SIMPLICITY OF THE FIRST EIGENVALUE OF \((p, q)\) NONLINEAR ELLIPTIC SYSTEM

FARID BOZORGNIA

Abstract. In this short note, the simplicity of the first eigenvalue of a nonlinear system is shown by an alternative proof; thereby, it states that the first eigenfunctions are unique up to modulo scaling.

1. Introduction

In recent years the eigenvalue problems for nonlinear elliptic system involving p-Laplace operator has been extensively studied \cite{5, 3}.

Let \(\Omega \subset \mathbb{R}^d\) be a connected, bounded and open domain with regular boundary \(\partial \Omega\). This paper is devoted to show the simplicity of the first eigenfunction \((u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)\) of the following eigenvalue system

\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{\alpha-1} |v|^{\beta-1} v \quad \text{in } \Omega, \\
-\Delta_q v &= \lambda |u|^{\alpha-1} |v|^{\beta-1} u \quad \text{in } \Omega,
\end{aligned}
\]

where \(p, q > 1\) and \(\alpha, \beta > 0\) are real numbers satisfying

\[
\frac{\alpha}{p} + \frac{\beta}{q} = 1.
\]

Here \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\). The first eigenvalue \(\lambda_1(p, q)\) of system \((1)\) is defined as the least real parameter \(\lambda\) for which both equations of \((1)\) have a nontrivial solution \((u, v)\) in \(W_0^{1,p}(\Omega) \times W^{1,q}(\Omega)\), where with \(u, v \neq 0\).

The coupled system \((1)\) has several applications. For instance, in the case where \(p > 2\), problem \((1)\) appears in the study of non-Newtonian fluids, pseudoplastics for \(1 < p < 2\), and in reaction-diffusion problems, flows through porous media, nonlinear elasticity, and glaciology for \(p = \frac{4}{3}\). We refer the interested readers to \cite{6} for more details.

In \cite{1, 4} it is shown that the first eigenvalue is simple and corresponding eigenfunction is non-negative. Moreover, the stability of the first eigenvalue with respect to \((p, q)\) is established. The proof given in this work is simpler compare with mentioned works.

2. The First Eigenvalue of Nonlinear Elliptic System

Consider the following system

\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{\alpha-1} |v|^{\beta-1} v \quad \text{in } \Omega, \\
-\Delta_q v &= \lambda |u|^{\alpha-1} |v|^{\beta-1} u \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

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where $p, q > 1$ and $\alpha, \beta > 0$ are real numbers satisfying
\[
\frac{\alpha}{p} + \frac{\beta}{q} = 1.
\]

**Definition 1.** The first eigenvalue $\lambda_1(p, q)$ of (2) is defined as the least real parameter $\lambda$ for which both equations of (2) have a nontrivial solution $(u, v)$ in the product Sobolev space $W_0^{1, p} \times W_0^{1, q}$ with $u \neq 0$ and $v \neq 0$.

Here by a solution to (2) we mean a pair $(u, v)$ such that
\[
\int_\Omega \vert \nabla u \vert^{p-2} \nabla u \cdot \nabla \phi \, dx + \int_\Omega \vert \nabla v \vert^{q-2} \nabla v \cdot \nabla \psi \, dx = \lambda \left( \int_\Omega |u|^{p-2} u \phi \, dx + \int_\Omega |v|^{q-2} v \psi \, dx \right),
\]
\[
\forall (\phi, \psi) \in W_0^{1, p}(\Omega) \times W_0^{1, q}(\Omega).
\]

The principal eigenvalue $\lambda_1(p, q)$ can be variationally characterized by minimizing the functional
\[
I(u, v) = \frac{\alpha}{p} \int_\Omega \vert \nabla u(x) \vert^p \, dx + \frac{\beta}{q} \int_\Omega \vert \nabla v(x) \vert^q \, dx,
\]
over the set
\[
C = \{(u, v) \in W_0^{1, p} \times W_0^{1, q} : \int_\Omega |u(x)|^{\alpha-1} |v(x)|^{\beta-1} u(x)v(x) \, dx = 1\},
\]
by definition
\[
\lambda_1(p, q) = \inf \{ I(u, v), (u, v) \in C \}.
\]

The pair $(u, v)$ is called an eigenvector. Note that that the solutions $(u, v)$ of (2) correspond to the critical points of the energy functional $I(u, v)$.

**Theorem 2.1.** Let $\lambda_1(p, q)$ be defined by (3), then $\lambda_1(p, q)$ is simple.

**Proof.** Let $(u, v)$ and $(\phi, \psi)$ be two normalized vectors of the first eigenfunction associated with $\lambda_1(p, q)$. We show that there exist real numbers $k_1, k_2$ such that $u = k_1 \phi$ and $v = k_2 \psi$. Our proof is based on proof given by Belloni and Kahlol (see [2] for scalar case). Note that the function defined below are admissible functions for problem (3):
\[
w_1 = \left( \frac{u^p + \phi^p}{2} \right)^{\frac{1}{p}} \quad \text{and} \quad w_2 = \left( \frac{v^q + \psi^q}{2} \right)^{\frac{1}{q}}.
\]
We have
\[
\vert \nabla w_1 \vert^p = \left( \frac{u^p + \phi^p}{2} \right)^p \nabla \log u + \phi \nabla \log \phi \bigg|_p,
\]
\[
\vert \nabla w_2 \vert^q = \left( \frac{v^q + \psi^q}{2} \right)^q \nabla \log v + \psi \nabla \log \psi \bigg|_q.
\]
Now by Jensen’s inequality for convex function $\theta(\cdot) = |\cdot|^p$ with
\[
\theta\left( \frac{\sum a_i x_i}{\sum a_i} \right) \leq \frac{\sum a_i \theta(x_i)}{\sum a_i},
\]
choose
\[
\left\{ \begin{array}{l}
a_1 = \frac{u^p}{u^p + \phi^p}, \\
x_1 = \nabla \log u,
\end{array} \right.
\]
\[
\left\{ \begin{array}{l}
a_2 = \frac{\phi^p}{u^p + \phi^p}, \\
x_2 = \nabla \log \phi.
\end{array} \right.
\]
Thus the following inequalities hold
\[
\vert \nabla w_1 \vert^p \leq \frac{1}{2} \vert \nabla u \vert^p + \frac{1}{2} \vert \nabla \phi \vert^p,
\]
\[
\vert \nabla w_2 \vert^q \leq \frac{1}{2} \vert \nabla v \vert^q + \frac{1}{2} \vert \nabla \psi \vert^q.
\]
The inequalities above are strict at points where \( \nabla \log u \neq \nabla \log \phi \) and \( \nabla \log v \neq \nabla \log \psi \). Now we have

\[
\lambda_1 \leq \frac{\alpha p \int_{\Omega} |\nabla w_1|^p dx + \beta q \int_{\Omega} |\nabla w_2|^q dx}{\int_{\Omega} w_1^\alpha w_2^\beta dx}.
\]

Note that by concavity we have

\[
w_1^\alpha w_2^\beta = \left( \frac{u^p + \phi^p}{2} \right)^\alpha \cdot \left( \frac{v^q + \psi^q}{2} \right)^\beta \geq \left( \frac{u^\alpha + \phi^\alpha}{2} \right) \cdot \left( \frac{v^\beta + \psi^\beta}{2} \right).
\]

It is easy to see that \( u, v, \phi, \psi \) can be chosen such that

\[
\int_{\Omega} u^\alpha v^\beta dx = \int_{\Omega} u^\alpha \psi^\beta dx = \int_{\Omega} \phi^\alpha v^\beta dx = \int_{\Omega} \phi^\alpha \psi^\beta dx = 1.
\]

Then the inequality in (4) reads as

\[
\lambda_1 \leq \frac{\alpha p (|\nabla u|^p + |\nabla \phi|^p) + \beta q (|\nabla v|^q + |\nabla \psi|^q)}{\int_{\Omega} u^\alpha v^\beta + u^\alpha \psi^\beta + \phi^\alpha v^\beta + \phi^\alpha \psi^\beta dx}.
\]

If \( \nabla \log u \neq \nabla \log \phi \) and \( \nabla \log v \neq \nabla \log \psi \) in a set of positive measure, then we would have strict inequality above, which is contradiction. This shows that \( u \) and \( v \) also \( \phi \) and \( \psi \) are constant multiplies of each other. \( \square \)

**References**

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