The effect of non-Markovian recovery on reversible failure propagation on networks

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Compared with the classical irreversible failure propagation, the spontaneous recovery model can better depict the failure propagation on network systems such as financial and transportation networks. In this paper, we study how the non-Markovian recovery process influences cascading failure dynamics in the spontaneous recovery systems. To this end, we compare two kinds of failure-recovery models: the non-Markovian recovery (NMR) model and the corresponding Markovian recovery (MR) model. We first develop a Pairwise approximation theory for the MR and NMR models, which can predict the time evolution, steady state, and hysteresis behaviour of failure-recovery systems more accurately than the classical mean-field (MF) method. We find that although the non-Markovian recovery mechanism does not essentially affect the steady state and hysteresis behaviour of failure-recovery systems, the two models exhibit distinct evolution processes. When investigating the effect of initial conditions on the phase transition and hysteresis behavior, we find in the hysteresis region, the NMR model exhibits a non-monotonic growth characteristic: with the increase of initial failed nodes, the steady state staying in relative low-failure phase first turns into relative high-failure phase, and then into relative low-failure phase again. We then use the MF theory to explain the phenomenon qualitatively and point out when considering non-Markovian dynamics, we should study the steady state not only from the perspective of the variation of parameters, but also from the initial conditions.

I. INTRODUCTION

Our daily lives are heavily dependent on the proper functioning of many man-made network systems such as electrical power grids, transportation networks, and financial networks. In these networks, cascading failure is a common dynamic process, where a failure of some nodes causes their neighbors to fail, and the successive failures eventually result in a large-scale failure of nodes. Classic examples include the collapse of power grids [1], traffic jams [2], economic depression [3], and so on. The malfunction of networks resulting from cascading failure does serious harm to our lives and social development [4–8]. Numerous studies have focused great attention on the causes of cascading failure, the impact of various network structures on failure propagation, and the performance of network robustness and vulnerability to failure propagation [9–20].

Most of the previous studies were interested in an irreversible failure propagation, where once a node fails, it cannot recover or return to the active state. However, an entire class of network systems such as financial and transportation networks can spontaneously recover their functions after their collapse [21–26]. In the reversible failure propagation, there are two types of failure and recovery of nodes [27]. For the first type, a node fails due to its internal failure (i.e., being independent of the states of its neighbors) and recovers spontaneously after a period of time. For example, a company fails due to its poor management, which causes a drop in its stock price, and after a period of improvement, its stock price rises again. For the second type, an external failure with enough of neighbor nodes failing causes a node to fail, and then the node recovers spontaneously after a period of time. The failed nodes in the second type often recover more easily because their failures result from the lack of neighbourhood support, rather than internal failures in the first type such as physical failures. A first-order phase transition, hysteresis behavior and phase-flipping phenomenon can be found in numerical simulations, and these dynamical characteristics were demonstrated by using the mean field (MF) theory [27]. The interesting phenomena have aroused interest of some researchers [28–32].

In the spontaneous recovery systems, two distinct dynamical processes have been considered. The first one assumes that the occurrence of an event is a Markovian process where an event occurs at a fixed rate, i.e., the inter-event times are exponentially distributed [33–35]. In other words, the occurrence process of the event is memoryless, where the current state of the system depends only on its last state. The second one, i.e., non-Markovian process, can be found to exist widely in many real dynamical systems such as human dynamics [36–38], biochemical reactions [39], and financial markets [7, 40]. Its memory feature makes the current state of the system depend not only on the last state of the system, but also on the previous states. Recently, increasing studies have devoted to study the impacts of non-Markovian process on the spreading dynamics [41, 42]. Its memory effect arouses a strong dynamical correlation among nodes, and often makes it difficult to tackle analytically [43–46].

Some spontaneous recovery processes have a non-
Markovian characteristic in the failure-recovery dynamical systems. In particular, the failures caused by internal causes must be restored within a specified time. For instance, in financial markets, the suspension period of a stock can not exceed a certain period of time. In this paper, we study how the non-Markovian spontaneous recovery process influences the failure propagation in the failure-recovery systems. Considering the mean-field theory proposed in the previous literature which has a large deviation from the simulation results, we propose a Pairwise approximation (PA) theory for the non-Markovian recovery (NMR) model [27], which can predict the simulation results better. In addition, we also propose a PA theory for the corresponding Markovian recovery (MR) model [47–49]. By comparing with the corresponding MR model, we investigate the effects of non-Markovian recovery process on the steady state, hysteresis behavior, and evolution processes of the systems. And we find an interesting phenomenon in numerical simulations. In the hysteresis region, the steady state of the MR model varies monotonously with the number of initial seeds. However, the corresponding NMR model exhibits a non-monotonic growth characteristic due to the non-Markovian spontaneous recovery mechanism. We then use a mean-field theory to analyze the reason qualitatively.

The paper is organized as follows. We introduces the NMR and the MR models in Sec. II. The Pairwise approximation theories are developed for the two models in Sec. III. Sec. IV shows the simulation results and the corresponding theoretical predictions. In Sec. V, we analyze the stability of phase transition and hysteresis behavior. Then, we use the mean-field theory to explain the reason of the new phenomenon observed in this section. In Sec. VI, we briefly summarize our conclusions and prospects.

II. SPONTANEOUS RECOVERY MODEL

To study the effects of spontaneous recovery mechanism on the failure-recovery systems, we focus on two spontaneous recovery models, i.e., the non-Markovian recovery (NMR) model and the Markovian recovery (MR) model. In the spontaneous recovery models, a node can be in one of two states, i.e., active state (or not failed, labeled as A-type node) and inactive state (or failed, labeled as I-type node). For a failed node in the inactive state, it will be labeled as X-type node if its failure results from internal reasons; and it will be labeled as Y-type node otherwise. The failure-recovery processes of the two models are as follows.

In the NMR model, an A-type node may spontaneously fail at a rate $\beta_1$ (i.e., become a X-type node); or it may fail at a rate $\beta_2$, when the number of its A-type neighbor nodes is less than or equal to a threshold $m$ (i.e., become a Y-type node) which means the lack of neighborhood support. Here $m$ is an integer. At the same time, a newly emerging X(Y)-type node will spontaneously recover after a time period $\tau_1(\tau_2)$ (i.e., become an A-type node). A schematic diagram of the failure-recovery processes can be found in Fig. 1.

Compared with internal reasons, external reasons cause active nodes to fail more frequently. For example, in financial markets, the stock prices of companies may fall due to the stock prices of other close companies declining. Without loss of generality, we assume $\beta_1 < \beta_2$. Additionally, the externally failed nodes tend to recover with less time than the internally failed nodes, as their failures result from the lack of neighborhood support rather than internal reasons. Hence we assume $\tau_1 > \tau_2$. But in the MR model, we assume a X(Y)-type node recovers spontaneously from failures at a constant rate $\mu_1(\mu_2)$, which is the only difference from the NMR model.

As mentioned, the two models are different only in the way of recovery processes. For the NMR model, the recovery process is non-Markovian, namely failed nodes recovering with memory effect. For example, at time $t$, the recovery quantity of X-type nodes is determined by the newly emerging quantity of X-type nodes at time $t - \tau_1$. However, the spontaneous recovery in the MR model is a Markovian process (i.e., memoryless), and the current recovery quantity of X-type nodes depends only on the quantity of X-type nodes at last step.

III. PAIRWISE APPROXIMATION THEORY

The Pairwise approximation (PA) method, which analyzes the flow of states probabilities of nodes and edges by considering the pairwise dynamic correlation, is used widely in epidemic and information spreading [50–53]. Here, in order to depict the dynamics of the MR and the NMR models more precisely, we develop a set of Pairwise approximation theories for the two models.

For the MR model, we denote $[U]_t$, and $[UV]_t$, with $U, V \in \{A, X, Y\}$ as a certain-type proportion of nodes and edges at time $t$, respectively. For example, the symbol $[A]_t$, $[X]_t$, and $[Y]_t$ represent the proportions of active nodes, X-type failed nodes and Y-type failed nodes at time $t$, respectively. The

FIG. 1. Schematic diagram of failure-recovery processes in the NMR and the MR model. In the propagation, an A-type node may spontaneously fail at rate $\beta_1$ (i.e., become a X-type node) being independent of the states of its neighbors, or fail at rate $\beta_2$ (i.e., become a Y-type node) when the number of its A-type neighbors $n_A$ is less than or equal to $m$, meaning the lack of neighborhood support. For the newly generated X and Y nodes, they will recover respectively from failures after time periods $\tau_1$ and $\tau_2$ for the NMR model, or recover respectively at rates $\mu_1$ and $\mu_2$ for the MR model.
symbol $[AX]_t$ represents the proportion of edges whose left end being an active node and right end being a $X$-type failed node at time $t$, respectively. Owing to the symmetry of edges, we have $[AX]_t = [XA]_t$.

The evolution equations for the proportions of nodes of various types can be written as

$$\frac{d[X]_t}{dt} = \beta_1 [A]_t - \mu_1 [X]_t$$

and

$$\frac{d[Y]_t}{dt} = \beta_2 E_t[A]_t - \mu_2 [Y]_t,$$

where $E_t$ represents the probability that an $A$-type node has the number of $A$-type neighbors $j \leq m$. As $[AA]_t/|A|_t$ denotes the probability that a neighbor of an $A$-type node is in the inactive state at time $t$, where $|A|_t = 1 - |X|_t - |Y|_t$, and $|AI|_t = [AX]_t + [AY]_t$, we know that the probability of an $A$-type node having $j$ $A$-type neighbors and $(k-j)$ $Y$-type neighbors can be calculated by $C^k_j (\frac{|AI|_t}{|A|_t})^k (\frac{1 - |AI|_t}{|A|_t})^j$.

Considering that $j$ could be any integer less than or equal to $m$, we have

$$E_t = \sum_{j=0}^{m} C^k_j (\frac{|AI|_t}{|A|_t})^k (\frac{1 - |AI|_t}{|A|_t})^j = \sum_{j=0}^{m} C^k_j (\frac{|AI|_t}{|A|_t})^k (\frac{1 - |AI|_t}{|A|_t})^j$$

In the evolution equation of $[X]_t/|Y]_t$, the first term means that the failure of $A$-type nodes results from the internal reasons (external reasons), which increases the proportion of $X$-type ($Y$-type) nodes. The second term represents the transition that $X$-type ($Y$-type) nodes recover, which decreases the proportion of $X$-type ($Y$-type) nodes.

Taking pairwise dynamical correlation into account (see details in Sec. I of Supplementary Material), the evolution equations for the edges in different states can be written as

$$\frac{d[AX]_t}{dt} = \mu_1 [XX]_t + \mu_2 [XY]_t + \beta_1 [AA]_t$$

and

$$\frac{d[AY]_t}{dt} = \mu_1 [AX]_t + \mu_2 (|YA]_t + [AY]_t)$$

$$-2(\beta_1 + \beta_2 E''_t)[AX]_t$$

where $E''_t = \sum_{j=0}^{m} C^k_j (|AI|_t)^k (1 - |AI|_t)^j$ represents the probability that an $A$-type node has the number of $A$-type neighbors $j \leq m$, except for the failed neighbor at the end of $A - A$ edge; and $E''_t$ represents the probability of an active node with the number of its active neighbors $n_A \leq m$, $E''_t$ denotes the probability of an $A$-type node in $A - X$ (or $A - Y$) edges satisfying $n_A \leq m$, and $E''_t$ denotes the probability of an $A$-type node connected with an $A$-type node satisfying $n_A \leq m$.

In the equation of $[AX]_t$, the first (second) term represents the transition that $X$-type ($Y$-type) nodes at left end of $X - X$ ($Y - X$) edges recover spontaneously (i.e., become $A$-type nodes), which contributes to the increment of $[AX]_t$. The third term represents the transition that $A$-type nodes at right end of $A - A$ edges fail due to the internal failure (i.e., become $X$-type nodes), which increases the proportion of $A - X$ edges. The forth term represents the transition that $X$-type nodes of $A - X$ edges recover spontaneously (i.e., become $A$-type nodes), which decreases the proportion of $A - X$ edges. The fifth term represents the transition that $A$-type nodes of $A - X$ edges fail due to the internal or external causes (i.e., become $X$-type or $Y$-type nodes), which decreases the proportion of $A - X$ edges. The fifth term represents the transition that $A$-type nodes of $A - A$ edges fail due to the internal or external causes (i.e., become $X$-type or $Y$-type nodes), which decreases the proportion of $A - A$ edges.

The evolution of the the states of the MR model can be described using nine differential equations (see the Sec. I in Supplementary Material). If we ignore the dynamical correlation between the states of nodes at the end of edges, we will have $[AI]_t = [A]_t[I]_t$, and thus the PA approach can be degenerated into the first-order mean field theory

$$\frac{d[X]_t}{dt} = \beta_1 [A]_t - \mu_1 [X]_t$$

and

$$\frac{d[Y]_t}{dt} = \beta_2 E_t[A]_t - \mu_2 [Y]_t,$$

where

$$E_t = \sum_{j=0}^{m} C^k_j (|I|_t)^k (1 - |I|_t)^j.$$
and \([U^\dagger V]\), where all the \(l_1, l_2\) represent the the passing time of the corresponding nodes in current state at time \(t\). The time evolutions for the proportions of \(X\)-type nodes can be given by

\[
[X^l]_{t+\Delta t} = \begin{cases} 
\beta_1 t[A]_t, & l = 0; \\
[X^{l-\Delta t}]_t, & l \in (0, \tau_1]; \\
0, & l \in (\tau_1, \infty).
\end{cases}
\]

The \(A\)-type nodes at time \(t\) will newly become \(X\) state with probability \(\beta_1\Delta t\) in the time period \([t, t + \Delta t]\), and thus \([X^0]_{t+\Delta t} = \beta_1 t[A]_t\). When the state age \(l \in (0, \tau_1]\), the \(X\)-type nodes with \(l\) at time \(t + \Delta t\) come from that of \(l - \Delta t\) at time \(t\), i.e., \([X^l]_{t+\Delta t} = [X^{l-\Delta t}]_t\). When \(l \in (\tau_1, \infty)\), the proportion of the \(X\)-type nodes at time \(t + \Delta t\) is equal to zero, since the state age \(l\) can not exceed the recovery time. Similarly, the equations of \([Y^l]_{t+\Delta t}\) can be expressed as

\[
[Y^l]_{t+\Delta t} = \begin{cases} 
\beta_2 t\Delta t E_t[A]_t, & l = 0; \\
[Y^{l-\Delta t}]_t, & l \in (0, \tau_2]; \\
0, & l \in (\tau_2, \infty),
\end{cases}
\]

where \(E_t = \sum_{j=0}^{m} C_{j}^{-k-j} \left(\frac{[A]_{j}}{[X]_{j}}\right)^{k-j} \left(1 - \frac{[A]_{j}}{[X]_{j}}\right)^{j}\), \(A\) represents the the passing time of the \(X\)-type node, and \(\beta\) is the rate of the recovery.

To capture the pairwise dynamical correlation (see details in Sec. I of Supplementary Material), the evolution equations for the edges in different states can be obtained as

\[
[AX^l]_{t+\Delta t} = \begin{cases} 
\beta_1 t[A]_t + \beta_1 t([X^{\tau_1} A]_t + [Y^{\tau_1} A]_t), & l = 0; \\
[X^{l-\Delta t}]_t + [Y^{\tau_1} X^{l-\Delta t}]_t + (1 - \beta_1 t - \beta_2 t E_t)[AX^{l-\Delta t}]_t, & l \in (0, \tau_1]; \\
0, & l \in (\tau_1, \infty),
\end{cases}
\]

and

\[
[AA]_{t+\Delta t} = (1 - \beta_1 t - \beta_2 t E_t)[AX^{\tau_1}]_t + [AX^{\tau_1}]_t + [Y^{\tau_2} A]_t + [A Y^{\tau_2}]_t + [X^{\tau_1} X^{\tau_1}]_t \\
+ [Y^{\tau_2} Y^{\tau_2}]_t + [X^{\tau_1} Y^{\tau_2}]_t + [Y^{\tau_2} X^{\tau_1}]_t + (1 - 2\beta_1 t - 2\beta_2 t E_t E''_t)[AA]_t, 
\]

where \(E'_t = \sum_{j=0}^{m} C_{j}^{-k-j} \left(\frac{[A]_{j}}{[X]_{j}}\right)^{k-j} \left(1 - \frac{[A]_{j}}{[X]_{j}}\right)^{j}\), \(E''_t = \sum_{j=0}^{m} C_{j}^{-k-j} \left(\frac{[A]_{j}}{[X]_{j}}\right)^{k-j} \left(1 - \frac{[A]_{j}}{[X]_{j}}\right)^{j}\) and \([AX]_t = \sum_{j=0}^{\tau_1} [AX^j]_t\). Similar to the MR model, we use \(E_t\),
\( E_t' \) and \( E_t'' \) as well to represent the probabilities that an active node satisfying the threshold condition \( n_A \leq m \) in different cases and will not repeat them.

The right-hand side of Eq. (11) describes the time evolution equations of \([AX^t]_t + \Delta t\) for different values of aged \( l \). When \( l = 0 \), the first term denotes that the \( A \)-type nodes at right end of \( A \)–\( X \) edges fail due to internal causes, which increases the proportion of \( A \)–\( X \) edges with \( l = 0 \) at time \( t + \Delta t \). The second (third) term denotes that \( A \)-type nodes connected with an \( X \)-type aged \( \tau_1 \) (\( Y \)-type aged \( \tau_2 \)) neighbor fail due to internal causes and meanwhile the \( X \)-type (\( Y \)-type) neighbor recovers spontaneously to \( A \) state because its state age reaches recovery time, which increases the proportion of the \( A \)–\( X \) edges with \( l = 0 \) at time \( t + \Delta t \). When \( 0 < l \leq \tau_1 \), the first (second) term denotes that the \( X \)-type aged \( \tau_1 \) (\( Y \)-type aged \( \tau_2 \)) nodes, connected with an \( X \)-type neighbor aged \( l - \Delta t \), recover spontaneously, which increases the proportion of \( A \)–\( X \) edges with \( l \in (0, \tau_1] \) at time \( t + \Delta t \). The third term denotes that there are no transitions on the states of the \( A \)–\( X \) edges, namely the \( A \)-type nodes in \( A \)–\( X \) edges do not fail during the time period \([t, t + \Delta t]\). When \( l > \tau_1 \), \([AX^t]_t + \Delta t\) is equal to zero. In the Eq. (12) of \([AA]_t + \Delta t\), the first term denotes that, in edges \( X^{\tau_1} - A \)–\( X^{\tau_1} \), \( Y^{\tau_2} - A \), and \( A - Y^{\tau_2} \), the \( A \)-type of states are not changed, but the states of the failed nodes have changed, which increases the proportion of \( A \)–\( A \) edges at time \( t + \Delta t \). The second term denotes that both ends of nodes in the \( I \)–\( I \) edges recover, which increases the proportion of \( A \)–\( A \) edges. The third term denotes that the states of both nodes in the \( A \)–\( A \) edges are not changed.

The complete equations can be found in Sec. I of Supplementary Material. The number of these equations is at least 
\[
(\frac{A^2}{2} + 1)^2 + (\frac{A^2}{2} + 1)^2
\]
which depends on the magnitude of step length \( \Delta t \) and the values of \( \tau_1 \) and \( \tau_2 \). In this paper, we set \( \Delta t = 0.01 \), \( \tau_1 = 100 \) and \( \tau_2 = 1 \), and the magnitude of the number of equations is thus at least \( 10^6 \).

If we ignore the dynamic correlation between nodes in the propagation, i.e., \([AA]_t = [A]_t[I]_t\), the above equations can be degenerated into the mean-field theory:

\[
[X]_t + \Delta t = \beta_1 \Delta t[A]_t + [X]_t - [X^{\tau_1}]_t, \tag{13}
\]

and

\[
[Y]_t + \Delta t = \beta_2 \Delta t[E]_t + [Y]_t - [Y^{\tau_2}]_t, \tag{14}
\]

where

\[
E_t = \sum_{j=0}^{m} C_k^{k-j}(I)_{t}^{k-j}(1 - [I]_{t})^{j}. \tag{15}
\]

Eq. (13) indicates that, in time period \([t, t + \Delta t]\), some \( A \)-type nodes turn into the \( X \)-type nodes due to the internal failure, namely \( \beta_1 \Delta t[A]_t \), and the \( X \)-type nodes (except for \( X^{\tau_1} \)) remain unchanged, namely \( [X]_t - [X^{\tau_1}]_t \). The equation of the \( Y \)-type nodes is similar to that of the \( X \)-type nodes.

IV. SIMULATION VERIFICATION

In this section, considering simplicity and feasibility of theoretical analysis, we choose Random Regular Networks (RRNs) as an example, where the network size \( N = 30,000 \) and the average degree \( k = 35 \). In the NMR model, we set the recovery time \( \tau_1 = 100(\tau_2 = 1) \) for \( X \)-type nodes (\( Y \)-type nodes), while makes recovery rate \( \mu_1 = 1/\tau_1 = 0.01 \) \( (\mu_2 = 1/\tau_2 = 1) \) in the MR model to obtain almost same scale of recovery time (see Sec. II in Supplementary Material for explanation). The threshold of the both models is set to \( m = 15 \). If there are no specific explanation, we fixed the external failure rate \( \beta_2 = 2 \) in the two models. We use synchronous updating simulation method, where the time interval of each step is \( \Delta t = 0.01 \). Initially, a fraction of nodes are chosen randomly as \( X \)-type nodes, and the remaining nodes are in \( A \) state.

Fig. 3 shows the dynamical behaviors of the MR model and the NMR model. As \( \beta_1 \) adiabatically increases (decreases), in Fig. 3 (a), the proportion of failed nodes \([I]_t\) in the steady state first increases (decreases) continuously and then jumps to a high (low) level. The stationary proportion of failed nodes \([I]_t\) in both models exhibits a hysteresis behavior analogous to phase transitions near a critical point. From Fig. 3 (a), we see that the PA theory can predict the simulation results more accurately than the MF theory. Fig. 3 (b) shows the \( (\beta_2, \beta_1) \) phase diagrams of the two models, which outlines regions of different phases and the hysteresis region. Compared with the MF method, the theoretical predictions from the proposed PA method are more consistent with the simulation results for the two models. It indicates that the spontaneous recovery systems exhibit a hysteresis behavior by either changing \( \beta_1 \) or \( \beta_2 \) adiabatically. Therefore, we fixed \( \beta_2 = 2 \) in the following paper for simplicity, if there is no specific declaration. It is worth noting that the two models have nearly the same stationary states and phase diagrams in Figs. 3 (a) and (b), which means that the non-Markovian recovery mechanism does not essentially affect the stationary states of failure-recovery systems.

We next investigate the time evolutions of the two failure-recovery systems. Taking the parameters in Fig. 3(a) and setting \( \beta_1 = 0.009 \) and \([I]_0 = 0 \), the time evolutions of \([I], [X] \) and \([Y] \) for the MR and NMR models are shown in Fig. 3(c) and Fig. 3(d), respectively. Obviously, the PA theory can predict the simulation results accurately. Comparing between Fig. 3(c) and Fig. 3(d), we see that even if the stationary values of \([I], [X] \) and \([Y] \) for the two models are almost equal, their evolution processes are different. In order to make a convenient comparison, we divide the time evolutions into different stages, e.g., three stages \([t_O, t_A], [t_A, t_B] \), and \([t_{B, t_C}] \) for the MR model in Fig. 3(c), and five stages \([t_O, t_A], [t_A, t_B], [t_{B, t_C}], [t_{C, t_D}], \) and \([t_{D, t_E}] \) for the NMR model in Fig. 3(d). In the early stages, i.e., \([t_O, t_A] \) and \([t_A, t_B] \), the evolution characteristics of two models are almost the same. In stage \([t_O, t_A] \), most of nodes are active and thus have the number of active neighbors not satisfying \( n_A \leq m \). Therefore, there is almost no \( Y \)-type node in the network, while some of the active nodes will become \( X \) state due to the inter-
FIG. 3. Comparison of dynamical behaviors between the MR and the NMR models. (a) the proportion of failed nodes in the steady state as a function of internal failure rate $\beta_1$ for the MR (orange squares) and the NMR (blue circles) models. The orange dotted line for the MR model and the blue dot-dashed line for the NMR model are obtained from the PA theory, and the gray dashed line (MR) and the gray dotted line (NMR) are obtained from the MF theory. Initially, we let $\beta_1 = 0$ and the initial failed nodes $[I]_0 = 0$ and then, when the system reaches a steady state, we begin to let $\beta_1$ increase adiabatically to $\beta_1 = 0.01$ by $\Delta \beta_1 = 0.0002$. After that, we let $\beta_1$ decrease adiabatically to 0 by $\Delta \beta_1 = 0.0002$. Here $\beta_2 = 2$. (b) The $(\beta_2, \beta_1)$ phase diagram for the MR and the NMR models, which outlines the regions of different phases and hysteresis. Orange squares and blue circles respectively represent the simulation results for the MR and NMR models. Orange solid line for the MR model and blue dot-dashed line for the NMR model are obtained from the MF theory. Gray dashed line and gray dotted line are theoretical predictions of the MR and NMR models from the MF theory, respectively. The temporal evolutions of the proportions of nodes in different states for the MR model (c) and the NMR model (d). Blue dashed line, green dot-dashed line, and purple dashed line respectively represent the projections of $I$, $X$, and $Y$ nodes from simulation results. The internal failure rate is set as $\beta_1 = 0.009$, which is pointed out by the black dot line in subfigure (a). The solid lines are obtained from the PA theory. All simulation results are averaged over $10^5$ dynamical realizations. Other parameters are $\mu_1 = 0.01, \mu_2 = 1, \tau_1 = 100, \tau_2 = 1$ and $m = 15$.

nal failure. In stage $[t_A, t_B]$, as the number of $X$-type nodes increases, more and more $A$-type nodes will fail due to external reasons that their active neighbors satisfy $n_A \leq m$, and thus $[Y]$ grows rapidly. In the late stages, For the MR model, there are less active nodes and more failed nodes in stage $[t_B, t_C]$ than the ones in stage $[t_A, t_B]$. The active nodes more easily turn to $Y$ state and $[Y]$ continues to increase, because the neighborhoods of most of the active nodes satisfy $n_A \leq m$. While $[X]$ begins to decrease slowly, since the $X$-type nodes constantly recover at a rate $\mu_1$. The system will reach a steady state when $t \to \infty$. In the late stage, the non-Markovian recovery mechanism makes the evolution of NMR model deviates from that of MR model. For the NMR model, the recovery proportions of $[X]$ and $[Y]$ at time $t$ are determined by the system state at time $t-\tau_1$ and $t-\tau_2$, respectively. The memory effect results in different competition processes between $X$-type and $Y$-type nodes in stages $[t_B, t_C]$, $[t_C, t_D]$ and $[t_D, t_E]$ (see details later).

V. Stability of Phase Transition and Hysteresis Behavior

As stated above, different competition processes between $X$-type and $Y$-type nodes dominate the time evolutions of the NMR model, which implies the the proportion of $X$-type nodes or $Y$-type nodes may markedly affect the cascading dynamics. Here we investigate how the proportion of initial failed nodes $[I]_0$ influences the phase transition of the NMR model.

We fix different initial values $[I]_0$ and observe how the proportion of failed nodes $[I]$ in steady state varies with the internal failure rate $\beta_1$. For the MR model in Fig. 4(a), when the initial value is small and large enough respectively, almost the
same hysteresis region can be found as in Fig. 3(a). In particular, the results for small $[I]_0$ (e., g., 0.01, 0.1, 0.23, 0.25) converge to the same curve and exhibit an explosive growth at a relatively large value of tipping point $\beta_c \approx 0.007$. While for a large enough value of $[I]_0 = 0.99$, an explosive growth of $[I]$ occurs at a relatively small value of $\beta_c \approx 0.003$. As shown in Fig. 4(c), as $[I]_0$ increases, $\beta_c$ first stabilizes around 0.007, and then sharply decreases to 0.003 at $[I]_0 \approx 0.4$. There is a large gap of $[I]$ in $\beta_c \in [0.003, 0.007]$ [see Fig. 4(a)], which is consistent with the hysteresis phenomenon in adiabatic processes [see Fig. 3(a)].

For the NMR model in Fig. 4(b), however, we see that the results for different $[I]_0$ can not converge to the same curve and the explosive growth occurs at different $\beta_c$. Interestingly, the tipping point $\beta_c$, as a function of $[I]_0$, shows a non-monotonic variation phenomenon: $\beta_c$ first decreases slowly in $[I]_0 \in [0, 0.23]$, then increases rapidly around $[I]_0 \approx 0.23$, and finally decreases very slowly with $[I]_0$ [see Fig. 4(d)]. This non-monotonic behaviour is clearly different from the hysteresis phenomenon in adiabatic processes. Moreover, we see that the PA theory can predict the simulated $\beta_c$ more accurately than the MF theory in Figs. 4(c) and (d).

Taking parameters in Figs. 4(a) and (b) and fixing $\beta_1 = 0.004$, we observe the stationary $[I]$ as a function of $[I]_0$ for the MR and NMR models, respectively. From Fig. 5(a), we see that as $[I]_0$ increases, the system in relative low-failure phase ($[I] = 0.33$) suddenly turns into relative high-failure phase ($[I] = 0.71$). But in Fig. 5(b), the steady state of the NMR model exhibits a non-monotonic behavior. As $[I]_0$ increases, the system in low-failure phase ($[I] = 0.33$) first turns into high-failure phase ($[I] = 0.71$) and then into low-failure phase ($[I] = 0.33$) again. In other words, more initial failed nodes can result in less failed nodes in steady state rather than more in the MR model, which is consistent with the results of $\beta_c$ in Fig. 4(d).

To understand this non-monotonic behavior, the MF theory can be easily applied to the theoretical analysis [47]. We next use the MF theory as a tool to analyze the causes of this phenomenon and make a qualitative explanation. To analyze the reasons, we take $[I]_0 = 0.05, 0.2, 0.3, 0.6$ [i.e., four dotted lines in Fig. 5(a) and (b)], and observe how the trajectories of both $[X]$ and $[Y]$ evolve with time $t$ for both models, which is obtained in Fig. 6. And the proportions of nodes in different states versus time can be found in Sec. 3 of Supplementary Material. As shown in Fig. 6, the light gray dot-dashed and dark gray dashed lines represent the solutions of $[X] = 0$ and $[Y] = 0$ from the MF theory of MR model, respectively, and the intersections of them are the the stationary solutions for the MR model. There are three steady-state solutions, where two of them are stable steady-state solutions and the rest is an unstable solution. Note that we set $\mu_1 = 1/\tau_1$ and $\mu_2 = 1/\tau_2$, and thus the NMR model has almost the same solutions of steady state with the corresponding MR model.

When the proportion of initial failed nodes is small enough, e., g., $[I]_0 = 0.05$ in Fig. 6(a), namely most of active nodes having active neighbors $n_A > m$, the active nodes can only fail internally rather than externally. Both of the two models will converge to a relative low-failure phase, where the trajectories evolve from $t_O$ to $t_A$.

When $[I]_0$ increases to 0.2 in Fig. 6(b), the evolution process for the MR model is similar to the case in Fig. 6(a), and the trajectory evolves from $t_O$ to $t_A$. This is because that the $X$-type nodes not only emerge at rate $\beta_1$ but also recover at rate $\mu_1$, which makes the system reaches a dynamic equilibrium of low-failure phase, and thus there are insufficient newly $X$-type nodes emerging to cause a large-scale external failures. While for the NMR model, there are always new $X$-type nodes emerging but no $X$-type nodes recover in the early stage $[t_O, t_A]$. Once the proportion of failed nodes, including initial $X$-type nodes and newly emerging $X$-type nodes in $[t_O, t_A]$, reaches a critical value, namely many $A$-type nodes with active neighbors satisfying $n_A \leq m$, $[Y]$ increases rapidly and the trajectory moves from $t_A$ to $t_A'$. In $[t_A', t_B')$, $[Y]$ decreases slowly due to a short recovery time $t_2 = 1$, while internal failures make $[X]$ increase slowly as $t_B' < 100$. In time interval $t \in [100, 100 + \Delta t)$, the age of initial failed nodes $([I]_0 = [X]_0 = 0.2)$ reaches the recovery time and these nodes will turn into $A$ state simultaneously. And thus $[X]$ decreases suddenly to a low value and the trajectory evolves sharply from $t_B'$ to $t_C'$. At this time, $[I]$ still stays around a high value, which makes more active nodes (including the original and new $X$-type nodes) satisfy $n_A \leq m$ and results in a rapid growth of $Y$-type nodes in $[t_C', t_D')$. At time $t_D'$, $[I]$ increases to a higher value, which makes $Y$ state be more competitive than $X$ state when they compete for $A$-type nodes, and thus the $A$-type nodes fail due to the external causes more easily. Therefore, $[X]$ and $[Y]$ (i.e., $X$) continues to increase (decrease) in $[t_D', t_E']$.

When $[I]_0 = 0.3$ in Fig. 6(c), the evolution process of the MR model is consistent with that in Fig. 6(b). For the NMR model, the evolution process is different from that in Fig. 6(b). In $[t_B', t_C')$, more initial failed nodes with $[I]_0 = [X]_0 = 0.3$ recover simultaneously because their ages (i.e., the current time) reach the recovery time $\tau_1 = 100$. At time $t_C'$, a relatively low value of $[I]$ makes less active nodes satisfy $n_A \leq m$. Meanwhile, a short recovery time makes all the current $Y$-type nodes turn to active state in $[t_C', t_D')$. After that, many active nodes fail internally and become $X$-type nodes, which exhibits that $[X]$ increases continuously and the trajectory moves from $t_D'$ to $t_A$.

When the initial value $[I]_0$ is large enough such as 0.6 in Fig. 6(d), many $A$-type nodes will fail externally and $[Y]$ will increase rapidly for the two models, as shown in the trajectory from $t_O$ to $A$. After that, for the MR model, the increment of $[Y]$ enhances the probability of active nodes becoming $Y$ state, while more and more $X$-type nodes recover and then turn to $A$ state. Its evolution exhibits the trajectory from $t_A$ to $t_B$, which is similar to the case in stage $[t_B, t_C]$ of Fig. 3(c). For the NMR model, the evolution process is similar to that in Fig. 6(c).

In summary, the essentially different recovery mechanisms make the competition processes between the $X$-type and the $Y$-type nodes being different for the two models. Specifically, the non-Markovian recovery makes the failure-recovery systems have a more complicated failure processes. The evolution of the failed nodes exhibits in three forms, e., g.,
FIG. 4. The effects of initial conditions on the phase transition of cascading failure. The proportion of failed nodes $[I]$ in the steady state as a function of internal failure rate $\beta_1$ for the MR model (a) and the NMR model (b), where blue circles, orange triangles down, green pentagons, red triangles up, purple diamonds represent the results for different initial proportions of failed nodes $[I]_0 = 0.01, 0.1, 0.23, 0.25, 0.99$, respectively. All the simulation results are implemented only once. Subfigures (c) and (d) respectively show the tipping point of explosive growth $\beta_c$ versus initial proportion of failed nodes $[I]_0$ for the MR model and the NMR model. The simulation results (circles) are averaged over 10 realizations, and the standard deviation is about $6E^{-5}$. The black dashed line and the gray dot-dashed line are theoretical predictions from the PA theory and MF theory, respectively. Other parameters are set as $\beta_2 = 2$, $\mu_1 = 0.01$, $\mu_2 = 1$, $\tau_1 = 100$, $\tau_2 = 1$, $m = 15$, and the network is RRN with $N = 30,000$ and $k = 35$.

Considering that some systems may have initial value $[Y]_0 \neq 0$, we plot the corresponding results of $[I]$ in the steady state for the two models in the $([X]_0, [Y]_0)$ plane corresponding to different initial values, as shown in Fig. 7. It shows that when observing how stationary $[I]$ varies with fixing $[Y]_0 \neq 0$ and changing $[X]_0$, a similar phenomenon can be found as in Fig. 5. Given that the structures of many real networks are heterogeneous, we use scale-free networks as an example to explore the stability of phase transition and hysteretic behavior in heterogeneous networks. Here we use the standard configuration model to generate the networks and the degree distribution is $P(k) = Ck^{-\gamma}$, where $C$ is a normalized constant and maximum degree $k_{\text{max}} \sim N^{1/(\gamma-1)}$. Besides, on heterogeneous networks, a fractional threshold is the more appropriate choice than an integer threshold. Thus as shown in Fig. 8, We choose the threshold equals 0.5 as an example and find the similar phenomena as in Fig. 4. How different initial conditions and network structure influence the cascading dynamics deserves further study.

VI. CONCLUSION

Non-Markovian process can be observed in many real systems and its memory feature brings more difficulties to the theoretical analysis of Non-Markovian dynamics. In this paper, we studied how the non-Markovian recovery affects the steady state, the hysteresis region and the temporal evolution dynamics of the failure-recovery system by comparing with the corresponding Markovian process. To this end, we developed the corresponding PA methods for the two models, which can match the simulation results more accurately than the MF methods. We found that although the non-Markovian recovery mechanism does not essentially affect the steady states and hysteresis behavior of failure-recovery systems, the two models exhibit distinct evolution processes, that is the memory effect of non-Markovian recovery results in different competition processes between $X$-type and $Y$-type nodes in different stages.
FIG. 5. The stationary $I$ as a function of $I_0$ for the MR model (a) and NMR model (b). The dashed line represents the MF theory results. Other parameters are $\beta_1 = 0.004, \beta_2 = 2, \mu_1 = 0.01, \mu_2 = 1, \tau_1 = 100, \tau_2 = 1, m = 15$, and the network is RRN with $k = 35$. The gray dotted vertical lines correspond to $[I]_0 = 0.05, 0.2, 0.3, 0.6$, respectively.

Considering the competition processes between these two types of failed nodes, we investigated how the proportion of initial failed nodes influences the phase transition and hysteresis behavior of the NMR model. We found that the different recovery characteristics make the two models behave quite different in both phase transition and hysteresis behavior. In the region outside of the hysteresis, the final steady-states are not significant different between the two models. However, in the hysteresis region, even if given the same initial conditions, the steady states of the two models can be totally different. In particular, as the fraction of initial failed nodes increases, the steady state of the MR model staying in relative low-failure phase turns into relative high-failure phase. While the NMR model exhibits a non-monotonic behavior: as the fraction of initial failed nodes grows, the steady state of the system staying in relative low-failure phase first turns into relative high-failure phase and then into relative low-failure phase again. In other words, more initial failed nodes lead to relative less failed nodes in the NMR model, rather than more in the corresponding MR model. We then analyzed the reasons by using the MF theory. The analyses show the competition between the $X$-type and the $Y$-type nodes exhibits three distinct processes under different initial conditions, which explains the non-monotonic behavior.

We noticed that even if both models have a mechanism of spontaneous recovery, the NMR model generates richer phenomena. We pointed out and emphasized that when considering non-Markovian dynamics, we should study the steady state not only from the perspective of the variation of parameters, but also from the initial conditions. We believe that the developed pairwise approximation theory and the interesting impact of the non-Markovian characteristics on the cascading dynamics is useful for the future studies, such as epidemic spreading, information diffusion, and synchronization. A possible extension of our theoretical framework could be to study the impact of different recovery-time distributions on the failure-recovery systems. Besides, the cascading dynamics on heterogeneous networks should be worth to explore.

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FIG. 6. The trajectories of both $[X]$ and $[Y]$ evolving with time $t$ under different initial values of $[I]_0$. The results are obtained from the MF theory. The solid red and blue lines are the results of the MR and NMR models, respectively. The light gray dot-dashed and dark gray dashed lines respectively represent the solutions of $\dot{[X]} = 0$ and $\dot{[Y]} = 0$ in the MF theory of the MR model, and the intersections of them are the steady-state solutions for the MR model. The parameters are the same as that in Fig. 5.

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FIG. 7. The stationary $[I]$ as a function of $[X]_0$ and $[Y]_0$ for the MR model (a) and NMR model (b). The results are obtained from the MF theory and $[X]_0$ and $[Y]_0$ satisfying $[X]_0 + [Y]_0 \leq 1$. The parameters are the same as that in Fig. 5.

Subfigures (a) and (b) respectively show the tipping point of explosive growth $\beta_c$ versus initial proportion of failed nodes $[I]_0$ for the MR model and the NMR model. The simulation results are averaged over 10 realizations, and the standard deviation is about $5E^{-3}$. Other parameters are set as $\beta_2 = 1.9, \mu_1 = 0.01, \mu_2 = 1, \tau_1 = 100, \tau_2 = 1$ and the fractional threshold is 0.5. The network is scale-free with $N = 30,000, \gamma = 3$ and mean degree $\langle k \rangle \approx 10$.

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Supplementary Material for the paper:
The effect of non-Markovian recovery on reversible failure propagation on networks

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I. PAIRWISE APPROXIMATION THEORY

A. The Markovian recovery (MR) model

We use symbols of the forms \([U]_t\) and \([UV]_t\) with \(U, V \in \{A, X, Y\}\) to denote the proportions of nodes and edges in different states at time \(t\), respectively. For example, the symbols \([A]_t, [X]_t\) and \([Y]_t\) represent the proportions of active nodes, \(X\)-type failed nodes and \(Y\)-type failed nodes at time \(t\), respectively. The symbol \([AX]_t\) represents the proportion of active nodes connected with a \(X\)-type failed node (i.e., the proportion of \(A - X\) edges) at time \(t\). We have \([AX]_t = [XA]_t\) due to symmetry.

The evolution equations for the proportions of various types of nodes and edges are given by

\[
\frac{d[A]_t}{dt} = \mu_1[X]_t + \mu_2[Y]_t - (\beta_1 + \beta_2 E_t)[A]_t, \quad (1)
\]

\[
\frac{d[X]_t}{dt} = \beta_1[A]_t - \mu_1[X]_t, \quad (2)
\]

\[
\frac{d[Y]_t}{dt} = \beta_2 E_t[A]_t - \mu_2[Y]_t, \quad (3)
\]

\[
\frac{d[AX]_t}{dt} = \mu_1[XX]_t + \mu_2[YX]_t + \beta_1[AA]_t - \mu_1[AX]_t - (\beta_1 + \beta_2 E'_{x,t})[AX]_t, \quad (4)
\]

\[
\frac{d[AY]_t}{dt} = \mu_1[XY]_t + \mu_2[YY]_t + \beta_2 E'_{y,t}[AA]_t - \mu_2[AY]_t - (\beta_1 + \beta_2 E'_{y,t})[AY]_t, \quad (5)
\]

\[
\frac{d[AA]_t}{dt} = \mu_1([XA]_t + [AX]_t) + \mu_2([YA]_t + [AY]_t) - 2(\beta_1 + \beta_2 E''_t)[AA]_t, \quad (6)
\]

\[
\frac{d[YY]_t}{dt} = \beta_2 E'_{y,t}([AY]_t + [YA]_t) - 2\mu_2[YY]_t, \quad (7)
\]

\[
\frac{d[AX]_t}{dt} = \beta_1([AX]_t + [XA]_t) - 2\mu_1[XX]_t, \quad (8)
\]

\[
\frac{d[AY]_t}{dt} = \beta_1[AY]_t + \beta_2 E'_{x,t}[XA]_t - (\mu_1 + \mu_2)[XY]_t, \quad (9)
\]

where

\[
E_t = \sum_{j=0}^{m} C_k^{k-j} \left( \frac{[AI]_t}{[A]_t} \right)^{k-j} \left( 1 - \frac{[AI]_t}{[A]_t} \right)^{j}, \quad (10)
\]

\[
E'_{x,t} = \sum_{j=0}^{m} C_k^{k-1-j} \left( \frac{[IAX]_t}{[AX]_t} \right)^{k-1-j} \left( 1 - \frac{[IAX]_t}{[AX]_t} \right)^{j}, \quad (11)
\]

\[
E'_{y,t} = \sum_{j=0}^{m} C_k^{k-1-j} \left( \frac{[IAY]_t}{[AY]_t} \right)^{k-1-j} \left( 1 - \frac{[IAY]_t}{[AY]_t} \right)^{j}, \quad (12)
\]

\[
E''_t = \sum_{j=0}^{m-1} C_k^{k-1-j} \left( \frac{[IAA]_t}{[AA]_t} \right)^{k-1-j} \left( 1 - \frac{[IAA]_t}{[AA]_t} \right)^{j}. \quad (13)
\]
In the evolution equation of $[X]_t([Y]_t)$, the first term represents the proportion of $A$-type nodes failing due to the internal reasons (external reasons), which increases the proportion of $X$-type ($Y$-type) nodes. And the second term represents the transition that $X$-type ($Y$-type) nodes recover, which decreases the proportion of $X$-type ($Y$-type) nodes.

In the equation of $[AX]_t$, the first (second) term represents the transition that $X$-type ($Y$-type) nodes at the left end of $X-X$ edges recover spontaneously (i.e., become $A$-type nodes), which increases the proportion of $A-X$ edges. The third term represents the transition that $A$-type nodes at the right end of $A-A$ edges have an internal failure (i.e., become $X$-type nodes), which increases the proportion of $A-X$ edges. The forth term represents the transition that $X$-type nodes at the end of $A-X$ edges fail due to the internal or external causes (i.e., become $X$-type or $Y$-type nodes), which decreases the proportion of $A-X$ edges. In the equation of $[AA]_t$, the first (second) term represents the transition that $X$-type ($Y$-type) nodes at the end of $A-X$ ($A-Y$) edges recover spontaneously (i.e., become $A$-type nodes), which increases the proportion of $A-A$ edges. The third term represents the transition that $A$-type nodes at the end of $A-A$ edges fail due to the internal or external causes (i.e., become $X$-type or $Y$-type nodes), which decreases the proportion of $A-A$ edges.

As shown in the above equations, given under different conditions, the probabilities of an active node satisfying the threshold condition are different, we use $E_t$, $E'_{x,t}$, $E'_{y,t}$, and $E''_t$ respectively to represent the probabilities. $E_t$ denotes the probability of an active node with $n_A \leq m$ ($n_A$ is the number of its active neighbors). $E'_{x,t}(E'_{y,t})$ denotes the probability of an $A$-type node in $A-X$ ($A-Y$) edges satisfying $n_A \leq m$. $E''_t$ denotes the probability of an $A$-type node connected with an $A$-type node satisfying $n \leq m$.

Using pairwise approximation $[UVW]_t = \frac{[UVW]_t}{[V]_t}$, we have $[AX]_t$, $[AY]_t$, [A]_t, and $[AA]_t$, namely $E'_{x,t} = E'_{y,t} = E_t$, and we have

$$E_t = \sum_{j=0}^{m} C_{k-j} \left( \frac{[A]_t}{[A]_t} \right)^{k-j} \left( 1 - \frac{[A]_t}{[A]_t} \right)^j,$$

$$E'_t = \sum_{j=0}^{m} C_{k-1-j} \left( \frac{[A]_t}{[A]_t} \right)^{k-1-j} \left( 1 - \frac{[A]_t}{[A]_t} \right)^j,$$

and

$$E''_t = \sum_{j=0}^{m-1} C_{k-1-j} \left( \frac{[A]_t}{[A]_t} \right)^{k-1-j} \left( 1 - \frac{[A]_t}{[A]_t} \right)^j.$$

There are nine equations to describe the failure propagation of the MR model.

### B. The non-Markovian recovery (NMR) model

To capture the memory effect of the non-Markovian recovery process, we write the NMR model in a form of difference equations by equalizing the non-Markovian recovery process to a series of Markovian processes. In the following below, each difference equation can describe the relationship of proportion of nodes or edges in different states between time $t+\Delta t$ and time $t$. Here, we use the notations $[U]_t$ and $[UV]_t$ with $U, V \in \{A, X, Y\}$ as well to represent the proportions of the nodes and the edges in different types at time $t$, respectively. In addition, we use the notations $[UV]_t$, $[U'V'c]_t$, and $[U'V']_t$, where all the $I$, $l_1$ and $l_2$ represent the the passing time of the corresponding nodes being in current state at time $t$. Due to symmetry, we have $[AX]_t = [XA]_t$. The evolution equations of the NMR model can be written as

$$[A]_{t+\Delta t} = [X^\tau]_t + [Y^\tau]_t + (1 - \beta_1 \Delta t - \beta_2 \Delta t E_t) [A]_t,$$

$$[X^I]_{t+\Delta t} = \begin{cases} 
\beta_1 \Delta t [A]_t, & l = 0; \\
[X^{l-\Delta t}]_t, & l \in (0, \tau_1); \\
0, & l \in (\tau_1, \infty),
\end{cases}$$
\[ [Y^l]_{t+\Delta t} = \begin{cases} 
\beta_2 \Delta t E_t[A]_t, & l = 0; \\
[Y^{l-\Delta t}]_t, & l \in (0, \tau_2); \\
0, & l \in (\tau_2, \infty), 
\end{cases} \] (19)

\[ [AX^l]_{t+\Delta t} = \begin{cases} 
\beta_1 \Delta t [AA]_t + \beta_1 \Delta t ([X^{\tau_1} A]_t + [Y^{\tau_2} A]_t), & l = 0; \\
[X^{\tau_1} X^{l-\Delta t}]_t + [Y^{\tau_2} X^{l-\Delta t}]_t + (1 - \beta_1 \Delta t - \beta_2 \Delta t E'_{x,t}) [AX^{l-\Delta t}]_t, & l \in (0, \tau_1); \\
0, & l \in (\tau_1, \infty), 
\end{cases} \] (20)

\[ [AY^l]_{t+\Delta t} = \begin{cases} 
\beta_2 \Delta t E''_t [AA]_t + \beta_2 \Delta t E''_{y,t} [Y^{\tau_2} A]_t + \beta_2 \Delta t E'_{x,t} [X^{\tau_1} A]_t, & l = 0; \\
[X^{\tau_1} Y^{l-\Delta t}]_t + [Y^{\tau_2} Y^{l-\Delta t}]_t + (1 - \beta_1 \Delta t - \beta_2 \Delta t E'_{y,t}) [AY^{l-\Delta t}]_t, & l \in (0, \tau_2); \\
0, & l \in (\tau_2, \infty), 
\end{cases} \] (21)

\[ [AA]_{t+\Delta t} = (1 - \beta_1 \Delta t - \beta_2 \Delta t E'_t) ([X^{\tau_1} A]_t + [AX^{\tau_1}]_t + [Y^{\tau_2} A]_t + [AY^{\tau_2}]_t) \\
+ ([X^{\tau_1} X^{\tau_1}]_t + [Y^{\tau_2} Y^{\tau_2}]_t + [X^{\tau_1} Y^{\tau_2}]_t + [Y^{\tau_2} X^{\tau_1}]_t) \\
+ (1 - 2 \beta_1 \Delta t - 2 \beta_2 \Delta t E''_t) [AA]_t, \] (22)

\[ [Y^{l_1} Y^{l_2}]_{t+\Delta t} = \begin{cases} 
0, & l_1 = 0 \text{ and } l_2 = 0; \\
\beta_2 \Delta t E''_{y,t} [AY^{l_2-\Delta t}]_t, & l_1 = 0 \text{ and } l_2 \in (0, \tau_2); \\
\beta_2 \Delta t E'_{y,t} [Y^{l_1-\Delta t} A]_t, & l_2 = 0 \text{ and } l_1 \in (0, \tau_2); \\
[Y^{l_1-\Delta t} Y^{l_2-\Delta t}]_t, & l_1 \text{ and } l_2 \in (0, \tau_2); \\
0, & l_1 \text{ or } l_2 \in (\tau_2, \infty), 
\end{cases} \] (23)

\[ [X^{l_1} X^{l_2}]_{t+\Delta t} = \begin{cases} 
0, & l_1 = 0 \text{ and } l_2 = 0; \\
\beta_1 \Delta t [AX^{l_2-\Delta t}]_t, & l_1 = 0 \text{ and } l_2 \in (0, \tau_1); \\
\beta_1 \Delta t [X^{l_1-\Delta t} A]_t, & l_2 = 0 \text{ and } l_1 \in (0, \tau_1); \\
[X^{l_1-\Delta t} X^{l_2-\Delta t}]_t, & l_1 \text{ and } l_2 \in (0, \tau_1); \\
0, & l_1 \text{ or } l_2 \in (\tau_1, \infty), 
\end{cases} \] (24)

\[ [X^{l_1} Y^{l_2}]_{t+\Delta t} = \begin{cases} 
0, & l_1 = 0 \text{ and } l_2 = 0; \\
\beta_1 \Delta t [AY^{l_2-\Delta t}]_t, & l_1 = 0 \text{ and } l_2 \in (0, \tau_2); \\
\beta_2 \Delta t E'_{x,t} [X^{l_1-\Delta t} A]_t, & l_2 = 0 \text{ and } l_1 \in (0, \tau_1); \\
[X^{l_1-\Delta t} Y^{l_2-\Delta t}]_t, & l_1 \in (0, \tau_1) \text{ and } l_2 \in (0, \tau_2); \\
0, & l_1 \in (\tau_1, \infty) \text{ or } l_2 \in (\tau_2, \infty), 
\end{cases} \] (25)

where

\[ E_t = \sum_{j=0}^{m} C_k^{k-j} \left( \frac{[AI]_t}{[A]_t} \right)^{k-j} \left( 1 - \frac{[AI]_t}{[A]_t} \right)^j, \] (26)

\[ E'_{x,t} = \sum_{j=0}^{m} C_{k-1}^{k-1-j} \left( \frac{[IAX]_t}{[AX]_t} \right)^{k-1-j} \left( 1 - \frac{[IAX]_t}{[AX]_t} \right)^j, \] (27)
\[ E_{y,t} = \sum_{j=0}^{m} C_{k-1-j}^k \left[ \frac{IAY}{t} \right]^{k-1-j} \left( 1 - \frac{IAY}{t} \right)^j, \]  
(28)

and

\[ E''_{t} = \sum_{j=0}^{m-1} C_{k-1-j}^{k-1} \left[ \frac{IAY}{t} \right]^{k-1-j} \left( 1 - \frac{IAY}{t} \right)^j, \]  
(29)

Besides, we have \([AX]^l_t = [X^l A]^t_t, [AY]^l_t = [Y^l A]_t, [X^l t Y^l z]_t = [Y^l z X^l t]_t, [AI]^l_t = \sum_{t=0}^{\tau_1} [AX]^l_t + \sum_{t=0}^{\tau_2} [AY]^l_t, [AI']^l_t = [AX]^l_t + [AY]^l_t].

In the evolution equation of \([Y^1 t]_{t+\Delta t}\), the proportion of the X-type nodes with \(l = 0\) at time \(t + \Delta t\) is equal to the proportion of the A-type nodes which fail due to internal causes in the time period \([t, t + \Delta t]\). The proportion of X-type nodes with \(0 < l \leq \tau_1\) is equal to the X-type nodes with \(0 < l - \Delta t \leq \tau_1\) at time \(t\). Since \(\Delta t\) cannot exceed the recovery time, the proportion of the X-type nodes with \(l > \tau_1\) is zero. Similarly, the equations of \([Y^1 t]_{t+\Delta t}\) can also be obtained.

For the equations of \([AX]^l_{t+\Delta t}\), when \(l = 0\), the first term means that the A-type nodes at right end of \(A - A\) edges fail due to internal causes, which increases the proportion of \(A - X^l\) edges with \(l = 0\) at time \(t + \Delta t\). The second (third) term means that A-type nodes connected with an X-type (Y-type) neighbor fail due to internal causes and meanwhile their X-type (Y-type) neighbor recover spontaneously (i.e., recover because of its recovery time reaching), which increases the proportion of the \(A - X^l\) edges with \(l = 0\). When \(0 < l \leq \tau_1\), the first (second) term denotes that X-type nodes connected with an X-type (Y-type) neighbor recover spontaneously as their recovery time are reached, which increases the proportion of \(A - X^l\) edges where \(l \in (0, \tau_1]\) at time \(t + \Delta t\). The third term denotes that there is no change in the states of the \(A - X^l - \Delta t\) edges, namely the A-type nodes in \(A - X^l - \Delta t\) edges not failing during the time period \([t, t + \Delta t]\). When \(l > \tau_1, [AX]^l_{t+\Delta t}\) is zero.

In the equation of \([AA]^l_{t+\Delta t}\), the first term denotes that, in edges \(X^{\tau_1} - A, A - X^{\tau_1}, Y^{\tau_2} - A\) and \(Y^{\tau_2} - A\), the states of A-type nodes are not changed, but the states of the failed nodes have changed, which increases the proportion of \(A - A\) edges at time \(t + \Delta t\). The second term denotes that both nodes at the end of \(I - l\) edges recover, which increases the proportion of \(A - A\) edges. The third term denotes that the states of both nodes at the end of \(A - A\) edges are not changed.

Similar to the MR model, we use \(E_t, E'_t, E''_t\) as well to represent the probabilities that an active node satisfies the threshold condition \(n \leq m\) in different cases and will not repeat them.

Using pairwise approximation \([UWV]^l_t = [UVW]^l_t [VW]^l_t\), we have \([IA]^l_t / [A]^l_\tau, [IAY]^l_t / [AY]^l_\tau = [AI]^l_t / [AI]^l_\tau\) and \([IAY]^l_t = [A]^l_\tau\), namely \(E_x.t = E'_x.t, E_y.t = E'_y.t, E''_t = E'_t\), and we have

\[ E_t = \sum_{j=0}^{m} C_{k-j}^k \left[ \frac{[AI]^l_t}{[AI]^l_\tau} \right]^{k-j} \left( 1 - \frac{[AI]^l_t}{[AI]^l_\tau} \right)^j, \]  
(30)

\[ E'_t = \sum_{j=0}^{m} C_{k-j}^{k-1} \left[ \frac{[AI]^l_t}{[AI]^l_\tau} \right]^{k-1-j} \left( 1 - \frac{[AI]^l_t}{[AI]^l_\tau} \right)^j, \]  
(31)

and

\[ E''_t = \sum_{j=0}^{m-1} C_{k-j}^{k-1} \left[ \frac{[AI]^l_t}{[AI]^l_\tau} \right]^{k-1-j} \left( 1 - \frac{[AI]^l_t}{[AI]^l_\tau} \right)^j, \]  
(32)

The number of equations is at least \((\frac{n}{2} + 1)^2 + (\frac{n}{2} + 1)^2\), which depends on the magnitude of step length \(\Delta t\) and the values of \(\tau_1\) and \(\tau_2\). In this paper, we take \(\Delta t = 0.1, \tau_1 = 100\) and \(\tau_2 = 1\), and the magnitude of the number of equations is at least \(10^6\).

II. THE RELATIONSHIP BETWEEN THE MR AND THE NMR MODELS

For the MR model, the equations in mean-field(MF) theory are written as

\[ \begin{cases} \frac{d[X]}{dt} = \beta_1 [A]_t - \mu_1 [X]_t, \\ \frac{d[Y]}{dt} = \beta_2 E_1 [A]_t - \mu_2 [Y]_t, \end{cases} \]  
(33)
where
\[
E_t = \sum_{j=0}^{m} C_k^{k-j} ([I]_t)^{k-j} (1 - [I]_t)^j. \quad (34)
\]

For the NMR model, the equations in MF theory are written as
\[
\begin{aligned}
[X]_{t+\Delta t} &= \beta_1 \Delta t[A]_t + [X]_t - [X^{\tau_1}]_t, \\
[Y]_{t+\Delta t} &= \beta_2 \Delta t E_t[A]_t + [Y]_t - [Y^{\tau_2}]_t,
\end{aligned} \quad (35)
\]
where
\[
E_t = \sum_{j=0}^{m} C_k^{k-j} ([I]_t)^{k-j} (1 - [I]_t)^j. \quad (36)
\]

Therefore, when the system reaches steady state (i.e., \( t \to \infty \)), we have \([\dot{A}]_t = 0\), \([\dot{X}]_t = 0\) and \([\dot{Y}]_t = 0\) and Eq. (35) can be written as
\[
\begin{aligned}
\beta_1 [A]_t - [X^{\tau_1}]_t &= 0, \\
\beta_2 E_t[A]_t - [Y^{\tau_2}]_t &= 0.
\end{aligned} \quad (37)
\]

From Eq. (37), we have \([X^{\tau_1}]_t = \beta_1 [A]_t = C_1\) and \([Y^{\tau_2}]_t = \beta_2 E_t[A]_t = C_2\), where \(C_1\) and \(C_2\) are constants. In addition, we have \([X^t]_t = [X^{\tau_1}]_t + \tau_1 - t = C_1\) and \([Y^t]_t = [Y^{\tau_2}]_t + \tau_2 - t = C_2\). Then we can get \([X]_t = \frac{\tau_1}{\tau_1} [X^{\tau_1}]_t\), namely \([X^{\tau_1}]_t = \frac{\tau_1}{\tau_1} [X]_t\). Similarly, we can get \([Y^{\tau_2}]_t = \frac{\tau_2}{\tau_2} [Y]_t\). Therefore, Eq. (37) can be rewritten as
\[
\begin{aligned}
\beta_1 [A]_t - \frac{1}{\tau_1} [X]_t &= 0, \\
\beta_2 E_t[A]_t - \frac{1}{\tau_2} [Y]_t &= 0.
\end{aligned} \quad (38)
\]

Comparing with the steady state of the MR model, that is
\[
\begin{aligned}
\beta_1 [A]_t - \mu_1 [X]_t &= 0, \\
\beta_2 E_t[A]_t - \mu_2 [Y]_t &= 0,
\end{aligned} \quad (39)
\]
we obtain that the steady states for the both models are almost equivalent, when fixing both \(\mu_1 = \frac{1}{\tau_1}\) and \(\mu_2 = \frac{1}{\tau_2}\).

### III. THE TRAJECTORY AND TIME EVOLUTION OF FAILED NODES
FIG. 1. Under different initial value $I_0$, the trajectories and time evolutions of proportions of $X$-type and $Y$-type nodes for the MR and the NMR models. The results are obtained from MF theory. When fixing $[X]_0 = 0.05$ and $[Y]_0 = 0$, the trajectories of $[X]$ and $[Y]$ are shown in (a), and the time evolutions for the MR and NMR models are respectively shown in (b) and (c). Subfigures (d), (e) and (f) are the results of $[X]_0 = 0.2$ and $[Y]_0 = 0$. Subfigures (g), (h) and (i) are the results of $[X]_0 = 0.3$ and $[Y]_0 = 0$. Subfigures (j), (k) and (l) are the results of $[X]_0 = 0.6$ and $[Y]_0 = 0$. The solid blue and red lines in first column are the results of the NMR and MR models, respectively. The light and dark gray dotted lines are the solutions of $\dot{X} = 0$ and $\dot{Y} = 0$ in the MF theory for the MR model, respectively. The intersections of them are the the steady-state solutions for the MR model. The solid blue, orange and green lines in second column and in third column are the results of $I$, $X$ and $Y$ for the MR model and NMR model, respectively. The evolutions from $t_{B'}$ to $t_{C'}$ and then to $t_{D'}$ are too fast to distinguish in (f), (i) and (l). The evolution from $t_{C}$ to $t_{A}$ is also too fast to distinguish in (k) and (l). Other parameters are $\beta_1 = 0.004$, $\beta_2 = 2$, $\mu_1 = 0.01$, $\mu_2 = 1$, $\tau_1 = 100$, $\tau_2 = 1$, $m = 15$ and the network is RRN with $k = 35$. 