Proportional Participatory Budgeting with Substitute Projects

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Abstract. Participatory budgeting is a democratic process for allocating funds to projects based on the votes of members of the community. However, most input methods of voters’ preferences prevent the voters from expressing complex relationships among projects, leading to outcomes that do not reflect their preferences well enough. In this paper, we propose an input method that begins to address this challenge, by allowing participants to express substitutes over projects. Then, we extend a known aggregation mechanism from the literature (Rule X) to handle substitute projects. We prove that our extended rule preserves proportionality under natural conditions, and show empirically that it obtains substantially more welfare than the original mechanism on instances with substitutes.

Keywords: Participatory Budgeting, Social Choice, Fairness

1 Introduction

Participatory budgeting (PB) is gaining increased attention from both researchers and practitioners and is actively in use in cities around the world \cite{17}. This process typically includes several steps. In the first step, citizens suggest and discuss different projects, resulting with a list of feasible projects and their cost estimation. Next, citizens vote on which of the projects they would like to fund. Finally, the votes are aggregated by a mechanism that selects a subset of projects to fund. Different input formats exist that allow voters to express their preferences over projects, such as approval voting, knapsack ranking and additive utility (specified later on).

In the real world, voters’ preferences may exhibit complex relationships between projects. For example, the utility from one project might depend on whether some other project is funded or not \cite{9,10}. Consider for example a city where there are 4 different suggestions to build a large parking lot in different places, as well as two unrelated projects (say, a playground and a library). The budget is sufficient for only four projects. There is a severe parking problem, so most citizens will assign higher utility to parking lots than to the other projects (or rank parking lots higher than other projects). As a result all 4 parking lots are likely to be funded and consume the entire budget, even if one or
two parking lots are sufficient to solve most of the parking shortage. A better outcome that leads to a higher social welfare would have been to fund only two parking lots, and in addition the playground and the library.

The above problem occurs since common mechanisms ignore the fact that the four parking lots are substitutes.

**Our contribution** We first extend the input format suggested by Jain et al. [9] for substitute projects. Then, we modify the iterative Rule X to a mechanism [13] called Substitute Rule X (SRX) by updating the utility of the projects for each voter according to their reported marginal utility functions.

On the theory side, we define a weakened version of proportionality notion from the literature to fit the case of substitutes, and prove SRX preserves proportionality under natural conditions.

We also show empirically that SRX obtains substantially higher welfare than RX on instances with substitutes while maintaining a fair allocation.

### 1.1 Related Work

Past work in PB defines and studies various notions of proportionality for evaluating aggregation mechanisms [12; 5; 6; 2; 11; 14; 16]. We formally define and discuss some of these notions in Section 4.

Other approaches in PB use social welfare to evaluate mechanisms, such as Goel et al. [8] which shows that when using the knapsack input format it is possible to achieve an outcome that maximizes the social welfare. Jain et al. [9] consider special cases where it is possible to find a polynomial time algorithm which maximizes the social welfare.

The tradeoff between welfare and proportionality is studied in [11]. This tradeoff has also been studied in the context of multi-winner elections [15] (which are essentially PB settings with unit cost for projects).

The literature suggests many different ways to represent voter preferences over a discrete set of projects (e.g., input format), such as approval voting [3; 2], knapsack voting [8; 7], ranking [4] and reporting utilities [12] for each of the projects. However, all of those methods ignore substitution relationships among projects. An exception is Jain et al. [9] [10] who tackle the issue of substitute projects. They suggest a way to describe interactions between projects using approval voting, while allowing projects to be substitutes or complimentary to each other. Jain et al. [9] suggest an aggregation mechanism for special cases which optimize social welfare in polynomial time, but disregard the proportionality of the outcome. In contrast, our mechanism, while also polynomial, improves the social welfare while keeping the outcome proportional.

In [10], the authors consider settings with “global” substitutes, i.e., a single partition shared by all voters. The focus of the paper is how to find partitions and not on choosing which projects to fund. Indeed, we adopt the assumption of global partition for our main results, but the model and some of our results extend beyond.
2 Preliminaries

For any \( a \in \mathbb{N} \), we use \( [a] \) to denote \( \{1, \ldots, a\} \). A PB instance is a tuple \( E = (V, A, \text{cost}, L) \), where:

1. \( N = [n] \) is a set of \( n \) voters participating in this election.
2. \( A = \{p_1, \ldots, p_M\} \) is a set of \( M \) projects.
3. \( \text{cost}: A \to \mathbb{R}^+ \) is a function specifying the cost of each project \( p \in A \). The cost for subset of alternatives \( T \subseteq A \) is \( \text{cost}(T) = \sum_{p \in T} \text{cost}(p) \).
4. \( L \in \mathbb{R}^+ \) is the total budget the voters have in order to fund the selected alternatives.

For each voter \( i \in V \) and bundle of projects \( B \subseteq A \), the term \( u_i(B) \) is the utility that voter \( i \) gets for bundle \( B \). For a group of voters \( S \subseteq V \) the utility is defined as \( u_S(B) = \sum_{i \in S} u_i(B) \). A bundle \( B \subseteq A \) of projects is considered feasible if \( \text{cost}(B) \leq L \).

2.1 Preference Modeling

The most expressive way to represent the voters’ preferences is using a utility function \( U_i : 2^A \to \mathbb{R} \) for each voter \( i \in V \). While this option enables us to express all possible preferences, querying the voters for the utility of every feasible bundle can be very tiresome and inaccurate. On the other extreme we can represent whether voters approve each project (approval voting), which is far less expressive than specifying a utility. This simple representation commonly used in the literature \[3, 2\]. Usually in the context of PB, \( k \)-approval is used, where the voters are limited to approving \( k \) projects. Between those two extreme ways of representing voters’ preferences, other representations exist which vary in their expressiveness. The first one is knapsack \[8, 7\], in which voters approve projects while limited to a given budget, rather than the number of projects. In another representation the voters rank the projects in decreasing order of preference \[4\]. Lastly, voters are requested to assign a utility to each individual project \[12\].

Unfortunately, none of the representations above allows for dependencies among projects. To address this issue Jain et al. \[9\] suggested an interaction structure, where the projects are divided upfront into partitions and voters have some utility function over each of the partitions.

We adopt this model with a different notation (for easier usage in our context) and generalize it by letting each voter define its own partitions instead of one global partition. Formally, we let each voter \( i \in [n] \) partition the projects into \( k_i \) partitions \( v_i = \{g_1, \ldots, g_{k_i}\} \) under conditions \( \bigcup_{g \in v_i} g \subseteq A \) and \( \forall k_1 \neq k_2 \in \{1, \ldots, k_i\}, g_{k_1} \cap g_{k_2} = \emptyset \). We allow each voter to have a different number of partitions.

We denote by \( A(i) := \bigcup_{j=1}^{k_i} g_j \) the set of all projects that voter \( i \) is interested in. For each \( i \in V, a \in A(i) \) we denote by \( g_i(a) \) the set of projects that \( i \) considers as substitutes for \( a \). For each partition \( g \in v_i \), voter \( i \) has a marginal utility function \( u_{i,g} : |A(i)| \to \mathbb{R}^+ \) such that \( u_{i,g}(j) \) is the utility to \( i \) for funding the \( j \)th
project in $g$. The utility of projects in a bundle $B \subseteq A$ is additive and defined as $u_i(B) := \sum_{g \in v_i} \sum_{j=1}^{|g \cap B|} u_{i,g}(j)$.

Although the above formalism does not constrain the shape of the utility function, this paper focuses on non-increasing marginal utility functions, in order to capture substitutes. We also require that the first project in each partition of substitutes has the same utility, as in approval voting. Formally, $\forall i \in V, \forall g \in v_i$, the utility $u_{i,g}(1) = I$ where $I$ is a constant (or equal to 0 if the voter does not want any of the projects in $g$).

We provide a few examples for marginal utility functions:

1. Unit demand: This is the strongest case, where the voter does not benefit at all from substitute projects. The voter wants exactly one project from each partition. Formally,
   $$\forall i \in V, \forall g \in v_i, u_{i,g}(j) = \begin{cases} I, & \text{if } j = 1 \\ 0, & \text{otherwise} \end{cases}$$

2. Minimal substitutes: This utility specifies that a voter strongly prefers projects that are not substitutes. The voter prefers to fund a desired project with no substitutes over funding any number of projects that are substitutes. This can be achieved by assigning a utility of $I$ to the first project in each partition, and utility of $\frac{I}{|A|}$ to any addition project in the same partition. Formally,
   $$\forall i \in V, \forall g \in v_i, u_{i,g}(j) = \begin{cases} I, & \text{if } j = 1 \\ \frac{I}{|A|}, & \text{otherwise} \end{cases}$$

3. PAV based utility [19]: The marginal utility for each partition is the reciprocal function. Formally: $\forall i \in V, \forall g \in v_i, u_{i,g}(j) = \frac{I}{j}$. The voter’s utility is then a sum of harmonic series:
   $$u_i(B) = \sum_{g \in v_i} \sum_{j=0}^{|g \cap B|} \frac{I}{j}.$$

3 Aggregating Votes

A PB mechanism $R$ maps a scenario $E$ to a subset of selected projects $R(E) \subseteq A$.

Our starting point is the Rule X (RX) algorithm recently introduced by Peters et al. [12]. RX is an iterative rule, which starts with “allocating” each voter an equal share of the budget $\frac{L}{|V|}$, and initialize an empty outcome $B = \emptyset$; then sequentially adds projects to $B$. At each step, in order to choose some project $p \in A \setminus B$, each voter needs to pay an amount that is proportional to her utility from the project, but no more than her remaining budget (note that with approval utilities this means only agents that approve the project pay). The total payment should cover the cost of the project.
Algorithm 1: qValue

Input:
1. project $p \in A$
2. $\forall i \in V, U_i(p)$

Result: q-value computation for project $p$

if $\sum_{i \in V, U_i(p) > 0} b_i(t) < \text{cost}(p)$ then
    return $\infty$
end

current_utility $\leftarrow \sum_{i \in V} U_i(p)$
cost_leftover $\leftarrow \text{cost}(p)$
removed_voters $\leftarrow \emptyset$

while True do
    current_g $\leftarrow$ cost_leftover/current_utility
    voter_removed $\leftarrow$ False
    for $i \in V \setminus \text{removed_voters}$ do
        if current_g $\times$ U_i(p) $>$ b_i(t) then
            current_utility $\leftarrow$ current_utility $-$ U_i(p)
cost_leftover $\leftarrow$ cost_leftover $-$ b_i(t)
removed_voters $\leftarrow$ removed_voters $\cup$ {i}
voter_removed $\leftarrow$ True
        end
    end
    if voter_removed == False then
        return (cost_leftover/current_utility)
    end
end

Formally, let $b_i(t)$ be the amount of money that voter $i$ is left with just before iteration $t$. We say that some project $p \in A$, is q-affordable if $\exists q \in \mathbb{R}_+$ such that

$$\sum_{i \in V} \min(b_i(t), U_i(p) \cdot q) \geq \text{cost}(p)$$

Where $U_i(p)$ is the utility of voter $i$ for project $p$.

If no candidate project is q-affordable for any $q$, Rule X terminates and returns $B$. Otherwise it selects project $p^{(t)} \notin B$ that is q-affordable for a minimum $q$, where individual payments are given by $c_i(p^{(t)}) := \min\{b_i(t), u_i(p^{(t)}) \cdot q\}$. Then we update the remaining budget as $b_i(t + 1) := b_i(t) - c_i(p^{(t)})$.

The pseudocode for calculating the qValue for a given project is shown in Algorithm 1. The pseudocode for RX is given in Algorithm 2 (without the highlighted line).

We emphasize that RX assumes additive utilities and does not take into account project substitutes (recall the parking lots example from the introduction). In order to handle this issue, we will now extend RX to Substitute Rule X (SRX) which takes into account project substitutes. This is by updating the marginal value of each project after every iteration.
Algorithm 2: (Substitutes) Rule X

Input:
1. $\forall i \in V, \forall g \in v_i, \forall p \in A, u_{i, g}(p)$
2. Budget $L$
3. $\forall p \in A, cost(p)$

Result: Feasible bundle $B \subseteq A$

$B_0 \leftarrow \emptyset$
$\forall i \in V : b_i(0) \leftarrow \frac{L}{|V|}$
t ← 1

while True do

$\forall i \in V, \forall p \in A : U_i(p) \leftarrow u_i(B_{t-1} \cup \{p\}) - u_i(B_{t-1})$ // only in SRX
$p^{(t)} \leftarrow \arg\min_{p \in A \setminus B_{t-1}}[qValue(p, U_{[|A|]}(p))]$
if $qValue(p^{(t)}, U_{[|A|]}(p^{(t)}) = \infty$ then
    return $B_{t-1}$
else
    $B_t \leftarrow B_{t-1} \cup \{p^{(t)}\}$
    $\forall i \in V : b_i(t) \leftarrow \max\{0, b_i(t-1) - U_i(p^{(t)}) \cdot q\}$
    t ← t + 1
end

end

The SRX algorithm can be obtained by adding the highlighted line to Algorithm 2. SRX works similarly to RX, with the exception that the marginal utility of every project is updated before each step.

The following example shows how each of the mechanisms works and the effect of using substitutes projects. The example will use both the RX and SRX rule using the minimal substitutes marginal utility function as described in the Preference Modeling section.

Example 1. Consider the PB scenario $\{V, A, cost, L\}$ where $V = \{v_1, v_2\}, A = \{a, b, c, d, e\}$, and preferences $v_1 = \{(a), (b), (c, d)\}$ and $v_2 = \{(a), (b, e), (c)\}$. Projects in the same partition are substitutes for the voters, e.g., projects $(c, d)$ are substitutes for voter 1. The budget $L = 2$ and cost function: $cost(a) = 1.1, cost(b) = cost(c) = 1, cost(d) = cost(e) = \frac{1}{3}$.

1. We begin with RX. Since RX does not distinguish between substitute projects, all projects with nonzero marginal utility are assigned utility 1. Project $a$ is 0.55-affordable; projects $b$ and $c$ are 0.5-affordable; projects $d$ and $e$ are $\frac{1}{3}$-affordable. RX takes the project with lowest qValue, therefore $d$ will be chosen (using lexicographic tie-breaking), followed by choosing project $e$ in the next iteration as the values didn’t change.

Each voter is left with a budget of $\frac{2}{3}$, and projects are still 0.55-affordable for $a$ and 0.5-affordable for $b$ and $c$. In the next iteration, project $c$ will be chosen and RX will terminate as there are no q-affordable projects. The final bundle of chosen projects is $B = \{c, d, e\}$ and the social welfare is $1+1+1+\frac{1}{3} = 3\frac{1}{3}$ (as voter 1 got two substitute projects) according minimal substitutes utilities.
2. We now use SRX with minimal substitutes marginal utilities: This procedure begins the same way as RX as no project was funded yet and all utilities are 1, therefore projects $d$ and $e$ are funded in the first two iterations.

Since project $b$ is substitute for $v_2$ and project $c$ is substitute for $v_1$, their utility changes to $u_2(b) = u_1(c) = \frac{1}{5}$ in the next iteration, and project $a$ that is still 0.55-affordable. However, projects $b$ and $c$ are now $\frac{2}{5}$-affordable. Project $a$ have the lowest qValue, therefore, it will be chosen, and SRX will terminate as no item is q-affordable anymore. The final bundle of chosen projects is $B = \{a, d, e\}$ and the social welfare is $1 + 1 + 1 + 1 = 4$ (project $a$ provides utility 1 for each voter, project $d$ provides utility 1 for $v_1$ and project $e$ provides utility 1 to $v_2$).

As can be seen from the example, when using RX, voter 1 gets two substitute projects, while when using SRX, the mechanism will prioritize using voter funds for projects that are not substitutes even if they are more costly.

4 Proportionality

Fairness is an important property of mechanisms for PB. One way to measure fairness is whether the mechanism is proportional, in the sense that any group of voters that could guarantee itself a certain amount of utility (by funding projects), is also entitled to some utility. There are various ways to formalize the proportionality requirement. The most strict proportionality requirement is the core, which requires that any set of agents gains (for each of its members) at least the utility they could guarantee with ‘their’ budget [5; 6].

Fain et al. [6] show that integral outcomes (containing indivisible projects only) might not exists in the core. Since it is still desired for a mechanism to be proportional in some level, researchers have defined weaker notions of proportionality. We will focus on two of these notions: Extended Justified Representation (EJR) [12], and Strong-Budget-Proportional-Justified-Representation (Strong-BPJR, which based on PJR for committee elections [1; 14]).

Recall that $A(i)$ contains all the desired projects of voter $i$.

We begin with the following definition:

**Definition 1 (T-cohesive group [12]).** Given a group of voters $S \subseteq V$ and set of projects $T \subseteq A$, we say that group $S$ is $T$-cohesive if $\frac{|S|}{|V|} \geq \text{cost}(T)$ and $T \subseteq \cap_{i \in S} A(i)$.

The following notions of proportionality relay on the definition of $T$-cohesive groups (a definition that we later revise).

**Definition 2 (Extended Proportionality Representation (EJR) [12]).** We say that mechanism $R$ is EJR if for every PB scenario $E$ and every $T$-cohesive group $S$ ($T \subseteq A$), it holds that $\exists i \in S$ such that $|A(i) \cap R(E)| \geq |T|$.

**Definition 3 (Strong-BPJR [2]).** We say that mechanism $R$ is Strong-BPJR if for every PB scenario $E$ and every $T$-cohesive group $S$ ($T \subseteq A$), it holds that $|(\cup_{i \in S} A(i)) \cap R(E)| \geq |T|$. 
Note that a mechanism that is EJR is also Strong-BPJR.

It is important to notice that even though Strong-BPJR is a weaker guaranty than EJR, it is still a useful measure of proportionality. This is because EJR requires funding $|T|$ projects desired by a single voter, while the rest of voters in $S$ may have no desired project funded. This means that even if all voters have $|T| - 1$ desired projects selected, EJR is still violated.

In contrast, Strong-BPJR guarantees that $|T|$ desired projects of the entire group will be funded, and in particular holds in the above example. This illustrates the advantage of Strong-BPJR, which allows the distribution of projects between voters.

T-cohesive groups are groups of voters who have similar preferences (they like the same projects $T$) and are able to fund those projects. Thus such groups are ‘entitled’ to representation. However, in the context of PB with substitute projects, this definition is too strong. Recall the parking lot example, where we have a group $S$ of voters that want all of the 4 parking lots as substitutes and they can fund them alone, this group is T-cohesive (where $T$ contains the 4 parking lots). Lets say that some voters in $S$ also want to fund the library, which costs as much as 3 parking lots. Since the library has no substitute, intuitively they will prefer to fund one parking lot and the library, instead of funding all of the parking lots. This example shows that group $S$ is better off with only 2 non-substitute projects instead of 4 substitute projects.

Insisting on the current definition of EJR or Strong-BPJR will require that some member of the group will have at least 4 desired projects selected, which is redundant and likely to come at the expense of other projects like the library, thereby resulting in a lower welfare overall. To handle such issue it is reasonable to weaken those notions.

We therefore modify the T-cohesive definition as follows:

**Definition 4 (T-cohesive group with substitutes).** Given a group of voters $S \subseteq V$ and set of projects $T \subseteq A$, we say that group $S$ is T-cohesive if $\frac{1}{|T|}|S| \geq \text{cost}(T)$ and $T \subseteq \cap_{i \in S} A(i)$ and $g_i(p_1) \neq g_i(p_2)$ for all $p_1, p_2 \in T$ s.t. $p_1 \neq p_2$.

Notice that both EJR and Strong-BPJR can be applied with either notion of T-cohesive groups. Unless stated otherwise, we use in the remainder of the paper only proportionality notions that rely on Def. 4.

It is important to notice that the new definitions are actually weaker and RX still holds them, but handles substitutes poorly in terms of welfare. We return to discussing welfare in Section 5 and in the remainder of this section we show that the loss of proportionality under the SRX mechanism is minimal.

### 4.1 Proportionality Analysis

This section will go over the two definitions of proportionality and present which of the definitions SRX holds, and under what constraints.

Unfortunately, without any assumptions, the extended rule does not satisfy neither EJR nor Strong-BPJR.
Theorem 1. SRX does not hold EJR and Strong-BPJR.

Proof. Given a PB scenario with 3 voters \{v_1, v_2, v_3\} and 5 projects \{a,b,c,d,e\}, where \(\text{cost}(a) = \text{cost}(b) = \text{cost}(c) = \text{cost}(d) = 1\), \(\text{cost}(e) = \frac{12}{11}\) and the voters’ preferences:

1. \(v_1: \{(b, c), (a, d), (e)\}\)
2. \(v_2: \{(b, a, c), (d), (e)\}\)
3. \(v_3: \{(a, b), (e), (d), (e)\}\)

The total budget given for the project is \(L = 3\).

To aggregate, we use SRX with minimal substitutes marginal utility function and lexicographic tie-breaking. At the first step projects \(a, b, c, d\) are \(\frac{1}{3}\)-affordable, while project \(e\) is \(\frac{1}{11}\)-affordable. Using tie breaking, SRX will choose to fund project \(a\), leaving each voter with \(\frac{2}{3}\).

After choosing project \(a\), the utility of the voters update accordingly \(u_1(d) = u_2(c) = u_3(b) = \frac{1}{5}\), resulting that projects \(a, b, c\) becoming \(\frac{5}{11}\)-affordable, while project \(e\) didn’t changed.

Therefore, SRX will choose to fund project \(e\), leaving the voters with not enough funds for any other project, and SRX stops with the final bundle \(\{a, e\}\).

Looking at the T-cohesive group \(S = \{v_1, v_2, v_3\}\) with \(T = \{b, c, d\}\), it hold that \(\forall i \in S, |A(i) \cap R(E)| = |\bigcup_{i \in S} A(i) \cap R(E)| = 2 < |T| = 3\), i.e. SRX is not EJR and Strong-PBJR. \(\square\)

As can be seen SRX does not holds EJR or Strong-PBJR. However, proportionality may hold in special cases of interest. Such cases are PB instances with global substitutes \[10\] where all voters have the same project partitions, and each voter can choose to either approve a partition or not.

Our main result is that assuming global substitutes is a sufficient condition to guarantee EJR outcomes:

Theorem 2. SRX holds EJR under global substitutes.

Recall that \(B_t\) is the set of projects selected until step \(t\) (included).

Lemma 1. Given T-cohesive group \(S\). Let \(T_t := \{a \in T : g(a) \cap B_t = \emptyset\}\) (i.e. projects in \(T\) that have not been selected and neither were their substitutes).

Let \(c_{t_{\text{min}}}\) be the price of the cheapest project in \(T_{t-1}\) divided by \(|S|\), if at step \(t\) it holds that \(T_t \neq \emptyset\) and \(\forall i \in S b_i(t) \geq c_{t_{\text{min}}}\) then each voter in \(S\), will pay at most \(c_{t_{\text{min}}}\).

Proof. In order to prove the lemma, lets look at step \(t\) of SRX where project \(p\) is chosen. There are two options for \(p\):

1. Project \(p\) is not approved by any voter from \(S\).

In this scenario all voters in \(S\) pays 0, therefore pays at most \(c_{t_{\text{min}}}\).
2. There is a group of voters \( \mathcal{S} \subseteq V \) which approve project \( p \), and there is some \( \mathcal{S}' \subseteq S \) such that \( \mathcal{S} \cap \mathcal{S}' \neq \emptyset \).

First, we will show that \( \forall p' \in T_i, i \in \mathcal{S}' \), \( p_i(p') \geq p_i(p) \), i.e., all voters in \( \mathcal{S}' \) pay for project \( p \) at most \( c_{t_{\min}} \). From the definition of \( T_i \), all projects in it have utility of \( I \) for all voters in \( S \), while the utility of project \( p \) can be either \( I \) or lower (according to the marginal utility) for voters in \( \mathcal{S}' \). Since \( p \) was chosen, it means it has the smallest qValue, therefore it needs to hold \( \forall i \in \mathcal{S}', p_i(p') = \min(b_i, I \cdot q_{p'}) \geq \min(b_i, I \cdot q_p) \geq \min(b_i, U_i(p) \cdot q_p) = p_i(p) \). This holds in particular for the cheapest project in \( T_i \) and therefore \( c_{t_{\min}} \geq p_i(p') \geq p_i(p) \).

\( \square \)

**Proof (of Theorem 2)**. Using Lemma 1 we know that at each step where \( T_i \neq \emptyset \) each voter in \( S \) will pay at most \( c_{t_{\min}} \) (and 0 if none of them approved the project). Notice that the size of \( T_i \) can decrease by at most one at each step a candidate which was approved by some voter in \( S \) was chosen (from the lemma).

Next, we will show that as long as \( \max_{i \in S} |A(i) \cap B_{t-1}| < |T| \) it holds that \( T_i \neq \emptyset \) on step \( t \).

Let us assume towards a contradiction that \( T_i = \emptyset \). This means that all projects in \( T \) or substitutes for them were chosen. However, because all voters have the same project partition, it means that for all \( i \in S \), \( A(i) \cap B_t \neq \emptyset \)—which is a contradiction.

Let \( P(k) \) be the total cost of the \( k \) cheapest projects in \( T \). We will show that as long \( T_i \neq \emptyset \) and \( \forall i \in S, |A(i) \cap B_t| \neq \emptyset \) it holds that each voter in \( S \) used at most \( \frac{P(A(i) \cap B_t)}{|S|} \) of his funds.

First, as long as \( T_i \neq \emptyset \) from Lemma 1 each voter in \( S \) will pay at most \( c_{t_{\min}} \) for the chosen project. Second, we saw that \( T_i \neq \emptyset \) as long as \( \max_{i \in S} |A(i) \cap B_t| < |T| \). From those two facts it follows that if no voter reached \( |T| \) projects, he paid for each of his approved-and-selected projects \( A(i) \cap B_t \) at most \( c_{t_{\min}} \) (in each respective step \( t' < t \) when the project was selected).

Now, at each step \( t' < t \) the chosen project \( p(t') \) falls into one of the categories:

1. \( p(t') \) is not a substitute project of any project in \( T \). In this case \( T_{t'} = T_{t'-1} \) and \( c_{t_{\min}} \) remains the same, so the next project can be funded with lower price.
2. \( p(t') \) has a substitute project \( p' \in T \). Due to global substitutes, it is not possible that the cost of the selected project was higher than that of \( p' \).

This shows that as long \( \max_{i \in S} |A(i) \cap B_t| < |T| \), it holds that each voter in \( S \) used at most \( \frac{P_i(A(i) \cap B_t)}{|S|} \) of his funds for any step \( t \). This means that there is some project in \( T_i \) that can be funded by the voters in \( S \) (because each voter in \( S \) have at least \( \frac{P_i(T)}{|S|} \) starting funds). Therefore, it is not possible for SRX to stop until \( \max_{i \in S} |A(i) \cap B_t| = |T| \) which equal to \( \max_{i \in S} |A(i) \cap B_t| \geq |T| \). This shows that SRX holds EJR according to Theorem 2.

\( \square \)

This shows that SRX holds EJR and Strong-BPJR under global substitutes. In addition, it is possible to show that for multi-winner instances (where all
Theorem 3. SRX holds Strong-BPJR under unit cost constraint if \( \forall i \in V, \forall g \in v_i, u_{i,g}(k) > 0 \) for all \( k \geq 0 \).

The proof for this theorem relies on the fact that the voters in \( S \) cannot waste more than 1 unit of their funds at each iteration. The full proof is given in Appendix 1.A.

To summarise this section, we saw that SRX holds EJR and Strong-PBJR under constraining global substitutes. In addition, SRX can also hold Strong-BPJR under unit-cost constraint (multi-winner instances).

5 Empirical Results

In this section we will run simulations in order to show that SRX improves welfare compared to RX while maintaining a fair allocation. We will focus on two types of utility profiles: Euclidean utilities with unit cost; and global substitutes—two scenarios for which we already know proportionality is guaranteed. Thus if there is a welfare improvement, we know it does not come at the expense of proportionality.

Euclidean utilities

In the first type of simulations, the scenarios are generated on a 2-dimensional Euclidean domain, limited to \( x,y \in [0,1] \) (similar to previous work in the field [18,16]), where voters and projects are represented as 2-dimensional Euclidean points.

The simulation includes 12 possible scenarios: the budget for each scenario can be 10, 20 or 30 units, and the number of categories for the projects can be 5, 15, 25 or 40. All projects have unit costs. For all of the scenarios we generate a PB instances using 100 projects and 100 voters, where their positions are sampled uniformly from \([0,1] \)^2.

After creating a PB instance, the voters preferences are created according to their distance from the projects (measured by Euclidean distance), where the voter has a positive preference for the 10 nearest projects. Desired projects from the same category are considered as substitutes (having utility according to minimal substitutes marginal utility), and all other projects have utility of 0.

Global substitutes

In the second type of simulations we have 96 possible scenarios according to the following \((k, \mu \text{ and } \sigma)\):

1. There are 100 projects which are divided randomly into partitions, each partition size is randomized uniformly from \([1,10]\), such that each partition represents substitute projects.

Note that while EJR may hold in some instances without global substitutes, it is generally computationally hard to verify.
2. There are 100 voters where each voter approves randomly $k$ partitions, where $k$ is chosen from $\{3, 5, 7, 9\}$ (which define the voter preference, and the utilities according to minimal substitutes).

3. The cost of each project is sampled from $N(\mu, \sigma^2)$ (allowing only projects with $0 \leq \text{cost} \leq \text{budget}$), where $\mu \in \{100, 150, 200, 250, 300, 350, 400, 450\}$ and $\sigma \in \{10, 20, 30\}$.

4. Each scenario is given a budget of 3000.

Evaluation

We use standard measures from the literature to evaluate the mechanisms [16]:

**Social Welfare (SW)** - The sum of utilities all voters get for the chosen bundle $B$ according to the defined utility function, i.e. $SW = \sum_{i \in V} u_i(B)$. Notice that $u_i(B)$ is defined according to minimal substitutes.

As mentioned at the start of the section, we look only at simulation which we know to promise proportionality. Therefore, we also check another measurement for fairness called anger ratio.

**Anger Ratio (AR)** - The percentage of the voters that did not receive any project with positive marginal utility. Formally: $AR = 100\frac{|\{v : A(i) \cap B = \emptyset\}|}{|V|}$.

For each type of simulation we run 1000 instances for each scenario, with a total of 12,000 instances for the euclidean utilities and 96,000 instances for the global substitutes. For each such instance we aggregate the votes using the RX mechanism, where the utility for any approved project is 1 (that is, ignoring any reported substitutes), and using minimal substitutes for SRX.

5.1 Simulation Results

Figure 1 shows how the SW and AR change as the number of categories is increasing (on average). As there are more categories, each voter will have less substitutes in their approved projects. This shows that on average SRX succeeds
in achieving higher social welfare compared to RX in all of the scenarios. As the results trend was the same for all budget sizes, the figure shows the results only for budget of 20. Results for all budget sizes can be seen at Appendix 1.B.

As can be seen in the left graph of Figure 1, SRX achieves higher social welfare (on average) in more scenarios compared to RX. When looking at the anger ratio in the right graph of the figure, there are more "happy" voters (who get at least one project they wanted) compared to RX. In the global substitutes simulations the results were similar and can be seen in detail in Appendix 1.B.

One exception is when the budget is limited to 10 (Euclidean utilities), and RX succeeds in achieving more happy voters (more details can be seen at Appendix 1.B).

When considering the simulations in Figure 1 we see that as the number of categories of projects grow, the closer the outcomes of RX and SRX become. This result makes sense as more categories lead to less substitute projects and both mechanisms reach similar social welfare and anger ratio.

Figure 2 presents the results for the Euclidean utilities scenario with unit costs from a different perspective. This figure shows the achieved social welfare (left) and anger ratio (right) by SRX and RX for each of the 1000 instances, visualising how well each of them did compared to the other and in how many instances. As can be seen in the left plot, SRX succeed in achieving better social welfare in more cases compare to RX (55% vs 9%) Moreover, in the cases where SRX achieves better SW, the difference compared to RX is greater than the other way.

When looking at the right plot, there are about the same amount of cases for each rule which succeed in getting better anger ratio (16% SRX better and 15% RX better). Actually, in the cases where the budget is 20, for all amount of categories the percentage of instances where each of the rules is better is
very close to each other (where SRX have a bit more instances). This result is interesting, as anger ratio can also be thought of as a way to quantify how proportional an outcome is, and SRX succeed to compete with RX even through it holds weaker formal notion of proportionality in this scenario.

While Figure 2 shows results only for 25 categories and budget of 20, other scenarios show the same trend. In addition, global substitutes simulations achieve even better results (having more instances where both the social welfare and anger ratio of SRX are better). More details about those cases can be seen in Appendix 1.B

One limitation of the simulations is that we chose very specific marginal utility function for substitute projects. If we would have used a different function for all partitions or different function for each partition, we might have gotten different results.

6 Conclusion And Future Work

|                  | Global substitutes | Unit cost |
|------------------|--------------------|-----------|
| EJR              | V                  | X         |
| Strong-BPJR      | V                  | V         |

Table 1. SRX proportionality summary

In this paper we proposed an input format that allows voters to express substitute relationships over projects in PB. We extended the existing aggregation mechanism Rule X to a new mechanism SRX that considers this input format. We showed that SRX satisfies two notions of proportionality from the literature, summarised in Table 1. First, it satisfies the Extended Justified Representing (EJR) notion of proportionality [12] under global substitutes constraints in which all voters share the same partition over substitutes. Second, it satisfies the Strong-BPJR [2] under conditions of unit costs for projects but without requiring global substitutes.

For both unit-cost and global substitutes scenarios we created synthetic simulations that satisfy EJR and Strong-BPJR proportionality. Our results show that the SRX mechanism achieves higher social welfare than the RX mechanism in all scenarios while still maintaining proportionality. In cases where SRX holds a weaker notion of proportionality than rule RX, SRX still succeeds in lowering voters’ “anger ratio” i.e., more voters receive at least one project they want.

There are several directions we plan for future research: First, we wish to define new notions of proportionality for the case of substitute projects that will guarantee the voters some utility, rather than the number of projects (using the marginal utilities). Second, while this paper focused on proportionality guarantees for SRX, we will study what guarantees it possible to get for the social welfare. This will be followed with far extensive simulations which will include further analysis of SRX using different marginal utility functions and comparing to more mechanisms from the literature, even when proportionality doesn’t hold. Lastly, we will study different family of marginal utility functions that can give better guarantees for both proportionality and social welfare.
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1. A Additional Proofs and notes

Proof for Theorem

Proof. First, let’s notice that at each step SRX have two options to choose a project $p \in A \setminus B_t$, either $p$ not approved by any voter in $S$ or approved by at least one voter $i \in S$.

If $p$ is not approved by any voter in $S$, it means that the funds group $S$ have, hasn’t changed, so we can ignore this step and look at the next one. If $p$ was approved by some voter $i \in S$, then happen two important things. First, the group $S$ of voter used at most 1 from their total funds in order to fund $p$, this is because all projects are unit cost. Secondly, Since $i \in S$ approved $p$, it means that $|\bigcup_{i \in S} A_i \cap B_t|$ increased by one.

This means that in each step, either group $S$ didn’t use their funds or they used at most 1 and they got one more approved project. But since group $S$ is $T$-cohesive, it means that their initial funds are at least $|T|$, this means that in order to finish their funds they must have funded at least $|T|$ projects, which means that $|\bigcup_{i \in S} A_i \cap B_t| \geq |T|$. 

1. B Simulation

Figure and Figure show how the size of the project in Euclidean utilities simulations affect the social welfare and anger ratio achieved by RX and SRX.
As expected the gap between SRX and RX is getting larger as there is a bigger budget, this is because when there is a small budget there isn’t a chance to decide between two projects where one of them is substitute to another funded project. Need to say, that in the case where the budget is too small for utilize the advantage of substitutes, RX succeed in achieving better anger ratio (not with big margin).

Figure 5 and Figure 6 shows how the difference in SW and AR changes when letting the voters to approve different number of projects and using different cost distribution for the projects in global substitutes simulations.

When considering those simulations, we see that as voters are allowed to approve more projects, the difference in social welfare between SRX and RX increases. This also makes sense, since when voters approve more project partitions, SRX can achieve better social welfare by splitting the chosen projects between more partitions. In contrast, the difference in anger ratio between SRX and RX decreases. In this case, only the voters that want completely different projects from the rest of the population will be hard to satisfy, therefore, there won’t much difference between SRX and RX.

In addition when looking at the effect of different cost distribution, the lower the $\sigma$ chosen for the cost, the higher the difference in voter satisfaction and the lower the difference in anger ratio. This happens because the bigger the $\sigma$ is, there are more cheaper projects, this way it is easier to ignore the the expensive ones and look only the cheap, helping avoid the need to choose between cheap substitute project compared to expensive not substitute project. Lastly,
we didn’t saw any significant change when using different mean for the cost, therefore, the figure shows only for mean=300.

Figure 7 and Figure 9 better demonstrates the social welfare SRX and RX succeed achieving in both types of simulations. As mentioned in Section 5, SRX succeed in achieving better social welfare in most instances. In addition the difference in social welfare when SRX is better is bigger than the other way. This is especially notable in Figure 9 where the few instances where RX is better, the difference in SW is pretty much negligible.

When look at the results from the unit cost scenario, it possible to better notice the effect of #categories and the budget. First, when the budget is small, there isn’t a lot of place to "manipulate" the outcome, and in most cases both mechanism result with the same SW. When the budget increase the more power SRX has when considering substitute projects, this is why there are more instances where SRX get better SW than RX when the budget 30 compared to 20.

When looking how the number of categories affect the social welfare, as said in Section 5 SRX are RX are getting closer results as the effect of substitute projects become less meaningful.

Lastly, in Figure 9, it is possible to see that SRX is better in almost all instances, and even when RX is better, it is by a small difference. This results is especially interesting due the fact that in global substitutes simulations, SRX holds EJR in all instances, same as RX. This show that SRX can achieve better outcomes compared to RX without losing level of proportionality in the process.
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Fig. 6. Average anger ratio achieved by RX and SRX for different cost distribution in global substitutes simulations.

Figure 6 and Figure 10 better demonstrates the anger ratio SRX and RX succeed achieving in both types of simulations. As mentioned in Section 5, SRX have very close percentage of instances where it achieve better anger ratio to percentage RX achieve.

Looking more closely at Figure 8, it possible to better notice the effect of #categories and the budget. Both when the number of categories increase and the budget decrease, SRX and RX are getting more similar anger ratios. This happen because when the number of categories increase, each voter will have less substitutes and the effect of SRX will be more negligible. In addition when having smaller budget, SRX can have less option to "manipulate" the outcome, as it is not enough to fund many substitutes.

Figure 10 demonstrate the disadvantage of relying on EJR, since there are cases where this definition is problematic. This is because it is possible to hold EJR when a very small amount of voters get several projects they want, while the rest of population doesn’t receive any project at all. The measurement of anger ratio present this well as can be seen in the Figure 10. In this scenario, both SRX and RX hold EJR, meaning they both are proportional, however, SRX have much more instances where every (or less voters compared to RX) voter get at least one project he wanted.

The results from the simulations help to show that SRX compete well against RX. It succeeds in achieving better social welfare, while still satisfying big part of the voters, even in cases where it doesn’t hold EJR.
Fig. 7. SRX vs RX social welfare for different instance of Euclidean utilities simulations.
Fig. 8. SRX vs RX anger ratio for different instance of Euclidean utilities simulations.
Fig. 9. SRX vs RX social welfare for different instance of global substitutes simulations.
Fig. 10. SRX vs RX anger ratio for different instance of global substitutes simulations.