On the semiprimitivity of free skew extensions of rings

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Abstract

Let $X$ be a set of noncommuting variables, and $G = \{\sigma_x\}_{x \in X}$, $D = \{\delta_x\}_{x \in X}$ be sequences of automorphisms and skew derivations of a ring $R$. It is proved that if the ring $R$ is semiprime Goldie, then the free skew extension $R[X; G, D]$ is semiprimitive.

Keywords: semiprime Goldie ring, skew derivation, free skew extension, Jacobson radical

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Introduction

A well-known result of S. A. Amitsur $[1]$ states that if the ring $R$ has no nil ideals then the polynomial ring $R[x]$ is semiprimitive. Subsequently, there has been a great deal of work examining the Jacobson radicals of more general ring extensions such as skew polynomial rings of automorphism type and of derivation type. For skew polynomial rings $R[x; \sigma]$ of automorphism type, it was shown in $[2]$ that even if $R$ is commutative and reduced, then the Jacobson radical $J(R[x; \sigma])$ can be nonzero. Many authors, including C. R. Jordan and D. A. Jordan $[9]$, A. D. Bell $[3]$, S. S. Bedi and J. Ram $[2],[14]$ and A. Moussavi $[12, 13]$ have extended Amitsur’s result to skew polynomial rings of the form $R[x; \sigma, \delta]$ and with certain additional conditions on $R$, where $\sigma$ is an automorphism (or monomorphism) of $R$, and $\delta$ is a $\sigma$-derivation of $R$. Very important and deep results on the Jacobson radical of skew polynomial rings of derivation type were obtained by A. Smoktunowicz (see $[16]$). For other results for such rings we refer to $[4]$, $[5]$, $[8]$, $[11]$ and $[17]$.

Recall that for a given unital ring $R$ with a ring endomorphism $\sigma$, a $\sigma$-derivation of $R$ is an additive map $\delta: R \to R$ satisfying the $\sigma$-Leibniz rule

$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y),$$

for $x, y \in R$. Then the skew polynomial ring $R[x; \sigma, \delta]$ can be described as the ring generated freely over $R$ by an element $x$ subject to the relation $xr = \sigma(r)x + \delta(r)$ for each $r \in R$. Throughout this paper we consider skew polynomial rings over an arbitrary set of noncommuting variables. More precisely, let $X$ be a nonempty set and suppose that for any $x \in X$ there exists a pair $(\sigma_x, \delta_x)$, where $\sigma_x: R \to R$ is a ring endomorphism and $\delta_x: R \to R$ is a $\sigma_x$-derivation. Put $G = \{\sigma_x\}_{x \in X}$ and $D = \{\delta_x\}_{x \in X}$.

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\( \mathcal{D} = \{ \delta_x \}_{x \in X} \). Let us emphasize that we do not assume that the mappings \( x \mapsto \sigma_x \) and \( x \mapsto \delta_x \) are injective. Next we denote by \( \langle X \rangle \) the free monoid generated by the set of free generators \( X \). We will write \( S = R[X; \mathcal{G}, \mathcal{D}] \) provided

1. \( S \) contains \( R \) as a subring;
2. \( X \subseteq S; \)
3. \( S \) is a free left \( R \)-module with basis \( \langle X \rangle; \)
4. \( xr = \sigma_x(r)x + \delta_x(r) \) for all \( x \in X \) and \( r \in R \).

Such a ring \( S \) we call a free skew extension of \( R \). If \( X = \{ x \}, \mathcal{G} = \{ \sigma \} \) and \( \mathcal{D} = \{ \delta \} \), then by Proposition 2.3 in \([7]\) \( S \) is the skew polynomial ring \( R[x; \sigma, \delta] \). In Section 1 we prove that free skew extensions \( S = R[X; \mathcal{G}, \mathcal{D}] \) always exist and they are unique up to isomorphism. The main result of the paper, which we will prove in Section 2, is

**Theorem A.** Let \( R \) be a semiprime left Goldie ring. Then for any set \( X \) of noncommuting variables and sequences \( \mathcal{G} = \{ \sigma_x \}_{x \in X}, \mathcal{D} = \{ \delta_x \}_{x \in X} \) of automorphisms and skew derivations of \( R \) the free skew extension \( R[X; \mathcal{G}, \mathcal{D}] \) is semiprimitive.

It is worth noting that A. D. Bell proved in \([3]\) that if \( \sigma \) is an automorphism and \( \delta \) is a \( \sigma \)-derivation of a semiprime Goldie ring \( R \), then \( R[x; \sigma, \delta] \) is a semiprimitive Goldie ring. This result was then extended by A. Moussavi \([12]\) and A. Moussavi, E. Hashemi \([13]\) to the case when \( \sigma \) is an injective endomorphism of \( R \). Observe that if the set \( X \) has at least two elements, then any free skew extension \( S = R[X; \mathcal{G}, \mathcal{D}] \) has infinite Goldie dimension. Indeed, if \( x, y \in X, x \neq y \), then the sum of left ideals \( \sum_{n=1}^{\infty} S_{xy^n} \) is direct, and so \( S \) is not a Goldie ring.

Notice that if \( \mathcal{G} = \{ id_R \}_{x \in X} \), then \( \mathcal{D} \) is a sequence of ordinary derivations of \( R \). This special case was considered in \([18]\) under the name of the Ore extension of derivation type, and denoted by \( R[X; \mathcal{D}] \). In Section 2 we apply Theorem A to free skew extensions of a certain class of rings, which includes rings with Krull dimension. In particular we obtain

**Corollary B.** Let \( R \) be an algebra over a field \( F \) of characteristic zero. If \( R \) has left Krull dimension, then for any set \( X \) of noncommuting variables and a sequence \( \mathcal{D} = \{ d_x \}_{x \in X} \) of derivations of \( R \) the Jacobson radical of the Ore extension \( R[X; \mathcal{D}] \) is nilpotent.

1. Preliminary results and definitions

In this Section we prove the existence and uniqueness of free skew extensions. To do this, we will apply the arguments from the book by K. R Goodearl and R. B Warfield \([7]\) used in the proof of the existence and uniqueness of skew polynomial rings \( R[x; \sigma, \delta] \).

**Proposition 1.1.** Given a unital ring \( R \), a set \( X \) of noncommuting variables and sequences \( \mathcal{G} = \{ \sigma_x \}_{x \in X}, \mathcal{D} = \{ \delta_x \}_{x \in X} \) of ring endomorphisms and skew derivations of \( R \), there exists a free skew extension \( R[X; \mathcal{G}, \mathcal{D}] \).

**Proof.** Let \( R\langle X \rangle \) be a monoid ring of the free monoid \( \langle X \rangle \) over \( R \). Then any element of \( R\langle X \rangle \) can be written uniquely in the form

\[
f(X) = \sum_{\Delta \in \langle X \rangle} r_{\Delta} \Delta \quad \text{(or shorter } f(X) = \sum r_{\Delta} \Delta),
\]

in which \( r_{\Delta} \) is a formal coefficient for each \( \Delta \in \langle X \rangle \).
where \( r_\Delta \in R \) and all but finitely many of these elements are zero. Let \( E = \text{End}_R(R(X)) \) be the ring of all endomorphisms of the additive group \( R(X) \). Then \( R \) can be treated as a subring of \( E \) via left multiplications \( r \mapsto r_x \); that is \( r_x(f(X)) = rf(X) \), where \( r \in R \), \( f(X) \in R(X) \).

For \( x \in X \) define \( \hat{x} \in E \) according to the rule

\[
\hat{x}(\sum r_\Delta \Delta) = \sum (\sigma_x(r_\Delta)x\Delta + \delta_x(r_\Delta)\Delta).
\]

Next, let \( S \) be the subring of \( E \) generated by \( R \cup \{ \hat{x} | x \in X \} \). Notice that for any \( r \in R \) and any \( f(X) = \sum r_\Delta \Delta \) in \( R(X) \)

\[
\hat{x}r(f(X)) = \hat{x}(\sum rr_\Delta \Delta) = \sum (\sigma_x(rr_\Delta)x\Delta + \delta_x(r_\Delta)\Delta)
\]

\[
= \sum (\sigma_x(r)\sigma_x(r_\Delta)x\Delta + \delta_x(r)r_\Delta \Delta + \sigma_x(r)\delta_x(r_\Delta)\Delta)
\]

\[
= \sigma_x(r)\sum (\sigma_x(r_\Delta)x\Delta + \delta_x(r_\Delta)\Delta) + \delta_x(r)\sum r_\Delta \Delta
\]

\[
= \sigma_x(r)\hat{x}(f(X)) + \delta_x(r)(f(X) = (\sigma_x(r)\hat{x} + \delta_x(r))(f(X)).
\]

Thus

\[
\hat{x}r = \sigma_x(r)\hat{x} + \delta_x(r) \quad \text{for all } x \in X \text{ and } r \in R \tag{1.1}
\]

For \( \Delta = x_1x_2 \ldots x_n \in \langle X \rangle \) let \( \hat{\Delta} = \hat{x}_1\hat{x}_2 \ldots \hat{x}_n \). The freeness of \( \langle X \rangle \) implies that \( \langle \hat{X} \rangle = \{ \hat{\Delta} | \Delta \in \langle X \rangle \} \) is a homomorphic image of the monoid \( \langle X \rangle \). The relations (1.1) imply that every element of \( S \) can be written in the form \( \sum r_\Delta \hat{\Delta} \), so \( S \) is generated by \( \langle \hat{X} \rangle \) as a left \( R \)-module. Observe that \( \hat{\Delta}(1) = \Delta \), so if \( \sum r_\Delta \hat{\Delta} \) is a zero map, then

\[
0 = (\sum r_\Delta \hat{\Delta})(1) = \sum r_\Delta \Delta.
\]

This forces that all coefficients \( r_\Delta \) are equal to 0, and hence \( S \) is a free left \( R \)-module with basis \( \langle \hat{X} \rangle \). Notice also that the property \( \hat{\Delta}(1) = \Delta \), implies that the monoid \( \langle \hat{X} \rangle \) is free, and clearly isomorphic to \( \langle X \rangle \). Thus the ring \( S \) satisfies all required conditions to be a free skew extension \( R[X; \mathcal{G}, \mathcal{D}] \).

Recall that the degree of a monomial \( \Delta = x_1x_2 \ldots x_n \in \langle X \rangle \) is defined to be \( n \) and denoted by \( \deg \Delta \). The uniqueness of free skew extensions \( R[X; \mathcal{G}, \mathcal{D}] \) will follow from the following universal mapping property

**Proposition 1.2.** Let \( S = R[X; \mathcal{G}, \mathcal{D}] \) be a free skew extension, where \( \mathcal{G} = \{ \sigma_x \}_{x \in X} \), \( \mathcal{D} = \{ \delta_x \}_{x \in X} \) are sequences of ring endomorphisms and skew derivations of \( R \), respectively. Suppose that \( T \) is a ring such that we have a ring homomorphism \( \psi: R \to T \), and a mapping \( \varphi: X \to T \) such that

\[
\varphi(x)\psi(r) = \psi(\sigma_x(r))\varphi(x) + \psi(\delta_x(r))
\]

for all \( x \in X \) and \( r \in R \). Then there is a unique ring homomorphism \( \overline{\psi}: S \to T \) such that \( \overline{\psi}|_R = \psi \) and \( \overline{\psi}(x) = \varphi(x) \) for \( x \in X \).
Proof. Let $\overline{\varphi}: \langle X \rangle \to T$ be a unique extension of $\varphi$ to a homomorphism of monoids. It is clear that the only possibility for $\overline{\psi}$ is the map given by the rule

$$
\overline{\psi}(\sum r_\Delta \Delta) = \sum \psi(r_\Delta)\overline{\varphi}(\Delta)
$$

Notice that $\overline{\psi}$ is additive, and so it is enough to show that $\overline{\psi}(ab) = \overline{\psi}(a)\overline{\psi}(b)$ for $a, b \in S$. Observe that if $b = \sum b_\Delta \Delta \in S$ and $x \in X$, then

$$
\overline{\psi}(xb) = \overline{\psi}(\sum \sigma_x(b_\Delta) x_\Delta) = \sum \psi(\sigma_x(b_\Delta)) \overline{\varphi}(x_\Delta) + \sum \psi(\delta_x(b_\Delta)) \overline{\varphi}(\Delta) = (\sum \psi(\sigma_x(b_\Delta)) \varphi(x) + \psi(\delta_x(b_\Delta))) \overline{\varphi}(\Delta) = \varphi(x) \overline{\psi}(b) = \overline{\varphi}(x) \overline{\psi}(b)
$$

Next it follows by induction on $\deg \Delta$ that $\overline{\psi}(\Delta b) = \overline{\varphi}(\Delta) \overline{\psi}(b)$ for all $\Delta$ in $\langle X \rangle$. Furthermore if $r \in R$, then

$$
\overline{\psi}(rb) = \sum \psi(r b_\Delta) \overline{\varphi}(\Delta) = \sum \psi(r) \psi(b_\Delta) \overline{\varphi}(\Delta) = \psi(r) \sum \psi(b_\Delta) \overline{\varphi}(\Delta) = \psi(r) \overline{\psi}(b).
$$

Consequently, if $a = \sum a_\Delta \Delta$, then

$$
\overline{\psi}(ab) = \sum \overline{\psi}(a_\Delta \Delta b) = \sum \psi(a_\Delta) \overline{\psi}(\Delta b) = \sum \psi(a_\Delta) \overline{\varphi}(\Delta) \overline{\psi}(b) = \overline{\psi}(a) \overline{\psi}(b).
$$

This finishes the proof. $\square$

As an immediate consequence of Proposition 1.2 we obtain the uniqueness of free skew extensions.

Corollary 1.3. Let $R$ be a ring, $\mathcal{G} = \{\sigma_x\}_{x \in X}$, $\mathcal{D} = \{\delta_x\}_{x \in X}$ be sequences of ring endomorphisms and skew derivations of $R$, respectively. Suppose that $\varphi: X \to X'$ is a bijection of sets, and put $\mathcal{G}' = \{\sigma_{\varphi^{-1}(x)}\}_{x' \in X'}$, $\mathcal{D}' = \{\delta_{\varphi^{-1}(x')}\}_{x' \in X'}$ where $x' = \varphi(x)$ for $x \in X$. If $S = R[X; \mathcal{G}, \mathcal{D}]$ and $S' = R[X'; \mathcal{G}', \mathcal{D}']$, then there exists a unique isomorphism $\psi: S \to S'$ such that $\psi|_R$ is the identity map on $R$ and $\psi(x) = x'$ for $x \in X$.

Let us fix a linear order $\prec$ on the set $X$. Then one can extend it lexicographically to the monoid $\langle X \rangle$; that is $x_{i_1} \cdots x_{i_k} \prec x_{j_1} \cdots x_{j_l}$ if and only if

- $k < l$ or
- $k = l$ and there exists $s$, $1 \leq s \leq k$ such that $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$, $i_s \neq j_s$ and $x_{i_s} \prec x_{j_s}$.

If $0 \neq f(X) = \sum r_\Delta \Delta \in R[X; \mathcal{G}, \mathcal{D}]$, then the set $\text{supp}(f(X)) = \{\Delta \mid r_\Delta \neq 0\}$ is called the support of $f(X)$. The largest monomial $\Delta_0$ in the support of $f(X)$ (with respect to $\prec$) is called the leading term of $f(X)$. The coefficient $r_{\Delta_0}$ of the leading term is called the leading coefficient of $f(X)$. Thus any nonzero element $f = f(X) \in R[X; \mathcal{G}, \mathcal{D}]$ has a unique decomposition

$$
f = r_{\Delta_0} + (f)_\prec \tag{1.2}
$$

where $\Delta_0$ is the leading term of $f$, $(f)_\prec$ is a finite sum of monomials $r_\Delta \Delta$ such that $\Delta \neq \Delta_0$ and $\Delta \prec \Delta_0$.

For a subset $A$ of $R$ we let $\text{l.ann}_R(A) = \{r \in R \mid rA = 0\}$, $\text{r.ann}_R(A) = \{r \in R \mid Ar = 0\}$ and $\text{ann}_R(A) = \{r \in R \mid rA = Ar = 0\}$ be the annihilators of $A$ in $R$. Recall that the ring $R$ is said to be left Goldie, if it contains no infinite direct sum of nonzero left ideals and satisfies the ascending chain condition on left annihilators.
2. Free skew extensions of semiprime Goldie rings

Throughout this Section we consider free skew extensions of automorphic type, that is we assume that \( \mathcal{G} = \{ \sigma_x \}_{x \in X} \) is a sequence of automorphisms and \( \mathcal{D} = \{ \delta_x \}_{x \in X} \) is a sequence of respective \( \sigma_x \)-derivations of a unital ring \( R \). Our goal is to prove that if the ring \( R \) is semiprime Goldie, then any free skew extension \( R[X; \mathcal{G}, \mathcal{D}] \) of automorphic type is semiprimitive.

We start with the following general lemma.

**Lemma 2.1.** Let \( I \) be an ideal of a semiprime ring \( R \) and let \( c \in I \) such that \( c \) is regular in \( I \). If \( M = \text{ann}_R(I) \), then

1. \( \text{l.ann}_R(c) = \text{r.ann}_R(c) = M \) and
2. \( \eta(c) \) is regular in the factor ring \( R/M \), where \( \eta: R \to R/M \) is the canonical epimorphism \( R \) onto \( R/M \).

**Proof.** We begin by observing that since \( R \) is semiprime, the left and right annihilators of \( I \) coincide and have zero intersection with \( I \). Since \( M = \text{ann}_R(I) \), it is certainly the case that \( Mc = cM = 0 \). Therefore to prove (1), we need to show that \( \text{l.ann}_R(c) \) and \( \text{r.ann}_R(c) \) are contained in \( M \). Suppose \( r \in R \) such that \( cr = 0 \); therefore \( crI = 0 \). Hence \( rI \) consists of elements of \( I \) which annihilate \( c \) on the right. However \( c \) is regular in \( I \), thus \( rI = 0 \). As a result \( r \in \text{ann}_R(I) = M \). An analogous argument shows that if \( rc = 0 \), then \( r \in \text{ann}_R(I) = M \). Thus both \( \text{l.ann}_R(c) \) and \( \text{r.ann}_R(c) \) are contained in \( M \).

Next suppose \( s \in R \) such that \( cs \in M \); therefore \( csI = 0 \). By the previous paragraph, we now have \( sI \subseteq I \cap \text{r.ann}_R(c) \subseteq I \cap M = 0 \). Thus \( s \in \text{ann}_R(I) = M \). This shows that the right annihilator of \( \eta(c) \) in \( R/M \) is zero and an analogous argument works for the left annihilator. Thus \( \eta(c) \) is regular in \( R/M \).

We continue with the following proposition.

**Proposition 2.2.** Let \( R \) be a semiprime left Goldie ring and let \( R[X; \mathcal{G}, \mathcal{D}] \) be a free skew extension of \( R \). If \( A \) is a nonzero ideal of \( R[X; \mathcal{G}, \mathcal{D}] \) and \( I \) is the additive subgroup of \( R \) generated by the leading coefficients of elements of \( A \) (with respect to a fixed linear order \( \prec \) on \( \langle X \rangle \) ), then

1. \( I \) is a two-sided ideal of \( R \),
2. \( \sigma_x(I) \subseteq I \), for every automorphism \( \sigma_x \) corresponding to \( x \in X \)
3. if \( M = \text{ann}_R(I) \), then \( \sigma_x(M) = M \) and \( \delta_x(M) \subseteq M \), for every \( x \in X \).

**Proof.** For (1), it suffices to show that if \( a \in I \) and \( r \in R \), then \( ra, ar \in I \). Since \( a \in I \),

\[
a = a_1 + \cdots + a_m,
\]

where each \( a_i \) is the leading coefficient of some \( \omega_i \in A \). Therefore \( ra_i \) is either 0 or a leading coefficient of \( r\omega_i \in A \). Thus

\[
ra = ra_1 + \cdots + ra_m,
\]

hence \( ra \) is the sum of leading coefficients of elements of \( A \) and \( ra \in I \).
According to (1.2), each \( \omega_i \) in the previous paragraph can be written as \( \omega_i = a_i \Delta_i + (\omega_i)_\prec \), where \( \Delta_i \) is the leading term of \( \omega_i \), and \( (\omega_i)_\prec \) is a finite sum of monomials \( r_\Delta \Delta \) such that \( \Delta \neq \Delta_i \) and \( \Delta \prec \Delta_i \). If \( s \in R \), then

\[
\Delta_i s = \pi(s) \Delta_i + (\Delta_i)_\prec,
\]

for some automorphism \( \pi \) of \( R \). More precisely, if \( \Delta = x_1 x_2 \ldots x_n \), then \( \pi = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_n} \).

Therefore

\[
\omega_i \pi^{-1}(r) = a_i r \Delta_i + (\omega_i \pi^{-1}(r))_\prec.
\]

Consequently, as above, \( a_i r \) is either 0 or the leading coefficient of \( \omega_i \pi^{-1}(r) \in A \). Thus

\[
ar = a_1 r + \cdots + a_m r,
\]

hence \( ar \) is the sum of leading coefficients of elements of \( A \) and \( ar \in I \).

For (2), let \( a, a_i, \omega_i \) be as in the proof of (1). Take \( x \in X \). Then \( x \omega_i \in A \) and

\[
x \omega_i = \sigma_x(a_i)(x \Delta_i) + (x \omega_i)_\prec.
\]

Therefore \( \sigma_x(a_i) \) is the leading coefficient of \( x \omega_i \in A \). Thus

\[
\sigma_x(a) = \sigma_x(a_1) + \cdots + \sigma_x(a_m),
\]

hence \( \sigma_x(a) \) is the sum of leading coefficients of elements of \( A \) and \( \sigma_x(a) \in I \).

For part (3), observe that (2) gives us the following descending chain of two-sided ideals of \( R \):

\[
I \supseteq \sigma_x(I) \supseteq \sigma_x^2(I) \supseteq \sigma_x^3(I) \supseteq \cdots.
\]

Since \( R \) is semiprime, if \( i \geq 0 \), the left and right annihilators of the ideal \( \sigma_x^i(I) \) are the same. In addition, since \( R \) is left Goldie, it satisfies the descending chain condition on annihilators of ideals.

Thus there exists \( n \in \mathbb{N} \) such that \( \text{ann}_R(\sigma_x^n(I)) = \text{ann}_R(\sigma_x^{n+1}(I)) \).

Whenever \( \tau \) is an automorphism of \( R \) and \( C \subseteq R \), we have \( \tau(r \text{ann}_R(C)) = r \text{ann}_R(\tau(C)) \). If \( n \in \mathbb{N} \) is such that \( \text{ann}_R(\sigma_x^n(I)) = \text{ann}_R(\sigma_x^{n+1}(I)) \) and if we let \( \sigma_x^n = \tau \), then

\[
\text{ann}_R(I) = \sigma_x^{-n}(\text{ann}_R(I)) = \sigma_x^{-n}(\text{ann}_R(\sigma_x^n(I))) = \sigma_x^{-n}(\text{ann}_R(\sigma_x^{n+1}(I))) = \sigma_x^{-n}(\text{ann}_R(\sigma_x(I))) = \sigma_x(\text{ann}_R(I)).
\]

The above equation shows that if \( M = \text{ann}_R(I) \), then \( M = \sigma_x(\text{ann}_R(I)) = \sigma_x(M) \).

Take any \( a, b \in I \) and \( m \in M \). Then \( \delta_x(a)b \in I \) and by (2) we obtain \( \sigma_x(a) \delta_x(b) \in I \). Thus

\[
0 = \delta_x(abm) = \delta_x(a)bm + \sigma_x(a) \delta_x(b)m + \sigma_x(a) \sigma_x(b) \delta_x(m) = \sigma_x(a) \sigma_x(b) \delta_x(m).
\]

Therefore \( (\sigma_x(I))^2 \delta_x(M) = 0 \) and

\[
(\sigma_x(I) \delta_x(M))^2 \subseteq (\sigma_x(I))^2 \delta_x(M) = 0.
\]

Since \( R \) is semiprime this tells us that the left ideal \( \sigma_x(I) \delta_x(M) \) is 0, hence

\[
\delta_x(M) \subseteq \text{ann}_R(\sigma_x(I)) = \sigma_x(M) = M.
\]

Thus \( \sigma_x(M) = M \) and \( \delta_x(M) \subseteq M \). □
We will need the following lemma.

**Lemma 2.3.** Let $X$ be a set of variables of cardinality $\text{card}(X) \geq 2$ and $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a family of finite sets consisting of elements of the free monoid $(X)$. Suppose that elements of $A_i$, $i = 1, 2, \ldots, m$, have the same degree $n_i$. Then there exist an integer $t \geq \max\{n_1 \mid 1 \leq i \leq m\}$ and elements $\nu_1, \ldots, \nu_m \in \langle X \rangle$ such that $\deg \nu_i = t - n_i$ and the sets $A_1 \nu_1, A_2 \nu_2, \ldots, A_m \nu_m$ are pairwise disjoint.

**Proof.** The lemma is obvious when the set $X$ is infinite. Suppose that $\text{card}(X) = d < \infty$. Without loss of generality we may assume that $n_1 \geq n_2 \geq \ldots \geq n_m$. It is clear that there are $d^s$ different elements of $(X)$ of degree $s$. Take $s$ such that $d^s > m$ and choose different elements $\omega_1, \omega_2, \ldots, \omega_m \in \langle X \rangle$ of degree $s$. Fix an element $x \in X$ and put $\nu_i = x^{n_1 - n_i} \omega_i$, $t = n_1 + s$. Then for any $a \in A_i \deg av_i = n_i + (n_1 - n_i) + s = t$. Furthermore, if $a \in A_i$, $b \in A_j$ and $i \neq j$, then $av_i = (ax^{n_1 - n_i})\omega_i \neq (bx^{n_1 - n_j})\omega_j = bv_j$. This proves that the sets $A_1 \nu_1, A_2 \nu_2, \ldots, A_m \nu_m$ are pairwise disjoint. $\square$

We can now prove our main result.

**Theorem 2.4.** Let $R$ be a semiprime left Goldie ring. Then for any set $X$ of noncommuting variables and sequences $\mathcal{G} = \{\sigma_x\}_{x \in X}$, $\mathcal{D} = \{\delta_x\}_{x \in X}$ of automorphisms and skew derivations of $R$ the free skew extension $R[X; \mathcal{G}, \mathcal{D}]$ is semiprimitive.

**Proof.** In light of the result of A. D. Bell [3] (mentioned in the Introduction) we may assume that the set $X$ contains at least two elements. Let us fix a linear order $\prec$ on the free monoid $(X)$. By way of contradiction, let us suppose $\mathcal{J}(R[X; \mathcal{G}, \mathcal{D}]) \neq 0$ and we will apply Proposition 2 with $A = \mathcal{J}(R[X; \mathcal{G}, \mathcal{D}])$. Next, we let $I$ be the ideal of leading coefficients in Proposition 2 and then let $M = \text{ann}_R(I)$. Recall that, for every $\sigma_x, \delta_x$ corresponding to some $x \in X$, we have $\sigma_x(M) = M$ and $\delta_x(M) \subseteq M$.

Since $M = \text{ann}_R(I)$, the sum $I + M$ is direct and is essential as both a left and right ideal of $R$. Every essential one-sided ideal of a semiprime Goldie ring contains a regular element, therefore there exist $c \in I$ and $d \in M$ such that $c + d$ is regular in $R$. If $0 \neq s \in I$ then $sM = Ms = 0$, hence

$$0 \neq s(c + d) = sc \quad \text{and} \quad 0 \neq (c + d)s = cs.$$  

Thus $c$ is regular in $I$. We can now apply Lemma 1 to $c \in I$ to conclude that $\text{l.ann}_R(c) = \text{r.ann}_R(c) = M$ and $\eta(c) = c + M$ is regular in the factor ring $R/M$.

Since $0 \neq c \in I$, there exist $a_1, \ldots, a_m$ such that $c = a_1 + \cdots + a_m$, where each $a_i$ is a leading coefficient of some $\omega_i \in \mathcal{J}(R[X; \mathcal{G}, \mathcal{D}])$. Using Lemma 2.3 we can multiply each $\omega_i$ by an appropriate monomial $\nu_i$ such that all the $\omega_i\nu_i$ have the same degree, say $t$, but none of the monomials of highest degree in the support of $\omega_i\nu_i$ appear in the support of $\omega_j\nu_j$, for $i \neq j$. According to [1, 2] we have

$$\omega_i\nu_i = a_i\Delta_i + (\omega_i\nu_i)_\prec, \quad \deg \Delta_i = t \quad \text{and} \quad \Delta_i \neq \Delta_j \quad \text{for} \quad i \neq j.$$  

Therefore, if

$$a(X) = \omega_1\nu_1 + \cdots + \omega_m\nu_m,$$  

then  

\[
\text{card}((X)) = d^s > m.
\]
we have $a(X) \in \mathcal{J}(R[X;\mathcal{G},\mathcal{D}])$ and

$$a(X) = a_1\Delta_1 + \cdots + a_m\Delta_m + \sum_i (\omega_i\nu_i)_x.$$  

Since none of the monomials $\Delta_i$ appear in the support of $\sum_i (\omega_i\nu_i)_x$, we have an irreducible decomposition

$$a(X) = a_1\Delta_1 + \cdots + a_m\Delta_m + f(X),$$

where $\deg f(X) \leq t$. In addition, since $a(X)x \in \mathcal{J}(R[X;\mathcal{G},\mathcal{D}])$ for all $x \in X$, we may assume that the $\nu_i$ were chosen to make the degree of $a(X)$ equal to some $t \geq 1$ and the constant term of $a(X)$ equal to 0.

Every element of $\mathcal{J}(R[X;\mathcal{G},\mathcal{D}])$ is quasi-invertible, therefore there exists an element $b(X) \in \mathcal{J}(R[X;\mathcal{G},\mathcal{D}])$ such that

$$a(X) + b(X) = a(X)b(X) = b(X)a(X).$$  \hspace{1cm} (2.3)

For every $\sigma_x, \delta_x$ corresponding to some $x \in X$, we have $\sigma_x(M) = M$ and $\delta_x(M) \subseteq M$, therefore the actions of $\sigma_x$ and $\delta_x$ induce actions on the factor $R/M$. In addition, we can examine the free skew extension $M[X;\mathcal{G},\mathcal{D}]$, and observe that $M[X;\mathcal{G},\mathcal{D}]$ is a two-sided ideal of $R[X;\mathcal{G},\mathcal{D}]$. We can now identify the factor ring $R[X;\mathcal{G},\mathcal{D}]/M[X;\mathcal{G},\mathcal{D}]$ with the free skew extension $(R/M)[X;\mathcal{G},\mathcal{D}]$. If we let $a(X)$ and $b(X)$ be the images of $a(X)$ and $b(X)$ in $(R/M)[X;\mathcal{G},\mathcal{D}]$, then equation (2.3) becomes

$$\overline{a(X)} + \overline{b(X)} = \overline{a(X)b(X)} = \overline{b(X)a(X)}.$$  \hspace{1cm} (2.4)

Recall that $c \notin M$, therefore at least one of $a_1, \ldots, a_m$ is not in $M$, hence $\overline{a(X)}$ also has degree $t \geq 1$. Thus equation (2.3) immediately implies that $\overline{b(X)}$ is not equal to zero in $(R/M)[X;\mathcal{G},\mathcal{D}]$. Now suppose $\overline{b(X)}$ has degree at least one. Then there exists $b' \in R$ and a monomial $\Delta$ of degree at least one such that $b' \notin M$ and $\overline{b\Delta}$ is the leading term of $\overline{b(X)}$. Therefore

$$\overline{b(X) \cdot a(X)} = \overline{b'\pi(a_1)\Delta\Delta_1 + \cdots + b'\pi(a_m)\Delta\Delta_m + g(X)},$$

where $\pi$ is the automorphism of $R$ equal to the product of the automorphisms corresponding to the $x \in X$ appearing in $\Delta$, and non of the monomials $\Delta\Delta_i$ is contained in the support of $\overline{g(X)} \in (R/M)[X;\mathcal{G},\mathcal{D}]$. Observe that, for $1 \leq i \leq m$, $b'\pi(a_i)$ must be the coefficient of $\Delta\Delta_i$ in $\overline{b(X) \cdot a(X)}$ since no other product of monomials from $\overline{a(X)}$ and $\overline{b(X)}$ could result in $\Delta\Delta_i$. If any of the $b'\pi(a_i)$ is nonzero in $R/M$, then the degree of $\overline{b(X) \cdot a(X)}$ exceeds the degree of $\overline{a(X)} + \overline{b(X)}$, contradicting equation (2.4).

As a result, $b'\pi(a_i) \in M$, for $1 \leq i \leq m$. This implies that

$$b'\pi(c) = b'\pi(a_1) + \cdots + b'\pi(a_m) \in M.$$  

Since $\pi(M) = M$, we have

$$\pi^{-1}(b')c = \pi^{-1}(b'\pi(c)) \in M.$$  

However, $\eta(c) = c + M$ is regular in $R/M$, hence $\pi^{-1}(b') \in M$. This immediately implies that $b' \in M$, contradiction that $b' \notin M$. 

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Having shown that $b(X)$ is nonzero and had degree less than 1 in $(R/M)[X;\mathcal{G},\mathcal{D}]$, it follows that
\[ cb(X) = cb_0 \neq 0, \]
for some $b_0 \in R$.

Multiplying equation (2.3) on the left by $c$ now gives us
\[ ca(X) + cb(X) = ca(X)b(X) = cb(X)a(X). \]
Since $cb(X) = cb_0$, examining the equation $ca(X) + cb(X) = cb(X)c(x)$ gives us
\[ ca(X) + cb_0 = cb_0a(X). \]

Since the constant term of $a(X)$ is 0, if we compare the constant terms of both sides of the previous equation, we obtain $cb_0 = 0$, contradicting that $cb_0 \neq 0$. Consequently, the ring $R[X;\mathcal{G},\mathcal{D}]$ is semiprimitive.

It is well known that any ring $R$ with left Krull dimension has a nilpotent prime radical $\mathcal{P}(R)$ (cf. [10], Corollary 6.3.8), and the factor ring $R/\mathcal{P}(R)$ is left Goldie (cf. [10], Proposition 6.3.5). These properties of rings with Krull dimension motivate the following general observation

**Corollary 2.5.** Let $\mathcal{G} = \{\sigma_x\}_{x \in X}$ and $\mathcal{D} = \{\delta_x\}_{x \in X}$ be families of automorphisms and skew derivations of the ring $R$ such that

1. the prime radical $\mathcal{P}(R)$ is nilpotent,
2. the factor ring $R/\mathcal{P}(R)$ is left Goldie,
3. $\mathcal{P}(R)$ is stable under $\delta_x$, for all $x \in X$.

Then the Jacobson radical $\mathcal{J}(R[X;\mathcal{G},\mathcal{D}])$ is nilpotent.

**Proof.** Since $\mathcal{P}(R)$ is stable under all $\sigma_x$ and $\delta_x$, $\mathcal{P}(R)[X;\mathcal{G},\mathcal{D}]$ is a nilpotent ideal of the skew extension $R[X;\mathcal{G},\mathcal{D}]$. In particular, $\mathcal{P}(R)[X;\mathcal{G},\mathcal{D}] \subseteq \mathcal{J}(R[X;\mathcal{G},\mathcal{D}])$. Observe that
\[ R[X;\mathcal{G},\mathcal{D}]/\mathcal{P}(R)[X;\mathcal{G},\mathcal{D}] \simeq (R/\mathcal{P}(R))[X;\mathcal{T},\mathcal{F}], \]
where $\bar{\sigma}_x$, $\bar{\delta}_x$ are induced automorphisms and skew derivations of $R/\mathcal{P}(R)$. The ring $R/\mathcal{P}(R)$ is semiprime Goldie, so by Theorem 2.3 the free skew extension $(R/\mathcal{P}(R))[X;\mathcal{T},\mathcal{F}]$ is semiprimitive. It means that $\mathcal{J}(R[X;\mathcal{G},\mathcal{D}]) = \mathcal{P}(R)[X;\mathcal{G},\mathcal{D}]$. \qed

The problem when the nil and prime radicals of a ring are stable under skew derivations is examined in [6]. In particular, Lemma 3 of [6] states that if $\delta$ is a $\sigma$-derivation of a ring $R$, then the nil radical $\mathcal{N}(R)$ is $\delta$-stable provided the element $\delta(a)$ is nilpotent for any $a \in \mathcal{N}(R)$. Notice that the condition (2) of Corollary 2.5 implies immediately the equality of radicals $\mathcal{N}(R) = \mathcal{P}(R)$. Indeed, it is well known that nil ideals of rings with ascending chain condition on left annihilators contain nonzero nilpotent ideals (see Lemma 2.3.2 of [10]). As a consequence, the condition (3) of Corollary 2.5 can be replaced by:

(3') $\delta_x(a)$ is nilpotent for all $x \in X$ and $a \in \mathcal{P}(R)$. 

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For the remainder of this paper, we will examine algebras over a field $F$ with $q$-skew derivations. Recall that a $\sigma$-derivation $\delta$ of $R$ is said to be a $q$-skew $\sigma$-derivation if there exists a nonzero element $q \in F$ such that $\delta \sigma = q \sigma \delta$. From Lemma 4 of [6] it follows that if $I$ is a $\sigma$-stable ideal of $R$, then for any $a_1, a_2, \ldots, a_n \in I$
\[
\delta^n(a_1a_2\ldots a_n) = (n!)_q \sigma^{n-1}\delta(a_1)\sigma^{n-2}\delta(a_2)\ldots \delta(a_{n-1})\delta(a_n) + w,
\]
where $w \in I$ and $(n!)_q = \prod_{i=1}^{n} (1 + q + \ldots + q^{i-1})$. Thus if $I^n = 0$, then the identity (2.5) gives that for any $a \in I$ $(n!)_q (\delta(a))^n \in I$, and hence $((n!)_q)^n (\delta(a))^n = 0$. As a consequence we obtain that $\delta$ satisfies (3') provided $(n!)_q \neq 0$ in $F$. Notice that $(n!)_q \neq 0$ in $F$ means that either $q$ is not a root of unity of degree $d \leq n$ or $n < \text{char } F$, when $q = 1$. Consequently, we obtain the following observation.

**Corollary 2.6.** Let $G = \{\sigma_x\}_{x \in X}$ and $D = \{\delta_x\}_{x \in X}$ be families of automorphisms and skew derivations of an $F$-algebra $R$ such that

1. the prime radical $\mathcal{P}(R)$ is nilpotent with index of nilpotency equal to $n$,
2. the factor ring $R/\mathcal{P}(R)$ is left Goldie,
3. for any $x \in X \delta_x \sigma_x = q_x \sigma_x \delta_x$, where $q_x \in F^*$ and $(n!)_{q_x} \neq 0$ in $F$.

Then the Jacobson radical $\mathcal{J}(R[X; G, D])$ is nilpotent.

Since the nil and prime radicals of algebras over fields of characteristic zero are stable under ordinary derivations (see also Proposition 2.6.28 in [15]), we obtain

**Corollary 2.7.** Let $R$ be an algebra over a field $F$ of characteristic zero. If $R$ has left Krull dimension, then for any set $X$ of noncommuting variables and a sequence $D = \{d_x\}_{x \in X}$ of derivations of $R$ the Jacobson radical of the Ore extension $R[X; D]$ is nilpotent.

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