algorithm 1. The main result of this section is Theorem 2.1. In particular, it improves the worst-case complexity bound \( O(n^2 \cdot 2^\omega) \) from Ferson et al. (2005) (see also Remark 2.7). The proof of Theorem 2.1 will be given in Section 2.1.

**Theorem 2.1** (properties of the FGKLA algorithm (Algorithm 1)).

(a) The FGKLA algorithm correctly solves (1.2).

(b) Let \( G = (V = \{1, \ldots, n\}, E) \) be an undirected graph where \( \{i, j\} \in E \) if and only if \( \mathbf{x}_i^{1/n} \cap \mathbf{x}_j^{1/n} \neq \emptyset \) (here, \( i \neq j \)). Let \( \omega \) be the size of the largest clique in \( G \). Then, FGKLA algorithm works in time \( O(n \log n + n \cdot 2^\omega) \).
**Definition 2.2.** The graph $G$ from Theorem 2.1 is referred to as FGKLA intersection graph with data $x_1, \ldots, x_n$.

2.1. Idea of the FGKLA algorithm. Since the quadratic form $V(x)$ is positive semidefinite, the maximum of (1.2) is attained in a vertex (an extremal point) of the feasible set

$$x = \{x \mid \underline{x} \leq x \leq \overline{x}\}.$$

There are $2^n$ vertices in total. In a vertex $x$ we have $x_i \in \{x_i = x_i^* - x_i^\Delta, \overline{x}_i = x_i^* + x_i^\Delta\}$ for every $i$. FGKLA algorithm reduces the number of vertices to be examined from $2^n$ to $O(n \cdot 2^\omega)$. The reduction is based on Lemma 2.3. A similar lemma was used in the original paper Ferson et al. (2005).

**Lemma 2.3.** Let $x, x' \in \mathbf{x}$ and let there exist $i \in \{1, \ldots, n\}$ such that

(i) $x_j = x'_j$ for all $j \neq i$ and

(ii) one of the following is satisfied:

(a) $x_i = \overline{x}_i$, $\mu[x] > x_i^* + \frac{1}{n} x_i^\Delta$ and $x_i' = \underline{x}_i$,

(b) $x_i = \underline{x}_i$, $\mu[x] < x_i^* - \frac{1}{n} x_i^\Delta$ and $x_i' = \overline{x}_i$.

Then $V(x) < V(x')$.

**Proof.** We prove the claim for Assumption (ii.a), i.e. $x_i = \overline{x}_i$, $\mu[x] > x_i^* + \frac{1}{n} x_i^\Delta$, $x_i' = \underline{x}_i$. The proof is analogous for (ii.b).

Let $J = \{1, \ldots, n\} \setminus \{i\}$. We want to prove $V(x) - V(x') < 0$. 

We have
\[
V(x) - V(x') = \frac{1}{n} \left( \bar{x}_i^2 + \sum_{j \in J} x_j^2 - \bar{x}_i^2 - \sum_{j \in J} x_j^2 
- \frac{1}{n} \left( (\bar{x}_i + \sum_{j \in J} x_j)^2 - (\bar{x}_i + \sum_{j \in J} x_j)^2 \right) \right)
= \frac{1}{n} \left( \bar{x}_i^2 - \bar{x}_i^2 - \frac{1}{n} \left( \bar{x}_i^2 - \bar{x}_i^2 + 2(\bar{x}_i - x_i) \sum_{j \in J} x_j \right) \right)
= \frac{1}{n} \left( 4x_i^* \bar{x}_i - \frac{1}{n} (4x_i^* \bar{x}_i) - \left( 4x_i^* \sum_{j \in J} x_j \right) \right)
= \frac{4}{n} x_i^* \left( x_i^* - \frac{1}{n} x_i^* - \frac{1}{n} \left( -\bar{x}_i + \sum_{j \in \{1, \ldots, n\}} x_j \right) \right)
< \frac{4}{n} x_i^* \left( x_i^* - \frac{1}{n} x_i^* + \frac{1}{n} x_i^* + \frac{1}{n} x_i^* - x_i^* - \frac{1}{n} x_i^* \right) = 0. \quad \square
\]

**Corollary 2.4.** Let \(x' \in \mathbf{x}\) be a maximizer and \(\mu' = \mu[x']\). Let \(X\) be the set of all vectors \(x \in \mathbf{x}\) satisfying:

(a) \(x_i = \underline{x}_i\) if \(\mu' > x_i^* + \frac{1}{n} x_i^\Delta\),

(b) \(x_i = \bar{x}_i\) if \(\mu' < x_i^* - \frac{1}{n} x_i^\Delta\), and

(c) \(x_i \in \{\underline{x}_i, \bar{x}_i\}\) if \(\mu' \in x_i^{1/n}\).

Then \(X\) contains a maximizer.

In cases (a) and (b) we say that variable \(x_i\) (or index \(i\)) is *determinable* with respect to \(\mu'\); in case (c), variable \(x_i\) (or index \(i\)) is *free* with respect to \(\mu'\).

Algorithm 1 works as follows. It builds the set \(A\) (Line 1) containing all endpoints \(x_i^* \pm \frac{1}{n} x_i^\Delta\) of the narrowed intervals \(x_i^{1/n}\), \(i = 1, \ldots, n\) (here, \(A\) acts as a set rather than a list, meaning that possible duplicities are removed), sorts them and denotes them by \(a_1 < \cdots < a_m\) (Line 2).
Algorithm 1 FGKLA algorithm

Input: $\mathbf{x} = x^* - x^\Delta \in \mathbb{Q}^n$, $\mathbf{F} = x^* + x^\Delta \in \mathbb{Q}^n$ s.t. $\mathbf{x} \leq \mathbf{F}$
1: $A := \{x_i^s + \frac{s_i}{n} x_i^\Delta \in [\mu[\mathbf{x}], \mu[\mathbf{F}]] \ | \ s_i \in \{\pm 1\}, \ i = 1, \ldots, n\};$
   \quad \quad m := |A|
2: sort $A$ and denote its elements by $a_1 < \cdots < a_m$
3: for $k \in \{1, \ldots, m\}$ do $B_{ak} := \emptyset$; $E_{ak} := \emptyset$
4: for $i \in \{1, \ldots, n\}$ do add $i$ to both $B_{x_i^s - \frac{1}{n} x_i^\Delta}$ and $E_{x_i^s + \frac{1}{n} x_i^\Delta}$
5: $V_1 := \mu[\mathbf{x}_1^2, \ldots, \mathbf{x}_n^2]}; V_2 := \mu[\mathbf{F}]; M := V_1 - V_2; L := \emptyset$
6: for $k \in \{1, \ldots, m\}$ do
7: \quad for $i \in B_{ak}$ do $L := L \cup \{i\}$
8: \quad examine all $2^{|L|}$ vertices with Algorithm 2
9: \quad for $i \in E_{ak}$ do $L := L \setminus \{i\}; V_1 := V_1 + \frac{1}{n}(\mathbf{x}_i)^2 - \frac{1}{n}(\mathbf{F}_i)^2$;
   \quad \quad $V_2 := V_2 - \frac{2}{n}x_i^\Delta$
10: end for
11: return $M$

Consider the set $\{\mu[\mathbf{x}] \ | \ \mathbf{x} \in \mathbf{F} = [\mu[\mathbf{x}], \mu[\mathbf{F}]]$ of all possible means. The endpoints from $A$ divide the interval $[\mu[\mathbf{x}], \mu[\mathbf{F}]]$ into at most $2n + 1$ regions
\[
[a_0 := \mu[\mathbf{x}], a_1), (a_1, a_2), \ldots, (a_{m-1}, a_m), (a_m, a_{m+1} := \mu[\mathbf{F}]].
\]
Thanks to Lemma 2.3 and Corollary 2.4, every region $(a_k, a_{k+1})$ contains means $\mu$ with the same set of free indices. For a region $(a_k, a_{k+1})$, we denote this set by $I(a_k, a_{k+1})$, i.e. $I(a_k, a_{k+1}) := \{i \in \{1, \ldots, n\} \ | \ x_i^{1/n} \cap (a_k, a_{k+1}) \neq \emptyset\}$. The set $A$ of endpoints contains the worst possible mean values with respect to the number of free indices. More precisely: for every $a_k$, $k = 1, \ldots, m$, all indices from $I(a_k-1, a_k) \cup I(a_k, a_{k+1})$ are free.

On Line 5 we examine the vertex $\mathbf{F}$. The value of $V(\mathbf{F})$ is computed and stored as $M$, the maximal value of $V$ found so far. Variables $V_1$ and $V_2$ will be useful in Line 2.

Then, Algorithm 1 takes means $a_1, \ldots, a_m$ one by one. For every mean, say $a_k$, it takes the set $B_{ak} = \{i \ | \ x_i^s - \frac{1}{n} x_i^\Delta = a_k\}$ of indices of narrowed intervals beginning in $a_k$ and inserts it to the set $L$ of free indices with respect to $a_k$ (Line 7). Indices $\{1, \ldots, n\} \setminus L$ are determinable with respect to $a_k$. This yields $2^{|L|}$ candidate vertices that are examined by Algorithm 2 called on Line 8.
Algorithm 2 Examining vertices corresponding to free indices in $L$

Input: list $L$ of free indices (variables $V_1, V_2$ are global)

1: $z := (0, \ldots , 0) \in \{0, 1\}^{|L|}; s := (1, \ldots , 1) \in \{\pm 1\}^{|L|}; c := 0$
2: while $c < 2^{|L|}$ do
3: for $i \in \{1, \ldots , |L|\}$ do
4: if $z_i = 0$ then goto Line 7
5: $z_i := 0$
6: end for
7: $z_i := 1; s_i := -s_i$ \hspace{1cm} (i is the value with which for cycle ends)
8: $V_1 := V_1 + \frac{1}{n}(x^*_L + s_i x^\Delta_L)^2 - \frac{1}{n}(x^*_L - s_i x^\Delta_L)^2; V_2 := V_2 + \frac{2}{n}s_i x^\Delta_L$
9: if $V_1 - V_2^2 > M$ then $M := V_1 - V_2^2$
10: $c := c + 1$
11: end while

Then, indices from the set $E_{a_k} = \{i \mid x^*_i + \frac{1}{n} x^\Delta_i = a_k\}$ of narrowed intervals ending in $a_k$ are removed from $L$. Intervals with these indices will be fixed to the lower endpoint for every upcoming $k' > k$ (Line 9 of Algorithm 1). The update of $V_1$ and $V_2$ will be explained later.

Algorithm 2 consecutively traverses all $2^{|L|}$ vertices of $x$ resulting from fixing either $x_i = x_i^*$ or $x_i = x_i^\Delta$ for the free indices $i \in L$. For every such vertex, say $x$, the variance $V(x)$ is computed. To make these computations cheap, the traversal of $L$ is performed in a way that two successive vertices $x, x'$ differ in just one component. Then Lemma 2.5 shows how to get $V(x')$ from $V(x)$ with $O(1)$ arithmetic operations. The variance is stored indirectly as variables $V_1$ and $V_2$; they can be easily updated when $x_i$ is switched to $x_i^*$, or vice versa.

**Lemma 2.5.** For $x \in \mathbb{R}^n$, we have $V(x) = V_1 - V_2^2$, where $V_1 = \mu[(x_1^2, \ldots , x_n^2)]$ and $V_2 = \mu[x]$. Furthermore, if $x'$ differs from $x$ in just one component, say $i$th, then

$$V(x') = V_1 + \frac{1}{n}((x')^2_i - x_i^2) - (V_2 + \frac{1}{n}(x'_i - x_i))^2.$$ 

Algorithm 2 is an adaptation of the algorithm from [Rohn (2006), 2006].
The proof of correctness can be found therein. In our variant, the variable $s \in \{\pm 1\}^{|L|}$ indicates the current vertex. In every iteration of while cycle, some $s_i$ is set to $-s_i$. The $i$th index $L_i$ is taken from $L$ (here we consider $L$ as a list rather than a set) and $x_{L_i}$ is switched to the other endpoint. For this new vertex, $V_1$ and $V_2$ are updated (Line 8) and the resulting variance $V$ is compared to the best value found so far (Line 9).

The following property of Algorithm 2 is crucial for the correctness and complexity of FGKLA algorithm: When Algorithm 2 ends, then $s = (1, \ldots, 1)$. The next proposition immediately follows.

**Lemma 2.6.** Let $V_1^*, V_2^*$ be the values of the global variables $V_1, V_2$ when Algorithm 2 starts and let $V_1^{**}, V_2^{**}$ be the values of $V_1, V_2$ when Algorithm 2 ends. Then $V_1^* = V_1^{**}$ and $V_2^* = V_2^{**}$.

In particular, this means that when entering Line 8 of Algorithm 1 we always start examining the free indices with $x_i \in \bar{x}_i$ for each $i \in L$.

Finally, Line 9 of Algorithm 1 removes intervals ending in $a_k$ from $L$. These intervals are going to be fixed to their lower endpoints in the following iterations. Since they are at the upper endpoint now, Line 9 updates $V_1$ and $V_2$ accordingly.

**Proof.** Proof of Theorem 2.1

a) **Correctness.** Let $x \in \mathbb{R}^n$ be a maximizer of (1.2). Since the maximum is attained in a vertex of the feasible set $\mathbf{x}$, we can assume $x_i \in \lbrace \underline{x}_i, \bar{x}_i \rbrace$ for all $i$. Moreover, thanks to Corollary 2.4, we can assume $x_i = \underline{x}_i$ for every $i$ such that $\mu[x] < x_i^* - \frac{1}{n} x_i^\Delta$ and $x_i = \bar{x}_i$ for every $i$ such that $\mu[x] > x_i^* + \frac{1}{n} x_i^\Delta$. Put all other indices to set $L'$, i.e. $L' = \lbrace i \in \lbrace 1, \ldots, n \rbrace \mid \mu[x] \in x_i^{1/n} \rbrace$. Set $k = \arg \max_{k \in \lbrace 1, \ldots, m \rbrace} \mu[x] - a_k$. Consider the set $L$ processed by Algorithm 2 in $k$th iteration of Algorithm 1. By construction, $L' \subseteq L$. Hence, the maximizer $x$ is among the examined vertices.
b) **Complexity.** On Line 2, the algorithm sorts 2n numbers with complexity $O(n \log n)$. Algorithm 2 is called at most $m$ times, where $m \leq 2n = O(n)$. Recall that $\omega$ is the size of the maximal clique of the FGKLA intersection graph. In the $k$th iteration of the for cycle on Lines 6 to 10 of Algorithm 1 we have $|L| = |\{i \mid a_k \in x_i^{1/n}\}|$. Thus $|L| \leq \omega$.

Algorithm 2 performs exactly $2^{|L|}$ iterations of the while cycle on Lines 2 to 11. Inside its iteration, there is the for cycle on Lines 3 to 6. The amortized time complexity of this for cycle is $O(1)$, because in its iteration it either sets some nonzero $z_i$ to 0 or stops iterating. Since $z_i$ is set to a nonzero value only $2^{|L|}$ times, the overall time of all courses of the for cycle is $O(2^{|L|})$.

Computing time in the remaining steps is negligible. In particular, note that since $B_{a_1}, \ldots, B_{a_m}$ are pairwise disjoint sets (the same holds true for $E_{a_1}, \ldots, E_{a_k}$), the total number of iterations of for cycles on Lines 7 to 9 is at most $n$ during the whole course of FGKLA algorithm.

The overall complexity is $O(n \log n + n \cdot 2^\omega)$. □

**Remark 2.7.** Aside of the implementation details (which are important for the reduced time complexity bound), our formulation of the algorithm differs from the original paper Ferson et al. (2005) also for another reason. The original formulation can lead to complexity $O(n^2 \cdot 4^\omega)$, for example if $\omega = \ell$ and if there are $\ell$ narrowed intervals ending in some $a_k$ and further $\ell$ narrowed intervals starting in $a_{k+1}$. However, a minor modification of the original formulation would be sufficient to achieve the time $O(n^2 \cdot 2^\omega)$.

### 3. A probabilistic model

This section is devoted to the main probabilistic result: on average, FGKLA algorithm works in “almost” linear time.

Here we use the statistical motivation of the problem as described in Section 1.3. Namely, in statistics, data are often assumed to form a random sample from a certain distribution. This
is exactly our probabilistic model: we assume that both centers of the intervals and their radii form two independent random samples from fairly general classes of distributions.

**Assumption 3.1** (the probabilistic model).

(A) The centers $x^*_1, \ldots, x^*_n$ are independent and identically distributed ("i.i.d.") random variables with a Lipschitz continuous cumulative distribution function ("c.d.f.") $\Phi^*(z)$. That is, there exists a constant $L > 0$ such that

$$
\Phi^*(\tilde{z}) - \Phi^*(z) \leq L(\tilde{z} - z) \quad \text{whenever} \quad \tilde{z} > z.
$$

(B) The radii $x^\Delta_1, \ldots, x^\Delta_n$ are i.i.d. nonnegative random variables with a finite moment of order $1 + \varepsilon$ for some $\varepsilon > 0$. In other words, we assume

$$
(3.2) \quad \gamma := E[(x^\Delta_i)^{1+\varepsilon}] < \infty.
$$

(C) The random variables $x^*_i, x^\Delta_i$ are independent.

**Theorem 3.3.** Denote by $\omega$ the size of the largest clique of the FGKLA intersection graph with data $[\underline{x}_i := x^*_i - x^\Delta_i, \overline{x}_i := x^*_i + x^\Delta_i]_{i=1,\ldots,n}$. If $n$ is sufficiently large, then

(a) $E2^\omega \leq 1 + n \frac{1}{\log \log n}$ and $E\omega \leq \frac{3}{2} \left(1 + \frac{\log n}{\log \log n}\right)$,

(b) $\Pr[\omega \geq \delta n] \leq e^{-n \log \log n}$ for any $\delta > 0$.

**Remark 3.4.** Proof of **Theorem 3.3** will be given in **Section 3.1**. Statement (b) should be understood more precisely as follows: for every $\delta > 0$ there exists $n_\delta$ such that $\Pr[\omega \geq \delta n] \leq e^{-n \log \log n}$ if $n \geq n_\delta$. 
Corollary 3.5 (main result). The average computing time is

\[ O(\mathbb{E}[n \log n + n \cdot 2^\omega]) = O(n \log n + n \cdot 2^\omega) = O(n \log n + n \cdot n^{\log \log n}) = O(n^{1+\epsilon}) \]

for an arbitrarily small \( \epsilon > 0 \). Moreover, the computing time is \( 2^{\Omega(n)} \) when \( \omega \) is linear in \( n \) and this event occurs with probability as small as \( O(e^{-n \log \log n}) \).

Remark 3.6. Assumption 3.1(B) on the distribution of radii is very mild; indeed, we need just something a little more than existence of the expectation (we even do not need finite variance). On the other hand, Lipschitz continuity of \( \Phi^* \) (Assumption 3.1(A)) is unavoidable; we will show what can happen without Lipschitz continuity in Section 4. We will also discuss there what happens when we relax the independence assumption (Assumption 3.1(C)) and what is the cost for dependence paid by existence of higher-order moments.

3.1. Proof of Theorem 3.3

Notation. For a random variable \( X \), its probability density function ("p.d.f.") is denoted by \( \varphi_X \). Denote by \( q_j \) the \( j \)th \( n \)-quantile of the distribution of centers: i.e., let \( q_1, \ldots, q_{n-1} \) satisfy

\[ \Phi^*(q_j) = \frac{j}{n}, \quad j = 1, \ldots, n-1. \]

Let \( I_1, I_2, \ldots, I_{n-1}, I_n \) stand for the intervals \( (-\infty, q_1], [q_1, q_2], \ldots, [q_{n-2}, q_{n-1}], [q_{n-1}, \infty) \). Consider the probabilities

\[ p_j^n := \Pr[x_i^{1/n} \cap I_j \neq \emptyset]; \]

observe that \( p_j^n \) does not depend on \( i \) by the i.i.d. assumptions. Probabilities \( p_1^n, \ldots, p_n^n \) may differ due to the different shape of \( \varphi_{x_i^*} \) around \( q_1, \ldots, q_{n-1} \). In the upcoming lines, we utilize the fact

\[ \varphi_{x_i^*}(z) \leq L \quad \forall z \]

implied by Lipschitz continuity of \( \Phi^*(z) \) to derive the following upper bound:
**Lemma 3.8.** For every $j = 1, \ldots, n$ we have $p_j^n \leq \frac{\alpha}{n}$, where

$$\alpha := 1 + 2L \left( 1 + \frac{\gamma}{\varepsilon} \right).$$

**Remark 3.10.** The value of $\alpha$ depends on the properties of distributions of centers and radii. The “hard” cases are those with $\alpha \gg 1$. Indeed, the difficult case is when $\varepsilon$ is close to zero (“radii can be large with a high probability”), $\gamma \gg 0$ (“radii are large on average”) and $L \gg 0$ (“the density of centers can have high peaks”, or “many centers can be close to one another”).

**Proof (of lemma Lemma 3.8).** Let $n$ and $i$ be fixed; we omit the index $n$ for brevity. For $j = 2, \ldots, n - 1$, we can decompose $p_j^n \equiv p_j$ as

$$p_j = \Pr[\mathbf{x}_i^{1/n} \cap I_j \neq \emptyset] = \Pr[x_i^* \in I_j] + \underbrace{\Pr[x_i^* + \frac{1}{n}x_i^\Delta \geq q_{j-1} \land x_i^* < q_{j-1}]}_{= p_j^-} + \underbrace{\Pr[x_i^* - \frac{1}{n}x_i^\Delta \leq q_j \land x_i^* > q_j]}_{= p_j^+}.$$

Observe that probability $p_j^-$ vanishes for $j = 1$, as well as probability $p_j^+$ does for $j = n$.

By definition of $I_j$, we have $\Pr[x_i^* \in I_j] = \frac{1}{n}$.

For $p_j^+$, we derive the upper bound $\text{(3.14)}$. Note that the bound does not depend on $j$. Note that the bound holds also true for $p_j^-$ by symmetry.

By Markov’s inequality and **Assumption 3.1(B)** we get

$$\Pr[x_i^\Delta \geq z] \leq \frac{\gamma}{z^{1+\varepsilon}}.$$

Setting $x'_j := n(x_i^* - q_j)$, from $\text{(3.7)}$ we have

$$\varphi_{x'_j}(z) \leq \frac{L}{n} \forall z.$$
Now
(3.14)
\[ p_j^+ = \Pr[x_i^* - \frac{1}{n}x_i^\Delta \leq q_j \land x_i^* > q_j] \]
\[ = \Pr[x_i^\Delta \geq x_j' \land x_j' > 0] \]
\[ = \int_0^\infty \Pr[x_i^\Delta \geq z \mid x_j' = z] \cdot \varphi_{x_j'}(z) \, dz \]
\[ = \int_0^\infty \Pr[x_i^\Delta \geq z] \cdot \varphi_{x_j'}(z) \, dz \]
\[ = \int_0^1 \Pr[x_i^\Delta \geq z] \cdot \varphi_{x_j'}(z) \, dz + \int_1^\infty \Pr[x_i^\Delta \geq z] \cdot \varphi_{x_j'}(z) \, dz \]
\[ \leq \int_0^1 \varphi_{x_j'}(z) \, dz + \int_1^\infty \frac{\gamma}{z^{1+\varepsilon}} \cdot \varphi_{x_j'}(z) \, dz \]
\[ \leq \frac{L}{n} + \frac{L\gamma}{n} \int_1^\infty \frac{1}{z^{1+\varepsilon}} \, dz \]
\[ = \frac{L}{n} + \frac{L\gamma}{\varepsilon n} \]
\[ = \frac{L}{n} \left(1 + \frac{\gamma}{\varepsilon}\right). \]

Finally,
\[ p_j \leq \frac{1}{n} + 2 \frac{L}{n} \left(1 + \frac{\gamma}{\varepsilon}\right) = \frac{1}{n} \left(1 + 2L \left(1 + \frac{\gamma}{\varepsilon}\right)\right) = \frac{\alpha}{n}. \]

Let us introduce indicator variables for all \( i, j = 1, \ldots, n \):

\[ W_{ij} = \begin{cases} 1 & \text{if } \mathbf{x}_{i/n}^j \cap I_j \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases} \]

Note that \( W_{ij} \) is alternatively distributed with parameter \( p_j^n \) and that the variables
(3.15) \[ W_{1j}, W_{2j}, \ldots, W_{n-1,j}, W_{nj} \]
are independent.

Hence, the sum of these variables follows binomial distribution, i.e.
\[ \sum_{i=1}^n W_{ij} =: E_j \sim \text{Bi}(n, p_j^n). \]
By introducing 
\[ \overline{E} \sim \text{Bi}(n, \frac{\alpha}{n}), \]
the estimate \( p_j^n \leq \frac{\alpha}{n} \) from Lemma 3.8 implies

\[ \Pr[E_j \geq z] \leq \Pr[\overline{E} \geq z] \quad (\forall z). \tag{3.16} \]

If \( \omega \geq \kappa \) for some \( \kappa \), at least \( \kappa \) intervals \( x_i^{1/n} \) have to share a common intersection. This common intersection belongs to one of intervals \( I_1, \ldots, I_n \), hence there exists \( j \) such that \( E_j \geq \kappa \). More precisely,

\[ \Pr[\omega \geq \kappa] \leq \Pr[E_1 \geq \kappa \lor \cdots \lor E_n \geq \kappa]. \tag{3.17} \]

Fact (3.16) allows us to use an estimate based on Penrose’s tail bound for binomial distribution.

**Lemma 3.18** (Tail bound for the binomial distribution \((\text{Penrose} \ 2003, \ p. \ 16))\). Let

\[ H(\xi) = 1 - \xi + \xi \log \xi. \tag{3.19} \]

If \( Z \sim \text{Bi}(n, \pi) \) and \( \kappa \geq n\pi \), then

\[ \Pr[Z \geq \kappa] \leq \exp \left( -n\pi \cdot H \left( \frac{\kappa}{n\pi} \right) \right). \]

If \( \kappa \geq \alpha \), we can extend the estimate (3.17) to the form

\[ \Pr[\omega \geq \kappa] \leq \Pr[E_1 \geq \kappa \lor \cdots \lor E_n \geq \kappa] \tag{3.20} \]

\[ \leq \sum_{j=1}^{n} \Pr[E_j \geq \kappa] \tag{3.21} \]

\[ \leq \sum_{j=1}^{n} \Pr[\overline{E} \geq \kappa] \tag{3.22} \]

\[ \leq n \exp \left[ -n \frac{\alpha}{n} \cdot H \left( \frac{\kappa}{n\alpha} \right) \right] \]

\[ = n \exp \left[ -\alpha \cdot H \left( \frac{\kappa}{\alpha} \right) \right]. \tag{3.23} \]
In (3.21) we used the union bound \( \Pr[Q \lor \cdots \lor Q_n] \leq \sum_{i=1}^n \Pr[Q_i] \) for any events \( Q_1, \ldots, Q_n \). In (3.22) we used (3.16) and the tail bound from Lemma 3.18.

Let

\[
(3.24) \quad c := \frac{1}{\log 2} \quad \text{and} \quad k_n := c \cdot \frac{\log n}{\log \log n}.
\]

If \( n \) is sufficiently large, we can estimate

\[
E2^\omega = \sum_{\ell=1}^n 2^\ell \cdot \Pr[\omega = \ell]
\]

\[
= \sum_{\ell=1}^{\lfloor k_n \rfloor} 2^\ell \cdot \Pr[\omega = \ell] + \sum_{\ell=\lfloor k_n \rfloor + 1}^n 2^\ell \cdot \Pr[\omega = \ell]
\]

\[
\leq \sum_{\ell=1}^{\lfloor k_n \rfloor} 2^{\lfloor k_n \rfloor} \cdot \Pr[\omega = \ell] + \sum_{\ell=\lfloor k_n \rfloor + 1}^n 2^\ell \cdot \Pr[\omega \geq \ell]
\]

\[
\leq 2^{k_n} \sum_{\ell=1}^{\lfloor k_n \rfloor} \Pr[\omega = \ell]
\]

\[
+ \sum_{\ell=\lfloor k_n \rfloor + 1}^n 2^\ell \cdot n \cdot \exp\left( -\alpha \cdot H\left( \frac{\ell}{\alpha} \right) \right)
\]

\[
=: u_\ell
\]

\[
\leq 2^{k_n}
\]

\[
\leq 2^{\frac{c \log n}{\log \log n}} + 2u_{\lfloor k_n \rfloor + 1}
\]

\[
(3.25) \quad \leq e^{(\log 2) \frac{c \log n}{\log \log n}} + 4 \cdot n^{1-\frac{c}{\log \log n}}
\]

\[
=: \zeta_n
\]

\[
(3.26) \quad \leq e^{(\log 2) \frac{c \log n}{\log \log n}} + 4 \cdot n^{1-\frac{c}{\log \log n}}
\]

\[
(3.27) \quad \leq n \frac{1}{\log \log n} + 1
\]

where \( K := \log \frac{8\alpha}{c} \). Clearly, \( 1 + n \frac{1}{\log \log n} = O(n^\epsilon) \) for every \( \epsilon > 0 \). Inequalities (3.25) and (3.26) follow from Lemma 3.28 showing basic properties of the sequence \( u_\ell \). Namely, it shows that it decreases
exponentially fast. The estimate $\zeta_n \leq \frac{1}{4}$ if $n$ is sufficiently large $n$ follows from the observation that $1 - c < 0$; thus $\zeta_n \xrightarrow{n \to \infty} 0$. 
Lemma 3.28. Let $k_n \geq 4e\alpha$ (here, $e = \exp(1)$).

(a) $\sum_{\ell=k_n+1}^{n} u_{\ell} < 2u_{[k_n]+1}$.

(b) $u_{[k_n]+1} \leq 2\zeta_n$.

Proof. To prove (a) we show that $u_{\ell} < \frac{1}{2}u_{\ell-1}$ for

$$\ell \geq 4e\alpha.$$  

It follows that

$$\sum_{\ell=k_n+1}^{n} u_{\ell} \leq 2u_{[k_n]+1},$$

since $\sum_{\ell=k_n+1}^{n} u_{\ell}$ can be bounded by the sum of a geometric sequence with quotient $\frac{1}{2}$. We have

$$u_{\ell} = n \cdot 2^\ell \cdot \exp\left(-\alpha \cdot H\left(\frac{\ell}{\alpha}\right)\right)$$

$$\quad = n \cdot 2^\ell \cdot \exp\left[-\alpha\left(1 - \frac{\ell}{\alpha} + \frac{\ell}{\alpha} \log \left(\frac{\ell}{\alpha}\right)\right)\right]$$

$$\quad = n \cdot 2^\ell \cdot \exp\left[-\alpha + \ell - \ell \log \ell + \ell \log \alpha\right]$$

$$\quad = ne^{-\alpha} \cdot \left(\frac{2e\alpha}{\ell}\right)^\ell$$

and

$$\frac{u_{\ell}}{u_{\ell-1}} = \left(\frac{\ell-1}{\ell}\right)^{\ell-1} \cdot \frac{2e\alpha}{\ell} < \frac{2e\alpha}{\ell} \leq \frac{1}{2};$$

the last inequality follows from [3.29].

For (b) we use the fact that $H(\xi) = 1 - \xi + \xi \log \xi$ is a nondecreasing function for $\xi \geq 1$. Thus

$$u_{[k_n]+1} = 2^{[k_n]+1} \cdot n \exp\left[-\alpha \cdot H\left(\frac{[k_n]+1}{\alpha}\right)\right]$$

$$\quad \leq 2 \cdot 2^{k_n} \cdot n \exp\left[-\alpha \cdot H\left(\frac{k_n}{\alpha}\right)\right]$$

$$\quad \leq 2 \cdot e^{k_n} \cdot n \exp\left[-\alpha \cdot H\left(\frac{k_n}{\alpha}\right)\right]$$

$$\quad = 2 \exp\left[\log n + k_n - \alpha \cdot \left(1 - \frac{k_n}{\alpha} + \frac{k_n}{\alpha} \log \left(\frac{k_n}{\alpha}\right)\right)\right]$$
\begin{align*}
&= 2 \exp \left[ \log n + k_n - \alpha + k_n \log k_n + k_n \log \alpha \right] \\
&\leq 2 \exp \left[ \log n + (2 + \log \alpha) k_n - k_n \log k_n \right] \\
&\leq 2 \exp \left[ \log n + (\log 8\alpha) k_n - k_n \log k_n \right] \\
&= 2 \exp \left[ \log n + (\log 8\alpha) \frac{c \log n}{\log \log n} - \frac{c \log n}{\log \log n} \log \frac{c \log n}{\log \log n} \right] \\
&= 2 \exp \left[ \log n + (\log 8\alpha) \frac{c \log n}{\log \log n} \\
&\quad - \frac{c \log n}{\log \log n} (\log c + \log \log n - \log \log \log n) \right] \\
&= 2 \exp \left[ \log n + (\log 8\alpha) \frac{c \log n}{\log \log n} \\
&\quad - \frac{(c \log c) \log n}{\log \log n} - c \log n + (c \log n) \frac{\log \log \log n}{\log \log n} \right] \\
&= 2 \exp \left[ (\log n) \left( (1 - c) + c \cdot \frac{\log 8\alpha - \log c + \log \log \log n}{\log \log n} \right) \right] \\
&= 2 \exp \left[ (\log n) \left( (1 - c) + c \cdot \frac{K + \log \log \log n}{\log \log n} \right) \right] \\
&= 2n^{1-c} \cdot n^{c \frac{K + \log \log \log n}{\log \log n}} = 2\zeta_n. \quad \square
\end{align*}

Remark 3.30. Note that a huge $n$ might be needed to achieve $\zeta_n \leq \frac{1}{4}$; the particular value depends on $\alpha$. However, if we admit a greater $c$, e.g. $c = 8\alpha$, we get $K = 0$ and $\zeta_n \leq n^{1-8\alpha+8\alpha} \approx n^{1-5\alpha}$, which tends to zero fast, so condition $\zeta_n \leq \frac{1}{4}$ is not at all restrictive even from the practical viewpoint. On the other hand, the exponent in $n^{\frac{c \log 2}{\log \log n}} + 1$ becomes a bit worse.

This shows that at the cost of a pair of worse constants, the method behaves well even for small $n$. 

In order to complete the proof of Theorem 3.3(a), we need to estimate $E\omega$. Using Jensen’s inequality we get

$$
E\omega \leq \frac{\log(E2^\omega)}{\log 2} \leq \frac{\log(1 + n\log\log n)}{\log 2} \leq \frac{3}{2} \left(1 + \log e^{\log\log n}\right) = \frac{3}{2} \left(1 + \frac{\log n}{\log\log n}\right).
$$

The proof of Theorem 3.3(b) is a corollary of the above theory. Indeed, using the notation from (3.20) – (3.23), definition of $H(\xi)$ from (3.19) and Lemma 3.18, we have

$$
\Pr[\omega \geq \delta n] e^{-n \log\log n} \leq \sum_{j=1}^{n} \Pr[E \geq \delta n] \cdot e^{n \log\log n} \leq n \exp \left[-\alpha \cdot H \left(\frac{\delta n}{\alpha}\right)\right] \cdot e^{n \log\log n} = \exp \left[\log n + n \log\log n - \alpha \cdot \left(1 - \frac{\delta n}{\alpha} + \frac{\delta n}{\alpha} \log \left(\frac{\delta n}{\alpha}\right)\right)\right] \leq \exp \left[\log n + n \log\log n + \delta n - \delta n \log \left(\frac{\delta}{\alpha}\right)\right] \xrightarrow{n \to \infty} 0,
$$

because the term $(\star)$ is of the order $n \log n$ and dominates all other terms in the limit. The proof of Theorem 3.3 is complete. 

Remark 3.31. The same proof method can be easily generalized to estimate, for example, the probability that the clique is as large as $n^\eta$ for a fixed $0 \leq \eta \leq 1$ (i.e., this is the event “the computing time exceeds $2^{n^\eta}$”). In this case we get $\Pr[\omega \geq n^\eta] \leq \exp(-n^\eta \log\log n)$. 


4. Concluding remarks and comments

4.1. “Unfriendly” distributions for the FGKLA algorithm: Why Lipschitz continuity \textbf{(Assumption 3.1(A))} is unavoidable. We show that if we drop the Lipschitz continuity assumption, we can get $E \omega \geq \pi n$ for some $\pi > 0$ and thus exponential computing time on average (using the fact that average computing time $\geq E 2^\omega \geq 2E \omega = 2^{\Omega(n)}$).

Non-continuous distributions. First consider $\Phi^*(z)$, the c.d.f. of $x^*_i$, with a discontinuity point $z_0$. Then $\pi := \Pr[x^*_i = z_0] > 0$. Setting

\[(4.1)\]

$U_i = \begin{cases} 1, & \text{if } z_0 \in \mathbf{x}_i^{1/n}, \\ 0 & \text{otherwise}, \end{cases}$

we get $E U_i = \Pr[U_i = 1] \geq \pi$ and $\omega \geq \sum_{i=1}^n U_i$ a.s. Thus

$E \omega \geq \sum_{i=1}^n E U_i \geq \pi n.$

Continuous non-Lipschitz distributions. We show that the misbehavior of the non-continuous distribution from the previous paragraph can be “simulated” by a non-Lipschitz continuous distribution. Let $z_0$ be a discontinuity point of $\Phi^*(z)$ from the last paragraph, let $\Phi_0 := \lim_{z \nearrow z_0} \Phi^*(z)$ and $\eta := \Phi^*(z_0 + 1) - \Phi_0$. Clearly $\eta > 0$. Consider another distribution of $x^*_i$ with c.d.f.

$\tilde{\Phi}^*(z) = \begin{cases} \Phi^*(z) & \text{if } z < z_0 \text{ or } z > z_0 + 1, \\ \Phi_0 + \eta \cdot (z - z_0)^\varepsilon & \text{if } z_0 \leq z \leq z_0 + 1 \end{cases}$

with $\varepsilon > 0$ arbitrarily small. Now $\tilde{\Phi}^*(z)$ is continuous (if there are more discontinuity points of $\Phi^*(z)$ outside $[z_0, z_0 + 1)$, a similar construction can be done in each of them). If $x^*_i = 1$ a.s. and $U_i$ has the same meaning as in $(4.1)$, we get

$E U_i = \Pr[U_i = 1] \geq \Pr[z_0 \leq x^*_i \leq z_0 + \frac{1}{n}] = \eta n^{-\varepsilon},$

and thus

$E \omega \geq \sum_{i=1}^n E U_i \geq \eta n^{1-\varepsilon}.$
Taking $\varepsilon$ close to zero, we get a clique with average size arbitrarily close to the order $n$.

### 4.2. The independence assumption (Assumption 3.1(C)) is also essential.

If we relax the independence assumption, we can get only a weaker estimate on $p_n^j$ than the bound $p_n^j = O(n^{-1})$ from Lemma 3.8. Said informally, we needed $p_n^j = O(n^{-1})$ in Lemma 3.18 to satisfy $np_n^j = O(1)$. Then, since $k_n$ grows unboundedly (although slowly), we were able to apply the tail bound for $n$ sufficiently large.

But in the dependent case we can derive only the bound

$$p_n^j = O(n^{-\frac{1}{2}}),$$

resulting in $np_n^j = O(n^{\frac{1}{2}})$. Then, $k_n$ would have to grow faster than $n^{\frac{1}{2}}$ to be able to apply the tail bound and we would get

$$E\omega \sim n^{\frac{1}{2}}$$

or even something worse. Then, the average computation time bound would be as poor as $2\sqrt{n}$. This is a high price for dependence. For specific extremal distributions, the situation can indeed be so bad, as shown in Section 4.3; but for “usual” distributions with enough moments the situation is much better, as explained in Section 4.4.

Let us show (4.2) without the assumption of independence of $x_i^*$ and $x_i^\Delta$. By Markov’s inequality we have

$$\zeta_n := \Pr[x_i^\Delta \geq n^{\frac{1}{2+\varepsilon}}] \leq \gamma n^{-\frac{1+\varepsilon}{2+\varepsilon}}$$

similarly as in (3.12): recall that we have only assumed the existence of a finite moment of order $(1 + \varepsilon)$ with value $\gamma$. We have

$$p_n^j = \Pr[x_i^* \in I_j] + p^+_j + p^-_j$$

$$\leq \frac{1}{n} + O(n^{-\frac{1}{2}}) + O(n^{-\frac{1}{2}})$$

$$= O(n^{-\frac{1}{2}}),$$
as the bounds for \( p_j^+ \) (and \( p_j^- \), similarly) can be obtained by setting \( x'_j = n(x_i^* - q_j) \) and using \((3.13)\):

\[
p_j^+ = \Pr[x'_j \leq x_i^\Delta \land x'_j \geq 0]
= \Pr[x'_j \leq x_i^\Delta \land x'_j \geq 0 \mid x_i^\Delta \geq n^{1/2+\varepsilon}] \cdot \zeta_n
+ \Pr[x'_j \leq x_i^\Delta \land x'_j \geq 0 \mid x_i^\Delta < n^{1/2+\varepsilon}] \cdot (1 - \zeta_n)
\leq \zeta_n + \Pr[x'_j < n^{1/2+\varepsilon} \land x'_j \geq 0]
\leq \gamma n^{-1/2+\varepsilon} + L n^{-1} n^{1/2+\varepsilon}
= \gamma n^{-1/2+\varepsilon} + L n^{-1/2+\varepsilon}
= O(n^{-1/2}).
\]

4.3. An extremal distribution. Unfortunately, the bounds from the previous sections cannot be generally improved. We show an example where Assumption 3.1(A) and Assumption 3.1(B) are satisfied, Assumption 3.1(C) is violated and a slightly weaker form of \((4.3)\) holds true — the clique is as large as \( n^{1/2-\varepsilon} \) on average, for an arbitrarily small \( \varepsilon > 0 \). Thus we can push the average computation time of FKGLA algorithm arbitrarily close to \( 2^{\sqrt{n}} \).

Let \( x_1^*, \ldots, x_n^* \sim \text{Unif}(0, 1) \) independent. Then, clearly, Assumption 3.1(A) is satisfied. Let \( 0 < \varepsilon < 1 \) (a choice with \( \varepsilon \) close to zero is interesting). Define

\[
 x_i^\Delta := (x_i^*)^{-1+\varepsilon}, \quad i = 1, \ldots, n.
\]

Assumption 3.1(B) is satisfied: indeed, the moment of order \( 1 + \varepsilon \) is finite, since

\[
 \mathbb{E}[(x_i^\Delta)^{1+\varepsilon}] = \mathbb{E}[(x_i^*)^{(\varepsilon-1)(\varepsilon+1)}]
= \mathbb{E}[(x_i^*)^{\varepsilon^2-1}]
= \int_0^1 x^{\varepsilon^2-1} \, dx
= \varepsilon^{-2} < \infty,
\]

and \( x_1^\Delta, \ldots, x_n^\Delta \) are independent.

For \( i = 1, \ldots, n \) define

\[
 U_i = \begin{cases} 
 1, & \text{if } 0 \in x_i^{1/n}, \\
 0, & \text{otherwise.}
\end{cases}
\]
We have
\[
E U_i = \Pr[U_i = 1] = \Pr[\frac{1}{n} x_i^\Delta \geq x_i^*] = \Pr[(x_i^*)^{-1+\varepsilon} \geq nx_i^*] = \Pr[(x_i^*)^{-2+\varepsilon} \geq n] = \Pr[x_i^* \leq n^{-\frac{1}{2-\varepsilon}}] = n^{-\frac{1}{2-\varepsilon}}.
\]
Obviously, \(\omega \geq \sum_{i=1}^{n} U_i\) a.s. Thus
\[
E \omega \geq \sum_{i=1}^{n} E U_i = n^{1-\frac{1}{2-\varepsilon}},
\]
which is close to \(n^{\frac{1}{2}}\) if \(\varepsilon\) is small.

4.4. FGKLA algorithm can benefit from high-order moments: A trade-off between dependence and existence of such moments. Note that the problem with dependence of the input random variables sketched in Section 4.2 is closely related to the value of \(\varepsilon\). Here we show that
if we assume the existence of high-order moments, we can push the bound on \(p_j^n\) close to the “desired” order \(O(n^{-1})\) and get good computation time of the FGKLA algorithm even in the dependent case. Indeed, if \(\tilde{\gamma} := E[(x_i^\Delta)^d] < \infty\) for some \(d\), then Markov’s inequality gives us \(\zeta_n := \Pr[x_i^\Delta \geq n^{\frac{1}{1+d}}] \leq \tilde{\gamma} n^{-\frac{d}{1+d}}\). Now we have
\[
p_j^+ = \Pr[x_j' \leq x_i^\Delta \land x_j' \geq 0] = \Pr[x_j' \leq x_i^\Delta \land x_j' \geq 0 \mid x_i^\Delta \geq n^{\frac{1}{1+d}}] \cdot \zeta_n + \Pr[x_j' \leq x_i^\Delta \land x_j' \geq 0 \mid x_i^\Delta < n^{\frac{1}{1+d}}] \cdot (1 - \zeta_n) \leq \zeta_n + \Pr[x_j' < n^{\frac{1}{1+d}} \land x_j' \geq 0] \leq \tilde{\gamma} n^{-\frac{d}{1+d}} + Ln^{-1} n^{\frac{1}{1+d}} \leq \tilde{\gamma} n^{-\frac{d}{1+d}} + Ln^{-\frac{d}{1+d}} = O(n^{-\frac{d}{1+d}}),
\]
which is close to \(n^{-1}\) if \(d\) is large.

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