Noncommutative Bialynicki-Birula Theorem

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Abstract

In this short note we prove that every maximal torus action on the free
algebra is conjugate to a linear action. This statement is the free algebra
analogue of a classical theorem of A. Bialynicki-Birula.

1 Actions of algebraic tori

In this note we consider algebraic torus actions on the affine space, according to
Bialynicki-Birula, and formulate certain noncommutative generalizations.

We begin by recalling a few basic definitions. Let $\mathbb{K}$ be an algebraically closed
field.

Definition 1.1. An algebraic group is a variety $G$ equipped with the structure of a
group, such that the multiplication map $m : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2$ and the
inverse map $\iota : G \to G : g \mapsto g^{-1}$ are morphisms of varieties.

Definition 1.2. A $G$-variety is a variety equipped with an action of the algebraic
group $G$,

$\alpha : G \times X \to X : (g, x) \mapsto g \cdot x,$

which is also a morphism of varieties. We then say that $\alpha$ is an algebraic $G$-action.
Let $K$ be our ground field, which is assumed to be algebraically closed. Let $Z = \{z_1, z_2, \ldots\} = \{z_i : i \in I\}$ be a finite or a countable set of variables (where $I = \{1, 2, \ldots\}$ is an index set), and let $Z^*$ denote the free semigroup generated by $Z$, $Z^* = Z^\setminus\{1\}$. Moreover let $F_I(K) = K \langle Z \rangle$ be the free associative $K$-algebra and $\hat{F}_I(K) = K \langle \langle Z \rangle \rangle$ be the algebra of formal power series in free variables.

Denote by $W = \langle Z \rangle$ the free monoid of words over the alphabet $Z$ (with 1 as the empty word) such that $|W| \geq 1$, for $|W|$ the length of the word $W \in Z^*$.

For an alphabet $Z$, the free associative $K$-algebra on $Z$ is

$$K \langle Z \rangle := \bigoplus_{W \in Z^*} K^W,$$

where the multiplication is $K$-bilinear extension of the concatenation on words, $Z^*$ denotes the free monoid on $Z$, and $K^W$ denotes the free $K$-module on one element, the word $W$. Any element of $K \langle Z \rangle$ can thus be written uniquely in the form

$$\sum_{k=0}^{\infty} \sum_{i_1, i_2, \ldots, i_k \in I} a_{i_1, i_2, \ldots, i_k} z_{i_1} z_{i_2} \cdots z_{i_k},$$

where the coefficients $a_{i_1, i_2, \ldots, i_k}$ are elements of the field $K$ and all but finitely many of these elements are zero.

In our context, the alphabet $Z$ is the same as the set of algebra generators, therefore the terms “monomial” and “word” will be used interchangeably.

In the sequel, we employ a (slightly ambiguous) short-hand notation for a free algebra monomial. For an element $z$, its powers are defined as usual. Any monomial $z_{i_1} z_{i_2} \cdots z_{i_k}$ can then be written in a reduced form with subwords $zz \ldots z$ replaced by powers.

We then write

$$z^I = z_{j_1}^{i_1} z_{j_2}^{i_2} \cdots z_{j_k}^{i_k}$$

where by $I$ we mean an assignment of $i_k$ to $j_k$ in the word $z^I$. Sometimes we refer to $I$ as a multi-index, although the term is not entirely accurate. If $I$ is such a multi-index, its absolute value $|I|$ is defined as the sum $i_1 + \cdots + i_k$.

For a field $K$, let $K^\times = K \setminus \{0\}$ denote the multiplicative group of its non-zero elements viewed as an algebraic $K$-group.

**Definition 1.3.** An $n$-dimensional algebraic $K$-torus is a group

$$T_n \simeq \left(K^\times \right)^n$$

(with obvious multiplication).

Denote by $A^n$ the affine space of dimension $n$ over $K$.

**Definition 1.4.** A (left) torus action is a morphism

$$\sigma : T_n \times A^n \to A^n.$$

that fulfills the usual axioms (identity and compatibility):

$$\sigma(1, x) = x, \quad \sigma(t_1, \sigma(t_2, x)) = \sigma(t_1 t_2, x).$$

An action $\sigma$ is **effective** if for every $t \neq 1$ there is an element $x \in A^n$ such that $\sigma(t, x) \neq x$. 

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In [1], Bialynicki-Birula proved the following two theorems.

**Theorem 1.5.** Any regular action of $T_n$ on $\mathbb{A}^n$ has a fixed point.

**Theorem 1.6.** Any effective and regular action of $T_n$ on $\mathbb{A}^n$ is a representation in some coordinate system.

The term ”regular” is to be understood here as in the algebro-geometric context of regular function (Bialynicki-Birula also considered birational actions). The last theorem says that any effective regular maximal torus action on the affine space is conjugate to a linear action, or, as it is sometimes called, **linearizable**.

An algebraic group action on $\mathbb{A}^n$ is the same as an action by automorphisms on the algebra

$$\mathbb{K}[x_1, \ldots, x_n]$$

of global sections. In other words, it is a homomorphism

$$\sigma : T_n \to \text{Aut} \mathbb{K}[x_1, \ldots, x_n].$$

An action is effective iff $\ker \sigma = \{1\}$.

The polynomial algebra is a quotient of the free associative algebra

$$F_n = \mathbb{K}\langle z_1, \ldots, z_n \rangle$$

by the commutator ideal $I$ (it is the two-sided ideal generated by all elements of the form $fg - gf$). From the standpoint of Noncommutative geometry, the algebra $\Gamma(X, \mathcal{O}_X)$ of global sections (along with the category of f.g. projective modules) contains all the relevant topological data of $X$, and various non-commutative algebras (PI-algebras included) may be thought of as global function algebras over ”noncommutative spaces”. Therefore, noncommutative analogue of the Bialynicki-Birula theorem is a subject of legitimate interest.

In this short note we establish the free algebra version of the Bialynicki-Birula theorem. The latter is formulated as follows.

**Theorem 1.7.** Suppose given an action $\sigma$ of the algebraic n-torus $T_n$ on the free algebra $F_n$. If $\sigma$ is effective, then it is linearizable.

## 2 Proof of Theorem 1.7

The proof proceeds along the lines of the original commutative case proof of Bialynicki-Birula.

If $\sigma$ is the effective action of Theorem 1.7, then for each $t \in T_n$ the automorphism

$$\sigma(t) : F_n \to F_n$$

is given by the $n$-tuple of images of the generators $z_1, \ldots, z_n$ of the free algebra:

$$(f_1(t, z_1, \ldots, z_n), \ldots, f_n(t, z_1, \ldots, z_n)).$$

Each of the $f_1, \ldots, f_n$ is a polynomial in the free variables.
Lemma 2.1. There is a translation of the free generators

\[ (z_1, \ldots, z_n) \mapsto (z_1 - c_1, \ldots, z_n - c_n), \quad (c_i \in \mathbb{K}) \]

such that (for all \( t \in \mathbb{T}_n \)) the polynomials \( f_i(t, z_1 - c_1, \ldots, z_n - c_n) \) have zero free part.

Proof. This is a direct corollary of Theorem 1.5. Indeed, any action \( \sigma \) on the free algebra induces, by taking the canonical projection with respect to the commutator ideal \( I \), an action \( \bar{\sigma} \) on the commutative algebra \( \mathbb{K}[x_1, \ldots, x_n] \). If \( \sigma \) is regular, then so is \( \bar{\sigma} \). By Theorem 1.5, \( \bar{\sigma} \) (or rather, its geometric counterpart) has a fixed point, therefore the images of commutative generators \( x_i \) under \( \bar{\sigma}(t) \) (for every \( t \)) will be polynomials with trivial degree-zero part. Consequently, the same will hold for \( \sigma \).

\[ \square \]

We may then suppose, without loss of generality, that the polynomials \( f_i \) have the form

\[ f_i(t, z_1, \ldots, z_n) = \sum_{j=1}^{n} a_{ij}(t)z_j + \sum_{j,l=1}^{n} a_{ijl}(t)z_jz_l + \sum_{k=3}^{N} \sum_{\|J\| = k} a_{i,J}(t)z_J \]

where by \( z_J \) we denote, as in the introduction, a particular monomial

\[ z_{i_1}^{k_1} z_{i_2}^{k_2} \ldots \]

(a word in the alphabet \( \{z_1, \ldots, z_n\} \) in the reduced notation; \( J \) is the multi-index in the sense described above); also, \( N \) is the degree of the automorphism (which is finite) and \( a_{ij}, a_{ijl}, \ldots \) are polynomials in \( t_1, \ldots, t_n \).

As \( \sigma_i \) is an automorphism, the matrix \([a_{ij}]\) that determines the linear part is non-singular. Therefore, without loss of generality we may assume it to be diagonal (just as in the commutative case [1]) of the form

\[ \text{diag}(t_1^{m_{11}} \ldots t_n^{m_{1n}}, \ldots, t_1^{m_{n1}} \ldots t_n^{m_{nn}}). \]

Now, just as in [1], we have the following

Lemma 2.2. The power matrix \([m_{ij}]\) is non-singular.

Proof. Consider a linear action \( \tau \) defined by

\[ \tau(t) : (z_1, \ldots, z_n) \mapsto (t_1^{m_{11}} \ldots t_n^{m_{1n}} z_1, \ldots, t_1^{m_{n1}} \ldots t_n^{m_{nn}} z_n), \quad (t_1, \ldots, t_n) \in \mathbb{T}_n. \]

If \( T_1 \subset T_n \) is any one-dimensional torus, the restriction of \( \tau \) to \( \mathbb{T}_1 \) is non-trivial. Indeed, were it to happen that for some \( \mathbb{T}_1 \),

\[ \tau(t)z = z, \quad t \in \mathbb{T}_1, \quad (z = (z_1, \ldots, z_n)) \]

then our initial action \( \sigma \), whose linear part is represented by \( \tau \), would be identity modulo terms of degree > 1:

\[ \sigma(t)(z_i) = z_i + \sum_{j,l} a_{ijl}(t)z_jz_l + \cdots. \]
Now, equality $\sigma(t^2)(z) = \sigma(t)(\sigma(t)(z))$ implies

$$\sigma(t)(\sigma(t)(z_i)) = \sigma(t) \left( z_i + \sum_{jl} a_{ijl}(t)z_j z_l + \cdots \right)$$

$$= z_i + \sum_{jl} a_{ijl}(t)z_j z_l + \sum_{jl} a_{ijl}(t)(z_j + \sum_{km} a_{jkm}(t)z_k z_m + \cdots)$$

$$(z_l + \sum_{k'lm'} a_{lk'm'}(t)z_{k'} z_{l'} + \cdots) + \cdots$$

$$= z_i + \sum_{jl} a_{ijl}(t^2)z_j z_l + \cdots$$

which means that

$$2a_{ijl}(t) = a_{ijl}(t^2)$$

and therefore $a_{ijl}(t) = 0$. The coefficients of the higher-degree terms are processed by induction (on the total degree of the monomial). Thus

$$\sigma(t)(z) = z, \quad t \in T_1$$

which is a contradiction since $\sigma$ is effective. Finally, if $[m_{ij}]$ were singular, then one would easily find a one-dimensional torus such that the restriction of $\tau$ were trivial.

Consider the action

$$\varphi(t) = \tau(t^{-1}) \circ \sigma(t).$$

The images under $\varphi(t)$ are

$$(g_1(z, t), \ldots, g_n(z, t)), \quad (t = (t_1, \ldots, t_n))$$

with

$$g_i(z, t) = \sum g_{i, m_1 \ldots m_n}(z) t_1^{m_1} \ldots t_n^{m_n}, \quad m_1, \ldots, m_n \in \mathbb{Z}.$$  

Define $G_i(z) = g_{i,0\ldots0}(z)$ and consider the map $\beta : F_n \to F_n$,

$$\beta : (z_1, \ldots, z_n) \mapsto (G_1(z), \ldots, G_n(z)).$$

**Lemma 2.3.** $\beta \in \text{Aut} F_n$ and

$$\beta = \tau(t^{-1}) \circ \beta \circ \sigma(t).$$

**Proof.** This lemma mirrors the final part in the proof in [1]. The conjugation is straightforward, since for every $s, t \in T_n$ one has

$$\varphi(st) = \tau(t^{-1}s^{-1}) \circ \sigma(st) = \tau(t^{-1}) \circ \tau(s^{-1}) \circ \sigma(s) \circ \sigma(t) = \tau(t^{-1}) \circ \varphi(s) \circ \sigma(t).$$

Denote by $\hat{F}_n$ the power series completion of the free algebra $F_n$, and let $\hat{\sigma}$, $\hat{\tau}$ and $\hat{\beta}$ denote the endomorphisms of the power series algebra induced by corresponding morphisms of $F_n$. The endomorphisms $\hat{\sigma}$, $\hat{\tau}$, $\hat{\beta}$ come from (polynomial) automorphisms and therefore are invertible.

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Let
\[ \hat{\beta}^{-1}(z_i) \equiv B_i(z) = \sum_J b_{i,J} z^J \]
(just as before, \( z^J \) is the monomial with multi-index \( J \)). Then
\[ \hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1}(z_i) = B_i(t_1^{m_{11}} \ldots t_n^{m_{1n}} G_1(z), \ldots, t_1^{m_{1n}} \ldots t_n^{m_{nn}} G_n(z)). \]

Now, from the conjugation property we must have
\[ \hat{\beta} = \hat{\sigma}(t^{-1}) \circ \hat{\beta} \circ \hat{\tau}(t), \]
therefore
\[ \hat{\sigma}(t) = \hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1} \]
and
\[ \hat{\sigma}(t)(z_i) = \sum_J b_{i,J}(t_1^{m_{11}} \ldots t_n^{m_{1n}})^{j_1} \ldots (t_1^{m_{1n}} \ldots t_n^{m_{nn}})^{j_n} G(z)^J; \]
here the notation \( G(z)^J \) stands for a word in \( G_i(z) \) with multi-index \( J \), while the exponents \( j_1, \ldots, j_n \) count how many times a given index appears in \( J \) (or, equivalently, how many times a given generator \( z_i \) appears in the word \( z^J \)).

Therefore, the coefficient of \( \hat{\sigma}(t)(z_i) \) at \( z^J \) has the form
\[ b_{i,J}(t_1^{m_{11}} \ldots t_n^{m_{1n}})^{j_1} \ldots (t_1^{m_{1n}} \ldots t_n^{m_{nn}})^{j_n} + S \]
with \( S \) a finite sum of monomials of the form
\[ c_L(t_1^{m_{11}} \ldots t_n^{m_{1n}})^{l_1} \ldots (t_1^{m_{1n}} \ldots t_n^{m_{nn}})^{l_n} \]
with \((j_1, \ldots, j_n) \neq (l_1, \ldots, l_n)\). Since the power matrix \( [m_{ij}] \) is non-singular, if \( b_{i,J} \neq 0 \), we can find a \( t \in \mathbb{T}_n \) such that the coefficient is not zero. Since \( \sigma \) is an algebraic action, the degree
\[ \sup_t \text{deg}(\hat{\sigma}) \]
is a finite integer \( N \). With the previous statement, this implies that
\[ b_{i,J} = 0, \quad \text{whenever } |J| > N. \]

Therefore, \( B_i(z) \) are polynomials in the free variables. What remains is to notice that
\[ z_i = B_i(G_1(z), \ldots, G_n(z)). \]
Thus \( \beta \) is an automorphism. \( \square \)

From Lemma 2.3 it follows that
\[ \tau(t) = \beta^{-1} \circ \sigma(t) \circ \beta \]
which is the linearization of \( \sigma \). Theorem 1.7 is proved.
3 Discussion

The noncommutative toric action linearization theorem that we have proved has several useful applications. In the work [6], it is used to investigate the properties of the group Aut $F_n$ of automorphisms of the free algebra. As a corollary of Theorem 1.7 one gets

**Corollary 3.1.** Let $\theta$ denote the standard action of $\mathbb{T}_n$ on $K[x_1, \ldots, x_n]$ – i.e., the action

$$\theta_t : (x_1, \ldots, x_n) \mapsto (t_1x_1, \ldots, t_nx_n).$$

Let $\tilde{\theta}$ denote its lifting to an action on the free associative algebra $F_n$. Then $\tilde{\theta}$ is also given by the standard torus action.

This statement plays a part, along with a number of results concerning the induced formal power series topology on Aut $F_n$, in the establishment of the following proposition (cf. [6]).

**Proposition 3.2.** When $n \geq 3$, any Ind-scheme automorphism $\varphi$ of Aut($K\langle x_1, \ldots, x_n \rangle$) is inner.

One could try and generalize the free algebra version of the Bialynicki-Birula’s theorem to other noncommutative situations. Another way of generalization lies in changing the dimension of the torus. In a complete analogy with further work of Bialynicki-Birula [2], we expect the following to hold.

**Conjecture 3.3.** Any effective action of $\mathbb{T}_{n-1}$ on $F_n$ is linearizable.

On the other hand, there is little reason to expect this statement to hold with further lowering of the torus dimension. In fact, even in the commutative case the conjecture that any effective toric action is linearizable, in spite of considerable effort (see [7]), proved negative (counterexamples in positive characteristic due to Asanuma, [8]).

Another direction would be to replace $\mathbb{T}$ by an arbitrary reductive algebraic group, however the commutative case also does not hold even in characteristic zero (cf. [9]).

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