Erasure List-Decodable Codes from Random and Algebraic Geometry Codes

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Abstract

Erasure list decoding was introduced to correct a larger number of erasures with output of a list of possible candidates. In the present paper, we consider both random linear codes and algebraic geometry codes for list decoding erasure errors. The contributions of this paper are two-fold. Firstly, we show that, for arbitrary $0 < R < 1$ and $\epsilon > 0$ $(R$ and $\epsilon$ are independent), with high probability a random linear code is an erasure list decodable code with constant list size $2^{O(1/\epsilon)}$ that can correct a fraction $1 - R - \epsilon$ of erasures, i.e., a random linear code achieves the information-theoretic optimal trade-off between information rate and fraction of erasure errors. Secondly, we show that algebraic geometry codes are good erasure list-decodable codes. Precisely speaking, for any $0 < R < 1$ and $\epsilon > 0$, a $q$-ary algebraic geometry code of rate $R$ from the Garcia-Stichtenoth tower can correct $1 - R + \sqrt{\frac{\epsilon}{q - 1}} + \frac{1}{2} - \epsilon$ fraction of erasure errors with list size $O(1/\epsilon)$. This improves the Johnson bound applied to algebraic geometry codes. Furthermore, list decoding of these algebraic geometry codes can be implemented in polynomial time.

Index Terms

Erasure codes, List decoding, Algebraic geometry codes, Generalized Hamming weights.

I. INTRODUCTION

Erasure codes have received great attentions for their wide applications in recovering packet losses in the internet and storage systems. In the model of erasure channel, errors are described as erasures, namely the receivers are supposed to know the positions where the erasures occurred. Compared with other communication channels such as adversarial noise channel, erasure channel is much simpler. Thus, we can expect better parameters for erasure channel than adversarial noise channel. Instead of the unique decoding, the model of list decoding for which a decoder allows to output a list of possible codewords that are consistent with the unerased portion of the received word.

The problem of list decoding for classical adversarial noise channel has been extensively studied (see [3], [7], [8], [9], [16], [18], [19], for example). A fundamental problem in list decoding is the tradeoff among the information rate, decoding radius (i.e., fraction of errors that can be corrected) and the list size. In other words, if we fix one of these three parameters, then one is interested in optimal tradeoff between the remaining two parameters. For instance, if the list size is fixed to be constant or polynomial in the length of codes, the problem becomes a tradeoff between information rate and decoding radius.

Definition 1.1: $(\tau, L)$-erasure list decodability) Let $\Sigma$ be a finite alphabet of size $q$, $L \geq 1$ an integer, and $\tau \in (0, 1)$. A code $C \subseteq \Sigma^n$ is said to be $(\tau, L)$-erasure list-decodable, if for every $r \in \mathbb{F}_q^{1-\tau)n}$, and any subset $T \subseteq \{1, 2, \cdots, n\}$ of size $(1-\tau)n$, one has

$$|\{c \in C | c_T = r\}| \leq L,$$

where $c_T$ is the projection of $c$ onto the coordinates indexed by $T$. In other words, given any received word with at most $\tau n$ erasures, there are at most $L$ codewords that are consistent with the unerased portion of the received word.

Known results

It is known that, for an erasure channel where the codeword symbols are randomly and independently erased with probability $\tau$, the capacity is $1 \tau$ (see [4]). Although erasure list decoding has been considered previously (see [6], [10], [8], [9]), a lot of problems still remain unsolved. Let us summarize some of previous results on erasure list decoding below.

(i) It was shown in [6] that, for any small $\epsilon > 0$ and $\tau \in (0, 1)$, a $(\tau, L)$-erasure list-decodable code of rate $1 \tau \epsilon$ must satisfy $L \Omega(1/\epsilon)$; and on the other hand, there exists a $(\tau, O(\exp(1/\epsilon)^\epsilon)$-erasure list-decodable code of rate $1 \tau \epsilon$.

(ii) In [7, Proposition 10.1], the Johnson bound for erasure decoding radius was derived. It says that, for any given $\epsilon > 0$, every $q$-ary code of relative distance $\delta < 1 - 1/q$ is $(\delta + \frac{\epsilon}{q - 1} - \epsilon, O(1/\epsilon))$-erasure list-decodable. This means that, with a constant list size, erasure decoding radius is enlarged by approximately $\frac{\epsilon}{q - 1}$ compared with unique erasure decoding whose decoding radius is $\delta$. On the other hand, it was shown further in [7, Proposition 10.2] that there exists a $q$-ary code of length $n$ and relative distance $\delta < 1 - 1/q$ that is not $(\delta + \frac{\epsilon}{q - 1} + \epsilon, 2^{\Omega(\epsilon^2 n)})$-erasure list-decodable for every small

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\( \epsilon > 0 \). This implies that the best bound on erasure list decoding radius of a \( q \)-ary code of relative minimum distance \( \delta \) is
\[
\delta + \frac{\epsilon}{q}. 
\]
(iii) In [6], Guruswami showed that, for any small \( \epsilon > 0 \), with high probability a random linear code of rate \( R = \Omega(\epsilon / \log(1/\epsilon)) \) is \((1 - \sigma, O(1/\sigma))\)-erasure list-decodable for every \( \sigma \) satisfying \( \epsilon \leq \sigma \leq 1 \). Furthermore, by the concatenation method Guruswami showed in [6] that, for any small \( \epsilon > 0 \), one can construct a family of concatenated (binary) \((1 - \epsilon, O(1/\epsilon))\)-erasure list-decodable codes of rate \( \Omega(\epsilon^2/\log(1/\epsilon)) \) in polynomial time. A slightly better rate was obtained for nonlinear codes over larger alphabet size in [9].

**Our results and comparison**

Our contributions of this paper are two-fold.

(i) Firstly, we show that, for arbitrary \( 0 < R < 1 \) and \( \epsilon > 0 \) \((R \) and \( \epsilon \) are independent), with high probability a random linear code is \((1 - R - \epsilon, 2^{O(1/\epsilon)})\)-erasure list-decodable, i.e., a random linear code achieves the information-theoretic optimal tradeoff between information rate and fraction of erasure errors that can be corrected. While Theorem 2 in [6] which was derived from [11] only shows existence of \((1 - R - \epsilon, 2^{O(1/\epsilon)})\)-erasure list-decodable codes for arbitrary \( 0 < R < 1 \) and \( \epsilon > 0 \).

(ii) Secondly, we show that algebraic geometry codes are good erasure list-decodable codes. Precisely speaking, for any \( 0 < \tau < 1 \) and \( \epsilon > 0 \), a \( q \)-ary algebraic geometry code from the Garcia-Stichtenoth tower has rate at least \( 1 - \tau - \frac{1}{\sqrt{q - 1}} + \frac{1}{q} - \epsilon \) and is \((\tau, O(1/\epsilon))\)-erasure list-decodable. Furthermore, list decoding of these algebraic geometry codes can be implemented in polynomial time. On the other hand, if we apply the Johnson bound given in [7, Proposition 10.1] to general algebraic geometry codes, we can only claim that a \( q \)-ary algebraic geometry code from the Garcia-Stichtenoth tower has rate \( 1 - \tau - \frac{1}{\sqrt{q - 1}} + \frac{1}{q} - \epsilon \) and is \((\tau, O(1/\epsilon))\)-erasure list-decodable. This rate is always smaller than our rate for any \( \tau \in (0, 1) \). This implies that the Johnson bound could be improved for some special class of codes although it is optimal in general.

**Open problems**

For adversarial error channel, it has been shown that, given decoding radius \( 0 < \tau < 1 \), the optimal rate for list decoding is \( R = 1 - H_q(\tau) \), where \( H_q(x) = x \log_q(q - 1) - x \log_q x - (1 - x) \log_q(1 - x) \) is the \( q \)-ary entropy function. More precisely speaking, for any small \( \epsilon > 0 \) and \( \tau \) with \( 0 < \tau < 1 - 1/q \), with high probability a random code is \((1 - H_q(\tau) - \epsilon, O(1/\epsilon))\)-list decodable. Furthermore, every \( q \)-ary \((1 - H_q(\tau) - \epsilon, L)\)-list-decodable code has list size at least \( \Omega(\log 1/\epsilon) \). It is still an open problem to determine if there exists a \( q \)-ary \((1 - H_q(\tau) - \epsilon, L)\)-list-decodable code with list size \( L \) smaller than \( O(\epsilon/\tau) \). Under the situation of erasure list decoding, the optimal rate \( R \) that one can achieve is \( R = 1 - \tau \). If we denote \( L_{\tau,q}(\epsilon) \) to be the smallest integer \( L \) for which there are \( q \)-ary \((\tau, L)\)-erasure list-decodable codes of rate at least \( 1 - \tau - \epsilon \) for infinitely many lengths \( n \), then it follows from our result and [6] that \( \Omega(\frac{1}{\tau}) \leq L_{\tau,q}(\epsilon) \leq 2^{O(1/\epsilon)} \). Now the first open problem is

**Open Problem 1:** Determine \( L_{\tau,q}(\epsilon) \).

In the literature, there are not many results on constructive bounds on erasure list decoding except for sufficiently large \( q \) or small rate [6], [10]. The second open problem would be

**Open Problem 2:** Narrow the rate gap between \( 1 - \tau - \frac{1}{\sqrt{q - 1}} + \frac{1}{q} \) and \( 1 - \tau \) by constructing erasure list-decodable codes explicitly, i.e., construct a \( q \)-ary \((\tau, L)\)-erasure list-decodable codes of rate \( R > 1 - \tau - \frac{1}{\sqrt{q - 1}} + \frac{1}{q} \) such that the list size \( L \) is either a constant or a polynomial in length.

**Organization**

The paper is organized as follows. In Section 2, we introduce some necessary notation and definitions and known results as well. Section 3 is devoted to random codes. In the last section, we show that algebraic geometry codes are good erasure list-decodable codes.

**II. Preliminaries**

In this paper, we only focus on linear codes. Recall that a \( q \)-ary \([n, k]_q \) linear code is an \( F_q \)-linear subspace of \( F_q^n \) with dimension \( k \), where \( F_q \) is a finite field with \( q \) elements and \( q \) is a prime power. \( n \) is called the length of the code and \( k \) is the dimension of the code. The information rate of the code \( C \) is defined as \( R = k/n \) which represents the efficiency of the code. Another important parameter of the code is the distance which represents the error correcting capability. The distance of a linear code \( C \) is defined to be the minimum Hamming weight of nonzero codewords of \( C \), denoted by \( d = d(C) \). The relative distance \( \delta = \delta(C) \) is defined to be the quotient \( d/n \).

From Definition 1.1, one knows that, in a \((\tau, L)\)-erasure list decodable code \( C \) of length \( n \), for every \( r \in F_q^{(1-\tau)n} \) and \( T \subseteq \{1, 2, \ldots, n\} \) with \( |T| = (1 - \tau)n \) the number of the codewords in the output list that are consistent with \( r \) at the coordinates indexed by \( T \) is at most \( L \). Thus, if \( C \) is linear, it is equivalent to saying that the number of the codewords
that are 0 at the coordinates indexed by T is at most L, i.e., $|\{c \in C|c_T = 0\}| \leq L$. Hence, an $[n, k, d]_q$-linear code is $((d-1)/n, 1)$-erasuere list-decodable, but not $(d/n, 1)$-erasuere list-decodable.

**Definition 2.1:** (Erasure list decoding radius (ELDR))

(i) For an integer $L \geq 1$ and a linear code $C$ of length $n$, we denote

$$\text{Rad}_L(C) := \max \{s \in \mathbb{Z}_{\geq 0} : C \text{ is } (s/n, L) \text{-erasuere list-decodable}\}.$$ 

(ii) For an infinite family $\mathcal{C} = \{C_i\}_{i \geq 1}$ of $q$-ary linear codes with length tending to $\infty$ and an integer $L \geq 1$, we denote

$$\text{ELDR}_L(\mathcal{C}) := \liminf_i \left\{\frac{\text{Rad}_L(C_i)}{n_i}\right\},$$

where $n_i$ is the length of $C_i$.

**Definition 2.2:** For an integer $L \geq 1$ and $0 \leq \tau \leq 1$, the maximum rate for linear $(\tau, L)$-erasuere list-decodable code families is defined to be

$$R_L(\tau) := \sup_{C : \text{ELDR}_L(\mathcal{C}) \geq \tau} R(C).$$

The notion of erasure list decoding for linear codes actually had already been studied in the form of generalized Hamming weight, see [17]. However, the explicit relationship between erasure list decoding and generalized Hamming weight had not been made clear until the work in [6]. The concept of generalized Hamming weight was initially introduced in [17] and later received great attention due to applications in cryptography, design of codes, $t$-resilient functions and so on [1].

**Definition 2.3:** (Generalized Hamming Weight) The $r$-th generalized Hamming weight of a code $C$, denoted by $d_r(C)$, is defined to be the size of the smallest support of an $r$-dimensional subcode of $C$, i.e.,

$$d_r(C) = \min\{|\text{Supp}(D)| : D \text{ is a subspace of } C \text{ of dimension } r\},$$

where $\text{Supp}(D) = \{i : \exists(c_1, \ldots, c_n) \in D, c_i \neq 0\}$.

Note that $d_1(C)$ is exactly the minimum distance $d$ of $C$. The characterization of erasure list decidability through generalized Hamming weight is given below.

**Lemma 2.4:** (see [6]) A linear code $C$ of length $n$ is $(s/n, L)$-erasuere list-decodable if and only if $d_r(C) > s$, where $r = \lfloor \log_q L \rfloor + 1$.

The link stated in Lemma 2.4 establishes a two-way bridge. Results for erasure list decoding can be derived directly from the existing results on generalized Hamming weight, and thus the applications of generalized Hamming weight are inherited. In the meanwhile, some new properties for generalized Hamming weight can be obtained as well if one can develop some fresh ideas on erasure list decoding.

In [6], Guruswami made use of the connection between generalized Hamming weight and erasure list decoding to establish some bounds for rate $R_L(\tau)$ through the existing bounds on generalized Hamming weight.

**Lemma 2.5:** (see [6]) One has

(i) For every integer $L \geq 1$ and every $\tau$, $0 \leq \tau \leq 1$,

$$R_L(\tau) \geq 1 - \frac{\tau}{r} \log_q q^r - 1 - \frac{H_q(\tau)}{r},$$

where $r = \lfloor \log_q L \rfloor + 1$. In particular, for any small $\epsilon > 0$ and $\tau \in (0, 1)$, there exists a $(\tau, O(\exp(\frac{1}{\epsilon^2})))$-erasuere list-decodable code of rate $1 - \tau - \epsilon$.

(ii) For small $\epsilon > 0$ and $\tau$ with $0 < \tau < 1$, a $(\tau, L)$-erasuere list-decodable code of rate $1 - \tau - \epsilon$ must satisfy $L \geq \Omega(\frac{1}{\epsilon^2})$.

### III. Random List Decodable Erasure Codes

Random $(1 - \epsilon, O(1/\epsilon))$-erasuere list-decodable codes of rate $R = \Omega(\epsilon/\log(1/\epsilon))$ was discussed in [6] by using a characterization of generator matrices of erasure list-decodable codes. However, the rate is quite small and actually is dependent on $\epsilon$. In this section, we are going to show that for any $0 \leq R \leq 1$ ($R$ is independent of $\epsilon$), with probability $1 - q^{-\Omega(n)}$ a random linear code $C$ of length $n$ and rate $R$ is $(1 - R - \epsilon, 2^{O(1/\epsilon)})$-erasuere list-decodable. Our approach is through a characterization of parity-check matrices of list-decodable codes.

**Proposition 3.1:** If $k/n \to R > 0$ when $n$ tends to $\infty$, then for a random matrix $H$ over $\mathbb{F}_q$ of size $(n - k) \times n$, the probability that $H$ is full-rank is approaching 1 when $n$ tends to $\infty$.

**Proof:** On one hand, it is easy to compute that the total number of random matrix $H$ over $\mathbb{F}_q$ of size $(n - k) \times n$ with full rank is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-k-1}).$$
On the other hand, the total number of matrices $H$ over $\mathbb{F}_q$ of size $(n-k) \times n$ is $q^{n(n-k)}$. Let $E$ denote the event that an $(n-k) \times n$ random matrix $H$ over $\mathbb{F}_q$ is full-rank, then

$$Pr(E) = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{n-k-1})}{q^{n(n-k)}}.$$ 

To show $\lim_{n \to \infty} Pr(E) = 1$, it suffices to show that $\lim_{n \to \infty} \ln Pr(E) \to 0$.

When $n$ tends to $\infty$, we have

$$0 \geq \ln \left( \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{n-k-1})}{q^{n(n-k)}} \right) = \sum_{i=k+1}^{n} \ln \left( 1 - \frac{1}{q^i} \right) \geq \sum_{i=k+1}^{n} \left( - \frac{2}{q^i} \right) \geq - \frac{2n}{q^k} \to 0.$$ 

This completes the proof. 

**Lemma 3.2:** Let $s$ be a positive integer, then an $[n,k]_q$ code $C$ is $(s/n, L)$-erasure-list-decodable if and only if any submatrix $H'_{(n-k)\times s}$ of the parity check matrix $H_{(n-k)\times n}$ of $C$ has rank at least $s - \lfloor \log_q L \rfloor$.

**Proof:** By Definition 3.1 and the fact that $C$ is a linear code, $C$ is $(s/n, L)$-erasure-list-decodable if and only if

$$|\{c \in C|c_T = 0\}| \leq L$$

for $T \subseteq \{1, 2, \ldots, n\}$ with size $n-s$. This implies that $C$ is $(s/n, L)$-erasure-list-decodable if and only if for any submatrix $H'_{(n-k)\times s}$ of $H_{(n-k)\times n}$,

$$|\{x \in \mathbb{F}_q^s | H'_{(n-k)\times s} \cdot x = 0\}| \leq L,$$

i.e., the solution space of $H'_{(n-k)\times s}$ has dimension at most $\lfloor \log_q L \rfloor$. Therefore, $H'_{(n-k)\times s}$ has rank at least $s - \lfloor \log_q L \rfloor$.

**Theorem 3.3:** For every small $\epsilon > 0$, a real $0 < R < 1$ and sufficiently large $n$, with probability at least $1 - q^{-\Omega(n)}$, a random linear code over $\mathbb{F}_q$ of length $n$ and rate $R$ is $(1-R-\epsilon, 2^{O(1)})$-erasuer-list-decodable.

**Proof:** Put $\ell = \lceil \frac{1}{\epsilon}(2-R\log_q 2 + 1) \rceil$ and $L = q^\ell$. Thus, $L = 2^{O(1)}$. We randomly pick a matrix $H_{(n-k)\times n}$. Then with probability approaching 1, $H_{(n-k)\times n}$ is full rank from Proposition 3.1. Let such a full rank matrix $H_{(n-k)\times n}$ be the parity check matrix of our linear code $C$. Then we are going to prove that with probability at most $q^{-\Omega(n)}$, $C$ is not $(s, L)$-erasuer-list-decodable for $s = \lfloor n-k-\epsilon n \rfloor$. By Lemma 3.2, this happens only if some $(n-k)\times s$ submatrix of $H$ has rank less than $s - \lfloor \log_q L \rfloor$.

Denote $n-k$ by $K$. Let $A$ denote the number of full-rank matrices $H_{(n-k)\times n}$ in which there exists $s = n-k - \epsilon n$ columns with rank at most $s - \ell$. Note that the total number of matrices of size $K \times s$ over $\mathbb{F}_q$ with rank at most $s - \ell$ is equal to $
abla_{s-\ell} (K) \cdot \binom{q^s-1}{s} \cdots \binom{q^{s-1}}{s} q^{(K-s)}$. Thus, we have

$$A \leq \binom{n}{s} q^{K(n-s)} \sum_{i=0}^{s-\ell} \binom{K}{i} (q^s-1) \cdots (q^i-1) q^{(K-i)s}$$

$$\leq 2^n \sum_{i=0}^{s-\ell} \binom{K}{i} q^{(s+K-i^2)+K(n-s)}$$

$$\leq 2^n \times q^{(s+K)(s-\ell)-(s-\ell)^2+K(n-s)} \sum_{i=0}^{s-\ell} \binom{K}{i}$$

$$\leq 2^n+K \times q^{(s+K)(s-\ell)-(s-\ell)^2+K(n-s)}$$

$$\leq q^{(n+K)\log_q 2} \cdot q^{(2K-\epsilon n)(K-\epsilon n-\ell)-(K-\epsilon n-\ell)^2+K(n-K+\epsilon n)}$$

$$< q^n (2-R) \log_2 2 \epsilon + K n.$$ 

Substituting the value of $\ell$ to the above equation, we have

$$\lim_{n \to \infty} \frac{A}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-k-1})} \leq \lim_{n \to \infty} \frac{A}{q^{n(n-k)n}} \times \lim_{n \to \infty} \frac{q^{(n-k)n}}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-k-1})} \leq \lim_{n \to \infty} q^{-n} = 0.$$ 

This implies that with probability at most $q^{-n}$, a random matrix $H_{(n-k)\times n}$ has full rank and a submatrix of size $(n-k)\times s$ of rank at most $s-\ell$. The claimed result follows from setting of our parameters.
IV. Algebraic Geometry Codes are Good Erasure List-Decodable Codes

In the previous section, we proved that random codes are good erasure list-decodable codes. There is still a lack of constructive results on erasure list decoding. Though Guruswami [6] presented a constructive result from concatenated codes, the rate is extremely small. In this section, we show that algebraic geometry (AG for short) codes are good erasure list-decodable codes and furthermore they can be list decoded in polynomial-time. As a preparation, we recall some basic results on AG codes first. Readers may refer to [15] for more details.

Let \( \mathcal{X} \) be a smooth, projective, absolutely irreducible curve of genus \( g(\mathcal{X}) \) (we will use \( g \) instead of \( g(\mathcal{X}) \) if there is no confusion in the context) defined over \( \mathbb{F}_q \). We denote by \( \mathbb{F}_q(\mathcal{X}) \) the function field of \( \mathcal{X} \). Denote by \( N(\mathcal{X}) \) the number of rational points of \( \mathcal{X} \). Let \( \mathcal{P} = \{P_1, \ldots, P_n\} \) be a set of \( n \) distinct rational points over \( \mathbb{F}_q \). Let \( G \) be a divisor such that \( \text{Supp}(G) \cap \{P_1, \ldots, P_n\} = \emptyset \). Define \( \mathcal{L}(G) \) as the Riemann-Roch space associated to \( G \) and denote \( \dim \mathcal{L}(G) = \ell(G) \). The algebraic geometry code \( C(G, \mathcal{P}) \) is defined as the image of \( \mathcal{L}(G) \) in \( \mathbb{F}_q^n \) under the following evaluation map

\[
C : \mathcal{L}(G) \longrightarrow \mathbb{F}_q^n, \quad f \mapsto (f(P_1), \ldots, f(P_n)).
\]

If \( n > \deg G \), then \( C(G, \mathcal{P}) \) is an \( [n, \geq \deg G - g + 1, \geq n - \deg G|_q] \)-AG code. Throughout this section, we always assume that \( n \) is bigger than \( \deg G \).

The gonality of a curve \( \mathcal{X} \) was introduced in [13]. It is defined to be the smallest degree of a nonconstant map from \( \mathcal{X} \) to the projective line. We denote the gonality of \( \mathcal{X} \) by \( t(\mathcal{X}) \). More specifically, if \( \mathcal{X} \) is defined over a field \( \mathbb{F}_q \) and \( \mathbb{F}_q(\mathcal{X}) \) is the function field of \( \mathcal{X} \), then \( t(\mathcal{X}) \) is the minimum degree of the field extensions of \( \mathbb{F}_q(\mathcal{X}) \) over a rational function field. It is easy to see that if \( g(\mathcal{X}) = 0 \), then \( t(\mathcal{X}) = 1 \). If \( g(\mathcal{X}) = 1 \) or 2, then \( t(\mathcal{X}) = 2 \). However, for general \( g \), the gonality is no longer determined by genus. In general, we have the following lower bound for \( t(\mathcal{X}) \).

**Lemma 4.1:** ([13]) Let \( \mathcal{X} \) be a curve defined over \( \mathbb{F}_q \) of genus \( g \) with \( N \) rational points. Then \( t(\mathcal{X}) \geq N/(q+1) \).

By using the lower bound on \( t(\mathcal{X}) \), one has the following proposition.

**Proposition 4.2:** \( C(G, \mathcal{P}) \) is \( \left( \frac{1}{n} (n - \deg(G) + \left\lceil \frac{n}{q+1} \right\rceil - 1), q \right) \)-erasure list-decodable.

**Proof:** Let \( \mathcal{P} \) be a set of \( n \) distinct rational points on \( \mathcal{X} \) that satisfies \( |\mathcal{P}| = \left\lceil \frac{n}{q+1} \right\rceil \geq n - \deg G|_q \). For any subset \( T \subseteq \{1, 2, \ldots, n\} \) of size \( n - s \), we claim that

\[
|\{c \in C(G, \mathcal{P})|c|_T = 0\}| \leq q.
\]

This is equivalent to proving that

\[
\dim \mathcal{L}\left(G - \sum_{i \in T} P_i\right) \leq 1.
\]

Suppose \( \dim \mathcal{L}\left(G - \sum_{i \in T} P_i\right) \geq 2 \), the one can choose a nonzero function \( f \in \mathcal{L}\left(G - \sum_{i \in T} P_i\right) \), then

\[
(f) + G - \sum_{i \in T} P_i \geq 0.
\]

Let \( H = (f) + G - \sum_{i \in T} P_i \geq 0 \). Then it is clear that

\[
\deg H = \deg \left(G - \sum_{i \in T} P_i\right) = \deg(G) + s - n \leq \left\lceil \frac{n}{q+1} \right\rceil - 1 \quad \text{and} \quad \dim \mathcal{L}(H) = \dim \mathcal{L}\left(G - \sum_{i \in T} P_i\right) \geq 2.
\]

Choose a function \( z \in \mathcal{L}(H) \setminus \mathbb{F}_q \), then \( [\mathbb{F}_q(\mathcal{X}) : \mathbb{F}_q(z)] \) is at most \( \deg(H) \leq \left\lceil \frac{n}{q+1} \right\rceil - 1 < \frac{N}{q+1} \). This contradicts Lemma 4.1.

Our desired result follows from Definition 1.1.

**Proposition 4.2** can be extended by the Grismer bound through the following lemma.

**Lemma 4.3:** If a divisor \( G \) satisfies \( \ell(G) \geq t \geq 1 \) and \( \deg G < N \), then \( \deg G \geq N \cdot \frac{q^{t-1} - 1}{q^t - 1} \), where \( N \) stands for the number of rational points on \( \mathcal{X} \).

**Proof:** Suppose \( P_1, \ldots, P_N \) are \( N \) distinct rational points on \( \mathcal{X} \). By the strong approximation theorem, there exists \( x \in \mathbb{F}_q(\mathcal{X}) \) such that \( \text{Supp}((x) + G) \cap \{P_1, \ldots, P_N\} = \emptyset \). Then \( \ell((x) + G) = \ell(G) \) and \( \deg((x) + G) = \deg(G) \). Thus, we can obtain an algebraic geometry code \( C((x) + G, \{P_1, \ldots, P_N\}) \) with parameters \( [N, \ell(G), d \geq N - \deg G|_q] \). By the Grismer bound [12], we have

\[
N \geq \ell(G) - 1 \sum_{i=0}^{\ell(G) - 1} \left\lfloor \frac{d}{q^i} \right\rfloor \geq \ell(G) - 1 \sum_{i=0}^{\ell(G) - 1} \left\lfloor \frac{d}{q^i} \right\rfloor \geq (N - \deg G) \sum_{i=0}^{\ell(G) - 1} \frac{1}{q^i}.
\]

Thus, the desired result follows from the above inequality.

**Theorem 4.4:** If \( G \) satisfies \( \ell(G) \geq t \geq 1 \) and \( \deg G < n \), then \( C(G, \mathcal{P}) \) is \( \left( \frac{1}{n} (n - \deg(G) + \left\lceil \frac{n - \deg(G)}{q^t - 1} \right\rceil - 1), q^{t-1} \right) \)-erasure list-decodable.
Proof: Let \( s \) be an integer satisfying \( s \leq n - \deg G + \left\lceil \frac{q^{t-1} - 1}{q-1} n \right\rceil - 1 \). For any \( T \subseteq \{1, 2, \ldots, n\} \) of size \( n - s \), we have
\[
\deg \left( G - \sum_{i \in T} P_i \right) = \deg G - |T| = \deg G - n + s \leq \left\lceil \frac{q^{t-1} - 1}{q-1} n \right\rceil - 1 < \frac{N \cdot q^{t-1} - 1}{q^t - 1}.
\]
By Lemma 4.3, we have
\[
\ell \left( G - \sum_{i \in T} P_i \right) \leq t - 1.
\]
Our desired result follows from Definition 1.1. \( \square \)

Remark 4.5: When \( t = 1 \), Theorem 4.4 shows that \( C(G, P) \) is \( \left( \frac{1}{n} (n - \deg(G) - 1), 1 \right) \)-erasurde list-decodable. For \( t = 2 \), we obtain the result of Proposition 4.2.

Combing Lemma 2.4 and Theorem 4.4, we immediately obtain the following lower bound on generalized Hamming weight of algebraic geometry codes.

Corollary 4.6: For \( 1 \leq t \leq \deg(G) - g + 1 \), the \( t \)-th generalized Hamming weight of \( C(G, P) \) satisfies
\[
d_t(C(G, P)) \geq n - \deg(G) + \left\lceil \frac{q^{t-1} - 1}{q^t - 1} n \right\rceil.
\]
Now we come to the main result of this section.

Theorem 4.7: Let \( q \) be a square. For any small \( \epsilon > 0 \) and \( \tau \) with \( 0 < \tau < 1 - \frac{1}{\sqrt{q-1}} + \frac{1}{q} - \epsilon \), there exists a family \( \{ C(G, P) \} \) of algebraic geometry code with length tending to \( \infty \) such that \( C(G, P) \) have rate at least \( 1 - \tau - \frac{1}{\sqrt{q-1}} + \frac{1}{q} - \epsilon \) and are \((\tau, q^{t-1})\)-eraser list-decodable. Furthermore, it can be list decoded in \( O((n \log n)^{3}) \) time, where \( n \) is the length of the code.

Proof: Choose a curve \( X/\mathbb{F}_q \) in the Garcia-Stichtenoth tower [5]. Then \( N(X)/g(X) \rightarrow \mathbb{Q} \). Let \( \mathcal{P} = \{ P_1, P_2, \ldots, P_n \} \) with \( n = N(X) - 1 \). Choose the last rational point \( P \) of \( X \) such that \( P \notin \mathcal{P} \). Put
\[
m := n - \lceil \tau n \rceil + \left\lceil \frac{q^{t-1} - 1}{q^t - 1} n \right\rceil - 1
\]
and \( G = mP \). By Theorem 4.4, \( C(G, \mathcal{P}) \) is \( \left( \frac{1}{n} (n - m + \left\lceil \frac{q^{t-1} - 1}{q^t - 1} n \right\rceil - 1), q^{t-1} \right) \)-erasurde list-decodable for any constant \( t \geq 1 \). Hence, \( C(G, \mathcal{P}) \) is \((\tau, q^{t-1})\)-eraser list-decodable. Pick \( \epsilon = \frac{1}{q} - \frac{q^{t-1} - 1}{q^t - 1} = \frac{q-1}{q(q^t - 1)} \). Moreover, the rate of \( C(G, \mathcal{P}) \) is at least
\[
\frac{1}{n} (m - g + 1) \rightarrow 1 - \tau - \frac{1}{\sqrt{q-1}} + \frac{q^{t-1} - 1}{q^t - 1} = 1 - \tau - \frac{1}{\sqrt{q-1}} + \frac{1}{q} - \epsilon.
\]
This proves the first statement of the theorem.

Finally by [14], we know that a basis of \( \mathcal{L}(G) \) can be found in \( O((n \log n)^3) \) time, where \( n \) is the length of the code. Assume that we have already found a basis \( f_1, \ldots, f_k \) of \( \mathcal{L}(G) \). Suppose that \( \mathbf{c} = (c_1, \ldots, c_n) \) was transmitted and \( \epsilon_T \) was received with \( T \subseteq \{1, 2, \ldots, n\} \) and \( |T| \geq (1 - \tau)n \). A function \( f \in \mathcal{L}(G) \) is in the list if and only if \( f(P_i) = c_i \) for all \( i \in T \). Let \( f = \sum_{i=1}^{k} \lambda_i f_i \) with \( \lambda_i \) being unknowns. Then one has \( \sum_{i=1}^{k} \lambda_i f_i(P_i) = c_i \) for all \( i \in T \). This is a system of linear equations with \( |T| \) equations and \( k \) unknowns. It can be solved in \( O(n^3) \) time. This completes the proof.

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