GENERIC PROPERTIES OF DISPERSION RELATIONS FOR DISCRETE PERIODIC OPERATORS

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Abstract. An old problem in mathematical physics deals with the structure of the dispersion relation of the Schrödinger operator \(-\Delta + V(x)\) in \(\mathbb{R}^n\) with periodic potential near the edges of the spectrum, i.e. near extrema of the dispersion relation. A well known and widely believed conjecture says that generically (with respect to perturbations of the periodic potential) the extrema are attained by a single branch of the dispersion relation, are isolated, and have non-degenerate Hessian (i.e., dispersion relations are graphs of Morse functions). In particular, the important notion of effective masses hinges upon this property.

The progress in proving this conjecture has been slow. It is natural to try to look at discrete problems, where the dispersion relation is (in appropriate coordinates) an algebraic, rather than analytic, variety. Moreover, such models are often used for computation in solid state physics (the tight binding model). Alas, counterexamples exist showing that the genericity fails in some discrete situations.

We consider a general periodic discrete operator depending polynomially on some parameters. We prove the natural dichotomy: the non-degeneracy of extrema either fails or holds in the complement of a proper algebraic subset of the parameters. Thus, a random choice of a point in the parameter space gives the correct answer “with probability one.”

We then use methods from computational and combinatorial algebraic geometry to prove the genericity conjecture for a particular diatomic \(\mathbb{Z}^2\)-periodic structure with many free parameters. Here several new approaches to the genericity problem introduced and many examples of both alternatives are provided.

1. Introduction

Consider a \(\mathbb{Z}^n\)-periodic self-adjoint elliptic operator \(L\) in \(\mathbb{R}^n\). The issue discussed below can be formulated and studied in a more general setting, but the reader can think of the Schrödinger operator \(L = -\Delta + V(x)\) with a real \(\mathbb{Z}^n\)-periodic potential \(V(x)\), which we assume to be sufficiently “nice\(^1\),” e.g., \(L_\infty\).

1.1. Dispersion relation and spectrum. We recall some notions from the spectral theory of periodic operators and solid state physics (see, e.g. [3,33,34,43,49]).

For \(k \in \mathbb{R}^n\) (called quasimomentum in physics) let us define the twisted Schrödinger operator \(L(k)\) to be \(L\) applied to functions \(u(x)\) on \(\mathbb{R}^n\) that are \(k\)-automorphic (also called Floquet, sometimes Bloch functions, with quasimomentum \(k\)), i.e.

\[
(1) \quad u(x + \gamma) = e^{ik \cdot \gamma} u(x) \text{ for any } \gamma \in \mathbb{Z}^n, 
\]

\(^1\)Assuming the potential even being \(C^\infty\) does not seem to make the problem we discuss any easier.
where $k \cdot \gamma = \sum_j k_j \gamma_j$. In other words, $u(x) = e^{i k \cdot x} p(x)$, where the function $p(x)$ is $\mathbb{Z}^n$-periodic.

Then $L(k)$ is an elliptic operator in a line bundle over the torus. The quasimomentum $k$ is well defined up to shifts by vectors from the lattice $2\pi \mathbb{Z}^n$ (the dual lattice $G^*$ to $G := \mathbb{Z}^n$, i.e. consisting of all vectors $k$ such that $k \cdot \gamma \in 2\pi \mathbb{Z}$ for any $\gamma \in G$). Thus, it is sufficient to restrict $k$ to the Brillouin zone $B = [-\pi, \pi)^n$.

It is often convenient to factor out the $G^*$-periodicity and consider instead of vectors $k \in \mathbb{C}^n$ the complex vectors $z$ with non-zero components

$$ z := e^{i k} := (e^{i k_1}, \ldots, e^{i k_n}) \in (\mathbb{C} \setminus \{0\})^n. $$

**Definition 1.** Vectors $z$ in (2) are called Floquet multipliers (the name comes from the Floquet theory for ODEs [17, 40, 50], where $n = 1$).

When the quasimomentum $k$ is real, the corresponding Floquet multiplier belongs to the unit torus

$$ \mathbb{T}^n := \{ z \in \mathbb{C}^n \mid |z_j| = 1, j = 1, \ldots, n \} \subset \mathbb{C}^n. $$

In terms of Floquet multipliers $z$, the Floquet functions satisfy

$$ u(x + \gamma) = z^\gamma u(x) \quad \text{for any } \gamma \in \mathbb{Z}^n, $$

where $z^\gamma = e^{i k \cdot \gamma} = e^{i \sum_j k_j \gamma_j}$. In other words, $u(x) = z^x p(x)$, where the function $p(x)$ is $\mathbb{Z}^n$-periodic.

1.2. Characters and quasimomenta. Floquet theory is a version of Fourier series expansion. One thus is interested in harmonics into which the expansion is done. These are the characters of the group $G$ of periods.

**Definition 2.** A character of a group $G$ is a homomorphism $\gamma : G \rightarrow \mathbb{C} \setminus \{0\}$, where the set $\mathbb{C} \setminus \{0\}$ of non-zero complex numbers is a group with respect to multiplication. In other words, $\gamma$ satisfies the following conditions:

$$ \gamma(e) = 1, \quad \text{where } e \text{ is the unit in } G, $$

$$ \gamma(g_1 g_2) = \gamma(g_1) \gamma(g_2) \quad \text{for any elements } g_1, g_2 \text{ of } G. $$

A unitary character is a character that maps $G$ into the unit circle $\mathbb{S} := \{ z \in \mathbb{C} \mid |z| = 1 \}$.

The following statement is well known (and easy to prove):

**Lemma 3.**

1. Every character of $G = \mathbb{Z}^n$ can be represented by a vector $k \in \mathbb{C}^n$ as follows:

$$ \gamma(g) = e^{i k \cdot g}, \quad g \in \mathbb{Z}^n, $$

where $k \cdot g = \sum_j k_j g_j$.

2. The character in (5) is unitary if and only if $k \in \mathbb{R}^n$.

3. Characters (unitary characters) provide all irreducible (unitary irreducible) representations of $\mathbb{Z}^n$.

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2In another Floquet multiplier $z$ incarnation, see the Definition 1 below, $L(z)$ acts in the trivial bundle over the torus, but the operator depends analytically on $z$
1.3. Floquet decomposition. The above discussion suggests the use of Fourier series. A standard argument justifies the decomposition of the (unbounded self-adjoint) operator $L$ in $L_2(\mathbb{R}^n)$ into the direct integral (see [33, 34, 43, 49])

$$L = \bigoplus \int_B L(k)dk.$$ 

As $L(k)$ is an elliptic operator in sections of a line bundle over the torus $\mathbb{R}^n/\mathbb{Z}^n$, it has discrete spectrum $\sigma(k) := \sigma(L(k))$ that consists of infinitely many eigenvalues, each of finite multiplicity,

$$\lambda_1(k) < \lambda_2(k) \leq \lambda_3(k) \leq \cdots \to \infty.$$ 

See [33, 34, 43, 49] for more details.

**Definition 4.** The (real) dispersion relation (or Bloch variety) $B_L$ of the periodic operator $L$ is the subset of $\mathbb{R}^n_k \times \mathbb{R}_\lambda$, where $(k, \lambda) \in B_L$ if and only if

$$Lu = \lambda u \text{ has a non-zero solution } u(x) = e^{ik \cdot x} p(x),$$

where $p(x)$ is periodic with the same group of periods as the operator.

The $j$th eigenvalue function $\lambda_j(k)$ is the $j$th band function.

Thus, the dispersion relation is the graph of the multiple-valued function $k \mapsto \sigma(L(k))$.

![Figure 1](image)

**Figure 1.** A dispersion relation. Three lower branches (band functions) are shown.

**Remark 5.** By allowing both the quasimomentum $k$ and the spectral parameter $\lambda$ to be complex, one defines the complex Bloch variety $B_{L, \mathbb{C}}$. In the complex domain, though, numbering the eigenvalues in their order becomes impossible. In many cases, they become branches of the same irreducible analytic function.

The following properties of the dispersion relation (Bloch variety) are well known:

**Proposition 6 ([33, 34]).**

1. The complex Bloch variety is an analytic subvariety of $\mathbb{C}^n_k \times \mathbb{C}_\lambda$. Namely, it is set of all zeros of an entire function $f(k, \lambda)$ of a finite exponential order\(^3\) on $\mathbb{C}^n_k \times \mathbb{C}_\lambda$.

2. The projection of the real Bloch variety onto the real $\lambda$-axis is the spectrum $\sigma(L)$ of the operator $L$ in $\mathbb{R}^n$.

\(^3\)In some instances, the exponential estimate becomes important, see Section 7.
(3) The projection of the graph of the \( j \)th band function into the real \( \lambda \)-axis is a finite closed interval called the \( j \)th spectral band \( I_j \). The spectral bands might overlap, or leave open spaces in between called spectral gaps, see Fig. 2.

![Spectral bands and gaps](image)

**Figure 2.** Spectral bands and gaps.

**Spectral edges** are the endpoints of spectral gaps (see Fig. 2). By Proposition 6, they correspond to some of the extremal values of band functions. We are interested in the generic (with respect to perturbation of the periodic potential or other parameters of the operator) structure of these extrema. The genericity can be understood in a variety of ways, e.g. holding for a second Baire category set of potentials in an appropriate (Banach) space of potentials (the most likely situation), or stronger one - for a dense open subset, or even stronger - in the exterior of an analytic (or even algebraic) subset in the space.

1.4. **The spectral edge conjecture.** An old conjecture, more or less explicitly formulated in a variety of sources, e.g. in [34, Conjecture 5.25] or in [18,33,41,42], deals with the structure of the dispersion relation of the Schrödinger operator in \( \mathbb{R}^n \) with periodic potential near the edges of the spectrum, i.e. near (some of) the extrema of the dispersion relation. This well known and widely believed conjecture says that generically the band functions are Morse functions. We make this precise below:

**Conjecture 7.** Generically (with respect to the potentials and other free parameters of the operator, e.g. metric in the Laplace-Beltrami operator), the extrema of band functions satisfy the following conditions:

1. Each extremal value is attained by a single band \( \lambda_j(k) \).
2. The loci of extrema are isolated.
3. The extrema are non-degenerate, i.e. at them the corresponding band functions have non-degenerate Hessians.

Notice that (3) implies (2).

This conjecture asserts that generically near a spectral edge the dispersion relation has a parabolic shape and thus resembles the dispersion relation at the bottom of the spectrum of the free operator \(-\Delta\). This in turn would trigger appearance of various properties analogous to those of the Laplace operator. One can mention, for instance, electron’s effective masses in solid state theory [3,30], Green’s function asymptotics [6,26,28,37], homogenization [8,14], Liouville type theorems [4,27,35,36,38], Anderson localization [1], perturbation of discrete spectra in gaps in general [11–13], and others.
Why would one conjecture this? Existence of a degenerate extremum of the dispersion relation is an “analytic equality type” restriction. It is thus natural to believe that it holds either almost never or almost always with respect to the potential and other parameters of the operator. It is unlikely that such a restriction for (almost) all periodic potentials would be not known. To put it differently, the common idea is that generically, the dispersion relation probably behaves like the spectrum of a “generic” family of self-adjoint matrices [5]. As we have already mentioned, this has been conjectured in various (explicit or implicit) forms by several authors.

The progress in proving this conjecture has been very slow. We summarize here briefly the successes achieved so far. It is known, that the expected parabolic structure always (not only generically) holds at the bottom of the spectrum of Schrödinger operator with a periodic electric potential [29] (which is not necessarily true if a magnetic field is involved [48]). In [31], the statement (1) of the conjecture was proven. The full conjecture was proven in [18] in $2D$ for any given number of bands and small smooth potentials. The statement (2) was proven in $2D$ [21] in a stronger form, even without the genericity clause.

1.5. “Extended” dispersion relations. Besides varying the quasimomentum $k$, one might vary other parameters of the operator, e.g. of the periodic potential $V \in L_\infty(W)$, or introduce a periodic diffusion coefficient (metric) in the operator: $-\nabla \cdot D(x) \nabla u + Vu$ (assuming that ellipticity is preserved). One can thus consider an extended dispersion relation in an appropriate Banach space of quadruples $(D,V,k,\lambda)$. As long as ellipticity is preserved, an analog of Proposition 6 still holds [34].

1.6. Discrete version. It is natural to first look at discrete problems. Then, the dispersion relation becomes (in appropriate coordinates) an algebraic variety [22, 32]. Such discrete formulations are frequently used, e.g. in solid state physics, when using the tight binding approximation [3, 30]. Some counterexamples to generic non-degeneracy in the discrete [21], as well as quantum graph [9,21] case have been known.

Calling a set generic if its complement is contained in a proper algebraic subset, we show that in the discrete periodic case, the following dichotomy holds: either the set of parameters for which there are degenerate critical points is generic, or there is a generic set of parameters for which there are no degenerate critical points. Thus, testing a “random” sample of parameters should provide an “almost surely correct” answer.

1.7. The structure of the paper: In Section 2 we provide a detailed description of the discrete case. The main result (Theorem 15) and its proof are presented in Section 3. A specific example is considered in Section 4, and we present three approaches to establishing Conjecture 7 for this example. These are based on a numerical computation, an “almost surely” verification, and then an actual proof that relies on an exact count of solutions. We use symbolic computation to find all maximal substructures of this example for which

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4The example in [21] deals with a two-atomic structure on $\mathbb{Z}^2$, where the potential has just two different possible values: $v_0$ attained at all vertices $(n,m) \in \mathbb{Z}^2$ such that $n + m$ is even, and $v_1 \neq v_0$ when $n + m$ is odd. Thus, the only free parameters are $v_0$ and $v_1$. The impression is that the number of parameters should be large enough in order for the genericity to hold. We conjecture (see Conjecture 17) and prove in a particular case (Theorem 26) that increasing the number of parameters cannot destroy generic non-degeneracy.
Conjecture 7 does not hold. Final remarks can be found in Section 7 and acknowledgments in Section 8.

2. Description of the discrete case

We consider a discrete situation, i.e. when the group $G = \mathbb{Z}^n$ acts on a graph $\Gamma$ with the set of vertices $V$ and edges $E$. We write $x \sim y$ for two vertices $x, y \in V$ when there exists an edge connecting them.

Let us start with making this notion precise.

2.1. Periodic graphs.

Definition 8. An infinite graph $\Gamma$ is said to be periodic (or $\mathbb{Z}^n$-periodic) if $\Gamma$ is equipped with an action of the free abelian group $G = \mathbb{Z}^n$, i.e. a mapping $(g, x) \in G \times \Gamma \mapsto gx \in \Gamma$, such that the following properties are satisfied:

1. **Group action:**
   - For any $g \in G$, the mapping $x \mapsto gx$ is a bijection of $\Gamma$;
   - $0x = x$ for any $x \in \Gamma$, where $0 \in G = \mathbb{Z}^n$ is the neutral element;
   - $(g_1g_2)x = g_1(g_2x)$ for any $g_1, g_2 \in G, x \in \Gamma$.

2. **Faithful:** If $gx = x$ for some $x \in \Gamma$, then $g = 0$.

3. **Discrete:** For any $x \in \Gamma$, there is a neighborhood $U$ of $x$ such that $gx \notin U$ for $g \neq 0$.

4. **Co-compact:** The space of orbits $\Gamma/G$ is finite. In other words, the whole graph can be obtained by the $G$-shifts of a finite subset.

5. **Structure preservation:**
   - $gu \sim gv$ if and only if $u \sim v$. In particular, $G$ acts bijectively on the set of edges.
   - If other parameters are present (e.g., weights at vertices or at edges), the action preserves their values.

A simple way to visualize this is to think of a graph $\Gamma$ embedded into $\mathbb{R}^n$ in such a way that it is invariant with respect to the shifts by integer vectors $g \in \mathbb{Z}^n \subset \mathbb{R}^n$, which produces an action of $\mathbb{Z}^n$ on $\Gamma$. When $n \geq 3$, using curved edges, the graph $\Gamma$ has such an equivariant embedding in $\mathbb{R}^n$ where the action is shifting by integer vectors in $\mathbb{R}^n$. When $n = 2$, edges of a graph may cross when embedded into $\mathbb{R}^2$, and when $n = 1$, severe overlapping will occur. In the cases when $n = 1$ or 2, the graph $\Gamma$ has an embedding in $\mathbb{R}^3$ that is periodic with respect to a copy of either $\mathbb{Z}$ or $\mathbb{Z}^2$ embedded in a coordinate ray or coordinate plane.

Definition 9. Due to co-compactness (4), there exists a finite part $W$ of $\Gamma$ such that

- The union of all $G$-shifts of $W$ covers $\Gamma$,
  \[ \bigcup_{g \in G} gW = \Gamma. \]
- Different shifted copies of $W$, i.e. $g_1W$ and $g_2W$ with $g_1 \neq g_2 \in G$, do not share any vertices.

Such a compact subset $W$ is called a fundamental domain for the action of $G$ on $\Gamma$.

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5 It is easy to modify the notions and proofs below for the case when multiple edges are allowed between a pair of vertices.
Examples of fundamental domains are shown in Figs. 3 and 4.

**Remark 10.** Note that a fundamental domain $W$ is not uniquely defined.

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2.2. Floquet-Bloch theory.

2.2.1. *Floquet transform on periodic graphs.* As in the continuous case, the standard idea of harmonic analysis suggests that, as long as we are dealing with a linear problem that commutes with an action of the abelian group $G = \mathbb{Z}^n$, Fourier series expansion$^6$ with respect to this group should simplify the problem. Its implementation leads to what is known as the **Floquet transform.** Indeed, what one needs to do is to expand functions on the graph $\Gamma$ into the unitary characters $\gamma_k$ or, equivalently, $\gamma_z$.

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$^6$I.e., expansion into irreducible representations.
Let $\Gamma$ be a $\mathbb{Z}_n$-periodic graph and $f$ be a finitely supported (or sufficiently fast decaying) function defined on the set of vertices $V$ of $\Gamma$.

**Definition 11.** We define the Floquet transform of $f$ as

\[ f(v) \mapsto \hat{f}(v, z) = \sum_{g \in \mathbb{Z}_n} f(gv)z^{-g}, \]

where $gv$ denotes the action of $g \in \mathbb{Z}_n$ on the vertex $v \in V$ and $z = (z_1, \ldots, z_n) \in (\mathbb{C} \setminus \{0\})^n$ is the Floquet multiplier.

Using quasimomenta instead of Floquet multipliers, (8) becomes

\[ f(v) \mapsto \hat{f}(v, e^{ik}) = \sum_{g \in \mathbb{Z}_n} f(gv)e^{-i(k \cdot g)}. \]

The reader can notice that (8)–(9) is just the Fourier transform with respect to the action of $G = \mathbb{Z}_n$ on the set $V$ of vertices.

Let us formulate some basic properties of the Floquet transform. The following statement follows by a direct inspection of (8)–(9).

**Lemma 12.** The following identities hold:

\[ \hat{f}(gv, z) = z^g \hat{f}(v, z), \]

\[ \hat{f}(gv, e^{ik}) = e^{ik \cdot g} \hat{f}(v, e^{ik}) \]

\[ \hat{f}(v, e^{i(k + \gamma)}) = \hat{f}(v, e^{ik}) \text{ for } \gamma \in G^* = 2\pi \mathbb{Z}_n. \]

The equalities (10)–(11) of the lemma show that, as one would expect, $G$-shifts after Floquet transform become multiplication by the corresponding characters. To put it differently, for a fixed $z$, the function $\hat{f}(v, z)$ on $\Gamma$ is automorphic with the character $z^g = e^{ik \cdot g}$. It also shows that the values of $\hat{f}(v, z)$ are determined completely if they are known for vertices $v$ from a fundamental domain $W$ as they can be extended to the whole graph using (10). We thus introduce the following notation.

**Definition 13.** Let $W$ be a (finite) fundamental domain of the action of the group $G = \mathbb{Z}_n$ on $\Gamma$. We will denote $\hat{f}(v, z)|_{v \in W}$ by $\hat{f}(z)$, where the latter expression is considered as a function of $z$ with values in the space of functions on $W$. In other words, $\hat{f}(z)$ takes values in $\mathbb{C}^{|W|}$.

We also see that the Floquet transformed function is a $G^*$-periodic function of the quasi-momentum $k$ according to the identity (12).

**Remark 14.** One can also interpret the Floquet transform as follows: one takes a function $\phi$ on $\Gamma$ and cuts $\Gamma$ into non-overlapping pieces by restricting $f$ to the shifted copies $gW$ of a fundamental domain $W$. These pieces are shifted back to $W$ and then are taken as (vector valued) Fourier coefficients of the Fourier series (9) that defines the Floquet transform.

2.2.2. Floquet transform of periodic difference operators. Let $A$ be a difference operator on a $\mathbb{Z}_n$-graph $\Gamma$. In other words, $A$ is an infinite $|V| \times |V|$ matrix. We assume that $A$ has finite order, meaning that in each row it has only finitely many non-zero entries (in other words, only finitely many neighbors of each vertex are involved). We also assume that $A$ is periodic, i.e.
commuting with the action of the group $\mathbb{Z}^n$. After Floquet transform this operator becomes
the operator of multiplication by a matrix $A(z)$ of size $|W| \times |W|$ depending rationally on the
Floquet multiplier $z$ (or analytically on the quasimomentum $k$).

As an example, let us consider the Laplace operator on the regular hexagonal lattice $2D$
lattice $\Gamma$ (see Fig. 5). The group $\mathbb{Z}^2$ acts on $\Gamma$ by the shifts by vectors $p_1e_1 + p_2e_2$, where

\[(p_1, p_2) \in \mathbb{Z}^2\] and vectors $e_1 = (3/2, \sqrt{3}/2)$, $e_2 = (0, \sqrt{3})$ are shown in Figure 5. We choose as
a fundamental domain (Wigner-Seitz cell) of this action the shaded parallelogram region $W$. Two black vertices $a$ and $b$ belong to $W$, while $b'$, $b''$, and $b'''$ lie in shifted copies of $W$. Three
edges $f, g, h$, directed as shown in the picture, belong to $W$.

We consider the Laplace operator

\[Af(v) = \sum_{w \sim v} f(w) - 3f(v).\]

One can find the “symbol” $A(z)$ (or $A(k)$ in terms of the quasimomenta) by applying $A$ to
functions $f$ automorphic with the character $z = e^{ik}$. Such a function is determined by its
values at the vertices $a$ and $b$. Indeed, since $b'$ is obtained from $b$ by a shift of $-e_1$, one has
$f(b') = z_1^{-1} f(b) = e^{-ik_1} f(b)$. Similarly, $f(b'') = z_2^{-1} f(b) = e^{-ik_2} f(b)$. One also finds that the
values at two neighbors of the vertex $b$ are $z_1 f(a) = e^{ik_1} f(a)$ and $z_2 f(a) = e^{ik_2} f(a)$. Hence,

\[
\begin{align*}
(Af)(a) &= -3f(a) + (z_1^{-1} + z_2^{-1} + 1) f(b) \\
&= -3f(a) + (e^{-ik_1} + e^{-ik_2} + 1) f(b), 	ext{ and} \\
(Af)(b) &= (z_1 + z_2 + 1)f(a) - 3f(b) \\
&= (e^{ik_1} + e^{ik_2} + 1)f(a) - 3f(b).
\end{align*}
\]
We thus obtain the expression for the “symbol” of $A$:

\begin{equation}
A(z) = \begin{pmatrix}
-3 & z_1^{-1} + z_2^{-1} + 1 \\
z_1 + z_2 + 1 & -3
\end{pmatrix}
\end{equation}

(14)

\begin{equation}
A(k) = \begin{pmatrix}
-3 & e^{-ik} + e^{-i k_2} + 1 \\
e^{i k_1} + e^{i k_2} + 1 & -3
\end{pmatrix}
\end{equation}

The matrix $A(z)$ depends rationally on the $z$-variables. In fact, it is a Laurent polynomial. Since variables $z$ belong to the unit torus $\mathbb{T}^n$, no singularities of $A(z)$ appear there. It is thus possible to multiply the matrix $A(z)$ by $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ with a sufficiently high power $m$, so the resulting matrix $\tilde{A}(z)$ has polynomial entries. E.g., in the above example, by multiplying by $z_1 z_2$ (i.e. $m = 1$), the dispersion relation in the $(z, \lambda)$ space can be given as follows

\begin{equation}
\{(z, \lambda) \mid \det \left( \tilde{A}(z) - \lambda z_1 z_2 \right) = 0 \}, \quad \text{where}
\end{equation}

\begin{equation}
\tilde{A}(z) = \begin{pmatrix}
-3 z_1 z_2 & z_1 + z_2 + z_1 z_2 \\
z_1^2 z_2 + z_1 z_2^2 + z_1 z_2 & -3 z_1 z_2
\end{pmatrix}
\end{equation}

(15)

The dispersion relation is an algebraic variety of codimension 1 in $\mathbb{C}^3$.

An analogous construction holds for general finite order periodic difference operators on graphs, with $z_1 z_2$ being replaced by the product $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$.

We can consider the equation

\begin{equation}
\Phi(z, \lambda) := \det \left( \tilde{A}(z) - \lambda z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \right) = 0
\end{equation}

(16)

as an implicit description of the graph of a multiple-valued function

$$F : z \mapsto \lambda$$

that shows dependence of the spectrum of $\tilde{A}$ on the parameters $z$.

The matrix $A$ depends polynomially on extra parameters $\alpha$ (e.g., weights at vertices and/or edges, potentials, etc.),

\begin{equation}
\Phi(\alpha, z, \lambda) := \det \left( \tilde{A}(\alpha, z) - \lambda z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \right) = 0.
\end{equation}

(17)

Correspondingly, we have a family of functions

$$F_\alpha := F(\alpha, \cdot) : z \mapsto \lambda.$$

Thus, the question can be reformulated as follows:

**Does non-degeneracy of all critical points of the function $F_\alpha$ on the torus hold generically with respect to the parameters $\alpha$?**

Establishing any weaker type of genericity would be valuable, in particular for the continuous (PDE) situation, but as our results show, in discrete case it should be understood in the strongest sense: as being valid outside of a proper algebraic subset. This is clearly not expected to happen for PDEs.
3. The critical point dichotomy

The matrix $A(\alpha, z, \lambda)$ introduced above, and thus $\tilde{A}(\alpha, z, \lambda)$ as well, has a very special structure, due to the periodicity. However, an important dichotomy holds in a very general situation, without any connection to the periodicity.

We now formulate and prove the following dichotomy statement.

**Theorem 15.** Let $U \subset \mathbb{C}^n$ be a neighborhood of the torus $\mathbb{T}^n$, $P(z)$ be a polynomial, and $A(\alpha, z)$ be a finite size matrix polynomially dependent on the parameters $\alpha \in \mathbb{C}^m$ and $z \in \mathbb{C}^n$. Consider for any $\alpha$ the equation

$$\Phi_\alpha(z, \lambda) := \det (A(\alpha, z) - \lambda P(z)I) = 0$$

as an implicit description of the graph of a multiple-valued function

$$F_\alpha: z \mapsto \lambda,$$

that shows dependence of the (weighted by $P(z)$) spectrum of $A$ on the parameters $z$.

Then the set $DV$ of points $\alpha$ where $F_\alpha$ has a degenerate critical point on (and near to) torus $\mathbb{T}^n$, either belongs to a proper algebraic subset of $\mathbb{C}^m$ or it contains the complement of such a set.

**Proof.** Let us first define the set $DC \subset \mathbb{C}^m \times U \times \mathbb{C}$ of points $(\alpha, z, \lambda)$, where one encounters a degenerate critical point of $F_\alpha$. Projecting into the $\alpha$-space $\mathbb{C}^m$, we obtain the set $DV$ of parameters $\alpha$ that we are interested in.

This can be easily done in terms of the function $\Phi_\alpha(z, \lambda)$, by using the condition that this function, as well as implicitly computed gradient and Hessian of $\lambda$ with respect to $z$ all vanish. This, indeed, can be done by implicit differentiation using the equation $\Phi_\alpha(z, \lambda) = 0$:

$$\begin{align*}
\Phi_\alpha(z, \lambda) &= 0, \\
\frac{\partial \lambda}{\partial z_j} &= 0 \text{ for all } j = 1, \ldots, n, \\
\det \left( \frac{\partial^2 \lambda}{\partial z_i \partial z_j} \right) &= 0.
\end{align*}$$

The gradient and Hessian of $\lambda$ with respect to $z$ can be obtained by implicitly differentiating equation $\Phi_\alpha(z, \lambda) = 0$. This produces rational expressions whose vanishing is equivalent to vanishing of their polynomial numerators. Thus, one obtains the system of $n + 2$ polynomial equations

$$\begin{align*}
\Phi_\alpha(z, \lambda) &= 0, \\
P_j(\alpha, z, \lambda) &= 0, \text{ for } j = 1, \ldots, n, \\
H(\alpha, z, \lambda) &= 0,
\end{align*}$$

which describes the set $DC$. We ask: how large can the projection $DV := \pi DC$ of $DC$ into the space $\mathbb{C}^m_\alpha$ of parameters $\alpha$ be? As a projection of an algebraic set, its closure is algebraic. If the dimension of the projection is less than $m$, then it is a proper subset of $\mathbb{C}^m$ and we have genericity of the parameters for which all critical points are nondegenerate. The alternative is that the closure of $DV$ is $\mathbb{C}^m$, so that for generic $\alpha$ there are degenerate critical points. In this last case, there may yet be a proper algebraic set of parameters $\alpha$ for which all critical points are nondegenerate. \qed
Our desire is to have the set $DV$ “small,” i.e. the first alternative of Theorem 15 to take place. However, as we have already mentioned, even in the case of “two-atomic” periodic discrete structure, this is not necessarily the case [21]. So, how can one tell in which of two options of the dichotomy we are in a particular case? While we do not have a complete answer to this, the following “random test” follows from Theorem 15. Let us pick a value of $\alpha$ “randomly” and compute the dispersion relation. If it has no degenerate critical points, then we know “with probability one” that $DV$ is contained in a proper algebraic subset, and thus non-degeneracy is generic. If instead we determine that $\alpha \in DV$, then we know that “with probability 1” degeneracy is generic. Indeed, the chances for a randomly selected point to belong to a given proper algebraic subset are zilch.

**Corollary 16.** If a random choice of parameters provides non-degeneracy, then “almost surely” non-degeneracy holds “almost always,” i.e. outside of a proper algebraic subset. Analogously, if a random choice of parameters produces a degenerate example, degeneracy “almost surely” holds generically.

We also formulate the following conjecture:

**Conjecture 17.** Generic non-degeneracy survives under extending the set of parameters, e.g. if in addition to varying the potential, we start varying the metric as well. In other words, changing more parameters cannot make the situation worse.

4. An example and three alternative approaches

One can ask whether one can avoid a random choice of parameters. The answer is “yes,” but it is not that easy to implement.

Namely, there are $n + 2$ polynomial equations (20) determining the set $DC$. If the codimension of $DC$ were exactly $n + 2$, then projecting onto the space $\mathbb{C}^m_\alpha$ along the $(n + 1)$-dimensional $U \times \mathbb{C}$ would produce a set $DV$ of at least codimension 1 and thus for generic parameters $\alpha$, all critical points are nondegenerate. This dimension-counting was part of our intuition behind Conjecture 7.

Unfortunately, the codimension of an algebraic set (or at list of some of its irreducible components) could be less than the number of the defining equations (in our case, $n + 2$). So, one can try to figure out the dimensions (and thus codimensions) of the irreducible components, which is, as the example below shows, sometimes possible, but far from being easy.

4.1. An example of a discrete structure. We consider the discrete periodic graph $\Gamma$ shown in Fig. 6. The square fundamental domain $W$ contains two vertices (“atoms”) $a$ and $b$ and nine edges shown with solid lines. We allow connections inside $W$ and to its four adjacent copies, introducing thus more free parameters, which hopefully would make the conjecture more likely to hold. No loops or multiple edges are allowed. Shifts of $W$ by integer linear combinations of basis vectors $e_1$ and $e_2$ tile the plane. We write $V$ and $E$ for the sets of vertices and edges of $\Gamma$ respectively.

The graph $\Gamma$ is equipped with a periodic weight function $\alpha$ (an analog of a metric, or an anisotropic diffusion coefficient) that assigns to each edge a non-negative number.
Figure 6. The fundamental domain $W$ of the considered diatomic structure. The basic period vectors are $e_1$ and $e_2$. The fundamental domain contains two atoms $a$ and $b$ and nine (solid) edges. The dotted edges and other atoms are obtained by shifting the fundamental domain by integer linear combinations of $e_1, e_2$. Numbers $\alpha_j$ are weights associated with the solid edges.

Remark 18. Notice that the graph is diatomic, like in [21], but the freedom of choosing parameters is nine-dimensional, while it was only two-dimensional in [21]. The intuition is that this should help the genericity.

Let us denote the set of non-negative real numbers by $\mathbb{R}_+$. Given $\alpha = (\alpha_1, \ldots, \alpha_9) \in \mathbb{R}_+^9$, we can assign the weights $\alpha_j, j = 1, \ldots, 9$, to the edges from the fundamental domain $W$ as shown in Fig. 6. The entire structure $\Gamma$ and all edge weights can be obtained from $W$ by $\mathbb{Z}^2$-shifts (with the basis $e_1,e_2$). Define a divergence\(^7\) (or Laplace-Beltrami) type operator $L_\alpha$ acting on the graph $\Gamma$ as follows:

\[
L_\alpha f(u) = \sum_{e=(u,v) \in E} \alpha(e)(f(u) - f(v)),
\]

where $u, v \in V$ and $\alpha(e)$ is the weight of edge $e$. When this does not lead to confusion, we will use the notation $L$ instead of $L_\alpha$.

For each $k = (k_1,k_2)$ from the Brillouin zone $B = [-\pi, \pi)^2$, let $L(k)$ be the Bloch Laplacial that acts as (21) on the set of functions defined on $\Gamma$ that satisfy the Floquet condition

\[
f(u + p_1 e_1 + p_2 e_2) = f(u)e^{i(p_1 k_1 + p_2 k_2)}
\]

\(^7\)This is a discrete analog of a second order divergence type elliptic partial differential operator.
for all \((p_1, p_2) \in \mathbb{Z}^2\) and all \(u \in V\). Such a function \(f\) is determined by its restriction to the fundamental domain \(W\). Due to the direct integral decomposition (6), one obtains
\[
\sigma(L) = \bigcup_{k \in B} \sigma(L(k)).
\]

Since there are two vertices inside the fundamental domain, each operator \(L(k)\), acts on a two-dimensional space of functions defined on the two atoms, and thus has a spectrum \(\sigma(L(k)) = \{\lambda_1(k), \lambda_2(k)\}\), where \(\lambda_1(k) \leq \lambda_2(k)\).

Our second main result is the following.

**Theorem 19.** The dispersion relation of the operator \(L_\alpha\) generically (i.e., outside of an algebraic subset of the parameters \(\alpha\)) satisfies all three conditions of Conjecture 7.

5. **Proof of Theorem 19**

We provide three arguments for Theorem 19. The first uses the paradigm of numerical algebraic geometry [47]. While it yields a detailed understanding of the set \(DC \subset \mathbb{C}_\alpha^0 \times (\mathbb{C} \setminus \{0\})^2 \times \mathbb{C}_\lambda\) of degenerate critical points of the dispersion relation, it is not a traditional proof as the results of the numerical computation are not certified in the sense of [24]. The second argument uses Theorem 15, and it is probabilistic in the sense of Corollary 16. We give a third argument which is a proof in the traditional sense. The value of these arguments is that they illustrate the possibilities of ideas and methods from computational algebraic geometry for studying such questions.

5.1. **Analytic reduction.** We express the condition that the dispersion relation of \(L_\alpha\) has a degenerate extremum in terms of the Floquet multipliers \(z = (z_1, z_2) := (e^{ik_1}, e^{ik_2})\) instead of quasimomenta \(k \in B\). To start, notice that:

\begin{equation}
\frac{\partial \lambda}{\partial k_j} = i e^{i k_j} \frac{\partial \lambda}{\partial z_j}, \text{ for } j = 1, 2,
\end{equation}

and

\begin{equation}
\frac{\partial^2 \lambda}{\partial k_1^2} \frac{\partial^2 \lambda}{\partial k_2^2} - \left( \frac{\partial^2 \lambda}{\partial k_1 \partial k_2} \right)^2 = e^{2i(k_1+k_2)} \left( \frac{\partial^2 \lambda}{\partial z_1^2} \frac{\partial^2 \lambda}{\partial z_2^2} - \left( \frac{\partial^2 \lambda}{\partial z_1 \partial z_2} \right)^2 \right).
\end{equation}

For each \(z \in T^2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}\), write \(\Lambda_\alpha(z)\) for the operator that acts as (21) on the set of functions defined on \(\Gamma\) that satisfy the condition

\[
f(u + p_1 e_1 + p_2 e_2) = z_1^{p_1} z_2^{p_2} f(u).
\]

The spectrum \(\sigma(\Lambda_\alpha(z))\) of \(\Lambda_\alpha(z)\) coincides with \(\sigma(L_\alpha(k))\) for \(k \in B\) such that \((z_1, z_2) = (e^{ik_1}, e^{ik_2})\).

From (22) and (23), if the dispersion relation of \(L_\alpha\) has a degenerate extremum, then there exist \(z = (z_1, z_2) \in T^2\) and \(\lambda \in \mathbb{R}\) such that \(\lambda \in \sigma(\Lambda_\alpha(z))\) with \(\lambda\) being a critical point,

\begin{equation}
\frac{\partial \lambda}{\partial z_j} = 0, \text{ for } j = 1, 2,
\end{equation}

for all \((p_1, p_2) \in \mathbb{Z}^2\) and all \(u \in V\). Such a function \(f\) is determined by its restriction to the fundamental domain \(W\). Due to the direct integral decomposition (6), one obtains
\[
\sigma(L) = \bigcup_{k \in B} \sigma(L(k)).
\]
and at which the Hessian vanishes,

\[
\frac{\partial^2 \lambda}{\partial z_1^2} \frac{\partial^2 \lambda}{\partial z_2^2} - \left( \frac{\partial^2 \lambda}{\partial z_1 \partial z_2} \right)^2 = 0.
\]

**Definition 20.** Let \( DC_\mathbb{R} \) be the set of \( (\alpha_1, \ldots, \alpha_9, z_1, z_2, \lambda) \in \mathbb{R}_+^9 \times \mathbb{T}^2 \times \mathbb{R} \) such that \( \lambda \in \sigma(\Lambda_\alpha(z)) \) and both (24) and (25) hold.

Note that \( DC_\mathbb{R} \) contains all points of the dispersion relation of \( \Lambda_\alpha \) where \( \lambda \) is a degenerate extremum. Consequently, its projection \( DV_\mathbb{R} \) into the space \( \mathbb{R}_+^9 \) of weights \( \alpha \) includes the set of weights for which there is a degenerate extremal value of a band function for \( \Lambda_\alpha \).

Our proof of Theorem 19 proceeds in three steps.

1. \( DC_\mathbb{R} \) is a subset of an algebraic variety \( DC \subset \mathbb{C}_\alpha^9 \times (\mathbb{C}\setminus\{0\})^2 \times \mathbb{C}_\lambda \).
2. \( DC \) has dimension eight.
3. The projection \( DV \) of \( DC \) to the \( \mathbb{C}_\alpha^9 \) of complex weights \( \alpha \) has dimension at most eight.

This implies that the semialgebraic set \( DV_\mathbb{R} \subset \mathbb{R}_+^9 \cap DV \) has positive codimension in \( \mathbb{R}_+^9 \), which will complete the proof of Theorem 19.

Steps (1), (2), and (3) are consequences of Lemmas 21, 22, and 23, proven below.

We derive equations that vanish on \( DC_\mathbb{R} \). For \( z \in \mathbb{T}^2 \), in the ordered basis \((f(a), f(b))\), the operator \( \Lambda_\alpha(z) \) is represented by the \( 2 \times 2 \) matrix \( A_\alpha(z) = (a_{ij})_{j,l=1}^2 \) with entries

\[
\begin{align*}
        a_{11} &= 2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_9 - \alpha_1(z_2 + z_2^{-1}) - \alpha_4(z_1 + z_1^{-1}), \\
        a_{12} &= -(\alpha_2 z_2 + \alpha_6 z_2^{-1} + \alpha_3 z_1 + \alpha_9 z_1^{-1} + \alpha_5), \\
        a_{21} &= -(\alpha_2 z_2^{-1} + \alpha_3 z_1^{-1} + \alpha_6 z_2 + \alpha_9 z_1 + \alpha_5), \\
        a_{22} &= \alpha_5 + 2\alpha_7 + 2\alpha_8 + \alpha_9 + \alpha_5 + \alpha_2 + \alpha_3 - \alpha_7(z_2 + z_2^{-1}) - \alpha_8(z_1 + z_1^{-1}).
\end{align*}
\]

The eigenvalues of the matrix \( A_\alpha(z) \) form the spectrum of \( \Lambda_\alpha(z) \). Dropping the subscript \( \alpha \), \( \lambda \) belongs to the spectrum of \( \Lambda(z) \) if and only if it satisfies the characteristic equation,

\[
\lambda^2 - \lambda \text{Tr} A(z) + \text{det} A(z) = 0.
\]

For \( j = 1, 2 \), implicit differentiation of (26) leads to an expression of \( \frac{\partial \lambda}{\partial z_j} \) as a rational function.

The vanishing of its numerator is equivalent to \( \frac{\partial^2 \lambda}{\partial z_j} = 0 \), giving the consequence of (24),

\[
\lambda \frac{\partial (\text{Tr} A)}{\partial z_j} - \frac{\partial (\text{det} A)}{\partial z_j} = 0, \text{ for } j = 1, 2.
\]

If we now compute \( \frac{\partial^2 \lambda}{\partial z_i \partial z_j} \) implicitly using (26) and that \( \frac{\partial \lambda}{\partial z_j} = \frac{\partial \lambda}{\partial z_i} = 0 \) (24), we obtain

\[
(2\lambda - \text{Tr} A(z)) \frac{\partial^2 \lambda}{\partial z_i \partial z_j} = \lambda \frac{\partial^2 \text{Tr} A(z)}{\partial z_i \partial z_j} - \frac{\partial^2 \text{det} A(z)}{\partial z_i \partial z_j}.
\]

Thus the vanishing of the Hessian determinant (25) implies that

\[
\left( \lambda \frac{\partial^2 (\text{Tr} A)}{\partial z_1^2} - \frac{\partial^2 (\text{det} A)}{\partial z_1^2} \right) \cdot \left( \lambda \frac{\partial^2 (\text{Tr} A)}{\partial z_2^2} - \frac{\partial^2 (\text{det} A)}{\partial z_2^2} \right)
\]

\[
- \left( \lambda \frac{\partial^2 (\text{Tr} A)}{\partial z_1 \partial z_2} - \frac{\partial^2 (\text{det} A)}{\partial z_1 \partial z_2} \right)^2 = 0.
\]
Let \( g_1, \ldots, g_4 \) be, respectively, the left hand sides of the characteristic equation (26), the two equations (27) for critical points of the dispersion relation, and the Hessian equation (28). These are rational functions in the variables \( \alpha_1, \ldots, \alpha_9, z_1, z_2, \lambda \) with an interesting structure. They are polynomials of degrees 2, 2, 2, and 4 in \( \alpha_1, \ldots, \alpha_9, \lambda \) (homogeneous in \( \alpha \)) and Laurent polynomials in \( z_1, z_2 \)—their denominators all have the form \( z_1^{n_1} z_2^{n_2} \) for some \( n_1, n_2 \in \mathbb{N} \). Thus \( g_1, \ldots, g_4 \) are Laurent polynomials that are defined on \( \mathbb{C}^9_{\alpha} \times (\mathbb{C}\setminus\{0\})^2 \times \mathbb{C}_\lambda \). Let \( DC \) be the algebraic variety defined by \( g_1 = g_2 = g_3 = g_4 = 0 \). This implies our first lemma.

**Lemma 21.** The set \( DC_R \) of points on the dispersion relation for the family \( L_\alpha \) having degenerate critical points is a subset of the set of real points of the algebraic variety \( DC \).

To study the variety \( DC \subset \mathbb{C}^9 \times (\mathbb{C}\setminus\{0\})^2 \times \mathbb{C} \), for each \( j = 1, \ldots, 4 \), let \( f_j \) be the numerator of \( g_j \). Then \( f_1, \ldots, f_4 \) are ordinary polynomials in \( \alpha, z, \lambda \). Let \( P \subset \mathbb{C}^{12} \) be the algebraic variety defined by the vanishing of \( f_1, \ldots, f_4 \). Then \( DC = P \cap (\mathbb{C}^9 \times (\mathbb{C}\setminus\{0\})^2 \times \mathbb{C}) \), but we may have \( DC \neq P \), as \( P \) may have components where \( z_1 z_2 = 0 \). Indeed that is the case.

**Lemma 22.** The dimension of \( P \) is nine and the dimension of \( DC \) is eight.

In Subsection 5.2 we describe the decomposition of \( P \) into irreducible components, which proves Lemma 22.

The dimension of the image of an algebraic variety \( X \) under a map is contained in an algebraic variety whose dimension is at most that of \( X \) \cite[Sect. I.6.3]{45}. Combined with Lemma 22, this implies our third lemma, which completes the proof of Theorem 19.

**Lemma 23.** The image \( DV \) of \( DC \) under the projection to \( \mathbb{C}^9_{\alpha} \) has dimension at most eight.

### 5.2. Numerical algebraic geometry verification.

We describe computations that establish Lemma 22 and therefore Theorem 19. They, as well as the derivations of Subsection 5.1, are archived on the website that accompanies this article.\(^8\)

We used the software Bertini \cite{7}, which is freely available and implements many algorithms in numerical algebraic geometry \cite{47}. We started with the polynomials \( f_1, \ldots, f_4 \), which are the numerators of (26), (27), and (28) and which define the algebraic variety \( P \subset \mathbb{C}^{12} \). Bertini used the algorithms of regeneration \cite{23}, numerical irreducible decomposition \cite{46}, and deflation \cite{39} to study \( P \), determining its decomposition into irreducible components, as well as the dimension and degree of each component, and the multiplicity of \( P \) along that component. A component is singular if the multiplicity is at least 2. For each component \( X \) of \( P \), it computes the points of \( Y \cap X \), where \( Y \) is a general affine linear subspace of \( \mathbb{C}^{12} \) in general position with \( \dim Y + \dim X = 12 \). The number of points is the degree of \( X \), and examining the coordinates of the points reveals information about the component \( X \). We sketch the consequence of that computation.

First, while \( P \) is defined by four equations in \( \mathbb{C}^{12} \), it has ten irreducible components of dimensions eight and nine. Specifically, it has seven components of dimension eight and three of dimension nine. On all three components of dimension nine one of \( z_1 \) or \( z_2 \) vanishes. One component has degree 4 and multiplicity 2, and on it \( z_1 = z_2 = 0 \). The other two components have multiplicity 1 and each has degree 8. On one, \( z_1 = 0 \) and on the other, \( z_2 = 0 \). These

\(^8\)www.math.tamu.edu/~sottile/research/stories/dispersion/
components do not lie in $DC$ as $z_1 z_2 \neq 0$ on $DC$. This already implies that $DC$ has dimension eight, and therefore implies Theorem 19.

The seven components of $P$ of dimension eight are all components of $DC$. One component has degree 744 and is non-singular. On all other components, we have $z_1^2 = z_2^2 = 1$. There are two further non-singular components of degree 8. On one, $(z_1, z_2) = (1, -1)$ and on the other, $(z_1, z_2) = (-1, 1)$. The remaining four components are singular. One has degree 8 and multiplicity 2, and on it, $(z_1, z_2) = (-1, -1)$. Another has degree 3 and multiplicity 2, and on it $(z_1, z_2) = (1, 1)$ and $\lambda$ is not constant. On the remaining two components, $\lambda = 0$ and $(z_1, z_2) = (1, 1)$. One has degree 3 and multiplicity 2, and the other has degree 1 and multiplicity 4.

This computation does not constitute a traditional proof, as Bertini does not certify its output. We give an alternative verification using the dichotomy of Theorem 15 that relies on a symbolic computation, and then a proof that uses geometric combinatorics.

5.3. Symbolic computation verification. We provide an “almost surely” verification of Theorem 19, using symbolic computation, which is exact and certified. For this, we select a point $\alpha \in \mathbb{R}^9$ and to check that $\alpha \notin DV$ and hence that $\alpha \notin DV$ by showing that $DC \cap (\{\alpha\} \times (\mathbb{C}\{0\})^2 \times \mathbb{C})$ is empty. An application of Theorem 15 then shows that Theorem 19 is almost surely valid, in the sense of Corollary 16. This can be done as in the following example.

Example 24. Let $\alpha = (1, 2, 3, 4, 5, 6, 7, 8, 1)$. Then there are no points $(z, \lambda) \in (\mathbb{C}\{0\})^2 \times \mathbb{C}$ such that $g_1, \ldots, g_4$ all vanish at $(\alpha, z, \lambda)$.

Proof. A complex number $z$ is non-zero ($z \in \mathbb{C}\{0\}$) if and only if there exists $u \in \mathbb{C}$ with $zu = 1$. For $\alpha = (1, 2, 3, 4, 5, 6, 7, 8, 1)$, a Gröbner basis computation in both Maple and Singular [19] shows that the ideal $I$ in $\mathbb{Q}[\lambda, z_1, z_2, u_1, u_2]$ generated by $f_1, \ldots, f_4, z_1 u_1 - 1, z_2 u_2 - 1$ contains 1. But then $I$ defines the empty set, by Hilbert’s Nullstellensatz.

Fig. 7 shows the dispersion relation for $L_{(1,2,3,4,5,6,7,8,1)}$. The horizontal plane is at $\lambda = 0$ and the domain is $k \in [-\frac{\pi}{2}, \frac{3}{2}\pi]$.

Figure 7. Dispersion relation for $L_{(1,2,3,4,5,6,7,8,1)}$. 
5.4. **Combinatorial algebraic geometry proof.** We now present a valid proof of Theorem 19. Since $A_\alpha(z)$ is a $2 \times 2$ matrix, its characteristic polynomial (26) is quadratic in $\lambda$ with leading coefficient 1, and thus the variety it defines in $\mathbb{C}^9 \times (\mathbb{C}\setminus\{0\})^2 \times \mathbb{C}$ (the dispersion relation) has the property that its projection to the $(\alpha, z)$ parameters $\mathbb{C}^9 \times (\mathbb{C}\setminus\{0\})^2$ is a proper map with each fiber consisting of either two points, or a single point of multiplicity 2.

Consider now the set $CP$ which is defined by the characteristic equation (26) and the two critical point equations (27). Then $CP$ consists of points $(\alpha, z, \lambda)$ such that $\lambda$ is a critical point of the dispersion relation. The critical points $(z, \lambda) \in (\mathbb{C}\setminus\{0\}) \times \mathbb{C}$ for any given $\alpha \in \mathbb{C}^9$ are the set of solutions to three equations in the three variables $z_1, z_2, \lambda$.

Let us explain this. The exponent of a monomial $z_1^{a_1} z_2^{a_2} \lambda^{a_3}$ is an integer vector $(a_1, a_2, a_3) \in \mathbb{Z}^3$. The exponents that occur in the nonzero terms in a Laurent polynomial $f$ form its support. Their convex hull $N(f)$ is the Newton polytope of $f$. The bound in Bernstein’s theorem is given by Minkowski’s mixed volume of the Newton polytopes of the equations.

The polynomials $f_1, f_2, f_3$ which define $CP$ have the following Newton polytopes.

For the characteristic equation $f_1$, this is the pyramid with vertex $(2, 2, 2)$ and base the square with vertices $(0, 0, 0), (2, 0, 0), (4, 2, 0), (2, 4, 0)$. The side length of each edge in the base is $2\sqrt{2}$ and its height is 2, so that its volume is $32/6$. The other two are reflections of each other. The first has base the hexagon with vertices

$$(0, 1, 0), (1, 0, 0), (3, 0, 0), (4, 1, 0), (3, 2, 0), (1, 2, 0),$$

and its other vertices are $(1, 1, 1)$ and $(3, 1, 1)$. If all polytopes are translated so that the centers of their bases are at the origin $(0, 0, 0)$, then the second two lie inside the pyramid.

**Lemma 25.** The mixed volume of the three polytopes (29) is 32.

**Proof.** This is a consequence of a result of Rojas [44, Cor. 9], which is explained in [16, Cor. 3.7]. Let the three translated polytopes be $P, Q,$ and $R$, with $P$ being the pyramid. Then $P = P \cup Q \cup R$.

Observe that at least one of the three polytopes ($P$) meets every vertex of $P$. Also, for every edge $e$ of $P$, at least two have an edge lying along $e$. Finally, for every facet $F$ of $P$, all three polytopes have a facet lying along $F$. A consequence of [44, Cor. 9] is that the mixed volume of $P, Q, R$ is $3! = 6$ times the volume of $P$, which is 32.

**Proof of Theorem 19.** For the point $\alpha = (1, 2, 3, 4, 5, 6, 7, 8, 1)$, we use maple or Singular to show that there are 32 nondegenerate critical points of the dispersion relation. By Lemma 25, this is the maximal number of critical points, and so we conclude that $\alpha$ is a regular value.
of the projection map $\pi: \mathbb{C}P \to \mathbb{C}^9$. Furthermore, there is a neighborhood $U$ of $\alpha$ such that over it the map $\pi$ is a 32-sheeted cover and therefore is proper near $\alpha$.

The set $DC$ of degenerate critical points is a closed subset of $\mathbb{C}P$ and thus its projection to $\mathbb{C}^9$ is proper near $\alpha$. Thus its image $DV$ is closed in a neighborhood of $\alpha$. Since $\alpha \notin DV$, this implies that the complement of $DV$ in $\mathbb{C}^9$ contains a neighborhood of $\alpha$. But any nonempty classical open subset of $\mathbb{C}^9$ is Zariski-dense, and therefore we conclude that the complement of $DV$ in $\mathbb{C}^9$ contains a nonempty Zariski open set, which completes the proof. □

An interesting outcome from this version of the proof is the following result:

**Theorem 26.** Under the conditions of Theorem 19, the statement of Conjecture 17 holds true. In other words, increasing the number of parameters in (e.g., adding more edges to) the two-atomic structure shown in Fig. 6 does not change the conclusion on genericity of Theorem 19.

Indeed, the crucial mixed-volume computation of Lemma 25 does not react to increasing the number of parameters $\alpha$.

6. Degenerate subgraphs

Lemma 24 and Corollary 16 provide an efficient method to study Conjecture 7 on sufficiently simple discrete periodic graphs. We illustrate this on a case study involving all $2^9$ subgraphs of the graph of Fig. 6, corresponding to choosing a subset $S$ of the nine edges.

Let $f_1, \ldots, f_4 \in \mathbb{Q}[\alpha, \lambda, z_1, z_2]$ be the polynomials that define the variety $P$ following Lemma 22. Given a subset $S$ of the nine edges, let $I_S \subset \mathbb{Q}[\alpha, \lambda, z_1, z_2, u_1, u_2]$ be the ideal generated by $z_1u_1 - 1$, $z_2u_2 - 1$, and the polynomials obtained from $f_1, \ldots, f_4$ by setting all parameters $\alpha_j$ equal to zero for $j \notin S$. Then for parameters $\alpha_S = (\alpha_i \mid i \in S)$, $I_S$ vanishes on the set $DC$ of degenerate critical points on the dispersion relation for the graph $\Gamma_S$ corresponding to $S$ with parameters $\alpha_S$.

We have a maple script that, for each subset $S$, evaluates the ideal $I_S$ at ten random instances of the parameters $\alpha_S$. If, for each of these instances of the parameters $\alpha_S$ it finds that $1 \notin I_S$ (so that the corresponding dispersion relation has a degenerate critical point), then it adds $S$ to a set $DSG$ of degenerate subgraphs. This set $DSG$ contains 87 subsets $S$. This set, according to Theorem 26 has the structure of a simplicial complex. That is, if $S \in DSG$ and $T \subset S$, then $T \in DSG$. There are eleven maximal subsets (simplices) $S$ in $DSG$, corresponding to eleven maximal subgraphs of the graph in Fig. 6 which always have degenerate dispersion relations. We display them in Fig. 8. Observe that each of these graphs are disconnected. Moreover, they consist either of one or more $\mathbb{Z}$-periodic graphs together with their disjoint copies under translation by $\mathbb{Z}$, (thus providing jointly a $\mathbb{Z}^2$-periodicity), or two disjoint isomorphic copies of a $\mathbb{Z}^2$-periodic graph. It is a simple exercise that both situations lead to degeneration (even in the continuous case). Thus all degenerations occur for “obvious” reasons only.

7. Conclusions and final remarks

(1) In the continuous case, the dispersion relations are not algebraic, and the operators $L(z)$ are unbounded and thus the projection of the set $DC$ is NOT a proper map. One
Figure 8. Maximal degenerate subgraphs of the graph of Fig. 6.

thus needs to deal with projections of sets of zeros of entire functions into subspaces, such as for instance in the classical theorem by Julia [25]. The situation there is complicated, and to get any reasonable results about projections of such analytic sets, one needs more information, e.g. assumptions on the growth of the defining function [2,20]. Although such growth estimates do exist (see, e.g. [33,34]), the authors have not succeeded in establishing similar results for, say, Schrödinger operators with periodic potentials. We, however, conjecture that an analog of the dichotomy Theorem 15 should hold, with genericity understood in the (unavoidably weaker) Baire category sense.
(2) It may be possible to extend our analysis in Section 5.4 to other periodic graphs. A starting point could be to determine the Newton polytopes and mixed volumes of the polytopes corresponding to the equations defining the set $CP$ of critical points of the dispersion relation.

(3) The initial impression is (see Conjecture 17) that one needs to have sufficiently many free parameters in the operator to expect generic non-degeneracy. It would be, however, interesting to have better understanding of what makes some discrete periodic problems degenerate. The examples of Section 6 may be instructive.

(4) There are 98 disconnected subgraphs of the graph of Fig. 6, which also form a simplicial complex. Of those that do not appear in Fig. 8, we show the three which are maximal in Fig. 9.

(5) Theorem 26 confirms the Conjecture 17 in our specific example. It would be very interesting to know to what extent the Conjecture holds in general.

(6) It is easy to create (see [10]), using non-trivial graph topology, compactly supported eigenfunctions. This is known to lead to appearance of flat components in the Bloch variety, and thus degenerate extrema. However, this situation is non-generic: it is destroyed by generic small variations of the lengths (weights) of edges.

(7) The reader should notice that we quickly abandon discussion of the spectral edges only and target a loftier goal - all critical points. It might be easier to understand the generic structures of (much fewer) spectral edges, but the authors have not figured out how to use this distinction.

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