Non-extremal black holes

of $N = 2, d = 4$ supergravity

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Abstract

We propose a generic recipe for deforming extremal black holes into non-extremal black holes and we use it to find and study the static non-extremal black-hole solutions of several $N = 2, d = 4$ supergravity models ($SL(2, \mathbb{R})/U(1)$, $\mathbb{C}P^3$ and $STU$ with four charges). In all the cases considered, the non-extremal family of solutions smoothly interpolates between all the different extremal limits, supersymmetric and not supersymmetric. This fact can be used to explicitly find extremal non-supersymmetric solutions also in the cases in which the attractor mechanism does not completely fix the values of the scalars on the event horizon and they still depend on the boundary conditions at spatial infinity.

We compare (supersymmetry) Bogomol’nyi bounds with extremality bounds, we find the first-order flow equations for the non-extremal solutions and the corresponding superpotential, which gives in the different extremal limits different superpotentials for extremal black holes. We also compute the entropies (areas) of the inner (Cauchy) and outer (event) horizons, finding in all cases that their product gives the square of the moduli-independent entropy of the extremal solution with the same electric and magnetic charges.

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Introduction

Black holes are among the most interesting objects that occur in theories of gravity that include or extend general relativity, such as supergravity and superstring theories because their thermal behavior (Hawking radiation and Bekenstein–Hawking entropy) provides a unique window into the quantum-mechanical side of these theories. Their study in the framework of supergravity and superstring theories has generated a huge body of literature, the largest part of which concerns extremal (mostly but not always supersymmetric) black holes.

There are several reasons for having a special interest in extremal black holes: the solutions are simpler to find, they are protected from classical and quantum corrections when they are supersymmetric, there is an attractor mechanism for the scalar fields of most of them [1, 2, 3, 4], their entropies are easier to interpret microscopically in the framework of superstring theory [5] etc. Much of the progress in their study has been facilitated by the explicit knowledge of general families of extremal supersymmetric solutions e.g. in $N = 2, d = 4$ supergravity theories, where we know how to find systematically all of them [6, 7, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] (see also the review Ref. [20]).

By contrast, only a few non-extremal black-hole solutions are known (for instance, in $N = 2, d = 4$ theories), partly because they are more difficult to find than their extremal counterparts, and partly because they do not enjoy so many special properties. It is, however, clear that non-extremal black holes are at least as interesting as the extremal ones from a physical point of view, because they are closer to those that we may one day be able to observe. Furthermore, in $d = 4$ dimensions adding any amount of angular momentum to extremal black holes causes the event horizon to disappear [17]. This does not happen in non-extremal black holes, at least as long as the angular momentum does not exceed a certain value.

In order to learn more about them it is necessary to have more examples available for their study. In this paper we are going to propose a procedure to find non-extremal solutions of $N = 2, d = 4$ supergravity theories by deforming in a prescribed way the supersymmetric extremal solutions that we know how to construct systematically. Another prescription has been proposed in the literature, namely the introduction of an additional harmonic function (called Schwarzschild factor in Ref. [28] and non-extremality factor in Ref. [29]), but it is unclear whether this method will work in all cases and for all models.

Our proposal makes crucial use of the formalism of Ferrara, Gibbons and Kallosh in Ref. [30], which turns out to be very convenient for our purposes. This formalism is based on the use of a particular radial coordinate $\tau$ that covers the exterior of the event horizon (which is always at $\tau = -\infty$ in these coordinates, suitable value for the study of attractors). Furthermore, in this formalism the equations of motion have been reduced to a very small number of ordinary differential equations in the variable $\tau$, which should simplify the task of finding solutions. In these equations there is a function of the scalars and the electric charges (the so-called black-hole potential), which plays a very important rôle, since its critical points are associated with possible extremal black-hole solutions. Then, using this formalism, we can also relate more easily the non-extremal solutions to the extremal solutions that have the same electric and magnetic...
charges.

We are going to test our proposal in a number of \( N = 2, d = 4 \) models and then study the main characteristics of the non-extremal solutions constructed. In this work we consider only regular static black holes.

This paper is organized as follows: in Section 1 we review essential facts concerning extremal and non-extremal black holes in the formalism and coordinates used by Ferrara, Gibbons and Kallosh in Ref. [30]. This will help us to establish our notation and conventions, find an ansatz for the non-extremal black holes based on the expressions for well-known solutions in these coordinates and show that these coordinates also cover the region that is bounded by the inner (Cauchy) horizon. In Section 2 we use the ansatz for the \( SL(2, \mathbb{R})/U(1) \) axion-dilaton model to deform the supersymmetric extremal solutions (which we review first in detail) into non-extremal solutions, from which we can obtain in adequate limits supersymmetric and non-supersymmetric extremal black holes. In Section 3 we do the same for the \( \mathbb{C}P^1 \) model. The black hole potential has flat directions and its non-supersymmetric critical points span a hypersurface in the moduli space. In other words: the attractor mechanism does not uniquely fix the values of the scalars on the horizon in terms of the electric and magnetic charges alone. Consequently the prescription of Ref. [31] for constructing full interpolating solutions from the horizon values of scalars by replacing charges with harmonic functions does not work. We will find these extremal non-supersymmetric solutions as limits of the non-extremal ones. In Section 4 we do the same for the well-known 4-charge solutions of the \( STU \) model. We show that there are 16 possible extremal limits, and discuss which of them are \( N = 2 \) and/or \( N = 8 \) supersymmetric. Section 5 contains our conclusions and directions for further work.

1 Extremal and non-extremal black holes

In this section we are going to review some well-known results on static extremal and non-extremal black-hole solutions, of which we will make use later. We will also study some examples of explicit non-extremal solutions in order to gain insight and formulate a general prescription for the deformation of supersymmetric extremal solutions into non-extremal solutions.

1.1 Introductory example: the Schwarzschild black hole

The prime example of a (non-extremal) black-hole is the Schwarzschild solution, which in Schwarzschild coordinates is given by

\[
ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2_{(2)} ,
\]

where \( d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta \, d\varphi^2 \) is the spherically symmetric metric of the unit 2-sphere. In this case, the "extremal limit" is Minkowski spacetime and the non-extremality parameter that goes to zero in the extremal limit, which we will denote from now on by \( r_0 \), is just the mass \( M \):

\[
r_0 = M .
\]
The event horizon is located at the Schwarzschild radius \( r_h = 2M \) and there is a curvature singularity at \( r = 0 \).

The coordinate transformation

\[
r = \left( \rho + \frac{r_0}{2} \right) / \rho,
\]

brings this solution to the spatially isotropic form

\[
ds^2 = \left( 1 - \frac{r_0/2}{\rho} \right)^2 \left( 1 + \frac{r_0/2}{\rho} \right)^{-2} dt^2 - \left( 1 + \frac{r_0/2}{\rho} \right)^4 \left( d\rho^2 + \rho^2 d\Omega_2^2 \right),
\]

in which the horizon is located at \( \rho_h = r_0/2 \).

In order to study the attractor behavior of different quantities on the event horizon of a black hole it is convenient to use a radial coordinate \( \tau \) such that \( \tau \to -\infty \) on the horizon. In the Schwarzschild black hole there seems to be no attractor behavior, but a coordinate \( \tau \) with this property can be readily found \[32\]:

\[
\rho = -\frac{r_0}{2 \tanh \frac{r_0}{2} \tau}
\]

and with it the Schwarzschild solution can be put in the form

\[
ds^2 = e^{2U} dt^2 - e^{-2U} \gamma_{\mu\nu} dx^\mu dx^\nu,
\]

\[
\gamma_{\mu\nu} dx^\mu dx^\nu = \frac{r_0^4}{\sinh^4 \frac{r_0}{2} \tau} d\tau^2 + \frac{r_0^2}{\sinh^2 \frac{r_0}{2} \tau} d\Omega_2^2,
\]

which is valid for the exterior of any static non-extremal black hole with different values of the function \( U(\tau) \). For the Schwarzschild black hole

\[
U = r_0 \tau,
\]

and the radial coordinate \( \tau \) takes values in the interval \( (-\infty, 0) \), whose limits correspond to the event horizon and spatial infinity, where the radius of the 2-spheres becomes infinitely large. In the interval \( (0, +\infty) \) the above metric describes a Schwarzschild solution with negative mass and a naked singularity at \( \tau \to +\infty \) (just transform \( \tau \to -\tau \)). In more general cases the interval \( (0, +\infty) \) will describe different patches of the black-hole spacetime.

Using the above general metric for static, non-extremal black holes, it can be shown \[33\] that the non-extremality parameter \( r_0 \) satisfies

\[
r_0^2 = 2ST,
\]

where \( S \) is the Bekenstein entropy and \( T \) is the Hawking temperature.

In the extremal limit \( r_0 \to 0 \) all static black holes are described by a metric of the same general form of Eq. \((1.6)\), but the 3-dimensional spatial metric reduces to

\[
\gamma_{\mu\nu} dx^\mu dx^\nu = \frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2} d\Omega_2^2,
\]
which, as can be seen by setting \( \tau = -1/r \), is the Euclidean metric of \( \mathbb{R}^3 \) in standard spherical coordinates. In the Schwarzschild case, \( U = 0 \) in the extremal limit and the full metric becomes Minkowski’s.

### 1.2 General results

In Ref. [30], in which the attractor behavior of general, static, \( d = 4 \) black-hole solutions was first studied, it was assumed that all of them could be written in the general form of Eq. (1.6), \( U \) being a function of \( \tau \) to be determined and \( r_0 \) (denoted by \( c \) in Ref. [30]) being a general non-extremality parameter whose value as a function of physical constants (mass, electric and magnetic charges and asymptotic values of the scalars) also has to be determined. The action considered in that reference (slightly adapted to our conventions)

\[
I = \int d^4x \sqrt{|g|} \left\{ R + G_{ij}(\phi) \partial_{\mu} \phi^i \partial^{\mu} \phi^j + 2\Im N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^{\Sigma \mu \nu} - 2\Re N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} \ast F^{\Sigma \mu \nu} \right\},
\]

(1.10)

and can describe the bosonic sectors of all 4-dimensional ungauged supergravities for appropriate \( \sigma \)-model metrics and kinetic matrices \( N_{\Lambda \Sigma}(\phi) \). The indices \( i, j, \ldots \) run over the scalar fields and the indices \( \Lambda, \Sigma, \ldots \) over the 1-form fields. Their numbers are related only for \( N \geq 2 \) supergravity theories.

Using the general form of the metric for a static non-extremal black hole, Eq. (1.6), as well as the conservation of the electric and magnetic charges, the equations of motion of the above generic action can be reduced to those of an effective mechanical system with variables \( U(\tau), \phi(\tau) \):

\[
U'' + e^{2U} V_{bh} = 0,
\]

(1.11)

\[
(U')^2 + \frac{1}{2} G_{ij} \phi^i' \phi^j' + e^{2U} V_{bh} = r_0^2,
\]

(1.12)

\[
(G_{ij} \phi^i')' - \frac{1}{2} \partial_i G_{jk} \phi^j \phi^k' + e^{2U} \partial_i V_{bh} = 0.
\]

(1.13)

Primes signify differentiation with respect to the inverse radial coordinate \( \tau \), which plays the role of the evolution parameter. The so-called black-hole potential is given by:

\[
-V_{bh}(\phi, Q) \equiv -\frac{1}{2} Q^M Q^N M_{MN} \equiv -\frac{1}{2} (p^\Lambda \ q_\Lambda \ \begin{pmatrix} (\mathcal{J} + 2\Re \mathcal{J} \mathcal{J}^{-1})_{\Lambda \Sigma} & -2(\Re \mathcal{J}^{-1})_{\Lambda \Sigma} \\ -2(\mathcal{J}^{-1} \Re \mathcal{J})_{\Lambda \Sigma} & (\mathcal{J}^{-1} \Lambda \Sigma) \end{pmatrix} \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix}),
\]

(1.14)

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2 Our conventions are those of Refs. [16][17].
3 We adopt the sign of the black-hole potential opposite to most of the literature on black-hole attractors, conforming instead to the conventions of Lagrangian mechanics.
where we replaced each symplectic pair of superscript and subscript indices $\Lambda, \Sigma, \ldots$ with a single Latin letter $M, N, \ldots$, and used the shorthand

\[ R_{\Lambda \Sigma} \equiv \Re m_{\Lambda \Sigma}, \quad J_{\Lambda \Sigma} \equiv \Im m_{\Lambda \Sigma}, \quad (\mathfrak{I}^{-1})^{\Lambda \Sigma} \mathfrak{J}_{\Sigma \Gamma} = \delta^{\Lambda \Gamma}. \] (1.15)

Eqs. (1.11) and (1.13), but not the constraint Eq. (1.12), can be derived from the effective action

\[ I_{\text{eff}}[U, \phi^i] = \int d\tau \left\{ (U')^2 + \frac{1}{2} \mathcal{G}_{ij} \phi^i \phi^j' - e^{2U} \mathcal{V}_{\text{bh}} + r_0^2 \right\}. \] (1.17)

In Ref. [30] it was shown that for regular extremal ($r_0 = 0$) black holes the values of the scalars on the event horizon $\phi^i_h$ are critical points of the black hole potential, i.e. they satisfy

\[ \partial_i \mathcal{V}_{\text{bh}} \big|_{\phi^i_h} = 0. \] (1.18)

These equations can be solved in terms of the charges but, if the black hole potential has flat directions, the equations will be underdetermined and their solution will have residual dependence on the asymptotic values of the scalars at spatial infinity ($\tau \to 0^-$):

\[ \phi^i_h = \phi^i_h(\phi^i_\infty, Q). \] (1.19)

Furthermore, it was shown that the value of the black-hole potential at the critical points gives the entropy:

\[ S = -\pi \mathcal{V}_{\text{bh}}(\phi, Q) \big|_{\phi^i_h} \] (1.20)

and that the near-horizon geometry is that of $\text{AdS}_2 \times S^2$ with the $\text{AdS}_2$ and $S^2$ radii both equal to $(-\mathcal{V}_{\text{bh}}(\phi^i_h))^{1/2}$. Even though the critical loci may not be isolated points, in which case the scalars will vary along the flat directions of the potential when one changes $\phi^i_\infty$, the stationary value itself will not be affected, hence the entropy depends on the charges only \[35\]. Each solution to Eq. (1.18) yields a possible set of values of the scalars on the event horizon and of the radii, thus a possible extremal black-hole solution.

In the general case one can prove the following extremality bound \[30\]:

\[ r_0^2 = M^2 + \frac{1}{2} \mathcal{G}(\phi^i_\infty) \Sigma^i \Sigma^j + \mathcal{V}_{\text{bh}}(\phi^i_\infty, Q) \geq 0, \] (1.21)

where $M, \Sigma^i$ are the mass and scalar charges defined by the behavior at spatial infinity ($\tau \to 0^-$).

\[ ^4 \text{The three equations (1.11)–(1.13) can be derived from a more general effective action, which is reparametrization invariant:} \]

\[ I_{\text{eff}}[U, \phi^i, e] = \int d\tau \left\{ e^{-1} \left[ (U')^2 + \frac{1}{2} \mathcal{G}_{ij} \phi^i \phi^j' \right] - e^{2U} \mathcal{V}_{\text{bh}} - r_0^2 \right\}, \] (1.16)

where $e(\tau)$ is an auxiliary einbein. We can recover the effective action Eq. (1.17) in the gauge $e(\tau) = 1$, in which the equation of motion of $e$ gives precisely the constraint Eq. (1.12). The constant term in Eq. (1.17) is usually ignored, as it is a total derivative.

\[ ^5 \text{In the absence of stationary points the scalars would be singular on the horizon. We do not consider such cases.} \]
\[ U \sim 1 + M \tau , \]
\[ \phi^i \sim \phi^i_\infty - \Sigma^i \tau . \]  

1.2.1 Flow equations

Whenever the potential term can be represented as a sum of squares of derivatives of a so-called (generalized) superpotential function \( Y(U, \phi^i, Q, r_0) \) of the warp factor \( U \) and the scalars \( \phi^i \),

\[- e^{2U} V_{bh}^2 = (\partial_U Y)^2 + 2 G^{ij} \partial_i Y \partial_j Y , \]

the effective action Eq. (1.17) also admits a rewriting as a sum of squares (up to a total derivative)

\[ I_{\text{eff}}[U, \phi^i] = \int d\tau \left\{ (U' \pm \partial_U Y)^2 + \frac{1}{2} G^{ij} (\phi'^j \pm 2 G^{i\ell} \partial_{\ell} Y) (\phi'^j \pm 2 G^{j\ell} \partial_{\ell} Y) \mp 2 Y' \right\} , \]

whose variation leads to first-order gradient flow equations, solving the second-order equations of motion [36].\(^6\)

\[ U'' = \partial_U Y , \]
\[ \phi'^i = 2 G^{ij} \partial_j Y . \]

Of the two signs in Eq. (1.24), only one, dependent on conventions, is physically admissible. We take \( \partial_U Y \) to be positive.) It is easy to see that

\[ \partial_i Y = 0 \Rightarrow \partial_i V_{bh} = 0 , \]

which sometimes simplifies the task of finding critical points of the black-hole potential. Observe also that when there is a generalized superpotential \( Y \), the mass and scalar charges are determined by its derivatives at spatial infinity \( \tau \to 0^- \):

\[ M = \lim_{\tau \to 0^-} \partial_U Y , \quad \Sigma^i = - \lim_{\tau \to 0^-} G^{ij} \partial_j Y . \]

The generalized superpotential \( Y(U, \phi^i, Q, r_0) \) has been proven [40, 41] to exist in theories whose scalar manifold (after timelike dimensional reduction) is a symmetric coset space, thus in particular for extended supergravities with more than 8 supercharges.

In the extremal cases, when there is a generalized superpotential function \( Y(U, \phi, Q) \), it factorizes into

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\(^6\) This generalized the results of Refs. [37] [38]. For first-order equations with a \( \tau \)-dependent superpotential see Refs. [39] [40].
\[ Y(U, \phi, Q) = e^U W(\phi, Q), \]

where \( W \) is called the superpotential. The flow equations take the form \[ U' = e^U W, \]

\[ \phi' = 2e^U G^{ij} \partial_j W. \]

In supergravities with more than 8 supercharges and in the extremal limit there is always at least one superpotential associated with the skew eigenvalues of the central charge, the above flow equations are related to the Killing spinor identities, and the corresponding extremal black-hole solutions are supersymmetric. However, in general there are extremal black-hole solutions that are not supersymmetric and satisfy the above flow equations for a different superpotential. We will discuss this point in more detail for \( N = 2 \) supergravity in the next section.

The stationary values of the superpotential

\[ \partial_i W \big|_{\phi_h} = 0 \]

give the the entropy:

\[ S = \pi |W(\phi_h, Q)|^2. \]

### 1.3 \( N = 2, d = 4 \) supergravity

In this paper we will focus on theories of ungauged \( N = 2, d = 4 \) supergravity coupled to \( n \) vector supermultiplets (that is, with \( \bar{n} = n + 1 \) vector fields \( A^\Lambda_{\mu}, \Lambda = 0, 1, \ldots, n \), taking into account the graviphoton).\(^7\) The \( n \) scalars of these theories, denoted by \( Z^i, i = 1, \ldots, n \) are complex and parametrize a special Kähler manifold with Kähler metric \( G^{ij}_{\ast} = \partial_i \partial_j K \), where \( K(Z, Z^\ast) \) is the Kähler potential, and the Eqs. (1.11)–(1.13) can be rewritten in the form

\[ U'' + e^{2U} V_{\text{bh}} = 0, \]

\[ (U')^2 + G_{ij} Z^{ij'} Z^{j\ast'\ast} + e^{2U} V_{\text{bh}} = r_0^2, \]

\[ Z^{i''} + G^{ij} \partial_k G_{ij} Z^{k\ast'} Z^{l'} + e^{2U} G^{ij} \partial_j V_{\text{bh}} = 0. \]

Furthermore, the black-hole potential takes the simple form

\(^7\)See, for instance, Ref. [43], the review [44], and the original works [45, 46] for more information on \( N = 2, d = 4 \) supergravities.
\[- V_{\text{bh}}(Z, Z^*, Q) = |Z|^2 + G^{ij} \mathcal{D}_i Z \mathcal{D}_j Z^*, \]

where

\[ Z = Z(Z, Z^*, Q) \equiv \langle \mathcal{V} | Q \rangle = -V^M Q^N \Omega_{MN} = p^A \mathcal{M}_A - q_A \mathcal{L}_A, \tag{1.38} \]

is the central charge of the theory, \( V^M = (\mathcal{L}^A, \mathcal{M}_A) \) is the covariantly holomorphic symplectic section, \((\Omega_{MN}) = \begin{pmatrix} 0 & \mathbb{I}_{n \times n} \\ -\mathbb{I}_{n \times n} & 0 \end{pmatrix}\) is the symplectic metric, and

\[ \mathcal{D}_i Z = e^{-K/2} \partial_i (e^{K/2} Z), \tag{1.39} \]

is the Kähler covariant derivative.

Since \( \mathcal{D}_i Z = 2(\mathcal{Z}/\mathcal{Z}^*)^{1/2} \partial_i |\mathcal{Z}| \), in \( N = 2 \) theories there is always at least one superpotential \( W = |\mathcal{Z}|, \tag{1.40} \)

and the associated flow equations (1.30), (1.31) for extremal black holes take the form

\[ U' = e^U |\mathcal{Z}|, \tag{1.41} \]
\[ Z'^{ij} = 2e^U G^{ij} \partial_j |\mathcal{Z}|. \tag{1.42} \]

It can be shown that these flow equations follow from the \( N = 2 \) Killing spinor identities and the corresponding extremal black-hole solutions are supersymmetric \[^8\] \(|\mathcal{Z}| \) is the only superpotential associated to supersymmetric solutions in \( N = 2 \) theories, but there can be more non-supersymmetric superpotentials \( W \).

Then, for \( N = 2 \) theories, the critical points of the black-hole potential (that we will loosely call attractors from now on) are of two kinds:

**Supersymmetric (or BPS) attractors**, for which

\[ \mathcal{D}_i Z \big|_{Z_h} = 0 \quad \text{or, equivalently} \quad \partial_i |\mathcal{Z}| \big|_{Z_h} = 0. \tag{1.43} \]

As we have mentioned, the extremal black-hole solutions associated to these attractors are supersymmetric and the functions \( U(\tau), Z^i(\tau) \) satisfy the above flow equations. Furthermore, according to the general results, the entropy is given by the value of the central charge at the horizon

\[ S = \pi |Z(Z_h, Z^*_h, Q)|^2 \tag{1.44} \]

[^8]: For a rigorous proof, see Ref. [17].
and the mass of the black hole is given by the value of the central charge at infinity (BPS relation)

\[ M = |Z(Z_\infty, Z_\infty^*, Q)|. \tag{1.45} \]

In this case, since at supersymmetric critical points the Hessian of the black hole potential \(-V_{bh}\) is proportional to the (positive definite) metric on the scalar manifold, these points must be minima \cite{30}. As a consequence, the scalars on the horizon take attractor values \(Z_h = Z_h(Q)\), determined only by the electric and magnetic charges and independent of the asymptotic boundary conditions (at least within a single “basin of attraction” \cite{34}). To put it differently: supersymmetric attractors are stable. As already remarked, the attractor mechanism may fail for certain choices of charges for which the horizon is singular (small black holes).

**Non-supersymmetric attractors** \cite{47, 48, 49}. They satisfy an equation of the form

\[ \partial_i W|_{Z_h} = 0, \tag{1.46} \]

for a superpotential function \(W(Z, Z^*, Q) \neq |Z| \tag{42}\), and the solution satisfies the corresponding flow equations \((1.30), (1.31)\). The entropy will be given by Eq. \((1.33)\) and the mass and scalar charges by Eqs. \((1.28)\):

\[ S = \pi |W(Z_h, Z_h^*, Q)|^2, \quad M = |W(Z_\infty, Z_\infty^*, Q)|, \quad \Sigma^i = -G^{ij} \partial_j W(Z_\infty, Z_\infty^*, Q). \tag{1.47} \]

One of the main differences with the supersymmetric case is that the stationary points of the black hole potential do not necessarily need to be minima. For models whose scalar manifold is a homogeneous space (in particular thus for all models embeddable in \(N > 2\) supergravity) the Hessian at these points (expressed in a real basis \cite{56, 57}), has non-negative eigenvalues, therefore such stationary points are also stable, but only up to possible flat directions \cite{58, 59}. It means that the attractor mechanism is no longer guaranteed to completely fix the values of the scalars on the horizon \(Z_h^i\), which may still depend on the asymptotic values \(Z_\infty^i\) as well as on the charges \(Q\), even though the entropy will only depend on the charges. In this sense one may speak of *moduli spaces of attractors* parametrized by (combinations of) the \(Z_\infty^i\), as opposed to the supersymmetric attractors, which are isolated points in the target space of the scalars.

Only in the supersymmetric case \(\Sigma^i = D^i Z^*|_{Z_\infty}\) and, therefore, the general extremality bound Eq. \((1.24)\) does not reduce to just the BPS bound \(r_0^2 = M^2 - |Z(Z_\infty, Z_\infty^*, Q)|^2 \geq 0\) (otherwise, all extremal black holes in \(N = 2\) supergravity would automatically be supersymmetric, which is not true). One of our goals is to study the general extremality bound and interpret it in terms of the central charge and other known quantities, explaining why and how it happens that supersymmetry always implies extremality, but not the other way around, as first shown in Ref. \cite{50} (see also \cite{51}).

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1.3.1 \( N = 2, d = 4 \) black-hole solutions

How are the complete black-hole solutions (or, equivalently, the variables \( U(\tau), Z^i(\tau) \)) found? For supersymmetric (and, therefore, extremal) \( N = 2 \) supergravity solutions there is a well-established method to construct systematically all the possible black-hole solutions [1, 2, 9, 10, 11, 12, 13, 14, 16]. We will follow the prescription given in Ref. [16]:

1. Introduce a complex function \( X(Z, Z^*) \) with the same Kähler weight as the canonical symplectic section \( \mathcal{V} \) so that the quotient \( \mathcal{V}/X \) is invariant under Kähler transformations.

2. Define the real symplectic vectors \( \mathcal{R} \) and \( \mathcal{I} \) by

\[
\mathcal{R} + i\mathcal{I} \equiv \frac{\mathcal{V}}{X}.
\]

The components of \( \mathcal{R} \) can always be expressed in terms of those of \( \mathcal{I} \) (by solving the stabilization equations of Refs. [10, 11], although in some cases the relations may be difficult to find explicitly.

3. The \( 2\bar{n} \) components of imaginary part \( \mathcal{I}^M \) are given by as many real harmonic functions in \( \mathbb{R}^3 \). For single-center, spherically symmetric, black-hole solutions, they must have the form:

\[
\mathcal{I}^M = \mathcal{I}_{\infty}^M - \frac{1}{\sqrt{2}} \mathcal{Q}^M \tau.
\]

Furthermore, in order not to have NUT charge (and have staticity) we must require [17]

\[
\langle \mathcal{I}_{\infty} | \mathcal{Q} \rangle = -\mathcal{I}_{\infty}^M \mathcal{Q}^N \Omega_{MN} = 0.
\]

The choice of \( \mathcal{I}^M \) determines the components \( \mathcal{R}^M \) according to the previous discussion.

4. The scalar fields are given by

\[
Z^i = \frac{\mathcal{L}^i}{\mathcal{L}^0} = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0},
\]

and the metric function \( U \) is given by

\[
e^{-2U} = \frac{1}{2|X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle = -\mathcal{R}^M \mathcal{I}^N \Omega_{MN}.
\]

We will not need the explicit form of the vector fields but they can be found in Ref. [17].

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9 This prescription does not depend on the Kähler gauge. A function playing the same role as \( X \), namely \( 1/Z^* \), was also introduced in Ref. [8].

10 Since the relations must remain the same at all points in space, it suffices to infer them on the horizon, where the stabilization equations reduce to the attractor equations of Ref. [3].

11 The factor \( 1/\sqrt{2} \) is required for the correct normalization of the charges (in particular, to have the same normalization of the charges used in the definition of the black-hole potential) and it was omitted in Ref. [17].
Some extremal but non-supersymmetric solutions can be constructed from the attractor values \[52, 31\] by replacing the electric and magnetic charges in the expressions for the scalars on the horizon by harmonic functions (the metric function is obtained in the same way from the entropy). It is not clear, however, that this is always applicable, in particular when there are moduli spaces of non-supersymmetric attractors, as in the \(\mathbb{CP}^n\) model (Section 3).

There is no general algorithm to construct non-extremal black-hole solutions either. In some cases, the introduction of an additional harmonic function (called Schwarzschild factor in Ref. [28] and non-extremality factor in Ref. [29]) appears to be enough, but the explicit non-extremal solution [53] seems to suggest that this prescription may not always work. In order to gain more insight into this problem, which is of our main interest in this paper, we are going to examine in detail more examples of non-extremal solutions. Then we will formulate a prescription to deform any static extremal supersymmetric black-hole solution of \(N = 2, d = 4\) supergravity into a non-extremal one and, next, we will apply it to several examples in the following sections.

1.4 Second example: the Reissner–Nordström black hole

Let us consider pure \(N = 2, d = 4\) supergravity, with the bosonic action

\[
I = \int d^4x \sqrt{|g|} \left[ R - F^2 \right],
\]

which corresponds to a canonical section and period matrix

\[
\mathcal{V} = \begin{pmatrix} \mathcal{L}_0^0 \\ \mathcal{M}_0 \end{pmatrix} = \begin{pmatrix} i \\ \frac{1}{2} \end{pmatrix}, \quad \mathcal{N}_{00} = -\frac{i}{2}.
\]

The central charge and black-hole potential are

\[
Z = \frac{1}{2} p - i q, \quad -V_{bh} = |Z|^2,
\]

and, since there are no scalars, it has no critical points.

The supersymmetric extremal black-hole solutions can be constructed using the mentioned algorithm of Ref. [16]. First, we introduce the function \(X\) and the two harmonic functions

\[
\mathcal{I}^0 = \Im(\mathcal{L}^0/X) = \mathcal{I}^0_\infty - \frac{p^0}{\sqrt{2}} \tau,
\]

\[
\mathcal{I}_0 = \Im(\mathcal{M}_0/X) = \mathcal{I}_0_\infty - \frac{q_0}{\sqrt{2}} \tau,
\]

where \(\mathcal{I}^0_\infty, \mathcal{I}_0_\infty\) are constants to be determined later.\(^{12}\) It is convenient to combine these two real harmonic functions into a single complex harmonic function

\[
\mathcal{H} \equiv \frac{1}{\sqrt{2}} (\mathcal{I}^0 + 2i\mathcal{I}_0) = \mathcal{H}_\infty - Z \tau.
\]

\(^{12}\)These constants are often set equal to 1 from the beginning, which is in general incorrect, as we are going to show.
Then, it is easy to see that the zero-NUT-charge condition Eq. (1.50) can be written in the form

$$N = \Im m(\mathcal{H}_\infty \mathcal{Z}^*) = 0.$$  (1.58)

The stabilization equations determine the real parts

$$R_0 = -2I_0,$$
$$R_0 = \frac{1}{2}I_0,$$  (1.59)

and then the metric function is given by

$$e^{-2U} = |\mathcal{H}|^2 = |\mathcal{H}_\infty|^2 - 2\Re(\mathcal{H}_\infty \mathcal{Z}^*)\tau + |\mathcal{Z}|^2\tau^2.$$  (1.60)

Asymptotic flatness requires $|\mathcal{H}_\infty|^2 = 1$ and indicates that $M = \Re(\mathcal{H}_\infty \mathcal{Z})$, and then we get the well-known extremal, dyonic, Reissner–Nordström (RN) solution:

$$\mathcal{H}_\infty = \frac{\mathcal{Z}}{|\mathcal{Z}|}, \quad M = |\mathcal{Z}|, \quad S = \pi |\mathcal{Z}|^2, \quad e^{-2U} = (1 - |\mathcal{Z}|\tau)^2.$$  (1.61)

Observe that $e^{-2U}$ ends up as the square of a real harmonic function, which we can call $H$.

The non-extremal RN solutions are, of course, known as well. In our conventions, and Schwarzschild-like coordinates, the metric takes the form

$$ds^2 = \frac{(r - r_+)(r - r_-)}{r^2}dt^2 - \frac{r^2}{(r - r_+)(r - r_-)}dr^2 - r^2d\Omega^2(2),$$  (1.62)

where

$$r_\pm = M \pm r_0,$$  (1.63)

are the values of $r$ at which the outer (event) horizon (+) and inner (Cauchy) horizon (−) are located, and

$$r_0^2 = M^2 - |\mathcal{Z}|^2,$$  (1.64)

is the non-extremality parameter.

In order to study this solution using the black-hole potential formalism we first need to reexpress it in terms of the coordinate $\tau$. As an intermediate step we reexpress it in terms of spatially isotropic coordinates

$$r = [\rho^2 + M\rho + r_0^2/4]/\rho,$$  (1.65)

so it takes the form ($\rho_\pm \equiv M \pm |\mathcal{Z}|$)

$$ds^2 = \frac{(1 - \frac{r_0/2}{\rho})^2 (1 + \frac{r_0/2}{\rho})^2}{(1 + \frac{\rho_/2}{\rho})^2 (1 + \frac{\rho_-/2}{\rho})^2}dt^2 - \left(1 + \frac{\rho_/2}{\rho}\right)^2 \left(1 + \frac{\rho_-/2}{\rho}\right)^2 (d\rho^2 + \rho^2d\Omega^2(2)).$$  (1.66)
For $M = |Z|$ ($\rho_- = r_0 = 0$) we recover the extremal solution just studied (with $\rho = -1/\tau$). Next, we change to the coordinate $\tau$ as in the Schwarzschild case with $M$ replaced by $r_0$

$$
\rho = -\frac{r_0}{2\tanh \frac{r_0}{2\tau}},
$$

(1.67)

to obtain a metric of the standard form Eq. (1.6) with

$$
e^{-2U} = e^{-2r_0\tau} \left[ \frac{r_+}{2r_0} - \frac{r_- e^{2r_0\tau}}{2r_0} \right]^2.
$$

(1.68)

This metric function contains a Schwarzschild factor $e^{-2r_0\tau}$, which is the only one that remains when the charge vanishes, and the square of a function which is not a harmonic function in $\mathbb{R}^3$ but can be seen as a deformation of the function $H = 1 - |Z|/\tau$:

$$
\lim_{r_0 \to 0} \left[ \frac{r_+}{2r_0} - \frac{r_- e^{2r_0\tau}}{2r_0} \right] = H.
$$

(1.69)

As in the Schwarzschild case, when the radial coordinate coordinate $\tau$ takes values in the interval $(-\infty, 0)$, whose limits correspond to the event horizon and spatial infinity, the metric covers the exterior of the horizon. The explicit relation between the original Schwarzschild-like radial coordinate $r$ and $\tau$ in that interval is

$$
\tau = \frac{r}{2r_0 \tanh \frac{r_0}{2\tau}} \left\{ \frac{-r_0}{(r - M) + \sqrt{(r - M)^2 - r_0^2}} \right\}, \quad r \in (r_+, +\infty).
$$

(1.70)

In the RN case, however, the same metric also covers the interior of the inner horizon when $\tau$ takes values in the interval $\left( \frac{2}{r_0} \tanh \sqrt{\frac{M - |Z|}{M + |Z|}}, +\infty \right)$, whose limits correspond to the singularity at the origin and the inner horizon. The explicit relation between the original Schwarzschild-like radial coordinate $r$ and $\tau$ in that interval is

$$
\tau = \frac{2}{r_0} \tanh \left\{ \frac{-r_0}{(r - M) - \sqrt{(r - M)^2 - r_0^2}} \right\}, \quad r \in (0, r_-).
$$

(1.71)

It is easy to see that $e^{2U}$ tends to zero in the two limits $\tau \to \pm \infty$ and that the coefficient of $d\Omega^2_{(2)}$ in the metric, which can be understood as the square radius of the spatial sections of the horizons

$$
\frac{(2r_0)^2 e^{-2U}}{(e^{\tau_+} - e^{-\tau_0})^2} = \left( \frac{r_+ - r_- e^{2r_0\tau}}{e^{2r_0\tau} - 1} \right)^2 \tau \to \infty \frac{r_+^2}{\tau_+}.
$$

(1.72)

This allows us to compute the areas and, therefore, the “entropies” associated with both horizons using the standard metric:

$$
S_\pm/\pi = (r_\pm)^2,
$$

(1.73)
and, using the general result Eq. (1.8), the temperatures

\[ T_\pm = \frac{r_0^2}{2S_\pm} = \frac{1}{2\pi} (r_0/r_\pm)^2. \] (1.74)

### 1.5 General prescription

The previous result suggests the following prescription for deforming extremal, static, supersymmetric solutions of \( N = 2, d = 4 \) supergravity into non-extremal solutions: if the supersymmetric solution is given by

\[
U(\tau) = U_\epsilon[H(\tau)], \quad Z^i(\tau) = Z^i_\epsilon[H(\tau)],
\] (1.75)

where \( U_\epsilon \) and \( Z^i_\epsilon \) are the functions of certain harmonic functions \( H_\alpha(\tau) = H_\alpha\infty - Q_\alpha \tau \) (\( \alpha \) being some index) that one finds following the standard prescription for supersymmetric black holes, then the non-extremal solution is given by

\[
U(\tau) = U_\epsilon[\hat{H}(\tau)] + r_0\tau, \quad Z^i(\tau) = Z^i_\epsilon[\hat{H}(\tau)],
\] (1.76)

where the harmonic functions \( H \) have been replaced by the hatted functions

\[
\hat{H}_\alpha = a_\alpha + b_\alpha e^{2r_0\tau}. \] (1.77)

This ansatz has to be used in the three equations (1.34), (1.35) and (1.36) to determine the actual values of the integration constants \( a_\alpha, b_\alpha \). In the following sections we are going to see how this ansatz works in particular models, showing that the original differential equations are solved by the ansatz if the integration constants satisfy certain algebraic equations that related them to the charges \( Q^M \) and non-extremality parameter \( r_0 \), and we will argue that it should always work, even if the algebraic equations for the integration constants are in general difficult to solve.

Observe that, since in most cases \( e^{-2U_\epsilon(H)} \) is homogenous of second degree in the harmonic functions, following the same steps as in the RN example, we expect to find the event horizon in the \( \tau \rightarrow -\infty \) limit and the inner horizon \( \tau \rightarrow +\infty \) limit, which will allow us to find the entropies and temperatures using Eq. (1.8).

### 2 Axion-dilaton black holes

The so-called axion-dilaton black holes\(^\text{13}\) are solutions of the \( \tilde{n} = 2 \) theory with prepotential

\[
\mathcal{F} = -i\lambda^{0} \chi^{1}. \] (2.1)

This theory has only one complex scalar that it is usually called \( \tau \) but we are going to call \( \lambda \) to distinguish it from the radial coordinate. This scalar is given by

\(^{13}\text{For references on these black-hole solutions see Refs. [54].}\)
\[ \lambda \equiv i \chi^1 / \chi^0. \]  

(2.2)

In terms of \( \lambda \) the period matrix is given by

\[
(N_{\lambda \Sigma}) = \begin{pmatrix}
-\lambda & 0 \\
0 & 1/\lambda
\end{pmatrix}
\]

(2.3)

and, in the \( \chi^0 = i/2 \) gauge, the Kähler potential and metric are

\[
\mathcal{K} = -\ln \Im m \lambda, \quad \mathcal{G}_{\lambda \lambda'} = (2 \Im m \lambda)^{-2}.
\]

(2.4)

The reality of the Kähler potential requires the positivity of \( \Im m \lambda \). Therefore, \( \lambda \) parametrizes the coset \( SL(2, \mathbb{R})/SO(2) \) and the action for the bosonic fields is

\[
I = \int d^4x \sqrt{|g|} \left\{ R + \frac{\partial_{\mu} \lambda \partial^{\mu} \lambda^*}{2(\Im m \lambda)^2} - 2 \Im m \lambda \left[(F^0)^2 + |\lambda|^{-2}(F^1)^2\right] + 2 \Re e \lambda \left[F^0 \star F^0 - |\lambda|^{-2}F^1 \star F^1\right]\right\}.
\]

(2.5)

This theory is a truncation of \( N = 4, d = 4 \) supergravity. After replacing the matter vector field \( A^1 \) by its dual \( (F_1 = \Im m \lambda \star F^1 + \Re e F^1) \) the action takes the more (manifestly) symmetric form

\[
I = \int d^4x \sqrt{|g|} \left\{ R + \frac{\partial_{\mu} \lambda \partial^{\mu} \lambda}{(2 \Im m \lambda)^2} - 2 \Im m \lambda \left[(F^0)^2 + (F_1)^2\right] + 2 \Re e \lambda \left[F^0 \star F^0 + F_1 \star F_1\right]\right\},
\]

(2.6)

in which it has been exhaustively studied \([32]–[53]\). In particular, the most general (non-extremal and rotating) black-holes of this theory were presented in Ref. \([53]\). A preliminary check shows that in the static case the metric and scalars are, in the coordinate \( \tau \), of the form of our deformation ansatz, but we want to reobtain the non-extremal solutions using the ansatz and the language and notation of \( N = 2, d = 4 \) supergravities.

In order to apply the formalism reviewed in the previous section, let us start by constructing the black-hole potential.

The canonically normalized symplectic section \( \mathcal{V} \) is, in a certain gauge,

\[
\mathcal{V} = \frac{1}{2(\Im m \lambda)^{1/2}} \begin{pmatrix}
i \\
\lambda \\
-i \lambda \\
1
\end{pmatrix},
\]

(2.7)

and, in terms of the complex combinations

\[
\Gamma_1 \equiv p^1 + iq_0, \quad \Gamma_0 \equiv q_1 - ip^0,
\]

(2.8)

the central charge and its holomorphic covariant derivative and the black-hole potential are
\[ Z = \frac{1}{2 \sqrt{3m\lambda}} [\Gamma_1^* - \Gamma_0^*\lambda] , \]

\[ \mathcal{D}_\lambda Z = \frac{i}{4(3m\lambda)^{3/2}} [\Gamma_1^* - \Gamma_0^*\lambda^*] , \]  
(2.9)

\[ -V_{bh} = \frac{1}{23m\lambda} [ |\Gamma_1|^2 - 2\Re(\Gamma_1\Gamma_0^*)\Re\lambda + |\Gamma_0|^2|\lambda|^2] . \]

It is convenient to define the charge

\[ \tilde{Z} = \frac{1}{2 \sqrt{3m\lambda}} [\Gamma_1^* - \Gamma_0^*\lambda^*] , \]
(2.10)

in terms of which

\[ G^{ij} \mathcal{D}_i Z \mathcal{D}_j Z^* = |\tilde{Z}|^2 , \]
(2.11)

so we can write

\[ -V_{bh} = |Z|^2 + |\tilde{Z}|^2 , \quad -\partial_\lambda V_{bh} = 2Z^* \mathcal{D}_\lambda Z \sim Z^* \tilde{Z} . \]
(2.12)

### 2.1 Flow equations

The potential term can be expanded in the following way:

\[ -\left[ e^{2U} V_{bh} - r_0^2 \right] = \Upsilon^2 + 4G^{\lambda\lambda^*} \Psi \Psi^* , \]
(2.13)

where

\[ \Upsilon = \frac{e^U}{\sqrt{2}} \sqrt{e^{-2U} r_0^2 + |Z|^2 + |\tilde{Z}|^2 + \sqrt{(e^{-2U} r_0^2 + |Z|^2 + |\tilde{Z}|^2)^2 - 4|Z|^2|\tilde{Z}|^2}} , \]
(2.14)

\[ \Psi = \frac{i e^{2U} Z^* \tilde{Z}}{4 \sqrt{3m\lambda} \Upsilon} . \]
(2.15)

The vector field generated by \((\Upsilon, \Psi, \Psi^*)\) is conservative or, in other words, can be written as a gradient of a generalized superpotential \(Y(U, \lambda, \lambda^*)\)

\[ (\Upsilon, \Psi, \Psi^*) = (\partial_U Y, \partial_\lambda Y, \partial_{\lambda^*} Y) , \]
(2.16)

if and only if it is irrotational (i.e. its curl vanishes). This is the case here, since

\[ \partial_U \Psi - \partial_\lambda \Upsilon = \partial_U \Psi^* - \partial_{\lambda^*} \Psi = \partial_\lambda \Psi^* - \partial_{\lambda^*} \Psi = 0 , \]
(2.17)
Attractor & $\Im m \lambda_h$ & $|Z_h|^2$ & $|\tilde{Z}_h|^2$ & $-V_{bh\, h}$ & $M$ & $\Sigma^\lambda$
\hline
$\lambda_{h \text{ susy}} = \Gamma_1/\Gamma_0$ & $\Im m(\Gamma_1 \Gamma_0^*)$ & $\Im m(\Gamma_1 \Gamma_0^*)$ & 0 & $\Im m(\Gamma_1 \Gamma_0^*)$ & $|Z_\infty|$ & $2ie^{i\text{Arg}\tilde{Z}_\infty} \Im m \lambda_\infty Z'_\infty$
\hline
$\lambda_{h \text{ nsusy}} = \Gamma_1^*/\Gamma_0^*$ & $-\Im m(\Gamma_1 \Gamma_0^*)$ & 0 & $-\Im m(\Gamma_1 \Gamma_0^*)$ & $-\Im m(\Gamma_1 \Gamma_0^*)$ & $|\tilde{Z}_\infty|$ & $2ie^{-i\text{Arg}\tilde{Z}_\infty} \Im m \lambda_\infty Z_\infty$
\hline

Table 1: Critical points of the axidilaton model. Here we are using the notation $Z_h \equiv Z(\lambda_h, \lambda_h^*, Q)$ etc. In the supersymmetric case the mass $M$ can be found in the explicit solution or from the saturation of the supersymmetric bound. Then, the scalar charge $\Sigma^\lambda$ follows from the general extremality bound (or from the knowledge of the explicit solution). In the non-supersymmetric case we do not have analogous arguments and we need the explicit solution, given in Section 2.3.1.

which could have been expected on the basis of the results mentioned in Section 1.2. The explicit form of the generalized superpotential can be in principle obtained by integrating Eq. (2.14), but in practice this turns out to be very complicated.

The flow equations (1.25, 1.26), in the conventions of Eq. (1.35), now take the form:

$$U' = \Upsilon, \quad (2.18)$$

$$\lambda' = 2G^{\lambda\lambda*} \Psi^* . \quad (2.19)$$

In the particular case of the Reissner–Nordström black hole (cf. Section 1.4), the first of these equations reduces to the one derived in [37] (and the second is not applicable, since there are no scalars). For extremal black holes, studied in greater detail below, one recovers Eq. (1.30, 1.31) with either $W = |Z|$ (the supersymmetric case) or $W = |\tilde{Z}|$.

### 2.2 The extremal case

#### 2.2.1 Critical points

The critical points of the black hole potential are those for which $Z = 0$ or $\tilde{Z} = 0$. They are two isolated points in moduli space and only the second is supersymmetric. The situation is summarized in Table 1.

As already said in Section 1.3, the supersymmetric stationary points of the black hole potential must be a minimum. Indeed, the Hessian evaluated this point in the real basis has the double eigenvalue.
\[ \frac{|\Gamma_0|^4}{2 \Im m(\Gamma^*_1 \Gamma^*_0)} = |\Gamma_0|^2 \mathcal{G}_{\lambda\lambda'}^{\text{susy}} |_{h} = \frac{(p^0)^2 + (q_1)^2}{2(p^0 p^1 + q_0 q_1)}. \quad (2.20) \]

Again referring to Section 1.3, one can expect also the non-supersymmetric extremal stationary point of our model to be stable (up to possible flat directions). This is confirmed by the direct calculation of the Hessian, which has the double eigenvalue

\[ - \frac{|\Gamma_0|^4}{2 \Im m(\Gamma^*_1 \Gamma^*_0)} = |\Gamma_0|^2 \mathcal{G}_{\lambda\lambda'}^{\text{nususy}} |_{h} = \frac{(p^0)^2 + (q_1)^2}{2(p^0 p^1 + q_0 q_1)}. \quad (2.21) \]

Observe that the supersymmetric stationary point and the non-supersymmetric extremal stationary point exist for mutually exclusive choices of charges and that in this example, given that \( \tilde{Z} \) differs from \( Z \) by complex conjugation in the numerator, one could have also used, with appropriate modifications, the general supersymmetric argument [30] to study the stability of the non-supersymmetric critical point.

### 2.2.2 Supersymmetric solutions

According to the general procedure, the supersymmetric solutions are built out of the four harmonic functions

\[ \mathcal{I}^M = \mathcal{I}^M_{\infty} - \frac{Q^M}{\sqrt{2}} \tau. \quad (2.22) \]

In this theory the stabilization equations can be easily solved and they lead to

\[ \mathcal{R} = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \mathcal{I}, \quad (2.23) \]

where \( \sigma^1 \) is the standard Pauli matrix, so

\[ e^{-2U} = \langle \mathcal{R} | \mathcal{I} \rangle = 2(\mathcal{I}^0 \mathcal{I}^1 + \mathcal{I}_0 \mathcal{I}_1), \quad \lambda = \frac{\mathcal{L}^1}{\mathcal{L}^0} \frac{X}{i}, \quad \mathcal{R} = \frac{\mathcal{I}^1 + i \mathcal{I}_0}{\mathcal{I}^0 - i \mathcal{I}^1}. \quad (2.24) \]

It is useful to define the complex harmonic functions

\[ \mathcal{H}_1 \equiv \mathcal{I}^1 + i \mathcal{I}_0 = \mathcal{H}_1^\infty - \frac{\Gamma_1}{\sqrt{2}} \tau, \quad \mathcal{H}_0 \equiv \mathcal{I}^0 - i \mathcal{I}_0 = \mathcal{H}_0^\infty - \frac{\Gamma_0}{\sqrt{2}} \tau, \quad (2.25) \]

in terms of which we have

\[ e^{-2U} = 2 \Im m(\mathcal{H}_1 \mathcal{H}_0^*), \quad \lambda = \frac{\mathcal{H}_1}{\mathcal{H}_0}. \quad (2.26) \]

The solution depends on the charges \( Q \) and on the two complex constants \( \mathcal{H}_1^\infty \) and \( \mathcal{H}_0^\infty \). A combination of them \( (\mathcal{H}_1^\infty / \mathcal{H}_0^\infty) \) is \( \lambda_\infty \) and the other combination is determined in terms of \( Q \) and \( \lambda_\infty \) by imposing asymptotic flatness.
\[ 2\Im (\mathcal{H}_1 \mathcal{H}'_0) = 1, \]  
which provides one real condition, and absence of NUT charge
\[ \Re (\mathcal{H}_1 \Sigma_0 - \mathcal{H}_0 \Sigma_1^*) = 0, \]
which is another real condition. These conditions have two solutions
\[ \mathcal{H}_1 = \lambda \mathcal{H}_0, \quad \mathcal{H}_0 = \mp \frac{i}{\sqrt{2\Im \lambda}} \frac{Z'_0}{|Z'_0|}, \quad Z_\infty \equiv Z(\lambda, \lambda^*, Q), \]
but, using them in the expression for the mass
\[ M = \frac{1}{\sqrt{2}} \Im (\mathcal{H}_1 \Sigma_0 - \mathcal{H}_0 \Sigma_1^*), \]
one finds that only the upper sign gives a positive mass, which turns out to be equal to \(|Z_\infty|\), as expected.

The complete solution is, therefore, given by the two harmonic functions
\[ H_{1}\text{susy} = -\frac{i\lambda}{\sqrt{2\Im \lambda}} \frac{Z'_0}{|Z'_0|} - \frac{\Sigma_1}{\sqrt{2}}, \quad H_{0}\text{susy} = -\frac{i}{\sqrt{2\Im \lambda}} \frac{Z'_0}{|Z'_0|} - \frac{\Sigma_0}{\sqrt{2}}. \]

### 2.2.3 Extremal non-supersymmetric solutions

According to the proposal made for the \( STU \) model in Ref. [31], the metric and scalar fields of the extremal non-supersymmetric solutions can be constructed by replacing the electric and magnetic charges of their attractor values by the harmonic functions that have those charges as coefficients, that is \( Q^M \) should be replaced by the real harmonic function
\[ H^M = H_\infty^M - \frac{1}{\sqrt{2}} Q^M \tau. \]

The constant parts of the harmonic functions cannot be the same as those of the supersymmetric solution, otherwise the prescription would lead to
\[ e^{-2U} = -2(\mathcal{I}^0 \mathcal{I}^1 + \mathcal{I}_0 \mathcal{I}_1), \quad \lambda = \frac{\mathcal{I}^1 - i\mathcal{I}_0}{\mathcal{I}_1 + i\mathcal{I}^0}, \]
or, in terms of the complex harmonic functions defined in Eq. (2.25)
\[ e^{-2U} = -2\Im (\mathcal{H}_1 \mathcal{H}'_0), \quad \lambda = \frac{\mathcal{H}'_1}{\mathcal{H}'_0}. \]

If we plug in these expressions the values of the harmonic functions determined before, we get inconsistent results, because the metric function \( e^{-2U} \) is that of the supersymmetric case and goes to \(-1\) at spatial infinity. Thus, the prescription given in Ref. [31] should be interpreted as a
replacement of the charges by harmonic functions with asymptotic values yet to be determined by imposing asymptotic flatness etc. In Section 2.3.1 we will determine the form of the extremal non-supersymmetric solutions by taking an appropriate extremal limit of the non-extremal solution.

2.3 Non-extremal solutions

Our ansatz of Section 1.5 for the non-extremal solution is

\[ e^{-2U} = e^{-2[U_e(H)+r_0\tau]} \], \quad \quad e^{-2U_e(H)} = 2\Im(\hat{H}_1\hat{H}_0^*), \quad \lambda = \lambda_e(H) = \hat{H}_1/\hat{H}_0, \quad (2.35) \]

where the deformed harmonic functions are assumed to have the form

\[ \hat{H}_\Lambda \equiv A_\Lambda + B_\Lambda e^{2r_0\tau}, \quad \Lambda = 1, 0, \quad (2.36) \]

The four complex constants \( A_\Lambda, B_\Lambda \) need to be determined by imposing on them the equations of motion (1.34)–(1.36), asymptotic flatness, absence of NUT charge plus the definitions of \( M \) and \( \lambda_\infty \). Solving the equations of motion is not as complicated a task as it may look at first sight. First of all, we observe that all the dependence of \( U \) and \( \lambda \) on \( \tau \) is of the form of the Schwarzschild factor \( e^{2r_0\tau} \), which we are going to denote by \( f \). Using the chain rule and combining the first two equations, we get

\[ \dot{U}_e - (\dot{U}_e)^2 - G_{ij}\dot{Z}^i\dot{Z}^j = 0, \quad (2.37) \]

\[ (2r_0)^2 \left[ f\ddot{U}_e + \dot{U}_e \right] + e^{2U_e}V_{bh} = 0, \quad (2.38) \]

\[ (2r_0)^2 \left[ f \left( \ddot{Z}^i + G^{ij} \partial_k G_{ij} \dot{Z}^k \dot{Z}^l \right) + \dot{Z}^i \right] + e^{2U_e}G^{ij} \partial_j V_{bh} = 0. \quad (2.39) \]

Secondly, \( U_e \) and \( \lambda \) only depend on \( f \) through the deformed harmonic functions and, therefore, by virtue of the chain rule:

\[ \dot{U}_e = \partial_\Lambda U_e B_\Lambda + \text{c.c.}, \]

\[ \ddot{U}_e = \partial_\Sigma \partial_\Lambda U_e B_\Lambda B_\Sigma + \partial_\Sigma \partial_\Lambda U_e B_\Lambda B_\Sigma + \text{c.c.}, \quad (2.40) \]

\[ \dot{Z}^i = \partial_\Lambda Z^i B_\Lambda + \partial_\Lambda Z^i B_\Lambda^*, \]

e tc., where \( \partial_\Lambda \equiv \partial/\partial \hat{H}_\Lambda \) and \( \partial_\Lambda^* \equiv \partial/\partial \hat{H}_\Lambda^* \). Then Eq. (2.37) becomes, after multiplication by a convenient global factor, a quadratic polynomial in the deformed harmonic functions with coefficients that are combinations of the integration constants \( B_\Lambda \). This is true for any \( N = 2 \)
model. For the axidilaton model, the polynomial turns out to be the square of a generalization of the condition of absence of NUT charge:

\[ \Re(\mathcal{H}_1 B_0^* - \mathcal{H}_0 B_1^*) = \Re(A_1 B_0^* - A_0 B_1^*) \, . \tag{2.41} \]

Setting this quantity to zero yields an algebraic equation for the integration constants, which is enough to solve the first equation. In a similar fashion we find that the other two differential equations are solved by our ansatz if the integration constants satisfy certain algebraic constraints that we summarize here:

\[ \Re(A_1 B_0^* - A_0 B_1^*) = 0 \, , \tag{2.42} \]

\[ |\Gamma_1|^2 A_0 B_0 + |\Gamma_0|^2 A_1 B_1 - \Re(\Gamma_1 \Gamma_0^*)(A_1 B_0 - A_0 B_1) = 0 \, , \tag{2.43} \]

\[ |\Gamma_1|^2 B_0^2 + |\Gamma_0|^2 B_1^2 - 2\Re(\Gamma_1 \Gamma_0^*)B_1 B_0 - 8i r_0^2 \Im(A_1 A_0^*)(A_1 B_0 - A_0 B_1) = 0 \, , \tag{2.44} \]

\[ \Re(A_0 B_0^*) + \frac{1}{8 r_0^2} |\Gamma_0|^2 = 0 \, , \tag{2.46} \]

\[ \Re(A_1 B_1^*) + \frac{1}{8 r_0^2} |\Gamma_1|^2 = 0 \, , \tag{2.47} \]

\[ \Re(A_0 B_1^* + A_1 B_0^*) + \frac{1}{4 r_0^2} \Re(\Gamma_1 \Gamma_0^*) = 0 \, , \tag{2.48} \]

and to which we must add the conditions of asymptotic flatness and the definitions of \( M \) and \( \lambda_\infty \):

\[ 2\Im[(A_1 + B_1)(A_0^* + B_0^*)] = 1 \, , \tag{2.49} \]

\[ 2 r_0 \Im[A_1 A_0^* - B_1 B_0^*] = M \, , \tag{2.50} \]

\[ \frac{A_1 + B_1}{A_0 + B_0} = \lambda_\infty \, . \tag{2.51} \]

From these equations we can derive a relation between the non-extremality parameter, mass, charge and moduli, which is convenient to write in this form:

\[ M^2 r_0^2 = (M^2 - |Z_\infty|^2)(M^2 - |\tilde{Z}_\infty|^2) \, . \tag{2.52} \]
This shows that there are two different extremal limits (supersymmetric and non-supersymmetric) and that the non-extremal family of solutions interpolates between these two limits. This will allow us to obtain the extremal non-supersymmetric solution in a clean way. Observe that in the context of $N = 4$ supergravity both extremal limits are supersymmetric \[60, 53\].

Expanding the above expression and comparing with the general result Eq. (1.21) one can find the scalar charge up to a phase. From the complete solution (see later) we obtain the exact result

$$
\Sigma^\lambda = \frac{2i \Im \lambda^\infty Z_\infty Z_\infty^*}{M}.
$$

(2.53)

Since the expressions for the metric function and the scalar are invariant if we multiply $H_1, H_0$ by the same phase, we can use this freedom to simplify the equations setting $\Im (A_0 + B_0) = 0$.

Under this assumption we find (we use a tilde to stress the fact that these are not the final values of the integration constants):

$$
\tilde{A}_1 = \frac{\lambda^\infty}{2 \sqrt{2} \Im \lambda^\infty} \left\{ 1 + \frac{1}{Mr_0} \left\{ M^2 + \frac{1}{2} V_{bh}^\infty + \frac{i}{2} \left[ \frac{1}{\lambda^\infty} |\Gamma_1|^2 - \Re(\Gamma_1 \Gamma_0^*) \right] \right\} \right\},
$$

(2.54)

$$
\tilde{B}_1 = \frac{\lambda^\infty}{2 \sqrt{2} \Im \lambda^\infty} \left\{ 1 - \frac{1}{Mr_0} \left\{ M^2 + \frac{1}{2} V_{bh}^\infty + \frac{i}{2} \left[ \frac{1}{\lambda^\infty} |\Gamma_1|^2 - \Re(\Gamma_1 \Gamma_0^*) \right] \right\} \right\},
$$

(2.55)

$$
\tilde{A}_0 = \frac{1}{2 \sqrt{2} \Im \lambda^\infty} \left\{ 1 + \frac{1}{Mr_0} \left\{ M^2 + \frac{1}{2} V_{bh}^\infty - \frac{i}{2} \left[ \lambda^\infty |\Gamma_0|^2 - \Re(\Gamma_1 \Gamma_0^*) \right] \right\} \right\},
$$

(2.56)

$$
\tilde{B}_0 = \frac{1}{2 \sqrt{2} \Im \lambda^\infty} \left\{ 1 - \frac{1}{Mr_0} \left\{ M^2 + \frac{1}{2} V_{bh}^\infty - \frac{i}{2} \left[ \lambda^\infty |\Gamma_0|^2 - \Re(\Gamma_1 \Gamma_0^*) \right] \right\} \right\},
$$

(2.57)

where we are using the shorthand notation $V_{bh}^\infty \equiv V_{bh}(\lambda^\infty, \lambda^{\ast \infty}, Q)$.

Then, the metric function can be put in the two alternative forms

$$
e^{-2U} = 1 \pm \frac{M}{r_0} (1 - e^{\pm 2r_0\tau}) + \frac{S_{\pm} \sinh^2 r_0 \tau}{\pi r_0^2},
$$

(2.58)

where $S_{\pm}$ are the entropies associated to the outer ($+$) and inner ($-$) horizons, given in Eqs. (2.63)–(2.65). In any of these two forms $e^{-2U}$ is a sum of manifestly positive terms when $r_0^2 > 2$ and $S_{\pm} > 0$, so all the singularities will be covered by the horizons when they exist. The conditions under which this happens will be studied later.

### 2.3.1 Supersymmetric and non-supersymmetric extremal limits

The hatted functions have the following extremal limits ($r_0 \to 0$):

1. The supersymmetric extremal limit, when $M \to |Z_\infty|$
\[ H_{1,0} \rightarrow \frac{M}{|Z|} H_{\text{susy}}^{1,0}, \quad (2.59) \]

with \( H_{\text{susy}}^{1,0} \) given in Eq. (2.31).

2. The non-supersymmetric extremal limit, when \( M \rightarrow |\tilde{Z}| \)

\[ H_{1,0} \rightarrow \frac{|Z|}{\tilde{Z}} H_{\text{nsusy}}^{1,0}, \quad (2.60) \]

with

\[ H_{\text{nsusy}}^{1} = -\frac{i \lambda_{\infty}}{\sqrt{2} m \lambda_{\infty}} \frac{\tilde{Z}}{|Z|} - \frac{\Gamma_{1}}{\sqrt{2}} \tau, \quad H_{\text{nsusy}}^{0} = -\frac{i}{\sqrt{2} m \lambda_{\infty}} \frac{\tilde{Z}}{|Z|} - \frac{\Gamma_{0}}{\sqrt{2}} \tau. \quad (2.61) \]

\( H_{\text{nsusy}}^{1,0} \) can be obtained by replacing everywhere in \( H_{\text{nsusy}}^{1,0} \) the complex charges \( \Gamma_{0,1} \) by their complex conjugates \( \Gamma_{0,1}^{*} \).

We stress that in this case, the metric function and scalar are still given by

\[ e^{-2U} = 2 \Im (H_{\text{nsusy}}^{1} H_{\text{nsusy}}^{0})^*, \quad \lambda = H_{\text{nsusy}}^{1} / H_{\text{nsusy}}^{0}, \quad (2.62) \]

and it is immediate to check that they lead to the non-supersymmetric attractor and entropy.

### 2.3.2 Physical properties of the non-extremal solutions

The “entropies” (one quarter of the areas) of the outer (+) and inner (−) horizon, placed at \( \tau = -\infty \) and \( \tau = +\infty \), respectively, are given by

\[ S_{\pm} = \frac{(M^{2} - |Z_{\infty}|^{2}) \pm (M^{2} - |\tilde{Z}_{\infty}|^{2}) \pm 2Mr_{0}}{\pi}. \quad (2.63) \]

They can also be written in the form

\[ S_{\pm} = \pi \left( \sqrt{N_{R}} \pm \sqrt{N_{L}} \right)^{2}, \quad (2.64) \]

with

\[ N_{R} \equiv M^{2} - |Z_{\infty}|^{2}, \quad N_{L} \equiv M^{2} - |\tilde{Z}_{\infty}|^{2}, \quad (2.65) \]

so the product of these “entropies” is manifestly moduli-independent:

\[ S_{+} S_{-} = \pi^{2} (N_{R} - N_{L})^{2} = \pi^{2} [\Im (\Gamma_{1} \Gamma_{0}^*)]^{2}. \quad (2.66) \]
From Ref. [53] we know exactly how these expressions are modified by the introduction of angular momentum $J \equiv \alpha M$: the entropies are given by

$$S_{\pm} = (M^2 - |Z_\infty|^2) \pm (M^2 - |\tilde{Z}_\infty|^2) \pm 2M\sqrt{r_0^2 - \alpha^2},$$

and can be put in the suggestive form of Eq. (2.64) with

$$N_{R,L} \equiv M^2 - \frac{1}{2}(|Z_\infty|^2 + |\tilde{Z}_\infty|^2) \pm \frac{1}{2}\sqrt{(|Z_\infty|^2 - |\tilde{Z}_\infty|^2)^2 + 4J^2}.$$

Again, the product of the two entropies is moduli-independent:

$$S_+ S_- = \pi^2 (N_R - N_L)^2 = \pi^2 \left\{ |\Im \left( \Gamma_1 \Gamma_0^* \right) |^2 + 4J^2 \right\}.$$

The temperatures $T_{\pm}$ can be computed from $S_{\pm}$ using Eq. (1.8).

In the two extremal cases, the scalar takes attractor values on the horizon, which are independent of its asymptotic value $\lambda_\infty$. In non-extremal black holes the scalar takes the horizon value

$$\lambda_{h}^{\text{ne}} = \frac{\lambda_\infty S_+/\pi + i|\Gamma_1|^2 - \lambda_\infty \Re(\Gamma_1 \Gamma_0^*)}{S_+/\pi - i|\lambda_\infty |\Gamma_0|^2 - \Re(\Gamma_1 \Gamma_0^*)},$$

which manifestly depends on $\lambda_\infty$, from which we conclude that the attractor mechanism does not work in this case.

We observe that if, in the general non-extremal case, $\lambda_\infty$ is set equal to one of the two attractor values, then $\lambda(\tau)$ is constant over the space. In other words: the non-extremal deformation of a double-extremal black hole also has constant scalars and, therefore, has the metric of the non-extremal Reissner–Nordström black hole.

In the evaporation of a non-extremal black hole of this theory only $M$ changes, while the charges and $\lambda_\infty$ remain constant\footnote{There are no particles carrying electric or magnetic charges in ungauged $N = 2, d = 4$ supergravity and there is no perturbative physical mechanism that can change the moduli, which are properties characterizing the vacuum.}. The value of $M$ will decrease until it becomes equal to \(\max(|Z_\infty|, |\tilde{Z}_\infty|)\). This value depends on the values of the charges and moduli in this way:

$$|Z_\infty| > |\tilde{Z}_\infty| \iff \cos \text{Arg}(\lambda_\infty/\lambda_{h}^{\text{ne}}) > \cos \text{Arg}(\lambda_\infty/\lambda_{h}^{\text{nsusy}}).$$

Hence, if the phase of $\lambda_\infty$ is closer to that of the supersymmetric attractor value $\Gamma_1/\Gamma_0$ than to that of the non-supersymmetric one $\Gamma_1^*/\Gamma_0^*$, the central charge $|Z_\infty|$ will be larger than $|\tilde{Z}_\infty|$ and the evaporation process will stop at the supersymmetric extremal limit and vice versa. However, in this analysis we must take into account that the imaginary part of $\lambda$ must be positive at any point, which means that $\Im \lambda_\infty > 0$ and only one of $\lambda_{h}^{\text{nsusy}}$ and $\lambda_{h}^{\text{nsusy}}$ will satisfy that condition for a given choice of electric and magnetic charges. Then, it is easy to see that if $\Im \lambda_{h}^{\text{nsusy}} > 0$, for any $\lambda_\infty$ satisfying $\Im \lambda_\infty > 0$, the above condition is met and the endpoint of the evaporation process should be the supersymmetric one and, if $\Im \lambda_{h}^{\text{nsusy}} > 0$ then the opposite will be true for any admissible $\lambda_\infty$.\footnote{There are no particles carrying electric or magnetic charges in ungauged $N = 2, d = 4$ supergravity and there is no perturbative physical mechanism that can change the moduli, which are properties characterizing the vacuum.}
We conclude that a family of non-extremal black hole solutions with given electric and magnetic charges \( Q \) and parametrized by \( r_0 \) is always attracted to one of the two extremal solutions in the evaporation process, independently of our choice of \( \lambda_\infty \). The same will happen to the non-extremal black holes of the model that we are going to consider next and which can be regarded as an extension of the axidilaton model.

3 Black holes of the \( \overline{\mathbb{C}P}^n \) model

This model is characterized by the prepotential

\[
\mathcal{F} = -\frac{4}{7} \eta_{\Lambda \Sigma} \chi^\Lambda \chi^\Sigma, \quad (\eta_{\Lambda \Sigma}) = \text{diag}(+ \cdots -),
\]

and has \( n \) scalars

\[
Z^i \equiv \chi^i / \chi^0,
\]

to which we add for convenience \( Z^0 \equiv 1 \), so we have

\[
(Z^\Lambda) \equiv (\chi^\Lambda / \chi^0) = (1, Z^i), \quad (Z_{\Lambda}) \equiv (\eta_{\Lambda \Sigma} Z^\Sigma) = (1, Z_i) = (1, -Z^i).
\]

This will simplify our notation. Thus, the Kähler potential and metric are given by

\[
\mathcal{K} = -\log (Z^\Lambda Z_{\Lambda}), \quad \mathcal{G}_{ij} = -e^\mathcal{K} (\eta_{ij} - e^\mathcal{K} Z_i Z_{j}^*), \quad \mathcal{G}^{ij} = -e^{-\mathcal{K}} (\eta^{ij} + Z_i Z_{j}^*).
\]

The covariantly holomorphic symplectic section reads

\[
\mathcal{V} = e^{\mathcal{K}/2} \left( \begin{array}{c} Z^\Lambda \\ -\frac{i}{2} Z_{\Lambda} \end{array} \right),
\]

and, in terms of the complex charge combinations

\[
\Gamma_{\Lambda} \equiv q_{\Lambda} + \frac{i}{2} \eta_{\Lambda \Sigma} p^\Sigma,
\]

the central charge, its holomorphic Kähler-covariant derivative and the black-hole potential are given by

\[
\mathcal{Z} = e^{\mathcal{K}/2} Z^\Lambda \Gamma_{\Lambda}, \quad \mathcal{D}_i \mathcal{Z} = e^{3\mathcal{K}/2} Z_i Z^\Lambda \Gamma_{\Lambda} - e^{\mathcal{K}/2} \Gamma_i, \quad |\tilde{\mathcal{Z}}|^2 = \mathcal{G}^{ij} \mathcal{D}_i \mathcal{Z} \mathcal{D}_j \mathcal{Z}^* = e^\mathcal{K} |Z^\Lambda \Gamma_{\Lambda}|^2 - \Gamma^* \Lambda \Gamma_{\Lambda}, \quad -V_{\text{bh}} = 2e^\mathcal{K} |Z^\Lambda \Gamma_{\Lambda}|^2 - \Gamma^* \Lambda \Gamma_{\Lambda}.
\]
3.1 Flow equations

Similarly as in the axion-dilaton model, the potential term can be expanded into

$$- \left[ e^{2U} V_{bh} - r_0^2 \right] = \Upsilon^2 + 4 G^{ij} \Psi_i \Psi_j^* ,$$  \hspace{1cm} (3.8)

where

$$\Upsilon = \frac{e^U}{\sqrt{2}} \sqrt{|Z|^2 + |\bar{Z}|^2 + e^{-2U} r_0^2 + \sqrt{ \left( |Z|^2 + |\bar{Z}|^2 + e^{-2U} r_0^2 \right)^2 - 4|Z|^2 |\bar{Z}|^2} ,$$  \hspace{1cm} (3.9)

$$\Psi_i = e^{2U} \frac{Z^* D_i Z}{\Upsilon} ,$$  \hspace{1cm} (3.10)

with the definitions Eqs. (3.7).

Since

$$\partial_U \Psi_i - \partial_i \Upsilon = \partial_i \Psi_j - \partial_j \Psi_i = \partial_{i^*} \Psi_j - \partial_j \Psi_{i^*} = 0 ,$$  \hspace{1cm} (3.11)

there exists a superpotential, whose gradient generates the vector field \((\Upsilon, \Psi_i, \Psi_j^*)\) and the first-order equations

$$U' = \Upsilon ,$$  \hspace{1cm} (3.12)

$$Z^{i'} = 2 G^{ij'} \Psi_{j^*} ,$$  \hspace{1cm} (3.13)

solve the second-order equations of motion.

3.2 The extremal case

3.2.1 Critical points

To find the critical points of the black-hole potential it is simpler to search for the zeros of

$$G^{ij'} \partial_j V_{bh} = 2 Z^\Lambda \Gamma_\Lambda \left( \Gamma^{*i} - \Gamma^{*0} Z^i \right) ,$$  \hspace{1cm} (3.14)

which has two factors that can vanish separately. The second factor vanishes only for the isolated point in moduli space

$$Z^i = \Gamma^{*i}/\Gamma^{*0} ,$$  \hspace{1cm} (3.15)

and corresponds to the supersymmetric attractor, whereas the first factor vanishes for the complex hypersurface of the moduli space defined by the condition

$$Z^i_\Lambda \Gamma_\Lambda = 0 .$$  \hspace{1cm} (3.16)
These points are associated with non-supersymmetric black holes (the central charge vanishes). The attractor behavior fixes only a combination of scalars on the horizon, but each of them individually still depends on the asymptotic values $Z_i^\infty$. The situation is summarized in Table 2.

As we mentioned earlier, the supersymmetric stationary point must be stable and, since the $\mathbb{CP}^n$ model is also based on a homogeneous manifold, the non-supersymmetric stationary points must be stable as well, even though, because the stationary locus is a submanifold of complex codimension $1$, rather than an isolated point, one expects $n - 1$ complex flat directions. In fact the Hessian in the real basis has one double eigenvalue

$$
\frac{4 \delta^{ij} \Gamma^k_i \Gamma^j_k}{1 - \delta_{kl} Z^*_h Z^*_h}.
$$

At first it may seem that for sufficiently large values of the scalars on the horizon the eigenvalue could become negative. The above expression, however, is proportional to a (multiple) eigenvalue of the scalar metric, hence the values for which the Hessian becomes negative semidefinite would also render the scalar metric negative definite and are consequently not physically admissible.

### 3.2.2 Supersymmetric solutions

The stabilization equations are solved by

$$
\mathcal{R}_\Lambda = \frac{i}{2} \eta_{\Lambda \Sigma} \mathcal{I}^\Sigma, \quad \mathcal{R}^\Lambda = -2 \eta^{\Lambda \Sigma} \mathcal{I}_\Sigma,
$$

so

$$
\mathcal{L}^\Lambda / X = \mathcal{R}^\Lambda + i \mathcal{I}^\Lambda = -2 \eta^{\Lambda \Sigma} (\mathcal{I}_\Sigma - \frac{i}{2} \eta_{\Sigma \Omega} \mathcal{I}^\Omega).
$$

Defining the complex combinations of harmonic functions

$$
\mathcal{H}_\Lambda \equiv \mathcal{I}_\Lambda + \frac{i}{2} \eta_{\Lambda \Sigma} \mathcal{I}^\Sigma \equiv \mathcal{H}_\Lambda^\infty - \frac{1}{\sqrt{2}} \Gamma_\Lambda \tau,
$$

Table 2: Critical points of the $\mathbb{CP}^n$ model.
where $\mathcal{H}_{\Lambda \infty}$ are the values at spatial infinity, we find that the metric function and scalar fields are given by

$$e^{-2U} = 2\mathcal{H}^* \mathcal{H}_\Lambda, \quad Z^i = \frac{L^i/X}{L^0/X} = \frac{\mathcal{H}^* i}{\mathcal{H}^0},$$

(3.21)

where we are using $\eta$ to raise and lower the indices of the complex harmonic functions.

The solution depends on the $\bar{n}$ complex charges $\Gamma_\Lambda$ and on the $n+1$ complex constants $\mathcal{H}_{\Lambda \infty}$. $n$ combinations of them are determined by the asymptotic values of the $n$ scalars

$$Z^i_\infty = \mathcal{H}^* i / \mathcal{H}^0_\infty,$$

(3.22)

and the remaining one is determined by the two real conditions of asymptotic flatness

$$2\mathcal{H}^*_\infty \mathcal{H}_\Lambda \infty = 1,$$

(3.23)

and absence of NUT charge

$$\Im \left( \mathcal{H}^*_\infty \Gamma_\Lambda \right) = 0.$$

(3.24)

The result is

$$\mathcal{H}^*_\infty = \pm e^{\mathcal{K}/z} \frac{Z^*_\infty}{|Z^*_\infty|} Z^*_\Lambda, \quad \mathcal{H}^*_0 = \frac{1}{\sqrt{2}} \Gamma_\Lambda \tau,$$

(3.25)

where $\mathcal{K}_\infty$ and $Z_\infty$ are the asymptotic values of the Kähler potential and central charge, although the positivity of the mass, which is given, as expected, by $M = |Z_\infty|$ allows only for the upper sign.

The complete supersymmetric solution is, therefore, given by the $\bar{n}$ complex harmonic functions

$$\mathcal{H}^*_{\Lambda \infty} = e^{\mathcal{K}/z} \frac{Z^*_\infty}{|Z^*_\infty|} Z^*_\Lambda \infty - \frac{1}{\sqrt{2}} \Gamma_\Lambda \tau,$$

(3.26)

that depend only on the $2n+1$ physical complex parameters $Z^i_\infty, \Gamma_\Lambda$.

In order to find the extremal non-supersymmetric solutions we will first obtain the general non-extremal ones and then we will take the extremal non-supersymmetric limit. We will see that this procedure works as in the axidilaton case because the non-extremal solutions interpolate between the different extremal limits.

### 3.3 Non-extremal solutions

Our ansatz for the non-extremal solution is again

$$e^{-2U} = e^{-2[U_\Lambda(\mathcal{H}) + r_0 \tau]}, \quad e^{-2U_\Lambda(\mathcal{H})} = 2\hat{\mathcal{H}}^* \hat{\mathcal{H}}_\Lambda, \quad Z^i = Z^i_\Lambda(\mathcal{H}) = \hat{\mathcal{H}}^* i / \hat{\mathcal{H}}^0,$$

(3.27)

where the hatted functions are assumed to have the form
As in the axidilaton model, we have to find the $2n$ complex constants $A_\Lambda, B_\Lambda$ by requiring that we have a solution to the equations of motion (2.37)–(2.39). It is not difficult to see that this happens if the following algebraic conditions are satisfied:

\begin{align}
\Re m(B^* A_\Lambda) &= 0, \\
A^* A^\Sigma \xi_{\Lambda \Sigma} &= 0, \\
(A^* B^\Sigma + B^* A^\Sigma) \xi_{\Lambda \Sigma} &= 0, \\
B^* B^\Sigma \xi_{\Lambda \Sigma} &= 0, \\
(2r_0)^2 (B_i^* A_0^* - B_0^* A_i^*) A^* A_\Lambda + (\Gamma^*_i A_0^* - \Gamma^*_0 A_i^*) A^* \Gamma_\Lambda &= 0, \\
-(2r_0)^2 (B_i^* A_0^* - B_0^* A_i^*) B^* A_\Lambda + (\Gamma^*_i B_0^* - \Gamma^*_0 B_i^*) B^* \Gamma_\Lambda &= 0, \\
(\Gamma^*_i A_0^* - \Gamma^*_0 A_i^*) A^* \Gamma_\Lambda + (\Gamma^*_i B_0^* - \Gamma^*_0 B_i^*) B^* \Gamma_\Lambda &= 0,
\end{align}

where we have defined

\[ \xi_{\Lambda \Sigma} \equiv 2 (\Gamma_\Lambda \Gamma^*_\Sigma + 8r_0^2 A_\Lambda B^*_\Sigma) - \eta_{\Lambda \Sigma} (\Gamma^*_\Omega \Gamma_\Theta + 8r_0^2 A^*_\Omega B^*_\Theta). \]

In order to fully identify the constants $A_\Lambda, B_\Lambda$ in terms of the physical parameters, we must add to the above conditions the requirement of asymptotic flatness and the definitions of mass $M$ and of the asymptotic values of the scalars $Z^i_\infty$:

\begin{align}
2(A^* + B^*) (A_\Lambda + B_\Lambda) &= 1, \\
4\Re [B^* (A_\Lambda + B_\Lambda)] &= 1 - M/r_0, \\
\frac{A^* i + B^* i}{A^* 0 + B^* 0} &= Z^i_\infty.
\end{align}

The condition of absence of NUT charge arises naturally as a consequence of the equations of motion (it is Eq. (3.29)).

To solve these equations we choose $A_0 + B_0$ to be real, as we did in the axidilaton case. Then, we find the following result:
\[ A_\Lambda = \pm \frac{e^{K_\infty/2}}{2\sqrt{2}} \left\{ Z_{\Lambda\infty}^* \left[ 1 + \frac{(M^2 - e^{K_\infty}|Z_{\infty}^*\Gamma^*|^2)}{M r_0} \right] + \frac{\Gamma_\Lambda Z_{\infty}^* \Gamma_{\Sigma}^*}{M r_0} \right\}, \quad (3.40) \]

\[ B_\Lambda = \pm \frac{e^{K_\infty/2}}{2\sqrt{2}} \left\{ Z_{\Lambda\infty}^* \left[ 1 - \frac{(M^2 - e^{K_\infty}|Z_{\infty}^*\Gamma^*|^2)}{M r_0} \right] - \frac{\Gamma_\Lambda Z_{\infty}^* \Gamma_{\Sigma}^*}{M r_0} \right\}, \quad (3.41) \]

\[ M^2 r_0^2 = (M^2 - |Z_{\infty}|^2)(M^2 - |\tilde{Z}_{\infty}|^2), \quad (3.42) \]

where \(|\tilde{Z}|^2\) is defined in Eq. (3.41) and we remind the reader that \(-V_{bh} = |Z|^2 + |\tilde{Z}|^2\).

With these values it is easy to see that the metric function \(e^{-2U}\) can be put in exactly the same form as in the axidilaton case, given in Eq. (2.58) where \(r_0\) and \(S_{\pm}\) are now those of the present case. This means that the metric is regular in all the \(r_0^2 > 0\) cases.

### 3.3.1 Supersymmetric and non-supersymmetric extremal limits

Again, there are two possible extremal limits in which \(r_0 \to 0\):

1. Supersymmetric, when \(M^2 \to |Z|^2 = e^{K_\infty}|Z_{\infty}^*\Gamma^*|^2\). In this limit we get

\[ \mathcal{H}_\Lambda \xrightarrow{M \to |Z_{\infty}|} \pm \frac{Z_{\infty}^*}{|Z_{\infty}|} \mathcal{H}_{\Lambda}^{\text{susy}}, \quad (3.43) \]

where \(\mathcal{H}_{\Lambda}^{\text{susy}}\) is given by Eq. (3.26). This determines the phase of \(A_0 + B_0\), which we set to zero at the beginning for simplicity, making use of the formal phase invariance of the solution.

2. Non-supersymmetric, when \(M^2 \to |\tilde{Z}|^2 = e^{K_\infty}|Z_{\infty}^*\Gamma^*|^2 - \Gamma^*\Sigma^*\Gamma_{\Sigma}^*\). In this limit we get

\[ \mathcal{H}_\Lambda \xrightarrow{M \to |Z_{\infty}|} \frac{e^{K_\infty/2}}{2\sqrt{2}} \left\{ Z_{\Lambda\infty}^* \left[ - \frac{1}{|Z_{\infty}|} \left[ - Z_{\Lambda\infty}^* \Gamma^*\Sigma^*\Gamma_{\Sigma}^* + \Gamma_\Lambda Z_{\infty}^* \Gamma_{\Sigma}^* \right] \tau \right] \right\}. \quad (3.44) \]

In this case we do not have an explicit solution to compare with and we cannot determine the phase of \(A_0 + B_0\). However, the metric and scalar fields do not depend on that phase and the above harmonic functions determine them completely.

It takes little time to see that in this case the entropy is

\[ S = -\pi \Gamma^*\Sigma^*\Gamma_{\Sigma}, \quad (3.45) \]

as expected, and that on the event horizon the scalars take the values

\[ Z_{\Lambda i}^* = \frac{\Gamma^i Z_{\infty}^* \Gamma_{\Lambda}^* - Z_{\infty}^* \Gamma^* \Gamma_{\Sigma}^*}{\Gamma^0 Z_{\infty}^* \Gamma_{\Gamma}^* - \Gamma^* \Omega^0 \Gamma_{\Omega}}, \quad (3.46) \]
which depend manifestly on the asymptotic values. It is easy to check that the horizon values satisfy the condition \( Z_h \Lambda \Gamma = 0 \).

Had we tried to implement the prescription of replacement of charges by harmonic functions in the extremal non-supersymmetric horizon values, it is difficult to see how the full solution with the above coefficients in the harmonic functions could have been recovered.

### 3.3.2 Physical properties of the non-extremal solutions

The entropies of the black-hole solutions of this model can also be put in the form Eqs. (2.63)–(2.65), where now \( Z_\infty \) and \( \tilde{Z}_\infty \) take the form corresponding to the present model. In both extremal limits we obtain finite entropies which are moduli-independent, even though in the extremal non-supersymmetric limit the values of the scalars on the horizon depend on the asymptotic boundary conditions according to Eq. (3.46). In the non-extremal case, the product of the entropies of the inner and outer horizon gives the square of the extremal entropy and, consequently, is moduli-independent.

Also in this case the non-extremal deformation of the double-extremal solutions have constant scalars: if the asymptotic values of the scalars in the general case coincide with their horizon attractor values in the extremal case, then the scalars are constant and the metric is that of the Reissner–Nordström solution.

The endpoint of the evaporation process of the non-extremal black holes of this model is completely determined by their electric and magnetic charges and is independent of the choice of asymptotic values \( Z_\infty \) for the scalars. Thus, if \( \Gamma^\Lambda \Gamma \Lambda > 0 \), which is the property that characterizes the supersymmetric attractor, then \( |Z_\infty| > |\tilde{Z}_\infty| \) and the evaporation process will stop when \( M = |Z_\infty| \), the supersymmetric case. The opposite will be true if \( \Gamma^\Lambda \Gamma \Lambda < 0 \). Again, we can speak of an attractive behavior in the evaporation process.

### 4 D0-D4 black holes

In this section we are going to obtain, following the procedure outlined in Section 1.5, the non-extremal deformation of the well-known supersymmetric D0-D4 black hole embedded in the STU model [61, 62, 63].

We have chosen this particular solution because the non-extremal case is manageable, yet general enough to be interesting. Furthermore, the well-known supersymmetric limit has a straightforward microscopic interpretation. This fact could be useful for obtaining a microscopic interpretation in the non-extremal case, although this interpretation may be difficult to find, since for non-extremal black holes we have neither supersymmetry nor attractor mechanism to protect the solution from the effects of a strong-weak change of the coupling.

The STU model is defined through the following prepotential:\textsuperscript{15}

\[
\mathcal{F} = \frac{X^1 X^2 X^3}{X^0},
\]

\textsuperscript{15}Sometimes it is convenient to use the symmetric tensor \( d_{ijk} = |\epsilon_{ijk}| \) so \( \mathcal{F} = \frac{1}{6} d_{ijk} X^i X^j X^k / X^0 \).
and has three scalars customarily defined as
\[ Z_1 \equiv \frac{X^1}{X^0} \equiv S, \quad Z_2 \equiv \frac{X^2}{X^0} \equiv T, \quad Z_3 \equiv \frac{X^3}{X^0} \equiv U, \] (4.2)
with Kähler potential (in the \( X^0 = 1 \) gauge) and metric given by
\[ e^{-K} = -8 \Im m S \Im m T \Im m U, \quad \mathcal{G}_{ij} = \frac{\delta_{(i)j}^*}{4(\Im m Z^{(i)})^2}. \] (4.3)

The covariantly holomorphic symplectic section is given by
\[ V = \left( \frac{\mathcal{L}^\Lambda}{\mathcal{M}_A} \right) = e^{K/2} \begin{pmatrix} 1 \\ Z^i \\ -\mathcal{F} \\ 3d_{ijk}Z^jZ^k \end{pmatrix} = \frac{1}{2\sqrt{2}\sqrt{-\Im m S \Im m T \Im m U}} \begin{pmatrix} 1 \\ S \\ T \\ U \end{pmatrix} - \mathcal{F} U \begin{pmatrix} S \\ T \\ U \end{pmatrix} + \sum_{i=1}^{3} (3d_{ijk}Z^jZ^k - q_i Z^i), \] (4.4)
and therefore, we have
\[ \mathcal{Z} = e^{K/2} W, \]
\[ \mathcal{D}_i \mathcal{Z} = \frac{ie^{K/2}}{2\Im m Z^{(i)}} W_{(i)}, \] (4.5)
\[ -V_{bh} = e^{K} \left\{ |W|^2 + \sum_{i=1}^{3} |W_i|^2 \right\}, \]

where
\[ W = W(S, T, U, Q) \equiv -p^0 \mathcal{F} - q_0 + \sum_{i=1}^{3} (3d_{ijk}p^iZ^jZ^k - q_i Z^i), \] (4.6)
\[ W_1 \equiv W(S^*, T, U, Q), \] (4.7)
\[ W_2 \equiv W(S, T^*, U, Q), \] (4.8)
\[ W_3 \equiv W(S, T, U^*, Q). \] (4.9)

The D0-D4 black holes that we are going to consider only have four non-vanishing charges which, when embedded in the \( STU \) model, correspond to three magnetic charges \( p^i, i = 1, \ldots, 3 \).
from the vector fields in the three vector multiplets, and the electric charge \( q_0 \) of the graviphoton. In this case the function \( W \) reduces to just
\[
W = W(S, T, U, Q) = 3d_{ijk}p^iZ^jZ^k - q_0. \tag{4.10}
\]

Before we analyze the supersymmetric solution, which eventually is going to be deformed, we discuss the flow equations.

### 4.1 Flow equations

As in Eq. (3.8), also here it is possible to expand the potential term into squares of
\[
Υ = \frac{ie^U}{16\Im Z(i)} \left( \sqrt{e^{-2U}r_0^2 + (\hat{q}_0)^2} - \sum_{j=1}^3 (-1)^{\delta_{ij}} \sqrt{e^{-2U}r_0^2 + (\hat{p}_j)^2} \right), \tag{4.11}
\]

where the (hatted) dressed charges are defined as
\[
\hat{p}^i = -4|p^{(i)}|M(i) = \sqrt{2}e^{K/2}d_{(i)jk}|p^{(j)}|\Im Z^j\Im Z^k, \quad \hat{q}_0 = 4|q_0|\mathcal{L}^0 = \sqrt{2}|q_0|e^{K/2}. \tag{4.13}
\]

The superpotential can be obtained explicitly by integrating Eq. (4.11) with respect to \( U \):
\[
Y = \Upsilon - \frac{r_0}{4} \left[ \ln \left( e^{-U}r_0^2 + r_0\sqrt{e^{-2U}r_0^2 + (\hat{q}_0)^2} \right) + \sum_{j=1}^3 \ln \left( e^{-U}r_0^2 + r_0\sqrt{e^{-2U}r_0^2 + (\hat{p}_j)^2} \right) \right], \tag{4.14}
\]

and the first-order flow equations take the form:
\[
U' = Y = \partial_U Y, \tag{4.15}
\]
\[
Z^{ij'} = 2g^{ij'}\Psi^{*j} = 2g^{ij'}\partial_{j'}Y. \tag{4.16}
\]

### 4.2 The extremal case

#### 4.2.1 Critical points

We start by computing the derivatives of the black-hole potential:
\[
-\partial Z^i V_{bh} = \frac{ie^K}{\Im Z^i} \left\{ W_1W^* + W_2W_3^* \right\} = 0. \tag{4.17}
\]
This equation and the other two that can be obtained by permuting $S$ with $T$ and $U$ we get the system

\[W_1 W^* + W_2^* W_3^* = 0,\]
\[W_2 W^* + W_1^* W_3^* = 0,\]
\[W_3 W^* + W_1^* W_2^* = 0,\]

(4.18)

that admits three kinds of solutions:

1. $W \neq 0$ and $W_i = 0 \ \forall i$. This is the $N = 2$ supersymmetric solution because $W_i = 0$ implies $D_i Z = 0$. It corresponds to an isolated point in moduli space.

2. $W_1 \neq 0$, $W = W_2 = W_3 = 0$ and the other two permutations of this solution. These three isolated points in moduli space are not $N = 2$ supersymmetric but correspond to $N = 8$ supersymmetric critical points since $W$ and the $W_i$’s are associated to the four skew eigenvalues of the central charge matrix of $N = 8$ supergravity [64]. Formally they can be obtained from the supersymmetric critical point by taking the complex conjugate of one of the complex scalars.

3. $|W| = |W_i| \ \forall i$ and $\text{Arg} \ W = \sum_{i=1}^3 \text{Arg} \ W_i - \pi$. These are only 4 real equations for the 3 complex scalars and admit a 2-parameter space of solutions which are not supersymmetric in either $N = 2$ or $N = 8$ supergravity [64]. The values of the scalars on the horizon will depend on two real combinations of their asymptotic values.

4.2.2 Supersymmetric solutions

Solving directly the equations $W_i = 0 \ \forall i$ is complicated, but we can find the supersymmetric attractor values if we can construct the supersymmetric solutions by the standard method. This requires solving the stabilization equations or the attractor equations on the horizon, which is not straightforward either, but has already been done in Ref. [65].

If $I^0 \neq 0$, the scalars and metric function of the supersymmetric extremal solutions are given in terms of the real harmonic functions $I^M$ by

\[Z^i = \frac{I^\Lambda I_\Lambda - 2 I^{(i)} I_{(i)}}{2 J_i} - i \frac{e^{-2U}}{4 J_{(i)}} \]

(4.19)

\[e^{-2U} = 2 \sqrt{4 I_0 I^1 I^2 I^3 - 4 I^0 I_1 I_2 I_3 + 4 \sum_{i<j} I^i I^j I^i I^j - (I^\Lambda I_\Lambda)^2},\]

(4.20)

where

36
\[ J_i = 3d_{ijk} \mathcal{T}^j \mathcal{T}^k - \mathcal{T}_i \mathcal{T}^0. \]  
(4.21)

If \( \mathcal{T}^0 = 0 \), the metric function \( e^{-2U} \) and the scalars \( Z^i \) are the restriction to \( \mathcal{T}^0 = 0 \) of the above expressions.

The harmonic functions have the general form Eq. (1.49) but, as usual, given the charges \( Q^M \), the asymptotic constants \( T^M_\infty \) are restricted by the condition of absence of NUT charge Eq. (1.50).

The simplest supersymmetric extremal D0-D4 black holes, the ones we are going to consider, have \( \mathcal{T}^0 = \mathcal{T}_i = 0 \) (\( \mathcal{T}^0 = \mathcal{T}_i = 0 \) implies \( p^0 = q_i = 0 \) but not the other way around). The scalars and metric function take the simple forms

\[ Z^i = -4ie^{2U} \mathcal{T}_0 \mathcal{T}^i, \]  
(4.22)

\[ e^{-2U} = 4 \sqrt{\mathcal{T}_0 \mathcal{T}^1 \mathcal{T}^2 \mathcal{T}^3}, \]  
(4.23)

and the condition of absence of NUT charge Eq. (1.50) is automatically satisfied for arbitrary values of the constants \( T^M_\infty \).

The regularity of the metric and scalar fields (whose imaginary part must be strictly positive in these conventions) for all \( \tau \in (-\infty, 0) \) implies

\[ \text{sign} \mathcal{T}_0_\infty = \text{sign} q_0, \quad \text{sign} \mathcal{T}^i_\infty = \text{sign} p^i, \forall i, \]  
(4.24)

and the reality of the metric function and negative definiteness of the imaginary parts of the scalars imply

\[ \mathcal{T}_0 \mathcal{T}^i > 0, \quad \mathcal{T}_0 \mathcal{T}^1 \mathcal{T}^2 \mathcal{T}^3 > 0, \]  
(4.25)

which leave us with just two options

\[ \mathcal{T}_0, \mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3 > 0, \quad q_0, p^1, p^2, p^3 > 0, \quad \mathcal{T}_0_\infty, \mathcal{T}^1_\infty, \mathcal{T}^2_\infty, \mathcal{T}^3_\infty > 0, \]  
(4.26)

\[ \mathcal{T}_0, \mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3 < 0, \quad q_0, p^1, p^2, p^3 < 0, \quad \mathcal{T}_0_\infty, \mathcal{T}^1_\infty, \mathcal{T}^2_\infty, \mathcal{T}^3_\infty < 0. \]

Therefore, in the supersymmetric solution we have two disconnected possibilities (in the sense that it is not possible to go from one to the other continuously without making the metric functions or the scalars singular).

Imposing asymptotic flatness and absence of NUT charge we find that the four harmonic functions can be written in terms of the physical parameters in the form

\[ \mathcal{T}_0 = s_0 \left\{ \frac{1}{4\sqrt{2}L_0^0} - \frac{1}{\sqrt{2}}|q_0|\tau \right\} = \frac{s_0}{4\sqrt{2}L_0^0} \left( 1 - \hat{q}_0 \tau \right), \]  
(4.27)

\[ \mathcal{T}^i = s^{(i)} \left\{ -\frac{1}{4\sqrt{2}M^{(i)}_\infty} - \frac{1}{\sqrt{2}}|p^{(i)}|\tau \right\} = -\frac{s^{(i)}}{4\sqrt{2}M^{(i)}_\infty} \left( 1 - \hat{p}^{(i)} \tau \right), \]  
(4.27)
where \( s_0, s_i \) are the signs of the charges \( q_0, p^i \) and \( \hat{q}_0, \hat{p}_\infty^i \) are the asymptotic values of the dressed charges defined in Eq. (4.13). These are positive by definition. On the other hand, as previously discussed, the signs \( s_0, s_i \) must be either all positive or all negative in the supersymmetric case.

Plugging these expressions into the metric function we can compute the entropy and the mass of the black hole, finding

\[
\frac{S}{\pi} = |Z(Z_h, Z^*_h, Q)|^2 = 2\sqrt{q_0 p^1 p^2 p^3},
\]

\[
M = |Z(Z_\infty, Z^*_\infty, Q)| = \frac{1}{4} (q_0 + \hat{p}_\infty^1 + \hat{p}_\infty^2 + \hat{p}_\infty^3).
\]

4.2.3 Extremal, non-supersymmetric solutions

According to the discussion of the critical points of the black-hole potential, we can obtain 3 non-supersymmetric extremal black holes by formally replacing one of the scalars by its complex conjugate. If we do it for \( Z_1 \), for instance, we get

\[
\Im Z^1 = -4e^{2U} I_0 I^1 \quad \rightarrow \quad +4e^{2U} I_0 I^1,
\]

\[
e^{-2U} = 4\sqrt{I_0 I^1 I^2 I^3},
\]

with \( \Im Z^1 \) strictly negative. This transformation is equivalent to the replacement of \( I^1 \) by \( -I^1 \) everywhere. To take into account these and also further possibilities, we write the extremal solutions in the form

\[
Z^{(i)} = -4s_0 s^{(i)} e^{2U} I_0 I^{(i)}, \quad e^{-2U} = 4\sqrt{s_0 s^1 s^2 s^3 I_0 I^1 I^2 I^3},
\]

where \( s_0, s^i \) are the signs of the respective harmonic functions (which coincide with those of the charges and those of the asymptotic constants). The possible choices and their relation to supersymmetry are given in Table 3.

The entropy of these solutions is given by

\[
\frac{S}{\pi} = 2\sqrt{s_0 s^1 s^2 s^3 q_0 p^1 p^2 p^3} = 2\sqrt{|q_0 p^1 p^2 p^3|},
\]

and the mass is still given by Eq. (4.29)

\[
M = \frac{1}{4} (\hat{q}_0 + \hat{p}_\infty^1 + \hat{p}_\infty^2 + \hat{p}_\infty^3),
\]

but it coincides with \( |Z(Z_\infty, Z^*_\infty, Q)| \) only for the first two choices of signs in Table 3. For the choices in the rows \( i+1 = 2, 3, 4 \) of the table, the mass equals \( e^{K/2}|W_i| \) (for them \( |Z(Z_\infty, Z^*_\infty, Q)| = 0 \)) and for the other eight combinations of signs the mass is numerically equal to \( 4|Z(Z_\infty, Z^*_\infty, Q)| \). Thus, for all these extremal black holes \( M > |Z(Z_\infty, Z^*_\infty, Q)| \).
Table 3: Possible sign choices for extremal black holes of the D0-D4 model. The first two possibilities (first row of the table) correspond to the $N = 2$ supersymmetric black holes. The six choices in the 2nd, 3rd and 4th rows correspond to the extremal black holes that are not supersymmetric in $N = 2$ supergravity but are supersymmetric when the theory is embedded in the $N = 8$ supergravity. The last 8 choices (4 rows) correspond to extremal black holes which are not supersymmetric in any theory.

4.3 Non-extremal D0-D4 black hole

According to the general prescription we describe the non-extremal solution with four functions $\hat{I}_0, \hat{I}_1, \hat{I}_2, \hat{I}_3$ of $\tau$, which we will denote collectively by $\hat{I}^\Lambda$ in this section and which we assume to be of the general form

$$\hat{I}^\Lambda = a^\Lambda + b^\Lambda e^{2r_0 \tau},$$  \hfill (4.34)

The metric factor and scalar fields are assumed to take the form

$$e^{-2U} = e^{-2(U + r_0 \tau)},$$  \hfill (4.35)

$$Z^i = -4ie^{2U} \hat{I}_0 \hat{I}^i,$$  \hfill (4.36)

where

$$e^{-2U_0} = 4\sqrt{\hat{I}_0 \hat{I}^1 \hat{I}^2 \hat{I}^3}.$$  \hfill (4.37)

Observe that the consistency of this ansatz requires that all the functions $\hat{I}^\Lambda$ must be simultaneously positive or negative. Furthermore, they must be finite in the interval $\tau \in (-\infty, 0)$, which implies that

$$\text{sign } a^\Lambda \neq \text{sign } b^\Lambda, \quad |a^\Lambda| > |b^\Lambda| \quad \forall \Lambda.$$  \hfill (4.38)

Plugging this ansatz into the Eqs. (2.37)–(2.39) we find that they are solved if the constants $a^\Lambda, b^\Lambda$ satisfy for each value of $\Lambda$
\[ a^{(\Lambda)}b^{(\Lambda)} = \frac{(p^{(\Lambda)})^2}{8r_0^2}. \] (4.39)

In order to determine all the constants in terms of the physical parameters we impose asymptotic flatness and use the definitions of mass and the asymptotic values of the scalars, which yield the additional relations (the condition of absence of NUT charge is automatically satisfied)

\[ \prod_{\Lambda} (a^{\Lambda} + b^{\Lambda}) = \frac{1}{16}, \] (4.40)

\[ \sum_{\Lambda} \frac{b^{\Lambda}}{a^{\Lambda} + b^{\Lambda}} = 2 \left( 1 - \frac{M}{r_0} \right), \] (4.41)

\[ \Im Z_i^\infty = -4(a_0 + b_0)(a^i + b^i). \] (4.42)

The solution to these equations that satisfies the finiteness condition Eq. (4.38) is

\[
\begin{pmatrix}
a_0 \\
b_0
\end{pmatrix} = \frac{\varepsilon}{8\sqrt{2}L_0^0} \left\{ 1 \pm \frac{1}{r_0} \sqrt{r_0^2 + (\hat{q}_0^\infty)^2} \right\},
\] (4.43)

\[
\begin{pmatrix}
a^i \\
b^i
\end{pmatrix} = -\frac{\varepsilon}{8\sqrt{2}M_i^\infty} \left\{ 1 \pm \frac{1}{r_0} \sqrt{r_0^2 + (\hat{p}_i^\infty)^2} \right\},
\] (4.44)

where the upper sign corresponds to the constant \( a \) and the lower to \( b \) and \( \varepsilon \) is the global sign of the functions \( \hat{T}^{\Lambda} \). We must stress that, unlike in the extremal case, this sign is not related to that of the charges.

### 4.3.1 Physical properties of the non-extremal solutions

The mass is given by

\[ M = \frac{1}{4} \sum_{\Lambda} \sqrt{r_0^2 + (\hat{p}_0^\Lambda)^2}, \] (4.45)

and it is evident that in the extremal limit it takes the value Eq. (4.33), while the entropies are given by

\[ \frac{S_+}{\pi} = \frac{A_+}{4\pi} = \sqrt{\prod_{\Lambda} \left( r_0^2 \pm \sqrt{r_0^2 + (\hat{p}_0^\Lambda)^2} \right)}, \] (4.46)

and take the value Eq. (4.32), since \( \prod_{\Lambda} |\hat{p}_0^\Lambda| = \prod_{\Lambda} |p^{(\Lambda)}| \). Observe that
\[
\frac{S_+ S_-}{\pi} = 4|q_0 p^1 p^2 p^3|,
\]  
(4.47)
which is the square of the moduli-independent entropy of all the extremal black holes.

It is highly desirable to have an explicit expression of the non-extremality parameter \( r_0 \) in terms of the physical parameters \( M, p^A, Z_i^\infty \), which, in turn, would allow us to express mass and entropy as functions of \( p^A, Z_i^\infty \) alone. Furthermore, such an expression would allow us to study the different extremal limits or relations between \( M \) and \( p^A \) and \( Z_i^\infty \) that make \( r_0 \) vanish. In the general case, solving Eq. (4.45) explicitly is impossible, though. We can, nevertheless, consider some particular examples, obtained by fixing the relative values of the dressed charges \( \hat{p}^i \) and \( \hat{q}_0^\infty \):

1. If \( \hat{p}_1^\infty = \hat{p}_2^\infty = \hat{p}_3^\infty = \hat{q}_0^\infty \), then Eq. (4.45) simplifies to:

\[
M = \sqrt{r_0^2 + (\hat{q}_0^\infty)^2},
\]
(4.48)
so

\[
r_0^2 = (M - \hat{q}_0^\infty) (M + \hat{q}_0^\infty),
\]
(4.49)
from which we conclude that we can reach the extremal limit \( M = \hat{q}_0^\infty \) in two different ways\(^{16}\) \( M = s_0 \hat{q}_0^\infty \) and \( M = -s_0 \hat{q}_0^\infty \). Which one is reached depends on \( s_0 = \text{sign} \ q_0 \). Whether this limit is supersymmetric or not will depend on the signs of the charges, as discussed in Table 3.

We can use Eq. (4.49) to express the entropy in terms of the mass, the charges, and the asymptotic values of the scalars at infinity in the familiar form:

\[
\frac{S_\pm}{\pi} = \left( \sqrt{N_R^{(1)}} \pm \sqrt{N_L^{(1)}} \right)^2,
\]
(4.50)
where

\[
N_R^{(1)} = M^2, \quad N_L^{(2)} = M^2 - \hat{q}_0^2 \infty.
\]
(4.51)

2. If \( \hat{p}_1^\infty = \hat{p}_2^\infty \) and \( \hat{p}_3^\infty = \hat{q}_0^\infty \), then the mass of the black hole is given by

\[
M = \frac{1}{2} \left[ \sqrt{r_0^2 + (\hat{p}_1^\infty)^2} + \sqrt{r_0^2 + (\hat{q}_0^\infty)^2} \right],
\]
(4.52)
and Eq. (4.52) can be inverted to obtain

\(^{16}\)We remind the reader that we have defined the dressed charges to always be positive.
\[ M^2 r_0^2 = \left( M^2 - \frac{(\hat{p}_\infty^1 + \hat{q}_0 \infty)^2}{4} \right) \left( M^2 - \frac{(\hat{p}_\infty^1 - \hat{q}_0 \infty)^2}{4} \right), \]  

(4.53)

from which we find four possible extremal limits:

\[
M = \begin{cases} 
\frac{1}{2} \left( s^1 \hat{p}_\infty^1 + s_0 \hat{q}_\infty \right), \\
\frac{1}{2} \left( s^1 \hat{p}_\infty^1 - s_0 \hat{q}_\infty \right), \\
-\frac{1}{2} \left( s^1 \hat{p}_\infty^1 + s_0 \hat{q}_\infty \right), \\
-\frac{1}{2} \left( s^1 \hat{p}_\infty^1 - s_0 \hat{q}_\infty \right). 
\end{cases} 
\]  

(4.54)

Which extremal limit will be attained if the mass diminishes in the process of evaporation depends on the signs of the charges \( s^1, s_0 \) but it will always be the largest possible value so that

\[ M = \frac{1}{2} (\hat{p}_\infty^1 + \hat{q}_0 \infty). \]  

(4.55)

In terms of the mass, the charges, and the asymptotic values of the scalars at infinity, the entropies are again given by

\[
S_\pm = \left( \sqrt{N_{R}^{(2)}} \pm \sqrt{N_{L}^{(2)}} \right)^2, 
\]  

(4.56)

where

\[
N_{R}^{(2)} = M^2 - \frac{(\hat{p}_\infty^1 + \hat{q}_0 \infty)^2}{4}, \quad N_{L}^{(2)} = M^2 - \frac{(\hat{p}_\infty^1 - \hat{q}_0 \infty)^2}{4}, \]  

(4.57)

and the product of the two entropies gives the moduli-independent entropy of the extremal black hole with the same charges, squared.

3. If \( \hat{p}_\infty^1 = \hat{p}_\infty^2 = \hat{p}_\infty^3 \), then the mass is given by

\[
M = \frac{1}{4} \left[ 3 \sqrt{r_0^2 + (\hat{p}_\infty^1)^2} + \sqrt{r_0^2 + (\hat{q}_\infty)^2} \right], 
\]  

(4.58)

This equation can be written in a polynomial form by squaring it several times, and then it can be solved for \( r_0^2 \)

\[
r_0^2 = \frac{1}{9} \left[ (\hat{q}_\infty)^2 - 9(\hat{p}_\infty^1)^2 + 20M^2 - 6\sqrt{2}(\hat{q}_\infty)^2M^2 - (\hat{p}_\infty^1)^2M^2 + 2M^4 \right]. \]  

(4.59)
From this equation we can obtain the extremal values of $M$:

$$M = \begin{cases} 
\frac{1}{4}(3s^1 \hat{p}^1_\infty + s_0 \hat{q}_0 \hat{q}_\infty), \\
\frac{1}{4}(3s^1 \hat{p}^1_\infty - s_0 \hat{q}_0 \hat{q}_\infty), \\
-\frac{1}{4}(3s^1 \hat{p}^1_\infty + s_0 \hat{q}_0 \hat{q}_\infty), \\
-\frac{1}{4}(3s^1 \hat{p}^1_\infty - s_0 \hat{q}_0 \hat{q}_\infty). 
\end{cases}$$

(4.60)

The extremal limit that will be reached first in the evaporation process will be that with the largest value of the mass

$$M = \frac{1}{4}(3\hat{p}^1_\infty + \hat{q}_0 \hat{q}_\infty),$$

(4.61)

and the supersymmetry will depend on the signs of the charges.

As in the previous examples, we can write the entropy in terms of $N^{(3)}_R$ and $N^{(3)}_L$, although in this case the expression for them is not very manageable. However, we can compute

$$\sqrt{S_+ S_- \over \pi} = N^{(3)}_R - N^{(3)}_L = (\hat{p}^1_\infty)^{3/2} \sqrt{\hat{q}_0 \hat{q}_\infty} = 2\sqrt{|q_0 \hat{q}_1 p^2 \hat{p}^2|}.$$  

(4.62)

Eq. (4.62) depends only on the charges and it is indeed the supersymmetric entropy, as already demonstrated in the general case (4.46) and (4.47).

In the general case, even though finding a closed-form explicit expression for $r_0 (Z^1_\infty, Q, M)$ is at best unfeasible, it is still possible to obtain the values $M_c = M(z^1_\infty, Q)$ at which extremality is reached by setting $r_0 = 0$ in Eq. (4.45). There are $2^4 = 16$ possible extremal limits given by

$$M = \begin{cases} 
\pm \frac{1}{4} (s_0 \hat{q}_0 + s^1 \hat{p}^1 + s^2 \hat{p}^2 + s^3 \hat{p}^3), \\
\pm \frac{1}{4} (s_0 \hat{q}_0 - s^1 \hat{p}^1 + s^2 \hat{p}^2 + s^3 \hat{p}^3), \\
\pm \frac{1}{4} (s_0 \hat{q}_0 + s^1 \hat{p}^1 - s^2 \hat{p}^2 + s^3 \hat{p}^3), \\
\pm \frac{1}{4} (s_0 \hat{q}_0 - s^1 \hat{p}^1 - s^2 \hat{p}^2 + s^3 \hat{p}^3), \\
\pm \frac{1}{4} (-s_0 \hat{q}_0 + s^1 \hat{p}^1 + s^2 \hat{p}^2 - s^3 \hat{p}^3), \\
\pm \frac{1}{4} (-s_0 \hat{q}_0 - s^1 \hat{p}^1 + s^2 \hat{p}^2 + s^3 \hat{p}^3), \\
\pm \frac{1}{4} (-s_0 \hat{q}_0 + s^1 \hat{p}^1 - s^2 \hat{p}^2 + s^3 \hat{p}^3), \\
\pm \frac{1}{4} (-s_0 \hat{q}_0 - s^1 \hat{p}^1 - s^2 \hat{p}^2 - s^3 \hat{p}^3). 
\end{cases}$$

(4.63)

for the $2^4$ possible choices of $s_0, s^1, s^2, s^3$ of the charges in Table 3. The first limit is $N = 2$ supersymmetric etc. In all cases, the extremal mass will be given by the same expression Eq. (4.33).

It is important to observe that the non-extremal solution has no constraints on the signs (or the absolute values) of the charges, hence it interpolates between the 16 discrete extremal limits.
5 Conclusions

In this paper we have constructed static non-extremal black-hole solutions of three \( N = 2, d = 4 \) supergravity models using a general prescription based on several well-known examples of non-extremal black holes. While we have given some arguments to justify why this prescription may always work for all models, we are far from having a general proof and more examples need to be considered \([66]\).

On the other hand, the non-extremal solutions we have found are interesting per se. They seem to share some important properties:

1. Even though in all the models considered there are several disconnected branches of extremal solutions, there is only one non-extremal solution that interpolates between all of them. All the extremal solutions are reachable by taking the appropriate extremal \((r_0 \to 0)\) limit. Furthermore, if we let \(M\) diminish while leaving the charges and asymptotic values of the scalars constant (as happens in the evaporation process in these theories), which extremal limit is attained depends on the charges alone.

2. There seems to be a unique non-extremal superpotential in each theory and, in the different extremal limits, it gives the different superpotentials associated to the different branches of extremal solutions.

3. The non-extremality parameter \(r_0\), expressed in terms of the mass, charges and asymptotic values of the scalars, holds a great deal of information about the theory because \(r_0\) vanishes whenever the value of the mass equals the value of any of the possible extremal superpotentials (some of which are the skew eigenvalues of the central charge matrix). Therefore, knowing this function \(r_0(Z_\infty, Q, M)\) we would know all the possible superpotentials. Unfortunately, there seems to be no a priori formula to determine it \([17]\) and sometimes (e.g. for the \(STU\) model) it is not possible to find it explicitly even when the full solution is known.

4. The metrics have generically two horizons at the values \(\tau = -\infty\) (the outer, event horizon) and \(\tau = +\infty\) (the inner, Cauchy horizon) whose areas and associated entropies are easily calculable and turn out to depend on the values of the scalars at infinity. The product of these two entropies is, in the three cases considered here, the square of the moduli-independent entropy of the extremal black hole that has the same charges.

5. The non-extremal solutions can be used to find some non-supersymmetric extremal solutions that cannot be constructed by the standard methods, as we have shown in the \(\mathbb{CP}^n\) model case.

If this prescription works also in more complicated cases, it will give us the opportunity to study how non-extremal black holes are affected by quantum corrections and perhaps will give us new insights into the microscopic interpretation of the black-hole entropy in non-extremal cases. Work in this direction is in progress.

\[\text{Eq. (1.21) requires the knowledge of the scalar charges } \Sigma^i(Z_\infty, Q, M), \text{ which we know how to compute only after we have the complete black-hole solution.}\]
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