A JSJ-TYPE DECOMPOSITION THEOREM FOR SYMPLECTIC FILLINGS OF CONTACT 3-MANIFOLDS

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Abstract. Let \((M, \xi)\) be a contact 3-manifold and \(T^2 \subset (M, \xi)\) a convex torus of a special type called a mixed torus. We prove a JSJ-type decomposition theorem for strong and exact symplectic fillings of \((M, \xi)\) when \((M, \xi)\) is cut along \(T^2\). As an application we show the uniqueness of exact fillings when \((M, \xi)\) is obtained by Legendrian surgery on a knot in \((S^3, \xi_{\text{std}})\) when the knot is stabilized both positively and negatively.

1. Introduction

A fundamental question in contact geometry is to determine the symplectic fillings of a given contact manifold, i.e. to what extent does the boundary determine its interior? The goal of this paper is to explain how to decompose the symplectic filling \((W, \omega)\) of a contact manifold \((M, \xi)\) when we decompose \(M = \partial W\) along a convex torus of a special type which we call a mixed torus, and to use this decomposition to show the uniqueness of some fillings of contact manifolds obtained as Legendrian surgeries. The reference to JSJ-type decompositions refers to this splitting along tori.

Recall that a strong symplectic filling of a contact manifold \((M, \xi)\) is a symplectic manifold \((W, \omega)\) such that \(\partial W = M, \omega = d\alpha\) near \(M\), and \(\alpha\) is a positive contact form for \(\xi\). An exact symplectic filling of \((M, \xi)\) is a strong symplectic filling \((W, \omega)\) such that \(\omega = d\alpha\) on all of \(W\).

Let us start with a partial list of known results classifying the number of exact symplectic fillings of a given contact manifold. A detailed survey can be found in \([O2]\).

• (Eliashberg [El]) \((S^3, \xi_{\text{std}})\) has a unique exact filling up to symplectomorphism.
• (Wendl [We]) \((T^3, \xi_1)\), where \(\xi_1\) is canonical contact structure on the unit cotangent bundle of \(T^2\), has a unique exact filling up to symplectomorphism (Stipsicz [St] had previously shown that, up to homeomorphism, there is a unique exact filling on \(\Sigma(2, 3, 5)\) and \((T^3, \xi_1))\).
• (McDuff [MD]) The standard tight contact structure on $L(p, 1)$ has a unique exact filling up to diffeomorphism for $p \neq 4$ and for $p = 4$ there are two.

• (Lisca [Li]) Lisca classified the fillings for $L(p, q)$ with the canonical contact structure.

• (Plamenevskaya and Van Horn-Morris [PV], Kaloti [Ka]) There is a unique filling for lens spaces of the form $L(p(m+1)+1, m+1)$ with virtually overtwisted contact structures. The case $L(p, 1)$ is shown in [PV] and the general case in [Ka].

• (Sivek and Van Horn-Morris [SV]) Fillings for the unit cotangent bundle of an orientable surface are unique up to s-cobordism, and similar results for non-orientable surfaces were proven by Li and Ozbagci [LO].

• (Akhmedov, Etnyre, Mark, Smith [AEMS]) It is not always the case that there is a unique exact filling, or even finitely many.

Before stating our main theorem, let us introduce mixed tori, and say what it means to split a contact manifold along a mixed torus. We will call a convex torus $T \subset (M, \xi)$ a mixed torus if there exists an embedding of $T^2 \times [0, 2]$ into $M$ so that $T = T^2 \times \{1\}$, the restriction of $\xi$ to $T^2 \times [0, 2]$ is virtually overtwisted, and each of $T^2 \times [0, 1]$ and $T^2 \times [1, 2]$ is a basic slice. (We will recall these contact geometry notions in Section 2.)

Now suppose we have a mixed torus $T \subset (M, \xi)$, and identify $T$ with $\mathbb{R}^2/\mathbb{Z}^2$ in such a way that the dividing curves of $T$ have slope $\infty$. For the embedding $T^2 \times [0, 2]$, we will denote the slope of the dividing curves of $T^2 \times \{i\}$ by $s_i$, and we may normalize the identification $T = \mathbb{R}^2/\mathbb{Z}^2$ so that $s_0 = -1$. With this identification understood, we will now define the closed contact manifold $(M', \xi')$ which results from splitting $M$ along $T$ with slope $s$, for any integer $s \in \mathbb{Z}$. Topologically, we obtain $M'$ by attaching a solid torus to $M \setminus T$ along each of the two torus components of $\partial (M \setminus T) = T_0 \sqcup T_1$:

$$M' = S_0 \cup_{\psi_0} (M \setminus T) \cup_{\psi_1} S_1.$$  

We will define the contact structure $\xi'$ to be $\xi$ on $M \setminus T$, while on $S_i \subset M'$ we use the unique tight contact structure determined by the characteristic foliation of $\partial S_i$. The remaining ambiguity lies in the maps $\psi_i: \partial S_i \to T_i$. These are chosen so that the image of the meridian of $S_i$ is a curve of slope $s$ in $T_i = \mathbb{R}^2/\mathbb{Z}^2$. The fact that $s$ must be an integer follows from the fact that the dividing set, which serves as a longitude, has slope $\infty$.

The result of splitting $M$ along $T$ with slope $s$ will not generally be a tight contact manifold. Indeed, from the slopes $s_0$ and $s_2$ we may...
determine a finite list of slopes $s$ for which $(M', \xi')$ will be tight. First, consider the Legendrian knot $L_1 \subset S_1$ given by the core of $S_1$. Along $L_1$, the contact planes will have slope $s$, and as we move towards $\partial S_1 = T_1$, these planes will make clockwise rotations towards the slope $s_1 = \infty$, as the dividing curves of $T_1$ have this slope. As we then continue pushing through the basic slice $T \times [1, 2]$, the contact planes continue their clockwise rotation towards $s_2$. Altogether, in this portion of $(M', \xi')$, the slope of the contact planes rotates from $s$ to $\infty$ to $s_2$. If $s > s_2$, then this rotation passes through an angle in excess of $\pi$, meaning that $\xi'$ is overtwisted. Moreover, if $s = s_2$, then restricting $\xi'$ to the union of $S_1$ with $T_2 \times [1, 2]$ produces a solid torus whose boundary dividing curves have slope 0 — that is, they are meridional. Such a solid torus is necessarily overtwisted, so if $\xi'$ is tight, then $s < s_2$. Similar reasoning — with a change of orientation — shows that $s > s_0 = -1$, meaning that $0 \leq s \leq s_2 - 1$, provided $\xi'$ is a tight contact structure.

Notice that if $(M', \xi')$ is the result of splitting $(M, \xi)$ along $T$ with some slope, then we may recover $(M, \xi)$ from $(M', \xi')$ by removing a pair of solid tori and identifying the dividing sets and meridians of their boundary tori. This relationship between $(M, \xi)$ and $(M', \xi')$ allows us to obtain a filling of $(M, \xi)$ from any filling of $(M', \xi')$ via round symplectic 1-handle attachment. A round symplectic 1-handle is a symplectic manifold-with-boundary diffeomorphic to $S^1 \times D^1 \times D^2$, carrying a Liouville vector field $Z$ which is inward-pointing along $S^1 \times S^0 \times D^2$ and outward-pointing along $S^1 \times D^1 \times S^1$. The vector field $Z$ induces the standard contact structure on the boundary solid tori $S^1 \times D^0 \times D^2$, and is thus attached to a symplectic filling by identifying two standard solid tori in the boundary of the filling. For instance, if $(W', \omega')$ is a strong symplectic filling of $(M', \xi')$, we may attach a round symplectic 1-handle to $(W', \omega')$ along standard neighborhoods of the core Legendrians $L_0 \subset S_0$ and $L_1 \subset S_1$ to obtain $(W, \omega)$, a strong symplectic filling of $(M, \xi)$. This construction will be explained in greater detail in Section 2.

We our now prepared to state our main theorem, which says that every filling of $(M, \xi)$ may be constructed as above.

**Theorem 1.1.** Let $(M, \xi)$ be a closed, cooriented 3-dimensional contact manifold and let $(W, \omega)$ be a strong (respectively, exact) symplectic filling of $(M, \xi)$. If there exists a mixed torus $T^2 \subset (M, \xi)$, witnessed by an embedding $T^2 \times [0, 2]$, then there exists a (possibly disconnected) symplectic manifold $(W', \omega')$ such that:

- $(W', \omega')$ is a strong (respectively, exact) filling of its boundary $(M', \xi')$;
(M′, ξ′) is the result of splitting (M, ξ) along T with some slope 0 ≤ s ≤ s2 − 1, where s2 is identified as above;

(W, ω) can be recovered from (W′, ω′) by round symplectic 1-handle attachment.

Remark 1.2. The condition that T2 be a mixed torus is essential; the theorem is not true if one assumes that T2 is just a convex torus with two homotopically essential dividing curves.

There are two results we must mention here to properly contextualize Theorem 1.1. The first is due to Eliashberg [El], who showed that if (M, ξ) is a closed contact 3-manifold obtained from (M′, ξ′) (which may be disconnected) via a connected sum, then every symplectic filling of (M, ξ) is obtained from such a filling of (M′, ξ′) by attaching a Weinstein 1-handle. Our result is in the same spirit, replacing Eliashberg’s holomorphic discs with holomorphic annuli, and thus considering an embedded torus in (M, ξ) rather than an embedded sphere.

This leads us to the next result of historical importance. Work of Jaco-Shalen [JS] and Johannson [J] established the JSJ decomposition for irreducible, orientable, closed 3-manifolds. For such a manifold M, this decomposition identifies a (unique up to isotopy) minimal collection T ⊂ M of disjoint, incompressible tori for which each component of M\N(T) is either atoroidal or Seifert-fibered, where N(T) ⊂ M is an open tubular neighborhood of T. One can think of this decomposition as a toroidal version of the prime decomposition of 3-manifolds, which allows us to write every compact, orientable 3-manifold as the connected sum of a unique collection of prime 3-manifolds. In the same way, our result takes the connected sums of Eliashberg and replaces them with toroidal decompositions.

We can use Theorem 1.1 to prove:

**Theorem 1.3.** Let L be an oriented Legendrian knot in a closed cooriented 3-manifold (M, ξ). Let (M′, ξ′) be the manifold obtained from (M, ξ) by Legendrian surgery on S+_S_−(L), where S_+ and S_- are positive and negative stabilizations, respectively. Then every exact filling of (M′, ξ′) is obtained from a filling of (M, ξ) by attaching a symplectic 2-handle along S+_S_−(L).

In particular the following corollary holds when (M, ξ) = (S^3, ξ_{std}), since (S^3, ξ_{std}) has a unique exact filling.

**Corollary 1.4.** If (M′, ξ′) is obtained from (S^3, ξ_{std}) by Legendrian surgery on S+_S_−(L), then (M′, ξ′) has a unique exact filling up to symplectomorphism.
Remark 1.5. Corollary [1.4] is not true if \( L \) is stabilized twice with the same sign. For example \( L(4,1) \) can be obtained from Legendrian surgery on a twice stabilized unknot but has two distinct fillings.

Kaloti and Li [KL] had previously shown the uniqueness up to symplectomorphism of exact fillings on manifolds obtained from Legendrian surgery along certain 2-bridge and twist knots and their stabilizations. Related results were shown by Lazarev for higher dimensions in [La]. While not stated in quite the same manner, the main result of Lazarev involves surgery on loose Legendrians. We observe that in dimensions greater than or equal to 5 all Legendrians which have been stabilized near a cusp edge are loose; their analog in dimension 3 is a Legendrian which has been stabilized both positively and negatively.

2. Background

2.1. Contact geometry preliminaries. A knot in \( L \subset (M, \xi) \) is called Legendrian if it is everywhere tangent to the contact structure \( \xi \). The front projection of a Legendrian knot in \( (\mathbb{R}^3, \ker(dz - ydx)) \) is its projection to the \( xz \)-plane. The stabilization of \( L \subset (M, \xi) \) is obtained by locally adding a zigzag in the front projection, there are two possibilities \( S^\pm \) as given in Figure 1.

An oriented properly embedded surface \( \Sigma \) in \( (M, \xi) \) is called convex if there is a vector field \( v \) transverse to \( \Sigma \) whose flow preserves \( \xi \). A convex surface \( \Sigma \) which is closed or compact with Legendrian boundary has a dividing set \( \Gamma_\Sigma \): The dividing set \( \Gamma_\Sigma(v) \) of \( \Sigma \) with respect to \( v \) is the set of points \( x \in \Sigma \) where \( v(x) \in \xi(x) \). The set \( \Gamma_\Sigma(v) \) is a disjoint union of properly embedded smooth curves and arcs which are transverse to the characteristic foliation \( \xi|_\Sigma \). If \( \Sigma \) is closed, there will only be closed curves \( \gamma \subset \Gamma_\Sigma(v) \). The isotopy type of \( \Gamma_\Sigma(v) \) is independent of the choice of \( v \) — hence we will slightly abuse notation and call it the dividing set of \( \Sigma \) and denote it \( \Gamma_\Sigma \). We will write \( \Gamma \) for \( \Gamma_\Sigma \) when there is no ambiguity in \( \Sigma \). Denote the number of connected components of \( \Gamma_\Sigma \) by \( \#\Gamma_\Sigma \). The set \( \Sigma \setminus \Gamma \) is \( R_+ \cup (-R_-) \), where \( R_+ \)
is the subsurface where the orientations of v (coming from the normal orientation of Σ) and the normal orientation of ξ coincide, and \( R_− \) is the subsurface where they are opposite.

A convex surface has a standard neighborhood \( \Sigma \times [-\epsilon, \epsilon] \subset (M, \xi) \) such that \( \Sigma = \Sigma \times \{0\} \) and on this neighborhood \( \alpha \) can be written as \( \alpha = g dt + \beta \), where \( g : \Sigma \to \mathbb{R} \) is a smooth function, \( \beta \) is a 1-form on \( \Sigma \), and \( \Gamma = \{g = 0\} \).

The standard neighborhood \( N(L) \) of a Legendrian knot \( L \) is a sufficiently small tubular neighborhood of \( L \) whose torus boundary is convex and whose dividing set has 2 components. If \( S_±(L) \) is the stabilization of \( L \), then \( N(S_±(L)) \) can be viewed as a subset of \( N(L) \). Fix an oriented identification \( \partial N(L) \simeq \mathbb{R}^2/\mathbb{Z}^2 \) such that \( \text{slope}(\Gamma_{\partial N(L)}) = \infty \) and \( \text{slope}(\text{meridian}) = 0 \). Then \( \text{slope}(\Gamma_{\partial N(S_±(L))}) = -1 \).

### 2.2. Bypasses.

A \textit{bypass disk} \( D \) for a Legendrian knot \( L \) is a convex disk whose boundary is the union of two Legendrian arcs \( a \) and \( b \) such that

- \( a = L \cap D \subset L \).
- Along \( a \) there are three elliptic singularities, two at the endpoints of \( a \) with the same sign, and one in the middle with the opposite sign.
- Along \( b \) there are at least 3 singularities all of the same sign.
- There are no other singularities in \( D \).

\textit{Remark 2.1.} We may define a new Legendrian knot \( L' = (L - a) \cup b \) and observe that \( L \) is a stabilization of \( L' \), with sign determined by the sign of the middle singularity on \( a \). We say that \( D \) is a \textit{stabilizing disk} for \( L' \).

The following theorem due to Honda [H] shows how a bypass changes the dividing set of a surface:

\textbf{Theorem 2.2 (\cite{H} Lemma 3.12).} Let \( \Sigma \) be a convex surface, \( D \) a bypass disk along \( a \subset \Sigma \). Inside any open neighborhood of \( \Sigma \cup D \) there is a one-sided neighborhood \( \Sigma \times [0, 1] \) such that \( \Sigma = \Sigma \times \{0\} \) and \( \Gamma_\Sigma \) is related to \( \Gamma_{\Sigma \times \{1\}} \) by Figure 2.

We say \( \Sigma \times \{1\} \) is obtained from \( \Sigma \) by a bypass attachment. If the endpoints of the Legendrian arc \( a \) lie on the dividing set \( \Gamma \) of \( \Sigma \) then we say the bypass is attached along \( \Gamma \).

### 2.3. Basic slices.

Identify \( T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2 \). Consider a tight \( (T^2 \times I, \xi) \), where \( I = [0, 1] \), with convex boundary where both boundary components have two homotopically non-trivial dividing curves. Let \( s_0 \) and
$s_1$ be the slopes of the dividing curves on $T^2 \times \{0\}$ and $T^2 \times \{1\}$ respectively. If the slopes of the dividing curves are connected by a single edge on the Farey tessellation and the slopes of all dividing curves on convex tori parallel to $T^2 \times \{0\}$ and $T^2 \times \{1\}$ have slopes on $[s_1, s_0]$ if $s_1 < s_0$ and on $[s_1, \infty] \cup [-\infty, s_0]$ if $s_0 < s_1$ then $(T^2 \times I, \xi)$ is called a basic slice. It was shown by Honda [H] that there are exactly two tight contact structures on a given basic slice. They are distinguished by their relative Euler class.

We would like to know when $T^2 \times [0, 2]$ is universally tight given that $T^2 \times [0, 1]$ and $T^2 \times [1, 2]$ are basic slices. Let $s_0, s_1, s_2$, the slopes of the dividing sets on $T^2 \times \{0, 1, 2\}$, be $-2, -1, 0$ respectively. Then $T^2 \times [0, 2]$ is universally tight if the relative Euler class $e(\xi, s)$ satisfies $PD(e(\xi, s)) = \pm (0, 2)$, where $s$ is a nowhere zero section of $\xi$ on the boundary.

**Definition 2.3.** A convex torus $T^2 \times \{1\} = T^2 \subset (M, \xi)$ is a mixed torus if there exist basic slices $T^2 \times [0, 1]$ and $T^2 \times [1, 2]$ such that $T^2 \times [0, 2]$ is not universally tight.

In other words, $T^2$ is a mixed torus if the basic slices $T^2 \times [0, 1]$ and $T^2 \times [1, 2]$ have opposite sign.

**2.4. Contact handles.** Let $D$ be a bypass disk, $\Sigma \subset (M, \xi)$ a convex surface. There is a correspondence between attaching $D$ to $\Sigma$ and attaching a pair of topologically canceling handles to a one-sided neighborhood $N(\Sigma) = \Sigma \times [0, \epsilon]$ of $\Sigma$. In particular, the coordinates on $N(\Sigma)$ may be chosen so that $\xi$ has the form $\ker(dt + \beta)$, for some $\beta \in \Omega^1(\Sigma)$. Then the boundary components $\Sigma \times \{0, \epsilon\}$ are convex, with dividing sets $\Gamma_{\Sigma} \times \{0, \epsilon\}$. By attaching a contact 1-handle followed by a contact 2-handle to $N(\Sigma)$ along $\Sigma \times \{\epsilon\}$, we obtain a neighborhood of the form $\Sigma \times [0, 1]$ as described in Theorem 2.2.

![Figure 2](image-url)
We now give a brief description of standard models for contact handles of index 1 and 2. Full details can be found in [O].

Each of our standard models may be realized as a subset of \((\mathbb{R}^3, \ker \alpha)\), where \(\alpha = dz + ydx + 2xdy\). Consider the subsets
\[
H_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + z^2 \leq \epsilon, y^2 \leq 1\}
\]
and
\[
H_2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + z^2 \leq 1, y^2 \leq \epsilon\},
\]
for some small \(\epsilon > 0\). We have a contact vector field
\[
Z = 2x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}
\]
on \((\mathbb{R}^3, \ker \alpha)\) which witnesses the convexity of \(\partial H_1\) and \(\partial H_2\). The respective dividing sets are
\[
\partial H_1 \cap \{z = 0\} \quad \text{and} \quad \partial H_2 \cap \{z = 0\}.
\]
Finally, we say that \((H_1, \ker \alpha)\) is our standard model for a contact 1-handle, with attaching region \(\partial H_1 \cap \{y = \pm 1\}\) — where \(Z\) is inward-pointing — while \((H_2, \ker \alpha)\) is our standard contact 2-handle, attached using \(-Z\), so that \(\partial H_2 \cap \{x^2 + z^2 = 1\}\) is our attaching region.

As with topological 1-handle attachment, we attach a contact 1-handle to a 3-dimensional contact manifold-with-convex-boundary \((M, \xi)\) by identifying the attaching region of \((H_1, \ker \alpha)\) with regular neighborhoods of a pair of points \(p, q \in \partial M\). The difference from topological handle attachment is that we require this identification to identify the dividing sets of the regular neighborhoods with that of the attaching region. This allows the contact structures on \((M, \xi)\) and \((H_1, \ker \alpha)\) to be identified, so that handle attachment yields a contact manifold. Contact 2-handle attachment is analogous.

There are circumstances in which we want handle attachment to extend not only our contact structure \(\xi\) on \(M\), but also a preferred contact form for \(\xi\). (This will be the case in our proof of Theorem 1.1.) This can be done, for instance, when \((M, \xi)\) carries the structure of a sutured contact manifold, as defined in [CGHH, Section 2]. When \((M, \xi)\) admits such a structure, there is a neighborhood \(U(\Gamma) \subset M\) of the dividing set \(\Gamma \subset \partial M\) on which the contact structure admits a standard form. The Reeb dynamics on this neighborhood will conform to those of \((H_1, \ker \alpha)\), allowing handle attachment to occur in a manner which preserves these dynamics.
2.5. **Legendrian surgery.** Let $L$ be a Legendrian knot in $(M, \xi)$ with standard neighborhood $N(L)$. Topologically Legendrian surgery is a $tb(L) - 1$ Dehn surgery on $L$ and we then take care that the contact structures agree on the boundary.

More precisely, pick an oriented identification of $\partial N(L)$ with $\mathbb{R}^2 / \mathbb{Z}^2$ so that $\pm (1,0)^T$ is the meridian and $\pm (0,1)^T$ corresponds to slope of $\Gamma_{N(L)}$. Identifying $\partial M \setminus N(L)$ with $-\partial N(L)$ we can define maps

$$\phi_\pm : \partial (D^2 \times S^1) \to \partial (M \setminus N(L))$$

on the topological level by

$$\phi(x, y) = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Let $M_\pm(L)$ be the manifold obtained by gluing $D^2 \times S^1$ to $M \setminus N(L)$ using this map. The contact structure $\xi$ restricts to a contact structure $\xi|_{M \setminus N(L)}$ on $M \setminus N(L)$ and the two dividing curves on $\partial (M \setminus N(L))$, as seen on $\partial (D^2 \times S^1)$, represent $(\mp 1, 1)$ curves. Thus, according to [K], there is a unique tight contact structure on $D^2 \times S^1$ having convex boundary with these dividing curves. Hence we may extend $\xi|_{M \setminus N(L)}$ to a contact structure $\xi_\pm$ on $M_\pm$. The contact manifold $(M_\pm, \xi_\pm)$ is said to be obtained from $(M, \xi)$ by $\pm 1$-contact surgery on $L$. The term **Legendrian surgery** refers to $-1$-contact surgery.

2.6. **Symplectization.** Let $(M, \xi)$ be a 3-dimensional contact manifold with contact form $\alpha$. The symplectization of $(M, \xi)$ is the symplectic manifold $(\mathbb{R} \times M, d(e^s\alpha))$, where $s$ is the $\mathbb{R}$ coordinate. Given a strong symplectic filling $(W, \omega)$ of $(M, \xi)$ we can form the completion $(\hat{W}, \hat{\omega})$ of $W$ by attaching $([0, \infty) \times M, d(e^s\alpha))$ to $M = \partial W$, where $\omega = d\alpha$ on $M \times \{0\}$. We will refer to $([0, \infty) \times M, d(e^s\alpha))$ as the symplectization part and $(W, \omega)$ as the cobordism part of the completion.

2.7. **Liouville hypersurfaces and convex gluing.** Theorem [1.1] relies on a result of Avdek [A]. This section reviews the necessary background for a 3-dimensional contact manifold $(M, \xi)$.

A **Liouville domain** is a pair $(\Sigma, \beta)$ where

1. $\Sigma$ is a smooth, compact manifold with boundary,
2. $\beta \in \Omega^1(\Sigma)$ is such that $d\beta$ is a symplectic form on $\Sigma$, and
3. the unique vector field $Z_\beta$ satisfying $d\beta(Z_\beta, \cdot) = \beta$ points out of $\partial \Sigma$ transversely.

The vector field $Z_\beta$ on $\Sigma$ described above is called the **Liouville vector field** for $(\Sigma, \beta)$.

**Remark 2.4.** A Liouville domain is an exact filling of its boundary.
Let \((M, \xi)\) be a 3-dimensional contact manifold and let \((\Sigma, \beta)\) be a 2-dimensional Liouville domain. A **Liouville embedding** \(i : (\Sigma, \beta) \to (M, \xi)\) is an embedding \(i : \Sigma \to M\) such that there exists a contact form \(\alpha\) for \((M, \xi)\) for which \(i^*\alpha = \beta\). The image of a Liouville embedding will be called a **Liouville submanifold** and will be denoted by \((\Sigma, \beta) \subset (M, \xi)\). We say that \((\Sigma, \beta) \subset (M, \xi)\) is a **Liouville hypersurface** in \((M, \xi)\).

One example of a Liouville hypersurface is the positive region of a convex surface.

Every Liouville hypersurface \((\Sigma, \beta) \subset (M, \xi)\) admits a neighborhood of the form

\[
N(\Sigma) = \Sigma \times [-\epsilon, \epsilon]
\]

on which \(\alpha = dt + \beta\) where \(t\) is a coordinate on \([-\epsilon, \epsilon]\). After rounding the edges \(\partial \Sigma \times (\partial [-\epsilon, \epsilon])\) of \(\Sigma \times [-\epsilon, \epsilon]\), we obtain a neighborhood \(N(\Sigma)\) of \(\Sigma\) for which \(\partial N(\Sigma)\) is a smooth convex surface in \((M, \xi)\) with contact vector field \(t\partial_t + Z_\beta\) and dividing set \(\{0\} \times \partial \Sigma\).

Fix a 2-dimensional Liouville domain \((\Sigma, \beta)\) and a (possibly disconnected) 3-dimensional contact manifold \((M, \xi)\). Let \(i_1^{}\) and \(i_2^{}\) be Liouville embeddings of \((\Sigma, \beta)\) into \((M, \xi)\) whose images, which we will denote by \(\Sigma_1^{}\) and \(\Sigma_2^{}\), are disjoint. Let \(\alpha\) be a contact form for \((M, \xi)\) satisfying \(\alpha|_{\Sigma_1} = \alpha|_{\Sigma_2} = \beta\).

Consider neighborhoods \(N(\Sigma_1), N(\Sigma_2) \subset M\) as described above. Taking coordinates \((x, z)\) on the boundary of each such neighborhood, where \(x \in \Sigma\) we may consider the mapping

\[
\Upsilon : \partial N(\Sigma_1) \to \partial N(\Sigma_2), \quad \Upsilon(x, z) = (x, -z).
\]

The map \(\Upsilon\) sends

(1) the positive region of \(\partial N(\Sigma_2)\) to the negative region of \(\partial N(\Sigma_1)\),

(2) the negative region of \(\partial N(\Sigma_1)\) to the positive region of \(\partial N(\Sigma_2)\),

and

(3) the dividing set of \(\partial N(\Sigma_1)\) to the dividing set of \(\partial N(\Sigma_2)\)

in such a way that we may perform a **convex gluing**. In other words, the map \(\Upsilon\) naturally determines a contact structure \(#((\Sigma, \beta), (i_1, i_2))\xi\) on the manifold

\[
#((\Sigma, \beta), (i_1, i_2))M := \left(M \setminus (N(\Sigma_1) \cup N(\Sigma_2))\right) / \sim
\]

where \(p \sim \Upsilon(p)\) for \(p \in N(\Sigma_1)\). Avdek then proves the following in [A]:

**Theorem 2.5** ([A, Theorem 1.9]). Let \((M, \xi)\) be a closed, possibly disconnected, \((2n + 1)\)-dimensional contact manifold. Suppose that there
are two Liouville embeddings $i_1, i_2 : (\Sigma, \beta) \to (M, \xi)$ with disjoint images. Then there is an exact symplectic cobordism $(W, \omega)$ whose negative boundary is $(M, \xi)$ and whose positive boundary is $\#(\Sigma, \beta) (M, \xi)$.

Avdek’s cobordism between $(M, \xi)$ and $\#((\Sigma, \beta), (i_1, i_2))(M, \xi)$ is constructed by attaching what he calls a symplectic handle to the compact symplectization $([0, 1] \times M, d(e^s \alpha))$ of $(M, \xi)$. Up to edge rounding, the symplectic handle modeled on the Liouville hypersurface $(\Sigma, \beta)$ has the form

$$(H_\Sigma, \omega_\beta) = ([{-1, 1}] \times \mathcal{N}(\Sigma), d\theta \wedge dz + d\beta),$$

where $\theta$ is the coordinate on $[-1, 1]$ and $\mathcal{N}(\Sigma)$ is an abstract copy of the neighborhood of $\Sigma$ described above. This handle admits a vector field $V_\beta$ satisfying $L_{V_\beta} \omega_\beta = \omega_\beta$ and which points transversely out of $\partial H_\Sigma$ along $[-1, 1] \times \partial \mathcal{N}(\Sigma)$ and into $\partial H_\Sigma$ along $\{\pm 1\} \times \mathcal{N}(\Sigma)$. This allows us to attach $(H_\Sigma, \omega_\beta)$ to the compact symplectization of $(M, \xi)$ along the neighborhoods $\mathcal{N}(\Sigma_1), \mathcal{N}(\Sigma_2) \subset \{1\} \times M$. For full details, see [A].

Remark 2.6. In case $(\Sigma, \beta) = (DT^* S^1, \lambda_{can})$, we call $(H_\Sigma, \omega_\beta)$ a round symplectic 1-handle. Attaching a round symplectic 1-handle to a symplectic cobordism is equivalent to attaching a Weinstein 1-handle, followed by a Weinstein 2-handle which passes over the 1-handle.

3. Proof of Theorem 1.1

Let $(M, \xi)$ be a contact manifold with a strong (resp. exact) symplectic filling $(W, \omega)$ and mixed torus $T^2 \subset M$. Let $(\hat{W}, \hat{\omega})$ be the completion of $(W, \omega)$ and $J$ an adapted almost complex structure on $\hat{W}$ (i.e. on $(R \times M, d(e^s \alpha))$, $J$ is $s$-invariant, takes $\partial_s$ to $R_\alpha$, and $\xi = \ker \alpha$ to itself and on $W$ is $\omega$-compatible). During the proof we will impose additional conditions on $J$ but the regularity will still be ensured by the automatic transversality results of Wendt [We3]. The proof of Theorem 1.1 proceeds as follows. First we will construct a 1-parameter family $S = \{u_t : (\mathbb{R} \times S^1, j) \to (\hat{W}, J) \mid du_t \circ j = J \circ du_t, t \in \mathbb{R}\}$ of finite energy embedded holomorphic cylinders in $(\hat{W}, \hat{\omega})$ such that

(C1) When $t \gg 0$ the images $\Sigma_t$ and $\Sigma_{-t}$ of the curves $u_t$ and $u_{-t}$ are in the symplectization $[0, \infty) \times M$.

(C2) When $t \gg 0$ their projections under the map $\pi : [0, \infty) \times M \to M$ are $R_+(T^2)$ and $R_-(T^2)$ respectively.

(C3) $\text{Im}(u_t) \cap \text{Im}(u_{t'}) = \emptyset$ if $t \neq t'$.

We then show that $S = \cup_{t \in \mathbb{R}} \Sigma_t$ sweeps out a properly embedded solid torus in $(\hat{W}, \hat{\omega})$. We finally cut $W$ along the solid torus $S' = W \cap S$
and modify the result to obtain a strong (resp. exact) filling of the cut open manifold.

Our first step is to standardize the contact form and almost complex structure on a neighborhood of $T^2$. We will essentially follow the holomorphic curve construction coming from open book decompositions of Wendl [We2]. We also note that this is essentially the same as the construction in [V, Section 4] except that Vaugon uses a sutured boundary condition instead of a convex boundary condition.

Lemma 3.1 ([We2, Section 3]). There is a choice of contact form $\alpha$ defined on a neighborhood of $T^2$ such that the components of $\Gamma_{T^2}$ are non-degenerate elliptic Reeb orbits of Conley-Zehnder index 1 with respect to the framing coming from $T^2$.

Proof. By the flexibility theorem, modulo a perturbation of the convex surface $T^2$, it suffices to construct an explicit model subject to the condition that $\Gamma_{T^2}$ consists of two parallel curves of slope $\infty$.

Let $N(\Gamma_{T^2})$ be a small neighborhood of $\Gamma_{T^2}$ and let $S^1 \times D^2_{\rho_0}$ (here $D^2_{\rho_0} = \{(\rho, \phi) | \rho \leq \rho_0\}$ with $\rho_0 > 0$ small) be a component of $N(\Gamma_{T^2})$. On $S^1 \times D^2_{\rho_0}$, let $\alpha = f(\rho)d\theta + g(\rho)d\phi$ such that the following conditions hold:

- The path $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ is a straight line segment in first quadrant with $(f(0), g(0)) = (c, 0)$ for some $c > 0$.
- $0 < -f'(\rho) \ll g'(\rho)$
- The maps $D^2_{\rho_0} \rightarrow \mathbb{R}$ defined by $(\rho, \phi) \mapsto f(\rho)$ and $(\rho, \phi) \mapsto g(\rho)/\rho^2$ are smooth at the origin.

Then the Reeb vector field is $R_{\alpha} = \frac{\partial}{\partial \theta} - \frac{f'(\rho)}{g'(\rho)} \frac{\partial}{\partial \phi}$ where $D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho)$. At $\rho = 0$ the Reeb field is $\partial_\theta$. Under these conditions $\rho = 0$ is a nondegenerate Reeb orbit of Conley-Zehnder index 1 with respect to the framing coming from $T^2$ and all other orbits in $S^1 \times D^2_{\rho_0}$ have much larger action.

On $N' := (T^2 \times [-\epsilon, \epsilon]) - N(\Gamma_{T^2})$ let $\alpha = dt + \beta$ such that $t \in [-\epsilon, \epsilon]$ and $\ker(\beta)$ directs the characteristic foliation on $T^2$. We can choose coordinates $(x, y)$ on $T^2$ such that $R^*_1 := R_+ - N(\Gamma_{T^2}) \simeq [-1, 1] \times S^1$ and $\beta = -ydx$. In order to match the contact forms on the overlaps of $N'$ and $N(\Gamma(T^2))$ we may need to take a diffeomorphism of $N'$ which restricts to the identity on $R^*_1$. $\square$

With the contact form $\alpha$ fixed, we will refer to the action $A_\alpha(\gamma)$ of a Reeb orbit $\gamma$, defined by

$$A_\alpha(\gamma) = \int_{\gamma} \alpha.$$
Let $e_1$ and $e_2$ be the elliptic Reeb orbits constructed in Lemma 3.1.

We now show how to extend $\alpha$ to the 1-sided neighborhood $N(T^2 \cup D)$ where $D$ is a bypass.

**Lemma 3.2.** Let $T^2 \subset (M, \xi)$ be a mixed torus with dividing set $\Gamma$. There exists a decomposition $N(T^2 \cup D) = N_1 \cup \Sigma N_2 \simeq T^2 \times [0, 1]$ and an extension of $\alpha$ to $N(T^2 \cup D)$ such that:

1. $N_i$ corresponds to contact $i$-handle attachment, for $i = 1, 2$;
2. $T^2 = T^2 \times \{0\}$ is convex with dividing set $e_1 \cup e_2$;
3. $T^2 \times \{1\}$ is convex with dividing set $e_4 \cup e_5$ which are elliptic orbits of Conley-Zehnder index 1 with respect to $T^2$;
4. $\Sigma$ is a genus 2 convex surface separating $N_1$ and $N_2$, meets $T^2 \times \{0\}$ along $e_1 \subset T^2 \times \{0\}$ and $T^2 \times \{1\}$ along $e_4$, and contains one other orbit, an elliptic orbit $e_3$ of Conley-Zehnder index 1 with respect to $\Sigma$;
5. the Reeb vector field $R_\alpha$ is positively transverse to $R_+$ and negatively transverse to $R_-$ for each of $T^2 \times \{0\}, T^2 \times \{1\}$, and $\Sigma$;
6. there exist hyperbolic orbits $h_2$ and $h_5$ in $N_1$ and $N_2$, respectively; they have Conley-Zehnder index 0 with respect to $T^2$;
7. the actions of $e_3$ and $e_4$ are larger than the actions of all other named orbits: $A_\alpha(e_3), A_\alpha(e_4) > A_\alpha(e_1), A_\alpha(e_2), A_\alpha(h_2), A_\alpha(h_5)$;
8. all other orbits contained in $N_1$ or $N_2$ have arbitrarily large action.

A schematic picture of the Reeb orbits in $N(T^2 \cup D)$ is given in Figure 3.

![Figure 3](image-url)

**Figure 3.** Sufficiently short Reeb orbits in $N(T^2 \cup D)$ which are strictly contained in $N_1$ and $N_2$. The $e_i$ are elliptic orbits and the $h_i$ are canceling hyperbolic orbits.
Figure 4. We construct $N_1^\epsilon$ by first thickening $T^2$ away from $e_1$, as on the left. We identify edges in the obvious way to form $T^2$, and we conflate $e_2$ with a parallel copy. We then perform a convex-to-sutured modification, which introduces the hyperbolic orbit $h_2$, as seen on the right. Though not drawn, the Reeb orbit $e_2$ persists on the right, becoming part of the interior of $N_1^\epsilon$.

Proof. Our goal is to extend the neighborhood of $T^2$ identified by Lemma 3.1 to a bypass neighborhood, and from the discussion in Section 2.4, we know that this can be accomplished by attaching to our neighborhood a canceling pair of contact 1- and 2-handles. We will extend the contact form $\alpha$ to this bypass neighborhood by extending it over each handle individually.

Let us write $T^2 \times [0, \epsilon]$ for (half of) the neighborhood produced by Lemma 3.1. Within this one-sided neighborhood we find $N_1^\epsilon$, a neighborhood obtained by “thickening $T^2$ away from $e_1$.” The boundary $\partial N_1^\epsilon$ has two torus components, which we abusively call $T^2 \times \{0\}$ and $T^2 \times \{\epsilon\}$, and which meet along the elliptic orbit $e_1$. In particular, $T^2 \times \{\epsilon\}$ is convex, with dividing set $\Gamma = \{e_1, e_2\}$. (Here we are conflating $e_2$ with a parallel copy in $T^2 \times \{\epsilon\}$.) The bypass $D$ is assumed to have endpoints on $e_2$. As discussed in Section 2.4, we may attach a contact handle to $N_1^\epsilon$ in a manner which respects the contact form whenever $N_1^\epsilon$ has a properly adapted neighborhood of $e_2 \subset T^2 \times \{\epsilon\}$. According to [CGHH, Lemma 4.1], we may modify $N_1^\epsilon$ by introducing a canceling hyperbolic orbit $h_2$ for $e_2$, as depicted in Figure 4. Following this modification, we attach a contact 1-handle to $N_1^\epsilon$ along a pair of points in $h_2$, extending the contact form $\alpha$ as we go, to yield the neighborhood $N_1$. This neighborhood has boundary components $T^2$ and $\Sigma$, where $\Sigma$ is a surface of genus 2 with dividing curves $e_1, e_3$, and $e_4$. We take the actions $\mathcal{A}_{\alpha}(e_3)$ and $\mathcal{A}_{\alpha}(e_4)$ to be much larger than $\mathcal{A}_{\alpha}(e_1), \mathcal{A}_{\alpha}(e_2)$, and $\mathcal{A}_{\alpha}(h_2)$.
We have now produced \(N_1 \subset N(T^2 \cup D)\), as depicted in Figure 3. Attaching a topologically canceling contact 2-handle to \(N_1\) along \(\Sigma\) will give \(N(T^2 \cup D)\), allowing us to identify \(N_2\). Our claims about the Reeb dynamics of \(N_2\) follow by viewing this region as the result of attaching a contact 1-handle to \(N_\epsilon \subset T^2 \times [1-\epsilon, 1] \subset N(T^2 \cup D)\), and repeating the analysis above. The last statement of the lemma follows from [V, Theorem 2.1], which investigates Reeb orbits near a bypass. In particular, while \(N(T^2 \cup D)\) may contain Reeb orbits not listed here, their actions will be strictly larger than those of \(e_1, e_2\). While the contact form \(\alpha\) may not, a priori, meet the hypotheses of [V, Theorem 2.1], [V, Proposition 2.2] allows us to modify \(\alpha\) so that [V, Theorem 2.1] may be applied.

\[\square\]

Lemma 3.3. Let \((B, \beta = -df \circ j)\) be a 2-dimensional Weinstein domain, where \(f : B \to \mathbb{R}\) is a Morse function such that \(\partial B\) is a level set of \(f\), and let \(\alpha = dt + \beta\) be a contact form on \([-\epsilon, \epsilon] \times B\), where \(t \in [-\epsilon, \epsilon]\). Then there is an adapted almost complex structure on \(\mathbb{R} \times [-\epsilon, \epsilon] \times B\) such that we can lift \(B\) to a holomorphic curve by the map \(u(x) = (f(x), 0, x)\).

Proof. The Liouville vector field \(X\) for \(\beta\) directs the characteristic foliation on \(B = \{0\} \times B\) and satisfies \(d\beta(X, \cdot) = \beta\) and \(\beta(X) = 0\). The Reeb vector field on \([-\epsilon, \epsilon] \times B\) is \(\partial_t\). The contact structure \(\ker(\alpha)\) is spanned by \(X\) and \(jX + g\partial_t\) for some function \(g : B \to \mathbb{R}\). Since \(0 = \alpha(jX + g\partial_t) = g + \beta(jX) = g + df(X)\) we have that \(g = -df(X)\).

We want the almost complex structure \(J\) to lift \(j\) so we specify

\[J(X) = X - df(X)\partial_t \quad J(\partial_t) = \partial_t.\]

In order to verify that \(u(x) = (f(x), 0, x)\) is \(J\)-holomorphic we verify

\[J(df(X), 0, X) = (df(jX), 0, jX).\]

Indeed,

\[(df(X), 0, X) = \beta(X) = (0, 0, jX)\]

and

\[J(df(X), 0, X) = (0, df(X), 0) + (0, -df(X), jX) = (0, 0, jX).\]

This shows that \(u\) is \(J\)-holomorphic. \(\square\)

We can lift the components \(R_+\) and \(R_-\) of \(T^2\) to Fredholm index 2 holomorphic curves in the symplectization \(\mathbb{R} \times M\) with positive ends at \(e_1\) and \(e_2\).

Lemma 3.4 ([We2, Prop. 7]). There are embedded holomorphic curves \(u_\pm : \mathbb{R} \times S^1 \to [0, \infty) \times M\) such that:
• $u_\pm$ are Fredholm regular and index 2.
• $u_\pm$ are positively asymptotic to $e_1$ and $e_2$.
• The image of $u_\pm$ under the projection $\pi : [0, \infty) \times M \to M$ is $R_\pm(T^2)$.

Proof. Consider the standard tight neighborhood $[-\epsilon, \epsilon] \times T^2$ of $T^2$. Let $R'_\pm$ be $R_\pm$ minus small collar neighborhoods of their respective boundary components. Then $\{0\} \times R'_+ \cup \{0\} \times R'_$ are Weinstein domains. By Lemma 3.3 they lift to holomorphic curves in the symplectization which have constant $s$ coordinate at the boundary.

We will construct holomorphic half cylinders in the standard neighborhood of Lemma 3.1 which are asymptotic to $e_1$ and $e_2$ which will glue to these lifts.

The vectors $v_1 = \partial_\rho$ and $v_2 = -g(\rho)\partial_\theta + f(\rho)\partial_\phi$ span the contact structure on $S^1 \times D^2$. Pick a smooth function $\beta(\rho) > 0$ and define $J$ by the condition $Jv_1 = \beta(\rho)v_2$. We will assume that $\beta(\rho) = 1$ outside a neighborhood of $\rho = 0$.

In conformal coordinates $(s, t)$, a map

$$ u(s, t) = (a(s, t), \theta(s, t), \rho(s, t), \phi(s, t)) $$

is $J$-holomorphic if

$$ a_s = f\theta_t + g\phi_t \quad \rho_s = \frac{1}{\beta D}(f'\theta_t + g'\phi_t) $$

$$ a_t = -f\theta_s - g\phi_s \quad \rho_t = -\frac{1}{\beta D}(f'\theta_s + g'\phi_s) $$

where $f$, $g$, $D$ and $\beta$ are all functions of $\rho(s, t)$. At the boundary the two equations on the right become

$$ \rho_s = -\theta_t, \quad \rho_t = \theta_s. $$

There are then solutions of the form

$$ u_{\phi_0} : [0, \infty) \times S^1 \to \mathbb{R} \times (S^1 \times \mathbb{D}) : (s, t) \mapsto (a(s, t), t, \rho(s), \phi_0) $$

for any choice of $\phi_0$, where $a(s)$ and $\rho(s)$ solve the ordinary differential equations

$$ \frac{da}{ds} = f(\rho), \quad \frac{d\rho}{ds} = \begin{cases} 
-\frac{1}{\beta(\rho)} \frac{f'(\rho)}{D(\rho)} & \text{if } \rho > \rho_0 \\
\frac{f'(\rho)}{\beta(\rho)D(\rho)} & \text{otherwise}
\end{cases} $$

Therefore there are holomorphic half cylinders $u_{\phi_0}$ for any choice of $\phi_0$. The conditions imposed on $f(\rho)$ and $g(\rho)$ imply that the curve $u_{\phi_0}$ with $\rho(0) = 1$ yields a holomorphic half-cylinder which is positively asymptotic to $e_1$ or $e_2$ as $s \to \infty$ and which has $a(s, t)$ and $\phi(s, t)$ constant near the boundary.
We want to glue these half cylinders to the lifts of \( R_+ \) and \( R_- \) to create the curves in the lemma. Consider \([-\epsilon, \epsilon] \times T^2 - N(\Gamma)\) where \( N(\Gamma) \) is the union of the standard neighborhood from Lemma 3.1. There is a diffeomorphism from \([-\epsilon, \epsilon] \times T^2 \) to \(([-\epsilon, \epsilon] \times T^2) - N(\Gamma)\) such that near the boundary \( t \to \phi \). Using this diffeomorphism we can then glue \( N(\Gamma) \) to \([-\epsilon, \epsilon] \times T^2\) such that the contact structures and Reeb orbits match at the boundary of each.

Let \( \phi_0 \) correspond to \( t = 0 \) under this diffeomorphism. Then we can glue the half cylinders asymptotic to \( e_1 \) and \( e_2 \) to the lifts of \( R_+ \) and \( R_- \) by specifying that \( a(1) = f_\pm(\partial R_\pm) \), where \( f_\pm \) is a Morse function on \( R_\pm \). These curves are Fredholm regular by automatic transversality cf. [We2, Proposition 7].

Since \( T^2 \) is mixed there is another bypass layer \( T^2 \times [-1, 0] \) stacked “on top” with \( T^2 \times [0, 1] \) as the “bottom layer”, see Figure 5. The orientation of the top layer is reversed because the bypass has opposite sign. Let \( P \) be a thrice-punctured sphere. We will construct holomorphic curves which represent the solid lines in Figure 5. The modification used to introduce hyperbolic orbits, taken from [CGHH, Lemma 4.1], provides a holomorphic curve between the hyperbolic orbit and its canceling elliptic orbit, so we need not construct these.

**Lemma 3.5.** There are embedded holomorphic curves

\[
  u_{i,j,k}^\pm : P \to [0, \infty) \times T^2 \times [-1, 1]
\]

and

\[
  u_{i,j}^\pm : \mathbb{R} \times S^1 \to [0, \infty) \times T^2 \times [-1, 1]
\]

for admissible \( \{i, j, k\} \) and \( \{i, j\} \) such that:

- \( u_{i,j,k}^\pm \) and \( u_{i,j}^\pm \) are Fredholm regular and have index 2 and
- \( u_{i,j,k}^\pm \) are positively asymptotic to \( e_i, e_j, \) and \( e_k \) and \( u_{i,j}^\pm \) are positively asymptotic to \( e_i \) and \( e_j \).

The admissible \( \{i, j, k\} \) and \( \{i, j\} \) are \{1, 7, 6\}, \{1, 3, 4\}, \{1, 6\}, \{1, 4\}, \{4, 5\}, \{6, 8\} and the \( u^+ \) and \( u^- \) are distinguished by whether the orientations of their projections to \( M \) agree with \( R_+ \) or \( R_- \) with respect to the orientation coming from \( T^2 \).

These curves are represented by solid lines in Figure 5. Note that we also have \( u_{1,2}^\pm \), produced by Lemma 3.4.

**Proof.** Recall that in a neighborhood of an elliptic orbit \( e_i \) there are holomorphic half cylinders of the form

\[
  u_{\phi_i} : [0, \infty) \times S^1 \to \mathbb{R} \times (S^1 \times D) \quad (s, t) \mapsto (a(s), t, \rho(s), \phi_i).
\]
Choose \( \phi_i \neq \phi_0 \) and let \( A_{\phi_i} \) be the image of \( \phi_i \). If \( P \) is a thrice-punctured sphere we can repeat the procedure of Lemma 3.4 to lift \( P \) minus the three ends to a holomorphic curve and glue the boundary to \( A_{\phi_i} \). These curves have \( \text{ind} = 2 \) by a straightforward index calculation and are Fredholm regular by [We2, Prop. 7].

Let \( \mathcal{M}(e_1, e_2) \) denote the moduli space of \( \text{ind} = 2 \) curves \( u : \mathbb{R} \times S^1 \to \mathbb{R} \times M \) which are positively asymptotic to \( e_1 \) and \( e_2 \) and represent the same homology class as \( u_+ \) or \( u_- \) — the curves identified in Lemma 3.4 — and let \( \mathcal{M}(e_1, e_2)/\mathbb{R} \) be the quotient by the \( \mathbb{R} \)-translation. We can now describe the compactification of this moduli space. One can learn about the analysis of such moduli spaces in, for instance, [We4].

**Lemma 3.6.** The compactification \( \overline{\mathcal{M}(e_1, e_2)}/\mathbb{R} \) is the disjoint union of two components \( \mathcal{N}_\pm \) containing the equivalence classes of \( u_\pm \) up to \( \mathbb{R} \) translation. The boundary \( \partial \mathcal{N}_\pm \) consists of

- a two-level building \( v_{1,\pm} \cup v_{0,\pm} \), where \( v_{1,\pm} \) is the top level consisting of a cylinder positively asymptotic to \( e_2 \) and negatively asymptotic to \( h_2 \) and \( v_{0,\pm} \) is the bottom level consisting of a cylinder positively asymptotic to \( e_1 \) and \( h_2 \) and
• another two-level building $v_1' \cup v_0'$ with $h_2$ replaced by $h_2'$.

Let $A_{\alpha}$ denote the $\alpha$-action of a Reeb orbit.

Proof. We begin by claiming that $M(e_1, e_2)/\mathbb{R}$ is not compact. We think of this moduli space as parametrizing curves $u : \mathbb{R} \times S^1 \to M$. Notice that these curves do not intersect, since the corresponding curves in $\mathbb{R} \times M$ are prohibited from intersecting by positivity of intersections. Now if $M(e_1, e_2)/\mathbb{R}$ contains a component diffeomorphic to $S^1$, then the curves in this component sweep out a neighborhood of $e_1$. That is, there is a neighborhood $N(e_1)$ of $e_1$, each of whose points lies on the image of some curve in $M(e_1, e_2)/\mathbb{R}$. But $N(e_1)$ must intersect $N_2$, meaning that some curve in $M(e_1, e_2)/\mathbb{R}$ intersects $\pi_M(u_{i, j, k}^\pm)$ for some admissible $\{i, j, k\}$. Because the curves in $M(e_1, e_2)$ are homologous to $u_+ \text{ or } u_-$, which do not intersect any $u_{i, j, k}^\pm$, this is a contradiction.

So $M(e_1, e_2)/\mathbb{R}$ is not compact, and the compactification $\overline{M(e_1, e_2)/\mathbb{R}}$ is therefore a disjoint union of closed intervals. We now investigate $\partial M(e_1, e_2)/\mathbb{R}$.

We may assume $A_{\alpha}(e_1) = A_{\alpha}(e_2)$. By [V, Theorem 2.1] the only Reeb orbits that could possibly have smaller action than $A_{\alpha}(e_i)$, $i = 1, 2$, are those in Figure 5. When enumerating the buildings which might appear in $\partial M(e_1, e_2)/\mathbb{R}$, these are the only orbits we need consider. Moreover, because the curves in $M(e_1, e_2)/\mathbb{R}$ are disjoint from the projections $\pi_M(u_{i, j, k}^\pm)$, their images must be contained in $N_1 \cup N_2'$.

With these restrictions, we find just four boundary elements. We see that $\partial M(e_1, e_2)/\mathbb{R}$ can contain a cylinder positively asymptotic to $e_2$ and negatively asymptotic to $h_2$ followed by a cylinder positively asymptotic to $e_1$ and $h_2$. The same is true for $h_2$ replaced by $h_2'$. These buildings are the desired boundary elements. Indeed, these are the only holomorphic buildings contained in $N_1$ or $N_2'$ having at least one level with a curve positively asymptotic to $e_2$. We conclude that $\overline{M(e_1, e_2)/\mathbb{R}}$ consists of two closed intervals, as desired. □

In order to cut along $T^2$ we need to push this index 2 family of curves into the filling $(W, \omega)$.

**Lemma 3.7.** There is a regular 1-parameter family

$$\mathcal{S} = \{u_t : (\mathbb{R} \times S^1, j) \to (\hat{W}, J) | du_t \circ j = J \circ du_t\}$$

of embedded holomorphic cylinders in $(\hat{W}, \hat{\omega})$ parametrized by $t \in \mathbb{R}$ satisfying

(C1) When $t \gg 0$ the images $\Sigma_t$ and $\Sigma_{-t}$ of the curves $u_t$ and $u_{-t}$ are in the symplectization $[0, \infty) \times M$. 
(C2) When $t \gg 0$ their projections under the map $\pi : [0, \infty) \times M \to M$ are $R_+(T^2)$ and $R_-(T^2)$ respectively.

(C3) $\text{Im}(u_t) \cap \text{Im}(u_{t'}) = \emptyset$ if $t \neq t'$.

Proof. Consider the ind = 1 family $\mathcal{M}_{\tilde{W}}(e_1, h_2)$ consisting of holomorphic cylinders in $\tilde{W}$ that limit to $e_1$ and $h_2$ at the positive ends and represent the same homology class as $v_{0,+}$ or $v_{0,-}$ from Lemma 3.6.

Our first claim is that $\mathcal{M}_{\tilde{W}}(e_1, h_2)$ contains a noncompact component which interpolates between $v_{0,+}$ and $v_{0,-}$. The failure of $\mathcal{M}_{\tilde{W}}(e_1, h_2)$ to be compact follows as in Lemma 3.6. Namely, the elements of $\mathcal{M}_{\tilde{W}}(e_1, h_2)$ are disjoint from the “walls” $u_{\pm}, u_{1,7,6}^\pm, u_{1,3,4}^\pm, u_{4,5}^\pm, u_{6,8}^\pm$ and their $\mathbb{R}$-translations, and this prohibits $\mathcal{M}_{\tilde{W}}(e_1, h_2)$ containing an $S^1$-family of curves whose projections to $M$ would encircle $e_1$.

Let us now consider the noncompact ends of $\mathcal{M}_{\tilde{W}}(e_1, h_2)$. Because ours is an ind = 1 family, we may ignore bubbling phenomena and consider the holomorphic buildings into which elements of $\mathcal{M}_{\tilde{W}}(e_1, h_2)$ might break. Say $w$ is the topmost element of such a building, with image in $\mathbb{R} \times M$. Because of the walls identified above, $\pi_M \circ w$ must be contained in $N_1, N'_2, N_2, N'_1$. We claim that $w = v_{0,+}$ or $w = v_{0,-}$. Assume without loss of generality that the slopes of $\Gamma_{T^2 \times \{0\}}$ and $\Gamma_{T^2 \times \{1\}}$ are 0 and 1, respectively. Under the identification $H_1(T^2 \times [-1,1]) \simeq H_1(T^2) \simeq \mathbb{Z}^2$, we can take $[e_1] = (0, -1)$ and $[e_2] = (0, 1)$. Then $[h_2'] = [h_2] = (0, 1)$, $[e_3] = (-1, 0)$, and $[e_4] = [e_5] = [h_5] = (1, 1)$. If $\text{Im}(\pi \circ w) \subset N_2$, then $w$ must have $e_1$ at the positive end; however, no nonnegative linear combination of $[e_3], [e_4], [h_5], [e_5]$ is homologous to $[e_1]$. If $\text{Im}(\pi \circ w) \subset N_1$, then either

1. $e_1$ is at the positive end
2. $h_2$ is at the positive end, or
3. both $e_1$ and $h_2$ are at the positive end.

The only possibility is $[h_2] = [e_3] + [e_4]$, but we are taking $A_b(h_2) < A_a(e_3) + A_a(e_4)$ which is a contradiction. This implies the claim.

So $\mathcal{M}_{\tilde{W}}(e_1, h_2)$ provides an interval of holomorphic curves interpolating between $v_{0,+}$ and $v_{0,-}$, and serves as the middle portion of our family $S$. For $t \gg 0$, take $u_t$ (resp. $u_{-t}$) to be a translation of $v_{0,+}$ (resp. $v_{0,-}$) by some $t + c$, where $c$ is a constant, viewed inside the symplectization part $[0, \infty) \times M$. This implies (C1). Because the curves $v_{0,+}$ and $v_{0,-}$ each have an end asymptotic to $h_2$ rather then $e_2$, item (C2) is not met precisely on the nose. But $e_2$ and $h_2$ are canceling orbits, so we may isotope $T^2$ so that $R_+(T^2) = \text{Im}(\pi \circ u_t)$ and $R_-(T^2) = \text{Im}(\pi \circ u_{-t})$ for $t \gg 0$.

We now prove (C3). For large $t \neq t'$ the images of $u(t)$ and $u(t')$ are disjoint so their intersection number $i(u_+(t); u_+(t')) = 0$. The
intersection number is a relative homology invariant, so positivity of intersections ensures that no new intersections are introduced as we push into $W$. □

Each of the holomorphic cylinders $u_t$ is embedded; we now claim that the family $S$ sweeps out a properly embedded solid torus in $\hat{W}$. Namely, we show that distinct cylinders are disjoint.

**Lemma 3.8.** The map $\iota : \mathbb{R} \times \mathbb{R} \times S^1 \to \hat{W}$ defined by

$$\iota(t, s, \theta) := u_t(s, \theta)$$

is an embedding.

**Proof.** As noted, each curve $u_t$ is an embedding, and thus we may think of $u_t'$, for $t'$ near $t$, as a holomorphic section of the normal bundle $N_{u_t}$. The first Chern number of this bundle is given by $c_1(N_{u_t}) = c_1(u_t^*T\hat{W})$, so we compute $c_1(u_t^*T\hat{W})$. We have (cf. [We3, Equation 1.1])

$$2c_1(u_t^*T\hat{W}) = \text{ind}(u_t) - \mu_{CZ}(u_t),$$

with the last term being a signed count of the Conley-Zehnder indices of the orbits to which $u_t$ is asymptotic. Our family of curves has index 1, and $\mu_{CZ}(e_1) = 1$, while $\mu_{CZ}(h_2) = 0$, so $c_1(u_t^*T\hat{W}) = 0$. We conclude that $N_{u_t}$ has trivial first Chern number, and thus that its holomorphic sections are zero-free. So $\iota$ is an embedding. □

We want to remove $S \cap W$ from $W$. In order to do this we first modify $W$ slightly. Consider $W_R = W \cup ([0, R] \times M)$, where $R$ is large so that there exist $u_T$ and $u_{-T}$ whose images are in $[0, \infty) \times M$ and whose $\pi$-projections after restricting to $[0, R] \times M$ are $R'_+$ and $R'_-$ which are $R_\pm$ minus small collar neighborhoods. Then form $W'_R = W_R - \tilde{\Gamma}(\Gamma_{T_2})$, where $\tilde{\Gamma}(\Gamma_{T_2})$ is a small (half-)tubular neighborhood of $\{R\} \times \Gamma_{T_2}$ in $W_R$. Note that $W'_R$ has corners; we will refer to the horizontal boundary $\partial_h W'_R = S^1 \times D^2 = \partial W'_R - \partial W_R$ as well as the vertical boundary $\partial_v W'_R = \partial W'_R - \partial_h W'_R$ of $W'_R$. We assume that $\{R\} \times R'_\pm = \{R\} \times R_\pm - \tilde{\Gamma}(\Gamma_{T_2})$.

**Lemma 3.9.** There exists an embedding $\Sigma \times [-T - 1, T + 1] \subset W'_R$ such that:

1. $\Sigma \times \{t\}$ is an annulus and is a symplectic submanifold of $W'_R$, for $t \in [-T - 1, T + 1]$;
2. $\Sigma \times \{(T + 1)\} = \{R\} \times R'_\pm$;
3. for $t \in [-T - 1, T + 1]$, $\partial \Sigma \times \{t\} = S^1 \times \gamma(t) \subset \partial_h W'_R$, where $\gamma(t)$ is a straight arc from $(-1, 0)$ to $(1, 0)$ in $D^2$. 


Proof. First note that the family $\Sigma_t, t \in [-T, T]$, restricted to $W'_R$, gives rise to an embedding $\Sigma \times [-T, T] \subset W'_R$ that satisfies the conditions of the lemma except for $\Sigma \times \{ \pm T \} = \{ R \} \times R'_\pm$. For $t \gg 0$ the curves $u_{\pm t}$ have the form $u_{\pm t}(x) = (f(x), 0, x)$ in $\mathbb{R} \times \mathbb{R} \times R'_\pm$ by Lemma 3.3. We can interpolate symplectically from $\Sigma_{\pm t} = \text{Im}(u_{\pm t})$ to $\Sigma_{\pm(t+1)} = R'_\pm$ through symplectic subsurfaces of the form $(c f(x), 0, x)$ for $c \in [0, 1]$. A slight modification of $\Sigma \times [-T-1, T+1]$ near $\partial \Sigma \times [-T-1, T+1]$ ensures the desired behavior of $\partial \Sigma \times \{ t \}$. □

Let $S' = \Sigma \times [-T-1, T+1]$ with coordinates $(x, t)$. This is a solid torus in $W'_R$, and we now work to normalize a neighborhood $N(S')$ of $S'$. In particular, we give $N(S')$ the structure of a symplectic handle, the removal of which will produce our new symplectic filling $W' = W'_R - N(S')$. In case the original filling is exact, we construct a Liouville form on $W'$.

Lemma 3.10. After slight adjustments of $S'$ and $W'_R$, there exist a neighborhood $N(S') = S' \times [-\epsilon, \epsilon]_w \subset W'_R$ and a 1-form $\lambda = \lambda_S + \lambda_B$ on $N(S')$ such that:

1. $\Sigma \times \{ -T-1, T+1 \} \times [-\epsilon, \epsilon] \subset \partial_x W'_R$ and $(\partial \Sigma) \times [-T-1, T+1] \times [-\epsilon, \epsilon] \subset \partial_x W'_R$;
2. $\lambda_S$ is the Liouville form for $R'_+$ and, after adjusting $\partial_x W'_R$, also agrees with the Liouville form for $R'_-$;
3. $\lambda_B$ is a 1-form on $B = [-T-1, T+1] \times [-\epsilon, \epsilon]$;
4. $d\lambda$ agrees with the symplectic form on $W'_R$;
5. $\lambda$ agrees with the Liouville form on $W'_R$ near $\partial W'_R$;
6. the Liouville vector field $X_{\lambda}$ points into $N(S')$ along $w = \pm \epsilon$.

Proof. Recall that if our original filling is exact, we have denoted the Liouville form on $W'_R$ by $\beta$. We let

$N(S') = \Sigma \times [-T-1, T+1]_t \times [-\epsilon, \epsilon]_w \subset W'_R$

be a neighborhood of $S'$, and, if our filling is exact, let $\Lambda$ denote the restriction of $\beta$ to $N(S')$. If our original filling is only assumed to be strong, we take $\Lambda$ to be any 1-form defined on $N(S')$ with $d\Lambda = \omega$. We will modify $N(S')$ and $\Lambda$ to achieve the desired properties. For $(t, w) \in [-T-1, T+1] \times [-\epsilon, \epsilon]$, we write $\Sigma_{t, w} = \Sigma \times \{ t \} \times \{ w \}$, and denote by $\lambda_{t, w}$ the restriction of $\Lambda$ to $\Sigma_{t, w}$.

Our first observation is that, using the natural identification of $\Sigma_{t, w}$ with $\Sigma$, we may assume that all of the 1-forms $\lambda_{t, w}$ agree on $\partial \Sigma$. Indeed, it is enough to show that the $\lambda_{t, w}$-length of the components of $\partial \Sigma$ is independent of $(t, w)$. By extending $\Sigma_{t, w}$ towards the dividing set (that is, by taking a smaller neighborhood of $\Gamma_{T^2}$ in the construction of $W'_R$),
we see that \( \Sigma_{t,w} \) may be taken to have arbitrarily large \( d\lambda_{t,w} \)-area, and we choose to extend each \( \Sigma_{t,w} \) in such a way that this area is independent of \( (t, w) \). Moreover, the Liouville forms \( \Lambda_{\pm(T+1),w} \) agree with those of \( R_{\pm} \), and thus with each other. We call this common 1-form \( \lambda_{\Sigma} \).

Now that we have normalized \( \Lambda \) on \( \partial(S \times [-T - 1, T + 1]) \times [-\epsilon, \epsilon] \), we begin the normalization process on

\[
S' = \Sigma \times [-T - 1, T + 1] \times \{0\}.
\]

We write \( \Lambda|_{S'} = \lambda_t + f_t dt \) for some smooth function \( f_t \), where \( \lambda_t := \lambda_{t,0} \).

So along \( S' \) we have

\[
d\Lambda = d\lambda_t + (d^2 f_t - \dot{\lambda}_t) dt,
\]

with \( d_\Sigma \) denoting the derivative in the \( \Sigma \)-direction. Now apply a \( \Sigma \)-fiberwise diffeomorphism to obtain \( d_\Sigma \lambda_t = d\lambda_{\Sigma} \) for all \( t \in [-T - 1, T + 1] \).

We choose this diffeomorphism so that it restricts to the identity on \( \Sigma \times \{\pm(T+1)\} \). Notice that the characteristic line field \( \ker d\Lambda \) is transverse to the \( \Sigma \)-fibers of \( S' \). This allows us to apply a diffeomorphism after which \( \ker d\Lambda \) is directed by \( \partial_t \). So we have the following normalization on \( S' \):

\[
(3.2) \quad d^2 f_t = \dot{\lambda}_t, \quad d_\Sigma \lambda_t = d\lambda_{\Sigma}, \quad \ker d\Lambda = \mathbb{R}\langle \partial_t \rangle,
\]

where we recall that \( \Lambda = \lambda_t + f_t dt \). Our next goal is to modify \( \Lambda \) so that \( \lambda_t = \lambda_{\Sigma} \) and \( f_t \) is independent of the \( \Sigma \)-coordinate.

Note that the first equation of \((3.2)\) allows us to write

\[
\lambda_t = \lambda_{\Sigma} + d_\Sigma h_t
\]

for some compactly supported function \( h_t \) on \( \Sigma \). We extend \( h \) to a neighborhood of \( S' \subset N(S') \) and observe that \( dh = 0 \) along \( t = \pm(T+1) \), since \( \lambda_{\pm(T+1)} = \lambda_{\Sigma} \). It follows that the Liouville form \( \Lambda' := \Lambda - dh \) is Liouville homotopic to \( \Lambda \). Notice that restricting to \( S' \) yields

\[
\Lambda' = \lambda_{\Sigma} + \left( f_t - \frac{\partial h}{\partial t} \right) dt \quad \text{and} \quad d\Lambda' = d\Lambda = d\lambda_{\Sigma}.
\]

It follows that \( f_t - \frac{\partial h}{\partial t} \) is independent of the \( \Sigma \)-coordinate, as desired.

By performing the Liouville homotopy \( \Lambda \to \Lambda' \) parametrically with respect to \( w \), we may write

\[
\Lambda' = \lambda_{\Sigma} + f dt + gw
\]

on all of \( N(S') \), for some smooth functions \( f \) and \( g \).

At last, we investigate the Liouville dynamics of \( (N(S'), \Lambda') \). A Moser argument allows us to assume that \( f, g \) are such that

\[
d\Lambda' = d\lambda_{\Sigma} + dt \wedge dw,
\]
and then the Liouville vector field $X$ for $\Lambda'$ is given by

$$X = X_\Sigma + g \partial_t - f \partial_w,$$

where $X_\Sigma$ is the Liouville vector field for $(\Sigma, \lambda_\Sigma)$. For $t = \pm(T + 1)$ we have $\pm g > 0$, and we homotope $g_0 := g$ to the function $g_1 := Nt$ for some $N \gg 0$, with the homotopy satisfying

- $\frac{dg_\tau}{d\tau}$ is nondecreasing with respect to $\tau$;
- $\pm g_\tau > 0$ along $t = \pm(T + 1)$.

This homotopy of $g$ gives us a Liouville homotopy $\Lambda_\tau = \lambda_\Sigma + fdt + g_\tau dw$. Notice that the $\partial_w$-component of the Liouville vector field associated to $\Lambda_\tau$ points out of $t = \pm(T + 1)$, thanks to our second condition on $g_\tau$. We have now modified our Liouville form to become

$$\Lambda_1 = \lambda_\Sigma + fdt + Ntdw.$$

The corresponding Liouville vector field points out of $N(S')$ along $t = \pm(T + 1)$, and we now add a correcting term near $S' = \{w = 0\}$ in order to achieve the desired behavior along the remainder of $\partial N(S')$. Namely, let $\varphi : [-\epsilon, \epsilon] \to [0, 1]$ vanish outside of $[-\epsilon/2, \epsilon/2]$, with $\varphi \equiv 1$ near 0. Then we define

$$\lambda := \Lambda_1 + \varphi(w)(w - f)dt,$$

noting that $d\lambda$ is symplectic and that $\lambda|_{S'} = \lambda_\Sigma + Ntdw$. By redefining $N(S')$ to be

$$N(S') = \Sigma \times [-T - 1, T + 1] \times [-\epsilon/4, \epsilon/4]$$

we obtain the desired standard neighborhood of $S'$. □

The following lemma clarifies that if the original filling is exact, then the Liouville form constructed above for $N(S')$ can indeed be constructed globally on $W' = W'_R - N(S')$.

Lemma 3.11. If $(W, \beta)$ is an exact filling, then there exists a 1-parameter family of Liouville forms $\beta_\tau, \tau \in [0, 1]$, on $W'_R$ such that $\beta_0 = \beta$ and $\beta_1 = \lambda$ on $N(S') \cap \{-\epsilon/2 \leq w \leq \epsilon/2\}$.

Proof. Since $d\beta$ and $d\lambda'$ agree on $N(S')$, there exists a function $f$ on $N(S')$ such that $\lambda' - \beta = df$. We can choose $f$ such that $f = 0$ on $\partial W'_R$. Next modify $f$ to $g$ on $N(S')$ such that $g = f$ for $w \in [-\epsilon/2, \epsilon/2]$ and $g = 0$ for $w = \pm \epsilon$; then extend $g$ by 0 to all of $W'_R$. Now consider the 1-parameter family of Liouville forms $\beta_\tau = \beta + \tau dg$. Clearly $\beta_0 = \beta$ and $\beta_1 = \lambda'$ on $N(S') \cap \{-\epsilon/2 \leq w \leq \epsilon/2\}$. □

Finally we explain how to obtain $W$ from $W' := W'_R - N(S')$. This construction uses Theorem 2.5. After cutting $(M, \xi)$ along the mixed torus, we can find two disjoint copies of $\Sigma$ inside $W'$. By construction
Σ is a Liouville domain. As discussed in Section 2.7, the proof of Theorem 2.5 proceeds by attaching a symplectic handle to a collar neighborhood of \((M, \xi)\) in \((W', \omega)\). After attaching this handle we obtain \((W, \omega)\) with convex boundary \(#(\Sigma, \beta) (M, \xi)\) as desired.

4. Proof of Theorem 1.3

We will now prove Theorem 1.3 using Theorem 1.1. Let \((M', \xi')\) be the contact manifold obtained from \((M, \xi)\) by Legendrian surgery on \(S_+ S_- (L)\).

Let \((W, \omega)\) be an exact filling of \((M', \xi')\). Consider the standard neighborhood \(N(S_- (L)) \subset M\) of \(S_- (L)\). Let \(V_1\) be the solid torus obtained from \(N(S_- (L))\) by Legendrian surgery along \(S_+ S_- (L)\). Let \(V_2 = M - N(S_- (L))\). Then \(M' = V_1 \cup V_2\).

The torus \(T = \partial N(S_- (L))\) is a mixed torus because stabilizing twice with opposite signs is equivalent to performing two bypasses with opposite signs. Theorem 1.1 then guarantees that we can decompose \(W\) into a manifold \(W'\) such that \(\partial W' = M_1 \sqcup M_2\), where \(M_1 = V_1 \cup \partial S' S'\) and \(M_2 = V_2 \cup \partial S' S'\). Notice that, by construction, \(\partial W'\) is disconnected. The contact structures on \(M_1\) and \(M_2\) are obtained by using the canonical tight contact structure on the solid torus \(S'\).

The choice of \(S'\) is not, a priori, unique, but we claim that it is so. Take an oriented identification of \(\partial N(S_- (L))\) with \(\mathbb{R}^2 / \mathbb{Z}^2\) such that the meridian of \(N(S_- (L))\) has slope 0 and \(\Gamma_{\partial N(S_- (L))}\) has slope \(\infty\). With respect to this identification, \(\Gamma_{\partial N(L)}\) has slope 1 and \(\Gamma_{\partial N(S_+ S_- (L))}\) has slope \(-1\). The meridian \(\mu_{V_1}\) of \(V_1\) has slope \(-1/2\). The boundary of the solid torus \(S'\) has the same dividing set as \(V_1\), but the meridian \(\mu(S')\) is undetermined. Since the shortest integer vector representing the meridian must form an integer basis with the shortest integer vector representing the dividing set, the possible choices for \(\mu(S')\) are of the form \((1, m)\) for \(m \in \mathbb{Z}\).

Observe that since \(M_i\) is fillable it must be tight. We want to compute which choices of \(\mu(S')\) yield tight contact structures on \(M_1\) and \(M_2\) using the classification of tight contact structures from [H]. The choices for \(\mu(S')\) are compiled in Table 1. First consider \(M_1\). On the \(S'\) part the contact planes rotate from the meridian of \(S'\) to the dividing set \(\Gamma\) in a counterclockwise manner viewed using the identification with \(\mathbb{R}^2 / \mathbb{Z}^2\) as in Figure 6 and on the \(V_1\) part they rotate from \(\Gamma\) to the meridian. Rotation by more than \(\pi\) results in an overtwisted contact structure which contradicts the fillability of \(M_1\). From Figure 6 we see that this eliminates the possibility \(m \leq -1\).
On $M_2$ we see that if $m > 1$ then the slopes of the dividing curves rotate more than $\pi$. If $m = 1$ then we can find a solid torus with convex boundary and boundary slope 0 by taking the union of $N(L) - N(S(L))$ with $S'$, which is then overtwisted by Giroux's flexibility theorem. This leaves $\mu(S') = (1, 0)$ as the only option. With this choice, $M_1 \simeq (S^3, \xi_{std})$ and $M_2 \simeq (M, \xi)$ and $M_1$ has a unique exact filling.

Now because $(S^3, \xi_{std})$ is supported by a planar open book decomposition, [Et, Theorem 4.1] tells us that $(S^3, \xi_{std})$ is not symplectically co-fillable. That is, there is no connected, symplectic manifold $(X, \omega)$ with disconnected convex boundary, one of whose boundary components is $(S^3, \xi_{std})$. Since $\partial W' = (S^3, \xi_{std}) \sqcup (M, \xi)$, we conclude that $W'$ is the disjoint union of a filling $(W_0, \omega_0)$ of $(M, \xi)$ and $(B^4, \omega_{std})$.

From Theorem 1.1 we know that the filling $(W, \omega)$ of $(M', \xi')$ is obtained from $(W', \omega')$ by attaching a round symplectic 1-handle. Our final claim is that this corresponds to attaching a symplectic 2-handle to $(W_0, \omega_0)$ along $S_+ S_-(L)$. Indeed, round symplectic 1-handle attachment amounts to the attachment of a Weinstein 1-handle, followed by a Weinstein 2-handle passing over that 1-handle. The 1-handle is attached along two copies of $B^3$, one taken from each copy of $S'$ in $W'$. The effect of this is to “cancel” $B^4$, leaving us with $(W_0, \omega_0)$. The two copies of $S'$ are joined by the 1-handle to form a single solid torus in $W_0$, the core curve of which is $S_+ S_-(L)$. The round 1-handle attachment is completed by attaching a symplectic 2-handle along this curve. This proves Theorem 1.3.

Acknowledgement: The author would especially like to thank Austin Christian for his invaluable help and discussions. The author would like to thank Burak Ozbagci for helpful discussions during his time at
Table 1. Choices of meridian for $\partial S'$ using the identification $N(S_-(L)) \simeq \mathbb{R}^2/\mathbb{Z}^2$. X's correspond to overtwisted contact structures.

A UCLA and for suggestions on this paper. The author would also like to thank John Etnyre for helpful suggestions.

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