Commutators of vector-valued intrinsic square functions on vector-valued generalized weighted Morrey spaces

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Abstract

In this paper, we will obtain the strong type and weak type estimates for vector-valued analogues of intrinsic square functions in the generalized weighted Morrey spaces $M^{\Phi,\varphi}_w(\mathbb{R}^n)$. We study the boundedness of intrinsic square functions including the Lusin area integral, Littlewood-Paley g-function and $g^*_\lambda$-function and their $k$th-order commutators on vector-valued generalized weighted Morrey spaces $M^{\Phi,\varphi}_w(l_2)$. In all the cases the conditions for the boundedness are given either in terms of Zygmund-type integral inequalities on $\varphi(x,r)$ without assuming any monotonicity property of $\varphi(x,r)$ on $r$.

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1 Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [2, 3] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let $K$ be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [9] states that the commutator operator $[b, K]f = K(bf) - bKf$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [6]-[8], [3], [10], [11]).

The classical Morrey spaces were originally introduced by Morrey in [32] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [10] [11] [18] [32]. Recently, Komori and Shirai [29] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some
classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev [21, 22] introduced the generalized weighted Morrey spaces $M_p,\varphi$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [25, 27, 28, 35]).

The intrinsic square functions were first introduced by Wilson in [40, 41]. They are defined as follows. For $0 < \alpha \leq 1$, let $C_\alpha$ be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi$’s support is contained in $\{ x : |x| \leq 1 \}$, $\int_{\mathbb{R}^n} \phi(x)dx = 0$, and for $x, x' \in \mathbb{R}^n$,

$$|\phi(x) - \phi(x')| \leq |x - x'|^\alpha.$$ 

For $(y, t) \in \mathbb{R}^{n+1}_+$ and $f \in L^{1,\text{loc}}(\mathbb{R}^n)$, set

$$A_\alpha f(t, y) \equiv \sup_{\phi \in C_\alpha} |f \ast \phi_t(y)|,$$

where $\phi_t(y) = t^{-n}\phi(y/t)$. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of $f$ by the formula

$$G_{\alpha,\beta}(f)(x) = \left( \int \int_{\Gamma_\beta(x)} (A_\alpha f(t, y))^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma_\beta(x) = \{ (y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < \beta t \}$. Denote $G_{\alpha,1}(f) = G_\alpha(f)$.

This function is independent of any particular kernel, such as Poisson kernel. It dominates pointwise the classical square function (Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ is depend of kernels with uniform compact support, there is pointwise relation between $G_{\alpha,\beta}(f)$ with different $\beta$:

$$G_{\alpha,\beta}(f)(x) \leq \beta^{\frac{2n}{n+1} + \alpha} G_\alpha(f)(x).$$

We can see details in [40].

The intrinsic Littlewood-Paley g-function and the intrinsic $g^*_\lambda$ function are defined respectively by

$$g_\alpha f(x) = \left( \int_0^\infty (A_\alpha f(y, t))^2 \frac{dt}{t} \right)^{1/2},$$

$$g^*_\lambda,\alpha f(x) = \left( \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^n (A_\alpha f(y, t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$ 

When we say that $f$ maps into $l_2$, we mean that $\tilde{f}(x) = (f_j)_{j=1}^\infty$, where each $f_j$ is Lebesgue measurable and, for almost every $x \in \mathbb{R}^n$

$$\|\tilde{f}(x)\|_{l_2} = \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{1/2}.$$
Let \( \vec{f} = (f_1, f_2, \ldots) \) be a sequence of locally integrable functions on \( \mathbb{R}^n \). For any \( x \in \mathbb{R}^n \), Wilson \[41\] also defined the vector-valued intrinsic square functions of \( \vec{f} \) by \( \|G_\alpha \vec{f}(x)\|_{l_2} \) and proved the following result.

**Theorem A.** Let \( 1 \leq p < \infty, \ 0 < \alpha \leq 1 \) and \( w \in A_p \). Then the operators \( G_\alpha \) and \( g^*_{\lambda,\alpha} \) are bounded from \( L^p_w(l_2) \) into itself for \( p > 1 \) and from \( L^1_w(l_2) \) to \( W L^1_w(l_2) \).

Moreover, in \[31\], Lerner showed sharp \( L^p_w \) norm inequalities for the intrinsic square functions in terms of the \( A_p \) characteristic constant of \( w \) for all \( 1 < p < \infty \).

Also Huang and Liu \[12\] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In \[38\] and \[39\], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In \[37\], Wang considered intrinsic functions and the commutators generated with BMO functions on weighted Morrey spaces. Let \( b \) be a locally integrable function on \( \mathbb{R}^n \). Setting

\[
A_{\alpha,b}^k f(t, y) \equiv \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)]^k \phi_t(y - z) f(z) dz \right|
\]

the \( k \)th-order commutators are defined by

\[
[b, G_\alpha]^k f(x) = \left( \int \int_{\Gamma(x)} (A_{\alpha,b}^k f(t, y))^2 \frac{dydt}{tn+1} \right)^{\frac{1}{2}},
\]

\[
[b, g_\alpha]^k f(x) = \left( \int_0^\infty (A_{\alpha,b}^k f(t, y))^2 \frac{dt}{t} \right)^{\frac{1}{2}}
\]

and

\[
[b, g^*_{\lambda,\alpha}]^k f(x) = \left( \int \int_{\mathbb{R}^n+1} \left( \frac{t}{t + |x - y|} \right)^{\lambda_n} (A_{\alpha,b}^k f(t, y))^2 \frac{dydt}{tn+1} \right)^{\frac{1}{2}}.
\]

A function \( b \in L^1_{loc}(\mathbb{R}^n) \) is said to be in \( BMO(\mathbb{R}^n) \) if

\[
\|b\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_{B(x, r)}| dy < \infty,
\]

where \( b_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} b(y) dy \).

By the similar argument as in \[14\] and \[37\], we can get

**Theorem B.** Let \( 1 < p < \infty, \ 0 < \alpha \leq 1 \), \( w \in A_p \) and \( b \in BMO(\mathbb{R}^n) \). Then the \( k \)th-order commutator operators \( [b, G_\alpha]^k \) and \( [b, g^*_{\lambda,\alpha}]^k \) are bounded from \( L^p_w(l_2) \) into itself.

In this paper, we will consider the boundedness of the operators \( G_\alpha, g_\alpha, g^*_{\lambda,\alpha} \) and their \( k \)th-order commutators on vector-valued generalized weighted Morrey spaces. Let \( \varphi(x, r) \) be a positive measurable function on \( \mathbb{R}^n \times \mathbb{R}_+ \) and \( w \) be
non-negative measurable function on \( \mathbb{R}^n \). For any \( \vec{f} \in L_w^{p,\text{loc}}(l_2) \), we denote by \( M_w^{p,\varphi}(l_2) \) the vector-valued generalized weighted Morrey spaces, if
\[
\|\vec{f}\|_{M_w^{p,\varphi}(l_2)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \| \vec{f}(\cdot) \|_{L_w^p(B(x,r))} < \infty.
\]
When \( w \equiv 1 \), then \( M_w^{p,\varphi}(l_2) \) coincide the vector-valued generalized Morrey spaces \( M_w^{p,\varphi}(l_2) \). There are many papers discussed the conditions on \( \varphi(x, r) \) to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [17] (see, also [18]), by Guliyev the following condition was imposed on the pair \( (\varphi_1, \varphi_2) \):
\[
\int_r^\infty \varphi_1(x, t) \frac{dt}{t} \leq C \varphi_2(x, r).
\]
where \( C > 0 \) does not depend on \( x \) and \( r \). Under the above condition, they obtained the boundedness of Calderón-Zygmund singular integral operators from \( M_w^{p,\varphi_1}(\mathbb{R}^n) \) to \( M_w^{p,\varphi_2}(\mathbb{R}^n) \). Also, in [11] and [20], Guliyev et. introduced a weaker condition: If \( 1 \leq p < \infty \), there exits a constant \( C > 0 \), such that, for any \( x \in \mathbb{R}^n \) and \( t > 0 \),
\[
\int_r^\infty \text{ess inf}_{t < s < \infty} \frac{\varphi_1(x, s) s^{\frac{n}{p} + 1}}{t} \frac{dt}{s} \leq C \varphi_2(x, r).
\]
If the pair \( (\varphi_1, \varphi_2) \) satisfies condition (1.1), then \( (\varphi_1, \varphi_2) \) satisfied condition (1.2). But the opposite is not true. We can see remark 4.7 in [20] for details.

Recently, in [21, 22] (see, also [25, 28, 35]), Guliyev introduced a weighted condition: If \( 1 \leq p < \infty \), there exits a constant \( C > 0 \), such that, for any \( x \in \mathbb{R}^n \) and \( t > 0 \),
\[
\int_r^\infty \text{ess inf}_{t < s < \infty} \frac{\varphi_1(x, s) w(B(x, s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi_2(x, r),
\]
In this paper, we will obtain the boundedness of the vector-valued intrinsic function, the intrinsic Littlewood-Paley \( g \) function, the intrinsic \( g_1^* \) function and their \( k \)-th order commutators on vector-valued generalized weighted Morrey spaces when \( w \in A_p \) and the pair \( (\varphi_1, \varphi_2) \) satisfies condition (1.3) or the following inequalities,
\[
\int_r^\infty \ln^k \left( e + \frac{t}{r} \right) \text{ess inf}_{t < s < \infty} \frac{\varphi_1(x, s) w(B(x, s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi_2(x, r),
\]
where \( C \) does not depend on \( x \) and \( r \). Our main results in this paper are stated as follows.

**Theorem 1.1.** Let \( 1 \leq p < \infty \), \( 0 < \alpha \leq 1 \), \( w \in A_p \) and \((\varphi_1, \varphi_2)\) satisfies condition (1.3). Then the operator \( G_\alpha \) is bounded from \( M_w^{p,\varphi_1}(l_2) \) to \( M_w^{p,\varphi_2}(l_2) \) for \( p > 1 \) and from \( M_w^{1,\varphi_1}(l_2) \) to \( W M_w^{1,\varphi_2}(l_2) \).
Theorem 1.2. Let \( 1 \leq p < \infty \), \( 0 < \alpha \leq 1 \), \( w \in A_p \), \( \lambda > 3 + \frac{\alpha}{p} \) and \( (\varphi_1, \varphi_2) \) satisfies condition (1.3). Then the operator \( g_{\lambda, \alpha}^* \) is bounded from \( M_{w, \varphi_1}^p(l_2) \) to \( M_{w, \varphi_2}^p(l_2) \) for \( p > 1 \) and from \( M_{w, \varphi_1}^1(l_2) \) to \( W M_{w, \varphi_2}^1(l_2) \).

Theorem 1.3. Let \( 1 < p < \infty \), \( 0 < \alpha \leq 1 \), \( w \in A_p \), \( b \in BMO \) and \( (\varphi_1, \varphi_2) \) satisfies condition (1.4). Then \([b, G_{\alpha}]^k \) is bounded from \( M_{w, \varphi_1}^p(l_2) \) to \( M_{w, \varphi_2}^p(l_2) \).

Theorem 1.4. Let \( 1 < p < \infty \), \( 0 < \alpha \leq 1 \), \( w \in A_p \), \( b \in BMO \) and \( (\varphi_1, \varphi_2) \) satisfies condition (1.4), then for \( \lambda > 3 + \frac{\alpha}{p} \), \([b, g_{\lambda, \alpha}^*]^k \) is bounded from \( M_{w, \varphi_1}^p(l_2) \) to \( M_{w, \varphi_2}^p(l_2) \).

In [40], the author proved that the functions \( G_{\alpha}f \) and \( g_{\alpha}f \) are pointwise comparable. Thus, as a consequence of Theorem 1.1 and Theorem 1.3 we have the following results.

Corollary 1.5. Let \( 1 \leq p < \infty \), \( 0 < \alpha \leq 1 \), \( w \in A_p \) and \( (\varphi_1, \varphi_2) \) satisfies condition (1.3), then \( g_{\alpha} \) is bounded from \( M_{w, \varphi_1}^p(l_2) \) to \( M_{w, \varphi_2}^p(l_2) \) for \( p > 1 \) and from \( M_{w, \varphi_1}^1(l_2) \) to \( W M_{w, \varphi_2}^1(l_2) \).

Corollary 1.6. Let \( 1 < p < \infty \), \( 0 < \alpha \leq 1 \), \( w \in A_p \), \( b \in BMO \) and \( (\varphi_1, \varphi_2) \) satisfies condition (1.4), then \([b, g_{\alpha}]^k \) is bounded from \( M_{w, \varphi_1}^p(l_2) \) to \( M_{w, \varphi_2}^p(l_2) \).

Remark 1.7. Note that, in the scalar valued case the Theorems 1.1 - 1.3 and Corollaries 1.5 - 1.6 was proved in [26] \((w \equiv 1)\) and [27]. Also, in the scalar valued case and \( w \equiv A_p \) and \( \varphi_1(x, r) = \varphi_2(x, r) \equiv w(B(x, r))^{\frac{\kappa - 1}{p}} \), \( 0 < \kappa < 1 \) Theorems 1.1-1.4 and Corollaries 1.5-1.6 was proved by Wang in [37, 38]. How as, if \( \varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa - 1}{p}} \), then the vector-valued generalized weighed Morrey space \( M_{w, \varphi}^p(l_2) \) coincide the vector-valued weighed Morrey space \( L_{w, \varphi}^p(l_2) \) and the pair \((w(B(x, r))^{\frac{\kappa - 1}{p}}, w(B(x, r))^{\frac{\kappa - 1}{p}}) \) satisfies the both conditions (1.3) and (1.4). Indeed, by Lemma 3.1 there exists \( C > 0 \) and \( \delta > 0 \) such that for all \( x \in \mathbb{R}^n \) and \( t > r \):

\[
w(B(x, t)) \geq C \left( \frac{t}{r} \right)^{n\delta} w(B(x, r)).
\]

Then

\[
\int_r^\infty \text{ess inf}_{t < s < \infty} \frac{w(B(x, s))^{\frac{\kappa - 1}{p}}}{w(B(x, t))^{1/p}} \frac{dt}{t} \leq \int_r^\infty \ln^k \left( e + \frac{t}{r} \right) \frac{\text{ess inf}_{t < s < \infty} w(B(x, s))^{\frac{\kappa - 1}{p}}}{w(B(x, t))^{1/p}} \frac{dt}{t} = \int_r^\infty \ln^k \left( e + \frac{t}{r} \right) w(B(x, t))^{\frac{\kappa - 1}{p}} \frac{dt}{t}
\]
\[
\leq \int_r^\infty \ln k \left( e + \frac{t}{r} \right) \left( \left( \frac{t}{r} \right)^n \right) w(B(x, r)) \frac{\kappa^{-1}}{p} \frac{dt}{t}
\]
\[
= w(B(x, r))^{\kappa^{-1}} \int_r^\infty \ln k \left( e + \frac{t}{r} \right) \left( \frac{t}{r} \right)^n \frac{\kappa^{-1}}{p} \frac{dt}{t}
\]
\[
= w(B(x, r))^{\kappa^{-1}} \int_1^\infty \ln k \left( e + \tau \right) \tau^{n\kappa^{-1}} \frac{d\tau}{\tau}
\]
\[
\approx w(B(x, r))^{\kappa^{-1}}.
\]

Throughout this paper, we use the notation \( A \lesssim B \) to mean that there is a positive constant \( C \) independent of all essential variables such that \( A \leq CB \). Moreover, \( C \) may be different from place to place.

## 2 Vector-valued generalized weighted Morrey spaces

The classical Morrey spaces \( M^{p,\lambda} \) were originally introduced by Morrey in [32] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [15, 30].

We denote by \( M^{p,\lambda}(l_2) \equiv M^{p,\lambda}(\mathbb{R}^n, l_2) \) the vector-valued Morrey space, the space of all vector-valued functions \( \vec{f} \in L^{p,\text{loc}}(l_2) \) with finite quasinorm

\[
\left\| \vec{f} \right\|_{M^{p,\lambda}(l_2)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{\kappa^{-1}}{p} \left\| \vec{f} \right\|_{L^p(B(x,r), l_2)},
\]

where \( 1 \leq p < \infty \) and \( 0 \leq \lambda \leq n \).

Note that \( M^{p,0}(l_2) = L^p(l_2) \) and \( M^{p,n}(l_2) = L^\infty(l_2) \). If \( \lambda < 0 \) or \( \lambda > n \), then \( M^{p,\lambda}(l_2) = \Theta \), where \( \Theta \) is the set of all vector-valued functions equivalent to 0 on \( \mathbb{R}^n \).

We define the vector-valued generalized weighted Morrey spaces as follows.

**Definition 2.1.** Let \( 1 \leq p < \infty \), \( \varphi \) be a positive measurable vector-valued function on \( \mathbb{R}^n \times (0, \infty) \) and \( w \) be non-negative measurable function on \( \mathbb{R}^n \). We denote by \( M^{p,\varphi}_w(l_2) \equiv M^{p,\varphi}_w(\mathbb{R}^n, l_2) \) the vector-valued generalized weighted Morrey space, the space of all vector-valued functions \( \vec{f} \in L^{p,\text{loc}}(l_2) \) with finite norm

\[
\left\| \vec{f} \right\|_{M^{p,\varphi}_w(l_2)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \left\| \vec{f} \right\|_{L^p_w(B(x,r), l_2)},
\]

where \( L^p_w(B(x,r), l_2) \) denotes the vector-valued weighted \( L^p \)-space of measurable functions \( f \) for which

\[
\left\| f \right\|_{L^p_w(B(x,r), l_2)} \equiv \left\| f \chi_{B(x,r)} \right\|_{L^p_w(\mathbb{R}^n)} = \left( \int_{B(x,r)} \left\| f(y) \right\|_{l_2}^p w(y) dy \right)^{\frac{1}{p}}.
\]
Furthermore, by $WM_{W}^{p,\varphi}(l_{2})$ we denote the vector-valued weak generalized weighted Morrey space of all functions $f \in WL_{W}^{p,loc}(l_{2})$ for which

$$\|\vec{f}\|_{WM_{W}^{p,\varphi}(l_{2})} = \sup_{x \in \mathbb{R}^{n}, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|\vec{f}\|_{WL_{W}^{p}(B(x, r), l_{2})} < \infty,$$

where $WL_{W}^{p}(B(x, r), l_{2})$ denotes the weak $L^{p}_{w}$-space of measurable functions $f$ for which

$$\|\vec{f}\|_{WL_{W}^{p}(B(x, r), l_{2})} \equiv \|\vec{f}\chi_{B(x, r)}\|_{WL_{W}^{p}(l_{2})} = \sup_{t > 0} \left( \int_{\{y \in B(x, r) : \|\vec{f}\chi_{B(x, r)}(y)\|_{l_{2}} > t\}} w(y)dy \right)^{\frac{1}{p}}.$$

**Remark 2.2.**

1. If $w \equiv 1$, then $M_{W}^{p,\varphi}(l_{2}) = M^{p,\varphi}(l_{2})$ is the vector-valued generalized Morrey space.

2. If $\varphi(x, r) \equiv w(B(x, r))^{\frac{n-1}{p}}$, then $M_{W}^{p,\varphi}(l_{2}) = L_{W}^{p,\kappa}(l_{2})$ is the vector-valued weighted Morrey space.

3. If $\varphi(x, r) \equiv v(B(x, r))^{\frac{1}{p}} w(B(x, r))^{-\frac{1}{p}}$, then $M_{W}^{p,\varphi}(l_{2}) = L_{v,w}^{p,\lambda}(l_{2})$ is the vector-valued two weighted Morrey space.

4. If $w \equiv 1$ and $\varphi(x, r) = r^{\frac{n}{p}}$ with $0 < \lambda < n$, then $M_{W}^{p,\varphi}(l_{2}) = L^{p,\lambda}(l_{2})$ is the vector-valued weak Morrey space and $WM_{W}^{p,\varphi}(l_{2}) = WL^{p,\lambda}(l_{2})$ is the vector-valued weak Morrey space.

5. If $\varphi(x, r) \equiv w(B(x, r))^{-\frac{1}{p}}$, then $M_{W}^{p,\varphi}(l_{2}) = L_{W}^{p}(l_{2})$ is the vector-valued weighted Lebesgue space.

### 3 Preliminaries and some lemmas

By a weight function, briefly weight, we mean a locally integrable function on $\mathbb{R}^{n}$ which takes values in $(0, \infty)$ almost everywhere. For a weight $w$ and a measurable set $E$, we define $w(E) = \int_{E} w(x)dx$, and denote the Lebesgue measure of $E$ by $|E|$ and the characteristic function of $E$ by $\chi_{E}$. Given a weight $w$, we say that $w$ satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball $B$, we have $w(2B) \leq Dw(B)$. When $w$ satisfies this condition, we write brevity $w \in \Delta_{2}$.

If $w$ is a weight function, we denote by $L^{p}_{w}(l_{2}) \equiv L^{p}_{w}(\mathbb{R}^{n}, l_{2})$ the vector-valued weighted Lebesgue space defined by finiteness of the norm

$$\|\vec{f}\|_{L^{p}_{w}(l_{2})} = \left( \int_{\mathbb{R}^{n}} \|\vec{f}(x)\|_{l_{2}}^{p} w(x)dx \right)^{\frac{1}{p}} < \infty,$$

and by $\|\vec{f}\|_{L^{\infty}_{w}(l_{2})} = \text{ess sup}_{x \in \mathbb{R}^{n}} \|\vec{f}(x)\|_{l_{2}} w(x)$ if $p = \infty$. 

We recall that a weight function $w$ is in the Muckenhoupt’s class $A_p$ $[33]$, $1 < p < \infty$, if

$$[w]_{A_p} := \sup_B [w]_{A_p(B)} = \sup_B \left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where the sup is taken with respect to all the balls $B$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls $B$ by Hölder’s inequality 

$$[w]_{A_p}^{1/p} = |B|^{-1} \|w\|^{1/p}_{L^p(B)} \|w^{-1/p}\|_{L^{p'}(B)} \geq 1.$$ 

For $p = 1$, the class $A_1$ is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$, $A_\infty = \bigcup_{1 \leq p < \infty} A_p$ and $[w]_{A_\infty} = \inf_{1 \leq p < \infty} [w]_{A_p}$.

**Lemma 3.1.** $[16]$ (1) If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$

$$w(\lambda B) \leq \lambda^p [w]_{A_p} w(B).$$

(2) If $w \in A_\infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$

$$w(\lambda B) \leq 2^n [w]_{A_\infty} w(B).$$

(3) If $w \in A_p$ for some $1 \leq p \leq \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any ball $B$ and a measurable set $S \subset B$,

$$\frac{w(S)}{w(B)} \leq C \left( \frac{|S|}{|B|} \right)^{\delta}.$$

We are going to use the following result on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t} \int_0^t g(r) d\mu(r), \quad 0 < t < \infty,$$

where $\mu$ is a non-negative Borel measure on $(0, \infty)$.

**Theorem 3.2.** $[4]$ The inequality

$$\operatorname{ess sup}_{t>0} \omega(t)Hg(t) \leq c \operatorname{ess sup}_{t>0} v(t)g(t)$$

holds for all functions $g$ non-negative and non-increasing on $(0, \infty)$ if and only if

$$A := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \frac{d\mu(r)}{\operatorname{ess sup}_{0 \leq s \leq r} v(s)} < \infty,$$

and $c \approx A$.  

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We also need the following statement on the boundedness of the Hardy type operator
\[(H_1g)(t) := \frac{1}{t} \int_0^t \ln^k \left(e + \frac{t}{r}\right) g(r) d\mu(r), \quad 0 < t < \infty,\]
where \(\mu\) is a non-negative Borel measure on \((0, \infty)\).

**Theorem 3.3.** The inequality
\[
\text{ess sup}_{t>0} \omega(t) H_1 g(t) \leq c \text{ess sup}_{t>0} v(t) g(t)
\]
holds for all functions \(g\) non-negative and non-increasing on \((0, \infty)\) if and only if
\[
A_1 := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \ln^k \left(e + \frac{t}{r}\right) \frac{d\mu(r)}{\text{ess sup}_{0<s<r} v(s)} < \infty,
\]
and \(c \approx A_1\).

Note that, Theorem 3.3 can be proved analogously to Theorem 4.3 in [19].

**Definition 3.4.** \(BMO(\mathbb{R}^n)\) is the Banach space modulo constants with the norm \(\| \cdot \|_*\) defined by
\[
\|b\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_{B(x, r)}| dy < \infty,
\]
where \(b \in L^1_{\text{loc}}(\mathbb{R}^n)\) and
\[
b_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} b(y) dy.
\]

**Lemma 3.5.** ([34], Theorem 5, p. 236) Let \(w \in A_\infty\). Then the norm \(\| \cdot \|_*\) is equivalent to the norm
\[
\|b\|_{*, w} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{w(B(x, r))} \int_{B(x, r)} |b(y) - b_{B(x, r), w}| w(y) dy,
\]
where
\[
b_{B(x, r), w} = \frac{1}{w(B(x, r))} \int_{B(x, r)} b(y) w(y) dy.
\]

**Remark 3.6.** (1) The John-Nirenberg inequality: there are constants \(C_1, C_2 > 0\), such that for all \(b \in BMO(\mathbb{R}^n)\) and \(\beta > 0\)
\[
|\{x \in B : |b(x) - B| > \beta}\} | \leq C_1 |B| e^{-C_2 \beta / \|b\|_*}, \quad \forall B \subset \mathbb{R}^n.
\]
For $1 \leq p < \infty$ the John-Nirenberg inequality implies that
\[
\|b\|_\ast \approx \sup_B \left( \frac{1}{|B|} \int_B |b(y) - b_B|^p dy \right)^{\frac{1}{p}}
\] (3.1)
and for $1 \leq p < \infty$ and $w \in A_\infty$
\[
\|b\|_\ast \approx \sup_B \left( \frac{1}{w(B)} \int_B |b(y) - b_B|^p w(y) dy \right)^{\frac{1}{p}}.
\] (3.2)

Note that, by the John-Nirenberg inequality and Lemma 3.1 (part 3) it follows that
\[
w(\{ x \in B : |b(x) - b_B| > \beta \}) \leq C_1 w(B) e^{-C_2 \beta \|b\|_\ast}
\] for some $\delta > 0$. Hence
\[
\int_B \left| b(y) - b_B \right|^p w(y) dy = p \int_0^{\infty} \beta^{p-1} w(\{ x \in B : |b(x) - b_B| > \beta \}) d\beta
\]
\[
\leq pC_1^p w(B) \int_0^{\infty} \beta^{p-1} e^{-C_2 \beta \|b\|_\ast} d\beta = C_3 w(B) \|b\|_\ast^p,
\]
where $C_3 > 0$ depends only on $C_1^p$, $C_2$, $p$, and $\delta$, which implies (3.2).

Also (3.1) is a particular case of (3.2) with $w \equiv 1$.

The following lemma was proved in [22].

Lemma 3.7. i) Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 \leq p < \infty$, $x \in \mathbb{R}^n$, $k > 0$ and $r_1, r_2 > 0$. Then
\[
\left( \frac{1}{w(B(x, r_1))} \int_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^k w(y) dy \right)^{\frac{1}{p}} \leq C \left( 1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \|b\|_\ast^k,
\]
where $C > 0$ is independent of $f$, $w$, $x$, $r_1$, and $r_2$.

ii) Let $w \in A_\rho$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, $k > 0$ and $r_1, r_2 > 0$. Then
\[
\left( \frac{1}{w^{1-p'}(B(x, r_1))} \int_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^{k p'} w(y)^{1-p'} dy \right)^{\frac{1}{p'}} \leq C \left( 1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \|b\|_\ast^k,
\]
where $C > 0$ is independent of $f$, $w$, $x$, $r_1$, and $r_2$. 

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4 Proofs of main theorems

Before proving the main theorems, we need the following lemmas.

Lemma 4.1. [37] For \( j \in \mathbb{Z}_+ \), denote
\[
G_{\alpha,2^j}(f)(x) = \left( \int_0^\infty \int_{|x-y| \leq 2^jt} (A_\alpha f(y,t)) \frac{dydt}{t^{n+1}} \right)^{\frac{1}{p}}
\]
Let \( 0 < \alpha \leq 1, 1 < p < \infty \) and \( w \in A_p \). Then any \( j \in \mathbb{Z}_+ \), we have
\[
\|G_{\alpha,2^j}(f)\|_{L_w^p} \lesssim 2^j \left( \frac{\|w\|_{A_p}}{2^j} \right) \|G_{\alpha}(f)\|_{L_w^p}.
\]
This lemma is easy from the following inequality which is proved in [40].
\[
G_{\alpha,\beta}(f)(x) \leq \beta^{\frac{3n}{2}+\alpha} G_{\alpha}(f)(x).
\]

By the similar argument as in [3], we can get the following lemma.

Lemma 4.2. Let \( 1 < p < \infty \), \( 0 < \alpha \leq 1 \) and \( w \in A_p \), then the commutators \([b, G_{\alpha}]^k\) is bounded from \( L_w^p(l_2) \) to itself whenever \( b \in BMO \).

Now we are in a position to prove theorems.

Lemma 4.3. Let \( 1 \leq p < \infty \), \( 0 < \alpha \leq 1 \) and \( w \in A_p \).

Then, for \( p > 1 \) the inequality
\[
\|G_{\alpha}\tilde{f}\|_{L_w^p(B(x_0,t),l_2)} \lesssim \left( w(B) \right)^{\frac{1}{p}} \int_{2r}^\infty \|\tilde{f}\|_{L_w^p(B(x_0,t),l_2)} \left( w(B(x_0,t)) \right)^{-\frac{1}{p}} \frac{dt}{t}
\]
holds for any ball \( B = B(x_0,r) \) and for all \( \tilde{f} \in L_w^{p,loc}(l_2) \).

Moreover, for \( p = 1 \) the inequality
\[
\|G_{\alpha}\tilde{f}\|_{W_{L_w^1}(B(x_0,t),l_2)} \lesssim w(B) \int_{2r}^\infty \|\tilde{f}\|_{L_w^1(B(x_0,t),l_2)} \left( w(B(x_0,t)) \right)^{-1} \frac{dt}{t},
\]
holds for any ball \( B = B(x_0,r) \) and for all \( \tilde{f} \in L1locl2 \).

Proof. The main ideas of these proofs come from [22]. For arbitrary \( x \in \mathbb{R}^n \), set \( B = B(x_0,r) \), \( 2B \equiv B(x_0,2r) \). We decompose \( \tilde{f} = \tilde{f}_0 + \tilde{f}_\infty \), where \( \tilde{f}_0(y) = \tilde{f}(y)\chi_{2B}(y) \), \( \tilde{f}_\infty(y) = \tilde{f}(y) - \tilde{f}_0(y) \). Then,
\[
\|G_{\alpha}\tilde{f}\|_{L_w^p(B(x_0,r),l_2)} \leq \|G_{\alpha}\tilde{f}_0\|_{L_w^p(B(x_0,r),l_2)} + \|G_{\alpha}\tilde{f}_\infty\|_{L_w^p(B(x_0,r),l_2)} := I + II.
\]

First, let us estimate \( I \). By Theorem A, we can obtain that
\[
I \leq \|G_{\alpha}\tilde{f}_0\|_{L_w^p(l_2)} \lesssim \|\tilde{f}_0\|_{L_w^p(l_2)} = \|\tilde{f}\|_{L_w^p(2B,l_2)}.
\]

(4.1)
On the other hand,

\[
\|\tilde{f}\|_{L^p(B, t^2, t^2)} \approx |B| \|\tilde{f}\|_{L^p(2B, t^2, t^2)} \int_0^\infty \frac{dt}{t^{n+1}}
\]

\[
\leq |B| \int_{2^r}^\infty \|\tilde{f}\|_{L^p(B(x_0, t^2), t^2)} \frac{dt}{t^{n+1}}
\]

\[
\lesssim w(B) \int_{2^r}^\infty \|\tilde{f}\|_{L^p(B(x_0, t^2), t^2)} \frac{dt}{t^{n+1}}
\]

\[
\lesssim w(B) \int_{2^r}^\infty \|\tilde{f}\|_{L^p(B(x_0, t^2), t^2)} \|w^{-1/p}\|_{L^p(B(x_0, t^2))} \frac{dt}{t^{n+1}}
\]

\[
\lesssim \frac{1}{t} \int_{2^r}^\infty \|\tilde{f}\|_{L^p(B(x_0, t^2), t^2)} \|w(B(x_0, t))\| \frac{dt}{t^{n+1}}
\]

Therefore from (4.1) and (4.2) we get

\[
I \lesssim w(B) \int_{2^r}^\infty \|\tilde{f}\|_{L^p(B(x_0, t^2), t^2)} \|w(B(x_0, t))\| \frac{dt}{t^{n+1}}
\]  

(4.3)

Then let us estimate II.

\[
\|f \ast \phi_t(y)\|_{l_2} = \left\| t^{-n} \int_{|y-z| \leq t} \phi\left(\frac{y-z}{t}\right) \tilde{f}_{\infty}(z) dz \right\|_{l_2} 
\]

\[
\leq t^{-n} \int_{|y-z| \leq t} \|\tilde{f}_{\infty}(z)\|_{l_2} dz.
\]

Since \(x \in B(x_0, r), (y, t) \in \Gamma(x)\), we have \(|z - x| \leq |z - y| + |y - x| \leq 2t\), and

\[
r \leq |z - x_0| - |x_0 - x| \leq |x - z| \leq |x - y| + |y - z| \leq 2t.
\]

So, we obtain

\[
\|G_\alpha f_{\infty}(x)\|_{l_2} \leq \left( \int_{\Gamma(x)} \left( t^{-n} \int_{|y-z| \leq t} \|\tilde{f}_{\infty}(z)\|_{l_2} dz \right) \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_{t > r/2} \int_{|x-y| < t} \left( \int_{|x-z| \leq 2t} \|\tilde{f}_{\infty}(z)\|_{l_2} dz \right) \frac{dy dt}{t^{3n+1}} \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( \int_{t > r/2} \left( \int_{|x-z| \leq 2t} \|\tilde{f}_{\infty}(z)\|_{l_2} dz \right) \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}}.
\]

By Minkowski and Hölder’s inequalities and \(|z - x| \geq |z - x_0| - |x_0 - x| \geq \frac{1}{2} |z - x_0|\),

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we have

\[ \| G_{\alpha} \tilde{f}_\infty (x) \|_{l_2} \leq \int_{\mathbb{R}^n} \left( \int_{t > |x - x_0|^{1/p}} \frac{dt}{t^{2n+1}} \right)^{1/2} \| \tilde{f}_\infty (z) \|_{l_2} \, dz \]

Thus,

\[ \int_{|z - x_0| > 2r} \| \tilde{f}(z) \|_{l_2} \, dz \leq \int_{|z - x_0| > 2r} \| \tilde{f}(z) \|_{l_2} \, dz \]

By combining (4.3) and (4.5), we have

\[ \| G_{\alpha} \tilde{f}_\infty \|_{L^p_x(B,l_2)} \leq w(B)^{1/p} \int_{2r}^{\infty} \| \tilde{f} \|_{L^p_x(B(x_0,t),l_2)} (w(B(x_0,t)))^{-1/p} \, \frac{dt}{t}. \] (4.4)

By combining (4.3) and (4.5), we have

\[ \| G_{\alpha} \tilde{f} \|_{L^p_x(B,l_2)} \leq w(B)^{1/p} \int_{2r}^{\infty} \| \tilde{f} \|_{L^p_x(B(x_0,t),l_2)} (w(B(x_0,t)))^{-1/p} \, \frac{dt}{t}. \] (4.5)

Proof of Theorem 1.1
By Lemma 3.3 and Theorem 3.2 we have for \( p > 1 \)

\[ \| G_{\alpha} \tilde{f} \|_{M^w_{p,\varphi} (l_2)} \leq \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2 (x_0, r)^{-1} \int_r^{\infty} \| \tilde{f} \|_{L^p_x(B(x_0,t),l_2)} (w(B(x_0,t)))^{-1/p} \, \frac{dt}{t} \]

\[ = \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2 (x_0, r)^{-1} \int_0^{l^{-1}} \frac{1}{r} \int_0^r \| \tilde{f} \|_{L^p_x(B(x_0,t^{-1}),l_2)} (w(B(x_0,t^{-1})))^{-1/p} \, \frac{dt}{t} \]

\[ = \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2 (x_0, r^{-1}-1) \frac{1}{r} \int_0^r \| \tilde{f} \|_{L^p_x(B(x_0,t^{-1}),l_2)} (w(B(x_0,t^{-1})))^{-1/p} \, \frac{dt}{t} \]

\[ \leq \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_1 (x_0, r^{-1})^{-1} (w(B(x_0, r^{-1})))^{-1/p} \| \tilde{f} \|_{L^p_x(B(x_0,r^{-1}),l_2)} \]

\[ = \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_1 (x_0, r)^{-1} (w(B(x_0, r)))^{-1/p} \| \tilde{f} \|_{L^p_x(B(x_0,r),l_2)} = \| \tilde{f} \|_{M^w_{p,\varphi} (l_2)} \]
and for $p = 1$

$$
\|G_\alpha \vec{f}\|_{W^{M_\alpha,\varphi_2}(l_2)} \lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} \int_r^\infty \|\vec{f}\|_{L^\infty(B(x_0, t), l_2)} \left(w(B(x_0, t))\right)^{-1} \frac{dt}{t}
$$

$$
= \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} \int_0^r \|\vec{f}\|_{L^\infty(B(x_0, t^{-1}), l_2)} \left(w(B(x_0, t^{-1}))\right)^{-1} \frac{dt}{t}
$$

$$
\lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_1(x_0, r)^{-1} \left(w(B(x_0, r^{-1}))\right)^{-1} \|\vec{f}\|_{L^\infty(B(x_0, r), l_2)}
$$

$$
= \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_1(x_0, r)^{-1} \left(w(B(x_0, r))\right)^{-1} \|\vec{f}\|_{L^\infty(B(x_0, r), l_2)} = \|\vec{f}\|_{W^{M_\alpha,\varphi_1}(l_2)}.
$$

**Lemma 4.4.** Let $1 \leq p < \infty$, $0 < \alpha \leq 1$, $\lambda > 3 + \frac{\alpha}{n}$ and $w \in A_p$. Then, for $p > 1$ the inequality

$$
\|g_{\lambda,\alpha}(\vec{f})\|_{L^p_w(B, l_2)} \lesssim \left(w(B)\right)^{\frac{1}{p}} \int_{2r}^\infty \|\vec{f}\|_{L^p_w(B(x_0, t), l_2)} \left(w(B(x_0, t))\right)^{-\frac{1}{p}} \frac{dt}{t}
$$

holds for any ball $B = B(x_0, r)$ and for all $\vec{f} \in L^{p,\text{loc}}(l_2)$.

Moreover, for $p = 1$ the inequality

$$
\|g_{\lambda,\alpha}(\vec{f})\|_{W^{L^p_w(B, l_2)}} \lesssim w(B) \int_{2r}^\infty \|\vec{f}\|_{L^p_w(B(x_0, t), l_2)} \left(w(B(x_0, t))\right)^{-1} \frac{dt}{t}
$$

holds for any ball $B = B(x_0, r)$ and for all $\vec{f} \in L^{1,\text{loc}}l_2$.

**Proof.** From the definition of $g_{\lambda,\alpha}(f)$, we readily see that

$$
\|g_{\lambda,\alpha}(\vec{f})(x)\|_{l_2} = \left\|\left( \int_0^\infty \int_{\mathbb{R}^n} \left( \frac{t}{t + |x - y|}\right)^{n\lambda} \left(A_\alpha \vec{f}(y, t)\right) \frac{2 dy dt}{t^{n+1}}\right)^{1/2} \right\|_{l_2}
$$

$$
\leq \left\|\left( \int_0^\infty \int_{|x - y| < t} \left( \frac{t}{t + |x - y|}\right)^{n\lambda} \left(A_\alpha \vec{f}(y, t)\right) \frac{2 dy dt}{t^{n+1}}\right)^{1/2} \right\|_{l_2}
$$

$$
+ \left\|\left( \int_0^\infty \int_{|x - y| \geq t} \left( \frac{t}{t + |x - y|}\right)^{n\lambda} \left(A_\alpha \vec{f}(y, t)\right) \frac{2 dy dt}{t^{n+1}}\right)^{1/2} \right\|_{l_2}
$$

$$
:= III + IV.
$$

First, let us estimate $III$.

$$
III \leq \left\|\left( \int_0^\infty \int_{|x - y| < t} \left( \frac{t}{t + |x - y|}\right)^{n\lambda} \left(A_\alpha \vec{f}(y, t)\right) \frac{2 dy dt}{t^{n+1}}\right)^{1/2} \right\|_{l_2} \leq \|G_\alpha \vec{f}(x)\|_{l_2}.
$$
Now, let us estimate IV.

\[
IV \leq \left\| \left( \sum_{j=1}^{\infty} \int_0^\infty \int_{2^{j-1}t \leq |x-y| \leq 2^jt} \left( \frac{t}{t+n+1} \right)^{n\lambda} \left( A_\alpha \tilde{f}(y,t) \right)^2 dydt \right)^{1/2} \right\|_{l_2}
\]

\[
\leq \left\| \left( \sum_{j=1}^{\infty} \int_0^\infty \int_{2^{j-1}t \leq |x-y| \leq 2^jt} 2^{-jn\lambda} \left( A_\alpha \tilde{f}(y,t) \right)^2 dydt \right)^{1/2} \right\|_{l_2}
\]

\[
\leq \sum_{j=1}^{\infty} 2^{-jn\lambda} \left\| \int_0^\infty \int_{|x-y| \leq 2^jt} \left( A_\alpha \tilde{f}(y,t) \right)^2 dydt \right\|_{l_2}^{1/2}
\]

\[
:= \sum_{j=1}^{\infty} 2^{-jn\lambda} \| G_{\alpha,2j}(\tilde{f})(x) \|_{l_2}
\]

Thus,

\[
\| g_{\alpha,\lambda}(\tilde{f}) \|_{L^p_w(B,B,l_2)} \leq \| G_{\alpha,0}(\tilde{f}) \|_{L^p_w(B,B,l_2)} + \sum_{j=1}^{\infty} 2^{-jn\lambda} \| G_{\alpha,2j}(\tilde{f}) \|_{L^p_w(B,B,l_2)}.
\] (4.6)

By Lemma 4.3, we have

\[
\| G_{\alpha,0}(\tilde{f}) \|_{L^p_w(B,B,l_2)} \leq \left( w(B) \right)^{\frac{1}{p}} \int_{2r}^{\infty} \| \tilde{f} \|_{L^p_w(B(x_0,t))} \left( w(B(x_0,t)) \right)^{-\frac{1}{p}} dt.
\] (4.7)

In the following, we will estimate \( \| G_{\alpha,2j}(\tilde{f}) \|_{L^p_w(B,B,l_2)} \). We divide \( \| G_{\alpha,2j}(\tilde{f}) \|_{L^p_w(B,B,l_2)} \) into two parts.

\[
\| G_{\alpha,2j}(\tilde{f}) \|_{L^p_w(B,B,l_2)} \leq \| G_{\alpha,2j}(\tilde{f}_0) \|_{L^p_w(B,B,l_2)} + \| G_{\alpha,2j}(\tilde{f}_\infty) \|_{L^p_w(B,B,l_2)},
\] (4.8)

where \( \tilde{f}_0(y) = \tilde{f}(y)\chi_{2B}(y), \tilde{f}_\infty(y) = \tilde{f}(y) - \tilde{f}_\infty(y) \). For the first part, by Lemma 4.1

\[
\| G_{\alpha,2j}(\tilde{f}_0) \|_{L^p_w(B,B,l_2)} \leq 2^{j(\frac{3n}{2}+\alpha)} \| G_\alpha(\tilde{f}_0) \|_{L^p_w(B,B,l_2)} \leq 2^{j(\frac{3n}{2}+\alpha)} \| f \|_{L^p_w(B,B,l_2)}
\]

\[
\leq 2^{j(\frac{3n}{2}+\alpha)} w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \| \tilde{f} \|_{L^p_w(B(x_0,t),l_2)} \left( w(B(x_0,t)) \right)^{-\frac{1}{p}} dt.
\] (4.9)

For the second part.

\[
\| G_{\alpha,2j}(\tilde{f}_\infty)(x) \|_{l_2} = \left\| \left( \int_0^\infty \int_{|x-y| \leq 2^jt} \left( A_\alpha \tilde{f}(y,t) \right)^2 dydt \right)^{1/2} \right\|_{l_2}
\]

\[
= \left\| \left( \int_0^\infty \int_{|x-y| \leq 2^jt} \left( \sup_{\phi \in \mathcal{C}_\alpha} |\tilde{f} * \phi_t(y)| \right)^2 dydt \right)^{\frac{1}{2}} \right\|_{l_2}
\]

\[
\leq \left( \int_0^\infty \int_{|x-y| \leq 2^jt} \left( \int_{|z-y| \leq t} \| \tilde{f}_\infty(z) \|_{l_2} dz \right)^2 dydt \right)^{\frac{1}{2}}.
\]
Since $|x - z| \leq |y - z| + |x - y| \leq 2^{j+1} t$, we get

$$
\| G_{\alpha,2}(\tilde{f}_\infty)(x) \|_{L^2_t} \leq \left( \int_0^\infty \int_{|x-y| \leq 2^j t} \left( \int_{|x-z| \leq 2^{j+1} t} \| \tilde{f}_\infty(z) \|_{L^2_z} \, dz \right)^2 \, dy \, dt \right)^{\frac{1}{2}}
\leq \left( \int_0^\infty \left( \int_{|x-z| \leq 2^{j+1} t} \| \tilde{f}_\infty(z) \|_{L^2_z} \, dz \right)^2 \frac{2^{j+1} dt}{t^{2n+1+1}} \right)^{\frac{1}{2}}
\leq 2^{\frac{3jn}{2}} \int_{R^n} \left( \int_{|x-z| > 2^j t} \| \tilde{f}(z) \|_{L^2_z} \frac{dt}{t^{n+1}} \right)^{\frac{1}{2}} \, dz.
$$

For $|z - x| \geq |x_0 - z| - |x - x_0| \geq |x_0 - z| - \frac{1}{2} |x_0 - z| = \frac{1}{2} |x_0 - z|$, so by Fubini’s theorem and Hölder’s inequality, we obtain

$$
\| G_{\alpha,2}(\tilde{f}_\infty)(x) \|_{L^2_t} \leq 2^{\frac{3jn}{2}} \int_{|x-z| > 2^j t} \| \tilde{f}(z) \|_{L^2_z} \, dz
\leq 2^{\frac{3jn}{2}} \int_{2^j t}^{\infty} \| \tilde{f}(z) \|_{L^2_z} \frac{dt}{t^{n+1}} \, dz
\leq 2^{\frac{3jn}{2}} \int_{2^j t}^{\infty} \| \tilde{f}(\cdot) \|_{L^2_z(B(x_0,t))} \| \tilde{f}(\cdot) \|_{L^{w^{-1}}_z(B(x_0,t))} \frac{dt}{t^{n+1}}
\leq 2^{\frac{3jn}{2}} \int_{2^j t}^{\infty} \| \tilde{f}(\cdot) \|_{L^\infty_z(B(x_0,t),t)} \left( w(B(x_0,t)) \right)^{-\frac{1}{w}} \frac{dt}{t^{n+1}}.
$$

So,

$$
\| G_{\alpha,2}(\tilde{f}_\infty) \|_{L^\infty_w(B,t^2)} \leq 2^{\frac{3jn}{2}} \left( \frac{w(B)}{j^{2n+1}} \int_{2^j t}^{\infty} \| \tilde{f}(\cdot) \|_{L^\infty_z(B(x_0,t),t)} \left( w(B(x_0,t)) \right)^{-\frac{1}{w}} \frac{dt}{t^{n+1}} \right)^{\frac{1}{2}}.
$$

Combining (4.8), (4.9) and (4.10), we have

$$
\| G_{\alpha,2}(\tilde{f}) \|_{L^\infty_w(B,t^2)} \leq 2^{j(\frac{3jn}{2} + \alpha)} \left( \frac{w(B)}{j^{2n+1}} \int_{2^j t}^{\infty} \| \tilde{f}(\cdot) \|_{L^\infty_z(B(x_0,t),t)} \left( w(B(x_0,t)) \right)^{-\frac{1}{w}} \frac{dt}{t^{n+1}} \right)^{\frac{1}{2}}.
$$

Thus,

$$
\| g_{\lambda,\alpha}^*(\tilde{f}) \|_{L^\infty_w(B,t^2)} \leq \left\| G_{\alpha} \tilde{f} \right\|_{L^\infty_w(B,t^2)} + \sum_{j=1}^{\infty} 2^{-\frac{jn}{2}} \left\| G_{\alpha,2}^j(\tilde{f}) \right\|_{L^\infty_w(B,t^2)}. \quad (4.12)
$$
Since \( \lambda > 3 + \frac{\alpha}{n} \), by (4.7), (4.11) and (4.12), we have the desired lemma. \( \square \)

**Proof of Theorem 1.2**

From inequality (4.13) we have

\[
\|s_{\lambda,\alpha}^*(\vec{f})\|_{M_{w}^{p,\varphi_2}(l_2)} \leq \|G_{\alpha}\vec{f}\|_{M_{w}^{p,\varphi_2}(l_2)} + \sum_{j=1}^{\infty} 2^{-j\frac{\alpha}{n}} \|G_{\alpha,2^j}(\vec{f})\|_{M_{w}^{p,\varphi_2}(l_2)}. \tag{4.13}
\]

By Theorem 1.1, we have

\[
\|G_{\alpha}\vec{f}\|_{M_{w}^{p,\varphi_2}(l_2)} \lesssim \|\vec{f}\|_{M_{w}^{p,\varphi_1}(l_2)}. \tag{4.14}
\]

In the following, we will estimate \( \|G_{\alpha,2^j}(\vec{f})\|_{M_{w}^{p,\varphi_2}(l_2)} \). Thus, by substitution of variables and Theorem 3.2, we get

\[
\|G_{\alpha,2^j}(\vec{f})\|_{M_{w}^{p,\varphi_2}(l_2)} \lesssim \|\vec{f}\|_{M_{w}^{p,\varphi_1}(l_2)}. \tag{4.15}
\]

Since \( \lambda > 3 + \frac{\alpha}{n} \), by (4.13), (4.14) and (4.15), we have the desired theorem.

**Lemma 4.5.** Let \( 1 < p < \infty, 0 < \alpha \leq 1, w \in A_p \) and \( b \in BMO \).

Then the inequality

\[
\|[b,G_{\alpha}]^k \vec{f}\|_{L_w^p(B,l_2)} \lesssim (w(B))^{\frac{1}{p}} \int_{2r}^{\infty} \ln^k \left( e^{\frac{t}{r}} \right) \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \left( w(B(x_0,t)) \right)^{-\frac{1}{p}} \frac{dt}{t}
\]

holds for any ball \( B = B(x_0,r) \) and for all \( f \in L_w^{p,\text{loc}}(l_2) \).

**Proof.** We decompose \( \vec{f} = \vec{f}_0 + \vec{f}_\infty \), where \( \vec{f}_0 = \vec{f} \chi_{2B} \) and \( \vec{f}_\infty = \vec{f} - \vec{f}_0 \). Then

\[
\|[b,G_{\alpha}]^k \vec{f}\|_{L_w^p(B,l_2)} \leq \|[b,G_{\alpha}]^k \vec{f}_0\|_{L_w^p(B,l_2)} + \|[b,G_{\alpha}]^k \vec{f}_\infty\|_{L_w^p(B,l_2)}.
\]

By Lemma 4.2, we have that

\[
\|[b,G_{\alpha}]^k \vec{f}_0\|_{L_w^p(B,l_2)} \lesssim \|b\|_w \|\vec{f}_0\|_{L_w^p(l_2)} = \|b\|_w \|\vec{f}\|_{L_w^p(2B,l_2)} \lesssim \|b\|_w r(B) \frac{1}{p} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \left( w(B(x_0,t)) \right)^{-\frac{1}{p}} \frac{dt}{t}.
\]

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Therefore
\[ \| [b, G_\alpha]^k \tilde{f}_\infty(x) \| _{L^p_w(B, l_2)} \leq \| A(\cdot) \| _{L^p_w(B)} + \| B(\cdot) \| _{L^p_w(B)}. \]

First, for \( A(x) \), we find that
\[ A(x) = |b(x) - b_{B,w}|^k \left( \int \int \sup_{\phi \in C_\alpha} \left| \int_\mathbb{R}^n \phi_t(y - z) \tilde{f}_\infty(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \]
\[ = |b(x) - b_{B,w}|^k \left\| G_\alpha \tilde{f}_\infty(x) \right\| _{l_2}. \]

By Lemma 3.7 and from the inequality (4.4), we can get
\[ \| A(\cdot) \| _{L^p_w(B)} = \left( \int _B |b(x) - b_{B,w}|^{kp} \left( \left\| G_\alpha \tilde{f}_\infty(x) \right\| _{l^2} \right)^p w(x) dx \right)^{\frac{1}{p}} \]
\[ \leq \left( \int _B |b(x) - b_{B,w}|^{kp} w(x) dx \right)^{\frac{1}{p}} \int _{2r} \| \tilde{f} \| _{L^p_w(B(x_0,t), l_2)} \left( w(B(x_0,t)) \right)^{-\frac{1}{p}} \frac{dt}{t} \]
\[ \leq \| b \| ^k \| w(B) \| ^{\frac{1}{p}} \int _{2r} \| \tilde{f} \| _{L^p_w(B(x_0,t), l_2)} \left( w(B(x_0,t)) \right)^{-\frac{1}{p}} \frac{dt}{t}. \]

For \( B(x) \), since \( |y - x| < t \), we get \( |x - z| < 2t \). Thus, by Minkowski’s inequality,
\[ B(x) \leq \left\| \left( \int \int \sup_{\phi \in C_\alpha} \left| \int _{|x - z| < 2t} |b_{B,w} - b(z)|^k \tilde{f}_\infty(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right\| _{l_2} \]
\[ \leq \left( \int _0 ^\infty \int _{|x - z| < 2t} |b_{B,w} - b(z)|^k \left\| \tilde{f}_\infty(z) \right\| _{l_2} ^2 \frac{dydt}{t^{2n+1}} \right)^{\frac{1}{2}} \]
\[ \leq \int _{|x_0 - z| > 2r} |b_{B,w} - b(z)|^k \left\| \tilde{f}_\infty(z) \right\| _{l_2} \frac{dz}{|x - z|^n}. \]
For $B(x)$, using the inequality $|z - x| \geq \frac{1}{2}|z - x_0|$, we have

$$
B(x) \lesssim \int_{|x_0 - z| > 2r} |b(z) - b_{B,w}|^k \|\hat{f}(z)\|_{L^p_{w}(B(x_0,t))} \frac{dz}{|x_0 - z|^n} \\
\lesssim \int_{|x_0 - z| > 2r} |b(z) - b_{B,w}|^k \|\hat{f}(z)\|_{L^p_{w}(B(x_0,t))} \int_{|x_0 - z|}^{\infty} dt \frac{t}{t^{n+1}} \\
\lesssim \int_{2r}^\infty \int_{2r \leq |x_0 - z| \leq t} |b(z) - b_{B,w}|^k \|\hat{f}(z)\|_{L^p_{w}(B(x_0,t))} dz \frac{dt}{t^{n+1}}.
$$

Applying Hölder’s inequality and by Lemma 3.7, we get

$$
\|B(\cdot)\|_{L^p_w(B)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^\infty \left( \int_{B(x_0,t)} |b(z) - b_{B,w}|^{kp'} w(z)^{1-p'} dz \right)^{\frac{1}{p'}} \|\hat{f}(\cdot)\|_{L^p_w(B(x_0,t))} \frac{dt}{t^{n+1}} \\
\lesssim \|b\| \ast w(B)^{\frac{1}{p}} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \|w^{-1/p}\|_{L^{p'}(B(x_0,t))} \|\hat{f}\|_{L^p_w(B(x_0,t))} \frac{dt}{t^{n+1}} \\
\lesssim \|b\| \ast w(B)^{\frac{1}{p}} \int_{2r}^\infty \ln \left( e + \frac{t}{r} \right) \|\hat{f}\|_{L^p_w(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t}.
$$

Thus,

$$
\|\ [b, G_\alpha]^k \hat{f} \|_{L^p_w(B, L^2)} \lesssim \|b\| \ast w(B)^{\frac{1}{p}} \int_{2r}^\infty \ln \left( e + \frac{t}{r} \right) \|\hat{f}\|_{L^p_w(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t}.
$$

**Proof of Theorem 1.3**

By substitution of variables, we obtain

$$
\|\ [b, G_\alpha]^k \hat{f} \|_{M^{\mu_1}^{\rho_2}(L^2)} \\
\lesssim \|b\| \ast \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} \int_{2r}^\infty \ln \left( e + \frac{t}{r} \right) \|\hat{f}\|_{L^p_w(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t} \\
\lesssim \|b\| \ast \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} \int_0^{r-1} \ln \left( e + \frac{1}{t r} \right) \|\hat{f}\|_{L^p_w(B(x_0,t^{-1}), L^2)} w(B(x_0,t^{-1}))^{-\frac{1}{p}} \frac{dt}{t} \\
= \sup_{x_0 \in \mathbb{R}^n, r > 0} \|b\| \ast \varphi_2(x_0, r^{-1})^{-1} r \int_0^{r} \ln \left( e + \frac{r}{t} \right) \|\hat{f}\|_{L^p_w(B(x_0,t^{-1}), L^2)} w(B(x_0,t^{-1}))^{-\frac{1}{p}} \frac{dt}{t} \\
\lesssim \|b\| \ast \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_1(x_0, r^{-1})^{-1} w(B(x_0, r^{-1}))^{-\frac{1}{p}} \|\hat{f}\|_{L^p_w(B(x_0,r^{-1}), L^2)} \\
= \|b\| \ast \|\hat{f}\|_{M^{\mu_1}^{\rho_1}(L^2)}.
$$

By using the argument as similar as the above proofs and that of Theorem 1.2 we can also show the boundedness of $[b, G_\alpha^\ast]^k$. 

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