Axisymmetric Consolidation of Unsaturated Soils with Vertical Drain with Radial and Vertical Drainage

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Abstract: In this paper, an analytical solution is presented to study the axisymmetric consolidation of a vertical drain inserted in a finite unsaturated soil layer. In the derivation, the decoupling process is employed to convert the governing equations of excess pore-air and pore-water pressures into essentially the same homogeneous partial differential equations (PDEs). Later, the solution with time is obtained by employing the separation variable method. Additionally, validation exercise is conducted by degenerating current solutions into that for the consolidation of vertical drains in saturated soils. Finally, the changing regularity of excess pore-air pressure, excess pore-water pressure and normalized settlement is studied against the ratios of air-water permeability, the ratios of radial-vertical permeability and depth. It is concluded that the solution presented in this paper is reliable and the research method has a reference to more complicated consolidation problems.

1. Introduction
The vertical drain has attained continuous attention for its high efficiency to facilitate the consolidation process of soil deposits in recent decades, and the consolidation mechanism for unsaturated vertical drain foundation is that the soil element is compressed due to the dissipation of air and water phases in the radial and vertical directions [1].

Groundwater levels vary in many areas due to the uncertainty of rainfall, which means that not all soil deposits can be treated as fully saturated soils. In recent decades, theories of the consolidation in unsaturated stratum have achieved a great progress through unremitting research of the scholars. For example, based on the traditional one-dimensional (1D) unsaturated consolidation hypothesis by Fredlund and Hasan [2], Dakshanamurthy et al. [3, 4] proposed two-dimensional (2D) and three-dimensional (3D) theories to research the consolidation behavior of unsaturated deposits. By adopting the decoupling method, the semi-analytical solutions of the unsaturated consolidation under free strain for vertical drain foundation with radial dissipation were derived by Qin et al. [5]. Later, Zhou et al. [6] achieved the analytical solutions for the axisymmetric consolidation with radial dissipation based on Qin et al. [5]. On the other hand, by Laplace transform and inverse Laplace transform theories, Ho and Fatahi [7] obtained the analytical solution for consolidation of vertical drain in unsaturated deposits under time-dependent loadings with both radial and vertical drainage. However, the complicated mathematical transform and inverse transform techniques will make these solutions impractical [6], and the solutions for the radial and vertical drainage of unsaturated vertical drain foundations that can be degenerated into those in saturated soils cannot be found in literature.

This paper presents an analytical solution for the consolidation of vertical drain in unsaturated soils with a more concise derivation process. Specifically, the decoupling process is conducted to transfer the coupling governing equations into equivalent homogeneous PDEs. And the separation variable method
is used to obtain the functions in analytical forms. It is noted that the derivation process proposed in this paper can be adopted in more complex problems unsaturated soils.

2. Mathematical Modeling

The consolidation modeling is shown as figure 1, a vertical drain with the radius $r_w$ is constructed in a finite unsaturated soil layer. A soil element is adopted to illustrate the radial and vertical flow of air and water phases. The radius of the vertical drain well is $r_w$, and the outer radius of the influenced area is $r_e$. The thickness of the vertical drain foundation is $H$. $k_{ar}$, $k_{wr}$, $k_{az}$ and $k_{wz}$ are the permeability coefficients of air and water phases flowing in radial and vertical directions, respectively.

![Fig. 1 The axisymmetric consolidation modeling of vertical drain in unsaturated soils](image)

2.1. Basic assumptions

Basic assumptions for the consolidation modeling of vertical drain foundation in this paper are made as follows:

1. The derivation is based on the free strain assumption;
2. Air and water flow independently and continuously;
3. Water phase and soil particles are incompressible;
4. The consolidation coefficients are invariant and constant;
5. Smear effect and drain resistance are neglected.

Actually, the third assumption is not suitable for all cases while it is reasonable for a small stress increment in the transitory process [1, 2].

2.2 Governing equations

Governing equations of air and water phases for axisymmetric consolidation in unsaturated soils are as follows [7]:

$$\frac{\partial u_a}{\partial t} = -C_a \frac{\partial u_a}{\partial t} - C_{aw} \left( \frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \frac{\partial u_a}{\partial r} + \frac{1}{r} \frac{\partial u_a}{\partial z} \right) - C_{zz} \frac{\partial^2 u_a}{\partial z^2}$$

(1)

$$\frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_w}{\partial t} - C_{wr} \left( \frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \frac{\partial u_w}{\partial r} + \frac{1}{r} \frac{\partial u_w}{\partial z} \right) - C_{wz} \frac{\partial^2 u_w}{\partial z^2}$$

(2)

Relevant parameters are as follows:

$$C_a = \frac{m_a^2}{m^{\text{at}}_a - m^{\text{at}}_w - u_{\text{atm}} n_0 (1 - S_0)} \left( \frac{\bar{u}_a}{\bar{u}_a^0} \right)^3$$

$$C_{wr} = \frac{k_{wr} RT}{g n_0 M \left( m^{\text{at}}_a - m^{\text{at}}_w - u_{\text{atm}} (1 - S_0) n_0 / (\bar{u}_a^0)^2 \right)}$$
where, $u_a$ and $u_w$ (kPa) are excess pore-air and pore-water pressures, respectively; $R$ is the universal air constant and $R = 8.314$ J/(mol·K); $T$ is the absolute temperature, $T = 293.6$ K; $u_{a0}$ (kPa) is the absolute initial excess pore-air pressure, and $u_{w0} = u_{a0} + u_{atm}$. $\gamma_w$ (kPa) is the value of the initial excess pore-air pressure, $u_{atm}$ (kPa) is the atmospheric pressure; $M$ is the air mass molecular, that is $M = 0.029$ kg/mol; $n_0$ and $S_0$ are the initial porosity and saturation, respectively; $m_1$ and $m_2$ (kPa$^{-1}$), respectively, are the coefficients related to the variations of net normal stress and suction; $\gamma_w$ is the unit weight of water, and $\gamma_w = 9.8$ kN/m$^3$; $m_w$ and $m_w$ (kPa$^{-1}$), respectively, are the coefficients related to the variations of net normal stress and suction.

2.3 Initial and boundary conditions

1) Initial conditions ($0 < z < H$)

$$u_a(r, z, 0) = u_{a0}, \quad u_w(r, z, 0) = u_{w0}$$

where, $u_a$ and $u_w$ are the initial excess pore-air and pore-water pressures, respectively.

2) Boundary conditions ($r_w < r < r_e$)

The boundary conditions are listed below.

$$u_a(r_w, z, t) = u_w(r_w, z, t) = 0$$

$$\frac{\partial u_a}{\partial r}(r_w, z, t) = \frac{\partial u_w}{\partial r}(r_w, z, t) = 0$$

$$u_a(r, 0, t) = u_w(r, 0, t) = 0$$

$$u_a(r, H, t) = u_w(r, H, t) = 0$$

The solutions can be obtained by adopting Eqs. (4a) to (4d) into governing equations (1) and (2).

3. Analytical Solutions

Eqs. (1) and (2) can also be transformed as follows:

$$\frac{\partial u_a}{\partial t} = A_{aw} \left( \frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \frac{\partial u_a}{\partial r} \right) + A_{aw} \left( \frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \frac{\partial u_w}{\partial r} \right) + A_{aw} \frac{\partial^2 u_a}{\partial z^2} + A_{aw} \frac{\partial^2 u_w}{\partial z^2}$$

$$\frac{\partial u_w}{\partial t} = W_{aw} \left( \frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \frac{\partial u_a}{\partial r} \right) + W_{aw} \left( \frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \frac{\partial u_w}{\partial r} \right) + W_{aw} \frac{\partial^2 u_a}{\partial z^2} + W_{aw} \frac{\partial^2 u_w}{\partial z^2}$$

where,

$$A_{aw} = \frac{C_{aw}}{1 - C_w C_w}, \quad A_{aw} = \frac{C_a C_w}{1 - C_a C_w}, \quad A_{aw} = -\frac{C_w}{1 - C_w C_w}, \quad A_{aw} = \frac{C_w C_w}{1 - C_w C_w},$$

$$W_{aw} = \frac{C_w C_w}{1 - C_w C_w}, \quad W_{aw} = -\frac{C_w}{1 - C_w C_w}, \quad W_{aw} = \frac{C_w C_w}{1 - C_w C_w}, \quad W_{aw} = -\frac{C_w}{1 - C_w C_w}.$$

Eqs. (5) and (6) can be converted into equivalent PDEs by introducing variables of $\Phi_1$ and $\Phi_2$:

$$\frac{\partial \Phi_1}{\partial t} + Q_2 \left( \frac{\partial^2 \Phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_1}{\partial r} \right) = 0$$

$$\frac{\partial \Phi_2}{\partial t} + Q_2 \left( \frac{\partial^2 \Phi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_2}{\partial r} \right) = 0$$

where, $\Phi_1 = u_a + q_a u_a$ and $\Phi_2 = q_w u_a + u_w$. Details of the derivation process and relevant parameters are presented in Appendix A.
By applying the separation variable method, Eqs. (7) and (8) can be expressed as followed:

\[ \Phi_{1mn}(t, r, z) = T_{1mn}(t) R_m(r) Z_n(z) \]  
\[ \Phi_{2mn}(t, r, z) = T_{2mn}(t) R_m(r) Z_n(z) \]  

where, \( Z_n(z) \), \( R_m(r) \), \( T_{1mn}(t) \) and \( T_{2mn}(t) \) are the eigenfunctions with respect to vertical flow, radial flow and time, respectively.

Eq. (9) can be transformed into the following eigenfunctions:

\[ \frac{d^2 Z_n(z)}{dz^2} + \nu_n^2 Z_n(z) = 0 \]  
\[ \frac{d^2 R_m(r)}{dr^2} + \frac{\lambda_m}{r} \frac{dR_m(r)}{dr} + \frac{\nu_m^2}{r^2} R_m(r) = 0 \]  
\[ T_{1mn}'(t) + (Q_m \lambda_m + Q_r \nu_m) T_{1mn}(t) = 0 \]  

where, \( \nu_n \) and \( \lambda_m \) are the eigenvalues of \( Z_n(z) \) and \( R_m(r) \), respectively.

Substituting Eqs. (3) and (4) into Eqs. (11) to (13) gives:

\[ \sin \left( \frac{n \pi}{H} z \right) D_m \left( \frac{\mu_m}{r_w} r \right) T_m(t) = 0 \]  
\[ \cos \left( \frac{n \pi}{H} z \right) D_m \left( \frac{\mu_m}{r_w} r \right) T_m(t) = 0 \]  
\[ T_{1mn}(t) = 4 \Phi_1 \left( 0, r, z \right) \frac{\alpha_m X_m}{\nu_m} e^{-(\Omega_m + \Omega_m) t} \]  
\[ T_{2mn}(t) = 4 \Phi_1 \left( 0, r, z \right) \frac{\alpha_m X_m}{\nu_m} e^{-(\Omega_m + \Omega_m) t} \]  

where,

\[ \Phi_1(t, r, z) = u_0^a + u_0^b q_{1z}, \quad \Phi_2(t, r, z) = u_0^a q_{1z} + u_0^a, \quad X_m = \left( r_w \right)^{1/2} \left[ J_1(\mu_m) Y_0(\mu_m) - J_0(\mu_m) Y_1(\mu_m) \right] \]  
\[ \gamma_m = \left( r_w \right)^{-1/2} \left[ J_0(\mu_m) Y_0(\mu_m) - J_1(\mu_m) Y_1(\mu_m) \right] \]  
\[ \alpha_m = 1 - (-1)^{n/2}, \quad \nu_m = (n \pi / H)^2, \quad m = 1, 2, 3, \ldots \]  

Accordingly,

\[ \frac{\partial e_a}{\partial t} = m_{1a} \frac{\partial (\sigma_a - u_a)}{\partial t} + m_2 \frac{\partial (u_a - u_a)}{\partial t} \]
where, \( m'_{1k} = m''_{1k} + m''_{1k} \), \( m''_{2k} = m''_{2k} + m''_{2k} \).

Bringing Eqs. (15) and (16) into Eq. (19) gives the normalized settlement of the vertical drain foundation in unsaturated soils:

\[
\frac{w'}{w_{\text{max}}} = \frac{\int_{r_0}^{r_0} 2\pi \varepsilon_{\text{v}} r dr dz}{\pi H \left( (r_v)^2 - (r_a)^2 \right) \varepsilon_{\text{v}, \text{max}}}
\]

(23)

4. Verification and Examples

4.1 Degeneration into the saturated soils

It should be appointed that the decoupling method proposed in this paper works well in transforming the solutions of unsaturated soils into the solution of saturated soils. When referring to the fully saturated soil deposits, all properties related to the air phase are equal to zero (i.e., \( u'_a = k'_a = m'_{1a} = m'_{2a} = C'_a = C''_a = C''_{aw} = C''_{aw} = 0 \)), so \( A_{ra} = A_{sa} = A_{rw} = A_{sw} = W_{ra} = W_{sw} = 0 \) can be achieved by definition. \( q_{12} = q_{21} = 0 \) can also be obtained by Eq. (A4).

By taking in \( q_{12} = q_{21} = 0 \) into Eqs. (7) and (8) and Eqs. (18) and (19), respectively, the governing equations and the solutions for excess pore pressures of vertical drain in saturated soils can be achieved as follows:

\[
\frac{\partial u_m}{\partial t} = \frac{k_{aw}}{\gamma'_m m'_w} \left( \frac{\partial^2 u_m}{\partial r^2} + \frac{1}{r} \frac{\partial u_m}{\partial r} \right) + \frac{k_{wa}}{\gamma'_w m'_w} \frac{\partial^2 u_m}{\partial z^2}
\]

(24)

\[
u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\alpha \pi u_m}{na \mu_n \pi} \sin \left( \frac{m\pi}{H} r \right) D_m \left( \frac{u_m}{r_0} \right) e^{-\left( \frac{k_{aw}}{\gamma'_m m'_w} + \frac{k_{wa}}{\gamma'_w m'_w} \right) r}
\]

(25)

Eqs. (24) and (25) are the traditional governing equation and solution for excess pore-water pressure of vertical drain in saturated, so the solutions can be perfectly transferred to that in saturated soils. \( \varepsilon_r \) is the volume compressibility of saturated soil. To sum up, the decoupling method proposed in this paper is valid in unsaturated soils.

4.2 Results of study

The mathematical modeling is shown in figure 1, a boundless vertical instantaneous loading \( q_0 = 100 \) kPa is uniformly scattered on the upper surface of the unsaturated vertical drain foundation. Other parameters are as follows:

\( u'_a = 20 \) kPa, \( u'_{aw} = 40 \) kPa, \( r_a = 0.5 \) m, \( r_r = 4.5 \) m, \( H = 10 \) m, \( u_{\text{atm}} = 101.3 \) kPa, \( k_{aw} = k_{wa} = 10^{-9} \) m/s, \( k_{w} = k_{w} = 10^{-10} \) m/s, \( S_r = 80\% \), \( n_0 = 50\% \), \( T = 293.6 \) K, \( m'_{1w} = -5 \times 10^{-5} \) kPa\(^{-1}\), \( m'_{2w} = -2 \times 10^{-4} \) kPa\(^{-1}\), \( m''_{1w} = -2.5 \times 10^{-4} \) kPa\(^{-1}\). \( m''_{2w} = -2 \times 10^{-4} \) kPa\(^{-1}\), \( m''_{3w} = -1 \times 10^{-4} \) kPa\(^{-1}\).

In this analysis, the consolidation regularity of the vertical drain in unsaturated soils is studied against the ratio of air-water permeability \( k_{aw}/k_{wa} \) (\( k_{wa} \) stands for \( k_{aw} \) and \( k_{wa} \), \( k_{aw} \) represents \( k_{aw} \) and \( k_{wa} \)), the ratio of radial-vertical permeability \( k_r/k_z \) (\( k_r \) includes \( k_{aw} \) and \( k_{wa} \), \( k_z \) represents \( k_{az} \) and \( k_{wz} \)) and depth. The initial excess pore pressures are computed by the method from Fredlund and Hasan [2]. In addition, the study is verified and investigated all at \( r = 3 \) m and \( z = 5 \) m, and parameters in the radial and vertical directions are seemed as equal (i.e., \( k_{aw} = k_{wa} = 10^{-9} \) m/s, \( k_{w} = k_{w} = 10^{-10} \) m/s) if they are not involved in the case study. Specifically, the parameters (i.e., \( m'_{1w}, m'_{2w}, m''_{1w}, m''_{2w} \), and \( k_{aw} \) and \( k_{wa} \)) in Ho and Fatahi [7] should be consistent in this paper in the comparison.
Figure 2 depicts the comparisons between the results from current solutions and those from Ho and Fatahi [7]. It is obvious that the results are identical from figure 2 and it can be concluded that the current analytical solution is correct and applicable. The dissipation process of excess pore pressures is illustrated under \( k_r/k_w = 0.1, 1, 10, 100 \) and 1000. It can be observed from figure 2(b) that the dissipation curves of excess pore-water pressure are reverse single S type when \( k_r/k_w < 1 \), whereas the curves are reverse double S type when \( k_r/k_w > 1 \). It is because that the dissipation type is more like that in saturated soils when \( k_r/k_w < 1 \), and the reduction of excess pore-air pressure will cause little effect on the excess pore-water pressure. However, when \( k_r/k_w > 1 \), the dissipation of excess pore-water pressure is induced by the recharge of air phase during the first stage, so the dissipation of excess pore-water pressure is dramatically influenced by the reduction of excess pore-air pressure in the first stage. When excess pore-air pressure dissipates to termination, excess pore-water pressure dissipates due to the discharge of water phase in the second stage.

The effects of the ratio \( k_r/k_z \) on the variation of the dissipation process are analyzed in figure 3 at \( k_r/k_z = 0.1, 1, 5, 10 \) and 20. In this part, \( k_w = 10^{-9} \text{ m/s} \) and \( k_w = 10^{-10} \text{ m/s} \) are applied. As obviously presented in figure 4, different values of \( k_r/k_z \) almost indicate that different dissipation curves share the same shape and can be translated reciprocally in the horizontal direction, and \( k_r/k_z \) at bigger values move the curves to left. In other words, bigger values of \( k_r/k_z \) can accelerate the dissipation process of excess pore pressures. It should be noted that different values of \( k_r/k_z \) will not change the relative size of permeability coefficients \( k_r \) and \( k_w \), so the increasing \( k_r/k_z \) advances any dissipation state during the whole dissipation process.
Fig. 4 Dissipation of (a) excess pore-air pressure and (b) pore-water pressure along the vertical direction with time.

The variations of excess pore pressures along the longitudinal direction at different time steps are presented in figure 4, and the research is conducted at \( t = 1 \times 10^3, 2 \times 10^3, 1 \times 10^4, 2 \times 10^4, 5 \times 10^4, 1 \times 10^5 \) and \( 2 \times 10^5 \) s for excess pore-air pressure while \( t = 1 \times 10^3, 1 \times 10^4, 5 \times 10^4, 1 \times 10^5, 2 \times 10^5, 1 \times 10^6, 1 \times 10^7, 5 \times 10^7 \) and \( 1 \times 10^8 \) s for excess pore-water pressure. It can be found in Fig. 4(a) that excess pore-air pressure in the area closer to the permeable boundaries (i.e., the top and bottom surfaces) dissipates much quicker and is equal to zero at the permeable boundaries, which is reasonable and consistent with practical engineering. In special, there are vertical parts in dissipation curves at \( t < 1 \times 10^4 \) s for excess pore-air pressure according to figure 4(a). It is because that the radial dissipation of excess pore-air pressure around the middle part (\( z = 0.5H \)) takes the main part, thus excess pore-air pressure along the vertical direction changes uniformly when it is early enough near the middle part. Later, the phenomenon blurs when \( t > 1 \times 10^4 \) s for excess pore-air pressure. As shown in figure 4(b), the dissipation curves for excess pore-water pressure can be divided into two similar independent parts, respectively (i.e., the first part at \( t < 2 \times 10^5 \) s and the second part at \( t > 2 \times 10^5 \) s). In fact, the phenomenon in figure 4(b) shares the same explanation with that in figure 2(b). In the first part, excess pore-water pressure dissipates because of the reduction of excess pore-air pressure, so the distribution type is similar with that of figure 4(a). When excess pore-air pressure dissipates to termination at \( t = 2 \times 10^5 \) s, the filling process is involved to adjust the nonlinear parts to be vertical again. After that, excess pore-water pressure dissipates due to the discharge of water phase in the second part.

Fig. 5 Normalized settlement under different (a) \( k_a/k_w \) and (b) \( k_r/k_z \) with time.

Figure 5 illustrates the impact of different values of \( k_a/k_w \) and \( k_r/k_z \) on normalized settlements, and results of the normalized settlement with \( k_a/k_w \) from Ho and Fatahi [7] are also applied to verify the correctness of the results obtained in this paper. As shown in figure 5 (a), the results are identical, and the vertical drain foundation subsides faster at bigger values of \( k_a/k_w \) (only with the variation of \( k_a \)), and the normalized settlement curves converge at the second stage (at around \( t > 10^7 \) s). It can also be found from figure 5(b) that the bigger value of \( k_r/k_z \) is, the quicker the vertical drain foundation settles.
Specifically, the whole process for settlement accelerates under bigger $k_{aw}$ and $k_{wr}$, which is different from that of figure 5(a). It can be analyzed from figure 5(a) that the types (i.e., reverse single and double S types) of settlement curves and explanation are similar with that of figure 2(b).

5. Conclusion Remarks

This paper presents a simpler while reliable analytical procedure to derive the analytical solution for the unsaturated consolidation of vertical drain foundation. Starting from the decoupling process to transfer the governing equations into a set of homogeneous PDEs, and based on the separation variable method, the connections of excess pore-air pressure, excess pore-water pressure and normalized settlement are established with $r, z, t$ in explicit forms. Afterwards, the final analytical solution can be obtained by bringing in the initial and boundary conditions. Finally, the proposed solution is verified reliable and efficient to solve the consolidation problems of vertical drain foundation by the analytical analysis and calculation process. The influence of different $k_{aw}/k_{wa}$, $k_{aw}/k_{wa}$ and depth on the consolidation behavior is examined. In addition, the solutions can degenerate into the solution for the saturated consolidation of vertical drain so that the derivation process can be used to solve consolidation problems with more complex boundaries in unsaturated soils.

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Appendix A. Derivation of $\Phi_1$ and $\Phi_2$

By adopting the two arbitrary constants $q_1$ and $q_2$, Eqs. (5) and (6) can be summarized as:

$$\begin{bmatrix} Q_{q_1} \\ Q_{q_2} \end{bmatrix} = \begin{bmatrix} A_{aw}q_1 + W_{aw}q_2 \\ A_{aw}q_1 + W_{aw}q_2 \end{bmatrix} \begin{bmatrix} A_{aw}q_1 + W_{aw}q_2 \\ A_{aw}q_1 + W_{aw}q_2 \end{bmatrix}$$  \hspace{1cm} (A1)

In order to give a true statement to the Eq. (A1), the $Q$ must satisfy the following condition:

$$\begin{bmatrix} (A_{sw} - Q_1)(W_{rw} - Q_1) - A_{sw}W_{rw} \\ (A_{sw} - Q_2)(W_{rw} - Q_2) - A_{sw}W_{rw} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (A2)

The roots $Q_1$ and $Q_2$ of the quadratic equation Eqs. (A2) can be expressed as:

$$\begin{bmatrix} Q_{1,2} \\ Q_{1,2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_{sw} + W_{rw} \pm \left[ (A_{sw} - W_{rw})^2 + 4A_{sw}W_{rw} \right]^{1/2} \\ A_{sw} + W_{rw} \pm \left[ (A_{sw} - W_{rw})^2 + 4A_{sw}W_{rw} \right]^{1/2} \end{bmatrix}$$  \hspace{1cm} (A3)

When $Q = Q_1$, the solutions of $q_1$ and $q_2$ in Eq. (A2) are $q_{11}$ and $q_{21}$, respectively. Similarly, $q_1$ and $q_2$ in Eq. (A3) are $q_{12}$ and $q_{22}$ when $Q = Q_2$, respectively.

$$\begin{bmatrix} q_{12} \\ q_{21} \end{bmatrix} = \begin{bmatrix} W_{rw}/(Q_{2w} - A_{aw}) \\ A_{aw}/(Q_{2w} - W_{rw}) \end{bmatrix} = \begin{bmatrix} W_{rw}/(Q_{2w} - A_{aw}) \\ A_{aw}/(Q_{2w} - W_{rw}) \end{bmatrix}$$  \hspace{1cm} (A4)

Generally, it is reasonably assumed that $q_{11} = q_{22} = 1$, so Eqs. (7) and (8) can be obtained by substituting Eqs. (5) and (6) into Eqs. (A1) and (A3), respectively.

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