Jaynes-Cummings model without rotating wave approximation.
Asymptotics of eigenvalues.

E.A.Tur

Department of Higher Mathematics
St–Petersburg State Institute of
Fine Mechanics and Optics (Technical University)
Sablinskaya 14, 197101 St–Petersburg, Russia
E-mail: Teduard@cards.lanck.net

Abstract. In this paper the perturbation theory with the frequency of transition in atom as perturbation parameter is constructed. The estimation of the reminder term of series of this perturbation theory is given. With the help of this perturbation theory we have found an exact asymptotics of eigenvalues of complete hamiltonian in the limit of high quantum numbers. It is shown that the counter-rotating terms keep a leading term but absolutely change a second term of this asymptotic.

1. Introduction.

The Jaynes-Cummings model without rotating wave approximation (RWA) is the elementary model describing an interaction of atom with a field. But despite of this it can not be solved exactly. This model without the RWA was considered by different methods in works [1-7]. The hamiltonian of this model has the form

\[ H = H_0 + g V = \omega_0 \sigma_0 + \omega a^+ a + g \sigma_1 (a + a^+) \]  

where \( a \) and \( a^+ \) are the photon creation and annihilation operators, \( g \) is the coupling constant, \( \omega \) and \( \omega_0 \) are the frequencies of mode and atomic transition respectively, \( \sigma_0 \) and \( \sigma_1 \) are the \( 2 \times 2 \) matrices of form

\[ \sigma_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

It is well known that the RWA formulas for eigenvalues take into account only zero and first order of the perturbation theory on the coupling constant \( g \). Therefore they are valid only at small relative coupling constant \( g/\omega \) and sufficiently small quantum index. More precisely, the validity of the RWA formulas for eigenvalues is defined by the condition \( g\sqrt{n}/\omega \ll 1 \). In the case of exact resonance (\( \omega = \omega_0 \)) the expression \( g\sqrt{n} \) defines the splitting of eigenvalues. In optics \( g/\omega \ll 1 \). Hence, unique opportunity to leave for limits of the RWA is the consideration of the highly exited states with sufficiently large quantum index \( n \). That is the RWA loses force at sufficiently large energies of a field mode. How the eigenvalues of the hamiltonian \( (1) \) and the splitting of them behave at the large quantum indexes? In the present paper we shall answer on this question by constructing the perturbation theory on the parameter \( \omega_0 \), which enters linearly in the hamiltonian \( (1) \). We shall show that this perturbation theory well describes not only an eigenvalues at \( \omega_0/\omega \ll 1 \) but also the highly laying eigenvalues at arbitrary \( g \) and \( \omega_0 \leq k \omega \), where \( k = \sqrt{3}/(2\pi) \simeq 0.23 \). We also give an estimation of the reminder term of series and find two first terms of asymptotic of eigenvalues on quantum index. It is interesting that the second term of this asymptotic is qualitatively differed from
the corresponding term in the RWA. This difference leads to the fact that the splitting of eigenvalues vanishes at the large quantum numbers, unlike the RWA case, when the splitting infinitely increases.

We remain open the question about the validity of the Jaynes-Cummings model itself at the limit of large average energy of a field mode, because then the manyphot transi on transitions between other levels become important. Nevertheless, we can hope, that the resonant levels give the basic contribution in atomic dynamics even for large average energy of a field mode.

2. Perturbation theory on the parameter $\omega_0$.

In work [6] we have shown that the hamiltonian of model (1) can be re presented in invariant subspaces by two Jacobi matrices of form

\[
H_1 = \begin{pmatrix}
\omega_0 & g\sqrt{1} & 0 & 0 & 0 & \ldots \\
g\sqrt{1} & 2\omega_0 + \omega & g\sqrt{2} & 0 & 0 & \ldots \\
0 & g\sqrt{2} & \omega_0 + 2\omega & g\sqrt{3} & 0 & \ldots \\
0 & 0 & g\sqrt{3} & 2\omega_0 + 3\omega & g\sqrt{4} & \ldots \\
0 & 0 & 0 & g\sqrt{4} & \omega_0 + 4\omega & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

\[
H_2 = \begin{pmatrix}
2\omega_0 & g\sqrt{1} & 0 & 0 & 0 & \ldots \\
g\sqrt{1} & \omega_0 + \omega & g\sqrt{2} & 0 & 0 & \ldots \\
0 & g\sqrt{2} & 2\omega_0 + 2\omega & g\sqrt{3} & 0 & \ldots \\
0 & 0 & g\sqrt{3} & \omega_0 + 3\omega & g\sqrt{4} & \ldots \\
0 & 0 & 0 & g\sqrt{4} & 2\omega_0 + 4\omega & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

Let us present the operators $H_1$ and $H_2$ as

\[
H_1 = A_0 + \omega_0 P_1, \quad H_2 = A_0 + \omega_0 P_2,
\]

where $A_0$ is the unbounded main operator without periodic modulation of main diagonal, $P_1$ and
The diagonal projectors

\[
P_2 = \begin{pmatrix}
\omega_0 & g\sqrt{1} & 0 & 0 & 0 \\
g\sqrt{1} \omega_0 + \omega & g\sqrt{2} & 0 & 0 & 0 \\
0 & g\sqrt{2} \omega_0 + 2\omega & g\sqrt{3} & 0 & 0 \\
0 & 0 & g\sqrt{3} \omega_0 + 3\omega & g\sqrt{4} & 0 \\
0 & 0 & 0 & g\sqrt{4} \omega_0 + 4\omega & \ldots
\end{pmatrix}
\]

(3)

\[
P_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
, \quad P_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

(4)

The operator \(A_0\) is the hamiltonian of shifted oscillator. It can be diagonalized with the help of Bogolubov’s transformation. Its eigenvalues and eigenvectors have the form

\[
A_0 |a_m\rangle = \lambda_m |a_m\rangle, \quad \lambda_m = \omega_0 + m\omega - g^2/\omega, \quad |a_m\rangle = \sum_{n=0}^{\infty} P_n^{(m)} |e_n\rangle, \quad \langle a_m|a_n\rangle = \delta_{m,n}.
\]

(5)

where \(|e_n\rangle\) is the basis of matrix representation (3), \(P_n^{(m)}\) are defined by Feynman-Schwinger’s formulas [8,9]

\[
P_n^{(m)} = \exp\left\{-\frac{g^2}{2\omega^2}\right\} \sqrt{\frac{n!}{m!}} \left(\frac{g}{\omega}\right)^{m-n} L_n^{m-n}(g^2/\omega^2)
\]

(6)

Here \(L_n^s\) are generalized Chebyshev-Laguerre’s polynomials

\[
L_n^s(x) = \frac{(n + s)!}{n!} \sum_{i=0}^{n} C_n^i (-1)^i \frac{x^i}{(i + s)!}, \quad C_n^i = \frac{n!}{i!(n - i)!}
\]

It is easy to verify that the expression (3) can be presented also in the form of contour integral

\[
P_n^{(m)} = \exp\left\{-\frac{g^2}{2\omega^2}\right\} \sqrt{\frac{m!}{n!}} \left(\frac{g}{\omega}\right)^{n-m} \frac{1}{2\pi i} \int_C x^{m-1} \left(\frac{1}{x} - 1\right)^n \exp\left\{\frac{g^2}{\omega^2 x}\right\} dx
\]

(7)

where \(C\) is the circle of unit radius with the centre in the origin of coordinates of a complex plane \(x\). This expression we will use further.
From (2) it follows that the operators $H_1$ and $H_2$ depend linearly on $\omega_0$. If we know the solution of spectral problem for the operator $A_0$, we can build the perturbation theory on the parameter $\omega_0$. Let us find the matrix form of the operators $P_1$ and $P_2$ in the basis of the operator $A_0$ eigenvectors. Let $U(g)$ be the orthogonal transformation from $|e_m\rangle$ at $|a_m\rangle$

$$|a_m\rangle = U(g) |e_m\rangle, \quad \langle e_k | U(g) | e_m \rangle = P_k^{(m)}, \quad U^T U = E$$

Using (4),(5) and (8), we have

$$P_{k,m}^{(1)} \equiv \langle a_k | P_1 | a_m \rangle = \langle e_k | U^T P_1 U | e_m \rangle = \langle e_k | P^{(1)} | e_m \rangle = \sum_{n-\text{odd}} P_n^{(k)} P_n^{(m)}$$

$$P_{k,m}^{(2)} \equiv \langle a_k | P_2 | a_m \rangle = \langle e_k | U^T P_2 U | e_m \rangle = \langle e_k | P^{(2)} | e_m \rangle = \sum_{n-\text{even}} P_n^{(k)} P_n^{(m)},$$

where $P^{(1)} = U^T P_1 U$ and $P^{(2)} = U^T P_2 U$ are the transformed projectors.

Let us consider for example the sum (9). Using the representation (7) and summarizing on odd values $n$, we come to the formula

$$P_{k,m}^{(1)} = \frac{1}{2} \delta_{k,m} - \frac{(-1)^k}{2} \exp \left\{ -\frac{2g^2}{\omega^2} \right\} \sqrt{m! \, k!} \left( \frac{g}{\omega} \right)^{m-k} \sum_{i=0}^{k} C_k^i (-1)^i \frac{(4g^2/\omega^2)^i}{(i + m - k)!}$$

The contour integrals in this expression can be calculated consistently with the help of residues. As a result, we obtain the following expression for $P_{k,m}^{(1)}$

$$P_{k,m}^{(1)} = \frac{1}{2} \delta_{k,m} - \frac{(-1)^k}{2} \exp \left\{ -\frac{2g^2}{\omega^2} \right\} \sqrt{m! \, k!} \left( \frac{g}{\omega} \right)^{m-k} \sum_{i=0}^{k} C_k^i (-1)^i \frac{(4g^2/\omega^2)^i}{(i + m - k)!}$$

Similarly, one can obtain and the expression for $P_{k,m}^{(2)}$, defined by the sum (11)

$$P_{k,m}^{(2)} = \frac{1}{2} \delta_{k,m} + \frac{(-1)^k}{2} \exp \left\{ -\frac{2g^2}{\omega^2} \right\} \sqrt{m! \, k!} \left( \frac{g}{\omega} \right)^{m-k} \sum_{i=0}^{k} C_k^i (-1)^i \frac{(4g^2/\omega^2)^i}{(i + m - k)!}$$

Comparing (11) and (12) with (5), we have

$$P_{k,m}^{(1)} = \frac{1}{2} \delta_{k,m} - \frac{(-1)^k}{2} P_k^{(m)} (2g)$$

$$P_{k,m}^{(2)} = \frac{1}{2} \delta_{k,m} + \frac{(-1)^k}{2} P_k^{(m)} (2g),$$
or in the operator form
\[ P^{(1)} = U^T(g) P_1 U(g) = \frac{1}{2} \left( E - B U(2g) \right) \] (15)
\[ P^{(2)} = U^T(g) P_2 U(g) = \frac{1}{2} \left( E + B U(2g) \right) , \] (16)

where \( B \) is the diagonal matrix with elements \( B_{m,k} = (-1)^k \delta_{m,k} \). Let us note that the matrices \( B \) and \( U \) satisfy to the identity
\[ [B U]^2 = E \]

The formulaes (13), (14) (or (11), (12)) allow to write at once the approximated expressions for eigenvalues taking into account only zero and first orders of the perturbation theory on \( \omega_0 \). The first order correction to an eigenvalues is defined by diagonal elements of perturbation. Taking into account the formula (5) for the eigenvalues of the operator \( A_0 \) and the expressions (13), (14), (6) (at \( k = m \)), we obtain the following approximated formulaes for eigenvalues \( \lambda_m^{(1)} \) and \( \lambda_m^{(2)} \) of the operators \( H_1 \) and \( H_2 \) respectively
\[ \lambda_m^{(1)} \simeq 3\omega_0/2 + m\omega - g^2/\omega - \frac{(1)^m \omega_0}{2} \exp \left\{ -\frac{2g^2}{\omega^2} \right\} L_m(4g^2/\omega^2) \]
\[ \lambda_m^{(2)} \simeq 3\omega_0/2 + m\omega - g^2/\omega + \frac{(1)^m \omega_0}{2} \exp \left\{ -\frac{2g^2}{\omega^2} \right\} L_m(4g^2/\omega^2) \]

This formulaes was obtained in work [5] with the help of a so-called "operator method".

Let us consider now the constructed perturbation theory series in more detail. We shall show that the two first term of this series give an exact asymptotic of an eigenvalues \( \lambda_m^{(1)} \) and \( \lambda_m^{(2)} \) at large quantum index \( m \).

3. Asymptotic of eigenvalues.

Let us consider, for example, an eigenvalues \( \lambda_m^{(2)} \) of the operator \( H_2 \). The proof of the appropriate formulas for \( \lambda_m^{(1)} \) is completely similarly. In what follows for brevity we shall omit the top index (2) at eigenvalues and write \( \lambda_m \) instead of \( \lambda_m^{(2)} \). The perturbation theory series for exact eigenvalue \( \lambda_m \) has the form
\[ \lambda_m = \sum_{k=0}^{\infty} \lambda_m^{(k)} , \quad \lambda_m^{(k)} \sim (\omega_0)^k \] (17)

General expression for \( \lambda_m^{(k)} \), in case when an operator depends linearly on the perturbation parametr and an eigenvalues are not degenerate (here, due to the simplicity of Jacobi matrix spectrum), has the form [10]
\[ \lambda_m^{(k)} = \frac{(-\omega_0)^k}{k} \sum_{n_1 + \ldots + n_k = k-1}^{n_i \geq 0} \text{tr} \left[ P S_m^{n_1} \ldots P S_m^{n_k} \right] , \quad k \geq 1 , \] (18)
where
\[
S_m^0 \equiv -|a_m\rangle\langle a_m|, \quad S_m^n = \sum_{i \neq m} \frac{|a_i\rangle\langle a_i|}{(\lambda_i^{(0)} - \lambda_m^{(0)})^n} = \frac{1}{\omega^n} \sum_{i \neq m} \frac{|a_i\rangle\langle a_i|}{(i - m)^n}, \quad n \geq 1
\] (19)

Here, we have omitted as well as above the top index (2) at the perturbation operator \( \mathbf{P}^{(2)} \) and used the formula (5) for the unperturbed eigenvalues \( \lambda_m^{(0)} \).

We have found already that
\[
\lambda_m^{(0)} = m\omega + \omega_0 - g^2/\omega
\] (20)
\[
\lambda_m^{(1)} = \omega_0/2 + \frac{(-1)^m \omega_0}{2} \exp \left\{ -\frac{2g^2}{\omega^2} \right\} L_m(4g^2/\omega^2) = \omega_0/2 + O(m^{-1/4}), \quad m \to \infty
\] (21)

Here, we have used the asymptotic of Chebyshev-Laguerre’s polynomials (see, for example, [11]).

Let us consider the second order correction \( \lambda_m^{(2)} \) which is defined by
\[
\lambda_m^{(2)} = \omega_0^2 \sum_{k \neq m} \frac{|P_{k,m}|^2}{\lambda_m^{(0)} - \lambda_k^{(0)}}
\]

According to (14) and (20), this expression can be presented in the form
\[
\lambda_m^{(2)} = \frac{\omega_0^2}{4\omega} \sum_{k \neq m} \frac{[P_k^{(m)}(2g)]^2}{m - k}
\]
The behaviour of this expression as \( m \to \infty \) is defined by the behaviour of sum
\[
t_m = \sum_{k \neq m} \frac{[P_k^{(m)}(2g)]^2}{m - k}
\] (22)

Let us show that \( t_m \to 0 \) as \( m \to \infty \). For this purpose, let us transform (22) to the form
\[
t_m = \sum_{n=1}^{\infty} \frac{C_{m,n}}{n},
\]
where the transformation matrix \( C_{m,n} \) is defined as follows
\[
C_{m,n} = \begin{cases} 
[P_m^{(m)}(2g)]^2 - [P_n^{(m)}(2g)]^2, & n \leq m \\
-[P_n^{(m)}(2g)]^2, & n > m
\end{cases}
\] (23)

The condition \( t_m \to 0 \) follows from \( \frac{1}{n} \to 0 \), if and only if the transformation \( C_{m,n} \) satisfies to the
following conditions (12), Theorem 4)

1. \[ \sum_n |C_{m,n}| < H, \] where \( H \) does not depend of \( m \)

2. \[ \lim_{m \to \infty} C_{m,n} = 0, \] for arbitrary \( n \)

Then \( C_{m,n} \) is the regular transformation. Let us prove the first condition. Taking into account (23), we have

\[ \sum_n |C_{m,n}| = \sum_{n=1}^{m} \left[ |P_{m-n}^{(m)}(2g)|^2 - |P_{n}^{(m)}(2g)|^2 \right] + \sum_{n=m+1}^{\infty} |P_{n}^{(m)}(2g)|^2 < \]

\[ < \left[ \sum_{n=0}^{\infty} |P_{n}^{(m)}(2g)|^2 \right] - |P_{m}^{(m)}(2g)|^2 \]

But since the values \( P_{n}^{(m)}(2g) \) are the matrix elements of the orthogonal transformation \( U(2g) \), the sum in square brackets is equal to unit identically. The diagonal matrix element \( P_{m}^{(m)}(2g) \), due to (6), equals to

\[ P_{m}^{(m)}(2g) = \exp \left\{ -\frac{2g^2}{\omega^2} \right\} L_m(4g^2/\omega^2), \]

and due to the asymptotic of Chebyshev-Laguerre's polynomials [11], tends to zero as \( m \to \infty \). Therefore, we have

\[ \sum_n |C_{m,n}| < 1 - \delta_m, \quad \delta_m \to 0 \text{ as } m \to \infty \]

And hence, the condition 1 in (24) is fulfilled.

Let us check now the validity of the second condition in (24). For this purpose, due to (23), it is necessary to consider the diagonal asymptotic of the non-diagonal matrix elements of the transformation \( U(2g) \). From (6), we have

\[ P_{m-n}^{(m)}(2g) = \exp \left\{ -\frac{2g^2}{\omega^2} \right\} \sqrt{\frac{(m-n)!}{m!}} \left( \frac{2g}{\omega} \right)^n L_{m-n}^n(4g^2/\omega^2) \]

Using the asymptotic of generalized Chebyshev-Laguerre's polynomials [11]

\[ L_n^s(x) = \pi^{-1/2} n^{s/2-1/4} x^{-s/2-1/4} e^{x/2} \left\{ \cos(2\sqrt{n}x - s\pi/2 - \pi/4) + O(n^{-1/2}) \right\}, \quad n \to \infty, \]

we obtain

\[ P_{m-n}^{(m)}(2g) \sim \frac{1}{m^{1/4}}, \quad m \to \infty, \quad \text{for arbitrary } n \]

Due to the symmetry of perturbation matrix \( P_{k,m} \), we obtain at once the same asymptotic and for \( P_{m+n}^{(m)}(2g) \)

\[ P_{m+n}^{(m)}(2g) \sim \frac{1}{m^{1/4}}, \quad m \to \infty, \quad \text{for arbitrary } n \]

According to (23), it follows that the condition 2 is also fulfilled. Therefore due to the above theorem, \( t_m \) and hence \( \lambda_m^{(2)} \) tend to zero as \( m \to \infty \):

\[ \lambda_m^{(2)} \to 0, \quad m \to \infty \] (25)
Let us consider the third order correction $\lambda^{(3)}_m$ to the eigenvalue $\lambda_m$. From (18) and (19) it follows that $\lambda^{(3)}_m$ is defined by expression

$$
\lambda^{(3)}_m = \frac{\omega^3_0}{\omega^2} \left[ \sum_{i,j \neq m} \frac{P_{m,i} P_{i,j} P_{j,m}}{(i-m)(j-m)} - P_{m,m} \sum_{i \neq m} \frac{|P_{m,i}|^2}{(i-m)^2} \right]
$$

Using (14), we have

$$
|\lambda^{(3)}_m| \leq \frac{(\omega_0/2)^3}{\omega^2} \left[ \sum_{i,j \neq m} |P_m(i,2g)\,| P_i(j,2g)\,| P_j(m,2g)| \right] \left[ \sum_{i \neq m} \frac{|P_{m,i}(2g)|^2}{(i-m)^2} \right]^{1/2} = \gamma_{i,m} \sigma_m
$$

Due to the orthogonality of the transformation $U(2g)$, as well as above, we have

$$
\gamma_{i,m} = \sqrt{1 - |P_i^{(m)}(2g)|^2} < 1, \quad \text{for arbitrary } i \text{ and } m,
$$

and therefore

$$
\sum_{j \neq m} \frac{|P_i(j,2g)| \,|P_j(m,2g)|}{|j-m|} < \sigma_m = \left[ \sum_{j \neq m} \frac{|P_j(m,2g)|^2}{|j-m|^2} \right]^{1/2}
$$

Using the theorem on regular transformation, just as it was made for $\lambda^{(2)}_m$, one can show that $\sigma_m \to 0$ as $m \to \infty$.

Thus for the first term in (26), we have the inequality

$$
\sum_{i,j \neq m} \frac{|P_{m,i}(2g)| \, |P_i(j,2g)| \, |P_j(m,2g)|}{|i-m| \, |j-m|} < \sigma_m \sum_{i \neq m} \frac{|P_{m,i}(2g)|}{|i-m|}
$$

Let us apply once again Cauchy’s inequality to the sum on $i$ in the right side of this inequality

$$
\sum_{i \neq m} \frac{|P_{m,i}(2g)|}{|i-m|} \leq \left[ \sum_{i \neq m} \frac{1}{|i-m|^2} \right]^{1/2} = \gamma_{m,m} f_m < f_m,
$$

$$
f_m = \left[ \sum_{i \neq m} \frac{1}{(i-m)^2} \right]^{1/2} = \left[ \sum_{k=1}^{m} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} \right]^{1/2} < \left[ 2 \sum_{k=1}^{\infty} \frac{1}{k^2} \right]^{1/2} = \sqrt{\frac{\pi^2}{3}} = \frac{\pi}{\sqrt{3}}
$$

It follows that

$$
\sum_{i,j \neq m} \frac{|P_{m,i}(2g)| \, |P_i(j,2g)| \, |P_j(m,2g)|}{|i-m| \, |j-m|} < \frac{\pi}{\sqrt{3}} \sigma_m,
$$

(27)
and the inequality (26) takes the form

\[ |\lambda_m^{(3)}| < \frac{(\omega_0/2)^3}{\omega^2} \sigma_m \left[ \frac{\pi}{\sqrt{3}} + \left| 1 + (-1)^m P_m^{(m)}(2g) \right| \sigma_m \right] \tag{28} \]

Since \( \pi/\sqrt{3} \simeq 1.81 > 1 \) and \( P_m^{(m)}(2g) \to 0 \) as \( m \to \infty \) that there exists such \( m_0 \) that

\[ \left| 1 + (-1)^m P_m^{(m)}(2g) \right| < \pi/\sqrt{3}, \quad m > m_0 \tag{29} \]

Despite of that \( |P_m^{(m)}(2g)| < 1 \) for arbitrary \( m \), we could not prove that the inequality (29) is valid for arbitrary \( m \). Taking into account that \( \sigma_m < 1 \) and (29), we obtain from (28)

\[ |\lambda_m^{(3)}| < \frac{(\omega_0/2)^3}{3\omega^2} 6 \frac{\pi}{\sqrt{3}} \sigma_m, \quad m > m_0; \quad \sigma_m \to 0, \quad m \to \infty \tag{30} \]

Here, number 6 is the number of components in the sum (18) for \( k = 3 \).

Let us note that since \( \sigma_m \to 0 \) as \( m \to \infty \), we could write instead of (30) more strong inequality, following from (28). But we shall write just the inequality (30), following from (29), because just this way can be used and for higher orders of a perturbation theory.

Let us consider the \( k \)-th order correction \( \lambda_m^{(k)} \). Using Cauchy’s inequality and condition (29) one can show as well as above that the absolute value of each term in the sum (18) is bounded by expression

\[ \frac{(\omega_0/2)^k}{k\omega^{k-1}} \left( \frac{\pi}{\sqrt{3}} \right)^{k-2} \sigma_m, \quad m > m_0, \tag{31} \]

and hence

\[ |\lambda_m^{(k)}| < \frac{(\omega_0/2)^k}{k\omega^{k-1}} \left( \frac{\pi}{\sqrt{3}} \right)^{k-2} N_k \sigma_m, \quad m > m_0, \quad k > 2, \tag{32} \]

where \( N_k \) is the number of terms in the sum (18), i.e. the number of solutions of the equation \( n_1 + \ldots + n_k = k - 1, \quad n_i \geq 0 \)

\[ N_k = \frac{(2k-2)!}{[(k-1)!]^2} \tag{33} \]

Let us show the estimation (31), for example, on the typical term, entering in (18) at \( k = 4 \)

\[ h_m = \frac{(\omega_0)^4}{4\omega^3} \text{tr} \left[ P S_m^0 P S_m^0 P S_m^1 P S_m^2 \right] = \frac{(\omega_0)^4}{4\omega^3} P_{m,m} \sum_{i,j \neq m} \frac{P_{m,i} P_{i,j} P_{j,m}}{(i-m)(j-m)^2} \]

Using (14), we have

\[ |h_m| < \frac{(\omega_0/2)^4}{4\omega^3} \left| 1 + (-1)^m P_m^{(m)}(2g) \right| \sum_{i,j \neq m} \frac{|P_m^{(i)}(2g)| |P_i^{(j)}(2g)| |P_j^{(m)}(2g)|}{|i-m||j-m|} \]
At last, using (29) and already obtained estimation (27), we obtain
\[ |h_m| < \frac{(\omega_0/2)^4}{4\omega^3} \left( \frac{\pi}{\sqrt{3}} \right)^2 \sigma_m, \quad m > m_0, \]
that is the estimation (31) for \( k = 4 \).

With the help of the unequality (32) we can estimate the reminder term of the series (17)
\[ \left| \lambda_m - \sum_{k=0}^{n} \lambda_m^{(k)} \right| = \left| \sum_{k=n}^{\infty} \lambda_m^{(k)} \right| < \frac{3\omega}{\pi^2} \sigma_m \sum_{k=n}^{\infty} \frac{N_k}{k} \left( \frac{\omega_0 \pi}{\omega \sqrt{3}} \right)^k, \quad m > m_0, \quad n > 2 \quad (34) \]

From (32) it follows that asymptoticaly
\[ N_k = \frac{(2k - 2)!}{[ (k - 1)! ]^2} \sim \frac{4^k}{\sqrt{k}}, \quad k \to \infty \]

It follows that the series in right part of (34) converges at \( \omega_0 \leq \omega \sqrt{3}/(2\pi) \). We can estimate it as follows. Since \( N_k < 2^{2k-2} \), we have
\[ \left| \lambda_m - \sum_{k=0}^{n} \lambda_m^{(k)} \right| < \frac{3\omega}{4\pi^2} \sigma_m \sum_{k=n}^{\infty} \left( \frac{2\omega_0 \pi}{\omega \sqrt{3}} \right)^k, \quad m > m_0, \quad n > 2 \]
or
\[ \left| \lambda_m - \sum_{k=0}^{n} \lambda_m^{(k)} \right| < \frac{3\omega}{4\pi^2} \sigma_m \frac{(2\omega_0 \pi)^n}{1 - \frac{2\omega_0 \pi}{\omega \sqrt{3}}}, \quad m > m_0, \quad n > 2 \quad (35) \]

Taking into account that \( \sigma_m \to 0 \) as \( m \to \infty \) and using (35), (29), (20) and (21), we obtain the following asymptotic of eigenvalues \( \lambda_m \)
\[ \lambda_m = m\omega + 3\omega_0/2 - g^2/\omega + o(1), \quad m \to \infty \quad (36) \]

Since the formulas (13), (14) differ only by sign, it easy to see that the same asymptotic takes place and for eigenvalues \( \lambda_m^{(1)} \) of the operator \( H_1 \). Thus, we have proved the following result

**Theorem**: If \( \omega_0 \leq \omega \sqrt{3}/(2\pi) \), then the eigenvalues \( \lambda_m^{(1)} \) and \( \lambda_m^{(2)} \) of the operators \( H_1 \) and \( H_2 \) have the asymptotic (36) and the reminder term of perturbation theory series have the estimation (35).
4. Conclusion.

In physical applications the main role plays not eigenvalues itself but a difference of neighbouring eigenvalues, determining in the resonant case $\omega_0 = \omega$ the splitting of originally degenerate levels

$$\Delta_m^{(1)} = \lambda_{2m+2}^{(1)} - \lambda_{2m+1}^{(1)}, \quad m = 0, 1, 2, \ldots$$

$$\Delta_m^{(2)} = \lambda_{2m+1}^{(2)} - \lambda_{2m}^{(2)}, \quad m = 0, 1, 2, \ldots$$

From (36) it follows directly that

$$\Delta_m^{(1,2)} \to \omega, \quad m \to \infty$$

It is in the sharp contradiction with the RWA. In the RWA an eigenvalues appropriate, for example, to $\lambda_m^{(2)}$ (in resonant case $\omega = \omega_0$), are defined by the expression

$$\lambda_m = \omega (2m + 2) \pm g \sqrt{2m + 1}, \quad m = 0, 1, 2, \ldots$$

Therefore in the RWA the splitting grows as $\sqrt{2m}$.

This change of splitting undoubtedly should change the time dynamics of quantum amplitudes, especially, when the average energy of a field mode is sufficiently large.

We have proved the asymptotic formula (36) only at the condition $\omega_0 \leq \omega \sqrt{3}/(2\pi)$. But the numerical calculations shows that it is valid and for $\omega_0 > \omega \sqrt{3}/(2\pi)$.

Acknowledgments.

I am grateful to Prof. S.N. Naboko and Prof. N.M. Bogolubov for their questions and useful remarks.

References

[1] Reik H. G., Nusser H., Amarante Ribeiro L. A., "Exact solution of non-adiabatic model hamiltonians in solid state physics and optics.”, J. of Phys. A, v. 15, n. 11, 1982, p. 3491.

[2] Graham R., Hohnerbach M., "Quantum chaos of the two-level atom.”, Phys. Lett. A, v. 101, n. 2, 1984, p. 61.

[3] Kus M., Lewenstein M., "Exact isolated solutions for the class of quantum optical systems.”, J. of Phys. A, v. 19, n. 2, 1986, p. 305.

[4] Lais P., Steimle T., "Squeezing in the Jaynes - Cummings model without the RWA”, Optics communications, v. 78, n. 5.6 , 1990, p. 346.

[5] Feranchuk I. D., Komarov L. I., Ulyanenkov A. P., "Two - level system in a one - mode quantum field : numerical solution on the basis of the operator method.”, J. of Phys. A, v. 29, 1996, p. 4035.
[6] Tur E.A., "Jaynes-Cummings model: Solution without rotating wave approximation", Optics and Spectroscopy, Vol. 89, n. 4, 2000, pp. 574-588.

[7] Tur E.A., "Energy Spectrum of the Hamiltonian of the Jaynes-Cummings Model without Rotating-Wave Approximation", Optics and Spectroscopy, Vol. 91, n. 6, 2001, pp. 899-902.

[8] Feynman R.P., Phys.Rev., 84, 1951, 108.

[9] Schwinger J., Phys.Rev., 91, 1953, 728.

[10] Kato T., Perturbation theory for linear operators, Springer-Verlag Berlin · Heidelberg · New York, 1966.

[11] Szego G., "Orthogonal polynomials", New York, 1939.

[12] Hardy G., Divergent series, Oxford, 1949.