Extended covariance under nonlinear canonical transformations in Weyl quantization

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Abstract

A theory of non-unitary-invertible as well as unitary canonical transformations is formulated in the context of Weyl’s phase space representations. Exact solutions of the transformation kernels and the phase space propagators are given for the three fundamental canonical maps as linear, gauge and contact (point) transformations. Under the nonlinear maps a phase space representation is mapped to another phase space representation thereby extending the standard concept of covariance. This extended covariance allows Dirac-Jordan transformation theory to naturally emerge from the Hilbert space representations of the Weyl quantization.
Non-linear canonical transformations (CT) played a crucial role in the context of transformation theory in the historical development of quantum mechanics. So profound the contribution of the transformation theory to the fundamental understanding of quantum mechanics is that it is just to compare it to the beginning of a new phase in analytical dynamics initiated by Poisson in the generalized coordinates and later by Jacobi, Poincaré, Appell and Hamilton in the development of the canonical formalism. While the development in the early phases of quantum mechanics was characterized by the configuration and phase space approaches, its later elaborations led to the conception of abstract Hilbert space through which the formerly important transformation theory approach lost its momentum. Contrary to the case with the well-formulated linear canonical transformations, formulating the nonlinear ones is made more challenging by such general problems as invertibility and uniqueness, unitarity versus non-unitarity, complicated \( \hbar \) dependences and, in many cases, the non-existence of the transformation generators in connection with the absence of the identity limit. They mediate a unique language with the path integral quantization at one extreme and the Fresnel’s geometrical optics on the other. Their unitary representations were first treated by Dirac as a first step towards the path integral quantization. In 1927 Weyl introduced a new (de)-quantization scheme based on a generalized operator Fourier correspondence between an operator \( \hat{F} = \mathcal{F}(\hat{p}, \hat{q}) \) and a phase space function \( f(p, q) \). To observe the Dirac correspondence as a special case, Weyl restricted the space of the operator to the Hilbert-Schmidt space where monomials such as \( \hat{p}^m \hat{q}^n \) acquire finite norm for all \( 0 \leq m, n \). Weyl’s formalism was then extended by the independent works of von Neumann, Wigner, Groenewold and Moyal to a general phase space correspondence principle. There has been some reviving interest in the nonlinear quantum canonical transformations and their classical limits. The goal of this paper is to formulate the phase space representations of the nonlinear quantum CT within Weyl’s (de)-quantization scheme. In addition, it is also shown that the Weyl quantization allows (contrary to some conventional belief, see Ref. [6]) a restricted covariance under certain types of nonlinear CTs. In the beginning of section II Weyl quantization is outlined and the relation between the canonical Moyal and Poisson brackets is examined. In part A therein the generators of CT are examined in the phase space in a general perspective where a new covariance of the phase space representations under nonlinear CT is introduced mainly for unitary transformations. In section III the integral (both in phase and Hilbert spaces) of the CT are introduced and applied to concrete examples. Section IV examines the covariance under nonlinear CT by a phase space propagator approach. In section V we show that the Weyl formalism admits phase space representations of non-unitary (invertible) CT as well. Section VI is where the exact solutions of linear symplectic, gauge and contact transformations are given and applied to a few physical examples. A large part of this section is devoted to the exact formulation of the contact CT.
II. WEYL QUANTIZATION AND CANONICAL TRANSFORMS

According to the Weyl scheme a Hilbert-Schmidt operator $\hat{F}$ is mapped one-to-one and onto to a phase space function $f(p, q)$ as

$$f(p, q) = Tr\{\hat{\Delta}(p, q) \hat{F}\} , \quad \hat{F} = \int d\mu(p, q) f(p, q) \hat{\Delta}(p, q)$$

where

$$\hat{\Delta}(p, q) = \int d\mu(\alpha, \beta) e^{-i(\alpha p + \beta q)/\hbar} e^{i(\alpha p + \beta q)/\hbar}$$

is an operator basis satisfying all the necessary conditions of completeness and orthogonality of the generalized Fourier operator expansion. The phase space function $f(p, q)$ is often referred to as the symbol of $\hat{F}$. The operator product corresponds to the non-commutative, associative star product

$$\hat{F} \hat{G} \quad \Leftrightarrow \quad f \star g$$

$$\hat{F} \hat{G} \hat{H} \quad \Leftrightarrow \quad f \star g \star h$$

where $\hat{F}, \hat{G}, \hat{H}$ and their respective symbols $f, g, h$ are defined by (1) and (2). The star product is a formal exponentiation of the Poisson bracket $\hat{D}(p, q)$ as

$$\star_{(p, q)} \equiv \exp\{\frac{i\hbar}{2} \hat{D}(p, q)\} = \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^n \frac{1}{n!} [\hat{D}(p, q)]^n , \quad \hat{D}(p, q) = \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q}$$

where the arrows indicate the direction that the partial derivatives act. Unless specified by arrows as in (3), their action is implied to be on the functions on their right. According to (3) and (4) the symbol of the commutator is defined by the Moyal bracket $[\hat{F}, \hat{G}] \quad \Leftrightarrow \quad \{f(p, q), g(p, q)\}_{q,p}^{(M)} = f \star_{q,p} g - g \star_{q,p} f$ which is of crucial importance in deformation quantization. In deformation quantization, the Moyal bracket is a representation of the quantum commutator in terms of a nonlinear partial differential operator and at the same time an $\hbar$-deformation of the classical Poisson bracket. The canonical commutation relation (CCR) between the canonical operators, say $\hat{P}, \hat{Q}$, is represented by the symbols of these operators respectively denoted by $P(p, q), Q(p, q)$, satisfying

$$[\hat{P}, \hat{Q}] = -i\hbar \quad \Rightarrow \quad \{P, Q\}_{q,p}^{(M)} = 2 \sum_{k=0}^{\infty} \left(\frac{i\hbar}{2}\right)^{2k+1} \frac{1}{(2k+1)!} P(p, q) \left[\hat{D}(q, p)\right]^{2k+1} Q(p, q)$$

$$= -i\hbar .$$

It is well known that, a large class of CT can be represented by not only unitary but also non-unitary (and invertible) operators whose action preserve the CCR. Counter examples to unitary transformations are abound and some of the distinct ones are connected with the multi-valued (non-invertable) or domain non-preserving (non-unitary and invertible) operators. A few examples can be given by the polar-phase-space (i.e. action-angle) and quantum Liouville transformation which are multi-valued transformations, or Darboux type transformations between iso-spectral Hamiltonians.
In this paper, we will reformulate the quantum canonical (unitary as well as non-unitary) transformations within the Weyl formalism paying specific attention to a particular class of these transformations in which both the old and the new phase space variables are independent of $\hbar$. The importance of this particular class is that, thinking of $\hbar$ as a free parameter, and restricting to the case in which $P(p,q)$ and $Q(p,q)$ are $\hbar$–independent, the only non-zero contribution to the $\hbar$ expansion of the canonical Moyal bracket in (3) is the first (i.e. $k = 0$) term

$$\{P, Q\}_q^p(M) = i\hbar \{P, Q\}^p_q + \mathcal{O}(\hbar^{2k+1})\bigg|_{1\leq k} \mapsto -i\hbar$$

yielding

$$\{P, Q\}_q^p(M) = i\hbar \{P, Q\}^p_q = -i\hbar,$$

where all $\mathcal{O}(\hbar^{2k+1})$ terms with $1 \leq k$ necessarily vanish. In Eq.’s (7) and (8) the superscript $P$ stands for the Poisson bracket. We also observe that (8) holds between the canonical pairs, whereas it is not generally true for arbitrary functions $f(p,q)$ and $g(p,q)$. Eq. (8) states an equivalence between the canonical Moyal and the canonical Poisson brackets for $\hbar$ independent transformations. In order to show that Eq. (8) is a restricted subclass of that \(\hat{p}\) valued phase space moments $\hat{\psi}$ functions decay sufficiently strongly at the boundaries to admit an infinite set of finite restrictions may arise if the size of the canonical algebra induced by these three generators is smaller than the full space of canonical transformations.

The result in (9) implies that an $\hbar$ independent quantum CT is also a classical CT, a result that was obtained by Jordan long time ago using a semiclassical approach. The converse of that $\hbar$-independent quantum CT implies classical CT is not always true. On the other hand, as will be observed in section VI that the restriction imposed by Eq. (9) is not severe as one observes that the three elementary classical transformations, i.e. linear, gauge and the contact transformations can generate an infinite number of varieties respecting Eq. (9). The restrictions may arise if the size of the canonical algebra induced by these three generators is smaller than the full space of canonical transformations.

A. The phase space images of canonical transformations

The Weyl formalism is restricted to a subspace of the Hilbert space in which the state functions decay sufficiently strongly at the boundaries to admit an infinite set of finite valued phase space moments $\hat{p}^m \hat{q}^n$ with non-negative integers $m,n$. If the moments are symmetrically ordered we denote them by $t_{m,n} = \{\hat{p}^m \hat{q}^n\}$. The $t_{m,n}$’s are simpler to represent
in the phase space and they correspond to the monomials $p^m q^n$. A function $f(p, q)$ which can be written as a double Taylor expansion in terms of the monomials $p^m q^n$ corresponds to a symmetrically ordered expansion of an operator $\hat{F}$ as

$$f(p, q) = \sum_{0 \leq (m,n)} f_{m,n} p^m q^n \quad \leftrightarrow \quad \hat{F} = \sum_{0 \leq (m,n)} f_{m,n} \hat{t}_{m,n}^{(0)}.$$  \hfill (10)

Symmetrically ordered monomials are Hermitian and they can be convenient in the expansion of other Hermitian operators. From now on we use the symmetrical ordering, unless specified.

The phase space representations are more convenient to use than operator algebra for keeping track of $\hbar$‘s. Since $\hat{t}_{m,n} \iff p^m q^n$, $\hbar$ dependences appear only in the phase space expansions representing non-symmetrical monomials. Suppose that the operator $\hat{F}$, which has the Weyl representation $f(p, q)$, is transformed by an operator $\hat{U}$ which has the Weyl representation $u(p, q)$ by $\hat{F}' = \hat{U}^{-1} \hat{F} \hat{U}$. Assume that the transformation is given in an exponential form $\hat{U}_A = e^{i\gamma \hat{A}}$ where $\hat{A} = A(\hat{p}, \hat{q})$ also has Weyl representation $a(p, q)$. Consider

$$A(\hat{p}, \hat{q}) = \sum_{n,m,r} a_{n,m,r} \hat{p}^n \hat{q}^m \hat{p}^r$$ \hfill (11)

where $a_{n,m,r}$ are some coefficients. We have

$$f'(p, q) = Tr\{ \hat{F}' \hat{\Delta} \} = Tr\{ \hat{F} \hat{U}_A \hat{\Delta} \hat{U}_A^{-1} \}$$

$$\hat{U}_A \hat{\Delta} \hat{U}_A^{-1} = \hat{\Delta} + (i\gamma) [\hat{A}, \hat{\Delta}] + \frac{(i\gamma)^2}{2!} [\hat{A}, [\hat{A}, \hat{\Delta}]] + \ldots$$ \hfill (12)

The right hand side of (12) can be represented by certain linear first order phase space differential operators producing the left and right action of $\hat{p}$ and $\hat{q}$ on $\hat{\Delta}$ as

$$\hat{p} \hat{\Delta}(p, q) = \left[ p + \frac{i\hbar}{2} \frac{\partial}{\partial q} \right] \hat{\Delta}(p, q), \quad \hat{\Delta}(p, q) \hat{p} = \left[ p - \frac{i\hbar}{2} \frac{\partial}{\partial q} \right] \hat{\Delta}(p, q),$$

$$\hat{q} \hat{\Delta}(p, q) = \left[ q - \frac{i\hbar}{2} \frac{\partial}{\partial p} \right] \hat{\Delta}(p, q), \quad \hat{\Delta}(p, q) \hat{q} = \left[ q + \frac{i\hbar}{2} \frac{\partial}{\partial p} \right] \hat{\Delta}(p, q).$$ \hfill (13)

Using Eq. (13), the first commutator in the expansion in (12) becomes

$$[\hat{\Delta}, \hat{\Delta}] = \sum_{n,m,r} a_{n,m,r} \left\{ \hat{p}_L^r \hat{q}_L^m \hat{p}_L^n - \hat{p}_R^r \hat{q}_R^m \hat{p}_R^n \right\} \hat{\Delta}(p, q)$$

$$\equiv \hat{V}_A^{(-)} \hat{\Delta}.$$ \hfill (14)

Note that, the orderings with respect to $(\hat{p}, \hat{q})$ and $(\hat{p}_L, \hat{q}_L)$ are opposite and those with respect to $(\hat{p}, \hat{q})$ and $(\hat{p}_R, \hat{q}_R)$ are the same. The right hand side of (12) can be obtained by infinitely iterating the commutator (14) which yields

$$\hat{U}_A \hat{\Delta} \hat{U}_A^{-1} = e^{i\gamma \hat{V}_A^{(-)}} \hat{\Delta}.$$ \hfill (15)
Using Eq. (13) in (12)  

\[
f'(p,q) = e^{i\gamma \hat{V}^{(-)}} f(p,q).
\] 

The action of the operator \( \hat{V}^{(-)} \) on \( \hat{\Delta} \) reproduces the commutator \([\hat{\Delta}, \hat{\Delta}]\). We denote it by \( \hat{A} \leftrightarrow \hat{V}^{(-)} \). It is also a linear vector space, i.e. if \( \hat{C} = \alpha \hat{A} + \beta \hat{B} \leftrightarrow \hat{V}^{(-)} = \alpha \hat{V}^{(-)} + \beta \hat{V}^{(-)} \). Using the Jacobi identity for \( \hat{V}^{(-)} = \hat{C}^{(-)} \), we have

\[
[\hat{A}, \hat{B}] \mapsto \hat{V}^{(-)}[\hat{A}, \hat{B}] = -[\hat{V}^{(-)} \hat{A}, \hat{V}^{(-)} \hat{B}]
\] 

as the relation acts on \( \hat{\Delta} \) from the left. Hence, if \( \hat{A}_i \) are generators of a Lie algebra then their images \( \hat{V}^{(-)} \) are generators of the image Lie algebra. There is a simple formulation of such transformations when \( \hat{A} \) in Eq. (14) is the symmetric monomial \( \hat{t}_{m,n} \) or a sum of such terms. In this case, the phase space operator \( \hat{V}^{(-)} \) is also a symmetrically ordered function of \( \hat{p}_L, \hat{q}_R, \hat{q}_L, \hat{q}_R \). We then have

\[
\hat{t}_{m,n} \hat{\Delta}(p,q) = \{\hat{p}_L^m \hat{q}_L^n\} \hat{\Delta}(p,q), \quad \hat{\Delta}(p,q) \hat{t}_{m,n} = \{\hat{p}_R^m \hat{q}_R^n\} \hat{\Delta}(p,q)
\] 

therefore

\[
[\hat{t}_{m,n}, \hat{\Delta}(p,q)] = \left[\{\hat{p}_L^m \hat{q}_L^n\} - \{\hat{p}_R^m \hat{q}_R^n\}\right] \hat{\Delta}(p,q) \equiv \hat{S}_{m,n}(p,q)
\]

where we used the specific notation \( \hat{S}_{m,n} \) for the image of the symmetric monomials \( \hat{t}_{m,n} \). By an infinite iteration\(^\text{14}\) Eq. (13) can be cast into

\[
\hat{U}_{m,n} \hat{\Delta} \hat{U}_{m,n}^{-1} = \exp\{i \gamma_{m,n} \hat{S}_{m,n}(p,q)\} \hat{\Delta}, \quad \text{where} \quad \hat{U}_{m,n}(\gamma_{m,n}) = \exp\{i \gamma_{m,n} \hat{t}_{m,n}\}
\]

and \( \gamma_{m,n} \in \mathbb{C} \) is completely arbitrary. Note that the symmetrical monomials \( \hat{t}_{m,n} \) and hence the differential left and right generators \( \{\hat{p}_L^m \hat{q}_L^n\} \) and \( \{\hat{p}_R^m \hat{q}_R^n\} \) are Hermitian. Therefore for real \( \gamma_{m,n} \) Eq. (20) implies unitarity. Also note that \( [\hat{p}_L, \hat{q}_L] = [\hat{q}_R, \hat{p}_R] = i \hbar \) and all other commutators between the left and right operators vanish.

The Weyl correspondence including the covariance under canonical transformations can now be summarized in the commuting diagram

\[
\begin{array}{c}
\hat{V}^{(-)} \\
\hat{\Delta} \\
\hat{F}^{-1} \\
\hat{U}_{\hat{A}} \\
\hat{F} \\
\end{array}
\begin{array}{c}
f(p,q) \quad \overset{Weyl}{\leftrightarrow} \\
\hat{V}^{(-)} f \\
\hat{F}^{-1} \\
\hat{U}_{\hat{A}} \\
\hat{F} \\
\end{array}
\]

The meaning of the diagram (21) can be facilitated by an example. Consider, for instance, the unitary transformation corresponding to \( \hat{U}_{2,1} \). Using Eq. (13) and (13) we find the corresponding differential generator \( \hat{S}^{(-)}_{2,1} \) as

\[
\hat{S}^{(-)}_{2,1} = i\hbar (2pq \frac{\partial}{\partial q} - p^2 \frac{\partial}{\partial p})
\]

which has an explicit overall \( \hbar \) dependence. Also note that \( \hat{S}^{(-)}_{2,1} \) is an Hamiltonian vector field. For any \( f(p,q) \) its action gives the Poisson bracket \( \hat{S}^{(-)}_{2,1} f(p,q) = \{f(p,q), p^2 q\}(F) \).
Let us consider for $f$ and $f'$ in the diagram (21) the canonical coordinates $(p, q)$ and $(P, Q)$. Then, using Eq. (22)

$$P(p, q) = e^{-i\gamma S_{2,1}^{(-)}} p = \frac{p}{1 + \gamma \hbar p}, \quad Q(p, q) = e^{-i\gamma S_{2,1}^{(-)}} q = q (1 + \gamma \hbar p)^2. \quad (23)$$

As the $\hbar$ dependence of Eq. (23) is concerned, it can be scaled out by redefining the free transformation parameter as $\gamma \to \gamma_{2,1}/\hbar$. It can be directly observed that the canonical transformation in Eq. (23) respects (8) and (9).

### III. INTEGRAL REPRESENTATIONS

Eq. (1) can be used for an invertible transformation $\hat{U}$ as

$$\hat{U} = \int d\mu(p, q) u(p, q) \hat{\Delta}(p, q). \quad (24)$$

If $\hat{U}$ is a unitary operator then $u(p, q)$ satisfies $u^*(p, q) = u^{-1}(p, q)$ where $*$ denotes the complex conjugation and the $u^{-1}$ is the symbol representing $\hat{U}^{-1}$ in the phase space. The inverse here is defined with respect to the star product, i.e. $u^* u^{-1} = u^{-1} u = 1$. The inner product is defined by

$$(\psi, \hat{U} \phi) = \int dq \psi^* (q) (\hat{U} \phi)(q). \quad (25)$$

Inserting $\psi^*(q) = \delta(q - y)$ in (25) and using the matrix elements $\langle y | \hat{\Delta}(p, q) | x \rangle$ one has the integral coordinate representations

$$(\hat{U} \varphi)(y) = \int dx e^{iF(y, x)} \varphi(x), \quad e^{iF(y, x)} = \int \frac{dp}{2\pi\hbar} e^{-ip(x-y)/\hbar} u(p, \frac{x+y}{2}) \quad (26)$$

or the mixed representation

$$(\hat{U} \varphi)(y) = \int \frac{dp_x}{2\pi\hbar} e^{iK(y, p_x)} \tilde{\varphi}(p_x), \quad e^{iK(y, p_x)} = \int dx e^{i[F(y, x) - p_x/\hbar]} \quad (27)$$

alternatively, the integral momentum representations,

$$(\hat{U} \varphi)(p_y) = \int \frac{dp_x}{2\pi\hbar} e^{iH(p_y, p_x)} \tilde{\varphi}(p_x), \quad e^{iH(p_y, p_x)} = \int dq e^{i[p_xq_y - p_yq]/\hbar} u(p_y + p_x/2, q) \quad (28)$$

or the other mixed case

$$(\hat{U} \varphi)(p_y) = \int dx e^{iL(p_y, x)} \varphi(x), \quad e^{iL(p_y, x)} = \int \frac{dp_x}{2\pi\hbar} e^{i[H(p_y, p_x) + xpx/\hbar]} \quad (29)$$

Note, that we have not assumed anything particular concerning unitarity or non-unitarity of $\hat{U}$. We now assume that $\hat{U}$ produces the canonical transformation

$$\hat{P} = \hat{U}^{-1} \hat{p} \hat{U}, \quad \hat{Q} = \hat{U}^{-1} \hat{q} \hat{U} \quad (30)$$

multiplying both sides by $\hat{U}$ on the left and using the correspondence in Eq. (3) we find
where \( \star = \star_{q,p} \) as defined in (3). Eq’s (31) and (32) are for most cases, highly nonlinear infinite order, pde’s and their exact solutions can only be found for a finite number of transformations.

Let us examine Eq’s (31) and (32) for a few well known cases. We first do it for the group of linear symplectic transformations \( SL_2(\mathbb{R}) \).

a) \( SL_2(\mathbb{R}) \):

In this case we have

\[
\begin{pmatrix} P \\ Q \end{pmatrix} = g \begin{pmatrix} p \\ q \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
\]

Directly using (33) in (31) and (32) one has

\[
u(p,q) = 2 \sqrt{a+d+2} \exp\left\{ \frac{2i}{a+d+2} \left[ b q^2 - c p^2 + (a-d)pq \right] \right\}, \quad Trg \neq -2
\]

where the normalization is chosen such that identity transformation corresponds to unity. By (26) this can be converted into the wavefunction transforming kernel

\[
e^{iF(y,x)} = \frac{e^{-i\pi/4}}{\sqrt{2\pi \hbar c}} e^{i\frac{a}{\hbar c}(ax^2+dx^2-2xy)}
\]

yielding the correct integral kernel for \( SL_2(\mathbb{R}) \) transformation including the normalization factor. The special cases such as \( Trg = -2 \) can be treated with additional limiting procedures which will not be considered here.

b) Linear Potential:

The second exactly solvable system is the linear potential model

\[
\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} p \\ q+ap^2 \end{pmatrix}, \quad a \in \mathbb{R}
\]

using (31) and (32) once more we find,

\[
u(p,q) = N_a \exp\left( -\frac{ia}{3\hbar} p^3 \right), \quad N_a \bigg|_{a=0} = 1
\]

which is more conveniently used in a mixed type of transformation kernel given by Eq. (27) as

\[
e^{iK(y,p_x)} = e^{\frac{a}{\hbar} \left( y p_x - \frac{a}{2} p_x^2 \right)}
\]

where \( N_a = 1 \) is used, yielding the correct solution of the linear potential model.

In both examples the unitary transformation kernel \( u(p,q) \) is closely related to the appropriate classical generating function of the canonical transform as remarked by Dirac. In the
early days of the quantum theory. A close look into (35) as well as (38) confirms that they are exponentiated versions of one of the four types of generating functions that one learns in the textbooks. Indeed, (35) is, after renaming $y \rightarrow Q$ and $x \rightarrow q$ as the new and the old coordinates

$$F(Q, q) = \frac{1}{2c} (a Q^2 + d q^2 - 2Qq),$$

(39)

which is just the classical generating function $F_{cl}(Q, q)$ for the linear symplectic transformations satisfying $p = \partial F_{cl}(Q, q) / \partial q$ and $P = -\partial F_{cl}(Q, q) / \partial Q$. However, a closer analogy can be hindered especially for those transformations which are infinitesimally close to the identity and the contact transformations (see section VI). Classically, $F_{cl}(Q, q)$ cannot be directly derived for both these cases by solving the partial differential equations written a few lines above. Despite that $F_{cl}(Q, q)$ is not defined, Eq. (35) is correctly described by a delta function at the identity limit

$$\lim_{g \rightarrow I} e^{iF(Q,q)} = \delta(Q - q) .$$

(40)

In section VI we examine the contact transformations for which $e^{iF(Q,q)}$ is always represented as a delta function [see Eq. (35)].

Returning to the linear potential model, the classical generator $K_{cl}(Q, p)$ is found from the equations $q = \partial K_{cl}(Q, p) / \partial p$ and $P = \partial K_{cl}(Q, p) / \partial Q$ and it has a well defined identity limit. Eq. (38) that was found for the linear potential model matches exactly with the exponentiated classical generator and agrees with Dirac’s exponentiation formula.

It was claimed that a finite CT can be written as a finite decomposition of the three elementary CT’s which are the phase space rotations (fractional Fourier $\hat{U}_F$), gauge (i.e. $\hat{U}_G$) and the contact (i.e. $\hat{U}_{ct}$) CT’s. For each of these elementary transformations Eq’s (31) and (32) have exact and $\hbar$-uncorrected [see Eq. (44) below] solutions $u_F, u_G$ and $u_{ct}$ respectively. Solving for an arbitrary finite CT is then equivalent to finding its correct finite decomposition in terms of an ordered star product of the elementary transformations which, may have the following pattern, $u = u_{F_1} \star u_{G_1} \star u_{F_2} \star u_{ct_1} \ldots$. This decomposition tremendously simplifies the solution of (31) and (32) where otherwise no exact solutions may be possible by direct computation. A simple example is

$$\left(\begin{array}{c} P \\ Q \end{array} \right) = \left(\begin{array}{c} -q \\ p + aq^2 \end{array} \right), \quad a \in \mathbb{R}$$

(41)

which involves a composite action of an initial Fourier transformation and the transformation in (38) as

$$\left(\begin{array}{c} p \\ q \end{array} \right) \overset{\text{Fourier}}{\rightarrow} \left(\begin{array}{c} -q \\ p \end{array} \right) \overset{\text{LP}}{\rightarrow} \text{Eq. (41)}$$

(42)

where LP stands for the linear potential. A direct attempt to solve for the kernel $u(p, q)$ corresponding to Eq. (11) produces a finite (and in more general cases an infinite) series of complicated $\hbar$-corrections. On the other hand we know that the solution is

$$u(p, q) = u_{LP}(p, q) \star u_{F}(p, q) .$$

(43)
where \( u_{LP} \) is given by Eq. (37) and \( u_F \) is found from Eq. (34) by inserting \( a = d = 0, b = -c = 1 \).

Eq. (1) provides some background we need in order to understand the solutions of (31) and (32) for the class of problems for which \( u(p, q) \) has no \( \hbar \) corrections. The \( \hbar \)-corrections to the CT generators was analysed in Ref.[15] in reference to a particular Hamiltonian. This concept can be made independent of a dynamical model by demanding that the solution of (31) and (32) yields integral kernels \( F, K, H, L \) in (26)-(29) which are all in the order of \( 1/\hbar \) independent from any class of Hamiltonians implied by

\[ u(p, q) = e^{2i\hbar T(p, q)}, \quad \frac{\partial T}{\partial \hbar} = 0 \]  

(44)

hence \( T(p, q) \) has no \( \hbar \) dependence. For the linear symplectic group [see Eq. (35)] and the gauge transformations [see Eq. (51) below] Eq. (14) is manifest. For the contact transformations there is a \( \hbar \)-correction to \( T(p, q) \) which is linear in \( \hbar \) (see section VI) and accounting for the weight factors introduced by the change of coordinate \( q \rightarrow Q(q) \).

By inspecting Eqs (31) and (32) one expects to find that the particular class of transformations for which

\[ u(p, q) *_{q, p} Q(p, q) = u(p, q) *_{q, p} Q, \]

\[ u(p, q) *_{q, p} P(p, q) = u(p, q) *_{q, p} P \]  

(45)

(46)

holds, yields \( \hbar \)-uncorrected solutions as in Eq. (44) for \( u(p, q) \). It is intuitive that the conditions in (45) and (46) are sufficient but not necessary for the \( \hbar \)-uncorrected solutions in (44). If Eq’s (45) and (46) hold, then

\[ (Q - \frac{i\hbar}{2} \frac{\partial p}{\partial q}) u(p, q) = (q + \frac{i\hbar}{2} \frac{\partial p}{\partial q}) u(p, q) \]  

(47)

\[ (P + \frac{i\hbar}{2} \frac{\partial q}{\partial p}) u(p, q) = (p - \frac{i\hbar}{2} \frac{\partial q}{\partial p}) u(p, q). \]  

(48)

Considering the general form in (44) the solution is

\[ \left( \frac{\partial}{\partial q} \right) T = \left( 2 + \partial_P p + \partial_Q q \right)^{-1} \left( \begin{array}{cc} 1 + \partial_Q q & -\partial_P q \\ -\partial_Q p & 1 + \partial_P p \end{array} \right) \left( \begin{array}{c} q - Q \\ P - p \end{array} \right) \]  

(49)

where it is required that all derivatives are finally expressed in \( (p, q) \) and the determinant of the matrix \( (2 + \partial_P p + \partial_Q q) \) is non-zero. The solution to (43) is clearly \( \hbar \) independent if the canonical transformation \( (p, q) \rightarrow (P, Q) \) is also independent of \( \hbar \). Eq’s (44) and (45) are manifestly satisfied for the linear symplectic transformations in Eq. (33). We next examine the CT generated by a gauge transformation

\[ \hat{U}_{0,.} = e^{i\tau \int^q dx A(x)} : \hat{p} \rightarrow \hat{p} - \tau \hbar A(q) \quad \rightarrow \quad P(p, q) = p - \tau \hbar A(q) \]  

\[ : \hat{q} \rightarrow \hat{q} \quad \rightarrow \quad Q(p, q) = q \]  

(50)

for which the matrix in (43) is simplified. Since \( \tau \) is a free parameter, it can be scaled: \( \tau \hbar \rightarrow \tau \). Inserting (50) in (44) we find

\[ T(p, q) = \frac{\tau}{2} \int^q dx A(x) \quad \rightarrow \quad u(p, q) = e^{i\tau \int^q dx A(x)}. \]  

(51)
The transformed wavefunction is then found by the use of (26) as
\[(\hat{U}_0, \varphi)(y) = e^{i\bar{h}\tau} \int^y dx A(x) \varphi(x)\] (52)
which is what one expects to find\[13\].

It is clear that Eq's (47) and (48) considerably increase the power of \((31)\) and \((32)\). However for some other transformations Eq’s (45) and (46) do not hold even when there are no \(\hbar\)-corrections in the solution of \(u(p,q)\). The contact transformations are good examples to these type of solutions for which we refer to section VI.

**IV. THE PHASE SPACE PROPAGATOR**

Let us assume that \(F(\hat{p}, \hat{q})\) is an operator and \(F'(\hat{P}, \hat{Q}) = F'(\hat{p}, \hat{q})\) describes a transformation of it under \(\hat{U} : \hat{p} \mapsto \hat{P}\) and \(\hat{U} : \hat{q} \mapsto \hat{Q}\). Then
\[
\mathcal{F}(\hat{p}, \hat{q}) = \sum_{m,n} f_{m,n} \hat{t}_{m,n} \quad \iff \quad \mathcal{F}(\hat{P}, \hat{Q}) = \sum_{m,n} f_{m,n} \hat{T}_{m,n}
\] (53)

where \(\hat{t}_{m,n} = \{\hat{p}^m \hat{q}^n\}\) and \(\hat{T}_{m,n} = \{\hat{P}^m \hat{Q}^n\}\) are symmetrically ordered monomials. Hence, the new Hamiltonian comes out of the transformation naturally as symmetrically ordered under the new operators. It is clear that \(\mathcal{F}(\hat{P}, \hat{Q})\) is not generally ordered symmetrically in the old operators \(\hat{p}, \hat{q}\). In principle, \(f'(p,q)\) can be found once \(f(p,q)\) and the transformation is known by the use of Eq. (16).

Another method to describe the CT is to define a *phase space propagator* \(G\) to relate \(f'\) and \(f\) as
\[
f'(p,q) = \int d\mu(s,t) G(p,q; s,t) f(s,t)
\] (54)
as discussed in Ref. [9]. Using (16) we find an equation of motion for the propagator
\[
e^{-i\gamma \hat{V}(-)} G(p,q; s,t) = (2\pi\hbar) \delta(p - s) \delta(q - t) .
\] (55)

Writing the delta functions in terms of the Cauchy integrals and inverting the operator on the left hand side, the propagator is
\[
G(p,q; s,t) = e^{i\gamma \hat{V}(-)} \int d\mu(x,k) e^{\hat{\bar{h}}^+[p-s]x+(q-t)k}
\] (56)
where \(e^{i\gamma \hat{V}(-)}\) is the image of the transformation as described in Eq. (16) which acts as a phase space evolution operator for the propagator. Eq. (56) determines the kernel if the transformation \((p,q) \mapsto (P,Q)\) is known. An important observation is that, the kernel in (56) is independent from the functions \(f', f\) and is uniquely given by the canonical transformation itself. Consider, for instance, the CT given by \(\hat{U}_{3,0}\) corresponding to \(m = 3, n = 0\) generating (36). The phase space CT generator is, by (19)
\[
\hat{S}_{3,0} = \hat{p}_L^3 - \hat{p}_R^3 = i\hbar (3p^2 \partial_q - \frac{\hbar^2}{4} \partial^3_q)
\] (57)
where \( \gamma_{3,0} = -a/3\hbar \). Inserting (57) into (56) one finds

\[
G(p, q; s, t) = \int d\mu(x, k) e^{\frac{i}{\hbar}[(p-s)x+(q-t)k]} e^{\frac{i}{\hbar}k(q+a_p^2)}
= \delta(p - s) \int dk e^{\frac{i}{\hbar}[(q-t+a_p^2)k+\frac{ak^3}{12}]} \quad (58)
\]

We demonstrate the use of the propagator in a concrete example by considering \( f(p, q) \) as the Wigner function \( W_\psi(p, q) \) of the state \( \psi \) where in the coordinate and momentum representations, respectively,

\[
W_\psi(p, q) = \int_\mathbb{R} dx e^{-\frac{i}{\hbar}px} \psi^*(q - \frac{x}{2}) \psi(q + \frac{x}{2}) = \int_\mathbb{R} \frac{dk}{2\pi\hbar} e^{\frac{i}{\hbar}k(q+a_p^2)} e^{i \frac{ak^3}{12} \psi^*(p + \frac{k}{2})} \tilde{\psi}(p - \frac{k}{2}). \quad (59)
\]

The Wigner function in (55) being the Weyl symbol of the quantum mechanical density matrix is subject to Eq. (54) under the action of a canonical transformations. The representation independent form of Eq's (59) is given by

\[
W_\psi(p, q) = Tr \{ \hat{A}(p, q) | \psi \rangle \langle \psi | \} \quad (60)
\]

Defining a transformed state \( \Psi \) as

\[
| \Psi \rangle = \hat{U}_{3,0}^\dagger | \psi \rangle \quad (61)
\]

and using (54) and (59)

\[
W_\Psi(p, q) = e^{-\frac{i}{\hbar}\delta_{3,0}} W_\psi(p, q) = e^{i(p^2 \hbar \frac{k^2}{12} \delta_{3,0})} W_\psi(p, q)
= \int \frac{dk}{2\pi\hbar} e^{\frac{i}{\hbar}k(q+a_p^2)} e^{i \frac{ak^3}{12} \psi^*(p + \frac{k}{2})} \tilde{\psi}(p - \frac{k}{2})
= \int \frac{dk}{2\pi\hbar} e^{\frac{i}{\hbar}k[(q-t)+a_p^2]} e^{i \frac{ak^3}{12} W_\psi(p, t)}
= \int d\mu(s, t) G(p, q; s, t) W_\psi(s, t) \quad (62)
\]

where the propagator is given by (58). We now derive (62) by transforming the wavefunctions in (54) in momentum representation using (28). Starting from (61) and applying it in (28) we have

\[
\Psi(p_x) = (\hat{U}_{3,0} \psi)(p_x) = \int \frac{dp_x}{2\pi\hbar} e^{iH(p_x, p_x)} \psi(p_x), \quad e^{iH(p_y, p_x)} = 2\pi\hbar \delta(p_x - p_y) e^{-\frac{i}{\hbar}p_x^2} \quad (63)
\]

We use (53) in

\[
W_\Psi(p, q) = W_{\psi, \psi}^0(p, q) = \int \frac{dk}{2\pi\hbar} e^{\frac{i}{\hbar}kq} \tilde{\psi}^*(p + \frac{k}{2}) \tilde{\psi}(p - \frac{k}{2}) \quad (64)
\]

to write (54) as

\[
W_\Psi(p, q) = \int \frac{dv}{2\pi\hbar} e^{\frac{i}{\hbar}qv} \left[ \int dp_x \delta(p_x - p - v/2) e^{\frac{iv}{\hbar}p_x^2} \right]
\left[ \int dk_x \delta(k_x - p + v/2) e^{-\frac{iv}{\hbar}k_x^2} \right] \tilde{\psi}^*(p + v/2) \tilde{\psi}(p - v/2)
= \int \frac{dk_x}{2\pi\hbar} e^{\frac{i}{\hbar}k_x[(q-t)+a^2]} e^{i \frac{ak_x^3}{12} W_\psi(p, t)}
= \int d\mu(s, t) G(p, q; s, t) W_\psi(s, t) \quad (65)
\]
which is identical to (62).

A yet another method for deriving the phase space propagator exists and it is particularly useful when $\hat{U}$ is harder to find than its kernel $u$ and $u^{-1}$. Starting from Eq. (54) and $u(p, q) \star f'(p, q) = f(p, q) \star u(p, q)$, we find that

$$f(p, q) = \int d\mu(s, t) \left[ u \star G \star u^{-1} \right] f(s, t), \quad \Rightarrow \quad G = (2\pi\hbar)^{-1} \delta(p - s)\delta(q - t) \star u$$

(66)

where $\star = \star_{(q, p)}$ is implied. It is observed that the two methods implied by Eq. (56) and (66) are equivalent for general cases.

The results in this section indicate that the concept of phase space propagator is an alternative way in the formulation of nonlinear canonical transformations. The integral transformation between the two Wigner functions derived in Ref. [14] is a specific example of the phase space propagator studied here.

V. BEYOND UNITARITY

The canonical transformations are not restricted by unitarity. The similarity transformation of the type

$$\hat{F} \rightarrow \hat{F}' = \hat{C}^{-1} \hat{F} \hat{C},$$

(67)

with $\hat{F}$ and $\hat{F}'$ being the original and the transformed canonical phase space operators, preserves the canonical commutation relations if $\hat{C}$ is invertible.

The unitarity of the transformation $\hat{U}$ studied in the previous sections is also not required for the Weyl formalism. However, the unitarity can be useful to have in the representations of the operators in the Hilbert space since it preserves the inner product. The invertible transformations are used in constructing different representations of the same system. Within the same Hilbert space the representations are connected by unitary transformations. The non-unitary, invertible ones are needed to build equivalences between different Hilbert spaces (isometries). In this section we will discuss the extended Weyl correspondence of the formalism introduced in sections III and IV to include the non-unitary versions of $u(p, q)$ [which we denote by $c(p, q)$ after Eq. (67)] of which the transformation properties are derived from Eq. (67). The Weyl correspondence for $\hat{C}$ implies that

$$\hat{C} = \int d\mu(p, q) c(p, q) \hat{\Delta}(p, q)$$

(68)

where $c(p, q)$ is the solution of the same equations (31) and (32).

Let us consider the particular case

$$\hat{C}^{(\alpha)} = e^{\alpha \hat{D}} \quad \hat{D} = i\hat{p} + g(\hat{q})$$

(69)

which is an exponential extension of the first order Darboux transformation. It is well-known that $\hat{D}$ in (69) intertwines between two iso-spectral separable Hamiltonians second order in $\hat{p}$ if $g$ satisfies certain properties. Here we will assume no specific conditions on
$g$ other than the infinite differentiability and the real valuedness. Since $\hat{C}$ is not unitary $c^{(\alpha)}(p, q)$ does not respect the unitarity condition [i.e. in general $(c^{(\alpha)}(p, q))^* \ast c^{(\alpha)}(p, q) \neq 1$] and the inverse is well defined as $c^{(-\alpha)}(p, q)$. Given (67), the transformed phase space variables under the action of (69) are

$$P(p, q) = p - i G(q) , \quad Q = q + \alpha$$

(70)

where $G(x) = g(x) - g(x + \alpha)$. Observe that (70) respects (9). Inserting (70) into (31) and (32) one finds that $c^{(\alpha)}(p, q)$ satisfies

$$\partial_p c^{(\alpha)} = \frac{i\alpha}{\hbar} c^{(\alpha)}$$

$$G(q + \frac{i\hbar}{2} \partial_p) c^{(\alpha)} = -\hbar \partial_q c^{(\alpha)} .$$

(71)

The solution of Eq's (71) is easily found as

$$c^{(\alpha)}(p, q) = e^{\frac{i\hbar}{\alpha} p - i \int^q dz G(z - \alpha/2)} .$$

(72)

Using Eq's (26) one can find the integral kernel as

$$e^{iF(y, x)} = \delta(\alpha - (x - y)) e^{\frac{i\hbar}{\alpha} \int^y dz G(z - \alpha/2)} .$$

(73)

Knowing Eq. (73) is sufficient to write the transformation for functions $\varphi(x)$ as

$$(\hat{C}^{(\alpha)} \varphi)(y) = e^{-\frac{i\hbar}{\alpha} \int^y dz [g(z + \alpha/2) - g(z - \alpha/2)]} \varphi(\alpha + y) .$$

(74)

It is observed that the intertwining operator $\hat{D}$ is the first order term of (74) in $\alpha$ which is found by

$$\frac{d}{d\alpha} (\hat{C}^{(\alpha)} \varphi)(y) \bigg|_{\alpha=0} = \left[ \frac{d}{dy} + g(y) \right] \varphi(y) = \hat{D} \varphi(y) .$$

(75)

A different version of Eq. (74) can be written as

$$\hat{C}^{(\alpha)}_2 = e^{i\frac{i\hbar}{\alpha} \hat{p}} e^{\frac{i\hbar}{\alpha} B(\hat{q})} e^{i\frac{i\hbar}{\alpha} \hat{p}}$$

(76)

which is equivalent to Eq (33) when $B'(x) = [g(x + \alpha/2) - g(x - \alpha/2)]/\alpha$ where prime indicates derivate with respect to $x$. This version is more convenient than (33) because of the fact that it can be more appropriately represented by successive applications of $\hat{U}_{m,n}(\gamma_{m,n})$'s in Eq (20) as

$$\hat{C}^{(\alpha)}_2 = \hat{U}_{1,0}(\frac{\alpha}{2\hbar}) \hat{U}_{0,0}(-i \frac{\alpha}{\hbar}) \hat{U}_{1,0}(\frac{\alpha}{2\hbar}) .$$

(77)

The corresponding phase space transformation is

$$P(p, q) = p + i\alpha B'(q + \alpha/2) , \quad Q(p, q) = q + \alpha .$$

(78)

The solution of $c^{(\alpha)}_2(p, q)$ corresponding to $\hat{C}^{(\alpha)}_2$ is found separately for each transformation factor in (77) as
\[ c_2^{(\alpha)}(p, q) = e^{i\frac{\alpha}{\hbar} p} * e^{i\frac{\alpha}{\hbar} B(q)} * e^{i\frac{\alpha}{\hbar} p} = e^{i\frac{\alpha}{\hbar} [p-iB(q)]} \]  

(79)

which is in a simpler form than (72) above.

The integral kernel in Eq (54) corresponding to this non-unitary transformation can also be found. Applying (56)

\[
G(p, q; s, t) = e^{-i\frac{\alpha}{\hbar} \hat{S}(\cdot, \cdot)} e^{-i\frac{\alpha}{\hbar} B(\cdot, \cdot)} e^{-i\frac{\alpha}{\hbar} \hat{S}(\cdot, \cdot)} \int d\mu(x, k) e^{i\frac{\alpha}{\hbar} [(p-s)x+(q-t)k]} \]

(80)

One parameter invertible canonical transformations (unitary or non-unitary) are only a small class in the entire canonical space having the privilege of abelian exponential representations. Spectrum non-preserving invertible transformations fall outside this class and they may be represented by transformations without an infinitesimal limit which will be examined in the next section.

VI. A FINITE SET FOR CANONICAL TRANSFORMATIONS

Three types of invertible phase space maps have been suggested as the elementary generators of the entire invertible classical canonical transformations as a) Fourier transformation (and all its independent powers): \( p \mapsto -q \) and \( q \mapsto p \), b) Gauge transformations: \( p \mapsto p + A(q) \) and \( q \mapsto q \), c) Contact transformations \( p \mapsto p/Q'(q) \) and \( q \mapsto Q(q) \). Our purpose here is not to attempt a proof (or counterproof) of whether successive actions of (a), (b) and (c) can generate the complete domain of the invertible quantum CT. We will be confined to give the explicit solutions of the transformation kernels and the phase space propagators for these three cases.

a) Fourier transformation is realized as a special case of linear symplectic transformations in section III when the \( g \) matrix is given by the \( 2 \times 2 \) Fourier matrix \((a = 0, b = -1, c = 1, d = 0)\). The unitary generator for the Fourier transformation is given by

\[
\hat{U}_F = e^{-i\frac{\pi}{4} \hat{V}_F^{(-)}}
\]

(81)

for which we can find the unitary transformation kernel by direct substitution of the parameters of \( g \) in Eq. (34) as

\[
u(p, q) = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} (p^2+q^2)}
\]

(82)

To find the phase space generator \( \hat{V}_F^{(-)} \) we apply the same method as Eq’s (13)-(20) yielding

\[
\hat{V}_F^{(-)} = (\hat{p}_L^2 + \hat{q}_L^2) - (\hat{p}_R^2 + \hat{q}_R^2) = 2i\hbar \left( p \partial_q - q \partial_p \right)
\]

(83)

The unitary transformation by \( \hat{U}_F \) corresponds to the action of \( e^{-i\frac{\pi}{4} \hat{V}_F^{(-)}} \) on the phase space functions.
b) The Gauge transformations have also been studied as examples of $\hbar$- uncorrected solutions of Eq’s (15) and (16) for (50). The unitary transformation kernel is given in Eq. (51). The phase space transformation generator $\hat{V}_G^{(-)}$ corresponding to this case is easily found from $\hat{U}_0,$ given by Eq. (50) as

$$\hat{V}_G^{(-)} = \int_{qR}^{qL} dx \ A(x) = \int_{q+\frac{\bar{p}}{2}}^{q-\frac{\bar{p}}{2}} dx \ A(x)$$

(84)

where $\hat{U}_0$ corresponds to the action of $e^{i\tau \hat{V}_G^{(-)}}$ on the phase space functions.

c) The contact transformations are more complicated than the first two cases which has to do with the fact that not all CT’s have infinitesimal generators and/or exponential representations. They nevertheless have exact implicit solutions for the transformation kernel $u(p,q)$ which we firstly examine at a general setting. Considering $q \mapsto Q(q)$, $p \mapsto p/Q'(q)$ the problem is to find a consistent solution to Eq’s (31) and (32) which are in this case ($t = i\hbar/2$)

$$Q(q-t \partial_p) u(p,q) = (q+t \partial_p) u(p,q)$$

(85)

$$\frac{1}{Q'(q-t \partial_p)} (p+t \partial_q) u(p,q) = (p-t \partial_q) u(p,q).$$

(86)

The solution is in the form

$$u(p,q) = e^{B - \frac{q}{\hbar} (A-q)}$$

(87)

where $A = A(q)$ and $B = B(q)$ are implicitly solved by

$$Q(x)|_{x=A(q)} = 2q - A(q),$$

(88)

$$B(q) = \ln \left( \frac{Q'(x)}{1 + Q'(x)} \right)_{x=A(q)}$$

(89)

where $Q'(x) = dQ(x)/dx.$ Since $Q(x)$ must be invertible we infer from (88) that if a solution exits for $A(q)$ it must also be invertible. Therefore $A(q)$ is a monotonic function of $q$, as $Q(q)$ is, within the range of invertibility. For a large number of cases the solutions of (88) and (89) are explicit and for some others explicit forms exist. We now examine a few cases.

a) $Q = e^q, P = e^{-q} p$ is a typical example for a spectrum non-preserving transformation that one encounters in the phase space representations of the radial dimension. For this transformation one obtains $e^{A(q)} = 2q - A(q)$ for which a numerical solution is necessary. We infer that for $-\infty < q < \infty$, $A(q)$ has the range $[0, \infty)$ and monotonically increasing with $q$. Once $A(q)$ is known the solution for $B$ is provided by $e^B = e^A/(1 + e^A)$. b) The inverse of (a) is $Q = \ln q, P = q p$ which is one of the three successive transformations in transforming the quantum Liouville Hamiltonian to a free particle. For this one must solve $A e^A = e^{2q}$ numerically. c) The third type of transformation is $Q = \frac{1}{\alpha} \sinh \alpha q$, $P = \frac{p}{\cosh \alpha q}$ or its trigonometric variates. This transformation is also seen in the study of the quantum Liouville problem. It is spectrum preserving and it has an identity limit $\alpha \to 0$. It is thus expected here that the transformation kernel is unitary (spectrum preserving with identity limit) and distinct from the previous examples. d) We have studied the contact CT generated by $\hat{U}_{2,1}$ in Sec. II.A as given by Eq. (24). This is the only explicitly solvable
model we study here. To find the explicit solution we first redefine \((p, q) \rightarrow (q, p)\) and \((P, Q) \rightarrow (Q, P)\) in (23) and then use (88). After finding the solution we switch back the coordinates. The final result for \(A\) and \(B\) is then the solution of a quadratic equation and is given by \(A(p) = \left[(p - 1) + \sqrt{1/\gamma^2 + p^2}\right]/\gamma\) and \(B(p) = \left[1 + (1 + \gamma A)^2\right]^{-1}\) where \(u(p, q)\) is given by Eq. (57), with \(p\) and \(q\) interchanged.

What does Eq. (87) correspond to in the Hilbert space? To answer that question the integral given by Eq. (87) with \(p\) is

\[p\]

formation for the fields the \(\delta\)

\[q\]

By the use of Eq’s (3) and (4), Eq. (95) implies

\[x\]

\[y\]

Defining \((A, q, p)\) coordinates. The final result for \((A, q, p)\) is given by

\[A\]

is given by

\[Q\]

is then the solution of a quadratic equation and \(\delta\)-function must be inverted for \(x\) as

\[\delta(2[y - Q(A(x_+))]) = \frac{2}{A'(x_+)Q'(x_+)} \delta(x - 2A^{-1}(Q^{-1}(y)) + y)\]  

(91)

The overall \(x_+\) dependent front factor in Eq. (90) is the inverse of the \(x_+\) dependent front factor in Eq. (71) which can be seen by using (88) and its derivative with respect to \(q\). After the cancellation of the front factors, the transformation for the field \(\varphi(x)\) is found as a point transformation of the field coordinate \(x\) as

\[(\hat{U}_{ct} \varphi)(y) = \varphi(2A^{-1}(Q^{-1}(y)) - y)\]  

(92)

in order to calculate the argument of \(\varphi\) on the right hand side we use Eq. (88) one more time. Defining \(y = Q(A(q))\) and inverting it, i.e. \(q = A^{-1}(Q^{-1}(y))\), we get \(2A^{-1}(Q^{-1}(y)) = Q^{-1}(y) + y\) for arbitrary \(y\). Using this in Eq. (91), (74) and (90) we find that

\[e^{iF(y,x)} = \delta(x - Q^{-1}(y))\]  

(93)

which implies for the field \(\varphi\)

\[(\hat{U}_{ct} \varphi)(y) = \varphi(Q^{-1}(y))\]  

(94)

as expected to be the adjoint action of \(\hat{U}_{ct}\) in the function space.

Now let us examine the phase space image \(\hat{V}_{ct}^{-1}\) of the contact transformations characterized by \(q \mapsto Q(q)\). In the operator space one expects the transformation to be

\[\hat{U}_{ct} : \hat{q} = Q(\hat{q}) , \quad \hat{U}_{ct} : \hat{p} = \frac{1}{2} \left[ \frac{1}{Q'(\hat{q})} \hat{p} + \hat{p} \frac{1}{Q'(\hat{q})} \right] .\]  

(95)

By the use of Eq’s (3) and (4), Eq. (95) implies \(q \mapsto Q(q) , \quad p \mapsto p/Q'(q)\) as desired. Those contact transformations that are not connected to the identity by a continuous parameter may not have explicit exponential forms. We will formulate the problem for the exponential ones which can be written as \(\hat{U}_{ct} = e^{i\gamma \hat{A}_{ct}}\). \(\hat{A}_{ct}\) is in the form

\[\hat{A}_{ct} = e^{i\gamma \hat{A}_{ct}}\]  

From Eq. (94) it is clear that \(\hat{A}_{ct}\) is in the form
where $\hat{F}(\hat{q})$ is a real valued operator. To find $\hat{V}_{ct}^{(-)}$ we use the correspondence in (17) using $\hat{A} = \hat{A}_F$ and $\hat{B} = \hat{A}_G$ where $\hat{A}_F$ and $\hat{A}_G$ are given by $\hat{A}_F = (\hat{p}^2 + \hat{q}^2)$ and $\hat{A}_G = \int dq'/Q'(q)$. 

\[
\hat{A}_F \mapsto \hat{V}_F^{(-)} , \quad \hat{A}_G \mapsto \hat{V}_G^{(-)} \quad \Rightarrow \quad [\hat{A}_F, \hat{A}_G] \mapsto \hat{V}_{[\hat{A}_F, \hat{A}_G]}^{(-)}
\]

(97)

where we define $\hat{V}_{[\hat{A}_F, \hat{A}_G]}^{(-)} \equiv \hat{V}_{ct}$ and

\[
[\hat{A}_F, \hat{A}_G] = -2i\hbar \left( \hat{p} \frac{1}{Q'(\hat{q})} + \frac{1}{Q'(\hat{q})} \hat{p} \right) \propto \hat{A}_{ct}
\]

(98)

which is the desired generator in (96) for $F(x) = 1/Q'(x)$. The overall constants are not important since they are renormalized by the parameters of the transformation. In order to find $\hat{V}_{[\hat{A}_F, \hat{A}_G]}^{(-)}$ the Eq. (98) must be acted upon $\Delta$ as implied by Eq. (14). We define $\hat{V}_{ct}^{(-)}$ without the overall constant factor as the image of Eq. (98) written as

\[
\hat{V}_{ct}^{(-)} = \frac{1}{2} \left( \hat{p}_L \frac{1}{Q'(\hat{q}_L)} + \frac{1}{Q'(\hat{q}_L)} \hat{p}_L - \frac{1}{Q'(\hat{q}_R)} \hat{p}_R - \frac{1}{Q'(\hat{q}_R)} \right)
\]

(99)

which is indeed the commutator $-[\hat{V}_{ct}^{(-)}, \hat{V}_{ct}^{(-)}]$ as expressed in (17). Once the phase space image transformations are found for the Fourier in (83), for the gauge in (84) and for the contact transformations in (89) the corresponding phase space propagators can be found by using (50).

\section*{VII. CONCLUSIONS}

In this work we introduced Weyl’s phase space representations of the nonlinear quantum canonical transformations. The formalism is independent from any particular dynamical model. The phase space generators (images), the transformation kernels and phase space propagators of invertible CT are introduced as equivalent aspects of the same formalism and their interrelations are examined.

Defining the gauge set $\{\hat{V}_G^{(-)}\}$ and the contact set $\{\hat{V}_{ct}^{(-)}\}$, the commutation relations defined between them are closed as

\[
[[\hat{V}_{ct}^{(-)}, \hat{V}_{ct}^{(-)}] = i\hbar \{\hat{V}_{ct}^{(-)}\}, \quad [[\hat{V}_G^{(-)}, \hat{V}_G^{(-)}] = 0
\]

(100)

where brackets symbolically mean that the commutator of the two sets involved is a member of the set on the right side of the equations. The operators $\hat{A}$ as discussed above Eq. (11) and their images $\hat{V}_{ct}^{(-)}$ as defined by Eq. (17) are generators of the same canonical transformation in the operator and phase spaces respectively, as guaranteed by Eq. (17). For all the three generators considered, $\hat{V}_F^{(-)}$, $\hat{V}_G^{(-)}$ and $\hat{V}_{ct}^{(-)}$, the resulting transformations respect Eq. (1). These observations strongly imply that the complete algebraic set of invertible classical CT may be isomorphically connected, via the Weyl correspondence, with a quantum algebraic
set at least for those transformations satisfying Eq. (9). Further work on the implications of Eq. (9) in a quantum-classical perspective may illustrate whether the number of such independent algebraic sets is finite. 18
It has been believed for a long time that Weyl quantization did not possess covariance under nonlinear CT. As the results in this work indicate, different Weyl representations can be connected by the nonlinear CT thereby extending the concept of covariance instead of breaking it. Another advantage in seeing this as an extended covariance is that Dirac’s transformation theory which is essentially a Hilbert space approach can be naturally merged with Weyl’s phase space approach bringing the theory of CT (particularly nonlinear and invertible) back to where it should belong.

Nearly as old as the quantum mechanics itself, the Weyl quantization remains to be one of the most active fields in a wide area of physics. Without need of mentioning its applications in quantum and classical optics, condensed matter physics and engineering, it has been put into a more general frame in the deformation quantization. 16 Recently, it also proved to be an essential part of the non-commutative quantum field and string theories in the presence of background gauge fields. 24 It is then natural to expect that the theory of canonical transformations, which is subject to progress within itself, may also find some applications in these new directions.

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