Formation Control of Multiple Underactuated Surface Vehicles Based on Prescribed-Time Method

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ABSTRACT This paper investigates the formation control problem of multiple underactuated surface vehicles (USVs). By utilizing input and state transformations, the dynamic model of the USV is converted into an equivalent system consisting of two cascade-connected subsystems. For one of the subsystems, by combining back-stepping method with time-varying scale functions, \( C_1 \) smooth prescribed-time control laws are presented. This approach is proved to stabilize the transformed subsystem in the fixed time and address the formation problem while guaranteeing global exponential stabilization. Simulations are given to demonstrate the effectiveness of the presented method.

INDEX TERMS Exponential stability, formation control, prescribed-time method, underactuated surface vehicles.

I. INTRODUCTION

Formation control of surface vehicles is particularly interesting and has many applications in practice. For example, several autonomous vehicles trace a target and enclose it; several vehicles search a target in a large water area cooperatively, and several vehicles with some sensors on them monitor a large area cooperatively for security, etc. The focus is on designing a control law that can drive a group of vehicles to perform the desired group behavior (e.g., formation, flocking, consensus, rendezvous, and agreement). In the literature, three types of formation control are taken into account for the underactuated surface vehicles. The first type is the formation of tracking with a leader. For this type, intuitively, the vehicles are required to come into the desired formation shape and track the leader [4]–[6]. The second type is the formation following, for which the vehicles are demanded to form a desired geometric pattern and move along a given path [7]. The last type is the formation without a leader, and the control objective is regulating the vehicles to the desired formation shape, not necessarily to track a leader or go along a path [8]. The three types of formation control problems are just like trajectory tracking, path following, and stabilization control problems in the conventional control sense [9].

Few works have been done for the third leader-less formation control of underactuated surface vehicles. This is because this problem cannot be solved directly by the tracking control technologies developed for leader-follower formation, or by the approaches developed for the consensus of linear multi-agent. What is more, although the underactuated surface vehicle is subject to nonholonomic constraint, its model cannot be transformed into the standard nonholonomic chained system, so the control schemes designed for the cooperative control of nonholonomic systems [11] cannot be applied here. To solve this problem, a smooth time-varying distributed control law is designed in [3] by using appropriate coordinate transformation and graph theory, which can guarantee that the vehicles converge to the desired geometric pattern with the same orientations and vanishing velocities. For solving the same formation problem, another smooth time-varying distributed control law is proposed in [8] achieving the asymptotic formation of underactuated vehicles with non-diagonal inertia/damping matrices. In [10], the time-varying control laws are proposed to avoid that the communication graph among the vessels is fixed and containing a spanning tree.
Above mentioned time-varying control approaches rely on asymptotic methods. Recently, researchers investigated finite/fixed-time control methods for stabilization of USVs such as [15], [16]. Compared with asymptotic control approaches, finite/fixed-time control methods not only have faster convergence speed but also better disturbance rejection properties and robustness [12], [13]. Most finite/fixed-time controllers use fractional power feedback of the form $x^p$ (with $p$ and $l$ being some positive odd integers). In contrast to asymptotic methods, fractional power terms can cause high-frequency oscillations in the system states, i.e. chattering. A number of methods for attenuating chattering have been proposed, such as boundary layer method [17] and high-order sliding-mode method [18]. The boundary layer method includes saturation function and sigmoid function methods. But it can only guarantee the existing condition of the sliding-mode outside a small boundary layer around the sliding-mode manifold, which will increase the steady-state tracking errors. The high-order sliding-mode method is to hide the discontinuity of control in its higher derivatives.

Fortunately, the prescribed-time control law proposed in [1] can guarantee fixed-time stability of the system without chattering. However, different from trajectory tracking and path following problems, stabilization requires all states converge to zero leads the existence of coupling nonlinear terms, which violates the Brockett necessary condition [21]. So that the USV system cannot be linearized into the strict chain system and hence, the prescribed-time method for the typical nonlinear system cannot be used directly to stabilize USVs. A prescribed-time control scheme of USVs to consider the nonlinear coupling of states might be of great significance.

This paper discusses the exponential formation stabilization of USVs with the prescribed-time approach. After introducing the model transformation in [20], a novel state transformation based on graph theory is proposed to convert the system into a simple cascade form of two subsystems. For the transformed system, we propose the prescribed-time control laws by combining the back-stepping method and scale functions that grow unbounded towards the terminal times. Different from the existing prescribed-time control method, we employ different scale functions for each virtual input respectively to decouple the states of USV. The proposed time-varying control laws are $C_1$ smooth and proved to guarantee global exponential convergence within a designated time independent of the initial conditions. Compared with the existing fixed/finite time methods of USV stabilization such as [16] and [15], the proposed $C_1$ smooth approach requires no Sign terms and hence decreases the chattering problem.

The paper is organized as follows. Section II is problem formulation and the objective. Section III gives input and state transformations. Section IV gives the control laws and stability confirmations. Numerical simulations and discussions are given in Section V.

## II. PROBLEM STATEMENT

### A. PRELIMINARIES

**Definition 1:** Consider the system defined by

$$\dot{z}(t) = f(t, z(t)) + r, t \in R_+, z(0) = z_0,$$  

(1)

where $z \in R^n$ is the state vector, $f : R_+ \times R^n \rightarrow R^n$ is a nonlinear vector field locally bounded in time. The origin of system (1) is said to be globally uniformly finite-time stable if it is globally asymptotically stable and there exists a locally bounded function $T : R^n \rightarrow R_+ \cup \{0\}$ such that $z(t, z_0) = 0$ for all $t \geq T(z_0)$, where $z(t, z_0)$ is an arbitrary solution of the Cauchy problem of (1). The function $T$ is called the settling-time function.

**Definition 2:** The origin of system (1) is said to be globally prescribed-time stable if it is globally finite-time stable and the settling-time $T$ is a user-assignable finite constant, i.e., $\forall 0 < T_p \leq T_{\max} < \infty$ ($T_p$ denotes the physically possible time range), $T$ can be prescribed such that $T_p \leq T \leq T_{\max}$. $\forall z_0 \in R^n$.

### B. SYSTEM MODELS

Consider $n$ underactuated surface vehicles [2]. Each surface vehicle has two propellers, which provide the force capable of affecting surge and torque capable of affecting yaw. Following the results in [3], the kinematics of the $j$th surface vehicle for $1 \leq j \leq n$ can be written as

$$\begin{pmatrix}
\dot{x}_j \\
\dot{y}_j \\
\dot{\psi}_j
\end{pmatrix} =
\begin{bmatrix}
\cos(\psi_j) & -\sin(\psi_j) & 0 \\
\sin(\psi_j) & \cos(\psi_j) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
x_j \\
y_j \\
\psi_j
\end{pmatrix}$$  

(2)

where $(x_j, y_j)$ denotes the coordinates of the center of mass of the $j$th surface vehicle in the earth-fixed frame, $\psi_j$ is the orientation of the $j$th vehicle, and $u_j$, $v_j$, $r_j$ are the velocities of the $j$th vehicle in the surge, sway and yaw, respectively. The dynamics of the $j$th surface vehicle can be written as:

$$\dot{u}_j = \frac{m_{2,j}}{m_{1,j}} v_j r_j - \frac{d_{1,j}}{m_{1,j}} u_j + \frac{1}{m_{1,j}} \tau_{1,j},$$  

(3a)

$$\dot{v}_j = -\frac{m_{1,j}}{m_{2,j}} u_j r_j - \frac{d_{2,j}}{m_{2,j}} v_j,$$  

(3b)

$$\dot{\psi}_j = \frac{m_{1,j} - m_{2,j}}{m_{3,j}} u_j v_j - \frac{d_{3,j}}{m_{3,j}} r_j + \frac{1}{m_{3,j}} \tau_{2,j},$$  

(3c)

where $m_{i,j}$ and $d_{i,j}$ are constants in this paper. $\tau_{1,j}$ is the surge control force and $\tau_{2,j}$ is the yaw control moment. States $x_j$, $y_j$, $\psi_j$, $u_j$, $v_j$, $r_j$ and $\tau_{1,j}$ are measurable. Both $\tau_{1,j}$ and $\tau_{2,j}$ are control inputs to design. For such USVs, an inertial frame and a body-fixed frame are used as depicted in Figure 1.
For the analysis that follows, we assume that:

**Assumption 1:** The environment forces due to wind, currents and waves can be neglected in the model of each underactuated surface vehicle.

**Assumption 2:** The inertia, added mass and damping matrices are diagonal for each underactuated surface vehicle.

### C. COMMUNICATION BETWEEN VEHICLES

During the control, each vehicle knows its own state and the states of some of the other vehicles by communication. If we consider each vehicle as a node, $G = (\Delta, E, A)$ be a directed graph of $N$ orders, where $\Delta = \{v_1, v_2, \ldots, v_N\}$ is a finite nonempty set of nodes, and $E \subseteq \Delta \times \Delta$ is the set of edges. The weighted adjacency matrix $A = a_{ij} \in R^{N \times N}$ is defined such that $a_{ij}$ is positive if $(v_i, v_j) \in E$, while $a_{ij} = 0$, otherwise. The Laplacian matrix $L = [l_{ij}]_{N \times N}$ is defined as $l_{ij} = \sum_{k=1, k \neq i}^N a_{ik}$ and $l_{ii} = -a_{ij}, i \neq j, j = 1 \ldots N$. A directed graph contains a directed spanning tree if there exists a directed path from the root to every other node in the graph. For more terminology on graph theory, interested readers may refer to [22], [23].

**Assumption 3:** The communication topology $G = (\Delta, E, A)$ contains a directed spanning tree.

### D. OBJECTIVE

Given a desired geometric pattern $p$ defined by constant vectors $[p_{x,i}, p_{y,i}]^T$ (1 ≤ j ≤ n) noting that rotation by an angle and translation of $p$ do not change its geometric form. We aim to design controllers $t_{1,j}$ and $t_{2,j}$ for each vehicle based on its relative state information with its neighbors, such that

$$\lim_{t \to \infty} [x_j - x_i, y_j - y_i] = S(\eta_1) \begin{bmatrix} p_{x,i} - p_{x,j} \\ p_{y,i} - p_{y,j} \end{bmatrix}, \quad (4a)$$

where $1 \leq i \neq j \leq m, \eta = [\eta_1, \eta_2, \eta_3, \eta_4]^T$ is a free constant vector,

$$S(\eta_1) = \begin{bmatrix} \cos(\eta_1) & -\sin(\eta_1) \\ \sin(\eta_1) & \cos(\eta_1) \end{bmatrix}. \quad (4b)$$

In the formation control problem, (4a) means that the group of surface vehicles converges to the desired geometric pattern $P$. Equation (4b) means that the desired formation is stationary, centered at the point $\eta_2, \eta_3$ and that each vehicle converges to the same orientation angle $\eta_4$. The vector $\eta$ may be predefined or not. In the formation control problem, the control law for vehicle $j$ is required to be designed based on the relative information between vehicle $j$ and vehicle $i$ for $i \in N_j$ such that the state of each vehicle converges to a stationary point. Noting Brockett’s necessary condition for stabilizing a non-holonomic system [21], there does not exist a feedback law for each vehicle which is a smooth function of its own state and the states of its neighbors such that the state of each vehicle converges to a stationary point. So, the defined formation control problem is challenging.

### III. TRANSFORMATION

To facilitate the control law design, we first transform systems (2) and (3) into a suitable form. Using the state transformation [3]:

$$z_{1,j} = (x_j - \epsilon_{1,j}) \cos(\psi_j) + (y_j - \epsilon_{2,j}) \sin(\psi_j), \quad (5a)$$

$$z_{2,j} = -(x_j - \epsilon_{1,j}) \sin(\psi_j) + (y_j - \epsilon_{2,j}) \cos(\psi_j) + \frac{m_{2,j}}{d_{2,j}} \psi_j, \quad (5b)$$

$$z_{3,j} = \psi_j, \quad z_{4,j} = v_j, \quad (5c)$$

$$z_{5,j} = -\frac{m_{1,j}}{d_{2,j}} u_j - z_{1,j}, \quad z_{6,j} = r_j, \quad (5d)$$

and the control input transformation

$$\alpha_{1,j} = \left(\frac{d_{1,j}}{d_{2,j}} - 1\right) u_j - z_{2,j} z_{6,j} - \frac{r_{1,j}}{d_{2,j}}, \quad (6a)$$

$$\alpha_{2,j} = \frac{m_{1,j}}{m_{3,j}} u_j v_j - \frac{d_{3,j}}{m_{3,j}} r_j + \frac{r_{2,j}}{m_{3,j}}, \quad (6b)$$

where $\epsilon_{1,j} = p_{x,j} - \sum_{i=1}^{m} p_{x,i}, \epsilon_{2,j} = p_{y,j} - \sum_{i=1}^{m} p_{y,i}, \epsilon_{3,j} = p_{x,j} - \sum_{i=1}^{m} p_{x,i}$ and $\epsilon_{4,j} = p_{y,j} - \sum_{i=1}^{m} p_{y,i}$. For $1 \leq j \leq m$, the time derivative of states $z_{1,j}, z_{1,j}^* z_{2,j}, z_{3,j}, z_{4,j}, z_{5,j}$ and $z_{6,j}$ can be

$$\dot{z}_{1,j} = -\frac{d_{2,j}}{m_{1,j}} z_{1,j} + \dot{z}_{r,j}, \quad (7a)$$

$$\dot{z}_{4,j} = -\frac{d_{3,j}}{m_{2,j}} z_{4,j} + \frac{d_{3,j}}{m_{2,j}} z_{5,j} (z_{1,j} + z_{5,j}), \quad (7b)$$

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where $z_{r,j} = -\frac{d_{2j}}{m_{2j}}z_{s,j} + z_{3,j}c_{6,j} - \frac{m_{2j}}{d_{2j}}z_{4,j}z_{6,j}$. For the transformation, we have the following result according to paper [3]:

**Lemma 1:** By transformations (5) and (6), if $(z_{2,j}, z_{3,j}, z_{5,j}, z_{6,j})$ and $\dot{z}_{s,j}$ are bounded, and

$$\lim_{t \to \infty} z_{2,j} = 0, \quad \lim_{t \to \infty} z_{3,j} = c_{4}, \quad \lim_{t \to \infty} z_{5,j} = c_{5}, \quad \lim_{t \to \infty} z_{6,j} = \exp.$$

then the formation control problem is solved with $\eta = [0, -c_{5}\cos(c_{4}), -c_{5}\sin(c_{4}), c_{4}]^T$, where $c_{4}$ and $c_{5}$ are constants and $\lim_{t \to \infty} z_{6,j} \to 0$ means that $z_{6,j}$ exponentially converges to zero. Furthermore, if $c_{4} = c_{5} = 0$, then the formation control problem is solved with $\eta = 0$.

Hence we can achieve the formation by stabilizing the transformed system (7). Although the state transformation in [3] can simplify discussions, topological relationships between USVs are not considered. To this end, we give the further state transformation based on the graph theory, aiming to achieve the USV formation by the consensus tracking control of states $z_{1,j}, z_{2,j}, z_{4,j}$ and $z_{5,j}$. Denote $\Theta_{i} = \{ \theta_{1,i}, \theta_{2,i}, \theta_{3,i}, \theta_{4,i}, \theta_{5,i}, \theta_{6,i} \}$, one has

$$\dot{\theta}_{1,i} = b_{1}z_{1,i} + \theta_{1,i}, \dot{\theta}_{2,i} = b_{2}z_{2,i} + \theta_{2,i}, \dot{\theta}_{3,i} = b_{3}z_{3,i} - c_{4}, \quad \dot{\theta}_{4,i} = b_{4}z_{4,i} + \theta_{4,i}, \dot{\theta}_{5,i} = b_{5}(z_{5,i} - c_{5}) + \theta_{5,i}, \dot{\theta}_{6,i} = z_{6,i},$$

where $\theta_{1,i} = l_{j}z_{1,i} - \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{x,j} - d_{x,i}) - \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{z,j} - d_{z,i}) - \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{y,j} - d_{y,i}) - \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{z,j} - d_{z,i})$, $\theta_{4,i} = l_{j}z_{4,i} - \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{x,j} - d_{x,i}) - \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{z,j} - d_{z,i}) + \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{y,j} - d_{y,i}) + \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{z,j} - d_{z,i}) + \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{x,j} - d_{x,i}) + \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{y,j} - d_{y,i}) + \sum_{j=1,i \neq j}^{n} a_{i,j}(d_{z,j} - d_{z,i})$.

For $1 \leq j \leq m$, we can derive from the equations (2), (3), (5) and (10) that

$$\dot{z}_{s,j} = \begin{pmatrix} -\frac{m_{2j}}{d_{2j}}z_{4,j}z_{6,j} \\
\frac{d_{2j}}{m_{2j}}z_{4,j} - d_{y,j}z_{6,j} \\
\frac{d_{2j}}{m_{2j}}(b_{j} + l_{j})z_{6,j} \\
\frac{d_{2j}}{m_{2j}}(b_{j} + l_{j})z_{6,j} \end{pmatrix},$$

$$\dot{z}_{p,j} = \sum_{i=1,i \neq j}^{n} a_{i,j}(\dot{z}_{6,i} - \dot{z}_{6,j})z_{5,j},$$

$$\dot{z}_{5,j} = (b_{j} + l_{j})z_{6,j} - \sum_{i=1,i \neq j}^{n} \dot{z}_{6,i}z_{5,j},$$

$$\dot{z}_{6,j} = \sigma_{1,j} - \sum_{i=1,i \neq j}^{n} \sigma_{1,i},$$

where $z_{s,j} = \sum_{i=1,i \neq j}^{n} a_{i,j}(\dot{z}_{6,i} - \dot{z}_{6,j})z_{5,j}$. For the proposed transformation, we have the following results.

**Lemma 2:** If states $\theta_{2,i}, \theta_{3,i}, \theta_{4,i}$ and $\theta_{6,i}$ converge to zero in fixed-time, then properties in Lemma 1 hold.

**Proof:** If states $\theta_{2,i}, \theta_{3,i}, \theta_{4,i}$ and $\theta_{6,i}$ converge to zero in fixed-time, then the constant $t_{s}$ exists such that $\forall t > t_{s}$, $|\theta_{2,i}, \theta_{3,i}, \theta_{4,i}, \theta_{6,i}| = 0$. According to equations (10), this means for all $t \geq t_{s}$,

$$z_{2,j} = 0, \quad z_{3,j} = c_{4}, \quad z_{5,j} = c_{5}, \quad z_{6,j} = 0, \quad \dot{z}_{5,j} = 0.$$ (13)

Hence we can have that $(z_{2,j}, z_{3,j}, z_{5,j}, z_{6,j})$ and $\dot{z}_{s,j}$ are bounded, and

$$\lim_{t \to \infty} z_{2,j} = 0, \quad \lim_{t \to \infty} z_{3,j} = c_{4}, \quad \lim_{t \to \infty} z_{5,j} = c_{5}, \quad \lim_{t \to \infty} z_{6,j} = c_{5}, \quad \lim_{t \to \infty} \dot{z}_{5,j} = 0.$$ (14)

By Lemma 1, the proof is completed.

**Remark 1:** Based on the graph theory, we propose the novel transformation. The similar to stabilization of the single USV, the formation system is divided into two chain systems. So that we can achieve the objective by stabilizing the transformed subsystem (12) to simply the control process.

**IV. STABILIZATION OF USVs**

In this section, we propose prescribed-time control laws to stabilize the transformed system (12) in the fixed-time with time-varying scale functions, which are proved to guarantee the formation exponentially stable.

**A. PRESCRIBED TIME THEORY**

Consider the following system

$$\dot{z}_{i} = \dot{z}_{i+1},$$

$$\dot{z}_{n} = f(z,t) + g(z,t)\tau,$$

where $z = [z_{1}, \ldots, z_{n}]^T$ is the state, $\tau \in R$ is the control input and $f(z,t)$, $g(z,t)$ are possibly non-vanishing. Then we have the following two lemmas, whose proofs are in the reference [1].

**Lemma 3 (Scaling Transformation):** The scaling transformation $z(t) \to w(t)$ given by

$$w = \mu_{1}^{n+1}P(\mu_{1})z,$$

where $\mu_{1} = \frac{T}{T_{s}^{m+1}}$, the matrix $P(\mu_{1})$ is a lower triangular matrix having elements $(P_{ij})$ given by

$$P_{ij}(\mu_{1}) = \tilde{P}_{ij}(\mu_{1})^{m+1}, \quad 1 \leq j \leq i \leq n,$$

$$\tilde{P}_{ij} = \begin{pmatrix} \frac{(i - 1)(n + m + i - j - 1)!}{T_{s}^{m+1}(n + m - 1)!} \end{pmatrix}.$$ (19)
yields the system
\[ w_1 = \mu(t)z_1(t), \]
\[ w_q(t) = dw_{q-1}(t)/dt, \quad q = 2, \ldots, n + 1. \]

**Lemma 4 (Inverse Transformation):** Given the transformation \( x(t) \rightarrow w(t) \) given by \( w(t) = \mu^{m+1}P(\mu_1)x \), in (32), the inverse transformation \( w(t) \rightarrow z(t) \) is given by
\[ z = \sigma^{m+1}(t)Q(\sigma)w, \]
where \( \sigma(t) = \frac{T}{T-t} \) is a monotonically decreasing linear function with the properties that \( \sigma(0) = 1 \) and \( \sigma(T) = 0 \), which means, in particular, that regulation is achieved in specified time \( T \), the inverse matrix \( Q(v) = P(\mu_1)^{-1} \) is a lower triangular matrix having elements \( q_{ij} \) given by
\[ q_{ij}(v) = \bar{q}_{ij}(v^{n+j-i}), \quad 1 \leq j \leq i \leq n, \]
\[ \bar{q}_{ij} = \frac{1}{i!}\left(-1\right)^{i-j}(n+m)!. \]

Furthermore, \( \bar{d} = \sup_{v \in [0,1]}|Q(\sigma)| \).

Denote
\[ r_1 = [w_1, \ldots, w_{n-1}]^T = J_1w \in \mathcal{R}^{n-1}, \]
\[ r_2 = \dot{r}_1 = [w_2, \ldots, w_n]^T = J_2w \in \mathcal{R}^{n-1}. \]
where
\[ J_1 = \left[I_{n-1}, 0_{(n-1)\times 1}\right], \quad J_2 = \left[0_{(n-1)\times 1}, I_{n-1}\right]. \]
and
\[ K_{n-1} = \left[k_1, \ldots, k_{n-1}\right]^T \in \mathcal{R}^{n-1}, \]
where \( k_{n-1} \) an appropriately chosen coefficient vector so that the polynomial \( s^{n-1} + k_{n-1}s^{n-2} + \cdots + k_1 \) and the matrix
\[ \Lambda = \begin{bmatrix} 0 & I_{n-2} \\ -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix} \]
am both Hurwitz. Now we replace the state \( w_n \) by the new variable \( z \) as
\[ z = w_n + K_{n-1}r_1. \]
This then results in
\[ \dot{r}_1 = \Lambda r_1 + e_{n-1}z, \]
where \( e_{n-1} = [0, \ldots, 0, 1]^T \in \mathcal{R}^{n-1}. \) Before proceeding, we note that the linear system (31) is ISS.

The derivative of the new state (30) is
\[ \dot{z} = \dot{w}_n + K_{n-1}^TJ_2w, \]
which, by substitution of \( \dot{w}_n = w_{n+1}, \dot{x}_n = x_{n+1}, \) and then (A.1), and writing out the \( k = 0 \) term from the sum, yields
\[ \dot{z} = \mu(\dot{x}_n + L_0 + L_1) = \mu(bu + f + L_0 + L_1), \]
with
\[ L_0 = \sum_{k=1}^{n} \frac{(n)!}{(k)!} \mu^{(k)}x_{n+1-k}, \quad L_1 = \sigma^{n+m}(t)K_{n-1}^TJ_2w. \]
In the following lemma, the quantity \( L_0 \) is expressed in terms of \( w \).

**Lemma 5 (Rewriting \( L_0 \)):** The quantity \( L_0 \) is expressed as
\[ L_0 = \sigma^{m}l_0(\sigma)w, \]
where \( l_0(v) = [l_{0,1}, l_{0,2}, \ldots, l_{0,n}] \), and for \( j = 1, 2, \ldots, n, \)
\[ l_{0,j}(v) = \tilde{l}_{0,j}\sigma^{-1}(t), \]
with
\[ \tilde{l}_{0,j} = \frac{n+m+n-j}{T^{n+1-j}} \sum_{i=0}^{n-j} \left( \frac{n(i+j-1)}{i!} \right) \left( \frac{(-1)^j(2n-m-i-j)!}{(n+m-j-1)!} \right). \]
Furthermore, \( l_0(\sigma) \) is bounded.

**Lemma 6:** The system (16) with the controller
\[ \tau = -\frac{1}{g(z,t)}[f(z,t) + L_0 + L_1 + kz], \]
has a globally fixed-time asymptotically stable equilibrium at the origin, with a prescribed convergence time \( T \), and there exist \( \tilde{M}, \tilde{d} > 0 \) such that
\[ |z(t)| \leq \sigma^{m+1}(t)\tilde{M}e^{-\tilde{d}(t)}|z_0|, \]
for all \( t \in [0,T) \). Furthermore, the control \( \tau \) remains bounded over \( [0,T) \) and, if \( f(z,t) \) is vanishing at \( z = 0, \) \( \tau \) also converges to zero as \( t \rightarrow T \).

**B. FIXED-TIME STABILIZATION**

Supported by the above theories, we discuss the USV. Define \( \Lambda, e_{3,j} \) and \( e_{6,i} \) as \( \Lambda = -\frac{K}{x_{3,j}} \Lambda, \quad \Lambda(0) = \Lambda_0 > 0 \) satisfying \( \Lambda_0 = \sqrt{\frac{T^2}{(T_2-T_1)^K-1}}, e_{3,j} = \vartheta_{3,j} - \alpha_{3,j}, \) and \( e_{6,j} = \vartheta_{6,j} - \alpha_{6,j}, e_{5,j} = \vartheta_{5,j} - \alpha_{5,j}, \) where \( 0 < T_1 < T_2 \) are positive constants, and \( K > 3 \) is an integer. In the backstepping design process, \( \alpha_{3,j} \) is the auxiliary function, \( \alpha_{5,i} \) and \( \alpha_{6,j} \) are virtual inputs,

\[ \alpha_{3,j} = \begin{cases} \Lambda, & t < T_2, \\ 0, & t \geq T_2, \end{cases} \]
\[ \alpha_{6,j} = \begin{cases} -\frac{K}{T_2-t} \Lambda, & t < T_2, \\ 0, & t \geq T_2, \end{cases} \]
\[ \alpha_{5,j} = \begin{cases} \vartheta_{5,j} - \vartheta_{2,j} \Lambda_0, & t < T_1, \\ 0, & t \geq T_1, \end{cases} \]
Then, we suggest a back-stepping approach to stabilize this triangular form to the origin.

**Theorem 1:** Consider the system (12), if states \( e_{3,j}, e_{5,j} \) and \( e_{6,j} \) converge to zero before time \( T_0 \), the following properties hold.

(i) States \( \vartheta_{2,j}, \vartheta_{3,j}, \vartheta_{5,j}, \vartheta_{6,j} \) converge to zero before the fixed-time \( T_2 \) where \( T_0 < T_1 \).
(ii). Virtual inputs $\omega_{5,j}$ and $\omega_{6,j}$ are $C^2$ smooth.

Proof: If control laws $\tau_1$ and $\tau_2$ ensure that $e_{3,j}$, $e_{5,j}$ and $e_{6,j}$ converge to zero before time $T_1$, then we can derive that

$$\omega_{3,j} = \begin{cases} \frac{(T_2 - t)^K}{T_2^K} \Lambda_0, & t < T_2, \\ 0, & t \geq T_2, \end{cases}$$

$$\omega_{6,j} = \begin{cases} -K \frac{(T_2 - t)^{K-1}}{T_2^K} \Lambda_0, & t < T_2, \\ 0, & t \geq T_2. \end{cases}$$

Due to $K > 1$, $\omega_{3,j}$ and $\omega_{6,j}$ are continuous at the point $t = T_1$. If the control law $\omega_{2,j}$ ensures that $e_{3,j}$ and $e_{6,j}$ can converge to zero before time $T_2 < T_1$, then in the time interval $[T_0, T_2]$, we can have that

$$\dot{\omega}_{3,j} = \frac{(T_2 - t)^K}{T_2^K} \Lambda_0, \quad \dot{\omega}_{6,j} = -K \frac{(T_2 - t)^{K-1}}{T_2^K} \Lambda_0.$$ (44)

According to the dynamics of subsystem $[\dot{\varphi}_{2,j}, \dot{\varphi}_{3,j}, \dot{\varphi}_{5,j}, \dot{\varphi}_{6,j}]$, the time derivative of $\varphi_{2,j}$ can be stated as

$$\dot{\varphi}_{2,j} = -K \frac{(T_2 - t)^{K-1}}{T_2^K} \Lambda_0 \dot{\varphi}_{5,j}.$$ (45)

If the control law $\varphi_{1,j}$ ensures $e_{5,j}$ can converge to zero before time $T_0$, in interval $[T_0, +\infty]$, we can have

$$\dot{\varphi}_{5,j} = \omega_{5,j} = \begin{cases} \frac{\dot{\varphi}_{2,j}}{T_1 - t} \Lambda_0, & t < T_1, \\ 0, & t \geq T_1. \end{cases}$$ (46)

This means that

$$\dot{\varphi}_{2,j} = \begin{cases} -K \frac{(T_2 - t)^{K-1}}{T_2^K} \Lambda_0^2 \frac{\dot{\varphi}_{2,j}}{T_1 - t}, & t < T_1, \\ 0, & t \geq T_1. \end{cases}$$ (47)

In time interval $[0, T_1]$,

$$\dot{\varphi}_{2,j} = -K \frac{(T_2 - t)^{K-1}}{T_2^K} \Lambda_0^2 \frac{\dot{\varphi}_{2,j}}{T_1 - t}.$$ (48)

According to Binomial Theorem, we can have $\forall \ t \in [0, T_1]$,

$$\dot{\varphi}_{2,j} = - \left[ \sum_{i=0}^{K-1} \binom{K}{k-1} \frac{(T_2 - T_1)^{K-1-i} (T_1 - t)^i}{T_1 - t} \right] \times \frac{K \Lambda_0^2}{T_2^K} \dot{\varphi}_{2,j}.$$ (49)

Therefore, after calculations,

$$\varphi_{2,j} = e^{\Lambda_1} \left( (T_2 - t) \frac{K \Lambda_0^2}{T_2^K} \varphi_{2,j}(0) \right),$$ (50)

where $\Lambda_1$ can be presented as:

$$\Lambda_1 = \frac{K \Lambda_0^2}{T_2^K} \sum_{i=1}^{K-1} \binom{K}{k-1} \frac{(T_2 - T_1)^{K-1-i} (T_1 - t)^i}{i}.$$ (51)

According to the above equations, we can have $\varphi_{2,j} (t = T_1) = 0$. Since in time interval $[T_1, +\infty]$, $\dot{\varphi}_{2,j} = 0$, we can conclude that $\forall \ t > T_1$, $\dot{\varphi}_{2,j} = 0$.

(ii). The expression of $\omega_{5,j}$ can be stated as

$$\omega_{5,j} = \begin{cases} \Lambda_0 e^{\Lambda_1} (T_2 - t) - \frac{K \Lambda_0^2}{T_2^K} (T_2 - T_1) e^{\Lambda_1} \varphi_{2,j}(0), & t < T_1, \\ 0, & t \geq T_1. \end{cases}$$ (52)

Since $\Lambda_0 = \sqrt{\frac{T_2^K}{(T_2 - T_1)^K - 1}}$, we can have

$$\omega_{5,j} = \begin{cases} \varphi_{2,j}(0) \Lambda_0 e^{\Lambda_1} (T_2 - t) - \frac{K \Lambda_0^2}{T_2^K} (T_2 - T_1) e^{\Lambda_1}, & t < T_1, \\ 0, & t \geq T_1. \end{cases}$$ (53)

$$\omega_{6,j} = \frac{-K (T_2 - t)^{K-1}}{T_2^K} \Lambda_0, \quad t < T_2,$$ (54)

Then the $(K - 2)$th order derivative of $\omega_{5,j}$ and $\omega_{6,j}$ are

$$\omega^{(K-2)}_{5,j} = (-1)^{K-2} \frac{T_2^K}{T_2^K} \varphi_{2,j}(0) \Lambda_0 \prod_{i=0}^{K-1} \frac{K}{K - i},$$ (55)

$$\omega^{(K-2)}_{6,j} = P \left( -1 \frac{T_2^K}{T_2^K} \varphi_{2,j}(0) \Lambda_0 \prod_{i=0}^{K-1} (K - i), \right)$$ (56)

where $\Lambda_1 = \frac{K \Lambda_0^2}{T_2^K} \sum_{i=1}^{K-1} \binom{K}{i} (T_2 - T_1)^{K-1-i} (T_1 - t)^i$. Hence $\omega_{5,j}$ and $\omega_{6,j}$ are $K - 2$ smooth. The proof is completed.

Remark 2: Compared with conventional time-varying functions, instead of terms $\sin(t)$ or $\cos(t)$, we introduce the scale function $\frac{1}{t^2}$ to provide the excitation conditions. So that the subsystem (12) can be stabilized to the origin before the fixed-time $T_2$. Note that different from previous fixed-time approaches, the proposed prescribed-time control laws contain no Sign terms. Hence the chattering can be decreased.

Remark 3: The parameter $K$ is to increase the smoothness of $\omega_{5,j}$ and $\omega_{6,j}$, which can affect the smoothness of control laws. Since we need control laws are $C_1$ smooth, the parameter $K$ requires to be equal or greater than 3.

Remark 4: Since the existence of nonlinear term $\dot{\varphi}_{2,j} = \omega_{5,j} \omega_{6,j}$ and the requirement that all states converge to zero, the system cannot be linearized and stabilized by prescribed-time methods for linear systems. Hence, this paper introduces different scale functions for the virtual inputs $\omega_{5,j}$ and $\omega_{6,j}$, to decouple the nonlinear term by designing terminal times respectively.

According to Theorem 1, based on the back-stepping method, virtual inputs $\omega_{5,j}$ and $\omega_{6,j}$ can stabilize the subsystem $[\dot{\varphi}_{2,j}, \dot{\varphi}_{3,j}]$ in the prescribed-time and ensure that $\alpha_{5,j}$ and $\alpha_{6,j}$ are $K = 2$ smooth. Next, we design the actual control inputs $\tau_{1,j}$ and $\tau_{2,j}$ such that $\dot{\varphi}_{2,j}$, $\dot{\varphi}_{3,j}$, $e_{3,j}$, $e_{5,j}$ and $e_{6,j}$ converge to zero in the prescribed-time. The dynamics of
\([e_3, e_5, e_6]\) can be stated as:

\[
\begin{align*}
\dot{e}_{3,j} &= e_{6,j}, \\
\dot{e}_{5,j} &= \dot{e}_{3,j} - \dot{\alpha}_{6,j}, \tag{57a}
\end{align*}
\]

where

\[
\begin{align*}
\dot{\alpha}_{5,j} &= \begin{cases} \\
\frac{\partial \phi_{3,j} \partial \lambda_0}{T_1 - t} - \frac{\partial \phi_{3,j} \lambda_0}{(T_1 - t)^2}, & t < T_1, \\
0, & t \geq T_1,
\end{cases} \\
\dot{\alpha}_{6,j} &= \begin{cases} \\
K(K - 1)\frac{1}{T_2} \lambda_0, & t < T_2, \\
0, & t \geq T_2,
\end{cases}
\tag{58, 59}
\end{align*}
\]

With Lemma 6, we are now in a position to present the prescribed-time control scheme

\[
\begin{align*}
\sigma_{1,j} &= \dot{\alpha}_{5,j} - \alpha_{a,j}, \\
\sigma_{2,j} &= \dot{\alpha}_{6,j} - \alpha_{b,j}.
\end{align*}
\tag{60, 61}
\]

Here \(\alpha_{a,j}\) and \(\sigma_{b,j}\) can be presented as:

\[
\begin{align*}
\sigma_{a,j} &= \begin{cases} \\
\frac{m + 1}{T_0 - t} e_{5,j} - \frac{T_0}{T_0 - t} e_{6,j}, & t < T_0, \\
\frac{(m + 3)(m + 2)}{T_0 - t} e_3 + \frac{T_0 - t}{(T_0 - t)^2} e_{3,j}, & t \geq T_0,
\end{cases} \\
\sigma_{b,j} &= \begin{cases} \\
\frac{(m + 1)}{(T_0 - t)^m + 1} e_{5,j}, & t < T_0, \\
\frac{(m + 2)}{(T_0 - t)^m + 1} e_{5,j} - \frac{T_0 - t}{(T_0 - t)^m + 1} e_{3,j}, & t \geq T_0.
\end{cases}
\end{align*}
\]

Theorem 2: Consider the system (12), control laws \(\sigma_{1,j}\) and \(\sigma_{2,j}\) in (60) and (61) ensure following properties hold.

(i). States \(\dot{\theta}_{3,j}, \dot{\theta}_{3,j}, \dot{\theta}_{5,j}, \dot{\theta}_{6,j}\) converge to zero before the fixed-time \(T_2\) where \(T_0 < T_1\).

(ii). Inputs \(\sigma_{1,j}\) and \(\sigma_{2,j}\) are \(C_1\) smooth.

Proof: (i). According to dynamics (57a-57b) and control laws (60-61), we can have that \(\forall t < T_0\),

\[
\begin{align*}
\dot{e}_{3,j} &= e_{6,j}, \\
\dot{e}_{5,j} &= -\frac{m + 1}{T_0 - t} e_{5,j} - \frac{\kappa T_0^{m+1}}{(T_0 - t)^{m+1}} e_{5,j}, \\
\dot{e}_{6,j} &= \frac{(m + 3)(m + 2)}{T_0 - t} e_3 - \frac{2(m + 2)e_6}{T_0 - t} - \frac{K_1}{T_0 - t} e_{3,j} + \frac{\kappa K_1 T_0^{m+2}}{(T_0 - t)^{m+2}} e_{3,j} - \frac{(m + 2)T_0^{m+2}}{(T_0 - t)^{m+2}} e_{3,j},
\end{align*}
\]

By results in Lemmas 3-6, this implies that \(e_{3,j}, e_{5,j}, e_{6,j}\) can be converged to zero in fixed-time \(T_0\).

(ii). The time derivative of \(\sigma_{a,j}\) and \(\sigma_{b,j}\) can be presented as:

\[
\begin{align*}
\dot{\sigma}_{a,j} &= \begin{cases} \\
-\frac{m + 1}{T_0 - t} e_{5,j} - \frac{T_0}{T_0 - t} e_{6,j}, & t < T_0, \\
\frac{(m + 1)}{(T_0 - t)^m + 1} e_{5,j}, & t \geq T_0,
\end{cases} \\
\dot{\sigma}_{b,j} &= \begin{cases} \\
\frac{(m + 1)}{(T_0 - t)^m + 1} e_{5,j} - \frac{T_0 - t}{(T_0 - t)^m + 1} e_{3,j}, & t < T_0, \\
\frac{(m + 2)}{(T_0 - t)^m + 1} e_{5,j} - \frac{(m + 2)T_0^{m+2}}{(T_0 - t)^{m+2}} e_{3,j}, & t \geq T_0.
\end{cases}
\end{align*}
\]

where \(\Delta_0\) can be present as

\[
\Delta_0 = \begin{cases} \\
\frac{(m + 3)(m + 2)}{T_0 - t} e_3 - \frac{T_0 - t}{(T_0 - t)^2} e_{3,j} + \frac{K_1}{T_0 - t} e_{3,j} + e_{6,j}, & t < T_0, \\
\frac{(m + 2)K_1 T_0^{m+2}}{(T_0 - t)^m + 1} e_{3,j} - \frac{(m + 2)T_0^{m+2}}{(T_0 - t)^m + 1} e_{3,j}, & t \geq T_0.
\end{cases}
\]

By Lemma 6, for all \(t \in [0, T_0]\), states \(e_{3,j}, e_{5,j}, e_{6,j}\) satisfy that

\[
|e_{k,j}(t)| \leq \sigma_{m+1}(t) \tilde{M} e^{-\tilde{M}(t)} |e_{k,j}(0)|, \quad k = 3, 5, 6.
\tag{63}
\]

Therefore, we can have that

\[
\lim_{t \to T_0^+} \sigma_{a,j} = \sigma_{b,j} = 0.
\tag{64}
\]

According to the above equations, we can have that \(\dot{\sigma}_{a,j}\) and \(\dot{\sigma}_{b,j}\) are continuous implying that \(\sigma_{a,j}\) and \(\sigma_{b,j}\) are \(C_1\) smooth. Additionally, by Theorem 1, we know \(\alpha_{a,j}\) and \(\alpha_{b,j}\) are \(C^2\) smooth, meaning \(\dot{\alpha}_{5,j}\) and \(\dot{\alpha}_{6,j}\) are \(C_1\) smooth. Hence inputs \(\sigma_{1,j}\) and \(\sigma_{2,j}\) are \(C_1\) smooth.

Remark 5: Although switching points exist in the control process, we can ensure the \(C_1\) smoothness by designing parameters, such as \(K\). So that the derivatives on the left and right sides of the switch points are equal.

It is interesting to note that, in contrast to most existing finite-time control methods, the proposed fixed-time control scheme, as shown in (60) and (61), is built not only upon
regular (rather than fractional power) state feedback but also upon time-varying (rather than constant) gain. It is such a structural feature that renders the convergence time not only finite but also user-assignable. Also, with the time-varying gain, the proposed control avoids excessively large initial driving force as encountered in many high and constant gain-based control methods, because the initial value of the time-varying gain here can be set as small as desired, rendering the initial control effort small. It should be mentioned that the nature of the time-varying gain as involved in the control scheme, although calling for gain updating according to the given simple analytical algorithm, does not cause any technical difficulty for implementation because the computation involved in updating the gain is even simpler than those involved in updating the parameters in traditional adaptive control.

This paper emphasizes to use $C_1$ smooth time-varying scaling functions achieving exponential formation control of USVs, based on which the closed system always has certain robustness against uncertainties and disturbances. Hence, due to space limitations, we focus on designing time-varying scale functions instead of disturbance/uncertainty rejection approaches. In addition, some previous robust and adaptive methods of nonlinear systems in [24]–[26] can be adopted to solve the disturbance and uncertainty problems. Besides, the neural network technology is combined with back-stepping or adaptive methods to achieve tracking control of nonlinear systems (e.g. [27]–[29]). However, due to differences of kinematics, both kinds of nonlinear approaches can not be utilized directly here. An improved back-stepping control law based on above methods can be considered as an alternative for formation control of USVs in the future.

C. CONTROL ALGORITHM

To sum up, with the main results, we give the following algorithm for the implementation of the presented method and results in USVs.

**Corollary 1:** Consider the systems (2) and (3), if control laws $\tau_{1,j}$ and $\tau_{2,j}$ are designed as

$$\begin{align*}
\tau_{1,j} &= (d_{1,j} - d_{2,j}) u_j - d_{2,j} z_{2,j} z_6,j - d_{2,j}\sigma_{1,j}, \quad (65a) \\
\tau_{2,j} &= -(m_{1,j} - m_{2,j}) u_j v_j + d_{3,j} r_j - m_{3,j}\sigma_{2,j}, \quad (65b)
\end{align*}$$

where $j = 1, 2, 3 \cdots n$, $\sigma_{1,j}$ and $\sigma_{2,j}$ are presented in (60) and (61), then the following properties hold. (i). The formation control problem (4) is solved. (ii). Control laws $\tau_{1,j}$ and $\tau_{2,j}$ are $C_1$ smooth.

**Proof:** (i). According to input transformation (6) and Theorem 2, control laws (65a) and (1) ensure the fixed-time stability of the subsystem (12). By Lemma 2, this indicates the exponential stability of system (11) and (12). So that the formation control problem is solved based on the results of Lemma 1. (ii). By Theorem 2, we can have that $\sigma_{1,j}$ and $\sigma_{2,j}$ are $C_1$ smooth, which can be developed to inputs $\tau_{1,j}$ and $\tau_{2,j}$.

| TABLE 2. Formation control algorithm of USVs. |
|---|
| 1. Choose parameters $p_1, p_2, p_3, p_4, p_5, p_6, \tau_0, \tau_1, \tau_2, K, m$ and $\kappa$. |
| 2. Take measurements $\omega_{i,j}, b_{m,1,j}, m_{2,j}, m_{3,j}, d_{1,j}, d_{2,j}, d_{3,j}, i, j = 1, 2, 3$. |
| 3. Compute the variables $x_{1,j}$, $x_{2,j}$, $x_{3,j}$, $x_{4,j}$, $x_{5,j}$, $x_{6,j}$ as $x_{1,j} = (x_{j} - (p_{x,j} - \sum_{i=1}^{n} p_{x,i})) \cos (\psi_{j})$, $x_{2,j} = (x_{j} - (p_{x,j} - \sum_{i=1}^{n} p_{x,i})) \sin (\psi_{j})$, $x_{3,j} = \psi_{j}$, $x_{4,j} = \psi_{j}$, $x_{5,j} = \frac{m_{1,j}}{m_{2,j}} - x_{1,j}$, $x_{6,j} = r_{j}$. |
| 4. Compute the variables $\theta_{1,j}, \theta_{2,j}, \theta_{3,j}, \theta_{4,j}, \theta_{5,j}$ as $\theta_{1,j} = b_{1,1,j} - \sum_{i=1}^{n} a_{1,j} x_{1,i} - \sum_{i=1}^{n} a_{1,j} x_{2,i} - \sum_{i=1}^{n} a_{1,j} x_{3,i} - \sum_{i=1}^{n} a_{1,j} x_{4,i} - \sum_{i=1}^{n} a_{1,j} x_{5,i} - \sum_{i=1}^{n} a_{1,j} x_{6,i}$, $\theta_{2,j} = b_{1,2,j} - \sum_{i=1}^{n} a_{2,j} x_{1,i} - \sum_{i=1}^{n} a_{2,j} x_{2,i} - \sum_{i=1}^{n} a_{2,j} x_{3,i} - \sum_{i=1}^{n} a_{2,j} x_{4,i} - \sum_{i=1}^{n} a_{2,j} x_{5,i} - \sum_{i=1}^{n} a_{2,j} x_{6,i}$, $\theta_{3,j} = \theta_{1,j} - \sum_{i=1}^{n} a_{3,j} x_{1,i} - \sum_{i=1}^{n} a_{3,j} x_{2,i} - \sum_{i=1}^{n} a_{3,j} x_{3,i} - \sum_{i=1}^{n} a_{3,j} x_{4,i} - \sum_{i=1}^{n} a_{3,j} x_{5,i} - \sum_{i=1}^{n} a_{3,j} x_{6,i}$, $\theta_{4,j} = \theta_{2,j} - \sum_{i=1}^{n} a_{4,j} x_{1,i} - \sum_{i=1}^{n} a_{4,j} x_{2,i} - \sum_{i=1}^{n} a_{4,j} x_{3,i} - \sum_{i=1}^{n} a_{4,j} x_{4,i} - \sum_{i=1}^{n} a_{4,j} x_{5,i} - \sum_{i=1}^{n} a_{4,j} x_{6,i}$. |
| 5. Design $\tau_{1,j}$ and $\tau_{2,j}$ as (65a) and (65b). |

To help readers grasp the essence of our formation control methodology, the structure of proposed algorithm can be described as five steps: 1). control parameter selection; 2). model parameter measurement; 3). differential homeomorphism transformation; 4). virtual controller design; 5). real controller design. Details can be seen in Table 2 and Figure 2.

**Remark 6:** Generally, time-varying control laws cannot guarantee the exponential stability of USVs. To this end, some special terms such as $e^{\kappa t}$ are used [3]. Differently, this paper solves this problem by fixed-time stabilize the...
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**FIGURE 4.** Paths and orientations of USVs with desired formation.

**FIGURE 5.** Responses of control inputs.

**TABLE 3.** Cases of initial states.

| Case | \( \psi_d \) | \( x_1(0) \) | \( x_2(0) \) | \( x_3(0) \) | \( y_1(0) \) | \( y_2(0) \) | \( y_3(0) \) | \( \psi_1(0) \) | \( \psi_2(0) \) | \( \psi_3(0) \) |
|------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1    | 0 rad       | -2 m       | -2 m        | 0 m         | -2 m        | -2 m        | 4 m         | 0 rad       | 0 rad       | 0 rad       |
| 2    | \( \frac{\pi}{2} \) | 4 m        | 8 m        | 0 m         | 1 m         | -1 m        | 1 m         | 0.5 rad     | 0.7 rad     | -0.1 rad    |

subsystem (12). Compared to the previous fixed/finite-time approaches [15], [16], the presented control laws are \( C_1 \) smooth.

For the explosion of terms in back-stepping method, some estimation approaches have been utilized, such as the command filtered technology. However, since it always causes filter errors in the closed system and our control laws have no or few explosion of terms, the command filtered back-stepping technology is not used here.

**V. SIMULATION**

**A. FORMATION STABILITY**

In this section, the effectiveness of the proposed control laws is shown by simulation. Consider three identical underactuated surface vehicles with the model parameters: \( m_{1,j} = 25.8 \), \( m_{2,j} = 33.8 \), \( m_{3,j} = 2.76 \), \( d_{1,j} = 12 \), \( d_{2,j} = 17 \), and \( d_{3,j} = 0.5 \) for \( j = 1, 2, 3 \). To illustrate the formation stability of USVs by our method. Different initial conditions of three systems are considered as two cases, which are presented in Table 3. The desired geometric pattern is a triangle defined by \[ [p_{x,1}, p_{y,1}] = [-1, -1], [p_{x,2}, p_{y,2}] = [1, -1], [p_{x,3}, p_{y,3}] = [0, 2] \]. Choose the control parameters as \( T_0 = 0.5, T_1 = 1.5, T_2 = 3, K = 3, \kappa = 1 \).

Assume the communication digraph \( G \) among the surface vessels is fixed and as in Figure 3, the cooperative control laws can be obtained by Corollary 1. Without loss of generality, we consider two cases of initial conditions, whose details are stated in Table 3. Figures 4(a) and 4(b) show the path of each vehicle, which converge to the desired geometric pattern. The simulation also shows that in both two cases, each \([\psi_j, u_j, v_j, r_j]\) converges to a constant vector, where velocities \( u_j, v_j \) and \( r_j \) converge to zero.
FIGURE 6. Comparisons of states $\varphi_{2j}$ under Case 1.

FIGURE 7. Comparisons of states $\varphi_{2j}$ under Case 2.

FIGURE 8. Comparisons of inputs $\omega_1$.

In Figures 5(a) and 5(b), responses of inputs illustrate that proposed control laws have no chattering problems.

B. COMPARISON

1) EFFECTS OF INITIAL CONDITIONS

To illustrate the effectiveness of our method under different initial conditions, we have the comparisons of states $\varphi_{2j}$ between our method and finite-time control laws of [15] in Cases 1 and 2. Note that the convergence of states $\varphi_{2j}$ mean that the stability of the transformed system (12), where $j = 1, 2, 3$.

Figures 6(a)-7(b) show that in both two cases, the state $\varphi_{2j}$ can be converged to zero before 3s by our approach, whose settling time is designed as $T_2$. However, with different
initial conditions, the finite-time control method has different settling times. This illustrates that by prescribed-time control laws, the state $\theta_{2,j}$ can be converged in the fixed-time, which is independent of initial conditions.

2) SMOOTHNESS COMPARISON

To evaluate the smoothness of our method, we consider the comparison of the input trajectories with the fixed-time control method in [16] under initial condition Case 1 in Figure 8. Obviously, due to the existence of Sign function, the conventional fixed-time control law can cause chattering. By introducing a time-varying scale function, the proposed prescribed-time control laws are $C_1$ smooth and have better smoothness on trajectories. This implies the proposed control laws relatively easy to implement in engineering due to the smoothness.

VI. CONCLUSION

In this paper, based on a global diffeomorphism transformation, by introducing time-varying scale functions, the prescribed-time control laws are proposed to stabilize the transformed system in the fixed-time. Then the control laws are proved to ensure the exponential formation stabilization of USVs.

As actuator saturation due to mechanical constraints may have significant impacts on the system transient behavior and even stability, stabilization of USVs subject to actuator saturation is still an open problem. Some intelligent control methods of nonlinear systems can.

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