Quark–lepton mass relation in a realistic $A_4$ extension of the Standard Model

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Abstract

We propose a realistic $A_4$ extension of the Standard Model involving a particular quark–lepton mass relation, namely that the ratio of the third family mass to the geometric mean of the first and second family masses is equal for down-type quarks and charged leptons. This relation, which is approximately renormalization group invariant, is usually regarded as arising from the Georgi–Jarlskog relations, but in the present model there is no unification group or supersymmetry. In the neutrino sector we propose a simple modification of the so-called Zee–Wolfenstein mass matrix pattern which allows an acceptable reactor angle along with a deviation of the atmospheric and solar angles from their bi-maximal values. Quark masses, mixing angles and CP violation are well described by a numerical fit.

1. Introduction

Supersymmetric Grand Unified Theories (SUSY GUTs) are very attractive from the theoretical point of view as they allow to obtain the SM from a single unified gauge group [1,2]. Apart from predicting for instance the quantization of electric charge, they reduce the number of free parameters. For example, they give the right value for the electroweak mixing angle and may provide a good framework for the understanding of the flavor problem. Indeed, several GUT models have been studied in the literature, having as prediction a mass relation between down-quark masses and the charged leptons. For instance in the SU(5) unified framework Georgi and Jarlskog have found the mass relation [3]

$$m_e = \frac{1}{3} m_d, \quad m_\mu = 3 m_s, \quad m_\tau = m_B,$$

which is in good agreement with data to first approximation, assuming that holds at the GUT scale, and taking into account renormalization group running to low energies, with suitable SUSY threshold effects. Such mass relations are very welcome since, by itself, the Standard Model sheds no light on the flavor problem.

However, the Large Hadron Collider has so far not found any evidence for Supersymmetry (SUSY) as indicated so far found any evidence for Supersymmetry (SUSY) as indicated within the simplest Grand Unified Theories (GUTs), namely those arising from a single Higgs doublet $H$. In particular the LHC has not found any evidence for Supersymmetry (SUSY) as indicated within the simplest Grand Unified Theories (GUTs), namely those which do not involve an intermediate scale such as $SU(5)$.

Here we advocate an alternative TeV-scale approach to the flavor problem employing just the Standard Model gauge symmetry, supplemented only by a non-Abelian discrete flavor symmetry. For the latter we adopt $A_4$, the discrete group of even permutations of four objects isomorphic to the group of symmetries of the tetrahedron. It is the smallest group containing triplet irreducible representations. Several $A_4$-based flavor models have been suggested [4–6], for reviews see Refs. [7–10]. Recently three of us have proposed an $SU(3) \otimes SU(2)_L \otimes U(1)$ model [11] based on the discrete family symmetry $A_4$ leading to the quark–lepton mass relation:

$$\frac{m_\tau}{\sqrt{m_\tau m_\mu}} \approx \frac{m_B}{\sqrt{m_B m_d}}.$$

It is clear that Eq. (2) provides an interesting generalization of Eq. (1) which is found to be in very good agreement with data. Given that it is approximately renormalization group invariant, it holds at all mass scales [11]. In contrast to the Georgi–Jarlskog relation, Eq. (2) arises just from the flavor structure of the model, and the fact from the existence of two Higgs doublets selectively

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coupled to the up- and down-type fermions.¹ Both of these were assigned to be $A_4$ triplets, with one of them coupling to the down-type quarks and charged leptons. Note that the model in [11] employed an extended Higgs sector with three families of supersymmetric Higgs doublets $H_d$ and $H_u$ for which there is presently no evidence, with present data being consistent with a single Higgs doublet. Moreover, the model predicted $V_{ub} = 0 = V_{cb}$. While providing a good starting point for the CKM matrix, a derivation of the full quark mixing was lacking.

In the present Letter we propose an alternative discrete family symmetry $A_4$ model, keeping the same motivation for introducing $A_4$ into the Standard Model [11], namely, to shed light on the flavor problem. Indeed the model presented here provides a fully realistic description of all quark and lepton masses and mixing angles, and in particular reproduces the successful quark–lepton mass relation in Eq. (2). In contrast to the previous construction in [11] its remains closer in spirit to the SM, since we do not assume supersymmetry nor unification, keeping a single Higgs doublet $H$ instead of multi-Higgs doublets. In particular, we assign right-handed up quarks to singlet representation of $A_4$ instead of triplet as in the original model. The $A_4$ flavor symmetry is broken by SM singlet flavons which distinguish up-type quarks from down-type quarks and charged leptons, with additional flavons in the neutrino sector, where we require extra Abelian discrete groups to distinguish these sectors. Assuming full explicit breaking of $A_4$ through suitable scalar flavon multiplets we show that this simple modification of the model in [11] can describe all the CKM mixing parameters. Moreover, it is straightforward to obtain also the so-called Zee–Wolfenstein mass matrix pattern in the neutrino sector using $A_4$ invariance [12]. However, since it predicts bi-maximal mixing, current neutrino oscillation data analysis [13] rule out such a pattern [14]. We propose a simple modification of the Zee–Wolfenstein model where all the mixing angle, as well as the reactor angle can be reproduced.

In the next section we introduce our model, in Section 3 we obtain our quark–lepton mass relation, in Section 4 we give the fit for the quark mixing parameters, in Section 5 we describe the neutrino mass generation mechanism and study its phenomenological implications, while in Section 6 we give our conclusions.

2. The model

The matter content and the flavor group assignment are given in Table 1. Note that all the fermions, apart of the $u_R$ fields, are assigned to triplets of $A_4$.² In the scalar sector we have one SM Higgs doublet and four flavon fields. With respect to the model of Ref. [11] we have extra Abelian symmetries, namely $Z_2^u$, $Z_2^d$ and $Z_3$. The reason for imposing such a symmetries is because our present model is not supersymmetric. We are replacing the SUSY-Higgs doublets $H^u$ and $H^d$ (triplet of $A_4$ in Ref. [11]) with scalar $SU_1(2)$-singlets (flavons) triplets of $A_4$ times the Standard Model Higgs doublet, namely

$$H^d \rightarrow H \Phi_d, \quad H^u \rightarrow H \Phi_u,$$

where $H = i \sigma_2 H^*$. It is clear that the $Z_2^u$ and $Z_2^d$ symmetries glue the $\Phi_d$ and $\Phi_u$ flavons fields to the up and down-quark sectors, respectively, while the extra $Z_3^f$ symmetry is used to separate the charged and neutral fermion sectors.

¹Such structure is required in supersymmetric models, but the mechanism proposed in Ref. [11] leading to Eq. (2) is more general, relying only of the two-doublet nature of the Higgs sector, as mentioned above. Here we abandon the use of supersymmetry.

²Therefore the present model cannot be embedded in any grand-unified framework.

The Lagrangian for quarks and charged leptons in our model is given by

$$\mathcal{L} = \frac{y_{\alpha\alpha'}}{M} (Q_{d,R})_\alpha Q_{d,L} H d_{\alpha'} + \frac{y_{\alpha'}}{M} (U_{R})_\alpha U_{L} d_{\alpha'} + \frac{y_{\beta}}{M} (Q_{\nu,L})_\beta Q_{\nu,R} \nu_{\beta'} + H.c.,$$

where $\alpha, \alpha'$ label $A_4$ triplets. We remark that the product of two $A_4$ triplets is given by $3 \times 3 = 1 + 1' + 1'' + 3 + 3$ where the two triplet contractions can be written as the symmetric and the antisymmetric ones and denoted as $3_1,$ $3_2.$³ Thus we have that $\alpha = 3_1, 3_2$ while $\alpha' = 3,$ implying that $y_{\alpha\alpha'}$ gives only two couplings $y_{d1} = y_{3_11}$ and $y_{d2} = y_{3_12}.$ On the other hand, $\beta$ and $\beta'$ can be $1,' 1''$ in such a way that $\beta \times \beta' = 1.$ Note that, while the $A_4$ flavor symmetry holds in the (non-renormalizable) Yukawa terms leading to charged fermion masses, we assume it to be completely broken in the scalar potential. Indeed, we assume that the scalar flavon multiplets get vacuum expectation values (vevs) in an arbitrary direction of $A_4,$ preserving none of its subgroups. This can be easily achieved just by including terms in the scalar potential which are SO(3) invariant as discussed in [15]. In this case the scalar flavon fields get vevs in arbitrary directions of $A_4,$ that is

$$\langle \phi_f \rangle \propto (v_{f1}, v_{f2}, v_{f3}).$$

where $v_{f1} \neq v_{f2} \neq v_{f3}$ and $f = u, d, \nu.$ To complete the model we need also to specify the mechanism of neutrino mass generation, see Section 5, below.

3. The charged lepton–quark mass relation

From the $A_4$ contraction rules (see Appendix A) and the fact that the charged leptons and down-type quarks are in the same $A_4$ representations, one sees that the charged leptons and down-type quark mass matrices have the form

$$M_f = \begin{pmatrix}
0 & y_{1f} v_f & y_{2f} v_f \\
y_{1f} v_f & 0 & y_{1f} v_f \\
y_{2f} v_f & y_{1f} v_f & 0
\end{pmatrix},$$

where $f = d, l.$ This special form is the same as obtained in Refs. [11,16]. With the redefinition of variables:

$$y_{1f} = a_l^{-1} / v_{f2}, \quad y_{2f} = b_l / v_{f2},$$

$$a_l = v_{f3} / v_{f2}, \quad r_l = v_{f1} / v_{f2},$$

the mass matrix for the mass matrix in Eq. (6) takes the form

$$M_f = \begin{pmatrix}
0 & a_l a_l^{-1} & b_l \\
a_l^{-1} & 0 & a_l r_l \\
0 & a_l r_l & 0
\end{pmatrix}. $$

Let us now consider the system given by the following three invariants

$$\det S^f = (m_1^2 m_2^2 m_{3f}^2)^2,$$

$$\text{Tr} S^f = m_1^2 + m_2^2 + m_{3f}^2,$$

(\text{Tr} S^f)^2 - \text{Tr} S^f S^f = (m_1^2 + m_{3f}^2)m_2^2 + m_1^2 m_{3f}^2.$$

³In $A_4$ there is only one triplet irreducible representation, here $3_1$ and $3_2$ are not different irreducible representations, but simply a way to indicate different contractions.
m f
a f
rl
rf
b f
γ
Table 1
Matter content of the model.

| L | l_R | Q | d_R | u_Ri | u_Rd | H | ψ_u | ψ_d | ψ_e | ε_
|---|----|---|-----|-------|-------|---|-------|-------|-------|---|
| A_4 | 3 | 3 | 3 | 1 | 1'' | 1' | 1 | 3 | 3 | 1 |
| d_f | + | + | + | + | - | - | - | + | + | + |
| b_f | + | - | + | - | + | + | + | - | + | + |
| z_f^2 | ω | ω^2 | 1 | 1 | 1 | 1 | 1 | 1 | ω | ω |

where $S^f = M_f M_f^T$. This system can be solved and we find

$$r_f \approx \frac{m_3}{\sqrt{m_1 m_2}} \sqrt{\alpha_f},$$  \hspace{1cm} (12)

$$a_f \approx \frac{m_2}{m_3} \sqrt{\frac{m_1 m_2}{\alpha_f}},$$  \hspace{1cm} (13)

$$b_f \approx \frac{m_1 m_2}{m_3^2} \sqrt{\frac{\alpha_f}{\alpha_f}},$$  \hspace{1cm} (14)

in the limit $r_f \gg a_f, 1$ and $r_f \gg b_f/a_f$. These equations are general in the sense that in the complex case, namely complex Yukawa couplings and vevs, the invariants, Eqs. (9)–(11) do not depend on the phases of the vevs $\nu_{ui}$. Indeed, the only dependence on the relative phase of the Yukawa couplings, $y^f_1$ and $y^f_2$ enters in the determinant, Eq. (9), and this is negligible in the above limit. From Eqs. (12), (13), (14), one finds simple relations for the second and third family masses, namely,

$$m_2^f \approx a_f r_f,$$

$$m_3^f \approx b_f r_f,$$

$$\frac{m_2^f}{m_3^f} \approx \frac{a_f}{b_f}.$$  \hspace{1cm} (15)

from which we require $a_f \ll b_f$ in order to account for the second and third family mass hierarchy. Moreover, since the charged leptons and down-type quarks couple to the same Higgs and flavons, we have\(^4\)

$$\alpha_f \approx a_f = \alpha_d,$$

so that, from Eq. (12), we obtain the mass relation [11]

$$m_e \approx m_b \sqrt{m_{e_1} m_{u_1}} \sqrt{m_{t_1} m_{d_1}}.$$  \hspace{1cm} (17)

Now we turn to the up-type quark sector. From the Lagrangian in Eq. (4), the up-quark mass matrix is given by

$$M_u = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}. \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (18)

In what follows we discuss the resulting structure of the quark mixing matrix.

4. Quark mixing: The CKM matrix

Recall that the down-type quark mass matrix takes the form in Eq. (6) while the up-type quark mass matrix has just been given in Eq. (18). Out of these matrices one finds the matrices $M_u^T \cdot M_u$ and $M_d^T \cdot M_d^T$. Their diagonalization results in two unitary matrices $V_u$ and $V_d$ for which one can obtain approximate analytical expressions. In the down sector, from $M_d^T \cdot M_d$, one finds,

$$V_{d_{12}} \approx \frac{m_d}{m_s} \frac{1}{\sqrt{\alpha^d}},$$  \hspace{1cm} (19)

$$V_{d_{13}} \approx \frac{m_s}{m_b} \sqrt{m_d m_s} \frac{1}{\sqrt{\alpha^d}},$$  \hspace{1cm} (20)

$$V_{d_{23}} \approx \frac{m_s m_s}{m_b} \frac{1}{\sqrt{\alpha^d}}.$$  \hspace{1cm} (21)

One sees that if $\alpha^d \sim O(1)$ the down sector gives about the Cabibbo angle in the 1–2 plane while the mixings in the 1–3 and 2–3 planes are negligible. On the other hand, from the up-quark sector one finds, approximately

$$M_u^T \cdot M_u \sim \begin{pmatrix} \lambda_2 \lambda_4 \lambda_2 \lambda_4 \lambda_2 \lambda_4 \lambda_2 \lambda_4 \lambda_2 \lambda_4 \end{pmatrix},$$  \hspace{1cm} (22)

with coefficients of order one in front of the $(i, j)$ if at least one of the Yukawa couplings is of order one.\(^5\) Thus for the up-quark matrix mixing factor $V_u$ we have that

$$V_{u_{23}} \approx \lambda_2,$$  \hspace{1cm} (23)

$$V_{u_{13}} \approx \lambda_4,$$  \hspace{1cm} (24)

$$V_{u_{12}} \approx \lambda_2,$$  \hspace{1cm} (25)

where we have assumed

$$V_{u_{12}} : V_{u_{23}} : V_{u_{31}} = 1 : \lambda_2 : \lambda_4.$$  \hspace{1cm} (26)

The overall quark mixing matrix is given by the product $V_u \cdot V_d$.

One sees how the Cabibbo angle arises from the down-type quark matrix mixing factor $V_d$, while the $V_{ub}$ and $V_{cb}$ CKM mixing angles arise from the up-quark matrix mixing factor $V_u$. Taking $\lambda \approx 0.2$ we obtain approximately the correct value for the mixing angle. However the order-one parameters are relevant in order to exactly determine the quark mixing angles. In order to obtain quantitative predictions for these we perform a global numerical fit. The experimental data used and the one $\sigma$ error bars are given in the second column of Table 2, taking the quark masses (at the scale of the $M_2$) from [18] and the quark mixing angles from [19]. The third column of Table 2 displays the values predicted by our model when the values of its parameters are those in Eqs. (27).

We note that the phases of the up couplings $y^u_i$ can be reabsorbed by transforming the right-handed fields $u_{Ri}$ while in the down sector not all the phases can be removed. For simplicity we

\(^4\) Note that this relation is natural in supersymmetric models [11,17]. Here it follows from the $Z_2$ assignments, namely the fact that the same flavon $\phi_i$ couple to down-quarks and charged leptons.

\(^5\) We note that the magnitude of the order-one parameters that enter implicitly in Eq. (22) is relevant in order to fit the quark mass hierarchy. In particular note that the Yukawa couplings in Eq. (18) have a hierarchical structure as determined from Eq. (27).
assume all the couplings to be real and we show how to make a
fit of the quark mixing parameters (including the complex phase)
and masses. We note that by taking \( \omega \) to be the only phase in
our parametrization one can fit for the CKM phase, namely the Jarlskog
invariant.\(^6\) In Table 2 we compare our theoretical predictions with
the current experimental values for the quark masses and CKM
mixing parameters. The theoretical predictions are for the values:

\[
y_1^v y_3^v = -297\,393 \text{ MeV},
\]
\[
y_1^v y_2^v = -15\,563 \text{ MeV},
\]
\[
y_1^v y_3^v = 277 \text{ MeV},
\]
\[
\frac{v_2^v}{v_3^v} = 1.03 \lambda^2,
\]
\[
\frac{v_1^v}{v_3^v} = 2.14 \lambda^4,
\]
\[
\alpha_d = 1.58,
\]
where \( \lambda \approx 0.2. \)

5. Neutrino masses and mixing

As in Ref. [11] here we consider an effective way to generate
neutrino masses by à la Weinberg by upgrading the standard
dimension-five operator to the flavon case, making it dimension-
six, that is\(^7\)

\[
y_v^v \frac{y_v^v}{\Lambda^2} [d L H \phi_v + y_v^v/L^2 \Lambda^2 [d L H \phi_v]],
\]

(28)

After electroweak symmetry breaking it gives to the following Ma-

gorana neutrino mass matrix

\[
m_\nu = \begin{pmatrix}
a & b & 0 \\
b & c & 0 \\
0 & 0 & d
\end{pmatrix},
\]

(29)

where \( a = y_{\nu_v}^v/\Lambda^2 v_2^v (\psi_{\nu_3}), b = y_{\nu_v}^v/\Lambda v_H (\psi_{\nu_2}), c = y_{\nu_v}^v/\Lambda^2 v_2^v (\psi_{\nu_1}) \)

and \( d = y_{\nu_v}^v/\Lambda v_H (\xi_v). \) Note that, unlike the charged fermion case, only the symmetric contractions are allowed from the first operator in Eq. (28). We remark that these parameters in the neutrino sector are unrelated to those in the charged fermion sector.

\(^6\) However, this does not constitute a geometrical origin of the phase since the Yukawa couplings are complex.

\(^7\) It is easy to formulate type-II [20–22] or inverse [23] seesaw variants of the model. However for simplicity here we keep the effective description presented above.

Table 2

| Observable | Experimental value | Model prediction |
|------------|--------------------|------------------|
| \( m_1 \) [MeV] | 2.9 ± 0.4 | 2.93 |
| \( m_2 \) [MeV] | 57.7 ± 16.6 | 62 |
| \( m_3 \) [MeV] | 2820 ± 50 | 2830 |
| \( m_\text{tot} \) [MeV] | 1.45 ± 0.45 | 1.63 |
| \( m_\text{sum} \) [MeV] | 635 ± 86 | 640 |
| \( m_\text{sum} \) [GeV] | 172.1 ± 0.6 ± 0.9 | 172.1 |
| \( |V_{ub}| \) | 0.22534 ± 0.00065 | 0.2253 |
| \( |V_{cb}| \) | 0.00351 ± 0.00014 | 0.00347 |
| \( |V_{ub}| \) | 0.0412 ± 0.0006 | 0.0408 |
| \( J \) | 2.96 ± 0.20 | 2.93 |

Fig. 1. Effective neutrino mass parameter \( m_\nu \) as function of the lightest neutrino mass. The gray shaded regions correspond to the flavor-generic normal hierarchy neutrino spectra. The blue points correspond to the prediction of our \( A_4 \) model. For comparison we give the current and future sensitivities for \( m_{\nu_e} \) [25,26] and \( m_\nu \) [27, 28], respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this letter.)

Taking the limit where \( d = 0 \) the neutrino mass matrix has the well-known Zee–Wolfenstein structure, which cannot reproduce the current neutrino oscillation data [13], since it gives to bi-maximal mixing [14]. The addition of the unit matrix contribution proportional to \( d \) introduces deviations from maximal atmospheric mixing proportional to \( b \) and also introduces a non-zero reactor angle \( \theta_{13} \approx (a – b)/(2d) \), while the solar angle is approximately given by \( \tan 2\theta_{12} \approx 2(a + b)/d \), which reduces to maximal solar mixing in the Zee–Wolfenstein limit \( d \to 0 \). One finds a strict correlation between the neutrinoless double beta decay rate and the magnitude of the parameter \( d \). In fact, as seen in Fig. 1 we find a (weak) lower bound for the neutrinoless double beta decay rate, despite the fact that the model has a normal hierarchy neutrino spectrum, which follows from the presence of the flavor symmetry.\(^8\) In our numerical scan we also obtain a restricted set of neutrino oscillation angles. For example, the curved-shaped region in the left panel of Fig. 2 (orange in color version) corresponds to the “predicted” atmospheric angle consistent with the currently allowed values of the solar angle at 3σ, from Ref. [13]. For comparison we display also the 1σ bands for \( \sin^2 \theta_{23} \) and \( \sin^2 \theta_{13} \). In the right panel in Fig. 2 we re-express our \( \sin^2 \theta_{23} \) “prediction” in terms of the lightest neutrino mass \( m_1 \), again keeping undis-
played oscillation parameters at 3σ. The existence of the above restrictions reflects the fact that, as a result of the flavor symmetry, we have in total less parameters than observables to de-
scribe.

6. Conclusions

We have proposed a realistic \( A_4 \) extension of the Standard Model leading to the quark–lepton mass relation given in Eq. (2). This successful and nearly renormalization invariant mass relation generalizes the Georgi–Jarlskog formula and arises outside the con-
text of unification. Quark masses, mixing angles and CP violation are properly accounted for, while in the neutrino sector we obtain a generalized Zee–Wolfenstein mass matrix giving an acceptable reactor angle along with a deviation of the atmospheric and solar angles from their bi-maximal values. As seen in Fig. 1 the neutrinoless double beta decay rate correlates sharply with this deviation.

\(^8\) Similar examples of \( A_4 \) models with similar features can be found in [24].
Appendix A. The product rules for the $A_4$ group

Here we adopt the $SO(3)$ basis for the generators of the $A_4$ group, which can be written as $S$ and $T$ with $S^2 = T^2 = (ST)^3 = 1$. $A_4$ has four irreducible representations, three singlets $1$, $1'$ and $1''$ and one triplet. In the basis where $S$ is real diagonal,

$$ S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (A.1) $$

The products of singlets are:

$$ 1 \otimes 1' = 1, \quad 1' \otimes 1'' = 1, $$

$$ 1' \otimes 1' = 1', \quad 1'' \otimes 1'' = 1' \quad (A.2) $$

one has the following triplet multiplication rules,

$$(ab)_1 = a_1 b_1 + a_2 b_2 + a_3 b_3, $$

$$(ab)'_1 = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3, $$

$$(ab)'_2 = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3, $$

$$(ab)_3 = (a_2 b_3, a_3 b_1, a_1 b_2), $$

$$(ab)_2 = (a_3 b_2, a_1 b_3, a_2 b_1). \quad (A.3) $$

where $\omega^3 = 1$, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

References

[1] H. Georgi, S. Glashow, Phys. Rev. Lett. 32 (1974) 438.
[2] J.C. Pati, A. Salam, Phys. Rev. D 10 (1974) 275.
[3] H. Georgi, C. Jarlskog, Phys. Lett. B 86 (1979) 297.
[4] K.S. Babu, E. Ma, J.W.F. Valle, Phys. Lett. B 552 (2003) 207, arXiv:hep-ph/0206292.
[5] G. Altarelli, F. Feruglio, Nucl. Phys. B 720 (2005) 64, arXiv:hep-ph/0504165.
[6] M. Holthausen, M. Lindner, M.A. Schmidt, arXiv:1211.5143, 2012.
[7] H. Ishimori, T. Kobayashi, H. Okada, Y. Shimizu, M. Tanimoto, JHEP 0904 (2009) 011, arXiv:0811.4683.
[8] M. Hirsch, D. Meloni, et al., arXiv:1201.5525, 2012.
[9] S. Morisi, J.W.F. Valle, Fortsch. Phys. 61 (2013) 466, http://dx.doi.org/10/1002/prop.2012.00125, arXiv:1206.6678, 2012.
[10] S.F. King, C. Luhn, Rept. Prog. Phys. 76 (2013) 056201, arXiv:1301.1340 [hep-ph].
[11] S. Morisi, E. Peinado, Y. Shimizu, J. Valle, Phys. Rev. D 84 (2011) 036003, arXiv:1104.1633.
[12] W. Grimus, H. Kuhbock, Phys. Rev. D 77 (2008) 055008, arXiv:0710.1585.
[13] D. Forero, M Tortola, J.W.F. Valle, Phys. Rev. D 85 (2012) 073012. This updates New J. Phys. 13 (2011) 105004 and New J. Phys. 13 (2011) 063004, by including the data presented at Neutrino 2012.
[14] C. Jarlskog, M. Matsuda, S. Skadhauge, M. Tanimoto, Phys. Lett. B 449 (1999) 240, arXiv:hep-ph/9812282.
[15] S.F. King, M. Malinsky, Phys. Lett. B 645 (2007) 351, arXiv:hep-ph/0610250.
[16] S. Morisi, E. Peinado, Phys. Rev. D 80 (2009) 113011, arXiv:0910.4389.
[17] F. Bazzocchi, S. Morisi, E. Peinado, J.W.F. Valle, A. Vicente, JHEP 1301 (2013) 033, arXiv:1202.1529 [hep-ph].
[18] K. Bora, arXiv:1106.5909, 2012.
[19] J. Beringer, et al., Particle Data Group, Phys. Rev. D 85 (2012) 010001.
[20] J. Schechter, J.W.F. Valle, Phys. Rev. D 22 (1980) 2227.
[21] G. Altarelli, F. Feruglio, Nucl. Phys. B 720 (2005) 64, arXiv:hep-ph/0504165.
[22] G. Altarelli, F. Feruglio, Nucl. Phys. B 720 (2005) 64, arXiv:hep-ph/0504165.
[23] G. Altarelli, F. Feruglio, Nucl. Phys. B 720 (2005) 64, arXiv:hep-ph/0504165.
[24] G. Altarelli, F. Feruglio, Nucl. Phys. B 720 (2005) 64, arXiv:hep-ph/0504165.