Equivalent weavings

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Abstract. A weaving is defined as the interlacement of the overlapping simple connected graphs. In particular, weavings may be constructed from colorings of the plane. A definition of equivalent weavings constructed from colorings of the plane as well as some examples are presented in this paper.

1. Introduction
In this paper, we present a definition for equivalent weavings. We begin with a summary of the method for constructing weavings of overlapping nets from 2-colorings of the plane and a definition of weavings as presented by Miro, Garciano and Zambrano in [2]. We then present a definition of combinatorial and weaving equivalence and give some examples of equivalent weavings from colorings of the plane.

In [1], Batten and Robson used weavings to study molecular entanglements. Meanwhile, Ramsden, Robins, and Hyde, in [4], showed that three-dimensional crystalline euclidean nets may be constructed from weavings on the hyperbolic plane.

The ultimate goal is to present a methodology for constructing weavings that can be used to construct three-periodic patterns using the methodology given in [4].

2. Methodology for constructing weavings
Let \( \triangle \) be any triangle on the plane \( \mathbb{P} \) with interior angles \( \pi/p, \pi/q, \pi/r \) where \( p, q, r \) are integers greater than or equal to 2.Repeatedly reflecting the triangle on its sides results in a triangle tiling \( T := T(p, q, r) \) of the plane by copies of the triangle \( \triangle \). Let \( P, Q, R \) denote the reflections on the side of \( \triangle \) opposite \( \pi/p, \pi/q \) and \( \pi/r \), respectively. The group \( G := G(p, q, r) \) of isometries generated by \( P, Q, R \) is called a triangle group with group presentation

\[
\langle P, Q, R \mid P^2 = Q^2 = R^2 = (QR)^p = (RP)^q = (PQ)^r \rangle.
\]

By construction, each \( t \in T \) is the image of \( \triangle \) under some unique element \( g \in G \). The group \( G \) acts transitively on the tiling. Hence, the mapping \( g \to g\triangle \) is a bijection. Note that,

(i) \( \bigcup_{g \in G} g\triangle = \mathbb{P} \) and

(ii) \( \triangle^o \cap g\triangle^o = \emptyset \), for every \( g \in G\setminus\{I\} \).
Thus, $\triangle$ is a fundamental region of the triangle group $G(p, q, r)$.

A coloring of $T$ is a labeling of the tiles in $T$ by colors. We use the method presented by Provido, De las Peñas and Felix in [3] to color triangle tilings.

Consider an index 2 subgroup $H$ of $G$ and let $\{g_1 = I, g_2\}$ be a complete set of left coset representatives of $H$ in $G$. Then $G = H \cup g_2H$ and $H \cap g_2H = \emptyset$. Consequently, $T = H \triangle \cup g_2H \triangle$. Also, since $H \cap g_2H = \emptyset$ and the mapping $g \mapsto g\triangle$ is a bijection, then $H \triangle \cap g_2H \triangle = \emptyset$. Suppose $c_1, c_2$ is a set of distinct colors, where $c_1$ represents the color white and $c_2$ represents the color black. A 2-coloring of $T$ can be defined by assigning color $c_1$ to $H\triangle$ and $c_2$ to $g_2H\triangle$. This results in a two-coloring of the triangle tiling $T$ by $H$ which we denote by $T_H(p, q, r)$.

![Figure 1](image1.png)

**Figure 1.** (a) The tiling $T(4, 4, 2)$, and (b) its 2-coloring by $H_2 = \langle R, Q, PQP, PRP \rangle$

In Fig 1, we illustrate a coloring of $T(4, 4, 2)$ by a subgroup $H_2 = \langle R, Q, PQP, PRP \rangle$ of $G(4, 4, 2)$.

In [2], we introduced a concept of black and white patches of a colored tiling $T_H$. Intuitively, a (black) $B$-patch of a colored tiling is composed of edge-adjacent black tiles of the same color all having a unique point in common. A (white) $W$-patch is similarly defined. For example, the $B$-patches of the colored tiling $T_{H_2}$ where $H_2 = \langle R, Q, PQP, PRP \rangle$ is illustrated in Figure 2(b). A $B$-patch is a union of edge-adjacent black tiles about the common vertex of 1- and 2-edges; $W$-patches are formed similarly.

![Figure 2](image2.png)

**Figure 2.** (a) Labeling the edges opposite the angles $\pi/p, \pi/q$, and $\pi/r$ by 0, 1 and 2, respectively, and (b) The $B$- and $W$-patches of $T_{H_2}$

We now construct two disjoint simple connected graphs, $N_B = (V, E)$ and $N_W = (V', E')$ which we refer to as $B$- and $W$-nets, respectively. The vertex sets $V_B$ and the $V_W$ are the set of centroids of the $B$-patches and $W$-patches of $T_H$ respectively. In Figure 2(b), the black and green points are $B$- and $W$-vertices of $T_{H_2}$. 
Given an index 2 subgroup $H = \langle h_1, h_2, ..., h_l \rangle$ of $G$, we can determine a constructible fundamental region $\triangle_H = \triangle \cup g\triangle$ with respect to the given the generating set of $H$. This is elaborated in [2]. The vertices in the constructible fundamental region are called the initial $B$-vertex $b := b_H$ and the initial $W$-vertex $w := w_H$ of $\triangle_H$.

Now for $b$ and each generator $h_i$, construct edges $\{b, h_i b\}$ and $\{b, h_i^{-1} b\}$ using black line segments. The $B$-motif $\triangle_{bH}$ consists of the $B$-vertex $b$ together with the segments of the edges $\{b, h_i b\}, \{b, h_i^{-1} b\}$ within $\triangle_H$. The $W$-motif $\triangle_{mH}$ is similarly constructed.

The elements of the set $E_B$ are of the form $h\{b, h_i b\} := \{hb, hh_i b\}$, such that $h \in H, h_i \in \{h^j_1, h^j_2, ..., h^j_l\}$ and $j \in \{1, -1\}$. The set $E_W$ is similarly defined. The above construction yields two disjoint nets $N_B = (V, E)$ and $N_W = (V', E')$. An overlapping net $O_H = N_B \cup N_W$ is the union of $N_B$ and $N_W$.

We call the points where the edges of the nets $N_B$ and $N_W$ intersect, the weaving points of the overlapping net and denote the set of all weaving points in $O_H$ by $\mathcal{O}_H$. We define a weaving map $\omega_H : \mathcal{O}_H \to \{\oplus, \ominus\}$ by first assigning the $\oplus$ or $\ominus$ values to the weaving points $p$ within the fundamental region $\triangle_H$ and since every element $q$ of $\mathcal{O}_H$ is of the form $q = hp$ for some $h \in H$, let $\omega(q) = \omega(p)$. Thus, we obtain a weaving $W_H = (O_H, \omega_H)$. Also, for a weaving point $(e_b, e_w) \in \mathcal{O}_H$, if $\omega(e_b, e_w) = \oplus$, then edge $e_b$ is above edge $e_w$ while edge $e_b$ is below edge $e_w$ if $\omega(e_b, e_w) = \ominus$. If $\omega_H(\mathcal{O}_H) = \{\oplus, \ominus\}$, then $W_H$ is a proper weaving. This method is outlined in Figure 3 and discussed in detail in [2].

![Weaving construction outline](image)

**Figure 3.** Weaving construction outline

Consider the weaving $W_{H_2} = (O_{H_2}, \omega)$ constructed from the index 2 subgroup $H_2 = \langle R, Q, PQP, PRP \rangle$ of $G(4, 4, 2)$.

Since $\omega_{H_2}(\mathcal{S}) = \{\oplus\}$, $W_{H_2}$ is not a proper weaving. Consider index 2 subgroups of $H_2$, whose fundamental regions consist of two copies of $\triangle_{H_2}$ resulting in additional weaving points. By inspection, only the subgroup $N = \langle R, PRP, QRQ, PQRQP \rangle$ of $H_2$ results in a proper weaving as shown in Figure 4.
Figure 4. (a) Nonproper weaving constructed by subgroup $H_2$ and (b) the weaving constructed by subgroup $N$ of $H_2$

3. The Combinatorial Equivalence of Overlapping Nets

We first introduce the concept of the combinatorial equivalence between two overlapping nets.

Definition 1. Let $\mathcal{O} = \mathcal{N}_1 \cup \mathcal{N}_2$ and $\mathcal{O}' = \mathcal{M}_1 \cup \mathcal{M}_2$ be two overlapping nets, where $\mathcal{N}_i = (\mathcal{U}_i, \mathcal{E}_i)$ and $\mathcal{M}_i = (\mathcal{V}_i, \mathcal{F}_i)$ for $i = 1, 2$. We say that the overlapping nets $\mathcal{O}$ and $\mathcal{O}'$ are combinatorially equivalent if and only if there exists a bijective mapping $\phi : \mathcal{U}_1 \cup \mathcal{U}_2 \rightarrow \mathcal{V}_1 \cup \mathcal{V}_2$ such that any vertices $u, v$ are adjacent in $\mathcal{O}$ if and only if $\phi(u)$ and $\phi(v)$ are adjacent in $\mathcal{O}'$, and any two edges $e_1 = \{u_1, u_2\}$ and $e_2 = \{u_1', u_2'\}$ intersect in $\mathcal{O}$ if and only if $\phi(e_1) = \{\phi(u_1), \phi(u_2)\}$ and $\phi(e_2) = \{\phi(v_1), \phi(v_2)\}$ intersect in $\mathcal{O}'$.

Proposition 1. The relation of combinatorial equivalence is an equivalence relation on the set of all overlapping nets on the plane.

Proof. Let $\mathcal{O} = \mathcal{N}_1 \cup \mathcal{N}_2$ be an overlapping net where $\mathcal{N}_i = (\mathcal{U}_i, \mathcal{E}_i)$. Clearly, the trivial isometry $I : \mathcal{U}_1 \cup \mathcal{U}_2 \rightarrow \mathcal{U}_1 \cup \mathcal{U}_2$ fixing each vertex is a bijection that preserves the adjacency of vertices and intersection of edges. Hence $\mathcal{O}$ is combinatorially equivalent to itself.

Now, suppose $\mathcal{O} = \mathcal{N}_1 \cup \mathcal{N}_2$ is combinatorially equivalent to $\mathcal{O}' = \mathcal{M}_1 \cup \mathcal{M}_2$. Then there exists a bijection $\phi : \mathcal{U}_1 \cup \mathcal{U}_2 \rightarrow \mathcal{V}_1 \cup \mathcal{V}_2$ which preserves adjacency of the vertices and intersection of edges. Using its inverse function $\phi^{-1}$, it follows that $\mathcal{O}'$ is combinatorially equivalent to $\mathcal{O}$.

Lastly, let $\mathcal{O}'' = \mathcal{L}_1 \cup \mathcal{L}_2$ be overlapping nets such that $\mathcal{L}_i = (\mathcal{V}_i, D_i)$ for $i = 1, 2$. Furthermore, suppose that there exist a bijective mapping $\phi : \mathcal{U}_1 \cup \mathcal{U}_2 \rightarrow \mathcal{V}_1 \cup \mathcal{V}_2$ such that $\mathcal{O}$ and $\mathcal{O}'$ are combinatorially equivalent and a bijective mapping $\alpha : \mathcal{V}_1 \cup \mathcal{V}_2 \rightarrow \mathcal{Y}_1 \cup \mathcal{Y}_2$ such that $\mathcal{O}'$ and $\mathcal{O}''$ are combinatorially equivalent. Since the product of two bijective functions is also bijective, then $\alpha \phi : \mathcal{U}_1 \cup \mathcal{U}_2 \rightarrow \mathcal{Y}_1 \cup \mathcal{Y}_2$ is a bijection. Furthermore, since $\mathcal{O}$ and $\mathcal{O}'$ are combinatorially equivalent, two vertices $u, v$ are adjacent in $\mathcal{U}_1 \cup \mathcal{U}_2$ if and only if $\phi(u)$ and $\phi(v)$ are adjacent in $\mathcal{V}_1 \cup \mathcal{V}_2$. Since $\mathcal{O}'$ and $\mathcal{O}''$ are combinatorially equivalent, then $\phi(u)$ is adjacent to $\phi(v)$ if and only if $\alpha \phi(u)$ and $\alpha \phi(v)$ are adjacent in $\mathcal{Y}_1 \cup \mathcal{Y}_2$. Thus, adjacency of vertices in $\mathcal{U}_1 \cup \mathcal{U}_2$ is preserved by $\alpha \phi$. A similar argument can be used to show that two edges $e_1$ and $e_2$ intersect in $\mathcal{O}$ if and only if $\alpha \phi(e_1)$ and $\alpha \phi(e_2)$ intersect in $\mathcal{O}''$. Thus, combinatorial equivalence of overlapping nets is transitive and therefore an equivalence relation.

As an example, consider the overlapping nets constructed from $H_2 = \langle R, Q, PQP, PRP \rangle$ and $H_4 = \langle RQR, RPR, P, Q \rangle$ as shown in Figure 5. If $T$ is a product of a $135^\circ$ about the origin indicated by the red dot and a translation with a scaling, then $T$ will be the bijective mapping such that $\mathcal{O}_{H_2}$ and $\mathcal{O}_{H_4}$ are combinatorially equivalent.
The relation of weaving equivalence is an equivalence relation.

Proposition 2. The relation of weaving equivalence is an equivalence relation.

Proof. Suppose \( \mathcal{W} = (\mathcal{O}, \omega) \) is a weaving where \( \mathcal{O} = \mathcal{N}_1 \cup \mathcal{N}_2 \) such that \( \mathcal{N}_i = (\mathcal{U}_i, \mathcal{E}_i) \) for \( i = 1, 2 \) and let \((e_b, e_w)\) be an element of the set of weaving points \( \mathcal{I} \) of \( \mathcal{O} \). Consider the mapping \( I : \mathcal{U}_1 \cup \mathcal{U}_2 \to \mathcal{U}_1 \cup \mathcal{U}_2 \) where \( I \) is the trivial isometry that sends \( \mathcal{O} \) to itself. Since \( I \) is a bijection, and preserves the adjacency of the vertices and the intersection of edges, then the overlapping net \( \mathcal{O} \) is combinatorially equivalent to itself. Also, since \( I(\mathcal{I}) = \mathcal{I} \), then \( \omega(e_b, e_w) = \omega(E(e_b, e_w)) \) for all \((e_b, e_w) \in \mathcal{I}\).

Suppose a weaving \( \mathcal{W} = (\mathcal{O}, \omega) \) is equivalent to a weaving \( \mathcal{W}' = (\mathcal{O}', \omega') \) where \( \mathcal{O}' = \mathcal{M}_1 \cup \mathcal{M}_2 \) and \( \mathcal{M}_i = (\mathcal{V}_i, \mathcal{F}_i) \) for \( i = 1, 2 \). Then there exists a bijective mapping \( \phi : \mathcal{U}_1 \cup \mathcal{U}_2 \to \mathcal{V}_1 \cup \mathcal{V}_2 \) such that \( \mathcal{O} \) and \( \mathcal{O}' \) are combinatorially equivalent. Since edges intersect in \( \mathcal{O} \) if and only if their images under \( \phi \) intersect in \( \mathcal{O}' \), then it follows that \( \phi(\mathcal{I}) = \mathcal{I}' \) where \( \mathcal{I}' \) is the set of weaving points in \( \mathcal{O}' \). From the second condition of the definition one may immediately deduce that \( \omega'(\phi(e_b, e_w)) = \omega(e_b, e_w) \) or \( \omega'(\phi(e_b, e_w)) = \omega(e_b, e_w) \) for all \((e_b, e_w) \in \mathcal{I}\).

Let \( \mathcal{W}'' = (\mathcal{O}'', \omega'') \) where \( \mathcal{O}'' = \mathcal{L}_1 \cup \mathcal{L}_2 \) and \( \mathcal{L}_i = (\mathcal{Y}_i, \mathcal{D}_i) \). Suppose that weaving \( \mathcal{W}' = (\mathcal{O}', \omega') \) is equivalent to weaving \( \mathcal{W}'' \) and \( \mathcal{W} \) is equivalent to \( \mathcal{W}' \). This implies that \( \mathcal{O} \) and \( \mathcal{O}' \) are combinatorially equivalent under a bijection \( \phi \) and \( \mathcal{O}' \) and \( \mathcal{O}'' \) are combinatorially equivalent under a bijection \( \alpha \). By the transitive property of combinatorial equivalence, \( \mathcal{O} \) and \( \mathcal{O}'' \) are combinatorially equivalent under the bijective mapping \( \alpha \phi \) from \( \mathcal{U}_1 \cup \mathcal{U}_2 \) to \( \mathcal{Y}_1 \cup \mathcal{Y}_2 \).

Since \( \phi(\mathcal{I}) = \mathcal{I}' \) where \( \omega(e_b, e_w) = \omega'(\phi(e_b, e_w)) \) or \( \omega(e_b, e_w) = \omega(\phi(e_b, e_w)) \), and \( \alpha(\mathcal{I}') = \mathcal{I}'' \) where \( \omega'(\phi(e_b, e_w)) = \omega''(\alpha \phi(e_b, e_w)) \) or \( \omega'(\phi(e_b, e_w)) = \omega''(\alpha \phi(e_b, e_w)) \) for all \((e_b, e_w) \in \mathcal{I}'\).

Then \( \alpha \phi(\mathcal{I}) = \mathcal{I}'' \) and \( \omega(e_b, e_w) = \omega''(\alpha \phi(e_b, e_w)) \) or \( \omega(\phi(e_b, e_w)) = \omega''(\alpha \phi(e_b, e_w)) \) for all \((e_b, e_w) \in \mathcal{I}'\).

This implies that \( \mathcal{W} = (\mathcal{O}, \omega) \) is equivalent to \( \mathcal{W}'' = (\mathcal{O}', \omega'') \). This shows that weaving equivalence is an equivalence relation.

The equivalence classes of weavings with respect to this equivalence relation are called weaving patterns.

4. The Weaving Equivalence

We now present a definition of equivalent weavings.

Definition 2. Two weavings \( \mathcal{W} = (\mathcal{O}, \omega) \) and \( \mathcal{W}' = (\mathcal{O}', \omega') \) are equivalent weavings if the following conditions are satisfied:

(i) there exists a bijection \( \phi : \mathcal{O} \to \mathcal{O}' \) under which \( \mathcal{O} \) and \( \mathcal{O}' \) are combinatorially equivalent, and

(ii) \( \omega(e_b, e_w) = \omega'(\phi(e_b, e_w)) \) or \( \omega(e_b, e_w) = \omega(\phi(e_b, e_w)) \) for all \((e_b, e_w) \in \mathcal{I} \) where \( \mathcal{I} = \emptyset \) and \( \mathcal{I} = \emptyset \).

\[ \text{(a) Overlapping net } \mathcal{O}_{H_2} \][\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{image1.png}
\caption{Overlapping nets by \( H_2 = \langle R, Q, PQP, PRP \rangle \) and \( H_4 = \langle RQR, RPR, P, Q \rangle \) of \( G(4, 4, 2) \)}
\end{figure} \]

\[ \text{(b) Overlapping net } \mathcal{O}_{H_4} \]
5. Examples
In Section IV, we noted that $\mathcal{O}_{H_2}$ and $\mathcal{O}_{H_4}$ are combinatorially equivalent. To show that $\mathcal{W}_{H_4} = \{\mathcal{O}_{H_4}, \omega\}$ and $\mathcal{W}_{N_4} = \{\mathcal{O}_{H_2}, \omega'\}$ satisfy condition 2 of equivalent weavings, consider Figure 6.

![Figure 6. Weavings constructed from subgroup $H_4 = \langle RQR, RPR, P, Q \rangle$ of $G(4,4,2)$ and an index 2 subgroup $N_4 = \langle R, PRP, QRQ, PQRQP \rangle$ of $H_2$, respectively.](image)

For this example, first consider the effect of the transformation $T$ to the motif $\triangle_{N_4}$ of $\mathcal{W}_{N_4}$ as shown in Figure 7.

![Figure 7. (a) Motif $\triangle_{N_4}$, (b) motif $\triangle_{N_4}$ rotated 135° about the origin and (c) motif $\triangle_{N_4}$ translated and scaled](image)

Notice that both motifs $\triangle_{H_4}$ and $\triangle_{N_4}$ of overlapping nets $\mathcal{O}_{H_4}$ and $\mathcal{O}_{N_4}$ have a $b$-vertex and a $w$-vertex. Figure 8 shows that the motifs have the same structure and the weaving points of one can be made to correspond with the weaving points of the other.

![Figure 8. Equivalent weavings by $H_4 = \langle RQR, RPR, P, Q \rangle$ and its index 2 subgroup $N_4 = \langle R, PRP, QRQ, PQRQP \rangle$, respectively.](image)

Since $\mathcal{O}_{H_4}$ and $\mathcal{O}_{N_4}$ are combinatorially equivalent and both weavings $\mathcal{W}_{H_4}$ and $\mathcal{W}_{N_4}$ are constructed by 4 reflection lines along the sides of the motif, then subgroup $H_4$ and $N_4$ result in
weavings such that \( \omega(e_b, e_w)_{H_4} = \omega'(\phi(e_b, e_w)) \forall (e_b, e_w)_{H_4} \in \mathcal{F}_{H_4} \). This implies that \( \mathcal{W}_{H_4} \) and \( \mathcal{W}_{N_4} \) are equivalent weavings.

The previous example illustrates that if one can find an isometry of the plane that sends a weaving to another weaving, then the two weavings are equivalent.

In our study, we are interested in listing down weaving patterns or non-equivalent weavings that we can derive from the subgroups of a triangle group. In fact, this is the motivation why we came up with the concept of equivalent weavings. In the next example, we give some weaving patterns derived from a subgroup of the triangle tiling \( G(6, 4, 2) \), which corresponds to a triangle tiling of the hyperbolic plane, since \( \frac{\pi}{6} + \frac{\pi}{4} + \frac{\pi}{2} < \pi \).

Now, consider the overlapping net constructed from the colored tiling \( T(6, 4, 2) \) by subgroup \( H_2 = \langle R, Q, PQP, PRP \rangle \) of \( G(6, 4, 2) \) as shown in Figure 9.

![Figure 9](image)

**Figure 9.** (a) The nonproper weaving constructed by \( H_2 \) from the colored tiling \( T(6, 4, 2) \) and (b) the only proper weaving constructed from an index 2 subgroup of \( H_2 \)

Since the motif \( \Delta_{mH_2} \) has only one weaving point, then clearly subgroup \( H_2 \) does not yield a proper weaving. This is what we call a No Weaving Scenario. When this scenario occurs, consider an index 2 subgroup \( N_i \) of \( H \), identify the motif \( \Delta_{N_i} \) of \( N_i \) in \( T(6, 4, 2) \) and assign values to the weaving points in \( \Delta_{N_i} \). This method of overcoming a No Weaving Scenario is discussed in detail in [2].

For the seven index 2 subgroups \( N_i \) of \( H_2 = \langle R, Q, PQP, PRP \rangle \), only subgroup \( N_2 = \langle Q, PQP, RQR, PRQRP \rangle \) results in a proper weaving as shown in Figure 9(b).

Consider the index 4 subgroups of \( H_2 = \langle R, Q, PQP, PRP \rangle \). For weavings constructed from the same overlapping net, we need only satisfy condition 2 of the definition on weaving equivalence. By inspection we have determined two nonequivalent weaving patterns from the index 4 subgroups \( K_i \) of \( H_2 \): \( K_1 = \langle Q, R, PQRQP, PQPRQRQP \rangle \) and \( K_2 = \langle Q, RQR, PRQRP \rangle \) as shown in Figure 10.
Figure 10. Weavings patterns from subgroups of $H_2 = \langle R, Q, PQP, PRP \rangle$

6. Recommendation and Conclusion
In this paper, we developed a concept of weaving equivalence. We used this concept to get a collection of weavings constructed using our methodology which are not combinatorially identical.

As a continuation of the study, we are looking at extending the definition of $B$- and $W$-vertices to include more points other than the centroids of the $B$- and $W$-patches. We also consider constructing weavings from $n$-colorings of triangle tilings for $n > 2$. Ultimately, our goal is to use these weavings to construct three-periodic patterns using the methodology given in [4].

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