On the density of $\sigma$-curvatures and multiplicity of solutions to the fractional Nirenberg problem

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Abstract

In this paper we prove some results on the density and multiplicity of solutions to the fractional Nirenberg problem. By modifying the minimax procedure introduced by Seré, Coti Zelati and Rabinowitz and combining the blow-up analysis argument, we obtain a $C^0$ density result and also the existence of infinitely many multi-bump solutions.

Key words: Fractional Nirenberg problem, Blow-up analysis, Multi-bump solution.

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1 Introduction

The Nirenberg problem, raised by Nirenberg in the years 1969-1970, asks on the standard sphere $(S^n, g_0)$ ($n \geq 2$), if one can find a conformally invariant metric $g$ such that the scalar curvature (Gauss curvature for $n = 2$) of $g$ is equal to the given function $K$. So the Nirenberg problem is also called the prescribed curvature problem on $S^n$. This problem is equivalent to solving

$$-\Delta_{g_0} w + 1 = K e^{2w} \quad \text{on } S^2,$$

and

$$-\Delta_{g_0} v + c(n)R_0 v = c(n)K v^{\frac{n+2}{n-2}} \quad \text{on } S^n \quad \text{for } n \geq 3,$$

where $\Delta_{g_0}$ is the Laplace-Beltrami operator on $(S^n, g_0)$, $c(n) = (n-2)/(4(n-1))$, $R_0 = n(n-1)$ is the scalar curvature of $(S^n, g_0)$ and $v = e^{\frac{n-2}{4}w}$. The Nirenberg problem has been studied extensively, we refer as examples [5–7, 13–19, 28, 30, 35, 38, 44–48, 54, 57, 60, 63, 64] and references therein. For more recent and further studies, see [4,55].

In this paper, we are concerned with the fractional Nirenberg problem with the Nirenberg’s equation in the fractional setting which constitutes in itself a branch in geometric analysis. This problem was naturally raised on $\sigma$-curvature: finding a new metric $g$ on the standard sphere $S^n$, $n \geq 2$, conformally equivalent to the standard one $g_0$, such that its $\sigma$-curvature is equal to a prescribed function $K$ on $S^n$. More precisely, we investigate the existence of solutions to the following nonlinear equation:

$$P_\sigma(v) = c(n, \sigma)K v^{\frac{n+2\sigma}{n-2\sigma}}, \quad v > 0 \quad \text{on } S^n,$$

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where $\sigma \in (0, 1)$, $P_\sigma = \Gamma(B + \frac{1}{2} + \sigma)/\Gamma(B + \frac{1}{2} - \sigma)$ is the $2\sigma$-order conformal Laplacian on $S^n$, $B = \sqrt{-\Delta_{g_0} + \frac{(n-1)^2}{4}}$, $c(n, \sigma) = \Gamma(\frac{n}{2} + \sigma)/\Gamma(\frac{n}{2} - \sigma)$, $\Gamma$ is the Gamma function, $K$ is a given function on $S^n$.

The operator $P_\sigma$ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection. Indeed, let

$$F : \mathbb{R}^n \to S^n \setminus \{\mathcal{N}\}, \quad x \mapsto \left(\frac{2x}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1}\right)$$

be the inverse of stereographic projection, where $\mathcal{N}$ is the north pole of $S^n$. Then it holds that

$$P_\sigma(\phi) \circ F = |J_F|^{-\frac{n+2\sigma}{n}}(-\Delta)^\sigma(|J_F|^{-\frac{2\sigma}{n}}(\phi \circ F)) \quad \text{for } \phi \in C^\infty(S^n),$$

where $|J_F| = (\frac{2}{1+|x|^2})^n$, and $(-\Delta)^\sigma$ is the fractional Laplacian operator defined by

$$(-\Delta)^\sigma u(x) = C_{n, \sigma} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\sigma}} \, dy,$$

where P.V. is the principal value and $C_{n, \sigma} = \pi^{-\frac{1}{2}(n + \frac{\sigma}{n})} \frac{\Gamma(\frac{n}{2} + \sigma)}{\Gamma(\frac{n}{2} - \sigma)}$. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing $C^\infty$ function in $\mathbb{R}^n$, and it can be equivalently defined as:

$$(-\Delta)^\sigma u(x) = -\frac{1}{2} C_{n, \sigma} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2\sigma}} \, dy,$$

see [27, 61]. Then let $u = |J_F|^{-\frac{n+2\sigma}{n}}(\phi \circ F)$, one can transfer Eq. (1.1) into the following equation with critical exponent

$$(-\Delta)^\sigma u = K(x)u^{\frac{n+2\sigma}{n}}, \quad u > 0 \quad \text{in } \mathbb{R}^n. \quad (1.2)$$

In general, the intertwining operator $P_\sigma$ can be well-defined for all $\sigma \in (0, \frac{n}{2})$ when $n \geq 2$, see e.g., Branson [8]. For $\sigma = 1$, $P_1 = -\frac{n(n-1)}{n-2} \Delta_{g_0} + n(n-1)$ is the well known conformal Laplacian associated with the classical Nirenberg problem. For $\sigma = 2$, $P_2 = \Delta_{g_0}^2 - \frac{1}{2}(n^2 - 2n - 4) \Delta_{g_0} + \frac{n-4}{16} n(n^2 - 4)$ is the well known Paneitz operator. Up to positive constants $P_k(1)$ is the scalar curvature associated to $g_0$ and $P_2(1)$ is the so-called $Q$-curvature. In fact, $P_1$ and $P_2$ are the first two terms of a sequence of conformally covariant elliptic operators $P_k$ which exists for all positive integers $k$ if $n$ is odd and for $k = \{1, \ldots, n/2\}$ if $n$ is even. These operators have been first introduced by Graham, Jenne, Mason and Sparling in [33]. In [34], Graham and Zworski showed that $P_k$ can be realized as the residues at $\sigma = k$ of a meromorphic family of scattering operators. Unlike the Laplacian, the fractional Laplacian is a non-local operator. In a seminal paper [11], Caffarelli and Silvester express the non-local operator $(-\Delta)^\sigma$, $\sigma \in (0, 1)$ on $\mathbb{R}^n$ as a generalized Dirichlet-Neumann map for an elliptic boundary value problem with local differential operators. Later on, Chang and González [12] showed that for any $\sigma \in (0, \frac{n}{2})$, the operator $P_\sigma$ can also be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold. The fractional operators $P_\sigma$ and their associated fractional order curvatures $P_\sigma(1)$ which will be called $\sigma$-curvatures have been the subject of many studies. On general manifolds, the prescribing $\sigma$-curvature problem was considered in [12, 31, 32, 34, 58] and references therein. Throughout the paper, we assume $\sigma \in (0, 1)$ and $n \geq 2$ without otherwise stated.

Problem (1.1) (or (1.2)) is a focus of research in the recent decades, and it continues to inspire new thoughts, see for example [1, 2, 20–23, 36, 37, 40, 41, 50–53, 56]. Fundamental progress was made by Jin, Li and Xiong in [40, 41], from which they obtained compactness and existence results by applying the blow-up analysis and the degree counting argument. It is also worth to noting that Jin, Li and
Xiong in [42] generalized the previous results to all $\sigma \in (0, \frac{1}{2})$ by using integral representations. Later on, the authors in [1, 2, 20, 22] obtained some existence criterions by establishing Euler-Hopf type index formula. Recently, there have been some works devoted to the multiplicity results, and those mainly use the Lyapunov-Schmidt reduction method (see e.g., [21, 23, 36, 50–53, 56]).

The aim of this paper is to investigate the number of positive solutions to Eq. (1.1) (or (1.2)) under various local assumptions on the prescribed function $K$. Basically speaking, we obtain a $C^0$ density result for the fractional Nirenberg problem (1.1) by constructing infinitely many multi-bump solutions to the corresponding perturbed equations. As a variation of this idea, the related problem (1.2) with $K(x)$ being periodic in one of the variables are also studied and infinitely many multi-bump solutions (modulo translations by its periods) are obtained under some flatness conditions. Furthermore, the solutions we construct in this paper concentrate at local maximum points of $K(x)$, whose distance is very large.

The first main theorem of this paper deals with the existence of multi-bump solutions to a perturbed fractional Nirenberg problem.

**Theorem 1.1.** Let $K \in L^\infty(\mathbb{S}^n)$ and assume that for some $\bar{x} \in \mathbb{S}^n$, $\bar{\varepsilon} > 0$, $K(\bar{x}) > 0$, and $K \in C^0(B_{\bar{\varepsilon}}(\bar{x}))$ ($B_{\bar{\varepsilon}}(\bar{x})$ denotes the geodesic ball in $\mathbb{S}^n$ of radius $\bar{\varepsilon}$ and centered at $\bar{x}$). Then for any $\varepsilon \in (0, \bar{\varepsilon})$, any integers $k \geq 1$ and $m \geq 2$, there exists $K_{\varepsilon,k,m} \in L^\infty(\mathbb{S}^n)$ with $K_{\varepsilon,k,m} - K \in C^0(\mathbb{S}^n)$, $\|K_{\varepsilon,k,m} - K\|_{C^0(\mathbb{S}^n)} < \varepsilon$ and $K_{\varepsilon,k,m} \equiv K$ in $\mathbb{S}^n \setminus B_{\varepsilon}(\bar{x})$, such that, for each $2 \leq s \leq m$, the equation
\[
P_{\sigma}(v) = c(n, \sigma)K_{\varepsilon,k,m}v^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n
\] (1.3)
has at least $k$ positive solutions with $s$ bumps.

A couple of remarks regarding Theorem 1.1 are in order.

**Remark 1.1.**
1. For the precise meaning of “$s$ bumps”, refer to the proof of Theorem 1.1 in Section 7. Roughly speaking, we say a solution has $s$ bumps if most of its mass is concentrated in $s$ disjoint regions. Since the number of bumps and the number of solutions can be chosen arbitrarily, we obtain the existence of infinitely many multi-bump solutions to Eq. (1.3).

2. One cannot expect to perturb any $K(x)$ near any point $\bar{x} \in \mathbb{S}^n$ in the $C^1$ sense to obtain the existence of solutions. This is evident if we take $K(x) = x_{n+1} + 2$ with $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$ and $\bar{x}$ to be different from the north and the south poles, since the perturbed function would violate the Kazdan-Warner type condition (see [40, Proposition A.1]).

3. The perturbation $K_{\varepsilon,k,m}$ can be constructed explicitly in Section 6 and it is a way of gluing approximate solutions into genuine solutions. The method is variational, rather than through Lyapunov-Schmidt reduction method as in [21, 23, 36, 50–53, 56], etc. The solutions we obtained have most of their mass in disjoint small balls centered at the maximum points of $K_{\varepsilon,k,m}$, which are far away from each other. This fact accounts for why there are infinitely many solutions solve Eq. (1.3). Moreover, the solutions we constructed are nearly bubble functions, we refer to Section 6 for more details.

4. If $K_{\varepsilon} = 1 + \varepsilon K(x)$ and $K(x)$ has at least two critical points satisfying some local conditions, Chen-Zheng [21] showed that Eq. (1.2) with $K = K_{\varepsilon}$ has two multi-bump solutions when $\varepsilon$ is small. Here we give a more general existence result since we can perturb any given positive continuous function in any neighborhood of any given point on $\mathbb{S}^n$ such that for the perturbed equations there exist many solutions.
(5) The main feature of Theorem 1.1 is that, even if a given function $K \in L^\infty(\mathbb{S}^n)$ in Theorem 1.1 cannot be realized as the $\sigma$-curvature of a metric $g$ conformal to $g_0$, nevertheless we can find a function $K'$ arbitrarily close to $K$ in $C^0(\mathbb{S}^n)$ which is the $\sigma$-curvature as many conformal metrics to $g_0$ as we want.

As a consequence of Theorems 1.1, we also have

**Corollary 1.1.** The smooth $\sigma$-curvature functions of metrics conformal to $g_0$ are dense in $C^0(\mathbb{S}^n)$.

Next we consider the related problem (1.2). Before stating the results, we introduce some notation.

Let $E$ be the completion of the space $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$
\|u\|_E := \left( \int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} u|^2 \, dx \right)^{1/2} = \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\sigma}} \, dx \, dy \right)^{1/2},
$$

see [27]. Denote the fractional critical Sobolev exponent $2^*_{\sigma} := \frac{2n}{n-2\sigma}$, it is well known that $E$ can be embedded into $L^{2^*_{\sigma}}(\mathbb{R}^n)$ and the sharp $\sigma$-order Sobolev inequality is

$$
S_{n,\sigma} \left( \int_{\mathbb{R}^n} |u|^{2^*_{\sigma}} \, dx \right)^{1/2^*_{\sigma}} \leq \left( \int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} u|^2 \, dx \right)^{1/2}. 
$$

For any $z \in \mathbb{R}^n$ and $\lambda > 0$, set

$$
\delta(z, \lambda)(x) := 2^{\frac{n-2\sigma}{\sigma}} \left( \frac{\Gamma(\frac{n+2\sigma}{2})}{\Gamma(\frac{n-2\sigma}{2})} \right)^{\frac{\sigma}{2}} \frac{\lambda}{\lambda^2 + |x - z|^2}^{\frac{n-2\sigma}{2}}. 
$$

Then $\delta(z, \lambda)$ is the solution to the problem

$$
(-\Delta)^{\sigma/2} u = u^{\frac{n+2\sigma}{n-2\sigma}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n 
$$

for every $(z, \lambda) \in \mathbb{R}^n \times (0, +\infty)$ (see, e.g., [32, 40]). Moreover, (1.5) and its non-zero constant multiples attain the sharp $\sigma$-order Sobolev inequality (1.4), see Lieb [49].

Let $D$ be the completion of $C_c^\infty(\mathbb{R}^{n+1}_+)$ with respect to the weighted Sobolev norm

$$
\|U\|_D := \left( \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla U(X)|^2 \, dX \right)^{1/2} 
$$

equipped with the natural inner product

$$
\langle U, V \rangle_D := \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} \nabla U \nabla V \, dX \quad \text{for} \quad U, V \in D,
$$

where $X = (x, t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$.

Throughout the paper, we write $\| \cdot \|_E$ (resp. $\| \cdot \|_D$) to denote the norm of $E$ (resp. $D$) and simply use $E^+$ (resp. $D^+$) to denote the set consisting of all positive functions of $E$ (resp. $D$).

We analyze Eq. (1.2) via the extension formulation for fractional Laplacians established by Caffarelli and Silvestre [11]. This is a commonly used tool nowadays, through which instead of Eq. (1.2) we can study a degenerate elliptic equation with a Neumann boundary condition in one dimension higher:

$$
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in} \ \mathbb{R}^{n+1}_+, \\
\partial_t u = N_{\sigma} K(x) U(x, 0)^{\frac{n+2\sigma}{n-2\sigma}} & \text{on} \ \mathbb{R}^n,
\end{cases}
$$

1.4
where \( u(x) = U(x, 0) \),
\[
\partial_0^\sigma U := - \lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(X),
\]
and \( N_\sigma = 2^{1-2\sigma} \Gamma(1 - \sigma)/\Gamma(\sigma) \). The extension will always refer to the cononical one:
\[
U(X) = P_\sigma[u] := \beta(n, \sigma) \int_{\mathbb{R}^n} \frac{t^{2\sigma}}{(|x - \xi|^2 + t^2)^{n+2\sigma}} u(\xi) \, d\xi,
\]
(1.10)
where \( \beta(n, \sigma) \) is a normalization constant. Namely, it is obtained by a Possion type integral. We always drop the harmless constant \( N_\sigma \) for brevity and refer to \( U = P_\sigma[u] \) in (1.10) to be the extension of \( u \). For more details, one may refer to [11].

Utilizing the above extension formulation, we know that \( \delta(z, \lambda) := P_\sigma[\delta(z, \lambda)] \) solves
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\partial_\sigma^\nu U = N_\sigma U(x, 0)^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } \mathbb{R}^n, \\
U(x, 0) = \delta(z, \lambda) & \text{on } \mathbb{R}^n,
\end{cases}
\]
and as an immediately consequence we have
\[
\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla \delta(z, \lambda)|^2 \, dX = N_\sigma \int_{\mathbb{R}^n} \delta(z, \lambda)^{2\sigma} \, dx.
\]
(1.11)
Moreover, the extremal functions of Sobolev trace inequality on \( H^1(t^{1-2\sigma}, \mathbb{R}^{n+1}_+) \)
\[
S_{n, \sigma}\left( \int_{\mathbb{R}^n} |U(x, 0)|^{2\sigma} \, dx \right)^{1/2} \leq \left( \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla U|^2 \, dX \right)^{1/2}
\]
(1.12)
have the form \( U(x, t) = \alpha \delta(z, \lambda) \) for any \( \alpha \in \mathbb{R} \setminus \{0\}, \, z \in \mathbb{R}^n \) and \( \lambda > 0 \). Here \( S_{n, \sigma} = N_\sigma^{-1/2} S_{n, \sigma} \) is the optimal constant and \( H^1(t^{1-2\sigma}, \mathbb{R}^{n+1}_+) \) is the closure of \( C^\infty_c(\mathbb{R}^{n+1}_+) \) under the norm
\[
\|U\|_{H^1(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)} := \left( \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} (|U|^2 + |\nabla U|^2) \, dX \right)^{1/2},
\]
see, e.g., [31,43]. Furthermore, it follows from (1.5), (1.6) and (1.11) that
\[
\|\delta(0, 1)\|_\sigma = \|\delta(z, \lambda)\|_\sigma, \quad S_{n, \sigma} = N_\sigma^{-1/2} \|\delta(0, 1)\|^{2\sigma/n}_{\sigma}
\]
(1.13)
for any \((z, \lambda) \in \mathbb{R}^n \times (0, +\infty)\). We call \( \delta(z, \lambda) \) and \( \delta(z, \lambda) \) bubbles.

Let \( K \in L^\infty(\mathbb{R}^n) \), we define the energy functional \( I_K : D \to \mathbb{R} \) by
\[
I_K(U) := \frac{1}{2} \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla U|^2 \, dX - \frac{1}{2\sigma} \int_{\mathbb{R}^n} K(x)|U(x, 0)|^{2\sigma} \, dx.
\]
(1.14)
Obviously a positive critical point give rise to a positive solution to Eq. (1.9) and thus a positive solution to Eq. (1.2).

For \( z_j = (z_{j1}, \ldots, z_{jm}) \in \mathbb{R}^n, \, \lambda_j \in (0, +\infty) \) and \( j = 1, \ldots, k \), let
\[
D_{z, \lambda, k} := \left\{ v \in D : \langle \delta(z_j, \lambda_j), v \rangle = \langle \frac{\partial \delta(z_j, \lambda_j)}{\partial \lambda_j}, v \rangle = \langle \frac{\partial \delta(z_j, \lambda_j)}{\partial z_j}, v \rangle = 0 \right\}
\]
for \( j = 1, \ldots, k \) and \( i = 1, \ldots, n \). Here and in the following, \( \langle \cdot, \cdot \rangle \) denotes the inner product (1.8).

Let \( K(x) \in L^\infty(\mathbb{R}^n) \), \( O^{(1)}, \ldots, O^{(m)} \subset \mathbb{R}^n \) are some open sets with \( \text{dist}(O^{(i)}, O^{(j)}) \geq 1 \) for any \( i \neq j \). If \( K \in C^0(\bigcup_{i=1}^m O^{(i)}) \), we define \( V(m, \varepsilon) := V(m, \varepsilon, O^{(1)}, \ldots, O^{(m)}, K) \) as the following open set in \( D \) for \( \varepsilon > 0 \):

\[
V(m, \varepsilon) := \left\{ U \in D : \exists \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m, \exists z = (z_1, \ldots, z_m) \in O^{(1)} \times \ldots \times O^{(m)}, \right.
\]

\[
\exists \lambda = (\lambda_1, \ldots, \lambda_m), \lambda_i > \varepsilon^{-1}, \forall i \leq m, \text{ such that }
\]

\[
|\alpha_i - K(z_i)|^{2\sigma-n}/4\sigma < \varepsilon, \forall i \leq m, \text{ and }
\]

\[
\|U - \varphi(\alpha, z, \lambda)\|_\sigma < \varepsilon\}
\]

where

\[
\varphi(\alpha, z, \lambda) := \sum_{i=1}^m \alpha_i \delta(z_i, \lambda_i).
\]

The open set \( V(m, \varepsilon) \) recodes the information of the concentration rate and the locations of points of concentration. Furthermore, the family of solutions we construct is of the form (after stereographic projection for Eq. (1.1))

\[
U = \sum_{i=1}^m \alpha_i \tilde{\delta}(z_i, \lambda_i) + v,
\]

where \( v \in D_{z, \lambda, m} \) and the contribution of the error term \( v \) can be negligible (see Proposition 4.1). Moreover, we construct multi-bump solutions near critical points of \( K(x) \) and the bumps can be chosen arbitrarily many. For this purpose, we assume that \( K(x) \) satisfies the following conditions:

\( (K_1) \) \( K(x) \) is periodic in at least one variable, that is, there is a positive constant \( T \), such that

\[ K(x_1 + lT, x_2, \ldots, x_n) = K(x_1, x_2, \ldots, x_n) \]

for any integer \( l \) and \( x \in \mathbb{R}^n \).

\( (K_2) \) Let \( \Sigma \) denote the set consisting of all the critical points \( z \) of \( K(x) \), satisfying (after a suitable rotation of the coordinate system depending on \( z \))

\[
K(x) = K(z) + \sum_{i=1}^n a_i |x_i - z_i|^\beta + R(|x - z|)
\]

(1.16)

for \( x \) close to \( z \), where \( a_i \) and \( \beta \) are some constants depending on \( z \), \( a_i \neq 0 \) for \( i = 1, \ldots, n \), \( \sum_{i=1}^n a_i < 0 \), \( \beta \in (n - 2\sigma, n) \), and \( R(y) \) is \( C^{[\beta]-1,1} \) and satisfies

\[
\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta+s} = o(1) \quad \text{as } y \to 0,
\]

where \( C^{[\beta]-1,1} \) means that up to \([\beta] - 1\) derivatives are Lipschitz functions, \([\beta]\) denotes the integer part \( \beta \) and \( \nabla^s \) denotes all possible partial derivatives of order \( s \).

We remark that the \( (K_2) \) type condition was originally introduced by Li in [46] and widely used in the fractional setting, see, e.g., [40, 56].

**Theorem 1.2.** Assume that \( K(x) \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) satisfies \( (K_1)-(K_2) \) and
(K₃) \( K_{\text{max}} := \max_{x \in \mathbb{R}^n} K(x) > 0 \) is achieved and the set \( K^{-1}(K_{\text{max}}) := \{x \in \mathbb{R}^n : K(x) = K_{\text{max}}\} \) has at least one bounded connected component, denoted as \( \mathcal{C'} \).

Then for any integers \( m \geq 2 \), Eq. (1.9) has infinitely \( m \)-bump solutions in \( D \) modulo translations by \( T \) in the \( x_1 \) variable. More precisely, for any \( \varepsilon > 0 \) and \( x^* \in \mathcal{C'} \), there exists some constant \( l^* > 0 \) such that for any integers \( l^{(1)}, \ldots, l^{(k)} \) satisfying \( 2 \leq k \leq m \), \( \min_{1 \leq i \leq k} |l^{(i)}| \geq l^* \) and \( \min_{i \neq j} |l^{(i)} - l^{(j)}| \geq l^* \), there is at least one solution \( U \) of Eq. (1.9) in \( V(k, \varepsilon, B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)})) \) with

\[
kc - \varepsilon \leq I_K(U) \leq kc + \varepsilon,
\]

where

\[
x^{(i)} = x^* + (l^{(i)}T, 0, \ldots, 0), \quad c = \frac{\sigma}{n} (K(x^*))^{(2\sigma-n)/2\sigma} (S_{n,\sigma})^{n/\sigma},
\]

and \( V(k, \varepsilon, B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)})) \) are some subsets of \( D \) defined in (1.15).

Several remarks involving the hypotheses of the above theorem are in order.

**Remark 1.2.**

(1) We can see from the description of (K₃) that there exists some bounded open neighborhood \( O \) of \( \mathcal{C'} \) such that \( K_{\text{max}} \geq \max_{x \in \partial O} K + \delta \), where \( \delta \) is some small positive number. This fact together with assumption (K₁) implies that \( K(x) \) has a sequence of local maximum points \( z_j \) with \( |z_j| \to \infty \) as \( j \to \infty \).

(2) Condition (K₃) is sharp in the sense that one can construct examples easily below to show that if (K₃) is not satisfied, Eq. (1.2) (or equivalently Eq. (1.9)) may have no nontrivial solutions, which shows that (K₃) is not merely a technical hypothesis.

(3) \( U \in V(k, \varepsilon, B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)})) \) implies that \( U \) has most of its mass concentrated in \( B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)}). \) In particular, if \( (l^{(1)}, \ldots, l^{(k)}) \neq (\tilde{l}^{(1)}, \ldots, \tilde{l}^{(k)}) \), the corresponding solutions \( U \) and \( \tilde{U} \) are different.

(4) \( c \) is the mountain pass value corresponding to Eq. (1.9) and \( S_{n,\sigma} \) is the sharp constant in (1.12).

(5) Note that the authors in [51, 56] apply the Lyapunov-Schmidt reduction method to obtain infinitely many multi-bump solutions clustered on some lattice points in \( \mathbb{R}^n \) under similar assumptions. Liu [52] use the same method to construct infinitely many concentration solutions to Eq. (1.2) under the assumption that \( K(x) \) has a sequence of strictly local maximum points moving to infinity. The solutions we constructed in Theorem 1.2, roughly speaking, concentrate at \( k \) different points and the distance between different concentrate points is very large.

**Example 1.1** (Nonexistence). Let \( K(x) \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), \( K(x) \) and \( \nabla K(x) \) are bounded in \( \mathbb{R}^n \), \( \frac{\partial K}{\partial x_2} \) is nonnegative but not identically zero. Then the only nonnegative solution of Eq. (1.2) in \( E \) is the trivial solution \( u \equiv 0 \).

**Proof.** Let \( u \geq 0 \) be any solution in \( E \). Multiplying (1.2) by \( \frac{\partial u}{\partial x_2} \) and using [40, Proposition A.1], we obtain \( \int_{\mathbb{R}^n} \frac{\partial K}{\partial x_2} u^{2^*} dx = 0 \). The hypotheses on \( K(x) \) imply that \( u \) is identically zero in an open set, hence \( u \equiv 0 \) by the unique continuation results (see, e.g., [39]).

In fact we can obtain more information on the solutions obtained in Theorem 1.2.

**Theorem 1.3.** Assume that \( K(x) \in L^\infty(\mathbb{R}^n) \) satisfies (K₁)-(K₂), and
There exist some positive constant $A_1$ and a bounded open set $O \subset \mathbb{R}^n$ such that

\[
K \in C^1(O),
\]
\[
1/A_1 \leq K(x) \leq A_1, \quad \forall x \in \overline{O},
\]
\[
\max_{x \in \overline{O}} K(x) = \sup_{x \in \mathbb{R}^n} K(x) > \max_{x \in \partial O} K(x).
\]

Then for any $\varepsilon > 0$, Eq. (1.9) has infinitely many solutions in $D^+$ satisfying

\[
c \leq I_K(U) \leq c + \varepsilon \quad \text{or} \quad 2c - \varepsilon \leq I_K(U) \leq 2c + \varepsilon
\] (1.17)

and

\[
\sup \{ \| U \|_{L^\infty(\mathbb{R}^{n+1})} : I_K'(U) = 0, U \text{ satisfies (1.17)} \} = \infty,
\]

where

\[
c = \frac{\sigma}{n} (\max_{\overline{O}} K)^{(2\sigma-n)/2\sigma} (S_{n,\sigma})^{n/\sigma}.
\]

More precisely, for any $\varepsilon > 0$, there exists $l^* > 0$ such that for any integers $l^{(1)}$ and $l^{(2)}$ satisfying $|l^{(1)} - l^{(2)}| \geq l^*$, there is at least one solution $U$ of (1.9) in $V(1, \varepsilon, O, K) \cup V(2, \varepsilon, O^{(1)}_l, O^{(2)}_l, K)$ with (1.17), where

\[
O^{(1)}_l = O + (l^{(1)}T, 0, \ldots, 0), \quad O^{(2)}_l = O + (l^{(2)}T, 0, \ldots, 0),
\]

and $V(1, \varepsilon, O, K)$, $V(2, \varepsilon, O^{(1)}_l, O^{(2)}_l, K)$ are some subsets of $D$ defined in (1.15).

Remark 1.3. By the stereographic projection from $\mathbb{S}^n \setminus \{N\}$ to $\mathbb{R}^n$, the solutions obtained in Theorem 1.2 and 1.3 can be lifted to a solution of Eq. (1.1) on $\mathbb{S}^n$ which is positive except at the north pole $N$. In this sense, Eq. (1.1) is solvable under the assumptions of Theorem 1.2 and 1.3.

For $n \geq 3$ and $\sigma = 1$, Theorem 1.1–1.3 was proved by Li [46]. In our earlier work [62], we deal with the case $\sigma = 1/2$ and obtain the density and multiplicity results. The main objective of this paper is to extend the above results to the nonlocal setting $\sigma \in (0, 1)$. Although certain parts of the proof can be obtained by minor modifications of the classical arguments in [44–46], there are plenty of technical difficulties which demand new ideas to handle the non-local terms. We will pay attention to, for instance, the following features.

- We largely depend on the extension result of Caffarelli and Silvester [11] to analyze solutions. It is a standard argument nowadays to study nonlocal problems. Because of the degeneracy of the extended problem (1.2), it is not easy to study the asymptotic behavior of the solution near the infinity, see Section 2 where some properties of solutions are obtained. Hence, in showing the decay property of rescaled solutions, we do not use potential analysis, but iteratively apply the rescaling argument based on the maximum principle.

- Suppose that $\sigma \in (0, 1) \setminus \{1/2\}$, a complicated function space needed while the energy minimization problem in $\sigma = 1/2$ can be solved via reflection method. Here we give an extension result for fractional Sobolev spaces defined on Lipschitz boundary and apply an embedding theorem of fractional Sobolev spaces into weighted Sobolev spaces, see Section 3.

- We construct solutions by modifying the minimax procedure in the classical case while dedicate estimates needed to run the gradient flow, see Section 4 and 5.
We study Eq. (1.1) (or (1.2)) by subcritical approach, for example, we refer to the reader [40,41]. It is worth noting that our methods continue in the direction pioneered in the earlier work [25,26,59]. These techniques provide, roughly speaking, ways of gluing “approximate solutions” together to obtain a genuine solution. There have been some works on “gluing approximate solutions” by using the Lyapunov-Schmidt reduction method (see e.g., [21,23,36,50–53,56]) where more precise information on the linearized problem is needed. However it seems that the methods in Sere [59], Coti Zelati-Rabinowitz [25,26] and Li [44,46] have provided an elegant way to glue approximate solutions for certain periodic differential equations where it is difficult to obtain as precise information as needed for applying the Lyapunov-Schmidt reduction method. Inspired by the above works, we apply these techniques and combine with the localization method in [11] to construct solutions corresponding to the nonlocal problem (1.2). As far as we know, this paper is the first attempt to modify the above mentioned gluing method towards equations with the fractional Laplacians in the Euclidean setting or the conformal Laplacian operators under a particular choice of the metric in constructing multi-bump solutions. This paper also overcomes the difficulty appearing in using Lyapunov-Schmidt reduction method to locate the concentrating points of the solutions.

Let us introduce some arrangements to prove the main results in this paper. Theorems 1.1–1.3 and Corollary 1.1 are derived in Section 7 from Proposition 7.1, a more general result on Eq. (1.2). To derive Proposition 7.1, we first study a compactified problem (Theorem 4.1) in Section 2–6. Then we derive Proposition 7.1 by using Theorem 4.1 and some blow-up analysis in [40]. Theorem 4.1 is a technical result in our paper, which is essential to make the variational gluing methods applicable.

The present paper is organized as the following. In Section 2, we establish some a priori estimates for solutions to degenerate elliptic equations. In Section 3, we consider a minimization problem on exterior domain. In Section 4, existence and multiplicity results for the subcritical case will be stated, and its proof will be sketched. In Section 5, we follow and refine the analysis of Bahri and Coron [5,6] to study the subcritical interaction of two well-spaced bubbles. The technical result Theorem 4.1 will be completed in Section 6 by applying minimax procedure as in Coti Zelati and Rabinowitz [25,26]. Finally, the main theorems are proved in Section 7 with the aid of blow-up analysis established by Jin, Li and Xiong [40].

Notation: We collect below a list of the main notation used throughout the paper.

- We use capital letters, such as $X = (x,t)$ (resp. $Y = (y,s)$) to denote an element of the upper half space $\mathbb{R}^{n+1}_+$, where $x \in \mathbb{R}^n$ (resp. $y \in \mathbb{R}^n$) and $t > 0$ (resp. $s > 0$).
- For a domain $D \subset \mathbb{R}^{n+1}$ with boundary $\partial D$, we denote $\partial' D$ as the interior of $\overline{D} \cap \partial \mathbb{R}^{n+1}$ in $\mathbb{R}^n = \partial \mathbb{R}^{n+1}_+$ and $\partial'' D = \partial D \setminus \partial' D$.
- For $X \in \mathbb{R}^{n+1}$, denote $B_r(X) := \{X \in \mathbb{R}^{n+1} : |X - X| < r\}$ and $B^+_r(X) := B_r(X) \cap \mathbb{R}^{n+1}_+$, where $| \cdot |$ is the Euclidean distance. If $X = (\tau,0) \in \partial \mathbb{R}^{n+1}_+$, $B_r(X) := \{x \in \mathbb{R}^n : |x - \tau| < r\}$. Hence $\partial' B^+_r(X) = B_r(\tau)$ if $X \in \partial \mathbb{R}^{n+1}_+$. Moreover, when $X = (\tau,0)$, we simply use $B_r(\tau)$ (resp. $B^+_r(\tau)$ and $B^+_r(X)$ for $B_r(X)$ (resp. $B^+_r(X)$ and $B_r(X)$) and will not keep writing the center $X$ if $X = 0$.
- For any weakly differentiable function $U(x,t)$ on $\mathbb{R}^{n+1}_+$, we denote $\nabla_x U = (\partial_{x_1} U, \ldots, \partial_{x_n} U)$ and $\nabla U = (\nabla_x U, \partial_t U)$.
- For any $\sigma \in (0,1)$ and $n \geq 2$, we denote $2^*_\sigma = \frac{2n}{n-2\sigma}$ and $H(x) = \left( \frac{2}{1+|x|} \right)^{n-2\sigma}/2$.
- $C > 0$ is a generic constant which can vary from line to line.
2 Some a priori estimates for degenerate elliptic equations

In this section we present some a priori estimates of positive solutions to the equation

\((-\Delta)^{\sigma} u = K(x)u^{\frac{n+2\sigma}{n-2\sigma}}\), \quad |x| \geq 1

with \(\tau \geq 0\) small. For this purpose, we use the extension formula (1.10) to consider the related degenerate elliptic equations. Before we present the main results in this section, we introduce some notation.

For any \(z \in \mathbb{R}^n\) and \(R > 0\) we define the weighted Sobolev space \(H^1(t^{1-2\sigma}, B_R^+(z))\) of functions \(U \in L^2(t^{1-2\sigma}, B_R^+(z))\) such that \(\nabla U \in L^2(t^{1-2\sigma}, B_R^+(z))\), endowed with the norm

\[
\|U\|_{H^1(t^{1-2\sigma}, B_R^+(z))} := \left( \int_{B_R^+(z)} t^{1-2\sigma} |\nabla U|^2 + |U|^2 \, dX \right)^{1/2}.
\]  

(2.1)

We say \(U \in H^1_{loc}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)\) if \(U \in H^1(t^{1-2\sigma}, B_R^+\setminus \{0\})\) for every \(R > 0\), and \(U \in H^1_{loc}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)\) if \(U \in H^1(t^{1-2\sigma}, B_R^+\setminus \{0\})\) for any all \(R > \varepsilon > 0\).

**Proposition 2.1.** Suppose that \(K \in L^\infty(\mathbb{R}^n\setminus B_1)\) satisfies \(\|K\|_{L^\infty(\mathbb{R}^n\setminus B_1)} \leq A_0\) for some positive constant \(A_0\). Then there exists some positive constants \(\mu_1 = \mu_1(n, \sigma, A_0)\) and \(C(n, \sigma, A_0)\) such that for any positive solution \(U(x,t)\) of

\[
\begin{cases}
\text{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\partial_t U = K(x)U(x,0) \frac{n+2\sigma}{n-2\sigma} & \text{on } \mathbb{R}^n \setminus B_1,
\end{cases}
\]

with \(\nabla U \in L^2(t^{1-2\sigma}, \mathbb{R}^{n+1}_+\setminus B_1^+)\), \(U(x,0) \in L^{2^*_\sigma}(\mathbb{R}^n \setminus B_1)\) and

\[
\int_{\mathbb{R}^{n+1}_+\setminus B_1^+} t^{1-2\sigma} |\nabla U|^2 \, dX \leq \mu_1,
\]

(2.2)

we have

\[
\sup_{X \in \mathbb{R}^{n+1}_+\setminus B_1^+} |X|^{n-2\sigma} U(X) \leq C(n, \sigma, A_0).
\]

**Proof.** Our arguments are in the spirit of those in [10, 44, 45]. By some appropriate extension of \(U(X)\) to \(B_1^+\), it follows from (1.12) and (2.2) that

\[
\left( \int_{\mathbb{R}^n \setminus B_1} |U(x,0)|^{2^*_\sigma} \, dx \right)^{2/2^*_\sigma} \leq C_0(n, \sigma) \int_{\mathbb{R}^{n+1}_+\setminus B_1^+} t^{1-2\sigma} |\nabla U|^2 \, dX \leq C_0(n, \sigma) \mu_1.
\]

(2.3)

Throughout the paper, we use \(C_0(n, \sigma) > 1\) denotes some universal constant only depend on \(n\) and \(\sigma\).

Now we perform a Kelvin transformation on \(U(X)\). Let

\[
\tilde{X} = (\tilde{x}, \tilde{t}) = \frac{X}{|X|^2}, \quad |X| \geq 1,
\]

\[
V(\tilde{X}) = \frac{1}{|X|^{n-2\sigma}} U\left( \frac{\tilde{x}}{|X|^2}, \frac{\tilde{t}}{|X|^2} \right).
\]
By (2.2) and (2.3), some calculations lead to
\[
\int_{B_1^+} \bar{t}^{1-2\sigma} |\nabla V|^2 \, d\bar{X} + \int_{B_1} |V(\bar{x}, 0)|^{2^*} \, d\bar{x} \leq C_0(n, \sigma) \mu_1.
\]
Moreover, \(V(\bar{X})\) satisfies
\[
\begin{cases}
\text{div}(\bar{t}^{1-2\sigma} \nabla V) = 0 & \text{in } \mathbb{R}_n^{n+1}, \\
\partial_\nu V = K(\frac{\bar{x}}{|\bar{x}|^2}) V(\bar{x}, 0)^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } B_1 \setminus \{0\}.
\end{cases}
\]

Therefore, it follows from [40, Lemma 2.8] that \(V(\cdot, 0) \in L^q(B_{9/10})\) for some \(q > \frac{n}{2\sigma}\). Using the Harnack inequality in [40, Proposition 2.6(iii)] (see also [9]), we just need to give an a priori bound of \(\|V(\cdot, 0)\|_{L^\infty(B_{1/2})}\) to complete the proof. We claim that there exists some positive constant \(C(n, \sigma, A_0)\) such that
\[
\|V(\bar{x}, 0)\|_{L^\infty(B_{1/2})} \leq C(n, \sigma, A_0). \tag{2.4}
\]

We prove (2.4) by contradiction argument. Suppose the contrary of (2.4), then there exists a sequence of \(\{K_j\}\) and \(\{U_j > 0\}\) satisfying
\[
\|K_j\|_{L^\infty(\mathbb{R}^n \setminus B_1)} \leq A_0,
\]
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U_j) = 0 & \text{in } \mathbb{R}_n^{n+1}, \\
\partial_\nu U_j = K_j(x) U_j(x, 0)^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } B_1 \setminus \{0\},
\end{cases}
\]
\[
\int_{B_1^+} \bar{t}^{1-2\sigma} |\nabla V_j|^2 \, d\bar{X} + \int_{B_1} |V_j(\bar{x}, 0)|^{2^*} \, d\bar{x} \leq C_0(n, \sigma) \mu_1,
\]
but
\[
\|V_j(\bar{x}, 0)\|_{L^\infty(B_{1/2})} \geq j,
\]
where \(V_j(\bar{X})\) is obtained by a Kelvin transformation on \(U_j(X)\) as before.

Note that, again by [40, Proposition 2.6(iii)], \(V_j\) is Hölder continuous in \(\overline{B}_{0,9}^+\), thus we can choose \(\bar{x}_j \in B_{0,9}\) such that
\[
(0.9 - |\bar{x}_j|)^{(n-2\sigma)/2} V_j(\bar{x}_j, 0) = \max_{|\bar{x}| \leq 0.9} (0.9 - |\bar{x}|)^{(n-2\sigma)/2} V_j(\bar{x}, 0).
\]
Let \(s_j = \frac{1}{2}(0.9 - |\bar{x}_j|) > 0\). Clearly we have
\[
s_j^{(n-2\sigma)/2} \sup_{B_{s_j}(\bar{x}_j)} V_j(\bar{x}, 0) \geq 3^{(2\sigma-n)/2} (0.9 - |\bar{x}_j|)^{(n-2\sigma)/2} V_j(\bar{x}_j, 0)
\]
\[
= 3^{(2\sigma-n)/2} \max_{|\bar{x}| \leq 0.9} (0.9 - |\bar{x}|)^{(n-2\sigma)/2} V_j(\bar{x}, 0)
\]
\[
\geq 3^{(2\sigma-n)/2} \max_{|\bar{x}| \leq 0.5} (0.9 - 0.5)^{(n-2\sigma)/2} V_j(\bar{x}, 0)
\]
\[
\rightarrow \infty \quad \text{as } j \rightarrow \infty,
\]
In conclusion, we obtain

\[ V_j(\bar{x}, 0) = (0.9 - |\bar{x}|)^{(2\sigma-n)/2} \max_{|\bar{x}| \leq 0.9} (0.9 - |\bar{x}|)^{(n-2\sigma)/2} V_j(\bar{x}, 0) \]

\[ \geq (0.9 - |\bar{x}|)^{(2\sigma-n)/2} \max_{|\bar{x}-\bar{x}_j| \leq s} (0.9 - |\bar{x}|)^{(n-2\sigma)/2} V_j(\bar{x}, 0) \]

\[ \geq (2s_j)^{(2\sigma-n)/2} (s_j)^{(n-2\sigma)/2} \max_{|\bar{x}-\bar{x}_j| \leq s_j} V_j(\bar{x}, 0) \]

\[ \geq 2^{(2\sigma-n)/2} \max_{|\bar{x}-\bar{x}_j| \leq s_j} V_j(\bar{x}, 0). \]

In conclusion, we obtain

\[ |\bar{x}_j| < 0.9, \]

\[ (s_j)^{(n-2\sigma)/2} \max_{|\bar{x}-\bar{x}_j| \leq s_j} V_j(\bar{x}, 0) \to \infty \quad \text{as} \quad j \to \infty, \]

\[ V_j(\bar{x}_j, 0) \geq 2^{(2\sigma-n)/2} \max_{|\bar{x}-\bar{x}_j| \leq s_j} V_j(\bar{x}, 0). \]

Now, consider

\[ W_j(\hat{x}, \hat{t}) = \frac{1}{V_j(\bar{x}_j, 0)} V_j\left(\hat{x} + \frac{\hat{x}}{V_j(\bar{x}_j, 0)^{\frac{2}{n-2\sigma}}}, \frac{\hat{t}}{V_j(\bar{x}_j, 0)^{\frac{2}{n-2\sigma}}}\right), \quad \hat{X} = (\hat{x}, \hat{t}) \in \Omega_j, \]

where

\[ \Omega_j := \left\{ (\hat{x}, \hat{t}) \in \mathbb{R}^{n+1}_+ : \left(\hat{x} + \frac{\hat{x}}{V_j(\bar{x}_j, 0)^{\frac{2}{n-2\sigma}}}, \frac{\hat{t}}{V_j(\bar{x}_j, 0)^{\frac{2}{n-2\sigma}}}\right) \in B_1 \setminus \{0\} \right\}. \]

Clearly, \( W_j(\hat{X}) \) satisfies

\[ \int_{B_{R_j}^+} \hat{t}^{1-2\sigma} |\nabla W_j|^2 \, d\hat{X} + \int_{B_{R_j}} |W_j(\hat{x}, 0)|^{2^*} \, d\hat{x} \leq C_0(n, \sigma) \mu_1, \]

\[ \left\{ \begin{array}{l}
\text{div}(\hat{t}^{1-2\sigma} \nabla W_j) = 0 \\
\partial_\nu W_j = K_j \left(\hat{x} + \frac{\hat{x}}{V_j(\bar{x}_j, 0)^{\frac{2}{n-2\sigma}}}\right) W_j(\hat{x}, 0)^{\frac{4+2\sigma}{n-2\sigma}} \quad \text{on} \quad \partial' \Omega_j, \\
W_j(0, 0) = 1,
\end{array} \right. \]

\[ W_j(\hat{x}, 0) \leq 2^{\frac{n-2\sigma}{2}}, \quad \forall |\hat{x}| < R_j, \]

where

\[ R_j := V_j(\bar{x}_j, 0)^{2/(n-2\sigma)} s_j \to \infty \quad \text{as} \quad j \to \infty. \]

By [40, Proposition 2.6], for any given \( \bar{T} > 0 \) we have

\[ 0 \leq W_j \leq C(\bar{T}) \quad \text{in} \quad B_{R_j/2} \times [0, \bar{T}), \]

where \( C(\bar{T}) \) depends only on \( n, \sigma \) and \( \bar{T} \). Then by Corollary 2.10 and Theorem 2.14 in [40], there exists some \( \alpha > 0 \) such that for every \( R > 1 \),

\[ \|W_j\|_{H^1(t^{1-2\sigma} B_R^+)} + \|W_j\|_{C^0(B_R)} + \|W_j(\cdot, 0)\|_{C^{2,\alpha}(B_R)} \leq C(R), \]

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where $C(R)$ is independent of $i$. Thus, after passing to a subsequence, we have, for some nonnegative function $W \in H^1_{\text{loc}}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+) \cap C^\alpha_{\text{loc}}(\mathbb{R}^{n+1}_+)$
\[
\begin{cases}
W_j \to W \quad \text{weakly in } H^1_{\text{loc}}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)
\vspace{0.5cm}
W_j \to W \quad \text{in } C^{\alpha/2}_{\text{loc}}(\mathbb{R}^{n+1}_+)
\vspace{0.5cm}
W_j(\cdot, 0) \to W(\cdot, 0) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n).
\end{cases}
\]

Let $\overline{K}$ be the weak * limit of \(\{K_j(\overline{x}_j + 1/V_j(\overline{x}_j, 0)) \}^{2/(n-2\sigma)}\) in $L^\infty_{\text{loc}}(\mathbb{R}^n)$, then we have $\|\overline{K}\|_{L^\infty(\mathbb{R}^n)} \leq A_0$. Moreover, $W(\hat{X})$ satisfies
\[
\begin{align*}
\text{div}(\hat{t}^{1-2\sigma} \nabla W) &= 0 & \text{in } \mathbb{R}^{n+1}_+,
\vspace{0.5cm}
\partial_{\nu}^\sigma W &= \overline{K}(\hat{x})W(\hat{x}, 0) \frac{n+2\sigma}{n-2\sigma} & \text{on } \mathbb{R}^n,
\vspace{0.5cm}
W(0, 0) &= 1,
\end{align*}
\]
\[(2.5)\]

Multiplying $(2.5)$ by $W$ and integrating by parts, we obtain
\[
\begin{align*}
\int_{\mathbb{R}^{n+1}_+} \hat{t}^{1-2\sigma} |\nabla W|^2 \, d\hat{X} + \int_{\mathbb{R}^n} |W(\hat{x}, 0)|^{2\sigma} \, d\hat{x} &\leq A_0 \left( \int_{\mathbb{R}^{n+1}_+} \hat{t}^{1-2\sigma} |\nabla W|^2 \, d\hat{X} \right)^{n/(n-2\sigma)} \mathcal{S}_{n, \sigma}^{-2\sigma},
\end{align*}
\]
where $\mathcal{S}_{n, \sigma}$ is defined in $(1.12)$. Therefore,
\[
1 \leq A_0 \left( \int_{\mathbb{R}^{n+1}_+} \hat{t}^{1-2\sigma} |\nabla W|^2 \, d\hat{X} \right)^{2\sigma/(n-2\sigma)} \mathcal{S}_{n, \sigma}^{-2\sigma}.
\]

This is a contradiction if we choose $\mu_1 = \mu_1(n, \sigma, A_0)$ to satisfy $A_0(C_0(n, \sigma)\mu_1)^{2\sigma/(n-2\sigma)} \mathcal{S}_{n, \sigma}^{-2\sigma} < 1$.

We complete the proof of $(2.4)$. \hfill $\Box$

**Proposition 2.2.** Let $\mu_1$ and $C(n, \sigma, A_0)$ be the positive constants in Proposition 2.1. Then for any $2 < l_1 < l_2 < \infty$, there exists a positive constant $R_1 = R_1(n, \sigma, A_0, \mu_1, l_1, l_2) > l_2$ such that for any $K \in L^\infty(B_{R_1} \setminus B_1)$ with $\|K\|_{L^\infty(B_{R_1} \setminus B_1)} \leq A_0$ and any positive solution $U(X)$ of
\[
\begin{align*}
\text{div}(t^{1-2\sigma} \nabla U) &= 0 & \text{in } \mathbb{R}^{n+1}_+,
\vspace{0.5cm}
\partial_{\nu}^\sigma U &= K(x)U(x, 0)^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } B_{R_1} \setminus B_1,
\end{align*}
\]
with
\[
\begin{align*}
\int_{B_{R_1}^+ \setminus B_1^+} t^{1-2\sigma} |\nabla U|^2 \, dX + \int_{B_{R_1} \setminus B_1} |U(x, 0)|^{2\sigma} \, dx &\leq \mu_1,
\end{align*}
\]
we have
\[
\sup_{X \in \mathbb{R}^{n+1}_+} |X|^{n-2\sigma} U(X) \leq 2C(n, \sigma, A_0).
\]

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Proof. Suppose the contrary, then for $R_j = l_2 + j$ with $j = 3, 4, 5, \ldots$, there exists a sequence of $\{K_j\}$ and $\{U_j > 0\}$ satisfying
\[
\|K_j\|_{L^\infty(B_{R_1} \setminus B_1)} \leq A_0,
\]
\[
\begin{cases}
  \text{div}(t^{1-2\sigma}\nabla U_j) = 0 & \text{in } \mathbb{R}^{n+1}_+,
  \\
  \partial_\nu U_j = K_j(x)U_j(x,0)\frac{n+2\sigma}{n-2\sigma} & \text{on } B_{R_j} \setminus B_1,
\end{cases}
\]
\[
\int_{B_{R_j}^+ \setminus B_1^+} t^{1-2\sigma}|\nabla U_j|^2 \, dX + \int_{B_{R_j} \setminus B_1} |U_j(x,0)|^{2\sigma} \, dx \leq \mu_1,
\]
but
\[
\sup_{X \in \mathbb{R}^{n+1}_+ \setminus B_1^+} |X|^{-2\sigma} U_j(X) > 2C(n, \sigma, A_0).
\]

Arguing as in the proof of Proposition 2.1, we know that for any $\mu \in (0,1)$, $\|U_j\|_{L^\infty(B_{R_j/2} \setminus B_{1+\mu})}$ is bounded by a constant independent of $j$. Let $U(X)$ be the $H^1_{loc}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ weak limit of $U_j$ (passing to a subsequence), we know that
\[
\begin{cases}
  \text{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+,
  \\
  \partial_\nu U = \overline{K}(x)U(x,0)\frac{n+2\sigma}{n-2\sigma} & \text{on } \mathbb{R}^n \setminus B_1,
\end{cases}
\]
\[
\sup_{X \in \mathbb{R}^{n+1}_+ \setminus B_1^+} |X|^{-2\sigma} U(X) \geq 2C(n, \sigma, A_0),
\]
where $\overline{K}(x)$ is the weak * limit of $K_j(x)$ in $L^\infty(B_{R_1} \setminus B_1)$ satisfying $\|\overline{K}\|_{L^\infty(B_{R_1} \setminus B_1)} \leq A_0$. However, it follows from Proposition 2.1 that
\[
\sup_{X \in \mathbb{R}^{n+1}_+ \setminus B_1^+} |X|^{-2\sigma} U(X) \leq C(n, \sigma, A_0).
\]
This contradicts to (2.6). \(\square\)

**Proposition 2.3.** Suppose that $l_2 > 100l_1 > 100$, $K \in C^1(B_{l_2} \setminus B_{l_1})$ and $\|K\|_{C^1(B_{l_2} \setminus B_{l_1})} \leq A_1$ for some positive constant $A_1$. Then for any positive solution $U(X)$ of
\[
\begin{cases}
  \text{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+,
  \\
  \partial_\nu U = K(x)U(x,0)\frac{n+2\sigma}{n-2\sigma} & \text{on } B_{l_2} \setminus B_{l_1},
\end{cases}
\]
satisfying
\[
\sup_{x \in B_{l_2} \setminus B_{l_1}} |x|^{-2\sigma} U(x,0) \leq A
\]
for some constant $A > 1$, we have
\[
\sup_{X \in \mathbb{R}^{n+1}_+ \setminus B_{l_1}^+} |X|^{n+1-2\sigma} |\nabla U| \leq C(n, \sigma, A_1, A),
\]
and
\[
\sup_{X \in \mathbb{R}^{n+1}_+ \setminus B_{l_1}^+} |X|^{n} t^{1-2\sigma} |\partial_t U| \leq C(n, \sigma, A_1, A),
\]
where $C(n, \sigma, A_1, A)$ is some positive constant depending only on $n, \sigma, A_1$ and $A$.  

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Proof. For any $r \in (4l_1, l_2/4)$, we have
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+,
\partial_\nu^\sigma U = K(x)U(x,0) \frac{n+2\sigma}{n-2\sigma} & \text{on } B_2 \setminus B_{1/2},
\end{cases}
\]
and $\sup_{x \in B_2 \setminus B_{1/2}} U(x,0) \leq (r/2)^{n-2\sigma} A$ by (2.7).

Let $V(X) = r^{n-2\sigma} U(rX)$, then it solves
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla V) = 0 & \text{in } \mathbb{R}^{n+1}_+,
\partial_\nu^\sigma V = K(rx)V(x,0) \frac{n+2\sigma}{n-2\sigma} & \text{on } B_2 \setminus B_{1/2}.
\end{cases}
\]
Note that $V(x,0) \leq 2^{\frac{2n-n}{\sigma}} A$ and $|K(rx)V(x,0)|^{\frac{n+2\sigma}{n-2\sigma}} \leq 2^{n+2\sigma} A_n A^{\frac{n-2\sigma}{n+2\sigma}}$ in the annulus $\{x \in \mathbb{R}^{n+1} : 1/2 \leq |x| \leq 2\}$, we deduce from [40, Proposition 2.6(iii)] and [9, Lemma 4.5] that
\[
\sup_{|X|=1} |\nabla_x V| \leq C(n, \sigma, A_1, A),
\]
and
\[
\sup_{|X|=1} |t^{1-2\sigma} \partial_t V| \leq C(n, \sigma, A_1, A).
\]
As a consequence,
\[
\sup_{|X|=r} |X|^{n+1-2\sigma} |\nabla_x U| \leq C(n, \sigma, A_1, A),
\]
and
\[
\sup_{|X|=r} |X|^n |t^{1-2\sigma} \partial_t U| \leq C(n, \sigma, A_1, A).
\]
This finishes the proof. \qed

Proposition 2.4. Let $\mu_1$, $R_1$ and $C(n, \sigma, A_0)$ be the constants in Proposition 2.2. Then for any $2 < l_1 < l_2 < \infty$, there exist some positive constants $\mu_2 = \mu_2(n, \sigma, A_0) \leq \mu_1$ and $\uppi = \uppi(n, \sigma, l_1, l_2, A_0)$ such that for any $0 \leq \tau \leq \uppi$, $K \in L^\infty(B_{R_1} \setminus B_1)$ with $\|K\|_{L^\infty(B_{R_1} \setminus B_1)} \leq A_0$, and any positive solution $U(X)$ of
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+,
\partial_\nu^\sigma U = K(x)U(x,0) \frac{n+2\sigma}{n-2\sigma} & \text{on } B_{2R_1} \setminus B_1,
\end{cases}
\]
with
\[
\int_{B_{2R_1} \setminus B_1^+} t^{1-2\sigma} |\nabla U|^2 \, dX + \int_{B_{2R_1} \setminus B_1} |U(x,0)|^{2\sigma} \, dx \leq \mu_2,
\]
we have
\[
\sup_{x \in \mathbb{R}^{n+1}_2 \setminus B_{l_1}^+} |X|^{n-2\sigma} U(X) \leq 3C(n, \sigma, A_0).
\]
Furthermore, if $K \in C^1(B_{R_1} \setminus B_1)$ with $\|K\|_{C^1(B_{R_1} \setminus B_1)} \leq A_1$ for some positive constant $A_1$, we have
\[
\sup_{x \in \mathbb{R}^{n+1}_2 \setminus B_{l_1}^+} |X|^{n+1-2\sigma} |\nabla_x U| \leq 2C(n, \sigma, A_1, A),
\]
and
\[ \sup_{X \in \Omega_2^+ \setminus \Omega_1^+} |X|^n |\partial_t U| \leq 2C(n, \sigma, A_1, A), \]

where \( C(n, \sigma, A_1, A) \) is the constant in Proposition 2.3 with \( A \) replaced by \( 3C(n, \sigma, A_0) \).

**Proof.** Suppose the contrary, then there exists a sequence \( 0 \leq \tau_j \to 0 \) and \( U_{\tau_j} > 0 \) satisfying
\[
\begin{align*}
\text{div}(t^{1-2\sigma} \nabla U_{\tau_j}) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\
\partial_{\nu} U_{\tau_j} &= K(x)U_{\tau_j}(x,0)\frac{n+2\sigma-\tau_j}{n-2\sigma} \quad \text{on } B_{2R_1} \setminus B_1, \\
\int_{\Omega_2^+ \setminus \Omega_1^+} t^{1-2\sigma} |\nabla U_{\tau_j}|^2 \, dX + \int_{B_{2R_1} \setminus B_1} |U_{\tau_j}(x,0)|^{2^*_\sigma} \, dx &\leq \mu_2,
\end{align*}
\]

but
\[ \sup_{X \in \Omega_2^+ \setminus \Omega_1^+} \{|X|^{n-2\sigma} U_{\tau_j}(X)| > 3C(n, \sigma, A_0). \]

Choosing \( \mu_2 \in (0, \mu_1) \) to be small, we use an argument similar to the proof of Proposition 2.1 to obtain (by passing to a subsequence)
\[ U_{\tau_j}(X) \to U(X) \quad \text{in } C^{\alpha/2}_{\text{loc}}(B_{2R_1}^+ \setminus B_1^+). \]

Moreover, \( U(X) \) satisfies
\[ \sup_{X \in \Omega_2^+ \setminus \Omega_1^+} \{|X|^{n-2\sigma} U(X)| \geq 3C(n, \sigma, A_0). \quad (2.9) \]

However, by Proposition 2.2 we have
\[ \sup_{X \in \Omega_2^+ \setminus \Omega_1^+} \{|X|^{n-2\sigma} U(X)| \leq 2C(n, \sigma, A_0), \]

which contradicts to (2.9). We finish the proof of (2.8). The other two terms can be managed in a similar manner by using Proposition 2.3 and contradiction argument, we omit it here. \( \square \)

### 3 A minimization problem on exterior domain

For any \( z_1, z_2 \in \mathbb{R}^n \) satisfy \( |z_1 - z_2| \geq 10 \), denote \( \Omega := \mathbb{R}^{n+1}_+ \setminus \{B_1^+(z_1) \cup B_1^+(z_2)\} \). We define \( D_\Omega \) by taking the closure of \( C^\infty_c(\overline{\Omega}) \) under the norm
\[
||U||_{D_\Omega} := \left( \int_\Omega t^{1-2\sigma} |\nabla U|^2 \, dX \right)^{1/2} + \left( \int_{\partial \Omega} |U(x,0)|^{2^*_\sigma} \, dx \right)^{1/2^*_\sigma}.
\]

Clearly, \( D_\Omega \) is a Banach space.

By some appropriate extension to \( \overline{B}_1^+(z_1) \cup \overline{B}_1^+(z_2) \) and using (1.12), we have a Sobolev trace type inequality on \( D_\Omega \):
Lemma 3.1. Let $D_{\Omega}$ be defined as above. There exists some positive constant $C(n, \sigma)$ such that for all $U \in D_{\Omega}$, there holds
\[
\left( \int_{\partial'\Omega} |U(x,0)|^{2_\sigma} \, dx \right)^{1/2_\sigma} \leq C(n, \sigma) \left( \int_{\Omega} t^{1-2\sigma} |\nabla U|^2 \, dX \right)^{1/2},
\]
where the constant $C(n, \sigma)$ depends only on $n$ and $\sigma$. In particular, it does not depend on $z_1, z_2$ provided $|z_1 - z_2| \geq 10$.

Its proof can be done as in [43, Lemma A.1] with minor modifications, so we omit it here.

Let $K \in L^\infty(\partial'\Omega)$ satisfy $\|K\|_{L^\infty(\partial'\Omega)} \leq A_0$ for some constant $A_0 > 0$. We define a functional on $D_{\Omega}$ by
\[
I_K,\Omega(U) := \frac{1}{2} \int_{\Omega} t^{1-2\sigma} |\nabla U|^2 \, dX - \frac{1}{2_\sigma - \tau} \int_{\partial'\Omega} K(x) H^\tau(x) |U(x,0)|^{2_\sigma - \tau} \, dx
\]
with $\tau \geq 0$ small. For any $U \in D_{\Omega}$, using Hölder inequality and Lemma 3.1, we have
\[
\begin{align*}
|I_K,\Omega(U) - \frac{1}{2} \int_{\Omega} t^{1-2\sigma} |\nabla U|^2 \, dX| &\leq A_0 C_0(n, \sigma) \left( \int_{\partial'\Omega} |U(x,0)|^{2_\sigma} \, dx \right)^{(2_\sigma - \tau)/2_\sigma} \\
&\leq A_0 C_0(n, \sigma) \left( \int_{\Omega} t^{1-2\sigma} |\nabla U|^2 \, dX \right)^{(2_\sigma - \tau)/2} ,
\end{align*}
\]
(3.1)
where $C_0(n, \sigma)$ denotes some universal constant which can vary from line to line.

Proposition 3.1. Let $D_{\Omega}$ be defined as above. There exist some constants $r_0 = r_0(n, \sigma, A_0) \in (0,1)$ and $C_1 = C_1(n, \sigma) > 1$ such that for any $z_1, z_2 \in \mathbb{R}^n$ with $|z_1 - z_2| \geq 10$, and $W \in W^{\sigma,2}(\partial'\Omega)$ with $r = \|W\|_{W^{\sigma,2}(\partial'\Omega)} \leq r_0$, the following minimum is achieved:
\[
\min \left\{ I_{K,\Omega}(U) : U \in D_{\Omega}, U|_{\partial'\Omega} = W, \int_{\Omega} t^{1-2\sigma} |\nabla U|^2 \, dX \leq C_1 r_0^2 \right\}.
\]

The minimizer is unique (denoted $U_W$) and satisfies $\int_{\Omega} t^{1-2\sigma} |\nabla U_W|^2 \, dX \leq C_1 r_0^2 / 2$. Furthermore, the map $W \mapsto U_W$ is continuous from $W^{\sigma,2}(\partial'\Omega)$ to $D_{\Omega}$. In particular, the constants $r_0$ and $C_1$ are independent of $z_1, z_2$ provided $|z_1 - z_2| \geq 10$.

Remark 3.1. (1) Following the terminology in [3], $W^{\sigma,2}(\partial'\Omega)$ stands for fractional order Sobolev space defined on the boundary $\partial'\Omega$, obtained by transporting (via a partition of unity and pull-back) the standard scale $W^{\sigma,2}(\mathbb{R}^n) = H^\sigma(\mathbb{R}^n)$. We refer the readers to [3, p.215] for more details.

(2) When $\sigma = 1/2$, the proof of Proposition 3.1 can be directly followed from [44, Proposition 2.1] or [45, Proposition 3.3].

Proof of Proposition 3.1. We claim that there exist some constant $C_1 = C_1(n, \sigma) > 0$ and $\overline{W} \in D_{\Omega}$ such that
\[
\int_{\Omega} t^{1-2\sigma} |\nabla \overline{W}|^2 \, dX \leq \frac{C_1}{8} r_0^2 \quad \text{and} \quad \overline{W}|_{\partial'\Omega} = W. \tag{3.2}
\]
To justify this, we first note that $\partial'\Omega$ is compact and smooth (mollifying the singularities of $\partial'\Omega$ if necessary), thus any open cover $\{U_j\}$ of $\partial'\Omega$ and the associated collection $\{\Psi_j\}$ of smooth maps
Thus, if we choose \( r \)

Therefore it is standard to conclude the existence of a unique local minimizer

Next we prove the existence of the minimizer. Write \( U = V + \overline{W}, V|_{\partial^c \Omega} = 0, J_{K,\Omega}(V) = I_{K,\Omega}(U) = I_{K,\Omega}(V + \overline{W}) \). We only need to minimize \( J_{K,\Omega}(V) \) for \( \int_{\Omega} t^{1 - 2\sigma} |\nabla V|^2 \, dX \leq 2C_1 r_0^2 \) due to the above argument. It is easy to see that if \( r_0 \) is small enough, then \( J_{K,\Omega} \) is strictly convex in the ball

Therefore it is standard to conclude the existence of a unique local minimizer \( V_W \).

Finally, set \( U_W = V_W + \overline{W} \), then \( U_W \) is a local minimizer. As discussed above, \( U_W \) satisfies \( \int_{\Omega} t^{1 - 2\sigma} |\nabla U_W|^2 \, dX \leq C_1 r^2/2 \). The uniqueness and the continuity of the map \( W \mapsto U_W \) follows from the strict local convexity of \( J_{K,\Omega} \).

\[ \square \]
4 Existence and multiplicity results for the subcritical case

We point out that due to the presence of the fractional Sobolev critical exponent, the corresponding Euler-Lagrange functional corresponding to Eq. (1.2) does not satisfy the Palais-Smale condition. One way to overcome such a difficulty is to consider the following subcritical approximation problem

\[
\begin{cases}
(-\Delta)^\sigma u = K(x)H^\tau(x)u^{\frac{n+2\sigma}{n-2\sigma} - \tau} & \text{in } \mathbb{R}^n, \\
\quad u \in E^+, 
\end{cases}
\]  

with \( \tau > 0 \) small. The aim in this section is to establish the existence and multiplicity results for the above subcritical type equations using the localization method introduced by Caffarelli and Silvestre [11] as stated in the introduction.

We first introduce some notation which is used throughout the paper.

Let \( \{K_l(x)\} \) be a sequence of functions satisfying the following conditions.

(i) There exists some positive constant \( A_1 > 1 \) such that for any \( l = 1, 2, 3, \ldots \),

\[
|K_l(x)| \leq A_1, \quad \forall x \in \mathbb{R}^n. \tag{4.2}
\]

(ii) For some integer \( m \geq 2 \), there exist \( z_l^{(i)} \in \mathbb{R}^n, 1 \leq i \leq m, R_l \leq \frac{1}{2} \min_{i \neq j} |z_l^{(i)} - z_l^{(j)}|, \) such that \( K_l \) is continuous near \( z_l^{(i)} \) and

\[
\begin{align*}
\lim_{l \to \infty} R_l &= \infty, \\
K_l(z_l^{(i)}) &= \max_{x \in B_{R_l}(z_l^{(i)})} K_l(x), \quad 1 \leq i \leq m, \\
\lim_{l \to \infty} K_l(z_l^{(i)}) &= a^{(i)}, \quad 1 \leq i \leq m, \\
K_{\infty}^{(i)}(x) &= (\text{weak } *) \lim_{l \to \infty} K_l(x + z_l^{(i)}), \quad 1 \leq i \leq m. \tag{4.3-4.5}
\end{align*}
\]

(iii) There exist some positive constants \( A_2, A_3 > 1, \delta_0, \delta_1 > 0, \) and some bounded open sets \( O_1^{(i)}, \ldots, O_m^{(i)} \subset \mathbb{R}^n \), such that, if we define for \( 1 \leq i \leq m, \)

\[
\bar{O}_l^{(i)} = \{ x \in \mathbb{R}^n : \text{dist}(x, O_l^{(i)}) < \delta_0 \}, \\
O_l = \bigcup_{i=1}^m O_l^{(i)}, \quad \bar{O}_l = \bigcup_{i=1}^m \bar{O}_l^{(i)},
\]

we have

\[
\begin{align*}
&z_l^{(i)} \in O_l^{(i)}, \quad \text{diam}(O_l^{(i)}) < R_l/10, \\
&K_l \in C^1(\bar{O}_l, [1/A_2, A_2]), \\
&K_l(z_l^{(i)}) \geq \max_{x \in \partial O_l^{(i)}} K_l(x) + \delta_1, \\
&\max_{x \in \bar{O}_l} |\nabla K_l(x)| \leq A_3. \tag{4.7-4.10}
\end{align*}
\]
For \( \varepsilon > 0 \) small, we define \( V_l(m, \varepsilon) := V(m, \varepsilon, O_l^{(1)}, \ldots, O_l^{(m)}, K_i) \) as in (1.15). Here and in the following, we are concerned with the case \( m = 2 \), since the more general result is similar in nature.

If \( U \) is a function in \( V_l(2, \varepsilon) \), one can find an optimal representation, following the ideas introduced in \([5,6]\). Namely, we have

**Proposition 4.1.** There exists \( \varepsilon_0 \in (0,1) \) depending only on \( A_1, A_2, A_3, n, \sigma, \delta_0, \) and \( m \), but independent of \( l \), such that, for any \( \varepsilon \in (0, \varepsilon_0] \), \( U \in V_l(2, \varepsilon) \), the following minimization problem

\[
\min_{(\alpha, z, \lambda) \in B_2} \left\| U - \sum_{i=1}^{2} \alpha_i \delta(z_i, \lambda_i) \right\| \sigma
\]

has a unique solution \((\alpha, z, \lambda)\) up to a permutation. Moreover, the minimizer is achieved in \( B_{2\varepsilon} \) for large \( l \), where

\[
B_\varepsilon = \left\{ (\alpha, z, \lambda) : \alpha = (\alpha_1, \alpha_2), 1/(2A_2^{(n-2\sigma)/4\sigma}) \leq \alpha_1, \alpha_2 \leq 2A_2^{(n-2\sigma)/4\sigma}, z = (z_1, z_2) \in O_l^{(1)} \times O_l^{(2)}, \lambda = (\lambda_1, \lambda_2), \lambda_1, \lambda_2 \geq \varepsilon^{-1} \right\}. 
\]

In particular, we can write \( U \) as follows:

\[
U = \sum_{i=1}^{2} \alpha_i \delta(z_i, \lambda_i) + v, 
\]

where \( v \in D_{\varepsilon, 2} \). In addition, the variables \( \{\alpha_i\} \) satisfy

\[
|\alpha_i - K_i(z_i)^{(2\sigma-n)/4\sigma}| = o_\varepsilon(1) \quad \text{for} \ i = 1, 2, 
\]

where \( o_\varepsilon(1) \to 0 \) as \( \varepsilon \to 0 \).

Proof. The proof is similar up to minor modifications to the corresponding statements in \([5,6]\), we omit it here.

In the sequel, we will often split \( U \), a function in \( V_l(2, \varepsilon) \), \( \varepsilon \in (0, \varepsilon_0] \), under the form

\[
U = \alpha_1^l \delta(z_1^l, \lambda_1^l) + \alpha_2^l \delta(z_2^l, \lambda_2^l) + v^l 
\]

after making the minimization (4.11). Proposition 4.1 guarantees the existence and uniqueness of \( \alpha_i = \alpha_i(u) = \alpha_i^l \), \( z_i = z_i(u) = z_i^l \) and \( \lambda_i = \lambda_i(u) = \lambda_i^l \) for \( i = 1, 2 \) (we omit the index \( l \) for simplicity).

For any \( K \in L^\infty(\mathbb{R}^n) \) and \( U \in D \), we define

\[
I_{K,\tau}(U) := \frac{1}{2} \int_{\mathbb{R}^{n+1}} t^{1-2\sigma} |\nabla U|^2 \, dx - \frac{1}{2^{*}_\sigma - \tau} \int_{\mathbb{R}^n} K(x) H^T(x) |U(x,0)|^{2^*_\sigma - \tau} \, dx 
\]

with \( \tau \geq 0 \) small. Clearly, \( I_K = I_{K,0} \), \( I_{K,\tau} \in C^2(D, \mathbb{R}) \).

To continue our proof, let \( \{\mathbf{\tau}_l\} \) be a sequence satisfying

\[
\lim_{l \to \infty} \mathbf{\tau}_l = 0, \quad \lim_{l \to \infty} (|z_1^{(1)}| + |z_2^{(2)}|) \mathbf{\tau}_l = 1. 
\]

We now give a lower bound energy estimate for some well-spaced bubbles.
Lemma 4.1. Let $\varepsilon_0$ be the constant in Proposition 4.1. Suppose that $\varepsilon_1 \in (0, \varepsilon_0)$ small enough and $l$ large enough, $0 \leq \tau \leq \tau_l$. Then there exists some constant $A_4 = A_4(n, \sigma, \delta_1, A_2) > 1$ such that for any $U \in V_l(2, \varepsilon_1)$ with $z_1(U) \in \widetilde{O}_l^{(1)}$, $z_2(U) \in \widetilde{O}_l^{(2)}$, and dist($z_1(U), \partial O_l^{(1)}$) < $\delta_1/(2A_3)$ or dist($z_2(U), \partial O_l^{(2)}$) < $\delta_1/(2A_3)$, we have

$$I_{K_i,\tau}(U) \geq c^{(1)} + c^{(2)} + 1/A_4,$$

where

$$c^{(i)} = \frac{\sigma}{n}(a^{(i)}(2\sigma-n)/2\sigma \cdot (S_{n,\sigma})^{n/\sigma}).$$  (4.15)

Proof. We assume that dist($z_1(U), \partial O_l^{(1)}$) < $\delta_1/(2A_3)$. It follows from (1.13), (4.12), and some direct computations that, for $\varepsilon_1 > 0$ small and $l$ large,

$$I_{K_i,\tau}(U) = \sum_{i=1}^{2} I_{K_i,\tau}(\alpha_i \delta(z_i, \lambda_i)) + o_{\varepsilon_1}(1)$$

$$= \sum_{i=1}^{2} \left\{ \frac{1}{2} K_i(z_i)^{(2\sigma-n)/2\sigma \cdot (\delta(z_i, \lambda_i))^2} \right\}$$

$$= \frac{1}{2\sigma} K_i(z_i)^{(2\sigma-n)/2\sigma \cdot (\delta(0,1))^2}$$

$$\geq \sum_{i=1}^{2} \left\{ \frac{1}{2} K_i(z_i)^{(2\sigma-n)/2\sigma \cdot (\delta(0,1))^2} \right\}$$

$$= \sum_{i=1}^{2} \frac{\sigma}{n} K_i(z_i)^{(2\sigma-n)/2\sigma \cdot (S_{n,\sigma})^{n/\sigma}}$$

Combining this estimate with the assumption dist($z_1(U), \partial O_l^{(1)}$) < $\delta_1/(2A_3)$, we obtain

$$I_{K_i,\tau}(U) \geq \frac{\sigma}{n} K_1(z_1^{(1)}) - \delta_1/2)^{(2\sigma-n)/2\sigma \cdot (S_{n,\sigma})^{n/\sigma}}$$

$$+ \frac{\sigma}{n} K_1(z_2^{(2)})^{(2\sigma-n)/2\sigma \cdot (S_{n,\sigma})^{n/\sigma}} + o_{\varepsilon_1}(1) + o(1)$$

$$\geq \sum_{i=1}^{2} c^{(i)} + 1/A_4,$$

where the choice of $A_4$ is evident thanks to (4.4), (4.5), (4.9), (4.10) and (4.15). The proof is now complete. \qed

From now on, the value of $A_4$ and $\varepsilon_1$ are fixed. The main result in this section can be stated as follows:

**Theorem 4.1.** Suppose that $\{K_l\}$ is a sequence of functions satisfying (i)–(iii). If there exist some bounded open sets $O^{(1)}, \ldots, O^{(m)} \subset \mathbb{R}^n$ and some constants $\delta_2, \delta_3 > 0$ such that for all $1 \leq i \leq m$,

$$\tilde{O}_l^{(i)} - z_l^{(i)} \subset O^{(i)} \quad \text{for all } l,$$  (4.16)
\[ \{ U \in D^+ : I'_{K_\infty^{(i)}}(U) = 0, \, c^{(i)} \leq I_{K_\infty^{(i)}}(U) \leq c^{(i)} + \delta_2 \} \cap V(1, \delta_3, O^{(i)}, K_\infty^{(i)}) = \emptyset. \] (4.17)

Then for any \( \varepsilon > 0 \), there exists integer \( l_{\varepsilon,m} > 0 \) such that for any \( l \geq l_{\varepsilon,m} \) and \( \tau \in (0, \tau_l) \), there exists \( U_l = U_{l,\tau} \in V_l(m, \varepsilon) \) which solves

\[
\begin{align*}
\text{div}(l^{-2\sigma} \nabla U_l) &= 0 & \text{in } \mathbb{R}^{n+1}, \\
\partial^\sigma U_l &= K_l(x)H^\tau(x)U_l(x,0) & \text{on } \mathbb{R}^n.
\end{align*}
\] (4.18)

Furthermore, \( U_l \) satisfies

\[
\sum_{i=1}^{m} c^{(i)} - \varepsilon \leq I_{K_l,\tau}(U_l) \leq \sum_{i=1}^{m} c^{(i)} + \varepsilon. \] (4.19)

We prove Theorem 4.1 by contradiction argument. For simplicity, we only consider the case of \( m = 2 \), since the changes for \( m > 2 \) are evident. Suppose the contrary of Theorem 4.1, i.e., for some \( \varepsilon^* > 0 \), there exists a sequence of \( l \to \infty, 0 < \tau_l < \tau \) such that Eq. (4.18), for \( \tau = \tau_l \), has no solution in \( V_l(2, \varepsilon^*) \) satisfying (4.19) with \( \varepsilon = \varepsilon^* \). A complicated procedure will be followed in order to yield a contradiction. It will be outlined now and the details will be given in the next two sections. The proof consists of two parts:

- **Part 1.** Under the contrary of Theorem 4.1, we obtain a uniform lower bounds of the gradient vectors in some certain annular region. It is a standard consequence of the Palais-Smale condition in variational argument.

- **Part 2.** We use variational method to construct an approximating “minimax” curve. Part 1 can be used to construct a deformation. In our setting, we follow the nonnegative gradient flow to make a deformation, which is an important process to obtain a counterexample.

Part 1 will be carried out in Section 5 and Part 2 in Section 6.

### 5 First part of the proof of Theorem 4.1

For \( \varepsilon_2 > 0 \), we denote \( \tilde{V}_l(2, \varepsilon_2) \) the set of functions \( U \) in \( D \) satisfies: there exist \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \), \( z = (z_1, z_2) \in O_l^{(1)} \times O_l^{(2)} \) and \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \) such that

\[
\begin{align*}
\lambda_1, \lambda_2 &> \varepsilon_2^{-1}, \\
|\lambda_i^\tau - 1| &< \varepsilon_2, \quad i = 1, 2, \\
|\alpha_i - K_l(z_i) (2^\sigma - \eta)/4^\sigma | &< \varepsilon_2, \quad i = 1, 2, \\
\| U - \sum_{i=1}^{2} \alpha_i \tilde{\delta}(z_i, \lambda_i)^{1+O(\eta)} \|_\sigma &< \varepsilon_2.
\end{align*}
\]

Throughout the paper, we denote \( p_l = \frac{n+2\sigma}{n-2\sigma} - \eta_l \).

**Lemma 5.1.** For \( \varepsilon_2 = \varepsilon_2(\varepsilon_1, \varepsilon^*, n, \sigma) > 0 \) small enough, we have, for \( l \) large enough,

\[
\tilde{V}_l(2, \varepsilon_2) \subset V_l(2, \varepsilon_2(1)) \subset V_l(2, \varepsilon_1) \cap V_l(2, \varepsilon^*),
\] (5.1)

where \( \alpha_{\varepsilon_2}(1) \to 0 \) as \( \varepsilon_2 \to 0 \).
Proof. It is easy to check (5.1) by using the definition of \( \tilde{V}_l(2, \varepsilon_2) \). Hence we omit it.

The following result is the crucial step in the proof of Theorem 4.1.

**Proposition 5.1.** Under the hypotheses of Theorem 4.1 and the contrary of the conclusion of Theorem 4.1, there exist some constants \( \varepsilon_2 \in (0, \min\{\varepsilon_0, \varepsilon_1, \varepsilon^*, \delta_3\}) \) and \( \varepsilon_3 \in (0, \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon^*, \delta_3\}) \) which are independent of \( l \) such that (5.1) holds for such \( \varepsilon_2 \), and there exist \( \delta_4 = \delta_4(\varepsilon_2, \varepsilon_3) > 0 \) and \( l'_{\varepsilon_2, \varepsilon_3} > 1 \) such that for any \( l \geq l'_{\varepsilon_2, \varepsilon_3} \), \( U \in \tilde{V}_l(2, \varepsilon_2) \backslash \tilde{V}_l(2, \varepsilon_2/2) \) with \( |I_{K_1, \tau_l}(U) - (c^{(1)} + c^{(2)})| < \varepsilon_3 \), we have

\[
\|I'_{K_1, \tau_l}(U)\| \geq \delta_4,
\]

where \( I'_{K_1, \tau_l} \) denotes the Fréchet derivative.

**Remark 5.1.** Proposition 5.1 will be used to construct an approximating “minimaxing” curve in Section 6.

Evidently, we have, under the contrary of Theorem 4.1, that for each \( l \),

\[
\inf\{\|I_{K_1, \tau_l}(U)\| : U \in \tilde{V}_l(2, \varepsilon_2) \backslash \tilde{V}_l(2, \varepsilon_2/2), |I_{K_1, \tau_l}(U) - (c^{(1)} + c^{(2)})| < \varepsilon_3\} > 0.
\]

We prove Proposition 5.1 by contradiction argument. Suppose the statement in the Proposition 5.1 is not true, then no matter how small \( \varepsilon_2, \varepsilon_3 > 0 \) are, there exists a subsequence (which is still denoted as \( \{U_l\} \)) such that

\[
\{U_l\} \subset \tilde{V}_l(2, \varepsilon_2) \backslash \tilde{V}_l(2, \varepsilon_2/2),
\]

\[
|I_{K_1, \tau_l}(U_l) - (c^{(1)} + c^{(2)})| < \varepsilon_3,
\]

\[
\lim_{l \to \infty}\|I'_{K_1, \tau_l}(U_l)\| = 0.
\]

However, under the above assumptions, we can prove that there exists another subsequence, still denoted by \( \{U_l\} \), such that \( U_l \in \tilde{V}_l(2, \varepsilon_2/2) \), which leads to a contradiction. The existence of such sequence needs some lengthy indirect analysis to the interaction of two “bubbles”. We break the proof of Proposition 5.1 into several claims.

First we write

\[
U_l = \alpha_1^l(\delta(z_1^l, \lambda_1^l)) + \alpha_2^l(\delta(z_2^l, \lambda_2^l)) + v_l
\]

after making the minimization (4.11). By Proposition 4.1 and some standard arguments in [5, 6, 44], if \( \varepsilon_2 > 0 \) small enough, we have

\[
(\lambda_1^l)^{-1}, (\lambda_2^l)^{-1} = o_{\varepsilon_2}(1),
\]

\[
|\alpha_i^l - K_i(z_i^l)^{2\sigma-n}/4\sigma| = o_{\varepsilon_2}(1), \quad i = 1, 2,
\]

\[
\|v_l\|_\sigma = o_{\varepsilon_2}(1),
\]

\[
\text{dist}(z_1^l, O^{(1)}_l), \quad \text{dist}(z_2^l, O^{(2)}_l) = o_{\varepsilon_2}(1).
\]

Next we will derive some elementary estimates of the interaction of two “bubbles” in (5.5). More precisely, we want to find another representative of \( U_l \) in (5.5), from which we can deduce its location and concentrate rate easily. Let us introduce a linear isometry operator first.

For \( z \in \mathbb{R}^n \), we define \( T_z : D \to D \) by

\[
T_zU(x, t) := U(x + z, t).
\]

It is easy to see that \( \|T_zU\|_\sigma = \|U\|_\sigma \).

We now give bubble’s profile of (5.5).
Claim 1. For $\varepsilon_2 > 0$ small enough, we have
\[ \lim_{l \to \infty} \lambda_1^l = \lim_{l \to \infty} \lambda_2^l = \infty. \]

Proof. Assume to the contrary that $\lambda_1^l = \lambda_1 + o(1) < \infty$ up to a subsequence. Here and in the following, let $o(1)$ denote any sequence tending to 0 as $l \to \infty$. Now the proof consists of three steps.

Step 1 (Construct a positive solution). One observes from (5.5) that
\[ T_{z_1}U_l = \alpha_1^l \tilde{\delta}(0, \lambda_1^l) + \alpha_2^l \tilde{\delta}(z_2^l - z_1^l, \lambda_2^l) + T_{z_1}v_l. \]
Then by Proposition 4.1, after passing to a subsequence, we have
\begin{align}
\lim_{l \to \infty} \alpha_1^l &= \alpha_1 \in \left[ \frac{1}{2} (A_2)^{(2\sigma-n)/4\sigma} - o_\varepsilon(1), 2(A_2)^{(2\sigma-n)/4\sigma} + o_\varepsilon(1) \right], \\
\lim_{l \to \infty} \alpha_2^l &= \alpha_2 \in \left[ \frac{1}{2} (A_2)^{(2\sigma-n)/4\sigma} - o_\varepsilon(1), 2(A_2)^{(2\sigma-n)/4\sigma} + o_\varepsilon(1) \right],
\end{align}
and
\[ T_{z_1}v_l \rightharpoonup w_0 \quad \text{weakly in } D \]
for some $w_0 \in D$. From the lower semi-continuity of the norm and (5.8), we infer that
\[ \|w_0\|_\sigma \leq \lim_{l \to \infty} \|T_{z_1}v_l\|_\sigma = o_\varepsilon(1). \]
Using the assumption (ii) (stated in the beginning of Section 4), we get
\[ \lim_{l \to \infty} |z_1^l - z_2^l| \geq \lim_{l \to \infty} R_l = \infty. \] (5.13)
Therefore,
\[ T_{z_1}U_l \rightharpoonup W := \alpha_1 \tilde{\delta}(0, \lambda_1) + w_0 \quad \text{weakly in } D. \] (5.14)
Obviously, $W \neq 0$ if $\varepsilon_2$ is small enough.

Next we prove that $W$ is a weak solution of the following equation
\begin{equation}
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla W) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\partial^\sigma_\nu W = T_\zeta K^{(1)}_\infty |W(x,0)|^{\frac{4\sigma}{n-2\sigma}} W(x,0) & \text{on } \mathbb{R}^n,
\end{cases}
\end{equation}
where $\zeta \in O^{(1)}$ with $\text{dist}(\zeta, \partial O^{(1)}) > \delta_0/2$. Note that we have abused notation a bit. Only in this proof we write $T_\zeta K^{(1)}_\infty = K^{(1)}_\infty (\cdot + \zeta)$.

For any $\phi \in C^\infty_c(\mathbb{R}^{n+1}_+)$, it follows from (5.4) that
\[ I'_{K_{11}^l} (U_l) (T_{z_1^l} \phi) = o(1) \|T_{z_1^l} \phi\|_\sigma = o(1) \|\phi\|_\sigma = o(1). \]
Summing up (4.6), (4.10), (4.14) and (5.14), we find that
\[ o(1) = \langle U_l, T_{z_1^l} \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot) H^{T_l}(\cdot)|U_l|^{p_l-1} U_l T_{z_1^l} \phi \]
\[ = \langle T_{z_1} U_l, \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot + z_1^l) H^{T_l}(\cdot + z_1^l)|T_{z_1} U_l|^{p_l-1} (T_{z_1} U_l) \phi \]
\[ = \langle T_{z_1} U_l, \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot + z_1^l) H^{T_l}(\cdot + z_1^l)|T_{z_1} U_l|^{p_l-1} (T_{z_1} U_l) \phi \]
\[ = \langle W, \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} T_\zeta K_\infty^{(1)} |W|^{\frac{4\sigma}{n-2\sigma}} W \phi + o(1), \]

where \( \zeta = \lim_{l \to \infty} (z_1^{(l)} - z_1^{(1)}) \) along a subsequence. This means \( W \) is a weak solution of (5.15).

The positivity of \( W \) can be verified from the following argument.

Let us write \( W = W^+ - W^- \), where \( W^+ = \max(W, 0) \) and \( W^- = \min(W, 0) \). It follows from (5.14), (5.12) and (1.12) that

\[ \int_{\mathbb{R}^n \times \{0\}} (W^-)^2 \sigma \, dx = o_{\varepsilon_2}(1). \]  

(5.16)

Multiplying (5.15) by \( W^- \) and integrating by parts, we have

\[ \int_{\mathbb{R}^n_+} |\nabla W^-|^2 \, dX = \int_{\mathbb{R}^n \times \{0\}} T_\zeta K_\infty^{(1)} (W^-)^{2\sigma} \, dx \]

\[ \leq o_{\varepsilon_2}(1) \left( \int_{\mathbb{R}^n \times \{0\}} (W^-)^{2\sigma} \, dx \right)^{2/2\sigma} \]

\[ \leq o_{\varepsilon_2}(1) \int_{\mathbb{R}^n_+} |\nabla W^-|^2 \, dX, \]  

(5.17)

where we used (5.16) in the first inequality and (1.12) in the second inequality. Hence, if \( \varepsilon_2 > 0 \) is small enough, we immediately obtain \( W^- \equiv 0 \), namely, \( W \geq 0 \). It follows from (5.15) and the strong maximum principle that \( W > 0 \).

**Step 2** (Energy bound estimates). We now begin to estimate the value of \( I_{T_\zeta K_\infty^{(1)}}(W) \) in order to obtain a contradiction. The estimate we are going to establish is

\[ c^{(1)} \leq I_{T_\zeta K_\infty^{(1)}}(W) \leq c^{(1)} + o_{\varepsilon_2}(1), \]

(5.18)

where \( o_{\varepsilon_2}(1) \to 0 \) as \( \varepsilon_2 \to 0 \).

Firstly, multiplying (5.15) by \( W^- \) and integrating by parts, we have

\[ \int_{\mathbb{R}^n_+} t^{1-2\sigma} |\nabla W|^2 \, dX = \int_{\mathbb{R}^n \times \{0\}} T_\zeta K_\infty^{(1)} W^{2\sigma} \, dx. \]

This implies that

\[ I_{T_\zeta K_\infty^{(1)}}(W) = \frac{1}{2} \int_{\mathbb{R}^n_+} t^{1-2\sigma} |\nabla W|^2 \, dX - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} T_\zeta K_\infty^{(1)} W^{2\sigma} \, dx \]

\[ = \frac{\sigma}{n} \int_{\mathbb{R}^n \times \{0\}} T_\zeta K_\infty^{(1)} W^{2\sigma} \, dx. \]

We thus conclude from (1.12) and the upper bound \( T_\zeta K_\infty^{(1)} \leq a^{(1)} \) that

\[ S_{n, \sigma} \leq \frac{\left( \int_{\mathbb{R}^n_+} t^{1-2\sigma} |\nabla W|^2 \, dX \right)^{1/2}}{\left( \int_{\mathbb{R}^n \times \{0\}} W^{2\sigma} \, dx \right)^{1/2\sigma}} \]

\[ \leq \frac{\left( \int_{\mathbb{R}^n_+} t^{1-2\sigma} |\nabla W|^2 \, dX \right)^{1/2}}{\left( \int_{\mathbb{R}^n \times \{0\}} T_\zeta K_\infty^{(1)} W^{2\sigma} \, dx \right)^{1/2\sigma}} (a^{(1)})^{1/2\sigma} \]

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\[ = \left( \int_{\mathbb{R}^n \times \{0\}} T_{t} K^{(1)}_{\infty} W^{2\sigma} \, dx \right)^{\sigma/n} (\alpha^{(1)})^{1/2\sigma}. \]

Therefore, we complete the proof of the first inequality in (5.18).

On the other hand, we deduce from (4.2) that

\[ |K^{(1)}_{\infty}(x)| \leq A_1, \quad \forall x \in \mathbb{R}^n. \]

Owing to (1.13), (5.5), (5.6), (5.8) and (5.13), we have

\[
I_{K_{i}, \tau_i}(U_l) = I_{K_{i}, \tau_i}(\alpha_{l}^j \delta(z_{1, l}^j, \lambda_{l}^j)) + I_{K_{i}, \tau_i}(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) + o_{\varepsilon}(1) \\
= I_{t} T_{l} K_{i, \tau_i}(\alpha_{l}^j \delta(0, \lambda_{l}^j)) + I_{K_{i}, \tau_i}(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) + o_{\varepsilon}(1) \\
= I_{t} T_{l} K_{i, \tau_i}(\alpha_{l}^j \delta(0, \lambda_{l}^j)) + I_{K_{i}, \tau_i}(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) + o_{\varepsilon}(1) + o(1) \\
= I_{t} K_{\infty}^{(1)}(W) + I_{K_{i}, \tau_i}(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) + o_{\varepsilon}(1) + o(1).
\]

Consequently,

\[ I_{t} K_{\infty}^{(1)}(W) = I_{K_{i}, \tau_i}(U_l) - I_{K_{i}, \tau_i}(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) + o_{\varepsilon}(1) + o(1). \quad (5.19) \]

Combining (5.4) and (5.5), we find

\[
o(1) = I_{K_{i}, \tau_i}(U_l)(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) = I_{K_{i}, \tau_i}(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) + o_{\varepsilon}(1) + o(1), \]

namely,

\[
\|\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)\|_{\sigma}^2 = \int_{\mathbb{R}^n} K_{i}(x) H_{\tau}(x)(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))^{2\sigma-\tau_i} \, dx + o_{\varepsilon}(1) + o(1), \quad (5.20)
\]

\[
I_{K_{i}, \tau_i}(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)) = \frac{\sigma}{n} \|\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)\|_{\sigma}^2 + o_{\varepsilon}(1) + o(1). \quad (5.21)
\]

From (1.13) and (5.11), we obtain

\[
\|\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j)\|_{\sigma}^2 \geq \left( \frac{1}{2} (A_2)^{(2\sigma-n)/4\sigma} - o_{\varepsilon}(1) \right) (S_{n, \sigma})^{n/\sigma} \\
> \frac{1}{4} (A_2)^{(2\sigma-n)/4\sigma} (S_{n, \sigma})^{n/\sigma}. \quad (5.22)
\]

Then, by (1.12), (4.3)–(4.5), (5.6), (5.13), and Hölder inequality, we have

\[
S_{n, \sigma} \leq \frac{\left( \int_{\mathbb{R}^n_{+1}} t^{1-2\sigma} |\nabla(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))|^2 \, dX \right)^{1/2}}{\left( \int_{\mathbb{R}^n} (\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))^{2\sigma} \, dx \right)^{1/2}} \\
\leq \frac{\left( \int_{\mathbb{R}^n_{+1}} t^{1-2\sigma} |\nabla(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))|^2 \, dX \right)^{1/2}}{\left( \int_{B_{Rt}(z_{2}^j)} (\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))^{2\sigma} \, dx \right)^{1/2} + o(1)} \\
\leq \frac{\left( \int_{\mathbb{R}^n_{+1}} t^{1-2\sigma} |\nabla(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))|^2 \, dX \right)^{1/2} \cdot K_{i}(z_{2}^j)^{1/2\sigma}}{\left( \int_{B_{Rt}(z_{2}^j)} K_{i}(x) H_{\tau}(x)(\alpha_{l}^{l} \delta(z_{2, l}^j, \lambda_{l}^j))^{2\sigma-\tau_i} \, dx \right)^{1/2} + o(1)}
\]

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\[
\frac{\left(\int_{\mathbb{R}^n} t^{1-2\sigma} |\nabla (\alpha_1^2 \delta(z_1^l, \lambda_2^l))|^2 \, dX\right)^{1/2}}{\left(\int_{\mathbb{R}^n} K_l(x) H^{n}(x) (\alpha_2^2 \delta(z_2^l, \lambda_2^l))^{2\sigma-n} \, dx\right)^{1/2} + o(1)}.
\]

Thus, using (5.20), we establish that
\[
S_{n,\sigma} \leq \|\alpha_2^2 \delta(z_2^l, \lambda_2^l)\|^2 \|\lambda_2^l\|^{2\sigma/n} . (a(2))^{1/2\sigma} + o(1).
\]

This together with (5.21) gives
\[
I_{K_{l,\tau}} (\alpha_2^2 \delta(z_2^l, \lambda_2^l)) \geq \frac{\sigma}{n} (a(2)) (2\sigma-n)/2\sigma (S_{n,\sigma})^{n/\sigma} + o_{\varepsilon_2}(1) + o(1)
\]
\[
= c(2) + o_{\varepsilon_2}(1) + o(1). 
\]

(5.23)

Putting (5.3), (5.19) and (5.23) together, we obtain the right hand side of (5.18).

**Step 3** (Completion of the proof). Finally, a contradiction arises from (4.17), (5.14)–(5.18), and the positivity of \( W \) for \( \varepsilon_2 > 0 \) small enough. This proves that \( \lim_{l \to \infty} \lambda_1^l = \infty \). Similarly we have \( \lim_{l \to \infty} \lambda_2^l = \infty \). Claim 1 has been established.

For any \( \lambda > 0 \) and \( z \in \mathbb{R}^n \), we define \( \mathcal{R}_{\lambda, z} : D \to D \) by
\[
\mathcal{R}_{\lambda, z} U(x, t) := \lambda^{2\sigma/(1-n)} U(x + z, \lambda t).
\]

It is clear that \( \mathcal{R}_{\lambda, z}^{-1} U(x, t) = \lambda^{2\sigma/(n-1)} U(\lambda(x - z), \lambda t) \).

**Lemma 5.2.** There exists some constant \( C = C(n, \sigma, A_2) > 0 \) such that for small \( \varepsilon_2 \) and large \( l \), we have
\[
(\lambda_1^l)^{\tau_l}, (\lambda_2^l)^{\tau_l} \leq C.
\]

**Proof.** Applying (5.4), we deduce that
\[
I'_{K_{l, \tau}} (U_l) (\delta(z_1^l, \lambda_1^l)) = o(1).
\]

(5.24)

Now an explicit calculation from (5.8), (5.13), Claim 1, and Proposition 4.1 yields that
\[
(\delta(z_1^l, \lambda_1^l), v_l) = 0,
\]
\[
(\bar{\delta}(z_1^l, \lambda_1^l), \delta(z_2^l, \lambda_2^l)) = o(1),
\]
\[
\int_{\mathbb{R}^n} K_l(z_1^l, \lambda_1^l) \delta(z_2^l, \lambda_2^l) \, d\nu_l = o(1),
\]
\[
\int_{\mathbb{R}^{n+1}} K_l(z_1^l, \lambda_1^l) \, d\nu_l = o_{\varepsilon_2}(1).
\]

Putting together the above estimates, we have
\[
(\alpha_1^l \nu_l) \int_{\mathbb{R}^n} K_l(z_1^l, \lambda_1^l) \, d\nu_l = \alpha_1^l \|\delta(z_1^l, \lambda_1^l)\|_{\sigma}^2 + o_{\varepsilon_2}(1) + o(1).
\]

(5.25)

Then the proof of the first term completed from (4.8), (4.14), (5.7), (5.9), (5.25), and Claim 1. Similarly we have \( (\lambda_2^l)^{\tau_l} \leq C \).
Without loss of generality, we can always assume that
\[ \lambda_1 \leq \lambda_2. \] (5.26)

A direct computation using (5.5) shows that
\[ \mathcal{T}_{l, \lambda_1, z_1} U_l = \bar{\alpha}_1 l \delta(0,1) + \bar{\alpha}_2 l (\lambda_1 (z_2 - z_1), \lambda_2 / \lambda_1) + \mathcal{T}_{l, \lambda_1, z_1} \eta_l, \] (5.27)
where
\[ \bar{\alpha}_1 = \alpha_1 (\lambda_1) \frac{(n-2\sigma)/2-2\sigma/(p_l-1)}{\lambda_1}, \]
\[ \bar{\alpha}_2 = \alpha_2 (\lambda_1) \frac{(n-2\sigma)/2-2\sigma/(p_l-1)}{\lambda_2}. \]

Then we can verify the existence of \( U_1 \in D \) and \( \xi_1 \in O^{(1)} \) such that
\[ \mathcal{T}_{l, \lambda_1, z_1} U_l \rightharpoonup U_1 \quad \text{weakly in } D, \] (5.28)
\[ \lim_{l \to \infty} (z_1 - z_1^{(1)}) = \xi_1, \] (5.29)
up to a subsequence.

Accordingly, by making use of (4.6), (4.10), (5.29) and (5.9), we have
\[ \lim_{l \to \infty} K_l(z_1) (2\sigma-n)/4\sigma = K_1^{(1)} (\xi_1) (2\sigma-n)/4\sigma. \] (5.30)

Now, one observes from (5.4) that for any \( \phi \in C^\infty_c (\mathbb{R}^{n+1}_+) \), we get
\[
o(1) = I'_{K_l, \tau_l} (U_l)(\mathcal{T}_{l, \lambda_1, z_1} \eta_l)
= \langle U_l, \mathcal{T}_{l, \lambda_1, z_1} \eta_l \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot) H^\tau_l(\cdot) |U_l|^{p_l-1} U_l \mathcal{T}_{l, \lambda_1, z_1} \eta_l \phi
= \langle \lambda_1 \rangle^{4\sigma/(p_l-1)+2\sigma-n} \left\{ \langle \mathcal{T}_{l, \lambda_1, z_1} U_l, \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot) (\lambda_1 + z_1) \right. \right.
\left. \times H^\tau_l(\cdot) |\lambda_1 + z_1| \mathcal{T}_{l, \lambda_1, z_1} U_l |^{p_l-1} \mathcal{T}_{l, \lambda_1, z_1} U_l \phi \right\}.
\]
Taking the limit \( l \to \infty \), and then using (4.6), (4.14), (5.9), (5.28), (5.29), and Lemma 5.2, we obtain
\[ \langle U_1, \phi \rangle - \int_{\mathbb{R}^n} K_1^{(1)} (\xi_1) |U_1(x,0)|^{4\sigma/2\sigma} U_1(x,0) \, dx = 0. \]

Namely, \( U_1 \) satisfies
\[
\begin{cases}
\operatorname{div}(t^{1-2\sigma} \nabla U_1) = 0 & \text{in } \mathbb{R}^{n+1}_+,
\partial^\sigma_\nu U_1 = K_1^{(1)} (\xi_1) |U_1(x,0)|^{4\sigma/2\sigma} U_1(x,0) & \text{on } \mathbb{R}^n.
\end{cases}
\] (5.31)

Moreover, we see from (5.27) that \( U_1 \) is not identically zero if \( \varepsilon_2 \) is small enough. We then argue as before to obtain \( U_1 > 0 \).

By the classification theorem of positive solutions of (5.31) (see [32, Proposition 1.3] or [40, Theorem 1.8]), there exists \( \lambda^* > 0 \) and \( z^* \in \mathbb{R}^n \) such that
\[ U_1 = K_1^{(1)} (\xi_1) (2\sigma-n)/4\sigma \delta(z^*, \lambda^*). \] (5.32)
Claim 2. For $l$ large enough, we have

$$|z^*| = o_{\varepsilon_2}(1), \quad |\lambda^* - 1| = o_{\varepsilon_2}(1), \quad (\lambda_1^l)^n = 1 + o_{\varepsilon_2}(1).$$

Proof. First of all, using Lemma 5.2, we find

$$\lim_{l \to \infty} (\lambda_1^l)^n = A_{\varepsilon_2, \varepsilon_3}$$

along a subsequence, where $A_{\varepsilon_2, \varepsilon_3}$ is a positive constant independent of $l$ for fixed $\varepsilon_2$ and $\varepsilon_3$. Thanks to\(^{(5.7)}\) and\(^{(5.30)}\), we obtain

$$\alpha_1^l = K_\infty^{(1)}(\xi_1)((2\sigma-n)/4\sigma + o_{\varepsilon_2}(1) + o(1)).$$

Note that

$$\tilde{\alpha}_1^l = \alpha_1^l(\lambda_1^l)^{(n-2\sigma)/2-2\sigma/(p_1-1)} = \alpha_1^l(\lambda_1^l)^{-(n-2\sigma)^2/8\sigma + o(1)}.$$ (5.33)

Then it can be computed that

$$\tilde{\alpha}_1^l = K_\infty^{(1)}(\xi_1)((2\sigma-n)/4\sigma (A_{\varepsilon_2, \varepsilon_3})^{-(n-2\sigma)^2/8\sigma} + o_{\varepsilon_2}(1) + o(1).$$ (5.34)

Moreover, by\(^{(5.6)}\),\(^{(5.13)}\) and\(^{(5.26)}\)--\(^{(5.28)}\), we have

$$\tilde{\alpha}_1^l \tilde{\delta}(0,1) + T_{t, \lambda_1^l, z_1^l} U_1 \rightarrow U_1 \quad \text{weakly in } D \quad \text{(5.35)}$$

Consequently, it follows from\(^{(5.8)}\),\(^{(5.34)}\),\(^{(5.35)}\), and Lemma 5.2 that

$$\|K_\infty^{(1)}(\xi_1)((2\sigma-n)/4\sigma (A_{\varepsilon_2, \varepsilon_3})^{-(n-2\sigma)^2/8\sigma} \tilde{\delta}(0,1) - K_\infty^{(1)}(\xi_1)((2\sigma-n)/4\sigma \tilde{\delta}(z^*, \lambda^*)\|_\sigma = o_{\varepsilon_2}(1) + o(1).$$

Finally, taking the limit $l \to \infty$, we get

$$|z^*| = o_{\varepsilon_2}(1), \quad \lambda^* = 1 + o_{\varepsilon_2}(1), \quad A_{\varepsilon_2, \varepsilon_3} = 1 + o_{\varepsilon_2}(1).$$

We define $\xi_t \in D$ by

$$T_{t, \lambda_1^l, z_1^l} U_t = U_1 + T_{t, \lambda_1^l, z_1^l} \xi_t.$$ (5.36)

It follows from\(^{(5.28)}\) that

$$T_{t, \lambda_1^l, z_1^l} \xi_t \rightarrow 0 \quad \text{weakly in } D \quad \text{(5.37)}$$

Claim 3. For $\varepsilon_2$ small enough, we have $\|I_{K_{t, \gamma}}^{K_{t, \gamma}}(\xi_t)\| = o(1)$.

Proof. For any $\phi \in C_c(\mathbb{R}^{n+1}_{+})$, it follows from\(^{(5.4)}\),\(^{(5.36)}\),\(^{(5.31)}\), and Lemma 5.2 that

$$o(1)\|\phi\|_\sigma = I_{K_{t, \gamma}}^{K_{t, \gamma}}(U_t)(T_{t, \lambda_1^l, z_1^l}^{-1} \phi)$$

$$= \langle U_t, T_{t, \lambda_1^l, z_1^l}^{-1} \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K_t(\cdot) H^n(\cdot | U_t)^{p_1-1} U_t T_{t, \lambda_1^l, z_1^l}^{-1} \phi$$

$$= (\lambda_1^l)^{(4\sigma/p_1-1)+2\sigma-n} \int_{\mathbb{R}^n \times \{0\}} K_t(\cdot, \lambda_1^l + z_1^l)$$

$$= o(1).$$

\(\Box\)
\( \times H^n(\cdot/\lambda'_1 + z'_1)|\mathcal{F}_{L_1, z'_1} U_1|^{p-1}(\mathcal{F}_{L_1, z'_1} U_1) \phi \) 
\[ = (\lambda'_1)^{4\sigma/(p-1)+2\sigma-n} \left\{ \langle U_1, \phi \rangle + \langle \mathcal{F}_{L_1, z'_1} \xi_1, \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) \right. \]
\[ \times H^n(\cdot/\lambda'_1 + z'_1)|\mathcal{F}_{L_1, z'_1} U_1|^{p-1}(\mathcal{F}_{L_1, z'_1} U_1) \phi \) 
\[ = I_{K_l, \tau_1}(\xi_1)(\mathcal{F}_{L_1, z'_1} \phi) + (\lambda'_1)^{4\sigma/(p-1)+2\sigma-n} \left\{ \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) H^n(\cdot/\lambda'_1 + z'_1)|\mathcal{F}_{L_1, z'_1} \xi_1|^{p-1}(\mathcal{F}_{L_1, z'_1} U_1) \phi \right\}. \] (5.38)

Then a direct calculation exploiting (4.14), (5.30), (5.32), Claim 2, Hölder inequality, and the fractional Sobolev embedding theorem shows 
\[ \left| \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) H^n(\cdot/\lambda'_1 + z'_1)|U_1|^{p-1} \phi - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) H^n(\cdot/\lambda'_1 + z'_1)|\mathcal{F}_{L_1, z'_1} \xi_1|^{p-1}(\mathcal{F}_{L_1, z'_1} U_1) \phi \right| = o(1) \| \phi \|_\sigma. \] (5.39)

Finally, by (5.38), (5.39), Lemma 5.2, and some elementary inequalities, we deduce that 
\[ |I_{K_l, \tau_1}(\xi_1)(\mathcal{F}_{L_1, z'_1} \phi)| = o(1) \| \phi \|_\sigma + O(1) \int_{\mathbb{R}^n \times \{0\}} \left( |\mathcal{F}_{L_1, z'_1} U_1|^{p-1} U_1 + |\mathcal{F}_{L_1, z'_1} \xi_1| U_1^{p-1} \right) \| \phi \|_\sigma \]
\[ = o(1) \| \phi \|_\sigma, \]
where the last inequality follows from (5.37), Claim 2, Hölder inequality, and the fractional Sobolev embedding theorem. Claim 3 has been established. \( \square \)

**Claim 4.** \( I_{K_l, \tau_1}(\xi_1) \leq c(2) + \varepsilon_3 + o(1) \).

**Proof.** By a change of variable and using Claim 2 and (5.36), some calculations lead to 
\[ I_{K_l, \tau_1}(U_1) = \frac{1}{2} \| U_1 \|_\sigma^2 - \frac{1}{p_l+1} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot) H^n(\cdot)|U_1|^{p_l+1} \]
\[ = (\lambda'_1)^{4\sigma/(p_l-1)+2\sigma-n} \left( \frac{1}{2} \| U_1 \|_\sigma^2 - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) \right. \]
\[ \times H^n(\cdot/\lambda'_1 + z'_1)|\mathcal{F}_{L_1, z'_1} U_1|^{p_l+1} \} + o(1) \]
\[ = (\lambda'_1)^{4\sigma/(p_l-1)+2\sigma-n} \left( \frac{1}{2} \| U_1 \|_\sigma^2 + \langle U_1, \mathcal{F}_{L_1, z'_1} \xi_1 \rangle + \frac{1}{2} \| \mathcal{F}_{L_1, z'_1} \xi_1 \|_\sigma^2 \right. \]
\[ - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) H^n(\cdot/\lambda'_1 + z'_1)|\mathcal{F}_{L_1, z'_1} \xi_1|^{p_l+1} \]
\[ - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) H^n(\cdot/\lambda'_1 + z'_1)|\mathcal{F}_{L_1, z'_1} \xi_1|^{p_l+1} \]
\[ - O(1) \int_{\mathbb{R}^n \times \{0\}} \left( |\mathcal{F}_{L_1, z'_1} U_1| + |\mathcal{F}_{L_1, z'_1} \xi_1| U_1^{p_l} \right) \} + o(1) \]
\[ = I_{K_l, \tau_1}(\xi_1) + (\lambda'_1)^{2(p_l+1)/(p_l-1)-n} \left( \frac{1}{2} \| U_1 \|_\sigma^2 - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda'_1 + z'_1) \right. \]
\[ \times H^n(\cdot/\lambda_1^l + z_1^l)U_1^{n+1} \right) + o(1), \]  
\hfill (5.40)

where we used (5.32) and (5.37) in the last equality.

We derive from (4.6), (5.29) and (1.12) that

\[
\frac{1}{2} \|U_1\|_\sigma^2 - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_1^l + z_1^l)H^n(\cdot/\lambda_1^l + z_1^l)U_1^{n+1} \\
= I_{K_\infty^{(1)}}(U_1) + o(1) \\
\geq \sigma K_\infty^{(1)}(\xi_1)^{(2\sigma-n)/2\sigma} (S_{n,\sigma})^{n/\sigma} + o(1) \\
\geq c^{(1)} + o(1), \]  
\hfill (5.41)

Claim 4 follows from (5.3), (5.40), (5.41), and the fact \((\lambda_1^l)^{4\sigma/(p_1-1)+2\sigma-n} \geq 1. \quad \square\)

From (5.5), (5.32) and (5.36), we have

\[ \xi_t = U_t - \mathcal{T}_{t,\lambda_1^l,z_1^l}^{-1} U_1 = \alpha_2^l \tilde{\delta}(l, \lambda_2^l) + w_t, \]  
\hfill (5.42)

where

\[ w_t = \alpha_2^l \tilde{\delta}(l, \lambda_1^l) - K_\infty^{(1)}(\xi_1)^{(2\sigma-n)/2\sigma} (\lambda_1^l)^{2\sigma/(p_1-1)-(n-2\sigma)/2\sigma} \tilde{\delta}(l, \lambda_1^l + z_1^l, \lambda_1^l) + v_t. \]

Now using (5.33) and Claim 2, we have, for large \(l\), that

\[ \|w_t\|_\sigma = o_{c_2}(1). \]  
\hfill (5.43)

We can simply repeat the previous arguments on \(\xi_t\) instead of on \(U_t\). For the reader’s convenience, we carry out some crucial steps.

A direct computation using (5.42) shows that

\[ \mathcal{T}_{t,\lambda_1^l,z_1^l} \xi_t = \mathcal{T}_{t,\lambda_1^l,z_1^l}^{-1} w_t, \]  
\hfill (5.44)

where

\[ \mathcal{T}_{t,\lambda_1^l,z_1^l} = \alpha_2^l \delta(t, \lambda_2^l) = o_2^l(\lambda_2^l)^{(n-2\sigma)/2\sigma/(p_1-1)}. \]  
\hfill (5.45)

Then we can verify the existence of \(U_2 \in D\) and \(\xi_2 \in O(2)\) such that

\[ \mathcal{T}_{t,\lambda_2^l,z_2^l} \xi_t \rightharpoonup U_2 \text{ weakly in } D, \]  
\hfill (5.46)

\[ \lim_{t \to \infty} (z_2^l - z_t^2(2)) = \xi_2, \]  
\hfill (5.47)

up to a subsequence.

Accordingly, by making use of (4.6), (4.10) and (5.47), we have

\[ \lim_{l \to \infty} K_l(z_2^l)^{(2\sigma-n)/2\sigma} = K_\infty^{(2)}(\xi_2)^{(2\sigma-n)/2\sigma}. \]  
\hfill (5.48)

For any \(\phi \in C_c^\infty(\mathbb{R}^{n+1}_+),\) it follows from Claim 3 and Lemma 5.2 that

\[ o(1) = I_{K_\infty^{(1)}}(\xi_2)(\mathcal{T}_{t,\lambda_2^l,z_2^l}^{-1} \phi) \\
= (\lambda_2^l)^{4\sigma/(p_1-1)+2\sigma-n} \left\{ (\mathcal{T}_{t,\lambda_2^l,z_2^l} \xi_t, \phi) - \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_2^l + z_2^l) \right\}. \]
\[
\times H_n(\lambda' + z'_{2})|\mathcal{F}_{t,\lambda_{2}',\xi_{2}}\xi_{i}|^{p-1}(\mathcal{F}_{t,\lambda_{2}',\xi_{2}}\phi)\}
\]

Taking the limit \(l \to \infty\) and arguing as before, we have
\[
\langle U_2, \phi \rangle - \int_{\mathbb{R}^n \times \{0\}} K^{(2)}_{\infty}(\xi_{2})|U_2|^{\frac{4\sigma}{4\sigma - n}} U_2 \phi = 0.
\]

Namely, \(U_2\) satisfies
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U_2) = 0 & \text{in } \mathbb{R}^{n+1}, \\
\partial_{x}^{\sigma} U_2 = K^{(2)}_{\infty}(\xi_{2})|U_2(x,0)|^{\frac{4\sigma}{4\sigma - n}} U_2(x,0) & \text{on } \mathbb{R}^n.
\end{cases}
\] (5.49)

Arguing as before, one can prove that, for \(\varepsilon_2\) small enough, \(U_2 > 0\) and for some \(z^{**} \in \mathbb{R}^n\) and \(\lambda^{**} > 0\), there holds
\[
U_2 = K^{(2)}_{\infty}(\xi_{2})^{(2\sigma-n)/4\sigma} \delta(z^{**}, \lambda^{**}).
\] (5.50)

**Claim 5.** For \(l\) large enough, we have
\[
|z^{**}| = o_{\varepsilon_2}(1), \ |\lambda^{**} - 1| = o_{\varepsilon_2}(1), \ (\lambda_{2}')^{\eta_1} = 1 + o_{\varepsilon_2}(1).
\]

**Proof.** Lemma 5.2 shows that, up to a subsequence,
\[
\lim_{l \to \infty} (\lambda_{2}')^{\eta_1} = B_{\varepsilon_2, \varepsilon_3},
\]
where \(B_{\varepsilon_2, \varepsilon_3}\) is a positive constant independent of \(l\) for fixed \(\varepsilon_2\) and \(\varepsilon_3\). One derive from (4.6), (4.10), (5.9) and (5.47) that
\[
\lim_{l \to \infty} K_l(z^{(2)}_2)^{(2\sigma-n)/4\sigma} = K^{(2)}_{\infty}(\xi_{2})^{(2\sigma-n)/4\sigma}.
\] (5.51)

Then it follows from (5.45) and (5.51) that
\[
\bar{\alpha}_2 = K^{(2)}_{\infty}(\xi_{2})^{(2\sigma-n)/4\sigma} (B_{\varepsilon_2, \varepsilon_3})^{-(n-2\sigma)^2/8\sigma} + o_{\varepsilon_2}(1) + o(1).
\] (5.52)

Consequently, by (5.43), (5.44), (5.46), (5.50), (5.52), and Lemma 5.2, we have
\[
\|K^{(2)}_{\infty}(\xi_{2})^{(2\sigma-n)/4\sigma} (B_{\varepsilon_2, \varepsilon_3})^{-(n-2\sigma)^2/8\sigma} \delta(0,1) - K^{(2)}_{\infty}(\xi_{2})^{(2\sigma-n)/4\sigma} \delta(z^{**}, \lambda^{**})\|_\sigma = o_{\varepsilon_2}(1) + o(1).
\]

Taking the limit \(l \to \infty\), we get
\[
|z^{**}| = o_{\varepsilon_2}(1), \ \lambda^{**} = 1 + o_{\varepsilon_2}(1), \ B_{\varepsilon_2, \varepsilon_3} = 1 + o_{\varepsilon_2}(1).
\]

We define \(\eta_l \in D\) by
\[
\mathcal{F}_{t,\lambda_{2}',\eta_{l}} = U_2 + \mathcal{F}_{t,\lambda_{2}',\xi_{2}} \eta_{l}.
\] (5.53)

Clearly,
\[
\mathcal{F}_{t,\lambda_{2}',\eta_{l}} \rightharpoonup 0 \quad \text{weakly in } D.
\] (5.54)

**Claim 6.** For \(\varepsilon_2\) small enough, we have \(\|I'_{K_{\lambda_{2}',\eta_l}}(\eta_l)\| = o(1)\).
Proof. We follow the same method as the proof of Claim 3 and only record some crucial steps.

For any \( \phi \in C_c^\infty(\mathbb{R}^{n+1}_+) \), by (5.49), (5.53), Claim 3, and Lemma 5.2, we have

\[
o(1)\|\phi\| = I_{K_1, \tau_l}(\xi_l)(\mathcal{T}_{l, \lambda_2, z_2}^{-1} \phi) \\
= I_{K_1, \tau_l}(\eta_l)(\mathcal{T}_{l, \lambda_2, z_2}^{-1} \phi) + (\lambda_2^{4/(p_l-1)}+2\sigma-n) \int_{\mathbb{R}^n \times \{0\}} K_2(\xi_l) U_2^{2\sigma-1} \phi \\
+ \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_2 + z_2^l) H^\tau_l(\cdot/\lambda_2 + z_2^l) |\mathcal{T}_{l, \lambda_2, z_2} \eta_l|^{p_l-1} |\mathcal{T}_{l, \lambda_2, z_2} |^{p_l-1} \phi \\
- \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_2 + z_2^l) H^\tau_l(\cdot/\lambda_2 + z_2^l) |\mathcal{T}_{l, \lambda_2, z_2} \xi_l|^{p_l-1} (\mathcal{T}_{l, \lambda_2, z_2} \xi_l) \right\} \tag{5.55}
\]

Then a direct calculation exploiting (4.14), (5.48), (5.50), Claim 2, Hölder inequality, and the fractional Sobolev embedding theorem shows

\[
\left| \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_2 + z_2^l) H^\tau_l(\cdot/\lambda_2 + z_2^l) U_2^{p_l} \phi - \int_{\mathbb{R}^n \times \{0\}} K_2(\xi_l) U_2^{2\sigma-1} \phi \right| = o(1)\|\phi\|_\sigma. \tag{5.56}
\]

Finally, by (5.55), (5.56), Lemma 5.2, and some elementary inequalities, we obtain

\[
|I_{K_1, \tau_l}(\eta_l)(\mathcal{T}_{l, \lambda_2, z_2}^{-1} \phi)| \\
= o(1)\|\phi\|_\sigma + O(1) \int_{\mathbb{R}^n \times \{0\}} \left\{ |\mathcal{T}_{l, \lambda_2, z_2} \eta_l|^{p_l-1} U_2 + |\mathcal{T}_{l, \lambda_2, z_2} \xi_l|^{p_l-1} \right\} \phi \\
= o(1)\|\phi\|_\sigma,
\]

the last step follows from (5.54), Claim 2, Hölder inequality, and the fractional Sobolev embedding theorem. Claim 6 has been established. \(\square\)

Claim 7. For \( \varepsilon_2 \) small enough, we have \( I_{K_1, \tau_l}(\eta_l) \leq \varepsilon_3 + o(1) \).

Proof. We follow the same method as the proof of Claim 4 and only record some crucial steps.

In view of Claim 5, (5.50), (5.53) and (5.54), we obtain

\[
I_{K_1, \tau_l}(\xi_l) = \frac{1}{2} \|\xi_l\|_\sigma^2 - \frac{1}{p_l+1} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot) H^\tau_l(\cdot) |\xi_l|^{p_l+1} \\
= (\lambda_2^{4/(p_l-1)}+2\sigma-n) \left\{ \frac{1}{2} \|\mathcal{T}_{l, \lambda_2, z_2} \xi_l\|_\sigma^2 - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_2 + z_2^l) \right. \\
\times H^\tau_l(\cdot/\lambda_2 + z_2^l) |\mathcal{T}_{l, \lambda_2, z_2} \xi_l|^{p_l+1} \right\} + o(1) \\
= I_{K_1, \tau_l}(\eta_l) + (\lambda_2^{4/(p_l-1)}+2\sigma-n) \left\{ \frac{1}{2} \|U_2\|_\sigma^2 - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_2 + z_2^l) \right. \\
\times H^\tau_l(\cdot/\lambda_2 + z_2^l) U_2^{p_l+1} \right\} + o(1). \tag{5.57}
\]

Then we derive from (4.6), (5.47) and (1.12) that

\[
\frac{1}{2} \|U_2\|_\sigma^2 - \frac{1}{2\sigma} \int_{\mathbb{R}^n \times \{0\}} K_l(\cdot/\lambda_2 + z_2^l) H^\tau_l(\cdot/\lambda_2 + z_2^l) U_2^{p_l+1} \\
= I_{K_1, \tau_l}(\xi_l)(U_2) + o(1) \tag{5.58}
\]

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\[
\geq \frac{\sigma}{n} K_n^{(2)}(\xi_2)^{(2\sigma-n)/2\sigma} (S_{n,\sigma})^{n/\sigma} + o(1)
\geq c^{(2)} + o(1).
\]

Claim 7 follows from (5.57), (5.59), Claim 4, and the fact \((\lambda_2^\alpha)^{2(p_1+1)(p_1-1)-n} \geq 1.
\]

Claim 8. For \(\varepsilon_2 > 0\) small enough, we have \(\eta \to 0\) strongly in \(D\).

Proof. It follows from Claim 6 and Claim 7 that

\[
\|\eta\|_\sigma^2 \leq \frac{n}{\sigma} \varepsilon + o(1).
\]

Suppose that Claim 8 does not hold, then along a subsequence we have

\[
\|\eta\|_\sigma^2 = 1 + o(1).
\]

We derive from (5.61), Hölder inequality, and Claim 6 that

\[
\|\eta\|_\sigma^2 \leq C(n, \sigma, A_1) \left( \int_{\mathbb{R}^n} |\eta(x, 0)|^{2\sigma} \, dx \right)^{(p_1+1)/2n} + o(1),
\]

and then

\[
\|\eta\|_\sigma^2 \leq C(n, \sigma, A_1) \int_{\mathbb{R}^n} |\eta(x, 0)|^{2\sigma} \, dx + o(1).
\]

It follows from (5.62) and (1.12) that

\[
S_{n,\sigma} \leq \frac{\left( \int_{\mathbb{R}^n} t^{1-2\sigma} |\nabla \eta|^2 \, dX \right)^{1/2}}{\left( \int_{\mathbb{R}^n} |\eta(x, 0)|^{2\sigma} \, dx \right)^{1/2}} \leq \frac{\left( \int_{\mathbb{R}^n} t^{1-2\sigma} |\nabla \eta|^2 \, dX \right)^{1/2} C(n, \sigma, A_1)^{1/2\sigma}}{\left( \int_{\mathbb{R}^n} t^{1-2\sigma} |\nabla \eta|^2 \, dX + o(1) \right)^{1/2\sigma}}.
\]

Thus,

\[
S_{n,\sigma} \leq C(n, \sigma, A_1)^{1/2\sigma} \|\eta\|_{\sigma}^{2\sigma/n}.
\]

However, (5.60) and (5.63) cannot hold at the same time if \(\varepsilon_2 > 0\) is small enough. Claim 8 has been established.

Rewriting (5.36) and (5.53), we have

\[
U_1 = \mathcal{F}_{l,\lambda_2^\alpha,\varepsilon_2}^{-1} U_1 + \mathcal{F}_{l,\lambda_2^\alpha,\varepsilon_2}^{-1} U_2 + \eta.
\]

Claim 9. For \(\varepsilon_2 > 0\) small enough, we have

\[
(\lambda_1^\alpha)^{\tau_1} = 1 + o_{\varepsilon_3}(1) + o(1), \quad (\lambda_2^\alpha)^{\tau_1} = 1 + o_{\varepsilon_3}(1) + o(1).
\]

Proof. We deduce from (5.40), (5.41), and Lemma 5.2 that

\[
I_{K_{l,\tau_1}}(U_1) \geq I_{K_{l,\tau_1}}(\xi_l) + (\lambda_1^{4\sigma/(p_1-1)+2\sigma-n} \varepsilon_c + o(1).
\]

Combining (5.57), (5.59), and Lemma 5.2, we are led to

\[
I_{K_{l,\tau_1}}(\xi_l) \geq I_{K_{l,\tau_1}}(\eta_l) + (\lambda_2^{4\sigma/(p_1-1)+2\sigma-n} c(2) + o(1).
\]

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Then we use Claim 8 to deduce that

\[ I_{K_l, \eta_l}(\eta_l) = o(1). \] (5.67)

Finally, we put together (5.3), (5.65)–(5.67) to obtain

\[ \sum_{i=1}^{2} \{ (\lambda_i^1)^{4\sigma/(p_l-1)+2\sigma-n} - 1 \} c(i) \leq \varepsilon_3 + o(1). \]

This completes the proof of Claim 9.

**Claim 10.** Let \( \delta_5 = \delta_1/(2A_3) \). Then if \( \varepsilon_2 > 0 \) is chosen to be small enough, we have, for large \( l \), that

\[ \text{dist}(z_1^l, \partial O_l^{(1)}) \geq \delta_5, \quad \text{dist}(z_2^l, \partial O_l^{(2)}) \geq \delta_5. \]

**Proof.** Suppose the contrary, that for a sequence of \( l \to \infty \), we may assume \( \text{dist}(z_1^l, \partial O_l^{(1)}) < \delta_5 \). In the following, we shall estimate the value of the \( I_{K_l, \eta_l}(U_l) \), which will allow us to reach a contradiction.

To compute the value of \( I_{K_l, \eta_l}(U_l) \), we first observe from (1.13), (5.5)–(5.8), Claim 1, Claim 2 and Claim 5 that

\[
I_{K_l, \eta_l}(U_l) = I_{K_l, \eta_l}(\alpha_1^l \hat{\delta}(z_1^l, \lambda_1^l)) + I_{K_l, \eta_l}(\alpha_2^l \hat{\delta}(z_2^l, \lambda_2^l)) + o_{\varepsilon_2}(1) \\
= I_{K_l, \eta_l}(K_l(z_1^l)^{(2\sigma-n)/4\sigma \hat{\delta}(z_1^l, \lambda_1^l)}) \\
+ I_{K_l, \eta_l}(K_l(z_2^l)^{(2\sigma-n)/4\sigma \hat{\delta}(z_2^l, \lambda_2^l)}) + o_{\varepsilon_2}(1) \\
= I_{K_l}(K_l(z_1^l)^{(2\sigma-n)/4\sigma}) + I_{K_l}(K_l(z_2^l)^{(2\sigma-n)/4\sigma}) + o_{\varepsilon_2}(1) + o(1) \\
= \sum_{i=1}^{2} K_l(z_i^l)^{(2\sigma-n)/2\sigma} n(S_{n, \sigma})^{\sigma/n} + o_{\varepsilon_2}(1) + o(1).
\]

An application of (4.9) and (4.10) shows

\[ K_l(z_1^l) \leq K_l(z_1^{(1)}) - \delta_1 + A_3 \delta_5 = K_l(z_1^{(1)}) - \delta_1/2. \] (5.68)

Therefore,

\[
I_{K_l, \eta_l}(U_l) \geq (K_l(z_1^{(1)}) - \delta_1/2)^{(2\sigma-n)/2\sigma} n(S_{n, \sigma})^{\sigma/n} \\
+ (K_l(z_1^{(2)})^{(2\sigma-n)/2\sigma} n(S_{n, \sigma})^{\sigma/n} + o_{\varepsilon_2}(1) + o(1) \\
= (a^{(1)} - \delta_1/2)^{(2\sigma-n)/2\sigma} n(S_{n, \sigma})^{\sigma/n} \\
+ (a^{(2)})^{(2\sigma-n)/2\sigma} n(S_{n, \sigma})^{\sigma/n} + o_{\varepsilon_2}(1) + o(1) \\
= c^{(1)} \left( \frac{a^{(1)}}{a^{(1)} - \delta_1/2} \right)^{(n-2\sigma)/2\sigma} + c^{(2)} + o_{\varepsilon_2}(1) + o(1) \\
> c^{(1)} + c^{(2)} + o_{\varepsilon_2}(1) + o(1) \] (5.69)

by utilizing (4.4), (4.5), (4.14), (5.6) and (5.68). However, if \( \varepsilon_2 > 0 \) is small enough and \( l \) large enough, (5.69) contradicts to (5.3). This completes the proof of Claim 10. \( \square \)
We are now in the position to prove Proposition 5.1.

Proof of Proposition 5.1. Applying (4.6), (4.10), (5.29), (5.32), and Claim 9, we deduce that

\[ \mathcal{T}_{l,\lambda_i,\varepsilon}^{-1} U_1 = (\lambda_1^l)^{2\sigma/(\nu-1)} U_1 (\lambda_1^l (x - z_1^l), \lambda_1^l) \]

\[ = (\lambda_1^l)^{2\sigma/(\nu-1)-(n-2\sigma)/2\sigma} K^{(1)}_\infty (\xi_1)^{(2\sigma-n)/4\sigma} \delta (z_1^l + z^*/\lambda_1^l, \lambda^* \lambda_1^l) \]

\[ = K^{(1)}_\infty (\xi_1)^{(2\sigma-n)/4\sigma} \delta (z_1^l + z^*/\lambda_1^l, \lambda^* \lambda_1^l) + o_{\varepsilon_3} (1) \]

Similarly, we have

\[ \mathcal{T}_{l,\lambda_i,\varepsilon}^{-1} U_2 = K_l (z_2^l + z^{**}/\lambda_2^l)^{(2\sigma-n)/4\sigma} \delta (z_2^l + z^{**}/\lambda_2^l, \lambda^{**} \lambda_2^l) + o_{\varepsilon_3} (1) + o(1). \]

Therefore, we can rewrite (5.64) as (see Claim 8 and the above)

\[ U_l = K_l (z_1^l + z^*/\lambda_1^l)^{(2\sigma-n)/4\sigma} \delta (z_1^l + z^*/\lambda_1^l, \lambda^* \lambda_1^l) \]

\[ + K_l (z_2^l + z^{**}/\lambda_2^l)^{(2\sigma-n)/4\sigma} \delta (z_2^l + z^{**}/\lambda_2^l, \lambda^{**} \lambda_2^l) + o_{\varepsilon_3} (1) + o(1). \]

We now fix the value of \( \varepsilon_2 \) to be small to make all the previous arguments hold and then make \( \varepsilon_3 > 0 \) small (depending on \( \varepsilon_2 > 0 \)) to make the following hold (using Claim 9):

\[ |(\lambda^* \lambda_1^l)^n - 1| \leq o_{\varepsilon_3} (1) + o(1) < \varepsilon_2/2, \]

\[ |(\lambda^{**} \lambda_2^l)^n - 1| \leq o_{\varepsilon_3} (1) + o(1) < \varepsilon_2/2. \]

From (5.70), (5.71), Claim 1 and Claim 9, we see that for \( \varepsilon_3 > 0 \) small, we have, for large \( l \),

\[ U_l \in \widetilde{V}_l (2, \varepsilon_2/2), \]

which contradicts to (5.2). This concludes the proof of Proposition 5.1.

\[ \square \]

6 Complete the proof of Theorem 4.1

In this section we will complete the proof of Theorem 4.1. Precisely, under the contrary of Theorem 4.1 and combining with the Proposition 5.1 established in Section 5, we will reach a contradiction after a lengthy indirect argument. The method we shall use is similar to that in [44], see also [24–26, 59], but we have to set up a framework to fit the fractional situation. The basic idea is as follows: Given finitely many solutions (at low energy), to translate their supports far apart and patch the pieces together create many multi-bump solutions. The authors [24–26, 59] have introduced the original and powerful ideas which permit the construction of such solutions via variational methods. In particular, they are able to find many homoclinic-type solutions to periodic Hamiltonian systems (see [25, 59]) and to certain elliptic equations of nonlinear Schrödinger type on \( \mathbb{R}^n \) with periodic coefficients (see [26]). Li has given a slight modification to the minimax procedure in [25, 26] and has applied it to certain problems where periodicity is not present, for example, the problem of prescribing scalar curvature on \( S^n \) (see [44–46]).

Inspired from the above, we modify the techniques of [24–26, 44, 59] to fit variational problems with nonlocal terms. To reduce overlaps, we will omit the proofs of several intermediate results which closely follow standard arguments, giving appropriate references. Let us start with defining a certain family of sets and minimax values and giving some notation.
For any \( z \in \mathbb{R}^n \), we define the space \( \mathcal{H}_0^1(t^{1-2\sigma}, \Sigma_R^+(z)) \) as the closure in \( H^1(t^{1-2\sigma}, B_R^+(z)) \) of \( C_c^\infty(\Sigma_R^+(z)) \) under the norm (2.1). It follows from the Hardy–Sobolev inequality in [29, Lemma 2.4] that \( \mathcal{H}_0^1(t^{1-2\sigma}, \Sigma_R^+(z)) \) can be endowed with the equivalent norms

\[
\|U\|_{\mathcal{H}_0^1(t^{1-2\sigma}, \Sigma_R^+(z))} := \left( \int_{B_R^+(z)} t^{1-2\sigma} |\nabla U|^2 \, dX \right)^{1/2}.
\]

In this section, we write \( \tau = \tau_l \), \( p = p_l \), and \( \mathcal{H}_0^1(t^{1-2\sigma}, \Sigma_R^+(z)) = \mathcal{H}_0^1(\Sigma_R^+(z)) \) to simplify the notation.

Now, we define

\[
\gamma_{l, \tau}^{(1)} = \{g^{(1)} \in C([0, 1], \mathcal{H}_0^1(\Sigma_{R_l}^+(z_l^{(1)}))) : g^{(1)}(0) = 0, I_{K_l, \tau}(g^{(1)}(1)) < 0 \},
\]

\[
\gamma_{l, \tau}^{(2)} = \{g^{(2)} \in C([0, 1], \mathcal{H}_0^1(\Sigma_{R_l}^+(z_l^{(2)}))) : g^{(2)}(0) = 0, I_{K_l, \tau}(g^{(2)}(1)) < 0 \},
\]

\[
c_{l, \tau}^{(1)} = \inf_{g^{(1)} \in \gamma_{l, \tau}^{(1)}} \max_{0 \leq \theta_1 \leq 1} I_{K_l, \tau}(g^{(1)}(\theta_1)),
\]

\[
c_{l, \tau}^{(2)} = \inf_{g^{(2)} \in \gamma_{l, \tau}^{(2)}} \max_{0 \leq \theta_2 \leq 1} I_{K_l, \tau}(g^{(2)}(\theta_2)).
\]

We have abused the notation a little by writing \( I_{K_l, \tau} \) as \( I_{K_l, \tau} : \mathcal{H}_0^1(\Sigma_{R_l}^+(z_l^{(1)})) \to \mathbb{R} \) and also \( I_{K_l, \tau} : \mathcal{H}_0^1(\Sigma_{R_l}^+(z_l^{(2)})) \to \mathbb{R} \).

**Proposition 6.1.** Let \( \{K_l\} \) be a sequence of functions satisfying (4.2), (4.4) and (4.5). Then there holds

\[
c_{l, \tau}^{(1)} = o(1), \quad (6.1)
\]

\[
c_{l, \tau}^{(2)} = o(1), \quad (6.2)
\]

where \( o(1) \to 0 \) as \( l \to \infty \).

**Proof.** We will only prove (6.1), because (6.2) can be justified in a similar manner.

From the definition of \( c_{l, \tau}^{(1)} \), we deduce that

\[
C^{-1}(n, \sigma, A_1) \leq c_{l, \tau}^{(1)} \leq C(n, \sigma, A_1) \quad (6.3)
\]

for some constant \( C(n, \sigma, A_1) > 0 \). Moreover, for any \( U \in \mathcal{H}_0^1(\Sigma_{R_l}^+(z_l^{(1)})) \setminus \{0\} \), one has

\[
c_{l, \tau}^{(1)} \leq \max_{0 \leq s < \infty} I_{K_l, \tau}(sU)
\]

\[
= \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{\left( \int_{B_R^+(z_l^{(1)})} t^{1-2\sigma} |\nabla U|^2 \, dX \right)^{(p+1)/(p-1)}}{\left( \int_{B_{R_l}^+(z_l^{(1)})} K_l(x)H^\tau(x) |U(x, 0)|^{p+1} \, dx \right)^{2/(p-1)}} \right).
\]

Let \( U = \eta(x + z_l^{(1)} + t)\tilde{\delta}^+(z_l^{(1)}, \lambda_l) \), where \( \eta \) is a nonnegative smooth cut-off function supported in \( B_1 \) and equal to 1 in \( B_{1/2} \), and \( \{\lambda_l\} \) is a sequence satisfying

\[
\lim_{l \to \infty} \lambda_l = \infty, \quad (\lambda_l)^\tau = 1 + o(1).
\]

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Then we obtain
\[ c^{(1)}_{l,\tau} \leq \frac{\sigma}{n}(a^{(1)})^{(2\sigma-n)/2\sigma}(S_{n,\sigma})^{n/\sigma} + o(1) \]
\[ = c^{(1)} + o(1). \]

The other side of the inequality can be proved as the following.

For any \( l, \tau \) fixed, it is well-known that there exists \( \{U_k\} \subset \mathcal{H}^1_0(\Sigma^+_R(z^{(1)}_l)) \) such that
\[ \lim_{k \to \infty} I_{K_l,\tau}(U_k) = c^{(1)}_{l,\tau}, \]
\[ I'_{K_l,\tau}(U_k) \to 0 \quad \text{in} \quad H^{-\sigma}(\Sigma^+_R(z^{(1)}_l)) \quad \text{as} \quad k \to \infty, \]
where \( H^{-\sigma}(\Sigma^+_R(z^{(1)}_l)) \) denotes the dual space of \( \mathcal{H}^1_0(\Sigma^+_R(z^{(1)}_l)) \). Namely, we have
\[ \frac{1}{2} \int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX - \frac{1}{p+1} \int_{B_R(z^{(1)}_l)} K_l(x) H^\tau(x)|U_k(x,0)|^{p+1} \, dx = c^{(1)}_{l,\tau} + o_k(1), \quad (6.5) \]
\[ \int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX = \int_{B_R(z^{(1)}_l)} K_l(x) H^\tau(x)|U_k(x,0)|^{p+1} \, dx + o_k(1), \quad (6.6) \]
where \( o_k(1) \to 0 \) as \( k \to \infty \). It follows that
\[ c^{(1)}_{l,\tau} = \frac{\sigma}{n} \int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX + o_k(1) + o(1). \quad (6.7) \]

By (1.12), (4.14), (6.3), (6.5) and (6.6), we have
\[ S_{n,\sigma} \leq \left( \frac{\int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX}{\int_{B_R(z^{(1)}_l)} |U_k(x,0)|^{2\sigma} \, dx} \right)^{1/2\sigma} \]
\[ \leq \frac{\left( \int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX \right)^{1/2}}{(1 + o(1)) \left( \int_{B_R(z^{(1)}_l)} |U_k(x,0)|^{p+1} \, dx \right)^{1/(p+1)}} \]
\[ \leq \frac{\left( \int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX \right)^{1/2}}{(1 + o(1)) \left( \int_{B_R(z^{(1)}_l)} K_l(x) H^\tau(x)|U_k(x,0)|^{p+1} \, dx \right)^{1/(p+1)}} \]
\[ = \left( \int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX \right)^{1/2} \frac{\sigma}{n} K_l(z^{(1)}_l)^{1/2\sigma} + o_k(1) + o(1), \]

namely,
\[ \lim_{k \to \infty} \int_{B^{+}_R(z^{(1)}_l)} t^{1-2\sigma} |\nabla U_k|^2 \, dX \geq K_l(z^{(1)}_l)^{(2\sigma-n)/2\sigma}(S_{n,\sigma})^{n/\sigma} + o(1). \quad (6.8) \]

We can deduce from (4.5), (6.7) and (6.8) that
\[ c^{(1)}_{l,\tau} \geq \frac{\sigma}{n}(a^{(1)})^{(2\sigma-n)/2\sigma}(S_{n,\sigma})^{n/\sigma} + o(1) = c^{(1)} + o(1). \]

This gives the proof of (6.1). \( \square \)
We define
\[
\Gamma_l = \{ G = \mathcal{G}^{(1)} + \mathcal{G}^{(2)} : \mathcal{G}^{(1)}, \mathcal{G}^{(2)} \text{ satisfy } (6.10) - (6.14) \},
\] (6.9)
\[
\mathcal{G}^{(1)}, \mathcal{G}^{(2)} \in C([0, 1]^2, D),
\] (6.10)
\[
\mathcal{G}^{(1)}(0, \theta_2) = \mathcal{G}^{(2)}(\theta_1, 0) = 0, \quad 0 \leq \theta_1, \theta_2 \leq 1,
\] (6.11)
\[
I_{K_i,\tau}(\mathcal{G}^{(1)}(1, \theta_2)) < 0, \quad I_{K_i,\tau}(\mathcal{G}^{(2)}(\theta_1, 1)) < 0, \quad 0 \leq \theta_1, \theta_2 \leq 1,
\] (6.12)
\[
supp \mathcal{G}^{(1)} \subset \mathcal{B}^{+}_{K_i}(\gamma^{(1)}_l), \quad \theta = (\theta_1, \theta_2) \in [0, 1]^2,
\] (6.13)
\[
supp \mathcal{G}^{(2)} \subset \mathcal{B}^{+}_{K_i}(\gamma^{(2)}_l), \quad \theta = (\theta_1, \theta_2) \in [0, 1]^2,
\] (6.14)
\[
\nu_{i,\tau} = \inf_{G \in \Gamma_l} \max_{\theta \in [0, 1]^2} I_{K_i,\tau}(G(\theta)).
\] (6.15)

**Remark 6.1.** Observe that if \( G = g^{(1)} + g^{(2)} \) with \( g^{(1)} \in \gamma^{(1)}_{l,\tau}, \ g^{(2)} \in \gamma^{(2)}_{l,\tau} \) and \( supp g^{(1)} \cap supp g^{(2)} = \emptyset \), then \( I_{K_i,\tau}(G) = I_{K_i,\tau}(g^{(1)}) + I_{K_i,\tau}(g^{(2)}) \).

**Proposition 6.2.** Suppose that \( \{ K_i \} \) is a sequence of functions satisfying (4.2), (4.4) and (4.5), then there holds \( \nu_{i,\tau} = c^{(1)}_{l,\tau} + c^{(2)}_{l,\tau} + o(1) \).

**Proof.** The first inequality \( \nu_{i,\tau} \geq c^{(1)}_{l,\tau} + c^{(2)}_{l,\tau} \) can be achieved from the definition of \( c^{(1)}_{l,\tau} \) and \( c^{(2)}_{l,\tau} \) with additional compactness argument on \([0, 1]^2\), we omit it here and refer to [25, Proposition 3.4] for details.

On the other hand, for \( 0 \leq \theta_1, \theta_2 \leq 1 \), let
\[
g^{(1)}_l(\theta_1) = \theta_1 C_1 K_i(z^{(1)}_l)^{(2\sigma - n)/4\sigma} \eta(x + z^{(1)}_l, t) \delta(z^{(1)}_l, \lambda_l),
\]
\[
g^{(2)}_l(\theta_2) = \theta_2 C_1 K_i(z^{(2)}_l)^{(2\sigma - n)/4\sigma} \eta(x + z^{(2)}_l, t) \delta(z^{(2)}_l, \lambda_l),
\]
where \( \lambda_l \) is defined in (6.4) and \( C_1 = C_1(n, \sigma, A_1, A_2) > 1 \) is a constant such that
\[
I_{K_i,\tau}(g^{(1)}_l(\theta_1)) < 0 \quad \text{and} \quad I_{K_i,\tau}(g^{(2)}_l(\theta_2)) < 0
\]
for large \( l \). We fix the value of \( C_1 \) from now on.

For \( \theta = (\theta_1, \theta_2) \in [0, 1]^2 \), let \( G_l(\theta) = g^{(1)}_l(\theta_1) + g^{(2)}_l(\theta_2) \). Observe that \( supp g^{(1)}_l(\theta_1) \cap supp g^{(2)}_l(\theta_2) = \emptyset \), a direct calculation shows that
\[
\max_{\theta \in [0, 1]^2} I_{K_i,\tau}(G_l(\theta)) = \max_{\theta_1 \in [0, 1]} I_{K_i,\tau}(g^{(1)}_l(\theta_1)) + \max_{\theta_2 \in [0, 1]} I_{K_i,\tau}(g^{(2)}_l(\theta_2)) + o(1)
\]
\[
\leq \max_{0 \leq s < \infty} I_{K_i,\tau}(s \eta(x + z^{(1)}_l, t) \delta(z^{(1)}_l, \lambda_l))
\]
\[
+ \max_{0 \leq s < \infty} I_{K_i,\tau}(s \eta(x + z^{(2)}_l, t) \delta(z^{(2)}_l, \lambda_l)) + o(1)
\]
\[
= \frac{\sigma}{n} \left( a^{(1)}_l \right)^{(2\sigma - n)/2\sigma} \left( S_{n,\sigma} \right)^{n/\sigma}
\]
\[
+ \frac{\sigma}{n} \left( a^{(2)}_l \right)^{(2\sigma - n)/2\sigma} \left( S_{n,\sigma} \right)^{n/\sigma} + o(1)
\]
\[
= c^{(1)}_{l,\tau} + c^{(2)}_{l,\tau} + o(1),
\]
where the last equality is due to Proposition 6.1. Therefore, \( \nu_{i,\tau} \leq c^{(1)}_{l,\tau} + c^{(2)}_{l,\tau} + o(1) \). This ends the proof. \(\square\)
In the following, under the contrary of Theorem 4.1, we can construct $H_l \in \Gamma_l$ for large $l$, such that

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(H_l(\theta)) < b_{l,\tau},$$

which contradicts to the definition of $b_{l,\tau}$. A lengthy construction is required to establish this fact and a brief sketch of it will be given now by the details.

**Step 1:** Choosing some suitably small number $\varepsilon_4 > 0$, we can construct $G_l \in \Gamma_l$ such that

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(G_l(\theta)) \leq b_{l,\tau} + \varepsilon_4,$$

and satisfies some further properties.

**Step 2:** By the negative gradient flow of $I_{K_l,\tau}$, $G_l$ is deformed to $U_l$ such that

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(U_l(\theta)) \leq b_{l,\tau} - \varepsilon_4.$$

If $U_l \in \Gamma_l$, we will reach a contradiction to the definition of $b_{l,\tau}$. However, $U_l$ is not necessarily in $\Gamma_l$ any more since the deformation may not preserve properties (6.13)-(6.14).

**Step 3:** Applying Propositions 2.4, 3.1 and 5.1, we modify $U_l$ to obtain $H_l \in \Gamma_l$ such that

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(H_l(\theta)) \leq b_{l,\tau} - \varepsilon_4/2.$$

All the three steps are completed for large $l$ only. Now we give the details to establish these steps.

**Step 1:** Construction of $G_l$. Let $G_l$ be the one we have just defined. We establish some properties of it which are needed.

**Lemma 6.1.** For any $\varepsilon \in (0,1)$, if $I_{K_l,\tau}(g_l^{(i)}(\theta_l)) \geq c_{l,\tau}^{(i)} - \varepsilon$ for $i = 1, 2$, then there exists some constants $\Lambda_1 = \Lambda_1(n, \sigma, \varepsilon, A_1, A_3) > 1$ and $C_0(n, \sigma) > 0$ such that for any $l \geq \Lambda_1$ and $0 \leq \theta_l, \theta_2 \leq 1$, we have $|C_1\theta_l - 1| \leq C_0(n, \sigma)\sqrt{\varepsilon}$, $i = 1, 2$.

**Proof.** We only take into account the case $i = 1$ since the other case can be covered in the same way. Let $s_1 = C_1\theta_l$, a direct calculation shows that

$$I_{K_l,\tau}(g_l^{(1)}(\theta_l))$$

$$= \frac{1}{2} s_1^2 K_l(z_l^{(1)})^{(2\sigma-n)/2\sigma} \|\eta(x + z_l^{(1)}, \lambda_l)\|^2_{2\sigma}$$

$$- \frac{1}{p + 1} s_1^{p+1} K_l(z_l^{(1)})^{(p+1)(2\sigma-n)/4\sigma}$$

$$\times \int_{\mathbb{R}^n} K_l(x)H^\tau(x)\|\eta(x + z_l^{(1)}, 0)\|_{p+1}^2 \, dx$$

$$= \left(\frac{1}{2} + o(1)\right) s_1^2 K_l(z_l^{(1)})^{(2\sigma-n)/2\sigma} \|\delta(0,1)\|^2_{2\sigma}$$

$$- \left(\frac{1}{p + 1} + o(1)\right) s_1^{p+1} K_l(z_l^{(1)})^{(2\sigma-n)/2\sigma} \|\delta(0,1)\|^2_{2\sigma}$$

$$= \left[\frac{n}{2\sigma} + o(1)\right] s_1^2 - \left(\frac{n-2\sigma}{2\sigma} + o(1)\right) s_1^{p+1} c_{l,\tau}^{(1)},$$

(6.16)

where Proposition 6.2 is used in the last step. Hence, using (6.16) and the hypothesis $I_{K_l,\tau}(g_l^{(1)}(\theta_l)) \geq c_{l,\tau}^{(1)} - \varepsilon$, we complete the proof. \qed
Lemma 6.2. For any \( \varepsilon \in (0, 1) \), there exists a constant \( \Lambda_2 = \Lambda_2(n, \sigma, \varepsilon, A_1, A_3) > \Lambda_1 \) such that for any \( l \geq \Lambda_2 \) and \( 0 \leq \theta_1, \theta_2 \leq 1 \), we have \( I_{K_l, \tau}(g_l^{(i)}(\theta)) \leq c_l^{(i)} + \varepsilon/10, \ i = 1, 2 \).

Proof. The proof is similar to Proposition 6.2. \( \square \)

Lemma 6.3. For any \( \varepsilon \in (0, 1) \), there exists a constant \( \Lambda_3 = \Lambda_3(n, \sigma, \varepsilon, A_1, A_3) > \Lambda_2 \) such that

\[
I_{K_l, \tau}(G_l(\theta)) \bigg|_{\theta \in \Theta_0[0,1]^2} \leq \max\{c_l^{(1)} + \varepsilon, c_l^{(2)} + \varepsilon\} \quad \text{for all } l \geq \Lambda_3.
\]

Proof. Lemma 6.3 follows immediately from Lemma 6.2. \( \square \)

Lemma 6.4. For any \( \varepsilon \in (0, 1/2) \), if \( I_{K_l, \tau}(G_l(\theta)) \geq c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon \), then there exists a constant \( C_0 = C_0(n, \sigma) > 1 \) such that for any \( l \geq \Lambda_3 \) and \( \theta \in [0,1]^2 \), we have \( |C_l \theta_i - 1| \leq C_0 \sqrt{\varepsilon}, i = 1, 2 \).

Proof. Since \( g_l^{(1)}(\theta_1) \) and \( g_l^{(2)}(\theta_2) \) have disjoint supports, after a direct calculation, we see that

\[
I_{K_l, \tau}(G_l(\theta)) = I_{K_l, \tau}(g_l^{(1)}(\theta_1)) + I_{K_l, \tau}(g_l^{(2)}(\theta_2)) + o(1).
\]

Then it follows from Lemma 6.2 and the hypothesis \( I_{K_l, \tau}(G_l(\theta)) \geq c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon \) that

\[
I_{K_l, \tau}(g_l^{(1)}(\theta_1)) \geq c_{l, \tau}^{(1)} - 2\varepsilon \quad \text{and} \quad I_{K_l, \tau}(g_l^{(2)}(\theta_2)) \geq c_{l, \tau}^{(2)} - 2\varepsilon. \tag{6.17}
\]

Lemma 6.4 follows immediately from (6.17) and Lemma 6.1. \( \square \)

Step 2: The deformation of \( G_l \). Let

\[
M_l = \sup \left\{ \|I_{K_l, \tau}(U)\| : U \in V_l(2, \varepsilon_1) \right\},
\]

\[
\beta_l = \text{dist} \left( \partial V_l(2, \varepsilon_2), \partial \widetilde{V}_l(2, \varepsilon_2/2) \right).
\]

One can see from the definition of \( M_l \) that there exists some constant \( C_2(n, \sigma, \varepsilon_2, A_1) > 1 \) such that

\[
M_l \leq C_2(n, \sigma, \varepsilon_2, A_1). \tag{6.18}
\]

It is also clear from the definition of \( \widetilde{V}_l(2, \varepsilon_2) \) that

\[
\beta_l \geq \frac{\varepsilon_2}{4}. \tag{6.19}
\]

By Lemma 6.4, we choose \( \varepsilon_4 \) to satisfy, for \( l \) large, that

\[
\varepsilon_4 < \min \left\{ \varepsilon_3, \frac{1}{2A_4}, \frac{\varepsilon_2\delta_4(\varepsilon_2, \varepsilon_3)^2}{8C_2(n, \sigma, \varepsilon_2, A_1)} \right\}. \tag{6.20}
\]

\[
I_{K_l, \tau}(G_l(\theta)) \geq c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4 \quad \text{implies that}
\]

\[
G_l(\theta) \in \widetilde{V}_l(2, \varepsilon_2/2), \ z_1(G_l(\theta)) \in O_l^{(1)}, \ z_2(G_l(\theta)) \in O_l^{(2)}. \tag{6.21}
\]

\( G_l(\theta) \) has been defined by now. We know from Lemma 6.2 that for \( l \) large enough,

\[
\max_{\theta \in [0,1]^2} I_{K_l, \tau}(G_l(\theta)) \leq c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} + \varepsilon_4.
\]
For any \( U_0 \in \tilde{V}_l(2, \varepsilon_2/2) \), we consider the negative gradient flow of \( I_{K_l, \tau} \),

\[
\begin{align*}
\frac{d}{ds} \xi(s, U_0) &= -I'_{K_l, \tau}(\xi(s, U_0)), \quad s \geq 0, \\
\xi(0, U_0) &= U_0.
\end{align*}
\]  

(6.22)

Under the contrary of Theorem 4.1, we know \( I_{K_l, \tau} \) satisfies the Palais-Smale condition. Furthermore, the flow defined above never stops before exiting \( \tilde{V}_l(2, \varepsilon^*) \).

We define \( U_l \in C([0, 1]^2, D) \) by the following.

- If \( I_{K_l, \tau}(G_l(\theta)) \leq c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4 \), we define \( s^*_l(\theta) = 0 \).

- If \( I_{K_l, \tau}(G_l(\theta)) > c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4 \), then according to (6.21), \( G_l(\theta) \in \tilde{V}_l(2, \varepsilon_2/2) \), \( z_1(G_l(\theta)) \in O_l^{(1)} \) and \( z_2(G_l(\theta)) \in O_l^{(2)} \). We define

\[
s^*_l(\theta) = \min\{s > 0 : I_{K_l, \tau}(\xi(s, G_l(\theta))) = c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4 \}.
\]

Now we set

\[
U_l(\theta) = \xi(s^*_l(\theta), G_l(\theta)).
\]

The existence and continuity of \( s^*_l(\theta) \) is guaranteed by the following lemma.

**Lemma 6.5.** For any \( U_0 \in \tilde{V}_l(2, \varepsilon_2/2) \) with \( z_1(U_0) \in O_l^{(1)} \), \( z_2(U_0) \in O_l^{(2)} \), and \( I_{K_l, \tau}(U_0) \in (c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4, c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} + \varepsilon_4) \), the flow line \( \xi(s, U_0) \) (s \geq 0) cannot leave \( \tilde{V}_l(2, \varepsilon_2) \) before reaching \( I_{K_l, \tau}^{-1}(c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4) \).

**Proof.** The proof can be done exactly in the same way as in [6, Lemma 5], so we omit it. \(\square\)

Lemma 6.5 is a local version of Lemma 5 in [6] since we only need the compactness property of the flow line in certain region and Proposition 5.1 has provided control of \( \|U_l'_{K_l, \tau}\| \) in the region.

We can see from Lemma 6.5 that \( s^*_l(\theta) \) is well defined. Since \( I_{K_l, \tau} \) has no critical point in \( \tilde{V}_l(2, \varepsilon_2) \cap \{U \in D : |I_{K_l, \tau}(U) - c^{(1)} - c^{(2)}| \leq \varepsilon_4\} \subset V_l(2, \varepsilon^*) \cap \{U \in D : |I_{K_l, \tau}(U) - c^{(1)} - c^{(2)}| \leq \varepsilon^*\} \) under the contradiction hypothesis, \( s^*_l(\theta) \) is continuous in \( \theta \) (see also, [45, Proposition 5.11] and [6, Lemma 5]), hence \( U_l \in C([0, 1]^2, D) \).

**Step 3: The construction of \( H_l \).** It follows from the construction of \( U_l \) that

\[
\max_{\theta \in [0, 1]^2} I_{K_l, \tau}(U_l(\theta)) \leq c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4.
\]  

(6.23)

Since the gradient flow does not keep properties (6.13)-(6.14), \( U_l(\theta) \) is not necessarily in \( \Gamma_l \) any more. It follows from Lemma 6.5 that if \( I_{K_l, \tau}(G_l(\theta)) > c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4 \), then the gradient flow \( \xi(s, G_l(\theta)) \) (s \geq 0) cannot leave \( \tilde{V}_l(2, \varepsilon_2) \) before reaching \( I_{K_l, \tau}^{-1}(c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4) \).

Using (6.21) and the above information we know that if \( I_{K_l, \tau}(G_l(\theta)) > c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4 \), then \( U_l(\theta) \in \tilde{V}_l(2, \varepsilon_2) \subset V_l(2, o_{\varepsilon_2}(1)) \) with \( z_1(U_l(\theta)) \in O_l^{(1)} \) and \( z_2(U_l(\theta)) \in O_l^{(2)} \). This implies that

\[
\int_{\Omega_l} t^{-1-2\sigma} |\nabla U_l(\theta)|^2 dX + \int_{\partial\Omega_l} |U_l(\theta)|^{2\sigma} dX = o_{\varepsilon_2}(1),
\]  

(6.24)

and

\[
\|U_l(\theta)\|_{W^{\sigma, 2}(\partial\Omega_l)} = o_{\varepsilon_2}(1),
\]  

(6.25)
where

\[ \Omega_l = \mathbb{R}^{n+1}_+ \backslash \{ B_r^+(z_{l}^{(1)}) \cup B_r^+(z_{l}^{(2)}) \}, \]

\[ r = 4(\text{diam } O^{(1)} + \text{diam } O^{(2)}), \]

\[ \text{diam } O^{(1)} = \text{sup}\{|x - y| : x, y \in O^{(1)}\}, \]

\[ \text{diam } O^{(2)} = \text{sup}\{|x - y| : x, y \in O^{(2)}\}. \]

By Proposition 3.1 (\( \varepsilon_2 > 0 \) sufficiently small), we can modify \( U_l(\theta) \) in \( \Omega_l \) after making the following minimization.

Let

\[ \varphi_l(\theta) = U_l(\theta)|_{\partial^a \Omega_l}. \]

Thanks to (6.24)-(6.25), we can apply Proposition 3.1 to obtain the minimizer \( U_{\varphi_l}(\theta) \)

\[ \min \left\{ I_{K_l, \Omega_l}(U) : U \in D_{\Omega_l}, U|_{\partial^a \Omega_l} = \varphi_l(\theta), \int_{\Omega_l} t^{1-2\gamma} |\nabla U|^2 \, dX \leq C_1 r_0^2 \right\}, \]

where \( D_{\Omega_l} \) is the closure of \( C_c^\infty(\Omega_l) \) under the norm

\[ ||U||_{D_{\Omega_l}} := \left( \int_{\Omega_l} t^{1-2\gamma} |\nabla U|^2 \, dX \right)^{1/2} + \left( \int_{\partial^a \Omega_l} |U(x, 0)|^{2\gamma} \, dx \right)^{1/2\gamma}, \]

and \( C_1 \) and \( r_0 \) are the constants given by Proposition 3.1.

For \( \theta \in [0, 1]^2 \), we define

\[ W_l(\theta)(X) = \begin{cases} U_l(\theta)(X), & X \in B_r^+(z_{l}^{(1)}) \cup B_r^+(z_{l}^{(2)}), \\ U_{\varphi_l}(\theta)(X), & X \in \mathbb{R}^{n+1}_+ \backslash \{ B_r^+(z_{l}^{(1)}) \cup B_r^+(z_{l}^{(2)}) \} = \Omega_l. \end{cases} \]

It follows from Proposition 3.1 that \( U_l \in C([0, 1]^2, D) \) and satisfies

\[ \max_{\theta \in [0, 1]^2} I_{K_l, \tau}(W_l(\theta)) \leq \max_{\theta \in [0, 1]^2} I_{K_l, \tau}(U_l(\theta)) \leq c_{l, \tau}^{(1)} + c_{l, \tau}^{(2)} - \varepsilon_4, \]

(6.26)

\[ \int_{\Omega_l} t^{1-2\gamma} |\nabla W_l(\theta)|^2 \, dX + \int_{\partial^a \Omega_l} |W_l(\theta)|^{2\gamma} \, dx = o_{\varepsilon_2}(1), \]

(6.27)

\[ \begin{cases} \text{div}(t^{1-2\gamma} \nabla W_l(\theta)) = 0 & \text{in } \Omega_l, \\ \partial^a_{\gamma} W_l(\theta) = K_l(x) H^T(x) W_l(\theta) & \text{on } \partial^a \Omega_l. \end{cases} \]

(6.28)

In order to construct the required \( H_l \), we introduce the following notation. First we write

\[ \Omega_1^l := (B_{l_1}^+(z_{l_1}^{(1)}) \backslash B_r^+(z_{l_1}^{(1)})) \cup (B_{l_1}^+(z_{l_1}^{(2)}) \backslash B_r^+(z_{l_1}^{(2)})), \]

\[ \Omega_2^l := (B_{l_2}^+(z_{l_2}^{(1)}) \backslash B_{l_1}^+(z_{l_1}^{(1)})) \cup (B_{l_2}^+(z_{l_2}^{(2)}) \backslash B_{l_1}^+(z_{l_1}^{(2)})), \]

\[ \Omega_3^l := (\mathbb{R}^{n+1}_+ \backslash B_{l_2}^+(z_{l_2}^{(1)})) \cap (\mathbb{R}^{n+1}_+ \backslash B_{l_2}^+(z_{l_2}^{(2)})). \]

Clearly, \( \Omega_l = \Omega_1^l \cup \Omega_2^l \cup \Omega_3^l \) for large \( l \).
For $l_2 > 100l_1 > 1000r$ (we determine the values of $l_1, l_2$ at the end), we introduce cut-off functions $\eta_l \in C_c^\infty(\mathbb{R}^{n+1})$ for large $l$,

$$
\eta_l(x, t) = \begin{cases} 
1, & (x, t) \in B_{l_1}(z_l^{(1)}) \cup B_{l_1}(z_l^{(2)}), \\
0, & (x, t) \in (\mathbb{R}^{n+1} \setminus B_{l_2}(z_l^{(1)})) \cap (\mathbb{R}^{n+1} \setminus B_{l_2}(z_l^{(2)})), \\
0, & \text{elsewhere}, 
\end{cases}
$$

and set

$$
H_l(\theta) = \eta_l(x, t)W_l(\theta).
$$

Next, we will prove that $H_l(\theta) \in \Gamma_l$, but the energy of $H_l(\theta)$ contradicts to $b_{l, r}$.

Multiplying $(1 - \eta_l)W_l(\theta)$ on both sides of (6.28) and integrating by parts, we have

$$
\int_{\Omega_l} t^{1-2\sigma} \nabla((1 - \eta_l)W_l(\theta)) \nabla W_l(\theta) \, dX = \int_{\partial \Omega_l} K_l(x)H^\tau(x) (1 - \eta_l(x, 0)) |U_l(\theta)|^{p+1} \, dx.
$$

A direct computation shows that

$$
\begin{align*}
\int_{\Omega_l} t^{1-2\sigma} |\nabla W_l(\theta)|^2 \, dX &- \int_{\partial \Omega_l} K_l(x)H^\tau(x)|U_l(\theta)|^{p+1} \, dx \\
&= \int_{\Omega_l} t^{1-2\sigma} W_l(\theta) \nabla \eta_l \nabla W_l(\theta) \, dX \\
&\quad - \int_{\Omega_l} t^{1-2\sigma} (1 - \eta_l)|\nabla W_l(\theta)|^2 \, dX \\
&\quad + \int_{\partial \Omega_l} K_l(x)(1 - \eta_l(x, 0))H^\tau(x)|U_l(\theta)|^{p+1} \, dx \\
&\geq - \int_{\Omega_l} \left[ \frac{10}{l_2 - l_1} t^{1-2\sigma} |W_l(\theta)||\nabla W_l(\theta)| + t^{1-2\sigma} |\nabla W_l(\theta)|^2 \right] \, dX \\
&\quad - 2A_1 \int_{\partial \Omega_l} |U_l(\theta)|^{p+1} \, dx \\
&\geq - \int_{\Omega_l} \frac{10}{l_2 - l_1} t^{1-2\sigma} |W_l(\theta)||\nabla W_l(\theta)| \, dX - 4A_1 \int_{\partial \Omega_l} |U_l(\theta)|^{p+1} \, dx.
\end{align*}
$$

Then by Proposition 2.4, there exists some constant $C_3(n, \sigma, A_1) > 1$ such that for $l$ large enough, we have

$$
|W_l(\theta)| \leq \frac{C_3(n, \sigma, A_1)}{|(x - z_l^{(i)}, t)|^{n-2\sigma}},
$$

$$
|\nabla_x W_l(\theta)| \leq \frac{C_3(n, \sigma, A_1)}{|(x - z_l^{(i)}, t)|^{n+1-2\sigma}},
$$

$$
t^{1-2\sigma} |\partial_t W_l(\theta)| \leq \frac{C_3(n, \sigma, A_1)}{|(x - z_l^{(i)}, t)|^{n+1-2\sigma}},
$$

(6.29)

(6.30)

(6.31)
for all \((x, t) \in \Sigma_{i_t}^{+} (z_t^{(i)}) \setminus \Sigma_{i_t}^{-} (z_t^{(i)})\), \(i = 1, 2\). Consequently,

\[
\int_{\Omega_t^+} t^{1-2\sigma} |\nabla W_t(\theta)|^2 \text{d}X - \int_{\partial \Omega_t^+} K_t(x) H^T(x)|U_t(\theta)|^{p+1} \text{d}x \\
\geq \left\{ \begin{array}{ll}
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{1}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1} \right] & \text{if } 0 < \sigma \leq \frac{1}{2}, \\
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{l_2^{l_2-l_1+2+2\sigma}}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1} \right] & \text{if } \frac{1}{2} < \sigma < 1,
\end{array} \right.
\] (6.32)

Thanks to (6.29)–(6.31), [40, Lemma A.4] and a density argument, we see from (6.26) and (6.32) that

\[
I_{K_t, \tau}(H_t(\theta)) = \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla (\eta W_t(\theta))|^2 \text{d}X \\
- \frac{1}{p+1} \int_{\mathbb{R}^n \times \{0\}} K_t(x) H^T(x)|\eta(x, 0)W_t(\theta)|^{p+1} \text{d}x \\
= \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla \eta|^2 |W_t(\theta)|^2 \text{d}X \\
+ \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} \eta W_t(\theta) \nabla \eta \nabla W_t(\theta) \text{d}X \\
+ \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\eta|^2 |\nabla W_t(\theta)|^2 \text{d}X \\
- \frac{1}{p+1} \int_{\mathbb{R}^n \times \{0\}} K_t(x) H^T(x)|\eta(x, 0)W_t(\theta)|^{p+1} \text{d}x \\
\leq I_{K_t, \tau}(W_t(\theta)) - \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} (1 - |\eta|^2) |\nabla W_t(\theta)|^2 \text{d}X \\
+ \frac{1}{p+1} \int_{\mathbb{R}^n \times \{0\}} K_t(x) H^T(x)(1 - |\eta(x, 0)|^{p+1}) |W_t(\theta)|^{p+1} \text{d}x \\
+ \left\{ \begin{array}{ll}
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{1}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} \right] & \text{if } 0 < \sigma \leq \frac{1}{2}, \\
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{l_2^{l_2-l_1+2+2\sigma}}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} \right] & \text{if } \frac{1}{2} < \sigma < 1,
\end{array} \right.
\] 
\leq c_t^{(1)} + \varepsilon_4 - \varepsilon - \frac{1}{2} \int_{\Omega_t^+} t^{1-2\sigma} |\nabla W_t(\theta)|^2 \text{d}X \\
+ \frac{1}{p+1} \int_{\partial \Omega_t^+} K_t(x) H^T(x)|U_t(\theta)|^{p+1} \text{d}x \\
+ \left\{ \begin{array}{ll}
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{1}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} \right] & \text{if } 0 < \sigma \leq \frac{1}{2}, \\
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{l_2^{l_2-l_1+2+2\sigma}}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} \right] & \text{if } \frac{1}{2} < \sigma < 1,
\end{array} \right.
\] 
\leq c_t^{(1)} + \varepsilon_4 - \varepsilon - \frac{1}{2} \int_{\Omega_t^+} t^{1-2\sigma} |\nabla W_t(\theta)|^2 \text{d}X \\
+ \frac{1}{p+1} \int_{\partial \Omega_t^+} K_t(x) H^T(x)|U_t(\theta)|^{p+1} \text{d}x \\
+ \left\{ \begin{array}{ll}
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{1}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} \right] & \text{if } 0 < \sigma \leq \frac{1}{2}, \\
C_0(n, \sigma) C_3(n, \sigma, A_1) \left[ \frac{l_2^{l_2-l_1+2+2\sigma}}{(l_2-l_1)^{l_2-l_1+2+2\sigma}} + \frac{1}{l_1^{l_2-l_1+2+2\sigma}} \right] & \text{if } \frac{1}{2} < \sigma < 1,
\end{array} \right.
\]
Now we choose $l_1 > 10r$, $l_2 > 200l_1$ to be large enough such that
\[
I_{K_l, \tau}(H_l(\theta)) \leq c^{(1)}_{l, \tau} + c^{(2)}_{l, \tau} - \frac{\varepsilon 4}{2}.
\] (6.33)

Then for $l$ large enough (depending on $l_1, l_2, \varepsilon$'s, $C$'s), we have

\[
H_l \in \Gamma_l.
\] (6.34)

Therefore, for $l$ sufficiently large, we have

\[
\max_{\theta \in [0,1]^2} I_{K_l, \tau}(H_l(\theta)) \leq c^{(1)}_{l, \tau} + c^{(2)}_{l, \tau} - \frac{\varepsilon 4}{2} < b_{l, \tau}.
\] (6.35)

However, (6.35) cannot hold by (6.34) and the definition of $b_{l, \tau}$. This completes the proof of Theorem 4.1.

### 7 Proof of main theorems

In this section we present our main result from which we deduce Theorems 1.1–1.3 and Corollary 1.1.

**Proposition 7.1.** Suppose that $\{K_l\}$ is a sequence of functions satisfying conditions (i)–(iii) (see Section 4) and (K2). Assume also that there exist some bounded open sets $O^{(1)}, \ldots, O^{(m)} \subset \mathbb{R}^n$ and some positive constants $\delta_2, \delta_3 > 0$ such that for any $1 \leq i \leq m$,

\[
\overline{O}^{(i)}_l - z_l^{(i)} \subset O^{(i)} \quad \text{for all } l,
\]

\[
\{U \in D^+: I'_{K^{(i)}_t}(U) = 0, c^{(i)} \leq I_{K^{(i)}_t}(U) \leq c^{(i)} + \delta_2\} \cap V(1, \delta_3, O^{(i)}, K^{(i)}_t) = \emptyset.
\]

Then for any $\varepsilon > 0$, there exists integer $\overline{t}_{\varepsilon, m} > 0$ such that for any $l \geq \overline{t}_{\varepsilon, m}$, there exists $U_l \in V_l(m, \varepsilon)$ which solves

\[
\begin{aligned}
\div(1^{1-2\sigma} \nabla U_l) &= 0 \quad \text{in } \mathbb{R}_{t+1}^n, \\
\partial^\sigma U_l &= K_l(x)U_l(x, 0) \frac{n+2\sigma}{n-2\sigma} \quad \text{on } \mathbb{R}^n.
\end{aligned}
\] (7.1)

Furthermore, $U_l$ satisfies

\[
\sum_{i=1}^m c^{(i)} - \varepsilon \leq I_{K_l}(U_l) \leq \sum_{i=1}^m c^{(i)} + \varepsilon.
\]

The proof of Proposition 7.1 is by contradiction arguments, depending on blow-up analysis for a family of equations (4.1) approximating Eq. (7.1). More precisely, if the sequence of subcritical solutions $U_{l, \tau}$ ($0 < \tau < \overline{t}_l$) obtained in Theorem 4.1 is uniformly bounded as $\tau \to 0$, some local estimates in [40] imply that there exists a subsequence converging to a positive solution of Eq. (7.1). However, a prior $\{U_{l, \tau}\}$ might blow up, we have to rule out this possibility. Note that $U_{l, \tau} \in V_l(m, \delta_{e_2}(1))$, which consists of functions with $m (m \geq 2)$ “bumps”, we apply some results of
Suppose that Definition 7.1.

Let \( \Omega \subset \mathbb{R}^n \) be a domain, \( \tau_i \geq 0 \) satisfy \( \lim_{i \to \infty} \tau_i = 0 \), \( p_i = \frac{n+2\sigma}{n-2\sigma} - \tau_i \), and let \( K_i \in C^{1,1}(\Omega) \) satisfy, for some constants \( A_1, A_2 > 0 \),

\[
1/A_1 \leq K_i(x) \leq A_1 \quad \text{for all } x \in \Omega, \quad \|K_i\|_{C^{1,1}(\Omega)} \leq A_2. \tag{7.2}
\]

Let \( u_i \in L^\infty(\Omega) \cap E \) with \( u_i \geq 0 \) in \( \mathbb{R}^n \) satisfy

\[
(-\Delta)^{\sigma} u_i = K_i u_i^{p_i} \quad \text{in } \Omega. \tag{7.3}
\]

We say that \( \{u_i\} \) blows up if \( \|u_i\|_{L^\infty(\Omega)} \to \infty \) as \( i \to \infty \).

**Definition 7.1.** Suppose that \( \{K_i\} \) satisfies (7.2) and \( \{u_i\} \) satisfies (7.3). We say a point \( \overline{\eta} \in \Omega \) is an isolated blow up point of \( \{u_i\} \) if there exist \( 0 < \overline{\eta} < \text{dist}(\overline{\eta}, \Omega), \overline{C} > 0 \), and a sequence \( y_i \) tending to \( \overline{\eta} \), such that, \( y_i \) is a local maximum of \( u_i \), \( u_i(y_i) \to \infty \) and

\[
u_i(y) \leq \overline{C}|y-y_i|^{-2\sigma/(p_i-1)} \quad \text{for all } y \in B_r(y_i).
\]

Let \( y_i \to \overline{\eta} \) be an isolated blow up point of \( u_i \), define

\[
\overline{\nu}_i(r) = \frac{1}{|\partial B_r(y_i)|} \int_{\partial B_r(y_i)} u_i, \quad r > 0,
\]

and

\[
\overline{w}_i(r) = r^{2\sigma/(p_i-1)} \overline{\nu}_i(r), \quad r > 0.
\]

**Definition 7.2.** We say \( y_i \to \overline{\eta} \in \Omega \) is an isolated simple blow up point, if \( y_i \to \overline{\eta} \) is an isolated blow up point, such that, for some \( \rho > 0 \) (independent of \( i \)), \( \overline{\nu}_i \) has precisely one critical point in \((0, \rho)\) for large \( i \).

Utilizing these notions, we now present some facts. By some standard blow up arguments, the blow up points cannot occur in \( \mathbb{R}^n \setminus \bigcup_{i=1}^m \overline{O}_i^{(1)} \) since the energy of \( \{U_{i,\tau}\} \) in the region is small by the fact that \( U_{i,\tau} \in V_l(m, \sigma_2(1)) \) and the definition of \( V_l(m, \sigma_2(1)) \). Hence the blow up points can occur only in \( \bigcup_{i=1}^m \overline{O}_i^{(1)} \). By the structure of functions in \( V_l(m, \sigma_2(1)) \) and some blow up arguments obtained in \([40, \text{Proposition 5.1}]\), there are at most \( m \) isolated blow up points, namely, the blow up occurs in \( \{\overline{\eta}_1, 0), \ldots, (\overline{\eta}_m, 0)\} \) for some \( \overline{\eta}_i \in \overline{O}_i^{(1)} \) \((1 \leq i \leq m)\). Futhermore, we conclude from \([40, \text{Proposition 4.16}]\) that an isolated blow up point has to be an isolated simple blow up point. From the structure of functions in \( V_l(m, \sigma_2(1)) \) we know that if the blow up does occur, there have to be exactly \( m \) isolated simple blow up points, see \([40, \text{Section 4}]\) for more details.

Let us consider this situation only, namely, \( \{\overline{\nu}_i = (\overline{\eta}_i, 0), 1 \leq i \leq m\} \) is the blow up set and they are all isolated simple blow up points. Moreover, in our situation, \( K_i(x) = K(x) H^{\tau_i}(x) \) is the sequence of functions in Eq. (7.3). We may assume that the blow up occurs at \( U_i = U_{i,\tau_i} \) and we can apply some blow-up analysis results in \([40]\) to \( U_i \). Here and in the following we suppress the dependence of \( l \) in the notation since \( l \) is fixed in the blow-up analysis.

We now complete the proof of Proposition 7.1 by checking balance via a Pohozaev type indentity.
**Proof of Proposition 7.1.** Let \( U_i \) be the extension of \( u_i \) (see (1.10)) corresponding to the solution of Eq. (7.3) with \( K_i = KH^n \). Let \( \overline{y} = \overline{y}_1 \) and \( \{y_i\} \) be the sequence as in Definitions 7.1 and 7.2. Applying the Pohozaev identity [40, Proposition 4.7] to \( U_i \), we obtain

\[
\int_{\partial B_k^\pm(y_i)} B'(Y, U_i, \nabla U_i, R, \sigma) + \int_{\partial B_k^\pm(y_i)} s^{1-2\sigma} B''(Y, U_i, \nabla U_i, R, \sigma) = 0,
\]

where

\[
B'(Y, U_i, \nabla U_i, R, \sigma) = \frac{n-2\sigma}{2} K_i U_i^{p_i+1} + \langle Y, \nabla U_i \rangle K_i U_i^{p_i},
\]

and

\[
B''(Y, U_i, \nabla U_i, R, \sigma) = \frac{n-2\sigma}{2} U_i \frac{\partial U_i}{\partial \nu} - R \frac{1}{2} |\nabla U_i|^2 + R \left| \frac{\partial U_i}{\partial \nu} \right|^2.
\]

We are going to derive a contradiction to (7.5), by showing that for small \( R > 0 \),

\[
\lim_{i \to \infty} \sup U_i(Y_i)^2 \int_{\partial B_k^\pm(y_i)} B'(Y, U_i, \nabla U_i, R, \sigma) \leq 0,
\]

and

\[
\lim_{i \to \infty} \sup U_i(Y_i)^2 \int_{\partial B_k^\pm(y_i)} s^{1-2\sigma} B''(Y, U_i, \nabla U_i, R, \sigma) < 0.
\]

Hence Proposition 7.1 will be established.

Let \( \mathcal{S} = \{ \overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_m \} \), applying Bôcher Lemma in [40, Lemma 4.10] and maximum principle, we deduce that

\[
U_i(Y_i)U_i(Y) \to G(Y) = \sum_{k=1}^m b_k |Y - \overline{Y}_k|^{2\sigma-n} + h(Y)
\]

in \( C^\alpha_{loc}(\mathbb{R}^{n+1} \setminus \mathcal{S}) \cap C^2_{loc}(\mathbb{R}^{n+1}) \)

and

\[
u_i(y_i)u_i(y) \to G(y, 0) = \sum_{k=1}^m b_k |y - \overline{y}_k|^{2\sigma-n} + h(y, 0)
\]

in \( C^2_{loc}(\mathbb{R}^n \setminus \{ \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_m \}) \)

as \( i \to \infty \), where \( b_k > 0 \) (1 \( \leq k \leq m \)) and \( h(Y) \) satisfies

\[
\begin{cases}
\text{div}(s^{1-2\sigma} \nabla h) = 0 & \text{in } \mathbb{R}^{n+1}, \\
\partial^\nu h = 0 & \text{on } \mathbb{R}^n.
\end{cases}
\]

In particular, in a small half punctured disc at \( Y_1 \), we have

\[
\lim_{i \to \infty} U_i(Y_i)U_i(Y) = b_1 |Y - Y_1|^{2\sigma-n} + b + w(Y),
\]

where \( b > 0 \) is a positive constant and \( w(Y) \) is a smooth function near \( Y_1 \) with \( w(Y_1) = 0 \).

Now if we choose \( R > 0 \) small enough, it is easy to verify (7.7) by

\[
\lim_{i \to \infty} \sup U_i(Y_i)^2 \int_{\partial B_k^\pm(y_i)} s^{1-2\sigma} B''(Y, U_i, \nabla U_i, R, \sigma)
\]

\[
= \int_{\partial B_k^\pm(y_i)} s^{1-2\sigma} B''(Y, G, \nabla G, R, \sigma)
\]

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\[
- \frac{(n - 2\sigma)^2}{2} h_1^2 \int_{\partial B_i^+} s^{1 - 2\sigma} \, dy \, ds + o_R(1) < 0.
\]

On the other hand, via integration by parts, we have
\[
\int_{\partial B_i^+(y_i)} B'(Y, U_i, \nabla U_i, R, \sigma) 
= \frac{n - 2\sigma}{2} \int_{B_1(y_i)} \mathcal{K}_i u_i^{p_i + 1} + \int_{B_1(y_i)} (y - y_i, \nabla u_i) \mathcal{K}_i u_i^{p_i} 
= \frac{n - 2\sigma}{2} \int_{B_1(y_i)} \mathcal{K}_i u_i^{p_i + 1} - \frac{n}{p_i + 1} \int_{B_1(y_i)} K_i u_i^{p_i + 1} 
- \frac{1}{p_i + 1} \int_{B_1(y_i)} (y - y_i, \nabla K_i) u_i^{p_i + 1} + \frac{R}{p_i + 1} \int_{\partial B_1(y_i)} K_i u_i^{p_i + 1},
\]
where [40, Proposition 4.9] is used in the last inequality. Hence (7.6) follows from [40, Corollary 4.15].

Finally, from the above argument we know that there will be no blow up occur under the hypotheses of Proposition 7.1 and hence Proposition 7.1 is established.

We are now ready to complete the proofs of the main results in our paper.

Proof of Theorem 1.2. Suppose Theorem 1.2 is false, then for some \( \overline{\varepsilon} > 0 \) (we can assume \( \overline{\varepsilon} \) to be very small) and \( k \in \{2, 3, 4, \ldots \} \), there exists a sequence of integers \( I^{(i)}_l, \ldots, I^{(k)}_l \), such that
\[
\lim_{l \to \infty} |I^{(i)}_l| = \infty,
\]
\[
\lim_{l \to \infty} |I^{(i)}_l - I^{(j)}_l| = \infty, \quad i \neq j,
\]
but Eq. (1.9) has no solution in \( V(K, \overline{\varepsilon}, B_{\overline{\varepsilon}}(x^{(1)}_l), \ldots, B_{\overline{\varepsilon}}(x^{(k)}_l)) \) which satisfies \( kc - \overline{\varepsilon}, K \leq lc + \overline{\varepsilon} \), where \( c = (\sigma/n)(K(x^*)^{2(\alpha - n)/2\alpha}(S_{n,\sigma})^{n/\sigma}) \) and \( x^{(i)}_l = x^* + (I^{(i)}_l T, 0, \ldots, 0) \).

For \( \varepsilon > 0 \) small, define
\[
K_l(x_1, x_2, \ldots, x_n) = K(x_1, x_2, \ldots, x_n), \\
O^{(i)}_l = B_{\varepsilon}(x^{(i)}_l), \quad \text{and} \quad \tilde{O}^{(i)}_l = B_{2\varepsilon}(x^{(i)}_l), \\
R_l = \min \{ |I^{(i)}_l|, |I^{(i)}_l - I^{(j)}_l| \}.
\]
\[
K^{(i)}_\infty(x_1, x_2, \ldots, x_n) = \lim_{l \to \infty} K(x_1, x_2, \ldots, x_n) = \lim_{l \to \infty} K(x_1, x_2, \ldots, x_n), \\
a^{(i)} = K(x^*)
\]
It is easy to see that \( K^{(i)}_\infty \) is \( T \)-periodic in \( x_1 \) and satisfies assumption \( (K_2) \) and
\[
K^{(i)}_\infty(x^*) = \sup_{x \in \mathbb{R}^n} K^{(i)}_\infty(x) > 0.
\]
Under our hypothesis, it follows from [40, Theorem 5.4] that it is impossible to have more than one blow-up point. Applying [40, Theorem 5.6] and Proposition 7.1, we immediately get a contradiction.
Proof of Theorem 1.3. Theorem 1.3 can be proved similarly to Theorem 1.2 before, we omit it here.

Proof of Theorem 1.1. Let \( \bar{x} \in S^n \) be the north pole and make a stereographic projection to the equatorial plane of \( S^n \), then (1.1) is transformed to (1.2), up to a harmless positive constant in front of \( K(x) \). Here \( K(x) \in L^\infty(\mathbb{R}^n) \) satisfies, for some constants \( A_1 > 1, R > 1, \) and \( K_\infty > 0, \)

\[
1/A_1 \leq K(x) \leq A_1, \\
K \in C^0(\mathbb{R}^n \setminus B_R), \\
\lim_{|x| \to \infty} K(x) = K_\infty.
\]

Let \( \psi \in C^\infty(\mathbb{R}^n) \) satisfies assumption \((K_2)\) and

\[
\|\psi\|_{C^2(\mathbb{R}^n)} < \infty, \\
\lim_{|x| \to \infty} \psi(x) =: \psi_\infty > 0, \\
\langle \nabla \psi, x \rangle < 0, \ \forall x \neq 0.
\]

It follows from (7.8)–(7.10) that \( \psi \) violates the Kazdan-Warner type condition (see [40, Proposition A.1]) and therefore

\[
(-\Delta)^\sigma u = \psi |u|^{\frac{2\sigma}{n-2\sigma}} u \quad \text{in} \quad \mathbb{R}^n
\]

has no nontrivial solution in \( E \).

For any \( \varepsilon \in (0, 1), k = 1, 2, 3, \ldots \) and \( m = 2, 3, 4, \ldots \), we choose an integer \( \tilde{k} \) such that for any \( 2 \leq s \leq m \), there holds \( C_{\tilde{k}}^s \geq k \), where \( C_{\tilde{k}}^s \) is a combination number. Then we choose \( e_1, e_2, \ldots, e_{\tilde{k}} \in \partial B_1 \) to be \( \tilde{k} \) distinct points. Let

\[
\delta_R = \max_{|x| \geq R} |K(x) - K_\infty| + \max_{|x| \geq R} |\psi(x) - \psi_\infty|, \quad R > 1,
\]

and \( \Omega^{(i)}_l \) be the connected component of

\[
\{ x : \varepsilon(\psi(x - le_i) - \psi_\infty) + K_\infty - \delta \sqrt{l} > K(x) \},
\]

which contains \( x = le_i \). Define

\[
R_l = \min_{1 \leq i \leq m} \sup \{ |x - le_i| : x \in \Omega^{(i)}_l \}, \\
K_{\varepsilon,k,m,l} = \begin{cases} 
\varepsilon(\psi(x - le_i) - \psi_\infty) + K_\infty - \delta \sqrt{l} & \text{if } x \in \Omega^{(i)}_l, \\
K(x) & \text{otherwise}.
\end{cases}
\]

It is easy to prove that

\[
\text{diam}(\Omega^{(i)}_l) \leq \sqrt{l} \quad \text{for large } l, \\
\lim_{l \to \infty} R_l = \infty.
\]

Now we consider the equation corresponding to \( K_{\varepsilon,k,m,l} \):

\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\partial^\sigma_{\nu} U = K_{\varepsilon,k,m,l} U(x, 0) \frac{r^{\frac{n+2\sigma}{n-2\sigma}}}{n-2\sigma} & \text{on } \mathbb{R}^n.
\end{cases}
\]
For any $2 \leq s \leq m$, we claim that, for $l$ large enough, Eq. (7.11) has at least $k$ solutions with $s$ bumps.

To verify it, let $e_{j_1}, \ldots, e_{j_s}$ be any $s$ distinct points among $e_1, \ldots, e_{\overline{m}}$. For $i = 1, 2, \ldots, s$, we define

$$z_i^{(i)} = le_{j_i},$$
$$O_i^{(i)} = B_1(z_i^{(i)}), \quad \overline{O}_i^{(i)} = B_2(z_i^{(i)}),$$
$$K_i^{(i)} = \varepsilon(\psi - \psi_\infty) + K_\infty,$$
$$a_i^{(i)} = \varepsilon(\psi(0) - \psi_\infty) + K_\infty.$$

Then using an argument similar to the proof of Theorem 1.2, we can prove that there exists at least a solution in $V_l(s, \varepsilon)$ for large $l$. It is also easy to see that if we choose a different set of $s$ points among $\{e_1, \ldots, e_{\overline{m}}\}$, we get different solutions since their mass are distributed in different regions by the definition of $V_l(s, \varepsilon)$. By the choice of $\overline{m}$, there are at least $k$ different sets of such points. Therefore Eq. (7.11) has at least $k$ solutions for large $l$. This gives the proof of the claim.

Finally, we fix $l$ large enough to make the above argument work for all $2 \leq s \leq m$ and set $K_{\varepsilon,k,m} = K_{\varepsilon,k,m,l}$. Thus there exists at least $k$ solutions with $s$ ($2 \leq s \leq m$) bumps to the following equations

$$(-\Delta)^s u = K_{\varepsilon,k,m} u^{\frac{n+2s}{n-2s}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n.$$  

Theorem 1.1 follows from the above after performing a sterographic projection on the original equation, we omit the details here.

**Proof of Corollary 1.1.** We can see from the proof in Theorem 1.1 that if $K \in C^\infty(\mathbb{S}^n)$, then $K_{\varepsilon,k,m} - K \in C^\infty(\mathbb{S}^n)$ can also be achieved.  

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