SECRET SHARING USING NON-COMMUTATIVE GROUPS
AND THE SHORTLEX ORDER

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ABSTRACT. In this paper we review the Habeeb-Kahrobaei-Shpilrain
secret sharing scheme \cite{5} and introduce a variation based on the shortlex
order on a free group. Drawing inspiration from adjustments to classical
schemes, we also present a method that allows for the protocol to remain
secure after multiple secrets are shared.

1. INTRODUCTION

Secret sharing is a cryptographic protocol by which a dealer distributes a
secret via shares to participants such that only certain subsets of participants
can together use their shares to recover the secret. A secret sharing scheme
begins with a dealer, a secret, participants, and an access structure. The
access structure determines which groups of participants have access to the
secret. The goal of the scheme is to distribute the secret to the participants
in such a way that only sets of participants within the access structure have
access to the secret. In this way, it is most often the case that no individual
participant can recover the secret on their own.

Secret sharing schemes are ideal tools for when the secret is both highly
important and highly sensitive. The fact that there are multiple shares, as
opposed to one private key in private key cryptography, makes the secret
less likely to be lost while allowing high levels of confidentiality. If any one
share is comprised the secret can generally still be recovered with the non-
comprised shares. Additionally, even though the secret is spread out over
multiple shares, recovering the secret is limited by the access structure, and
so the secret remains secure. Secret sharing has applications in multiparty
encryption, Byzantine agreement, and threshold encryption among others.
See \cite{1} for a survey on secret sharing and its applications in cryptography
and computer science.

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2. Formal Definition

A secret sharing scheme consists of a dealer, \( n \) participants, \( P_1, \ldots, P_n \), and an access structure \( A \subseteq 2^{\{P_1, \ldots, P_n\}} \) such that for all \( A \in A \) and \( A \subseteq B \), \( B \in A \).

To share a secret \( s \), the dealer runs an algorithm:

\[
\text{Share}(s) = (s_1, \ldots, s_n)
\]

and then distributes each share \( s_i \) to \( P_i \).

In order to recover the secret, participants can run the algorithm \( \text{Recover} \) which has the property that for all \( A \in A \):

\[
\text{Recover}(\{s_i : i \in A\}) = s
\]

and if \( A \notin A \) then running \( \text{Recover} \) is either computationally infeasible or impossible.

As such, only groups of participants in \( A \) can access the secret. The monotonicity of \( A \) is also apparent in that if \( A \in A \) and \( A \subseteq B \) then the set of participants in \( A \) could also recover the secret for \( B \). A secret sharing scheme is called perfect if \( \forall A \notin A \) the shares \( s_i \in A \) together give no information about \( s \).

3. Shamir’s Secret Sharing Scheme

One of the more common access structures one sees in secret sharing is the \((k,n)\) threshold:

\[
A = \{ A \in 2^{\{P_1, \ldots, P_n\}} : |A| \geq k \}
\]

Namely, \( A \) consists of all subsets of the \( n \) participants of size \( k \) or greater. We call a secret sharing scheme that has \( A \) as a \((k,n)\) threshold a \((k,n)\) threshold scheme. The problem of discovering a perfect \((k,n)\) threshold scheme was solved independently by G. Blakely [2] and A. Shamir [13] in 1979.

In the Shamir Secret Sharing Scheme, the secret is an element in \( \mathbb{Z}_p \) where \( p \) is a prime number larger than the number of participants. Given a secret \( s \), the dealer generates the shares for a \((k,n)\) threshold by doing the following:
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- The dealer randomly selects \( a_1, \ldots, a_{k-1} \in \mathbb{Z}_p \) such that \( a_{k-1} \neq 0 \) and constructs the polynomial \( f(x) = a_{k-1}x^{k-1} + \cdots + a_1x + s \).
- For each participant \( P_i \) the dealer publishes a corresponding \( x_i \in \mathbb{Z}_p \). The dealer then distributes the share \( s_i = f(x_i) \) to each \( P_i \) over a private channel.

Any subset of \( k \) participants can then reconstruct the polynomial \( f(x) \) by using polynomial interpolation and then finding \( f(0) = s \). This method finds \( s \) uniquely as any degree \( k - 1 \) polynomial is uniquely determined by the \( k \) shares. The shares are consistent because each \( (x_i, f(x_i)) \) is a point on the polynomial \( f(x) \) and thus any \( k \) shares will reconstruct the same polynomial. In order to reconstruct a polynomial \( f(x) = a_0 + a_1x + \cdots a_{k-1}x^{k-1} \) given points \( (x_1, f(x_1)), \ldots, (x_k, f(x_k)) \) one can solve for the coefficients column in the following system of linear equations:

\[
\begin{pmatrix}
  x_1^{k-1} & \cdots & x_1 & 1 \\
  x_2^{k-1} & \cdots & x_2 & 1 \\
  \vdots  & \ddots & \vdots & \vdots \\
  x_k^{k-1} & \cdots & x_k & 1 \\
\end{pmatrix}
\begin{pmatrix}
  a_{k-1} \\
  a_{k-2} \\
  \vdots \\
  a_0 \\
\end{pmatrix}
= \begin{pmatrix}
  f(x_1) \\
  f(x_2) \\
  \vdots \\
  f(x_k) \\
\end{pmatrix}
\]

The above method of interpolation demonstrates that Shamir’s scheme is perfect. If there were less than \( k \) shares, then the system of equations above would have more equations than unknowns, and there would not be a unique solution for \( a_0 \).

4. SECRET SHARING USING NON-COMMUTATIVE GROUPS

Given a set of letters \( X = \{x_1, x_2, \ldots, x_n\} \) we define the free group generated by \( X \), \( F(X) \), as the set of reduced words in the alphabet \( X^{\pm 1} = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\} \), where a word is reduced if there are no subwords of the form \( x_i^{-1}x_i \) or \( x_ix_i^{-1} \). Given a set of words \( R \subset F(X) \) we define \( \langle \langle R \rangle \rangle \) as the smallest normal subgroup of \( F(X) \) containing \( R \) and define the group \( G = \langle X | R \rangle = F(X)/\langle \langle R \rangle \rangle \). We call \( R \) the set of relators of \( G \).

A group \( G = \langle X | R \rangle \) has a solvable word problem if there exists an algorithm to determine if any word \( w \in G \) is trivial. Habeeb-Kahrobaei-Shpilrain (HKS) secret sharing [5] uses a group with an efficiently solvable word problem to create an \((n, n)\) threshold scheme which can be extended to a \((k, n)\) threshold scheme using the method of Shamir.
4.1. \((n, n)\) Threshold. In this case the secret, \(s\), is an element of \(\{0, 1\}^k\) which we view as a column vector. The setting is initialized by making a set of generators \(X = \{x_1, \cdots, x_n\}\) public. To distribute the shares the dealer does the following:

- Distributes to each \(P_i\) over a private channel a set of words \(R_i\) in the alphabet \(X^\pm\) that define the group \(G_i = \langle X \mid R_i \rangle\)
- Randomly generates the shares \(s_i \in \{0, 1\}^k\) for \(i = 1, \cdots, n - 1\) and \(s_n = s - \sum_{j=0}^{n-1} s_j\) where the addition is bitwise addition in \(\mathbb{F}_2^k\).
- Publishes words \(w_{ji}\) over the alphabet \(X^\pm\) such that a word \(w_{ji}\) is trivial in \(G_i\) if \(s_{ji} = 1\) and non-trivial if \(s_{ji} = 0\).

Since the \(G_i\) have efficiently solvable word problem, the participant \(P_k\) can determine which of the \(w_{jk}\) are trivial or non-trivial and can independently recover \(s_k\). To recover the secret, the \(P_i\) add the \(s_i\) and find \(s\). Note that even though the \(w_{ji}\) are sent over an open channel, the shares remain secure since the \(R_i\) are private. Therefore no other participant can recover \(s_i\) from the \(w_{ji}\) since only \(P_i\) knows \(G_i\).

4.2. \((k, n)\) Threshold. One can extend the above scheme to a \((k, n)\) threshold via Shamir’s scheme. As is the case with Shamir’s scheme, the secret \(s\) is an element of \(\mathbb{Z}_p\) and the shares, \(s_i\), correspond to points on a polynomial of degree \(k - 1\) with constant term \(s\). The shares are distributed and reconstructed in an identical manner as above by viewing the \(s_i\) in their binary form. The trivial and non-trivial words are sent to each \(P_i\) so that they reconstruct each \(s_i\) in its binary form. After recovering their shares any element of the access structure can use polynomial interpolation to find \(s\):

- The dealer randomly selects \(a_1, \cdots, a_{k-1} \in \mathbb{Z}_p\) such that \(a_{k-1} \neq 0\) and constructs the polynomial \(f(x) = a_{k-1}x^{k-1} + \cdots + a_1x + s\)
- For each participant \(P_i\) the dealer publishes a corresponding \(x_i \in \mathbb{Z}_p\). The dealer then converts each \(s_i = f(x_i)\) into binary. And thus, each \(s_i\) can be viewed as a column vector of length \(l = \log_2 p + 1\)
- As was the case in the \((n, n)\) scheme, the dealer distributes the \(s_i\) over an open channel by sending each \(P_i\) the words \(w_{1i}, \cdots, w_{li}\) over the alphabet \(X^\pm\) such that \(w_{ji}\) is trivial in \(G_i\) if \(s_{ji} = 1\) and non-trivial if \(s_{ji} = 0\).
- The participants reconstruct their own \(s_i\) and can recover the secret using polynomial interpolation.
Some advantages this secret sharing scheme has over Shamir’s scheme include the fact that after the $R_i$ are distributed, one can still use them to send out and reconstruct more secrets rather than having to privately distribute new shares each time a different secret is picked. Private information has to only be sent once initially for an arbitrary amount of secrets to be shared due to the method of distributing the shares. Despite this, the scheme is vulnerable to an adversary determining the relators by potentially seeing patterns in words they learn are trivial. Namely, after a participant reveals their share (possibly while recovering the secret) an adversary could potentially determine which of the $w_{ji}$ were trivial and possibly determine the group presentation of $G_i$ which would allow them to construct $P_i$’s share on their own. We will discuss this weakness later, but using certain methods it is unlikely that any adversary could feasibly do this. Another advantage to this scheme is that since it is based on the Shamir secret sharing protocol it can benefit from the large amount of research done on Shamir’s scheme. For instance, the verification methods or proactive secret sharing protocols from [14] and [6] can still be used in this scheme.

4.3. Small Cancellation Groups. In this section we introduce a candidate group for the above secret sharing scheme.

A word $w$ is cyclically reduced if it is reduced in all of its cyclic permutations. Note that this only occurs if the word is freely reduced, it has no subwords of the form $x_i^{-1}x_i$ or $x_ix_i^{-1}$, and the first and last letters are not inverses of each other.

A set of words $R$ is called symmetrized if each word is cyclically reduced and the entire set and their inverses are closed under cyclic permutation. If $R$ is viewed as a set of relators, symmetrizing $R$ does not change the resulting group as the closure $\bar{R}$ under cyclic permutations and inverses is a subset of the normal closure.

Given a set $R$ we say that $v$ is a piece if it is a maximal initial subword of two different words, namely if there exist $w_1, w_2 \in R$ such that $w_1 = vr_1$ and $w_2 = vr_2$. A group $G = \langle X | R \rangle$ satisfies the small cancellation condition $C'(\lambda)$ for $0 < \lambda < 1$ if for all $r \in R$ such that $r = vw$ where $v$ is a piece, then $|v| < \lambda |r|$. 
Small cancellation groups satisfying $C'(\frac{1}{6})$ have the additional property that their word problem is solvable in quadratic time making them an ideal candidate for the HKS secret sharing scheme. Moreover, it can be seen from their definition that if the number of generators and the length of the relators are large compared to the number of relators, it is likely that there will be small cancellation since the probability that any two words have a large maximal initial segment is low. After generating a random set of relators satisfying the above properties, it is also fast to symmetrize the set and then find the pieces and check that they are no larger than one sixth of the word. As such, it is fast to create such groups by repeatedly randomly generating relators, symmetrizing, and checking to see if they satisfy the $C'(\frac{1}{6})$ condition. There are other groups that have an efficient word problem that could also function as candidate groups, but small cancellation groups have the advantage of being efficient to generate randomly.

4.4. Secret Sharing and the Shortlex Ordering. Let $X = \{x_1, \cdots, x_n\}$ and $G = \langle X \rangle$. A shortlex ordering on $G$ is induced by an order on $X^{\pm 1}$ as follows. Given reduced $w = x_{i_1} \cdots x_{i_p}$ and $l = x_{j_1} \cdots x_{j_k}$ with $w \neq l$ then $w < l$ if and only if:

- $|w| < |l|
- \text{ or if } p = k \text{ and } x_{i_a} < x_{j_a} \text{ where } a = \min \{x_{i_a} \neq x_{j_a}\}$

For example, let $X = \{x, y\}$ and give $X^{\pm}$ the ordering $x < x^{-1} < y < y^{-1}$. Then some of the first words in order would be:

\[
e < x < x^{-1} < y < y^{-1} < x^2 < xy < xy^{-1} < x^{-2} < x^{-1}y < x^{-1}y^{-1} < yx < yx^{-1} < y^2 < y^{-1}x < y^{-1}x^{-1} < y^{-2} < x^3 < x^2y < x^2y^{-1} < xyx < xyx^{-1} < \cdots\]

This method of counting group elements can be used to combine group theory with the numerical aspects present in many other cryptographic schemes. For instance, the shortlex ordering can be used to modify the secret sharing scheme above:

- In this case, the dealer publishes the letters $X$ and over a private channel sends a set of words, $R_i$ in $X^{\pm 1}$ to each $P_i$ such that $G_i = \langle X|R_i \rangle$ is a group with an efficient algorithm to reduce words to some normal form with respect to the $R_i$.
- The dealer chooses a secret $s \in \mathbb{Z}_p$ for some large prime $p > n$ and generates a random polynomial, $f$ in $\mathbb{Z}_p[x]$ with constant term $s$
The dealer assigns a public $x_i \in \mathbb{Z}_p$ to each participant, computes $f(x_i)$, and finds $s_i \in F(X)$ such that $s_i$ is the $f(x_i)$th word in $F(X)$. Note that $x_i$ is not a generator of $G$, but rather the $x$-coordinate associated to each participant’s share.

The dealer publishes a word $w_i$ that reduces to $s_i$ in $G_i$. This can be done efficiently by interspersing conjugated products of relators between the letters of $s_i$.

Each participant $P_i$ computes their share by reducing $w_i$ to get $s_i$ and then computing its position in $F(X)$.

Using their shares they find the secret using polynomial interpolation.

The main advantage of this new method is that participants need only reduce one word rather than a number of words corresponding to the length of the secret. $C'(\frac{1}{n})$ continue to be an ideal platform for this protocol because reducing words to a minimal length has polynomial time complexity using a deterministic version of Dehn’s algorithm. In fact, it is the same algorithm used to solve the word problem. In general, being able to reduce words in this fashion is more general than being able to solve the word problem and in some cases may be more complex.

It is important to note a few things about this scheme:

- $s_i$ must be in some cannonical or reduced form in $G_i$. If a random $f(x_i)$ does not correspond to a cannonical or reduced word, the dealer can always assign $P_i$ a different $x_i$.
- Some reduction algorithms can be done in multiple ways given the same initial conditions, so it is important to fix a protocol so that whatever process $P_i$ uses to reduce $w_i$ terminates at $s_i$.
- $C'(\frac{1}{n})$ have the property that any reduced word of length less than half of any relator is in a unique minimal length form. Hence, if the lengths of the $s_i$ are restricted, then it is not even necessary to worry about the first two concerns. The shortlex order in general grows exponentially with respect to the length of a word, which means that if the generating set grows linearly, the amount of words of a given length grows exponentially. Therefore in most cases the lengths of the $s_i$ can be small when the corresponding $f(x_i)$ are large.
4.5. **Efficiency.** Each step in modified HKS scheme can be done efficiently. As mentioned previously, generating $C'(\frac{1}{6})$ groups can be done quickly by repeatedly generating set of relators and checking to see if they satisfy the necessary small cancellation condition. See [7] for algorithms to efficiently generate random group elements. The necessary computations using the shortlex ordering can be done using basic combinatorial formulas that are very fast for a computer to evaluate. Additionally, the $w_i$ can be created efficiently using methods from [11] and then reduced in quadratic time with Dehn’s algorithm. Moreover, when using $C'(\frac{1}{6})$ groups, it is not necessary to check that each $w_i$ reduces to $s_i$ in $G_i$ as long as the length of $s_i$ is less than the length of half of any of the relators. Hence each additional step to the standard Shamir’s scheme can be done efficiently. This is also an improvement over the standard HKS scheme since the amount of words that need to be reduced is independent of the length of the secret, making it possible for larger secrets to be distributed efficiently.

4.6. **Updating Relators.** The main security concern for this cryptoscheme is the possibility of an adversary discovering a participant’s set of relators. This can either be done using information gained from combining shares, but even potentially just from the public $w_i$. As more secrets are shared, the original set of relators becomes less secure. Moreover, information may be discovered either by breaking into wherever a participant stores their relators or if partial information was discovered during the initial step. In this section we present a method to refresh a participant’s relator set using the same inherent security assumptions necessary for the cryptoscheme, namely that at least one round of secret sharing is secure.

To do this we add steps that can take place before any new secret is sent out:

- For each $P_i$ the dealer creates a set of words, $R'_i$, in $X^{\pm 1}$ such that $G_i = \langle X \mid R'_i \rangle$ satisfies the same desired properties.
- In order to distribute each $r \in R'_i$, the dealer pads $r$ with relators in $R_i$ as done previously and publishes them.
- $P_i$ then reduces $r$ by using the relators in $R_i$. 
- After the full set of words in $R'_i$ is published and reduced, $P_i$ deletes the original $R_i$ and sets $R_i := R'_i$.

If these steps are done before an adversary can gain adequate information about relators, then after an update phase the information an adversary has
gained will be largely rendered useless. Also note that a single secret can be kept secure over a long period of time using the methods in [6]. In this case, it is important that the words in $R_i'$ are reduced with respect to the original $R_i$. As such, $R_i$ and $R_i'$ are not completely unrelated, but as the relators become updated each additional time, they will have less and less to do with the original set of relators.

5. Conclusion

In this paper we propose a modification of the HKS secret sharing scheme using the shortlex ordering on free groups. It improves the original scheme by removing the relation of the number of times each participant has to solve the word problem to the length of the secret. As such, larger secrets can be shared efficiently and the overall scheme is more efficient. Moreover, it shares the advantage over Shamir's scheme that multiple secrets can be shared given the same initial private information. We also introduce a method to update relators so that the scheme remains secure when arbitrarily many secrets are shared and that does not involve more private information being distributed.

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References

[1] Amos Beimel. Secret-sharing schemes: a survey. In Proceedings of the Third international conference on Coding and cryptology, IWCC’11, pages 11–46, Berlin, Heidelberg, 2011. Springer-Verlag.
[2] G.R. Blakley. Safeguarding cryptographic keys. In Proceedings of the 1979 AFIPS National Computer Conference, pages 313–317, Monval, NJ, USA. AFIPS Press.
[3] Joan Feigenbaum, editor. Advances in Cryptology - CRYPTO '91, 11th Annual International Cryptology Conference, Santa Barbara, California, USA, August 11-15, 1991, Proceedings, volume 576 of Lecture Notes in Computer Science. Springer, 1992.
[4] Paul Feldman. A practical scheme for non-interactive verifiable secret sharing. In Proceedings of the 28th Annual Symposium on Foundations of Computer Science, SFCS ’87, pages 427–438, Washington, DC, USA, 1987. IEEE Computer Society.
[5] Maggie Habeeb, Delaram Kahrobaei, and Vladimir Shpilrain. A secret sharing scheme based on group presentations and the word problem. Contemp. Math., Amer. Math. Soc., 582:143–150, 2012.
[6] Amir Herzberg, Markus Jakobsson, Stanislaw Jarecki, Hugo Krawczyk, and Moti Yung. Proactive public key and signature systems. In Proceedings of the 4th ACM conference on Computer and communications security, CCS ’97, pages 100–110, New York, NY, USA, 1997. ACM.

[7] Derek F. Holt, Bettina Eick, and Eamonn A. O’Brien. Handbook of Computational Group Theory. CRC Press, 2005.

[8] S.M. Jarecki. Proactive Secret Sharing and Public Key Cryptosystems. Massachusetts Institute of Technology, Department of Electrical Engineering and Computer Science, 1996.

[9] Jonathan Katz and Yehuda Lindell. Introduction to Modern Cryptography. Chapman and Hall/CRC Press, 2007.

[10] Ueli M. Maurer, editor. Advances in Cryptology - EUROCRYPT ’96, International Conference on the Theory and Application of Cryptographic Techniques, Saragossa, Spain, May 12-16, 1996, Proceeding, volume 1070 of Lecture Notes in Computer Science. Springer, 1996.

[11] Alexei Myasnikov, Vladimir Shpilrain, and Alexander Ushakov. Group-based Cryptography. Springer, 2008.

[12] Torben P. Pedersen. Non-interactive and information-theoretic secure verifiable secret sharing. In Feigenbaum [3], pages 129–140.

[13] Adi Shamir. How to share a secret. Commun. ACM, 22(11):612–613, 1979.

[14] Markus Stadler. Publicly verifiable secret sharing. In Maurer [10], pages 190–199.

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