The Novikov–Veselov hierarchy of equations and integrable deformations of minimal Lagrangian tori in $\mathbb{C}P^2$

A. E. Mironov

Abstract
We associate a periodic two-dimensional Schrödinger operator to every Lagrangian torus in $\mathbb{C}P^2$ and define the spectral curve of a torus as the Floquet spectrum of this operator on the zero energy level. In this event minimal Lagrangian tori correspond to potential operators. We show that Novikov–Veselov hierarchy of equations induces integrable deformations of minimal Lagrangian torus in $\mathbb{C}P^2$ preserving the spectral curve. We also show that the highest flows on the space of smooth periodic solutions of the Tzizéica equation are given by the Novikov–Veselov hierarchy.

1 Introduction
The surface $\Sigma$ in $\mathbb{C}P^2$ is called Lagrangian if the restriction of the Fubini-Studi form on $\Sigma$ is equal to zero. Let $S^5$ be a unit sphere in $\mathbb{C}^3$, and $\mathcal{H} : S^5 \to \mathbb{C}P^2$ — a Hopf bundle. Define a conformal Lagrangian immersion $\varphi : \Omega \to \mathbb{C}P^2$ of the domain $\Omega \subset \mathbb{R}^2$ as a composition $r : \Omega \to S^5$ and $\mathcal{H}$.

The following lemma holds:

Lemma 1 The components $r_j$ of the vector function $r$ satisfy the Schrödinger equation

$$Lr_j = \partial^2_x r_j + \partial^2_y r_j + i(\beta_x \partial_x r_j + \beta_y \partial_y r_j) + 4e^\nu r_j = 0$$

where $2e^\nu(dx^2 + dy^2)$ is an induced metric on the surface $\varphi(\Omega)$, and $\beta(x, y)$ — a Lagrangian angle defined by the equality

$$e^{i\beta} = dz_1 \wedge dz_2 \wedge dz_3(\sigma),$$
Lemma 1 allows the following definition.

A Lagrangian torus defined by a doubly periodic conformal mapping

\[ \varphi = \mathcal{H} \circ r : \mathbb{R}^2 \to \mathbb{C}P^2 \]

is called finite-gap if the corresponding Schrödinger operator \( L \) with periodic coefficients is finite-gap on a zero energy level, i.e. if the Bloch functions (common eigenfunctions of \( L \) and of the translation operators) of the operator \( L \) on a zero energy level are parametrized by a Riemannian surface \( \Gamma \) of finite genus. The Riemannian surface \( \Gamma \) is called the spectrum of a Lagrangian torus, and its genus — the torus' spectral genus.

The concept of a spectrum and of the torus' spectral genus was first introduced by Taimanov [2] for an arbitrary smooth torus in \( \mathbb{R}^3 \). Here, a Dirac operator serves as an analogue to the Schrödinger operator.

Since the mapping \( \varphi \) is doubly periodic, the components of the vector function \( r \) are Bloch functions of the operator \( L \).

Finite-gap Schrödinger operators with respect to one energy level were first introduced by Dubrovin, Krichever and Novikov [3]. The authors of [3] also indicate data of an inverse problem. These are used to recover the potential and the magnetic field, and to deduce explicit formulae.

The Lagrangian angle of minimum Lagrangian surfaces in \( \mathbb{C}P^2 \) is constant (see, for example, [4]); thus the potential Schrödinger operators

\[ L = \partial_x^2 + \partial_y^2 + 4e^v \]

correspond to them. In this case Lemma 1 is analogous to the assertion that the components of the vector functions that define a conformal minimal immersion of a plane domain into the three-dimensional Euclidean space are harmonic functions.

Clifford’s torus in \( \mathbb{C}P^2 \) has the spectral genus 0. Castro and Urbano [5] and Joyce [6] have constructed examples of minimal Lagrangian tori of spectral genera 2 and 4. Note that the spectral genus of a finite-gap minimal Lagrangian torus is even because the spectral curve of a finite-gap potential Schrödinger operator has a holomorphic involution [1].

Sharipov [7] proved that the metric of a minimal torus in \( S^5 \) satisfies the Tzizéica equation. He also constructed finite-gap solutions of this equation.
Actually, as already noted in [8], the Sharipov construction is suitable for the
collection of all minimal Lagrangian tori in $\mathbb{C}P^2$. For this it is necessary to
apply the Hopf mapping to the mapping from $\mathbb{R}^2$ into $S^5$ that was constructed
by Sharipov.

Since work [7] does not discuss the problem of periodicity of the con-
strued mappings, it leaves unclear the question whether there exist min-
imal Lagrangian tori of arbitrary spectral genus. This gap was filled by a
work of Carberry and McIntosh [9] which proved that for any spectral genus
$g$ there exists a $(\frac{g}{2} - 2)$-dimensional set of minimal Lagrangian tori in $\mathbb{C}P^2$.

By the methods used in [10] it can be shown that all minimal smoothly
immersed Lagrangian tori are finite-gap. In fact, the metric of a minimal
Lagrangian torus fulfills the Tzizéica equation

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4e^{-2\varphi} - 4e^\varphi.$$

Mikhailov [11] proved that the Tzizéica equation is integrable (as an in-
determinate Hamiltonian system). Therefore, all its smooth real doubly
periodic solutions are finite-gap. Finite-gap solutions are characterized by
stationarity with respect to some higher flow [12]. In our case this follows
from the fact that the function $\partial_t \varphi$, where $t_i$ is the highest time, satisfies the
elliptic equation

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 8e^{-2\varphi} + 4e^\varphi)\partial_t \varphi = 0$$

on the torus $\mathbb{R}^2/\Lambda$, where $\Lambda$ is a lattice of periods. Since the spectrum of
the elliptic operator on a torus is discrete, the functions $\partial_t \varphi$ are linearly
dependent and there exists a peak time with respect to which $\varphi$ is stationary.

The work’s basic result consists of the following. Let the map

$$r : \mathbb{R}^2 \to S^5$$

define a finite-gap minimal Lagrangian torus $T \subset \mathbb{C}P^2$ of spectral genus
$g > 4$. Then the following is true:

**Theorem 1** There is a mapping $\tilde{r}(t), t = (t_1, t_2, \ldots), \tilde{r}(0) = r,$ defining a
deformation of torus $T$ in the class of minimal Lagrangian tori in $\mathbb{C}P^2$. The
map $\tilde{r}$ satisfies the equations

$$L\tilde{r} = \frac{\partial^2}{\partial x^2} \tilde{r} + \frac{\partial^2}{\partial y^2} \tilde{r} + 4e^\varphi \tilde{r} = 0,$$
$$\partial_{t_n} \tilde{r} = A_n \tilde{r},$$
where \( A_n \) are operators of order \((2n + 1)\) on the variables \((x, y)\). Deform the potential \( \tilde{V} = 4e^\tilde{v}, \) \( \tilde{v}(0) = v \), according to the Novikov–Veselov hierarchy

\[
\frac{\partial L}{\partial t_n} = [L, A_n] + B_n L,
\]

where \( B_n \) are operators of order \((2n - 1)\) on the variables \((x, y)\). The deformations \( \tilde{r}(t) \) preserve the spectrum of torus \( T \) and its conformal type.

Thus, the highest flows on the space of smooth periodic solutions of the Tzizéica equation are given by the Novikov–Veselov hierarchy.

It can be shown that the deformations corresponding to the Novikov–Veselov hierarchy’s first equation leave the torus geometrically unchanged on it’s spot, while from the second equation the deformations nontrivial.

We suggest, using the first equation of the Novikov–Veselov hierarchy one can construct deformations of any arbitrary Lagrangian torus, with the minimal tori being immovable.

As was shown by Taimanov [14], the local deformations of surfaces in \( \mathbb{R}^3 \) introduced in [13], under the action of the modified Novikov–Veselov equation transform tori into tori preserving the Wilmore functional. Distinct from our construction, [13] defines the deformation of the tori not by the deformation of a radius–vector, but of a Gauss mapping. The proof that the surface remains closed under the action of such deformations substantially uses the characteristics of the modified Veselov–Novikov equation. However, in our case the closure of the surfaces follows from the explicit form \( \tilde{r}(t) \) (see below).

The proof of Theorem 1 is based on Lemma 1 and the Sharipov construction [6].

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## 2 Proof of Theorem 1

Since the map \( \varphi \) is Lagrangian and conformal, it is easily verified [4] that

\[
< r, r_x >= < r, r_y >= < r_x, r_y > = 0, \quad |r_x|^2 = |r_y|^2 = 2e^v,
\]

where \( < ., . > \) is the Hermitian product in \( \mathbb{C}^3 \). Thus, from the definition of the Lagrangian angle \( \beta \) we obtain

\[
R = \left( \begin{array}{c} r \\ e^{-\frac{\beta}{2}} \frac{r_x}{|r_x|} \\ e^{-\frac{\beta}{2}} \frac{r_y}{|r_y|} \end{array} \right) = \left( \begin{array}{ccc} r^1 & r^2 & r^3 \\ \frac{1}{\sqrt{2}}e^{-\frac{\beta}{2}}r^1_x & \frac{1}{\sqrt{2}}e^{\frac{\beta}{2}}r^2_x & \frac{1}{\sqrt{2}}e^{-\frac{\beta}{2}}r^3_x \\ \frac{1}{\sqrt{2}}e^{-\frac{\beta}{2}}r^1_y & \frac{1}{\sqrt{2}}e^{\frac{\beta}{2}}r^2_y & \frac{1}{\sqrt{2}}e^{-\frac{\beta}{2}}r^3_y \end{array} \right) \in SU(3),
\]
where \( r^1, r^2 \) and \( r^3 \) are components of the vector \( r \). The matrix \( R \) satisfies the equations

\[
R_x = AR, \quad R_y = BR, \tag{1}
\]

where matrices \( A \) and \( B \) have the form

\[
A = \begin{pmatrix}
0 & \sqrt{2}e^{\frac{\beta_x}{2}}i & 0 \\
-\sqrt{2}e^{-\frac{\beta_y}{2}} & if & -\frac{v_y}{2} + i(h + \frac{\beta_y}{2}) \\
0 & \frac{v_y}{2} + i(h + \frac{\beta_y}{2}) & -i f
\end{pmatrix} \in \text{su}(3),
\]

\[
B = \begin{pmatrix}
0 & 0 & \sqrt{2}e^{\frac{\beta_x}{2}}i \\
0 & ih & \frac{v_x}{2} + i(-f + \frac{\beta_x}{2}) \\
-\sqrt{2}e^{-\frac{\beta_y}{2}} & -\frac{v_x}{2} + i(-f + \frac{\beta_x}{2}) & -ih
\end{pmatrix} \in \text{su}(3),
\]

\( f(x, y) \) and \( h(x, y) \) are some functions. From the zero curvature equation

\[ A_y - B_x + [A, B] = 0 \]

follows the next lemma (see [15])

**Lemma 2** The following equations hold:

\[
2G_y + 2F_x = (\beta_{xx} - \beta_{yy})e^v,
\]

\[
2F_y - 2G_x = (\beta_y v_x + \beta_x v_y)e^v,
\]

\[
\Delta v = 4(\mathcal{F}^2 + \mathcal{G}^2)e^{-2v} - 4e^v - 2(\mathcal{F}_x + \mathcal{G}_y)e^{-v},
\]

where \( \mathcal{F} = fe^v, \mathcal{G} = he^v \).

From (1) we obtain the equalities

\[
\begin{align*}
r_{xx} &= \frac{1}{2}(-4e^v r + r_x(2if + v_x + i\beta_x) + r_y(2ih - v_y + i\beta_y)), \\
r_{yy} &= \frac{1}{2}(-4e^v r + r_x(-2if - v_x + i\beta_x) + r_y(-2ih + v_y + i\beta_y)).
\end{align*}
\]

From these equalities follows Lemma 1.

Below we consider minimal Lagrangian tori. From Lemma 2 we obtain \( \Delta \mathcal{F} = \Delta \mathcal{G} = 0 \); consequently, since functions \( \mathcal{F} \) and \( \mathcal{G} \) are doubly periodic, \( \mathcal{F} \) and \( \mathcal{G} \) are constants and from Lemma 2 follows the Tzizéica equation.
Consider the following equations with the spectral parameter $\lambda$
\[
\partial_z R(\lambda) = A(\lambda) R(\lambda), \quad \partial_{\bar{z}} R(\lambda) = B(\lambda) R(\lambda),
\]
where $z = x + iy$,
\[
A(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & v_z & -\frac{i}{\lambda} e^{-v} \\ -e^v & 0 & 0 \end{pmatrix}, \quad B(\lambda) = \begin{pmatrix} 0 & 0 & 1 \\ -e^v & 0 & 0 \\ 0 & -i\lambda e^{-v} & v_z \end{pmatrix},
\]
\[
R(\lambda) = \begin{pmatrix} r(\lambda) \\ r_z(\lambda) \\ r_{\bar{z}}(\lambda) \end{pmatrix}.
\]
For $\lambda = 1$ equations (2) are equivalent to the equations (1) ($\beta = 0$), and the zero curvature equation for matrices $A(\lambda)$ and $B(\lambda)$ are equivalent to the Tziz\'eika equation for any $\lambda$. In the case of the finite gap solutions of the equations (2) there is a matrix $W(x, y, \lambda)$ rationally depending on $\lambda$ [16] such that
\[
W_z = [A(\lambda), W], \quad W_{\bar{z}} = [B(\lambda), W].
\]
The coefficients of the rational function on $\lambda$ and $\mu$
\[
Q(\lambda, \mu) = \det(W - \mu E),
\]
where $E$ is a unit matrice, do not depend on $x$ and $y$. The spectrum of a minimal Lagrangian torus is given in the $(\lambda, \mu)$–plane by the equation $Q(\lambda, \mu) = 0$. Hence, the spectrum is the three–sheeted cover of the $\lambda$-plane, i.e. the spectrum is a trigonal curve.

For the construction of the minimal finite-gap Lagrangian tori, recall the following construction. Finite-gap real potential Schrödinger operators are built on the following spectral data [1]: $\Gamma$ is a nonsingular Riemannian surface of even genus $g = 2g_0$, two marked points $\infty_1, \infty_2 \in \Gamma$, a nonspecial divisor $D = P_1 + \ldots + P_g$, local parameters $k_1^{-1}$ and $k_2^{-1}$ near points $\infty_1$ and $\infty_2$. Surface $\Gamma$ should have a holomorphic involution
\[
\sigma : \Gamma \to \Gamma, \quad \sigma^2 = 1,
\]
with two fixed points $\infty_1$ and $\infty_2$ such that
\[
\sigma(k_1^{-1}) = -k_2^{-1}, \quad D + \sigma D = \infty_1 + \infty_2 + K,
\]
where \( s = 1, 2 \), \( K \) is a canonical class on \( \Gamma \). In order for \( L \) to be real, the surface \( \Gamma \) must have an antiholomorphic involution commutative to \( \sigma \)

\[
\tau : \Gamma \to \Gamma, \quad \tau^2 = 1,
\]
such that

\[
D = \tau(D), \quad \tau(\infty_1) = \infty_2, \quad k_1(\tau(P)) = \overline{k_2(P)}.
\]

There is a unique function \( \psi(P, x, y) \) called Baker–Akhiezer function which is meromorphic on \( \Gamma \setminus \{\infty_1, \infty_2\} \) and has simple poles on divisor \( D \) and the following asymptotics

\[
\psi(P, x, y) = \exp(k_1z) \left( 1 + \frac{\xi(x, y)}{k_1} + \ldots \right), \quad P \to \infty_1,
\]

\[
\psi(P, x, y) = \exp(k_2z) \left( 1 + \frac{\eta(x, y)}{k_2} + \ldots \right), \quad P \to \infty_2.
\]

Function \( \psi \) satisfies the Schrödinger equation

\[
\partial_x^2 \psi + \partial_y^2 \psi + 4e^v \psi = 0
\]

where \( e^v = -\xi z - \eta z \).

Below we explain the Sharipov construction to build finite-gap solutions of the Tzizéica equation [7] (in [7] instead of the anti-holomorphic involution \( \tau \) we consider, in our terminology, an anti-holomorphic involution \( \sigma \tau \)). Let curve \( \Gamma \) have a meromorphic function \( \lambda \) with the divisor of zeros and poles \( 3\infty_1 - 3\infty_2 \) such that

\[
\lambda(\sigma(P)) = -\lambda(P), \quad \lambda(\tau(P))\overline{\lambda(\sigma(P))} = 1. \tag{3}
\]

Choose \( k_1 \) and \( k_2 \) such that in the vicinity of \( \infty_1 \) and \( \infty_2 \) function \( \lambda \) has the form

\[
\lambda = ik_1^{-3}, \quad P \to \infty_1,
\]

\[
\lambda = \frac{k_2^3}{i}, \quad P \to \infty_2.
\]

The choice of such spectral data provides for the smoothness and realness of the potential of the Schrödinger operator. From the uniqueness of the Baker–Akhiezer function follow the equalities

\[
\psi_{zz} = \frac{\xi z}{\xi z^2} \psi_z + \frac{k_1^3}{\xi z} \psi_y = v_z \psi_z - \frac{i}{\lambda} e^{-v} \psi_z, \tag{4}
\]

7
\[ \psi_{zz} = \frac{k_z^2}{\eta_z} \psi_z + \frac{\eta_z}{\eta_z} \psi = -e^{-v}i \lambda \psi_z + v \psi_z, \quad (5) \]

\[ \psi_{zz} = \xi_z \psi = \eta_z \psi = -e^v \psi, \quad (6) \]

\[ \psi(P) = \psi(\tau(P)), \quad \psi_z(P) = \psi_z(\tau(P)), \quad \psi_{zz}(P) = \psi_{zz}(\tau(P)). \quad (7) \]

Consider the function

\[ F(P, Q) = <e(P), e(Q)>, \]

where \( e(P) = (\psi(P), \psi_z(P)e^{-\bar{z}}, \psi_{zz}(P)e^{-\bar{z}}) \). From (7) obtain

\[ F(P, Q) = \psi(P)\psi(\tau(Q)) + \psi_z(P)\psi_z(\tau(Q))e^{-v} + \psi_{zz}(P)\psi_{zz}(\tau(Q))e^{-v}. \]

From (3)–(6) obtain

\[ F_z(P, Q) = -ie^{-2v} \left( \frac{1 - \lambda(P)\lambda(Q)}{\lambda(P)} \right) \psi_z(P)\psi_z(\tau(Q)), \]

\[ F_{\bar{z}}(P, Q) = -ie^{-2v} \left( \frac{\lambda(P)\lambda(Q) - 1}{\lambda(Q)} \right) \psi_z(P)\psi_z(\tau(Q)). \]

Function \( \lambda \) gives a three–sheeted cover of \( \mathbb{C}P^1 \) by the curve \( \Gamma \). Let \( \lambda(P_1) = \lambda(P_2) = \lambda(P_3) = 1 \). Then the function \( F(P_i, P_j) \) does not depend on \( x \) and \( y \).

Put

\[ r_j = C_j \psi(P_j), \]

where \( C_j = \frac{1}{|\psi(P_j)|} \). In this case the equations (1) are fulfilled, where matrices \( A \) and \( B \) belong to the Lie algebra su(3) (see [7]).

By the following asymptotics, the Baker–Akheizer function defines the integrable deformations of torus \( T \):

\[ \psi(P, x, y, t) = \exp(k_1 z + \sum_{n=1}^{\infty} k_1^{2n+1} t'_n + \cdots), \quad P \to \infty_1, \]

\[ \psi(P, x, y, t) = \exp(k_2 \bar{z} + \sum_{n=1}^{\infty} k_2^{2n+1} t'_n + \cdots), \quad P \to \infty_2, \]

where \( t'_j = t_j + it_j \). The function \( \psi(P, x, y, t) \) has the properties (4)–(7).
The function $\psi$ can be extracted in terms of a Prym theta-function of the involution $\sigma$ (see [1]). There is a basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$ on $\Gamma$ such that

$$\sigma(a_i) = a_{i+g_0}, \quad \sigma(b_i) = -b_{i+g_0}, \quad i = 1, \ldots, g_0$$

and a corresponding basis of Abelian differentials $\omega_1, \ldots, \omega_g$, with the properties

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}.$$ 

The Prym variety of $(\Gamma, \sigma)$ is

$$P = \mathbb{C}^{g_0}/\{2\pi i \mathbb{Z}^{g_0} + \Omega \mathbb{Z}^{g_0}\},$$

where the components of the symmetric matrix $\Omega$ are periods of the following differentials: $\Omega_{ij} = \int_{b_j} \eta_i$, $\eta_i = \omega_i + \omega_{i+g_0}$. Let $\eta(P)$ denote the map

$$\eta : \Gamma \to P, \quad \eta(P) = \left( \int_{P_0}^{P} \eta_1, \ldots, \int_{P_0}^{P} \eta_{g_0} \right),$$

where $P_0 \in \Gamma$ is some fixed point. By $\Omega_k$ and $\tilde{\Omega}_k$, $k = 0, 1, \ldots$ we denote meromorphic differentials on $\Gamma$ with unique poles in $\infty_1$ and $\infty_2$ of the form $d(k_s^{-s/(2k+1)})$, $s = 1, 2$ and normalized by $\int_{a_j} \Omega_k = \int_{a_j} \tilde{\Omega}_k = 0$. Let

$$V_k = \left( \int_{b_1} \Omega_k, \ldots, \int_{b_{g_0}} \Omega_k \right), \quad \tilde{V}_k = \left( \int_{b_1} \tilde{\Omega}_k, \ldots, \int_{b_{g_0}} \tilde{\Omega}_k \right).$$

The theta-function of the Prym variety is defined by the convergent series

$$\theta(z) = \sum_{n \in \mathbb{Z}^{g_0}} \exp \left( \frac{1}{2} \langle \Omega n, n \rangle + \langle z, n \rangle \right),$$

$z = (z_1, \ldots, z_{g_0}) \in \mathbb{C}^{g_0}$. The theta-function has the properties of periodicity

$$\theta(z + 2\pi i n + \Omega m) = \exp \left( -\frac{1}{2} \langle \Omega m, m \rangle + \langle z, m \rangle \right) \theta(z),$$

$m, n \in \mathbb{Z}^{g_0}$. The function $\psi$ has the following form (see [1]):

$$\psi = \frac{\theta(\eta(P) + zV_0 + \bar{z}\tilde{V}_0 + t'_1V_1 + \bar{t}'_1\tilde{V}_1 + \ldots - e)}{\theta(\eta(P) - e)\theta(zV_0 + \bar{z}\tilde{V}_0 + t'_1V_1 + \bar{t}'_1\tilde{V}_1 + \ldots - e)}.$$
\[
\times \exp \left( z \left( \int_{P_0}^{P} \Omega_0 - \alpha_0 \right) + \bar{z} \int_{\infty_1}^{P} \tilde{\Omega}_0 + t'_1 \left( \int_{P_0}^{P} \Omega_1 - \alpha_1 \right) + \bar{t}_1 \int_{\infty_1}^{P} \tilde{\Omega}_1 + \ldots \right),
\]
\[
\alpha_j \text{ are some constants,} \ e \in \mathbb{C}^{g_0} \text{ is some vector. Put}
\]
\[
\tilde{r}_j(t) = C_j(t) \psi(P_j, x, y, t),
\]
where \( C_j(t) = \frac{1}{|\psi(P_j, x, y, t)|} \).

From the formula for the function \( \psi \) follows that if the map \( \mathcal{H} \circ \tilde{r} \) is periodic for \( t = 0 \), then it is periodic with the same periods for any \( t \). The function \( \tilde{r}_j \) satisfies the equations of Theorem 1 (see [1]). Theorem 1 is proven.

We give an example of a Riemannian surface with the involutions \( \sigma \) and \( \tau \). Let \( \Gamma \) be a smooth supplement of the surface given in the \((\lambda, \mu)\)-plane by the equation
\[
\mu^3 = \mu Q_1(\lambda) + Q_2(\lambda),
\]
where
\[
Q_1(\lambda) = q_{-2k}\lambda^{-2k} + \ldots + q_{2k}\lambda^{2k}, \quad \bar{q}_{-j} = q_j,
\]
\[
Q_2(\lambda) = p_{-(2n+1)}\lambda^{-(2n+1)} + \ldots + p_{2n+1}\lambda^{2n+1}, \quad \bar{p}_{-j} = -p_j.
\]
The surface \( \Gamma \) has the holomorphic involution
\[
\sigma = (\lambda, \mu) = (-\lambda, -\mu)
\]
with two fixed points \( \infty_1 = (0, \infty) \) and \( \infty_2 = (\infty, \infty) \) and the anti-holomorphic involution
\[
\tau(\lambda, \mu) = \left( -\frac{1}{\lambda}, -\bar{\mu} \right).
\]
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Andrey Mironov
Sobolev Institute of Mathematics SB RAS
Pr. acad. Koptyuga 4, 630090, Novosibirsk, Russia.
E-mail: mironov@math.nsc.ru