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A NOTE ON EXTREMAL TORIC ALMOST KÄHLER METRICS

EVELINE LEGENDRE

Abstract. An almost Kähler structure is extremal if the Hermitian scalar curvature is a Killing potential [29]. When the almost complex structure is integrable it coincides with extremal Kähler metric in the sense of Calabi [8]. We observe that the existence of an extremal toric almost Kähler structure of involutive type implies uniform K-stability and we point out the existence of a formal solution of the Abreu equation for any angle along the invariant divisor. Applying the recent result of Chen–Cheng [10] and He [27], we conclude that the existence of a compatible extremal toric almost Kähler structure of involutive type on a compact symplectic toric manifold is equivalent to its relative uniform K–stability (in a toric sense). As an application, using [5], we get the existence of an extremal toric Kähler metric in each Kähler class of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2))$.

1. Introduction

The objects and problems of toric Kähler geometry have been fruitfully translated in terms of convex affine geometry in the works of Abreu [1], Guillemin [25], Donaldson [15], Apostolov and al. [3] with important applications in the very hard and central problem of Calabi extremal Kähler metrics [5]. In particular, Donaldson used this theory to prove the celebrated Yau–Tian–Donaldson conjecture, [37, 36, 14], for toric surfaces with vanishing Futaki invariant in [14, 15, 17, 18]. There is a relative version of this conjecture due to Székelyhidi [34] which is more relevant in the presence of symmetries and for general extremal (non constant scalar curvature) Kähler metrics. This conjecture predicts that given a complex compact manifold $(M^{2n}, J)$ with a Kähler class $\Omega$ and a maximal compact torus $T \subset \text{Aut}(M, J)$, the existence of an invariant extremal Kähler metrics in $\Omega$ is equivalent to the ”relative K–stability” of $(M^{2n}, J, \Omega)$ in a sense to be determined precisely but which would be related to an algebro-geometric notion of stability.

We recall briefly the toric counterpart of this theory, with more details in Section 2, as it was developed by Donaldson [14]. In the toric setting, $(M^{2n}, J, \Omega)$ is invariant by a compact torus $T = T^n$ and characterized completely by a convex polytope $P$, open and relatively compact in $t^*$, the dual of the Lie algebra $t$ of $T$, together with an affine measure $\sigma \in M(P)$ on the boundary of $P$. The $K$–stability (relative to $T$) is related to the positivity of a certain functional

$$L_{(P, \sigma)}(f) = \int_{\partial P} f\sigma - \frac{1}{2} \int_P fA_\sigma dx$$

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on a set $\tilde{C}$ of convex functions $f$ on $P$, see Definition 3.1. In this definition, $dx = dx_1 \wedge \cdots \wedge dx_n$ is a Lebesgue measure on $t^* \simeq \mathbb{R}^n$ and $A_\sigma \in \text{Aff}(t^*)$ is the extremal affine function, see §2.4. Following [14, 35], if there exists $\lambda > 0$ such that

$$L_{(P,\sigma)}(f) \geq \lambda \int_{\partial P} f \sigma$$

for any "normalized" $f$ in $\tilde{C}$ then $(P,\sigma)$ is uniformly $K$–stable and $K$–stable if $\lambda = 0$ is the only possible choice, see Definition 3.1.

The $K$–stability or uniform $K$–stability only depends on $P$ and $\sigma$ and we define

$$u_{Ks}(P) = \{ \sigma \in M(\partial P) | (P,\sigma) \text{ is uniformly } K\text{–stable} \},$$

$$Ks(P) = \{ \sigma \in M(\partial P) | (P,\sigma) \text{ is } K\text{–stable} \}.$$

Of course we have $u_{Ks}(P) \subset Ks(P)$.

Compatible Kähler structures are essentially parametrized by a set of convex functions $S(P,\sigma) \subset C^\infty(P), \text{ called symplectic potentials and satisfying some boundary condition, recalled in } \S2.2, \text{ depending on } \sigma$. Given $u \in S(P,\sigma)$, the associated Kähler structure $(g_u,J_u)$ is extremal in the sense of Calabi if it satisfies the following so-called Abreu equation

$$S(H^u) = - \sum_{i,j=1}^n \frac{\partial^2 u_{ij}}{\partial x_i \partial x_j} \in \text{Aff}(t^*)$$

where $H^u = (u_{ij}) = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^{-1}$ is the inverse Hessian of $u$ for a flat connection on $t^* \simeq \mathbb{R}^n$.

The relative version of the Yau–Tian–Donaldson conjecture for toric manifold is generalized following [14] by the prediction that, given a simple relatively compact polytope $P \subset \mathbb{R}^n$, one should have

$$\{ \sigma \in M(\partial P) | \exists u \in S(P,\sigma) \text{ such that } S(H^u) \in \text{Aff}(t^*) \} = Ks(P).$$

Some experts think that the stability condition must be strengthened and one of the suggestion, see [35 11], is to conjecture that

$$\{ \sigma \in M(\partial P) | \exists u \in S(P,\sigma) \text{ such that } S(H^u) \in \text{Aff}(t^*) \} = u_{Ks}(P).$$

As we argue in §3.2 by combining Chen–Li–Sheng work [11] and the recent progress of Chen–Cheng [10] and He [27], with Donaldson [14] and Zhou–Zhu [38] results this conjecture is indeed true.

**Theorem 1.1.** Given any compact convex labelled simple polytope $(P,\sigma)$,

$$\exists u \in S(P,\sigma) \text{ such that } S(H^u) \in \text{Aff}(t^*)$$

if and only if $(P,\sigma)$ is uniformly $K$–stable (i.e $\sigma \in u_{Ks}(P)$).

In the constant scalar curvature case, that is when $A_\sigma(P,\sigma)$ is a constant, this last statement is Theorem 1.8 of Chen–Cheng in [10] given that Donaldson showed in [14 Proposition 5.2.2] that uniform K-stability of $(P,\sigma)$ is equivalent to the $L^1$–stability of Chen and Cheng. Theorem 1.1 above is an application of He’s recent important result [27].

**Remark 1.2.** To pass from Theorem 1.1 to a positive resolution of the relative version of the Yau–Tian–Donaldson conjecture one would need to show that the uniform stability of a labelled polytope is equivalent to the stability with respect to toric degenerations, see Remark 3.2.
Observe that (2) is a non-linear 4-th order PDE problem on \( \phi \) but only a linear second order PDE problem on \( H \). Denote \( \mathcal{AK}(P, \sigma) \) the set of matrix-valued function \( H : P \to GL(\mathbb{R}^n) \) symmetric, positive definite and satisfying some boundary condition depending on \( \sigma \) detailed in [2,3]. Then one can define a smooth toric almost Kähler structure \((g_H, J_H)\) on \((M, \omega)\) as explained in [3, 29] and recalled in §2.3. Such an almost Kähler structure \((g_H, J_H)\) is extremal in the sense of Lejmi if it satisfies the Abreu equation (6), that is

\[
S(H) = - \sum_{i,j=1}^{n} \frac{\partial^2 H_{ij}}{\partial x_i \partial x_j} \in \text{Aff}(t^*).
\]

Lejmi studied the notion of extremal toric almost Kähler metrics in [29] and showed that a large and interesting part of them (the involutive type ones) is in one-to-one correspondence with \( \mathcal{AK}(P, \sigma) \).

Chen–Li–Sheng proved that existence of a toric Calabi extremal Kähler metrics implies that the toric variety is uniformly \( K \)-stable, proving one side of the conjecture for toric manifolds [11]. In this note we observe and explain that their proof works equally well for extremal almost Kähler metrics and prove that

**Proposition 1.3.** For any simple relatively compact \( P \subset \mathbb{R}^n \), we have

\[
\{ \sigma \in M(\partial P) \mid \exists H \in \mathcal{AK}(P, \sigma) \text{ such that } S(H) \in \text{Aff}(t^*) \} \subset uKs(P).
\]

In particular, if \((M, J, g, \omega)\) is a compact toric Kähler manifold such that \((M, \omega)\) admits a compatible extremal toric almost Kähler metrics of involutive type then \((M, J, [\omega])\) is uniformly \( K \)-stable with respect to toric degenerations.

As a direct consequence of this last Proposition and Theorem 1.1 above we get

**Corollary 1.4.** The existence of an extremal toric almost Kähler metric of involutive type compatible with \( \omega \) implies the existence of a compatible extremal toric Kähler metric.

**Remark 1.5.** It is unlikely that in general, for compact Kähler manifold of non-toric type, the existence of an extremal almost Kähler metric \((M, J, \omega)\) implies uniform \( K \)-stability of \((M, J)\) or the existence of an extremal Kähler metric compatible with \( \omega \). However, as pointed out in [28], a certain notion of stability could generalize the conjecture and theory to almost Kähler metrics.

In [5], for any \( k_2, k_1 > 0 \) and any toric symplectic form \( \omega \) on the total space of the projective bundle \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)) \to \mathbb{P}^1 \), they construct explicit examples of almost Kähler metrics compatible with \( \omega \). One can check directly that these metrics are of involutive type. As an application of Corollary 1.4 we get the following.

**Corollary 1.6.** Each Kähler class of \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)) \) admits a compatible extremal toric Kähler metric.

The convex affine geometry point of view has been exploited successfully to provide a complete understanding of the situation, confirming the relative version of the Yau–Tian–Donaldson conjecture, when the moment polytope is a convex quadrilateral in [6, 7, 31, 33] (in particular for toric compact orbisurfaces with second betti number equal 2) including explicit solution or destabilizing test configuration whenever they exist. A key ingredient of the aforementioned papers is an explicit

\[\text{(6)}\]

\[S(H) = - \sum_{i,j=1}^{n} \frac{\partial^2 H_{ij}}{\partial x_i \partial x_j} \in \text{Aff}(t^*).\]

\[\text{(7)}\]

\[\{ \sigma \in M(\partial P) \mid \exists H \in \mathcal{AK}(P, \sigma) \text{ such that } S(H) \in \text{Aff}(t^*) \} \subset uKs(P).\]

\[\text{In particular, if } (M, J, g, \omega) \text{ is a compact toric Kähler manifold such that } (M, \omega) \text{ admits a compatible extremal toric almost Kähler metrics of involutive type then } (M, J, [\omega]) \text{ is uniformly } K \text{-stable with respect to toric degenerations.}\]

\[\text{As a direct consequence of this last Proposition and Theorem 1.1 above we get}\]

\[\text{Corollary 1.4. The existence of an extremal toric almost Kähler metric of involutive type compatible with } \omega \text{ implies the existence of a compatible extremal toric Kähler metric.}\]

\[\text{Remark 1.5. It is unlikely that in general, for compact Kähler manifold of non-toric type, the existence of an extremal almost Kähler metric } (M, J, \omega) \text{ implies uniform } K \text{-stability of } (M, J) \text{ or the existence of an extremal Kähler metric compatible with } \omega. \text{ However, as pointed out in [28], a certain notion of stability could generalize the conjecture and theory to almost Kähler metrics.}\]

\[\text{In [5], for any } k_2, k_1 > 0 \text{ and any toric symplectic form } \omega \text{ on the total space of the projective bundle } \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)) \to \mathbb{P}^1, \text{ they construct explicit examples of almost Kähler metrics compatible with } \omega. \text{ One can check directly that these metrics are of involutive type. As an application of Corollary 1.4 we get the following.}\]

\[\text{Corollary 1.6. Each Kähler class of } \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)) \text{ admits a compatible extremal toric Kähler metric.}\]

\[\text{The convex affine geometry point of view has been exploited successfully to provide a complete understanding of the situation, confirming the relative version of the Yau–Tian–Donaldson conjecture, when the moment polytope is a convex quadrilateral in [6, 7, 31, 33] (in particular for toric compact orbisurfaces with second betti number equal 2) including explicit solution or destabilizing test configuration whenever they exist. A key ingredient of the aforementioned papers is an explicit}\]

\[\text{Here uniform K-stability should be understand as defined above, see Remark 3.2.}\]
formal solution $H_{A,B} : P \to \text{Sym}^2(t^*)$ depending on 2 polynomials $A$ and $B$ on one variable satisfying the boundary condition depending on $\sigma$ and satisfying the second order PDE corresponding to the extremal equation of Calabi. One of the main observations of [6, 7, 31, 33] is that $H_{A,B}$ is positive definite if and only if the labelled polytope $(P, \sigma)$ is $K$–stable and if and only if $H_{A,B}$ is the inverse Hessian of a symplectic potential.

A complete answer, like the one given for convex quadrilateral is certainly out of reach for convex polytope in general. However, we point out in this note that some parts of the strategy of [6, 7, 31, 33] may be extended in general thanks to the following observation.

**Proposition 1.7.** For any simple labelled polytope $(P, \sigma)$, there exists an infinite dimensional family of formal extremal solutions $H : P \to \text{Sym}^2(t^*)$ of equation (6) satisfying the boundary condition associated to $\sigma$. Whenever one of these solutions is positive definite on the interior of $P$, $(P, \sigma)$ is uniformly $K$–stable.

We discuss in [34] consequences of this last result and open problems in relation with the relative toric version of the Yau–Tian–Donaldson conjecture.

In the next section we gather facts, definition, key results and recall brief explanations on the topic of toric extremal (almost) Kähler metrics. Section 3 contains the proof of Propositions 1.3 and 1.7.

**Acknowledgement** The fact that the statement of Theorem 1.1 should follow more or less directly by the works of [14, 27, 38] has been pointed out to me by Vestislav Apostolov. I also thank Mehdi Lejmi for comments on a previous version and the anonymous referee for careful reading.

2. LABELLED POLYTOPE AND TORIC (ALMOST) KÄHLER GEOMETRY

2.1. Rational labelled polytopes and toric symplectic orbifolds.

**Notations.** In the sequel a polytope $P$ refers to an open, convex, polyhedral, simple and relatively compact subset of an affine space $t^* \simeq \mathbb{R}^n$. Simple means that each vertex is the intersection of exactly $n$ facets (where $n$ is the dimension of $t^*$). We order and denote the facets $F_1, \ldots, F_d \subset \overline{P}$. Choosing a non-zero inward normal vector $\vec{n}_s \in t$ to each facet $F_s$, we can write

$$P = \{ x \in t^* \mid \ell_{\vec{n},s}(x) > 0, \quad s = 1, \ldots, d \}$$

where $\ell_{\vec{n},s}$ is the unique affine-linear function on $t^*$ such that $d\ell_{\vec{n},s} = \vec{n}_s$ and

$$F_s = \ell_{\vec{n},s}^{-1}(0) \cap \overline{P}.$$  

**Definition 2.1.** Let $P \subset t^*$ be a polytope as above.

(a) A labelling for $P$ is an ordered set of non-zero vectors $\vec{n} = (\vec{n}_1, \ldots, \vec{n}_d) \in (t)^d$ each $\vec{n}_s$ being normal to the facet $F_s$ and inward to $P$. A labelled polytope is a pair $(P, \vec{n})$.

(b) A rational labelled polytope is a triple $(P, \vec{n}, \Lambda)$ where $(P, \vec{n})$ is a labelled polytope and $\Lambda \subset t$ is a lattice containing the labels $\vec{n}_1, \ldots, \vec{n}_d$.

(c) A Delzant polytope is a pair $(P, \Lambda)$ where $\Lambda \subset t$ is a lattice containing a set of labels $\vec{n} = (\vec{n}_1, \ldots, \vec{n}_d)$ such that for each vertex $\{p\} = \cap_{s \in I_p} F_s$ the set $\{\vec{n}_s \mid s \in I_p\}$ is a $\mathbb{Z}$–basis of $\Lambda$.  

We denote by \( N(P) := \{ \vec{n} = (\vec{n}_1, \ldots, \vec{n}_d) \in (t)^d \mid (P, \vec{n}) \text{ labelled polytope} \} \). Obviously \( N(P) \simeq \mathbb{R}^d_{>0} \). We will also be working on the dual space \( M(P) \) of measures \( \sigma \) on \( \partial P \) such that there exists a labelling \( \vec{n} \in N(P) \) satisfying
\[
\vec{n}_s \wedge \sigma = -dx \quad \text{on } F_s
\]
where \( dx = dx_1 \wedge \cdots \wedge dx_n \) is a fixed affine invariant volume form on \( t^* \). Again \( M(P) \simeq \mathbb{R}^d_{>0} \) and \( \sigma \in M(P) \) is determined by its restriction to the facets of \( P \). We write (formally) \( \sigma = (\sigma_1, \ldots, \sigma_d) \) where \( \sigma_s = \sigma|_{F_s} \) is an affine invariant \((n-1)\)-form on the hyperplane supporting \( F_s \).

**Remark 2.2.** Fixing \( dx = dx_1 \wedge \cdots \wedge dx_n \) once and for all, we get a bijection \( N(P) \simeq M(P), \vec{n} \mapsto \sigma \vec{n} \) with inverse \( \sigma \mapsto \vec{n}_\sigma \) given by the relation (8). In the following we use both notation \((P, \sigma)\) or \((P, \vec{n})\) for the labelled polytope \((P, \vec{n}, \Lambda)\).

### 2.1.2. Delzant–Lerman–Tolman correspondence.
Delzant showed that compact toric symplectic manifolds are in one to one correspondence with Delzant polytopes via the momentum map [13] and Lerman–Tolman [32] extended the correspondence to orbifolds by introducing rational labelled polytope. They are many ways to construct the corresponding (compact) toric symplectic orbifold \((M, \omega, T := t/\Lambda)\) from the data \((P, \vec{n}, \Lambda)\). We recall only the one we will use which, as far as we know, has been developed in [20] [16] [30].

1. **Local toric charts:** Each vertex \( p \) of \( P \) is the intersection of \( n \) facets thus corresponds to a subset \( I_p \subset \{ 1, \ldots, d \} \) of \( n \) indices which in turn corresponds to a basis of \( t \) namely \( \{ \vec{n}_s \mid s \in I_p \} \) that induces a sublattice \( \Lambda_p = \text{span}_\mathbb{Z}\{ \vec{n}_i \mid i \in I_p \} \) of \( \Lambda \). Considering the torus \( T_p = t/\Lambda_p \) we get a (non-compact) toric symplectic manifold
\[
(M_p := \bigoplus_{s \in I_p} \mathbb{C} \vec{n}_s \simeq \mathbb{C}^n, \omega_{std}, T_p)
\]
by identifying \( T_p \simeq \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) via which \( T_p \) acts on \( \mathbb{C}^n \). The momentum map \( x_p : M_p \to t^* \) is given
\[
x_p(z) = p + \frac{1}{2} \sum_{i \in I_p} |z_i|^2 \alpha_s
\]
where \( \{ \alpha_{\vec{n}_s} \mid i \in I_p \} \subset t^* \) is the dual basis of \( \{ \vec{n}_s \mid i \in I_p \} \).

2. **Gluing over \( P \times T \):** Now using the exact sequence
\[
\Lambda/\Lambda_p \to T_p \xrightarrow{\phi_p} T
\]
where \( T = t/\Lambda \) we get a way to glue equivariantly the (uniformizing) chart \( M_p \) over \( P \times T \) seen as a toric symplectic manifold with momentum map \( x \) being the projection on the first factor, see [30] for more details.

In this construction, \((M, \omega)\) is obtained as the compactification of \((P \times T, dx \wedge d\theta)\). Here \( dx \wedge d\theta \) is the canonical symplectic form of \( P \times T \) coming from the one of the universal cover \( P \times t \subset t^* \times t \). In particular, we get directly a set of action angle coordinates \((x, \theta)\) on the set where the action is free \( M = P \times T = x^{-1}(P) \). These coordinates are usually constructed with the help of a Kähler metric [9] and one can prove that they are well defined up to an equivariant symplectomorphism.

### 2.2. Symplectic potentials and toric Kähler metrics.
Let \((M, \omega, T)\) be a compact toric symplectic orbifold associated with the rational labelled polytope \((P, \vec{n}, \Lambda)\). In particular \( x : M \to \overline{p} \) is the momentum map. We fix a set of action angle coordinates \((x, \theta)\) on the set \( M \) where the torus action is free. The
Proposition 2.3. \[1, 3, 15, 25\] For any strictly convex function \(u \in C^\infty(P)\),
\[
 g_u = \sum_{i,j} u_{ij} dx_i \otimes dx_j + u^{ij} d\theta_i \otimes d\theta_j,
\]
with \((u_{ij}) = \text{Hess } u\) and \((u^{ij}) = (u_{ij})^{-1}\), is a smooth Kähler structure on \(P \times T\) compatible with the symplectic form \(dx \wedge d\theta\). Conversely, any \(T\)-invariant compatible Kähler structure on \((P \times T, dx \wedge d\theta)\) is of this form.

Moreover, the metric \(g_u\) is the restriction of a smooth (in the orbifold sense) toric Kähler metric on \((M, \omega)\) if and only if
\begin{enumerate}
  \item \(u \in C^0(P)\) whose restriction to \(P\) or to any face’s interior (except vertices), is smooth and strictly convex;
  \item \(u - u_{\vec{n}}\) is the restriction of a smooth function defined on an open set containing \(\overline{P}\) where
\end{enumerate}
\[
 u_{\vec{n}} = \frac{1}{2} \sum_{s=1}^d \ell_{\vec{n},s} \log \ell_{\vec{n},s}
\]
is the so-called Guillemin potential.

The functions \(u\) satisfying the conditions of the previous Proposition are called symplectic potentials and we denote the set of such as \(S(P, \vec{n})\) or \(S(P, \vec{\sigma}_n)\). In sum, the set of smooth compatible toric (orbifold) Kähler metrics on \((M, \omega, T)\) is in one-to-one correspondence with the quotient of \(S(P, \vec{n})\) by \(\text{Aff}(t^*, \mathbb{R})\), acting by addition. The correspondence is explicit and given by \([9]\).

Remark 2.4. The Guillemin potential \(u_{\vec{n}}\) lies in \(S(P, \vec{n})\) and corresponds to the Guillemin Kähler metric on the toric symplectic orbifold in the rational case.

The boundary conditions (1) and (2) of Proposition 2.3 appear when comparing the metrics \(g_u\) and \(g_{u_{\vec{n}}}\) on the charts \(M_p\) as defined in \([2, 12]\).

Remark 2.5. Passing from symplectic to complex point of views is direct in toric geometry. Given \(u \in S(P, \sigma)\) the map \((x, \theta) \mapsto (\nabla u)_x + \sqrt{-1} \theta\) provides the complex coordinates as the coordinates on the universal covering of the big orbit \(\tilde{M} \simeq (\mathbb{C}^*)^n\), see e.g.\([10]\). In these coordinates the Kähler potential of the Kähler form \(\omega\) is the Legendre transform of \(u\).

2.3. Toric almost Kähler metrics. An almost Kähler structure \((g, J, \omega)\) on \(M^{2n}\) has everything of a Kähler structure but the endomorphism \(J \in \Gamma(\text{End}(TM))\), is not necessarily integrable. That is, \(g\) is a Riemannian metric, \(\omega\) is a symplectic form, and \(J \in \Gamma(\text{End}(TM))\) squares to minus the identity and they satisfy the following compatibility relation:
\[
g(J\cdot, J\cdot) = g(\cdot, \cdot) \quad g(J\cdot, \cdot) = \omega(\cdot, \cdot).
\]
A toric almost Kähler metric \((g, J)\) is then an almost Kähler metric on a toric symplectic manifold/orbifold \((M, \omega, T)\) such that \((g, J)\) is compatible with \(\omega\) and \(g\) (equivalently \(J\)) is invariant by the torus \(T\).

Let \((M, \omega, T)\) be a toric symplectic manifold with a momentum map \(x : M \to t^*\) and moment polytope \(\overline{P} = x(M)\) labelled by \(\vec{n} \in \mathbb{N}(P)\). We use notation layed
in [2.1.1] and fix a set of affine coordinates \( x = (x_1, \ldots, x_n) \) on \( \mathfrak{t}^* \). In [29], the author proves among other things that \( T^- \)-invariant almost Kähler structures compatible with \((M, \omega)\) and such that the \( g^-\)-orthogonal distribution to the orbit is involutive (we call it toric almost Kähler structure of involutive type) are parametrized by symmetric bilinear forms

\[
H : \mathcal{T} \to \text{Sym}^2(\mathfrak{t}^*)
\]

satisfying some conditions pointed out in [3] that we now recall.

(i) **Smoothness** \( H \) is the restriction on \( \mathcal{T} \) of a smooth \( \text{Sym}^2(\mathfrak{t}^*) \)-valued function defined on an open neighborhood of \( \mathcal{T} \).

(ii) **Boundary condition** For any point \( y \) in interior of a codimension 1 face \( F_s \subset \mathcal{T} \), we have

\[
H_y(\vec{n}_s, \cdot) = 0
\]

\[
dH_y(\vec{n}_s, \vec{n}_s) = 2\vec{n}_s.
\]

(iii) **Positivity** For any point \( y \) in interior \( \mathcal{F} \) of a face \( F \subset \mathcal{T} \), \( H \) is positive definite as \( \text{Sym}^2(T_yF) \)-valued function.

**Proposition 2.6.** [3, 29] Let \((M, \omega, T)\) be a toric symplectic manifold and \((g, J)\) be a compatible \( T^- \)-invariant almost Kähler metric of involutive type compatible with \( \omega \). Then the symmetric bilinear form defined for \( a, b \in \mathfrak{t} \) and \( x \in \mathcal{T} \) by \( H_x(a, b) := g_p(X_a, X_b) \) for any \( p \in M \) such that \( x(p) = x \), satisfies the conditions (i), (ii) and (iii). Moreover, for any such symmetric bilinear form \( H : \mathcal{T} \to \text{Sym}^2(\mathfrak{t}^*) \) satisfying conditions (i), (ii) and (iii) there is a unique compatible \( T^- \)-invariant almost Kähler metric \((g_H, J_H)\) of involutive type satisfying \( H_x(p)(a, b) = g^{YH}_p (X_a, X_b) \) for any \( p \in M \). With respect to action angle coordinates \((x, [\theta])\) on \( \mathfrak{t}^* \times T \cong \hat{M} \), the metric \( g \) is given as

\[
g = \sum_{i,j} G_{ij} dx_i \otimes dx_j + H_{ij} d\theta_i \otimes d\theta_j,
\]

where \( G = (G_{ij}) = H^{-1} \).

**Remark 2.7.** Condition (12) implies that \( H(u_s, \cdot) : \mathcal{T} \to \mathbb{R} \) vanishes on \( F_s \) and in particular is constant. Then for all \( y \in \mathcal{F}_s \), we have

\[
(dH)_y(u_s, \cdot) \in \mathfrak{t}^* \otimes (T_y\mathcal{F}_s)^0 = \mathfrak{t}^* \otimes \mathcal{R}u_s
\]

where \((T_y\mathcal{F}_s)^0 = \mathcal{R}u_s\) denotes the annihilator of \( T_y\mathcal{F}_s \subset T_y(\mathfrak{t}^*) = \mathfrak{t}^* \) in \( \mathfrak{t} \). Therefore condition (13) is that the trace of \((dH)_y(u_s, \cdot)\) equals 2.

Fixing an affine invariant volume form \( dx = dx_1 \wedge \cdots \wedge dx_n \), the labelling \( \vec{n} \in \mathbb{N}(P) \) corresponds to a measure \( \sigma \in \mathbb{M}(P) \) as defined in [2.1.1]. Observe that the Boundary Condition above (i.e. condition (ii) namely (12) and (13)) implies that

\[
\sigma = \frac{1}{2} \sum_{i,j=1}^n (-1)^i H_{i,j} dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n.
\]

Assuming condition (12) holds condition (13) is equivalent to (15).

\[\text{When a set of coordinates is fixed, we use the notation } f_{,i} = \frac{\partial}{\partial x_i} f, f_{,ij} = \frac{\partial^2}{\partial x_i \partial x_j} f \ldots\]
Thanks to Proposition 2.6 we can parametrize the space of compatible toric almost Kähler metrics of involutive type as

$$\mathcal{AK}(P, \sigma) := \{ H : \overline{P} \to \text{Sym}^2(t^*) | H \text{ satisfies conditions (i), (ii) and (iii)} \}.$$ 

The inverse \((u^{ij})\) of the Hessian of symplectic potential \(u \in S(P, \bar{n})\) can be extended as a bilinear form \(H^u \in \mathcal{AK}(P, \sigma)\). Observe also that for \(H_0, H_1 \in \mathcal{AK}(P, \sigma)\) we have

$$H_t = tH_1 + (1-t)H_0 \in \mathcal{AK}(P, \sigma) \quad \forall t \in [0, 1].$$

The space \(\mathcal{AK}(P, \sigma)\) is then a convex infinite dimensional set of metrics.

2.4. The extremal vector field. Given a symplectic potential \(u \in S(P, \bar{n})\) the scalar curvature of the Kähler metric \(\gamma_u\) is given by the pull back to \(\overline{P}\) of the following expression, called the Abreu formula

\[ S(H^u) = -\sum_{i,j=1}^{n} \partial^2 u^{ij} / \partial x_i \partial x_j \]

as proved in [1] by direct computation. The function \(16\) extends as a smooth function on \(\overline{P}\) because the boundary condition (2) of Proposition 2.3 implies that \((u^{ij}) \in \Gamma(P, t^* \otimes t^*)\) extends as a smooth bilinear form on \(\overline{P}\), see [3]. It is shown in [29] that the suitable connection one should consider in case of almost Kähler metrics is the Chern connection (which do not coincide with the Levi-Civita connection in the non-Kähler setting). It turns out that the formulas in the toric case coincide in the sense that for \(H \in \mathcal{AK}(P, \sigma)\), the Hermitian scalar curvature is the pull-back of

\[ S(H) := -\sum_{i,j=1}^{n} H_{ij,ij}. \]

Calabi’s extremal Kähler metrics are characterized by the condition that the Hamiltonian vector field of the scalar curvature is a Killing vector field [8] and extremal almost Kähler metric are defined with the same condition on the Hermitian scalar curvature [29]. Therefore, here, they correspond to the \(H \in \mathcal{AK}(P, \sigma)\) such that

\[ S(H) \in \text{Aff}(t^*, \mathbb{R}). \]

As observed by Donaldson in [18], picking an invariant volume form \(dx = dx_1 \wedge \ldots \wedge dx_n\) on \(t^*\), the \(L^2\)-projection of \(S(H^u)\) on \(\text{Aff}(t^*, \mathbb{R})\) does not depend on the choice of \(u \in S(P, \bar{n})\). This fact holds for \(H \in \mathcal{AK}(P, \sigma)\) and is the effect of a more general theory of invariant developed in [21] [22] [29] which in the toric case follows from integration by parts. Indeed, using the condition (ii) of definition of \(\mathcal{AK}(P, \sigma)\) we have that for any \(f \in \text{Aff}(t^*, \mathbb{R})\) and \(H \in \mathcal{AK}(P, \sigma)\)

\[ \int_{\overline{P}} S(H) f dx = 2 \int_{\partial_{\overline{P}}} f \sigma_{\bar{n}}. \]

These computations do not require the existence of a lattice containing \(\bar{n}_{\sigma}\), the labelling associated to \(\sigma_{\bar{n}} \in \mathbf{M}(P)\) (see [2, 11]), or of a compact toric symplectic orbifold anywhere. Summing up these facts we get the following key result.
Proposition 2.8. [21, 23, 18, 29] For any labelled polytope \((P, \sigma)\), there exists a unique affine function \(A_{P, \sigma} \in \text{Aff}^t, \mathbb{R}\) such that
\[
\int_P A_{P, \sigma} f \, dx = \int_P S(H) f \, dx = 2 \int_{\partial P} f \sigma
\]
for any \(f \in \text{Aff}^t, \mathbb{R}\) and any \(H \in \mathcal{AK}(P, \sigma)\). Moreover, if there exists \(H \in \mathcal{AK}(P, \sigma)\) such that the metric \(g^H\) is extremal almost Kähler in the sense of Calabi (and Lejmi) then
\[
S(H) = A_{P, \sigma}.
\]

Remark 2.9. A direct corollary of the last Proposition is that the functional \(L_{(P, \sigma)}\) vanishes identically on affine-linear function.

Remark 2.10. The function \(A_{P, \sigma}\) depends linearly on \(\sigma \in M(P)\).

2.5. Extremal Kähler metrics unicity and an open condition. Uniqueness of extremal toric Kähler metric in a given class for a fixed torus is not an issue thanks to the proof of Guan in [24], using the convexity of the K–energy functional over geodesics. His proof works very well on symplectic potentials in \(S(P, \vec{n})\) as soon as \(P\) is compact using the works of [18], see e.g. [31, §2.2.1], because \(S(P, \vec{n})\) is a convex set with respect to smooth geodesics for the Mabuchi metric (which, here, are the affine lines \((1-t)u_0 + tu_1\)) defined on the space of Kähler metrics [24]. Therefore, we get the following unicity result.

Proposition 2.11. Let \((P, \vec{n})\) be a labelled polytope. If \(u_0, u_1 \in S(P, \vec{n})\) satisfy \(S(u_0) = S(u_1) = A_{P, \vec{n}}\) then \(u_1 - u_0\) is the restriction to \(P\) of an affine linear function on \(t^d\).

Donaldson proved in [17] that the set of labelling \(\vec{n} \in N(P)\) for which the Abreu’s equation has a solution is open in \(N(P)\), the \(d\)–dimensional open cone of labellings of \(P\) in \(t^d\).

Proposition 2.12 (Donaldson [17]). Let \((P, \sigma)\) be a labelled polytope. Assume that there is a potential \(u \in S(P, \vec{n}_\sigma)\) satisfying the Abreu equation. Then there exists an open neighborhood \(U \subset M(P)\) of \(\sigma\) such that for each \(\tilde{\sigma} \in U\) there exists a potential \(\tilde{u} \in S(P, \vec{n}_\tilde{\sigma})\) satisfying the Abreu equation.

The statement in [17] is not exactly the one above but the proof works in this degree of generality. The argument is standard. The linearisation of \(u \mapsto S(H^u)\) is an elliptic operator. To get around the lack of compacity of \(P\), Donaldson argue that the system of charts associated to the vertices, see [23, 24] provides the kind of compactification needed. This idea is developped with details in [30].

3. Uniform K–stability and Extremal almost Kähler metrics

3.1. Uniform K–stability and Chen–Li–Sheng result. Consider the functional
\[
\mathcal{L}_{(P, \sigma)}(f) = \int_{\partial P} f \sigma - \frac{1}{2} \int_P f A_\sigma \, dx
\]
which can be defined on various spaces of functions on \(\mathcal{P}\); for example \(C^0(\mathcal{P})\). From Proposition 2.8 we get that \(\mathcal{L}_{(P, \sigma)}\) vanishes identically on the space of affine-linear functions.
Following [14], we define the set $\mathcal{C}_\infty$ of continuous convex functions on $\mathcal{P}$, which are smooth on the interior, we have $\mathcal{S}(P, \sigma) \subset \mathcal{C}_\infty$ for all $\sigma \in \mathcal{M}(P)$. We fix $p_\sigma \in P$, the set of a normalized functions is

$$\tilde{\mathcal{C}} := \{ f \in \mathcal{C}_\infty \mid f(p) \geq f(p_\sigma) = 0 \quad \forall p \in P \}.$$ 

Note that the only affine-linear function in $\tilde{\mathcal{C}}$ is the trivial one.

**Definition 3.1.** A labelled polytope $(P, \sigma)$ is uniformly $K$-stable if there exists $\lambda > 0$ such that

$$\mathcal{L}_{(P, \sigma)}(f) \geq \lambda \int_{\partial P} f\sigma$$

for any $f \in \tilde{\mathcal{C}}$.

**Remark 3.2.** Let $\mathcal{T}(P)$ be the set of continuous piecewise linear convex functions on $\mathcal{P}$, that is $f \in \mathcal{T}(P)$ if there are $f_1, \ldots, f_m \in \text{Aff}(t^*, \mathbb{R})$ such that $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ for $x \in \mathcal{P}$. Given a lattice $\Lambda \subset t$, we define $\mathcal{T}(P, \Lambda) \subset \mathcal{T}(P)$, the set of continuous piecewise linear convex functions on $P$ taking integral values on the dual lattice $\Lambda^* \subset t^*$. When $(P, \eta, \Lambda)$ is rational Delzant and its vertices lie in the dual lattice $\Lambda^* \subset t^*$, the associated symplectic manifold $(M, \omega)$ is rational (that is $[\omega] \in H^2_{dR}(M, \mathbb{Q})$) and for any compatible toric complex structure $J$ on $M$ the Kähler manifold $(M, J, k[\omega])$ (for some $k$ big enough) is polarized by a line bundle $L^k \to M$. In this situation, Donaldson presents in [14] a way to associate a test configuration $(X_f, \mathcal{L}_f)$ over $(M, L)$ to any function $f \in \mathcal{T}(P, \Lambda)$ such that the Donaldson–Futaki invariant of $(X_f, \mathcal{L}_f)$ coincides, up to a positive multiplicative constant, with $\mathcal{L}_{(P, \sigma)}(f)$. These test configurations are called *toric degenerations* in [14] and [38]. The Yau–Tian–Donaldson conjecture predicts that if $\Lambda_{P, \vec{n}}$ is a constant and there exists a solution $u \in \mathcal{S}(P, \vec{n})$ of the Abreu equation (2) then

$$\mathcal{L}_{(P, \sigma)}(f) \geq 0$$

for any $f \in \mathcal{T}(P, \Lambda)$ with equality if and only $f$ is affine-linear.

Observe that the map $f \mapsto \int_{\partial P} f\sigma$ is a norm on $\tilde{\mathcal{C}}$. Therefore, Definition 3.1 coincides with the notion of uniform $K$-stability in the sense of Székelyhidi [35] but with a different norm and adapted to the toric situation. Moreover, this is the notion of stability in Definition 3.1 that Chen–Li–Sheng used in [11] to prove that

**Theorem 3.3.** [11] If $(P, \sigma)$ is a labelled polytope and that there exists a solution $u \in \mathcal{S}(P, \sigma)$ of the Abreu equation (2) then $(P, \sigma)$ is uniformly $K$-stable.

**Proof of Proposition 1.3.** Our Proposition 1.3 follows by observing that in the proof of the last Theorem, Chen–Li–Sheng only use the fact that the Hessian and inverse Hessian $H^u$ of the solution $u \in \mathcal{S}(P, \sigma)$ are positive definite on the interior of $P$. One important step for their proof is to show that: a labelled polytope $(P, \sigma)$ is uniformly $K$-stable if and only if $\mathcal{L}_{(P, \sigma)}(f) \geq 0$ on some compactification $\mathcal{C}_K$ of $\tilde{\mathcal{C}}$. But this is general and does not need any hypothesis on the existence of a solution of the Abreu equation. This latter hypothesis is only needed for Lemma 5.1 of [11]. The crucial observation is the following, if $H : \mathcal{P} \to \text{Sym}^2(t^*)$ satisfies equation (6), that is $\mathcal{S}(H) = -\sum_{i,j=1}^n H_{ij,ij} = \Lambda_{(P, \sigma)}$ then the boundary conditions (ii) of (6) implies that

$$\mathcal{L}_{(P, \sigma)}(f) = \int_P \langle H, \text{Hess} f \rangle dx$$

(21)
whenever \( f \) is twice differentiable. Formula (\ref{21}) goes back to \cite{14}.

Therefore, let \( H \) be a solution of equation (\ref{6}), then for any interval \( I \subset P \) and sequence of convex functions \( f_k \in C_{\infty} \subset C^\infty(P) \) converging locally uniformly to \( f \) then we have, using (\ref{21}) and weak convergence of Monge–Ampère measures, that

\[
\mathcal{L}_{(P,\sigma)}(f_k) \geq \tau m_1(f)
\]

where \( m_1(f) \) is the Monge–Ampère measure induced by \( f \) on \( I \) and \( \tau \) is a positive constant independant of \( k \). This is the claim of Lemma 5.1 of \cite{11} from which one can derive Proposition \cite{13} using the same argument than \cite{11} in the last paragraph of their section 5.

\[ \square \]

### 3.2. Uniform \( K \)–stability implies the existence of an extremal toric Kähler metric

In this paragraph we will put together the work of Donaldson \cite{14}, He \cite{27} and Zhou–Zhu \cite{38} to prove that

**Proposition 3.4.** Let \((P,\sigma)\) be a compact convex simple polytope. If \((P,\bar{n})\) is uniformly K–stable then there exists \( u \in \mathcal{S}(P,\sigma) \) such that

\[
S(H^u) = A_{(P,\sigma)}.
\]

Given a compact group \( K \subset \text{Aut}_0(M,J) \) containing the extremal vector field (the Hamiltonian Killing version of it) in its Lie algebra center and a fixed \( J \)–compatible \( K \)–invariant Kähler metric \( \omega \), one can define the modified Mabuchi \( K \)–energy as a functional \( \mathcal{K} \) on the space of \( K \)–invariant Kähler potentials \( \mathcal{H}_K := \{ \phi \in C^\infty(M)^K \mid \omega + d\bar{\partial} \phi > 0 \}. \) This functional is important because it detects the \( K \)–invariant extremal Kähler metrics in \((M,J,[\omega])\). Let \( K = K_0 \) be a compact subgroup of \( \text{Aut}_0(M,J) \) whose complexified Lie algebra \( \mathfrak{h}_0 \) is the reduced part of \( \mathfrak{h} := \text{Lie}\text{Aut}_0(M,J) \). Denote \( G_0 \) the complexification of \( K_0 \) in \( \text{Aut}_0(M,J) \). An important ingredient in this theory is a certain distance \( d_1,G_0 \) on \( \mathcal{H}_K \) introduced by Darvas \cite{12} and corresponding to the \( L^1 \)–norm on \( T_0 \mathcal{H}_K \). That is for \( \psi \in T_0 \mathcal{H}_K \), the norm \( \int_M |\psi|^2 \omega_0^n \) allows to compute the lenght of curves and then \( d_1(\phi_0,\phi_1) \) is the infimum of the lenght of the curves joining \( \phi_0 \) and \( \phi_1 \). Then \( d_1,G_0(\phi_0,\phi_1) = \inf_{g \in G_0} d_1(\phi_0, g^* \phi_1) \).

**Theorem 3.5.** \cite[Theorem 4 of He \cite{27}]{27} There is a \( K_0 \)–invariant extremal Kähler metrics in \((M,J,[\omega])\) if and only if the modified Mabuchi \( K \)–energy is bounded below on \( \mathcal{H}_K \) and proper with respect to \( d_1,G_0 \).

On a toric manifold, following Donaldson \cite{14}, it is more natural to define the \( K \)–energy on the space of symplectic potentials as follows. Let \((P,\sigma)\) be a labelled compact simple polytope with extremal affine function \( A_{P,\bar{n}} \in \text{Aff}(t^*,\mathbb{R}) \) and \( u \in \mathcal{S}(P,\sigma) \), the modified Mabuchi \( K \)–energy (of the corresponding Kähler potential) is

\[
\mathcal{F}_{(P,\sigma)}(u) = - \int_P \log \det(u_{ij}) \, dx + \mathcal{L}_{(P,\sigma)}(u).
\]

Indeed, direct calculation shows that the critical points of this functional on \( \mathcal{S}(P,\bar{n}) \) are the symplectic potentials satisfying

\[
S(H^u) = A_{(P,\sigma)}.
\]

This allows us to translate He’s Theorem (recalled in Theorem 3.5 above) in terms of \((P,\sigma)\) only. As explained in \cite{30}, when it concerns \( T \)–invariant objects \((T \subset K_0 \) in the toric case), analytic proofs eg. estimates of Chen–Cheng \cite{10}, translate without problems using the smooth local complex charts (which do exist for any simple
labeled polytope) and the compactness of $\mathcal{P}$. Then to prove Proposition 3.4 it is sufficient to show that $\mathcal{F}_{(P,\sigma)}$ is bounded below on $\overline{C}$ and that it is proper with respect to $d_{1,\mathcal{G}_0}$.

The first condition is given by Donaldson.

**Lemma 3.6.** [Lemma 3.2 of Donaldson] (13) If $(P,\sigma)$ is uniformly K-stable then $\mathcal{F}_{(P,\sigma)}$ is bounded below on $\overline{C}$.

We will derive the second using the following result.

**Lemma 3.7.** [Lemma 2.3 of Zhou–Zhu] (38) If $(P,\sigma)$ is uniformly K-stable then there exist real positive constants $C,D$ such that

$$ (23) \quad \mathcal{F}_{(P,\sigma)}(u) \geq C \int_P u dx - D $$

for all $u \in \overline{C}$.

Given two normalized symplectic potentials $u_0, u_1 \in S(P,\sigma) \cap \overline{C}_\infty$, we consider the curve $u_t = tu_1 + (1 - t)u_0 \in S(P,\sigma)$ and the curve given by its Legendre transform $\tilde{\phi}_t : t \to \mathbb{R}$ (which is a curve of Kähler potentials in the sense that $(\omega = d\phi_t, J)$ is biholomorphically isometric to $(\omega, J_{u_t})$ on $M$, see eg. (11) (16) (30)).

Thanks to the normalization we have $\int_P u dx = \int_R |u| dx$ for $u \in \overline{C}$ and $\dot{u}_t(x) = -\phi_t((\nabla u_t)_x)$ thus

$$ \int_P |u_0| dx + \int_P |u_1| dx \geq \int_P |u_1 - u_0| dx = \int_0^1 \int_P |\dot{u}_t| dx dt $$

$$ = \int_0^1 \int_P |\dot{\phi}_t((\nabla u_t)_x)| dx dt = \int_0^1 \int_0^x |\dot{\phi}_t(y)| \det(D\nabla \phi_t)_y dy dt $$

where the last equality uses the change of variables into complex coordinates, see Remark 2.5. This is used to get the expression

$$ \int_0^1 \int_0^x |\dot{\phi}_t(y)| \det(D\nabla \phi_t)_y dy dt = \frac{1}{(2\pi)^n} \int_0^1 \int_M |\dot{\phi}_t| \omega_{\phi_t}^n dt. $$

Now, the right hand side of the last expression is the Darvas length (12) of the curve $\phi_t$ connecting two Kähler potentials $\psi_0 := \phi_0 - \phi$ and $\psi_1 := \phi_1 - \phi$ in $\mathcal{H}_{K\alpha}$, therefore

$$ \frac{1}{(2\pi)^n} \int_0^1 \int_M |\dot{\phi}_t| \omega_{\phi_t}^n dt \geq d_1(\psi_0, \psi_1) \geq d_{1,\mathcal{G}_0}(\psi_0, \psi_1). $$

Summing up, for any $u_1 \in S(P,\sigma) \cap \overline{C}_\infty$, we have that

$$ \int_P |u_0| dx + \int_P u_1 dx \geq d_{1,\mathcal{G}_0}(\psi_{u_0}, \psi_{u_1}) $$

with $\psi_u$ being the Kähler potential corresponding to the metric associated to $u$. In particular, fixing $u_0$ and substituting to $u_1$ a sequence $u_{1,k}$ such that $d_{1,\mathcal{G}_0}(\phi_{u_0}, \phi_{u_{1,k}}) \to +\infty$ we get that $\int_P u_{1,k} dx \to +\infty$ which, using Zhou–Zhu properness Lemma 3.7 implies that

$$ \mathcal{F}_{(P,\sigma)}(u_{1,k}) \to +\infty. $$

This, with Lemma 3.3 above, is enough to fulfill He’s condition and get that there exists a torus invariant extremal Kähler metric. That is, it concludes the proof of Proposition 3.4 which, together with Theorem 3.3 of Chen–Li–Sheng (11) gives Theorem (11).
3.3. **Extremal almost Kähler metrics.** In this note we are interested in the $H \in \mathcal{AK}(P,\sigma)$ satisfying the Abreu equation \([20]\). We will consider the following set of formal solutions

$$W(\sigma) := \{H : \overline{P} \to \text{Sym}^2(t^*) | H \text{ satisfies conditions (i), (ii) and } S(H) = A_{P,\sigma}\}$$

$$\mathcal{W} := \bigsqcup_{\sigma \in M(P)} W(\sigma).$$

The only thing a $\text{Sym}^2(t^*)$–valued function $H \in W$ misses to define an extremal toric almost Kähler metric in the sense of Lejmi is the positivity (that is condition (iii)). Therefore

$$W^+ := \mathcal{AK}(P,\sigma) \cap W(\sigma)$$

parametrizes the space of extremal toric almost Kähler metrics of involutive type on $P \times t$ with boundary conditions imposed by the condition (ii) with respect to $\sigma$ (see \([15]\)). Translated in our notation, Lejmi proved in \([29]\), see also \([14]\), that the set $W^+(\sigma)$ is either empty or infinite dimensional.

**Proposition 3.8.** Let $P$ be a simple polytope. For any labelling $\sigma \in M(P)$ the set $W(\sigma)$ is not empty. Moreover, the set

$$\{\sigma \in M(P) | W^+(\sigma) \neq \emptyset\}$$

is a non-empty open convex cone in $M(P)$.

**Proof.** First, note that the Abreu equation is linear on $W$ and that the boundary condition data $\sigma \in M(P)$ depends linearly on the $\text{Sym}^2(t^*)$–valued function thanks to \([15]\). Therefore, it is sufficient to find an open set $U \subset M(P)$ of $\sigma$'s such that $W(\sigma)$ is not empty to prove the first assertion. Indeed, in this case $U$ would contain a basis $\{\sigma_s\}_{s=1,\ldots,d} \subset U$ and any $\tilde{\sigma} \in M(P)$ is such $\sigma = \sum_{s=1}^{d} a_s \sigma_s$ with $a_s \in \mathbb{R}$. Picking any solution $H_s \in W(\sigma_s)$ we have $\sum_{s=1}^{d} a_s H_s \in W(\tilde{\sigma})$. According to \([30]\) for each polytope there exists $\sigma_{KE} \in M(P)$, unique up to dilatation, and a symplectic potential $u_{KE} \in S(P,\vec{n}_{\sigma_{KE}})$ such that the metric $g_{u_{KE}}$ is Kähler–Einstein on $P \times t$ with respect to the natural symplectic structure on $t^* \times t$. In particular, $H_{u_{KE}}$ is a solution of Abreu’s equation and thus $H_{u_{KE}} \in W^+(\sigma_{KE})$. Thanks to Donaldson openness result, see Proposition 2.12 above, there exists an open set $U \subset M(P)$ of $\sigma$'s such that $W^+(\sigma)$ is not empty. The second assertion follows the same argument with a special care for positive definite condition. \(\square\)

**Proposition 1.7** is a direct consequence of the last proposition.

3.4. **The space of formal solutions.**

**Proposition 3.9.** [Donaldson \([14]\)] Let $(P,\sigma)$ be a labelled polytope. Assume the set $W^+(\sigma)$ is non empty. Then the functional $N : W^+(\sigma) \to \mathbb{R}$ defined by

$$N(H) = \int_P \log(\det H) \, dx$$

is concave and the critical point, if it exists, is the inverse of a Hessian of a potential $u \in S(P,\vec{n}_{\sigma})$.

The union of the $W^+(\sigma)$ is a convex cone

$$W^+ := \bigsqcup_{\sigma \in M(P)} W^+(\sigma).$$
From the observation (15), the map \( m : \mathcal{W}^+ \to \mathbf{M}(P) \) taking \( H \in \mathcal{W}^+ \) to the measure \( m(H) = \sigma \in \mathbf{M}(P) \) is well-defined. The "fibers" of \( m \) are the \( \mathcal{W}^+ (\sigma) \). Proposition 1.3 implies that the image of the map \( m \) lies into \( \text{uKs}(P) \).

Note that \( \mathcal{W}^+ \) contains the inverse Hessians of the extremal Kähler potentials, that is the union over \( \mathbf{M}(P) \) of \( \mathcal{K}\mathcal{W}^+ (\sigma) := \{ H^u | u \in \mathcal{S}(P,u_\sigma), H^u \in \mathcal{W}^+ (\sigma) \} \). When non-empty, \( \mathcal{K}\mathcal{W}^+ (\sigma) \) contains a unique point, the maximum of \( H \) on \( \mathcal{W}^+ (\sigma) \) thanks to Proposition 3.3. Since \( N \) is continuous on \( \mathcal{W}^+ \), \( \mathcal{K}\mathcal{W}^+ := \bigcup_{\sigma \in \mathbf{M}(P)} \mathcal{K}\mathcal{W}^+ (\sigma) \) is connected. The relative toric version of the Yau–Tian–Donaldson conjecture is then equivalent to

(i) \( \mathcal{K}\mathcal{W}^+ \) meets each fiber \( \mathcal{W}^+ (\sigma) \),
(ii) \( m \) is onto.

The assertion (i) is that if there exists an extremal toric almost Kähler metric compatible with \( \omega \) then there exists an extremal toric Kähler metric and assertion (ii) is that if \( (P,\sigma) \) is uniformly \( K \)-stable then there exists an extremal toric almost Kähler metric compatible with \( \omega \). This is Corollary 1.4.

4. Miscellaneous

4.1. The normal and the angle. Let \( \vec{m} = (\vec{m}_1, \ldots, \vec{m}_d) \) and \( \vec{n} = (\vec{n}_1, \ldots, \vec{n}_d) \) be two distinct sets of labels on the same polytope \( P \subset \mathfrak{t}^* \) and assume that \( (P,\vec{m},\Lambda) \) is rational Delzant and thus associated to a compact toric symplectic manifold \( (M,\omega,T = t/\Lambda) \) through the Delzant–Lerman–Tolman correspondence. For any \( u \in \mathcal{S}(P,\vec{n}) \) the metric \( g_u \), see (9), defines a smooth Kähler metric on \( P \times \mathfrak{t}^* \cong M = x^{-1}(P) \) compatible with \( \omega \). However, since \( u \notin \mathcal{S}(P,\vec{m}) \) the metric \( g_u \) is not the restriction of a smooth metric on \( M \). The behavior of \( g_u \) along the boundary of \( M \) has been analysed in [30] and we recall the conclusion below.

Recall that \( \vec{m}_s \) and \( \vec{n}_s \) are inward to \( P \) and normal to the facet \( F_s \). We denote \( a_s > 0 \) the real number such that

\[
 a_s \vec{n}_s = \vec{m}_s.
\]

Note that the boundary condition of \( \mathcal{S}(P,\vec{n}) \) depends on the labelling via the Guillemin potential \( u_{\vec{n}} \), see Remark 2.4. Also, all the potentials in \( \mathcal{S}(P,\vec{n}) \) have the same behavior along \( \partial P \) and for every \( u \in \mathcal{S}(P,\vec{n}) \), \( g_u \) differs from \( g_{u_{\vec{n}}} \) only by the addition of a smooth tensor on \( T \times \mathfrak{t}^* \times T \). Therefore, without loss of generality, we pick \( u_{\vec{n}} \in \mathcal{S}(P,\vec{n}) \) to understand that behavior.

The metric \( g_{u_{\vec{n}}} \) which is smooth on \( M = P \times T = x^{-1}(P) \), has a

- singularity of cone angle type and angle \( 2a_s \pi \) along \( x^{-1}(\hat{F}_s) \), if \( a_s < 1 \);
- smooth extension on \( x^{-1}(P \cup \hat{F}_s) \), if \( a_s = 1 \);
- singularity characterized by a large angle \( 2a_s \pi > 2\pi \) along \( x^{-1}(\hat{F}_s) \), if \( a_s > 1 \).

where, here, we have adopted the terminology in [19].

**Proposition 4.1.** [30] Let \( (M,\omega,T) \) be a toric compact symplectic manifold with labelled moment polytope \( (P,\vec{m},\Lambda) \) and momentum map \( x : M \to \mathfrak{t}^* \). For any labelling \( \vec{n} \) of \( P \), any potential \( u \in \mathcal{S}(P,\vec{n}) \) provides a Kähler metric \( g_u \), defined via (9), smooth and compatible with \( \omega \) on \( M = x^{-1}(P) \) and with cone angle singularity \( 2\pi(\vec{n}_s/\vec{m}_s) \) transverse to the divisor \( x^{-1}(\hat{F}_s) \). Conversely, any compatible
T–invariant Kähler metric smooth outside a divisor D and with cone angle singularity transverse to D is of this form.

It is straightforward to extend the argument proving the last proposition to almost Kähler metric. Indeed we just compared the behaviour of the Hessian and inverse Hessian of \( u \vec{n} \) and \( u \vec{m} \). Therefore, any \( H \in \mathcal{A}K(P, \sigma \vec{n}) \) defines an almost Kähler metrics on \( \mathring{M} \) and with cone angle singularity \( 2\pi(\vec{n}_s/\vec{m}_s) \) transverse to the divisor \( x^{-1}(\mathring{F}_s) \).

4.2. The constant scalar curvature case. In case \((P, \vec{n}, \Lambda)\) is rational and associated to a compact toric symplectic orbifold \((M, \omega, T)\) via the Delzant–Lerman–Tolman correspondence and assuming we fix a compatible toric Kähler struc-

ture \((g_u, J_u)\) (so that \( u \in S(P, \vec{n}) \)) then the classical Futaki invariant evaluated on the real holomorphic vector field \( J_u X_f \) induced by the affine linear function \( f \in \text{Aff}(t^*, \mathbb{R}) \) is defined in [21] to be

\[
\text{Fut}(M, [\omega])(f) := \frac{2}{\int_P dx} \left( \int_{\partial P} f d\sigma_{\vec{n}} \int_P dx - \int_P f dx \int_{\partial P} d\sigma_{\vec{n}} \right).
\]

This observation is a motivation to introduce the functional

\[
\text{Fut}(P, \vec{n})(f) := \int_{\partial P} f d\sigma_{\vec{n}} \int_P dx - \int_P f dx \int_{\partial P} d\sigma_{\vec{n}},
\]

which in the rational case, up to a multiplicative positive constant, is the classical Futaki invariant restricted to the complex Lie algebra \( t \oplus Jt \). Moreover, in the case the classical Futaki invariant vanishes, equivalently when \( A_{\sigma} \) is a constant (which is then \( A_{\sigma} = 2 \int_{\partial P} \sigma_u/\int_P dx \) then

\[
\text{Fut}(P, \vec{n})(f) = \frac{2}{\int_P dx} \mathcal{L}(P, \sigma)(f)
\]

for any \( f \in \text{Aff}(t^*, \mathbb{R}) \).

**Corollary 4.2.** Given any labelled polytope \((P, \vec{n})\), if there exists a symplectic potential \( u \in S(P, \vec{n}) \) such that \( g_u \) has constant scalar curvature then \( \text{Fut}(P, \vec{n}) \) vanishes identically on \( \text{Aff}(t^*, \mathbb{R}) \).

Let \( \eta \) and \( \vec{n} \) be labellings for the same polytope \( P \). Then, for each \( s = 1, \ldots, d \), \( \eta_s \) and \( \vec{n}_s \) are inward to \( P \) and normal to the facet \( F_s \) and so there is a real number \( a_s > 0 \) such that

\[
a_s \vec{n}_s = \eta_s.
\]

When restricted on \( F_s \), we have \( d\sigma_{\vec{n}} = a_s d\sigma_{\eta} \). Therefore,

\[
\text{Fut}(P, \vec{n})(f) = \int_P dx \sum_s a_s \int_{F_s} f d\sigma_{\eta} - \int_P f dx \sum_s a_s \int_{F_s} d\sigma_{\eta}.
\]
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and thus

\[ \text{Fut}(P, \vec{n})(f) = \text{Fut}(P, \eta)(f) \]

\[ - \int_P dx \sum_s (1 - a_s) \int_{F_s} f \, d\sigma_{\eta} + \int_P f \, dx \sum_s (1 - a_s) \int_{F_s} d\sigma_{\eta}. \]

(27)

Note that, whenever \((P, \eta, \Lambda)\) is rational Delzant and thus associated to a compact toric symplectic manifold \((M, \omega, T = t/\Lambda)\) through the Delzant–Lerman–Tolman correspondence, the last expression coincides, up to some multiplicative positive constant, with the log Futaki invariant (relative to the torus \(T\)) defined in [19]. Indeed, consider the case where \(a_1 = \beta\) and \(a_s = 1\) for \(s = 2, \ldots, d\) then we recover from (27) that

\[ \text{Fut}_{D, \beta}(\Xi_f, [\omega]) = \frac{2(2\pi)^n \text{Fut}(P, \vec{n})(f)}{\int_M \omega^n} \]

where we follow the notation of [26] with \(D = x^{-1}(F_1)\).

Observe from (26) that the vanishing of the Futaki invariant imposes linear conditions on the labelling normals.

**Proposition 4.3.** Given a polytope \(P \subset t^*\) of dimension \(n\) with \(d\) facets, there exists a \((d - n)\)-dimensional cone \(C(P) \subset t^d\) of labelling \(\vec{n} \in C(P)\) such that \(\text{Fut}(P, \vec{n})\) vanishes identically on \(\text{Aff}(t^*, \mathbb{R})\).

In [30] the last proposition follows non trivial consideratio, we give an elementary proof here.

**Proof.** Put coordinates \(x = (x_1, \ldots, x_n)\) on \(t^*\) and translate \(P\) if necessary so that \(\int_P x_i \, dx = 0\) for any \(i = 1, \ldots, n\). The result follows if the linear map \(\mathbb{R}^d \to \mathbb{R}^n\) defined by

\[ \mathbb{R}^d \ni a \mapsto \left( \sum_{s=1}^d a_s \left( \int_P x_i \, dx \int_{F_s} d\sigma_{\vec{n}} - \int_P dx \int_{F_s} x_i \, d\sigma_{\vec{n}} \right) \right)_{i=1, \ldots, n} \]

(29)

is onto and his kernel meets the positive quadrant of \(\mathbb{R}^d\). With the suitable coordinate chosen the rhs of (29) is up to non-zero multiplicative constant

\[ \left( \sum_{s=1}^d a_s \int_{F_s} x_1 d\sigma_{\vec{n}}, \ldots, \sum_{s=1}^d a_s \int_{F_s} x_n d\sigma_{\vec{n}} \right) \in \mathbb{R}^n. \]

This is onto by convexity of \(P\), indeed, for any coordinates \(x_i\) there is a facet of \(P\) on which \(x_i\) is sign definite. Basic consideration on barycenter and the observation that \(0 \in P\) imply that the kernel of the map [29] contains an element of the positive quadrant of \(\mathbb{R}^d\). \(\square\)

**References**

[1] M. Abreu *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math. 9 (1998), 641–651.

[2] M. Abreu *Kähler metrics on toric orbifolds*, J. Differential Geom. 58 (2001), 151–187.

[3] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, C. Tønnesen-Friedman *Hamiltonian 2–forms in Kähler geometry. II. Global classification*, J. Differential Geom. 68 (2004), 277–345.
A NOTE ON EXTREMAL TORIC ALMOST KÄHLER METRICS

[4] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, C. Tønnesen-Friedman
Hamiltonian 2-forms in Kaehler geometry III: Extremal metrics and stability, Invent. Math. 173 (2008), 547–601.

[5] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, C. Tønnesen-Friedman
Extremal Kaehler metrics on projective bundles over a curve, Adv. Math. 227 (2011), 2385–2424.

[6] V. Apostolov, D. M. J. Calderbank, P. Gauduchon
Ambitoric geometry I: Einstein metrics and extremal ambikaehler structures
Journal fü r die reine und angewandte Mathematik, 721 (2016), 109–147.

[7] V. Apostolov, D. M. J. Calderbank, P. Gauduchon
Ambitoric geometry II: Extremal toric surfaces and Einstein 4-orbifolds
Ann. Sci. Ecole Norm. Supp. (4) 48 (2015), 1075–1112.

[8] E. Calabi
Extremal Kähler metrics. II.
Differential geometry and complex analysis, I. Chavel, H. M. Farkas Eds., 95–114, Springer, Berlin, 1985.

[9] D.M.J. Calderbank, L. David, P. Gauduchon
The Guillemin formula and Kähler metrics on toric symplectic manifolds
J. Symplectic Geom. 1 (2003) 767–784.

[10] X.X. Chen. J. Cheng
On the constant scalar curvature Kähler metrics (III), General automorphism group
ArXiv1801.05907v1

[11] B. Chen, A.-M. Li, L.Sheng
Uniform K-stability for extremal metrics on toric varieties
Journal of diff. equation, 257, 5, 2014, 1487–1500

[12] T. Darvas
The Mabuchi completion of the space of Kähler potentials
Amer. J. Math. 139 (2017), no. 5, 1275–1313.

[13] T. Delzant
Hamiltoniens périodiques et images convexes de l’application moment
Bull. Soc. Math. France 116 (1988), 315–339.

[14] S. K. Donaldson
Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), 289–342.

[15] S. K. Donaldson
Interior estimates for solutions of Abreu’s equation, Collect. Math. 56 (2005), 103–142.

[16] S. K. Donaldson
Kähler geometry on toric manifolds, and some other manifolds with large symmetry
Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM) 7, 29–75, Int. Press, Somerville, MA, 2008.

[17] S. K. Donaldson
Extremal metrics on toric surfaces: a continuity method, J. Differential Geom. 79 (2008), 389–432.

[18] S. K. Donaldson
Constant scalar curvature metrics on toric surfaces, Geom. Funct. Anal. 19 (2009), 83–136.

[19] S. K. Donaldson
Kähler metrics with cone singularities along a divisor, Essays in mathematics and its applications, 49–79, Springer, Heidelberg, 2012.

[20] J.J. Duistermaat, A. Pelayo
Reduced phase space and toric variety coordinatizations of Delzant spaces. Math. Proc. Cambridge Philos. Soc. 146 (2009), no. 3, 695–718.

[21] A. Futaki
An obstruction to the existence of Einstein Kähler metrics, Invent. Math. 73 (1983), no. 3, 437–443.

[22] A. Futaki, T. Marubbi
Bilinear forms and extremal Kähler vector fields associated with Kähler classes, Math. Ann. 301 (1995), 199–210.

[23] P. Gauduchon
Extremal Kähler metrics: An elementary introduction, In preparation.

[24] D. Guan
On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles, Math. Res. Lett. 6 (1999), 547–555.

[25] V. Guillemin
Kähler structures on toric varieties, J. Diff. Geom. 40 (1994), 285–309.

[26] Y. Hashimoto
Scalar curvature and Futaki invariant of Kähler metrics with cone singularities along a divisor, arXiv:math.DG/15008.02640v1.

[27] W. He
On Calabi’s extremal metric and properness, arXiv:math.DG/1801.071536

[28] J. Keller, M. Lejmi
On the lower bounds of the L2-norm of the Hermitian scalar curvature, arxiv:math.DG/1702.01810

[29] M. Lejmi
Extremal almost-Kähler metrics, Int. J. Math. 21 (2010), No. 12.

[30] E. Legendre
Toric Kähler-Einstein metrics and convex compact polytopes, Journal of Geometric Analysis, 26 (1), 2016, 399–427.

[31] E. Legendre
Toric geometry of convex quadrilaterals, J. Symplectic Geom. 9 (2011), 343–385.
[32] E. Lerman, S. Tolman Hamiltonian torus actions on symplectic orbifolds and toric varieties, Trans. Amer. Math. Soc. 349 (1997), 4201–4230.
[33] L.M. Sektnan An investigation of stability on certain toric surfaces, arXiv:1610.09419 [math.DG]
[34] G. Székelyhidi Extremal metrics and K-stability Bull. London Math. Soc. 39 (2007), 76–84
[35] G. Székelyhidi Extremal metrics and K-stability (PhD Thesis) arXiv:math/0611002
[36] G. Tian Kähler–Einstein metrics with positive scalar curvature, Inventiones Math. 130 No 1 (1997), 1–37.
[37] S.T. Yau Open problems in Differential Geometry, Proc. Symp. Pure Math., 54 (1993), 1–18.
[38] B. Zhou, X. Zhu K-stability on toric manifolds, Proc. Amer. Math. Soc. 136 (2008), 3301–3307.

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