Abstract

Let $R$ be a commutative ring. We introduce the notion of support of an object in an $R$-linear triangulated category. As an application, we study the non-existence of Bridgeland stability condition on $R$-linear triangulated categories.

Keywords

Affine schemes · Stability conditions · Linear triangulated categories

Mathematics Subject Classification

14R10 · 13E10

1 Introduction

Inspired by Douglas’s work for $\Pi$-stability (cf. [14]), Bridgeland [11] introduces a notion of stability condition on a triangulated category. He also shows that the set $\text{Stab } D$ of stability conditions on a triangulated category $D$ has a natural topology and that each connected component is a complex manifold. Hence we refer to the set $\text{Stab } D$ as the space of stability conditions on $D$.

In mirror symmetry, the space is important as follows: Suppose that the category $D$ is the bounded derived category $D^b(M)$ of coherent sheaves on a Calabi–Yau manifold $M$. Then the moduli space of complex structures on the mirror manifold of $M$ are expected to be described by the space $\text{Stab } D^b(M)$. In fact, if $M$ is an elliptic curve, the moduli space is isomorphic to a quotient of $\text{Stab } D^b(M)$ by [11]. Other cases (for instance, the case of K3 surfaces) are discussed in [13].

Also in algebraic geometry, the space of stability conditions gives a powerful tool to study birational geometry for moduli spaces of complexes and much work has been done in this direction. For instance, moduli spaces on K3 surfaces or abelian surfaces are studied by many authors (Arcara and Bertram in [1] or [2], Bayer and Macri in [6] and [7] or Minaminde, Yanagida and Yoshioka in [21]. The case of other surfaces are studied by [3, 18, 23], or [24].
Though we have so far assumed that the space is non-empty, the existence of stability conditions on a triangulated category is a sensitive question, even if the category is the derived category $\text{D}^b(X)$ of a smooth projective variety $X$. If $\dim X \leq 2$, then $\text{Stab} \text{D}^b(X)$ does exists [11, 12] and [2]. In the case of $\dim X = 3$, Bayer, Macri, and Toda [10] proposed a generalized Bogomolov inequality and show that the inequality implies the existence of stability conditions. The inequality was shown to be correct in many cases, (for instance, the projective 3-fold $\mathbb{P}^3$ by [19], quadratic 3-folds by [26], Fano three folds of Picard rank one by [17], and abelian 3-folds by [8] and by [20]), nevertheless Schmidt provided a counterexample in [27]. Following Schmidt’s example, Bernardara et al. [9] and Piyaratne [25] independently proposed a modification of the inequality and show that the modified one also implies the existence.

Let us discuss our main case where stability conditions do not exist. Since a stability condition on a triangulated category naturally induces a bounded $t$-structure (see also Remark 3.2), if the category has no bounded $t$-structure, then there are no stability condition. Such a category can be found by using schemes with singularities or varieties which are non-proper. For instance, as mentioned by Antieau, Gepner, and Heller [4], if the scheme $X$ is a nodal cubic curve, then the triangulated category $\text{D}^\text{perf}(X)$ of perfect complexes on $X$ has no bounded $t$-structure and hence no stability condition. They also conjecture that the category of perfect complexes on a finite dimensional Noetherian scheme $X$ has a bounded $t$-structure if and only if $X$ is regular. If the scheme $X$ is affine, the conjecture holds by Smith [28]. Recently Neeman [22] studies the conjecture and proposes a generalization.

Let us introduce another example for the non-existence. Suppose that $X$ is an affine scheme $\text{Spec} \ R$ of a Noetherian ring $R$. Then $\text{Stab} \text{D}^b(X)$ is non-empty if and only if $\dim X = 0$ by the author [15]. Unlike the case of perfect complexes, the category $\text{D}^b(X)$ has a natural bounded $t$-structure. Instead of the non-existence of $t$-structures, our proof is based on the supports of complexes in $\text{D}^b(X)$.

The aim of this paper is to extend the results in [15] to more general $R$-linear triangulated categories. The main theorem is the following:

**Theorem 1.1** Let $R$ be a Noetherian ring and let $X \to \text{Spec} \ R$ be a proper morphism of schemes. If the dimension of the image is positive, then $\text{D}^b(X)$ and $\text{D}^\text{perf}(X)$ have no stability condition.

Recall that the support $\text{Supp} \ E$ of a complex $E$ of (quasi-)coherent sheaves is the union of the support of each cohomology $H^i(E)$. To develop the argument in [15], we give a generalization of the support for objects in an $R$-linear triangulated category $\text{D}$. Precisely if $E$ is an object in the $R$-linear category $\text{D}$, we define $\text{Supp}_R E$ as the support of the $R$-module $\text{Hom}_\text{D}(E, E)$ of endomorphisms. The generalized support $\text{Supp}_R E$ coincides with $\text{Supp} E$ if $E$ is a bounded complex of (not necessarily finite) $R$-modules.

Using the generalized support $\text{Supp}_R E$, we give a sufficient condition for the non-existence of stability conditions under a certain finiteness assumption on $\text{D}$. The properness in Theorem 1.1 guarantees the finiteness.

Finally we would like to mention a related work. For a flat family of projective variety $\mathcal{X} \to S$, Bayer et al. [5] introduce families of stability conditions, which are analogous to the notion of relatively ample line bundle on the family $\mathcal{X} \to S$, to study Kuznetsov’s non-commutative K3 surfaces. From an algebraic point of view, Theorem 1.1 gives the importance of their work as follows. A global stability condition on $\text{D}^b(\mathcal{X})$ cannot exist, but families of stability conditions on the fibers may. In fact, if the dimension of the fiber is smaller than 3, a family exists by [5].
2 R-Linear Triangulated Categories

From now on, $R$ is a commutative ring. Recall that the support of an $R$-module $N$ is the set of prime ideals $p$ such that the localization of $N$ at $p$ is non-zero. The support of the module $N$ is denoted by $\text{Supp} \ N$. Namely we have

$$\text{Supp} \ N = \{ p \in \text{Spec} \ R \mid R_p \otimes_R N \neq 0 \}.$$

**Definition 2.1** Let $\mathbf{D}$ be an $R$-linear triangulated category and $M$ an object in $\mathbf{D}$.

(1) We denote by $\mu : R \to \text{Hom}_\mathbf{D}(M, M)$ the morphism defined by the following composition and the value of $r \in R$ is denoted by $\mu_r$:

$$R \times \{ \text{id}_M \} \subset R \times \text{Hom}_\mathbf{D}(M, M) \to \text{Hom}_\mathbf{D}(M, M).$$  \hfill (2.1)

(2) The support of the object $M$ is defined as the support of $\text{Hom}_\mathbf{D}(M, M)$ as an $R$-module and is denoted by $\text{Supp}_R M$:

$$\text{Supp}_R M := \text{Supp} \text{Hom}_\mathbf{D}(M, M).$$

**Lemma 2.2** Let $\mathbf{D}$ be an $R$-linear triangulated category. Suppose that a distinguished triangle $A \xrightarrow{i} B \xrightarrow{p} C$ in $\mathbf{D}$ satisfies $\text{Hom}_\mathbf{D}^0(A, C) = \text{Hom}_\mathbf{D}^{-1}(A, C) = 0$.

(1) Any morphism $\varphi : B \to B$ uniquely induces morphisms $\varphi_A : A \to A$ and $\varphi_C : C \to C$ such that $i \cdot \varphi_A = \varphi_B \cdot i$ and $p \cdot \varphi_B = \varphi_C \cdot p$.

(2) If $\varphi = 0$ then the morphisms $\varphi_C$ and $\varphi_A$ are zero.

**Proof** We obtain the diagram of exact sequences of $R$-modules:

$$\text{Hom}_\mathbf{D}(A, A) \xrightarrow{\alpha} \text{Hom}_\mathbf{D}(B, B) \xrightarrow{i^*} \text{Hom}_\mathbf{D}(A, B) \xrightarrow{p_*} \text{Hom}_\mathbf{D}(B, C) \xrightarrow{\gamma} \text{Hom}_\mathbf{D}(A, C)$$

The assumptions imply both $\alpha$ and $\gamma$ are isomorphisms. Then $\varphi_A$ and $\varphi_C$ are given by

$$\varphi_A = \alpha^{-1}(i_* \varphi) \text{ and } \varphi_C = \gamma^{-1}(p_* \varphi).$$

The second assertion is obvious from the above diagram. \hfill $\square$

**Proposition 2.3** Let $\mathbf{D}$ be an $R$-linear triangulated category. Suppose that a distinguished triangle $A \xrightarrow{i} B \xrightarrow{p} C$ in $\mathbf{D}$ satisfies $\text{Hom}_\mathbf{D}^0(A, C) = \text{Hom}_\mathbf{D}^{-1}(A, C) = 0$.

Then the following holds:

$$\text{Supp}_R A \cup \text{Supp}_R C \subset \text{Supp}_R B$$  \hfill (2.2)

**Proof** Suppose that $p \notin \text{Supp}_R B$. It is enough to show that $p \notin \text{Supp}_R A$ and $p \notin \text{Supp}_R C$. By the assumption $p \notin \text{Supp}_R B$, there exists $r \in R - p$ such that $r \cdot \text{id} = \mu_r = 0$ in $\text{End}(B)$. \hfill $\heartsuit$
Since $D$ is $R$-linear, the endomorphism $\mu_r \in \text{End}(B)$ also induces the endomorphisms of $A$ and $C$ which make the following diagram commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B & \xrightarrow{p} & C \\
\downarrow{\mu_r} & & \downarrow{\mu_r} & & \downarrow{\mu_r} \\
A & \xrightarrow{i} & B & \xrightarrow{p} & C
\end{array}
$$

By Lemma 2.2, all vertical arrows are zero. Thus the localizations $\text{End}(A) \otimes_R R_p$ and $\text{End}(C) \otimes_R R_p$ are zero. \qed

**Corollary 2.4** Let $R$ be an $R$-linear triangulated category. The $i$th cohomology of $E \in D$ with respect to a bounded $t$-structure $(D^{\leq 0}, D^{\geq 1})$ on $D$ is denoted by $H^i(E)$. Then the following holds:

$$
\text{Supp}_R H^i(E) \subset \text{Supp}_R E.
$$

**Proof** Set $p$ and $q$ by

$$
p = \max\{i \in \mathbb{Z} \mid H^i(E) \neq 0\}, \quad \text{and}
q = \min\{i \in \mathbb{Z} \mid H^i(E) \neq 0\}.
$$

The proof is by induction on $p - q$. If $p - q = 0$, then the assertion is clear.

Taking the filtration with respect to the $t$-structure, we obtain the following triangle

$$
E^{p-1} \longrightarrow E \longrightarrow H^p(E)[-p].
$$

where $E^{p-1} \in D^{\leq 0}[1-p]$. Lemma 2.2 implies $\text{Supp}_R E^{p-1} \subset \text{Supp}_R E$ and $\text{Supp}_R H^p(E) \subset \text{Supp}_R E$. Then the assumption of induction implies the desired assertion. \qed

**Lemma 2.5** Let $D$ be an $R$-linear triangulated category. Consider a diagram of distinguished triangle in $D$:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B & \xrightarrow{p} & C \\
\downarrow{\psi_A} & & \downarrow{\psi_B} & & \downarrow{\psi_C} \\
A & \xrightarrow{i} & B & \xrightarrow{p} & C
\end{array}
$$

If $\psi_A = 0$ and $\psi_C = 0$ then the composite $\psi_B^2$ is zero.

**Proof** Since $p \cdot \psi_B = \psi_C \cdot p = 0$, there exists a morphism $\varphi : B \to A$ such that $i \cdot \varphi = \psi_B$. Thus we see

$$
\psi_B^2 = \psi_B \cdot i \cdot \varphi = i \cdot \psi_A \cdot \varphi = 0.
$$

\qed

**Lemma 2.6** Let $A \xrightarrow{i} B \xrightarrow{p} C$ be a distinguished triangle in an $R$-linear triangulated category $D$. Then the following holds:

$$
\text{Supp}_R B \subset \text{Supp}_R A \cup \text{Supp}_R C.
$$
Proof  Take a prime ideal $p$ such that $p \notin \text{Supp}_R A$ and $p \notin \text{Supp}_R C$. Then there exists $r_A$ (resp. $r_C$) in $R - p$ such that $r_A \cdot \text{id}_A = 0$ (resp. $r_C \cdot \text{id}_C = 0$). Then $r_0 = r_A r_C$ satisfies $r_0 \text{id}_A = 0$ and $r_0 \text{id}_C = 0$. Since the category $D$ is $R$-linear, the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\mu_{r_0} & & \mu_{r_0} \\
A & \xrightarrow{i} & B
\end{array}
\quad
\begin{array}{ccc}
& & p \\
& \mu_{r_0} & \\
& & \\
& & p \\
C & \xrightarrow{\mu_{r_0}} & C
\end{array}
$$

By Lemma 2.5, we see $\mu^2_{r_0} = \mu_{r_0} : B \to B$ is the zero morphism. Hence $\text{id}_B \in \text{End}(B)$ is zero via localization to $p \in \text{Spec } R$. Thus we see $p \notin \text{Supp}_R B = \text{Supp } \text{End}(B)$.

Corollary 2.7  Let $D$ be an $R$-linear triangulated category. The $i$th cohomology of $E \in D$ with respect to a bounded $t$-structure on $D$ is denoted by $H^i(E)$. The following holds:

$$\text{Supp}_R E = \bigcup_{i \in \mathbb{Z}} \text{Supp}_R H^i(E).$$

Proof  By Corollary 2.4, it is enough to show $\text{Supp}_R E \subset \bigcup_{i \in \mathbb{Z}} \text{Supp}_R H^i(E)$. Set $p$ and $q$ by

$$
p = \max \{ i \in \mathbb{Z} \mid H^i(E) \neq 0 \}, \quad\text{and}
q = \min \{ i \in \mathbb{Z} \mid H^i(E) \neq 0 \}.
$$

The proof is by the induction on $p - q$. Similarly to the proof of Corollary 2.4, we have the distinguished triangle

$$
E^{p-1} \to E \to H^p(E)[-p].
$$

Lemma 2.6 implies

$$
\text{Supp}_R E \subset \text{Supp}_R E^{p-1} \cup \text{Supp}_R H^p(E).
$$

Then the assumption of the induction implies

$$
\text{Supp}_R E^{p-1} = \bigcup_{i \in \mathbb{Z}} H^i(E^{p-1})
$$

which completes the proof.

In the last of this section, we show that the generalized support coincides with “usual supports” of complexes of $R$-modules. So let us suppose a triangulated category is the unbounded derived category $D(\text{Mod } R)$ of (not necessarily finite) $R$-modules. Recall that the support of an object $E$ in $D(\text{Mod } R)$ is the union of the supports of the $i$th cohomology $H^i(E)$:

$$\text{Supp } E := \bigcup_{i \in \mathbb{Z}} \text{Supp } H^i(E).$$

Lemma 2.8  Let $R$ be commutative ring and let $E$ be a complex of $R$-modules.

1. We have $\text{Supp } E \subset \text{Supp}_R E$.
2. If the complex $E$ is bounded, then $\text{Supp } E = \text{Supp}_R E$. 

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Proof Suppose \( p \notin \text{Supp}_R E \). Then there exists \( r \in R - p \) such that \( r \cdot \text{id} = \mu_r \) is zero in \( \text{Hom}(E, E) \). Taking cohomology with respect to the standard \( t \)-structure, \( \mu_r : E \to E \) induces the multiplication \( H^i(\mu_r) : H^i(E) \to H^i(E) \). Note that \( H^i(\mu_r) \) is also the multiplication morphism \( \mu_r \) on \( H^i(E) \). Since \( \mu_r \in \text{End}(E) \) is zero, so does \( \mu_r = H^i(\mu_r) \in \text{End}(H^i(E)) \) for any \( i \in \mathbb{Z} \) by Lemma 2.2. Hence \( p \) is not in \( \text{Supp} H^i(E) \) for any \( i \in \mathbb{Z} \). Thus we have

\[
\text{Supp} E = \bigcup_{i \in \mathbb{Z}} \text{Supp} H^i(E) \subset \text{Supp}_R E.
\]

Suppose that \( E \) is bounded and take a \( p \in \text{Spec} \ (R - \bigcup_{i \in \mathbb{Z}} \text{Supp} H^i(E) \). Then there exists \( r_i \in R - p \) such that \( \mu_{r_i} : H^i(E) \to H^i(E) \) is zero for each \( i \) with \( H^i(E) \neq 0 \). Since \( E \) is bounded, let \( s \) be the product of such \( r_i \).

Then the morphism \( \mu_s \in \text{End}(H^i(E)) \) is zero for any \( i \in \mathbb{Z} \). By Lemma 2.9 below, there exists \( N \in \mathbb{N} \) such that the \( N \)-th composite \( \mu_s^N = \mu_{sN} \) is the zero morphism of \( E \). Hence we see \( p \notin \text{Supp}_R E \).

Lemma 2.9 Let \( E \) be a bounded complex of \( R \)-modules. For an \( s \in R \), if \( \mu_s \in \text{End}(H^i(E)) \) is zero for any \( i \in \mathbb{Z} \), then there exists \( N \in \mathbb{N} \) such that the composite \( \mu_s^N = \mu_{sN} \) is zero in \( \text{End}(E) \).

Proof Set \( p := \max \{ i \in \mathbb{Z} \mid H^i(E) \neq 0 \} \) and \( q := \{ i \in \mathbb{Z} \mid H^i(E) \neq 0 \} \). The proof is by induction on \( p - q \).

If \( p - q = 0 \), then \( N \) is taken to be 1. Suppose the assertion holds for \( p - q = \ell - 1 \). Taking truncation of \( E \), we have the following distinguished triangle:

\[
E^{p-1} \longrightarrow E \longrightarrow H^p(E)[-p].
\]

By the assumption on induction, there exists \( N \in \mathbb{N} \) such that \( \mu_s^N : E^{p-1} \to E^{p-1} \) is zero. Since \( \mu_s^N : H^p(E) \to H^p(E) \) is zero, Lemma 2.5 implies that the endomorphism \( \mu_s^{2N} : E \to E \) is zero. \( \square \)

3 Stability Conditions and \( R \)-Linear Categories

The aim in this section is to study a property of the generalized supports for \( \sigma \)-stable objects with respect to a stability condition \( \sigma \). We would like to start this section with a brief review of stability conditions. All the details are in the original article [11] due to Bridgeland.

3.1 Stability Conditions

Definition 3.1 A stability condition \( \sigma = (Z, P) \) on a triangulated category \( D \) is a pair of a group homomorphism \( Z : K_0(D) \to \mathbb{C} \) called a central charge and a collection \( P \) of full subcategories \( P(\phi) \) of \( D \) for \( \phi \in \mathbb{R} \) satisfying the following condition:

1. For any \( E \in P(\phi) \), we have the value of the class \([E]\) by \( Z \) is in \( \mathbb{R}_{>0} \cdot \exp(\sqrt{-1} \pi \phi) \).
2. An object \( E \in D \) is in \( P(\phi) \) if and only if \( E[1] \) is in \( P(\phi + 1) \) for any \( \phi \in \mathbb{R} \).
3. If \( \phi < \psi \) then \( \text{Hom}_D(E, F) = 0 \) for any \( E \in P(\phi) \) and \( F \in P(\psi) \).
(4) for any object $E$ in $\mathbf{D}$, there exists a finite sequence of distinguished triangles

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n = E$$

such that each mapping cone $A_i$ of the morphism $E_{i-1} \rightarrow E_i$ is in $\mathcal{P}(\phi_i) \setminus \{0\}$ and that the sequence $\{\phi_i\}_{i=1}^n$ is decreasing.

**Remark 3.2** We would like to summarize basics for stability conditions.

1. The collection of full subcategories $\mathcal{P} = \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$ is called the *slicing of $\mathbf{D}$*.
2. One can show the sequence (3.1) is unique up to isomorphisms and is called the *Harder-Narasimhan filtration* of $E \in \mathbf{D}$. We refer to the natural number $n$ in (3.1) as the *length* of the Harder-Narasimhan filtration of $E$. By the definition, the length of $E$ is 1 if and only if $E$ is in $\mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$.
3. Given a stability condition $\sigma = (Z, \mathcal{P})$ and an interval $I \subset \mathbb{R}$, set a full subcategory $\mathcal{P}(I)$ by the extension closure containing $\bigcup_{\phi \in I} \mathcal{P}(\phi)$. Since the incusion $\mathcal{P}((-\infty, 0])[1] \subset \mathcal{P}((-\infty, 0])$ holds, the pair $(\mathcal{P}((-\infty, 0]), \mathcal{P}((0, \infty)))$ gives a bounded $t$-structure on $\mathbf{D}$ and the category $\mathcal{P}((0, 1]) = \mathcal{P}((-\infty, 1]) \cap \mathcal{P}((0, \infty))$ is abelian.
4. One can show that the full subcategory $\mathcal{P}(\phi)$ is abelian for all $\phi \in \mathbb{R}$. A non-zero object in $\mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$ is said to be $\sigma$-*semistable* with the phase $\phi$. Moreover if a $\sigma$-semistable object $A$ with the phase $\phi$ does not have non-trivial subobject $F$ in $\mathcal{P}(\phi)$, then $A$ is said to be $\sigma$-*stable*.
5. A stability condition $\sigma = (Z, \mathcal{P})$ is said to be *locally finite* if for any $\phi \in \mathbb{R}$, there exists an $\epsilon > 0$ such that the quasi-abelian category $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is finite length. By the definition of $\sigma$, any object $E \in \mathbf{D}$ is given by a successive extension of finite $\sigma$-semistable objects. If the stability condition is locally finite, any semistable object is given by a successive extension of finite $\sigma$-stable objects.
6. The set of locally finite stability conditions on $\mathbf{D}$ is denoted by $\text{Stab} \mathbf{D}$. The set $\text{Stab} \mathbf{D}$ has a natural topology and each connected component of $\text{Stab} \mathbf{D}$ has a complex structure by [11, Theorem 1.2].

**Remark 3.3** The support property (cf. [16, Section 1.2]) is often required in Definition 3.1. To distinguish the difference, we refer to a stability condition with the support property as a Bridgeland stability condition. Since the support property implies the locally finiteness, the set of Bridgeland stability conditions is a subset of $\text{Stab} \mathbf{D}$.

The support property yields a nice property for deformations of stability conditions and hence is important. However, we do not require it since we do not discuss deformations.

Though the following lemma is very well-known for experts, we write down it since it is very crucial for our purpose.

**Lemma 3.4** Let $\sigma = (Z, \mathcal{P})$ be a locally finite stability condition on an $R$-linear triangulated category $\mathbf{D}$. If an object $A \in \mathbf{D}$ is $\sigma$-stable with phase $\phi$, then any non-zero endomorphism $A \rightarrow A$ is an isomorphism.
Proof Since $\text{Hom}_D(A, A)$ is stable under the shift functor $[1]: D \to D$, we may assume $\phi \in (0, 1)$ without loss of generality. Let $\text{Im} \phi$ be the image of $\phi: A \to A$ in the abelian category $\mathcal{P}((0, 1])$. Since $\text{Im} \phi$ is a quotient of $A$ and a subobject of $A$, if $\text{Im} \phi$ is not $A$, then we have

$$\arg Z(A) < \arg Z(\text{Im} \phi) < \arg Z(A),$$

which clearly gives a contradiction. Hence $\text{Im} \phi$ must be isomorphic to $A$. Then this implies both the kernel and the cokernel of $\phi$ are zero, and hence $\phi$ is an isomorphism. 

\[ \square \]

### 3.2 Isomorphism Property

We wish to study a property of $\text{Supp}_R A$ for a $\sigma$-stable object $A$ in an $R$-linear triangulated category $D$. The following definition reflects properties of stable objects.

**Definition 3.5** Let $D$ be an $R$-linear triangulated category. An object $M \in D$ satisfies the isomorphism property if the following holds:

(Ism) $\forall r \in R$, $\mu_r : M \to M$ is an isomorphism if $\mu_r$ is non-zero.

For $\text{Supp}_R E$ to be well-behaved, we need a finiteness assumption on the triangulated category $D$.

**Definition 3.6** Let $D$ be an $R$-linear triangulated category. $D$ is said to be finite over $R$ if the $R$-module $\text{Hom}_D(E, E)$ is finite for any object $E \in D$.

**Lemma 3.7** Assume that the commutative ring $R$ is an integral domain whose Krull dimension $\dim R$ is positive and an $R$-linear triangulated category $D$ is finite. If an object $M$ in $D$ satisfies

(a) the morphism $\mu_r : M \to M$ is an isomorphism for any $r \in R \setminus \{0\}$,

then $M$ is zero.

**Proof** It is enough to show $\text{Hom}_D(M, M) = 0$, since the category $D$ is additive.

If $\text{Hom}_D(M, M)$ is non-zero, then we have $\text{Supp}_R M = \text{Spec} R$ since $\text{Hom}_D(M, M)$ contains $R$ by the assumption (a). On the other hand, there exists a non-unit $r$ in $R$. Then the assumption implies that $\mu_r$ gives an isomorphism of $M$. Hence we have the following isomorphism

$$\mu_r^*: \text{Hom}_D(M, M) \to \text{Hom}_D(M, M).$$

Hence the $R$-module $\text{Hom}_D(M, M)$ satisfies $\text{Hom}_D(M, M) \otimes R/(r) = 0$ which implies $\text{Supp}_R M \cap \text{Spec} R/(r) = \emptyset$ since $\text{Hom}_D(M, M)$ is finitely generated. This contradicts the fact $\text{Supp}_R M = \text{Spec} R$. 

\[ \square \]

**Lemma 3.8** Suppose an object $M$ in a finite $R$-linear triangulated category $D$ satisfies the condition (Ism). Then the following holds:

1. The ideal $\text{ann}(\text{End}(M))$ of $R$ is prime.
2. If $M$ is non-zero, then $\text{ann}(\text{End}(M)) = \text{ann}(f)$ for any $f \in \text{End}(M) \setminus \{0\}$, that is, the $R$-module $\text{End}(M)$ has a unique associated prime.
3. If $M$ is non-zero, then $\dim \text{Supp}_R M = 0$. 

\[ \square \]
Proof We prove the first assertion. Let \( a \) and \( b \) be in \( R \). Suppose the product \( ab \) is in \( \text{ann}(\text{End}(M)) \) and \( a \notin \text{ann}(\text{End}(M)) \). Then the endomorphism \( \mu_a \) is non-zero and hence is invertible. Thus \( \mu_b \) is zero by \( \mu_{ab} = \mu_a \mu_b \). Hence \( b \in \text{ann}(\text{End}(M)) \).

To prove the second assertion, take \( a \in \text{ann}(f) \). If \( \mu_a \) is non-zero, then \( \mu_a \) is an isomorphism by the assumption. This contradicts \( a \in \text{ann}(f) \). Thus \( \mu_a \) is the zero-morphism and we have \( \text{ann}(f) \subset \text{ann}(\text{End}(M)) \). The opposite inclusion is clear.

We finally prove the assertion (3). Put \( p = \text{ann}(\text{End}(M)) \). \( \text{End}(M) \) is an \( R/p \)-module and \( M \) satisfies the condition (Ism). If the prime ideal \( p \) is not maximal, then \( \dim R/p > 0 \) and Lemma 3.7 implies \( M = 0 \). Hence \( p \) has to be maximal and \( \dim \text{Supp}_R M = 0 \) since \( \text{Supp}_R M \) is a subset of \( \text{Spec } R/p \).

\[ \text{Lemma 3.9} \quad \text{Let } D \text{ be a finite } R \text{-linear triangulated category and let } \sigma \text{ be a locally finite stability condition on } D \text{. If an object } E \in D \text{ is } \sigma \text{-stable then } \dim \text{Supp}_R E = 0. \]

Proof If \( E \) is \( \sigma \)-stable, then the condition (Ism) holds by Lemma 3.4. Hence Lemma 3.8 implies \( \dim \text{Supp}_R E = 0 \).

\[ \text{Proposition 3.10} \quad \text{Let } D \text{ be a finite } R \text{-linear triangulated category and } \sigma \text{ be a locally finite stability condition on } D \text{. If an object } E \in D \text{ is } \sigma \text{-semistable, then } \dim \text{Supp}_R E = 0. \]

Proof Let \( \mathcal{P}(\phi) \) be the slicing of \( \sigma \) with phase \( \phi \). Recall that \( \mathcal{P}(\phi) \) is an abelian category. Since \( \sigma \) is locally finite, any object \( E \in \mathcal{P}(\phi) \) is given by a successive extension of finite \( \sigma \)-stable objects.

The proof is induction on the number of stable factors of \( E \in \mathcal{P}(\phi) \). If the number is 1, the assertion follows from Lemma 3.9 since \( E \) is \( \sigma \)-stable.

Now \( E \) is not \( \sigma \)-stable but \( \sigma \)-semistable. Take a \( \sigma \)-stable subobject \( A \) of \( E \). Then the quotient \( E/A \) satisfies the assumption on the induction. Hence Lemma 2.6 implies

\[ \text{Supp}_R E \subset \text{Supp}_R A \cup \text{Supp}_R E/A. \]

and the assumption on induction implies \( \dim \text{Supp}_R E/A = 0 \). Then \( \text{Supp}_R E \) is a subset of a zero-dimensional set, and this gives the proof.

\[ \text{Theorem 3.11} \quad \text{Let } D \text{ be a finite } R \text{-linear triangulated category. Suppose that there exists a locally finite stability condition } \sigma \text{ on } D \text{. For any object } E \in D \text{, the dimension of the support } \text{Supp}_R E \text{ is zero. Moreover any cohomology } H^1(E) \text{ of } E \text{ has zero dimensional.} \]

Proof The proof is by induction on the length \( \ell(E) \) of the Harder-Narasimhan filtration of \( E \) with respect to \( \sigma \). If \( \ell(E) = 1 \), the assertion follows from Proposition 3.10.

Suppose that the assertion holds for \( \ell(E) - 1 \). Taking the last term of the Harder-Narasimhan filtration of \( E \), we obtain the distinguished triangle

\[ E_{n-1} \longrightarrow E \longrightarrow A_n, \quad (3.2) \]

where \( A_n \) is \( \sigma \)-semistable and \( \ell(E_{n-1}) = \ell(E) - 1 \). Thus we see \( \dim \text{Supp}_R E_{n-1} = 0 \) and \( \dim \text{Supp}_R A_n = 0 \). Then Lemma 2.6 implies the desired assertion. The last assertion follows from Corollary 2.7.

\[ \text{Corollary 3.12} \quad \text{Let } D \text{ be a finite } R \text{-linear triangulated category. If there exists an object } M \in D \text{ such that } \dim \text{Supp}_R M > 0, \text{ then there exists no locally finite stability condition on } D. \]

Proof If there exists a locally finite stability condition on \( D \), the dimension the support of any object in \( D \) is zero. This contradicts Theorem 3.11.
Corollary 3.13  Let \( f : \mathcal{X} \rightarrow \text{Spec } R \) be a proper morphism to the affine Noetherian scheme Spec \( R \). If the dimension of the image of \( f \) is positive, then both \( \text{Stab } \mathcal{D}^b(\mathcal{X}) \) and \( \text{Stab } \mathcal{D}^{\text{perf}}(\mathcal{X}) \) are empty.

Proof  Recall that \( \mathcal{D}^{\text{perf}}(\mathcal{X}) \) is a full subcategory of \( \mathcal{D}^b(\mathcal{X}) \) and the structure sheaf \( \mathcal{O}_X \) is in \( \mathcal{D}^{\text{perf}}(\mathcal{X}) \).

Let \( Z \) be the image of \( f \). Note that \( Z \) is a closed subscheme of Spec \( R \). Then the morphism \( \tilde{f} : \mathcal{X} \rightarrow Z \) is proper and hence \( \mathcal{D}^b(\mathcal{X}) \) is linear and finite over the ring \( H^0(Z, \mathcal{O}_Z) \). Since \( \tilde{f} \) is surjective, the ring \( H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) \) contains \( H^0(Z, \mathcal{O}_Z) \). Then the assumption implies

\[
\dim \text{Supp } H^0(Z, \mathcal{O}_Z) \mathcal{O}_\mathcal{X} \geq \dim \text{Supp } H^0(Z, \mathcal{O}_Z) > 0.
\]

Then Corollary 3.12 implies the desired assertion.

Remark 3.14  In [15], we show that the bounded derived category of a affine Noetherian scheme Spec \( R \) does not have a locally finite stability condition if \( \dim R > 0 \). Corollary 3.13 gives a generalization.

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Appendix A by Hiroyuki Minamoto

Inspired by the main body this paper, we prove the following theorem.

Theorem A.1  Let \( X \) be a Noetherian scheme. Assume that the Krull dimension \( \dim \Gamma(X, \mathcal{O}_X) \) of the ring \( \Gamma(X, \mathcal{O}_X) \) is positive. Then both \( \text{Stab } \mathcal{D}^b(X) \) and \( \text{Stab } \mathcal{D}^{\text{perf}}(X) \) are empty.

We need preparations.

Lemma A.2  Let \( X \) be a Noetherian scheme, \( s \in \Gamma(X, \mathcal{O}_X) \) a global section, and \( Z := \text{Spec } \mathcal{O}_X/(s) \) the vanishing locus of the section \( s \). For an object \( M \in \mathcal{D}^b(X) \), we denote by \( \mu^M_s \) the multiplication morphism \( M \rightarrow M \) defined in Definition 2.1. Then for \( M \in \mathcal{D}^b(X) \), the following assertion hold:

1. If \( \mu^M_s \) is zero, then \( \text{Supp } M \subset Z \).
2. If \( \mu^M_s \) is an isomorphism, then \( \text{Supp } M \subset X \setminus Z \).

Proof  (1) By the assumption, the support of \( i \)-th cohomology \( H^i(M) \) of \( M \) with respect to standard \( t \)-structure is contained in \( Z \) since the cohomology \( H^i(\mu^M_s) \) of the morphism \( \mu^M_s \) is zero for all \( i \in \mathbb{Z} \). This gives the proof.

(2) By the assumption, the multiplication morphism \( H^i(\mu^M_s) : H^i(M) \rightarrow H^i(M) \) is an isomorphism for each \( i \in \mathbb{Z} \). Hence the localization \( H^i(\mu^M_s)_x : H^i(M)_x \rightarrow H^i(M)_x \) of the
morphism $H^i(\mu_s^M)$ on each $x \in X$ is an isomorphism of $\mathcal{O}_{X,x}$ modules. If $x$ is in $Z$, then clearly the germ $s_x$ is in the maximal ideal $m_{X,x}$ of $\mathcal{O}_{X,x}$. Hence we see $m_{X,x}H^i(M)_x = H^i(M)_x$. By Nakayama’s lemma, we have $H^i(M)_x$ is zero and hence $\text{Supp} \ M \subset X \setminus Z$. 

**Proposition A.3** Let $X$ be a connected Noetherian scheme, and $Z$ the vanishing locus $\text{Spec} \mathcal{O}_X/\langle s \rangle$ of a global section $s \in \Gamma(X, \mathcal{O}_X)$. Assume that $s$ is neither nilpotent nor invertible. Then both $\text{Stab} \mathbf{D}^b(X)$ and $\text{Stab} \mathbf{D}^{\text{perf}}(X)$ are empty.

**Proof** We deal with $\mathbf{D}^b(X)$. The same proof works for $\mathbf{D}^{\text{perf}}(X)$.

Suppose to the contrary that $\mathbf{D}^b(X)$ has a locally finite stability condition. Then by the argument of the proofs of Proposition 3.10 and Theorem 3.11, there exists finite non-zero objects $M_1, \ldots, M_n \in \mathbf{D}^b(X)$ having the property (Ism) such that $\mathcal{O}_X$ is in the thick hull of $M_1, \ldots, M_n$.

We set

$I = \{i \mid 1 \leq i \leq n, \mu_s^{M_i} \text{ is an isomorphism}\}, \text{ and } J = \{j \mid 1 \leq j \leq n, \mu_s^{M_j} \text{ is zero}\}$.

Note that $I \cup J = \{1, 2, \ldots, n\}$. We set $Y_I := \bigcup_{i \in I} \text{Supp} \ M_i$ and $Y_J := \bigcup_{j \in J} \text{Supp} \ M_j$. Then we see $Y_I \subset X \setminus Z$ and $Y_J \subset Z$ by Lemma A.2. On the other hand, we have $X = \text{Supp} \mathcal{O}_X \subset \bigcup_{i=1}^n \text{Supp} \ M_i = Y_I \sqcup Y_J \subset Y_I \sqcup Z$ and hence $X = Y_I \sqcup Z$. Observe that $Y_I$ is closed, since the set $I$ is finite and $M_i$ have finitely generated cohomology groups. Since $X$ is connected by the assumption, we have either $Z = X$ or $Z = \emptyset$. However the condition $Z = X$ contradicts to the assumption $s$ is non-nilpotent and the condition $Z = \emptyset$ contradicts to the assumption that $s$ is non-invertible.

We proceed a proof of the main theorem of appendix.

**Proof of Theorem A.1** Take a connected component $X'$ of $X$ such that $\dim \Gamma(X', \mathcal{O}_{X'}) \geq 1$. Then it has a global section which is neither nilpotent nor invertible. Then the assertion follows from Proposition A.3.

**References**

1. Arcara, D., Bertram, A.: Reid’s theorem and Thaddeus pairs revisited. In: Grassmannians, Moduli Spaces and Vector Bundles, volume 14 of Clay Math. Proc., pp. 51–68. Amer. Math. Soc., Providence, RI (2011)

2. Arcara, D., Bertram, A.: Bridgeland-stable moduli spaces for $K$-trivial surfaces. J. Eur. Math. Soc. (JEMS) 15(1):1–38, With an appendix by Max Lieblich (2013)

3. Arcara, Daniele, Bertram, Aaron, Coskun, Izzet, Huizenga, Jack: The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability. Adv. Math. 235, 580–626 (2013)

4. Antieau, Benjamin, Gepner, David, Heller, Jeremiah: $K$-theoretic obstructions to bounded $t$-structures. Invent. Math. 216(1), 241–300 (2019)

5. Bayer, Arend, Lahoz, Martí, Macrì, Emanuele, Nuer, Howard, Perry, Alexander, Stellari, Paolo: Stability conditions in families. Publ. Math. Inst. Hautes Études Sci. 133, 157–325 (2021)

6. Bayer, Arend, Macrì, Emanuele: MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones. Lagrangian fibrations. Invent. Math. 198(3), 505–590 (2014)

7. Bayer, Arend, Macrì, Emanuele: Projectivity and birational geometry of Bridgeland moduli spaces. J. Am. Math. Soc. 27(3), 707–752 (2014)

8. Bayer, Arend, Macrì, Emanuele, Stellari, Paolo: The space of stability conditions on abelian threefolds, and on some Calabi–Yau threefolds. Invent. Math. 206(3), 869–933 (2016)

9. Bernardara, M., Macrì, E., Schmidt, B., Zhao, X.: Bridgeland stability conditions on Fano threefolds. Épijournal Geom. Algébrique, 1:Art. 2, 24, (2017)

10. Bayer, Arend, Macrì, Emanuele, Toda, Yukinobu: Bridgeland stability conditions on threefolds I: Bogomolov–Gieseker type inequalities. J. Algebraic Geom. 23(1), 117–163 (2014)
11. Bridgeland, T.: Stability conditions on triangulated categories. Ann. Math. (2) 166(2), 317–345 (2007)
12. Bridgeland, Tom: Stability conditions on K3 surfaces. Duke Math. J. 141(2), 241–291 (2008)
13. Bridgeland, T.: Spaces of stability conditions. In: Algebraic Geometry—Seattle 2005. Part 1, volume 80 of Proc. Sympos. Pure Math., pp. 1–21. Amer. Math. Soc., Providence, RI (2009)
14. Douglas, M.R.: Dirichlet branes, homological mirror symmetry, and stability. In: Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), pp. 395–408. Higher Ed. Press, Beijing (2002)
15. Kawatani, K.: Stability conditions on affine noetherian schemes, arXiv:2009.14466 (2020)
16. Kontsevich, M., Soibelman, Y.: Stability structures, motivic Donaldson–Thomas invariants and cluster transformations, arXiv:0811.2435 (2008)
17. Li, Chunyi: Stability conditions on Fano threefolds of Picard number 1. J. Eur. Math. Soc. (JEMS) 21(3), 709–726 (2019)
18. Li, Chunyi, Zhao, Xiaolei: Birational models of moduli spaces of coherent sheaves on the projective plane. Geom. Topol. 23(1), 347–426 (2019)
19. Macrì, Emanuele: A generalized Bogomolov–Gieseker inequality for the three-dimensional projective space. Algebra Number Theory 8(1), 173–190 (2014)
20. Maciocia, Antony, Piyaratne, Dulip: Fourier–Mukai transforms and Bridgeland stability conditions on abelian threefolds. Algebra Geom. 2(3), 270–297 (2015)
21. Minamide, Hiroki, Yanagida, Shintarou, Yoshioka, Kota: The wall-crossing behavior for Bridgeland’s stability conditions on abelian and K3 surfaces. J. Reine Angew. Math. 735, 1–107 (2018)
22. Neeman, A.: Bounded t-structures on the category of perfect complexes, arXiv:2202.08861 (2022)
23. Nuer, H.: Projectivity and birational geometry of Bridgeland moduli spaces on an Enriques surface. Proc. Lond. Math. Soc. (3) 113(3), 345–386 (2016)
24. Nuer, H., Yoshioka, K.: MMP via wall-crossing for moduli spaces of stable sheaves on an Enriques surface. Adv. Math. 372, 107283 (2020)
25. Piyaratne, D.: Stability conditions, Bogomolov-Gieseker type inequalities and fano 3-folds, arXiv:1705.04011 (2017)
26. Schmidt, Benjamin: A generalized Bogomolov–Gieseker inequality for the smooth quadric threefold. Bull. Lond. Math. Soc. 46(5), 915–923 (2014)
27. Schmidt, Benjamin: Counterexample to the generalized Bogomolov–Gieseker inequality for threefolds. Int. Math. Res. Not. IMRN 8, 2562–2566 (2017)
28. Smith, H.: Bounded t-structures on the category of perfect complexes over a Noetherian ring of finite Krull dimension. Adv. Math., 399:Paper No. 108241 (2022)