Multiplication is an open bilinear mapping in the Banach algebra of functions of bounded Wiener $p$-variation

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Abstract

Let $BV_p[0,1]$, $1 ≤ p < ∞$, be the Banach algebra of functions of bounded $p$-variation in the sense of Wiener. Recently, Kowalczyk and Turowska [8] proved that the multiplication in $BV_1[0,1]$ is an open bilinear mapping. We extend this result for all values of $p ∈ [1,∞)$.

Keywords: Multiplication in a Banach algebra, open bilinear mapping, Banach algebra of functions of bounded Wiener $p$-variation.

1. Introduction

Let $\mathbb{A}$ be a Banach algebra with a Banach algebra norm $\| \cdot \|_\mathbb{A}$. We denote by $B_\mathbb{A}(a,\varepsilon)$ the open ball in $\mathbb{A}$ centered at $a$ of radius $\varepsilon > 0$, that is,

$$B_\mathbb{A}(a,\varepsilon) := \{ b \in \mathbb{A} : \|a - b\|_\mathbb{A} < \varepsilon \}.$$

We say that the multiplication in $\mathbb{A}$ is a bilinear mapping locally open at a pair $(a, b) ∈ \mathbb{A}^2 := \mathbb{A} × \mathbb{A}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$B_\mathbb{A}(a \cdot b, \delta) \subset B_\mathbb{A}(a, \varepsilon) \cdot B_\mathbb{A}(b, \varepsilon),$$

where

$$B_\mathbb{A}(a,\varepsilon) \cdot B_\mathbb{A}(b,\varepsilon) := \{ c \cdot d ∈ \mathbb{A} : c ∈ B_\mathbb{A}(a,\varepsilon), d ∈ B_\mathbb{A}(b,\varepsilon) \}.$$

Following [8], the multiplications in $\mathbb{A}$ is called an open bilinear mapping if it is locally open at every pair $(a, b) ∈ \mathbb{A}^2$.

Note that the multiplication might not be an open bilinear mapping even in very simple situations. For instance, if $\mathbb{A} = C[0,1]$ is the algebra of real continuous functions with the supremum norm

$$\|f\|_\infty := \sup_{x ∈ [0,1]} |f(x)|,$$  \hspace{1cm} (1.1)

then for the function $g = x - 1/2$ one has $g^2 ∈ (B_\mathbb{A}(g,1/2))^2 \setminus \text{int}((B_\mathbb{A}(g,1/2))^2)$, where int$(S)$ denotes the interior of a set $S$ (see [8]). Thus, the multiplication is not an open bilinear mapping in the algebra $C[0,1]$.

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This result was extended in [10] to the case of the algebra $C^n[0,1]$ of $n$ times continuously differentiable functions.

The aim of this paper is to show that the multiplication is an open bilinear mapping in the Banach algebra $BV_p[0,1]$, $1 \leq p < \infty$, of functions of bounded Wiener $p$-variation, extending the recent result by Kowlačzyk and Turowska [8] for $p = 1$ to all values $p \in [1,\infty)$.

Let us recall the definition of functions of bounded Wiener $p$-variation. Suppose that $0 \leq \alpha \leq \beta \leq 1$. Let $\mathcal{P}[\alpha,\beta]$ be the set of all partitions $P = \{t_0,\ldots,t_m\}$ of the segment $[\alpha,\beta]$ of the form

$$\alpha = t_0 < t_1 < \cdots < t_m = \beta.$$ 

Following [12] and [2, Definition 1.31], for a given a real number $p \in [1,\infty)$, a partition $P = \{t_0,\ldots,t_m\} \in \mathcal{P}[\alpha,\beta]$ and a function $f : [\alpha,\beta] \to \mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, the nonnegative number

$$\text{Var}_p(f,P,[\alpha,\beta]) := \sum_{j=1}^m |f(t_j) - f(t_{j-1})|^p$$

is called the Wiener $p$-variation of $f$ on $[\alpha,\beta]$ with respect to $P$, while the (possibly infinite) number

$$\text{Var}_p(f,[\alpha,\beta]) := \sup\{\text{Var}_p(f,P,[\alpha,\beta]) : P \in \mathcal{P}[\alpha,\beta]\},$$

where the supremum is taken over all partitions of $[\alpha,\beta]$, is called the total Wiener $p$-variation of $f$ on $[\alpha,\beta]$. Let

$$BV_p[0,1] := \{f : [0,1] \to \mathbb{F} \in \{\mathbb{R},\mathbb{C}\} : \text{Var}_p(f,[0,1]) < \infty\}$$

be the set of all functions of bounded Wiener $p$-variation. It is well known that $BV_p[0,1]$ is a Banach algebra with respect to the pointwise multiplication and the norm

$$\|f\|_{BV_p} := \|f\|_{\infty} + (\text{Var}_p(f,[0,1]))^{1/p},$$

(1.2)

where $\|f\|_{\infty}$ is given by (1.1) (for instance, this result follows from [2, Theorem 3.7 and Corollary 3.8] with $\Phi(t) = t^p$, $1 \leq p < \infty$).

**Theorem 1.1 (Main result).** Let $1 \leq p < \infty$. Then the multiplication in the Banach algebra $BV_p[0,1]$ is an open bilinear mapping.

The paper is organized as follows. In Section 2 following the main lines of the proof of [8, Theorem 2.4], we show that the multiplication in a Banach algebra continuously embedded into the Banach algebra $B[0,1]$ of bounded functions and satisfying natural assumptions (the so-called symmetry property, the inverse closedness property and the selection principle) is locally open at every pair of functions $(F,G)$ such that $|F| + |G|$ is bounded away from zero. We call such functions $F$ and $G$ jointly nondegenerate. Further, we show that the Banach algebra $BV_p[0,1]$ of functions of bounded $p$-variation in the Wiener sense and the Banach algebra $A_pBV[0,1]$ of functions of bounded variation in the Shiba-Waterman sense (see [3, 4, 11]) satisfy the hypotheses of the above result. In Section 3 we extend [8, Lemma 2.1] from the setting of $BV_1[0,1]$ to the setting of $BV_p[0,1]$ with an arbitrary $p \geq 1$. We should note that the passage from $p = 1$ to an arbitrary $p \geq 1$ is not trivial. In Section 4 with the aid of the main result of Section 3 and following the scheme of the proof of [8, Theorem 2.2], we show that an arbitrary pair of functions $(F,G) \in (BV_p[0,1])^2$ can be approximated by a pair of jointly nondegenerate functions $(F_1,G_1) \in (BV_p[0,1])^2$ such that $F \cdot G = F_1 \cdot G_1$. In Section 5 we prove Theorem 1.1 combining the results of Sections 2 and 4. We conclude the paper with the conjecture that multiplication is an open bilinear mapping also in the Banach algebra $A_pBV[0,1]$ of functions of bounded variation in the sense of Shiba-Waterman.

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2. Local openness of multiplication in algebras of bounded functions

Let \( B[0, 1] \) denote the Banach algebra of all bounded functions \( f : [0, 1] \to F \), where \( F \in \{ \mathbb{R}, \mathbb{C} \} \), with the norm given by \( \| \cdot \| \). We say that functions \( f, g \in B[0, 1] \) are jointly nondegenerate if

\[
\inf_{x \in [0, 1]} (|f(x)| + |g(x)|) > 0.
\]

Let \( \mathcal{F}[0, 1] \) be a Banach algebra equipped with a norm \( \| \cdot \|_\mathcal{F} \) and continuously embedded into the algebra \( B[0, 1] \). We will say that the algebra \( \mathcal{F}[0, 1] \) satisfies the symmetry property if for every function \( f \in \mathcal{F}[0, 1] \), its complex conjugate \( \overline{f} \) also belongs to \( \mathcal{F}[0, 1] \) and \( \| \overline{f} \|_\mathcal{F} = \| f \|_\mathcal{F} \). It is clear that every real algebra \( \mathcal{F}[0, 1] \) has the symmetry property.

Further, we will say that \( \mathcal{F}[0, 1] \) satisfies the inverse closedness property if for every sequence of functions \( \{ f_n \} \) satisfying

\[
\inf_{x \in [0, 1]} |f(x)| > 0
\]

implies that \( 1/f \in \mathcal{F}[0, 1] \) and

\[
\left\| \frac{1}{f} \right\|_\mathcal{F} \leq \left( \inf_{x \in [0, 1]} |f(x)| \right)^{-2} \| f \|_\mathcal{F}.
\]

Finally, we will say that \( \mathcal{F}[0, 1] \) satisfies the selection principle if from every sequence of functions \( \{ f_n \} \) satisfying

\[
\sup_{n \in \mathbb{N}} \| f_n \|_\mathcal{F} < \infty
\]

one can extract a subsequence \( \{ f_{n_k} \} \) that converges pointwise on \([0, 1]\) to a function \( f \in \mathcal{F}[0, 1] \).

**Theorem 2.1.** Let \( \mathcal{F}[0, 1] \) be a Banach algebra continuously embedded into the Banach algebra \( B[0, 1] \). Suppose that the algebra \( \mathcal{F}[0, 1] \) satisfies the symmetry property, the inverse closedness property and the selection principle. Then the multiplication in \( \mathcal{F}[0, 1] \) is locally open at every pair of jointly nondegenerate functions \((F, G) \in (\mathcal{F}[0, 1])^2\).

**Proof.** The proof is analogous to that of \( \cite{3} \) Theorem 2.4. Since \( \mathcal{F}[0, 1] \) is continuously embedded into \( B[0, 1] \), there is a constant \( C \geq 1 \) such that for all \( f \in \mathcal{F}[0, 1] \),

\[
\sup_{x \in [0, 1]} |f(x)| \leq C \| f \|_\mathcal{F}.
\]

(2.1)

Without loss of generality, we can suppose that \( \varepsilon \in (0, 1) \). Take

\[
\delta := \min \left\{ \frac{1}{2} \frac{1}{2} \inf_{x \in [0, 1]} |F(x)| + |G(x)| \right\}
\]

and

\[
K := 2 \max \left\{ \| F \|_\mathcal{F}, \| G \|_\mathcal{F}, 1 \right\}.
\]

(2.2)

(2.3)

Let \( h \in \mathcal{F}[0, 1] \) be such that

\[
\| h \|_\mathcal{F} < \varepsilon, \quad \frac{\delta^8}{128CK^6}.
\]

(2.4)

Consider

\[
F_0 := F, \quad G_0 := G, \quad h_0 := h
\]

and define sequences \( \{ F_n \}_{n=0}^\infty, \{ G_n \}_{n=0}^\infty, \) and \( \{ h_n \}_{n=0}^\infty \) inductively by

\[
F_{n+1} := F_n + h_n \cdot \overline{G_n} = \frac{F_n + h_n}{|F_n|^2 + |G_n|^2}, \quad G_{n+1} := G_n + h_n \cdot \overline{F_n} = \frac{G_n + h_n}{|F_n|^2 + |G_n|^2}, \quad h_{n+1} := -h_n^2 \cdot \frac{F_nG_n}{(|F_n|^2 + |G_n|^2)^2}.
\]

(2.5)

(2.6)

(2.7)

(2.8)
We claim that for \( n \in \mathbb{N} \cup \{0\} \),

(i) \[ F_n G_n + h_n = FG + h, \]

(ii) \[ \|F_n\|_F \leq \frac{K}{2} + 1 - 2^{-n}, \quad \|G_n\|_F \leq \frac{K}{2} + 1 - 2^{-n}, \]

(iii) \[ \inf_{x \in [0,1]} (|F_n(x)| + |G_n(x)|) \geq \delta + \delta \cdot 2^{-n}, \]

(iv) \[ \|h_n\|_F \leq \varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6}. \]

We will prove these claims by induction. It follows from (2.6) that

\[ F_0 G_0 + h_0 = FG + h. \]

We obtain from (2.8), (2.9) that

\[ \|F_0\|_F = \|F\|_F \leq \frac{K}{2}, \quad \|G_0\|_F = \|G\|_F \leq \frac{K}{2}, \quad \|h_0\|_F = \|h\|_F \leq \varepsilon \cdot \frac{\delta^8}{128CK^6}. \]

That is, (i)–(iv) are satisfied for \( n = 0 \).

Now we assume that (i)–(iv) are fulfilled for some \( n \in \mathbb{N} \cup \{0\} \). Then, taking into account (2.8), we see that \( K/2 \geq 1 \) and

\[ F_n G_n + h_n = FG + h, \quad \|F_n\|_F \leq \frac{K}{2} + 1 - 2^{-n} < K, \]

\[ \|G_n\|_F \leq \frac{K}{2} + 1 - 2^{-n} < K, \]

\[ \inf_{x \in [0,1]} (|F_n(x)| + |G_n(x)|) \geq \delta + \delta \cdot 2^{-n} > \delta, \]

\[ \|h_n\|_F \leq \varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6}. \]

Let us show that (i)–(iv) are fulfilled for \( n + 1 \).

(i) It follows from (2.8), (2.9) that

\[ F_{n+1} G_{n+1} + h_{n+1} = \left( F_n + \frac{h_n \cdot \overline{G_n}}{|F_n|^2 + |G_n|^2} \right) \left( G_n + \frac{h_n \cdot \overline{F_n}}{|F_n|^2 + |G_n|^2} \right) - \frac{h_n^2 \cdot \overline{F_n} \overline{G_n}}{(|F_n|^2 + |G_n|^2)^2} \]

\[ = F_n G_n + h_n \frac{F_n F_n + G_n G_n}{|F_n|^2 + |G_n|^2} + h_n^2 \frac{F_n G_n}{(|F_n|^2 + |G_n|^2)^2} - h_n^2 \frac{F_n G_n}{(|F_n|^2 + |G_n|^2)^2} \]

\[ = F_n G_n + h_n = FG + h. \]

Hence, (i) is satisfied for \( n + 1 \).

(ii) Since \( F[0,1] \) is a Banach algebra satisfying the symmetry property, we obtain from (2.10) and (2.11) that

\[ \|F_n\|^2 + |G_n|^2 \leq \|F_n \cdot \overline{F_n}\|_F + \|G_n \cdot \overline{G_n}\|_F \leq \|F_n\|_F \|F_n\|_F + \|G_n\|_F \|G_n\|_F \]

\[ = \|F_n\|^2 + \|G_n\|^2 \leq K^2 + K^2 = 2K^2. \]
It follows from (2.12) that for every $x \in [0, 1]$,
\[
\delta^2 \leq \left( |F_n(x)| + |G_n(x)| \right)^2 = |F_n(x)|^2 + 2|F_n(x)| \cdot |G_n(x)| + |G_n(x)|^2 \leq 2 \left( |F_n(x)|^2 + |G_n(x)|^2 \right).
\]

Hence
\[
\inf_{x \in [0, 1]} \left( |F_n(x)|^2 + |G_n(x)|^2 \right) \geq \frac{\delta^2}{2}.
\] (2.15)

Taking into account that $F[0, 1]$ is a Banach algebra with the symmetry property, it follows from (2.6) and (2.10)–(2.11) that
\[
\|F_{n+1}\| \leq \|F_n\| \|x\| \|G_n\| \leq \left( \frac{K}{2} + 1 - 2^{-n} \right) + \|h_n\| \|x\| K \|F_n|^2 + |G_n|^2 \| \leq \left( \frac{K}{2} + 1 - 2^{-n} \right) + \|h_n\| \|x\| K \left( \frac{1}{|F_n|^2 + |G_n|^2} \right). (2.16)
\]

Since $F[0, 1]$ has the inverse closedness property, we deduce from (2.14)–(2.15) that
\[
\left\| \frac{1}{|F_n|^2 + |G_n|^2} \right\| \leq \left( \inf_{x \in [0, 1]} \left( |F_n(x)|^2 + |G_n(x)|^2 \right) \right)^{-2} \|F_n|^2 + |G_n|^2 \| \leq \left( \frac{2}{\delta^2} \right)^2 2K^2 = \frac{8K^2}{\delta^4}. (2.17)
\]

Combining (2.16)–(2.17) and taking into account that $\varepsilon \in (0, 1)$ and $C \geq 1$, we obtain
\[
\|F_{n+1}\| \leq \frac{K}{2} + 1 - 2^{-n} + \frac{8K^3}{\delta^4} \cdot \varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^n} < \frac{K}{2} + 1 - 2^{-n} + 2^{-n} \cdot \frac{\delta^4}{16} (2.18)
\]

It follow from (2.2)–(2.3) that $\delta \leq 1 \leq K/2$. Therefore
\[
\frac{\delta^4}{16K^3} = \frac{\delta}{16} \left( \frac{\delta}{K} \right)^3 \leq \frac{\delta}{16} \cdot \frac{1}{8} = \frac{\delta}{128} < \frac{1}{2} \] (2.19)

In view of (2.18)–(2.19) we obtain
\[
\|F_{n+1}\| \leq \frac{K}{2} + 1 - 2^{-n} + 2^{-n-1} = \frac{K}{2} + 1 - 2^{-n-1}.
\]

Analogously it can be shown that
\[
\|G_{n+1}\| \leq \frac{K}{2} + 1 - 2^{-n-1}.
\]

Thus, (ii) is fulfilled for $n + 1$.

(iii) Since $F[0, 1]$ is a Banach algebra and $\varepsilon \in (0, 1)$, it follows from (2.6), (2.1), (2.11), (2.13), (2.17), and (2.19) that for $x \in [0, 1]$,
\[
|F_n(x)| \leq |F_{n+1}(x)| + |h_n(x)| \frac{|G_n(x)|}{|F_n(x)|^2 + |G_n(x)|^2} \leq |F_{n+1}(x)| + \|h_n\| \|x\| |G_n| \| \left( \frac{1}{|F_n|^2 + |G_n|^2} \right). (2.20)
\]

Hence
\[
|F_{n+1}(x)| > |F_n(x)| - 2^{-n-2}\delta, \quad x \in [0, 1].
\]

Analogously,
\[
|G_{n+1}(x)| > |F_n(x)| - 2^{-n-2}\delta, \quad x \in [0, 1].
\] (2.21)
We conclude from (2.2) and (2.20)–(2.21) that
\[
\inf_{x \in [0,1]} (|F_{n+1}(x)| + |G_{n+1}(x)|) \geq \inf_{x \in [0,1]} (|F_n(x)| + |G_n(x)|) - 2 \cdot 2^{-n-2}\delta \\
\geq \delta + \delta \cdot 2^{-n} - \delta \cdot 2^{-n-1} = \delta + \delta \cdot 2^{-n-1}.
\]

Hence (iii) is fulfilled for \( n \).

(iv) Since \( F[0,1] \) is a Banach algebra with the symmetry property, \( \varepsilon \in (0,1) \) and \( C \geq 1 \), it follows from (2.8), (2.10)–(2.11), (2.13) and (2.17) that
\[
\|h_{n+1}\|_F \leq \|h_n\|^2_2 \|F_n\|_F \|G_n\|_F \left(\frac{1}{\|F_n\|^2 + |G_n|^2}\right)^2 = \|h_n\|^2_2 \|F_n\|_F \|G_n\|_F \left(\frac{1}{\|F_n\|^2 + |G_n|^2}\right)^2 \\
\leq \left(\varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6}\right)^2 K^2 \left(\frac{8K^2}{\delta^4}\right)^2 = \varepsilon^2 \cdot 2^{-2n-1} \cdot \frac{\delta^8}{128CK^6} < \varepsilon \cdot 2^{-n-1} \cdot \frac{\delta^8}{128CK^6}.
\]

Hence (iv) is fulfilled for \( n \).

Thus, we have verified properties (i)–(iv) by induction for all \( n \in \mathbb{N} \cup \{0\} \).

In view of (ii), the terms of the sequences \( \{F_n\}_{n=0}^\infty \) and \( \{G_n\}_{n=0}^\infty \) have uniformly bounded norms. By the selection principle, there exist a subsequence \( \{F_{n_k}\}_{k=0}^\infty \) of \( \{F_n\}_{n=0}^\infty \) and a subsequence \( \{G_{n_k}\}_{k=0}^\infty \) of \( \{G_n\}_{n=0}^\infty \) such that for every \( x \in [0,1] \),
\[
\lim_{k \to \infty} F_{n_k}(x) = f(x), \quad \lim_{k \to \infty} G_{n_k}(x) = g(x),
\]
where \( f, g \in F[0,1] \). It follows from (2.1) and (iv) that for all \( x \in [0,1] \),
\[
\lim_{n \to \infty} |h_n(x)| \leq C \lim_{n \to \infty} \|h_n\|_F \leq \frac{\delta \delta^8}{128CK^6} \lim_{n \to \infty} 2^{-n} = 0.
\]

In view of (i) and (2.22)–(2.23), we obtain for \( x \in [0,1] \),
\[
f(x)g(x) = \lim_{k \to \infty} F_{n_k}(x)G_{n_k}(x) = \lim_{k \to \infty} (F_{n_k}(x)G_{n_k}(x) + h_{n_k}(x)) = F(x)G(x) + h(x).
\]

Since
\[
f(x) - F(x) = \lim_{k \to \infty} (F_{n_k}(x) - F(x)) = \lim_{k \to \infty} \sum_{j=0}^{n_k} (F_{j+1}(x) - F_j(x)) = \sum_{n=0}^\infty (F_{n+1}(x) - F_n(x)),
\]
\( F[0,1] \) is a Banach algebra with the symmetry property, \( \varepsilon \in (0,1) \) and \( C \geq 1 \), we obtain from (2.6), (2.11), (2.13), (2.17), and (2.19) that
\[
\|f - F\|_F \leq \sum_{n=0}^\infty \|F_{n+1} - F_n\|_F \leq \sum_{n=0}^\infty \|h_n\|_F \|G_n\|_F \left(\frac{1}{\|F_n\|^2 + |G_n|^2}\right)^2 \leq \sum_{n=0}^\infty \varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6} \cdot K \cdot \frac{8K^2}{\delta^4} = \frac{\varepsilon \delta^4}{10CK^3} \sum_{n=0}^\infty 2^{-n} < \varepsilon.
\]

Analogously we can show that
\[
\|g - G\|_F < \varepsilon.
\]

So, for every \( h \in F[0,1] \) satisfying (2.24), there exist \( f \) and \( g \) in \( F[0,1] \) such that (2.22) and (2.20) hold, and \( FG + h = fg \) (see (2.24)). This means that
\[
B_{F[0,1]}(F \cdot G, \eta) \subset B_{F[0,1]}(F, \varepsilon) \cdot B_{F[0,1]}(G, \varepsilon)
\]
with \( \eta := \varepsilon \cdot \frac{\delta^8}{128CK^6} \). Hence, the multiplication in the Banach algebra \( F[0,1] \) is locally open at the pair \((F, G) \in (F[0,1])^2\).
Corollary 2.2. Let $1 \leq p < \infty$. Then the multiplication in $BV_p[0,1]$ is locally open at every pair of jointly nondegenerate functions $(F,G) \in (BV_p[0,1])^2$.

**Proof.** We have to verify the hypotheses of Theorem 2.1. The definitions of the norms (1.2) and (1.1) immediately imply that the Banach algebra $BV_p[0,1]$ is continuously embedded into the Banach algebra $B[0,1]$ (with the embedding constant 1) and that the algebra $BV_p[0,1]$ satisfies the symmetry property. It follows from the Helly-type selection theorem [2, Theorem 2.49] with $\Phi(t) = t^p$, $1 \leq p < \infty$, that $BV_p[0,1]$ satisfies the selection principle.

Let us show that $BV_p[0,1]$ has the inverse closedness property. Take a function $f \in BV_p[0,1]$ such that

$$\inf_{x \in [0,1]} |f(x)| > 0$$

and a partition $P = \{t_0, \ldots, t_m\} \in \mathcal{P}[0,1]$. Then $f(t_j) \neq 0$ for $j \in \{0, \ldots, m\}$ in view of (2.27) and

$$\text{Var}_p(1/f, P, [0,1]) = \sum_{j=1}^m \frac{|f(t_j)|^p}{\text{Var}(f, [0,1])} = \sum_{j=1}^m \left| f(t_j) - f(t_{j-1}) \right|^p f(t_j) f(t_j)$$

$$\leq \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2p} \text{Var}_p(f, P, [0,1]).$$

Therefore

$$\text{Var}_p(1/f, [0,1]) \leq \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2p} \text{Var}_p(f, [0,1]).$$

On the other hand,

$$\|1/f\|_{\infty} = \sup_{x \in [0,1]} |1/f(x)| = \left( \inf_{x \in [0,1]} |f(x)| \right)^{-1}.$$ (2.29)

Combining (2.28) and (2.29), we arrive at the following:

$$\|1/f\|_{BV_p} = \|1/f\|_{\infty} + \left( \text{Var}_p(1/f, [0,1]) \right)^{1/p}$$

$$\leq \left( \inf_{x \in [0,1]} |f(x)| \right)^{-1} + \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2} \left( \text{Var}_p(f, [0,1]) \right)^{1/p}$$

$$\leq \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2} \left( \|f\|_{\infty} + \left( \text{Var}_p(f, [0,1]) \right)^{1/p} \right)$$

$$= \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2} \|f\|_{BV_p}.$$ (2.30)

Thus $BV_p[0,1]$ satisfies the inverse closedness property. It remains to apply Theorem 2.1. □

Let us show that the hypotheses of Theorem 2.1 are also satisfied in the case of Banach algebras of functions of generalized variation in the Shiba-Waterman sense. Shiba [4] introduced the class $\Lambda_pBV[0,1]$ with $1 \leq p < \infty$, extending the concept of the bounded A-variation in the sense of Waterman [11]. Let $\Lambda = \{\lambda_i\}_{i=1}^\infty$ be a nondecreasing sequence of positive numbers such that $\sum_{i=1}^\infty \frac{1}{\lambda_i} = +\infty$ and let $1 \leq p < \infty$. A function $f : [0,1] \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is said to be of bounded $\Lambda_p$-variation in the Shiba-Waterman sense if

$$\text{Vap}_{\Lambda_p}(f, [0,1]) := \sup \sum_{i=1}^n \frac{|f(I_i)|^p}{\lambda_i} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping intervals on $[0,1]$ and $f(I_i) := f(\sup I_i) - f(\inf I_i)$. Let $\Lambda_pBV[0,1]$ be the set of all functions $f : [0,1] \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ of bounded $\Lambda_p$-variation. Kantorowitz [3, Theorem 1] proved that $\Lambda_pBV[0,1]$ is a Banach algebra with respect to the pointwise multiplication and the norm

$$\|f\|_{\Lambda_pBV} := \|f\|_{\infty} + \left( \text{Vap}_{\Lambda_p}(f, [0,1]) \right)^{1/p}.$$ (2.31)
Corollary 2.3. Let $1 \leq p < \infty$. Then the multiplication in $\Lambda_p BV[0,1]$ is locally open at every pair of jointly nondegenerate functions $(F, G) \in (\Lambda_p BV[0,1])^2$.

Proof. As in the proof of the previous corollary, we have to verify the hypotheses of Theorem 2.1. The definitions of the norms (2.31) and (1.1) immediately imply that the Banach algebra $\Lambda_p BV[0,1]$ is continuously embedded into the Banach algebra $B[0,1]$ (with the embedding constant 1) and that the algebra $\Lambda_p BV[0,1]$ satisfies the symmetry property. The selection principle for the algebra $\Lambda_p BV[0,1]$ is proved in [6, Theorem 3.2].

If $f \in \Lambda_p BV[0,1]$ satisfies (2.27), then for every interval $I \subset [0,1]$,

$$|(1/f)(I)| \leq \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2} |f(I)|.$$ 

Therefore

$$\text{Var}_{\Lambda_p}(1/f, [0,1]) \leq \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2p} \text{Vap}_{\Lambda_p}(f, [0,1]).$$ 

(2.32)

Combining (2.32) and (2.29), similarly to (2.30), we obtain

$$\|1/f\|_{\Lambda_p BV} \leq \left( \inf_{x \in [0,1]} |f(x)| \right)^{-2} \|f\|_{\Lambda_p BV}.$$ 

Thus $\Lambda_p BV[0,1]$ satisfies the inverse closedness property. It remains to apply Theorem 2.1. □

3. Key lemma

The aim of this section is to prove an extension of [8, Lemma 2.1] for the Banach algebras $BV_p[0,1]$ with arbitrary $p \in [1, \infty)$.

Let us start with several elementary inequalities.

Lemma 3.1. Let $1 \leq p < \infty$. Then

$$(1 + x)^p \leq 1 + p2^{p-1}x \quad \text{for all} \quad x \in [0,1].$$ 

(3.1)

Proof. Integrating both sides of the inequality

$$(1 + t)^{p-1} \leq 2^{p-1}, \quad t \in [0,1]$$

from 0 to $x$, one gets

$$\frac{1}{p} ((1 + x)^p - 1) \leq 2^{p-1}x,$$

which is equivalent to (3.1). □

Lemma 3.2. Let $1 \leq p < \infty$. Then

$$(a + b)^p \leq a^p + \max\{p, 2\} 2^{p-1}b \quad \text{for all} \quad a, b \in [0,1].$$ 

(3.2)

Proof. If $a = 0$ then (3.2) holds because $b^p \leq b$. Suppose $a > 0$. If $b \leq a$, then it follows from Lemma 3.1 that

$$(a + b)^p = a^p \left( 1 + \frac{b}{a} \right)^p \leq a^p \left( 1 + p2^{p-1} \frac{b}{a} \right) = a^p + p2^{p-1}a^{p-1}b \leq a^p + p2^{p-1}b.$$ 

(3.3)

If $b > a$, then

$$(a + b)^p < (2b)^p = 2^p b^p < a^p + 2^p b^p \leq a^p + 2^p b.$$ 

(3.4)

Estimate (3.2) follows from (3.3) and (3.4). □
Corollary 3.3. Let $1 \leq p < \infty$ and $u, v \in \mathbb{C}$ be such that $|u - v|, |v| \leq 1$. Then
\[ |u - v|^p \geq |u|^p - \max\{p, 2\} 2^{p-1}|v|. \tag{3.5} \]

Proof. Using (3.2) with $a = |u - v|$ and $b = |v|$, one gets
\[ |u|^p \leq (|u - v| + |v|)^p \leq |u - v|^p + \max\{p, 2\} 2^{p-1}|v|, \]
which immediately implies (3.5). \hfill \Box

The following lemma is a special case of the desired result for functions with values in the segment $[0, 1]$.

Lemma 3.4. Let $1 \leq p < \infty$ and let $f \in BV_p([0,1]$ be such that $f : [0,1] \to [0,1]$. For any $\varepsilon > 0$ there exist $\eta > 0$ such that if
\[ 0 \leq x_1 < x_2 < \cdots < x_m \leq 1 \quad \text{and} \quad f(x_j) < \eta, \ j = 1, \ldots, m, \tag{3.6} \]
then
\[ \left( \sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p} < \varepsilon. \tag{3.7} \]

Proof. Choose a partition $0 = y_1 < y_2 < \cdots < y_n = 1$ such that
\[ \sum_{k=1}^{n-1} |f(y_{k+1}) - f(y_k)|^p > \text{Var}_p(f, [0,1]) - \frac{\varepsilon^p}{2}. \]
Set
\[ \eta = \min \left\{ 1, \frac{\varepsilon^p}{n(p + 2)2^{p+1}} \right\}. \]
Suppose (3.6) holds. If $[y_k, y_{k+1}]$ contains some of the points $x_1, \ldots, x_m$, let
\[ j_k := \min\{j : x_j \in [y_k, y_{k+1}]\}, \quad J_k := \max\{j : x_j \in [y_k, y_{k+1}]\}. \]
Note that since $f \geq 0$, one has
\[ (f(y_k))^p + (f(y_{k+1}))^p \geq (\max\{f(y_k), f(y_{k+1})\})^p \geq |f(y_{k+1}) - f(y_k)|^p. \]
Then using Corollary 3.3, one gets
\[
|f(x_{j_k}) - f(y_k)|^p + |f(x_{j_k} - 1) - f(x_{j_k})|^p + \cdots + |f(x_{j_k}) - f(x_{j_k-1})|^p + |f(y_{k+1}) - f(x_{j_k})|^p \\
\geq (f(y_k))^p - \max\{p, 2\} 2^{p-1}f(x_{j_k}) + \sum_{j=j_k}^{J_{k-1}} |f(x_{j+1}) - f(x_j)|^p + (f(y_{k+1}))^p - \max\{p, 2\} 2^{p-1}f(x_{j_k}) \\
\geq |f(y_{k+1}) - f(y_k)|^p - \max\{p, 2\} 2^p\eta + \sum_{j=j_k}^{J_k} |f(x_{j+1}) - f(x_j)|^p - \eta^p \\
\geq |f(y_{k+1}) - f(y_k)|^p - (p + 2)2^p\eta + \sum_{j=j_k}^{J_k} |f(x_{j+1}) - f(x_j)|^p,
\]
where we take $f(x_{m+1}) = 0$ if $J_k = m$. In the last inequality above, we have used the following inequality
\[ \max\{p, 2\} + 1 \leq p + 2. \]
Summing over \( k \) from 1 to \( n - 1 \), one obtains
\[
\text{Var}_p(f, [0, 1]) \geq \sum_{k=1}^{n-1} |f(y_{k+1}) - f(y_k)|^p - (n - 1)(p + 2)2^p \eta + \sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p
\]
\[
> \text{Var}_p(f, [0, 1]) - \frac{\varepsilon p}{2} - \frac{\varepsilon p}{2} + \sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p,
\]
which proves (3.7). \( \square \)

We are now in a position to prove the main result of this section. For \( p = 1 \) the following lemma was proved in [3, Lemma 2.1].

**Lemma 3.5 (Key lemma).** Let \( 1 \leq p < \infty \) and \( f \in BV_p[0, 1] \). For any \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that if
\[
0 \leq x_1 < x_2 < \cdots < x_m \leq 1 \quad \text{and} \quad |f(x_j)| < \delta \quad \text{for} \quad j \in \{1, \ldots, m\},
\]
then
\[
\left( \sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p} < \varepsilon.
\]

**Proof.** There is nothing to prove if \( f = 0 \). So, we assume that \( f \neq 0 \). Let \( M := \|f\|_{\infty} \), \( f_0 := \frac{1}{M} f \). Let \( u \) and \( v \) be the real and the imaginary parts of \( f_0 \). Hence \( f_0 = u + iv \). Consider the functions
\[
w_1 = u_+ := \max\{u, 0\} = \frac{|u| + u}{2}, \quad w_2 = u_- := (-u)_+ = \frac{|u| - u}{2} = u_- - u
\]
and \( w_3 = v_+, \ w_4 = v_- \). Then \( f_0 = w_1 - w_2 + i(w_3 - w_4) \) and
\[
0 \leq w_l \leq \|f_0\|_{\infty} = 1 \quad \text{for} \quad l \in \{1, 2, 3, 4\}.
\]
Since \( |a_+ - b_+| \leq |a - b| \) for all \( a, b \in \mathbb{R} \), one also has
\[
\text{Var}_p(w_l, [0, 1]) \leq \text{Var}_p(f_0, [0, 1]) = \frac{1}{M^p} \text{Var}_p(f, [0, 1]) \quad \text{for} \quad l \in \{1, 2, 3, 4\}.
\]
Take an arbitrary \( \varepsilon > 0 \). It follows from Lemma 3.4 that for every \( l \in \{1, 2, 3, 4\} \), there exists \( \eta_l > 0 \) such that
\[
0 \leq x_1 < x_2 < \cdots < x_m \leq 1 \quad \text{and} \quad w_l(x_j) < \eta_l, \quad j = 1, \ldots, m
\]
imply
\[
\left( \sum_{j=1}^{m-1} |w_l(x_{j+1}) - w_l(x_j)|^p \right)^{1/p} < \frac{\varepsilon}{4M}.
\]
Let \( \eta := M \min\{\eta_l : l = 1, 2, 3, 4\} \). If
\[
0 \leq x_1 < x_2 < \cdots < x_m \leq 1 \quad \text{and} \quad |f(x_j)| < \eta, \ j = 1, \ldots, m,
\]
then
\[
w_l(x_j) < \frac{1}{M} \eta \leq \eta_l, \quad j = 1, \ldots, m,
\]
and it follows from the above that
\[
\left( \sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p} = M \left( \sum_{j=1}^{m-1} |f_0(x_{j+1}) - f_0(x_j)|^p \right)^{1/p} \\
\leq M \sum_{l=1}^{4} \left( \sum_{j=1}^{m-1} |w_l(x_{j+1}) - w_l(x_j)|^p \right)^{1/p} < M \sum_{l=1}^{4} \frac{\varepsilon}{4M} = \varepsilon,
\]
which completes the proof. \qed

4. Approximating in $BV_p[0, 1]$ an arbitrary pair of functions by a pair of jointly nondegenerate functions

Let us start this section with two simple lemmas.

**Lemma 4.1.** Let $1 \leq p < \infty$ and $f \in BV_p[0, 1]$. Then $f$ possesses a limit from the left and from the right at each point. Moreover $f$ has a most countably many discontinuities.

This statement can be proved as in the case $p = 1$ (see, e.g., [4, Proposition 1.32 and Corollary 1.33]).

**Lemma 4.2.** Let $1 \leq p < \infty$, $\rho > 0$, and $f : (a, b) \rightarrow \mathbb{C}$ be such that
\[
\inf_{x \in (a, b)} |f(x)| < \rho.
\]
Then
\[
\sup_{x \in (a, b)} |f(x)| \leq \rho + \sup_{[\alpha, \beta] \subset (a, b)} \left( \text{Var}_p(f, [\alpha, \beta]) \right)^{1/p}. \tag{4.1}
\]

**Proof.** There exists $x_0 \in (a, b)$ such that $|f(x_0)| < \rho$. Consider an arbitrary $x \in (a, b)$. Let $I_x \subset (a, b)$ be the segment with the endpoints $x$ and $x_0$. By [4, Proposition 1.32(c)],
\[
|f(x) - f(x_0)| \leq \left( \text{Var}_p(f, I_x) \right)^{1/p} \leq \sup_{[\alpha, \beta] \subset (a, b)} \left( \text{Var}_p(f, [\alpha, \beta]) \right)^{1/p}.
\]
Hence
\[
|f(x)| \leq |f(x_0)| + \sup_{[\alpha, \beta] \subset (a, b)} \left( \text{Var}_p(f, [\alpha, \beta]) \right)^{1/p} < \rho + \sup_{[\alpha, \beta] \subset (a, b)} \left( \text{Var}_p(f, [\alpha, \beta]) \right)^{1/p}.
\]
Since $x \in (a, b)$ is arbitrary,
\[
\sup_{x \in (a, b)} |f(x)| \leq \rho + \sup_{[\alpha, \beta] \subset (a, b)} \left( \text{Var}_p(f, [\alpha, \beta]) \right)^{1/p},
\]
which completes the proof. \qed

The next theorem says that an arbitrary pair of functions in $(BV_p[0, 1])^2$ can be approximated by a pair of jointly nondegenerate functions with the same product.

**Theorem 4.3.** Suppose that $1 \leq p < \infty$. For every $\varepsilon > 0$ and every pair of functions $(F, G) \in (BV_p[0, 1])^2$ there is a pair of jointly nondegenerate functions $(F_1, G_1) \in (BV_p[0, 1])^2$ such that $F \cdot G = F_1 \cdot G_1$ and
\[
\|F - F_1\|_{BV_p} < \varepsilon, \quad \|G - G_1\|_{BV_p} < \varepsilon.
\]
Proof. The idea of the proof is borrowed from the proof of [8, Theorem 2.2]. Fix $\varepsilon > 0$. By Lemma 3.5, we can find some $\delta > 0$ such that for every partition

$$0 \leq x_1 < x_2 < \cdots < x_m \leq 1,$$

we have

$$|F(x_j)| < \delta \text{ for } j \in \{1, \ldots, m\} \Rightarrow \left( \sum_{j=1}^{m-1} |F(x_{j+1}) - F(x_j)|^p \right)^{1/p} < \frac{\varepsilon}{48} \quad (4.2)$$

and

$$|G(x_j)| < \delta \text{ for } j \in \{1, \ldots, m\} \Rightarrow \left( \sum_{j=1}^{m-1} |G(x_{j+1}) - G(x_j)|^p \right)^{1/p} < \frac{\varepsilon}{48}. \quad (4.3)$$

Take

$$\eta := \min \left\{ \delta, \frac{\varepsilon}{24}, \frac{1}{2} \sup_{x \in [0,1]} (|F(x)| + |G(x)|) \right\}. \quad (4.4)$$

By the representation theorem for open sets on the real line (see, e.g., [1, Theorem 3.11]), the interior of the set $\{x \in [0,1] : |F(x)| + |G(x)| < \eta\}$ is the union of at most countable collection of disjoint open intervals. Let $A_0$ be the collection of those open intervals $U = (a, b)$, $a < b$, in this union such that

$$\inf_{x \in U} (|F(x)| + |G(x)|) < \frac{\eta}{2}.$$

We claim that there are only finitely many intervals in $A_0$. Indeed, assume the contrary:

$$A_0 = \{U_i = (a_i, b_i) : i \in \mathbb{N}, a_i < b_i\}.$$

Without loss of generality, we can assume that $b_i \leq a_{i+1}$ for every $i \in \mathbb{N}$. Let $H := |F| + |G|$. By the definition of the infimum, for every $i \in \mathbb{N}$, there exists $x_i \in (a_i, b_i)$ such that $H(x_i) < \eta/2$. On the other hand, there is at least one point $y_i$ such that $b_i \leq y_i \leq a_{i+1}$ and $H(y_i) \geq \eta$. Hence

$$\text{Var}_p(H, [0,1]) \geq \sum_{i=1}^{\infty} |H(y_i) - H(x_i)|^p \geq \sum_{i=1}^{\infty} \left( \eta - \frac{\eta}{2} \right)^p = +\infty,$$

which is impossible since $H = |F| + |G| \in BV_p[0,1]$. Thus, for some $N \in \mathbb{N}$, we have

$$A_0 = \{(a_1, b_1), \ldots, (a_N, b_N)\}.$$ 

Let

$$\rho := \min \left\{ \frac{\eta}{2}, \frac{\varepsilon}{48N} \right\} \quad (4.5)$$

and let $A$ be the part of $A_0$ consisting of the intervals $(a_i, b_i)$ such that

$$\inf_{x \in (a_i, b_i)} (|F(x)| + |G(x)|) < \rho. \quad (4.6)$$

Relabelling $(a_i, b_i) \in A$ if necessary, we can assume

$$A = \{(a_1, b_1), \ldots, (a_n, b_n)\},$$

where $n \leq N$.

For $i \in \{1, \ldots, n\}$, put

$$c_i := \max \left\{ \sup_{x \in (a_i, b_i)} |F(x)|, \frac{\varepsilon}{24n} \right\}, \quad d_i := \max \left\{ \sup_{x \in (a_i, b_i)} |G(x)|, \frac{\varepsilon}{24n} \right\}. \quad (4.7)$$
It follows from definitions [4.7], [4.13] and the definition of the collection \( A \) that
\[
\max_{1 \leq i \leq n} \max \{ c_i, d_i \} \leq \frac{\varepsilon}{24}. \tag{4.8}
\]

Taking into account the definition of the collection \( A \) and [4.3], we see that for every \( i \in \{1, \ldots, n\} \), every interval \([\alpha, \beta] \subset (a_i, b_i)\) and every its partition \( \alpha = x_1 < \cdots < x_m = \beta \), one has \( |F(x_j)| < \delta \) and \( |G(x_j)| < \delta \) for \( j \in \{1, \ldots, m\} \). Then [4.2]–[4.3] imply that
\[
\sum_{i=1}^{n} \sup_{[\alpha, \beta] \subset (a_i, b_i)} Var_p(F, [\alpha, \beta]) \leq \left( \frac{\varepsilon}{48} \right)^p, \tag{4.9}
\]
\[
\sum_{i=1}^{n} \sup_{[\alpha, \beta] \subset (a_i, b_i)} Var_p(G, [\alpha, \beta]) \leq \left( \frac{\varepsilon}{48} \right)^p. \tag{4.10}
\]

It follows from Lemma [4.2], definition [4.5], estimates [4.9]–[4.10], and the inequality
\[
(t + \tau)^p \leq 2^{p-1} (t^p + \tau^p), \quad t, \tau \geq 0 \tag{4.11}
\]
that
\[
\sum_{i=1}^{n} \left( \sup_{x \in (a_i, b_i)} |F(x)| \right)^p \leq \sum_{i=1}^{n} \left( \rho + \sup_{[\alpha, \beta] \subset (a_i, b_i)} (Var_p(F, [\alpha, \beta]))^{1/p} \right)^p \leq \sum_{i=1}^{n} 2^{p-1} \left( \frac{\varepsilon}{48N} \right)^p + \sup_{[\alpha, \beta] \subset (a_i, b_i)} Var_p(F, [\alpha, \beta]) \leq \left( \frac{\varepsilon}{24} \right)^p, \tag{4.12}
\]
and
\[
\sum_{i=1}^{n} \left( \sup_{x \in (a_i, b_i)} |G(x)| \right)^p \leq \left( \frac{\varepsilon}{24} \right)^p.
\]

Combining [4.7] and [4.12], we see that
\[
\left( \sum_{i=1}^{n} c_i^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} \left( \sup_{x \in (a_i, b_i)} |F(x)| \right)^p \right)^{1/p} + \sum_{i=1}^{n} \left( \frac{\varepsilon}{24n} \right)^p \leq \left( \frac{\varepsilon}{24} \right)^p + \frac{\varepsilon}{24} = \frac{\varepsilon}{12} \tag{4.13}
\]
and, similarly,
\[
\left( \sum_{i=1}^{n} d_i^p \right)^{1/p} \leq \frac{\varepsilon}{12}. \tag{4.14}
\]

Define \( f, g : [0, 1] \to \mathcal{F} \in \{ \mathbb{R}, \mathbb{C} \} \) by
\[
f(x) := \begin{cases} 
F(x), & x \notin \bigcup_{i=1}^{n} (a_i, b_i), \\
c_i + d_i, & x \in (a_i, b_i), \quad i \in \{1, \ldots, n\},
\end{cases} \tag{4.15}
\]
\[
g(x) := \begin{cases} 
G(x), & x \notin \bigcup_{i=1}^{n} (a_i, b_i), \\
F(x)G(x) / c_i + d_i, & x \in (a_i, b_i), \quad i \in \{1, \ldots, n\}.
\end{cases} \tag{4.16}
\]
It follows from (4.7)–(4.8) and (4.10) that

\[ \| F - f \|_\infty = \max_{1 \leq i \leq n} \sup_{x \in (a_i, b_i)} |F(x) - (c_i + d_i)| < \max_{1 \leq i \leq n} 2(c_i + d_i) \leq 2 \left( \frac{\varepsilon}{24} + \frac{\varepsilon}{24} \right) = \frac{\varepsilon}{6} \]  

and

\[ \text{Var}_p(F - f, [0, 1]) \leq \sum_{i=1}^{n} \sup_{[\alpha, \beta] \subset (a_i, b_i)} \text{Var}_p(F - (c_i + d_i), [\alpha, \beta]) \]

\[ + \sum_{i=1}^{n} \lim_{x \to a_i} |F(x) - (c_i + d_i)|^p + \sum_{i=1}^{n} \lim_{x \to b_i} |F(x) - (c_i + d_i)|^p \]

\[ \leq \sum_{i=1}^{n} \sup_{[\alpha, \beta] \subset (a_i, b_i)} \text{Var}_p(F, [\alpha, \beta]) + 2 \sum_{i=1}^{n} \sup_{x \in (a_i, b_i)} (|F(x) + c_i + d_i|)^p \]

\[ < \sum_{i=1}^{n} \sup_{[\alpha, \beta] \subset (a_i, b_i)} \text{Var}_p(F, [\alpha, \beta]) + 4p \sum_{i=1}^{n} (c_i + d_i)^p. \]  

Combining (4.17)–(4.18) with (4.10) and (4.13)–(4.14), we see that

\[ \| F - f \|_{\text{BV}_p} = \| F - f \|_\infty + \left( \text{Var}_p(F - f, [0, 1]) \right)^{1/p} \leq \frac{\varepsilon}{6} + \left( \frac{\varepsilon}{48} \right)^{1/p} + \frac{\varepsilon}{6} + \left( \frac{\varepsilon}{24} \right)^{1/p} < \frac{\varepsilon}{6} + \frac{\varepsilon}{24} + \frac{\varepsilon}{3} = \frac{7\varepsilon}{8}. \]  

Analogously, it follows from (4.7)–(4.8) and (4.10) that

\[ \| G - g \|_\infty = \max_{1 \leq i \leq n} \sup_{x \in (a_i, b_i)} \left| G(x) - \frac{F(x)G(x)}{c_i + d_i} \right| \]

\[ \leq \max_{1 \leq i \leq n} \left( \sup_{x \in (a_i, b_i)} |G(x)| + \sup_{x \in (a_i, b_i)} |G(x)| \sup_{x \in (a_i, b_i)} \frac{|F(x)|}{c_i + d_i} \right) \]

\[ \leq \max_{1 \leq i \leq n} \left( \frac{d_i + d_i \cdot c_i}{c_i + d_i} \right) < 2 \max_{1 \leq i \leq n} d_i \leq \frac{2\varepsilon}{24} = \frac{\varepsilon}{12}. \]  

If \( i \in \{1, \ldots, n\} \) and \([\alpha, \beta] \subset (a_i, b_i)\), then taking into account inequality (4.11) and definitions (4.7), we get

\[ \text{Var}_p \left( \frac{1}{c_i + d_i} \right), [\alpha, \beta] \]

\[ \leq 2^{p-1} \left\{ \sup_{x \in [\alpha, \beta]} |G(x)|^p \cdot \text{Var}_p \left( 1 - \frac{F}{c_i + d_i}, [\alpha, \beta] \right) + \text{Var}_p(F, [\alpha, \beta]) \cdot \sup_{x \in [\alpha, \beta]} \left| 1 - \frac{F(x)}{c_i + d_i} \right|^p \right\} \]

\[ \leq 2^p \left\{ \left( \frac{\sup_{x \in (a_i, b_i)} |G(x)|}{c_i + d_i} \right)^p \cdot \text{Var}_p(F, [\alpha, \beta]) + \text{Var}_p(F, [\alpha, \beta]) \cdot \left( 1 + \sup_{x \in (a_i, b_i)} \frac{|F(x)|}{c_i + d_i} \right)^p \right\} \]

\[ \leq 2^p \left\{ \left( \frac{d_i}{c_i + d_i} \right)^p \cdot \text{Var}_p(F, [\alpha, \beta]) + \text{Var}_p(F, [\alpha, \beta]) \left( 1 + \frac{c_i}{c_i + d_i} \right)^p \right\} \]

\[ \leq 2^p \text{Var}_p(F, [\alpha, \beta]) + 4^p \text{Var}_p(F, [\alpha, \beta]). \]
Further, definitions (4.7) imply that for \( i \in \{1, \ldots, n\} \),
\[
\lim_{x \to a_i^+} |G(x) \left( 1 - \frac{F(x)}{c_i + d_i} \right)|^p + \lim_{x \to b_i^-} |G(x) \left( 1 - \frac{F(x)}{c_i + d_i} \right)|^p \\
\leq 2 \sup_{x \in (a_i, b_i)} |G(x)|^p \cdot \sup_{x \in (a_i, b_i)} \left| 1 - \frac{F(x)}{c_i + d_i} \right|^p \leq 2d_i^p \left( 1 + \frac{c_i}{c_i + d_i} \right)^p \leq 2^{p+1} d_i^p \leq 4p d_i^p.
\]

(4.22)

It follows from (4.21)–(4.22) that
\[
\text{Var}_p(G - g, [0, 1]) \leq \sum_{i=1}^{n} \sup_{x \in [a_i, b_i]} \text{Var}_p \left( G \left( 1 - \frac{F}{c_i + d_i} \right), [\alpha, \beta] \right) + 4p \sum_{i=1}^{n} \sup_{x \in [a_i, b_i]} \text{Var}_p(F_i([\alpha, \beta]) + 4p \sum_{i=1}^{n} d_i^p.
\]

(4.23)

Combining (4.20) and (4.23) with (4.9)–(4.10) and (4.14), we see that
\[
\|G - g\|_{BV, p} = \|G - g\|_{\infty} + \left( \text{Var}_p(G - g, [0, 1]) \right)^{1/p} < \frac{\varepsilon}{12} + \left( \frac{2p}{48} \right)^p \leq \left( \frac{\varepsilon}{48} \right)^p + 4p \sum_{i=1}^{n} d_i^p \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{3} < \varepsilon.
\]

(4.24)

It follows from (4.19) and (4.24) that \( f, g \in BV_p[0, 1] \), whence
\[
h := |f| + |g| \in BV_p[0, 1].
\]

In view of Lemma (4.11), the set \( J \) of jumps of \( h \) is at most countable. Let \( \partial S \) and \( \text{int}(S) \) denote the boundary and the interior of a set \( S \subset [0, 1] \), respectively. Consider the sets
\[
S_\eta := \{ x \in [0, 1] : h(x) < \eta \}, \quad B_\eta := \{ x \in [0, 1] : h(x) \geq \eta \}.
\]

Note that in view of the choice of \( \eta \) in (4.24), the set \( B_\eta \) is nonempty. Then we have \( \partial(S_\eta) \setminus J \subset B_\eta \). Consider the set
\[
J_\eta := \partial(S_\eta) \setminus B_\eta \subset J.
\]

We have
\[
[0, 1] = B_\eta \cup S_\eta = B_\eta \cup \text{int}(S_\eta) \cup J_\eta,
\]

(4.25)

where the sets \( B_\eta \), \( \text{int}(S_\eta) \) and \( J_\eta \) are pairwise disjoint.

We claim that the set
\[
J_\eta := \{ y \in J_\eta : h(y) < \eta/2 \}
\]

is finite. Indeed, since \( J_\eta^p \subset J_\eta \subset J \), the set \( J_\eta^p \) is at most countable. Assume the contrary, that is, that the set \( J_\eta^p \) is infinite. Let \( J_\eta^p = \{ y_j \}_{j=1}^{\infty} \), and \( y_j < y_{j+1} \) for all \( j \in \mathbb{N} \). Then for every \( j \in \mathbb{N} \), there exists \( x_j \in B_\eta \) such that \( y_{j-1} < x_j < y_{j+1} \). Therefore
\[
\text{Var}_p(h, [0, 1]) \geq \sum_{j=1}^{\infty} \left| h(x_j) - h(y_{j+1}) \right|^p \geq \sum_{j=1}^{\infty} \left( \frac{\eta}{2} \right)^p = +\infty,
\]

which is impossible since \( h \in BV_p[0, 1] \). Thus, the set \( J_\eta^p \) is finite.
Consider the (obviously, finite) set
\[ J_0^0 := \{ y \in J_0^\eta : h(y) = 0 \}. \]
Let \( k \) be the cardinality of \( J_0^0 \). Define the functions \( F_1, G_1 : [0, 1] \to \mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \} \) by
\[
F_1(x) := \begin{cases} 
  f(x), & x \in [0, 1] \setminus J_0^0, \\
  \frac{\varepsilon}{24k}, & x \in J_0^0,
\end{cases}
\] (4.26)
and
\[
G_1(x) := g(x), & x \in [0, 1].
\] (4.27)

It is clear that
\[
f(x) = g(x) = 0, \quad x \in J_0^0.
\] (4.28)

It follows from (4.15)–(4.16) and (4.26)–(4.28) that
\[
F(x)G(x) = f(x)g(x) = F_1(x)G_1(x), \quad x \in [0, 1].
\] (4.29)

Moreover,
\[
\|F_1 - f\|_{BV_p} = \|f\|_\infty + \left( \text{Var}_p(F_1 - f, [0, 1]) \right)^{1/p} = \frac{\varepsilon}{24k} + \left( 2k \left( \frac{\varepsilon}{24k} \right)^p \right)^{1/p} \leq \frac{2k + 1}{24k} \varepsilon \leq \frac{\varepsilon}{8}.
\] (4.30)

Combining (4.19) and (4.30), we have
\[
\|F - F_1\|_{BV_p} \leq \|F - f\|_{BV_p} + \|f - F_1\|_{BV_p} < \frac{7\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon.
\] (4.31)

In view of (4.24) and (4.27), we have
\[
\|G - G_1\|_{BV_p} = \|G - g\|_{BV_p} < \varepsilon.
\] (4.32)

For a set \( S \subset [0, 1] \), let
\[
I(S) := \inf_{x \in S} (|F_1(x)| + |G_1(x)|).
\]

Then it follows from (4.26)–(4.27) that
\[
I_1 := I(B_0) = \inf_{x \in B_0} (|f(x)| + |g(x)|) \geq \eta > 0,
\] (4.33)
\[
I_2 := I(J_0^0 \setminus J_0^\eta) = \inf_{y \in J_0^0 \setminus J_0^\eta} (|f(y)| + |g(y)|) \geq \frac{\eta}{2} > 0,
\] (4.34)
\[
I_3 := I(J_0^\eta \setminus J_0^0) = \min_{y \in J_0^\eta \setminus J_0^0} (|f(y)| + |g(y)|) > 0
\] (4.35)
(recall that the set \( J_0^\eta \setminus J_0^0 \) is finite), and
\[
I_4 := I(J_0^0) \geq \frac{\varepsilon}{24k} > 0.
\] (4.36)

By the definition of the collection \( A \) and definitions (4.15)–(4.16) and (4.26)–(4.27), we have
\[
I_5 := I \left( \text{int}(S_\eta) \setminus \left( \bigcup_{i=1}^n (a_i, b_i) \right) \right) = \inf_{x \in \text{int}(S_\eta) \setminus \left( \bigcup_{i=1}^n (a_i, b_i) \right)} (|F(x)| + |G(x)|) \geq \rho > 0
\] (4.37)
(see (4.5) and (4.6)) and, in view of (4.7), we see that

\[ I_6 := I \left( \bigcup_{i=1}^n (a_i, b_i) \right) \geq \min_{1 \leq i \leq n} \inf_{x \in (a_i, b_i)} \left( |f(x)| + |g(x)| \right) \geq \min_{1 \leq i \leq n} (c_i + d_i) \geq \frac{\varepsilon}{12n} > 0. \quad (4.38) \]

It follows from (4.25) and (4.33)–(4.38) that

\[ I([0, 1]) \geq \min_{1 \leq j \leq 6} I_j > 0. \]

Thus, functions \( F_1, G_1 \in BV_p[0, 1] \) are jointly nondegenerate. Combining this observation with (4.29) and (4.31)–(4.32), we arrive at the conclusion of the theorem. \( \square \)

5. Proof of the main result and final remarks

**Proof of Theorem 5.1**

Take an arbitrary pair \( (F, G) \in (BV_p[0, 1])^2 \). Fix \( \varepsilon > 0 \). It follows from Theorem 4.3 that there exists a pair of jointly nondegenerate functions \( (F_1, G_1) \in (BV_p[0, 1])^2 \) such that

\[ F \cdot G = F_1 \cdot G_1 \quad (5.1) \]

and

\[ \|F - F_1\|_{BV_p} < \varepsilon/2, \quad \|G - G_1\|_{BV_p} < \varepsilon/2. \quad (5.2) \]

By Corollary 2.2, there exists a \( \delta > 0 \) such that

\[ B_{BV_p[0, 1]}(F_1 \cdot G_1, \delta) \subset B_{BV_p[0, 1]}(F_1, \varepsilon/2) \cdot B_{BV_p[0, 1]}(G_1, \varepsilon/2). \quad (5.3) \]

Combining (5.1)–(5.3), we arrive at the following:

\[ B_{BV_p[0, 1]}(F \cdot G, \delta) \subset B_{BV_p[0, 1]}(F, \varepsilon/2) \cdot B_{BV_p[0, 1]}(G_1, \varepsilon/2) \subset B_{BV_p[0, 1]}(F, \varepsilon) \cdot B_{BV_p[0, 1]}(G_1, \varepsilon). \]

Thus, the multiplication in the Banach algebra \( BV_p[0, 1] \) is locally open at the pair \( (F, G) \). Since \( (F, G) \in (BV_p[0, 1])^2 \) is an arbitrary pair, we conclude that the multiplication in \( BV_p[0, 1] \) is an open bilinear mapping. \( \square \)

Let \( 1 \leq p < \infty \) and \( \Lambda_p BV[0, 1] \) be the Banach algebra of all functions of bounded variation in the Shiba-Waterman sense. We conclude the paper with the following.

**Conjecture 5.1.** The multiplication in the Banach algebra \( \Lambda_p BV[0, 1] \) is an open bilinear mapping.

In view of Corollary 2.3, to confirm this conjecture, one has to prove that every pair of functions \( (f, g) \in (\Lambda_p BV[0, 1])^2 \) can be approximated by a pair of jointly nondegenerate functions \( (f_1, g_1) \in (\Lambda_p BV[0, 1])^2 \) such that \( f \cdot g = f_1 \cdot g_1 \).

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