TORSION IN THE COHOMOLOGY OF TORUS ORBIFOLDS

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Abstract. We study torsion in the integral cohomology of a certain family of $2n$-dimensional orbifolds $X$ with actions of the $n$-dimensional compact torus. Compact simplicial toric varieties are in our family. For a prime number $p$, we find a necessary condition for the integral cohomology of $X$ to have no $p$-torsion. Then we prove that the necessary condition is sufficient in some cases. We also give an example of $X$ which shows that the necessary condition is not sufficient in general.

Introduction

A toric variety is a normal complex algebraic variety of complex dimension $n$ with an algebraic action of $(\mathbb{C}^*)^n$ having a dense orbit. A toric variety is not necessarily compact and may have singularity. The famous theorem of Danilov-Jurkiewicz gives an explicit description of the integral cohomology ring of a compact smooth toric variety in terms of the associated fan. It in particular says that the integral cohomology groups are torsion-free and concentrated in even degrees.

The analogous result holds for a compact simplicial toric variety $X$ (simplicial means that $X$ is an orbifold) but with rational coefficients. S. Fischli and A. Jordan studied the integral cohomology groups $H^*(X)$ in their dissertations [7], [11] using spectral sequences. Their results give an explicit computation of $H^k(X)$ and $H^{2n-k}(X)$ for $k \leq 3$ under some conditions. Based on their results, M. Franz developed Maple package torhom [8] to compute those cohomology groups. One can see that $H^*(X)$ has torsion in general while it has no torsion when $X$ is a weighted projective space ([12]). Therefore we are naturally led to ask when $H^*(X)$ has torsion or no torsion.

The orbit space $Q$ of a compact simplicial toric variety $X$ by the restricted action of the $n$-dimensional compact torus $T$ is a nice manifold with corners (sometimes called a manifold with faces). All faces of $Q$ (even $Q$ itself) are contractible and $Q$ is often homeomorphic to a simple polytope as manifolds with corners. MacPherson showed that $X$ is homeomorphic to the quotient space $(Q \times T)/\sim$ under some equivalence relation $\sim$ defined using the primitive vectors in the one-dimensional cones in the fan of $X$ (see [9]). The one-dimensional cones correspond to the facets of $Q$ so that one can think of the primitive vectors as a map

$$v: \{Q_1, Q_2, \ldots, Q_m\} \to \mathbb{Z}^n \quad (Q_i's \text{ are facets of } Q).$$
The map \( v \) satisfies some linear independence condition and a map satisfying the condition is called a characteristic function on \( Q \) (see Definition in Section 1). Note that there are many characteristic functions which do not arise from compact simplicial toric varieties.

Bahri-Sarkar-Song [1] consider the quotient space \( X(Q,v) = (Q \times T)/\sim \). Although they restrict their concern to \( Q \) being a simple polytope, the characteristic function \( v \) used to define the equivalence relation \( \sim \) is arbitrary; so the quotient space does not necessarily arise from a compact simplicial toric variety. They give a sufficient condition for \( H^*(X(Q,v)) \) to be torsion-free in terms of \( Q \) and \( v \). They also give a Danilov-Jurkiewicz type description for the ring structure of \( H^*(X(Q,v)) \) when it is torsion-free.

In this paper, we also consider the quotient space \( X = X(Q,v) = (Q \times T)/\sim \) where \( v \) is arbitrary as above but our \( Q \) is a compact connected nice manifold with corners and not necessarily a simple polytope. When \( Q \) has a vertex (equivalently \( X \) has a \( T \)-fixed point), our \( X \) is a torus orbifold in the sense of \([10]\). We give an explicit description of \( H^k(X) \) and \( H^{2n-k}(X) \) for \( k \leq 2 \) under some condition on \( Q \). Motivated by the explicit description of \( H^{2n-1}(X) \), we introduce a positive integer \( \mu(Q_I) \) depending on the characteristic function \( v \) for each \( Q_I = \bigcap_{i \in I} Q_i \), where \( I \) is a subset of \( \{1, \ldots, m\} \) and we understand \( Q_I = Q \) when \( I = \emptyset \) and \( \mu(Q_I) = 1 \) when \( Q_I = \emptyset \). The \( \mu(Q_I)'s \) are all one when \( X \) has no singularity. Here is a summary of our results, which follows from Propositions 5.2, 7.1, 7.2 and 7.3.

**Theorem.** Let \( Q \) be a connected nice manifold with corners of dimension \( n \geq 1 \). Let \( p \) be a prime number and suppose that every face of \( Q \) (even \( Q \) itself) is acyclic with \( \mathbb{Z}/p \)-coefficients. If \( H^*(X(Q,v)) \) has no \( p \)-torsion, then \( \mu(Q_I) \) is coprime to \( p \) for every \( Q_I \). The converse holds when the face poset of \( Q \) is isomorphic to the face poset of one of the following:

1. the suspension \( \emptyset^n \) of the \((n-1)\)-simplex \( \Delta^{n-1} \), i.e. \( \emptyset^n \) is obtained from \( \Delta^{n-1} \times [-1,1] \) by collapsing \( \Delta^{n-1} \times \{1\} \) and \( \Delta^{n-1} \times \{-1\} \) to a point respectively,
2. \( \Delta^n \),
3. \( \Delta^{n-1} \times [-1,1] \).

**Remark.** (1) When \( n \geq 3 \), there are many nice manifolds with corners \( Q \) which have the same face posets as \( \emptyset^n \), \( \Delta^n \) or \( \Delta^{n-1} \times [-1,1] \) but not homeomorphic to them. For instance, one can produce such \( Q \) by taking connected sum of them and integral homology \( n \)-spheres with non-trivial fundamental groups.

(2) The \( n \)-simplex \( \Delta^n \) and the prism \( \Delta^{n-1} \times [-1,1] \) can be obtained from the suspension \( \emptyset^n \) by performing a vertex cut once and twice respectively. So, the reader might think that the converse mentioned in the theorem above would hold for \( Q \) obtained from \( \emptyset^n \) by performing a vertex cut repeatedly. However, we will see in Section 8 that this is not true for \( Q \) obtained from \( \emptyset^3 \) by performing a vertex cut four times.

The paper is organized as follows. In Section 1 we set up notations. In Section 2 we compute \( H^{2n-k}(X) \) \((k \leq 2)\) for the quotient space \( X = (Q \times T)/\sim \) using the idea in Yeroshkin’s paper [17]. Namely, we delete a small neighborhood of the singular set in \( X \) to obtain a smooth manifold and investigate the relation of the cohomology groups between \( X \) and the smooth manifold. In Section 3 we show that the quotient map \( X \to Q \) induces an isomorphism on their fundamental groups when \( Q \) has a vertex.
In Section 4 we apply the results in Sections 2 and 3 to the case when \( n = 2 \) and 3. In Section 5 we introduce \( \mu(Q_I) \) and find a necessary condition for \( H^*(X) \) to have no \( p \)-torsion. In Section 6 we recall Theorem on Elementary Divisors and deduce two facts used in Section 7. In Section 7 we prove that the necessary condition obtained in Section 5 is sufficient for \( Q \) mentioned in the theorem above. Section 8 gives an example mentioned in the remark above. In the appendix we will observe that a result of Fischli or Jordan on \( H^{2n-1}(X) \) and the torsion part of \( H^{2n-2}(X) \) agrees with our Proposition 2.2 when \( X \) is a compact simplicial toric variety.

1. Setting and notation

In this section, we set up some notations and give some remarks. Let \( Q \) be a connected manifold with corners of dimension \( n \) (see [6, p.180] for the precise definition of a manifold with corners). Then faces are defined and a codimension-one face is called a facet. We assume that \( Q \) is nice, which means that every codimension-\( k \)-face is a connected component of intersections of \( k \) facets. The teardrop, which is homeomorphic to the 2-disk, is a manifold with corners but not nice (see [6, p.181]). A simple polytope is a nice manifold with corners and any intersection of faces is connected unless it is empty. However, intersections of faces of a nice manifold with corners are not necessarily connected. For instance, a 2-gon, that is the suspension of the two facets consists of two vertices.

Let \( S^1 \) be the unit circle group of the complex numbers \( \mathbb{C} \) and \( T \) be an \( n \)-dimensional connected compact abelian Lie group. As is well-known, \( T \) is isomorphic to \((S^1)^n\). We set

\[ N := \text{Hom}(S^1, T) \cong \mathbb{Z}^n. \]

Let \( Q \) have \( m \) facets and we denote them by \( Q_1, \ldots, Q_m \).

**Definition.** A function \( v: \{Q_1, \ldots, Q_m\} \to N \) is called a characteristic function on \( Q \) if it satisfies the following two conditions:

1. \( v(Q_i) \) is primitive for each \( i \in [m] := \{1, \ldots, m\} \)
2. whenever \( Q_I = \bigcap_{i \in I} Q_i \) is nonempty for \( I \subset [m] \), \( v(Q_i)'s \) \( (i \in I) \) are linearly independent over \( \mathbb{Q} \).

We denote by \( \hat{N} \) the sublattice of \( N \) generated by \( v_1, \ldots, v_m \).

We call \( v(Q_i)'s \) the characteristic vectors and abbreviate \( v(Q_i) \) as \( v_i \). Condition (2) above implies that when \( Q \) has a vertex, rank \( \hat{N} = n \). It also implies that when \( Q_I \neq \emptyset \), the toral subgroup of \( T \) generated by \( v_i(S^1)'s \) \( (i \in I) \), denoted by \( T_I \), is of dimension \( |I| \) where \( |I| \) is the cardinality of \( I \).

To the pair \((Q, v)\) we associate a quotient space

\[ X(Q, v) := (Q \times T)/\sim \]

with the equivalence relation \( \sim \) on the product \( Q \times T \) defined by

\[ (q, t) \sim (q', t') \text{ if and only if } q = q' \text{ and } t^{-1}t' \in T_I \]

where \( I \) is the subset of \([m]\) such that \( Q_I \) is the smallest face of \( Q \) containing \( q = q' \). The space \( X(Q, v) \) has a \( T \)-action induced from the natural \( T \)-action on \( Q \times T \). The orbit space of \( X(Q, v) \) by the \( T \)-action is \( Q \) and the quotient map

\[ \pi: X(Q, v) \to Q = X(Q, v)/T \]
is induced from the projection map $Q \times T \to Q$. Then it is not difficult to see the following facts (see [13] for example). A $T$-fixed point in $X(Q,v)$ corresponds to a vertex of $Q$, so $X(Q,v)$ has a $T$-fixed point if and only if $Q$ has a vertex. If $v_i$’s ($i \in I$) are a part of a basis of $N$ for every $I$ with $Q_I \neq \emptyset$, then $X(Q,v)$ is a manifold but otherwise $X(Q,v)$ is an orbifold. The singularity of $X(Q,v)$ lies in the union of $\pi^{-1}(Q_I)$ over all $I$ with $|I| \geq 2$.

As mentioned in the Introduction, if $X$ is a compact simplicial toric variety of complex dimension $n$ so that $X$ has an algebraic action of $(\mathbb{C}^*)^n$ having a dense orbit, then the orbit space $Q$ of $X$ by the compact $n$-dimensional subtorus $T$ of $(\mathbb{C}^*)^n$ is a nice manifold with corners and $X$ is homeomorphic to $X(Q,v)$ where $v_i$’s are primitive edge vectors of the fan associated to $X$. Moreover, faces of $Q$ (even $Q$ itself) are all contractible, which follows from the existence of the residual action of $(\mathbb{C}^*)^n / T$ on $Q = X/T$.

2. $H^{2n-k}(X(Q,v))$ for $k \leq 2$

In this section, we abbreviate $X(Q,v)$ as $X$ and all (co)homology groups will be taken with $\mathbb{Z}$-coefficients unless otherwise stated. When $n = 1$, $Q$ is a closed interval if $Q$ has a vertex and a circle otherwise, and $X$ is homeomorphic to $S^2$ or a torus accordingly. We will assume $n \geq 2$ in this section. Remember that $\pi: X \to Q$ is the quotient map.

Let $Q^{(n-2)}$ be the union of $Q_I$ over all $I$ with $|I| \geq 2$ and we assume $Q^{(n-2)} \neq \emptyset$. The singular set of $X$ lies in $\pi^{-1}(Q^{(n-2)})$ as remarked in Section [11]. Let $Q'$ be a “small closed tubular neighborhood” of $Q^{(n-2)}$ of $Q$ and set $X' := \pi^{-1}(Q')$.

**Lemma 2.1.** $H^{2n-k}(X) \cong H_k(X \setminus \text{Int } X')$ for $k \leq 2$.

**Proof.** Note that $H^r(X') = 0$ for $r \geq 2n - 3$ because $X'$ is homotopy equivalent to $\pi^{-1}(Q^{(n-2)})$ and $\dim \pi^{-1}(Q^{(n-2)}) = 2n - 4$. Therefore, the exact sequence in cohomology for the pair $(X, X')$ yields an isomorphism

$$H^{2n-k}(X, X') \cong H^{2n-k}(X) \quad \text{for } k \leq 2.$$  

(2.1)

On the other hand,

$$H^{2n-k}(X, X') \cong H^{2n-k}(X \setminus \text{Int } X', \partial X') \quad \text{by excision}$$

$$\cong H_k(X \setminus \text{Int } X') \quad \text{by Poincaré-Lefschetz duality.}$$

(2.2)

(Note that $X \setminus \text{Int } X'$ is a manifold with boundary $\partial X'$.) The lemma follows from (2.1) and (2.2). \qed

**Proposition 2.2.** $H^{2n}(X) \cong \mathbb{Z}$ and $H^{2n-1}(X) \cong H_1(Q) \oplus N/\hat{N}$. If $H_1(Q_i) = 0$ for every $i$, then

$$H^{2n-2}(X) \cong \mathbb{Z}^{m - \text{rank } \hat{N}} \oplus H_2(Q) \oplus (H_1(Q) \otimes H_1(T)) \oplus (\wedge^2 N/\hat{N} \wedge N).$$

**Remark.** When $Q$ has a vertex, $\text{rank } \hat{N} = n$ as remarked in Section [11]. Moreover, when $Q$ has a vertex and $n = 2$, the last term $\wedge^2 N/\hat{N} \wedge N$ above is zero. Indeed, since we may assume $N = \mathbb{Z}^2$ and $\hat{N} = \langle e_1, ae_2 \rangle$ with some integer $a$, $\hat{N} \wedge N = \langle e_1 \wedge e_2 \rangle = \wedge^2 N$, where $\{e_1, e_2\}$ denotes the standard base of $\mathbb{Z}^2$. 
Proof. The statement for $H^{2n}(X)$ follows immediately from Lemma 2.1.

We shall prove the statement for $H^{2n-1}(X)$. Let $Q^0 := (\text{Int } Q) \cap (Q \setminus Q')$ and $Q^1$ be the intersection of $(Q \setminus Q')$ and a small open neighborhood of $\partial Q$ in $Q$.

Since $\pi^{-1}(Q^0) \simeq Q \times T$, $\pi^{-1}(Q^1) \simeq \bigsqcup_{i=1}^m (Q_i \times T/v_i(S^1))$, $\pi^{-1}(Q^0) \cap \pi^{-1}(Q^1) \simeq \bigsqcup_{i=1}^m (Q_i \times T)$, $\pi^{-1}(Q^0 \cup Q^1) = X \setminus X'$, the Mayer-Vietoris exact sequence in homology for the triple $(X \setminus X', \pi^{-1}(Q^0), \pi^{-1}(Q^1))$ yields the following exact sequence:

$$
\bigoplus_{i=1}^m H_2(Q_i \times T) \xrightarrow{\partial} H_2(Q \times T) \oplus \bigoplus_{i=1}^m H_2(Q_i \times T/v_i(S^1)) \to H_2(X \setminus X')
$$

(2.3) 

$$
\bigoplus_{i=1}^m H_1(Q_i \times T) \xrightarrow{\partial} H_1(Q \times T) \oplus \bigoplus_{i=1}^m H_1(Q_i \times T/v_i(S^1)) \to H_1(X \setminus X')
$$

(2.4) 

$$
\bigoplus_{i=1}^m H_0(Q_i \times T) \xrightarrow{\partial} H_0(Q \times T) \oplus \bigoplus_{i=1}^m H_0(Q_i \times T/v_i(S^1)).
$$

As is easily seen, $f_0$ is injective; so

$$\text{ker } \varphi_1 = \text{coker } f_1.$$

(2.5) 

We write $f_1$ as $(\psi_1, \varphi_1)$ according to the decomposition of the target space. Since

$$\varphi_1 : \bigoplus_{i=1}^m H_1(Q_i \times T) \to \bigoplus_{i=1}^m H_1(Q_i \times T/v_i(S^1)),$$

which is $f_1$ composed with the projection on the second factor, is surjective, one has

$$\text{coker } f_1 \cong H_1(Q \times T)/\psi_1(\text{ker } \varphi_1).$$

Since $H_1(Y \times T) = H_1(Y) \oplus H_1(T)$ for any topological space $Y$, elements in $\text{ker } \varphi_1$ are of the form $(c_1v_1, \ldots, c_m v_m)$ with integers $c_i$, where $H_1(T)$ is identified with
$N = \text{Hom}(S^1, T)$ in a natural way. It follows that
\begin{equation}
(2.6) \quad H_1(Q \times T)/\psi_1(\ker \varphi_1) \cong H_1(Q) \oplus N/\hat{N}.
\end{equation}
The statement for $H^{2n-1}(X)$ in the proposition follows from $\text{(2.4), (2.5), (2.6)}$ and Lemma $\text{2.1}$.

The computation of $H^{2n-2}(X)$ is similar to that of $H^{2n-1}(X)$. We write $f_2$ as $(\psi_2, \varphi_2)$ similarly to $f_1$. Since $H_1(Q_i) = 0$ for any $i$ by assumption, $\ker f_1$ is a free abelian group of rank $m - \text{rank} \hat{N}$ as is easily seen; so it follows from $\text{(2.3)}$ that
\begin{equation}
(2.7) \quad H_2(X \setminus X') \cong \mathbb{Z}^{m-\text{rank} \hat{N}} \oplus \text{coker} f_2.
\end{equation}
Similarly to $\varphi_1$, the map
\begin{equation}
(2.8) \quad \varphi_2 : \bigoplus_{i=1}^m H_2(Q_i \times T) \to \bigoplus_{i=1}^m H_2(Q_i \times T/v_i(S^1))
\end{equation}
is surjective; so
\begin{equation}
(2.9) \quad \text{coker} f_2 \cong H_2(Q \times T)/\psi_2(\ker \varphi_2).
\end{equation}
Here,
\begin{equation}
(2.10) \quad H_2(Y \times T) = H_2(Y) \oplus (H_1(Y) \otimes H_1(T)) \oplus H_2(T)
\end{equation}
for any topological space $Y$ by the Künneth formula. Therefore, since $H_1(Q_i) = 0$ by assumption, it follows from $\text{(2.8) and (2.10)}$ that $\ker \varphi_2$ is contained in $\bigoplus_{i=1}^m H_2(T)$.

We note that $H_2(T)$ and $H_2(T/v_i(S^1))$ can be identified with $\wedge^2 N$ and $\wedge^2 (N/v_i)$ respectively and the kernel of the projection $\wedge^2 N \to \wedge^2 (N/v_i)$ is $(v_i) \wedge N$. Therefore
\begin{equation}
\text{coker} f_2 \cong H_2(Q) \oplus (H_1(Q) \otimes H_1(T)) \oplus (\wedge^2 N/\hat{N} \wedge N)
\end{equation}
This together with $\text{(2.7) and (2.9)}$ proves the statement for $H^{2n-2}(X)$ in the proposition. \hfill \Box

3. Fundamental Groups

For a subset $I$ of $[m]$, we define
\begin{equation}
T^m_I := \{(h_1, \ldots, h_m) \in T^m \mid h_j = 1 \quad (\forall j \notin I)\}.
\end{equation}
and consider a space
\begin{equation}
Z_Q := (Q \times T^m)/\sim_e
\end{equation}
where $\sim_e$ is the equivalence relation on the product $Q \times T^m$ defined by
\begin{equation}
(q, s) \sim_e (q', s') \text{ if and only if } q = q' \text{ and } s^{-1}s' \in T^m_I
\end{equation}
and $I$ is the subset of $[m]$ such that $Q_I$ is the smallest face of $Q$ containing $q = q'$.

We note that $Z_Q$ locally admits a smooth structure. Indeed, since $Q$ is a manifold with corners, any point of $Q$ has a neighborhood $U$ homeomorphic to $(\mathbb{R}_{>0})^r \times \mathbb{R}^{n-r}$ for some $0 \leq r \leq n$ and it follows from the construction of $Z_Q$ that the inverse image of $U$ by the projection map $\kappa : Z_Q \to Q$ is homeomorphic to $\mathbb{C}^r \times \mathbb{R}^{n-r} \times T^{m-r}$.

Therefore $Z_Q$ locally admits a smooth structure and hence is a topological manifold.

Remark. When $Q$ is a simple polytope, $Z_Q$ is called a moment-angle manifold and it is known that $Z_Q$ admits a smooth structure and is 2-connected (see $\text{[3] or [4]}$). Moreover, the moment-angle manifold $Z_Q$ is homotopy equivalent to $\mathbb{C}^m - Z$ defined in $\text{[5]}$ (see Theorem 4.7.5 in $\text{[4]}$), where $Z$ is the union of coordinate subspaces in $\mathbb{C}^m$ determined by $Q$. 
Lemma 3.1. The projection map $\kappa: Z_Q \to Q$ induces an isomorphism $\kappa_*: \pi_1(Z_Q) \cong \pi_1(Q)$ on the fundamental groups.

Proof. Similarly to the above argument, one can see that $\kappa^{-1}(Q_i)$, where $Q_i$ is a facet of $Q$, is a locally smooth closed manifold. Moreover, it is a locally smooth codimension two submanifold of $Z_Q$. Indeed, a closed tubular neighborhood of $Q_i$ in $Q$ can be identified with $Q_i \times [0,1]$, and $\rho_i: \kappa^{-1}(Q_i \times \{1\}) \to \kappa^{-1}(Q_i)$, where $\rho_i$ is induced from $((q,1),t) \to (q,t)$ for $q \in Q_i = Q_i \times \{0\} \subset Q_i \times [0,1] \subset Q$ and $t \in T^m$, is a principal $S^1$-bundle, and the total space $E_i$ of the associated complex line bundle can be identified with a closed tubular neighborhood of $Z_i := \kappa^{-1}(Q_i)$ in $Z_Q$.

Since $Z_i$ is a locally smooth closed codimension two submanifold of $Z_Q$, the transversality argument can be applied. Therefore, if a continuous map $f: S^1 \to Z_Q$ meets $Z_i$, then one can slightly push $f$ in the fiber direction of $E_i$ so that the deformed $f$ does not meet $Z_i$. Applying this deformation to $f$ for every $i$, we see that $f$ is homotopic to a continuous map whose image lies in $\kappa^{-1}(\text{Int } Q) = \text{Int } Q \times T^m$. This means that the inclusion map $\iota_i: \text{Int } Q \times T^m \to Z_Q$ induces an epimorphism $\iota_*: \pi_1(\text{Int } Q \times T^m) = \pi_1(\text{Int } Q) \times \pi_1(T^m) \to \pi_1(Z_Q)$.

Since $\text{Int } Q$ is homotopy equivalent to $Q$, we may replace $\text{Int } Q$ by $Q$ above and we have a sequence

$$ \pi_1(Q) \times \pi_1(T^m) \xrightarrow{\iota_*} \pi_1(Z_Q) \xrightarrow{\kappa_*} \pi_1(Q), $$

where the composition $\kappa_* \circ \iota_*$ agrees with the projection on the first factor, so that the kernel of $\iota_*$ is contained in the second factor $\pi_1(T^m)$.

Let $S_i$ be the $i$-th $S^1$-factor of $T^m$ and choose a point $q_i \in (Q_i \times \{1\}) \cap \text{Int } Q$. Then $\iota((q_i \times S_i))$ is a fiber of the principal $S^1$-bundle $\rho_i: \kappa^{-1}(Q_i \times \{1\}) \to Z_i = \kappa^{-1}(Q_i)$, so it shrinks to a point in $Z_i$. Therefore $\pi_1(T^m)$ is in the kernel of the epimorphism $\iota_*$ and this implies the lemma. \qed

We recall a result from Bredon’s book \cite{Bredon}.

Lemma 3.2. \cite{Bredon} Corollary 6.3 on p.91. If $X$ is an arcwise connected $G$-space, $G$ compact Lie, and if there is an orbit which is connected (e.g., $G$ connected or $X^G \neq \emptyset$), then the quotient map $X \to X/G$ induces an epimorphism on their fundamental groups.

The characteristic map $v: \{Q_1, \ldots, Q_n\} \to \text{Hom}(S^1, T)$ defines a homomorphism $T^m \to T$, denoted $v$ again. Note that $v(T^m)$ is a subtorus of $T$ of dimension rank $N$, in particular, $v$ is surjective if and only if rank $N = \text{rank } N$ (this is the case when $Q$ has a vertex). The product map $id \times v: Q \times T^m \to Q \times T$ induces a continuous map $V: Z_Q = Q \times T^m / \sim_e \to Q \times T/ \sim = X(Q,v) = X$ and it further induces an injective continuous map $\tilde{V}: Z_Q / \text{ker } v \to X$, so that $\tilde{V}$ is a homeomorphism if $v$ is surjective since the spaces are compact and Hausdorff.

Proposition 3.3. If $Q$ has a vertex, then $\pi_*: \pi_1(X) \cong \pi_1(Q)$. 

Theorem 3.4. \textit{If $Q$ has a vertex and $H_1(Q) = H_2(Q) = 0$, then $H^1(X) = 0$ and $H^2(X) \cong \mathbb{Z}^{m-n}$.}

\textbf{Proof.} By Proposition 3.3, $\pi_1(X) \cong \pi_1(Q)$ and hence $H_1(X) \cong H_1(Q)$. Therefore $H_1(X) = 0$ since $H_1(Q) = 0$ by assumption and hence $H^1(X) = 0$ and $H^2(X)$ has no torsion by the universal coefficient theorem. On the other hand, since $X$ is an orbifold, Poincaré duality holds with $\mathbb{Q}$-coefficients. Therefore the rank of $H^2(X)$ is equal to that of $H^{2n-2}(X)$, that is $m - n$ by Proposition 2.2 and its subsequent remark. \hfill $\Box$

4. LOW DIMENSIONAL CASES

A nice manifold with corners $Q$ is called \textit{face-acyclic} if every face of $Q$ (even $Q$ itself) is acyclic. We note that if $Q$ is face-acyclic, then $Q$ must have a vertex. Indeed, let $F$ be a face of $Q$ of minimum dimension. Then $F$ has no boundary because the boundary of $F$ must consist of faces of smaller dimensions, so $F$ is a closed manifold. But since $F$ is acyclic, this means that $F$ is a point. Therefore $Q$ has a vertex.

We shall apply the previous results when $Q$ is face-acyclic and $n = \dim Q$ is 2 or 3. The following corollary follows from Proposition 2.2 and Corollary 3.4.
Corollary 4.1. Suppose that $Q$ is face-acyclic and $\dim Q = 2$, that is, $Q$ is an $m$-gon ($m \geq 2$). Then we have

$$H^j(X) \cong \begin{cases} 
\mathbb{Z} & (j = 0, 4) \\
\mathbb{Z}^{m-2} & (j = 2) \\
N/\hat{N} & (j = 3) \\
0 & \text{(otherwise)}. 
\end{cases}$$

Example. Let $a$ be a positive integer. Take $Q$ to be a 2-simplex, $N = \mathbb{Z}^2$ and

$$v_1 = (2a, 1), \ v_2 = (0, 1), \ v_3 = (-a, -1).$$

Then $\hat{N} = \langle ae_1, e_2 \rangle$ and $N/\hat{N} \cong \mathbb{Z}/a$. The space $X$ is not a weighted projective space when $a \geq 2$ since it has torsion in cohomology, where $\{e_1, e_2\}$ denotes the standard base of $\mathbb{Z}^2$ as before.

Corollary 4.2. Suppose that $Q$ is face-acyclic and $\dim Q = 3$. Then

$$H^j(X) \cong \begin{cases} 
\mathbb{Z} & (j = 0, 6) \\
\mathbb{Z}^{m-3} & (j = 2) \\
0 \text{ or some torsion group} & (j = 3) \\
\mathbb{Z}^{m-3} \oplus \wedge^2 N/(\hat{N} \wedge N) & (j = 4) \\
N/\hat{N} & (j = 5) \\
0 & \text{(otherwise)}. 
\end{cases}$$

Proof. Since $Q$ is face-acyclic, $Q$ has a vertex as remarked at the beginning of this section; so all the statements except for $j = 3$ follows from Proposition 2.2 and Corollary 4.1. In order to prove the statement for $j = 3$, it suffices to show $H^3(X; \mathbb{Q}) = 0$ and this is equivalent to showing that the euler characteristic of $X$ is $2m - 4$ (note that we know the rank of $H^3(X)$ except for $j = 3$).

Since $Q$ is face-acyclic and of dimension 3, the boundary of $Q$ is a 2-sphere, every 2-face of $Q$ is a 2-disk and the number of 2-faces is $m$ by definition. Let $V$ be the number of vertices of $Q$. Then the number of edges of $Q$ is $3V/2$ and hence we obtain an identity $V - 3V/2 + m = 2$ by Euler’s formula, which implies $V = 2m - 4$. On the other hand, it is known that the euler characteristic of $X$ is equal to that of the $T$-fixed point set $X^T$ (see [2] Theorem 10.9 in p.163). In our case $X^T$ is isolated and corresponds to the vertices of $Q$. Therefore, the euler characteristic of $X$ is equal to $V$, that is $2m - 4$. \qed

Example. It happens that $\hat{N} \wedge N = \wedge^2 N$ even if $\hat{N} \neq N$. For instance, take $Q$ to be a 3-simplex, $N = \mathbb{Z}^3$ and

$$v_1 = (0, 0, 1), \ v_2 = (2, 0, 1), \ v_3 = (0, 1, 1), \ v_4 = (-2, -1, -1).$$

Then

$$\hat{N} = (2e_1, e_2, e_3), \quad \hat{N} \wedge N = \langle e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 \rangle = \wedge^2 N,$$

where $\{e_1, e_2, e_3\}$ denotes the standard base of $\mathbb{Z}^3$.

Corollary 4.2 says that if $\hat{N} = N$, then $H^j(X)$ has no torsion except $j = 3$. However, $H^3(X)$ can be nontrivial (so, a nontrivial torsion group) when $\hat{N} = N$. We shall give such an example below. One can also find many such examples using Maple package torhom.
Example. Let $a$ be a positive integer and take the following five primitive vectors in $\mathbb{Z}^3$:  
\begin{align*}
  v_+ &= (0, 0, 1), \\
  v_1 &= (2a, 1, 0), \\
  v_2 &= (0, 1, 0), \\
  v_3 &= (-a, -1, 0), \\
  v_- &= (1, 0, -1).
\end{align*}

Then $\hat{N} = N$. We consider the complete simplicial fan $\Delta$ having the following six 3-dimensional cones  
\[ \angle v_+ v_1 v_2, \angle v_+ v_1 v_3, \angle v_+ v_2 v_3, \angle v_- v_1 v_2, \angle v_- v_1 v_3, \angle v_- v_2 v_3 \]
where $\angle v_i v_j v_k$ $(i \in \{+,-\}, i,j \in \{1,2,3\})$ denotes the cone spanned by $v_i$, $v_j$, and $v_k$. Let $X$ be the compact simplicial toric variety associated to the fan $\Delta$. Let $C$ be a subset of $\mathbb{Z}^3$. Then one can see that $H_1(C; \mathbb{Z})$ vanishes unless $(p, q) = (0, 3)$ by Corollary 4.1. Therefore all the differentials except  
\[ d_{0,3}^2: E_{0,3}^2 \rightarrow E_{2}^{2,2} \quad \text{and} \quad d_{0,4}^2: E_{0,4}^2 \rightarrow E_{2}^{2,3} \]
are trivial. Here, $E_{0,3}^2 = H^0(\mathbb{CP}^1; \mathbb{H}^3(F))$ is trivial or a torsion group by Corollary 4.1, while $E_{2,2}^2 = H^2(\mathbb{CP}^1; \mathbb{H}^2(F)) = H^2(F)$ is a free abelian group again by Corollary 4.1. So $d_{0,3}^2$ must be trivial. Hence we have $H^3(X) \cong \mathbb{Z}/a$ again by Corollary 4.1 (see Example after Corollary 4.1) and hence we have $H^3(X) \cong \mathbb{Z}/a$. On the other hand, since $\hat{N} = N$ as remarked above, $H^1(X)$ has no torsion for $j \neq 3$ by Corollary 4.2.

5. A necessary condition for no $p$-torsion

Let $I$ be a subset of $[m]$ with $Q_I \neq \emptyset$. Although $Q_I$ is not necessarily connected, we understand that $Q_I$ stands for a connected component of $Q_I$ in this section for notational convenience. Then the characteristic function $v_i$ associates a characteristic function $v_I$ on $Q_I$ as follows. Since $v_i$’s $(i \in I)$ are linearly independent over $\mathbb{Q}$, they span a $|I|$-dimensional linear subspace of $N \otimes \mathbb{R}$ and its intersection with $N$ is a rank $|I|$ sublattice of $N$, denoted $N_I$. Then $N(I) := N/N_I$ is a free abelian group of rank $n - |I|$ and we denote the projection map from $N$ to $N(I)$ by $\pi_I$. If $Q_I \cap Q_J$ is nonempty for $j \in [m]\setminus I$, then its connected components are facets of $Q_I$, and any facet of $Q_I$ is of this form. The element $\pi_I(v_j) \in N(I)$ is not necessarily primitive and we define $v_I(Q_I \cap Q_J)$ to be the primitive vector in $N(I)$ which has the same direction as $\pi_I(v_j)$, where $Q_I \cap Q_J$ also stands for a connected component of $Q_I \cap Q_J$. Then one can see that $v_I$ is a characteristic function on $Q_I$. Similarly to $\hat{N}$, one can define a sublattice $\hat{N}(I)$ of $N(I)$ using $v_I$. We allow $I = \emptyset$ and understand $Q_\emptyset = Q$.  

\[ E_{2}^{p,q} = H^p(\mathbb{CP}^1; \mathbb{H}^q(F)) \]
and $E_{2}^{p,q} = 0$ unless $p = 0, 2$ and $q = 0, 2, 3, 4$ by Corollary 4.1. Therefore all the differentials except  
\[ d_{0,3}^2: E_{0,3}^2 \rightarrow E_{2}^{2,2} \quad \text{and} \quad d_{0,4}^2: E_{0,4}^2 \rightarrow E_{2}^{2,3} \]
are trivial. Here, $E_{0,3}^2 = H^0(\mathbb{CP}^1; \mathbb{H}^3(F)) = H^3(F)$ is trivial or a torsion group by Corollary 4.1, while $E_{2,2}^2 = H^2(\mathbb{CP}^1; \mathbb{H}^2(F)) = H^2(F)$ is a free abelian group again by Corollary 4.1. So $d_{0,3}^2$ must be trivial. Hence we have $H^3(X) \cong \mathbb{Z}/a$ again by Corollary 4.1 (see Example after Corollary 4.1) and hence we have $H^3(X) \cong \mathbb{Z}/a$. On the other hand, since $\hat{N} = N$ as remarked above, $H^1(X)$ has no torsion for $j \neq 3$ by Corollary 4.2.
$N(\emptyset) = N$ and $\hat{N} (\emptyset) = \hat{N}$. We define
\[
\mu(Q) := \begin{cases} 
|N(I)/\hat{N}(I)| & \text{when } Q_I \neq \emptyset, \\
1 & \text{when } Q_I = \emptyset.
\end{cases}
\]
Here $|N(I)/\hat{N}(I)|$ is not necessarily finite. For instance, take $Q = S^1 \times [-1, 1]$ and assign characteristic vectors $(1, 0)$ and $(-1, 0)$ to the facets $S^1 \times \{1\}$ and $S^1 \times \{-1\}$ respectively. Then $N/\hat{N}$ is an infinite cyclic group and hence $|N(I)/\hat{N}(I)|$ is infinite for $I = \emptyset$. One can easily construct a similar example such that $|N(I)/\hat{N}(I)|$ is infinite for some $I \neq \emptyset$.

**Remark.** When $|I| = n$, $N(I) = \{0\}$; so $\mu(Q) = 1$. When $|I| = n - 1$, $N(I)$ is of rank one and $\hat{N}(I)$ is generated by a primitive vector; so $\hat{N}(I) = N(I)$ and hence $\mu(Q) = 1$ in this case too. Another case which ensures $\mu(Q) = 1$ is the following. Let $q$ be a vertex of $Q$. Then there is a subset $J$ of $[m]$ with $|J| = n$ such that $q \in Q_J$. If $\{v_j\}_{j \in J}$ is a base of $N$, then $\mu(Q) = 1$ for every subset $I$ of $J$, which easily follows from the definition of $\mu(Q)$. We note that for a prime number $p$, $H^*(X(Q, v); \mathbb{Z})$ has no $p$-torsion if and only if $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$, which follows from the universal coefficient theorem (see [15] Corollary 56.4).

**Lemma 5.1.** [2] Theorem 2.2 on pp.376-377. Let a group $G$ of prime order $p$ act on a finite dimensional space $X$ with $A \subset X$ closed and invariant. Suppose that $G$ acts trivially on $H^*(X, A; \mathbb{Z})$. Then
\[
\sum_{i \geq 0} \text{rk } H^{k+2i}(X^G, X^G \cap A; \mathbb{Z}/p) \leq \sum_{i \geq 0} \text{rk } H^{k+2i}(X, A; \mathbb{Z}/p).
\]

**Proposition 5.2.** If $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$, then $H_1(Q_1; \mathbb{Z}/p) = 0$ and $\mu(Q_1)$ is finite and coprime to $p$ for every $I$.

**Proof.** We abbreviate $X(Q, v)$ as $X$ as before. Since $H^{odd}(X; \mathbb{Z}/p) = 0$, we have $H^{odd}(X^G; \mathbb{Z}/p) = 0$ for every $p$-subgroup $G$ of $T_I$ by repeated use of Lemma 5.1. In fact, let $G$ be an order $p$ subgroup of $S^1$. The induced action of $G$ on $H^*(X)$ is trivial because $G$ is contained in the connected group $S^1$. Then $\text{rk } H^{odd}(X^G; \mathbb{Z}/p) \leq \text{rk } H^{odd}(X; \mathbb{Z}/p)$ by Lemma 5.1 applied with $A = \emptyset$. Therefore, $H^{odd}(X^G; \mathbb{Z}/p) = 0$ by assumption. Repeating the same argument for $X^G$ with the induced action of $S^1/G$, which is again a circle group, we conclude that $H^{odd}(X^G; \mathbb{Z}/p) = 0$ for any $p$-subgroup $G$ of $S^1$.

For a positive integer $k$, let $G_k$ be the $p$-subgroup of $T_I$ consisting of all elements of order at most $p^k$. Then $G_k \subset G_k'$ for $k \leq k'$ and the union $\bigcup_{k=1}^{\infty} G_k$ is dense in $T_I$. Therefore $X^{G_k} = X^{T_I}$ if $k$ is sufficiently large. Since $X_I = \pi^{-1}(Q_I)$ is a connected component of $X^{T_I}$, this shows that $H^{odd}(X_I; \mathbb{Z}/p) = 0$. But $H^{2(n-|I|)-1}(X_I)$ is isomorphic to $H_1(Q_I) \oplus N(I)/\hat{N}(I)$ by Proposition 2.2 and hence the universal coefficient theorem implies the proposition. \[\square\]

\[\text{1}\
\text{Detailed explanation about this assertion. Since the set of isotropy groups of } X \text{ is finite, there is a positive integer } r \text{ such that } X^{G_k} = X^{G_r} \text{ for every } k \geq r. \text{ Since } G_r \text{ is a subgroup of } T_I, \text{ we have } X^{G_r} \supset X^{T_I}. \text{ We shall prove the opposite inclusion. Let } x \in X^{G_T}. \text{ The isotropy group } T_x \text{ at } x \text{ contains } G_k \text{ for every } k \geq r \text{ because } X^{G_T} = X^{G_r} \text{ but since } T_x \text{ is a closed subgroup of } T, T_x \text{ must contain the closure of } \bigcup_{k=r}^{\infty} G_k, \text{ that is } T_I. \text{ Therefore } x \in X^{T_I} \text{ and hence } X^{G_r} = X^{T_I}.\]
When $H^{\text{odd}}(X(Q,v);\mathbb{Z}/p) = 0$, Proposition 5.2 gives a constraint on the topology of $Q_I$, that is $H_1(Q_I;\mathbb{Z}/p) = 0$. It is proved in [14] that if $X(Q,v)$ is a manifold and $H^{\text{odd}}(X(Q,v);\mathbb{Z}) = 0$, then $Q$ is face-acyclic. This implies that there will be more constraints on the topology of $Q_I$ when $H^{\text{odd}}(X(Q,v);\mathbb{Z}/p) = 0$, to be more precise, we expect that $Q$ is face $p$-acyclic which means that (every component of) $Q_I$ is acyclic with $\mathbb{Z}/p$-coefficients for every $I$. Therefore, in order to consider the converse of Proposition 5.2 it would be appropriate to assume that $Q$ is face $p$-acyclic. We will prove in Section 7 that the converse holds in some cases while we will see in Section 8 that the converse does not hold in general.

6. THEOREM ON ELEMENTARY DIVISORS

We recall the Theorem on Elementary Divisors and deduce two facts from it, which will play a role in the next section.

**Theorem 6.1** (Theorem on Elementary Divisors, see [10]). Let $N'$ be a submodule of rank $n'$ in $N = \mathbb{Z}^n$. Then there are bases $\{u'_1, \ldots, u'_{n'}\}$ of $N'$ and $\{u_1, \ldots, u_n\}$ of $N$ such that $u'_i = \epsilon_i u_i$ with some integer $\epsilon_i$ for $i = 1, 2, \ldots, n'$ and $\epsilon_1 \epsilon_2 \cdots \epsilon_{n'}$. Moreover if $A = (a_1, \ldots, a_k)$ is an $n \times k$ integer matrix whose column vectors $a_1, \ldots, a_k$ generate $N'$ and

$$\delta_i := \gcd(\det B \mid B \text{ is an } i \times i \text{ submatrix of } A),$$

then $\delta_i = \delta_{i-1} \epsilon_i$ for $i = 1, 2, \ldots, n'$. In particular, if $n' = n$, then $\delta_n = |N/N'|$.

We deduce two facts from Theorem 6.1.

**Lemma 6.2.** Let $A$ be an $n \times n$ integer matrix of rank $n$ and $\hat{A} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the epimorphism induced from $A$. Then $\ker \hat{A} \cong \text{coker } A$.

**Proof.** By Theorem 6.1 we may think of $A$ as the diagonal matrix with diagonal entries $\epsilon_1, \ldots, \epsilon_n$. Then one easily sees that $\ker \hat{A}$ and $\text{coker } A$ are both isomorphic to $\prod_{i=1}^{n} \mathbb{Z}/\epsilon_i$, proving the lemma. \qed

Let $a_1, \ldots, a_{n+1}$ be elements of $\mathbb{Z}^n$ which generate a sublattice $\langle a_1, \ldots, a_{n+1} \rangle$ of rank $n$ and set $d_i := |\text{det}(a_{j})_{j \neq i}|$ for $i \in [n+1]$. It follows from Theorem 6.1 that

$$\delta_n = \gcd(d_1, \ldots, d_{n+1}) = |\mathbb{Z}^n/(a_1, \ldots, a_{n+1})|.$$  

Suppose that $a_{n+1}$ is primitive. Let $\bar{a}_k (k \neq n+1)$ be the projection image of $a_k$ on $\mathbb{Z}^n/(a_1, \ldots, a_{n+1})$ and let $a'_k$ be the primitive vector in the quotient lattice $\mathbb{Z}^n/(a_1, \ldots, a_{n+1})$ which has the same direction as $\bar{a}_k$ when $\bar{a}_k$ is nonzero and $a'_0$ be the zero vector when so is $\bar{a}_k$. Set $d'_i := \text{det}(a'_1, \ldots, a'_j, \ldots, a'_n)$. With this understood we have the following.

**Lemma 6.3.** $\gcd(d_1, \ldots, d_n) | d_{n+1}$, i.e., $\gcd(d_1, \ldots, d_n) = \gcd(d_1, \ldots, d_{n+1})$. Moreover, $\gcd(d'_1, \ldots, d'_n) | \gcd(d_1, \ldots, d_{n+1})$.

**Proof.** Theorem 6.1 applied with $N'$ generated by $a_{n+1}$ says that there is a basis $\{u_1, \ldots, u_n\}$ of $N = \mathbb{Z}^n$ such that $a_{n+1} = \epsilon_1 u_1$ with some integer $\epsilon_1$. But since $a_{n+1}$ is primitive, we have $\epsilon_1 = \pm 1$. Therefore, we may assume that $a_{n+1} = (0, \ldots, 0, 1)^T$ through a linear transformation of $\mathbb{Z}^n$. We have

$$d_{n+1} = |\text{det}(a_1, \ldots, a_n)| = \sum_{j=1}^{n} a_j^n \tilde{a}_j^n$$

where $a_j^n$ is the $(n,j)$ entry of the matrix $(a_1, \ldots, a_n)$ and $\tilde{a}_j^n$ is its cofactor. Since $a_{n+1} = (0, \ldots, 0, 1)^T$, $\tilde{a}_j^n$ agrees with $d_j = |\text{det}(a_1, \ldots, a_j, \ldots, a_{n+1})|$ up to sign.
Therefore $\tilde{a}_j^n$ is divisible by $\gcd(d_1, \ldots, d_n)$ for every $j$ and this together with (6.2) implies the former statement in the lemma.

Since $a_{n+1} = (0, \ldots, 0, 1)^T$, $\mathbb{Z}^n/\langle a_{n+1} \rangle$ can naturally be identified with $\mathbb{Z}^{n-1}$ and we have

\[(6.3) \quad d_j = |\det(\tilde{a}_1, \ldots, \tilde{a}_j, \ldots, \tilde{a}_{n+1})| = |\det(\tilde{a}_1, \ldots, \tilde{a}_j, \ldots, \tilde{a}_n)| \quad \text{for} \quad j = 1, 2, \ldots, n\]

where $\tilde{a}_k$ ($k = 1, 2, \ldots, n$) is the projection image of $a_k$ on $\mathbb{Z}^n/\langle a_{n+1} \rangle = \mathbb{Z}^{n-1}$. Since $\tilde{a}_k$ is a positive scalar multiple of $a'_k$, $d_j' = |\det(a'_1, \ldots, a'_j, \ldots, a'_n)|$ divides the latter term in (6.3) above and hence $d_j$. This together with the former statement in the lemma implies the latter statement in the lemma. $\square$

7. CONVERSE OF PROPOSITION 5.2 IN THREE CASES

In this section we show that if $Q$ is face $p$-acyclic and has the same face poset as one of the following:

**Case 1:** the suspension $\Diamond^n$ of an $(n-1)$-simplex $\Delta^{n-1}$ (see the Introduction),

**Case 2:** the $n$-simplex $\Delta^n$,

**Case 3:** the prism $\Delta^{n-1} \times [-1, 1]$,

then the converse of Proposition 5.2 holds, i.e. if $\mu(Q)$ is finite and coprime to $p$ for every $I$, then $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$.

First we establish Case 1. Then we reduce Case 2 to Case 1 by collapsing a face of $Q$ to a point. In Case 3, according to the characteristic function $v$, we collapse one or two faces of $Q$ to a point reducing Case 3 to Case 2 or Case 1. The argument then becomes much more complicated than that reducing Case 2 to Case 1. It would be interesting to see whether this inductive argument works for an arbitrary product of simplices.

Let $q$ be a vertex of $Q$. Then $q$ lies in $Q_I$ for some $I \subset [m]$ with $|I| = n$. We set

$$d_Q(q) := |\det((v_i)_{i \in I})|$$

where $v_i = v(Q_i)$ as before.

**Case 1.** In this case $Q$ has two vertices, say $q$ and $q'$, and $d_Q(q) = d_Q(q') = \mu(Q)$.

**Proposition 7.1.** Suppose that $Q$ is face $p$-acyclic, has the same face poset as $\Diamond^n$ and $\mu(Q)$ is coprime to $p$. Then $X(Q, v)$ has the same cohomology as $S^{2n}$ with $\mathbb{Z}/p$-coefficients, in particular $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$.

**Proof.** When $n = 1$, $Q$ is a closed interval and $X(Q, v)$ is homeomorphic to $S^2$; so the proposition holds when $n = 1$. In the following we assume $n \geq 2$, so that $Q$ has $n$ facets.

Let $T^n = (S^1)^n$. Then $\text{Hom}(S^1, T^n)$ is naturally isomorphic to $\mathbb{Z}^n$ and we identify them. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{Z}^n$ and $e: \{Q_1, \ldots, Q_n\} \to \mathbb{Z}^n = \text{Hom}(S^1, T^n)$ be the characteristic function assigning $e_i$ to $Q_i$. Then we have a $T^n$-space $X(Q, e)$ which is actually a manifold because $\{e_i\}_{i=1}^n$ is a basis of $\mathbb{Z}^n$.

The characteristic vectors $v_i \in N = \text{Hom}(S^1, T)$ define an epimorphism $\tilde{v}: T^n \to T$ sending $(h_1, \ldots, h_n)$ to $\prod_{i=1}^n v_i(h_i)$. One can see that the surjective map from $Q \times T^n$ to $Q \times T$ sending $(q, t)$ to $(q, \tilde{v}(t))$ descends to a $\tilde{v}$-equivariant map from $X(Q, e)$ to $X(Q, v)$ and further descends to a homeomorphism

$$X(Q, e)/\ker \tilde{v} \approx X(Q, v).$$
Here $|\ker \tilde{v}| = |N/\hat{N}|$ by Lemma 6.2 and it is coprime to $p$ by assumption. Moreover, since $\ker \tilde{v}$ is a subgroup of the connected group $T^n$ acting on $X(Q,e)$, the induced action of $\ker \tilde{v}$ on $H^*(X(Q,e);\mathbb{Z}/p)$ is trivial. Therefore we have

$$H^*(X(Q,e)/\ker \tilde{v};\mathbb{Z}/p) \cong H^*(X(Q,e);\mathbb{Z}/p)$$

(see [2, Theorem 2.4 in p.120]) and hence it suffices to prove that $X(Q,e)$ has the same cohomology as $S^{2n}$ with $\mathbb{Z}/p$-coefficients.

Since $Q$ has the same face poset as $\hat{Q}^n$ and every face of $\hat{Q}^n$ is contractible, there is a face preserving map $f : Q \to \hat{Q}^n$ which induces an isomorphism on the face posets. Since $Q$ is face $p$-acyclic, $f$ induces an isomorphism on cohomology with $\mathbb{Z}/p$-coefficients at each face. Similarly to the definition of $e$, one has a characteristic function on $\hat{Q}^n$, also denoted by $e$. Then the map from $Q \times T^n$ to $\hat{Q}^n \times T^n$ sending $(q,t)$ to $(f(q),t)$ descends to a map

$$X(Q,e) \to X(\hat{Q}^n,e)$$

which induces an isomorphism on cohomology with $\mathbb{Z}/p$-coefficients. Since $X(\hat{Q}^n,e)$ is homeomorphic to $S^{2n}$, this proves the desired result.

\[ \square \]

**Case 2.** Since $Q$ has the same face poset as the $n$-simplex $\Delta^n$, $Q$ has $n+1$ facets $Q_1, \ldots, Q_{n+1}$ and $n+1$ vertices $q_1, \ldots, q_{n+1}$. We number them in such a way that $q_i$ is the unique vertex not contained in $Q_i$. It follows from (6.1) and Lemma 6.3 that

$$\mu(Q) = \gcd(d_Q(q_1), \ldots, d_Q(q_{n+1})) = \gcd(d_Q(q_1), \ldots, d_Q(q_i), \ldots, d_Q(q_{n+1}))$$

and

$$\mu(Q_i)$$

divides $\mu(Q)$ for any $i \in [n+1]$.

In fact, the former identity in (7.1) follows from (6.1). The latter identity with $i = n+1$ follows from Lemma 6.3 but the same proof of Lemma 6.3 works for any $i$ and proves the desired identity. Similarly, the last assertion in (7.1) also follows from (the proof of) Lemma 6.3

**Proposition 7.2.** Suppose that $Q$ is face $p$-acyclic, has the same face poset as $\Delta^n$ and $\mu(Q)$ is coprime to $p$. Then $H^{odd}(X(Q,v);\mathbb{Z}/p) = 0$.

**Proof.** We abbreviate $X(Q,v)$ as $X$. We prove the proposition by induction on $n$. When $n = 1$, $Q$ is a closed interval and $X$ is homeomorphic to $S^2$; so the proposition holds in this case. We assume that the proposition holds for any face $p$-acyclic $(n-1)$-dimensional manifold with corners satisfying the assumption in the proposition. For every $i$, $Q_i$ has the same face poset as $\Delta^{n-1}$ and $\mu(Q_i)$ by (7.1), so $H^{odd}(X_i;\mathbb{Z}/p) = 0$ by the induction assumption, where $X_i = \pi^{-1}(Q_i)$ and $\pi : X \to Q$ is the quotient map. On the other hand, since $\mu(Q) = \gcd(d_Q(q_1), \ldots, d_Q(q_{n+1}))$ is coprime to $p$ by assumption, $d_Q(q_i)$ is coprime to $p$ for some $i$. For such $i$, $Q_i$ is face $p$-acyclic, has the same face poset as $\hat{Q}^n$ and $\mu(Q/Q_i) = d_Q(q_i)$ is coprime to $p$, so $H^{odd}(X/X_i;\mathbb{Z}/p) = 0$ by Proposition 7.1. These together with the exact sequence

$$\to H^{odd}(X/X_i;\mathbb{Z}/p) \to H^{odd}(X;\mathbb{Z}/p) \to H^{odd}(X_i;\mathbb{Z}/p) \to$$

show $H^{odd}(X;\mathbb{Z}/p) = 0$. \[ \square \]

**Case 3.** We denote the facets of $Q$ corresponding to $\Delta^{n-1} \times \{\pm 1\}$ by $Q_\pm$ and the others by $Q_1, \ldots, Q_n$. Accordingly, we abbreviate the characteristic vectors $v(Q_\pm)$ as
We abbreviate Proof. n on poset as ∆. Proposition 7.4.

In Case 3.

On the other hand, since v loss of generality we may assume (7.2) and the exact sequence H by Proposition 7.2.

of Q images of v (7.3) Hq by definition and hence (7.4) p coprime to Q/Q by Proposition 7.4. There is a vertex p of Q such that dQ(q) is coprime to p.

We will prove this lemma later. It suffices to prove the following for our purpose in Case 3.

Proposition 7.4. Suppose that Q is face p-acyclic, has the same face poset as ∆ n× [−1, 1]. If µ(Q) is coprime to p and either µ(Q+) or µ(Q−) is coprime to p, then there is a vertex q of Q such that dQ(q) is coprime to p.

Proof. We abbreviate X(Q, v) as X and denote by Xε (ε = + or −) the inverse image of Qε by the quotient map π: X → Q. Since Qε is face p-acyclic, has the same face poset as ∆ n× and µ(Qε) is coprime to p by assumption, we have

Hodd(Xε; ℤ/p) = 0

by Proposition 7.2.

By Lemma 7.3 there is a vertex q of Q such that dQ(q) is coprime to p. Without loss of generality we may assume q = q−, i.e. dQ(q−) is coprime to p. Since we have (7.2) and the exact sequence

Hodd(X/Q; ℤ/p) → Hodd(X; ℤ/p) → Hodd(X/Q; ℤ/p) →,

it suffices to prove

Hodd(X/Q; ℤ/p) = 0.

We consider two cases.

Case a. The case where det(v1, . . . , vn) = 0. In this case, the characteristic function v on Q induces a characteristic function on Q/Q+, denoted v+, and X/X+ = X(Q/Q+, v+). We note that Q/Q+ is face p-acyclic and has the same face poset as ∆ n since Q is face p-acyclic and has the same poset as ∆ n× [−1, 1]. Moreover, since q− is a vertex of Q/Q+, and dQ/Q+(q−) = dQ(q−) is coprime to p, µ(Q/Q+) is coprime to p. Therefore, (7.3) follows from Proposition 7.2.

Case b. The case where det(v1, . . . , vn) = 0.

Claim. There is a vertex q of Qn such that dQn(q) is coprime to p, so µ(Qn) is coprime to p.

Proof. Write vi = (v1, . . . , vn)T and v− = (v1, . . . , vn)T. Since vn is primitive, we may assume vn = (0, . . . , 0, 1)T by Theorem 6.1. Denote by ̃ vi and ̃ v− the projection images of vi and v− on ℤn/(vn) and by v′ i and v′− the primitive vectors which have the same directions as ̃ vi and ̃ v− respectively. Then

dQn(q−i) = |det(v′ 1, . . . , v′ i, . . . , v′ n−1, v′−)|

by definition and hence

(7.4) dQn(q−i)|det( ̃ v1, . . . , ̃ vi, . . . , ̃ vn−1, ̃ v−)|

On the other hand, since vn = (0, . . . , 0, 1)T, we have

det(v1, . . . , vn) = det( ̃ v1, . . . , ̃ vn−1)
and the left hand side above is zero by assumption. It follows that
\[ d_Q(q_n^-) = |\det(v_1, \ldots, v_{n-1}, v_n)| \]
\[ = |v_n^n \det(\tilde{v}_1, \ldots, \tilde{v}_{n-1}) + \sum_{j=1}^{n-1} v_j^n (-1)^{n-j} \det(\tilde{v}_1, \ldots, \tilde{v}_j, \tilde{v}_{n-1}, \tilde{v}_n)| \]
\[ = \sum_{j=1}^{n-1} v_j^n (-1)^{n-j} \det(\tilde{v}_1, \ldots, \tilde{v}_j, \tilde{v}_{n-1}, \tilde{v}_n) \]
where the second identity above is the expansion of \(\det(v_1, \ldots, v_{n-1}, v_n)\) with respect to the \(n\)th row. By (7.6) \(\gcd(d_Q(q_1^-), \ldots, d_Q(q_{n-1}^-))\) divides the last term above. Since \(d_Q(q_n^-)\) is coprime to \(p\), this means that \(d_Q(q_i^-)\) is coprime to \(p\) for some \(i\), proving the claim.

Now we shall prove (7.3) by induction on the dimension \(n\) of \(Q\). When \(n = 1\), \(Q\) is a closed interval, \(X = S^2\) and \(X_+\) is a point; so (7.3) holds in this case. We assume \(n \geq 2\) in the following. Let \(X_n\) be the inverse image of \(Q_n\) by the quotient map \(\pi: X \to Q\). The face poset of \(Q_n\) is the same as that of \(\Delta^{n-2} \times [-1, 1]\) and \(Q_n\) is face \(p\)-acyclic. The facets corresponding to \(\Delta^{n-2} \times \{\pm 1\}\) are \(Q_n \cap Q_{\pm}\) and \(\mu(Q_n \cap Q_{\pm})\) are coprime to \(p\) by (7.4) because \(\mu(Q_{\pm})\) are coprime to \(p\) by assumption. Moreover, \(\mu(Q_n)\) is also coprime to \(p\) by the claim above. Therefore,
\[ H_{\text{odd}}(X_n/(X_n \cap X_+); \mathbb{Z}/p) = 0 \]
by the induction assumption.

The quotient \(Q/(Q_n \cup Q_{\pm}) =: \tilde{Q}\) is face \(p\)-acyclic and \(\tilde{Q}\) has the same face poset as \(Q^n\). The characteristic function \(v\) on \(Q\) induces a characteristic function on \(\tilde{Q}\), denoted \(\tilde{v}\), because \(q_n^-\) is a vertex of \(Q\) and \(d_Q(q_n^-) = d_Q(q_n^-)\) is coprime to \(p\), in particular nonzero. The quotient space \(X_n/(X_n \cap X_+)\) is a subspace of \(X/X_+\) and
\[ (X/X_+) \big/ (X_n/(X_n \cap X_+)) = X(\tilde{Q}, \tilde{v}). \]
Since \(d_Q(q_n^-) = \mu(\tilde{Q})\) is coprime to \(p\), \(H_{\text{odd}}(X(\tilde{Q}, \tilde{v}); \mathbb{Z}/p) = 0\) by Proposition 7.1. This together with (7.5), (7.6) and the exact sequence
\[ \rightarrow H_{\text{odd}}((X/X_+) \big/ (X_n/(X_n \cap X_+)); \mathbb{Z}/p) \rightarrow H_{\text{odd}}(X/X_+; \mathbb{Z}/p) \]
implies (7.3).

Now it remains to prove Lemma 7.3.

Proof of Lemma 7.3. We may assume that \(\mu(Q_+)\) is coprime to \(p\). We may also assume that \(v_+ = (0, \ldots, 0, 1)^T\) by Theorem 6.1 through some identification of \(N\) with \(\mathbb{Z}^n\). Suppose that
\[ p|d_Q(q) \text{ for all vertices } q \text{ of } Q \]
and we will deduce a contradiction in the following.

By Lemma 6.3 \(\det(v_1, \ldots, v_n)\) is divisible by \(\gcd(d_Q(q_1^n), \ldots, d_Q(q_n^n))\), so it follows from (7.7) that
\[ p|\det(v_1, \ldots, v_n). \]
We write \(v_i = (v_{i1}^1, \ldots, v_{in})^T \in \mathbb{Z}^n\) for \(i = 1, 2, \ldots, n\).
Claim 1. There is an $i \in [n]$ such that $p \not| v_i^j$ for all $j \neq n$.

Proof. Since $v_+ = (0, \ldots, 0, 1)^T$, we naturally identify the quotient lattice $\mathbb{Z}^n/\langle v_+ \rangle$ with $\mathbb{Z}^{n-1}$ and then the projection image $\bar{v}_i$ of $v_i$ on the quotient lattice $\mathbb{Z}^{n-1}$ is $(v_i^1, \ldots, v_i^{n-1})$. Set $s_i = \gcd(v_i^1, \ldots, v_i^{n-1})$. Then $\bar{v}_i/s_i =: v_i'$ is primitive. Since $d_Q(q)$ is assumed to be divisible by $p$ for all vertices $q$ of $Q$, we have

\begin{equation}
\tag{7.9}
 p \mid \det(v_i, \ldots, v_{i-1}, v_+) \quad \text{for every subset } \{i_1, \ldots, i_{n-1}\} \text{ of } [n].
\end{equation}

Here, since $v_+ = (0, \ldots, 0, 1)^T$, we have

\begin{equation}
\tag{7.10}
 \det(v_i, \ldots, v_{i-1}, v_+) = \det(\bar{v}_i, \ldots, \bar{v}_{i-1}) = (\prod_{k=1}^{n-1} s_{ik}) \det(v'_i, \ldots, v'_{i-1}).
\end{equation}

Now suppose that $s_i$ is not divisible by $p$ for any $i$. Then it follows from (7.9) and (7.10) that $p \mid \det(v'_i, \ldots, v'_{i-1})$ for every subset $\{i_1, \ldots, i_{n-1}\}$ of $[n]$. Since $\mu(Q_+)$ agrees with the greatest common divisor of all $\det(v'_i, \ldots, v'_{i-1})$ by (6.1), this shows that $p\mid \mu(Q_+)$ which contradicts the assumption that $\mu(Q_+)$ is coprime to $p$. Therefore $p\mid s_i$, for some $i$, proving the claim.

Claim 2. $p \mid \det(v_i, \ldots, v_{i-2}, v_-, v_+)$ for every subset $\{i_1, \ldots, i_{n-2}\}$ of $[n]$.

Proof. Since $v_+ = (0, \ldots, 0, 1)^T$, we have

\begin{equation}
\tag{7.11}
 \det(v_i, \ldots, v_{i-2}, v_-, v_+) = \det(\bar{v}_i, \ldots, \bar{v}_{i-2}, \bar{v}_-)
\end{equation}

where $\bar{v}_- = (v_1^1, \ldots, v_1^{n-1})^T$ is the projection image of $v_-$ on the quotient $\mathbb{Z}^{n}/\langle v_+ \rangle = \mathbb{Z}^{n-1}$. We shall observe that the right hand side in (7.11) is divisible by $p$. Without loss of generality we may assume that the $i$ in Claim 1 is $n$, so that $p \not| v_n^j$ for all $j \neq n$. We consider two cases.

Case a. The case where $n \in \{i_1, \ldots, i_{n-2}\}$. Since $\bar{v}_n = (v_1^1, \ldots, v_1^{n-1})^T$ and $p \not| v_n^j$ for all $j \neq n$, the right hand side in (7.11) is divisible by $p$.

Case b. The case where $n \notin \{i_1, \ldots, i_{n-2}\}$. In this case, we consider the expansion of $\det(v_i, \ldots, v_{i-2}, v_-, v_n)$ with respect to the last column. Since $v_n = (v_1^1, \ldots, v_1^{n-1})^T$ and $p \not| v_n^j$ for all $j \neq n$, we have

\begin{equation}
\tag{7.12}
 |\det(v_i, \ldots, v_{i-2}, v_-, v_n) | \equiv |v_n^j \det(\bar{v}_i, \ldots, \bar{v}_{i-2}, \bar{v}_-)| \quad (\text{mod } p).
\end{equation}

Here the left hand side above is $d_Q(q)$ for $q = (\bigcap_{k=1}^{n-2} Q_{i_k}) \cap Q_+ \cap Q_n$, so it is divisible by $p$ by (7.7). Moreover, $v_n^j$ is not divisible by $p$ because otherwise every entry of $v_n$ is divisible by $p$ and this contradicts $v_n$ being primitive. It follows from (7.12) that the right hand side in (7.11) is divisible by $p$ in this case, too.

This completes the proof of the claim.

Now (7.7), (7.8) and Claim 2 show that all $n \times n$ minors of $(v_1, \ldots, v_n, v_-, v_+)$ are divisible by $p$ and hence $p \mid \mu(Q)(= |N/N|)$ by Theorem 6.1. This contradicts the assumption that $\mu(Q)$ is coprime to $p$, proving the lemma.

8. Example

In this section we shall give an example of a compact simplicial toric variety showing that the converse of Proposition 5.2 does not hold in general.

Let $Q$ be the 3-dimensional simple polytope with the 7 facets $Q_+, Q_-, Q_1, \ldots, Q_5$, where $Q_4$ and $Q_5$ are triangles obtained by cutting two vertices of a prism, shown in Figure 1 below. The polytope $Q$ can be obtained from $\vartriangle^3$ by performing a vertex cut four times.
Let $d$ be a positive integer. To the 7 facets $Q_1, \ldots, Q_5, Q_+, Q_-$, we respectively assign the following vectors:

\[
\begin{align*}
v_1 &= (1, 0, 0) \\
v_2 &= (-1, d, -d) \\
v_4 &= (0, 1, 0) \\
v_5 &= (d, 1 - d, -d) \\
v_+ &= (0, 0, 1) \\
v_- &= (1, -1, -1),
\end{align*}
\]

giving a characteristic function $v$ on $Q$. There are ten vertices in $Q$. At each vertex, there are exactly three facets meeting and the determinant of the three vectors assigned to the facets is nonzero, indeed their absolute values are as follows:

\[
\begin{align*}
|\det(v_1, v_4, v_+)| &= |\det(v_2, v_4, v_+)| = |\det(v_1, v_5, v_-)| = 1 \\
|\det(v_1, v_2, v_4)| &= |\det(v_1, v_3, v_+)| = |\det(v_1, v_3, v_-)| = d \\
|\det(v_1, v_2, v_5)| &= d(2d - 1) \\
|\det(v_2, v_3, v_-)| &= d(d + 1) \\
|\det(v_2, v_3, v_+)| &= 2d. \\
\end{align*}
\]

(Precisely speaking, the vectors are regarded as column vectors here by taking transpose.) Therefore, at each vertex, the cone spanned by the three vectors is 3-dimensional and has the origin as the apex. One can also check that

\[
\begin{align*}
v_4 &= (v_1 + v_2 + dv_+)/d \\
v_5 &= ((d + 1)v_1 + v_2 + (2d - 1)v_-)/2d \\
v_+ &= -(2v_1 + v_2 + v_3)/d \\
v_- &= ((d + 3)v_1 + v_2 + 2v_3)/d.
\end{align*}
\]

Since $d$ is a positive integer, this shows that $-v_+$ is in the cone $\angle v_1v_2v_3$ and $v_4$ is in the cone $\angle v_1v_2v_+$ while $v_-$ is in the cone $\angle v_1v_2v_3$ and $v_5$ is in the cone $\angle v_1v_2v_-$ (see Figure 2), where $\angle uvw$ denotes the cone spanned by vectors $u, v, w$. This implies that the ten 3-dimensional cones have no overlap and cover the entire $\mathbb{R}^3$, giving a complete simplicial fan so that the quotient space $X = X(Q, v)$ is homeomorphic to a compact simplicial toric variety.

We shall check that $\mu(Q_I) = 1$ for each face $Q_I$ of $Q$, where $\mu(Q_I)$ is defined in Section 5. As remarked in Section 5, $\mu(Q_I) = 1$ when $|I| = 2$ or 3. Clearly $\tilde{N} = N(= \mathbb{Z}^3)$. Therefore it suffices to check $\mu(Q_I) = 1$ when $|I| = 1$. At vertices $Q_1 \cap Q_4 \cap Q_+, Q_2 \cap Q_4 \cap Q_+$ and $Q_1 \cap Q_5 \cap Q_-$, we have

\[
|\det(v_1, v_4, v_+)| = |\det(v_2, v_4, v_+)| = |\det(v_1, v_5, v_-)| = 1.
\]
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Figure 2. Each vector \( v_i \) is denoted by a point in \( \mathbb{R}^2 \cup \{\infty\} \) and a segment connecting \( v_i, v_j \) corresponds to the 2-dimensional cone spanned by them and a triangle formed by \( v_i, v_j, v_k \) corresponds to the 3-dimensional cone spanned by them.

and hence \( \mu(Q_I) = 1 \) for every \( I \) with \( |I| = 1 \) except \( I = \{3\} \) again by the remark in Section 5. In order to see \( \mu(Q_3) = 1 \), we note that \( \{v_3, v_4, v_+\} \) is a base of \( N \) and

\[
\begin{align*}
v_1 &= -v_3 - dv_4, \\
v_2 &= v_3 + 2dv_4 - dv_+.
\end{align*}
\]

Therefore, the images of \( v_1 \) and \( v_2 \) by the quotient map \( \pi_{\{3\}}: N \to N(\{3\}) = N/(v_3) \) are \((-d,0)\) and \((2d,-d)\) with respect to the base \( \{\pi_{\{3\}}(v_4), \pi_{\{3\}}(v_+)\} \). Thus the corresponding primitive vectors are \((-1,0)\) and \((2,-1)\) which form a base of \( N(\{3\}) \).

Hence \( \mu(Q_3) = 1 \).

We shall compute \( H^3(X) \). Take a plane in \( \mathbb{R}^3 \) which meets the facets \( Q_1, Q_2, Q_3 \) transversally and does not meet the other facets of \( Q \). Cutting \( Q \) along the plane, we divide \( Q \) into two polytopes, denoted \( P_+ \) and \( P_- \) containing \( Q_+ \) and \( Q_- \) respectively. Let \( \pi: X \to Q \) be the quotient map and set

\[
Y_\varepsilon := \pi^{-1}(P_\varepsilon) \quad \text{for} \quad \varepsilon = \pm, \quad Y := Y_+ \cap Y_-, \quad P := P_+ \cap P_-.
\]

The quotient space \( P_\varepsilon/P \) can be regarded as a prism. The characteristic function \( v \) on \( Q \) induces a characteristic function on \( P_\varepsilon/P \), denoted \( w_\varepsilon \), and \( X/Y_+ = Y_-/Y \) (resp. \( X/Y_- = Y_+/Y \)) is homeomorphic to \( X(P_\varepsilon/P, w_-) \) (resp. \( X(P_\varepsilon/P, w_+) \)). The same argument as above shows that \( \mu \) takes 1 on all faces of the prism \( P_\varepsilon/P \), so

(8.1) \( H^*(X, Y_\varepsilon) \) and \( H^*(Y_\varepsilon, Y) \) are torsion free and vanish in odd degrees

by Proposition 7.1.

Let \( \tilde{Q} \) be a nice manifold with corners obtained from \( Q \) by collapsing \( Q_4 \cup Q_+ \) and \( Q_5 \cup Q_- \) to a point respectively. The \( \tilde{Q} \) has three facets coming from \( Q_1, Q_2, Q_3 \) and the characteristic function \( v \) on \( Q \) induces a characteristic function \( \tilde{v} \) on \( \tilde{Q} \). Since

\[
\begin{align*}
v_1 &= (1, 0, 0), \\
v_2 &= (-1, d, -d), \\
v_3 &= (-1, -d, 0),
\end{align*}
\]
one can see that $H^4(X(\tilde{Q}, \tilde{v})) \cong \mathbb{Z}/d$ by Corollary 12 and since $X(\tilde{Q}, \tilde{v})$ is homeomorphic to the suspension of $Y$, we obtain

(8.2) \quad H^3(Y) \cong \mathbb{Z}/d.

Now, consider the exact sequence in cohomology for the pair $(Y_+, Y)$:

(8.3) \quad \rightarrow H^3(Y_+, Y) \rightarrow H^3(Y_+) \rightarrow H^3(Y) \rightarrow H^4(Y_+, Y) \rightarrow .

Since $H^3(Y_+, Y) = 0$ and $H^4(Y_+, Y)$ is torsion free by (8.1) and $H^3(Y)$ is a torsion group by (8.2), it follows from the exact sequence (8.3) that

(8.4) \quad H^3(Y_+) \cong H^3(Y) \cong \mathbb{Z}/d.

Next, consider the exact sequence in cohomology for the pair $(X, Y_+)$:

(8.5) \quad \rightarrow H^3(X, Y_+) \rightarrow H^3(X) \rightarrow H^3(Y_+) \rightarrow H^4(X, Y_+) \rightarrow .

Similarly to the above argument, $H^3(X, Y_+) = 0$ and $H^4(X, Y_+)$ is torsion free by (8.1) and $H^3(Y_+)$ is a torsion group by (8.4), so it follows from the exact sequence (8.5) that

$H^3(X) \cong H^3(Y_+) \cong \mathbb{Z}/d.$

Thus $X = X(Q, v)$ is the desired example when $d \geq 2$.

**Appendix**

In this appendix, we observe that when $X$ is a compact simplicial toric variety of complex dimension $n$, a result of Fischli [7] or Jordan [11] implies that $H^{2n-1}(X) \cong N/N$ and $\text{Tor} H^{2n-2}(X) \cong \wedge^2 N/(N \wedge N)$, where $\text{Tor} H^{2n-2}(X)$ denotes the torsion part of $H^{2n-2}(X)$. This result agrees with Proposition 2.2 since $Q$ is contractible in this case.

Let $\Delta$ be a simplicial complete fan of dimension $n$ and let $X$ be the associated compact simplicial toric variety. Let $M$ be the free abelian group dual to $N$. Since $N = \text{Hom}(S^1, T)$, $M$ can be thought of as $\text{Hom}(T, S^1)$. According to [7 Theorem 2.3] or [11 Theorem 2.5.5],

$H^{2n-1}(X) \cong \text{coker } \delta_1, \quad \text{Tor } H^{2n-2}(X) \cong \text{coker } \delta_2,$

where

(8.6) \quad \delta_r : \bigoplus_{\tau \in \Delta^{(1)}} \wedge^{n-r}(\tau^\perp \cap M) \rightarrow \wedge^{n-r}M \quad (r = 1, 2)

is the sum of inclusion maps with signs, $\Delta^{(1)}$ denotes the set of one-dimensional cones in $\Delta$ and $\tau^\perp$ denotes the subspace of $M \otimes \mathbb{R}$ which vanish on $\tau$.

We shall interpret the above in terms of $N$. Let $\sigma$ be a cone of dimension $n-k$ in $\Delta$. Then we have

(8.7) \quad \wedge^\ell(\sigma^\perp \cap M) \cong \text{Hom}(\wedge^{k-\ell}(\sigma^\perp \cap M), \mathbb{Z}) \quad (: \text{ rank } \sigma^\perp \cap M = k) \cong \wedge^{k-\ell}(N/N_\sigma) \quad (: \text{ } N/N_\sigma \text{ is dual to } \sigma^\perp \cap M) \cong (\wedge^{n-k}N_\sigma) \wedge (\wedge^{k-\ell} N)

where $N_\sigma$ is the intersection of $N$ with the subspace of $N \otimes \mathbb{R}$ spanned by $\sigma$. The last isomorphism above is given as follows. Choose a base $\rho_1, \ldots, \rho_{n-k}$ of $N_\sigma$. Since $N_\sigma$ is of rank $n-k$, $\wedge^{n-k}N_\sigma$ is a free abelian group of rank one and $\rho_1 \wedge \cdots \wedge \rho_{n-k}$
is its generator. For \( w \in N \), we denote by \([w]\) the element of \( N/N_\sigma \) determined by \( w \). Then the following correspondence

\[
[w_1] \wedge \cdots \wedge [w_{k-\ell}] \rightarrow \rho_1 \wedge \cdots \wedge \rho_{n-k} \wedge w_1 \wedge \cdots \wedge w_{k-\ell}
\]

is well defined and gives the desired isomorphism from \( \wedge^{k-\ell}(N/N_\sigma) \) to \( (\wedge^{n-k}N_\sigma) \wedge (\wedge^{k-\ell}N) \). This isomorphism is independent of the choice of the base \( \rho_1, \ldots, \rho_{n-k} \) up to sign. Namely, the isomorphism (8.7) depends only on the choice of orientations on \( M \) (or \( N \)) and \( \sigma \).

Applying (8.7) to \( \sigma = \tau \in \Delta^{(1)} \) and \( \sigma = 0 \), we obtain

\[
\wedge^{n-1}(\tau^\perp \cap M) \cong N_{r}, \quad \wedge^{n-1} M \cong N,
\]

\[
\wedge^{n-2}(\tau^\perp \cap M) \cong N_{r} \wedge N, \quad \wedge^{n-2} M \cong \wedge^{2} N.
\]

Since \( \delta_r \) is the sum of inclusion maps with signs, the image of \( \delta_1 \) (resp. \( \delta_2 \)) in (8.6) can be identified with \( \hat{N} \) (resp. \( \hat{N} \wedge N \)) and hence

\[
H^{2n-1}(X) \cong E_2^{n-1} \cong N/\hat{N}, \quad \text{Tor } H^{2n-2}(X) \cong E_2^{n-2} \cong \wedge^2 N/(\hat{N} \wedge N).
\]

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