Numerical simulation of two- and three-dimensional gravity-capillary waves

A S Dosaev and Yu I Troitskaya
Institute of Applied Physics of the Russian Academy of Sciences, 46 Ul'yanov Street, 603950, Nizhny Novgorod, Russia
E-mail: dosaev@appl.sci-nnov.ru

Abstract. Dynamics of gravity-capillary deep water waves is studied within the framework of potential Euler equations with parameterizations for wind forcing and viscous decay. Conformal mapping based technique for two dimensions and its generalization for weakly three-dimensional waves are employed to model evolution of nonlinear decimeter-range waves and generation of parasitic capillary ripples.

1. Introduction
In nature wind waves of gravity-capillary range are special in many respects. Steep decimeter-range waves develop a characteristic pattern of parasitic capillary ripples on their forward slopes. Parasitic ripples are generated by a narrow pressure distribution associated with sharp crest of the underlying wave [1], and are therefore very sensitive to a change of its parameters. Ripples are drawing energy from the underlying wave and dissipating due to viscousity, making decimeter-range waves subject to nonlinear damping. On the other hand, decimeter waves are also subject to strong wind forcing. Understanding the dynamics of gravity-capillary waves is important for remote sensing, because short ripples are the main scatterers of EM waves in the microwave range, and for studying long waves generation mechanisms, because momentum transfer from atmosphere to ocean occurs mainly due to the interaction of wind with short waves.

We perform numerical simulation of gravity-capillary waves within the framework of fully nonlinear equations of motion (Euler equations) for potential flow with a free surface using parameterizations for wind forcing and viscous decay. For two-dimensional flows a powerful technique of conformal transformations is available, and numerical methods based on it have high accuracy and computational efficiency [2]. Another common approach to modeling potential waves, applicable to both two- and three-dimensional motions, is to use boundary integral equation methods. They, however, have high computational costs. Here to take an account of three-dimensional effects we employ a quasi-three-dimensional model put forward by Ruban [3]. It is based on the method of conformal transformations and allows an efficient implementation using a Fast Fourier Transform. The model assumes narrow directional distribution of waves while not imposing any limitations on their steepness. We model gravity-capillary waves dynamics in both two- and three-dimensional setups and study evolution of horizontal asymmetry of wave profiles.
2. Basic equations

Within our approach we assume water to be an ideal incompressible fluid and the wave motion to be potential. In the Cartesian coordinate frame with the \( x \) axis coinciding with the unperturbed water surface and the \( y \) axis directed upward, governing equations for the free surface \( y = \eta \) and velocity potential \( \varphi \) are:

\[
\Delta \varphi = 0, \quad (1)
\]

\[
\eta_t + \nabla \perp \eta \cdot \nabla \perp \varphi = \varphi_y, \quad (2)
\]

\[
\varphi_t + \frac{1}{2} |\nabla \varphi|^2 + p + T\kappa + g\eta = 0 \quad (3)
\]

where \( \nabla \perp \) denotes horizontal component of gradient, \( T \) is surface tension coefficient and \( \kappa \) is free surface curvature. At infinity the deep water boundary condition holds:

\[ |\nabla \varphi| \to 0 \quad \text{as} \quad y \to -\infty \]

Other effects, such as viscous damping and wind forcing, are incorporated into the pressure term \( p \). Viscous dissipation is modeled in the spirit of [4] by the term

\[ p_{\text{visc}} = -4\nu \Delta_s \varphi, \]

where \( \Delta_s \) denotes the Laplacian operator on the free surface, \( \nu \) is kinematic viscosity. Wind input is required in order to balance energy losses due to dissipation; here we parameterize it as

\[ p_{\text{wind}} = 0.04u_2^2 \frac{\partial \eta}{\partial x}, \]

where \( u_2^2 \) is wind friction velocity, so that it corresponds to the Miles’ generation for the small slopes [5].

2.1. Numerical model for 2D waves

To solve the system (1)-(3) we employ a method based on the use of conformal mapping. The flow domain is mapped into the lower half-plane of complex variable \( w = u + iv \):

\[ x + iy = z(u + iv, t), \quad (4) \]

so that the function \( z(w) \) is analytic in the flow domain. In the new independent variables \( (u, v) \) the Laplace equation (1) for the potential \( \varphi \) retains its form, while the free surface becomes a straight line, being mapped onto the real axis \( v = 0 \). That makes solving the Laplace equation trivial. "Complex potential” \( \theta = \varphi + iv \), where \( \psi \) is the stream function and harmonic conjugate of \( \varphi \), becomes an analytic function in the half-plane \( v < 0 \). On the real axis \( \psi(u) = \hat{H}\varphi(u) \), where \( \hat{H} \) is Hilbert transform.

Evolution equations for the mapping \( z \) and potential \( \varphi \) can be derived from (1)-(3):

\[
z_t = z_u (\hat{H} - i) \frac{\hat{H}\varphi_u}{|z_u|^2}, \quad (5)
\]

\[
\varphi_t = -\frac{\varphi_u^2 - (\hat{H}\varphi_u)^2}{2|z_u|^2} + \varphi_u \hat{H} \left( \frac{\hat{H}\varphi_u}{|z_u|^2} \right) - p - T\kappa - g\eta
\]

The equations that we have beed actually solving numerically can be obtained by a further change of variables

\[ R = \frac{1}{z'}, \quad V = \frac{i\theta'}{z'}, \]
which leads to a system with improved numerical stability (Dyachenko equations [6]):

\[ \begin{align*}
R_t &= i(U'R - UR'), \\
V_t &= i(UV' - RP'(|V|^2 + 2p + 2T\kappa)) + g(R - 1),
\end{align*} \tag{6} \]

where prime denotes differentiation with respect to \( w \),

\[ U = \mathcal{P} (RV^* + R^*V), \quad \mathcal{P} = \frac{1}{2} (1 + i\hat{\mathcal{H}}), \]

asterisk denotes complex conjugate. The system (6) was integrated numerically using Dormand-Prince method of Runge-Kutta family with adaptive step size.

2.2. Numerical model for 3D waves

As was shown by Ruban [3], method of conformal transformations can be applied to a problem of weakly three-dimensional waves’ evolution, and the corresponding approximate equations of motion retain some of the good computational properties of the two-dimensional system (5). The method is based on the assumption of narrow directional distribution of waves, while no limitation is imposed onto waves’ steepness.

A new horizontal axis \( q \) pointed perpendicular to the “main” horizontal direction \( x \) is introduced, so that conformal mapping \( z \) and potential \( \varphi \) become functions of \( q \):

\[ x + iy = z(u + iv, q, t), \quad \varphi = \varphi(u, v, q, t) \]

The characteristic scale of change along the \( q \) direction is assumed to be large: \( \epsilon^2 = l_x/l_q \ll 1 \). The case of \( \epsilon = 0 \) corresponds to a purely two-dimensional motion. From a perturbation theory with regard to small parameter \( \epsilon \) the first order correction to (5) is obtained in [3], [7] together with some possible regularizations for the resulting equations. The resulting formulas are somewhat too cumbersome to be quoted here, but the three-dimensional corrections to the RHS of (5) can still be computed using Fast Fourier Transforms and local operations in either coordinate- or wavenumber representation.

It remains an open question to what extent those approximate equations of motion can describe the dynamics of waves of gravity-capillary or capillary ranges.

3. Evolution of gravity-capillary waves

In our numerical experiments we prescribe the initial conditions as a pure Stokes wave and then observe its evolution under the action of wind forcing. In a strictly periodic setup, and if the wind forcing is not too strong so the dominant wave does not grow to the point of breaking, the evolution leads to a stationary configuration where the wind input is balanced by the nonlinear damping due to parasitic ripples. An example of such process is shown on figure 1.

Very short gravity-capillary waves represent an interesting special case, when the resulting stationary configuration depends not only on the magnitude of wind forcing, but also on the way the wind friction velocity changes in time before arriving at the final value. Two possible stationary configurations for \( \lambda = 5 \) cm waves can be seen on figure 4. The difference vanishes for longer waves.

An important property of these stationary wave profiles is that they are asymmetric with regard to the vertical axis, that is, their front slope is steeper than the rear. This feature may be important to remote sensing, because it affects the inclination of the array of scatterers (that is, capillary ripples) to the horizon. The front-back asymmetry of wave profile \( \eta \) can be quantified as [8]

\[ A = \frac{\langle (\hat{\mathcal{H}}\eta)^3 \rangle}{\sqrt{\langle \eta^2 \rangle^3}}, \]
Figure 1. Evolution of 2d wave profile under the action of wind: $\lambda = 7$ cm, $u_\ast = 10$ cm/s

where $\hat{H}$ is Hilbert transform. Steeper front slope corresponds to negative values of $A$. As the figure 2 shows, after a few large oscillations (during which asymmetry can be positive or negative) the field settles at a moderate value of negative asymmetry.

Evolution of individual waves in a modulated wavetrain can differ substantially from the periodic scenario. Figure 5 shows an example of a three-dimensional simulation, where the initial conditions were prescribed as a Stokes wave modulated in both longitudinal and transversal directions. Corresponding evolution of profile asymmetry (in the $q = 0$ plane) is shown on figure 3. It can be seen that the waves in a modulated wavetrain oscillate in a similar manner as in the strictly periodic two-dimensional setup, but the modulation prevents their arrival at a stationary configuration.

Acknowledgements
This research was supported by RFBR grant No. 18-35-00658.

References
[1] Longuet-Higgins M 1995 Journal of Fluid Mechanics 301 79–107
[2] Chalikov D and Sheinin D 2005 Journal of Computational Physics 210 247–273
[3] Ruban V P 2005 Physical Review E 71 055303
[4] Ruvinsky K D, Feldstein F I and Freidman G I 1991 Journal of Fluid Mechanics 230 339–353
[5] Plant W J 1982 Journal of Geophysical Research: Oceans 87 1961–1967
[6] Zakharov V E, Dynchenko A I and Vasilyev O A 2002 European Journal of Mechanics-B/Fluids 21 283–291
[7] Ruban V P 2010 The European Physical Journal Special Topics 185 17–33
[8] Elgar S 1987 IEEE transactions on acoustics, speech, and signal processing 35 1725–1726
Figure 4. Stationary gravity-capillary wave profiles obtained in different setups; $\lambda = 5$ cm, $u_* = 13.5$ cm/s

Figure 5. Numerical simulation of a modulated three-dimensional gravity-capillary wave, $\lambda = 7$ cm, $ak \sim 0.25$