Boundary feedback stabilization of quasilinear hyperbolic systems with partially dissipative structure

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Abstract

In this paper, we study the boundary feedback stabilization of a quasilinear hyperbolic system with partially dissipative structure. Thanks to this structure, we construct a suitable Lyapunov function which leads to the exponential stability to the equilibrium of the $H^2$ solution. As an application, we also obtain the feedback stabilization for the Saint-Venant-Exner model under physical boundary conditions.

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1. Introduction and main results

Many models in physics, mechanics and other fields, including gas and fluid dynamics for instance, are described as hyperbolic equations. Control problems, particularly the stability and stabilization problems, of hyperbolic systems have been widely studied for decades (see [2, 3] and the references therein).

One classical approach to establish the asymptotic stability of hyperbolic system is the characteristic method. In the framework of $C^1$-solution, dissipative boundary conditions that lead to exponential stability of quasilinear hyperbolic systems without source terms have been found in [8, 15].

Another important approach to design boundary feedback controls is the Backstepping method. It has been used to stabilize exponentially the inhomogeneous quasilinear hyperbolic

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system in $H^2$ norm (see [6, 12]). One can refer to [14] for many successful examples about feedback stabilization with this approach.

The third powerful approach is the Lyapunov function method. A strict Lyapunov function is introduced in [16] to achieve the exponential stability of a class of symmetric linear hyperbolic systems. Similar Lyapunov functions are used for quasilinear homogeneous hyperbolic systems in the framework of $H^2$-solution in [4].

If the hyperbolic system is inhomogeneous, the Lyapunov function approach can still be applied (see for instance [7, 9]). However, the nonzero source term change a lot the stability properties. With a source term, a simple quadratic Lyapunov function ensuring exponential stability for the $L^2$ norm (or $H^p$ norm) does not always exist no matter what the boundary conditions are. In [1], the authors study a linear $2 \times 2$ hyperbolic system and found a necessary and sufficient condition for simple quadratic Lyapunov function. Later in Chapter 6 of [2], the authors give a sufficient (but a priori non-necessary) condition such that the exponential stability of the system for the $H^p$ norm with $p \geq 2$ is achieved. We refer to [10] for a relevant result in $C^1$-norm or $C^p$-norm. Naturally, these conditions all include one interior condition which requires a good coupling structure of the hyperbolic system, compared to the homogeneous case. However, as mentioned in their papers, this interior condition (typically a differential matrix inequality) is not straightforward to be checked in a specific model.

Different from the above, Herty and Yong study the boundary feedback stabilization of one-dimensional linear hyperbolic systems with a relaxation term in [11]. The key assumption is a structural stability condition which is introduced from [17] and is satisfies in many physical models. Later, in [18], Yong shows that under this structural stability condition, the boundary feedback stabilization result is also available for a class of one-dimensional linear hyperbolic system with vanishing eigenvalues.

Motivated by [11], in this paper, we consider a one-dimensional quasilinear hyperbolic system with the same relaxation structure. Thanks to the partial dissipation in the structural stability condition, we establish the local exponential stability of this nonlinear system for the $H^2$-norm. The main strategy is to construct a strict Lyapunov function together with a perturbation argument based on linear approximation. Compared to the result in [11], we provide an explicit sufficient condition on the gains of stabilizing boundary feedback control. As an application, we also obtain the boundary feedback stabilization of the Saint-Venant-Exner model proposed in [13] under physical boundary conditions.

Precisely, we are concerned with the boundary feedback stabilization of the following one-dimensional quasilinear hyperbolic system

$$
U_t + A(U)U_x = Q(U), \quad t \in (0, \infty), x \in (0, 1) \tag{1.1}
$$

where $U = (u_1, \ldots, u_n)^T$ is the unknown vector function of $(t, x)$, $A : \mathbb{R}^n \mapsto \mathcal{M}_{n,n}(\mathbb{R})$ is a smooth matrix function and $Q : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth vector function.
Let $U^* \in \mathbb{R}^n$ be an equilibrium of \( (1.1) \), i.e.,
\[
Q(U^*) = 0.
\] (1.2)

Without loss of generality, we may assume $U^* = 0$, otherwise one can consider $U - U^*$ as the unknown functions.

We first assume that the system \( (1.1) \) is hyperbolic in a neighborhood of $U = 0$, i.e., the matrix $A(U)$ has $n$ real eigenvalues
\[
\Lambda_r(U) < 0 < \Lambda_s(U) \quad (r = 1, \cdots, m; \ s = m + 1, \cdots, n),
\] (1.3)
and it has a complete set of left eigenvectors $L_i(U) = (L_{i1}(U), \cdots, L_{in}(U))$, $(i = 1, \cdots, n)$, i.e.,
\[
L_i(U)A(U) = \Lambda_i(U)L_i(U) \quad (i = 1, \cdots, n).
\] (1.4)

Let
\[
L(U) = \begin{pmatrix}
L_1(U) \\
\vdots \\
L_n(U)
\end{pmatrix}
\quad \text{and} \quad
\Lambda(U) = \begin{pmatrix}
\Lambda_-(U) & 0 \\
0 & \Lambda_+(U)
\end{pmatrix}
\] (1.5)

where
\[
\Lambda_-(U) = \text{diag}\{\Lambda_1(U), \cdots, \Lambda_m(U)\}, \quad \text{and} \quad
\Lambda_+(U) = \text{diag}\{\Lambda_{m+1}(U), \cdots, \Lambda_n(U)\}.
\] (1.6)

Then
\[
L(U)A(U) = \Lambda(U)L(U).
\] (1.7)

It is easy to see that system \( (1.1) \) is hyperbolic if and only if there is a symmetric positive definite matrix $A_0(U)$, such that
\[
A_0(U)A(U) = \Lambda^T(U)A_0(U).
\] (1.8)

Then it follows that
\[
(L^{-1}(0))^T A_0(0)L^{-1}(0)A(0) = \Lambda(0)(L^{-1}(0))^TA_0(0)L^{-1}(0).
\] (1.9)

Consequently, there exist two symmetric positive definite matrices $X_1(0) \in \mathcal{M}_{m,m}(\mathbb{R})$ and $X_2(0) \in \mathcal{M}_{n-m,n-m}(\mathbb{R})$ such that
\[
(L^{-1}(0))^T A_0(0)L^{-1}(0) = \begin{pmatrix}
X_1(0) & 0 \\
0 & X_2(0)
\end{pmatrix}.
\] (1.10)

Moreover, we assume the system possesses the following partially dissipative structure in a neighborhood of $U = 0$:
There exist invertible matrices $\mathbf{P}(U) \in \mathcal{M}_{n,n}(\mathbb{R})$ and $\mathbf{S}(U) \in \mathcal{M}_{r,r}(\mathbb{R})$ with $0 < r \leq n$, such that
\[
\mathbf{P}(U)\mathbf{Q}_U(U) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{S}(U) \end{pmatrix} \mathbf{P}(U), \tag{1.11}
\]
\[
\mathbf{A}_0(U)\mathbf{Q}_U(U) + \mathbf{Q}_U^T(U)\mathbf{A}_0(U) \leq -\mathbf{P}^T(U) \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_r \end{pmatrix} \mathbf{P}(U). \tag{1.12}
\]

Here $\mathbf{Q}_U(U)$ stands for the Jacobian matrix of $\mathbf{Q}$ with respect to $U$, $\mathbf{I}_r$ denotes the $r \times r$ identity matrix. Let us point out that the above assumptions (1.11) and (1.12) are called \textit{structural stability conditions} in [17, 11], which are commonly satisfied in lots of physical models.

Let
\[
\xi(t, x) = \mathbf{L}(0)U(t, x) \tag{1.13}
\]
be the linearized diagonal variable and denote
\[
\xi(t, x) = \begin{pmatrix} \xi_-(t, x) \\ \xi_+(t, x) \end{pmatrix} \tag{1.14}
\]
where $\xi_-(t, x) \in \mathbb{R}^m$ and $\xi_+(t, x) \in \mathbb{R}^{n-m}$. According to the theory on the well-posedness of the quasilinear hyperbolic system, the typical boundary conditions are given as follows
\[
\begin{pmatrix} \xi_+(t, 0) \\ \xi_-(t, 1) \end{pmatrix} = \mathbf{K} \begin{pmatrix} \xi_+(t, 1) \\ \xi_-(t, 0) \end{pmatrix}, \quad t \in (0, \infty), \tag{1.15}
\]
where the feedback matrix
\[
\mathbf{K} = \begin{pmatrix} \mathbf{K}_{00} & \mathbf{K}_{01} \\ \mathbf{K}_{10} & \mathbf{K}_{11} \end{pmatrix}, \tag{1.16}
\]
$\mathbf{K}_{00} \in \mathcal{M}_{n-m,n-m}(\mathbb{R})$, $\mathbf{K}_{01} \in \mathcal{M}_{n-m,m}(\mathbb{R})$, $\mathbf{K}_{10} \in \mathcal{M}_{m,n-m}(\mathbb{R})$ and $\mathbf{K}_{11} \in \mathcal{M}_{m,m}(\mathbb{R})$ are all matrices with constant elements.

Finally, the initial condition is prescribed as
\[
U(0, x) = U_0(x), \quad x \in (0, 1), \tag{1.17}
\]
with $U_0 \in H^2((0, 1); \mathbb{R}^n)$ in a neighborhood of $U = 0$.

Regarding the well-posedness of the solutions to the problem (1.1), (1.15) and (1.17), we have the following proposition

\textbf{Proposition 1.1.} There exists $\delta_0 > 0$ such that, for every $U_0 \in H^2((0, 1); \mathbb{R}^n)$ satisfying
\[
\|U_0\|_{H^2((0,1);\mathbb{R}^n)} \leq \delta_0, \tag{1.18}
\]
and the $C^1$ compatibility conditions at the points $(t, x) = (0, 0), (0, 1)$, the problem (1.1), (1.15) and (1.17) has a unique maximal classical solution

$$U \in C^0([0, T); H^2((0, 1); \mathbb{R}^n))$$  \hfill (1.19)

with $T \in (0, +\infty]$. Moreover, if

$$||U(t, \cdot)||_{H^2((0,1); \mathbb{R}^n)} \leq \delta_0, \quad \forall t \in [0, T),$$  \hfill (1.20)

then $T = +\infty$.

Our main result is the following theorem.

**Theorem 1.1.** Assume that the hyperbolic system (1.1) has no vanishing characteristic speed and it possesses the partially dissipative structure, i.e., (1.3), (1.8), (1.11) and (1.12) hold. Let $K$ be chosen such that matrices

$$\begin{pmatrix} X_2(0)\Lambda_+(0) & 0 \\ 0 & -X_1(0)\Lambda_-(0) \end{pmatrix} - K^T \begin{pmatrix} X_2(0)\Lambda_+(0) & 0 \\ 0 & -X_1(0)\Lambda_-(0) \end{pmatrix} K$$  \hfill (1.21)

and

$$\begin{pmatrix} e^{-\Lambda+(0)}\Lambda_+(0) & 0 \\ 0 & -\Lambda_-(0) \end{pmatrix} - K^T \begin{pmatrix} \Lambda_+(0) & 0 \\ 0 & -e^{-\Lambda_-(0)}\Lambda_-(0) \end{pmatrix} K$$  \hfill (1.22)

are both positive definite. Then, the closed-loop system (1.1), (1.15) and (1.17) is locally exponentially stable for the $H^2$-norm, i.e., there exist positive constants $\delta$, $C$ and $\nu$, such that the solution to the system (1.1), (1.15) and (1.17) satisfies

$$||U(t, \cdot)||_{H^2((0,1); \mathbb{R}^n)} \leq Ce^{-\nu t}||U_0||_{H^2((0,1); \mathbb{R}^n)}, \quad t \in [0, +\infty),$$  \hfill (1.23)

provided that

$$||U_0||_{H^2((0,1); \mathbb{R}^n)} \leq \delta$$  \hfill (1.24)

and the $C^1$ compatibility conditions are satisfied at $(t, x) = (0, 0)$ and $(0, 1)$.

**Remark 1.1.** Theorem 1.1 still holds if the assumption (1.11) is extended to a more general case that

$$P(U)Q(U)P^{-1}(U) = \begin{pmatrix} S_{11}(U) & S_{12}(U) \\ S_{21}(U) & S_{22}(U) \end{pmatrix},$$  \hfill (1.25)

where $S_{22}(U) \in \mathcal{M}_{r,r}(\mathbb{R})$ with $0 < r \leq n$ is an invertible matrix, $|S_{11}(0)|_\infty$ and $|S_{21}(0)|_\infty$ are sufficiently small.

**Remark 1.2.** Theorem 1.1 still holds if the assumption (1.12) is extended to a more general case that

$$A_0(U)Q(U) + Q_0^T(U)A_0(U) \leq -P^T(U) \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{R}(U) \end{pmatrix} P(U),$$  \hfill (1.26)

where $\mathbf{R}(U) \in \mathcal{M}_{r,r}(\mathbb{R})$ is a symmetric positive definite matrix.
Remark 1.3. The conditions on the feedback matrix (1.21) and (1.22) are satisfied provided that $|K|_\infty$ is sufficiently small. Particularly, if $K$ is chosen as (1.16) where
\[
K_{00} = \kappa_+ I_{n-m}, \quad K_{11} = \kappa_- I_m, \quad K_{01} = 0, \quad K_{10} = 0,
\]
with two constants $\kappa_+$ and $\kappa_-$ satisfying
\[
\kappa_+^2 < \exp\left(-\max_{s=m+1,\ldots,n} \Lambda_s(0)\right) \quad \text{and} \quad \kappa_-^2 < \exp\left(\min_{r=1,\ldots,m} \Lambda_r(0)\right).
\]

Remark 1.4. Let us emphasize that the partially dissipative structure (1.11) and (1.12) combined with the dissipative boundary conditions (1.21) and (1.22) can be included by the interior and boundary stability conditions proposed in [2, Theorem 6.6]. However, the interior conditions on stability are typically differential matrix inequality, while the conditions proposed in this paper are all algebraic conditions which are more straightforward to be checked.

Remark 1.5. It is also worthy of mentioning that the stability conditions (both interior and boundary conditions) for the nonlinear hyperbolic systems depend on the topology and in particular that the stability in $H^2$ norm does not imply the stability in $C^1$ norm (see [5]).

The paper is organized as follows: in Section 2, we introduce a new quasilinear hyperbolic system with a simpler structure under a transformation of the unknown functions. Then in Section 3, we construct a weighted $H^2$-Lyapunov function to prove the exponential stability of the new system which implies immediately Theorem 1.1. The proofs of related lemmas will be given in Section 4. Finally, in Section 5, the main result is applied to the Saint-Venant-Exner model for moving water on an open canal under physical boundary conditions.

2 Transformation of the system

In this section, we introduce a new hyperbolic system with a partially dissipative but simpler structure under a transformation of the unknown functions. In this way, the exponential stability of the original system is reduced to that of the new system.

Let
\[
V = P(0)U.
\]

Then, the system (1.1) can be reduced to
\[
V_t + A(V)V_x = B(V),
\]
where
\[
A(V) = P(0)A(P^{-1}(0)V)P^{-1}(0) \quad \text{and} \quad B(V) = P(0)Q(P^{-1}(0)V).
\]
Clearly, \( V = 0 \) is an equilibrium of (2.2) and the Jacobian matrix of \( B \) with respect to \( V \) at the equilibrium can be calculated as
\[
B_{V}(0) = P(0)QU(0)P^{-1}(0).
\] (2.4)

Let
\[
L(V) = L(P^{-1}(0)V)P^{-1}(0) \quad \text{and} \quad \Lambda(V) = A(P^{-1}(0)V).
\] (2.5)

It is easy to check that \( L(V) \) is the matrix composed of the left eigenvectors of \( A(V) \), i.e.,
\[
L(V)A(V) = \Lambda(V)L(V),
\] (2.6)
which implies that system (2.2) is a hyperbolic system without vanishing characteristic speeds.

Obviously, we have
\[
\Lambda(0) = \Lambda(0) = \text{diag}\{\Lambda_1(0), \ldots, \Lambda_n(0)\}.
\] (2.7)

Let
\[
A_0(V) = (P^{-1}(0))^T A_0(P^{-1}(0)V)P^{-1}(0).
\] (2.8)

Obviously, \( A_0(V) \) is a symmetric positive definite matrix satisfying
\[
A_0(V)A(V) = A^T(V)A_0(V).
\] (2.9)

Thanks to (2.4) and (2.8), the partially dissipative structure (1.11)-(1.12) for the original system (1.1) implies the following partially dissipative but simpler structure for system (2.2) at the equilibrium \( V = 0 \).
\[
B_{V}(0) = \begin{pmatrix} 0 & 0 \\ 0 & S(0) \end{pmatrix},
\] (2.10)
\[
A_0(0)B_{V}(0) + B_{V}^T(0)A_0(0) \leq - \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix}.
\] (2.11)

According to the structure (2.10) and (2.11), we write \( V(t, x) \) as
\[
V(t, x) = \begin{pmatrix} v_1(t, x) \\ v_2(t, x) \end{pmatrix} \quad \text{with} \quad v_1 \in \mathbb{R}^{n-r}, \quad v_2 \in \mathbb{R}^r
\] (2.12)
for further use.

From (1.13) and (2.5), we can see that the linear diagonal variables \( \xi(t, x) \) now becomes
\[
\xi(t, x) = L(0)V(t, x),
\] (2.13)
with (1.14), which implies that the boundary conditions are still given by (1.15).

The initial condition for the variable \( V \) is given by
\[
V(0, x) = V_0(x) = P(0)U_0(x), \quad x \in (0, 1).
\] (2.14)

In order to prove Thereom 1.1, it suffices to establish the \( H^2 \)-stabilization for the system (2.2), (2.14) and (1.15).
3 Proof of Theorem 1.1

In this section, we can find suitable conditions on the feedback matrix $K$ such that the closed-loop system (2.2), (2.14) and (1.15) is exponentially stable in $H^2((0,1))$-norm. Then Theorem 1.1 follows immediately.

Let $V_0$ with small $H^2((0,1); \mathbb{R}^n)$ norm be such that the $C^1$ compatibility conditions at $(t,x) = (0,0)$ and $(0,1)$ are satisfied. Let also $V \in C^0([0,T], H^2((0,1); \mathbb{R}^n))$ be the maximal classical solution of the problem (2.2), (2.14) and (1.15). We remark that we only prove the stabilization result for smooth solutions while the conclusion follows easily from an density classical solution of the problem (2.2), (2.14) and (1.15).

Motivated by [2] and [11], we construct a weighted Lyapunov function as follows:

$$V(t) \triangleq L_0(t) + L_1(t) + L_2(t)$$

with

$$L_0(t) \triangleq \int_0^1 V^T (\alpha A_0(V) + L^T(V)e^{-\Lambda(0)x}L(V))V \, dx,$$

$$L_1(t) \triangleq \int_0^1 V^T (\alpha A_0(V) + L^T(V)e^{-\Lambda(0)x}L(V))V_t \, dx,$$

$$L_2(t) \triangleq \int_0^1 V^T (\alpha A_0(V) + L^T(V)e^{-\Lambda(0)x}L(V))V_{tt} \, dx,$$

where $e^{-\Lambda(0)x} = \text{diag}\{e^{-\Lambda_1(0)x}, \cdots , e^{-\Lambda_n(0)x}\}$, $\alpha > 0$ is a constant to be chosen later.

For the simplicity of statements, we denote the $\| \cdot \|_{L^2(0,1)}$ norm as $\| \cdot \|$, $\| \cdot \|_{C^0([0,1])}$ norm as $| \cdot |_{0}$.

By definition of the Lyapunov function $L(t)$, $L(t)$ is equivalent to the energy $\|V\|^2 + \|V_t\|^2 + \|V_{tt}\|^2$ if $|V(t,\cdot)|_0$ is small. On the other hand, Differentiation of system (2.2) with respect to $t$ and $x$ gives that

$$V_{tt} + A(V)V_{tx} = B_V(V)V_t - (A'(V)V_t)V_x,$$

$$V_{tx} + A(V)V_{xx} = B_V(V)V_x - (A'(V)V_x)V_x,$$

in which $A'(V)V_t, A'(V)V_x$ are matrices with entries $\frac{\partial A_i(V)}{\partial x_j}V_t, \frac{\partial A_i(V)}{\partial x_j}V_x$ respectively. Then it is easy to see that if $L(t)$ is equivalent to the energy $\|V(t,\cdot)\|^2_{H^2((0,1); \mathbb{R}^n)}$ if $|V(t,\cdot)|_1$ is small.

Next, we turn to estimate the time derivative of $L(t)$. For this purpose, we can establish the following lemmas with the assumptions (1.21)-(1.22) on the feedback matrix $K$. The proof of the lemmas will be given in Section 4.

**Lemma 3.1.** There exist positive constants $\alpha_0, \beta_0, \gamma_0$ and $\delta_0$ independent of $V$ such that if $|V(t,\cdot)|_0 \leq \delta_0$,

$$L'_0(t) \leq -\alpha_0 \|v_1\|^2 + (\beta_0 - \alpha)\|v_2\|^2 + \gamma_0 \int_0^1 (|V|^3 + |V|^2|V_t|) \, dx,$$

where $v_1, v_2$ is defined in (2.12).
Lemma 3.2. There exist positive constants $\alpha_1, \beta_1, \gamma_1$ and $\delta_1$ independent of $V$ such that if $|V(t, \cdot)|_0 \leq \delta_1$,
\[ L'_1(t) \leq -\alpha_1||v_{1t}||^2 + (\beta_1 - \alpha)||v_{2t}||^2 + \gamma_1 \int_0^1 (|V_1|^3 + |V||V_t|^2) \, dx. \] (3.8)

Lemma 3.3. There exist positive constants $\alpha_2, \beta_2, \gamma_2$ and $\delta_2$ independent of $V$ such that if $|V(t, \cdot)|_1 \leq \delta_2$,
\[ L'_2(t) \leq -\alpha_2||v_{1t}||^2 + (\beta_2 - \alpha)||v_{2t}||^2 + \gamma_2 \int_0^1 (|V_1||V_t|^2 + |V||V_t|^2 + |V_t|^2|V_{tt}|) \, dx. \] (3.9)

With the help of Lemmas 3.1, 3.2 and 3.3 we are ready to prove Theorem 1.1.

Let the constants $\alpha > \max\{\beta_0, \beta_1, \beta_2\}$ and $\delta_4 \leq \min\{\delta_0, \delta_1, \delta_2, \delta_3\}$. The combination of (3.7)–(3.9) yields that there exist positive constants $\beta$ and $\gamma$ such that
\[ L'(t) \leq -\beta L(t) + \gamma |V(t, \cdot)|_1 L(t), \] (3.10)
if $|V(t, \cdot)|_1 \leq \delta_4$.

Let $\delta_5 \triangleq \min\{\delta_4, \frac{\beta}{2}\}$. If we assume in a priori that $|V(t, \cdot)|_1 \leq \delta_5$ for $t \in (0, T)$, we get
\[ L'(t) \leq -\frac{\beta}{2} L(t), \quad t \in (0, T) \] (3.11)
which implies that $L(t)$ decays exponentially
\[ L(t) \leq e^{-\frac{\beta t}{2}} L(0), \quad t \in (0, T). \] (3.12)

Using the equivalence of the energy $||V(t, \cdot)||^2_{H^2((0,1)\times\mathbb{R}^n)}$ and $L(t)$, we obtain
\[ ||V(t, \cdot)||_{H^2((0,1)\times\mathbb{R}^n)} \leq C_1 e^{-\frac{\beta t}{2}} ||V_0||_{H^2((0,1)\times\mathbb{R}^n)}, \quad \forall t \in [0, T) \] (3.13)
for some constant $C_1 > 0$.

Note also the Sobolev inequality implies
\[ |V(t, \cdot)|_1 \leq C_2 ||V(t, \cdot)||_{H^2((0,1)\times\mathbb{R}^n)} \leq C_2 C_1 ||V_0||_{H^2((0,1)\times\mathbb{R}^n)}, \quad \forall t \in [0, T). \] (3.14)

Let now
\[ \delta = \frac{\delta_5}{C_2 C_1}, \] (3.15)
then the a priori estimate on $|V(t, \cdot)|_1 \leq \delta_5$ indeed holds in $[0, T)$ if $||V_0||_{H^2((0,1)\times\mathbb{R}^n)} \leq \delta$. Therefore (3.13) follows immediately. According to Proposition 1.1 we finally conclude that the inequality (3.13) indeed holds for $T = +\infty$.

Consequently, it follows from (2.1) that the solution $U$ to the problem (1.1), (1.15) and (1.17) is locally exponentially stable for the $H^2$-norm. This concludes the proof of Theorem 1.1.
4 Proof of the Lemmas

4.1 Proof of Lemma 2.1

We calculate the time-derivative of $\mathbb{I}_0(t)$ defined by (3.2),
\[ \mathbb{I}_0'(t) = \int_0^1 2 \left( \alpha V^T A_0(V) V_t + V^T L^T(V) e^{-\Lambda(0)x} L(V) V_t \right) \, dx + \mathcal{O} \left( \int_0^1 |V|^2 |V_t| \, dx; |V(t, \cdot)|_0 \right). \] (4.1)

Here and hereafter $\mathcal{O}(X; Y)$ denotes the terms that for $X \geq 0$, $Y \geq 0$, there exist $C > 0$ and $\varepsilon > 0$ independent of $V$, $V_t$ and $V_{tt}$, satisfying
\[ Y \leq \varepsilon \Rightarrow |\mathcal{O}(X; Y)| \leq CX. \] (4.2)

Substituting the system (2.2) into (4.1), we have
\[ \mathbb{I}_0'(t) = \mathbb{I}_0 + J_0 + \mathcal{O} \left( \int_0^1 |V|^2 |V_t| \, dx; |V(t, \cdot)|_0 \right) \] (4.3)

where
\[ \mathbb{I}_0 \triangleq \int_0^1 2 \alpha \left( V^T A_0(V) B(V) - V^T A_0(V) A(V) V_x \right) \, dx, \] (4.4)
\[ J_0 \triangleq \int_0^1 2 \left( V^T L^T(V) e^{-\Lambda(0)x} L(V) B(V) - V^T L^T(V) e^{-\Lambda(0)x} L(V) A(V) V_x \right) \, dx. \] (4.5)

Let’s first estimate the term $\mathbb{I}_0$. Using (2.9), (2.11) and integrations by parts, we have
\[
\mathbb{I}_0 = \int_0^1 \alpha V^T \left( A_0(0) B_V(0) + B^T_V(0) A_0(0) \right) V - \left( \alpha V^T A_0(V) A(V) V \right)_x \, dx \\
+ \mathcal{O} \left( \int_0^1 (|V|^3 + |V|^2 |V_t|) \, dx; |V(t, \cdot)|_0 \right) \\
\leq -\alpha \|v_2\|^2 + \left[ -\alpha V^T A_0(V) A(V) V \right]_1^1 + \mathcal{O} \left( \int_0^1 (|V|^3 + |V|^2 |V_t|) \, dx; |V(t, \cdot)|_0 \right). \] (4.6)

Then we turn to estimate the term $J_0$. Linear approximation together with (2.6) implies
\[
J_0 = \int_0^1 2 V^T L^T(0) e^{-\Lambda(0)x} L(0) B_V(0) V \, dx - \int_0^1 V^T L^T(0) e^{-\Lambda(0)x} \Lambda^2(0) L(0) V \, dx \\
- \int_0^1 \left( V^T L^T(V) e^{-\Lambda(0)x} \Lambda L(V) V \right)_x \, dx + \mathcal{O} \left( \int_0^1 (|V|^3 + |V|^2 |V_t|) \, dx; |V(t, \cdot)|_0 \right). \] (4.7)

Note that the matrix $L^T(0) e^{-\Lambda(0)x} L(0)$ is symmetric and positive definite. We denote it as following block matrix $M$
\[
M(x) \triangleq \begin{pmatrix}
M_{11}(x) & M_{12}(x) \\
M_{12}^T(x) & M_{22}(x)
\end{pmatrix} \] (4.8)
according to the block matrix $B_V(0)$. Then by Cauchy-Schwarz inequality, we get for all $\varepsilon > 0$, that
\[
\int_0^1 2V^T L^T(0)e^{-\Lambda(0)x}L(0)B_V(0)V \, dx = \int_0^1 2v_1^2 M_{12}(x)S(0)v_2 + 2v_2^T M_{22}(x)S(0)v_2 \, dx \\
\leq \varepsilon ||v_1||^2 + C_\varepsilon ||v_2||^2, \quad (4.9)
\]
where $C_\varepsilon > 0$ is constant depending on $\varepsilon$. Because $L^T(0)e^{-\Lambda(0)x}A^2(0)L(0)$ is positive definite, there exists a constant $c_0 > 0$ such that
\[
- \int_0^1 V^T L^T(0)e^{-\Lambda(0)x}A^2(0)L(0)V \, dx \leq -c_0 ||V||^2. \quad (4.10)
\]
Thus, it follows from (4.7), (4.9)-(4.10) and integrations by parts that
\[
J_0 \leq \varepsilon c_0 ||v_1||^2 + (C_\varepsilon - c_0)||v_2||^2 + \left[ - V^T L^T(V)e^{-\Lambda(0)x}A(V)L(V)V \right]_0^1 \\
+ \mathcal{O}\left( \int_0^1 (|V|^3 + |V|^2|V_t|) \, dx; |V(t, \cdot)|_0 \right). \quad (4.11)
\]
Let $\varepsilon = \frac{\omega}{\lambda}$. Then combining (4.3), (4.6) and (4.11) yields that there exist positive constants $\alpha_0, \beta_0$ independent of $V$, such that
\[
L'_0(t) \leq -\alpha_0 ||v_1||^2 + (\beta_0 - \alpha)||v_2||^2 + \mathcal{E}_0 + \mathcal{O}\left( \int_0^1 (|V|^3 + |V|^2|V_t|) \, dx; |V(t, \cdot)|_0 \right), \quad (4.12)
\]
where the boundary term
\[
\mathcal{E}_0 = -V^T(t, x)\left( \alpha A_0(V)A(V) + L^T(V)e^{-\Lambda(0)x}A(V)L(V) \right)V(t, x) \bigg|_0^1. \quad (4.13)
\]
It remains to estimate $\mathcal{E}_0$. Using (2.6), (2.13) and the linear approximation, we have
\[
\mathcal{E}_0 = -\xi^T(t, x)\left( \alpha (L^{-1}(0))^T A_0(0)L^{-1}(0)\Lambda(0) + e^{-\Lambda(0)x}\Lambda(0) \right)\xi(t, x) \bigg|_0^1 \\
+ \mathcal{O}\left( |V(t, 0)|^3 + |V(t, 1)|^3; |V(t, 0)| + |V(t, 1)| \right). \quad (4.14)
\]
Noting (2.5) and (2.8), (1.10), we easily obtain that
\[
(L^{-1}(0))^T A_0(0)L^{-1}(0) = (L^{-1}(0))^T A_0(0)L^{-1}(0) = \begin{pmatrix} X_1(0) & 0 \\ 0 & X_2(0) \end{pmatrix}. \quad (4.15)
\]
Substituting the boundary condition (1.15) and (4.15) into (4.14), we thus get
\[
\mathcal{E}_0 = -\begin{pmatrix} \xi_+(t, 1) \\ \xi_-(t, 0) \end{pmatrix}^T \mathbf{G} \begin{pmatrix} \xi_+(t, 1) \\ \xi_-(t, 0) \end{pmatrix} + \mathcal{O}\left( |V(t, 0)|^3 + |V(t, 1)|^3; |V(t, 0)| + |V(t, 1)| \right) \quad (4.16)
\]
where $\mathbf{G}$ is a symmetric matrix defined as
\[
\mathbf{G} \triangleq \alpha \begin{pmatrix} \begin{pmatrix} X_2(0)\Lambda_+(0) \\ 0 \end{pmatrix} & 0 \\ \begin{pmatrix} X_2(0)\Lambda_+(0) \\ 0 \end{pmatrix} & -X_1(0)\Lambda_-(0) \end{pmatrix} - \mathbf{K}^T \begin{pmatrix} \begin{pmatrix} X_2(0)\Lambda_+(0) \\ 0 \end{pmatrix} & 0 \\ \begin{pmatrix} X_2(0)\Lambda_+(0) \\ 0 \end{pmatrix} & -X_1(0)\Lambda_-(0) \end{pmatrix} \mathbf{K}
\]
Using (2.13) and the boundary condition (1.15), we have

\[ + \left[ \begin{array}{cc} e^{-\Lambda^+(0)} & 0 \\ 0 & -\Lambda^-(0) \end{array} \right] - K^T \left[ \begin{array}{cc} \Lambda^+(0) & 0 \\ 0 & -e^{-\Lambda^-(0)} \end{array} \right] K \right]. \quad (4.17) \]

Consequently, (4.16) becomes

\[ V(t, 0) = L^{-1}(0)\xi(t, 0) = L^{-1}(0) \left( \begin{array}{c} 0 \\ K_{00} K_{01} \end{array} \right) \left( \begin{array}{c} \xi_+(t, 1) \\ \xi_-(t, 0) \end{array} \right) \quad (4.18) \]

\[ V(t, 1) = L^{-1}(0)\xi(t, 1) = L^{-1}(0) \left( \begin{array}{cc} K_{10} & K_{11} \\ I_{n-m} & 0 \end{array} \right) \left( \begin{array}{c} \xi_+(t, 1) \\ \xi_-(t, 0) \end{array} \right) \quad (4.19) \]

Consequently, (4.16) becomes

\[ B_0 = -\left( \begin{array}{c} \xi_+(t, 1) \\ \xi_-(t, 0) \end{array} \right)^T G \left( \begin{array}{c} \xi_+(t, 1) \\ \xi_-(t, 0) \end{array} \right) + \mathcal{O}\left( \left( |V(t, 0)| + |V(t, 1)| \right) \left( \begin{array}{c} \xi_+(t, 1) \\ \xi_-(t, 0) \end{array} \right)^2 \right) ; |V(t, 0)| + |V(t, 1)| \right). \quad (4.20) \]

Note that the assumptions (1.21)–(1.22) on the boundary feedback matrix \( K \) implies the symmetric matrix \( G \) is positive definite. Therefore, there exist \( \delta_0 > 0 \) and \( \gamma_0 > 0 \) such that the boundary term \( B_0 \leq 0 \), and furthermore the estimate (3.7) holds if \( |V(t, \cdot)|_0 < \delta_0 \). This concludes the proof of Lemma 3.1.

### 4.2 Proof of Lemma 2.2

By (3.3), the time-derivative of \( L_1(t) \) can be expressed as

\[ L_1'(t) = \int_0^1 2\left( \alpha V_t^T A_0(V)V_t + V_t^T L_t^T (V)e^{-\Lambda^0 x}L(V)V_t \right) dx + \mathcal{O}\left( \int_0^1 |V_t|^3 dx ; |V(t, \cdot)|_0 \right). \quad (4.21) \]

Substituting the term of \( V_t \) derived from (3.5) into (4.21), we have

\[ L_1'(t) = \int_0^1 2\alpha V_t^T A_0(V) \left( B_V(V)V_t - (A'(V)V_t)V_x - A(V)V_{tx} \right) dx \]
\[ + \int_0^1 2V_t^T L_t(V)e^{-\Lambda^0 x}L(V) \left( B_V(V)V_t - (A'(V)V_t)V_x - A(V)V_{tx} \right) dx \]
\[ + \mathcal{O}\left( \int_0^1 |V_t|^3 dx ; |V(t, \cdot)|_0 \right) \]

Thus, by using integrations by parts and some straightforward calculations, we get

\[ L_1'(t) = \int_0^1 \alpha V_t^T \left( A_0(0)B_V(0) + B_V(0)^T A_0(0) \right)V_t - V_t^T L_t^T (0)e^{-\Lambda^0 x}A^2(0)L(0)V_t dx \]
\[ + \int_0^1 2V_t^T L_t^T (0)e^{-\Lambda^0 x}L(0)B_V(0)V_t dx - V_t^T \left( \alpha A_0(V)A(V) + L_t(V)e^{-\Lambda^0 x}A(V)L(V) \right)V_t^1 \]
\[ + \mathcal{O}\left( \int_0^1 (|V_t|^3 + |V||V_t|^2) dx ; |V(t, \cdot)|_0 \right). \]
Similarly as the analysis of $\mathbb{L}_t'(t)$ in the proof of Lemma 3.1 we obtain that there exist positive constants $\alpha_1$ and $\beta_1$, such that

$$\mathbb{L}_t'(t) \leq -\alpha_1 \|v_{1t}\|^2 + (\beta_1 - \alpha)\|v_{2t}\|^2 + \mathbb{B}_1 + \mathcal{O}\left(\int_0^1 (|V_t|^3 + |V| |V_t|^2) \, dx; |V(t, \cdot)|_0\right),$$  \hspace{1cm} (4.22)

where the boundary term $\mathbb{B}_1$ is

$$\mathbb{B}_1 = -\xi_t^T(t, x) \left(\alpha(L^{-1}(0))^T A_0(0)L^{-1}(0)A(0) + e^{-\Lambda(0)x} \Lambda(0)\right) \xi_t(x, t) \bigg|_0^1 + \mathcal{O}\left(\|V(t, 0)||V(t, 0)||V(t, 1)||V_t(1)||^2; |V(t, 0)| + |V(t, 1)|\right).$$  \hspace{1cm} (4.23)

Taking the time-derivative (1.15) and (1.18)-(4.19), we can easily express $(V_t(t, 0), V_t(t, 1))$ in terms of $(\xi_{+t}(t, 1), \xi_{-t}(t, 0))$, thus the boundary term $\mathbb{B}_1$ can be rewritten as

$$\mathbb{B}_1 = -\begin{pmatrix} \xi_{+t}(t, 1) \\ \xi_{-t}(t, 0) \end{pmatrix}^T \mathbf{G} \begin{pmatrix} \xi_{+t}(t, 1) \\ \xi_{-t}(t, 0) \end{pmatrix} + \mathcal{O}\left(\|V(t, 0)| + |V(t, 1)|\right) \left| \begin{pmatrix} \xi_{+t}(t, 1) \\ \xi_{-t}(t, 0) \end{pmatrix} \right|^2; |V(t, 0)| + |V(t, 1)|.$$

(4.24)

Note again with the assumption (1.21) can imply that $\mathbf{G}$ is positive definite. Therefore, there exists $\delta_1 > 0$ and $\gamma_1 > 0$ such that $\mathbb{B}_1 \leq 0$, and furthermore the estimate (3.8) holds if $|V(t, \cdot)|_0 < \delta_1$. The finishes the proof of Lemma 3.2

### 4.3 Proof of Lemma 2.3

Calculating the time-derivative of $\mathbb{L}_2(t)$ gives

$$\mathbb{L}_2'(t) = \int_0^1 2 \left(\alpha V_t^T A_0(V) V_{tt} + V_t^T L^T(V) e^{-\Lambda(0)x} \Lambda(0) V_{tt} \right) \, dx + \mathcal{O}\left(\int_0^1 |V_t||V_{tt}|^2 \, dx; |V(t, \cdot)|_0\right).$$

(4.25)

Differentiating system (3.5) with respect to $t$ and combining (2.2) and (3.5), we have,

$$V_{tt} + A(V) V_{tx} = B_V(V) V_t + (B_V(V))_t V_t - 2(A'_t(V) V_t) V_{tx} - (A'_{tt}(V) V_t) V_x.$$  \hspace{1cm} (4.26)

Substituting the term of $V_{tt}$ derived from (1.26) into $\mathbb{L}_2'(t)$, we do integration by parts and linear approximation, as in the proof of Lemma 3.1 and 3.2, to deduce that

$$\mathbb{L}_2'(t) = \int_0^1 \alpha V_t^T \left(\alpha A_0(0) B_V(0) + B_V(0)^T A_0(0)\right) V_t - V_t^T L^T(0) e^{-\Lambda(0)x} \Lambda(0) L(0) V_t \, dx$$

$$+ \int_0^1 2 V_t^T L^T(0) e^{-\Lambda(0)x} L(0) V_t \, dx - V_t^T \left(\alpha A_0(0) A(V) + L^T(V) e^{-\Lambda(0)x} \Lambda(V) L(V)\right) V_t \bigg|_0^1 + \mathcal{O}\left(\int_0^1 |V||V_{tt}|^2 + |V_t||V_{tt}|^2 + |V_t|^2|V_{tt}| \, dx; |V(t, \cdot)|_1\right).$$

(4.27)

Thanks to the partially dissipative structure (2.10) and (2.11) and the fact that $L^T(0) e^{-\Lambda(0)x} \Lambda(2)(0) L(0)$ is positive definite, there exist positive constants $\alpha_2$ and $\beta_2$ independent of $V$ such that

$$\mathbb{L}_2'(t) \leq -\alpha_2 \|v_{1tt}\|^2 + (\beta_2 - \alpha)\|v_{2tt}\|^2 + \mathbb{B}_2$$
where \( \mathbb{B}_2 \) denotes the boundary term derived from integration by parts. Taking the second time-derivative of (1.15) and (4.18)-(4.19), we can rewrite the boundary term \( \mathbb{B}_2 \) as

\[
\mathbb{B}_2 = - \begin{pmatrix} \xi_{+u}(t,1) \\ \xi_{-u}(t,0) \end{pmatrix}^T \mathbf{G} \begin{pmatrix} \xi_{+u}(t,1) \\ \xi_{-u}(t,0) \end{pmatrix} + \mathcal{O} \left( (|V(t,0)| + |V(t,1)|) \left( \left| \frac{\xi_{+u}(t,1)}{|V(t,0)|} \right| + \left| \frac{\xi_{-u}(t,0)}{|V(t,1)|} \right| \right)^2 \right).
\]

(4.29)

Note again that \( \mathbf{G} \) is positive definite if the assumptions (1.21)-(1.22) hold. Therefore, there exists \( \delta_2 > 0 \) and \( \gamma_2 > 0 \) such that \( \mathbb{B}_2 \leq 0 \) and furthermore the estimate (3.9) holds if \( |V(t, \cdot)|_1 < \delta_2 \). This ends the proof of Lemma 3.3.

5 Application to Saint-Venant-Exner equations

We now consider the Saint-Venant-Exner equations for a moving bathymetry on a sloping channel with a rectangular cross-section:

\[
\begin{align*}
H_t + V H_x + H V_x &= 0, \\
B_t + a V^2 V_x &= 0, \\
V_t + V V_x + g H_x + g B_x &= g S_b - C_f V^2.
\end{align*}
\]

(5.1)

Here \( H = H(t, x) \) is the water depth, \( B = B(t, x) \) is the elevation of the sediment bed and \( V = V(t, x) > 0 \) is the average velocity of water. Moreover, \( g \) is the gravity constant, constant \( S_b \) is the bottom slope of the channel, constant \( C_f \) means the friction coefficient and constant \( a \) is a parameter that includes porosity and viscosity effects on the sediment dynamics (see [13]).

Let \((H_*, B_*, V_*)^T\) with \( H_*>0, B_*>0 \) and \( V_*>0 \) be a constant equilibrium of (5.1), i.e.,

\[
g S_b H_* = C_f (V_*)^2 > 0.
\]

(5.2)

Let \( U \triangleq (h, b, v)^T \) be the deviations of the states:

\[
h(t, x) = H(t, x) - H_*, \quad b(t, x) = B(t, x) - B_*, \quad v(t, x) = V(t, x) - V_*.
\]

(5.3)

Then the Saint-Venant-Exner equations (5.1) can be rewritten in the form of (1.1) with

\[
\mathbf{A}(U) = \begin{pmatrix} v + V_* & 0 & h + H_* \\ 0 & 0 & a(v + V_*)^2 \\ g & g & v + V_* \end{pmatrix}, \quad \mathbf{Q}(U) = \begin{pmatrix} 0 & 0 \\ g S_b - C_f (v + V_*)^2 \end{pmatrix}.
\]

(5.4)

In order to apply Theorem 1.1 we will verify that the hyperbolic system satisfies the partially dissipative structure. For simplicity, we only show that (1.3), (1.8), (1.11) and (1.12) are satisfied at the equilibrium \( U = 0 \).
First, the matrix $A(0)$ has three eigenvalues $\lambda_i$ ($i = 1, 2, 3$) satisfying
\[ \lambda^3 - 2V_*\lambda^2 + (V_*^2 - gaV_*^2 - gH_*)\lambda + gaV_*^3 = 0. \]  
(5.5)

Thus we get the following relations
\[ \lambda_1\lambda_2\lambda_3 = -gaV_*^3, \]
\[ \lambda_1 + \lambda_2 + \lambda_3 = 2V_*, \]
\[ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = V_*^2 - gaV_*^2 - gH*. \]  
(5.6)

Here $\lambda_1, \lambda_3$ are the characteristic velocities of the water flow and $\lambda_2$ is the characteristic velocity of the sediment motion. Therefore, (5.2) and (5.6) yield that matrix $A(0)$ has no vanishing eigenvalues with
\[ \lambda_1 < 0 < \lambda_2 \leq \lambda_3. \]  
(5.7)

Due to the fact that the sediment motion is much slower than the water flow, we can make the following reasonable assumptions that
\[ \lambda_2 < -\lambda_1 \quad \text{and} \quad 0 < \lambda_2 < \frac{3}{2}V_*, \]  
(5.8)

which leads to the following relations
\[ \lambda_1 - \frac{3}{2}V_* < 0, \quad \lambda_2 - \frac{3}{2}V_* < 0, \quad \lambda_3 - \frac{3}{2}V_* = \frac{1}{2}V_* - \lambda_1 - \lambda_2 > 0. \]  
(5.9)

Next, we choose an invertible matrix $P(0)$ and a symmetric positive definite matrix $A_0(0)$, such that (1.8), (1.11) and (1.12) are satisfied. Inspired by [11], we take
\[ P(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{V_*}{2H_*} & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_0(0) = \begin{pmatrix} 4gH_* + 2agV_*^2 & -\frac{g}{2H_*} & -\frac{V_*}{2H_*} \\ -\frac{g}{2H_*} & \frac{3g}{2V_*} & 0 \\ -\frac{V_*}{2H_*} & 0 & 1 \end{pmatrix}. \]  
(5.10)

According to (5.6) and (5.9), it is easy to verify that
\[ \det(A_0(0)) = \frac{3g^2}{2aH_*V_*^2} + \frac{g^2}{2H_*^2} - \frac{3g}{8a^2H_*^2} = \frac{g}{aH_*^2V_*^3} \prod_{i=1}^{3} \left( \lambda_i - \frac{3}{2}V_* \right) > 0. \]  
(5.11)

Therefore, $A_0(0)$ is symmetric positive definite. It follows also that $A_0(0)A(0) = A^T(0)A_0(0)$.

Note that
\[ Q_U(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_f \frac{V_*^2}{H_*} & 0 & -2C_f \frac{V_*}{H_*} \end{pmatrix}. \]  
(5.12)

Direct computations give that
\[ P(0)Q_U(0)P^{-1}(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -C_f \frac{V_*}{H_*} \end{pmatrix}. \]  
(5.13)
\[(P^{-1}(0))^T (A_0(0)Q_U(0) + Q_U^T(0)A_0(0))P^{-1}(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4Ct \end{pmatrix} \] (5.14)

i.e., the partially dissipative structure indeed holds.

Let \( L_i(0) \) be the left eigenvectors corresponding to \( \lambda_i \) \( (i = 1, 2, 3) \), then we have

\[
L(0) = \begin{pmatrix} L_1(0) \\ L_2(0) \\ L_3(0) \end{pmatrix} = \begin{pmatrix} \frac{g}{\lambda_1 - V_s} & \frac{g}{\lambda_2} & 1 \\ \frac{g}{\lambda_3 - V_s} & \frac{g}{\lambda_3} & 1 \end{pmatrix} \] (5.15)

and further

\[
L^{-1}(0) = \begin{pmatrix} \frac{\lambda_1H_r}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} & \frac{\lambda_2H_r}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} & \frac{\lambda_3H_r}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\ \frac{aV_s^2(\lambda_1 - V_s)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} & \frac{aV_s^2(\lambda_2 - V_s)}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} & \frac{aV_s^2(\lambda_3 - V_s)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\ \frac{\lambda_1(\lambda_2 - V_s)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} & \frac{\lambda_2(\lambda_3 - V_s)}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} & \frac{\lambda_3(\lambda_3 - V_s)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{pmatrix} \] (5.16)

Consequently, we have

\[
(L^{-1}(0))^T A_0(0)L^{-1}(0) = \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{pmatrix} \] (5.17)

where

\[
X_{11} = \frac{\lambda_1(\lambda_1 - \frac{3}{2}V^*)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad X_{22} = \frac{\lambda_2(\lambda_2 - \frac{3}{2}V^*)}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}, \quad X_{33} = \frac{\lambda_3(\lambda_3 - \frac{3}{2}V^*)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \] (5.18)

For the Saint-Venant-Exner equations [5.1], we introduce the following physical boundary conditions

\[
\begin{aligned}
&b(t, 0) = 0, \\
v(t, 0) = -k_1h(t, 0), \\
v(t, 1) = -k_2(h(t, 1) + b(t, 1)).
\end{aligned} \] (5.19)

where \( k_1 \) and \( k_2 \) are two feedback parameters.

Then the boundary conditions [5.19] can be rewritten in the form of its linearized diagonal variable \( \xi \) defined by [1.13] as

\[
\begin{pmatrix} \xi_2(t, 0) \\ \xi_3(t, 0) \\ \xi_1(t, 1) \end{pmatrix} = K \begin{pmatrix} \xi_2(t, 1) \\ \xi_3(t, 1) \\ \xi_1(t, 0) \end{pmatrix}
\]

with

\[
K = \begin{pmatrix} 0 & 0 & \pi_2(k_1) \\ 0 & 0 & \pi_3(k_1) \\ \chi_2(k_2) & \chi_3(k_2) & 0 \end{pmatrix}
\] (5.20)

where \( \pi_j \) and \( \chi_j \) \( (j = 2, 3) \) are the following quantities depending on \( k_1 \) and \( k_2 \):

\[
\pi_j(k_1) = \frac{\lambda_1 - V_s}{\lambda_j - V_s} \cdot \frac{g - k_1(\lambda_j - V_s)}{g - k_1(\lambda_1 - V_s)} \quad (j = 2, 3),
\] (5.21)
\[ \chi_2(k_2) = \frac{\lambda_2(\lambda_3 - \lambda_1)(\lambda_2 - V_*)}{\lambda_1(\lambda_3 - \lambda_2)(\lambda_1 - V_*)} \cdot \frac{g + k_2(\lambda_2 - V_*)}{g + k_2(\lambda_1 - V_*)}, \]  
\[ \chi_3(k_2) = \frac{\lambda_3(\lambda_1 - \lambda_2)(\lambda_3 - V_*)}{\lambda_1(\lambda_3 - \lambda_2)(\lambda_1 - V_*)} \cdot \frac{g + k_2(\lambda_3 - V_*)}{g + k_2(\lambda_1 - V_*)}. \]  

Consequently, the dissipative boundary conditions (5.21) and (5.22) yield

\[ \pi^2(k_1) \frac{X_{22}}{X_{11}} \frac{\lambda_2}{|\lambda_1|} + \pi^2(k_1) \frac{X_{33}}{X_{11}} \frac{\lambda_3}{|\lambda_1|} \leq 1, \]  
\[ \chi^2(k_2) \frac{X_{11}}{X_{22}} \frac{|\lambda_1|}{\lambda_2} + \chi^2(k_2) \frac{X_{11}}{X_{33}} \frac{|\lambda_1|}{\lambda_3} \leq 1 \]  

and

\[ \pi^2(k_1) \frac{\lambda_2}{|\lambda_1|} + \pi^2(k_1) \frac{\lambda_3}{|\lambda_1|} \leq 1, \]  
\[ \chi^2(k_2) e^{\lambda_2 - \lambda_1} \frac{|\lambda_1|}{\lambda_2} + \chi^2(k_2) e^{\lambda_3 - \lambda_1} \frac{|\lambda_1|}{\lambda_3} \leq 1. \]  

Let

\[ \beta_j = \max \left\{ \frac{X_{jj}}{X_{11}}, 1 \right\}, \eta_j = \max \left\{ \frac{X_{11}}{X_{jj}}, e^{\lambda_j - \lambda_1} \right\}, \quad (j = 2, 3). \]  

Then, (5.24), (5.25), (5.26) and (5.27) are satisfied if

\[ \pi^2(k_1) \frac{\lambda_2}{|\lambda_1|} + \pi^2(k_1) \frac{\lambda_3}{|\lambda_1|} \leq 1, \]  
\[ \chi^2(k_2) e^{\lambda_2 - \lambda_1} \frac{|\lambda_1|}{\lambda_2} + \chi^2(k_2) e^{\lambda_3 - \lambda_1} \frac{|\lambda_1|}{\lambda_3} \leq 1. \]  

Finally we conclude by Theorem 1.1 that

**Theorem 5.1.** If the boundary feedback parameters \( k_1 \) and \( k_2 \) satisfy (5.29) and (5.30), respectively, then the constant equilibrium \((H_*, B_*, V_*)^T\) of the Saint-Venant-Exner system (5.1), (5.19) is locally exponentially stable for the \( H^2 \)-norm.

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