DYNAMICS OF THE FOOD-CHAIN POPULATION IN A POLLUTED ENVIRONMENT WITH IMPULSIVE INPUT OF TOXICANT

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Abstract. Some industrial behaviors, such as wasting outputs and inadequately treated and stored hazardous materials, may pollute our environment, so some populations in the polluted habitats are at the edge of extinction. In this work, we develop a mathematical model that validates the dynamics of the food-chain population in a polluted environment with impulsive toxicant input. Based on the model, we obtain a sufficient condition for the extinction of populations. When the concentration of toxicants surpasses the threshold, it will contribute to the extinction of populations in the related environment. Also, sufficient conditions for the permanence of populations are obtained in our analysis. Several numerical simulations validate the theoretical conclusions and further reflect the influence of toxicants.

1. Introduction. Chemostat, one useful cultivation device of microorganism, has been investigated from 1950 by Monod [23], Novick and Szilard [26]. The input of fresh nutrient in the chemostat can be regarded as the continuous turnover of the resources in the ecological system and the output from the chemostat can be viewed as the loss of the populations, such as mortality, predation and harvest, in nature. Hence, chemostat can be utilized to simulate dynamics of populations in lake [14] or some other ecological environments [34]. Furthermore, chemostat has continuously been studied by many researchers either from experimental or theoretical aspects [7, 8, 15, 21, 22, 24, 27, 34, 37, 38].

In nature, the changes of some populations may be correlated to each other, such as in host-parasite and prey-predator systems. The most typical system is the food chain model [3, 4, 5, 10, 11, 12, 13, 17, 18, 28, 29, 36, 39] in the ecological system that the prey supports the predator and then the super-predator exploits the predator. Bi-trophic food chain, such as prey-predator model, is the simplest case [4, 18, 39] compared with the tri-trophic food chain system [3, 10, 13]. Tri-trophic food web system reflects the basic relationship and structure of populations in the ecosystem, such as aquatic environment with phytoplankton, zooplankton, and fish. Through studying the chemostat with food web, many researchers investigated the
dynamics of all populations in the food chain \[3, 4, 6, 10, 18, 39\]. Zhang et al. \[39\] considered a bi-trophic food chain system in the chemostat with periodically input and output and determined that the period of pulse contributes to bifurcations when its value crosses the critical value. Boer et al. \[3\] and Gragnani et al. \[10\] investigated a tri-trophic food chain system in the chemostat and concluded that the change of the concentration of nutrient input induces bifurcations in the system.

Contaminants from the agricultural or industrial behaviors may alter the ecological structure and also release toxicants into the population \[2, 9, 25, 30, 31\]. Bacelar et al. \[2\] pointed out that the contaminant may cause massive killing in some ecosystems if the pollutant in the system is released accidentally. The indirect influence of the contaminant transmission through the food web may alter the population’s abundance or the community position in the ecosystem \[9\]. With the effect of the contaminants, the structure of population and further ecosystem may change into an unstable state \[32, 33, 35\], where the species diversity will decline. Many researchers investigated the influence of toxicants on the population with the help of chemostat, which can be viewed as the abstract ecosystem and food chain schematic \[1, 16, 19, 20, 21, 40\]. Zhao et al. \[40\] studied one chemostat system in the polluted environment and determined that the toxicant in the system may contribute to the eradication and permanence of microorganisms.

Food web system \[3, 10, 11, 13, 17, 18, 28, 36, 39, 40\] and contaminant effect \[1, 2, 9, 16, 19, 25, 30, 31, 32, 33, 35, 40\] in the ecosystem are separately investigated by authors. However, there exist a few works \[16, 19, 20, 21, 40\] analyzing both phenomena together. With the biological schematic of the chemostat, we analyze the influence of contaminant on the populations with food chain system. This paper is organized as follows. In Section 2, we will explain the mathematical system for studying the dynamics of the food-chain population in a polluted environment with impulsive toxicant input. Some preliminary results for the system are listed in Section 3. In Sections 4, we will determine the sufficient condition for the global stability of the equilibrium of extinction. The sufficient conditions for the permanence of the populations in the system are given in Section 5. Several numerical examples are provided in Section 6 to validate the theoretical conclusions obtained in Sections 4 and 5. A brief discussion is in Section 7.

2. **Mathematical model.** Bacelar et al. \[2\] incorporated the influence of contaminant into their system through adding extra value into the mortality rate of the population and analyzed the effect of toxicity in the system. Compared with this approach, some researchers investigated the influence of contaminant on the dynamics of population in the ecosystem with impulsive input of the toxicant \[16, 19, 20, 21, 40\]. Zhao et al. \[40\] utilized the following impulsive equation to describe the dynamics of the toxicant in the system

\[
\begin{align*}
\dot{c}(t) &= -Qc(t), \quad \bar{t} \neq nT, \ n \in Z^+ , \\
\Delta c(t) &= Qf, \quad \bar{t} = nT, \ n \in Z^+ ,
\end{align*}
\]

where \(c(t)\) is the concentration of toxicant in the system, \(f\) is the input rate of toxicant and \(Q\) is the dilution rate for the system. Here, \(\Delta y(t) = y(t^+) - y(t)\). For this model, they only considered the toxicant in the system. Moreover, Meng et al. \[21\] investigated the influence of toxicant in the system through modeling the dynamics of the toxicant with the following impulsive equation:
The biological meaning of other parameters can be viewed in [3, 10].

Canale’s chemostat model [3, 10]:

\[
\begin{aligned}
\dot{x}_0(t) &= D(S^0 - x_0) - \frac{a_1 x_0 x_1}{b_1 + x_0}, \\
\dot{x}_1(t) &= \epsilon_1 \frac{a_2 x_0 x_1}{b_2 + x_1} - \frac{a_3 x_2 x_1}{b_3 + x_1} - (d_1 + \epsilon_1 D)x_1, \\
\dot{x}_2(t) &= \frac{a_2 x_1 x_2}{b_2 + x_1} - \frac{a_3 x_3 x_2}{b_3 + x_2} - (d_2 + \epsilon_2 D)x_2, \\
\dot{x}_3(t) &= \frac{a_3 x_2 x_3}{b_3 + x_2} - (d_3 + \epsilon_3 D)x_3,
\end{aligned}
\]

where \(x_0(t)\) is the concentration of nutrient in the system and \(x_i(t)\) \((i = 1, 2, 3)\) are the concentration of prey, predator and super-predator in the system respectively. The biological meaning of other parameters can be viewed in [3, 10].

**Figure 1.** Schematic diagram of system. Fresh nutrient \(S(t)\) is supplied in the chemostat, which represents the supplement of natural resources in the ecosystem. Prey population \(x_1(t)\) consumes the nutrient \(S(t)\) in the system and further provides food to the predator population \(x_2(t)\). Finally, the super predator population \(x_3(t)\) exploits the predator population \(x_2(t)\). The toxicant effect from the ecosystem with impulsive input may disturb the population dynamics and reduce the populations. Besides the dilution process of the chemostat for the nutrient \(S(t)\) and populations \(x_i(t)\) \((i = 1, 2, 3)\), the mortality process of the populations \(x_i(t)\) will also alter the dynamics of populations.

In this work, we utilize the approach in system (2) to deal with the interactions between populations in the food chain. To model the effect of toxicant on the dynamics of populations, we further introduce two variables, \(T_0(t)\) and \(T_e(t)\) that represent the concentrations of toxicant in the population and the environment, respectively. On the basis of the discussion above and the diagram in Figure 1, we...
build a mathematical model to study the food chain system with toxicant effect in the chemostat:

\[
\begin{align*}
\frac{dS(t)}{dt} &= d(S^0 - S(t)) - \frac{m_1 S(t)x_1(t)}{n_1 + S(t)}, \\
\frac{dx_1(t)}{dt} &= \frac{m_1 S(t)x_1(t)}{n_1 + S(t)} - d_1 x_1(t) - r_1 T_0(t) x_1(t) - \frac{m_2 x_1(t)x_2(t)}{n_2 + x_1(t)}, \\
\frac{dx_2(t)}{dt} &= \frac{m_2 x_1(t)x_2(t)}{n_2 + x_1(t)} - d_2 x_2(t) - r_2 T_0(t) x_2(t) - \frac{m_3 x_2(t)x_3(t)}{n_3 + x_2(t)}, \\
\frac{dx_3(t)}{dt} &= \frac{m_3 x_2(t)x_3(t)}{n_3 + x_2(t)} - d_3 x_3(t) - r_3 T_0(t) x_3(t), \\
\frac{dT_0(t)}{dt} &= kT_e(t) - gT_0(t), \\
\frac{dT_e(t)}{dt} &= -vT_e(t), \\
\Delta S(t) &= 0, \Delta x_i(t) = 0 (i = 1, 2, 3), \Delta T_0(t) = 0, \Delta T_e(t) = \Lambda, t = nT,
\end{align*}
\]  

(3) for \( n \in \mathbb{Z}^+ \). Similar to previous models, we define \( \Delta y(t) = y(t^+) - y(t) \) in system (3). The three terms \(-r_i T_0(t) x_i(t) (i = 1, 2, 3)\) in system (3) represent the decrease of the populations \( x_i(t) \) induced by the toxicants in the population and the last two equations along with the related impulsive conditions reveal the alteration of toxicants in the population and the environment. The biological meanings of the variables and the parameters used in system (3) are listed in Table 1. For the initial condition of the nutrient and the populations, we assume that \( S(0), x_1(0), x_2(0) \) and \( x_3(0) \) are less than \( S^0 \). Since \( d_i \) represents the sum of dilution rate of the system

| Term          | Biological meaning                                                                 |
|---------------|-------------------------------------------------------------------------------------|
| \( S(t) \)   | Concentration of nutrient at time \( t \)                                           |
| \( x_i(t) \)  | Concentration of prey, predator and super predator at time \( t \) \( (i = 1, 2, 3) \) |
| \( T_0(t) \)  | Concentration of toxicant in the population at time \( t \)                          |
| \( T_e(t) \)  | Concentration of toxicant in the environment at time \( t \)                        |
| \( S^0 \)     | Concentration of input nutrient                                                     |
| \( d \)       | Dilution rate of the system                                                         |
| \( d_i \)     | Sum of dilution rate of system and                                                   |
|               | mortality rate of populations \( (i = 1, 2, 3) \)                                   |
| \( m_i \)     | Maximum growth rate \( (i = 1, 2, 3) \)                                             |
| \( n_i \)     | Half saturation constant \( (i = 1, 2, 3) \)                                        |
| \( r_i \)     | Rate of decrease of the growth rate for population \( x_i \) \( (i = 1, 2, 3) \)    |
| \( k \)       | Environmental toxicant uptake rate per unit mass population                          |
| \( g \)       | Population net ingestion and depuration rates of toxicant                            |
| \( v \)       | Loss rate of toxicants from the environment through volatilization                  |
| \( \Lambda \) | Impulsive input constant for toxicant at time \( nT \), \( n \in \mathbb{Z}^+ \)       |

and the mortality rate of populations, we assume that \( d_i \geq d \) for all \( i = 1, 2, 3 \) in this paper. In the following sections, we will investigate the dynamics of all populations in system (3), that is, determining the conditions of extinction and permanence of
populations with the influence of toxicants from the environment. Some notations and preliminary results are given in the following section.

3. Some notations and preliminary results. In this section, we will provide some notations and some basic lemmas for the proofs of the results in the later sections. Throughout this work, we assume that $S(t), x_i(t) (i = 1, 2, 3)$ and $T_0(t)$ are continuous at $t = nT$ for $n \in \mathbb{Z}^+$. $T_0(t)$ is left continuous at $t = nT$ and $T_0(nT^+) = \lim_{t\to nT^+} T_0(t)$. Moreover, let $(R^+)^6 = \{x = (x_1, x_2, \ldots, x_6) \in R^6$ such that $x_i > 0$ $(i = 1, 2, \ldots, 6)\}$, and define $(g(t)) = \frac{1}{T} \int_0^T g(u) du$.

Definition 3.1. The population $x_i(t)$ $(i = 1, 2, 3)$ in the device is considered to be extinct if $\lim_{t \to \infty} x_i(t) = 0$. On the contrary, the population $x_i(t)$ $(i = 1, 2, 3)$ is considered to be persistent if there is a positive constant $\lambda$ such that $\liminf_{t \to \infty} (x_i(t)) \geq \lambda > 0$.

We first discuss about the following subsystem of system (3)

$$\begin{align*}
\frac{dT_0(t)}{dt} &= kT_0(t) - gT_0(t), \\
\frac{dT_i(t)}{dt} &= -\nu T_i(t), \\
\Delta T_0(t) &= 0, \Delta T_i(t) = \Lambda, t = nT, n \in \mathbb{Z}^+.
\end{align*}$$

(4)

For subsystem (4), we have the following result.

Lemma 3.2. There exists a unique $T$-periodic solution $(T_0^*(t), T_i^*(t))$ such that for each solution $(T_0(t), T_i(t))$ of the system (4), we have that $T_0(t) \to T_0^*(t)$ and $T_i(t) \to T_i^*(t)$ when $t \to \infty$. Meanwhile, $T_0(t) > T_0^*(t)$ and $T_i(t) > T_i^*(t)$ for all $t > 0$ if $T_0(0) > T_0^*(0)$ and $T_i(0) > T_i^*(0)$. Here,

$$\begin{align*}
T_0^*(t) &= T_0^*(0)e^{-g(t-nT)} + \frac{k\Lambda e^{-\nu(t-nT)}}{(g-\nu)(1-e^{-\nu T})}, \\
T_i^*(t) &= \frac{\Lambda e^{-\nu T}}{1-e^{-\nu T}}, \\
T_0(0) &= \frac{k\Lambda e^{-\nu T}}{(g-\nu)(1-e^{-\nu T})}, \\
T_i(0) &= \frac{\Lambda}{1-e^{-\nu T}},
\end{align*}$$

(5)

where $t \in (nT, (n+1)T]$ and $n \in \mathbb{Z}^+$. The proof of this lemma is similar to the proofs of Lemma 3.1 in [20] and Lemma 2.1 in [21], and hence we omit it here.

Next, we will show that the solution of system (3) is bounded above when $t$ is large enough.

Lemma 3.3. Let $\beta_0 = \frac{k\Lambda}{\nu g T}$. Given any initial value $(S(0), x_i(0), T_0(0), T_0^+(0)) \in (R^+)^6$ $(i = 1, 2, 3)$, the positive solution of system (3) satisfies

$$\limsup_{t \to \infty} S(t) \leq S^0, \quad \limsup_{t \to \infty} x_i(t) \leq S^0 \quad (i = 1, 2, 3), \quad \lim_{t \to \infty} \langle T_0(t) \rangle = \beta_0.$$ 

Proof. On the basis of the first four equations in system (3), we have

$$\begin{align*}
\frac{d(S(t) + x_1(t) + x_2(t) + x_3(t))}{dt} &= d(S^0 - S) - (d_1 + r_1 T_0) x_1 - (d_2 + r_2 T_0) x_2 \\
&\quad - (d_3 + r_3 T_0) x_3 \\
&\leq dS^0 - d(S + x_1 + x_2 + x_3).
\end{align*}$$
Consider the following differential equation
\[
\begin{cases}
\frac{dz(t)}{dt} = dS^0 - dz(t), \\
z(0) = S(0) + x_1(0) + x_2(0) + x_3(0).
\end{cases}
\]

It follows from basic calculations and the comparison theorem of differential equation that
\[
\limsup_{t \to \infty} S(t) + x_1(t) + x_2(t) + x_3(t) \leq \limsup_{t \to \infty} z(t) = S^0,
\]
which implies
\[
\limsup_{t \to \infty} S(t) \leq S^0, \quad \limsup_{t \to \infty} x_i(t) \leq S^0 \ (i = 1, 2, 3).
\]

Now, we will prove that
\[
\lim_{t \to \infty} (T_0(t)) = \frac{kA}{vgT} = \beta_0.
\]

By Lemma 3.2, we have \( T_0(t) \to T^*_0(t) \), and
\[
\lim_{t \to \infty} \frac{1}{T} \int_0^T T_0(s)ds = \frac{1}{T} \int_0^T T^*_0(s)ds.
\]

The expression of \( T^*_0(t) \) in Lemma 3.2 checks that
\[
\int_0^T T^*_0(s)ds = \frac{kA}{v(g-v)} - \frac{kA}{g(g-v)} = \frac{kA}{vg} = \beta_0T.
\]

Hence, we have
\[
\lim_{t \to \infty} (T_0(t)) = \beta_0.
\]

The proof of Lemma 3.3 is completed. \( \square \)

4. Stability of extinction equilibrium. System (3) has an extinction equilibrium \( E_0 = (S^0, 0, 0, 0, T^*_0(t), T^*_e(t)) \), where \( T^*_0(t) \) and \( T^*_e(t) \) are shown in Lemma 3.2. In this section, we will discuss about the stability of this equilibrium and have the following result.

**Theorem 4.1.** If \( \frac{m_1S^0}{(m+S^0)(d_1+r_1S^0)} < 1 \), all populations in the system will be extinct and there exists a unique globally asymptotically stable periodic solution \( E_0 = (S^0, 0, 0, 0, T^*_0(t), T^*_e(t)) \) for system (3).

**Proof.** In accordance with Lemma 3.2, we can validate that system (3) has a unique boundary periodic solution \( E_0 \), where all population are extinct. To investigate the stability of \( E_0 \), we first let
\[
(z_1(t), z_2(t), z_3(t), z_4(t), z_5(t), z_6(t)) = (S(t), x_1(t), x_2(t), x_3(t), T_0(t), T_e(t)) - E_0.
\]

By applying linearization to system (3), we can obtain the following linearized system
\[
\begin{align*}
\frac{dz_1(t)}{dt} &= -dz_1 - \frac{m_1S^0}{n_1 + S^0} z_2, \\
\frac{dz_2(t)}{dt} &= \frac{m_1S^0}{n_1 + S^0} z_2 - d_1 z_2 - r_1 T^*_0(t) z_2, \\
\frac{dz_3(t)}{dt} &= -d_2 z_3 - r_2 z_0^*(t) z_3, \\
\frac{dz_4(t)}{dt} &= d_3 z_4 - r_3 T^*_0(t) z_4,
\end{align*}
\]
\[
\frac{dz_5(t)}{dt} = kz_5 - gz_5, \\
\frac{dz_6(t)}{dt} = -vz_6.
\]

The stability of the equilibrium \( E_0 \) is determined by the eigenvalues of the following matrix
\[
A = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 \exp(kT)
\end{pmatrix},
\]
where
\[
\lambda_1 = e^{-dT} < 1,
\]
\[
\lambda_2 = \exp \left\{ \int_0^T \frac{m_1 S^0}{n_1 + S^0} - d_1 - r_1 T_0^* (s) \, ds \right\},
\]
\[
\lambda_3 = \exp \left\{ \int_0^T -d_2 - r_2 T_0^* (s) \, ds \right\} < 1,
\]
\[
\lambda_4 = \exp \left\{ \int_0^T -d_3 - r_3 T_0^* (s) \, ds \right\} < 1,
\]
\[
\lambda_6 = e^{-gT} < 1, \quad \lambda_6 = e^{-vT} < 1.
\]

Based on the Floquet theory, the periodic solution \((S^0, 0, 0, 0, T_0^* (t), T_0^* (t))\) is locally stable if \( \lambda_2 < 1 \). If \( \frac{m_1 S^0}{(n_1 + S^0)(d_1 + r_1 T_0^*)} < 1 \), we get that \( \frac{m_1 S^0}{n_1 + S^0} - d_1 - r_1 T_0^* < 0 \) then \( \lambda_2 < 1 \), which implies that \((S^0, 0, 0, 0, T_0^* (t), T_0^* (t))\) is locally stable. Next, we prove that \( E_0 \) is globally stable. Under the condition \( \frac{m_1 S^0}{(n_1 + S^0)(d_1 + r_1 T_0^*)} < 1 \), we have
\[
\int_0^T \frac{m_1 S^0}{n_1 + S^0} - d_1 - r_1 T_0^* (s) \, ds < 0.
\]
We can select a sufficiently small constant \( \epsilon > 0 \) such that
\[
\int_0^T \frac{m_1 S^0}{n_1 + S^0} - d_1 - r_1 T_0^* (s) + r_1 \epsilon \, ds < 0. \tag{7}
\]
According to Lemma 3.2, we know that \( \lim_{t \to \infty} T_0(t) = T_0^* (t) \). Therefore, there is \( T_1 > 0 \) such that \( T_0^* (t) - \epsilon < T_0(t) < T_0^* (t) + \epsilon \) for all \( t > T_1 \). We have the following inequality for the second equation of system (3) when \( t > T_1 \)
\[
\frac{dx_1(t)}{dt} \leq \left[ \frac{m_1 S^0}{n_1 + S^0} - d_1 - r_1 (T_0^* (t) - \epsilon) \right] x_1.
\]
Consider the following system
\[
\begin{align*}
\frac{dx(t)}{dt} &= \left[ \frac{m_1 S^0}{n_1 + S^0} - d_1 - r_1 (T_0^* (t) - \epsilon) \right] y(t), \quad t \neq nT, \\
y(t^+) &= y(t), \quad t = nT, \\
y(0^+) &= x_1(0). \tag{8}
\end{align*}
\]
After taking integration from \( nT \) to \( (n+1)T \) on both sides, we obtain
\[
y((n+1)T) = y(nT) \exp \left\{ \int_0^T \left[ \frac{m_1 S^0}{n_1 + S^0} - d_1 - r_1 (T_0^* (s) - \epsilon) \right] \, ds \right\},
\]
which yields
\[ y(nT) = x_1(0) \exp \left\{ n \int_0^T \left[ \frac{m_1 S_0}{n_1 + S_0} - d_1 - r_1(T_0^*(s) - \epsilon) \right] ds \right\}. \]

Inequality (7) implies that \( \lim_{n \to \infty} y(nT) = 0 \). In addition, we have
\[ y(t) = y(nT) \exp \left\{ \int_{nT}^t \left[ \frac{m_1 S_0}{n_1 + S_0} - d_1 - r_1(T_0^*(s) - \epsilon) \right] ds \right\}, \quad t \in (nT, (n + 1)T]. \]

Since \( \exp \left\{ \int_{nT}^t \left[ \frac{m_1 S_0}{n_1 + S_0} - d_1 - r_1(T_0^*(s) - \epsilon) \right] ds \right\} \) is bounded, we have \( \lim_{t \to \infty} y(t) = 0 \).

According to the comparison theorem of differential equation, we prove that
\[ \lim_{t \to \infty} \sup x_1(t) \leq \lim_{t \to \infty} \sup y(t) = 0. \]

Since \( x_1(t) \geq 0 \), we can validate that
\[ \lim_{t \to \infty} x_1(t) = 0. \]
Consequently, if \( \frac{m_1 S_0}{(n_1 + S_0)(d_1 + r_1 \beta_0)} < 1 \), \( E_0 \) is globally asymptotically stable. \( \square \)

5. Persistence of population. In this section, we will discuss about the conditions for different equilibria with non-zero populations. System (3) has an equilibrium \( E_1 = (S^*(t), x_1^*(t), 0, 0, T_0^*(t), T_1^*(t)) \), which means that only population \( x_1(t) \) will be persistent in the system. In Lemma 3.2, we have the expression of \( (T_0^*(t), T_1^*(t)) \).

We have the following result for system (3) that guarantees the permanence of \( x_1(t) \) in the system.

**Theorem 5.1.** If
\[ \frac{m_1 S_0}{n_1 + S_0} - d_1 - \frac{m_2 S_0}{n_2} - r_1 \beta_0 > 0, \]
\[ \frac{m_2 S_0}{n_2 + S_0} - d_2 - r_2 \beta_0 < 0, \]
then \( \lim \inf_{t \to \infty} x_1(t) > 0 \) and \( \lim \sup_{t \to \infty} \frac{\ln x_1(t)}{t} < 0 \), which means that only \( x_1(t) \) is persistent in system (3).

**Proof.** First, we take integration from 0 to \( t \) for the first two equations in system (3) and divide \( t \) on both sides to obtain
\[ \delta_1(t) = \frac{S(t) - S(0)}{t} + \frac{x_1(t) - x_1(0)}{t} \geq dS_0 - d\langle S(t) \rangle - \left[ d_1 + r_1 T_0^* + \frac{m_2 S_0}{n_2} \right] \langle x_1(t) \rangle, \]
where \( T_0^* = \max_{0 \leq t \leq T} T_0^*(t) \), then we have
\[ \langle S(t) \rangle \geq S_0 - \frac{d_1 + r_1 T_0^* + \frac{m_2 S_0}{n_2}}{d} \langle x_1(t) \rangle - \frac{\delta_1(t)}{d}. \]

Let \( V_1(t) = \ln x_1(t) + x_1(t) \). Lemma 3.3 implies that \( V_1(t) \) is bounded above when \( t \) is large enough. We have
\[ \frac{dV_1}{dt} = \frac{m_1 S}{n_1 + S} - d_1 - r_1 T_0 - \frac{m_2 x_2}{n_2 + x_1} + \frac{m_1 S x_1}{n_1 + S} - d_1 x_1 - r_1 T_0 x_1 - \frac{m_2 x_1 x_2}{n_2 + x_1} \]
\[ \geq \frac{m_1 S}{n_1 + S} S - d_1 - r_1 T_0 - \frac{m_2 S_0}{n_2} - \left( d_1 + r_1 T_0^* + \frac{m_2 S_0}{n_2} \right) x_1. \]
After taking integration from 0 to $t$ and dividing $t$ on both sides, we obtain
\[
\frac{V_1(t)}{t} - \frac{V_1(0)}{t} \geq \frac{m_1}{n_1 + S^0} \langle S(t) \rangle - d_1 - r_1(T_0(t)) - \frac{m_2 S^0}{n_2} \left( d_1 + r_1 T_0^* + \frac{m_2 S^0}{n_2} \right) \langle x_1(t) \rangle,
\]
which implies based on the inequality (9)
\[
\left( d_1 + r_1 T_0^* + \frac{m_2 S^0}{n_2} \right) \left( \frac{m_1}{d(n_1 + S^0)} + 1 \right) \langle x_1(t) \rangle \geq \frac{m_1 S^0}{n_1 + S^0} - d_1 - \frac{m_2 S^0}{n_2} - r_1(T_0(t)) + \frac{V_1(0)}{t} - \frac{m_1 \delta(t)}{d(n_1 + S^0)}.
\]

Since $x_1(t) \leq S^0$, and $V_1(t)$ is bounded above when $t$ is large enough, we have
\[
\lim_{t \to \infty} \frac{V_1(t)}{t} \leq 0, \quad \lim_{t \to \infty} \frac{V_1(0)}{t} = \lim_{t \to \infty} \delta(t) = 0.
\]
Consequently, taking inferior limit on both sides for inequality (10), we have
\[
\liminf_{t \to \infty} \langle x_1(t) \rangle \geq \frac{d(n_1 + S^0)}{(d_1 + r_1 T_0^* + \frac{m_2 S^0}{n_2})[m_1 + d(n_1 + S^0)]} \times \left[ \frac{m_1 S^0}{n_1 + S^0} - d_1 - \frac{m_2 S^0}{n_2} - r_1 \beta_0 \right].
\]
If $\frac{m_1 S^0}{n_1 + S^0} - d_1 - \frac{m_2 S^0}{n_2} - r_1 \beta_0 > 0$, we have
\[
\liminf_{t \to \infty} \langle x_1(t) \rangle > 0.
\]
For studying the population $x_2$, we consider the third equation of system (3) and have
\[
\frac{d \ln x_2(t)}{dt} = \frac{m_2 x_1}{n_2 + x_1} - d_2 - r_2 T_0 - \frac{m_3 x_3}{n_3 + x_2} \leq \frac{m_2 S^0}{n_2 + S^0} - d_2 - r_2 T_0,
\]
which implies that
\[
\frac{\ln x_2(t) - \ln x_2(0)}{t} \leq \frac{m_2 S^0}{n_2 + S^0} - d_2 - r_2(T_0(t)).
\]
If $\frac{m_2 S^0}{n_2 + S^0} - d_2 - r_2 \beta_0 < 0$, taking superior limit on the both sides for the inequality above yields
\[
\limsup_{t \to \infty} \frac{\ln x_2(t)}{t} \leq \frac{m_2 S^0}{n_2 + S^0} - d_2 - r_2 \beta_0 < 0,
\]
which means the extinction of $x_2(t)$ in the system. The proof is completed. \hfill \qed

Next, we consider the equilibrium with the permanence of both $x_1(t)$ and $x_2(t)$, $E_2 = (S^*(t), x_1(t), x_2^*(t), 0, T_0^*(t), T_2^*(t))$. For this case, we have the following theorem.

**Theorem 5.2.** If
\[
A = \min \left\{ \frac{m_1}{n_1 + S^0}, \frac{m_2}{n_2 + S^0} - d_1 \right\} > 0,
\]
where

\[ B = \frac{AdS^0}{d_1} - \left[ \frac{Ar_1S^0}{d_1} + r_1 + r_2 + r_1S^0 \right] \beta_0 - d_1 - d_2 - \frac{m_3S^0}{n_3} > 0, \]

\[ C = \frac{m_3S^0}{n_3 + S^0} - d_3 - r_3\beta_0 < 0, \]

then we can obtain that \( \lim\inf_{t \to \infty} \langle x_2(t) \rangle > 0 \) and \( \lim\sup_{t \to \infty} \frac{\ln x_1(t)}{t} < 0 \) for system (3), which implies that both \( x_1(t) \) and \( x_2(t) \) are persistent and \( x_3(t) \) will die out in system (3).

**Proof.** First, we take integration from 0 to \( t \) for the first three equations in system (3) and divide \( t \) on both sides to obtain

\[ \delta_2(t) = \frac{S(t) - S(0)}{t} + \frac{x_1(t) - x_1(0)}{t} + \frac{x_2(t) - x_2(0)}{t} \]

\[ \geq dS^0 - d\langle S(t) \rangle - d_1\langle x_1(t) \rangle - r_1S^0\langle T_0(t) \rangle - \left( d_2 + r_2T_0^* + \frac{m_3S^0}{n_3} \right) \langle x_2(t) \rangle, \]

which implies

\[ \delta_2(t) \geq dS^0 - d\langle (S(t)) + \langle x_1(t) \rangle \rangle - r_1S^0\langle T_0(t) \rangle - \left( d_2 + r_2T_0^* + \frac{m_3S^0}{n_3} \right) \langle x_2(t) \rangle. \] (11)

Let \( V_2(t) = V_1(t) + \ln x_2(t) + x_2(t) \). Lemma 3.3 implies that \( V_2(t) \) is bounded above when \( t \) is large enough. We obtain that if \( A > 0 \)

\[ \frac{dV_2(t)}{dt} \geq A(S + x_1) - d_1 - d_2 - \frac{m_3S^0}{n_3} - \left( \frac{m_2}{n_2} + m_2 + d_2 + r_2T_0^* + \frac{m_3S^0}{n_3} \right) x_2 \]

\[ - (r_1 + r_2 + r_1S^0)T_0. \]

Integration from 0 to \( t \) and dividing \( t \) on both sides yield

\[ \frac{V_2(t) - V_2(0)}{t} \geq A\langle S(t) \rangle + \langle x_1(t) \rangle - d_1 - d_2 - \frac{m_3S^0}{n_3} - (r_1 + r_2 + r_1S^0)\langle T_0(t) \rangle \]

\[ - \left( \frac{m_2}{n_2} + m_2 + d_2 + r_2T_0^* + \frac{m_3S^0}{n_3} \right) \langle x_2(t) \rangle. \]

The inequality above and inequality (11) validate that

\[ \alpha x_2(t) \geq \frac{AdS^0}{d_1} - \left[ \frac{Ar_1S^0}{d_1} + r_1 + r_2 + r_1S^0 \right] \langle T_0(t) \rangle \]

\[ - d_1 - d_2 - \frac{m_3S^0}{n_3} - \frac{A\delta_2(t)}{d_1} - \frac{V_2(t) - V_2(0)}{t}, \]

where \( \alpha = \frac{Ad_2 + r_2T_0^* + \frac{m_3S^0}{n_3}}{d_1} + \frac{m_2}{n_2} + m_2 + d_2 + r_2T_0^* + \frac{m_3S^0}{n_3} \). By Lemma 3.3 and the inequality above, we have that if \( A > 0 \) and \( B > 0 \), then

\[ \lim\inf_{t \to \infty} \langle x_2(t) \rangle \geq \frac{B}{\alpha} > 0, \]

which implies the permanence of \( x_2(t) \) in the system. Since the population \( x_2(t) \) is larger than zero for a long term behavior only if \( x_1(t) > 0 \), the permanence of \( x_2(t) \) in the system implies the permanence of \( x_1(t) \).

For studying the population \( x_3 \), we consider the fourth equation of system (3) and have

\[ \frac{d\ln x_3(t)}{dt} = \frac{m_3x_2}{n_3 + x_2} - d_3 - r_3T_0 \]
If $C < 0$, which yields
\[
\ln x_3(t) - \ln x_3(0) \leq \frac{m_3S^0}{n_3 + S^0} - d_3 - r_3T_0(t),
\]
which yields
\[
\frac{\ln x_3(t) - \ln x_3(0)}{t} \leq \frac{m_3S^0}{n_3 + S^0} - d_3 - r_3(T_0(t)).
\]
If $C < 0$, taking superior limit for the inequality above on both sides to obtain that
\[
\lim \sup_{t \to \infty} \frac{\ln x_3(t)}{t} \leq \frac{m_3S^0}{n_3 + S^0} - d_3 - r_3\beta_0 < 0,
\]
which means the extinction of $x_3(t)$ in the system. The proof of this theorem is completed. □

Next, we determine the condition for the permanence of all populations $x_i(t)$ ($i = 1, 2, 3$).

**Theorem 5.3.** If
\[
D = \min \left\{ \frac{m_1}{n_1 + S^0} - d_1, \frac{m_2}{n_2 + S^0} - d_2, \frac{m_3}{n_3 + S^0} - d_3 - \frac{m_2}{n_2} \right\} > 0,
\]
\[
E = \frac{DdS^0}{\max\{d_1, d_2\}} - \frac{D(r_1S^0 + r_2S^0)\beta_0}{\max\{d_1, d_2\}} - d_1 - d_2 - d_3 - (r_1 + r_2 + r_3 + r_1S^0 + r_2S^0)\beta_0 > 0,
\]
then $\lim \inf_{t \to \infty} x_3(t) > 0$, which implies that all populations $x_i(t)$ ($i = 1, 2, 3$) are persistent in the system.

**Proof.** First, we take integration from 0 to $t$ for the first four equations in system (3) and divide $t$ on both sides to obtain that
\[
\delta_3(t) = \frac{S(t) - S(0)}{t} + \frac{x_1(t) - x_1(0)}{t} + \frac{x_2(t) - x_2(0)}{t} + \frac{x_3(t) - x_3(0)}{t}
\geq dS^0 - \max\{d_1, d_2\}[\langle S(t) \rangle + \langle x_1(t) \rangle + \langle x_2(t) \rangle] - (r_1S^0 + r_2S^0)\langle T_0(t) \rangle
\quad - (d_3 + r_3T_0^*)\langle x_3(t) \rangle,
\]
which implies
\[
\frac{1}{\max\{d_1, d_2\}}[dS^0 - (r_1S^0 + r_2S^0)\langle T_0(t) \rangle] - (d_3 + r_3T_0^*)\langle x_3(t) \rangle - \delta_3(t).
\]
Let $V_3(t) = V_2(t) + \ln x_3(t) + x_3(t)$. Lemma 3.3 implies that $V_3(t)$ is bounded above when $t$ is large enough. We obtain that if $D > 0$
\[
\frac{dV_3(t)}{dt} \geq \frac{m_1S}{n_1 + S^0} + \left( \frac{m_2}{n_2 + S^0} - d_1 \right) x_1 + \left( \frac{m_3}{n_3 + S^0} - d_2 - \frac{m_2}{n_2} \right) x_2 - d_1 - d_2 - d_3 - (r_1 + r_2 + r_3 + r_1S^0 + r_2S^0)T_0 - \left( \frac{m_3}{n_3} + d_3 + r_3T_0^* \right) x_3
\geq D(S + x_1 + x_2) - d_1 - d_2 - d_3 - (r_1 + r_2 + r_3 + r_1S^0 + r_2S^0)T_0
\quad - \left( \frac{m_3}{n_3} + d_3 + r_3T_0^* \right) x_3.
\]
Integration on both sides from 0 to $t$ and dividing $t$ validate that
\[
\frac{V_3(t) - V_3(0)}{t} \geq D(\langle S(t) \rangle + \langle x_1(t) \rangle + \langle x_2(t) \rangle) - (r_1 + r_2 + r_3 + r_1S^0 + r_2S^0)\langle T_0(t) \rangle
\leq \frac{m_3S^0}{n_3 + S^0} - d_3 - r_3T_0(t).
\]
Together with inequality (12), we have
\[
\frac{V_3(t) - V_3(0)}{t} \geq \frac{DdS^0}{\max\{d_1, d_2\}} - \frac{D(r_1S^0 + r_2S^0)}{\max\{d_1, d_2\}} \langle T_0(t) \rangle - d_1 - d_2 - d_3
\]
\[
- \left[ \frac{D(d_3 + r_3T_0^*)}{\max\{d_1, d_2\}} + \frac{m_3}{n_3} + d_3 + r_3T_0^* \right] \langle x_3(t) \rangle
\]
\[
- (r_1 + r_2 + r_3 + r_1S^0 + r_2S^0) \langle T_0(t) \rangle - \frac{D\delta_3(t)}{\max\{d_1, d_2\}}
\]
By Lemma 3.3 and the inequality above, we obtain that if \( D > 0 \) and \( E > 0 \), we have
\[
\liminf_{t \to \infty} \langle x_3(t) \rangle \geq \frac{E}{\max\{d_1, d_2\} + \frac{m_3}{n_3} + d_3 + r_3T_0^*} > 0,
\]
which implies the permanence of \( x_3(t) \) in the system. Since the population \( x_3(t) \) is larger than zero for a long term behavior only if \( x_2(t) > 0 \) and \( x_1(t) > 0 \), the permanence of \( x_3(t) \) in the system implies the permanence of \( x_1(t) \) and \( x_2(t) \). \( \square \)

6. Numerical simulations. In the previous sections, we have studied the conditions for the extinction and the persistence of the populations. Here, several simulations will be provided to validate the results discussed in the former sections.

First, we select \( k = 1, g = 1, v = 1, \Lambda = 0.2, T = 1, r_1 = 0.01, r_2 = 0.02, r_3 = 0.05, d = 0.4, d_1 = 0.5, d_2 = 0.6, d_3 = 0.8, S^0 = 1, m_1 = 1.5, n_1 = 3, m_2 = 1.6, n_2 = 3, m_3 = 3, n_3 = 1.5 \). This setting implies that
\[
\beta_0 = \frac{k\Lambda}{vgT} = 0.2, \quad \frac{m_1S^0}{(n_1 + S^0)(d_1 + r_1\beta_0)} = 0.747 < 1.
\]
According to Theorem 4.1, the extinction equilibrium \( E_0 = (S_0^0, 0, 0, T_0^*(t), T_e^*(t)) \) is global asymptotically stable. The corresponding simulation is shown in Figure 2. Here, \( \beta_0 \) is the average concentration of input toxicant per impulsive period for population \( x_1(t) \), which, with the parameters \( r_1 \) and \( d_1 \), determines the loss of the population \( x_3(t) \) in the system. \( \frac{m_1S^0}{n_1 + S^0} < d_1 + r_1\beta_0 \) implies that the number of growth for \( x_1(t) \) is less than the number of loss. Under this situation, \( x_1(t) \) will go to zero as \( t \) is increasing, and it further implies the extinction of \( x_2(t) \) and \( x_3(t) \).

With the values of \( k, g, v, \Lambda, T, r_1, r_2, r_3, S^0 \) used before, and let \( d = 0.12, d_1 = 0.15, d_2 = 0.18, d_3 = 0.2, m_1 = 2, n_1 = 2.6, m_2 = 0.5, n_2 = 2.9, m_3 = 0.8, n_3 = 2 \), we have
\[
\beta_0 = \frac{k\Lambda}{vgT} = 0.2, \quad \frac{m_1S^0}{n_1 + S^0} - d_1 - \frac{m_2S^0}{n_2} - r_1\beta_0 = 0.2311 > 0,
\]
\[
\frac{m_2S^0}{n_2 + S^0} - d_2 - r_2\beta_0 = -0.0558 < 0.
\]
Consequently, the conditions in Theorem 5.1 are satisfied. It implies that the population \( x_1(t) \) will be persistent and other populations are extinct as shown in Figure 3. In this case, the time series of \( T_0(t) \) and \( T_e(t) \) are same with Figure 2(c) and (d).
Figure 2. Numerical simulation of the solution for the case that 
\[ \frac{m_1 S_0}{n_1 + S_0} = 0.747 < 1. \] (a) Temporal dynamic of the solution \( S(t) \). (b) Temporal dynamics of the solutions \( x_i(t) \) \((i = 1, 2, 3)\). (c) Temporal dynamic of the solution \( T_0(t) \). (d) Temporal dynamic of the solution \( T_e(t) \).

Figure 3. Numerical simulation of the solution for the case that 
\[ \frac{m_1 S_0}{n_1 + S_0} - d_1 - \frac{m_2 S_0}{n_2 + S_0} - r_1 \beta_0 = 0.2311 > 0 \text{ and } \frac{m_2 S_0}{n_2 + S_0} - d_2 - r_2 \beta_0 = -0.0558 < 0. \] (a) Temporal dynamic of the solution \( S(t) \). (b) Temporal dynamics of the solutions \( x_i(t) \) \((i = 1, 2, 3)\).
Next, we consider the permanence of $x_1(t)$ and $x_2(t)$. With the same values of $k, g, v, \Lambda, T, r_1, r_2, r_3, S_0$ used before, we set $d = 0.1, d_1 = 0.2, d_2 = 0.25, d_3 = 0.5, m_1 = 2.4, n_1 = 0.2, m_2 = 4.9, n_2 = 0.6, m_3 = 0.7, n_3 = 2.4$. We have

$$\beta_0 = \frac{k\Lambda}{vgT} = 0.2,$$

$$A = \min\left\{ \frac{m_1}{n_1 + S_0} , \frac{m_2}{n_2 + S_0} - d_1 \right\} = 2 > 0,$$

$$B = \frac{AdS_0}{d_1} - \left[ \frac{Ar_1S_0}{d_1} + r_1 + r_2 + r_1S_0 \right] \beta_0 - d_1 - d_2 - \frac{m_3S_0}{n_3} = 0.2303 > 0,$$

$$C = \frac{m_3S_0}{n_3 + S_0} - d_3 - r_3\beta_0 = -0.3041 < 0.$$

According to Theorem 5.2, we know that $x_2(t)$ is persistent as $A > 0$ and $B > 0$ are satisfied. The simulation is displayed in Figures 4 and 5.

![Figure 4](image-url)

**Figure 4.** Numerical simulation of the solution for the case that $A > 0$, $B > 0$ and $C < 0$. (a) Temporal dynamic of the solution $S(t)$. (b) Temporal dynamic of the solution $x_1(t)$. (c) Temporal dynamic of the solution $x_2(t)$. (d) Temporal dynamic of the solution $x_3(t)$.

Figure 4 displays the dynamics of $S(t)$ and $x_i(t)$ ($i = 1, 2$). The numerical simulation reflects the periodic oscillations for this case in the dynamics of the survival populations $x_1(t), x_2(t)$ and nutrient $S(t)$ and population $x_3(t)$ will die out since the extinction condition ($C < 0$) is satisfied in this case. Figure 5 displays the portrait phases that further demonstrate the periodic phenomenon for populations $x_1(t)$ and $x_2(t)$. The portrait phases validate that the population in the system will fluctuate around some nonnegative constant and when the number of one population declines, the other population will increase as shown in Figure 5(a).

Now, if we consider the same values of $k, g, v, \Lambda, T, r_1, r_2, r_3$ as before and choose $d = 0.1, d_1 = 0.12, d_2 = 0.15, d_3 = 0.15, S_0 = 1.3, m_1 = 3, n_1 = 0.8, m_2 = 2, n_2 = 1.6, m_3 = 5, n_3 = 1.1$, we obtain that

$$\beta_0 = \frac{k\Lambda}{vgT} = 0.2,$$
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Figure 5. Numerical simulation of the solution for the case that \( A > 0, B > 0 \) and \( C < 0 \). (a) Portrait phase of \( x_1(t) \) and \( x_2(t) \). (b) Portrait phase of \( S(t) \) and \( x_i(t) \) \((i = 1, 2)\).

\[
D = \min \left\{ \frac{m_1}{n_1 + S^0}, \frac{m_2}{n_2 + S^0} - d_1, \frac{m_3}{n_3 + S^0} - d_2 - \frac{m_2}{m_2} \right\} = 0.5697 > 0,
\]
\[
E = \frac{DdS^0}{\max\{d_1, d_2\}} - \frac{D(r_1S^0 + r_2S^0)\beta_0}{\max\{d_1, d_2\} - d_1 - d_2 - d_3}
- (r_1 + r_2 + r_3 + r_1S^0 + r_2S^0)\beta_0 = 0.0634 > 0.
\]

According to Theorem 5.3, the population \( x_3(t) \) is persistent in the system, which further implies the permanence of \( x_1(t) \) and \( x_2(t) \) as shown in Figure 6.

Figure 6. Numerical simulation of the solution for the case that \( D > 0 \) and \( E > 0 \). (a) Temporal dynamic of the solution \( S(t) \). (b) Temporal dynamic of the solution \( x_1(t) \). (c) Temporal dynamic of the solution \( x_2(t) \). (d) Temporal dynamic of the solution \( x_3(t) \).

The dynamics of solutions \( S(t) \) and \( x_i(t) \) \((i = 1, 2, 3)\) reflect that the concentration of nutrient \( S(t) \) and the number of population \( x_i(t) \) \((i = 1, 2, 3)\) are changing periodically in this case, which forms the limit cycle (see portrait phases shown
in Figure 7) and supports the persistence of populations $x_i(t)$ $(i = 1, 2, 3)$ if the conditions in Theorem 5.3 are satisfied.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Numerical simulation of the the solution for the case that $D > 0$ and $E > 0$. (a) Portrait phase of $S(t)$ and $x_1(t)$. (b) Portrait phase of $S(t)$ and $x_2(t)$. (c) Portrait phase of $S(t)$ and $x_3(t)$. (d) Portrait phase of $x_1(t), x_2(t)$ and $x_3(t)$.}
\end{figure}

7. Conclusion. Contaminants from the agricultural or industrial behaviors in the ecosystem, especially the aquatic environment, contribute to the alteration of population dynamics and some populations may be extinct due to contaminant in the ecosystem. In this paper, a chemostat system with food web was utilized to investigate the dynamics of populations in the polluted environment with impulsive input of the toxicant [2].

The conditions for the extinction and the permanence of the population were obtained in this work. From the expression of the conditions, we can understand how to control the period of the impulsive input of toxicant (measured by $T$) for preventing the population extinction. In the modern world, zero release of toxicants is impossible during industrial production. Our work provides us a quantitative way to decide the toxic substances control acting to reduce the effect of toxicants and balancing industrial development and environmental protection. Here, we mainly focus on the negative effect of toxicants. However, other factors, such as stochastic effect [21], may play an essential role in the permanence of the population. Based on this framework, the interactions of more factors can be easily included, and systematically studied in further investigation.

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