Matter coupling to degenerate spacetimes in first order gravity

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Abstract
We develop a systematic study of the equations of motion in the first order gravity with matter fields for degenerate metrics. Like the Hilbert–Palatini action functional for pure gravity, the action functionals for matter fields used are first order. These are defined for both invertible and non-invertible metrics. Description for invertible metrics is equivalent to second order gravity theory with matter. For degenerate metrics the theory describes a different phase. The analysis for tetrads with one zero eigen value in theory with scalar, Abelian vector gauge and fermion matter fields is presented in detail.

Keywords: first order gravity, degenerate spacetime, alternate gravity theory

1. Introduction
The usual standard description of gravity based on Einstein–Hilbert action functional is the second order formulation constructed with invertible metrics (det $g_{\mu\nu} \neq 0$). Matter fields are also coupled here with action functionals which are defined for non-degenerate metrics. There is another theory of gravity based on Hilbert–Palatini action given in terms tetrads $e^I_\mu$ and $SO(1,3)$ connection fields $\omega^{\mu}_{\nu I}$ where both of these are treated as independent in the action functional. Tetrads introduced in the original Cartan formalism provide a local orthonormal frame of the tangent space at a spacetime point of a manifold. This geometric framework requires the tetrads to be invertible. This first order formulation of gravity with invertible tetrad fields is known to be equivalent to that described by the Einstein–Hilbert action. However, as is well known, writing the first order Hilbert–Palatini action functional in the differential form language transparently exhibits a remarkable property that this action functional does not contain the inverse tetrad and hence, as a field theory, is not only defined for invertible tetrads but also for degenerate tetrad fields. Thus this action functional does provide a broader field theoretic framework to study gravity theories with degenerate metrics and has been used for this purpose in many a study. This first order formulation would then differ from the standard second order formulation in the important aspect: as a field theory this first order
formulation is defined for both non-degenerate \((\det e^\mu_\nu \neq 0)\) and degenerate \((\det e^\mu_\nu = 0)\) metrics. Additional phase here characterized by degenerate tetrads has significantly different structure.

Interest in degenerate metrics has a long history [1–7]. Quantum field theory with first order Hilbert–Palatini action would include contributions from configurations with both non-degenerate and degenerate tetrads in the functional integral. There is a priori no reason to exclude degenerate tetrad field configurations from such a field theory. Further, degenerate spacetimes have also been invoked in the discussion of topology change [7, 8]. Topology changes may occur in quantum theory of spacetime. It is also possible that these may originate even in classical theory [7].

Recently a systematic detail study of non-invertible tetrads configurations in generalized first order gravity has been developed [9, 10]. For degenerate tetrads the theory is shown to possess solutions of vacuum equations of motion which generically exhibit presence of torsion. This special property follows even in absence of any matter fields. Examples of solutions which have degenerate metric in a region and non-degenerate metric in another portion of the spacetime have also been found.

The analysis in the first order gravity in [9, 10] was done without presence of any matter fields. To extend this to include matter fields, we need to introduce matter field action functionals which are also first order. For a fermion field, the standard action functional used is already first order. On the other hand, usually used actions for other fields like scalar and gauge fields have second order forms and these are defined only for invertible metrics. There is a straightforward procedure to construct first order action functionals from the second order actions by applying the general Ostrogradsky construction for lowering the number of derivatives by introducing additional auxiliary field variables [11]. The first order matter actions so constructed then turn out to have a special structure which allows us to define them for both invertible and non-invertible tetrads. In the following, we shall display this property of the matter action functionals explicitly. For invertible metrics, these lead to equations of motion which are exactly equivalent to those obtained from the second order action functionals and hence, at classical level, are exactly equivalent to the usual second order formulations. For degenerate metrics there is a different structure which will be studied here in detail for tetrads with one zero eigenvalue.

The article has been organized as follows. In section 2, we discuss coupling of a scalar matter field in first order gravity by writing a first order action for the scalar field. The action functional is defined for both invertible and non-invertible tetrads. For pedagogical clarity, we explicitly display that, for non-degenerate tetrads, this theory is exactly equivalent to the standard theory of scalar matter field coupled to second order Einstein–Hilbert gravity. Next we present the analysis for degenerate tetrads with one zero eigenvalue displaying the detail structure of the equations of motion. Section 3 contains the analysis for first order gravity theory containing a vector gauge field. First order action functional for the vector field introduced is defined for both non-degenerate and degenerate tetrads. For invertible tetrads, the theory as expected is equivalent to the second order gravity theory coupled to Maxwell electromagnetism. We present a detail analysis for degenerate tetrads. In section 4, we extend the discussion to first order gravity with fermions. Lastly section 5 contains some concluding remarks.

2. First order scalar field action

We describe the coupling of a scalar field in first order gravity through the action:

\[ S = S_{\text{HPA}} + S_{\text{scalar}} \]  

(1)
where $S_{\text{HP}}$ is the Hilbert–Palatini action functional with cosmological constant ($\Lambda$) term:

$$S_{\text{HP}} = \frac{1}{8\kappa^2} \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu} e^t_{\mu} e^t_{\nu} \left( R^{KL}_{\alpha\beta}(\omega) - \frac{\Lambda}{3} e^K_{\alpha} e^K_{\beta} \right)$$  \hspace{1cm} (2)$$

and

$$R^{KL}_{\alpha\beta}(\omega) = \partial_{(\alpha} \omega_{\beta)}^{KL} + \omega_{[\alpha}^{KM} \omega_{\beta]}^{L}$$ \hspace{1cm} (3)$$

is the field strength of the $SO(1,3)$ gauge field $\omega^{IJ}$. In this action independent fields are tetrad $e^I_{\mu}$ and connection $\omega^{IJ}_{\mu}$. The matter action functional for the scalar field coupled to the tetrad is:

$$S_{\text{scalar}} = \frac{1}{6} \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu} e^K_{\alpha} e^K_{\beta} \left[ \partial_{\beta} \phi B^\ell_\beta + \frac{1}{8} e^t_\mu \left( B^M_{\beta} B_M - m^2 \phi^2 \right) \right].$$  \hspace{1cm} (4)$$

This action functional contains two independent fields, the scalar field $\phi$ and $B$. $e^I_{\mu}$ and $\omega^{IJ}_{\mu}$.

Note here the Greek indices ($\mu, \nu, \alpha, \beta$) indicate the spacetime coordinates and Latin letters ($I, J, K, L, M$) label internal $SO(1,3)$ indices which are raised and lowered by the flat metric $\eta_{IJ} = \text{diag} (-1, 1, 1, 1) = \eta_{IJ}$. Completely antisymmetric epsilon symbols take constant values 0 and $\pm 1$ with $\epsilon^{0123} = +1$ and $\epsilon_{0123} = +1$.

Like the Hilbert–Palatini action $S_{\text{HP}}$, the matter action functional $S_{\text{scalar}}$ is first order and is defined for both invertible and non-invertible tetrads. Inverse tetrad does not appear anywhere in these expressions. However, as we shall see below, this matter action functional is exactly equivalent to standard second order action for the scalar field $\phi$ of mass $m$ for non-degenerate tetrads.

We now obtain Euler–Lagrange equations of motion by varying the total action (1) with respect to all the independent fields, $e^I_{\mu}$, $\omega^{IJ}_{\mu}$, $\phi$, and $B^M$. Variations with respect to $B^M$ and $\phi$ yield respectively:

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu} e^K_{\alpha} e^K_{\beta} \left[ \partial_{\beta} \phi \delta_\beta^t + \frac{1}{4} e^t_\mu B_M \right] = 0$$ \hspace{1cm} (5)$$

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu} \left[ \partial_{\beta} \left( e^K_{\alpha} e^K_{\beta} B^t_\beta \right) + \frac{m^2}{4} e^t_\mu e^K_{\alpha} e^K_{\beta} \phi \right] = 0.$$ \hspace{1cm} (6)$$

Next, variations of action (1) with respect to the connection field $\omega^{IJ}_{\mu}$ and tetrad field $e^I_{\mu}$ lead respectively to the Euler–Lagrange equations of motion:

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu} e^K_{\alpha} D_\alpha(\omega)e^K_{\beta} = 0$$ \hspace{1cm} (7)$$

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu} \left[ R^{KL}_{\alpha\beta}(\omega) - \frac{2\Lambda}{3} e^K_{\alpha} e^K_{\beta} \right] = -4\kappa^2 T^\mu_I$$ \hspace{1cm} (8)$$

where

$$T^\mu_I \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu} e^K_{\alpha} e^K_{\beta} \left[ \partial_{\beta} \phi B^L_\beta + \frac{1}{6} e^L_{\beta} \left( B_M B_M^M - m^2 \phi^2 \right) \right].$$ \hspace{1cm} (9)$$

Here $D_\mu(\omega)e^K_{\alpha} \equiv \partial_\mu e^K_{\alpha} + \omega^{IJ}_{\mu} e^K_{\alpha}$ is the $SO(1,3)$ gauge covariant derivative of the tetrad.

From equation (8), by applying a gauge covariant derivative, we obtain:
\[ 4\kappa^2 D_\mu(\omega) T^\mu_I = -\epsilon^{\mu\nu\alpha\beta} e_{IJKL} D_\mu(\omega) e^I_\nu R_{\alpha\beta}^{KL}(\omega) \]  

(10)

where the covariant derivative is: \( D_\mu(\omega) T^\mu_I \equiv \partial_\mu T^\mu_I + \omega^I_\mu T^\mu_J \). To obtain this equation we have used the equation of motion (7) and also the Bianchi identity:

\[ D_{[\mu}(\omega) R_{\nu\alpha\beta]}(\omega) \equiv 0. \]  

(11)

Note that, like the action functional (1), the Euler–Lagrange equations of motion (5)–(8) obtained from it are defined for both invertible and non-invertible tetrads. We shall now analyze these for non-degenerate and degenerate tetrads separately.

2.1. Invertible tetrads

For invertible tetrads, it is well known that this first order theory is exactly same as the second order theory of a scalar field coupled to gravity. For the sake of completeness, we shall briefly outline the demonstration of this fact here.

For tetrads with non-zero determinant \( e \equiv \det e^I_\mu \),

\[ \epsilon^{\mu\nu\alpha\beta} e_{IJKL} e^I_\mu e^J_\nu e^K_\alpha e^L_\beta = 24 e \neq 0, \]  

(12)

the inverse tetrad \( e^I_\mu \) is defined through relations:

\[ e^I_\mu e^J_\nu = \delta^I_\nu, \quad e^I_\mu e^J_\mu = \delta^J_\mu. \]

For the spacetime metric \( g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ} \), the inverse is \( g^{\mu\nu} = e^I_\mu e^J_\nu \eta^{IJ} \) and \( g \equiv \det g_{\mu\nu} = -e^2 \).

Using equation (12) and the identity \( \epsilon^{\mu\nu\alpha\beta} e_{IJKL} e^I_\mu e^K_\nu e^L_\beta = 6ee^\mu_\mu \), it is straightforward to check that the scalar action functional (4) can be written as:

\[ S_{\text{scalar}} = \int d^4x \left[ e^I_\mu \partial_\mu \phi B^I + \frac{1}{2} (B_M B^M - m^2 \phi^2) \right] \]  

(13)

and first order Euler–Lagrange equations of motion (5) and (6) can be respectively recast as:

\[ B_I = -e^I_\mu \partial_\mu \phi \]  

(14)

\[ \partial_\mu \left( ee^\mu_\nu B^\nu \right) + em^2 \phi = 0. \]  

(15)

Equation (14) is a constraint reflecting the fact that \( B_I \) is not an independent field. Using this constraint in the matter action (13), we obtain

\[ S_{\text{scalar}} = -\frac{1}{2} \int d^4x e \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right] \]  

(16)

which is the standard second order action for a scalar field \( \phi \) of mass \( m \) in curved spacetime. Again using the constraint (14) in the first order equation of motion (15) leads to

\[ -\partial_\mu (eg^{\mu\nu} \partial_\nu \phi) + em^2 \phi = 0 \]  

(17)

which is the standard second order equation of motion for scalar field in curved spacetime. This equation can also be obtained directly by varying the second order scalar field action (16) with respect to \( \phi \).
For non-degenerate tetrads, the equations of motion (7) is equivalent to the no-torsion condition: $D^\mu(\omega) e_\beta^\mu = 0$. It is well known that this equation can be solved for the 24 connection fields in terms of tetrads:

$$\omega_{\mu}^\nu = \omega_{\mu}^\nu(e) \equiv \frac{1}{2} \left( e^{\alpha \nu} \partial_{[\mu} e_{\alpha]}^\mu - e^{\alpha \mu} \partial_{[\nu} e_{\alpha]}^\nu \right).$$

(18)

With this connection, equations of motion (8) can be easily seen to be equivalent to the standard second order Einstein field equations for gravity with scalar matter and cosmological constant. The stress–energy tensor for the scalar field matter

$$\tilde{T}^{\mu \nu} \equiv e^\alpha_\mu \tilde{T}^{\nu}_\alpha$$

obtained from equation (9), has the standard form of the second order theory:

$$T^{\mu}_\alpha \equiv e^\alpha_\mu \tilde{T}^{\nu}_\alpha = \frac{1}{2} e^\beta_\nu \partial_\beta \phi - \frac{1}{2} e^\mu_\nu \left( (\partial \phi)^2 + m^2 \phi^2 \right).$$

(19)

where $(\partial \phi)^2 \equiv g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$. Here we have used the constraint (14) in writing the last step. Lastly, for invertible tetrads, we notice that equation (10) represents the conservation equation:

$$D^\mu(\omega) T^{\nu}_\mu = D^\mu(\omega) (e \tilde{T}^{\nu}_\mu) = 0.$$

Thus first order theory with scalar matter is exactly equivalent to the standard theory of gravity for invertible tetrads. However, first order theory has an additional phase containing solutions with degenerate tetrads.

2.2. Non-invertible tetrads

We shall now study the case where the tetrad $e_\mu^I$ has one zero eigenvalue. We parameterize this tetrad as:

$$e_\mu^I = \begin{pmatrix} 0 & 0 & e_a \\ 0 & e_a \end{pmatrix}$$

(20)

where $e_\mu^I = e_\mu^a = 0$ and the $3 \times 3$ block of triads $e_a^i$ ($i = 1, 2, 3; a = x, y, z$) is invertible with $\text{det} e_a^i \equiv \tilde{e} \neq 0$. Inverse triad will be denoted by $\tilde{e}_a^i \delta_i^a = \delta_i^a$, $\tilde{e}_a^i e_a^j = \delta_i^j$. The degenerate metric is:

$$g_{\mu \nu} = e_\mu^I e_\nu^J \eta_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ab} \end{pmatrix}; \quad g_{ab} = e_a^i e_b^j.$$  

Let us now analyze the Euler–Lagrange equations of motion (5)–(8) for the degenerate tetrad (20). The matter Euler–Lagrange equations of motion (5) and (6) lead to:

$$\partial_\mu \phi = 0; \quad \partial_\mu (e B^0) = 0$$

(21)

where we have used the identity $6\tilde{e} = e^{abc} \epsilon_{ijk} \tilde{e}_a^i \tilde{e}_b^j \tilde{e}_c^k$. Thus this set of Euler–Lagrange equations of motion make the scalar field $\phi$ and $(e B^0)$ time independent. Note that there are no constraints on $\partial_\mu \phi$ and $B^0$.

Next we shall study Euler–Lagrange equations of motion (7) and (8) following closely the discussions in [9]. For degenerate tetrad (20), twenty four equations of motion in (7) can be broken into four sets of 3, 3, 9, and 9 equations respectively as follows:

$$e^{abc} \epsilon_{ijk} e_a^i D_\beta(\omega) e_b^j = 0$$

(22)
\[ \epsilon^{abc} \epsilon_{ijk} e^d_a D_b(\omega) e^c_j = 0 \]  
(23)

\[ \epsilon^{abc} \epsilon_{ijk} e^d_b D_i(\omega) e^c_j = 0 \]  
(24)

\[ \epsilon^{abc} \epsilon_{ijk} e^d_c D_k(\omega) e^c_i = 0. \]  
(25)

The last equation (25) is solved by

\[ \omega^{ij}_t = 0. \]  
(26)

Next, equation (24) implies that \( D_i(\omega)e_\mu^\nu \equiv \partial_\nu e_\mu^\nu + \omega^{ij}_t e_j^\nu = 0 \), which can be solved for \( \omega^{ij}_t \) as:

\[ \omega^{ij}_t = \hat{e}^\mu_j \partial_\nu e_\mu^\nu = \epsilon_i^\mu \partial_\nu e_\mu^\nu = -\epsilon_i^\mu \partial_\nu e_\nu^\mu = -e_i^\mu \partial_\nu \hat{e}^\nu_j. \]

Note that \( \partial_\nu g_{\mu\nu} \equiv D_i(\omega)e_\mu^\nu e_\nu^\rho = 0 \).

This implies that \( t \)-dependence of the triad fields \( e_\mu^\nu \) is only a gauge artifact and hence can be rotated away completely by an internal space \( O(3) \) rotation. Thus we make a gauge choice such that

\[ \partial_\nu e_\mu^\nu = 0 \text{ and hence } \omega^{ij}_t = 0. \]  
(27)

Equation (23) can be solved by:

\[ \omega^{ij}_\mu \equiv M^i_j = e^j_\mu M^\mu_j \text{ with } M^\mu_j = M_j^\mu. \]  
(28)

These fix three components of \( \omega^{ij}_\mu \) represented by the antisymmetric part of the matrix \( M^\mu_j \), \( M^\mu_j - M^\mu_i = 0 \), leaving six components in the symmetric matrix \( M^j_i \) undetermined. Lastly, equation (22) is solved by:

\[ \omega^{ij}_\mu = \hat{\omega}^{ij}_\mu(e) + \kappa^{ij}_\mu = \hat{\omega}^{ij}_\mu(e) + \epsilon_\mu^{\rho\beta} N_i^\rho \cdot N_j^\beta = e_\mu^i N_i^\rho \cdot e_\mu^\rho N_j^\beta = N^{ij}_\mu, \]

\[ \hat{\omega}^{ij}_\mu(e) \equiv \frac{1}{2} \left( \epsilon_\mu^{\rho\beta} \partial_\beta e_\mu^\rho - \epsilon_\mu^{\rho\beta} \partial_\rho e_\mu^\beta - e_\mu^i \epsilon_\mu^{\beta\gamma} \partial_\beta e_\mu^\gamma \right). \]  
(29)

Here \( \hat{\omega}^{ij}_\mu(e) \) is the torsion-free connection satisfying

\[ D_{[\nu}(\omega) e_{\mu]}^\nu = 0. \]  
(30)

Thus, finally, of all the twenty four components of the gauge fields \( \omega^{ij}_\mu \), we have fixed twelve by equations of motion (22)–(25). Rest twelve represented by two \( 3 \times 3 \) symmetric matrices \( M^i_j \) and \( N^{ij}_\mu \) introduced in equations (28) and (29) are left undetermined.

Now we shall analyze the last Euler–Lagrange equation of motion (8) for non-invertible tetrads. We start by listing various components of \( T^I_\mu \) of equation (9) for the degenerate tetrad (20):

\[ T^0_0 \equiv \dot{e} \dot{T}^0_0 = \dot{e} \left[ \epsilon^i_k \partial_\mu \phi B^k + \frac{1}{2} (B_M B^M - m^2 \phi^2) \right] \]  
(31)

\[ T^i_0 \equiv \dot{e} \dot{T}^i_0 = -\dot{\phi} \partial_\mu B^0 \]  
(32)

\[ T^0_a \equiv \dot{e} \dot{T}^0_a = -\dot{e} \partial_\mu \phi B^a \]  
(33)

\[ T^a_i \equiv \dot{e} \dot{T}^a_i = \dot{e} \partial_\mu \phi B^a. \]  
(34)

Because of the matter equations of motion (21) the scalar field \( \phi \) does not have any \( t \) dependence. This makes the last two components (33) and (34) vanish:
\[T^a_0 = 0, \ T^a_i = 0.\] 

For degenerate tetrads (20), it is convenient to split the sixteen equations of motion in (8) into four sets of 1, 3, 3 and 9 equations as follows:

\[
\kappa^2 T^0_0 \equiv \dot{\kappa} \dot{T}^0_0 = -\frac{1}{2} \dot{\kappa} \hat{\epsilon}^b_i \dot{\epsilon}^c_j R_{bc}^{ij}(\omega) - 2\Lambda
\]  

\[
\kappa^2 T^i_i \equiv \dot{\kappa} \dot{T}^i_i = \dot{\epsilon}^b_i \dot{\epsilon}^c_j R_{bc}^{ij}(\omega)
\]  

\[
\kappa^2 T^0_a \equiv \dot{\kappa} \dot{T}^0_a = \dot{\epsilon}^b_i \dot{\epsilon}^c_j R_{bc}^{ij}(\omega)
\]  

\[
\kappa^2 T^a_a \equiv \dot{\kappa} \dot{T}^a_a = \dot{\epsilon}^b_i \dot{\epsilon}^c_j R_{bc}^{ij}(\omega).
\]

We now use equations (31)–(35) in these equations. Using \(T^0_0 = 0\) and \(T^a_i = 0\) in the equations of motion (38) and (39) respectively lead to:

\[
\dot{\epsilon}^a_i R_{ab}^{ij}(\omega) = 0
\]  

\[
R_{ab}^{0j}(\omega) = \partial_t M_{ij} = 0
\]

where we have used \(\omega^a_i = 0\) from equation (26) and \(\omega_{ab}^{ij} \equiv M_{ij}^a\) from equation (28).

Next, equation of motion (37) and equation (32) imply:

\[
\dot{\epsilon}^b_i R_{ab}^{0j}(\omega) = \dot{\epsilon}^c_j D_{[a}(\omega)M_{b]}^i = \kappa^2 \epsilon^i_a \dot{T}^i_i = -\kappa^2 \partial_t \phi B^0
\]

where we have used equation (28) with \(M^0 = M^0\) and equation (29) which implies \(\dot{\epsilon}^b_i \kappa_{ab} = 0\) due to the symmetric nature of the matrix \(N^0\).

Lastly we study the equation of motion (36). For this, breaking \(\omega_{ab}^{ij}\) into torsion-free part \(\tilde{\omega}_{ab}(\epsilon)\) and contorsion part as in equation (29), we find that

\[
R_{ab}^{ij}(\omega) = \tilde{R}_{ab}^{ij}(\omega) + \epsilon^{ik} D_{[a}(\omega)N_{b]}^{ij} = N_{[a}^{,ij}N_{b]}^{,j} + M_{[a}^{ij}M_{b]}^{,j}
\]

\[
= \tilde{R}_{ab}^{ij}(\omega) - \epsilon^{ik} \epsilon^j_a D_{[b}(\omega)N^{ik} + (M^{il}M^{jk} - N^{jl}N^{ik})\epsilon^j_a \epsilon^l_b
\]

where \(\tilde{R}_{ab}^{ij}(\omega) \equiv \partial_t \omega_{ab}^{ij} + \omega_{[a}^{,ik} \omega_{b]}^{,kj}\). Using this fact in the equation of motion (36), we obtain the constraint:

\[
\dot{\epsilon}^b_i \dot{\epsilon}^c_j R_{bc}^{ki}(\omega) = (M^{il}M^{jk} - M^{ik}M^{jl}) + (N^{ik}N^{jl} - N^{jl}N^{ik}) - 2\Lambda
\]

\[
= -2\kappa^2 \dot{T}^0_0 = -2\kappa^2 \left[ \dot{\epsilon}^b_i \partial_t \phi B^0 + \frac{1}{2} (B_{ab}B_{ab} - m^2 \phi^2) \right]
\]

where we have used the property that matrix \(N^0\) is symmetric.

Thus, we have four new constraint equations in (40)–(43) in addition to those in (26)–(29) obtained earlier. Note that constraint (40) does not give us any additional information as it is identically satisfied when equations (26)–(29) are used. This can readily be seen by noting that \(R_{ab}^{ij}(\omega) = \partial_t \omega_{ab}^{ij} = \partial_t \kappa_{ab}^{ij}\) when constraints (26), (27) and (29) hold. Now \(\dot{\epsilon}^b_i R_{ab}^{0j}(\omega) = \partial_t \dot{\epsilon}^b_i \kappa_{ab}^{ij} = 0\) because \(\dot{\epsilon}^b_i \kappa_{ab}^{ij} \equiv \epsilon^{ik} \epsilon^j_a \dot{\epsilon}^l_b N^l_k = \epsilon^{ik} N^{lk} \equiv 0\) due to the symmetric character of the matrix \(N^0\).

A particular solution of constraints (41) and (42) is provided by:

\[\text{Note that the sign of } M_{[a}^{ij}M_{b]}^{,j} \text{ term is positive as against that in the Euclidean gravity studied earlier [9]. This sign is due to the Lorentzian nature of the internal metric } \eta^{ij}.\]
\[ M^i_a = \lambda \epsilon^i_a, \quad \Leftrightarrow \quad M^j = \lambda \delta^{ji} \]  
(44)

where \[ \partial \lambda = 0, \quad \partial_a \lambda = \frac{\kappa^2}{2} \epsilon^i_a \bar{T}^i_0 = - \frac{\kappa^2}{2} \partial_a \phi B^0. \]  
(45)

Using this in the constraint (43) leads to the master constraint:

\[ \hat{e}^k \hat{e}^j R^j_k(\hat{\omega}) + 6\lambda^2 - 2\Lambda - \xi = -2\kappa^2 \bar{T}^i_0 \]

\[ = -2\kappa^2 \left[ \hat{e}^k \partial_\xi \phi B^k + \frac{1}{2} \left( B_{ik} B^{ik} - m^2 \phi^2 \right) \right] \]

(46)

where \[ \xi = N^{ik} N^{il} - N^{jl} N^{lk}. \]  
(47)

Note from equation (46), we have

\[ \partial \xi = 2\kappa^2 \partial \bar{T}^i_0 = \kappa^2 \left[ \hat{e}^k \partial_\xi \phi + B_k \right] \partial B^k \]

(48)

where we use the fact that, by equations of motion, all fields except \( N^0 \) (and hence \( \xi \)) and \( B_i \) are \( t \) independent in (46).

This completes our analysis of all the Euler–Lagrange equations of motion for degenerate tetrad (20). The connection fields \( \omega^i_j \) are all given by (26)–(29) and (44), (45). Further, we have the master constraint (46) relating geometric quantities to the matter fields.

Lastly, we analyze the equation (10). For degenerate tetrads (20), this equation is identically satisfied for \( I = i \); both the left-hand side and right-hand side are zero. For \( I = 0 \), this equation is exactly the same as (48).

Note that, for the invertible tetrads, Euler–Lagrange equations of motion (7) contain enough equations that can be solved for all the twenty four components of the connection \( \omega^i_j \). On the other hand, for the degenerate tetrads, there are not enough independent equation in (7) to solve for all the components of the connection. As presented in (26)–(29), only twelve of these are fixed with other twelve as represented by two symmetric matrices \( M^0 \) and \( N^0 \) are left undetermined. The set of Euler–Lagrange equations in (8), yields two more equations for \( M^0 \) as presented in (41) and (42) which can be solved for \( M^0 \) as in (44) and (45). Next, of the six components in symmetric \( N^0 \), three off diagonal independent elements can be thought of as gauge degrees of freedom as these can be rotated away by \( SO(3) \) gauge rotations. Rest three (the three eigenvalues) are still undetermined. Of these only one gauge invariant combinations represented by \( \xi \) of equation (47) appears in the Euler–Lagrange equation (46).

We conclude this section with one final remark. Note that, for the non-degenerate tetrads, Euler–Lagrange equations of motion (5)–(8) lead to real dynamical equations of motion containing second order time derivatives for the physical degrees of freedom. These are represented by the Einstein field equations for the tetrads and the second order dynamical equation (17) for the scalar field. Unlike this, in the degenerate phase, the Euler–Lagrange equations of motion (5)–(8) do not yield any dynamical equations of motion for the fields which would otherwise have to contain second order time-derivatives. Here we have only constraint equations for the fields. The degenerate tetrad phase does not explore any dynamics for both the geometric as well as for the matter fields, but only describes possible global or topological degrees of freedom.
3. First order Abelian gauge field action

Now we consider vector gauge fields coupled to gravity. The discussion will be developed in detail for $U(1)$ vector gauge field. Generalization to more general non-Abelian vector gauge fields is straight forward.

For an Abelian vector gauge field $A_\mu$ coupled to gravity, we start with the action:

$$S = S_{HP} + S_{EM}$$  \hspace{1cm} (49)

where $S_{HP}$ is the Hilbert–Palatini action functional with cosmological constant (2) and the matter action functional is

$$S_{EM} = \frac{1}{8} \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} e^K_\alpha e^L_\beta \left[ F_{\mu\nu} B^{IJ} + \frac{1}{12} e^K_\alpha e^L_\beta B_{MN} B^{MN} \right].$$  \hspace{1cm} (50)

Here $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the vector gauge field $A_\mu$ and six additional fields are introduced through $B_{MN}$ which is antisymmetric in the $SO(1,3)$ internal space labels: $B_{MN} = -B_{NM}$. Like $S_{HP}$, the matter field action functional $S_{EM}$ is first order and is defined for both non-degenerate and degenerate tetrads.

Varying the action functional (49) with respect to $B_{MN}$ and $A_\mu$, respectively leads to the following Euler–Lagrange equations of motion:

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} e^K_\alpha e^L_\beta \left[ F_{\mu\nu} \delta^I_M \delta^J_N + \frac{1}{6} e^K_\alpha e^L_\beta B_{MN} \right] = 0$$  \hspace{1cm} (51)

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} \partial_\nu \left( e^K_\alpha e^L_\beta B^{IJ} \right) = 0.$$  \hspace{1cm} (52)

Note that these equations of motion are defined for both invertible and non-invertible tetrads. As we shall see below, for non-degenerate tetrads, these two first order equations are equivalent to standard Maxwell equations of motion for electromagnetic field in curved space time.

Now varying the action (49) with respect to the connection $\omega^{IJ}_\mu$ and tetrad $e^I_\mu$ respectively leads to the following Euler–Lagrange equations of motion:

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} e^K_\alpha D^I_\nu (\omega) e^L_\beta = 0$$  \hspace{1cm} (53)

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} e^I_\nu \left[ R_{\alpha\beta}^{KL}(\omega) - \frac{2\Lambda}{3} e^K_\alpha e^L_\beta \right] = -4\kappa^2 T^\mu_I$$  \hspace{1cm} (54)

which are same as the equations (7) and (8) obtained for the scalar matter theory except for $T^\mu_I$ on the right side of the second equation which, for the Abelian vector field matter, is now given by:

$$T^\mu_I = \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} e^K_\nu \left( F_{\alpha\beta} B^{KL} + \frac{1}{6} e^K_\alpha e^L_\beta B_{MN} B^{MN} \right).$$  \hspace{1cm} (55)

Also from (54), using equation of motion (53) and Bianchi identity, we notice that this $T^\mu_I$ for the vector matter field has to obey the following equation:

$$4\kappa^2 D^I_\mu(\omega) T^\mu_I = -\epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL} D^I_\nu (\omega) e^K_\nu R_{\alpha\beta}^{KL}(\omega).$$  \hspace{1cm} (56)

Like the action functional (49) and the matter equations of motion (51) and (52), the gravity equations of motion (53) and (54) are also defined for both invertible and non-invertible tetrads.
3.1. Non-degenerate tetrads

For invertible tetrads, \( \det e_I^\mu \equiv e \neq 0 \), we can use the identities:
\[
\epsilon_{\mu\nu\alpha\beta} \epsilon_{IJKL} e^K_\alpha e^L_\beta = 2e e_I^\mu e_J^\nu
\]
and
\[
\epsilon_{\mu\nu\alpha\beta} \epsilon_{IJKL} e^K_\alpha e^L_\beta e^I_\mu e^J_\nu = 24e
\]
to rewrite the matter action functional (50) as:
\[
S_{\text{EM}} = \frac{1}{2} \int d^4x \left( e_I^\mu e_J^\nu F_{\mu\nu} B^IJ + \frac{1}{2} B_{MN} B^{MN} \right)
\]  
(57)
and Euler–Lagrange equations of motion (51) and (52) respectively can be written as:
\[
e_I^\mu e_J^\nu F_{\mu\nu} + B_{MN} = 0
\]  
(58)
\[
\partial_\nu \left( e_I^\mu e_J^\nu B^{IJ} \right) = 0.
\]  
(59)
The first equation reflect the fact that \( B_{MN} \) are not independent fields. Use this constraint equation in the second equation (59) to obtain the second order equation as:
\[
\partial_\nu \left( e g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \right) = 0.
\]  
(60)
Substitute the constraint (58) in the matter action (57) to write it as:
\[
S_{\text{EM}} = -\frac{1}{4} \int d^4x e F_{\mu\nu} F^{\mu\nu} g^{\alpha\beta} g^{\alpha\beta}
\]  
(61)
which is the standard second order form of the action functional with (60) as the equation of motion for electromagnetic field in curved spacetime.

For invertible tetrads, using identities \( \epsilon_{\mu\nu\alpha\beta} \epsilon_{IJKL} e^K_\alpha e^L_\beta = 6e e_I^\mu \), equation (55) can be written as:
\[
T^I_\mu \equiv \frac{e}{4} \left[ -4F_{IK} B^{IK} + e^\nu \left( 2F_{KL} B^{KL} + B_{KL} B^{KL} \right) \right]
\]
where \( F_{IK} \equiv e^\nu e^K_\nu F_{\mu\nu} \) and \( B^{IJ} \equiv e^K_\nu B^{KL} \). This, on using the constraint (58), becomes:
\[
T^I_\mu \equiv e \tilde{T}_{I'}^\mu \equiv e \left[ F_{IK} - \frac{1}{4} e^\nu F_{\alpha\beta} F^{\alpha\beta} \right].
\]  
(62)
The Euler–Lagrange equation of motion (53) is the same as that in the case of scalar matter field discussed earlier and, for invertible tetrads, is solved exactly by the torsion-free connection fields given in terms of the tetrads, \( \omega_{IJ}^\mu = \omega_{IJ}^\mu (e) \) as in equation (18). Also for this torsion-free connection, from equation (56) we notice that \( T^I_\mu \) of equation (62) has to satisfy the condition:
\[
D_\mu (\omega) T^I_\mu \equiv D_\mu (\omega) \left( e \tilde{T}_{I'}^\mu \right) = 0.
\]  
(63)
Same discussion as was done for the scalar matter field case earlier, leads us to realize that Euler–Lagrange equations of motion (54) for invertible tetrads, with connection fields as given in equation (18), are exactly the standard second order Einstein field equations for gravity with electromagnetic matter and cosmological constant obtained in the usual second order formalism. Note that \( \tilde{T}_{\mu\nu} = e^{\alpha\beta} \tilde{T}_{I'}^\mu \equiv \left( F_{\mu\alpha} F^{\alpha\nu} - \frac{1}{4} g_{\alpha\beta} F_{\alpha\beta} F^{\alpha\beta} \right) \) is the standard stress–energy tensor for the electromagnetic field and equation (63) represents the conservation condition for this stress–energy tensor.
3.2. Degenerate tetrads

Now we analyze the general Euler–Lagrange equations of motion (51)–(54) for tetrads with one zero eigenvalue as in (20). Matter equations of motion (51) and (52) for this degenerate tetrad imply:

\[ F_{ia} = 0; \quad \partial_j \left( \dot{e}_j^a \mathcal{B}^{0i} \right) = 0, \quad \partial_a \left( \dot{e}_a^i \mathcal{B}^0 \right) = 0. \]  

We have no conditions on \( F_{ab} \) and \( B_{ij} \).

Euler–Lagrange equation of motion (53) here is the same as that for the scalar case, and hence the connection components here are the same:

\[
\omega_a^{0i} \equiv M_a^i \equiv e_a^i M^0 \text{ with } M^0 = M^j ;
\]

\[
\omega_a^{ij} = \tilde{\omega}_a^{ij}(e) + \kappa_a^{ij} \equiv \varpi_a^{ij}(e) + \epsilon^{ijk} N_a^k ,
\]

\[ N_a^k = e_a^l N^k_l \text{ with } N^k_l = N^l_k ,
\]

\[
\varpi_a^{ij}(e) \equiv \frac{1}{2} \left( \tilde{e}_b^j \partial_{[a} e^l_b - \tilde{e}_b^j \partial_{[a} e^l_b - e_a^l \tilde{e}_b^j \partial_{[b} e^l_c \right) \]  

(65)

where we have made a gauge choice to make the triads \( e_a^i \) as \( t \)-independent. Use these solutions to write the matter Euler–Lagrange equations of motion (64) as:

\[ F_{ia} = 0; \quad \partial_j \mathcal{B}^{0i} = 0, \]

\[ \dot{e}_i^a e_i^a (\mathcal{B}) = \dot{e}_i^a \left( \partial_a \mathcal{B}^0 + \omega_a^{0i} \mathcal{B}^0 \right) = \dot{e}_i^a \left( \partial_a \mathcal{B}^0 + \omega_a^{0i} \mathcal{B}^0 \right) = 0 \]  

(66)

where we have used \( D_a(\omega) \left( \dot{e} \tilde{e}^a \right) = D_a(\omega) \left( \dot{e} \tilde{e}^a \right) = 0 \) to obtain the last equation.

The various components of \( T_i^a \) of (55) for degenerate tetrad (20) now can be written as:

\[
T_0^i \equiv \dot{e} \tilde{T}_0^i = \frac{\epsilon}{2} \left( \tilde{e}_k^b \tilde{e}_l^c F_{bc} + \frac{1}{2} B_{MN} B^{MN} \right),
\]

\[
T_i^a \equiv \dot{e} \tilde{T}_i^a = \tilde{e}_b^i \tilde{e}_l^c F_{bc} B_{kl} ,
\]

\[
T_0^a \equiv \dot{e} \tilde{T}_0^a = \tilde{e}_b^a \tilde{e}_l^c F_{bc} B_{kl} ,
\]

\[
T_i^a \equiv \dot{e} \tilde{T}_i^a = \tilde{e}_b^i \tilde{e}_l^c F_{bc} B_{kl} .
\]

(67)

Since \( F_{ee} = 0 \) due to the matter equation of motion above, the last two equations here are:

\[ T_0^a \equiv \dot{e} \tilde{T}_0^a = 0, \quad T_i^a \equiv \dot{e} \tilde{T}_i^a = 0. \]  

(68)

Like in the scalar matter theory, the sixteen Euler–Lagrange equations of motion in (54) for degenerate tetrads are:

\[ \kappa^2 T_0^i \equiv \dot{e} \kappa^2 \tilde{T}_0^i = - \frac{1}{2} \epsilon \left[ \tilde{e}_b^k \tilde{e}_l^j R_{bc}^{kl}(\omega) - 2 \Lambda \right] \]  

(69)

\[ \kappa^2 T_i^a \equiv \dot{e} \kappa^2 \tilde{T}_i^a = \tilde{e}_b^i \tilde{e}_l^c R_{bc}^{kl}(\omega) \]  

(70)

\[ \kappa^2 T_0^a \equiv \dot{e} \kappa^2 \tilde{T}_0^a = \tilde{e}_b^a \tilde{e}_l^c R_{bc}^{kl}(\omega) \]  

(71)

\[ \kappa^2 T_i^a \equiv \dot{e} \kappa^2 \tilde{T}_i^a = \tilde{e}_b^i \tilde{e}_l^c R_{bc}^{0k}(\omega) \]  

(72)
but with $T_{\mu}^{\nu}$ now for vector gauge matter fields as in (67). As earlier, equations (71) and (72) respectively imply:

\[
\hat{e}_{i}^{a} R_{\mu i}^{\mu} (\omega) = 0 \tag{73}
\]

\[
R_{\mu i}^{\mu} (\omega) = \partial_{2} M_{a}^{i} = 0 \tag{74}
\]

where we have used (68) and (65). Next, use the second equation in (67) in the equation of motion (70), to obtain the constraint:

\[
\hat{e}_{i}^{b} R_{ab}^{0} (\omega) = \hat{e}_{i}^{b} D_{\mu}^{a} (\hat{\omega}) M_{a}^{i} = \kappa^{2} \hat{e}_{i}^{b} T_{i}^{0} = - \kappa^{2} \hat{e}_{i}^{b} F_{ab}^{b} B^{0}. \tag{75}
\]

Lastly, using the first equation in (67) in the equation of motion (69), we have

\[
\hat{e}_{i}^{b} e_{i}^{l} R_{ab}^{kl} (\omega) - 2 \Lambda = - 2 \kappa^{2} \hat{T}_{0}^{l} = - \kappa^{2} \left( \hat{e}_{i}^{b} e_{i}^{l} F_{bc}^{b} B^{kl} + \frac{1}{2} B_{MN} B^{MN} \right)
\]

which, using $\omega_{a}^{\mu} = \bar{\omega}_{a}^{\mu} (e) + e_{i}^{\mu} N_{i}^{k}$, can be further seen to be:

\[
\hat{e}_{i}^{b} e_{i}^{l} R_{ab}^{kl} (\bar{\omega}) - \left( M_{a}^{b} M_{b}^{k} - M_{b}^{j} M_{a}^{kl} \right) + \left( N_{i}^{b j} N_{i}^{j k} - N_{i}^{b j} N_{i}^{j kl} \right) - 2 \Lambda = - 2 \kappa^{2} \hat{T}_{0}^{l} = - \kappa^{2} \left( \hat{e}_{i}^{b} e_{i}^{l} F_{bc}^{b} B^{kl} + \frac{1}{2} B_{MN} B^{MN} \right). \tag{76}
\]

Note that (73) holds identically for configurations satisfying the constraints (65) and a particular solution to the constraints (74) and (75) is given by

\[
M_{a}^{i} = \lambda e_{a}^{i}, \quad \partial_{2} \lambda = 0, \quad \partial_{a} \lambda = \frac{\kappa^{2}}{2} \hat{e}_{a}^{i} T_{i}^{0} = - \frac{\kappa^{2}}{2} \hat{e}_{i}^{b} F_{ab} B^{0}. \tag{77}
\]

Using this in equation (76), we have the final master constraint:

\[
\hat{e}_{i}^{b} e_{i}^{l} R_{ab}^{kl} (\bar{\omega}) + 6 \lambda^{2} - 2 \Lambda - \xi = - 2 \kappa^{2} \hat{T}_{0}^{l} = - \kappa^{2} \left( \hat{e}_{i}^{b} e_{i}^{l} F_{bc}^{b} B^{kl} + \frac{1}{2} B_{MN} B^{MN} \right)
\]

where $\xi \equiv N_{i}^{b j} N_{i}^{j k} - N_{i}^{b j} N_{i}^{j kl}$.

Note that, from the constraint (78), using $\partial_{2} F_{bc} \equiv - (\partial_{a} F_{ct} + \partial_{c} F_{ta}) = 0$ due to matter equation of motion, we have:

\[
\partial_{2} \xi = 2 \kappa^{2} \partial_{2} \hat{T}_{0}^{l} = \kappa^{2} \left( \hat{e}_{i}^{b} e_{i}^{l} F_{bc}^{b} + B_{kl} \right) \partial_{b} B^{kl}. \tag{79}
\]

This equation is equivalent to equation (56) for $I = 0$ for degenerate tetrads (20). For $I = i$, equation (56) is identically satisfied, as both left-hand and right-hand side are zero for degenerate tetrads.

4. Fermion action functional

Here, for a fermion coupled to gravity in the first order formulation, we start with the following action:

\[
S = S_{HP} + S_{F} \tag{80}
\]
where $S_{HPA}$ is the Hilbert–Palatini action (2) and the fermion matter action is:

$$S_F = \frac{1}{6} \int d^4 x \epsilon_{\mu\nu\rho\sigma} e_{\mu} e_{\nu} e_{\rho} e_{\sigma} \left[ \frac{i}{\hbar} \bar{\psi} \gamma^\mu D_\mu(\omega) \psi - \frac{i}{2} \bar{D}_\mu(\omega) \psi \gamma^\mu \gamma^\nu \psi + \frac{m}{4} \epsilon_{\mu} \bar{\psi} \psi \right]$$

(81)

where $SO(1, 3)$ covariant derivatives are $D_\mu(\omega) \psi \equiv \partial_\mu \psi - \frac{i}{2} \omega^{\mu}_{\nu \sigma} \sigma_{\nu \sigma} \psi$ and $\bar{D}_\mu(\omega) \psi \equiv (D_\mu(\omega)^\dagger \psi) \gamma^0 \equiv \partial_\mu \bar{\psi} + \frac{i}{2} \omega^\mu_{\nu \sigma} \bar{\psi} \sigma_{\nu \sigma}$ with $\sigma_{\nu \sigma} = \frac{1}{2} \{ \gamma^\nu, \gamma^\sigma \}$. Note that fermion action $S_F$, like Hilbert–Palatini action (2), is defined for both invertible and non-invertible tetrads.

Varying the total action (80) with respect to the independent fields, $\bar{\psi}, \psi$, connection $\omega^{\mu}_{\nu \sigma}$ and tetrad $e^\mu_\nu$, respectively leads to the Euler–Lagrange equations of motion:

$$e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} e_{\mu} e_{\nu} e_{\rho} e_{\sigma} \left[ \frac{i}{\hbar} \bar{\psi} \gamma^\mu D_\mu(\omega) \psi + \frac{m}{4} \epsilon_{\mu} \bar{\psi} \psi \right] + \frac{3i}{2} e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} S e^I e_I e^J e_J \gamma^{IJ} \psi = 0$$

(82)

$$e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} e_{\mu} e_{\nu} e_{\rho} e_{\sigma} \left[ -i D_\mu(\omega) \psi \gamma^\nu \gamma^\mu \psi + \frac{m}{4} \epsilon_{\mu} \bar{\psi} \psi \right] - \frac{3i}{2} e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} S e^I e_I e^J e_J \gamma^{IJ} \psi = 0$$

(83)

$$e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} e_{\mu} e_{\nu} e_{\rho} e_{\sigma} \left[ \gamma^0 D_\mu(\omega) e^\mu_\nu \right] + \frac{\kappa^2}{2} e^\rho_\mu e^\sigma_\nu e^\sigma_\beta e^\nu_\alpha e^0_\gamma \gamma^{\gamma}_{\beta \gamma} \psi = 0$$

(84)

$$e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} e_{\mu} e_{\nu} e_{\rho} e_{\sigma} R^{KL}_{\alpha \beta} (\omega) - \frac{2\Lambda}{3} e^K_\alpha e^L_\beta \right] = -4\kappa^2 T^\mu_\mu$$

(85)

where $2S e^I e_I \equiv D_\mu(\omega) e^\mu_\nu$ is the torsion and

$$T^\mu_\mu = \frac{1}{4} e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} e_{\mu} e_{\nu} e_{\rho} e_{\sigma} \left[ \frac{i}{\hbar} \bar{\psi} \gamma^\mu D_\mu(\omega) \psi - \frac{i}{2} \bar{D}_\mu(\omega) \psi \gamma^\nu \psi + \frac{m}{4} \epsilon_{\mu} \bar{\psi} \psi \right].$$

(86)

Apply covariant derivative to equation (85) and use Bianchi identity and equation (84) to obtain the constraint on this $T^\mu_\mu$ as:

$$4\kappa^2 D_\mu(\omega) T^\mu_\mu = - e^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} D_\mu(\omega) e^\mu_\nu. R^{KL}_{\alpha \beta} (\omega).$$

(87)

Like in the earlier cases, all the Euler–Lagrange equations of motion above are defined for both non-degenerate and degenerate tetrads.

4.1. Invertible tetrads

Fermions have been well studied in literature in first order formulation of gravity with non-degenerate tetrads. We shall briefly summarize the discussion only for completeness here.

For invertible tetrads, it is straightforward to check that the fermion action (81) can be rewritten in the standard form usually used in earlier studies:

$$S_F = \int d^4 x \ e \left[ \frac{i}{2} \bar{\psi} e^\mu_\nu \gamma^\mu D_\mu(\omega) \psi + \frac{i}{2} \bar{D}_\mu(\omega) \psi e^\mu_\nu \gamma^\nu \psi + \frac{m}{4} \epsilon_{\mu} \bar{\psi} \psi \right]$$

(88)

and the fermion equations of motion (82) and (83) take the form:

2 Our gamma matrices satisfying the Clifford algebra $\gamma^0 \gamma^\nu + \gamma^\nu \gamma^0 = -2\eta^\nu$, with $\eta^\nu = \eta_{\mu \nu} = \text{diag}(-1, 1, 1, 1)$. are $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^\nu = \begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix}$, with $\gamma_3 = -i \sigma_3 \gamma^0 \gamma^3 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^\nu)^\dagger = -\gamma^\nu$, $(\sigma^\nu)^\dagger = \sigma^\nu$, $(\sigma^\nu)^\dagger = -\sigma^\nu$ and $\sigma_{\nu \sigma} = \delta_{\nu \sigma} \gamma^0 - \gamma^\nu \gamma^\sigma + \sigma^\nu \gamma^\sigma = i d_{\nu \kappa \lambda} \gamma_\mu \gamma^{\nu \kappa \lambda} + \sigma_{\nu \sigma} \gamma^{\nu \kappa \lambda} = 1, 0, 1, 2, 3$.
\[ i e^\mu_I \gamma^I D_\mu (\omega) \psi + m \psi - i e^\nu_I S^I_{\mu \nu} \gamma^\nu \psi = 0 \]
\[- i e^\mu_I D_\mu (\omega) \overline{\psi} \gamma^I + m \overline{\psi} + i e^\mu_I S^I_{\mu \nu} \overline{\psi} \gamma^\nu = 0. \quad (89) \]

Next, for invertible tetrads, the equation of motion (84) can be simplified to
\[ 2S_{\alpha \beta I} \equiv D_{\beta I} (\omega) e^\beta_{\alpha I} = - \kappa^2 2 \epsilon_{IJKL} e^\mu_{\alpha I} \overline{\psi} \gamma^5 \gamma^I \gamma^J \psi \quad (90) \]
indicating presence of matter induced torsion. This equation can be solved for the 24 connection components as:
\[ \omega^\mu_{IJ} = \omega^\mu_{IJ} (e) + \kappa^2_{IJ} \quad (91) \]
where \( \omega^\mu_{IJ} (e) \) is the torsion-free connection (18), \( D_{\beta I} (\omega(e)) e^\beta_{\mu I} = 0 \), and contorsion is:
\[ \kappa^2_{IJ} = \tfrac{\kappa^2}{4} \epsilon_{IJKL} e^\mu_{\alpha K} \overline{\psi} \gamma^5 \gamma^I \psi. \quad (92) \]

For invertible tetrads with connection fields given by (91) and (92), it is straightforward to see that the equations of motion (85) are second order Einstein field equations with cosmological constant and now matter stress–energy \( \tilde{T}_\mu^\nu \) of (86) for fermion fields as:
\[ \tilde{T}_\mu^\nu \equiv \tilde{T}_\mu^\nu = e \left[ e^\mu_i \left( \frac{1}{2} \overline{\psi} e^5_J \gamma^J D_\mu (\omega) \psi - \frac{1}{2} D_\mu (\omega) \overline{\psi} e^5_J \gamma^J \psi + m \overline{\psi} \psi \right) \right] \]
\[ - e^\nu_i e^5_j \left( \frac{1}{2} \overline{\psi} \gamma^5 D_\mu (\omega) \psi - \frac{1}{2} D_\mu (\omega) \overline{\psi} \gamma^5 \psi \right). \quad (93) \]

However, it is well known, the second order theory so obtained is not exactly same as the standard second order theory obtained from Einstein–Hilbert action with fermion matter described by an action obtained by minimal coupling prescription. The difference lies in equations (90)–(92) reflecting presence of fermion dependent torsion in the theory discussed above. This is in contrast to the standard second order theory of gravity with fermions which is torsion free.

4.2. Non-invertible tetrads

We now analyze the fermion theory for tetrads (20) with one zero eigen value. The fermion Euler–Lagrange equations of motion (82) and (83) for this degenerate tetrad can be written as:
\[ D_\mu (\omega) \psi + e^\mu_i \gamma^0 \left( S^0_{\mu \gamma} - S^{0 \gamma}_{\mu} \right) \psi = 0 \]
\[ D_\mu (\omega) \overline{\psi} + e^\mu_i \overline{\psi} \left( S^{0 \gamma}_{\mu} - S^0_{\mu \gamma} \right) \gamma^0 = 0. \quad (94) \]

We break twenty four Euler–Lagrange equations of motion in (84) into four sets of 3, 3, 9 and 9 equations as:
\[ e^{abc} \epsilon_{ijl} e^l_a D_b (\omega) e^c_j = 0 \quad (95) \]
\[ e^{abc} \left( \epsilon_{ijk} e^k_a D_b (\omega) e^c_j - \frac{\kappa^2}{2} e^a_m e^b_e e^c_k \overline{\psi} \gamma^5 \gamma^k \psi \right) = 0 \quad (96) \]
\[ e^{abc} \epsilon_{ijk} e^j I D_l (\omega) e^k_c = 0 \quad (97) \]

\[ \text{Class. Quantum Grav. 36 (2019) 065014} \]
\[ e^{abc} e_{ik} e_{jk} D_i(\omega) e^o_k = 0. \tag{98} \]

Except for (96), all these equations are the same as those obtained for the scalar and vector theories above. So we can follow similar discussion as earlier to solve these. The solution of equations (95), (97) and (98) is given by:

\[
\begin{aligned}
\partial_t e_a^i &= 0; \quad \omega_a^{0i} = 0, \quad \omega_a^{ij} = 0; \\
\omega_a^{ij} &= \omega_a^{ij}(e) + \kappa_a^{ij} \equiv \omega_a^{ij}(e) + e^{ik} N_a^k, \\
N_a^k &= e^i_k N^k \text{ with } N^k = N^{ij}, \\
\omega_a^{ij}(e) &\equiv \frac{1}{2} \left( e^i_k \partial_{(a} e^j_{b)} - e^j_k \partial_{(a} e^i_{b)} - e^i_k e^j_l \partial_{(a} e^i_{c)} \right) \tag{99}
\end{aligned}
\]

where, without loss of generality, we have made the gauge choice to make the triads \( e_a^i \) independent of \( t \). The only change appears in the connection component \( \omega_a^{0i} \) obtained from (96) which is now:

\[
\begin{aligned}
\omega_a^{0i} &\equiv M_a^i = e_a^i M^{ij} = e_a^i (M^{ij} + \bar{M}^{ij}) \equiv e_a^i (M_a^{ij} + \bar{M}_a^{ij}), \\
M^{ij} &\equiv \frac{1}{2} (M^{ij} + \bar{M}^{ij}), \quad \bar{M}^{ij} \equiv \frac{1}{2} (M^{ij} - \bar{M}^{ij}) = \frac{\kappa^2}{2} e^{ik} \bar{\psi} \gamma_k \gamma_l \psi. \tag{100}
\end{aligned}
\]

Thus, three components of \( M^{ij} \) represented by the antisymmetric part \( \bar{M}^{ij} \) are fixed in terms of the fermions, but other six components in the symmetric part \( M^{ij} \) are not determined by the equations of motion. These are in addition to the six undetermined fields contained in the symmetric matrix \( N^{ij} \) of (99). This solves all the equations of motion in (95)–(98). Note that equation (99), implies \( 2S_{ab}^{ij} \equiv D_i(\omega) e_a^i - D_a(\omega) e_i^0 = D_i(\omega) e_a^i = 0 \). This, in turn, for the fermion equations of motion (94) implies:

\[
D_i(\omega) \psi = \partial_i \psi = 0. \tag{101}
\]

Various components of \( T_{ij}^{\mu} \) of (86) for degenerate tetrads (20) can be written as:

\[
\begin{aligned}
T_0^i \equiv \dot{e} T_0^i &= \dot{e} \left[ \frac{i}{2} e^a_i \left( \bar{\psi} \gamma^0 D_a(\omega) \psi - \bar{D}_a(\omega) \psi \gamma^0 \psi \right) + m \bar{\psi} \psi \right] \tag{102} \\
T_i^i \equiv \dot{e} T_i^i &= - \frac{i}{2} \dot{e} e^a_i \left( \bar{\psi} \gamma^0 D_a(\omega) \psi - D_a(\omega) \bar{\psi} \gamma^0 \psi \right) \tag{103} \\
T_0^a \equiv \dot{e} T_0^a &= - \frac{i}{2} \dot{e} e^a_i \left( \bar{\psi} \gamma^0 D_i(\omega) \psi - D_i(\omega) \bar{\psi} \gamma^0 \psi \right) \tag{104} \\
T_i^a \equiv \dot{e} T_i^a &= \frac{i}{2} \dot{e} e^a_i \left( \bar{\psi} \gamma^0 D_i(\omega) \psi - D_i(\omega) \bar{\psi} \gamma^0 \psi \right). \tag{105}
\end{aligned}
\]

We use the solutions (99)–(101) in these equations:

\[
\begin{aligned}
T_0^i \equiv \dot{e} T_0^i &= \dot{e} \left[ \frac{i}{2} e^a_i \left( \bar{\psi} \gamma^0 D_a(\omega) \psi - \bar{D}_a(\omega) \psi \gamma^0 \psi \right) + m \bar{\psi} \psi \\
&\quad + \frac{1}{2} N_a \bar{\psi} \gamma_5 \gamma_0 \psi - \frac{2}{\kappa^2} \bar{M}_a^{ij} \bar{M}^{ij} \right] \tag{106} \\
T_i^i \equiv \dot{e} T_i^i &= - \frac{i}{2} \dot{e} e^a_i \left( \bar{\psi} \gamma^0 D_a(\omega) \psi - D_a(\omega) \bar{\psi} \gamma^0 \psi + \frac{2i}{\kappa^2} \epsilon^{imn} N_a^{mn} \right) \tag{107}
\end{aligned}
\]
\[ T_0^a \equiv \dot{T}_0^a = 0 \]  
\[ T_i^a \equiv \dot{T}_i^a = 0. \]  

Now we analyze the last set of sixteen Euler–Lagrange equations contained in \( (85) \). As earlier, we break these into four sets of 1, 3, 3 and 9 equations as:

\[ \kappa^2 T_0^t \equiv \hat{e} \kappa^2 T_0^t = -\frac{1}{2} \hat{e} \left[ \hat{e}^b \hat{e}^c R_{bc}^{kl}(\omega) - 2\Lambda \right] \]  
\[ \kappa^2 T_i^t \equiv \hat{e} \kappa^2 T_i^t = \hat{e} \hat{e}^c R_{bc}^{0i}(\omega) \]  
\[ \kappa^2 T_0^a \equiv \hat{e} \kappa^2 T_0^a = \hat{e} \hat{e}^a \hat{e}^b R_{bc}^{kl}(\omega) \]  
\[ \kappa^2 T_i^a \equiv \hat{e} \kappa^2 T_i^a = \hat{e} \hat{e}^a \hat{e}^b R_{bc}^{0i}(\omega) \]

where now various components of \( T_{\mu}^I \) are given by equations \( (106)-(109) \). The last two equations are exactly the same as earlier for the scalar and vector gauge matter field cases and hence we have:

\[ \hat{e}^b R_{0i}^b(\omega) = 0 \]  
\[ R_{0i}^b(\omega) = \partial_i M^i = 0 \]

where we have used \( \omega_i^0 = 0 \) and \( \omega_i^y = 0 \) in the second equation. From equations \( (111) \) and \( (107) \), we have:

\[ \hat{e}^b R_{0i}^b(\omega) = \kappa^2 \hat{e}^a \hat{e}^b \hat{T}_i^a \]
\[ = -\frac{i}{2} \kappa^2 \left( \overline{\psi} \gamma^0 D_a(\omega) \psi - \overline{D_a(\omega)} \psi \gamma^0 \psi \right) \]
\[ = -\frac{i}{2} \kappa^2 \left( \overline{\psi} \gamma^0 D_a(\omega) \psi - \overline{D_a(\omega)} \psi \gamma^0 \psi \right) + \epsilon^{lmn} N_{ij}^a M^{mn}. \]  

Note that

\[ \hat{e}^b R_{0i}^b(\omega) = \hat{e}^b D_{[\omega]}(\omega) M_{ij}^l = \hat{e}^b D_{[\omega]}(\omega) M_{ij}^l + \epsilon^{lmn} N_{ij}^a M^{mn}. \]

Using this in \( (116) \), we have

\[ \hat{e}^b D_{[\omega]}(\omega) M_{ij}^l = -\frac{i}{2} \kappa^2 \left( \overline{\psi} \gamma^0 D_a(\omega) \psi - \overline{D_a(\omega)} \psi \gamma^0 \psi \right). \]  

We break \( M_{ij}^l \) as \( M_{ij}^l = \tilde{M}_{ij}^l + \check{M}_{ij}^l \) where \( \tilde{M}_{ij}^l \equiv \hat{e}^b \hat{e}^a \hat{e}^b M_{ij}^l \) and \( \check{M}_{ij}^l \equiv \hat{e}^b \hat{e}^a \hat{e}^b M_{ij}^l \) as the symmetric and antisymmetric parts of the matrix \( M^{ml} \). For the antisymmetric \( M^{0i} \) given in terms of the fermions as in \( (100) \), it is straightforward to check that it satisfies the following equation:

\[ \hat{e}^b D_{[\omega]}(\omega) \tilde{M}_{ij}^l = -\frac{\kappa^2}{4\hat{e}} g_{ab} \epsilon^{bcd} \partial_c \left( \overline{\psi} \gamma_d \gamma^i \psi \right) \]

where \( \gamma_d \equiv \hat{e}^d \gamma^i \). Substitute this in \( (117) \) to obtain the constraint on the symmetric part \( \check{M}_{ij}^l \) as:
\[
\hat{e}^b D_{[\mu}(\bar{\omega})M_{\nu]} = \frac{\kappa^2}{4\epsilon} g_{ab} \epsilon^{bcd} \partial_c (\bar{\psi}\gamma_5 \gamma_d \psi) - \frac{i\kappa^2}{2} \left( \bar{\psi}\gamma^0 D_a(\bar{\omega})\psi - D_a(\bar{\omega})\psi \gamma^0 \psi \right). \tag{119}
\]

Now we are left to analyze the Euler–Lagrange equation of motion (110). This we do in the same manner as in earlier cases of scalar and vector gauge field matter to obtain:

\[
\hat{e}^b \xi^c \bar{R}_{bc}(\bar{\omega}) - \left( M^{dk} M^{jk} - M^{dk} M^{ij} \right) + \left( N^{dk} N^{ik} - N^{dk} N^{kl} \right) - 2\Lambda = -2\kappa^2 T^0_0 = -2\kappa^2 \left[ \frac{i}{2} \hat{e}^b \left( \bar{\psi}\gamma^0 D_a(\bar{\omega})\psi - D_a(\bar{\omega})\psi \gamma^0 \psi \right) \right] + m\bar{\psi}\psi + \frac{1}{2} N_{ll} \bar{\psi}\gamma_5 \gamma_0 \psi - \frac{2}{\kappa^2} \tilde{M}^i \tilde{M}^j. \tag{120}
\]

As in earlier cases, equation (114) has no additional information beyond that already contained in (99). Thus we are left with (115), (119) and (120) as the set of constraints. A particular solution of the constraints (115) and (119) is given by:

\[
\bar{M}_{ia} = \lambda \epsilon^i_a, \quad \partial_t \lambda = 0, \quad \partial_a \lambda = \frac{\kappa^2}{8\epsilon} g_{ab} \epsilon^{bcd} \partial_c (\bar{\psi}\gamma_5 \gamma_d \psi) - \frac{i\kappa^2}{4} \left( \bar{\psi}\gamma^0 D_a(\bar{\omega})\psi - D_a(\bar{\omega})\psi \gamma^0 \psi \right). \tag{121}
\]

This leaves us with the master constraint (120).

Note that, from equation (120), we have:

\[
\partial_t \xi = \kappa^2 \partial_t N_{[ll} \bar{\psi}\gamma_5 \gamma_0 \psi, \quad \xi \equiv N^{[ll} N^{ll} - N^{[ll} N^{kl]} \tag{122}
\]

where we have used the fact that, due to equations of motion, all but \(N^{[ll} \) in equation (120) are \(t\) independent. This equation can also be written as:

\[
N^{lm} \partial_t N_{[ml]} = \left( N^{[ll} - \frac{\kappa^2}{2} \bar{\psi}\gamma_5 \gamma_0 \psi \right) \partial_t N^{[ll].}
\]

Equation (122) is equivalent to equation (87) for \(I = 0\) for the degenerate tetrads (20). For \(I = i\), equation (87) is identically satisfied for configurations obeying the equations of motion listed above.

5. Concluding remarks

We have extended the discussion of degenerate metrics in first order gravity by including matter fields. This has been done using first order action functionals for the matter fields. Like the Hilbert–Palatini action, these matter actions are defined for both invertible and non-invertible tetrads. The Euler–Lagrange equations of motion obtained from such constructed actions are also defined for invertible as well as non-invertible metrics. This provides an appropriate field theoretic framework to study degenerate metrics. For degenerate metrics, the theory exhibits a new phase with very different structure which has been displayed through a detail analysis for non-invertible tetrads with one zero eigen value. The matter fields considered are scalar, \(U(1)\).
vector gauge field and fermions. Generalization to other matter fields like non-Abelian vector gauge fields is straight forward and can be done in a similar manner. Also this analysis can be extended to study degenerate tetrads with more that one zero eigen values in a similar spirit.

It may be emphasized here that for pure gravity and also with scalar or gauge matter fields, the first order formulation with invertible tetrads and the standard second second order formalism are classically exactly equivalent. Even, quantum mechanically, the torsion-free second order formulation without any matter fields is already known to be equivalent to the first order theory with non-degenerate metrics [12]. However, as the field theory based on Hilbert–Palatini action does contain the addition sector with degenerate metrics, the quantum theory would have to include the effects of this phase also in the path integrals.

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References

[1] Einstein A and Rosen N 1935 Phys. Rev. 48 73
[2] Hawking S 1978 Nucl. Phys. B 144 349
[3] Henneaux M 1979 Bull. Soc. Math. Belg. 31 47
  Henneaux M, Pilati M and Teitelboim C 1982 Phys. Lett. B 110 123
  Pilati M 1982 Phys. Rev. D 26 2645
  Pilati M 1983 Phys. Rev. D 28 729
[4] Tseytlin A A 1982 J. Phys. A: Math. Gen. 15 L105
[5] Ashtekar A 1987 Phys. Rev. D 36 1587
  Bengtsson I 1989 Int. J. Mod. Phys. A 04 5527
  Koshti S and Dadhich N 1989 Class. Quantum Grav. 6 L223
  Bengtsson I 1990 Class. Quantum Grav. 7 27
  Bengtsson I 1991 Class. Quantum Grav. 8 1847
  Varadarajan M 1991 Class. Quantum Grav. 8 L235
[6] Bengtsson I and Jacobson T 1997 Class. Quantum Grav. 14 3109
  Bengtsson I and Jacobson T 1998 Class. Quantum Grav. 15 3941
  Jacobson T and Romano J D 1992 Class. Quantum Grav. 9 L119
  Romano J D 1993 Phys. Rev. D 48 5676
  Reisenberger M P 1995 Nucl. Phys. B 457 643
  Yoneda G, Shinkai H and Nakamichi A 1997 Phys. Rev. D 56 2086
[7] Horowitz G T 1991 Class. Quantum Grav. 8 587
[8] Wheeler J A 1957 Ann. Phys. 2 604
  Wheeler J A 1962 Geometrodynamics (New York: Academic)
  Geroch R P 1967 J. Math. Phys. 8 782
[9] Kaul R K and Sengupta S 2016 Phys. Rev. D 93 084026
  Kaul R K and Sengupta S 2016 Phys. Rev. D 94 104047
[10] Kaul R K and Sengupta S 2017 Phys. Rev. D 96 104011
  Sengupta S 2017 Phys. Rev. D 96 104031
[11] Kibble T W B 1961 J. Math. Phys. 2 212
  Okubo S and Tosa Y 1979 Phys. Rev. D 20 462
  Sudermanney K 1982 Constrained Dynamic (Berlin: Springer)
Capovilla R, Dell J and Jacobson T 1991 *Class. Quantum Grav.* **8** 59
Koshti S 1992 *Class. Quantum Grav.* **9** 1937
McKeon D G C 1994 *Can. J. Phys.* **72** 601
Deser S 2006 *Class. Quantum Grav.* **23** 5773
Kiriushecheva N and Kuzmin S V 2007 *Class. Quantum Grav.* **24** 1371
Kiriushecheva N, Kuzmin S V and McKeon D G C 2012 *Can. J. Phys.* **90** 165

[12] Aros R, Contreras M and Zanelli J 2003 *Class. Quantum Grav.* **20** 2937