Probabilistic implementation of Hadamard and Unitary gates

Wei Song*, Ming Yang, Zhuo-Liang Cao

School of Physics & Material Science, Anhui University, Hefei, 230039, P. R. of China

Abstract

We show that the Hadamard and Unitary gates could be implemented by a unitary evolution together with a measurement for any unknown state chosen from a set $A = \{ |\Psi_i\rangle, |\overline{\Psi_i}\rangle \}$ ($i = 1, 2$) if and only if $|\Psi_1\rangle, |\Psi_2\rangle, |\overline{\Psi_1}\rangle, |\overline{\Psi_2}\rangle$ are linearly independent. We also derive the best transformation efficiencies.

Key words: Probabilistic implementation; Hadamard gate

PACS: 03.67.-a; 03.65.-w

Manipulation and extraction of quantum information are important tasks in building quantum computer. Unlike classical information there are several limitations on the basic operations that one can perform on quantum information. Linearity of quantum mechanics unveils that we cannot duplicate an unknown quantum state accurately[1]. This has been proven by Wootters and Zurek[1] and Dieks[2] which called the quantum no-cloning theorem. Though exact cloning is not possible, in the literature various cloning machines have been proposed [3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22] which operate either in a deterministic or probabilistic way. Corresponding to the quantum no-cloning theorem, Pati and Braunstein [23] demonstrated that the linearity of quantum mechanics also forbids one to delete one unknown state ideally against a copy [23], which is called the quantum no-deleting principle. Notably, Zurek [24] further verified the existence of limitations on cloning and deleting completely an unknown state and pointed out the importance of studying approximate and probabilistic deletion corresponding to cloners [3,4,10]. And some probabilistic and state-dependent deleting machines have

* Corresponding author.

Email addresses: wsong@mars.ahu.edu.cn (Wei Song), zlcao@mars.ahu.edu.cn (Zhuo-Liang Cao).
been established [25,26,27]. In addition, it is found that one cannot flip a spin of unknown polarization, because the flip operator $V$ defined as

$$V |\Psi\rangle = |\bar{\Psi}\rangle$$  \hspace{1cm} (1)$$

is not unitary but anti-unitary, where if qubit $|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$ belongs to two-dimensional Hilbert space, then its opposite direction is defined as $|\bar{\Psi}\rangle = \beta^* |0\rangle - \alpha^* |1\rangle$, but there is no physical operation capable of implementing such a transformation. This is called the quantum no-complementing principle [12,28,29]. It is also shown that if we are given an arbitrary state $|\Psi\rangle \in H^2$ of an unknown qubit and a blank state $|\Sigma\rangle \in H^2$, there does not exist an isometric operator $U$ such that

$$U |\Psi\rangle |\Sigma\rangle = |\Psi\rangle |\bar{\Psi}\rangle,$$  \hspace{1cm} (2)$$

holds [29], which is called the quantum no-anti-cloning property. Lately, Pati considered the question if we are given an unknown qubit pointed in some arbitrary direction $n$ in a state $|\Psi\rangle$ or in the direction $-n$ in a state $|\bar{\Psi}\rangle$ can we design a logic gate that will transform these inputs as follows [29]:

$$|\Psi\rangle \rightarrow \frac{1}{\sqrt{2}} \left( |\Psi\rangle + |\bar{\Psi}\rangle \right),$$

$$|\bar{\Psi}\rangle \rightarrow \frac{1}{\sqrt{2}} \left( |\Psi\rangle - |\bar{\Psi}\rangle \right),$$  \hspace{1cm} (3)$$

where $|\Psi\rangle$ is an unknown state $|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle \in H^2$, with $\alpha$ and $\beta$ being unknown complex numbers and $|\alpha|^2 + |\beta|^2 = 1$, and $|\bar{\Psi}\rangle$ is the complement of $|\Psi\rangle$. They proved that there is no universal Hadamard gate defined by (3) for an unknown qubit that will create an equal superposition of the original state $|\Psi\rangle$ and its complement state $|\bar{\Psi}\rangle$. They also show that it is not possible to design a unitary transformation that will create an unequal superposition of the original qubit with its complement which will transform the inputs as follows [29]:

$$|\Psi\rangle \rightarrow a |\Psi\rangle + b |\bar{\Psi}\rangle,$$

$$|\bar{\Psi}\rangle \rightarrow b^* |\Psi\rangle - a^* |\bar{\Psi}\rangle,$$  \hspace{1cm} (4)$$

where $a, b$ are known complex numbers and $|a|^2 + |b|^2 = 1$.

Analogous to the probabilistic cloning we will prove that the above transformation defined by Eq. (3) and Eq. (4) could be implemented by a unitary
evolution together with a measurement if $|\Psi_i\rangle$ and $|\bar{\Psi}_i\rangle$ are chosen from a linearly independent set. Firstly we will consider how to implement the transformation defined by Eq. (3). The result is the following theorem:

Theorem 1. There exists a unitary operator $U$ such that for any unknown state chosen from a set $A = \{|\Psi_i\rangle, |\bar{\Psi}_i\rangle\}$ ($i = 1, 2$)

$$U (|\Psi_i\rangle |P_0\rangle) = \sqrt{\gamma_i} \frac{1}{\sqrt{2}} \left(|\Psi_i\rangle + |\bar{\Psi}_i\rangle\right) |P_0\rangle + \sum_{j=1}^{2} a_{ij} |\Phi_A^{(j)}\rangle |P_j\rangle,$$

$$U (|\bar{\Psi}_i\rangle |P_0\rangle) = \sqrt{\delta_i} \frac{1}{\sqrt{2}} \left(|\Psi_i\rangle - |\bar{\Psi}_i\rangle\right) |P_0\rangle + \sum_{j=1}^{2} b_{ij} |\bar{\Phi}_A^{(j)}\rangle |P_j\rangle,$$

for some real numbers $\gamma_i \geq 0, \delta_i \geq 0, a_{ij} \geq 0$ and $b_{ij} \geq 0$ with $i, j = 1, 2$, if and only if $|\Psi_1\rangle, |\Psi_2\rangle, |\bar{\Psi}_1\rangle, |\bar{\Psi}_2\rangle$ are linearly independent, where $|P_0\rangle, |P_1\rangle, |P_2\rangle$ denoting the probe states are orthonormal, and the states $|\Phi_A^{(j)}\rangle$’s of system A are normalized but unnecessarily orthogonal, $|\Psi_i\rangle$ and $|\bar{\Phi}_A^{(j)}\rangle$ are complement of $|\Psi_i\rangle$ and $|\Phi_A^{(j)}\rangle$ respectively. To prove the existence of the unitary operator $U$ described by Eq. (5) and Eq. (6), we first notice the following lemma [10].

Lemma 1. If two sets of states $|\phi_1\rangle, |\phi_2\rangle, ..., |\phi_n\rangle$, and $|\bar{\phi}_1\rangle, |\bar{\phi}_2\rangle, ..., |\bar{\phi}_n\rangle$ satisfy the condition

$$\langle \phi_i | \phi_j \rangle = \langle \bar{\phi}_i | \bar{\phi}_j \rangle, (i = 1, 2, ..., n; j = 1, 2, ..., n),$$

there exists a unitary operator $U$ to make $U |\phi_i\rangle = |\bar{\phi}_i\rangle, (i = 1, 2, ..., n)$.

We should also need to know the following fact: The linear independence of $\{|\Psi_i\rangle, |\bar{\Psi}_i\rangle\}$ ($i = 1, 2$) implies that $\{|\Psi_i\rangle\}$ ($i = 1, 2$) and $\{|\bar{\Psi}_i\rangle\}$ ($i = 1, 2$) are also of linear independence.

Now we prove the above theorem by using the lemma.

The $2 \times 2$ inter-inner-products of Eq. (5) and Eq. (6) yield the matrix equation

$$X^{(1)} = \sqrt{\Gamma} X^{(2)} \sqrt{\Gamma^+} + A A^+,$$

$$X^{(3)} = \sqrt{A^+} X^{(4)} \sqrt{A} + B B^+,$$

where the $2 \times 2$ matrixes $A = [a_{ij}], B = [b_{ij}], X^{(1)} = [\langle \psi_i | \psi_j \rangle], X^{(2)} = \frac{1}{2} \left(\langle \psi_i | \psi_j \rangle + \langle \psi_i | \bar{\psi}_j \rangle + \langle \bar{\psi}_i | \psi_j \rangle + \langle \bar{\psi}_i | \bar{\psi}_j \rangle\right)$, $X^{(3)} = \left[\langle \bar{\psi}_i | \bar{\psi}_j \rangle\right]$ and
\[ X^{(4)} = \left[ \frac{1}{2} \left( \langle \Psi_i | \Psi_j \rangle - \langle \Psi_i | \bar{\Psi}_j \rangle - \langle \bar{\Psi}_i | \Psi_j \rangle + \langle \bar{\Psi}_i | \bar{\Psi}_j \rangle \right) \right]. \]

The diagonal efficiency matrix \( \Gamma \) is defined by \( \Gamma = \text{diag} \left( \gamma_1, \gamma_2 \right) \), hence \( \sqrt{\Gamma} = \sqrt{\Gamma^+} = \text{diag} \left( \sqrt{\gamma_1}, \sqrt{\gamma_2} \right) \).

Lemma 1 shows that if Eq. (8) and Eq. (9) is satisfied with a diagonal positive-definite matrix \( \Gamma \) and \( \Lambda \), the unitary evolution (5) and (6) can be realized in physics. Let us prove the existing of Eq. (8) firstly. Following the Lemma 2 in Ref. [10], since the states \(|\Psi_1\rangle, |\Psi_2\rangle\) are linearly independent, then the matrix \( X^{(1)} \) is positive definite. From the fact we know \( \{ |\bar{\Psi}_i\rangle \} (i = 1, 2) \) is linearly independent. As a consequence, one can show that the matrix \( X^{(2)} \) is positive definite. From continuity, for small enough but positive \( \gamma_i \), the matrix \( X^{(1)} - \sqrt{\Gamma} X^{(2)} \sqrt{\Gamma^+} \) is also positive definite. Therefore there is unitary matrix \( V \) such that:

\[ V^+ \left( X^{(1)} - \sqrt{\Gamma} X^{(2)} \sqrt{\Gamma^+} \right) V = \text{diag} \left( m_1, m_2 \right), \quad (10) \]

for some real numbers \( m_i > 0 \) \( (i = 1, 2) \). Now we choose

\[ A = V \text{diag} \left( \sqrt{m_1}, \sqrt{m_2} \right) V^+. \quad (11) \]

Equation (8) is thus satisfied with a diagonal positive-definite efficiency matrix \( \Gamma \). Within the same approach of verify Eq. (8), we could prove that Eq. (9) could also be satisfied with a diagonal positive-definite efficiency matrix \( \Lambda \). In general we have \( \Gamma \neq \Lambda \). Thus we complete the proof of Theorem 1. Actually we may find that if \(|\Psi_1\rangle, |\Psi_2\rangle\) are linearly dependent, then the matrix \( X^{(1)} \) is only positive-semidefinite. With a diagonal positive-definite matrix \( \Gamma \), in general, \( X^{(1)} - \sqrt{\Gamma} X^{(2)} \sqrt{\Gamma^+} \) is no longer a positive-semidefinite matrix. But the matrix \( AA^+ \) is positive-semidefinite. So Eq. (8) could not be satisfied. With similar procedure, we could prove that if \(|\bar{\Psi}_1\rangle, |\bar{\Psi}_2\rangle\) are linearly dependent, then Eq. (9) also could not be satisfied. These show that the Hadamard gate defined in Eq. (3) could not be probabilistically realized by any unitary reduction operation for any unknown state chosen from the linearly dependent states \(|\Psi_1\rangle, |\Psi_2\rangle, |\bar{\Psi}_1\rangle, |\bar{\Psi}_2\rangle\).

In the following, we derive the best possible efficiencies able to be attained. The general unitary evolution of the system \( AP \) can be decomposed as

\[ U (|\Psi_i\rangle |P_0\rangle) = \sqrt{\gamma_i} \frac{1}{\sqrt{2}} \left( |\Psi_i\rangle + |\bar{\Psi}_i\rangle \right) |P^{(i)}\rangle + \sqrt{1 - \gamma_i} |\Phi_{AP}^{(i)}\rangle, \quad (12) \]

\[ U (|\bar{\Psi}_i\rangle |P_0\rangle) = \sqrt{\delta_i} \frac{1}{\sqrt{2}} \left( |\Psi_i\rangle - |\bar{\Psi}_i\rangle \right) |P^{(i)}\rangle + \sqrt{1 - \delta_i} |\Phi_{AP}^{(i)}\rangle, \quad (i = 1, 2), \quad (13) \]

where \(|P_0\rangle\) and \(|P^{(i)}\rangle\) are normalized states of the probe \( P \) and \(|\Phi_{AP}^{(i)}\rangle\) and \(|\Phi_{AP}^{(2)}\rangle\) are two normalized states of the composite system \( AP \). \(|\Phi_{AP}^{(i)}\rangle\) is the
complement of $|\Phi_{AP}^{(i)}\rangle$. Without loss of generality, the coefficients in Eq. (12) and Eq. (13) are assumed to be positive real numbers. Obviously, Eq. (5) and Eq. (6) are special case of Eq. (12) and Eq. (13) with $|P^{(i)}\rangle = |P_0\rangle$ and $|\Phi_{AP}^{(i)}\rangle$ having a special decomposition. We denote the subspace spanned by the state $|P^{(i)}\rangle$ by the symbol $S_0$. During the transformation, after the unitary evolution a measurement of the probe with a postselection of the measurement results projects its state into the subspace $S_0$. After this projection, the state of the system A should be \( \frac{1}{\sqrt{2}} \left( |\Psi_i\rangle + |\bar{\Psi}_i\rangle \right) \) in Eq. (12) and be \( \frac{1}{\sqrt{2}} \left( |\Psi_i\rangle - |\bar{\Psi}_i\rangle \right) \) in Eq. (13), so all the states $|\Phi_{AP}^{(i)}\rangle$ ought to lie in a space orthogonal to $S_0$. This requires

\[
|P^{(i)}\rangle \langle P^{(i)}| \Phi_{AP}^{(i)}\rangle = 0, \quad (i = 1, 2). \tag{14}
\]

With the condition (14), inter-inner-products of Eq. (12) and Eq. (13) yield the following two matrix equations:

\[
X^{(1)} = \sqrt{\Gamma} X_P^{(2)} \sqrt{\Gamma^+} + \sqrt{I_n - \Gamma Y_1 \sqrt{I_n - \Gamma^+}}, \tag{15}
\]

\[
X^{(3)} = \sqrt{\Lambda} X_P^{(4)} \sqrt{\Lambda^+} + \sqrt{I_n - \Lambda Y_2 \sqrt{I_n - \Lambda^+}}, \tag{16}
\]

where the $2 \times 2$ matrix $Y_1 = \begin{bmatrix} \langle \Phi_{AP}^{(i)} | \Phi_{AP}^{(j)} \rangle \end{bmatrix}$, $Y_2 = \begin{bmatrix} \langle \Phi_{AP}^{(i)} | \Phi_{AP}^{(j)} \rangle \end{bmatrix}$, $X_P^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \left( |\Psi_i\rangle \langle \Psi_j| + |\bar{\Psi}_i\rangle \langle \bar{\Psi}_j| \right) \end{bmatrix}$, $X_P^{(4)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \left( |\Psi_i\rangle \langle \Psi_j| - |\bar{\Psi}_i\rangle \langle \bar{\Psi}_j| \right) \end{bmatrix}$. Following the proof of Lemma 2 in Ref. [10], $Y_1$ and $Y_2$ are positive-semidefinite matrices, thus $\sqrt{I_n - \Gamma Y_1 \sqrt{I_n - \Gamma^+}}$ and $\sqrt{I_n - \Lambda Y_2 \sqrt{I_n - \Lambda^+}}$ are positive-semidefinite matrices. So $X^{(1)} = \sqrt{\Gamma} X_P^{(2)} \sqrt{\Gamma^+}$ and $X^{(3)} = \sqrt{\Lambda} X_P^{(4)} \sqrt{\Lambda^+}$ should also be positive-semidefinite. On the other hand, if $X^{(1)} = \sqrt{\Gamma} X_P^{(2)} \sqrt{\Gamma^+}$ and $X^{(3)} = \sqrt{\Lambda} X_P^{(4)} \sqrt{\Lambda^+}$ are positive-semidefinite matrices, following the proof of Theorem 1, Eq. (15) and Eq. (16) can be satisfied with a special choice of $|\Phi_{AP}^{(i)}\rangle$, then lemma 1 shows that the transformation defined by Eq. (3) could be implemented by a unitary evolution together with a measurement such that for any unknown state chosen from a set $A = \{ |\Psi_i\rangle, |\bar{\Psi}_i\rangle \} (i = 1, 2)$. We thus get the following theorem:

**Theorem 2.** There exists transformation defined by Eq. (3) for any unknown state chosen from a set $A = \{ |\Psi_i\rangle, |\bar{\Psi}_i\rangle \} (i = 1, 2)$ with diagonal efficiency matrix $\Gamma$ and $\Lambda$ if and only if the matrixes $X^{(1)} = \sqrt{\Gamma} X_P^{(2)} \sqrt{\Gamma^+}$ and $X^{(3)} = \sqrt{\Lambda} X_P^{(4)} \sqrt{\Lambda^+}$ are positive-semidefinite.

The semipositivity of the matrixes $X^{(1)} = \sqrt{\Gamma} X_P^{(2)} \sqrt{\Gamma^+}$ and $X^{(3)} = \sqrt{\Lambda} X_P^{(4)} \sqrt{\Lambda^+}$ give a series of inequalities about the efficiencies $\gamma_i$ and $\delta_i$. The best possible
transformation efficiencies \( \gamma_i \) and \( \delta_i \) are obtained by solving these inequalities. For example, if there are four states \(|\Psi_1\rangle\), \(|\Psi_2\rangle\), \(|\bar{\Psi}_1\rangle\), and \(|\bar{\Psi}_2\rangle\), we can show that the transformation efficiencies \( \gamma \) and \( \delta \) satisfy

\[
\frac{\gamma_1 + \gamma_2}{2} \leq \frac{1 - |\langle \Psi_1 | \Psi_2 \rangle|}{1 - \frac{1}{2} |\langle \Psi_1 | \Psi_2 \rangle + \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle + \langle \bar{\Psi}_1 | \Psi_2 \rangle + \langle \bar{\Psi}_2 | \bar{\Psi}_1 \rangle|},
\]

(17)

\[
\frac{\delta_1 + \delta_2}{2} \leq \frac{1 - |\langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle|}{1 - \frac{1}{2} |\langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle - \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle - \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle + \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle|},
\]

(18)

For the state chosen from polar great circle, we have

\[
\langle \Psi_1 | \Psi_2 \rangle = \frac{1}{2} \left( \langle \Psi_1 | \Psi_2 \rangle + \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle + \langle \bar{\Psi}_1 | \Psi_2 \rangle + \langle \bar{\Psi}_2 | \bar{\Psi}_1 \rangle \right)
\]

\[
= \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle = \frac{1}{2} \left( \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle - \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle - \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle + \langle \bar{\Psi}_1 | \bar{\Psi}_2 \rangle \right),
\]

(19)

under the condition Eq. (19), Eq. (17) and Eq. (18) reduce to \( \frac{\gamma_1 + \gamma_2}{2} \leq 1 \) and \( \frac{\delta_1 + \delta_2}{2} \leq 1 \), then the transformation efficiencies \( \gamma \) and \( \delta \) could both reach unity which accords with the results of Ref. [29].

In the following we will show how to implement the transformation defined by Eq. (4). The result is the following theorem:

**Theorem 3.** There exists a unitary operator \( U \) such that for any unknown state chosen from a set \( A = \{|\Psi_i\rangle, |\bar{\Psi}_i\rangle\} (i = 1, 2) \)

\[
U \left( |\Psi_i\rangle |P_0\rangle \right) = \sqrt{\gamma_i} \frac{1}{\sqrt{2}} \left( a |\Psi_i\rangle + b |\bar{\Psi}_i\rangle \right) |P_0\rangle + \sum_{j=1}^{2} a_{ij} |\Phi_A^{(j)}\rangle |P_3\rangle, \quad (20)
\]

\[
U \left( |\bar{\Psi}_i\rangle |P_0\rangle \right) = \sqrt{\delta_i} \frac{1}{\sqrt{2}} \left( b^* |\Psi_i\rangle - a^* |\bar{\Psi}_i\rangle \right) |P_0\rangle + \sum_{j=1}^{2} b_{ij} |\Phi_A^{(j)}\rangle |P_3\rangle, \quad (21)
\]

for some real numbers \( \gamma_i \geq 0, \delta_i \geq 0, a_{ij} \geq 0, \) and \( b_{ij} \geq 0 \) with \( i, j = 1, 2, \) if and only if \(|\Psi_1\rangle, |\Psi_2\rangle, |\bar{\Psi}_1\rangle, |\bar{\Psi}_2\rangle\) are linearly independent. The inter-inner-products of Eq. (20) and Eq. (21) yield the following two matrix equations

\[
X^{(1)} = \sqrt{\Gamma} \sqrt{X^{(5)}} \sqrt{\Gamma^+} + AA^+, \quad (22)
\]

\[
X^{(3)} = \sqrt{\Lambda} \sqrt{X^{(6)}} \sqrt{\Lambda^+} + BB^+, \quad (23)
\]

where the \( 2 \times 2 \) matrixes \( X^{(5)} = \left[ |a|^2 \langle \Psi_i | \Psi_j \rangle + a^* b \langle \bar{\Psi}_i | \bar{\Psi}_j \rangle + ab^* \langle \bar{\Psi}_i | \Psi_j \rangle + |b|^2 \langle \bar{\Psi}_i | \bar{\Psi}_j \rangle \right] \),
and $X^{(6)} = \left[ |a|^2 \langle \Psi_i | \Psi_j \rangle - a^* b \langle \Psi_i | \bar{\Psi}_j \rangle - a b^{*} \langle \bar{\Psi}_i | \Psi_j \rangle + |b|^2 \langle \bar{\Psi}_i | \bar{\Psi}_j \rangle \right]$ the following procedure of verifying theorem 3 is similar to that of Theorem 1, and we omit the details. Within the same approach of verifying Theorem 2, we have the following corollary:

Corollary. There exists transformation defined by Eq. (4) for any unknown state chosen from a set $A = \{ |\Psi_i \rangle, |\bar{\Psi}_i \rangle \}$ with diagonal efficiency matrixes $\Gamma$ and $\Lambda$ if and only if the matrixes $X^{(1)} - \sqrt{\Gamma} X^{(6)} \sqrt{\Gamma^+}$ and $X^{(3)} - \sqrt{\Lambda} X^{(6)} \sqrt{\Lambda^+}$ are positive-semidefinite.

It should be pointed out that our probabilistic quantum information processing machine could also be comprehended in the concept of quantum program[30], which was originally discussed by Nielsen and Chuang [30]. The theory of quantum program has been developed in Ref. [31,32,33,34,35,36]. In Ref. [36] M. Hillery et al show that an arbitrary SU(2) transformation of qubits can be encoded in program state of a universal programmable probabilistic quantum processor. The probability of success of this processor can be enhanced by systematic correction of errors via conditional loops. Our probabilistic quantum information processing machine could also be regarded as a special probabilistic programmable quantum processor. There are two inputs of this machine: a data state, which is chosen from a set $A = \{ |\Psi_i \rangle, |\bar{\Psi}_i \rangle \}$, and a program state. In our machine, the program state is represented by the probe $P$. The instructions of the quantum program are to perform the Hadamard and Unitary gates defined by Eq. (3) and Eq. (4) for the input qubits. The transformation efficiencies of our probabilistic quantum information processing machines dependent on the value of $a$ and $b$, whereas the probabilistic quantum processor constructed in Ref. [36] has the same transformation efficiencies for arbitrary SU(2) rotation. So we conclude that our machine is not universal for arbitrary unitary operation, but it might reach higher transformation efficiencies for some special unitary operation compared to the probabilistic quantum processor constructed in Ref. [36].

To conclude, we have constructed some probabilistic quantum information processing machines to implement the Hadamard and Unitary gates defined by Eq. (3) and Eq. (4). As we know, these gates are very useful in various quantum algorithms and information processing protocols, while linearity of quantum mechanics does not allow linear superposition of an unknown qubit with its complement. Our results show that these quantum operations can be implemented perfectly with non-zero probabilities. The best transformation efficiencies are derived. We expect our results to play a fundamental role in future understanding of quantum information theory. There still leaves an interesting question that is how to construct an approximately universal machine to realize the transformation defined by Eq. (3) and Eq. (4) in a deterministic way.
Acknowledgement

We thank Yong-Sheng Zhang for helpful discussions and suggestions. This work is supported by Anhui Provincial Natural Science Foundation under Grant No: 03042401, the Key Program of the Education Department of Anhui Province under Grant No: 2004kj005zd and the Talent Foundation of Anhui University.

References

[1] W. K. Wootters and W. H. Zurek, Nature. 299, 802 (1982).
[2] D. Dieks, Phys. Lett. A. 92, 271 (1982).
[3] V. Buzek and M. Hillery, Phys. Rev. A. 54, 1844 (1996).
[4] N. Gisin and S. Massar, Phys. Rev. Lett. 79, 2153-2156 (1997).
[5] D. Bruss, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. 81, 2598-2601 (1998).
[6] R. F. Werner, Phys. Rev. A. 58, 1827-1832 (1998).
[7] M. Keyl, R. F. Werner, J. Math. Phys. 40, 3283-3299 (1999).
[8] Y.-J. Han, Y.-S. Zhang, G.-C. Guo, Phys. Rev. A. 66, 052301 (2002).
[9] H. Fan, K. Matsumoto, X.-B Wang, and M. Wadati, Phys. Rev. A. 65, 012304 (2002).
[10] L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. 80, 4999-5002 (1998).
[11] L.-M. Duan and G.-C. Guo, Phys. Lett. A. 243, 261-264 (1998).
[12] V. Buzek, M. Hilery, and R. F. Werner, Phys. Rev. A. 60, 2626-2629 (1999).
[13] A. Chefles and S. M. Barnett, Phys. Rev. A. 60, 136-144 (1999).
[14] D. Bruss, M. Cinchetti, G. M. D'Ariano, C. Macchiavello, Phys. Rev. A. 62, 12302 (2000).
[15] N. J. Cerf, A. Ipe, and X. Rottenberg, Phys. Rev. Lett. 85, 1754-1757 (2000).
[16] J. Fiurasek, S. Iblisdir, S. Massar, and N. J. Cerf, Phys. Rev. A. 65, 040302 (2002).
[17] S. L. Braunstein, N. J. Cerf, S. Iblisdir, P. vanLoock, and S. Massar, Phys. Rev. Lett. 86, 4938-4941 (2001).
[18] V. Buzek, S. L. Braunstein, M. Hillery, and D. Bruss, Phys. Rev. A. 56, 3446-3452 (1997).
[19] C.-W. Zhang, C.-F. Li, Z.-Y. Wang, and G.-C. Guo, Phys. Rev. A. 62, 042302 (2000).
[20] M. Horodecki, A. Sen, and U. Sen, quant-ph/0403169.
[21] J. Fiurasek, quant-ph/0403165.
[22] Z.-L. Cao, W. Song, Phys. Lett. A. 325, 309-314 (2004).
[23] A. K. Pati and S. L. Braunstein, Nature. 404, 164 (2000).
[24] W. H. Zurek, Nature. 404, 131 (2000).
[25] J. Feng, Y.-F. Gao, J.-W. Cao, J.-S. Wang, M.-S. Zhan, Phys. Lett. A. 292, 12 (2001).
[26] D.-W. Qiu, Phys. Rev. A. 65, 052329 (2002).
[27] W. Song, M. Yang, and Z.-L. Cao, Phys. Lett. A. 327, 123-128 (2004).
[28] N. Gisin, S. Popescu, Phys. Rev. Lett. 83, 432 (1999).
[29] A. K. Pati, Phys. Rev. A. 66, 062319 (2002).
[30] M. A. Nielsen, I. L. Chuang, Phys. Rev. Lett. 79, 321 (1997).
[31] G. Vidal, L. Masanes, and J. I. Cirac, Phys. Rev. Lett. 88, 047905 (2002).
[32] M. Hillery, V. Buzek, and M. Ziman, Fortschr. Phys. 49, 987 (2001).
[33] M. Hillery, V. Buzek, and M. Ziman, Phys. Rev. A. 65, 022301 (2002).
[34] M. Hillery, M. Ziman, and V. Buzek, Phys. Rev. A. 66, 042302 (2002).
[35] M. Dusek, and V. Buzek, Phys. Rev. A. 66, 022112 (2002).
[36] M. Hillery, M. Ziman, and V. Buzek, Phys. Rev. A. 69, 042311 (2004).