Bispectral operators of prime order

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Abstract

The aim of this paper is to solve the bispectral problem for bispectral operators whose order is a prime number. More precisely we give a complete list of such bispectral operators. We use systematically the operator approach and in particular - Dixmier ideas on the first Weyl algebra. When the order is 2 the main theorem is exactly the result of Duistermaat-Gr"unbaum . On the other hand our proofs seem to be simpler.

0 Introduction

Bispectral operators have been introduced by F.A.Gr"unbaum (cf. [G1, G2]) in his studies on applications of spectral analysis to medical imaging.

In the present paper we give complete classification of bispectral operators of prime order. We start with some definitions and results that are needed to state our results, as well as to make clear the connection with other research.

An ordinary differential operator \( L(x, \partial_x) \) is called bispectral if it has an eigenfunction \( \psi(x, z) \), depending also on the spectral parameter \( z \), which is at the same time an eigenfunction of another differential operator \( \Lambda(z, \partial_z) \) now in the spectral parameter \( z \). In other words we look for operators \( L, \Lambda \) and a function \( \psi(x, z) \) satisfying equations of the form:

\[
L \psi = f(z) \psi, \quad (0.1)
\]

\[
\Lambda \psi = \theta(x) \psi. \quad (0.2)
\]

Although, as mentioned above, the study of bispectral operators has been stimulated by certain problems of computer tomography, later it turned out that they are connected to several actively developing areas of mathematics and physics - the KP-hierarchy, infinite-dimensional Lie algebras and their representations, particle systems, automorphisms of algebras of differential operators, non-commutative geometry, etc. (see e.g. [BHY1, BHY2, BHY3, BW, BW1, BW2, DG, K, W1, W2, MZ], as well as the papers in the proceedings volume of the conference in Montréal [BP]).

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In the fundamental paper [DG], Duistermaat and Grünbaum raised the problem to find all bispectral operators and completely solved it for operators \( L \) of order two. The complete list is as follows. If we present \( L \) as a Schrödinger operator
\[
L = \left( \frac{d}{dx} \right)^2 + u(x),
\]
the potentials \( u(x) \) of bispectral operators, apart from the obvious Airy \((u(x) = ax)\) and Bessel \((u(x) = cx^{-2})\) ones, are organized into two families of potentials \( u(x) \), which can be obtained by finitely many "rational Darboux transformations"

1. from \( u(x) = 0 \),
2. from \( u(x) = -(\frac{1}{4})x^{-2} \).

Thus the classification scheme prompted by the paper [DG] is by the order of the operators. G. Wilson [W1] introduced another classification scheme - by the rank of the bispectral operator \( L \) (see the next section for definitions). In the above cited paper [W1] (see also [W2]) Wilson gave a complete description of all bispectral operators of rank 1 (and any order). In the terminology of Darboux transformations (see [BHY1]) all bispectral operators of rank 1 are those obtained by rational Darboux transformations on the operators with constant coefficients, i.e. \( L = p(\partial_x) \in \mathbb{C} \). In the above mentioned papers [DG, W1] the classification is split into two, more or less independent parts. First, there is an explicit construction of families of bispectral operators of a given class (order 2 in [DG]; rank 1 in [W1]). The construction can be given in terms of Darboux transformations of "canonical" operators (a notion that needs clarification, see the last section for some comments). A second part should be to give a proof that, if an operator (in the corresponding class) is a bispectral one, then it belongs to the constructed families.

In the last few years there has been increased activity [BHY1, BHY3, KR, Z] in the direction of constructing classes of bispectral operators ([BHY1, BHY3, KR, Z]). For a survey on this subject, see [BP, H1] and the references therein. To the best of our knowledge, all known up to now families of bispectral operators can be constructed by the methods of [BHY1, BHY3]. For a simplified exposition of these results, see the first part of [H1]. A challenging problem is to prove that all the bispectral operators have already been found. A natural approach would be to divide the differential operators into suitable classes, e.g. - by order as in [DG] or by rank and to try to isolate the bispectral ones amongst them. In [HM] we have proposed another classification scheme - that is to consider the operators with a fixed type of singularity at infinity. The main result of that paper is the classification of bispectral operators possessing the simplest type of singularity at infinity - the Fuchsian one. My opinion is that all mentioned above classification schemes may help each other as seen from the main results here.

In the present paper we return to the initial classification scheme - that of [DG]. We give a list of several families that contains all bispectral operators whose order is a prime number. Before stating the results we introduce some definitions and notations which will be used also throughout the paper. We are going to consider operators, normalized as follows:
It is well known that with the above normalization all the coefficients of $L$ are rational functions (see [DG, W1] or the next section).

Now we can formulate the main result of the present paper.

**Theorem 0.1** An operator $L$, whose order is a prime number, is bispectral if and only if it belongs to one of the following sets:

1. Generalized Airy operators:
   \[ A = \partial^p + \sum_{j=1}^{p-2} a_j \partial^j - x, \quad a_j \in \mathbb{C}; \]  \[ (0.4) \]

2. Generalized Bessel operators:
   \[ B = x^{-p} (x \partial - \beta_1) \ldots (x \partial - \beta_p), \quad \beta_j \in \mathbb{C}; \]  \[ (0.5) \]

3. Operators with constant coefficients:
   \[ C = \partial^p + \sum_{j=1}^{p-2} a_j \partial^j, \quad a_j \in \mathbb{C}; \]  \[ (0.6) \]

4. Operators, obtained by monomial Darboux transformations from the Bessel operators having the property that at least one difference $\beta_i - \beta_j \in p\mathbb{Z}, i \neq j$;

5. Polynomial Darboux transformations from operators with constant coefficients.

**Remark 0.2**

1. For $p = 2$ this is just the content of the classical result of Duistermaat-Grünbaum [DG]. Indeed, in that case we have $\beta_1 - \beta_2 \in 2\mathbb{Z}$. In [DG] this part of the theorem is formulated in a form close to this one. To obtain the potential $u(x) = -\frac{1}{4}x^{-2}$ mentioned above in 2) and corresponding to $\beta_1 = \beta_2 = 1/2$ we have to perform monomial Darboux transformation. Having in mind that a composition of monomial Darboux transformations is again monomial Darboux transformation we get 2). The potential $u(x) = 0$ from 1) corresponds to the only operator in 3., Theorem 1.3 for $p = 2$.

2. The different notions of Darboux transformations used here are explained in the next section.

Theorem 0.1 is in fact a consequence of two slightly more general results. Their importance lies in the fact that they can be used for an induction process in further classification (see [H2]). In any case the proofs do not simplify when restricted to operators of prime order and seem to be more natural as performed here (see below).

At the end we briefly review the organization of the paper. In Section 1. we recall some definitions and results, by now standard for the problem, with the purpose to fix the notation and the terminology. Section 2. treats the case of operators with
bounded (at infinity) coefficients $V_j$. We begin the section with an auxiliary result. We show that from the normalized operator $L$ as in [1.3] and satisfying only the so-called ”ad-condition” with the polynomial $\theta$, the function $f(z)$ can be chosen naturally and then one can build the operator $\Lambda$ already normalized only in terms of $L$ and $\theta$. This will be needed after that to prove the following theorem for operators with bounded coefficients:

**Theorem 0.3** Let the rank of the bispectral operator $L$ with bounded coefficients equals its order. Then it is a monomial Darboux transformation of a Bessel operator.

The essential part of the proof is to establish the vanishing of the coefficients $V_j$ at infinity. Then the result is contained in [HM].

In section 3. we consider operators with increasing coefficients. We first obtain a normal form for for their ”leading terms” (subsection 3.1). Here we exploit once again (as in [HM]) crucial ideas of Dixmier analysis of the first Weyl algebra [Dx]. When the order is a prime number we get the non-vanishing (at infinity) part of $L$ to be (generalized) Airy operator $A$. In the next subsection we develop some version of wave (pseudo-differential) operators, expanded in negative powers of Airy operators. This tool turns to be enough to prove (for operators of any order, not only prime) in subsection 3.3 the theorem:

**Theorem 0.4** Let the bispectral operator $L = A \pm \text{(vanishing at infinity perturbation)}$. Then the perturbation is zero.

We hope the careful reader has noticed some similarity with the analysis in [DG] for the latter class. We believe that our proof is simpler and more transparent (and for this reason works for higher order operators). One thing that we certainly benefited from [DG] is to realize that behind their calculations of the normal form of second order operators there stands Dixmier analysis on the first Weyl algebra $A_1$. On the other hand in both steps we use different approaches.

The results of the present papers have been announced in the second part of the survey paper [H1].

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1 Preliminaries

In this section we have collected some terminology, notations and results relevant for the study of bispectral operators. Our main concern is to introduce unique notation which will be used throughout the paper and to make the paper self contained.
There are also few results which cannot be found formally elsewhere, but in fact are reformulations (in a form suitable for the present paper) of statements from other sources.

1.1

In this subsection we recall some definitions, facts and notation from Sato’s theory of KP-hierarchy \[S, DJKM, SW\] needed in the paper. For a complete presentation of the theory we recommend also \[Di, vM\]. We start with the notion of the wave operator \(K(x, \partial_x)\). This is a pseudo-differential operator

\[K(x, \partial_x) = 1 + \sum_{j=1}^{\infty} a_j(x) \partial_x^{-j},\]  

(1.1)

with coefficients \(a_j(x)\) which could be convergent or formal power (Laurent) series. In the present paper we will consider \(a_j\) most often as formal Laurent series in \(x^{-1}\). The wave operator defines the (stationary) Baker-Akhiezer function \(\psi(x, z)\):

\[\psi(x, z) = K(x, \partial_x) e^{xz}.\]  

(1.2)

From (1.1) and (1.2) it follows that \(\psi\) has the following asymptotic expansion:

\[\psi(x, z) = e^{xz} (1 + \sum_{j=1}^{\infty} a_j(x) z^{-j}), \quad z \to \infty.\]  

(1.3)

Introduce also the pseudo-differential operator \(P\):

\[P(x, \partial_x) = K \partial_x K^{-1}.\]  

(1.4)

The following spectral property of \(P\), crucial in the theory of KP-hierarchy, is also very important for the bispectral problem:

\[P \psi(x, z) = z \psi(x, z).\]  

(1.5)

When it happens that some polynomial of \(P\), say \(f(P)\), is a differential operator, we get that \(\psi(x, z)\) is an eigenfunction of an ordinary differential operator \(L = f(P)\):

\[L \psi = f(z) \psi.\]  

(1.6)

It is possible to introduce the above objects in many different ways, starting with any of them (and with other, not introduced above). For us it would be important also to start with given differential operator \(L\):

\[L(x, \partial_x) = \partial_x^N + V_{N-2}(x) \partial_x^{N-2} + \ldots + V_0(x).\]  

(1.7)

One can define the wave operator \(K\) as:

\[LK = K f(\partial).\]  

(1.8)
An important notion, connected to an operator $L$ is the algebra $A_L$ of operators commuting with $L$ (see [Kr, BC]). This algebra is commutative one. The wave function $\psi(x, z)$ (defined in (1.2)) is a common wave function for all operators $M$ from $A_L$:

$$M \psi(x, z) = g_M(z) \psi(x, z).$$

We define also the algebra $A_L$ of all functions $g_M(z)$ for which (1.9) holds for some $M \in A_L$. Obviously the algebras $A_L$ and $A_L$ are isomorphic. Following [Kr] we introduce the rank of the algebra $A_L$ as the greatest common divisor of the orders of the operators in $A_L$.

1.2

Here we shall briefly recall the definition of Bessel wave function. Let $\beta \in \mathbb{C}^N$ be such that

$$\sum_{i=1}^{N} \beta_i = \frac{N(N-1)}{2}.$$  \hspace{1cm} (1.10)

**Definition 1.1 [F, Z, BHY1]** Bessel wave function is called the unique wave function $\Psi_\beta(x, z)$ depending only on $xz$ and satisfying

$$L_\beta(x, \partial_x)\Psi_\beta(x, z) = z^N \Psi_\beta(x, z),$$

where the Bessel operator $L_\beta(x, \partial_x)$ is given by (1.3). Because the Bessel wave function depends only on $xz$, (1.11) implies

$$D_x \Psi_\beta(x, z) = D_z \Psi_\beta(x, z),$$

$$L_\beta(z, \partial_z)\Psi_\beta(x, z) = x^N \Psi_\beta(x, z).$$

Next we define monomial and polynomial Darboux transformations of Bessel operators. The definitions are slight modification of the definitions given in [BHY1]. Let $h(L_\beta)$ be a polynomial in a Bessel operator.

**Definition 1.2** We say that the operator $\tilde{L}$ is polynomial Darboux transformation of $L_\beta$ if there exist differential operators $P(x, \partial_x)$, $Q(x, \partial_x)$ and a polynomial $h$ such that

$$h(L_\beta) = QP$$

$$\tilde{L} = PQ$$

and the operator $P(x, \partial_x)$ has the form

$$P(x, \partial_x) = x^{-n} \sum_{k=0}^{n} p_k(x^N)D_x^k,$$

where $p_k$ are rational functions, $p_n \equiv 1$. 

6
We will use the following definition of monomial Darboux transformations.

**Definition 1.3** We say that the operator $\tilde{L}$ is a *monomial Darboux transformation* of the Bessel operator $L_\beta$ iff it is a polynomial Darboux transformation with $h(L_\beta) = L_\beta^d$, $d \in \mathbb{N}$.

**Remark 1.4** In [DG] the authors work with *rational Darboux transformations*. It is easy to show that a composition of rational Darboux transformations is a monomial Darboux transformation.

We end this subsection by reformulating (in a weaker form) the main results, which we need from [BHY1, HM].

**Theorem 1.5** The polynomial Darboux transformations of the Bessel operators are bispectral operators.

**Theorem 1.6** If a bispectral operator $L$ has vanishing coefficients at infinity, it is a monomial Darboux transformation of a Bessel operator.

1.3

Here we recall several simple properties of bispectral operators following [DG], [W1]. As we have already mentioned in the introduction we are going to study ordinary differential operators $L$ of arbitrary order $N$ which are normalized as in (0.3), i.e. with $V_N = 1$ and $V_{N-1} = 0$. Assuming that $L$ is bispectral means that we have also another operator $\Lambda$, a wave function $\psi(x,z)$ and two other functions $f(z)$ and $\theta(x)$, such that the equations (0.1) and (0.2) hold. The following lemma, due to [DG], has been fundamental for all studies of bispectral operators.

**Lemma 1.7** There exists a number $m \in \mathbb{N}$, such that

$$ (\text{ad } L)^{m+1} \theta = 0. $$

For its simple proof, see [DG], [W1]. To the best of my knowledge this is the only property of bispectral operators that is used in their studies. It is widely believed that the condition (1.17), called the ad-condition is equivalent to bispectrality (provided (0.3) holds). In what follows we assume only that the ad-conditon holds, i.e. we are not going to use the existence of $f(z)$, $\Lambda$ and $\psi$. For us it would be important to construct them only from $L$ and $\theta$ at least formally. This will be done in the next section.

We will consider that $m$ is the minimal number with this property. An important corollary of the above lemma is the following result.

**Lemma 1.8** Let the operator $L$ be normalized as in (0.3). Then

(i) The function $\theta(x)$ is a polynomial.

(ii) The coefficients $\alpha_j$ in the expansion (1.1) of the wave operator $K$ are rational functions.
Proof. We repeat the simple proof following [W1] as the proof introduces some notions needed later. From the equation (1.17) it follows that

\[(\text{ad } \partial_x^N)^{m+1}(K^{-1}\theta K) = 0.\]

On the other hand the kernel of the operator \((\text{ad } \partial_x^N)^{m+1}\) consists of all pseudo-differential operators whose coefficients are polynomials in \(x\) of degree at most \(m\). This gives that

\[\theta(x)K = K\Theta,\]

with a pseudo-differential operator \(\Theta:\)

\[\Theta = \Theta_0 + \sum_{1}^{\infty} \Theta_j \partial_x^{-j}\]

whose coefficients \(\Theta_j\) are polynomials of degree at most \(m\). We have \(\theta(x) = \Theta_0(x)\). This gives (i). Comparing the coefficients at \(\partial_x^{-j}\) we find that all the coefficients \(\alpha_j(x)\) of \(K\) are rational functions. ✷

Remark 1.9 We notice that at least one of the coefficients \(\Theta_j\) has degree exactly \(m\), where \(m\) from Lemma 1.7 is minimal. This fact will be used later.

The last lemma has as an obvious consequence one of the few general results, important in all studies of bispectral operators. Noticing that the coefficients of \(L\) are polynomials in the derivatives of \(\alpha_j(x)\) we get

**Lemma 1.10** The coefficients of \(L\) are rational functions.

Remark 1.11 Obviously the same results hold for the pair \(\Lambda, f(z)\), when imposing the conditions (0.3) on \(\Lambda\). But here we need to derive this statement from conditions only on \(L\) and a suitable choice of \(f(z)\). This will be done in the next section.

2 Operators with bounded coefficients

In this section we are going to prove our main result for operators with coefficients bounded near infinity. As mentioned earlier we will consider slightly more general situation - operators for which the rank and the order coincide. The main part of the proof is to show that in this case the coefficients are in fact vanishing at infinity and hence the result follows from the main theorem in [HM].

First I would like to fix the notation. Put

\[L = \partial^p + \sum_{j=0}^{p-2} W_j(x) \partial^j,\]

where \(W_j = c_j + V_j(x)\), \(c_j\) are constants and \(V_j(x) = \mathcal{O}(x^{-1})\). Next define the polynomial \(f(z)\) to be:
\[ f(z) = z^p + \sum_{j=0}^{p-2} c_j z^j. \]  

(2.2)

In this way we can rewrite our operator in the form

\[ L = f(\partial) + \sum_{j=0}^{p-2} V_j(x) \partial^j, \]  

(2.3)

where \( V_j(x) = O(x^{-1}) \).

Our first goal is to show that starting with the normalized bispectral operator \( L \) the above choice of \( f(z) \) leads to a normalized operator \( \Lambda \). The construction is formal, i.e. the wave function is a formal series, but this is enough for what follows. As all the auxiliary results (the three lemmas below) are slight modification of corresponding results from [HM] we omit their proofs. It is well known that one can present the operator \( L \) in the form:

\[ L = K f(\partial) K^{-1} \]  

(2.4)

where the polynomial \( f(z) \) is defined in (2.2). In the next lemma, following [DG] we find the simplest restrictions on the coefficients of the wave operator \( K \) and on \( L \).

**Lemma 2.1**  
(i) The coefficients \( V_j(x), j = N-2,...0 \) of \( L \) vanish at \( \infty \) at least as \( x^{-2} \).

(ii) The coefficients \( \alpha_j, j = 1,\ldots \) of the wave operator \( K \) vanish at least as \( x^{-1} \).

This lemma allows us to introduce following [BHY4] an anti-isomorphism \( b \) between the algebra \( \mathcal{B} \) of pseudo-differential operators \( P(x, \partial_x) \) in the variable \( x \) and the same algebra \( \mathcal{B}' \) but in the variable \( z \). More precisely \( \mathcal{B} \) consists of those pseudo-differential operators

\[ P = \sum_{k}^{\infty} p_j(x^{-1}) \partial_x^{-j}, \]

for which there is a number \( n \in \mathbb{Z} \) (depending on \( P \)) such that all expressions \( x^n p_j(x^{-1}), j = k, k+1,\ldots \) are formal power series in \( x^{-1} \). The involution

\[ b : \mathcal{B} \rightarrow \mathcal{B}' \]

is defined by

\[ b(P)e^{xz} = Pe^{xz} = \sum_{k}^{\infty} z^{-j} p_j(\partial_z^{-1}) e^{xz}, \quad \text{for} \quad P \in \mathcal{B} \]  

(2.5)

i.e. \( b \) is just a continuation of the standard anti-isomorphism between two copies of the Weyl algebra. In what follows we will use also the anti-isomorphism

\[ b_1 : \mathcal{B} \rightarrow \mathcal{B}', \quad b_1(P) = b(\text{Ad}_K P). \]  

(2.6)
Obviously $b$ and $b_1$ can be considered as involutions of $\mathcal{B}$ and without any ambiguity we can denote the inverse isomorphisms $b^{-1}, b_1^{-1}: \mathcal{B}' \to \mathcal{B}$ by the same letters. Since the operators $K$ and $\Theta = K^{-1}\theta K$ are from $\mathcal{B}$ we can define two operators $S$ and $\Lambda$ as follows:

\begin{align*}
S(z, \partial_z) &= b(K(x, \partial_x)), \\
\Lambda(z, \partial_z) &= b(\Theta).
\end{align*}

(2.7) (2.8)

Explicitly one has

\[ S = \sum_{j=0}^{\infty} z^{-j} a_j(\partial_z) = \sum_{j=0}^{\infty} a_j(z)\partial_z^{-j}, \quad a_0 = 1 \]

(2.9) and also

\[ \Lambda(z, \partial_z) = \sum_{j=0}^{\infty} z^{-j} \Theta_j(\partial_z) = \sum_{i=0}^{m} \Lambda_i(z)\partial_z^i, \]

(2.10)

where $\Lambda_m \neq 0$ (see Remark 1.9) and the coefficients $\Lambda_i$ and $a_j$ should be viewed as formal power series. As in [HM] we can prove

**Lemma 2.2** The coefficients $a_j$ of the operator $S$ are rational functions.

From the last lemma it follows that $\Lambda$ is normalized as required in (1.3). Denote temporarily by $r$ the degree of the polynomial $\theta$, i.e. if $\theta(x) = z^r + \ldots$.

**Lemma 2.3** With the choice of $f(z)$ as in (2.2) the coefficients $\Lambda_i$ of the operator $\Lambda$ are rational functions and $\Lambda$ satisfies (1.2). The degree of $\theta$ is $m$ and

\[ \Lambda_m = 1, \quad \Lambda_{m-1} = 0. \]

(2.11)

The point in the last lemma is that the normalization of $\Lambda$ is a consequence of the suitable choice of the polynomial $f(z)$. Of course the wave function is only formal but this suffices for the proof of the main theorem.

Now we are ready to give the classification of operators with bounded near infinity coefficients of operators with the same rank and order.

Let us fix the notation. Put

\[ L = \partial^N + \sum_{j=0}^{N-2} V_j(x)\partial^j, \]

(2.12)

where $V_j = c_j + W_j(x)$, $c_j$ are constants and $W(x) = \mathcal{O}(x^{-1})$. As before define the polynomial $f(z)$ to be:

\[ f(z) = z^N + \sum_{j=0}^{N-2} c_j z^j. \]

(2.13)

With this choice of $f(z)$ as we know (see ) the operator $\Lambda$ is with rational coefficients. Now we are ready to prove the main result of this section.
Theorem 2.4 If the rank of the operator $L$ with bounded at infinity coefficients equals its order $N$ then all constants $c_j = 0$, i.e. the coefficients $V_j = \mathcal{O}(x^{-1})$.

Proof. We are going to use again the ad-condition (1.3). Choose the number $m$ so that $\text{ad}^m_{f(z)}(\Lambda) \neq 0$ and $\text{ad}^{m+1}_{f(z)}(\Lambda) = 0$. Simple computation shows that

$$ad^m_{f(z)}(\Lambda) = (-1)^m m!(f'(z))^m$$  \hfill (2.14)

On the other hand we have

$$ad^m_{L}(\theta) = Q \neq 0$$  \hfill (2.15)

Obviously

$$[L, Q] = 0.$$  \hfill (2.16)

This gives that $Q \in A_L$. From this and from the fact that the rank of $L$ is equal to its order $N$ we get that $Q$ is a polynomial in $L$:

$$Q = q_r L^r + q_{r-1} L^{r-1} + \ldots, \quad q_j \in \mathbb{C},$$  \hfill (2.17)

where the coefficient $q_r \neq 0$. Using the involution $b_1$ we get

$$b_1(Q) = q_r f^r(z) + \sum_{j=0}^{r-1} q_j f^j(z)$$  \hfill (2.18)

Using (2.14) we have the following string of identities:

$$b_1(Q) = b_1(\text{ad}^m_{L}(\theta)) = (-1)^m (\text{ad}^m_{f(z)}(\Lambda)) = m!(f'(z))^m.$$  \hfill (2.19)

In this way (2.18) and (2.19) yield

$$q_r f^r(z) + \sum_{j=0}^{r-1} q_j f^j(z) = m!(f'(z))^m.$$  \hfill (2.20)

Now we are going to compare both the degrees and the first two coefficients of the two hand-sides of (2.20). First notice that comparing the degrees of the leading terms gives:

$$(N - 1)m = rN.$$  

This gives that $m$ is divisible by $N$, i.e. $m = sN$ and $r = s(N-1)$. Next, comparing the coefficients at the highest degree we get $q_r = m! N^m$. Suppose that some of the coefficients $c_j$ of $f(z)$ are not zero . Denote the second non-zero term after $z^r$ by $c_k z^k$. Computing the coefficient at the second non-zero terms in both sides (2.20) we get $k = N - 1$. This is a contradiction to the normalizing condition $c_{N-1} = 0$.  \hfill $\blacksquare$

From the above theorem and from the main result in [HM] (see also Section 1.2, Theorem 1.6) we get the proof of Theorem 0.3.
3 Operators with increasing coefficients

3.1 Normal forms

Let $L$ be a bispectral operator, normalized as in (0.3), i.e.

$$L = \partial_x^N + \sum_{j=0}^{N-2} V_j(x) \partial_x^j. \quad (3.1)$$

We will consider that for some $j$ the corresponding coefficient $V_j$ is increasing at infinity, i.e. has Laurent expansion at infinity of the form:

$$V_j(x) = r_j \sum_{m=-\infty}^\infty a_{j,m} x^m, \quad (3.2)$$

where $r_j > 0$ and $a_{j,r_j} \neq 0$. We call $r_j$ the order of $V_j$. In what follows we are going to use several properties shared by the first Weyl algebra $A_1$ and the larger algebra $\mathcal{R}[\partial]$ of differential operators with rational coefficients. Following [Dx] we define filtration in $\mathcal{R}[\partial]$. Let $\rho, \sigma \in \mathbb{R}$. We put $wt(x) = \rho$, $wt(\partial) = \sigma$. Next the weight of $L$ is given by the following definition:

**Definition 3.1** Assume that $L = V_n \partial^n + V_{n-1} \partial^{n-1} + \cdots + V_0$ is an arbitrary element of $\mathcal{R}[\partial]$. For each term $V(x) \partial_x^i$ define its weight

$$v_{\rho,\sigma}(V(x) \partial_x^i) = \rho(\text{ord}V) + \sigma i.$$

Then the number

$$v_{\rho,\sigma}(L) := \max_{0 \leq i \leq n} v_{\rho,\sigma}(V_i(x) \partial^i)$$

will be called $(\rho, \sigma)$-order of $L$.

The second definition associates to each differential operator from $\mathcal{R}[\partial]$ a $(\rho, \sigma)$-homogeneous polynomial.

**Definition 3.2** Assume the notation of the previous definition and denote by $I(L)$ the set \{ $i \in \{0, 1, \cdots, n\} | v_{\rho,\sigma}(V_i \partial^i) = v_{\rho,\sigma}(L)$\}. The polynomial $f \in \mathbb{C}[x, x^{-1}, y]$ defined as:

$$f(x, y) = \sum_{i \in I} a_i x^{\text{ord}V_i} y^i, \quad (3.3)$$

where $a_i \in \mathbb{C}$ are uniquely determined from the expansion

$$V_i = a_i x^{\text{ord}V_i} + (\text{lower order terms}),$$

will be called polynomial associated with $L$. The operator $L_0 =: f(x, \partial_x)$ will be called the homogeneous part of $L$. Here, as usual the columns :: denote normal ordering, i.e. the differentiation is pushed to the right.
Consider again the bispectral operator (3.1). In what follows we assume that $\rho$ and $\sigma$ are positive integers. It is always possible to choose them in such a way that the polynomial $f$ associated with $L$ has at least two terms of the kind

$$f = y^N + \alpha x^k y^p + \ldots, \quad \alpha \neq 0,$$

where $k > 0$, $p \geq 0$. For this purpose one can use the Newton polygon. More precisely denote by $E(L)$ the set of points $(m,j)$, such that $a_{m,j} \neq 0$, where $a_{m,j}$ is from (3.2). Consider the plane with the points of $E(L)$ and take the convex closure of $E(L)$ (the Newton polygon). Then draw a line passing through the point $(0,N) \in E(L)$ and another point, say $(k,j) \in E(L)$ with $k > 0$ and such that the Newton polygon remains below the line. This line is unique - all other points $(k_1,j_1) \in E(L)$ with the same property lie on it. Then one can find a non-zero solution in integers of the equation $N\sigma = k\rho + j\sigma$. In our situation both $\rho$ and $\sigma$ are positive as $j < N$ and at least one $V_j$ is increasing. Notice that with this choice the polynomial $f(x,y) \in \mathbb{C}[x,y]$.

Denote also by $g(x) = x^l$ the polynomial associated to $\theta(x)$. Our goal is to find severe restrictions on the polynomial $f$. We are going to use the following result which is a slight modification of a particular case of the fundamental Proposition 7.3 from [Dx].

**Lemma 3.3** Suppose the element $L \in \mathcal{R}[\partial]$ acts on an element $G$ nilpotently, i.e.

$$\text{ad}^m_L(G) = 0, \quad m \geq 1$$

Let $f$ and $g$ be the polynomials associated to $L$ and $G$ and $f$ contains at least two terms. If $\rho$ and $\sigma$ are positive integers and $v_{\rho,\sigma}(L) > \rho + \sigma$, then one of the following cases holds:

(a) $f^s = g^r$, \hspace{1cm} (3.5)

where $s$ and $r$ are the weights of $g$ and $f$,

(b) $\sigma > \rho$, $\rho$ divides $\sigma$ and

$$f = X^n(X^m + \mu Y)^k,$$ \hspace{1cm} (3.6)

(c) $\rho > \sigma$, $\sigma$ divides $\rho$ and

$$f = Y^n(Y^m + \mu X)^k$$ \hspace{1cm} (3.7)

(d) $\rho = \sigma$ and

$$f = (Y + \lambda x)^n(Y + \mu X)^k$$ \hspace{1cm} (3.8)

where $n \geq 0$, $k \geq 0$, $m > 0$ and $\mu, \lambda \in \mathbb{C}$.

**Remarks on the proof of the lemma.** The proof essentially repeats that of lemma 7.3 from [Dx]. We mention the minor differences. While in [Dx] all the polynomials $(\rho,\sigma)$-associated with the elements of the Weyl algebra belong to $\mathbb{C}[x,y]$, here we
work in $\mathbb{C}[x, x^{-1}, y]$. The fact that the polynomials belong to $\mathbb{C}[x, y]$ is needed in [81x] mainly to speak about their roots in some algebraic closure of $\mathbb{C}(y)$ (respectively - $\mathbb{C}(x)$), when considered as polynomials in $x$ (respectively in $y$). But the ring considered here has the same property.

**Lemma 3.4** Suppose that $k > 1$ (k is from (3.4)). Then either $v_{\rho, \sigma}(L) > \rho + \sigma$ or $L$ cannot act nilpotently on $\theta$.

**Proof.** Writing (3.4) in the form $f = y^{p}(y^{N-p} + \alpha x^{k}) + \ldots$ we notice that one can take $\rho = N - p$ and $\sigma = k \geq 2$ First suppose $N = k = 2$ and $p = 0$. Then $L$ cannot be nilpotent (see [81x]). We give a slightly different proof here. One can easily see that if $v_{\rho, \sigma}(\theta) = 2l > 0$, then $v_{\rho, \sigma}ad_{L}(\theta) = 2l$ unless $f^{s} = g^{2}$. But this is not possible as the polynomial associated with $ad_{L}(\theta)$ contains a term with $y$ of degree one. Denote by $g_{m}$ the polynomial associated to $ad_{L}^{m}(\theta)$. By induction on $m$ we see that the resulting applications of $ad_{L}$ always contain terms where the power of $y$ is less than 2. This shows that the equation $f^{s} = g^{2}$ is impossible.

Now we can suppose that either $p > 0$ or $\max(N, k) \geq 3$ (we recall that both $N \geq 2$ and $k \geq 2$). In the first case we have $v_{\rho, \sigma}(L) = N\sigma > kN \geq N + k > N - p + k > \rho + \sigma$. If $p \geq 0$ then $v_{\rho, \sigma}(L) = N\sigma = Nk > N + k \geq \rho + \sigma$. The strict inequality is a consequence of the fact that $\max(N, k) \geq 3$. □

**Lemma 3.5** Let $L$ act on $\theta$ nilpotently. The polynomial $f$ has the form

$$f = (y^{r} - x)^{k}, \quad r > 1. \tag{3.9}$$

**Proof.** If the term with highest power $k$ in (3.4) is 1 then $f$ has the form $f = y^{n}(y^{r} - \lambda x)$. In the case when $k \geq 2$ from Lemma 3.3 we know that $f = y^{n}(y^{r} - \lambda x)^{k}$ or $f = (y - \lambda x)^{\alpha}(y - \mu x)^{\beta}$, $\alpha + \beta = k$, i.e. we have one of the cases b), c) or d) as $f^{s}$ is impossible. By applying the automorphism $\Phi_{1,1}^{(\mu)}$ of $A_{1}$ (here $\Phi_{1,1}^{(\mu)}(x) = x$, $\Phi_{1,1}^{(\mu)}(y) = y + \mu x$) to $f$ and to $x^{l}$ we reduce the last case to the previous ones, i.e. we assume that $f = y^{n}(y^{r} - \lambda x)^{k}$, where $n \geq 1$ and $k \geq 1$. Without loss of generality we can assume that $\lambda = 1$. Our goal will be to show that if $n \geq 1$ and $k \geq 1$ then $ad_{L}$ cannot be zero for any $s \in \mathbb{N}$. We will show that at least some of the terms with highest weight will be preserved. Suppose that $n \geq 1$, $k \geq 1$. First we continue the automorphisms $\Phi_{1,r}$ from $A_{1}$ to its skew-field. Notice that they preserve the filtration. For this reason we are going to work only with the homogeneous part as the the rest of the terms have no impact on it. Apply the automorphism $\Phi_{r,1}$ of $A_{1}$ to $f$ and to $x^{l}$. Using that $\Phi_{r,1}(\partial) = \partial$ and $\Phi_{r,1}(x) = x + \partial^{r}$ we obtain

$$\Phi_{r,1}(\partial^{n}(\partial^{r} - x)^{k}) = (-1)^{k}\partial^{n}x^{k},$$

$$\Phi_{r,1}(x^{l}) = (x + \partial^{r})^{l}.$$ Consider now $ad_{(\partial^{n}x^{k})}^{l}(x + \partial^{r})^{l}$. Write

$$(x + \partial^{r})^{l} = \sum c_{j}x^{j}\partial^{r(l-j)} + \ldots,$$
where by ... we denote the lower weight terms. Then by linearity

$$ad_{\partial^n x^k}^s(x + \partial^r)^l = \sum_{j=0}^l c_j^l ad_{\partial^n x^k}^s(x^j \partial^{(l-j)}) + \ldots$$ (3.10)

Let $k \geq n$. Simple computation gives that

$$ad_{\partial^n x^k}^s(x^l) = \prod_{j=0}^{s-1} [nl + (k - n)j] \partial^{s(n-1)} x^{l+s(k-1)} + \ldots$$

As $n \geq 1, \ l \geq 1, \ k - n \geq 0$ the coefficient at the term of highest power in $x$ is positive for any $s \geq 1$, which shows that (3.10) cannot be zero for any $s$. Now suppose that $n \geq k \geq 1$. Consider

$$ad_{\partial^n x^k}^s(\partial^r) = [(-1)^s \prod_{j=0}^{s-1} [lrk + (n - k)j]] \partial^{s(n-1)+lr} x^{s(k-1)} + \ldots$$

By the same argument the coefficient at the highest power in $\partial$ is not zero for any $s$. This shows that either $n = 0$ or $k = 0$. But from the assumption (2.7) it follows that $k$ cannot be zero. □

**Remark 3.6** Note that the above result gives a normal form for the leading terms of all bispectral operators with increasing coefficients of any order.

Now assume that the order $N$ of $L$ is a prime number. This gives that $k = 1, \ N = r$. Using Lemma 3.5 we obtain that:

**Lemma 3.7** The operator $L$ has the form

$$L = \partial^N + \sum_{j=1}^{N-2} a_j \partial^j - x + \sum_{j=0}^{N-2} W_j(x) \partial^j,$$ (3.11)

where $\lim_{x \to \infty} W_j(x) = 0$ and $a_j \in \mathbb{C}$.

We will call the operator

$$A = \partial^N + \sum_{j=1}^{N-2} a_j \partial^j - x$$

the principal part of $L$. Following the terminology of [BHY1] $A$ is the (generalized) Airy operator.

**3.2 Airy PDO’s**

Let

$$A = \partial^N + \sum_{j=1}^{N-2} a_j \partial^j - x$$ (3.12)
be the generalized Airy operator. Our aim here is to develop a calculus of pseudo-differential operators written in terms of inverse powers of Airy operators in complete analogy with the standard one, described in sect.1.1. All the results of the section are obtained for any order \( N \), i.e. without assuming that the order is prime. Let \( \Phi(x) \) be a nonzero function in \( \text{Ker} A \), i.e.

\[
A \Phi(x) = 0. \quad (3.13)
\]

Then the function \( \Psi(x, z) = \phi(x + z) \) satisfies the equations:

\[
A(x, \partial_x)\Psi(x, z) = z\Psi(x, z), \quad (3.14)
\]

\[
A(z, \partial_z)\Psi(x, z) = x\Psi(x, z). \quad (3.15)
\]

Obviously the function \( \Psi(x, z) \) satisfies also the equation

\[
\partial_x \Psi(x, z) = \partial_z \Psi(x, z). \quad (3.16)
\]

The equations (3.14)-(3.16) define an anti-involution \( b \) on the Weyl algebra \( A_1 \), acting on the generators \( A, \partial_x \) of \( A_1 \) by

\[
b(A(x, \partial_x)) = z, \quad (3.17)
\]

\[
b(\partial_x) = \partial_z. \quad (3.18)
\]

From (3.18) one easily finds that

\[
b(x) = A(z, \partial_z), \quad (3.19)
\]

which would be used later.

Now define the algebra \( B_1 \) of pseudo-differential operators of the type:

\[
P(x, \partial_x) = \sum_{j=-m}^{\infty} a_j(x, \partial_x)A^{-j}, \quad (3.20)
\]

with operator coefficients of the form:

\[
a_j = \sum_{k=0}^{N-1} \alpha_{j,k}(x)\partial^k, \quad (3.21)
\]

where the functions \( \alpha_{j,k}(x) \) are formal Laurent series:

\[
\alpha_{j,k}(x) = \sum_{s=r}^{\infty} \beta_{j,k}^{(s)}x^{-s} \quad (3.22)
\]

and the index \( r \) depends only on \( P \) (but not on \( j! \)). Then the anti-automorphism \( b \) can be continued on \( B_1 \) as
\[ b(P) = \sum_{j=-m}^{\infty} z^{-j} \sum_{k=0}^{N-1} \sum_{s=r}^{\infty} \beta_{j,k}^{(s)} \partial_x^k A^{-s} = \sum_{s=r}^{\infty} b_s(z, \partial_z) A^{-s}(z). \]  

(3.23)

Introduce the "wave operator" \( K \) as follows:

\[ K = 1 + \sum_{j=1}^{\infty} m_j(x, \partial_x) A^{-j}, \]  

(3.24)

where

\[ m_j(x, \partial_x) = \sum_{k=0}^{N-1} \alpha_{j,k}(x) \partial_x^k. \]  

(3.25)

Let \( L \) be a differential operator of the form:

\[ L = A^l + V_{l-1} A^{l-1} + \ldots, \]  

(3.26)

where

\[ V_j(x, \partial) = \sum_{k=0}^{N-1} V_{j,k}(x) \partial^k. \]  

(3.27)

Then one can find a wave operator (not unique) \( K \) of the form (3.24) so that

\[ L = K A^l K^{-1}. \]  

(3.28)

The coefficients \( \alpha_{j,k}(x) \) from (3.26) of the expansion of \( K \) can be found by induction from the equation:

\[ L K = K A^l \]  

(3.29)

Multiplying the above equation from the right by \( A, A^2, \ldots \) one computes the coefficients \( \alpha_{j,k}(x) \) in the expansion (3.24) - (3.26) of \( K \) in terms of the functions \( V_{j,k} \).

We would particularly be interested in the case \( l = 1 \).

In what follows we assume that the operator \( L \) is bispectral, it satisfies equation of the form (0.1), together with an equation of the form (0.2). In that case as in [DG, W1] one can prove the following lemma.

**Lemma 3.8** The coefficients \( a_{j,k} \) of the operator \( K \) are rational functions.

**Proof.** We mimic the well known proof (see [DG, W1]). Write

\[ (ad_L)^m(\theta) = 0. \]  

(3.30)

This is equivalent to

\[ (ad(A^l))^m(K^{-1} \theta K). \]  

(3.31)

Put

\[ \Theta = K^{-1} \theta K = \sum_{j=0}^{\infty} \theta_j A^{-j}, \]  

(3.32)

where
\[ \theta_j = \sum_{k=0}^{N-1} \theta_{j,k} \partial^k. \]

This gives

\[ (adA^l)^m(\theta_j) = 0. \]

The leading terms of the above equation give

\[ \theta_j^{(m)} j, N-1 \partial^m (Nl - 1) + \theta_j^{(m)} j, N-2 \partial^m (Nl - 1) + \ldots + \theta_j^{(m)} j, 0 \partial^m (Nl - 1) + \ldots = 0 \]

Here in the brackets containing the coefficients at \( \partial^m (Nl - 1) + N - s \), \( s = 2, \ldots, N \) the dots after \( \theta_j, N - s \) denote expressions of derivatives of \( \theta_j, N - r \), \( r < s \) of order not lower than \( m \). Then obviously by induction we get that all \( \theta_j^{(m)} j, n - 1 \equiv 0 \), \( n = 1, \ldots, N \), which shows that they are polynomials of degree \( d \leq m \). Then the computation of the coefficients \( a_j(x) \) of \( K \) is performed as in [DG, W1] (see also sect.2). We see that they are rational functions. \( \Box \)

### 3.3 Proof of the main theorem for operators with increasing coefficients

Let \( L \) be an operator of order \( N \) with Airy principal part

\[ A = \partial^N + a_{N-2} \partial^{N-2} + \ldots + a_1 \partial - x, \]

i.e.

\[ L = A + \sum_{j=0}^{N-2} V_j(x) \partial^j = A + V(x, \partial_x) \]

and

\[ \lim_{x \to \infty} V_j(x) = 0. \]

Using the techniques of the previous subsection we present \( L \) in the form

\[ L = A + V = KAK^{-1} \]

Our goal will be to show that bispectrality, and in particular - the rationality of the coefficients \( \alpha_{j,k}(x) \) in the expansion (3.24) and (3.25) of \( K \), implies that the perturbation \( V(x, \partial_x) \equiv 0 \), which is equivalent to \( \alpha_{j,k} \equiv 0 \) for all \( j, k \). But first we need some notation and auxiliary results.

From the equation
\( LK = KA, \quad (3.34) \)

we can compute recursively the coefficients \( m_j \) of the operator \( K \). For this we will need some formulas to compare the coefficients of the two sides of (3.34). Introduce the operators \( b_j, c_j, U_j, W_j, j = 1, 2, \ldots \) by:

\[
[A, m_j] = b_j A + c_j; \quad (3.35)
\]

\[
V(x, \partial)m_j = U_j A + W_j. \quad (3.36)
\]

We will need to order the monomials in \( m \) and related to it expressions as follows:

**Definition 3.9** We say that the monomial \( x^{r_1} \partial^{k_1} \) is of higher order than the monomial \( x^{r_2} \partial^{k_2} \) if \( r_1 > r_2 \) or \( r_1 = r_2 \) and \( k_1 > k_2 \). We will call the number \( r \) the height of \( m \), if the highest order term of \( m \) is of the type \( c x^r \partial^k, \quad c \neq 0 \). We denote this number by \( ht(m) \).

In other words we use lexicographic ordering in the set of the monomials \( x^r \partial^k \) but the order is only the power of \( x \).

In what follows we are going to use the abbreviation l.o.t. (for lower order terms) compared to some operator \( m \) with the meaning that they are lower than at least one of the terms in \( m \). We will need also the following lemma:

**Lemma 3.10** In the above formulas we have

(i)
\[
b_j = \sum_{k=1}^{N-1} N \alpha_{j,k} \partial^{k-1} + \text{l.o.t.} \quad (3.37)
\]

\[
c_j = N \alpha_{j,0} \partial^{N-1} + \sum_{k=1}^{N-1} (x N \alpha_{j,k} + k \alpha_{j,k}) \partial^{k-1} + \text{l.o.t.} \quad (3.38)
\]

(ii) \( ht(c_j) = ht(b_j) + 1 \).

**Proof.** The proof is straightforward computation. To avoid two indices we will suppress the dependence on \( j \) (it is irrelevant at that moment). We have

\[
A \circ m = m\left( \sum_{k=1}^{N-1} a_k \partial^k \right) - x m + \sum_{k=0}^{N-1} N \alpha_k \partial^{N+k-1} + \text{l.o.t.} \quad (3.39)
\]

\[
m \circ A = m\left( \sum_{k=1}^{N-1} a_k \partial^k \right) - x m - \sum_{k=0}^{N-1} k \alpha_k \partial^{k-1}. \quad (3.40)
\]

Subtracting (3.40) from (3.39) we get

\[
[A, m] = \sum_{k=0}^{N-1} N \alpha_k \partial^{N+k-1} + \sum_{k=0}^{N-1} a_k \partial^{k-1} + \text{l.o.t..} \quad (3.41)
\]
Split the first sum into two parts as follows. One of them contains derivatives from \( N \) to \( 2N - 2 \); the second will contain the rest of the them. Then we have for the first part

\[
\sum_{k=0}^{N-1} N\alpha_k \partial^{N+k-1} = (\sum_{k=1}^{N-1} N\alpha'_k \partial^{k-1}) \partial^N
\]

Next use the identity \( \partial^N = A - \sum_{k=1}^{N-2} a_k + x \) to get

\[
N\alpha_0 \partial^{N-1} + \sum_{k=1}^{N-1} N\alpha'_k \partial^{N+k-1} = N\alpha_0 \partial^{N-1} + (\sum_{k=1}^{N-1} N\alpha'_k \partial^{k-1}) A + x(\sum_{k=1}^{N-1} N\alpha'_k \partial^{k-1}) + \ldots
\]

where the dots represent terms with derivatives of \( \alpha_k \). Then we repeat the same procedure to all terms (including the ones in l.o.t. from (3.41)) containing \( \partial^k \) with \( k \geq N \). After finite number of steps we get (3.37) and (3.38). The second part of the lemma follows immediately from the first one.

Lemma 3.11
(i) The equation (3.34) is equivalent to the equations:

\[
b_1 + V + U_1 = 0 \tag{3.42}
\]

\[
b_{j+1} + c_j + W_j + U_{j+1} = 0, \quad j = 1, \ldots \tag{3.43}
\]

(ii) The coefficients of \( V \) and \( b_1 \) behave at infinity as \( x^{-2} \)

Proof. The first part is simply comparing the coefficients. Indeed, writing in detail (3.34) we get

\[
A + Am_1 A^{-1} + \ldots + V + Vm_1 A^{-1} + \ldots = A + m_1 + \ldots
\]

Using (3.35) and (3.36) we can simplify the last equation to

\[
V + b_1 + U_1 + \ldots = 0,
\]

where \( \ldots \) denote the purely pseudo-differential part. This gives (3.42). Multiplying (3.34) by \( A, A^2 \), etc. from the right and arguing in the same manner we get (3.43).

To prove the second part we use (3.37) with \( j = 1 \) and (3.42). Notice that the leading terms of \( b_1 \) are derivatives of a rational functions. Being equal to the leading terms of \( V \) they vanish. Hence they vanish at least of order \( x^{-2} \).

Lemma 3.12
The following inequalities hold:

\[
ht(U_j) \leq ht(b_j) - 1, \tag{3.44}
\]

\[
ht(W_j) \leq ht(c_j) - 1. \tag{3.45}
\]
Proof. The proof is similar to that of Lemma 3.10. Using (3.36) we obtain

\[ V m_j = \sum_{k=1}^{2N-1} \sum_{s=k}^{N-1} V_s \alpha_{j,k-s} \partial^k = \sum_{k=0}^{2N-1} \tilde{V}_{j,k} \partial^k \]

For \( k = N, \ldots, N - 1 \) put \( \tilde{V}_{j,k} = U_{j,k-N} \). As above split the sum into two parts, the first one containing the terms with \( \partial^k, k \geq N \):

\[ V m_j = \sum_{k=0}^{N-1} \tilde{U}_{j,k} \partial^k \partial^N + \sum_{k=0}^{N-1} \tilde{V}_{j,k} \partial^k \]

Again use the identity \( \partial^N = A - \sum_{k=1}^{N-2} a_k + x \) several times to get the first sum in the form:

\[ \sum_{k=0}^{N-1} \tilde{U}_{j,k} \partial^k + \text{l.o.t.} A + x \sum_{k=0}^{N-1} \tilde{U}_{j,k} \partial^k + \text{l.o.t.} \]

Then obviously we have:

\[ U_j = \sum_{k=0}^{N-1} \tilde{U}_{j,k} \partial^k + \text{l.o.t.} \quad (3.46) \]

\[ W_j = \sum_{k=0}^{N-1} \tilde{V}_{j,k} \partial^k + x \sum_{k=0}^{N-1} \tilde{U}_{j,k} \partial^k + \text{l.o.t.} \quad (3.47) \]

Now using that the order at infinity of \( V \) is \( x^{-2} \) we get that \( \text{ht}(U_j) \leq \text{ht}(m) - 2 \), \( \text{ht}(W_j) \leq \text{ht}(m_j) - 1 \). From the last inequalities we get (3.44) and (3.45). \( \square \)

Now we are ready to finish the proof of the Theorem 3.4.

Proof. of Theorem 3.4. We recall that we have to show that \( V \equiv 0 \). Assume that some of the coefficients \( V_j \) are not zero. Then we shall compute the leading terms of the operators \( b_j \) recursively using (3.43) and Lemma 3.11 and taking into account the estimates (3.44) and (3.45). First notice that that the highest order term in \( b_1 \) is of the type \( \alpha_1 x^{s_1} \partial^k, \alpha_1 \neq 0 \), with \( k < N \) and \( s_1 < 0 \). Suppose that the highest order term in \( b_j \) is \( \alpha_j x^{s_j} \partial^k, \alpha_j \neq 0 \). Then the highest order term in \( b_{j+1} \) is computed, using (3.43), (3.37) and (3.38) to be \( \alpha_{j+1} x^{s_{j+1}} \partial^k \) with \( \alpha_{j+1} = -\alpha_j (N(s_{j+1})+k)/N(s_{j+1}) \). Having in mind \( k < N \) we get that \( \alpha_{j+1} \neq 0 \). After a finite number of steps we will get that for some \( j \) the corresponding \( s_j = -1 \). But this contradicts the fact that the highest order term of \( b_j \) is a derivative of a rational function. \( \square \)

4 Final remarks on the proof and comments
4.1 Proof of Theorem 0.1

Essentially we already have performed the proof of the main theorem. We just have to notice that when the order of $L$ is prime and there are coefficients increasing at infinity Lemma 3.7 and Theorem 0.4 give that the operator is Airy and hence bispectral.

If the coefficients of $L$ are bounded (at infinity) and the order is prime, then using the fact the the rank divides the order we get that either the rank of $L$ is 1 or it is equal to its order. The latter case is treated in Theorem 0.3. If the rank is one then this is the main result of [W1]. Finally the inverse part, i.e. that all the operators listed in Theorem 0.1 are bispectral is the main result of [BHY1].

4.2 Comments

Here I would like to make some speculations on eventual continuation of the classification. It seems to me that the methods of [BHY3] (see also [H1] for more details) will be enough to construct all bispectral operators. Assuming that then the classification should be: 1) find all "basic" bispectral operators; 2) show that Darboux transforms reduce any bispectral operator to a "basic" one.

Having in mind the constructions in [DG, BHY1, KR, W1] it seems natural to consider basic those operators that have as few singularities as possible and generate their centralizers. Then in view of the main result of [H2] one class of operators that certainly should be considered as "basic" is the class of bispectral operators $L$ in the Weyl algebra that together with some other operator $Q$ satisfy the "canonical commutation relation" (CCR):

$$[L, Q] = 1$$  \hspace{1cm} (4.1)

More precisely they are basic because according to [H2] all bispectral operators in the Weyl algebra are simply polynomials in operators $L$ that satisfy (4.1). It is tempting to believe that the CCR is enough for bispectrality:

**Conjecture 4.1** If the operators $L$ and $Q$ satisfy the CCR (4.1) then they are bispectral.

This conjecture seems to be a difficult one as it is easily shown (cf. [H2]) to be equivalent to the famous conjecture of Dixmier-Kirillov:

**Conjecture 4.2** If the operators $L, Q$ satisfy the CCR (4.1) then they generate the Weyl algebra $A_1$. In other words, any endomorphism of the Weyl algebra is an automorphism.

The Bessel operators as well as other examples from [BHY3] show that one needs also to add to the list of the basic operators $L$ the ones that together with some other operator $Q$ satisfy the following "string" equation

$$[L, Q] = L$$  \hspace{1cm} (4.2)

Our, maybe insufficient experience, suggests that the above two classes contain all basic bispectral operators. A more precise conjecture is the following one:
Conjecture 4.3 Every bispectral operator is a polynomial Darboux transformation of a bispectral operator satisfying either (4.1) or (4.2).

To make the classification of the basic bispectral operators more explicit introduce the following notation. Denote by $B_\alpha$ the algebra spanned by a Bessel operator $L_\alpha$, $x^N$ and $D$. Then

Conjecture 4.4 (1) A bispectral operator satisfying (4.1) belongs the Weyl algebra. (2) A bispectral operator satisfying (4.2) belongs to one of the algebras $B_\alpha$.

Some progress could be achieved if one finds analogs for $B_\alpha$ of the results from [H2]. One difficulty would be to extend Dixmier results for these algebras.

All the above conjectures aim to a complete classification of bispectral operators. A more modest goal is to find the bispectral operators with some properties. Here is a conjecture in this direction.

Conjecture 4.5 An operator $L$ with bounded at infinity coefficients is bispectral if and only if it is a polynomial Darboux transformation of a Bessel operator.

This conjecture seems natural in view of the "if" part, obtained in [BHY1]. It will be very useful either to give a proof of Wilson’s result about rank one operators without using algebraic-geometric arguments or modify his proof in higher rank situation. Despite of the many interesting results for higher rank solutions of KP-hierarchy I have not found a construction suitable for our purposes. In any case the above conjecture seems to me to be within the reach of the existing tools unlike the previous ones.

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24
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