On variational principles in coupled strain-gradient elasticity

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Received 12 November 2021; accepted 3 February 2022

Abstract
Strain-gradient elasticity is a special case of high-gradient theories in which the potential energy density depends on the first and second gradient of the displacement field. The presence of a coupling term in the material law leads to a non-diagonal quadratic form of the stored energy, which makes it difficult for the derivation of fundamental theorems. In this article, two variational principles of the minimum of potential and complementary energies are argued in the context of the coupled strain-gradient elasticity theory. The basis of the proofs of both variational principles is the equivalent transformation of the strain and strain-gradient energy density that allows to avoid the complication related to the presence of the fifth-rank coupling tensor $C_5$ in the equation for the potential energy density and leads to diagonalization of the quadratic form of the stored energy. This transformation enables to inverse Hook’s law, to determine compliance tensors, and to obtain closed-form relation for the complementary energy. After that the proofs of both principles of a minimum of potential and complementary energies are provided in the usual manner adopted in the classical theory of elasticity.

Keywords
Variational principles, coupled strain-gradient elasticity, principle of the minimum of potential energy, principle of the minimum of complementary energy

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1. Introduction

Gradient continuum theories have a long history. Early works in this regard are Cosserat and Cosserat [1] and Hellinger [2]. Then in the 60th and early 70th of the last century, they were generalized in [3–8]. Next resurgence of interest to higher-gradient theories took place in numerous papers due to necessity to overcome some restrictions of the classical elasticity [9–25].

Unlike gradient continuum theories, classical elasticity has a sound basis. There are many theorems on which we can rely, many of them are summarized in Gurtin [26]. These are theorems regarding the existence and uniqueness of solutions, auxiliary theorems like Betty’s reciprocal theorem, Clapeyron’s work theorem, and de Saint Venant’s principle, variational formulations like the minimum principle of the elastic potential, the minimum principle of complementary energy, corresponding maximum principles, the principles of Hellinger-Reissner, de Veubeke-Hu-Washizu, and some others, Hooke’s law is fully explored for all anisotropy classes [27]. For all of these, representation theorems for the stiffness tetrad can be found. So, regarding the linear theory [26], can be considered a concluding work.

In gradient elasticity, the proofs of fundamental theorems are presented mainly for the uncoupled case, that is, when the coupling tensor of fifth-order is assumed to be equal to zero. However, by dropping the coupling tensor of fifth order, it is restricted to centro-symmetric materials. This excludes 30% of all crystals [28] that may be piezo-active, and one can even construct isotropic materials that are not centro-symmetric, like an entangled bundle of coil springs of equal chirality.

The uniqueness theorem for isotropic, uncoupled gradient elasticity is given by Mindlin and Eshel [7]. The proof is based on the assumption of positive definiteness of the stiffness tensors of the fourth- and sixth-orders. Mindlin’s uniqueness proof holds for mixed boundary conditions, and there are no restrictions regarding the shape of the domain. Conditions of positive definiteness in absence of coupling terms were presented also in previous works [5,29]. The inequality constraints obtained by Mindlin [5] and Dell’Isola et al. [29] have been extended for the case of coupled strain-gradient elasticity by Nazarenko et al. [30]. To this end, it has been used so-called block diagonalization of the composite stiffness [31]. Such an equivalent transformation makes it possible to obtain the necessary conditions of positive definiteness and convexity of the hemitropic strain and strain-gradient energy, including the coupling stiffness $C_5$. The uniqueness of solution for the case of coupled strain-gradient elasticity is proven by Nazarenko et al. [32], which is based on the decoupling of the strain and strain-gradient energy due to block diagonalization and is applicable to any symmetry of stiffness tensors.

The principles of the minimum of the elastic potential (PMEP), which is minimized over the displacement field, and the principle of the minimum of the complementary potential (PMCP), which is minimized over the stress field, are elementary variational formulations of the elastostatic boundary value problem [26]. They serve as starting point for the construction of approximate solutions, they are needed for different mean value theorems, and they give upper bounds for the stored elastic energy and the global stiffness of elastic systems. All of this has important practical applications, for numerical solution methods like the FEM (approximate solutions) and as bounds for effective stiffness’s in homogenization theory (upper bounds). Likewise, maximum principles have been formulated. In homogenization theory, they give with the absence of body forces the lower bounds for the stored energy and hence for the effective stiffness.

Following Gurtin [26], the origin of the PMEP is hard to trace and goes probably back to the middle of the 19th century to Green and Kirchhoff. It can be derived from the virtual power principle by replacing the virtual stress power by the virtual change of the elastic energy. With appropriate assumptions regarding the test function, the principle of virtual power becomes the first variation of the elastic potential. Alternatively one can start from the local balance of momentum, multiply by a test functions and integrate. The independent function is the displacement field in the PMEP. Since the displacement method is much more popular than the force method (or flexibility method), it is fundamental to many finite-element implementations.

The PMCP is from the beginning of the 20th century. Its derivation involves stricter assumptions (star convex domains) and is hence less general than the PMCP. It requires statically admissible stress fields, of which the one that satisfies the stress boundary conditions also minimizes the complementary potential.

Since the starting point is the virtual power, the generalization of the PMEP to gradient elasticity is relatively simple, see e.g. previous works [33–36]. The extension of the PMCP is less clear. There are some works, like Polizzotto [35] (section 3.1 therein) who examines a stress-gradient elasticity. The change of the independent variable from one displacement field to several stress fields ($T_2$ and $T_3$) is
rather complicated. It is not necessary in Polizzotto’s study [35], since he considers a stress field and its gradient.

In this work, we generalize and derive the PMEP and the PMCP for the case of the coupled strain-gradient elasticity. Representing of the complementary energy in the coupled strain-gradient elasticity is not a trivial problem itself [see for details Nazarenko et al.’s study 31]. Using the block diagonalization technique presented by Nazarenko et al. [31] and Nazarenko et al. [30] we reduce the potential and complementary energies to uncoupled form and then derive the principle of a minimum of the potential and complementary energies.

The structure of this paper is as follows. The next section briefly outlines the main relations for potential and complementary energies, their representation in decoupled form and variational equation of the stress equilibrium with the boundary conditions. The principles of minimum potential energy and minimum complementary energy are proven (novel) after that. Finally, the conclusions and discussion are given in the last section.

2. Coupled strain-gradient elasticity

2.1. Potential energy

For a first strain-gradient theory, the strain and strain-gradient energy can be written as

\[ w = \frac{1}{2} E_2 \cdot C_4 \cdot E_2 + E_2 \cdot C_5 \cdot \cdots \cdot E_3 + \frac{1}{2} E_3 \cdot C_6 \cdot \cdots \cdot E_3 \]  

where \( E_2 \) and \( E_3 \) are the strains and the second gradient of displacement calculated as a function of the displacement field \( u(x) \)

\[ E_2 = \text{sym}(u \otimes \nabla), \quad E_3 = u \otimes \nabla \otimes \nabla, \]  

and \( C_4, C_5, \) and \( C_6 \) are the stiffnesses. Here the number of indices corresponds to the tensor rank of \( E_2, E_3, C_4, C_5, C_6 \). Scalars, vectors, second- and higher-rank tensors are denoted by italic letters (like \( a \) or \( A \)), bold minuscules (like \( a \)), bold majuscules (like \( A \)), and blackboard bold majuscules (like \( A \)), respectively.

\( \nabla \) is the three-dimensional nabla operator \( \nabla = \partial / \partial x_i e_i, (i = 1, 2, 3) \), where \( e_1, e_2, e_3 \) are an orthonormal base vectors. The repeating indices imply a summation. \( \otimes \) denotes the dyadic product. A gradient of the displacement \( u \otimes \nabla \) is defined as

\[ u \otimes \nabla = \frac{\partial u_i}{\partial x_j} e_i \otimes e_j = u_{i,j} e_i \otimes e_j . \]  

The dots denote scalar contractions, where the double and triple scalar contractions are determined with regard to orthonormal base vectors \( e_i \) as follows

\[ e_i \otimes e_j \cdot e_k \otimes e_l \cdot e_m \otimes e_n = \delta_{ik} \delta_{jl} \delta_{mn} . \]  

The tensors \( E_2, E_3 \) and \( C_4, C_5, C_6 \) have the following index symmetries

\[ E_{ij} = E_{ji} \]  

\[ E_{ijk} = E_{ikj} \]  

\[ C_{ijkl} = C_{iklj} = C_{ijlk} \]  

\[ C_{ijklm} = C_{jiklm} = C_{ijklm} \]  

\[ C_{ijklmn} = C_{injlk} = C_{ikljm} = C_{ijklm} . \]
Remark 1. Both variants

\[ u \otimes \nabla \otimes \nabla \]

and

\[ \text{sym}(u \otimes \nabla) \otimes \nabla \]

can be treated as the strain gradient. In the first case, a third-order tensor with one index symmetry has 18 independent components. Linearized rotations removed from

\[ u \otimes \nabla \]

are saved in accordance with the compatibility conditions \( \mathbf{E} \), while the second one

\[ \text{sym}(u \otimes \nabla) \otimes \nabla \]

is a strain gradient.

Equation (1) can be equivalently transformed as follows

\[
w = \frac{1}{2} E_2^m \cdot \mathbf{C}_4 \cdot E_2^m + \frac{1}{2} E_3 \cdot C_6 \cdot E_3, \]

in order to present the strain and strain-gradient contributions as decoupled (see Nazarenko et al. [30]). Here the superscript \( m \) indicates the modified strain and stiffness tensors

\[
E_2^m = E_2 + E_3 \cdots \mathbf{C}_5^T \cdot \mathbf{C}_4^{-1}
\]

and

\[
C_6^m = C_6 - \mathbf{C}_5^T \cdot \mathbf{C}_4^{-1} \cdot \mathbf{C}_5.
\]

We define the transpose of \( \mathbf{C}_5 \) by

\[
E_2 : \mathbf{C}_5 \cdots E_3 = E_3 \cdots \mathbf{C}_5^T : E_2,
\]

that is, in terms of indices

\[
C_{ijklm}^T = C_{klmij}.
\]

The decoupled matrix representation of the potential energy density can also be achieved by splitting a modified second gradient of displacement and modified stiffness tensor of fourth-rank (see Nazarenko et al. [31]):

\[
w = \frac{1}{2} E_2 \cdot C_4^m \cdot E_2 + \frac{1}{2} E_3^m \cdot C_6 \cdots E_3.
\]

As indicated above, the superscript \( m \) corresponds the modified tensors, which are specified as

\[
E_3^m = E_3 + E_2 \cdot \mathbf{C}_5 \cdots C_6^{-1}
\]

and

\[
C_4^m = C_4 - C_5 \cdots C_6^{-1} \cdots \mathbf{C}_5^T.
\]

Remark 2. It should be pointed out that the equations for the potential energy density (1), (11), (16) are equivalent and that both modifications (11), (16) can be applied for arbitrary material symmetry classes (see Nazarenko et al. [31]).
Remark 3. It should be noted that equations similar to equations (16) and (18) are presented by Lurie et al. [37] equations (8) and (9). Equations (8) and (9) in Lurie et al. [37] are not given quite correctly and should be considered taking into account equation (10).

2.2. Complementary energy

2.2.1. Terminology and the special case of linear elasticity. The following different energies are usually distinguished:

The strain energy is the stored energy as a function of the strains. In terms of scalar strains and stresses it is

\[ w(\varepsilon_0) = \int_0^{\varepsilon_0} \sigma(\varepsilon) \, d\varepsilon. \] (19)

For a linear stress strain relation

\[ \sigma = E \varepsilon \] (20)

it is

\[ w(\varepsilon_0) = \frac{E}{2} \varepsilon_0^2. \] (21)

Geometrically, it is the area below the stress–strain curve up to the point \((\sigma, \varepsilon)\).

The final value of energy is the product \(\sigma_0 \varepsilon_0\). It has no physical meaning but appears when performing a Legendre transformation. Geometrically, it is the area of the rectangle spanned by the point \((\sigma, \varepsilon)\) and the origin.

The complementary energy is the difference between the final value of energy and the strain energy,

\[ w^* = \sigma_0 \varepsilon_0 - w(\varepsilon_0). \] (22)

For linear stress–strain relations, it is equal to the strain energy in value

\[ w^* = \frac{E}{2} \varepsilon_0^2 - \frac{E}{2} \varepsilon_0^2 = \frac{E}{2} \varepsilon_0^2. \] (23)

It is usually associated with a change of the independent variable from the strains to the stresses. Then

\[ w^*(\sigma) = \frac{E^{-1}}{2} \sigma \] (24)

is the Legendre transform of the function \(w(\varepsilon)\). Due to the linearity of the stress–strain relation (or the quadratic form of the strain energy), the Legendre transform is simply obtained using the material law \(\sigma = E \varepsilon\) to change the independent variable. This generalizes to the 3D case. The Legendre transform of the quadratic form

\[ w(E_2, E_3) = \frac{1}{2} E_2 : C_4 : E_2 + \frac{1}{2} E_3 \cdots C_6 \cdots E_3 \] (25)

is

\[ w^*(T_2, T_3) = \frac{1}{2} T_2 : C_4^{-1} : T_2 + \frac{1}{2} T_3 \cdots C_6^{-1} \cdots T_3 \] (26)

with

\[ T_2 = C_4 : E_2 \] (27)

\[ T_3 = C_6 \cdots E_3, \] (28)
which is why we can obtain the complementary energy once we can invert the stress–strain relation after getting rid of \( C_5 \).

### 2.2.2. The coupled strain-gradient elasticity.

\[
\frac{w'}{C} = \frac{1}{2} T_2 \cdot S_4 \cdot T_2 + T_2 \cdot S_5 \cdot T_3 + \frac{1}{2} T_3 \cdot S_6 \cdot T_3 ,
\]

where \( S_4, S_5, S_6 \) are the compliance tensors and have the same symmetry as \( C_4, C_5, C_6 \). The stresses and the double stresses are given by

\[
T_2 = \frac{\partial w}{\partial E_2} = C_{4} \cdot E_2 + C_5 \cdot E_3,
\]

\[
T_3 = \frac{\partial w}{\partial E_3} = C_{5}^T \cdot E_2 + C_6 \cdot E_3.
\]

This is the generalized Hookean law. The inverse is

\[
E_2 = \frac{\partial w'}{\partial T_2} = S_{4} \cdot T_2 + S_5 \cdot T_3 ,
\]

\[
E_3 = \frac{\partial w'}{\partial T_3} = S_{5}^T \cdot T_2 + S_6 \cdot T_3 .
\]

It has been demonstrated by Nazarenko et al. [31] that the both modified equations for the potential energy density equations (11), (16) can be used in order to specify the compliance tensors \( S_4, S_5, S_6 \) and to determine the stress energy density. In the same article the compliance tensors \( S_4, S_5, S_6 \) are derived as

\[
S_4 = C_4^{-1} + C_4^{-1} \cdot C_5 \cdot \ldots \cdot (C_6^m)^{-1} \cdot \ldots C_5^T \cdot C_4^{-1},
\]

\[
S_5 = C_4^{-1} \cdot C_5 \cdot \ldots \cdot (C_6^m)^{-1},
\]

\[
S_6 = (C_6^m)^{-1},
\]

or

\[
S_4 = (C_6^m)^{-1},
\]

\[
S_5 = (C_4^{-1})^{-1} \cdot C_5 \cdots C_6^{-1},
\]

\[
S_6 = C_6^{-1} + C_6^{-1} \cdot C_5 \cdots (C_6^m)^{-1} \cdots C_5^T \cdots C_6^{-1},
\]

where \( C_6^m \) and \( C_4^m \) are determined according to equations (13), (18). Both presentations for \( S_4, S_5, S_6 \) are identical.

### 2.3. Variational equation of the stress equilibrium and the boundary conditions

The equilibrium equations and the natural boundary conditions for the model under consideration can be obtained using the Lagrange variational principle

\[
\delta L = \delta a - \delta \int_V w dV = 0
\]
where $\mathcal{L}$ is the Lagrangian, $\delta a$ is the work of external forces and double forces defined in equations (57) and (58), and $w$ is a strain and strain-gradient energy density equation (1). The variation of the potential energy density is defined as

$$
\delta w(\mathbf{E}_2, \mathbf{E}_3) = \frac{\partial w}{\partial \mathbf{E}_2} \cdot \delta \mathbf{E}_2 + \frac{\partial w}{\partial \mathbf{E}_3} \cdot \delta \mathbf{E}_3, \quad (41)
$$

or we obtain

$$
\delta w = T_2 \cdot \delta \mathbf{E}_2 + T_3 \cdot \delta \mathbf{E}_3, \quad (42)
$$

with the stresses $T_2$ and the double stresses $T_3$ determined in equations (30) and (31). Then the total potential energy density for an arbitrary variation of the displacement field $\mathbf{u}$ can be written down as

$$
\delta \int \nu \, w \, dV = \int \nu \, (T_2 \cdot \delta \mathbf{E}_2 + T_3 \cdot \delta \mathbf{E}_3) \, dV \\
= \int \nu \, [T_2 \cdot (\delta \mathbf{u} \otimes \nabla) + T_3 \cdot (\delta \mathbf{u} \otimes \nabla \otimes \nabla)] \, dV. \quad (43)
$$

Applying the chain rule and the divergence theorem in the form

$$
\int _\nu \mathbf{v} \cdot \nabla \, dV = \int _S \mathbf{v} \cdot \mathbf{n} \, dS \quad (44)
$$

gives

$$
\int _\nu T_2 \cdot (\delta \mathbf{u} \otimes \nabla) \, dV \\
= - \int _\nu \delta \mathbf{u} \cdot T_2 \cdot \nabla \, dV + \int _S \delta \mathbf{u} \cdot T_2 \cdot \mathbf{n} \, dS, \quad (45)
$$

and

$$
\int _\nu T_3 \cdot (\delta \mathbf{u} \otimes \nabla \otimes \nabla) \, dV = - \int _\nu \delta \mathbf{u} \otimes \nabla \cdot T_3 \cdot \nabla \, dV + \int _S \delta \mathbf{u} \otimes \nabla \cdot T_3 \cdot \mathbf{n} \, dS \\
= \int _\nu \delta \mathbf{u} \cdot T_3 \cdot \nabla \cdot \nabla \, dV - \int _S \delta \mathbf{u} \cdot T_3 \cdot \nabla \cdot \mathbf{n} \, dS + \int _S \delta \mathbf{u} \otimes \nabla \cdot T_3 \cdot \mathbf{n} \, dS \quad (46)
$$

Given the identities equations (45) and (46), equation (43) can be presented as the sum of volume and surface integrals

$$
\delta \int \nu \, w \, dV = \int _\nu (T_2 \cdot \delta \mathbf{E}_2 + T_3 \cdot \delta \mathbf{E}_3) \, dV \\
= - \int _\nu \delta \mathbf{u} \cdot (T_2 - T_3 \cdot \nabla) \cdot \nabla \, dV \\
+ \int _S \delta \mathbf{u} \cdot (T_2 - T_3 \cdot \nabla) \cdot \mathbf{n} \, dS \quad (47)
$$
It should be noted that $\delta \mathbf{u} \otimes \nabla$ is not independent of $\delta \mathbf{u}$ on $S$. Indeed, knowing $\delta \mathbf{u}$ on $S$, it is possible to determine the surface gradient of $\delta \mathbf{u}$. Following Toupin [3] and Mindlin [6], we decompose the gradient of $\delta \mathbf{u}$ into its normal and tangential parts

$$\delta \mathbf{u} \otimes \nabla = \delta \mathbf{u} \otimes (\nabla_n + \nabla_s).$$

Here

$$\nabla_n = \mathbf{n} \otimes \nabla = \nabla \cdot \mathbf{n} = \frac{\partial}{\partial x_j} n_j,$$

and

$$\nabla_s = \nabla \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}).$$

Decomposing the gradient of $\delta \mathbf{u}$ in the last term of equation (47) into its normal and tangential parts [3,6,32]

$$\delta \mathbf{u} \otimes \nabla \cdot (\mathbf{T}_3 \cdot \mathbf{n}) = \delta \mathbf{u} \otimes \nabla_n \cdot (\mathbf{T}_3 \cdot \mathbf{n}) + \delta \mathbf{u} \otimes \nabla_s \cdot (\mathbf{T}_3 \cdot \mathbf{n}).$$

We convert it into the sum of only independent variations $\delta \mathbf{u}$ and $\delta \mathbf{u} \otimes \nabla_n$. By employing the chain rule, the surface gradient can be presented in the following form:

$$\delta \mathbf{u} \otimes \nabla_s \cdot (\mathbf{T}_3 \cdot \mathbf{n}) = (\delta \mathbf{u} \cdot \mathbf{T}_3 \cdot \mathbf{n}) \cdot \nabla_s - \delta \mathbf{u} \cdot (\mathbf{n} \cdot \mathbf{T}_3) \cdot \nabla_s,$$

and using the surface divergence theorem in the form

$$\int_S \mathbf{v} \cdot \nabla_s dS = \int_S (\mathbf{n} \cdot \nabla_s) \mathbf{v} \cdot dS + \oint_C \mathbf{v} \cdot \mathbf{m} dC,$$

with $\mathbf{v} = \delta \mathbf{u} \cdot \mathbf{T}_3 \cdot \mathbf{n}$ in equation (53) gives

$$\int_S (\delta \mathbf{u} \cdot \mathbf{T}_3 \cdot \mathbf{n}) \cdot \nabla_s dS = \int_S \delta \mathbf{u} \cdot (\mathbf{T}_3 \cdot \mathbf{n} \otimes \mathbf{n}) (\mathbf{n} \cdot \nabla_s) dS$$

$$+ \oint_C \delta \mathbf{u} \cdot (\mathbf{T}_3 \cdot \mathbf{n} \otimes \mathbf{m}) dC,$$

where $C$ is the union of all edges of the domain $V$, $\mathbf{t}$ is the unit tangent to the edge, and $\mathbf{m} \ (= \mathbf{t} \times \mathbf{n})$ is the unit outward normal to $C$ tangent to $\mathbf{t}$.

Accounting for the identity equation (54) and equation (51), the last term in the variation of the total potential energy from equation (47) is

$$\int_S \delta \mathbf{u} \otimes \nabla \cdot \mathbf{T}_3 \cdot \mathbf{n} dS$$

$$= \int_S \{ \delta \mathbf{u} \otimes \nabla_n \cdot \mathbf{T}_3 \cdot \mathbf{n} + (\delta \mathbf{u} \cdot \mathbf{T}_3 \cdot \mathbf{n}) \cdot \nabla_s$$

$$- \delta \mathbf{u} \cdot (\mathbf{T}_3 \cdot \mathbf{n}) \cdot \nabla_s \} dS.$$
\[
\delta \int_{\Omega} w dV = - \int_{\Omega} \delta u \cdot (T_2 - T_3 \cdot \nabla) \cdot \nabla dV \\
+ \int_{\partial \Omega} \delta u \cdot \left\{ (T_2 - T_3 \cdot \nabla - T_3 \cdot \nabla_s) \cdot n \right\} dS \\
+ \int_{\partial \Omega} \delta u \otimes \nabla_n \cdot T_3 \cdot n dS \\
+ \oint_{\Sigma} \delta u \cdot (T_3 \cdot n \otimes m) dC .
\]

The variation of the total potential energy equation (47) can be written down as

\[
\delta \int_{\Omega} w dV = - \int_{\Omega} \delta u \cdot (T_2 - T_3 \cdot \nabla) \cdot \nabla dV \\
+ \int_{\partial \Omega} \delta u \cdot \left\{ (T_2 - T_3 \cdot \nabla - T_3 \cdot \nabla_s) \cdot n \right\} dS \\
+ \int_{\partial \Omega} \delta u \otimes \nabla_n \cdot T_3 \cdot n dS \\
+ \oint_{\Sigma} \delta u \cdot (T_3 \cdot n \otimes m) dC .
\]

The admissible form of the work of external forces and double forces is governed by the variation of the strain and strain-gradient energy:

\[
\delta a = \int_{\Omega} \delta u \cdot f dV \\
+ \int_{\partial \Omega} (\delta u \cdot p + \delta u \otimes \nabla_n \cdot \mathbf{R}) dS + \oint_{C} \delta u \cdot \mathbf{c} dC ,
\]

where \( f \) is a body force per unit volume, \( p \) is a surface traction, \( \mathbf{R} \) is a surface normal double force on \( S \) (e.g. Eremeyev et al. [38]), and \( \mathbf{c} \) is a line force on edge \( C \), or

\[
\delta a = \int_{\Omega} \delta u \cdot f dV \\
+ \int_{\partial \Omega} (\delta u \cdot p + D(\delta u) \cdot \mathbf{r}_n) dS + \oint_{C} \delta u \cdot \mathbf{c} dC .
\]

Here \( \mathbf{r}_n \) is a double traction in normal direction on \( S \) (e.g. Bertram [39])

\[
\mathbf{r}_n = \mathbf{R} \cdot \mathbf{n} ,
\]

and

\[
D \mathbf{u} = (\mathbf{u} \otimes \nabla) \cdot \mathbf{n} = \frac{\partial u_i}{\partial x_j} n_j e_i .
\]
The variation of the potential energy for all admissible functions \( \delta u \) in according to equation (40) is

\[
\delta \int_V w \, dV = \int_V \delta u \cdot f \, dV + \int_S (\delta u \cdot p + D\delta u \cdot r_n) \, dS + \oint_C \delta u \cdot \mathbf{e} \, dC. \tag{61}
\]

Accounting for equation (56), we obtain the stress equilibrium equation:

\[
(T_2 - T_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = 0, \tag{62}
\]

and the natural (static) boundary conditions for

- The prescribed vector field of the tractions on the part of the body surface \( S_d \)

\[
p_{pr} = (T_2 - T_3 \cdot \nabla - T_3 \cdot \nabla_s) \cdot \mathbf{n} + T_3 \cdot [(\mathbf{n} \cdot \nabla_s) \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \nabla_s], \tag{63}
\]

- The prescribed double tractions in normal direction on the \( S_d \)

\[
r_{n,pr} = T_3 \cdot \mathbf{n} \otimes \mathbf{n}, \tag{64}
\]

- The prescribed line forces on edge on the part of edge \( C_d \)

\[
c_{pr} = T_3 \cdot \mathbf{n} \otimes \mathbf{m}. \tag{65}
\]

Here, the subscript \( pr \) is short for “prescribed.”

The kinematic boundary conditions in terms of the displacement \( \mathbf{u} \) on the part of the body surface \( S_g \) \((S_d \cup S_g = S)\) and on the part of the surface edge follow from equation (61)

\[
\mathbf{u}_{pr} = \mathbf{u} \text{ on } S_g, \tag{66}
\]

\[
D\mathbf{u}_{pr} = D\mathbf{u} \text{ on } S_g, \tag{67}
\]

\[
\mathbf{u}_{pr} = \mathbf{u} \text{ on } C_g. \tag{68}
\]

3. Variational principles

In this section, we present the proof of two variational principles for the coupled strain-gradient elasticity theory. One is the principle of the minimum potential energy, the other is the principle of the minimum complementary energy.

3.1. Principle of minimum potential energy

**Theorem 1.** Let us consider the linear elastic gradient material with the positive definite potential energy density, let \( \mathfrak{X} \) be the set of all (compatible) vector fields \( \mathbf{u}(\mathbf{x}) \) on the body which fulfil the displacement
boundary conditions on the part of the body surface \( S_g \) \((S_d \cup S_g = S)\) and on the part of the surface edge \( C_g \) \((C_d \cup C_g = C)\). We define the following functional \( \Phi : \mathfrak{A} \to \mathbb{R} \)

\[
\Phi(u) = \int_V w dV - \int_V u \cdot f dV
- \int_{S_d} (u \cdot p_{pr} + Du \cdot r_{npr}) dS - \int_{C_d} u \cdot c_{pr} dC,
\]

where the following fields are prescribed:

- The body force \( f \) in the interior of the body \( V \),
- The tractions \( p_{pr} \) on the body surface \( S_d \),
- The double tractions in normal direction \( r_{npr} \) on the body surface \( S_d \),
- The line forces on edge \( c_{pr} \) on the edge \( C_d \) of the body surface \( S_d \),

where \( p_{pr} \), \( r_{npr} \) and \( c_{pr} \) are defined in equations (63) – (65). Here \( w \) is the potential energy density described by equation (1), and \( T_2 \) and \( T_3 \) are the stresses and the double stresses.

Then the functional \( \Phi \) obtains a minimum for the solution \( u^0 \) of the mixed boundary value problem, that is,

\[
\Phi(u^0) \leq \Phi(u) \quad \forall u \in \mathfrak{A},
\]

If equality holds, then \( u^0 \) and \( u \) differ only by a rigid body displacement

\[
u(x) - u^0(x) = u_c + \Omega \cdot (x - x^0),
\]

where \( u_c \) and \( x^0 \) are two constant vectors and \( \Omega \) is a constant antisymmetric tensor.

**Proof.** Let \( u^0 \) be the solution, and \( \delta u = u - u^0 \). Then \( \delta u \) fulfils zero-displacement boundary conditions on \( S_g \). Let

\[
\delta E_2 = \text{sym}(\delta u \odot \nabla), \quad \delta E_3 = \delta u \odot \nabla \odot \nabla
\]

and

\[
\delta T_2 = C_4 \cdot \delta E_2 + C_5 \cdots \delta E_3, \\
\delta T_3 = C_5^T \cdot \delta E_2 + C_6 \cdots \delta E_3.
\]

Then

\[
\int_V w(u) dV - \int_V w(u^0) dV = \int_V w(\delta u) dV + \int_V \{E_2^0 \cdot C_4 \cdot \delta E_2 + E_2^0 \cdot C_5 \cdots \delta E_3
+ \delta E_2 \cdot C_5 \cdots E_3^0 + E_2^0 \cdot C_6 \cdots \delta E_3\} dV,
\]

where

\[
E_2^0 = E_2(u^0) = \text{sym}(u^0 \odot \nabla), \\
E_3^0 = E_3(u^0) = u^0 \odot \nabla \odot \nabla.
\]
Given that
\[ T_2 = C_4 \cdot E_2 + C_5 \cdot E_3, \]
\[ T_3 = C_5^T \cdot E_2 + C_6 \cdot E_3, \] (77)
and the identity
\[ \delta E_2 \cdot C_5 \cdot E_3 = E_3 \cdot C_5^T \cdot \delta E_2, \] (78)
We have
\[ \int_V w(u) \, dV - \int_V w(u^0) \, dV = \int_V w(\delta u) \, dV + \int_V \{ T_2^0 \cdot \delta E_2 + T_3^0 \cdot \delta E_3 \} \, dV. \] (79)
The last term of the above equation can be written in accordance with equations (43) and (56) in terms of the displacement variation for zero-displacement boundary conditions on \( S_g \)
\[ \int_V (T_2^0 \cdot \delta E_2 + T_3^0 \cdot \delta E_3) \, dV \]
\[ = - \int_V \delta u \cdot (T_2^0 - T_3^0 \cdot \nabla) \cdot \nabla \, dV \]
\[ + \int_{S_d} \delta u \cdot \{ (T_2^0 - T_3^0 \cdot \nabla) - T_3^0 \cdot \nabla_s \} \cdot n \, dS_d \]
\[ + \int_{S_d} \delta u \cdot \nabla_n \cdot T_3^0 \cdot n \, dS_d \]
\[ + \oint \delta u \cdot [ T_3^0 \cdot n \otimes m ] \, dC_d. \] (80)
Thus
\[ \Phi(u) - \Phi(u^0) = \int_V w(u) \, dV - \int_V u \cdot f \, dV \]
\[- \int_{S_d} (u \cdot p_{pr} + D(u \cdot r_{npr}) \, dS_d - \oint u \cdot c_{pr} \, dC_d \]
\[- \int_V w(u^0) \, dV + \int_V u^0 \cdot f \, dV \]
\[ + \int_{S_d} (u^0 \cdot p_{pr} + D(u^0 \cdot r_{npr}) \, dS_d + \oint u^0 \cdot c_{pr} \, dC_d \]
\[ = \int_V w(\delta u) \, dV + \int_V (\delta u \otimes \nabla \cdot T_2^0) \]
\[ + \delta u \otimes \nabla \otimes \nabla \cdot \mathbb{T}_d^0 \) dV - \int_V \delta u \cdot f dV \]

\[ - \int_{S_d} (\delta u \cdot p_{pr} + D\delta u \cdot r_{n pr}) dS_d - \int_{C_d} \delta u \cdot c_{pr} dC_d \]

\[ = \int_V w(\delta u) dV - \int_V \delta u \cdot (T_2^0 - T_3^0 \cdot \nabla) \cdot \nabla dV \]

\[ + \int_{S_d} (\delta u \cdot p^0 + D\delta u \cdot r^0_{n}) dS_d + \int_{C_d} \delta u \cdot e^0 dC_d \]

\[ - \int_V \delta u \cdot f dV \quad (81) \]

\[ - \int_{S_d} (\delta u \cdot p_{pr} + D\delta u \cdot r_{n pr}) dS_d - \int_{C_d} \delta u \cdot c_{pr} dC_d \]

\[ = \int_V w(\delta u) dV - \int_V \delta u \cdot \{ (T_2^0 - T_3^0 \cdot \nabla) \cdot \nabla + f \} dV \]

\[ + \int_{S_d} \{ \delta u \cdot (p^0 - p_{pr}) + D\delta u \cdot (r^0_{n} - r_{n pr}) \} dS_d \]

\[ + \int_{C_d} \delta u \cdot (e^0 - c_{pr}) dC_d , \]

with

\[ p^0 = (T_2^0 - T_3^0 \cdot \nabla - T_3^0 \cdot \nabla s) \cdot n \]

\[ + T_3^0 \cdot [ (n \cdot \nabla s) n \otimes n - n \otimes \nabla s ] \]

\[ r^0_{n} = T_3^0 \cdot n \otimes n , \]

\[ e^0 = T_3^0 \cdot n \otimes m . \]

Since \( u^0 \) solves the mixed boundary value problem, the three last integrals in equation (81) are zero. Because of the positive definiteness of \( w \) we have \( w(\delta u) \geq 0 \) and therefore

\[ \Phi(u) \geq \Phi(u^0) . \]

Equality holds if and only if

\[ \delta E_2 = \text{sym}(\delta u \otimes \nabla) = 0 , \]

or

\[ \text{sym}(u \otimes \nabla) = \text{sym}(u^0 \otimes \nabla) . \]

Therefore, the two displacement fields can only differ by an (infinitesimal) rigid body motion

\[ \delta u(x) = u(x) - u^0(x) = u_c + \Omega \cdot (x - x^0) , \]

where \( u_c \) and \( x^0 \) are two constant vectors and \( \Omega \) is a constant antisymmetric tensor. If the displacements are prescribed for at least three points that do not lie on a straight line, then

\[ u(x) = u^0(x) . \]
3.2. Principle of minimum complementary energy

**Theorem 2.** Let consider the linear elastic gradient material with positive definite complementary energy, let $\mathcal{C}$ be the binary set of symmetric stress fields $T_2$ and of couple stress fields $T_3$ that obey the equilibrium conditions

$$ (T_2 - T_3 \cdot \nabla) \cdot \nabla + f = 0, \quad (88) $$

and the natural (static) boundary conditions for the following prescribed fields:

- The vector field of the tractions $p_{pr}$ on the part of the surface of the body $S_d$,
- The double tractions in normal direction $r_{npr}$ on $S_d$,
- The line forces on edge $c_{pr}$ on the part of edge $C_d$,

with $p_{pr}$, $r_{npr}$ and $c_{pr}$ defined in equations (63) – (65). Then the functional

$$ \Psi(T_2, T_3) = \int_V w^* dV - \int_{S_d} (u_{pr} \cdot p + D u_{pr} \cdot r_n) dS_g $$

(89)

with the complementary elastic energy $w^*$ equation (29) obtains for the solution $T_2^0, T_3^0$ of the mixed boundary value problem a minimum in $\mathcal{C}$, that is

$$ \Psi(T_2^0, T_3^0) \leq \Psi(T_2, T_3) \quad \forall T_2, T_3 \in \mathcal{C}. \quad (90) $$

If equality holds, then $T_2^0 = T_2$ and $T_3^0 = T_3$. Here $S_4, S_5, S_6$ are the compliance tensors, the stresses $T_2$, and the double stresses $T_3$ are defined in equations (30), (31).

**Proof.** Let $T_2^0, T_3^0$ be the solution of the mixed boundary value problem, and $\delta T_2 = T_2 - T_2^0$ and $\delta T_3 = T_3 - T_3^0$. $\delta T_2$ and $\delta T_3$ fulfil zero-traction boundary conditions on $S_g$ and satisfies

$$ (\delta T_2 - \delta T_3 \cdot \nabla) \cdot \nabla = 0. \quad (91) $$

Then

$$ \int_V w^*(T_2, T_3) dV - \int_V w^*(T_2^0, T_3^0) dV $$

$$ = \int_V \left\{ \frac{1}{2} \delta T_2 \cdot S_4 - \delta T_2 \cdot S_4 \cdot T_2 
+ \delta T_2 \cdot S_5 \cdot T_2 + T_2 \cdot S_5 \cdot \delta T_3 + \delta T_2 \cdot S_5 \cdot T_3 
+ \frac{1}{2} \delta T_3 \cdot S_6 \cdot \delta T_3 + \delta T_3 \cdot S_6 \cdot \delta T_3 
\right\} dV $$

(92)

$$ = \int_V \left\{ w^*(\delta T_2, \delta T_3) + \delta T_2 \cdot (S_4 \cdot T_2 + S_5 \cdot T_3) 
+ \delta T_3 \cdot (S_6 \cdot T_3 + S_6^T \cdot T_2) \right\} dV. $$

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Given that
\[ E_2 = S_4 \cdot T_2 + S_5 \cdots T_3, \]
\[ E_3 = S_4^T \cdot T_2 + S_6 \cdots T_3, \]

We obtain
\[ \int_V w^* (T_2, T_3) dV - \int_V w^* (T_2^0, T_3^0) dV = \int_V \{ w^* (\delta T_2, \delta T_3) + \delta T_2 \cdot E_2 + \delta T_3 \cdots E_3 \} dV. \]

Accounting for equation (2), we obtain (see details in Nazarenko et al. [32])
\[ \int_V (\delta T_2 \cdot E_2 + \delta T_3 \cdots E_3) dV = -\int_V u \cdot (\delta T_2 - \delta T_3 \cdot \nabla) \cdot \nabla dV + \int_S u \cdot \{(\delta T_2 - \delta T_3 \cdot \nabla - \delta T_3 \cdot \nabla_s) \cdot n + \delta T_3 \cdot [n \cdot \nabla_s] n \otimes n - n \otimes \nabla_s] \} dS + \int_S u \otimes \nabla_n \cdot \delta T_3 \cdot n dS + \int_C u \cdot [\delta T_3 \cdot n \otimes m] dC. \]

Thus, since \( u = u_{pr} \) in \( S_g \) \((S_g = S \backslash S_d)\) we have with equation (91) and the zero-traction boundary conditions on \( S_d \)
\[ \int_V (\delta T_2 \cdot E_2 + \delta T_3 \cdots E_3) dV = \int_{S_g} u_{pr} \cdot \{(\delta T_2 - \delta T_3 \cdot \nabla - \delta T_3 \cdot \nabla_s) \cdot n + \delta T_3 \cdot [n \cdot \nabla_s] n \otimes n - n \otimes \nabla_s] \} dS_g + \int_{S_g} u_{pr} \otimes \nabla_n \cdot \delta T_3 \cdot n dS_g + \int_{C_g} u_{pr} \cdot [\delta T_3 \cdot n \otimes m] dC_g = \int_{S_g} (u_{pr} \cdot \delta p + D_{u_{pr}} \cdot \delta r_n) dS_g + \int_{C_g} u_{pr} \cdot \delta c dC_g, \]
where

\[ \delta p = (\delta T_2 - \delta T_3 \cdot \nabla - \delta T_3 \cdot \nabla_s) \cdot n \]
\[ + \delta T_3 \cdot [(n \cdot \nabla_s) n \otimes n - n \otimes \nabla_s], \]
\[ \delta r_n = \delta T_3 \cdot n \otimes n, \]
\[ \delta c = \delta T_3 \cdot n \otimes m. \]

Finally

\[ \Psi(T_2, T_3) - \Psi(T_2^0, T_3^0) = \int_V w^* (T_2, T_3) \, dV \]
\[ - \int_{S_g} (u_{pr} \cdot p + D u_{pr} \cdot r_n) \, dS_g - \int_{C_g} u_{pr} \cdot c \, dC_g \]
\[ - \int_V w^* (T_2^0, T_3^0) \, dV + \int_{S_g} (u_{pr} \cdot p^0 + D u_{pr} \cdot r_n^0) \, dS_g \]
\[ + \int_{C_g} u_{pr} \cdot c^0 \, dC_g, \]

with

\[ p^0 = p(T_2^0, T_3^0), \quad r_n^0 = r_n(T_2^0, T_3^0), \quad c^0 = c(T_2^0, T_3^0). \]

Accounting for equations (94) and (96) we obtain

\[ \Psi(T_2, T_3) - \Psi(T_2^0, T_3^0) = \]
\[ \int_V w^* (\delta T_2, \delta T_3) \, dV - \int_{S_g} (u_{pr} \cdot \delta p + D u_{pr} \cdot \delta r_n) \, dS_g \]
\[ - \int_{C_g} u_{pr} \cdot \delta c \, dC_g \]
\[ + \int_{S_g} (u_{pr} \cdot \delta p + D u_{pr} \cdot \delta r_n) \, dS_g + \int_{C_g} u_{pr} \cdot \delta c \, dC_g \]
\[ = \int_V w^* (\delta T_2, \delta T_3) \, dV. \]

Therefore, since the positive definiteness of \( w^* \), we have \( w^* (\delta T_2, \delta T_3) \geq 0 \) and consequently

\[ \Psi(T_2, T_3) \geq \Psi(T_2^0, T_3^0), \]

and equality

\[ \Psi(T_2, T_3) = \Psi(T_2^0, T_3^0) \]

holds only if

\[ \delta T_2 = 0, \quad \delta T_3 = 0, \]

or

\[ T_2 = T_2^0, \quad T_3 = T_3^0. \]
Indeed, it is shown by Nazarenko et al. [31] that the complementary energy equation (29) can be presented in a modified form

\[ w'(T_2, T_3) = \frac{1}{2} T_2 \cdot C_4^{-1} \cdot T_2 + \frac{1}{2} T_3^m \cdot \cdot \cdot \cdot (C_6^m)^{-1} \cdot \cdot \cdot T_3^m, \]  

(105)

or

\[ w'(\delta T_2, \delta T_3) = \frac{1}{2} \delta T_2 \cdot C_4^{-1} \cdot \delta T_2 + \frac{1}{2} \delta T_3^m \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (C_6^m)^{-1} \cdot \cdot \cdot \delta T_3^m, \]  

(106)

where \( C_6^m \) is defined in equation (13) and \( T_3^m \) and \( \delta T_3^m \) are determined as

\[ T_3^m = T_3 - C_T \cdot C_4^{-1} \cdot C_5 \cdot T_2, \]
\[ \delta T_3^m = \delta T_3 - C_T \cdot C_4^{-1} \cdot \delta T_2. \]  

(107)

It is evident that \( w'(\delta T_2, \delta T_3) = 0 \) only if \( \delta T_2 = 0 \) and \( \delta T_3 = 0 \). Since \( \delta T_2 = 0 \), we obtain that \( \delta T_3 = 0 \).

4. Conclusion

The variational formulations of the minimum of potential and complementary energies are considered for the coupled strain-gradient elasticity theory. They play an important role in the construction of approximate solutions by numerical methods like FEM and in the evaluation of the upper and lower bounds, like the Voigt and Reuss bounds [40,41] in homogenization theory.

The coupling term in the equations for the stored and complementary energies equations (1) and (29) complicate the proof of the minimum character of these principles. To overcome this, the equation for the potential energy is rewritten with substitute quantities. It leads to decoupling or diagonalization of the quadratic form of the potential energy. Such a diagonalization makes it possible to convert Hooke’s law, to obtain the relations for compliance tensors, to derive a closed-form expression for complementary energy and to prove that the solution of the boundary value problem is indeed minimizes the potential and complementary energies even in the presence of fifth-rank coupling tensor \( C_5 \). An additional complication in the proof of the minimum of complementary energy is related to the change of the independent variable from one displacement field to several stress fields \( (T_2, T_3) \).

It should be noted that the proofs are general and hold for any symmetry of stiffness tensors including non-centro-symmetric materials, even though in terms of the substitute quantities, the coupling stiffness is zero.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: Authors gratefully acknowledge the financial support by the German Research Foundation (DFG) via Project AL 341/51-1.

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References

[1] Cosserat, F. and Cosserat, E. *Théorie des corps déformables*. Paris: A. Hermann et fils, 1909.
[2] Hellinger, E. Die Allgemeinen Ansätze der Mechanik der Kontinua. In: Klein, F. and Müller, C (eds) *Encyklopädie der mathematischen Wissenschaften*, vol. IV–4, Heft 5. Leipzig: Teubner, 1913, pp. 601–694.
[3] Toupin, RA. Elastic materials with couple-stresses. *Arch Ration Mech Anal* 1962; 11(1): 385–414.
[4] Green, AE, and Rivlin, RS. Multipolar continuum mechanics. *Arch Ration Mech Anal* 1964; 17(2): 113–147.
[5] Mindlin, RD. Micro-structure in linear elasticity. *Arch Ration Mech Anal* 1964; 16(1): 51–78.
[6] Mindlin, R. Second gradient of strain and surface-tension in linear elasticity. *Int J Solids Struct* 1965; 1: 417–438.
[7] Mindlin, R., and Eshel, N. On first strain-gradient theories in linear elasticity. *Int J Solids Struct* 1968; 4: 109–124.
[8] Germain, P. The method of virtual power in continuum mechanics—part 2: microstructure. *SIAM J Appl Math* 1973; 25(3): 556–575.
[9] Askes, H, Suiker, ASJ, and Sluys, LJ. A classification of higher-order strain-gradient models: linear analysis. *Arch Appl Mech* 2002; 72: 171–188.
[10] Cordero, NM, Forest, S, and Busso, EP. Second strain gradient elasticity of nano-objects. *J Mech Phys Solids* 2016; 97: 92–124.
[11] Forest, S, Cordero, NM, and Busso, EP. First vs Second gradient of strain theory for capillarity effects in an elastic fluid at small length scales. *Comput Mater Sci* 2011; 50(4): 1299–1304.
[12] Forest, S, and Bertram, A. Formulations of strain gradient plasticity. In: Altenbach, H, Maugin, G, and Eremeyev, V (eds) *Mechanics of generalized continua: Advanced structured materials*, vol. 7. Berlin: Springer, 2011, pp. 137–149.
[13] Eugster, SR, dell’Isola, F, and Steigmann, DJ. Continuum theory for mechanical metamaterials with a cubic lattice substructure. *Math Mech Comp Syst* 2019; 7(1): 75–98.
[14] Georgiadis, HG, and Anagnostou, DS. Problems of the Flamant–Boussinesq and Kelvin type in dipolar gradient elasticity. *J Elast* 2008; 90; 71–98.
[15] Javili, A, dell’Isola, F, and Steinmann, P. Geometrically nonlinear higher-order strain-gradient elasticity with energetic boundaries. *J Mech Phys Solids* 2013; 61: 2381–2401.
[16] Placidi, L, Barchiesi, E, Turco, E, et al. A review on 2D models for the description of pantographic fabrics. *ZAMM: J Appl Math Mech / Zeitschrift Angew Math Mech* 2016; 67(5): 121.
[17] Rahali, Y, Giorgio, I, Ganghoffer, J, et al. Homogenization À la piola produces second gradient continuum models for linear pantographic lattices. *Int J Eng Sci* 2015; 97: 148–172.
[18] Abdoul-Anziz, H, and Seppecher, P. Strain gradient and generalized continua obtained by homogenizing frame lattices. *Math Mech Comp Syst* 2018; 6(3): 213–250.
[19] Volkov-Bogorodsky, DB, Evrushenkov, YG, Zubov, VI, et al. Calculation of deformations in nanocomposites using the block multipole method with the analytical-numerical account of the scale effects. *Comput Math Math Phys* 2006; 46: 1234–1253.
[32] Nazarenko, L., Glüge, R., and Altenbach, H. Uniqueness theorem in coupled strain gradient elasticity with mixed boundary conditions. *Contin Mech Thermodyn* 2022; 34: 93–106.

[33] Kirchner, N., and Steinmann, P. A unifying treatise on variational principles for gradient and micromorphic continua. *Philos Mag* 2005; 85(33–35): 3875–3895.

[34] Polizzotto, C. Gradient elasticity and nonstandard boundary conditions. *Int J Solids Struct* 2003; 40(26): 7399–7423.

[35] Polizzotto, C. A unifying variational framework for stress gradient and strain gradient elasticity theories. *Eur J Mech: A/ Solids* 2015; 49: 430–440.

[36] Gao, XL, and Park, SK. Variational formulation of a simplified strain gradient elasticity theory and its application to a pressurized thick-walled cylinder problem. *Int J Solids Struct* 2007; 44(22): 7486–7499.

[37] Lurie, SA, Belov, PA, Solyaev, YO, et al. Symmetry and applied variational models for strain gradient anisotropic elasticity. *Nanosci Technol: Int J* 2021; 12(1): 75–99.

[38] Eremeyev, VA, Lurie, SA, Solyaev, YO, et al. On the well posedness of static boundary value problem within the linear dilatational strain gradient elasticity. *Zeitschrift Angew Math Phys* 2020; 71(6): 182.

[39] Bertram, A. *Compendium on gradient materials.* 4th ed. 2019, https://www.lkm.tu-berlin.de/fileadmin/lg49/publikationen/bertram/Compendium_on_Gradient_Materials_2019_neu.pdf

[40] Voigt, W. Ueber die Beziehung zwischen den beiden Elastizitätsconstanten isotroper Körper. *Annal Phys* 1889; 274(12): 573–587.

[41] Reuss, A. Berechnung der Fließgrenze von Mischkristallen auf Grund der Plastizitätsbedingung für Einkristalle. *ZAMM: J Appl Math Mech / Zeitschrift Angew Math Mech* 1929; 9(1): 49–58.