A systematic way to find and construct exact finite dimensional
matrix product stationary states

Y. Hieida
Computer and Network Center, Saga University,
Saga 840-8502, JAPAN
*E-mail: hieida@cc.saga-u.ac.jp

T. Sasamoto
Department of Mathematics and Informatics,
Faculty of Science, Chiba University,
1-33 Yayoi-cho, Inage, Chiba 263-8522, Japan

We explain how to construct matrix product stationary states which are
composed of finite-dimensional matrices. Our construction explained in this
article was first presented in a part of Ref. 1 for general models. In this article,
we give more details on the treatment than in the above-mentioned reference,
for one-dimensional asymmetric simple exclusion process (ASEP).

Keywords: stationary state, matrix product state, asymmetric simple exclusion
process (ASEP), PASEP

1. Introduction

Evolution of the probabilities of configurations in a class of one-dimensional
stochastic models is described by the master equation like this:

\[ \frac{d}{dt} \vec{P}_L(t) = -H \vec{P}_L(t) , \]  

where \( H \) is a transition rate matrix and \( \vec{P}_L(t) \) is a column vector whose
component \( P(C; t) \) is the probability of finding the \( L \)-site system in a con-
figuration \( C \) at time \( t \). If \( \vec{P}_L(t) \) is independent of \( t \), we obtain from Eq. (1)

\[ 0 = H \vec{P}_L . \]  

We call the solution \( \vec{P}_L \) of this equation the \textbf{stationary state}. Usually, as
\( t \rightarrow \infty \), \( \vec{P}_L(t) \) reaches a stationary state \( \vec{P}_L \). So, once we have the stationary
state, we can calculate, for example, the density profile, correlations, etc.
in the long-time limit. From this viewpoint, obtaining stationary states is important in nonequilibrium statistical mechanics.

For small $L$, one can solve Eq. (2) numerically and study its properties exactly. But, in general, obtaining stationary state for any system size $L$ as a function of $L$ is a difficult task.

It is known that, for some one-dimensional stochastic models, their stationary states can be written in a special form, namely, in the form of matrix products. We call this the matrix product stationary state (MPSS). We consider the situation in which this matrix is independent of system size; in this case, we can construct its stationary state for any system size.

Thus we want to find matrices which compose a MPSS. Although, the dimensions of those matrices are generally infinite for a model whose interaction range is finite, the dimensions can be finite for some models including the ASEP under some conditions for model parameters. We call a MPSS for the former (resp. latter) case an infinite (resp. a finite)-dimensional MPSS. To our knowledge, there is no systematic way of constructing an infinite-dimensional MPSS. But, for a finite-dimensional MPSS, we found a systematic way of finding them, which we explain in this article.

Our construction explained in this article was first presented in a part of Ref. 1, where the above-mentioned construction and a systematic way of checking the validity of the obtained MPSS for an arbitrary system size were explained for general stochastic models on the one-dimensional lattice. Main purpose of this article is to give more details on the treatment in Ref. 1. In order to explain our construction concretely, we use in this article only the one-dimensional asymmetric simple exclusion process (ASEP).

It is interesting that numerical matrix product states in stochastic models (for example, Ref. 8) also play an important role in the method of density matrix renormalization group (DMRG). This article, however, does not treat such a numerical method.

2. Model

The ASEP in this article is defined on the one-dimensional lattice whose size is $L$ (Fig. 1). Each site can take two states, namely, a site is either empty or occupied by one particle. The time of the model is continuous. A particle in the bulk hops to the left (resp. right) neighbor site with a rate $q$ (resp. 1) if the left (resp. right) neighbor site is empty. At the leftmost site, a particle

---

*aThe ASEP in this article is also called Partially ASEP (PASEP).
is injected with a rate $\alpha$ if the site is empty. A particle at the rightmost site is removed with a rate $\beta$ if the site is occupied by a particle.

As for the master equation, for example, $\vec{P}_{L=4}(t)$ in Eq. (1) has the following components:

$$
\vec{P}_{L=4}(t) = \begin{pmatrix}
P(0, 0, 0, 0; t) \\
P(0, 0, 0, 1; t) \\
P(0, 0, 1, 0; t) \\
P(0, 0, 1, 1; t) \\
P(0, 1, 0, 0; t) \\
P(0, 1, 0, 1; t) \\
P(0, 1, 1, 0; t) \\
P(0, 1, 1, 1; t) \\
P(1, 0, 0, 0; t) \\
P(1, 0, 0, 1; t) \\
P(1, 0, 1, 0; t) \\
P(1, 0, 1, 1; t) \\
P(1, 1, 0, 0; t) \\
P(1, 1, 0, 1; t) \\
P(1, 1, 1, 0; t) \\
P(1, 1, 1, 1; t)
\end{pmatrix} \tag{3}
$$

where $\tau_k \ (1 \leq k \leq 4)$ in the component $P(\tau_1, \tau_2, \tau_3, \tau_4; t)$ of Eq. (3) represents the number of a particle at the $k$-th site. The transition rate matrix
in Eq. (1) for the ASEP is defined as follows:

$$H := h^{(L)} + \sum_{k=1}^{L-1} h_k + h^{(R)}.$$  \hspace{1cm} (4)

In this equation, $h^{(L)}$ (resp. $h^{(R)}$) which expresses injections of particles at the leftmost site (resp. removals of particles at the rightmost site), is defined by

$$h^{(L)} := h^{\text{left}} \otimes I \otimes (L-1) \quad \text{and} \quad h^{(R)} := I \otimes (L-1) \otimes h^{\text{right}},$$ \hspace{1cm} (5)

where

$$h^{\text{left}} := \begin{pmatrix} \alpha & 0 \\ -\alpha & 0 \end{pmatrix}$$ \hspace{1cm} (6)

$$h^{\text{right}} := \begin{pmatrix} 0 & -\beta \\ 0 & \beta \end{pmatrix}.$$ \hspace{1cm} (7)

Eq. (6) (resp. Eq. (7) ) is expressed in a basis of states whose order is $\tau_1 = 0, 1$ (resp. $\tau_L = 0, 1$). In Eq. (5), we introduce a shorthand notation of direct products of the two-dimensional identity matrix $I$:

$$I^{\otimes n} := I \otimes I \otimes \cdots \otimes I.$$ \hspace{1cm} (8)

In Eq. (4), $h_k$, which describes hopping process between $k$-th and $(k+1)$-th sites is defined

$$h_k := I^{\otimes (k-1)} \otimes h_{\text{int}} \otimes I^{\otimes (L-k-1)},$$ \hspace{1cm} (9)

where

$$h_{\text{int}} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ \hspace{1cm} (10)

This is expressed in a basis of states whose order is $\tau_k, \tau_{k+1} = (0, 0), (0, 1), (1, 0), (1, 1)$.

### 3. Matrix product stationary states (MPSSs)

It is possible that some models including the ASEP have a special form of a stationary state, namely, a matrix-product stationary state (MPSS). For
example, “the vector Eq. (3) can be represented by a MPSS” means that there exist \( \langle W |, E, D \rangle \) and \( | V \rangle \) such that Eq. (3) is equal to
\[
\frac{1}{Z_4} \langle W | (E^{\otimes 4}) | V \rangle.
\] (11)

Generally, the MPSS for the ASEP is the stationary state which is equal to the following form
\[
\frac{1}{Z_L} \langle W | (E^{\otimes L}) | V \rangle.
\] (12)

In Eq. (11) and Eq. (12),
- \( \langle W | \) and \( | V \rangle \) are an \( M \)-dimensional row vector and an \( M \)-dimensional column vector, respectively.
- \( E \) and \( D \) are \( M \)-dimensional square matrices.
- \( Z_L \) is the normalization constant whose definition is
\[
Z_L := \langle W | (E + D)^L | V \rangle.
\] (13)

Hereafter, we call Eq. (12) an \textit{M-dimensional MPSS}. It should be noted that \( M \) is not always finite (in general, \( M = \infty \)) as explained in Sec. 1.

In the following section 4, we derive necessary conditions for the existence of an \( M = 1, 2 \)-dimensional MPSS for the ASEP by using its stationary states for finite system sizes. In the subsequent section (Sec. 5), we explain how to find \( \langle W |, E, D \rangle \) and \( | V \rangle \) in a two-dimensional MPSS assuming the existence of the MPSS.

4. Necessary conditions for the existence of an \( M = 1, 2 \)-dimensional MPSS

In this section, we derive necessary conditions for the existence of an \( M = 1, 2 \)-dimensional MPSS for the ASEP \(^b\).

As a tool of our derivation, we introduce a matrix form of a stationary state \( \tilde{P}_L \). This form has the same components as \( \tilde{P}_L \). The difference between the matrix form and \( \tilde{P}_L \) exists in ways of arranging the components. For example, the matrix form \( P^{2,2} \) is defined by rearranging components of

\(^b\)Ref. 1 describes also derivation for general \( M \)-dimensional MPSSs which some models with \( N \) states per site have. The ASEP in this article corresponds to \( N = 2 \).
According to this definition, the matrix form $P_{2,2}$ converted from the MPSS (Eq. (12) with $L = 4$) is

$$P_{2,2} = \frac{1}{Z_4} \begin{pmatrix}
\langle W|EEEE|V \rangle & \langle W|EEED|V \rangle & \langle W|EDEE|V \rangle & \langle W|EDDD|V \rangle \\
\langle W|EDEE|V \rangle & \langle W|EDDE|V \rangle & \langle W|EDEE|V \rangle & \langle W|EEEE|V \rangle \\
\langle W|DDEE|V \rangle & \langle W|DDDE|V \rangle & \langle W|DDDE|V \rangle & \langle W|DEDD|V \rangle \\
\langle W|DDEE|V \rangle & \langle W|DDED|V \rangle & \langle W|DDDE|V \rangle & \langle W|DEDD|V \rangle
\end{pmatrix},$$  
(15)

Generally, the matrix form $P_{m,n}$ converted from the MPSS (Eq. (12)) can be written down as

$$P_{m,n} := \frac{1}{Z_{m+n}} \langle W| (E \otimes m D \otimes n)|V \rangle,$$  
(16)

with $m + n = L$.

The rank of a matrix form is important to our derivation. Suppose, by elementary transformations, the $2^m \times 2^n$ matrix $P_{m,n}$ can be transformed into a matrix whose form is

$$\begin{pmatrix} I_r & B \\ O & I_r \end{pmatrix},$$  
(17)

where $I_r$, $O$ and $B$ represents the $r \times r$ identity matrix, an $(2^m - r) \times 2^n$ zero matrix and an $r \times (2^n - r)$ matrix, respectively. Then, we can tell that the rank of $P_{m,n}$ is $r$ because elementary transformations do not change ranks of matrices and the rank of Eq. (17) is $r$. Therefore, the rank of $P_{2,2}$ in Eq. (14) is generally greater than 2. However, we can prove the fact that "if the stationary state can be written in a two(resp. one)-dimensional MPSS, the rank of $P_{2,2}$ is at most 2(resp. 1)."  
(18)

We give a proof of this fact only for the case of a two-dimensional MPSS.  

^cThis proof is slightly different from one in Ref. 1.
From Eq. (16) with \( m = n = 2 \), we obtain
\[
Z_4 P^{2,2} := \langle W | (E D)^{\otimes 2} | V \rangle.
\] (19)

The right hand side of this equation can be written as a matrix product
\[
AB
\] (20)
where
\[
A := \langle W | (E D)^{\otimes 2} , B := (E D)^{\otimes 2} | V \rangle.
\] (21)

\( A \) is a \( 4 \times 2 \) matrix and \( B \) is a \( 2 \times 4 \) matrix. Please note that both of the rank of \( A \) and the rank of \( B \) is at most 2. And linear algebra tells us that the rank of \( AB \) is less than or equal to the smaller number of the rank of \( A \) and the rank of \( B \). This concludes the rank of \( P^{2,2} \) is not greater than 2.

Using the fact (18), let us calculate concretely the necessary condition of existence of an \( M(=1,2) \)-dimensional MPSS for
\[
\alpha > 0 , \beta > 0 .
\] (22)

First, we obtain \( \tilde{P}_{L=4} \) by solving Eq. (2) with \( L = 4 \). And then we convert the vector form \( \tilde{P}_{L=4} \) into the matrix form \( P^{2,2} \). In the following, we denote the element in the \( i \)-th row and the \( j \)-th column of a matrix \( A \) by \( (A)_{i,j} \).

According to Maple, which is one of computer algebra systems, we get
\[
(Z_4 P^{2,2})_{1,1} = \left( \frac{\beta}{\alpha} \right)^4 ,
\] (23)

Because this is nonzero element (see Eq. (22)), we can multiply the first row of \( P^{2,2} \) by the inverse of Eq. (23). Then, we subtract the first row of the resultant matrix multiplied by \( (P^{2,2})_{i,1} \) from the \( i \)-th row. Thus we obtain a matrix \( A^{(2)} \) like this:
\[
\begin{pmatrix}
1 & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{pmatrix},
\] (24)

where the elements denoted by “*” mean some expressions. The second diagonal element of \( A^{(2)} \) is this:
\[
(A^{(2)})_{2,2} = -\frac{\beta^2 (\alpha + q) (\alpha - 1 + \beta + q)}{\alpha^2} \propto (\alpha - 1 + \beta + q).
\] (25)
case 1 First, we treat the case
\[ \alpha - 1 + \beta + q \neq 0. \] (27)
Multiplying the second row of \( A^{(2)} \) by the inverse of \( (A^{(2)})_{2,2} \) in Eq. (25), we obtain the matrix \( \tilde{A}^{(2)} \). Then, we subtract the second row (of \( \tilde{A}^{(2)} \)) multiplied by \( (A^{(2)})_{i,2} \) \( (i = 1, 3, 4) \) from the \( i \)-th row of \( \tilde{A}^{(2)} \). Thus we obtain the matrix, which we call \( A^{(3)} \), like this:
\[
\begin{pmatrix}
1 & 0 & * & * \\
0 & 1 & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & * & *
\end{pmatrix}.
\] (28)
It should be noted that any of the elements of the third row of this matrix is zero. \( (A^{(3)})_{4,3} \) is
\[
\left( A^{(3)} \right)_{4,3} = \frac{(q + 1) \left( \alpha \beta + \alpha q - q + q^2 + q \beta \right) (\alpha - 1 + \beta + q) \beta}{\alpha + q} \alpha \beta + \alpha q - q + q^2 + q \beta \left( \alpha - 1 + \beta + q \right)
\] (29)
This is nonzero if we assume Eq. (27) and
\[
\alpha \beta + \alpha q - q + q^2 + q \beta \neq 0.
\] (30)
We interchange the 3rd row and the 4th row of \( A^{(3)} \) and call the resultant matrix \( A^{(3)}' \).
We can multiply the 3rd row of \( A^{(3)}' \) by the inverse of \( (A^{(3)}')_{3,3} \) \( \neq 0 \), we can multiply the 3rd row of \( A^{(3)}' \) by the inverse of \( (A^{(3)}')_{3,3} \). We call the obtained matrix \( \tilde{A}^{(3)}' \). Then, we subtract the 3rd row (of \( \tilde{A}^{(3)}' \)) multiplied by \( (A^{(3)}')_{i,3} \) from the \( i \)-th row. Thus we obtain the matrix
\[
A^{(4)} := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 - \frac{\beta + q}{\beta} \\
0 & 0 & 1 & \beta^{-1} \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (32)
Comparing this with Eq. (17), we know the rank of \( A^{(4)} \) is 3.
case 2 We go back to Eq. (30) and treat the case where Eq. (31) does not hold. In this case,
\[ \alpha \beta + \alpha q - q + q^2 + q \beta = 0. \] (33)
We solve this with respect to \( \alpha \) and obtain
\[ \alpha = -\frac{q(-1 + \beta + q)}{\beta + q}. \] (34)
Substituting this into \( A^{(3)} \), we obtain
\[
\begin{bmatrix}
1 & 0 & \frac{q(-1+\beta+q)^2}{\beta(\beta+q)} & -\frac{q(-1+\beta+q)^2}{\beta^2(\beta+q)} \\
0 & 1 & -\frac{q^2\beta-\beta-2q+q^2}{\beta+q} & -\frac{2q^2+3q\beta-2q-\beta^2}{\beta(\beta+q)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (35)
Comparing this with Eq. (17), we know the rank of Eq. (35) is 2.

case 3 We go back to Eq. (26) and treat the case where Eq. (27) does not hold. In this case,
\[ \alpha - 1 + \beta + q = 0. \] (36)
We solve this with respect to \( \alpha \) and obtain
\[ \alpha = 1 - \beta - q \] (37)
Substituting this into \( A^{(2)} \), we obtain
\[
\begin{bmatrix}
1 & -\frac{1+\beta+q}{\beta} & -\frac{1+\beta+q}{\beta} \frac{(-1+\beta+q)^2}{\beta^2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (38)
Comparing this with Eq. (17), we know the rank of Eq. (38) is 1.

In summary,
- the rank of \( P^{2,2} \) is 3 if \( \alpha - 1 + \beta + q \neq 0 \) and \( \alpha \beta + \alpha q - q + q^2 + q \beta \neq 0 \) are satisfied. \(^d\)

\(^d\)The reason why the rank is not 4 in this case is that the ASEP has infinite-dimensional MPSSs with
\[ (DE - qED) \propto (E + D), \quad D \mid V \propto \mid V \] (39)
and thus \( EE \mid V \}, ED \mid V \}, DE \mid V \} \) and \( DD \mid V \} \) can be written as linear combinations of (at most) three independent vectors.
the rank of $P^{2,2}$ is 2 if $\alpha - 1 + \beta + q \neq 0$ and
\[\alpha \beta + \alpha q - q + q^2 + q \beta = 0\] (40)
are satisfied.

the rank of $P^{2,2}$ is 1 if
\[\alpha - 1 + \beta + q = 0\] (41)
is satisfied.

So we can conclude that according to the fact\(18\), Eq. (40) (resp. Eq. (41)) is the necessary condition of existence of a two (resp. one)-dimensional MPSS. These conditions agree with the known result.\(^5,6,12,13\)

5. How to find two-dimensional matrices and vectors in MPSSs for the ASEP

For the ASEP, the MPSS is constructed according to Eq. (12) with matrices ($E$ and $D$) and vectors ($\langle W |$ and $| V \rangle$). Hereafter we call the matrices and vectors in the MPSS "the set of matrices".

In this section, we explain our way of finding the set of matrices in the two-dimensional MPSS for the ASEP when $\hat{P}_L$ can be expressed as a MPSS. It should be noted that our way of finding is not restricted to two-dimensional MPSS for the ASEP. Our way can be applied to finite $M(\geq 2)$-dimensional MPSSs for models which have the finite numbers $N(\leq M)$ of states per site.\(^1\)

A similarity transformation plays a crucial role in our way of finding. So we begin with an explanation of the transformation in the MPSS.

Let us consider a similarity transformation for the set of matrices
\[\tilde{E} := S^{-1}ES, \quad \tilde{D} := S^{-1}DS, \quad \langle \tilde{W} | := \langle W | S, \quad | \tilde{V} \rangle := S^{-1} | V \rangle,\] (42)
where $S$ is a $2 \times 2$ matrix. Please note that, in this section, $E$ and $D$ are $2 \times 2$ matrices and $\langle W |$ and $| V \rangle$ are two-dimensional vectors. The transformation Eq. (42) does not change a matrix form Eq. (16) including, for example, $P^{2,2}$ in Eq. (15).

We make a choice $S$ in Eq. (42) as \(^c\)
\[S := \begin{pmatrix} \kappa & \mu \\ \lambda & \nu \end{pmatrix},\] (44)
\(^c\)The following set of Eqs. (44) and (45) is what the equation
\[S := (E|V) \ (D|V),\] (43)
which is Eq. (2.28) in Ref. 1, means.
where
\[
\left( \begin{array}{c}
\kappa \\
\lambda
\end{array} \right) := E \ | \ V \rangle , \quad \left( \begin{array}{c}
\mu \\
\nu
\end{array} \right) := D \ | \ V \rangle .
\] (45)

Our choice, namely, the set of Eqs. (44) and (45), plays another crucial role in our way of finding. Because of
\[
S^{-1}S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
S^{-1} \left( \begin{array}{c}
\kappa \\
\lambda
\end{array} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad S^{-1} \left( \begin{array}{c}
\mu \\
\nu
\end{array} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
\] (46)

Using Eqs. (42) and (45) and (46), we can derive
\[
\tilde{E} | \tilde{V} \rangle = S^{-1} E | V \rangle = S^{-1} \left( \begin{array}{c}
\kappa \\
\lambda
\end{array} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .
\] (48)

\[
\therefore \tilde{E} | \tilde{V} \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .
\] (49)

Similarly, we can derive also the equation
\[
\tilde{D} | \tilde{V} \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
\] (50)

Using Eqs. (49) and (50), we can derive the equations for $\tilde{E}$ and $\tilde{D}$
\[
Z_3 P^{1,2}[1 : 2, 1 : 2] = (Z_2 P^{1,1}) \tilde{E}
\] (52)
and
\[
Z_3 P^{1,2}[1 : 2, 3 : 4] = (Z_2 P^{1,1}) \tilde{D}.
\] (53)

In Eqs. (52) and (53), we introduce the notation $A[b : c, d : e]$ for a submatrix of a matrix $A$, which is constructed by selecting the row range from the $b$-th row to the $c$-th row and the column range from the $d$-th column and $e$-th column.

The following set of Eqs. (49) and (50) is what the equation
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left( \tilde{E} | \tilde{V} \rangle \tilde{D} | \tilde{V} \rangle \right),
\] (47)
which is Eq. (2.29) in Ref. 1, means.

The following set of Eqs. (52) and (53) is what the equation
\[
Z_3 P^{1,2} = Z_2 P^{1,1} \left( \tilde{E} \tilde{D} \right),
\] (51)
which is Eq. (2.35) in Ref. 1, means.
The derivation of Eqs. (52) and (53) is postponed in the end of this section. 

Obtaining \( \tilde{E} \) and \( \tilde{D} \) from Eqs. (52) and (53) under Eq. (40) is, of course, easy: By solving Eq. (2) for \( L = 2 \) under Eq. (40), we obtain \( \tilde{P}_{L=2} \). Then we convert the matrix form \( Z_2 \tilde{P}^{1,1} \) from \( Z_2 \tilde{P}_{L=2} \) as we converted Eq. (14) from \( \tilde{P}_{L=4} \). Similarly, we obtain the matrix form \( Z_3 \tilde{P}^{1,2} \). By multiplying the inverse of the \( 2 \times 2 \) matrix \( Z_2 \tilde{P}^{1,1} \) from the left of both sides of Eq. (52)(resp. Eq. (53)), we can obtain the solution \( \tilde{E} \) (resp. \( \tilde{D} \)).

Now that we know \( \tilde{E} \) and \( \tilde{D} \), we can obtain \( |\tilde{V}\rangle \) (resp. \( \langle \tilde{W}| \) ) from Eq. (49) or Eq. (50)(resp. the set of Eqs. (59) and (56) or the set of Eqs. (59) and (60). We can show\(^1\) that the thus obtained \( \tilde{E}, \tilde{D}, |\tilde{V}\rangle \) and \( \langle \tilde{W}| \) are equivalent to the known results.\(^5,6,12,13\)

In the remaining part of this section, we derive only Eq. (52)(we can derive Eq. (53) in the similar way).

By the definition Eq. (16) and the similarity transformation Eq. (42), we have

\[
Z_2 \tilde{P}^{1,1} = \langle \tilde{W}| \left( \begin{matrix} \tilde{E} & \tilde{D} \end{matrix} \right) |\tilde{V}\rangle. \tag{54}
\]

Therefore

\[
Z_2 \tilde{P}^{1,1} = \left( \begin{matrix} \langle \tilde{W}| \tilde{E}^2 |\tilde{V}\rangle & \langle \tilde{W}| \tilde{E} \tilde{D} |\tilde{V}\rangle \\ \langle \tilde{W}| \tilde{D} \tilde{E} |\tilde{V}\rangle & \langle \tilde{W}| \tilde{D}^2 |\tilde{V}\rangle \end{matrix} \right). \tag{55}
\]

Using

\[
\langle \tilde{W}| \tilde{E} =: (x_1, y_1), \tag{56}
\]

\( \langle \tilde{W}| \tilde{E}^2 |\tilde{V}\rangle \) in Eq. (55) is equal to \( x_1 \) in Eq. (56) because

\[
\langle \tilde{W}| \tilde{E}^2 |\tilde{V}\rangle = \langle \tilde{W}| \tilde{E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_1, \tag{57}
\]

where the first (resp. second) equal sign in Eq. (57) holds because of Eq. (49).

---

\(^b\)The derivation starts at the paragraph containing Eq. (54).

\(^i\)In this article, we assume that the necessary condition Eq. (40) for existence of a two-dimensional MPSS is also sufficient condition.
(resp. Eq. (56)). In the similar manner, using also Eq. (50), we can show
\[
Z_2 P^{1,1} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix},
\]
where we introduced
\[
\langle \tilde{W} | \tilde{D} \rangle =: (x_2, y_2).
\]
(60)
By the same way as the derivation of Eq. (54), we have
\[
Z_3 P^{1,2} = \langle \tilde{W} | \tilde{E} \rangle \left( \begin{array}{c} \tilde{E} \\ \tilde{D} \end{array} \right) \otimes | \tilde{V} \rangle.
\]
(61)
From this equation, we obtain the left part of
\[
Z_3 P^{1,2}[1 : 2, 1 : 2] = \left( \begin{array}{c} \langle \tilde{W} | \tilde{E} | \tilde{V} \rangle \\ \langle \tilde{W} | \tilde{D} \tilde{E} | \tilde{V} \rangle \\ \langle \tilde{W} | \tilde{D} \tilde{E} \tilde{D} | \tilde{V} \rangle \end{array} \right).
\]
(62)
Using Eq. (49), the left upper element in the right hand side of this equation can be transformed as
\[
\langle \tilde{W} | \tilde{E}^3 | \tilde{V} \rangle = \left( \begin{array}{c} \langle \tilde{W} | \tilde{E}^2 \rangle \end{array} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
(63)
Using Eq. (50), the right upper element in the right hand side of Eq. (62) can be transformed as
\[
\langle \tilde{W} | \tilde{D} \tilde{E} | \tilde{V} \rangle = \left( \begin{array}{c} \langle \tilde{W} | \tilde{E}^2 \rangle \end{array} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
(64)
Eqs. (63) and (64) mean that components of the first row of
\[
Z_3 P^{1,2}[1 : 2, 1 : 2]
\]
is equal to components of \( \langle \tilde{W} | \tilde{E}^3 \rangle = (x_1, y_1) \tilde{E} \). Similarly, we can show that components of the second row of \( Z_3 P^{1,2}[1 : 2, 1 : 2] \) is equal to components of \( (x_2, y_2) \tilde{E} \). Therefore we obtain
\[
Z_3 P^{1,2}[1 : 2, 1 : 2] = \begin{pmatrix} x_1, y_1 \\ x_2, y_2 \end{pmatrix} \tilde{E},
\]
(65)
\footnote{The set of Eqs. (59), (56) and (60) is what the equation
\[
Z_2 P^{1,1} = \left( \begin{array}{c} \langle \tilde{W} | \tilde{E} \\ \langle \tilde{W} | \tilde{D} \end{array} \right)
\]
which is Eq. (2.30) in Ref. 1, means.}
that is,

\[ Z_3 P^{1,2}[1 : 2, 1 : 2] \equiv \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \tilde{E}. \quad (66) \]

This equation and Eq. (59) means Eq. (52). In the similar way, we can show Eq. (53).

6. Summary

In this article, after a brief introduction, we described the asymmetric simple exclusion process (ASEP) in Sec. 2 and a matrix product stationary state (MPSS) for the ASEP in Sec. 3. Furthermore, we have explained:

(i) a systematic way to find necessary conditions for the existence of an \( M (= 1, 2) \)-dimensional MPSS (Eq. (12)) in Sec. 4;

(ii) a systematic way by which the two-dimensional matrices \((E \text{ and } D)\) and vectors \((|W \rangle \text{ and } |V \rangle)\) in the MPSS (Eq. (12)) can be constructed from the stationary states for the two-site system and the three-site system if the condition in (i) is also a sufficient condition.

The method (i) (resp. (ii)) is applicable to \( M (\geq 2) \)-dimensional MPSSs (Eq. (12)) for not only the ASEP but also models which have \( N (\geq 2) \) (resp. \( N (\leq M) \)) states per site.¹

For other examples and details, please see Ref. 1. This reference explains also a systematic way to check the validity of the obtained MPSS for arbitrary system sizes in the restricted models.

Acknowledgments

The authors would like to thank Tomotoshi Nishino for informing us of the workshop, where partial contents written in this article was presented.

References

1. Y. Hieida and T. Sasamoto, *Journal of Physics A: Mathematical and General* 37, p. 9873 (2004).
2. R. A. Blythe and M. R. Evans, *Journal of Physics A: Mathematical and Theoretical* 40, p. R333 (2007).
3. K. Krebs and S. Sandow, *Journal of Physics A: Mathematical and General* 30, p. 3165 (1997).
4. K. Klauck and A. Schadschneider, *Physica A: Statistical Mechanics and its Applications* 271, 102 (1999).
5. F. H. L. Essler and V. Rittenberg, *Journal of Physics A: Mathematical and General* **29**, p. 3375 (1996).
6. K. Mallick and S. Sandow, *Journal of Physics A: Mathematical and General* **30**, p. 4513 (1997).
7. G. M. Schütz, *Exactly solvable models for many-body systems far from equilibrium in Phase Transitions and Critical Phenomena* vol 19 ed C Domb and J L Lebowitz (Academic, London, 2001).
8. Y. Hieida, *Journal of the Physical Society of Japan* **67**, 369 (1998).
9. S. R. White, *Phys. Rev. Lett.* **69**, 2863 (Nov 1992).
10. S. R. White, *Phys. Rev. B* **48**, 10345 (Oct 1993).
11. Peschel I et al (ed), *Lecture Note in Physics: Density-Matrix Renormalization* (Springer, Berlin, 1999).
12. T. Sasamoto, *Journal of Physics A: Mathematical and General* **32**, p. 7109 (1999).
13. T. Sasamoto, *Journal of the Physical Society of Japan* **69**, 1055 (2000).