Kaluza-Klein Pistons with non-Commutative Extra Dimensions

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Abstract

We calculate the scalar Casimir energy and Casimir force for a $R^3 \times N$ Kaluza-Klein piston setup in which the extra dimensional space $N$ contains a non-commutative 2-sphere, $S_{FZ}$. The cases to be studied are $T^d \times S_{FZ}$ and $S_{FZ}$ respectively as extra dimensional spaces, with $T^d$ the $d$ dimensional commutative torus. The validity of the results and the regularization that the piston setup offers are examined in both cases. Finally we examine the 1-loop corrected Casimir energy for one piston chamber, due to the self interacting scalar field in the non-commutative geometry. The computation is done within some approximations. We compare this case for the same calculation done in Minkowski spacetime $M^D$. A discussion on the stabilization of the extra dimensional space within the piston setup follows at the end of the article.

Introduction

Casimir pistons have received much attention in recent years [2, 13, 12, 18, 28, 29, 20, 15]. This is due to the very attractive quantitative features that the piston setup has. Most of these are concentrated on the fact that although the Casimir energy of a scalar field between parallel plates can be singular, in the piston setup the Casimir force is regular. Thus we can have closed form formulas for the Casimir force.

The Casimir piston can be realized by three parallel plates in which the one in the middle can be movable. Suppose between the plates a scalar field exists. The boundary conditions on the plates can be either Dirichlet or Neumann. In most cases it was found that the movable plate is attracted to the nearest end. This holds for rectangular cavities. However when closed cylinder with arbitrary cross section is considered with Dirichlet piston and

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Neumann walls, a repulsive force was found [17]. On the contrary a three dimensional piston with all surfaces perfectly conducting is attracted to the closest wall [16].

The parallel plates Casimir force taking into account extra compact spatial dimensions has been used to put restrictions on the size of the extra dimensions [21]. Thus the calculation of Casimir force on a piston in the presence of compact extra dimensions is a very interesting task. The incorporation of additional Kaluza-Klein dimensions in pistons was made in [2, 3, 12]. In reference [2] the scalar Casimir force was computed for a three dimensional piston with arbitrary Kaluza-Klein extra dimensions. It was found that for both Neumann and Dirichlet boundary conditions, the piston is attracted to the closest wall. This statement was true for arbitrary cross section of the piston and also for arbitrary geometry and topology of the extra compact dimensions. This is a very serious argument. As in most cases the scalar Casimir energy was singular, but due to the piston setup the Casimir force was finite and found in closed form using zeta regularization techniques [24, 25].

The argument that the scalar Casimir force for the piston is independent on the topology and the geometry of the compact extra dimensions is very serious as we said. This holds for compact Riemann spaces with arbitrary curvature. In this article we shall check this argument in the case where the extra dimensional manifold consists of non-commuting extra dimensions.

The study for non-commuting extra dimensions was motivated by string theories studies of D0 branes in constant background Ramond-Ramond (RR) Chern-Simons field [4, 9, 14]. It was shown that this system leads to a non commutative 2-sphere which is known as fuzzy sphere. We shall discuss on this later on.

Beside the above reasoning for using them, non-commuting extra dimensions can solve a problem in field theories with compact extra dimensions. In most cases one must explain why extra dimensions are so small compared to the other three spatial dimensions [14]. Also explain why the extra compact dimensions did not inflate as the other three dimensions. This was first dealt by Appelquist and Chodos [30]. They found a negative Casimir energy in the bulk five dimensional model they studied. This resulted in the shrinking of the extra dimension. In addition in order the extra dimension does not collapse to the Planck scale, one must find a mechanism for radius stabilization. It is very natural to expect that radius stabilization of our field theory can easily occur if there exists a natural minimum length scale in our theory. One probable scale is the Planck scale. This is not so good news for extra dimensions experiments. However if we assume that the extra dimensions can be of Tev scale then another intrinsic length must be found. Non-commutative extra dimensions offer this possibility. Non-commutativity naturally introduces a minimum volume, that of the Moyal cell, which is proportional to the non-commutativity parameter (we shall discuss on these issues in the following sections) which has dimension of length [14]. Thus if someone considers the ordinary compact extra dimensions and additionally some extra dimensional non-commutative compact space (or just the non-commutative space), this could lead to the stabilization of the extra spaces. This is a strong motivation for using non-commutative extra dimensions [14]. The stabilization of the extra dimensional space in the case of the piston is a very interesting task but we shall not deal with it in this article but we will discuss some aspects at the end of
the article. We shall only examine the question if the scalar Casimir force for the piston is independent on the topology and the geometry of the compact extra dimensions, in the case non-commutative extra dimensions exist.

Let us mention here that the non-commutativity spoils Lorentz invariance. However having non-commutativity in compact extra spatial dimensions is not a serious opposition to observations so far [14].

In this paper we shall study the massive self interacting $\Phi^4$ scalar field theory in a piston setup. We shall assume that the spacetime has additional compact dimensions with topology $T^d \times S_{FZ}$, with $T^d$ the commutative $d$ dimensional torus and $S_{FZ}$ the 2 dimensional non-commutative sphere, or the fuzzy sphere. The total space has topology $R^3 \times T^d \times S_{FZ}$. The piston will be assumed to have two infinite dimensions and also the two plates will be at distance $L$. The piston will be $\alpha$ from the one plate and $L - \alpha$ from the other (we follow reference [2]). The distance from the plate shall be generally denoted $a$ and apply the general result that we will find for the $\alpha$ and $L - \alpha$ cases. The scalar field shall be assumed to have Dirichlet boundary conditions on the plates. Our purpose is to find the Casimir energy and Casimir force for the self interacting massive scalar field in the piston plates setup, at tree and at 1-loop level. The 1-loop level contributions will be calculated in a specific approximation which serves to obtain an analytic result.

The outline is as follows. In section 1 we shall review the necessary features of the non-commutative 2-sphere. The presentation will be enough detailed so the presentation is self contained (we follow references [14, 9, 4, 11]). In section 2 the eigenvalues of the self interacting $\Phi^4$ scalar field in the piston setup shall be given. In section 3 and 4 we compute the Casimir energy for the piston setup and also the Casimir force for $T^d \times S_{FZ}$ and $S_{FZ}$ as extra dimensional space. In section 5 we add one loop corrections to the Casimir energy. In section 6 the conclusions follow.

1 The Fuzzy Sphere, Matrix Models and D0 Branes

1.1 D0 Branes Action in the Presence of RR Field

To start we shall present the string theory application of the fuzzy sphere that strongly motivates non-commutative geometry [9]. Consider N D0 branes in a constant background RR four form field, $F^{(4)}_{tijk} = -2\lambda_N \epsilon_{tijk}$. The Born-Infeld action for the D0 branes (that is for Dirichlet branes with zero spatial dimensions) in the presence of the constant RR field (in leading order within the matrix theory approximation) is [9, 4],

$$S = T_0 \text{Tr}\left(\frac{1}{2} X_i^2 + \frac{1}{4} [X_i, X_j] [X_i, X_j] - \frac{i}{3} \lambda_N \epsilon_{ijk} X_i [X_j, X_k] \right)$$

where the last two terms are the leading order Born-Infeld potential. In relation (1) the $X_i$ are $N \times N$ matrices and $T_0$ is the zero brane tension [9]. The static equations of motion for the above action are,

$$[X_j, \left([X_i, X_j] - i \lambda_N \epsilon_{ijk} X_i [X_j, X_k]\right)] = 0.$$
The energy of the extrema is,

\[ E = -T_0 \text{Tr} \left( \frac{1}{4} [X_i, X_j] [X_i, X_j] - \frac{i}{3} \lambda_N \epsilon_{ijk} X_i [X_j, X_k] \right) \tag{3} \]

The above equation clearly admits the commuting matrices solution,

\[ [X_i, X_j] = 0 \tag{4} \]

with energy \( E = 0 \). A non-commuting ansatz solution for equation (2) is given by the \( N \times N \) matrix representation of \( SU(2) \) algebra,

\[ [X_i, X_j] = i \lambda_N \epsilon_{ijk} X_k \tag{5} \]

with,

\[ X_i = \lambda N J_i, \tag{6} \]

The matrices \( J_i \) define the \( N \) dimensional \( SU(2) \) representation we mentioned previously, and are labelled by the total spin \( N/2 \). The non-commutativity parameter is of dimension length and can be taken positive. Within the D0 interaction context, the solution minimizing equation (2) will be the irreducible \( N \times N \) \( SU(2) \) representation given by the non-commuting matrices \( X_i \), satisfying equation (5). As we will describe shortly, relation (5) suggests the fuzzy sphere geometry. Indeed the matrices satisfy,

\[ X_1^2 + X_2^2 + X_3^2 = r_N^2. \tag{7} \]

We shall discuss on the meaning of \( r_N \) shortly. Thus we saw that the appearance of non-commutativity in string theory arises naturally within the \( N \) D0 brane action in the presence of a constant RR Chern Simons term. The ground state of this system has the fuzzy sphere geometry. Indeed the non-commutative setup we described has minimum energy compared to all other representations of the \( SU(2) \) algebra. So the D0 brane system condense into a non-commutative configuration. This can be shown to be a bound state of a spherical D2 brane and \( N \) D0 branes (for details see [9]).

### 1.2 The Fuzzy Sphere

The infinite dimensional algebra \( A_\infty \) contains all the information about the standard sphere \( S^2 \). It is generated by \( x = (x_1, x_2, x_3) \in R^3 \), with,

\[ [x_i, x_j] = 0, \tag{8} \]

and

\[ \sum_{i=1}^{3} x_i^2 = \rho^2 \tag{9} \]

The non-commutative sphere is connected with the non-commutative algebra \( A_N \) [4 11]. This algebra is generated by \( x = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \) with,

\[ [\hat{x}_i, \hat{x}_j] = i \lambda_N \epsilon_{ijk} \hat{x}_k, \tag{10} \]
and as before,
\[ \sum_{i=1}^{3} \hat{x}_i^2 = \rho^2 \]  
(11)

The parameter \( \lambda_N \) characterizes the non-commutativity as we seen before. Writing the above in terms of \( \hat{J}_i = \frac{1}{\lambda_N} \hat{x}_i \),
\[ [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk} \hat{J}_k. \]  
(12)

Also the Casimir operator \( C \) of the group is,
\[ C = \sum_{i=1}^{3} \hat{J}_i^2 = \rho^2 \lambda_N^{-2} \]  
(13)

For the case in interest (that is for the \( N \times N \) irreducible representation of the \( SU(2) \) algebra), the Casimir operator takes the value,
\[ C = \frac{N}{2} \left( \frac{N}{2} + 1 \right). \]  
(14)

Thus the \( \lambda \) and \( N \) are related by,
\[ \rho \lambda_N^{-1} = \sqrt{\frac{N}{2} \left( \frac{N}{2} + 1 \right)}. \]  
(15)

We saw earlier the parameter \( r_N \). Now for the greater irreducible representation, this parameter is equal to,
\[ r_N \lambda_N^{-1} = \sqrt{\frac{N}{2} \left( \frac{N}{2} + 1 \right)}. \]  
(16)

We can see that the radius of the non-commutative 2-sphere is finite and depends on the representation of the \( SU(2) \) algebra through the parameter \( N \). Also it depends on the non-commutativity parameter \( \lambda_N \). Thus the radius of the fuzzy sphere is fixed and quantized in terms of the non-commutativity parameter.

Finally the Laplacian on the non-commutative sphere is given by \( \Delta = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 \). We shall use this in the following when we obtain the eigenvalues of the total space.

### 2 Eigenvalues and eigenfunctions

Consider the scalar field \( \Phi \) in space topology \( R^3 \times T^d \times S_{FZ} \), where as we seen \( S_{FZ} \) is the fuzzy sphere. In this section we compute the eigenvalues at tree level (that is Kaluza-Klein spectrum). The scalar field is supposed to obey Dirichlet boundary conditions on the piston boundaries, that is,
\[ \Phi(x^{D-1}, 0, y) = \Phi(x^{D-1}, \alpha, y) = \Phi(x^{D-1}, L, y) = 0. \]  
(17)

In the above equation, \( D \) is the total dimensionality of the non-compact dimensions (in the end \( D = 4 \)), \( x \) denotes the non-compact dimensions and \( y \) denotes the \( d \) dimensional
torus coordinates. We suppose the torus has equal radii for all the dimensions. The non-commutative sphere is described by the non-commutative coordinates $X = (X_1, X_2, X_3)$ of relation (5) which are the $N$ dimensional irreducible representation of the $SU(2)$ algebra. The radius $r_N$ is fixed as we seen before, see relation (16). The action for the scalar field reads,

$$S = \int d^D x \int d^d y \int_{T^r} \left[ \frac{1}{2} \Phi(\Box_x + \Box_y + \Delta + m^2) \Phi + \frac{g}{4!} \Phi^4 \right].$$ (18)

In the above $\Delta = \sum J_i^2$ is the Laplacian, where $J_i$ as in relation (12). Also the integral $\int_{T^r}$ is defined as an integration over the fuzzy sphere as follows,

$$\int_{T^r} F(X) = \frac{4\pi}{N+1} \text{Tr}[F(X)]$$ (19)

where $X$ is defined before. The scalar field $\Phi$ has the following harmonic expansion,

$$\Phi(x, y, X_i) = \int \frac{d^{D-1} p}{(2\pi)^{D-1}} \sum_{n=1}^{\infty} \sum_{\tilde{n}} \sum_{J, m} b_n^J(\tilde{p}, \tilde{n}) e^{i\tilde{p} \cdot \tilde{n}} e^{i2\pi \tilde{n} \cdot \tilde{y}} Y^J_m \sin\left(\frac{n\pi}{a}\right).$$ (20)

In the above $\tilde{n} = (n_1, n_2, ..., n_d)$ and $Y^J_m$ the usual spherical harmonics. The eigenfrequencies of the scalar field in the cavity at tree order are given by,

$$(w^{J, n}_{\tilde{p}, \tilde{n}})^2 = \tilde{p}^2 + (n\pi/a)^2 + (\tilde{n}/R)^2 + J(J + 1) + m^2.$$ (21)

Also in the case of the fuzzy sphere as the extra dimensional space the eigenvalues are,

$$(w^{J, n}_{\tilde{p}})^2 = \tilde{p}^2 + (n\pi/a)^2 + J(J + 1) + m^2.$$ (22)

### 3 Casimir piston for $T^d \times S^F_Z$ extra dimensional space

Let us calculate the Casimir energy and Casimir force for the piston setup in the case the extra dimensional is the $d$-dimensional commutative torus and the fuzzy sphere. We shall do the calculations for general $s$ and $D$. This will give us the opportunity to have a general form of the results for all space dimensions. In the end we have in mind that $s = -\frac{1}{2}$ and $D = 3$. The Casimir energy reads,

$$E_c(s, a) =$$

$$\frac{1}{4\pi^2} \int d^{D-1} p \sum_{J=0}^{N} \sum_{n=1}^{\infty} \sum_{n_1, n_2, ..., n_d = -\infty}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + (\frac{n\pi}{a})^2 + \frac{d}{R} \right]^s.$$ (23)

Upon integrating over the continuous dimensions using,

$$\int dk^{D-1} \frac{1}{(k^2 + A)^s} = \pi^\frac{D-1}{2} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \frac{1}{A^{s - \frac{D-1}{2}}}.$$ (24)
relation (23),

\[ E_c(s, a) = \frac{1}{4\pi^2} \frac{\pi^{D-1}}{2} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \times \]

\[ \sum_{J=0}^{N} \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty} \left[ \left( \frac{n\pi}{a} \right)^2 + \sum_{k=1}^{d} \left( \frac{n_k}{R} \right)^2 + J(J+1) + m^2 \right]^{\frac{D-1}{2} - s}. \]  

Using the identity,

\[ \sum_{n=1}^{\infty} f(n) = \frac{1}{2} \left( \sum_{n=-\infty}^{\infty} f(n) - f(0) \right) \]  

which holds for an even function of \( n \), which is our case. Thus relation (25) can be written,

\[ E_c(s, a) = \frac{1}{8\pi^2} \frac{\pi^{D-1}}{2} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \times \]

\[ \left( \sum_{J=0}^{N} \sum_{n=-\infty}^{\infty} \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty} \left[ \left( \frac{n\pi}{a} \right)^2 + \sum_{k=1}^{d} \left( \frac{n_k}{R} \right)^2 + J(J+1) + m^2 \right]^{\frac{D-1}{2} - s} \right) \]

\[ - \sum_{J=0}^{N} \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty} \left[ \left( \sum_{k=1}^{d} \left( \frac{n_k}{R} \right)^2 + J(J+1) + m^2 \right]^{\frac{D-1}{2} - s} \right) \]

The above relation can be written in terms of the inhomogeneous Epstein zeta function [24, 25, 26, 27],

\[ Z_{d}^{v_2}(s; w_1, \ldots, w_d) = \sum_{n_1 \ldots n_N = -\infty}^{\infty} \left[ w_1 n_1^2 + \ldots + w_d n_d^2 + v^2 \right]^{-s}. \]  

as follows,

\[ E_c(s, a) = \frac{1}{8\pi^2} \frac{\pi^{D-1}}{2} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \left[ \sum_{J=0}^{N} Z_{d+1}^{w_2}(s - \frac{D-1}{2}; \frac{\pi^2}{a^2}, \frac{1}{R^2}, \ldots, \frac{1}{R^2}) \right] \]

\[ - \sum_{J=0}^{N} Z_{d+1}^{w_2}(s - \frac{D-1}{2}; \frac{1}{R^2}, \ldots, \frac{1}{R^2}) \]

with \( w_2^2 = J(J+1) + m^2 \). Upon using the following expansion for the inhomogeneous Epstein zeta [24, 25, 26, 27],

\[ Z_{d}^{w_2}(s; a_1, \ldots, a_d) = \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} w^{d-2s} \]

\[ + \frac{2\pi^s w_2^{d-s}}{\Gamma(s)\sqrt{a_1 a_2 \ldots a_d}} \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty} \left[ \sum_{k=1}^{d} \left( \frac{n_k}{a_k} \right)^2 \right]^{\frac{1}{2}(s - \frac{d}{2})} \times \]

\[ K_{d-2s}^{\frac{d}{2}}(2\pi w \left[ \sum_{k=1}^{d} \frac{n_k^2}{a_k} \right]^{\frac{1}{2}}) \]
the Casimir energy reads,

\[ E_c(s, a) = \frac{1}{8\pi^2} \pi^{\frac{D-1}{2}} \Gamma\left(s - \frac{D-1}{2}\right) \times \]

\[ \sum_{J=0}^{N} \left\{ \frac{R^d a \pi^d \Gamma\left(s - \frac{D-1}{2} - \frac{d+1}{2}\right)}{\Gamma\left(s - \frac{D-1}{2}\right)} \left(\sqrt{J(J+1) + m^2}\right)^{d+1-2\left(s - \frac{D-1}{2}\right)} \right. \]

\[ + \frac{2\pi^{s - \frac{D-1}{2}} a \left(\sqrt{J(J+1) + m^2}\right)^{\frac{d+1}{2} - \left(s - \frac{D-1}{2}\right)}}{\Gamma\left(s - \frac{D-1}{2}\right)} \times \]

\[ \sum_{n, n_1, n_2, ..., n_d = -\infty}^{\infty} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left(\frac{n a}{\pi}\right)^2 \right]^{\frac{1}{2} - \left(s - \frac{D-1}{2}\right)} \times \]

\[ K_{\frac{d+1}{2} - \left(s - \frac{D-1}{2}\right)} \left(2\pi \sqrt{J(J+1) + m^2} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left(\frac{n a}{\pi}\right)^2 \right]^{\frac{1}{2}} \right) \]

\[ - \frac{R^d a \pi^d \Gamma\left(s - \frac{D-1}{2} - \frac{d}{2}\right)}{\Gamma\left(s - \frac{D-1}{2}\right)} \left(\sqrt{J(J+1) + m^2}\right)^{d-2\left(s - \frac{D-1}{2}\right)} \]

\[ - \frac{2\pi^{s - \frac{D-1}{2}} a \left(\sqrt{J(J+1) + m^2}\right)^{\frac{d}{2} - \left(s - \frac{D-1}{2}\right)}}{\Gamma\left(s - \frac{D-1}{2}\right)} \times \sum_{n_1, n_2, ..., n_d = -\infty}^{\infty} \left[ \sum_{k=1}^{d} (n_k R)^2 \right]^{\frac{1}{2} - \left(s - \frac{D-1}{2} - \frac{d}{2}\right)} \times \]

\[ K_{\frac{d}{2} - \left(s - \frac{D-1}{2}\right)} \left(2\pi \sqrt{J(J+1) + m^2} \left[ \sum_{k=1}^{d} (n_k R)^2 \right]^{\frac{1}{2}} \right) \}

It is obvious that the above expression contains singularities for the case \( D = 3 \) and \( s = -\frac{1}{2} \). The anomalies are contained to the two gamma functions, namely, \( \Gamma\left(s - \frac{D-1}{2} - \frac{d+1}{2}\right) \) and \( \Gamma\left(s - \frac{D-1}{2} - \frac{d}{2}\right) \). The first is singular for any \( d = \)even value and the second is singular for any \( d = \)odd value. However the piston setup offers a nice way out from these singularities. Firstly for computing the force we shall need the derivative over \( a \), which will eliminate the last two terms of (31). Thus one singularity can be avoided. Also the other singularity can be avoided only for the piston setup, due to linearity to \( a \). Taking the derivative over \( a \) and adding the contributions from the two chambers, namely the \( a \) and the \( L - a \), the singularity is cancelled. Thus the total Casimir energy for the piston configuration is regular. This is similar to the case studied in [22]. The reason behind this cancellation is of course the piston configuration and the independence of the 'non-homogeneity' parameter of the sums (in our case \( J(J+1) + m^2 \)) from \( a \). We shall see in a following section that when the 'non-homogeneity' parameter is \( a \) dependent, the Casimir energy and the Casimir force are singular even for the piston setup.

Adding the contributions from the two chambers, the Casimir force,

\[ F_c = -\frac{\partial E_c(s, L - \alpha)}{\partial \alpha} - \frac{\partial E_c(s, \alpha)}{\partial \alpha}, \] (32)
is equal to,

\[ F_c(s) = -\frac{1}{8\pi^2} \frac{\pi^{D-1}}{2} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \times \]

\[ \sum_{J=0}^{N} \frac{\partial}{\partial a} \left\{ \frac{2\pi^s - \frac{D-1}{2} R^d a \left( \sqrt{J(J+1) + m^2} \right)^{\frac{D+1}{2} - (s - \frac{D-1}{2})}}{\Gamma(s - \frac{D-1}{2})} \times \right. \]

\[ \left. \sum_{n,n_1,n_2,\ldots,n_d=-\infty}^{\infty} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n a}{\pi} \right)^2 \right] \frac{1}{2} (s - \frac{D-1}{2} - \frac{D+1}{2}) \times \right. \]

\[ K_{\frac{D+1}{2} - (s - \frac{D-1}{2})} \left( 2\pi \sqrt{J(J+1) + m^2} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n a}{\pi} \right)^2 \right] \right)^{\frac{1}{2}} \]

\[ + \frac{2\pi^s - \frac{D-1}{2} R^d (L-a) \left( \sqrt{J(J+1) + m^2} \right)^{\frac{D+1}{2} - (s - \frac{D-1}{2})}}{\Gamma(s - \frac{D-1}{2})} \times \]

\[ \left. \sum_{n,n_1,n_2,\ldots,n_d=-\infty}^{\infty} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n (L-a)}{\pi} \right)^2 \right] \frac{1}{2} (s - \frac{D-1}{2} - \frac{D+1}{2}) \times \right. \]

\[ K_{\frac{D+1}{2} - (s - \frac{D-1}{2})} \left( 2\pi \sqrt{J(J+1) + m^2} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n (L-a)}{\pi} \right)^2 \right] \right)^{\frac{1}{2}} \}

From the beginning we considered the non-commutativity parameter to be of order 1. This will be our fundamental length scale and we shall use this to compare length scales. We shall now compute the above Casimir force in special cases. The most interesting case is when \( a \gg R \) and \( L - a \gg R \). In this case \( a \gg 1 \) and \( R \ll 1 \). We shall take the asymptotic limit of the Bessel functions after taking the derivative of the above expressions.

Using,

\[ \frac{K_{\nu}(xz)}{z^{\nu}} = \frac{1}{2} \int_{0}^{\infty} e^{-\frac{t}{2}(x^2 + z^2)} t^{\nu+1} dt, \]

(34)

the derivative of \( \frac{K_{\nu}(xz)}{z^{\nu}} \) appearing in (33) is equal to,

\[ \frac{\partial}{\partial z} \left( \frac{K_{\nu}[x(a^2 + b)^c]}{[(a^2 + b)^c]^\nu} \right) = -xz c a (a^2 + b)^{c-1} K_{\nu-1} \frac{[x(a^2 + b)^c]}{[(a^2 + b)^c]^\nu+1} \]

(35)
Using (35) the total Casimir force for the two chambers of the piston reads,

\[
F_c(s) = -\frac{1}{8\pi^2} \frac{\Gamma(s) \Gamma(s - D - 1)}{\Gamma(s - D - 1/2)} \times \\
\sum_{j=0}^{N} \left\{ 2\pi^{s-D-1/2} R^d \left( \sqrt{J(J+1) + m^2} \right)^{d+1/2 - (s-D-1/2)} \times \\
\sum_{n,n_1,n_2,...,n_d=-\infty}^{\infty} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n a}{\pi} \right)^2 \right]^{1/2(s-D-1/2)} \times \\
K_{d+1/2-(s-D-1/2)} \left( 2\pi \sqrt{J(J+1) + m^2} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n (L-a)}{\pi} \right)^2 \right]^{1/2} \right) \\
2\pi^{s-D-1/2} R^d \left( \sqrt{J(J+1) + m^2} \right)^{d+1/2 - (s-D-1/2)} \times \\
\sum_{n,n_1,n_2,...,n_d=-\infty}^{\infty} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n (L-a)}{\pi} \right)^2 \right]^{1/2(s-D-1/2)} \times \\
K_{1+D+1/2-(s-D-1/2)} \left( 2\pi \sqrt{J(J+1) + m^2} \left[ \sum_{k=1}^{d} (n_k R)^2 + \left( \frac{n (L-a)}{\pi} \right)^2 \right]^{1/2} \right) \\
\right\}
\]

The above relation is not so "friendly". However within approximation \( a \gg R \) and \( L-a \gg R \), the argument of the Bessel functions above are large. Thus we can use the asymptotic
expansion,

\[ K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left( 1 + \frac{\nu - 1}{8z} + \ldots \right). \]  

(37)

It is obvious that the leading contributions comes when keeping only the \( n_i = 1 \) terms in the summations. So finally the Casimir force equals to,

\[
F_\nu(s) = - \sum_{J=0}^{N} \frac{1}{4 \pi s^2} R^d \frac{\left( \sqrt{J(J+1) + m^2} \right)^{d+1 - (s - D - \frac{1}{2})} \Gamma(s)}{\pi^2} \times \\
\left\{ \begin{array}{c}
\frac{1}{2} \left[ dR^2 + \frac{a^2}{\pi^2} \right] \frac{1}{2} (s - D - \frac{1}{2} - \frac{d+1}{2} - 2) e^{-2\pi(J+1)/m^2} \sqrt{dR^2 + \frac{a^2}{\pi^2}} \\
- \frac{1}{2} dR^2 + \frac{(L-a)^2}{\pi^2} \frac{1}{2} (s - D - \frac{1}{2} - \frac{d+1}{2} - 2) e^{-2\pi(J+1)/m^2} \sqrt{dR^2 + \frac{(L-a)^2}{\pi^2}} \\
- a^2 \left( \frac{d+1}{2} - (s - D - \frac{1}{2}) \right) \left( \sqrt{J(J+1) + m^2} \right) \times \\
\frac{1}{2 \pi^2} dR^2 + \frac{a^2}{\pi^2} \frac{1}{2} (s - D - \frac{1}{2} - \frac{d+1}{2} - 2) e^{-2\pi(J+1)/m^2} \sqrt{dR^2 + \frac{a^2}{\pi^2}} \\
+ (L-a)^2 \left( \frac{d+1}{2} - (s - D - \frac{1}{2}) \right) \times \\
\frac{1}{2 \pi^2} \left( \sqrt{J(J+1) + m^2} \right) \left[ dR^2 + \frac{(L-a)^2}{\pi^2} \right] \frac{1}{2} (s - D - \frac{1}{2} - \frac{d+1}{2} - 2) e^{-2\pi(J+1)/m^2} \sqrt{dR^2 + \frac{(L-a)^2}{\pi^2}} \right\} 
\]

Let us now discuss the behavior of the Casimir force as a function of \( a \). First notice that the parameter \( R^d \) in the first line in combination with the exponentials make the Casimir force really small in magnitude. This is to be contrasted to the case when only the fuzzy sphere is the extra dimensional space.

Now in order to be correct with the values that \( a \) is allowed to take we mention that (38) is obtained in the limit \( a \gg 1 \) and \( L-a \gg 1 \). Thus the extremely small values for \( a \) are excluded. So we shall examine the values of \( a \) that change the sign of \( L-a \), that is \( a < \frac{L}{2} \) and \( a > \frac{L}{2} \). It is obvious that when \( a \gg 1 \) and \( L-a \gg 1 \) and \( a < \frac{L}{2} \) the most dominating terms in relation (38) are the last two terms. For \( D = 3 \) and \( s = \frac{1}{2} \) and for \( a < \frac{L}{2} \), the Casimir force is negative because the first of the two last terms is dominating. So as in the cases we mentioned in the introduction, the piston is attracted to the closest end. Now when \( a > \frac{L}{2} \) the most dominating term of the last two is the second one. Thus the Casimir force is positive. So the Casimir force is repulsive in this case.

Before ending this section let us discuss on a topic. Notice that the last two terms of relation (38) are multiplied by \( \sqrt{J(J+1) + m^2} \). When the fuzzy sphere is excluded from the calculations, one can say that due to this term in the small mass limit the last two terms cease to be dominating. So the Casimir force changes sign. However in our case this does not happens because the finite values of order \( \sim 1 \) that \( J(J+1) \) can take. Thus even in the small mass limit, the sign of the Casimir force does not change from the behavior we described previously.
4 Casimir piston with a fuzzy sphere extra dimension

We now consider a fuzzy sphere as the extra dimensional space. We again deal the case with general \( s \) and \( D \). In order to get the case of our interest and study the singularities, at the end we put \( s = -\frac{1}{2} \) and \( D = 3 \), as previously. The Casimir energy in this case is,

\[
E_c(s, a) = \frac{1}{4\pi^2} \int d^{D-1}p \sum_{J=0}^{N} \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + \left( \frac{n\pi}{a} \right)^2 + J(J+1) + m^2 \right]^{-s}.
\]  

(39)

which upon integrating over the continuous modes and using (26), we obtain,

\[
E_c(s, a) = \frac{1}{8\pi^2 \pi^{D-1} \Gamma(s - \frac{D-1}{2}) \Gamma(s)} \times
\]

\[
\left( \sum_{J=0}^{N} \sum_{n=-\infty}^{\infty} \left[ \frac{(n\pi/a)^2 + J(J+1) + m^2}{a^2} \right]^{\frac{D-1}{2} - s} - \sum_{J=0}^{N} \left[ J(J+1) + m^2 \right]^{\frac{D-1}{2} - s} \right)
\]

Upon using as before relation (28),

\[
E_c(s, a) = \frac{1}{8\pi^2 \pi^{D-1} \Gamma(s - \frac{D-1}{2}) \Gamma(s)} \times
\]

\[
\left( \sum_{J=0}^{N} Z_{\frac{D-1}{2}}^w (s - \frac{D-1}{2} + \frac{\pi^2}{a^2}) - \sum_{J=0}^{N} \left[ J(J+1) + m^2 \right]^{\frac{D-1}{2} - s} \right)
\]

Note that \( w_j = J(J+1) + m^2 \), as previously. Using (30), relation (41) becomes,

\[
E_c(s, a) = \frac{1}{8\pi^2 \pi^{D-1} \Gamma(s - \frac{D-1}{2}) \Gamma(s)} \times
\]

\[
\sum_{J=0}^{N} \left\{ a \pi^{-\frac{1}{2}} \Gamma(s - \frac{D-1}{2} - \frac{1}{2}) \left( \sqrt{J(J+1) + m^2} \right)^{1-2(s - \frac{D-1}{2})} \\
+ 2\pi^{s-\frac{D-1}{2}} a \left( \sqrt{J(J+1) + m^2} \right)^{\frac{1}{2} - (s - \frac{D-1}{2})} \\
\times \sum_{n=-\infty}^{\infty} \left[ \frac{n a}{\pi} \right]^{(s - \frac{D-1}{2} - \frac{1}{2})} K_{\frac{1}{2} - (s - \frac{D-1}{2})} \left( 2n a \sqrt{J(J+1) + m^2} \right) \\
- \left[ J(J+1) + m^2 \right]^{\frac{D-1}{2} - s} \right\}
\]

It is obvious that due to the gamma function \( \Gamma(s - \frac{D-1}{2} - \frac{1}{2}) \), for the values \( s = -\frac{1}{2} \) and \( D = 3 \), a singularity appears in the first term of the above expression. However when we compute the Casimir force, as in the previous section, the linearity of this term to \( a \) and the addition of the other piston chamber will cancel this singular term. Following the
procedure of the previous section we obtain the Casimir force (we use the limit $L - a \gg 1$ and $a \gg 1$),

$$F_c(s) = - \sum_{J=0}^{N} \frac{1}{4} \pi^{s-2} \left( \sqrt{J(J+1)} + m^2 \right)^{1/2} (s - \frac{D-1}{2}) \frac{\Gamma(s)}{\Gamma(s)} \times \frac{(s - \frac{D-1}{2}) - \frac{D}{2}}{\Gamma(s)} e^{-2\pi J(J+1)+m^2}$$

As in the previous section case, the same arguments hold for the allowed values of $a$. Thus when $D = 3$ and $s = -\frac{1}{2}$ and for $a < \frac{L}{2}$ and $L - a \gg 1$ and $a \gg 1$, the two terms of relation (43) dominate making the Casimir force negative. Thus the piston is attracted to the nearest end. However when $a > \frac{L}{2}$ the force is positive and thus repulsive. So the behavior of the sign is the same as in the previous section case where an extra dimensional commutative torus and the fuzzy sphere was the extra dimensional space.

Note that the difference with the torus case of the previous section is that the Casimir force is not as small was the force for $T^d \times S_{FZ}$, because the $R^d$ term does not appear here. However the force is exponentially suppressed as in the previous section.

5 Computation of Casimir energy adding one loop self energy corrections

We discussed in the introduction the motivation to use non-commutative extra dimensions. The need comes from the stabilization they offer to the commutative extra dimensions. A calculation that follows naturally in this setup is the calculation of the Casimir energy. Also great interest comes from the computation of the Casimir energy by adding one loop corrections to the self energy of the scalar field due to non-commutative extra dimension. This was computed in reference [4] for the case $M^D \times T^d \times S_{FZ}$. We present a correct brief calculation in the appendix. $M^D$ represents the $D$-dimensional Minkowski spacetime. The inclusion of higher loop corrections due to self interactions of scalar field in a spacetime with non-commutative extra dimensions is frequently done in such theories (see [4, 14, 9]). However in the case of interest a problem arises when $D$=even. The Casimir
energy is computed with the approximation $\frac{1}{R^2} \gg m^2$ and $\frac{1}{R^2} \gg N(N+1)$. Within this approximation the one loop correction to the Casimir energy, as we prove in detail in the appendix, when $D=\text{even}$, it contains singularities. Also when $d=\text{odd}$ the Casimir energy also contains singularities. Indeed the Casimir energy reads,

$$E_c(s) = \frac{1}{4\pi^2} \frac{D-1}{2} \frac{\Gamma(s-D/2)}{\Gamma(s)} \times$$

$$\left\{ \frac{R^{d-2}}{(2\pi)^d} \frac{\Gamma(s-D/2-1)}{\Gamma(s-D/2)} w^{d-2(s-D/2)} \right\}$$

$$+ \frac{2\pi^{d/2}}{\Gamma(s-D/2)} \sum_{n_1,n_2,...,n_d=-\infty}^{\infty} \left[ \sum_{k=1}^d \left( \frac{n_k R}{2\pi} \right)^2 \right]^{D/2} \left( s-D/2-s \right) K_{D/2-s} \left( 2\pi w \left[ \sum_{k=1}^d \left( \frac{n_k R}{2\pi} \right)^2 \right]^{D/2} \right)$$

$$\times$$

$$A = \left( 3N^2 + 6N + 1 \right) Z_d(2-D) \Gamma(1-D/2) \left( \frac{1}{2\pi} \right)^{2-D}.$$

In the above relations the problems are clearly seen (note that $s = -D/2$ in the end of the calculation). Thus the most phenomenologically interesting case, $D = 4$, is out of the question. This motivated us to calculate the Casimir energy in the case of two parallel plates separated at a distance $a$ and maybe generalize the calculation to a piston setup to see if the usual regularization the Casimir piston offers to the force, hold here too (this is because the dimensionality of the infinite dimensions reduces to three). We computed the one loop corrections for the cases of $S_F Z$ and $S_F Z \times T^d$ as extra dimensional space. In the case with $S_F Z \times T^d$, the most easy and most interesting case, from a computational point of view, arises when $R = \frac{a}{\pi}$. All the calculations are done within the approximations, $\frac{1}{R^2} \gg m^2$ and $\frac{1}{R^2} \gg N(N+1)$.

### 5.1 One loop corrections to the Casimir energy of a piston chamber with $S_F Z$ as extra dimensional space

We shall consider firstly the case where the extra dimensional space is only the fuzzy sphere. We deal this case first because it is simpler. With the incorporation of the 1-loop self energy corrections due to self interactions in non-commutative geometry, the Casimir energy for the piston chamber reads,

$$E_c(s,a) = \frac{1}{4\pi^2} \int d^{D-1}p \sum_{J=0}^{N} \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + \left( \frac{n\pi}{a} \right)^2 + J(J+1) + m^2 + \Sigma \right]^{-s}.$$

In the above expression, $\Sigma$ is the one loop corrections and equals to,

$$\Sigma = \Gamma^{(2)}_{\text{planar}} + \Gamma^{(2)}_{\text{nonplanar}}$$
with,
$$\Gamma_{\text{planar}}^{(2)} = \frac{g}{12\pi} I^P, \quad (48)$$
and,
$$\Gamma_{\text{nonplanar}}^{(2)} = \frac{g}{24\pi} I^{NP}. \quad (49)$$

$I^P$ and $I^{NP}$ are the planar and non-planar contributions respectively and are, (we follow references [14] and [4]),

$$I^P = \int \frac{d^D \vec{p}}{(2\pi)^D} \sum_{n=1}^{N} \sum_{J=0}^{\infty} \frac{2J + 1}{\vec{p}^2 + (n\pi/a)^2 + J(J+1) + m^2} \quad (50)$$

and the non-planar contribution,

$$I^{NP} = \int \frac{d^D \vec{p}}{(2\pi)^D} \sum_{n=1}^{N} \sum_{J=0}^{\infty} (-1)^{J+j+N} \frac{(2J + 1)(N+1)}{\vec{p}^2 + (n\pi/a)^2 + J(J+1) + m^2} \left\{ \begin{array}{ccc} N/2 & N/2 & j \\ N/2 & N/2 & J \end{array} \right\}. \quad (51)$$

The index $j$ is to be contracted with the symbol in curly brackets. This symbol corresponds to the Wigner symbol of the $SU(2)$ algebra, see [14]. We now compute $I^P$ and $I^{NP}$. We will need the following properties of the Wigner symbol,

$$\sum_{N} (2N+1) \left\{ \begin{array}{ccc} A & B & N \\ C & D & P \end{array} \right\} \left\{ \begin{array}{ccc} A & B & N \\ C & D & Q \end{array} \right\} = \frac{1}{2P+1} \delta_{PQ} \quad (52)$$

and also,

$$\sum_{N} (-1)^{N+P+Q}(2N+1) \left\{ \begin{array}{ccc} A & B & N \\ C & D & P \end{array} \right\} \left\{ \begin{array}{ccc} A & B & N \\ D & C & Q \end{array} \right\} = \left\{ \begin{array}{ccc} A & C & Q \\ B & C & P \end{array} \right\} \quad (53)$$

An interesting computational case arises in the approximation,

$$\pi^2 a^2 \gg N(N+1), \quad \frac{\pi^2}{a^2} \gg m^2 \quad (54)$$

This approximation is applicable in the case when the plate separation is very small in magnitude. Note that within this approximation we cannot take the $N \to \infty$ limit, which corresponds to the continuum two dimensional commutative sphere [14]. With approximation (54), the planar contribution becomes,

$$I^P = \pi \frac{D}{2} a^{2-D} \Gamma(1 - \frac{D}{2}) \sum_{n=1}^{\infty} \sum_{J=0}^{N} \frac{2J + 1}{(n\pi)^2 (2J+1)} \quad (55)$$

Remember that in the end we take $D = 3$ again. The integration over momenta differs on the initial calculation of the Casimir energy. This is because the loop corrections
include total spacetime integration and not just space integration (in all the cases the integrations are assumed to be Euclidean). So the planar and non-planar contributions (see relations (50) and (51)) are integrated over $D$ continuous dimensions which are, $D - 1$ space dimensions and 1 time dimension. Using the Riemann zeta function, the planar contribution becomes,

$$I^P = \pi \frac{D}{2} a^{2-D} \Gamma \left(1 - \frac{D}{2}\right)(N^2 + 2N)\zeta(2 - D)$$  \hspace{1cm} (56)$$

In the same way the non planar contribution reads,

$$I^{NP} = \pi \frac{3D}{2} a^{2-D} \Gamma \left(1 - \frac{D}{2}\right)(N + 1)^2\zeta(2 - D)$$  \hspace{1cm} (57)$$

Thus relation (47) becomes,

$$\Sigma = g_{\frac{24}{4}} \pi \frac{3D}{2} a^{2-D} \Gamma \left(1 - \frac{D}{2}\right)(3N^2 + 6N + 1)\zeta(2 - D)$$  \hspace{1cm} (58)$$

and in more compact form

$$\Sigma = \sigma_1 a^{2-D}$$  \hspace{1cm} (59)$$

and $\sigma = \frac{g_{\frac{24}{4}}}{2} \pi \frac{3D}{2} \Gamma \left(1 - \frac{D}{2}\right)(3N^2 + 6N + 1)\zeta(2 - D)$. Within the approximation (54), the Casimir energy reads,

$$E_c(s, a) = \frac{1}{8\pi^2} \pi \frac{D}{2} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n=1}^{\infty} \left[ \frac{n\pi}{a} \right]^2 + \sigma_1 a^{2-D} \right]^{\frac{D-1}{2} - s}$$  \hspace{1cm} (60)$$

As in previous, using (30), (26),

$$E_c(s, a) = \frac{1}{8\pi^2} \pi \frac{D}{2} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \left\{ \frac{a}{\frac{D}{2}} \frac{\Gamma(s - \frac{D-1}{2} - \frac{1}{2})}{\Gamma(s - \frac{D-1}{2})} \left( \sqrt{\sigma_1 a^{2-D}} \right)^{1 - 2(s - \frac{D-1}{2})} \right. \right.$$

$$\left. + \frac{2\pi^{s - \frac{D-1}{2}} a}{\Gamma(s - \frac{D-1}{2})} \left( \sqrt{\sigma_1 a^{2-D}} \right)^{\frac{1}{2} - (s - \frac{D-1}{2})} \times \right.$$

$$\left. \sum_{n=-\infty}^{\infty} \left[ \frac{n a}{\pi} \right]^{(s - \frac{D-1}{2} - \frac{1}{2})} K_{\frac{1}{2} - (s - \frac{D-1}{2})} \left( 2n a \sqrt{\sigma_1 a^{2-D}} \right) - \left[ \sigma_1 a^{2-D} \right]^{\frac{D-1}{2} - s} \right\}$$

The last relation clearly diverges for $D = 3$ and $s = -\frac{1}{2}$. Unfortunately computing the Casimir force and adding the contributions from the two chambers still does not yield a regular expression. This is because the linearity of the first term of (61) is lost.
5.2 One loop Casimir energy of a piston chamber with $T^d \times S_{FZ}$ as extra dimensional space

We now compute the 1-loop self energy corrected Casimir energy for the $a$ piston chamber in the case the extra dimensional space is $T^d \times S_{FZ}$. The most easy and interesting case is when,

$$R = \frac{a}{\pi} \quad (62)$$

We now compute the planar and non planar graphs within this approximation. The planar contribution is,

$$I^P = \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty} \sum_{J=0}^{\infty} \sum_{k=1}^{D-1} \frac{2J + 1}{p_k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{2\pi n}{R}\right)^2 + J(J + 1) + m^2} \quad (63)$$

Within the approximation (54), and using the same techniques as in previous we obtain,

$$I^P = \pi^{3D-2}\Gamma(1 - \frac{D}{2})(N^2 + 2N)\left(\zeta(2 - D) + \frac{1}{2}Z_{1+d}(1 - \frac{D}{2})\right)a^{2-D} \quad (64)$$

In the same way, the non-planar contribution reads,

$$I^{NP} = \pi^{3D-2}\Gamma(1 - \frac{D}{2})(N + 1)^2\left(\zeta(2 - D) + \frac{1}{2}Z_{1+d}(1 - \frac{D}{2})\right)a^{2-D} \quad (65)$$

So the one loop correction to the self energy is,

$$\Sigma = \sigma_2 a^{2-D} \quad (66)$$

with,

$$\sigma_2 = \frac{g}{24\pi^{3/2}}\Gamma(1 - \frac{D}{2})(3N^2 + 6N + 1)\left(\zeta(2 - D) + \frac{1}{2}Z_{1+d}(1 - \frac{D}{2})\right) \quad (67)$$

Now the one loop self energy corrected Casimir energy in this case reads,

$$E_c(s, a) = \frac{1}{4\pi^2} \int d^{D-1}p \sum_{J=0}^{N} \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty} \left[\sum_{k=1}^{D-1} p_k^2 + \left(\frac{n\pi}{a}\right)^2 + \sum_{k=1}^{d} \left(\frac{n_k\pi}{a}\right)^2 + \Sigma\right]^{-s} \quad (68)$$
Following the steps of the previous sections, we obtain finally,

\[
E_c(s, a) = \frac{1}{8\pi^2} \frac{\Gamma(s - D - 1)}{\Gamma(s)} \times \\
\left\{ a^{d+1} \pi^d \frac{\Gamma(s - D - 1 - d/2)}{\pi^{d+1} \Gamma(s - D/2 - 1/2)} \left( \sqrt{\sigma^2 a^{2-D}} \right)^{d+2(s-D-1)} \right. \\
+ 2\pi^s \frac{D-1}{D-2} a^{d+1} \left( \sqrt{\sigma^2 a^{2-D}} \right)^{d+1/2 - (s-D-1/2)} \\
\left. \times \sum_{n,n_1,n_2,...,n_d=-\infty}^{\infty} \left[ \sum_{k=1}^{d+1} \left( \frac{n_k a}{\pi} \right)^{2} \right] \frac{1}{2} \right\}
\]

It is obvious that for \( s = -\frac{1}{2} \) and \( D = 3 \) relation (69) contains singularities. Actually the singularities are contained in the gamma functions \( \Gamma(s - D - 1 - d/2) \) and \( \Gamma(s - D/2 - 1 - d/2) \) for \( d \) even and odd. Unfortunately the Casimir energy is divergent for all \( D \) values. Also the Casimir force is also divergent for all \( d \) values. We expected the piston configuration would lead to a regularization of the Casimir force which is ill defined in the case the plates are excluded. However even in the piston case this does not happen because the singularities of the Casimir energy are not linear to \( a \) as happened in the previous sections, see for example relations (61) and (42) and no cancellation due to piston setup occurs. Thus there is no consistent way to incorporate the one loop corrections to the scalar mass within the approximation (62) and (54) even in the piston setup. Maybe the problem is the approximations or the dimensionality of spacetime, as in the Minkowski case (see appendix).

6 Conclusions

We have examined how the Casimir force for a piston behaves when non-commutative extra dimensions are included in the calculations. Particularly we have examined the cases when the extra dimensional space is \( T^d \times S_{FZ} \) and \( S_{FZ} \), with \( T^d \) the commutative
\(d\) torus and \(S_{FZ}\) the two dimensional non-commutative sphere. For the case \(T^d \times S_{FZ}\), when the two piston chambers are much larger than the compact torus radius, the Casimir force behaves as follows,

- When \(a > \frac{L}{2}\) the force is repulsive.
- When \(a < \frac{L}{2}\) the force is attractive and the piston is attracted to the nearest end.

In the case of the fuzzy sphere as extra dimensional space the results are the same. Also we concluded that the small mass limit will not affect our results due to the non-commutativity effects. Thus this behavior for \(a \gg 1\) and \(L - a \gg 1\) remain true for all the dimensions of the extra dimensional torus. Additionally the force in the case \(T^d \times S_{FZ}\) is very suppressed due to the \(R^d\) multiplying the whole expression. This suppression does not appear in the case of \(S_{FZ}\). We also examined the incorporation of one loop corrections to the self energy due to the self interactions of the scalar field in the non-commutative space. The most mathematically interesting case was when \(R = \frac{a}{\pi}\). We did this for one piston chamber. We expected to obtain a regular expression for the Casimir force. Without the plate separation the Casimir energy was singular for \(D = 4\) and generally for \(D=\text{even}\). Thus we hoped that the introduction of plates would solve the problem. However this did not happen, but contrary the result is completely singular. Even the addition of the other chamber of the piston did not change the result. We proved that this happens when the "mass" of the scalar field is \(a\) dependent. We have tried also Neumann boundary conditions but the results did not change.

Before closing we discuss on the stabilization issue. The non-commutative space having intrinsically a scale, the Moyal shell, one expects to stabilize the commutative extra dimensions (of course if only non-commutative dimensions appear they are by definition stabilized). Thus we should check on this for the piston setup. This could be studied if we computed the force with respect to \(R\),

\[
F_R = -\frac{\partial E_c}{\partial R}.
\] (70)

Unfortunately the expression contains singularities, as we can see by taking the derivatives over \(R\) of relations (31) and (42). Also this remains true when only the commutative torus is the only extra dimensional space. Thus we must find a regularization method in order to answer this problem. We hope to comment on this soon.

**APPENDIX**

We will now compute the Casimir energy in the case the extra dimensions are \(T^d \times S_{FZ}\) and for the four dimensional spacetime (for the one loop corrections see [4]). We shall use the approximation,

\[
\frac{\pi^2}{R^2} \gg N(N+1), \quad \frac{\pi^2}{R^2} \gg m^2.
\] (71)
The one loop corrected Casimir energy is equal to,

\[ E_c(s) = \int d^{D-1}p \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty'} \sum_{J=0}^{N} \left[ \sum_{k=1}^{d} \left( \frac{n_k}{R} \right)^2 + \vec{p}^2 + J(J+1) + m^2 + \Sigma \right]^{-s}. \tag{72} \]

In this case at the end of the calculation, \( s = -\frac{1}{2} \) and \( D = 4 \) contrary to the piston case where \( D = 3 \). \( \Sigma \) equals to the self energy 1-loop correction and as previously equals to,

\[ \Sigma = \frac{g}{12\pi} I^P + \frac{g}{24\pi} I^{NP}. \tag{73} \]

As before,

\[ I^P = \pi^D \Gamma\left(1 - \frac{D}{2}\right) \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty'} \sum_{J=0}^{N} \sum_{k=1}^{d} \left( \frac{n_k R}{2\pi} \right)^2 + J(J+1) + m^2. \tag{74} \]

Note that in this case the approximation (71) is valid if the zero modes of the commutative extra dimensions are excluded for the summation, contrary to the piston case. After calculations, \( I^P \) equals to,

\[ I^P = \pi^D \left( \frac{R}{2\pi} \right)^{2-D} (N^2 + 2N) Z_d(2-D) \Gamma\left(1 - \frac{D}{2}\right) \tag{75} \]

and in the same way, \( I^{NP} \),

\[ I^{NP} = \pi^D \left( \frac{R}{2\pi} \right)^{2-D} (N + 1)^2 Z_d(2-D) \Gamma\left(1 - \frac{D}{2}\right). \tag{76} \]

Finally,

\[ \Sigma = AR^{2-D}, \tag{77} \]

with,

\[ A = \frac{g}{24} \pi^D \left( 3N^2 + 6N + 1 \right) Z_d(2-D) \Gamma\left(1 - \frac{D}{2}\right) \left( \frac{1}{2\pi} \right)^{2-D}. \tag{78} \]

For more details see [4]. Now the Casimir energy, after integrating over the continuous \( D - 1 \) dimensions (that is 3 dimensions),

\[ E_c(s) = \frac{1}{4\pi^2} \pi^{D-1} \frac{\Gamma\left(s - \frac{D-1}{2}\right)}{\Gamma(s)} \sum_{n_1, n_2, \ldots, n_d = -\infty}^{\infty'} \left[ \sum_{k=1}^{d} \left( \frac{2\pi n_k R}{\Gamma(s) \Gamma\left(s - \frac{D-1}{2}\right)} \right)^2 + AR^{2-D} \right]^{-s}. \tag{79} \]
Including the zero mode in the sum and using the homogeneous Epstein zeta, the above becomes,

\[ E_c(s) = \frac{1}{4\pi^2} \sum_{\ell=1}^{D-1} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \times \]

\[ \left\{ \frac{\Gamma(s - \frac{D-1}{2} - \frac{d}{2})}{\Gamma(s - \frac{D-1}{2})} \right\} w^{d-2(s - \frac{D-1}{2})} + \frac{2\pi^{s-D-\frac{d}{2}} w^{d-2(s-D-\frac{d}{2})}}{\Gamma(s-D-\frac{d}{2})} \sum_{n_1,n_2,...,n_d=-\infty}^{\infty} \left[ \sum_{k=1}^{d} \left( \frac{n_k R}{2\pi} \right)^2 \right] \times \]

\[ K\left( s - \frac{D-1}{2} \right) \left( 2\pi w \left[ \sum_{k=1}^{d} \left( \frac{n_k R}{2\pi} \right)^2 \right] \right) - (AR^{2-D} \frac{D-1}{2} - s) \],

and \( w^2 = AR^{2-D} \). From relation (78) and (80) it is obvious that the only singularity free case is when \( D = \text{odd} \) and \( d = \text{even} \). Unfortunately this does not correspond to our world because \( D = 4 \). This is why we did the calculation for the piston configuration as we pointed in previous sections. In the piston setup singularities don’t appear (at least in the force) and the continuous dimensions are reduced by one. So we could expect that the ill calculation we just made could be done. Unfortunately this was not true as we saw in the previous sections.

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