On Nonlinear Part of Filled-Section in Splicing

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1 Introduction

The purpose of this paper is to define the nonlinear part of the filled section $\Psi_N = (\Psi_{N,-}, \Psi_{N,+})$ denoted by $N = (N_-, N_+)$ within the framework of the usual analysis of Banach manifolds rather than in the setting of $\text{Sc}$-analysis of polyfold theory in [1, 2]. The main difficulty for this is that the simple choice made in [2] for $\Phi_-^a$ using a linear operator leads to a filled-section with loss of differentiability so that the theory for filled-sections has to be formulated in $\text{Sc}$-analysis.

To overcome this difficulty, recall that for each fixed gluing parameter $a = (R, \theta) \in [R_0, \infty) \times S^1$, $N_a^\pm : L_{k,\delta}^p (C_-, E) \times L_{k,\delta}^p (C_+, E) \to L_{k-1,\delta}^p (C_-, E) \times L_{k-1,\delta}^p (C_+, E)$ is obtained from $\Phi_{\pm,N}^a : L_{k,\delta}^p (S_a^\pm, E) \to L_{k-1,\delta}^p (S_a^\pm, E)$ by the conjugation by the total gluing map $T^a : C_- \cup C_+ \to S^a$. Here $C_\pm \simeq (0, \pm \infty) \times S^1$, $E = \mathbb{C}^n$ and $L_{k,\delta}^p$-maps here are the $L_{k}^p$-maps that decay exponentially along the ends of the half cylinders $C_\pm$ with the decay rate $0 < \delta < 1$.

Since $\Phi_+$ is required to be the usual $\overline{\partial}_J$-operator in the Gromov-Witten theory, the nonlinear part $\Phi_{\pm,N}^a$ then is given by $\Phi_{\pm,N}^a (v_+) = J(v_+) \partial_s v_+$, where $v_+ : S_+^a \to \mathbb{C}^n$ is a $L_{k,\delta}^p$-map with the domain $S_+^a \simeq [-R, R] \times S^1$ with cylindrical coordinate $(t, s)$. To get desired $\Phi_{N}^a$ without loss of loss of differentiability, the key observation is that the choice of $\Phi_{\pm,N}^a : L_{k,\delta}^p (S_a^\pm, E) \to L_{k-1,\delta}^p (S_a^\pm, E)$ with $S_a^\pm \simeq \mathbb{R}^1 \times S^1$ should be $\Phi_{-N}^a (v_-)^{\prime\prime} = \Phi_{+N}^a (v_-)^{\prime\prime} = J(v_+) \partial_s v_-$ with $J(v_+)$ being considered as almost complex structure along
$v_-$. To make sense out of this senseless identity, we note that the sub-cylinder with $t \in (-d - l, d + l)$ is where the splicing matrix $T_\beta$ is not a constant. It is also the place where the loss of differentiability takes place for the choice of $\Phi^a_{-N}$ similar to the one in [2]. Since the sub-cylinder is contained in both $S^a_-$ and $S^a_+$, we may define the almost complex structure $J(v_-)$ along $(-d - l, d + l) \times S^1$ to be $J(v_+)$. More generally, we need to define a new extended gluing $\hat{v}_+: \hat{S}^a_+ \to E$ with $\hat{S}^a_+ \simeq (-\infty, \infty) \times S^1 \simeq S^a_-$ (see the definition in next section). Denote the identification map by $\Gamma : S^a_- \to \hat{S}^a_+$, which will transfer the almost complex structure $J(\hat{v}_+)$ into $J(\hat{v}_+ \circ \Gamma)$ considered as an almost complex structure along $v_-$. Let $J(v) = J(\hat{v}_+ \circ \Gamma) \oplus J(v_+)$ be the corresponding total almost complex structure along $v = (v_-, v_+)$. Then $\Phi^a_N$ is defined to be $\Phi^a_N(v) = J(v) \partial_N v$. The idea of this construction is to enforce the commutativity of the splicing matrix $T_\beta$ and the total almost complex structure $J(v)$ along $v$: $J(v) \circ T_\beta = T_\beta \circ J(v)$. One can see from the proof of the main theorem below that this commutativity is essentially equivalent to the requirement of no loss of differentiability for $\Phi_N$.

Then the main theorem of this paper is the following theorem.

**Theorem 1.1** Using the gluing profile $R = e^{1/r} - e^{1/r_0}$, the filled-section $\Psi_N = \{\Psi^R_N\}$ above with $r \in [0, r_0)$ (hence $R \in (R_0, \infty)$ for $R_0 = (R, \theta)$ and $\theta \in S^1$) is of class $C^1$. Consequently, the filled-section $\Psi = \Psi_L + \Psi_N : L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_+, E) \times D_{r_0} \to L^p_{k-1,\delta}(C_-, E) \times L^p_{k-1,\delta}(C_+, E)$ is of class $C^1$.

Clearly the argument in these sequence of papers can be generalized to deal with general nonlinear equations with quasi-linear principal "symbols" as well as higher order equations of similar types. Applications of this kind will be given somewhere else.

**Remark 1.1** (A) For any given positive integer $m$, the $C^m$-smoothness of the filled-section $\Psi$ can be proved by essentially the same argument but using different length and center functions $l = L_m(R) = R^{m/(m+1)} \cdot \ln^2 R$ and $d = 3l = 3L_m(R)$. For $C^1$-smoothness here, the function $l = L_1(R)$.

(B) One may assume that $\Phi_{-N}$ has the general form $\Phi_{-N}(v) = A(v_-) \cdot \partial_s v_-$ with $A(v_-)$ being a $\text{End}(E)$-valued section over $v_-$ which is to be chosen. Let $J_A(v) = A(v_-) \oplus J(v_+)$. Then one can show that for generic $J$, the resulting $\Psi_N$ has no loss of differentiability implies the commutativity $J_A(v) \circ T_\beta = T_\beta \circ J_A(v)$ so that upto the choices of the transfer map $\Gamma$, our definition for $\Psi_{-N}$ is essentially the only possible choice with the desired properties.
The main theorem will be proved in Sec. 3 after recalling the basic definitions in the splicing in Sec. 2.

Like [3], we will only deal with the case for $L_{k,\delta}^p$-maps with fixed ends at their double points. Throughout this paper we will assume that (i) $p > 2$ and (ii) $k - 2/p > 1$. Unlike [3], this last condition is needed for the estimates in Sec. 3 in order to prove the main theorem above. If one insists using Hilbert space (hence $p = 2$), then the condition above becomes $k \geq 3$.

## 2 Basic definitions of the splicing

In this section we recall the definitions of the splicing in [3]. These definitions are tailored for defining the filled-section in the setting of Banach analysis (comparing with the definitions in [2]).

### 2.1 Total gluing of the nodal surface $S$

Let $S = C_- \cup_{d_- = d_+} C_+$ with the double point $d_- = d_+$ with $(C_\pm, d_\pm)$ being the standard disk $(D, 0)$. Identify $(C_-, d_-)$ with $((-\infty, 0) \times S^1, -\infty \times S^1) = (L_- \times S^1, -\infty \times S^1)$ canonically upto a rotation by considering the double point $d_-$ as the $S^1$ at $-\infty$ of the half cylinder $L_- \times S^1$. Here we have denoted the negative half line $(-\infty, 0)$ by $L_-$. Similarly $(C_+, d_+) \simeq ((0, \infty) \times S^1, \infty \times S^1) = (L_+ \times S^1, \infty \times S^1)$.

- **Cylindrical coordinates on $C_\pm$**:

  By the identification $C_\pm \simeq L_\pm \times S^1$, each $C_\pm$ has the cylindrical coordinates $(t_\pm, s_\pm) \in L_\pm \times S^1$.

  Let $a = (R, \theta) \in [0, \infty] \times S^1$ be the gluing parameter. To defined the total gluing/deformation $S^a = S^{(R, \theta)}$ with gluing parameter $R \neq \infty$, we introduce the $a$-dependent cylindrical coordinates $(t^{a, \pm}, s^{a, \pm})$ on $C_\pm$ by the formula $t_\pm = t^{a, \pm} \pm R$ and $s_\pm = s^{a, \pm} \pm \theta$. In the following if there is no confusion, we will denote $t^{a, \pm}$ by $t$ and $s^{a, \pm}$ by $s$ for both of these $a$-dependent cylindrical coordinates.

  Thus the $t$-range for $L_-$ is $(-\infty, R)$ and the $t$-range for $L_+$ is $(-R, \infty)$ with the intersection $L_- \cap L_+ = (-R, R)$.

- **Total gluing $S^a = (S^a_-, S^a_+)$**:

  In term of the $a$-dependent cylindrical coordinates $(t, s)$, $C_- = (-\infty, R) \times S^1$ and $C_+ = (-R, \infty) \times S^1$. 

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Then $S_1^a$ is defined to be the finite cylinder of length $2R$ obtained by gluing $(-1,R) \times S^1 \subset C_-$ with $(-R,1) \times S^1 \subset C_+$ along the ”common” region $(-1,1) \times S^1$ by the identity map in term of he $a$-dependent coordinates $(t,s)$. Similarly, $S^n_a$ is the infinite cylinder defined by gluing $(-\infty,1) \times S^1 \subset C_-$ with $(1,\infty) \times S^1 \subset C_+$ along $(-1,1) \times S^1$ by the identity map.

Geometrically, both $S^n_a$ are obtained by first cutting each $C_\pm$ along the circle at $t=0$ into two sub-cylinders, then gluing back the sub-cylinders in $C_-$ with the corresponding ones in $C_+$ along the same circle with a relative rotation of angle $2\theta$. Set $S^\infty = S$.

Now the cylindrical coordinates $(t_\pm,s_\pm)$ on $C_\pm$ as well as the $a$-dependent cylindrical coordinates $(t,s)$ become the corresponding ones on each $S^n_a$ with the relation: $t_\pm = t \pm R$ and $s_\pm = s \pm \theta$.

### 2.2 Splicing matrix $T_\beta$

To defined $T_\beta$, we need to choose a length function depending $R$. For the purpose of this paper, the length function $L(R) = L_1(R) = R^{1/2} \cdot ln^2 R$.

The splicing matrix $T_\beta$ used in this paper is defined by using a pair of cut-off function $\beta = (\beta_-,\beta_+)$ depending on the two parameters $(l,d)$ that parametrize the group of affine transformations $\{t \rightarrow lt + d\}$ of $\mathbb{R}^1$ defined as follows.

Fix a smooth cut-off function $\alpha_- : \mathbb{R}^1 \rightarrow [0,1]$ with the property that $\alpha_-(t) = 1$ for $t < -1$, $\alpha(t) = 0$ if $t > 1$ and $\alpha' \leq 0$. Let $\alpha_+ = 1 - \alpha$. Fix $l_0 > 1$ and $d_0 > 1$. Then $\beta_\pm = \{\beta_{\pm,l,d}\} : \mathbb{R}^1 \times [l_0,\infty) \times [d_0,\infty) \rightarrow [0,1]$ defined by $\beta_\pm(t,l,d) = \alpha_\pm((t \pm d)/l)$, or $\beta_{\pm,l,d} = \rho_l \circ \tau_{\pm,d}\alpha$. Here the translation and multiplication operators are defined by $\tau_d(\xi)(t) = \xi(t + d)$ and $\rho_l(\xi)(t) = \xi(t/l)$ respectively.

The pair $(l,d)$ will be the functions on $R$, $(l = L(R), d = d(R))$ with $d = 3l$ and $l$ defined above.

Clearly $\beta_\pm$ is a smooth cut-off function with the following two properties:

- $P_1$ : for $k \leq k_0$ the $C^0$-norm of the $k$-th derivative $\|\beta^{(k)}\|_{C^0} \leq C/l^k$, where $C = \|\alpha\|_{C^{k_0}}$;

- $P_2$ : under the assumption that $d \geq 3l$, the support of $\beta'_-$ is contained in the interval $(d-l,d+l)$ with $\beta_- = 1$ on $(-\infty,d-l]$ and $\beta_- = 0$ on $[d+l,\infty)$; and $\beta'_+$ is contained in the interval $[-d-l,-d+l]$ with $\beta_+ = 1$ on $[-d+l,\infty)$ and $\beta_+ = 0$ on $(-\infty,-d-l]$.

The splicing matrix then is defined by
\[ T_\beta = \begin{bmatrix} \beta_- & -\beta_+ \\ \beta_+ & \beta_- \end{bmatrix}. \]

Note that from \( P_2 \), on \((-d+l, d-l)\), \( \beta_- = \beta_+ = 1 \). Then for \( t \) in the three intervals \((-\infty, -d-l), (-d+l, d-l) \) and \((d+l, \infty)\), \( T_\beta(t) \) are the following constant matrices

\[ M_1 = Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{ and } M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]

Note that \( \beta_\pm(t) < 1 \) implies that \( \beta_\mp(t) = 1 \) so that the determinant

\[ 1 \leq D = \det \begin{bmatrix} \beta_- & -\beta_+ \\ \beta_+ & \beta_- \end{bmatrix} = \beta_-^2 + \beta_+^2 \leq 2. \]

This implies that \( T^a = (\ominus_a, \ominus_a) \) defined below is invertible uniformly.

### 2.3 Total gluing \( T^a \) of maps and sections

Let \( C^\infty(C_\pm, E) \) be the set of \( E \)-valued \( C^\infty \) functions on \( C_\pm \), where \( E = \mathbb{C}^n \). Similarly \( C^\infty(S^a_\pm, E) \) consists of all \( E \)-valued smooth functions on \( S^a_\pm \).

Then \( T^a = (T^a_-, T^a_+) : C^\infty(C_- E) \times C^\infty(C_+ E) \to C^\infty(S^a_- E) \times C^\infty(S^a_+ E) \) is defined as follows.

In matrix notation, for each \((\xi_-, \xi_+) \in C^\infty(C_- E) \times C^\infty(C_+ E) \) considered as a column vector,

\[ T^a((\xi_-, \xi_+)) = \begin{bmatrix} T^a_-(\xi_-, \xi_+) \\ T^a_+(\xi_-, \xi_+) \end{bmatrix} = (\xi_- \ominus_a \xi_+, \xi_- \ominus \ominus_a \xi_+) \]

\[ = \begin{bmatrix} \beta_- & -\beta_+ \\ \beta_+ & \beta_- \end{bmatrix} \begin{bmatrix} \tau_a \xi_- \\ \tau_a \xi_+ \end{bmatrix}. \]

The inverse of the total gluing, \((T^a)^{-1} = (T^a_-, T^a_+)^{-1} : C^\infty(S^a_- E) \times C^\infty(S^a_+ E) \to C^\infty(C_- E) \times C^\infty(C_+ E) \) is defined as following: for a pair of the \( E \)-valued functions \((\eta_-, \eta_+) \in C^\infty(S^a_- E) \times C^\infty(S^a_+ E)\),

\[ (T^a)^{-1}(\eta_-, \eta_+) = (\ominus_a \ominus_a)^{-1}(\eta_-, \eta_+) = \begin{bmatrix} \tau_a & 0 \\ 0 & \tau_{-a} \end{bmatrix} \cdot \frac{1}{D} \begin{bmatrix} \beta_- & \beta_+ \\ -\beta_+ & \beta_- \end{bmatrix} \begin{bmatrix} \eta_- \\ \eta_+ \end{bmatrix}. \]
Note that the map \((u_-, u_+) \in C^\infty(L_- \times S^1, E) \times C^\infty(L_+ \times S^1, E)\) satisfies the asymptotic condition that \(u_-(-\infty) = u_+(\infty)\) that is corresponding to the condition that \(T^a(u_-, u_+)(-\infty) = -T^a(u_-, u_+)(+\infty)\). In other words \(T^a\) maps these two subspaces each other isomorphically as before.

### 2.4 Definition of the transfer map \(\Gamma^R\)

First fix the length function \(l = L(R) = \sqrt{R(lnR)^2}\) and the center function \(d = 3l\) with \(R \geq R_0\) for a fixed \(R_0 >> 1\).

- **Extended gluing:** For the gluing parameter \(a = (R, \theta)\), extended gluing
  
  \[u_\ast \hat{\oplus}_a u_+ : \hat{\mathcal{S}}_+^a := \mathbb{R}^1 \times S^1 \to E \]
  
  is the connection matrix introduced in [3].

  
  Let \(\Phi^a\) be the identity map in term of the \((t, s)\)-coordinates for both \(\mathcal{S}^-_a\) and \(\hat{\mathcal{S}}^a_+\).

  
  In the following \(u_\ast \hat{\oplus}_a u_+\) will be denoted by \(\hat{\mathcal{S}}^a_+\) and \(u_\ast \hat{\oplus}_a u_+\) by \(u^a_+\).

  
  Thus in \(t\)-coordinate, for \(-d-l-1 < t < d+l+1\), \(\hat{\mathcal{S}}^a_+ = u^a_+\), for \(t > d+l+2\), \(\hat{\mathcal{S}}^a_+ = \tau_a u_+\) and for \(t < -d-l-2\), \(\hat{\mathcal{S}}^a_+ = \tau_{-a} u_-\).

  
  Although, \(\Gamma^a\) does not has effect on the total gluing \(T^a\), it transfers the complex structure \(J(\hat{\mathcal{S}}^a_+)\) along the map \(\hat{\mathcal{S}}^a_+ \to M\) to a complex structure \(J(\hat{\mathcal{S}}^a_+ \circ \Gamma^a)\) along \(\mathcal{S}^-_a \to M\) by composing with the map \(\Gamma^a : \mathcal{S}^-_a \to \hat{\mathcal{S}}^a_+\).

  
  Then \(\Phi^a_\circ (\hat{\mathcal{S}}^a_+) = \partial_t \hat{\mathcal{S}}^a_+ + J(v_-) \partial_s \hat{\mathcal{S}}^a_+\) is defined to be the usual \(\bar{\partial}_J\)-operator. The new \(\Phi^a_\circ\) is still a \(\bar{\partial}_J\)-operator but with the above almost complex structure on \(\mathcal{S}^-_a\). The connection matrix introduced in [3].

  
  We note that in order to makes sense for the definition of \(\Phi^a_\circ\), we need to know that \(\hat{\mathcal{S}}^a_+\) is determined by \(v^a\). Indeed, given \(v^a\), there is an unique pair \((u_-, u_+) = (T^a)^{-1}(v^a, v^a_+)\) such that \(T^a(u_-, u_+) = (v^a, v^a_+).\) Then we can construct the extended gluing \(\hat{\mathcal{S}}^a_+ =: u_\ast \hat{\oplus}_a u_+\) as above.

  
  Let \(\Phi^a = (\Phi^a_-, \Phi^a_+)\). Then \(\Psi^a =: (T^a)^{-1} \circ \Phi^a \circ T^a\).
3 The nonlinear part $N$ of $\Psi$

Since the splicing matrix $T_\beta$ is $s$-independent and the translation operator $\tau_\theta$ appeared in the total gluing map $T^a$ commutes with both $\partial_t$ and $\partial_s$, $\tau_\theta$ does not affect analysis here in any essential way. In the most part of the rest of this section we will only give the details for the results using $T^R$ and state the corresponding ones using $T^a$.

We now derive the formula for the nonlinear part $\Psi^R_N$.

The nonlinear part $N^R$:

The nonlinear part of $\Psi^R$ denoted by $N^R : C^\infty(C_-, E) \times C^\infty(C_+, E) \to C^\infty(C_-, E) \times C^\infty(C_+, E)$, is defined as follows.

For $(u_-, u_+) \in C^\infty(C_-, E) \times C^\infty(C_+, E)$ with $T^R(u_-, u_+) = (v_-, v_+)$, $N^R(u_-, u_+) = (T^R)^{-1}(J(v_+ \circ \Gamma^R)\partial_s v_-, J(v_+)\partial_s v_+)$. In matrix notation $N^R((u_-, u_+)) =$

$$\begin{bmatrix}
\tau^R & 0 \\
0 & \tau_{-R}
\end{bmatrix} \cdot \frac{1}{D} \begin{bmatrix}
\beta_- & \beta_+ \\
-\beta_+ & \beta_-
\end{bmatrix} \begin{bmatrix}
J(v^R_+ \circ \Gamma^R) & 0 \\
0 & J(v^R_+)
\end{bmatrix} \begin{bmatrix}
\beta_- & \beta_+ \\
-\beta_+ & \beta_-
\end{bmatrix} \frac{\partial_s}{\tau_{-R}u_-} \begin{bmatrix}
\tau_{-R}u_- \\
\tau_{-R}u_+
\end{bmatrix}.$$

Here $v^R_+ = \beta_+\tau_{-R}u_- + \beta_-\tau_{-R}u_+$. 

We note that starting from the last term of the last identity, the result of each matrix multiplication can be interpreted as a pair of function defined on the common domain $\mathbb{R}^1$ in $t$-coordinate even they have different domains. This follows from the way that the total gluing map $T^R$ and its inverse are defined. It can also be seen from our next computation of $N^R((u_-, u_+))$ by restricting the pair of functions to three type of subintervals in $t$-variable.

- The $N^R$ over the interval $(-d-l, d+l)$:

  On the $t$-interval $(-d-l, d+l)$ where the non-trivial part of $\beta$ is lying on $\hat{v}^R_+ = v^R_+$ and in $t$-coordinate $\Gamma^R$ is the identity map so that $J(\hat{v}^R_+ \circ \Gamma^R) = J(v^+)$). Then all the terms of the following identity are well-defined functions on $(-d-l, d+l) \times S^1$ in $(t, s)$ coordinate, and we have that on $(-d-l, d+l) \times S^1$
\[
\frac{1}{D} \cdot \begin{bmatrix}
\beta_- & \beta_+ \\
-\beta_+ & \beta_-
\end{bmatrix} \begin{bmatrix}
J(\hat{v}_+^R \circ \Gamma^R) & 0 \\
0 & J(v_+^R)
\end{bmatrix} \begin{bmatrix}
\beta_- & -\beta_+ \\
\beta_+ & \beta_-
\end{bmatrix} \begin{bmatrix}
\partial_s \tau_R u_- \\
\partial_s \tau_R u_+
\end{bmatrix}
\]

\[
= \frac{1}{D} \cdot \begin{bmatrix}
\beta_- & \beta_+ \\
-\beta_+ & \beta_-
\end{bmatrix} \begin{bmatrix}
J(v_+^R) & 0 \\
0 & J(v_+^R)
\end{bmatrix} \begin{bmatrix}
\beta_- & -\beta_+ \\
\beta_+ & \beta_-
\end{bmatrix} \begin{bmatrix}
\partial_s \tau_R u_- \\
\partial_s \tau_R u_+
\end{bmatrix}.
\]

Thus

\[
N^R((u_-, u_+) = \begin{bmatrix}
\tau_R & 0 \\
0 & \tau_R
\end{bmatrix} \begin{bmatrix}
J(v_+^R)\partial_s \tau_R u_- & J(v_+^R)\partial_s \tau_R u_+
\end{bmatrix} = \begin{bmatrix}
\tau_R J(v_+^R)\partial_s u_- & \tau_R J(v_+^R)\partial_s u_+
\end{bmatrix}.
\]

\[
N^R((u_-, u_+) = (\tau_R J(v_+^R)\partial_s u_-, \tau_R J(v_+^R)\partial_s u_+).
\]

*The $N^R$ away from the interval $(−d − l + 1, d + l − 1)$:

We already know from last subsection that away from $(−d − l, d + l)$, $T_\beta$ is given by $M_1 = id$ and $M_3$. We may assume that this is true away from the interval $(−d − l + 1, d + l − 1)$. Hence for $t > d + l − 1, D = 1$ and

\[
\frac{1}{D} \cdot \begin{bmatrix}
\beta_- & \beta_+ \\
-\beta_+ & \beta_-
\end{bmatrix} \begin{bmatrix}
J(\hat{v}_+^R \circ \Gamma^R) & 0 \\
0 & J(v_+^R)
\end{bmatrix} \begin{bmatrix}
\beta_- & -\beta_+ \\
\beta_+ & \beta_-
\end{bmatrix} \begin{bmatrix}
\partial_s \tau_R u_- \\
\partial_s \tau_R u_+
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
J(\hat{v}_+^R \circ \Gamma^R) & 0 \\
0 & J(v_+^R)
\end{bmatrix} \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
\partial_s \tau_R u_- \\
\partial_s \tau_R u_+
\end{bmatrix}.
\]

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\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
J(\dot{v}_+^R \circ \Gamma^R) & 0 \\
0 & J(v_+^R)
\end{bmatrix}
\begin{bmatrix}
-\partial_s \tau_R u_+ \\
\partial_s \tau_R u_-
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
-J(\dot{v}_+^R \circ \Gamma^R)\partial_s \tau_R u_+ \\
J(v_+^R)\partial_s \tau_R u_-
\end{bmatrix}
= \begin{bmatrix}
J(v_+^R)\partial_s \tau_R u_- \\
J(\dot{v}_+^R \circ \Gamma^R)\partial_s \tau_R u_+
\end{bmatrix}
\].

Note that in above computation, each step makes sense even the pair of functions are not defined on the same domain.

Hence for \( t > d + l - 1 \), \( N^R(u_-, u_+) = (\tau_R J(v_+^R)\partial_s u_-, \tau_- R J(\dot{v}_+^R \circ \Gamma^R)\partial_s u_+) \).

For \( t < -d - l + 1 \), both matrices become the identity matrix with \( D = 1 \) so that

\[
\frac{1}{D} \cdot \begin{bmatrix}
\beta_- & \beta_+ \\
-\beta_+ & \beta_-
\end{bmatrix}
\begin{bmatrix}
J(\dot{v}_+^R \circ \Gamma^R) & 0 \\
0 & J(v_+^R)
\end{bmatrix}
\begin{bmatrix}
\partial_s \tau_R u_- \\
\partial_s \tau_R u_+
\end{bmatrix}
\]

\[
= \begin{bmatrix}
J(\dot{v}_+^R \circ \Gamma^R) & 0 \\
0 & J(v_+^R)
\end{bmatrix}
\begin{bmatrix}
\partial_s \tau_R u_- \\
\partial_s \tau_R u_+
\end{bmatrix}
= \begin{bmatrix}
J(\dot{v}_+^R \circ \Gamma^R)\partial_s \tau_R u_- \\
J(v_+^R)\partial_s \tau_R u_+
\end{bmatrix}
\].

Hence for \( t < -d - l + 1 \), \( N^R(u_-, u_+) = (\tau_R J(\dot{v}_+^R \circ \Gamma^R)\partial_s u_-, \tau_- R J(v_+^R)\partial_s u_+) \).

In summary, we have proved

**Lemma 3.1** In \((t, s)\)-coordinate, \( N^R(u_-, u_+) = (\tau_R J(\dot{v}_+^R \circ \Gamma^R)\partial_s u_-, \tau_- R J(v_+^R)\partial_s u_+) \) for \( t < -d - l + 1 \), \( N^R(u_-, u_+) = (\tau_R J(v_+^R)\partial_s u_-, \tau_- R J(\dot{v}_+^R \circ \Gamma^R)\partial_s u_+) \) for \(-d - l < t < d + l \) and \( N^R(u_-, u_+) = (\tau_R J(v_+^R)\partial_s u_-, \tau_- R J(\dot{v}_+^R \circ \Gamma^R)\partial_s u_+) \) for \( t > d + l - 1 \).

Since on the overlap regions, \( \Gamma^R \) is the identity map, we have the following

**Corollary 3.1** On the overlap regions, \( N^R \) is defined by the same formula. There is no loss of differentiability in \( N^R \).

We note that in the above lemma the two components \( N^R_{\pm}(u_-, u_+) \) already expressed their own natural coordinates \( t_\pm \) coming from \( u_\pm \), but we still use the \( t \)-coordinate to divide the domains. It is more convenient to the division of the domains for \( N^R(u_-, u_+) = (N^-_R(u_-, u_+), N^+_R(u_-, u_+)) \) in the natural coordinates \( t_\pm \). By abusing the notations, in next lemma coordinates \( t_\pm \) are still denoted by \( t \).
Note that for \( d + l - 1 < t < R \), in \( t_- = t - R \)-coordinate with \( t_\) = \(-R + d + l - 1 < t_\) < 0, which corresponds to the right half the region the usual gluing \( u_- \oplus_R u_+ \) coming from \( u_- \). For such a \( t \), \( v_+^R = \beta_+ \tau_R u_- + \beta_- \tau_R u_+ = \tau_R u_- \) so that the term \( \tau_R J(v_+^R) = J(\tau_R \circ \tau_R u_-) = J(u_-) \). Similarly for \( t < -d - l - 2 \), it is in \( t_- < -R - d - l - 2 \), the term \( \hat{v}_+^R \circ \Gamma^R = \tau_R u_- \) again so that \( \tau_R J(\hat{v}_+^R \circ \Gamma^R) = (\tau_R \circ \tau_R u_-) = J(u_-) \).

Then we have

**Lemma 3.2**

\[
N_-^R(u_-, u_+) = \begin{cases} 
\tau_R J(\hat{v}_+^R \circ \Gamma^R) \partial_s u_- , & \text{if } t < -R - d - l - 2, \\
\tau_R J(v_+^R) \partial_s u_- & \text{if } -R - d - l - 3 < t < -R + d + l + 1 \\
\tau_R J(v_+^R) \partial_s u_- & \text{if } -R + d + l < t < 0 
\end{cases}
\]

\[
= \begin{cases} 
\tau_R J(\hat{v}_+^R \circ \Gamma^R) \partial_s u_- , & \text{if } t < -R - d - l - 2, \\
\tau_R J(v_+^R) \partial_s u_- & \text{if } -R - d - l - 3 < t < -R + d + l + 1, \\
\tau_R J(v_+^R) \partial_s u_- & \text{if } -R + d + l < t < 0 
\end{cases}
\]

Similarly,

\[
N_+^R(u_-, u_+) = \begin{cases} 
\tau_- R J(\hat{v}_+^R \circ \Gamma^R) \partial_s u_+ , & \text{if } t > R + d + l + 2, \\
\tau_- R J(v_+^R) \partial_s u_+ & \text{if } R - d - l - 1 < t < R + d + l + 3 \\
\tau_- R J(v_+^R) \partial_s u_+ & \text{if } 0 < t < R - d - l 
\end{cases}
\]

\[
= \begin{cases} 
\tau_- R J(\hat{v}_+^R \circ \Gamma^R) \partial_s u_+ , & \text{if } t > R + d + l + 2, \\
\tau_- R J(v_+^R) \partial_s u_+ & \text{if } R + d + l - 1 < t < R + d + l + 3, \\
\tau_- R J(v_+^R) \partial_s u_+ & \text{if } R - d - l - 1 < t < R + d + l \\
\tau_- R J(v_+^R) \partial_s u_+ & \text{if } 0 < t < R - d - l 
\end{cases}
\]

\[
= \begin{cases} 
\tau_- R J(\hat{v}_+^R \circ \Gamma^R) \partial_s u_+ , & \text{if } t > R + d + l + 2, \\
\tau_- R J(v_+^R) \partial_s u_+ & \text{if } R - d - l - 3 < t < R + d + l + 3, \\
\tau_- R J(v_+^R) \partial_s u_+ & \text{if } 0 < t < R - d - l - 2 
\end{cases}
\]
In the last identity, we rearrange the division of the domain so that away from \([R - d - l - 3, R + d + l + 3]\) of length \(2(d + l + 3)\), \(N_+^R(u_-, u_+)\) is just \(N_+^\infty(u_-, u_+) = J(u_+)\partial_s u_+\) so that the error term \(E^R(u_-, u_+) =: N_+^R(u_-, u_+) - N_+^\infty(u_-, u_+) = \{\tau_R J(\hat{v}_+^R \circ \Gamma_R) - J(u_+)\} \partial_s u_+ = \{J(\hat{v}_+^R \circ \Gamma_R \circ \tau_R) - J(u_+)\}\partial_s u_+\) is localized on \([R - d - l - 3, R + d + l + 3]\) \(\times S^1\).

The formulas for \(N_+^{R_0}(u_-, u_+)\) can be obtained similarly by simply replacing \(R\) by \(R_0\) in above formulas.

**Theorem 3.1** \(N = \{N^R\} : L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times [R_0, \infty] \to L^p_{k-1,\delta}(C_-, E) \times L^p_{k-1,\delta}(C_-, E)\) is continuous.

**Proof:**

From above formula, the continuity of \(N\) for \(R \neq \infty\) is clear. To see the continuity of \(N\) at \(R = \infty\), we only need to consider \(N_+\). Then it follows from the estimate proved below in this section that for \(t \in [R - d - l - 3, R + d + l + 3]\), the term

\[
\| J(\hat{v}_+^R \circ \Gamma_R \circ \tau_R) - J(u_+) \|_{C^k-1} \leq C(\beta, \gamma) \cdot \| J \|_{C^{k-1}} \cdot e^{-\delta R/2} \cdot \| u \|_{k,p,\delta}(1 + e^{-\delta R/2} \cdot \| u \|_{k,p,\delta})
\]

so that

\[
\| N_+^R(u_-, u_+) - N_+^\infty(u_-, u_+) \|_{k-1,p,\delta} = \| \{J(\hat{v}_+^R \circ \Gamma_R \circ \tau_R) - J(u_+)\} \partial_s u_+ \|_{k-1,p,\delta}
\]

\[
\leq \| \{J(\hat{v}_+^R \circ \Gamma_R \circ \tau_R) - J(u_+)\} \|_{C^{k-1}} \| \partial_s u_+ \|_{k-1,p,\delta}
\]

\[
\leq C \cdot e^{-\delta R/2} \cdot \| u \|_{k,p,\delta}^2(1 + e^{-\delta R/2} \cdot \| u \|_{k,p,\delta}).
\]

This together with the fact that \(N^\infty\) is continuous implies that \(N\) is continuous. Indeed

\[
\| N_+^R(u_1) - N_+^\infty(u_2) \|_{k-1,p,\delta} \leq \| N_+^R(u_1) - N_+^\infty(u_1) \|_{k-1,p,\delta} + \| N_+^\infty(u_1) - N_+^\infty(u_2) \|_{k-1,p,\delta}
\]

\[
\leq C \cdot e^{-\delta R/2} \cdot \| u_1 \|_{k,p,\delta}^2(1 + e^{-\delta R/2} \cdot \| u_1 \|_{k,p,\delta}) + \| N_+^\infty(u_1) - N_+^\infty(u_2) \|_{k-1,p,\delta},
\]

which is less than

\[
\epsilon + \| N_+^\infty(u_1) - N_+^\infty(u_2) \|_{k-1,p,\delta}
\]

with any given \(\epsilon > 0\) and any fixed \(u_1\) by taking \(R\) sufficiently large. The conclusion then follows from the continuity of \(N^\infty\). \(\square\)
Theorem 3.2  \[ N = \{N^R\} : L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times (R_0, \infty) \rightarrow L^p_{k-1,\delta}(C_-, E) \times L^p_{k-1,\delta}(C_-, E) \text{ is of class } C^1. \]

Theorem 3.3  The derivative \[ DN = \{DN^R\} : L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times (R_0, \infty) \rightarrow L(L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times \mathbb{R}^1, L^p_{k-1,\delta}(C_-, E) \times L^p_{k-1,\delta}(C_-, E)) \] can be extended continuously over \( r = 0 \) (hence \( R = \infty \)).

Corollary 3.2  The derivative \[ DN = \{DN^{r_0}\} : L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times D_{r_0} \rightarrow L(L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times \mathbb{R}^2, L^p_{k-1,\delta}(C_-, E) \times L^p_{k-1,\delta}(C_-, E)) \text{ is of class } C^1. \]

To prove the theorems, we use the formula above for \( N^R(u_-, u_+) = (N_-(u_-, u_+), N^R_+(u_-, u_+)) \). Since the two parts \( N^R \) are in the same natural, we only need to deal with \( N^R_+(u_-, u_+) \).

Let \( N = \{N^R, R \in [0, \infty]\} : L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_+, E) \times [R_0, \infty) \rightarrow L^p_{k-1,\delta}(C_-, E) \times L^p_{k-1,\delta}(C_+, E). \)

Denote \( L^p_{k,\delta}(C_\pm, E) \) by \( W_\pm \) and \( W = W_- \times W_+ \).

In these notation, \( N_+ : W \times [0, \infty) \rightarrow L^p_{k-1,\delta}(C_+, E) \).

\( \bullet \) Partial derivative \( D_W N_+ \):

Consider the case \( R \neq \infty \) first.

By the last formula above, \( D_W N_+ : W \times [0, \infty) \rightarrow L(L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times \mathbb{R}^1, L^p_{k-1,\delta}(C_+, E)) \) along \( (u_-, u_+)-\)direction is given by the following formula:

\[
(D_W N_+)_{u_-, u_+}(\xi_-, \xi_+) = \left\{ \begin{array}{ll}
\partial_J v^R \circ \Gamma^R \circ \tau_-(\xi) \partial_s u_+ + \tau_- R J(\hat{v}^R \circ \Gamma^R) \partial_s \xi_+ \\
if R - d - l - 3 < t < R + d + l + 3,

\partial_J u_+(\xi_+) \partial_s u_+ + J(u_+) \partial_s \xi_+ \\
if t \notin [R - d - l - 3, R + d + l + 3]
\end{array} \right.
\]

Thus away from \( [R - d - l - 3, R + d + l + 3] \times S^1 \), \( (D_W N_+)_{u_-, u_+}(\xi_-, \xi_+) = \partial_J u_+(\xi_+) \partial_s u_+ + J(u_+) \partial_s \xi_+; \)

Let \( \rho : [0, \infty) \rightarrow [0, 1] \) be a smooth cut-off function such that

\[
\rho(t) = \left\{ \begin{array}{ll}
1 & \text{if } t \in [R - d - l - 3, R + d + l + 3],

0 & \text{if } t \notin [R - d - l - 2, R + d + l + 2]
\end{array} \right.
\]

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Then $D_W N_+ = D_W N_+^\infty + \rho D_W E_+^R$ with $E_+^R =: N_+^R - N_+^\infty$, so that $\rho D_W E_+^R$ becomes a map $\rho D_W E_+ : W \times [0, \infty) \to L(L^p_{k, \delta}(C_-, E) \times L^p_{k, \delta}(C_-, E) \times R^1, L^p_{k-1, \delta}(C_+, E))$.

Since $D_W N_+^\infty$ is continuous proved, for instance, by Floer, we only need to consider $\rho D_W E_+^R$.

Now we derive a more explicit formula for $D_W E$. To this end, note that for $R - d - l - 3 < t < R + d + l + 3$,

$$ (D_W E_+)_{u-, u+, R}(\xi_-, \xi_+) $$

$$ = \{ \partial J_{\hat{u}^R} \circ \Gamma^R \circ \tau^{-1} \circ R D_W(\hat{v}^R \circ \Gamma^R \circ \tau^{-1})(\xi) - \partial J_{u_+}(\xi_+) \} \partial_s u_+ + \{ J(\hat{v}^R \circ \Gamma^R \circ \tau^{-1}) - J(u_+) \} \partial_s \xi_+. $$

Recall that $\hat{v}^R = u_{-} \hat{R} u_{+} = \gamma_-, d \gamma_+ d u_{-} \circ R u_{+} + (1 - \gamma_-) \tau^{-1} u_{-} + (1 - \gamma_+ d) \tau^{-1} u_{-} = \gamma_-, d \gamma_+ d (\hat{\beta}_- \tau^{-1} R u_{-} + \hat{\beta}_- \tau^{-1} R u_{+}) + (1 - \gamma_-) \tau^{-1} R u_{-} + (1 - \gamma_+ d) \tau^{-1} R u_{+} = (\gamma_+ \beta_+ + (1 - \gamma_-)) u_{-} \circ \tau^{-1} R + (\gamma_- \beta_- + (1 - \gamma_+)) u_{+} \circ \tau^{-1} R$.

In the last identity above we have denoted $\gamma_\pm, \pm d$ by $\gamma_\pm$.

Hence

$$ \hat{v}^R_{+} \circ \Gamma^R \circ \tau^{-1} = \hat{v}^R_{+} \circ \tau^{-1} $$

$$ = \{ (\gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau^{-1} R \} u_{-} \circ \tau^{-1} R + \{ (\gamma_- \beta_- + (1 - \gamma_+)) \circ \tau^{-1} R \} u_{+}. $$

**Lemma 3.3**

$$ D_W(\hat{v}^R_{+} \circ \Gamma^R \circ \tau^{-1})(\xi) $$

$$ = \{ (\gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau^{-1} R \} \xi_{-} \circ \tau^{-1} R + \{ (\gamma_- \beta_- + (1 - \gamma_+)) \circ \tau^{-1} R \} \xi_{+}. $$

$$ D_R(\hat{v}^R_{+} \circ \Gamma^R \circ \tau^{-1}) = -2 \{ (\gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau^{-1} R \} \partial_t u_{-} \circ \tau^{-1} R $$

$$ + \partial_R \{ (\gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau^{-1} R \} u_{-} \circ \tau^{-1} R \partial_t u_{+} \circ \tau^{-1} R + \partial_R \{ (\gamma_- \beta_- + (1 - \gamma_+)) \circ \tau^{-1} R \} u_{+} \circ \tau^{-1} R. $$

**Lemma 3.4** The function $F_1 : W \times [0, \infty) \to L(L^p_{k, \delta}(C_-, E) \times L^p_{k, \delta}(C_-, E) \times R^1, L^p_{k-1, \delta}(C_+, E))$ defined by

$$ F_1(\xi, R) = \rho D_W(\hat{v}^R_{+} \circ \Gamma^R \circ \tau^{-1})(\xi) $$

$$ = \rho \cdot \{ (\gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau^{-1} R \} \xi_{-} \circ \tau^{-1} R + \{ (\gamma_- \beta_- + (1 - \gamma_+)) \circ \tau^{-1} R \} \xi_{+} $$

is continuous.
The proof of the lemma follows from a few facts below that will be used repeatedly:

(A) $F_+: W \times [0, \infty) \to L(L^p_{k,\delta}(C_+, E), L^p_{k-1,\delta}(C_+, E))$ defined by $F(u, R)(\xi) = \tau_R\xi$ is continuous. There is a corresponding function $F_-$. 

(B) Any smooth function $f_\pm$ such as $f_\pm = \beta_\pm' : C_\pm \to \mathbb{R}^1$ gives rise a $C^\infty$-map $F_\pm : \mathbb{R}^1 \to C^m(C_\pm, \mathbb{R}^1)$ defined by $F_\pm(R) = f \circ \tau_R$ for any $m$. In particular, we may assume that $m >> k$.

(C) Any smooth section such as $J : B \subset M \to \text{End}(E)$ with $B$ being a small ball in $M$ gives rise a $C^\infty$-map $F_{J,\pm} : L^p_{k,\delta}(C_\pm, B) \to L^p_{k,\delta}(C_\pm, \text{End}(E))$ again defined by the composition $u_\pm \to J \circ u_\pm$.

(D) The paring

$L^p_{k}(C_\pm, E) \times L(L^p_{k,\delta}(C_\pm, E), L^p_{k,\delta}(C_\pm, E)) \to L^p_{k,\delta}(C_\pm, E)$ is bounded bilinear and hence smooth as long as the space $L^p_{k}(C_\pm, E)$ forms Banach algebra.

(E) For $m >> k$, $L(L^p_{k,\delta}(C_\pm, E), L^p_{k,\delta}(C_\pm, E))$ is a $C^m(C_\pm, \mathbb{R}^1)$-module and the multiplication map

$C^m(C_\pm, \mathbb{R}^1) \times L(L^p_{k,\delta}(C_\pm, E), L^p_{k,\delta}(C_\pm, E)) \to L(L^p_{k,\delta}(C_\pm, E), L^p_{k,\delta}(C_\pm, E))$

is bounded bilinear and hence smooth.

The proofs for (B), (C) and (E) are straightforward and (D) is stated in the first chapter of Lang’s book \[4\] for general Banach spaces.

The property (A) is well-known and was proved for instance in \[3\]. In fact, what is needed here is a modified version of (A) in Lemma 4.4 of \[3\]. Given this modified version of (A), the proof of the lemma is almost identical to the corresponding statement in \[3\]. We leave the straightforward verification to the readers.

**Corollary 3.3** For $R \neq \infty$, $D_W N$ is continuous.

To prove that $D_W N$ can be extended continuously over $R = \infty$, we only need to show this for $\rho D_W E_+ : W \times [0, \infty) \to L(L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_-, E) \times \mathbb{R}^1, L^p_{k-1,\delta}(C_+, E))$, which follows from the estimates below.

Recall that we only consider the space $L^p_{k,\delta}(C_\pm, E)$ of $L^p_k$-maps that $\delta$-exponentially decay so that the estimates are only applicable to the case with fixed ends. In particular, the values $u_{\pm}(d_{\pm})$ of each map at the double point is fixed and set to be $0 \in E$. 

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Then we have for $t \in [R-d-l-3, R+d+l+3]$, 
\[
\hat{v}_+^R \circ \Gamma^R \circ \tau_-(t, s) = \hat{v}_+^R \circ \tau_R(t, s)
\]
\[
= \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)u_-(t-2R, s) + \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)u_+(t, s)
\]
\[
= \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)u_-(t, s) + \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)u_+(t, s)
\]
with $t_\tau \in [-R-d-l-3, -R+d+l+3]$. 

Similarly 
\[
D_W(\hat{v}_+^R \circ \Gamma^R \circ \tau_R)(\xi)(t, s)
\]
\[
= \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)\xi_-(t-2R, s) + \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)\xi_+(t, s)
\]
\[
= \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)\xi_-(t, s) + \{(\gamma-\gamma_+\beta_+ + (1-\gamma_+)) \circ \tau_R\}(t, s)\xi_+(t, s)
\]
with $t_\tau \in [-R-d-l-3, -R+d+l+3]$.

**Lemma 3.5** For $t \in [R-d-l-3, R+d+l+3]$, 

(1) 
\[
\|\xi_+\|_{C^{k-1}} \leq C \cdot e^{-\delta(R-d-l-3)} \|\xi_+\|_{k,p,\delta},
\]
with $C = C_k$ independent of $R$.

(II) 
\[
\|\hat{v}_+^R \circ \Gamma^R \circ \tau_R\|_{C^{k-1}} \leq C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot \|u\|_{k,p,\delta}.
\]

(III) 
\[
\|D_W(\hat{v}_+^R \circ \Gamma^R \circ \tau_R)(\xi)\|_{C^{k-1}} \leq C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot \|\xi\|_{k,p,\delta}.
\]

(IV) 
\[
\|J(\hat{v}_+^R \circ \Gamma^R \circ \tau_R) - J(u_+)\|_{C^{k-1}}
\]
\[
\leq C(\beta, \gamma) \cdot \|J\|_{C^{k-1}} \cdot e^{-\delta R/2} \cdot \|u\|_{k,p,\delta}(1 + e^{-\delta R/2} \cdot \|u\|_{k,p,\delta}).
\]

**Proof:**

(I) For $t \in [R-d-l-3, R+d+l+3]$, applying Sobolev embedding to each sub-cylinder of length 1 inside $[R-d-l-3, R+d+l+3]$, there is a constant $C = C_k$ independent of $R$ such that 
\[
\|\xi_+\|_{C^{k-1}} \leq C\|\xi_+\|_{[R-d-l-3,R+d+l+3]} \|_{k,p} \leq e^{-\delta(R-d-l-3)} \|\xi_+\|_{k,p,\delta}.
\]

(II) For $t \in [R-d-l-3, R+d+l+3]$,
\[
\|\hat{v}_+^R \circ \Gamma^R \circ \tau_R\|_{C^{k-1}} \leq C\{1 + \|\beta\|_{C^{k-1}} + \|\gamma\|_{C^{k-1}}\}^3 \cdot \|u_\tau\|_{[-R-d-l-3,-R+d+l+3]} \|_{C^{k-1}}
\]
\[
+ \|u_+\|_{R-d-l-3,R+d+l+3} \|C^{k-1}\)
\leq C(\|\beta\|_{C^{k-1}}, \|\gamma\|_{C^{k-1}}) \cdot e^{-\delta(R-d-l-3)} \cdot (\|u_-\|_{k,p,\delta} + \|u_+\|_{k,p,\delta})
\leq C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot (\|u_-\|_{k,p,\delta} + \|u_+\|_{k,p,\delta}) = C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot \|u\|_{k,p,\delta}.
\]

Here and below, we use \(C(\beta, \gamma)\) to denote the polynomial function \(C(\|\beta\|_{C^{k-1}}, \|\gamma\|_{C^{k-1}})\) of positive integer coefficients in \(\|\beta\|_{C^{k-1}}\) and \(\|\gamma\|_{C^{k-1}}\), which is \(C(\{\|\beta\|_{C^{k-1}} + \|\gamma\|_{C^{k-1}}\})^3\) in this case.

(III) Similarly, for \(t \in [R-d-l-3, R+d+l+3]\), the term
\[
\|D_w(\hat{\nu}_+^R \circ \Gamma^R \circ \tau_{-R})(\xi)\|_{C^{k-1}} \leq C\{1+\|\beta\|_{C^{k-1}} + \|\gamma\|_{C^{k-1}}\}^3 \|\xi_-\|_{[-R-d-l-3,-R+d+l+3]} \|C^{k-1}\)
+ \|\xi_+\|_{[R-d-l-3,R+d+l+3]} \|C^{k-1}\)
\leq C' \|\beta\|_{C^{k-1}} \cdot e^{-\delta(R-d-l-3)} \cdot (\|\xi_-\|_{k,p,\delta} + \|\xi_+\|_{k,p,\delta})
\leq C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot (\|\xi_-\|_{k,p,\delta} + \|\xi_+\|_{k,p,\delta}) = C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot \|\xi\|_{k,p,\delta}.
\]

(IV) For \(t \in [R-d-l-3, R+d+l+3]\), the term
\[
\|J(\hat{\nu}_+^R \circ \Gamma^R \circ \tau_{-R}) - J(u_+)\|_{C^{k-1}}
\leq \|[J(\hat{\nu}_+^R \circ \Gamma^R \circ \tau_{-R}) - J(u_+)]\|_{C^{k-1}} \|\xi_-\|_{[-R-d-l-3,-R+d+l+3]} \|C^{k-1}\)
+ \|\nabla (J(\hat{\nu}_+^R \circ \Gamma^R \circ \tau_{-R}))\|_{C^{k-1}} \|\nabla (J((u_+))\|_{C^{k-1}} \|\xi_-\|_{k,p,\delta} + \|\xi_+\|_{k,p,\delta})
\leq C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot (\|\xi_-\|_{k,p,\delta} + \|\xi_+\|_{k,p,\delta}) = C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot \|\xi\|_{k,p,\delta}.
\]

\[
\|D(\hat{\nu}_+^R \circ \Gamma^R \circ \tau_{-R}) - D(u_+)^{\#}\|_{C^{k-2}}
\leq C(\beta, \gamma) \cdot e^{-\delta R/2} \cdot \|\xi\|_{k,p,\delta}.
\]

Therefore we have
Proposition 3.1

\[ \|\cdot (D_W E_+)^{u_-, u_+, R}\|_0 \sim e^{-\delta R/2} C(\beta, \gamma) \cdot \|\cdot \|_{C^{k-1}} \cdot \|J\|_{C^k} \cdot \|u\|_{k,p,\delta} \cdot (1 + \|u\|_{k,p,\delta}). \]

Proof:

\[
\|\cdot (D_W E_+)^{u_-, u_+, R}\|_0 = \sup_{\|\xi\|_{k,p,\delta} \leq 1} \|\cdot (D_W E_+)^{u_-, u_+, R} (\xi^-, \xi^+)\|_{k-1,p,\delta}
= \sup_{\|\xi\|_{k,p,\delta} \leq 1} \|\cdot D J_{\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}} D_W (\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}) (\xi) - DJ_{u_+} (\xi^+)\| \cdot \partial_s u_+
+ \{J(\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}) - J(u_+)\}\cdot \partial_s \xi_+ \|_{k-1,p,\delta}
\leq \|\cdot \|_{C^{k-1}} \cdot \{\|\cdot \|_{C^k} \cdot \{\|\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}\|_{C^{k-1}} + \|u_+\|_{C^{k-1}}\}\cdot \sup_{\|\xi\|_{k,p,\delta} \leq 1} \{\|D_W (\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}) (\xi)\|_{C^{k-1}} + \|\xi\|_{R-d-l-3, R+d+l+3}\|_{C^{k-1}}\}\cdot \|\partial_s u_+\|_{k-1,p,\delta}
+ \{J(\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}) - J(u_+)\}|_{C^{k-1}} \cdot \sup_{\|\xi\|_{k,p,\delta} \leq 1} \|\partial_s \xi_+\|_{k-1,p,\delta}\]
\]

(by the estimates in the previous lemma for \(D_W (\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}) (\xi)\|_{C^{k-1}}
and \|\xi\|_{R-d-l-3, R+d+l+3}\|_{C^{k-1}})

\[
\leq \|\cdot \|_{C^{k-1}} \cdot \{\|\cdot \|_{C^k} \cdot \{\|\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}\|_{C^{k-1}} + \|u_+\|_{C^{k-1}}\}\cdot \sup_{\|\xi\|_{k,p,\delta} \leq 1} \{(1 + C(\beta, \gamma)) e^{-\delta R^2/2} \|\xi\|_{k,p,\delta}\}\cdot \|\partial_s u_+\|_{k-1,p,\delta} + \|J(\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}) - J(u_+)\|_{C^{k-1}}\}
\]

(by the estimate in the previous lemma for \(J(\hat{\nu}_+^R \circ \Gamma_R \circ \tau_{-R}) - J(u_+)\|_{C^{k-1}}\))

\[
\sim \|\cdot \|_{C^{k-1}} \cdot \{\|\cdot \|_{C^k} \cdot \{u_+^2\|_{k,p,\delta} \cdot (1 + C(\beta, \gamma)) \cdot e^{-\delta R^2/2}
+ C(\beta, \gamma) \cdot \{\|\cdot \|_{C^k} \cdot e^{-\delta R^2/2} \cdot \|u_+\|_{k,p,\delta} (1 + \|u_+\|_{k,p,\delta})\}\}\}
\sim e^{-\delta R^2/2} C(\beta, \gamma) \cdot \|\cdot \|_{C^{k-1}} \cdot \|J\|_{C^k} \cdot \|u_+\|_{k,p,\delta} \cdot (1 + \|u_+\|_{k,p,\delta}).
\]

Next consider \(D_{R_0} N_+: L^p_{k,\delta}(C^+, E) \times [R_0, \infty) \rightarrow L^p_{k-1,\delta}(C^+, E)\).
Recall that

\[
N_+(R_0, u_-, u_+) = \begin{cases} J(u_+) \partial_s u_+, & \text{if } t > R + d + l + 2, \\
\tau_{-R_0} J(\hat{\nu}_+^R \circ \Gamma_R) \partial_s u_+, & \text{if } R + d - l - 3 < t < R + d + l + 3, \\
J(u_+) \partial_s u_+, & \text{if } 0 < t < R + d - l - 2. 
\end{cases}
\]

Hence we have
Lemma 3.6 For \( R \neq \infty \),
\[
D_R N_+(R, u_-, u_+) =
\begin{cases}
D_{\hat{v}_+^R \circ \Gamma^R \circ \tau_R}(D_R \{ \hat{v}_+^R \circ \Gamma^R \circ \tau_R \}) \partial_s u_+, & \text{if } t \in [R - d - l - 3, R + d + l + 3], \\
0, & \text{if } t \not\in [R - d - l - 2, R + d + l + 2]
\end{cases}
\]
and \( D_R N_+(R, u_-, u_+) = \)
\[
D_{\hat{v}_+^R \circ \Gamma^R \circ \tau_R \circ \tau_R}(D_R \{ \hat{v}_+^R \circ \Gamma^R \circ \tau_R \}) \partial_s u_+, & \text{if } t \in [R - d - l - 3, R + d + l + 3], \\
0, & \text{if } t \not\in [R - d - l - 2, R + d + l + 2]
\]
Recall here
\[
\hat{v}_+^R \circ \Gamma^R \circ \tau_R = \hat{v}_+^R \circ \tau_R
\]
\[
= \{(\gamma_- - \gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau_R\}u_- \circ \tau_2R + \{(\gamma_- - \gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau_R\}u_+,
\]
and
\[
D_R(\hat{v}_+^R \circ \Gamma^R \circ \tau_R) = -2\{(\gamma_- - \gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau_R\} \partial_t u_- \circ \tau_2R
\]
\[
- \{(a_- \gamma_- \gamma_+ + a_+ \gamma_- \gamma_+ \beta_+ - a_- \gamma_+) \circ \tau_R\} u_- \circ \tau_2R
\]
\[
- \{(a_- \gamma_- \gamma_+ + a_+ \gamma_- \gamma_+ \beta_+ - a_+ \gamma_+) \circ \tau_R\} u_+.
\]
Similarly but simpler
\[
D_R(\hat{v}_+^R \circ \Gamma^R \circ \tau_R) = -2\{(\gamma_- - \gamma_+ \beta_+ + (1 - \gamma_-)) \circ \tau_R\} \partial_s u_- \circ \tau_2R_{\theta}.
\]
Here \( a_\pm = \partial_{\theta}(l(R) + d(R)). \) Recall that we may assume that \( l(R) + d(R) = R^{1/2} \cdot \ln^2 R \) for \( R \in [R_0, \infty) \) and \( R_0 >> 0. \)
Then \( |a_\pm| \sim R^{-1/2} \cdot \{\ln^2 R - 2 \ln R\} \sim R^{-1/2} \cdot \ln^2 R. \)
From this formula, using the fact that \( F : L^p_{k,\delta}(C_+, E) \times [R_0, \infty) \rightarrow L^p_{k-1,\delta}(C_+, E) \) given by \( F(u, R) = Du \circ \tau_R \) is continuous together with the general properties labeled as (B), (C) and (D) before, it is clear that the next lemma is true.

Lemma 3.7 For \( R \neq \infty \), \( D_R N_+ \) is continuous

Proposition 3.2 Assume further that \( k - 2/p > 2 \) so that \( \| \nabla u \|_{C^0} \leq C \| u \|_{k,p}. \)
Then \( D_R N_+ (D_r N_+) \) extends over \( R = \infty \) (\( r = 0 \)) continuously with respect to the gluing profile \( R = e^{1/r}. \) So does \( D_R N_+. \) Consequently, \( N : L^p_{k,\delta}(C_-, E) \times L^p_{k,\delta}(C_+, E) \times D_{r_0} \rightarrow L^p_{k-1,\delta}(C_-, E) \times L^p_{k-1,\delta}(C_+, E) \) is of class \( C^1. \)
Proof:
This follows from the following two estimates, in which the effect of $|a_\pm|$ will be ignored since $|a_\pm| \sim R^{-1/3} \ll 1$.
We state each as a lemma.

Lemma 3.8 For $t \in [R - d - l - 3, R + d + l + 3]$,

$$
\|DJ_{\hat{v}_R^{R \circ \Gamma_R \circ \tau_R \circ \tau_R}}(DR\{\hat{v}_R^R \circ \Gamma_R \circ \tau_R\})\partial_s u_+\|_{k-1,p,\delta} \qquad \leq \|DJ\|_{C^{k-1}} \cdot c(\beta, \gamma) \cdot \|u\|_{k,p,\delta} \cdot (I + II).
$$

Here $I \sim e^{-\delta R/2} \cdot \|u_+\|_{k,p,\delta} \cdot \|u_-\|_{k,p,\delta}$ and $II \sim e^{-\delta R/2} \|u_+\|_{k,p,\delta}^2$.

Proof:

For $t \in [R - d - l - 3, R + d + l + 3]$,

$$
\|DJ_{\hat{v}_R^{R \circ \Gamma_R \circ \tau_R \circ \tau_R}}(DR\{\hat{v}_R^R \circ \Gamma_R \circ \tau_R\})\partial_s u_+\|_{k-1,p,\delta} \leq \|DJ\|_{C^{k-1}} \cdot c(\beta, \gamma) \cdot \|u\|_{k,p,\delta} \cdot \|\partial_s u_+\|_{k-1,p,\delta} \cdot \left\{\|u_- \circ \tau_R\|_{C^0} + \|\partial_t u_- \circ \tau_R\|_{C^0}\right\} \\
+\|\partial_s u_+\|_{k-2,p,\delta} \cdot \left\{\|u_- \circ \tau_R\|_{C^{k-2}} + \|\partial_t u_- \circ \tau_R\|_{C^{k-2}}\right\} \\
+\|\partial_s u_+\|_{C^0} \cdot \left\{\|u_- \circ \tau_R\|_{k-1,p,\delta} + \|\partial_t u_- \circ \tau_R\|_{k-1,p,\delta}\right\} \\
+\|DJ\|_{C^{k-1}} \cdot c(\beta, \gamma) \cdot \|u\|_{k,p,\delta} \cdot \left\{\|\partial_s u_+\|_{k-1,p,\delta} \cdot \|u_+\|_{C^0} \\
+\|\partial_s u_+\|_{k-2,p,\delta} \cdot \|u_+\|_{C^{k-2}} + \|DJ\|_{C^{k-1}} \cdot c(\beta, \gamma) \cdot \|u\|_{k,p,\delta} \cdot (I + II).
$$

Estimate for $I$ with $t \in [R - d - l - 3, R + d + l + 3]$:

$$
I \sim \|\partial_s u_+\|_{k-1,p,\delta} \cdot \|u_- \circ \tau_R\|_{[R-d-l-3,R+d+l+3]}\|_{C^1} \\
+\|\partial_s u_+\|_{k-2,p,\delta} \cdot \|u_- \circ \tau_R\|_{[R-d-l-3,R+d+l+3]}\|_{C^{k-1}} + \|\partial_s u_+\|_{[R-d-l-3,R+d+l+3]}\|_{C^0} \cdot \|u_- \circ \tau_R\|_{k,p,\delta} \\
\sim \|\partial_s u_+\|_{k-1,p,\delta} \cdot \|u_- \circ \tau_R\|_{[R-d-l-3,R+d+l+3]}\|_{k,p} \\
+\|\partial_s u_+\|_{k-2,p,\delta} \cdot \|u_- \circ \tau_R\|_{[R-d-l-3,R+d+l+3]}\|_{k,p} \\
+\|\partial_s u_+\|_{[R-d-l-3,R+d+l+3]}\|_{C^0} \cdot \|u_- \circ \tau_R\|_{[R-d-l-3,R+d+l+3]}\|_{k,p,\delta} \\
\leq \|u_+\|_{k,p,\delta} \cdot \|u_- \circ \tau_R\|_{[-R-d-l-3,-R+d+l+3]}\|_{k,p} \\
+\|\partial_s u_+\|_{[R-d-l-3,R+d+l+3]}\|_{C^0} \cdot e^{\delta(R+d+l+3)} \|u_- \circ \tau_R\|_{[R-d-l-3,R+d+l+3]}\|_{k,p}
$$
Lemma 3.9 from the following estimate.

Proof:

Finally we need to show that $D_{\theta}N$ extends over $R = \infty$.

Estimate for $II$ with $t \in [R - d - l - 3, R + d + l + 3]$:

$$II \sim u_+ ||_{k,p,\delta} \cdot || u_+ ||_{C^k} \sim u_+ ||_{k,p,\delta} \cdot || u_+ ||_{[R-d-l-3, R+d+l+3]} ||_{k-1,p} \leq u_+ ||_{k,p,\delta} \cdot e^{-\delta R/2} \cdot || u_+ ||_{k,p,\delta}.$$ 

It follows from the estimate in above lemma that $D_{R}N$ extends over $R = \infty$.

Finally we need to show that $D_{\theta}N$ extends over $R = \infty$. This follows from the following estimate.

Lemma 3.9 For $t \in [R - d - l - 3, R + d + l + 3]$,

$$|| D_{\theta}N_{+}(R_{\theta}, u_{-}, u_{+}) ||_{k-1,p,\delta} \leq || D_{J} ||_{C^k} \cdot c(\beta, \gamma) \cdot || u ||_{k,p,\delta} \cdot III$$

with $III \sim e^{-\delta R/2} \cdot || u_+ ||_{k,p,\delta} \cdot || u_+ ||_{k,p,\delta}$.

Proof:

For $t \in [R - d - l - 3, R + d + l + 3]$, recall

$$D_{\theta}(\hat{v}_{+}^{R_{\theta}} \circ \Gamma^{R_{\theta}} \circ \tau_{-R_{\theta}}) = -2\{(\gamma_{-} - \gamma_{+} + (1 - \gamma_{-})) \circ \tau_{-R}\} \partial_{s}u_{-} \circ \tau_{-2R_{\theta}}.$$ 

Then $|| D_{\theta}N_{+}(R_{\theta}, u_{-}, u_{+}) ||_{k-1,p,\delta} =$

$$|| D_{J} \hat{v}_{+}^{R_{\theta}} \circ \Gamma^{R_{\theta}} \circ \tau_{-R_{\theta}} \circ \tau_{-2R_{\theta}} (D_{\theta}(\hat{v}_{+}^{R_{\theta}} \circ \Gamma^{R_{\theta}} \circ \tau_{-R_{\theta}})) \partial_{s}u_{+} ||_{k-1,p,\delta}$$

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\[ \leq \| DJ \|_{C^{k-1}} \cdot c(\beta, \gamma) \cdot \| u \|_{k,p,\delta} \{ \| \partial_s u_+ \|_{k-1,p,\delta} \cdot \| \partial_s u_- \circ \tau_{-2R_0} \|_{C^0} \]
\[ + \| \partial_s u_+ \|_{k-2,p,\delta} \cdot \| \partial_s u_- \circ \tau_{-2R_0} \|_{C^{k-2}} \]
\[ + \| \partial_s u_+ \|_{C^0} \cdot \| \partial_s u_- \circ \tau_{-2R_0} \|_{k-1,p,\delta} \}. \]
\[ \sim \| DJ \|_{C^{k-1}} \cdot c(\beta, \gamma) \cdot \| u \|_{k,p,\delta} \cdot \| \partial_s u_- \circ \tau_{-2R_0} \|_{C^{k-2}} \]
\[ + \| \partial_s u_+ \|_{C^0} \cdot \| \partial_s u_- \circ \tau_{-2R_0} \|_{k-1,p,\delta} \}. \]

\[ = \| DJ \|_{C^{k-1}} \cdot c(\beta, \gamma) \cdot \| u \|_{k,p,\delta} \cdot III. \]

For \( t \in [R - d - l - 3, R + d + l + 3] \),

\[ III \sim \| u_+ \|_{k,p,\delta} \cdot \| \partial_s u_- \circ \tau_{-2R_0} \|_{[R - d - l - 3, R + d + l + 3]} \|_{k-1,p} \]
\[ + \| \partial_s u_+ \|_{[R - d - l - 3, R + d + l + 3]} \|_{k-1,p,\delta} \]
\[ \leq \| u_+ \|_{k,p,\delta} \cdot \| \partial_s u_- \|_{[-R - d - l - 3, -R + d + l + 3]} \|_{k-1,p} \]
\[ + \| \partial_s u_+ \|_{[-R - d - l - 3, -R + d + l + 3]} \|_{k-1,p,\delta} \]
\[ \leq \| u_+ \|_{k,p,\delta} \cdot e^{\delta(-R + d + l + 3)} \| \partial_s u_- \|_{k-1,p,\delta} \]
\[ + e^{\delta(-R + d + l + 3)} \| \partial_s u_+ \|_{k-1,p,\delta} \cdot e^{\delta(R + d + l + 3)} \| \partial_s u_- \|_{k-1,p,\delta} \]
\[ \sim e^{-\delta R/2} \cdot \| u_- \|_{k,p,\delta} \cdot \| u_+ \|_{k,p,\delta}. \]

By the last lemma in [3], this finishes the proof of the Theorem 1.1.

References

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