Quantum mechanics lessons for standard cosmology

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Abstract. By recalling the relevance of the Sturm-Liouville theory has had on the solutions of quantum mechanics problems, here it is explored the possibility of getting some insight to the solutions for a standard cosmology model for inflation, from a time independent Schrödinger type equation derived from the equations of motion for a single scalar field in a flat space time with a FRW metric and a cosmological constant.

1. Introduction
In 1925 Erwin Schrödinger proposed his model for wave mechanics; his first task was to describe the dynamics of an electron in a hydrogen atom. Ever since his model was proposed, it was found that the second order differential equations appearing in some of the most interesting time independent problems of quantum mechanics (QM) had been studied in the century before this model was born, and that they pertain to the set of equations studied in the Sturm-Liouville (SL) theory. Such is the case of the time independent equation for the hydrogen atom, whose radial solutions are the generalized Laguerre polynomials, and where the angular part, as in any central problem, is solved in terms of the associated Legendre polynomials, as part of the solution of Laplace’s equation; another case is the simple harmonic oscillator, whose solutions come in terms of the Hermite polynomials. In fact, the boundary conditions imposed in the QM problems is what implies the appearance of discrete solutions, an aspect which is usually cited as the quantization of the problem, or the quantization of the parameter in question; the energy, for example.

Some of the special functions found on QM problems were initially used to solve non-mechanical problems: the Laguerre polynomials, named after Edmond Laguerre (1834-1886), had been used to numerically compute the Gaussian quadrature of some integrals, while the Hermite polynomials, named after Charles Hermite (1822-1901), conform a sequence that arises in various probability problems, numerical computations of Gaussian quadratures, and can also be used to describe Dirac’s Delta function. While the hydrogen atom is the only QM problem analytically solved, there are very few second order ordinary differential equations with known analytical solutions, the most important for physics being those pertaining to the SL theory, where hermiticity is the most important property.

On the other hand, the general relativity (GR) equations are in some cases reduced to second order ordinary differential equation, and the special functions of the SL theory again immediately become the candidates for analytical solutions. As QM inherited some properties from the SL theory with regards to the analytic solutions, one may ask what could be learned from the QM solutions that could give us some insight into the GR problems.
Consider for example the case of a single scalar field (SF) as the source for inflation, a useful mechanism to understand the inflationary period that provided the initial conditions of the early Universe from which the large scale structure that we see can be obtained. In this scenario, the functional form of the field and the potential driving its dynamics is rather arbitrarily inserted into the equations, since there is no unique prescription or phenomenology to determine them. But, for a given potential function, it may be possible to consider the equivalent QM problem and from its dynamics gain some insight to the problem.

2. Bifurcations in a QM phase space portrait

Let us first consider the classical picture of a one dimensional time independent QM problem,[1] where the particle is subjected to a potential well, with $U(x) < 0$ inside the well region, $a < x < b$, and $U = 0$ elsewhere, and where we are interested in the bounded states solutions of the problem, with $E < 0$. The Schrödinger equation in this case is

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + U(x) \right] \psi(x) = E \psi(x).$$  \hspace{1cm} (1)

where we set $\hbar = m = 1$. Now let us redefine the problem by changing the coordinate $x$ by $t$ and call it time, and let the function $\psi(t)$ be the new coordinate $q(t)$ and its time derivative its associated momentum $p(t) = \frac{d\psi}{dt}$. Then, the following equations hold

$$\frac{dq}{dt} = p$$

$$\frac{dp}{dt} = 2[U(t) - E].$$  \hspace{1cm} (2)

Therefore, for this QM problem we have an associated classical picture, where the dynamics are determined by the Hamiltonian $H(p,q) = p^2/2 + (E - U(t))q^2$, with the boundary conditions $q(\pm \infty) = 0$. The dynamics could be determined from the following matrix relation

$$\frac{d\bar{q}}{dt} = \Lambda(t)\bar{q}(t)$$  \hspace{1cm} (3)

where

$$\Lambda(t) = \begin{bmatrix} 0 & 1 \\ 2(U(t) - E) & 0 \end{bmatrix}$$  \hspace{1cm} (4)

is the dynamics generator matrix and

$$\bar{q}(t) = \begin{bmatrix} q \\ p \end{bmatrix}$$  \hspace{1cm} (5)

Outside of the potential well, where $U(t) \equiv 0$, the generator becomes time independent

$$\Lambda = \begin{bmatrix} 0 & 1 \\ 2|E| & 0 \end{bmatrix} \text{ for } t \leq a \text{ or } t \geq b$$  \hspace{1cm} (6)

leading to the explicit solutions for eq.(3)

$$\bar{q}(t) = \begin{cases} e^{\Lambda(t-a)}\bar{q}(a) & \text{for } t \leq a \\ e^{\Lambda(t-b)}\bar{q}(b) & \text{for } t \geq b \end{cases}$$  \hspace{1cm} (7)
Moreover, since \( \Lambda^2 = 2|E|1 \), it possesses two eigenvalues, \( \lambda_\pm = \pm \sqrt{2|E|} \), with the corresponding eigenvectors
\[
\vec{e}_+ = \frac{1}{\sqrt{2|E|}} \quad \text{and} \quad \vec{e}_- = -\frac{1}{\sqrt{2|E|}}
\] (8)

Therefore, for any given value of the parameter \( E \) the time evolution preserve the fixed directions \( \vec{e}_\pm \). Hereafter we assume that the phase space trajectory began in the direction \( \vec{e}_+ \). In the QM case, \( E \) is called an eigenvalue of the Schrödinger equation (1) iff there exists a non-trivial trajectory that vanishes both at \( t \to -\infty \) and \( t \to +\infty \), that is, if after traversing the potential well the phase space trajectory that originated in the direction \( \vec{e}_+ \) at \( t < a \) is driven to a trajectory that moves in the \( \vec{e}_- \) direction for \( t > b \). However, this is the most unusual case, since there exist only a countable set of values for which this occurs, while there is an infinite number of values of the parameter \( E \) for which the trajectories diverge in the direction \( \vec{e}_+ \) after traversing the potential well. The phase space trajectories are depicted in Fig.1.

![Figure 1. The bifurcations of phase space trajectories. The fixed directions \( \vec{e}_\pm \) determine the eigenvalue problem.](image)

An interesting aspect of this picture is that the asymptotes directions determined by \( \vec{e}_\pm \) are fixed, no matter what the form of the potential well be, and that they determine a constant ratio between the values of the phase space coordinates, \( q \) and \( p \) for large \( |t| \). This is also true if the potential function has two different constant values outside of the well.

We shall now consider a QM analogue of standard cosmology, and apply there some of the notions introduced here.

3. The QM equivalent of standard cosmology
Consider now an homogeneous and isotropic universe, i.e., a FRW background with a scalar homogeneous field \( \phi(t) \) minimally coupled to gravity and nonzero cosmological constant
\[
\int d^4x \sqrt{-g} \left[ R + \Lambda + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right],
\] (9)
where \((\nabla \phi)^2 = g^{\mu\nu} \partial_\mu \partial_\nu\), and \(V(\phi)\) is the potential energy of the field. In order to describe the dynamics of the scalar field during inflation the usual treatment is performed [2]-[7], leading to the pair of equations

\[
3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \Lambda \tag{10}
\]

\[
\ddot{\phi} + 3H \dot{\phi} = -\frac{dV(\phi)}{d\phi}, \tag{11}
\]

where dot means derivative with respect to time, and we set \(M_{Pl} = c = 1\). The time derivative of eq.(10) is related to eq.(11) through the momentum equation

\[
\dot{H} = -\frac{1}{2} \dot{\phi}^2. \tag{12}
\]

With the use of eqs. (11) and (12) the dynamics of the model may be described by the single equation:

\[
3H^2 + \dot{H} = V(\phi) + \Lambda. \tag{13}
\]

which can be recognized as a Riccati equation for the Hubble parameter \(H(t)\). Now, since the time independent one dimensional Schrödinger equation can be transformed into a Riccati equation, its appearance here suggests the introduction of a QM analogue to inflationary cosmology. This has been proposed earlier, leading however to different approaches: a nonlinear Schrödinger equation [8], or a case of cosmology with a perfect fluid but in the context of classical mechanics ([9, 10,]). We have developed our own treatment before [11], and we shall relate it here to the previous section.

The Riccati equation (13) is transformed into a Schrödinger equation when the Hubble parameter \(H(t)\) is related to a wave function \(\psi(t)\) through

\[
H = \frac{1}{3} \frac{\psi(t)}{\dot{\psi}(t)}, \tag{14}
\]

taking the form

\[
\left[-\frac{d^2}{dt^2} + 3V(t)\right] \psi(t) = -3\Lambda \psi(t). \tag{15}
\]

This transformation immediately brings some analogies between the two models, QM and inflationary cosmology, in particular when the Hubble parameter is not singular, implying that \(\psi(t)\) has to be an at least \(C^1\) class function. In that case, we have to consider the ground state solutions of all known exactly solvable bound state problems of QM, where the SF potential \(V(\phi)\) and the cosmological constant \(\Lambda\) become the QM potential \(U(x)\) and ground state energy eigenvalue \(E_g\), respectively,

\[
3V(\phi(t)) + 3\Lambda \leftrightarrow 2U(x) - 2E_g \tag{16}
\]

We have realized this analogies before [11]. The general algebraic procedure is very simple; let us recall here the case of the QM simple harmonic oscillator. In this case, when we substitute the ground state eigenfunction \(\psi_g(x) = c_0 \exp\{-x^2/2\}\) into eq.(14) we find that \(H(t) = -\frac{x^2}{2} t\); then, use eq.(12) we find that \(\phi(t) = \sqrt{2w} t\), which together with eq.(16) gives us that \(V(\phi) = \frac{w}{2} \phi^2\). If we simply replace \(w = 2\lambda\) we have finally that \(V(\phi) = \lambda \phi^2\). \(^1\) It is amazing that the same type of functional dependence appears in both models. Other combinations of QM typical potentials

\(^1\) In this case \(U(x) = \frac{1}{2} w^2 x^2 = 2\lambda^2 x^2\).
and inflationary cosmology potentials can be found elsewhere, for one dimensional and radial problems.[11]

It is interesting that the exact analytic solution for the $\lambda \phi^2$ potential differs very much from the required behavior for an inflationary model, where the main reason would be that typically this potential is used together with the slow roll approximation, which imposes special conditions on the dynamics of the field ($\ddot{\phi} = 0$, for example.) To begin with, a cosmological constant is needed, or the exact potential would need to be $\lambda \phi^2 + const.$ [12] Also, the QM analogue cited here for $V(\phi) = \lambda \phi^2$, gives a value of $H(t) = -\frac{2\lambda}{3} t$ for the Hubble parameter, implying that the equation for the dynamics of the field, eq.(11), be

$$\ddot{\phi} - 2\lambda t \dot{\phi} + 2\lambda \phi = 0 \quad (17)$$

whose solution clearly is $\phi(t) = const. \times t$, whereas a constant Hubble parameter, the slow roll case, would derive in a equation where oscillations are still possible.

\[\text{Figure 2. Different forms of } a(t) \text{ obtained from one QM problem.}\]

In Fig.(2) different curves for the scale factor obtained from the QM analogy of this problem are depicted. The dashed curves correspond to $a(t)$ obtained from the ground state and first excited state eigenfunctions, $\psi_0(x)$ and $\psi_1(x)$, of the harmonic oscillator problem. The solid curves draw the scale factor for three different cosmological constants, derived from the energy eigenvalue $E$, with $E < E_0$, $E_0 < E < E_1$ and $E_1 < E < E_2$, whose wave functions diverge towards $+\infty$, $-\infty$ and to $+\infty$ again, the first one without nodes and the other with increasing number of nodes. Therefore, only the wave function with $E < E_0$ could lead to a solution with $a(t) > 0$ for all $t$, describing a cosmological solution for an ever expanding universe (bold solid line.) There is an continuum of solutions with $E < E_0$ that diverge to $+\infty$ as $t$ grows, and for all of them the slope $\frac{a}{t}$ is determined by the cosmological constant $\Lambda$, the “energy” parameter of the corresponding QM problem. Besides the analogies found between the two models [11], this may be the most interesting aspect of the relation between both models, since there are implications for the form of the Hubble parameter, as is stated in eq.(14).

4. Conclusion

In this work it is shown that two completely separate models, one being a cosmological model for inflation and the other a one dimensional quantum mechanics time independent model, may have more in common than what is expected, and that their relations could give us some insight to the description of some of the most relevant parameters in standard cosmology; in the present
case, the Hubble parameter is being studied with respect to the way bifurcations are produced in a QM bounded problem, showing that these parameter is highly related to the cosmological constant.

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