On the Asymptotics of Solutions of the Klein – Gordon – Fock Equation with Meromorphic Coefficients in the Neighborhood of Infinity

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Abstract. In this paper we investigate the construction of uniform asymptotics of solutions for the Klein-Gordon-Fock boundary value problem with meromorphic coefficients as \( t \to \infty \). The difficulty in solving these problems is due to the fact that infinity, as known, is an irregular singular point of such equations. In the case of holomorphic coefficients, the order of degeneracy will be at most two, and in the case of meromorphic coefficients it can be arbitrary, and it will depend on the order of the poles of the meromorphic coefficients. The main method for constructing asymptotics in the neighbourhood of an irregular singular point is the method of resurgent analysis, which is based on the Laplace-Borel integral transform. Using this method, in this paper, a uniform asymptotics of the problem under consideration is constructed in a neighborhood of infinity for arbitrary meromorphic (or holomorphic) coefficients.

Key Words. Asymptotics, Klein-Gordon-Fock operator, meromorphic coefficients.

1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be an arbitrary open set in \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \), and \( \Omega \cup \partial \Omega = \overline{\Omega} \) is the closure of \( \Omega \); \( Q = \Omega \times (t \geq 0) \) is a cylinder in the space of points \((x, t) \in \mathbb{R}^{n+1}, x \in \Omega, 0 \leq t < \infty \).

Let us construct uniform asymptotics of solutions of the boundary-value problem for the Klein–Gordon–Fock equation with meromorphic (or holomorphic) coefficients as \( t \to \infty \).

Here, we consider the boundary value problem for function \( u(x, t) \):

\[
\left( \frac{d^2}{dt^2} + a_0(t) \right) u(x, t) + c_0(t) u(t, x) = 0, \tag{1}
\]

\[
\left( \frac{\partial u(x, t)}{\partial t} + \beta \frac{\partial u(x, t)}{\partial n} \right) = 0, \tag{2}
\]

where \( a(x) \) and \( \beta(x) \) are infinitely differentiable functions with uniformly bounded derivatives on the boundary \( \partial \Omega \), \( a(x) \geq 0 \) and \( \beta(x) \geq 0 \) such that \( a(x) + \beta(x) > 0 \).

It is assumed that the functions \( a(t) \) and \( c(t) \) are meromorphic (or is holomorphic) functions in a neighborhood of infinity, which means that there exists an exterior of the circle \( |t| > R \) such that the functions \( a(t) \) and \( c(t) \) are expanded in it in the Laurent series:

\[
a^0(t) = t^m \sum_{j=0}^{\infty} a_j t^{-j}, \quad c^0(t) = t^k \sum_{j=0}^{\infty} c_j t^{-j}, \tag{3}
\]

where \( m \in \mathbb{Z}, k \in \mathbb{Z} \); in this case we can always choose \( m \) and \( k \) so that \( a_0 \neq 0, c_0 \neq 0 \). And on condition \( m = k \), this number can be chosen so that \( c_0 + \lambda a_0 \neq 0 \).
Indeed, if \( m \leq 0 \) and \( k \leq 0 \), then series (3) are a Taylor series at a neighborhood of infinity, if \( m > 0 \) and \( k > 0 \) are a Laurent series. In present work, we finally construct the asymptotics of solutions to Eq. (1) with boundary condition (2) as \( t \to \infty \).

Using the method of separating variables \( u(x,t) = Y(x)v(t) \), we get the Sturm-Liouville problem for \( Y(x) \):

\[
\Delta Y(x) + \lambda Y(x) = 0,
\]

and the equation for \( v(t) \):

\[
\left( \frac{d}{dt} \right)^2 v(t) - a^n(t)\lambda v(t) + c^0(t)v(t) = 0.
\]

The solution of the Sturm – Liouville problem for the Laplace operator is well described in [1].

The study of the asymptotics of solutions of differential equations at the neighborhood of of their regular and irregular singular points is one of the classical problems of the analytical theory of linear differential equations. The problem of constructing asymptotics of solutions of differential equations at the neighbourhood of regular singular points was considered, for example, in [2]. It was shown that at the neighbourhood of a regular singular point, the asymptotics of the solution of an ordinary differential Eq. (5) is conormal:

\[
\sum_j r^j \sum_{i=0}^k a_j^i \ln^i \frac{1}{t},
\]

where \( a_j^i, s_j \) are some complex numbers.

Let’s proceed to constructing of the asymptotic of Eq. (5) at \( t \to \infty \). By make the substitution of variable \( t = \frac{1}{r} \) Eq. (5) can be written in the form:

\[
\left( -r^2 \frac{d}{dr} \right)^2 v(r) + (c(r) - a(r)\lambda) v(r) = 0,
\]

where

\[
a(r) = \frac{1}{r^m} \sum_{j=0}^\infty a_j r^j, \quad c(r) = \frac{1}{r^r} \sum_{j=0}^\infty c_j r^j.
\]

As a rule, infinity is an example of an irregular singularity. One of the first papers in which the study of asymptotics in a neighborhood of an infinitely distant singular point is the Thomé paper [18]. In this paper, it was shown that the asymptotics of solutions can contain formal, generally speaking, asymptotic series. Further, in the paper of Poincaré [3] it was shown that these series are asymptotic. Earlier Sternberg in [4] considered the case of non-Fuchsian type conditions for linear equations and their systems. Here, we construct the asymptotics of solutions to the non-Fuchsian equations in the form of products of the corresponding exponentials by divergent power series:

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i^j b_i^j r^j r^i.
\]

The main method used to construct the asymptotics in the neighborhood of an irregular singular point is the resurgent analysis method, which is based on the Laplace-Borel integral transform. The foundations of this method were proposed by J. Ecalle [5] at the end of the 20th century, and now this method is being developed and applied to various problems with irregular singularities, an example of this in application is this article. By applying this method, in this paper we have constructed a uniform asymptotics of the problem under consideration, in a neighborhood of infinity for arbitrary meromorphic (or holomorphic) coefficients.

2. Definitions and auxiliary statements

Let us introduce some notions of the resurgent analysis. By \( S_{R,\epsilon} \) we denote the sector \( S_{R,\epsilon} = \{ r | -\epsilon < \arg r < \epsilon, |r| < R \} \).
**Definition 1.** [17] We will say that the function \( f \) is analytical on \( S_{\rho,\alpha} \), and is of exponential growth no more than \( k \), if there are non-negative constants \( C \) and \( \alpha \) such that in the sector \( S_{\rho,\alpha} \), the following inequality is valid:

\[
|f| < C e^{\alpha r^k}.
\]

By \( E_k(S_{\rho,\alpha}) \) we denote the space of functions holomorphic in the domain \( S_{\rho,\alpha} \) and of \( k \)-exponential growth in zero; by \( E(C) \) – the space of integer functions of exponential growth.

**Definition 2.** [9] The \( k \)-Laplace–Borel transform of the function \( f(r) \in E_k(S_{\rho,\alpha}) \) is the mapping \( B_k : E_k(S_{\rho,\alpha}) \rightarrow E(\Omega_{\rho,\alpha})/E(C) \):

\[
\tilde{f} = B_k f = \frac{1}{2\pi i} \int_{r_0} e^{\alpha p r^k} f(r) \, \frac{dr}{r^k} ,
\]

where \( r_0 \) denotes an arbitrary point of the sector.

The inverse \( k \)-Laplace–Borel transform is defined by the formula:

\[
B_k^{-1} \tilde{f} = \frac{k}{2\pi} \int_0^{2\pi} e^{\alpha p r^k} \tilde{f}(p) dp .
\]

**Definition 3.** [9] The function \( \tilde{f} \) is called infinitely extendable, if for any number \( R \), there is a discrete set of points \( Z_r \) in \( C \) such that the function \( \tilde{f} \) is analytically extended from the initial domain of definition along any path with a length smaller than \( R \), which does not pass through \( Z_r \).

**Definition 4.** [9] The element \( f \) of the space \( E_k(S_{\rho,\alpha}) \) is called the \( k \)-resurgent function, if its \( k \)-Laplace–Borel transform \( \tilde{f} = B_k f \) is infinitely extendable.

**Theorem 1.** [9, 10] Let \( f \) be a resurgent function, then the solution of the equation

\[
H(r) \left( -r^2 \frac{d}{dr} r^k \right) u = f
\]

is a resurgent function. If the polynomial \( H_0(p) \) has simple roots at the points \( p_1, \ldots, p_m \), then the asymptotic behavior of the solution of the homogeneous equation has the form

\[
u(r) \approx \sum_{j=1}^{m} \exp \left( \frac{p_j}{r^k} \right) r^{\frac{j}{2}} \sum_{i=0}^{\infty} B_i^j r^i ,
\]

where the sum is taken by the union of all the roots of the polynomial \( H_0(p) \).

For equations with \( k + 1 \)-order degeneracy, where \( k \in \mathbb{N} \), namely, for equations of the form

\[
H \left( r^2, \frac{1}{k} r^{k+1} \frac{d}{dr} \right) u = 0
\]

in the case when the roots of the main symbol are simple, the asymptotic behavior have the form

\[
u(r) \approx \sum_{j=1}^{m} \exp \left( \frac{p_j}{r^k} + \sum_{i=1}^{m} \frac{\alpha^i}{r^{i-k}} \right) r^{\frac{j}{2}} \sum_{i=0}^{\infty} b_i^j r^i .
\]

If \( k + 1 = \frac{m}{n} \), \( m \in \mathbb{N} \), \( k \in \mathbb{N} \), \( m > k \), the asymptotic behavior of the solution will be:

\[
u \approx \sum_{j=1}^{m} \exp \left( \frac{P_j}{r^k} + \sum_{i=1}^{m-k-1} \frac{\alpha^i}{r^{i-k}} \right) r^{\frac{j}{2}} \sum_{i=0}^{\infty} b_i^j r^i .
\]

A proof of this theorem can be found in [6–8].

For equations with arbitrary meromorphic coefficients, a proof of infinite extendability was obtained in [6, 7, 9]. This result allows us to apply the methods of resurgent analysis to the construction in spaces of functions of exponential growth of uniform asymptotics of solutions to linear differential equations with holomorphic coefficients.
3. Main results
Consider the case: \( k \leq 0, m \leq 0 \). Then

\[
a(r) = \sum_{j=0}^{\infty} a_j r^{l-m}, \quad c(r) = \sum_{j=0}^{\infty} c_j r^{l-k}.
\]

From the difference: \( (c(r) - a(r)k) = b_0 + rb_1 + r^2 \sum_{j=0}^{\infty} b_{i+2} r^i \), we obtain

\[
b_0 = \left( c_0 r^{-k} - \lambda a_0 r^{-m} \right) \text{,} \quad b_1 = \frac{c(r) - a(r)k - b_0}{r} \text{.}
\]

**Lemma 1.** In case \( b_0 \neq 0 \), then all the asymptotics of solutions to Eq. (5) at a neighborhood of infinity with respect to \( t \) in the space of exponentially growing functions have the form:

\[
v(t) \approx \left( e^{p_1 t} \sum_{i=0}^{\infty} A_i t^{-i} + e^{p_2 t} \sum_{i=0}^{\infty} A_i' t^{-i} \right),
\]

where \( p_i \), \( i = 1, 2 \), are the roots of the second order polynomial \( p^2 + b_0 \), and \( \sigma_i, i = 1, 2 \), are determined by the formula:

\[
\sigma_i = -\frac{b_i}{2 p_i}.
\]

In case \( b_0 = 0 \), \( b_1 \neq 0 \), the asymptotics of the solution to Eq. (5) has the form

\[
v(t) \approx t^{\frac{1}{2}} \left( e^{p_1 t} \sum_{i=0}^{\infty} A_i t^{-i} + e^{p_2 t} \sum_{i=0}^{\infty} A_i' t^{-i} \right),
\]

where \( p_i \), \( i = 1, 2 \), are the roots of the polynomial \( p^2 + 4 b_1 \), and \( \sum_{i=0}^{\infty} A_i' t^{-i} \), \( j = 1, 2 \), are asymptotic series.

And finally in case \( b_0 = b_1 = 0 \), the asymptotics of the solution to Eq. (5) has the form:

\[
v(t) \approx \text{conormal form representation}.
\]

Now consider the cases: \( k \geq m, k > 0, \) and \( k > m, m > 0 \). Introduce the notation \( l = \max(k, m) \). If \( l = k \), then \( b_0 = \left( c_0 - \lambda r^{-k} a_0 \right) \text{;} \) and if \( l = m \), then \( b_0 = \left( r^{-m} c_0 - \lambda a_0 \right) \text{.}

**Lemma 2.** In case \( k = 2n \), the asymptotics of the solution to Eq. (6) have the form

\[
v(r) \approx \exp \left( \frac{p_1}{r^{n+1}} + \sum_{i=0}^{\infty} \frac{\alpha_i}{r^{n+1-i}} \right) r^{\sigma_1} \sum_{i=0}^{\infty} b_i r^i + \exp \left( \frac{p_2}{r^{n+1}} + \sum_{i=0}^{\infty} \frac{\alpha_i'}{r^{n+1-i}} \right) r^{\sigma_2} \sum_{i=0}^{\infty} b_i' r^i.
\]

In case \( k = 2n+1 \) the asymptotics of the solution to Eq. (6) have the form

\[
v(r) \approx \exp \left( \frac{p_1}{r^{n+2}} + \sum_{i=1}^{2n+1} \frac{\alpha_i}{r^{n+2-i}} \right) r^{\sigma_1} \sum_{i=1}^{\infty} v_i r^i + \exp \left( \frac{p_2}{r^{n+2}} + \sum_{i=1}^{2n+2} \frac{\alpha_i'}{r^{n+2-i}} \right) r^{\sigma_2} \sum_{i=1}^{\infty} v_i' r^i.
\]

Here \( p_1, p_2 \) are the roots of the second order polynomial \( H_0(p) = \left( \frac{2}{2+k} \right)^2 b_0 + p^2 \), and \( \alpha_i, \sigma_j, j = 1, 2 \), are some numbers, methods of calculating which, see[6 - 8].

**Theorem 2.** The asymptotics of solutions to problem (1), (2) in the space of exponentially growing functions in a neighborhood of infinity in \( t \) can be represented as a linear combination of functions \( u_j(x, t) \), that is, \( u_j(x, t) \approx v_j(t) Y_j(X) \), \( j = 0, 1, \ldots \).

Note that in some papers [14], [15], where authors consider the problem of wave propagation in a medium whose velocity characteristics vary under an external action in the three-dimensional case. In the same papers, we study the problem of constructing asymptotics of the solutions for a wave equation with a variable time-dependent coefficient multiplying the Laplacian; this coefficient is assumed to be a meromorphic function in a neighborhood of infinity.
In [10-13], various boundary value problems were investigated for elliptic equations and systems, such as the elasticity system, the Stokes system, and the biharmonic (or polyharmonic) equation in unbounded domains with the condition of boundedness of the weighted energy or the Dirichlet integral, where asymptotic expansions of solutions of elliptic operators are presented, including in the form of co-normal asymptotics.

4. Conclusion

The main result of the paper is the construction of uniform asymptotics of solutions of the boundary value problem for the Klein–Gordon–Fock equation with meromorphic coefficients in a neighborhood of infinity. Obviously, using the results of this article, one can also construct uniform asymptotics of this problem in a neighborhood of zero. In the future, the methods used in this article can be applied to the construction of uniform asymptotics of solutions of arbitrary second-order equations in a neighborhood of regular and irregular singular points, and also, to the construction of asymptotics for solutions of various equations of mathematical physics. Also, a further direction of research is to determine the conditions for the convergence of asymptotic series included in the expressions for the asymptotics of solutions.

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