Coherent multi-spin exchange in a quantum-dot spin chain

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Heisenberg exchange coupling between neighboring electron spins in semiconductor quantum dots provides a powerful tool for quantum information processing and simulation. Although so far unrealized, extended Heisenberg spin chains can enable long-distance quantum information transfer and the generation of non-equilibrium quantum states. In this work, we implement simultaneous, coherent exchange coupling between all nearest-neighbor pairs of spins in a quadruple quantum dot. The main challenge in implementing simultaneous exchange is the nonlinear and nonlocal dependence of the exchange couplings on gate voltages. Through a combination of electrostatic simulation and theoretical modeling, we show that this challenge arises primarily due to lateral shifts of the quantum dots during gate pulses. Building on this insight, we develop two models, which can be used to predict the confinement gate voltages for a desired set of exchange couplings. We achieve simultaneous and independent control of all three exchange couplings in a quadruple quantum dot. We demonstrate two-, three-, and four-spin exchange oscillations, and our data agree with simulations.

INTRODUCTION

A unique and valuable feature of electron-spin qubits in quantum dots [1, 2] is the voltage-controlled nearest-neighbor Heisenberg exchange coupling. Exchange coupling results from the interplay of the electronic confinement potential and the Pauli exclusion principle, which prevents any two electrons from having the same quantum numbers. On a basic level, exchange enables two- [1, 3–5] and three-qubit gates [6] for single-spin qubits. Exchange also allows rapid and high-fidelity initialization and readout of pairs of spins. As a result, exchange coupling underlies the operation of electron spin qubits consisting of two [7, 8], three [9–12], or more [13] electrons. Superexchange [14, 15] in multi-electron systems and extended exchange-coupled spin chains can enable new forms of quantum-information transfer [16, 17] and the generation of many-body entangled states [18]. Recent experiments exploiting pulsed exchange in spin chains point to the feasibility of these proposals [19]. Heisenberg spin chains are also predicted to generate non-equilibrium quantum phenomena [20, 21].

Until now, a spin chain with multiple independently adjustable exchange couplings has not been demonstrated in semiconductor quantum dots. Recently, several technological advances, including overlapping gate architectures [22–24], barrier-controlled exchange coupling [25, 26], and virtual gates [27–30] have combined to make this a possibility. In this work, we synthesize these results to demonstrate coherent multi-spin exchange in a GaAs quadruple quantum dot array. As discussed further below, a significant challenge is the nonlinear and non-local dependence of the exchange couplings in quantum-dot arrays on the confinement gate voltages, which we show results from electronic wavefunction shifts during exchange pulses.

We successfully model our data using the Heitler-London expression for exchange between two spins [31], assuming that the barrier-gate pulses used to induce exchange primarily shift the locations of the electrons. The parameters we extract by fitting our data to the Heitler-London model agree well with electrostatic simulations of the confinement potential of our device. We also show that a simpler, exponential model also fits our data well and can be used to predict gate voltages for independent control of exchange couplings. We demonstrate two-, three- and four-spin exchange in our four-dot device. Our results are applicable to Si qubits and extensible to longer arrays of spin qubits, an encouraging prospect for quantum information processing and the exploration of Heisenberg spin chain physics.

DEVICE

We use a quadruple quantum dot in a GaAs/AlGaAs heterostructure with overlapping-gates [Fig. 1 (a)]. Two extra quantum dots placed above the main-dot array serve as charge sensors, and are configured for rf-refl moteometry [32, 33]. We tune the confinement potential using “virtual gates” [27–30, 34] such that each dot contains only one electron. We define virtual plunger gate voltages $P_1$, $P_2$, $P_3$, and $P_4$ as linear combinations of the physical plunger gate voltages ($p_1$, $p_2$, $p_3$, $p_4$).
\( p_4 \) such that changes to \( P_i \) are proportional to changes in the electrochemical potential of dot \( i \). We also define virtual barrier-gate voltages \( B_1, B_2, \) and \( B_3 \) as the voltage applied to the corresponding physical barrier \( (b_1, b_2, \) and \( b_3) \) together with a linear combination of physical plunger voltages chosen such that the chemical potentials of the dots are unchanged by the barrier pulse. These virtual gates are related to the actual voltages via the measured capacitance matrix \( A \) through \( G = A \cdot g \), where \( G = [P_1, P_2, P_3, B_1, B_2, B_3]^T \) and \( g = [p_1, p_2, p_3, p_4, b_1, b_2, b_3]^T \). In the following, we will use the term “virtual gate voltages” to mean pulses away from the idling tuning of the device, which is the symmetric operating point \([25, 26]\) of the four-dot array with one electron in each dot.

For initialization and readout, we configure the four-spin array into two pairs. We refer to spins 1 and 2 as the left pair, and spins 3 and 4 as the right pair. We initialize the array in the product state \(|↑↑↑↑\rangle\) via adiabatic separation of singlets in the hyperfine gradient \([7, 19]\). Here the arrows indicate the spin states of all four spins. Alternatively, we can prepare a polarized triplet state \(|↑↑⟩\) and potential of our device in COMSOL using the Thomas-Fermi approximation \([38]\). Figure 2(a) shows the

\[
\begin{align*}
H &= \frac{\hbar}{4} \sum_{i=1}^{3} J_i (\sigma_i \cdot \sigma_{i+1}) + \frac{\hbar}{2} \sum_{i=1}^{4} B^z_i \sigma^z_i. \\

\end{align*}
\]

Here \( J_i \) is the exchange coupling strength (with units of frequency) between dots \( i \) and \( i + 1 \), \( \sigma_i = [\sigma_i^x, \sigma_i^y, \sigma_i^z] \) is the Pauli vector describing the components of spin \( i \), and \( \hbar \) is Planck’s constant. \( B^z_i \) is the \( z \)-component magnetic field experienced by each spin, and it includes both a large 0.5 T external magnetic field and the smaller hyperfine field. The quantization axis (\( z \)-direction) is defined by the external magnetic field direction. The \( x \)- and \( y \)-components of the hyperfine field are neglected in this Hamiltonian since their sizes are negligible compared to the external magnetic field. \( B^z_i \) also has units of frequency.

**THE EFFECT OF POSITION SHIFTS ON EXCHANGE**

A single non-zero exchange coupling \( J_i \) is easily tuned by adjusting the voltage on the relevant barrier \( B_i \) \([25, 26]\). When we extend the interaction to more than two spins by pulsing another virtual barrier gate \( B_j \), however, the original exchange coupling \( J_i \) is strongly affected. For example, a large pulse to \( B_3 \) nominally induces a nonzero \( J_3 \). But adding an additional pulse to \( B_2 \) during the evolution rapidly and nonlinearly reduces \( J_3 \) before eventually turning on a \( J_2 \) \([Figs. 1(b)]\). In fact, \( J_3 \) reduces to nearly zero before \( J_2 \) turns on.

We now show that this effect primarily results from lateral shifts of the quantum dots during a barrier-gate pulse. We have self-consistently calculated the electron density and potential of our device in COMSOL using the Thomas-Fermi approximation \([38]\). Figure 2(a) shows the

![Electrostatic simulations. (a) Linecuts of the simulated potential associated with dots 2 and 3 vs. the barrier voltage pulse \( B_2 \). The left dip is the potential of dot 2, and the right dip is the potential of dot 3. The dots move closer together as \( B_2 \) increases. The dashed lines are guides to the eye. (b) Fitted parameters of the simulated double-dot potential vs. \( B_2 \). Based on the simulated potential of dot 1 (not shown), we also find that dots 1 and 2 move farther apart during the sample pulse.](image)
potential associated with dots 2 and 3 tuned to single-occupancy as a function of the barrier voltage $B_2$. Our simulations include compensation pulses on the plunger gates to match our use of virtual gates in the actual experiment. The dots clearly move toward each other as $B_2$ increases.

To obtain more detailed information about this process, we fit our two-dimensional simulated potentials to an equation of the form [31]:

$$V(x, y) = -V_0 \left[ \exp \left( \frac{-(x-a)^2}{l_x^2} \right) + \exp \left( \frac{-(x+a)^2}{l_x^2} \right) \right] \times \exp \left( \frac{y^2}{l_y^2} \right).$$

Here $x$ and $y$ are coordinates in the plane of the two-dimensional electron gas, and $V_0$, $a$, $l_x$, $l_y$ characterize the potential wells of the dots. Double Gaussians of this type are commonly used to model double dots, but usually a separate barrier term is included, as in Ref. [31]. Our simulated potential is shallow enough that a separate barrier term is not required to reproduce the potential we simulate. By fitting our simulated potentials to Eq. 2, we extract how the parameters $a$, $l_x$, and $V_0$ vary with $B_2$ [Fig. 2(b)]. The distances between dots 2 and 3, $(2a)_{23}$ and dots 1 and 2, $(2a)_{12}$ change approximately linearly during the barrier pulse, but in opposite directions, because dot 2 moves closer to dot 3 but farther from dot 1. Based on the simulations, we calculate that $a_2$ changes by about -0.3 $\mu$mV$^{-1}$, and $a_1$ changes by about 0.19 $\mu$mV$^{-1}$. Other parameters of the confinement potential change as well during the barrier pulse.

Reference [31] computes the exchange coupling between two quantum dots in a potential of the form Eq. 2 in the Heitler-London (HL) framework. At zero magnetic field, the result is

$$J_{HL}(V_0, a) = \frac{2S^2}{1 - S^2} \left\{ \frac{\hbar^2 a^2}{m l_0^2} - \frac{2V_0 l_x l_y}{\sqrt{(l_x^2 + l_y^2)(l_x^2 + l_0^2)}} \right\} \times \left[ \exp \left( \frac{-(2a)^2}{l_x^2 + l_y^2} \right) - 2 \exp \left( \frac{-a^2}{l_x^2 + l_0^2} \right) \right]$$

$$- \sqrt{\pi} \left\{ \frac{2}{4\pi \epsilon_0 l_0} \left[ 1 - S l_0 \left( \frac{a^2}{l_0^2} \right) \right] \right\}.$$  

Here $S = \exp(-a^2/l_0^2)$, $l_0 = \sqrt{\hbar/m\omega_0}$, with $\omega_0 = \sqrt{V_0/m l_0^2}$, $m$ is the electron effective mass, $\epsilon_0$ is the permittivity of free space, $\epsilon$ is the dielectric constant of the material, and $l_0$ is the zeroth order modified Bessel function. In writing this equation, we have assumed that the minima of the double-dot potential occur at $x = \pm a$. We have also ignored the magnetic-field-dependent terms, because for the magnetic field used here (0.5T), the effective magnetic confinement is still weaker than the electrostatic confinement.

To determine if lateral position shifts can indeed explain our data, we experimentally measure how $B_j$ affects $J_1$ in our device (Fig. 3). We then fit our data to Eq. 3. To parameterize the effect of the $B_j$, we allow for $a_1 = a_1 B_1 + a_2 B_2 + a_3 B_3$, where the $a_j$ are fit parameters. We also fit for $V_0$, but we constrain $l_x = l_y = 100$ nm, which is approximately the value we obtain from our simulations. The fitted values are $V_0 = 11.4$ meV, $a_1 = -0.40 \mu$mV$^{-1}$, $a_2 = 0.25 \mu$mV$^{-1}$, and $a_3 = 0.01 \mu$mV$^{-1}$. The simulated and fitted values of $V_0$ are on the same order of magnitude. This level of agreement is reasonable, considering that the simulation includes the potential associated with the electron density in the quantum dots. Our simulated values of the quantum-dot position shifts (-0.3 $\mu$mV$^{-1}$ and 0.19 $\mu$mV$^{-1}$) described above agree well with the fitted values of $a_1$ and $a_2$. This agreement supports our hypothesis that lateral position shifts cause the observed trends in our data.

**FIG. 3.** $J_1$ vs. $B_j$. (a) $J_1$ vs. $B_1$. (b) $J_1$ vs $B_2$, for $B_1 = 60$ mV. (c) $J_1$ vs. $B_3$, for $B_1 = 60$ mV. The black data points in each panel are obtained from the fast Fourier transform of a dataset similar to Fig. 1(a). In (a)-(c), the dark blue line is the fit to the exponential model, and the light blue line is the fit to the HL model. Panels (d)-(f) show the difference between the fits and the data for the two models.

**MODELING THE DEPENDENCE OF EXCHANGE ON ALL BARRIER GATES**

The nonlinear and nonlocal dependence of exchange couplings on the barrier gate voltages, which results from position shifts of the quantum dots, poses a challenge to implementing simultaneous exchange coupling between all dots in an extended array. Previous work has investigated how to adjust multiple interdot tunnel couplings iteratively [28, 30]. Here, we discuss two different models, which allow us to determine the virtual gate voltages.

$$V(x, y) = -V_0 \left[ \exp \left( \frac{-(x-a)^2}{l_x^2} \right) + \exp \left( \frac{-(x+a)^2}{l_x^2} \right) \right] \times \exp \left( \frac{y^2}{l_y^2} \right).$$

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given a set of target exchange couplings. In contrast to previous iterative approaches, our approach generates a predictive model. As discussed further below, predictive tuning of exchange couplings in extended spin chains is especially helpful when multiple exchange couplings are present, because the observed spin oscillation frequencies do not correspond with the bare two-spin Heisenberg couplings. We also use our model to control coherent exchange coupling instead of incoherent electron tunneling. Finally, our approach has the advantage that it enables calibrating exchange couplings at the symmetric operating point, where tunnel couplings cannot easily be determined.

Our general approach is to measure how all of the $J_j$ depend on the $B_j$ and then to fit the parameters of a nonlinear model to the data. Using these fit parameters, we create a model which allows us to generate a set of virtual gate voltages $G(j)$, for a set of target exchange values $j = [j_1, j_2, j_3]$, where the $j_i$ are the desired exchange values. We compute the actual gate voltages using the transformation described above. We validate this model by inducing exchange between two, three, and four spins and compare our observations with simulations, and we find good agreement.

To calibrate the models, we begin by inducing one strong exchange coupling $J_j \gg J_{j \neq i}$ and measuring the effect of the $B_j$ on that exchange coupling. For example, to measure how $J_1$ depends on $B_1$, we initialize the array as discussed above with $B_j = 0$. Then we pulse $B_1$ from 25 to 65 mV, and we record exchange oscillations at each pulse height. Setting $B_1 = 60$ mV and $B_3 = 0$ mV, we then pulse $B_2$ from 0 to 35 mV, and we record exchange oscillations. Setting $B_1 = 60$ mV and $B_2 = 0$ mV, we sweep $B_3$ from 0 to 40 mV and again record exchange oscillations. The pulses on $B_3$ are not sufficient to induce substantial $J_2$ due to the large pulse height on $B_1$. The pulses on $B_3$ do induce substantial $J_3$, but $J_1$ is not affected by the next-nearest-neighbor exchange coupling. We extract the oscillation frequencies through a fast Fourier transform of the data (Fig. 3). We repeat this process for the other $J_i$ (see Supplementary Material).

The resulting $J_i$ vs $B_j$ data may be fitted to a set of equations related to Eq. 3, of the form $J_i = J_{HL}(V_0^i, a_i)$, where

$$a_1 = a + \alpha_{11}B_1 + \alpha_{12}B_2 + \alpha_{13}B_3$$

$$a_2 = a + \alpha_{21}B_1 + \alpha_{22}B_2 + \alpha_{23}B_3$$

$$a_3 = a + \alpha_{31}B_1 + \alpha_{32}B_2 + \alpha_{33}B_3,$$

where the $V_0^i$ and the $\alpha_{ij}$ are fit parameters. As discussed above, we constrain $l_p$ and $l_o$ to be the values found from simulations. These equations model our data quite well (Fig. 3), and the parameters we extract from the fits agree reasonably with our simulations. Values of $V_0^i$ range from 6.4 to 11.4 meV, and values of $\alpha_{ii}$ range from -0.40 to -0.43 $\mu$V$^{-1}$. Once the model is calibrated and the parameters found, we choose target exchange values $j$. We then numerically solve the set of equations $j_i = J(V_0^i, a_i)$ for the interdot separations $a_i$, and then we invert Eqs. 4-6 to find the desired barrier gate voltages.

While this model (the “HL model”) originates from a microscopic theory, the exchange coupling is a highly non-linear function of the potential parameters, and some a-priori knowledge of the quantum-dot confinement potential is desirable. Using the HL model in practice also involves numerically solving non-linear equations, which can be susceptible to errors. An alternative, more robust, model for the dependence of the $J_i$ on the $B_j$ is motivated by the realization that the part of the expression for $J_i$ in Eq. 3 that is most sensitive to the inter-dot separation is the factor $S_i^2 \approx \exp(-2a_i^2/l_0^2)$. Setting $a_i = a + \gamma_i$, where $\gamma_i \ll a \approx 100$ nm, and $\gamma_i \ll l_0 \approx 32$ nm we have

$$J_i \sim \exp(-2a_i^2/l_0^2) \approx \exp(-2a_i^2/l_0^2) \exp(-4a_\gamma l_0^2/a_i^2).$$

If $\gamma_i = \alpha_{11}B_1 + \alpha_{12}B_2 + \alpha_{13}B_3$ as before, we expect the $J_i$ to depend approximately exponentially on the $B_j$. Thus, we introduce the following “exponential model” for our data:

$$J_1 = \beta_1 \exp(\delta_{11}B_1 + \delta_{12}B_2 + \delta_{13}B_3 + \gamma_1)$$

$$J_2 = \beta_2 \exp(\delta_{21}B_1 + \delta_{22}B_2 + \delta_{23}B_3 + \gamma_2)$$

$$J_3 = \beta_3 \exp(\delta_{31}B_1 + \delta_{32}B_2 + \delta_{33}B_3 + \gamma_3).$$

Here $\delta_{ij}$, $\beta_i$, $\gamma_i$ are fit parameters. Empirically, the fit parameters $\gamma_i$ are required for two reasons. As discussed above, the exchange couplings are not pure exponential functions of the barrier gates. Second, the hyperfine gradient can increase the measured oscillation frequency above the bare exchange frequency. Including $\gamma_i$ in the fit allows us to accommodate these deviations from pure exponential behavior.

The exponential model also matches our data quite well (Fig. 3). Typical values of $\beta_i$ and $\gamma_i$ are on the order of 10 MHz, and values of $\delta_{ij}$ range from 93 to 114 V$^{-1}$. From Eq. 7, we expect that $\delta_{ij} \approx -4a_{\gamma ij}/l_0^2$. Using $a = 100$ nm, $l_0 = 32$nm, and taking a typical value of $\alpha_{ii} = -0.4\mu$V$^{-1}$, we expect $\delta_{ii} \approx 156$V$^{-1}$, which agrees reasonably well with our fitted values. We have also conducted measurements to confirm that the $\delta_{ij}$ do not depend significantly on the barrier gate voltages, supporting the form of the exponential model (see Supplementary Material).

With the model parameters in hand, we choose a set of target exchange values $j$. Setting $J = j$, we invert Eqs. 8-10 to find the required virtual barrier gate voltages:

$$B_1 \left[ \begin{array}{ccc} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{array} \right]^{-1} \log \left( \frac{(j_1 - \gamma_1)/\beta_1}{(j_2 - \gamma_2)/\beta_2} \right) \log \left( \frac{(j_3 - \gamma_3)/\beta_3}{(j_4 - \gamma_4)/\beta_4} \right).$$
FIG. 4. Two- and three-spin exchange. (a) Two-spin exchange oscillations obtained by linearly sweeping $j_3$ from 0 to 200 MHz. Inset: FFT of the data. (b) Three-spin exchange oscillations obtained by linearly sweeping $j_3$ from 10 to 150 MHz and fixing $j_2$ at 70 MHz. Inset: FFT of the data. Theoretical predictions of the exchange oscillation frequencies are overlaid in red. (c) Simulated three-spin exchange oscillations corresponding to the data in (b). Inset: simulated FFT.

For both the exponential and HL models, we require that the virtual plunger gates remain fixed at the symmetric operating point, and then we transform the virtual gate voltages to physical gate voltages using the capacitance matrix through $g = A^{-1} \cdot G$, as discussed above.

MODEL VALIDATION

In practice we prefer to use the exponential model because it features a robust inversion process and fits our data as well as the HL model (Fig. 3). (See the Supplementary Material for a comparison of the gate voltages generated by these two models.) We first validate our approach by sweeping $j_3$, the target exchange between spins 3 and 4, linearly from 0 to 200 MHz [Fig. 4(a)] with $j_1 = 0$ and $j_2 = 0$. The observed oscillation frequency matches our expectation. We also perform the same test on qubits 1 and 2, and qubits 2 and 3, and observe good agreement between the measured and target frequencies.

In generating the data of Fig. 4(a), we did not simultaneously require that $j_1 = j_2 = 0$. The presence of small MHz-level residual exchange couplings would cause our model to generate large negative values of $B_1$ and $B_2$ in this case. Instead, we fixed $B_1 = B_2 = 0$.

Next we induce three-spin exchange between spins 2, 3, and 4. The array is initialized in the $|↑↑↓↑⟩$ state. Preparing $|↑↑⟩$ on the left side ensures that spins 2, 3, and 4 remain in the $s_z = +\frac{1}{2}$ subspace, regardless of the sign of the local magnetic field gradient. We fix $j_1 = 0$, $j_2 = 70$ MHz, and we sweep $j_3$ linearly from 10 to 150 MHz, and we measure oscillations on the right pair. We compare our data to simulated predictions (see Supplementary Material for details on the simulation). The observed frequencies closely match our expectation, confirming that we can correctly set the target frequencies [Figs. 4(b)-(c)]. Note the presence of three distinct frequencies in the measured spectrum of Fig. 4(b). These frequencies are not the bare Heisenberg couplings. Instead, they result from the different energy splittings between the singlet-like and triplet-like states of three spins [9]. The theoretically predicted [9] low-lying energy splittings of three exchange-coupled spins are shown in red in Fig. 4(b), and they overlap nicely with our measurements [9]. This theoretical prediction assumes zero magnetic gradient between the dots. The presence of a hyperfine gradient in our device causes the experimental data to slightly deviate from the theoretical predictions [Fig. 4(c)]. However, this deviation is relatively insignificant for exchange strengths above 10 MHz.

The model parameters we extract from the fits alone do not suffice to accurately generate the target three-spin exchange frequencies for both the exponential and HL models. We empirically find that the voltages $G(j)$ generate actual exchange frequencies $J < j$ when two or more of the $j_i$ are non-zero (see Supplementary Material). To overcome this challenge, we make minor adjustments to the fitted model parameters and compare the observed three-spin exchange frequencies with simulations. We repeat this process for a few iterations until the experimental data match closely the simulated predictions, usually within about 10 MHz. In the exponential model, we normally need to modify the values of $\beta_i$ and $\gamma_i$, while the values of $\delta_{ij}$ can remain mostly unchanged. For the HL model, we usually need to increase slightly the confinement strength $V_0$. See the Supplementary Material for a comparison of the fitted and adjusted parameters and gate voltages.

The need for a modified parameter set for three-spin exchange may originate for the following reasons. First, we calibrate our model when only one of the $J_i$ is large. However, three-spin exchange requires multiple large $J_i$, and this requires several large, simultaneous barrier-gate pulses. Second, our assumption that only linear position shifts induce exchange likely breaks down at large
gate voltages. Indeed, Fig. 2(b) shows that both the characteristic size and confinement energy change during barrier gate pulses. Third, generating three-spin exchange involves simultaneous large voltage pulses on several plunger and barrier gates, and any errors in our measured capacitance matrix will cause errors in the exchange couplings.

Despite the need for an additional set of parameters for three-spin exchange, the data in Fig. 4 show that our model can still be used in this regime. Indeed, this additional set of parameters also suffices to induce four-spin exchange, because the exchange coupling has vanishing dependence on the next-nearest-neighbor barrier gates [Figs. 3(c)]. To calibrate the two sets of three-spin exchange parameters needed for four-spin exchange, we first tune the model for exchange between spins 2-3-4, which yields precise values of $\alpha_{23}$, $\alpha_{32}$, and $\alpha_{13}$, together with $\beta_2$, $\beta_3$, $\gamma_2$, and $\gamma_3$. To calibrate the model for exchange between spins 1-2-3, we leave these parameters fixed, and tune $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$, $\beta_1$, and $\gamma_1$. With the parameters tuned in this way, we induce simultaneous exchange coupling between all four spins in the array by initializing the array in the $|\downarrow\uparrow\uparrow\uparrow\rangle$ state, and we sweep all three target exchange frequencies linearly from 10 to 40 MHz. The oscillations are measured on both the left and the right pairs [Figs. 5(a)-(b)]. The experimental data and the simulated predictions [Figs. 5(c)-(d)] match closely. The agreement between the experimental data and the simulation shows we have good control over all exchange couplings. The maximum simultaneous exchange is limited to about 40 MHz, because larger barrier pulses disrupt the tuning of the device.

**DISCUSSION**

A significant source of error in our model is the fluctuating nuclear hyperfine field, which is of order 10 MHz in our device (see Supplementary Material). Empirically, our model can generate exchange couplings which are accurate to about 10 MHz, suggesting that the hyperfine fields are a significant source of error. Our model becomes increasingly inaccurate when exchange frequencies approach the hyperfine field values.

Although different sets of parameters are needed for two- and three-spin exchange, the range of validity of the model is sufficient to accurately predict voltages for a wide range of exchange values in either of these cases. To improve on this model, it is likely that additional effects beyond lateral position shifts need to be accounted for. As discussed above, it is likely that other parameters of the dots are changing or that the capacitance matrix may require further refinement.

In the future, it seems likely that fabricating devices with extremely narrow barrier gates may help to reduce the voltages needed for simultaneous exchange between multiple electrons in quantum-dot spin chains, although large pulses may still be required to adjust the exchange from very small to very large values. It is less clear that the challenge of position shifts can be avoided with narrow barrier gates. In any case, we expect that electrostatic simulations can be used effectively to guide the design of future quantum-dot spin chains. We also expect that the use of electrostatic simulations to guide quantum-dot spin-qubit tuning and operation will become a valuable tool as quantum-dot devices increase in complexity.

**CONCLUSION**

In summary, we have demonstrated simultaneous coherent exchange between two, three, and four spins in a quadruple quantum dot. We have also shown that lateral position shifts of the quantum dots during barrier pulses present a significant hurdle to implementing simultaneous exchange between multiple electron spins. Using a phenomenological model based on a microscopic theory, we can predict the virtual gate voltages required to generate a set of target exchange frequencies. Our method is also scalable and applicable to Si qubits. This method enables us to generate a four-site Heisenberg spin chain,
which is an exciting prospect for the exploration of the physics associated with interacting spin chains.

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[1] D. Loss and D. P. DiVincenzo, Phys. Rev. A 57, 120 (1998).

[2] B. E. Kane, Nature 393, 133 (1998).

[3] D. P. DiVincenzo, D. Bacon, J. Kempe, G. Burkard, and K. B. Whaley, Nature 408, 339 (2000).

[4] K. C. Nowack, M. Shafiei, M. Laforest, G. E. D. K. Prawiroatmodjo, L. R. Schreiber, C. Reichl, W. Wegscheider, and L. M. K. Vandersypen, Science 333, 1269 (2011).

[5] D. M. Zajac, A. J. Sigillito, M. Russ, F. Borjans, J. M. Taylor, G. Burkard, and J. R. Petta, Science 359, 439 (2017).

[6] M. J. Gullans and J. R. Petta, Phys. Rev. B 100, 085419 (2019).

[7] J. R. Petta, A. C. Johnson, J. M. Taylor, E. A. Laird, A. Yacoby, M. D. Lukin, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Science 309, 2180 (2005).

[8] S. Fedorov, E. Bluhm, D. Mahalu, V. Umansky, and A. Yacoby, Nature Physics 5, 903 (2009).

[9] E. A. Laird, J. M. Taylor, D. P. DiVincenzo, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Phys. Rev. B 82, 075403 (2010).

[10] J. Medford, J. Beil, J. M. Taylor, S. D. Bartlett, A. C. Doherty, E. I. Rashba, D. P. DiVincenzo, H. Lu, A. C. Gossard, and C. M. Marcus, Nature Nanotechnology 8, 654 EP (2013), article.

[11] K. Eng, T. D. Ladd, A. Smith, M. G. Borselli, A. A. Kiselev, B. H. Fong, K. S. Holabird, T. M. Hazard, B. Huang, P. W. Deelman, I. Milosavljevic, A. E. Schmitz, R. S. Ross, M. F. Gyure, and A. T. Hunter, Science Advances 1, 1500214 (2015).

[12] Y.-P. Shim and C. Tahan, Phys. Rev. B 93, 121410 (2016).

[13] M. Russ, J. R. Petta, and G. Burkard, Phys. Rev. Lett. 121, 177701 (2018).

[14] T. A. Baart, T. Fujita, C. Reichl, W. Wegscheider, and L. M. K. Vandersypen, Nature Nanotechnology 12, 26 (2016).

[15] F. K. Malinowski, F. Martins, T. B. Smith, S. D. Bartlett, A. C. Doherty, P. D. Nissen, S. Fallahi, G. C. Gardner, M. J. Manfra, C. M. Marcus, and F. Kuemmeth, Nature Communications 10, 1196 (2019).

[16] S. Bose, Phys. Rev. Lett. 91, 207901 (2003).

[17] S. Bose, Contemporary Physics 48, 13 (2007).

[18] M. Friesen, A. Biswas, X. Hu, and D. Lidar, Phys. Rev. Lett. 98, 230503 (2007).

[19] Y. P. Kandel, H. Qiao, S. Fallahi, G. C. Gardner, M. J. Manfra, and J. M. Nichol, Nature 576, 553 (2019).

[20] E. Barnes, D.-L. Deng, R. E. Throckmorton, Y.-L. Wu, and S. Das Sarma, Phys. Rev. B 93, 085420 (2016).

[21] E. Barnes, J. M. Nichol, and S. E. Economou, Phys. Rev. B 99, 035311 (2019).

[22] S. J. Angus, A. J. Ferguson, A. S. Dzurak, and R. G. Clark, Nano Letters 7, 2051 (2007).

[23] D. M. Zajac, T. M. Hazard, X. Mi, K. Wang, and J. R. Petta, Applied Physics Letters 106, 223507 (2015).

[24] D. M. Zajac, T. M. Hazard, X. Mi, E. Nielsen, and J. R. Petta, Phys. Rev. Applied 6, 054013 (2016).

[25] F. Martins, F. K. Malinowski, P. D. Nissen, E. Barnes, S. Fallahi, G. C. Gardner, M. J. Manfra, C. M. Marcus, and F. Kuemmeth, Phys. Rev. Lett. 116, 116801 (2016).

[26] M. D. Reed, B. M. Maune, R. W. Andrews, M. G. Borselli, K. Eng, M. P. Jura, A. A. Kiselev, T. D. Ladd, S. T. Merkel, I. Milosavljevic, E. J. Pritchett, M. T. Rakher, R. S. Ross, A. E. Schmitz, A. Smith, J. A. Wright, M. F. Gyure, and A. T. Hunter, Phys. Rev. Lett. 116, 110402 (2016).

[27] T. A. Baart, M. Shafiei, T. Fujita, C. Reichl, W. Wegscheider, and L. M. K. Vandersypen, Nature Nanotechnology 11, 330 EP (2016).

[28] T. Hengseng, T. Fujita, L. Janssen, X. Li, C. J. Van Diepen, C. Reichl, W. Wegscheider, S. Das Sarma, and L. M. K. Vandersypen, Nature Nanotechnology 5, 70 EP (2017).

[29] C. Volk, A. M. J. Zwerger, U. Mukhopadhyay, P. T. Een-debak, C. J. van Diepen, J. P. Dehollain, T. Hengseng, T. Fujita, C. Reichl, W. Wegscheider, and L. M. K. Vandersypen, npj Quantum Information 5, 29 (2019).

[30] A. R. Mills, M. M. Feldman, C. Monical, P. J. Lewis, K. W. Larson, A. M. Mounce, and J. R. Petta, Applied Physics Letters 115, 113501 (2019).

[31] R. de Sousa, X. Hu, and S. Das Sarma, Phys. Rev. A 64, 042307 (2001).

[32] D. J. Reilly, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Applied Physics Letters 91, 162101 (2007).

[33] C. Barthel, M. Kjaergaard, J. Medford, M. Stopa, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Phys. Rev. B 81, 161308 (2010).

[34] A. R. Mills, D. M. Zajac, M. J. Gullans, F. J. Schupp, T. M. Hazard, and J. R. Petta, Nature Communications 10, 1063 (2019).

[35] L. A. Orona, J. M. Nichol, S. P. Harvey, C. G. L. Bttcher, S. Fallahi, G. C. Gardner, M. J. Manfra, and A. Yacoby, Phys. Rev. B 98, 125404 (2018).

[36] C. Barthel, D. J. Reilly, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Phys. Rev. Lett. 103, 160503 (2009).

[37] S. A. Studenikin, J. Thorgrimsson, G. C. Aers, A. Kam, P. Zawadzki, Z. R. Wasilewski, A. Bogan, and A. S. Sachrajda, Applied Physics Letters 101, 233101 (2012).

[38] A. Frees, J. K. Gamble, D. R. Ward, R. Blume-Kohout, M. Eriksson, M. Friesen, and S. Coppersmith, Phys. Rev. Applied 11, 024063 (2019).
Supplementary Material for
Coherent multi-spin exchange in a quantum-dot spin chain

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Device

The quadruple quantum dot device is fabricated on a GaAs/AlGaAs heterostructure, with three layers of overlapping Al confinement gates. An additional grounded top gate covers the main device area for smoothing the potential anomalies caused by fabrication defects. The two-dimensional electron gas (2DEG) resides at the GaAs and AlGaAs interface, 91 nm below the semiconductor surface. The 2DEG density \( n = 1.5 \times 10^{11} \text{cm}^{-2} \) and mobility \( \mu = 2.5 \times 10^6 \text{cm}^2/\text{Vs} \) were measured at \( T = 4 \text{K} \). The device is cooled in a dilution refrigerator with base temperature of approximately 10mK. An external magnetic field \( B = 0.5 \text{T} \) is applied parallel to the 2DEG, normal to the axis connecting the quantum dots.

Measurement of Hyperfine Gradient

We can measure the hyperfine gradients between neighboring quantum dots by inducing singlet-triplet oscillations and measuring the oscillation frequencies. First we prepare \( |\mathcal{S}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \) states on both the left and the right sides in the (2,0,0,2) charge configuration, and diabatically separate them into the (1,1,1,1) configuration. The rapid charge
separation allows both sides to retain their singlet states (rather than evolving to product states). The left side undergoes singlet-triplet oscillations with frequency $\Delta B_{12} = |B_i^1 - B_i^2|$, and the right side oscillates with frequency $\Delta B_{34} = |B_i^3 - B_i^4|$. $B_i^i$ is defined as in Eq. (1) in the main text. After both sides evolve for a variable amount of time around the gradients, we project them onto the $\{|S\rangle, |T\rangle\}$ basis via diabatic charge transfer back to the (2,0,0,2) configuration. The measurements on the left side show singlet-triplet oscillations under $\Delta B_{12}$, and the measurements on the right side show oscillations under $\Delta B_{34}$. To measure the gradient $\Delta B_{23}$, we create an entangled state between qubits 2 and 3 via a $\sqrt{SWAP}$ gate [1]. We first initialize the array into the $|\uparrow\uparrow\downarrow\uparrow\rangle$ state, and then a $\sqrt{SWAP}$ gate between qubits 2 and 3 generates the entangled state $\frac{1}{\sqrt{2}}(|S_{23}\rangle + i |T_{0,23}\rangle)$, which undergoes singlet-triplet oscillations with frequency $\Delta B_{23} = |B_2^2 - B_3^3|$, while qubits 1 and 4 remain unchanged. After some evolution time, another $\sqrt{SWAP}$ is applied between qubits 2 and 3 to preserve the coherence. The array is then adiabatically transferred back to the (2,0,0,2) configuration, and the oscillations are measured on the right side. Each data point is averaged for 128 iterations, and we repeat the measurement 2048 times. We extract the oscillation frequencies from the FFT of the data, and we create histograms of the measured frequencies [Supplementary Fig. 1]. From the histograms we can obtain the hyperfine gradient fluctuations. In our device, the mean values of the gradient fluctuations appear to slowly drift over time, but the standard deviations remain relatively constant. We incorporate the experimentally obtained standard deviations to the simulation, but adjust the mean values in the simulation to better match the experimental data.

Simulation

We generate all simulated data by numerically integrating the Schrödinger equation and calculating the singlet return probability for both the left side (qubits 1 and 2) and the right side (qubits 3 and 4). The Hamiltonian is given in Eq (1) in the main text. Our simulation includes errors from state preparation, relaxation during readout, readout fidelity, charge noise, and hyperfine field fluctuations. To simulate the errors in singlet load and adiabatic charge separation, we create the two-electron state

$$|\tilde{S}\rangle = s_1 |S\rangle + s_2 |T_0\rangle + s_3 |T_+\rangle + s_4 |T_-\rangle,$$  

(1)
where $|s_1|^2 = f_s$ is the singlet load fidelity, and $|s_2|^2 = |s_3|^2 = |s_4|^2 = \frac{1}{3}(1 - f_s)$. Here $|T_0⟩ = \frac{1}{\sqrt{2}} (|↑↓⟩ + |↓↑⟩)$, $|T_+⟩ = |↑↑⟩$, and $|T_-⟩ = |↓↓⟩$. The error associated with the process of charge separation is neglected, since the dominating error source is the singlet loading error. In the case of the $T_+$ load, we simulate the state to be

$$|\tilde{T}⟩ = t_1 |S⟩ + t_2 |T_0⟩ + t_3 |T_+⟩ + t_4 |T_-⟩ ,$$

where the coefficients are calculated following the process described in Ref. [2]. In both cases, random phases are assigned to the coefficients for each realization of the simulation. To simulate the errors from relaxation and readout fidelity, we calculate the final singlet return probability for each side to be

$$\tilde{P}_S = (1 - g - 2r)P_S + g + r ,$$

where $P_S$ is the true singlet return probability after evolution, $r = 1 - f_m$ is the probability of misidentifying singlet as triplet (or vice versa) as a result of noise, and $g = 1 - \exp(-t_m/T_1)$ is the probability of the triplet relaxing to the singlet during measurements. Here $f_m$ is the measurement fidelity, $t_m$ is the measurement time, and $T_1$ is the relaxation time. To account for charge noise and hyperfine field fluctuations, we vary the values of $J_i$ and $B_i^z$ in the Hamiltonian between simulation runs. The values in each run are sampled according to a Gaussian distribution. Each set of simulated data is averaged over 4096 realizations of charge noise, hyperfine field fluctuations, and state preparation errors.

**Exchange Coupling Model**

We use the exponential model to describe the dependence of the exchange couplings on all barrier gate voltages (Supplementary Fig. 2). The parameters are obtained by fitting Eqs. (8)-(10) in the main text to exchange oscillation frequencies extracted from the calibration data. The model with these initial parameters accurately predicts the oscillation frequencies in two-qubit exchange interactions, but fails to do so for three- and four-qubit interactions. To obtain a set of parameters that is suitable for multi-qubit exchange oscillations, we modify the initial parameters by comparing the three-qubit exchange data with the simulation. Eventually we have two sets of parameters: one for two-qubit exchange oscillations, and the other one for multi-qubit exchange oscillations. A comparison between the two sets of parameters are shown in Supplementary Table I and Supplementary Fig. 2.
We show the robustness of the exponential model parameters in Supplementary Fig. 3. We induce the exchange coupling \(J_3\) by increasing \(B_3\), and at the same time we fix \(B_2\) to a non-zero value (without inducing \(J_2\)). We repeat the experiment while varying \(B_2\), and fit \(J_3\) to an exponential function of \(B_3\) and obtain \(\delta_{33}\) for each value of \(B_2\). Finally we monitor how \(\delta_{33}\) changes with \(B_2\). We also run the same test for \(\delta_{32}\) and monitor how it changes with \(B_3\) (Supplementary Fig. 3). Both \(\delta_{32}\) and \(\delta_{33}\) show minimal dependence on the nearest-neighbor barrier gate voltages, which indicates the functional form for our model is justified.

Table I shows typical fitted and modified parameters for the exponential model. Supplementary Fig. 5 shows all fits to the calibration data using the HL model.

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[1] Yadav P. Kandel, Haifeng Qiao, Saeed Fallahi, Geoffrey C. Gardner, Michael J. Manfra, and John M. Nichol, “Coherent spin-state transfer via heisenberg exchange,” Nature 573, 553–557 (2019).

[2] Lucas A. Orona, John M. Nichol, Shannon P. Harvey, Charlotte G. L. Bøttcher, Saeed Fallahi, Geoffrey C. Gardner, Michael J. Manfra, and Amir Yacoby, “Readout of singlet-triplet qubits at large magnetic field gradients,” Phys. Rev. B 98, 125404 (2018).
Supplementary Table I. Comparison of the initial and the modified parameters used for modeling.

| parameters   | $\beta$ (MHz) | $\delta \times 10^{-3}$ (V$^{-1}$) | $\gamma$ (MHz) |
|--------------|---------------|----------------------------------|----------------|
| initial      | 0.244         | 0.1142 -0.0717 -0.0026            | -0.604         |
|              | 1.424         | -0.0099 0.1085 -0.0184            | -4.993         |
|              | 10.17         | 0.0005 -0.0468 0.0932            | -6.882         |
| modified     | 0.125         | 0.1135 -0.0713 -0.0022            | -8             |
|              | 1.15          | -0.0099 0.1084 -0.0184            | -0.25          |
|              | 7.5           | 0.0005 -0.0462 0.0907            | -0.5           |
Supplementary Figure 1. Measurements of hyperfine gradients. (a) Absolute values of the FFT of $\Delta B_{12}$ oscillations, with extracted frequencies shown in red. Only 64 out of 2048 repetitions are shown. (b) Extracted $\Delta B_{12}$ frequency distribution. (c) Absolute values of the FFT of $\Delta B_{23}$ oscillations. (d) $\Delta B_{23}$ frequency distribution. (e) Absolute values of the FFT of $\Delta B_{34}$ oscillations. (f) Extracted $\Delta B_{34}$ frequency distribution.
Supplementary Figure 2. Full calibration data, initial, and modified fits for the exponential model. Blue data points are experimentally measured two-qubit exchange oscillation frequencies, red lines are initial exponential fit, and green lines are predicted frequencies using the model with modified parameters.
Supplementary Figure 3. Robustness of exponential model parameters. a Dependence of \( J_3 \) on the barrier \( B_3 \), with different values of \( B_2 \). Data points are experimentally obtained frequencies, and solid lines are exponential fits. b Change in the fit parameter \( \alpha_{33} \) with \( B_2 \). c Dependence of \( J_3 \) on the barrier \( B_2 \), with different values of \( B_3 \). d Change in the fit parameter \( \alpha_{32} \) with \( B_3 \).
Supplementary Figure 4. Full calibration data, initial, and modified fits for the HL model. Blue data points are experimentally measured two-qubit exchange oscillation frequencies, red lines are initial exponential fit, and green lines are predicted frequencies using the model with modified parameters. To generate the modified parameters, we adjusted the HL parameters to generate voltages to match the voltages used in the three-spin exchange experiment. We only adjust parameters for $J_2$ and $J_3$. 
Supplementary Figure 5. Comparison of gate voltages predicted by initial and modified fits for both the exponential and HL models. The gate voltages correspond to the three-spin exchange data presented in the main text.