The mathematical model of vertical shaft support stability for ensuring the ecological safety of mine working

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Abstract. The analytical dependencies modeling the stress-strain state of the vertical shaft support for the porous structure materials whose compressed skeleton has simultaneously elastic and plastic properties are obtained. The spatial stability of the ground state of the considered cylindrical support of mine working is explored within the framework of the exact three dimensional equations of stability. The realization of the given models ensures the increase of efficiency and ecological safety of mine works.

1. Introduction

At present, the construction and protection of underground structures of various kinds remain the topical issue. As well, there are a number of related problems that need to be solved, such as environmental protection from pollution, different aspects of seismic safety and seismic survey [1, 11, 12]. These problems are usually solved by creating an underground structure support. A support is a complex engineering construction that needs significant time and financial expenditures. The main economic factor in erecting a support is its structural dimensions. In some cases, for example, at sinking stable rocks, a support can be considered only as a cover protecting a shaft from destruction [1]. In other cases, when underground building takes place in large depths or in difficult mining and geological conditions (permafrost, high seismicity, neotectonic phenomena etc.), in weak and unstable rocks a support should be modeled as a load-bearing construction [1, 5]. If the condition of beneficial combination of the depth and material strength is violated, the stability of mine workings and their supports gets the features of the complex engineering and scientific problem [2-4].

The purpose of underground structures calculation is determining the stress-strain state of these constructions elements and specifying their strength and stability conditions. According to the calculation results, the rational support structures and the optimal cross-sectional sizes providing the reliable performance of constructions at minimum cost are being chosen.

To solve all these problems, it’s necessary to have knowledge about destruction and stability of underground structure support.

The destruction of underground structure support can happen as a result of the following two situations:

1) the stress-strain state reaches the strength limits;
2) the stress-strain state reaches the critical value corresponding to the loss of stability (failure) of support.

The solution of the first problem is based on comparison of the stress-strain state, which can be found in analytical or numerical form with the material ultimate strength. In the second case, the initial phase of stability problem solution is to find the ground stress-strain state of the structure in the analytical form.

The characteristic feature of deep underground structure support is the plastic range of stress between elastically deformed part of support and its internal contour. That’s why in modeling the buckling of support, it’s necessary to use the models that simultaneously consider the elastic and plastic properties of materials as well as their internal structure.

2. Subcritical stress-strain state of the vertical shaft support with consideration of the initial porosity and plasto-elastic properties of the completely compressed matrix.

The deformation of porous material with a characteristic value $\varepsilon_0$, determined by the specific volume of pores, is divided into two interconnected stages [7]. The first stage is the elastic deformation of the porous medium; the second one is the inelastic deformation of the completely compressed matrix with strengthening elasto-plastic properties. At the first stage the dependence between stresses and deformations is taken from Hooke’s law for the compressible body

$$
\sigma_j^\beta = \left\{ \begin{array}{ll}
\lambda_1 e_{\alpha}^\beta g_{\alpha j}^\beta + 2\mu_1 e_{\alpha j}^\beta, \\
-\varepsilon_0^\alpha < \varepsilon_0,
\end{array} \right.
$$

where $\sigma_j^\beta$, $e_j^\beta$, $g_j^\beta$ are mixed components of stress tensors, elastic deformations, and metric tensor respectively, $\lambda_1$, $\mu_1$ are Lame constants for the compressible body.

At the stage of deforming the material with the completely compressed matrix the connection between stresses and elastic deformations is taken from Hooke’s law for the incompressible body [8]

$$
S_j^\beta = \left\{ \begin{array}{ll}
2(\mu_0 + \mu_1) e_j^\beta - 2\mu_0 \left( e_j^\beta \right)_0 + \frac{2}{3} \mu_0 \varepsilon_0 g_j^\beta, \\
-\varepsilon_0^\alpha = \varepsilon_0,
\end{array} \right.
$$

where $S_j^\beta$ are the components of the deviator stress tensor; $\left( e_j^\beta \right)_0$ are the elastic deformations having accumulated in the body by the moment of the complete compression of pores, i.e. if the second condition is fulfilled (2); $\mu_0 + \mu_1$ is the shear modulus of the incompressible body with the completely compressed skeleton.

In the zone of inelastic deformation of the material with the completely compressed matrix the model of incompressible strengthening plasto-elastic body [9] with the loading function is assumed

$$
F = \left( S_j^\beta - c e_j^\beta \right) \left( S_j^\beta - c e_j^\beta \right) - k^2,
$$

where $c$ and $k$ are the coefficient of strengthening and the yield stress of material respectively, $e_j^\beta$ are the components of plastic deformation tensor.

In the zone of inelastic deformation the total deformation consists of elastic and plastic components and is determined by the equality
\[ \epsilon_j^p = \epsilon_j^e + \epsilon_j^p, \]  
herewith the plastic and elastic components satisfy the conditions of incompressibility
\[ \epsilon_{nn}^p = 0, \quad \epsilon_{nn}^e = -\epsilon_0. \]

Here in (2), (4), (5) and farther «\(p\)» and «\(e\)» superscripts denote their belonging to the plastic and elastic areas of the compressed skeleton deformation.

Below the problem of determining the ground stress-strain state of vertical shaft support is considered. The vertical opening support is modeled by the cylindrical body with the outer \(b\) and inner \(a\) radiiuses (figure 1). The pressure of mountain massif on the support is replaced by the compressive load with \(q_b\) intensity that is uniformly distributed on the outer surface. The compressive load with \(q_a\) intensity uniformly distributed on the inner surface models the liquid and gas pressure on the support.

\[ \text{Figure 1. The circular cylindrical support of the vertical shaft under the radial compression.} \]

Due to the fact that the considered section of the support is far enough from both the day surface and the opening bottom, the respective boundary effects are not considered in determining the stress-strain state.

The axisymmetric stress-strain state of the circular cylindrical support of the vertical shaft within the framework of plane deformation state in the cylindrical coordinate system \((r, \theta, z)\) is modeled by the following relations of geometric linear theory:

- the equilibrium equation
  \[ \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_{\theta\theta}}{r} = 0; \]  
- the Cauchy relations
  \[ \epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r}, \]  
where \(u\) is the radial component of the displacement vector;
- stress boundary conditions on the inner and outer surfaces of support
  \[ \sigma_r |_{r=a} = -q_a, \quad \sigma_r |_{r=b} = -q_b, \quad (q_a > 0, \quad q_b > 0). \]
The dependencies (1), (2) between stresses and deformations under the taken assumptions are respectively rewritten in the form

\[ \sigma_r = \left( \lambda_1 + 2 \mu_1 \right) \varepsilon_r + \lambda_4 \varepsilon_\theta, \quad \sigma_\theta = \lambda_4 \varepsilon_r + \left( \lambda_1 + 2 \mu_1 \right) \varepsilon_\theta, \quad \sigma_z = \lambda_1 \left( \varepsilon_r + \varepsilon_\theta \right), \] (9)

\[ S_r = 2(\mu_0 + \mu_1) \varepsilon_r - 2\mu_0 \varepsilon_\theta + \frac{2}{3} \lambda_4 \varepsilon_\theta, \quad S_\theta = 2(\mu_0 + \mu_1) \varepsilon_\theta - 2\mu_0 \varepsilon_r + \frac{2}{3} \lambda_4 \varepsilon_r, \quad S_z = \frac{2}{3} \mu_4 \varepsilon_\theta. \] (10)

Herewith the condition of complete compression has the form

\[ \varepsilon_r^e + \varepsilon_\theta^e = -\varepsilon_\theta^0. \] (11)

For the considered axisymmetric case of plane deformation the relations (3)–(5) are rewritten respectively in the forms

\[ \varepsilon_r^0 + \varepsilon_0^\theta = -\varepsilon_\theta^0, \quad \varepsilon_r^0 = \varepsilon_\theta^0 + \varepsilon_\theta^0 = 0. \] (12)

Displacement and stresses continuity conditions on the plastic-elastic interface \( \gamma \) of completely compressed skeleton have the form

\[ (u_r - u_\theta^0) \big|_{\gamma} = 0, \quad \left( \sigma_r^0 - \sigma_\theta^0 \right) \big|_{\gamma} = 0, \quad \left( \sigma_\theta^0 - \sigma_\theta^e \right) \big|_{\gamma} = 0. \] (15)

Then, according to (6)–(9) the stress-strain state of the cylindrical support at the stage of elastic deformation of the material with internal structure that is realized upon fulfillment of the condition

\[ q_0 = \varepsilon_0 \left( \lambda_1 + 1 \right) \left( 1 - a^2 \right) + q_a \cdot f(\varepsilon_0) a^2, \] (16)

is determined by the relations

\[ u = \frac{q_b - q_a a^2}{2(\lambda_1 + 1)(a^2 - 1)} r + \frac{(q_b - q_a) a^2}{2(a^2 - 1)} \frac{1}{r}, \quad \varepsilon_r = \frac{q_b - q_a a^2}{2(\lambda_1 + 1)(a^2 - 1)} \frac{1}{a^2 - 1} + \frac{(q_b - q_a) a^2}{2(a^2 - 1)} \frac{1}{r^2}, \]

\[ \sigma_r = \frac{a^2 \left( r^2 - 1 \right)}{r^2 \left( 1 - a^2 \right)} + q_b \frac{a^2 - r^2}{r^2 \left( 1 - a^2 \right)}, \quad \sigma_\theta = \frac{a^2 \left( r^2 + 1 \right)}{r^2 \left( 1 - a^2 \right)} - q_b \frac{r^2 + a^2}{r^2 \left( 1 - a^2 \right)}, \quad \sigma_z = \frac{\lambda_1 \left( q_a a^2 - q_b \right)}{(\lambda_1 + 1)(1 - a^2)} \] (17)

In (16), (17), and farther all the relations are written in the dimensionless form, herewith all the values with the dimension of length belong to the radius \( b \), and the values with the dimension of stresses to the value \( \mu_1 \).

From (17), taking into account (11), we’ll find out that the stress-strain state at the moment of complete closure of pores that takes places under the loads satisfying the condition

\[ q_b = \varepsilon_0 \left( \lambda_1 + 1 \right) \left( 1 - a^2 \right) + q_a \cdot f(\varepsilon_0) a^2. \] (18)

is determined in the form

\[ u_0 = \frac{-\varepsilon_0}{2} r + \frac{q_a \cdot f(\varepsilon_0) - \varepsilon_0 \left( \lambda_1 + 1 \right)}{2 a^2} \frac{1}{r}, \quad \varepsilon_{r0} = \frac{-\varepsilon_0}{2} - \frac{q_a \cdot f(\varepsilon_0) - \varepsilon_0 \left( \lambda_1 + 1 \right) a^2}{2 r^2}, \quad \varepsilon_{\theta0} = \frac{-\varepsilon_0}{2} + \frac{q_a \cdot f(\varepsilon_0) - \varepsilon_0 \left( \lambda_1 + 1 \right) a^2}{2 r^2}, \]

\[ \sigma_{r0} = -\varepsilon_0 \left( \lambda_1 + 1 \right) - \frac{q_a \cdot f(\varepsilon_0) - \varepsilon_0 \left( \lambda_1 + 1 \right) a^2}{r^2}, \quad \sigma_{\theta0} = \frac{-\varepsilon_0 (\lambda_1 + 1) + (q_a \cdot f(\varepsilon_0) - \varepsilon_0 (\lambda_1 + 1)) a^2}{r^2}, \quad \sigma_{z0} = -\lambda_4 \varepsilon_0. \] (19)
where  
\[ f\left(\varepsilon_0\right) = \begin{cases} 1, & \text{if } \varepsilon_0 \neq 0 \\ 0, & \text{if } \varepsilon_0 = 0 \end{cases}, \]  
«0» subscript under the movement, stresses, and deformations components means that the given values have been calculated by the moment of the complete compression of pores.

The stress-strain state of the vertical opening support at the stage of inelastic deformation of the material with the completely compressed matrix that is realized upon fulfillment of the condition

\[ q_b > \varepsilon_0\left(\lambda_1 + 1\right)(1-a^2) + q_a \cdot f\left(\varepsilon_0\right)a^2. \]  
(20)

according to (6)–(15), (19) is determined in the form

- in the elastic range \((\gamma < r < 1)\)

\[ \sigma_r = \chi r^2 \sqrt{k^2 - \frac{1}{3}\varepsilon_0^2} \left(1 - \frac{1}{r^2}\right) -q_b, \quad \sigma_\theta = \chi r^2 \sqrt{k^2 - \frac{1}{3}\varepsilon_0^2} \left(1 + \frac{1}{r^2}\right) - q_b; \]  
(21)

- in the plastic range \((a < r < \gamma)\)

\[ \varepsilon_r^p = -\varepsilon_\theta^p = \frac{\chi \sqrt{k^2 - \varepsilon_0^2/3}}{c + 2\mu} \left(1 - \frac{\gamma^2}{r^2}\right), \]

\[ \sigma_r = -q_a + \chi \sqrt{k^2 - \frac{1}{3}\varepsilon_0^2} \left(\frac{\gamma^2}{a^2} - \frac{\gamma^2}{r^2} + \frac{2\mu}{c + 2\mu} \left(1 - \frac{r^2}{a^2} + 2\ln\frac{r}{a}\right)\right), \]

\[ \sigma_\theta = -q_a + \chi \sqrt{k^2 - \frac{1}{3}\varepsilon_0^2} \left(\frac{\gamma^2}{a^2} + \frac{\gamma^2}{r^2} + \frac{4\mu}{c + 2\mu} \left(1 - \frac{3 - \gamma^2}{a^2} - \frac{\gamma^2}{r^2} + \ln\frac{r}{a}\right)\right). \]  
(22)

Movements and complete deformations in the elastic and plastic regions are determined by the relations

\[ u = \frac{D}{r} - \frac{\varepsilon_0}{2} r, \quad \varepsilon_r = -\frac{D}{r^2} - \frac{\varepsilon_0}{2}, \quad \varepsilon_\theta = \frac{D}{r^2} - \frac{\varepsilon_0}{2}. \]  
(23)

Here in (21)–(23)

\[ D = \frac{\chi r^2 \sqrt{k^2 - \varepsilon_0^2/3} + \mu_0 \left(q_a \cdot f\left(\varepsilon_0\right) - \varepsilon_0\left(\lambda_1 + 1\right)a^2\right)}{2\mu}, \quad \chi = \left(q_a - q_b\right), \quad \mu = 1 + \mu_0. \]

The radius \(\gamma\) of the plastic-elastic interface is determined by solution of the equation

\[ q_b - q_a + \chi \sqrt{k^2 - \frac{1}{3}\varepsilon_0^2} \left(\frac{\gamma^2}{a^2} - \gamma^2 + \frac{2\mu}{c + 2\mu} \left(1 - \frac{\gamma^2}{a^2} + 2\ln\frac{\gamma}{a}\right)\right) = 0. \]  
(24)

The figures present the results of numerical experiment that was carried out within the framework of the obtained solutions describing the stress-strain state of vertical opening support at the stage of inelastic deformation of the material.

In the figures 2 and 3 curves 1 correspond to \(k = 8.5 \cdot 10^{-3}\), curves 2 - \(k = 0.01\), curve 3 - \(k = 0.012\). In the figures 4 and 5 curves 1 correspond to \(\varepsilon_0 = 10^{-4}\), curves 2 - \(\varepsilon_0 = 7 \cdot 10^{-3}\), curve 3 - \(\varepsilon_0 = 0.01\). The relative values of other geometrical and physical-mechanical parameters, unless otherwise stated, were the following

\(a = 0.5, q_a = 0.001, q_b = 0.012, c = 0.005, \lambda_4 = 3, \mu_4 = 1, k = 0.01, \varepsilon_0 = 0.001, \mu = 2\).
3. The research of spatial buckling mode of the vertical shaft monolithic support.

Below we solve the spatial stability problem of heterogeneous plasto-elastic equilibrium of the vertical shaft cylindrical support ground state (20)-(24), accepting the generic conception of continuous loading [6]. The perturbed state of the investigated construction is described by the following relations of the three-dimensional linearized stability theory of deformable bodies:

- the equilibrium equation for the areas of plastic and elastic deformation of the material of the support with the completely compressed matrix

$$\nabla_i \left( \sigma^j + \sigma^{0j}_\alpha \nabla^\alpha u_j \right) = 0,$$

(25)

where $\nabla$ is the symbol of covariant derivation, values without «0» superscript correspond to the perturbed components, values with «0» superscript correspond to the components of the ground unperturbed state defined by the relations (20)-(24);

- the boundary conditions on the inner and outer surfaces of the support

$$N_i \left( \sigma^j + \sigma^{0j}_\alpha \nabla^\alpha u_j \right) = 0,$$

(26)
where $N_i$ are the components of unit normal vector to the surface;

- the matching conditions on the elastic plastic interface $\gamma$ have the form

\[
\begin{bmatrix}
N_i\left(\sigma_i^0 + \sigma_0\nabla^m u_i\right)
\end{bmatrix} = 0, \quad \left[ u_j \right] = 0.
\]  

(27)

where the square brackets indicate the difference of the corresponding values belonging to the elastic and plastic areas;

- the connection between amplitude values of stresses and displacements for the compressed skeleton with plasto-elastic properties and the property of farther incompressibility in the plastic and elastic areas

\[
\sigma_i^0 = \left( x_{\alpha\delta} g^{\alpha\beta} \nabla_\alpha u_\beta + p \right) g_i^\beta + \left( 1 - g_i^\beta \right) g_{\beta\rho} \mu \left( \nabla_\rho u_\beta + \nabla_\beta u_\rho \right), \quad X_{\beta\rho} , \quad \Sigma_\alpha ,
\]  

(28)

where

\[
x_{\alpha\delta} = 2 \mu g_{\alpha\delta} - v f_{\alpha\delta}, \quad f_{ij} = S_{ij} - \epsilon_{ij}, \quad \nu = \begin{cases} 4\mu^2 / k^2 (2\mu + c) & \text{in the plastic range,} \\ 0 & \text{in the elastic range} \end{cases}
\]  

(29)

- the condition of incompressibility

\[\nabla^m u_\alpha = 0 .\]  

(30)

Thus, the failure of vertical shaft monolithic support is based on investigating the closed system of equations (25)-(30) if there is the elastic-plastic material behavior interface. It is a partial differential equation system in terms of the amplitude values of displacement vector with the coordinates $(u, v, w)$ and the hydrostatic pressure $p$ for plastic and elastic domains of the support. The nontrivial solution of this problem corresponds to the ground state instability. We will seek a double trigonometric series solution in elastic and plastic deforming areas

\[
u = \sum_{n}^{\infty} \sum_{m}^{\infty} A_{nm}(r) \cos(m \theta) \cos(nz), \quad \nu = \sum_{n}^{\infty} \sum_{m}^{\infty} B_{nm}(r) \sin(m \theta) \cos(nz),
\]

\[
u = \sum_{n}^{\infty} \sum_{m}^{\infty} C_{nm}(r) \cos(m \theta) \sin(nz), \quad p = \sum_{n}^{\infty} \sum_{m}^{\infty} D_{nm}(r) \cos(m \theta) \cos(nz),
\]  

(31)

where $n$ and $m$ are the wave formation parameters.

As the system (25)-(30) is a linear homogeneous system, hence, it can be written with the same values $n, m$ for each member. That’s why further we don’t write $m$ and $n$ indices for notational convenience, without loss of generality.

Inserting the perturbations (31) into the boundary value problem and taking into account (28)-(30) after a number of transformations, we will obtain the boundary value problem in the terms of functions $A(r), B(r), D(r)$.

It’s not possible to obtain an exact analytical solution of the formulated boundary value problem. The approximate solution can be obtained by the finite-difference method [10]. As a result we have the finite system of homogeneous algebraic equations, linear with respect to parameters $A_{nm}, B_{nm}, D_{nm}$. In calculating the determinant, along with finding the ground stress-strain state for each area of elastic and plastic deformation of support (20)-(23) we must take into account the equation that determines the position of the elastic-plastic interface $\gamma$ entirely covering the internal contour of the support. The minimization is for the following parameters: the difference interval, the wave formation parameters on the contour $m$ and generatrix $n$, the material and structure parameters $\lambda_j$. Thus, we have a multidimensional $q_a$-optimization problem depending on the parameters $m, n$, if the resulting algebraic system determinant is equal to zero.

The results of numerical experiment are presented in figures 6 and 7.
Figure 6. Results of calculating physical-mechanical parameters of the material.

Figure 7. Results of calculating geometrical parameters of the material.

Figure 6 presents the dependence of critical value of internal pressure $q_a$ on the standard internal radius $a$ of monolithic support at different material strengthening values of support with completely compressed matrix. Herewith, curve 1 corresponds to $c=0.01$, curve 2 - $c=0.04$, curve 3 - $c=0.08$. The relative value of external pressure in this case is $q_b=0.03$. Figure 7 shows the dependence of critical value of internal loading $q_a$ on the relative value of rock pressure with the intensity $q_b$ at different values of the internal radius $a$ of the monolithic support. Herewith, curve 1 corresponds to $a=0.6$, curve 2 - $a=0.5$, curve 3 - $a=0.4$.

The values of wave formation parameters $m=n=3$ correspond to all presented dependencies. Dimensionless values of other physical-mechanical and geometrical parameters, unless otherwise stated, were the following $k=0.007$, $\lambda = 2$, $\mu = 1$, $\varepsilon = 8 \cdot 10^{-4}$.

The numerical calculation analysis has shown that the area of stability reduces, when the value of external pressure on the support with the intensity $q_b$ increases; the bulking failure of the circular cylindrical support of the vertical shaft at a great distance from the day surface occurs in the axisymmetric form with three half-waves on the contour and generatrix; the values of internal critical pressure corresponding to the failure of the cylindrical support under consideration reduce both with the increased relative thickness of support and with increased relative material strengthening factor $c$ of support with completely compressed matrix.

4. Conclusions

The results presented in the work are the following.

1. The influence of physical-mechanical parameters of support material (characteristic value determined by the specific volume of pores, the material fluidity limit, the strengthening factor, etc.), the external loads and the geometrical parameters of the support on distribution of displacements and
stress fields as well as on the radius of plastic-elastic boundary in the cylindrical support under the
consideration has been evaluated within the framework of the obtained model.
2. The numerical experiment is based on the developed mathematical model of the vertical shaft
support failure. According to the experiment the following conclusions have been made:
– plastic-elastic interface in the monolithic cylindrical support of vertical opening can significantly influence its stability;
– understanding the ground (subcritical) stress-strain state of support construction is not enough
to predict its operational reliability, because on large depths the buckling failure of the support
ground state precedes the exhaustion of bearing capacity;
– the critical value of internal pressure corresponding to the buckling failure of monolithic
cylindrical support with circular cross-sectional shape significantly depends on the physical-
mechanical as well as geometrical parameters of construction. Herewith, the critical value of
internal pressure reduces, i.e. the area of stability extends both with the increased strengthening
factor $c$ and the increased fluidity limit $k$ of the support material;
– with growing value of external pressure on the support, which can be observed at the
increased depth of mining output, the value of critical pressure on the internal surface of support
enlarges;
– the optimal thickness of vertical shaft support considerably depends both on the value of
external pressure, determined by the pressure of rock massif on the support, and on the value of
pressure, evenly distributed on the internal surface of support. Herewith, with growing rock massif
pressure on the support its optimal thickness increases, and with growing loading on the internal
surface of support the optimal thickness of the latter can be reduced.

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