Chiral spin chain interfaces as event horizons

Matthew D. Horner, Andrew Hallam, and Jiannis K. Pachos
School of Physics and Astronomy, University of Leeds, Leeds, LS2 9JT, United Kingdom
(Dated: July 20, 2022)

The interface between different quantum phases of matter can give rise to novel physics, such as exotic topological phases or non-unitary conformal field theories. Here we investigate the interface between two spin chains in different chiral phases. Surprisingly, the mean-field theory description of this interacting composite system is given in terms of Dirac fermions in a curved space-time geometry. In particular, the boundary between the two phases represents a black hole horizon. We demonstrate that this representation is faithful both analytically, by employing bosonisation to obtain a Luttinger liquid model, and numerically, by employing Matrix Product State methods. A striking prediction from the black hole equivalence emerges when a quench, at one side of the interface between two opposite chiralities, causes the other side to thermalise with the Hawking temperature for a wide range of parameters and initial conditions.

Interfaces of quantum systems offer a fertile environment for rich and exotic physics to emerge that often cannot be met without the support of bulk systems. For example, domain walls between fractional quantum Hall states can give rise to parafermions, anyons with non-Abelian statistics [1], while domain walls between 2D Heisenberg models can give rise to deconfined fractional excitations [2]. Moreover, higher order topological phases support gapless edge states at boundary defects [3–5] and non-unitary conformal field theories can emerge at the boundary of interacting field theories [6]. The complexity of interfaces, especially for interacting systems, is so high that simple and effective modelling is instrumental to obtaining a qualitative and quantitative understanding of their behaviour.

Here we investigate the interface between two spin-1/2 chains that are in different chiral phases. We consider the XY model Hamiltonian supplemented by a three-spin chiral operator, i.e., our system is intrinsically interacting. Such chiral systems are of much interest as they exhibit a rich spectrum of quantum correlations [7] and can give rise to skyrmionic configurations [8]. We find that such a chiral interface can be effectively modelled by a black hole event horizon. Specifically, we demonstrate that the mean field theory of the system can be modelled as a Dirac fermion in the curved background geometry of a black hole, where the event horizon is positioned at the interface between the two phases. The chiral phase is identified with the inside of the black hole, whilst the non-chiral phase is identified with the asymptotically flat spacetime exterior to the black hole.

To establish the validity of the mean field theory that gives rise to the back hole description and further understand the behaviour of the interacting chiral system, we model it both numerically with Matrix Product State (MPS) methods and analytically by bosonising the system and using the theory of Luttinger liquids. In this way, we show that the horizon description is quantitatively and qualitatively faithful to the one provided by mean field theory. We further demonstrate that the interface is between two conformal field theories with central charge $c = 2$ inside the black hole and $c = 1$ outside, thus identifying the change in the fermionic degrees of freedom across the interface.

To test the faithfulness of the description of our system with black hole physics, we use the mean field description to investigate the time evolution following a quench that propagates through the horizon. We take an interface between two opposite chiralities that models a black hole and white hole opposite from each other. We demonstrate numerically that a pulse in one chiral phase is transmitted through the interface as thermal radiation to the other chiral phase, i.e., the pulse is thermalised when it passes through the event horizon. The temperature of the thermal radiation is approximated well by the Hawking temperature for a wide range of coupling profiles and initial conditions. Hence, the horizon physics can provide a high level modelling of the chiral interface that accurately predicts the evolution of the interacting chiral phases across a phase boundary. We envision that gravity at extreme curvatures can provide an elegant formalism that can efficiently model several strongly interacting systems and their interfaces in higher dimensions.

In this work we consider a chain of $N$ spin-1/2 particles with Hamiltonian

$$H = \sum_{n=1}^{N} \left[ -\frac{u}{2} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) + \frac{v}{4} \chi_n \right], \quad (1)$$

where $u, v \in \mathbb{R}$, $\{\sigma_n^x, \sigma_n^y, \sigma_n^z\}$ are the Pauli matrices of the $n$th spin and $\chi_n$ is the spin chirality given by the three-spin interaction

$$\chi_n = \sigma_n \cdot \sigma_{n+1} \times \sigma_{n+2}, \quad (2)$$

where $\sigma_n$ is the vector of Pauli matrices of the $n$th spin [7, 9]. The chirality operator is a measure of the solid angle spanned by three neighbouring spins.

The model given in Eq. (1) can be mapped to an interacting fermionic Hamiltonian via a Jordan-Wigner transformation. After application of self-consistent mean field (MF) theory to Eq. (1) (see Supplementary Material) we obtain

$$H_{\text{MF}} = \sum_{n=1}^{N} \left( -u c_n^\dagger c_{n+1} - \frac{iv}{2} c_n^\dagger c_{n+2} + \text{h.c.} \right). \quad (3)$$
system we derive the corresponding field theoretic description. The continuum limit of $H_{MF}$ is found by Taylor expanding $H_{MF}$ in momentum space about $p_0$, the Fermi point where the energy becomes zero, to first order in $p$. This yields a Dirac Hamiltonian on a $(1+1)$D spacetime with metric

$$ds^2 = \left(1 - \frac{v^2}{u^2}\right) dt^2 - \frac{2v}{u^2} dtdx - \frac{1}{u^2} dx^2,$$

which is the Schwarzschild metric written in Gullstrand-Painlevé coordinates [15]. If the coupling $v$ is upgraded to a sufficiently slowly-varying function $v(x)$, such as in Fig. 1(b), then this continuum description remains valid and an event horizon is located at the critical point $x_h$, where $|v(x_h)| = |u|$, which corresponds to the location of critical tilt in Fig. 1(c). The corresponding Hawking temperature is given by [12]

$$T_H = \frac{1}{2\pi} \left| \frac{dv(x_h)}{dx} \right| .$$

Therefore, the original spin model of Eq. (1) with inhomogeneous couplings is effectively described by a $(1+1)$D black hole [10–12, 15].

To investigate the nature of quantum phases supported by Eq. (1), and the transitions between them, we consider the case of homogeneous couplings $u$ and $v$. Beginning with the free fermion model derived using the mean field approximation of Eq. (3) reveals that for small $v$ the system is in a disordered, gapless, XY phase, while as $v$ increases it passes through a second-order phase transition into a gapless chiral phase, corresponding to a non-zero ground state chirality $\langle \chi_n \rangle$, as seen in Fig. 2(a), so the chirality behaves as an order parameter. In the mean-field approximation the chiral phase transition is located at $|v| = |u|$, coinciding with the critical tilting of the Dirac cones and the appearance of additional Fermi points, as shown in Fig. 1(c). Near the critical point, 

$$\langle \chi_n \rangle \sim (v - v_c)^\gamma,$$

with critical point $v_c = u$ and critical exponent $\gamma = 1$. On the other hand, studying the full spin model of Eq. (1) using finite DMRG [16], we estimate that the phase transition of the full model is located at $v_c \approx 1.12u$ with a critical exponent of $\gamma \approx 0.39$. A comparison between the chirality of the full spin model at various system sizes and the mean-field model for inhomogeneous and homogeneous couplings can be seen in Fig. 1(b) and Fig. 2(a) respectively, showing that the chirality behaves as an order parameter and the effectiveness of the mean field approximation.

To gain further insight into the nature of the chiral phase transition, we consider the behaviour of the bipartite entanglement entropy as $v$ is increased. As discussed above, the model is gapless for all $v$, i.e. it can be described by a conformal field theory (CFT). In this case we...
FIG. 2. (a) The average ground state chirality, ⟨χ⟩ = \sum_n(χ_n)/N, for the mean-field (MF) model (N = 500) and spin model found using DMRG (N = 200, D = 300) where u = 1. (b) The central charge, c, obtained for the MF model (N = 500) and the spin model found using DMRG (v ≤ u: N = 200, D = 300 and v > u: N = 160, D = 800). (c)-(d) The Fermi velocities, v_u and v_R respectively (u = 1) derived from the mean field (MF) and Luttinger liquid descriptions compared to the numerical results of the MPS excitation ansatz for the spin model at bond dimension D = 36 in the thermodynamic limit.

expect the ground state entanglement entropy of a partition of L \ll N spins to obey the Cardy formula

\[ S_L = \frac{c}{3} \ln L + S_0, \quad (8) \]

where c is the central charge of the CFT and S_0 is a constant [17]. Using this formula, we estimate the central charge as a function of v for the full spin model and the mean field approximation. In Fig. 2(b) we see that c ≈ 1 in the XY phase which jumps to c ≈ 2 in the chiral phase, with good agreement between the spin and mean field results. We can clearly interpret this in the mean-field model: the additional Fermi points appearing when |v| > |u| cause the model to transition from a c = 1 CFT with a single Dirac fermion to a c = 2 = 1 + 1 CFT with two Dirac fermions, as seen by the additional Fermi points of the dispersion in Fig. 1(c). This can also be understood from the lattice structure of the MF model, as seen in Fig. 1(a), where for |v| \ll |u| a single zig-zag fermionic chain dominates (c = 1) while for |v| \gg |u| two fermionic chains dominate, corresponding to the stands of the ladder, thus effectively doubling the degrees of freedom (c = 2).

As we have seen numerically, the mean field approximation can faithfully reproduce many of the features of the fully interacting model, especially for |v| < |u|, suggesting that the interactions are not significant. We investigate this further by bosonising the spin Hamiltonian for |v| < |u|, mapping it to a Luttinger liquid [18–20]. After a Jordan-Wigner transformation, the spin Hamiltonian takes the form \[ H = H_{\text{MF}} + H_{\text{int}}, \] where \( H_{\text{MF}} \) is the quadratic mean-field Hamiltonian of Eq. (3) and \( H_{\text{int}} \) is an interaction term containing quartic terms. For |v| < |u|, the single-band mean field dispersion relation of Eq. (4) suggests that the spin model has two Fermi points located at \( p_{R,L} = \pm \pi/2 \), with the Fermi velocities \( v_{R,L} = 2(\pm u - v) \). By expanding around these Fermi points and bosonising the interaction terms using the methods of [18–20], we find the fully interacting Hamiltonian

\[ H = u \int dx \left[ \Pi^2 + (\partial_x \Phi)^2 \right], \quad (9) \]

where the fields obey the canonical commutation relations [\( \Phi(x), \Pi(y) = i\delta(x - y) \)]. The interactions of \( H_{\text{int}} \) simply rescaled the Fermi velocities \( v_{R,L} \rightarrow v'_{R,L} = 2(\pm u - v (1 - 2/\pi)) \), but leaves the Luttinger parameter unchanged at \( K = 1 \) (see Supplementary Material), suggesting the model remains a non-interacting free fermion model.

The dispersion of the full spin model as a function of v for |v| < |u| can be calculated using the MPS excitation ansatz working in the thermodynamic limit [21]. The full dispersion features unequal left- and right-moving Fermi velocities whose magnitudes change oppositely with v—the signature of tilting of the cones—and the appearance of additional Fermi points, similar to the mean field model. In Fig. 2(c) and 2(d), the Fermi velocities \( v_{L,R} \) obtained from the mean field approximation, the Luttinger liquid model and the spin Hamiltonian are compared. We see that the Luttinger liquid model is much more accurate than the mean field approximation. We expect that the quantitative disagreement and the observed asymmetry in the change to the left and right velocities to be lifted at higher order in perturbation theory.

It has been shown in the literature that many analogue gravitational systems in condensed matter will exhibit a Hawking-like effect [10–12, 22–32]. Reversing the argument, we would like to see if the Hawking radiation can effectively describe quenched time evolutions across the chiral interface. To be able to simulate large system sizes and long evolution times, we resort to the mean field description of Eq. (3) rather than the full spin model. Consider an open, inhomogeneous system with couplings \( u(x) = 1 \) and

\[ v(x) = \alpha \tanh[\beta(x - x_\text{h})], \quad (10) \]

where \( \alpha, \beta \in \mathbb{R} \) and \( x_\text{h} \) is the centre of the system. Here, we take x as the unit cell coordinate in order to align with our continuum limit conventions (see Supplementary Material). This produces a positive and negative chiral region separated by a small zero-chirality region in the centre of the system. For large enough \( \alpha \) and \( \beta \), the zero-chirality region has arbitrarily small size, so the system effectively models an interface between two oppositely polarized chiral phases, corresponding to a black hole-white hole interface respectively in the continuum.
FIG. 3. Thermalisation of the mean field (MF) model after a quench to the Hawking temperature $T_H$. (a) the lattice wavefunction $\psi_n$ on the right half of the system $(n \in [n_h, N])$ transmitted through the horizon, for the couplings $u = 1$ and $v$ given by Eq. (10) with $\alpha = 20$, $\beta = 0.1$, and the horizon at $n_h = N/2$ with $N = 500$. The particle tunnels across at $t \approx 2$ and a small wavepacket escapes into the other half, which we interpret as Hawking radiation. (b) A snapshot of the overlap $-\ln P$ vs. the energy of the state $E$ at time $t = 4.5$. The system thermalises shortly after the particle passes through the interface, displaying a linear dependence on $E$, where the gradient is given by $1/T$. (c) The extracted temperature $T$ of the radiation vs. $\alpha$ extracted after a short time $t = 4.5$. $T$ grows linearly with $\alpha$, very close to the predictions of the Hawking formula $T_H \approx \alpha \beta / 2\pi$.

Following the method of Ref. [22], we initialise a single-particle state $|n_0\rangle = c_{n_0}^{\dagger}|0\rangle$ on the $n_0$th lattice site inside the left half of the system, and let the wavefunction evolve freely across the boundary into the other half via the Hamiltonian $H_{\text{MF}}$, as shown in Fig. 3(a). We then measure the overlap of the wavefunction with localised energy modes that exist only on the other side of the boundary as

$$P(k, t) = |\langle k | e^{-iHt} | n_0 \rangle|^2,$$

where $|k\rangle$ are the single-particle eigenstates of the Hamiltonian $H_{\text{out}}$, where $H_{\text{out}}$ is the Hamiltonian of Eq. (3) truncated to the outside region only. This method utilises the result that Hawking radiation can be viewed as quantum tunnelling [33].

We find numerically that the interface between the two chiral phases thermalises the wavefunction: once the wavefunction evolves across the interface, the distribution takes the form $P(k, t) \propto e^{-E(k)/T}$, where $T$ is some effective temperature. Fig. 3(b) shows the distribution $P(k, t)$ at time $t = 4.5$ for a system with parameters $N = 500$, $n_h = 250$, $\alpha = 20$ and $\beta = 0.1$, where we prepared the particle at $n_0 = 230$. It is clear that $P(k, t)$ follows a Boltzmann distribution at some temperature $T$, where the gradient of the line is given by $1/T$. We observe the system strongly thermalises. The value $\beta = 0.1$ is taken to suppress the effects from having finite lattice spacing and finite system size. In Fig. 3(c), we present the dependence of the measured temperature $T$ on the magnitude $\alpha$ for $\beta = 0.1$. We see it closely follows the predictions of the Hawking formula $T_H = \alpha \beta / 2\pi$, obtained from Eq. (6) and Eq. (10), for a wide range of couplings, $\alpha$, thus accurately modelling the physics of the chiral interface. The thermalisation to the Hawking $T$ breaks down when $\alpha < 4$ as the couplings will not be sharp enough to provide a sufficient interface, whereas for large $\alpha$ the couplings vary too fast for the continuum approximation to be valid, which is where the black hole physics should emerge.

The Hawking temperature $T_H = \alpha \beta / 2\pi$ of our system is a very simple analytic formula that describes a complex thermalisation process. In particular, it does not depend on the initial value $n_0$, i.e., the position where quench starts, nor the horizon location $n_h$ which effectively gives the size ratio between the two chiral phases in our system. To verify these properties, we numerically determine the dependence of $T$ on $n_0$ (Fig. 4(a)) and $n_h$ (Fig. 4(b)). We see that the measured temperature $T$ is largely insensitive to the initial position of the particle $n_0$. The black hole description only fails if $n_0$ is initially too close or too far away from the interface or when the interface $n_h$ is too close the edges of the system. In all these cases boundary effects start to contribute and the exterior region which the overlap $P(k, t)$ is measured in becomes too small. These two observations show that the thermalisation across the interface is robust and will aid in any potential experimental realisation of the model.
In this work, we demonstrated that that low-energy behaviour of a chiral fermionic phase can be described by Dirac fermions on a black hole background. The interface of such a chiral phase with a non-chiral phase gives rise to an event horizon with quantum properties modelled by Hawking radiation. This analogy allowed us to predict the time evolution of a quenched system with a chiral interface which should thermalise to the Hawking temperature $T_H$. We demonstrated that this is indeed the case for a wide variety of quenches, positions of the interface and coupling parameters, thus providing a faithful high level description of chiral interfaces. To demonstrate the faithfulness of our approach we employed mean field theory, matrix product state techniques and bosonisation to fully characterise the phase diagram of the model. We envision that this bridge between chiral systems and black holes can facilitate the quantum simulation of Hawking radiation, e.g. with cold atom technology [29, 31, 34]. Moreover, our investigation opens the way for modelling certain strongly correlated systems by effective geometric theories with extreme curvature, thus providing an intuitive tool for their analytical investigation.

Acknowledgements: We thank Patricio Salgado-Rebolledo, Joe Barker and Diptiman Sen for insightful discussions. M.D.H., A.H. and J.K.P. acknowledge support by EPSRC (Grant No. EP/R020612/1).

[1] D. Clarke, J. Alicea, and K. Shtengel, Exotic non-abelian anyons from conventional fractional quantum Hall states, Nat Commun 3, 1348 (2013).
[2] C. D. Batista and S. A. Trugman, Exact ground states of a frustrated 2d magnet: Deconfined fractional excitations at a first-order quantum phase transition, Phys. Rev. Lett. 93, 217202 (2004).
[3] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Quantized electric multipole insulators, Science 357, 61 (2017).
[4] J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Reflection-symmetric second-order topological insulators and superconductors, Phys. Rev. Lett. 119, 246401 (2017).
[5] M. Ezawa, Higher-order topological insulators and semimetals on the breathing kagome and pyrochlore lattices, Phys. Rev. Lett. 120, 026801 (2018).
[6] P. Dey and A. Söderberg, On analytic bootstrap for interface and boundary cft, J. High Energ. Phys. 2021 (7).
[7] D. I. Tsomokos, J. J. García-Ripoll, N. R. Cooper, and J. R. Pachos, Chiral entanglement in triangular lattice models, Phys. Rev. A 77, 012106 (2008).
[8] Y. Tikhonov, S. Kondovych, J. Mangeri, M. Pavlenko, L. Baudry, A. Sené, A. Galda, S. Nakhmanson, O. Heinonen, A. Razumay, I. Luk’yanchuk, and V. M. Vinokur, Controllable skyrmion chirality in ferroelectrics, Sci Rep 10, 8657 (2020).
[9] C. D’Cruz and J. K. Pachos, Chiral phase from three-spin interactions in an optical lattice, Phys. Rev. A 72, 043608 (2005).
[10] G. Volovik and K. Zhang, Lifshitz transitions, type-II dirac and weyl fermions, event horizon and all that, J. Low Temp. Phys. 189, 276–299 (2017).
[11] G. Volovik and P. Hulttala, Fermionic microstates within the painlevé-gullstrand black hole, J. Exp. Theor. Phys. 94, 853–861 (2002).
[12] G. Volovik, Black hole and hawking radiation by type-II weyl fermions, Jpett Lett. 104, 645–648 (2016).
[13] H. Nielsen and M. Ninomiya, A no-go theorem for regularizing chiral fermions, Physics Letters B 105, 219 (1981).
[14] H. Nielsen and M. Ninomiya, Absence of neutrinos on a lattice: (ii). intuitive topological proof, Nuclear Physics B 193, 173 (1981).
[15] G. E. Volovik, The Universe in a Helium Droplet, 2nd ed. (Clarendon Press, 2003) p. 424.
[16] U. Schollwöck, The density-matrix renormalization group in the age of matrix product states, Annals of physics 326, 96 (2011).
[17] P. Calabrese and J. Cardy, Entanglement entropy and quantum field theory, Journal of Statistical Mechanics: Theory and Experiment 2004, P06002 (2004).
[18] T. Giamarchi, Quantum Physics in One Dimension (Oxford University Press, 2003).
[19] E. Miranda, Introduction to bosonization, Brazilian Journal of Physics 33 (2002).
[20] S. Aditya and D. Sen, Bosonization study of a generalized statistics model with four fermi points, Phys. Rev. B 103, 235162 (2021).
[21] J. Haegeman, T. J. Osborne, and F. Verstraete, Post-matrix product state methods: To tangent space and beyond, Phys. Rev. B 88, 075133 (2013).
[22] R.-Q. Yang, H. Liu, S. Zhu, L. Luo, and R.-G. Cai, Simulating quantum field theory in curved spacetime with quantum many-body systems, Phys. Rev. Research 2, 023107 (2020).
[23] D. Sabsovich, P. Wunderlich, V. Fleurov, D. I. Pikulin, R. Ilan, and T. Meng, Hawking fragmentation and hawking attenuation in weyl semimetals, Phys. Rev. Research 4, 013055 (2022).
[24] D. Maertens, N. Bultinck, and K. Van Acoleyen, Hawking radiation on the lattice as universal (floquet) quench dynamics (2022).
[25] H. Huang, K.-H. Jin, and F. Liu, Black-hole horizon in the dirac semimetal zn_2znxs, Phys. Rev. B 98, 121110 (2018).
[26] H. Liu, J.-T. Sun, C. Song, H. Huang, F. Liu, and S. Meng, Fermionic analogue of high temperature hawking radiation in black phosphorus, Chinese Physics Letters 37, 067101 (2020).
[27] S. Guan, Z.-M. Yu, Y. Liu, G.-B. Liu, L. Dong, Y. Lu, Y. Yao, and S. A. Yang, Artificial gravity field, astrophysical analogues, and topological phase transitions in strained topological semimetals, npj Quantum Materials 2, 23 (2017).
[28] A. Retzker, J. I. Cirac, M. B. Plenio, and B. Reznik, Methods for detecting acceleration radiation in a bose-einstein condensate, Phys. Rev. Lett. 101, 110402 (2008).
[29] J. Rodríguez-Laguna, L. Tarruell, M. Lewenstein, and A. Celi, Synthetic unruh effect in cold atoms, Phys. Rev.
Appendix A: Mean field theory and its results

1. Jordan-Wigner transformation

In this work we study a modification of the 1D spin-1/2 XY model. The Hamiltonian is given by

$$H = \frac{1}{2} \sum_{n=1}^{N} \left[ -\frac{u}{2} \left( \sigma_{n}^{x} \sigma_{n+1}^{x} + \sigma_{n}^{y} \sigma_{n+1}^{y} \right) + \frac{v}{4} \chi_{n} \right] \equiv H_{\text{HX}} + H_{\chi}, \quad (A1)$$

where $u, v \in \mathbb{R}$, $\{\sigma_{n}^{x}, \sigma_{n}^{y}, \sigma_{n}^{z}\}$ are the Pauli matrices of the $n$th spin and $\chi_{n}$ is the spin chirality given by the three-spin interaction $\chi_{n} \equiv \sigma_{n} \cdot (\sigma_{n+1} \times \sigma_{n+2})$ [7, 9], where $\sigma_{n}$ is the vector of Pauli matrices of the $n$th spin. We apply periodic boundary conditions $\sigma_{n} = \sigma_{n+N}$ throughout, however we always have the thermodynamic limit $N \rightarrow \infty$ in mind.

In order to make analytic progress with this model, we map from the language of spins to the language of fermions by applying a Jordan-Wigner transformation defined as

$$\sigma_{n}^{+} = \exp \left( -i\pi \sum_{m<n} c_{m}^{\dagger} c_{m} \right) c_{n}^{\dagger}, \quad \sigma_{n}^{-} = \exp \left( i \pi \sum_{m<n} c_{m}^{\dagger} c_{m} \right) c_{n}, \quad \sigma_{n}^{z} = 1 - 2c_{n}^{\dagger} c_{n}, \quad (A2)$$

where $\sigma_{n}^{\pm} = (\sigma_{n}^{x} \pm i \sigma_{n}^{y})/2$ and $c_{n}$ are fermionic operators obeying the commutation relations $\{c_{n}, c_{m}^{\dagger}\} = \delta_{nm}$ and $\{c_{n}, c_{m}\} = \{c_{n}^{\dagger}, c_{m}^{\dagger}\} = 0$. Using the definition of $\sigma_{n}^{\pm}$, we have the useful identities

$$\sigma_{n}^{+} \sigma_{m+1}^{-} + \sigma_{m}^{+} \sigma_{m+1}^{+} = 2 \sigma_{m+1}^{+} \sigma_{m+1}^{-} + \text{h.c.}, \quad (A3)$$

$$\sigma_{n}^{+} \sigma_{m+1}^{-} - \sigma_{m}^{+} \sigma_{m+1}^{+} = 2i \sigma_{m+1}^{+} \sigma_{m+1}^{-} + \text{h.c.}. \quad (A4)$$

The first identity allows us to rewrite $H_{\text{XY}}$ straight away, whilst the second identity allows us rewrite the chirality $\chi_{n}$ as

$$\chi_{n} = \epsilon_{abc} \sigma_{n}^{a} \sigma_{n+1}^{b} \sigma_{n+2}^{c}$$

$$= (\sigma_{n}^{x} \sigma_{n+1}^{y} - \sigma_{n}^{y} \sigma_{n+1}^{x}) \sigma_{n+2}^{z} + (\sigma_{n+1}^{x} \sigma_{n+2}^{y} - \sigma_{n+1}^{y} \sigma_{n+2}^{x}) \sigma_{n}^{z} + (\sigma_{n+2}^{x} \sigma_{n+2}^{y} - \sigma_{n+2}^{y} \sigma_{n+2}^{x}) \sigma_{n}^{z} + 2i (\sigma_{n+1}^{x} \sigma_{n+2}^{y} + \sigma_{n+1}^{y} \sigma_{n+2}^{x} + \sigma_{n+2}^{x} \sigma_{n+1}^{y} + \sigma_{n+2}^{y} \sigma_{n+1}^{x}) + \text{h.c.}. \quad (A5)$$

With this, the Hamiltonian of Eq. (A1) takes the form

$$H = \sum_{n} \left[ -u \sigma_{n}^{+} \sigma_{n+1}^{-} + \frac{iv}{2} \left( \sigma_{n}^{+} \sigma_{n+1}^{-} \sigma_{n+2}^{+} + \sigma_{n+1}^{+} \sigma_{n+2}^{-} \sigma_{n+2}^{+} + \sigma_{n+2}^{+} \sigma_{n+2}^{-} \sigma_{n+1}^{+} \right) \right] + \text{h.c.}. \quad (A6)$$

Now the Hamiltonian is in a convenient form, we apply a Jordan-Wigner transformation. We have

$$\sigma_{n+1}^{+} \sigma_{n+1}^{-} = c_{n+1}^{\dagger} c_{n+1}, \quad (A7)$$

$$\sigma_{n+2}^{+} \sigma_{n+2}^{-} = c_{n+2}^{\dagger} \exp (-i\pi c_{n+1}^{\dagger} c_{n+1}) c_{n}, \quad (A8)$$

therefore the Hamiltonian transforms to

$$H = \sum_{n} \left[ -u c_{n}^{\dagger} c_{n+1} + \frac{iv}{2} \left( c_{n}^{\dagger} c_{n+1} \sigma_{n+2}^{+} + c_{n+1}^{\dagger} c_{n+2} \sigma_{n}^{+} - c_{n}^{\dagger} c_{n+2} \right) \right] + \text{h.c.}, \quad (A9)$$
where the final term loses its $\sigma^z_{n+1}$ because $\sigma^z_{n+1} = \exp(i\pi c_{n+1}^\dagger c_{n+1})$ which cancels with the exponential obtained from the Jordan-Wigner transformation in Eq. (A8). We also swap the final term for its Hermitian conjugate which picks up a minus sign.

For a system with periodic boundary conditions, after applying the Jordan-Wigner transformation, we would pick up boundary terms which couple both lattice sites $n = N$ and $n = N - 1$ to the first lattice site $n = 1$, however this term contributes an order $O(1/N)$ correction to the Hamiltonian which can we can safely ignore as we assume we work in the thermodynamic limit for large $N$ [35].

2. Self-consistency equation

The Jordan-Wigner transformation brings the Hamiltonian to an interacting fermionic Hamiltonian in Eq. (A9) due to the four-fermion interaction terms contained in, for example, $c_n^\dagger c_{n+1}^\dagger \sigma^z_{n+2}$, therefore this Hamiltonian cannot be diagonalised easily. In order to make progress, we apply mean field theory to transform this Hamiltonian into a non-interacting quadratic Hamiltonian. We replace the operators $\sigma^z_n$ with their expectation values as $\sigma^z_n \rightarrow \langle \sigma^z_n \rangle \equiv Z$, where the expectation value is done with respect to the ground state of the mean field Hamiltonian and we assume translational invariance to drop the index $n$. This gives us the mean field Hamiltonian

$$H_{MF}(Z) = \sum_n \left[ -(u - ivZ)c_n^\dagger c_{n+1} - \frac{iv}{2} c_n^\dagger c_{n+2} \right] + \text{h.c.} \equiv \sum_{n,m} h_{nm}c_n^\dagger c_m$$

(A10)

which is now a quadratic Hamiltonian as a function of $Z$ that can be diagonalised exactly using by diagonalising the single-particle Hamiltonian $h_{ij}$. In order for this to be self-consistent, we require

$$\langle \Omega(Z) | \sigma^z_n | \Omega(Z) \rangle = Z,$$

(A11)

where $|\Omega(Z)\rangle$ is the ground state of $H_{MF}(Z)$, obtained by occupying all of the negative energy single-particle modes of $h$. Solving this equation numerically, we find that the solution is $Z = 0$. This could also be deduced as $H_{MF}(Z)$ has no chemical potential, therefore the ground state should be at half-filling $\langle c_n^\dagger c_n \rangle = 1/2$ which corresponds to $Z = 0$. Hence, our mean-field Hamiltonian is given by

$$H_{MF} = \sum_n \left( -uc_n^\dagger c_{n+1} - \frac{iv}{2} c_n^\dagger c_{n+2} \right) + \text{h.c.}$$

(A12)

3. Correlations

For the case of the homogeneous model (constant $u$ and $v$) the Hamiltonian of Eq. (A12) has translational symmetry so can be diagonalised exactly with a discrete Fourier transform

$$c_n = \frac{1}{\sqrt{N}} \sum_{p \in \text{B.Z.}} e^{ipn} c_p$$

(A13)
where B.Z. = [−π, π) and k is quantised as k = 2mπ/N for m ∈ ℤ. This yields
\[ H_{MF} = \sum_p E(p)c_p^\dagger c_p, \quad E(p) = -2u \cos(p) + v \sin(2p), \quad (A14) \]

where \( E(p) \) is the dispersion as shown in Fig. (5) for two different values of \( v \). The Fermi points of this model are the points for which \( E(p) = 0 \). We have the usual Fermi points at \( p_{RL, L} = ±\frac{\pi}{2} \), but if \( |v| > |u| \) we have an additional two crossings at \( p_1 = \sin^{-1} \left( \frac{v}{u} \right) \) and \( p_2 = \pi - p_1 \). Note that the Fermi velocities at each point \( v_p = E'(p_p) \), where \( \mu \) labels the Fermi points, are unequal—the signature of tilting cones as we shall see later. These additional zero energy crossings are a result of the Nielsen-Ninomiya theorem which states that the number of left-movers and right-movers in a lattice model must be equal and they break up the negative energy portion of the Brillouin zone into two disconnected regions as shown by the shaded portions in Fig. (5).

The correlation matrix is defined as \( C_{nm} = \langle \Omega_{MF} | c_n^\dagger c_m | \Omega_{MF} \rangle \), where \( |\Omega_{MF}\rangle \) is the ground state of the Hamiltonian of Eq. (A12). Mapping to momentum space with a discrete Fourier transform of Eq. (A13), we can write
\[ C_{nm} = \frac{1}{N} \sum_{p,q \in BZ} e^{-ipn} e^{iqm} \langle \Omega | c_p^\dagger c_q | \Omega \rangle = \frac{1}{N} \sum_{p : E(p) < 0} \frac{1}{2\pi} \int_{p : E(p) < 0} dp e^{-ip(n-m)}, \quad (A15) \]

where in the second equality we used the fact that the ground state \( |\Omega\rangle \) has all negative energy states occupied, so \( \langle \Omega | c_p^\dagger c_q | \Omega \rangle = \delta_{pq} \theta(-E(p)) \) and in the final equality we took the thermodynamic limit by moulding the sum into a Riemann sum with \( \Delta p = 2\pi/N \) and taking the limit as \( N \to \infty \).

In the following calculations we assume that \( u, v > 0 \). For \( v < u \) the correlation function is given by
\[ C_{nm} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dp e^{-ip(n-m)} = \frac{\sin \left( \frac{(n-m)\pi}{2} \right)}{\pi(n-m)}, \quad (A16) \]

which is independent of \( v \) and is the same result obtained for the XY model (\( v = 0 \)). For \( v > u \), the negative energy portion of the Brillouin zone splits into two disconnected regions as shown in Fig. 5 so the integral splits into two as
\[ C_{nm} = \frac{1}{2\pi} \left( \int_{-\pi/2}^{\pi/2} dp + \int_{\pi/2}^{\pi/2} dp \right) e^{-ip(n-m)} = \frac{i}{2\pi(n-m)} \left\{ (-1)^{n-m} e^{i\pi(n-m)} + e^{-i\pi(n-m)} - 2 \cos \left( \frac{(n-m)\pi}{2} \right) \right\} \quad (A17) \]

which is now a function of \( v \) and is complex in general.

4. Phase transitions

First, we look at the ground state energy density. The density in the thermodynamic limit is given by
\[ \rho_0 = \lim_{N \to \infty} \frac{1}{N} \sum_{p : E(p) < 0} E(p) = \frac{1}{2\pi} \int_{p : E(p) < 0} dp E(p) \quad (A18) \]

where we took the thermodynamic limit by using the standard trick of moulding the sum into a Riemann sum and taking the limit. Evaluating the integrals for both \( v < u \) and \( v > u \) yields
\[ \rho_0 = \begin{cases} \frac{-2u}{\pi} & v \leq u \\ \frac{-1}{\pi} \left( \frac{u^2}{v} + v \right) & v > u \end{cases} \quad (A19) \]

We see that \( \frac{\partial^2 \rho_0}{\partial v^2} \) is discontinuous at \( v = u \) and hence this point corresponds to a second-order phase transition.

Now we look at the chirality. If we expand out the chirality operator of the Hamiltonian in Eq. (A9) by using the definition \( \sigma_n^z = 1 - 2c_n^\dagger c_n \), we find
\[ \chi_n = 2i(t_n^1 c_{n+1} + c_n^1 c_{n+2} + c_{n+2}^1 c_n) - 4i(c_n^1 c_{n+1} c_{n+2} + c_n^1 c_{n+1}^1 c_{n+2}) + h.c., \quad (A20) \]

which contains quartic terms. We are interested in the ground state expectation value of this quantity, where the ground state \( |\Omega_{MF}\rangle \) is the ground state of the mean field Hamiltonian Eq. (A12). As \( H_{MF} \) is a free Hamiltonian, we can employ Wick’s theorem to evaluate the chirality. A contraction between two operators is defined as
\[ \widehat{AB} = AB - :AB:, \quad (A21) \]
where the colons denote normal ordering with respect to the ground state. From this, we see that contractions between fermions give
\[
\langle \sigma^z_{n,n} \rangle = \langle c_n^\dagger c_m \rangle, \quad \langle c_n^\dagger c_m \rangle = \langle c_n^\dagger c_m \rangle^\dagger = 0.
\] (A22)

Using Wick’s theorem, the quartic terms in the chirality are given by
\[
c_n^\dagger c_{n+1}^\dagger c_{n+2}^\dagger c_{n+2} = \langle c_n^\dagger c_{n+1}^\dagger c_{n+2}^\dagger c_{n+2}^\dagger \rangle - \langle c_n^\dagger c_{n+1}^\dagger \rangle \langle c_{n+2}^\dagger c_{n+2}^\dagger \rangle + \langle c_{n+1}^\dagger c_{n+2}^\dagger \rangle \langle c_n^\dagger c_n^\dagger \rangle^\dagger + \text{(partially contracted terms)}:
\] (A23)

and similarly for the second quartic term in Eq. (A20), where we have anti-commuted fermions next to each other before contracting. Using the fact that the ground state expectation value of any normal-ordered operator is zero, the partially contracted terms vanish and we find the chirality is given by
\[
\langle \chi_n \rangle = 2i(C_{n,n+1} + C_{n+1,n+2} + C_{n+2,n}) - 4i(C_{n,n+1}C_{n+2,n+2} - C_{n+1,n}C_{n,n+2} + C_{n+1,n+2}C_n, C_n - C_{n+1,n}C_{n,n+2}) + \text{c.c.}
\] (A24)

We see the chirality takes the form \(\langle \chi_n \rangle = iz + \text{c.c.} = -2\text{Im}(z)\), so it is a function of the imaginary part of the correlation matrix. Therefore, a necessary condition for chirality is that our correlation matrix must be complex, so straight away we can deduce that the chirality will be zero for \(|v| < |u|\) as the correlation matrix here is real, as given by Eq. (A16). Using the correlation matrix Eq. (A17), we find that the chirality is given by
\[
\langle \chi_n \rangle = \begin{cases} 
0 & v \leq u \\
\frac{i}{2} \left(1 - \frac{v^2}{u^2}\right) \left(1 - \frac{4u}{w^2}\right) & v > u
\end{cases}
\] (A25)

so we see that the chirality behaves as an order parameter. Close to the critical point \(v = u\) we have
\[
\chi_j(v) \approx \chi_j(u) + (v - u)\chi_j'(u) \propto v - u,
\] (A26)

so the critical exponent is equal to 1.

In order to achieve a complex correlation matrix, and hence chirality, we require a Hamiltonian that breaks inversion symmetry and has complex next-to-nearest-neighbour correlations, the simplest of which is our Hamiltonian.

Appendix B: Luttinger model

1. Particle-hole symmetry

Let us return to the full spin model of Eq. (A1). After a Jordan-Wigner transformation, we arrived at the interacting Hamiltonian
\[
H = \sum_n \left(-uc_n^\dagger c_{n+1} - \frac{iv}{2} c_n^\dagger c_{n+2} + \frac{iv}{2} \sum_n (c_n^\dagger c_{n+1} \sigma_{n+2}^z + c_n^\dagger c_{n+2} \sigma_{n+1}^z) + \text{h.c.} \right) \equiv H_0 + H_{\text{int}},
\] (B1)

where \(\sigma_n^z = 1 - 2c_n^\dagger c_n\). This fully interacting Hamiltonian has particle-hole symmetry under the transformation
\[
c_n \rightarrow U^\dagger c_n U = (-1)^n c_n^\dagger, \quad c_n^\dagger \rightarrow U^\dagger c_n^\dagger U = (-1)^n c_n.
\] (B2)

Let us look at the consequences of this symmetry. Exact diagonalisation of the spin Hamiltonian reveals that the ground state is non-degenerate for an even number of lattice sites so we fix \(N \in 2\mathbb{N}\) to avoid any subtleties due to degeneracy, therefore our ground state will be an eigenstate of \(U\) with a phase of \(\pm 1\) as \(U^2 = \mathbb{I}\). Suppose we calculated the ground state density, we have
\[
\langle \Omega | c_n^\dagger c_n | \Omega \rangle = \langle \Omega | U^\dagger c_n^\dagger c_n U | \Omega \rangle
\]
\[
= (-1)^n \langle \Omega | c_n^\dagger c_n | \Omega \rangle
\]
\[
= \langle \Omega | (1 - c_n^\dagger c_n) | \Omega \rangle
\]
\[
= 1 - \langle \Omega | c_n^\dagger c_n | \Omega \rangle
\] (B3)

\[
\Rightarrow \langle \Omega | c_n^\dagger c_n | \Omega \rangle = \frac{1}{2}
\]
which is our usual half-filling result. Now, for the nearest-neighbour correlations we have

\[
\langle \Omega | c_n^\dagger c_{n+1} | \Omega \rangle = \langle \Omega | U^\dagger c_n^\dagger c_{n+1} U | \Omega \rangle \\
= (-1)^{2n+1} \langle \Omega | c_n^\dagger c_{n+1}^\dagger | \Omega \rangle \\
= \langle \Omega | c_{n+1}^\dagger c_n | \Omega \rangle \\
= \langle \Omega | c_n^\dagger c_{n+1} | \Omega \rangle^* \tag{B4}
\]

therefore the nearest-neighbour correlators are real. We use these results in the following calculation.

For a product of two operators, normal ordering amounts to subtracting off the ground state expectation value as \( A : = \langle \Omega | A | \Omega \rangle \). We can use this to simplify the interaction term of Eq. (B1) which prepares us for bosonisation later. We have

\[
e_n^\dagger c_{n+1} = :c_n^\dagger c_{n+1}: + \langle \Omega | c_n^\dagger c_{n+1} | \Omega \rangle \equiv :c_n^\dagger c_{n+1}: + \alpha, \tag{B5}
\]

where we have defined the correlation \( \alpha = \langle \Omega | c_n^\dagger c_{n+1} | \Omega \rangle \). Similarly, we have

\[
\sigma_n^\dagger = 1 - 2c_n^\dagger c_n = 1 - 2( : c_n^\dagger c_n : + \langle \Omega | c_n^\dagger c_n | \Omega \rangle) = -2 : c_n^\dagger c_n :, \tag{B6}
\]

where we used the half filling result \( \langle \Omega | c_n^\dagger c_n | \Omega \rangle = \frac{1}{2} \). From this, we can substitute this into the interaction Hamiltonian of Eq. (B1) to give

\[
H_{\text{int}} = -iv \sum_n \left[ ( : c_n^\dagger c_{n+1} : + \alpha) : c_{n+2}^\dagger c_{n+2} : + \left( : c_{n+1}^\dagger c_{n+2} : + \alpha \right) : c_n^\dagger c_n : \right] + \text{h.c.}
= -iv \sum_n \left( : c_n^\dagger c_{n+1} : c_{n+2}^\dagger c_{n+2} : + : c_{n+1}^\dagger c_{n+2} : c_n^\dagger c_n : \right) + \text{h.c.}, \tag{B7}
\]

where we used the fact that \( \alpha \) is real and \( : c_n^\dagger c_n : \) is Hermitian to get rid of \( \alpha \).

\section{Expanding the fields about the Fermi points}

For the phase \(|v| < |u|\), the mean field theory agrees extremely well with the total spin model and demonstrates that the additional chirality term interaction is irrelevant for ground state properties, whereby the model behaves as if it is the XY model \((v = 0)\). In this phase, the mean field also suggests that the model has two Fermi points at \( p_{R,L} = \pm \frac{\pi}{2} \). Therefore, we expand our fields as

\[
\frac{c_n}{\sqrt{\Omega}} = \sum_{\mu=R,L} e^{ip_\mu a_n} \psi_\mu(x_n), \tag{B8}
\]

where the sum is over the Fermi points, \( \psi_\mu(x) \) is a continuous field sampled at discrete lattice sites and we have reinstated the lattice spacing \( a \).

First, we substitute the expansion of Eq. (B8) into \( H_0 \) of Eq. (B1) to give

\[
H_0 = \sum_{\mu,\nu} \sum_n a e^{-i(p_\mu - p_\nu) a_n} \left[ -ue^{ip_\mu a_n} \psi_\mu(x_n) \psi_\nu(x_{n+1}) - \frac{iv}{2} e^{2ip_\mu a_n} \psi_\mu(x_n) \psi_\nu(x_{n+2}) \right] + \text{h.c.}. \tag{B9}
\]

We now discard any oscillating term in the Hamiltonian as these integrate to zero, so we requires \( p_\mu = p_\nu \) in the first phase. This yields

\[
H_0 = \sum_\mu \sum_n a \left[ -ue^{ip_\mu a_n} \psi_\mu^\dagger (\psi_\mu + a \partial_x \psi_\mu + O(a^2)) - \frac{iv}{2} e^{2ip_\mu a_n} \psi_\mu^\dagger (\psi_\mu + 2a \partial_x \psi_\mu + O(a^2)) \right] + \text{h.c.}
= -i \sum_\mu \sum_n a^2 \left( \pm u \psi_\mu^\dagger \partial_x \psi_\mu - v \psi_\mu^\dagger \partial_x \psi_\mu \right) + O(a^3) + \text{h.c.} \tag{B10}
\]

\[
\rightarrow -2i \sum_\mu \int dx \psi_\mu^\dagger \partial_x \psi_\mu, 
\]
where in the second line \( \pm \) corresponds to \( \mu = R, L \). We have renormalised the couplings as \( au \to u \) and \( av \to v \). The Fermi velocities are given by \( v_{RL} = 2(\pm u - v) \).

We now repeat the procedure for the interaction term \( H_{\text{int}} \) of Eq. (B1). We substitute in the expansion of Eq. (B8) into \( H_{\text{int}} \) to give

\[
H_{\text{int}} = -iv \sum_{\mu,\nu,\alpha,\beta} \sum_{n} e^{-i(p_{\mu} - p_{\nu} + p_{\alpha} - p_{\beta})n} e^{i(p_{\mu} - 2(p_{\nu} - p_{\beta})n)} e^{-i(p_{\mu} - 2p_{\nu})n} \langle \psi_{\mu}^\dagger \psi_{\nu} \rangle : \psi_{\alpha}^\dagger \psi_{\beta} : + O(a^3) + \text{h.c.} , \tag{B11}
\]

where we have expanded all fields to zeroth order in \( a \) to ensure the Hamiltonian retains order \( a^2 \) and renormalised the couplings as \( av \to v \). We discard any term that oscillates which requires \( p_{\mu} - p_{\nu} + p_{\alpha} - p_{\beta} = 2n\pi/a \) for \( n \in \mathbb{Z} \). With this we find only four terms survive giving us

\[
H_{\text{int}} = 2v \int dx \left( \rho_R^2 + \rho_R \rho_L - \rho_L \rho_R - \rho_R^2 \right) + \text{h.c.} = 4v \int dx \left( \rho_R^2 - \rho_L^2 \right) , \tag{B12}
\]

where we have defined the normal-ordered densities \( \rho_{\mu} = : \psi_{\mu}^\dagger \psi_{\mu} : \).

3. Bosonising the Hamiltonian

If we pull everything together, the normal-ordered Hamiltonian is given by

\[
: H : = : H_0 + H_{\text{int}} : = -i \sum_{\mu = R, L} \int dx \left( v_{\mu} : \psi_{\mu}^\dagger \partial_x \psi_{\mu} : \pm 4v : \rho_{\mu}^2 : \right) , \tag{B13}
\]

where the \( \pm \) corresponds to \( R \) and \( L \) respectively. Following Ref. [19], we map the fermionic fields \( \psi_{\mu} \) to bosonic fields \( \phi_{\mu} \) with the mapping

\[
\psi_{R,L} = F_{R,L} \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\frac{2N_{R,L}}{\pi} x} e^{-i\frac{\pi}{2} \phi_{R,L}} , \quad \rho_{R,L} = N_{R,L} L + \frac{1}{\sqrt{2\pi}} \partial_x \phi_{R,L} , \tag{B14}
\]

where \( N_{R,L} \) are defined as the normal ordered number operators for the right- and left-moving excitations respectively, \( L = Na \) is the system’s length, \( F_{R,L} \) are a pair of Klein factors and \( \alpha \) is a cutoff. The bosonic fields obey the commutation relations

\[
[\phi_{RL}(x), \phi_{RL}(y)] = \pm \frac{i}{2} \delta_{\mu\nu} \text{sgn}(x - y) , \tag{B15}
\]

whilst pairs of fields about different Fermi points commute. The fermionic fields and densities obey the useful identities

\[
: \psi_{R,L}^\dagger \partial_x \psi_{R,L} : = \pm \frac{i}{2} \partial_x \phi_{R,L} , \quad \rho_{R,L} = \pm \frac{1}{\sqrt{2\pi}} \partial_x \phi_{R,L} , \tag{B16}
\]

where we have taken \( L \to \infty \). With this, the Hamiltonian is mapped to

\[
: H : = \int dx \left( \frac{1}{2} \left[ |v_R| : (\partial_x \phi_R)^2 : + |v_L| : (\partial_x \phi_L)^2 : \right] + \frac{2v}{\pi} \left[ (\partial_x \phi_R)^2 - (\partial_x \phi_L)^2 \right] \right) \tag{B17}
\]

where the renormalised Fermi velocities are given by

\[
v_{R,L}' = 2 \left( \pm u - v \left( 1 - \frac{2}{\pi} \right) \right) . \tag{B18}
\]

As the Fermi velocities of the model are not equal, we must generalise the bosonisation procedure of Ref. [19]. Define the canonical transformation

\[
\Phi = \sqrt{\frac{N}{2}} \left( |v_L'| \phi_L - |v_R'| \phi_R \right) , \quad \Theta = \sqrt{\frac{N}{2}} \left( |v_L'| \phi_L + |v_R'| \phi_R \right) , \tag{B19}
\]
where $\mathcal{N}$ is a constant to ensure the fields obey the correct commutation relations. Just as for the case of equal Fermi velocities in Ref. [19], we require the fields $\Phi$ and $\Theta$ to obey the commutation relations

$$[\Phi(x), \Theta(y)] = -\frac{i}{2} \text{sgn}(x - y). \quad (B20)$$

In terms of our canonical transformation, we have

$$\mathcal{N} = \frac{v'}{2} \left( |v'_L| |\phi_L(x), \phi_L(y)\rangle - |v'_R| |\phi_R(x), \phi_R(y)\rangle \right)$$

$$= \frac{v'}{2} \left( -\frac{i}{2} |v'_L| \text{sgn}(x - y) - \frac{i}{2} |v'_R| \text{sgn}(x - y) \right)$$

$$= -\frac{i}{4} (|v'_L| + |v'_R|) \text{sgn}(x - y),$$

therefore we require

$$\mathcal{N} = \frac{2}{|v'_L| + |v'_R|} = \frac{1}{2u}. \quad (B21)$$

Inverting the canonical transformation of Eq. (B19), we have

$$\sqrt{|v'_L|} \phi_- = \sqrt{u} (\Theta + \Phi), \quad \sqrt{|v'_R|} \phi_+ = \sqrt{u} (\Theta - \Phi). \quad (B23)$$

Substituting this back into the bosonised Hamiltonian of Eq. (B17), we have

$$: H : = u \int dx \left[ : (\partial_x \Phi)^2 : + : (\partial_x \Phi)^2 : \right]. \quad (B24)$$

If we differentiate the commutator $[\Phi(x), \Theta(y)]$ with respect to $y$, we find $[\Phi(x), \partial_y \Theta(y)] = i \delta(x - y)$, so we can identify the canonical momentum as $\Pi(x) = \partial_x \Theta(x)$. Therefore, the Hamiltonian takes the form of the free boson

$$: H : = u \int dx \left[ : \Pi^2 : + : (\partial_x \Phi)^2 : \right], \quad (B25)$$

which is exactly the same result obtained from bosonising the XY model ($v = 0$) which demonstrates that the interactions for $|v| < |u|$ are irrelevant in the ground state. According to the theory of Luttinger liquids, this implies that $K = 1$ which is the sign of non-interacting fermions [18].

**Appendix C: Emergent Black Hole**

In order to make the link with relativity, we now label the lattice sites as alternating between sub-lattices $A$ and $B$ by introducing a two-site unit cell. We can rewrite the mean field Hamiltonian of Eq. (A12) as

$$H_{MF} = \sum_n \left[ -ua_n^\dagger (b_n + b_{n-1}) - i\frac{v}{2} (a_n^\dagger a_{n+1} + b_n^\dagger b_{n+1}) \right] + \text{h.c.,} \quad u, v \in \mathbb{R}, \quad (C1)$$

where the Fermions obey the commutation relations $\{a_n, a_m^\dagger\} = \{b_n, b_m^\dagger\} = \delta_{nm}$, while all other commutators vanish. The index $n$ now labels the unit cells. We Fourier transform the fermions with the definition

$$a_n = \frac{1}{\sqrt{N_c}} \sum_{p \in \text{B.Z.}} e^{ipn} a_p \quad (C2)$$

and similarly for $b_n$, where $N_c = N/2$ is the number of unit cells in the system and $a_c = 2a$ is the unit cell spacing for a given lattice spacing $a$. Applying this to the Hamiltonian, we arrive at

$$H_{MF} = \sum_p \chi_p^\dagger h(p) \chi_p, \quad (C3)$$
where we have defined the two-component spinor $\chi_\mu = (a_\mu, b_\mu)^T$ and the single-particle Hamiltonian

$$h(p) = \begin{pmatrix} g(p) & f(p) \\ f^*(p) & g(p) \end{pmatrix}, \quad f(p) = -u(1 + e^{-i a_\mu p}), \quad g(p) = v \sin(a_\mu p).$$

(C4)

The dispersion relation is given by

$$E(p) = g(p) \pm |f(p)| = v \sin(a_\mu p) \pm u \sqrt{2 + 2 \cos(a_\mu p)}.$$  

(C5)

In Fig. 6, we see that the parameter $v$ has the effect of tilting the cones.

The Fermi points of the dispersion are located at $p_0 = \frac{\pi}{a_c}$ and $p_1 = \frac{1}{a_c} \arccos(1 - \frac{2v^2}{u^2})$ however $p_1$ only exists if $|v| \leq |u|$, where the cone is located at $p_1$. Let’s focus on the cone: we take the continuum limit by Taylor expanding the single-particle Hamiltonian $h(p)$ about the Fermi point $p_0$. We have

$$h(p_0 + p) = a_\mu u \sigma^\mu p - a_\mu v \bar{\sigma}^\mu p + O(p^2) \equiv e_\mu^i \alpha^a p_i.$$  

(C6)

where we have defined the coefficients $e_0^a = -a_\mu v, e_1^a = a_\mu u$ and the matrices $\alpha^0 = \mathbb{I}, \alpha^1 = \sigma^y$. Now we take the continuum limit $a_\mu \to 0$ and the thermodynamic limit $N_\sigma \to \infty$ so both the real space lattice and Brillouin zone become isomorphic to the real line. During the limiting process, we renormalise the couplings $a_\mu u \to u$ and $a_\mu v \to v$, where $u$ and $v$ are non-zero and finite and we define the continuum limit coordinate $x = na_\mu$. Note that, due to the bipartite labelling of the lattice, the coordinate $x$ labels the unit cells. Therefore, the continuum limit Hamiltonian after an inverse Fourier transform to real space is given by

$$H = \int dx \chi^\dagger(x) \begin{pmatrix} -ie_\mu^i \alpha^a \bar{\partial}_i & \end{pmatrix} \chi(x)$$

(C7)

where we have defined $A\partial_\mu B = \frac{i}{2} (A\partial_\mu B - (\partial_\mu A)B)$ which only acts on spinors, the Dirac alpha and beta matrices $\alpha^a = (1, \sigma^y)$ and $\beta = \sigma^z$.

The action corresponding to the Hamiltonian of Eq. (C7) is given by

$$S = \int_M d^{1+1}x \chi^\dagger(x) \begin{pmatrix} \bar{\partial}^i + ie_\mu^i \alpha^a \partial_i & \end{pmatrix} \chi(x)$$

$$= \int_M d^{1+1}x \chi(x) \bar{\epsilon}_a \gamma^a \partial_\mu \chi(x),$$  

(C8)

where we have defined the Dirac gamma matrices $\gamma^0 = \beta = \sigma^z$ and $\gamma^i = \beta \alpha^i$, and $\bar{\chi} = \chi^\dagger \sigma^z$. We see that this action corresponds to the action of a Dirac spinor $\chi$ on a flat Minkowski spacetime with space-dependent parameters $e_\mu^a$.

The quantities $e_\mu^a$ look very similar to a tetrad basis if we were doing field theory on a curved spacetime. The Dirac action on curved space is given by

$$S = \int_M d^{1+1}x |e| \left[ \frac{i}{2} (\bar{\psi} \gamma^\mu D_\mu \psi - \overline{D_\mu \psi} \gamma^\mu \psi) - m \bar{\psi} \psi \right],$$  

(C9)

where $D_\mu = \partial_\mu + \omega_\mu$ is the covariant derivative, $\gamma^\mu = e_\mu^a \gamma^a$ and $\omega_\mu = \frac{1}{8} \omega_{\alpha \beta} (\gamma^\alpha, \gamma^\beta)$, where $e_\mu^a$ and $\omega_{\alpha \beta}$ are the components of the the vielbein and spin connection respectively [36]. Comparing Eq. (C8) to Eq. (C9), we see that
we can interpret the continuum limit of the lattice model as a curved space field theory if we define the spinor $\psi$ related to $\chi$ via

$$\chi = \sqrt{|e|} \psi,$$

where $\psi$ is a spinor field which propagates on a spacetime with tetrad

$$e^\mu_a = \begin{pmatrix} 1 - v & -v \\ 0 & u \end{pmatrix}, \quad e^\mu_a = \begin{pmatrix} 1 & v/u \\ 0 & 1/u \end{pmatrix},$$

and Dirac gamma matrices $\gamma^0 = \sigma^z$ and $\gamma^1 = -i\sigma^x$ which obey the anti-commutation relations $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, where $\eta^{ab} = \text{diag}(1, -1)$. The vielbein corresponds to the metric $g_{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$, which is explicitly given by

$$g_{\mu\nu} = \begin{pmatrix} 1 - v^2/u^2 & -v/u^2 \\ -v/u^2 & 1/u^2 \end{pmatrix},$$

or equivalently in terms of differentials

$$ds^2 = \left(1 - \frac{v^2}{u^2}\right) dt^2 - \frac{2v}{u^2} dt dx - \frac{1}{u^2} dx^2.$$

This is the Schwarzschild metric expressed in Gullstrand-Painleve coordinates [15] which is sometimes known as the acoustic metric. We refer to this metric as an internal metric of the model as it depends upon the internal couplings of the Hamiltonian and not the physical geometry of the lattice.

In order to bring the metric Eq. (C13) into standard form, we employ the coordinate transformation $(t, x) \mapsto (\tau, x)$ via

$$\tau(t, x) = t + \int_{x_0}^x dz \frac{v(z)}{u^2 - v^2(z)},$$

which maps the metric to

$$ds^2 = \left(1 - \frac{v^2}{u^2}\right) d\tau^2 - \frac{1}{u^2 \left(1 - \frac{v^2}{u^2}\right)} dx^2,$$

which is a metric in Schwarzshild form. If we upgrade $u$ and $v$ to slowly-varying functions of position, then the preceding calculation is still valid and the event horizon is therefore located at the point $x_h$ where $|v(x_h)| = |u(x_h)|$.

In this project, we fix $u(x) = 1$ so it aligns with the standard Schwarzschild metric in natural units. Using the Hawking formula for the temperature of a black hole, the temperature is given by [12]

$$T_H = \frac{1}{2\pi |v'(x_h)|}.$$  

Appendix D: Integrability and thermalisation

It is important to note that, as has been discussed elsewhere, e.g. [23], the Hawking temperature obtained in this study is observed through scattering processes rather than in the equilibration values of observables. The effective thermalisation observed through $H_{\text{MF}}$ takes place at very short time-scales after release of the particle. If we allow the system to evolve for a long time, it will not equilibrate to a thermal state, but instead it will equilibrate to a generalised Gibbs ensemble [37, 38]—this is because the mean field Hamiltonian $H_{\text{MF}}$ is integrable. An initial analysis of the energy-level statistics of the full spin Hamiltonian of Eq. (1) suggests to us that the model may be integrable—showing Poisson level statistics—however we leave a systematic study of this to future work.