Time machines and the Principle of Self-Consistency as a consequence of the Principle of Stationary Action (II): the Cauchy problem for a self-interacting relativistic particle

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Abstract

We consider the action principle to derive the classical, relativistic motion of a self-interacting particle in a 4-D Lorentzian spacetime containing a wormhole and which allows the existence of closed time-like curves. In particular, we study the case of a pointlike particle subject to a ‘hard-sphere’ self-interaction potential and which can traverse the wormhole an arbitrary number of times, and show that the only possible trajectories for which the classical action is stationary are those which are globally self-consistent. Generically, the multiplicity of these trajectories (defined as the number of self-consistent solutions to the equations of motion beginning with given Cauchy data) is finite, and it becomes infinite if certain constraints on the same initial data are satisfied. This confirms the previous conclusions (for a non-relativistic model) by Echeverria, Klinkhammer and Thorne that the Cauchy initial value problem in the presence of a wormhole ‘time machine’ is classically ‘ill-posed’ (far too many solutions). Our results further extend the recent claim by Novikov et al. that the ‘Principle of self-consistency’ is a natural consequence of the ‘Principle of minimal action.’
1 Introduction

Since the original works by [1], there has been a long debate over the issue whether the laws of physics might allow for the existence of closed time-like curves (CTCs) inside our universe [2–14]. Macroscopic CTCs might be easily realized as a semiclassical consequence of the ‘quantum foam’ structure of spacetime at Planck scales (see, e.g., Refs. [15]–[16].) Close to this scale, it is conjectured that spacetime might allow for non-trivial topological fluctuations, for example in the form of wormholes, intuitively speaking 4-d ‘handle-like’ geometries, whose two ‘mouths’ join distant regions of spacetime. The matter content which is required to produce a static wormhole should violate the averaged weak energy condition (AWEC) [3-6], and other properties might be important as well (cf. Ref. [18]). Classically, the AWEC condition might be a problem, since ordinary classical energy densities are positive. It is not yet clear whether quantum effects could anyway preserve the AWEC for generic cases, or if vacuum polarization divergences in the quantum theory can actually destabilize the wormhole (see, e.g., Refs. [19, 20]). However, if the laws of physics actually permit the existence of traversable wormholes, then generic relative motions of the two wormhole’s mouths, or equivalently generic gravitational redshifts at the mouths due to external gravitational fields, could in principle produce CTCs looping through it [4, 7-8]: when traversed from mouth to mouth, the wormhole acts as a ‘time machine’ allowing one to travel into the past or into the future. For spacetimes with CTCs, past and future are no longer ‘globally’ distinct [9, 10-11]. In particular, as originally pointed out in Ref. [12], events on CTCs should causally influence each other along the ‘loops in time’ in a self-adjusted, consistent way. This requirement has been explicitly formulated as the ‘Principle of self-consistency,’ according to which the only solutions to the laws of physics that can occur locally in the real universe are those which are globally self-consistent [7, 9, 12–14]. For further discussion and references about these and other wormhole related topics we refer the interested reader to Ref. [21].

In a recent paper [22] (from now on referred to as TM-I) we showed that the ‘Principle of self-consistency’ actually needs not to be imposed as an independent assumption which is necessary in order to make sense of spacetimes with CTCs, but instead can be seen as a direct consequence of the more fundamental ‘Principle of minimal action.’ In particular, in TM-I we considered the simple model of a nonrelativistic particle which is constrained to have fixed initial and final positions, to loop through the wormhole once and to interact with itself by means of a ‘hard-sphere’, elastic potential. We used the action principle to derive the classical trajectories, and found that the only possible solutions which minimize the action are those which are

\[ \text{Recently, also dynamical wormhole solutions which satisfy the weak and dominant energy conditions have been found, see, e.g., [17].} \]

\[ \text{Just in order to avoid possible confusion, throughout the whole paper we will use the word ‘classical’ as opposed to ‘quantum’, and ‘nonrelativistic’ as opposed to ‘relativistic’.} \]
globally self-consistent. In the case of coplanar motion with respect to the wormhole’s mouths, the possible, globally self-consistent trajectories in which the particle’s copies collide are of three types, depending on the possible ways in which momentum is exchanged at the collision event. These results led us to formulate the conjecture that the ‘Principle of self-consistency’ should be a consequence of the ‘Principle of minimal action’ for all physical phenomena. The extension of this model to the case of 3-d motion has been also considered in Ref. [23] (from now on referred to as TM-II).

In a previous series of papers [10, 11], a similar model for the motion of a non-relativistic ‘billiard ball’-like particle in the spacetime containing a wormhole ‘time machine’ was considered in the context of a Cauchy initial value problem, where the initial path and speed of the particle are assumed to be fixed (for a general discussion also including the case of a scalar field, see Ref. [9]). In the case of elastic self-interaction of the particle, it was shown [11] that generic classes of Cauchy data with initial velocity $|\vec{v}_i| > D/\tau$ (where $D$ and $\tau$ respectively denote the wormhole mouths’ separation and the time displacement into the past after the wormhole traversal) have multiple, and even infinite numbers of globally self-consistent solutions to the equations of motion (trajectories where the particle is initially at rest far from the wormhole have, in fact, multiplicity one), with no evidence for non self-consistent trajectories.

The present paper is to be intended as a generalization of the analysis made in the previous papers TM-(I,II) to the case of the Cauchy initial value problem for the motion of a ‘billiard ball’, pointlike, relativistic particle multiply looping through a wormhole ‘time machine’ and (elastically) self-interacting via a ‘hard-sphere’ potential. In particular, we address again the issues of the existence of globally self-consistent solutions to the equations of motion in the presence of the wormhole ‘time machine’, of their derivation from a ‘Principle of stationary action’, and therefore argue against the actual need of imposing the ‘Principle of self-consistency’ as an extra independent assumption in the laws of physics. We then make further contact with the non-relativistic analysis of Ref. [10], checking for the multiplicity of the globally self-consistent solutions, and for the classical well-posedness of the Cauchy initial value problem.

The outlay of the paper is the following. We start in Section 2 by introducing the main formulas for the kinematics, the dynamical equations and conservation laws for the Cauchy problem in the model of a short range, ‘hard-sphere’ potential (effectively treating the particle as a pointlike ‘billiard ball’). In Section 3 we state the main lines for the analysis of the stationary points of the action describing the classical motion of the particle in three spatial dimensions, separately analyzing the cases without and with collisions. In particular, we show the local solutions to the conservation laws of relativistic energy and momentum for the case of coplanar motion of the particle with respect to the wormhole mouths. Then, in Section 4, we analyze the globally self-consistent solutions to the equations of motion in the case of an inelastic self-collision of ‘billiard ball’-like particles was made in Ref. [11].
self-consistent solutions to the Cauchy initial value problem (distinguishing among the various possibilities of the initial Cauchy data for the modulus and direction of the velocity of the particle), study their multiplicity and show (explicit and detailed formulas for the case of a coplanar motion of the particle with respect to the wormhole’s mouths are presented in the Appendix A.2) that the action is stationary along all these trajectories, therefore concluding that the ‘Principle of self-consistency’ is a direct consequence of the ‘Principle of stationary action’. We conclude in Section 3 with some discussion and comparison with previous results [9]-[10] presented in the context of a similar Cauchy initial value problem for the motion of a non-relativistic particle in the presence of a wormhole ‘time machine’.

2 The model

We consider the relativistic motion of a self-interacting particle of mass $m$ in the background with a wormhole ‘time machine’ and with a fixed set of initial Cauchy data. Similarly as done in Refs. TM-(I,II), the mouths of the wormhole are here treated as pointlike and to be infinitely heavy, so that we can neglect the recoil effect on the geometry when the particle traverses the wormhole. In particular, we suppose that the mouths are at rest in some reference frame, and consider the problem using this frame. Spacetime outside the ‘time machine’ is approximated to be Minkowskian. Our discussion is essentially independent of other features defining the internal structure of the wormhole (although it is consistent, e.g., with the choice of ‘traversal rules’ suggested in Ref. [10]). Finally, we depart from Refs. TM-(I,II) by allowing for motions in which the particle can traverse the wormhole an arbitrary number ($n$) of times.

The motion can be schematically described in the following way. The particle is assumed to start at time $t_i$ in the position $\vec{r}_i$, with velocity $\vec{v}_i$, and to enter the first mouth (B) of the wormhole at time $\bar{t}_1 + \tau$ (position $\vec{r}_B$). Then it exits from the other mouth (A) at the earlier time $\bar{t}_1$ (position $\vec{r}_A$) and moves towards mouth B, where it enters at time $\bar{t}_2 + \tau$, again exits from mouth A at time $\bar{t}_2$ and so on, making $n$ of such wormhole traversals (i.e., exiting mouth A at time $\bar{t}_k$ and reentering mouth B at time $\bar{t}_{k+1} + \tau$, for $k = 1, \ldots, n-1$). The particle definitively leaves the wormhole from mouth A at time $\bar{t}_n$. For the particle itself (in its proper time), each motion through the wormhole happens almost instantaneously, as the path length of the wormhole handle is assumed to be infinitely short. According to an external observer, instead, after each of the traversals of the ‘time machine’ the particle travels back in time by the amount $\Delta t = -\tau$, where by definition $\tau > 0$.

$^d$These boundary conditions will be ‘relaxed’ only in Section 4.1.c and Appendix A.2.c, where we will consider the motion of a particle initially heading towards mouth A (with Cauchy data $\vec{r}_i$, $\vec{v}_i$ at $t_i$), entering it at time $\bar{t}_n$ and, after making $n$ wormhole traversals (excluding mouth B at time $\bar{t}_k + \tau$ and reentering mouth A at time $\bar{t}_{k-1}$, for $k = 2, \ldots, n$), finally exiting from mouth B at time $\bar{t}_1 + \tau$. In this case, after each wormhole traversal, the particle travels forward in time by the amount $\Delta t = +\tau$. 

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We first analyze the generic motion of the particle in three spatial dimensions, and then discuss the detailed trajectories for the case of coplanar motion with respect to the wormhole’s mouths.

The conditions for the wormhole to act as a ‘time machine’ during each of its traversal and also after a multiple number $n$ of traversals are, respectively,

$$
\tau > |\vec{r}_B - \vec{r}_A|/c,
\bar{t}_1 + \tau > \bar{t}_n,
$$

where $c$ is the speed of light.

In the case of a general potential $V$ for the self-interaction of the particle, and depending on the relations between the times $\bar{t}_k$, $\bar{t}_{k+1} + \tau$ ($k = 1, \ldots, n-1$) and the times $t_i$, $\bar{t}_i + \tau$ and $\bar{t}_n$, at each time $t$, in principle, one might have from a minimum of one to a maximum of $n + 1$ copies of the relativistic particle existing at the same time, copies which are treated as independent objects having multiple, self-interactions between themselves and with the wormhole mouths. If the particle enters mouth A (at time $\bar{t}_n$) and finally appears from mouth B in the future (at time $\bar{t}_1 + \tau$), at any time $t$ there is at most one copy of the particle in the space external to the wormhole. In particular, we can describe the trajectories of the copies of the particle in the following way: $\vec{r}_1$ for the copy moving from the initial position $\vec{r}_i$ to the first wormhole mouth entrance (position $\vec{r}_A$), $\vec{r}_k$ ($k = 2, \ldots, n$) for the copies performing the multiple wormhole passages (between wormhole mouth’s positions $\vec{r}_A$ and $\vec{r}_B$), and finally $\vec{r}_{n+1}$ for the last copy definitely leaving the second mouth (position $\vec{r}_A$).

We then assume the Cauchy initial data for the relativistic particle,

$$
\bar{t}_1(t_i) \equiv \vec{r}_i, \\
\bar{t}_1(\bar{t}_i) \equiv \vec{v}_i
$$

(initial position and velocity), to be fixed in a region devoid of CTCs (cf., e.g., Ref. [10], where the initial path and velocity are fixed instead).

The time displacement $\tau$ in the wormhole, as well as the positions of the wormhole’s mouths are also assumed to be known. The wormhole entrance and exit conditions on the position of the particle are formally summarized as the constraints:

$$
\bar{r}_{k+1}(\bar{t}_k) \equiv \vec{r}_A ; \ k = 1, \ldots, n, \\
\bar{r}_k(\bar{t}_k + \tau) \equiv \vec{r}_B ; \ k = 1, \ldots, n.
$$

For simplicity, we also choose to work in the ansatz:

$$
\bar{t}_n > t_i.
$$

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*In Section 4.1.c and Appendix A.2.c, for the case of collisionless motion of a particle traversing the wormhole from mouth A to mouth B and travelling forward in time, we will also consider the modified boundary conditions $\bar{r}_{n-k+1}(\bar{t}_k) = \vec{r}_A$ and $\bar{r}_{n-k+2}(\bar{t}_k + \tau) = \vec{r}_B$, for $k = 1, \ldots, n$.

†Actually, since the relation between the times $t_i$ and $\bar{t}_n$ turns out not to be relevant for the kinematics of the particle, the choice [10] is made mainly for illustrative purposes.
2.1 ‘Billiard ball’ pointlike particle

In principle it would be possible to write down the classical action describing a general interaction among the particle copies and the wormhole mouths. However, as in general the problem of solving the equations of motion by minimizing such an action turns out not to be straightforward (see also Ref. TM-I), we will not address it here.

Instead, the nature of the trajectories can be greatly simplified assuming that the particle and wormhole mouths are not charged and by working with the model of a short range, ‘hard-sphere’ self-interaction potential for the particle

\[ V(r) = \dot{V} \theta(r_s - r); \quad \dot{V} \to \infty, \quad r_s \to 0, \]

where we have introduced the variable

\[ \bar{r}(t) \equiv \bar{r}_{n+1}(t) - \bar{r}_1(t); \quad r \equiv |\bar{r}|. \]

In this model, the particle behaves like a (small) ‘billiard ball’ and we essentially neglect the interaction of the copies of the particle along most of their motion in Region II, except at the point of the (eventual) elastic collision. The effect of the potential is limited to an infinitesimally small period of time around \( t_0 \), the time of the (eventual) collision. We want to make clear that, since treating the particle like a ‘billiard ball’ instantaneously interacting with itself under the action of a ‘hard sphere’ potential would implicitly assume that the velocity of sound is infinite, the choice of the ansatz (5) can be made consistent with a relativistic description of the motion only if the particle is treated as pointlike. In this ansatz one can then also exclude (neglecting relativistic ‘retarded’ effects in the potential) eventual interactions among copies \( k = 2, \ldots, n \) of the particle and among these and the other two copies \( k = 1, n+1 \), as this only requires certain restrictions on the time displacement produced by the wormhole (see Eq. (24) in Section 3.1) and on the geometry of the trajectories in Region II (see Section 3.3.1). As a consequence of this choice, the actual ordering between the times \( \bar{t}_k, \bar{t}_{k+1} + \tau \) \((k = 1, \ldots, n - 1)\) and \( t_i, \bar{t}_n, \bar{t}_1 + \tau \) becomes essentially irrelevant for the purpose of our analysis. We exclude the possibilities of more than one collision between the two copies of the particle in Region II, and also Cauchy data such that the particle is initially moving along the direction of the line connecting the two wormhole mouths.

In fact, proceeding along lines similar to Refs. TM-(I,II), it seems convenient to describe the motion of the relativistic particle in the presence of the ‘time machine’ using the following four main separate regions:

\[ \text{For instance, one can assume that both the particle and the wormhole are ‘dressed’ with electromagnetic charge. In this case one should take into account the interactions between the particle copies and the wormhole mouths - when a charged particle enters or exits the wormhole the charge of the wormhole mouths also changes. The charged particles will then have, in general, non zero acceleration and emit radiation. Radiation is also expected to propagate through the wormhole throat. Finally, due to relativistic effects, all interactions should be described in terms of ‘retarded’ times.} \]
I) $t_i < t < \bar{t}_n$: only the first copy of the particle with position $\vec{r}_1(t)$ is present;

II) $\bar{t}_n < t < \bar{t}_1 + \tau$: two copies of the particle with positions $\vec{r}_1(t)$ and $\vec{r}_{n+1}(t)$, interacting via the potential $V$, are present;

III) $t > \bar{t}_1 + \tau$: only the $(n + 1)$-th copy of the particle with position $\vec{r}_{n+1}(t)$ is present;

IV) $\bar{t}_k < t < \bar{t}_{k+1} + \tau$, $k = 1, ..., n - 1$: from one to $n - 1$, non self-interacting copies with position $\vec{r}_k(t)$ ($k = 1, ..., n - 1$), moving between mouth A and mouth B are present (regardless of the eventual presence of one or both of the $k = 1, n + 1$ copies of the particle in Regions I-III).

The total action describing such a motion is the sum of the actions of the single paths in each separate region (subject to obvious continuity conditions for the position of the copies of the particle at times $\bar{t}_k$ and $\bar{t}_k + \tau$, $k = 1, ..., n$), i.e.,

$$ S = -mc^2 \left\{ \int_{t_i}^{\bar{t}_n} d\tilde{t} \left[ \gamma_1(\tilde{t}) \right]^{-1} + \int_{\bar{t}_1 + \tau}^{t} d\tilde{t} \left[ \gamma_{n+1}(\tilde{t}) \right]^{-1} + \sum_{k=1}^{n-1} \int_{\bar{t}_k}^{\bar{t}_{k+1} + \tau} d\tilde{t} \left[ \gamma_{k+1}(\tilde{t}) \right]^{-1} \right\} $$

$$ - \int_{t_n}^{\bar{t}_1 + \tau} d\tilde{t} \left\{ mc^2 \left[ \gamma_1(\tilde{t}) \right]^{-1} + \left[ \gamma_{n+1}(\tilde{t}) \right]^{-1} + V(\vec{r}(\tilde{t})) \right\} $$

$$ \equiv S_1(t_i, \bar{t}_n) + S_{n+1}(\bar{t}_1, \bar{t}_n, t) + S_2 + S_{1, n+1}(\bar{t}_n, \bar{t}_1 + \tau). \quad (7) $$

The general procedure consists in imposing the principle of stationarity of the action, considering continuous paths for which the initial and final coordinates of the particle (respectively, at times $t_i$ and $t$) are (provisorily) held fixed. One then obtains the classical equations of motion in each of the four Regions I, II, III and IV, which can be solved separately subject to the Cauchy initial conditions $^{24}$.

**Regions I and III.** By the variation of the action $S_1$ in Eq. $^{12}$ with respect to $\vec{r}_1$ in the first region, and of $S_{n+1}$ with respect to $\vec{r}_{n+1}$ in the third region, we find

$$ \frac{\delta S_1}{\delta \vec{r}_1} = 0 \quad \Rightarrow \quad (\gamma_1 \ddot{\vec{r}}_1) = 0, \quad (8) $$

Equations $(8)$ clearly represent linear motion.

**Region II.** By varying the action $S_{1, n+1}$ in Eq. $^{12}$ with respect to $\vec{r}_1$ and $\vec{r}_{n+1}$, we have the following equations of motion

$$ \frac{\delta S_{1, n+1}}{\delta \vec{r}_1} = 0 \quad \Rightarrow \quad mc^2 (\gamma_1 \ddot{\vec{r}}_1) = V'(r) \dot{\vec{r}}, $$

$$ \frac{\delta S_{1, n+1}}{\delta \vec{r}_{n+1}} = 0 \quad \Rightarrow \quad mc^2 (\gamma_{n+1} \ddot{\vec{r}}_{n+1}) = -V'(r) \dot{\vec{r}}. \quad (9) $$

$^{k}$Note that the Euler-Lagrange equations, although formally derived from Eq. $^{12}$ by a variational principle which assumes fixed initial and final particle positions, are second order differential equations which can be solved by giving the initial coordinate and velocity of the particle. It is in this sense that one can consider the Cauchy initial value problem for the classical trajectories by starting from the action principle, see Ref. $^{24}$.
If we further introduce the relativistic momenta
\[ \vec{p}_k \equiv m\gamma_k \dot{\vec{r}}_k ; \ k = 1, n + 1, \] (10)
then, summing Eqs. (9) we obtain
\[ [\dot{\vec{p}}_1 + \dot{\vec{p}}_{n+1}](t) = 0. \] (11)

Moreover, noting that the action (7) is invariant with respect to time translations, we can make use of the variational principle to show that (see, e.g., Ref. [24])
\[ \sum_{i=1, n+1} \left[ \dot{\vec{r}}_i \frac{\delta S_{1,n+1}}{\delta \dot{\vec{r}}_i} - \mathcal{L}_{1,n+1} \right] = \mathcal{E} \] (12)
(where \( \mathcal{E} = \text{const} \)), from which we obtain the first integral for the relativistic motion
\[ mc^2\gamma_1(t) + mc^2\gamma_{n+1}(t) + V[\vec{r}(t)] = \mathcal{E}. \] (13)

Equations (11) and (13) are nothing but the relativistic laws of momentum and energy conservation for the motion of the two copies of the particle in Region II.

Moreover, for \( r > r_s \sim 0 \) we have \( V(\vec{r}) = 0, \) and Eqs. (11) and (13) state that, everywhere in Region II except at the point of eventual collision, the motion of the two copies of the particle is also linear.

**Region IV.** By the variation of the action \( S_{\Sigma} \) in Eq. (4) with respect to \( \vec{r}_k \) we get the following equations of motion
\[ \frac{\delta S_{\Sigma}}{\delta \vec{r}_k} = 0 \Rightarrow (\gamma_k \dot{\vec{r}}_k) = 0 ; \ k = 2,...n. \] (14)

Eqs. (14) clearly imply that also the motion in Region IV is linear for each of the copies \( k = 2,...n \) of the relativistic particle. In details, in terms of the relativistic \( \gamma \)-factors we have that
\[ \gamma_k(\bar{t}_{k-1}) = \gamma_k(\bar{t}_k + \tau) = \text{const} ; \ k = 2,...n. \] (15)

Finally, taking the variation of the action (4) with respect to \( \bar{t}_k, \) and excluding the possibility of collisions on the verge of the wormhole’s mouths (in other words, assuming that \( V[\vec{r}(\bar{t}_k)] = V[\vec{r}(\bar{t}_k + \tau)] = 0, \ k = 1,...n, \) we get the set of conditions
\[ \frac{\partial S}{\partial \bar{t}_k} = 0 \Rightarrow \gamma_k(\bar{t}_k + \tau) = \gamma_{k+1}(\bar{t}_k) = \gamma_1'(\bar{t}_1 + \tau) = \gamma_2(\bar{t}_1) ; \ k = 2,...n, \] (16)

\( ^i \)Taking the potential (3), it is clear that Eq. (13) is not well defined at the point \( r = r_s. \) However, it is possible to show (similarly as done in Ref. TM-I, see Appendix A.1) that the total relativistic energy \( m(\gamma_1 + \gamma_{n+1})c^2 \) is conserved before and after the collision.

\( ^j \)Anticipating the discussion and notation of Section 3.3, in Eqs. (16)-(17) we introduce a prime to distinguish the velocity of the first copy of the particle before its first entrance into the wormhole mouth B and after the eventual collision with its \((n+1)\)-th copy (finally exiting from mouth A), from its - unprimed - velocity before the same eventual collision.
which, on use of Eqs. (15), finally gives

$$|\vec{v}_k| = |\vec{v}_k'| = |\vec{v}_{n+1}|; \quad k = 2, \ldots, n,$$

stating that the energy of the particle at each entrance and exit from the wormhole mouths must be conserved.

In conclusion, for the motion of a relativistic particle constrained to traverse the wormhole an arbitrary number ($n$) of times and to have a given set of fixed initial Cauchy data, we have to distinguish between the two cases:

i) **trajectories without self-collision.** In this case, the first copy of the particle moves linearly from the initial position $\vec{r}_i$ (and velocity $\vec{v}_i$) at time $t_i$ until it enters the wormhole mouth B at time $\bar{t}_1 + \tau$. Similarly, after the $n$ wormhole traversals, the $(n+1)$-th copy of the particle moves linearly from mouth A at time $\bar{t}_n$ along a certain direction which is specified by the internal wormhole geometry.

ii) **trajectories with self-collision.** In this case, instead, the motion for the first ($(n+1)$-th) copy of the particle is linear from the initial position $\vec{r}_i$ at time $t_i$ (from the wormhole mouth A at time $\bar{t}_n$) up to the collision event, with coordinates

$$\vec{r}_1(t_0) = \vec{r}_{n+1}(t_0) \equiv \vec{r}_0.$$  

After the collision, the motion of the first copy of the particle is again linear up to the wormhole mouth B at time $\bar{t}_1 + \tau$, and also the $(n+1)$-th copy moves linearly. Of course, the directions of the trajectories for the first ($(n+1)$-th) copy of the particle will be, in general, different before and after the collision (see Sections 3.3 and 3.3.1).

In the case of a short-range potential, therefore, the stationarity problem is simplified as the trajectories will depend only on the parameters

$$\bar{t}_1, \quad \bar{t}_n, \quad t_0, \quad \vec{r}_0$$

and, of course, on the initial Cauchy data $\vec{r}_i, \quad \vec{v}_i, \quad t_i$.

The problem is now to look for the stationary points (if any) of the action (16), evaluated along the classical trajectories (8), (9) and (14), with respect to the parameters (19).

### 3 The trajectories

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$k$As already remarked at the beginning of Section 2, in Section 4.1.c and Appendix A.2.c we will also consider the case of a particle linearly moving from $\vec{r}_i$ (velocity $\vec{v}_i$) at time $t_i$, to mouth A at time $\bar{t}_n$, and then finally exiting (after $n$ wormhole traversals) and linearly moving from mouth B at time $\bar{t}_1 + \tau$. 

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3.1 Motion in Region IV

On account of our assumptions about at most a single self-collision of the particle in Region II, the analysis of the motion in Region IV is necessary mainly in order to determine the relationship between the times of first entrance and last exit of the particle itself from the wormhole.

Solving explicitly for the trajectories in Region IV, i.e. Eqs. (14)-(15), subject to the conditions (17) and to the boundary conditions (3), we get

$$\vec{r}_k(t) = |\vec{v}'_1|\frac{(\vec{r}_B - \vec{r}_A)}{|\vec{v}'_1|}(t - \bar{t}_{k-1}) + \vec{r}_A; \ k = 2,...n.$$  \hspace{1cm} (20)

In particular, the times of entrance and exit, respectively, from the wormhole mouths B and A, are related by the obvious recurrence relations

$$\bar{t}_k + \tau = \bar{t}_{k-1} + \frac{|\vec{r}_B - \vec{r}_A|}{|\vec{v}'_1|}; \ k = 2,...n.$$  \hspace{1cm} (21)

Iterating Eq. (21) for $k = 2,...n$, we can finally get the relation between the time of first entrance of the particle into the wormhole mouth B ($\bar{t}_1 + \tau$) and the time of last exit from mouth A ($\bar{t}_n$) as

$$\bar{t}_n = \bar{t}_1 + (n-1)\left[\frac{|\vec{r}_B - \vec{r}_A|}{|\vec{v}'_1|} - \tau\right].$$  \hspace{1cm} (22)

From inspection of Eq. (22), it is straightforward to check that the condition of the wormhole acting as a ‘time machine’ (i.e., that $\bar{t}_1 + \tau > \bar{t}_n$, see the last of Eqs. (1)) even after its multiple traversals ($n > 1$) by the particle turns out as:

$$\tau > \left(\frac{n-1}{n}\right)|\vec{r}_B - \vec{r}_A|/|\vec{v}'_1|.$$  \hspace{1cm} (23)

Finally, from Eq. (21) we can also easily get the condition for no self-collision between the $k = 2,...n$ copies of the relativistic particle in Region IV, i.e.

$$\tau \neq \frac{|\vec{r}_B - \vec{r}_A|}{|\vec{v}'_1|}.$$  \hspace{1cm} (24)

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*For the case of a particle first entering mouth A, traversing the wormhole $n$ times and finally exiting mouth B without any self-collision (Section 4.1.c and Appendix A.2.c), Eqs. (20)-(22) should be replaced by: $\vec{r}_k(t) = |\vec{v}'_i|(|\vec{r}_A - \vec{r}_B|)(t - \bar{t}_{n-k+2} - \tau)/|\vec{r}_B - \vec{r}_A| + \vec{r}_B, \ \bar{t}_k + \tau = \bar{t}_{k-1} - |\vec{r}_B - \vec{r}_A|/|\vec{v}'_i| \ (for \ k = 2,...n) \ \text{and} \ \bar{t}_1 = \bar{t}_n + (n-1)[|\vec{r}_B - \vec{r}_A|/|\vec{v}'_i| + \tau], \ \text{for which of course the condition} \ \bar{t}_1 + \tau > \bar{t}_n \ \text{is trivially satisfied.}
3.2 Motion in Region II: no collision

We first consider the variational problem in Region II for the case of ‘no collision’ (i.e., when the separation \( r \) between the two copies of the particle is always greater than \( r_s \)). In this case the trajectories can be parametrized as

\[
\vec{r}_1(t), \quad t_i < t < \bar{t}_1 + \tau, \\
\vec{r}_{n+1}(t), \quad t > \bar{t}_n,
\]

(25)

with the (constant) velocities subject to the conditions (see Eq. (9) for \( V = 0 \) and Eq. (17))

\[
|\vec{v}_1| = |\vec{v}_{n+1}| = |\vec{v}_i|, \quad \vec{v}_{n+1} \equiv |\vec{v}_i|\vec{k}_{n+1}
\]

(26)

(with \( \vec{k}_{n+1} \) a unit vector). Then, the solutions of the equations of motion (9), subject to the Cauchy data (2) and the boundary conditions (3), are given by the linear trajectories

\[
\vec{r}_1(t; \bar{t}_1) = \vec{v}_i(t - \bar{t}_1 - \tau) + \vec{r}_B, \\
\vec{r}_{n+1}(t; \bar{t}_1) = |\vec{v}_i|\vec{k}_{n+1}(t - \bar{t}_n) + \vec{r}_A,
\]

(27)

where \( \bar{t}_n \) is a function of \( \bar{t}_1 \) via Eq. (22).

The variational problem is now extremely simple. In fact, the action (7) evaluated along the classical trajectories (27) is a function only of the (fixed) Cauchy data, i.e.

\[
S_{cl,n-c} = -mc^2\left[\gamma_i^{-1} \int_{t_i}^{\bar{t}_1+\tau} d\bar{t} + \gamma_{n+1}^{-1} \int_{\bar{t}_n}^{t} d\bar{t}\right] + S_{\Sigma,cl}
\]

(28)

(where, as in the non-relativistic case discussed in Ref. TM-I, the contribution from the potential term is zero, see Appendix A.1) and is stationary. The only ‘unknown’ in the problem is \( \bar{t}_1 \), which can be fixed by further imposing the boundary condition (3) on the stationary trajectories (27) (see Appendix A.2).

3.3 Motion in Region II: self-collision

In the case of self-collision under the action of the ‘hard-sphere’ potential (5), as we already remarked in Section 3, the motion is also linear everywhere except at the
event of the impact. Besides the general conditions (1), in order that the motion with self-collision in Region II is self-consistent, we must impose the further, obvious, time ordering conditions

\[ t_0 > \tilde{t}_n, \]
\[ \tilde{t}_1 + \tau > t_0, \]
\[ t_0 > t_i. \]  \hfill (29)

To clearly identify the trajectories before and after the collision, it is then convenient to use the following notation (see Ref. TM-I)

\[ \vec{r}_1(t), \quad t_i < t < t_0, \]
\[ \vec{r}_1'(t), \quad t_0 < t < \tilde{t}_1 + \tau, \]  \hfill (30)

for the first copy of the particle, and

\[ \vec{r}_{n+1}(t), \quad \tilde{t}_n < t < t_0, \]
\[ \vec{r}_{n+1}'(t), \quad t > t_0, \]  \hfill (31)

for the \((n + 1)\)-th copy of the particle, with \(\tilde{t}_n\) to be seen as a function of \(\tilde{t}_1\) through Eq. (22), as before.

It is easy to show that the solutions of the classical equations of motion (9), subject to the Cauchy initial data (2) and to the boundary conditions (3), are given by

\[
\vec{r}_1(t; \tilde{t}_1, t_0, \vec{r}_0) = \vec{r}_0 + \vec{v}_i(t - t_0),
\]
\[
\vec{r}_{n+1}(t; \tilde{t}_1, t_0, \vec{r}_0) = \vec{r}_0 + (\vec{r}_0 - \vec{r}_A)(t - t_0)/(t_0 - \tilde{t}_n),
\]
\[
\vec{r}_1'(t; \tilde{t}_1, t_0, \vec{r}_0) = \vec{r}_0 + (\vec{r}_B - \vec{r}_0)(t - t_0)/(\tilde{t}_1 + \tau - t_0),
\]
\[
\vec{r}_{n+1}'(t; \tilde{t}_1, t_0, \vec{r}_0) = \vec{r}_0 + \vec{v}_{n+1}'(t - t_0).
\]  \hfill (32)

Then, with the new notation (30)–(31), the total action \(S_{cl,c}\) for the collision case, evaluated along the classical trajectories (32), (again the contribution of the potential term is zero) more simply reads

\[
S_{cl,c}(t_0, \tilde{t}_1, \vec{r}_0) = -mc^2 \left[ \int_{t_i}^{t_0} \vec{v}_i dt + \int_{t_0}^{\tilde{t}_1 + \tau} \vec{v}_{i+\tau} dt 
+ \int_{t_n}^{t_0} \vec{v}_{n+1} dt + \int_{t_0}^{\tilde{t}_1} \vec{v}_{n+1}' dt \right] + S_{\Sigma,cl}
= -mc \left[ \sqrt{(t_0 - t_i)^2 c^2 - (\vec{r}_0 - \vec{r}_i)^2}
+ \sqrt{(\tilde{t}_1 + \tau - t_0)^2 c^2 - (\vec{r}_0 - \vec{r}_B)^2[n\tau/(\tilde{t}_1 + \tau - t_0)]}
+ \sqrt{(t - t_0)^2 c^2 - (\vec{r}(t) - \vec{r}_0)^2} \right].
\]  \hfill (33)

The variational problem for the collision case consists in looking for the stationary points of the action (33) with respect to the parameters \(\vec{r}_0, t_0, \) and \(\tilde{t}_1,\) i.e.

\[
\frac{\partial S_{cl,c}}{\partial \vec{r}_0} = 0; \quad \frac{\partial S_{cl,c}}{\partial t_0} = 0; \quad \frac{\partial S_{cl,c}}{\partial \tilde{t}_1} = 0.
\]  \hfill (34)
From Eqs. (17) and (32)-(34) we find the following conditions

\[ \vec{u}_i + \vec{u}_{n+1} = \vec{u}'_1 + \vec{u}'_{n+1}, \]
\[ \gamma_i + \gamma_{n+1} = \gamma'_1 + \gamma'_{n+1}, \]
\[ \gamma'_1 = \gamma_{n+1}, \] (35)

where we have introduced the relativistic velocities

\[ \vec{u}^{(l)}_k \equiv \gamma_k \vec{v}^{(l)}_k \quad ; \quad k = 1, n + 1. \] (36)

Eqs. (35) respectively represent the conservation laws for relativistic momentum and energy during the collision, and the conservation of relativistic energy at the first entrance and last exit of the particle from the two wormhole’s mouths.

Equations (35) can in principle be solved either directly in the \( \vec{r}_0, t_0, \) and \( \tilde{t}_1 \) variables, or in terms of the velocity variables (for instance, in 2-d, considering \( \vec{v}'_1 \) and \( \vec{v}'_{n+1} \) as unknowns and \( \vec{v}_i \) and \( \vec{v}_{n+1} \) as parameters).

Using velocities as our unknowns and introducing the quantities (see Refs. TM-(I,II))

\[ \vec{a} \equiv \vec{u}_i - \vec{u}'_{n+1} = \vec{u}'_1 - \vec{u}_{n+1}, \]
\[ \vec{b} \equiv \vec{u}'_1 + \vec{u}_{n+1}, \]
\[ \vec{c} \equiv \vec{u}_i + \vec{u}'_{n+1}, \] (37)

Eqs. (35) can be easily transformed into the equivalent system of conditions

\[ \vec{a} \cdot \vec{b} = 0, \]
\[ \vec{a} \cdot \vec{c} = 0. \] (38)

For the motion in three spatial dimensions, the most general solution of the conservation laws (35) is thus given by

\[ \vec{u}_i = \frac{1}{2}(\vec{c} + \vec{a}), \]
\[ \vec{u}_{n+1} = \frac{1}{2}(\vec{b} - \vec{a}), \]
\[ \vec{u}'_1 = \frac{1}{2}(\vec{b} + \vec{a}), \]
\[ \vec{u}'_{n+1} = \frac{1}{2}(\vec{c} - \vec{a}). \] (39)

for any arbitrary \( \vec{a} \) which is orthogonal to arbitrary \( \vec{b} \) and \( \vec{c} \). Then, in principle, in the case of a generic three dimensional motion, Eqs. (38) (with \( \vec{a} \) given by the first of Eqs. (37)) can be solved, using Eqs. (32), for \( \vec{r}_0, t_0, \) and \( \tilde{t} \) and therefore for the complete trajectories (see Ref. TM-II). We shall not do that here, but only consider the simpler case of two dimensional spatial motion.
3.3.1 Coplanar motion

The solutions of the conservation Eqs. (35) for the case of two dimensional, relativistic motion of the copies of the particle coplanar with respect to the wormhole mouths can be deduced from the generic three dimensional solutions (39) by restricting to the following ansatz for \( \vec{a}, \vec{b}, \vec{c} \) (see Refs. TM-(I,II))

\[ i) \quad \vec{c} = \epsilon \vec{b}; \quad \text{or} \quad ii) \quad \vec{a} = \vec{0}, \quad (40) \]

where \( \epsilon \) is an arbitrary constant.

a) Cauchy data with generic relativistic velocity: \( \gamma_i \sim \text{finite} \)

The ansatz \( i) \) of Eq. (40) (for \( \epsilon \neq -1 \)) corresponds to the case of \[ \]

- **‘Mirror exchange of velocities’ rule:**

\[
\begin{align*}
(u'_1)_x &= (u_{n+1})_x, \\
(u'_{n+1})_x &= (u_i)_x, \\
(u'_1)_y &= -(u_{n+1})_y, \\
(u'_{n+1})_y &= -(u_i)_y,
\end{align*}
\]

expressed in components in the frame whose \( x \)-axis is along the direction of \( \vec{u}_i + \vec{u}_{n+1} = \vec{u}'_1 + \vec{u}'_{n+1} \), i.e. where

\[
(u_i)_y + (u_{n+1})_y = 0. \quad (42)
\]

In terms of velocities, the ‘mirror exchange’ solution can also be written as

\[
\begin{align*}
(v'_1)_x &= (v_{n+1})_x, \\
(v'_{n+1})_x &= (v_i)_x, \\
(v'_1)_y &= -(v_{n+1})_y, \\
(v'_{n+1})_y &= -(v_i)_y, \\
\gamma_i(v_i)_y + \gamma_{n+1}(v_{n+1})_y &= 0.
\end{align*}
\]

It is quite easy to check that this kind of trajectories is possible only when the relativistic particle is initially (at time \( t_i \)) pointing away from both wormhole mouths.

For the particular value \( \epsilon = -1 \), the ansatz \( i) \) of Eq. (40) no longer corresponds to the ‘topology’ of the ‘mirror exchange solutions,’ but to the case of

\[ ^p\]For this classification of the classical trajectories see Refs. [10] and TM-(I,II).
• ‘Collinear velocities’ rule:

\[ \vec{u}_{n+1} = -\vec{u}_i, \]
\[ \vec{u}'_1 = -\vec{u}'_{n+1}, \]
\[ |\vec{u}_{n+1}| = |\vec{u}'_1| = |\vec{u}'_{n+1}| = |\vec{u}_i|, \]

which is also equivalent, in terms of velocities, to

\[ \vec{v}_{n+1} = -\vec{v}_i, \]
\[ \vec{v}'_1 = -\vec{v}'_{n+1}, \]
\[ |\vec{v}_{n+1}| = |\vec{v}'_1| = |\vec{v}'_{n+1}| = |\vec{v}_i|. \]

These solutions are ‘degenerate’ in the sense that the velocities \( \vec{v}_i \) and \( \vec{v}_{n+1} \) must be along the direction identified by \( \vec{r}_i \) and \( \vec{r}_A \) (i.e. the velocity of the first copy of the particle must be initially pointing towards the wormhole mouth A), and similarly the velocities \( \vec{v}'_1 \) and \( \vec{v}'_{n+1} \) must be along the direction identified by \( \vec{r}_0 \) and \( \vec{r}_B \) (i.e. the velocity of the second copy of the particle after the collision must be outwards pointing from the wormhole mouth B).  

The ansatz \( ii) \) of Eq. (40) corresponds, instead, to the case of

• ‘Velocity exchange’ rule:

\[ \vec{u}'_1 = \vec{u}_{n+1}, \]
\[ \vec{u}'_{n+1} = \vec{u}_i, \]

or, in terms of velocities,

\[ \vec{v}'_1 = \vec{v}_{n+1}, \]
\[ \vec{v}'_{n+1} = \vec{v}_i. \]

However, from a simple analysis of the ‘topology’ of the motion (for a motion with ‘velocity exchange’ the trajectories \( \vec{r}'_1 \) and \( \vec{r}_{n+1} \) should lie along the direction of the line connecting the two wormhole mouths, i.e. parallel to the trajectories \( \vec{r}_k \), \( k = 2, \ldots n \)), it is easy to show that the condition of admitting only one self-collision of the particle in Regions I-IV requires \( n = 1 \), i.e. only one wormhole traversal is allowed in this case.

Finally, there is also the ‘trivial’ solution in which the velocities of the copies of the particle do not change before and after the collision. This actually corresponds to the ‘no collision’ case which we already considered in Section 3.2.

\[ \text{The solution } \vec{c} = -\vec{b}, \vec{a} = \vec{0} \text{ is a ‘doubly degenerate’ case, as it corresponds to one dimensional spatial motion of the copies of the particle along the line connecting the wormhole’s mouths, and is not considered here.} \]
b) Cauchy data with ultrarelativistic velocity I: $\gamma_i \to \infty$; $\gamma_k \sim \text{finite}$, $k \neq i$

In the case of initial Cauchy data with ultrarelativistic velocity (but the velocities of all other copies of the particle only relativistic, i.e. with $\gamma_k \sim \text{finite}$ for $k \neq i$), most of the possible solutions to the energy and momentum conservation equations (35) remain the same as those discussed in the previous paragraph, i.e. in principle one can have ‘collinear exchange’ and ‘no collision’ motion with an arbitrary number of wormhole traversals, or ‘velocity exchange’ motion with a single wormhole traversal, depending on the direction of $\vec{v}_i \equiv c \vec{k}_i$ ($|\vec{k}_i| = 1$). The only slight difference is with the case of ‘mirror exchange’ (i.e., $\vec{v}_i$ directed away from both wormhole mouths A and B), whose only non trivial solution is given by Eq. (43) replaced by

$$
\begin{align*}
(v'_1)_x = (v_{n+1})_x &\simeq \text{finite} < c, \\
(v'_{n+1})_x = (v_i)_x &\simeq c[1 - (\epsilon_2)^2/2], \\
(v'_1)_y = -(v_{n+1})_y &\simeq \text{finite} < c, \\
(v'_{n+1})_y = -(v_i)_y &\simeq \epsilon_1 c, \\
\gamma_i(v_i)_y = -\gamma_{n+1}(v_{n+1})_y &\equiv \epsilon = \text{finite},
\end{align*}
$$

with $|\epsilon_1| < |\epsilon_2| \ll 1$ and $|\epsilon_2|/|\epsilon_1| \simeq (1 + c^2/\epsilon^2)^{1/2}$.

c) Cauchy data with ultrarelativistic velocity II: $\gamma_k \to \infty$, $k = i,...n+1$

The case in which both the initial velocity and that of all other copies of the particle are ultrarelativistic is again similar to the case of initial Cauchy data with generic relativistic velocity, but only ‘collinear exchange’, ‘no collision’ (eventually with multiple wormhole traversals) and ‘velocity exchange’ (only with one wormhole traversal) motions are in principle possible, while the ‘mirror exchange’ solutions are no longer possible. This is easily seen from the fact that, for the case in which all $\gamma_k \to \infty$ ($k = i,...n+1$), introducing for the velocities the notation

$$
\vec{v}_k \equiv c\vec{k}_k \quad ; \quad |\vec{k}_k| = 1 \quad ; \quad k = 1,...n+1,
$$

the conservation Eqs. (35) reduce to the single condition

$$
\vec{k}_i + \vec{k}_{n+1} = \vec{k}'_1 + \vec{k}'_{n+1},
$$

which only admits solutions with ‘velocity’ and ‘collinear’ exchange ‘topologies’, but not ‘mirror’ type ones.
4 The Cauchy initial value problem

We turn now to the global analysis of the stationarity problem for the action (7), representing the motion of a relativistic particle passing through a wormhole ‘time machine’ an arbitrary number of times, self-colliding once and subject to a given set of fixed initial Cauchy data.

In the previous Section we discussed the possible solutions to the local energy and momentum conservation laws which can be deduced by imposing the stationarity of the action (7). Our task is now to verify whether there exist globally self-consistent motions which are stationary points for the action (7) for a generic set of Cauchy initial data, in particular for any given modulus and direction of the initial velocity (and for any given initial position \( \vec{r}_i \)). In doing this we will be proving that, for the model of a relativistic particle self-interacting via the ‘hard-sphere’ potential (4) and constrained to traverse a wormhole ‘time machine’, the whole set of classical trajectories which are globally self-consistent are those which can be derived by simply imposing the ‘Principle of stationary action’. In this sense, we would be thus extending to the case of relativistic motion of the particle the result, previously obtained in Refs. TM-(I,II) for the case of the non-relativistic motion, that the ‘Principle of self-consistency’ is a consequence of the more general ‘Principle of stationary action’.

Finally, we will also study the multiplicity (defined as the number of self-consistent classical trajectories beginning with a fixed set of initial Cauchy data, see Ref. [10]) of these stationary points, and compare our results with those obtained in the non-relativistic model of Ref. [10].

Our discussion of the Cauchy problem will be performed by considering separately the possible values of the modulus of the initial velocity of the relativistic particle (i.e., ‘generic relativistic’, when \( \gamma_i \sim \text{finite} \), ‘ultrarelativistic (I)’, when \( \gamma_i \rightarrow \infty \) but \( \gamma_k \sim \text{finite} \) for \( k \neq i \), and ‘ultrarelativistic (II)’, when all \( \gamma_k \rightarrow \infty \) for \( k = i, \ldots, n + 1 \) and, for each of these cases, the various possible directions of the same velocity (i.e., particle initially pointing away from both wormhole mouths, particle pointing towards mouth B and particle pointing towards mouth A). In the following subsections we summarize the main results of such an analysis, while for a more detailed discussion of the trajectories for each of the fixed initial Cauchy data we refer to Appendix A.2.

4.1 Cauchy data with generic relativistic velocity:

\( \gamma_i \sim \text{finite} \)

a) \( \vec{v}_i \) pointing away from mouths A and B

If the initial velocity of the particle is smaller than \( c \) and pointing away from both wormhole mouths, a simple analysis of the various solutions of the energy and momentum conservation equations (B5) shows that the only possible globally self-consistent ‘topologies’ for the particle motion are those corresponding to the self-collision of the particle in Region II with ‘mirror exchange’ or ‘velocity exchange’ of velocities (plus
the ‘trivial’ motion in which the particle does not enter the wormhole, with multiplicity one). As it is shown in Appendix A.2, for these kinds of initial Cauchy data a unique (non trivial) stationary point for the action, and therefore a unique classical motion (Eqs. (62) and (71)-(72) for the ‘mirror exchange’ case, and Eqs. (86)-(87) for the ‘velocity exchange’ case), is possible for any fixed number \( n \) (only \( n = 1 \) for the ‘velocity exchange’ case) of wormhole traversals by the relativistic particle, subject to the constraints (63), (73)-(74) and (78)-(79) (for the ‘mirror exchange’ case, in the ‘gauge’ fixed by Eqs. (59) and (64)), or (88) and (90)-(91) (for the ‘velocity exchange’ case, in the ‘gauge’ fixed by Eqs. (85)).

Since both ‘mirror exchange’, ‘velocity exchange’ and ‘no entrance’ motions are possible in this case, and the number of possible wormhole traversals is in principle arbitrary, the multiplicity of the trajectories for this case is at least one (in general finite, as discussed in details in Appendix A.2), and it becomes infinite (again, in the frame fixed by Eqs. (59) and (64)) provided that, e.g., the constraints (80)-(81) and (83) are satisfied (see the discussion in Appendix A.2). Thus, generally, the Cauchy initial value problem is classically ill-posed (far too many solutions).

b) \( \vec{v}_i \) pointing towards mouth B
When the relativistic particle is initially moving towards wormhole mouth B, the only possible, globally self-consistent, ‘topology’ for the particle motion is that corresponding to the case of ‘no collision’. Also in this case it can be shown (see Appendix A.2 for details) that there is a unique stationary point for the action, and therefore a unique classical trajectory (Eqs. (95)-(96)), for any fixed number \( n \) of wormhole traversals by the relativistic particle (provided condition (97) holds).

Since the number of wormhole traversals is arbitrary, the multiplicity of the trajectory is in general finite (at least one), and it becomes infinite if, e.g., condition (98) is satisfied. Again, generally the Cauchy initial value problem is classically ill-posed.

We would like to stress that, among the ‘no collision’ cases, we are also including trajectories which are treated like that only because of our approximation that the wormhole mouths are pointlike. Indeed, if the Cauchy initial data are specified in such a way that the original and the final copies come to the same space point simultaneously, then there might be a collision, but only a ‘glancing’ one. After such a ‘gentle’ collision the original copy would move along a slightly altered trajectory which still takes it into mouth B because of the finite size of this mouth. In other words, this case would actually correspond to the class of the so called ‘dangerous trajectories’ described in Ref. [10]. Only when we take the limit of pointlike mouths we should treat this case among the ‘no collision’ ones.

c) \( \vec{v}_i \) pointing towards mouth A
Finally, when the relativistic particle is initially heading towards wormhole mouth
A, only ‘collinear exchange’ or ‘no collision’ motions are possible. In this case, the trajectory for which the action (7) is stationary is also unique (Eqs. (103)-(104), in the ‘gauge’ (101), for the ‘collinear exchange’ motion, and Eqs. (95) - with subscripts A and B exchanged - and (110) for the ‘no collision’ motion), for each fixed number \( n \) of wormhole traversals (provided, for the ‘collinear exchange’ case, the constraints (103)-(106) of Appendix A.2 are satisfied).

Since the ‘no collision’ motion is always possible for arbitrary \( n \) (only provided \(|\vec{v}_i| < c\)), in this case the multiplicity of solutions to the equations of motion is always infinite. The Cauchy initial value problem is always classically ill-posed.

We should note here that for the ‘no collision’ trajectories the multiplicity of solutions is in some sense fictitious and an artifact of our approximation of treating the wormhole mouths as pointlike. In particular, if we take into account the finite size of the mouths and we fix the wormhole internal structure such as, e.g., the traversal rules of Ref. [10] are satisfied, then for different \( n \)’s one would need slightly different Cauchy initial data. On the other hand, the finite size of the mouths would remove the degeneracy of the ‘no collision’ trajectories, since also the motions with ‘mirror exchange’ collisions would become possible.

4.2 Cauchy data with ultrarelativistic velocity I: 
\[
\gamma_i \rightarrow \infty \quad ; \quad \gamma_k \sim \text{finite} \quad , \quad k \neq i
\]

The discussion about the cases of a particle traversing the wormhole ‘time machine’ and whose initial velocity is ultrarelativistic (but the velocities of all other copies of the particle are only relativistic, i.e. with \( \gamma_k \sim \text{finite} \) for \( k \neq i \)) is essentially the same as that made in the previous Section, with the only caveat that everywhere in the formulas for the trajectories one should, of course, substitute \( \vec{v}_i \) by \( c\vec{k}_i \) (\(|\vec{k}_i| = 1\)) and, for the ‘mirror exchange’ type of collision, use the ‘gauge’ fixing condition (92) of Appendix A.2 instead of Eq. (64). For any given Cauchy data, any given ‘topology’ of the motion (‘no collision’, ‘mirror exchange’, ‘velocity exchange’ and ‘collinear exchange’) and any fixed number of wormhole traversals, the multiplicity of the globally self-consistent classical trajectory for which the action is stationary is one, and therefore generically finite when allowing for an arbitrary number of such traversals (provided certain constraints are satisfied). For case a), i.e. an initial velocity pointing away from both wormhole mouths, the multiplicity of classical trajectories is at least one and becomes infinite if the constraints (78)-(74) and, e.g., (83) are satisfied. For case b), i.e. an initial velocity pointing towards mouth B, condition (78) reduces to the first of conditions (11) for the existence of the ‘time machine’. Therefore, the multiplicity of the classical trajectories is always infinite in this case. In case c) the multiplicity is also always infinite. For any choice of the initial data (except when both any of the constraints (83), (73)-(74), (81) - in the ‘gauge’ (58), (64) - and any

\[ ^{1}\text{In particular, the ‘no collision’ case distinguishes itself from all other motions described in this paper since the particle travels forward in time while traversing the wormhole.} \]
of the constraints (88), (90)-(91) - in the ‘gauge’ (85) - are violated, in which case the multiplicity of trajectories is one and the Cauchy problem well defined) the Cauchy initial value problem is classically ill-posed.

4.3 Cauchy data with ultrarelativistic velocity II:

\[ \gamma_k \to \infty \quad , \quad k = i, \ldots, n + 1 \]

The only change with respect to the discussion in the previous Section for the case in which the velocities of all the copies of the particle (including the initial velocity) are ultrarelativistic, is in the impossibility of having ‘mirror exchange’ type of solutions to the equations of motion (see Eq. (50) in Section 3.3.1). Therefore, for the case of Cauchy data in which the particle is initially pointing away from both wormhole mouths, the only possible trajectories are the trivial one where the particle never enters the wormhole (multiplicity one) and that corresponding to ‘velocity exchange’ (multiplicity one if conditions (88) and (90)-(91) hold). For all choices of the initial data (except those violating any of the constraints (88), (90)-(91), for which the multiplicity of trajectories is one and the Cauchy problem well defined) the Cauchy initial value problem remains classically ill-defined.

We have thus proved that, for the model of a particle which is constrained to traverse (albeit an arbitrary number of times) a wormhole ‘time machine’ geometry, to self-interact via the ‘hard-sphere’ potential (6) and subject to any of the initial Cauchy data described in Sections 4.1-4.3, the whole set of classical trajectories which are globally self-consistent can be directly and simply recovered by imposing the ‘Principle of stationary action’. However, the multiplicity of trajectories starting with fixed initial data is generically finite, if not even infinite, thus making the Cauchy initial value problem classically ill-posed.

5 Discussion

The analysis of the Cauchy initial value problem for some simple physical systems, e.g. represented by free classical fields or self-interacting particles, evolving in the background of a spacetime containing CTCs has been recently tackled by the authors of Refs. [9-11]. In particular, the authors of Ref. [9] considered the Cauchy initial value problem for the evolution of a classical, massless scalar field \( \phi \) in the presence of CTCs, showing that in general the ‘Principle of self-consistency’ constrains those initial data for the field \( \phi \) which are posed in the future of the Cauchy horizon where the CTCs reside. These constraints appear to be mild in the sense that the initial data can be fixed arbitrarily in some neighbourhood of any event, while being adjusted elsewhere in order to guarantee a globally self-consistent evolution. The

\[ \text{footnote:} \text{ For the description of the classical and quantum initial value problems for chronology violating spacetimes in which space consists of a finite number of space points, see also Ref. [23].} \]
authors of Ref. [10], instead, mainly focused their attention to the Cauchy initial value problem for the motion of a non-relativistic, classical 'billiard ball'-like particle in the background of a wormhole ‘time machine’. Allowing for an arbitrary number of wormhole traversals by the particle, it was claimed that almost all initial trajectories have infinite multiplicity and make the Cauchy initial value problem ill-posed (far too many solutions). In particular, for a fixed initial path and speed (i.e., fixed $|\vec{v}_i|$, $\psi_A$ and $h$, the last two quantities respectively representing the angle between $\vec{v}_i$ and the mouths line of centers, and the initial impact parameter along $\vec{v}_i$ with respect to mouth B) of the particle, coplanar motion with respect to the wormhole mouths, and when the constraints

\begin{align}
|\vec{v}_i| > |\vec{r}_B - \vec{r}_A|/\tau,
L \gg |\vec{r}_B - \vec{r}_A|; \quad L \gg h
\end{align}

are satisfied, it was shown that the number of solutions for a single ‘mirror exchange’ collision is infinite. Only the trajectory for which the particle is initially at rest far from the wormhole appears to have multiplicity one, while no evidence was found for trajectories with zero multiplicity. The analysis was also generalized to the case of slightly non coplanar motion, confirming that infinite multiplicity of solutions should be generic.

Similarly, in this paper we considered the Cauchy initial value problem for the motion of a pointlike relativistic particle which can traverse a wormhole ‘time machine’ several times and self-interacts via a short range, ‘hard-sphere’ potential. We showed that all of the trajectories of the relativistic particle which are globally self-consistent can be found by simply imposing the ‘Principle of stationary action’, thus confirming our earlier conjecture (formulated in the context of a fixed boundary data problem for a non-relativistic motion, Refs. TM-(I,II)) that the ‘Principle of self-consistency’ is a natural consequence of more fundamental physical principles. In particular, we also addressed the issue of the classical multiplicity of such trajectories, finding that it is always at least one, in general finite and, if certain constraints are satisfied by the initial Cauchy data, even infinite. The set of constraints leading to infinite multiplicity of solutions is consistent with that previously found in Ref. [10] for the case of non-relativistic motion. However, as explained in more details in Appendix A.2, the results of our analysis are apparently in contrast with the claim of Ref. [10] that infinite multiplicity of trajectories is a generic property of any set of initial Cauchy data. The existence of a non empty set, among our solutions, of trajectories having finite multiplicity can be shown to be the direct consequence of the different definition of the initial data for the Cauchy problem (the authors of Ref. [10] only fix the initial path and leave one of the components of $\vec{r}_i$ as an arbitrary parameter of the

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4\(\)Since $L$ is the distance, along the $\vec{v}_i$ direction, from the collision to the point of closest approach to mouth B, this means that the collision occurs far from the wormhole.

5\(\)The issue of the possible existence of zero multiplicity, i.e. globally self-inconsistent, classical trajectories in the background of a wormhole ‘time machine’ has been also addressed in Refs. [26].
motion) and, moreover, of the kinematical constraints on the velocities of the particle \(|\vec{v}_k| < c, k = i, n + 1\), for a particle initially pointing away from both wormhole mouths A and B) or of the conditions for the existence of the ‘time machine’ (for a particle initially moving towards mouth B). These constraints apparently were not considered in Ref. [10].

On the other hand, our conclusion that the Cauchy initial value problem for the motion of a relativistic ‘billiard ball’-like particle in the spacetime containing CTCs is generally ill-posed (too many solutions) extends the non-relativistic results of Ref. [10].

Even if in our analysis, which is purely classical, we have considered for simplicity only those motions entailing at most a single self-collision, we expect that multiple-collision solutions should further increase the multiplicity of trajectories, in agreement with what suggested in Ref. [10].

Finally, in Refs. [10] and [27] it was claimed that, when taking into account the effects of quantum mechanics, the classically, non-relativistic ill-posed Cauchy problem should become quantum mechanically well-posed, in the sense that the probability distributions for the outcomes of measurements (predicted in the context of a path integral, sum-over-histories formulation, see Ref. [28]) turn out to be unique. The question whether similar quantum effects can also ‘beneficially’ apply to the ill-posedness of the Cauchy initial value problem for the relativistic model presented here is an interesting and still open issue which we hope to address in a future publication.

Acknowledgements

This work is supported in part by Danish Natural Science Research Council through grant N9401635 and in part by Danmarks Grundforskningsfond through its support for the establishment of the Theoretical Astrophysics Center. A.C.’s research was partly funded by TAC and also by a JSPS postdoctoral fellowship, under contract No. P95196.

*Although the quantum evolution of self-interacting states propagating in spacetimes with CTCs might be non-unitary, see, e.g., Refs. [29, 25] and [6].
A Appendix

A.1 The ‘hard-sphere’ potential

In this appendix we prove some interesting properties for the ‘hard-sphere’ potential of Eq. (3).

- Energy conservation

For the potential (3), the energy conservation Eq. (13) is apparently ill defined at the collision event. What, however, this equation clearly says is that, in the region $r < r_s$, the motion is not classically allowed, as the total relativistic energy would be infinitely large and negative. Noting that the relativistic $\gamma$ factors satisfy the following properties

$$ (\gamma_k \dot{r}_k) = \gamma_k^3 \dot{\gamma}_k, $$

$$ \dot{\gamma}_k = \dot{r}_k \cdot \dot{r}_k \gamma_k^3/c^2, $$

and assuming that the classical motion proceeds until $r = r_s$ (i.e., the collision takes place at time $t_0$), we can now integrate the left hand sides of Eqs. (9) around $r_s$, obtaining

$$ m \int_{r_k(t_0-\delta t_1)}^{r_k(t_0+\delta t_2)} d\gamma_k \cdot (\gamma_k \dot{r}_k) = m \int_{t_0-\delta t_1}^{t_0+\delta t_2} dt \dot{r}_k \cdot \dot{\gamma}_k = mc^2 \int_{t_0-\delta t_1}^{t_0+\delta t_2} dt \dot{\gamma}_k $$

for $k = 1, n + 1$ and where $\delta t_1, \delta t_2 > 0$. Summing Eqs. (52) for $k = 1$ and $k = n + 1$ we then get

$$ I = m \left[ \int_{r_1(t_0-\delta t_1)}^{r_1(t_0+\delta t_2)} d\gamma_1 \cdot (\gamma_1 \dot{r}_1) + \int_{r_{n+1}(t_0-\delta t_1)}^{r_{n+1}(t_0+\delta t_2)} d\gamma_{n+1} \cdot (\gamma_{n+1} \dot{r}_{n+1}) \right] $$

$$ = mc^2 \{[\gamma_1(t_0 + \delta t_2) + \gamma_{n+1}(t_0 + \delta t_2)] - [\gamma_1(t_0 - \delta t_1) + \gamma_{n+1}(t_0 - \delta t_1)]\}. $$

Moreover, when integrating on the right hand sides of Eqs. (9), one obtains

$$ I = V \left[ \int_{r_{n+1}(t_0-\delta t_1)}^{r_{n+1}(t_0+\delta t_2)} d\gamma_{n+1} \cdot \dot{r}_n \delta(r - r_s) - \int_{r_1(t_0-\delta t_1)}^{r_1(t_0+\delta t_2)} d\gamma_1 \cdot \dot{r}_1 \delta(r - r_s) \right] $$

$$ = V \int_{r_s+\epsilon_1}^{r_s+\epsilon_2} dr \delta(r - r_s) = 0 $$

(with $\epsilon_1, \epsilon_2 > 0$). Comparing Eqs. (54) and (55) we finally get the conservation law for the relativistic energy before and after the collision

$$ m(\gamma_1 + \gamma_{n+1})c^2 = \text{const}, $$

as anticipated.

*These results are valid for the generic case of motion in three spatial dimensions.*
Zero contribution to classical action

Let us consider the contribution of the potential (5) to the action (7) evaluated along the classical trajectories (27) or (32).

In the ‘no collision’ case, denoting with \( t_m \) and \( r_m \) the time and position of minimum distance between the two copies of the particle, we have, for the potential (5),

\[
\int_{\bar{t}_n}^{\bar{t}_1 + \tau} dt \left[ V(r) \right]_{cl} = \hat{V} \left[ \int_{\bar{t}_n}^{t_m} dt \theta(r_s - r) + \int_{\bar{t}_1 + \tau}^{r_m} \frac{dr}{\hat{r}} \theta(r_s - \hat{r}) \right] = 0, \quad (57)
\]

since obviously, in the ‘no collision’ case, \( r_m, r(\bar{t}_n) \) and \( r(\bar{t}_1 + \tau) \) are always greater than \( r_s \).

The proof proceeds along similar lines for the collision case (see, i.e., Ref. TM-I).

In conclusion, the contribution of the ‘hard-sphere’ potential \( V \) to the classical action can always be neglected both in the collision and ‘no collision’ cases.

A.2 Trajectories for the coplanar motion

a) \( \vec{v}_i \) pointing away from mouths A and B

i) Generic relativistic velocity: \( \gamma_i \sim \text{finite} \)

\bullet ‘Mirror exchange of velocities’

When the Cauchy data for the relativistic particle are such that the initial velocity is pointing away from both wormhole mouths, with \( \gamma_i \sim \text{finite} \), one possible local solution of the conservation Eqs. (35) is given by the ‘mirror exchange’ type of collision. In this case, assuming \( n \) traversals of the particle through the wormhole ‘time machine’, and in the frame specified by Eq. (12) of Section 3.3.1, we can write down in components the trajectories for the two copies of the particle in Region II as

\[
\begin{align*}
x_1(t) &= (v_i)_x t + (b_1)_x, \\
y_1(t) &= (v_i)_y t + (b_1)_y, \\
x_{n+1}(t) &= (v_{n+1})_x t + (b_{n+1})_x, \\
y_{n+1}(t) &= -\gamma_i (v_i)_y t / \gamma_{n+1} + (b_{n+1})_y, \\
x'_1(t) &= (v_{n+1})_x t + (b'_1)_x, \\
y'_1(t) &= \gamma_i (v_i)_y t / \gamma_{n+1} + (b'_1)_y, \\
x'_{n+1}(t) &= (v_i)_x t + (b'_{n+1})_x, \\
y'_{n+1}(t) &= -(v_i)_y t + (b'_{n+1})_y.
\end{align*}
\]
We then further fix our coplanar coordinate system by choosing the collision point $\mathbf{r}_0$ as its origin, i.e. we take
\[ x_0 = y_0 = 0, \tag{59} \]
and also define, for simplicity of notation, the following quantity
\[ \rho \equiv (n - 1) \frac{|\mathbf{r}_B - \mathbf{r}_A|}{|\mathbf{v}_{n+1}|}. \tag{60} \]
Since the ‘unknowns’ in the problem (58) are twelve ($(b_1)_x, (b_1')_x, (b_{n+1+1})_x, (b_{n+1+1})_y, (v_{n+1})_x, (v_{n+1})_y, t_0$ and $\bar{t}_1$), while the conditions to be imposed coming from the Cauchy and boundary data (Eqs. (2), (3) and (18)) are fourteen, the solutions will be constrained. In particular, it is quite straightforward to formally solve for the trajectory parameters as
\begin{align*}
(b_1)_x &= (b_{n+1})_x = x_i - (v_i)x_i, \\
(b_1)_y &= -b_{n+1})_y = y_i - (v_i)y_i, \\
(b_{n+1+1})_x &= (b_{n+1})_x = (x_A - x_B)[t_i - x_i/(v_i)x]/(n\tau - \rho), \\
(b_{n+1+1})_y &= -b_{n+1})_y = (y_A + y_B)[t_i - x_i/(v_i)x]/(n\tau - \rho), \\
(v_{n+1+1})_x &= (v_i)x, \\
(v_{n+1+1})_y &= -(v_i)y, \\
(v_{n+1+1})_x &= (v_{n+1})_x = (x_B - x_A)/(n\tau - \rho), \\
(v_{n+1+1})_y &= -(v_{n+1})_y = -(y_A + y_B)/(n\tau - \rho),
\end{align*}
(61)
with the times of collision and of first entrance of the particle into wormhole mouth B respectively given by
\begin{align*}
t_0 &= t_i - x_i/(v_i)x, \\
\bar{t}_1 &= t_0 - \tau + \frac{x_B}{(x_B - x_A)}(n\tau - \rho), \tag{62}
\end{align*}
while the constraints can be written as
\begin{align*}
(v_i)x/(v_i)y &= x_i/y_i, \\
y_B/x_B &= -y_A/x_A, \tag{63}
\end{align*}
plus the ‘gauge’ condition
\[ (y_A + y_B)\gamma_{n+1}/(n\tau - \rho) = (v_i)y_i. \tag{64} \]
At this level, the expressions for $([b \div v]_{n+1})_{x,y}$ in Eqs. (61), and for $\bar{t}_1$ in the second of Eqs. (62), are still formal and implicit, since they depend on the parameter $\rho$, which, by its definition (60), is a function of the velocity $|\mathbf{v}_{n+1}|$. In order to solve
for the components \((v_{n+1})_x\) and \((v_{n+1})_y\), we first rewrite the last two of Eqs. (61) introducing the explicit form of \(\rho\), Eq. (60), and obtaining

\[
\begin{align*}
(v_{n+1})_x(n-1)|\vec{r}_B - \vec{r}_A| &= |\vec{v}_{n+1}|[x_A - x_B + n\tau (v_{n+1})_x], \\
(v_{n+1})_y(n-1)|\vec{r}_B - \vec{r}_A| &= |\vec{v}_{n+1}|[y_A + y_B + n\tau (v_{n+1})_y].
\end{align*}
\]

(65)

Dividing member by member the two equations in (65), we have

\[
\begin{align*}
(v_{n+1})_x(n-1)|\vec{r}_B - \vec{r}_A| &= |\vec{v}_{n+1}|[x_A - x_B + n\tau (v_{n+1})_x], \\
(v_{n+1})_y &= (v_{n+1})_y (y_A + y_B)/(x_A - x_B).
\end{align*}
\]

(66)

The system of equations (66) can be finally solved for \((v_{n+1})_{x,y}\) squaring the last of Eqs. (60), using the first of Eqs. (60) and the second of conditions (63) in order to eliminate the dependence on \(\vec{v}_{n+1}\), and finally making use of the following constraints

\[
\begin{align*}
x_B(n\tau - \rho)/(x_B - x_A) &> 0, \\
x_A(n\tau - \rho)/(x_B - x_A) &< 0,
\end{align*}
\]

(67)

which are a consequence of the first two of the time ordering constraints (29) applied to the ‘mirror exchange’ solutions (62) (moreover, the last of constraints (29) gives \(x_i/(v_i)_x < 0\)). The explicit result which is consistent with the conditions (67) is given by

\[
\begin{align*}
(v_{n+1})_x &= [(1 - n)|\vec{r}_B - \vec{r}_A| x_A + (x_B - x_A)r_A]/n\tau r_A, \\
(v_{n+1})_y &= y_A(v_{n+1})_x/x_A.
\end{align*}
\]

(68)

In particular, noting from the second of Eqs. (68) that we have

\[
|\vec{v}_{n+1}| = -(v_{n+1})_x r_A/x_A,
\]

(69)

we can also explicitly write down for the parameter \(\rho\)

\[
n\tau - \rho = \frac{n\tau (x_B - x_A)r_A}{[(1 - n)|\vec{r}_B - \vec{r}_A| x_A + (x_B - x_A)r_A]},
\]

(70)

and, therefore, from Eq. (62), we find the explicit solution for \(\tilde{t}_1\) as

\[
\tilde{t}_1 = t_0 - \tau + \frac{n\tau x_B r_A}{[(1 - n)|\vec{r}_B - \vec{r}_A| x_A + (x_B - x_A)r_A]}.
\]

(71)

Using all these results into Eqs. (68), the final explicit form of the classical trajectories for the motion with ‘mirror exchange’ collision can be written as

\[
\begin{align*}
x_1(t) &= x_{n+1}'(t) = (v_i)_x [t - t_i + x_i/(v_i)_x], \\
y_1(t) &= -y_{n+1}'(t) = (v_i)_y x_1(t)/(v_i)_x, \\
x_{n+1}(t) &= x_1(t) = [(1 - n)|\vec{r}_B - \vec{r}_A| x_A + (x_B - x_A)r_A] x_1(t)/n\tau r_A (v_i)_x, \\
y_{n+1}(t) &= -y_1'(t) = y_A x_{n+1}(t)/x_A.
\end{align*}
\]

(72)
Moreover, since for such trajectories one can easily show that $n\tau - \rho > 0$, it is straightforward to check that the possible solution for the system of constraints (67) is given by

$$x_A / x_B < 0.$$  \hfill (73)

Finally, we have a further condition coming from the kinematical limit $|\vec{v}_{n+1}| < c$ which, when used into Eqs. (68) and (69), leads to the final constraint

$$\left[\frac{(n - 1)|\vec{r}_B - \vec{r}_A|x_A + (x_A - x_B)r_A}{n\tau x_A}\right] < c.$$  \hfill (74)

Moreover, from Eqs. (68)-(69) and the constraint (73) it is easy to see that the condition (23) for the existence of the ‘time machine’ is also trivially satisfied.

From our analysis we can conclude that the Cauchy initial value problem for the motion of a relativistic particle constrained to traverse a wormhole ‘time machine’ $n$-times ($n$ fixed), to have an initial velocity pointing away from both wormhole mouths A and B and self-interacting via a ‘mirror exchange’ collision, has a unique, globally self-consistent solution, given by the first of Eqs. (62) and Eqs. (71)-(72), provided that the initial Cauchy data are chosen to satisfy, i.e., the first of conditions (63) and condition (74) holds. Moreover, the second of conditions (63) and condition (64) can be interpreted as fixing the actual coordinates of the collision point in an arbitrary frame. This is because such conditions are not written in an invariant form under a global change of coordinates in the 2-d spatial frame where the wormhole mouths are at rest, so that changing the values of, e.g., $x_A$ and $y_A$ (leaving the other coordinates fixed) in (63) and (74) is actually equivalent to change the position of $\vec{r}_0$ relative to the wormhole mouths A and B. The trajectories are further restricted to satisfy the constraint (73).

The multiplicity of the trajectories can be seen to depend on conditions (64) and (74), which involve the initial velocity $\vec{v}_i$ (condition (74) implicitly depends on $\vec{v}_i$ via Eq. (64)). In particular, it is instructive to rewrite the constraint (74), using Eqs. (63), (68) and (71), extracting the initial velocity $|\vec{v}_i|$ as a function of $n$ and the other trajectory parameters, i.e.

$$|\vec{v}_i|^2 = c^2[\alpha f(n)/(1 + \alpha f(n))],$$  \hfill (75)

where we have defined the function

$$f(n) \doteq [(n - 1)a + b]^2/[n^2d - [(n - 1)a + b]^2]$$  \hfill (76)

and the quantities

$$\alpha \doteq y^2_A r^2_i / y^2_i r^2_A > 0,$$
$$a \doteq |\vec{r}_B - \vec{r}_A| > 0,$$
$$b \doteq (1 - x_B / x_A)r_A > 0,$$
$$d \doteq c^2r^2 > 0.$$  \hfill (77)
It is easy to check from Eq. (75) that the velocity $|\vec{v}_i|$ is a monotonic, decreasing function of the number $n$ of wormhole traversals, with extrema $[|\vec{v}_i|_{\text{min}}]_{n \to \infty} = c\{\alpha a^2/[d + (\alpha - 1)a^2]\}^{1/2}$, $[|\vec{v}_i|_{\text{max}}]_{n=1} = c\{\alpha b^2/[d + (\alpha - 1)b^2]\}^{1/2}$.

In other words, only for those initial Cauchy data having a velocity

$$\text{Min}[c, c g(x_A, x_B - x_A)] > |\vec{v}_i| > c g(r_A, \vec{r}_B - \vec{r}_A),$$

with

$$g(\mu, \nu) = \frac{|y_A| |\nu| r_i}{\sqrt{c^2 \tau^2 y_i^2 \mu^2 + (x_i^2 y_A^2 - x_A^2 y_i^2)\nu^2}},$$

the constraint (74) is solvable, therefore fixing the coordinates of $\vec{r}_0$ relatively to A and B. For instance, if the condition

$$|x_i y_A| \leq |y_i x_A|$$

holds, then the lower bound on $|\vec{v}_i|$ simplifies as

$$|\vec{v}_i| > |\vec{r}_B - \vec{r}_A|/\tau.$$

Finally, let us consider the constraint (74), starting from the case in which $x_A > 0$. In this case, the constraint (74) can be rewritten as

$$n[|\vec{r}_B - \vec{r}_A| - c\tau]x_A < (x_B - x_A)r_A + |\vec{r}_B - \vec{r}_A|x_A.$$  

Then, since for the wormhole to act as a ‘time machine’ the first of conditions (1) must hold, it is easy to show that Eq. (82) can be satisfied independently of the number $n$ of wormhole traversals provided that the following sufficient condition is satisfied

$$\left(1 - \frac{x_B}{x_A}\right)\frac{r_A}{c\tau} < 1,$$

which also implies, due to the second of Eqs. (77), $r_A < c\tau$. A similar argument works for the case of $x_A < 0$, leading again to condition (83).

In other words, as remarked below Eq. (74), one can see that a change in $n$ actually corresponds, via Eq. (74), to a change in the position of one of the two coordinates of the collision point $\vec{r}_0$ relative to the wormhole mouths A and B. However, only for those initial velocities satisfying the constraint (78) (and constraint (74)) there is a possible solution to the Cauchy initial value problem, each with a different location of the collision point.

Since, for a given $\vec{v}_i$, the constraint (74) can be in general satisfied only up to a finite number of wormhole traversals, this means that there is only a finite number of possible locations for the collision point, and that the multiplicity of the solutions is in general finite. On the other hand, if, e.g., the constraint (83) is also satisfied by the initial velocity, the kinematical constraint (74) is always satisfied (independently of $n$) and a change in $n$ (for any $n \geq 1$) only affects the constraint (74), which causes
a change in the coordinates of \( \vec{r}_0 \) relative to the mouths A and B. In particular, when \( n \to \infty \) and assuming \( |\vec{v}_i| \) fixed, one can also read Eq. (73) as giving, e.g., \( y_A \) as a function of \( |\vec{v}_i| \) (this \( y_A \) is finite, as in general also \( x_A, x_i \) and \( y_i \) are, see Eqs. (63), and there are no further constraints on the initial Cauchy data). Then, only provided that, e.g., the constraints (71), (74) and (73) are satisfied, the multiplicity of trajectories becomes infinite. We finally note that Eq. (81) is the same condition assumed by the authors of Ref. [10] (their Eq. (3.4)) to prove that non-relativistic trajectories with a single ‘mirror exchange’ self-collision have infinite multiplicity. However, we have here extended the analysis of Ref. [10] by showing that the stricter condition (78) must be satisfied by the initial data in order that the motion is globally self-consistent. Moreover, we have shown that initial Cauchy data not satisfying either condition (78) or (74) lead to a motion with zero or finite multiplicity, apparently contradicting the claim of Ref. [10] about infinite multiplicity of trajectories being generic. The origin of this ‘mismatch’ can be easily traced back partly to the different choice of initial Cauchy data made by the authors of Ref. [10] (fixed \( \vec{v}_i \) and impact parameter) for describing the particle motion, and partly to the kinematical constraint (74), which was simply not considered in Ref. [10]. It can be shown, in fact, that the choice of data made in Ref. [10] (which, in our framework, is essentially equivalent to fix \( \vec{v}_i \) and only one component of \( \vec{r}_i \)) leads to the same set of trajectories and constraints as our Eqs. (71)-(72), (74) and (78), with the only difference that now the value of \( t_0 \) is undetermined and, consequently, the trajectories themselves are implicit functions of \( t_0 \) (for instance, \( x_1(t) = (v_i)_x(t - t_0) \)). The point is that with this choice of initial Cauchy data, provided the kinematical constraint (74) and the gauge condition (64) are satisfied at least for \( n = 1 \), since one of the components of \( \vec{r}_i \) can be still chosen arbitrarily (it is a free parameter of the motion), the multiplicity of the trajectories always turns out to be infinite.

• ‘Velocity exchange’

The other possible nontrivial motion when the initial velocity of the particle is heading away from both mouths A and B of the wormhole is that for which the particle self-collides with ‘velocity exchange’ in Region II and traverses the wormhole only once (see the discussion in Section 3.3.1).

Taking into account the solution (17) for the conservation equations at the collision point, the trajectories can be parametrized as

\[
\begin{align*}
\vec{r}_1(t) &= \vec{v}_i t + \vec{b}_1, \\
\vec{r}_1'(t) &= \vec{v}_2 t + \vec{b}_1', \\
\vec{r}_2(t) &= \vec{v}_2 t + \vec{b}_2, \\
\vec{r}_2'(t) &= \vec{v}_i t + \vec{b}_2'.
\end{align*}
\]

Choosing to work in the coordinate frame where

\[
\begin{align*}
\vec{r}_0 &= \vec{0}, \\
y_A &= y_B = 0,
\end{align*}
\]
and imposing on the trajectories (84) the Cauchy and boundary data (2) and (3), after some simple algebraic steps we finally find the following result

\[ \vec{r}_1(t) = \vec{r}_2'(t) = \vec{v}_i(t - t_i) + \vec{r}_i, \]
\[ x_2(t) = x_1'(t) = (x_B - x_A)[t - t_i + x_i/(v_i)_x]/\tau, \]
\[ y_2(t) = y_1'(t) = 0, \]

with

\[ t_0 = t_i - x_i/(v_i)_x, \]
\[ \bar{t}_1 = t_0 + x_A\tau/(x_B - x_A), \]

and subject to the following constraint on the initial Cauchy data

\[ x_i/y_i = (v_i)_x/(v_i)_y. \]

Finally, there are the time ordering conditions coming from Eqs. (29) which, for the case of ‘velocity exchange’ collision, read as

\[ x_B/(x_B - x_A) > 0, \]
\[ x_A/(x_B - x_A) < 0, \]
\[ x_i/(v_i)_x < 0, \]

and which can be solved if the following constraints hold

\[ x_A/x_B < 0 \]

and

\[ x_i/(v_i)_x < 0. \]

There are no further constraints coming from the kinematical conditions \(|\vec{v}_i|, |\vec{v}_2| < c\) (in fact, these are equivalent to the condition for the existence of the ‘time machine’, i.e. the first of Eqs. (1)).

Thus, the Cauchy initial value problem for the motion of a relativistic particle initially heading away from both wormhole mouths A and B, self-interacting via a ‘velocity exchange’ collision and traversing the wormhole ‘time machine’ only once has a unique, globally self-consistent solution, given by Eqs. (86) and (87), provided that the initial Cauchy data are chosen to satisfy conditions (88) and (91), and the further constraint (90) also holds. Of course the multiplicity of solutions is one in this case (zero if any of conditions (88)-(89) does not hold). Similarly to the ‘mirror exchange’ case, it can be easily shown that the choice of initial data made in Ref. [10] would formally lead to the same set of Eqs. (86)-(91), with \(t_0\) undetermined, and generally to infinite multiplicity of trajectories.

In conclusion, the Cauchy initial value problem for the motion of a relativistic particle initially heading away from both wormhole mouths A and B is generally
classically ill-posed (i.e. there are far too many solutions, the trivial one with ‘no traversal’ of the wormhole, plus one with ‘velocity exchange’ and multiple, possibly infinite with ‘mirror exchange’). Only for those initial data not satisfying both any of the constraints (63), (73)-(74), (78) (in the ‘gauge’ (59), (64)) and any of the constraints (88), (90)-(91) (in the ‘gauge’ (85)) the trivial motion with ‘no traversal’ is the unique possible trajectory and the Cauchy initial value problem is classically well defined.

**ii) Ultrarelativistic velocity I:** $\gamma_i \to \infty$ ; $\gamma_k \sim \text{finite}$ , $k \neq i$

In this case the analysis made in the previous paragraph goes through almost step by step, from the explicit formulas for the times (Eqs. (62) and (71) for the ‘mirror exchange’ case, and Eqs. (87) for the ‘velocity exchange’ case) and the trajectories (Eqs. (72) for ‘mirror exchange’ and Eqs. (86) for ‘velocity exchange’), to the set of constraints (Eqs. (88), (89)-(91) and (88)-(91) for ‘mirror exchange’, Eqs. (88) and (90)-(91) for ‘velocity exchange’), albeit the formal substitution of $\vec{v}_i$ by $c\vec{k}_i$ ($|\vec{k}_i| = 1$).

The only difference is that now, for the case of ‘mirror exchange’, condition (64) is being replaced by

$$\frac{(y_A + y_B)}{\sqrt{c^2(n\tau - \rho)^2 - (x_A - x_B)^2}} \simeq \left| \frac{\epsilon_1}{\epsilon_2} \right| \text{sign}[(v_i)_y].$$

(92)

Also the results about uniqueness of the (globally self-consistent) stationary point at $n$ fixed (for each of the possible ‘topologies’ of the trajectories), and about multiplicity (generically finite, infinite when the constraints (78) and (83) hold) are still valid in the ultrarelativistic case (I). Generally, the Cauchy initial value problem is classically ill-posed.

**iii) Ultrarelativistic velocity II:** $\gamma_k \to \infty$ , $k = i, \ldots n + 1$

Finally, in the case in which all the velocities are ultrarelativistic, only the trajectories with ‘no traversal’ and ‘velocity exchange’ (with one wormhole traversal) are possible in general (see the discussion in Section 3.3.1). In particular, for those initial Cauchy data not obeying any of conditions (88)-(89), only the trivial ‘no traversal’ motion is possible, with multiplicity one, and in this case the Cauchy initial value problem is classically well-posed. Otherwise, the multiplicity of solutions is two and the Cauchy initial value problem is classically ill-posed.
b) $\vec{v}_i$ pointing towards mouth B

i) Generic relativistic velocity: $\gamma_i \sim \text{finite}$

- 'No collision'

  The only possible motion when the particle is initially heading towards wormhole mouth B is that for which the particle enters the wormhole but experiences 'no collision'\(^\dagger\). In this case, the trajectories which are solutions of the equations of motion (9) or (33) can be explicitly parametrized in the following way

$$\vec{r}_1(t) = \vec{v}_i t + \vec{b}_1 ; \quad t_i < t < \bar{t}_1 + \tau,$$

$$\vec{r}_{n+1}(t) = \vec{v}_{n+1} t + \vec{b}_{n+1} ; \quad t > \bar{t}_n,$$

where all the velocities are constant and we can write, due to the conservation Eq. (17),

$$\vec{v}_{n+1} \equiv |\vec{v}_{n+1}| \vec{k}_{n+1} = |\vec{v}_i| \vec{k}_{n+1} ; \quad |\vec{k}_{n+1}| = 1.$$  

(93)

Imposing in Eqs. (93) the initial Cauchy data (2), the boundary data (3) and also using Eq. (22) relating $\bar{t}_n$ to $\bar{t}_1$, we can easily find the explicit form of the trajectories as

$$\vec{r}_1(t) = \vec{r}_i + |\vec{v}_i| (\vec{r}_B - \vec{r}_i)(t - t_i)/|\vec{r}_B - \vec{r}_i|,$$

$$\vec{r}_{n+1}(t) = \vec{r}_A + \vec{k}_{n+1} (|\vec{v}_i|(t - t_i + n\tau) + (1 - n)|\vec{r}_B - \vec{r}_A| - |\vec{r}_B - \vec{r}_i|),$$

and, for the time of first entrance into wormhole mouth B

$$\bar{t}_1 = t_i - \tau + \frac{|\vec{r}_B - \vec{r}_i|}{|\vec{v}_i|}.$$  

(96)

The condition for the existence of the 'time machine' (after a fixed - but otherwise arbitrary - number $n$ of wormhole traversals), holds provided that the initial Cauchy data satisfy (see Eq. (23) with $|\vec{v}'_1| = |\vec{v}_i|$)

$$c > |\vec{v}_i| > (n - 1)|\vec{r}_B - \vec{r}_A|/n\tau$$  

(97)

(for $n = 1$ condition (27) is replaced by $|\vec{v}_i| < c$ and the first of constraints (1)).

In conclusion, the Cauchy initial value problem for the motion of a relativistic particle constrained to traverse a wormhole 'time machine' $n$-times ($n$ fixed) and to have an initial velocity pointing towards wormhole mouth B has a unique, globally self-consistent solution provided that the initial Cauchy data are chosen to satisfy condition (27). Moreover, if the initial data are also chosen to satisfy the stronger constraint

$$|\vec{v}_i| > |\vec{r}_B - \vec{r}_A|/\tau,$$  

(98)

\(^\dagger\)Regarding the degeneracy of this case related to our approximations see the end of Section 4.1.b.
then the constraint (97) is easily seen to be satisfied for arbitrary \( n \), thus implying that the multiplicity of the trajectories becomes infinite (in this case the kinematical condition \( |\vec{v}_i| < c \) does not give any further constraint if the first of conditions (1) holds). Otherwise, following a reasoning similar to that of the previous paragraph (i.e., the case of initial Cauchy data with velocity pointing away from mouths A and B and having a ‘mirror exchange’ collision), it is possible to show that the multiplicity is always finite (at least one if \( |\vec{v}_i| < c \) and \( \tau > |\vec{r}_B - \vec{r}_A|/c \)). Again, the condition (98) for infinite multiplicity of classical trajectories is the same as that found by the authors of Ref. [10], although, contrarily to their claims, we have shown here that infinite multiplicity is not generic. Moreover, fixing the initial data as in Ref. [14] would now imply that the impact parameter is zero, leading formally to the same trajectories (95), but with \( \vec{t}_1 \) undetermined. In the ansatz of Ref. [14] it is easy to see that even just one wormhole traversal, i.e. when the first of conditions (1) for the existence of the ‘time machine’ and the constraint \( |\vec{v}_i| < c \) are satisfied, would be enough to guarantee infinite multiplicity of solutions. Generally, the Cauchy initial value problem is classically ill-posed with both kinds of initial data. Only when condition (97) is not satisfied (but still \( |\vec{v}_i| < c \) and \( \tau > |\vec{r}_B - \vec{r}_A|/c \)) there is a unique possible trajectory (with one wormhole traversal) and the Cauchy initial value problem is classically well defined.

ii) Ultrarelativistic velocity I/II: \( \gamma_i \to \infty \); \( \gamma_k \sim \text{finite} \), \( k \neq i \); \( \gamma_k \to \infty \), \( k = i, \ldots, n+1 \)

When the initial Cauchy velocity becomes ultrarelativistic (and similarly when the velocities of all copies of the particle, \( \vec{v}_k, k = i, \ldots, n+1 \), are ultrarelativistic), the previous results remain formally valid (with, of course, \( \vec{v}_k \) duly replaced by \( c\vec{v}_k, |\vec{v}_k| = 1 \), for \( k = i, \ldots, n+1 \)), but, since condition (98) becomes equivalent to the condition of existence of the ‘time machine’ (first of Eqs. (1)), the multiplicity of the trajectories is always infinite. For all choices of initial data (with \( |\vec{v}_i| < c \) and \( \tau > |\vec{r}_B - \vec{r}_A|/c \)), the Cauchy initial value problem is classically ill-posed.

c) \( \vec{v}_i \) pointing towards mouth A

i) Generic relativistic velocity: \( \gamma_i \sim \text{finite} \)

- ‘Collinear velocities’

One of the two possible motions in the case of a relativistic particle which is initially heading towards wormhole mouth A is that for which the particle self-collides in Region II under a ‘collinear exchange’ of velocities, and traverses the wormhole an arbitrary number \( n \) of times. [*]

*Regarding the degeneracy of this case because of our approximations see the end of Section 4.1.c.
Using the ‘collinear exchange’ rules (15) and introducing the following notation for the velocities of the copies of the particle in Region II
\[
\vec{v}_i \equiv |\vec{v}_i| \vec{k}_i, \quad \vec{v}_{n+1} \equiv -|\vec{v}_i| \vec{k}_i, \quad \vec{v}_{n+1}' \equiv -|\vec{v}_i| \vec{k}_i', \quad \vec{v}_{n+1}'' \equiv -|\vec{v}_i| \vec{k}_i''
\]
(99)
(where the \( \vec{k}_k'(0), k = i, 1 \) are unit vectors), we can write for the trajectories
\[
\vec{r}_1(t) = |\vec{v}_i| \vec{k}_i t + \vec{b}_1,
\]
\[
\vec{r}_{n+1}'(t) = |\vec{v}_i| \vec{k}_i' t + \vec{b}_1',
\]
\[
\vec{r}_{n+1}(t) = -|\vec{v}_i| \vec{k}_i t + \vec{b}_{n+1},
\]
\[
\vec{r}_{n+1}'(t) = -|\vec{v}_i| \vec{k}_i' t + \vec{b}_{n+1}'.
\]
(100)
In particular, working in the coordinate frame where
\[
\vec{r}_0 = \vec{0},
\]
\[
y_A = y_i = 0,
\]
(101)
for which we have
\[
\vec{k}_i = (-[\text{sign}(x_i)], 0),
\]
\[
\vec{k}_i' = (x_B/r_B, y_B/r_B) ; \quad r_B \equiv |\vec{r}_B|,
\]
(102)
one can impose the Cauchy and boundary conditions, Eqs. (2) and (3), on Eqs. (100) and finally find for the trajectories
\[
x_1(t) = -x_{n+1}(t) = x_i - [\text{sign}(x_i)] |\vec{v}_i|(t - t_i),
\]
\[
y_1(t) = y_{n+1}(t) = 0,
\]
\[
x_{1}'(t) = -x_{n+1}'(t) = -[\text{sign}(x_i)] x_B x_1(t)/r_B,
\]
\[
y_{1}'(t) = -y_{n+1}'(t) = -[\text{sign}(x_i)] y_B x_1(t)/r_B,
\]
(103)
and for the times \( t_0 \) and \( \bar{t}_1 \)
\[
t_0 = t_i + |x_i|/|\vec{v}_i|,
\]
\[
\bar{t}_1 = t_i - \tau + |r_B + |x_i||/|\vec{v}_i|.
\]
(104)
It is very easy to check that the solutions (104) automatically satisfy the time ordering constraints of Eqs. (29) (and consequently condition (23) for the existence of the ‘time machine’).

Moreover, since, as in the case of ‘mirror exchange’ or ‘velocity exchange’ collisions, the Cauchy and boundary conditions turn out to be more than the parameters in the trajectories, these are constrained by the following condition
\[
|\vec{v}_i| = [x_A] + r_B + (n - 1)|\vec{r}_B - \vec{r}_A|/n\tau.
\]
(105)
The discussion about the existence and multiplicity of stationary points for the action in the case of fixed Cauchy data for a particle initially moving towards the wormhole mouth A, self-interacting via ‘collinear exchange’ of velocities in Region II and traversing the wormhole n times is essentially similar to that done in the case of a ‘mirror exchange’ type of collision. In particular, for each fixed n, the Cauchy initial value problem has a unique, globally self-consistent solution, given by Eqs. (103) and (104), provided that the initial Cauchy data are chosen to satisfy the constraint (105) (and, of course, the kinematic condition $|\vec{v}_i| \leq c$). Again the initial velocity $|\vec{v}_i|$ is a monotonic decreasing function of n in Eq. (105), with extrema

$$\min[c, \left(\|x_A\| + r_B\right)/\tau] > |\vec{v}_i| > \left|\vec{r}_B - \vec{r}_A\right|/\tau.$$  \hspace{1cm} (106)

When the initial Cauchy data are fixed such that constraint (106) is satisfied, then Eq. (105) is solvable and it can be interpreted, for arbitrary and varying n, as fixing one of the coordinates of the collision point relatively to the fixed wormhole mouths.

For example, if

$$\tau \geq \frac{r_B + |x_A|}{c},$$  \hspace{1cm} (107)

the kinematical condition $|\vec{v}_i| < c$ is automatically satisfied independently of the number n of wormhole traversals. However, in the limit that $n \to \infty$, one sees from Eq. (105) that $|x_A|$ and, consequently, $|x_i - x_A|$ (for $x_i$ fixed in the frame (101)), become infinitely large. This implies that, using arguments similar to those given for case a) (‘mirror exchange’ or ‘velocity exchange’ collisions), if the constraints (106) and (107) are satisfied, the multiplicity of the trajectory for ‘collinear exchange’ motion is in general finite, unless the particle starts its motion in a region infinitely far from the wormhole, in which case the multiplicity becomes infinite. Finally, the multiplicity is zero if condition (106) is not satisfied. The lower bound on the initial velocity coming from condition (106) is the same as conditions (81) and (98), which lead to infinitely many solutions in cases a) and b). Again, the kinematical condition $|\vec{v}_i| < c$ limits the set of initial data whose trajectories have infinite multiplicity.

• ‘No collision’

The other possible motion when $\vec{v}_i$ is pointing towards mouth A is that for which the particle directly enters mouth A and finally exits from mouth B (after n wormhole traversals), travelling forward in time and without experiencing any self-collision. The analysis of the kinematics for this motion is similar to that done for the ‘no collision’ case b).

In particular, using the following relation between the time of first entrance into

---

2 The case of ‘collinear exchange’ of velocities was not considered in Ref. [10]. Assuming their ansatz for the initial data (now it is enough to give $\vec{v}_i$, while the information about $h$ is redundant), one can show that t₀ is undetermined (function of the arbitrary $x_i$) and that multiplicity is infinite provided that condition (106) holds.
wormhole mouth A \( \bar{t}_n \) and the time of last exit from mouth B \( \bar{t}_1 + \tau \)

\[ \bar{t}_1 = \bar{t}_n + (n - 1) \left[ \frac{\vec{r}_B - \vec{r}_A}{|\vec{v}_i|} + \tau \right], \]  

(108)

and imposing the boundary conditions (see also the footnote on pag. 6)

\[ \vec{r}_1(\bar{t}_n) = \vec{r}_A, \]
\[ \vec{r}_{n+1}(\bar{t}_1 + \tau) = \vec{r}_B \]

(109)

and the usual Cauchy initial data \( (2) \), one can easily find that the trajectories are given again by the ‘no collision’ formulas \( (95) \), provided one interchanges \( \vec{r}_B \) with \( \vec{r}_A \) everywhere. Moreover, one also finds for the time \( \bar{t}_1 \)

\[ \bar{t}_1 = t_i + \frac{|\vec{r}_A - \vec{r}_i|}{|\vec{v}_i|} + (n - 1) \left[ \frac{|\vec{r}_B - \vec{r}_A|}{|\vec{v}_i|} + \tau \right], \]  

(110)

for which, clearly, the second of conditions \( (1) \) for the existence of the time machine is trivially satisfied.

Therefore, for each fixed number \( n \) of wormhole traversals, the Cauchy initial value problem for a particle initially heading towards mouth A and exiting from mouth B without having self-collision has a unique and globally self-consistent solution. Since there are no constraints restricting the initial Cauchy data (apart from the requirement \( |\vec{v}_i| < c \)), the multiplicity of this set of trajectories is always infinite.

The choice of initial Cauchy data made as in Ref. [10] would again formally lead to the same Eqs. \((95)\) for the trajectories, but with \( \bar{t}_1 \) undetermined, and to infinite multiplicity of solutions even for one single wormhole traversal.

In conclusion, the Cauchy initial value problem for the case of a particle initially heading towards mouth A is classically ill-posed (the multiplicity of each classical trajectory is always infinite).

**ii) Ultrarelativistic velocity I/II:** \( \gamma_i \to \infty \); \( \gamma_k \sim finite \), \( k \neq i \); \( \gamma_k \to \infty \), \( k = i, ... n+1 \)

Also in this case (with equations of the previous paragraph still formally valid, modulo the substitution \( \vec{v}_k \to c \vec{k}_k, |\vec{k}_k| = 1 \), for \( k = 1, ... n+1 \) the multiplicity of trajectories is always infinite and the Cauchy initial value problem is ill-posed.
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