The size of the largest antichains in products of linear orders

D. Bouyssou*a, T. Marchantb*, M. Pirlotc

*aLAMSADE, UMR 7243, CNRS, Université Paris-Dauphine, PSL Research University, 75016 Paris, France
*bGhent University, Dunantlaan 1, 9000 Ghent, Belgium
*cUniversité de Mons, rue de Houdain 9, 7000 Mons, Belgium

Abstract

We present new exact and asymptotic results about the size of the largest antichain in the product of \( n \) linear orders.

Keywords: maximal antichain, Sperner, multichoice cooperative game

1. Introduction

Antichains in the poset \( \{0, 1\}^n \) equipped with the standard partial ordering are well-studied and have many different interpretations (e.g. Ersek Uyanik et al. [2017]). An expression for the maximal size of such antichains in \( \{0, 1\}^n \) is given by a classical theorem of Sperner [1928]. If we consider the more general poset \( \{1, \ldots, m\}^n \) also equipped with the standard partial ordering, an expression for the size of the largest antichain is given in Sander [1993]. Sander also provides asymptotic results when \( m \) is fixed and \( n \) goes to infinity. The interest of Sander in this problem arose form a recreational mathematics problem posed in Motek [1986]. Actually, antichains and, hence, maximal antichains, in the poset \( \{1, \ldots, m\}^n \) are of interest in many domains. For instance in game theory, Hsiao and Raghavan [1993] define a multichoice cooperative game as a real-valued mapping on \( \{1, \ldots, m\}^n \), where \( n \) is the number of players and \( \{1, \ldots, m\} \) denotes the set of ordered actions that each player can take. A profile in such a game is a vector \( x = (x_1, \ldots, x_n) \in \{1, \ldots, m\}^n \) and represents the actions taken by each agent. A winning profile is such that the value of the game at that profile is 1. A winning profile \( x \) is minimal if there is no other winning profile \( y \) such that \( y \leq x \). If a game is monotone, then the set of all minimal winning profiles is an antichain. Besides, Grabisch [2016] shows that antichains in \( \{1, \ldots, m\}^n \) play an important role in the analysis of these

*Authors are listed alphabetically and have equally contributed.
*Corresponding author

Email addresses: bouyssou@lamsade.dauphine.fr (D. Bouyssou), thierry.marchant@ugent.be (T. Marchant), marc.pirlot@umons.ac.be (M. Pirlot)

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multichoice games. Our personal interest in antichains in the poset \( \{1, \ldots, m\}^n \) stems from the analysis of a multicriteria sorting model presented in Fernández et al. (2017). In this context, the size of maximal antichains corresponds to the maximum number of profiles needed to represent a twofold ordered partition in their model, whenever such a representation is possible. Another paper about antichains in \( \{1, \ldots, m\}^n \) is Tsai (2018); it presents an upper bound for the number of antichains (a generalization of Dedekind numbers).

In the present paper, we will extend Sander’s results in two directions. First, we will present an exact expression for the size of the largest antichain in the heterogeneous product \( \prod_{i=1}^{n} \{1, \ldots, m_i\} \). Then, we will provide asymptotic results for the size of the largest antichain in \( \{1, \ldots, m\}^n \) when \( n \) is fixed and \( m \) goes to infinity.

2. Notation and definitions

Let \( P \) be a set and \( \leq \) be a binary relation defined on \( P \), satisfying (i) reflexivity (\( \forall x \in P, x \leq x \)), (ii) antisymmetry (\( \forall x, y \in P, x \leq y \) and \( y \leq x \Leftrightarrow x = y \)) and (iii) transitivity (\( \forall x, y, z \in P, x \leq y \) and \( y \leq z \Rightarrow x \leq z \)). The pair \((P, \leq)\) is called a partially ordered set (poset) \(^1\). When there is no ambiguity, the poset \((P, \leq)\) is simply denoted by \( P \). For all \( x, y \) belonging to a poset \( P \), we say that \( x \) and \( y \) are comparable if \( x \leq y \) or \( y \leq x \). A chain of \( P \) is a totally ordered subset of \( P \). A linear order on \( P \) is a poset such that \( P \) is a chain. An antichain of \( P \) is a subset of pairwise incomparable elements. A largest antichain is an antichain of maximal cardinality.

Let \((P, \leq_p)\) and \((Q, \leq_q)\) be two posets. The product poset \((P \times Q, \leq)\) is defined to be the set of all pairs \((a, b), a \in P, b \in Q\), with the order given by \((a, b) \leq (a', b')\) if and only if \((a \leq_p a')\) and \((b \leq_q b')\). Let \( n \) be a positive integer and \( \mathbf{m} = (m_1, \ldots, m_n) \) be an element of \( \mathbb{N}^n \), where \( \mathbb{N} \) denotes the set of positive integers. For any \( a \in \mathbb{N} \), let \([a]\) denote the set \( \{1, \ldots, a\} \). For any \( i \in [n] \), the poset \((\{m_i\}, \leq)\), where \( \leq \) is the usual ordering of the natural numbers, is a linear order (also called a chain). The product of these \( n \) linear orders is the poset \((\prod_{i=1}^{n} [m_i], \leq)\) where \( \leq \) is defined as follows: for all \( x, y \in \prod_{i=1}^{n} [m_i] \), \( x \leq y \) iff \( x_i \leq y_i \) for all \( i \in [n] \). The poset \((\prod_{i=1}^{n} [m_i], \leq)\) is also called a direct product of chains (Caspard et al., 2012). When \( \mathbf{m} \) is such that \( m_i = m \) for all \( i \in [n] \), then the Cartesian product \( \prod_{i=1}^{n} [m_i] \) is homogeneous and can be written as \([m]^n\).

The size of the largest antichains in \( \prod_{i=1}^{n} [m_i] \) and \([m]^n\) will respectively be denoted by \( s(\mathbf{m}) \) and \( S(m, n) \). Sperner (1928) has proved that the size of the largest antichain in \([2]^n\) is

\[
S(2, n) = \binom{n}{n/2}.
\]

When \( n \) is large, a convenient approximation for \( S(2, n) \) is obtained using Stirling’s formula: \( S(2, n) \sim 2^n \sqrt{2/\pi n} \). Later, Sander (1993) has proved that the

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\(^1\) Most definitions about posets are taken from Proctor et al. (1980).
size of the largest antichain in \([m]^n\) is

\[
S(m, n) = \sum_{j=0}^{\lfloor g/m \rfloor} (-1)^j \binom{n}{j} \left( \frac{n - 1 + g - mj}{n - 1} \right),
\]

(1)

with \(g = \lfloor n(m-1)/2 \rfloor\). Sander has also provided a bound and some asymptotic results for \(S(m, n)\) when \(m\) is fixed. Notice that \(S(m, n)\) corresponds to Sequence A077042 in the Online Encyclopedia of Integer Sequences (OEIS 2019).

Section 3 is devoted to the general case of heterogeneous products and presents some exact results about \(s(m)\). In Section 4, we consider the special case of homogeneous products and we present a new exact result about \(S(m, n)\) and also an asymptotic result when \(n\) is fixed.

3. Heterogeneous product

Let us define \(m_I = \sum_{i \in I} m_i\) and

\[
h = \left\lfloor \frac{n + \sum_{i \in \mathbb{N}} m_i}{2} \right\rfloor.
\]

(2)

Our result about heterogeneous products is the following.

**Theorem 1.** For all \(\overline{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n\),

\[
s(\overline{m}) = \sum_{I \subseteq [n]: m_I < h - n} \binom{h - m_I - 1}{n - 1} (-1)^{|I|}.
\]

(3)

Before proving this result, we recall some definitions and results about posets. Let \((P, \leq)\) be a poset. For any \(x, y \in P\), we say that \(y\) covers \(x\) in \(P\) iff \(x < y\) and there is no \(z\) such that \(x < z < y\). A ranking (or grading) of a poset \(P\) is a partition of \(P\) into (possibly empty) sets \(P_i\) \((i \in \mathbb{Z})\) such that, for each \(i\), every element in \(P_i\) is covered only by elements in \(P_{i+1}\). The set \(P_i\) is called the \(i\)th rank of \(P\). If a poset admits a ranking, then we say that it is ranked (or graded).

The Whitney numbers of a ranked poset \(P\) are \(\{p_i : i \in \mathbb{Z}\}\), where \(p_i\) is the cardinality of \(P_i\). Let \(p_k\) be a largest Whitney number of a ranked poset; we say \(P\) is rank-unimodal if \(p_i \geq p_{i-1}\), for \(i \leq k\) and \(p_i \geq p_{i+1}\), for \(i \geq k\). The Whitney numbers are said to be symmetric, or \(P\) is said to be rank-symmetric, if there exists a \(d\) such that \(p_i = p_{d-i}\) for all \(i\). Let \(P\) and \(Q\) be ranked posets and \(R = P \times Q\). The rankings of \(P\) and \(Q\) induce the following ranking on \(R\):

\[
R_i = \bigcup_i (P_i \times Q_{d-i}),
\]

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2This bound is later rediscovered by Mattner and Roos (2008) in a different context. In addition, Mattner and Roos (2008) note that (1) can be found in de Moivre (1756) as the solution of a probability problem.
and $R$ has Whitney numbers

$$r_l = \sum_{i \in \mathbb{Z}} p_i q_{l-i}. \tag{4}$$

A $k$-family is a subset of $P$ containing no chain of size $k + 1$. Equivalently, a $k$-family is a union of $k$ (possibly empty) antichains, so that if $P$ is ranked, any union of $k$ ranks is a $k$-family. $P$ is said to be Sperner if the rank of largest size is an antichain of maximum size, and $P$ is $k$-Sperner if the union of the $k$ largest ranks is a $k$-family of maximum size. $P$ is strongly Sperner if it is $k$-Sperner for all $k \geq 1$. $P$ is a Peck poset if it is strongly Sperner, rank-unimodal, and rank-symmetric. Theorem 3.2 in Proctor et al. (1980) shows that the product of two Peck posets is a Peck poset.

**Proof of Theorem 1** In this proof, for the sake of brevity, we use $X$ to denote the poset $(\Pi_{i=1}^{n} [m_i], \preceq)$ or the set $\Pi_{i=1}^{n} [m_i]$. For each $i \in [n]$, the poset $([m_i], \leq)$ is a Peck poset. Hence, $n - 1$ applications of Theorem 3.2 in Proctor et al. (1980) show that $X$ is also a Peck poset. Let us consider a ranking of $X$ such that the minimal element $(1, \ldots, 1)$ in $X$ has rank $n$, which is the sum of the coordinates of the minimal element. The maximal element $(m_1, \ldots, m_n)$ has rank $\sum_{i \in [n]} m_i$. Let $p_{j}^{l}$ be the $j$th Whitney number of $([m_i], \leq)$; it is equal to 1 for each non-empty rank. Since $X$ is rank-unimodal, and rank-symmetric, a maximal Whitney number corresponds to the median rank $h$ (defined by (2)). Because of the Sperner property, this rank is also an antichain of maximum size and $n - 1$ applications of (4) show that the size of this antichain is

$$s(m) = \sum_{i_{n-1} \in \mathbb{Z}} \ldots \sum_{i_{2} \in \mathbb{Z}} \sum_{i_{1} \in \mathbb{Z}} p_{i_{1}}^{1} p_{i_{2}}^{2} \ldots p_{i_{n-1}}^{n-1} p_{h-i_{1}-i_{2}-\ldots-i_{n-1}}^{n}.$$

Define $i_n = h - \sum_{j \in [n-1]} i_j$ and we obtain

$$s(m) = \sum_{i_{n-1} \in \mathbb{Z}} \ldots \sum_{i_{2} \in \mathbb{Z}} \sum_{i_{1} \in \mathbb{Z}} p_{i_{1}}^{1} p_{i_{2}}^{2} \ldots p_{i_{n-1}}^{n-1} p_{i_{n}}^{n} = \sum_{\sum_{i \in [n]} i = h} 1. \tag{5}$$

Hence, an antichain of maximum size in $X$ is the set

$$A = \{ x \in X : \sum_{i \in [n]} x_i = h \}$$

and a generating function for $s(m)$ is defined as follows, for all $x \in \mathbb{R}$, $|x| < 1$,

$$f(m, x) = (x^1 + \ldots + x^{m_1}) \times \ldots \times (x^1 + \ldots + x^{m_n}).$$

We also have

$$f(m, x) = \left( \frac{x}{1-x} \right)^n \left( \sum_{l \subseteq [n]} (-1)^{|l|} x^{m_l} \right).$$
We denote by $c_p(A(x))$ the coefficient of $x^p$ in the polynomial $A(x)$. Hence $s(m) = c_h(f(m, x))$. A property of products of polynomials, which extends to absolutely convergent series, is:

$$c_p(A(x)B(x)) = \sum_{k=0}^{p} c_k(A(x)) \times c_{p-k}(B(x)).$$

In our case, we have

$$s(m) = c_h(f(m_1, \ldots, m_n, x))$$

$$= c_h \left( \frac{x}{1-x} \right)^n \times \sum_{I \subseteq [n]} (-1)^{|I|} x^{m_I}$$

$$= \sum_{j=0}^{h} c_j \left( \frac{x}{1-x} \right)^n \times c_{h-j} \left( \sum_{I \subseteq [n]} (-1)^{|I|} x^{m_I} \right).$$

For $|x| < 1$, we have

$$c_j \left( \frac{x}{1-x} \right)^n = c_j(x + x^2 + x^3 + \ldots)^n$$

and $c_j \left( \frac{x}{1-x} \right)^n$ is thus the number of $n$-tuples of positive integers whose sum is equal to $j$. It is \cite{MacMahon1893} p.835. We also have

$$c_{h-j} \left( \sum_{I \subseteq [n]} (-1)^{|I|} x^{m_I} \right) = \sum_{I \subseteq [n]: m_I = h-j} (-1)^{|I|}.$$

Hence

$$s(m) = \sum_{j=0}^{h} \binom{j-1}{n-1} \sum_{I \subseteq [n]: m_I = h-j} (-1)^{|I|}$$

and, since $\binom{j-1}{n-1} = 0$ whenever $j < n$, we obtain

$$s(m) = \sum_{j=n}^{h} \binom{j-1}{n-1} \sum_{I \subseteq [n]: m_I = h-j} (-1)^{|I|}.$$

Permuting the order of the two sums yields

$$s(m) = \sum_{I \subseteq [n]: m_I \leq h-n} \binom{h-m_I-1}{n-1} (-1)^{|I|}.$$

\qed
Table 1 illustrates how $s(m)$ varies as a function of $m$. In particular, we see that increasing one of the $m_i$’s way above the others has a limited impact. When all components of $m$ are identical, it is easy to show that (3) coincides with Sander’s expression (1). In that case, Sander’s expression is computationally more efficient than ours.

This section about heterogeneous products does not contain any asymptotic result because it does not seem relevant to let one of the parameters, say $m_5$, go to infinity while keeping the other parameters constant.

4. Homogeneous product

Let $h = \lceil n(m + 1)/2 \rceil$. Our result about homogeneous products is the following.

**Theorem 2.** For all $n \geq 2$, if $n(m + 1)$ is even, then $S(m, n)$ is equal to

$$m^{n-1} - 2 \sum_{r=n-1}^{h-m-1} \sum_{i=0}^{r+1} (-1)^i \binom{n-1}{i} \binom{r-im-1}{r-im-n+1}. \quad (6)$$

Otherwise, $S(m, n)$ is equal to

$$m^{n-1} - 2 \sum_{r=n-1}^{h-m-1} \sum_{i=0}^{r+1} (-1)^i \binom{n-1}{i} \binom{r-im-1}{r-im-n+1}$$

$$- \sum_{i=0}^{r+1} (-1)^i \binom{n-1}{i} \binom{h-im-1}{h-im-n+1}. \quad (7)$$

**Proof.** We only prove (6). The proof of (7) is similar. We have seen in the proof of Theorem 1 that an antichain of maximum size in $\Pi_{i \in [n]} [m_i]$ is the set $A = \{x \in X : \sum_{i \in [n]} x_i = h\}$ where $h = \lceil n+\sum_{i \in [n]} m_i \rceil$. Hence, if $n(m+1)$ is even, then an antichain of maximum size in $[m]^n$ is the set $A = \{x \in [m]^n : \sum_{i \in [n]} x_i = h\}$, with $h = n(m+1)/2$. Since $1 \leq x_n \leq m$, if we project the set $A$ on $[m]^{n-1}$ by dropping the last coordinate $x_n$, we obtain the set $A' = \{y \in [m]^{n-1} : h - m \leq \sum_{i \in [n-1]} y_i \leq h - 1\}$. Since no $x, y \in A$ are comparable, we know that no distinct $x, y \in A$ project on the same element in $[m]^{n-1}$. Hence $|A'| = |A|$ and $S(m, n)$ is equal to

$$|A'| = m^{n-1} - \left| \{y \in [m]^{n-1} : \sum_{i \in [n-1]} y_i \leq h - m - 1\} \right|$$

$$- \left| \{y \in [m]^{n-1} : \sum_{i \in [n-1]} y_i \geq h\} \right|$$

$$= m^{n-1} - 2 \left| \{y \in [m]^{n-1} : \sum_{i \in [n-1]} y_i \leq h - m - 1\} \right|. \quad (6)$$
where the last equality holds because

\[ n(m+1)/2 - m - 1 - \min_{y \in [m]^{n-1}} \sum_{i \in [n-1]} y_i = \max_{y \in [m]^{n-1}} \sum_{i \in [n-1]} y_i - h. \]

Let us rewrite \( \{ y \in [m]^{n-1} : \sum_{i \in [n-1]} y_i \leq h - m - 1 \} \) as the union of several sets:

\[ \bigcup_{r=n-1}^{h-m-1} A_r \text{ where } A_r = \{ y \in [m]^{n-1} : \sum_{i \in [n-1]} y_i = r \}. \]

Clearly, for any \( r \neq s \), \( A_r \cap A_s = \emptyset \) and

\[ S(m, n) = m^{n-1} - 2 \sum_{r=n-1}^{h-m-1} |A_r|. \] (8)

Let

\[ B_r = \{ y \in \mathbb{N}_+^{n-1} : \sum_{i \in [n-1]} y_i = r \}, \]

\[ C^l_r = \{ y \in \mathbb{N}_+^{n-1} : \sum_{i \in [n-1]} y_i = r \text{ and } y_l > m \} \text{ for } l \in [n-1] \]

and \( C^l_r = B_r \setminus C^l_r \). Then \( A_r = \bigcap_{l \in [n-1]} C^l_r \) and, thanks to the inclusion-exclusion principle,

\[
|A_r| = |B_r| - \left| \bigcup_{l \in [n-1]} C^l_r \right|
= |B_r| - \sum_{ \emptyset \neq J \subseteq [n-1] } (-1)^{|J|-1} \left| \bigcap_{l \in J} C^l_r \right|
= |B_r| - \sum_{ [i] : 1 \leq i \leq n } (-1)^{i-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) \left| \bigcap_{l \in [i]} C^l_r \right|,
\]

where the last equality holds because all dimensions play the same role. The set \( B_r \) is a regular \((n-2)\)-dimensional simplex. Its cardinality is equal to the \((r-n+1)\)-th simplicial polytope number in \( n-2 \) dimensions [Kim 2002], that is

\[ |B_r| = \left( \begin{array}{c} r-1 \\ r-n+1 \end{array} \right). \] (9)

The set \( \bigcap_{l \in [i]} C^l_r \) is the set of all elements of \( \mathbb{N}_+^{n-1} \) such that at least \( i \) components are strictly larger than \( m \). If \( r < i(m-1)+n-1 \), then \( \bigcap_{l \in [i]} C^l_r \) is empty because it is not possible to have at least \( i \) components strictly larger than \( m \). Hence

\[
|A_r| = |B_r| - \sum_{ [i] : 3 \leq j \leq i } (-1)^{i-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) \left| \bigcap_{l \in [i]} C^l_r \right|,
\]

(10)
where \( j \) is the largest integer such that \( r \geq j(m-1)+n-1 \). If \( r \geq i(m-1)+n-1 \), then
\[
\left| \bigcap_{l \in [i]} C_l^r \right| = |B_{r-im}| = \left( \frac{r-im-1}{r-im-n+1} \right). \tag{11}
\]
Combining (8), (9), (10) and (11) concludes the proof. \( \square \)

Expressions (6) and (7) are less elegant than Sander’s expression (1). They are also computationally less efficient. Indeed the first summation in (6) has approximately \( nm/2 \) terms while the only summation in (1) has approximately \( n/2 \) terms. Expressions (6) and (7) are nevertheless interesting because they allow us to derive an asymptotic result for \( S(m,n) \) when \( n \) is fixed and \( m \to \infty \) (see Theorem 3). This was not possible with (1).

When \( n < 5 \), expressions (6) and (7) reduce to particularly simple expressions.

**Corollary 1.**

\[
S(m,2) = m;
\]
\[
S(m,3) = \begin{cases} 
3m^2/4 & \text{if } m \text{ is even} \\
3m^2 + 1/4 & \text{if } m \text{ is odd};
\end{cases}
\]
\[
S(m,4) = \frac{2m^3 + m}{3}.
\]

For \( n = 2, 3 \) or 4, the asymptotic behaviour of \( S(m,n) \) is easy to derive from this corollary, while the general case is covered by our next result.

**Theorem 3.** For all \( n \geq 2 \), when \( m \to \infty \), \( S(m,n) \) is equal to \( m^{n-1} g(n) + \mathcal{O}(m^{n-2}) \) where \( g(n) \) is equal to
\[
1 - 2 \sum_{j=0}^{n-4} \sum_{i=0}^{j} (-1)^i \frac{(1+j-i)^{n-1} - (j-i)^{n-1}}{i! \, (n-1-i)!}, \tag{12}
\]
when \( n \) is even, or to
\[
1 - 2 \sum_{j=0}^{n-5} \sum_{i=0}^{j} (-1)^i \frac{(1+j-i)^{n-1} - (j-i)^{n-1}}{i! \, (n-1-i)!}
\]
\[
- 2 \sum_{i=0}^{n-3} (-1)^i \frac{(n/2 - 1 - i)^{n-1} - (n/2 - i)^{n-1}}{i! \, (n-1-i)!}, \tag{13}
\]
when \( n \) is odd.
Proof. We only prove (12). Expression (6) for $S(m, n)$ can also be written as

$$m^{n-1} - 2 \left( \sum_{j=0}^{n-1} \sum_{r=n-1+j(m-1)}^{j} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right)$$

$$+ \sum_{r=n-1+\frac{m-1}{2}}^{\infty} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right) \cdot (14)$$

For $n$ fixed, $g(n)$ is the limit for $n \to \infty$ of (14) divided by $m^{n-1}$, that is

$$1 - \lim_{m \to \infty} \frac{2}{(n-2)!m^{n-1}} \left( \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right)$$

$$+ \sum_{r=n-1+\frac{m-1}{2}}^{\infty} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right) \cdot (14)$$

Since $\lim_{a \to \infty} \frac{a^b}{(a-b)!} = 1$, this is also equal to

$$1 - \lim_{m \to \infty} \frac{2}{(n-2)!m^{n-1}} \left( \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right)$$

$$+ \sum_{r=n-1+\frac{m-1}{2}}^{\infty} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right) \cdot (14)$$

or, since $\sum_{d=a}^{b} d^c = (c+1)^{-1}[d^{c+1}]_a + \mathcal{O}(d^c)$ as $d \to \infty$,

$$1 - \lim_{m \to \infty} \frac{2}{(n-1)!m^{n-1}} \left( \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right)$$

$$+ \sum_{r=n-1+\frac{m-1}{2}}^{\infty} (-1)^i \binom{n-1}{i} \binom{r-im-1}{i} \right) \cdot (14)$$

We then substitute $r$ with the summation bounds, we simplify and we take the limit, keeping only the coefficients of the highest power of $m$ (i.e. $m^{n-2}$) and we obtain (12). The second highest power of $m$ is $m^{n-2}$ and this completes the proof.

Expressions (12) and (13) are easy to compute for $n$ between 2 and approximately 100. Beyond 100, using Stirling’s approximation for factorials helps.
For very large $n$, the computation time becomes prohibitive because of the large number of terms in the summations.

Numerical estimations of $g(n)$ are given in Table 2 and Figure 1 illustrates how quickly $S(m, n)/m^{n-1}$ converges to $g(n)$ for $n = 5$ and 10.

Notice that, when $n < 5$, Corollary 1 provides an asymptotic expression for $S(m, n)$ that is tighter than that resulting from Theorem 3.

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Table 1: The size of the largest antichains with \(m_1, \ldots, m_n\) levels on \(n\) attributes.

| \(m_1, \ldots, m_n\) | size |         |         |         |         |         |
|-----------------------|------|---------|---------|---------|---------|---------|
| 5,5                   | 5    | 5, 5, 5 | 19      | 5, 5, 10| 25      | 5, 5, 100| 25      |
| 5, 5, 10              | 210  | 5, 5, 10, 10 | 250  | 5, 5, 10, 20 | 250  | 5, 5, 10, 100 | 250  |
| 10, 10, 10            | 670  | 10, 10, 10, 10 | 1000  | 10, 10, 10, 20 | 1000  | 10, 10, 10, 100 | 1000  |
| 100, 100, 100, 100    | 666700 | 10, \ldots, 10 (10 times) | 432457640 | 100, \ldots, 100 (10 times) | 430438025018583040 |

Table 2: Some values of \(g(n)\). Fractions are exact.

| \(n\) | \(g(n)\) | \(n\) | \(g(n)\) |
|-------|---------|-------|---------|
| 2     | 1       | 8     | 0.47936507936507944 |
| 3     | 3/4     | 9     | 0.45292096819196426 |
| 4     | 2/3     | 10    | 0.43041776895943573 |
| 5     | 115/192 | 20    | 0.30669310173797590 |
| 6     | 11/20   | 30    | 0.25104851499027436 |
| 7     | 0.5110243055555556 | 100   | 0.12337408033008801 |

Table 2: Some values of \(g(n)\). Fractions are exact.
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