On the Estimation of the Convergence Rate in the Janashia-Lagvilava Spectral Factorization Algorithm

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The spectral factorization problem originated in Wiener’s and Kolmogorov’s papers and its solution has an immediate impact on various fields of applied sciences. This is the reason that there are numerous attempts of different mathematicians to create a sound computational method of such factorization. The Janashia-Lagvilava matrix spectral factorization algorithm [1] is the first method which uses the theory of Hardy spaces $H_p$ for solution of the problem. This naturally turned out to be very fruitful since the problem itself is posed in this branch of mathematics.

A matrix spectral density

\[ S(z) = \begin{pmatrix} a(z) & b(z) \\ b(z) & c(z) \end{pmatrix}, \]

$z \in \mathbb{T}$, which admits the spectral factorization $S(z) = \chi^+(z)(\chi^+(z))^\ast$ is approximated in this algorithm by the matrix-function

\[ S_N(z) = \begin{pmatrix} f_1^+(z) & 0 \\ \varphi_N(z) & f^+(z) \end{pmatrix} \begin{pmatrix} f_1^+(z) & \varphi_N(z) \\ 0 & f^+(z) \end{pmatrix}, \]

where $f_1^+(z)$ and $f^+(z)$ are scalar spectral factors of $a(z)$ and $\frac{\det S(z)}{a(z)}$, respectively, and $\varphi_N(z) = \sum_{n=-N}^{\infty} c_n(\varphi)z^n$ ($\varphi = b/f_1^+$ and $c_n(\varphi)$ is the $n$-th Fourier coefficient of $\varphi$). The spectral factorization $S_N(z) = \chi_N^+(z)(\chi_N^+(z))^\ast$ is written explicitly in [1] and it is proved that

\[ \|\chi_N^+(z) - \chi^+(z)\|_{H_2} \rightarrow 0. \]

We have obtained a quantitative estimation of the convergence in (2) under the certain restriction on (1), namely when $S^{-1}(z) \in L_1$.

**Theorem.** $\|\chi_N^+ - \chi^+\|_{H_2} \leq (1 + \sqrt{2}\|S\|_{L_1}\|S^{-1}\|_{L_1})\|\varphi_N - \varphi\|_{L_2}$.

This type of estimation might be crucial in specific cases for actual computation of the spectral factor within a given accuracy.

**References**

[1] G. Janashia and E. Lagvidava, “A method of approximate factorization of positive definite matrix functions”, *Studia Math.*, Vol. 137, pp. 93–100, 1999.