UNBIASED ESTIMATION AND BACKTESTING OF RISK IN THE CONTEXT OF HEAVY TAILS

MARCEL PITERA AND THORSTEN SCHMIDT

ABSTRACT. While the estimation of risk is an important question in the daily business of banks and insurances, it is surprising that efficient procedures for this task are not well studied. Indeed, many existing plug-in approaches for the estimation of risk suffer from an unnecessary bias which leads to the underestimation of risk and negatively impacts backtesting results, especially in the small sample environment. In this article, we consider efficient estimation of risk in practical situations and provide means to improve the accuracy of risk estimators and their performance in backtesting. In particular, we propose an algorithm for bias correction and show how to apply it for generalized Pareto distributions. Moreover, we propose new estimators for value-at-risk and expected shortfall, respectively, and illustrate the gain in efficiency when heavy tails exist in the data.

Keywords: value-at-risk, expected shortfall, estimation of risk measures, bias, risk estimation, backtesting, unbiased estimation of risk measures, generalized Pareto distribution.

1. Introduction

Risk measures are a central tool in capital reserve evaluation and quantitative risk management, see McNeil et al. (2010) and references therein. Given the immense amount of literature linked to various risk measurement aspects, it is surprising that efficient statistical estimation procedures for risk measures have rarely been discussed in the literature. In most cases plug-in estimation procedures act as the reference framework, see Cont et al. (2013) and Davis (2016) for exemplary contributions. In this article, we focus on an economic notion of unbiasedness for risk measures which turns out to be important whenever the risk measure needs to be estimated. Unbiasedness in the context of risk measures is the generalization of the well-known statistical unbiasedness to the risk landscape. This concept was introduced in Pitera and Schmidt (2018). Here, we extend those results and study efficiency of the frequently used plug-in estimators with focus set on backtesting. Moreover, we consider for the first time unbiased estimation of risk measures in the important class of generalized Pareto distributions (GPD) and propose two new and efficient estimators for value-at-risk and expected shortfall, respectively.

Generally speaking, a monetary risk measure quantifies the risk associated with the future profits and losses (P&L) of a financial position in monetary terms. More precisely, the risk measure $\rho$ associates to $X$, a random variable describing portfolio’s P&L, the amount of money, $\rho(X)$, which has to be added to the position to make it acceptable. There are many possible choices for such a risk measure. For example, this could refer to Value-at-Risk (VaR), which corresponds to a low quantile of $X$, or the Expected Shortfall (ES), which is an average over VaR. For these risk measures, it is clear that the output depends only on the distribution of $X$; see Föllmer and Schied (2011).

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While in most theoretical risk measure studies the distribution of $X$ is considered known and fixed, in all practical applications the distribution of $X$ is unknown and needs to be estimated. This changes the picture in a dramatic way and typically leads to a systematic underestimation of risk capital, which increases with heavy tails, small sample size, and high significance level. In Pitera and Schmidt (2018), we have already evaluated the performance of the classical estimators of VaR and observed a significantly higher rate of overshoots than expected. This was the case even in the perfect independent and identically distributed (light-tailed) data setting.

This article aims to introduce a general procedure for bias reduction in the estimation of risk measures to improve the efficiency of the estimators and to avoid risk underestimation. We do so by introducing methods for improving the accuracy of existing estimators and their performance in backtesting. A particular focus is laid on the setting with heavy tails.

This article is organized as follows. In Section 2 we introduce the theoretical background for risk unbiasedness, risk sufficiency and backtesting, while in Section 3 we show that risk bias can be reduced when estimating risk. Section 4 illustrates the performance on simulated examples under GPD distribution and Section 5 concludes.

2. Estimating risk

Consider the estimation of the riskiness of a future position measured by a risk measure, say $\rho$, that is applied to the position’s future P&L, say $X$. For the estimation purposes we are given a historical data $X_1, \ldots, X_n$ linked to $X$. In the considered case, this data is assumed to be independent and identically distributed (i.i.d.); this is a frequent assumption when the Historical Simulation approach is used, see Jorion (2001). Therefore, in an economic context, where the environment often undergoes rapid changes, the sample size should be relatively small. Otherwise, regime switches and change points need to be taken into account.

We begin with the formal introduction of the risk measure estimation framework. An estimator of the risk measure $\rho$ for $X$ is simply a (measurable) function of the available data, which we denote by $\hat{\rho} = \hat{\rho}(X_1, \ldots, X_n)$. The future position is secured by adding the estimated economic capital $\hat{\rho}$ leading to the secured position $Y := X + \hat{\rho}$.

The secured position is again a random variable and we are interested in its riskiness. Quite intuitively, we can measure the risk of $Y$ by applying risk measure $\rho$. If the risk measure is law-invariant, the quantified risk depends on the underlying distribution (which we do not know). Consequently, the risk of the secured position should be acceptable for all possible distributions. Let $\Theta$ denote the corresponding parameter space and let each $\theta \in \Theta$ identify a certain distribution choice (of both $X$ and $Y$). Having this in mind, we introduce a family of risk measures $\mathcal{R} := \{\rho_{\theta} : \theta \in \Theta\}$ which we use for the validation of the secured position. In other words, $\rho_{\theta}$ is used to quantify the risk of $X$ (and $Y$) under $\theta \in \Theta$.

As in Pitera and Schmidt (2018), we call an estimator $\hat{\rho}$ risk unbiased with respect to the family of risk measures $\mathcal{R}$, if for all $\rho \in \mathcal{R}$,

$$\rho(X + \hat{\rho}) = 0.$$  \hfill (2.1)

In addition, we call $\hat{\rho}$ risk sufficient if $\rho(X + \hat{\rho}) \leq 0$ for all $\rho \in \mathcal{R}$.

For brevity, we typically write unbiased instead of risk unbiased and sufficient instead of risk sufficient. We hope the risk of confusing this notion with the statistical concepts of unbiasedness and sufficiency is small. Note the concept of risk unbiasedness is the proper generalization of the statistical bias to non-linear measures as in the case of risk measurement. From this perspective, apart from risk unbiasedness, it is natural to require additional efficiency properties from a good
risk estimator. For example, this could relate to statistical consistency or minimal expected risk. We refer to Section 4 and Example 7.3 in Pitera and Schmidt (2018) for additional details.

An unbiased estimator is of course (risk) sufficient. But, a sufficient estimator is allowed to overestimate the risk (but not to underestimate it). When the risk measures are monotone, an unbiased estimator is a sufficient estimator requiring only minimal capital.

Again, note that $\hat{\rho} = \hat{\rho}(X_1, \ldots, X_n)$ is a random variable. Hence, we can not use cash-additivity to split $X$ and $\hat{\rho}$ in (2.1). In the case of normally distributed random variables one would consider $\theta = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times \mathbb{R}_{>0}$. Then, unbiasedness means that the risk of the secured position $Y$ is zero for all possible choices of $\mu \in \mathbb{R}$ and $\sigma > 0$. In Section 2.2.1 we present a closed form formula for the unbiased estimator in the Gaussian setting.

Remark 2.1 (Relation to statistical bias). When the risk measure $\rho$ corresponds to the expectation, Equation (2.1) coincides with the statistical bias, i.e. the condition $E_\theta[\hat{\rho}] = -E_\theta[X]$, for all $\theta \in \Theta$. Here $E_\theta$ is the expectation under $\theta \in \Theta$. Of course, the arithmetic mean turns out to be the unbiased estimator in this case, both in the statistical sense and in the risk sense.

Remark 2.2 (Risk unbiasedness for distribution-based risk measures). Typically, we are given a distribution-based risk measure (often called law-invariant), i.e. a risk measure which only depends on the distribution of the underlying losses. Such a risk measure can be represented as function $R : \mathbb{F} \to \mathbb{R} \cup \{+\infty\}$, where $\mathbb{F}$ denotes the set of all probability distributions. VaR and ES are prominent examples. Denote by $F_\theta$ the distribution of $X$ in scenario $\theta \in \Theta$. Then, the risk of $X$ is given by $\rho_\theta(X) = R(F_\theta)$. However, we are interested in the risk of the secured position $Y$. By $G_\theta$ we denote the distribution of the secured position for scenario $\theta$ (in the i.i.d. case this can be obtained from the convolution of $F_\theta$ with the distribution of the estimator $\hat{\rho}$). Consequently, $\hat{\rho}$ is unbiased if

$$R(G_\theta) = 0, \quad \theta \in \Theta. \quad (2.2)$$

2.1. Relation between backtesting and risk bias. In the following we show that the notions of unbiasedness and sufficiency have an intrinsic connection to backtesting. In particular, we show how the backtesting framework representation discussed in Moldenhauer and Pitera (2019) can be placed into the context of unbiasedness.

Consider a distribution-based risk measure family $(R_\alpha)_{\alpha \in [0,1]}$. We assume that this family is increasing with respect to $\alpha$. Of course, this could correspond to VaR or ES frameworks, where $\alpha \in (0,1]$ is the underlying risk level. In particular, we want to estimate the risk of $X$ at the pre-defined reference risk level $\alpha_0 \in (0,1]$ (often 0.01 or 0.025). Within the backtesting framework, a risk estimator $\hat{\rho}$ is estimated several times and compared to the realized P&L. In this regard, we assume that we are given a sample of $n + m$ observations, denoted by $x_{-n-1}, \ldots, x_0, x_1, \ldots, x_m$. Then, we compute two quantities from the historic data. First, we compute $m$ risk estimators $\hat{\rho}_1, \ldots, \hat{\rho}_m$ where $\hat{\rho}_t = \hat{\rho}(x_{t-n}, \ldots, x_{t-1})$. Second, we select a corresponding realization of the (future) P&L, which is in our case is given by $x_t$, in order to confront it with the projected risk $\hat{\rho}_t$.

The aim of backtesting is the validation of the estimator $\hat{\rho}$. The $t$-th day projected risk $\hat{\rho}_t$ is estimated using data available up to the previous day. For brevity, we define the $t$-th day secured position by $y_t := x_t + \hat{\rho}_t$ and set $y := (y_t)$. A natural choice for validation is the performance measure that is dual to the family of risk measures $(R_\alpha)$. The empirical counterpart of the dual performance measure, defined in Cherny and Madan (2009), is given by

$$B(y) := \inf\{ \alpha \in (0,1] : R_{\alpha}^{\text{emp}}(y) \leq 0\}, \quad (2.3)$$
where $R_{\alpha}^{\text{emp}}(y) := R_{\alpha}(\hat{F}(y))$ with $\hat{F}(y)$ denoting the empirical distribution of $y$. Intuitively speaking, we look for the smallest value of $\alpha \in (0, 1]$ that makes the secured sample $y$ acceptable, based on the empirical distribution.

The inner part of Equation (2.3), $R_{\alpha}^{\text{emp}}(y) \leq 0$, is the empirical counterpart of sufficiency. As a consequence, if the risk estimator is sufficient, one expects $B(y)$ to not exceed the reference level $\alpha_0$ which should in turn indicate good performance of the secured position. Moreover, the minimum in (2.3) is achieved for $\alpha$ for which the empirical equivalent of unbiasedness is satisfied. Thus, for an unbiased estimator, the value of $B(y)$ should be close to the reference $\alpha_0$ value. Note that in (2.3) we implicitly assumed that $y$ is i.i.d. at least when the risk of secured position is being estimated (under correct model specification).

In practice, to account for potential sample bias, model misspecification, non i.i.d. sample properties, etc., some variation in $B(y)$ is acceptable. The typical approach is to use so called traffic-light approach with certain fixed thresholds imposed on the value of $B(y)$ which we will introduce now.

2.1.1. Backtesting value-at-risk. The classical regulatory backtest for VaR counts the empirical number of overshoots and compares them to the expected number of overshoots. More precisely, given the VaR-secured positions $y$, the (average) breach count statistic

$$T(y) := \frac{1}{m} \sum_{t=1}^{m} \mathbb{1}_{\{y_t < 0\}}$$

is used in the regulatory backtesting, see BCBS (2009). For the regulatory backtesting we use yearly time series (250 historical observations): the model is in the green zone, if $T(y) < 0.02$, in the yellow zone, if $0.02 \leq T(y) < 0.04$, and in the red zone, if $T(y) \geq 0.04$. This corresponds to less than five, between five and nine, and ten or more breaches, respectively, when the non-averaged value of $T(y)$ is considered; see BCBS (1996) for more details.

2.1.2. Backtesting expected shortfall. The backtesting statistic based on (2.3) can also be used for ES backtesting. For the ES-secured sample $y$, we compute analogously the empirical mean of the overshooting samples,

$$G(y) := \frac{1}{m} \sum_{t=1}^{m} \mathbb{1}_{\{y_{(1)} + \ldots + y_{(t)} < 0\}}.$$  

The performance statistic (2.5) simply measures the cumulative breach count and answers a simple question: how many (worst-case) scenarios do we need to consider to know that the aggregated loss does not exceed the aggregated capital reserve, see Moldenhauer and Pitera (2019) for details.

2.2. Risk estimation and backtesting under normality. In this section we show that many popular risk estimators are risk biased, even in an asymptotic sense under the Gaussian setting. A risk biased estimation leads to a systematic underestimation of the risk which is in turn reflected in a bad backtesting performance. To illustrate this fact, we present two simple examples.

2.2.1. Value-at-risk under normality. Consider the case where the observed sample is i.i.d. and normally distributed. We are interested in VaR at level $\alpha_0 = 1\%$. For any day $t$, we consider the Gaussian plug-in estimator by plugging in $\hat{\mu}_t$ and $\hat{\sigma}_t$ into the theoretical value-at-risk gaussian formula, i.e. we set

$$\hat{V@R}_{t}^{\text{plug-in}} := \hat{\mu}_t + \hat{\sigma}_t \Phi^{-1}(0.01);$$

here $\hat{\mu}_t$ and $\hat{\sigma}_t$ correspond to the maximum-likelihood estimators for $\mu$ and $\sigma$, respectively, obtained using past $n$ observations and $\Phi$ is the cumulative distribution function of the standard normal
distribution. The estimator given in (2.6) does not satisfy property (2.1) and therefore is biased. However, as shown in Pitera and Schmidt (2018), the estimator given by

\[ \hat{V}@R_{t}^{u} := \hat{\mu}_{t} + \hat{\sigma}_{t} \sqrt{\frac{n+1}{n} t_{n-1}^{-1}(0.01)}, \]

is unbiased; here \( t_{n-1} \) refers to t-student distribution function with \( n-1 \) degrees of freedom.

If we refrain from normality, it is natural to consider the empirical quantile as estimator for value-at-risk, which we call empirical estimator\(^1\) denoted by \( \hat{V}@R_{t}^{\text{emp}} \). An (unknown) reference for the estimators is the true risk, given by

\[ \hat{V}@R_{t}^{\text{true}} := \mu + \sigma \Phi^{-1}(0.01). \]

We are now ready to perform backtesting and numerically illustrate the impact of risk bias. We construct the backtests for the aforementioned estimators according to Equation (2.4): for \( z \in \{\text{plug-in, u, emp, true}\} \) the average number of exceptions up to time \( m \) is given by

\[ T^{z} := \frac{1}{m} \sum_{t=1}^{m} \mathbb{1}_{\{x_{t} + \hat{V}@R_{t}^{z} < 0\}}. \]

For a numerical illustration we keep the window length of historical data, \( n \), fixed and consider \( m \to \infty \). For the case of standard normal random variables (i.e. \( \mu = 0, \sigma = 1 \)), we plot the resulting backtests in Figure 1 (left side).

\(^1\)To estimate empirical VaR we used the standard quantile function built into R software with default (type 9) setting.
Ideally, the value $T^z = T^z(m)$ should converge to theoretical value 1% for an efficient estimator. However, Figure 1 shows that this is neither the case for the plug-in estimator nor the empirical estimator. It holds, however, for the unbiased estimator, and – of course – for the true risk.

Under normality, the asymptotic exception rate for the plug-in estimator could be in fact directly computed and is strictly related to estimator bias. For $n = 250$, we obtain

$$
\lim_{m \to \infty} T^{\text{plug-in}}(m) = t_{n-1} \left( \sqrt{\frac{n}{n+1}} \Phi^{-1}(0.01) \right) \approx 1.05\%.
$$

The simulations also show that the asymptotic exception rate of the empirical estimator is larger and oscillates around 1.35%.

**Remark 2.3 (Some consequences).** While the empirical estimator asymptotic exception rate (1.35%) might not look severe it does have a serious monitoring consequence. Under the correct model setting for VaR at level 1%, the individual breach probability should be 1% which leads to probability of reaching the yellow or red traffic-light zone (having more than 4 exceptions in the annual backtest) equal to approximately 10%. On the other hand, if the individual breach probability is equal to 1.35%, the probability of reaching yellow or red zone is equal to approximately 25%. In consequence, the monitoring will detect problems with the model twice as often as it should, at least if daily risks (rather than three month average risks) are used for backtesting.

### 2.2.2. Expected shortfall under normality.

We continue with the setting of Section 2.2.1. The classical ES plug-in estimator is given by

$$
\hat{\text{ES}}^{\text{plug-in}}_t := \hat{\mu}_t + \hat{\sigma}_t \phi\left(\Phi^{-1}(\alpha)\right),
$$

and we are interested in $\alpha_0 = 0.025$ as is customary for the standard expected shortfall backtest. The unbiased estimator can be obtained by a linear rescaling of the plug-in estimator. It is given by

$$
\hat{\text{ES}}^u_t := \hat{\mu}_t + c_{250} \cdot \hat{\sigma}_t \phi\left(\Phi^{-1}(\alpha)\right);$$

the constant $c_{250} = 1.0077$ was obtained using approximation scheme introduced in Example 5.4 in Pitera and Schmidt (2018).

In addition, we consider the empirical estimator for expected shortfall given by

$$
\hat{\text{ES}}^{\text{emp}}_t := \frac{\sum_{i=t-n}^{t-1} x_i \mathbb{1}\{x_i + \text{VaR}^{\text{emp}}_t \leq 0\}}{\sum_{i=t-n}^{t-1} \mathbb{1}\{x_i + \text{VaR}^{\text{emp}}_t \leq 0\}}.
$$

For reference, we denote the true (but unknown) value of the expected shortfall by

$$
\text{ES}_t^{\text{true}} := \mu + \sigma \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha}.
$$

For the backtest of expected shortfall, we rely on (2.5). Hence, for every $z \in \{\text{plug-in, u, emp, true}\}$ we consider

$$
G^z = G^z(m) := \frac{1}{m} \sum_{t=1}^{m} \mathbb{1}\{y^z_{(1)} + \ldots + y^z_{(n)} < 0\},
$$

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2We get $1 - F_{B(n,p)}(4) \approx 10\%$, where $B(n,p)$ is the Bernoulli distribution with $n = 250$ trials and probability of success equal to $p = 1\%$. 
where, for each $m$, the order statistic $y_{(k)}^z$ corresponds to the $k$-th order statistic of the associated secured position $y_t = x_t + \hat{ES}_t^z$, $t = 1, \ldots, m$. On the right hand side of Figure 1 we present the values of our numerical study, which confirm the findings from the value-at-risk case.

While in the Gaussian framework we were able to compute explicitly an unbiased estimator, this is no longer the case in more general situations. The next example studies the case of risk estimation under GPD distributional assumptions.

2.3. Risk estimation and backtesting under generalized Pareto distributions. In the context of heavy tails, a closed form for the unbiased estimator is no longer available and one has to rely on numerical procedures. We introduce such an approach in the following chapter. Here we describe the currently existing estimators for generalized Pareto distributions and illustrate their critical underestimation of risk.

Again, we consider an i.i.d. sample, but this time we assume that $X_t$ is taken from a GPD distribution (left tail) with threshold $u \in \mathbb{R}$, shape $\xi \in \mathbb{R}$, and scale $\beta \geq 0$; we also set $p := P(X \geq u)$. In practical applications, the threshold $u$ is often considered as known while the parameters $\xi$ and $\beta$ have to be estimated.

2.3.1. Value-at-risk under GPD. Under the assumption of a fixed and known distribution it is well-known how to compute the value-at-risk under GPD, we refer to McNeil et al. (2010) for details. Indeed, basic calculations yield the true risk value

$$V_{\text{true}} := -u + \frac{\beta}{\xi} \left( \alpha^{-\xi} - 1 \right).$$

Denoting by $\hat{\xi}_t$ and $\hat{\beta}_t$ (with additional $t$ referring to the estimation period) the standard estimators for $\xi$ and $\beta$ we readily obtain the plug-in estimator for value-at-risk in the GPD case,

$$V_{\text{plug-in}} := -u + \frac{\hat{\beta}_t}{\hat{\xi}_t} \left( \alpha^{-\hat{\xi}_t} - 1 \right).$$

When the threshold $u$ needs to be estimated additionally, we would replace $u$ by $\hat{u}_t$.

To numerically asses an existing bias in the estimation, we additionally consider the empirical estimator $V_{\text{emp}}$, which is of course the same as in the previous section. We consider a fixed parameter set $u = 1$, $\xi = 0.05$, $\beta = 0.7$, and $p = 0.2$. Having $p = 0.2$ means that only 20% of the data lie beyond the threshold $u$ and the observed sample consists only of the data above the threshold. The corresponding periods have to be adjusted accordingly: we consider a learning period of length $n = 250 \cdot p = 50$ and reference level $\alpha_0 = 0.01/p = 0.05$.

For $z \in \{\text{plug-in}, \text{emp}, \text{true}\}$, we construct the secured position and perform the backtest according to Equation (2.8). The results are shown on the left hand side of Figure 2. The figure considers a fixed window size together with increasing sample size, i.e. $m \to \infty$. We observe that the bias vanishes for the true value-at-risk, as expected there is no bias once the true distribution is known. On the contrary, both the plug-in estimator and the empirical estimator show a clear bias. The asymptotic risk level reached by $V_{\text{plug-in}}$ is around 6% (instead of 5%), while for $V_{\text{emp}}$ it is even close to 7%. The latter means that in 7% of the observed cases the estimated risk capital is not sufficient to cover the occurred losses which corresponds to a significant underestimation of the present risk.

2.3.2. Expected shortfall under GPD. Similarly to the value-at-risk, under the the assumption of a fixed and known distribution, expected shortfall can be computed explicitly, we refer to McNeil et al. (2010) for details. Indeed, basic calculations yield the true risk value

$$ES_{\text{true}} := -u + \frac{\beta}{\xi^2} \left( \alpha^{-\xi^2} - 1 \right).$$

Denoting by $\hat{\xi}_t$ and $\hat{\beta}_t$ (with additional $t$ referring to the estimation period) the standard estimators for $\xi$ and $\beta$ we readily obtain the plug-in estimator for expected shortfall in the GPD case,

$$ES_{\text{plug-in}} := -u + \frac{\hat{\beta}_t}{\hat{\xi}_t^2} \left( \alpha^{-\hat{\xi}_t^2} - 1 \right).$$

When the threshold $u$ needs to be estimated additionally, we would replace $u$ by $\hat{u}_t$.
et al. (2010) for details. The expected shortfall computes to (which is then the true risk)

\[ \text{ES}_{t}^{\text{true}} := \frac{\text{V@R}_{t}^{\text{true}}}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi} \]

Denoting by \( \hat{\xi}_t \) and \( \hat{\beta}_t \) (with additional \( t \) referring to the estimation period) the standard estimators for \( \xi \) and \( \beta \) we readily obtain the plug-in estimator for expected shortfall in the GPD case,

\[ \text{ES}_{t}^{\text{plug-in}} := \frac{\hat{\text{V@R}}_{t}^{\text{plug-in}}}{1 - \hat{\xi}_t} + \frac{\hat{\beta}_t - \hat{\xi}_t u}{1 - \hat{\xi}_t} \]

In addition we compute backtesting results for the empirical ES estimator \( \hat{\text{ES}}^{\text{emp}} \) defined as in (2.9). We consider the same fixed parameter set as for the computation of the value-at-risk estimators: \( u = 1, \xi = 0.05, \beta = 0.7, \) and \( p = 0.2 \) together with a learning period of length \( n = 250 \cdot p = 50 \) and reference level \( \alpha_0 = 0.01/p = 0.05. \) The results for the backtesting statistic \( G^z \) introduced in Equation (2.10), where \( z \in \{\text{plug-in}, \text{emp}, \text{true}\} \), are presented in Figure 2. Similar to the computations for the value-at-risk we observe no bias for the true risk, but a significant bias for the plug-in estimator and an even larger bias for the empirical estimator.

The presented examples illustrate that using biased risk estimators leads to a systematic underestimation of risk. This important defect becomes more pronounced with heavier tails of the underlying distribution, increased significance (i.e. decreasing \( \alpha \)), or reduced sample size. It can be shown that the risk bias negatively effects the predictive accuracy as defined in Gneiting (2011), i.e. that the consistent VaR score of the unbiased estimator (2.7) is better than standard Gaussian plug-in estimator (2.6); we refer to Section 8.3 in Pitera and Schmidt (2018) for more details.

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**Figure 2.** Backtesting results under GPD for increasing observation length \( m \) up to 100,000 days and a rolling window of length \( n = 250. \) **Left:** Average number of exceptions \( T^z \) at level \( \alpha_0 = 1\% \). We present the statistic for \( \hat{\text{V@R}}_{t}^{\text{emp}} \) (empirical), \( \hat{\text{V@R}}_{t}^{\text{plug-in}} \) (plug-in), and under the assumption of full knowledge of the underlying distribution, \( \text{V@R}_{t}^{\text{true}} \) (true). **Right:** Empirical mean of the overshoots \( G^z \) at level 2.5\%. We present the statistic for \( \hat{\text{ES}}_{t}^{\text{emp}} \) (empirical), \( \hat{\text{ES}}_{t}^{\text{plug-in}} \) (plug-in), and under the assumption of full knowledge of the underlying distribution, \( \text{ES}_{t}^{\text{true}} \) (true). Only the true (and unknown) risk is close to theoretical exception rate.
3. Bias reduction

Motivated by the findings in the previous section we introduce a general scheme for improving the efficiency of an existing estimator by reducing its bias. We start from the observation that in the Gaussian case the unbiased estimator given in (2.7) is obtained from the plug-in estimator in (2.6) by appropriately rescaling the parameter estimates, from \((\hat{\mu}, \hat{\sigma})\) to
\[
\left(\hat{\mu}, \sqrt{\frac{n + 1}{n} \frac{t_{n-1}(\alpha)}{\Phi^{-1}(\alpha)} \hat{\sigma}}\right).
\]

More generally, if we estimate the d-dimensional parameter \(\theta\) by the estimator \(\hat{\theta}\), we look for a modification of the estimated parameter from \(\hat{\theta}\) to \(a \circ \hat{\theta}\), where \(\circ\) is the componentwise multiplication and choose \(a\) such that the bias is minimized. We propose a bootstrapping algorithm which achieves this.

3.1. Local risk sufficiency. We start by relaxing the unbiasedness criterion to a slightly weaker notion, which we call local risk unbiasedness (and, in an analogous way, risk sufficiency). The idea is not to require unbiasedness on the full parameter set \(\Theta\), but on a subset \(\Theta_0\), which will be chosen dependent on the data at hand.

For simplicity, we focus on distribution-based risk measures and use the notation introduced in Remark 2.2. Recall from Remark 2.2 that we denoted the distribution of \(X\) under scenario \(\theta \in \Theta\) by \(F_\theta\) and the distribution of the secured position \(Y = X + \hat{\rho}\) by \(G_\theta\). Given a distribution-based reference risk measure \(R\) we define the maximal risk bias of the estimator \(\hat{\rho}\) as
\[
B_*(\hat{\rho}) := \sup_{\theta \in \Theta} R(G_\theta).
\]

If \(B_*(\hat{\rho}) = 0\), then \(\hat{\rho}\) is risk sufficient since
\[
\rho_\theta(X + \hat{\rho}) \leq B_*(\hat{\rho}) = 0, \quad \theta \in \Theta.
\]

For practical applications it might be too restrictive to assume unbiasedness, since this requires the above property for all \(\theta \in \Theta\) and we consider the subset \(\Theta_0 \subseteq \Theta\). We call \(\hat{\rho}\) locally risk sufficient (on \(\Theta_0\)), if
\[
B_*(\hat{\rho}, \Theta_0) := \sup_{\theta \in \Theta_0} R(G_\theta) = 0. \tag{3.1}
\]

In the case we consider here, the data sample and the future P&L are i.i.d., such that \(X\) and \(\hat{\rho}\) are independent. Denoting by \(H_\theta\) the distribution of \(\hat{\rho}\) under scenario \(\theta \in \Theta\), we can rewrite (3.1) as
\[
B_*(\hat{\rho}, \Theta_0) = \sup_{\theta \in \Theta_0} R(F_\theta \ast H_\theta), \tag{3.2}
\]
where \(\ast\) is the convolution operator. Let \(\hat{\theta}\) denote the estimator of \(\theta\), and let \(\hat{\rho}^{\text{plug-in}} = R(F_{\hat{\theta}})\) denote the plug-in risk estimator. Then, the local bias of the plug-in estimator can be computed via Equation (3.2),
\[
B_*^{\text{plug-in}}(\Theta_0) := \sup_{\theta \in \Theta_0} R(F_{\hat{\theta}} \ast H_{\hat{\theta}}^{\text{plug-in}}); \tag{3.3}
\]
here \(H_{\hat{\theta}}^{\text{plug-in}}\) denotes to the distribution of the plug-in estimator \(\hat{\rho}^{\text{plug-in}}\) in the case where the sample random variables \(X_1, \ldots, X_n\) are i.i.d. from the distribution associated with \(\theta\). This distribution will often be difficult to obtain and we propose to estimate it by bootstrap. In principle, the family of estimated distributions could be different from the family of scenarios for the testing risk measures, but we will not exploit this possibility here.
Algorithm 1 (The bootstrapped risk estimator).

Begin by an estimation step.

1. Estimate $\hat{\theta}$ (e.g. using MLE approach) from $X_1, \ldots, X_n$.

The next steps perform the bootstrap.

2. For each $i = 1, \ldots, B$, simulate an i.i.d. sample $X_i$ of size $n$ from the estimated distribution $\hat{F}_\theta(X)$ and compute estimator $\hat{\theta}_i$ for each $X_i$.

3. For any $a \in \mathbb{R}^d$ (or a suitable chosen subset $A$) do the following:
   
   (3a) estimate the distribution of $H_{\text{plug-in}}^{a \circ \hat{\theta}}$ directly from sample $(\hat{\theta}_1, \ldots, \hat{\theta}_B)$, e.g. by a kernel density estimation. Denote the estimated distribution by $\tilde{F}(a)$.
   
   (3b) Compute the convolution of $F_{\hat{\theta}} \ast \tilde{F}(a)$.

Finally, compute the optimal choice of the parameter $a$:

4. Calculate $a^* := \arg\min_{a \in A} \left| R(F_{\hat{\theta}} \ast H_{\text{plug-in}}^{a \circ \hat{\theta}}) \right|$, $a$.

5. Set the bootstrapped risk estimator to $\hat{\rho}^b := R(F_{a^* \circ \hat{\theta}})$.

It should be noted that the local bias-minimising shift might be non-unique. Generally speaking, it is better to shift the shape parameters than the location parameters: If $\Theta = \mathbb{R}^2$ and the first coordinate is the location parameter (mean), then this can be achieved by choosing $A = (1, \mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1\}$.

Figure 3. The algorithm for bootstrapping bias reduction.

3.2. The bootstrapping bias reduction. Starting from the observation that plug-in estimators are biased in general, our idea is to tweak the risk estimator $\hat{\theta}$ a little bit to obtain a modified estimator which has a smaller absolute value of bias. To achieve this, we replace $\hat{\theta}$ by $a \circ \hat{\theta} = (a_1 \hat{\theta}_1, \ldots, a_d \hat{\theta}_d)$ and choose $a \in \mathbb{R}^d$ such that the absolute value of bias is minimized locally at $\Theta_0 = \{\hat{\theta}\}$, i.e. we set

$$a^* = \arg\min_{a \in \mathbb{R}^m} \left| R(F_{\hat{\theta}} \ast H_{\text{plug-in}}^{a \circ \hat{\theta}}) \right|.$$ 

(3.4)

Finally, by replacing initial risk estimate $R(F_{\hat{\theta}})$ by its bootstrapped counterpart $R(F_{a^* \circ \hat{\theta}})$, we reduce the bias of the estimator. The corresponding algorithm is summarized in detail in Figure 3.

Note that in (3.4) we considered the absolute value of bias to simplify the evaluation process. Alternatively, one could consider the positive bias penalty function (3.3) and introduce supplementary optimality criterion that minimises the expected value of estimated risk. Note that such framework could be also desirable from the economic perspective since in practical setting we want to be secured with minimal capital.

To illustrate how Algorithm 1 works let us calculate the value of $a^*$ for two reference VaR cases that were already considered before, i.e. for plug-in normal and plug-in GPD estimators. Inspired by the Gaussian case, (2.7), we apply the shift only to the scale parameters. In the GPD case we consider an exogenous threshold $u$ which we set to $u = 0$. Also, we fix the rolling window length to $n = 50$, use i.i.d. samples, and consider VaR at level 5%. In the normal case, we compute $a^*$ for the adjusted plug-in estimator

$$\hat{\mu} + (a^* \cdot \hat{\sigma}) \Phi^{-1}(0.01),$$

for various choices of true $\mu \in \mathbb{R}$ and $\sigma > 0$, while for the GPD case we consider adjusted plug-in estimator

$$\frac{(a^* \cdot \hat{\beta})}{\xi} \left( \alpha - \hat{\xi} - 1 \right),$$

for various exemplary choices of underlying true parameters $\xi \in \mathbb{R} \setminus \{0\}$ and $\beta > 0$. We set $B = 20,000$ for bootstrap sampling, compare the algorithm detailed in Figure 3.
The results on an exemplary representative parameter grid are presented in Figure 4. Two main conclusions could be made. First, the local bias adjustment for the normal case does not depend on the underlying parameters. This was in fact already shown in Pitera and Schmidt (2018), where the closed form formula for the global unbiased estimator has been presented, see also Section 2.2.1. Note that the value of bias adjustment the algorithm recovered is in perfect agreement with the theoretical value, i.e. it is very close to

$$\sqrt{\frac{n+1}{n}t_{n-1}^{-1}(p)/\Phi^{-1}(p)} \approx 1.029,$$

which is essentially the ratio between above the mean values of (2.6) and (2.7). Second, in the GPD case, it could be observed that the bias value adjustment looks like a (monotonic) function of $\xi$ parameter. This is aligned with the intuition as the $\beta$ parameter is used for (affine) scaling which should not impact (relative) bias adjustment (as it does not in the Normal case).

4. Performance of the bias reduction

In this section we present two examples of the local bias minimization when the underlying returns are distributed as GPD. In the first case we consider value-at-risk, while in the second we focus on expected shortfall.

To assess the performance of the bias reduction, we simulate samples from the GPD families with different parameter settings and compare the performance of the plug-in estimators with its locally unbiased equivalent constructed via Algorithm 1. For completeness, in both cases we also show the results for true risk and empirical estimators.

4.1. Performance metrics. To measure the performance efficiently, we introduce three supplementary performance metrics. First, we are interested in the average of the estimated regulatory capital. When other criteria are satisfied, this allows to chose the estimator which requires minimal regulatory capital. As in the previous section, we assume that we have $m$ estimates $\hat{\rho}_1, \ldots, \hat{\rho}_m$ at hand. We define the mean risk value (MRV) by

$$MRV := \frac{1}{m} \sum_{t=1}^{m} \hat{\rho}_t. \quad (4.1)$$
Small values of MRV indicate that (on average) smaller capital reserves are required. Ideally, this metric should be close to the true risk. Since parameters need to be estimated we expect that this model uncertainty is reflected by an increase in MRV. MRV can also be seen as an approximation of the standard (statistical) estimator bias.

The second metric we introduced is linked to the standard deviation from the mean risk values. We introduce its standard deviation (SD)

$$SD := \sqrt{\frac{1}{m} \sum_{t=1}^{m} (\hat{\mu}_t - \text{MRV})^2}.$$  

The third metric we are interested in is the already introduced performance measure that is dual to the underlying risk family. For VaR, this corresponds to the exception rate statistic, i.e. to the test statistic $G$ defined in (2.5),

$$T := \frac{1}{m} \sum_{t=1}^{m} 1_{\{y_t < 0\}}, \quad \text{and} \quad G := \frac{1}{m} \sum_{t=1}^{m} 1_{\{y_t + \cdots + y_{t+n} \leq 0\}}.$$  

For the VaR case, we include two additional metrics. The first one is based on the rolling window traffic-light backtest: fix the length of an additional rolling window, i.e. $N = 50$. We count the average number of overshoots in the interval $[s, s + N - 1]$ by $T_1(s) := \sum_{t=s}^{s+N-1} 1_{\{y_t < 0\}}$. The average Non-Green Zone (NGZ) classifier is defined as

$$\text{NGZ} := \frac{1}{m - N} \sum_{s=1}^{m-N} 1_{\{T_1(s) \geq z_\alpha\}}$$

where $z_\alpha$ is the exception number threshold based on Bernoulli trial 95% quantile value for $n$ trials and probability of success equal to $\alpha \in (0, 1)$. Under the correct model assumption the sequence $(1_{\{y_t < 0\}})$ forms a Bernoulli process with probability of success equal to $\alpha$. Consequently, under correct model specification $T_1(t_0) \approx T_{N_0}$ should be linked to a Bernoulli trial of length $n$. For example, for $\alpha = 0.05$ and $n = 50$, we get $F_{B(50,0.05)}(4) \approx 0.89$ and $F_{B(50,0.05)}(5) \approx 0.96$, which would lead to threshold $z_\alpha = 5$. In a perfect setting, NGZ should be close to $1 - F_{B(n,p)}(z_\alpha - 1)$.

Finally, the second VaR additional metric is based on the Diebold-Mariano comparative test statistic with unbiased estimator being the reference one. For simplicity, to avoid the discussion about joint-elicitability and (statistical) relevance of the output metrics (see e.g. Fissler et al. (2015)) we decided to introduce this metric only for VaR, where the framework is much simpler. The test statistic is given by

$$DM := \sqrt{n} \cdot \frac{\hat{\mu}(d)}{\hat{\sigma}(d)},$$

where $d = (d_t)_{t=1}^n$ is the time series of comparative errors between two estimators, $\hat{\mu}(d)$ is the sample mean comparative error, and $\hat{\sigma}(d)$ is the sample standard deviation of the comparative error, respectively; see Osband and Reichelstein (1985) for details. In the VaR case, the $t$-th day comparative error is given by $d_t := S_1(-V\hat{\mu}^R_{t}^{z_1}, x_t) - S_1(-V\hat{\mu}^R_{t}^{z_2}, x_t)$, where $z_1$ and $z_2$ denote certain estimators (such as plug-in estimator, empirical estimator, or true risk) $S_1$ is a strictly consistent VaR scoring function given by $S_1(r, x) = (\mathbb{1}_{\{r \geq x\}} - \alpha)(r - x)$, and $x_t$ is the $t$-th day realized sample value.

4.2. Bias reduction for value-at-risk in the GPD framework. We consider three different sets of GPD parameters $(u, \xi, \beta)$ and estimate VaR using three different (conditional) reference $\alpha$
levels. The parameter sets we picked are
\[
\begin{align*}
\theta_0 &= (0.978, 0.212, 0.869), \\
\theta_1 &= (2.2, 0.388, 0.545), \\
\theta_2 &= (0.40028, 1.19, 0.774),
\end{align*}
\]
while the \( \alpha \) thresholds are equal to 5\%, 7.5\%, and 10\%, respectively. In first two
cases the learning period is equal to \( n = 50 \), while in the last case it is equal to \( n = 42 \).
The first setting corresponds to the \( t \)-student distribution with 3 degrees of freedom and 20\% percentage
threshold; the second setting corresponds to market data calibrated parameters from S&P500 index – see Table 5 in Gilli
and Kellezi (2006); the third setting comes from Table 5 in Moscadelli (2004) and is related to an
operational risk framework. Note that in the third set we have \( \xi > 1 \) which implies that the first
moment is infinite. For simplicity, in all cases, we fix \( u \), estimate \( \beta \) and \( \xi \), and shift only \( \beta \) when
Algorithm 1 is used. For computational purposes we fixed bootstrap sample size to \( B = 50,000 \), see Step (2) of Algorithm 1 for details.

We consider six different VaR estimators: the empirical estimator, the plug-in GPD estimator, true
risk, and the following three estimators:

First, for reference, we adjust the GPD plug-in estimator \( \hat{V}_\alpha \) by a fixed multiplier \( a \). We
compute \( a \) using the true values of \( \xi \) and \( \beta \) – which is not available in practice. But this estimator
will serve as a benchmark for the other estimators. We set
\[
\hat{V}_\alpha^{\text{true}} := u + a \cdot (\hat{V}_\alpha^{\text{plug-in}} - u),
\]
where \( a \) is computed using true values of \( \xi \) and \( \beta \) in the first step of Algorithm 1; note that adjusting
\( \hat{\beta} \) by \( a \) results in (4.3). Here, we slightly modified the objective function in step (4) to allow a small
positive bias and reduce the size of \( a \). Namely, in all cases we allowed bias to be equal to 10\% of
the estimator standard error. For the first two datasets, this corresponds to approximately 1\% of
the true risk value, while for the third dataset this corresponds to 10\% of the true risk value. Note
that in the third case we have \( \xi > 1 \) which increase the standard error. The correction was done
to slightly bring down capital without (considerably) impacting exception thresholds.

In addition we consider the following two unbiased estimators. While the first one results directly
in applying the ideas from the previous section on bootstrapping, we propose with the second
estimator an improved estimator which adapts better to the tail behavior of the GPD.

The first unbiased estimator, called \( \text{bootstrapping estimator} \), is computed using the bootstrapping
algorithm detailed in Figure 3 applied to each sample separately:
\[
\hat{V}_\alpha^{\text{boot}} := u + \hat{a}_t \cdot (\hat{V}_\alpha^{\text{plug-in}} - u),
\]
where \( \hat{a}_t \) is the local bias correction for \( t \)-th day estimates \( \hat{\xi}_t \) and \( \hat{\beta}_t \).

The second unbiased estimator, called \( \text{splitting estimator} \), splits between the bootstrapping
estimator and the plug-in estimator. In this estimator, we do not to apply the bias correction if
the estimated risk was breaching the 10\% upper quantile of the plug-in risk estimator to avoid high
sensitivity to outliers. More precisely, the splitting estimator is defined as
\[
\hat{V}_\alpha^{\text{splitting}} := \begin{cases} 
\hat{V}_\alpha^{\text{boot}} & \text{if } V_\alpha^{\text{plug-in}} \leq z \\
\hat{V}_\alpha^{\text{plug-in}} & \text{if } V_\alpha^{\text{plug-in}} > z,
\end{cases}
\]
where \( z \) is equal to the 10\%-upper quantile of the aggregated estimated capital sample (\( V_\alpha^{\text{plug-in}} \)).
One could see \( V_\alpha^{\text{splitting}} \) as a more realistic version of \( V_\alpha^{\text{boot}} \) in which the risk manager decides
not to apply the risk bias correction if he thinks that the estimated capital is severely overestimated in the first place.

As in Section 2.2.1, we follow a backtesting rolling window approach with total number of simulations equal to $N = n + m = 100,000$. The results of the simulation are presented in Table 1. It can be seen, that only for the unbiased estimators (and of course for the true VaR and the estimator using additional information, true2) the exception rate $(T)$ and non-green zone $(NGZ)$ frequency is close to the expected/desired values. Indeed, both empirical and GPD plug-in estimators have higher (than expected) exception rate and enter yellow (or red) zone more frequently. Summarizing, the unbiased estimators $\hat{V}_{\alpha}^{\text{boot}}$ and $\hat{V}_{\alpha}^{\text{splitting}}$ outperform the empirical and plug-in estimators in all respects. If we had to decide between those two measures, the Diebold-Mariano tests suggest that the splitting estimator has a better performance.

### 4.3. Bias reduction for expected shortfall in the GPD framework.

In this setting we repeat the previous simulation example in the context of expected shortfall. First, we consider the first two parameter sets from Equation (4.2) since, for $\theta_2$, we have $\xi > 1$ such that expected shortfall is no longer finite. The results are presented in Table 2.

Two generic remarks should be made. First, the bias adjustment for expected shortfall is large in comparison to the adjustment in the value-at-risk case, computed for the same confidence threshold

---

| Parameters | VaR | $\alpha$ | n | T | NGZ | DM | MRV (SD) |
|------------|-----|----------|---|---|-----|----|---------|
| GPD $u = 0.978$ | emp | plug-in | 0.050 | 50 | 0.066 | 0.20 | -11.8 | 4.47 (0.92) |
| $\xi = 0.212$ | true | true2 | 0.051 | 18.7 | 4.61 (0.00) |
| $\beta = 0.869$ | boot | splitting | 0.051 | -24.5 | 4.89 (0.92) |
| GPD $u = 2.200$ | emp | plug-in | 0.075 | 50 | 0.091 | 0.13 | -10.7 | 4.59 (0.71) |
| $\xi = 0.388$ | true | true2 | 0.077 | 18.2 | 4.63 (0.00) |
| $\beta = 0.545$ | boot | splitting | 0.077 | -21.7 | 4.81 (0.68) |
| GPD $u = 0.400$ | emp | plug-in | 0.100 | 42 | 0.119 | 0.08 | -13.3 | 10.57 (7.16) |
| $\xi = 1.190$ | true | true2 | 0.101 | 12.0 | 9.82 (4.51) |
| $\beta = 0.774$ | boot | splitting | 0.111 | -24.1 | 10.80 (6.09) |

Table 1. The table presents numerical results the estimation of value-at-risk in three cases of a GPD distribution. It reports the empirical (emp), the plug-in, the estimator knowing $\xi$ and $\beta$ (true), the bootstrapping estimator using $\xi$ and $\beta$ for computing $\alpha$ (true2), and the two locally unbiased estimators: the bootstrapping estimator (boot) and the splitting estimator (splitting). The reported values are the cumulative exception rate $T$, the non-green zone (NGZ), the Diebold-Mariano test statistic with true2 as competitive approach, and the mean risk value MRV (with standard deviation in brackets).
The table presents numerical results for the estimation of expected shortfall in two cases of a GPD distribution. It reports the empirical (emp), the plug-in, the estimator knowing $\xi$ and $\beta$ (true), the bootstrapping estimator using $\xi$ and $\beta$ for computing $\alpha$ (true2), and the two locally unbiased estimators: the bootstrapping estimator (boot) and the splitting estimator (splitting). The reported values are the cumulative aggregated exception rate statistic $G$, the non-green zone (NGZ), and the mean risk value MRV (with standard deviation in brackets).

$\alpha \in (0, 1)$. Something we expected since expected shortfall takes into account the whole tail of the distribution. In particular, the closer $\xi$ is to 1, the bigger the bias, and the multiplier is close to 70% for $\xi \approx 0.8$.

We illustrate this, with a comparison plot with VaR and ES bias adjustments computed for all estimated parameter values for the second dataset, see Figure 5. Please note that the simulations are in perfect agreement with results in Figure 4, i.e. $\xi$ is the main bias determination driver. Second, as expected, the performance (measured in terms of $G$) for empirical and plug-in estimators is significantly worse in comparison to the unbiased estimators which tend to underestimate the risk. This is aligned with the results presented for the VaR case.

$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Parameters} & \text{ES} & \alpha & n & G & \text{MRV (SD)} \\
\hline
\text{GPD} & \text{emp} & 0.100 & 6.14 (1.83) \\
& \text{plug-in} & 0.078 & 6.73 (2.01) \\
& \text{true} & 0.051 & 6.70 (0.00) \\
& \text{true2} & 0.050 & 7.88 (2.41) \\
& \text{boot} & 0.056 & 8.06 (3.17) \\
& \text{splitting} & 0.057 & 7.66 (2.22) \\
\hline
\text{GPD} & \text{emp} & 0.136 & 6.80 (2.40) \\
& \text{plug-in} & 0.121 & 7.08 (2.44) \\
& \text{true} & 0.077 & 7.07 (0.00) \\
& \text{true2} & 0.079 & 8.35 (3.08) \\
& \text{boot} & 0.091 & 8.59 (4.21) \\
& \text{splitting} & 0.092 & 8.03 (2.65) \\
\hline
\end{array}$

Figure 5. The heatplots present local bias adjustment shift for GPD plug-in estimator for VaR (left) and ES (right), both at level 7.5%. The underlying sample size in both cases is equal to $n = 50$. The values are computed for all parameter pairs that were estimated for 100 000 simulated data based on Dataset 2 specification.

$^{3}$In Regulatory framework VaR at 1% was replaced by ES at 2.5% which should essentially result in the same level of bias, at least in the Gaussian setting.
5. Concluding Remarks

This article showed how to compute unbiased estimators for value-at-risk and expected shortfall in the case of Gaussian and GPD underlying distribution. We proposed a new estimator for value-at-risk and expected shortfall, which we call splitting estimator, and showed it is outperforming the existing plug-in estimators in the considered simulations.

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Institute of Mathematics, Jagiellonian University, Lojasiewicza 6, 30-348 Cracow, Poland
Email address: marcin.pitera@im.uj.edu.pl

Dep. of Mathematical Stochastics, University of Freiburg, Eckerstr.1, 79104 Freiburg, Germany
Email address: thorsten.schmidt@stochastik.uni-freiburg.de