POSITION VECTORS OF NUMERICAL SEMIGROUPS

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Abstract. We provide a new way to represent numerical semigroups by showing that the position of every Apéry set of a numerical semigroup \( S \) in the enumeration of the elements of \( S \) is unique, and that \( S \) can be re-constructed from this “position vector.” We extend the discussion to more general objects called numerical sets, and show that there is a one-to-one correspondence between \( m \)-tuples of positive integers and the position vectors of numerical sets closed under addition by \( m + 1 \). We consider the problem of determining which position vectors correspond to numerical semigroups.

1. Introduction

We let \( \mathbb{N} \) and \( \mathbb{N}_0 \) denote the positive and nonnegative integers, respectively. A numerical semigroup \( S \) is a subsemigroup of \( \mathbb{N}_0 \) that contains 0 and has finite complement in \( \mathbb{N}_0 \). For two elements \( u \) and \( u' \) in \( S \), \( u \preceq_S u' \) if there exists an \( s \in S \) such that \( u + s = u' \). This defines a partial ordering on \( S \). The minimal elements in \( S \setminus \{0\} \) with respect to this ordering form a unique minimal set of generators for \( S \), which is denoted by \( \{a_1, a_2, \ldots, a_\nu\} \) where \( a_1 < a_2 < \cdots < a_\nu \). The semigroup \( S = \{\sum_{i=1}^\nu c_i a_i : c_i \geq 0\} \) is represented using the notation \( S = \langle a_1, \ldots, a_\nu \rangle \). Since the minimal generators of \( S \) are distinct modulo \( a_1 \), the set of minimal generators is finite. Furthermore, having finite complement in \( \mathbb{N}_0 \) is equivalent to \( \gcd\{a_i : 1 \leq i \leq \nu\} = 1 \).

The number of minimal generators of a semigroup \( S \) is called the embedding dimension of \( S \), and is denoted by \( \nu = \nu(S) \). The element \( a_1 \) is called the multiplicity of \( S \), and is also denoted by \( e_0(S) \). When \( S \neq \mathbb{N}_0 \), we always have \( 2 \leq \nu(S) \leq e_0(S) \).

For \( 0 \neq n \in S \), the Apéry set of \( S \) with respect to \( n \) is the set
\[
\text{Ap}(S, n) = \{ w \in S : w - n \notin S \}.
\]

Every numerical semigroup containing \( n \) has a unique Apéry set with respect to \( n \) from which much can be gleaned. Indeed, in [3] the Apéry set is described as the most versatile tool in numerical semigroup theory.

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We take the representation by an Apéry set a step further. Consider the semigroup $S = \langle 4, 7, 9 \rangle$. We have $\text{Ap}(S, 4) = \{0, 7, 9, 14\}$. If we enumerate the elements of $S$ so that $S = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$, where $\lambda_i < \lambda_j$ whenever $i < j$, then $\text{Ap}(S, 4) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_8\}$. We can say that the position of the Apéry set in the enumeration is given by $(0, 2, 4, 8)$. It will be convenient to remove 0 from this vector and consider the difference of the components. For example, we represent $S = \langle 4, 7, 9 \rangle$ (as a semigroup containing 4) with the vector $(2, 2, 4, 8)$ instead of $(0, 2, 4, 8)$. This new vector has nonnegative integer components and stores equivalent information about the semigroup. We make the following definition.

**Definition 1.1.** Let $S = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ be a numerical semigroup containing $n \neq 0$ such that $\lambda_i < \lambda_j$ whenever $i < j$. If $\text{Ap}(S, n) = \{\lambda_0, \lambda_{x_1}, \lambda_{x_2}, \ldots, \lambda_{x_{n-1}}\}$, then the $(n-1)$-tuple $(x_1, x_2 - x_1, x_3 - x_2, \ldots, x_{n-1} - x_{n-2})$ is called the position vector of $S$ with respect to $n$, and is denoted by $\text{pv}_n(S)$.

We show in Corollary 2.11 that if $S$ and $T$ are semigroups containing $n \neq 0$, then $S = T$ if and only if $\text{pv}_n(S) = \text{pv}_n(T)$. Thus, a position vector is a representation of the semigroup. This is a rather remarkable fact. Consider the semigroup $\langle 4, 7, 9 \rangle$. Since the position vector is $(2, 2, 4)$ (which means that the elements of $\text{Ap}(\langle 4, 7, 9 \rangle, 4)$ are in positions 0, 2, 4, and 8 in the enumeration), no other semigroup can have an Apéry set with elements in those same positions. This is certainly not true for minimal generating sets: the position of the minimal generators of $\langle 4, 7, 9 \rangle$ and $\langle 4, 6, 9 \rangle$ are the same, namely the first, second, and fourth elements in the enumeration. Nonetheless, the position vectors are $(2, 2, 4)$ and $(2, 2, 5)$ respectively.

Not every vector of positive integers is the position vector of a numerical semigroup. In Section 2 we extend our consideration to more general objects than numerical semigroups, which we call numerical sets, and show that there is a one-to-one correspondence between elements of $\mathbb{N}^{n-1}$ and the position vectors of numerical sets closed under addition by $n$. Although we do not stress the fact in this paper, a numerical set $I$ can always be interpreted as a relative ideal of a semigroup contained in $I$, and so, in a sense, we have not extended beyond the theory of numerical semigroups.

Among the position vectors of numerical sets, we examine the problem of determining which represent numerical semigroups in Section 3.

2. **Position vectors of numerical sets**

As stated in the introduction, we need to work with objects more general than numerical semigroups.

**Definition 2.1.** A numerical set $I$ is a subset of $\mathbb{N}_0$ that contains 0 and has finite complement in $\mathbb{N}_0$.

**Remark 2.2.** A numerical set $I$ is closed under addition if and only if it is a numerical semigroup. When $I$ is not closed under addition, it is a relative ideal of the numerical semigroup $I - I$. Thus, we can think of the numerical sets as a certain collection of relative ideals which includes numerical semigroups. See [1] for information on relative ideals and [5, 6] for numerical sets.
We need to define the position vector of a numerical set as we did for numerical semigroups in the introduction. We begin with a preliminary definition.

**Definition 2.3.** For \( n \neq 0 \), let \( \Gamma_n \) be the collection of numerical sets \( I \) such that \( n + i \in I \) for all \( i \in I \).

For \( I \in \Gamma_n \), we can define the Apéry set with respect to \( n \) as

\[
\text{Ap}(I, n) = \{ w \in I : w - n \notin I \}.
\]

As with numerical semigroups, it is not difficult to see that \( \text{Ap}(I, n) \) contains exactly \( n \) elements of \( I \) including 0. Now we can define the position vector of the numerical set.

**Definition 2.4.** Let \( I = \{ \lambda_0, \lambda_1, \lambda_2, \ldots \} \) be in \( \Gamma_n \) such that \( \lambda_i < \lambda_j \) whenever \( i < j \). If \( \text{Ap}(I, n) = \{ \lambda_0, \lambda_{x_1}, \lambda_{x_2}, \ldots, \lambda_{x_{n-1}} \} \), the \((n-1)\)-tuple \((x_1, x_2-x_1, x_3-x_2, \ldots, x_{n-1}-x_{n-2})\) is called the **position vector** of \( I \) with respect to \( n \), and is denoted by \( \text{pv}_n(I) \).

A numerical set has multiple position vectors, but no two can have the same length. Therefore, \( f : \Gamma_n \mapsto \mathbb{N}^{n-1} \) such that \( f(I) = \text{pv}_n(I) \) is a well-defined function. The goal of this section is to prove that \( f \) is a one-to-one correspondence, which is proven in Theorem 2.10.

We first establish Proposition 2.6, which contains a result about permutations on the set \( 1, 2, \ldots, m \).

**Definition 2.5.** Let \( \pi = [\pi_1 \cdots \pi_m] \) be a permutation of the set \( \{1, \ldots, m\} \). Then the **conversion vector** of \( \pi \) is \( r = (r_1, \ldots, r_m) \), where \( r_i = |\{ j : j < i \text{ and } \pi_j < \pi_i \}| \).

Notice that in Definition 2.5, we have \( 0 \leq r_i < i - 1 \) for all \( 1 \leq i \leq m \). Moreover, Proposition 2.6 reveals that every vector \((r_1, \ldots, r_m)\) with this restriction is the conversion vector of a unique permutation. This result is similar to a well-known result about inversion vectors of permutations, see [7][8].

**Proposition 2.6.** For a fixed integer \( m \geq 1 \), let \( r = (r_1, r_2, \ldots, r_m) \) be a vector such that \( 0 \leq r_i \leq i - 1 \), for \( 1 \leq i \leq m \). Then there is a unique permutation \( \pi \) of the set \( \{1, \ldots, m\} \) for which \( r \) is its conversion vector.

**Proof.** Since there are exactly \( m! \) such vectors and \( m! \) permutation of length \( m \), it suffices to show that each vector is the conversion vector of some permutation. The uniqueness follows by counting.

We will proceed by induction on \( m \). If \( m = 1 \), then \( r = (0) \) and \( \pi = [1] \) has \( r \) as its conversion vector. Now assume that \( m \geq 2 \) and that there is a permutation \( \sigma = [\sigma_1 \sigma_2 \cdots \sigma_{m-1}] \) with the conversion vector \((r_1, r_2, \ldots, r_{m-1})\). For each \( 1 \leq i \leq m - 1 \), define

\[
\delta_i = \begin{cases} 
1 & \text{if } \sigma_i > r_m \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \pi = [(\sigma_1 + \delta_1)(\sigma_2 + \delta_2) \cdots (\sigma_{m-1} + \delta_{m-1})(r_m + 1)] \). It is not difficult to see that \( \pi \) is a permutation, and that \((\sigma_i + \delta_i) < (\sigma_j + \delta_j)\) if and only if \( \sigma_i < \sigma_j \), for \( 1 \leq i \leq m - 1 \). Thus the conversion vector of \( \pi \) is \( r \). \( \square \)
Example 2.7. Consider the permutation [42351]. We can directly observe that the conversion vector is (0, 0, 1, 3, 0). Conversely, since the proof in Proposition 2.6 is constructive, we can recursively recover the permutation as follows:

\[
\begin{align*}
[(0 + 1)] &= [1] \\
[(1 + 1)(0 + 1)] &= [21] \\
[(2 + 1)(1 + 0)(1 + 1)] &= [312] \\
[(3 + 0)(1 + 0)(2 + 0)(3 + 1)] &= [3124] \\
[(3 + 1)(1 + 1)(2 + 1)(4 + 1)(0 + 1)] &= [42351].
\end{align*}
\]

If the elements of the Apéry set of a numerical set \(I\) are known, the position vector can be determined without considering the enumeration of all the elements of \(I\). To see this, let \(\text{Ap}(I, n) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}\). We write \(\lambda_i = nk_i + r_i\), where \(0 \leq r_i < n\). Notice that the elements of \(\text{Ap}(I, n)\) form a complete residue system modulo \(n\) and \(\pi_0 = 0\). Thus, \(\pi = [\pi_1 \cdots \pi_{n-1}]\) is a permutation of the set \(\{1, \ldots, n-1\}\) with a corresponding conversion vector \(r = (r_1, r_2, \ldots, r_{n-1})\).

**Theorem 2.8.** Let \(I \in \Gamma_n\) and \(\text{Ap}(I, n) = \{w_0, w_1, \ldots, w_{n-1}\}\), where \(w_i < w_j\) whenever \(i < j\). We write \(w_i = nk_i + r_i\), with \(0 \leq r_i < n\). Let \(r = (r_1, \ldots, r_{n-1})\) be the conversion vector of \(\pi = [\pi_1 \pi_2 \cdots \pi_{n-1}]\). Then \(\text{pv}_n(I) = (v_1, v_2, \ldots, v_{n-1})\), where \(v_1 = k_1 + 1\) and \(v_i = i(k_i - k_{i-1}) + (r_i - r_{i-1})\), for \(2 \leq i \leq n - 1\).

**Proof.** Let \(I = \{\lambda_0, \lambda_1, \ldots\}\) and \(w_i = \lambda_{x_i}\). Since \(0 = w_0 = \lambda_{x_0}\), we have \(x_0 = 0\). Next we show that \(x_i = ik_i - \sum_{j=0}^{i-1} k_j + r_i + 1\), for \(1 \leq i \leq n - 1\). To do this, we compute the number of elements in \(I\) that are strictly less than \(nk_i\) for \(1 \leq i \leq n - 1\). The sequence \((k_0, k_1, \ldots, k_{n-1})\) is non-decreasing, and we set \(l\) to be the largest index such that \(k_l = k_i\). Now \(s \in I\) and \(nk_l \leq s \leq nk_l + (n-1)\) if and only if \(s = \lambda_{x_j} + n(k_l - k_j)\) for some \(0 \leq j \leq l\). Thus

\[
|\{0, 1, 2, \ldots, nk_l - 1\} \cap I| = |\{0, 1, 2, \ldots, nk_l - 1\} \cap I| \\
= \sum_{j=0}^{l}(k_l - k_j) \\
= \sum_{j=0}^{i-1}(k_i - k_j) \\
= ik_i - \sum_{j=0}^{i-1} k_j.
\]
Next, \( s \in I \) and \( nk_i \leq s < \lambda_{x_i} \) if and only if \( s = \lambda_{x_j} + n(k_i - k_j) = nk_i + \pi_j \), for some \( 0 \leq j < i \) and \( \pi_j < \pi_i \). There are \( r_i + 1 \) such elements. Therefore, we have

\[
    x_i = | \{ \lambda_0, \lambda_1, \ldots, \lambda_{x_i-1} \} | \\
    = ik_i - \sum_{j=0}^{i-1} k_j + r_i + 1
\]

Recall that, by definition, \( v_i = x_i - x_{i-1} \). Thus, the result now follows. \( \square \)

Now we show that \( f: \Gamma_n \mapsto \mathbb{N}^{n-1} \) such that \( f(I) = pv_n(I) \) is a one-to-one correspondence by constructing the inverse map.

**Setup 2.9.** Let \( v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{N}^{n-1} \). We will define two recursive sequences as follows:

1. \( t_1 = 0 \) and \( t_i = (v_i + t_{i-1}) \mod i \), for \( 2 \leq i \leq n - 1 \).
2. \( l_1 = v_1 - 1 \) and \( l_i = l_{i-1} + \frac{v_i + t_{i-1} - t_i}{i} \), for \( 2 \leq i \leq n - 1 \).

Notice that \( 0 \leq t_i \leq i - 1 \) for \( 1 \leq i \leq n - 1 \). Thus, \( (t_1, t_2, \ldots, t_{n-1}) \) is the conversion vector of a permutation \( \sigma = [\sigma_1 \sigma_2 \cdots \sigma_{n-1}] \) according to Proposition 2.6. Now let

\[
    \mathcal{A}_v = \{ 0 \} \cup \{ nl_i + \sigma_i : 1 \leq i \leq n - 1 \}.
\]

Since \( \sigma \) is a permutation on the set \( \{ 1, \ldots, n-1 \} \), \( \mathcal{A}_v \) is a complete residue system modulo \( n \) that contains 0. Thus, \( \mathcal{A}_v \) generates a numerical set \( I(\mathcal{A}_v) = \{ w + kn : w \in \mathcal{A}_v, k \geq 0 \} \) in \( \Gamma_n \). We set \( g : \mathbb{N}^{n-1} \mapsto \Gamma_n \) such that \( g(v) = I(\mathcal{A}_v) \).

**Theorem 2.10.** The functions \( f: \Gamma_n \mapsto \mathbb{N}^{n-1} \) given by \( f(I) = pv_n(I) \) and \( g : \mathbb{N}^{n-1} \mapsto \Gamma_n \) by \( g(v) = I(\mathcal{A}_v) \) are inverse functions. Therefore, the function \( f \) is a one-to-one correspondence between numerical sets closed under addition by the element \( n \) and \( (n-1) \)-tuples of positive integers.

**Proof.** Let \( I \in \Gamma_n \) with \( Ap(I, n) = \{ 0, w_1, \ldots, w_{n-1} \} \), and write \( w_i = nk_i + \pi_i \), where \( 0 \leq \pi_i < n \). Also let \( v = pv_n(I) \) and \( \mathcal{A}_v = \{ 0 \} \cup \{ nl_i + \sigma_i : 0 \leq i \leq n - 1 \} \) as defined in Setup 2.9. It suffices to show that \( \pi_i = \sigma_i \) and \( k_i = l_i \) for all \( 1 \leq i \leq n - 1 \).

As noted before, \( [\pi_1 \pi_2 \cdots \pi_{n-1}] \) is a permutation, and let \( r = (r_1, r_2, \ldots, r_{n-1}) \) be its conversion vector. By Theorem 2.8, \( r_1 = 0 \), and \( r_i = (v_i + r_{i-1}) - i(k_i - k_{i-1}) \). Since \( 0 \leq r_i \leq i - 1 \), it follows that \( r_i = (v_i + r_{i-1}) \mod i \), for \( 1 \leq i \leq n - 1 \). Referring to Setup 2.9, we find that \( r_i = t_i \), for \( 1 \leq i \leq n - 1 \). Thus, \( r \) is the conversion vector of both \( [\pi_1 \pi_2 \cdots \pi_{n-1}] \) and \( [\sigma_1 \sigma_2 \cdots \sigma_{n-1}] \), and we conclude the two permutations are equal.

Again, by Theorem 2.8 and Setup 2.9, \( k_1 = v_1 + 1 = l_1 \), and for \( 2 \leq i \leq n - 1 \),

\[
    k_i - k_{i-1} = \frac{v_i + r_{i-1} - r_i}{i} \\
    = \frac{v_i + t_{i-1} - t_i}{i} \\
    = \frac{l_i - l_{i-1}}{i}
\]

We conclude that \( k_i = l_i \), for \( 1 \leq i \leq n - 1 \). This shows that \( g \circ f = id_{\Gamma_n} \), and similarly, we obtain that \( f \circ g = id_{\mathbb{N}^{n-1}} \). Therefore, the function \( f \) is a one-to-one correspondence. \( \square \)
The next corollary is really a restatement of Theorem 2.10.

**Corollary 2.11.** Every numerical set closed under addition by the element $n \in \mathbb{N}$ has a unique position vector of length $n - 1$. In particular, no two numerical semigroups have the same position vector. Moreover, every vector of length $n - 1$ with entries in $\mathbb{N}$ is the position vector of a numerical set closed under addition by the element $n$.

The next example demonstrates what we have developed in this section.

**Example 2.12.** Let $S = \langle 6, 16, 20, 21, 29 \rangle$. This is a numerical semigroup containing 6, and hence $S \in \Gamma_6$. We can compute

$$\text{Ap}(S, 6) = \{0, 16, 20, 21, 29\} = \{0, 6(2) + 4, 6(3) + 2, 6(3) + 3, 6(4) + 5, 6(6) + 1\}.$$  

The permutation $[42351]$ has conversion vector $(0, 0, 1, 3, 0)$. Thus,

\[
\begin{align*}
  v_1 &= 2 + 1 = 3 \\
  v_2 &= 2(3 - 2) + (0 - 0) = 2 \\
  v_3 &= 3(3 - 3) + (1 - 0) = 1 \\
  v_4 &= 4(4 - 3) + (3 - 1) = 6 \\
  v_5 &= 5(6 - 4) + (0 - 3) = 7.
\end{align*}
\]

So, we have $pv_6(S) = (3, 2, 1, 6, 7)$.

Conversely, suppose we start with $v = (3, 2, 1, 6, 7) \in \mathbb{N}^5$. According to Setup 2.9 and Theorem 2.10, the $r_i$'s are $(0, 0, 1, 3, 0)$ and the $k_i$'s $(2, 3, 3, 4, 6)$. From the conversion vector $(0, 0, 1, 3, 0)$, we construct the corresponding permutation $[42351]$ (see Example 2.7). Now,

\[
\begin{align*}
  w_0 &= 0 \\
  w_1 &= 6(2) + 4 = 16 \\
  w_2 &= 6(3) + 2 = 20 \\
  w_3 &= 6(3) + 3 = 21 \\
  w_4 &= 6(4) + 5 = 29 \\
  w_5 &= 6(6) + 1 = 37.
\end{align*}
\]

Thus, $S = \{w_i + c_i n : c_i \geq 0\} = \langle 6, 16, 20, 21, 29 \rangle$.

3. **Position vectors of numerical semigroups**

Now that we have a one-to-one correspondence between $\mathbb{N}^{n-1}$ and numerical sets closed under addition by $n$ established in the previous section, we want to know which position vectors correspond to semigroups. We provide a method for solving this problem and give an explicit answer for semigroups containing small numbers.

We begin with a necessary and sufficient condition for a complete residue system modulo $n$ that contains 0 to be the Apéry set of a numerical semigroup. This result is similar to others contained in [2, 4], and the proof is omitted.
**Lemma 3.1.** Let $A = \{w_0, w_1, \ldots, w_{n-1}\}$ be a complete residue system modulo $n$ that contains $0$, where $w_0 < w_1 < \cdots < w_{n-1}$. Then, $I(A) = \{w + kn : w \in A, k \geq 0\}$ is a numerical semigroup if and only if $w_i + w_j \geq w_i$ whenever $w_i + w_j \equiv w_1 \mod n$ with $0 < i \leq j < l$.

We can now translate Lemma 3.1 into a condition concerning the position vector, but first we need a few preliminary results.

**Lemma 3.2.** Let $I \in \Gamma_n$ be a numerical set with Apéry set $Ap(I, n) = \{w_0, w_1, \ldots, w_{n-1}\}$, where $w_0 < w_i < \cdots < w_{n-1}$. We set $w_i = nk_i + \pi_i$ for $0 \leq i \leq n - 1$. If $v = (v_1, \ldots, v_{n-1})$ is the position vector of $I$, then $k_1 = v_1 - 1$ and

$$k_i - k_{i-1} = \left\lfloor \frac{v_i - 1}{i} \right\rfloor + \gamma_i,$$

where $\gamma_1 = 0$ and

$$\gamma_i = \begin{cases} 0 & \text{if } \pi_{i-1} < \pi_i \\ 1 & \text{if } \pi_{i-1} > \pi_i \end{cases},$$

for $2 \leq i \leq n - 1$.

**Proof.** It follows from Theorem 2.8 that $k_1 = v_1 - 1$ and $v_i = i(k_i - k_{i-1}) + (r_i - r_{i-1})$, for $2 \leq i \leq n - 1$, where $r = (r_1, \ldots, r_{n-1})$ is the conversion vector of $\pi = [\pi_1 \cdots \pi_{n-1}]$. We rewrite this as $v_i - 1 = i(k_i - k_{i-1} - \gamma_i) + (i\gamma_i + r_i - r_{i-1} - 1)$. If we show that $0 \leq i\gamma_i + r_i - r_{i-1} - 1 < i$ whenever $2 \leq i \leq n - 1$, then it follows that

$$k_i - k_{i-1} = \left\lfloor \frac{v_i - 1}{i} \right\rfloor + \gamma_i.$$

First suppose that $r_i > r_{i-1}$. By the definition of the conversion vector, we have $\pi_{i-1} < \pi_i$ and recall that $0 \leq r_j \leq j - 1$ for all $1 \leq j \leq n - 1$. Thus, $\gamma_i = 0$ and $0 \leq r_i - r_{i-1} - 1 \leq i - 2$. Next, suppose that $r_i \leq r_{i-1}$ so that $\pi_{i-1} > \pi_i$. Then $\gamma_i = 1$ and $1 \leq i + r_i - r_{i-1} - 1 \leq i - 1$, which completes the proof. \qed

The next proposition will lead to a convenient equivalence relation on the elements of $\mathbb{N}^{n-1}$, i.e., the position vectors of numerical sets in $\Gamma_n$.

**Proposition 3.3.** Let $v = (v_1, \ldots, v_m)$ and $z = (z_1, \ldots, z_m)$ be two position vectors with associated conversion vectors and permutations denoted by $r$ and $\pi$ for $v$, and $s$ and $\sigma$ for $z$. Then the following are equivalent:

1. $v_i \equiv z_i \mod i$ for all $1 \leq i \leq m$
2. $r = s$
3. $\pi = \sigma$.

**Proof.** For (1) implies (2), by Setup 2.9, the conversion vector depends on the position vector modulo $i$ for the $i$-th entry. The equivalence of (2) and (3) follows from Proposition 2.6. Lastly, (2) implies (1) since, according to Theorem 2.8, $v_i \equiv r_i - r_{i-1} \mod i$. \qed
Definition 3.4. We say two elements \( v = (v_1, \ldots, v_m) \) and \( z = (z_1, \ldots, z_m) \) of \( \mathbb{N}^m \) are congruent, denoted by \( v \sim z \), if \( v_i \equiv z_i \mod i \) for all \( 1 \leq i \leq m \). In this case, \( v \) and \( z \) have the same associated permutation \( \pi \) and are said to be in the permutation class defined by \( \pi \).

We can now present the main result of this section.

Theorem 3.5. Let \( (v_1, v_2, \ldots, v_m) \) be a vector of positive integers in the permutation class defined by \( [\pi_1 \pi_2 \cdots \pi_m] \). Also set \( \gamma_i \) as in Lemma 3.2 and 

\[
u_i = \left\lfloor \frac{v_i - 1}{i} \right\rfloor.
\]

Then \( (v_1, v_2, \ldots, v_m) \) is the position vector of a numerical semigroup if and only if

\[
\sum_{x=1}^{i}(u_x + \gamma_x) + \frac{\pi_i + \pi_j - \pi_l}{m + 1} \geq \sum_{x=j+1}^{l}(u_x + \gamma_x),
\]

whenever \( 0 < i \leq j < l \) and \( \pi_i + \pi_j \equiv \pi_l \mod (m + 1) \).

Proof. Let \( (v_1, v_2, \ldots, v_m) \) be the position vector of a numerical semigroup \( S \) with Apéry set \( \text{Ap}(S, m+1) = \{w_0, \ldots, w_m\} \), where \( w_i = (m+1)k_i + \pi_i \) for \( 1 \leq i \leq m \). If \( 0 < i \leq j < l \) and \( \pi_i + \pi_j \equiv \pi_l \mod (m + 1) \), then \( w_i + w_j \equiv w_l \mod (m + 1) \) and by Lemma 3.1 we have \( w_i + w_j \geq w_l \). Thus,

\[
k_i + \frac{\pi_i + \pi_j - \pi_l}{m + 1} \geq k_l - k_j
\]

\[
\sum_{x=1}^{i}(k_x - k_{x-1}) + \frac{\pi_i + \pi_j - \pi_l}{m + 1} \geq \sum_{x=j+1}^{l}(k_x - k_{x-1})
\]

\[
\sum_{x=1}^{i}(u_x + \gamma_x) + \frac{\pi_i + \pi_j - \pi_l}{m + 1} \geq \sum_{x=j+1}^{l}(u_x + \gamma_x).
\]

Essentially reversing these steps provides the converse argument and finishes the proof. \( \square \)

The next example applies Theorem 3.5 to numerical semigroups containing 3.

Example 3.6. Every numerical set closed under addition by 3 belongs to one of two permutation classes, namely one defined by permutation \([12]\) or the permutation \([21]\). We consider these two cases separately:

1. For the permutation \([12]\), we have \( \pi_1 + \pi_1 \equiv \pi_2 \mod 3 \), \( \gamma_1 = 0 \), and \( \gamma_2 = 0 \). Thus,

\[
u_1 + \frac{1 + 1 - 2}{3} \geq u_2 \quad u_1 \geq u_2.
\]
For the permutation \([21]\), we have
\[
\pi_1 + \pi_1 \equiv \pi_2 \mod 3,
\gamma_1 = 0,\text{ and } \gamma_2 = 1.
\]
Thus,
\[
u_1 + \frac{2 + 2 - 1}{3} \geq u_2 + 1
\]
\[
u_1 \geq u_2.
\]
We conclude that the vector \((v_1, v_2)\) corresponds to a numerical semigroup if and only if
\[
u_1 \geq u_2,\text{ or equivalently, }
\]
\[
v_1 - 1 \geq \left\lfloor \frac{v_2 - 1}{2} \right\rfloor.
\]

Using the method derived from Theorem 3.5 and demonstrated in Example 3.6, we summarize the computational results for semigroups containing \(n\) where \(2 \leq n \leq 5\). We omit the details.

**Theorem 3.7.** Let \((v_1, v_2, \ldots, v_{n-1})\) be an \((n-1)\)-tuple of positive integers and set
\[
u_i = \left\lfloor \frac{v_i - 1}{i} \right\rfloor.
\]
The following is a list necessary and sufficient conditions for \(v\) to be the position vector of the a numerical semigroup containing \(n\) for \(2 \leq n \leq 5\).

- \(n=2: (v_1)\) with no restriction
- \(n=3: (v_1, v_2)\) such that \(u_1 \geq u_2\).
- \(n=4: (v_1, v_2, v_3)\) with restrictions given in Table 1

| ~ to one of | satisfying |
|------------|------------|
| \((1,1,1), (1,2,3)\) | \(u_1 \geq u_2\) and \(u_1 \geq u_3\) |
| \((1,1,2), (1,2,2)\) | \(u_1 \geq u_3\) |
| \((1,2,1)\) | \(u_1 \geq u_2 + u_3\) |
| \((1,1,3)\) | \(u_1 \geq u_2 + u_3 + 1\) |

**Table 1.** Restrictions for a 3-tuple to represent a semigroup

- \(n=5: (v_1, v_2, v_3, v_4)\) with restrictions given in Table 2

In Theorem 3.7, \(n\) is an element of the semigroup. By adding an extra restriction, we can force \(n\) to be the multiplicity.

**Proposition 3.8.** Let \(S\) be a semigroup containing \(n\) with position vector \((v_1, v_2, \ldots, v_{n-1})\). Then \(n\) is the multiplicity of \(S\) if and only if \(v_1 > 1\).

**Proof.** We always have \(v_1 \geq 1\). If \(v_1 = 1\), then the first nonzero element of \(S\) is not in \(\text{Ap}(S, n)\). Thus the first nonzero element of \(S\) cannot be smaller than \(n\), and so \(n\) is the multiplicity of \(S\). If \(v_1 = 1\), then the first nonzero element of \(S\) is in \(\text{Ap}(S, n)\). Thus the first nonzero element of \(S\) is smaller than \(n\), and \(n\) is not the multiplicity of \(S\). \(\square\)

**Remark 3.9.** It follows that in Theorem 3.7, we can add the restriction \(u_1 > 0\) to ensure that \(n\) is the multiplicity of \(S\).
Table 2. Restrictions for a 4-tuple to represent a semigroup

| ~ to one of | satisfying |
|------------|------------|
| (1, 1, 1, 1), (1, 1, 2, 2), (1, 2, 2, 3), (1, 2, 3, 4) | \( u_1 \geq u_2, u_1 \geq u_3, u_1 \geq u_4, \) and \( u_1 + u_2 \geq u_3 + u_4 \) |
| (1, 2, 1, 2), (1, 2, 3, 1) | \( u_1 \geq u_2 \) and \( u_1 \geq u_3 + u_4 \) |
| (1, 1, 1, 2), (1, 2, 1, 4) | \( u_1 \geq u_2 + u_3, u_1 \geq u_4, \) and \( u_1 + u_2 \geq u_3 + u_4 \) |
| (1, 2, 3, 3), (1, 1, 3, 1) | \( u_1 \geq u_2 + u_3 + 1, u_1 \geq u_4, \) and \( u_1 + u_2 \geq u_3 + u_4 \) |
| (1, 1, 1, 3), (1, 2, 3, 2), (1, 2, 2, 1), (1, 1, 2, 4), (1, 1, 2, 3), (1, 2, 2, 2) | \( u_1 \geq u_2 + u_3 + u_4 + 1 \) |
| (1, 1, 3, 4) | \( u_1 \geq u_2 + u_3 + u_4 + 2 \) |
| (1, 2, 1, 1) | \( u_1 \geq u_2 + u_3 + u_4 \) |
| (1, 1, 2, 1), (1, 2, 1, 3) | \( u_1 \geq u_2 + u_3 \) and \( u_1 \geq u_3 + u_4 \) |
| (1, 2, 2, 4), (1, 1, 3, 2) | \( u_1 \geq u_2 + u_3 + 1 \) and \( u_1 \geq u_3 + u_4 + 1 \) |

References

[1] Barucci, V., Dobbs, D., Fontana, M.: Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains. Mem. Amer. Math. Soc. 125, vii-77 (1997).
[2] Branco, M., García-García, J., García-Sánchez, P. A., Rosales, J.: Systems of inequalities and numerical semigroups. J. London Math. Soc. (2) 65, 611-623 (2002).
[3] García-Sánchez, P. A., Rosales, J. C.: Numerical semi groups. Developments in Mathematics, 20. Springer, New York (2009). ISBN 978-1-4419-0159-0.
[4] Kaplan, N.: Counting numerical semigroups by genus and some cases of a question of Wilf. J. Pure Appl. Algebra, (5) 216, 1016-1032 (2009).
[5] Marzuola, J., Miller, A.: Counting numerical sets with no small atoms. J. Combin. Theory Ser. A 117 (2010), no. 6, 650-667.
[6] Pellikaan, R., Torres, F.: On Weierstrass semigroups and the redundancy of improved geometric Goppa codes. IEEE Trans. Inform. Theory 45(7) (1999), 2512–2519.
[7] Pemmaraju, S. V., Skiena, S.: “Permutations and combinations” in computational discrete mathematics: combinatorics and graph theory with Mathematica. Cambridge University Press (2003). ISBN 978-0-521-1-2146-0.
[8] Thompkins, C.: Machine attacks on problems whose variable are permutations. Proc. Symposia Applied Mathematics. McGraw-Hill, New York (1956).