Quantized alternate current on curved graphene

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Based on the numerical solution of the quantum lattice Boltzmann method in curved space, we predict the onset of a quantized alternating current on curved graphene sheets. Such numerical prediction is verified analytically via a set of semi-classical equations relating the Berry curvature to real space curvature. The proposed quantised oscillating current on curved graphene could form the basis for the implementation of quantum information processing algorithms.

I. INTRODUCTION

In recent years, the most puzzling features of quantum mechanics, such as entanglement and non-local "spooky action at distance", long regarded as a sort of extravagant speculative speculations, have received spectacular experimental confirmation, [1–4]. Besides their deep fundamental implications, such phenomena may also open up transformative scenarios for material science and related applications in quantum computing and telecommunications, [5, 6]. Along with such burst of experimental activity, a corresponding upsurge of theoretical and computational methods has also emerged in the last two decades, including, among others, new quantum-many body techniques, quantum simulators [7–10] and quantum walks [11].

Quantum walks were first introduced by Ahronov and collaborators in 1993, [12], just a few months before the appearance of the first quantum lattice Boltzmann scheme [13], which was only recently recognized to be a quantum walk too, [14]. Quantum walks, [15, 16], are currently utilized to investigate exotic states of quantum matter [17] and to design new materials and technologies for quantum engineering applications [18].

Quantum walks can also help exploring the emergence of classical behaviour in the limit of a vanishing De Broglie length [19]. Likewise, quantum cellular automata, [20–22], can be used for simulating complex systems in analogy with their classical counterparts.

Finally, quantum walks have also shown connections with topological aspects of quantum mechanics, most notably the Berry phase [23]. Indeed, Berry connection and Berry curvature can be understood as a local gauge potential and gauge field, respectively and they define a Berry phase as introduced in 1984 [23]. The Berry phase has important implications as an analytic tool in topological phases of matter [24] and, under suitable conditions, it can also be related to real space curvature [25], thus providing a potential bridge between the classical and quantum descriptions of a given system.

As of quantum materials, graphene presents one of the most promising cases for realizing a new generation of quantum devices[26–28]. Indeed, since its discovery [29], this flatland wonder-material has not ceased to surprise scientists with its amazing mechanical and electronic behaviour. For example, stacking graphene sheets at specific angles has shown spectacular indications of superconductivity and other exotic properties [30].

Tunable transport properties are a basic requirement in electronic devices and specifically in graphene [31]. Furthermore, it has been shown that graphene sheets can be curved in such a way as to trap particles [32], thus opening further prospects for technological applications based on localized quantum states.

In this work, we propose the generation of a quantized oscillating current on curved graphene, which could be used in conjunction with trapped fermions for the realization of quantum cellular automata.

Electron transport is simulated by numerically solving the Dirac equation in curved space [32, 33] using an extension to curved space of the quantum lattice Boltzmann method [34]. In addition, a simpler representation of the system is solved analytically through a set of semi-classical equations of motion, relating Berry to real space curvature.

The paper is organized as follows. First, we introduce the Dirac equation and its extension to curved space and specifically deformed graphene. In the subsequent section, we present the results of numerical simulations and finally we conclude with a summary and outlook section.

A detailed description of the numerical model is provided in the Appendix (Appendix A).

II. THE DIRAC EQUATION IN CURVED SPACE AND GRAPHENE

The Dirac equation in curved space can be written in compact notation as follows:

\[(i\gamma^\mu D_\mu - m)\Psi = 0, \quad (1)\]

in natural units \(h = c = 1\), where \(m\) is the particle mass, the index \(\mu = 0, 1, 2\) runs over 2D space-time. In the
above, \( \Psi = (\Psi^+, \Psi^-) = (\psi_1^+, \psi_2^+, \psi_1^-, \psi_2^-) \in \mathbb{C}^4 \) denotes the Dirac four-spinor, and \( \gamma^\mu = \gamma^\alpha e_\alpha^\mu \) are the generalized \( \gamma \)-matrices, where \( \gamma^\alpha \in \mathbb{C}^{4 \times 4} \) are the standard \( \gamma \)-matrices (in Dirac representation). The symbol \( e_\alpha^\mu \) is the tetrad (first index: flat Minkowski, second index: curved spacetime).

Here, the tetrad is defined by \( e_\alpha^\mu g_{\mu\nu} e_\beta^\nu = \eta_{\alpha\beta} \), where \( g_{\mu\nu} \) denotes the metric tensor and \( \eta_{\alpha\beta} \) is the Minkowski metric. The tetrad basis is chosen such that the standard Dirac matrices can be utilised with no need to transform to a new coordinate basis. The symbol \( D_\mu \) denotes the covariant spinor derivative, defined as \( D_\mu \Psi = \partial_\mu \Psi + \Gamma_\mu \Psi \), where \( \Gamma_\mu \) denotes the spin connection matrices given by \( \Gamma_\mu = -\frac{1}{2} \omega^{\alpha\beta}_\mu \sigma_{\alpha\beta} \), where \( \sigma_{\alpha\beta} = \frac{1}{2} [\gamma_\alpha, \gamma_\beta] \) and \( \omega^{\alpha\beta}_\mu = \epsilon^\alpha_{\nu\mu} \nabla_\nu e^\beta \). The Dirac equation in curved space describes quantum relativistic Dirac particles (e.g. electrons) moving on arbitrary manifold trajectories.

A. Theory of strained graphene

Using the tight binding Hamiltonian to describe the bi-partite lattice of graphene, it is established that in the low-energy limit, the dispersion relation is linear, as described by the Dirac cones at the corners of the first Brillouin zone, which can be described by the following Dirac Hamiltonian:

\[
H_D = -i \int \Psi^\dagger \gamma_0 \gamma_i \partial_i \Psi \, d^2 x, \tag{2}
\]

in natural units, where \( \Psi \) is in the chiral representation.

In the context of graphene, the general Dirac spinor is defined as \( \Psi = (\psi^K_A, \psi^K_B) = (\psi_1^K, \psi_2^K, \psi_1^K, \psi_2^K) \), for sub-lattices \( A, B \) and valleys \( K, K' \).

The equation of motion stemming from this Hamiltonian is precisely the Dirac equation.

In this work, we consider a static space-time metric, with trivial time components

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ij} \end{pmatrix},
\]

where the latin indices run over the spatial dimensions.

This simplifies the Dirac equation Eq. (1) to

\[
\partial_i \Psi + \sigma^\alpha e_\alpha^i (\partial_i + \Gamma_i) \Psi = 0 - i \gamma_0 m \Psi, \tag{3}
\]

with \( \sigma^\alpha = \gamma^0 \gamma^\alpha \). After addition of external vector and scalar potentials \( A_i(x) \) and \( V(x) \) respectively, as explained in Ref. [35], the Dirac equation takes the following form:

\[
\partial_i \Psi + \sigma^\alpha e_\alpha^i (\partial_i + \Gamma_i - i A_i) \Psi = -i \gamma_0 (m - V) \Psi. \tag{4}
\]

Defining the Dirac current as \( J^\mu = \Psi \gamma^\mu \Psi \), the charge density conservation law can be written as \( \partial_\mu J^\mu + \nabla_j J^j = 0 \), where \( \rho = \Psi \dagger \Psi \in \mathbb{R} \) and the \( J^j = \Psi \gamma^j \Psi \in \mathbb{R} \).

The standard Dirac Hamiltonian for Eq. (4) equation is given by:

\[
H_D = -i \int \Psi^\dagger \sigma^\alpha e_\alpha^i (\partial_i + \Gamma_i - i A_i) \Psi \sqrt{g} d^2 x, \tag{5}
\]

For the case of graphene, the effective Hamiltonian reads as follows [36]:

\[
H_D = -i \int \Psi^\dagger \sigma^\alpha (v^a_i \partial_i + \Gamma^a - i A^a_\alpha) \Psi d^2 x, \tag{6}
\]

where \( v^a_i = \delta_{ai} + u_{ai} - \beta e_{ai} \) is the space-dependent Fermi velocity, \( \Gamma^a = \frac{1}{2} \partial_i v^a_j \) is a complex gauge vector field which guarantees the hermicity of the Hamiltonian and \( A^a_\alpha \) is a strain-induced pseudo-vector potential, given by \( A^a_\alpha = (A^a_x, A^a_y) = \frac{\beta}{2a} (\varepsilon_{xx} - \varepsilon_{yy}, -2 \varepsilon_{xy}) \). Furthermore, \( \beta \) is the material-dependent electron Gruneisen parameter, \( a \) the lattice spacing and \( \varepsilon_{ij} = u_{ij} + \frac{1}{2} \partial_i h \partial_j h \) is the general strain tensor, with in-plane, \( u_{ij} \) and out of plane, \( h \) deformations.

Comparing this to the standard Dirac Hamiltonian in curved space Eq. (5), we can match both Hamiltonians \( H_D \) and \( H_D^\prime \) by fulfilling the following relations:

\[
v^a_i = \sqrt{g} e_i^a, \quad \Gamma^a = \sqrt{g} e_i^a \Gamma_i, \quad A^a_\alpha = \sqrt{g} e_i^a A_i. \tag{7}
\]

All three relations above can be simultaneously fulfilled by an effective metric tensor derived from the explicit expression of the tetrad [35].

The numerical solutions are obtained with the Quantum Lattice Boltzmann Method, as described in Appendix A and Ref. [35].

III. QUANTIZED ALTERNATING CURRENT

GRAPHENE STRIP

To investigate the potential of curvature on curved graphene sheets, we propose a periodic system with alternating current (AC) behaviour, which is quantized according to its shape. The system geometry is initialized by the discrete mapping (or chart),

\[
h^\alpha(x, y) = \begin{pmatrix} x \\ y \sin(\theta x/2) \end{pmatrix} \tag{8}
\]

with \( x \in \{0, 2\pi \}, \ y \in \{-L_y/2, L_y/2\}, \) \( L_y \) being the domain size in the \( y \) dimension, see Fig. 1. The boundaries are periodic along the \( x \)-direction and closed at \( -L_y/2, L_y/2 \).

The initial condition is given by a Gaussian wavepacket of the form:

\[
\Psi(r, k) = \frac{1}{\sqrt{2\pi\sigma^2}} \left( \frac{1}{\lambda \epsilon^{\theta \theta}} \right)^e^{-\frac{|r|^2}{4\lambda^2} + i k \cdot r}, \tag{9}
\]

where \( \lambda = \pm 1 \) is the band index, \( \theta = \arctan(k_y/k_x) \), \( \sigma \) is a measure of the width, \( r = (x, y) \), \( x, y \) are the two
space coordinates and $k = (k_x, k_y)$, $k_x$, $k_y$ represent the $x$ and $y$ momenta, respectively.

The initial values are taken as $k_x = 1$, $k_y = 0$ and $\lambda = 1$. In the simulations, we consider a rectangular sheet with periodic boundary conditions on a grid of size $L_x \times L_y = 256 \times 128$ or $20nm \times 5nm$, while the external potential $A_x$ is set to zero. Therefore, the subsequent motion is purely curvature-driven.

The discretization of the real space shape of the graphene strip, is plotted in Fig. 1(a). The norm of the motion is purely curvature-driven. A potential and $x$-direction: expected, with no clear indication of motion along the time-steps. plotted in Fig. 1(b) for the initial and a few subsequent steps.

As one can appreciate, the wave-packet spreads as expected, with no clear indication of motion along the $y$ direction. The position of the center of charge density along the $y$ direction:

$$\bar{y} = \left( \frac{\int_{area} y(\rho(t) - \rho(0))dA}{\int_{area} \rho(t)dA} \right)$$

is plotted as a function of time in Fig. 2, where $dA = dx dy$. A small but significant oscillation along the $y$ direction is observed.

These oscillations can be understood as the geometrical equivalent of the Bloch oscillations and they are a consequence of the sinusoidal, periodic domain, with the frequency quantized in units of the parameter $\eta$. For a slowly perturbed Hamiltonian and expanding around the wave-packet center $c$ (initialized to $(0, 0)$ here), $H = H_c + \Delta H$, assuming a periodic system described by a Bloch wavefunction, the semi-classical equations of motion are given by [37]:

$$\dot{r}_c = \frac{\partial \varepsilon}{\partial k} - (\Omega_{kr} \cdot \dot{r}_c + \Omega_{kk} \cdot \dot{k}_c) - \Omega_{kt} \tag{11}$$

$$\dot{k}_c = \frac{\partial \varepsilon}{\partial r} - (\Omega_{rk} \cdot \dot{k}_c + \Omega_{rr} \cdot \dot{r}_c) - \Omega_{rt} \tag{12}$$

where $r_c, k_c$ are the center of mass position and momentum of the wavepacket, $r, k$ are the position and momentum vectors, $t$ is time, $\varepsilon$ is the band energy and $\Omega_{kr} = (\Omega_{kr})_{\alpha\beta} = \partial k_\alpha A_{r_\beta} - \partial r_\beta A_{k_\alpha}$ is the Berry curvature and $A_r$ the Berry connection.

As shown in the Appendix B, the Berry phase, and thus Berry curvature, can be related directly to the spin connection $\Gamma_\mu$ through $A^{(n)}_\mu (R) = i \langle \Psi(R)|\partial R|\Psi(R)\rangle \rightarrow A_{R_\alpha} = \text{Tr} (\Psi |\Gamma_i |\Psi \rangle$ for some parameter space $R$ and eigen-function index $n$.

The non zero terms of Eqs. (11,12) for the specific geometry are $\dot{r}_c = \partial \varepsilon / \partial k \approx v_f$, $k_c = \Omega_{rr} \cdot \dot{r}_c$, which imply:

$$\frac{\partial k_\alpha}{\partial r_\beta} = (\Omega_{rr})_{\alpha\beta}. \tag{13}$$

For small-amplitude local wave-packets:

$$\delta k_\alpha = \int (\Omega_{rr})_{\alpha\beta} dr_\beta = \int (\partial r_\alpha A_{r_\beta} - \partial r_\beta A_{r_\alpha}) dr_\beta \tag{14}$$

and thus $\delta k_y \propto \sin(\eta x)$ and $\delta k_z \propto \cos(\eta x)$.

Therefore, the oscillations can be explained in terms of a real space Berry curvature, jointly with the classical geodesic equation on the corresponding manifold.

The frequency of these Bloch-like oscillations is quantized according to $\eta$. Finally, as the center of mass density is equivalent to a driven oscillating current, the system might be implemented as a periodic, quantized oscillating current device.
IV. CONCLUSIONS AND OUTLOOK

We have proposed the realization of a quantized alternating current on a curved graphene sheet. The oscillating current is numerically computed through the quantum lattice Boltzmann method in curved space and verified analytically via a set of semi-classical equations relating the Berry curvature to real space curvature. We interpret this result as a geometrical analogue of the Bloch oscillations, quantized according to the geometrical period $\eta$.

Building on these results, more complex and adjustable graphene devices can be envisaged in the context of curvature-based design. For example, the proposed quantized oscillating current on graphene, in conjunction with trapping quantum dots [32], could form the building block for quantum information processing algorithms.

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Appendix A: Curved-space quantum lattice Boltzmann

The quantum lattice Boltzmann (QLB) method used for solving the Dirac equation as minimally coupled to curved space is an extension of the original method developed by Succi et al. [34]. The method exploits the conceptual similarities between the Dirac equation and the Boltzmann equation on the lattice. We present here the QLB method for a three-dimensional manifold, with straightforward usage to lower dimensional systems, [38–40].

1. The Dirac equation

The classical Boltzmann equation for a particle density distribution function $f(x_a, v_a, t)$ is given by

$$\partial_t f + v^i \partial_i f = C[f] - F^a \partial_{v^a} f, \quad (A1)$$

the left-hand side describes the advection of the distribution function, velocity $v^a$, whereas the right-hand side describes the collisions between particles and the effect of external forces $F^a$. Furthermore, the Dirac equation in curved space in Eq. 1 can be cast into a kinetic theory form,

$$\partial_t \Psi + \sigma^a \partial_{v^a} \Psi = C\Psi + \mathcal{F}\psi. \quad (A2)$$

Therefore, similarly to the Boltzmann equation, the left hand side represents the 'free streaming' step along matrix valued 'velocities' $\sigma^a$ while the right hand side contains a 'Collision' and a 'Forcing' term.

The collision term of Eq. A2 is represented by

$$\mathcal{C} = -(im\gamma^0 + \sigma^a e^a_i \Gamma_i), \quad (A3)$$

where $m$ is the fermion mass. The 'forcing term' is given by:

$$\mathcal{F} = -\sigma^a (e^a_i - \delta^a_i) \partial_{v^a}. \quad (A4)$$

where the symbols have their usual meaning. The partial derivative of the Dirac equation is distributed between the streaming part and the forcing term, resulting in a lattice-compatible classical streaming operator of the form $\partial_t + \gamma^a \partial_{v^a}$, where $\gamma^a \in \mathbb{Z}$. The forcing term is a consequence of the generalized Dirac matrices $\gamma^i = e^a_i \gamma^a$ and captures the bulk of the curvature effects. The partial derivative in Eq. A4 is approximated by a local lattice finite difference scheme.

2. Diagonal streaming operator

In order to obtain a diagonal streaming operator the complex $\sigma$-matrices have to be diagonalized first, which yields a diagonal velocity matrix with eigenvalues $v^a = \pm 1$. The diagonalization is achieved by suitable "rotation matrices":

$$X^a_\gamma \sigma^a X_a = \begin{cases} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{cases} = \gamma^0 \quad \text{for } a = 0, 1, 2,$$

where the unitary transformation matrices $X_1, X_2, X_3$ are given by:
The streaming and collision operations are performed in successive steps using operator splitting, since the simultaneous diagonalization of the three σ matrices is not possible:

\[
\Psi(t + \frac{\Delta t}{D}) = \exp\left(-\Delta t \sigma^a \partial_a + \frac{\Delta t}{D} (\mathcal{C} + \mathcal{F})\right) \Psi(t),
\]

\[
\Psi(t + \frac{2\Delta t}{D}) = \exp\left(-\Delta t \sigma^a \partial_a + \frac{\Delta t}{D} (\mathcal{C} + \mathcal{F})\right) \Psi(t + \frac{\Delta t}{D}),
\]

\[
\Psi(t + \Delta t) = \exp\left(-\Delta t \sigma^a \partial_a + \Delta t (\tilde{\mathcal{C}} + \tilde{\mathcal{F}})\right) \Psi(t + \frac{2\Delta t}{D}),
\]

where \(D = 3\) denotes the spatial dimensions. Each streaming step can be diagonalized by left-multiplying with \(X_a^\dagger\):

\[
X_a^\dagger \Psi(t + \frac{\Delta t}{D}) = \exp\left(-\Delta t \sigma^a \partial_a + \Delta t (\tilde{\mathcal{C}} + \tilde{\mathcal{F}})\right) \Psi_a(t),
\]

(A5)

with the definitions:

\[
\tilde{\Psi}_a := X_a^\dagger \Psi, \quad \tilde{\mathcal{F}}_a := \frac{1}{2} X_a^\dagger \mathcal{F} X_a, \quad \tilde{\mathcal{C}}_a := \frac{1}{2} X_a^\dagger \mathcal{C} X_a,
\]

for \(a = 1, 2, 3\) (no Einstein summation is used here). The exponential approximated as

\[
\exp\left(-\Delta t \sigma^a \partial_a + \Delta t (\tilde{\mathcal{C}} + \tilde{\mathcal{F}})\right) \approx \left(1 - \Delta t \sigma^a \partial_a + \Delta t (\tilde{\mathcal{C}}_a + \tilde{\mathcal{F}}_a)\right) + \left(1 - \frac{\Delta t}{2} \tilde{\mathcal{C}}_a\right)^{-1} \left(1 + \frac{\Delta t}{2} \tilde{\mathcal{C}}_a\right).
\]

The expansion of the collision operator \(e^{\Delta t \hat{\mathcal{C}}_a}\) is unitary and thus conserves exactly the probability of the wavefunction. The streaming \(e^{-\Delta t \sigma^a \partial_a}\) and forcing \(e^{\Delta t \hat{\mathcal{F}}_a}\) operators are not expanded, as this is prohibited by the derivative. A simple 2nd-order expansion is performed, limiting the probability norm to \(\Delta t^2\) accuracy. The operator splitting implies an error of order \(O(\Delta t^2)\), as \(e^{\Delta t X} \cdot e^{\Delta t Y} = e^{\Delta t (X+Y)} + 1/2! \Delta t^2 [X,Y] = e^{\Delta t (X+Y)} + O(\Delta t^3)\).

The manifold is described by a chart \(h\) defined in linear space, discretized on a regular rectangular lattice. The curved space quantum lattice Boltzmann method evolves the four-spinor \(\Psi = (\Psi^+, \Psi^-) = (\Psi_1^+, \Psi_2^+, \Psi_1^-, \Psi_2^-)\) from \(t\) to \(t+\delta t\). Once the operators are split, the following algorithm is performed in sequence for each lattice direction \(n_a\), where \(n_1 = (1, 0), n_2 = (0, 1)\) and \(a = 1, 2\).

1. **Rotation**: The spinor is rotated by \(X_a\)

\[
\tilde{\Psi}_a(x, t) = X_a^\dagger \Psi(x, t).
\]

(A6)

2. **Collisions and curvature**: The collision and force operators are applied to the rotated spinor,

\[
\tilde{\Psi}_a^\dagger(x, t) = \left(\Delta t \tilde{\mathcal{F}}_a + (1 - \frac{\Delta t}{2} \tilde{\mathcal{C}}_a)^{-1} (1 + \frac{\Delta t}{2} \tilde{\mathcal{C}}_a)\right) \tilde{\Psi}_a(x, t),
\]

where \(\tilde{\Psi}_a(x, t)\) denotes an auxiliary field,

\[
\tilde{\mathcal{F}}_a \tilde{\Psi}_a(x, t) = (e_a^\dagger - e_a) \left(\tilde{\Psi}_a(x + n_i \Delta t, t) - \tilde{\Psi}_a(x, t)\right),
\]

(A7)

where \(n_i\) is the lattice direction and \(\mathcal{C}\) is the collision term, Eq. (A3). The upper sign applies to the spin-up components \((\Psi_1^+, \Psi_2^+\) and the lower sign to the spin-down ones \((\Psi_1^-, \Psi_2^-)\).

3. **Streaming**: The spinor components are streamed to the closest grid points along the lattice direction \(\pm n_a\),

\[
\tilde{\Psi}_a(x, t + \frac{\Delta t}{2}) = \tilde{\Psi}_a^\dagger(x \mp n_a \Delta t, t).
\]

(A8)

4. **Inverse Rotation**: The spinor is rotated back via \(X_a\),

\[
\Psi_a(x, t + \frac{\Delta t}{2}) = X_a \tilde{\Psi}_a(x, t + \frac{\Delta t}{2}).
\]

(A10)

5. Repeat steps 2–4 for the next spatial direction

The external potentials \(V(x)\), scalar, and \(A(x)\), vector are added to the collision operator Eq. (A7), such that

\[
\tilde{\mathcal{C}}_a = \frac{1}{2} X_a^\dagger \mathcal{C} X_a = \frac{i}{\sqrt{g}} \left[\Delta m - V(x)\right] \gamma^0 X_a - \gamma^0 e_a^\dagger \mathcal{A}_i - i \gamma^5 e_a^\dagger.
\]

(A11)

The simulation for strained graphene is carried out with modified Eqs. (A7,A8), according to the following scheme:

\[
\tilde{\mathcal{C}}_a \rightarrow \sqrt{g} \tilde{\mathcal{C}}_a, \quad e_a^\dagger \rightarrow \sqrt{g} e_a^\dagger.
\]

The additional factor \(\sqrt{g}\) originates from the volume element of the Hamiltonian Eq. (6).

Appendix B: Berry phase relation to the spin connection

To solve the Dirac equation, minimally coupled to curvature, Eq. 1, with \(A_i = 0\) and assuming that the
wave-packet has a negligible spread, \( \delta r \rightarrow 0 \), the connection component of the covariant derivative can be absorbed into the wavefunction, so that

\[
\Psi \rightarrow \Psi \exp \left( i \int_{r_c}^{r_c + \delta r} \Gamma_i dr \right) \tag{B1}
\]

where \( r_c \) is the center of mass position and \( \Gamma_i \) is the spin-connection matrix. For a Gaussian wavepacket with spread \( \sigma \) and momentum \( k \), the wavefunction Eq. (B1) takes the following form

\[
\Psi(r, k) = \frac{1}{\sqrt{2\pi\sigma^2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} e^{i \int \Gamma_i dr} e^{-\frac{|r - r_c|^2}{4\sigma^2} + ik \cdot r}. \tag{B2}
\]

This wave-function minimally couples the standard Dirac equation to curved space through the spin connection. Defining the Berry connection as

\[
A^i_n(R) = i \langle \Psi(R) | \partial_R | \Psi(R) \rangle \tag{B3}
\]

for some parameter space \( R \) and eigen-function \( n \). The Berry phase can be calculated from the complete loop integral of the connection, according to:

\[
\gamma = \oint_0^{2\pi} A(R) g^{11} dR. \tag{B4}
\]

In a similar manner to the treatment of the Aharonov-Bohm effect from Berry [37], we define the slow and fast coordinates \( R \) and \( r \) respectively, such that \( \Psi(R, r) \rightarrow \Psi(r - R) \). The wave-function takes then the form

\[
\Psi_r(R - r) = \frac{1}{\sqrt{2\pi\sigma^2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} e^{i \int \Gamma_r dr} e^{-\frac{|R - r|^2}{4\sigma^2} + ik(R - r)}. \tag{B5}
\]

From Eq. B3 the explicit form of the wavefunction implies that \( A^i = \text{Tr} \Gamma_i \). The implication of these result is that the Berry connection and curvatures can be directly related to the real space affine connection and Ricci curvature tensor under suitable conditions.

As a consequence, the phase change of a wavepacket moving around a closed loop, can be calculated from the Berry phase. Integrating naively around a closed loop

\[
\gamma = \oint_0^{2\pi} \text{Tr} \langle \Psi_r | \partial_r | \Psi_r \rangle g^{11} dR, \tag{B6}
\]

where \( \text{Tr} \) denotes the trace of the resulting matrix and takes into account the spinorial character of the Dirac wavefunction.