Chiral anomaly for local boundary conditions

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Abstract

It is known that in the zeta function regularization and in the Fujikawa method chiral anomaly is defined through a coefficient in the heat kernel expansion for the Dirac operator. In this paper we apply the heat kernel methods to calculate boundary contributions to the chiral anomaly for local (bag) boundary conditions. As a by-product some new results on the heat trace asymptotics are also obtained.

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1 Introduction

Chiral anomaly, which was discovered more than 30 years ago [1], still plays an important role in particle physics. On smooth manifolds without boundaries many successful approaches to the anomalies exist1. Modern developments in theoretical physics require anomaly calculations on branes and domain walls (see, e.g., [3]). Taking into account also more traditional applications as the bag model of hadrons [7], we see that understanding the chiral anomaly in the presence of boundaries or singularities is an important task.

1The reader may consult, for example, ref. [2] or the original papers [3][4][5].
In the case of non-trivial background fields, and especially in the presence of boundaries or singularities, the heat kernel technique seems to be the most adequate one for analysing the one-loop effects (see \[8\] for a recent review). The heat kernel approach to the anomalies is essentially equivalent to the Fujikawa approach \[4\] and to the calculations based on the finite-mode regularization \[9\], but it can be more easily extended to complicated geometries.

In the present paper we consider an euclidean version of the bag boundary conditions \[7\] (see also \[10\]). Although these are the most simple and physically most natural boundary conditions, chiral anomaly for these conditions has not been calculated so far\(^2\). This can be explained by the fact that the calculation is indeed rather involved. We use the zeta function regularization and our approach to the anomaly follows closely the paper \[11\]. Mathematical foundations for analysing spectral geometry of the Dirac operator with local boundary conditions were developed in \[12\]. A simple overview of spectral properties of Dirac operator on manifolds with boundary can be found in \[13\].

To calculate the anomaly we employ the following strategy. First we relate the anomaly to a heat kernel coefficient for the square of the Dirac operator. Then we generalise the problem and consider the heat kernel for an arbitrary operator of the Laplace type with mixed boundary conditions and with a matrix valued smearing function. This generalisation allows us to use the heavy machinery of the heat kernel expansion for such operators. In particular, important information follows from two simple examples of boundary value problems in one and two dimensions. In this way we are able to calculate first five heat kernel coefficients. Chiral anomaly is then obtained simply by substituting explicit expressions for the connection, the potential, and other quantities in terms of the background vector and axial vector fields.

In the next section we remind some basic formulae regarding the zeta function regularization, chiral anomaly, and local (bag) boundary conditions. Sec. 3 is devoted to the relationship between Dirac and Laplace operators. The heat kernel coefficients are calculated in sec. 4, which is the main technical part of this paper. In sec. 5 we return to the Dirac operator and finally calculate boundary contributions to chiral anomaly in two and four dimensions. Our results and their possible extensions are discussed in sec. 6. Appendix \[A\] contains the heat kernel coefficients for mixed boundary conditions with a scalar smearing function. Appendix \[B\] gives details of the particular case calculations used in sec. 4.

\(^2\)There exist calculations of global chiral anomaly (i.e. for a position-independent chiral transformation parameter) for some other types of boundary operators, and also calculation of the scale anomaly for bag boundary conditions. Literature on these topics is rather large, so that for uniformity we quote nobody here. A literature survey can be found in \[8\].
2 Boundary conditions and chiral anomaly

Let us consider an \( n \)-dimensional Riemannian manifold \( M \) with boundary \( \partial M \). Dirac \( \gamma \)-matrices satisfy the Clifford commutation relation

\[
\{ \gamma^\mu, \gamma^\nu \} = -2g^{\mu\nu}.
\] (1)

The \( \gamma \)-matrices defined in this way are anti-hermitian, \( \gamma^\mu \dagger = -\gamma^\mu \). We also need the chirality matrix which will be denoted \( \gamma^5 \) independently of the dimension of \( M \). As usual, \( \gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5 \) and \( \gamma^5 \dagger = \gamma^5 \). We assume that \( n \) is even. We fix the sign of \( \gamma^5 \) by choosing

\[
\gamma^5 = \frac{i^{n(n+1)/2}}{n!} \epsilon^{\mu\nu\ldots\rho} \gamma_\mu \gamma_\nu \cdots \gamma_\rho.
\] (2)

Consider the Dirac operator

\[
\hat{D} = \gamma^\mu \left( \partial_\mu + V_\mu + iA_\mu \gamma_5 - \frac{1}{8} [\gamma_\rho, [\gamma_\sigma, \sigma^\mu_{\rho\sigma}]] \right)
\] (3)

in external vector \( V_\mu \) and axial vector \( A_\mu \) fields. We suppose that \( V_\mu \) and \( A_\mu \) are anti-hermitian matrices in the space of some representation of the gauge group. \( \sigma^\mu_{\rho\sigma} \) is the spin-connection\(^3\).

The Dirac operator transforms covariantly under infinitesimal local gauge transformations:

\[
\delta_\lambda A_\mu = [A_\mu, \lambda]
\]
\[
\delta_\lambda V_\mu = \partial_\mu \lambda + [V_\mu, \lambda]
\]
\[
\hat{D} \rightarrow \hat{D} + [\hat{D}, \lambda]
\] (4)

and under infinitesimal local chiral transformations:

\[
\tilde{\delta}_\varphi A_\mu = \partial_\mu \varphi + [V_\mu, \varphi],
\]
\[
\tilde{\delta}_\varphi V_\mu = -[A_\mu, \varphi],
\]
\[
\hat{D} \rightarrow \hat{D} + i\{\hat{D}, \gamma^5 \varphi\}.
\] (5)

The parameters \( \lambda \) and \( \varphi \) are anti-hermitian matrices.

We adopt the zeta-function regularization\(^4\) and write the effective action for the Dirac fermions as

\[
W = -\ln \det \hat{D} = -\frac{1}{2} \ln \det \hat{D}^2 = \frac{1}{2} \zeta'(0) + \frac{1}{2} \ln(\mu^2)\zeta(0),
\] (6)

\(^3\)The spin-connection must be included even on a flat manifold if the coordinates are not Cartesian.

\(^4\)Our approach to the effective action for fermions is close to that in the papers [11, 14]. We refer to these works for a more detailed derivation of chiral anomaly (with somewhat more accurate treatment of the zero mode problem).
where

\[ \zeta(s) = \text{Tr}(\hat{D}^{-2s}) , \]  

(7)

prime denotes differentiation with respect to \( s \), and Tr is the functional trace.

It is easy to show that the effective action (6) is gauge invariant, \( \delta \lambda W = 0 \), and that the variation of \( W \) under an infinitesimal chiral transformation reads

\[ \mathcal{A} := \tilde{\delta}_\varphi W = -2i\text{Tr}(\gamma^5 \varphi \hat{D}^{-2s})|_{s=0} . \]  

(8)

Let us define an integrated heat kernel for a second order elliptic partial differential operator \( L \) by the equation:

\[ K(Q,L,t) := \text{Tr} \left( Q \exp(-tL) \right) , \]  

(9)

where \( Q(x) \) is a matrix valued function. For the boundary conditions we consider in this paper (see eq. (23) below) there exists an asymptotic expansion \([19]\) as \( t \to 0 \):

\[ K(Q,L,t) \simeq \sum_{k=0}^{\infty} a_k(Q,L) t^{(k-n)/2} . \]  

(10)

The heat kernel is related to the zeta function by the Mellin transformation:

\[ \text{Tr}(\gamma^5 \varphi \hat{D}^{-2s}) = \Gamma(s)^{-1} \int_0^\infty dt \, t^{s-1} K(\gamma^5 \varphi, \hat{D}^2, t) . \]  

(11)

In particular\(^5\),

\[ \mathcal{A} = -2ia_n(\gamma^5 \varphi, \hat{D}^2) . \]  

(12)

The same expression for the anomaly follows also from the Fujikawa approach\([4]\).

We shall need some basic notions from differential geometry. Let \( R_{\mu\nu\rho\sigma} \) be the Riemann tensor, and let \( R_{\mu\nu} = R^\sigma_{\mu\nu\sigma} \) be the Ricci tensor. With our sign convention the scalar curvature \( R = R^\mu_\mu \) is +2 on the unit sphere \( S^2 \). Curvature does not play any important role in our calculations. However, we shall see below that curved space offers no complications in our approach compared to the flat case.

If the manifold \( M \) has a boundary, boundary conditions should be imposed on the spinor field \( \psi \). We need several basic definitions regarding differential geometry of manifolds with boundary. Let \( \{ e_j \} \), \( j = 1, \ldots, n \) be a local orthonormal frame for the tangent space to the manifold and let on the boundary \( e_n \) be an inward pointing normal vector. Then \( \{ e_a \} \), \( a = 1, \ldots, n-1 \) can be identified with a local orthonormal frame for the tangent space to the boundary. The frame \( \{ e_j \} \) will be used to transform curved (world) indices \( \mu, \nu, \ldots, \sigma \) to “flat” indices and back. For example, for a vector \( v_\mu \) this transformation reads: \( v_j = e^a_j v_\mu, \)

\(^5\)A rather formal way to derive this equation consists in integration of the asymptotic expansion \([10]\).
\[ v_a = e_a^\mu v_{\mu}, \quad v_n = e_n^\mu v_{\mu}. \] In Euclidean space there is no distinction between flat upper and lower indices.

The extrinsic curvature is defined by the equation
\[ L_{ab} = \Gamma_{ab}^n, \tag{13} \]
where \( \Gamma \) is the Christoffel symbol. For example, on the unit sphere \( S^{n-1} \) which bounds the unit ball in \( \mathbb{R}^n \) the extrinsic curvature is \( L_{ab} = \delta_{ab} \).

We impose local\(^6\) boundary conditions:
\[ \Pi_- \psi|_{\partial M} = 0, \quad \Pi_- = \frac{1}{2} (1 - \gamma_5 \gamma_n), \tag{14} \]
which are nothing else than a Euclidean version of the MIT bag boundary conditions \( [7] \). For these boundary conditions \( \Pi_\perp^\dagger = \Pi_- \), and the normal component of the fermion current \( \psi^\dagger \gamma_n \psi \) vanishes on the boundary.

An important comment on chiral transformations of the boundary conditions \( [14] \) is in order. Finite version of the infinitesimal transformation \( (5) \) reads:
\[ \hat{D} \rightarrow \hat{D} = e^{i\psi \gamma_5} \hat{D} e^{i\psi \gamma_5}. \tag{15} \]
This relation yields the following transformation law for the boundary projector:
\[ \Pi_- \rightarrow \Pi_-^{[\phi]} = e^{-i\phi \gamma_n} \Pi_- e^{i\phi \gamma_n} = \frac{1}{2} (1 - \gamma_5 \gamma_n e^{2i\phi \gamma_5}), \tag{16} \]
so that the boundary condition
\[ \Pi_-^{[\phi]} \psi|_{\partial M} = 0 \tag{17} \]
remains consistent with \( [14] \) and \( [15] \). Eq. \( (17) \) represents an Euclidean version \( [16] \) of chiral bag boundary conditions \( [17] \). The boundary conditions \( (17) \) are considerably more complicated than \( (14) \). Even such fundamental property of \( (17) \) as the strong ellipticity (which ensures, for example, existence of only simple poles of the zeta function) has been established only recently \( [18] \). Fortunately, as we stay at the level of linear perturbations, the condition \( (14) \) is enough for our purposes. However, already the Wess-Zumino consistency conditions \( [3] \) which imply two consequent chiral transformations require a more general setting of the chiral bag \( (17) \).

3 Dirac and Laplace operators

To calculate the anomaly \( (12) \) it is convenient to consider a more general problem of calculation of the coefficients \( a_k(Q, L) \) (see eq. \( (11) \)) for general matrix valued

\(^6\)Locality means that the projector \( \Pi_- \) acts at each point of the boundary independently. An example of non-local boundary conditions can be found in \([13]\).
function $Q$ and general operator $L$ of Laplace type. Any operator of Laplace type can be expanded locally as

$$L = -(g^{\mu\nu} \partial_\mu \partial_\nu + a^\sigma \partial_\sigma + b), \quad (18)$$

where $a$ and $b$ are some matrix valued functions. One can always introduce a connection $\omega_\mu$ and another matrix valued function $E$ so that $L$ takes the form:

$$L = -(g^{\mu\nu} \nabla_\mu \nabla_\nu + E) \quad (19)$$

Here $\nabla_\mu$ is a sum of covariant Riemannian derivative with respect to metric $g_{\mu\nu}$ and connection $\omega_\mu$. One can, of course, express $E$ and $\omega$ in terms of $a^\mu$, $b$ and $g_{\mu\nu}$:

$$\omega_\mu = \frac{1}{2} g_{\mu\nu}(a^\nu + g^{\rho\sigma} \Gamma^\nu_{\rho\sigma}), \quad (20)$$

$$E = b - g^{\mu\nu}(\partial_\nu \omega_\mu + \omega_\mu \omega_\nu - \omega_\rho \Gamma^\rho_{\mu\nu}) \quad (21)$$

For the future use we introduce also the field strength for $\omega$:

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]. \quad (22)$$

The connection $\omega_\mu$ will be also used to construct covariant derivatives. It will be convenient to work with flat indices (denoted by Latin letters) which we have introduced in the previous section. The subscript $;i...jk$ will be used to denote repeated covariant derivatives with the connection $\omega$ and the Christoffel connection on $M$. The subscript $;a...bc$ will denote repeated covariant derivatives containing $\omega$ and the Christoffel connection on the boundary. Difference between these two covariant derivatives is measured by the extrinsic curvature $\Omega^2$. For example, $E_{;ab} = E_{;ab} - L_{ab}E_n$.

Let us now turn to boundary conditions. We assume given two complementary projectors $\Pi_\pm$, $\Pi_- + \Pi_+ = I$ and define mixed boundary conditions by the relations

$$\Pi_- \psi|_{\partial M} = 0, \quad (\nabla_n + S) \Pi_+ \psi|_{\partial M} = 0, \quad (23)$$

where $S$ is a matrix valued function on the boundary. In other words, the components $\Pi_- \psi$ satisfy Dirichlet boundary conditions, and $\Pi_+ \psi$ satisfy Robin (modified Neumann) ones.

It is convenient to define

$$\chi = \Pi_+ - \Pi_- \quad (24)$$

Now we have to calculate the geometric quantities introduced above for generic $L$ in the particular case $L = \hat{D}^2$. After lengthy but straightforward calculation
one obtains from (3), (20) – (22) (see also [22] for the abelian case):

$$\omega_\mu = V_\mu - \frac{i}{2} [\gamma_\mu, \gamma_\nu] A^\nu \gamma_5 - \frac{1}{8} [\gamma_\rho, \gamma_\sigma] \sigma^{\rho, \sigma}_{\mu},$$  \hspace{1cm} (25)

$$E = -\frac{1}{2} \gamma^\mu \gamma^\nu V_{\mu\nu} + (n - 3) \gamma^\mu \gamma^\nu A_\mu A_\nu - A_\mu A^\mu - \frac{1}{4} R + i D_\mu A^\mu \gamma_5,$$  \hspace{1cm} (26)

$$\Omega_{\mu\nu} = i \gamma^\kappa (D_\mu A_\kappa) \gamma_\nu \gamma_5 - i \gamma^\kappa (D_\nu A_\kappa) \gamma_\mu \gamma_5 + i A_{\mu\nu} \gamma_5 - [A_\mu, A_\nu] + V_{\mu\nu}$$

$$+ \frac{1}{4} \gamma^\kappa \gamma^\tau R_{\kappa\tau\mu\nu} - [A_\mu, A_\kappa] \gamma^\kappa \gamma_\nu + [A_\kappa, A_\nu] \gamma^\kappa \gamma_\mu -$$

$$- \gamma^\kappa A_\kappa \gamma_\mu \gamma_\tau A_\tau \gamma_\nu + \gamma^\kappa A_\kappa \gamma_\gamma_\mu \gamma_\tau A_\tau \gamma_\nu$$  \hspace{1cm} (27)

with the notations $$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu],$$  $$A_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu, D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho + [V_\mu, A_\nu].$$ Note, that the gauge covariant derivative $$D_\mu$$ differs from $$\nabla_\mu$$ defined above.

Since $$\hat{D}$$ is a first order differential operator it was enough to fix the boundary conditions (14) on a half of the components. To proceed with a second order operator $$L = \hat{D}^2$$ we need boundary conditions on the remaining components as well. They are defined by the consistency condition:

$$\Pi_- \hat{D} \psi |_{\partial \mathcal{M}} = 0,$$  \hspace{1cm} (28)

which is equivalent to the second (Robin) boundary condition in (23) with

$$S = -\frac{1}{2} \Pi_+ L_{aa}. $$  \hspace{1cm} (29)

## 4 Asymptotic expansion of the heat kernel

### 4.1 General strategy

In this section we study the short $$t$$ asymptotics (10) for an arbitrary operator $$L$$ of Laplace type. Two particular cases of the expansion (10) are known. The heat kernel coefficients $$a_k, k = 0, 1, 2, 3, 4, 5$$ for a scalar $$Q = f I$$ (where $$f$$ is a function and $$I$$ is the unit operator) are presented in Appendix A. The case of arbitrary $$Q$$ but pure Dirichlet or Neumann boundary conditions (i.e. when either $$\Pi_+$$ or $$\Pi_-$$ is zero) was studied in [23]. Here we need a combination of these two cases. Namely, we are interested in mixed boundary conditions and a matrix valued $$Q$$. According to the general theory [19] the coefficients $$a_k(Q, L)$$ are locally computable. This means that each $$a_k(Q, L)$$ can be represented as a sum of volume and boundary integrals of local invariants constructed from $$Q, \Omega, E$$, the curvature tensor, and their derivatives. Boundary invariants may also include $$S, L_{ab}$$ and $$\chi$$. Total mass dimension of such invariants should be $$k$$ for the volume terms and $$k - 1$$ for the boundary ones.
The following property of the heat kernel coefficients \cite{19} will be useful in the calculations. Let us define a shifted operator

\[ L_\epsilon = L - \epsilon Q. \]  

(30)

Then

\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \text{Tr}(\exp(-tL_\epsilon)) = t\text{Tr}(Q \exp(-tL)). \]  

(31)

By expanding both sides of this equation in a power series of \( t \) one obtains:

\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} a_{k+2}(1, L_\epsilon) = a_k(Q, L). \]  

(32)

All geometric quantities (metric, effective connection, boundary conditions, etc.) corresponding to \( L_\epsilon \) are the same as for the unperturbed operator \( L \) except for the potential \( E \) which receives a shift,

\[ E_\epsilon = E + \epsilon Q. \]  

(33)

Therefore, variation of the heat kernel coefficients w.r.t. \( \epsilon \) is equivalent to the variation of \( E_\epsilon \). Equation (32) together with the heat kernel coefficients for scalar smearing function presented in Appendix A allow to calculate the coefficients \( a_k(Q, L) \) for \( k = 0, 1, 2, 3 \):

\[ a_0(Q, L) = (4\pi)^{-n/2} \int_M d^n x \sqrt{g} \text{tr}(Q). \]  

(34)

\[ a_1(Q, L) = \frac{1}{4} (4\pi)^{-(n-1)/2} \int_{\partial M} d^{n-1} x \sqrt{h} \text{tr}(\chi Q). \]  

(35)

\[ a_2(Q, L) = \frac{1}{6} (4\pi)^{-n/2} \left\{ \int_M d^n x \sqrt{g} \text{tr} \left( 6QE + QR \right) \right. \]

\[ + \left. \int_{\partial M} d^{n-1} x \sqrt{h} \text{tr} \left( 2QL_{aa} + 12QS + 3\chi Q;_n \right) \right\}. \]  

(36)

\[ a_3(Q, L) = \frac{1}{384} (4\pi)^{-(n-1)/2} \int_{\partial M} d^{n-1} x \sqrt{h} \text{tr} \left\{ Q(-24E + 24\chi E \chi \right. \]

\[ + 48\chi E + 48E \chi - 12\chi_a \chi_a + 12\chi_{aa} - 6\chi_a \chi_a \chi + 16\chi R \]

\[ + 8\chi R_{aan} + 192S^2 + 6L_{aa}S + (3 + 10\chi) L_{aa} L_{bb} \]

\[ + (6 - 4\chi) L_{ab} L_{ab} + Q;_n (96S + 192S^2) + 24\chi Q;_nn \}. \]  

(37)

Since the boundary terms in \( a_6(1, L) \) are not known, we have to adopt a different strategy to calculate \( a_4(Q, L) \). The volume part of \( a_4(Q, L) \) is already known \cite{24}, so that we have to define the boundary contributions only. First we have to write down all possible local boundary invariants of dimension 3 with arbitrary coefficients. Boundary invariants are traces over “internal” indices of
local polynomials constructed from \( R, E, \chi, \Omega, L, S \) and \( Q \) and from their derivatives. All \( a, b, c, \ldots \) indices must be contracted in pairs. Note, that the normal index \( n \) must not be contracted. This reflects specific symmetry of the spectral problem in the presence of a boundary which selects direction of the normal. Next, we have to use various properties of the heat kernel expansion to define these constants. In the particular case \( Q = If \) we should restore the known result \([61]\). Of course, for a matrix valued \( Q \) there are considerably more different invariants than in the scalar case since one has to take into account non-commutativity of \( Q \) with \( E, \chi, \Omega \) etc. Therefore, for example, the invariants \( \text{tr} (QE) \) and \( \text{tr} (Q\chi E\chi) \) are different although they coincide in the limit \( Q = If \). To restrict the number of invariants (and the computational complexity) from now on we consider the case

\[ S = 0, \quad L_{ab} = 0 \]  \hspace{1cm} (38)

only.

Since the Riemann tensor does not have internal (spinorial or gauge) indices, it commutes with \( Q \) and \( \chi \). Therefore, the particular case \( Q = If \) allows to restore all curvature dependent boundary terms in \( a_4 \):

\[
a_4(Q, L)[\text{boundary}] = \frac{1}{360} (4\pi)^{-n/2} \int_{\partial M} d^{n-1}x \sqrt{h} \text{tr} (Q(12R_{n} + 30\chi R_{n}) \\
+ 30Q_{n} \chi R + O(R^0)) . \tag{39}
\]

To control the invariants containing \( E \) and \( \Omega \) we use the property \([20, 19]\) that the constants appearing in front of all invariants depend on the dimension of the manifold only through an overall factor \((4\pi)^{-n/2}\). This property makes it possible to use low dimensional particular case calculations to define the heat kernel coefficients in arbitrary dimension.

### 4.2 Particular case calculations

To define the terms in \( a_4 \) which depend on \( \chi, E \) and \( \Omega \) it is enough to consider the case of the simplest geometry \( M = \mathbb{R}_+ \times \mathbb{R}^{n-1} \) with flat metric. It is important that the mass dimension of the boundary integrand in \( a_4(Q, L) \) is three. Therefore, terms containing both \( E \) and \( \Omega \) cannot appear. For this reason, \( E \) and \( \Omega \) terms can be considered separately. As well, \( \Omega_{ab} \) cannot enter the invariants since there is no rank two antisymmetric tensor of dimension one. Consequently, we may restrict ourselves to the case

\[ \omega_n = 0 . \] \hspace{1cm} (40)

To simplify the calculations we also impose

\[ \chi = \text{const.} \] \hspace{1cm} (41)
Then $\chi a$ will be represented by a commutator $[\omega_a, \chi]$, and $\Omega an = -\partial_n \omega_a$. These two invariants are independent on the boundary.

We shall need a bi-local heat kernel $K(x, z; t)$ which is defined as a solution of the heat equation

$$(\partial_t + L)K(x, z; t) = 0$$

with the initial condition

$$K(x, z; 0) = \delta(x, z).$$

Because of the restrictions (38) and (40) the boundary conditions simplify to

$$\Pi_- K(x, z; t)|_{\partial M} = 0, \quad \partial_n \Pi_+ K(x, z; t)|_{\partial M} = 0.$$  \hspace{1cm} (44)

This bi-local kernel is related to the “localised” one (cf. (9)) by the equation:

$$K(Q, L, t) = \int_M d^n x \sqrt{\text{det} g} (Q(x)K(x, x; t)).$$ \hspace{1cm} (45)

We stress, that $K(x, z; t)$ is a distribution.

The fundamental solution of “free” heat conduction equation

$$(\partial_t - \partial^2_x) K_0(x, z; t) = 0$$

on $M = \mathbb{R}^n$ is well known:

$$K_0(x^a, x^n, y^a, y^n; t) = (4\pi t)^{-n/2} \exp \left( -\frac{\sum_{a=1}^{n-1} (x^a - y^a)^2 + (x^n - y^n)^2}{4t} \right).$$ \hspace{1cm} (47)

From this kernel one can construct a solution of (46) on $M = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ which satisfies the conditions (44) at $x^n = 0$:

$$K_\chi(x, z; t) = K_0(x^a, x^n, z^a, z^n; t) + \chi K_0(x^a, x^n, z^a, -z^n; t).$$ \hspace{1cm} (48)

The operators which we consider in this section can be represented in the following form:

$$L = -\partial^2_x - P,$$

where $P$ is a first or zeroth order differential operator.

It is easy to show that the kernel $K(x, z; t)$ defined by the equation

$$K(x, z; t) = K_\chi(x, z; t) + \int_0^t d\tau \int_M dy K_\chi(x, y; t - \tau) P(y)K(y, z; \tau)$$ \hspace{1cm} (50)

satisfies both the full heat equation (42) and the boundary conditions (44).
The equation (50) admits a solution\(^7\) in terms of power series in \(P\):

\[
K(x, z; t) = K_\chi(x, z; t) + \sum_{p=1}^{\infty} \int_0^t d\tau_p \int_0^{\tau_p} d\tau_{p-1} \ldots \int_0^{\tau_2} d\tau_1 \int_M dy_p \ldots \int_M dy_1 
\times K_\chi(x, y_p; t - \tau_p) P(y_p) K_\chi(y_p, y_{p-1}; \tau_p - \tau_{p-1}) \ldots P(y_1) K_\chi(y_1, z; \tau_1).
\] (51)

A remarkable feature of (51) is that only a finite number of terms contribute to each \(a_k(Q, L)\) for fixed \(k\). This equation will be used to calculate the heat kernel expansion for two particular choices of \(L\) (see below).

We start with a one-dimensional example

\[
L_1 = -\partial_x^2 - E(x),
\] (52)

so that \(P_1(y) = E(y)\). \(E\) is an arbitrary matrix valued function on \(M = \mathbb{R}_+\).

Details of the calculation can be found in Appendix \textbf{B}. To the first order in \(E\) we have:

\[
K(Q, L_1, t) = \frac{t^2}{(4\pi)^2} \int_0^\infty dx \text{tr} \left( Q E + t \frac{1}{6} Q E_{;\mu} \right) 
+ \frac{t}{(4\pi)^0} \text{tr} Q \left( -\frac{1}{16} (E - \chi E \chi) + \frac{1}{8} (\chi E + E \chi) \right)_{x=0} 
+ \frac{t^2}{(4\pi)^2} \text{tr} \left[ Q \left( \frac{1}{12} (E_{;n} + \chi E_{;n} \chi) + \frac{1}{4} (\chi E_{;n} + E_{;n} \chi) \right) 
+ Q_{;n} \left( -\frac{1}{12} (E - \chi E \chi) + \frac{1}{4} (\chi E + E \chi) \right) \right]_{x=0}
+ O(t^3).\] (53)

The first and the second lines of (53) can serve as a consistency check of our calculations (cf. [33, 37]). The rest of (53) defines uniquely all terms in the boundary part of \(a_4(Q, L)\) containing \(E\) subject to the restrictions (38). Indeed, no other invariants of dimension 3 can appear. For example, \(\text{tr} (Q E_{;\alpha})\) is not allowed since it contains a non-contracted tangential index. \(\chi_{;n}\) cannot appear since \(\chi\) is defined on the boundary only and, hence, may be differentiated only tangentially.

In order to define the term in \(a_4(Q, L)\) containing \(\Omega\) and/or tangential derivatives \(\chi\) we consider a two dimensional example \(M = \mathbb{R} \times \mathbb{R}_+\) and \(L_2 = -(\partial_\alpha + \omega_\alpha)^2\). In addition to (40) we also suppose \(\partial_\alpha \omega_\alpha = 0\). Clearly, this condition does not exclude any relevant invariant. Then \(P_2(y) = 2\omega_\alpha (y^n) \partial_\alpha + \omega_\alpha^2 (y^n)\). This time

\(^7\)On manifolds without boundary a similar expansion served as a starting point for the covariant perturbation theory of ref. [25]. On manifolds with boundary similar perturbation series were used [26] to evaluate dependence of the heat kernel on the boundary conditions.
we need the first and second order terms in the expansion (51). All necessary
technical tools can be found again in Appendix B. The result reads:

\[ K(Q, L, t) = t^2 \int d^2 x \operatorname{tr} Q \left( -\frac{1}{32} \chi_{\alpha} \chi_{\alpha} + \frac{1}{32} \chi_{\alpha a} \right. \]
\[ \left. - \frac{1}{64} \chi_{\alpha a} \chi_{\alpha a} \right) \]
\[ + \frac{t}{4\pi} \int d^2 x \operatorname{tr} Q \frac{1}{12} \Omega_{\mu \nu} \Omega^{\mu \nu} \]
\[ + \frac{t}{4\pi} \int d^2 x \operatorname{tr} \left\{ Q \left( \frac{1}{20} \chi_{\alpha} \Omega_{\alpha n} + \frac{1}{30} \chi_{\alpha a} \Omega_{\alpha n} \chi \right. \right. \]
\[ \left. + \frac{1}{12} \Omega_{\alpha n} \chi_{\alpha} + \frac{1}{60} \chi_{\alpha a} \chi_{\alpha a} \right) \}
\[ + Q_{,\mu n} \left( -\frac{1}{20} \chi_{\alpha} \chi_{\alpha} + \frac{1}{12} \chi_{\alpha a} - \frac{1}{60} \chi_{\alpha a} \chi_{\alpha} \right) \} + O(t^3). \]

Next we collect individual contributions contained in (39), (53) and (54) to obtain:

\[ a_4(Q, L) = \frac{1}{360} (4\pi)^{-n/2} \left\{ \int_M d^n x \sqrt{g} \operatorname{tr} \left\{ Q \left( 60 E_{,\mu} + 60 R E + 180 E^2 \right. \right. \right. \]
\[ \left. + 30 \Omega_{\mu \nu} \Omega^{\mu \nu} + 12 R_{,\mu} + 5 R^2 - 2 R_{\mu \nu} R^{\mu \nu} + 2 R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \right) \}
\[ + \int_{\partial M} d^{n-1} x \sqrt{h} \operatorname{tr} \left\{ Q \left\{ 30 E_{,n} + 30 \chi E_{,n} + 90 \chi E_{,n} + 90 E_{,n} \chi \right. \right. \right. \]
\[ \left. + 18 \chi_{\alpha} \Omega_{\alpha n} + 12 \chi_{\alpha} \Omega_{\alpha n} \chi + 18 \Omega_{\alpha n} \chi_{\alpha} - 12 \chi_{\alpha} \Omega_{\alpha n} \chi_{\alpha} \right. \right. \right. \]
\[ \left. + 6 \left[ \chi_{\alpha a} \Omega_{\alpha n} \chi_{\alpha a} + 54 \chi_{\alpha a} \Omega_{\alpha n} \chi_{\alpha a} \right] + 30 [ \chi_{\alpha}, \Omega_{\alpha n} ] + 12 R_{,n} + 30 R_{,n} \right) \]
\[ + Q_{,\mu n} \left( -30 E + 30 \chi E + 90 \chi E + 90 E \chi \right. \right. \]
\[ \left. - 18 \chi_{\alpha a} \chi_{\alpha} + 30 \chi_{\alpha a} \chi_{\alpha} + 30 \chi R + 30 \chi Q_{,\mu n} \right) \} \}. \]

Let us remind that we have imposed a restriction on boundary conditions and on the extrinsic curvature of the boundary.

5 Calculation of the anomaly

5.1 Dimension two

The remaining part of the calculation is rather simple. One has to substitute (38) with (25) - (27), (29) and (14) in (8). The result reads:

\[ \mathcal{A} = -\frac{1}{\pi} \int_M d^2 x \sqrt{g} \operatorname{tr} \left( \varphi \left( \frac{1}{2} e^{\mu \nu} (V_{\mu \nu} + [A_\mu, A_\nu]) - D_\mu A_\mu \right) \right). \]
Surprisingly, there is no “genuine” boundary contribution here, and this is our main result in two dimensions. For constant $\varphi$ it is consistent with an earlier calculation \[14\]. The volume term is very well known (see, e.g. \[2\]). Note, that in two dimensions axial vectors can be transformed to vectors, and, therefore, the whole anomaly may be generated by just the first term under the integral in \(56\).

5.2 Dimension four

Chiral anomaly in four dimensions is calculated the same way except for that we have to use eq. \(55\). Now the anomaly contains two contributions.

$$A = A_v + A_b .$$

The volume part

$$A_v = \frac{-1}{180 (2\pi)^2} \int_M d^4x \sqrt{g} \text{tr} \varphi \left( -120 [D_\mu V^{\mu \nu}, A_\nu] ight. + 60 [D_\mu A_\nu, V^{\mu \nu}] - 60 D_\mu D^\mu D_\nu A_\nu + 120 \{ \{ D_\mu A_\nu, A_\nu \}, A^\mu \} 
+ 60 \{ D_\mu A^\mu, A_\nu A_\nu \} + 120 A_\mu D_\nu A_\nu A^\mu + 30 [[A_\mu, A_\nu], A^{\mu \nu}] 
+ \epsilon_{\mu \nu \rho \sigma} \left\{ -45 i V^{\mu \nu} V^{\rho \sigma} + 15 i A^{\mu \nu} A^{\rho \sigma} - 30 i (V^{\mu \nu} A^\rho A^\sigma + A^\mu A^\nu V^{\rho \sigma}) 
- 120 i A^\mu V^{\nu \rho} A^\sigma + 60 i A^\mu A^\nu A^\rho A^\sigma \right\} 
- 60 (D_\sigma A_\nu) R^{\nu \sigma} + 30 (D_\mu A^\mu) R 
- \frac{15i}{8} \epsilon_{\mu \nu \rho \sigma} R^{\mu \nu} R^{\rho \sigma} \right)$$

is known\(^8\). The boundary part

$$A_b = \frac{-1}{180 (2\pi)^2} \int_{\partial M} d^3x \sqrt{h} \text{tr} \left( 12 i \epsilon^{abc} \{ A_b, \varphi \} D_a A_c 
+ 24 \{ \varphi, A^a \} \{ A_a, A_n \} - 60 [A^a, \varphi] (V_{na} - [A_n, A_a]) 
+ 60 (D_n \varphi) D_\mu A^\mu \right)$$

is new. It has been derived under two restrictions \(58\). Note, that in the present context, the first condition \(S = 0\) actually follows from the second one \(L_{ab} = 0\) due to \(29\). We shall analyse physical consequences of \(59\) in a future publication.

\(^8\)The flat space part of \(58\) can be found e.g. in \[10\]. The term with $R \ast R$ follows from the local index theorem and has a very long history \[27\]. We are not aware of any works which considered chiral anomaly for non-abelian axial vector field in curved space, so that the terms \((DA)R\) have a chance to be new. There are certain similarities between \(58\) and chiral anomaly in the Riemann-Cartan space \[28\].
6 Conclusions

We have calculated boundary contributions to chiral anomaly for local (bag) boundary conditions in two and four dimensions (in four dimensions we have supposed that the boundary is totally geodesic, $L_{ab} = 0$). As a by-product we obtained explicit expressions for several heat kernel coefficients with mixed boundary conditions and with a matrix-valued smearing function. These heat kernel expressions are rather universal. By choosing a bit different expressions for $Q$ and for the fields (25) - (27) one can easily extend our results to other anomaly-like expression relevant for hadron physics (see, e.g., [29]) and, probably, even to supersymmetry [30].

We have found no specific boundary contributions to the anomaly in two dimensions. In four dimensions there are boundary terms in the anomaly, which must have important physical consequences both in hadron physics and in the standard model.

Our present paper is the first one which treats local chiral anomaly in the presence of boundaries in generic background vector and axial vector fields. Therefore, our results may be improved or extended in many directions.

It is clear that in order to bring our results closer to physical applications we have to lift the restriction $L_{ab} = 0$ in four dimensions (which excludes, for example, spherical boundaries in flat space). This is just a technical problem which can be solved by the same methods as presented above. Another problem is to extend our results beyond the linear order in the chiral transformation parameter. Such an extension requires chirally transformed boundary conditions (17). We are going to address these two problems in the near future.

Brane-world and domain wall configurations lead to interactions confined at a singular surface. Mathematically such interactions are described at the one-loop level by some “gluing conditions” which relate boundary values of the functions and their normal derivatives on two sides of the singular surface. Such cases may be treated by the same methods as presented in this paper. Moreover, a lot of important information on the heat kernel expansion for gluing conditions is contained in the heat kernel expansion for the boundary conditions case (see, e.g., [31]).

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A Heat kernel expansion with mixed boundary conditions

Here we give explicit expressions for the heat kernel coefficients \( a_k \), \( k = 0, 1, 2, 3, 4 \) for an operator of Laplace type subject to mixed boundary conditions \( [20] \) (see also \([21]\) for minor corrections in \( a_4 \)), \( f \) is a scalar function.

\[
\begin{align*}
a_0(f, L) &= (4\pi)^{-n/2} \int_M d^n x \sqrt{g} \text{tr} (f) \\
a_1(f, L) &= \frac{1}{4} (4\pi)^{-(n-1)/2} \int_{\partial M} d^{n-1} x \sqrt{h} \text{tr} (\chi f), \\
a_2(f, L) &= \frac{1}{6} (4\pi)^{-n/2} \left\{ \int_M d^n x \sqrt{g} \text{tr} (6 f E + f R) + \int_{\partial M} d^{n-1} x \sqrt{h} \text{tr} (2 f L_{aa} + 3 \chi f_m + 12 f S) \right\}. \\
a_3(f, L) &= \frac{1}{384} (4\pi)^{-(n-1)/2} \int_{\partial M} d^{n-1} x \sqrt{h} \text{tr} \left\{ f (96 \chi E + 16 \chi R + 8 f \chi R_{an} + (13 \Pi_+ - 7 \Pi_-) L_{aa} L_{bb} + (2 \Pi_+ + 10 \Pi_-) L_{ab} L_{ab} + 96 S L_{aa} + 192 S^2 - 12 \chi_{a\chi:a} + f_{;n} ((6 \Pi_+ + 30 \Pi_-) L_{aa} + 96 S) + 24 \chi f_{;nn} \right\}. \\
a_4(f, L) &= \frac{1}{360} (4\pi)^{-n/2} \left\{ \int_M d^n x \sqrt{g} \text{tr} \left\{ f (60 E_{\mu}^{\mu} + 60 E + 180 E^2 + 30 \Omega_{\mu\nu} \Omega^{\mu\nu} + 12 R_{\mu}^{\mu} + 5 R^2 - 2 R_{\mu\nu} R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}) \right\} + \int_{\partial M} d^{n-1} x \sqrt{h} \text{tr} \left\{ f \left\{ (240 \Pi_+ + 120 \Pi_-) E_{;n} + (42 \Pi_+ - 18 \Pi_-) R_{;n} + 24 L_{aa;bb} + 240 E L_{aa} + 20 R L_{aa} + 4 R_{an} L_{bb} - 12 R_{ab} L_{ab} + 4 R_{abc} L_{ac} + \frac{1}{21} \left\{ (280 \Pi_+ + 40 \Pi_-) L_{aa} L_{bb} L_{cc} + (168 \Pi_+ - 264 \Pi_-) L_{ab} L_{bc} L_{ac} + (224 \Pi_+ + 320 \Pi_-) L_{ab} L_{bc} L_{ac} \right\} + 720 E + 120 S R + 144 S L_{aa} L_{bb} + 48 S L_{ab} L_{ab} + 480 S^2 L_{aa} + 480 S^3 + 120 S_{;aa} + 60 \chi_{a\chi:a} \Omega_{;an} - 12 \chi_{b\chi:a} L_{bb} - 24 \chi_{c\chi:b} L_{ab} - 120 \chi_{a\chi:b} S \right\} + f_{;n} (180 \chi E + 30 \chi R + \frac{1}{7} \left\{ (84 \Pi_+ - 180 \Pi_-) L_{aa} L_{bb} + (84 \Pi_+ + 60 \Pi_-) L_{ab} L_{ab} \right\} + 72 S L_{aa} + 240 S^2 - 18 \chi_{a\chi:a} + f_{;nn} (24 L_{aa} + 120 S) + 30 \chi f_{;\mu}^{\mu n} \right\} \right\}. \quad (64)
\end{align*}
\]

The coefficient \( a_5 \) can be found in \([32]\). Here we present only those terms which enter \([32]\) with \( k = 3 \). Namely, we put \( f = 1 \) and neglect all terms which
do not contain $E$.

\[ a_5(1, L) = \frac{1}{5760} (4\pi)^{-(m-1)/2} \int_{\partial M} d^{n-1}x \sqrt{\text{tr}} (360\chi E_{nn} + 1440E_n S) \]
\[ + 720\chi E^2 + 180\chi aa E + 240\chi RE - 120\chi R_{nn} E + 2880E^2 \]
\[ + (270 - 180\chi) L_{aa} E_n + 1440L_{aa} S E + (45 + 150\chi) L_{aa} L_{ab} E \]
\[ + (90 - 60\chi) L_{ab} L_{ab} E - 180(E^2 - \chi E \chi E) - 180\chi a \chi a E \]
\[ - 90\chi a \chi a E + O(E^0). \]  

(65)

**B Particular case calculation: details**

Let us consider in detail calculations of the expansion (53) for the operator (52).

First we use the (51) to represent the first order terms in $E$ of $K(Q, L_1; t)$ as

\[
\frac{1}{4\pi} \int_0^t d\tau \int_0^\infty dx \int_0^\infty dy \frac{1}{\sqrt{\tau t - \tau}} \text{tr} \left\{ Q(x) \left( e^{-\frac{(x-y)^2}{4\tau t}} + \chi e^{-\frac{(x+y)^2}{4\tau t}} \right) \right. 
\]
\[
\times E(y) \left( e^{-\frac{(x-y)^2}{4\tau(t-\tau)}} + \chi e^{-\frac{(x+y)^2}{4\tau(t-\tau)}} \right) \}.
\]  

(66)

Next we integrate over $\tau$ with the help of the relation

\[
\int_0^t d\tau \frac{\exp(-a^2/\tau) \exp(-b^2/(t-\tau))}{\sqrt{\tau(t-\tau)}} = \pi \cdot \text{erfc} \left( \frac{|a| + |b|}{\sqrt{t}} \right).
\]  

(67)

The parameters \(a\) and \(b\) are equal either to \((x-y)/2\) or to \((x+y)/2\). Note, that “reflected” terms depending on \((x+y)\) are always multiplied by $\chi$.

It is convenient to consider 3 different types of contributions separately.

1. **Terms without \(\chi\).**

   In this case $a = b = (x - y)/2$. Let us change the variables

   \[
   x = k + r\sqrt{t}, \quad y = k \quad \text{for} \ x > y \\
   y = k + r\sqrt{t}, \quad x = k \quad \text{for} \ x < y.
   \]  

   (68)

   In both cases $k, r \in [0, +\infty[$. Then we use a small $t$ expansion:

   \[
   \int_0^{+\infty} dr \text{erfc}(r)f(r\sqrt{t}) = \frac{f(0)}{\sqrt{\pi}} + \frac{f'(0)\sqrt{t}}{4} + \frac{f''(0)t}{6\sqrt{\pi}} + \ldots
   \]  

   (69)

   which is valid for any smooth function which decays sufficiently fast at
infinity. We obtain:

\[
\int_0^{+\infty} dx \int_0^{+\infty} dy \text{erfc} \left( \frac{|x-y|}{\sqrt{t}} \right) \frac{Q(x)E(y)}{4} = \int_0^{+\infty} dk \int_0^{+\infty} dr \text{erfc}(r) \frac{t^2}{4} \left( Q(k + \sqrt{tr})E(k) + Q(k)E(k + \sqrt{tr}) \right)
\]

\[
= \frac{t^2}{2\sqrt{\pi}} \int_0^{+\infty} dy \text{tr} \left( Q(y)E(y) + \frac{t}{6} Q(y)E''(y) \right) - \frac{t}{16} \text{tr} Q(0)E(0)
\]

\[
+ \frac{t^2}{2\sqrt{\pi}} \text{tr} \left( \frac{1}{12} Q(0)E'(0) - \frac{1}{12} Q'(0)E(0) \right) + O(t^2).
\]  

(70)

2. Terms with two \( \chi \). In this case \( a = b = (x+y)/2 \). We change the variables:

\[
x = r\sqrt{t}\cos \phi, \quad y = r\sqrt{t}\sin \phi,
\]

so that \( r \in [0, \infty[, \phi \in [0, \pi/2] \). We use (67) and then integrate over \( r \) and \( \phi \) assuming \( t \) is small. The resulting asymptotic expansion reads

\[
\int_0^{+\infty} dx \int_0^{+\infty} dy \text{erfc} \left( \frac{|x+y|}{\sqrt{t}} \right) \frac{Q(x)\chi E(y)\chi}{4} = \frac{t}{16} \text{tr} Q(0)\chi E(0)\chi
\]

\[
+ \frac{t^2}{2\sqrt{\pi}} \text{tr} \left( \frac{1}{12} Q'(0)\chi E(0)\chi + \frac{1}{12} Q(0)\chi E'(0)\chi \right) + O(t^2)
\]

(72)

3. Terms with single \( \chi \). We have \( a = (x-y)/2, b = (x+y)/2 \). We use the variables:

\[
x = r\sqrt{t}(\cos \phi + \sin \phi), \quad y = r\sqrt{t}\cos \phi \quad \text{for} \ x > y;
\]

\[
y = r\sqrt{t}(\cos \phi + \sin \phi), \quad x = r\sqrt{t}\cos \phi \quad \text{for} \ x < y.
\]

(73)

Acting as above we obtain:

\[
\int_0^{+\infty} dx \int_0^{+\infty} dy \text{erfc} \left( \frac{|x+y| + |x-y|}{2\sqrt{t}} \right) \frac{t}{8} \text{tr} Q(0)\left( \chi E(0) + E(0)\chi \right) +
\]

\[
= \frac{t^2}{2\sqrt{\pi}} \text{tr} \left( \frac{1}{4} Q(0)\left( \chi E'(0) + E'(0)\chi \right) + \frac{1}{4} Q'(0)\left( \chi E(0) + E(0)\chi \right) \right) + O(t^2)
\]

(74)

The sum of (70), (72) and (74) yields (53).
Calculations of the $\omega$ terms can be carried out in a similar way. The following integral is useful:

$$\int_0^t dp \int_0^p d\tau \frac{\exp\left(-\frac{a^2}{t-p}\right) \exp\left(-\frac{b^2}{p-\tau}\right) \exp\left(-\frac{c^2}{\tau}\right)}{\sqrt{t-p} \sqrt{p-\tau} \sqrt{\tau}} =$$

$$= -2\pi^{3/2} \cdot (|a| + |b| + |c|) \cdot \text{erfc} \left( \frac{|a| + |b| + |c|}{\sqrt{t}} \right) + 2\pi \sqrt{t} \cdot \exp \left( -\frac{(|a| + |b| + |c|)^2}{t} \right)$$

(75)

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