YANG-MILLS, COMPLEX STRUCTURES
AND CHERN’S LAST THEOREM

To The Memory of Shiing-Shen Chern

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Recently Shiing-Shen Chern suggested that the six dimensional sphere $S^6$ has no complex structure. Here we explore the relations between his arguments and Yang-Mills theories. In particular, we propose that Chern’s approach is widely applicable to investigate connections between the geometry of manifolds and the structure of gauge theories. We also discuss several examples of manifolds, both with and without a complex structure.
1 Introduction

The existence of a complex structure has a central role in many physical problems. It is of particular importance to supersymmetric quantum field and string theories, where geometry plays an essential role. For example Ricci flat Kähler manifolds are fundamental constituents in superstring compactification \cite{1}, and mirror symmetry and duality relations involve structures that relate complex geometry to symplectic geometry \cite{2}.

Sometimes it can be quite difficult to determine whether a manifold admits a complex structure. A notorious example in this respect is the six dimensional sphere $S^6$. The existence of a proof that it admits no complex structure has been both widely advertised \cite{1}, \cite{3} and openly challenged \cite{4}. Very recently Chern attempted to settle this controversy. At the age of 93 he outlined a proof that $S^6$ has no complex structure \cite{5}. Sadly, he passed away before presenting a complete proof of his Last Theorem. Chern’s approach employs differential geometry in a form reminiscent of Yang-Mills theories. Thus we expect that generalizations of his approach will eventually lead to novel and deep connections between the geometry of manifolds and the structure of gauge theories.

The scope of the present article is to broaden Chern’s approach. We explain how it can be applied to study the properties of gauge theories and the existence of complex structures in a general class of manifolds. As concrete examples we consider several familiar gauge theory models with physical relevance. These describe the two dimensional sphere $S^2$, where the existence of a complex structure is related to the structure of Dirac monopole. The four dimensional sphere $S^4$, where the lack of a complex structure can be directly inferred from the properties of a Yang-Mills instanton. And the two (complex) dimensional projective space $\mathbb{CP}^2$ whose complex structure reflects the properties of the $SU(2) \times U(1)$ Lie subgroup in a $SU(3)$ Yang-Mills theory.

For a vector space that has $2n$ real dimensions, a complex structure is a linear endomorphism $\mathbb{J}$ that squares to $\mathbb{J}^2 = -1$. If the vector space is a tangent space of a $2n$-dimensional manifold $\mathcal{M}$, the set of endomorphisms $\mathbb{J}_x$ on the tangent bundle $T\mathcal{M}$ equips the manifold with an almost complex structure. The almost complex structure $\mathbb{J}_x$ is integrable and $\mathcal{M}$ is a complex manifold, iff \cite{4}

$$d = \partial + \bar{\partial}$$

The integrability can also be stated as

$$\partial^2 = \bar{\partial}^2 = 0$$

In analogy, on the cotangent bundle $T^*\mathcal{M}$ of the manifold $\mathcal{M}$ we can employ $\mathbb{J}_x$ to introduce globally defined $(1,0)$ one-forms $\omega_k \ (k = 1, \ldots, n)$. If these
one-forms on $T^*\mathcal{M}$ can be represented as holomorphic linear combinations of one-forms $dw^k = du^k + idv^k$ where $(u^k, v^k)$ are local coordinates on $\mathcal{M}$, the almost complex structure $\mathbb{J}_x$ is integrable [4].

In a complementary description (see e.g. [5], [6]) the integrability of an almost complex structure is formulated directly in terms of the non-linearly independent type $(1, 0)$ one-forms $\omega_k$ such that

i) The almost complex structure on a manifold $\mathcal{M}$ is a locally decomposable $(n, 0)$-form $\Omega$,
$$
\Omega = \omega_1 \wedge \ldots \wedge \omega_n
$$

ii) $\Omega$ is non-degenerate,
$$
\Omega \wedge \bar{\Omega} \neq 0
$$

iii) The $(1, 0)$-forms $\omega_k$ satisfy the integrability condition
$$
\Omega \wedge d\omega_k = 0
$$

which can also be written in the alternative form
$$
d\omega_k = 0 \mod (\omega_l)
$$

From [4] it is obvious that any two-dimensional almost complex structure is integrable. A familiar example is the two dimensional sphere $S^2 \cong \mathbb{C}P^1$ which inherits its (unique) complex structure from the complex plane $\mathbb{C}$ by stereographic projection. But in higher dimensions a given almost complex structure on a manifold $\mathcal{M}$ is not necessarily integrable, and an almost complex manifold does not need to be a complex manifold. Examples with non-integrable almost complex structures can be constructed whenever the real dimensionality $2n$ of a manifold is at least four [4]. Furthermore, unlike $S^2$ the four dimensional sphere $S^4$ does not even admit an almost complex structure, a nontrivial fact that follows from the properties of its Chern class and index theorems. Indeed, this lack of even an almost complex structure is a property shared by all higher dimensional spheres $S^{2n}$ whenever $n > 3$. But the case $n = 3$ is exceptional: The six dimensional sphere $S^6$ is an almost complex manifold and it remains an open problem whether or not it is actually a complex manifold.

2 Chern’s Last Theorem

There is a good reason why $S^2$ and $S^6$ are the only spheres that can have an almost complex structure. This is due to the fact, that $\mathbb{R}^3$ and $\mathbb{R}^7$ are the only
vector spaces where one can define an antisymmetric bilinear cross product of vectors (see, e.g. [3]). This existence of a vector cross product on $\mathbb{R}^3$ and $\mathbb{R}^7$ reflects the fact that besides real and complex numbers, quaternions and octonions are the only normed division algebras.

In the present section we consider the seven dimensional case $\mathbb{R}^7$, where the vector cross product derives from the multiplicative properties of imaginary octonions [5]. Indeed, as a linear space imaginary octonions coincide with $\mathbb{R}^7$. Thus the unit sphere $S^6 \subset \mathbb{R}^7$ is isomorphic with the space of unit imaginary octonions, and it acquires a natural almost complex structure from the action of the octonionic vector cross product in $\mathbb{R}^7$. Explicitely this octonionic almost complex structure on $S^6$ is constructed as follows: Let $\hat{e}_1, \ldots, \hat{e}_7$ be the basis of imaginary unit octonions i.e. each of them is a square root of $-1$. A generic point $y \in \mathbb{R}^7$ can be identified with

$$ y = y_1\hat{e}_1 + \ldots + y_7\hat{e}_7 $$

A point $x$ on the unit sphere $S^6 \subset \mathbb{R}^7$ becomes then identified with

$$ x = x_1\hat{e}_1 + \ldots + x_7\hat{e}_7 ; \quad \sum_i x_i^2 = 1 $$

The octonionic multiplication of vectors in $\mathbb{R}^7$, which we denote by $\times$, is defined in terms of the octonionic multiplicative rule of the basis elements $\hat{e}_k$. This is a totally antisymmetric bilinear automorphism of the form

$$ \hat{e}_i \times \hat{e}_j = \sum_{k=1}^7 c^{ijk}\hat{e}_k \quad (5) $$

We can choose the octonionic basis so that the only nonvanishing components of the totally antisymmetric octonionic tensor $c^{ijk}$ are [5]

$$ c^{123} = c^{147} = c^{165} = c^{246} = c^{257} = c^{354} = c^{367} = 1 \quad (6) $$

This reflects the index cycling and the index doubling symmetries of the octonionic product,

$$ \hat{e}_i \times \hat{e}_j = \hat{e}_k \quad \Rightarrow \quad \hat{e}_{i+1} \times \hat{e}_{j+1} = \hat{e}_{k+1} \quad (mod \ 7) \quad (7) $$

$$ \hat{e}_i \times \hat{e}_j = \hat{e}_k \quad \Rightarrow \quad \hat{e}_{2i} \times \hat{e}_{2j} = \hat{e}_{2k} \quad (8) $$

Consider a $y \in \mathbb{R}^7$ which is orthogonal to a given $x \in S^6 \subset \mathbb{R}^7$ in the sense of the standard $\mathbb{R}^7$ vector inner product; in the octonionic basis we realize this inner product with

$$ < \hat{e}_i, \hat{e}_j > = \delta_{ij} $$

3
The octonionic (cross) product \( x \times y \) determines a mapping

\[
y = y_i \hat{e}_i \rightarrow \mathcal{J}_x(y) \equiv x \times y = c^{ijk} x_i y_j \hat{e}_k
\]

and due to the total antisymmetry of the \( c^{ijk} \), we have

\[
\mathcal{J}_x(x) = 0
\]

The mapping (9) is clearly a linear automorphism \( \mathcal{J}_x : y \rightarrow x \times y \) of the tangent bundle \( TS^6 \). Explicitly, the action of \( \mathcal{J}_x \) on the tangent bundle coincides with that of the matrix

\[
\mathcal{J}_x = \begin{pmatrix}
0 & x_3 & -x_2 & -x_5 & x_4 & -x_7 & x_6 \\
-x_3 & 0 & x_1 & -x_6 & x_7 & x_4 & -x_5 \\
x_2 & -x_1 & 0 & -x_7 & -x_6 & x_5 & x_4 \\
x_5 & x_6 & x_7 & 0 & -x_1 & -x_2 & -x_3 \\
-x_4 & -x_7 & x_6 & x_1 & 0 & -x_3 & x_2 \\
x_7 & -x_4 & -x_5 & x_2 & x_3 & 0 & -x_1 \\
-x_6 & x_5 & -x_4 & x_3 & -x_2 & x_1 & 0
\end{pmatrix}
\]

(10)

Since \(|x| = 1\) we clearly have

\[
(\mathcal{J}_x^2)_{ij} = -\delta_{ij} + x_i x_j
\]

(11)

This implies that for any tangent vector \( y \in TS^6 \)

\[
(\mathcal{J}_x^2)_{ij} y_j = -y_i
\]

which confirms that \( \mathcal{J}_x \) indeed defines an almost complex structure on \( S^6 \).

The present (octonionic) almost complex structure on \( S^6 \) is not integrable. This can be verified by a direct computation of (1), for example with the one-forms

\[
\eta_a = idx_a + \sum_{b \neq a} J_{ab} dx_b
\]

with

\[
J = \mathcal{J}_x|_{x_7 = \pm \sqrt{1 - \sum_{a=1}^{6} x_a^2}}
\]

But it is in principle possible that in addition to a non-integrable complex structure there can exist an integrable one. The deformation theory of (almost) complex structures is described by the Kodaira-Spencer theory which we note, is also relevant to the topological type-B string theory [10].

Recently Chern proposed [5] that none of the almost complex structures on \( S^6 \) can be integrable, and argued that from this it follows that \( S^6 \) does
not admit any complex structure. His approach is based on the fact that the almost complex structure (9) is invariant under the natural action of the 14-dimensional exceptional Lie group $G_2$ on $S^6$. Indeed, the group $G_2$ is the automorphism group of octonions and as a manifold it is a principal $SU(3)$-bundle over $S^6$, i.e. locally

$$G_2 \sim S^6 \times SU(3)$$

In order to outline Chern’s approach, we denote by $g_i$ ($i, j, k = 1, \ldots, 14$) the Lie algebra generators of Lie($G_2$), and by $h_a$ ($a, b, c = 1, \ldots, 8$) we denote the generators of its $su(3) = \text{Lie}(SU(3))$ subalgebra. Let $m_s$ ($s, t, u = 1, \ldots, 6$) be the remaining generators of Lie($G_2$) so that as vector spaces

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

Under this decomposition the commutation relations of the $G_2$ Lie algebra acquire the form [9]

$$[h_a, h_b] = f^{abc} h_c$$

(12)

$$[h_a, m_s] = t^{ast} m_t$$

(13)

$$[m_s, m_t] = t^{sta} h_a + k^{stu} m_u$$

(14)

where all structure constants are totally antisymmetric. Explicitly, we have for the nonvanishing $k_{stu}$ [9]:

$$k^{136} = -k^{145} = k^{235} = k^{246} = -\frac{1}{\sqrt{3}}$$

Since these are nontrivial, we conclude that $su(3)$ is not a symmetric subalgebra of the Lie($G_2$) Lie algebra. It is this nontriviality of $k_{stu}$ that ultimately leads to the lack of integrability in the (invariant) almost complex structures on the sphere $S^6$.

The relevant geometry of the Lie group $G_2$ can be described by the ensuing Maurer-Cartan equation, which we write as a flatness condition for a Lie($G_2$)-valued Yang-Mills connection [5], [11]

$$F = dA + A \wedge A = 0 \iff A = g^{-1} dg$$

(15)

Using the fact that $G_2$ is a $SU(3)$-bundle over $S^6$, we decompose the Lie($G_2$)-valued Yang-Mills connection $A$ into a linear combination

$$A = \kappa + \rho$$
where \( \kappa \) takes values in the \( su(3) \) subalgebra and \( \rho \) is a linear combination of the remaining six generators \( m_\kappa \).

In terms of \( \kappa \) and \( \rho \) the components of the Maurer-Cartan equation acquire the form

\[
F_\kappa = d\kappa^a + f^{abc} \kappa^b \wedge \kappa^c = -t^{ast} \rho^s \wedge \rho^t \tag{16}
\]

\[
d\rho^s = -2t^{sat} \kappa^s \wedge \rho^t - k^{stu} \rho^t \wedge \rho^u \tag{17}
\]

Here \( F_\kappa \) is the \( SU(3) \) curvature two-form, and the one-forms \( \rho^a \) form a basis for the cotangent bundle \( T^*S^6 \). Since \( S^6 \) is an almost complex manifold these Maurer-Cartan equations can be represented in a manifestly complex form. For this we introduce a holomorphic polarization on \( T^*S^6 \) and present the \( \rho^a \) as linear combinations of the ensuing \((1,0)\)-forms \( \theta^\alpha \) and \((0,1)\)-forms \( \bar{\theta}^\alpha \) where now \( \alpha, \beta, \gamma = 1, 2, 3 \). Explicitly, one can choose

\[
\begin{align*}
\theta^1 &= \rho^1 + i\rho^2 \\
\theta^2 &= \rho^4 + i\rho^3 \\
\theta^3 &= \rho^6 - i\rho^5
\end{align*}
\]

for the basis of \( \text{Lie}(G_2) \) algebra described in [9]. The Maurer-Cartan equation (17) now acquires the form

\[
d\theta^\alpha = t^{\alpha\beta\gamma} \kappa^\beta \wedge \theta^\gamma + \frac{2}{\sqrt{3}} \epsilon^{\alpha\beta\gamma} \bar{\theta}^\beta \wedge \bar{\theta}^\gamma \tag{18}
\]

where the almost complex structure of \( S^6 \subset G_2 \) is manifest. But since the last term in (18) involves only the \((0,1)\)-forms \( \bar{\theta}^\alpha \) we have

\[
\Omega \wedge d\theta^\alpha \equiv \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge d\theta^\alpha \neq 0 \quad (\alpha = 1, 2, 3)
\]

which establishes that the integrability condition (3), (4) is not obeyed. As a consequence the present almost complex structure is not integrable.

In [5] Chern suggests that the existence of the almost complex structure determined by the present one-forms \((\bar{\theta}^\alpha, \bar{\theta}^\alpha) \in T^*S^6 \) makes it impossible to endow \( S^6 \) with an integrable almost complex structure. For this, he assumes that there is a complete set of one-forms \((\omega, \bar{\omega})\) that defines an integrable almost complex structure on \( S^6 \). By completeness one can represent these one-forms as linear combinations of the present one-forms \( \theta, \bar{\theta} \),

\[
\omega^\alpha = p^\alpha_\beta \theta^\beta + q^\alpha_\beta \bar{\theta}^\beta \tag{19}
\]

Using linear algebra Chern argues [5] that any possible choice of \( p^\alpha_\beta \) and \( q^\alpha_\beta \) in (19) leads to a conflict between the integrability condition (11) and the Maurer-Cartan equations (18) for \( \theta^\alpha, \bar{\theta}^\alpha \). A completion of his arguments should prove that the six dimensional sphere \( S^6 \) has no complex structure.
3 On General Structure

Chern’s approach [5] to the almost complex structure on $S^6$ employs natural structures in Yang-Mills theory. Consequently we expect that his approach has a much wider applicability. To inspect this, we consider a Yang-Mills theory with a gauge group $\mathcal{H}$ on a manifold $\mathcal{M}$ so that together these two combine into a total space which is another Lie group $\mathcal{G}$. A physical perspective to this structure could be, that we have a $\mathcal{G}$-invariant gauge theory where the total group contains the gauge part $\mathcal{H}$ and the part corresponding to the vacuum or space-time manifold $\mathcal{M}$. In a sense, the manifolds $\mathcal{M}$ and $\mathcal{H}$ are then dual to each other within the gauge group $\mathcal{G}$. The duality relation is a mapping between these two manifolds and it is determined by the flatness condition

$$ F = dA + A \wedge A = 0 $$ (20)

for the Lie($\mathcal{G}$)-valued Yang-Mills connection $A$

If we write $A = g^{-1}dg$ where $g \in \mathcal{G}$, the flatness condition (20) leads to the standard form of the Maurer-Cartan equation for $\mathcal{G}$. But instead we now decompose $A$ as follows,

$$ A = g^{-1}dg = \kappa + \vartheta $$ (21)

Here $\kappa$ is the projection of $A$ to the Lie algebra of the subgroup $\mathcal{H}$ and $\vartheta$ denotes the remaining ”space-time” components of $A$ in Lie($\mathcal{G}$). We decompose the Lie algebra of $\mathcal{G}$ into the ensuing vector space sum

$$ \mathfrak{g} = \mathfrak{h} + \mathfrak{m} $$

so that $\kappa$ is a linear combination of generators in $\mathfrak{h}$, while $\vartheta$ is a linear combination of generators in $\mathfrak{m}$. Under this decomposition the Lie algebra of $\mathcal{G}$ in general acquires the form

$$ [\mathfrak{h}_a, \mathfrak{h}_b] = f^{abc}\mathfrak{h}_c $$ (22)

$$ [\mathfrak{h}_a, \mathfrak{m}_s] = C^{ast}\mathfrak{m}_t $$ (23)

$$ [\mathfrak{m}_s, \mathfrak{m}_t] = C^{sta}\mathfrak{h}_a + C^{stu}\mathfrak{m}_u $$ (24)

With these, the flatness condition (20) (Maurer-Cartan equation) becomes

$$ F_\kappa^a \equiv d\kappa^a + f^{abc}\kappa^b \wedge \kappa^c = -C^{ast}\vartheta^s \wedge \vartheta^t $$ (25)
which one can now interpret as a certain ”duality relation” between the curvature tensor $F_\kappa$ of the $\mathcal{H}$ invariant Yang-Mills theory with base manifold $\mathcal{M}$ and a “monopole equation” for $\vartheta$ on $\mathcal{M}$, coupled to those components of $\kappa^a$ for which $C^{sat} \neq 0$. We note that a comparison with the functional form of Cartan’s first structure equation [12] (see also equation (52) below) suggests that the $\vartheta \wedge \bar{\vartheta}$ term on the r.h.s. of (26) is a contribution from a torsion connection, and we see that it is directly related to the non symmetric structure of the Lie algebra decomposition (24).

One can verify that these equations are gauge covariant w.r.t. gauge transformations in the $\mathcal{H}$ subgroup. Indeed, if $h \in \mathcal{H}$ the ensuing gauge transformation acts as follows,

$$\kappa \rightarrow h^{-1}\kappa h + h^{-1}dh$$

$$F_\kappa \rightarrow h^{-1}F_\kappa h$$

$$\vartheta \rightarrow h^{-1}\vartheta h$$

and using this in (25), (26) we find that the equations are covariant under the $\mathcal{H}$ gauge transformations.

The equations (25), (26) can be employed to investigate various properties of the $\mathcal{H}$ gauge theory and the geometry of the manifold $\mathcal{M}$. Of particular interest to us here is whether $\mathcal{M}$ admits an almost complex structure and whether this structure can be integrable. For this we recall that the $\vartheta$ span the cotangent bundle $T^*\mathcal{M}$. If $\mathcal{M}$ is an almost complex manifold, we can introduce a holomorphic polarization on $T^*\mathcal{M}$ to represent the $\vartheta$ as linear combinations of the one-forms which are either of type $(1,0)$ or type $(0,1)$. For an almost complex $\mathcal{M}$, the Maurer-Cartan equations (25), (26) should then acquire a complex decomposition. On the other hand, if $\mathcal{M}$ has no almost complex structure the Maurer-Cartan equations do not admit any complex decomposition, in any holomorphic polarization on $T^*\mathcal{M}$.

Suppose now the manifold $\mathcal{M}$ admits an almost complex structure, and that $\theta^\alpha$, $\bar{\theta}^\alpha$ are one-forms of the type $(1,0)$ and $(0,1)$ respectively in the given holomorphic polarization on $T^*\mathcal{M}$. In such a basis the Maurer-Cartan equation (26) acquires the manifestly complex form

$$(\delta^\alpha_\beta d + C^{\alpha\beta\gamma}_a \kappa^a \wedge) \theta^\beta = -C^{\alpha\beta\gamma}_+ \theta^\beta \wedge \theta^\gamma - C^{\alpha\beta\gamma}_- \bar{\theta}^\beta \wedge \bar{\theta}^\gamma - C^{\alpha\beta\gamma}_{+-} \bar{\theta}^\beta \wedge \bar{\theta}^\gamma$$

see also (20), where the $C$-coefficients on the r.h.s. are in general complex-valued functions that display the following antisymmetries w.r.t. last two indices

$$C^{\alpha\beta\gamma}_{++} + C^{\alpha\beta\gamma}_{++} = C^{\alpha\beta\gamma}_{-+} + C^{\alpha\beta\gamma}_{-+} = 0$$
From (4) we conclude that the integrability of the almost complex structure becomes translated into the condition

\[ C^{\alpha\beta\gamma} = 0 \]

Clearly this condition is invariant under linear transformations of the \( \theta^\alpha \). As a consequence the lack of integrability in the almost complex structure becomes related to the nontriviality of the structure constant \( C^{stu} \) in (24), which we already noted in connection of \( \mathbb{S}^6 \); see (14). This suggests that if the embedding of \( \mathcal{H} \) in \( \mathcal{G} \) is not symmetric the almost complex structure of \( \mathcal{M} \) can not be integrable.

We conclude this section with the following observations:

- If \( C^{ast} \) vanishes the \( \mathcal{H} \)-curvature \( F_\kappa \) is flat, and \( \kappa \) is the Maurer-Cartan form of \( \mathcal{H} \).
- If \( C^{ast} \) is nontrivial, the connection \( \kappa \) is nontrivial. The duality relation (25) then allows us to relate the Chern classes of the principal \( \mathcal{H} \) bundle over \( \mathcal{M} \) to the solutions of the “monopole equation” defined directly on \( \mathcal{M} \), since according to (25)

\[
\text{Det} \left( 1 + \frac{1}{4\pi} F^a \theta_a \right) = \text{Det} \left( 1 - \frac{1}{4\pi} \theta_a C^{ast} \vartheta^s \wedge \vartheta^t \right) \tag{27}
\]

- If \( C^{ast} \) and \( C^{stu} \) both vanish \( \vartheta \) is closed and \( F_\kappa \) is flat.
- If \( C^{ast} \) vanishes but \( C^{stu} \) does not, \( F_\kappa \) is flat and (26) becomes

\[
F_\vartheta = d\vartheta^s + C^{stu} \vartheta^t \wedge \vartheta^u = 0
\]

Hence \( F_\vartheta \) is in general a curvature with a torsion connection, since in general the \( m_s \) do not define a Lie algebra.

- Finally, for the symmetric case, when \( C^{stu} \) vanishes but \( C^{ast} \neq 0 \), the curvature \( F_\kappa \) is nontrivial and the \( \vartheta \) are covariantly constant with respect to \( \mathcal{H} \) and the Maurer-Cartan equations have the form

\[
F_\kappa^a = -C^{ast} \vartheta^s \wedge \vartheta^t \tag{28}
\]

\[
D_\kappa \vartheta = 0 \tag{29}
\]

where we remind that \( D_\kappa \) is a covariant derivative with respect to those components \( \kappa^a \) for which \( C^{ast} \neq 0 \). The functional form of these equations resembles the Seiberg-Witten equations [13], [14]. In particular, if as in [15] we expand \( \vartheta \) in terms of a hyperplane this resemblance becomes manifest. Indeed, the duality considered here generalize the dualities considered in [15].
4 Two Dimensional Sphere

It is a well known fact, that in three dimensions the familiar cyclic cross product of the orthonormal basis vectors \((\hat{i}, \hat{j}, \hat{k}) \in \mathbb{R}^3\) coincides with the multiplicative structure of imaginary quaternions. Furthermore, as the sphere \(S^2 \subset \mathbb{R}^3\) is in a one-to-one correspondence with three dimensional unit vectors, its (almost) complex structure is a manifestation of the multiplicative properties of unit imaginary quaternions. We now consider this case as an illustrative but simple example of the previous constructions.

We introduce a \(U(1) \simeq S^1\) gauge theory on \(S^2\). Since locally \(S^2 \times S^1 \simeq S^3\) and \(S^3 \simeq SU(2)\), we can identify

\[
\mathcal{M} \simeq S^2
\]

\[
\mathcal{H} \simeq U(1)
\]

\[
\mathcal{G} \simeq SU(2)
\]

so that our starting point is the \(SU(2)\) Maurer-Cartan equation on \(S^2\).

As in Section 2, we first relate the (almost) complex structure on \(S^2 \subset \mathbb{R}^3\) to the vector cross product on \(\mathbb{R}^3\). For this we consider a point \(x \in S^2\), and for any non-vanishing \(y \in T_x(S^2)\) so that

\[
< x, y > = 0
\]

we define

\[
\mathbb{J}_x(y) = x \times y
\]

(Notice that \(\mathbb{J}_x(x) = 0\).) Then

\[
\mathbb{J}_x^2(y) = x \times (x \times y) = x(x, y) - (x, x)y = -y
\]

Thus \(\mathbb{J}_x^2 = -1\) and it defines an almost complex structure.

The matrix realization of \(\mathbb{J}_x\) is

\[
\mathbb{J}_x = \begin{pmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{pmatrix}
\]

(32)

Since \(|x| = 1\) this clearly satisfies

\[
(\mathbb{J}_x^2)_{ij} = -\delta_{ij} + x_i x_j
\]

in full parallel with the construction in Section 2.
Consider now the Maurer-Cartan form for $G = SU(2)$,

$$A = g^{-1}dg = \begin{pmatrix} \kappa & \vartheta \\ -\bar{\vartheta} & -\kappa \end{pmatrix}$$  (34)

where we have used the holomorphic polarization to represent $\vartheta$ in a complex basis. The Maurer-Cartan equations (25), (26) become

$$F = d\kappa = \vartheta \wedge \bar{\vartheta}$$  (35)

$$(d+2\kappa) \wedge \vartheta = 0$$  (36)

where the (almost) complex structure is manifest: Comparing with (1) it is obvious that we can identify $\Omega = \vartheta$ and (36) clearly implies (1). The condition (2) is also satisfied since $F$ in (35) is nonvanishing, as its integral over $M \simeq S^2$ is the first Chern class of the U(1) bundle which is nontrivial as it relates to the area two-form on the base $S^2$

$$\text{Ch}_1(F) = \frac{1}{2\pi} \int F = \frac{1}{2\pi} \int \vartheta \wedge \bar{\vartheta}$$  (37)

The nontriviality of this first Chern class can also be seen explicitly, by considering the following two parametrizations of the group manifold $G \simeq SU(2)$. We describe this manifold either by using the coordinates

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$$  (38)

where $|\alpha|^2 + |\beta|^2 = 1$, or by employing the angular parametrization

$$\alpha = \cos \frac{\theta}{2} e^{\frac{i}{4}(\psi+\phi)} \equiv \cos \frac{\theta}{2} e^{i\psi_+}$$  (39)

$$\beta = \sin \frac{\theta}{2} e^{\frac{i}{4}(\psi-\phi)} \equiv \sin \frac{\theta}{2} e^{i\psi_-}$$  (40)

Explicitly, we find for the Maurer-Cartan form

$$g^{-1}dg = \begin{pmatrix} \kappa & \vartheta \\ -\bar{\vartheta} & -\kappa \end{pmatrix} = \begin{pmatrix} \bar{\alpha}d\alpha + \beta d\bar{\beta} - \beta d\bar{\alpha} & \bar{\beta}d\alpha - ad\bar{\beta} + ad\bar{\alpha} \\ \bar{\beta}d\alpha - ad\bar{\beta} + ad\bar{\alpha} & \bar{\alpha}d\alpha + \beta d\bar{\beta} - \beta d\bar{\alpha} \end{pmatrix}$$  (41)

$$= \begin{pmatrix} \frac{i}{2} (\cos \theta d\psi + d\phi) & \frac{1}{2} e^{-i\phi}(d\theta + i \sin \theta d\psi) \\ \frac{1}{2} e^{i\phi}(d\theta - i \sin \theta d\psi) & -\frac{i}{2} (\cos \theta d\psi + d\phi) \end{pmatrix}$$  (43)
It is a familiar result that \((39), (40)\) describe two hemispheres of \(S^2\) which are parameterized by angles \((\theta, \psi_\pm)\) and \(\psi_+ - \psi_- = \phi\) that is we have the Hopf bundle, or Dirac monopole, with unit charge. Indeed, in the present case the equations \((25), (26), (35), (36)\) have a simple physical interpretation as the familiar relation between the vortex condensate \(\vartheta\) in the background of the abelian gauge field \(\kappa\), and the ensuing magnetic flux; see also [15].

We note that the Maurer-Cartan forms are complex-valued one forms on \(T^*S^2\) with coordinates \((\theta, \psi)\),

\[
\vartheta = \frac{1}{2} e^{-i\phi}(d\theta + i \sin \theta d\psi)
\]

\[
\bar{\vartheta} = \frac{1}{2} e^{i\phi}(d\theta - i \sin \theta d\psi)
\]

and

\[
\vartheta \wedge \bar{\vartheta} = -\frac{i}{2} \sin \theta d\theta \wedge d\psi
\]

is the volume two-form on \(S^2\), and the integral \((37)\) is clearly nontrivial. In terms of the cartesian coordinates on \(\mathbb{R}^3\) we then have

\[
e^{i\phi} \vartheta = idx_1 + \sum_{a=1,2} J_{1a} dx_a
\]

where

\[
J = \mathbb{J}_{x^1, x^2, x^3, x^4, x^5} \sqrt{1 - x_1^2 - x_2^2}
\]

5 \(S^4\) And Instantons

The four dimensional sphere \(S^4\) is a well known example of a manifold with no almost complex structure. But there does not seem to be an easy, direct way to see this. Instead one exploits the properties of characteristic classes: The tangent bundle \(TS^4\) is nontrivial, with a nontrivial Euler class. Consequently for an almost complex structure the second Chern class \(\text{Ch}_2\) of the complex tangent bundle \(TS^4\) should be nontrivial. This Chern class is proportional to the first Pontryagin class of the underlying real tangent bundle. But this vanishes since \(S^4\) is a hypersurface in \(\mathbb{R}^5\) with a trivial normal bundle. As a consequence \(TS^4\) cannot be a complex bundle, i.e. \(S^4\) cannot have any almost complex structure.

We now describe how this conclusion emerges from the present formalism. Using the fact that

\[
SO(D + 1)/SO(D) \simeq S^D
\]

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we select
\[ M \simeq S^4 \]
\[ H \simeq SO(4) \]
\[ G \simeq SO(5) \]

With \( \gamma_i \ (i = 1, \ldots, 5) \) the five dimensional Euclidean Dirac matrices
\[ \{ \gamma_i, \gamma_j \} = 2\delta_{ij} \]

the Lie algebra of \( SO(5) \) is represented by the antisymmetric matrices
\[ M_{ij} = \frac{1}{4i} [\gamma_i, \gamma_j] \quad (i > j) \]

Explicitly,
\[ [M_{ij}, M_{kl}] = f^{ij(kl)(mn)} M_{mn} = -i(\delta_{jk} M_{il} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik} + \delta_{il} M_{jk}) \] (44)

The \( \mathfrak{h} \simeq so(4) \) subalgebra is generated by \( M_{ij} \) for \( i, j = 1, 2, 3, 4 \). The \( so(5) \) generators \( M_{ij} \) with either \( i = 5 \) (or \( j = 5 \)) determine the embedding \( m \simeq T S^4 \subset so(5) \). This is a symmetric embedding since the \( C^{stu} \) in (24) vanish. This suggests that if there exists a holomorphic polarization on \( T^* S^4 \) which allows us to write the present Maurer-Cartan equation (26) in a manifestly complex form, the ensuing almost complex structure on \( S^4 \) could be integrable. But as we see below, the absence of any almost complex structure on \( S^4 \) manifests itself as an obstruction for representing the equation (26) in a holomorphic polarization.

We now proceed to inspect the present \( SO(5) \sim S^4 \times SO(4) \) version of the Maurer-Cartan equations. Explicitly, the Maurer-Cartan equations
\[ d\rho^m = f^{(ij)(kl)(5m)} A^{ij} \wedge A^{kl} \]

have the form
\[ d\rho^1 = -i\rho^2 \wedge \kappa^{21} - i\rho^3 \wedge \kappa^{31} - i\rho^4 \wedge \kappa^{41} \] (45)
\[ d\rho^2 = -i\rho^1 \wedge \kappa^{21} + i\rho^3 \wedge \kappa^{32} + i\rho^4 \wedge \kappa^{42} \] (46)
\[ d\rho^3 = -i\rho^4 \wedge \kappa^{43} + i\rho^1 \wedge \kappa^{31} + i\rho^2 \wedge \kappa^{32} \] (47)
\[ d\rho^4 = -i\rho^3 \wedge \kappa^{43} - i\rho^2 \wedge \kappa^{42} - i\rho^1 \wedge \kappa^{41} \] (48)
It is a relatively straightforward exercise in linear algebra to show that it is impossible to represent the $d\rho^m$ in any polarization which allows these equations to be written in a manifestly complex form. For example, if we identify

$$\theta^{12} = \rho^1 + i\rho^2$$

$$\theta^{34} = \rho^3 + i\rho^4$$

we find

$$d\theta^{12} - \theta^{12} \wedge \kappa^{21} = -\frac{1}{2}(\theta^{34} + \bar{\theta}^{34}) \wedge (\kappa^{32} - i\kappa^{31}) + \frac{i}{2}(\theta^{34} - \bar{\theta}^{34}) \wedge (\kappa^{42} + i\kappa^{41})$$

(49)

where the r.h.s. is clearly inconsistent with a holomorphic polarization. The same conclusion persists if following [5] we substitute in (45)-(48) an arbitrary linear combination of the form

$$\psi^a = p^a_b \theta^b + q^a_b \bar{\theta}^b$$

where $p^a_b$ and $q^a_b$ are a priori unknown functions. Consequently the fact that $S^4$ does not admit any almost complex structure manifests itself in the fact that the Maurer-Cartan equations do not allow for a complex decomposition.

From (25) we can compute the second Chern number of the $SO(4)$ principal bundle over $S^4$. Since

$$f^{(ij)(k5)(m5)} = i$$

we have

$$F^{(ij)}_\kappa = -2i \sum_{i<j} \rho^i \wedge \rho^j$$

and thus the second Chern number for the $SO(4)$ connection $\kappa$ is computed by the volume four-form on $S^4$,

$$\text{Ch}_2(\kappa) = -\frac{1}{8\pi^2} \int \text{Tr} F^2_\kappa = -\frac{1}{24\pi^2} \int \rho^1 \wedge \rho^2 \wedge \rho^3 \wedge \rho^4$$

(50)

We observe that the structure of the equations (45)-(48) coincides with the structure of the equations that describe the familiar Yang-Mills instanton bundle with base $S^4$, fibre $SU(2) \simeq S^3$ and the total space $S^7$. Indeed, this is expected since the construction of the instanton bundle employs the natural $O(4) \simeq SU(2) \times SU(2)$ connection $(\omega_{0i}, \omega_{ij})$ with $i, j = 1, 2, 3$, on the sphere $S^4$. To show this, we recall the de Sitter metric on $S^4$ with unit radius (see e.g. [12]),

$$ds^2 = \frac{dr^2 + r^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{(1 + r^2)^2} = \sum_{a=0}^{3} (e^a)^2$$
where \( \varpi \) for \( i = 1, 2, 3 \) and \( e^a \) for \( a = 0, 1, 2, 3 \) are defined by

\[
(1 + r^2) \begin{pmatrix} e^0 \\ e^1 \\ e^2 \\ e^3 \end{pmatrix} = \begin{pmatrix} dr \\ r \varpi_1 \\ r \varpi_2 \\ r \varpi_3 \end{pmatrix} = \frac{1}{r} \begin{pmatrix} x & y & z & t \\ -t & -z & y & x \\ z & -t & -x & y \\ -y & x & -t & z \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}
\]

(51)

The \( e^a \) define the \( S^4 \) vierbein, and if we introduce the natural connection

\[
\omega^1_0 = \omega^2_3 = \varpi_1, \quad \omega^2_0 = \omega^3_1 = \varpi_2, \quad \omega^3_0 = \omega^1_2 = \varpi_3
\]

the component form of the ensuing first Cartan structure equation (for vanishing torsion \( T^a \))

\[
d e^a + \omega^a_b \wedge e^b = T^a = 0
\]

(52)

coincides with the functional form of \((45)-(48)\). Explicitly,

\[
d e^1 = e^2 \wedge \omega^1_2 + e^3 \wedge \omega^2_3 + e^0 \wedge \omega^1_0 \quad (cyclic)
\]

which clearly reveals that we can identify

\[
e^i \sim \rho^i \quad \& \quad e^0 \sim \rho^4
\]

(53)

The absence of an almost complex structure on \( S^4 \) is then synonymous to the fact, that the \( S^4 \) Cartan structure equation \((52)\) is not consistent with any holomorphic polarization.

Indeed, this lack of almost complex structures on \( S^4 \) is encoded in the explicit form of the BPST instanton solution: The self-dual and anti-self-dual combinations of the connection one-form \( \omega_{ab} \)

\[
A^a_{\pm} = \mp \omega_{0a} - \epsilon_{abc} \omega_{bc}
\]

exactly coincide with the components of the Yang-Mills connection one-form that describes the \( SU(2) \) BPST single-instanton solution. Furthermore, using the relation between \((52)\) and \((45)-(48)\), we get from \((51)\) the second Chern number of the BPST instanton in terms of the volume four-form on the base manifold \( S^4 \),

\[
\text{Ch}_2(A_{\pm}) \propto \int e^0 \wedge e^1 \wedge e^2 \wedge e^3
\]

This reproduces \((50)\) when we recall the identification \((53)\).
6 Complex Structure of \( \mathbb{CP}^2 \)

Complex projective spaces \( \mathbb{CP}^n \) with \( n \geq 1 \) are familiar examples of manifolds with nontrivial complex structures that can be described explicitly, in terms of complex coordinates. Here we explain how these complex structures can also be understood from the point of view of the present formalism. For this we consider in detail the simplest nontrivial case of \( \mathbb{CP}^2 \).

There are several ways to introduce complex coordinates on the manifold \( \mathbb{CP}^2 \). A particularly instructive one \cite{16} is obtained by embedding

\[
(Z_0, Z_1, Z_2) \mapsto \left( \frac{Z_2 \bar{Z}_0}{|Z_0|^2 + |Z_1|^2 + |Z_2|^2}, \frac{Z_0 \bar{Z}_1 - Z_1 \bar{Z}_2}{|Z_0|^2 + |Z_1|^2 + |Z_2|^2} \right)
\]

of \( \mathbb{CP}^2 \) into \( \mathbb{R}^7 \), which is considered as a subspace of \( \mathbb{C}^4 \). The ensuing complex coordinates on \( \mathbb{CP}^2 \) are then ratios of the homogeneous coordinates \( (Z_0, Z_1, Z_2) \). In contrast to \( S^4 \) where one adds a single infinity point to \( \mathbb{R}^4 \), in the case of \( \mathbb{CP}^2 \) one adds at infinity the two-dimensional cycle \( \mathbb{CP}^1 \simeq S^2 \), which gives an additional possibility of regluing the complex co-ordinates \(^1\).

Clearly, this is a generalization of the familiar representation of \( \mathbb{CP}^1 \simeq S^2 \subset \mathbb{R}^3 \) in terms of a stereographic projection.

In order to describe the complex structure of \( \mathbb{CP}^2 \) in terms of a Yang-Mills theory, we start from the fact that locally

\[ SU(3) \sim U(2) \times \mathbb{CP}^2 \]

so that we have the identifications

\[
\mathcal{M} \simeq \mathbb{CP}^2 \\
\mathcal{H} \simeq U(2) \\
\mathcal{G} \simeq SU(3)
\]

We then consider the Maurer-Cartan equation of the \( SU(3) \) Yang-Mills theory. In the standard Gell-Mann basis the subalgebra \( u(2) \sim su(2) \times u(1) \) is generated by the matrices \( \lambda_a \) for \( a = 1, 2, 3, 8 \). Since the \( C^{stu} \) in \cite{24} now

\(^1\)The open set of \( \mathbb{CP}^2 \) around the ”origin” can be described in terms of the co-ordinates \( (z_1 = Z_1/Z_0, z_2 = Z_2/Z_0) \). At \( z_2 \to \infty \) one should use the complex co-ordinates \( u = (u_1, u_2) \): \( u_1 = z_1/z_2 = Z_1/Z_2 \) and \( u_2 = 1/z_2 = Z_0/Z_2 \). Contrarily, at \( z_1 \to \infty \) one should use instead the co-ordinates \( v = (v_1, v_2) \), where \( v_1 = 1/z_1 = Z_0/Z_1 \) and \( v_2 = z_2/z_1 = Z_2/Z_1 \). In the ”symmetric” \( \mathbb{CP}^2 \) case at \( z \to \infty \) the whole \( \mathbb{CP}^1 \) is present, where \( u \) and \( v \) complex co-ordinates van be ”reglued” into each other, however if we ”remove” this line (the \( S^4 \) case) the whole picture becomes singular, since one cannot just put \( u = v \).
vanish the embedding of $U(2) \subset SU(3)$ is symmetric which suggests that an almost complex structure can be integrable.

We introduce the holomorphic polarization

\begin{align*}
\theta_1 &= \rho_4 + i\rho_5 \\
\theta_2 &= \rho_6 + i\rho_7
\end{align*}

This leads to the following version of the Maurer-Cartan equation (25)

\begin{align*}
F^1_{\kappa} + iF^2_{\kappa} &= -\frac{i}{2}\theta_1 \wedge \bar{\theta}_2 \\
F^3_{\kappa} &= -\frac{i}{4}\theta_1 \wedge \bar{\theta}_1 + \frac{i}{4}\theta_2 \wedge \bar{\theta}_2 \\
F^8_{\kappa} &= i\frac{\sqrt{3}}{4}\theta_1 \wedge \bar{\theta}_1 - i\frac{\sqrt{3}}{4}\theta_2 \wedge \bar{\theta}_2
\end{align*}

and for (26) we get

\begin{align*}
d\theta_1 &= -i\kappa_3 \wedge \theta_1 - i\sqrt{3}\kappa_8 \wedge \theta_1 - i(\kappa_1 + i\kappa_2) \wedge \theta_2 \\
d\theta_2 &= i\kappa_3 \wedge \theta_2 - i\sqrt{3}\kappa_8 \wedge \theta_2 - i(\kappa_1 - i\kappa_2) \wedge \theta_1
\end{align*}

Here the almost complex structure of $\mathbb{C}P^2$ is manifest. Furthermore, comparing with (4) we conclude that this almost complex structure is indeed integrable.

We note that the equations (54)-(56) have the structure of (28). The equations (57), (58) can also be combined into the functional form (29),

\[ D_{\kappa} \Phi = 0 \]

when we define the $U(2) \sim U(1) \times SU(2)$ covariant derivative

\[ D_{\kappa} = I \otimes d + iI \otimes \sqrt{3}\kappa_8 + i\sigma_i \otimes \kappa_i \]

where the $\sigma_i$ are Pauli matrices, and

\[ \Phi = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \]

Finally, the Chern classes of the $U(2)$ Yang-Mills connection $\kappa$ can be computed directly from the cohomology classes of $\mathbb{C}P^2$, from (54)-(56) we find for the second Chern class

\[ \text{Ch}_2 = \frac{1}{8\pi^2} \int (\text{Tr} F^2_{\kappa} - \text{Tr} F_{\kappa} \text{Tr} F_{\kappa}) \propto \int \theta_1 \wedge \theta_2 \wedge \bar{\theta}_1 \wedge \bar{\theta}_2 \]

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7 Conclusions

In conclusion, Chern’s attempt to prove his Last Theorem which states that $S^6$ has no complex structure has a natural interpretation in terms of Yang-Mills theory. Unfortunately, the completion of his proof appears to be highly nontrivial but to argue to the contrary it seems to us, that an explicit realization of a complex structure needs to be constructed. Here we have proposed how his arguments can be viewed in a wider perspective, and considered several examples. In particular, we have suggested that Chern’s approach relates directly to the study of physically relevant issues, such as the unified description of space-time and gauge symmetry and geometrical structure of an unbroken vacuum in a spontaneously broken gauge theory. One may also hope that this leads to a better understanding of the relation between the Yang-Mills theory and Kodaira-Spencer theory of gravity, relevant for the study of topological strings. Consequently we expect that further investigations of the relations between the geometry of manifolds, complex structures, and the Yang-Mills theory along these lines could lead to interesting and both physically and mathematically relevant observations.

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