Multilevel Monte Carlo simulation of a diffusion with non-smooth drift.

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Abstract. We show that Lasso and Bayesian Lasso are very close when the sparsity is large and the noise is small. Then we propose to solve Bayesian Lasso using multivalued stochastic differential equation. We obtain three discretizations algorithms, and propose a method for calculating the cost of Monte-Carlo (MC), multilevel Monte Carlo (MLMC) and MCMC algorithms.

Keywords: Lasso, MCMC, MLMC, PMALA, EDS.

1. Introduction

Let \( y = Ax + \sigma w \) be the classical linear regression problem see e.g. \cite{31} and the references herein, (see also \cite{11 12 13} for some new applications). Here \( p \) and \( n \) is a couple of positive integers, \( y \in \mathbb{R}^n \) are the observations, \( x \in \mathbb{R}^p \) is the unknown signal to recover, \( w \in \mathbb{R}^n \) is the standard noise, \( \sigma \) is the size of the noise and \( A \) is a known matrix which maps the signal domain \( \mathbb{R}^p \) into the observation domain \( \mathbb{R}^n \). The matrix \( A \) is in general ill-conditioned (e.g. in the case \( n < p \)) which makes difficult to use the least squares estimate. Penalization is a popular way to compute an approximation of \( x \) from the observations \( y \). The general framework proposes to recover the vector \( x \) using the posterior probability distribution function proportional to

\[
\exp\left(-P(x) - \frac{\|Ax - y\|^2}{2\sigma^2}\right).
\]

Here \( \| \cdot \| \) denotes the Euclidean norm. This requires to define a penalization \( P \) to enforce some prior information on the signal \( x \). The term \( \frac{\|Ax - y\|^2}{2\sigma^2} \) reflects Gaussian prior on the noise \( w \). The parameter \( \sigma^2 > 0 \) reflects the noise level.

The \( l^1 \) penalization is the sum of the absolute values \( P(x) = \alpha \|x\|_1 \) of the components of \( \alpha x \). The parameter \( \alpha > 0 \) reflects the sparsity level of the variable

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The Lasso := arg min\{α∥x∥_1 + ∥Ax−y∥^2/2, x ∈ \mathbb{R}^p\} was first introduced in [31]. It is also called Basis Pursuit De-Noising method [8]. It was introduced to induce sparsity in the variable x. A large number of theoretical results has been provided for the \ell^1\penalization see e.g. [9, 14, 23] and the references herein.

We will suppose that \alpha = 2\beta and \sigma^2 = 1/2\beta. It follows that the posterior PDF is equal to

\frac{1}{Z_\beta} \exp \left(-2\beta F(x)\right), \quad (1)

where

F(x) = ∥x∥_1 + ∥Ax−y∥^2/2 \quad (2)

and Z_\beta is the partition function, i.e.

Z_\beta = \int_{\mathbb{R}^p} \exp \left(-2\beta F(x)\right) dx.

Bayes estimator of x is equal to

m_\beta := \int_{\mathbb{R}^p} x \exp \left(-2\beta F(x)\right) \frac{dx}{Z_\beta}. \quad (3)

Lasso is the maximum a posteriori estimator

Lasso = arg min \{F(x): x ∈ \mathbb{R}^p\}. \quad (4)

In the sequel X_\beta will denote \{F(x): x ∈ \mathbb{R}^p\}. Hence Bayes estimator (3) is the mathematical expectation

\mathbb{E}[X_\beta]. \quad (5)

In the first part of this work we show how Bayes estimator converges to Lasso as \beta → +∞. In the second part we consider for fixed \beta the random vector X_\beta as the limit of a multivalued stochastic process (x(T)) (Langevin diffusion with non-smooth drift) as T → +∞. We propose to approximate Bayes estimator \mathbb{E}[X_\beta] by the mathematical expectation \mathbb{E}[x(T)] for large T. We obtain three discretizations algorithms. Two among them are known as unadjusted Langevin algorithm (ULA) ([22]) and STMALA ([15]). We calculate the latter mathematical expectation \mathbb{E}[x(T)] using Monte Carlo (MC), Multilevel Monte Carlo (MLMC) and MCMC methods. We propose a method for calculating the cost of MC, MLMC and MCMC.

2. Lasso estimator properties

First, we need some notations. For each x ∈ \mathbb{R}^p, the sub-differential sgn(x) = \partial∥x∥_1 is the set of the column vector \xi ∈ \mathbb{R}^p such that the component \xi_i = sgn(x_i) = 1 if x_i > 0, \xi_i = sgn(x_i) = -1 if x_i < 0 and \xi_i ∈ [-1, 1] if x_i = 0.

We will denote, for each subset J ⊂ \{1, \ldots, p\} and for each vector v ∈ \mathbb{R}^p, v(J) = (v(i) : i ∈ J) ∈ \mathbb{R}^{|J|}. Here |J| denotes the cardinality of J. The notation
\(v \leq w\) means \(v(i) \leq w(i)\) for all \(i = 1, 2, \ldots, p\). The scalar product is denoted by \(\langle \cdot, \cdot \rangle\), and \((e_i: \ i = 1, 2, \ldots, p)\) denotes the canonical basis of \(\mathbb{R}^p\).

Now we recall a well known properties of Lasso estimator see e.g. [32].

**lemma:** The vector \(x(y)\) is a minimizer of the map \(x \to F(x) = \|x\|_1 + \frac{\|Ax-y\|^2}{2}\) if and only if the vector
\[
\xi := A^*(y - Ax(y)) \in sgn(x(y)).
\]
The vectors \(\xi, Ax(y)\) and the \(l^1\)-norm \(\|x(y)\|_1\) are constant on the set of Lasso estimators. Moreover, the set of Lasso is convex and compact. Here \(A^*\) denotes the transpose of the matrix \(A\).

We introduce the sets
\[
I = \{i \in \{1, \ldots, p\} : \ |\xi_i| < 1\}, \quad (7)
\]
\[
\partial I = \{i \in \{1, \ldots, p\} : \ |\xi_i| = 1\}. \quad (8)
\]
Observe that the support \(\{i \in \{1, \ldots, p\} : \ x_i(y) \neq 0\}\) of any Lasso \(x(y)\) is contained in \(\partial I\), and \(I\) is contained in the set \(\{i \in \{1, \ldots, p\} : \ x_i(y) = 0\}\) of the null components of \(x(y)\). For each subset \(J\) of \(\{1, \ldots, p\}\), \(A_J\) denotes the submatrix of \(A\) having its columns indexed by \(J\).

From "equation (6)" it is easy to show that the injectivity of \(A_{\partial I}\) implies the uniqueness of Lasso. In fact, under this hypothesis the system
\[
\xi_{\partial I} = A_{\partial I}^T y - A_{\partial I}^T A_{\partial I} x_{\partial I}(y)
\]
has a unique solution. As the support of any Lasso \(x(y)\) is contained in \(\partial I\), then Lasso is unique.

In the sequel for each \(x \in \mathbb{R}^p\),
\[
\pi(x) = \arg \min \{\|x - x(y)\| : \ x(y) \in \text{Lasso}\}.
\]

**prop:** The random positive number \(\|X_\beta - \pi(X_\beta)\|\) converges to 0 in probability as \(\beta \to +\infty\).

**proof:** The proof is similar to Theorem 4.1. in [1]. It works as following.

Let \(\delta > 0\), and \(\eta > 0\) such that
\[
\inf \{F(x) : \ |x - \pi(x)| \geq \delta\} > M(\eta) = \sup \{F(x) : \ |x - \pi(x)| \leq \eta\},
\]
where \(F\) is given by "equation (2)". We have
\[
P(\|X_\beta - \pi(X_\beta)\| \geq \delta) = \frac{\int_{\|x-\pi(x)\| \geq \delta} \exp(-\beta F(x)) dx}{\int \exp(-\beta F(x)) dx} \leq \frac{\int_{\|x-\pi(x)\| \geq \delta} \exp(-\beta F(x) - M(\eta)) dx}{\int_{\|x-\pi(x)\| \leq \eta} \exp(-\beta F(x) - M(\eta)) dx}.
\]

From the estimate
\[
\int_{\|x-\pi(x)\| \geq \delta} \exp(-\beta (F(x) - M(\eta))) dx \leq \int_{\|x-\pi(x)\| \geq \delta} \exp(-(F(x) - M(y))) dx < +\infty
\]
and the bounded convergence theorem, the numerator \( \int_{||x - \pi(x)|| \leq \delta} \exp(-\beta(F(x) - M(\eta)))dx \to 0 \) as \( \beta \to +\infty \). The denominator

\[
\int_{||x - \pi(x)|| \leq \eta} \exp(-\beta(F(x) - M(\eta)))dx \to \int_{||x - \pi(x)|| \leq \eta} dx.
\]

It follows that

\[
P(\|X_\beta - \pi(X_\beta)\| \geq \delta) \leq \frac{\int_{||x - \pi(x)|| \geq \delta} \exp(-\beta(F(x) - M(\eta)))dx}{\int_{||x - \pi(x)|| \leq \eta} dx} \to 0
\]
as \( \beta \to +\infty \).

Now we are interested in the speed of convergence of \( X_\beta - \pi(X_\beta) \to 0 \) as \( \beta \to +\infty \). The first step of this convergence is based on the following.

**Prop:** Let \( x(y) \) be any Lasso estimator and \( m = F(x(y)) \) be the minimum of the objective function \( F(x) \) "equation (2)". The function \( F(x) - m \) is equal to

\[
\sum_{i=1}^{p} |x_i|(1 - sgn(x_i)\xi_i) + \frac{\|A(x - x(y))\|^2}{2}.
\]

And then

\[
\sum_{i=1}^{p} |x_i|(1 - sgn(x_i)\xi_i) = \sum_{i \in I} |x_i|(1 - sgn(x_i)\xi_i) + 2 \sum_{i \in \partial I : sgn(x_i)\xi_i = -1} |x_i|.
\]

Here \( \xi \) is defined by "equation [6]", \( I \) and \( \partial I \) are defined by "equation [7]", and "equation [10]".

**Proof:** From the equality \( \|Ax - y\|^2 = \|A(x - x(y))\|^2 + 2\langle A(x - x(y)), Ax(y) - y \rangle + \|Ax(y) - y\|^2 \), we have

\[
F(x) = \frac{\|Ax - x(y)\|^2}{2} + \langle A(x - x(y)), Ax(y) - y \rangle + \frac{\|Ax(y) - y\|^2}{2}.
\]

From the equality \( \xi = A^*(y - Ax(y)) \), we have

\[
\langle x - x(y), A^*(Ax(y) - y) \rangle = -\langle x - x(y), \xi \rangle = -\langle x, \xi \rangle + \|x(y)\|_1.
\]

Now formulas "equation [9]" and "equation [10]" are an easy consequence of the formula "equation [11]".

Now, we are interested in the asymptotic independence of the components \( (X_\beta(i) : i \in I), (X_\beta(i) : i \in \partial I) \) as \( \beta \to +\infty \). We are going to solve this problem when \( A^*_{\partial I}A_{\partial I} \) is invertible. In this case Lasso is a singleton \{x(y)\}.

The support of \( x(y) \) is \( S = \{i : x_i(y) \neq 0\} \). The complementary of \( S \) is \( I_0 = \{i : x_i(y) = 0\} \). The boundary of \( \partial I_0 = \{i : x_i(y) = 0, |\xi_i| = 1\} \). The family
\((S, I_0 \setminus \partial I_0, \partial I_0)\) is a partition of \(\{1, 2, \ldots, p\}\). In the sequel \(\mathbb{R}^p\) is considered as the set of the sequences \((x_i : i \in (I_0 \setminus \partial I_0) \cup \partial I_0) \cup S)\) indexed by \((I_0 \setminus \partial I_0) \cup \partial I_0 \cup S)\). The notation \(\mathbb{R}^j\) will denotes the set of the sequences \((x_j : j \in J)\) with values in \(\mathbb{R}\).

Observe that \(I = I_0 \setminus \partial I_0\) ”equation (7)”, and \(\partial I = S \cup \partial I_0\) ”equation (10)”. For \(i \in S\) and for \(x_i\) near \(x_i(y)\), we have \(\text{sgn}(x_i) = \xi_i\). In this case the equality ”equation (10)” becomes
\[
\sum_{i \in I_0 \setminus \partial I_0} |x_i|(1 - \text{sgn}(x_i)\xi_i) + 2 \sum_{i \in \partial I_0 : \text{sgn}(x_i) = -1} |x_i|.
\]

Now we decompose \(X_\beta\) as following. Each partition \(\partial I_0^- \cup \partial I_0^+\) of \(\partial I_0\) defines two sets
\[
\Delta^- := \Delta(\partial I_0^-) = \{x \in \mathbb{R}^p : x_i \xi_i = -1, \forall i \in \partial I_0^\}.,
\]
\[
\Delta^+ := \Delta(\partial I_0^+) = \{x \in \mathbb{R}^p : x_i \xi_i = 1, \forall i \in \partial I_0^\}.
\]

We have
\[
\mathbb{R}^p = \bigcup_{\partial I_0^- \cup \partial I_0^+ = \partial I_0} \Delta^- \cap \Delta^+.
\]

It follows that for each suitable function \(f\)
\[
\mathbb{E}[f(X_\beta)] = \sum_{\partial I_0^- \cup \partial I_0^+ = \partial I_0} \mathbb{E}[f(X_\beta) | X_\beta \in \Delta^- \cap \Delta^+] \mathbb{P}(X_\beta \in \Delta^- \cap \Delta^+).
\]

The main result of this section is the following.

**prop:** We have for each partition \(K^- \cup K^+ = \partial I_0\) with \(K^- \neq \emptyset\) that
\[
\mathbb{P}(X_\beta \in \Delta(K^-) \cap \Delta(K^+)) \rightarrow 0 \quad \text{as} \quad \beta \rightarrow +\infty.
\]

**proof:** We suppose without loosing any generality for all \(i \in \partial I_0\) that \(\xi_i = 1\). From ”equation (2)”, we have for large \(\beta\) that
\[
\mathbb{P}(X_\beta \in \Delta(K^-) \cap \Delta(K^+)) \approx \frac{A(\beta, \delta, K^-, K^-)}{\sum_{\partial I_0^+ \cup \partial I_0^- = \partial I_0} A(\delta, \partial I_0^+ \cup \partial I_0^-)},
\]
where \(\delta\) is small and
\[
A(\beta, \delta, \partial I_0^+, \partial I_0^-) = \int_{[x \in \Delta^- \cap \Delta^+, \|x - x(y)\|_{\infty} \leq \delta]} \exp (- \beta G(x))dx,
\]
\[
G(x) = \sum_{i \in I_0 \setminus \partial I_0} |x_i|(1 - \xi_i \text{sgn}(x_i)) + 2 \sum_{i \in \partial I_0^-} |x_i| + \frac{\|A(x - x(y))\|^2}{2}.
\]

We recall that by hypothesis \(K^- \neq \emptyset\), but in the denominator the sum \(\sum_{\partial I_0^+ \cup \partial I_0^- = \partial I_0}\) contains the case \(\partial I_0^- = \emptyset\).
We use the new variables

\[ u_i = \beta x_i, \quad i \in I_0 \setminus \partial I_0^+, \]

\[ v_i = \sqrt{\beta} (x_i - x_i(y)), \quad i \in S \cup \partial I_0^+, \]

and then we obtain

\[ A(\beta, \delta, \partial I_0^+ \setminus \partial I_0^-) = \beta^{-|I_0 \setminus \partial I_0^+| - |S| + |\partial I_0^+|} C(\beta, \delta, \partial I_0^+ \setminus \partial I_0^-), \]

where

\[ C(\beta, \delta, \partial I_0^+ \setminus \partial I_0^-) = \int \exp \left( -G(u, v, \beta, \partial I_0^+ \setminus \partial I_0^-) \right) du dv, \]

with

\[ G(u, v, \beta, \partial I_0^+ \setminus \partial I_0^-) = \sum_{i \in I_0 \setminus \partial I_0^+} |u_i| (1 - \xi_i \text{sgn}(u_i)) + \frac{\|A_{S \cup \partial I_0^+} v_{S \cup \partial I_0^+} + \beta^{-1/2} A_{I_0^+ \setminus \partial I_0^-} u_{I_0^+ \setminus \partial I_0^-}\|^2}{2}. \]

Observe that \( G(u, v, \beta, \partial I_0^+ \setminus \partial I_0^-) \) converges to

\[ G(u, v, \partial I_0^+ \setminus \partial I_0^-) = \sum_{i \in \partial I_0^+ \setminus \partial I_0^-} |u_i| (1 - \xi_i \text{sgn}(u_i)) + \frac{2}{2} \sum_{i \in \partial I_0^-} |u_i| + \frac{\|A_{S \cup \partial I_0^+} v_{S \cup \partial I_0^+}\|^2}{2} \]

as \( \beta \to +\infty \), and then \( C(\beta, \delta, \partial I_0^+ \setminus \partial I_0^-) \) converges to the following positive constant

\[ C(\partial I_0^+, \partial I_0^-) := \int \exp \left( -G(u, v, \partial I_0^+, \partial I_0^-) \right) du dv \]

as \( \beta \to +\infty \). By observing that \( |\partial I_0^-| = |\partial I_0^+| \) is the minimizer of

\[ |\partial I_0^-| \to |I_0| - \frac{|\partial I_0^+|}{2} + \frac{|S|}{2}, \]

it follows that for \( K^- \not= \emptyset \),

\[ A(\beta, \delta, K^+, K^-) \]

converges to 0 as \( \beta \to +\infty \).

As a consequence we derive that as \( \beta \to +\infty \),

\[ \mathbb{P}(X_\beta(i) \xi_i = 1, \forall i \in \partial I_0) \to 1, \]

and then we get the following.

**Theo:** If \( A^*_{\partial I} A_{\partial I} \) is invertible, then the components

\[ \left( (\beta X_\beta(i), i \in I_0 \setminus \partial I_0), (\sqrt{\beta} (X_\beta(i) - x_i(y)) : i \in S \cup \partial I_0) \right) \]
are asymptotically independent as $\beta \to +\infty$. Their asymptotic PDF are proportional respectively to
\[
\prod_{i \in I_0 \setminus \partial I_0} \exp \left( -|x_i|(1 - \text{sgn}(x_i)\xi) \right), \\
\exp \left( -\|A_{S \cup \partial I_0}(x - x(y))_{S \cup \partial I_0}\|^2_2 \right).
\]

3. Bayesian Lasso and multivalued diffusion

First we solve rigorously the following stochastic differential equation
\[
dx = -[\partial \|x\|_1 + A^*(Ax - y)]dt + dw, \quad (13)
\]
where $w$ is the standard Brownian motion. Second we show that the solution of "equation (13)" is ergodic with the stationary probability density "equation (1)" with $\beta = 1$.

3.1. Yosida approximation

Let $\varphi : \mathbb{R}^p \to (-\infty, +\infty]$ be a proper l.s.c. convex function, and $\mathcal{P} (\mathbb{R}^p)$ be the set of subsets of $\mathbb{R}^p$. The sub-differential $\partial \varphi$ is the map from $\mathbb{R}^p \to \mathcal{P} (\mathbb{R}^p)$ defined by
\[
\partial \varphi(x) = \{ v \in \mathbb{R}^p : \varphi(x + h) \geq \varphi(x) + \langle h, v \rangle, \forall h \in \mathbb{R}^p \}.
\]
The domain
\[
\text{Dom}(\partial \varphi) = \{ x : \partial \varphi(x) \neq \emptyset \}.
\]
A sequence of single valued approximations for the subdifferential $\partial \varphi(x)$ is based on Yosida approximation. For each $\varepsilon > 0$ and $z \in \mathbb{R}^p$, the equation
\[
x = z + \varepsilon \partial \varphi(z)
\]
has a unique solution denoted by
\[
z = (I + \varepsilon \partial \varphi)^{-1}(x) \\
:= \text{prox}_{\varepsilon \varphi}(x).
\]
The map $\text{prox}_{\varepsilon \varphi} : \mathbb{R}^p \to \text{Dom}(\partial \varphi)$ is called proximal function. The Yosida approximation of the sub-differential $\partial \varphi$ is the application
\[
\beta_\varepsilon(x) := \frac{x - \text{prox}_{\varepsilon \varphi}(x)}{\varepsilon}.
\]
The following are well known see e.g. [21].

prop: We have
\begin{enumerate}
\item $\text{prox}_{\varepsilon \varphi}$ is a contraction from $\mathbb{R}^p$ to $\text{Dom}(\partial \varphi)$.
\item $\beta_\varepsilon$ is monotone on the whole $\mathbb{R}^p$, i.e.
\[
\langle \beta_\varepsilon(x_1) - \beta_\varepsilon(x_2), x_1 - x_2 \rangle \geq 0,
\]
for all $x_1, x_2 \in \mathbb{R}^p$, and is Lipschitz continuous with the constant $\frac{1}{\varepsilon}$.
\end{enumerate}
(iii) For every \( x \in \mathbb{R}^p, \beta_\varepsilon(x) \in \partial \varphi(prox_{\varepsilon \varphi}(x)). \)

**prop:** For each \( \varepsilon > 0 \), the map
\[
\varphi_\varepsilon(x) = \min\{\varphi(z) + \frac{\|x - z\|^2}{2\varepsilon}\}
\]
is called the Yosida approximation of the function \( \varphi \). We have
(i) \( \varphi_\varepsilon \) is convex with the domain \( \mathbb{R}^p \).
(ii) \( \varphi_\varepsilon \) is of class \( C^1 \) with \( \nabla \varphi_\varepsilon = \beta_\varepsilon \).
(iii) The infimum defining \( \varphi_\varepsilon(x) \) is attained at \( prox_{\varepsilon \varphi}(x) \), and
\[
\varphi_\varepsilon(x) = \frac{\varepsilon}{2} \| \beta_\varepsilon(x) \|^2 + \varphi_\varepsilon(prox_{\varepsilon \varphi}(x)).
\]
(iv) Letting \( \varepsilon \downarrow 0 \), we have \( \varphi_{\varepsilon} \uparrow \varphi(x) \) for all \( x \in \mathbb{R}^p \).

In the case \( \varphi(x) = \|x\|_1 \), we have
\[
prox_{\alpha \varphi}(x) = (x + \alpha)1_{[x \leq -\alpha]} + (x - \alpha)1_{[x > \alpha]},
\]
and
\[
\varphi_\varepsilon(x) = \min \left\{ \sum_{i=1}^p |z_i| + \frac{\|z - x\|^2}{2\varepsilon} \right\}
\]
\[
= \sum_{i=1}^p \min \left\{ |z_i| + \frac{|z_i - x_i|^2}{2\varepsilon} \right\}
\]
\[
= \sum_{i=1}^p \left[ (|x_i| - \frac{\varepsilon}{2})1_{[|x_i| \geq \varepsilon]} + \frac{|x_i|^2}{2\varepsilon}1_{[|x_i| \leq \varepsilon]} \right].
\]
The gradient
\[
\nabla \varphi_\varepsilon(x) = \beta_\varepsilon(x)
\]
\[
= \text{sgn}(x)1_{[|x| \geq \varepsilon]} + \frac{x}{\varepsilon}1_{[|x| \leq \varepsilon]}.
\]
Finally
\[
prox_{\alpha \varphi}(x) = (x + \alpha)1_{[x \leq -\alpha - \varepsilon]} + \frac{\varepsilon x}{\alpha + \varepsilon}1_{[|x| \leq \alpha + \varepsilon]} + (x - \alpha)1_{[x \geq \alpha + \varepsilon]}.
\]

3.2. Multivalued stochastic differential equation

Now, we come back to Multivalued stochastic differential equation. Let \( w \) be the standard Brownian motion on \( \mathbb{R}^p \) and \( b : \mathbb{R}^p \rightarrow \mathbb{R}^p \) be a smooth map. A solution of the \( \mathbb{R}^p \)-multivalued stochastic differential equation (abbreviated MSDE)
\[
dx_t = -\partial \varphi(x_t)dt - b(x_t)dt + dw_t
\]
is a couple of continuous adapted stochastic processes \( t \in [0, +\infty) \rightarrow (x(t), l(t)) \) with values in \( \mathbb{R}^p \times \mathbb{R}^p \), and such that \( l(0) = 0 \), \( t \rightarrow l(t) \) has bounded variation on each compact interval and
\[
dx_t = -dl_t - b(x_t)dt + dw_t,
\]
Suppose that \( \text{prop:} \) equation (13) is given by the couple \((x, t)\) and \("t \to (\alpha(t), \beta(t))\) such that \(\beta(t) \in \partial \varphi(\alpha_t)\). Observe that if \(dl_t = l'_t dt\), then \(l'_t \in \partial \varphi(x_t)\).

It’s known that if
\[
\|b(x_1) - b(x_2)\| \leq C\|x_1 - x_2\|, \quad \forall x_1, x_2,
\]
\[
\|b(x)\| \leq C(1 + \|x\|), \quad \forall x,
\]
then there exists a unique solution \((x, l)\). See e.g. [6], [7], [5], [20], [4], [28]. It follows that "equation (13)" has a unique solution \((x, l)\). In general the measure \(dl_t\) is not absolutely continuous with respect to the Lebesgue measure \(dt\). However we are going to show that \(dl_t\) is absolutely continuous in the case "equation (13)". We recall two methods for constructing the solution \(x\) of "equation (13)".

1) By choosing \(\varphi(x) = \|x\|_1, b(x) = A^*(Ax - y)\), then the solution of "equation (13)" is the unique couple \((x, l)\) of continuous maps such that \(l(0) = 0, t \to l(t)\) has bounded variation on each compact interval and
\[
dx(t) = -[dl_t + A^*(Ax(t) - y)dt] + dw_t, \quad \frac{dl(t)}{dt} \in \partial \|x(t)\|_1. \quad (15)
\]

2) By choosing \(\varphi(x) = \|x\|_1 + \frac{\|Ax - y\|^2}{2}, b(x) = 0\), then the solution of "equation (13)" is given by the couple \((x(t), k(t))\) such that
\[
dx(t) = -dk(t) + dw(t), \quad k(t) \in \partial \varphi(x(t)).
\]

The uniqueness of the solution of "equation (13)" implies that \(dk(t) = dl_t + A^*(Ax(t) - y)dt\). Now, we are going to show that \(l\) is absolutely continuous. For this aim we recall Skorokhod problem [7]. Let \(f\) be any continuous function from \([0, T] \to \mathbb{R}^d\), and \(\psi : \mathbb{R}^p \to \mathbb{R}\) be any convex function. Then there exists a unique couple \((x, k)\) of continuous maps such that \(k(0) = 0, t \to k(t)\) has bounded variation on each compact interval,
\[
x(t) = f(t) - k(t), \quad \forall t \geq 0, \quad (16)
\]
and the measure \(\langle x(t) - \alpha(t), dk(t) - \beta(t)dt \rangle\) is nonnegative for all continuous trajectory \(t \to (\alpha(t), \beta(t))\) such that \(\beta(t) \in \partial \psi(\alpha(t))\). Now we ready to announce our result.

prop: Suppose that
\[
m = \sup\{\|v\| : \quad v \in \bigcup_{x \in \mathbb{R}^p} \partial \psi(x)\}
\]
is finite. Then the function \(l\) solution of Skorokhod problem "equation (16)" is absolutely continuous.

proof: Let \(e \in \mathbb{R}^p\) such that \(\|e\| = 1, \gamma > 0\) and \(v \in \partial \psi(\gamma e)\) having the smallest Euclidean norm. As \((x, k)\) is the solution of Skorokhod problem, then
\[
\langle x(t) - \gamma e, dl(t) \rangle \geq \langle x(t) - \gamma e, vd(t) \rangle \geq -m(\|x(t)\| + \gamma) dt.
\]
For each $0 \leq s < t$, we have

$$\langle l(t) - l(s), e \rangle = \int_s^t \langle e, dl(u) \rangle$$

$$= \gamma^{-1} \int_s^t \langle x(u), dl(u) \rangle - \gamma^{-1} \int_s^t \langle x(u) - \gamma e, dl(u) \rangle$$

$$\leq \gamma^{-1} \int_s^t \langle x(u), dl(u) \rangle + m \gamma^{-1} \int_{t_i}^{t_{i+1}} \|x(u)\| du + m(t - s).$$

From the latter inequality and

$$\|l(t) - l(s)\| = \sup \{ \langle l(t) - l(s), e \rangle : e \in \mathbb{R}^p, \|e\| = 1 \},$$

and by tending $\gamma \to +\infty$, we get

$$\|l(t) - l(s)\| \leq m(t - s).$$

Which achieves the proof.

By choosing $f(t) = x_0 - \int_0^t A^*(Ax(s) - y) ds + w_t$, we derive that $(x, l)$ "equation (15)" is the solution of Skorokhod problem. As the hypothesis "equation (17)" is satisfied for $\psi(x) = \|x\|_1$, with $m = 1$, then $l$ is absolutely continuous. Finally the solution of "equation (13)" satisfies

$$x(t) = x(0) - \int_0^t [v(s) + A^*(Ax(s) - y)] ds + w(t), \quad (18)$$

and $v(t) \in \partial \|x(t)\|_1, \|v(t)\| \leq 1$, dt a.e. Moreover we can show that a.s. for $i = 1, \ldots, p$ that $x_i(t) \neq 0$ and $v_i(t) = sgn(x_i(t))$, dt a.e. The "equation (18)" becomes

$$dx(t) = \frac{1}{2} \nabla \ln (\rho(x(t))) dt + dw(t), \quad (19)$$

where

$$\rho(x) := \frac{1}{Z} \exp \left( -2 \|x\|_1 - \|Ax - y\|^2 \right). \quad (20)$$

The equation "equation (19)" is known as distorted Brownian motion [18] with the generalized Schrödinger operator

$$H = -\frac{1}{2} \Delta - \frac{1}{2} \left( \sum_{i=1}^p \delta(x_i) + Trace(A^*A) \right) + \frac{1}{2} \|sgn(x) + A^*(Ax - y)\|^2.$$ 

Here $\Delta$ is Laplacian operator and $\delta$ denotes the Dirac measure at 0.

### 3.3. Transition probabilities in the one dimensional case

In the one dimensional case

$$dx(t) = -\lambda sgn(x(t)) dt + dw(t), \quad x(0) = x_0, \quad \lambda > 0$$

is known as bang-bang Brownian motion [25], or the diffusion with V potential [26]. In this case Schrödinger operator has the form

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} (1 - \delta).$$
The transition probabilities \( p^\lambda(x, t \mid x_0, 0) \) of the bang-bang Brownian motion is known \([3]\). We can calculate it using Girsanov Formula, and the trivariate density of Brownian motion, its local time and occupation times \([19]\). We obtain

\[
p^\lambda(x, t \mid x_0, 0) = q^\lambda(x, t \mid x_0, 0) \lambda \exp(-2\lambda |x|) \]

where

\[
q^\lambda(x, t \mid x_0, 0) = \exp\left(\lambda (|x_0| + |x|) - \frac{t \lambda^2}{2} \right) \gamma_t(x - x_0) + F\left(\frac{\lambda t - (|x| + |x_0|)}{\sqrt{t}}\right),
\]

\[
F(x) = \int_{-\infty}^x \frac{\exp(-\frac{u^2}{2})}{\sqrt{2\pi}} \, du,
\]

\[
\gamma_t(u) = \frac{\exp(-\frac{u^2}{2t})}{\sqrt{2t\pi}}.
\]

Observe that \( p^\lambda(x, t \mid x_0, 0) \to \lambda \exp(-2\lambda |x|) \) as \( t \to +\infty \) for all \( x_0 \). Hence, the MSDE

\[
dx(t) = -\lambda \text{sgn}(x(t)) \, dt + dw(t)
\]

is ergodic with the invariant density \( \lambda \exp(-2\lambda |x|) \).

4. Sampling using multivalued SDE

As we said before, the solution \((x(t))\) of "equation (13)" is ergodic. It follows that \( \lim_{T \to +\infty} x(T) \) has the probability distribution \( \rho \) "equation (20)". If we dispose of a trajectory \( t \in [0, T] \to x_t \) for large \( T \), then for any \( \rho \)-integrable function \( h \),

\[
\frac{1}{T} \int_0^T h(x_t) \, dt \approx \mathbb{E}[h(x(T))] \approx \int_{\mathbb{R}^p} h(x) \rho(x) \, dx.
\]

Hence for large \( T \) the expectation \( \mathbb{E}[x(T)] \) of the solution "equation (13)" is close to Bayes estimator "equation (3)". We will approximate \( \mathbb{E}[x(T)] \) using numerical schemes of "equation (13)" and the timestep

\[
\Delta t_l = 2^{-l} T,
\]

with the level \( l = l_s, l_s + 1, \ldots \). In all the sequel the small level \( l_s := \frac{\ln(T)}{\ln(2)} + 1 \).

Having a numerical scheme \((x_L(sc, k) : \ k = 1, \ldots, 2^L)\) such that \( \mathbb{E}[x_L(sc, 2^L)] \to \mathbb{E}[x(T)] \) as \( L \to +\infty \), we need to calculate \( \mathbb{E}[x_L(sc, 2^L)] \) for large \( L \). To achieve this goal we use Monte Carlo (MC) and multilevel Monte Carlo (MLMC) algorithms. We will discuss the efficiency of MC and MLMC estimates. We will mimic the results obtained in [27] for Coulomb collisions, and propose a method for calculating the cost.

5. MC Efficiency and computational cost

Given a sample \((x_l^k(sc, 2^l) : \ k = 1, \ldots, N_l)\) of \( x_l(sc, 2^l) \) having the size \( N_l \), we define

\[
\hat{x}_l^{N_l}(sc, 2^l) = \frac{1}{N_l} \sum_{k=1}^{N_l} x_l^k(sc, 2^l), \quad l, \text{ and } N_l \text{ are fixed},
\]
The computational cost that is known. To make the scheme as efficient as possible, "equation (26)". Applying the method of Lagrange multipliers of obtaining (\(x_k^l(sc, 2^l)\) : \(k = 1, \ldots, N_l\)) is the product of the number of timestep \(\frac{T}{\Delta t_l} = 2^l\) and the number of samples \(N_l\). Namely,

\[K(N_l, \Delta t_l) = N_l \frac{T}{\Delta t_l} = N_l2^l.\]

To make the scheme as efficient as possible, \(K\) must be minimal subject to the constraint "equation (26)". Applying the method of Lagrange multipliers

\[L(N_l, \Delta t_l, \lambda) = N_l \frac{T}{\Delta t_l} + \lambda \left( e(sc, \Delta t_l) + \frac{Var_l(sc)}{N_l} - \eta^2 \right),\]

we get the optimal choice

\[
\frac{T}{\Delta t_l} - \lambda \frac{Var_l(sc)}{N_l^2} = 0,
\]
\[-N_i \frac{T}{(\Delta t_i)^2} + \lambda \frac{\partial e}{\partial t_i}(sc, \Delta t_i) = 0,\]
\[e(sc, \Delta t_i) + \frac{\text{Var}_i(sc)}{N_i} = \eta^2.\]  
(27)

It follows that
\[\frac{\partial e(sc, \Delta t_i)}{\Delta t_i} = \frac{\eta^2 - e(sc, \Delta t_i)}{\Delta t_i},\]  
(28)
\[e(sc, \Delta t_i) < \eta^2.\]  
(29)

We propose to solve the latter system numerically as follows. In all the sequel we estimate \(\hat{x}(T)\) by \(\hat{x}_L(sc, 2^L)\) with \(L = 16\). Hence we obtain the following approximation:
\[\|\hat{x}_L(sc, 2^L) - \hat{x}_L(sc, 2^l)\|^2 \approx e(sc, \Delta t_i).\]  
(30)

Second
\[\frac{\partial e(sc, \Delta t_i)}{\Delta t_i} \approx \frac{e(sc, \Delta t_i) - e(sc, \Delta t_{i+1})}{T2^{l-1}}.\]

The "equation (28)" becomes
\[3e(sc, \Delta t_i) - 2e(sc, \Delta t_{i+1}) \approx \eta^2.\]

Now we calculate for \(l \geq l_s\) the quantity
\[3e(sc, \Delta t_i) - 2e(sc, \Delta t_{i+1})\]  
(31)
until it becomes close to \(\eta^2\) and
\[e(sc, \Delta t_i) < \eta^2.\]  
(32)

Having \(l\), we calculate \(\text{Var}_i(sc)\) by
\[\sum_{i=1}^{p} \frac{1}{N} \sum_{k=1}^{N} \left| x_{t,i}(sc, 2^l) - \frac{1}{N} \sum_{k=1}^{N} x_{t,i}(sc, 2^l) \right|^2.\]  
(33)

Having \(l\) and \(\text{Var}_i(sc)\) we calculate the optimal sample size \(N_i\) using the "equation (27)" and then we derive the optimal cost \(K_l\).

6. MLMC Efficiency and computational cost

Multilevel Monte Carlo (MLMC) was initially developed for financial mathematics [16], [17] and now used in a disparate areas.

Multilevel Monte Carlo considers multilevels. In our study we consider the levels \(l = l_s, l_s + 1, \ldots, l_m < L = 16\). The smallest level \(l_s\) is choosen such that \(\Delta t_{l_s} = \frac{1}{2}\).

We generate a sample \((x_{t,i}^k(sc, 2^l) : k = 1, \ldots, N_i)\) of size \(N_i\) of \(x_{t,i}(sc, 2^l)\), and for each \(l = l_s + 1, \ldots, l_m\), we generate from the same underlying stochastic path and initial conditions the samples \((x_{t,i}^k(sc, 2^l) : k = 1, \ldots, N_i)\) (\(x_{t,i}(sc, 2^{l-1})\)) respectively of \(x_{t,i}(sc, 2^l)\) and \(x_{t,i}(sc, 2^{l-1})\). Moreover, the samples \((x_{t,i}^k(sc, 2^{l}) : k =\)
1, \ldots, N_{l_s}), (x_t^k(s, 2^l), x_{l-1}^k(s, 2^{l-1}) : k = 1, \ldots, N_l) for \ l = l_s + 1, \ldots, l_m have to be independent. Using the telescoping sum

\[ \hat{x}_{lm}^s(sc, 2^{lm}) = \hat{x}_{ls}^s(sc, 2^{ls}) + \sum_{l=l_s+1}^{l_m} (\hat{x}_l(sc, 2^l) - \hat{x}_{l-1}(sc, 2^{l-1})), \]

MLMC proposes the estimate

\[ \hat{x}_{lm}^{Nl}(sc, 2^{lm}) = \hat{x}_{ls}^{Nl}(sc, 2^{ls}) + \sum_{l=l_s+1}^{l_m} (\hat{x}_l^{Nl}(sc, 2^l) - \hat{x}_{l-1}^{Nl}(sc, 2^{l-1})) \]

of \( \hat{x}_{lm}(sc, 2^{lm}) := E[\hat{x}_{lm}(sc, 2^{lm})]. \)

We introduce for each level \( l \) and sample size \( N_l \) the following notations:

\[ \hat{x}_l^{Nl}(sc, 2^l) - \hat{x}_{l-1}^{Nl}(sc, 2^{l-1}) := \delta \hat{x}_l^{Nl}(sc, 2^l). \]

It follows that

\[ \hat{x}_{lm}^{Nl}(sc, 2^{lm}) = \hat{x}_{ls}^{Nl}(sc, 2^{ls}) + \sum_{l=l_s+1}^{l_m} (\hat{x}_l^{Nl}(sc, 2^l) - \hat{x}_{l-1}^{Nl}(sc, 2^{l-1})) \]

\[ := \hat{x}_{ls}^{Nl}(sc, 2^{ls}) + \sum_{l=l_s+1}^{l_m} \delta \hat{x}_l^{Nl}(sc, 2^l), \]

(34)

where \( \delta \hat{x}_l^{Nl}(sc, 2^l) := \hat{x}_l^{Nl}(sc, 2^l) - \hat{x}_{l-1}^{Nl}(sc, 2^{l-1}). \)

An accurate estimate \( \hat{x}_{lm}^{Nl}(sc, 2^{lm}) \) of \( \hat{x}(T) \) is one for which the mean square error

\[ MSE := E\left[\|\hat{x}(T) - \hat{x}_{lm}^{Nl}(sc, 2^{lm})\|^2\right] = \|\hat{x}(T) - \hat{x}_{lm}(sc, 2^{lm})\|^2 + Var(\hat{x}_{lm}^{Nl}(sc, 2^{lm})) \]

is small.

If we set \( V_l = Var(x_{ls}(sc, 2^{ls})) \), and for \( l = l_s + 1, \ldots, l_m, \)

\[ V_l = Var(\delta x_l(sc, 2^l)), \]

(35)

then

\[ Var(\hat{x}_{lm}^{Nl}(sc, 2^{lm})) = \sum_{l=l_s}^{l_m} \frac{V_l}{N_l}, \]

and

\[ MSE = \|\hat{x}(T) - \hat{x}_{lm}(sc, 2^{lm})\|^2 + \sum_{l=l_s}^{l_m} \frac{V_l}{N_l}. \]

If

\[ \|\hat{x}(T) - \hat{x}_{lm}(sc, 2^{lm})\|^2 := e(sc, \Delta t_{lm}) + \sum_{l=l_s}^{l_m} \frac{V_l}{N_l} = \eta^2, \]

(36)

then efficiency of MLMC is equivalent to minimize

\[ K = \sum_{l=l_s}^{l_m} K_l = \sum_{l=l_s}^{l_m} N_l \frac{T}{\Delta t_l}, \]

(37)
under the constraint "equation (36)".
We estimate for \( l_s \leq l < L = 16 \), \( \| \hat{x}(T) - \hat{x}_l(sc, 2^l) \|^2 \) by \( \| \hat{x}_L(sc, 2^L) - \hat{x}_l(sc, 2^l) \|^2 \) and then we are interested in the set \( l(\eta) \) of levels \( l \) such that
\[
e(s, \Delta t_l) = \| \hat{x}_L(sc, 2^L) - \hat{x}_l(sc, 2^l) \|^2 \approx \frac{\eta^2}{2}.
\]
For each \( l_{opt} \in l(\eta) \), the "equation (36)" becomes
\[
\sum_{l=l_s}^{l_{opt}} V_l = \eta^2 - e(s, \Delta t_{l_{opt}}).
\]
Having the optimal \( l_{opt} \), the minimization of \( K "equation (37)" \) under the constraint (38) is solved by Lagrange multiplier
\[
\partial_{N_l}(K + \lambda \left( \frac{V_l}{N_l} + \sum_{l=l_s+1}^{l_{opt}} \frac{V_l}{N_l} - (\eta^2 - e(s, \Delta t_{l_{opt}})) \right)) = 0, \quad l = l_s, \ldots, l_{opt}.
\]
Hence
\[
2^l = \frac{\lambda V_l}{N_l^2}, \quad l = l_s, \ldots, l_{opt},
\]
\[
\sum_{l=l_s+1}^{l_{opt}} \frac{V_l}{N_l} = \eta^2 - e(s, \Delta t_{l_{opt}}).
\]
It follows for \( l = l_s, \ldots, l_{opt} \), that \( N_l = \sqrt{\lambda V_l 2^{-l}} \), and then
\[
\sum_{l=l_s}^{l_{opt}} \frac{\sqrt{V_l 2^{l/2}}}{\sqrt{\lambda}} = \eta^2 - e(s, \Delta t_{l_{opt}}).
\]
Having \( l_{opt} \), we estimate \( V_{l_s} \), and \( (V_l : l = l_s + 1, \ldots, l_{opt}) \) by
\[
\hat{V}_{l_s} = \sum_{i=1}^{p} \frac{1}{N} \sum_{k=1}^{N} |x_{l_s,i}^k - \hat{x}_{l_s,i}^k|^2,
\]
\[
\hat{V}_l := \sum_{i=1}^{p} \frac{1}{N} \sum_{k=1}^{N} |\delta x_{l,i}^k - \delta \hat{x}_{l,i}^k|^2, \quad l = l_s + 1, \ldots, l_{opt}.
\]
Hence for \( l = l_s, \ldots, l_{opt} \)
\[
N_l = \frac{1}{\eta^2 - e(s, \Delta t_{l_{opt}})} \sqrt{V_l 2^{-l}} \sum_{k=l_s}^{l} \sqrt{V_k 2^k}.
\]
Now, we are going to present our schemes.

7. Semi-implicit Euler schemes

Numerical approximation has been tackled in [3], [21], [2], [24], see also [29, 30]. Semi-implicit Euler scheme (SIES) of "equation (14)" is given by
\[
x_l(k+1) - x_l(k) = -\nabla \varphi(x_l(k+1)) \Delta t_l - b(x_l(k)) \Delta t_l + \sqrt{\Delta t_l} n_l(k+1),
\]
where \((n(k+1) \colon k = 0, 1, \ldots, 2^l - 1)\) is a sequence of i.i.d. standard Gaussian vectors. Known \(x_l(k)\) and \(n(k+1)\), we have

\[
x_l(k+1) = \text{prox}_{\Delta t_l \varphi}(x_l(k) - b(x_l(k)) \Delta t_l + \sqrt{\Delta t_l} n(k+1)) \quad (42)
\]

The weak and the strong convergence property of the scheme "equation (42)" to the solution "equation (14)" are defined respectively in terms of

\[
e_w(\Delta t_l) = \| E[ x(T) - x_l(2^l) ] \|, \quad (43)
\]
\[
e_s(\Delta t_l) = E \left[ \| x(T) - x_l(2^l) \|^2 \right]^{\frac{1}{2}}. \quad (44)
\]

From (5) the strong error "equation (44)" is estimated by

\[
O((\Delta t_l \ln(\frac{1}{\Delta t_l}))^{\frac{1}{4}}). \quad (45)
\]

By setting \(\varphi(x) = \|x\|_1 + \frac{\|Ax - y\|^2}{2}\), and \(b = 0\), EES1 of (13) is given by

\[
x_l(k+1) = \text{prox}_{\Delta t_l \varphi}(x_l(k)) + \sqrt{\Delta t_l} n_{k+1}.
\]

The proximal \(\text{prox}_{\Delta t_l \varphi}(x^{(k)})\) is not computable, but for large \(l\), we have

\[
\text{prox}_{\Delta t_l \varphi}(x) \approx \text{prox}_{\Delta t_l \|\cdot\|_1} \left( x + A^*(y - Ax) \Delta t_l \right).
\]

Finally we get

\[
x_l(k+1) = \text{prox}_{\Delta t_l \|\cdot\|_1}(x_l(k) + A^*(y - Ax_l(k)) \Delta t_l) + \sqrt{\Delta t_l} n_{k+1}, \quad (46)
\]

known as PULA algorithm [22].

8. Explicit Euler scheme

8.1. Algorithm EES1

By setting \(\varphi(x) = \|x\|_1 + \frac{\|Ax - y\|^2}{2}\), and \(b = 0\), EES1 of (13) is given by

\[
x_l(k+1) = \text{prox}_{\Delta t_l \varphi}(x_l(k)) + \sqrt{\Delta t_l} n_{k+1}.
\]

The proximal \(\text{prox}_{\Delta t_l \varphi}(x^{(k)})\) is not computable, but for large \(l\), we have

\[
\text{prox}_{\Delta t_l \varphi}(x) \approx \text{prox}_{\Delta t_l \|\cdot\|_1} \left( x + A^*(y - Ax) \Delta t_l \right).
\]

Finally we get

\[
x_l(k+1) = \text{prox}_{\Delta t_l \|\cdot\|_1}(x_l(k) + A^*(y - Ax_l(k)) \Delta t_l) + \sqrt{\Delta t_l} n_{k+1}, \quad (46)
\]

known as PULA algorithm [22].

8.2. Algorithm EES2

By setting \(\varphi(x) = \|x\|_1\), and \(b(x) = A^*(Ax - y)\), we obtain our new scheme

\[
x_l(k+1) = \text{prox}_{\Delta t_l \|\cdot\|_1}(x_l(k)) + A^*(y - Ax_l(k)) \Delta t_l + \sqrt{\Delta t_l} n_{k+1}. \quad (47)
\]

9. Numerical implementation

As an illustration we consider the case \(p = 10\), \(n = 7\) and the entries of the matrix \(A\) are independent Bernoulli random variables with values \(\pm \frac{1}{\sqrt{n}}\), and \(w \sim N(0, \frac{1}{2} I_n)\). We simulate the vector \(x(\text{true})\) from the PDF \(\exp(-2\|x\|_1)\). We get the data \(y := Ax(\text{true}) + w\) from a realization of \(A\) and \(w\). The time horizon \(T = 10\) the maximal level \(L = 16\) and the smallest level \(l_s = 5\).
9.1. Graphics of Trajectories of each scheme

For each scheme $sc$ and for each level $l = l_s, l_s + 1, l_s + 2$, we plot the trajectories $k \in [0, 2^l] \to \mathbf{x}_l(sc, k)$. For the largest level $L = 16$ we plot only the first component.

**Figure 1.** The chains of SIES, EES1 and EES2 for $l = l_s$.

**Figure 2.** The chains of SIES, EES1 and EES2 for $l = l_s + 1$. 
Figure 3. The chains of SIES, EES1 and EES2 for $l = l_s + 2$.

Figure 4. The first component of the chains SIES, EES1 and EES2 for $l = 16$.

9.2. The Cost of each scheme using MC

We approximate for each scheme $x(T)$ by $x_L^T(sc, 2^L)$ with $L = 16$, and we look for the optimal level $l_{opt}$ and the optimal sample size $N_{opt}$ such that

$$MSE := \mathbb{E} \left[ \left\| \mathbb{E}[x_L^T(sc, 2^L)] - \frac{1}{N_{opt}} \sum_{k=1}^{N_{opt}} x_i^k(sc, 2^L) \right\|^2 \right] = \eta^2.$$
We need for \( l = l_1, \ldots, L - 2 \) to calculate \( e(sc, \Delta t_l) := \| \mathbf{E}[\mathbf{x}_L(sc, 2^L)] - \mathbf{E}[\mathbf{x}_l(sc, 2^l)] \|^2 \). Using Monte-Carlo with \( N = 1000 \), we obtain by

\[
e(sc, \Delta t_l) \approx \| \frac{1}{N} \sum_{k=1}^{N} \mathbf{x}_L^k(sc, 2^L) - \frac{1}{N} \sum_{k=1}^{N} \mathbf{x}_l^k(sc, 2^l) \|^2.
\]

Table 1 shows the numerical values of \( e(sc, \Delta t_l) \) for each scheme and for \( l = 5, \ldots, 13 \).

| \( l \) | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|------|---|---|---|---|---|----|----|----|----|
| \( e(SIES, \Delta t_l) \) | 0.0050 | 0.0080 | 0.0071 | 0.0022 | 0.0054 | 0.0066 | 0.0056 | 0.0043 | 0.0022 |
| \( e(EES1, \Delta t_l) \) | 0.0380 | 0.0025 | 0.0069 | 0.0043 | 0.0016 | 0.0039 | 0.0027 | 0.0032 | 0.0022 |
| \( e(EES2, \Delta t_l) \) | 0.0107 | 0.0041 | 0.0044 | 0.0042 | 0.0054 | 0.0111 | 0.0041 | 0.0048 | 0.0065 |

By fixing \( \eta^2 \geq \max(e(sc, \Delta t_l), sc = SIES, EES1, EES2, l = 5, \ldots, 13) \), the constraint \( e(sc, \Delta t_l) \leq \eta^2 \) holds for each level \( l = 5, \ldots, 13 \). The optimal level \( l_{opt} \) is such that \( 3e(sc, \Delta t_l) - 2e(sc, \Delta t_{l+1}) \approx \eta^2 \). Having \( l_{opt} \) we calculate

\[
Var_{l_{opt}}(sc) := \sum_{i=1}^{p} Var(x_{l_{opt},i}(sc, 2^{l_{opt}})),
\]

\[
\approx \sum_{i=1}^{p} \frac{1}{N} \sum_{k=1}^{N} \left| x_{l_{opt},i}(sc, 2^{l_{opt}}) - \frac{1}{N} \sum_{k=1}^{N} x_{l_{opt},i}(sc, 2^{l_{opt}}) \right|^2,
\]

and we derive the optimal \( N_{opt}(sc) = \frac{Var_{l_{opt}}(sc)}{\eta^2-e(sc, \Delta t_{l_{opt}})} \).

The Figure 5 shows how to find graphically the optimal level \( l_{opt} \).

We summarize for the three schemes in the Table 2 the values of \( l_{opt}, N_{opt} \) and their cost. The scheme SIES has the lowest cost.

| \( l_{opt} \) | \( N_{opt} \) | Cost |
|------|----|-----|
| SIES | 7  | 70  | 8938 |
| EES1 | 7  | 81  | 10427 |
| EES2 | 10 | 83  | 85035 |

Table 2. Optima level and cost of MC for each scheme.

9.3. Computational cost of MLMC

In the Figure (6) for each scheme we plot \( l \to e(sc, \Delta t_l) \) "equation (25)". We derive graphically the optimal level \( l_{opt} \).
We summarize for the three schemes in the Table 2 the values of $l_{opt}$, $N_{l_s}(opt), \ldots, N_{l_{opt}}(opt)$ and their cost. Like MC method the scheme SIES has the lowest cost.

**N.B.** For each $l_{opt}$, the optimal sample sizes are $N_{5-l_{opt}} := N_{5(opt)}, \ldots, N_{l_{opt}}(opt)$, e.g. for the scheme SIES $l_{opt} = 6$ and $N_{5-6} = 74, 20$. 

**Figure 5.** Graphical determination of $l_{opt}$ for the schemes SIES, EES1 and EES2.

**Figure 6.** Graphical identification of $l_{opt}$ for each sheme.
Table 3. Optimal level and cost of MLMC for each scheme.

| Scheme | $lopt$ | $N_{5-lopt}$ | Cost  |
|--------|--------|-------------|-------|
| SIES   | 6      | 74          | 3639.18 |
| EES1   | 7      | 132         | 8962.85 |
| EES2   | 10     | 167         | 18029.47 |

10. Markov chain Monte Carlo method MCMC

Using the ergodicity we suppose that the PDF of $\mathbf{x}(T)$ is approximated by $\rho(\mathbf{x}) = Z^{-1} \exp(-2\|\mathbf{x}\|_1 - \|A\mathbf{x} - \mathbf{y}\|^2)$. For the error $\eta^2$ fixed the cost of MCMC is the sample size $N$ such that

$$\mathbb{E}\left[\|\mathbb{E}[\mathbf{x}(T)] - \frac{1}{N} \sum_{k=1}^{N} MCMC(k)\|^2\right] \approx \eta^2.$$ 

Here $MCMC$ is a trajectory of the Markov Chain Monte Carlo having the target $\rho$.

We recall how MCMC works. Let $k \rightarrow MC(k)$ be a Markov chain having the transition probability density $\pi(x_2 | x_1) > 0$ for all $x_1, x_2 \in \mathbb{R}^p$. We construct from $MC$ a new Markov chain $k \rightarrow MCMC(k)$ having the transition probability

$$\alpha \pi(x_2 | x_1) dx_2 + (1 - \alpha) \delta_{x_1}(x_2)$$

where

$$\alpha = \min\left(1, \frac{\rho(x_2)\pi(x_1 | x_2)}{\rho(x_1)\pi(x_2 | x_1)}\right).$$

The new Markov chain $MCMC$ is ergodic and has $\rho(\mathbf{x})$ as its invariant probability density function. We propose the Markov chains $MC(k) := \mathbf{x}_{lopt}(sc, k)$ for $sc = EES1, EES2$ and $MC(k) = RW(k, \sigma^2)$. Here $RW(k, \sigma^2)$ denotes the Gaussian random walk, each step has the variance $\sigma^2$. We obtain three MCMC chains: $MCMC_{prox}(EES1), MCMC_{prox}(EES2), MCMC_{RW}$. Observe that $MCMC_{prox}(EES1)$ is known as PMALA \[22\]. Table 4 shows the cost of each method.

10.1. Computational cost of MCMC

In the table 4, we indicate the different costs of MC, $MCMC_{prox}$ and $MCMC_{RW}$. We create for each $N, M$ MCMC chains ($MCMC^n(k) : k = 1, \ldots, N, i = 1, \ldots, M$). We approximate $\mathbb{E}\left[\|\mathbb{E}[\mathbf{x}(T)] - \frac{1}{N} \sum_{k=1}^{N} MCMC(k)\|^2\right]$ by $\frac{1}{M} \sum_{i=1}^{M} \mathbb{E}[\mathbf{x}_L(sc, 2^L)] - \frac{1}{N} \sum_{k=1}^{N} MCMC^i(k)\|^2$.

Table 4 shows that the $MCMC_{RW}$ corresponding to the proposal distribution $\mathcal{N}(0, 0.3)$ is the winner. But it loses against MLMC with the scheme SIES (see Table 2).

Concluding remark. In this work we studied the approximation of Bayesian Lasso using MC, MLMC and MCMC methods and three schemes Semi-implicit Euler scheme (SIES), and two Explicit Euler schemes EES1 and EES2. Furthermore, we
Table 4. Cost of MC, MCMC\textsubscript{prox} and MCMC\textsubscript{RW} for EES1 and EES2 schemes.

|        | Cost (MC) | Cost (MCMC\textsubscript{prox}) | Cost (MCMC\textsubscript{RW}) | Cost (MCMC\textsubscript{RW}) |
|--------|-----------|----------------------------------|-------------------------------|-------------------------------|
| EES1   | 10427     | 5340                             | 3990 (σ² = 0.3)               | 17230 (σ² = 0.8)             |
| EES2   | 85035     | 6200                             | 3890 (σ² = 0.3)               | 16230 (σ² = 0.8)             |

proposed a method for calculating the cost of each method and each scheme. We showed that the winner is MLMC with the scheme (SIES).

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