Abstract

Scalar radiation, represented by a massless scalar field in a Robertson-Walker metric, is taken into account. By using a weak non minimum vacuum definition, the radiation temperature as a time dependent function is obtained. When the universe evolution is nearly but non equal to $t^{1/2}$, it is possible to fit the temperature of the microwave background. A particular massive case is compared with the massless one. When the mass of the matter field is next to the Planck one and the time is going to infinite, a similar result to the Hawking radiation of the blackhole is obtained.
I. Introduction

The problem of the vacuum definition in Quantum Field Theory in Curved Spacetime is discussed since more than two decades ago [1]. The non trivial Riemannian connection, which appears in the field equation, under changes of the Cauchy surfaces, avoid the invariance, of the expansion in normal modes of the solution. The modification of the basis of solutions is represented by the Bogoliubov transformations, which are related with the particle creation between the space like Cauchy surfaces “in” and “out”. The conditions defining the vacuum in both surfaces were studied, for example, in refs. [2], [3] and [4], where some of the properties of the flat vacuum were generalized to curve geometries. The properties of the vacua are formulated via the Feynman propagator. However there are not cases where a time dependent definition is given. In the present work we will return to the pioneer article of Parker [5], where a time depending formalism of the Bogoliubov transformation is developed. Then we will consider Bogoliubov transformations driven by the field equation. The condition that defines the vacuum on the “out” surface will be the minimal vacuum definition, used in ref.[3], but without imposing the minimization of the energy. That condition is incompatible with the dynamical particle creation, as it will be proved. Moreover we will see that a Planckian distribution emerges from the field equation without additional conditions.

II. Field equation and Bogoliubov transformation

Let us consider the action for a massive scalar field with arbitrary coupling given by

\[ S = \frac{1}{2} \int \sqrt{-g} d^4x (\partial_\mu \varphi \partial^\mu \varphi - (m^2 + \xi R) \varphi^2) \]  (1)

in a spacetime described by

\[ ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2) \]  (2)

Therefore the variation of the action results in the field equation

\[ (\nabla_\mu \partial^\mu + m^2) \varphi = 0 \]  (3)

(with \( \mu = 0, 1, 2, 3 \)).

One can expand \( \varphi \) using a Fourier representation. A discretization, in the same form as in ref.[5], can be used, so we introduce the periodic boundary condition
\( \varphi(x + nL, t) = \varphi(x, t) \), where \( n \) is a vector with integer Cartesian components and \( L \) a length which goes to infinity at the end of the calculation. Then, given a basis of solutions of the field eq.(3) \( \{ \psi_k(x) \} \cup \{ \psi_k^*(x) \} \), one can expand \( \varphi \) in the form

\[
\hat{\varphi}(x, t) = \sum_k [A_k \psi_k(x) + A_k^\dagger \psi_k^*(x)]
\]

where \( A_k \) and \( A_k^\dagger \) are annihilation-creation operators which satisfy the bosonic commutation relations

\[
[A_k, A_{k'}] = 0, \quad [A_k^\dagger, A_{k'}^\dagger] = 0, \quad [A_k, A_{k'}^\dagger] = \delta_{k,k'}
\]

Moreover these operators act on the \( \vert 0 \rangle \) vacuum in the way

\[
A_k \vert 0 \rangle = 0, \quad A_k^\dagger \vert 0 \rangle = \vert 1_k \rangle, etc.
\]

Due to the symmetry of the spacetime, it is possible to separate the dependence in the spatial coordinate \( x \) and the temporal one; then we write

\[
\psi_k(x) = h_k(t) \exp ik \cdot x
\]

replacing the last equation in eq.(3), \( h_k(t) \) satisfy

\[
\ddot{h}_k(t) + Q_k^2(t)h_k(t) = 0
\]

\[
Q_k^2(t) := \omega^2 - \frac{9}{4}H^2 - \frac{3}{2} \dot{H} + \xi R
\]

in eq.(9) \( R \) is the curvature scalar, given in the R-W metric as \( R = 6(2H^2 + \dot{H}) \), with \( H = \dot{a}/a \) the Hubble coefficient and \( \omega^2 = k^2/a^2 + m^2 \).

Following Parker [5] we introduce the time depending operators \( a_k(t) \) and \( a_k^\dagger(t) \), which satisfy eq.(5) for for any time, with the boundary condition

\[
A_k := a_k(t = t_0)
\]

and operate on the time depending Fock space, as follows:

\[
a_k(t) \vert 0, t \rangle = 0, \quad a_k^\dagger(t) \vert 0, t \rangle = \vert 1_k, t \rangle, etc.
\]
Now the $\hat{\phi}$ is expanded using the time depending operators, i.e.:

$$\hat{\phi}(x, t) = \sum_k [a_k(t)\phi_k(x) + a_k^\dagger(t)\phi^*_k(x)]$$  \hspace{1cm} (12)

with $\phi$ as a generalization of the normal modes of flat spacetime:

$$\phi_k(x) = \frac{1}{(La(t))^{3/2}\sqrt{2W}} \exp i(kx - \int_{t_0}^t W(k, t')dt')$$ \hspace{1cm} (13)

Replaying the ansatz used in ref.\[5\] we introduce the complex c-number functions of $k$ and $t$; $\alpha(k, t)$ and $\beta(k, t)$, such that

$$a_k(t) = \alpha(k, t)^*A_k + \beta(k, t)A_k^\dagger$$ \hspace{1cm} (14)

Then the particle creation operator $a_k^\dagger(t)a_k(t)$ satisfies

$$n_k := |\beta_k|^2 = <0|a_k^\dagger(t)a_k(t)|0>$$ \hspace{1cm} (15)

where $n$ is the mean value of the created particles.

Following with a brief review of the ref. [5], which is necessary to understand the main results, we replace eq.(14) in eq.(12) and by comparison with eqs (4) and (7) we obtain

$$h(k, t) = \frac{1}{(La(t))^{3/2}(2W(k, t))^{1/2}}[\alpha(k, t)^*e^{-i\int_{t_0}^t Wdt'} + \beta(k, t)^*e^{i\int_{t_0}^t Wdt'}]$$ \hspace{1cm} (16)

Because that the functions $\psi$ and $\psi^*$, defined by eq.(7), form a basis of solutions, then the Bogoliubov coefficients $\alpha$ and $\beta$, must satisfy

$$|\alpha|^2 - |\beta|^2 = 1$$ \hspace{1cm} (17)

Eq. (17) is equivalent to the parametrization:

$$\alpha(k, t) = e^{-i\gamma_\alpha(k, t)} \cosh \theta(k, t)$$ \hspace{1cm} (18a)

$$\beta(k, t) = e^{i\gamma_\beta(k, t)} \sinh \theta(k, t)$$ \hspace{1cm} (18b)
where $\gamma_\alpha$ and $\gamma_\beta$ are arbitrary functions, and $\theta$ (the Bogoliubov angle) is related with the particle creation number by

$$n_k(t) = \sinh^2 \theta(k, t)$$

(19)

Putting eqs (16) and (18) in eq. (8), performing the separation between real and imaginary parts, we have the system of equations:

$$\cosh \theta [(1 + \tanh \theta \cos \Gamma) M^2 + 2W\dot{\gamma}_\alpha] = 0$$

(20a)

$$\sinh \theta [M^2 \sin \Gamma + 2W\dot{\theta}] = 0$$

(20b)

where

$$M^2 := -\frac{1}{2}(\dot{W}/W) + \frac{1}{4}(\ddot{W}/W)^2 - W^2 + Q^2$$

(21)

and

$$\Gamma := \gamma_\alpha + \gamma_\beta - 2\int_{t_0}^t W dt'$$

(22)

In particular eqs(20) are satisfied when

$$\sinh \theta = 0$$

$$\cosh \theta = 1$$

the first equation corresponds to non-creation of particles, which is consistent with eq.(20a), because in that case it results

$$M^2 + 2W\dot{\gamma}_\alpha = 0$$

but from eq.(18a) it is $\gamma_\alpha = 0$, then $M^2 = 0$. In any other case we have the system of equations:
\[(1 + \tanh \theta \cos \Gamma)M^2 + 2W\dot{\gamma}_\alpha = 0 \quad (23a)\]

\[M^2 \sin \Gamma + 2W\dot{\theta} = 0 \quad (23b)\]

as in this case is \(M^2 \neq 0\), we can rewrite eq.(23a) in the form

\[\tanh \theta \cos \Gamma = -(1 + 2 \frac{W}{M^2} \dot{\gamma}_\alpha) \quad (24)\]

from eqs (19) and (17) we can write

\[\tanh \theta = \left( \frac{n}{n + 1} \right)^{1/2} \]

By replacing the last equation in eq. (24) and putting at square both members of that equation, we get

\[\frac{n}{n + 1} \cos^2 \Gamma = (1 + 2 \frac{W}{M^2} \dot{\gamma}_\alpha)^2 \quad (25)\]

Therefore from eq.(25) it is obtained

\[n = \frac{1}{f - 1} \quad (26)\]

with

\[f = \frac{\cos^2 \Gamma}{(1 + 2 \frac{W}{M^2} \dot{\gamma}_\alpha)^2} \quad (27)\]

As we can see from eq.(27) \(f = f(\gamma_\alpha, \dot{\gamma}_\alpha, \gamma_\beta, W, \dot{W})\), then we have superabundance of variables, we can choose some of them in a convenient way. If we use unities such that \(c = k_B = \bar{h} = 1\), we can define temperature by the equalization of the function \(f\) to the Planckian functional form;

\[f = \exp(\epsilon_k/T) \quad (28)\]

with \(\epsilon_k\) the energy by mode, \(T\) the temperature. From eq. (28) we can calculate the temperature \(T\) as a function of the particle model used, as we will see in the next section.
III. Time depending temperature

To get an expression for the temperature, first we must calculate the energy by mode $\epsilon_k$. That can be obtained from the total energy, i.e.

$E = <0|\hat{H}|0>$  \hspace{1cm} (29)

$\hat{H}$ is the metric hamiltonian, which can be defined by means of the 00 component of the energy-momentum operator $\hat{T}_{\mu\nu}$, which is obtained by the variation of the action respect to the metric [1]:

$\hat{H} = \int a^3 d^3 x \hat{T}_{00}$  \hspace{1cm} (30)

$\hat{T}_{00}$ is a functional of the field operator $\hat{\phi}$. In the discretized formulation is $\int d^3 x = L^3$. Following ref.[6] we can drop the oscillatory term in the $k$ variable, obtaining an expression for the energy similar to the one corresponding to a set of independent quantum oscillators:

$E = \sum_k \left( \frac{1}{2} + n_k \right) \epsilon_k$  \hspace{1cm} (31)

with (see [4]);

$\epsilon_k = \frac{1}{2W} \left\{ \frac{1}{4} \left( \frac{\dot{W}}{W} + 3(1 - 4\xi)H \right)^2 + 6\xi H^2 (1 - 6\xi) + W^2 + \omega^2 \right\}$  \hspace{1cm} (32)

Then from eq.(28) we can look for a temperature as a function of the vacuum definition (or the particle model), in the form

$T = \frac{\epsilon_k}{2 \ln |\cos \Gamma/(1 + 2\dot{\gamma}/M^2)|}$  \hspace{1cm} (33)

In order to have well defined the logarithm, the argument must be bigger than zero. Using the arbitrariness in the phase $\Gamma$, we can choose

$\Gamma \equiv \pi/4$  \hspace{1cm} (34)

Then the field equations turn to be
\[(1 + \frac{1}{\sqrt{2}} \tanh \theta) M^2 + 2W \dot{\gamma}_\alpha = 0 \quad (35a)\]

\[\frac{1}{\sqrt{2}} M^2 + 2W \dot{\theta} = 0 \quad (35b)\]

By the integration of eq.(35b) and by the replacing in eq.(35a) we obtain

\[\dot{\gamma}_\alpha = \frac{-M^2}{2W} [1 + \frac{1}{\sqrt{2}} \tanh \left( B - \frac{1}{2\sqrt{2}} \int_{t_0}^{t} \frac{M^2}{W} dt \right) ] \quad (36)\]

B is an integration constant, which is related with the particles that are present at the time \( t = t_0 \), i.e.

\[n(t = t_0) = \sinh^2 B\]

Replacing eq.(36) in eq.(33), we have

\[T = \frac{\epsilon_k}{2 \ln[\coth[u]]} \quad (37)\]

with

\[u := \frac{1}{2\sqrt{2}} \int_{t_0}^{t} \frac{M^2}{W} dt' - B \quad (38)\]

where \( \epsilon_k = \epsilon_k[W] \) is given by eq.(32) and \( M^2 = M^2[W] \) is given by eq.(21). In order to introduce the particle model we must to give the function \( W \). In our case we will propose

\[W = \omega \quad (39)\]

It is interesting to note that eq.(37) is very similar to the one obtained in a parametric photon pair production [7]. That photons has the same statistical that our field modes interacting with the curved geometry.

Eq.(39) means that at the observation time \( t \), the function \( W \) satisfies the zero WKB order [2],[3]. Then the modes \( \phi \) are analogous to the ones of flat spacetime. This is motivated by the fact that at the present time the universe is very flat. Then if we look at eq.(12) we have formally an expression for the field operator similar than the one corresponding to scalar bosonic field in flat space time. We cannot use the minimization of energy criterium, as we will see latter. That criterium is only compatible with an in-out process, but not when the dynamics is considered.
IV. Calculation of T in some particular cases

We will now study some particular cases in order to test the physical behavior of eq.(37).

i) Firstly let us considered the massless minimally coupled case \((m = 0, \xi = 0)\).

Then from eq.(39) we have

\[ W = \frac{k}{a} \]  

(40)

We also suppose that at the beginning there are not particles, then \(B = 0\) in eq.(38). Performing the calculation it results

\[ \epsilon_k = \frac{a}{2k} (H^2 + 2k^2/a^2) \]  

(41)

\[ u = -\frac{1}{2\sqrt{2}} \int_{t_0}^{t} a(2H^2 + \dot{H})dt' \]  

(42)

For the universe evolution given by

\[ a(t) = a_0 \left( \frac{t}{t_0} \right)^\alpha \]  

(43)

then

\[ \epsilon_k = \frac{k}{a_0} \left( \frac{a_0^2 \alpha^2}{2k^2 t_0^2} x^{2-\alpha} + x^\alpha \right) \]  

(44)

\[ u = \frac{1}{2\sqrt{2}} \frac{a_0 \alpha (2\alpha - 1)}{k t_0 (\alpha - 1)} (1 - x^{1-\alpha}) \]  

(45)

where \(x := t_0/t\).

From eqs (45) and (37) it results that \(T\) can be well defined when \(\alpha \in [0, 1/2]\).

The semiclassical approach is valid until the Planck time [1], therefore if we choose \(t_0\) as the Planck time the variable \(x \in [0, 1]\).

From fig.1 we can see that if there are not particles at the initial time, the temperature begins from zero and increases to reach a maximum, after that goes to zero (in the massless case). The qualitative behavior is similar in all the interval of \(\alpha\), in the fig. 1 we compare the cases \(\alpha = 1/4\) (dot-dashed line) and \(\alpha = 0.49\) (full line). The calculation was simplified by using the relation

\[ t_0 \frac{k}{a_0} \sim 1 \]  

(46)
We are thinking the universe, at the Planck time, as analogous to a radioactive nucleus with \( t_0 \) as the mean time life and \( \Delta \epsilon_k \sim k/a_0 \) the band width of the energy [8]. That approach means that it is necessary to wait a time \( t_0 \) in order to generate a particle of energy \( k/a_0 \).

ii) Now we consider the massive minimally coupled case \( (m \neq 0, \xi = 0) \). Then instead of eq. (45) we have

\[
u = \frac{a_0 \alpha (1 - 2\alpha)}{2\sqrt{2kt_0}}(I_1(1) - I_1(x)) + \frac{m^2 \alpha (1 - 2\alpha)}{4\sqrt{2k^3t_0}}(I_2(1) - I_2(x))
\]

\[+ \frac{3m^4 a_0^5 \alpha}{8\sqrt{2t_0 k^5}}(I_3(1) - I_3(x)) \tag{47}\]

where

\[
I_1(x) = \int_x^\infty \frac{x^{-\alpha}}{(1 + \lambda x^{-2\alpha})^{1/2}} \, dx, \quad I_2(x) = \int_x^\infty \frac{x^{-3\alpha}}{(1 + \lambda x^{-2\alpha})^{3/2}} \, dx,
\]

\[
I_3(x) = \int_x^\infty \frac{x^{-5\alpha}}{(1 + \lambda x^{-2\alpha})^{5/2}} \, dx,
\]

with \( \lambda := m^2 a_0^2 / k^2 \) an adimensional parameter. In order to compare with the massless case we can do the calculation for the particular case \( \alpha = 1/4 \). Using \( \lambda \sim 1 \) (which corresponds to \( m \simeq m_p \)) to consider an extremal case. The result is also shown in fig. 1 (dashed line). It is interesting to note that when the time goes to infinity, in the massive case, the temperature goes to a constant approximately equal to \( m_p / 4 \). That result is analogous to the Hawking one of the black hole with mass \( m_p \) (except that the \( 2\pi \) factor not appears), remember that \( T_H = 1/8\pi GM \) [9].

V. Conclusions and comments

The main result is that from the field equation and with the more straightforward generalization of the definition, for the modes in Quantum Field Theory in flat spacetime, we can get a time depending temperature expression. It is really the more straightforward generalization because we stress for the observation time \( t \) the correspondence

\[W \rightarrow \omega\]

where \( \omega \) for \( t = cte \) corresponds to the frequency of the field mode in the Minkowski space.
As we can see from fig. 1, the temporal behavior of the $T$ function, for the scalar radiation, is physically reasonable, and lead us to fit the microwave background by means of the fine tuning of the $\alpha$ parameter, which must be very close but not equal to 1/2, as it results from eq.(45).

For the massive case the temperature goes to a constant value when the time increase to infinity (see the dashed line in fig. 1).

Let me do now a brief comment about the minimization energy conditions as a time depending constraint. For an arbitrary coupling the minimization energy condition (or hamiltonian diagonalization condition) [4] is:

\[ W^2 = 6\xi(1 - 6\xi)H^2 + \omega^2 \]  
\[ \frac{\dot{W}}{W} = -3(1 - 4\xi)H \]

this equations, for all the time, in the massive case, has not physical sense because the scalar factor of the universe must depend on the mass and the mode, i.e. \( a = a(k,m,t) \). Still in the massless case it has not physical meaning when \( \xi \neq 0,1/6 \). If \( \xi = 0 \) and \( m = 0 \), eq. (48b) \( \Rightarrow a(t) = \text{constant} \). Only when \( \xi = 1/6 \) (conformal coupling) eq. (48b) is satisfied without any condition on \( a(t) \). But if we consider eq. (21), from the conditions given by eqs (48) it results

\[ M^2 = -12\xi H^2(1 - 6\xi) \]  
(49)

Then when \( \xi = 1/6 \) we have \( M^2 = 0 \), therefore from eq.(20b) we have

\[ \dot{\theta} = 0 \]  
(50)

So from eq. (19) we can say that in this case the particle creation is null. Then when the dynamics is considered, the criterium given by eqs. (48) does not work.

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FIG. 1 Dimensionless temperature $T/b$, with $b = k/2a_0$, vs the parameter $x = t_0/t$. The full line corresponds to the massless case with $\alpha = 0.49$, the dashed line to the massless with $\alpha = 1/4$ and the dot-dashed line to the massive case with $\alpha = 1/4$. 