A flexible theoretical framework to describe multimode photon-subtraction

M. F. Melalkia, L. Brunel, S. Tanzilli, J. Etesse & V. D’Auria
Université Côte d’Azur, CNRS, Institut de Physique de Nice, Parc Valrose, 06108 Nice Cedex 2, France
E-mail: virginia.dauria@univ-cotedazur.fr
6 August 2021

Abstract. This work establishes a versatile theoretical framework that explicitly describes non-Gaussian states that are obtained, in an heralded fashion, by applying a single-photon subtraction to a multimode resource. The treatment focuses on the very common experimental situation in which no mode-selective operation is available. The obtained theoretical framework allows retrieving, given a multimode input state, optimal conditions for heralding as well as for non-Gaussian feature detection, thus providing a powerful toolbox for experiments’ design. The application of the proposed approach to the case study of Schrödinger kitten preparation starting from a frequency multimode squeezed state illustrates the pertinence and impact of the derived theoretical tools.

1. Introduction

In the context of emerging quantum technologies, continuous variable (CV) quantum optics offers a wide panel of applications, ranging from quantum metrology [1] to quantum communication and computing [2,3]. A large number of CV demonstrations rely on non-Gaussian states, i.e. on quantum states represented by non-Gaussian functions in the quantum phase space [4], that besides being mandatory for important quantum information tasks, such as CV quantum computation [5], are essential for the generation of complex quantum states such as hybrid optical entanglement [6].

The quest for experimentally available non-Gaussian states has led to a flourishing research line that exploits the subtraction (or addition) of single-photons from input squeezed states as a powerful strategy for the conditional preparation of key resources such as Schrödinger kitten- and Fock- states [7]. At the same time, in many practical cases, squeezed states required in such protocols exhibit multimode features. This is the case, in particular, for squeezing generated via spontaneous parametric down conversion (SPDC) in non-linear media that, depending on chosen working conditions, can be highly multimode in
the frequency domain \([8, 9]\). Mastering such multimode behaviour thus becomes essential to guarantee high-quality generation and detection of pure non-Gaussian states. In this spirit, single-mode engineered SPDC sources \([10]\) as well as mode-selective state manipulation \([11]\) have been experimentally demonstrated. This work focuses, instead, on the very common situation in which source engineering and mode-selective operations are not possible or experimentally not available; it establishes a flexible theoretical framework able, given a frequency multimode input state, to explicitly predict the state that can be obtained after a non mode-selective photon subtraction, as well as to assess the best conditions to detect its non-Gaussian properties. Moreover, our framework is not only restricted to the case of frequency multimode states but remains general: it can for instance also be applied to the case of spatial multimode states. The original formalism is free from \textit{a priori} hypotheses on the shape or on the number of modes involved in the different stages of the subtraction protocol. Accordingly, beside providing a strong insight on a huge number of reported realistic cases, it supplies a precious toolbox for the conception, design and optimisation of future experiments. To prove its impact, the derived framework is applied to the case study of the Schrödinger kitten heralded generation, for which it allows retrieving optimal working conditions, including in the very practical scenario where input multimode squeezed light is provided by SPDC on lithium niobate compatible with integrated systems \([12]\). Developed methods can easily be extended to the special case of mode-selective operations already treated in the literature \([13]\) and can also be applied to further manipulation of non-Gaussian states via cascaded photon subtraction.

The paper is structured as follows. Section 2 provides a multimode description of single-photon subtraction as well as the explicit expression of the heralded non-Gaussian state as it is measured by a homodyne detection. In section 3, the obtained formalism is applied to the case of Schrödinger kitten generation. Different experimental configurations for both heralding and detection stages are discussed and compared. Derived results adopt the formalism of multimode features of multiple spectral components. This choice makes it possible to acquire easy physical intuitions of obtained results and discuss the common cases of non-Gaussian state generation via single-photon subtraction from squeezed states generated by SPDC \([9, 13]\). All numerical simulations refer to the manipulation of squeezing emitted in the C-band of classical telecommunication, compatible with future practical applications to fibre-based quantum communication.

2. Photon-subtraction on multimode states

Following a very common strategy, multimode states are treated in terms of the so-called supermodes \([14]\). These permit rewriting the output of a multimode quantum optical source as a tensor product of independent single mode states, each being described by a spectral envelop \(\psi_k(\omega)\) \([8, 14, 15]\). Supermode envelops \(\{\psi_k(\omega)\}\) form an orthonormal
basis \((\int \psi_k(\omega)\psi_k^*(\omega)d\omega = \delta_{k,l}, \delta_{k,l} \text{ being the Kronecker delta})\) and their associated bosonic operators \(\{\hat{A}_k\}\) can be easily written as:

\[
\hat{A}_k = \int \psi_k^*(\omega) \hat{a}(\omega) d\omega, \tag{1}
\]

where \(\hat{a}(\omega)\) are the bosonic operators associated with the individual spectral components of the source output.

The multimode state, generically indicated as \(|\psi\rangle\), undergoes the typical subtraction scheme shown in Fig. 1; it is sent towards a subtraction beam-splitter (BS) with reflection coefficient \(r_s \ll 1\). The reflected beam goes towards the heralding path, while the transmitted one carries the heralded state. In experiments, light in the heralding path is sent to a bucket single-photon detector (SPD), unable to distinguish light from different modes and whose detection signal heralds a successful photon subtraction and the preparation of a desired non-Gaussian state [7]. At the same time, the subtraction BS, a priori, acts in a similar way on all spectral components of \(|\psi\rangle\) or, equivalently, on all supermodes \(\{\psi_k(\omega)\}\). This makes it impossible to associate a photon detection event with a subtraction operation on a specific supermode and can eventually lead to mixed heralded states [16]. To comply with such a situation, a certain mode selectivity on the heralding path is generally obtained by adding an optical frequency filter before the photon-counting detector [7, 16, 17, 18]. The filter action can be modelled as a BS: different scenarios can thus be considered based on the shape of the transmission coefficient of the filter BS.

**Perfect mode-selective single-photon subtraction.** Fig. 1-a represents a mode-selective photon-subtraction. In this case, the filter is modelled as a mode-selective beam splitter, able to transmit and send towards the bucket single-photon detector only one given detection mode \(\psi_k^/(\omega)\). Depending on the chosen mode, it is possible to manipulate the original multimode state in a controlled manner or even to entangle formerly independent supermodes \(\{\psi_k(\omega)\}\) [11]. In particular, in the special case in which the detection mode matches one of the supermode envelops (i.e. \(\psi_k^/ = \psi_k\)), Schrödinger kitten or Fock states can be heralded. The description of mode-selective operation has been already treated in the literature [13] and will not be described in this work. However, it is pertinent to observe that original methods and results reported in the following paragraphs can be applied to the mode-selective regime by considering for the filter BS a discrete filter transmission coefficient \(t_k\) equal to 0 for all modes excepted for a specific \(\psi_k^/(\omega)\) and by accordingly adapting all related expressions.

**Non mode-selective single-photon subtraction.** Experimentally, mode-selective single-photon subtractions require adapted, and quite complex, non-linear optical stages [11]. In the large majority of reported experimental works, standard (passive) optical filters are used in the heralding path [7]. In such a non-mode selective operation, the filter can be modelled
Figure 1. Subtraction scheme applied to a multimode input state $|\psi\rangle$ in case of (a) mode-selective and (b) non-mode selective operations. The working supermode basis for each spatial mode is explicitly indicated in both cases. A hybrid Schrödinger / Heisenberg approach is adopted, as the operators do not evolve on the subtracting beam-splitter, but do evolve on the filter beam-splitter. Mode $\hat{b}_{\text{out}}$ represents the spatial mode of the heralding photons downstream of the filter that are directed toward the single-photon detector (SPD). Mode $\hat{c}_{\text{out}}$ represents photons that are rejected by the filter and thus disregarded. After the photon-counting operation, mode $\hat{a}$ carries the heralded non-Gaussian state $\hat{\rho}_{\text{out}}$. 
as a beam-splitter whose real transmission coefficient \( t(\omega) \) depends on the optical frequency. No \textit{a priori} hypothesis on the filter transmission profile is imposed.

As observed in Fig. 1-a, a hybrid Schrödinger/Heisenberg approach is adopted [19]: the bosonic operators do not evolve on the subtracting beam-splitter, but do evolve on the filter beam-splitter in the heralding path. Such a strategy allows introducing a global evolution operator, \( \hat{U} \), that permits describing the state obtained when both the subtraction and filter beam-splitter are explicitly considered:

\[
\hat{U} = e^{\theta \int (\hat{a}^\dagger(\omega)\hat{b}(\omega) - \hat{a}(\omega)\hat{b}^\dagger(\omega)) d\omega}. \tag{2}
\]

\[
\approx \hat{1} + \theta \int \left( \hat{a}^\dagger(\omega)\hat{b}(\omega) - \hat{a}(\omega)\hat{b}^\dagger(\omega) \right) d\omega, \tag{3}
\]

where \( \sin \theta/2 = r_s \ll 1 \) gives the reflection coefficient of the subtraction BS. Spatial modes are named as indicated in Fig. 1. Note that operator \( \hat{U} \) correctly encompasses all the spectral components of the field. In the previous expression, each operator \( \hat{b}(\omega) \) undergoes the transformation due to the filtering stage and can be rewritten in terms of the bosonic operators \( \hat{b}_{\text{out}}(\omega) \) and \( \hat{c}_{\text{out}}(\omega) \) that describe transmitted and reflected fields after the filter BS, respectively:

\[
\hat{b}(\omega) = t(\omega)\hat{b}_{\text{out}}(\omega) + r(\omega)\hat{c}_{\text{out}}(\omega), \tag{4}
\]

with \( t(\omega)^2 + r(\omega)^2 = 1 \). Expression (3) can thus be written as:

\[
\hat{U} \approx \hat{1} + \theta \int \left( \hat{a}^\dagger(\omega)t(\omega)\hat{b}_{\text{out}}(\omega) + \hat{a}^\dagger(\omega)r(\omega)\hat{c}_{\text{out}}(\omega) - \hat{a}(\omega)t^*(\omega)\hat{b}_{\text{out}}^\dagger(\omega) - \hat{a}(\omega)r^*(\omega)\hat{c}_{\text{out}}^\dagger(\omega) \right) d\omega. \tag{5}
\]

To highlight multimode features, \( \hat{U} \) can conveniently be expressed in terms of bosonic operators associated with the supermodes \( \{\psi_k(\omega)\} \) by inverting relation (1) and by defining, by analogy with \( \hat{A}_k \), operators \( \hat{B}_{\text{out},k} \) and \( \hat{C}_{\text{out},k} \) associated with the spatial modes downstream of the filter BS:

\[
\hat{a}(\omega) = \sum_k \psi_k(\omega)\hat{A}_k \tag{6a}
\]

\[
\hat{b}_{\text{out}}(\omega) = \sum_k \psi_k(\omega)\hat{B}_{\text{out},k} \tag{6b}
\]

\[
\hat{c}_{\text{out}}(\omega) = \sum_k \psi_k(\omega)\hat{C}_{\text{out},k}. \tag{6c}
\]

Transformation (6) lead to the appearance in Eq. (5) of terms of the kind:

\[
\hat{a}^\dagger(\omega)t(\omega)\hat{b}_{\text{out}}(\omega) = \sum_{k,k'} \psi^*_k(\omega)\hat{A}^\dagger_k t(\omega)\psi_{k'}(\omega)\hat{B}_{\text{out},k'}. \tag{7}
\]
and analogous. However, filtered supermode functions \( \{ t(\omega) \psi_k(\omega) \} \) are no longer appropriate to define optical modes as they are not orthogonal to each other or to the original \( \{ \psi_k(\omega) \} \). This reflects the fact that the contributions of the original input supermodes are mixed together by the mode-insensitive filter. It is thus \textit{a priori} impossible to factorise the action filter BS into that of multiple BS acting each on only a given supermode \( \psi_k(\omega) \).

\textit{Supermode decomposition}

Products involving the filter transmission and the supermodes functions can adequately be decomposed in a suitable orthonormal basis \( \{ \psi_k(\omega) \} \) whose support corresponds to spectral regions for which \( t(\omega) > 0 \). The use of such new modes permits to conveniently describe the detection of single photons downstream of the filter. However, in principle, the set of \( \{ \psi_k(\omega) \} \) is sufficient \textit{only} to describe the state of systems that actually passed through the filter; as-a-matter-of-fact, in a conditional preparation scheme, this restricts the use of such functions to the sole description of the heralding photon detection. To overcome this limitation, in this work, the basis \( \{ \psi_k(\omega) \} \) is completed with a set of orthonormal functions \( \{ \psi^\perp_k(\omega) \} \). Functions \( \{ \psi_k(\omega), \psi^\perp_k(\omega) \} \) are orthogonal to each other and have disjoint supports. Their properties are detailed in Appendix A. The complete orthonormal basis of \( \{ \psi_k(\omega), \psi^\perp_k(\omega) \} \) can now be used to decompose all spectral modes involved in the non-Gaussian manipulation scheme, including those of the subsystem carrying the heralded state. In particular, the supermode functions can be written as:

\[
\psi_k(\omega) = \sum_n p_{kn} \psi_n^\parallel(\omega) + q_{kn} \psi_n^\perp(\omega),
\]

where

\[
p_{kn} = \int \psi_k(\omega) (\psi_n^\parallel(\omega))^* d\omega \quad \text{and} \quad q_{kn} = \int \psi_k(\omega) (\psi_n^\perp(\omega))^* d\omega.
\]

The specific shape of \( \{ \psi_k^\parallel(\omega), \psi_k^\perp(\omega) \} \) depends on that of the input supermodes and on the filter profile \( t(\omega) \). In the case of continuous and regular functions \( t(\omega) \), such as those associated with Lorentzian-shaped or Gaussian profiles, the sets of \( \{ \psi_n^\parallel(\omega) \} \) can be in principle sufficient to obtain a complete basis. In all cases in which \( t(\omega) = 0 \) in one or more spectral regions, both \( \{ \psi_n^\parallel(\omega) \} \) and \( \{ \psi_n^\perp(\omega) \} \) are non-null and must be taken into account to obtain a complete basis. This concept is illustrated in Fig. 2 where the first \( \{ \psi_n^\parallel(\omega) \} \) and \( \{ \psi_n^\perp(\omega) \} \) are represented for the case of a Gaussian and of a rectangular filter.

Relation (8) allows re-expressing the bosonic operators associated with the supermodes, providing also the single mode squeezed states in terms of the bosonic operators that are
Figure 2. (Left): Examples of the first three $\{\psi_n(\omega)\}$ in the case of a filter with a Gaussian transmission coefficient $t(\omega)$ centred around 1560 nm and with FWHM of 5 nm ($\approx 600$ GHz). (Right) Examples of the first three (a) $\{\psi_n(\omega)\}$ and (b) $\{\psi_n(\omega)\}$, in the case of a filter with a rectangular profile centred around 1560 nm with a FWHM of 5 nm. For the rectangular filter, the coefficient $t(\omega)$ goes exactly to zero outside the 5 nm transmission bandwidth. The input state supermode are assumed to be Hermit functions. For both filters, $\psi_0(\omega)$ is given by $\int t(\omega) \psi_0(\omega) d\omega$ with a Gaussian $\psi_0(\omega)$ function. $\{\psi_{n>0}(\omega), \psi_{n}^{\perp}(\omega)\}$ are constructed by Gram-Schmidt process.

associated only with the filter supermodes $\psi_n(\omega)$ and $\psi_n(\omega)$. It comes:

$$\hat{A}_k = \sum_n p_{kn} \hat{A}_n^\parallel + q_{kn} \hat{A}_n^\perp \quad \iff \quad \hat{A}_n^\parallel = \sum_k p_{kn} \hat{A}_k, \quad \hat{A}_n^\perp = \sum_k q_{kn} \hat{A}_k.$$  \hspace{1cm} (10)

Expressions in terms of the coefficients $p_{kn}$ and $q_{kn}$ identical to Eqs. (10) can be obtained for operators $\hat{B}_{out}(\omega)$ and $\hat{C}_{out}(\omega)$ and lead to operators $\hat{B}_{out,n}^\parallel, \hat{B}_{out,n}^\perp, \hat{C}_{out,n}^\parallel$ and $\hat{C}_{out,n}^\perp$, respectively.

By exploiting the properties (A.7) and (A.8) of functions $\{\psi_n(\omega), \psi_n(\omega)\}$, the evolution operator can now be written as:

$$\hat{U} \approx 1 + \theta \sum_{kl} \left[ T_{lk} \hat{B}_{out,l}^\parallel \left( \hat{A}_k^\parallel \right)^\dagger - T_{lk}^* \left( \hat{B}_{out,l}^\parallel \right)^\dagger \hat{A}_k \right]$$

$$\quad + \theta \sum_{kl} \left[ R_{lk} \hat{C}_{out,l}^\parallel \left( \hat{A}_k^\parallel \right)^\dagger - R_{lk}^* \left( \hat{C}_{out,l}^\parallel \right)^\dagger \hat{A}_k \right]$$

$$\quad + \theta \sum_k \left[ \hat{C}_{out,k}^\parallel \left( \hat{A}_k^\parallel \right)^\dagger - \left( \hat{C}_{out,k}^\perp \right)^\dagger \hat{A}_k \right],$$  \hspace{1cm} (11)

with:

$$T_{lk} = \int \left( \psi_k^\parallel(\omega) \right)^* t(\omega) \psi_l^\parallel(\omega) d\omega \quad \text{and} \quad R_{lk} = \int \left( \psi_k^\parallel(\omega) \right)^* r(\omega) \psi_l^\perp(\omega) d\omega.$$  \hspace{1cm} (12)

The explicit shape of $T_{lk}, R_{lk}$ depends on chosen experimental conditions via $t(\omega)$ and $\psi_l^\parallel(\omega)$. Note that in Eq. (11) operators $\hat{C}_{out}$ are associated with the spectral components reflected
by the filter and will be eventually traced out; accordingly, the most relevant contribution is given by the term $T_{ab} B_{\text{out},1} \left( \hat{A}_k \right)^\dagger$ and its hermitian conjugate.

As shown by the explicit expression of $\hat{U}$, the introduction of the filter BS defines a new basis of modes and associated bosonic operators that permit to describe the whole subtraction process, although, in general, they do not necessarily match the original $\{ \psi_k(\omega) \}$ and corresponding $\{ \hat{A}_k \}$. Due to such a mismatch, any operation performed in a single detection mode $\psi_k(\omega)$ can have an impact distributed on all original supermodes $\{ \psi_k(\omega) \}$ and can even entangle them. This very general result has been experimentally demonstrated in Ref. [11] for the mode-selective case. In the special case of non-mode selective operation, expressing $\hat{U}$ in terms of the parallel modes (as shown in Eq. (11)) shows that a photon-counting operation on a single $\psi_k(\omega)$ also affects in a coherent way all parallel modes of the heralded state and entangle them too. In the language of single-photon subtraction, measuring a photon-counting event on a single $\psi_k(\omega)$ would correspond to herald the delocalised subtraction of a single-photon on a set of $\{ \psi_k \}$. Note that, since standard photon-counting detectors do not discriminate among signals coming from different $\psi_k(\omega)$, a photon-counting event downstream of the filter eventually leads to a mixed state as discussed in the next paragraph.

**Heralded single-photon subtracted state**

The above formalism allows explicitly describing the action of the operator $\hat{U}$ on any generic protocol input $|\psi\rangle$ at the port $a$ of the subtraction beam-splitter. The results obtained in this section refer to the case of a pure input state $\hat{\rho}_{\text{in}} = |\psi\rangle\langle\psi|$; however, the entire treatment can be easily extended to a mixed input. The state at the input port $b$ of the subtraction beam-splitter is assumed to be the vacuum state $|0\rangle$. The mode entering the subtraction beam-splitter will thus be indicated as $\hat{\rho} = \hat{\rho}_{\text{in}} \otimes |0\rangle\langle0|$.

The state after the filter BS but right before the heralding detection stage can be written as $\hat{U} \hat{\rho} \hat{U}^\dagger$, with $\hat{U}$ given by Eq. (11). As discussed, in general, the photon-counting detector on the heralding path is a bucket one unable to distinguish signals that originate from different parallel modes $\{ \psi_k(\omega) \}$. Accordingly, the overall heralded state is given by the mixture:

$$\hat{\rho}_{\text{out}} = \frac{1}{P} \text{Tr}_{B_{\text{out}}} \left[ \hat{\Pi}_{B_{\text{out}}} \hat{U} \hat{\rho} \hat{U}^\dagger \right],$$

where $P = \text{Tr}[\hat{\Pi}_{B_{\text{out}}} \hat{U} \hat{\rho} \hat{U}^\dagger]$ and $P \cdot \theta^2$ is the protocol success probability. $\text{Tr}_{B_{\text{out}}}$ is a partial trace over all detection modes $\{ \psi_n(\omega) \}$ of $\hat{b}_{\text{out}}$. The operator $\hat{\Pi}_{B_{\text{out}}}$ reads as:

$$\hat{\Pi}_{B_{\text{out}}} = \sum_n |1\rangle\langle1|_{B_{\text{out}}},$$

where $P = \text{Tr}[\hat{\Pi}_{B_{\text{out}}} \hat{U} \hat{\rho} \hat{U}^\dagger]$ and $P \cdot \theta^2$ is the protocol success probability. $\text{Tr}_{B_{\text{out}}}$ is a partial trace over all detection modes $\{ \psi_n(\omega) \}$ of $\hat{b}_{\text{out}}$. The operator $\hat{\Pi}_{B_{\text{out}}}$ reads as:
and is given by the superposition of projectors $|1⟩⟨1|_{\hat{g}_{\text{out}}}$, each describing the detection of a single photon in a specific mode $\psi_n^{\text{out}}(\omega)$. Note that, without any loss of generality, the bucket detector on the heralding path is assumed to be able to perfectly project its input on a single-photon state. The impact of poor or no photon number resolving ability on heralded state preparation is discussed in Ref. [20].

By making use of the explicit expression of $\hat{U}$ reported in Eq. (11) as well as of relations (10), it is easy obtain the exact expression the density matrix $\hat{\rho}_{\text{out}}$ in terms of the bosonic operators associated with the supermodes:

$$\hat{\rho}_{\text{out}} = \frac{1}{P} \sum_{n,k} \gamma_{k,n} \hat{A}_k \hat{\rho}_{\text{in}} \hat{A}_n^\dagger,$$  

(15)

where $P$ can be expressed as:

$$P = \sum_n \gamma_{n,n} \text{Tr}[\hat{A}_n \hat{\rho}_{\text{in}} \hat{A}_n^\dagger].$$  

(16)

The coefficients $\gamma_{k,n}$ can be written in a compact form as functions of the filter BS transmission:

$$\gamma_{k,n} = \int |t(\omega)|^2 \psi_k(\omega)\psi_n^*(\omega) d\omega.$$  

(17)

Expression (15) describes in a very simple and general form the mixed state heralded by a single-photon subtraction from the initial input state $\hat{\rho}$ in all cases where no mode-selective operation is possible on the heralding path. As anticipated, this corresponds to the extremely common experimental situation in which passive, linear, filters are employed on the heralding path [7, 16, 17, 18]. The special case of mode selective single-photon subtraction in a given supermode $\psi_n(\omega)$ corresponds to $\gamma_{k,n} = \delta_{k,n}\delta_{\bar{n},\bar{n}}$ in (15) and correctly leads to a pure output state $\hat{\rho}_{\text{out}} = \frac{1}{P} \hat{A}_{\bar{n}} \hat{\rho}_{\text{in}} \hat{A}_{\bar{n}}^\dagger$.

Continuous variable detection of the heralded state

Non-Gaussian properties of the heralded state $\hat{\rho}_{\text{out}}$ can be retrieved by sending it to a homodyne detector (HD) to make it beat with a classical local oscillator beam (LO). This technique permits a full reconstruction of the produced states via quantum homodyne detection. Note that, in experiments, the HD is generally triggered by the photon-counting events detected on the heralding path. As widely discussed in the literature [7, 11], the homodyne detector acts as a logic AND gate, in which the properties of the LO select the mode that will be actually measured. In particular, by expressing the amplitude of the LO beam as $\alpha_{\text{LO}}(\omega) = \sum_k c_k \psi_k(\omega)$, it is simple to prove that the homodyne detection measures the observable:

$$\hat{X}_{\text{H}}(\varphi) = \sum_{k=0}^{\infty} c_k \hat{X}_k(\varphi),$$  

(18)
where $\varphi$ is the LO phase and $\sum_k |c_k|^2 = 1$. Observable $\hat{X}_k(\varphi)$ is the quadrature
\[ \hat{X}_k(\varphi) = \frac{1}{\sqrt{2}} \left( e^{-i\varphi} \hat{A}_k + e^{i\varphi} \hat{A}_k^\dagger \right) \]
associated with the supermode $\psi_k(\omega)$. Amplitude and phase quadratures, $\hat{X}$ and $\hat{Y}$, correspond to $\varphi = 0, \pi/2$, respectively. Depending on the chosen spectral profile of the LO, $\alpha_{LO}(\omega)$, it is thus possible to observe the quadratures of a given supermode or of a combination of them. This possibility is extremely relevant when it comes to choosing how to measure the multimode features or to optimise the detection of a chosen mode. By exploiting Eqs. (15) and (18), the Wigner function of the state detected by the homodyne detector can be written as:
\[ W_H(x,y) = \frac{1}{2\pi^2 P} \sum_{k,n} \gamma_{k,n} \int \ldots \int e^{i2x'y} e^{-i(x-x'-c\cdot x)z} \langle \hat{A}_k \hat{\rho}_{in} \hat{A}_n^\dagger | x + 2x' c \rangle dxdzdxd', \] (19)
where $c^t = (c_0, ..., c_k, ...)$ with $|x\rangle = |x_0, ..., x_k, ...\rangle$, and where $k$ represents the supermode index.

3. Heralded generation of Schrödinger kitten states

The formalism established in the previous section describes in a general way the heralded state when single-photon subtraction is performed in a non-mode selective way on an arbitrary input state $\hat{\rho}$. This section focuses on the interesting case study of the heralded preparation of Schrödinger kitten states [7]. The typical heralded state generation and detection scheme is represented in Fig. 3.

Ideally, the protocol for generating small size Schrödinger cats relies on photon subtraction from a single-mode squeezed vacuum state $\hat{\rho}_{sq,k} = |\psi\rangle\langle\psi|_{sq,k}$ in a well defined supermode $\psi_k(\omega)$ and exhibiting a low squeezing level [7]. The corresponding state Wigner function is given by:
\[ W_T(x,y) = \frac{e^{-(x^2/s+sy^2)}}{\pi} \left[ \frac{2}{s} x^2 + 2sy^2 - 1 \right], \] (20)
where $s = \frac{1+\mu}{1-\mu}$ depends on the squeezing parameter $\zeta$ via $\mu = \tanh \zeta$. In many experimental situations, squeezed vacuum states are obtained by means of spontaneous parametric down conversion of an optical pump in a non-linear optical crystal. However, to produce single-mode squeezing, SPDC sources demand to be engineered according to specific characteristics, that cannot be obtained for all wavelengths and non-linear materials [10]. In general, the SPDC process can induce frequency/time correlations among optical components thus leading to a multimode behaviour [11]. In this situation, the squeezed states produced by the SPDC process is rather described by $\hat{\rho}_{in} = \otimes_k \hat{\rho}_{sq,k}$, i.e. by the product of independent pure single-mode squeezed vacuum states each in a different supermode $\psi_k(\omega)$. The explicit shape of the supermodes depends on the spectral profile of the SPDC pump beam and phase
Figure 3. Heralded preparation and detection of Schrödinger kitten states via single photon subtraction from a squeezed state produced by spontaneous parametric down conversion. Depending on the SPDC working condition, a multipartite squeezed state, $\hat{\rho}_{in}$, is produced. An optical passive filter of FWHM $\Delta \lambda_F$ is set before the SPD on the heralding path of the scheme, so as to implement a non-mode selective single-photon subtraction. The SPD detection signal is used to herald the production of the non-Gaussian state $\hat{\rho}_{out}$ as well as to trigger the homodyne detection. The measured state, represented by its Wigner function $W_H(x, y)$, strongly depends on the spectral profile $\alpha(\omega)$ of the homodyne local oscillator.

matching condition: by assuming a pump beam with a Gaussian spectrum, in the weak squeezing regime, $\{\psi_k(\omega)\}$ are given by Hermite functions and have the same global phase factor (leading to real $\gamma_{k,n}$) [8]. The squeezing parameter $\zeta_k$ of individual $\hat{\rho}_{sq,k}$ decreases as the supermode index $k$ increases. The effective number of excited modes can be quantified in terms of the Schmidt number [21],

$$K = \left( \sum_k \zeta_k^2 \right)^2 \sum_k \zeta_k^4.$$  

The Schmidt number $K$ is equal to 1 in the single-mode case and it increases with the number of supermodes with non-negligible excitation. In experiments, it depends on the SPDC pump spectral width as well as on the non-linear medium opto-geometrical properties at the chosen working wavelengths [21]. The output state described by (15) allows dealing with the general case of a single-photon subtraction at the output of a multimode SPDC process. Correspondingly, the Wigner function of the heralded state, as measured by the homodyne detector, can be expressed in terms of the mean photon number of squeezed states,
\[ n_k = \sinh^2(\zeta_k), \text{ and of } \mu_k = \tanh \zeta_k, \text{ as:} \]
\[
W_H(x, y) = e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} \left[ \sum_k \gamma_{k,k} n_k + \frac{1}{2\sigma_x^2} \left( \frac{x^2}{\sigma_x^2} - 1 \right) \sum_k n_k \mu_k \mu_n c_k c_n \right] + \frac{1}{2\sigma_y^2} \left( \frac{y^2}{\sigma_y^2} - 1 \right) \sum_k \gamma_{k,n} \mu_k \mu_n c_k c_n (1 + \mu_k)(1 + \mu_n), \tag{22}\]

with
\[
\sigma_x^2 = \frac{1}{2} \sum_n c_n^2 \frac{1 + \mu_n}{1 - \mu_n}, \quad \sigma_y^2 = \frac{1}{2} \sum_n c_n^2 \frac{1 - \mu_n}{1 + \mu_n}. \tag{23}\]

Expression (22) allows explicitly evaluating the impact on the measured state of both the filter profile, via the \(\{\gamma_{k,n}\}\), and of chosen the LO shape via the \(\{c_n\}\). The multimode features of the protocol input state are taken into account via \(n_k\) and \(\mu_k\).

**Measured heralded state without filtering on the heralding path**

The case of heralded states obtained when no filter is used on the heralding path provides a good intuition of the relevance of the filtering stage applied in the protocol. The absence of filter is described by \(t(\omega) = 1\) for all \(\omega\), i.e., according to Eq. (17), by \(\gamma_{k,n} = \delta_{k,n}\). Trivially, in this case, parallel modes \(\{\psi_k(\omega)\}\) exactly correspond to \(\{\psi_k(\omega)\}\) (see Eq. (A.1)). As indicated by Eq. (15), the protocol output state is a perfect mixture of photon subtracted states of the kind \(\hat{A}_n \hat{\rho}_m \hat{A}_n^\dagger\). Correspondingly, the Wigner function of the detected state can be expressed as:
\[
W_{H,\text{NF}}(x, y) = \sum_k p_k \left[ c_k^2 W_k^{(1)}(x, y) + (1 - c_k^2) W_{\text{SVS}}(x, y) \right], \tag{24}\]

with \(p_k = \frac{n_k}{\sum_l n_l}\) the probability of subtracting a photon from the mode \(k\). The first term of the previous expression represents the Wigner function of a single-photon subtracted squeezed state in supermode \(\psi_k(\omega)\) as seen by the HD:
\[
W_k^{(1)}(x, y) = e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} \left[ \frac{(1 + \mu_k)}{2\sigma_x^2(1 - \mu_k)} \left( \frac{x^2}{\sigma_x^2} - 1 \right) + \frac{(1 - \mu_k)}{2\sigma_y^2(1 + \mu_k)} \left( \frac{y^2}{\sigma_y^2} - 1 \right) + 1 \right]. \tag{25}\]

This state correctly corresponds to the target state of Eq. (20) when the homodyne LO is in the supermode \(\psi_k(\omega)\). In general cases, the effect of imperfect mode matching with the LO is included in \(\sigma_x\) and \(\sigma_y\).

The other term corresponds to squeezed vacuum contributions as detected by the homodyne detector.
\[
W_{\text{SVS}}(x, y) = e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} \tag{26}\]

12
Figure 4. Properties of the measured heralded state $W_{H,NF}(x,y)$ in the case of no filter on the heralding path. Left: Wigner function negativity $N_g$ of $W_{H,NF}(x,y)$ as a function of the local oscillator spectral bandwidth and the multimode character of the SPDC source providing the input state $\hat{\rho}_m$ via $K$; The LO spectrum is assumed to be Gaussian with a FWHM $\Delta\lambda_{LO}$. The white curve indicates the case of a local oscillator perfectly matched with the first SPDC supermode (i.e. $|c_0|^2 = 1$). Right: Probability $p_n$ of detecting an heralding photon from the supermode $\psi_n(\omega)$ for different SPDC working condition. Probabilities $p_n$ are directly proportional to the excitation of the squeezed supermodes $\{\psi_n(\omega)\}$. In all the simulations, the bandwidth of the SPDC pump beam is set to 0.5 nm. $K$ depends on the SPDC pump spectral width as well as on the the non-linear medium opto-geometrical parameters at the chosen working wavelengths [21].

Equation (24) expresses the fact that, in the case of no filter, photon-counting events cannot be associated to a specific supermode (nor to any subset of them); as a consequence, the state measured by the HD only occasionally corresponds to the desired non-Gaussian one. A physical intuition of this idea is provided by the case of a LO in a specific supermode $\alpha(\omega) = \psi_n(\omega)$, i.e. $c_k^2 = \delta_{k,n}$. In this condition:

$$W_{H,NF}(x,y)|_{c_k^2=1} = p_nW_n^{(1)}(x,y) + \sum_{k \neq n} p_k W_{SVS}(x,y). \tag{27}$$

The measured heralded state is equal to $W_n^{(1)}(x,y)$ only with a probability $p_n$, corresponding to the probability of subtracting a photon in the LO mode $\psi_n(\omega)$. All remaining cases lead to the detection of squeezed vacuum with a probability $\sum_{k \neq n} p_k = 1 - p_n$.

State non-classicality can also be evaluated in terms of their Wigner function negativity, defined as the volume of the negative part of the Wigner function [22]:

$$N_g = \frac{1}{2} \left[ \iint |W_H(x,y)| dx dy - 1 \right]. \tag{28}$$

The negativity for the ideal case of a single-photon subtraction on a single-mode squeezed vacuum states is computed to be $N_g = 2e^{-1/2} - 1 \approx 0.213$. 

13
Fig. 3-left shows $N_g$ for the detected heralded state as a function of the local oscillator spectral width and of the Schmidt number $K$. To comply with many experimental cases, in which the LO beam is directly provided by the output of a pulsed laser. The LO profile, $\alpha_{LO}(\omega)$, is assumed to be Gaussian with FWHM of $\Delta\lambda_{LO}$. As shown by the figure, optimal $N_g$ values are obtained for low $K \approx 1$ and when the Gaussian LO profile $\alpha_{LO}(\omega)$ is chosen so as to match exactly the first SPDC supermode, i.e. $\alpha_{LO}(\omega) = \psi_0(\omega)$ and $|c_0|^2 = 1$. This case is represented in the figure by the white line. When the number of supermodes at the SPDC output increases ($K > 1$), $N_g$ rapidly degrades. This behaviour can be understood by observing Fig. 3-right, that represents the probability $p_k$ of subtracting a photon in the supermode $k$; as it can be seen, the probability $p_0$ is generally higher than $p_{k>0}$ and the single-photon subtraction mostly takes place on the first supermode ($k = 0$). Accordingly, choosing the LO profile identical to the first supermode ($\alpha_{LO}(\omega) = \psi_0(\omega)$) corresponds to optimising, in Eq. (27), the detection of the non-Gaussian state $W_n(1)$ that shows a negative Wigner function. Note that, to measure high negativities, contributions with $p_{k>0}$, that lead in Eq. (27) to the detection of squeezed vacuum, should be minimised. This condition is easily respected when $K \approx 1$. However, as soon as the Schmidt number increases, the distribution of $p_k$ becomes flatter, thus explaining why, for high $K$ values, $N_g$ progressively decreases along the white line.

**Rectangular or Gaussian spectral filters in the heralding path**

As discussed in the previous sections, the actual spectral profile of the function $t(\omega)$ determines the shape of the filter supermodes $\{\psi^+_k(\omega), \psi^-_k(\omega)\}$ and, via Eq. (17), the explicit shape of the coefficients $\gamma_{k,n}$ to be included in $W_H(x,y)$ of Eq. (22). Accordingly, different filter profiles and FWHM can have a strong effect on the heralded states as well as on the LO profile that allows optimising the detection of non-Gaussian features. From the experimental side, different filter spectral profiles can be chosen depending on the specific situation. Plug-and-play filters with a rectangular profile, transmitting light only within a certain bandwidth are widely used in guided-wave experiments, where broadband SPDC emission is generated in single-pass configuration in non-linear waveguides [10, 12]. This scenario covers dense wavelength multiplexers (DWM) and Bragg filters and it is quite common when quantum optical states are produced at classical telecommunication wavelengths so as to be compatible with low loss fibre networks. Alternatively, filters with $t(\omega)$ described by continuous and strictly non-null functions, such as Gaussian or Lorentzian, can be used. This is the case, for instance, of bulk configurations in which DWM or Bragg filters with desired characteristics in terms of transmission or spectral width are not available and optical cavities are instead employed on the heralding path [18]. In the following, the cases of a rectangular filter and of a Gaussian-shaped $t(\omega)$ will be considered and compared. The explicit expressions of $\gamma_{k,n}$ for these cases are reported in Appendix C. As for the previous section, a Gaussian LO profile
Figure 5. Negativities $N_g$ of the measured heralded Wigner function $W_H(x,y)$ in the case of a rectangular filter on the heralding path expressed as functions of the local oscillator spectral bandwidth and Schmidt number $K$. Two filter bandwidths are considered: left: 5 nm ($\approx 600$ GHz) FWHM and right: 1 nm ($\approx 100$ GHz) FWHM (b). The white curve indicates the case of a local oscillator perfectly matching the first SPDC supermode (i.e. $|c_0|^2 = 1$).

$\alpha_{LO}(\omega)$ will be considered.

Fig. 5 gives the Wigner function negativity for the detected heralded state in the case of a rectangular spectral filter in the heralding path. The first $\{\psi_{\perp n}(\omega)\}$ and $\{\psi_{n}(\omega)\}$ for this kind of $t(\omega)$ are reported in Fig. 2 for the case of a filter FWHM of 5 nm. Two different filter bandwidths are considered (5 nm and 1 nm). As a general consideration, for both FWHMs, $N_g$ decreases when the number of excited supermodes increases ($K > 1$). In the case of a wide filter (Fig. 5-left), the degradation is quite fast, with a behaviour similar to the one discussed for the case of no filter in the previous section. The optimal LO is still very close to the case of $\alpha_{LO}(\omega) = \psi_0(\omega)$, matching the first SPDC supermode (i.e. $|c_0|^2 = 1$, see the white curve). The situation dramatically changes when the width of the filter is reduced (Fig. 5-right). In this case, the protocol performances are much robust against the multimode character of the SPDC source and high negativity values can be obtained also for higher $K > 1$, provided the FWHM $\Delta\lambda_{LO}$ for the Gaussian LO is suitably chosen. Remarkably, the optimal $\alpha_{LO}(\omega)$ profile does not match any of the supermodes $\{\psi_k(\omega)\}$, nor any of the filter-defined functions $\{\psi_{\perp n}(\omega)\}$ and $\{\psi_{n}(\omega)\}$ (see Fig. 2). Its FWHM $\Delta\lambda_{LO}$ can be easily computed numerically. The possibility of using a Gaussian LO in combination with a narrowband filter opens the way to simple experimental realisations, free from spectral shaping stages on the local oscillator path and mode-selective single-photon subtraction. In this regard, it is pertinent to underline that, although, to our knowledge, it has never been explicitly explained, the use of narrowband filters on the heralding path represents a widely reported strategy to improve the quality of single-photon subtracted states in pulsed regime [7].
The qualitative behaviour discussed for the case of a rectangular filter is found also in the case of a Gaussian $t(\omega)$. In this case, only the $\{\psi_n^{K}(\omega)\}$ are non-null. Their profile, similar but not exactly matching the one of the original supermodes, is represented in Fig. 2. As for the rectangular filter, reducing the FWHM of the Gaussian $t(\omega)$ has the effect of making the protocol robuster against higher $K$ and allows identifying an optimal LO bandwidth $\Delta\lambda_{LO}$. A comparison among the different filter shapes is shown in Fig. 6 in the case of an SPDC pump with spectral width of 0.5 nm. The figure left represents the negativity of the Wigner function in the case of no spectral filtering in the heralding path (black), as well as of 5 nm (red) and 1 nm (green) filter bandwidths for $K = 1.77$. The straight lines represent Gaussian spectral filters whereas the dashed lines rectangular spectral filters. As expected, the narrowband spectral filters improve the negativity of the homodyne detected state. Also, the optimal LO bandwidth is slightly shifted to lower values when the filter FWHM decreases. The narrower is the filter, the smaller is the difference between the behaviour obtained with a rectangular or a Gaussian filter profile. As shown in Fig. 6-right similar results can be found for different values of $K > 1$, where optimal measured negativities (i.e. corresponding to the optimal LO profiles) are reported as functions of $K$ and for different filter widths. It is relevant to stress that, the pertinence of narrowband filters in heralding schemes is equally confirmed in the context of low squeezing and heralded single photon sources, where pure single photon state can be retrieved, even from an highly multimode squeezing, when delta-like $t(\omega)$ are employed. The case of heralded single photon sources is discussed in detail in the Appendix D.
Discussed results provide a powerful tool to model single-photon subtraction in a multimode context and to optimise experimental conditions for the generation and detection of high quality non-Gaussian states. As for an example, in what follows, the developed treatment is applied to the practical situation of Schrödinger kitten states obtained from single-photon subtraction from squeezed light emitted by a realistic source. A particularly interesting case is that of a SPDC process occurring in integrated photonic circuits on Lithium Niobate (LN), where on-chip generation and active manipulation of optical quantum states at telecom wavelengths can be provided by the association of the material non linear optical- and electro-optical properties [12, 23]. At the same time, in pulsed regime, LN dispersion properties prevent from working in single mode conditions and generated squeezed light can exhibit a high number of modes \( K > 1 \). Fig. 7 considers a typical case of multimode light generated around 1560nm via type-0 SPDC in a 10mm-long LN crystal pumped by a pulsed beam at 780nm with a spectral width of 0.5nm. In such conditions, the Schmidt number can be calculated to be \( K \approx 9 \) [8]. Fig. 7 shows the obtained \( N_g \) for different filtering and LO choices. As it can be seen, a filter of 1nm allows reaching almost optimal \( N_g \) values for experimentally accessible Gaussian LO profiles with \( \Delta \lambda_{LO} \approx 2 \) nm. This corresponds to optical pulses in picosecond regime, that can be easily employed in guided-wave configurations, with the possibility of limiting parasitic non-linear optical effects and chromatic dispersion in optical fibres. For a subtraction BS reflectivity \( r_2^s \approx 0.05 \), the corresponding protocol success probability is on the order of 0.01, in agreement with typical values reported in the literature [7].

![Fig. 7](image)

**Figure 7.** *Left:* Negativity of the Wigner function as function of the LO spectral width, for \( K \approx 9 \) as for frequency degenerate SPDC around 1560nm, pumped with optical pulses at 780nm with a width of 0.5nm. The SPDC crystal is taken to be 10mm-long. *Right:* Optimal \( \Delta \lambda_{LO} \) and \( \Delta \lambda_F \) required to ensure, for a given \( K \), a Fidelity with the target state of 0.95. Curves refer to a single-photon subtraction scheme (red): with 5nm spectral filter in the heralding path; (green) or with 1nm spectral filter in the heralding path. (Straight lines): Gaussian spectral filters; (dashed lines): rectangular spectral filters.

The same results are obtained in terms of the Fidelity between the measured heralded
state and the target one of Eq. (20) (see Appendix B for the explicit calculation):

\[ \mathcal{F} = 2\pi \int \int W_H(x, y)W_T(x, y)dxdy. \] (29)

In the simulations, the squeezing parameter of the target state is assumed to be equal to \( \zeta_0 \), i.e. to the squeezing of the first spectral supermode \( (k = 0) \) and set to give -3 dB of noise reduction with respect to the standard quantum level; such a value is commonly used in Schrödinger kitten generation [7]. Fig. 7-right shows, for different \( K \), the LO bandwidth and the filter FWHM that allows reaching a fidelity with the target state of 0.95. As it can be seen for \( K=9 \), this leads to a filter FWHM of \( \approx 1 \text{nm} \) and to a \( \Delta \lambda_{LO} \approx 2 \text{nm} \).

4. Conclusions

This work presents a versatile and practical theoretical framework allowing to describe single-photon subtraction in the case where mode-selective operation is not available or not easy to implement. In this condition a simple spectral filter is considered on the heralding path of the scheme. The explicit shape of the heralded state, as measured by a homodyne detector, is provided as a function of the multimode character of the protocol input state, of the spectral profile of the filter and of the local oscillator shape. Based on the desired target features, this allows choosing adequate working conditions for future practical realisations, thus representing a powerful tool for optimally designing future experiments. The strength of the model is illustrated by applying it to the generation of Schrödinger kittens, obtained from single-photon subtraction from a multimode squeezed state. Different working conditions have been investigated and discussed, to determine the features of heralded non-Gaussian states that can be actually obtained and retrieved when single-mode generation or state manipulation are not easy to be implemented.
Appendix A. Properties of functions \( \{\psi_n^\parallel(\omega), \psi_n^\perp(\omega)\} \)

In this appendix, we provide some main properties of the functions \( \{\psi_n^\parallel(\omega)\} \) and \( \{\psi_n^\perp(\omega)\} \) that are used in the main text. The basic idea is to decompose the functions \( t(\omega)\psi_k(\omega) \) in an orthonormal basis \( \{\psi_n^\parallel(\omega)\} \):

\[
t(\omega)\psi_k(\omega) = \sum_n a_{kn} \psi_n^\parallel(\omega), \quad \text{with} \quad a_{kn} = \int t(\omega)\psi_k(\omega) (\psi_n^\parallel(\omega))^* \, d\omega, \quad (A.1)
\]

and to complete this basis with the set \( \{\psi_n^\perp(\omega)\} \). The functions \( \{\psi_n^\perp(\omega)\} \) are orthogonal to each other and to the \( \{\psi_n^\parallel(\omega)\} \), such that the ensemble of \( \{\psi_n^\parallel(\omega), \psi_n^\perp(\omega)\} \) gives a complete orthonormal basis. By inverting the relations (9), one obtains:

\[
\psi_n^\parallel(\omega) = \sum_k p_{kn}^* \psi_k(\omega) \quad (A.2a)
\]
\[
\psi_n^\perp(\omega) = \sum_k q_{kn}^* \psi_k(\omega). \quad (A.2b)
\]

Also, by noting that

\[
\sum_k p_{kn}^* p_{kn_2} = \sum_k q_{kn}^* q_{kn_2} = \delta_{n_1,n_2} \quad (A.3a)
\]
\[
\sum_k p_{kn_1}^* q_{kn_2} = \sum_k q_{kn_1}^* p_{kn_2} = 0, \quad (A.3b)
\]

it can be proved, thanks to (8) that

\[
\sum_k \psi_k(\omega)\psi_k^*(\omega') = \delta(\omega - \omega') \implies \sum_l \psi_l^\parallel(\omega) (\psi_l^\parallel(\omega'))^* + \psi_l^\perp(\omega) (\psi_l^\perp(\omega'))^* = \delta(\omega - \omega'),
\]

thus providing a completeness relations for the ensemble of functions \( \{\psi_n^\parallel(\omega), \psi_n^\perp(\omega)\} \).

By multiplying (A.2a) and (A.2b) by \( t(\omega) \) and remembering definition (A.1) one obtains:

\[
t(\omega)\psi_n^\parallel(\omega) = \sum_l \left( \sum_k p_{kn}^* a_{kl} \right) \psi_l^\parallel(\omega) = \sum_l T_{nl} \psi_l^\parallel, \quad (A.5a)
\]
\[
t(\omega)\psi_n^\perp(\omega) = \sum_l \left( \sum_k q_{kn}^* a_{kl} \right) \psi_l^\parallel(\omega). \quad (A.5b)
\]

Then:

\[
\int |t(\omega)|^2 |\psi_n^\perp(\omega)|^2 d\omega = \int \left[ |t(\omega)|^2 \psi_n^\perp(\omega) \right] (\psi_n^\perp(\omega))^* d\omega. \quad (A.6)
\]
However, according to the two relations (A.5), \[ |t(\omega)|^2 |\psi_n(\omega)|^2 \] only has components along \( \psi_n^\parallel \), which implies that \( \int |t(\omega)|^2 |\psi_n(\omega)|^2 d\omega = 0 \), and, as a consequence, that:

\[
\forall \omega, \quad t(\omega)\psi_n^\perp(\omega) = 0. \tag{A.7}
\]

This result is quite strong, and implies that \( \psi_n^\perp(\omega) \) is only nonzero where \( r(\omega) = 1 \), i.e. in all region for which the filter transmission coefficient \( t(\omega) \) goes exactly to 0. This also means that

\[
\forall \omega, \quad r(\omega)\psi_n^\perp(\omega) = \psi_n^\perp(\omega). \tag{A.8}
\]

Note also that (A.1) implies that if \( t(\omega) = 0 \) then \( \psi_n^\parallel(\omega) = 0 \), for regular enough functions. This implies that \( \psi_n^\perp(\omega) \) and \( \psi_n^\parallel(\omega) \) have disjoint supports.

Appendix B. Fidelity of the measured state with a target non-Gaussian state

The quality of the detected state can be evaluated by its fidelity with a monomode photon subtracted squeezed vacuum as target state as given by the overlap formula:

\[
\mathcal{F} = 2\pi \int \int W_H(x,y)W_t(x,y)\,dxdy \tag{B.1}
\]

By replacing (22) and (20) in (B.1) we find that:

\[
\mathcal{F} = \frac{2}{P\sigma_x\sigma_y\sqrt{(\frac{1}{\sigma_x^2} + \frac{2}{s}) (\frac{1}{\sigma_y^2} + 2s)}} \left[ -A + \left( \frac{2A}{s} - B \right) \frac{1}{\sigma_x^2 + 2s} + (2As - C) \frac{1}{\sigma_y^2 + 2s} \right.
\]

\[
+ \frac{2B}{s} \left( \frac{3}{\sigma_x^2 + 2s} \right) + 2Cs \left( \frac{3}{\sigma_y^2 + 2s} \right) + \left( \frac{2C}{s} + 2Bs \right) \left( \frac{1}{\sigma_x^2 + 2s} \right) \left( \frac{1}{\sigma_y^2 + 2s} \right) \right]
\]

\[
\left. \left. \right] \right), \tag{B.2}
\]

with

\[
A = P - \frac{1}{2\sigma_x^2} \sum_{k,n} \gamma_{k,n} \frac{\mu_k\mu_n c_k c_n}{(1 - \mu_k)(1 - \mu_n)} - \frac{1}{2\sigma_y^2} \sum_{k,n} \gamma_{k,n} \frac{\mu_k\mu_n c_k c_n}{(1 + \mu_k)(1 + \mu_n)} \tag{B.3}
\]

\[
B = \frac{1}{2\sigma_x^2} \sum_{k,n} \gamma_{k,n} \frac{\mu_k\mu_n c_k c_n}{(1 - \mu_k)(1 - \mu_n)} \tag{B.4}
\]

\[
C = \frac{1}{2\sigma_y^2} \sum_{k,n} \gamma_{k,n} \frac{\mu_k\mu_n c_k c_n}{(1 + \mu_k)(1 + \mu_n)}. \tag{B.5}
\]
Appendix C. Gaussian and rectangular filters on the heralding path

In the case of a Gaussian shaped JSA, the explicit shape of the supermode is given as in [8]:

\[ \psi_k(\omega) \propto \sqrt{\frac{\tau_s}{\pi 2^k k!}} H_k \left[ \tau_s (\omega - \frac{\omega_p}{2}) \right] e^{-\tau_s^2 (\omega - \frac{\omega_p}{2})^2 / 2}, \] (C.1)

where \( \omega_p \) is the SPDC pump central frequency and \( \tau_s \) is given by the pump and by the process \( K \) [8]. In general,

\[ \gamma_{k,n} = \int |t(\omega)|^2 \psi_k(\omega) \psi_n^*(\omega) d\omega, \] (C.2)

where \( \psi_k(\omega) \) is the signal mode \( k \) given by the equation C.1 with a phase factor which does not depend on the mode order \( k \) [8].

Using a Gaussian filter with transmittance \( t(\omega) \) can be described by:

\[ t(\omega) = e^{-\frac{4 \ln(2)}{\Delta \omega_F^2} (\omega - \frac{\omega_p}{2})^2}, \] (C.3)

where \( \Delta \omega_F \) is the FWHM bandwidth. Accordingly, by defining \( \nu = \frac{8 \ln(2)}{\Delta \omega_F^2} + \tau_s^2 \), the coefficients \( \gamma_{k,n} \) are:

\[ \gamma_{k,n} = \frac{\tau_s}{\sqrt{\nu 2^{k+n} k! n! \pi}} \int e^{-\nu (\omega - \frac{\omega_p}{2})^2} H_k \left[ \tau_s (\omega - \frac{\omega_p}{2}) \right] H_m \left[ \tau_s (\omega - \frac{\omega_p}{2}) \right] d\omega \] (C.4)

By using the integration formula given in [24]:

\[ \int e^{-y^2} H_k(ay) H_n(ay) dy = \sqrt{\pi} \sum_{m=0}^{\min[k,n]} 2^m m! \left( \begin{array}{c} k \\ m \end{array} \right) \left( \begin{array}{c} n \\ m \end{array} \right) (1 - a^2)^{k+n-2m} H_{k+n-2m}(0) \] (C.5)

it is possible to write:

\[ \gamma_{k,n} = \frac{\tau_s}{\sqrt{\nu 2^{k+n} k! n! \pi}} \sum_{m=0}^{\min[k,n]} 2^m m! \left( \begin{array}{c} k \\ m \end{array} \right) \left( \begin{array}{c} n \\ m \end{array} \right) (1 - \frac{\tau_s^2}{\nu})^{k+n-2m} H_{k+n-2m}(0) \] (C.6)

As the Hermite-Gauss polynomial verify:

\[ H_k(0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (-1)^\frac{k}{2} \frac{k!}{(\frac{k}{2})!} & \text{if } k \text{ is even} \end{cases} \] (C.7)

The explicit expression of the \( \gamma_{k,n} \) for a Gaussian filter is:

\[ \gamma_{k,n} = \begin{cases} 0 & \text{if } k + n \text{ is odd} \\ \frac{\tau_s}{\sqrt{\nu 2^{k+n} k! n!}} \sum_{m=0}^{\min[k,n]} 2^m m! \left( \begin{array}{c} k \\ m \end{array} \right) \left( \begin{array}{c} n \\ m \end{array} \right) (\frac{\tau_s^2}{\nu} - 1)^{\frac{k+n}{2} - m} (k + n - 2m)! \frac{(k + n - 2m)!}{(\frac{k+n}{2} - m)!} & \text{if } k + n \text{ is even} \end{cases} \] (C.8)
Following a similar reasoning, the case of a rectangular filter can also be described. In this case:

\[
t(\omega) = \begin{cases} 
    T_0 & \text{if } \omega \in \left[-\frac{\Delta \omega F}{2}, +\frac{\Delta \omega F}{2}\right] \\
    0 & \text{elsewhere}
\end{cases}
\]  

(C.9)

Correspondingly, the coefficients \(\gamma_{k,n}\) are:

\[
\gamma_{k,n} = \begin{cases} 
    0 & \text{if } k + n \text{ is odd} \\
    \frac{1}{\sqrt{2^{k+n}k!n!}} \int_{-\tau_s \Delta \omega F}^{+\tau_s \Delta \omega F} e^{-x^2} H_k(x)H_n(x)dx & \text{if } k + n \text{ is even}
\end{cases}
\]  

(C.10)

Appendix D. Heralded single photon production in a non mode-selective configuration

The impact of the filter bandwidth on heralded state preparation can also be discussed in the context of heralded single-photon sources. In this case, a particularly common configuration relies on pairs of single photons produced by a SPDC process in the weak pumping regime \[16\]; by deterministically separating the paired photons, for instance thanks to their polarisation, it is indeed possible to use the detection of one photon to herald the presence of its twin. The quality of produced states strongly depends on the possibility of engineering the SPDC process so as to work in single-mode configuration (\(K=1\)), a condition that often corresponds to sources exploiting type II SPDC \[16\]. Here, we will consider instead the case of degenerate SPDC, such that paired photons have parallel polarisations and that single-mode SPDC is not experimentally accessible; this configuration is for instance very common in integrated quantum optics \[12, 23\]. In the special case of weak pumping approximation, light at the output of the degenerate SPDC can be approximated as in the state:

\[
|\psi\rangle \approx |0\rangle + A \int \int d\omega d\omega' f(\omega, \omega') \hat{a}^{\dagger}(\omega)\hat{a}^{\dagger}(\omega'),
\]  

(D.1)

where the joint spectral amplitude \(f(\omega, \omega')\) (JSA) depends on the SPDC pump spectral width and on the process working condition via the phase-matching. The factor \(A\) gathers the constants linked with the non-linear process, such as pump power and non-linear susceptibility (here \(A << 1\) as we are in the weak pumping regime). The bosonic operators verify the commutation rule \([\hat{a}(\omega), \hat{a}^{\dagger}(\omega')] = \delta(\omega - \omega')\) and \([\hat{a}(\omega), \hat{a}(\omega')] = 0\). Note the the case of perfectly single-mode regime (i.e. \(K=1\)) corresponds to a factorisable JSA, \(f(\omega, \omega') = g(\omega)h(\omega')\) \[8\]. The state after the subtraction beamsplitter and the filtering stage can be obtained by employing the expressions (3) and (5). In the special case considered here, dealing with single photon regime, vacuum contributions are traced out by the single-photon detection process and the state right before the detectors can be written as:

\[
|\psi'\rangle \propto \int \int \int d\omega d\omega' d\omega'' f(\omega, \omega') t(\omega'') \hat{b}^{\dagger}_{\text{out}}(\omega'') \hat{b}^{\dagger}(\omega) \hat{a}^{\dagger}(\omega') |0\rangle,
\]  

(D.2)
where \( \hat{b}_{\text{out}}(\omega) \) indicates spectral components at frequency \( \omega \) downstream of the filter defined as in Eq. (D.1). As in the main text, the heralded state can be obtained by introducing a POVM operator describing the detection of light at the SPD. In the case considered, here, without introducing the description in terms of the supermodes, the single photon detection is associated with the POVM operator:

\[
\Pi_{\text{out}} = \int d\omega \ket{1_{\text{out}}(\omega)} \bra{1_{\text{out}}(\omega)},
\]

where \( \ket{1_{\text{out}}(\omega)} = \hat{b}_{\text{out}}(\omega) \ket{0} \) and describe, in a very general way, single photons downstream of the filter. By tracing over the degrees of freedom of the detected mode (i.e. those related to \( \hat{b}_{\text{out}} \)), the heralded state can be written as:

\[
\hat{\rho}_{\text{out}} \propto \int d\omega d\omega' d\Omega d\Omega' f(\omega, \omega') t(\omega'') \hat{a}(\omega'') \hat{a}^\dagger(\omega') \ket{0} \bra{0} \hat{a}(\Omega) \hat{a}(\Omega') \hat{a}^\dagger(\omega'') t(\omega'') f^*(\Omega, \Omega').
\]

A particularly interesting case is represented by the limit of an extremely narrowband filter, that we describe here by a transmission profile \( t(\omega) = \delta(\omega - \omega_0) \). In this limit, the previous expression reads as:

\[
\hat{\rho}_{\text{out}} \propto \int d\omega d\omega' d\Omega d\Omega' f(\omega, \omega') \hat{a}(\omega_0) \hat{a}^\dagger(\omega') \ket{0} \bra{0} \hat{a}(\Omega) \hat{a}(\Omega') \hat{a}^\dagger(\omega_0) f^*(\Omega, \Omega').
\]

By exploiting the commutation relations among bosonic operators at different frequency, it is possible to re-express the output state as:

\[
\hat{\rho}_{\text{out}} \propto \int d\omega d\omega' [f(\omega, \omega_0) f^*(\omega_0, \omega') + f(\omega_0, \omega) f^*(\omega_0, \omega')] \hat{a}^\dagger(\omega) \ket{0} \bra{0} \hat{a}^\dagger(\omega') + \int d\omega d\omega' [f(\omega, \omega_0) f^*(\omega', \omega_0) + f(\omega_0, \omega) f^*(\omega', \omega_0)] \hat{a}^\dagger(\omega) \ket{0} \bra{0} \hat{a}^\dagger(\omega')
\]

\[
= \int d\omega d\omega' G(\omega) G^*(\omega') \hat{a}^\dagger(\omega) \ket{0} \bra{0} \hat{a}^\dagger(\omega'),
\]

where we introduced \( G(\omega) = f(\omega, \omega_0) + f(\omega_0, \omega') \). By introducing the \( \hat{A} = \int d\omega G(\omega) \hat{a}^\dagger(\omega) \) Eq. (D.6) can be re-expressed as:

\[
\hat{\rho}_{\text{out}} \propto \hat{A}^\dagger \ket{0} \bra{0} \hat{A}.
\]

This relation correctly describes a pure single photon state heralded in the mode described by the operator \( \hat{A} \). This confirms the results found in the very general case described in the main text, about the purity and the quality of heralded state with narrowband filters. This result provides an analytical proof that optimal filtering is as narrow as possible for the generation of odd Schrödinger cat states with vanishingly small amplitude (single photon Fock states). This reasoning must be put in parallel with the well-established result that a pure signal photon can be heralded by the detection of a narrow filtered idler photon from a non-degenerate SPDC source.
[1] V. Giovannetti, S. Lloyd, and L. Maccone, “Quantum metrology,” Phys. Rev. Lett., vol. 96, p. 010401, 2006.

[2] S. L. Braunstein and P. van Loock, “Quantum information with continuous variables,” Rev. Mod. Phys., vol. 77, p. 513, 2005.

[3] A. Ferraro, S. Olivares, and M. G. A. Paris, Gaussian states in continuous variable quantum information. Bibliopolis, Napoli, 2005.

[4] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, “Gaussian quantum information,” Rev. Mod. Phys., vol. 84, pp. 621–669, May 2012.

[5] Q. Zhuang, P. W. Shor, and J. H. Shapiro, “Resource theory of non-gaussian operations,” Phys. Rev. A, vol. 97, p. 052317, May 2018.

[6] E. Gouzien, F. Brunel, S. Tanzilli, and V. D’Auria, “Scheme for the generation of hybrid entanglement between time-bin and wavelike encodings,” Phys. Rev. A, vol. 102, p. 012603, Jul 2020.

[7] A. I. Lvovsky, P. Grangier, A. Ourjoumtsev, V. Parigi, M. Sasaki, and R. Tualle-Brouri, “Production and applications of non-gaussian quantum states of light,” arXiv:2006.16985v1, 2020.

[8] W. Wasilewski, A. Lvovsky, K. Banaszek, and C. Radzewicz, “Pulsed squeezed light: Simultaneous squeezing of multiple modes,” Physical Review A, vol. 73, no. 6, p. 063819, 2006.

[9] M. Sasaki and S. Suzuki, “Multimode theory of measurement-induced non-gaussian operation on wideband squeezed light: Analytical formula,” Physical Review A, vol. 73, no. 4, p. 043807, 2006.

[10] A. Eckstein, A. Christ, P. J. Mosley, and C. Silberhorn, “Highly efficient single-pass source of pulsed single-mode twin beams of light,” Phys. Rev. Lett., vol. 106, p. 013603, Jan 2011.

[11] Y.-S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps, “Non-gaussian quantum states of a multimode light field,” Nat. Phys., vol. 16, pp. 144–147, 2020.

[12] F. Mondain, T. Lunghi, A. Zavatta, E. Gouzien, F. Doutre, M. De Micheli, S. Tanzilli, and V. D’Auria, “Chip-based squeezing at a telecom wavelength,” Photonics Research, vol. 7, no. 7, pp. A36–A39, 2019.

[13] V. Averchenko, C. Jacquard, V. Thiel, C. Fabre, and N. Treps, “Multimode theory of single-photon subtraction,” New Journal of Physics, vol. 18, no. 8, p. 083042, 2016.

[14] G. Patera, N. Treps, C. Fabre, and G. J. de Valcárcel, “Quantum theory of synchronously pumped type i optical parametric oscillators: generation of multiple, squeezed frequency combs below threshold,” Eur. Phys. J. D, vol. 56, p. 123, 2010.

[15] E. Gouzien, S. Tanzilli, V. D’Auria, and G. Patera, “Morphing supermodes: A full characterization for enabling multimode quantum optics,” Phys. Rev. Lett., vol. 125, p. 103601, Aug 2020.

[16] A. M. Brańczyk, T. Ralph, W. Helwig, and C. Silberhorn, “Optimized generation of heralded fock states using parametric down-conversion,” New Journal of Physics, vol. 12, no. 6, p. 063001, 2010.

[17] A. Christ, C. Lupo, M. Reichelt, T. Meier, and C. Silberhorn, “Theory of filtered type-ii pdc in the continuous-variable domain: Quantifying the impacts of filtering,” Phys. Rev. A, vol. 90, p. 023823, Aug 2014.

[18] W. Asavanant, K. Nakashima, Y. Shiozawa, J.-I. Yoshikawa, and A. Furusawa, “Generation of highly pure schroedinger’s cat states and real-time quadrature measurements via optical filtering,” Opt. Express, vol. 25, pp. 32227–32242, 2017.

[19] A. Ourjoumtsev, “Theoretical and experimental study of quantum coherent superpositions and of non-gaussian entangled states of the light,” PhD Thesis, 2008.

[20] V. D’Auria, O. Morin, C. Fabre, and J. Laurat, “Effect of the heralding detector properties on the conditional generation of single-photon states,” Eur. Phys. J. D, vol. 66, p. 249, 2012.

[21] A. Christ, B. Brecht, W. Maurer, and C. Silberhorn, “Probing multimode squeezing with correlation functions,” New J. Phys., vol. 13, p. 033027, 2011.

[22] A. Kenfack and K. Życzkowski, “Negativity of the wigner function as an indicator of non-classicality,” Journal of Optics B: Quantum and Semiclassical Optics, vol. 6, no. 10, p. 396, 2004.
[23] F. Lenzini, J. Janousek, O. Thearle, M. Villa, B. Haylock, S. Kasture, L. Cui, H.-P. Phan, D. V. Dao, H. Yonezawa, P. K. Lam, E. H. Huntington, and M. Lobino, “Integrated photonic platform for quantum information with continuous variables,” *Science Advances*, vol. 4, no. 12, 2018.

[24] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. Academic press, 2014.