On recoverability of discrete time signals from sparse observations

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Abstract

The paper investigates recoverability of discrete time signals represented by infinite sequences from incomplete observations. It is shown that there exist wide classes of signals that are everywhere dense in the space of square-summable signals and such that signals from these classes feature robust linear recoverability of their finite traces under very mild restrictions on the location of the observed data. In particular, the case arbitrarily sparse and non-periodic subsequences of observations are not excluded.

Keywords: discrete time signals, sampling, signal recovery, prediction, Z-transform, spectrum degeneracy.

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1 Introduction

In general, possibility of recovery of a signal from a sample is usually associated with constraints that ensures an uniqueness of recovery for the classes of underlying signals such as restrictions on the spectrum support or signal sparsity. Analysis of these classes can also lead to recovery methods for noise contaminated signals; the corresponding recovery algorithms can be applied to the projections of the underlying processes on a recoverable class of signals. For example, in continuous time setting, band-limited functions can be recovered without error from a discrete sample taken with a sampling rate that is at least twice the maximum frequency present in the signal (the Nyquist critical rate). This defines the class of recoverable functions and the set of observations required for the recovery.
Clearly, a process of a general type cannot be approximated by band-limited processes with a
preselected band or by processes with a sparse spectrum with a preselected degree of sparsity.
This leads to major limitations for data compression and recovery. For example, consider
data compression via approximation of a continuous time function by samples of band-limited
functions. A closer approximation would require wider spectrum band for these band-limited
functions or more frequent sampling; respectively, a closer approximation leads to less efficient
data compression.

Therefore, there is an important problem of finding wide enough classes of recoverable
processes for different choices set of available observations.

For example, for continuous time signals with certain structure, it was found that the
restrictions imposed by the Nyquist rate could be excessive for signal recovery; see e.g. [1, 10].
In particular, a sparse enough subsequence or a semi-infinite subsequence can be removed from
an oversampling sequence [7, 13]. There is also a so-called Papoulis approach [12] allowing
to reduce the sampling rate with additional measurements at sampling points. Very wide
uniqueness classes of continuous time signals with unlimited spectrum support were considered
in the framework of the approach based on the so-called Landau’s criterion; see. e.g., [8, 9, 11],
and a recent literature review in [11].

For finite discrete time signals, some paradigm changing results were obtained in [2, 3]
and consequent papers in the so-called compressive sensing setting. This approach explores
sparsity of signals, i.e. restrictions on the number of nonzero members of the underlying finite
sequences.

In general, there is a difference between the problem of uniqueness of recovery and the
problem of existence of a stable recovery algorithm. As was emphasized in [9], the uniqueness
results do not imply stable data recovery. For example, any sampling below the Landau’s
critical rate cannot be stable. The Landau’s rate mentioned here is a generalization of the
critical Nyquist rate for the case of stable recovery, non-equidistant sampling and disconnected
spectrum gaps.

The present paper considers infinite discrete time signals. It is shown that there exist wide
classes of signals that are everywhere dense in the space of square-summable signals and such
that signals from these classes feature robust recoverability of finite traces under very mild
restrictions on the location of the observed data (Theorem II below). In particular, the case
arbitrarily sparse and non-periodic subsequences of observations are not excluded. This result
represent a generalization of results [5, 6] obtained for some special sets of observed points and
special types of spectrum degeneracy.
The paper presents the required predicting kernels explicitly via their Z-transforms.

The paper is organized as following. Section 2 presents some definitions and preliminary results on predictability of sequences. Section 3 presents the main result. Section 4 contains the proofs. Section 5 presents some discussion.

2 Definitions and background

Let \( \mathbb{T} \triangleq \{ z \in \mathbb{C} : |z| = 1 \} \), and let \( \mathbb{Z} \) be the set of all integers. Let \( \mathbb{Z}^- = \{ t \in \mathbb{Z} : t \leq 0 \} \), and let \( \mathbb{Z}^+ = \{ t \in \mathbb{Z} : t > 0 \} \).

We denote by \( \ell_r \) the set of all sequences \( x = \{ x(t) \} \subset \mathbb{C} \), \( t = 0, \pm 1, \pm 2, \ldots \) such that \( \|x\|_{\ell_r} = (\sum_{t=-\infty}^{\infty} |x(t)|^r)^{1/r} < +\infty \) for \( r \in [1, \infty) \) or \( \|x\|_{\ell_{\infty}} = \sup_t |x(t)| < +\infty \) for \( r = +\infty \).

For \( x \in \ell_1 \) or \( x \in \ell_2 \), we denote by \( X = \mathbb{Z}x \) the Z-transform

\[
X(z) = \sum_{t=-\infty}^{\infty} x(t)z^{-t}, \quad z \in \mathbb{C}.
\]

Respectively, the inverse \( x = \mathbb{Z}^{-1}X \) is defined as

\[
x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega})e^{i\omega t}d\omega, \quad t = 0, \pm 1, \pm 2, \ldots
\]

We have that \( x \in \ell_2 \) if and only if \( \|X(e^{i\omega})\|_{L_2(-\pi, \pi)} < +\infty \). In addition, \( \|x\|_{\ell_{\infty}} \leq \|X(e^{i\omega})\|_{L_1(-\pi, \pi)} \).

For \( \rho > 0 \), we denote \( B_{\rho}(\ell_2) = \{ x \in \ell_2 : \|x\|_{\ell_2} \leq \rho \} \).

For a finite set \( S \), we denote by \( |S| \) the number of its elements.

We denote by \( \mathbb{I} \) is the indicator function.

The setting for the recovery problem

Let disjoint subsets \( \mathcal{M} \) and \( \mathcal{T} \) of \( \mathbb{Z} \) be given, and let \( \mathcal{V} \triangleq \mathbb{Z} \setminus (\mathcal{M} \cup \mathcal{T}) \).

We are interested in the problem of recovery values \( \{ x(t) \}_{t \in \mathcal{T}} \) from observations \( \{ x(s) \}_{s \in \mathcal{M}} \) for \( x \in \ell_2 \), possibly, in the presence of a contaminating noise. We consider linear estimates only.

**Definition 1** Let \( \mathcal{X} \subset \ell_2 \) be a set of signals. Consider a problem of recovery \( \{ x(t) \}_{t \in \mathcal{T}} \) from observations on \( \mathcal{M} \) of noise contaminated sequences \( x = \bar{x} + \xi \), where \( \bar{x} \in \mathcal{X} \), and where \( \xi \in \ell_2 \) represents a noise. We say that \( \mathcal{X} \) allows finitely robust \( (\mathcal{M}, \mathcal{T}) \)-recovery if there exists a sequence
of mappings \( h_n : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \), \( n = 1, 2, \ldots \), such that \( \sup_{t \in \mathcal{T}} \| h_n(t, \cdot) \|_{\ell_2} < +\infty \) and that, for any \( r > 0 \) and \( \varepsilon > 0 \), and any finite set \( I \subset \mathcal{T} \), there exists \( \rho > 0 \) and \( N > 0 \) such that

\[
\sup_{t \in \mathcal{T} \cap I} |x(t) - \hat{x}_n(t)| \leq \varepsilon \quad \forall \tilde{x} \in \mathcal{X} \cap B_r(\ell_2), \; \eta \in B_{\rho}(\ell_2), \; n > N,
\]

where \( x = \tilde{x} + \eta \) and

\[
\hat{x}_n(t) = \sum_{s \in \mathcal{M}, \; |s| \leq N} h_n(t, s)x(s).
\]  

**Proposition 1** If a set \( \mathcal{X} \) features finitely robust \((\mathcal{M}, \mathcal{T})\)-recoverability, then this set features finitely robust \((\bar{\mathcal{M}}, \bar{\mathcal{T}})\)-recoverability for any disjoint subsets \( \bar{\mathcal{M}} \) and \( \bar{\mathcal{T}} \) of \( \mathbb{Z} \) such that \( \mathcal{M} \subset \bar{\mathcal{M}} \) and \( \mathcal{T} \subset \bar{\mathcal{T}} \).

**Proof of Proposition** Let \( \{h_n\}_{n=1}^\infty \) be such as required for \((\mathcal{M}, \mathcal{T})\)-recoverability in Definition. Then the conditions of Definition hold the pair \((\bar{\mathcal{M}}, \bar{\mathcal{T}})\) if one selects the corresponding functions \( \bar{h}_n(t, s) = h_n(t, s)\mathbb{I}_{\{s \in \mathcal{M}\}} \).

3 The main results

**Theorem 1** Assume that any of the following conditions holds:

(A) \( |\mathcal{M} \cap \mathbb{Z}^-| = +\infty \) and \( |\mathcal{T} \cap \mathbb{Z}^-| < +\infty \).

(B) \( |\mathcal{M} \cap \mathbb{Z}^+| = +\infty \) and \( |\mathcal{T} \cap \mathbb{Z}^+| < +\infty \).

(C) \( |\mathcal{M} \cap \mathbb{Z}^-| = +\infty \) and \( |\mathcal{M} \cap \mathbb{Z}^+| = +\infty \).

Then there exists a set of processes \( \mathcal{B}_{\mathcal{M}, \mathcal{T}} \subset \ell_2 \) that features finitely robust \((\mathcal{M}, \mathcal{T})\)-recoverability and such that, for any \( x \in \ell_2 \) and any \( \varepsilon > 0 \), there exists \( \hat{x} \in \mathcal{B}_{\mathcal{M}, \mathcal{T}} \) such that \( \| \hat{x} - x \|_{\ell_2} \leq \varepsilon \) and \( \| \hat{x} \|_{\ell_2} \leq \| x \|_{\ell_2} \).

**Remark 1** Theorem assumes that the set \( \mathcal{M} \) is infinite but its statement actually implies that an estimate of a finite trace of unknown values can be obtained using a finite set of observations, even contaminated by a noise, as described in Definition. On the other hand, to ensure that the recovery error does not exceed a preselected value, one would need to include a sufficiently large number of observations.

**Remark 2** In the proof of Theorem below, the required predicting kernels are presented explicitly via their Z-transforms.
Some examples where Theorem I holds

(i) If $M = \{k, \; k \in \mathbb{Z}^-\}$ and $\mathcal{I} = \mathbb{Z}^+$, then condition (A) of Theorem I is satisfied.

(ii) If $M = \{km, \; k \in \mathbb{Z}^-\}$ and $\mathcal{I} = \mathbb{Z}^+$, where $m \in \mathbb{Z}^+$ is given, then condition (A) of Theorem I is satisfied.

(iii) If $M = \{kd, \; k \in \mathbb{Z}^-\}$ and $\mathcal{I} = \mathbb{Z}^+$, where $m \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$ are given, then condition (A) of Theorem I is satisfied.

(iv) If $M = \{kd, \; k \in \mathbb{Z}^+\}$ and $\mathcal{I} = \{k \in \mathbb{Z}, \; k < s\}$, where $d \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$ are given then condition (B) of Theorem I is satisfied.

(v) If $M = \{m|k|^d \text{sign} k, \; k \in \mathbb{Z}\}$ and $\mathcal{I} = \mathbb{Z}$, where $m \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$, then condition (C) of Theorem I is satisfied.

In the cases (i)-(iv), the recovery problem is a predicting problem. In the cases (ii)-(v), the subsequence of observations can be arbitrarily sparse. In the cases (iii)-(v), the sequences of observations are non-periodic, and there are infinitely growing gaps between observations.

4 Proof of Theorem I

Case A

Let us prove first that Theorem I holds if condition (A) is satisfied. We refer it as Case (A). Let $\theta = -1 + \min_{t \in \mathcal{I}} t$. By condition (A) of the theorem, $\theta > \infty$.

By Proposition I it suffices to consider the case where $M \subset \{t \in \mathbb{Z} : t \leq \theta\}$. Let us assume that this is the case.

Let the sequence $\{\tau(k)\}_{k \in \mathbb{Z}}$ be such that $\tau(k - 1) < \tau(k)$ for all $k$,

$$M = \{\tau(k)\}_{k=\infty}^\theta, \quad \tau(k) = k \quad \text{for} \quad k > \theta.$$ \hspace{1cm} (3)

Let us consider a mapping $f_\tau : \ell_2 \to \ell_2$ such that $y(k) = x(\tau(k))$ for all $k \in \mathbb{Z}$ for $y = f_\tau(x(\cdot))$.

For $\delta > 0$, let $J(\delta) \triangleq \{\omega \in (-\pi, \pi] : |e^{i\omega} - 1| \leq \delta\}$.

Let us define a mapping $g_\delta : \ell_2 \to \ell_2$ such that $\hat{y} = g_\delta(y) = \mathbb{Z}^{-1}\hat{Y}$, where $\hat{Y}(e^{i\omega}) = \mathcal{Y}(e^{i\omega})1_{\{\omega \in J(\delta)\}}$ and $Y = \mathbb{Z}y$.

Let $\mathcal{B}^y$ be the set of all $\hat{y} = g_\delta(y)$ for all $y \in \ell_2$ and all $\delta > 0$.

Let $\mathcal{M}^\theta = \{k \in \mathbb{Z}, \; k \leq \theta\}$ and $\mathcal{I}^\theta = \{k \in \mathbb{Z}, \; k > \theta\}$. 

5
Let us introduce kernels $h_n(t, s)$, $t, s \in \mathbb{Z}$, defined explicitly as

$$h_n(t, \cdot) = \mathcal{Z}^{-1}H_{n,t},$$

where

$$H_{n,t}(z) = (zV_n(z))^{t-\theta}, \quad V_n(z) \triangleq 1 - \exp\left(-\frac{\gamma_n}{z + \alpha_n}\right), \quad z \in \mathbb{C}. $$

Here $\alpha_n = 1 - \gamma_n^{-\hat{r}}$, where $\hat{r} > 0$ is fixed and where $\gamma_n \to +\infty$.

The equations for $h_n$ are based on the predictors introduced in [4]; they represent a reformulation of the corresponding equations in [6] adjusted for the special case where the spectrum degeneracy is located in a neighbourhood of $z = -1$.

By Lemma 2 from [4], the set $\mathcal{B}^y$ features finitely robust $(\mathcal{M}^y, \mathcal{T}^y)$-recoverability in the sense of Definition 1 with some the kernels $\{h_n\}_{n>0}$, such that the required estimate of $y_n(t)$ is

$$\hat{y}_n(t) = \sum_{s \in \mathcal{M}^y, |s| \leq N_y} h_n(t, s)y(s) = \sum_{s = -N_y}^{\theta} h_n(t, s)y(s)$$

Here $N_y > 0$ is a sufficiently large number.

Since $y_n(t) = x_n(t)$ for $t > \theta$ the corresponding estimate $\hat{x}_n(t)$ for $t > \theta$ can be presented as

$$\hat{x}_n(t) = \hat{y}_n(t) = \sum_{s \in \mathcal{M}, s \geq -N} h_n(t, \tau(s))x(\tau(s)).$$

Here $N = -\tau(-N_y)$.

Let us define a mapping $p_\delta : \ell_2 \to \ell_2$ such that $x = p_\delta(\bar{x})$ is defined such that

$$x(\tau(k)) = \hat{y}_\delta(k), \quad \text{if} \quad k \leq \theta, \quad x(s) = \bar{x}(s) \quad \text{if either} \quad s > \theta \quad \text{or} \quad s \notin \mathcal{M},$$

where $\hat{y}_\delta = g_\delta(y)$ and $y = f_\tau(\bar{x})$.

We construct the sought set $\mathcal{B}_{\mathcal{M}, \mathcal{T}}$ as the set of all $x = p_\delta(\bar{x})$ for all $\bar{x} \in \ell_2$ and all $\delta > 0$.

It follows from the definitions and from the established recoverability of the set $\mathcal{B}^y$ that the set $\mathcal{B}_{\mathcal{M}, \mathcal{T}}$ features finitely robust $(\mathcal{M}, \mathcal{T})$-recoverability in the sense of Definition 1 such that the
required estimate can be presented as

\[ \hat{x}_n(t) = \sum_{s \in M, |s| \leq N_1} h_n(t, \tau(s))x(s). \]

Case B

Similarly to the Case A, we obtain that condition (B) is sufficient to ensure that the statement of the theorem holds. For this, we can just repeat the proof adjusted to the use of backward prediction. We refer it as Case (B).

Case C

Let us prove that condition (C) is sufficient to ensure that the statement of the theorem holds.

Let \( M_\pm = M \cap \mathbb{Z}^\pm \) and \( T_\pm = T \cap \mathbb{Z}^\pm \). Further, for \( \tilde{x} \in \ell_2 \), let \( \tau_+, y_+, x_+ \), and \( B^y_+ \), be defined similarly to \( \tau, y, x = p_\delta(\tilde{x}) \), and \( B^y \), respectively, defined for Case (A) with \( \theta = 0 \). By the result obtained for the Case (A), it follows that the class \( B_{M_+, T_-} \) features finitely robust \((M_+, T_-)\)-recoverability, i.e., the conditions of Definition 1 hold, with the estimates

\[ \hat{x}_{n,+}(t) = \sum_{s \in M_-, s \geq -N} h_{n,+}(t, \tau_+(s))x_-(s), \quad t \in T_+. \]

Here kernels \( h_{n,+} \) are such as required in Definition 1.

Further, for \( \tilde{x} \in \ell_2 \), let \( \tau_-, y_-, x_- \), and \( B^y_- \), be defined similarly to \( \tau, y, x = p_\delta(\tilde{x}) \), and \( B^y \), respectively, for Case (B) with \( \theta = 0 \). By the theorem statement for Case (B), it follows that the class \( B_{M_+, T_-} \) features finitely robust \((M_+, T_-)\)-recoverability, i.e., the conditions of Definition 1 hold, with the estimates

\[ \hat{x}_{n,-}(t) = \sum_{s \in M_+, s \leq N} h_{n,-}(t, \tau_-(s))x_-(s), \quad t \in T_. \]

Here kernels \( h_{n,-} \) are such as required in Definition 1.

We construct the sought set \( B_{M, T} \) as the class of processes \( x \in \ell_2 \) that can be represented as

\[ x_n(t) = x_{n,+}(t)I_{\{t \in M_+\}} + x_{n,-}(t)I_{\{t \in M_-\}} \]

for some \( x_{n,+}(t) \in B_{M_+, T_-} \) and \( x_{n,-}(t) \in B_{M_-, T_+} \).
Clearly, an estimate (1) for a given $\varepsilon$ holds for sufficiently large $\bar{n}$ and small $\rho$, since similar estimates hold for $\sup_{t \in M_{\pm}} |\hat{x}_{n,\pm}(t) - x(t)|$.

Let us show that $\hat{x}_n$ can be represented via (2) for with some choice of appropriate mappings $h_n : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$. By the definitions, it follows that, for $t \in T$,

$$\hat{x}_n(t) = \sum_{s \in M_{-}, s \geq -N} h_{n,+}(t, \tau_+(s))x_-(s) + \sum_{s \in M_{+}, s \leq N} h_{n,-}(t, \tau_-(s))x_-(s).$$

Hence

$$\hat{x}_n(t) = \sum_{s \in M, |s| \leq N} h_n(t, s)x(s),$$

where

$$h_n(t, s) = \sum_{s \in M_{-}} h_{n,+}(t, \tau_+(s)) + \sum_{s \in M_{+}} h_{n,-}(t, \tau_-(s)).$$

This gives representation (2). This completes the proof for the Case (C) as well as the proof of Theorem 1. □

5 Discussion

Theorem 1 provides an existence result. The conditions on the choice of the sets $M$ and $T$ imposed by Theorem 1 are quite mild. There are no restrictions on the sparsity of $M$ or on the choices of $r, I, \varepsilon, \delta$ presented in Definition 1 and in the definition of $J(\delta)$. For example, the set $M$ can have arbitrarily located gaps, in particular, it can have periodic gaps as well as non-periodic gaps.

The proof of Theorem 1 provides explicitly a recovery algorithm based on a similar algorithm [6] focused on a special case of periodic set $M$; some problems for its numerical implementation are outlined therein. Some experiments are described in [6]. In particular, if $M$ is too sparse, or $I$ and $r$ are too large, or $\varepsilon$ and $\delta$ are too small, then the corresponding predicting kernels will be too large and too heavy-tailed for implementation on standard computers. The numerical examples in [6] show that an effective numerical implementation of this algorithm would require significant efforts.

There are other choices of the sets $B_{M,T}$ and of the recovery algorithms in the proof of Theorem 1. The proof above uses sets of band-limited processes with spectrum gaps $J(\delta)$. Alternatively, they could be replaced by processes featuring spectrum degeneracy at a single
point only, as is allowed in Lemma 2 [6]. For some important special cases, other linear predicting kernels could be more effective. For example, the recovering operators suggested in [5] would be preferable in the case where the set \( \mathcal{I} \) is finite and where the underlying processes belong to \( \ell_1 \). The linear recovery operators from [6] can be used in the case where the \( \mathcal{M} \) is a periodic subsequence of \( \mathbb{Z} \). An approaches from the proofs of [7, 13] could lead to a different proof of Theorem 1. However, it is unclear if the numerical feasibility can be improved by any of these possible modification.

We leave these questions for the future research.

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