Update Bandwidth for Distributed Storage

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Abstract

In this paper, we consider the update bandwidth in distributed storage systems (DSSs). The update bandwidth, which measures the transmission efficiency of the update process in DSSs, is defined as the total amount of data symbols transferred in the network when the data symbols stored in a node are updated. This paper contains the following contributions. First, we establish the closed-form expression of the minimum update bandwidth attainable by irregular array codes. Second, after defining a class of irregular array codes, called Minimum Update Bandwidth (MUB) codes, which achieve the minimum update bandwidth of irregular array codes, we determine the smallest code redundancy attainable by MUB codes. Third, the code parameters, with which the minimum code redundancy of irregular array codes and the smallest code redundancy of MUB codes can be equal, are identified, which allows us to define MR-MUB codes as a class of irregular array codes that simultaneously achieve the minimum code redundancy and the minimum update bandwidth. Fourth, we introduce explicit code constructions of MR-MUB codes and MUB codes with the smallest code redundancy. Fifth, we establish a lower bound of the update complexity of MR-MUB codes, which can be used to prove that the minimum update complexity of irregular array codes may not be achieved by MR-MUB codes. Last, we construct a class of \((n = k + 2, k)\) vertical maximum-distance separable (MDS) array codes that can achieve all of the minimum code redundancy, the minimum update bandwidth and the optimal repair bandwidth of irregular array codes.

I. INTRODUCTION

Some distributed storage systems (DSSs) adopt replication policy to improve reliability. However, the replication policy requires a high level of storage overhead. To reduce this overhead while maintain reliability, the erasure coding has been used in DSSs, such as Google File System [1] and Microsoft Azure Storage [2]. A main issue of erasure codes in DSSs is the required bandwidth to repair failure node(s). To tackle this issue, many linear block codes, such as regenerating codes [3], [4] and locally repairable codes (LRCs) [5], [6], were proposed in recent years. When the original data symbols change, the coded symbols stored in a DSS must be updated accordingly.
Since performing updates consumes both bandwidth and energy, a higher level of update efficiency is favorable for erasure codes in scenarios where updates are frequent.

The update process in a DSS has two important phases, which are symbol transmission among nodes and the symbol updating (i.e., reading-out and writing-in) in each node. Thus, the update efficiency should include the transmission efficiency and the I/O efficiency. Lots of update-efficient codes [7]–[14] have been proposed to minimize the update complexity from the viewpoint of the I/O efficiency in the update process. These works basically define the update complexity as the average number of coded symbols (i.e., parity symbols) that must be updated when any single data symbol is changed. Clearly, when we consider updating one data symbol, as the number of symbols transmitted between nodes is at most one, the less nodes affected by an update of a symbol, the better the transmission efficiency. Thus, the transmission efficiency problem seems simple and less important. However, when we consider updating many or even all symbols in a node simultaneously, the transmission efficiency problem becomes complicated and significant, and the above definition of the update complexity is not well suitable in this case. To our best knowledge, there is no work discussing the transmission efficiency in the update process of a DSS.

In this paper, we introduce a new metric, called the update bandwidth, to measure the transmission efficiency in the update process of erasure codes applied in DSSs. It is defined as the average amount of symbols that must be transmitted among nodes when the data symbols stored in a node are updated. As the storage capacity of a node is very large nowadays, we need to divide the data into small blocks of data symbols to encode. Since each block is often self-contained in its structure, it is justifiably more efficient to have an updating operation to operate on a block as a whole. In other words, when any data symbol in a block is required to be updated, all data symbols in the block are involved in this single updating operation. As the update bandwidth is the main focus in this paper, without loss of generality, we consider the simplest setting that there is only one coded block in each node in our analysis.

The update model that we consider is described as follows. Assume that there are \( n \) nodes \( \{N_i\}_{i=1}^n \) in the network. Node \( N_i \) stores data vector \( x_i \) and parity vector \( p_i \), where the former consists of data symbols, while the parity symbols are placed in the latter. Fig. 1 demonstrates the update procedure when the data vector \( x_1 \) is updated to \( x_1^* \). In the update procedure, \( N_1 \) first calculates \( n-1 \) intermediate vectors \( \{\Delta p_{1,i}\}_{i=2}^n \), and then send \( \Delta p_{1,i} \) to \( N_i \) respectively for \( i = 2, 3, \ldots, n \). After receiving \( \Delta p_{1,i} \), \( N_i \) computes the updated parity vector \( p_i^* \) from \( \Delta p_{1,i} \) and the old parity vector \( p_i \). This completes the update procedure. Notably, this update model is similar to the one adopted in [15], in which partial-updating schemes for erasure-coded storage are considered. Our general update model will be given formally in Section II-C.

It is worthy mentioning that the codes with the minimum update complexity (i.e., I/O efficiency) may not achieve the minimum update bandwidth, and vice versa. To show that, Fig. 2 presents two \( (n = 4, k = 2) \) maximum distance separable (MDS) array codes, where the elements in the \( i \)-th column are the symbols stored in node \( N_i \) and the number of symbols in each node is \( \alpha = 4 \). In Fig. 2(a) the first row and the third row form an instance of a \( 2 \times 4 \) P-code [11], and the second row and the fourth row form another instance of a \( 2 \times 4 \) P-code. Thus, Fig. 2(a) is an instance of a \( 4 \times 4 \) P-code. Furthermore, Fig. 2(b) is an instance of our codes proposed in Section V. In Figs.
Fig. 1. An instance of the considered update model, where the data symbols stored in \( N_1 \) are updated, and \( N_i \) has the data vector \( x_i \) and the parity vector \( p_i \). When updating the data symbols stored in \( N_1 \), \( N_1 \) sends intermediate symbols \( \{\Delta p_{1,i}\}_{i=2,\ldots,n} \) respectively to all other nodes such that they can calculate the new parity vectors.

\[
\begin{array}{cccc}
  x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} \\
  x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} \\
  x_{3,1} + x_{4,1} & x_{1,1} + x_{3,1} & x_{2,1} + x_{4,1} & x_{1,1} + x_{2,1} \\
  x_{3,2} + x_{4,2} & x_{1,2} + x_{3,2} & x_{2,2} + x_{4,2} & x_{1,2} + x_{2,2}
\end{array}
\]

(a)

Fig. 2. (a) presents an instance, where gray rows contain the data symbols, of a \( 4 \times 4 \) P-code with optimal update complexity 2 and update bandwidth 4; (b) presents an instance, where gray rows contain the data symbols, of a proposed \( (n = 4, k = 2) \) codes, which has update complexity larger than 2 and minimum update bandwidth 3. Note that the instance presents in (b) also has the optimal repair bandwidth.

\[
\begin{array}{cccc}
  x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} \\
  x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} \\
  x_{4,1} + x_{3,2} & x_{1,1} + x_{4,2} & x_{2,1} + x_{1,2} & x_{3,1} + x_{2,2} \\
  x_{2,1} + x_{2,2} + x_{3,2} & x_{3,1} + x_{3,2} + x_{4,2} & x_{4,1} + x_{4,2} + x_{1,2} & x_{1,1} + x_{1,2} + x_{2,2}
\end{array}
\]

(b)

\( x_{i,j} \) and \( x_{i,j} \) the data symbols \( \{x_{i,j}\}_{i=1,\ldots,4,j=1,2} \) are arranged in the first two gray rows, and the last two rows are occupied by parity symbols. It can be verified that the data symbols can be recovered by accessing any two columns of the codes in Figs. 2(a) and 2(b) and hence \( k = 2 \). It is known that P-codes [11] achieve the minimum update complexity when \( n - k = 2 \). Hence, when updating a data symbol in Fig. 2(a) we must update at least \( n - k = 2 \) parity symbols. For example, when updating \( x_{1,1} \), the third symbol in the second column \( x_{1,1} + x_{3,1} \) and the third symbol in the fourth column \( x_{1,1} + x_{2,1} \) need to be updated. However, in Fig. 2(b), when updating a data symbol, two or three parity symbols need to be updated, i.e., the corresponding update complexity is larger than 2. For example, when updating \( x_{1,1} \), the third symbol in the second column \( x_{1,1} + x_{4,2} \) and the fourth symbol in the fourth column \( x_{1,1} + x_{1,2} + x_{2,2} \) need to be updated. Yet, the updating of \( x_{1,2} \) requires the modification of
both parity symbols in the third column (i.e., \(x_{2,1} + x_{1,2}\) and \(x_{4,1} + x_{4,2} + x_{1,2}\)) and the fourth symbol in the fourth column (i.e., \(x_{1,1} + x_{1,2} + x_{2,2}\)).

Next, we consider the update bandwidth. Suppose that the two data symbols in the first node in Fig. 2(a) are updated, i.e., \(x_{1,j}\) are updated to \(x^*_1,j\), \(j = 1, 2\). The first node should send two symbols \(\Delta x_{1,1}\) and \(\Delta x_{1,2}\) to both nodes 2 and 4, where \(\Delta x_{i,j} = x^*_{i,j} - x_{i,j}\). Thus, the required bandwidth is four. It is easy to check that the required bandwidth of updating two data symbols of any other node is also four. Therefore, the update bandwidth of the 4×4 P-code is four. Next we show that the update bandwidth of the code in Fig. 2(b) is three. When two data symbols in node 1 in Fig. 2(b) are updated, we only need to send \(\Delta x_{1,1}\) to node 2, \(\Delta x_{1,2}\) to node 3, and \((\Delta x_{1,1} + \Delta x_{1,2})\) to node 4. Therefore, the required update bandwidth is three. We can verify that the required update bandwidth when updating any other node in Fig. 2(b) is also three. Consequently, the update bandwidth of the code in Fig. 2(b) is better than that of the 4×4 P-code in Fig. 2(a). We will show in Section IV that the code in Fig. 2(b) achieves the minimum update bandwidth among all \((4, 2)\) irregular array codes with two data symbols per node.

Other than update complexity and update bandwidth, the repair bandwidth, defined as the amount of symbols downloaded from the surviving nodes to repair the failed node, is also an important consideration in DSSs. The repair problem was first brought into the spotlight by Dimakis et al. [3]. It can be anticipated that a well-designed code with both minimum update bandwidth and optimal repair bandwidth is attractive for DSSs. Surprisingly, the code in Fig. 2(b) also achieves the optimal repair bandwidth among all \((4, 2)\) MDS array codes. One can check that we can repair the four symbols stored in node 1 by downloading the six underlined symbols in Fig. 2(b), i.e., \(\{x_{2,1}, x_{2,2}\}\) from node 2, \(\{x_{3,2}, x_{2,1} + x_{1,2}\}\) from node 3, and \(\{x_{4,1}, x_{1,1} + x_{1,2} + x_{2,2}\}\) from node 4. Thus, the repair bandwidth of node 1 is six, which is optimal for the parameters of \(n = 4, k = 2\) and \(\alpha = 4\) [3]. We can verify that the repair bandwidth of any other node in Fig. 2(b) is also six. Therefore, the repair bandwidth of the code in Fig. 2(b) is optimal.

The contributions of this paper are as follows.

- We introduce a new metric, i.e., update bandwidth, and emphasize its importance in scenarios where storage updates are frequent.
- We consider irregular array codes with a given level of protection against block erasures [16], and establish the closed-form expression of the minimum update bandwidth attainable for such codes.
- Referring the class of irregular array codes that achieve the minimum update bandwidth as MUB codes, we next derive the smallest code redundancy attainable by MUB codes.
- Comparing the smallest code redundancy of MUB codes with the minimum code redundancy of irregular array codes derived in [16], we identify a class of MUB codes, called MR-MUB codes, that can achieve simultaneously the minimum code redundancy of irregular array codes and the minimum update bandwidth of irregular array codes.
- Systematic code constructions for MR-MUB codes and for MUB codes with the smallest code redundancy are both provided.
- We establish a lower bound of the update complexity of MR-MUB codes, by which we confirm that the update complexity of irregular array codes may not be achieved by MR-MUB codes.
We construct an \((n, k = n - 2)\) MR-MUB code with the optimal repair bandwidth for all nodes via the transformation in [17], confirming the existence of the irregular array codes that can simultaneously achieve the minimum code redundancy, the minimum update bandwidth and the optimal repair bandwidth for all codes.

The rest of this paper is organized as follows. Section II introduces the notations used in this paper and the proposed update model. Section III establishes the necessary condition for the existence of an irregular array code. In Section IV, via the form of linear programings, we determine the minimum update bandwidth of irregular array codes and the smallest code redundancy of MUB codes. Section V presents the explicit constructions of MR-MUB codes and MUB codes. Section VI derives a lower bound of the update complexity of MR-MUB codes. Section VII devises a class of \((n = k + 2, k)\) MR-MUB codes with the optimal repair bandwidth for all nodes. Section VIII concludes this work.

II. Preliminary

A. Definition

We first introduce the notations used in this paper. Let \([n] \triangleq \{1, \ldots, n\}\) for a positive integer \(n\). \((a_i)_{i \in [n]}\) denotes an index set \((a_1, a_2, \ldots, a_n)\). Let \([x_{i,j}]_{i \in [m], j \in [n]}\) denote an \(m \times n\) matrix whose entry in row \(i\) and column \(j\) is \(x_{i,j}\). \(\text{wt}(v)\) denotes the weight of vector \(v\), i.e., the number of nonzero elements in vector \(v\). \(M^T\) represents the transpose of matrix \(M\). \(\text{row}(M)\), \(\text{col}(M)\) and \(\text{rank}(M)\) represent the number of rows of \(M\), the number of columns of \(M\) and the rank of \(M\), respectively. \(M^{-1}\) denotes the inverse matrix of \(M\), provided \(M\) is invertible. \(|S|\) denotes the cardinality of a set \(S\). \(\mathbb{F}_q\) denotes the finite field of size \(q\), where \(q\) is a power of a prime. For two discrete random variables \(X\) and \(Y\), their joint probability distribution is denoted as \(P_{XY}(x, y)\). \(H_q(X)\) denotes the \(q\)-ary entropy of \(X\), and \(I_q(X; Y)\) denotes the \(q\)-ary mutual information between \(X\) and \(Y\), where \(q\) is the base of the logarithm. We consider linear codes throughout the paper and the main notations used in this paper are listed in Table I.

B. Irregular array code

An irregular array code [16], [18] can be represented as an irregular array. Formally, given a positive integer \(n\) and two column vectors \(m = [m_1 \ldots m_n]^T\) and \(p = [p_1 \ldots p_n]^T\), where \(m_i, p_i \geq 0\) for \(i \in [n]\), the codeword of an irregular array code \(C\) over \(\mathbb{F}_q\) is denoted as

\[
C = (c_1, c_2, \ldots, c_n),
\]

where the column vector \(c_i\) contains \(m_i\) data symbols and \(p_i\) parity symbols. Specifically, we denote

\[
c_i = \begin{bmatrix} x_i \\ p_i \end{bmatrix}, \quad i \in [n],
\]

where \(x_i\) is the \(i\)-th data vector that contains \(m_i\) data symbols and \(p_i\) is the \(i\)-th parity vector that contains \(p_i\) parity symbols. Since \(x_i\) contains data symbols, we can naturally consider that \(x_i\) is uniformly distributed over \(\mathbb{F}_q^{m_i}\), and \(x_i\) and \(x_j\) are independent for \(i \neq j \in [n]\). As such, we have

\[
H_q(x_i) = m_i \quad \forall i \in [n],
\]
Table I

Main notations used in this paper

| Notation | Description |
|----------|-------------|
| $n$      | The number of nodes |
| $m_i$    | The number of data symbols in node $i$ |
| $p_i$    | The number of parity symbols in node $i$ |
| $m$      | $m = [m_1 \ldots m_n]^T$ |
| $P$      | $P = [p_1 \ldots p_n]^T$ |
| $x_i$    | The $i$-th data vector |
| $p_i$    | The $i$-th parity vector |
| $c_i$    | $c_i = [x_i^T \ P_i]^T$, the $i$-th column vector |
| $C$      | $C = (c_1, c_2, \ldots, c_n)$, the codeword of an irregular array code |
| $M_{i,j}$| The construction matrix |
| $A_{i,j}$, $B_{i,j}$ | A full rank decomposition of $M_{i,j}$, i.e., $M_{i,j} = B_{i,j}A_{i,j}$ |
| $\mathcal{E}$ | A subset of $[n]$ with $|\mathcal{E}| = n-k$, where the elements in it are denoted as $e_i$ with $i \in [n-k]$ |
| $\bar{\mathcal{E}}$ | $\bar{\mathcal{E}} = [n] \setminus \mathcal{E}$, where the elements in it are denoted as $\bar{e}_i$ with $i \in [k]$ |
| $C_{\mathcal{E}}, X_{\mathcal{E}}, P_{\mathcal{E}}$ | $C_{\mathcal{E}} = [e_1^T \ldots e_{|\mathcal{E}|-k}^T]$, $X_{\mathcal{E}} = [x_1^T \ldots x_{|\mathcal{E}|-k}^T]$, $P_{\mathcal{E}} = [p_1^T \ldots p_{|\mathcal{E}|-k}^T]^T$ |
| $B$      | The number of data symbols |
| $R$      | The number of parity symbols, i.e., code redundancy |
| $\gamma_{i,j}$ | The minimum number of symbols sent from node $i$ to node $j$ when updating the data symbols in node $i$ |
| $\gamma$ | The average required bandwidth when updating a node, i.e., update bandwidth |
| $\gamma_{\text{min}}$ | The minimum update bandwidth among all irregular array codes |
| $R_{\text{min}}$ | The minimum code redundancy among all irregular array codes |
| $R_{\text{data}}$ | The smallest code redundancy for irregular array codes with update bandwidth equal to $\gamma_{\text{min}}$ |
| $\theta$ | The average number of parity symbols affected by a change of a single data symbol, i.e., update complexity |

\[ I_q(x_i; x_j) = 0 \quad \forall i, j \in [n], i \neq j. \quad (4) \]

As all symbols in $p_i \in \mathbb{F}_q^{p_i}$ may not be independent, we can only obtain

\[ H_q(p_i) \leq p_i \quad \forall i \in [n]. \quad (5) \]

The storage redundancy (i.e., code redundancy) of $C$ is the total number of parity symbols, i.e., $R = \sum_{i=1}^n p_i$.

An example of irregular array codes is illustrated in Fig. 3 where the gray cells contain the data symbols. In this example, we have $m = [4 \ 2 \ 2 \ 0]^T$, $p = [2 \ 3 \ 3 \ 3]^T$ and $R = 11$. In addition, the first column of the irregular array code in Fig. 3 stores four data symbols $x_1 = [x_{1,1} \ x_{1,2} \ x_{1,3} \ x_{1,4}]^T$ and two parity symbols $p_1 = [x_{2,1} + x_{2,2} \ x_{3,2}]^T$.

When $m_i + p_i = m_j + p_j$ for any $i \neq j \in [n]$, the irregular array code $C$ is reduced to a regular array code. When $m_i = m_j$ and $p_i = p_j$ for all $i \neq j \in [n]$, $C$ is called a vertical array code. As an example, both codes in Fig. 2 are vertical array codes. When $p_i = 0$ for $i \in [k]$ and $m_j = 0$ for $k < j \leq n$, $C$ is called a horizontal array code. If we can retrieve all the data symbols by accessing any $k$ columns, and there is a set of $k-1$ columns which we can not retrieve all the data symbols from, then the code $C$ is parameterized as an $(n, k, m)$ irregular array code.
γ have full rank, which in turns validates γ with full rank. Furthermore, we first prove that

\textbf{Proof.}

\begin{theorem}
\end{theorem}

row $A_{i,j}$ is a $(4, 2, m)$ irregular MDS array code with $m = [4 2 0]^T$ and $p = [2 3 3 3]^T$.

We will demonstrate in Section \[\text{IV-C}\] that the code in Fig. 3 can only be reconstructed by accessing at least any two columns, and hence it is a $(4, 2, m)$ irregular array code. In fact, the code in Fig. 3 is also an MUB code with the smallest code redundancy (cf. Section \[\text{IV-C}\]). In the subsection that follows, we will introduce the update model and the update bandwidth of $(n, k, m)$ irregular array codes.

\[\text{C. Update model and update bandwidth}\]

In an $(n, k, m)$ irregular array code $C$, each parity symbol can be generated as a linear combination of all data symbols. Thus, the parity symbols in each column can be obtained from

$$p_j = \sum_{i=1}^{n} M_{i,j} x_i \quad j \in [n],$$

where $M_{i,j}$ is a $p_j \times m_i$ matrix, called construction matrix. Apparently, when the data vector in node $i$ is updated from $x_i$ to $x_i^*$, node $j$ with $j \in [n] \setminus \{i\}$ needs to update its parity vector via $p_j^* = p_j + M_{i,j} \Delta x_i$, where $\Delta x_i = x_i^* - x_i$. Such update process can be divided into two steps. First, node $i$ calculates the intermediate vector $A_{i,j} \Delta x_i$, and sends these symbols to node $j$. Second, node $j$ calculates $\Delta p_j = p_j^* - p_j$ from the intermediate vector via a linear transformation, i.e., $\Delta p_j = B_{i,j} A_{i,j} \Delta x_i$. As a result of (6), the two matrices $A_{i,j}$ and $B_{i,j}$ corresponding to the linear transformations respectively performed by node $i$ and node $j$ must satisfy $M_{i,j} = B_{i,j} A_{i,j}$. Based on the above update model, the number of symbols sent from node $i$ to node $j$ is $\text{row}(A_{i,j})$.

Denoting

$$\gamma_{i,j} \triangleq \min_{A_{i,j}, B_{i,j}} \{\text{row}(A_{i,j}) | M_{i,j} = B_{i,j} A_{i,j}\}, \text{ for } i \neq j,$$

as the minimum amount of symbols sent from node $i$ to node $j$ when updating the data symbols stored in node $i$, we have the following theorem.

\[\text{Theorem 1.} \ \text{row}(A_{i,j}) = \gamma_{i,j} \text{ if, and only if, } \text{rank}(A_{i,j}) = \text{rank}(B_{i,j}) = \text{row}(A_{i,j}), \text{ and both } A_{i,j} \text{ and } B_{i,j} \text{ are with full rank. Furthermore, } \gamma_{i,j} = \text{rank}(M_{i,j}).\]

\[\text{Proof.} \ \text{We first prove that } \text{row}(A_{i,j}) = \gamma_{i,j} \text{ implies } \text{rank}(A_{i,j}) = \text{rank}(B_{i,j}) = \text{row}(A_{i,j}), \text{ and both } A_{i,j} \text{ and } B_{i,j} \text{ have full rank, which in turns validates } \gamma_{i,j} = \text{rank}(M_{i,j}).\]
Assuming that \( \text{row}(\mathbf{A}_{i,j}) \) is with the minimum value, i.e., \( \text{row}(\mathbf{A}_{i,j}) = \gamma_{i,j} \), we show by contradiction that \( \text{row}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{A}_{i,j}) \). Suppose \( \text{rank}(\mathbf{A}_{i,j}) < \text{row}(\mathbf{A}_{i,j}) \). Then, there is an invertible matrix \( \mathbf{R}_{i,j} \) satisfying \( \mathbf{R}_{i,j} \mathbf{A}_{i,j} = \begin{bmatrix} \mathbf{A}_{i,j}' \end{bmatrix} \), where \( [0] \) is a \( (\text{row}(\mathbf{A}_{i,j}) - \text{rank}(\mathbf{A}_{i,j})) \times m \_i \) zero matrix, and \( \text{rank}(\mathbf{A}_{i,j}') = \text{row}(\mathbf{A}_{i,j}') = \text{rank}(\mathbf{A}_{i,j}) \). We thus have \( \mathbf{M}_{i,j} = \mathbf{B}_{i,j} \mathbf{R}_{i,j}^{-1} \mathbf{A}_{i,j} = \mathbf{B}_{i,j} \mathbf{R}_{i,j}^{-1} \begin{bmatrix} \mathbf{A}_{i,j}' \end{bmatrix} \), which implies \( \mathbf{M}_{i,j} = \mathbf{B}_{i,j}' \mathbf{A}_{i,j}' \) with \( \mathbf{B}_{i,j}' \) being the first rank(\( \mathbf{A}_{i,j} \)) columns of \( \mathbf{B}_{i,j} \mathbf{R}_{i,j}^{-1} \). However, \( \text{rank}(\mathbf{A}_{i,j}') = \text{rank}(\mathbf{A}_{i,j}) < \gamma_{i,j} \) contradicts to the definition of \( \gamma_{i,j} \). We therefore confirm that if \( \text{row}(\mathbf{A}_{i,j}) = \gamma_{i,j} \), then \( \text{row}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{A}_{i,j}) \). Similarly, we can show that if \( \text{col}(\mathbf{B}_{i,j}) = \gamma_{i,j} \), then \( \text{col}(\mathbf{B}_{i,j}) = \text{rank}(\mathbf{B}_{i,j}) \). As \( \text{row}(\mathbf{A}_{i,j}) = \text{col}(\mathbf{B}_{i,j}) \), we conclude that \( \text{row}(\mathbf{A}_{i,j}) = \gamma_{i,j} \) implies \( \text{rank}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{B}_{i,j}) = \text{row}(\mathbf{A}_{i,j}) = \text{col}(\mathbf{B}_{i,j}) \), and both \( \mathbf{A}_{i,j} \) and \( \mathbf{B}_{i,j} \) have full rank. An immediate consequence of the above proof is that this pair of \( \mathbf{A}_{i,j} \) and \( \mathbf{B}_{i,j} \) is a minimizer of \( \gamma_{i,j} \). By Sylvester’s rank inequality, we have

\[
\text{rank}(\mathbf{B}_{i,j}) + \text{rank}(\mathbf{A}_{i,j}) - \text{row}(\mathbf{A}_{i,j}) = \gamma_{i,j} \leq \text{rank}(\mathbf{M}_{i,j}).
\]

(8)

It can also be inferred that

\[
\text{rank}(\mathbf{M}_{i,j}) \leq \min\{\text{rank}(\mathbf{B}_{i,j}), \text{rank}(\mathbf{A}_{i,j})\} = \gamma_{i,j}.
\]

(9)

Hence,

\[
\gamma_{i,j} = \text{rank}(\mathbf{M}_{i,j}).
\]

(10)

We next show the converse statement, i.e., if both \( \mathbf{A}_{i,j} \) and \( \mathbf{B}_{i,j} \) are with full rank and \( \text{rank}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{B}_{i,j}) = \text{row}(\mathbf{A}_{i,j}) \), then \( \text{row}(\mathbf{A}_{i,j}) = \gamma_{i,j} \). Given \( \text{rank}(\mathbf{B}_{i,j}) = \text{row}(\mathbf{A}_{i,j}) \), we obtain by Sylvester’s rank inequality that

\[
\text{rank}(\mathbf{B}_{i,j}) + \text{rank}(\mathbf{A}_{i,j}) - \text{row}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{A}_{i,j}) \leq \text{rank}(\mathbf{M}_{i,j}) = \gamma_{i,j},
\]

(11)

which, together with \( \gamma_{i,j} = \text{rank}(\mathbf{M}_{i,j}) \leq \text{rank}(\mathbf{A}_{i,j}) \), establishes \( \text{row}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{A}_{i,j}) = \gamma_{i,j} \). This completes the proof.

We next show the converse statement, i.e., if both \( \mathbf{A}_{i,j} \) and \( \mathbf{B}_{i,j} \) are with full rank and \( \text{rank}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{B}_{i,j}) = \text{row}(\mathbf{A}_{i,j}) \), then \( \text{row}(\mathbf{A}_{i,j}) = \gamma_{i,j} \). Given \( \text{rank}(\mathbf{B}_{i,j}) = \text{row}(\mathbf{A}_{i,j}) \), we obtain by Sylvester’s rank inequality that

\[
\text{rank}(\mathbf{B}_{i,j}) + \text{rank}(\mathbf{A}_{i,j}) - \text{row}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{A}_{i,j}) \leq \text{rank}(\mathbf{M}_{i,j}) = \gamma_{i,j},
\]

(11)

which, together with \( \gamma_{i,j} = \text{rank}(\mathbf{M}_{i,j}) \leq \text{rank}(\mathbf{A}_{i,j}) \), establishes \( \text{row}(\mathbf{A}_{i,j}) = \text{rank}(\mathbf{A}_{i,j}) = \gamma_{i,j} \). This completes the proof.

Theorem 1 indicates that \( \gamma_{i,j} = \text{rank}(\mathbf{M}_{i,j}) \) is the minimum amount of symbols required to be sent from node \( i \) to node \( j \) when updating the data symbols stored in node \( i \). We thus define the update bandwidth \( \gamma \) for a code \( C \) as the average required bandwidth, i.e.,

\[
\gamma = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j}.
\]

(12)

By Theorem 1 the update bandwidth \( \gamma \) can be achieved by adopting two full-rank matrices that fulfill \( \mathbf{M}_{i,j} = \mathbf{B}_{i,j} \mathbf{A}_{i,j} \), where \( \mathbf{B}_{i,j} \) is a \( p \_j \times \gamma_{i,j} \) matrix and \( \mathbf{A}_{i,j} \) is a \( \gamma_{i,j} \times m \_i \) matrix. In the rest of the paper, the full-rank matrices \( \mathbf{B}_{i,j} \) and \( \mathbf{A}_{i,j} \) used in our update model are fixed as the ones with rank \( \gamma_{i,j} \).

D. Encoding aspect of the update model

The update model in the previous subsection can also be equivalently characterized via an encoding aspect from [7]. Specifically, we can first calculate

\[
\mathbf{p}_{i,j} = \mathbf{A}_{i,j} \mathbf{x}_i \quad \forall i, j \in [n], i \neq j.
\]

(13)
Similar to (5), since symbols in $p_{i,j} \in F_q^{\gamma_{i,j}}$ are possibly dependent, we can only obtain

$$H_q(p_{i,j}) \leq \gamma_{i,j} \quad \forall i,j \in [n], i \neq j.$$  \hfill (14)

Then, (6) can be rewritten as

$$p_j = \sum_{i=1, i \neq j}^n B_{i,j} p_{i,j} \quad \forall j \in [n].$$  \hfill (15)

As a result, the parity symbols are the coded symbols from two sets of encoding matrices $\{A_{i,j}\}_{i,j \in [n]}$ and $\{B_{i,j}\}_{i,j \in [n]}$. This encoding aspect of the update model will be adopted in later sections. Since the number of symbols passed from (13) to (15) is

$$\sum_{i=1}^n \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j} = n \gamma,$$

the average number of symbols transmitted among all nodes during the encoding process is equal to the update bandwidth $\gamma$.

### III. NECESSARY CONDITION FOR THE EXISTENCE OF AN IRREGULAR ARRAY CODE

In this section, we provide a necessary condition for the parameters $\{p_j\}_{j \in [n]}$ and $\{\gamma_{i,j}\}_{i \neq j \in [n]}$ such that an $(n, k, m)$ irregular array code $C$, where retrieval of data symbols can only be guaranteed by any other $k$ columns but not by any other $k-1$ columns, exists (cf. Theorem 2 and Corollary 1). For simplicity, we use $H(\cdot)$ and $I(\cdot; \cdot)$ to represent $H_q(\cdot)$ and $I_q(\cdot; \cdot)$ in this section.

Some notations used in the proofs below are first introduced (cf. Table I). For a subset $E \subset [n]$ with $|E| = n-k$, the elements in $E$ are denoted as $e_i$ with $i \in [n-k]$. Similarly, denote the elements in $\bar{E} \triangleq [n] \setminus E$ as $\bar{e}_i$ with $i \in [k]$. Let $X_E \triangleq [x_{e_1}^T \ldots x_{e_{n-k}}^T]^T$ and $X_{\bar{E}} \triangleq [x_{\bar{e}_1}^T \ldots x_{\bar{e}_k}^T]^T$, and $C_E$, $C_{\bar{E}}$, $P_E$ and $P_{\bar{E}}$ are similarly defined. Equation (6) can then be rewritten using these notations as

$$p_j = \sum_{i \in [n-k]} M_{e_i,j} x_{e_i} + \sum_{i \in [k]} M_{e_i,j} x_{\bar{e}_i}.$$  \hfill (16)

Thus, we can write

$$P_E = \begin{bmatrix} p_{e_1} \\ p_{e_2} \\ \vdots \\ p_{\bar{e}_k} \end{bmatrix} = \begin{bmatrix} M_{e_1,e_1} & \ldots & M_{e_1,e_{n-k}} \\ M_{e_2,e_1} & \ldots & M_{e_2,e_{n-k}} \\ \vdots & \ddots & \vdots \\ M_{\bar{e}_k,e_1} & \ldots & M_{\bar{e}_k,e_{n-k}} \end{bmatrix} X_E + \begin{bmatrix} M_{e_1,\bar{e}_1} & \ldots & M_{e_1,\bar{e}_k} \\ M_{e_2,\bar{e}_1} & \ldots & M_{e_2,\bar{e}_k} \\ \vdots & \ddots & \vdots \\ M_{\bar{e}_k,\bar{e}_1} & \ldots & M_{\bar{e}_k,\bar{e}_k} \end{bmatrix} X_{\bar{E}}.$$  \hfill (17)

Let

$$M_E \triangleq \begin{bmatrix} M_{e_1,e_1} & \ldots & M_{e_1,e_{n-k}} \\ M_{e_2,e_1} & \ldots & M_{e_2,e_{n-k}} \\ \vdots & \ddots & \vdots \\ M_{\bar{e}_k,e_1} & \ldots & M_{\bar{e}_k,e_{n-k}} \end{bmatrix}, \quad M_{\bar{E}} \triangleq \begin{bmatrix} M_{e_1,\bar{e}_1} & \ldots & M_{e_1,\bar{e}_k} \\ M_{e_2,\bar{e}_1} & \ldots & M_{e_2,\bar{e}_k} \\ \vdots & \ddots & \vdots \\ M_{\bar{e}_k,\bar{e}_1} & \ldots & M_{\bar{e}_k,\bar{e}_k} \end{bmatrix}.$$  \hfill (18)

Then, from (17) and (18), we establish

$$P_E = M_E X_E + M_{\bar{E}} X_{\bar{E}}.$$  \hfill (19)

In the following, we provide four lemmas that will be useful in characterizing a necessary condition for the existence of an $(n, k, m)$ irregular array code in Theorem 2.
Lemma 1. Given a matrix $A \in \mathbb{F}_q^{a \times b}$ and a random column vector $b \in \mathbb{F}_q^b$, we have $H(AB) \leq \text{rank}(A)$.

Proof. The lemma trivially holds when $\text{rank}(A) = 0$ or $\text{rank}(A) = \text{row}(A)$. Here, we provide the proof subject to $\text{row}(A) > \text{rank}(A) > 0$. Given a matrix $A \in \mathbb{F}_q^{a \times b}$, there is an invertible matrix $R$ such that $RA = [A']_{\text{row}}$, where $[0]$ is a $(\text{row}(A) - \text{rank}(A)) \times \text{col}(A)$ zero matrix. Then, given $A'b$, we can determine $Ab$ via $Ab = R^{-1} [A' \ b]$, and vice versa. Thus, we have $H(AB|A'b) = H(A'b|Ab) = 0$. As $I(A'b; Ab) = H(A'b) - H(A'b|Ab) = H(AB) - H(AB|A'b)$, we conclude $H(AB) = H(A'b) \leq \text{row}(A'b) = \text{rank}(A)$. This completes the proof. □

The next three lemmas associate $m$, $p$ and $\{\gamma_{i,j} = \text{rank}(M_{i,j})\}_{i,j \in [n]}$ through $\text{rank}(M_E)$.

Lemma 2. Given any $E \subset [n]$ with $|E| = n - k$, if each codeword $C \in E$ can be determined uniquely by $C_E$, then we have $\sum_{i \in E} m_i \leq \text{rank}(M_E)$.

Proof. If the knowledge of $C_E$ can reconstruct the entire $C$, then $H(X_E|C_E) = 0$. Thus, we have

$$I(C_E; X_E) = H(X_E) - H(X_E|C_E) = H(X_E) = \sum_{i \in E} m_i. \tag{20}$$

Since $I(X_E; X_E) = 0$ as indicated by (4), we get

$$I(C_E; X_E) = I(P_E; X_E|X_E) = I(X_E; X_E) + I(P_E; X_E|X_E) = I(P_E; X_E|X_E). \tag{21}$$

We then obtain from (19) and (21) that

$$I(P_E; X_E|X_E) = I(M_E X_E + M_E X_E; X_E|X_E)$$

$$= H(M_E X_E + M_E X_E|X_E) - H(M_E X_E + M_E X_E|X_E, X_E)$$

$$= H(M_E X_E|X_E)$$

$$= H(M_E X_E),$$

which, together with (20), (21) and Lemma 1 implies

$$\sum_{i \in E} m_i = I(C_E; X_E) = I(P_E; X_E|X_E) = H(M_E X_E) \leq \text{rank}(M_E). \tag{22}$$

□

Lemma 3. $\text{rank}(M_E) \leq \sum_{j \in [k]} \min\{p_j, \sum_{i \in E} \gamma_{i,j}\}$.

Proof. We first note from (18) that

$$\text{rank}(M_E) \leq \sum_{j \in [k]} \text{rank}([M_{e_1, \bar{e}_j} \ldots M_{e_{n-k}, \bar{e}_j}]). \tag{22}$$

Using $\gamma_{i,j} = \text{rank}(M_{i,j})$ from Theorem 1, we obtain

$$\text{rank}([M_{e_1, \bar{e}_j} \ldots M_{e_{n-k}, \bar{e}_j}]) \leq \sum_{i \in [n-k]} \text{rank}(M_{e_i, \bar{e}_j}) = \sum_{i \in [n-k]} \gamma_{e_i, \bar{e}_j}. \tag{23}$$

Next, we note that

$$\text{rank}([M_{e_1, \bar{e}_j} \ldots M_{e_{n-k}, \bar{e}_j}]) \leq \text{row}([M_{e_1, \bar{e}_j} \ldots M_{e_{n-k}, \bar{e}_j}]) = p_{\bar{e}_j}. \tag{24}$$
The lemma then follows from (26) and (27).

In parallel to (23) and (24), we next derive

\[
\text{rank}(\{M_{e_1, e_j} \ldots M_{e_{n-k}, e_j}\}) \leq \min \left\{ p_{e_j}, \sum_{i \in [n-k]} \gamma_{e_i, e_j} \right\}.
\]  

(25)

The validity of the lemma can thus be confirmed by (22) and (25).

**Lemma 4.** \(\text{rank}(M) \leq \sum_{i \in E} \min \{ m_i, \sum_{j \in E} \gamma_{i,j} \} \).

**Proof.** The proof of this lemma is similar to that of Lemma 3. First from (18), we establish

\[
\text{rank}(M) = \text{rank}(M^T_e) \leq \sum_{i \in [n-k]} \text{rank}(\{M^T_{e_i, e_1} \ldots M^T_{e_i, e_k}\}).
\]

(26)

In parallel to (23) and (24), we next derive \(\text{rank}(\{M^T_{e_1, e_j} \ldots M^T_{e_k, e_j}\}) \leq \sum_{j \in [k]} \gamma_{e_1, e_j} \) and \(\text{rank}(\{M^T_{e_1, e_i} \ldots M^T_{e_k, e_i}\}) \leq \text{row}(\{M^T_{e_1, e_i} \ldots M^T_{e_k, e_i}\}) = m_{e_1}, \) which immediately gives

\[
\text{rank}(\{M^T_{e_1, e_j} \ldots M^T_{e_k, e_j}\}) \leq \min \left\{ m_{e_1}, \sum_{j \in [k]} \gamma_{e_1, e_j} \right\}.
\]  

(27)

The lemma then follows from (26) and (27).

After establishing the above four lemmas, we are now ready to prove the main result in this section.

**Theorem 2.** Given any \(E \subset [n]\) with \(|E| = n - k\), if each codeword \(C \in \mathcal{C} \) can be determined uniquely by \(C_E\), then the following inequalities must hold:

\[
\sum_{j \in E} \gamma_{i,j} \geq m_i \quad \forall i \in E,
\]

(28)

\[
\sum_{i \in E} m_i \leq \sum_{j \in E} \min \left\{ p_{e_j}, \sum_{i \in E} \gamma_{i,j} \right\}.
\]

(29)

**Proof.** Inequality (29) is an immediate consequence of Lemmas 2 and 3.

The inequality in (28) can be proved by contradiction. Suppose \(\sum_{j \in E} \gamma_{u,j} < m_u\) for some \(u \in E\). Then, we can infer from Lemma 4 that

\[
\text{rank}(M_{e_j}) \leq \sum_{i \in E \setminus \{u\}} \min \left\{ m_i, \sum_{j \in E} \gamma_{i,j} \right\} + \sum_{j \in E \setminus \{u\}} \gamma_{u,j} < \sum_{i \in E \setminus \{u\}} m_i + m_u = \sum_{i \in E} m_i,
\]

(30)

which contradicts to Lemma 2. Consequently, inequality (28) must hold for every \(i \in E\).

For completeness, we conclude the section by reiterating the result in Theorem 2 in the following corollary.

**Corollary 1.** An \((n, k, m)\) irregular array code \(C\) with construction matrices \(\{M_{i,j}\}_{i,j \in [n]}\) and numbers of parity symbols specified in \(p\) must fulfill (28) and (29) for every \(E \subset [n]\) with \(|E| = n - k\).

**IV. LOWER BOUNDS FOR CODE REDUNDANCY AND UPDATE BANDWIDTH**

In this section, three lower bounds will be established, which are lower bounds respectively for code redundancy and update bandwidth, and a lower bound for code redundancy subject to the minimum update bandwidth. Their
achievability by explicit constructions of irregular array codes under $k \mid m_i$ for all $i \in [n]$ will be shown in Section V. Without loss of generality, we assume in this section that

$$m_1 \geq m_2 \geq \cdots \geq m_n \geq 0. \quad (31)$$

### A. Minimization of code redundancy

Theorem 2 indicates that a lower bound for the code redundancy of an $(n, k, m)$ irregular array code can be obtained by solving the linear programming problem below.

**Linear Programming 1.** To minimize $R = \sum_{i=1}^{n} p_i$, subject to (28) and (29) among all $E \subset [n]$ with $|E| = n - k$.

Since the object function of Linear Programming 1 is only a function of $p$, a code redundancy $R$ is attainable due to a choice of $p$, if there exists a set of corresponding $\{\gamma_{i,j}\}_{i \neq j \in [n]}$ that can validate both (28) and (29). A valid selection of such $\{\gamma_{i,j}\}_{i \neq j \in [n]}$ for a given $p$ is to persistently increase $\{\gamma_{i,j}\}_{i \neq j \in [n]}$ until both (28) and

$$p_j \leq \sum_{i \in E} \gamma_{i,j} \quad \forall j \in \bar{E}$$

are satisfied for arbitrary choice of $E \subset [n]$ with $|E| = n - k$. As a result, we can disregard (28) and reduce (29) to

$$\sum_{i \in E} m_i \leq \sum_{j \in \bar{E}} p_j, \quad (33)$$

leading to a new linear programming setup as follows.

**Linear Programming 2.** To minimize $R = \sum_{i=1}^{n} p_i$, subject to (33) among all $E \subset [n]$ with $|E| = n - k$.

**Lemma 5.** Linear Programming 1 is equivalent to Linear Programming 2.

**Proof.** It is obvious that all minimizers of Linear Programming 1 satisfy the constraint in Linear Programming 2. On the contrary, given a minimizer $p$ of Linear Programming 2, we can assign $\gamma_{i,j} = \max\{m_i, p_j\}$ to satisfy the constraints in Linear Programming 1. Thus, Linear Programming 1 and Linear Programming 2 are equivalent. \qed

**Remark.** Note that Linear Programming 2 which was first given in [16], is not related to the update bandwidth $\gamma$ of an irregular array code, while the proposed setup in Linear Programming 1 is. Thus, the latter setup can be used to determine an irregular array code of update-bandwidth efficiency by replacing the object function $R$ with update bandwidth $\gamma$. However, for the minimization of code redundancy, the two linear programming settings are equivalent as confirmed in Lemma 5.

To solve Linear Programming 2, Tosato and Sandell [16] introduced a water level parameter $\mu$, defined as

$$\mu = \max \left\{ m_{n-k}, \left\lfloor \frac{B}{k} \right\rfloor \right\}, \quad (34)$$

where $B \triangleq \sum_{i \in [n]} m_i$ is the total number of data symbols, and by following the assumption in (31), $m_{n-k}$ is the $(n - k)$-th largest element in vector $m$. It was shown in [16] that the minimum code redundancy equals

$$R_{\text{min}} = \sum_{i=1}^{n-k} (\mu - m_i)_+ + m_i, \quad (35)$$
which can only be achieved by those \( p \)'s satisfying

\[
\begin{cases}
p_i = [\mu - m_i]_+ & \text{for } 1 \leq i \leq n - k, \\
p_i \leq [\mu - m_i]_+ & \text{for } n - k < i \leq n, \\
\sum_{i=n-k+1}^{n} p_i = \sum_{i=1}^{n-k} m_i,
\end{cases}
\]

(36)

where \([x]_+ = \max\{0, x\}\). The class of \((n, k, m)\) irregular array codes conforming to (36) is called irregular MDS array codes \[16\]. In particular, when \(k \mid B\) and \(m_1 \leq B/k\), (35) and (36) can be respectively reduced to

\[
R_{\min} = \frac{(n - k)B}{k},
\]

(37)

and

\[
p_i = \frac{B}{k} - m_i \quad \forall i \in [n].
\]

(38)

B. Minimization of update bandwidth

We now turn to the determination of the minimum update bandwidth. As similar to Linear Programming 1, a lower bound for the update bandwidth of an \((n, k, m)\) irregular array code can be obtained using the linear programming below.

\[\text{Linear Programming 3. To minimize } \gamma = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j}, \text{ subject to } (28) \text{ and } (29) \text{ among all } \mathcal{E} \subset [n] \text{ with } |\mathcal{E}| = n - k.\]

Since \(p\) is not used in the above object function, a choice of \(\{\gamma_{i,j}\}_{i \neq j \in [n]}\) is feasible for the minimization of \(\gamma\) as long as there is a corresponding \(p\) that validates both (28) and (29). A valid selection of such \(p\) for given \(\{\gamma_{i,j}\}_{i \neq j \in [n]}\) is to set \(p_j = \sum_{i \in [n]} \gamma_{i,j}\), which reduces (29) to a consequence of (28), i.e.,

\[
\sum_{i \in \mathcal{E}} m_i \leq \sum_{j \in \mathcal{E}} \sum_{i \in \mathcal{E}} \gamma_{i,j}.
\]

(39)

As a result, by following an analogous proof to that used in Lemma 5, Linear Programming 3 can also be solved through the following equivalent setup.

\[\text{Linear Programming 4. To minimize } \gamma = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j} \text{ subject to } (28) \text{ among all } \mathcal{E} \subset [n] \text{ with } |\mathcal{E}| = n - k.\]

The solution of Linear Programming 4 is then given in the following theorem.

\[\text{Theorem 3. (Minimum update bandwidth) The minimum update bandwidth determined through Linear Programming 4 is given by}\]

\[
\gamma_{\min} = \frac{B}{n} + \frac{(n - k - 1)}{n} \sum_{i \in [n]} \left\lceil \frac{m_i}{k} \right\rceil.
\]

(40)

Under \(k < n - 1\), the minimum update bandwidth can only be achieved by the assignment that satisfies for every \(i \in [n]\),

\[
\sum_{u \in [w_i]} \gamma_{i,j_u(i)} = w_i \left\lceil \frac{m_i}{k} \right\rceil, \text{ and } \gamma_{i,j_u(i)} = \left\lceil \frac{m_i}{k} \right\rceil \text{ for } u \in [n - 1] \setminus [w_i],
\]

(41)
that (41) is the only assignment that can achieve γ update bandwidth no less than the \( k \) separately the situation of proof.

**Proof.** The proof is divided into four steps. First, we show all choices of \( \{\gamma_{i,j}\}_{i \neq j \in [n]} \) satisfying (28) yield an update bandwidth no less than the \( \gamma_{\min} \) given in (40). Second, we verify (41) can achieve \( \gamma_{\min} \). Third, we prove that (41) is the only assignment that can achieve \( \gamma_{\min} \) under \( k < n - 1 \). Last, we complete the proof by considering separately the situation of \( k = n - 1 \).

**Step 1.** Fix a set of \( \{\gamma_{i,j}\}_{i \neq j \in [n]} \) satisfying (28). Since (28) holds for arbitrary \( \mathcal{E} \), we can let \( \bar{\mathcal{E}} = \{j_1(i), \ldots, j_k(i)\} \) and obtain

\[
\sum_{u \in [k]} \gamma_{i,j_u(i)} \geq m_i. \tag{44}
\]

Noting that \( \{\gamma_{i,j_u(i)}\}_{u \in [n-1]} \) is in ascending order (cf. (42)), and that \( \gamma_{i,j} \) is a non-negative integer, we obtain from (44) that

\[
\gamma_{i,j_u(i)} \geq \left\lceil \frac{m_i}{k} \right\rceil. \tag{45}
\]

We continue to derive

\[
\sum_{j \in [n]\setminus\{i\}} \gamma_{i,j} = \sum_{u \in [k]} \gamma_{i,j_u(i)} + \sum_{u \in [n-1]\setminus[k]} \gamma_{i,j_u(i)} \geq \sum_{u \in [k]} \gamma_{i,j_u(i)} + (n-k-1)\gamma_{i,j_u(i)}. \tag{46}
\]

Combining (44), (45) and (46) gives

\[
\sum_{j \in [n]\setminus\{i\}} \gamma_{i,j} \geq m_i + (n-k-1)\left\lceil \frac{m_i}{k} \right\rceil, \tag{47}
\]

which implies

\[
\gamma = \frac{1}{n} \sum_{i \in [n]} \sum_{j \in [n]\setminus\{i\}} \gamma_{i,j} \geq \frac{B}{n} + \frac{(n-k-1)}{n} \sum_{i \in [n]} \left\lceil \frac{m_i}{k} \right\rceil = \gamma_{\min}. \tag{48}
\]

**Step 2.** Next, we confirm (41) is a valid choice of \( \{\gamma_{i,j}\}_{i \neq j \in [n]} \) that achieves \( \gamma_{\min} \). The validity of (28) can be confirmed by

\[
w_i = k \left\lceil \frac{m_i}{k} \right\rceil - m_i < k \left( \left\lceil \frac{m_i}{k} \right\rceil + 1 \right) - m_i = k \tag{49}
\]

and

\[
\sum_{j \in \bar{\mathcal{E}}} \gamma_{i,j} \geq \sum_{i \in [k]} \gamma_{i,j_u(i)} = w_i \left\lceil \frac{m_i}{k} \right\rceil + (k-w_i)\left\lceil \frac{m_i}{k} \right\rceil \tag{50}
\]

\[
= k \left\lceil \frac{m_i}{k} \right\rceil - \left( k \left\lceil \frac{m_i}{k} \right\rceil - m_i \right) \left( \left\lceil \frac{m_i}{k} \right\rceil - \left\lfloor \frac{m_i}{k} \right\rfloor \right) \tag{51}
\]

\[
= \begin{cases} 
    k \left\lceil \frac{m_i}{k} \right\rceil - 0, & k \mid m_i \\
    k \left\lceil \frac{m_i}{k} \right\rceil - (k \left\lceil \frac{m_i}{k} \right\rceil - m_i), & k \nmid m_i 
\end{cases} \tag{52}
\]

\[
= m_i. \tag{53}
\]
Hence, we can derive based on (42) and (53) that
\[
\sum_{j \in [n] \setminus \{i\}} \gamma_{i,j} = \sum_{u \in [k]} \gamma_{i,j,u} + \sum_{u \in [n-1] \setminus [k]} \gamma_{i,j,u} = m_i + (n - k - 1) \left\lceil \frac{m_i}{k} \right\rceil,
\]
which immediately gives
\[
\gamma = \frac{1}{n} \sum_{i \in [n]} \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j} = \frac{B}{n} + \frac{(n - k - 1)}{n} \sum_{i \in [n]} \left\lfloor \frac{m_i}{k} \right\rfloor = \gamma_{\text{min}}.
\]

**Step 3.** It remains to show by contradiction that no other assignment of \( \{ \gamma_{i,j} \}_{i \neq j \in [n]} \) can achieve \( \gamma_{\text{min}} \). The task will be done under \( k < n - 1 \) in this step. The situation of \( k = n - 1 \) will be separately considered in the next step.

Suppose \( \{ \gamma_{i,j}' \}_{i \neq j \in [n]} \) also achieves \( \gamma_{\text{min}} \). We then differentiate among four cases.

**Case 1:** If there is \( i' \in [n] \) such that \( \gamma_{i',j_{w_{i'}+1}(i')} > \left\lceil \frac{m_{i'}}{k} \right\rceil \), then we obtain from (42) and (49) that \( \gamma_{i',j_k(i')} > \left\lceil \frac{m_{i'}}{k} \right\rceil \), which together with (44) implies
\[
\sum_{j \in [n] \setminus \{i'\}} \gamma_{i',j} = \sum_{u \in [k]} \gamma_{i',j,u(i')} + \sum_{u \in [n-1] \setminus [k]} \gamma_{i',j,u(i')} > m_{i'} + (n - k - 1) \left\lfloor \frac{m_{i'}}{k} \right\rfloor.
\]

Because \( \{ \gamma_{i,j}' \}_{i \neq j \in [n]} \) fulfills (28) and hence validates (47), we have
\[
\gamma = \frac{1}{n} \sum_{i \in [n]} \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j}'
\]
\[
= \frac{1}{n} \left( \sum_{j \in [n] \setminus \{i'\}} \gamma_{i',j} + \sum_{i \in [n] \setminus \{i'\}} \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j} \right)
\]
\[
> \frac{1}{n} \left( \left( m_{i'} + (n - k - 1) \left\lceil \frac{m_{i'}}{k} \right\rceil \right) + \sum_{i \in [n] \setminus \{i'\}} \left( m_i + (n - k - 1) \left\lfloor \frac{m_i}{k} \right\rfloor \right) \right)
\]
\[
= \gamma_{\text{min}},
\]
contradicting to the assumption of \( \{ \gamma_{i,j}' \}_{i \neq j \in [n]} \) achieving \( \gamma_{\text{min}} \).

**Case 2:** If there is \( i' \in [n] \) such that \( \gamma_{i',j_{w_{i'}+1}(i')} < \left\lceil \frac{m_{i'}}{k} \right\rceil \) and \( w_{i'} + 1 = k \), then we can infer from (42) that
\[
\gamma_{i',j_{w_{i'}+1}(i')} \leq \left\lceil \frac{m_{i'}}{k} \right\rceil,
\]
and hence
\[
\sum_{u \in [k]} \gamma_{i',j_u(i')} = \sum_{u \in [w_{i'}]} \gamma_{i',j_u(i')} + \gamma_{i',j_{w_{i'}+1}(i')} + \sum_{u \in [k] \setminus [w_{i'}+1]} \gamma_{i',j_u(i')}
\]
\[
< w_{i'} \left\lceil \frac{m_{i'}}{k} \right\rceil + \left\lfloor \frac{m_{i'}}{k} \right\rfloor + \left( k - w_{i'} - 1 \right) \left\lfloor \frac{m_{i'}}{k} \right\rfloor = m_{i'}
\]
where the first strict inequality in (63) is due to \( \gamma_{i',j_{w_{i'}+1}(i')} < \left\lceil \frac{m_{i'}}{k} \right\rceil \), and the last equality follows from a similar derivation to (53). The inequality (63) then contradicts to (44).
Case 3: If there is $i' \in [n]$ such that $\gamma'_{i',j_{w_{i'}+1}(i')} < \lceil \frac{m_{i'}}{k} \rceil$ and $w_{i'} + 1 < k$, then we must have $\gamma'_{i',j_k(i')} > \lceil \frac{m_{i'}}{k} \rceil$. This is because in case $\gamma'_{i',j_k(i')} \leq \lceil \frac{m_{i'}}{k} \rceil$ under $\gamma'_{i',j_{w_{i'}+1}(i')} < \lceil \frac{m_{i'}}{k} \rceil$ and $w_{i'} + 1 < k$, we can obtain from [42] that
\[
\gamma'_{i',j_{w_{i'}+1}(i')} \leq \lceil \frac{m_{i'}}{k} \rceil,
\]
and hence a similar derivation to [63] gives
\[
\sum_{u \in [k]} \gamma'_{i',j_u(i')} = \sum_{u \in [w_{i'}]} \gamma'_{i',j_u(i')} + \sum_{u \in [k] \setminus [w_{i'} + 1]} \gamma'_{i',j_u(i')} < w_{i'} \left\lfloor \frac{m_{i'}}{k} \right\rfloor + \left( k - w_{i'} - 1 \right) \left\lceil \frac{m_{i'}}{k} \right\rceil = m_{i'},
\]
The inequality [66] then contradicts to [44], and therefore $\gamma'_{i',j_k(i')} > \lceil \frac{m_{i'}}{k} \rceil$. We continue to derive based on [44] that
\[
\sum_{j \in [n] \setminus \{i'\}} \gamma'_{i',j} = \sum_{u \in [k]} \gamma'_{i',j_u(i')} + \sum_{u \in [n-1] \setminus [k]} \gamma'_{i',j_u(i')} > m_{i'} + (n - k - 1) \left\lceil \frac{m_{i'}}{k} \right\rceil,
\]
based on which the same contradiction as [59] can be resulted.

Case 4: The previous three cases indicate that $\gamma'_{i',j_{w_{i'}+1}(i')} = \lceil \frac{m_{i'}}{k} \rceil$ for all $i' \in [n]$. Now if there is $i' \in [n]$ and $w_{i'} < u' \leq n - 1$ such that $\gamma'_{i',j_{u'}(i')} < \gamma'_{i',j_{w_{i'}+1}(i')}$, then we again use [44] to obtain
\[
\sum_{j \in [n] \setminus \{i'\}} \gamma'_{i',j} = \sum_{u \in [k]} \gamma'_{i',j_u(i')} + \sum_{u \in [n-1] \setminus [k]} \gamma'_{i',j_u(i')} > m_{i'} + (n - k - 1) \left\lceil \frac{m_{i'}}{k} \right\rceil,
\]
based on which the same contradiction as [59] can, again, be resulted.

The above four cases conclude that $\gamma'_{i,j_n(i)} = \lceil \frac{m_i}{k} \rceil$ for $u \in [n - 1] \setminus [w_i]$ and $i \in [n]$. Finally, [47] implies
\[
\sum_{j \in [n] \setminus \{i\}} \gamma'_{i,j} = \sum_{u \in [w_i]} \gamma'_{i,j_u(i)} + (n - w_i - 1) \left\lceil \frac{m_i}{k} \right\rceil \geq m_i + (n - k - 1) \left\lceil \frac{m_i}{k} \right\rceil.
\]
Since the sum of the left-hand-side of [70] is equal to the sum of the right-hand-side of [70], which is exactly $\gamma_{\min}$, we must have
\[
\sum_{u \in [w_i]} \gamma'_{i,j_u(i)} + (n - w_i - 1) \left\lceil \frac{m_i}{k} \right\rceil = m_i + (n - k - 1) \left\lceil \frac{m_i}{k} \right\rceil,
\]
which in turn gives
\[
\sum_{u \in [w_i]} \gamma'_{i,j_u(i)} = m_i + (w_i - k) \left\lceil \frac{m_i}{k} \right\rceil = w_i \left\lceil \frac{m_i}{k} \right\rceil,
\]
where the last equality can be confirmed similarly as [53].

Step 4. Last, we prove [43]. Note that the proofs in Steps 1 and 2 remain valid under $k = n - 1$, but some derivations in Step 3, e.g., [56], may not be applied when $k = n - 1$. In fact, when $k = n - 1$, a

\[\gamma_{i,j} = \begin{cases} 5, & (i, j) \in \{(1, 3), (2, 3), (3, 2)\} \\ 0, & (i, j) \in \{(1, 2), (2, 1), (3, 1)\} \end{cases}\]
larger class of assignments on $\{\gamma_{i,j}\}_{i \neq j \in [n]}$ can achieve $\gamma_{\text{min}}$. We show (43) by contradiction. Suppose $\{\gamma'_{i,j}\}_{i \neq j \in [n]}$ achieves $\gamma_{\text{min}}$ but satisfies $\sum_{j \in [n] \setminus \{i'\}} \gamma'_{i',j} > m_i$ for some $i' \in [n]$. Then,

$$\gamma = \frac{1}{n} \left( \sum_{i \in [n] \setminus \{i'\}} \sum_{j \in [n] \setminus \{i\}} \gamma'_{i,j} + \sum_{j \in [n] \setminus \{i'\}} \gamma'_{i',j} \right)$$

(74)

$$> \frac{1}{n} \left( \sum_{j \in [n] \setminus \{i'\}} m_i + m_{i'} \right)$$

(75)

$$= \frac{B}{n} = \gamma_{\text{min}},$$

(76)

which leads to a contradiction.

\[ \square \]

C. Determination of the smallest code redundancy subject to $\gamma = \gamma_{\text{min}}$

In Theorem 3, the class of optimal $\{\gamma_{i,j}\}_{i \neq j \in [n]}$ that achieve $\gamma_{\text{min}}$ is also determined. In particular, when $k < n - 1$ and $k \mid m_i$ for every $i$, we have that

$$\gamma_{i,j} = \frac{m_i}{k} \quad \forall i \neq j \in [n]$$

(77)

uniquely achieves $\gamma_{\text{min}}$. This facilitates our finding the smallest code redundancy attainable subject to $\gamma = \gamma_{\text{min}}$ as formulated in Linear Programming 5 below.

**Linear Programming 5.** To minimize $R = \sum_{i=1}^{n} p_i$ subject to (29) and (41) among all $E \subset [n]$ with $|E| = n - k$, provided $1 \leq k < n - 1$ and $k \mid m_i$ for all $i \in [n]$.

**Theorem 4.** The solution of Linear Programming 5 is given by

$$R_{\text{sma}} \triangleq \frac{(n-1)}{k} \sum_{i=1}^{n-k} m_i + \frac{(n-k)}{k} m_{n-k+1},$$

(78)

where by following the assumption in (31), $m_i$ is the $i$-th largest element in vector $\mathbf{m}$. The smallest code redundancy subject to $\gamma = \gamma_{\text{min}}$ is uniquely achieved by

$$p_j = \begin{cases} 
\frac{1}{k} \sum_{i=1}^{n-k+1} m_i - m_j & \text{for } 1 \leq j \leq n - k, \\
\frac{1}{k} \sum_{i=1}^{n-k} m_i & \text{for } n - k < j \leq n.
\end{cases}$$

(79)

**Proof.** We first prove by contradiction that

$$p_j \geq \sum_{i \in E} \gamma_{i,j} = \frac{1}{k} \sum_{i \in E} m_i \quad \forall E \text{ and } \forall j \in \bar{E}. $$

(80)

Suppose there are $E \subset [n]$ with $|E| = n - k$ and $j' \in \bar{E}$ such that

$$p_{j'} < \sum_{i \in E} \gamma_{i,j'} = \frac{1}{k} \sum_{i \in E} m_i. $$

(81)

can also achieve $\gamma_{\text{min}} = 5$. This justifies our separate consideration of the case of $k = n - 1$. 

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Then, (29) results in a contradiction as follows:

\[
\sum_{i \in E} m_i \leq \min \left\{ p_j', \sum_{i \in E} \gamma_{i,j'} \right\} + \sum_{j \in E \setminus \{j'\}} \min \left\{ p_j, \sum_{i \in E} \gamma_{i,j} \right\}
\]

(82)

\[
< \sum_{j \in E} \sum_{i \in E} \gamma_{i,j}
\]

(83)

\[
= \sum_{j \in E} \sum_{i \in E} \frac{m_i}{k} = \sum_{i \in E} m_i,
\]

(84)

where (84) follows from (77). Thus, (80) holds for arbitrary \( E \subset [n] \setminus \{j\} \). As a result, we have

\[
p_j \geq \max_{E \subset [n] \setminus \{j\} \mid |E|=n-k} \frac{1}{k} \sum_{i \in E} m_i = \frac{1}{k} \left( \sum_{i=1}^{n-k+1} m_i - m_j \right) \quad \text{for } 1 \leq j \leq n-k,
\]

(85)

\[
p_j \geq \frac{(n-1)}{k} \sum_{i=1}^{n-k} m_i + \frac{(n-k)}{k} m_{n-k+1} = R_{\text{sma}}.
\]

(86)

Since any \( \{p_j\}_{j \in [n]} \) that satisfies (85) with strict inequality for some \( j \in [n] \) cannot achieve \( R_{\text{sma}} \), the smallest code redundancy subject to \( \gamma = \gamma_{\text{min}} \) is uniquely achieved by the one that fulfills (85) with equality.

The contradiction proof in (84) requires \( \sum_{j \in E} \gamma_{i,j} = m_i \), which is guaranteed by (41) when \( k \mid m_i \) for all \( i \in [n] \). However, without \( k \mid m_i \) for all \( i \in [n] \), the \( \sum_{j \in E} \gamma_{i,j} \) in (41) may not achieve \( m_i \) but generally lies between \( m_i \) and \( k \left\lfloor \frac{m_i}{k} \right\rfloor \). Our preliminary study indicates that the general formula of \( R_{\text{sma}} \) for arbitrary \( k < n-1 \) and arbitrary \( m \) does not seem to have a simple expression but depends on the pattern of \( w = [w_1 \ w_2 \ \cdots \ w_n]^T \).

Theorem 5 only deals with \( w = [0 \ 0 \ \cdots \ 0]^T \). The establishment of the smallest code redundancy for cases that allow \( k \mid m_i \) is left as a future research.

Surprisingly, in the particular case of \( k = n-1 \), we found \( R_{\text{sma}} = R_{\text{min}} \) due to the fact that \( \sum_{j \in E} \gamma_{i,j} = m_i \) is guaranteed by (43).

**Linear Programming 6.** To minimize \( R = \sum_{i=1}^{n} p_i \) subject to (29) and (43) among all \( E \subset [n] \) with \( |E| = n-k \), provided \( k = n-1 \).

**Theorem 5.** The solution of Linear Programming 6 is given by the \( R_{\text{min}} \) in (35), which can only be achieved by those \( p \)'s satisfying (36).

**Proof.** It suffices to prove that Linear Programming 2 and Linear Programming 6 are equivalent under \( n-k = 1 \).

We first note that under \( n-k = 1 \), all feasible \( p \) and \( \{\gamma_{i,j}\}_{i \neq j \in [n]} \) satisfying (29) and (43), i.e.,

\[
m_i = \sum_{j \in [n] \setminus \{i\}} \gamma_{i,j} \leq \sum_{j \in [n] \setminus \{i\}} \min\{p_j, \gamma_{i,j}\} \quad \forall i \in [n],
\]

(87)

must validate (33), i.e.,

\[
m_i \leq \sum_{j \in [n] \setminus \{i\}} p_j \quad \forall i \in [n].
\]

(88)

On the contrary, for every \( p \) that fulfills (88), we can always construct \( \{\gamma_{i,j}\}_{i \neq j \in [n]} \) with \( \gamma_{i,j} \leq p_j \) such that (87) holds. Thus, Linear Programming 2 is equivalent to Linear Programming 6.
V. EXPLICIT CONSTRUCTIONS OF MUB AND MR-MUB CODES

A. MR-MUB and MUB codes

Based on the previous section, we can now define two particular classes of irregular array codes.

**Definition 1.** A Minimal Update Bandwidth (MUB) code is an \((n, k, m)\) irregular array code with update bandwidth equal to \(\gamma_{\text{min}}\).

**Definition 2.** A Minimum Redundancy and Minimum Update Bandwidth (MR-MUB) code is an \((n, k, m)\) irregular array code, of which the code redundancy and the update bandwidth are equal to \(R_{\text{min}}\) and \(\gamma_{\text{min}}\), respectively.

Note that the existence of MR-MUB codes for certain parameters \(n, k\) and \(m\) is not guaranteed. In certain cases, we can only have \(R_{\text{sma}} > R_{\text{min}}\), i.e., the smallest code redundancy subject to \(\gamma = \gamma_{\text{min}}\) is strictly larger than the minimum code redundancy among all irregular array codes. An example is given in Fig. 3 where we can obtain from (78) that the smallest code redundancy of \((4, 2, m = [4 \ 2 \ 2 \ 0]^T)\) irregular array codes is equal to

\[
R_{\text{sma}} = \frac{3}{2} \sum_{i=1}^{2} m_i + \frac{2}{2} m_3 = \frac{3}{2} (4 + 2) + 2 = 11,
\]

while the minimum code redundancy in (35) is given by

\[
R_{\text{min}} = ([4 - 4]_+ + 4) + ([4 - 2]_+ + 2) = 8.
\]

It can be verified that the code redundancy of the irregular array code in Fig. 3 achieves \(p_1 + p_2 + p_3 + p_4 = 11 = R_{\text{sma}}\).

To confirm that the code in Fig. 3 is an MUB code, we note that the update of the first node has to send \(\Delta x_{1,1}\) to node 2, \(\Delta x_{1,3}\) and \(\Delta x_{1,4}\) to node 3, \((\Delta x_{1,1} + \Delta x_{1,3})\) and \((\Delta x_{1,2} + \Delta x_{1,4})\) to node 4, respectively. Thus, the required update bandwidth for node 1 is 6. Similarly, we can verify that the required bandwidths of the second, the third and the fourth nodes are 3, 3 and 0, respectively. As a result, \(\gamma = \frac{1}{4}(6 + 3 + 3 + 0) = 3\), which equals \(\gamma_{\text{min}}\) in (40).

Two particular situations, which guarantee the existence of MR-MUB codes, are \(k = 1\) and \(k = n - 1\). In the former situation, we can obtain from (78) and (35) that

\[
R_{\text{sma}} = R_{\text{min}} = \frac{(n - k)}{k} B = \frac{(n - k)}{k} \sum_{i \in [n]} m_i,
\]

while the latter has been proven in Theorem 5. For \(1 < k < n - 1\), however, it is interesting to find that an MR-MUB code exists only when \(m\) is either an extremely balanced all-equal vector or an extremely unbalanced all-zero-but-one vector, which is proven in the next theorem under \(k \mid m_i\) for all \(i \in [n]\).

**Theorem 6.** Under \(1 < k < n - 1\) and \(k \mid m_i\) for all \(i \in [n]\), \((n, k, m)\) MR-MUB codes exist if, and only if, one of the two situations occurs:

\[
\begin{cases}
  m_i = \frac{B}{n} \forall i \in [n], \\
  p_j = \frac{(n-k)}{nk} B \forall j \in [n].
\end{cases}
\]
We then distinguish between two cases:

\[
\begin{align*}
    m_1 &= B, \text{ and } m_i = 0 \text{ for } 2 \leq i \leq n, \\
p_1 &= 0, \text{ and } p_j = \frac{B}{k} \text{ for } 2 \leq j \leq n.
\end{align*}
\] (93)

In either situation, \( \{\gamma_{i,j}\}_{i \neq j \in [n]} \) follows from (71).

**Proof.** The theorem can be proved by simply equating the two \( p_1 \)'s that respectively achieve \( R_{\text{min}} \) and \( R_{\text{sma}} \). Specifically, (36) indicates that \( R_{\text{min}} \) is achieved by \( p_1 = [\mu - m_1]^+ \), where \( \mu \) is given in (34). From (79), \( R_{\text{sma}} \) is reached when

\[
p_1 = \frac{1}{k} \left( \sum_{i=2}^{n-k+1} m_i - m_1 \right) = \frac{1}{k} \sum_{i=2}^{n-k+1} m_i.
\] (94)

We thus have

\[
p_1 = [\mu - m_1]^+ = \frac{1}{k} \sum_{i=2}^{n-k+1} m_i.
\] (95)

We then distinguish between two cases: \( p_1 = 0 \) and \( p_1 > 0 \).

Consider \( p_1 = \frac{1}{k} \sum_{i=2}^{n-k+1} m_i = 0 \), which from (31), immediately leads to \( m_1 = B \) and \( m_2 = m_3 = \cdots = m_n = 0 \). Thus, we obtain from (34) and (36) that \( \mu = \frac{B}{k} \) and \( p_j = \frac{B}{k} \) for \( 2 \leq j \leq n \). As anticipated, this \( p \) also satisfies (79) and validates \( R_{\text{min}} = R_{\text{sma}} \).

Next, we consider \( p_1 = [\mu - m_1]^+ > 0 \), which leads to \( \mu > m_1 \). As \( m_{n-k} \leq m_1 \) from (34), we have \( \mu = \frac{B}{k} > m_1 \). Thus, (95) becomes

\[
B \frac{1}{k} - m_1 = \frac{1}{k} \sum_{i=2}^{n-k+1} m_i,
\] (96)

which implies

\[
(k-1)m_1 = m_{n-k+2} + \cdots + m_n.
\] (97)

We can then conclude from (31) that \( m_1 = m_2 = \cdots = m_n \). The verification of \( R_{\text{min}} = R_{\text{sma}} \) straightforwardly follows.

In practice, it may be unusual to place all data symbols in one node. Thus, we will focus on the construction of MR-MUB codes that follows (92) in the next subsection. In other words, the \( (n,k,m = [m m \cdots m]^T) \) MR-MUB codes considered in the rest of the paper are \( (n,k) \) vertical MDS array codes with each node containing \( m \) data symbols and \( p = \frac{(n-k)m}{k} \) parity symbols subject to \( k \mid m \).

Note that Theorem 6 seems limited in its applicability since (92) simply shows vertical MDS codes can achieve both \( R_{\text{min}} \) and \( \gamma_{\text{min}} \) under a particular case of \( k \mid m \). However, without the condition of \( k \mid m \), vertical MDS array codes may not form a sub-class of MR-MUB codes. This can be justified by two observations. First, it can be verified from (36) that the fulfillment of both \( p_i = [\mu - m_i]^+ \) for \( 1 \leq i \leq n-k \) and \( \sum_{i=n-k+1}^{n} p_i = \sum_{i=1}^{n-k} m_i \) under each \( p_i = p \) and each \( m_i = m \) requires \( k \mid nm \). Thus, under \( k \nmid nm \), vertical MDS array codes cannot achieve the minimum code redundancy, and hence cannot be MR-MUB codes. Second, when \( k \mid nm \) but \( k \nmid m \), examples and counterexamples for vertical MDS array codes being able to achieve simultaneously \( R_{\text{min}} \) and \( \gamma_{\text{min}} \).
can both be constructed. Hence, we conjecture that $k \mid m$ is also a necessary condition for vertical MDS array codes being MUB codes, provided $k \nmid n$.

Theorem 6 only deals with the situation of $1 < k < n - 1$. For completeness, the next corollary incorporates also the two particular cases of $k = 1$ and $k = n - 1$.

**Corollary 2.** Under $1 \leq k < n$ and $k \mid m$, an $(n, k, m1)$ MR-MUB code must parameterize with

$$p_j = \frac{(n-k)}{k}m \neq p \ \forall j \in [n],$$

(100)

$$\gamma_{i,j} = \frac{m}{k} \ \forall i \neq j \in [n],$$

(101)

where $1 \equiv [1 \ 1 \ \cdots \ 1]^T$ is the all-one vector.

**Proof.** We only substantiate the corollary for $k = 1$ and $k = n - 1$ since the situation of $1 < k < n - 1$ have been proved in Theorem 6. The validity of (100) under $k = 1$ and $k = n - 1$ can be confirmed by (38). We can also obtain from (77) that (101) holds under $k = 1$. It remains to verify (101) under $k = n - 1$ by contradiction.

Fix $k = n - 1$. Suppose there is a $j' \in [n] \setminus \{i\}$ such that $\gamma_{i,j'} < \frac{m}{k} = p_{j'}$. A contradiction can be established from (29) as follows:

$$m = m_i \leq \sum_{j \in [n] \setminus \{i\}} \min \{p_j, \gamma_{i,j}\} \leq \sum_{j \notin [n] \setminus \{i,j'\}} p_j + \gamma_{i,j'} < \sum_{j \notin [n] \setminus \{i\}} p_j = m.$$

(102)

Accordingly, $\gamma_{i,j} \geq \frac{m}{k}$ for all $i \neq j \in [n]$, which implies

$$\sum_{j \in [n] \setminus \{i\}} \gamma_{i,j} \geq \frac{m}{k}(n - 1) = m.$$

(103)

By noting from (43) that the inequality in (103) must be replaced by an equality, (101) holds under $k = n - 1$. 

**B. Construction of MR-MUB codes**

For the construction of $(n, k, m1)$ MR-MUB code, denoted as $C_0$ for convenience, we require $x_i \in \mathbb{F}_q^m$ and $p_j \in \mathbb{F}_q^p$ with $p = \frac{(n-k)}{k}m$ for $i, j \in [n]$. The construction of $\{A_{i,j}\}_{i \neq j \in [n]}$ and $\{B_{i,j}\}_{i \neq j \in [n]}$ associated with $C_0$ are then addressed as follows.

A supporting example follows when $n = 6$, $k = 3$ and $m = 4$, where setting $p_i = p = 4$ and

$$\gamma_{i,j} = \begin{cases} 1, & j \in \{(i \mod 6) + 1, [(i + 1) \mod 6] + 1\} \\ 2, & \text{otherwise} \end{cases}$$

(98)

for $i \neq j \in [n]$ fulfills both (29) and (29), and achieves simultaneously $R_{\text{min}}$ and $\gamma_{\text{min}}$.

A counterexample exists when $n = 9$, $k = 6$ and $m = 2$. From (36), we know $R_{\text{min}}$ can only be achieved by adopting $p_j = 1$ for $j \in [n]$. By (11), the achievability of $\gamma_{\text{min}}$ requires $\gamma_{i,j_1} = \gamma_{i,j_2} = \gamma_{i,j_3} = \gamma_{i,j_4} = 0$ and $\gamma_{i,j_5} = \gamma_{i,j_6} = \gamma_{i,j_7} = \gamma_{i,j_8} = 1$ for every $i \in [n]$. Then, the pigeon hole principle implies that there is $j'$ such that $\{\gamma_{i,j'}\}_{i \in [n] \setminus \{j'\}}$ contains at least four $0$'s. Let $\gamma_{i,j'} = \gamma_{i,j'} = \gamma_{i,j'} = \gamma_{i,j'} = 0$ and $E = \{i_1, i_2, i_3\}$, where $j' \notin \{i_1, i_2, i_3, i_4\}$. A violation to (29) can thus be obtained as follows:

$$\sum_{i \in E} m_i = |E|m = 6$$

$$\sum_{j \in \hat{E}} \min \{p_j, \sum_{i \in E} \gamma_{i,j}\} = \min \{p_{j'}, \sum_{i \in E} \gamma_{i,j'}\} + \sum_{j \in \hat{E} \setminus \{j'\}} \min \{p_j, \sum_{i \in E} \gamma_{i,j}\} \leq 5.$$ 

(99)

Consequently, $(9, 6)$ vertical MDS array codes with each node having $m = 2$ data symbols cannot be MR-MUB codes.
First, we construct \( \{ A_{i,j} \}_{i \neq j \in [n]} \) of dimension \( \frac{n}{k} \times m \). Choose an \((n-1,k)\) MDS array code \( \mathcal{M} \) over \( \mathbb{F}_q \) with encoding function \( F : \mathbb{F}_q^{m} \rightarrow \mathbb{F}_q^{\frac{n}{k} \times (n-1)} \), where \( \frac{n}{k} \) is the number of rows of the MDS array code, and \( m \) is the number of data symbols in each row. As an example, \( \mathcal{M} \) can be a Reed-Solomon (RS) code subject to \( q \geq n-1 \). Denote \( F_i \equiv F(x_i) \). Then, \( \{ p_{i,j} \}_{i \neq j \in [n]} \) defined in [13], as well as \( \{ A_{i,j} \}_{i \neq j \in [n]} \), can be characterized via

\[
P_{i,(i+j-1) \text{ mod } n} + 1 = A_{i,(i+j-1) \text{ mod } n} x_i = (F_i)_j \quad \forall j \in [n],
\]

(104)

where \((F_i)_j \) is the \( j \)-th column of the matrix \( F_i \). This indicates that

\[
F_i = [p_{i,i+1} \ldots p_{i,n} p_{i,1} \ldots p_{i,i-1}] \quad \forall i \in [n].
\]

(105)

Next, we construct \( \{ B_{i,j} \}_{i \neq j \in [n]} \) of dimension \( p \times \frac{n}{k} \). Choose a \( p \times \frac{(n-1)m}{k} \) matrix \( V \) over \( \mathbb{F}_q \) such that arbitrary selection of \( p \) columns of \( V \) form an invertible matrix. For example, \( V \) can be a Vandermonde matrix subject to \( q \geq \frac{(n-1)m}{k} \). We then let

\[
B_{(i+j-1) \text{ mod } n} + 1 = [(V)_{(i-1) \text{ mod } n} + 1 (V)_{(i-2) \text{ mod } n} + 2 \ldots (V)_{(i-1) \text{ mod } n}] \quad \forall i \in [n-1] \text{ and } j \in [n],
\]

(106)

which implies that

\[
V = \begin{bmatrix} B_{j+1,j} & \ldots & B_{n,j} & B_{1,j} & \ldots & B_{j-1,j} \end{bmatrix}.
\]

(107)

Note that the right-hand-side of (107) remains constant regardless of \( j \in [n] \). Thus, we can obtain from (15) that

\[
P_j = V \begin{bmatrix} p_{j+1,j}^T & \ldots & p_{n,j}^T & p_{1,j}^T & \ldots & p_{j-1,j}^T \end{bmatrix}^T.
\]

(108)

We now prove the code so constructed is an MR-MUB code.

**Theorem 7.** \( C_O \) is an \((n,k,m)\) MR-MUB code.

**Proof.** The proof requires verifying two properties, which are i) \( C_O \) being an \((n,k,m)\) array code, and ii) \( C_O \) achieving \( R_{\text{min}} \) and \( \gamma_{\text{min}} \).

First, we justify i), i.e., \( C_O \) satisfying that given any set \( E \subset [n] \) with \( |E| = n - k \), the codeword \( C \) of \( C_O \) can be reconstructed from \( C_E \). When \( C_E \) is given, both \( X_E \) and \( P_E \) are known, and so are \( \{ p_{e_i,e_j} \}_{i \neq j \in [k]} \) according to (13). We can then establish from (15) that

\[
p_{e_j} - \sum_{i=1, i \neq j}^{k} B_{e_i,e_j} p_{e_i,e_j} = \sum_{i=1}^{n-k} B_{e_i,e_j} p_{e_i,e_j} = \begin{bmatrix} e_1,e_j & \ldots & B_{e_{n-k},e_j} \end{bmatrix} \begin{bmatrix} p_{e_1,e_j} \\ \vdots \\ p_{e_{n-k},e_j} \end{bmatrix} \quad \forall j \in [k],
\]

(109)

Since \( p_{e_j} - \sum_{i=1, i \neq j}^{k} B_{e_i,e_j} p_{e_i,e_j} \) is known and any \( p \) columns of \( V \), as defined in (107), forms an invertible matrix, we can obtain \( \{ p_{e_i,e_j} \}_{i \in [n-k], j \in [k]} \) by left-multiplying (109) by \( [B_{e_1,e_j} \ldots B_{e_{n-k},e_j}]^{-1} \). With the knowledge of \( k \) columns \( \{ p_{e_i,e_j} \}_{j \in [k]} \) of \( F_{e_i} \) in (105), we can recover \( x_{e_i} \) via the decoding algorithm of the \((n-1,k)\) MDS array code \( \mathcal{M} \). By this procedure, \( \{ x_i \}_{i \in [n]} \) can all be recovered.

Next, we verify ii). From (105), we have \( p_{i,j} \in \mathbb{F}_q^m \) and hence \( \gamma_{i,j} = \frac{m}{k} \), which leads to \( \gamma = \gamma_{\text{min}} \) as pointed out in (101). In addition, (108) shows \( p_j = \frac{(n-k)m}{k} \) for \( j \in [n] \), and hence \( R_{\text{min}} \) is achieved as addressed in (109). The justification of the two required properties of \( C_O \) is thus completed. \( \square \)
The \((4, 2, 2)\) MR-MUB code in Fig. 2(b) can be constructed via the proposed procedure. First, with \(x_i = [x_{i,1} \ x_{i,2}]^T\), \(\mathcal{M}\) is chosen as a \((3, 2)\) parity-check code over \(\mathbb{F}_q\), which gives

\[
\mathbf{F}_i = \begin{bmatrix} x_{i,1} & x_{i,2} & x_{i,1} + x_{i,2} \end{bmatrix} \quad \forall i \in [n].
\] (110)

Thus, from (105), we have

\[
p_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}
\]

which satisfies that the selection of any two columns forms an invertible matrix. By (108), we have

\[
p_1 = \mathbf{V} [\mathbf{p}_{2,1}^T \mathbf{p}_{3,1}^T \mathbf{p}_{4,1}^T] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{2,1} + x_{2,2} & x_{3,2} & x_{4,1} \\ x_{2,1} + x_{2,2} + x_{3,2} \end{bmatrix} = \begin{bmatrix} x_{3,2} + x_{4,1} \\ x_{2,1} + x_{2,2} + x_{3,2} \end{bmatrix}.
\] (112)

\(p_2, p_3\) and \(p_4\) can be similarly obtained and can be found in Fig. 2(b).

We now demonstrate via this example how erased nodes can be systematically recovered based on the chosen \(\mathcal{M}\) and \(\mathbf{V}\). Suppose nodes 1 and 2 are erased. As knowing from (108) that

\[
p_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{4,3} \\ p_{1,3} \\ p_{2,3} \end{bmatrix}.
\] (113)

we perform (109) to obtain

\[
p_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} p_{4,3} \\ p_{1,3} \\ p_{2,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{1,3} \\ p_{2,3} \end{bmatrix}.
\] (114)

Since \(p_3\) is known and \(p_{4,3}\) can be obtained from \(x_4\) via \(p_{4,3} = \mathbf{A}_{4,3} x_4\), we can recover \(p_{1,3}\) and \(p_{2,3}\) via

\[
\begin{bmatrix} p_{1,3} \\ p_{2,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \left( p_3 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} p_{4,3} \right).
\] (115)

The recovery of \(p_{1,4}\) and \(p_{2,4}\) can be similarly done via

\[
\begin{bmatrix} p_{1,4} \\ p_{2,4} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \left( p_4 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} p_{3,4} \right).
\] (116)

| \(\{p_{i,j}\}_{i \neq j \in [n]}\) of the MR-MUB code presented in Fig. 2(b) where the element in the \(i\)-th row and the \(j\)-th column is \(p_{i,j}\). |
|-----------------|-----------------|-----------------|-----------------|
| \(x_1,1\)       | \(x_1,2\)       | \(x_1,1 + x_1,2\) |
| \(x_2,1 + x_2,2\)| \(x_2,1\)       | \(x_2,2\)       |
| \(x_3,1 + x_3,2\)| \(x_3,1\)       | \(x_3,2\)       |
| \(x_4,1\)       | \(x_4,2\)       | \(x_4,1 + x_4,2\) |
We then note from (105) that $F_1 = F(x_1) = [p_{1,2} \ p_{1,3} \ p_{1,4}]$ is a codeword of $\mathcal{M}$, corresponding to $x_1$, and its second and third columns are just recovered via (115) and (116). By equating the second and the third columns of $F_1$ with (110), the recovery of $x_1$ is done. We can similarly recover $x_2$ by using the recovered $p_{2,3}$ and $p_{2,4}$ in (113) and (116). The recovery of the two erased nodes is thus completed.

C. Construction of MUB codes with the smallest code redundancy

We continue to propose a construction of $(n, k, m)$ MUB codes with the smallest code redundancy, and denote the code to be constructed as $C_U$ for notational convenience. This can be considered a generalization of the code construction in the previous subsection.

For the construction of $C_U$, we require $x_i \in \mathbb{F}_q$ and $p_j \in \mathbb{F}_q$ with $\{p_j\}_{j \in [n]}$ specified in (79) for $i, j \in [n]$. The construction of $\{A_{i,j}\}_{i \neq j \in [n]}$ and $\{B_{i,j}\}_{i \neq j \in [n]}$ associated with $C_U$ are then addressed as follows.

First, for $i \neq j \in [n]$, we construct $A_{i,j}$ of dimension $\frac{m_i}{k} \times m_i$. For each $i \in [n]$, choose an $(n-1, k)$ MDS array code $\mathcal{M}_i$ over $\mathbb{F}_q$ with encoding function $F_i : \mathbb{F}_q^{m_i} \rightarrow \mathbb{F}_q^{\frac{m_i}{k} \times (n-1)}$, where $\frac{m_i}{k}$ is the number of rows of the MDS array code, and $m_i$ is the number of data symbols in each row. Denote $F_i \triangleq F_i(x_i)$. Then, $\{p_{i,j}\}_{i \neq j \in [n]}$ defined in (13), as well as $\{A_{i,j}\}_{i \neq j \in [n]}$, can be characterized via

$$p_{i,[((i+j)-1) \mod n]+1} = A_{i,[((i+j)-1) \mod n]+1} x_i = (F_i)_j \quad \forall j \in [n].$$

This indicates that

$$F_i = [p_{i,i+1} \ldots p_{i,n} \ p_{i,1} \ldots p_{i,i-1}] \quad \forall i \in [n].$$

Next, for $i \neq j \in [n]$, we construct $B_{i,j}$ of dimension $p_j \times \frac{m_j}{k}$. Choose a $p_j \times \sum_{i \in [n] \setminus \{j\}} \frac{m_i}{k}$ matrix $V_j$ over $\mathbb{F}_q$ such that arbitrary selection of $p_j$ columns of $V_j$ form an invertible matrix. We then get $\{B_{i,j}\}_{i \neq j \in [n]}$ from

$$V_j = \begin{bmatrix} B_{j+1,j} & \ldots & B_{n,j} & B_{1,j} & \ldots & B_{j-1,j} \end{bmatrix} \quad \forall j \in [n].$$

Thus, we can obtain from (15) that

$$p_j = V_j \begin{bmatrix} p_{j+1,j}^T \ldots p_{n,j}^T \ p_{1,j}^T \ldots p_{j-1,j}^T \end{bmatrix}^T.$$  

We now prove the code so constructed is an MUB code with the smallest code redundancy.

**Theorem 8.** $C_U$ is an $(n, k, m)$ MUB code with the smallest code redundancy.

**Proof.** Similar to the proof of Theorem 7 the substantiation of this theorem requires verifying two properties: i) $C_U$ is an $(n, k, m)$ array code, and ii) $C_U$ achieves $R_{\text{sm}}$ and $\gamma_{\text{min}}$.

First, we justify i), i.e., $C_U$ satisfying that given any set $E \subset [n]$ with $|E| = n - k$, the codeword $C$ of $C_U$ can be reconstructed from $C_E$. When $C_E$ is given, both $X_E$ and $P_E$ are known, and so are $\{p_{e_i, e_j}\}_{i \neq j \in [k]}$ according to (13). We can then establish from (15) that

$$p_{e_j} = \sum_{i=1, i \neq j}^k B_{e_i, e_j} p_{e_i, e_j} = \begin{bmatrix} B_{e_1, e_j} & \ldots & B_{e_{n-k}, e_j} \end{bmatrix} \begin{bmatrix} p_{e_1, e_j} \\ \vdots \\ p_{e_{n-k}, e_j} \end{bmatrix} \quad \forall j \in [k].$$  

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According to (79), we have
\[
\text{row}(\mathbf{B}_{\bar{e}_i,\bar{e}_j}, \ldots \mathbf{B}_{\bar{e}_{n-k},\bar{e}_j}) = p_j \geq \sum_{i \in [n-k]} \frac{m_{e_i}}{k} = \sum_{i \in [n-k]} \text{col}(\mathbf{B}_{\bar{e}_i,\bar{e}_j}).
\] (122)

Since any \(p_j\) columns of \(\mathbf{V}_j\), as defined in (119), forms an invertible matrix, we obtain from (122) that \([\mathbf{B}_{\bar{e}_i,\bar{e}_j}, \ldots \mathbf{B}_{\bar{e}_{n-k},\bar{e}_j}]\) is of full column rank, and hence \(\{\mathbf{p}_{e_i,\bar{e}_j}\}_{i \in [n-k], j \in [k]}\) can be solved via (121). With the knowledge of \(k\) columns \(\{\mathbf{p}_{e_i,\bar{e}_j}\}_{j \in [k]}\) of \(\mathbf{F}_{e_i}\) in (118), we can recover \(\mathbf{x}_e\) via the decoding algorithm of the \((n-1,k)\) MDS array code \(\mathcal{M}_{e_i}\). By this procedure, \(\{\mathbf{x}_i\}_{i \in [n]}\) can all be recovered.

Next, we verify \(ii\). From (118), we have \(\mathbf{p}_{i,j} \in \mathbb{F}_q^m\) and hence \(\gamma_{i,j} = \frac{m_i}{m}\), which leads to \(\gamma = \gamma_{\text{min}}\) as pointed out in (77). In addition, (120) shows \(\{p_j\}_{j \in [n]}\) follows (79), and hence \(R_{\text{ana}}\) is achieved as addressed in Theorem 4. The justification of the two required properties of \(\mathcal{C}_U\) is thus completed. \(\square\)

We demonstrate that the \((4,2,4) = [4 \ 2 \ 0 \ 0]^T\) MUB code in Fig. 3 can be constructed via the proposed procedure. First, with \(\mathbf{x}_1 = [x_{1,1} \ldots x_{1,4}]^T\), \(\mathcal{M}_1\) is chosen as a \((3,2)\) MDS array code, which encodes \(\mathbf{x}_1\) into
\[
\mathbf{F}_1 = \begin{bmatrix} x_{1,1} & x_{1,3} & x_{1,1} + x_{1,3} \\ x_{1,2} & x_{1,4} & x_{1,2} + x_{1,4} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_{1,2} & \mathbf{p}_{1,3} & \mathbf{p}_{1,4} \end{bmatrix}.
\] (123)

For \(i = 2, 3\), \(\mathcal{M}_i\) is chosen to be a \((3,2)\) parity check code over \(\mathbb{F}_q\), as the one in (110). Since \(m_4 = 0\), \(\{\mathbf{p}_{4,j}\}_{j \in \{4\}}\) are null vectors. The resulting \(\{\mathbf{p}_{i,j}\}_{i \neq j \in [n]}\) are listed in Table III.

Next, we obtain from (79) that \(p_1 = 2\) and \(p_2 = p_3 = p_4 = 3\), and specify
\[
\mathbf{V}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_2 = \mathbf{V}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{V}_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},
\] (124)

where the selection of any \(p_i\) columns from \(\mathbf{V}_i\) forms an invertible matrix. By (120) and Table III, we have
\[
\mathbf{p}_1 = \begin{bmatrix} x_{2,1} + x_{2,2} \\ x_{3,2} \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} x_{1,1} \\ x_{1,2} \\ x_{1,1} + x_{3,2} \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} x_{1,3} \\ x_{1,4} \\ x_{2,1} \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_4 = \begin{bmatrix} x_{1,1} + x_{1,3} + x_{3,1} \\ x_{1,2} + x_{1,4} + x_{3,1} \\ x_{2,2} + x_{3,1} \end{bmatrix},
\] (125)
as presented in Fig. 3.

\begin{table}[h!]
\centering
\caption{\{\(\mathbf{p}_{i,j}\)\}_{i \neq j \in [n]} of the MUB code presented in Fig. 3 where the element in the \(i\)-th row and the \(j\)-th column is \(\mathbf{p}_{i,j}\).}

| \(\mathbf{p}_{i,j}\) | \(x_{1,1}\) | \(x_{1,2}\) | \(x_{1,3}\) | \(x_{1,4}\) | \(x_{2,1}\) | \(x_{2,2}\) | \(x_{3,1}\) | \(x_{3,2}\) |
|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(x_{2,1} + x_{2,2}\) | null | \(x_{2,1}\) | \(x_{2,2}\) | \null | \null | \null | \null |
| \(x_{3,2}\) | \(x_{3,1} + x_{3,2}\) | null | \(x_{3,1}\) | \null | \null | \null | \null |
| \null | \null | \null | \null | \null | \null | \null | \null |
\end{table}
Based on this example, the systematic recovery of erased nodes can be demonstrated as follows. Suppose nodes 1 and 2 are erased. Then, through \((120), (121)\) and \((124)\), we have

\[
p_4 = \begin{bmatrix} p_{1,3} \\ p_{2,3} \end{bmatrix}, \quad \text{and} \quad p_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad p_{5,4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{1,4} \\ p_{2,4} \end{bmatrix}.
\]

We can thus obtain \(p_{1,3}, p_{1,4}, p_{2,3}\) and \(p_{2,4}\). By noting \(p_{1,3} = [x_{1,3} \ x_{1,4}]^T\) and \(p_{1,4} = [x_{1,1} + x_{1,3} \ x_{1,2} + x_{1,4}]^T\), the recovery of \(x_1\) is done via the erasure correcting of \(M_1\). We can similarly recover \(x_2\) from \(p_{2,3}\) and \(p_{2,4}\). The recovery of the two erased nodes is therefore completed.

VI. UPDATE COMPLEXITY OF MR-MUB CODES

The update complexity of an array code, denoted as \(\theta\), is defined as the average number of parity symbols affected by updating a single data symbol \([8]\). For an \((n, k, m)\) irregular array codes, a definition-implied lower bound for update complexity is \(\theta \geq n - k\). This lower bound can be easily justified by contradiction. If \(\theta < n - k\), then at most \((n - k - 1)\) nodes are affected when updating a data symbol, which leads to a contradiction that this data symbol cannot be reconstructed by the remaining \(k\) unaffected nodes.

Previous results on update complexity indicate that the lower bound \(n - k\) is not attainable by \((n, k)\) horizontal MDS array codes with \(1 < k < n - 1\) \([8]\). Later, Xu and Bruck \([10]\) introduced an \((n, k)\) vertical MDS array code that can achieve \(\theta = n - k\). Since the proposed \((n, k, m)\) MR-MUB codes in the previous section are a class of vertical MDS array codes, a query that naturally follows is whether or not the update complexity of MR-MUB codes can reach the definition-implied lower bound. Unfortunately, we found the answer is negative under \(k > 1\), and will show in Theorem 9 that the update complexity of MR-MUB codes is lower-bounded by \(n - k + \frac{k - 1}{k}\).

In order to facilitate the presentation of the result in Theorem 9, five lemmas are addressed first. The first lemma indicates it suffices to consider the MR-MUB codes with \(\{M_{i,i}\}_{i \in [n]}\) being zero matrices; hence, we do not need to consider \(\{M_{i,i}\}_{i \in [n]}\) in the calculation of update complexity (cf. Lemma 3). The second lemma shows that for the determination of a lower bound of update complexity, we can focus on the decomposition of \(M_{i,j} = B'_{i,j}A'_{i,j}\) with \(A'_{i,j}\) containing an \(\gamma_{i,j} \times \gamma_{i,j}\) identity submatrix. As a result, \(B'_{i,j}\) is a submatrix of \(M_{i,j}\) and the column weights of \(M_{i,j}\) are lower-bounded by the column weights of \(B'_{i,j}\) (cf. Lemma 7). The next two lemmas then study the column weights of general \(B_{i,j}\) that is not necessarily a submatrix of \(M_{i,j}\) (cf. Lemmas 8 and 9). The last lemma accounts for the number of non-zero columns in \(M_{i,j}^{(\ell)} \triangleq [(M_{i,1})_\ell \ldots (M_{i,i-1})_\ell \ldots (M_{i,i+1})_\ell \ldots (M_{i,n})_\ell]\), where \((M_{i,j})_\ell\) denotes the \(\ell\)-th column of matrix \(M_{i,j}\).

**Lemma 6.** For any \((n, k, m)\) irregular array code \(C\) with construction matrices \(\{M_{i,j}\}_{i,j \in [n]}\), we can construct another \((n, k, m)\) irregular array code \(C'\) with \(\{M'_{i,i} = [0]\}_{i \in [n]}\) such that both codes have the same code redundancy and update bandwidth.

**Proof.** Let the construction matrices of \(C'\) be defined as

\[
M'_{i,j} = \begin{cases} 
M_{i,j} & i \neq j; \\
0 & i = j.
\end{cases}
\]
Then, there exists an invertible mapping between codewords of $C'$ and $C$, i.e.,

$$ c'_j = \begin{bmatrix} x_j \\ p'_j \end{bmatrix} = \begin{bmatrix} x_j \\ p_j - M_{j,j}x_j \end{bmatrix} = \begin{bmatrix} I & 0 \\ -M_{j,j} & I \end{bmatrix} \begin{bmatrix} x_j \\ p_j \end{bmatrix} = \begin{bmatrix} I & 0 \\ -M_{j,j} & I \end{bmatrix} c_j \quad \text{for } j \in [n], $$

(128)

where $I$ denotes an identity matrix of proper size. A consequence of (128) is that all data symbols can be retrieved by accessing any $k$ columns of the corresponding codeword of $C$ if, and only if, the same can be done by accessing any $k$ columns of the corresponding codeword of $C'$. As $C$ is an $(n, k, m)$ irregular array code, we confirm that $C'$ is also an $(n, k, m)$ irregular array code. Since $\gamma'_{i,j} = \text{rank}(M'_{i,j}) = \text{rank}(M_{i,j}) = \gamma_{i,j}$ with $i \neq j \in [n]$, the update bandwidth of $C'$ remains the same as that of $C$ according to (12). The relation of $p'_j = p_j - M_{j,j}x_j$ indicates $p'_j = \text{row}(p'_j) = \text{row}(p_j) = p_j$ for $j \in [n]$, confirming $C'$ and $C$ have the same code redundancy. The lemma is therefore substantiated. \hfill \Box

For an $(n, k, m)$ irregular array codes, the number of symbols affected by the update of the $\ell$-th symbol in $x_i$ is

$$ \theta_{i}^{(\ell)} = \sum_{j \in [n] \setminus i} \text{wt}((M_{i,j})_{\ell}), $$

(129)

and we can now omit $M_{i,i}$ due to Lemma 6. The update complexity $\theta$ of an irregular array code is therefore given by

$$ \theta = \frac{1}{B} \sum_{i \in [n]} \sum_{\ell \in [m_j]} \theta_{i}^{(\ell)}. $$

(130)

Since the update complexity is only related to the column weights of construction matrices, the next lemma provides a structure to be considered in the calculation of $\theta$ in (130).

**Lemma 7.** There exists a full rank decomposition of construction matrix $M_{i,j} = B'_{i,j} A'_{i,j}$ such that $A'_{i,j}$ contains a $\gamma_{i,j} \times \gamma_{i,j}$ identity submatrix.

**Proof.** The existence of a full rank decomposition $M_{i,j} = B_{i,j} A_{i,j}$ has been confirmed in Section II-C. As $A_{i,j}$ is with full row rank, there exists an invertible matrix $R_{i,j}$ such that $A'_{i,j} = R_{i,j} A_{i,j}$, where $A'_{i,j}$ contains a $\gamma_{i,j} \times \gamma_{i,j}$ identity submatrix. We can then obtain a new full rank decomposition $M_{i,j} = B'_{i,j} A'_{i,j}$ with $B'_{i,j} = B_{i,j} R_{i,j}^{-1}$. \hfill \Box

When $A'_{i,j}$ contains a $\gamma_{i,j} \times \gamma_{i,j}$ identity submatrix, $B'_{i,j}$ must be a submatrix of $M_{i,j}$. Thus, the column weights of $M_{i,j}$ are lower-bounded by the column weights of $B'_{i,j}$.

This brings up the study of the next two lemmas, which hold not just for a submatrix $B'_{i,j}$ of $M_{i,j}$ but for general full-rank decomposition $B_{i,j}$.

**Lemma 8.** Given an $(n, k, m)$ MR-MUB code with construction matrices $\{M_{i,j} = B_{i,j} A_{i,j}\}_{i \neq j \in [n]}$. $B_{\mathcal{E},j} \triangleq [B_{e_1,j} \ldots B_{e_{n-k},j}]$ is an invertible matrix for every $\mathcal{E} \subset [n]$ with $|\mathcal{E}| = n - k$ and for every $j \notin \mathcal{E}$.

**Proof.** First, we note respectively from (101) and (100) that $\gamma_{i,j} = \frac{m}{k}$ and $p_j = \frac{(n-k)m}{k}$. Hence, $B_{e_i,j}$ is an $(n-k)\frac{m}{k} \times \frac{m}{k}$ matrix, implying $B_{\mathcal{E},j}$ is an $(n-k)\frac{m}{k} \times (n-k)\frac{m}{k}$ square matrix. We then prove the lemma by contradiction.
Suppose $B_{\mathcal{E},j}$ is not invertible for some $\mathcal{E}$ with $|\mathcal{E}| = n-k$ and some $j \notin \mathcal{E}$. Then, $\text{rank}(B_{\mathcal{E},j}) < \frac{(n-k)m}{k}$. According to (13) and (15), we have

$$P_j = \sum_{\ell \in \mathcal{E}} B_{\ell,j}P_{\ell,j} + \sum_{i \in [n-k]} B_{e,i,j}P_{e,i,j}$$

$$= \sum_{\ell \in \mathcal{E}} B_{\ell,j}A_{\ell,j}X_\ell + \sum_{i \in [n-k]} B_{e,i,j}P_{e,i,j}$$

$$= [B_{e_1,j}A_{e_1,j} \cdots B_{e_{k-1},j}A_{e_{k-1},j}]X_\mathcal{E} + B_{\mathcal{E},j}P_{\mathcal{E},j},$$

(131) where $P_{\mathcal{E},j} \triangleq [p_{e_1,j}^T \ldots p_{e_{n-k},j}^T]^T$. This implies

$$H(p_j \mid X_\mathcal{E}) = H(B_{\mathcal{E},j}P_{\mathcal{E},j} \mid X_\mathcal{E}) \leq H(B_{\mathcal{E},j}P_{\mathcal{E},j}) \leq \text{rank}(B_{\mathcal{E},j}),$$

(134) where the last inequality follows from Lemma [1]. We then derive based on (21) that

$$I(C_\mathcal{E};X_\mathcal{E}) = I(X_\mathcal{E};P_{\mathcal{E}} \mid X_\mathcal{E}) = H(P_\mathcal{E} \mid X_\mathcal{E})$$

$$\leq \sum_{\ell \in \mathcal{E} \setminus \{j\}} H(p_\ell \mid X_\mathcal{E}) + H(p_j \mid X_\mathcal{E})$$

$$\leq \sum_{\ell \in \mathcal{E} \setminus \{j\}} H(p_\ell) + H(p_j \mid X_\mathcal{E})$$

$$\leq \sum_{\ell \in \mathcal{E} \setminus \{j\}} p_\ell + \text{rank}(B_{\mathcal{E},j})$$

$$< (n-k)m = H(X_\mathcal{E}),$$

(139) where the last strict inequality holds due to $\text{rank}(B_{\mathcal{E},j}) < \frac{(n-k)m}{k}$. The derivation in (139) indicates that $X_\mathcal{E}$ cannot be reconstructed from $C_\mathcal{E}$, leading to a contradiction to the definition of $(n,k,m1)$ array codes.

Lemma 9. For an $(n,k,m1)$ MR-MUB code with construction matrices $\{M_{i,j} = B_{i,j}A_{i,j}\}_{i \neq j \in [n]}$, $B_j \triangleq [B_{1,j} \cdots B_{j-1,j} B_{j+1,j} \cdots B_{n,j}]$ (140) must contain at least $\frac{(k-1)m}{k}$ columns whose weight is no less than 2.

Proof. Lemma [8] shows $B_{\mathcal{E},j}$ is invertible for arbitrary $\mathcal{E}$ with $|\mathcal{E}| = n-k$; hence, $B_j$ in (140) contains no zero column, and also has no identical columns. As each column of $B_j$ consists of in (140) contains no zero column, and also has no identical columns. As each column of $B_j$ consists of $\frac{(n-k)m}{k}$ components, the number of weight-one columns of $B_j$ must be at most $\frac{(n-k)m}{k}$. We thus conclude that there are at least

$$\text{col}(B_j) - \frac{(n-k)m}{k} = \frac{(n-1)m}{k} - \frac{(n-k)m}{k} = \frac{(k-1)m}{k}$$

(141) columns of $B_j$ with weights no less than 2. This completes the proof.

As previously mentioned, since the above two lemmas hold for the full-rank submatrix $B'_{i,j}$ of $M_{i,j}$, a lower bound on update complexity can thus be established.

Corollary 3. The construction matrices $\{M_{i,j}\}_{i \neq j \in [n]}$ of an $(n,k,m1)$ MR-MUB code must have at least $\frac{(k-1)mn}{k}$ columns with weights no less than 2.
Proof. Lemma \[9\] holds for those full-rank submatrices \(\{B'_{i,j}\}_{i \neq j \in [n]}\) of \(\{M_{i,j}\}_{i \neq j \in [n]}\). Thus, there are at least \(\frac{(k-1)n}{k}\) columns in \([M_{1,j} \ldots M_{j-1,j} M_{j+1,j} \ldots M_{n,j}]\), which have weights larger than 1. Consequently, the number of columns with weights no less than 2 in \(\{M_{i,j}\}_{i \neq j \in [n]}\) is at least \(\frac{(k-1)n}{k}\). \(\square\)

Lemma 10. Fix an \((n,k,m)\) MR-MUB code. For every \(i \in [n]\) and every \(\ell \in [m]\), there are at least \(n-k\) columns with non-zero weights in \(M_{i}^{(\ell)} \triangleq [(M_{i,1})_{\ell} \ldots (M_{i,i-1})_{\ell} \ldots (M_{i,i+1})_{\ell} \ldots (M_{i,n})_{\ell}]\).

Proof. Denote the data symbol in the \(\ell\)-th row of \(x_i\) as \(x_{i,\ell}\). If there were \(k\) zero columns in \(M_i^{(\ell)}\), then we can find \(k\) parity vectors that are functionally independent of \(x_{i,\ell}\) according to \([6]\), which implies we can find \(k\) nodes that cannot be used to reconstruct \(x_{i,\ell}\). A contradiction to the definition of \((n,k,m)\) MR MUB codes is obtained. \(\square\)

Considering Lemma \([10]\) holds for every \(i \in [n]\) and \(\ell \in [m]\), an immediate consequence is summarized in the next corollary.

Corollary 4. For an \((n,k,m)\) MR-MUB code, there are at least \((n-k)mn\) columns with nonzero weights in all construction matrices \(\{M_{i,j}\}_{i \neq j \in [n]}\).

Corollaries \([8]\) and \([4]\) then lead to the main result in this section.

Theorem 9. The update complexity \(\theta\) of \((n,k,m)\) MR-MUB codes is lower-bounded by \(n-k+\frac{k-1}{k}\).

Proof. Denote by \(\theta(\ell)\) the number of columns exactly with weight \(\ell\) in all construction matrices \(\{M_{i,j}\}_{i \neq j \in [n]}\). Since \(row(M_{i,j}) = \frac{(n-k)m}{k}\), it is obvious that \(\theta(\ell) = 0\) for \(\ell > \frac{(n-k)m}{k}\). Let \(\Theta(\ell) \triangleq \sum_{i=\ell}^{\frac{(n-k)m}{k}} \theta(i)\). We then derive from \((130)\) that

\[
\theta = \frac{1}{nm} \sum_{\ell \in \left[\frac{(n-k)m}{k}\right]} \ell \cdot \theta(\ell) = \frac{1}{nm} \sum_{\ell \in \left[\frac{(n-k)m}{k}\right]} \Theta(\ell) \geq \frac{\Theta(1) + \Theta(2)}{nm}. \tag{142}
\]

As Corollaries \([3]\) and \([4]\) imply \(\Theta(2) \geq \frac{(k-1)mn}{k}\) and \(\Theta(1) \geq (n-k)mn\), respectively, \((142)\) indicates that \(\theta \geq n-k+\frac{k-1}{k}\). \(\square\)

VII. A CLASS OF MR-MUB CODES WITH THE OPTIMAL REPAIR BANDWIDTH

A. Generic transformation for code construction

Consider \((n,k)\) MDS regular array codes with each node having exactly the same number of symbols, denoted as \(\alpha\). Hence, \(m_i + p_i = \alpha\) for every \(i \in [n]\). Let \(\beta_i\) be the amount of symbols that needs to be downloaded from all other \(n-1\) nodes when repairing node \(i\). Then, it is known \([3]\) that for all \((n,k)\) MDS regular array code designs, \(\beta_i \geq \frac{(n-1)\alpha}{(n-k)}\) for every \(i \in [n]\). As a consequence of this universal lower bound for every \(\beta_i\), an \((n,k)\) MDS regular array code is said to be with the optimal repair bandwidth for all nodes if \(\beta_i = \frac{(n-1)\alpha}{(n-k)}\) for every \(i \in [n]\).

In 2018, Li et al. \([17]\) proposed a generic transformation that converts a nonbinary \((n,k)\) MDS regular array code with node size \(\alpha\) into another \((n,k)\) MDS regular array code with node size \(\alpha' = (n-k)\alpha \) over the same field \(F_q\) such that 1) some chosen \((n-k)\) nodes have the optimal repair bandwidth \(\frac{(n-1)\alpha'}{(n-k)} = (n-1)\alpha\), and 2) the
normalized repair bandwidth\(^3\) of the remaining \(k\) nodes are preserved. Additionally, after applying the transformation \(⌈\frac{n}{n-k}⌉\) times, a nonbinary \((n,k)\) MDS regular array code can be converted into an \((n,k)\) MDS regular array code with all nodes achieving the optimal repair bandwidth.

In this section, using the transformation in [17], an \((n,k = n - 2, 2^k \frac{n}{n-k} m \mathbf{1})\) regular array code that achieves the optimal repair bandwidth for all nodes is constructed from an \((n,k = n - 2, m \mathbf{1})\) MR-MUB code under \(k \mid m\).
We will then prove in Theorem 11 that the transformed \((n,k = n - 2, 2^k \frac{n}{n-k} m \mathbf{1})\) regular array code also have the minimum code redundancy and the minimum update bandwidth and hence is an MR-MUB code.

For completeness, we restate the generic transform in [17] in the form that is necessary in this paper in the following theorem. Similar to [17], the symbols of the codes we construct are over \(\mathbb{F}_q\) with \(q > 2\), where the elements of \(\mathbb{F}_q\) are denoted as \(\{0,1,g,\ldots,g^{q-2}\}\) and \(g\) is a primitive element of \(\mathbb{F}_q\).

**Theorem 10.** *(Generic transform for \((n = k + 2, k)\) regular array codes [17])* Let \(C^{(0)} \triangleq [c_1^{(0)} \ldots c_n^{(0)}]\) and \(C^{(1)} \triangleq [c_1^{(1)} \ldots c_n^{(1)}]\) be codewords of a nonbinary \((n = k + 2, k)\) MDS regular array code \(C\) over \(\mathbb{F}_q\) with node size \(\alpha\), where the data symbols used to generate \(C^{(0)}\) and \(C^{(1)}\) can be different. Denote by \(\beta_i\) the repair bandwidth of \(C\) for node \(i\). Then,

\[
C' = \begin{bmatrix}
c^{(0)}_1 & \ldots & c^{(0)}_k & c^{(0)}_{k+1} & c^{(0)}_{k+2} + c^{(1)}_{k+2} \\
c^{(1)}_1 & \ldots & c^{(1)}_k & c^{(0)}_{k+2} + g c^{(1)}_{k+2} & c^{(1)}_{k+1}
\end{bmatrix} \in \mathbb{F}_q^{2\alpha \times (k+2)},
\]

are codewords of an \((n = k + 2, k)\) MDS regular array code \(C'\) with node size \(\alpha' = 2\alpha\), and its repair bandwidth for node \(i\) satisfies

\[
\beta'_i = \begin{cases} 
2\beta_i, & \text{for } i \in [k] \\
\frac{(n-1)\alpha'}{n-k} = (n-1)\alpha, & \text{for } k < i \leq n = k + 2.
\end{cases}
\]

It is worth noting that the last two nodes of the transformed code \(C'\) have achieved the universal lower bound and therefore is with the optimal repair bandwidth. Furthermore, it can be inferred from (145) that if the code \(C\) before transformation is already with the optimal repair bandwidth for every node, then the repair bandwidths of \(C'\) are also optimal for all nodes.

**B. MR-MUB code construction with the optimal repair bandwidth**

According to Theorem 10, given an \((n,k = n - 2, m \mathbf{1})\) MR-MUB code \(\mathcal{C}\) under \(k \mid m\), we can construct an \((n,n - 2, 2m \mathbf{1})\) regular array code \(\mathcal{C}'\) that satisfies 1) the last two nodes are with the optimal repair bandwidth, and 2) the remaining \(k\) nodes preserve the same normalized repair bandwidths as their corresponding nodes of \(\mathcal{C}\).

In order to distinguish between the codewords before and after transformation, we will use

\[
\begin{align*}
\{ y_i = \begin{bmatrix} y_i^{(0)} \\ y_i^{(1)} \end{bmatrix} \}_{i \in [n]} & \quad \text{and} \quad \{ q_i = \begin{bmatrix} q_i^{(0)} \\ q_i^{(1)} \end{bmatrix} \}_{i \in [n]} 
\end{align*}
\]

\(^3\)The normalized repair bandwidth for a node is defined as

\[
\frac{\text{the number of symbols downloaded for repairing this node}}{\text{the number of symbols repaired}}
\]
to denote the data vectors and the parity vectors of the transformed code $C'$, respectively. Data vectors and parity vectors of the base code $C$ are respectively denoted as $\{x_i^{(t)}\}_{i \in [n], t \in \{0,1\}}$ and $\{p_i^{(t)}\}_{i \in [n], t \in \{0,1\}}$. We then have the following correspondence between $\{y_i^{(t)}, q_i^{(t)}\}_{i \in [n], t \in \{0,1\}}$ and $\{x_i^{(t)}, p_i^{(t)}\}_{i \in [n], t \in \{0,1\}}$:

$$
C' = \begin{bmatrix}
    y_1^{(0)} & \ldots & y_k^{(0)} & y_{k+1}^{(0)} & \ldots & y_n^{(0)} \\
    q_1^{(0)} & \ldots & q_k^{(0)} & q_{k+1}^{(0)} & \ldots & q_n^{(0)} \\
    y_1^{(1)} & \ldots & y_k^{(1)} & y_{k+1}^{(1)} & \ldots & y_n^{(1)} \\
    q_1^{(1)} & \ldots & q_k^{(1)} & q_{k+1}^{(1)} & \ldots & q_n^{(1)}
\end{bmatrix},
$$

(147)

which implies

$$
\begin{align*}
    x_{k+1}^{(1)} &= y_{k+1}^{(1)} \\
    y_{k+1}^{(0)} &= y_{k+1}^{(0)}
\end{align*}
$$

(148)

We then present the main theorem in this section.

**Theorem 11.** $C'$ (whose codewords are defined in (147)) is an $(n, k = n - 2m)$ MR-MUB code over $\mathbb{F}_q$.

**Proof.** Recall from (100) that each node of $C$ has $\alpha = m + p = m + \frac{(n-k)}{k}m = \frac{nm}{k}$ symbols. Thus, from (147), each node of $C'$ contains $\alpha' = 2\alpha = \frac{2nm}{k}$ symbols.

Since each node of the transformed code $C'$ has $p' = 2p$ parity symbols, its code redundancy achieves the minimum value given in (37). It remains to show $C'$ also achieves the minimum update bandwidth.

Using the notations in Section II-C where the encoding matrices of $C$ are denoted as $\{A_{i,j}\}_{i,j \in [n]}$ and $\{B_{i,j}\}_{i,j \in [n]}$, we consider the update of node $i$ of $C'$ for $i \in [k]$. From (147), we need to compute

$$
\Delta y_i^{(t)} = y_i^{(t)*} - y_i^{(t)} \quad \text{for } \ell = 1, 2,
$$

(149)

where we add a star in the superscript to denote the value of a vector after this updating. Then, we must renew $q_j^{(t)}$ for $j \in [k] \setminus \{i\}$ based on $\Delta y_i^{(t)}$ according to $q_j^{(t)} + B_{i,j}A_{i,j}\Delta y_i^{(t)}$. The correspondence in (147) then indicates $q_j^{(t)} = p_j^{(t)}$ and $\Delta y_i^{(t)} = \Delta x_i^{(t)}$, i.e.,

$$
q_j^{(t)*} = p_j^{(t)*} = p_j^{(t)} + B_{i,j}A_{i,j}\Delta x_i^{(t)} = q_j^{(t)} + B_{i,j}A_{i,j}\Delta x_i^{(t)}.
$$

(150)

Accordingly, node $i$ shall send both $A_{i,j}\Delta y_i^{(0)} = A_{i,j}\Delta x_i^{(0)}$ and $A_{i,j}\Delta y_i^{(1)} = A_{i,j}\Delta x_i^{(1)}$ to node $j \in [k] \setminus \{i\}$, which implies $\gamma_{ij}^{(t)} = 2\gamma_{ij}$ for $i \neq j \in [k]$. The renew of $q_{k+1}$ requires sending $A_{i,k+1}\Delta x_i^{(0)}$ and $A_{i,k+2}(\Delta x_i^{(0)} + g\Delta x_i^{(1)})$ to node $k + 1$ since

$$
q_{k+1}^{(0)*} = q_{k+1}^{(0)} + B_{i,k+1}A_{i,k+1}\Delta y_i^{(0)} = p_{k+1}^{(0)} + B_{i,k+1}A_{i,k+1}\Delta x_i^{(0)},
$$

(151)

and

$$
q_{k+1}^{(1)*} = p_{k+1}^{(0)} + B_{i,k+2}\Delta x_i^{(0)} + g\Delta x_i^{(1)} = q_{k+1}^{(1)} + B_{i,k+2}\Delta x_i^{(0)} + g\Delta x_i^{(1)}.
$$

(152)
Thus, $\gamma'_{i,k+1} = \gamma_{i,k+1} + \gamma_{i,k+2}$ for $i \in [k]$. We can similarly obtain $\gamma'_{i,k+2} = \gamma_{i,k+1} + \gamma_{i,k+2}$ for $i \in [k]$ when concerning the adjustment of $q_{k+2}$ due to the update of $y_i$.

We next consider the update of $y_{k+1}$. Again, we compute $\Delta y_{k+1}^{(\ell)} = y_{k+1}^{(\ell)} - y_{k+1}^{(\ell)}$ for $\ell = 0, 1$. Note that all of $x_{k+1}^{(0)}, x_{k+2}^{(0)}$ and $x_{k+2}^{(1)}$ are involved in this update. Since $y_{k+2}^{(0)} = x_{k+2}^{(0)} + x_{k+2}^{(1)}$ remains unchanged, we have $\Delta y_{k+2}^{(0)} = \Delta x_{k+2}^{(0)} + \Delta x_{k+2}^{(1)} = 0$, which together with (148) implies $\Delta x_{k+1}^{(0)} = \Delta y_{k+1}^{(0)}$ and $\Delta x_{k+2}^{(1)} = (g - 1)^{-1} \Delta y_{k+1}^{(1)}$. As a result, for $j \in [k]$, the new parity vectors $q_j^{(1)}$ are renewed according to

$$q_j^{(1)*} = p_j^{(0)*} + B_{k+1,j} A_{k+1,j} \Delta x_{k+1}^{(0)} + B_{k+2,j} A_{k+2,j} \Delta x_{k+2}^{(0)}$$

(153)

and

$$q_j^{(1)*} = p_j^{(1)*} + B_{k+2,j} A_{k+2,j} \Delta x_{k+2}^{(1)}$$

(154)

which indicates node $k + 1$ should send $A_{k+1,j} \Delta y_{k+1}^{(0)}$ and $A_{k+2,j} \Delta y_{k+1}^{(1)}$ to node $j$ to renew its parity vector; hence, $\gamma'_{k+1,j} = \gamma_{k+1,j} + \gamma_{k+2,j}$ for $j \in [k]$. Concerning the renew of $q_{k+2}$, we derive

$$q_{k+2}^{(0)*} = p_{k+2}^{(0)*} + p_{k+2}^{(1)*}$$

$$= p_{k+2}^{(0)*} + B_{k+2,k+1} A_{k+2,k+1} \Delta x_{k+1}^{(0)} + B_{k+3,k+2} A_{k+3,k+2} \Delta x_{k+2}^{(0)} + B_{k+2,k+2} A_{k+2,k+2} \Delta x_{k+2}^{(1)}$$

(155)

and

$$q_{k+2}^{(1)*} = p_{k+1}^{(1)*} + B_{k+2,k+1} A_{k+2,k+1} \Delta x_{k+2}^{(1)}$$

(156)

which indicates node $k + 1$ should send $A_{k+1,k+2} \Delta y_{k+1}^{(0)}$ and $A_{k+2,k+1} \Delta y_{k+1}^{(1)}$ to node $k + 2$; hence, $\gamma'_{k+1,k+2} = \gamma_{k+1,k+2} + \gamma_{k+2,k+1}$.

Last, we consider the update of $y_{k+2}$, and can similarly obtain $\gamma'_{k+2,j} = \gamma_{k+1,j} + \gamma_{k+2,j}$. for $j \in [k]$ and $\gamma'_{k+2,k+1} = \gamma_{k+1,k+2} + \gamma_{k+2,k+1}$.
We summarize the matrix of $γ'_{i,j}$ for $i \neq j \in [n]$ as follows.

$$
\begin{bmatrix}
γ'_{1,2} & \cdots & γ'_{1,k} & γ'_{1,k+1} & γ'_{1,k+2} \\
γ'_{2,1} & \cdots & γ'_{2,k} & γ'_{2,k+1} & γ'_{2,k+2} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
γ'_{k,1} & \cdots & γ'_{k,k-1} & γ'_{k,k+1} & γ'_{k,k+2} \\
γ'_{k+1,1} & \cdots & γ'_{k+1,k} & γ'_{k+1,k+1} & γ'_{k+1,k+2} \\
γ'_{k+2,1} & \cdots & γ'_{k+2,k} & γ'_{k+2,k+1} & γ'_{k+2,k+2}
\end{bmatrix}
$$

(157)

Since $C$ is a $(n = k + 2, k, m\mathbf{1})$ MR-MUB code, we know from (101) that $γ_{i,j} = \frac{2m}{k}$ for $i \neq j \in [n]$. We then conclude from (157) that $γ'_{i,j} = \frac{2m}{k}$ for $i \neq j \in [n]$. Consequently, $C'$ is an $(n, n - 2, 2^{\lceil n/2 \rceil} m\mathbf{1})$ MR-MUB code over $\mathbb{F}_q$, which can be confirmed by Theorem 3.

By Theorem 10, we can optimize the repair bandwidth of two selected nodes at a time, and reapply the transformation $\left\lceil \frac{n}{k} \right\rceil$ times to obtain an $(n, k = n - 2, 2^{\lceil n/2 \rceil} m\mathbf{1})$ MR-MUB code with optimal repair bandwidth for all nodes as long as $k \mid m$.

Although the transformation in (17) holds for general $k$, a further generation of Theorem 11 to general $k$ satisfying, e.g., $k < n - 2$, cannot be done by following a similar procedure to the current proof, and the transformed code may not be an MR-MUB code. Hence, what we have proven in Theorem 11 is a particular case that guarantees the transformed code is an MR-MUB code if the code before transformation is an MR-MUB code.

VIII. Conclusion

In this paper, we introduced a new metric, called the update bandwidth, which measures the transmission efficiency in the update process of $(n, k, m)$ irregular array codes in DSSs. It is an essential measure in scenarios where updates are frequent. The closed-form expression of the minimum update bandwidth $γ_{\text{min}}$ was established (cf. Theorem 9), and the code parameters, using which the minimum update bandwidth (MUB) can be achieved, were identified. These code parameters then constitute the class of MUB codes. As code redundancy is also an important consideration in DSSs, we next investigated the smallest code redundancy attainable by MUB codes (cf. Theorems 4 and 5).

We then seek to construct a class of irregular array codes that achieves both the minimum code redundancy and the minimum update bandwidth, named MR-MUB codes. The code parameters for MR-MUB codes are therefore determined (cf. Theorem 6). An interesting result is that under $1 < k < n - 1$ and $k \mid m_i$ for $i \in [n]$, MR-MUB codes can only be vertical MDS codes unless $m = [m_1 \cdots m_n]$ containing only a single non-zero component. The explicit construction of MR-MUB codes was thus focused on $(n, k)$ vertical MDS codes, i.e., $(n, k, m\mathbf{1})$ MR-MUB codes.
codes (cf. Section V-B). A further generalization of the MR-MUB code construction was subsequently proposed for a class of MUB codes with the smallest code redundancy (cf. Section V-C).

At last, we studied the update complexity and repair bandwidth of MR-MUB codes. Through the establishment of a lower bound for the update complexity of MR-MUB codes (cf. Theorem 9), we found MR-MUB codes may not simultaneously achieve the minimum update complexity. However, an \((n, k = n - 2, m_1)\) MR-MUB code with the optimal repair bandwidth for all nodes can be constructed via the transformation in [17] (cf. Theorem 11).

There are some challenging issues remain unsolved.

1) Determine the smallest update bandwidth attainable by irregular MDS array codes [16], defined as the irregular array codes with the minimum code redundancy.

2) Determine the smallest code redundancy attainable by MUB codes when the condition of \(k \mid m_i\) for \(i \in [n]\) is violated.

3) Examine whether \(k \mid m\) is also a necessary condition for vertical MDS array codes being MUB codes, provided \(k \not\mid n\).

4) Check whether the lower bound for the update complexity of \((n, k, m_1)\) MR-MUB codes in Theorem 9 can be improved or achieved.

5) Study the optimal repair bandwidth of MR-MUB codes under \(n - k \geq 3\).

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