k-DEPENDENCE, DISJOINT MATCHINGS, AND AN EXTENSION OF A THEOREM OF FAVARON

GREGORY J. PULEO

Abstract. A vertex set $D$ in a graph $G$ is $k$-dependent if $G[D]$ has maximum degree at most $k - 1$, and $k$-dominating if every vertex outside $D$ has at least $k$ neighbors in $D$. Favaron [2] proved that if $D$ is a $k$-dependent set maximizing the quantity $k|D| - |E(G[D])|$, then $D$ is $k$-dominating. We extend this result, showing that such sets satisfy a stronger property: given any ordering $<$ of $V(G) - D$, there is a $k$-edge-chromatic subgraph of $G$ in which every vertex outside $D$ has degree at least $k - d_v - (D)$, where $d_v(D)$ is the number of earlier neighbors of $v$ in $V(G) - D$. Since any vertex outside $D$ may be taken as a minimal element of $<$, this implies, in particular, that $D$ is $k$-dominating. We apply this result to prove that when $G$ is chordal and $D$ is as above, then $G$ has a $k$-edge-chromatic subgraph in which every vertex outside $D$ has degree $k$.

1. Introduction

A vertex set $D$ in a graph $G$ is independent if the induced subgraph $G[D]$ has no edges. A vertex set is dominating if every vertex of $G$ either lies in the set, or has a neighbor in the set. Ore [10] observed that any maximal independent set is also a dominating set: by the maximality of the independent set, every vertex outside the set must have a neighbor in the set. Thus, $\gamma(G) \leq \alpha(G)$ for any graph $G$, where $\gamma(G)$ is the size of a smallest dominating set and $\alpha(G)$ is the size of a largest independent set.

Fink and Jacobson [3, 4] generalized the notions of independence and domination as follows:

Definition 1.1. For positive integers $k$, a vertex set $D \subset V(G)$ is $k$-dependent if the induced subgraph $G[D]$ has maximum degree at most $k - 1$. A vertex set $D$ is $k$-dominating if $|N(v) \cap D| \geq k$ for all $v \in V(G) - D$.

Fink and Jacobson posed the following question: letting $\gamma_k(G)$ denote the size of a smallest $k$-dominating set in $G$ and letting $\alpha_k(G)$ denote the size of a largest $k$-dependent set in $G$, is it true that $\gamma_k(G) \leq \alpha_k(G)$ for all $k$? Setting $k = 1$ yields the original inequality $\gamma(G) \leq \alpha(G)$. However, for $k > 1$ it is no longer true that every maximal $k$-dependent set is $k$-dominating. Favaron [2] answered the question of Fink and Jacobson, using a different notion of “optimality” for $k$-dependent sets:

Theorem 1.2 (Favaron [2]). If $D$ is a $k$-dependent set maximizing the quantity $k|D| - |E(G[D])|$ (over all $k$-dependent sets), then $D$ is a $k$-dominating set.

Since any set of at most $k$ vertices is a $k$-dependent set, it follows that every graph has a set of vertices which is both $k$-dependent and $k$-dominating, which yields $\gamma_k(G) \leq \alpha_k(G)$. Theorem 1.2 motivates the following definition.

Date: July 10, 2014.
Definition 1.3. For any $k$-dependent set $D$, define $\phi_k(D) = k|D| - |E(G[D])|$. A $k$-optimal set is a $k$-dependent set maximizing $\phi_k$.

The notation $\phi_k(D)$ is borrowed from the survey paper [1]. Our goal in this paper is to extend Theorem 1.2 by proving that $k$-optimal sets satisfy a property stronger than $k$-domination:

**Theorem 1.4.** Let $D$ be a $k$-optimal set in a graph $G$, let $X = V(G) - D$, and let $H$ be the maximal bipartite subgraph of $G$ with partite sets $D$ and $X$. If $<$ is an ordering of $X$, then $H$ has a $k$-edge-chromatic subgraph $M$ such that $d_M(v) + d^-(v) \geq k$ for all $v \in X$, where $d^-(v) = |\{w \in N(v) \cap X : w < v\}|$.

We call $d^-(v)$ the backdegree of $v$. In particular, since we can take any vertex $v \in V(G) - D$ to be minimal in $<$, Theorem 1.4 implies that any $k$-optimal set is $k$-dominating. In fact, we have the following stronger corollary, obtained by taking the vertices of an independent set to be minimal in $<$:

**Corollary 1.5.** Let $D$ be a $k$-optimal set in a graph $G$. For any independent set $S$ disjoint from $D$, there are $k$ disjoint matchings of $S$ into $D$, each saturating $S$.

The rest of the paper is organized as follows. In Section 2 we build some intuition by exploring the $k = 1$ case. Based on a result that holds in the $k = 1$ case, we pose the following conjecture:

**Conjecture 1.6.** If $D$ is a $k$-optimal set in a graph $G$, then $G$ has a $k$-edge-chromatic subgraph in which every vertex of $V(G) - D$ has degree $k$.

In Section 3 we prove a generalization of a theorem of Lebensold concerning disjoint matchings in bipartite graphs. In Section 4 we apply the results of Section 3 to prove Theorem 1.4. Finally, in Section 5 we apply Theorem 1.4 to prove that Conjecture 1.6 holds for chordal graphs. The proof uses a variant of edge list coloring.

2. **The Case $k = 1$**

When $k = 1$, things are simpler: a 1-dependent set is just an independent set, so a 1-optimal set is just a maximum-size independent set. The statement of the theorem can also be simplified: when $k = 1$, the only vertices of $V(G) - D$ for which the theorem says anything are those of backdegree 0, which form an independent set. Thus, the $k = 1$ case of Theorem 1.4 is equivalent to the following proposition, which we prove using Hall’s Theorem [7].

**Proposition 2.1.** Let $D$ be a maximum-size independent set. If $T$ is an independent set disjoint from $D$, then there is a matching of $T$ into $D$ that saturates $T$.

**Proof.** Assume that no such matching exists; we will obtain an independent set $D'$ with $|D'| > |D|$. If no such matching exists, then by Hall’s Theorem, there is a set $S \subseteq T$ such that $|N(S) \cap D| < |S|$. Let $D' = (D - N(S)) \cup S$. Since $S$ and $D$ are independent and since we have deleted $N(S)$, the set $D'$ is also independent. Since $|N(S) \cap D| < |S|$, we have $|D'| > |D|$, as desired. 

The proof of Theorem 1.4 uses a similar strategy: assuming that the desired $k$-edge-chromatic subgraph does not exist, we obtain a set of vertices for which
a similar Hall-type condition fails, and we use this set of vertices to construct a “better” \( k \)-dependent set.

Proposition 2.1 also has an interesting corollary, which we have not been able to generalize to arbitrary \( k \).

**Corollary 2.2.** If \( D \) is a maximum-size independent set in a graph \( G \), then \( G \) has a matching that saturates every vertex of \( V(G) - D \).

**Proof.** Let \( M_1 \) be a maximal matching in \( V(G) - D \), and let \( S \) be the set of vertices in \( V(G) - D \) not saturated by \( M_1 \). Since \( M_1 \) is a maximal matching, \( S \) is an independent set. By Proposition 2.1 there is a matching \( M_2 \) of \( S \) into \( D \) that saturates \( S \). Thus, \( M_1 \cup M_2 \) is a matching that saturates \( V(G) - D \). \( \square \)

Theorem 1.4 suggests that a similar result should hold for higher \( k \):

**Conjecture 2.3.** If \( D \) is a \( k \)-optimal set in a graph \( G \), then \( G \) has a \( k \)-edge-chromatic subgraph in which every vertex of \( V(G) - D \) has degree \( k \).

In Section 5, we prove Conjecture 2.3 for chordal graphs, applying Theorem 1.4 to a simplicial elimination order on \( V(G) - D \). In general, it seems difficult to find a suitable order to use in Theorem 1.4.

### 3. Extending Lebensold’s Theorem

Lebensold [8] proved the following generalization of Hall’s Theorem [7]. As Brualdi observed in his review of [8], the theorem is equivalent to a theorem of Fulkerson [5] concerning disjoint permutations in 0,1-matrices. An alternative proof of the theorem, using matroid theory, is due to Murty [9].

**Theorem 3.1** (Lebensold [8]). An \( X, D \)-bigraph has \( k \) disjoint matchings from \( X \) into \( D \), each saturating \( X \), if and only if

\[
\sum_{v \in D} \min\{k, |N(v) \cap S|\} \geq k |S|
\]

for every subset \( S \subset X \).

We extend the theorem to find necessary and sufficient conditions for the existence of a \( k \)-edge-chromatic subgraph in which the vertices of \( X \) are allowed to have different degrees.

**Lemma 3.2.** Let \( H \) be an \( X, D \)-bigraph, and write \( X = \{v_1, \ldots, v_t\} \). Let \( k \) be a positive integer and let \( d_1, \ldots, d_t \) be nonnegative integers with all \( d_i \leq k \). The following are equivalent:

1. \( H \) has a \( k \)-edge-chromatic subgraph \( M \) such that \( d_M(v_i) \geq d_i \) for all \( i \);
2. For every subset \( S \subset X \),

\[
\sum_{v \in D} \min\{k, |N(v) \cap S|\} \geq \sum_{v_i \in S} d_i.
\]

Theorem 3.1 is the special case of Lemma 3.2 obtained when all \( d_i = k \). We prove Lemma 3.2 using Theorem 3.1 so Theorem 3.1 is self-strengthening in this sense.
Proof. For each $i$, let $D_i$ be a set of size $k - d_i$, with all sets $D_i$ disjoint from each other and disjoint from $V(H)$, and let $D' = D \cup D_1 \cup \cdots \cup D_i$. Let $H'$ be the $X, D'$-bigraph obtained from $H$ by making the vertices in $D_i$ adjacent only to $v_i$. Consider the following two statements:

(1') $H'$ has $k$ edge-disjoint matchings, each saturating $X$;
(2') For every subset $S \subseteq X$,

$$\sum_{v \in D'} \min\{k, |N(v) \cap S|\} \geq k |S|.$$ 

By Theorem 3.1, (1') is equivalent to (2'). We prove that (1) is equivalent to (1') and (2) is equivalent to (2').

If $M_1, \ldots, M_k$ are edge-disjoint matchings in $H'$ each saturating $X$, then their restriction to $H$ yields a $k$-edge-chromatic subgraph $M$ of $H$ with each $d_M(v_i) \geq d_i$. Conversely, any such subgraph of $H$ can be extended to $k$ edge-disjoint matchings in $H'$. Thus, (1) is equivalent to (1').

Elements of $D_i$ each contribute 1 to the sum in (2') when $v_i \in S$, and contribute 0 otherwise. This yields

$$\sum_{v \in D'} \min\{k, |N(v) \cap S|\} = \sum_{v_i \in S} (k - d_i) + \sum_{v \in D} \min\{k, |N(v) \cap S|\},$$

so (2) is equivalent to (2').

4. Proof of Theorem 1.4

We first define an operation that we will need in order to prove Theorem 1.4. The definition is based on Favaron’s proof of Theorem 1.2. In this section, when $T$ is a vertex set and $v$ is a vertex, we often write $N_T(v)$ for $N(v) \cap T$, and likewise for $d_T(v)$ and $d^*_T(v)$. When $k$ is a nonnegative integer, we write $[k]$ for the set $\{1, \ldots, k\}$.

Definition 4.1. When $D$ is a $k$-dependent set and $v$ is a vertex of $V(G) - D$ such that $|N_D(v)| < k$, we define the set $D \oplus v$ as follows. Let $A = \{v \in N_D(v): d_X(w) = k - 1\}$, and let $T$ be a maximal independent set in $A$. We define $D \oplus v$ to be the set $(D - T) \cup \{v\}$.

The following lemma can be directly extracted from Favaron’s proof of Theorem 1.2.

Lemma 4.2 (Favaron). If $D$ is $k$-dependent and $v$ is a vertex of $V(G) - D$ with $|N_D(v)| < k$, then $D \oplus v$ is a $k$-dependent set with $\phi_k(D \oplus v) = \phi_k(D) + k - |N_D(v)|$.

Since $D \oplus v$ is $k$-dependent whenever $D$ is, we can define the following iterated version of the $\oplus$ operator:

Definition 4.3. Suppose that $D$ is a $k$-dependent set, $Z$ is a set disjoint from $D$, and $<$ is an order on $Z$ such that $d^*_Z(v) + d_D(v) < k$ for all $v \in Z$. We define the set $D \oplus < Z$ as follows: let $z_1, \ldots, z_i$ be the vertices of $Z$, written in order according to $<$. Let $D_0 = D$, and for $i \in [\ell]$, let $D_i = D_{i-1} \oplus z_i$. The set $D \oplus < Z$ is defined as $D_\ell$.

Strictly speaking, the definition of $\oplus <$ depends on the choice of the independent set $T$ when we apply $\oplus$; however, these choices can be made arbitrarily. We now obtain an analogue of Lemma 1.2.
The relationship among the various sets is illustrated in Figure 1. Let
\[ \phi \]
We claim that
define sets \( B, E, A, C \),
\[ (1) \]
simpler inequality
\[ |N(v_i) \cap D_{i-1}| \leq |N_D(v)| + d_Z^\phi(v) < k. \]
Thus, repeatedly applying Lemma 4.2 yields
\[ \phi_k(D \oplus_\prec Z) = \phi_k(D) + k |Z| - \sum_{v \in Z} (|N_D(v)| + d_Z^\phi(v)) \]
\[ \geq \phi_k(D) + k |Z| - \sum_{v \in Z} (|N_D(v)| + d_Z^\phi(v)). \]
\[ \square \]

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let \( D \) be a \( k \)-dependent set, let \( X = V(G) - D \), and let \( < \) be an ordering on \( X \). Assuming that there is no \( k \)-edge-chromatic subgraph with the desired properties, we construct a \( k \)-dependent set \( D' \) with \( \phi_k(D') > \phi_k(D) \).

Let \( v_1, \ldots, v_t \) be the vertices of \( X \), written in order according to \( < \). Since there is no \( k \)-edge-chromatic subgraph with the desired properties, applying Lemma 3.2 with \( d_i = \max\{0, k - d^-(v_i)\} \) shows that there is a set \( S \subset X \) such that
\[ \sum_{v \in D} \min\{k, |N(v) \cap S|\} < \sum_{v_i \in S} \max\{0, k - d^-(v_i)\}. \]

We may assume that \( d^-(v_i) \leq k \) for all \( v_i \in S \), since vertices with \( d^-(v_i) > k \) may be removed from \( S \) without causing the above inequality to fail. This gives the simpler inequality
\[ (1) \]
\[ \sum_{v \in D} \min\{k, |N(v) \cap S|\} < k |S| - \sum_{v_i \in S} d^-(v_i). \]
Define sets \( B, E, A, C \) by
\[ B = \{v \in D: |N(v) \cap S| \leq k - 1\}, \]
\[ E = \{v \in S: |N(v) \cap B| + d^-(v) \leq k - 1\}, \]
\[ A = D - B, \]
\[ C = S - E. \]

The relationship among the various sets is illustrated in Figure 1. Let \( D' = B \oplus_\prec E \).

We claim that \( \phi_k(D') > \phi_k(D) \). By Corollary 4.4, \( D' \) is \( k \)-dependent and
\[ \phi_k(D') \geq \phi_k(B) + k |E| - \sum_{v \in E} (|N_B(v)| + d^-(v)). \]
Now, observe that
\[ \sum_{v \in D} \min\{k, |NS(v)|\} = k |A| + \sum_{v \in B} |NS(v)|. \]
Thus, from (1),

\[ k |A| + \sum_{v \in B} |N_S(v)| < k |S| - \sum_{v \in S} d^-(v). \]

Counting the edges incident to \( B \) from the endpoints in \( S \) yields

\[
\sum_{v \in B} |N_S(v)| = \sum_{v \in C} |N_B(v)| + \sum_{v \in E} |N_B(v)| \geq k |C| - \sum_{v \in C} d^-(v) + \sum_{v \in E} |N_B(v)|,
\]

where we have used the fact that \( |N_B(v)| + d^-(v) \geq k \) for \( v \in C \). Therefore, (2) yields

\[
k |A| + k |C| - \sum_{v \in C} d^-(v) + \sum_{v \in E} |N_B(v)| < k |S| - \sum_{v \in S} d^-(v),
\]

which rearranges to

\[
0 < -k |A| + k |E| - \sum_{v \in E} (|N_B(v)| + d^-(v)),
\]

using twice the fact that \( S - C = E \). Since \( \phi_k(B) \geq \phi_k(D) - k |A| \), applying Corollary 4.4 yields

\[
\phi_k(D') \geq \phi_k(D) - k |A| + k |E| - \sum_{v \in E} (|N_B(v)| + d^-(v)) > \phi_k(D).
\]

Thus, when \( D \) is \( k \)-optimal, a \( k \)-edge-chromatic subgraph with the desired properties exists. \( \square \)

5. CHORDAL GRAPHS AND \( k \)-EDGE-CHROMATIC SUBGRAPHS

In this section, we apply Theorem 1.4 to prove the following special case of Conjecture 2.3.

**Theorem 5.1.** If \( D \) is a \( k \)-optimal set in a chordal graph \( G \), then \( G \) has a \( k \)-edge-chromatic subgraph in which every vertex of \( V(G) - D \) has degree \( k \).

In order to prove this theorem, we need to introduce some new definitions. Our main definition is a variant on list coloring.

**Definition 5.2.** A list assignment on a graph \( G \) is a function \( L \) that assigns to each vertex \( v \) a set of colors \( L(v) \). A (proper) partial edge coloring is \( L \)-saturating if for every vertex \( v \) and every color \( c \in L(v) \), some edge incident to \( v \) receives the color \( c \). A graph \( G \) is \( L \)-saturable if it has an \( L \)-saturating partial edge coloring. When \( f \) is a function from \( V(G) \) into the nonnegative integers, we say that \( G \) is \( f \)-saturable if \( G \) is \( L \)-saturable whenever \( |L(v)| \leq f(v) \) for all \( v \).
The connection between Conjecture \[2.3\] and Definition \[5.2\] is given by the following lemma.

**Lemma 5.3.** Let \( D \) be a \( k \)-optimal set in a graph \( G \), and let \( X = V(G) - D \). If \( X \) has an ordering \(<\) such that \( G[X] \) is \( d^- \)-saturable, then \( G \) has a \( k \)-edge-chromatic subgraph in which all vertices of \( X \) have degree \( k \).

**Proof.** Let \(<\) be any such ordering of \( X \), and let \( M' \) be the \( k \)-edge-chromatic subgraph given by Theorem \[1.3\]. Write \( M' = M'_1 \cup \cdots \cup M'_k \), where each \( M_i \) is a matching, and for each \( v \in X \), let \( L(v) \) be the set of all indices \( i \in [k] \) such that \( v \) is not saturated by \( M'_i \). The degree condition on \( M' \) yields \( |L(v)| = \min\{d^-(v), k\} \).

Since \( G[X] \) is \( d^- \)-saturable, there is a partial edge coloring \( \phi \) of \( G[X] \) that is \( L \)-saturating. For \( i \in [k] \), let \( M^*_i \) be the set of edges that receive color \( i \) in \( \phi \). Now we combine the matchings \( M_1, \ldots, M_k \) with the matchings \( M^*_1, \ldots, M^*_k \). For each \( i \in [k] \), let \( M_i \) be the set defined by

\[
M_i = M^*_i \cup \{e \in M'_i : e \cap e^* = \emptyset \ \text{for all} \ e^* \in M^*_i \}.
\]

By construction, each edge set \( M_i \) is a matching. Furthermore, every vertex of \( v \) is incident to an edge in each \( M_i \), since if \( v \) is not incident to any edge of \( M'_i \), then \( i \in L(v) \), so that \( v \) is incident to an edge in \( M^*_i \). Thus, the \( k \)-edge-chromatic subgraph \( M_1 \cup \cdots \cup M_k \) has the desired properties. \( \square \)

Thus, in order to prove Theorem \[5.1\] it suffices to prove the following lemma. A **simplicial elimination order** is a vertex ordering \(<\) such that when the vertices of \( G \) are written \( v_1, \ldots, v_n \) in order according to \(<\), then for each \( i \), the neighborhood of \( v_i \) in the graph \( G - \{v_1, \ldots, v_{i-1}\} \) is a clique. A graph is chordal if and only if it has a simplicial elimination order \[6\].

**Lemma 5.4.** If \(<\) is a simplicial elimination order on an \( n \)-vertex graph \( G \), then \( G \) is \( d^- \)-saturable.

**Proof.** We use induction on \( n \). When \( n = 1 \), there is nothing to prove, so assume that \( n > 1 \) and the claim holds for smaller \( n \). Let \( L \) be a list assignment with \(|L(v)| \leq d^-(v)\) for all \( v \). By adding extra colors if necessary, we may assume that \(|L(v)| = d^-(v)\) for all \( v \).

Let \( v \) be the minimum vertex in \(<\), and let \( w_1, \ldots, w_t \) be the neighbors of \( v \), written in order according to \(<\). Since \( N(v) \) is a clique, we have \( d^-(w_i) \geq i \) for each \( i \in [t] \). Thus, we may choose distinct colors \( c_1, \ldots, c_t \) such that \( c_i \in L(w_i) \) for each \( i \).

Let \( G' = G - v \). Since the restriction of \(<\) to \( G' \) is still a simplicial elimination order, the induction hypothesis says that \( G' \) is \( d^{c^*}_{G'} \)-saturable, where \( d^{c^*}_{G'}(w) \) is the backdegree of \( w \) in \( G' \). Furthermore, for all \( w \in V(G') \), we have

\[
d^{c^*}_{G'}(w) = \begin{cases} d^{c^*}_{G'}(w) - 1, & \text{if } w \in N(v); \\ d^{c^*}_{G'}(w), & \text{otherwise}. \end{cases}
\]

Let \( L' \) be the list assignment on \( G' \) given by

\[
L'(w) = \begin{cases} L(w) - c_i, & \text{if } w = w_i; \\ L(w), & \text{if } w \notin N(v). \end{cases}
\]

Now \(|L'(w)| = d^{c^*}_{G'}(w)\) for all \( w \in V(G) \), so by the induction hypothesis, \( G' \) has a partial edge coloring \( \phi' \) that is \( L' \)-saturating.
The coloring $\phi'$ is almost $L$-saturating in $G$, except that each vertex $w_i$ may fail to be incident to the color $c_i$. We extend $\phi'$ to a partial edge coloring of $G$ by coloring each edge $vw_i$ with color $c_i$ if the color $c_i$ is not already incident to $w_i$, and leaving the edge $vw_i$ uncolored otherwise. Since the colors $c_i$ are distinct, the resulting partial edge coloring $\phi$ is still proper; thus, $\phi$ is $L$-saturating. Since $L$ was arbitrary, it follows that $G$ is $d^-$-saturable. \hfill $\square$

Since every induced subgraph of a chordal graph is again chordal, Lemma 5.4 implies that the ordering required by Lemma 5.3 can always be found.

**References**

1. Mustapha Chellali, Odile Favaron, Adriana Hansberg, and Lutz Volkmann, *k*-domination and $k$-independence in graphs: a survey, *Graphs Combin.* **28** (2012), no. 1, 1–55. MR 2863534 (2012k:05005)
2. Odile Favaron, On a conjecture of Fink and Jacobson concerning $k$-domination and $k$-dependence, *J. Combin. Theory Ser. B* **39** (1985), no. 1, 101–102. MR 805459 (86k:05064)
3. John Frederick Fink and Michael S. Jacobson, *n*-domination in graphs, *Graph theory with applications to algorithms and computer science* (Kalamazoo, Mich., 1984), Wiley-Intersci. Publ., Wiley, New York, 1985, pp. 283–300. MR 812671 (87e:05086)
4. _______, On $n$-domination, $n$-dependence and forbidden subgraphs, *Graph theory with applications to algorithms and computer science* (Kalamazoo, Mich., 1984), Wiley-Intersci. Publ., Wiley, New York, 1985, pp. 301–311. MR 812672 (87e:05087)
5. D. R. Fulkerson, The maximum number of disjoint permutations contained in a matrix of zeros and ones, *Canad. J. Math.* **16** (1964), 729–735. MR 0168583 (29 #5843)
6. D. R. Fulkerson and O. A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math. **15** (1965), 835–855. MR 0186421 (32 #3881)
7. Philip Hall, On representatives of subsets, *J. London Math. Soc.* **10** (1935), no. 1, 26–30.
8. Kenneth Lebensold, Disjoint matchings of graphs, *J. Combinatorial Theory Ser. B* **22** (1977), no. 3, 207–210. MR 0450138 (56 #8435)
9. U. S. R. Murty, An application of Rado’s theorem to disjoint matchings in bipartite graphs, *J. London Math. Soc.* (2) **17** (1978), no. 2, 193–194. MR 491057 (80f:05054)
10. Oystein Ore, *Theory of graphs*, American Mathematical Society Colloquium Publications, Vol. XXXVIII, American Mathematical Society, Providence, R.I., 1962. MR 0150753 (27 #740)