On the relation between quantum walks and zeta functions

Norio Konno · Iwao Sato

Abstract
We present an explicit formula for the characteristic polynomial of the transition matrix of the discrete-time quantum walk on a graph via the second weighted zeta function. As applications, we obtain new proofs for the results on spectra of the transition matrix and its positive support.

Keywords
Quantum walk · Transition matrix · Ihara zeta function

Mathematics Subject Classification (2000)
05C50 · 05C60 · 81P68

1 Introduction
As a quantum counterpart of the classical random walk, the quantum walk has recently attracted much attention for various fields. The review and book on quantum walks are Ambainis [1], Kempe [10], Kendon [11], Konno [12], for examples. Quantum walks of graphs were applied in graph isomorphism problems. Graph isomorphism problems determine whether two graphs are isomorphic. Shiau et al. [17] first pointed out the deficiency of the simplest classical algorithm and continuous-time one particle quantum random walks in distinguishing some non-isomorphic graphs.
Emms et al. [4] introduced a graph-spectral technique induced by discrete-time quantum walks to distinguish two non-isomorphic graphs that are cospectral with respect to standard matrix representations. Gambel et al. [6] developed a method of characterizing the additional power that quantum walks of interacting particles have for distinguishing non-isomorphic regular graphs. Emms et al. [3] treated spectra of the transition matrix and its positive support of the discrete-time quantum walk on a graph, and showed that the third power of the transition matrix outperforms the graph spectra methods in distinguishing strongly regular graphs. Godsil and Guo [7] gave new proofs of the results of Emms et al. [3].

Already, the Ihara zeta function of a graph obtained various success related to graph spectra. Ihara zeta functions of graphs started from Ihara zeta functions of regular graphs by Ihara [9]. Originally, Ihara [9] presented $p$-adic Selberg zeta functions of discrete groups, and showed that its reciprocal is an explicit polynomial. Serre [16] pointed out that the Ihara zeta function is the zeta function of the quotient $T/\Gamma$ (a finite regular graph) of the one-dimensional Bruhat-Tits building $T$ (an infinite regular tree) associated with $GL(2, k_p)$. A zeta function of a regular graph $G$ associated with a unitary representation of the fundamental group of $G$ was developed by Sunada [19, 20]. Hashimoto [8] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara’s result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial. Various proofs of Bass’ Theorem were given by Stark and Terras [18], Foata and Zeilberger [5], Kotani and Sunada [13]. Sato [15] defined a new zeta function of a graph by using not an infinite product but a determinant.

Ren et al. [14] found an interesting relationship between the Ihara zeta function and the discrete-time quantum walk on a graph, and showed that the support of the transition matrix of the discrete-time quantum walk is equal to the Perron-Frobenius operator (the edge matrix) related to the Ihara zeta function. Based on this analysis, Ren et al. explained that the Ihara zeta function can not distinguish cospectral regular graphs.

It is known that a necessary condition of localization is the existence of degenerate eigenvalues of the transition matrix in the case of a three-state quantum walk (see Sect. 1.2 of [12], for example). Our main result (Theorem 4.1) gives an explicit formula for the characteristic polynomial of the transition matrix of the quantum walk. So the result is useful for investigating dynamics of the walk and would be helpful to the study on transmission of the quantum information.

The rest of the paper is organized as follows. Section 2 gives the definition of the transition matrix of the discrete-time quantum walk on a graph, and review results on it. In Sect. 3, we define the Ihara zeta function and the second weighted zeta function of a graph, and present their determinant expressions. In Sect. 4, we present our main result (Theorem 4.1) of this paper. As a corollary, we give another proof of a result by Emms et al. [3] on spectra of the transition matrix. In Sect. 5, we present another proof of a result by Emms et al. [3] on spectra of the positive support of the transition matrix.
2 Definition of the transition matrix of a quantum walk on a graph

Graphs treated here are finite. Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges $uv$ joining two vertices $u$ and $v$. For $uv \in E(G)$, an arc $(u, v)$ is the oriented edge from $u$ to $v$. Set $D(G) = \{(u, v), (v, u)| uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$. The degree $\text{deg} v = \text{deg}_G v$ of a vertex $v$ of $G$ is the number of edges incident to $v$. For a natural number $k$, a graph $G$ is called $k$-regular if $\text{deg}_G v = k$ for each vertex $v$ of $G$.

A discrete-time quantum walk is a quantum analog of the classical random walk on a graph whose state vector is governed by a matrix called the transition matrix. Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$. Set $d_j = du_j = \text{deg} v_j$ for $j = 1, \ldots, n$. The transition matrix $U = U(G) = (U_{ef})_{e,f \in D(G)}$ of $G$ is defined by

$$
U_{ef} = \begin{cases} 
\frac{2}{d_{t(f)}}(\frac{2}{d_{o(e)}}) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\
\frac{2}{d_{t(f)}} - 1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
$$

We introduce the positive support $F^+ = (F^+_ij)$ of a real matrix $F = (F_{ij})$ as follows:

$$
F^+_ij = \begin{cases} 
1 & \text{if } F_{ij} > 0, \\
0 & \text{otherwise}.
\end{cases}
$$

Let $G$ be a connected graph. If the degree of each vertex of $G$ is not less than two, i.e., $\delta(G) \geq 2$, then $G$ is called an md2 graph.

The transition matrix of a discrete-time quantum walk in a graph is closely related to the Ihara zeta function of a graph. We stare a relationship between the discrete-time quantum walk and the Ihara zeta function of a graph by Ren et al. [14].

**Theorem 2.1** (Ren, Aleksic, Emms, Wilson and Hancock [14]) Let $B - J_0$ be the Perron-Frobenius operator (or the edge matrix) of a simple graph subject to the md2 constraint, where the edge matrix is defined in Sect. 3. Let $U$ be the transition matrix of the discrete-time quantum walk on $G$. Then the $B - J_0$ is the positive support of the transpose of $U$, i.e.,

$$
B - J_0 = \left( T U \right)^+,
$$

where $T U$ is the transpose of $U$.

3 The Ihara zeta function of a graph

Let $G$ be a connected graph. Then a path $P$ of length $n$ in $G$ is a sequence $P = (e_1, \ldots, e_n)$ of $n$ arcs such that $e_i \in D(G), t(e_i) = o(e_{i+1})(1 \leq i \leq n - 1)$, where indices are treated mod $n$. Set $|P| = n, o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, $P$ is
called an \((o(P), t(P))-\)path. We say that a path \(P = (e_1, \ldots, e_n)\) has a backtracking if \(e_{i+1}^{-1} = e_i\) for some \(i (1 \leq i \leq n - 1)\). A \((v, w)\)-path is called a \(v\)-cycle (or \(v\)-closed path) if \(v = w\). The inverse cycle of a cycle \(C = (e_1, \ldots, e_n)\) is the cycle \(C^{-1} = (e_n^{-1}, \ldots, e_1^{-1})\).

We introduce an equivalence relation between cycles. Two cycles \(C_1 = (e_1, \ldots, e_m)\) and \(C_2 = (f_1, \ldots, f_m)\) are called equivalent if there exists \(k\) such that \(f_j = e_{j+k}\) for all \(j\). The inverse cycle of \(C\) is in general not equivalent to \(C\). Let \([C]\) be the equivalence class which contains a cycle \(C\). Let \(B'\) be the cycle obtained by going \(r\) times around a cycle \(B\). Such a cycle is called a power of \(B\). A cycle \(C\) is reduced if \(C\) has no backtracking. Furthermore, a cycle \(C\) is prime if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph \(G\) corresponds to a unique conjugacy class of the fundamental group \(\pi_1(G, v)\) of \(G\) at a vertex \(v\) of \(G\).

The Ihara zeta function of a graph \(G\) is a function of \(t \in \mathbb{C}\) with \(|t|\) sufficiently small, defined by

\[
Z(G, t) = Z_G(t) = \prod_{[C]} \left(1 - t^{|C|}\right)^{-1},
\]

where \([C]\) runs over all equivalence classes of prime, reduced cycles of \(G\).

Let \(G\) be a connected graph with \(n\) vertices and \(m\) edges. Two \(2m \times 2m\) matrices \(B = B(G) = (B_{ef})_{e,f \in D(G)}\) and \(J_0 = J_0(G) = (J_{ef})_{e,f \in D(G)}\) are defined as follows:

\[
B_{ef} = \begin{cases} 
1 & \text{if } t(e) = d(f), \\
0 & \text{otherwise},
\end{cases} \\
J_{ef} = \begin{cases} 
1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]

Then the matrix \(B - J_0\) is called the edge matrix of \(G\).

**Theorem 3.1** (Hashimoto [8]; Bass [2]) Let \(G\) be a connected graph. Then the reciprocal of the Ihara zeta function of \(G\) is given by

\[
Z(G, t)^{-1} = \det \left( I_{2m} - t(B - J_0) \right) = \left(1 - t^2\right)^{r-1} \det \left( I_n - tA(G) + t^2(D - I_n) \right),
\]

where \(I_n\) is the \(n \times n\) identity matrix, \(r\) and \(A(G)\) are the Betti number and the adjacency matrix of \(G\), respectively, and \(D = (d_{ij})\) is the diagonal matrix with \(d_{ii} = \deg v_i, V(G) = \{v_1, \ldots, v_n\}\).

Let \(G\) be a connected graph and \(V(G) = \{v_1, \ldots, v_n\}\). Then we consider an \(n \times n\) matrix \(W = (w_{ij})_{1 \leq i, j \leq n}\) with \(ij\) entry nonzero complex number \(w_{ij}\) if \((v_i, v_j) \in D(G)\), and \(w_{ij} = 0\) otherwise. The matrix \(W = W(G)\) is called the weighted matrix of \(G\). Furthermore, let \(w(v_i, v_j) = w_{ij}, v_i, v_j \in V(G)\) and \(w(e) = w_{ij}, e = (v_i, v_j) \in D(G)\). For each path \(P = (e_1, \ldots, e_r)\) of \(G\), the norm \(w(P)\) of \(P\) is defined as follows:

\[
w(P) = w(e_1)w(e_2) \cdots w(e_r).
\]

Let \(G\) be a connected graph with \(n\) vertices and \(m\) edges, and \(W = W(G)\) a weighted matrix of \(G\). A \(2m \times 2m\) matrix \(B_w = B_w(G) = (B_{ef}^{(w)})_{e,f \in D(G)}\) is defined as follows:

\[
B_w(G)_{ef} = \begin{cases} 
1 & \text{if } t(e) = d(f), \\
0 & \text{otherwise},
\end{cases} \\
J_{ef} = \begin{cases} 
1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]
The second weighted zeta function of $G$ is defined by

$$Z_1(G, w, t) = \det (I_{2m} - t(B_w - J_0))^{-1}.$$

If $w(e) = 1$ for any $e \in D(G)$, then the second weighted zeta function of $G$ is the Ihara zeta function of $G$.

**Theorem 3.2** (Sato [15]) Let $G$ be a connected graph, and let $W = W(G)$ be a weighted matrix of $G$. Then the reciprocal of the second weighted zeta function of $G$ is given by

$$Z_1(G, w, t)^{-1} = \left(1 - t^2\right)^{m-n} \det \left(I_n - tW(G) + t^2(D_w - I_n)\right),$$

where $n = |V(G)|$, $m = |E(G)|$ and $D_w = (d_{ij})$ is the diagonal matrix with $d_{ii} = \sum_{o(e) = v_i} w(e)$, $V(G) = \{v_1, \ldots, v_n\}$.

**4 The characteristic polynomial of the transition matrix**

We present a formula for the characteristic polynomial of $U$. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then the $n \times n$ matrix $T(G) = (T_{uv})_{u, v \in V(G)}$ is given as follows:

$$T_{uv} = \begin{cases} 1/(\deg_G u) & \text{if } (u, v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 4.1** Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges. Then, for the transition matrix $U$ of $G$, we have

$$\det(\lambda I_{2m} - U) = \left(\lambda^2 - 1\right)^{m-n} \det \left(\lambda^2 + 1\right)I_n - 2\lambda T(G) \right) \\
= \left(\lambda^2 - 1\right)^{m-n} \det \left(\lambda^2 + 1\right)D - 2\lambda A(G) \right)$$

where $\lambda \in \mathbb{C}$.

**Proof** Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$. Set $d_j = d_{v_j} = \deg v_j$ for each $j = 1, \ldots, n$. Then we consider a $2m \times 2m$ matrix $B_d = (B_{ef})_{e, f \in D(G)}$ given by

$$B_{ef} = \begin{cases} 2/d_{o(f)} & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.2, we see

$$\det \left(I_{2m} - t(B_d - J_0)\right) = \left(1 - t^2\right)^{m-n} \det \left(I_n - tW_d(G) + t^2(D_d - I_n)\right),$$
where \( W_d(G) = (w_{uv})_{u,v \in V(G)} \) and \( D_d = (d_{uv})_{u,v \in V(G)} \) are given as follows:

\[
  w_{uv} = \begin{cases} 
    2/d_u & \text{if } (u, v) \in D(G), \\
    0 & \text{otherwise,}
  \end{cases} \quad
  d_{uv} = \begin{cases} 
    2 & \text{if } u = v, \\
    0 & \text{otherwise.}
  \end{cases}
\]

Note that

\[
  d_j \times (2/d_j) = 2 \ (1 \leq j \leq n).
\]

Thus,

\[
  \det \left( I_{2m} - t \left( T B_d - T J_0 \right) \right) = \left( 1 - t^2 \right)^{m-n} \det \left( I_n - t W_d(G) + t^2 I_n \right).
\]

But, we have

\[
  T B_d - T J_0 = U \quad \text{and} \quad W_d(G) = 2 T(G).
\]

Therefore,

\[
  \det \left( I_{2m} - t U \right) = \left( 1 - t^2 \right)^{m-n} \det \left( \left( 1 + t^2 \right) I_n - 2t T(G) \right).
\]

Now, let \( t = 1/\lambda \). Then we get

\[
  \det \left( I_{2m} - \frac{1}{\lambda} U \right) = \left( 1 - \frac{1}{\lambda^2} \right)^{m-n} \det \left( \left( 1 + \frac{1}{\lambda^2} \right) I_n - \frac{2}{\lambda} T(G) \right).
\]

Thus,

\[
  \det \left( \lambda I_{2m} - U \right) = \left( \lambda^2 - 1 \right)^{m-n} \det \left( \left( \lambda^2 + 1 \right) I_n - 2\lambda T(G) \right).
\]

Next, we have

\[
  T(G) = D^{-1} A(G).
\]

Then it follows that

\[
  \det \left( \lambda I_{2m} - U \right) = \left( \lambda^2 - 1 \right)^{m-n} \det \left( \left( \lambda^2 + 1 \right) I_n - 2\lambda D^{-1} A(G) \right)
  = \left( \lambda^2 - 1 \right)^{m-n} \det \left( \lambda^2 + 1 \right) \det \left( D - 2\lambda A(G) \right).
\]

Since \( \det D^{-1} = 1/(d_{v_1} \cdots d_{v_n}) \),

\[
  \det \left( \lambda I_{2m} - U \right) = \frac{\left( \lambda^2 - 1 \right)^{m-n} \det \left( \lambda^2 + 1 \right) \det \left( D - 2\lambda A(G) \right)}{d_{v_1} \cdots d_{v_n}}.
\]
We can express the spectra of the transition matrix $U$ by means of those of $T(G)$ (see [3]). Let $\text{Spec}(F)$ be the spectra of a square matrix $F$.

**Corollary 4.2** (Emms, Hancock, Severini and Wilson [3]) *Let $G$ be a connected graph with $n$ vertices and $m$ edges. The transition matrix $U$ has $2n$ eigenvalues of the form

$$\lambda = \lambda_T \pm i \sqrt{1 - \lambda_T^2},$$

where $\lambda_T$ is an eigenvalue of the matrix $T(G)$. The remaining $2(m - n)$ eigenvalues of $U$ are $\pm 1$ with equal multiplicities.*

**Proof** By Theorem 4.1, we have

$$\det(\lambda I_{2m} - U) = (\lambda^2 - 1)^{m-n} \prod_{\lambda_T \in \text{Spec}(T(G))} \left( \lambda^2 + 1 - 2\lambda_T \lambda \right).$$

Solving $\lambda^2 + 1 - 2\lambda_T \lambda = 0$, we obtain

$$\lambda = \lambda_T \pm i \sqrt{1 - \lambda_T^2}.$$

The result follows. $\square$

Emms et al. [3] determined the spectra of the transition matrix $U$ by examining the elements of the transition matrix of a graph and using the properties of the eigenvector of a matrix. And now, we could explicitly obtain the spectra of the transition matrix $U$ from its characteristic polynomial.

**5 The positive support of the transition matrix of a graph**

By Theorem 2.1, we express the spectra of the positive support $U^+$ of the transition matrix of a regular graph $G$ by means of those of the adjacency matrix $A(G)$ of $G$ (see [3]).

**Theorem 5.1** (Emms, Hancock, Severini and Wilson [3]) *Let $G$ be a connected $k$-regular graph with $n$ vertices and $m$ edges, and $\delta(G) \geq 2$. The positive support $U^+$ has $2n$ eigenvalues of the form

$$\lambda = \frac{\lambda_A}{2} \pm i \sqrt{k - 1 - \frac{\lambda_A^2}{4}},$$

where $\lambda_A$ is an eigenvalue of the matrix $A(G)$. The remaining $2(m - n)$ eigenvalues of $U^+$ are $\pm 1$ with equal multiplicities.*

**Proof** Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$. Since $G$ is $k$-regular, we have $D = kI_n$. By
Theorems 2.1 and 3.1, we obtain
\[
\det(I_{2m} - tU^+) = \det(I_{2m} - t(T\mathbf{B} - T\mathbf{J}_0)) = \det(I_{2m} - t(B - J_0))
\]
\[
= (1 - t^2)^{m-n} \det(I_n - tA(G) + t^2(D - I_n))
\]
\[
= (1 - t^2)^{m-n} \det(I_n - tA(G) + t^2(k - 1)I_n).
\]

Now we put \( t = 1/\lambda \). Then we have
\[
\det(I_{2m} - \frac{1}{\lambda}U^+) = \left(1 - \frac{1}{\lambda^2}\right)^{m-n} \det\left((1 + \frac{k - 1}{\lambda^2})I_n - \frac{1}{\lambda}A(G)\right).
\]

Thus,
\[
\det(\lambda I_{2m} - U^+) = \left(\lambda^2 - 1\right)^{m-n} \det\left((\lambda^2 + k - 1)I_n - \lambda A(G)\right)
\]
\[
= \left(\lambda^2 - 1\right)^{m-n} \prod_{\lambda_A \in \text{Spec}(A(G))} \left(\lambda^2 + k - 1 - \lambda A\lambda\right).
\]

Solving \( \lambda^2 + k - 1 - \lambda A\lambda = 0 \), we get
\[
\lambda = \frac{\lambda A}{2} \pm i\sqrt{k - 1 - \frac{\lambda^2 A}{4}}.
\]

The result follows. \( \square \)

Godsil and Guo [7] presented a new proof of Theorem 5.1 by using linear algebraic technique.

From an argument in the first part of this proof, we easily obtain the following formula of the characteristic polynomial of \( U^+ \):

**Proposition 5.2**

\[
\det(\lambda I_{2m} - U^+) = \left(\lambda^2 - 1\right)^{m-n} \det\left((\lambda^2 + k - 1)I_n - \lambda A(G) + D\right).
\]

Emms et al. [3] found the eigenvalues of the positive support \( (U^2)^+ \) of the second power \( U^2 \) of the transition matrix \( U \) of a regular graph. Furthermore, Godsil and Guo [7] expressed \( (U^2)^+ \) in terms of \( U^+ \) by using linear algebraic technique, and presented another proof of the result of Emms et al. [3]. Emms et al. [3] determined a necessary and sufficient condition for any array of \( (U^3)^+ \) of a strongly regular graph to be equal to 1, and proposed the following conjecture:

**Conjecture 5.3** (Emms, Hancock, Severini and Wilson [3]) *Let \( G \) and \( H \) be strongly regular graphs with the same set of parameters. Then \( G \cong H \) if and only if \( \text{Spec}((U(G))^3)^+) = \text{Spec}((U(H))^3)^+) \).*
If we apply the linear algebraic technique of Godsil and Guo [7] to \((U^3)^+\), and approach the characteristic polynomial of \((U^3)^+\), then it might be likely to determine the spectra of \((U^3)^+\). In consequence, it might bring a progress toward a settlement of this conjecture. Related to this conjecture, to find an explicit formula for the characteristic polynomial of the positive support \((U^n)^+\) for any \(n\) would be one of the interesting future problems.

Acknowledgments The first author was partially supported by the Grant-in-Aid for Scientific Research (C) of Japan Society for the Promotion of Science (Grant No. 21540118). The second author was partially supported by the Grant-in-Aid for Scientific Research (C) of Japan Society for the Promotion of Science (Grant No. 19540154).

References

1. Ambainis, A.: Quantum walks and their algorithmic applications. Int. J. Quantum Inf. 1, 507–518 (2003)
2. Bass, H.: The Ihara-Selberg zeta function of a tree lattice. Int. J. Math. 3, 717–797 (1992)
3. Emms, D., Hancock, E.R., Severini, S., Wilson, R.C.: A matrix representation of graphs and its spectrum as a graph invariant. Electron. J. Comb. 13, R34 (2006)
4. Emms, D., Severini, S., Wilson, R.C., Hancock, E.R.: Coined quantum walks lift the cospectrality of graphs and trees. Pattern Recognit. 42, 1988–2002 (2009)
5. Foata, D., Zeilberger, D.: A combinatorial proof of Bass’ evaluations of the Ihara-Selberg zeta function for graphs. Trans. Am. Math. Soc. 351, 2257–2274 (1999)
6. Gamble, J.K., Friesen, M., Zhou, D., Joynt, R., Coppersmith, S.N.: Two particle quantum walks applied to the graph isomorphism problem. Phys. Rev. A 81, 52313 (2010)
7. Godsil, C., Guo, K.: Quantum walks on regular graphs and eigenvalues. arXiv:1011.5460 (2010)
8. Hashimoto, K.: Zeta functions of finite graphs and representations of \(p\)-adic groups. In: Adv. Stud. Pure Math. vol.15, pp. 211–280, Academic Press, New York (1989)
9. Ihara, Y.: On discrete subgroups of the two by two projective linear group over \(p\)-adic fields. J. Math. Soc. Jpn. 18, 219–235 (1966)
10. Kempe, J.: Quantum random walks: an introductory overview. Contemp. Phys. 44, 307–327 (2003)
11. Kendon, V.: Decoherence in quantum walks: a review. Math. Struct. Comput. 17, 1169–1220 (2007)
12. Konno, N.: Quantum walks. In: Lecture notes in mathematics. vol. 1954, pp. 309–452, Springer, Heidelberg (2008)
13. Kotani, M., Sunada, T.: Zeta functions of finite graphs. J. Math. Sci. Univ. Tokyo 7, 7–25 (2000)
14. Ren, P., Aleksic, T., Emms, D., Wilson, R.C., Hancock, E.R.: Quantum walks, Ihara zeta functions and cospectrality in regular graphs. Quantum Inf. Proc. 10, 405–417 (2011)
15. Sato, I.: A new Bartholdi zeta function of a graph. Int. J. Algebra 1, 269–281 (2007)
16. Serre, J.-P.: Trees. Springer, New York (1980)
17. Shiau, S.-Y., Joynt, R., Coppersmith, S.N.: Physically-motivated dynamical algorithms for the graph isomorphism problem. Quantum Inf. Comput. 5, 492–506 (2005)
18. Stark, H.M., Terras, A.A.: Zeta functions of finite graphs and coverings. Adv. Math. 121, 124–165 (1996)
19. Sunada, T.: \(L\)-Functions in geometry and some applications. In: Lecture notes in mathematics. vol. 1201, pp. 266–284, Springer, New York (1986)
20. Sunada, T.: Fundamental Groups and Laplacians (in Japanese). Kinokuniya, Tokyo (1988)