Factorization numbers of finite rank 3 abelian $p$-groups

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Abstract

In this short note we give a formula for the factorization number $F_2(G)$ of a finite rank 3 abelian $p$-group $G$. This extends a result in our previous work [9].

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Key words: factorization number, subgroup commutativity degree, Möbius function, finite abelian $p$-group.

1 Introduction

Let $G$ be a finite group, $L(G)$ be the subgroup lattice of $G$ and $H$, $K$ be two subgroups of $G$. If $G = HK$, then $G$ is said to be factorized by $H$ and $K$ and the expression $G = HK$ is said to be a factorization of $G$. Denote by $F_2(G)$ the factorization number of $G$, that is the number of all factorizations of $G$.

The starting point for our discussion is given by the papers [6, 1], where $F_2(G)$ has been computed for certain classes of finite groups. Then, in [9], we have obtained explicit formulas for the factorization numbers of an elementary abelian $p$-group and of a rank 2 abelian $p$-group. These are based on the connection between $F_2(G)$ and the subgroup commutativity degree $sd(G)$ of $G$ (see [7, 8]), namely

$$sd(G) = \frac{1}{|L(G)|^2} \sum_{H \leq G} F_2(H).$$
Obviously, by applying the well-known Möbius inversion formula to the above equality, one obtains

\[ F_2(G) = \sum_{H \leq G} sd(H)|L(H)|^2 \mu(H, G). \]

In particular, if \( G \) is abelian, then we have \( sd(H) = 1 \) for all \( H \in L(G) \), and consequently

\[ F_2(G) = \sum_{H \leq G} |L(H)|^2 \mu(H, G) = \sum_{H \leq G} |L(G/H)|^2 \mu(H). \]

This formula will be used in the following to calculate the factorization number of a rank 3 abelian \( p \)-group.

First of all, we recall a theorem due to P. Hall [2] (see also [4]), that permits us to compute explicitly the Möbius function of a finite \( p \)-group.

**Theorem 1.** Let \( G \) be a finite \( p \)-group of order \( p^n \). Then \( \mu(G) = 0 \) unless \( G \) is elementary abelian, in which case we have \( \mu(G) = (-1)^n p^{(n)} \).

We also need to know the total number of subgroups of a finite rank 3 abelian \( p \)-group. It has been determined by using different methods in [3, 5, 10].

**Theorem 2.** The total number of subgroups of \( \mathbb{Z}_{p^{\lambda_1}} \times \mathbb{Z}_{p^{\lambda_2}} \times \mathbb{Z}_{p^{\lambda_3}} \), where \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1 \), is

\[ f(\lambda_1, \lambda_2, \lambda_3) = \frac{A}{(p^2 - 1)^2(p - 1)}, \]

where

\[
A = (\lambda_3 + 1)(\lambda_1 - \lambda_2 + 1)p^{\lambda_2 + \lambda_3 + 5} + 2(\lambda_3 + 1)p^{\lambda_2 + \lambda_3 + 4} - 2(\lambda_3 + 1)(\lambda_1 - \lambda_2)p^{\lambda_2 + \lambda_3 + 3} \\
- 2(\lambda_3 + 1)p^{\lambda_2 + \lambda_3 + 2} + (\lambda_3 + 1)(\lambda_1 - \lambda_2 - 1)p^{\lambda_2 + \lambda_3 + 1} - (\lambda_1 + \lambda_2 - \lambda_3 + 3)p^{2\lambda_3 + 4} \\
- 2p^{2\lambda_3 + 3} + (\lambda_1 + \lambda_2 - \lambda_3 - 1)p^{2\lambda_3 + 2} + (\lambda_1 + \lambda_2 + \lambda_3 + 5)p^2 + 2p - (\lambda_1 + \lambda_2 + \lambda_3 + 1).
\]

We are now able to give the main result of our note.
Theorem 3. The following equality holds

\[ F_2(\mathbb{Z}_{p^{\lambda_1}} \times \mathbb{Z}_{p^{\lambda_2}} \times \mathbb{Z}_{p^{\lambda_3}}) = -p^3 f^2(\lambda_1-1, \lambda_2-1, \lambda_3-1) + p(f^2(\lambda_1-1, \lambda_2-1, \lambda_3) + pf^2(\lambda_1-1, \lambda_2, \lambda_3-1) + p^2 f^2(\lambda_1, \lambda_2-1, \lambda_3-1)) - (f^2(\lambda_1-1, \lambda_2, \lambda_3) + pf^2(\lambda_1, \lambda_2-1, \lambda_3)) + p^2 f^2(\lambda_1, \lambda_2, \lambda_3-1)) + f^2(\lambda_1, \lambda_2, \lambda_3), \]

where the quantities \( f(\lambda_1, \lambda_2, \lambda_3) \) are given by (3).

We remark that for \( \lambda_3 = 0 \) the above equality leads to the formula in Theorem 3 of [9]. It simplifies in the particular case \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda \).

Corollary 4. We have

\[ F_2(\mathbb{Z}_{p^{\lambda}} \times \mathbb{Z}_{p^{\lambda}} \times \mathbb{Z}_{p^{\lambda}}) = -p^3 f^2(\lambda-1, \lambda-1, \lambda-1) + p(1+p+p^2) f^2(\lambda, \lambda-1, \lambda-1) - (1+p+p^2) f^2(\lambda, \lambda, \lambda-1) + f^2(\lambda, \lambda, \lambda). \]

However, even in this case, an explicit formula for the factorization number is too difficult to be written, but we can do it for small values of \( \lambda_1, \lambda_2, \lambda_3 \).

Examples.

a) \( F_2(\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p}) = 9p^6 + 15p^5 + 21p^4 + 20p^3 + 11p + 13. \)

b) \( F_2(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}) = 5p^8 + 7p^7 + 16p^6 + 15p^5 + 21p^4 + 16p^3 + 20p^2 + 11p + 13. \)

2 Proof of Theorem 3

It is well-known that \( G = \mathbb{Z}_{p^{\lambda_1}} \times \mathbb{Z}_{p^{\lambda_2}} \times \mathbb{Z}_{p^{\lambda_3}} \) has a unique elementary abelian subgroup of order \( p^3 \), say \( M \), and that

\[ G/M \cong \Phi(G) \cong \mathbb{Z}_{p^{\lambda_1-1}} \times \mathbb{Z}_{p^{\lambda_2-1}} \times \mathbb{Z}_{p^{\lambda_3-1}}, \]

where \( \Phi(G) \) is the Frattini subgroup of \( G \). Moreover, all elementary abelian subgroups of \( G \) are contained in \( M \). Denote by \( M_i \), \( i = 1, 2, ..., p^2 + p + 1 \), the subgroups of order \( p \) and by \( M'_i \), \( i = 1, 2, ..., p^2 + p + 1 \), the subgroups of order \( p^2 \) in \( M \). Then every quotient \( G/M_i \) is isomorphic to a maximal subgroup of \( G \), which are: \( p^2 \) of type \( (\lambda_1, \lambda_2; \lambda_3 - 1) \), \( p \) of type \( (\lambda_1, \lambda_2 - 1, \lambda_3) \), and
1 of type \((\lambda_1 - 1, \lambda_2, \lambda_3)\). Similarly, every quotient \(G/M'_i\) is isomorphic to a subgroup of index \(p^2\) of \(G\) that contains \(\Phi(G)\), and these are: \(p^2\) of type \((\lambda_1, \lambda_2 - 1, \lambda_3 - 1)\), \(p\) of type \((\lambda_1 - 1, \lambda_2, \lambda_3 - 1)\), and \(1\) of type \((\lambda_1 - 1, \lambda_2 - 1, \lambda_3)\). In this way the equality (2) becomes

\[
F_2(G) = |L(G/M)|^2 \mu(M) + \sum_{i=1}^{p^2+p+1} |L(G/M_i)|^2 \mu(M_i)
\]

\[
+ \sum_{i=1}^{p^2+p+1} |L(G/M'_i)|^2 \mu(M'_i) + |L(G)|^2 \mu(1),
\]

in view of Theorem 1. Since we have \(\mu(M) = \mu(\mathbb{Z}_p^3) = -p^3\), \(\mu(M_i) = \mu(\mathbb{Z}_p) = -1\), \(\mu(M'_i) = \mu(\mathbb{Z}_p^2) = p\), \(\forall i = 1, 2, ..., p^2 + p + 1\), and \(\mu(1) = 1\), one obtains

\[
F_2(G) = -p^3 |L(\mathbb{Z}_{p^{\lambda_1 - 1}} \times \mathbb{Z}_{p^{\lambda_2 - 1}} \times \mathbb{Z}_{p^{\lambda_3 - 1}})|^2 + p \left( |L(\mathbb{Z}_{p^{\lambda_1 - 1}} \times \mathbb{Z}_{p^{\lambda_2 - 1}} \times \mathbb{Z}_{p^{\lambda_3}})|^2 + p^2 |L(\mathbb{Z}_{p^{\lambda_1}} \times \mathbb{Z}_{p^{\lambda_2 - 1}} \times \mathbb{Z}_{p^{\lambda_3 - 1}})|^2 \right)
\]

Under the notation of Theorem 2 this leads to the desired formula.  

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