Lattice QCD with dynamical quarks

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Abstract. An overview is given of lattice QCD with dynamical quarks taking as a case study the QCDSF collaboration results for two and three flavours of dynamical quarks for hadronic structure. We first sketch the lattice simulation and the idea of numerical simulations and why it is a difficult numerical problem. Measuring correlation functions on the generated configuration then gives access to hadronic masses and matrix elements. Several examples are given of results ranging from the mass spectrum and decay constants to the computation of low moments of structure functions.

1. Introduction and the problem of QCD

The goal of the lattice QCD programme is a numerical computation of strong interaction properties of particle physics such as the determination of the fundamental parameters of the QCD Lagrangian and physical properties of mesons/baryons such as masses, decay constants, (nucleon) matrix elements, form factors, generalised parton distributions, (GPDs), ... all directly from the underlying theory. In this article we present a case study from the QCDSF Collaboration of what these calculations involve concentrating on the light quark sector: $u$, $d$, $s$. Many other groups in Europe, USA, Japan ..., cover these and other aspects of lattice gauge theories (LGTs) and a good overview of the state of the field can be found from the plenary and parallel talks in the yearly Lattice Conference, the latest in the series being Lattice 2010, [1]. A recent review of LGTs and hadronic physics is given in [2]. There are also several introductory texts on lattice gauge theories, [3, 4, 5, 6, 7].

QCD is a non-abelian gauge theory, locally invariant under the ‘colour’ group $SU(3)$ with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \sum_q \bar{q}(i\gamma^\mu D_\mu - m_q)q,$$  \hspace{1cm} (1)

a generalised QED with 6 quarks and 8 gluons, where the gluons unlike the photon can now interact with themselves. At high energy scales, it is found that perturbation theory is valid, with loosely bound quarks – ‘asymptotic freedom’ while at low energy scale the coupling constant becomes large and binds quarks into hadrons (either mesons, $q\bar{q}$ states or baryons, $qqq$ states) – ‘confinement’. QCD is thus fundamentally different to QED and possesses a rich dynamical content.

How do we get predictions from QCD? There are two classes: perturbative methods based on asymptotic freedom and non-perturbative methods such as using a chiral Lagragians (write down the most general Lagrangian compatible with QCD symmetries and expand in $m_\pi, p \ll 4\pi f_\pi$);
heavy quark effective theories; sum rules; potential models (quarkonium) and lattice simulations. Of these it is fair to say that only the lattice approach directly tests the validity of the QCD Lagrangian directly, eq. (1). While perturbative methods are much the main connection between QCD and experiment they need additional hadronisation assumptions, lattice QCD is the only known direct bridge from theory which in principle covers all scales.

What would we like to determine? Some illustrative possibilities are

- The fundamental parameters of the QCD Lagrangian such as the running coupling constant, $\alpha_s(Q)$, and quark masses, $m^R_q$. Quark masses may be defined through the axial or vector currents

$$\partial_\mu A^R_\mu = (m^R_{q_1} + m^R_{q_2}) P^R, \quad \partial_\mu V^R_\mu = (m^R_{q_1} - m^R_{q_2}) S^R,$$

(2)

(where $A_\mu = \bar{q}_1 \gamma_\mu \gamma_5 q_2$ etc.).

- The hadron spectrum – both meson octet pseudoscalar/vector and baryon octet and decuplet multiplets of particles. As an example the meson vector octet and baryon octet are depicted in Fig. 1.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{vector_baryon_octets}
\caption{Vector meson ($\rho$, $K^*$, $\omega$) and baryon ($n$, $p$, $\Lambda$, $\Sigma$, $\Xi$) octets, left, right panels.}
\end{figure}

- Decay constants – a typical example is the semi-leptonic decay $\pi^- \rightarrow \mu^- \nu$ as shown in Fig. 2 with amplitude

$$T_{fi} = (-ig_W)^2 \cos \theta \bar{\pi}(K^-) \gamma_\lambda (1 - \gamma_5) v_\nu(K^-) \times \frac{i}{m_W^2} \times \langle 0|\pi\gamma_\lambda (1 - \gamma_5)d|\pi^- (0) \rangle,$$

(3)

where $f_\pi$ being defined by $\langle 0|\pi\gamma^\mu(1 - \gamma_5)\pi|\rangle = m_\pi f_\pi$. 

\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{pion_decay}
\caption{Decay of a pion.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{beta_decay}
\caption{\beta-decay.}
\end{figure}
• β decay where a neutron decay to a proton \( n \rightarrow p + e^- + \bar{\nu}_e \) (or \( d \rightarrow u + e^- + \bar{\nu}_e \)) as depicted in Fig. 3 with amplitude

\[
T_{fi} = \left( -ig_W^2 \cos \theta_c \right) \overline{u}_e(p_e) \gamma_\mu (1 - \gamma_5) \nu_e(p_e) \times \frac{1}{m_W^2} \times \langle p(p) | \overline{u} \gamma^\mu (1 - \gamma_5) d | n(\bar{p}) \rangle ,
\]

Decomposition of the blob in Fig. 3 is

\[
\langle p(p) | \overline{u} \gamma^\mu (1 - \gamma_5) d | n(\bar{p}) \rangle = \overline{u}_p(p) \gamma^\mu (1 - \gamma_5 g_A) u_n(p) .
\]

Isospin invariance gives the matrix element

\[
\langle \vec{p}, \vec{s} | \overline{u} \gamma^\mu \gamma^5 u - \frac{2}{3} \overline{d} \gamma^\mu d | \vec{p}, \vec{s} \rangle = 2 s^\mu g_A ,
\]

which is the convenient expression used for a lattice computation of \( g_A \).

• Form factors for example in electron–proton scattering as shown in Fig. 4 with

\[
T_{fi} = e^2 \overline{u}_e(p_e) \gamma^\mu u_e(p_e) \frac{1}{q^2} \langle p' | J^\mu (\vec{q}) | \bar{p} \rangle , \quad J^\mu = \frac{2}{3} \overline{u} \gamma^\mu u - \frac{1}{2} \overline{d} \gamma^\mu d + . . . .
\]

The proton blob can be decomposed into two form factors \( F_1(q^2) \), \( F_2(q^2) \)

\[
\langle p' | J^\mu (\vec{q}) | \bar{p} \rangle = \overline{u}(p') \left[ \gamma^\mu F_1(q^2) + i \sigma^{\mu \nu} \frac{q^\nu}{2m_N} F_2(q^2) \right] u(\bar{p}) ,
\]

where \( F_1^p(0) = 1 \) (due to vector current conservation and \( F_2^p(0) = \mu^p - 1 \) the anomalous magnetic moment (in units of \( e/(2m_N) \)) of the nucleon. From the form factor we can find the (charge) radii of the nucleon, defined by

\[
r_i^2 = -6 \frac{d F_i(q^2)}{dq^2} \bigg|_{q^2=0} , \quad i = 1, 2 .
\]

• High Energies / Hard Processes – DIS (deep inelastic scattering) which are described by the parton model and structure functions \( F \) as shown in Fig. 5 where \( ep \rightarrow eX \) when \( Q^2 \equiv -q^2 \rightarrow \infty \), with \( x = -q^2 / 2\nu = \text{fixed} \) (\( \nu = p \cdot q \rightarrow \infty \sim \text{energy transfer} \)). In this case \( \alpha_s \) is small, so there are ‘quasifree’ quarks (and gluons) – partons with \( F \equiv F(x) \) only – Bjorken scaling. \( x \) is fraction of nucleon momentum carried by parton \( p_{part} = xp_N \). With
QCD we have additional ln $Q^2$ scaling violations. Quantitatively using the OPE (operator product expansion) for both unpolarised/polarised processes

$$\int_0^1 dx x^{n-2} F_2^{p-n}(x, Q^2) = E_{NS}^{\overline{\text{MS}}} (\alpha_s) v^{\overline{\text{MS}}} (n_{eV}, N_S)(Q), \quad n \text{ even},$$

where NS, non-singlet: $p - n$ or $u - d$

$$\langle N|O_{q; \mu_1\cdots\mu_n}\rangle - \text{tr}[N] = 2v_n^{(q)} [p^{\mu_1}\cdots p^{\mu_n} - \text{tr}]$$

$$O_{q; \mu_1\cdots\mu_n} = \bar{q} \gamma^{\mu_1} i\mathcal{D} \cdots i\mathcal{D}^{\mu_n} q .$$

Alternatively the matrix elements can be given in terms of parton densities: $q, \bar{q}, g$ with

$$\langle x^{n-1} \rangle \equiv v_n^{(q)\overline{\text{MS}}} (\mu) = \int_0^1 dx x^{n-1} \left[ q^{\overline{\text{MS}}}(x, \mu) + (-1)^n \bar{q}^{\overline{\text{MS}}}(x, \mu) \right] ,$$

where $q$ is the probability of finding a quark with a fraction $x$ of the nucleon momentum. Naively, we would expect for the nucleon, the $u$ and $d$ to be $\delta$-functions at $x \sim \frac{1}{3}$ and it is indeed found that $u, d$ are smeared out $\delta$-functions around this $x$ value.

- A more general approach connecting these various methods is given by GPDs (which we shall not consider further here).

So in general we need computation of non-perturbative quantities: such as hadron masses, $m_H$ and matrix elements $\langle H|\hat{O}|H \rangle$.

2. The lattice approach

The lattice approach first Euclideanises and discretises space-time with a lattice spacing $a$. The path integral thus becomes a partition function which is a $N_S \times N_T \times d \times (n_c^2 - 1)$ integral. For example for a $\sim 32^3 \times 64 \times 4 \times 8$ lattice this means that we have $a \sim O(50,000,000)$ dimensional integral. The only known technique for such high dimensional integrals is the Monte Carlo method for finding expectation values of

$$\langle O \rangle = \frac{1}{Z} \int [dU] [dq \bar{q}] O e^{-(S_G + S_F)} .$$

The continuum $a \to 0$ limit is equivalent to $\xi \to \infty$ i.e. a second order phase transition. In Fig. 6 we depict this limit. Note that $1\text{fm} = 10^{-13}\text{m}$ is a typical hadronic scale ie $\sim$ nucleon radius.

![Figure 6. The lattice box.](image)

What do we choose for the action? It should have the classical limit $a \to 0$ of eq. (1). For the gluon action the choice is relatively easy. Defining the gauge links by $U_\mu(x) = e^{igA_\mu(x)}$ then a
closed loop (or sum of closed loops) with suitable coefficient is sufficient, being gauge invariant. The simplest choice is

\[ S_G = \frac{1}{3} \beta \sum_{\text{plaq}} \text{tr}_c (1 - U_{\mu\nu}^{\text{plaq}}) \rightarrow \int d^4x \frac{1}{4} F_{\mu\nu}^2 + O(a^2). \]  

(14)

For the fermion action we require

\[ S_F = \sum_{q,n} \bar{q} \mathcal{M}_n q \rightarrow \int d^4x \sum_q \bar{q}(\gamma_\mu D_\mu + m_q)q + O(a^p). \]  

(15)

But there is an immediate fundamental problem: a naive discretisation of the Dirac action leads to $2^4$ continuum flavours (rather than just one). Several types of fermion discretisation have been proposed to ameliorate this problem, as summarised in Table 1 where the pros and cons

|                | chiral symmetry | flavour symmetry | numerical simulation |
|----------------|-----------------|------------------|---------------------|
| Wilson / O(α)-improved Wilson (clover) | violated | recovered in the continuum | correct | medium expensive |
| twisted mass | violated | 2 flavours | medium expensive |
| staggered | exact $U(1)$ from $U(4)$ | 4 flavours | fast |
| Ginsparg-Wilson: overlap / domain wall | exact at finite $a$ | correct | most expensive |

Table 1. Various possibilities for lattice fermions.

are illustrated. The QCDSF Collaboration has used clover fermions where each quark in the action is given by

\[ S_F^{(q)} = a^4 \sum_x \left\{ \bar{q}(D_W + m_0)q + \frac{1}{4} \kappa c_{sw} \sum_{\mu\nu} \bar{q}\sigma_{\mu\nu} F_{\mu\nu} q \right\}, \]

with

\[ D_W = \frac{1}{2} \sum_\mu \left( \gamma_\mu (\Delta_\mu + \Delta_\mu^*) - a(\Delta_\mu \Delta_\mu^*) \right). \]  

(16)

(\(\Delta_\mu, \Delta_\mu^*\) are the forward and backward derivatives including the gauge link.) The mass, \(m_0\) is given by \(m_0 = m_q + m_c\) with

\[ m_q(c_{sw}) = \frac{1}{2\alpha} \left( \frac{1}{\kappa} - \frac{1}{\kappa_c(c_{sw})} \right), \quad m_c(c_{sw}) = \frac{1}{2\alpha} \left( \frac{1}{\kappa_c(c_{sw})} - \frac{1}{1/8} \right). \]  

(17)

So to find the chiral limit when the quark mass vanishes we need to tune to a critical kappa, \(\kappa_c(c_{sw})\) which corresponds to an additive mass renormalisation \(m_c(c_{sw})\). Discretisation errors
can be reduced from $O(a)$ to $O(a^2)$, if it can be arranged for one physical quantity (eg mass ratio) to have no $O(a)$ terms this fixes $c_{sw}(g^2)$, then all other masses are automatically improved to $O(a^2)$. Matrix elements, however, require further improvement operators.

Correlation functions are calculated by averaging over gauge field configurations with a weight given by the Boltzmann factor

$$⟨O⟩ = ∫[dU]O(U)p^∗(U), \quad p^∗(U) = \frac{1}{Z}e^{-S(U)},$$

generated using Monte Carlo methods. The Markov Chain, with $Q$ being the transition matrix is given by

$$p_{n+1}(U′) = ∫[dU]p_n(U)Q(U,U′), \quad ∫[dU′]Q(U,U′) = 1.$$

In ‘equilibrium’, $p_n → p^*$ and the Markov chain equation is satisfied by $p^*(U)Q(U,U′) = p^*(U′)Q(U′,U)$ or ‘detailed’ balance

$$Q(U,U′) = \text{prob. of choosing a Candidate config.} \times \text{prob. of Accepting new config.}$$

$$= p_C(U → U′) × p_A(U → U′).$$

For dynamical fermions the algorithm of choice is the Hybrid Monte Carlo algorithm. First (as with all algorithms) the Grassmann fermions must be integrated out (possible as we have quadratic Grassmann fields)

$$∫[dq][dψ]e^{qMq} = \det M = ∫[dφ][dφ^∗]e^{-φ^† M^{-1} ϕ}.$$  

Then

- $p_C$: candidate config $(U′, π′)$

$$p_C[(U, π) → (U′, π′)] = e^{-\frac{1}{2}π^2}[⟨U′, π′⟩ − ⟨U, π⟩],$$

using discretised Hamilton’s equations (leapfrog in computer time)

- $p_A$: accept/reject using Metropolis algorithm

$$p_A[(U, π) → (U′, π′)] = \min \left(1, e^{-ΔH} \right), \quad ΔH = H(U′, π′) − H(U, π).$$

Monte Carlo methods are like doing an experiment (in the computer) and consideration must be taken of computing the error bars,

$$\overline{O} = \frac{1}{N_{conf}} \sum_{n=1}^{N_{conf}} O_n, \quad \text{error} \propto \frac{1}{\sqrt{N_{conf}}}. (22)$$

The next step is to compute the appropriate masses/matrix elements. In Figs. 7, 8 we illustrate the computation required, the masses being given by

$$⟨M(t)M^†(0)⟩ = |⟨0|\hat{M}|M⟩|^2e^{-m_M t}, \quad ⟨B(t)\overline{B}(0)⟩ ∝ e^{-m_B t}. (23)$$

Similarly in Figs. 9, 10 we indicate the computation of baryonic quark bilinear matrix elements,
Figure 7. The meson two point correlation functions.

Figure 8. The baryon two point correlation functions.

Figure 9. The quark line connected baryon three point function.

Figure 10. The quark line disconnected baryon three point function.

Using

\[ R(t, \tau; \vec{p}) = \frac{\langle N(t; \vec{p}) O(\tau; \vec{0}) N(0; \vec{0}) \rangle}{\langle N(t; \vec{p}) N(0; \vec{0}) \rangle} \propto \langle N(\vec{p}) | \hat{O} | N(\vec{p}) \rangle. \]  

(24)

(For form factors, this result must be generalised to allow for momentum transfer.) Practically, due to the CPU cost, only the quark line connected part of the matrix element is calculated, which means that we are restricted to non-singlet matrix elements, for example

\[ \langle N | [uOu - dOd] | N \rangle. \]

A lattice computation requires the limits:

- Many independent configurations \( N_{\text{conf}} \to \infty \) typically at present \( N_{\text{conf}} \sim 1000 \)
- Large box \( L \to \infty \) typically at present \( L \sim 2 - 3 \text{fm} \)
- Continuum extrapolation \( (a \to 0) \) typically at present \( a \sim 0.11 - 0.08 - 0.06 \text{fm} \)
- Extrapolation \( (m_{q}^{r} \to m_{q}^{\text{phys}}, \ q = u, \ d, \ s) \) typically at present \( m_{\pi}^{2} \sim 400 - 200 \text{MeV} \).

All these limits make LGT a very difficult computation. It is easy to see that we still have a long way to go. Taking \( m_{\pi}L = 4 \) (a reasonable value to minimise finite size effects) and \( p_{\text{min}} \equiv 2\pi/L = m_{\pi} \), then this gives Fig. 11. Either we are forced to work with a small \( a \) but with ‘large’ \( m_{\pi} \) (no continuum extrapolation needed but a chiral extrapolation) or ‘coarse’ \( a \) with \( m_{ps} \sim 140 \text{MeV} \) (no chiral extrapolation necessary but continuum extrapolation). It should be noted that small changes in \( a, m_{\pi} \) or \( L \) produce huge changes in cost as we expect approximately

\[ \text{flops / config} \sim \left( \frac{L}{3 \text{fm}} \right)^{5} \times \left( \frac{0.1 \text{fm}}{a} \right)^{6} \times \left( \frac{1}{am_{\pi}} \right)^{1}. \]  

(25)

Currently Teraflop machines are used. It is not all gloom though: reached pion masses have been dropping rapidly in the last few years.
Figure 11. Plot of spatial lattice size, $N_S$ versus lattice spacing, $a$ (where $L = N_S a$).

Figure 12. The path to the physical point: $n_f = 2$ and $2 + 1$ flavours, left, right panels.

The path to the physical pion mass has to be chosen. While for two (degenerate) quark flavour masses, there is little to decide; as depicted in Fig. 12 for $2 + 1$ flavours, we have a plane, so many paths are possible. We have decided to start on a point on the $SU(3)$ flavour symmetry line and to keep the singlet quark mass constant, [8],

$$\frac{1}{3}(2m_l^R + m_s^R) = \text{const.},$$

as also sketched in Fig. 12. In this case we find results shown in Fig. 13.

To determine the scale (i.e. the value of $a$ in femotometers), there are various possibilities, we have chosen to use a flavour singlet quantity such as the centre of mass of the baryon octet, $X_N = \frac{1}{3}(m_N + m_\Sigma + m_\Xi)$.

After the simulation we still need to renormalise the matrix elements. Several methods have been proposed, the most popular general purpose method proposed was in [9] which mimics (continuum) perturbation theory in a certain ‘MOM’ scheme. We sketch the procedure here. The amputated Green’s function is first found (in the Landau gauge which maximizes $\sum_{x,\mu}(1 - \frac{1}{3}\text{Re} Tr U_{\mu}(x))$). Using a ‘momentum source’, [10], we first compute the correlation functions

$$C_{O,\alpha\beta}^{ab}(p) = \frac{1}{V} \sum_{xyz} e^{-ip(x-y)} \langle q_{\alpha}^a(x)O(z)q_{\beta}^b(y) \rangle_{q, U},$$

and then find the 1PI vertex function

$$\Gamma_O(p) = S^{-1}(p)C_O(p)S^{-1}(p),$$
Figure 13. \((2m_K^2 - m_\pi^2)/X^2\) versus \(m_\pi^2/X^2\) (with \(X^2 = (2m_K^2 + m_\pi^2)/3\)). The dashed black line, \(y = x\) represents the \(SU(3)\) flavour symmetric line. Shown are points on the flavour symmetric line (orange) followed by results with \(\overline{m} = \text{const.}\). Physical values are denoted by stars.

(where \(S\) is the quark propagator). The renormalised operator (and quark wavefunction) are

\[ O_R = Z_O O, \quad q_R = \sqrt{Z_q q}, \]

so the renormalised 1PI correlation is then \(\Gamma_{O,R}(p) = S_{O,R}^{-1}(p) C_{O,R}(p) S_{O,R}^{-1}(p) = Z_{O}^{-1} Z_{O} \Gamma_{O}(p)\). \(Z_O\) is fixed from the renormalisation condition at some scale \(p^2 = \mu^2\):

\[ \frac{1}{2} \text{tr} \left[ \Gamma_{O,R}(p) \Gamma_{O,R}^{-1}(p) \right] |_{p^2=\mu^2} = 1. \]

This has solved the problem, but phenomenologists use the (perturbative) \(\overline{MS}\) scheme, so we have to

- Convert the \(\text{MOM}\) scheme to the renormalisation group invariant or \(RGI\) form (which is a scale and scheme independent number)
- Measure the \(RGI\) constant
- Convert back to the \(\overline{MS}\) scheme

The disadvantage is that some perturbative knowledge used. The RGI form is found from

\[ O^{RGI} = Z_O^{RGI} O = \Delta Z_O^S(\mu) O^S(\mu), \quad S = \text{MOM}, \overline{MS}, \]

with

\[ [\Delta Z_O^S(\mu)]^{-1} = [2b_0 g^S(\mu)^2]^{-\frac{d_{\overline{MS}}}{2b_0}} \exp \left\{ \int_0^{g^S(\mu)} d\xi \left[ \frac{\beta^S(\xi)}{\beta^G(\xi)} + \frac{d_{\overline{MS}}}{b_0 \xi} \right] \right\}. \]

This allows us to measure a plateau and then to run to \(2 \text{ GeV}\). For \(n_f = 2\) flavours typical results are shown in Figs. 14 and 15.
3. Illustrative results

We first consider the strange quark mass. We can define the (bare) quark mass through either the axial or vector currents, which must then be renormalised, as in eq. (2) and section 2 and then extrapolated to the physical point. For \( n_f = 2 \), \[12\] we find the results shown in Fig. 16.

We next consider the vector meson spectrum. A flavour expansion about the symmetric point (Gell-Mann–Okubo) gives the constrained fits

\[
\begin{align*}
m_\rho &= m_0 + 2\alpha\delta m + (\beta_0 + 2\beta_1)\delta m^2 \\
m_K^* &= m_0 - \alpha\delta m + (\beta_0 + 5\beta_1 + 9\beta_2)\delta m^2 \\
m_\phi &= m_0 - 4\alpha\delta m + (\beta_0 + 8\beta_1)\delta m^2 ,
\end{align*}
\]

where \( \delta m_q = m_q - \overline{m} \). We see that the linear terms are described by 1 coefficient; the quadratic terms with 3 coefficients. In Fig. 17 we show the octet multiplet as a ‘fan’ plot. Very little curvature is seen. A similar result is obtained for the baryon spectrum. We also show in Fig. 17 the baryon octet ‘fan’ plot.

In Fig. 18 we show the ratio \( f_K/f_\pi \), while in Fig. 19, \( g_A \) is plotted, \[13\].

The \( F_1 \) form factors and \( r_1 \) charge radius, (for \( n_f = 2 \) flavours) are given by eq. (8) and are shown in Figs. 20, 21, \[14\]. It is also seen that there are small \( O(a^2) \) scaling violations and that
all the ‘activity’ appears to be taking place close to the physical point. Form factor radii are defined in eq. (9) and $r_1$ (for example) determined from a fit

$$F_1(q^2) = \frac{1}{1 + q^2/m_1^2},$$

is shown in Fig. 21. (The dashed line is from $\chi$-PT.) It is seen that $r_1^{LAT} < r_1^{EXPT}$ (and this is also true for $r_2$) – so probably we are (still) missing dynamics due to the ‘pion cloud’.

**Figure 17.** $m_{\rho O}/X_{\rho}$ ($\rho O = \rho, K^*, \phi_4$) and $m_{N O}/X_N$ ($N O = N, \Lambda, \Sigma, \Xi$) against $m_\pi^2/X_\pi^2$ (left, right panels respectively), together with the constrained fit of e.g. eq. (33).

**Figure 18.** Plot of $f_K/f_\pi$ versus $m_\pi^2/X_\pi^2$.

**Figure 19.** Plot of $g_A/f_\pi$ against $m_\pi^2$.

**Figure 20.** Plot of $F_1(Q^2)$ versus $Q^2 = -q^2$.

**Figure 21.** Plot of $r_1^2$ versus $m_\pi^2$. 
Another interesting quantity is the lowest moment of the structure function $F_{NS}^{2}$ of the nucleon, as shown in Fig. 22. It is apparent that the recent $n_f = 2$ (left) results, [14] and older $n_f = 0$ (right) results, [15] are very similar, and that there is also a challenge to understand this and the discrepancy to the phenomenological value.

4. Conclusions
We first emphasise that the lattice approach is a fundamental test of QCD. In this proceedings we have sketched the lattice programme (taking the QCDSF Collaboration as an example) from beginning to end. The approach is very much like an experiment – the precision can be improved by increasing the data set. Lattice gauge theory, LGT, can compute hadron masses and, more interestingly is capable of computing matrix elements. We need to take several limits however, each of which is painful, such as a large box, quark mass extrapolation to the physical point and a continuum extrapolation, $a^2 \to 0$. We have showed here some results for masses, decay constants, form factors (moments of) structure functions. Of course there are many other quantities under active consideration such as hadron decays, excited nucleon states, nucleon distribution amplitudes, GPDs, charm and $b$-quark physics finite temperature QCD, ‘technicolor’ experiments to name only a few. This will all need much faster machines for the programme to reach its goals.

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