A universal weasel
without large cardinals in $V$

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0 Introduction.

In [4], Steel constructs an $\Omega + 1$ iterable premouse, called $K^c$, of height $\Omega$ which is universal in the sense that it wins the coiteration against every coiterable premouse of height $\leq \Omega$. Here, $\Omega$ is a fixed measurable cardinal, and Steel works in the theory "ZFC + $\Omega$ is measurable + there's no inner model with a Woodin cardinal." In [1], Jensen shows that "$\Omega$ is measurable" can be relaxed to "$\Omega$ is inaccessible" here. Universal weasels are needed for the purpose of isolating $K$, the core model.

It would be desirable to replace $\Omega$ by $\text{OR}$ here, where $\text{OR}$ is the class of all ordinals, and to get rid of having to assume $\Omega$ ($\text{OR}$, that is) to be "large." I.e., we would like to prove the existence of a universal weasel in the theory "ZFC + there's no inner model with a Woodin cardinal." This would be a first step towards proving the existence of $K$ in that theory (cf. the discussion in the introduction to [2]).

This note solves the problem of constructing a universal weasel. We prove:

Theorem 0.1 Assume ZFC + there’s no inner model with a Woodin cardinal. There is then a universal weasel.

We warn the reader that some care is necessary in order to arrive at the appropriate notion of "universal" so as to make [1] not false for the wrong reasons (this has to do with iteration trees of length $\text{OR}$, and will be discussed below).

The key new idea here is to weaken the concept of "countably certified" of [4] Def. 1.2 which is crucial for the construction of $K^c$. Whereas the iterability proof of [4] can be checked to still go through with this weaker requirement on new extenders to be added to the $K^c$-sequence, an argument from [1] can be varied to prove the universality of $K^c$.

We do not know whether the $K^c$ constructed here satisfies a useful version of weak covering. (It can be shown that it does not necessarily satisfy weak covering at every (countably closed) singular cardinal.)
1 The existence of $K^c$.

We build upon [2] and [4]; in particular, we use the concept of “premouse” as isolated there.

Let $\mathcal{L} = \{\dot{\epsilon}, \dot{U}\}$ be the language for structures of the type $(N; E, U)$, where $E$ is binary and $U$ is unary. For the purposes of this paper let us introduce the following.

**Definition 1.1** Let $\Psi$ be a (first order) formula in the language $\mathcal{L}$. Then $\Psi$ is said to be $r\Sigma_3$ (restricted $\Sigma_3$) iff

$$\Psi \equiv \exists v_0 (\Phi_0 \land \Phi_1),$$

where $\Phi_0$ is $\Sigma_2$ in the language $\{\dot{\epsilon}\}$, and $\Phi_1$ is $\Sigma_0$ in the language $\mathcal{L}$.

$\mathcal{L}$ will be an appropriate language for the models witnessing certifiability.

**Definition 1.2** Let $\mathcal{M}$ be an active premouse, $F$ the extender coded by $\hat{F}^\mathcal{M}$, $\kappa = \text{c.p.}(F)$, and $\nu = \dot{\nu}^\mathcal{M} =$ the natural length of $F$. We say that $\mathcal{M}$ (or, $F$) is countably certified iff for all $\vec{A} = (A_k: k < \omega)$ with $\forall k \exists n = n(k) A_k \in \mathcal{P}([\kappa]^n) \cap \mathcal{M}$ there are $N'$ and $\vec{B} = (B_k: k < \omega)$ such that

(a) $\omega V_k \subset V_{\kappa}$ (i.e., $\text{cf}(\kappa) > \omega$), $N'$ is transitive, $\omega N' \subset N'$, and $V_{\nu+1} \subset N'$,

(b) $(V_{\kappa}; \in, \vec{A}) \prec_{r\Sigma_3} (N'; \in, \vec{B})$, and

(c) for all $k < \omega$, $B_k \cap [\nu]^n(k) = i_F(A_k) \cap [\nu]^n(k)$, where $i_F: \mathcal{M} \rightarrow \Sigma_1 \text{Ult}_0(\mathcal{M}, F)$ is the canonical embedding.

In this definition, we confuse $\vec{A}$ (and $\vec{B}$) with $\{(k, u): u \in A_k\}$ (and $\{(k, u): u \in B_k\}$, resp.).

It is easy to see that if $\mathcal{M}$ is countably certified in the sense of [4] Def. 1.2 then it is countably certified in the sense of [4] Def. 1.2.

We construct the models $\mathcal{N}_\xi$ and $\mathcal{M}_\xi$ as on p. 6 f. of [4], except that we don’t require (2) and (3) at all in Case 1 and that in (1) of Case 1 we understand “countably certified” in the sense of [4] Def. 1.2 rather than [4] Def. 1.2.

We now have to prove [4] Thm. 2.5, the assertion that if $\mathcal{N}_\theta$ exists then collapses of countable submodels of $\mathcal{C}_k(\mathcal{N}_\theta)$ are countably iterable for every $k < \omega$ (cf. [4] for the exact statement). As on pp. 12 ff. we’ll prove this in a simplified case, for trees of length $\omega$. We’ll leave it as an easy exercise for the reader to check that the proof of [2] Thm. 9. 14 can be varied in much the same way as the proof from [2] pp. 12 ff. in the light of our new meaning of “countably certified.”

**Lemma 1.3** Let $\sigma: \mathcal{P} \rightarrow \mathcal{N}_\eta$ with $\mathcal{N}_\eta \models \text{ZFC}$, and let $T$ be an iteration tree on $\mathcal{P}$ of length $\omega$ such that $\mathcal{D}^T = \emptyset$. Then there are $b$ and $\sigma'$ such that $b$ is a cofinal branch through $T$ and $\sigma': \mathcal{M}_b^T \rightarrow \mathcal{N}_\eta$ with $\sigma' \circ \pi_{0b}^T = \sigma$.  

\[2\]
Proof. For any $\tau: P \to Q$ we denote by 

$$U(\tau, Q)$$

the tree of attempts to find $b$, $\tau'$ such that $b$ is a cofinal branch through $T$ and $\tau': M_0^T \to Q$ with $\tau' \circ \pi_{00} = \tau$. We let $U(\tau, Q)$ consists of $\phi: M_0^T \to Q$, and if $\phi: M_1^T \to Q$ and $\phi': M_k^T \to Q$ then we put $\phi \leq_{U(\tau, Q)} \phi'$ iff $i \leq_T k$ and $\phi' \circ \pi_{ik} = \phi$.

Let us assume that $U(\sigma, N_\eta)$ is well-founded (in the obvious sense). We aim to derive a contradiction. Let us write $\nu_i = c.p.(E_i^T)$, and $\nu_i = \text{natural length of } E_i^T$.

We closely follow [4] p. 12 ff. We are going to define

$$(\sigma_i, Q_i, R_i : i < \omega)$$

such that the following requirements are met, for all $j < i < \omega$. (In what follows, the $\tau(\cdot, \cdot)$'s are the functions from [3] Lemma 3.1.)

1. $R_i$ is a transitive model of $ZFC^-$ with $\omega R_i \subset R_i$,
2. $\sigma_i: \mathcal{P}_i \to Q_i$, where $Q_i$ is an "$N$-model" of $R_i$,
3. $V_{\sigma_j(\nu_j)+1}^R = V_{\sigma_j(\nu_j)+1}^{R_i}$ and $\sigma_j(\nu_j) \leq \sigma_i(\nu_i)$,
4. $\tau(\cdot, \cdot) \circ \sigma_j \mid \nu_j = \sigma_i \mid \nu_j$,
5. if $U = U(\sigma_i \circ \pi_{0i}, Q_i)$ then $U$ is well-founded and $R_i$ has (in order type) at least $|\sigma_i|_U$ many cutoff points, and
6. $i > 0 \Rightarrow R_i \in R_{i-1}$.

It is (6) which gives the desired contradiction.

To commence, we let $\sigma_0 = \sigma$, $Q_0 = N_\eta$, and $R_0 = H_\theta$ for some large enough $\theta$.

Suppose now we are given $(\sigma_l, Q_l, R_l : l \leq i)$. We want to construct $\sigma_{i+1}$, $Q_{i+1}$, and $R_{i+1}$.

Set $F = \sigma_i(E_i^T)$, $\kappa = c.p.(F)$, and $\nu = \text{the natural length of } F$. Let us cheat by assuming $F$ is the top extender of $Q_i$. (If not, we have to consider the top extender of the target model of $\tau(\cdot, \cdot): J_{lh(F)}^Q \to Q$ instead; a similar cheating appears in [4] p. 12 ff.) By (2), $F$ is countably certified inside $R_i$. Let $A = (A_k : k < \omega)$ be an enumeration of

$$\mathcal{P}([\kappa]^{<\omega}) \cap Q_i \cap \text{ran}(\sigma_i).$$

By (1), $\vec{A} \in R_i$, and hence there are inside $R_i$ objects $N$, $N'$, and $\vec{B}$ such that

1. $N = V^R_\kappa$, $\omega N \subset N$, $N'$ is transitive, $\omega N' \subset N'$, and $V^N_{\nu+1} \subset N'$,
2. $(N; \in, \vec{A}) \prec_{\Sigma_3} (N'; \in, \vec{B})$, and

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1. $N = V^R_\kappa$, $\omega N \subset N$, $N'$ is transitive, $\omega N' \subset N'$, and $V^N_{\nu+1} \subset N'$,
2. $(N; \in, \vec{A}) \prec_{\Sigma_3} (N'; \in, \vec{B})$, and
(c) for all $k < \omega$, $B_k \cap [\nu]<\omega = i_F(A_k) \cap [\nu]<\omega$.

It now clearly suffices to prove the following

**Main Claim.** In $N'$, there are $\sigma$, $Q$, and $R$ such that

1. $R$ is a transitive model of $\text{ZFC}^-$ with $\omega R \subset R$,
2. $\sigma: \mathcal{P}_{i+1} \to Q$, where $Q$ is an “$N$-model” of $R$,
3. $V_{\sigma(\nu_i)+1}^R = V_{\sigma(\nu_i)+1}^{N'}$ (and hence $= V_{\sigma(\nu_i)+1}^R$),
4. $\sigma \upharpoonright \nu_i = \sigma_i \upharpoonright \nu_i$, and
5. if $U = U(\sigma \circ \pi_{0i+1}, Q)$ then $U$ is well-founded and $R$ has (in order type) at least $|\sigma|_U$ many cutoff points.

Notice that the assertion of the Main Claim is $\Sigma^N_2(\{T, \sigma_i(\nu_i), \sigma_i \upharpoonright \nu_i\})$. Let

$$\pi : (M; \epsilon, \vec{C}) \to (N'; \epsilon, \vec{B}),$$

where $M$ is countable (and hence $(M; \epsilon, \vec{C}) \in N' \cap N$), and $\pi$ is $\Sigma_2$-elementary w.r.t. $\{\epsilon\}$ and $\Sigma_0$-elementary w.r.t. $\mathcal{L}$. The fact that $(M; \epsilon, \vec{C})$ can be embedded into $N'$ in such a fashion is a $r\Sigma_3$-fact, and hence by $(N; \epsilon, \vec{A}) \prec_{r\Sigma_3} (N'; \epsilon, \vec{B})$ there is some

$$\pi' : (M; \epsilon, \vec{C}) \to (N; \epsilon, \vec{A})$$

such that $\pi'$ is $\Sigma_2$-elementary w.r.t. $\{\epsilon\}$ and $\Sigma_0$-elementary w.r.t. $\mathcal{L}$. In order to finish the proof of the Main Claim (and thus of [1.3]), it now suffices to verify the following

**Claim.** In $N$, there are $\sigma$, $Q$, and $R$ such that

1. $R$ is a transitive model of $\text{ZFC}^-$ with $\omega R \subset R$,
2. $\sigma: \mathcal{P}_{i+1} \to Q$, where $Q$ is an “$N$-model” of $R$,
3. $V_{\pi\sigma^{-1}(\sigma(\nu_i)+1)}^R = V_{\pi\sigma^{-1}(\sigma(\nu_i)+1)}^{N'}$,
4. $\sigma \upharpoonright \nu_i = \pi' \circ \pi^{-1}(\sigma \upharpoonright \nu_i)$, and
5. if $U = U(\sigma \circ \pi_{0i+1}, Q)$ then $U$ is well-founded and $R$ has (in order type) at least $|\sigma|_U$ many cutoff points.

Let $j = T\text{-pred}(i + 1)$. We define $\sigma': \mathcal{P}_{i+1} \to Q_j$ by

$$\pi_{ji+1}(f)(a) \mapsto \sigma_j(f)(\pi' \circ \pi^{-1}(\sigma_i(a_i))).$$

To see that this is well-defined and elementary we argue as follows.
\[ \mathcal{P}_{i+1} \models \Phi(\pi_{ji+1}(f)(a)) \]
\[ \iff \{ u : \mathcal{P}_j \models \Phi(f(u)) \} \in (E^f_i)_a \]
\[ \iff \sigma_i(\{ u : \mathcal{P}_j \models \Phi(f(u)) \}) \in F_{\sigma_i(a)} \].

Let \( \sigma_i(\{ u : \mathcal{P}_j \models \Phi(f(u)) \}) = A_k \). So \( i_F(A_k) \cap [\nu]^{<\omega} = B_k \cap [\nu]^{<\omega} \), and we may continue as follows.

\[ \iff \sigma_i(a) \in i_F(A_k) \]
\[ \iff \sigma_i(a) \in B_k \]
\[ \iff \pi' \circ \pi^{-1}(\sigma_i(a)) \in A_k. \]

But \( A_k = \sigma_i(\{ u : \mathcal{P}_j \models \Phi(f(u)) \}) = \sigma_j(\{ u : \mathcal{P}_j \models \Phi(f(u)) \}) = \{ u : Q_j \models \Phi(\sigma_j(f)(u)) \} \), and hence

\[ \iff Q_j \models \Phi(\sigma_j(f)(\pi' \circ \pi^{-1}(\pi_i(a)))) \].

We’ll have that \( \sigma' \circ \pi_{0i+1} = \sigma_j \circ \pi_{0j} \), and so \( \sigma' \in U(\sigma_j \circ \pi_{0j}, Q_j) \). Moreover, clearly,

\[ \epsilon = |\sigma'|_{U(\sigma_j \circ \pi_{0j}, Q_j)} < |\sigma_j|_{U(\sigma_j \circ \pi_{0j}, Q_j)} \]

and hence by (5) we may let \( \theta = \) the \( \epsilon \)th cutoff point of \( \mathcal{R}_j \). Working inside \( \mathcal{R}_j \), we may thus set

\[ \mathcal{R} = \text{the transitive collapse of the closure of } V_{\pi' \circ \pi^{-1}(\pi_i(\nu))} \cup \{ Q_j \} \]
under Skolem functions for \( V_{\theta}^{\mathcal{R}_j} \) and \( \omega \)-sequences,

\[ Q = \text{the image of } Q_j \text{ under the collapse, and} \]
\[ \sigma = \text{the image of } \sigma' \text{ under the collapse.} \]

It is now straightforward that we have shown the Claim.

\[ \square (L8) \]

Of course by standard arguments the previous sketch also shows that \( K^c \) exists unless there is a non-tame premouse, say.
Assume that there is no inner model with a Woodin cardinal. By the results in §1 together with Lemma 2.4 (b) we then have that $K^c$ is $< OR$ iterable. However, it may be the case that there is a definable tree on $K^c$ of length $OR$ with no cofinal branch.

This discussion leads us to the following.

**Definition 2.1** A weasel $W$ is universal iff whenever $(\mathcal{T}, \mathcal{U})$ is a coiteration of $W$ with some premouse $\mathcal{M}$ (using padded trees) with $lh(\mathcal{T}) = lh(\mathcal{U}) = OR + 1$ then $\mathcal{M}$ is a weasel, $\mathcal{D}^U \cap [0, OR]_U = \emptyset$, $\pi^U_{OR}$ "$OR \subset OR$, and $\mathcal{M}^U_{OR} \geq \mathcal{M}^T_{OR}$. 

N.B.: “$W$ is universal" is a schema which cannot be expressed by a single sentence in the language of ZFC.

I do not know if there is a notion of “universal" which is more useful.

Let us say that a premouse $\mathcal{M}$ is below superstrong iff for all $F = E^\mathcal{M}_\alpha \neq \emptyset$ we have that the natural length of $F$ is strictly less than $i_F(c.p.(F))$. We're now going to show:

**Theorem 2.2** Assume ZFC+ every premouse is below superstrong. Then $K^c$ is universal, if it exists.

**Proof.** Deny. Set $W = K^c$, and $W_\alpha = J^W_{\alpha + \omega}$ for $\alpha \in OR$. By a slight refinement (due to Zeman and the author) of an argument of Jensen (cf. [1]) there is then a (definable) class $C \subset OR$, club in $OR$, together with a commuting system $(\pi_{\alpha \beta} : \alpha \leq \beta \in C)$ of maps such that for all $\alpha \leq \beta \in C'$ do we have that $\pi_{\alpha \beta} : W_\alpha \to W_\beta$ is cofinal with $\pi_{\alpha \beta} \upharpoonright \alpha = id$ and $\pi_{\alpha \beta}(\alpha) = \beta$, and such that $(W_\alpha, \pi_{\alpha \beta} : \alpha \leq \beta \in C' \cap \kappa + 1)$ is the direct limit of $(W_\alpha, \pi_{\alpha \beta} : \alpha \leq \beta \in C \cap \kappa)$ for limit points $\kappa$ of $C$.

Let $n < \omega$ be large enough. There is then a (definable) $C' \subset C$, again club in $OR$, such that $N^\kappa = (V_\kappa; \in, C \cap \kappa, (W_\alpha, \pi_{\alpha \beta} : \alpha \leq \beta \in C \cap \kappa)) \prec_{\Sigma_n} (V; \in, C, (W_\alpha, \pi_{\alpha \beta} : \alpha \leq \beta \in C))$ for all $\kappa \in C'$. Pick $\kappa < \lambda \in C'$, both limit points of $C$, with $\omega V_\kappa \subset V_\kappa$ and $\omega V_\lambda \subset V_\lambda$ (i.e., $cf(\kappa) > \omega$ and $cf(\lambda) > \omega$).

Let $\bar{A} = (A_k; k < \omega)$ with $\forall k \exists m = m(k) A_k \in \mathcal{P}([\kappa]^m) \cap W$. Let $\alpha < \kappa$ be such that $A_k \in ran(\pi_{\alpha \kappa})$ for all $k < \omega$. Set $\bar{A}_k = \pi_{\alpha \kappa}^{-1}(A_k)$. Notice that $A_k$ is definable over $N^\kappa$ by

$$u \in A_k \iff \text{for all but boundedly many } \beta \in (C \cap \kappa) \setminus \alpha, u \in \pi_{\alpha \beta}(\bar{A}_k).$$
Define $B_k$ over $N^\lambda$ by

$$u \in B_k \iff \text{for all but boundedly many } \beta \in (C \cap \lambda) \setminus \alpha, \ u \in \pi_{\alpha \beta}(\bar{A}_k).$$

Then obviously $\pi_{\kappa \lambda}(A_k) = B_k$, for all $k < \omega$. It is also easy to verify that

$$(V_\kappa; \in, \bar{A}) \prec_{r\Sigma_3} (V_\lambda; \in, \bar{B}).$$

(Notice that if a formula is $\Sigma_0(\Sigma_p)$ then it is equivalent to a $\Sigma_p$ formula over models of $\Sigma_p$-replacement.)

Now let $F$ be the extender derived from $\pi_{\kappa \lambda}$, and let $\nu$ be its natural length. By our smallness assumption, $\nu < \lambda$. Let $\gamma = \nu^+ < \lambda$. A straightforward induction as in the proof of [2] Lemma 11.4 shows that

$$(J_\gamma^W, \bar{F})$$

satisfies the initial segment condition, and is hence a premouse. But we have shown that $F$ is countably certified. Thus $F = E_\gamma^W$, contradicting the fact that $\gamma$ is a cardinal of $W$.

□ (2.2)

Notice that 0.1 is now an immediate corollary of 2.2 together with what we showed in §1. By well-known arguments, we might in fact replace “there’s no inner model with a Woodin cardinal” by “every premouse is tame,” say, in the statement of 0.1.

3 $\omega$-completeness and countable certifiability.

We now want to discuss the relation between being $\omega$-closed and being countably certified (in our new sense).

**Definition 3.1** Let $M$ be an active premouse, $F$ the extender coded by $\hat{F}^M$, $\kappa = c.p.(F)$, and $\nu = \nu^M = \text{the natural length of } F$. We say that $F$ is strongly $\omega$-closed iff $\forall \ (a_n, X_n : n < \omega)$ with

$$\forall n < \omega \ \exists k < \omega \ (a_n \in \nu^k \land A_n \in \mathcal{P}([\kappa]^k) \cap M)$$

there is some transitive $N$ with

$$\omega N \subset N \land V_{\nu+1} \subset N$$
such that for all 
\[ \pi : M \rightarrow \Sigma_2 N \]
with \( M \) countable and transitive there is 
\[ \pi' : M \rightarrow \Sigma_2 V_\kappa \]
such that 
\[ \pi' \circ \pi^{-1} \mid \bigcup_{n<\omega} a_n \rightarrow \kappa \]
witnesses that \( F \) is \( \omega \)-complete w.r.t \( (a_n, X_n : n < \omega) \), i.e., 
\[ \forall n < \omega \ ( X_n \in F_{a_n} \Rightarrow \pi' \circ \pi^{-1}(a_n) \in X_n ) . \]

Recall that such \( F \) is \( \omega \)-complete iff for all \( (a_n, X_n : n < \omega) \) as in 3.1 there is an order-preserving \( \tau : \bigcup_n a_n \rightarrow \kappa \) with \( \forall n < \omega \ ( X_n \in F_{a_n} \Rightarrow \tau(a_n) \in X_n ) \). Trivially, if \( F \) is strongly \( \omega \)-closed then \( F \) is \( \omega \)-closed. Strong \( \omega \)-closedness requires that \( \tau \) is realized as the restriction of some \( \pi' \circ \pi^{-1} \) as above. We also have the following facts, which are easy to verify.

If \( F \) is countable certified in the sense of [4] Defn. 1.2, then \( F \) is countably certified in the sense of [4], and then \( F \) is strongly \( \omega \)-closed. We can still run the iterability proof for countable submodels of \( C_k(N_\theta) \) if we relax the requirement that new extenders be countably certified to that they be strongly \( \omega \)-complete. Of course, the new \( K^c \) is then still universal.

**References**

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