Minimal free resolution of the associated graded ring of certain monomial curves

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Abstract. In this article, we give the explicit minimal free resolution of the associated graded ring of certain affine monomial curves in affine 4-space based on the standard basis theory. As a result, we give the minimal graded free resolution and compute the Hilbert function of the tangent cone of these families.

1. Introduction

In this article, we study the minimal free resolution of the associated graded ring of the local ring $A$ of a monomial curve $C \subset A^4$ corresponding to an arithmetic sequence based on the standard basis theory. The associated graded ring $G = gr_m(A) = \bigoplus_{i=0}^{\infty} (m^i/m^{i+1})$ of $A$ with maximal ideal $m$ is a standard graded $k$-algebra. Since it corresponds to the important geometric construction, it has been studied to get comprehensive information on the local ring (see [13, 12, 7, 8, 9]. Because the minimal finite free resolution of a finitely generated $k$-algebra is a very useful tool to extract information about the algebra, finding an explicit minimal free resolution of a standard $k$-algebra is a basic problem. This difficult problem has been extensively studied in the case of affine monomial curves [16, 14, 4, 10, 2].

We recall that a monomial affine curve $C$ has a parametrization

$$x_0 = t^{m_0}, x_1 = t^{m_1}, \ldots, x_n = t^{m_n}$$

where $m_0, m_1, \ldots, m_n$ are positive integers with $\gcd(m_0, m_1, \ldots, m_n) = 1$. The additive semigroup, which is denoted by

$$< m_0, m_1, \ldots, m_n > = \{ \sum_{0 \leq i \leq n} Nm_i \mid N = \{0, 1, 2, \ldots\}\}$$

generated minimally by $m_0, m_1, \ldots, m_n$, i.e., $m_j \notin \sum_{0 \leq i \leq n; i \neq j} Nm_i$ for $i \in \{0, \ldots, n\}$.
Assume that \( m_0, m_1, \ldots, m_n \) be positive integers such that \( 0 < m_0 < m_1 < \ldots < m_n \) and \( m_i = m_0 + id \) for every \( 1 \leq i \leq n \), where \( d \) is the common difference, i.e. the integers \( m_i \)'s form an arithmetic progression. The monomial curve which is defined parametrically by

\[
    x_0 = t^{m_0}, \quad x_1 = t^{m_1}, \quad \ldots, \quad x_n = t^{m_n}
\]

such that \( 0 < m_0 < m_1 < \ldots < m_n \) form an arithmetic progression is called a certain monomial curve.

In order to study the associated graded ring of a monomial curve \( C \) at the origin, it is possible to consider either the associated graded ring of \( A = k[[t^{m_0}, t^{m_1}, \ldots, t^{m_n}]] \) with respect to the maximal ideal \( m = (t^{m_0}, t^{m_1}, \ldots, t^{m_n}) \) which is denoted by \( \text{gr}_m(k[[t^{m_0}, t^{m_1}, \ldots, t^{m_n}]] \) or the ring \( k[x_0, x_1, \ldots, x_n]/I(C)^* \), where \( I(C)^* \) is the ideal generated by the polynomials \( f^* \) for \( f \) in \( I(C) \), where \( f^* \) is the homogeneous summand of \( f \) of the least degree, since they are isomorphic. We recall that \( I(C)^* \) is the defining ideal of the tangent cone of the curve \( C \) at the origin.

Our main aim in this paper is to give an explicit minimal free resolution of the associated graded ring for certain monomial curves in affine 4-space. Even if one can obtain the numerical invariants of the minimal free resolution of the tangent cone of certain monomial curves in \( \mathbb{A}^4 \) by using the Theorem 4.1 and Proposition 4.6 in [16], we give the minimal free resolution of the tangent cone of certain monomial curves in affine 4-space in an explicit form by giving a new proof based on the standard basis theory. Using the standard basis theory and knowing the minimal generating set of binomial generators of the defining ideal of certain monomial curve explicitly from [11], we find the minimal generators of the tangent cone of a certain monomial curve in affine 4-space. By knowing the minimal generators, we show the Cohen-Macaulayness of the tangent cones of these families of curves. We obtain explicit minimal free resolution by using Schreyer’s theorem but prove it using the Buchsbaum-Eisenbud theorem [3]. Finally, we give the minimal graded free resolutions and as a corollary compute the Hilbert function of the tangent cones for these families. All computations have been carried out using SINGULAR [6].

2. Minimal generators of the associated graded ring

In this section, we find the minimal generators of the tangent cone of the certain monomial curve \( C \) having the defining ideal as in Theorem 4.5 in [11] in affine 4-space. First, we recall the theorem which gives the construction of the minimal set of generators for the defining ideal of certain affine monomial curve in \( \mathbb{A}^4 \).

Let \( m_0 < m_1 < m_2 < m_3 \) be positive integers with \( \text{gcd}(m_0, m_1, m_2, m_3) = 1 \) and assume that \( m_0, m_1, m_2, m_3 \) form an arithmetic progression with common difference \( d \). Let \( R = k[x_0, x_1, x_2, y] \) be a polynomial ring over the field \( k \). We use \( y \) instead of \( x_3 \) by following the same notation in [14, 15, 11]. Let \( \phi : R \to k[t^{m_0}, t^{m_1}, t^{m_2}, t^{m_3}] \) be the \( k \)-algebra homomorphism defined by

\[
    \phi(x_0) = t^{m_0}, \quad \phi(x_1) = t^{m_1}, \quad \phi(x_2) = t^{m_2}, \quad \phi(y) = t^{m_3}
\]
and \( I(C) = \text{Ker}(\phi) \). Let us write \( m_0 = 3a + b \) such that \( a \) and \( b \) are positive integers \( a \geq 1 \) and \( b \in [1,3] \). In \([14]\), the following theorem is given as a definition.

**Theorem 2.1.** \([11]\) Let

\[
\begin{align*}
\xi_{11} &:= x_1^2 - x_0 x_2, \\
\varphi_i &:= x_{i+1} x_2 - x_i y, \text{ for } 0 \leq i \leq 1, \\
\psi_j &:= x_{b+j} y^a - x_0^{a+d} x_j, \text{ if } 1 \leq b \leq 2 \text{ and } 0 \leq j \leq 2 - b, \\
\theta &:= y^{a+1} - x_0^{a+d} x_{3-b}.
\end{align*}
\]

\( \{ \xi_{11}, \varphi_i, \psi_j, \theta \} \) is a minimal generating set for the defining ideal \( I(C) \).

We will prove for \( b = 1, 2, \) and 3, respectively. By using the notation in \([5]\), we denote the leading monomial of a polynomial \( f \) by \( LM(f) \), the S-polynomial of the polynomials \( f \) and \( g \) by \( spoly(f, g) \) and the Mora’s polynomial weak normal form of \( f \) with respect to \( G \) by \( NF(f \mid G) \).

**Case \( b = 1 \)**.

From the minimal generating set \( G \) in Theorem 2.1 we obtain

\[
G = \{ \xi_{11}, \varphi_0 = x_1 x_2 - x_0 y, \varphi_1 = x_2^2 - x_1 y, \\
\psi_0 = x_1 y^a - x_0^{a+d+1}, \psi_1 = x_2 y^a - x_0^{a+d} x_1, \theta = y^{a+1} - x_0^{a+d} x_2 \}.
\]

Recalling that the ordering is the negative degree reverse lexicographical ordering, we have

\[
\begin{align*}
LM(\xi_{11}) &= x_1^2, \quad LM(\varphi_0) = x_1 x_2, \quad LM(\varphi_1) = x_2^2, \\
LM(\psi_0) = x_1 y^a, \quad LM(\psi_1) = x_2 y^a \quad \text{and} \quad LM(\theta) = y^{a+1}.
\end{align*}
\]

We begin with \( \xi_{11} \) and \( \varphi_0 \). \( LM(\xi_{11}) = x_1^2 \) and \( LM(\varphi_0) = x_1 x_2 \). We compute \( spoly(\xi_{11}, \varphi_0) = x_0 x_1 y^a - x_0 x_2^2 \). Among the leading monomials of the elements of \( G \), only \( LM(\varphi_1) \) divides \( LM(spoly(\xi_{11}, \varphi_0)) = x_0 x_2^3 \). Also \( ecart(\varphi_1) = ecart(spoly(\xi_{11}, \varphi_0)) = 0 \). \( spoly(\varphi_1, spoly(\xi_{11}, \varphi_0)) = 0 \) implies \( NF(spoly(\xi_{11}, \varphi_0) \mid G) = 0 \).
Next, we choose $\xi_{11}$ and $\varphi_1$. Since $\text{lcm}(LM(\xi_{11}), LM(\varphi_1)) = LM(\xi_{11}).LM(\varphi_1)$, then $NF(\text{spoly}(\xi_{11}, \varphi_1) \mid \{\xi_{11}, \varphi_1\}) = 0$. This implies that $NF(\text{spoly}(\xi_{11}, \varphi_1) \mid G) = 0$. In the same manner, $NF(\text{spoly}(\xi_{11}, \psi_1) \mid G) = 0$, $NF(\text{spoly}(\xi_{11}, \theta) \mid G) = 0$, $NF(\text{spoly}(\varphi_0, \theta) \mid G) = 0$, $NF(\text{spoly}(\varphi_1, \psi_0) \mid G) = 0$ and $NF(\text{spoly}(\varphi_1, \theta) \mid G) = 0$.

Now, we compute S-polynomial of $\xi_{11}$ and $\psi_0$. $\text{spoly}(\xi_{11}, \psi_0) = x_0^{a+d+1}x_1 - x_0x_2y^a$. Among the leading monomials of the elements of $G$, only $LM(\psi_1)$ divides $LM(\text{spoly}(\xi_{11}, \psi_0)) = x_0x_2y^a$. Also ecart($\psi_1$) = ecart($\text{spoly}(\xi_{11}, \psi_0)$) = $d$. $\text{spoly}(\psi_1, \text{spoly}(\xi_{11}, \psi_0)) = 0$ implies $NF(\text{spoly}(\xi_{11}, \psi_0) \mid G) = 0$.

Again, we compute S-polynomial of $\varphi_0$ and $\varphi_1$. $\text{spoly}(\varphi_0, \varphi_1) = x_1^2y - x_0x_2y$. Among the leading monomials of the elements of $G$, only $LM(\xi_{11})$ divides $LM(\text{spoly}(\varphi_0, \varphi_1)) = x_1^2y$. Also ecart($\xi_{11}$) = ecart($\text{spoly}(\varphi_0, \varphi_1)$) = 0. $\text{spoly}(\xi_{11}, \text{spoly}(\varphi_0, \varphi_1)) = 0$ implies $NF(\text{spoly}(\varphi_0, \varphi_1) \mid G) = 0$.

Now choose $\varphi_0$ and $\psi_0$. Then, S-polynomial of $\varphi_0$ and $\psi_0$ is $\text{spoly}(\varphi_0, \psi_0) = x_0^{a+d+1}x_2 - x_0y^{a+1}$. Once again, only $LM(\theta)$ divides $LM(\text{spoly}(\varphi_0, \psi_0)) = x_0y^{a+1}$ among the leading monomials of the elements of $G$. Also ecart($\theta$) = ecart($\text{spoly}(\varphi_0, \psi_0)$) = $d$. $\text{spoly}(\theta, \text{spoly}(\varphi_0, \psi_0)) = 0$ implies $NF(\text{spoly}(\varphi_0, \psi_0) \mid G) = 0$.

Similarly, $\text{spoly}(\varphi_0, \psi_1) = x_0^{a+d}x_1^2 - x_0^{a+1}y$. Again, as in the previous case $LM(\theta)$ divides $LM(\text{spoly}(\varphi_0, \psi_1)) = x_0y^{a+1}$. Also ecart($\theta$) = ecart($\text{spoly}(\varphi_0, \psi_1)$) = $d$. $\text{spoly}(\theta, \text{spoly}(\varphi_0, \psi_1)) = x_0^{a+d}x_1^2 - x_0^{a+1}x_2$. Among the leading monomials of the elements of $G$, only $LM(\xi_{11}) = x_1^2$ divides $LM(\text{spoly}(\theta, \text{spoly}(\varphi_0, \psi_1))) = x_0^{a+d}x_1^2$. ecart($\xi_{11}$) = ecart($\text{spoly}(\theta, \text{spoly}(\varphi_0, \psi_1))$) = 0. $\text{spoly}(\xi_{11}, \text{spoly}(\theta, \text{spoly}(\varphi_0, \psi_1))) = 0$ implies $NF(\text{spoly}(\varphi_0, \psi_1) \mid G) = 0$.

Similarly, we compute $\text{spoly}(\varphi_1, \psi_1) = x_0^{a+d}x_1x_2 - x_1y^{a+1}$. Among the leading monomials of the elements of $G$, only $LM(\psi_0)$ and $LM(\theta)$ divides $LM(\text{spoly}(\varphi_1, \psi_1)) = x_1y^{a+1}$. Note that ecart($\psi_0$) = ecart($\theta$) = $d$. Firstly, beginning with $\psi_0$, $\text{spoly}(\psi_0, \text{spoly}(\varphi_1, \psi_1)) = x_0^{a+d}x_1x_2 - x_0^{a+1}y$. Among the leading monomials of the elements of $G$, $LM(\varphi_1) = x_1x_2$ divides $LM(\text{spoly}(\psi_0, \text{spoly}(\varphi_1, \psi_1)))$. Also ecart($\varphi_1$) = ecart($\text{spoly}(\psi_0, \text{spoly}(\varphi_1, \psi_1))$) = 0. $\text{spoly}(\varphi_1, \text{spoly}(\psi_0, \text{spoly}(\varphi_1, \psi_1))) = 0$. Secondly, taking $\theta$, $\text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_1)) = 0$. Thus, $NF(\text{spoly}(\varphi_1, \psi_1) \mid G) = 0$.

We continue by computing $\text{spoly}(\psi_0, \psi_1) = x_0^{a+d}x_2 - x_0^{a+d+1}x_2$. $LM(\xi_{11}) = x_1^2$ divides $LM(\text{spoly}(\psi_0, \psi_1)) = x_0^{a+d}x_2$. Also ecart($\xi_{11}$) = ecart($\text{spoly}(\psi_0, \psi_1)$) = 0. $\text{spoly}(\xi_{11}, \text{spoly}(\psi_0, \psi_1)) = 0$ implies $NF(\text{spoly}(\psi_0, \psi_1) \mid G) = 0$.

In the same manner, $\text{spoly}(\psi_0, \theta) = x_0^{a+d}x_1x_2 - x_0^{a+d+1}y$. $LM(\varphi_0) = x_1x_2$ divides $LM(\text{spoly}(\psi_0, \theta)) = x_0^{a+d}x_1x_2$. Also ecart($\varphi_0$) = ecart($\text{spoly}(\psi_0, \theta)$) = 0. $\text{spoly}(\varphi_0, \text{spoly}(\psi_0, \theta)) = 0$ implies $NF(\text{spoly}(\psi_0, \theta) \mid G) = 0$.

Finally, we compute $\text{spoly}(\psi_1, \theta) = x_0^{a+d}x_2^2 - x_0^{a+d}x_1y$. $LM(\varphi_1) = x_2^2$ divides $LM(\text{spoly}(\psi_1, \theta)) = x_0^{a+d}x_2^2$. Also ecart($\varphi_1$) = ecart($\text{spoly}(\psi_1, \theta)$) = 0. $\text{spoly}(\varphi_1, \text{spoly}(\psi_1, \theta)) = 0$ implies $NF(\text{spoly}(\psi_1, \theta) \mid G) = 0$.

Case $b = 2$.

As in the previous case, we obtain by the minimal generating set $G$ in Theorem 2.1

$$G = \{\xi_{11} = x_1^2 - x_0x_2, \ \varphi_0 = x_1x_2 - x_0y, \ \varphi_1 = x_2^2 - x_1y, \ \varphi_1 = x_2^2 - x_1y, \ \varphi_1 = x_2^2 - x_1y\}$$
\[ \psi_0 = x_2 y^a - x_0^{a+d+1}, \quad \theta = y^{a+1} - x_0^{a+d} x_1 \} \]

\[ LM(\xi_{11}) = x_1^2, \quad LM(\varphi_0) = x_1 x_2, \quad LM(\varphi_1) = x_2^2, \quad LM(\psi_0) = x_2 y^a \text{ and } LM(\theta) = y^{a+1} \]

with respect to the negative degree reverse lexicographical ordering.

We begin with \( \xi_{11} \) and \( \varphi_0 \). This case is exactly the same as in \( b = 1 \).

Next, we choose \( \xi_{11} \) and \( \varphi_1 \). As in the first case, since \( \text{lcm}(LM(\xi_{11}), LM(\varphi_1)) = LM(\xi_{11}).LM(\varphi_1) \), then \( NF(\text{spoly}(\xi_{11}, \varphi_1) \mid \{\xi_{11}, \varphi_1\}) = 0 \). Therefore, this implies that \( NF(\text{spoly}(\xi_{11}, \varphi_1) \mid G) = 0 \). In the same manner, \( NF(\text{spoly}(\xi_{11}, \psi_0) \mid G) = 0, \quad NF(\text{spoly}(\xi_{11}, \theta) \mid G) = 0, \quad NF(\text{spoly}(\varphi_0, \theta) \mid G) = 0 \) and \( NF(\text{spoly}(\varphi_1, \theta) \mid G) = 0 \).

Again, we compute S-polynomial of \( \varphi_0 \) and \( \varphi_1 \). This one is also the same as in the previous case.

Now choose \( \varphi_0 \) and \( \psi_0 \). Then, S-polynomial of \( \varphi_0 \) and \( \psi_0 \) is \( \text{spoly}(\varphi_0, \psi_0) = x_0^{a+d+1} x_1 - x_0 y^{a+1} \). Once again, only \( LM(\theta) = y^{a+1} \) divides \( \text{LM}(\text{spoly}(\varphi_0, \psi_0)) = x_0 y^{a+1} \) among the leading monomials of the elements of \( G \). Also, \( \text{ecart}(\theta) = \text{ecart}(\text{spoly}(\varphi_0, \psi_0)) = d. \text{spoly}(\theta, \text{spoly}(\varphi_0, \psi_0)) = 0 \) implies \( NF(\text{spoly}(\varphi_0, \psi_0) \mid G) = 0 \).

Similarly, we compute \( \text{spoly}(\varphi_1, \psi_0) = x_0^{a+d+1} x_2 - x_1 y^{a+1} \). Among the leading monomials of the elements of \( G \), only \( LM(\theta) \) divides \( \text{LM}(\text{spoly}(\varphi_1, \psi_0)) = x_1 y^{a+1} \). ecart(\( \theta \)) = ecart(\( \text{spoly}(\varphi_1, \psi_0) \)) = d. spoly(\( \theta, \text{spoly}(\varphi_1, \psi_0) \)) = \( x_0^{a+d+1} x_2 - x_0^{a+d} x_1^2 \). Since spoly(\( \theta, \text{spoly}(\varphi_1, \psi_0) \)) is not zero, again among the leading monomials of the elements of \( G \), \( LM(\xi_{11}) = x_1^2 \) divides \( \text{LM}(\text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0))) = x_0^{a+d} x_1^2 \). ecart(\( \xi_{11} \)) = ecart(\( \text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0)) \)) = 0. spoly(\( \xi_{11}, \text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0)) \)) = 0. Thus, \( NF(\text{spoly}(\varphi_1, \psi_0) \mid G) = 0 \).

Finally, we compute \( \text{spoly}(\psi_0, \theta) = x_0^{a+d} x_1 x_2 - x_0^{a+d+1} y \). \( LM(\varphi_0) = x_1 x_2 \) divides \( \text{LM}(\text{spoly}(\psi_0, \theta)) = x_0^{a+d} x_1 x_2 \). Also ecart(\( \varphi_0 \)) = ecart(\( \text{spoly}(\psi_0, \theta) \)) = 0. spoly(\( \varphi_0, \text{spoly}(\psi_0, \theta) \)) = 0 implies \( NF(\text{spoly}(\psi_0, \theta) \mid G) = 0 \).

**Case \( b = 3 \).**

Finally, by writing 3 instead of \( b \) in the minimal generating set \( G \) in Theorem 2.1, we obtain

\[ G = \{ \xi_{11} = x_1^2 - x_0 x_2, \quad \varphi_0 = x_1 x_2 - x_0 y, \quad \varphi_1 = x_2^2 - x_1 y, \quad \theta = y^{a+1} - x_0^{a+d+1} \} \]

In the same manner, \( LM(\xi_{11}) = x_1^2, \quad LM(\varphi_0) = x_1 x_2, \quad LM(\varphi_1) = x_2^2 \) and \( LM(\theta) = y^{a+1} \) with respect to the negative degree reverse lexicographical ordering \( >_{ds} \).

As in the previous cases, we begin with \( \xi_{11} \) and \( \varphi_0 \) and this case is exactly the same as in \( b = 1 \).

In the same manner, \( NF(\text{spoly}((\xi_{11}, \varphi_1) \mid G) = 0, \quad NF(\text{spoly}(\xi_{11}, \theta) \mid G) = 0, \quad NF(\text{spoly}(\varphi_0, \theta) \mid G) = 0 \) and \( NF(\text{spoly}(\varphi_1, \theta) \mid G) = 0 \).

Finally, the computation of the S-polynomial of \( \varphi_0 \) and \( \varphi_1 \) also results as in the case \( b = 1 \).

Therefore, if \( b = 1, 2 \) and 3, we conclude that the set \( G \) is a standard basis with respect to the negative degree reverse lexicographical ordering \( >_{ds} \).

□

We can now find the minimal generating set of the tangent cone by using the above lemma.
PROPOSITION 2.4. Let \( C \) be a certain monomial curve having parametrization
\[
x_0 = t^{m_0}, \quad x_1 = t^{m_1}, \quad x_2 = t^{m_2}, \quad y = t^{m_3}
\]
m_0 = 3a + b for positive integers \( a \geq 1 \) and \( b \in [1, 3] \) and \( 0 < m_0 < m_1 < m_2 < m_3 \) form an arithmetic progression with common difference \( d \) and let the generators of the defining ideal \( I(C) \) be given by the set \( G \) in Theorem 2.1. Then the defining ideal \( I(C)^* \) of the tangent cone is generated by the set \( G^* \) consisting of the least homogeneous summands of the binomials in \( G \).

Proof. By the Lemma 2.3
\[
G := \left\{ \begin{array}{ll}
\{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\psi_0, \psi_1\} \cup \{\theta\} & \text{if } b = 1, \\
\{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\psi_0\} \cup \{\theta\} & \text{if } b = 2, \\
\{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\theta\} & \text{if } b = 3.
\end{array} \right.
\]
as in Theorem 2.1 is a standard basis of \( I(C) \) with respect to a local degree ordering \( >_{ds} \) with respect to \( x_0 > x_1 > x_2 > y \). Then, from [5] Lemma 5.5.11, \( I(C)^* \) is generated by the least homogeneous summands of the elements in the standard basis. Thus, \( I(C)^* \) is generated by if \( b = 1 \)
\[
G^* = \{\xi_{11} = x_1^2 - x_0x_2, \ \varphi_0^* = x_1x_2 - x_0y, \ \varphi_1^* = x_2^2 - x_1y, \ \psi_0^* = x_1y^a, \ \psi_1^* = x_2y^a, \ \theta^* = y^{a+1}\},
\]
if \( b = 2 \)
\[
G^* = \{\xi_{11} = x_1^2 - x_0x_2, \ \varphi_0^* = x_1x_2 - x_0y, \ \varphi_1^* = x_2^2 - x_1y, \ \psi_0^* = x_2y^a, \ \theta^* = y^{a+1}\},
\]
and if \( b = 3 \)
\[
G^* = \{\xi_{11} = x_1^2 - x_0x_2, \ \varphi_0^* = x_1x_2 - x_0y, \ \varphi_1^* = x_2^2 - x_1y, \ \theta^* = y^{a+1}\}.
\]

THEOREM 2.5. Let \( C \) be a certain monomial curve having parametrization
\[
x_0 = t^{m_0}, \quad x_1 = t^{m_1}, \quad x_2 = t^{m_2}, \quad y = t^{m_3}
\]
m_0 = 3a + b for positive integers \( a \geq 1 \) and \( b \in [1, 3] \) and \( 0 < m_0 < m_1 < m_2 < m_3 \) form an arithmetic progression with common difference \( d \). The certain monomial curve \( C \) with the defining ideal \( I(C) \) as in Theorem 2.1 has Cohen-Macaulay tangent cone at the origin.

Proof. We can apply the Theorem 2.1 in [1] to the generators of the tangent cone which are given by the set
if \( b = 1 \)
\[
G^* = \{\xi_{11} = x_1^2 - x_0x_2, \ \varphi_0^* = x_1x_2 - x_0y, \ \varphi_1^* = x_2^2 - x_1y, \ \psi_0^* = x_1y^a, \ \psi_1^* = x_2y^a, \ \theta^* = y^{a+1}\},
\]
if \( b = 2 \)
\[
G^* = \{\xi_{11} = x_1^2 - x_0x_2, \ \varphi_0^* = x_1x_2 - x_0y, \ \varphi_1^* = x_2^2 - x_1y, \ \psi_0^* = x_2y^a, \ \theta^* = y^{a+1}\},
\]
and if \( b = 3 \)
\[
G^* = \{ \xi_{11}^* = x_1^2 - x_0 x_2, \; \varphi_0^* = x_1 x_2 - x_0 y, \; \varphi_1^* = x_2^2 - x_1 y, \; \theta^* = y^{a+1} \}. 
\]
All of these sets are Gröbner bases with respect to the reverse lexicographic order with \( x_0 > y > x_1 > x_2 \). Since \( x_0 \) does not divide the leading monomial of any element in \( G^* \) in all three cases, the ring \( k[x_0, x_1, x_2, y]/I(C)^* \) is Cohen-Macaulay from Theorem 2.1 in [11]. Thus, \( R = \text{gr}_m(k[[t^{m_0}, t^{m_1}, t^{m_2}, t^{m_3}]] \approx k[x_0, x_1, x_2, y]/I(C)^* \) is Cohen-Macaulay.

3. Minimal free resolution of the associated graded ring

In this section, we study the minimal free resolution of \( \text{gr}_m(k[[t^{m_0}, t^{m_1}, t^{m_2}, t^{m_3}]] \) of the certain monomial curve \( C \) in affine 4-space.

**Theorem 3.1.** Let \( C \) be a certain affine monomial curve in \( \mathbb{A}^4 \) having parametrization \( x_0 = t^{m_0}, \ x_1 = t^{m_1}, \ x_2 = t^{m_2}, \ y = t^{m_3} \) \( m_0 = 3a + b \) for positive integers \( a \geq 1 \) and \( b \in \{1, 3\} \) and \( 0 < m_0 < m_1 < m_2 < m_3 \) form an arithmetic progression with common difference \( d \). Then the sequence of \( R \)-modules
\[
0 \rightarrow R^\beta_3(b) \xrightarrow{\phi_3(b)} R^\beta_2(b) \xrightarrow{\phi_2(b)} R^\beta_1(b) \xrightarrow{\phi_1(b)} R \xrightarrow{\phi} G \rightarrow 0
\]
is a minimal free resolution for the tangent cone of \( C \), where
\[
\beta_1(b) = \begin{cases} 
6 & \text{if } b = 1, \\
5 & \text{if } b = 2, \\
4 & \text{if } b = 3,
\end{cases} \quad \beta_2(b) = \begin{cases} 
8 & \text{if } b = 1, \\
5 & \text{if } b = 2, \\
5 & \text{if } b = 3,
\end{cases} \quad \beta_3(b) = \begin{cases} 
3 & \text{if } b = 1, \\
1 & \text{if } b = 2, \\
2 & \text{if } b = 3.
\end{cases}
\]

and \( \phi \)’s denote the canonical surjections and the maps between \( R \)-modules depend on \( b \)

\[
\phi_1(b = 1) = \left( g_1 = x_1^2 - x_0 x_2 \; \; g_2 = x_1 x_2 - x_0 y \; \; g_3 = x_2^2 - x_1 y \; \; g_4 = x_1 y^a \; \; g_5 = x_2 y^a \; \; g_6 = y^{a+1} \right)
\]
\[
\phi_2(b = 1) = \left( \begin{array}{cccccccc}
x_2 & y^a & -y & 0 & 0 & 0 & 0 & 0 \\
-x_1 & 0 & x_2 & y^a & 0 & 0 & 0 & 0 \\
x_0 & 0 & -x_1 & 0 & y^a & 0 & 0 & 0 \\
0 & -x_1 & 0 & -x_2 & 0 & x_2 & y & 0 \\
0 & 0 & 0 & -x_2 & -x_1 & 0 & y & 0 \\
0 & 0 & 0 & x_0 & x_1 & 0 & -x_1 & -x_2 \\
\end{array} \right),
\]
\[
\phi_3(b = 1) = \left( \begin{array}{cccc}
y^a & 0 & 0 & 0 \\
-x_2 & y & 0 & 0 \\
0 & y^a & 0 & 0 \\
x_1 & -x_2 & 0 & 0 \\
-x_0 & x_1 & 0 & 0 \\
0 & -x_2 & y & 0 \\
0 & x_1 & -x_2 & 0 \\
0 & -x_0 & x_1 & 0 \\
\end{array} \right),
\]
\[
\phi_1(b = 2) = \left( g_1 = x_1^2 - x_0 x_2 \; \; g_2 = x_1 x_2 - x_0 y \; \; g_3 = x_2^2 - x_1 y \; \; g_4 = x_2 y^a \; \; g_5 = y^{a+1} \right)
\]
that rank $\varphi$ contains a regular sequence of length 3.

It is easy to show that

$$\phi_2(b = 2) = \begin{pmatrix} x_2 & -y & 0 & 0 & 0 \\ -x_1 & x_2 & y^a & 0 & 0 \\ x_0 & -x_1 & 0 & y^a & 0 \\ 0 & 0 & -x_1 & -x_2 & y \\ 0 & 0 & x_0 & x_1 & -x_2 \end{pmatrix},$$

$$\phi_3(b = 2) = \begin{pmatrix} g_3 = y^{a+1} \\ g_4 = x_2y^a \\ -g_3 = -x_2^2 + x_1y \\ g_2 = x_1x_2 - x_0y \\ g_1 = x_1^2 - x_0x_2 \end{pmatrix},$$

$$\phi_1(b = 3) = \begin{pmatrix} 1 \end{pmatrix}$$

$$\phi_2(b = 3) = \begin{pmatrix} x_2 & y^{a+1} & -y & 0 & 0 \\ -x_1 & 0 & x_2 & y^{a+1} & 0 \\ x_0 & 0 & -x_1 & 0 & y^{a+1} \\ 0 & -x_1^2 + x_0x_2 & 0 & -x_1x_2 + x_0y & -x_2^2 + x_1y \end{pmatrix},$$

$$\phi_3(b = 3) = \begin{pmatrix} y^{a+1} \\ -x_2 \\ 0 & y^{a+1} \\ -x_0 & x_1 \\ -x_2 \end{pmatrix}.$$

**Proof.** We will prove the theorem for the three cases, $b = 1, 2, \text{ and } 3.$

**Case $b = 1.$**

It is easy to show that $\phi_1(1)\phi_2(1) = \phi_2(1)\phi_3(1) = 0$ which proves that the above sequence is a complex. To prove the exactness, we use Corollary 2 of Buchsbaum-Eisenbud theorem in [3]. We have to show that rank $\varphi$ contains a regular sequence of length $i$ for all $1 \leq i \leq 3.$ rank $\phi_1(1) = 1$ is trivial. We want to show that rank $\phi_2(1) = 5$. Since the columns of the matrix $\phi_2(1)$ are related by the generators of the defining ideal $I(C)$, note that all $6 \times 6$ minors of $\phi_2(1)$ are zero. $\phi_2(1)$ has a non zero divisor in the kernel. By McCoy’s theorem rank $\phi_2(1) \leq 5$. The determinants of $5 \times 5$ minors of $\phi_2(1)$ are $x_0g_6^2$ when the 6th row and the columns 3, 5 and 6 are deleted, and $x_1g_2^2$ when the 2nd row and the columns 2, 5 and 8 are deleted. Since $\{x_0g_6^2, x_1g_2^2\}$ are relatively prime, $I(\phi_2(1))$ contains a regular sequence of length 2. Also, among the $3 \times 3$ minors of $\phi_3(1)$, we have $\{-x_0g_1, -x_1g_2, -x_2g_3\}$. They are relatively prime, so $I(\phi_3(1))$ contains a regular sequence of length 3.

**Case $b = 2.$**

Clearly $\phi_1(2)\phi_2(2) = \phi_2(2)\phi_3(2) = 0$ and rank $\phi_1(2) = 1$ and rank $\phi_3(2) = 1$. We have to show that rank $\phi_2(2) = 4$ and $I(\phi_2(1))$ contains a regular sequence of length $i$ for all $1 \leq i \leq 3.$ Among the $4 \times 4$ minors of $\phi_2(2)$, $I(\phi_2(2))$ contains $\{-g_1^2, -g_2^2\}$. These two determinants
constitute a regular sequence of length 2, since they are relatively prime.

**Case b = 3.**

As in the previous cases, we have to show that rank \( \phi_1(3) = 1 \), rank \( \phi_2(3) = 3 \) and rank \( \phi_3(3) = 2 \), and also that \( I(\phi_i(1)) \) contains a regular sequence of length \( i \) for all \( 1 \leq i \leq 3 \). rank \( \phi_1(3) = 1 \) is trivial. We have to show that rank \( \phi_2(3) = 3 \). \( \phi_2(3) \) has a non zero divisor in the kernel. By McCoy’s theorem rank \( \phi_2(3) \leq 3 \). Among the \( 3 \times 3 \) minors of \( \phi_2(3) \), \( I(\phi_2(3)) \) contains \( \{g_1^2, g_2^2\} \) which is a regular sequence of length 2, since they are relatively prime. Also, among the \( 2 \times 2 \) minors of \( \phi_3(3) \), we have \( \{g_1, -g_2, g_3\} \). They are relatively prime, so \( I(\phi_3(3)) \) contains a regular sequence of length 3.

**Corollary 3.2.** Under the hypothesis of Theorem 3.1., the minimal graded free resolution of the associated graded ring \( G \) is given by

if \( b = 1 \)

\[
0 \longrightarrow R(-(a+3))^3 \xrightarrow{\phi_1(3)} R(-3)^2 \bigoplus R(-(a+2))^6 \xrightarrow{\phi_2(3)} R(-2)^3 \bigoplus R(-(a+1))^3 \xrightarrow{\phi_3(3)} R
\]

if \( b = 2 \)

\[
0 \longrightarrow R(-(a+4))^2 \xrightarrow{\phi_3(3)} R(-3)^2 \bigoplus R(-(a+2))^3 \xrightarrow{\phi_2(3)} R(-2)^3 \bigoplus R(-(a+1))^2 \xrightarrow{\phi_1(3)} R
\]

if \( b = 3 \)

\[
0 \longrightarrow R(-(a+4))^2 \xrightarrow{\phi_3(3)} R(-3)^2 \bigoplus R(-(a+3))^3 \xrightarrow{\phi_2(3)} R(-2)^3 \bigoplus R(-(a+1))^2 \xrightarrow{\phi_1(3)} R
\]

If \( H_G(i) = \dim_k(m^i/m^{i+1}) \) is the Hilbert function of \( G \), then

**Corollary 3.3.** Under the hypothesis of Theorem 3.1., the Hilbert function of the associated graded ring \( G \) is given by

if \( b = 1 \)

\[
H_G(i) = \binom{i + 3}{3} - 3 \binom{i + 1}{3} - 3 \binom{i - a + 2}{3} + 2 \binom{i}{3} + 6 \binom{i - a + 1}{3} - 3 \binom{i - a}{3}
\]

if \( b = 2 \)

\[
H_G(i) = \binom{i + 3}{3} - 3 \binom{i + 1}{3} - 2 \binom{i - a + 2}{3} + 2 \binom{i}{3} + 3 \binom{i - a + 1}{3} - \binom{i - a - 1}{3}
\]

if \( b = 3 \)

\[
H_G(i) = \binom{i + 3}{3} - 3 \binom{i + 1}{3} - \binom{i - a + 2}{3} + 2 \binom{i}{3} + 3 \binom{i - a}{3} - 2 \binom{i - a - 1}{3}
\]
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