ZEROS OF DEDEKIND ZETA FUNCTIONS UNDER GRH

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Abstract. Assuming GRH, we prove an explicit upper bound for the number of zeros of a Dedekind zeta function having imaginary part in \([T-a, T+a]\). We also prove a bound for the multiplicity of the zeros.

1. Introduction and results

Let \(K\) be a number field of dimension \(n_K\) and let \(\Delta_K\) be the absolute value of its discriminant. Let \(n_K(T; a)\) denote the number of zeros \(\rho = \beta + i\gamma\) of the Dedekind zeta function \(\zeta_K\) with \(|\gamma - T| \leq a\) and which are nontrivial, i.e., with \(0 < \beta < 1\).

An upper bound can be deduced via the equality \(n_K(T; a) = \frac{1}{2} \left( N_K(T + a) - N_K(T - a) \right) \) where \(N_K(T)\) counts the nontrivial zeros with imaginary part in \([-T, T]\). In this way from the explicit bound for \(N_K(T)\) recently proved by Trudgian [13] it follows that

\[
(1) \quad n_K(T; a) \leq \left( \frac{a}{\pi} + 0.230 \right) \left( \log \Delta_K + n_K \log T \right) + \text{lower order terms} \quad \forall T > a + 1,
\]

where the remaining terms are explicit, have lower order as a function of \(T\), and can be estimated independently of the discriminant. The constant 0.230 in \(1\) is the parameter \(D_1\) in [11, Table 2] for \(\eta = 10^{-3}\) and is nearly the infimum \(1/(\pi \log 4)\) for \(D_1\). The remainder term in the formula for \(N_K(T)\) is explicit but has the classical size \(O(\log T)\). This changes if one assumes the Generalized Riemann Hypothesis, since the works of Littlewood [8], Selberg [10] and Lang [7] show that in this case the remainder term drops to \(O\left( \frac{\log T}{\log \log T} \right)\), so that now one gets

\[
(2) \quad n_K(T; a) = \frac{a}{\pi} \left( \log \Delta_K + n_K \log T \right) + \text{lower order terms}.
\]

Recent computations of Carneiro, Chandee and Milinovich [9] show that for the Riemann zeta function the constant in the remainder term of \(N_Q(T)\) is \(\left( \frac{1}{4} + o(1) \right) \frac{\log T}{\log \log T}\), at most. Applying Lang’s heuristic [7], the general case should be similar to \(\left( 1 + o(1) \right) \frac{\log \Delta_K + n_K \log T}{\log \log T}\) and thus the lower order terms in \(1\) are probably such that

\[
n_K(T; a) \leq \left( \frac{a}{\pi} + \frac{c}{\log \log T} \right) \left( \log \Delta_K + n_K \log T \right) + \text{lower order terms}
\]

with an absolute constant \(c \lesssim 1\). Due to the very slow decay of the function \(1/\log \log T\), this tentative formula would improve on a result of type \(1\) only for very large \(T\). As a consequence, for numerical applications it is interesting to work...
out a totally explicit bound for \( n_{K}(T; a) \) under GRH, with an asymptotically non-optimal but small constant in front of the main term \( \log \Delta_{K} + n_{K} \log T \), and possibly small constants in every other position. In this spirit, in this paper we prove the following results: the first one for the zeros in the window \([T-a, T+a]\), the second for the multiplicity of a zero.

**Theorem 1.** Assume GRH. Then one has

\[
\begin{align*}
\tag{3} n_{K}(T; a) & \leq \frac{a}{2} \tilde{f}_{K}\left(\frac{1}{2} + \frac{a}{4} + iT\right) \quad \forall a \in (0, 2), \; T \geq 10 + a \\
\end{align*}
\]

and

\[
\begin{align*}
\tag{4} n_{K}(T; 0^{+}) & \leq \frac{3}{10}(2\sigma - 1)\tilde{f}_{K}(\sigma + iT) \quad \forall \sigma \in \left(\frac{1}{2}, 1\right), \; T \geq 10, \\
\end{align*}
\]

where

\[
\tilde{f}_{K}(\sigma + iT) := Q + 2\left(\frac{n_{K}}{1 - \sigma} + \frac{\log(\frac{\pi - 1}{\pi})}{\sigma} + \frac{0.64}{2\sigma - 1} + 1.37\right)Q^{2 - 2\sigma} + \left(\frac{0.14}{2\sigma - 1} - 20\right)n_{K} \\
\]

and \(Q := \log \Delta_{K} + (\log T + 20)n_{K} + 11\).

The main term of the bound (3) is \(\frac{a}{2}(\log \Delta_{K} + n_{K}\log T)\). It improves on the bound (1) for \(n_{K}(T; a)\) whenever \(a \leq 1.265\ldots\). For example, we get

\[
n_{K}\left(T; \frac{1}{2}\right) \leq \frac{1}{4}Q + (1.4n_{K} + 2.2)Q^{3/4} - 4n_{K} \quad \forall T \geq 10.5
\]

and

\[
n_{K}(T; 1) \leq \frac{1}{2}Q + (4n_{K} + 2.9)\sqrt{Q} - 9n_{K} \quad \forall T \geq 11.
\]

The bound (4) for the multiplicity is stronger than what we can deduce from (3) in the limit \(a \to 0^{+}\) (to compare the results take \(a = (4\sigma - 2)\) in (3)). Moreover, every \(\sigma < 0.883\) in (4) improves on what one can deduce from (1) in the limit \(a \to 0^{+}\); for example, for \(\sigma = 3/4\) we get

\[
n_{K}(T; 0^{+}) \leq \frac{3}{20}Q + (1.2n_{K} + 0.9)\sqrt{Q} - 2.9n_{K} \quad \forall T \geq 10.
\]

However, for a better result the form of (4) suggests to try with a \(\sigma\) such that \(2\sigma - 1 \to 0\). In fact, a proper choice of \(\sigma\) proves the following claim.

**Corollary 2.** Assume GRH. Let \(Q\) be as in Theorem 1 and let \(L := \log Q\). Suppose \(T \geq 10\), then

\[
n_{K}(T; 0^{+}) \leq \left(0.3\log L + 0.4 + 0.2\log^{2}\frac{L}{L} + \log^{L}\frac{L}{L}(1.9n_{K} + 0.9)\right)\frac{Q}{\log Q}.
\]

**Proof.** Let \(\epsilon := 2\sigma - 1\). From (4) we get

\[
\frac{10}{3Q}n_{K}(T; 0^{+}) \leq \epsilon + \left(\frac{4\epsilon}{1 - \epsilon}n_{K} + \frac{2}{\pi}\epsilon \log \epsilon + 2.76\epsilon + 1.28\right)Q^{-\epsilon} + \frac{0.14n_{K}}{Q}
\]

and setting \(\epsilon = \frac{\log L}{\log Q}\) (and using \(T \geq 10 \implies Q \geq 33 \implies \epsilon \leq 0.36\)) we get

\[
\frac{10}{3Q}n_{K}(T; 0^{+}) \leq \frac{\log L}{\log Q} + \left(6.3\frac{\log L}{\log Q}n_{K} + 2\frac{\log^{2}L}{\pi \log Q} + 2.76\frac{\log L}{\log Q} + 1.28\right)\frac{1}{\log Q}. \quad \square
\]
As recalled before, we already know that under GRH the multiplicity of \( \frac{1}{2} + iT \) is \( O\left( \frac{\log T}{\log \log T} \right) \), thus Corollary 2 is weaker than the best known result, but only by the presence of an extra \( \log L \), i.e., a triple log in \( T \), in the numerator. Moreover, it is uniform in \( K \), and totally explicit.

Due to the presence of the extra factor \( \log L \), every explicit bound of the form \( n_K(T; 0^+) \leq c \frac{\log T}{\log \log T} \) based on Corollary 2 holds only for a bounded range; nevertheless, the following result shows that for the Riemann zeta function this range is extremely large, even for a small value of \( c \).

**Corollary 3.** Assume RH and \( \log T \leq 10^{70593} \). Then

\[
n_Q(T; 0^+) \leq 4 \frac{\log T}{\log \log T},
\]

**Proof.** By Corollary 2 it is sufficient to prove that

\[
\left( \frac{3 \log L}{10} + 2.8 \frac{\log L}{L} + 0.2 \frac{\log^2 L}{L} + 0.4 \right) \frac{Q}{\log Q} \leq 4 \frac{\log T}{\log \log T}
\]

with \( Q = \log T + 31 \) and \( L = \log Q \). Taking account of the fact that the zeros \( \rho \) of the Riemann zeta function with \( |\Im(\rho)| < 10^{10} \) are simple \( [2] \) (actually this has been verified up to \( 10^{12} \) \( [4] \), but we prefer to base our result on a doubly checked computation), we may assume \( \log T \geq 23 \). In terms of \( T \), Inequality (5) is difficult, however, we can verify that both

\[
\frac{3}{10} \log L + 2.8 \frac{\log L}{L} + 0.2 \frac{\log^2 L}{L} + 0.4 \leq 2
\]

and

\[
\frac{Q}{\log Q} \leq 2 \frac{\log T}{\log \log T}
\]

hold for \( 23 \leq \log T \leq 10^{55} \) (the first one as a function of \( L \), the second one as a function of \( \log T \)). Thus we can assume \( \log T \geq 10^{55} \). Under this hypothesis (5) is implied by

\[
\frac{3}{10} \log L + 2.8 \frac{\log L}{L} + 0.2 \frac{\log^2 L}{L} + 0.4 \leq 4(1 - 10^{-10})
\]

which holds for \( L \leq 162546.6 \).



Theorem 1 is proved in two steps, following an idea which we have introduced in \([5]\): let \( f_K(s) := \sum \Re\left( \frac{2}{s-\rho} \right) \), where the sum is on the set of nontrivial zeros of \( \zeta_K \). First we exploit the fact that the terms appearing in the sum defining \( f_K \) are all positive and depend on the zeros of \( \zeta_K \) to find a suitable combination of values of \( f_K \) providing an upper bound for \( n_K(T; a) \) and \( n_K(T; 0^+) \); then we bound \( f_K(s) \) with \( \tilde{f}_K(s) \) in the critical strip. Both steps depend on GRH.

To bound \( f_K \) we will use a preliminary explicit upper bound for \( n_K(T; 1) \) which we deduced from a crude version of \([1]\). A virtuous circle appears here because the argument could be iterated producing better and better bounds. However, preliminary considerations suggest that the improvement is quite marginal and affects only the secondary constants.
For \( \text{Re}(s) > 1 \) we have
\[
-\frac{\zeta'_K}{\zeta_K}(s) = \sum_{n=1}^{\infty} \tilde{\Lambda}_K(n)n^{-s} \quad \text{with} \quad \tilde{\Lambda}_K(n) = \begin{cases} \sum_{p|n, f_p|k} \log Np & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases}
\]
where \( p \) is a prime number, \( p \) a prime ideal in \( K \) above \( p \), \( Np \) its absolute norm and \( f_p \) its residual degree. The formula for \( \tilde{\Lambda}_K \) shows that \( \tilde{\Lambda}_K(n) \leq n_K\Lambda(n) \) for every integer \( n \), so that
\[
\left| \frac{\zeta'_K}{\zeta_K}(s) \right| \leq -n_K \frac{\zeta'}{\zeta}(\sigma) \quad \forall \sigma = \text{Re}(s) > 1.
\]

The functional equation for \( \zeta_K \) reads
\[
\xi_K(1 - s) = \xi_K(s),
\]
where
\[
\xi_K(s) := s(s - 1)\Delta_K^{s/2} \Gamma_K(s)\zeta_K(s)
\]
and, with \( r_1 \) and \( r_2 \) the number of real and complex embeddings of \( K \),
\[
\Gamma_K(s) := \left[ \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \right]^{r_2} \left[ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right]^{r_1+r_2}.
\]
Since \( \xi_K(s) \) is an entire function of order 1 and does not vanish at \( s = 0 \), one has
\[
\xi_K(s) = e^{A_K+B_Ks} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho}
\]
for some constants \( A_K \) and \( B_K \), where \( \rho \) runs through all the zeros of \( \xi_K(s) \), which are precisely those zeros \( \rho = \beta + i\gamma \) of \( \zeta_K \) for which \( 0 < \beta < 1 \). We recall that the zeros are symmetric with respect to the real axis, as a consequence of the fact that \( \zeta_K(s) \) is real for \( s \in \mathbb{R} \).

Differentiating (8) and (10) logarithmically we obtain the identity
\[
\frac{\zeta'_K}{\zeta_K}(s) = B_K + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) - \frac{1}{2} \log \Delta_K - \left( \frac{1}{s} + \frac{1}{s - 1} \right) - \frac{\Gamma'_{\xi_K}}{\Gamma_{\xi_K}}(s),
\]
valid identically in the complex variable \( s \).

Stark [11, Lemma 1] proved that the functional equation (7) implies that \( B_K = -\sum_{\rho} \text{Re}(\rho^{-1}) \), and that once this information is available one can use (11) and the definition of the gamma factor in (9) to prove that the function \( f_K(s) := \sum_{\rho} \text{Re}\left(\frac{2}{s - \rho}\right) \) can be exactly computed via the alternative representation
\[
f_K(s) = 2\text{Re}\frac{\zeta'_K}{\zeta_K}(s) + \log \frac{\Delta_K}{\pi^{n_K}} + \text{Re}\left(\frac{2}{s} + \frac{2}{s - 1}\right) + (r_1+r_2)\text{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + r_2\text{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right).
\]

The relevance of this function for our problem comes from two facts: it is a sum on zeros each one appearing with the weight \( \text{Re}(\frac{2}{s - \rho}) \) which is positive under GRH whenever \( \text{Re}(s) > \frac{1}{2} \), and it can be computed via the alternative formula (12) which does not involve the zeros. For example, assuming GRH we get
\[
n_K(T; a) \leq \frac{1}{c(\sigma)} \sum_{\rho} \text{Re}\left(\frac{2}{\sigma + iT - \rho}\right) = \frac{1}{c(\sigma)} f_K(\sigma + iT) \quad T > a
\]
with \( c(\sigma) := \frac{2\sigma-1}{(\sigma-\frac{1}{2})^2 + a^2} \), which is a lower bound for the weight of the zeros counted by \( n_K(T; a) \). By (12) the part depending on the discriminant in \( c(\sigma)^{-1}f_K(\sigma + iT) \) is simply \( c(\sigma)^{-1} \log |\Delta_K| \), hence to bound the contribution of this parameter to \( n_K(T; a) \) we need to choose \( \sigma \) such that \( c(\sigma) \) is maximum. This happens when \( \sigma = \frac{1}{2} + a \), giving the bound

\[
(13) \quad n_K(T; a) \leq af_K\left(\frac{1}{2} + a + iT\right) \quad \forall T > a.
\]

Let \( \frac{1}{2} + iT \) be a zero for \( \zeta_K \), and let \( \nu \) be its multiplicity. Then \( f_K\left(\frac{1}{2} + a + iT\right) \sim 2\nu a^{-1} \) as \( a \to 0 \). Thus the previous formula overestimates the multiplicity of zeros by a factor two. The following argument improves (13) adding greater flexibility to the choice of the weight. Let \( g: \mathbb{R} \to \mathbb{R}^+ \) be any map and \( \mu \) be a measure in \( \mathbb{R} \) such that

\[
(14) \quad g(\gamma) \leq \int_{\mathbb{R}} \frac{d\mu(t)}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \quad \forall \gamma \in \mathbb{R}.
\]

Then summing over all zeros and assuming GRH we have

\[
\sum_\rho g(\gamma) \leq \sum_\rho \int_{\mathbb{R}} \frac{d\mu(t)}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} = \frac{1}{2\sigma - 1} \int_{\mathbb{R}} \sum_\rho \frac{2\sigma - 1}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} d\mu(t),
\]

producing the bound

\[
(15) \quad \sum_\rho g(\gamma) \leq \frac{1}{2\sigma - 1} \int_{\mathbb{R}} f_K(\sigma + it) d\mu(t).
\]

Moreover, suppose that \( \mu \) is symmetric with respect to a point \( t_0 \) and such that for some real \( c \) the measure \( \mu + c\delta_{t_0} \) is positive, where \( \delta_{t_0} \) is the Dirac measure at \( t_0 \). Furthermore, let \( f_K(\sigma + it) \) be a concave upper bound for \( f_K(\sigma + it) \) in the support of \( \mu \), then we immediately deduce that

\[
(16) \quad \sum_\rho g(\gamma) \leq \frac{\mu(\mathbb{R})}{2\sigma - 1} f_K(\sigma + it_0).
\]

The argument has a variational flavor: finding the minimum for \( \frac{\mu(\mathbb{R})}{2\sigma - 1} \) in the set of (symmetric around \( t_0 \) and) positive (outside \( t_0 \)) measures \( \mu \) satisfying (14).

We apply the argument with \( g(\gamma) := \chi_{|T-a,T+a|}(\gamma) \) and \( T > a \), so that

\[
\sum_\rho g(\gamma) = n_K(T; a).
\]

We have experimented with several possible measures, but actually our best results come from a very simple choice. In fact, we set

\[
(17) \quad d\mu(t) := \sum_{j=-2}^{2} c_j \delta_{T-b_j}(t),
\]

the sum of five of Dirac’s deltas, with \( c_{-j} = c_j \) and \( b_{-j} = -b_j \) for every \( j \), in order to make \( d\mu \) symmetric around \( T \). With this choice (14) becomes

\[
(18) \quad \chi_{[-a,a]}(\gamma) \leq \sum_{j=-2}^{2} \frac{c_j}{\alpha^2 + (\gamma - b_j)^2} \quad \forall \gamma \in \mathbb{R},
\]
where $\alpha := \sigma - \frac{1}{2}$; we have also removed the parameter $T$ via the shift $T + \gamma \to \gamma$.

We are interested in a combination of parameters producing a small value for

$$\frac{\mu(\mathbb{R})}{2\sigma - 1} = \frac{c_0 + 2c_1 + 2c_2}{2\alpha}.$$  

For the moment we have not yet determined any set of values for the parameters, however, we can make a simple test proving that our strategy has a good chance of producing something interesting. Suppose that $1/2 + iT$ is a zero and let $\nu$ be its multiplicity, then $f_{\nu}(s) \sim 2\Re(\nu(s - 1/2 - iT)^{-1})$ as $s$ goes to $1/2 + iT$. By (15) with the measure (17) and letting $\sigma \to 1/2$ we get

$$n_{\nu}(T; 0^+) \leq \frac{1}{2\sigma - 1} \sum_{j=-2}^{2} c_j f_{\nu}(\sigma + iT - ib_j) = \nu \sum_{j=-2}^{2} \frac{c_j}{\alpha^2 + b_j^2} + R(\sigma),$$

where the remainder $R(\sigma)$ is $O(\sum |b_j|)$. Hence the function on the right-hand side of (19) is substantially $\nu$ if we require that the constants $c_j$ produce an equality in (18) when $\gamma = 0$ and the $b_j$’s are small. In this way we improve on what comes from the elementary argument (13).

We have six parameters: $\alpha$, $b_1$ and $b_2$ and the three $c_j$’s. Equation (18) shows a homogeneity in $a$: once we have found a set of parameters $\alpha$, $b_j$ and $c_j$ for $a = 1$, the parameters $a\alpha$, $ab_j$ and $a^2c_j$ can be used for any $a$. We thus suppose $a = 1$. Moreover, we set $\alpha = \frac{1}{2}$ and to fix the value of the other parameters we impose the equality in (18) for $\gamma = 0$ (ensuring the good asymptotic estimate for the multiplicity), $\gamma = \pm \frac{3}{5}$ and $\gamma = \pm 1$: values $\frac{1}{4}$ for $\alpha$ and $\pm \frac{3}{5}$ have been chosen by trial and error and produce our best result, but are essentially arbitrary. The equalities form a linear system in the $c_j$’s which may be explicitly solved in terms of $b_1$ and $b_2$.

Then we impose two extra conditions: the first one requiring that the zero of the difference of both sides of (18) in $\gamma = \pm \frac{3}{5}$ has at least order two, the second one requiring that the zero in $\gamma = 0$ has at least order four. Due to the symmetry, these conditions correspond to the following two equations:

$$\begin{cases}
\sum_{j=-2}^{2} \frac{c_j(\frac{3}{5} - b_j)}{(\frac{1}{16} + (\frac{3}{5} - b_j)^2)^2} = 0, \\
\sum_{j=-2}^{2} \frac{c_j(\frac{1}{16} - 3b_j^2)}{(\frac{1}{16} + b_j^2)^3} = 0.
\end{cases}$$

(20)

Observe that the equation

$$\sum_{j=-2}^{2} \frac{c_j}{\frac{1}{16} + (\gamma - b_j)^2} = 1$$

(21)

(which is an algebraic equation in $\gamma$ of degree 10) has at least the roots $0$ (multiplicity $\geq 4$), $\pm \frac{3}{5}$ (multiplicity $\geq 2$, each) and $\pm 1$ (multiplicity $\geq 1$, each). Hence there are no other roots, and comparing the two sides of (21) as $\gamma \to \infty$ we conclude that (18) holds in $[-1, 1]$. Moreover, (18) is also true for $|\gamma| > 1$ if the $c_j$’s are not negative.

Equation (20) can be seen as an algebraic system in the variables $b_1^2$ and $b_2^2$. Using the resultant of the polynomials, we can check that there is a unique solution
such that $0 \leq b_1 \leq b_2$ and it is

$$b_1 = 0.355\ldots, \quad b_2 = 0.875\ldots,$$

giving

$$c_1 = 0.0200\ldots, \quad c_2 = 0.0491\ldots, \quad c_3 = 0.0651\ldots.$$

For this combination we get

$$\frac{\mu(\mathbb{R})}{2\sigma - 1} \leq \frac{1}{2}$$

such that, coming back to generic $a$, via (16) yields

$$(22) \quad n_K(T; a) \leq \frac{a}{2} \tilde{f}_K\left(\frac{1}{2} + \frac{a}{4} + iT\right) \quad \forall T > a,$$

where for the moment $\tilde{f}_K$ denotes any concave upper bound of $f_K$.  

Our second result (4) is proved with a slightly different argument. It needs the formulas for the $c_j$’s to be made explicit in terms of the other parameters, thus we further simplify our definition of the measure imposing $c_2 = 0$ in (17), i.e., setting

$$d\mu(t) := c_1 \delta_{T+b}(t) + c_0 \delta_T(t) + c_1 \delta_{T-b}(t).$$

We fix the $c_j$’s in such a way as to get an equality in (18) for $\gamma = 0$ and $\gamma = \pm a$; this happens for

$$(23) \quad \left\{ \begin{array}{l}
c_0 = -\alpha^6 + (3b^2 - 2a^2)\alpha^4 + (3b^2 - a^2)a^2\alpha^2, \\
c_1 = \alpha^6 + (2a^2 + 3b^2)\alpha^4 + (a^4 + 3b^4)\alpha^2 + (a^2 - b^2)^2b^2.
\end{array} \right.$$

The formulas in (23) show that $c_1$ is always positive, but that $c_0$ may be negative for some values of the parameters. However, for these $c_j$’s the function appearing on the right-hand side of (18) can be written as the sum of squares and thus is always positive. As a consequence only the range $[-a, a]$ has to be considered for (18). Previously we have taken advantage of the homogeneity of the problem in the $a$ parameter, but for the present application it is useful to work out a formula allowing the limit $a \to 0$ without simultaneously sending $\alpha$ to 0. As a consequence we set $b^2 = \frac{a^2}{2}$, giving

$$\left\{ \begin{array}{l}
c_0 = -2\frac{(2a^2 - a^2)(\alpha^2 + a^2)\alpha^2}{a^2(10\alpha^2 + 3a^2)}, \\
c_1 = \frac{(2a^2 + a^2)(4a^4 + 12a^2\alpha^2 + a^4)}{4a^2(10\alpha^2 + 3a^2)},
\end{array} \right.$$

which produces

$$\frac{c_0 + 2c_1}{2\alpha} = \frac{24a^4 + 18a^2\alpha^2 + a^4}{4(10\alpha^2 + 3a^2)\alpha}$$

without assuming any proportionality between $\alpha$ and $a$. Once again the choice $b^2 = \frac{a^2}{2}$ is the result of a trial and error procedure. Inequality (18) holds as the equality

$$\frac{c_1}{\alpha^2 + (\gamma - \frac{a}{\sqrt{2}})^2} + \frac{c_0}{\alpha^2 + \gamma^2} + \frac{c_1}{\alpha^2 + (\gamma + \frac{a}{\sqrt{2}})^2} = 1$$
in six complex points (multiplicity included). The definitions of the $c_j$’s give the roots $\gamma = 0$ (multiplicity $\geq 2$) and $\gamma = \pm a$; the remaining roots solve

$$\gamma^2 = \frac{a^4 - 36\alpha^4}{20a^2 + 6a^2}.$$ 

Inequality (18) is satisfied in the range $\gamma \in [-a, a]$ if and only if these two extra solutions are either 0 or nonreal. This is what happens as long as $a^2 < 6\alpha^2$, and since $n_K(T; 0^+) \leq n_K(T; a)$ for every $a$, we deduce that

$$n_K(T; 0^+) \leq \frac{24a^4 + 18a^2\alpha^2 + a^4}{4(10\alpha^2 + 3a^2)\alpha} \tilde{f}_K(\sigma + iT),$$

where again $\tilde{f}_K$ denotes any concave upper bound of $f_K$. Sending $a \to 0$ on the right-hand side we conclude that

$$n_K(T; 0^+) \leq \frac{3}{10}(2\sigma - 1) \tilde{f}_K(\sigma + iT) \quad \forall T > 0, \quad \forall \sigma > \frac{1}{2}.$$ 

In the next section we will prove that $f_K(\sigma + it) \leq \tilde{f}_K(\sigma + it)$ where $\tilde{f}_K$ is the function given in Theorem 1. With (16) this suffices to prove (3) and (4) from (22) and (24) respectively, because $\tilde{f}_K(\sigma + it)$ is a concave map in the $t$ variable.

### 3. Bounds in the critical strip

The following two lemmas collect some elementary inequalities involving the gamma function which we will need later, their proofs are in the Appendix.

**Lemma 4.** Let $s = \sigma + it$ with $\sigma \geq 0$ and $|t| \geq \sigma + 2$. Then

$$\text{Re}\left(\frac{\Gamma'}{\Gamma}(s)\right) \leq \log |s - 1/2|.$$ 

**Lemma 5.** Let $u \in [-3/4, -1/4]$, then

$$\int_R |\Gamma(u + iy)| dy \leq 4.73.$$ 

Furthermore, suppose $|t| \geq 10$, then

$$\int_R |\Gamma(u + i(t - y))| \log(1 + |y|) dy \leq 4.73 \log(1 + |t|),$$

$$\int_R \frac{|\Gamma(u + i(t - y))|}{|1 + 4iy|} dy \leq 0.171,$$

$$\int_R \frac{|\Gamma(u + i(t - y))|}{1 + \alpha y^2} dy \leq \begin{cases} 0.013 & \text{if } \alpha = 1, \\ 0.007 & \text{if } \alpha = 2. \end{cases}$$

Let $R_K(T)$ be defined via

$$N_K(T) =: \frac{T}{\pi} \log \left(\frac{T}{2\pi e}\right)^{n_K} \Delta_K + R_K(T).$$

Trudgian [13, Th. 2] proved unconditionally that for every $T \geq 1$ one has

$$|R_K(T)| \leq \tilde{R}_K(T) := d_1 W_K(T) + d_2 n_K + d_3$$

with

$$W_K(T) := \log \Delta_K + n_K \log \left(\frac{T}{2\pi}\right).$$
and \( d_1 = 0.317, d_2 = 6.333 + 0.317 \log(2\pi) \leq 6.9157 \) and \( d_3 = 3.482 \). We use this result first to bound \( n_K(t; 1) \), and later to bound certain finite sums over zeros (see Lemma 7 below).

**Lemma 6.** For \( t \in \mathbb{R} \),

\[
\begin{align*}
(28a) \quad n_K(t; 1) &\leq 0.636 W_K(t) + 6.92 n_K + 3.49 & \forall |t| > 1, \\
(28b) \quad n_K(t; 1) &\leq 0.954 \log \Delta_K + 5.19 n_K + 3.49 & \forall |t| \leq 1.
\end{align*}
\]

**Proof.** The symmetry of roots allows us to assume \( t \geq 0 \). For \( t \geq 2 \), using (27):

\[
n_K(t; 1) = \frac{1}{2}(N_K(t + 1^+) - N_K(t - 1^-))
\]

\[
\leq \frac{1}{2 \pi} ((t + 1) \log \left( \left( \frac{t + 1}{2 \pi e} \right)^n \Delta_K \right) - (t - 1) \log \left( \left( \frac{t - 1}{2 \pi e} \right)^n \Delta_K \right))
\]

\[+ d_1 \left( W_K(t + 1) + W_K(t - 1) \right) + d_2 n_K + d_3.\]

Since \((t + 1) \log(t + 1) - (t - 1) \log(t - 1) \leq 2(1 + \log t)\) and \( W_K(t) \) is a concave map, we get

\[
\leq \frac{1}{2 \pi} (2n_K(1 + \log t) + 2 \log \Delta_K - 2n_K \log(2\pi e)) + d_1 W_K(t) + d_2 n_K + d_3
\]

\[= \left( \frac{1}{\pi} + d_1 \right) W_K(t) + d_2 n_K + d_3.\]

Introducing the values for \( d_j \)'s we get (28a) for \( t \geq 2 \).

For \( 1 < t \leq 2 \), we have

\[
n_K(t; 1) \leq \frac{1}{2} N_K(t + 1^+) \leq \frac{1}{2} N_K(3)
\]

\[
\leq \left( \frac{3}{2 \pi} + \frac{d_1}{2} \right) \log \Delta_K + \left( \frac{3}{2 \pi} \log \left( \frac{3}{2 \pi e} \right) + \frac{d_1}{2} \log \left( \frac{3}{2 \pi} \right) + \frac{d_2}{2} \right) n_K + \frac{d_3}{2}.
\]

This bound is a bit smaller than what we get extrapolating to \( t \in (1, 2] \) the formula we have found for \( t > 2 \). Not needing its full strength, we estimate \( n_K(t; 1) \) for \( t \in (1, 2] \) with that bound, thus proving (28a).

For \( 0 \leq t \leq 1 \), we have

\[
n_K(t; 1) \leq N_K(t + 1^+) \leq N_K(2)
\]

\[
\leq \left( \frac{2}{\pi} + d_1 \right) \log \Delta_K + \left( \frac{2}{\pi} \log \left( \frac{1}{\pi e} \right) + d_1 \log \left( \frac{1}{\pi} \right) + d_2 \right) n_K + d_3
\]

giving (28b) when the values for \( d_j \)'s are introduced. \( \square \)

**Lemma 7.** Let \( c > 0 \) and \( t \in \mathbb{R} \), with \( |t| > c + 1 \). Let \( u > 0 \), then

\[
\sum_{|\gamma - t| \leq c} \frac{1}{|u + i(\gamma - t)|} \leq \left( \frac{\text{arcsinh}(c/u)}{\pi} + \frac{d_1}{u} \right) W_K(|t|) + \frac{d_2 n_K + d_3}{u}.
\]

**Proof.** Without loss of generality, we can assume \( t > 0 \). We write the sum as an integral in the density of zeros:

\[
\sum_{|\gamma - t| \leq c} \frac{1}{|u + i(\gamma - t)|} = \int_{t-c}^{t+c} \frac{d(N_K(\gamma))}{2|u + i(\gamma - t)|}.
\]
where the factor $\frac{1}{2}$ appears because only zeros with positive imaginary part matter, since $t - c > 1$. By (26) this is

$$= \int_{-c}^c \frac{W_K(\gamma + t)}{2\pi |u + i\gamma|} \, d\gamma + \int_{-c}^{c^+} \frac{d(R_K(\gamma + t))}{2|u + i\gamma|}. $$

$W_K$ is a concave map, thus

$$\int_{-c}^c \frac{W_K(\gamma + t)}{2\pi |u + i\gamma|} \, d\gamma \leq \int_{-c}^c \frac{d\gamma}{2\pi |u + i\gamma|} W_K(t) = \frac{\text{arcsinh}(c/u)}{\pi} W_K(t).$$

Moreover, integrating by parts and using the upper bound $|R_K| \leq \tilde{R}_K$ in (27) we get

$$\left| \int_{-c}^{c^+} \frac{d(R_K(\gamma + t))}{2|u + i\gamma|}\right| \leq \frac{\tilde{R}_K(t + c) + \tilde{R}_K(t - c)}{2(u + c^2)^{1/2}} + \frac{1}{2} \int_{-c}^c \frac{\tilde{R}_K(\gamma + t) |\gamma| \, d\gamma}{(u^2 + \gamma^2)^{3/2}}$$

and since $\tilde{R}_K$ is also a concave map for positive arguments, we get

$$\leq \frac{\tilde{R}_K(t)}{u(u + c^2)^{1/2}} + \frac{\tilde{R}_K(t)}{2} \int_{-c}^c \frac{|\gamma| \, d\gamma}{(u^2 + \gamma^2)^{3/2}} = \frac{\tilde{R}_K(t)}{u}. \quad \square$$

**Lemma 8.** For every real $t$ one has

$$\left| \frac{\Gamma'_K}{\Gamma_K} \left( \frac{1}{4} + it \right) - \frac{\Gamma'_K}{\Gamma_K} \left( 2 + it \right) \right| \leq \frac{10n_{GK}}{|1 + 4it|^2}.$$  

**Proof.** From the definition of $\Gamma_K$ in (39) we have

$$\frac{\Gamma'_K}{\Gamma_K} \left( \frac{1}{4} + it \right) - \frac{\Gamma'_K}{\Gamma_K} \left( 2 + it \right) = r_2 \left( \Gamma \left( \frac{5}{8} + i\frac{t}{2} \right) - \Gamma \left( \frac{3}{2} + i\frac{t}{2} \right) \right)$$

$$+ (r_1 + r_2) \left( \Gamma \left( \frac{1}{8} + i\frac{t}{2} \right) - \Gamma \left( 1 + i\frac{t}{2} \right) \right).$$

From the functional equation $s\Gamma(s) = \Gamma(s + 1)$ we get

$$\Gamma \left( \frac{8}{5} + i\frac{t}{2} \right) \frac{\Gamma'}{\Gamma} \left( 1 + i\frac{t}{2} \right) = -\frac{8}{1 + 4it} + \Gamma \left( \frac{9}{8} + i\frac{t}{2} \right) - \Gamma \left( 1 + i\frac{t}{2} \right),$$

now we proceed as in Lemma 4 from the Euler–Maclaurin summation formula for $\log \Gamma(s)$ the difference of the logarithmic derivatives is at most

$$\left| \log \left( \frac{8}{5} + i\frac{t}{2} \right) \right| + \frac{1}{2} \frac{1/8}{(8 + 1/4)} + \frac{1}{2} \frac{1/12}{(8 + 1/2)^2} + \frac{1}{3} \int_0^{+\infty} \frac{du}{u + 1 + it/2|}.$$  

The absolute value of the logarithm is lower than $-\log \left( 1 - \frac{1/8}{1 + it/2} \right)$, and the integral is $4/(4 + t^2 + 2\sqrt{4 + t^2})$. Using these bounds one proves that (29) is bounded by $10/|1 + 4it|$. The same argument applied to the other difference of gamma functions completes the proof. \square

In order to prove that $f_K(\sigma + it) \leq \tilde{f}_K(\sigma + it)$ we need an upper bound for $-\frac{\zeta_K}{\zeta_K}(s)$ in the critical strip, and for this purpose we follow the argument used in [12, Th. 14.4] for the Riemann zeta function. In our setting, however, the argument will be considerably complicated by the need of good explicit constants. Let $\sigma = \text{Re}(s) \in (\frac{1}{2}, 1)$ and let $\delta$ be a parameter in $(0, 1)$. We get

$$-\sum_{n=1}^{\infty} \frac{\Lambda_K(n)}{n^s} e^{-\delta n} = \frac{1}{2\pi i} \int_{-2-i\infty}^{2+i\infty} \frac{\zeta'_K(z)}{\zeta_K} \Gamma(z - s) \delta^{s-z} \, dz.$$
Moving the integration line to \( \text{Re}(z) = \frac{1}{4} \) we get the equality

\[
-\frac{\zeta_K^I}{\zeta_K}(s) = \sum_{n=1}^{\infty} \frac{\tilde{\Lambda}_K(n)}{n^s} e^{-\delta n} - \Gamma(1-s) \delta^{s-1} + \sum_{\rho} \Gamma(\rho-s) \delta^{s-\rho} + \frac{1}{2\pi i} \int_{1/4-i\infty}^{1/4+i\infty} \frac{\zeta_K'}{\zeta_K}(z) \Gamma(z-s) \delta^{s-z} \, dz
\]

(30)

here \( I \) is the value of the original integral, \( II \) comes from the pole of \( \zeta_K \) at \( z = 1 \), \( III \) from the nontrivial zeros, and \( -\zeta_K^I/(z) \) from the pole of \( \Gamma(z-s) \) at \( z = s \): the Cauchy theorem is applicable here since \( -\zeta_K^I/(z) \) grows polynomially along the vertical lines, while the gamma function decays exponentially. The following lemmas provide bounds for \( I-IV \) and will be combined into a suitable bound for \( \frac{\zeta_K^I}{\zeta_K}(s) \) in Lemma 15.

**Lemma 9 (Bound of I).** Assume RH. Let \( \sigma \in (\frac{1}{2}, 1) \) and \( \delta > 0 \). Then

\[
\sum_{n=1}^{\infty} \frac{\tilde{\Lambda}_K(n)}{n^\sigma} e^{-\delta n} \leq n_K \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} e^{-\delta n} \leq \left( \frac{\delta^{\sigma-1}}{1-\sigma} + \frac{0.07}{2\sigma - 1} + 4 \right) n_K.
\]

**Proof.** The first inequality is an immediate consequence of the inequality \( \tilde{\Lambda}_K(n) \leq n_K \Lambda(n) \). For the second one, let \( \psi^{(1)}(x) := \int_0^x \psi(u) \, du = \sum_{n \leq x} \Lambda(n) (x-n) \). Then

\[
\left| \psi^{(1)}(x) - \frac{x^2}{2} \right| \leq 0.0462x^{3/2} + 1.838x \quad \forall x \geq 1,
\]

(see [5], Th. 1.1); there the claim is stated for \( x \geq 3 \), but it actually holds for \( x \geq 1 \). Thus, by partial summation we get

\[
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} e^{-\delta n} = \int_{2^{-}}^{+\infty} \psi^{(1)}(x) \left[ \frac{e^{-\delta x}}{x^\sigma} \right]'' \, dx
\]

which we write as

\[
= \int_{2^{-}}^{+\infty} \frac{x^2}{2} \left[ \frac{e^{-\delta x}}{x^\sigma} \right]'' \, dx + \int_{2^{-}}^{+\infty} \left[ \psi^{(1)}(x) - \frac{x^2}{2} \right] \left[ \frac{e^{-\delta x}}{x^\sigma} \right]'' \, dx
\]

\[
= (2 + \sigma + 2\delta) \frac{e^{-2\delta}}{2\sigma} + \delta^{\sigma-1} \int_{2^{-}}^{+\infty} \frac{e^{-u}}{u^\sigma} \, du + \int_{2^{-}}^{+\infty} \left[ \psi^{(1)}(x) - \frac{x^2}{2} \right] \left[ \frac{e^{-\delta x}}{x^\sigma} \right]'' \, dx
\]

\[
\leq (2 + \sigma + 2\delta) \frac{e^{-2\delta}}{2\sigma} + \delta^{\sigma-1} (1-\sigma) + \int_{2^{-}}^{+\infty} \left[ \psi^{(1)}(x) - \frac{x^2}{2} \right] \left[ \frac{e^{-\delta x}}{x^\sigma} \right]'' \, dx.
\]

The function \( e^{-\delta x} x^{-\sigma} \) is completely monotone, thus

\[
\int_{2^{-}}^{+\infty} x^{3/2} \left[ \frac{e^{-\delta x}}{x^\sigma} \right]'' \, dx = \int_{2^{-}}^{+\infty} x^{3/2} \left[ \frac{e^{-\delta x}}{x^\sigma} \right]'' \, dx
\]

\[
= \sqrt{2} \left( \frac{3}{2} + \sigma + 2\delta \right) \frac{e^{-2\delta}}{2\sigma} + \frac{3}{4} \int_{2^{-}}^{+\infty} e^{-\delta x} x^{-1/2-\sigma} \, dx.
\]

The last integral is at most \( \min(e^{-2\delta} \delta^{-1}, e^{-2\delta} (\sigma-1/2)^{-1}) \), but later we will choose \( \sigma \) and \( \delta \) such that the minimum comes from the term in \( \sigma \), thus we write

\[
\leq \sqrt{2} \left( \frac{3}{2} + \sigma + 2\delta \right) \frac{e^{-2\delta}}{2\sigma} + \frac{3}{2} \frac{e^{-2\delta}}{2\sigma - 1}.
\]
Moreover,
\[ \int_2^{+\infty} x \left( \frac{e^{-\delta x}}{x^\sigma} \right)'' \, dx = \int_2^{+\infty} x \left( \frac{e^{-\delta x}}{x^\sigma} \right)'' \, dx = (1 + \sigma + 2\delta) \frac{e^{-2\delta}}{2\sigma}, \]
hence, recalling (31) and using the inequality \( \Gamma(1 - \sigma) \leq (1 - \sigma)^{-1} \) for \( \sigma \in (0, 1) \), we get
\[
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} e^{-\delta n} \leq (2 + \sigma + 2\delta) \frac{e^{-2\delta}}{2\sigma} + \frac{\delta^{\sigma - 1}}{1 - \sigma} + 0.0462 \left( \sqrt{2} \left( \frac{3}{2} + \sigma + 2\delta \right) \frac{e^{-2\delta}}{2\sigma} + \frac{3}{2} \frac{e^{-2\delta}}{2\sigma - 1} \right) + 1.838 \frac{1 + \sigma + 2\delta}{2\sigma - 1} \leq \frac{\delta^{\sigma - 1}}{1 - \sigma} + 0.07 + 4. \]

\textbf{Lemma 10 (Bound of II).} Let \( \sigma \in (\frac{1}{2}, 1) \), \( |t| \geq 2 \) and \( \delta \in (0, 1) \). Then
\[ |\Gamma(1 - s)\delta^{s - 1}| \leq \sqrt{2\pi} e^{-\frac{\pi}{2}|t|}\delta^{\sigma - 1}. \]

**Proof.** It is sufficient to prove that \( |\Gamma(s)e^{\pi s/2}| \leq \sqrt{2\pi} \) for any \( s \in D := \{ s \colon \text{Re}(s) \in [0, \frac{1}{2}], \text{Im}(s) \geq 10 \} \). By the Phragmén-Lindelöf principle it is sufficient to prove it for \( s \in \partial D \). The claim for \( \text{Re}(s) = \frac{1}{2} \) follows immediately from the reflection formula \( \Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin \pi s} \). The claim for the other two lines may be proved using the Euler–Maclaurin formula for the gamma function. \( \square \)

\textbf{Lemma 11 (Bound of III).} Assume GRH. Let \( \sigma \in (\frac{1}{2}, 1) \), \( |t| \geq 10 \) and \( \delta \in (0, 1) \). Then
\[ \left| \sum_{\rho} \Gamma(s - \rho)\delta^{s - \rho} \right| \leq \delta^{\sigma - \frac{1}{2}} \left[ \left( \frac{\log(\frac{1}{2\sigma - 1})}{\pi} + \frac{0.64}{2\sigma - 1} + 0.82 \right) W_\delta(t) + \left( \frac{13.9}{2\sigma - 1} + 1.6 \right) n_\delta + \frac{6.9}{2\sigma - 1} + 0.8 \right]. \]

**Proof.** We are assuming GRH, thus \( \left| \sum_{\rho} \Gamma(s - \rho)\delta^{s - \rho} \right| \leq \delta^{\sigma - \frac{1}{2}} \sum_{\rho} |\Gamma(s - \rho)| \). We separate the contribution of zeros close to \( t \), since in this case the weight \( \Gamma(s - \rho) \) is large because of the pole of \( \Gamma \) at 0. We chose 2 as threshold value, which appears to be near the optimal value \( \approx 2.3 \). Since \( |(u + iv)\Gamma(u + iv)| \leq 1 \) for \( u \in (0, 1/2) \) and every \( v \in \mathbb{R} \), we have for every \( t > 3 \) (and setting \( u := \sigma - \frac{1}{2} \in (0, \frac{1}{2}) \)),
\[ \sum_{|\gamma - t| \leq 2} |\Gamma(u + i(\gamma - t))| \leq \sum_{|\gamma - t| \leq 2} \frac{1}{|u + i(\gamma - t)|} \]
so that by Lemma 7 we get
\[ (32) \sum_{|\gamma - t| \leq 2} |\Gamma(u + i(\gamma - t))| \leq \left( \frac{\text{arcsinh}(\frac{1}{2\sigma - 1})}{\pi} + \frac{2d_1}{2\sigma - 1} \right) W_\delta(t) + \frac{2d_2 n_\delta + 2d_3}{2\sigma - 1}. \]
To estimate the contribution of zeros with \( |\gamma - t| \geq 2 \) we use the bound \( |\Gamma(u + iv)| \leq \sqrt{2\pi}e^{-\frac{\pi}{2}|v|} \) for \( u \in (0, 1/2) \), \( |v| \geq 2 \) proved in Lemma 10. Thus we get (assuming \( t > 3 \)) that
\[ \sum_{|\gamma - t| \geq 2} |\Gamma(s - \rho)| \leq \sqrt{2\pi}e^{-\pi} \sum_{j=0}^{\infty} e^{-j\pi} \left( n_\delta(t + 2j + 3; 1) + n_\delta(t - 2j - 3; 1) \right). \]
Without loss of generality, we can assume $t \in \mathbb{R} \setminus \mathbb{Z}$; then the claim for $t \in \mathbb{Z}$ will follow by continuity. Under this hypothesis the quantity $|t - 2j - 3|$ is smaller than 1 only for $j = \lceil \frac{t-2}{2} \rceil$. Thus from (28a) and (28b) we deduce that

$$\sum_{|\gamma-t| \geq 2} |\Gamma(s-\rho)| \leq \sqrt{2\pi} e^{-\pi} \sum_{j=0}^{\infty} e^{-j\pi} (0.64 \log \Delta_K + (0.64 \log(t + 2j + 3) + 5.75)n_K + 3.49)$$

$$+ \sqrt{2\pi} e^{-\pi} \sum_{j=0}^{\infty} \sum_{j \neq j} e^{-j\pi} (0.64 \log \Delta_K + (0.64 \log(|t-2j-3|) + 5.75)n_K + 3.49)$$

$$+ \sqrt{2\pi} e^{-\pi} e^{-j\pi} (0.96 \log \Delta_K + 5.19n_K + 3.49)$$

$$\leq \frac{\sqrt{2\pi} e^{-\pi}}{1 - e^{-\pi}} (1.28 \log \Delta_K + 11.5n_K + 6.98) + 0.32\sqrt{2\pi} e^{-\frac{4}{3}}(t-2) \log \Delta_K$$

$$+ 0.64\sqrt{2\pi} e^{-\pi} \left( \sum_{j=0}^{\infty} e^{-j\pi} \log(t + 2j + 3) + \sum_{j=0}^{\infty} e^{-j\pi} \log(|t-2j-3|) \right) n_K. \tag{33}$$

To bound the sums we use the inequalities $\log(t + 2j + 3) \leq \log t + \frac{2j+3}{t}$ for the first one, and $\log(|t-2j-3|) \leq \log t$ when $j \leq J$ and $\log(|t-2j-3|) \leq \log t + \frac{2(j-J)}{t}$ when $j > J$ for the second, with $J := \lfloor t - \frac{3}{2} \rfloor$. Thus,

$$\sum_{j=0}^{\infty} e^{-j\pi} \log(t + 2j + 3) + \sum_{j=0}^{\infty} e^{-j\pi} \log(|t-2j-3|)$$

$$\leq \sum_{j=0}^{\infty} e^{-j\pi} \left( \log t + \frac{2j+3}{t} \right) + \sum_{j=0}^{\infty} e^{-j\pi} \log t + \frac{2}{t} \sum_{j=J+1}^{\infty} e^{-j\pi} (j - J)$$

$$= \frac{2 \log t + 3}{1 - e^{-\pi}} + 2e^{-\pi}(1 + e^{-j\pi}) \frac{1}{t(1 - e^{-\pi})^2}.$$

Moving this bound into (33) we get for $t \geq 10$ that

$$\sum_{|\gamma-t| \geq 2} |\Gamma(s-\rho)| \leq 0.145 \log \Delta_K + (0.145 \log t + 1.33)n_K + 0.8 \leq 0.15W_K(t) + 1.6n_K + 0.8,$$

that with (32) gives

$$\sum_{\rho} |\Gamma(s-\rho)| \leq \left( \frac{\text{arcsinh}(\frac{4}{\sigma - 1})}{\pi} + \frac{2d_1}{2\sigma - 1} + 0.15 \right) W_K(t) + \frac{2d_2n_K + 2d_3}{2\sigma - 1} + 1.6n_K + 0.8.$$

We get the claim using the bound $\text{arcsinh}(4/u) \leq \log(1/u) + \text{arcsinh} 4$ which holds for $u \in (0, 1]$, and the known values for the $d_j$’s. \hfill \Box

To bound $IV$ efficiently we split it in two

$$IV = \frac{1}{2\pi i} \int_{1/4-i\infty}^{1/4+i\infty} \sum_{|\gamma-y| \leq 1} \frac{\Gamma(z-s)}{z-\rho} \delta^{s-z} \, dz$$

$$+ \frac{1}{2\pi i} \int_{1/4-i\infty}^{1/4+i\infty} \left[ \frac{\zeta'}{\zeta} (z) - \sum_{|\gamma-y| \leq 1} \frac{1}{z-\rho} \right] \Gamma(z-s) \delta^{s-z} \, dz =: IVa + IVb,$$

where $y := \text{Im}(z)$, which we estimate separately.
**Lemma 12** (Bound of IVa). Assume GRH. Let $\sigma \in \left(\frac{1}{2}, 1\right)$, $|t| \geq 10$ and $\delta \in (0, 1)$. Then

$$|IVa| \leq \frac{\delta^{\sigma - 1/4}}{2\pi} \left(9.16 \log \Delta_K + (9.16 \log(|t| + 1) + 114.03)n_K + 65.88\right).$$

**Proof.** It is sufficient to prove that

$$\int_{\mathbb{R}} \sum_{|\gamma - y| \leq 1} \frac{\Gamma(u + i(y - t))}{|\frac{1}{4} + i(\gamma - y)|} \ dy \leq 9.16 \log \Delta_K + (9.16 \log(|t| + 1) + 89.73)n_K + 65.88$$

when $u \in [-\frac{3}{2}, -\frac{1}{2}]$ and $t \geq 10$. We split the integral into the regions $|y| \geq 2$ where the sum on the zeros is estimated simply by $4n_K(y; 1)$, and the remaining part $|y| < 2$ where the sum is estimated simply by $4n_K(y; 1)$, getting

$$\int_{\mathbb{R}} \sum_{|\gamma - y| \leq 1} \frac{\Gamma(u + i(y - t))}{|\frac{1}{4} + i(\gamma - y)|} \ dy \leq \int_{|y| < 2} 4n_K(y; 1)\Gamma(u + i(y - t)) \ dy + \int_{|y| \geq 2} \left(\frac{\arcsinh 4}{\pi} + 4d_1\right)W_K(|y|) + 4(d_2n_K + d_3)\left|\Gamma(u + i(y - t))\right| \ dy.$$

Now we restore the part of the integral with $|y| < 2$, getting

$$= \int_{|y| < 2} \left(4n_K(y; 1) - \left(\frac{\arcsinh 4}{\pi} + 4d_1\right)W_K(|y| + 1)ight)\left(\Gamma(u + i(y - t))\right) \ dy + \int_{\mathbb{R}} \left(\frac{\arcsinh 4}{\pi} + 4d_1\right)W_K(|y| + 1) + 4(d_2n_K + d_3)\left|\Gamma(u + i(y - t))\right| \ dy.$$

The exponential decay of the gamma function and the assumption $t \geq 10$ allow us to bound the first integral trivially, without affecting the strength of the result. From (28a) and (28b) we get

$$4n_K(y; 1) - \left(\frac{\arcsinh 4}{\pi} + 4d_1\right)W_K(|y| + 1) - 4(d_2n_K + d_3) \leq 2\log \Delta_K$$

in $y < 2$. Moreover, $\left|\Gamma(u + iv)\right| \leq 2 \cdot 10^{-6}$ when $u \in [-\frac{3}{4}, -\frac{1}{4}]$ and $|v| \geq 8$. With bounds (25a) and (25b) these facts prove that (34) is bounded by

$$2 \cdot 10^{-5} \log \Delta_K + 4.73\left(\frac{\arcsinh 4}{\pi} + 4d_1\right)W_K(|t| + 1) + 4(d_2n_K + d_3)$$

which is the claim, once the values for the $d_j$’s are considered. \qed

The following lemma bounds the integrand in IVb.

**Lemma 13.** Assume GRH. For $s = \frac{1}{4} + it$ with $t \notin \mathbb{Z}$ we have

$$\left|\zeta_K^c(s) - \sum_{\rho \leq 1} \frac{1}{s - \rho}\right| \leq \left(2.18 + \frac{3.2}{1 + t^2}\right)\log \Delta_K + 11.7 + \frac{7}{1 + 2t^2}$$

$$+ \left(2.18 \log(|t| + 1) + 21.6 + \frac{10}{|1 + 4it|}\right)n_K.$$
Proof. We can assume \( t > 0 \). We subtract (11) at \( s = \frac{1}{4} + it \) and \( 2 + it \), obtaining

\[
\frac{\zeta(s)}{\zeta(s)} - \frac{\zeta(s)}{\zeta(s)}(2 + it) = \sum_\rho \left( \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \right)
- \frac{\Gamma'(s)}{\Gamma(s)}(s) + \frac{\Gamma'(s)}{\Gamma(s)}(2 + it) - \left( \frac{1}{s} + \frac{1}{s - 1} - \frac{1}{2 + it} - \frac{1}{1 + it} \right).
\]

We use (6) and Lemma 8 to estimate \( \frac{\zeta'}{\zeta}(2 + it) \) and the gamma factors respectively, and the bound \( |\frac{1}{s} - \frac{1}{2 + it} + \frac{1}{s - 1} - \frac{1}{1 + it}| \leq \frac{7}{1 + 2t^2} \). In this way we get

\[
(35) \quad \left| \frac{\zeta'}{\zeta}(s) - \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho} \right| \leq n_{\zeta}(2) + \sum_{|\gamma - t| > 1} \left| \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \right|
+ \sum_{|\gamma - t| \leq 1} \left| \frac{1}{2 + it - \rho} \right| + \frac{10n_{\zetaube}(t + 2j; 1) + n_{\zetaube}(t - 2j; 1)}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||}.
\]

Moreover, for the first sum on the right-hand side of (35) we have

\[
(36) \quad \sum_{|\gamma - t| > 1} \left| \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \right| = \sum_{|\gamma - t| > 1} \frac{1}{|s - \rho||2 + it - \rho|}
\leq \sum_{j=1}^{\infty} \frac{n_{\zetaube}(t + 2j; 1) + n_{\zetaube}(t - 2j; 1)}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||}.
\]

By hypothesis \( t \) is not an integer, then \( |t - 2j| \) is in \((0, 1)\) only for \( j = \text{int}(\frac{t + 1}{2}) \). Thus using the bound in (28a) for \( j \neq j \) and (28b) when \( j = j \) we deduce that (36) is

\[
\leq \sum_{j=1}^{\infty} \frac{0.64W_{\zetaube}(t + 2j) + 6.92n_{\zetaube} + 2 \cdot 3.49}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} + \frac{0.64W_{\zetaube}(t + 2j) + 6.92n_{\zetaube} + 3.49 + (0.96 \log \Delta_{\zetaube} + 5.19n_{\zetaube} + 3.49)}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} |j = j \text{ if } t > 1|.
\]

Using \( \log(t + 2j) + \log(|t - 2j|) \leq 2 \log(2 + 1) \) (for \( t \geq 0 \) and \( j \geq 1 \):

\[
\leq 2 \sum_{j=1}^{\infty} \frac{0.64(\log \Delta_{\zetaube} + \log (\frac{2(t + 1)}{2\pi})n_{\zetaube}) + 6.92n_{\zetaube} + 3.49}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} + \frac{0.64(\log \Delta_{\zetaube} + \log (\frac{1+2j}{2\pi})n_{\zetaube}) + 6.92n_{\zetaube} + 3.49 + (0.96 \log \Delta_{\zetaube} + 5.19n_{\zetaube} + 3.49)}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} |j = j \text{ if } t > 1|.
\]

Suppose \( t > 1 \). Then restoring the missing term in the first sum we get that (36) is bounded above by

\[
\leq 2 \sum_{j=1}^{\infty} \frac{0.64(\log \Delta_{\zetaube} + \log (\frac{2(t + 1)}{2\pi})n_{\zetaube}) + 6.92n_{\zetaube} + 3.49}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} + \frac{0.64(\log(2\pi(t + 2j)) - 2 \log(2(t + 1)))n_{\zetaube} + (0.96 - 0.64) \log \Delta_{\zetaube} + (5.19 - 6.92)n_{\zetaube}}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} |j = j|.
\]

Since

\[
\sum_{j=1}^{\infty} \frac{1}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} \leq 0.76, \quad \sum_{j=1}^{\infty} \frac{\log(2j)}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} \leq 0.82
\]

we get that (36) is bounded above by

\[
\leq 2 \sum_{j=1}^{\infty} \frac{0.64(\log \Delta_{\zetaube} + \log (\frac{2(t + 1)}{2\pi})n_{\zetaube}) + 6.92n_{\zetaube} + 3.49}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} + \frac{0.64(\log(2\pi(t + 2j)) - 2 \log(2(t + 1)))n_{\zetaube} + (0.96 - 0.64) \log \Delta_{\zetaube} + (5.19 - 6.92)n_{\zetaube}}{1 + i(2j - 1)||\frac{3}{2} + i(2j - 1)||} |j = j|.
\]
and since for $t \geq 1$ one has
\[
\log(2\pi(t + 2j)) - 2\log(2(t + 1)j) + 5.19 - 6.92 < 0 \quad \text{for } j = \bar{j},
\]
we get
\[
\frac{1}{|\frac{1}{4} + i(2j - 1)||\frac{3}{2} + i(2j - 1)|} \leq \frac{5.4}{1 + t^2} \quad \text{for } j = \bar{j},
\]
which is
\[
(37) \quad \leq \left(1 + \frac{1.8}{1 + t^2}\right) \log \Delta_K + (\log(t + 1) + 9.8)n_K + 5.31.
\]
Suppose $0 < t < 1$. Then the term for $j = \bar{j}$ disappears and (36) is
\[
(38) \quad \leq \log \Delta_K + (\log(t + 1) + 9.8)n_K + 5.31.
\]
Finally, we note that
\[
\sum_{|\gamma - t| \leq 1} \left| \frac{1}{2 + it - \rho} \right| \leq \frac{2}{3} n_K(t; 1)
\]
which can be bounded with Lemma 6. The claim follows putting all together in (35) and using (37) and (28a) for $t > 1$, and (38) and (28b) for $0 < t < 1$. The proof concludes by noticing that the first bound is worse than the second in $0 < t < 1$ and that therefore its range can be extended to $t > 0$. □

Lemma 14 (Bound of $IV$). Assume GRH. Let $\sigma \in (\frac{1}{2}, 1)$, $|t| \geq 10$ and $\delta \in (0, 1)$. Then
\[
|IV| \leq (3.11 \log \Delta_K + (3.11 \log(|t|) + 35)n_K + 20)\delta^{\sigma - 1/4}.
\]
Proof. By Lemmas 5 and 13 we get
\[
|IVb| \leq \frac{\delta^{\sigma - 1/4}}{2\pi} \left((4.73 \cdot 2.18 + 0.013 \cdot 3.2) \log \Delta_K + 4.73 \cdot 11.7 + 0.007 \cdot 7
\]
\[
+ (4.73 \cdot 2.18 \log(|t| + 1) + 4.73 \cdot 21.6 + 0.171 \cdot 10)n_K\right).
\]
We get the claim adding $|IVA|$ as estimated in Lemma 12. □

We are finally able to prove the bound of $\zeta_K'(s)$ in the critical strip.

Lemma 15. Assume GRH. Let $\sigma \in (\frac{1}{2}, 1)$ and $|t| \geq 10$. Then
\[
\left| \frac{\zeta_K'}{\zeta_K}(s) \right| \leq \left(\frac{n_K}{1 - \sigma} + \frac{\log(\frac{1}{2\sigma - 1})}{\pi} + \frac{0.64}{2\sigma - 1} + 1.37\right)Q^{2\sigma - 2} + \left(\frac{0.07}{2\sigma - 1} + 4\right)n_K
\]
with $Q := \log \Delta_K + (\log |t| + 20)n_K + 11$.
Proof. From (30) and Lemmas [9][10][11] and [14] we get

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq \delta^{\frac{1}{2}} \left( \left( \frac{\log(\frac{1}{\sigma-1})}{\pi} + \frac{0.64}{2\sigma - 1} + 0.82 \right) W_\delta(t) + \left( \frac{13.9}{2\sigma - 1} + 1.6 \right) n_\delta + \frac{6.9}{2\sigma - 1} + 0.8 \right) + \sqrt{2\pi e^{-\frac{1}{2}}} |\delta^{\sigma|t|} + (3.11 \log \Delta_\delta + (3.11 \log |t| + 35) n_\delta + 20) \delta^{-1/4} + \left( \frac{\delta^{-1}}{1 - \sigma} + \frac{0.07}{2\sigma - 1} + 4 \right) n_\delta,$$

which we simplify to

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq \left( \frac{\log(\frac{1}{\sigma-1})}{\pi} + \frac{0.64}{2\sigma - 1} + 0.82 \right) Q \delta^{\sigma^{-1/2}} + \sqrt{2\pi e^{-\frac{1}{2}}} |\delta^{\sigma|t|} + 3.11 Q \delta^{-1/4} + \left( \frac{\delta^{-1}}{1 - \sigma} + \frac{0.07}{2\sigma - 1} + 4 \right) n_\delta,$$

where $Q := \log \Delta_\delta + (\log |t| + 20) n_\delta + 11$ (thus $W_\delta \leq Q - 21.8 n_\delta - 11$).

We get the claim by setting $\delta := Q^{-2}$ and with some minor simplifications which come from the assumption $|t| \geq 10$ and the lower bound $Q \geq 33$. \hspace{1cm} \Box

Finally, the inequality $f_\delta(s) \leq \tilde{f}_\delta(s)$ with the $\tilde{f}_\delta(s)$ given in Theorem [11] follows from plugging the estimates of Lemmas [9] and [15] in (12), and simplifying the resulting inequality using the bound $n_\delta \log \left( \frac{\sqrt{\sigma^{2} + t^{2}}}{2\pi} \right) + \frac{2\sigma}{\sigma^{2} + t^{2}} + \frac{2\sigma - 2}{(\sigma - 1)^{2} + t^{2}} \leq n_\delta \log t$ which holds when $\sigma \in (\frac{1}{2}, 1)$ and $t \geq 1$.

APPENDIX

Proof of Lemma [4] Using the explicit formula [11] Th. 1.4.2] for $\log \Gamma(s)$ coming from the Euler–MacLaurin summation formula one gets

$$\Gamma'(s) - \log \left( s - \frac{1}{2} \right) = -\log \left( 1 - \frac{1}{2s} \right) - \frac{1}{2s} - \frac{1}{12s^{2}} + \frac{1}{120s^{4}} + \int_{0}^{+\infty} B_{4}\{u\} \frac{1}{(s + u)^{5}} \, du$$

$$= \frac{1}{24s^{2}} + \sum_{n=3}^{+\infty} \frac{1}{n^{2n}s^{n}} + \frac{1}{120s^{4}} + \int_{0}^{+\infty} B_{4}\{u\} \frac{1}{(s + u)^{5}} \, du,$$

where $B_{4}(x) = x^{4} - 2x^{3} + x^{2} - \frac{1}{30}$. Thus, if $t$ is positive we get

$$\text{Re} \left( \frac{\Gamma'(s)}{\Gamma(s)} - \log \left( s - \frac{1}{2} \right) \right) \leq \frac{1}{24} \text{Re} \left( \frac{1}{s^{2}} \right) + \sum_{n=3}^{+\infty} \frac{1}{n^{2n}|s|^{n}} + \frac{1}{120|s|^{4}} + \int_{0}^{+\infty} \frac{|B_{4}\{u\}|}{|s + u|^{5}} \, du$$

$$\leq \frac{1}{24} \text{Re} \left( \frac{1}{s^{2}} \right) + \frac{1}{120|s|^{4}} \left( 2|s| - 1 \right) + \frac{1}{120|s|^{4}} + \frac{1}{30} \int_{0}^{+\infty} \frac{du}{(u^{2} + t^{2})^{5/2}}$$

$$\leq \frac{1}{24} \text{Re} \left( \frac{1}{s^{2}} \right) + \frac{1}{120|s|^{4}} \left( 2t - 1 \right) + \frac{1}{120|s|^{4}} + \frac{1}{45t^{4}}.$$

This is a rational function of $\sigma$ and $t$, and with elementary arguments one proves that it is negative for $\sigma \geq 0$ and $t \geq \sigma + 2$. 
Proof of Lemma 5. The map $u \to \int_{\mathbb{R}} |\Gamma(u + iv)| \, dv$ is log-convex for $u \in (-1, 0)$ as a consequence of a general inequality of Hardy, Ingham and Pólya (see [9, Ch. 11, Prop. 4]), and the uniform exponential decay of $\Gamma(u + iv)$ for $v \to \infty$. Hence,

$$\int_{\mathbb{R}} |\Gamma(u + iy)| \, dy \leq \max \left\{ \int_{\mathbb{R}} |\Gamma\left(-\frac{3}{4} + iy\right)| \, dy, \int_{\mathbb{R}} |\Gamma\left(-\frac{1}{4} + iy\right)| \, dy \right\}$$

for any $u \in [-3/4, -1/4]$. The last two integrals are bounded, respectively, by 4.43 and 4.73 (an effective version of the Stirling bound as [11, Cor. 1.4.4] may be used to prove that the contribution of the range $y \in \mathbb{R}\setminus[-10, 10]$ to the integral is smaller than $10^{-6}$, and then, using the monotonicity of $y \mapsto |\Gamma(u + iy)|$, a Riemann sum with 10000 points produces the result). This proves (25a).

Without loss of generality, we can assume $t > 10$. By (25a), in order to prove (25b) it is sufficient to show that

$$\int_{\mathbb{R}} |\Gamma(u + i(t - y))| \log \left(\frac{1 + |y|}{1 + t}\right) \, dy$$

is negative. Let

$$F_1(u, v) := -\int_{\mathbb{R}}^{+\infty} |\Gamma(u + iw)| \, dw,$$

$$F_2(u, v) := -\int_{\mathbb{R}}^{+\infty} F_1(u, w) \, dw = \int_{\mathbb{R}}^{+\infty} (w - v) |\Gamma(u + iw)| \, dw,$$

so that $\partial_v F_1(u, v) = |\Gamma(u + iv)|$ and $\partial_v F_2(u, v) = F_1(u, v)$. We split (39) into three ranges $(-\infty, 0] \cup [0, t] \cup [t, +\infty)$, getting

$$\int_{\mathbb{R}} |\Gamma(u + i(t - y))| \log \left(\frac{1 + |y|}{1 + t}\right) \, dy = \int_{-\infty}^{0} |\Gamma(u + i(t - y))| \log \left(\frac{1 - y}{1 + t}\right) \, dy$$

$$+ \int_{0}^{t} |\Gamma(u + i(t - y))| \log \left(\frac{1 + y}{1 + t}\right) \, dy + \int_{t}^{+\infty} |\Gamma(u + i(y - t))| \log \left(\frac{1 + y}{1 + t}\right) \, dy$$

where in the last term we have used the equality $|\Gamma(u - iv)| = |\Gamma(u + iv)|$ to ensure the positivity of the imaginary part of the argument of the gamma function. Then an integration by parts shows that it is

$$= -\int_{-\infty}^{0} \frac{F_1(u, t - y)}{1 - y} \, dy + \int_{0}^{t} \frac{F_1(u, t - y)}{1 + y} \, dy - \int_{t}^{+\infty} \frac{F_1(u, y - t)}{1 + y} \, dy.$$

A second integration by parts produces

$$= 2F_2(u, t) - \int_{-\infty}^{0} \frac{F_2(u, t - y)}{(1 - y)^2} \, dy - \int_{0}^{t} \frac{F_2(u, t - y)}{(1 + y)^2} \, dy - \int_{t}^{+\infty} \frac{F_2(u, y - t)}{(1 + y)^2} \, dy.$$

Function $F_2$ being positive, this is

$$\leq 2F_2(u, t) - \frac{1}{(1 + t)^2} \int_{0}^{t} F_2(u, t - y) \, dy = 2F_2(u, t) - \frac{1}{(1 + t)^2} \int_{0}^{t} F_2(u, y) \, dy$$

which is negative if and only if

$$2(1 + t)^2 F_2(u, t) \leq \int_{0}^{t} F_2(u, y) \, dy.$$

This inequality holds when $t$ is large enough because the left-hand side decreases to 0 as a function of $t$. In order to prove that this happens already for $t \geq 10$ we use
an effective version of the Stirling bound (see [1, Cor. 1.4.4]) giving, when \( u < 0 \) and \( v > 0 \),
\[
|\Gamma(u + iv)| = \sqrt{2\pi}|u + iv|^{u-1/2}e^{-\frac{\pi}{2}v}e^{-u + v \arctan(u/v)}e^{R(u,v)}
\]
with \(|R(u,v)| \leq \frac{1}{8\pi}(\frac{\pi}{2} + \arctan(v/u))\). Thus, if furthermore \( u \in [-3/4, -1/4] \) and \( v \geq 10 \), we get
\[
|F_2(u + iv)| \leq \frac{4}{\pi^2}2^{3/4}e^{3/4}v^{-3/4}e^{\pi\pi v}e^{-\frac{\pi}{2}v}.
\]
We get that \( \forall v \geq 10 \) the left-hand side of (40) is lower than \( 2 \cdot 10^{-5} \). On the contrary,
\[
\int_0^t F_2(u,y) \, dy = \frac{1}{2} \int_0^t w^2|\Gamma(u + iw)| \, dw \geq \frac{1}{2} \int_0^1 w^2|\Gamma(u + iw)| \, dw \\
\geq \frac{1}{6} \min_{u \in [-3/4,-1/4]} |\Gamma(u + iw)|.
\]
The maximum modulus principle for holomorphic functions (applied to \( 1/\Gamma(z) \)) shows that the minimum is reached at the boundary of the region \([-3/4, -1/4] \times [0, 1]\), and it is easy to verify that here \( |\Gamma(z)| \) is always larger than 0.4 thus the right-hand side of (40) is larger than 0.06.

For (25c) we note that \( |\Gamma(u + iv)| \leq 5.3e^{-\pi|v|/2} \) for every \( u \in [-3/4, -1/4] \) and every \( v \) (this can be proved as Lemma [10]), thus it is sufficient to bound
\[
F(t) := \int_{\mathbb{R}} \frac{e^{-\frac{\pi}{2}|t-y|}}{|1+4iy|} \, dy.
\]
It is the unique bounded solution of the differential equation \( F''(t) - \frac{\pi^2}{4} F(t) = -\frac{\pi}{1+4it} \). A numerical check shows that \( F(10) = 0.032 \ldots \) is larger than \( \frac{\pi^2}{4\pi+40} = 0.031 \ldots \). Suppose that there exists \( t_0 > 10 \) with \( F(t_0) > F(10) \). Then there exists also a point \( t_1 > t_0 \) with \( F(t_1) = \max_{t \in [10, +\infty]} F(t) \), because \( F(+\infty) = 0 \). Then \( F''(t_1) \leq 0 \), because \( F \) is a \( C^2 \) map in \((10, +\infty)\). Thus from the differential equation we get
\[
\frac{\pi^2}{4} F(t_1) \leq \frac{\pi}{1+4it_1} < \frac{\pi}{1+40i} < \frac{\pi^2}{4} F(10)
\]
which violates the definition of \( t_1 \). This proves that \( F(10) = \max_{t \in [10, +\infty]} F(t) \), so that
\[
\int_{\mathbb{R}} \frac{|\Gamma(u + i(t-y))|}{|1+4iy|} \, dy \leq 5.3 F(t) \leq 5.3 F(10) \leq 0.171.
\]

The same argument may be applied to prove (25d), since \( F_\alpha(t) := \int_{\mathbb{R}} \frac{e^{-\frac{\pi}{2}|t-y|}}{1+\alpha y^2} \, dy \) is the unique solution of the differential equation \( F''_\alpha(t) - \frac{\pi^2}{4} F_\alpha(t) = -\frac{\pi}{1+\alpha y^2} \) which is bounded for \( t \to \pm \infty \), and \( F_1(10) = 0.0129 \ldots > \frac{4/\pi}{1+10^2} = 0.0126 \ldots \), and \( F_2(10) = 0.0065 \ldots > \frac{4/\pi}{1+2 \cdot 10^2} = 0.0063 \ldots \).

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