Exact Solutions to the Schrödinger Equation for the potential $V(r) = ar^2 + br^{-4} + cr^{-6}$ in 2D

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Abstract

Making use of an ansatz for the eigenfunctions, we obtain an exact closed form solution to the non-relativistic Schrödinger equation with the anharmonic potential, $V(r) = ar^2 + br^{-4} + cr^{-6}$ in two dimensions, where the parameters of the potential $a, b, c$ satisfy some constraints.

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1. Introduction

The exact solutions to the fundamental dynamical equations play crucial roles in physics. It is well-known that the exact solutions to the Schrödinger equation are possible only for the several potentials and some approximation methods are frequently applied to arrive at the solutions. On the other hand, in recent years, the higher order anharmonic potentials have attracted much more attention to physicists and mathematicians[1-3]. Interest in these anharmonic oscillator-like interactions stems from the fact that, in many case, the study of the relevant Schrödinger equation, for example in atomic and molecular physics, provides us with insight into the physical problem in question.

Recall that in the three-dimensional spaces, rough speaking, there are two main methods to be used to deal with the anharmonic potentials $V(r) = ar^2 + br^{-4} + cr^{-6}$. One[4, 5] is based on an ansatz for the eigenfunctions to obtain an exact solution with this potential. This method undoubtedly provides an exact solution for the ground state but sometimes with some constraints on the parameters of the potential. The other[6, 7] is relied on a Laurent series ansatz for the eigenfunctions, which converts the Schrödinger equation into a difference equation and then the continued fraction solutions are defined. This method, however, does not give any constrains for the parameters of the potential.

The reasons why we write this paper are as follows. On the one hand, with the advent of growth technique for the realization of the semiconductor quantum wells, the quantum mechanics of low-dimensional systems has become a major research field. Almost all of the computational technique developed for the three-dimensional problems has already been extended to lower dimensions. On the other hand, the study of the potential $V(r) = ar^2 + br^{-4} + cr^{-6}$ in two dimensions has never been appeared in the literature. We now attempt to research it in two dimensions.

This paper is organized as follows. In Sec. 2, we study the ground state of the Schrödinger equation with this potential using an ansatz for the eigenfunctions. The first excited state will be discussed by the same way in Sec. 3. The some constraints
on the parameters of the potential $a, b, c$ are given in Secs. 2 and 3. The figures for the unnormalized radial functions are plotted in the last section.

2. The ground states

Throughout this paper the natural unit $\hbar = 1$ and $\mu = 1/2$ are employed. Consider the Schrödinger equation with a potential $V(r)$ that depends only on the distance $r$ from the origin

$$H\psi = \left(\frac{1}{r} \frac{\partial}{\partial r} \frac{r}{\partial \psi} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}\right) \psi + V(r)\psi = E\psi,$$

where the potential

$$V(r) = ar^2 + br^{-4} + cr^{-6}, \quad a > 0, \quad c > 0.$$  

Owing to the symmetry of the potential, let

$$\psi(r, \varphi) = r^{-1/2} R_m(r) e^{\pm im\varphi}, \quad m = 0, 1, 2, \ldots,$$

where the radial wave function $R_m(r)$ satisfies the radial equation

$$\frac{d^2 R_m(r)}{dr^2} + \left[ E - V(r) - \frac{m^2 - 1/4}{r^2}\right] R_m(r) = 0,$$

where $m$ and $E$ denote the angular momentum and energy, respectively. For the solution of Eq. (4a), we make an ansatz \cite{4, 5} for the ground state

$$R_{m0}(r) = \exp[p_{m0}(r)],$$

where

$$p_{m0}(r) = \frac{1}{2} \alpha r^2 + \frac{1}{2} \beta r^{-2} + \kappa \ln r.$$  

After calculating, we arrive at the following equation

$$\frac{d^2 R_{m0}(r)}{dr^2} - \left[ \frac{d^2 p_{m0}(r)}{dr^2} + \left( \frac{dp_{m0}(r)}{dr} \right)^2 \right] R_{m0}(r) = 0.$$  

Compare Eq. (4b) with Eq. (4a) and obtain the following set of equations

$$\alpha^2 = a, \quad \beta^2 = c.$$

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\[ \kappa^2 - \kappa - 2\alpha\beta = m^2 - 1/4, \]  
\[ 3\beta - 2\beta\kappa = b, \]  
\[ E = -(2\kappa + 1)\alpha. \] 

It is easy to obtain the values of parameters for \( p_{m0}(r) \) from the Eqs. (7a) and (7b) written as

\[ \alpha = \pm \sqrt{a}; \quad \beta = \pm \sqrt{c}; \quad \kappa = \frac{1}{2} \pm \sqrt{m^2 + 2\sqrt{ac}}. \]  

In order to retain the well-behaved solution at the origin and at infinity, we choose positive sign in \( \kappa \) and negative signs in \( \alpha \) and \( \beta \). According to these choices, the Eq. (7c) leads to the following constraint on the parameters of the potential,

\[ (b + 2\sqrt{c})^2 - 4c(m^2 + 2\sqrt{ac}) = 0. \]  

The eigenvalue \( E \), however, will be given from Eq. (7d) as

\[ E = \sqrt{a} \left( 4 + \frac{b}{\sqrt{c}} \right). \]  

Now, the corresponding eigenfunctions Eq. (5) can be read as

\[ R_{m0}(r) = N_0 r^\kappa \exp \left[ \frac{-1}{2}(\sqrt{a}r^2 + \sqrt{c}r^{-2}) \right], \]  

where \( N_0 \) is the normalized constant. Here and hereafter \( \kappa = 1/2 + \sqrt{m^2 + 2\sqrt{ac}}. \)

3. The first excited states

With the same spirit, we make an ansatz for the eigenfunctions corresponding to the first excited state in the potential (2) as follows

\[ R_{m1}(r) = f_m(r) \exp[p_{m1}(r)], \]  

with \( f_m(r) \) given by

\[ f_m(r) = a_1 + a_2 r^2 + a_3 r^{-2}, \]  

and \( p_{m1}(r) \) given by

\[ p_{m1}(r) = \frac{1}{2} \alpha_1 r^2 + \frac{1}{2} \beta_1 r^{-2} + \kappa_1 \ln r. \]
For short, it is readily to see from Eqs. (12) and (13) that the radial function \( R_{m_1}(r) \) has the following relation

\[
R_{m_1}(r)'' - \left[ p_{m_1}(r)'' + (p_{m_1}(r)')^2 + \left( \frac{f_m(r)'' + 2p_{m_1}(r)f_m(r)'}{f_m(r)} \right) \right] R_{m_1}(r) = 0, \quad (4c)
\]

where the prime denotes the derivative of the radial function with respect to the variable \( r \). Calculating Eq. (4c) carefully and comparing it with Eq. (4a), we obtain

\[
a_2[E - \sqrt{a}(2\kappa_1 + 5)] = 0, \quad a_3[b - \sqrt{c}(2\kappa_1 - 7)] = 0, \quad (14a)
\]

\[
a_1[E - \sqrt{a}(2\kappa_1 + 1)] = a_2[m^2 - 1/4 + 2\sqrt{ac} - \kappa_1^2 - 3\kappa_1 - 2], \quad (14b)
\]

\[
a_1[m^2 - 1/4 + 2\sqrt{ac} - \kappa_1^2 + \kappa_1] = a_2[b - \sqrt{c}(2\kappa_1 + 1)] + a_3[-E + \sqrt{a}(2\kappa_1 - 3)], \quad (14c)
\]

\[
a_1[b - \sqrt{c}(2\kappa_1 - 3)] = -a_3[m^2 - 1/4 + 2\sqrt{ac} - \kappa_1^2 + 5\kappa_1 - 6], \quad (14d)
\]

\[
\alpha_1 = \pm \sqrt{a}, \quad \beta_1 = \pm \sqrt{c}. \quad (14e)
\]

Hence, if the angular momentum \( m \) of the first excited state is the same as that of the ground state, we obtain from Eq. (14)

\[
\kappa_1 = \frac{b + 7\sqrt{c}}{2\sqrt{c}}, \quad E_1 = \sqrt{a}(5 + 2\kappa_1), \quad (15a)
\]

\[
a_1 = 0, \quad a_2 = \sqrt{a}, \quad a_3 = -\sqrt{c}, \quad (15b)
\]

\[
\beta_1 = -\sqrt{c}, \quad \alpha_1 = -\sqrt{a}, \quad (15c)
\]

\[
b = -6\sqrt{c}, \quad (15d)
\]

where the constants both \( \alpha_1 \) and \( \beta_1 \) are chosen negative signs in order to hold well-behaved nature of the solution at \( r \to 0 \) and \( r \to \infty \). Equation (15d) is another constrain on the parameters of the potential.

At last, the eigenvalue \( E_1 \) and eigenfunctions \( R_{m_1}(r) \) for the first excited state with the potential (2) may be read from Eqs. (15a) and (12) as follows

\[
E_1 = \sqrt{a}\left(12 + \frac{b}{\sqrt{c}}\right), \quad (16)
\]

\[
R_{m_1}(r) = N_1(a_2r^2 + a_3r^{-2})r^{\kappa_1}\exp\left[-\frac{1}{2}(\sqrt{ar^2} + \sqrt{cr^{-2}})\right], \quad (17)
\]
where $N_1$ is the normalized constant for the first excited state and $\kappa_1$ is given by Eq. (15a).

As a matter of fact, the normalized constants $N_0$ and $N_1$ can be calculated in principle from the normalized relation

$$\int_0^\infty |R_{mi}|^2 dr = 1, \quad i = 0, 1. \quad (18)$$

Considering the values of the parameters of the potential, we fix them as follows. The value of parameter $a$ is first fixed, for example $a = 1.0$, the values of the parameter $c$ and $b$ are determined by the constraints Eq. (9) and Eq. (15d) for $m = 0$. By this way, the corresponding parameters turn out to $a = 1.0, c = 4, b = -12, \kappa = -1.5, \kappa_1 = 0.5, a_2 = 1, a_3 = -2$. The ground state and the first excited state energies corresponding to these values are obtained as $E_0 = -2$ and $E_1 = 6$, respectively. Actually, when we study the properties of the ground state and the first excited state, as we know, the unnormalized radial wave functions will not affect the main features of the wave functions. We have plotted the unnormalized radial wave functions $R_i^0, (i = 0, 1)$ in fig. 1 and fig. 2 for the ground state and the first excited states, respectively. Comparing them with the figures for the ground state and the first excited state in three dimensions, respectively, it is easy to find that they are different from each other. The reason is that the parameters of the potential $b,c$ are not same as those in three dimensions, which origins from the different constraints on the parameters of the potential, even if the parameter $a$ is same both in two dimensions and in three dimensions.

To summarize, we discuss the ground state and the first excited state for the Schrödinger equation with the potential $V(r) = ar^2 + br^{-4} + cr^{-6}$ using a simpler ansatz for the eigenfunctions and simultaneously two constrains for the parameters of the potential are arrived at from the compared equations, which then results in the variety for the energy eigenvalue and eigenfunctions with the varieties of the parameters of the potential. This simple and intuitive method is easy to be generalized. The other studies to the sextic potential and the octic potential as well as the inverse potential in two dimensions are in progress.
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