Nonlinear Sigma Model for Normal and Superconducting Systems: A Pedestrian Approach

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The nonlinear $\sigma$ model (NL$\sigma$M) epitomises a field-theoretical approach to (interacting) electrons in disordered media. These lectures are aimed at the audience who might have vaguely heard about the existence of the NL$\sigma$M but know very little of what is that, even less so of why it should be used and next to nothing of how it can be applied. These what, why and mainly how are the subject of the present lectures. In the first part, after a short description of why to be bothered, the NL$\sigma$M is derived from scratch in a relatively simple (but still rather mathematical) way for non-interacting electrons in the presence of disorder, and some illustration of its perturbative usage is given. In the second part it is generalised, not without some leap of faith, to include the Coulomb repulsion and superconducting pairing.

I. INTRODUCTION

Starting from the seminal paper by Wegner, a field-theoretical description based on the nonlinear $\sigma$ model (NL$\sigma$M) has become one of the main analytical approaches to quantum transport and thermodynamics in disordered electronic systems [2–24]. Quantum effects are due to interference of electron scattering from different impurities. These effects are relevant when the scattering mean free path $\ell$ is small compared to the dephasing length $L_\phi$ which characterises the scale above which interference is destroyed by an irreversible phase-breaking in electron’s wave function due to inelastic interactions. As $L_\phi$ diverges when $T\to 0$, the quantum interference effects are always important at low temperatures. Although the quantum corrections to transport coefficients are small in metals as a power of the small ratio $\lambda/\ell$ ($\lambda$ is the Fermi wavelength), they are essential in electron transport as they govern a nontrivial dependence on temperature, frequency or external fields [25,26]. Furthermore, the electron-electron interaction in the presence of the scattering from impurities also results in similar (small in $\lambda/\ell$) corrections both to the transport coefficients and to thermodynamic quantities (see Refs.27,28 for reviews), in a stark contrast to the case of pure normal metals where the interaction manifests itself only via the Fermi liquid parameters, which renormalise electrons spectral coefficients. The NL$\sigma$M gives analytical tools for describing both the weak disorder regime ($\lambda_F/\ell$) and a regime when the disorder increases with $\lambda_F/\ell$ approaching a critical value of order 1 where electronic states at the Fermi energy are localised [29,30].

There are two main characteristic features in the $\sigma$-model approach to the disordered system: (i) one performs the ensemble averaging over all configurations of disorder at the first stage of calculations; (ii) one separates “fast” and “slow” degrees of freedom and deals only with the latter. In the present context, “fast” implies energies of order $\epsilon_F$ and momenta of order $p_F$ while “slow” means energies smaller than $h/\tau_0 \equiv h\nu_F/\ell$ and momenta smaller than $h/\ell$. Such a scale separation works for a weak disorder. Quite often, the weak disorder condition $\lambda_F/\ell$ is formulated as $g_0 \gg 1$ where $g$ is the dimensionless conductance (loosely, the conductance measured in units of $e^2/h$) and $g_0 \propto (\epsilon_F \tau_0)^{d-1}$ is the dimensionless conductance of a cube of size $\ell^d$ in a $d$-dimensional system. The slow modes in a system of size $L \gg \ell$ describe diffusive propagation of electrons, and the standard NL$\sigma$M is the theory of interacting diffusive modes (diffusons) [31].

The functional of the $\sigma$ model has the following form:

$$F[Q;\omega] = \int d^d r \text{tr} \left[ \frac{\pi \hbar D}{8} (\nabla Q)^2 - \frac{i \pi \hbar \omega}{4} \Lambda Q \right], \quad (1)$$

where $D = v_F^2 \tau_0 / d$ is the diffusion coefficient, $\nu_0$ is the one-electron density of states, $d$ is the spatial dimensionality and $\omega$ is the frequency. The Hermitian matrix field $Q(r)$ (whose symmetry and dimensions depend on this or that particular way of the ensemble averaging) obey the following constraints which make the model nonlinear:

$$Q^2 = 1, \quad \text{Tr} \, Q = 0. \quad (2)$$

Writing down the above functional hardly explains for non-initiated what is the meaning of the model. It allows me, however, to formulate the main goals of these lectures:

1. To outline a pedestrian but consistent step-by-step derivation of the above functional for non-interacting electrons in a random potential.
2. To generalise the model, with the help of one or two leaps of faith, for including the Coulomb repulsion and pairing attraction of electrons.

3. To illustrate using one or two relatively simple examples how the NLσM works, both for noninteracting and interacting electrons.

But before attending to this programme, it is useful to give some ideas of why the NLσM is to be used at all, or where there are its advantages as compared to the usage of other analytical means, like a straightforward diagrammatic approach or a semi-phenomenological random matrix theory (whose applicability to the description of non-interacting electrons in disordered systems in a so-called ergodic regime has been rigorously established with the help of the NLσM).

The first application of the NLσM was a derivation of the renormalization-group (RG) equations of the one-parameter scaling theory of the Anderson localisation, illustrated on Fig. 1. The $\beta$ function, $\beta(g) \equiv d \ln g/d \ln (L/\ell)$, depends only on the dimensionless conductance $g$ and describes how it changes with scale or disorder. For $d = 3$, conductance increases with the system size for $\beta(g) > 0$ and decreases for $\beta(g) < 0$, becoming an ideal metal or ideal insulator, respectively, in the limit $L \to \infty$. In this case $\beta(g) = 0$ corresponds to the critical point for the metal-insulator transition. For $d = 2$ and $d = 1$, $\beta(g)$ is always negative so that no truly metallic state can exist in the thermodynamic limit. However, scaling considerations alone could only prove\(^\text{25}\) that $\beta(g) = (d - 2)g$ for $g \gg 1$, thus leaving some uncertainty in the marginal $d = 2$ case. The uncertainty has almost immediately been resolved\(^\text{26}\) with the help of the diagrammatic perturbation theory by proving that $g/g_0 = 1 - g^{-1}_0 \ln L/\ell$ for $d = 2$, i.e. that the correction to the classical behaviour, $\beta(g) = 0$, was indeed negative. Assuming also the renormalisibility, supported by the cancellation of all the corrections to $g$ of order $g^{-1}_0 \ln L/\ell$\(^\text{2}\) also found in Ref.\(^\text{26}\), would support the large-$g$ limit of the one-parameter picture on Fig. 1, which, however, has been confirmed directly only within the NLσM. The result $\beta(g) = -1/g + o(1/g^2)$ found in [2-5] means that all the main logarithmic contributions, $g^{-1}_0 \ln L/\ell^n$, should be cancelled in the perturbation theory, while diagrammatically such a cancellation has only been demonstrated\(^\text{26}\) for $n = 2$.

Having been originally designed for describing the electron motion in the diffusive regime, the NLσM has later been generalised both for a treatment of the (non-perturbative) ergodic regime,\(^\text{5,6}\), where it was used to justify the applicability of the random matrix theory for the description of electrons in small metallic grains, as had been suggested earlier\(^\text{32}\), and for a new approach to the ballistic chaotic regime.\(^\text{12,14,15,16}\) Thus, the NLσM covers all the variety of conditions in electronic disordered systems (see Fig. 2). Moreover, this an indispensable analytical tool to describe changes in properties of the disordered system approaching, with decreasing $g$, the region of the metal-insulator transition in $d > 2$ or of the crossover to the strong localisation in $d = 2$.

Further extensions of the NLσM have been related to the emergence of mesoscopics.\(^\text{14,15,16,37}\). When it became clear that fluctuations of practically all observable quantities in sub-micron and nano-samples are essential so that the ensemble-averaged quantities alone are not sufficient for a full characterisation of disordered systems, the NLσM turned out to be a natural tool for finding entire distributions of different observable quantities\(^\text{11,12,13,14}\), and describing their change with increasing disorder\(^\text{11}\).

The NLσM has also been extended to include interactions\(^\text{7,8,9,10,17,18,19,20,21}\), which allowed to reproduce earlier perturbative results\(^\text{2,26}\) with a proper account for the Landau Fermi-liquid parameters, and to derive the renormalization group equations describing (at least at a qualitative level) a metal–insulator transition

![FIG. 1: Schematic shape of the $\beta$ function in the one-parameter scaling theory](image-url)
in disordered interacting systems. The NLσM has been applied also to superconducting systems, in particular to include effects of Coulomb interaction, e.g., lowering the transition temperature $T_c$ by disorder – in a (well understood) deviation from the Anderson theorem, and also to describe inhomogeneous or granular superconducting systems. The interest in this approach has been greatly enhanced by the recent discovery of an apparent metal-insulator transition in 2D disordered systems in zero magnetic field.

The starting point for all further considerations will be the TOE model given below. First I introduce a functional integral representation for a simplest variant of the model that does not include interactions; then I will show how to derive the NLσM from such a representation by excluding “fast” degrees of freedom; after that I will extend the NLσM by including the Coulomb and pairing interactions.

II. FROM THE TOE MODEL TO A FUNCTIONAL INTEGRAL

Here TOE stands for “Theory of Everything”. In contrast to the high energy physics, there is no problem to write down the TOE Hamiltonian in context of electrons in solids:

$$\hat{H} = \sum_i \frac{\hat{p}_i^2}{2m} + \sum_i V_i + \frac{1}{2} \sum_{ij} V_{ij}.$$

(3)

Here the second term represents schematically both lattice and disorder potential, while the last represents all possible two-particle interactions (for example, the Coulomb repulsion or the Cooper attraction). In order to be really the TOE model, the above Hamiltonian should also include spin terms, as later in this paper in relation to superconductivity. Naturally, either with or without spin terms, there is no hope for a rigorous approach to, let alone the exact solution of, this model for an arbitrary disorder/interaction strength. Moreover, using this model for many problems in strongly correlated systems is no more fruitful than starting entirely ab initio. However, when the aim is to describe low-temperature properties of interacting electrons in disorder environment, which are governed by low-energy modes only, the TOE model is perfectly adequate and one does not need to go to a next hierarchical level. This does not make the model easily treatable, let alone straightforward. It became clear about 20 years ago that the interplay of disorder and interactions makes disordered conductors drastically different from pure metals, where the interactions reveal itself only in renormalisation of Fermi liquid parameters. In the perturbative regime the electron-electron interaction leads to nontrivial corrections to both transport coefficients and thermodynamic properties. These corrections govern numerous physical effects, like zero-bias anomaly or negative anomalous magnetoresistance, which have been summarised, e.g., in reviews.

FIG. 2: Different regimes for disordered systems: the ergodic regime corresponds to the times longer than $L^2/D$, the time of diffusion through the sample, while the ballistic to the times shorter than $\tau_{el}$; at the critical disorder, $g \sim 1$, the former regime shrinks to the energies of order or smaller than the level spacing $\delta$ and the later to the energies of order of $\varepsilon_F$, while the diffusive regimes evolves to a critical one embracing all the energies between $\delta$ and $\varepsilon_F$.33.
Note that in the present context a small perturbation parameter is \( g^{-1} \ll 1 \), independently of a value of the interaction parameter \( c^2/\hbar v_F \); in contrast to pure metals the condition of high density \( (\lambda_F \ll a_B \equiv \hbar^2/m e^2) \) is not necessary for a diffusive weakly disordered system, when \( \lambda_F, a_B \ll \ell \ll L \). A set of interesting questions is related to behaviour of the system when the disorder is increasing.

The most consistent treatment of the TOE model which includes describing its properties with increasing disorder can be achieved within the NLσM that can be derived from Hamiltonian \( \hat{H} \) for any reasonable interaction \( V_{ij} \), including both Coulomb and pairing interactions. However, disorder makes the model nontrivial even in the absence of any interaction. So the first task here is a derivation of the NLσM in the most straightforward case of \( V_{ij} = 0 \). A non-trivial step in this derivation would be the ensemble averaging over all realisations of disorder. But before coming to that, it is necessary to represent Green’s functions as functional integral.

### A. Introducing Green’s functions

Any physical quantity (any response function) can be represented, at least in the absence of interaction, as a product of one-particle Green’s functions. To this end, let’s first write down the standard formal representation for the one-particle Green’s function \( \hat{G}(\varepsilon) \), defined by \((\varepsilon - \hat{H})\hat{G}(\varepsilon) = \hat{1}\). For \( V_{ij} = 0 \), one formally diagonalise the Hamiltonian as \( \hat{H} |\alpha\rangle = \varepsilon_\alpha |\alpha\rangle \). Using the completeness condition, \( \sum_\alpha |\alpha\rangle \langle \alpha| = \hat{1} \), one has (with \( \hbar = 1 \) from now on)

\[
\hat{G}^\pm \equiv (\varepsilon - \hat{H} \pm i\delta)^{-1} = (\varepsilon - \hat{H} \pm i\delta)^{-1} \sum_\alpha |\alpha\rangle \langle \alpha| = \sum_\alpha \frac{|\alpha\rangle \langle \alpha|}{\varepsilon - \varepsilon_\alpha \pm i\delta} \tag{4}
\]

where infinitesimal \( \delta \) is introduced to avoid divergences at \( \varepsilon = \varepsilon_\alpha \). On recognises that \( \hat{G}^\pm \) are the standard retarded and advanced Green’s functions which can be re-written in the real space representation as

\[
G^\pm_\varepsilon (r, r') \equiv \langle r | \hat{G}^\pm | r' \rangle = \sum_\alpha \frac{|r\rangle \langle \alpha| \langle \alpha| r'\rangle}{\varepsilon - \varepsilon_\alpha \pm i\delta} = \sum_\alpha \frac{\varphi_\alpha^*(r) \varphi_\alpha(r')}{\varepsilon^\pm - \varepsilon_\alpha} \tag{5}
\]

In the presence of disorder, one does not know either eigenfunctions, \( \varphi_\alpha(r) \), or eigenvalues, \( \varepsilon_\alpha \), of \( \hat{H} \). The task is to represent the average products of \( G \) without any reference to unknown parameters. To this end, I shall first represent \( G \) as a “functional integral” which is just a multiple product of Gaussian integrals.

### B. From Gaussian to functional integrals

The first step is representing the denominator of Green’s function as a Gaussian integral. Noting that (with \( \varepsilon^+ \equiv \varepsilon + i\delta \))

\[
I_0 = \int_{-\infty}^{\infty} dx \, e^{i(x^+ - \varepsilon_\alpha)x^2} = \sqrt{\frac{\pi}{i(x^+ - \varepsilon_\alpha)}}
\]

one represents \((\varepsilon^+ - \varepsilon_\alpha)^{-1}\) as \((i/\pi)I_0^2\). (One assumes from now on that energy is dimensionless, which will make all the “action” functionals below also dimensionless). Similar representation would be, of course, possible for \((\varepsilon^- - \varepsilon_\alpha)^{-1}\) but not just for \((\varepsilon - \varepsilon_\alpha)^{-1}\): the convergence in \( I_0 \) is ensured by the presence of \( i\delta \). Representing \( I_0^2 \) as a product of integrals over \( dx \) and \( dy \) and changing variables to \( c = x + iy \) and \( c^* = x - iy \), one finds

\[
\frac{1}{\varepsilon^+ - \varepsilon_\alpha} = \int \frac{dc^* dc}{2\pi} \, e^{ic^*(\varepsilon^+ - \varepsilon_\alpha)c^*} \equiv Z_0. \tag{6}
\]

This gives the following representation of \( \det(\varepsilon - H)^{-1} = \prod_\alpha (\varepsilon^+ - \varepsilon_\alpha)^{-1} \) as a product of Gaussian integrals:

\[
\det(\varepsilon - H)^{-1} = \prod_\alpha \int \frac{dc^*_\alpha dc_\alpha}{2\pi} \, e^{ic^*_\alpha(\varepsilon^+ - \varepsilon_\alpha)c_\alpha} \equiv \int Dc^* Dc \, e^{\sum_\alpha c^*_\alpha(\varepsilon^+ - \varepsilon_\alpha)c_\alpha}, \tag{7}
\]

where \( Dc^* Dc \) is a symbolic notation for the measure of integration, which is just a product over all \( dc^*_\alpha dc_\alpha \). Of course, the above determinant is defined as a product of an infinite number of (well-defined) integrals, and as such, it may appear to be ill-defined. However, if one does not pursue mathematical rigour, this is quite a
convenient representation which will be used as a building block for that for Green’s functions, the latter will be free from trivial divergences that may appear in calculating $\det(\varepsilon - \hat{H})$.

Let us represent (exactly) the sum in the exponent in Eq. (4) by an integral. Using the orthonormality condition for the eigenfunctions of $\hat{H}$, one obtains

$$
\sum_{\alpha} c_{\alpha}(\varepsilon^+ - \varepsilon_{\alpha})c_{\alpha} = \sum_{\alpha\beta} c\bar{\alpha}(\varepsilon^+ - \varepsilon_{\alpha})c_{\beta} \int d^{d}r \varphi^{*}_{\alpha}(r) \varphi_{\beta}(r)
$$

$$
= \int d^{d}r \sum_{\alpha} c\bar{\alpha}(\varepsilon^+ - \hat{H}) \sum_{\beta} c\bar{\beta} \varphi_{\beta} \equiv \int d^{d}r \psi^{*}(r) \left( \varepsilon^+ - \hat{H} \right) \psi(r) \tag{8}
$$

As $c$ and $c^{*}$ were variables of integration which took all possible values in the complex plane, the complex conjugate fields $\psi$ and $\psi^{*}$ take all the possible values for any argument $r$. Using the identity (5), the determinant of Eq. (7) can be represented as the following functional integral over the fields $\psi^{*}$ and $\psi$

$$
Z_{1} \equiv \det \left( \varepsilon^+ - \hat{H} \right)^{-1} = \int D\psi^{*} D\psi e^{iS_{+}}, \quad S_{+} \equiv \int d^{d}r \psi^{*}(r) \left( \varepsilon^+ - \hat{H} \right) \psi(r) \tag{9}
$$

(An index in $Z_{1}$ is introduced here to make some notations in what follows more consistent). The measure of integration is written here symbolically as $D\psi^{*} D\psi$ in understanding that, using Eq. (5), one can always reduce the $\psi$-integral to a (relatively) well defined integral over $c$ and $c^{*}$. To illustrate how it works I shall substitute this integration measure and the definition of fields $\psi$ and $\psi^{*}$, Eq. (8), into the functional average $-i \langle \psi^{*} \psi \rangle_{+} \equiv \frac{1}{Z_{1}} \int \psi^{*}(r) e^{i\phi^{+}} D\psi D\psi^{*}$, thus demonstrating that this functional average provides a representation for the Green function of Eq. (5):

$$
-\frac{i}{Z_{1}} \sum_{\alpha\alpha'} \varphi_{\alpha}^{*}(r') \varphi_{\alpha}(r) \int \prod_{\beta} \frac{dc_{\beta} dc_{\bar{\beta}}}{2\pi i} \frac{c^{*}_{\alpha} c_{\alpha} e^{i\varepsilon_{\alpha}} c_{\beta}}{\varepsilon^{+} - \varepsilon_{\alpha} c_{\beta}}
\text{= 0 unless } \alpha = \alpha' \tag{7}
$$

$$
= \frac{i}{Z_{1}} \sum_{\alpha} \frac{\varphi^{*}_{\alpha}(r') \varphi_{\alpha}(r)}{(\varepsilon^{+} - \varepsilon_{\alpha})^{2}} \prod_{\beta \neq \alpha} \frac{1}{\varepsilon^{+} - \varepsilon_{\beta}} = \frac{1}{Z_{1}} \sum_{\alpha} \frac{\varphi^{*}_{\alpha}(r') \varphi_{\alpha}(r)}{\prod_{\beta \neq \alpha \varepsilon^{+} - \varepsilon_{\beta}}}
\equiv \frac{1}{Z_{1}} \sum_{\alpha} \frac{\varphi^{*}_{\alpha}(r') \varphi_{\alpha}(r)}{\varepsilon^{+} - \varepsilon_{\alpha}} \tag{9}
$$

However, the functional integral would be composible, had one every time needed to return to the $c$ representation in order to make a calculation. It is sufficient to realise once and forever that the functional integral is a shorthand notation for a product of Gaussian integrals, so that all the standard operations for Gaussian integrals can be directly translated into the language of functional integrals. To illustrate such a translation, let us note that in the above calculation the following elementary integral has been used:

$$
I_{0} = \int \frac{dc_{\alpha} dc_{\bar{\alpha}}}{2\pi i} c^{*}_{\alpha} c_{\alpha} e^{i\varepsilon_{\alpha}} = \frac{i}{(\varepsilon^{+} - \varepsilon_{\alpha})^{2}} \equiv \frac{i}{\varepsilon^{+} - \varepsilon_{\alpha}} \int \frac{dc_{\alpha} dc_{\bar{\alpha}}}{2\pi i} c^{*}_{\alpha} c_{\alpha} e^{i\varepsilon_{\alpha}} \tag{9}
$$

where $Z_{0}$ is the integral (9). There are many ways to get such a straightforward result. Let us choose that which is directly generalisable to the functional integral, representing $I_{0}$ above using parametrical derivatives with respect to the “source fields” $h$ and $h^{*}$

$$
I_{0} = \frac{i}{Z_{0}} \int \frac{dc^{*} dc}{2\pi i} c^{*} e^{i\varepsilon_{\alpha}} A_{c} = \frac{1}{Z_{0}} \frac{\partial^{2}}{\partial h^{*} \partial \bar{h}} \int \frac{dc^{*} dc}{2\pi i} e^{i(c^{*} A_{c} + h^{*} c^{*} + c h)}
$$

$$
= \frac{i}{Z_{0}} \frac{\partial^{2}}{\partial h^{*} \partial \bar{h}} \int \frac{dc^{*} dc}{2\pi i} e^{i(c^{*} A_{c} - h^{*} A^{-1} h)} = - \frac{\partial^{2}}{\partial h^{*} \partial \bar{h}} e^{-i h^{*} A^{-1} h} = \frac{i}{A} \tag{10}
$$

where a shift of variables, $c \rightarrow c - A^{-1} h$ and $c^{*} \rightarrow c^{*} - h^{*} A^{-1}$, has been used. It was implied that both $h$ and $h^{*}$ are put to zero after calculating the derivatives above. Now I apply practically the same procedure to the functional integral defined above in order to represent Green’s functions.
C. Green's function as a functional integral

To show directly that \(-i \langle \psi^* \psi \rangle_+\) represents Green's function, I express it as a functional derivative:

\[
-i \langle \psi^*(r_1) \psi(r_2) \rangle_+ = \frac{i}{Z_1} \delta^2 \int D\psi D\psi^* e^{iS_+} \langle \psi^*(r_1) \psi(r_2) \rangle \tag{11}
\]

\[
S_+(h, h^*) = \int \left[ \bar{\psi}^*(r) \left( \varepsilon^+ - \hat{H} \right) \psi(r) + h^*(r) \psi(r) + \psi^*(r) h(r) \right] d^d r.
\]

Noting that, by definition, \( (\varepsilon^+ - \hat{H})^{-1} = \hat{G}_\varepsilon \) and making a shift of variables similar to that above, \( \psi \rightarrow \psi - \hat{G} \cdot h \) and \( \psi^* \rightarrow \psi^* - h^* \cdot \hat{G} \) (with \((\hat{G} \cdot h)(r)\) being a convolution in the \( r \) space) one reduces \( S_+(h, h^*) \) in the functional integral to the following expression:

\[
S_+(h, h^*) \rightarrow S_+ - \int [h^*(r) G_\varepsilon^+(r, r') h(r')] d^d r d^d r',
\]

so that performing the functional differentiation one finds, indeed, that

\[
G_\varepsilon^+(r_1, r_2) = \frac{-i}{Z_1} \int D\psi D\psi^* \langle \psi^*(r_1) \psi(r_2) \rangle e^{iS_+} = \frac{1}{Z_1} \left. \frac{\delta^2 Z_1(h^*, h)}{\delta h^*(r_1) \delta h(r_2)} \right|_{h=0} \tag{12}
\]

\[
Z_1(h^*, h) = \int D\psi D\psi^* e^{iS_+(h^*, h)}.
\]

That completes the representation of Green's function as a functional integral. If one thinks that the above derivation was not "pedestrian" enough, it would be sufficient to compare it to the standard way of introducing functional integrals (see, e.g., Ref. 41). Note that (11) is not an action as usually understood: in the absence of time integral, it does not even have the dimensionality of \( \hbar \), and is simply assumed dimensionless. Unfortunately, such a simplified way is only possible for a quadratic Hamiltonian which can be straightforwardly diagonalized. Only the presence of disorder makes it nontrivial: \( \hat{H} \) in Eq. 4 contains a random potential \( V \). The ensemble averaging over the distribution of \( V \) would result in an effective functional containing higher powers of \( \psi^* \) and \( \psi \). If \( \hat{H} \) is nonlinear to begin with, like the Hamiltonian of the TOE model 4, one should choose a hard way to get a functional representation for Green's functions. However, there is an alternative. After having learned a bit how to deal with the functional integrals that emerged for the noninteracting problem, one would be able to use some leap of faith to extend considerations to include interactions. Such an approach would not be self-contained but would remain, in my opinion, reasonably consistent and believable.

A certain extension related to future considerations is useful right now. Particle statistics are totally irrelevant for one-particle problems, but in extending considerations to the interacting particles one shall be mainly interested in fermions. The functional integration described above could be extended to the case of interacting bosons rather than fermions as the commuting classical fields in the functional integrals correspond to operators with certain commutation relations in the Hamiltonian representation. As fermions are described by operators with certain anticommutation relations, it is not that surprising that the functional integral representation for fermions must be formulated in terms of anticommuting classical fields. There exists a very solid mathematical foundation of the theory of anticommuting classical variables4,11 reduced to a physical level of rigour in different derivations 4,5,11 of the NLσM. However, all one needs for the purpose of present considerations is a relatively moderate leap of faith that allows one to introduce a fermion representation at a one-particle level.

D. Fermions do it like bosons

but upside down: instead of having \( \det^{-1} \) in the partition function for bosons 9, one has \( \det \) in the appropriate fermionic representation:

\[
Z_1^+ \equiv \det (\varepsilon^+ - \hat{H}) = \int D\bar{\psi} D\psi e^{iS_+}, \quad S_+ \equiv \int \left[ \bar{\psi}^*(r) \left( \varepsilon^+ - \hat{H} \right) \psi(r) \right] d^d r, \tag{13}
\]

\[
G_\varepsilon^+(r_1, r_2) = \frac{-i}{Z_1^+} \int D\bar{\psi} D\psi \bar{\psi}(r_1) \psi(r_2) e^{iS_+} \tag{14}
\]
where \( \tilde{\psi} \) and \( \psi \) are classical anticommuting mutually conjugate fields. The conjugation requires some care. For the bosonic classical fields it was implied that \( \psi, \psi^* = a \pm ib \), with \( a \) and \( b \) being real fields. For the fermionic fields it is more convenient to introduce the conjugation operation in matrix notations:

\[
\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix}, \quad \tilde{\psi} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\varphi^* & \varphi \end{pmatrix}, \quad (\varphi^* )^* = -\varphi, \quad \tilde{\psi} \equiv (C\psi)^T
\]

The last equation above defines the so called “charge-conjugation” matrix \( C \) (which has nothing to do with the electric charge). The symmetry imposed by such a conjugation will not be used but in section [13] in what follows, it is sufficient to know that the classical fermionic fields anticommute (which implies not only \( \varphi \varphi^* = -\varphi^* \varphi \) but also \( \varphi^2 = \varphi^2 = 0 \)) and the partition function (13) is given by \( \det(\varepsilon - \hat{H}) \) rather than by \( \det^{-1}(\varepsilon - \hat{H}) \).

Now everything is ready for the ensemble averaging over disorder and the only question to ask is what to average. The averaged one-particle Green’s function \( \langle G \rangle \) is not an interesting object: in the absence of interactions disorder does not effect physical quantities which are directly expressible via \( \langle G \rangle \), such as density of states. Most of response functions of interest such as density-density correlations, conductance, mesoscopic fluctuations of any observable, etc., can be normally expressed via the averaged products of retarded and advanced Green’s functions. So the simplest product to average over disorder is \( G_+^\omega G_{-\omega}^{-} \), with \( G^+ \) represented by Eq. (13) and \( G^- \) represented as

\[
G_{-\omega}(r_1, r_2) = -\frac{i}{Z_1} \int D\tilde{\psi} D\psi \tilde{\psi}(r_1)\psi(r_2) e^{iS_-},
\]

where \( S_- \) differs from \( S_+ \) in Eq. (13) by substituting \( \varepsilon \sim \omega^+ \equiv \varepsilon - (\omega + i\delta) \) for \( \varepsilon^+ \equiv \varepsilon + i\delta \). In general, one averages a closed loop, i.e. a product like \( G(r_1, r_2)G(r_2, r_3)\ldots G(r_n, r_1) \). This allows one to introduce quite an economical representation, using one source field for a product \( \psi(r)\bar{\psi}(r) \) of two fields taken from the adjacent Green’s function:

\[
K_\omega(r, r') \equiv G_{\varepsilon + \omega^+ / 2}(r, r')G_{\varepsilon - \omega^- / 2}(r', r) = \frac{\delta^2 \ln Z_1(\eta)}{\delta \eta_{12}(r) \delta \eta_{21}(r')} \Bigg|_{\eta=0}
\]

\[
Z_1(\eta, \omega) = \int D\tilde{\psi} D\psi e^{iS_1(\eta, \omega)}
\]

\[
S_1(\eta, \omega) \equiv \int \left\{ \bar{\psi}(r'') \left(-\hat{\xi} - V + \frac{1}{2} \omega^+ \Lambda \right) \psi(r'') + \bar{\psi} \eta \psi \right\} d^dr''.
\]

Here \( \hat{\xi} \equiv p^2/2m - \varepsilon_F \); the frequency arguments have been shifted for convenience; \( \bar{\psi} \equiv (\bar{\psi}^+, \bar{\psi}^-) \), with \( \bar{\psi}^\pm \) having been used in representations for \( G^\pm \), respectively; \( \psi \) is a corresponding vector column; \( \eta \) is an arbitrary 4 \times 4 matrix field (whose diagonal elements, \( \eta^{++} \) and \( \eta^{--} \), are redundant and can be put to zero from the very beginning, if convenient, while \( \eta^{+-} \) and \( \eta^{-+} \) are 2 \times 2 matrices by themselves, as each \( \psi^\pm \) is a two-component field, Eq. (13): the constant matrix \( \Lambda \equiv \text{diag}(1, 1, -1, -1) \) is necessary as \( \omega^+ \) enters with the opposite sign in the actions \( S^\pm \); \( Z_1 \) is the functional integral in the r.h.s. of Eq. (17) calculated at \( \eta = 0 \). As the \( \eta \)-part of the action is not affected by the averaging, the resulting functional would be applicable to any product of \( G^+ \) and \( G^- \) subject to a proper extension of the matrix space (each \( G \) would require its own pair of fields in the functional representation) and an appropriate choice of the source fields \( \eta \).

### III. AVERAGING OVER DISORDER

For simplicity, one assumes the random potential \( V(r) \) in Eqs. (9) and (13) to be a Gaussian white noise with \( \langle V(r) \rangle = 0 \) and

\[
\langle V(r)V(r') \rangle = \frac{1}{2\pi\eta_0\tau_{el}} \delta(r-r'),
\]

which means that \( V(r) \) is taken from the following distribution:

\[
P \{ V(r) \} = N \exp \left[ -\pi\eta_0\tau_{el} \int V^2(r) d^dr \right],
\]
where $\mathcal{N}$ is a normalisation factor. Although the choice of a white-noise Gaussian potential makes calculations considerably easier, a generalisation to a slowly varying potential would not basically change results\cite{16,24} provided that the mean free path $\ell$ is substituted by the transport length $\ell_{tr}$. With this choice, the disorder averaging $\langle \ldots \rangle$ reduces to taking a functional average with the distribution $\frac{1}{Z_1\left(\eta=0\right)}$. Had one to average $Z_1$, this would be a straightforward Gaussian integral again. It is not straightforward to average in $Z_1$, though. If one first takes derivatives with respect to the source fields in Eq. (17), it is the presence of the denominator, $1/Z_1\left(\eta=0\right)$, that makes the averaging tricky.

This denominator is the partition function given by Eq. (13) in the bosonic and by Eq. (16) in the fermionic representation. Comparing these expressions, one can see that using a mixed, supersymmetric fermion-boson representation would cancel out this denominator. This idea turned out to be extremely fruitful and led to enormous progress in theory of non-interacting particles in disordered and chaotic systems. Numerous achievements of the supersymmetry method have been summarised in a book\cite{43,44} and a recent review\cite{45}. However, there is no simple way of applying this method to a many-particle system of interacting electrons, so that some other method is required.

After a few earlier attempts\cite{46,47,48}, it has recently been demonstrated\cite{18,19} that the Keldysh technique\cite{47,48} provides a viable route to a field-theoretical description. In this technique $Z = 1$ by construction, which makes the ensemble-averaging straightforward. However, this technique remains quite cumbersome and, although it might be indispensable in describing non-equilibrium problems, a considerably simpler alternative exists for interacting electrons in disordered systems at equilibrium (i.e. in the linear response regime for all transport problems).

This alternative is a so called replica trick, which is possibly the oldest method\cite{49,50} used for the disorder averaging. It is based on the following identity:

$$\ln Z_1 = \lim_{n \to 0} \frac{Z_1^n - 1}{n}. \tag{21}$$

To apply it one replicates the fields in Eq. (17), $\overline{\psi} \mapsto \overline{\Psi} \equiv (\overline{\psi}_1^+, \ldots, \overline{\psi}_n^+, \overline{\psi}_1^−, \ldots, \overline{\psi}_n^−)$, thus replacing it by the following:

$$K_\omega(r, r') = \lim_{n \to 0} \frac{1}{2n^2} \text{tr} \left\{ \frac{\delta^2 Z(\eta)}{\delta \eta(r) \delta S(r')} \right\}_{\eta=0}, \quad Z \equiv Z_n = Z_1^n = \int \mathcal{D}\overline{\Psi}D\psi e^{iS_n(\eta)}. \tag{22}$$

The functional integration in Eq. (22) implies a product of $n$ identical integrals; $S_n$ is given by the same expression (18) as $S_1$ but in terms of the replicated fields and with $\Lambda \equiv \text{diag}(1_{2n}, -1_{2n})$ being a block-diagonal $4n \times 4n$ matrix; $\eta$ is a block-off-diagonal $4n \times 4n$ matrix with $\eta^{++} = \eta^{--} = 0_n$ and $\eta^{+-} = \eta^{-+}$ being arbitrary $2n \times 2n$ matrices:

$$S(\eta) \equiv \int \left\{ \overline{\Psi}(r) \left[ -\hat{\xi} - V(r) + \frac{i}{2}\omega^+ \Lambda + \eta \right] \right\} d^d r. \tag{23}$$

An extra factor of $1/2n$ in Eq. (22) compared with Eq. (21) is due to the fact that the trace there gives the sum of $2n$ identical terms. From now on I will be dealing only with the replicated $Z$ and $S$ and thus omit the index $n$ there.

As the representation (22) does not contain a logarithm, one only needs to average $Z$ over the disorder distribution of Eq. (20), which, for any integer $n$, is a straightforward Gaussian integration that involves only the $V$-dependent part of $Z$:

$$\int \mathcal{D}V e^{\int -\frac{1}{\pi\nu_0\tau_{el}} V^2 + i\overline{\Psi}V \psi} d^d r = \exp \left[ -\frac{1}{4\pi\nu_0\tau_{el}} \int \left( \overline{\Psi} \psi \right)^2 d^d r \right]. \tag{24}$$

Thus one arrives at the following effective action:

$$S(\eta) \equiv \int \left\{ \overline{\Psi}(r) \left[ -\hat{\xi} + \frac{i}{2}\omega^+ \Lambda + \eta \right] \Psi(r) + \frac{i}{4\pi\nu_0\tau_{el}} \left( \overline{\Psi} \psi \right)^2 \right\} d^d r. \tag{25}$$

After calculating $K_\omega(r, r')$ for any positive integer $n$ one needs to take the replica limit $n \to 0$ which is by no means a well defined mathematical operation.

However, as long as one is interested in perturbative corrections only (including the RG corrections), one can talk about the replica method, not a trick, as illustrated in Fig. 3. All the perturbative corrections in the standard diagrammatic techniques are just crossings (corresponding to scattering from impurities) connected
by solid lines corresponding to Green’s function, like diagrams (a) – (c), where the dashed lines connecting the crosses represent the pair correlations of the random potential Eq. (19) resulting from the ensemble averaging. The field theory with the functional Eq. (25) with any integer \( n \) exactly reproduces all such diagrams but also yields those like (d) and (e) with vacuum loops. However, each loop contains a sum of \( n \) identical replicated contributions which lead to an extra factor of \( n \) thus making such diagrams to vanish in the replica limit \( n \to 0 \). Note that in the supersymmetric technique such diagrams vanish as each loop is the sum of fermionic and bosonic contributions having the opposite signs. However, in the latter approach it would be difficult to include diagrams like (f) that contains a loop corresponding to the electron-hole excitation in the presence of the interaction, whereas the replica approach would encompass such contributions rather naturally (section VI).

The functional of Eq. (25) is fully equivalent to the original model of Eq. (3) in the absence of interactions. However, there is little to gain from it without further approximations. The most important one is based on a reduced description that excludes all details of electronic motion at scales smaller than the mean free path \( \ell \).

IV. THE NL\( \sigma \)M FUNCTIONAL

A. Hubbard–Stratonovich decoupling

This is the name of the operation performed in order to exclude all “fast” modes of electron motion. Its first step is nothing more than the Gaussian integration like that in Eqs. (10) and (24), but the integration which is taken “backwards”. In Eq. (24), one integrates over the random potential \( V \) thus arriving at the action Eq. (25) that does not depend on \( V \) (and thus contains no randomness) but is no longer quadratic (i.e. integrable) in the fermionic fields. In order to decouple the quartic term in Eq. (25), one introduces a new field, \( Q \), which allows one to “unwind” the integration in Eq. (24) and represent the quartic term as a functional integral over \( Q \), thus making the action quadratic in the fermionic fields again and finally integrating over \( \Psi \) to arrive at the action in terms of the new field \( Q \). Of course, one could perform the \( \Psi \)-integration from the very beginning obtaining the action in terms of the disorder potential \( V \); however, such an action would not be very useful.

The introduction of the field \( Q \) is useful if it exploits fully the configurational space defined by the conjugate fields \( \Psi \) and \( \overline{\Psi} \) and excludes fast degree of freedom; to this end, \( Q \) is chosen as a matrix field having the same rank and symmetry as \( \Psi \otimes \overline{\Psi} \) (the latter is just a notation for an outer product of two vectors that results in a \( 4n \times 4n \) matrix). Noting that \( \text{tr}(\Psi \otimes \overline{\Psi})^2 = -\langle \overline{\Psi}\Psi \rangle^2 \) and choosing \( Q \) to be dimensionless, one can write the following identity that “unwinds” the integration Eq. (24):

\[
\exp \left[ -\frac{1}{4\pi\nu_0\tau_{el}} \int (\overline{\Psi}\Psi)^2 d^4r \right] = \int \frac{DQ}{\mathcal{N}_Q} \exp \left\{ \int \left[ -\frac{\pi\nu_0}{8\tau_{el}} \text{tr} Q^2 + \frac{i}{2\tau_{el}} \overline{\Psi}Q\Psi \right] d^4r \right\}.
\]

(26)

The normalisation integral \( \mathcal{N}_Q \) is just the same \( Q \)-integral at \( \overline{\Psi} = \Psi = 0 \) (and it is anyway irrelevant in the replica limit) and the Gaussian integration both here and in Eq. (25) is performed with the help of a shift of variables fully analogous to those in Eqs. (10) and (11). After this, the \( \Psi \)-dependent part of the action is, indeed, quadratic, with the term \( \frac{\pi}{2\tau_{el}} \overline{\Psi}Q\Psi \) being added into the brackets in Eq. (25). The Gaussian integration over the fermionic field in Eq. (26) is performed exactly like in Eq. (13) (the rank of matrices does not matter) and
FIG. 4: Diffuson (a) and cooperon (b) ladders: in diagrammatic techniques either pictures with triple dashed lines or those with wavy lines can be used to denote the ladders; the latter are more convenient in the present context as they are in direct correspondence with diagrammatic notations in the NLOσM, section V; they should not be confused with interaction lines, Fig. 5.

results, after using the remarkable matrix identity ln det $A = \text{tr} \ln A$, in the effective action $\mathcal{F}$ which depends only on $Q$:

$$\mathcal{F}[Q] = \frac{1}{L^d} \int d^d r \left\{ \frac{\pi}{8\tau_0 \delta_0} \text{tr} Q^2 - \frac{1}{2} \text{tr} \ln \left[ \frac{\omega^2}{2} + \eta - \xi + \frac{i}{2\tau_{el}} Q \right] \right\},$$

(27)

where $\delta_0 \equiv 1/(\nu_0 L^d)$ is the mean level spacing. The action has been renamed into $\mathcal{F}$ in order to stress that the functional averaging in Eq. (26) is performed with the Euclidian weight $e^{-\mathcal{F}}$ rather than with the oscillating one, $e^{iS}$. The partition function $Z$, Eq. (22), is now given by

$$Z = \int DQ \exp (-\mathcal{F}[Q]),$$

(28)

and the measure of integration will be specified later. The matrix $Q$ is Hermitian as it has the same symmetry as the product $\Psi \otimes \overline{\Psi}$ which is Hermitian due to the definition of the fields in Eq. (15). This is not the only symmetry of $Q$, as the charge conjugation introduced in Eq. (15) is essential and explains the appearance of the factor $1/2$ before the tr ln in Eq. (27), mysterious for generations of students.

If a diligent student had taken a course of action described between Eqs. (26) and (27), he would have duly arrived at Eq. (27) but without this $1/2$. Such a result, moreover, would be formally correct but with a matrix $Q$ ill suited for describing the slow variables. The point is that the field $Q$ should be chosen in such a way that it would correspond only to the slow part of $\text{tr} (\Psi \otimes \overline{\Psi})^2$ in the l.h.s. of Eq. (26), while the fast Fermi oscillations contained in each field $\psi$ were suppressed. Such a slow part could be chosen in two ways: it corresponds either to the $\Psi \overline{\Psi}$ pairing – with a small transferred momentum in the Fourier space, or to the $\overline{\Psi} \Psi$ (and $\Psi \overline{\Psi}$) pairing – with the small sum momentum. The charge conjugation introduced in Eq. (15) allows automatically for the existence of the two pairing channels, provided that the field $Q$ is chosen to have the same charge-conjugation symmetry as the product $\Psi \otimes \overline{\Psi}$, i.e. $Q = CQ^*C^{-1}$. Allowing also for the fact that $Q$ is Hermitian (which is, of course, necessary for at least a superficial convergency of the $Q$ integral in Eq. (26)), one can represent the $4n \times 4n$ matrix field $Q$ as

$$Q = \left( \begin{array}{cc} d & c \\ c^T & d^T \end{array} \right), \quad d^\dagger = d, \quad c^T = -c,$$

(29)

where $d$ and $c$ are $2n \times 2n$ matrices in the replica space, corresponding to the decoupling of the $\Psi$ pairs with the small transverse or small sum momentum, respectively. Then, in going from Eq. (26) to Eq. (27), one integrates independently over fields $d$ and $c$ which leads to the appearance of the factor $1/2$ in the latter equation. Some familiarity with the standard diagrammatics may help one to recognise that $d$ corresponds to a diffuson – the sum of impurity ladder diagrams with a small transfer momentum, while $c$ corresponds to a cooperon – the sum of ladder diagrams with a small sum momentum, Fig. 4. To make it a working analogy, one needs to use the fact that in Eq. (27) $Q$ is a slow varying field. To this end, one should expand the functional (27) in gradients of this field.
B. Gradient Expansion

The first step is to define a class of stationary, spatially-homogeneous (i.e. “infinitely slow”) fields for which the functional $Q$ satisfies a saddle-point condition. By varying the functional with respect to the field $Q$, one finds

$$- i \nu_0 Q = \langle r | G | r \rangle \equiv \left\langle r \left| \left( \frac{\omega}{2} \Lambda + \eta - \hat{\xi} + \frac{i}{2 \tau_{el}} Q \right)^{-1} \right| r \right\rangle.$$

(30)

This saddle-point condition is obviously satisfied at zero frequency $\omega$ and in the absence of the source field $\eta$ by any spatially-homogeneous matrix $Q$ of a proper rank and symmetry that obey the condition $Q^2 = \mathbb{1}$. This condition allows one to invert explicitly the matrix in the square brackets in Eq. (30): in the Fourier momentum space $\xi \equiv p^2/2m - \varepsilon_F$ becomes just a scalar variable of integration so that one finds

$$G(p, \omega=0) = \left[ -\xi + \frac{i}{2 \tau_{el}} Q \right]^{-1} = \frac{1}{\xi^2 + \frac{4 \tau_{el}}{\varepsilon_F}} \left( \xi + \frac{i}{2 \tau_{el}} Q \right).$$

(31)

After this, the corresponding integration over all momenta in the r.h.s. of Eq. (30) is straightforward in the pole approximation (justified provided that the weak disorder condition, $\varepsilon_F \tau_{el} \gg 1$, is fulfilled), when one replaces this integration by $\nu_0 \int_{-\infty}^{\infty} d\xi$.

The condition $Q^2 = \mathbb{1}$ is convenient to parameterise as

$$Q = U^\dagger \Lambda U,$$

(32)

where $U$ belongs to a so called symplectic group $Sp(2n)$. This means that $U^\dagger U = 1$, and each $U$ is a “real quaternion” matrix, i.e. can be represented as $2n \times 2n$ matrix with matrix elements of the structure $a + bi + c j + dk$, where $a, b, c, d$ are real $n \times n$ matrices and $i, j, k$ are the so called quaternion units, represented as $i \tau$ with $\tau$ being the set of $2 \times 2$ Pauli’s matrices in the “charge-conjugate” space defined by Eq. (15). Such a choice of $U$ preserves the charge-conjugate symmetry of matrix $Q$, Eq. (29). However, each $Q$ can be represented in the form (32) in infinitely many ways, as the “gauge transformation” $U \mapsto SU$ leaves $Q$ unchanged for any $S$ commuting with $\Lambda = \text{diag}(1_n, -1_n)$. Such a matrix $S$ is a block-diagonal matrix $S = \text{diag}(S_n, \tilde{S}_n)$, with each $S_n$ belonging to the symplectic group $Sp(n)$, i.e. $S \in Sp(n) \otimes Sp(n)$. To make the representation (32) single-valued, one needs to factorise out matrices $S$ (i.e. to fix the gauge). As a result of such a factorisation, one ends up with a coset space: $U \in Sp(2n)/Sp(n) \otimes Sp(n)$ for matrices $U$ that makes a single-valued representation for $Q$ in Eq. (32) (as matrices $Q$ and $U$ are now in one-to-one correspondence, $Q$ belongs to the equivalent coset space, although its matrix representation would be obviously different). A general geometric structure of such a coset space is not difficult to understand and use (see, e.g., an excellent book51 addressed specifically to physicists). However, it won’t be necessary since in what follows an explicit single-valued parameterisations of matrices $U$ and $Q$ will be used, which could be straightforwardly verified without any benefit of knowledge of differential geometry.

Note that $Q = \Lambda$ is a solution to the saddle point equation (30) at nonzero $\omega$. What is usually not stressed is that the saddle-point equation is satisfied not only by $\Lambda$ but by any spatially-homogeneous diagonal matrix commuting with $\Lambda$ with eigenvalues equal to $\pm 1$. This must be taken into account in considering solutions with replica-symmetry breaking, the latter being the necessary conditions of obtaining non-perturbative result within the replica approach.22,23. However, the existence of this wide class of the saddle-point solutions at $\omega \neq 0$ is irrelevant for most applications and, in particular, for the expansion of the functional of Eq. (27) near the saddle point of Eq. (30). In such an expansion, the transverse modes with $Q^2 \neq 1$ can be neglected as, due to the first term in the functional (27) they would incur a gap of order of $1/\tau_{el} \varepsilon_F \gg 1$ (for a homogeneous excitation) and would be thus suppressed. Then the first term gives just an irrelevant constant that could be neglected (it vanishes in the replica limit as $\text{Tr} 1 \propto n$ but such a term would be irrelevant even in a finite-$n$ $\sigma$ model.) Terms with a smaller gap of order $\omega/\delta$ would appear upon the expansion of the second, $\text{Tr} \ln$ term. Keeping them as just is as long as $\omega \tau_{el} \ll 1$ which will be assumed throughout. The expansion of the $\text{Tr} \ln$ term will also produce gapless gradient terms with the magnitude of order $(1/\tau_{el} \varepsilon_F)^2 (q\ell)^2$ in the momentum space: keeping them while neglecting the $Q^2 \neq 1$ modes is justified as long as $q\ell \ll 1$. Therefore, the two conditions of the applicability of the standard NLoM are

$$\omega \tau_{el} \ll 1 \quad q\ell \ll 1,$$

(33)
which should be taken together with the weak disorder condition, \( g \gg 1 \). Therefore, the standard NL(\( \sigma \)M is applicable for the description of the diffusive regime, at scales \( r \gg \ell \) and times \( t \gg \tau_{el} \). These conditions can be relaxed and the NL(\( \sigma \)M can be generalised\(^{12,13,14,15,16}\) for considering also ballistic regime \( (r \ll \ell) \) but such a generalisation will not be considered in the present lectures. Now the expansion of the tr ln term in Eq. (27) is straightforward by making use of the condition \(^{29}\) and applying the similarity transformation (omitting for now the source field \( \eta \)):

\[
\mathcal{F} = -\frac{1}{2} \text{Tr} \ln \left[ -\hat{\xi} + \frac{i}{2\tau_{el}} Q + \frac{\omega^+}{2} \Lambda \right] = -\frac{1}{2} \text{Tr} \ln \left[ -U \hat{\xi} U^\dagger + \frac{i}{2\tau_{el}} \Lambda + \frac{\omega}{2} U \Lambda U^\dagger \right]
\]

\[
= -\frac{1}{2} \text{Tr} \ln \left\{ \mathcal{G}_0^{-1} - U \left[ \hat{\xi}, U^\dagger \right] + \frac{\omega}{2} U \Lambda U^\dagger \right\}
\]

\[
= \text{const} - \frac{\omega}{2} \text{Tr} \left( \mathcal{G}_0 U \Lambda U^\dagger \right) + \frac{1}{4} \text{Tr} \left( \mathcal{G}_0 U \left[ \hat{\xi}, U^\dagger \right] \mathcal{G}_0 U \left[ \hat{\xi}, U^\dagger \right] \right),
\]

where \( \mathcal{G}_0 \) is Green’s function \(^{61}\) with \( Q = \Lambda \) and the symbol Tr implies the integration over \( r \) as well as the summation over matrix indices. The real part of the first term is an irrelevant \( n \)-proportional constant, while its imaginary part reduces to \(-(i\omega/4) \text{Tr} \left( \Lambda U^\dagger U \right) = -i \omega \text{Tr} \left( \Lambda Q \right)\), with the coefficient of proportionality \( \alpha = \Lambda \mathbb{M} \mathcal{G}_0 (r, r) = \Lambda \mathbb{M} \int \mathcal{G}_0 (p) \frac{d^d p}{(2\pi)^d} = \nu \int d\xi \left( \xi^2 + 1 \right) = \nu \pi \). Thus, the first term in the expansion \(^{61}\) gives the gradientless term in the NL(\( \sigma \)M functional \(^{11}\). In the second term in the expansion \(^{61}\), the operator \( \hat{\xi} = (-1/2m) \nabla^2 - \varepsilon_F \) in the commutator \( \left[ \hat{\xi}, U^\dagger \right] \), reduces to \( \nu_F n \cdot \nabla (n \text{ is a unit vector}) \) so that Fourier-transforming it into the momentum space and performing the angular integration, one finds this term equal to

\[
\frac{\nu_F^2}{4} \int \left\{ \left\langle \mathcal{G}_0 (r) (n \cdot \mathcal{A})_{r,r} \right\rangle \left( \mathcal{G}_0 (r) (n \cdot \mathcal{A})_{r} \right) \right\} d^dr d^dr' = \frac{\nu_F^2}{4d} \sum_{p,q} \left\{ \mathcal{G}_{0,p} A_q \cdot \mathcal{G}_{0,p+q} A_q \right\}
\]

where \( \mathcal{A}(r) \equiv U(r) \nabla U^\dagger (r) = -(\nabla U)^\dagger \). Neglecting higher-order terms in \( q \), one has \( \mathcal{G}_{0,p+q} \rightarrow \mathcal{G}_{0,p} \); substituting \( \sum_p \rightarrow \nu \int d\xi \), one reduces the above expression to

\[
\frac{\nu_F^2 \tau_{el}}{2d} \sum_q \int \frac{\left[ \mathcal{A}_q \cdot (\xi + i\Lambda) \mathcal{A}_q \right]}{(\xi^2 + 1)^2} d\xi = \frac{\pi \nu D}{8} \sum_q \text{tr} [\mathcal{A}_q, \Lambda]^2 = \frac{\pi \nu D}{8} \text{Tr} (\nabla Q)^2,
\]

where \( D = \nu_F^2 \tau_{el}/d \) is the diffusion coefficient, and the identity \( \nabla Q = U^\dagger [\mathcal{A}, \Lambda] U \) was used, that follows from the representation \(^{29}\) and the definition of \( \mathcal{A} \). This completes the derivation of the NL(\( \sigma \)M functional \(^{11}\), the matrix field \( Q \) obeying constraints \(^{29}\) or, equivalently, \(^{29}\). Including the symmetry-breaking factors, like a magnetic field, is straightforward as I will demonstrate below when deriving the NL(\( \sigma \)M for interacting electrons. The source fields \( \eta \) were left out of the expansion, but including them is simple: as seen from Eq. (25), they enter on par with the term \( \omega \Lambda \), so that their inclusion would result in additional gradientless terms, \( \text{Tr} \eta Q \). In a similar way, one can include any other source fields, e.g. those necessary for direct evaluation of the conductance \(^{11}\).

The expansion \(^{61}\) was made to the first non-vanishing powers in the small parameters \(^{61}\). The higher order terms, both in gradients and in the (symmetry-breaking) frequency may also be included. They are required in order to obtain tails of different distribution function or to describe the long-time relaxation in the framework of the replica NL(\( \sigma \)M\(^{11}\), but this goes far beyond present considerations.

What is left to do, before turning to the interaction case, is to illustrate how the NL(\( \sigma \)M works. The model is nonlinear because of the presence of the constraint \(^{29}\) – so such an illustration should show how the constraint may be explicitly resolved. From all the examples listed in the introduction, I will consider only one – that of mesoscopic fluctuations\(^{11,34,35}\).

V. MESOSCOPIC FLUCTUATIONS WITHIN THE NL(\( \sigma \)M

The most famous of sample-to-sample (mesoscopic) fluctuations are, of course, the universal conductance fluctuations\(^{11,34,35}\), whose full distribution was first calculated within the replica NL(\( \sigma \)M\(^{11}\). For the illustrative purposes of this section, it is more convenient to consider mesoscopic fluctuations of density of states (DoS), as this is the simplest quantity to calculate, whether diagrammatically\(^{29}\) or within the NL(\( \sigma \)M\(^{11}\). The main simplification within the NL(\( \sigma \)M is that one does not need to introduce any source field, since all the moments of
the DoS fluctuations, e.g. variance, can be obtained by differentiating the partition function \(\text{Eq. (28)}\) with respect to frequency \(\omega\):

\[
\frac{\langle (\delta\nu)^2 \rangle}{\nu_0^2} = -\frac{1}{2} \lim_{n \to 0} \left( \frac{\delta_0}{\pi n} \right)^2 \frac{\partial^2}{\partial \omega^2} \left[ Z(\omega) \right] \bigg|_{\omega=0}.
\]

(36)

Here \(\delta \nu \equiv \nu - \nu_0\), \(\nu_0\) is the disorder-independent average DoS, and \(\delta_0 \equiv 1/\nu_0 L^d\) is the mean level spacing. To get Eq. (36), one first expresses \(\nu\) in terms of Green’s functions,

\[
\nu(\varepsilon) = \frac{i}{2\pi L^d} \int \left[ G^+_{\varepsilon} (r, r) - G^-_{\varepsilon} (r, r) \right] d^dr
\]

and using the representation of \(G^\pm\) in terms of functional integrals, Eqs. (13)–(18) one obtains the above expression as

\[
\nu = \frac{1}{2\pi L^d} \frac{1}{Z_1} \int D\bar{\psi} D\psi \ e^{iS_1(\eta, \omega)} \text{Tr} \bar{\psi}(r)\Lambda \psi(r) = -\frac{i}{\pi L^d} \frac{\partial Z_1(\omega)}{\partial \omega}.
\]

The expression for \(\langle (\delta\nu)^2 \rangle\) is then obtained by the replication and the ensemble averaging of the product of two DoS, Eq. (22), as described above. Then \(Z(\omega)\) is finally expressed as the partition function for the NL\(\sigma M\), Eq. (28). It is implied in Eq. (36) that one considers the fluctuations in an open sample, where an inevitable infrared divergence is cut off at \(\omega \sim E_T\) (which allows one to put \(\omega \to 0\)) where the Thouless energy \(E_T = D/L^2\) is inversely proportional to the diffusion time through the sample. For a closed sample, Eq. (36) at finite \(\omega\) can be used for calculating the correlation function \(\langle \nu(\varepsilon + \omega) \nu(\varepsilon) \rangle\) but for \(\omega \lesssim \delta_0\) considerations should be more subtle (see (22)) than what follows.

The simplest task is to calculate \(Z(\omega)\) perturbatively. In order to make any calculation, one has to resolve the constraints (2), i.e. to express \(Q\) in terms of unconstrained matrices. For the purpose of constructing regular perturbation series, it is most convenient to parameterise \(Q\) as follows:

\[
Q = (\mathbb{1} - W/2)\Lambda(\mathbb{1} - W/2)^{-1}.
\]

(37)

Here \(W\) is anti-Hermitian matrix, off-diagonal in the “retarded-advanced” space, labelled by + or – indices as described between Eqs. (21) and (23):

\[
W = \begin{pmatrix}
0 & W^{+-} \\
W^{-+} & 0
\end{pmatrix} \equiv \begin{pmatrix}
0 & B \\
-B^+ & 0
\end{pmatrix}
\]

(38)

Here the \(2n \times 2n\) matrices \(B\) are unconstrained but for the fact that they obey the charge conjugation symmetry, Eq. (13). This means that they are “real quaternion” matrices, as described after Eq. (22): \(B = b^\mu \mathbb{1} + ib^\mu \tau_\mu\), with \(\tau_\mu\) being Pauli’s matrices in the charge conjugate space and all \(b^\mu\) \((\mu = 0, 1, 2, 3)\) being totally unconstrained \(n \times n\) real matrices. The parametrisation (37) makes \(Q\) automatically obeying both constraints in Eq. (2), \(Q^2 = \mathbb{1}\) and \(\text{tr} Q = 0\). The structure (38) makes \(W\) anticommuting with \(\Lambda\), thus ensuring that \(Q\) is Hermitian. The functional integration in Eq. (28) should now be performed over independent matrix components of \(B\). A particular convenience of the parametrisation (37) is in the fact that in the \(n = 0\) replica limit, the Jacobian of the change from \(Q(r)\) to \(B(r)\) is 1. To perform the integration, one should (again!) reduce it to Gaussian, by representing \(F = F_0 + (F - F_0)\). Here the quadratic in \(W\) functional \(F_0\) is obtained by expanding the NL\(\sigma M\) functional (11) to the lowest order in \(W\):

\[
F_0 = -\frac{\pi \nu_0 D}{8} \int \text{tr} (\nabla W)^2 d^dr = \frac{\pi \nu_0 D}{4} \text{Tr} |\nabla B|^2.
\]

(39)

The \(\omega\)-proportional term is not included into \(F_0\) (although it should, if one were interested in frequency-dependent observables), since in Eq. (36) one needs only quadratic in \(\omega\) terms. They are obtained by expanding \(e^{-\langle F - F_0 \rangle}\) in Eq. (28) for \(Z(\omega)\) in powers of \((F - F_0)\) and further expanding \(Q = \Lambda(1 + W + \frac{1}{2}W^2 + \ldots\) according to Eq. (38), and finally performing the functional integration

\[
\langle \cdots \rangle_0 \equiv \frac{\int DB(\cdots) \exp(-F_0)}{\int DB \exp(-F_0)}.
\]

(40)
As this integration is Gaussian, the result would be a product of all pair averages. The later can be found immediately, as Eq. (43) ensures that the the functional average (44) splits into a product of Gaussian integrals over independent matrix elements $b^d_{ij}$ of $B$:

$$
\left\langle b^d_{ij}(r)b^d_{km}(r') \right\rangle_0 = \Pi(r-r')\delta^{ij}\delta_{km},
$$

where the propagator $\Pi(r-r')$ obeys the diffusion equation, $\pi\nu_0 D\nabla^2\Pi(r-r') = \delta(r-r')$, with proper boundary conditions (open sample). For an $L^d$ cube, $\Pi$ can be defined via its Fourier transform as

$$
\Pi(r-r') = \frac{\delta_0}{\pi} \sum_q \frac{1}{Dq^2} e^{iq(r-r')}.
$$

Eq. (41) is relatively easy to derive but very inconvenient to use, since all matrices that enter the averaging should be explicitly split into their components in the retarded-advanced space, like in Eq. (42), and in the charge conjugate space, like in Eq. (29). But all these components can be assembled together once and forever by casting the averaging formula (41) into those (two) written explicitly in the full matrix space of $Q_{\Omega}$:

$$
\left\langle \text{tr}\left( W_r P W_{r'} R \right) \right\rangle_0 = \Pi(r-r')\left\{ \text{tr}(\text{APAR}^+ - PR^+) + \text{tr}(\text{AP})(\text{AR}) - \text{tr}(P)(\text{tr}(R)) \right\}
$$

$$
\left\langle \text{tr}\left( W_r P \right) \text{tr}\left( W_{r'} R \right) \right\rangle_0 = \Pi(r-r')\text{tr}(\text{APAR} - \text{APAR}^+ - PR + PR^+) \tag{43}
$$

Here $P$ and $R$ are arbitrary matrices of the same rank as $W$ (or $Q$). Equations (43) allow one to calculate any perturbative (or RG) expression. Let us use it for calculating $Z(\omega)$ up to $\omega^2$ and thus finding the DoS variance (36).

In the lowest perturbative order one needs to calculate the functional average $\frac{1}{2} \left\langle F^2_\omega \right\rangle_0$, where $F_\omega$ is the 2nd term in Eq. (1). Expanding $Q$, one finds that

$$
F_\omega \equiv \frac{i\pi\nu_0\omega}{4} \text{Tr} \Lambda Q = \frac{i\pi\nu_0}{8} \left[ \text{Tr} (\omega W^2) + \frac{1}{2} \text{Tr} (\omega W^4) + \ldots \right] \tag{44}
$$

since the zeroth term is an irrelevant $n$-proportional constant, and all the odd order terms vanish as $\text{tr} W = 0$. It is convenient to put $\omega$ inside the $\text{tr}$: in general, source fields that are necessary to calculate any other quantity but DoS are matrix field, and in a similar expansion they would have been inside the trace. Now to the lowest order

$$
Z(\omega) = -\frac{\pi^2\nu_0^2}{128} \left\langle \text{tr} (\omega W^2) \text{Tr} (\omega W^2) \right\rangle_0. \tag{45}
$$

One applies the averaging formulae (43) by choosing all possible pairings and performing them one after another (in an arbitrary order), considering at each step all matrices $W$ (but two active, i.e. participating in the chosen pairing) as fixed. There are only two different pairing schemes possible in Eq. (45): one either make two inter-trace, or two intra-trace pairings. In the first case, one applies first of the formulae (43), with $P = R = \mathbb{1}$; then only the last term does not vanish and equals $16n^2\pi$. Making two such inter-trace averaging one finds the contribution proportional to $n^4$ which vanishes in Eq. (45) in the replica limit. Therefore, one needs only the intra-trace averages which is convenient to represent diagrammatically, Fig. 5 (a). In such a representation, each trace corresponds to a (hatched) polygon with vertices representing either matrices $W$ or the source field $\omega$ (no more than one vertex in each polygon). At the first step of averaging, one applies the second of Eq. (43) with $P \equiv \omega W(r)$ and $R \equiv \omega W(r')$. Taking into account a combinatorial factor of 2 (each $W$ in the first trace can be paired with either of two in the second trace) and using that $W$ is anti-Hermitian and commutes with $\Lambda$, one finds

$$
\left\langle \text{tr} \omega W^2 \text{Tr} \omega W^2 \right\rangle_0 = -8\omega^2 \int G_{r-r'} \left\langle \text{tr} W_r W_{r'} \right\rangle_0 d^drd^d r' = 8\omega^2 (\text{tr} 1)^2 \int G_{r-r'} L^d d^d r r' = 128\omega^2 n^2 L^d \int G_{r-r'} = \frac{128\omega^2 n^2}{\pi^2\nu_0^2} \sum_q \frac{1}{(Dq^2)^2}.
$$

Combining this with Eq. (45) and substituting the result into Eq. (36) one recovers the well-known result

$$
\left\langle \delta\nu^2 \right\rangle_{\nu_0^2} = \left( \frac{\delta_0}{\pi} \right)^2 \sum_q \frac{1}{(Dq^2)^2} \frac{C_d}{g^2}. \tag{46}
$$
where $g = 2\pi^2 v_0 D L^{d-2}$ is the dimensionless conductance of a cube of volume $L^d$, and $C_d$ is a numerical coefficient dependent on the sample shape and dimensionality.

Exactly the same technique could be applied for calculating the two-level correlation function, \(\langle \nu(\varepsilon) \nu(\varepsilon + \omega) \rangle / \nu^2 - 1\). In this case $\omega$ plays a dual role of the frequency and the source field. The only technical change is that the $T \rho W^2$ should be included into the zeroth approximation, Eq. (39), which results in replacing $Dq^2$ by $Dq^2 - i\omega$ in the denominator of Eq. (42). In the diffusive regime of relatively large $\omega$, i.e. $\omega \gtrsim D/L^2$, the result (44) remains roughly valid, with substituting $L_\omega \equiv (D/\omega)^{1/2}$ for $L$ into the expression for $g$. However, this would fail in the two-dimensional case where for $L_\omega \ll L$ the coefficient $C_2$ in Eq. (21) vanishes due to the analytic properties of the diffusion propagator $\Pi(\omega, q)$\(^{53}\). In this case, the main perturbative contribution is due to the two-loop diagrams, Fig. 5(b, c). The hatched pentagon in diagram (b) corresponds to the 2\(\text{nd}\) term in the expansion (11). The hatched square in diagram (c) results from the average $\langle F_0^2 F_4^{(4)} \rangle_0$ with $F_4^{(4)}$ obtained by expanding $\nabla Q^2$ to the 4\(\text{th}\) power of $W$. Such expansions give all the necessary coefficients corresponding to the Hikami boxes of the standard diagrammatic technique, so that one is left with calculating two-loop integrals containing 3 or 4 diffusion propagators. Note that in the diagrammatic technique the diagrams $b$ and $c$ cancel each other while the only contribution results from the third possible 2-loop diagram\(^{53}\) describing the weak localisation correction to one of the diffusons. Such a diagram, on the contrary, vanishes within the parametrisation (37) while the identical result is given by the remaining two-loop diagrams $b$, $c$.

Finally note that the RG calculations are also based on the averaging formulae (43) which could also be appropriately modified to include cases of partially broken symmetry, either due to the presence of a magnetic field or magnetic impurities (unitary symmetry) or to the presence of spin-orbit scattering (symplectic symmetry). In all cases, the same averaging formulae are also applicable to the case when the NLS\(\sigma\)M also encompasses interactions.

### VI. COULOMB AND PAIRING INTERACTIONS IN THE SIGMA MODEL

The NLS\(\sigma\)M works perfectly for perturbative problems and – in certain cases – does reasonably well even for non-perturbative problems within the model of non-interacting electrons in the presence of disorder. Since the inclusion of interactions beyond the mean-field approach is impossible within the supersymmetric method, the replica approach is still the most reliable tool for models with interactions.

The original fermionic replica $\sigma$ model\(^{55}\) has been generalised by Finkelstein\(^7\) to include the interactions. The interest in this approach has been greatly enhanced by the recent discovery\(^{39}\) of an apparent metal-insulator transition in 2D disordered systems in zero magnetic field. It is not at all clear whether the observed effects are, indeed, due the transition\(^{54,55}\), and if they are – whether such a transition is, indeed, driven by interactions. However, a possibility of having the metal-insulator transition in a 2D disordered interacting system (schematically described by the TOE model\(^7\)), while it is absent in a disordered noninteracting system where all states are localised, is intriguing by itself. Such a possibility is arguably a driving force in a considerable revival of interest in Finkelstein’s $\sigma$ model (see, e.g., Refs. 17,18,19,20,21). At the moment, there is no clear evidence whether Keldysh techniques employed in Refs. 19,20 would give any edge over the replica method described in these lectures. So all further considerations are also limited to the replica approach.

Below I describe how to include the BCS pairing interaction and the Coulomb interaction (for simplicity, in a singlet channel only, as the inclusion of a triplet channel is technically almost identical) into the derivation of NLS\(\sigma\)M given in section IV. One inevitable complication is that such a derivation should be done in the
temperature technique, and here some leap of faith is necessary in order to come to any result within a relatively short paper. The resulting \( \sigma \) model looks similar to that for the non-interacting model, although its usage is considerably mode difficult. Still, I give a relatively simple example of such a usage to derive some known result for the BCS model. Note that the derivation follows that in Ref.\cite{22}, which is similar in spirit to that in Ref.\cite{23} but rather different from the original derivation in Ref.\cite{24}. The derivation below is also made as parallel as possible to that of the NL\( \sigma \)M in the absence of interactions.

### A. Hubbard–Stratonovich decoupling

Since the pair interaction, in contrast to the elastic scattering from impurities, does not conserve single-particle energy, the effective interaction functional should be dynamical. As usual, it is convenient to introduce imaginary time \( \tau \) implying everywhere the thermodynamic Gibbs averaging (together with the averaging over quenched disorder where applicable). Thus one considers thermodynamic Green’s functions related (in the Matsubara frequency representation) to the retarded and advanced Green’s functions of the previous sections by the standard procedure of analytical continuation\cite{40}. Then the effective functional corresponding to the interaction term in Eq. (48) can be written as

\[
\mathcal{F}_{\text{int}} = \frac{1}{2} \int dx d\tilde{x} \bar{\psi}_s(x) \psi_s(x') V_{xx'} \bar{\psi}_{s'}(x') \psi_{s'}(x),
\]

where \( x \equiv \{ r, \tau \} \), and all the fermionic fields are anti-periodic in imaginary time \( \tau \) with period \( 1/T \), \( s \equiv \{ \uparrow, \downarrow \} \) is the spin index, \( V_{xx'} \equiv \delta(\tau - \tau') V_s(r - r') \), and \( V_s \) represents the Coulomb interaction. The fields are still assumed to have the same matrix structure, Eq. (15), as in the noninteracting case, necessary to take into account both the slow modes (diffusion and cooperon) arising from the impurity scattering. After the replication necessary for treating the random potential, all the fields in Eq. (47) naturally have the same replica index and the action would be just a sum of all the actions (17) with different replica indices. Similarly, the BCS functional describing the attraction in the Cooper channel can be written as

\[
\mathcal{F}_{\text{sc}} = \lambda_0 \int dx \bar{\psi}_\uparrow(x) \psi_\downarrow(x) \psi_\downarrow(x) \psi_\uparrow(x),
\]

where \( \lambda_0 \) is the BCS coupling constant. The total action is now the sum of

The Hubbard–Stratonovich decoupling of the functional (47) - (48) is similar to that for the disorder functional \cite{20}, i.e. some new matrix fields are introduced to decouple the quartic terms via the Gaussian integration like in Eq. (21). To perform this simultaneously with averaging over disorder, one needs to replicate all the fermionic fields as described after Eq. (21). All the quartic interaction terms in the action are naturally diagonal in the replica indices.

To allow for all the slow modes in the interaction functional (47) one should introduce three matrix fields: the fields \( \Phi \) and \( \hat{f} \) to take account of a small-angle scattering (a singlet channel) and large-angle scattering, and yet another one that corresponds to the Coulomb repulsion in the Cooper channel. The last one would lead to the standard renormalization of the BCS attraction (see, e.g., Ref.\cite{24}) and is not considered here; further, it is assumed that systems under considerations still have an effective attraction in the Cooper channel after such a renormalization. The interaction in the Cooper channel is then decoupled with the help of another matrix field, \( \Delta \). Then the decoupling has the following form:

\[
e^{-\mathcal{F}_{\text{int}}} = \int \mathcal{D}\Phi \exp \left\{ -\frac{1}{2} \int dx dx' \Phi(x) V_{xx'}^{-1} \Phi(x') + i \int dx \bar{\Psi}_s(x) \Phi(x) \Psi_{s}(x) \right\}
\]

\[
+ \int \mathcal{D}\hat{f} \exp \left\{ -\frac{1}{2} \int dx dx' \text{tr} \left[ \hat{f}(x) V_{xx'}^{-1} \hat{f}(x') \right] + i \int dx \bar{\Psi}_s(x) f_{ss'}(x) \Psi_{s'}(x) \right\},
\]

\[
e^{-\mathcal{F}_{\text{sc}}} = \int \mathcal{D}\Delta \exp \left\{ -\frac{1}{\lambda} \int dx |\Delta(x)|^2 + i \int dx \left[ \Delta(x) \bar{\Psi}_\uparrow(x) \Psi_\downarrow(x) - \bar{\Delta}(x) \bar{\Psi}_\downarrow(x) \Psi_\uparrow(x) \right] \right\}
\]

After performing the Gaussian integration here, together with that in Eq. (21), one obtains instead of the action (21) the following effective action that depends both on \( \sigma \) (which is a new name for \( Q \), the latter being reserved for a later usage) and on all the new fields:

\[
\tilde{\mathcal{F}} = \mathcal{F}_{\text{fields}} + \frac{\pi \nu}{8\tau} \text{Tr} \sigma^2 - \frac{1}{2} \text{Tr} \ln \left[ \xi - i \left( \frac{1}{2\tau} \sigma + \Phi + \Delta + \hat{\epsilon} \right) - \hat{f} \right].
\]
Here $F_{\text{fields}}$ is the quadratic in $\Phi$, $f$ or $\Delta$ part of the action \[59\]; the operator $\varepsilon$ equals $i\tau_3 \partial_x$ in the imaginary time representation with $\tau_3$ being the appropriate Pauli matrix in the charge-conjugate space defined by Eq. \[58\] (it becomes a diagonal matrix of fermionic Matsubara frequencies, $\varepsilon_n = \pi(2n + 1)T$, in the frequency representation); $\text{Tr}$ refers to a summation over all the matrix indices and to an integration over $r$ and $\tau$ (or summation over Matsubara frequencies in the frequency representation). The triplet channel, described by the field $f$, is quite important: in particular, it can lead to the delocalization in the presence of disorder\textsuperscript{7,17}. Nevertheless, such effects will not be considered here and this term will be ignored from now on – mainly for simplicity (also, it does not contribute when considering effects of the Coulomb interaction in superconducting systems). The field $\sigma$ has the same structure as $Q$ in Eqs. \[52\] and \[53\], apart from explicitly including the $2 \times 2$ spin sector and replacing the $2 \times 2$ retarded-advanced sector by the dependence on the imaginary time $\tau$ (since it is diagonal in $\tau$, it becomes a matrix field in the Matsubara frequencies). In the absence of the fields $\Phi$ and $\Delta$, the saddle-point solution for the action \[50\] is formally the same as in the zero-temperature techniques of section IV, Eq. \[52\], but the matrix $\Lambda$ has a non-unit structure in the Matsubara (instead of advanced-retarded) sector:

\[
\sigma = U^\dagger \Lambda U, \quad \Lambda = \text{diag} \{ \text{sgn} \varepsilon \}. \tag{51}
\]

The field $\Phi$, corresponding to the singlet part of the Coulomb interaction, is diagonal in all the sectors. The ”order-parameter” field $\Delta$ is Hermitian and self-charge-conjugate, diagonal in the replica indices and coordinates $r$ and $\tau$, and has the following structure in the spin and time-reversal space:

\[
\hat{\Delta}(x) = |\Delta(x)| e^{\frac{i}{2} \chi(x) \hat{\tau}_3} \hat{\tau}_2^p \otimes \hat{\tau}_2^p e^{-\frac{i}{2} \hat{\chi}(x) \hat{\tau}_3}, \tag{52}
\]

where $|\Delta|$ and $\chi$ are the amplitude and the phase of the pairing field $\Delta(r, \tau)$, $\hat{\tau}_i$ and $\hat{\tau}_i^p$ are Pauli matrices ($i = 0, 1, 2, 3$ with $\hat{\tau}_0 = 1$) that span the charge-conjugate and spin sectors, respectively.

When $\Phi$ and $\Delta$ are included, the saddle-point equation for the functional \[50\] can be formally written similarly to Eq. \[50\] for the non-interacting zero-temperature case:

\[
-i \nu M(r) = \left\langle r \left| \left[ -\hat{\varepsilon} + \frac{i}{2 \tau_{\text{cl}}} \sigma + i \left( \hat{\varepsilon} + \hat{\Delta} + \Phi \right) \right]^{-1} \right| r \right\rangle. \tag{53}
\]

Now $\sigma = \Lambda$ does no longer represent the saddle point solution for $\varepsilon \neq 0$. Still, one can derive an effective functional by expanding the above $\text{Tr} \ln$ within the manifold $S(2)$ in the symmetry-breaking field $\varepsilon + \Delta + \Phi$ and in gradients of $\sigma$, as has been done in the original works by Finkelstein. An alternative is to make first a similarity transformation around $\Lambda$ within the manifold $S(2)$ to find the saddle point solution $\sigma_{sp}$ in the presence of the fields. This can be formally done with the help of matrix $U_0$ that diagonalizes the Hermitian field $\hat{\varepsilon} + \hat{\Delta} + \Phi$:

\[
\hat{\varepsilon} + \hat{\Delta} + \Phi = U_0^\dagger \lambda U_0, \quad \sigma_{sp} = U_0^\dagger \Lambda U_0. \tag{54}
\]

Here $U_0$ (which is yet undetermined but fixed) belongs to the same symmetry group that defines the manifold $S(2)$ in the absence of the interaction fields. By substituting Eq. \[50\] into the saddle-point equation \[50\], one can easily verify that this is, indeed, a spatially-homogeneous solution, provided that $\lambda \tau_{\text{cl}} \gg 1$ which will always be the case for a dirty superconductor ($\Delta \tau_{\text{cl}} \gg 1$) or a dirty metal (dimensionless conductance $q \gg 1$).

Now one can perform the expansion of $\text{Tr} \ln$, Eq. \[50\], in gradients of $Q$ and in the symmetry-breaking fields represented by the eigenvalues $\lambda$ and the matrix $U_0$, Eq. \[50\], by employing the following parametrisation:

\[
\sigma = U_0^\dagger Q U_0, \quad Q = U^\dagger \Lambda U, \tag{55}
\]

where $Q$ represents the saddle-point manifold in the metallic phase and $\sigma$ is obtained from $Q$ by the same rotation \[55\] as $\sigma_{sp}$ is obtained from the metallic saddle point $\Lambda$. Therefore, $Q$ is defined, as in the metallic phase, on the coset space $S(2n)/S(n) \otimes S(n)$ where, depending on the symmetry, $S$ represents the unitary, orthogonal or symplectic group. In the noninteracting case of section IV, only the case of the symplectic group, corresponding (unfortunately) to the orthogonal symmetry, was described, but a generalisation to the other two cases is straightforward and described in almost any paper on the NLorM for disordered systems. The parameterisation \[55\] simplifies considerably all the subsequent derivations and leads to a new variant of the nonlinear $\sigma$ model for interacting systems\textsuperscript{21} which can be more convenient for many applications than the original one\textsuperscript{20}. After substituting $\sigma = U_0^\dagger U^\dagger \Lambda U U_0$, Eq. \[55\], into Eq. \[50\], one obtains the following representation for the $\text{Tr} \ln$ term:

\[
\delta F = -\frac{1}{2} \text{Tr} \ln \{ \hat{G}_0^{-1} + U_0 [\hat{\varepsilon}, U_0^\dagger] - i (U \lambda U^\dagger) \},
\]
where one can also include an external magnetic field with the vector potential $\mathbf{A}$:

$$G_0 \equiv \left( \xi - \frac{i}{2\tau_{\text{el}}} \lambda \right)^{-1}, \quad \hat{\xi} \equiv \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - \varepsilon_F.$$

The expansion to the lowest powers of gradients and $\lambda$ is now straightforward and similar to that for the noninteracting case. It results after some calculations in the following effective action:

$$\bar{\mathcal{F}} = \frac{1}{2} \text{Tr}[\Phi V^{-1}\Phi] + \frac{1}{2\lambda_0} \sum_{\omega} \int \text{d}r |\Delta_\omega|^2 + \frac{\pi\nu}{2} \text{Tr} \left[ \frac{D}{4} (\partial Q)^2 - \lambda Q \right]. \quad (56)$$

The long derivative in Eq. (56) is defined as

$$\partial Q \equiv \nabla Q + \left[ A_0 - ieA_{3,0}, Q \right] \equiv \partial_0 Q + [A_0, Q], \quad (57)$$

where the matrix $A_0$ is given by

$$A_0 = U_0 \partial_0 U_0^\dagger, \quad (58)$$

and $\partial_0 \equiv \nabla - [ieA_{3,0}, \ldots]$ is the long derivative (57) in the absence of the pairing field $\Delta$. Both $U_0$ and $\lambda$ should be found from the diagonalization of $\lambda + \Delta + \Phi$, Eq. (52). Although such a diagonalization cannot be done in general, it will be straightforward in many important limiting cases. For $\Delta = \Phi = 0$, the field $A_0$ vanishes, $\partial \to \partial_0$, and $\lambda \to \varepsilon$, so that the functional (50) goes over to that of the standard nonlinear $\sigma$ model for non-interacting electrons.

The action (50) is most general in the current context. It allows one to develop a fully self-consistent approach to superconductivity of dirty metals in the presence of Coulomb interaction. However, any application requires a set of further simplifications. As a simple illustration, I will show below how to use the model in a dirty superconductor near the metal-superconductor transition in the absence of Coulomb interaction.

**B. Ginzburg-Landau Functional**

In the vicinity of the metal-superconductor transition one can expand the action (50) (in the absence of the Coulomb field $\Phi$) in the pairing field $\Delta$. A further simplification is possible in the weak disorder limit, $g \gg 1$, when the field $Q$ can be integrated out to obtain an effective action for the $\Delta$-field only. In the quadratic in $\Delta$ approximation, the kernel of this action will give, with due account for the disorder, an effective matrix propagator of the pairing field which governs properties of a disordered superconducting sample near the transition.

To integrate over $Q$, one splits the action (50) into $\bar{\mathcal{F}} = \mathcal{F} + \mathcal{F}_{\Delta}$ where $\mathcal{F}$ is the standard NL$\sigma$M functional as in the metallic phase. Then one makes a cumulant expansion, i.e. first expands $e^{-\mathcal{F}} e^{\mathcal{F}_{\Delta}}$ in powers of $\mathcal{F}_{\Delta}$, then performs the functional averaging with $e^{-\mathcal{F}_0}$ (denoted below by $\langle \ldots \rangle_Q$) and finally re-exponentiates the results. Apart from the last step, this is exactly as described in section IV for the noninteracting NL$\sigma$M. The expansion involves only the first and second order cumulants since the higher order cumulants generate terms of higher order in $\Delta$. Then the only contributions to the action quadratic in $\Delta$ are

$$\mathcal{F}_{\text{eff}}[\Delta] = \frac{1}{\lambda_0 T} \sum_{\omega} \int \text{d}r |\Delta_\omega|^2 - \frac{\pi\nu}{2} \left\langle \text{Tr} (\lambda - \varepsilon)Q \right\rangle_Q$$

$$- \left\langle \frac{\pi\nu D}{8} \text{Tr} [A_0, Q]^2 + \frac{(\pi\nu D)^2}{8} \left\langle \text{Tr} Q\partial_0 Q A_0 \right\rangle_Q \right\rangle_Q. \quad (59)$$

Expanding $\lambda$ and $A_0$ to the lowest power in $\Delta$ and performing the functional averaging one finds the action quadratic in $\Delta$ as follows:

$$\mathcal{F}_{\text{eff}}[\Delta] = \frac{\nu}{T} \sum_{\omega} \int \text{d}r \Delta_\omega^+(r) \langle r | \hat{\mathcal{K}}_\omega | r' \rangle \Delta_\omega^-(r'), \quad (60)$$

with the operator $\hat{\mathcal{K}}_\omega$ given by

$$\hat{\mathcal{K}}_\omega = \frac{1}{\lambda_0 T} - 2\pi T \sum_{\epsilon (\omega - \varepsilon < 0)} \left\{ \hat{\Pi}_{\omega} + \frac{1}{\pi \nu} \frac{\Pi_{\omega}^{\dagger}(2\varepsilon - \omega)(\varepsilon - \omega)^2}{(2\varepsilon - \omega)^2} \right\}. \quad (61)$$
Here $\Pi^c_{|\omega|}(r, r') = \langle r | \hat{\Pi}^c | r' \rangle$ are the cooperon and diffuson propagators. At $\omega = 0$ the later coincides with that of the NLèM for non-interacting electrons, Eq. (12). In general,

$$\hat{\Pi}^c_{|\omega|} = \left( \hat{\mathcal{C}} + |\omega| \right)^{-1},$$

where $\mathcal{C} = -D(\nabla - 2i eA)^2$ defines the cooperon modes; $\hat{\Pi}^d$ is obtained from $\hat{\Pi}^c$ by putting the external vector potential $A = 0$. In the last term in Eq. (61), $\hat{\Pi}^d_{|\omega|}(0) = \hat{\Pi}^d_{|\omega|}(r, r)$; this term may be obtained by expanding in $g^{-1}$ the cooperon propagator with the renormalised diffusion coefficient,

$$\hat{\mathcal{C}} \to \left[ 1 - \frac{1}{\pi \nu} \hat{\Pi}^d_{|\omega|}(0) \right] \hat{\mathcal{C}},$$

which is a weak localisation correction to free cooperon propagator $\Pi^c_{|\omega|}(r, r')$.

The summation over Matsubara frequencies in Eq. (61) yields

$$\hat{K}_\omega = \ln \frac{T}{T_0} + \psi \left( \frac{1}{2} + \frac{\omega - \hat{\mathcal{C}}}{4\pi T} \right) - \psi \left( \frac{1}{2} \right) - \frac{a_\omega \hat{\mathcal{C}}}{4\pi T},$$

where $T_0 \equiv T_{c0}(B = 0)$ is the transition temperature of the clean superconductor in the absence of a magnetic field and $\psi$ is the digamma function. The weak localisation correction is proportional to the coefficient $a_\omega$ given by

$$a_\omega(T) = \frac{1}{\pi \nu V} \sum_q \frac{1}{Dq^2} \left\{ \psi' \left( \frac{1}{2} + \frac{\omega + 4\pi T}{Dq^2} \right) - \psi' \left( \frac{1}{2} + \frac{\omega}{4\pi T} \right) \right\}.$$

For $\omega = 0$ the coefficient $a_0 \equiv a_{\omega = 0}(T)$ can be simplified in the two limits:

$$a_0 = \begin{cases} \frac{\psi'(1/2)}{\pi \nu L^d} \sum_{L < q < L^{-1}} \frac{1}{Dq^2}, & L \gg L_T, \\ -\frac{\psi''(1/2)}{8\pi^2 \nu L^d T}, & L \ll L_T, \end{cases}$$

where $L_T \equiv \sqrt{D/T}$ is the thermal smearing length.

The instability of the normal state (i.e. a transition into the superconducting state) occurs when the lowest eigenvalue of the operator $\hat{K}_\omega$ becomes negative. The eigenfunctions of this operator coincide with the eigenfunctions of the cooperon operator $\hat{\mathcal{C}}$. The lowest eigenvalue of $\hat{\mathcal{C}}$ is known to be $C_0 = DB/\phi_0$, where $\phi_0$ is the flux quanta. This ground state cooperon eigenfunction corresponds to the lowest eigenvalue $K_0$ of the operator $\hat{K}_\omega$. The condition $K_0 = 0$ implicitly defines the line $T_c(B)$ in the $(T, B)$-plane where the transition occurs:

$$\ln \frac{T_c}{T_0} + \psi \left( \frac{1}{2} + \frac{C_0}{4\pi T_c} \right) - \psi \left( \frac{1}{2} \right) = \frac{a_0 C_0}{4\pi T_c}.$$  

The term in the r.h.s. of Eq. (65) describes a $1/g$-correction to the main result. This weak localisation is linear in the magnetic field $B$ and vanishes as $B \to 0$ as expected (the Anderson theorem). In a nonzero magnetic field the weak localisation correction to $T_c$ is positive which has a very simple explanation. The superconductivity is destroyed by the magnetic field when $\Phi(\xi) \gtrsim \Phi_0$, where $\Phi(\xi)$ is the flux over the area with linear size of the order of the coherence length $\xi \sim \sqrt{D/T}$ and $\Phi_0$ is the flux quanta. The weak localisation corrections reduce $D$ and thus $\xi$. Therefore, one needs a stronger field to destroy the quantum coherence. The same reasoning explains the growth of $T_c$ in a fixed magnetic field.

Note finally that it would be straightforward to include the leading weak-localization corrections in all orders of $g^{-1}$ in by calculating the $Q$-averages in Eq. (61) via the renormalization group. This would lead to the renormalization of $D$ in the cooperon propagator (62), thus changing the shape of the $T_c(B)$ curve. However, the value of $T_c(0)$ will again remain unaffected, since the superconducting instability is defined by the onset of
the homogeneous zero mode in the operator $\hat{\mathcal{K}}$, Eq. (63), which does not depend on the value of the diffusion coefficient in the cooperon propagator.

It is worth stressing that the Anderson theorem reflects certain properties of a model rather than those of real superconductors. If one allows for Coulomb interaction, then the critical temperature of the superconducting transition is no longer independent of disorder. Combined effects of the interaction and disorder lead to corrections to the transition temperature proportional (in a slightly simplified way) to $g^{-1} \ln^3(T_c - \tau_0)$. For a relatively weak disorder, the system still remains superconducting at $T \to 0$ while for a sufficiently strong disorder the above corrections would suppress the superconducting pairing at any temperature and make a 2D system insulating at $T \to 0$. Such a mechanism of suppressing $\Delta$ by disorder gives a possible scenario for a widely observed superconductor–insulator (SI) transition in two-dimensional structures [58–61] where $|\Delta|$ remains finite and the superconductivity is suppressed by the loss of the phase coherence due to the fluctuations of the phase $\chi$ of the order parameter. These “phase-only” models can also be derived under certain parametrically controlled assumptions from the NLoM developed in this section. However, the scope of these lectures does not allow me to go into any further demonstrations.

In conclusion, I have demonstrated in detail how to derive from scratch the NLoM for non-interacting electrons in disordered media, have shown some example of its perturbative usage, and in a much more succinct mode, have illustrated how to derive and use it in the presence of the pairing and Coulomb interactions between electrons.

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The NLσM has been also generalised\textsuperscript{12,13,14,15,16} for ballistic systems with $L \lesssim \ell$ but such a generalisation will not be considered in the present lectures.

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