Kac-Moody algebras and Lie algebras of regular vector fields on tori

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January 30, 2022

Abstract: We consider the problem of representing the Kac-Moody algebra \( g(N) \) specified by an \( r \times r \) indecomposable generalised Cartan matrix \( N \) as vector fields on the torus \( \mathbb{C}^*^r \). It is shown that, if the representations are of a certain form, this is possible if and only if \( g(N) \cong \mathfrak{sl}(r+1, \mathbb{C}) \) or \( \tilde{\mathfrak{sl}}(r, \mathbb{C}) \). For \( \mathfrak{sl}(r+1, \mathbb{C}) \) and \( \tilde{\mathfrak{sl}}(r, \mathbb{C}) \), discrete families of representations are constructed. These generalise the well-known discrete families of representations of \( \mathfrak{sl}(2, \mathbb{C}) \) as regular vector fields on \( \mathbb{C}^* \).

1 Introduction

Let \( H, X_1, X_{-1} \) be a standard basis of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \). As is well-known, \( \mathfrak{sl}(2, \mathbb{C}) \) can be represented by regular vector fields on \( \mathbb{C}^* \) according to the formulae:

\[
R(H) = 2z \frac{d}{dz}, \quad R(X_1) = z \left( z \frac{d}{dz} \right), \quad R(X_{-1}) = \frac{1}{z} \left( -z \frac{d}{dz} \right),
\]

or more generally \((n \in \mathbb{Z}^*)\) according to:

\[
R_n(H) = 2z \frac{d}{nz d}, \quad R_n(X_1) = z^n \left( z \frac{d}{nz d} \right), \quad R_n(X_{-1}) = \frac{1}{z^n} \left( -z \frac{d}{nz d} \right).
\]

Replacing \( z \) by \( e^{i\theta} \) gives representations of \( \mathfrak{sl}(2, \mathbb{C}) \) by complex vector fields on the circle \( U(1) \). In this paper we consider the problem of obtaining analogous families
of representations by vector fields on $\mathbb{C}^{*r}$ and $U(1)^r$ for any Kac-Moody algebra with $r \times r$ indecomposable generalised Cartan matrix $N$.

In order to motivate the type of representations we will be interested in, observe that the homomorphisms $R_n$ above can be very simply expressed in terms of (i) the value of $R$ on the Cartan subalgebra $\langle H \rangle$ (i.e. $R(H) = 2z \frac{d}{dz}$) and (ii) a function which transforms as a simple root vector (i.e. $z$ which satisfies $2z \frac{d}{dz}(z) = 2z$).

Let $\hat{g}(N)$ be the Kac-Moody algebra (see [2.1] below for details) generated by \{\(H_a, X_{\pm i} : 1 \leq a \leq r + s, 1 \leq i \leq r\}\) with indecomposable generalised Cartan matrix $N$ and suppose given (i) a homomorphism $F : \langle H_a : 1 \leq a \leq r + s \rangle \rightarrow \text{Der}(A)$ and (ii) invertible elements $v_1, \ldots, v_r \in A$ transforming as simple root vectors (here $A$ is a complex commutative algebra). The problem we consider in this paper is: can $F$ be extended to a homomorphism $\hat{F} : \hat{g}(N) \rightarrow \text{Der}(A)$ such that the $\hat{F}(X_{\pm i})$ are of the form $v_i^{\pm 1}\delta_{\pm i}$ with $\delta_{\pm i} \in \text{Im} F$? Our main result is the following:

**Theorem 1** (a) $F$ can be extended as above iff $\text{Ker}F = Z$ (the centre of $\hat{g}(N)$) and $N$ is the generalised Cartan matrix of either $\text{sl}(r + 1, \mathbb{C})$ or $\text{sl}(r, \mathbb{C})$ (in Kac notation $A_r$ or $A_{r-1}^{(1)}$).

(b) In these cases, the extensions of $F$ are in one to one correspondence with the set of $r \times r$ matrices $A = [A_{ij}]$ satisfying $[\frac{A_{ij}}{A_{jj}}] + [\frac{A_{ji}}{A_{jj}}] = N$ and $\frac{A_{ji}}{A_{jj}} = 0$ or $-1$ if $i \neq j$.

We show further that there is a $\mathbb{Z}^{*r}$-parameter family of homomorphisms $\hat{F}^{A_{n_1, \ldots, n_r}} : \hat{g}(N) \rightarrow \text{Der}(A)$ associated to each such extension $\hat{F}^A$. The relationship between $\hat{F}^{A_{n_1, \ldots, n_r}}$ and $\hat{F}^A$ is analogous to the relationship between $R_n$ and $R$.

This theorem enables us (cf section 4) to prove that there exist families of embeddings of $\text{sl}(r + 1, \mathbb{C})$ and $\tilde{\text{sl}}(r, \mathbb{C})/\mathbb{Z}$ into the Lie algebra of (regular) vector fields on $\mathbb{C}^{*r}$ and $U(1)^r$ which generalise the embeddings of $\text{sl}(2, \mathbb{C})$ given above. Explicit formulae in the cases $\text{sl}(2, \mathbb{C})$, $\tilde{\text{sl}}(2, \mathbb{C})/\mathbb{Z}$, $\text{sl}(3, \mathbb{C})$ and $\text{sl}(3, \mathbb{C})/\mathbb{Z}$ are given in section 4.

The authors thank D. Bennequin and R.J. Stanton for useful conversations and remarks.

## 2 Preliminaries

As general references for this section one can consult the book [Kac] of V.Kac and the review article [Mac] of I.G. Macdonald.

Let $N = [n(i, j)]_{1 \leq i, j \leq r}$ be a generalised Cartan matrix (i.e. $n(i, j) \in \mathbb{Z}$, $n(i, i) = 2$, $n(i, j) \leq 0$ if $i \neq j$, $n(i, j) = 0$ iff $n(j, i) = 0$) and suppose $\text{corank}(N) = s$. Consider $3r + s$ independent variables $\{H_a, X_i, X_{-i} : 1 \leq a \leq \text{corank}(N)\}$ and consider the following relations:

\begin{align*}
\langle H_a, H_b \rangle &= \sum_{i} n(i, a+b)X_i, \\
\langle X_i, X_j \rangle &= \sum_{a} n(a, i-j)H_a, \\
\langle X_i, H_a \rangle &= \sum_{a} n(i, a)X_a
\end{align*}
Then suppose \( h' = \bigoplus_{1 \leq a \leq r} \mathbb{C}H_a \) \( (1) \) and independent variables \( \{H_a, X_i, X_{-i} : 1 \leq a \leq r + s, 1 \leq i \leq r\} \) subject to the relations:

(a) \( [H_a, H_b] = 0 \);
(b) \( [X_i, X_{-j}] = H_i \) and \( [X_i, X_{-j}] = 0 \) if \( i \neq j \);
(c) \( [H_a, X_j] = \alpha_j(H_a)X_j \) and \( [H_a, X_{-j}] = -\alpha_j(H_a)X_{-j} \);
(d) \( ad^{-n(i,j)+1}(X_i)(X_j) = 0 \) if \( i \neq j \);
(e) \( ad^{-n(i,j)+1}(X_{-i})(X_{-j}) = 0 \) if \( i \neq j \).

It is well-known (\cite{Mo}) that, up to isomorphism, the Lie algebra \( \hat{g}(N, \alpha_1, \ldots, \alpha_r) \) is independent of the choice of \( \alpha_1, \ldots, \alpha_r \in \mathfrak{h}^* \) satisfying (5) and that its centre is \( Z \). Henceforth, we write \( \hat{g}(N) \) for \( \hat{g}(N, \alpha_1, \ldots, \alpha_r) \).

It is also well-known that \( \hat{g}(N) \) is finite-dimensional if and only if all principal minors of \( N \) are \( > 0 \) and then \( \hat{g}(N) \) is semisimple (\cite{Sc}). In this case \( s = 0 \) and \( N \) is the Cartan matrix \( n(i,j) = \alpha_j(H_i) \) corresponding to a choice of simple roots \( \alpha_1, \ldots, \alpha_r \) and coroots \( H_1, \ldots, H_r \) (\cite{Sc}).

One can also associate to the above data the Kac-Moody algebra \( g(N) \) (\cite{K}). This is a quotient of \( \hat{g}(N) \) and whether or not the two are isomorphic in general is an open question. However if \( N \) is symmetrizable then \( \hat{g}(N) \) is isomorphic to the Kac-Moody algebra \( g(N) \) (\cite{G-K}).

By definition, \( \hat{g}(N) \) has the following universal property:

**Proposition 2.2** Let \( \mathfrak{l} \) be a Lie algebra. A linear map \( F : \mathfrak{h} \oplus \mathfrak{r}_+ \oplus \mathfrak{r}_- \to \mathfrak{l} \) such that \( \{F(H_a), F(X_i), F(X_{-i}) : 1 \leq a \leq r + s, 1 \leq i \leq r\} \) satisfy the relations of (2) extends to a unique Lie algebra homomorphism \( \hat{F} : \hat{g}(N) \to \mathfrak{l} \).
3 Statement of the problem

Let $\mathcal{A}$ be a complex commutative algebra and let

$$\text{Der}(\mathcal{A}) = \{D \in \text{End}(\mathcal{A}) : D(ab) = D(a)b + aD(b) \ \forall a, b \in \mathcal{A}\}.$$ 

We suppose given:

(A) a Lie algebra homomorphism $F : \mathfrak{h} \to \text{Der}(\mathcal{A})$;
(B) invertible elements $v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_r}$ in $\mathcal{A}$ such that

$$F(H)(v_{\alpha_i}) = \alpha_i(H)v_{\alpha_i}.$$ 

Example 3.1 Let $L_r = \mathbb{C}[z_1, \frac{1}{z_1}, \ldots, z_r, \frac{1}{z_r}]$ be the algebra of Laurent polynomials in $z_1, \ldots, z_r$. This is the algebra of regular functions on $\mathbb{C}^* \times \cdots \times \mathbb{C}^r$. Define $F : \mathfrak{h} \to \text{Der}(L_r)$ by setting

$$F(H)(z_i) = \alpha_i(H)z_i, \quad F(H)(\frac{1}{z_i}) = -\alpha_i(H)\frac{1}{z_i},$$

and extending by the derivation rule (cf [53] for the explicit formula). Then $F$ and $z_1, \ldots, z_r$ satisfy (A) and (B).

Remark 3.2 By (B), $\ker F \subseteq \mathbb{Z} = \ker \alpha_1 \cap \cdots \cap \ker \alpha_r$, and the $\alpha_i$ are well-defined on $\text{Im} F$.

Remark 3.3 $v_{\alpha_1}, \ldots, v_{\alpha_r}$ are linearly independent since $\alpha_1, \ldots, \alpha_r$ are distinct.

Remark 3.4 If $\lambda_1, \ldots, \lambda_r \in \mathbb{C}^*$, then $F$ and $\lambda_1 v_{\alpha_1}, \ldots, \lambda_r v_{\alpha_r}$ satisfy (A) and (B) above.

In the rest of the article we will usually write $H$ for $F(H)$ to ease notation.

If $\alpha = n_1 \alpha_1 + n_2 \alpha_2 + \cdots + n_r \alpha_r$ is a $\mathbb{Z}$-linear combination of the $\alpha_i$, we set

$$v_\alpha = v_{\alpha_1}^{n_1}v_{\alpha_2}^{n_2} \cdots v_{\alpha_r}^{n_r}.$$ 

By the derivation rule it is clear that for all $H \in \mathfrak{h}$

$$H(v_\alpha) = \alpha(H)v_\alpha.$$
The commutator of \( v_\alpha H \) and \( v_\beta H' \) is given by
\[
\text{ad}(v_\alpha H)(v_\beta H') = [v_\alpha H, v_\beta H'] = v_{\alpha+\beta}(H)H' - \alpha(H')H,
\]
and a simple induction shows that for \( k \geq 2 \),
\[
\text{ad}^k(v_\alpha H)(v_\beta H') = v_{\beta+k\alpha}(H)(\beta + \alpha)(H) \ldots (\beta + (k-1)\alpha)(H)H' \\
- k\alpha(H')(\beta + \alpha)(H) \ldots (\beta + (k-2)\alpha)(H)H).
\]

The problem we address is:

**PROBLEM 1**: Is it possible to find \( 2r \) elements \( \delta_1, \delta_2, \ldots, \delta_r, \delta_{-1}, \delta_{-2}, \ldots, \delta_{-r} \) in \( \text{Im} F \) such that
\[
\{ H_a, X_i = v_\alpha, \delta_i, X_{-i} = v_{-\alpha}, \delta_{-i} : 1 \leq a \leq r+s, 1 \leq i \leq r \}
\]

satisfy the relations of (2.1)?

In fact it will turn out that solving Problem 1 is equivalent to solving the following a priori simpler problem:

**PROBLEM 2**: Is it possible to find \( 2r \) elements \( \delta_1, \delta_2, \ldots, \delta_r, \delta_{-1}, \delta_{-2}, \ldots, \delta_{-r} \) in \( \text{Im} F \) such that
\[
\{ H_a, X_i = v_\alpha, \delta_i, X_{-i} = v_{-\alpha}, \delta_{-i} : 1 \leq a \leq r+s, 1 \leq i \leq r \}
\]

satisfy the relations (a),(b) and (c) of (2.1)?

The relations (a) and (c) are satisfied by construction so one only has consider the relations (b),(d) and (e) for Problem 1 and (b) for Problem 2. Note that the \( \{ H_a : r+1 \leq a \leq r+s \} \) are not involved in these relations. By (2.2), if there is a solution to Problem 1 then \( H_a, X_i, X_{-i} \) generate a Lie subalgebra of \( \text{Der}(A) \) isomorphic to a quotient of \( \hat{\mathfrak{g}}(N) \).

### 3.1 Necessary conditions

We first show that the existence of a solution to Problem 2 imposes restrictions on the generalised Cartan matrix \( N \) and the map \( F : \mathfrak{h} \rightarrow \text{Der}(A) \).

**Theorem 3.5** Suppose there exist elements \( \delta_1, \delta_2, \ldots, \delta_r, \delta_{-1}, \delta_{-2}, \ldots, \delta_{-r} \) in \( \text{Im} F \) such that
\[
\{ H_a, X_i = v_\alpha, \delta_i, X_{-i} = v_{-\alpha}, \delta_{-i} : 1 \leq a \leq r+s, 1 \leq i \leq r \}
\]
satisfy the relations (b) of (2.1)

Set \( A_{ij} = \alpha_i(\delta_j) \) for \( 1 \leq i, j \leq r \) (this makes sense by (2.3)). Then:
(i) $A_{ii} \neq 0$ for $1 \leq i \leq r$.

(ii) $\delta_{-i} = \frac{1}{\alpha_i}(-H_i + \frac{1}{\alpha_i} \delta_i)$ for $1 \leq i \leq r$.

(iii) $\frac{A_{ii}}{A_{jj}} = 0$ or $-1$ if $i \neq j$.

(iv) $\frac{A_{ii}}{A_{jj}} + \frac{A_{ji}}{A_{ii}} = n(j, i)$.

(v) The matrix $A' = \left[\frac{A_{ii}}{A_{jj}}\right]$ has the property that if the $(i, j)$ entry is $-1$ then the $i^{th}$ row and $j^{th}$ column of $A'$ have no other entries equal to $-1$ (their other off diagonal entries are $0$ by (iii)).

(vi) If $\frac{A_{ii}}{A_{jj}} = \frac{A_{ji}}{A_{ii}} = -1$ then the generalised Cartan matrix contains the factor

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

and $F(H_i) + F(H_j) = 0$.

Proof.

Rewriting the relations 2.1(b) in terms of the $\delta_i$ we get the following identities in $\text{Im } F$:

$$-\alpha_i(\delta_i)\delta_{-i} - \alpha_i(\delta_{-i})\delta_i = H_i$$

(8)

$$-\alpha_j(\delta_i)\delta_{-j} - \alpha_i(\delta_{-j})\delta_i = 0 \quad \text{(if } i \neq j).$$

(9)

**Remark 3.6** Note that $\lambda_1 \delta_1, \ldots, \lambda_r \delta_r, \frac{1}{\alpha_1} \delta_{-1}, \ldots, \frac{1}{\alpha_r} \delta_{-r}$ ($\lambda_i \neq 0$) and $-\delta_{-1}, \ldots, -\delta_{-r}, -\delta_1, \ldots, -\delta_r$ are also solutions of this system.

Applying $\alpha_i$ to the first equation (this is legitimate by 3.2) gives

$$-2\alpha_i(\delta_i)\alpha_i(\delta_{-i}) = \alpha_i(H_i) = 2,$$

(10)

and hence

$$\alpha_i(\delta_i)\alpha_i(\delta_{-i}) = -1,$$

(11)

whence $\alpha_i(\delta_i) \neq 0$. This proves (i).

Substituting in (8) we get the formula (ii) for $\delta_{-i}$ in terms of $\delta_i$:

$$\delta_{-i} = \frac{1}{\alpha_i(\delta_i)}(-H_i + \frac{1}{\alpha_i(\delta_i)} \delta_i).$$

(12)

Applying $\alpha_i$ and $\alpha_j$ to (8) gives for all $1 \leq i \neq j \leq r$

$$\alpha_i(\delta_{-j})(\alpha_j(\delta_i) + \alpha_i(\delta_i)) = 0$$

(13)

$$\alpha_j(\delta_i)(\alpha_j(\delta_{-j}) + \alpha_i(\delta_{-j})) = 0,$$

(14)
and since by (12) for all $1 \leq i, j \leq r$

$$\alpha_j(\delta_{-i}) = \frac{1}{\alpha_i(\delta_i)}(-n(i,j) + \frac{\alpha_j(\delta_i)}{\alpha_i(\delta_i)})$$

(15)

these equations are equivalent to for all $1 \leq i \neq j \leq r$

$$(-n(j,i) + \frac{A_{ij}}{A_{jj}})(A_{ji} + A_{ii}) = 0$$

(16)

$$A_{ji}(-1 - n(j,i) + \frac{A_{ij}}{A_{jj}}) = 0.$$  

(17)

From this it follows that for $1 \leq i \neq j \leq r$, either

$$A_{ji} = 0 \quad \text{and} \quad A_{ij} = A_{jj}n(j,i)$$

(18)

or

$$A_{ji} = -A_{ii} \quad \text{and} \quad A_{ij} = A_{jj}(1 + n(j,i)).$$

(19)

Hence $\frac{A_{ji}}{A_{ii}} = 0$ or $-1$ if $i \neq j$ and

$$\frac{A_{ij}}{A_{jj}} + \frac{A_{ji}}{A_{ii}} = n(j,i).$$

(20)

This proves (iii) and (iv) and the generalised Cartan matrix is symmetric with off diagonal entries equal to either 0, $-1$ or $-2$.

Substituting (20) in (15) gives for $1 \leq i, j \leq r$

$$\alpha_j(\delta_{-i}) = -\frac{A_{ij}}{A_{jj}A_{ii}}$$

(21)

and hence (8) becomes

$$A_{ji}(\delta_{-j} - \frac{1}{A_{jj}A_{ii}}) = 0 \quad \text{(if } i \neq j).$$

(22)

If we now apply $\alpha_k$ to this equation we get

$$A_{ji} \left(\frac{A_{jk}}{A_{kk}} + \frac{A_{ki}}{A_{ii}}\right) = 0 \quad \text{(if } i \neq j \text{ and } k \text{ arbitrary).}$$

(23)

We already know that either $\frac{A_{ii}}{A_{ii}} = 0$ or $-1$ if $i \neq j$. Suppose $\frac{A_{ii}}{A_{ii}} = -1$. Then we must have

$$\frac{A_{jk}}{A_{kk}} + \frac{A_{ki}}{A_{ii}} = 0 \quad \forall 1 \leq k \leq r.$$  

(24)
But if \( k \neq i \) and \( k \neq j \), \( \frac{A_{jk}}{A_{kk}} \leq 0 \) and \( \frac{A_{ki}}{A_{ii}} \leq 0 \); hence
\[
A_{jk} = A_{ki} = 0 \quad \text{when} \quad \frac{A_{ij}}{A_{ii}} = -1, \ k \neq i \text{ and } k \neq j.
\]
(25)
This means that the matrix \( A' = \left[ \frac{A_{ij}}{A_{jj}} \right] \) has the property that if an entry \( \frac{A_{ij}}{A_{jj}} = -1 \) then the row and the column of \( A' \) containing that entry have no other entries equal to \(-1\) (i.e., their other off diagonal entries are 0). This proves (v).

If \( n(i, j) = -2 \), we must have \( \frac{A_{ij}}{A_{jj}} = -1 \) and \( \frac{A_{ji}}{A_{ii}} = -1 \) by (iv). By (v) and (iv) the principal submatrix corresponding to \((i, j)\) defines a decomposition of the generalised Cartan matrix and the factor corresponding to \((i, j)\) is the matrix
\[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
\]
. This proves the first part of (vi).

Moreover, by (22)
\[
\delta_{-j} - \frac{1}{A_{jj}A_{ii}} \delta_i = 0 \quad \text{and} \quad \delta_{-i} - \frac{1}{A_{jj}A_{ii}} \delta_j = 0,
\]
(26)
which by (12) implies
\[
H_j = \frac{1}{A_{jj}} \delta_j - \frac{1}{A_{ii}} \delta_i \quad \text{and} \quad H_i = \frac{1}{A_{ii}} \delta_i - \frac{1}{A_{jj}} \delta_j.
\]
(27)
From this it follows that
\[
H_i + H_j = 0,
\]
(28)
or more precisely \( F(H_i + H_j) = 0 \) since \( H_i \) was an abbreviated notation for the image of \( H_i \in \mathfrak{h} \) under the given homomorphism \( F : \mathfrak{h} \to Der(A) \).

QED

**Corollary 3.7** If \( N \) is indecomposable and there is a solution to Problem 2 then the Lie algebra \( \hat{g}(N) \) is isomorphic to either \( A_r \) or \( A_r(1) \).

Proof. This is a consequence of the following lemma:

**Lemma 3.8** Let \( N \) be an \( r \times r \) indecomposable generalised Cartan matrix. There exists an \( r \times r \) matrix \( A' \) such that
\begin{itemize}
  \item[(a)] \( A'_{ii} = 1 \) for \( 1 \leq i \leq r \)
  \item[(b)] \( A'_{ij} = 0 \) or \( -1 \) if \( i \neq j \)
  \item[(c)] \( A'_{ij} + A'_{ji} = n(j, i) \)
  \item[(d)] if the \((i, j)\) entry of \( A' \) is \(-1\) then the \(i^{th}\) row and \( j^{th}\) column of \( A' \) have no other entries equal to \(-1\)
\end{itemize}

iff \( N \) is the generalised Cartan matrix of \( A_r \) \((r \geq 1)\) or \( A_{r-1}(1) \) \((r \geq 2)\). For \( A_1 \) or \( A_1^{(1)} \) there is one matrix satisfying these conditions and for \( A_r \) \((r \geq 2)\) or \( A_{r-1}(1) \) \((r \geq 3)\) there are two.
Proof. Suppose there exists a matrix $A'$ with the above properties. Then by (b) and (c), $N$ is symmetric and the off diagonal entries of $N$ are either $0, -1$ or $-2$.

If $n(i, j) = -2$ for some $(i, j)$ then $N$ contains the factor $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and since $N$ is indecomposable, $N = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ which is the generalised Cartan matrix of $A^{(1)}_1$. In this case $A' = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is the only possibility.

Suppose that $N$ has no $-2$ entry. Then the Dynkin diagram of $N$ has $r$ vertices and two vertices $i$ and $j$ are connected iff $n(i, j) = -1$. No two vertices are connected by more than one line and since $N$ is indecomposable the diagram is connected.

If $r = 1$, $N$ is clearly the Cartan matrix of $A_1$ and there is only one matrix $A'$ satisfying (a), (b), (c) and (d). Suppose $r \geq 2$. The matrix $A'$ enables us to orient each line in the diagram: we put an arrow from $i$ to $j$ if $A'_{ij} = -1$ and this is a well-defined orientation since by (c), $A'_{ij}$ and $A'_{ji}$ cannot both be equal to $-1$ or $0$. But by (d), for each vertex $i$ there is no more than one arrow ending at $i$ and no more than one arrow beginning at $i$. Hence the Dynkin diagram of $N$ is globally oriented and must be the Dynkin diagram of either $A_r$ $(r \geq 2)$ or $A^{(1)}_{r-1}$ $(r \geq 3)$. There are then exactly two matrices satisfying (a), (b), (c) and (d): $A'$ and $A'$ (corresponding to the two possible global orientations of the Dynkin diagram).

For the converse, note that the (generalised) Cartan matrix of $A_r$ can be written

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & 0 & -1 \end{pmatrix}$$

(29)

and that the generalised Cartan matrix of $A^{(1)}_{r-1}, r \geq 2$ can be written

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & -1 \\ -1 & 2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & \ldots & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & \ldots & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & \ldots & -1 \\ -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & 0 & -1 \end{pmatrix}$$

(30)

The matrices on the righthand sides of these equations clearly satisfy (a), (b), (c) and (d).
QED

For future reference we introduce the following terminology:

**Definition 3.9** (i) Let $N$ be the generalised Cartan matrix of $A_r$ or $A^{(1)}_{r-1}$. An $r \times r$ matrix $A = [A_{ij}]$ is called a solution matrix of $N$ if the matrix $[\frac{A_{ij}}{A_{jj}}]$ satisfies (a), (b), (c) and (d) of 3.8. If in addition $A_{ii} = 1$ for all $1 \leq i \leq r$, $A$ is called a normalised solution matrix of $N$.

(ii) If $A$ is a solution matrix we set $I_A = \{(i, j) : \frac{A_{ji}}{A_{ii}} = -1\}$. Note that $I^t(IA) = t(I^tA)$.

From (29) and (30) we see that if $A$ is a solution matrix of $A_r$ then (up to transposition)

$$I^A = \{(1, 2), (2, 3), \ldots (r - 1, r)\}$$

and that if $A$ is a solution matrix of $A^{(1)}_{r-1}$

$$I^A = \{(1, 2), (2, 3), \ldots (r - 1, r), (r, 1)\}.$$  

This has the following consequence:

**Corollary 3.10** If $N$ is indecomposable and there is a solution to Problem 2 then $\text{Ker} F = Z$.

Proof. By 3.2, $\text{Ker} F \subseteq Z$. If $N$ is the generalised Cartan matrix of $A_r$ then $Z = \{0\}$ and there is nothing to prove. If $N$ is the generalised Cartan matrix of $A^{(1)}_{r-1}$ then it is known (K, Mac) that $Z \subset H$ is spanned by $H_1 + H_2 + \cdots + H_r$. But by (27)

$$\forall (i, j) \in I^A, \quad H_j = \frac{1}{A_{jj}} \delta_j - \frac{1}{A_{ii}} \delta_i$$

and hence by (32)

$$H_1 + H_2 + \cdots + H_r = 0,$$

which, recalling our abbreviated notation, really means $F(H_1 + H_2 + \cdots + H_r) = 0$. QED

**Remark 3.11** The only property of the generalised Cartan matrix $N$ used in this section was that its diagonal elements are invertible complex numbers. This was needed (cf 10, 11) to prove that $A_{ii} \neq 0$. 

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3.2 Equivalence of Problem 1 and Problem 2

Theorem 3.12 Suppose there exist elements $\delta_1, \delta_2 \ldots \delta_r, \delta_{-1}, \delta_{-2} \ldots \delta_{-r}$ of $\text{Im } F$ such that

$$\{ H_a, X_i = v_{\alpha_i} \delta_i, X_{-i} = v_{-\alpha_i} \delta_{-i} : 1 \leq a \leq r + s, 1 \leq i \leq r \}$$

satisfy the relations (b) of 2.7 Then

$$\{ H_a, X_i = v_{\alpha_i} \delta_i, X_{-i} = v_{-\alpha_i} \delta_{-i} : 1 \leq a \leq r + s, 1 \leq i \leq r \}$$

also satisfy the relations (d) and (e) of 2.7.

Corollary 3.13 Problem 1 has a solution iff Problem 2 has a solution.

Proof. To prove that the relations (d) of 2.1 are satisfied, we have to show that

$$\text{ad}^{-n(i,j)+1}(X_i)(X_j) = 0 \quad \text{if } i \neq j,$$

which is equivalent to

$$\text{ad}^{-n(i,j)+1}(v_{\alpha_i} \delta_i)(v_{\alpha_j} \delta_j) = 0 \quad \text{if } i \neq j.$$

By 3.3(iii) and (iv), if the relations (b) of 2.1 are satisfied then the Cartan matrix is symmetric and $n(i,j) = 0, -1$ or $-2$ if $i \neq j$. Setting $\alpha = \alpha_i$, $H = \delta_i$, $\beta = \alpha_j$ and $H' = \delta_j$ in the formula (7) we get

$$\text{ad}^{-n(i,j)+1}(v_{\alpha_i} \delta_i)(v_{\alpha_j} \delta_j) = v_{\alpha_i+\alpha_j}(A_{ji}\delta_j - A_{ij}\delta_i)$$

(35)

if $n(i,j) = 0$,

$$\text{ad}^{-n(i,j)+1}(v_{\alpha_i} \delta_i)(v_{\alpha_j} \delta_j) = v_{\alpha_j+2\alpha_i}(A_j(A_{ji} + A_{ii})\delta_j - 2A_{ij}A_{ji}\delta_i)$$

(36)

if $n(i,j) = -1$ and

$$\text{ad}^{-n(i,j)+1}(v_{\alpha_i} \delta_i)(v_{\alpha_j} \delta_j) = v_{\alpha_j+3\alpha_i}(A_j(A_{ji} + A_{ii})(A_{ji} + 2A_{ii})\delta_j - 3A_{ij}A_{ji}(A_{ji} + A_{ii})\delta_i).$$

(37)

if $n(i,j) = -2$.

In the first case we have $A_{ij} = A_{ji} = 0$ by 3.3(iii) and 3.3(iv) and hence by (35)

$$\text{ad}^{-n(i,j)+1}(v_{\alpha_i} \delta_i)(v_{\alpha_j} \delta_j) = 0.$$

In the second case either $A_{ij} = -A_{jj}$ and $A_{ji} = 0$ or $A_{ij} = 0$ and $A_{ji} = -A_{ii}$, again by 3.3(iii) and 3.3(iv). Thus by (36)

$$\text{ad}^{-n(i,j)+1}(v_{\alpha_i} \delta_i)(v_{\alpha_j} \delta_j) = 0.$$
if \( n(i, j) = -1 \).
In the third case \( A_{ij} = -A_{jj} \) and \( A_{ji} = -A_{ii} \) by (3.5)(iii) and (3.5)(iv). Thus by (37),
\[
ad^{-n(i,j)+1}(v_{\alpha_i}\delta_i)(v_{\alpha_j}\delta_j) = 0
\]
if \( n(i, j) = -2 \). Hence in all cases the relations (d) of 2.1 are satisfied.
To prove that the relations (e) of 2.1 are satisfied we have to show that
\[
ad^{-n(i,j)+1}(v_{-\alpha_i}\delta_i)(v_{-\alpha_j}\delta_j) = 0 \quad \text{if } i \neq j.
\]
Setting \( B_{ij} = -\alpha_i(\delta_j) \), by (7) this is equivalent to
\[
B_{ji}\delta_{-j} - B_{ij}\delta_{-i} = 0 \quad \text{(38)}
\]
if \( n(i, j) = 0 \), to
\[
B_{ji}(B_{ji} + B_{ii})\delta_{-j} - 2B_{ij}B_{ji}\delta_{-i} = 0 \quad \text{(39)}
\]
if \( n(i, j) = -1 \) and to
\[
B_{ji}(B_{ji} + B_{ii})(B_{ji} + 2B_{ii})\delta_{-j} - 3B_{ij}B_{ji}(B_{ji} + B_{ii})\delta_{-i} = 0 \quad \text{(40)}
\]
if \( n(i, j) = -2 \).
But by (21), we have forall \( 1 \leq i, j \leq r \),
\[
\frac{B_{ij}}{B_{ii}} = \frac{A_{ij}}{A_{jj}}. \quad \text{(41)}
\]
This means the \( B_{ij} \) satisfy the properties (i), (iii) and (iv) of 3.3 and the equations (38), (39) and (40) follow from them exactly as the equations (35), (36) and (37) followed from the properties of the \( A_{ij} \) in the above proof of the relations (d) of 2.1.
QED

3.3 Sufficient conditions
Let \( N \) be the Cartan matrix of either \( A_r \) (\( r \geq 1 \)) or \( A_r^{(1)} \) (\( r \geq 2 \)) and let \( A = [A_{ij}] \) be a solution matrix of \( N \) (cf 3.9). We suppose given a complex commutative algebra \( \mathcal{A} \) and:
(A) a Lie algebra homomorphism \( F : \mathfrak{h} \to \text{Der}(\mathcal{A}) \) such that \( \ker F = \mathbb{Z} \);
(B) invertible elements \( v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_r} \) in \( \mathcal{A} \) such that
\[
F(H)(v_{\alpha_i}) = \alpha_i(H)v_{\alpha_i}.
\]
As before, we will write \( H \) for \( F(H) \) to ease notation.
By hypothesis $F$ factors to an isomorphism $F : \mathfrak{h}/Z \to \text{Im } F$ and therefore \{\alpha_1, \ldots, \alpha_r\} factors to a basis of $(\text{Im } F)^*$. Hence there exist unique $\delta_1, \delta_2 \ldots \delta_r$ in $\text{Im } F$ such that $[\alpha_i(\delta_j)] = [A_{ij}]$. For $1 \leq i \leq r$ set
\[
\delta_{-i} = \frac{1}{A_{ii}} (-H_i + \frac{1}{A_{ii}} \delta_i).
\] (42)

**Theorem 3.14** \{H_i, X_i = v_\alpha \delta_i, X_{-i} = v_{-\alpha} \delta_{-i} : 1 \leq i \leq r\} satisfy the relations (b) of 2.1 and hence Problem 1 has a solution.

**Proof.** By (8) and (9), the relations (b) of 2.1 are satisfied iff
\[
-\alpha_i(\delta_i) \delta_{-i} - \alpha_i(\delta_{-i}) \delta_i = H_i
\] (43)
\[
-\alpha_j(\delta_i) \delta_{-j} - \alpha_i(\delta_{-j}) \delta_i = 0 \quad \text{if } i \neq j.
\] (44)
Equation (43) is an immediate consequence of (42). Also by (42), (44) is equivalent to
\[
A_{jj} (H_j - \frac{\delta_j}{A_{jj}} + \frac{\delta_i}{A_{ii}}) = 0 \quad \text{if } i \neq j,
\] (45)
which, since $(i, j) \in I^A \Leftrightarrow \frac{A_{ii}}{A_{jj}} = -1$ and otherwise $A_{jj} = 0$, is equivalent to
\[
\forall (i, j) \in I^A, \quad H_j - \frac{\delta_j}{A_{jj}} + \frac{\delta_i}{A_{ii}} = 0.
\] (46)
The lefthand side is in $\text{Im } F$ so this equation is equivalent to
\[
\forall (i, j) \in I^A, \forall k : 1 \leq k \leq r, \quad \alpha_k (H_j - \frac{\delta_j}{A_{jj}} + \frac{\delta_i}{A_{ii}}) = 0
\] (47)
since $\alpha_1, \ldots, \alpha_r$ is a basis of $(\text{Im } F)^*$. But for all $1 \leq i, j, k \leq r$ we have
\[
\alpha_k (H_j - \frac{\delta_j}{A_{jj}} + \frac{\delta_i}{A_{ii}}) = n(j, k) - \frac{A_{kj}}{A_{jj}} + \frac{A_{ki}}{A_{ii}}
\] (48)
\[
= \frac{A_{jk}}{A_{kk}} + \frac{A_{ki}}{A_{ii}} \quad \text{(by 3.8(c)).}
\] (49)
Suppose $\frac{A_{ii}}{A_{kk}} = -1$. Then for $k = i$ or $k = j$ this is obviously 0 and for $k \neq i, k \neq j$, both terms vanish by property 3.8(d). This proves the theorem.

**QED**

In conclusion:

**Corollary 3.15** If $N$ is the generalised Cartan matrix of $A_r$ or $A^{(1)}_{r-1}$ and $F : \mathfrak{h} \to \text{Der}(A)$ satisfies $\text{Ker } F = Z$ then Problem 1 has a solution. The set of solutions is in one to one correspondence with the set of solution matrices of $N$. 

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Remark 3.16 If $[A_{ij}]$ is a solution matrix (SM), $[\mu_j A_{ij}]$ is also a SM where the $\mu_j$ are arbitrary non-zero complex numbers. This gives a principal $\mathbb{C}^*^r$ action on the space of SMs and the quotient is the space of normalised SMs. One clearly has an isomorphism of principal $\mathbb{C}^*^r$ bundles:

Solution Matrices $\cong$ Normalised Solution Matrices $\times \mathbb{C}^*^r$

given by:

$$[A_{ij}] \rightarrow ([\frac{A_{ij}}{A_{jj}}], (A_{11}, A_{22}, \ldots, A_{rr})).$$

In the case of $A_{1}$ and $A_{1}^{(1)}$ there is one normalised solution matrix which is symmetric. In the other cases there are two, one being the transpose of the other (cf 3.8).

The space of SMs has a natural involution $[A_{ij}] \mapsto [A_{ji}]$ which commutes with the principal $\mathbb{C}^*^r$ action. In terms of solutions $(\delta_{1}, \ldots, \delta_{r}, \delta_{-1}, \ldots, \delta_{-r})$ this corresponds to (cf. 3.6)

$$(\delta_{1}, \ldots, \delta_{r}, \delta_{-1}, \ldots, \delta_{-r}) \mapsto (-\delta_{-1}, \ldots, -\delta_{-r}, -\delta_{1}, \ldots, -\delta_{r}).$$

Remark 3.17 Since $Z = \bigcap_i \ker\alpha_i = \langle H_1 + \cdots + H_r \rangle$, we can find a basis $H_1', \ldots, H_r', H_1 + \cdots + H_r$ of $\mathfrak{h}$ such that $\alpha_i(H_j') = \delta_{ij}$ for $1 \leq i, j \leq r$. For $(n_1, \ldots, n_r) \in \mathbb{Z}^r$, define $F_{n_1, \ldots, n_r} : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{A})$ by

$$F_{n_1, \ldots, n_r}(H_i') = \frac{1}{n_i} F(H_i')$$

$$F_{n_1, \ldots, n_r}(H_1 + \cdots + H_r) = 0.$$  

One checks that $F_{n_1, \ldots, n_r}$ is independent of the choice of basis and that $F_{n_1, \ldots, n_r}$ and $v_1^{n_1}, \ldots, v_r^{n_r} \in \mathfrak{A}$ satisfy properties (A) and (B) above. Hence each SM defines an extension of $F_{n_1, \ldots, n_r}$ to a homomorphism $\hat{F}_{n_1, \ldots, n_r} : \hat{\mathfrak{g}}(N) \rightarrow \text{Der}(\mathfrak{A}).$

4 Examples: $sl(r)$, $\tilde{sl}(r)$ and Lie algebras of vector fields on tori

In this section $N$ is the generalised Cartan matrix of either $A_r$ ($r \geq 1$) or $A_1^{(1)}$ ($r \geq 2$). The Lie algebra $\hat{\mathfrak{g}}(N)$ is therefore isomorphic to either $A_r$ or $A_1^{(1)}$ (i.e to $sl(r+1)$ or $sl(r)$).

If $M$ is a manifold, $C^\infty(M)$, the space of smooth complex-valued functions on $M$, is a commutative algebra and $\text{Der}(C^\infty(M))$ is isomorphic to the Lie algebra
of vector fields $\mathcal{X}(M)$ on $M$. Suppose that on $M$ one can find invertible functions $v_1, \ldots, v_r$ and commuting vector fields $D_1, \ldots, D_r$ such that
\[ D_i(v_j) = \delta_{ij} v_j. \] (52)

Then if we define $F : \mathfrak{h} \to \mathcal{X}(M)$ by
\[ F(H_a) = \sum_{k=1}^{k=r} \alpha_k(H_a) D_k \quad (1 \leq a \leq r + s), \] (53)
it is clear that $\text{Ker} F = Z$ and that
\[ F(H_a)(v_j) = \alpha_j(H_a) v_j. \]

Hence $F$ satisfies the hypotheses (A) and (B) of subsection 3.3 and each choice of a solution matrix $A$ defines an extension of this map to a Lie algebra homomorphism $\hat{F}^A : \hat{\mathfrak{g}}(N) \to \mathcal{X}(M)$.

$A_r$ ($r \geq 1$) is simple so $\hat{F}^A$ is injective in that case. The Lie algebra $A_{r-1}^{(1)}$ ($r \geq 2$) is not simple but it is known (Proposition 1.7(b) in [K]) that every ideal either contains the derived algebra $A_{r-1}^{(1)}$ or is contained in the centre $Z$. It is clear that $\text{Ker} \hat{F}^A$ does not contain $A_{r-1}^{(1)}$ and therefore $\text{Ker} \hat{F}^A = Z$. Hence $\hat{F}^A : \hat{\mathfrak{g}}(N) \to \mathcal{X}(M)$ factors to an injection of $A_{r-1}^{(1)}/Z$ into $\mathcal{X}(M)$.

To describe $\hat{F}^A$ more explicitly one has to calculate the $\delta_i$ and $\delta_{-i}$ associated to a solution matrix $[A_{ij}]$ (cf. subsection 3.3). From equation (53) it is easy to see that
\[ \alpha_i(D_j) = \delta_{ij} \quad \forall 1 \leq i, j \leq r, \]
and hence that for $1 \leq i \leq r$,
\[ \delta_i = \sum_{j=1}^{j=r} A_{ji} D_j \quad \text{and} \quad \delta_{-i} = -\sum_{j=1}^{j=r} \frac{A_{ij}}{A_{ii}A_{jj}} D_j. \] (54)

By subsection 3.3 this proves the

**Proposition 4.1** With the notation above, the $\mathbb{C}$- linear map $F^A : \mathfrak{h} \oplus \mathfrak{r}_+ \oplus \mathfrak{r}_- \to \mathcal{X}(M)$ given by
\[ F^A(H_a) = \sum_{j=1}^{j=r} \alpha_j(H_a) D_j \quad (1 \leq a \leq r + s) \] (55)
\[ F^A(X_i) = A_{ii} v_i \sum_{j=1}^{j=r} \frac{A_{ij}}{A_{ii}} D_j \quad (1 \leq i \leq r) \] (56)
\[ F^A(X_{-i}) = -\frac{1}{A_{ii} v_i} \sum_{j=1}^{j=r} \frac{A_{ij}}{A_{jj}} D_j \quad (1 \leq i \leq r) \] (57)

extends to a Lie algebra homomorphism $\hat{F}^A : \hat{\mathfrak{g}}(N) \to \mathcal{X}(M)$ with kernel $Z$. 

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If \( n_1, \ldots, n_r \in \mathbb{Z}^* \), the vector fields \( \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \) and the functions \( v_1^{n_1}, \ldots, v_r^{n_r} \) also satisfy (52) (cf 3.17) and hence, repeating the above calculations in this case, we get the

**Corollary 4.2** Let \( n_1, \ldots, n_r \) be non-zero integers. The \( \mathbb{C} \)-linear map \( F_{n_1, \ldots, n_r}^A : \mathfrak{h} \oplus \mathfrak{r}_+ \oplus \mathfrak{r}_- \rightarrow \mathcal{X}(M) \) given by

\[
F_{n_1, \ldots, n_r}^A(H_a) = \sum_{j=1}^{j=r} \alpha_j(H_a) \frac{D_j}{n_j} \quad (1 \leq a \leq r + s)
\]

\[
F_{n_1, \ldots, n_r}^A(X_i) = A_{ii}v_i^{n_i} \sum_{j=1}^{j=r} \frac{A_{jj} D_j}{A_{ii} n_j} \quad (1 \leq i \leq r)
\]

\[
F_{n_1, \ldots, n_r}^A(X_{-i}) = -\frac{1}{A_{ii} v_i^{n_i}} \sum_{j=1}^{j=r} \frac{A_{ij} D_j}{A_{jj} n_j} \quad (1 \leq i \leq r)
\]

extends to a Lie algebra homomorphism \( \hat{F}_{n_1, \ldots, n_r}^A : \mathfrak{g}(N) \rightarrow \mathcal{X}(M) \) with kernel \( Z \).

In particular, taking \( M = \mathbb{C}^{*r} \), \( v_j = z_j \) (the coordinate functions) and \( D_j = z_j \frac{\partial}{\partial z_j} \) for \( 1 \leq j \leq r \), we can construct homomorphisms from \( \mathfrak{sl}(r + 1) \) or \( \mathfrak{sl}(r) \) to the Lie algebra of regular vector fields on \( \mathbb{C}^{*r} \). Similarly, taking \( M = U(1)^r \), \( v_j = e^{i\theta_j} \) and \( D_j = -i \frac{\partial}{\partial \theta_j} \) for \( 1 \leq j \leq r \), we can construct homomorphisms from \( \mathfrak{sl}(r + 1) \) or \( \tilde{\mathfrak{sl}}(r) \) to the Lie algebra complex vector fields on the compact torus \( U(1)^r \). We give now the complete formulae for \( \hat{F}_{n_1, \ldots, n_r}^A \) for \( \mathfrak{sl}(2) \), \( \mathfrak{sl}(3) \), \( \tilde{\mathfrak{sl}}(2) \) and \( \tilde{\mathfrak{sl}}(3) \).

**Example 4.3** If \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \) and \( H_1, X_1, X_{-1} \) is the standard basis, a solution matrix \( [A_{ij}] \) is a complex number \( \lambda \neq 0 \). The embeddings \( \hat{F}_n^A : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathcal{X}(M) \) are then given by

\[
H_1 \mapsto 2 \frac{D_1}{n}, \quad X_1 \mapsto \lambda v_1^{n} \frac{D_1}{n}, \quad X_{-1} \mapsto -\frac{1}{\lambda} v_1^{-n} \frac{D_1}{n}.
\]

If \( M = \mathbb{C}^* \), \( v_1 = z \) and \( D_1 = z \frac{d}{dz} \) this gives the well-known embeddings:

\[
H_1 \mapsto \frac{1}{n} z \frac{d}{dz}, \quad X_1 \mapsto \frac{\lambda}{n} z^{n+1} \frac{d}{dz}, \quad X_{-1} \mapsto -\frac{1}{n\lambda} z^{-n+1} \frac{d}{dz}
\]

of \( \mathfrak{sl}(2, \mathbb{C}) \) in \( \mathcal{X}(\mathbb{C}^*) \). The representations \( \hat{F}_n^\pm \) can be obtained geometrically by restricting the derivative of the action of \( \text{PSL}(2, \mathbb{C}) \) on \( \mathcal{P}_1(\mathbb{C}) \) to an appropriate affine coordinate chart.

**Example 4.4** (See also [R]). If \( \mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}) \), let \( \{ H_1, H_2, X_{\pm i} : 1 \leq i \leq 3 \} \) be a standard basis, i.e.: \( H_1, H_2 \) span a Cartan subalgebra and are the coroots
corresponding to the simple roots $\alpha_1, \alpha_2$; the positive roots are $\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2$; the Cartan matrix $N = [\alpha_j(H_i)]_{1 \leq i,j \leq 2}$ is the matrix \[
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\] \{ $H_i, X_{\pm i} : 1 \leq i \leq 2$ \} satisfy the relations of [27] and $X_{\pm 3} = \pm [X_{\pm 1}, X_{\pm 2}]$. The solution matrices of $N$ are of the form
\[
A = \begin{pmatrix}
A_{11} & \frac{1}{2}(-1 + \varepsilon)A_{22} \\
\frac{1}{2}(-1 + \varepsilon)A_{11} & A_{22}
\end{pmatrix},
\]
where $A_{11}, A_{22} \in \mathbb{C}^n$ are arbitrary and $\varepsilon = \pm 1$. The embeddings $\hat{\mathcal{A}}_{n_1,n_2}^A : sl(3, \mathbb{C}) \rightarrow \mathcal{X}(M)$ are then given by
\[
\begin{align*}
X_1 &\mapsto A_{11}v_1^{n_1} \left( \frac{D_1}{n_1} + \frac{1}{2}(-1 + \varepsilon)\frac{D_2}{n_2} \right) \\
X_2 &\mapsto A_{22}v_2^{n_2} \left( \frac{1}{2}(-1 - \varepsilon)\frac{D_1}{n_1} + \frac{D_2}{n_2} \right) \\
X_3 &\mapsto A_{11}A_{22}v_1^{n_1}v_2^{n_2} \left( \frac{-1}{2}(-1 - \varepsilon)\frac{D_1}{n_1} + \frac{1}{2}(-1 + \varepsilon)\frac{D_2}{n_2} \right)
\end{align*}
\]
\[
\begin{align*}
X_{-1} &\mapsto -\frac{1}{A_{11}}v_1^{n_1} \left( \frac{D_1}{n_1} + \frac{1}{2}(-1 - \varepsilon)\frac{D_2}{n_2} \right) \\
X_{-2} &\mapsto -\frac{1}{A_{22}}v_2^{n_2} \left( \frac{1}{2}(-1 + \varepsilon)\frac{D_1}{n_1} + \frac{D_2}{n_2} \right) \\
X_{-3} &\mapsto \frac{1}{A_{11}A_{22}}v_1^{n_1}v_2^{n_2} \left( \frac{-1}{2}(-1 + \varepsilon)\frac{D_1}{n_1} + \frac{1}{2}(-1 - \varepsilon)\frac{D_2}{n_2} \right)
\end{align*}
\]
\[
\begin{align*}
H_1 &\mapsto 2\frac{D_1}{n_1} - \frac{D_2}{n_2} \\
H_2 &\mapsto -\frac{D_1}{n_1} + 2\frac{D_2}{n_2}
\end{align*}
\]
If $M = \mathbb{C}^{n_2}$, the representations $\hat{\mathcal{A}}_{\pm 1, \pm 1}^A$ can be obtained geometrically by restricting the derivative of the action of $PSL(3, \mathbb{C})$ on $P_2(\mathbb{C})$ to an appropriate affine coordinate chart.

To give the explicit formulæ for the embeddings of the Lie algebras $\tilde{sl}(2, \mathbb{C})/Z$ and $\tilde{sl}(3, \mathbb{C})/Z$ (or $A_1^{(1)}$ and $A_2^{(1)}$ in Kac notation) it is convenient to first describe them using loop algebras ([K], [Mac]). Recall that in terms of [27], $\tilde{sl}(r, \mathbb{C})$ is constructed from generators $\{H_0, \ldots, H_r, \bar{X}_{\pm 0}, \ldots, \bar{X}_{\pm (r-1)}\}$, the generalised Cartan matrix [30] and linear forms $\bar{\alpha}_0, \ldots, \bar{\alpha}_{r-1}$ on $\langle \bar{H}_0, \ldots, \bar{H}_r \rangle$ whose values on $\langle \bar{H}_0, \ldots, \bar{H}_{r-1} \rangle$ are given by the Cartan matrix and whose values on $\bar{H}_r$ are given by $\bar{\alpha}_i(\bar{H}_r) = \delta_{i0}$.

The underlying vector space of the loop algebra $L(sl(r, \mathbb{C}))$ is the tensor product of $sl(r, \mathbb{C})$ with Laurent series in $t$ and its bracket is
\[
[t^m X, t^n Y] = t^{m+n}[X,Y] \quad (m, n \in \mathbb{Z}, X, Y \in sl(r, \mathbb{C})).
\]
The operator $d : L(sl(r, \mathbb{C})) \rightarrow L(sl(r, \mathbb{C}))$ given by
\[
d(t^m X) = mt^m X
\]
is a derivation and one can form the semi-direct product \( \tilde{L}(sl(r, \mathbb{C})) = L(sl(r, \mathbb{C})) \oplus \mathbb{C}d \) with the bracket

\[
[\xi \oplus \lambda d, \eta \oplus \mu d] = [\xi, \eta] + \lambda d(\eta) - \mu d(\xi).
\] (67)

It is well-known (\cite{Ka}, \cite{Ma}) that the Lie algebras \( \tilde{sl}(r, \mathbb{C}) \slash Z \) and \( \tilde{L}(sl(r, \mathbb{C})) \) are isomorphic. Explicitly, the map \( \Psi : \mathfrak{h} \oplus \mathfrak{r}_+ \oplus \mathfrak{r}_- \to \tilde{L}(sl(r, \mathbb{C})) \) extends to an isomorphism \( \tilde{\Psi} : sl(r, \mathbb{C}) \slash Z \to \tilde{L}(sl(r, \mathbb{C})) \) where

\[
\begin{align*}
\Psi(H_i) &= d \\
\Psi(\tilde{H}_i) &= H_i \quad (\text{if } 1 \leq i \leq r-1) \\
\Psi(\tilde{H}_0) &= -H_\phi = -(H_1 + \cdots + H_{r-1}) \\
\Psi(\tilde{X}_{\pm i}) &= X_{\mp i} \quad (\text{if } 1 \leq i \leq r-1) \\
\Psi(\tilde{X}_{\pm 0}) &= t^{\pm 1}X_{\mp \phi}.
\end{align*}
\] (68)

Here \( \{H_1, \ldots, H_{r-1}, X_{\pm 1}, \ldots, X_{\pm (r-1)}\} \) is a standard generating set of \( sl(r, \mathbb{C}) \) corresponding to simple roots \( \alpha_1, \ldots, \alpha_{r-1}, \phi = \alpha_1 + \cdots + \alpha_{r-1} \) is the highest root and \( X_{\pm \phi} \) are root vectors satisfying \( [X_{\phi}, X_{-\phi}] = H_\phi \), the coroot corresponding to \( \phi \).

**Example 4.5** If \( \{H_1, X_{\pm 1}\} \) is a standard basis of \( sl(2, \mathbb{C}) \) then \( \{d, t^m H_1, t^n X_{\pm 1} : m, n \in \mathbb{Z}\} \) is a basis of \( \tilde{L}(sl(2, \mathbb{C})) \). The Cartan matrix of \( \tilde{sl}(2, \mathbb{C}) \) is \( \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \) and the solution matrices are of the form \( \begin{pmatrix} A_{00} & -A_{11} \\ -A_{00} & A_{11} \end{pmatrix} \). The embedding \( \hat{F}^A_{n_0,n_1} \circ \Psi^{-1} : \tilde{L}(sl(2, \mathbb{C})) \to \mathcal{X}(M) \) is given by the formulae:

\[
\begin{align*}
t^m X_1 &\to A_{11}(A_{00}A_{11})^m(v_0^{n_0}v_1^{n_1})^mv_1^{n_1}(-D_{n_0} + D_{n_1}) \\
t^m X_{-1} &\to -\frac{1}{A_{11}}(A_{00}A_{11})^m(v_0^{n_0}v_1^{n_1})^mv_1^{-n_1}(-D_{n_0} + D_{n_1}) \\
t^m H_1 &\to (A_{00}A_{11})^m(v_0^{n_0}v_1^{n_1})^m(-2\frac{D_{n_0}}{n_0} + 2\frac{D_{n_1}}{n_1}) \\
d &\to \frac{1}{n_0}D_0.
\end{align*}
\] (69)

Note that if we set \( T = A_{00}A_{11}v_0^{n_0}v_1^{n_1} \) this implies that

\[
\hat{F}^A_{n_0,n_1} \circ \Psi^{-1}(t^mS) = T^m\hat{F}^A_{n_0,n_1} \circ \Psi^{-1}(S) \quad \forall S \in sl(2, \mathbb{C}).
\]

**Example 4.6** If \( \{H_1, H_2, X_{\pm i} : 1 \leq i \leq 3\} \) is a standard basis of \( sl(3, \mathbb{C}) \) then \( \{d, t^k H_1, t^l H_2, t^m X_i, t^n X_{-i} : k, l, m, n \in \mathbb{Z}, 1 \leq i \leq 3\} \) is a basis of \( \tilde{L}(sl(3, \mathbb{C})) \). The Cartan matrix of \( \tilde{sl}(3, \mathbb{C}) \) is

\[
\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}
\]
and the solution matrices are of the form

\[
A = \begin{pmatrix}
A_00 & \frac{1}{2}(-1 - \varepsilon)A_{11} & \frac{1}{2}(-1 + \varepsilon)A_{22} \\
\frac{1}{2}(-1 + \varepsilon)A_{00} & A_{11} & \frac{1}{2}(-1 - \varepsilon)A_{22} \\
\frac{1}{2}(-1 - \varepsilon)A_{00} & \frac{1}{2}(-1 + \varepsilon)A_{11} & A_{22}
\end{pmatrix}
\]

By (66) and 7.3, the images of \(\{d, H_1, H_2, X_{\pm1}, X_{\pm2}, t^{\pm1}X_{\pm3}\}\) under the embedding \(\tilde{\mathcal{F}}^A_{n_0, n_1, n_2} \circ \Psi^{-1} : \tilde{\mathcal{L}}(sl(3, \mathbb{C})) \to \mathcal{X}(M)\) are given by the formulae:

\[
\begin{align*}
X_1 & \mapsto A_{11}v_1^{n_1} \left(\frac{1}{2}(-1 - \varepsilon)\frac{D_{n_0}}{n_0} + \frac{D_{n_1}}{n_1} + \frac{1}{2}(-1 + \varepsilon)\frac{D_{n_2}}{n_2}\right) \\
X_2 & \mapsto A_{22}v_2^{n_2} \left(\frac{1}{2}(-1 + \varepsilon)\frac{D_{n_0}}{n_0} + \frac{1}{2}(-1 - \varepsilon)\frac{D_{n_1}}{n_1} + \frac{1}{2}\right) \\
\frac{1}{2}X_3 & \mapsto \frac{-1}{A_{00}v_0^{n_0}} \left(\frac{D_{n_0}}{n_0} + \frac{1}{2}(-1)\frac{D_{n_1}}{n_1} + \frac{1}{2}(-1 + \varepsilon)\frac{D_{n_2}}{n_2}\right) \\
X_{-1} & \mapsto \frac{-1}{A_{11}v_1^{n_1}} \left(\frac{1}{2}(-1 + \varepsilon)\frac{D_{n_0}}{n_0} + \frac{D_{n_1}}{n_1} + \frac{1}{2}(-1)\frac{D_{n_2}}{n_2}\right) \\
X_{-2} & \mapsto \frac{-1}{A_{22}v_2^{n_2}} \left(\frac{1}{2}(-1 - \varepsilon)\frac{D_{n_0}}{n_0} + \frac{1}{2}(-1 + \varepsilon)\frac{D_{n_1}}{n_1} + \frac{1}{2}\right) \\
tX_{-3} & \mapsto A_{00}v_0^{n_0} \left(\frac{D_{n_0}}{n_0} + \frac{1}{2}(-1 + \varepsilon)\frac{D_{n_1}}{n_1} + \frac{1}{2}(-1 - \varepsilon)\frac{D_{n_2}}{n_2}\right)
\end{align*}
\]

(70)

Calculating the images of \(X_{\pm3} = \pm[X_{\pm1}, X_{\pm2}]\) gives

\[
X_3 \mapsto -A_{11}A_{22}v_1^{n_1}v_2^{n_2}\varepsilon \left(\frac{D_{n_0}}{n_0} + \frac{1}{2}(-1 - \varepsilon)\frac{D_{n_1}}{n_1} + \frac{1}{2}(-1 + \varepsilon)\frac{D_{n_2}}{n_2}\right)
\]

(71)

\[
X_{-3} \mapsto \varepsilon \frac{A_{11}A_{22}v_1^{n_1}v_2^{n_2}}{A_{00}v_0^{n_0}} \left(\frac{D_{n_0}}{n_0} + \frac{1}{2}(-1 + \varepsilon)\frac{D_{n_1}}{n_1} + \frac{1}{2}(-1 - \varepsilon)\frac{D_{n_2}}{n_2}\right)
\]

(72)

This gives the formulae for the restriction of \(\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A : \tilde{\mathcal{L}}(sl(3, \mathbb{C})) \to \mathcal{X}(M)\) to the subset \(sl(3, \mathbb{C}) = \langle H_1, H_2, X_{\pm i} : 1 \leq i \leq 3\rangle\). Note that setting \(D_0 = 0\) one recovers the formulae (64). Now observe that if we set \(T = \varepsilon A_{00}A_{11}A_{22}v_0^{n_0}v_1^{n_1}v_2^{n_2}\) then

\[
\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(X_{-3}) = \frac{1}{T}\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(tX_{-3})
\]

(73)

\[
\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(X_3) = T\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}\left(\frac{1}{t}X_3\right)
\]

(74)

\[
\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(S)(T) = 0 \quad \forall S \in sl(3, \mathbb{C})
\]

(75)

\[
\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(d)(T) = T.
\]

(76)

From this it follows that \(\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1} : \tilde{\mathcal{L}}(sl(3, \mathbb{C})) \to \mathcal{X}(M)\) is given by the formulae for \(\tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(d), \tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(H_i), \tilde{\mathcal{F}}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(X_{\pm i})\) above.
and

\[ \hat{F}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(t^m S) = T^m \hat{F}_{n_0, n_1, n_2}^A \circ \Psi^{-1}(S) \forall S \in sl(3, \mathbb{C}). \quad (77) \]

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