It is proved that for every odd $n \geq 1039$ there are two words $u(x, y), v(x, y)$ of length $\leq 658n^2$ over the group alphabet $\{x, y\}$ of the free Burnside group $B(2, n)$, which generate a free Burnside subgroup of the group $B(2, n)$. This implies that for any finite subset $S$ of the group $B(m, n)$ the inequality $|S_t| > 4 \cdot 2.9^{\frac{1}{658s^2}}$ holds, where $s$ is the smallest odd divisor of $n$ that satisfies the inequality $s \geq 1039$.

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Introduction. For an arbitrary finite subset $S$ of a given group $G$, denote by $S'$ the set of all possible products of the form $a_1 \cdots a_t$, where $a_i \in S$. In [1] it is proved that for an arbitrary finite subset of a free group not contained in any cyclic subgroup there exist constants $c, \delta > 0$ such that $|S'^3| > c|S|^{1+\delta}$. S. R. Safin [2] showed that there exist constants $c_n > 0$ such that for any finite subset $S$ of a free group not contained in any cyclic subgroup the inequality $|S'| > c_t \cdot |S|^{(t+1)/2}$ holds for all positive integers $t$. Other interesting results on additive combinatorics can be found in [3, 4].

Our goal is the following theorem.

**Theorem 1.** For any finite symmetric subset $S$ of a free Burnside group $B(m, n)$ and $t \geq 2$ the inequality $|S'^t| > 4 \cdot 2.9^{\frac{1}{658s^2}}$ holds, where $s$ is the smallest odd divisor of $n$ satisfying the inequality $s \geq 1039$.

Recall that a relatively free group of rank $m$ in the variety of all groups, satisfying the identity $x^n = 1$, is denoted by $B(m, n)$ and is called a free periodic or free Burnside group of period $n$ and rank $m$. More simply

$$B(m, n) = \langle a_1, a_2, \ldots, a_m; x^n = 1 \rangle.$$
Auxiliary Lemmas. Consider the words
\[ w(x, y) = [x, yxy^{-1}] \]
and
\[ W(x, y) = [w(x, y)^d, xw(x, y)^d]^{-1}, \]
where \( d = 191 \). Denote
\[ u(x, y) = W(x, y)^{200}w(x, y)W(x, y)^{200}w(x, y)^2...W(x, y)^{200}w(x, y)^{n-1}W(x, y)^{200}, \]
\[ v(x, y) = W(x, y)^{300}w(x, y)W(x, y)^{300}w(x, y)^2...W(x, y)^{300}w(x, y)^{n-1}W(x, y)^{300}. \]

Lemma 1. Let \( n \geq 665 \) be an arbitrary odd number. If \( a \) and \( b \) do not commute in \( B(m, n) \) and \( a^p \neq 1 \), then \( w(a^p, b) \neq 1 \).

Proof. Suppose that \( w(a^p, b) = [a^p, ba^p b^{-1}]^{B(m,n)} = 1 \). According to Theorem VI.3.1 [5], there exists an element \( z \) of order \( n \) and integers \( r \) and \( s \) such that \( a^p = z^r \) and \( ba^p b^{-1} = z^s \). From the equality \( b^r z^r b^{-1} = z^s \) it follows that \( b \) belongs to the normalizer of the subgroup \( \langle z^r \rangle_{B(m,n)} \). Hence, \( |\langle z^r, b \rangle_{B(m,n)}| \leq |\langle z^r \rangle| \cdot |\langle b \rangle| \leq n^2 \).

Any finite subgroup of \( B(m,n) \) is cyclic (see VII.1.8 [5]). So, the subgroup \( \langle z^r, b \rangle_{B(m,n)} = \langle a^p, b \rangle_{B(m,n)} \) is cyclic. In particular, \( b \) belongs in the centerizer of \( a^p \). By Theorem VI.3.2 [5] the centralizer of any non trivial element of \( B(m,n) \) is cyclic. Since the elements \( a \) and \( b \) belong to the centralizer \( a^p \), they lie in the same cyclic subgroup, and so commute.

The contradiction obtained proves Lemma 1.

Lemma 2. Let \( n \geq 1039 \) be an arbitrary odd number. If \( a \) and \( b \) do not commute in the group \( B(m, n) \) and \( a \) is conjugate to a power of some elementary period \( E \) of rank \( \gamma \), then for some \( p = 2^k \), where \( 0 < k < 9 \), the element \( w(a^p, b) \) is conjugate to some elementary period \( E \) of rank \( \beta \geq \gamma + 1 \).

Proof. Let for some word \( T \) we have \( a = TE^r T^{-1} \) in \( B(m, n) \). Replacing \( E \) with \( E^{-1} \) if necessary, we can assume that \( 1 \leq r \leq \frac{n-1}{2} \). Let us first show that for some \( 186 \leq s \leq \frac{n+1}{2} - 148 \) and some integer \( 0 \leq k \leq 9 \) we have the congruence
\[ r \cdot 2^k \equiv s \pmod{n}. \]

Indeed, for \( 186 \leq r \leq \frac{n+1}{2} - 148 \) one can choose \( k = 0 \), and if \( \frac{186}{2^k} \leq r \leq \frac{372}{2^k} \), where \( k = 1, \ldots, 8 \), then \( 186 \leq r \cdot 2^k \leq 372 \leq \frac{n+1}{2} - 148 \) (since \( n \geq 1039 \)).

If \( \frac{n+1}{2} - 148 \leq r \leq \frac{n-1}{2} \), then \( 1 \leq n-2r \leq 295 \leq \frac{n+1}{2} - 148 \) and we can use the previous reasoning (again replacing \( E \) with \( E^{-1} \)). Thus, for some \( p = 2^k, 0 \leq k \leq 9 \), we get \( a^p = TE^r T^{-1} = TE^r T^{-1} \), where \( 186 \leq s \leq \frac{n+1}{2} - 148 \).

By Lemma 2.8 [6] the period \( E \) can be chosen minimizied, and by virtue of VI.2.4 and IV.3.12 [5] we can assume that \( T^{-1} b T \in \mathcal{M}_{\gamma} \cap \mathcal{M}_{\gamma+1} \). By Lemma 2, we have \( T^{-1} w(a^p, b) T \neq 1 \) in the group \( B(m, n) \), so \( [E^r, T^{-1} bTE^r T^{-1} b^{-1} T] \neq 1 \), and
Then the words we have the qualities word are a basis of a free Burnside subgroup of rank 2 of the group $B$ according to Lemma 7.2 [6].

Lemma 3. Let $n \geq 1003$ be an arbitrary odd number. Assume that $a$ and $b$ do not commute in the group $B(m,n)$, the element $a$ is conjugate of the power of some elementary period $E$ of rank $\gamma$, and for some $p$ the element $w(a^p,b)$ is conjugate to some elementary period of rank $\beta \geq \gamma + 1$. Then $W(a^p,b) \neq 1$ in $B(m,n)$.

Proof. By the condition we have $a = TE'T^{-1}$ for some elementary period $E$ of rank $\gamma$ and $w(a^p,b) = UAU^{-1}$, where $A$ is an elementary period of some rank $\beta > \gamma$. Suppose that $W(a^p,b) = [w(a^p,b)^d, a^pw(a^p,b)^d a^{-p}]_{B(m,n)} = 1$. Then by Theorem VI.3.1 [5], one can find an element $c$ of order $n$ and integers $t$ and $s$ such that $UA^d U^{-1} = c'$ and $a^p UA^d U^{-1} a^{-p} = c^t$. From here, as in Lemma 2, it follows that $(c', a^p)_{B(m,n)}$ is a cyclic group. Since the element $c'$ has the order $n$ (because the elementary period $A$ has the order $n$ and $(d, n) = 1$), it turns out that some power of the elementary period $E$ of rank $\gamma$ is conjugate of some power of elementary period $A$ of rank $\beta \geq \gamma + 1$ in the group $B(m,n)$. This contradicts Lemma 6.6 [6]. Hence $W(a^p,b) \neq 1$ in $B(m,n)$.

Lemma 3 is proved.

A Theorem on Free Subgroups.

Theorem 2. If $n \geq 1039$ is an arbitrary odd number and $a$ and $b$ are two non commuting elements of the group $B(2,n)$, then for some $p = 2^k$, where $0 \leq k \leq 9$, the words $u(x,y)$ and $v(x,y)$ are defined by equalities (1) and (2).

Proof. The starting point for proving Theorem 2 is the following assertion, proved in [7] (see also [8,9]).

Lemma 4. Theorem [7]. Let the commutator $[A^d, Z^{-1} B^d Z]$ be equal to the elementary period $C$ of rank $\alpha$ in the group $B(2,n)$, where $A$ is the minimized elementary period of rank $\gamma$, $B$ is the minimized elementary period of rank $\beta$, $Z \in \mathcal{M}_{\alpha - 1}$ ($\gamma \leq \beta \leq \alpha - 1$), $d = 191$ and $n \geq 1003$ are arbitrary odd numbers. Then the words

$$u_1 := C^{200} A^{200} A^2 \cdots A^n C^{200}$$

and

$$u_2 := C^{300} A^{300} A^2 \cdots A^n C^{300}$$

are a basis of a free Burnside subgroup of rank 2 of the group $B(2,n)$.

Proof. By VI.2.5 [5], the element $a$ is conjugate of a power of some elementary period $E$ of rank $\gamma \geq 1$ in the group $B(2,n)$. By Lemma 3, for some word $U$, for some $p = 2^k$, and for some elementary period $A$ of rank $\beta > \gamma$ we have the qualities $w(a^p,b) = UAU^{-1}$ and $W(a^p,b) = [UA^d U^{-1}, a^p U A^d U^{-1} a^{-p}]$ in $B(2,n)$. By virtue of Lemma 2.8 [6], the period $A$ can be considered to be minimized. By Lemma 3 $W(a^p,b) \neq 1$ in $B(2,n)$. By virtue of VI.2.4 and IV.3.12 [5], we can assume that $U^{-1} a^{pU} \in \mathcal{M}_{\beta} \cap \mathcal{M}_{\beta + 1}$. According to 3.2 [6], choose some reduced form $G$ of the commutator $[A^d, (U^{-1} a^{pU}) A^d (U^{-1} a^{-p} U)]$, which, according to
Lemma 7.2 [6], is an elementary period of some rank $\delta \geq \beta + 1$. By virtue of relation (3.6) from [6], the commutator $[A^d, (U^{-1}a^pU)A^d(U^{-1}a^{-p}U)]$ and its reduced form $G$ are related by equality $G^{B(m,n,\delta-1)} = t[A^d, U^{-1}a^pUA^dU^{-1}a^{-p}U]_{t^{-1}}$ for some $t \in \Theta(A, A_1)$ (see Definitions 2.3 and 3.1 [6]), where $A_1$ is a cyclic shift of the word $A$. It follows from II.3.5 and II.6.13 [5] that $A_1$ is also a minimized elementary period of rank $\gamma$, while VI.2.4 and IV.3.12 [5] for some $Z \in \mathcal{M}_{4-1} \cap \mathcal{A}_{\delta}$ we have $G = [A_1^d, ZA^dZ^{-1}]$, where $Z = tU^{-1}a^pU$. Applying Lemma 4, we conclude that the words

$$G^{200}A_1G^{200}A_1^2G^{200}A_1^{n-1}G^{200}$$

freely generate a free Burnside subgroup of rank 2 of the group $B(2,n)$. It remains to note that $U^{-1}A_1tU^{-1} = w(a^p, b), U^{-1}GtU^{-1} = W(a^p, b)$ in $B(2,n)$ and consequently we get

$$u(a^p, b) = (U^{-1})(G^{200}A_1G^{200}A_1^2G^{200}A_1^{n-1}G^{200})(U^{-1})^{-1},$$
$$v(a^p, b) = (U^{-1})(G^{300}A_1G^{300}A_1^2G^{300}A_1^{n-1}G^{300})(U^{-1})^{-1}.$$

Theorem 2 is proved.

**Proof of Theorem 1.** Let us proceed to the Proof of the Theorem 1. First, we estimate the word lengths $u(x, y)$ and $v(x, y)$, where

$$u(x, y) = W(x, y)^{200}w(x, y)W(x, y)^{200}w(x, y)^2...W(x, y)^{200}w(x, y)^{n-1}W(x, y)^{200},$$
$$v(x, y) = W(x, y)^{300}w(x, y)W(x, y)^{300}w(x, y)^2...W(x, y)^{300}w(x, y)^{n-1}W(x, y)^{300},$$

$$w(x, y) = [x, yxy^{-1}]$$

and

$$W(x, y) = [w(x, y)^d, xw(x, y)^dx^{-1}].$$

In this case, all words will be considered as positive words. Since $w(x, y) = xyy^{-1}x^{-1}y^{-1}$, then $|w(x, y)|_{\{x, y\}} = 4$ (via $|w(x, y)|_{\{x, y\}}$ denote the length of the word $w$ in the group alphabet $\{x, y\}$). Similarly, for any positive words $A = A(x, y), B = B(x, y)$ we have $|w(A, B)|_{\{x, y\}} = 4(|A| + |B|)$. Consequently,

$$|w(a^p, b)|_{\{a, b\}} = 2(p + 1).$$

Further we have

$$|W(a^p, b)|_{\{a, b\}} = 4(2d(p + 1) + 1),$$

and

$$|u(a^p, b)|_{\{a, b\}} = 200n|W(a^p, b)|_{\{a, b\}} + \frac{n(n-1)}{2}|w(a^p, b)|_{\{a, b\}}.$$  

Similarly,

$$|v(a^p, b)|_{\{a, b\}} = 300n|W(a^p, b)|_{\{a, b\}} + \frac{n(n-1)}{2}|w(a^p, b)|_{\{a, b\}}.$$  

Taking into account the equalities (3), (4), we finally get:

$$|u(a^p, b)|_{\{a, b\}} = 200n(4d(p + 1) + 1) + \frac{n(n-1)}{2}(2p + 1),$$

(5)
\[ |v(a^p, b)_{\{a,b\}}| = 300n(4d(p+1)+1) + \frac{n(n-1)}{2}2(p+1). \]  
\[ (6) \]

Since \( d = 191 \), \( n \geq 1039 \) and \( p \leq 2^9 \), from (5) and (6) it is easy to derive the following estimates:

\[ |u(a^p, b)_{\{a,b\}}| \leq 513n^2 + 44^3, |v(a^p, b)_{\{a,b\}}| \leq 513n^2 + 48^3 \leq 658n^2. \]

Recall that, by virtue of Theorem 2, the words \( u(a^p, b), v(a^p, b) \) generate a free Burnside group of rank 2. By S. I. Adyan’s theorem, the group \( B(2,n) \) has exponential growth. More precisely, according to Theorem 2.15, Chap. VI [5] the set \( \{u,v\}^k \) contains \( \gamma(k) > 4 \cdot 2.9^{k-1} \) pairwise distinct elements. This means that the set \( S^t \), where \( t \geq 658s^2 \), contains \( \gamma \left( \left\lfloor \frac{t}{658s^2} \right\rfloor \right) \) pairwise distinct element, where \( s \) is the smallest odd divisor of \( n \), satisfying the inequality \( s \geq 1039 \). Thus,

\[ |S^t| > 4 \cdot 2.9^{\frac{1}{s \ln s}}. \]

Theorem is proved.

\[ \square \]

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СТЕПЕНЬ ПОДМНОЖЕСТВ СВОБОДНЫХ ПЕРИОДИЧЕСКИХ ГРУПП

Доказано, что для каждого нечетного $n \geq 1039$ существуют два слова $u(x,y), v(x,y)$ длины $\leq 2^{22}n^3$ над групповым алфавитом $\{x, y\}$ свободной бернсайдовой группы $B(2, n)$, порождающие свободную подгруппу группы $B(2, n)$. Отсюда следует, что для любого конечного подмножества $S$ группы $B(m, n)$ выполняется неравенство $|S| > 4 \cdot 2.9^{\frac{2}{10s^3}}$, где $s$ – наименьший нечетный делитель числа $n$, удовлетворяющий неравенству $s \geq 1039$. 

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