Outcome determinism in measurement-based quantum computation with qudits

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Abstract
In measurement-based quantum computing (MBQC), computation is carried out by a sequence of measurements and corrections on an entangled state. Flow, and related concepts, are powerful techniques for characterising the dependence of the corrections on previous measurement outcomes. We introduce flow-based methods for MBQC with qudit graph states, which we call $\mathbb{Z}_d$-flow, when the local dimension is an odd prime. Our main results are a proof that $\mathbb{Z}_d$-flow is a necessary and sufficient condition for a strong form of outcome determinism. Along the way, we find a suitable generalisation of the concept of measurement planes to this setting and characterise the allowed measurements in a qudit MBQC. We also provide a polynomial-time algorithm for finding an optimal $\mathbb{Z}_d$-flow whenever one exists.

Keywords: measurement-based quantum computation, flow, qudit, determinism

In measurement-based quantum computation (MBQC), one starts with an entangled resource state (usually graph states [BR01]), and computation is carried out by sequential measurements where at each stage the measurement choice depends on previous results [RB01, RB02, DK06, DKP07]. This adaptivity is necessary to combat the randomness induced by measurements and

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make the overall computation deterministic. It also plays an important role both foundationally [dGK11, Rau+11] and in analysing trade-offs in optimizing computations [BK09, MHM15].

Causal flow [DK06], and its generalisation gflow [Bro+07], are graph-theoretical tools for characterising and analysing adaptivity for graph states in MBQC. They have proven a powerful tool including optimizing adaptive measurement patterns [Esl+18]; translating between MBQC and the circuit picture [DP10, MHM15, Bac+21], with application for parallelising quantum circuits [BK09]; to construct schemes for the verification of blind quantum computation [FK17, Man+17]; to extract bounds on the classical simulatability of MBQC [MK14]; to prove depth complexity separations between the circuit and measurement-based models of computation [BK09, MHM15]; to study trade-offs in adiabatic quantum computation [AMA14]; and recently for applying ZX-calculus techniques [Bac+21, de +20], including to circuit compilation and optimisation [Dun+20, KvdW20].

In recent years, there has been increased interest in computing, and quantum information in general, over higher-dimensional qudit systems, as opposed to qubits [Wan+20]. The added flexibility of increased dimension allows for shorter circuits in computation [Gok+19, Kik+20, BDC20], and improves fault-tolerance thresholds for error correction [CGL99, Ros+18, Joo+19], the noise tolerance in quantum key distribution [BT00, Cer+02, Dod+21] and the robustness of certain quantum algorithms [ZR07, PP12]. Furthermore, many physical systems exist which naturally encode qudits and a wide variety of experiments have been made demonstrating the possibility for quantum control of qudits [Kli+03, Wal+06, BW08, Nee+09, Smi+13, Ben+15, Erh+18, Gao+19, Lu+20, Chi+22] including the generation of qudit cluster states which are central to MBQC [Kim+15, Kue+17, Rei+19]. This has motivated the translation of MBQC and related models into qudits [Zho+03, Hal07, Pro15, Pro16, Pro+17], which naturally leads to the question: can the flow techniques above be extended to the qudit setting?

In this work we make such a generalisation, when the local dimension of the qudits is an odd prime. To do so, we reformulate gflow as a particularly nice matricial condition which generalises well to higher dimensions, and which we then call \( Z_d \)-flow. This linear algebraic formulation makes possible proofs which would be very clumsy in the original formulation. In section 1 we review quantum computation with qudits and introduce our computational model, a qudit version of the measurement calculus [DKP07] which further generalises the measurement calculus of Proctor [Pro15]. This requires careful consideration of the measurements which can be allowed as part of the MBQC, and results in our novel concept of measurement spaces. We also review the various determinism conditions present in the literature and in particular robust determinism [PS17]. In section 2 we introduce \( Z_d \)-flow, and show that it is sufficient to obtain a robustly deterministic MBQC. We also prove a converse: any robustly deterministic MBQC is shown to have a \( Z_d \)-flow. The proof of this converse requires substantially new methods when compared to the qubit case. Then, in section 3, we consider two worked-out examples of robustly deterministic MBQC based on \( Z_d \)-flow. Finally, in section 4 we present a polynomial-time algorithm for determining if a given graph has a \( Z_d \)-flow, and further prove that it always produces \( Z_d \)-flows of minimal depth (if it succeeds). These results are the first step in a characterisation of outcome determinism in MBQC for a large class of finite-dimensional quantum systems.

1. Preliminaries

The starting point for (and simplest example of) measurement-based quantum computation is the gate teleportation protocol [ZLC00]. In the case of qubits, or two-dimensional quantum systems, it is schematically described by the following quantum circuit:
This circuit (read left to right) describes a computation acting on an arbitrary input quantum state $|\psi\rangle$ belonging to a two-dimensional Hilbert space $\mathbb{C}^2$. This quantum state is then entangled with an auxiliary state $|+\rangle = \frac{1}{\sqrt{2}} |1\rangle$,

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$  \hfill (1)

by acting on the joint state $|\psi\rangle \otimes |+\rangle \in \mathbb{C}^4$ with the controlled-$Z$ gate:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  \hfill (2)

The controlled-$Z$ is depicted in the quantum circuit as the vertical line joining the two horizontal wires corresponding to the input and auxiliary states. The first subsystem of this entangled state is then measured, and a correction $X$ is applied to the auxiliary subsystem depending on the binary outcome $m$ of the measurement. The overall operation implemented by this sequence of elementary operations or gates can be straightforwardly seen to be a unitary operation acting on the input state $|\psi\rangle$ [Nie06] and which we will describe shortly.

First, let us turn our attention to the internals of the measurement. It is described by two parts: first, apply a unitary matrix $U$, then measure the Pauli $X$ operator. The $X$ operator is given by the Hermitian and unitary matrix:

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$  \hfill (3)

which therefore describes a valid quantum measurement. Now, the overall operation implemented by the whole circuit is unitary only under the following assumption: it must hold that the unitary matrix $U$ commutes with the $Z$ Pauli matrix described by:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (4)

In other words, we must have $[U, Z] = 0$. Then, the overall operation implemented by the quantum circuit is described by the map $|\psi\rangle \mapsto HU|\psi\rangle$, where

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$  \hfill (5)

is called the Hadamard operator. Simply put, one can apply the unitary $HU$ to the input $|\psi\rangle$ by this entangle-measure-correct procedure while retaining a degree of control over the actual unitary implemented through the choice of $U$. Measurements are in general described by quantum channels, which are not unitary, yet by making use of the outcome-dependant correction $X^m$ (where $m$ is the outcome obtained from the measurement), we can recover a unitary channel for the overall circuit.

Repeating this protocol sequentially makes it possible to implement more general unitaries: whenever $[U_1, Z] = 0 = [U_2, Z]$,
and it becomes clear that it should be possible to base an entire computational architecture around such measurement-based schemes for implementing unitaries. Moreover, something interesting happens if one commutes the entangling operations to the start of the circuit: This circuit is equivalent to the previous one in the transformation it implements, yet operationally distinct: we have recovered the entangle-measure-correct scheme of the gate teleportation protocol! First, we entangle the input with a set of auxiliary states, then perform a sequence of measurements and corresponding corrections on the resulting entangled state to implement the desired unitary. This is desirable since entangling operations are amongst the more difficult to implement physically, and so one would like to incur their cost up front. Notice however, that the correction we now need to make for the first measurement in order to make the protocol unitary is different than before.

How far can one extend this entangle-measure-correct scheme? It turns out to be possible to turn any quantum circuit into such an MBQC, by chaining together gate teleportations and an analogous protocol for implementing the $E$ gate on a pair of inputs [DKP07, Pro15, Zho+03]. However, we want to treat MBQC as its own computational model, rather than one built out of converting individual quantum gates to fragments of MBQCs which are then composed (sequentially and in parallel). For this, the circuit description language is rather clumsy, since tracking which measurement outcomes control which correction rapidly gets out of hand with larger computations: we need a better way of describing MBQCs. In the case of qubits, such a language is given by the measurement calculus [DKP07]. Formulating the calculus is a more subtle affair than might be apparent. On the one hand, there are other teleportation protocols, and these make use of different types of measurements and corrections. For example, the $Z$-rotation gadget allows one to apply a different unitary through a slightly different protocol: where now we must have $[U, X] = 0$ for the protocol to work, and we measure the $Z$ Pauli instead. On the other hand, it is considerably trickier to figure out what corrections are needed in order to make a computation in the calculus into a unitary since now much more general computations are allowed. An answer to the latter question is given by flow conditions such as causal flow [DK06] and gflow [Bro+07]. As we shall see in the following section, the gate teleportation protocol has a generalisation to qudits, giving rise to analogous questions. We first want to describe a measurement calculus for qudits which encompasses all such protocols at the very least, and also be sufficiently
general to implement any reversible quantum circuit. Therefore, we need to understand what measurements, or measurement devices, should be allowed. This will lead us to the definition of measurement spaces, which precisely characterise those measurements for which a correction is possible. Then, the remainder of the article is concerned with the question of generalising gflow to the setting of qudits.

1.1. Quantum computing with qudits

We now begin to introduce the operations which are specific to quantum computation with higher-dimensional quantum systems, or qudits. Many of the gates introduced in the previous subsection admit qudit versions that, at least on an algebraic level, behave analogously to their qubit alternatives. For instance, the Hadamard operator can be generalised to an operational also known as the Fourier transform on a finite group which is commonly called the Hadamard operator for qudits [Wan+20]. As such, we will suggestively use the same notation for these qudit versions of qubit operators, which also include the $X$ and $Z$ Pauli matrices. The qubit versions do not play any role in the rest of this article, so the reader can from this point on always assume that we are referring to the qudit version defined in this section.

Throughout this paper, $d$ denotes an arbitrary odd prime, and $\mathbb{Z}_d = \mathbb{Z} / d\mathbb{Z}$ the ring of integers with arithmetic modulo $d$. We also put $\omega := e^{i\pi/d}$, and let $\mathbb{Z}_d^*$ be the group of units of $\mathbb{Z}_d$. Since $d$ is prime, $\mathbb{Z}_d$ is a field and $\mathbb{Z}_d^* = \{1, \ldots, d-1\}$ as a set.

The Hilbert space of a qudit [Got99, Wan+20] is then $\mathcal{H} = \text{span}\{ |m\rangle \mid m \in \mathbb{Z}_d \} \cong \mathbb{C}^d$, and we write $U(\mathcal{H})$ the group of unitary operators acting on $\mathcal{H}$. We have the following standard operators on $\mathcal{H}$, also known as the clock and shift operators:

$$Z(m) := \omega^m|m\rangle \quad \text{and} \quad X(m) := |m+1\rangle \quad \text{for any} \quad m \in \mathbb{Z}_d. \tag{6}$$

In particular, note that $ZX = \omega XZ$. We call any operator of the form $\omega^k X^a Z^b$ for $k, a, b \in \mathbb{Z}_d$ a Pauli operator, although we will often drop the phase $\omega$ as it is of little importance in most cases. We say a Pauli operator is trivial if it is proportional to the identity. The Paulis are further related by the Hadamard gate:

$$H|m\rangle = \frac{1}{\sqrt{d}} \sum_{n \in \mathbb{Z}_d} \omega^{mn} |n\rangle \quad \text{s.t.} \quad HXH^\dagger = Z \quad \text{and} \quad HZH^\dagger = X^{-1}. \tag{7}$$

Equations (6) and (7) imply that both $X$ and $Z$, and in fact every Pauli has spectrum $\{\omega^k \mid k \in \mathbb{Z}_d\}$.

We also use the controlled-Z gate, which acts on $\mathcal{H} \otimes \mathcal{H}$,

$$E(m)|n\rangle := \omega^{mn} |m\rangle |n\rangle. \tag{8}$$

We can now give the qudit gate teleportation protocol, which the reader will immediately recognise from the qubit case given above:
for any unitary \( U \) such that \([U, Z] = 0\), and where \( P_X \) is the projection-valued measure associated to the \( X \) Pauli (that is, we measure in the eigenbasis of \( X \)), \(|0 : X\rangle\) is an eigenvector of \( X \) associated with eigenvalue 1 (this notation will become clear in the next section), and the 2-qudit gate is the controlled-\( Z \).

It is important to emphasise a key difference between the qudit and the qubit case: when \( d \neq 2 \), none of these operators are self-inverse. In fact, if \( Q \) is a Pauli and \( I \) the identity operator on \( H \), we have:

\[
Q^d = I, \quad E^d = I \otimes I \quad \text{and} \quad H^d = I. \tag{9}
\]

As a result, they are not self-adjoint either, something which needs to be taken into account when describing measurements.

1.1.1. Measurement spaces. For qubit MBQC, it is well established that by using a Pauli \( X \), \( Y \) or \( Z \) as an acausal correction operator, it is possible to perform MBQC on graph states where the measurements are taken from the plane on the Bloch sphere orthogonal to the correction \([\text{Bro} + \text{07}]\). Since there are three Pauli operators for qubits, this yields three allowable measurement planes for MBQC.

This interpretation is not as clear in the qudit case, partly because Pauli operators are self-adjoint only in the case \( d = 2 \), but mostly because the geometry of the Bloch ‘space’ is not as intuitive in the general case. A qudit-measurement will be described by a unitary matrix \( M \): given its spectral decomposition \( M = \sum \lambda_i P_i \), an \( M \)-measurement is the projective measurement \( f_{\lambda_i} \in \mathbb{Z}_d \). In the context of MBQC, we would like to have a distinguished measurement outcome, the one that does not need corrections, so we assume that all measurements have a fixpoint.

**Definition 1.** \( M \) is a fixpoint unitary on \( \mathcal{H} \) if \( M^\dagger M = MM^\dagger = I \) and there is a non-zero \(|\phi\rangle \in \mathcal{H}\) such that \( M|\phi\rangle = |\phi\rangle \). Given \((a, b) \in \mathbb{Z}_d^2 \setminus \{(0, 0)\}\), the measurement space \( \mathcal{M}(a, b) \) is defined as \( \mathcal{M}(a, b) := \{\text{fixpoint unitaries } M \text{ s.t. } X^a Z^b M = \omega M X^a Z^b\} \).

It should be pointed out that the commutation relation used to define the measurement space \( \mathcal{M}(a, b) \) is somewhat arbitrary. We could have chosen instead to use the relation

\[
X^a Z^b M = \omega^p M X^a Z^b \quad \text{for some } \quad p \in \mathbb{Z}_d^\ast. \tag{10}
\]

However, nothing is lost by considering only \( p = 1 \), since if \( M \) verifies equation (10), then

\[
X^{p^{-1}a} Z^{p^{-1}b} M = \omega^p M X^{p^{-1}a} Z^{p^{-1}b} = \omega M X^{p^{-1}a} Z^{p^{-1}b}, \tag{11}
\]

(where this calculation is formally carried out using \( p^{-1} = \frac{d \pm 1}{p} \)) which implies that \( M \in \mathcal{M}(p^{-1}a, p^{-1}b) \).

In fact, this construction is very analogous to one used in qudit quantum error correction where the \( M \) are called detectable errors \([\text{Got}99]\). The main point of this definition is that the Pauli \( X^a Z^b \) can be used to translate the eigenvectors of any measurement in the corresponding measurement space:
Proposition 2. If $M \in \mathcal{M}(a, b)$ for some non-trivial Pauli $Q = X^a Z^b$, then the spectrum of $M$ is \{ $\omega_m \mid m \in \mathbb{Z}_d$ \}, each eigenvalue has multiplicity 1, and $M$ is special unitary. Denoting $|0 : M \rangle$ the fixpoint of $M$, then $|m : M \rangle = Q^{-m} |0 : M \rangle$ is an eigenvector of $M$ associated with eigenvalue $\omega^m$.

Proof. By assumption, if $M \in \mathcal{M}(Q)$ then $M$ has a fixpoint $M|0 : M \rangle = |0 : M \rangle$. Then, it follows from the commutation relation that
\[
MQ|0 : M \rangle = \omega^{-1} QM|0 : M \rangle = \omega^{-1} Q|0 : M \rangle,
\]
so $Q|0 : M \rangle$ is an eigenvector of $M$ associated with eigenvalue $\omega^{-1}$. Repeating this procedure, we find that $|k : M \rangle = Q^{-k} |0 : M \rangle$ is an eigenvector of $M$ associated with eigenvalue $\omega^k$, and a counting argument shows that each of these eigenvalues must have multiplicity 1. Now, we have that $\text{det}(M) = \prod_{k \in \mathbb{Z}_d} \omega^k = 1$. □

This means that the Pauli $Q$ can be used as correction for any measurement in the corresponding measurement space, as is described in section 2. As in the qubit case, pairs of measurements within the same measurement space $\mathcal{M}(a, b)$ are still related to each other by rotations around the ‘correction’ axis $X^{a_l} Z^{b_l}$.

Proposition 3. Let $Q = X^a Z^b$ be a non-trivial Pauli operator and $N \in \mathcal{M}(a, b)$. Then $M \in \mathcal{M}(a, b)$ if and only if there is a special unitary $U \in SU(d)$ such that $M = UNU^\dagger$ and $[U, Q] = 0$.

Proof. (⇒) If $M \in \mathcal{M}(Q)$ then $\text{sp}(M) = \text{sp}(N) = \{ \omega^k \mid k \in \mathbb{Z}_d \}$ and each eigenvalue has multiplicity one. It follows that $M$ and $N$ are similar so that there is a unitary $U$ such that $M = UNU^\dagger$.

Furthermore, by proposition 2, the eigenvector $|k : M \rangle$ of $M$ can be obtained as $Q^{-k} |0 : M \rangle$, from which it also follows that $|k + 1 : M \rangle = Q|k : M \rangle$. But, we also have $MU|k + 1 : N \rangle = UNU^\dagger U|k + 1 : N \rangle = \omega^{k+1} U|k + 1 : N \rangle$ from which it follows that
\[
QU|k : N \rangle = Q|k + 1 : M \rangle = U|k + 1 : N \rangle = UQ|k : N \rangle.
\]
This is true for any $k \in \mathbb{Z}_d$, and since $N$ is unitary its eigenvectors form a basis for $\mathcal{H}$. We deduce that $QU = UQ$.

Finally, it is clear we can choose $U$ to be special unitary, since for any unit norm $\lambda \in \mathbb{C}$,
\[
(\lambda U)|N(\lambda U)^\dagger = |\lambda|^2 UNU^\dagger = UNU^\dagger.
\]

(⇐) Let $M = UNU^\dagger$ such that $[U, Q] = 0$, then we have
\[
MQ = UNU^\dagger Q = UNQU^\dagger = \omega QNU^\dagger = \omega QUNU^\dagger = \omega QM.
\]
Furthermore, $M$ and $N$ have the same spectrum, and in particular $M$ has a fixpoint since $N$ does. Then, $M \in \mathcal{M}(Q)$. □

In turn, this allows us to recover a parametrisation of measurement spaces much closer to the qubit case, where a measurement is given by angles relative to a reference Pauli axis of the Bloch sphere.

Corollary 4 (Measurement angles). For any non-zero $(a, b) \in \mathbb{Z}_d^2$, a measurement $M \in \mathcal{M}(a, b)$ is characterised by $d - 1$ angles $\theta = (\theta_1, \ldots, \theta_{d-1}) \in [0, 2\pi)^{d-1}$, up to a choice of reference axis $P \in \mathcal{M}(a, b)$.

Proof. Fix some $P \in \mathcal{M}(a, b)$, then by the proposition, every $M \in Q^\dagger$ is such that $M = UPU^\dagger$, and in particular $[U, Q] = 0$. This implies that in the eigenbasis of $Q$, $U$ takes the form of a diagonal matrix $\text{diag}(e^{i\theta_k} \mid k \in \mathbb{Z}_d)$ with $\theta_k \in [0, 2\pi)$. Since $\text{det}(U) = 1$, we have that $\sum_{k=0}^{d-1} \theta_k = 0$. 

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and one of these phases is redundant. Then, $U$ and by extension, $M$, is uniquely determined by the $d - 1$ phases $\{\theta_k\}_{k=1}^{d-1}$ (and the arbitrary choice of $P$).

As is the case for qubits, the choice of reference axes (one per measurement space) is entirely arbitrary, so we assume for the rest of the article that some fixed choice has been made for each measurement space.

### 1.1.2. Measurement patterns.

Given that we are interested in procedures with an emphasis on measurements and corrections conditioned on the outcomes of measurements, the quantum circuit description of computations is not very practical for our needs. Instead, we describe an MBQC by a sequence of commands, called a measurement pattern. The description of measurements hinges on the characterisation of measurement spaces in corollary 4. We suppose some arbitrary choice of reference is made for each measurement space, then:

**Definition 5 ([DKP07]).** A measurement pattern on a register $V$ of qudits consists in a finite sequence of $V$-indexed commands chosen from:

- $N_u$: initialisation of a qudit $u$ in the state $|0:X\rangle = H|0\rangle$;
- $E_{a,v}^\lambda$: application of $E^\lambda$ on qudits $u$ and $v$ for some $\lambda \in \mathbb{Z}_d$, with $u \neq v$;
- $M_{u}^{a,b}(\vec{\theta})$: measurement of qudit $u$ in the measurement space $\mathcal{M}(a,b)$ with angles $\vec{\theta}$;
- $X_u^m$ and $Z_u^m$: Pauli corrections depending on the outcome $m$, of the measurement of vertex $v$.

A measurement pattern is *runnable* if no commands act on non-inputs before they are initialised (except initialisations) or after they are measured, and no commands depend on the outcome of a measurement before it is made.

### 1.2. A graph-theoretical representation

Following Perdrix and Sanselme [PS17], we show that measurement patterns can be equivalently represented as labelled graphs. Then, following [Zho+03], these measurement patterns are Universal for all qudit quantum circuits.

**Notation.** A $\mathbb{Z}_d$-graph $G$ is a loop-free undirected $\mathbb{Z}_d$-edge-weighted graph on a set $V$ of vertices. We will identify the graph $G$ with its symmetric adjacency matrix $G \in \mathbb{Z}_d^{V \times V}$ (for some arbitrary ordering of the rows and columns). If $A, B \subseteq V$, we will also denote $G[A,B]$ the submatrix of $G$ obtained by keeping only the rows corresponding to elements of $A$ and the columns corresponding to elements of $B$. If $A \subseteq V$, then we denote $1_A \in \mathbb{Z}_d^V$ the column vector whose $u$th element is 1 if $u \in A$, 0 otherwise. Similarly we consider $\mathbb{Z}_d$-multisets of vertices where each vertex occurs with a multiplicity in $\mathbb{Z}_d$ and we will identify the $\mathbb{Z}_d$-multiset with column vectors in $\mathbb{Z}_d^V$.

The size of a multiset is defined by $|A| = \sum_{u \in V} A(u) \in \mathbb{Z}_d$, $x^\top$ is the transpose of $x$. Given a Pauli operator $P$ and a multiset $A$, let $P_A := \bigotimes_{u \in A} P_A^u$.

The commands of a measurement pattern verify the following identities, for every $u, v \in V$ such that $u \neq v$:

$$
\begin{align*}
X_u Z_v &= Z_v X_u, & X_u Z_u &= Z_u X_u, \\
X_u M_v &= M_v X_u, & Z_u M_v &= M_v Z_u, \\
E_{u,v} X_u &= X_u Z_v E_{u,v}, & E_{u,v} Z_u &= Z_u E_{u,v}. 
\end{align*}
$$

(15)
where we use the notation $A \simeq B$ to mean that there is a phase $e^{i\alpha}$ such that $A = e^{i\alpha} B$. It has been shown that any runnable measurement pattern can be rewritten using these commutation relations to the standard form [DKP07, Pro15]:

$$
\mathcal{P} \simeq \left( \prod_{v \in O^E} X^{m_v}_{x(v)} Z^{m_v}_{x(v)} M^v \mathcal{O}_{\diamond} (\bar{\theta}_v) \right) \left( \prod_{\langle u,v \rangle \in G} E^{G}_{u,v} \right) \left( \prod_{v \in F} N_v \right),
$$

(16)

where $x, z$ are functions $O^E \rightarrow \mathbb{Z}_d^V$, $m_v$ is the outcome of the measurement $M_v$, $G$ is the adjacency matrix of a $\mathbb{Z}_d$-graph, and $\langle u, v \rangle \in G$ identifies an edge in the graph $G$. The leftmost product is ordered according to the order $\prec$ in which the qudits are measured.

The functions $x, z$ also implicitly describe a measurement order: we can measure a qudit as soon as it is no longer needed for any future measurements. Formally, this is the transitive closure of the relation $\{(u, v) \mid x(v)_u \neq 0 \text{ or } z(v)_u \neq 0\}$, which gives a strict partial order on $O^E$. The measurement order $\prec$ must agree with this order if the pattern is runnable, and any order which does yields a measurement pattern.

This motivates the following definition [DK06, Bro+07, Bac+21]:

**Definition 6.** An open $\mathbb{Z}_d$-graph is a triple $(G, I, O)$ where $G$ is a $\mathbb{Z}_d$-graph over $V$, and $I, O \subseteq V$ are distinguished sets of vertices which identify inputs and outputs in an MBQC.

A labelled open $\mathbb{Z}_d$-graph is a tuple $(G, I, O, \lambda)$ where $(G, I, O)$ is an open $\mathbb{Z}_d$-graph and $\lambda : O^F \rightarrow \mathbb{Z}_d^V \setminus \{(0, 0)\}$ assigns a measurement space to each measured vertex.

Given the form (16), it is clear that we can describe a runnable measurement pattern by a tuple $(G, I, O, \lambda, x, z, M)$, where $(G, I, O, \lambda)$ is a labelled open $\mathbb{Z}_d$-graph, $x, z$ are functions $O^F \rightarrow \mathbb{Z}_d^V$, and $M$ is a function $O^F \rightarrow U(\mathcal{H})$ such that $M(u) \in M(\lambda(u))$ for all $u \in O^F$. $M$ gives the measurement to be made at each non-output vertex, and $x, z$ describe corresponding outcome-dependent corrections. The measurements are made in any order that agrees with the order imposed by $x, z$ described above. Note that the labelling $\lambda$ is technically required since the syntax of measurements in equation (16) depend on the labelling, but as we shall see in the next section, once the choice of $M$ is made, $\lambda$ has no effect on the actual computation carried out by the MBQC (the semantics of the measurement pattern).

### 1.3. Determinism

An MBQC $(G, I, O, \lambda, x, z, M)$ describes an inherently probabilistic computation with $d \times |O^E|$ possible branches (one for each set of measurement outcomes). Given an input state $|\psi\rangle \in \mathcal{H} \otimes I$ and set of outcomes $\bar{m} \in \mathbb{Z}_d^O$, the corresponding branch is given by:

$$
A_{\bar{m}}(|\psi\rangle) := \left( \prod_{\bar{m}_v \in \mathcal{O}_{\bar{m}}}^{\mathcal{P}} X^{m_v}_{x(v)} Z^{m_v}_{x(v)} M_v \mathcal{O}_{\diamond} (\bar{\theta}_v) \right) \left( \prod_{\langle u,v \rangle \in G} E^{G}_{u,v} \right) \left( \prod_{v \in F} N_v \right) |\psi\rangle \otimes \left( 0 : X \right).
$$

(17)

The branch maps give a Kraus decomposition for the CPTP map $\mathcal{H} \otimes I \rightarrow \mathcal{H} \otimes O$ implemented by the MBQC:

$$
\rho \mapsto \sum_{\bar{m} \in \mathbb{Z}_d^O} A_{\bar{m}}\rho A_{\bar{m}}^\dagger
$$

(18)

[DKP07] worked in the qubit setting but their proof is purely symbolic. Rewriting $U^\lambda = \prod_{\bar{m}_v \in \mathcal{O}_{\bar{m}}}^{\mathcal{P}} U$, where $U$ is any unitary from equation (15), and applying their standardisation procedure results in a pattern of the form (16). This process is formally described in [Pro15].
A measurement pattern is said to be deterministic if the output does not depend on the outcomes of the measurements. This is equivalent to saying that all branches (17) are proportional, in which case the pattern is described by the single Kraus operator $K_{\bar{G}}$, corresponding to obtaining outcome 0 for all measurements. This is by construction a correction-less branch since we have then obtained the ‘preferred’ outcome of each measurement. However, a problem comes up if $K_{\bar{G}} = 0$, in which case two deterministic MBQCs can have the same open graph but implement different maps. See [DK06, Bro+07, PS17] for examples.

To exclude these pathological cases, a stronger determinism condition was introduced by Danos and Kashefi [DK06]: a measurement pattern is strongly deterministic if all branch maps are equal up to a global phase. In particular, strongly deterministic measurement patterns implement isometries.

Now, the original purpose of flow was to obtain sufficient and necessary conditions for deciding when such an MBQC is deterministic. However, a characterisation of strong determinism is still an open question, even in the case of qubits. Instead, we restrict our attention to a yet stronger form of determinism, which is both more tractable and arguably more practical [PS17]:

**Definition 7 (Robust determinism).** $(G, I, O, \lambda, x, z)$ is robustly deterministic if for any $\prec$-lowerset $S \subseteq O^2$ and any $M : S \rightarrow U(\mathcal{H})$ such that $M(u) \in M(\lambda(u))$, the MBQC $(G, I, O \cup S^c, \lambda|x, z|S, M)$ is strongly deterministic.

Robust determinism is equivalent to the uniformly and stepwise strong determinism of Browne et al [Bro+07] in the qubit case.

### 1.4. Graph states

For an open graph $(G, I, O)$ and an arbitrary input state $|\phi\rangle \in \mathcal{H}^\otimes I$, we write

$$|G(\phi)\rangle = \left( \prod_{u, v \in V} E_{u,v}^{G_{\phi}} \right) \left( |\phi\rangle \bigotimes_{u \in F} |0 : X_i u\rangle \right),$$

(19)

which we call an open graph state. Open graph states are resource states for the MBQCs which we describe in this paper: comparing with equations (16) and (17), we see that every runnable measurement pattern first generates an open graph state of this form, then performs a sequence of measurements on that state. In the case $I = \emptyset$, we recover the well-known qudit graph states [Zho+03, MMP13]. The stabilisers of an open graph state are given by:

**Proposition 8 (Open graph stabilisers).** Let $(G, I, O)$ be an open graph, and $Q$ a product of Paulis. Then, $Q |G(\phi)\rangle = |G(\phi)\rangle$ for all $|\phi\rangle \in \mathcal{H}^\otimes I$ if and only if there is a multiset $\Lambda \in \mathbb{Z}_d^V$ such that $A_v = 0$ for all $v \in I$ and $Q = \omega^{\sum_{v \in V} X_v} X_{\Lambda G V}$. 

**Proof.** The stabilisers of $|G(\phi)\rangle$ are simply the stabilisers of $|\phi\rangle \bigotimes_{u \in F} |0 : X_i u\rangle$ conjugated by $E_G$. It is clear that the stabiliser group of $|0 : X\rangle$ is generated by $X^m$ for all $m \in \mathbb{Z}_d$. Since no Pauli stabilises every $|\psi\rangle \in \mathcal{H}$, it follows that the stabiliser group of $|\phi\rangle \bigotimes_{u \in F} |0 : X_i u\rangle$ is of the form $X_\Lambda$ for some $\Lambda \in \mathbb{Z}_d^V$ such that $A_v = 0$ if $v \in I$. Now, we have

$$E_G X_{\Lambda G V} = \left( \prod_{u, v \in V} E_{u,v}^{G_{\phi}} \right) X_\Lambda \left( \prod_{u, v \in V} E_{u,v}^{G_{\phi}} \right) = X_\Lambda \left( \prod_{v \in V} Z_{\Lambda G V}^{A_v} \right) = X_{\Lambda G V},$$

(20)

If $\prec$ is a partial order on $V$, then a $\prec$-lowerset is a subset $S \subseteq V$ such that if $u \prec v$ for some $v \in S$, then $u \in S$. 

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[7] If $\prec$ is a partial order on $V$, then a $\prec$-lowerset is a subset $S \subseteq V$ such that if $u \prec v$ for some $v \in S$, then $u \in S$. 

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so that
\[ E_G X_k E_G^\dagger = E_G \prod_{k \in V} X_k^A E_G^\dagger = X_k^{A_k} \prod_{v \in V} Z_v^{G_v A_k} \]
(21)

\[ = \prod_{k \in V} X_k^{A_k} Z_{G_k A_k} = \omega \sum_{\zeta \in \beta_k G_k} \sum_{v \in \zeta} X_k^{A_k} Z_{\sum_{\zeta \in \beta_k G_k}} Z_{G_k A_k} \]
(22)

\[ = \omega A_k^{G_k} X_k^{A_k} Z_{G_k A_k} \]
(23)
as claimed.

\[ \square \]

2. \( \mathbb{Z}_d \)-flow and outcome determinism

This leads us to the statement of our novel flow condition. It is a strict generalisation of gflow to qudits, which must take into account the additional freedom in open graphs described above.

**Definition 9.** \((G,I,O,\lambda)\) has a \( \mathbb{Z}_d \)-flow \((C,\Lambda)\) if \( C \) is a matrix in \( \mathbb{Z}_d^{V \times V} \) and \( \Lambda \) a totally ordered partition of \( V \) such that

(a) \( \forall u \in O^*, \lambda(u) = (C_{uu}, (GC)_{uu}) \);

(b) \( C[I,V] = 0 \) and \( C[V,O] = 0 \);

(c) for any \( M,N \in \Lambda \),
   
   1. \( C[M,M] \) and \( (GC)[M,M] \) are diagonal;
   
   2. whenever \( M \prec N \), \( C[M,N] = (GC)[M,N] = 0 \).

We call \( \Lambda \) a layer decomposition of \((G,I,O,\lambda)\) for \( C \) and the elements of \( \Lambda \) are layers.

If \((G,I,O,\lambda)\) is a labelled open graph with \( \mathbb{Z}_d \)-flow \((C,\Lambda)\), then we obtain a runnable MBQC \((G,I,O,\lambda,x_C,z_C)\) by imposing

\[ x_C(v) := (C_{\bullet v} - \lambda(v)1_{\{v\}}) \quad \text{and} \quad z_C(v) := ((GC)_{\bullet v} - \lambda(v)1_{\{v\}}), \]
(24)

where \( M_{\bullet v} \) is the \( v \)-th column of a matrix \( M \).

The layer decomposition \( \Lambda \) describes a (partial) measurement order for the non-output qudits: the qudits can be measured in any totalisation of the order induced on \( O^* \) by the order of \( \Lambda \), and qudits within the same layer can be measured simultaneously. This order is a (not necessarily strict) extension of the order \( \prec \) induced by \( x_C,z_C \) as described in the previous section.

The \( u \)-th columns (minus the \( u \)-th element) of \( C \) and \( GC \) describe where to apply \( X \) and \( Z \) corrections for the measurement of vertex \( u \in V \), respectively. The elements in the \( u \)-th columns above \( u \) then correspond to qudits that have already been measured, and must be zero for there to be no unwanted back-action. The elements in the \( u \)-th columns that corresponds to vertices in the same layer as \( u \) must also be \( 0 \), since those vertices can be measured before \( u \). These considerations impose condition (c).

Condition (b) follows from the fact that the outputs are not measured and thus no correction is needed. Furthermore, we cannot apply \( X \) corrections at an input vertex, since the measurement pattern we introduce relies on the fact that \( X_u N_u = N_u \). Finally, the \( u \)-th element of the \( u \)-th column describes what correction is applied at vertex \( u \), \( X_{uu}^{A(u)} Z_{(GC)_{uu}} \), when we follow this procedure. This correction must match the measurement space assigned to \( u \) so that we can use the back-action to perform the correction, which implies condition (a).
Then this MBQC is deterministic and implements an isometry:

**Theorem 10.** Suppose the graph state \((G, I, O, \lambda)\) has \(\mathbb{Z}_d\)-flow \((C, \Lambda)\), then the MBQC \((G, I, O, \lambda, x_C, z_C)\) is runnable and robustly deterministic. Furthermore, for a given choice of measurements \(\mathcal{M}\), it realises the isometry

\[
\mathcal{H}^\otimes I \longrightarrow \mathcal{H}^\otimes O
\]

\[
|\phi\rangle \mapsto \bigotimes_{u \in O^f} \langle 0 : M(u) | E_G \left( |\phi\rangle \bigotimes_{u \in Z^f} | 0 : X \rangle \right). \tag{25}
\]

**Proof.** Assume \((G, I, O, \lambda)\) has a \(\mathbb{Z}_d\)-flow \((C, \Lambda)\). We perform the measurements in the order given by any totalisation of the order induced by \(\Lambda\) on \(V\). We measure qudit \(u\) with a \(M\)-measurement, and we obtain a classical outcome \(s_u \in \mathbb{Z}_d\). Let \(Q_u := X^u_s Z^u_{GC(s)}\) then by lemma 3, the action of any measurement in \(\mathcal{M}(\lambda(u))\) correspond to the application on qudit \(u\) of the projector \(\langle m : M = \langle 0 : M | Q_u^m \rangle\). Thus a correction must consist in simulating the application of \(Q^{-s_u}\) on \(u\). The definition of \(\mathbb{Z}_d\)-flow implies that \(C\) and \(GC\) must be lower triangular, so that \(X_{(C(u)) \setminus \{u\}} Z_{GC(u) \setminus \{u\}}\) acts only on unmeasured qudits, where \(A \setminus \{u\}\) removes all the occurrences of \(u\) in \(A\):

\[
A \setminus \{u\} = v \mapsto \begin{cases} 0 & \text{if } u = v, \\ A(v) & \text{otherwise}. \end{cases} \tag{26}
\]

Then we have that:

\[
X^u_{(C(u)) \setminus \{u\}} Z^u_{GC(u) \setminus \{u\}} |G\rangle = X^u_{(C(u)) \setminus \{u\}} Z^u_{GC(u) \setminus \{u\}} Q^u_{\lambda^{-s_u}} |G\rangle = X^u_{(C(u)) \setminus \{u\}} Z^u_{GC(u) \setminus \{u\}} Q^u_{\lambda^{-s_u}} |G\rangle. \tag{27}
\]

As a consequence, the correction \(X^u_{(C(u)) \setminus \{u\}} Z^u_{GC(u) \setminus \{u\}}\) is runnable and makes the computation uniformly deterministic.

Since all the branch maps are equal, the computation is strongly deterministic, and since we have considered only a single measurement and the associated corrections, it is stepwise deterministic.

In [DK06] it was shown that if a measurement pattern is strongly deterministic then it implements an isometry. Since we correct each measurement to the outcome \(m = 0\), it is clear that the final isometry is given by

\[
\mathcal{H}^\otimes I \longrightarrow \mathcal{H}^\otimes O : \prod_{u \in O^f} \langle 0 : M_u | E_G | N_u \rangle \quad \text{as claimed.} \quad \square
\]

### 2.1. Recovering other flow conditions

\(g\)-flow was originally formulated in terms of a partial order on the vertices to be measured [Bro+07]. It is easy to see that the order of the layer decomposition \(\Lambda\) induces a (non-unique) partial order on the vertices \(V\). Given a partial order \(\prec\) on the vertices \(V\), then there is of course a (also non-unique) ordered partition of \(V\) that agrees with \(\prec\). Since either of these orders are only used to describe the measurement order for the vertices of the graph, we can write the \(\mathbb{Z}_d\)-flow condition in terms which are closer to [Bro+07] (stated without proof):

**Lemma 11 (Partial order \(\mathbb{Z}_d\)-flow).** \((G, I, O, \lambda)\) has a \(\mathbb{Z}_d\)-flow if and only if there exists a matrix \(C \in \mathbb{Z}_d^{V \times V}\) and a partial order \(\prec\) on \(V\) such that

\(a) \forall u \in O^f, \lambda(u) = (C_{uv}, (GC)_{uv}); \)
\(b) \ C_{uv} = 0\) whenever \(u \in I\) or \(v \in O\);
(c) when the columns and rows of $G$ and $C$ are ordered according to any totalisation of $\prec$, $C$ and $GC$ are lower triangular.

It is straightforward from this formulation to recover the gflow condition, since the parity conditions in the original formulation correspond to linear equations over $\mathbb{Z}_2$ (also stated without proof):

**Proposition 12.** An open $\mathbb{Z}_2$-graph $(G, I, O, \lambda)$ has a gflow if and only if it has a $\mathbb{Z}_2$-flow.

Although the semantics are subtly different, we also can also recover CV-flow [BM21] as a special case of our definition:

**Proposition 13.** An open $\mathbb{R}$-graph $(G, I, O, \lambda)$ such that $\lambda(O^c) = \{0, 1\}$ has a CV-flow if and only if it has an $\mathbb{R}$-flow.

### 2.2. The converse result

It has been shown in the qubit case that any measurement pattern that is robustly deterministic is such that the underlying open graph has a gflow [Bro+07]. We generalise this result to the case of qudits:

**Theorem 14.** If $(G, I, O, \lambda, x, z)$ is a robustly deterministic MBQC on $\mathbb{Z}_d$, then the underlying open $\mathbb{Z}_d$-graph $(G, I, O, \lambda)$ has a $\mathbb{Z}_d$-flow $(C, \Lambda)$ such that $x = x_C$ and $z = z_C$.

Our proof of theorem 14 relies crucially on the following lemma:

**Lemma 15.** Let $(G, I, O, \lambda)$ be an open graph, $|\psi\rangle, |\psi'\rangle \in (\mathcal{H} \otimes V)$, and $R \subseteq V$. For any $M : R \rightarrow U(\mathcal{H})$ such that, for each $u \in V$, $M(u) \in M(\lambda(u))$, and $m \in \mathbb{Z}_d^R$, put $|m : M\rangle = \bigotimes_{r \in R} |m_r : M(r)\rangle$. If, for every such $m$ and $M$, we have

$$\langle \tilde{m} : M | \phi \rangle \simeq \langle \tilde{m} : M | \phi' \rangle \quad \text{and} \quad \| \langle \tilde{m} : M | \phi \rangle \| = \frac{1}{\sqrt{d^{|R|}}} = \| \langle \tilde{m} : M | \phi' \rangle \|,$$

(28)

then there is a subset $L \subseteq R$, $\bar{x}, \bar{y} \in \mathbb{Z}_d^L$ and $|\psi\rangle \in \mathcal{H} \otimes V^L$ such that

$$|\phi\rangle \simeq |\psi\rangle \bigotimes_{n \in L} |x_n : Q_n\rangle \quad \text{and} \quad |\phi'\rangle \simeq |\psi\rangle \bigotimes_{n \in L} |y_n : Q_n\rangle.$$

(29)

It has a rather lengthy and technical proof left to appendix A. Instead, we provide a sketch of the proof:

**Proof (sketch).** The proof takes place in three parts:

(a) We show that, under the hypotheses of the lemma, both $|\phi\rangle$ and $|\phi'\rangle$ must have a Schmidt decomposition which is ‘diagonal in the $Q$ basis’, i.e. of the form:

$$|\phi\rangle = \sum_{x \in \mathbb{Z}_d} c_x |x : Q\rangle \otimes |\psi_x\rangle.$$

(30)

This is the content of lemma 25, and is proved by reworking a subset of the family equations of the right side of (28) into a more amenable form. We then carefully pick out a further subset using the orthogonality of trigonometric functions to find a system which forces the Schmidt decomposition.
(b) We use the Schmidt decomposition to simplify the equations on the left side of (28), resulting in yet another system of equations which is used to prove the lemma in the subcase where the subset $R$ consists in a singleton. This is lemma 26.
(c) An induction on the size of $R$ is used to prove the lemma in full generality.

In the original gfow articles [Bro+07, Dan+09], this lemma was not taken into account in full generality, yielding an incomplete proof of the converse result. The proof of the converse is however fixed in [PSM] for the qubit case, and we draw inspiration from that work in our generalisation.

**Proof of theorem 14.** Let $\prec$ be the order on $O^\ell$, and consider the last measurement made according to some totalisation of $\prec$. Suppose it is made at vertex $u$. Let $\mathbf{M} : O^\ell \to U(\mathcal{H})$ be such that $\mathbf{M}(v) \in \mathcal{M}(\lambda(v))$ for all $v \in O^\ell$. Performing the measurement with outcome $m$, there is a corresponding correction $X_m^m Z_m^m$ that acts only on outputs, and which induces the branch map:

$$|G(\phi)\rangle \mapsto X_m^m Z_m^m \left( \bigotimes_{v \in O^\ell \setminus \{u\}} \langle 0 : \mathbf{M}(v) \rangle \right) |G(\phi)\rangle$$

$$= X_m^m Z_m^m \left( \bigotimes_{v \in O^\ell} \langle 0 : \mathbf{M}(v) \rangle \right) Q_v^m |G(\phi)\rangle,$$

$$= \left( \bigotimes_{v \in O^\ell} \langle 0 : \mathbf{M}(v) \rangle \right) X_m^m Z_m^m Q_v^m |G(\phi)\rangle,$$

where $Q_u = X_u^{\lambda(u)} Z_u^{\lambda(u)}$ is the measurement error of the measurement of vertex $u$.

Since the overall computation is robustly deterministic by assumption, for any such choice of measurements $\mathbf{M}$, we must have

$$\left( \bigotimes_{v \in O^\ell} \langle 0 : \mathbf{M}(v) \rangle \right) X_m^m Z_m^m Q_v^m |G(\phi)\rangle \approx \left( \bigotimes_{v \in O^\ell} \langle 0 : \mathbf{M}(v) \rangle \right) |G(\phi)\rangle.$$

(32)

In particular, for any $\mathbf{M}(v) \in \mathcal{M}(\lambda(v))$, by proposition 3 we have $Q_v^{-m} \mathbf{M}(v) Q_v^m \in \mathcal{M}(\lambda(v))$, and

$$\langle 0 : Q_v^{-m} \mathbf{M}(v) Q_v^m \rangle = \langle 0 : \mathbf{M}(v) \rangle Q_v^m = \langle m : \mathbf{M}(v) \rangle.$$

(33)

It follows that for any choice of measurements $\mathbf{M}$ and any $\tilde{m} \in \mathbb{Z}_d^O$,

$$\langle \tilde{m} : \mathbf{M} |X_{\tilde{m}}^m Z_{\tilde{m}}^m Q_v^m |G(\phi)\rangle \approx \langle \tilde{m} : \mathbf{M} |G(\phi)\rangle,$$

(34)

so by lemma 15, there is a subset $L \subseteq O^\ell$, vectors $\tilde{x}, \tilde{y} \in \mathbb{Z}_d^{[L]}$ and a state $|\psi\rangle \in \mathcal{H}^\otimes V \setminus L$ such that

$$X_{\tilde{m}}^m Z_{\tilde{m}}^m Q_v^m |G(\phi)\rangle \approx |\psi\rangle \bigotimes_{n \in L} |x_n : Q_n\rangle \quad \text{and} \quad |G(\phi)\rangle \approx |\psi\rangle \bigotimes_{n \in L} |y_n : Q_n\rangle.$$
Then,
\[
X_{x(u)}^{m}Z_{z(u)}^{m}Q_{n}^{m} |\psi\rangle \bigotimes_{n \in L} |x_{n} : Q_{n}\rangle \simeq X_{x(u)}^{m}Z_{z(u)}^{m}Q_{n}^{m} |G(\phi)\rangle \\
\simeq |G(\phi)\rangle \simeq |\psi\rangle \bigotimes_{n \in L} |y_{n} : Q_{n}\rangle.
\]  
(36)

If \( u \notin L \), then since the corrections only act on outputs this implies that
\[
(X_{x(u)}^{m}Z_{z(u)}^{m}Q_{n}^{m} |\psi\rangle \bigotimes_{n \in L} |x_{n} : Q_{n}\rangle \simeq |\psi\rangle \bigotimes_{n \in L} |y_{n} : Q_{n}\rangle,
\]
so we must have \( x_{n} = y_{n} \) for all \( n \in L \).

If \( u \in L \),
\[
(X_{x(u)}^{m}Z_{z(u)}^{m}Q_{n}^{m} |\psi\rangle \bigotimes_{n \in L} |x_{n} : Q_{n}\rangle \\
\simeq (X_{x(u)}^{m}Z_{z(u)}^{m} |\psi\rangle) \otimes Q_{n}^{m} |x_{n} : Q_{n}\rangle \bigotimes_{n \notin L \setminus \{u\}} |x_{n} : Q_{n}\rangle \\
\simeq (X_{x(u)}^{m}Z_{z(u)}^{m} |\psi\rangle) \otimes \omega^{\text{max}} Q_{n}^{m} |x_{n} : Q_{n}\rangle \bigotimes_{n \notin L \setminus \{u\}} |x_{n} : Q_{n}\rangle,
\]
which also implies that \( x_{n} = y_{n} \) for all \( n \in L \). Then,
\[
X_{x(u)}^{m}Z_{z(u)}^{m}Q_{n}^{m} |G(\phi)\rangle \simeq |G(\phi)\rangle,
\]
(39)
and \( X_{x(u)}^{m}Z_{z(u)}^{m}Q_{n} \) stabilises the graph state for any \( m \in \mathbb{Z}_{d} \), up to a phase \( e^{i\alpha} \). By proposition 8 there is some multiset \( C_{uv} \in \mathbb{Z}_{d}^{\mathcal{V}} \) such that
\[
e^{i\alpha} X_{x(u)}^{m}Z_{z(u)}^{m} Q_{n} = \omega^{\text{C}_{uv} \cdot \mathcal{G} \cdot \mathcal{C}_{uv}} X_{x(u)}^{m}Z_{z(u)}^{m} \quad \text{and} \quad (C_{uv})_{v} = 0 \quad \text{if} \quad v \notin I.
\]  
(40)

The corrections act only on outputs, so that the factor of \( X_{x(u)} Z_{z(u)}^{-1} \) acting on \( u \) must be \( Q_{u} S \). This implies that \( X_{x(u)} Z_{z(u)}^{-1} \simeq Q_{u} \) so that \( \lambda(u) = ((C_{uv})_{u}, (G_{uv})_{u}) \), and furthermore, that \( (C_{uv})_{v} = (GC_{uv})_{v} = 0 \) if \( v \notin I \cup \{u\} \) since \( X_{x(u)}^{m}Z_{z(u)}^{m} \) acts only on outputs. Furthermore, tensor products of Paulis form a basis of the space of linear operators, so that we must have
\[
x(v) := (C_{uv} - \lambda(v)1_{\{v\}}) \quad \text{and} \quad z(v) := ((GC_{uv} - \lambda(v)) \mathbb{1}_{\{v\}}).
\]
(41)

Now, consider the open graph \((G, I, O \cup \{u\})\). Since \((G, I, O, \lambda, x, z)\) is robustly deterministic, we can repeat the same procedure on the new open graph
\[
(G, I, O \cup \{u\}, \lambda |_{(O \cup \{u\})^{\mathcal{V}}} x |_{(O \cup \{u\})^{\mathcal{V}}} z |_{(O \cup \{u\})^{\mathcal{V}}}),
\]
(42)
obtaining \( C_{v} \) for the last measured vertex \( v \) in \( \mathcal{O}^{\ell} \setminus \{u\} \). This procedure eventually terminates, and we end up with a column vector \( C_{uw} \) for each \( w \in \mathcal{O}^{\ell} \). Let \( C \in \mathbb{Z}_{d}^{\mathcal{V} \times \mathcal{V}} \) be the matrix whose \( uv \)th column is \( C_{uv} \), or 0 if \( u \in O \). Then from the equations in lemma 21 we see that the pair \((C, <)\) gives a \( \mathbb{Z}_{d}\)-flow for \((G, I, O, \lambda)\) by lemma 11. Furthermore, it is also clear from equations (24) and (41) that \( x = x_{C} \) and \( z = z_{C} \).
3. Some worked-out examples

This section contains a pair of examples of MBQC with $\mathbb{Z}_d$-flow. In order to present these example, we must introduce some notation for depicting open graphs. Unsurprisingly, these will be represented as graphs with some additional data which give the different commands of a measurement pattern:

\begin{equation}
\begin{array}{c}
\begin{array}{c}
(0, 1) \\
\hline
\end{array}
\end{array}
\end{equation}

The additional data consist in:

- **Vertex colouring.** Each of the vertices represents a qudit in the open graph state (as defined in section 1.4): square vertices are inputs to the MBQC, and white vertices are outputs. As a result, each circular vertex corresponds to a qudit that is initialised in the auxiliary state $|0 : X\rangle$.

- **$\mathbb{Z}_d$-weighted edges.** The edges in the graph correspond to entangling operations between the corresponding qudits in the measurement pattern. Since for qudits, we can apply $E^{\lambda}$ for any $\lambda \in \mathbb{Z}_d$, these edges are labelled with the corresponding weight of the entangling gate. In the example, these are $a, b, c \in \mathbb{Z}_d$.

- **Vertex labels.** Every vertex which is not an output must be measured, so we must describe a measurement space for each non-output (black) vertex in the graph. These are given labelling each such vertex by a pair $(x, y) \in \mathbb{Z}_d \times \mathbb{Z}_d \setminus \{(0, 0)\}$, which imposes that a measurement be performed on the corresponding qudit chosen from the measurement space $\mathcal{M}(x, y)$.

Thus, the simple graph depicted above corresponds to a measurement pattern of the form:

\begin{equation}
M_4^{(0)}(\vec{\beta})M_1^{(1)}(\vec{\alpha})E_{1,2}^aE_{2,4}^bE_{3,4}^cN_2N_3N_4
\end{equation}

for some choice of measurement angles $\vec{\alpha}, \vec{\beta} \in [0, 2\pi)^d$ and where we have enumerated the qudits clockwise starting from the top left:

\begin{equation}
\begin{array}{c}
\begin{array}{c}
1 \\
\hline
\end{array}
\end{array}
\end{equation}

The adjacency matrix for this graph, for this ordering of the vertices, is then given by:

\begin{equation}
G = \begin{pmatrix}
0 & a & 0 & 0 \\
0 & 0 & 0 & b \\
a & 0 & 0 & c \\
0 & b & c & 0
\end{pmatrix}.
\end{equation}

Since all qudit initialisations commute with other initialisations, and all entangling operations commute with other entangling operations, no ambiguity is introduced by the fact that the graphical depiction does not impose any order for these operations. Note however, that we...
have not described the corrections to perform to recover outcome determinism, and furthermore that the correction operations do not commute with the different measurements. This information is precisely what is captured in the additional data provided by a \( \mathbb{Z}_d \)-flow for the open graph. In fact, equation (44) is slightly inaccurate, since the measurements are ordered by finding a \( \mathbb{Z}_d \)-flow and this order is not encoded in the graph of equation (43).

3.1. Example 1

Let’s now take the example of equation (43), and check if it has a \( \mathbb{Z}_d \)-flow. For this, we need to find an ordering for the two measurements along with corresponding corrections. In other words, we want to find a Pauli operator acting only on the outputs (since these are the only remaining unmeasured qudits) whose back-action on the measured vertices corresponds exactly to the required correction.

Let’s consider the back-actions which result from enacting Pauli operators on the outputs:

(a) if we perform an \( X^m \) correction on qudit 2, we can see by simplifying by an open graph stabiliser, that the back-action on measured vertices is:

\[
\begin{align*}
\begin{array}{c}
\text{a} \\
\vdots
\end{array} & \begin{array}{c}
\text{X}^m \\
\vdots
\end{array} = \begin{array}{c}
\text{Z}^{-am} \\
\vdots
\end{array} \begin{array}{c}
\text{a} \\
\vdots
\end{array} \begin{array}{c}
\text{X}^m \\
\vdots
\end{array} \begin{array}{c}
\text{X}^{-m} \\
\vdots
\end{array} = \begin{array}{c}
\text{Z}^{-am} \\
\vdots
\end{array} \begin{array}{c}
\text{a} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-bm} \\
\vdots
\end{array} \\
\begin{array}{c}
\text{c} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-bm} \\
\vdots
\end{array} \begin{array}{c}
\text{c} \\
\vdots
\end{array}
\end{align*}
\]

(b) similarly, if we perform an \( X^n \) correction on qudit 3, the back-action on measured vertices is:

\[
\begin{align*}
\begin{array}{c}
\text{a} \\
\vdots
\end{array} & \begin{array}{c}
\text{X}^n \\
\vdots
\end{array} = \begin{array}{c}
\text{Z}^{-cn} \\
\vdots
\end{array} \begin{array}{c}
\text{b} \\
\vdots
\end{array} \begin{array}{c}
\text{c} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-cn} \\
\vdots
\end{array} \\
\begin{array}{c}
\text{a} \\
\vdots
\end{array} & \begin{array}{c}
\text{Z}^{-cn} \\
\vdots
\end{array} \begin{array}{c}
\text{a} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-cn} \\
\vdots
\end{array} \\
\begin{array}{c}
\text{c} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-cn} \\
\vdots
\end{array} \begin{array}{c}
\text{c} \\
\vdots
\end{array}
\end{align*}
\]

(c) and if we perform an \( Z^o \) correction on qudit 2, the back-action is:

\[
\begin{align*}
\begin{array}{c}
\text{a} \\
\vdots
\end{array} & \begin{array}{c}
\text{Z}^o \\
\vdots
\end{array} = \begin{array}{c}
\text{Z}^{-b^{-1}o} \\
\vdots
\end{array} \begin{array}{c}
\text{c} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-c^{-1}b^{-1}o} \\
\vdots
\end{array} \begin{array}{c}
\text{a} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-b^{-1}o} \\
\vdots
\end{array} \begin{array}{c}
\text{b} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-c^{-1}b^{-1}o} \\
\vdots
\end{array} \begin{array}{c}
\text{c} \\
\vdots
\end{array}
\end{align*}
\]

We can eliminate the resulting correction on qudit 3 since it has not been (and is never) measured. In fact, this third correction is perhaps better understood as the compound operation:

\[
\begin{align*}
\begin{array}{c}
\text{a} \\
\vdots
\end{array} & \begin{array}{c}
\text{Z}^{-bo} \\
\vdots
\end{array} = \begin{array}{c}
\text{Z}^{-b^{1}o} \\
\vdots
\end{array} \begin{array}{c}
\text{b} \\
\vdots
\end{array} \begin{array}{c}
\text{Z}^{-c^{-1}b^{-1}o} \\
\vdots
\end{array} \begin{array}{c}
\text{c} \\
\vdots
\end{array}
\end{align*}
\]

which implements an \( X \) correction on qudit 4.

Now, what corrections do we actually want to implement? We want to correct a measurement of vertex 1 in the measurement space \( \mathcal{M}(0, 1) \). By linearity of the corrections, it suffices
to consider the case when the measurement outcome is 1, and the corresponding correction is a \( Z \) Pauli acting on qudit 1. Similarly, we want to correct a measurement of vertex 4 in the measurement space \( \mathcal{M}(1,0) \)—the corresponding correction is an \( X \) Pauli acting on qudit 4.

We can assemble the corrections from equations (47)–(50) into a system of equations:

\[
\begin{align*}
-ma & = 1 \\
-bm - cn & = 0 \\
o & = 1
\end{align*}
\]  

(51)

where the first equation encodes the correction for qudit 1, the second equation encodes the fact that we do not want to induce an additional back-action \( Z \) on qudit 4 from our correction, and the last equation encodes the correction for qudit 4.

This system of equations admits a solution

\[
\begin{align*}
m & = -a^{-1} \\
n & = a^{-1}bc^{-1} \\
o & = 1
\end{align*}
\]  

(52)

for any choice of (prime) dimension \( d \), so long as \( a \) and \( b \) are non-zero. Accordingly, we can actually perform both corrections simultaneously, and there is no need to order the two measurements—they can also be performed simultaneously. We formalise this methodology for finding \( \mathbb{Z}_d \)-flow in the following section, resulting in a polynomial-time algorithm for \( \mathbb{Z}_d \)-flow.

Let’s now see how this fits into the definition of \( \mathbb{Z}_d \)-flow. In definition 9, we describe a second matrix \( C \) which must verify certain properties with respect to the adjacency matrix \( G \) (equation (46)). The \( u \)th column of \( C \) simultaneously describes a hypothetical set of \( X \) corrections to perform on the open graph state. The resulting \( Z \) corrections can be obtained by applying the adjacency matrix \( G \) to this column vector.

Since we want to apply an \( Z \) correction to qudit 1, we must therefore have \( C_{1,1} = 0 \) (no \( X \) correction is applied to qudit 1) and \((GC)_{1,1} = 1\). Similarly, we must have \( C_{4,4} = 1 \) and \((GC)_{4,4} = 0\). We do not want to make any correction on output vertices, so the corresponding columns of \( C \) and \( GC \) must be 0. Finally, we do not want to make back-action acting on previously-measured and already-corrected vertices. Put otherwise, we want all the corrections commands we need to implement for a given measurement to act only on unmeasured vertices, so that they can actually be implemented. If we order the vertices according to a valid measurement order, we therefore want \( C \) and \( GC \) to be lower diagonal, since elements above the diagonal then correspond to precisely such unwanted back-actions. In our case, since there is no forced order between the measurements of qudits 1 and 4, we can order the vertices and thus the columns of \( G, C \) and \( GC \) in the order \( 1 \prec 4 \prec 2 \prec 3 \) (leaving the outputs for last since they are never measured).

Then, the product

\[
\begin{pmatrix}
0 & 0 & a & 0 \\
0 & b & c & 0 \\
a & b & 0 & 0 \\
0 & c & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-a^{-1} & 0 & 0 & 0 \\
-a^{-1}bc^{-1} & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -b & 0 & 0 \\
0 & -c & 0 & 0
\end{pmatrix}
\]  

(53)
can be checked to verify all of these considerations, and by extension the properties required in definition 9. It therefore describes a valid $\mathbb{Z}_d$-flow for the graph of equation (43). It encodes the same information as the set of equations (51), as can be seen by identifying $m$, $n$ and $o$:

$$
\begin{bmatrix}
0 & 0 & a & 0 \\
0 & b & c & 0 \\
a & b & 0 & 0 \\
0 & c & 0 & 0
\end{bmatrix}_G
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
m & 0 & 0 & 0 \\
n & 0 & 0 & 0
\end{bmatrix}_C
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

(54)

3.2. Example 2

We now turn our attention to a slightly more complicated example in the specific case of $d = 5$ of an MBQC based on a graph with $\mathbb{Z}_5$-flow and describe the actual measurement pattern in more detail. Consider the open graph:

and where all the measurements are made in $\mathcal{M}(0, 1)$, that is, $\lambda(v) = (0, 1)$ for each non-output $v \in O^c$.

For the sake of brevity, we omit the step-by-step description of the flow-finding algorithm which we carried out in the previous example. It can be straightforwardly, if somewhat laboriously, verified (by checking the properties of definition 9) that the open graph has a simple $\mathbb{Z}_5$-flow given by the layer decomposition

$$
\Lambda_2 = I \quad \Lambda_1 \quad \Lambda_0 = O
$$

(56)
and correction matrix

\[
\begin{pmatrix}
3 & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & 4 \\
3 & 1 & 0 \\
1 & 1 & 1 \\
0 & 0 & 4 \\
3 & 1 & 0 \\
2 & 0 & 4 \\
0 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
3 & 2 & 0 \\
1 & 0 & 1 \\
0 & 4 & 1 \\
3 & 2 & 0 \\
2 & 4 & 0 \\
0 & 0 & 0 \\
3 & 2 & 0 \\
2 & 4 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

\( G \) \hspace{1cm} \( C \)

where the empty matrix elements are taken to be 0.

We assume the corresponding graph state has been produced, and consider only the measurements and corrections described by the flow. The first step of the measurement procedure is to measure the vertices in the first layer in the measurement order, here \( \Lambda_2 \). We label the measurement outcomes \( m_1, m_2, m_3 \) from top to bottom, and the corresponding corrections are described by the submatrices

\[
C[\Lambda_2^6, \Lambda_2] = 
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 4 & 1 \\
4 & 3 & 4 & 0 \\
\end{pmatrix}
\]  \hspace{1cm} \hspace{1cm}

(57)

The first column of each matrix gives the \( \text{X} \) and \( \text{Z} \) corrections respectively for the first measurement, so that we can correct for the measurement outcome \( m_1 \) by implementing the following operations:

\[
C[\Lambda_2^6, \Lambda_2] = \begin{pmatrix}
3 & 2 & 0 \\
2 & 4 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\hspace{1cm} \text{and} \hspace{1cm}
(GC)[\Lambda_2^6, \Lambda_2] = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 4 & 1 \\
\end{pmatrix}
\]

(58)

The first column of each matrix gives the \( \text{X} \) and \( \text{Z} \) corrections respectively for the first measurement, so that we can correct for the measurement outcome \( m_1 \) by implementing the following operations:

\[
\begin{align*}
&X^{3m_1} \quad 3 \quad 1 \\
&X^{2m_1} \quad 2 \\
&X^{2m_1} \quad 4 \\
&Z^{m_1} \quad Z^{m_1} \quad Z^{m_1} \quad Z^{m_1}
\end{align*}
\]

(59)
at the corresponding qudits in the graph state. The back-action of these corrections on the first vertex is precisely the operation $z^{m_1}$ needed to eliminate the measurement error. Similarly, we can implement the corrections for the second and third vertices by doing the corrections:

\begin{align}
X^{2m_2} & \quad \text{and} \quad Z^{4m_2} \\
X^{4m_2} & \quad Z^{4m_2}
\end{align}

respectively. By linearity of the correction operators, we can implement these three corrections in one go:

\begin{align}
X^a & \quad Z^d \\
X^b & \quad Z^e \\
X^c & \quad Z^f
\end{align}

where
\begin{align}
a &= 3m_1 + 2m_2 \\
b &= 3m_1 + 2m_2 \\
c &= 2m_1 + 4m_2 \\
d &= m_1 \\
e &= m_1 + 4m_2 + m_3 \\
f &= 4m_1 + 3m_2 + 4m_3
\end{align}

The corrections for the second layer in the measurement order, $\Lambda_1$, are given by

\begin{align}
C[\Lambda^c_1, \Lambda_1] &= \begin{pmatrix} 1 & 4 & 2 \\ 4 & 4 & 2 \\ 4 & 2 & 3 \end{pmatrix} \\
(GC)[\Lambda^c_1, \Lambda_1] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align}

Writing $m_4, m_5, m_6$ for the measurement outcomes from top to bottom, the corrections for the entire layer can then be implemented by:

\begin{align}
X^{m_4+4m_5+2m_6} & \\
X^{4m_4+4m_5+2m_6} & \\
X^{4m_4+2m_5+3m_6}
\end{align}
4. A polynomial-time algorithm for $\mathcal{Z}_d$-flow

```
input: A labelled open graph $(G, I, O, \lambda)$
output: A $\mathcal{Z}_d$-flow $(C, \lambda)$ or fail
1: Procedure $Z$-Flow $(G, I, O, \lambda)$
2: find $L := \{ u \in V \mid (\forall v \in V) : G_{uv} = 0 \}$ \hfill \triangleright Isolated vertices
3: layer$(0) := O \cup L$
4: $C := O_{V \setminus [V]}$
5: return $Z$-Flow-aux $(G, I, O \cup L, \lambda|_{O \cup L}, C, \text{layer}, 1)$
6: procedure $Z$-Flow-aux $(G, I, O, \lambda, C, \text{layer}, k)$
7: $L := \emptyset$ \hfill \triangleright Vertices which we are correcting in this layer
8: for all $v \in O$ do
9: $(a, b) := \lambda(v)$
10: solve in $\mathcal{Z}_d: G[O, O \setminus I]\bar{a} = b1_{\{v\}} - aG[O, \{v\}]$
11: if there is a solution $\bar{a}$ then
12: $L := L \cup \{v\}$ \hfill \triangleright Assign $v$ to the current layer
13: $C(O \setminus I, \{v\}) := \bar{a}$ \hfill \triangleright The corrections for vertex $v$
14: $C(\{v\}, \{v\}) := a$
15: if $L = \emptyset$ then \hfill \triangleright If we cannot correct for additional vertices, either:
16: if $O = \emptyset$ then
17: return $(C, \text{layer})$ \hfill \triangleright we have found a $\mathcal{Z}_d$-flow; or,
18: else
19: fail \hfill \triangleright there is no $\mathcal{Z}_d$-flow.
20: else
21: layer$(k) := L$
22: return $Z$-Flow-aux $(G, I, O \cup L, \lambda|_{O \cup L}, C, \text{layer}, k + 1)$
```

4.1. Correctness

This algorithm is strongly inspired by the analogous algorithm for finding gfows for $\mathcal{Z}_2$-graphs [MP08], and its generalisations to multiple measurement planes [Bac+21] and to finding CV-flows [BM21].

**Theorem 16.** $(G, I, O, \lambda)$ has a $\mathcal{Z}_d$-flow if and only if the algorithm above returns a valid $\mathcal{Z}_d$-flow.

**Proof.** It is clear the algorithm terminates, since at each call to $Z$-Flow-Aux, the algorithm either passes vertices from $V \setminus O$ to $O$, returns a $\mathcal{Z}_d$-flow, or fails. Since $V$ is finite, there are a finite number of recursions after which the algorithm either returns a $\mathcal{Z}_d$-flow or fails.

(Outputs a valid $\mathcal{Z}_d$-flow) Suppose the algorithm terminates with a pair $(H, C)$. We need to show this defines a valid $\mathcal{Z}_d$-flow. Consider the function $Z$-Flow-Aux at a given call, and let $H' := G(O, O \setminus I)$ and $h := G(O, v)$.

The output columns of $C$ are 0, and since the solution vector $x$ from line 11 never contains an input, the input rows of $C$ are also 0. Hence condition (a) is satisfied.

Similarly, the solution vector $x$ only has rows labelled by vertices $v \geq u$, so $C$ is lower triangular by construction. If the linear equation in line 11 is satisfied, then the entries above $(HC)_{uw}$ in the $u$th column of $HC$ will be 0. Hence (b) is satisfied. Indeed, for any $u > v$, $(HC)_{uw} = \sum_{w < v} H_{uw} C_{vw} = \sum_{w < u} H_{uw} C_{vw} + H_{uw} C_{uv} + \sum_{w > u} H_{uw} C_{vw} = h_u a + \sum_{w > u} H'_{uw} x_w$. Since $C_{uw} = a$, $H_{uw} = h_u$, $\forall w < u, C_{uw} = 0$, and $\forall w > u, H_{uw} = H'_{uw}$ and $C_{uw} = x_w$. As a consequence $(HC)_{uw} = (ah + bH'_{uw}) = (ah + b(u) - ah)y$.

Finally, for a non-output $u$, $C_{uw} = a$. As a consequence $C$ is an $\mathcal{Z}_d$-flow for $H$. 

22
(Outputs a valid layer decomposition) Let \( L \) be as in line 21 of the algorithm for some call \( k \) to \( Z\text{-Flow-Aux} \). It is clear that the equation line 10 does not depend on the vertices in \( L \) which appear before or after \( \{ v \} \) and therefore \( L \) is independent of the order in which the elements of \( \mathcal{L} \) are found. As a result, the output of the algorithm is invariant any permutation of the vertices in \( L \). Since this corresponds tautologically to a permutation of the layer \( V_L \) output by the algorithm, and every permutation that preserves the partition can be written as a product of such permutations, the \( Z_d \)-flow found by the graph is invariant under permutations that preserve the layers (whenever the algorithm succeeds).

(Outputs a \( Z_d \)-flow whenever there is one) Assume the algorithm fails, that is, for some call to \( Z\text{-Flow-Aux} \), line 10 has no solution for any remaining unfinished vertices. Let \( \overline{D} \) be the third parameter at that function call, and further assume that \( D \) is a \( Z_d \)-flow for \((G, I, O, \lambda) \).

Let \( \mathcal{D} \) be the matrix obtained by replacing the columns in \( D \) corresponding to \( \overline{D} \) with zeros and permuted such that the columns \( \overline{D} \) appear last. Then, \( \mathcal{D} \) is a \( Z_d \)-flow for \((G, I, \overline{D}, \lambda_{\overline{D}}) \). Let \( v \in \overline{D} \) be the last column before \( \overline{D} \), and put \( c := \mathcal{D} I \{ \overline{D}, \{ v \} \} \). Then,

\[
(G\mathcal{D}, \mathcal{D} \setminus I(c))_{\mathcal{D}} = \sum_{j \in \mathcal{D}, \overline{D}, \{ v \}} G_{ij} c_j = \sum_{j \in \mathcal{D}} G_{ij} \overline{D}_{jv} = \sum_{j \in \mathcal{D}} G_{ij} D_{jv} = \sum_{j \in \overline{D}} G_{ij} D_{jv} = \sum_{j \in \mathcal{V}} G_{ij} D_{jv} - \sum_{j \in \mathcal{D}} G_{ij} D_{jv} = (GD)_{\mathcal{V}} - \sum_{j \in \mathcal{D}} G_{ij} D_{jv} = b\delta_{u,v} - G_{uv} D_{uv} = b\delta_{u,v} - aG_{uv}.
\]

As a result, we see that \( c \) verifies the equation of line 10, which contradicts the failure of the algorithm. It follows that \((G, I, O, \lambda) \) cannot have a \( Z_d \)-flow if the algorithm fails. By contraposition, if \((G, I, O, \lambda) \) has a \( Z_d \)-flow, the algorithm succeeds. \( \Box \)

The core of the algorithm is the loop line 8. Letting \( n = \left| \mathcal{V} \right|, \ell = |O| \) and \( \ell' = |O \setminus I| \) at a given call to \( Z\text{-Flow-Aux} \), note that \( \ell' \leq \ell \leq n \). The loop amounts to solving \( n - \ell \) systems of \( n - \ell \) equations in \( \ell \) variables. Let \( x_i \) be the right hand side of equation line 10. Solving the system can be done by transforming the matrix

\[
[\mathcal{G}(\mathcal{D}, \mathcal{O} \setminus I) | x_{\ell+1} | \cdots | x_{n-\ell}]
\]

to upper echelon form. This can be done in time \( O(n^3) \) by Gaussian elimination, and backsubstituting to find the corresponding \( c_i \) to each \( x_i \) takes time \( O(n^2) \) or for all solutions \( O(n^3) \) since there are at most \( n \) backsubstitutions to perform. Finally, since each call to \( Z\text{-Flow-Aux} \) either eliminates a vertex or terminates, the algorithm recurses at most \( n \) times. The total complexity is therefore \( O(n^4) \).

Note this procedure can also be adapted to find a \( Z_d \)-flow for any labelling, rather than one fixed in advance. First, note that, for the existence of a \( Z_d \)-flow, it suffices to choose measurement planes up to a scalar factor. That is, \((C, \lambda) \) is a \( Z_d \)-flow for \((G, I, O, \lambda) \) with \( \lambda(u) = (a, b) \) if and only if it is a \( Z_d \)-flow for \((G, I, O, \lambda') \) where \( \lambda'(u) = (ka, kb) \). Hence we can solve for measurement planes at the same time as \( C \) by either fixing \( a = 1 \) and solving for \( b \) in the equation line 10 of the algorithm, or for non-inputs, fixing \( b = 1 \) and solving for \( a \).

4.2. Depth optimality

Our proofs follow the structure of [MP08], which introduced the idea of optimising gflows starting from the last layer and working back. The idea is to find corrections for as many measured vertices as possible at the part of the MBQC when there are the most constraints on possible corrections: when the only vertices left unmeasured are the outputs. This motivates the
following definition which allows us to conveniently manipulate layer decompositions ‘from
the back’:

**Definition 17.** Let \((C, \Lambda)\) be a \(\mathbb{Z}_d\)-flow for an open graph \((G, I, O, \lambda)\). Then, the depth of \((C, \Lambda)\)
 is \(|\Lambda| - 1\). Furthermore, we define an \(\mathbb{N}\)-indexing of the elements of \(\Lambda\) by:
\[
\Lambda_k := \text{max}(\Lambda \setminus \{\Lambda_n \mid n < k\}),
\]
where we note that \(\Lambda_m < \Lambda_n\) as elements of \(\Lambda\) if and only if \(n > m\), and \(\Lambda_k \neq \emptyset\) if and only if \(k\)
is less than or equal to the depth of \((C, \Lambda)\).

This definition of the depth of a \(\mathbb{Z}_d\)-flow corresponds to the intuitive interpretation: all
measurements (and corresponding corrections) within a layer can be made concurrently, therefore
there is an implementation that runs the MBQC in \(|\Lambda| - 1\) rounds of measurements (since the
outputs are not measured).

Now, we can use this definition to compare the depths of different \(\mathbb{Z}_d\)-flows:

**Definition 18.** Let \((C, \Lambda)\) and \((D, \Phi)\) be \(\mathbb{Z}_d\)-flows for an open graph \((G, I, O, \lambda)\). Then \((C, \Lambda)\)
is more delayed than \((D, \Phi)\) if for each \(k\),
\[
|\bigcup_{n=0}^{k} \Lambda_n| \geq |\bigcup_{n=0}^{k} \Phi_n|.
\]
and this inequality is strict for at least one \(k\). It is maximally delayed if there is no layer decom-
position which is more delayed.

Then, we can give a complete characterisation of the layer decompositions of maximally
delayed \(\mathbb{Z}_d\)-flows, which turn out to be uniquely defined:

**Proposition 19.** If \((C, \Lambda)\) is a maximally delayed \(\mathbb{Z}_d\)-flow for an open graph \((G, I, O, \lambda)\), then
\(\Lambda_0 = O \cup \{u \in V \mid (\forall v \in V) : G_{uv} = 0\}\) and for \(k > 0\),
\[
\Lambda_k = \left\{ u \in \{O \cup \bigcup_{1<n<k} \Lambda_n\}^d \mid \exists c \in \mathbb{R}^V \text{ s.t. } (c_u, (Gc)_u) = \lambda(u)
\right. \forall v \notin (O \cup \bigcup_{1<n<k} \Lambda_n) \cup \{u\}, c_v = (Gc)_v = 0 \right\}.
\]
In particular, if \((C, \Lambda)\) and \((D, \Phi)\) are maximally delayed \(\mathbb{Z}_d\)-flows for the same open graph,
then \(\Lambda = \Phi\).

To make this proof we need the following three lemmas, which have somewhat cumbersome
proofs that we have chosen to leave to appendix B:

**Lemma 20.** If \((C, \Lambda)\) is a maximally delayed \(\mathbb{Z}_d\)-flow for an open graph \((G, I, O, \lambda)\), then
\(\Lambda_0 = O \cup \{u \in V \mid (\forall v \in V) : G_{uv} = 0\}\), i.e. the union of the outputs and isolated vertices of
\((G, I, O, \lambda)\).

**Lemma 21.** If \((C, \Lambda)\) is maximally delayed for \((G, I, O, \lambda)\), then
\[
\Lambda_1 = \left\{ u \in O^d \mid \exists c \in \mathbb{Z}_d^{|V|} \text{ s.t. } (c_u, (Gc)_u) = \lambda(u)
\right. \forall v \notin O \cup \{u\}, c_v = (Gc)_v = 0 \right\}.
\]

**Lemma 22.** If \((C, \Lambda)\) is a maximally delayed \(\mathbb{Z}_d\)-flow of \((G, I, O, \lambda)\), \((D, \Phi)\) is a maximally
delayed \(\mathbb{Z}_d\)-flow of \((G, I, O \cup \Lambda_1, \lambda \mid_{(O \cup \Lambda_1)^c})\), where

- \(D\) is the matrix obtained by replacing the columns of \(C\) corresponding to \(\Lambda_1\) with zeros;
\[ \Phi_k := \begin{cases} \Lambda_k \cup O & \text{if } k = 0; \\ \Lambda_{k+1} & \text{otherwise}. \end{cases} \] (71)

Proof of proposition 19. \( \Lambda_0 \) must take the form given in lemma 20. Then, a recursive application of lemmas 21 and 22 shows that the layer decomposition of a maximally delayed \( \mathbb{Z}_d \)-flow is uniquely defined.

Since the open graph obtained from lemma 22 and used to calculate \( \Lambda_k \) with lemma 21 is \( (G, I, O, S)_{1 < n < k} \Lambda_n, \lambda |(O \cup I)_{1 < n < k} \Lambda_n) \), it is clear that \( \Lambda_k \) must take the form claimed.

This can be understood from the following principle. If a correction exists for the measurement of a vertex that acts only on outputs, then this correction can be performed at any point during the MBQC, since the outputs are never measured and therefore always available for corrections. As a result, we can delay this measurement as much as possible, to the penultimate layer, to give ourselves as much flexibility as possible in corrections for previous layers. As a result, we can put all vertices whose corrections act only on outputs in the final layer. Any vertices which do not verify this property must be in a layer which precedes the penultimate layer, since at the time they are measured there must be non-output vertices which are left unmeasured. Then, it suffices to show that there is a minimal depth \( \mathbb{Z}_d \)-flow that is maximally delayed to obtain:

Proposition 23. A maximally delayed \( \mathbb{Z}_d \)-flow for an open graph \( (G, I, O, \lambda) \) has minimal depth.

Proof. First, note that if \( (C, \Lambda) \) is more delayed than \( (D, \Phi) \), then in particular,

\[ |V| = | \bigcup_{n=0}^{\infty} \Lambda | \Lambda_n | \geq | \bigcup_{n=0}^{\infty} \Lambda | \Phi_n |, \] (72)

so that \( |\Lambda| \leq |\Phi| \). Assume now that \( (D, \Phi) \) has minimal depth, then any \( \mathbb{Z}_d \)-flow that is more delayed has the same depth. It follows that either \( (D, \Phi) \) is maximally delayed and has minimal depth, or there is a maximally delayed \( \mathbb{Z}_d \)-flow that is more delayed than \( (D, \Phi) \) thus has the same depth. But by proposition 19, every maximally delayed \( \mathbb{Z}_d \)-flow has the same layer decomposition for a given open graph, so that every maximally delayed \( \mathbb{Z}_d \)-flow has minimal depth.

Note however that a minimal depth decomposition is not necessarily maximally delayed. For example, we can always measure the entirety of the inputs first without changing the depth, but this measurement order is not always maximally delayed since this allows us to move inputs into earlier layers. Since the algorithm is constructed such that it finds a maximally delayed \( \mathbb{Z}_d \)-flow:

Theorem 24. The algorithm outputs a \( \mathbb{Z}_d \)-flow of minimal depth.

Proof. Assume that the algorithm succeeds with output \( (D, \Phi) \). We show that this output \( \mathbb{Z}_d \)-flow \( (D, \Phi) \) is maximally delayed. Firstly, we show that the output flow has \( \Phi_1 = \Lambda_1 \) from lemma 21. We know that \( \Phi_1 \subseteq \Lambda_1 \) and it is clear from the definition of the algorithm that

\[ \Phi_1 = \left\{ u \in O^c \mid \exists \vec{c} \in \mathbb{Z}_d^{O \setminus \{u\}} \text{ s.t. } G[O^c, O \setminus \{u\}], \vec{c} = b_1 \{u\} - aG[O^c, \{u\}] \right\}. \] (73)
Let \( \mathbf{u} \in \Lambda_1 \), that is there is some vector \( \mathbf{c} \in \mathbb{Z}_d^{|V|} \) such that
\[
\begin{align*}
\left\{ (\mathbf{c}_u, (G\mathbf{c})_u) = \lambda(u) \\
v \not\in O \cup \{u\}, c_v = (G\mathbf{c})_v = 0
\end{align*}
\]
(74)

Then, for any \( v \in O^c \),
\[
(G[O^c, V]\mathbf{c})_v = \sum_{j \in V} G_{yj}c_j = \sum_{j \in O \cup \{u\}} G_{yj}c_j = \sum_{j \in O} G_{yj}c_j + aG_{vu} = (G\mathbf{c})_v + aG_{vu} = b\delta_{vu} + aG_{vu}
\]
(75)
from which we see that \( \mathbf{u} \in \Phi_1 \) whence \( \Phi_1 = \Lambda_1 \).

Now, \( \Phi_2 \) is calculated in the next call to Z-Flow-Aux where the open graph passed as argument is \((G, I, O \cup \Phi_1, \lambda|_{(O \cup \Phi_1)})\). Using the same argument as for \( \Phi_1 \), \( \Phi_2 \) must match the layer \( \Lambda_2 \) obtained by applying lemma 21 to the \( \mathbb{Z}_d \)-flow resulting from 22.

Then, using the same recursion as in the proof of proposition 19, we see that \((D, \Phi)\) is maximally delayed. It follows from proposition 23 that the \( \mathbb{Z}_d \)-flow output by the algorithm has optimal depth.

5. Conclusion and future work

We have defined a flow condition suitable for qudit MBQC, and shown that it is both sufficient and necessary to obtain robustly deterministic measurement patterns. We have also presented an algorithm for finding such flows, which furthermore always finds a correction strategy of optimal depth. We leave two main open questions for future work.

Firstly, we have only considered the case of Hilbert spaces of prime dimension, and leave non-prime dimensions untreated. Most of our results generalise straightforwardly to the case of power-of-prime fields, with the exception of theorem 14 (or more specifically, lemma 26 which needs a new proof), but this means taking a somewhat unconventional, ‘Galoisian’ choice for the Pauli group \([\text{App09}]\). If the more conventional choice is made, the Pauli group is built on the cyclic group \( \mathbb{Z}_d \), and no field structure can be imposed if \( d \) is not prime. Then, even the definition of measurement spaces needs to be adapted.

Secondly, an important tool for studying MBQC is reversible circuit extraction. Work in this direction was started in \([\text{BM21}]\) where an extraction algorithm was found for measurement patterns where all the measurements belong to \( \mathcal{M}(0, 1) \) (the measurement space of \( Z \)), but no extraction algorithm is known for all measurement spaces.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Proof of lemma 15

Lemma 25. Let $| \phi \rangle$ be a state of a register $V$ of qudits, $Q$ a Pauli operator and fix some $v \in V$. If for every measurement $M \in \mathcal{M}(Q)$ of the qudit $v$ and every $m \in \mathbb{Z}_d$, we have
\[ \| \langle m : M | \phi \rangle \| = \frac{1}{\sqrt{d}} \] (76)
then $| \phi \rangle$ has a Schmidt decomposition of the form
\[ | \phi \rangle = \sum_{x \in \mathbb{Z}_d} c_x | x : Q \rangle \otimes | \psi_x \rangle, \] (77)
where $| x : Q \rangle$ is an eigenvector of $Q$ associated with eigenvalue $\omega^x$, and we take the coefficients $c_x$ to be real and non-negative.

Proof. Pick some $M \in \mathcal{M}(Q)$, we can write
\[ | \phi \rangle = \sum_{m \in \mathbb{Z}_d} | m : M \rangle | \phi_m \rangle \quad \text{where} \quad | \phi_m \rangle := \langle m : M | \phi \rangle \] (78)
and $\| \langle m : M | \phi \rangle \| = \frac{1}{\sqrt{d}}$.

Letting $\{ | \psi_m \rangle \}$ be the collection of vectors obtained by orthonormalising $\{ | \phi_m \rangle \}$, we can expand $| \phi \rangle$ in this basis:
\[ | \phi \rangle = \frac{1}{\sqrt{d}} \sum_{m,n \in \mathbb{Z}_d} \Psi_{mn} | m : M \rangle | \psi_n \rangle, \quad \text{and for any} \ m \in \mathbb{Z}_d, \quad | \Psi_{mn} |^2 = 1 \] (79)
where $\Psi$ is therefore a $d \times p$ matrix such that $p$ is the dimension of the subspace of $\mathcal{H}$ generated by the $| \phi_m \rangle$ and we denote $\Psi_{mn}$ the $m$th line vector of $\Psi$.

We know that, for every rotation $U$ in $SU(d)$ preserving $Q$ and every $M \in \mathcal{M}(Q)$, $UMU^\dagger$ is also in $\mathcal{M}(Q)$. The group of all such rotations acts on $\Psi$ from the left via the Hilbert space representation, and this action is generated by the rotations of the form $V_M^{-1} R_{k,l}(\xi)V_M$, where $V_M$ is the $d$-dimensional discrete Fourier transform matrix\(^8\) in the eigenbasis of $M$, and $R_{k,l}(\xi)$ is the diagonal matrix given by $k \in \mathbb{Z}_d$, $l \in \mathbb{Z}_d^*$ and $\xi \in \mathbb{R}$, by
\[ R_{k,l}(\xi)_{mn} := \begin{cases} e^{-i \xi} & \text{if} \ m = k; \\
 e^{i \xi} & \text{if} \ m = k + l; \\
 1 & \text{otherwise}. \end{cases} \] (80)

According to equation (76), applying a rotation preserving $Q$ to $\nu$ preserves the outcomes’ probabilities. As such, we deduce that the action of rotations $V_M^{-1} R_{k,l}(\xi)V_M$ on matrix $\Psi$ will preserve the norm of its line vectors. Namely, for every $k \in \mathbb{Z}_d$, $l \in \mathbb{Z}_d^*$ and $\xi \in \mathbb{R}$,
\[ | \Psi_{mn} |^2 = \| (D_{k,l,\xi} \Psi)_{mn} \|^2 \] where, $D_{k,l,\xi} := V_M^{-1} R_{k,l}(\xi)V_M$. (81)

Below, we explicit the right side of this equality to find which $\Psi$ satisfy equation (81). First, we compute the transformed matrix’ line vectors:
\[ (D_{k,l,\xi} \Psi)_{mn} = \Psi_{mn} + \frac{1}{d} \sum_{\alpha \in \mathbb{Z}_d} \Psi_{\alpha m} \left( \phi(k(m - \alpha))(e^{-i \xi} - 1) + \phi((k + l)(m - \alpha))(e^{i \xi} - 1) \right) \] (82)

\(^8\) Explicitly, $V_M$ is given by $\langle m : M | v \rangle | n : M \rangle = \phi(mn)$. 

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\[
\psi_m \pm P_{m,1,1}^k \sin \xi + P_{m,1,2}^k (\cos \xi - 1)
\]  

where
\[
P_{m,1,1}^k := \frac{2}{d} \sum_{\alpha \in \mathbb{Z}_d} \psi_{\alpha \cdot k} \omega^{(k + \frac{1}{2})(m - \alpha)} \sin \left( \frac{\pi l}{d} (m - \alpha) \right) \quad \text{and} \quad
\]
\[
P_{m,1,2}^k := \frac{2}{d} \sum_{\alpha \in \mathbb{Z}_d} \psi_{\alpha \cdot k} \omega^{(k + \frac{1}{2})(m - \alpha)} \cos \left( \frac{\pi l}{d} (m - \alpha) \right).
\]

We rewrite equation (81) as,
\[
\| \psi_m \|^2 = \| (D_{l,1,1} \psi)_m \|^2
\]
\[
= \| \psi_m + P_{m,1,1}^k \sin \xi + P_{m,1,2}^k (\cos \xi - 1) \|^2
\]
\[
= \| \psi_m \|^2 + A + B \sin \xi + C \cos \xi + D \cos 2 \xi + E \sin 2 \xi,
\]
from which we deduce:
\[
A + B \sin \xi + C \cos \xi + D \cos 2 \xi + E \sin 2 \xi = 0.
\]

We specify these five alphabetic constants for our kind reader while emphasizing that only the expression of \( D \) will be used thereafter:
\[
A := \frac{3}{2} \left( \| P_{m,1,2}^k \|^2 - 2 R (\psi_m P_{m,1,2}^{k*,}) + \frac{1}{2} \| P_{m,1,1}^k \|^2 \right),
\]
\[
B := 2 R (\psi_m P_{m,1,1}^k) - 2 R (P_{m,1,1}^k P_{m,1,2}^{k*,})
\]
\[
C := 2 R (\psi_m P_{m,1,2}^k) - 2 \| P_{m,1,2}^k \|^2
\]
\[
D := \frac{1}{2} \left( \| P_{m,1,2}^k \|^2 - \| P_{m,1,1}^k \|^2 \right)
\]
\[
E := 2 R (P_{m,1,1}^k P_{m,1,2}^{k*,})
\]

where \( P_{m,1,1}^{k,*} \) denotes the complex conjugate of \( P_{m,1,1}^k \).

We know that \( \{ \cos(n \xi), \sin(n \xi) \}_{n \in \mathbb{Z}} \) forms an orthogonal set in the space of periodic functions of period \( 2\pi \) with respect to the Hermitian form \( \langle f, g \rangle := \int_0^\pi f^* (t) g (t) dt \), and as such, the five alphabetic constants of the left side of equation (89) must be zero.

We develop the two terms of the constant \( D \), \( \forall m, k \in \mathbb{Z}_d, l \in \mathbb{Z}_d^*, \) and obtain:
\[
\| P_{m,1,1}^k \|^2 = \frac{4}{d^2} \sum_{\alpha, \alpha' \in \mathbb{Z}_d} \psi_{\alpha \cdot k} \psi_{\alpha' \cdot k} \omega^{(k + \frac{1}{2})(\alpha - \alpha')} \cos \left( \frac{\pi l}{d} (m - \alpha') \right) \cos \left( \frac{\pi l}{d} (m - \alpha) \right),
\]
\[
\| P_{m,1,2}^k \|^2 = \frac{4}{d^2} \sum_{\alpha, \alpha' \in \mathbb{Z}_d} \psi_{\alpha \cdot k} \psi_{\alpha' \cdot k} \omega^{(k + \frac{1}{2})(\alpha - \alpha')} \sin \left( \frac{\pi l}{d} (m - \alpha') \right) \sin \left( \frac{\pi l}{d} (m - \alpha) \right).
\]

Using the addition formulas of trigonometry, we deduce,
\[
D = \frac{2}{d^2} \sum_{\alpha, \alpha' \in \mathbb{Z}_d} \psi_{\alpha \cdot k} \psi_{\alpha' \cdot k} \omega^{(k + \frac{1}{2})(\alpha - \alpha')} \cos \left( \frac{\pi l}{d} (2m - \alpha - \alpha') \right) = 0.
\]
We introduce the following change of variables $2\beta := \alpha + \alpha'$ and $2\beta' := \alpha - \alpha'$, such that we obtain,

$$
\forall k, n \in \mathbb{Z}_d \text{ and } l \in \mathbb{Z}_d^*, \quad \sum_{\beta, \beta' \in \mathbb{Z}_d} \omega^{2(k+\frac{i}{2})\beta'} \Psi_{\beta+\beta'}^* \Psi_{\beta-\beta'} \cos \left( \frac{2\pi l}{d} (m - \beta) \right) = 0. \quad (93)
$$

Now, for any $l \in \mathbb{Z}_d^*$, the square matrix given by $\Omega_{k, \beta'} := \omega^{2(k+\frac{i}{2})\beta'}$ is invertible. As a consequence, we deduce from the previous equation that for all $m \in \mathbb{Z}_d$ and $l \in \mathbb{Z}_d^*$,

$$
\sum_{\beta \in \mathbb{Z}_d} \Psi_{\beta+\beta'}^* \Psi_{\beta-\beta'} \cos \left( \frac{2\pi l}{d} (m - \beta) \right) = 0. \quad (94)
$$

Developing the cosine, we obtain

$$
\cos \left( \frac{2\pi l m}{d} \right) \sum_{\beta \in \mathbb{Z}_d} \Psi_{\beta+\beta'}^* \Psi_{\beta-\beta'} \cos \left( \frac{2\pi l \beta}{d} \right) + \sin \left( \frac{2\pi l m}{d} \right) \times \sum_{\beta \in \mathbb{Z}_d} \Psi_{\beta+\beta'}^* \Psi_{\beta-\beta'} \sin \left( \frac{2\pi l \beta}{d} \right) = 0,
$$

from which we deduce, using again the argument used in equation (89), that for all $l \in \mathbb{Z}_d^*$,

$$
\sum_{\beta \in \mathbb{Z}_d} \Psi_{\beta+\beta'}^* \Psi_{\beta-\beta'} \cos \left( \frac{2\pi l \beta}{d} \right) = 0, \quad (96a)
$$

$$
\sum_{\beta \in \mathbb{Z}_d} \Psi_{\beta+\beta'}^* \Psi_{\beta-\beta'} \sin \left( \frac{2\pi l \beta}{d} \right) = 0. \quad (96b)
$$

These equations force the following conclusion: for all $\beta \in \mathbb{Z}_d$, the Hermitian product of $\Psi_{\beta+\beta'}$ and $\Psi_{\beta-\beta'}$ depends only of $\beta'$, namely:

$$
\Psi_{\beta+\beta'}^* \Psi_{\beta-\beta'} = r_{\beta'}. \quad (97)
$$

At this point, we define a 'Fourier transform' of our line vectors $\Psi_m$ as

$$
\Psi_F^\cdot := \frac{1}{\sqrt{d}} \sum_{m \in \mathbb{Z}_d} \Psi_m \omega^{m \gamma}. \quad (98)
$$

This transformation is invertible as:

$$
\Psi_m = \frac{1}{\sqrt{d}} \sum_{\gamma \in \mathbb{Z}_d} \Psi_F^\gamma \omega^{-m \gamma}, \quad (99)
$$

so that going back to $|\phi\rangle$,

$$
|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{m, n \in \mathbb{Z}_d} \Psi_{mn} |m\rangle |\psi_n\rangle \quad (100)
$$

$$
= \frac{1}{d} \sum_{m, n \in \mathbb{Z}_d} \left( \sum_{\gamma \in \mathbb{Z}_d} \Psi_F^\gamma \omega^{-m \gamma} \right) |m\rangle |\psi_n\rangle \quad (101)
$$

$$
= \frac{1}{d} \sum_{m, n \in \mathbb{Z}_d} \omega^{-m \gamma} |m\rangle \sum_{n \in \mathbb{Z}_d} \Psi_F^\gamma |\psi_n\rangle \quad (102)
$$
\[
\sum_{\gamma \in \mathbb{Z}_d} | - \gamma : Q | \psi^F_{\gamma i} >,
\]
(103)

where \( | \psi^F_{\gamma i} \rangle \). Making good use of equation (97), we find that for \( \gamma_1, \gamma_2 \in \mathbb{Z}_d \)

\[
\langle \psi^F_{\gamma_1} | \psi^F_{\gamma_2} \rangle = \sum_{n \in \mathbb{Z}_d} \Psi^*_{\gamma_1 n} \Psi_{\gamma_2 n} \langle \psi_n | \psi_n \rangle.
\]
(104)

\[
= \frac{1}{d} \sum_{n \in \mathbb{Z}_d} \left( \sum_{m_1, m_2 \in \mathbb{Z}_d} \Psi^*_{m_1 n} \Psi_{m_2 n} \omega^{-m_1 \gamma_1 + m_2 \gamma_2} \right)
\]
(105)

\[
= \frac{1}{d} \sum_{m_1, m_2 \in \mathbb{Z}_d} \left( \sum_{n \in \mathbb{Z}_d} \Psi^*_{m_1 n} \Psi_{m_2 n} \right) \omega^{-m_1 \gamma_1 + m_2 \gamma_2}
\]
(106)

\[
= \frac{1}{d} \sum_{m_1, m_2 \in \mathbb{Z}_d} \frac{1}{d} \delta_{\gamma_1, \gamma_2}.
\]
(107)

according to equation (97),

\[
= \frac{1}{d} \sum_{m_1, m_2 \in \mathbb{Z}_d} r_{m_1 - m_2} \omega^{-m_1 \gamma_1 + m_2 \gamma_2}
\]
(108)

\[
= \frac{1}{d} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_d} r_{\alpha_1} \omega^{-(\alpha_1 + \alpha_2) \gamma_1 + (\alpha_1 - \alpha_2) \gamma_2}
\]
(109)

summing over \( \alpha_1 \),

\[
= \sum_{\alpha_2 \in \mathbb{Z}_d} r_{\alpha_2} \omega^{-(\gamma_1 + \gamma_2) \delta_{\gamma_1, \gamma_2}}.
\]
(110)

The family \( \{ | \psi^F_{\gamma i} \rangle \}_{\gamma \in \mathbb{Z}_d} \) forms an orthogonal family. Note that, depending on the value of the \( r_{\alpha} \), some \( | \psi^F_{\gamma i} \rangle \) can be of norm 0. Nevertheless, whenever the condition of equation (76) is met, we have a valid Schmidt decomposition of \( | \phi \rangle \) of the form

\[
\frac{1}{\sqrt{d}} \sum_{\gamma \in \mathbb{Z}_d} | - \gamma : Q | \psi^F_{\gamma i} >.
\]
(111)

\[ \square \]

**Lemma 26.** Let \( | \phi \rangle, | \phi' \rangle \) be two states of a register \( V \) of qudits, \( x \in \mathbb{Z}_d^2 \) be non-zero and fix some \( n \in N \). If for every measurement \( M \in \mathcal{M}(x) \) of the qudit \( n \) and every \( m \in \mathbb{Z}_d \), we have

\[
\langle m : M | \phi \rangle \simeq \langle m : M | \phi' \rangle \quad \text{and} \quad \| \langle m : M | \phi \rangle \| = \frac{1}{\sqrt{d}} = \| \langle m : M | \phi' \rangle \|,
\]
(112)

then at least one of the following holds:

(a) \( | \phi \rangle \simeq | \phi' \rangle \);
(b) \( | \phi \rangle \) and \( | \phi' \rangle \) are separable and there are \( x, y \in \mathbb{Z}_d \), \( | \psi \rangle \in \mathcal{H}_V \setminus \{ x \} \) such that

\[
| \phi \rangle = | x : Q \rangle \otimes | \psi \rangle \quad \text{and} \quad | \phi' \rangle \simeq | y : Q \rangle \otimes | \psi \rangle,
\]
(113)

where \( | x : Q \rangle \) is the eigenvector of \( Q \) associated with eigenvalue \( \omega^x \).
Proof. Assume that both \(|\phi\), \(|\phi'\rangle\) have Schmidt rank 1. According to the previous lemma, we can write both states as  
\begin{equation}
|\phi\rangle = |x: Q\rangle \otimes |\psi_x\rangle \quad \text{and} \quad |\phi'\rangle = |y: Q\rangle \otimes |\psi_y\rangle ,
\end{equation}
using equation (112),
\begin{equation}
|\psi_x\rangle = \sqrt{d} (0: M|\phi\rangle) = e^{ia} \sqrt{d} (0: M|\phi'\rangle) = e^{ia} |\psi'_x\rangle ,
\end{equation}
and we are clearly in subcase (2) of the main lemma.

Now, assuming the Schmidt rank along the partition \(\{y; V \setminus \{v\}\}\) of both \(|\phi\rangle\) and \(|\phi'\rangle\) is greater than or equal to 2. According to the previous lemma,
\begin{equation}
|\phi\rangle = \sum_{x \in \mathbb{Z}_d} c_x |x: Q\rangle \otimes |\psi_x\rangle \quad \text{and} \quad |\phi'\rangle = \sum_{x \in \mathbb{Z}_d} c'_x |x: Q\rangle \otimes |\psi'_x\rangle .
\end{equation}

Then, for any \(m, k, l \in \mathbb{Z}_d\), and any \(\xi \in \mathbb{T}_d\), we have
\begin{equation}
\langle m: M|\phi\rangle = e^{i\alpha_n} \langle m: M|\phi'\rangle ,
\end{equation}
\begin{equation}
\langle m: M|D_{k,l,\xi}|\phi\rangle = e^{i\beta(k,l,\xi,m)} \langle m: M|D_{k,l,\xi}|\phi'\rangle ,
\end{equation}
where \(D_{k,l,\xi}\) is defined as in equation (81) and \(\beta\) is a function of the different parameters which define the rotation. Developing the right-hand side of the previous equation we find
\begin{equation}
\langle m: M|D_{k,l,\xi}|\phi'\rangle = \sum_x c'_x \langle m: M|D_{k,l,\xi}|x: Q\rangle |\psi'_x\rangle
\end{equation}
\begin{equation}
= \sum_x \left[ \omega^{mx} + \frac{1}{d} \sum_n \omega^{nx} \left( \omega^{k(m-n)}(e^{-i\xi} - 1) + \omega^{(k+l)(m-n)}(e^{i\xi} - 1) \right) \right] c'_x |\psi'_x\rangle .
\end{equation}

Likewise, for the left-hand side, we have for any \(m, k, l \in \mathbb{Z}_d\), and any \(\xi \in \mathbb{T}_d\),
\begin{equation}
\langle m: M|D_{k,l,\xi}|\phi\rangle
\end{equation}
\begin{equation}
= \langle m: M|\phi\rangle + \frac{1}{d} \sum_n \left( \omega^{j(m-n)}(e^{-i\xi} - 1) + \omega^{(j+k)(m-n)}(e^{i\xi} - 1) \right) \langle n: M|\phi\rangle
\end{equation}
\begin{equation}
= e^{i\alpha_n} \langle m: M|\phi'\rangle + \frac{1}{d} \sum_n \left( \omega^{j(m-n)}(e^{-i\xi} - 1) + \omega^{(j+k)(m-n)}(e^{i\xi} - 1) \right) e^{i\alpha_n} \langle n: M|\phi'\rangle
\end{equation}
\begin{equation}
= \sum_x \left[ e^{i\alpha_n} \omega^{mx} + \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{nx} \left( \omega^{k(m-n)}(e^{-i\xi} - 1) + \omega^{(k+l)(m-n)}(e^{i\xi} - 1) \right) \right] c_x |\psi_x\rangle ,
\end{equation}
where we have used equation (117) between the first two lines. By identifying components along the orthonormal basis elements \(\{ |\psi_x\rangle\}\) and removing terms where \(c'_x = 0\), we can write equation (118) as
\begin{equation}
\begin{aligned}
e^{i\beta(j,k,\xi,m)} & \left( \omega^{mx} + \frac{1}{d} \sum_n \omega^{nx} \omega^{k(m-n)}(e^{-i\xi} - 1) + \frac{1}{d} \sum_n \omega^{nx} \omega^{(k+l)(m-n)}(e^{i\xi} - 1) \right) \\
& = \omega^{mx} e^{i\alpha_n} + \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{nx} \omega^{k(m-n)}(e^{-i\xi} - 1) + \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{nx} \omega^{(k+l)(m-n)}(e^{i\xi} - 1) .
\end{aligned}
\end{equation}

Since \(|\phi'\rangle\) has Schmidt rank of at least 2, we can find \(y, z \in \mathbb{Z}_d\) such that \(y \neq z\), \(c'_y \neq 0\) and \(c'_z \neq 0\). For the next part, let \(k = y\) and \(l = z - y\) such that the phase of \(D_{k,l,\xi}\) is applied on the two non-zero components.
From now on, we note $\beta(\xi, m) := \beta(y, z - y, \xi, m)$. Taking the coefficients along $|\psi_i^\xi\rangle$, we rewrite the previous equation, for any $\xi \in \mathbb{T}$ and $m \in \mathbb{Z}_d$, as

$$e^{i\beta(\xi, m)} \omega^{my} e^{-i\xi} = \omega^{my} e^{i\alpha_n} + \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{ny} \omega^{y(m-n)} (e^{-i\xi} - 1)$$

$$+ \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{ny} \omega^{y(m-n)} (e^{i\xi} - 1),$$

(126)

taking the coefficients along $|\psi_i^\xi\rangle$ we extract a different equation,

$$e^{i\beta(\xi, m)} \omega^{mc} e^{i\xi} = \omega^{mc} e^{i\alpha_n} + \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{mc} \omega^{y(m-n)} (e^{i\xi} - 1)$$

$$+ \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{mc} \omega^{y(m-n)} (e^{-i\xi} - 1).$$

(127)

Finally, for any $\xi \in \mathbb{T}$ and $m \in \mathbb{Z}_d$,

$$e^{i\beta(\xi, m)} = \omega^{xc} e^{i\alpha_n} + \frac{1}{d} \sum_n e^{i\alpha_n} (1 - e^{-i\xi}) + \frac{1}{d} \sum_n e^{i\alpha_n} (1 - e^{i\xi}) (e^{2i\xi} - e^{i\xi})$$

(128)

and

$$e^{i\beta(\xi, m)} = e^{-i\xi} e^{i\alpha_n} + \frac{1}{d} \sum_n e^{i\alpha_n} \omega^{y(z-y)(m-n)} (e^{-i\xi} - e^{i\xi}) + \frac{1}{d} \sum_n e^{i\alpha_n} (1 - e^{-i\xi}).$$

(129)

So, the right sides of both equations are equal. However, we can use again the argument below equation (90), $\{e^{m\xi}\}_{m \in \mathbb{N}}$ is an orthogonal set in the space of periodic functions. As such, taking the terms in $e^{2i\xi}$ and $e^{i\xi}$,

$$\sum_n e^{i\alpha_n} \omega^{y(z-y)(m-n)} = 0$$

(130a)

$$e^{i\alpha_n} - \frac{1}{d} \sum_n e^{i\alpha_n} - \sum_n e^{i\alpha_n} \omega^{y(z-y)(m-n)} = 0.$$  

(130b)

Instantly, we get, for all $m$,

$$e^{i\alpha_n} = \frac{1}{d} \sum_n e^{i\alpha_n} \wedge \text{in particular} \quad e^{i\alpha_n} = e^{i\alpha_n}.$$  

We note this common phase $\alpha_n$ and this implies by direct calculation that

$$c_i^x |\psi_i^\xi\rangle = e^{i\alpha_n} c_i^x |\psi_i^\xi\rangle.$$  

(132)

Based on this result, we can conclude that we are in subcase (1) of the lemma:

$$|\phi\rangle = \sum x c_i^x |x \rangle \otimes |\psi_i^\xi\rangle,$$

(133)

$$= \sum x |x \rangle \otimes (c_i^x |\psi_i^\xi\rangle),$$

(134)

$$= \sum x |x \rangle \otimes (e^{i\alpha_n} c_i^x |\psi_i^\xi\rangle),$$

(135)

$$= e^{i\alpha_n} \sum x c_i^x |x \rangle \otimes |\psi_i^\xi\rangle,$$

(136)

$$= e^{i\alpha_n} |\phi_i^\xi\rangle,$$

(137)

as desired.
We have shown that any choice of \(|\psi\), \(|\psi'\rangle\) which verify the conditions of equation (112) must fall into either subcase (1) or (2) of the lemma, and we are done.

**Lemma 15.** Let \((G, I, O, \lambda)\) be an open graph, \(|\phi\rangle, |\phi'\rangle \in (H^{\otimes V})_1\) and \(R \subseteq V\). For any \(M : R \to U(H)\) such that, for each \(u \in V\), \(M(u) \in M(\lambda(u))\), and \(\tilde{m} \in Z_d^R\), put \(|\tilde{m} : M\rangle = \bigotimes_{r \in R} |m_r : M(r)\rangle\). If, for every such \(\tilde{m}\) and \(M\), we have

\[
\langle \tilde{m} : M | \phi \rangle \simeq \langle \tilde{m} : M | \phi' \rangle \quad \text{and} \quad \| (\tilde{m} : M | \phi \rangle \| = \frac{1}{\sqrt{d^{\vert R\vert}}} = \| (\tilde{m} : M | \phi' \rangle \|, \quad (28)
\]

then there is a subset \(L \subseteq R, x, y \in Z_d^L\) and \(|\psi\rangle \in H^{\otimes L}\) such that

\[
|\phi\rangle \simeq |\psi\rangle \bigotimes_{n \in L} |x_n : Q_n\rangle \quad \text{and} \quad |\phi'\rangle \simeq |\psi\rangle \bigotimes_{n \in L} |y_n : Q_n\rangle. \quad (29)
\]

**Proof.** The proof proceeds by induction on the size of \(R\). The case \(|R| = 0\) is trivial, and the case \(|R| = 1\) is lemma 26. Assume the statement is true for some non-empty \(R\), if \(R = V\) we are done since the induction cannot continue. If this is not the case, pick \(u \in V \setminus R\). If

\[
(m : M(u) | \otimes (\tilde{m} : M | \phi) \rangle \simeq ((m : M(u) | \otimes (\tilde{m} : M | \phi') \rangle
\]

and

\[
\| \sqrt{d^{\vert R\vert}} (m : M(u) | \otimes (\tilde{m} : M | \phi) \rangle \| = \frac{1}{\sqrt{d}}
\]

hold for all \(m \in Z_d^R\), then by lemma 26 we have one of the following cases:

(a) \(|\tilde{m} : M\rangle |\phi \rangle \simeq (\tilde{m} : M) |\phi' \rangle \quad \text{and} \quad \| (\tilde{m} : M | \phi \rangle \| = \frac{1}{\sqrt{d^R}} \quad \text{for any} \quad \tilde{m} \in Z_d^R \quad \text{so that by the induction hypothesis we are done.}

(b) For each \(\tilde{m} \in Z_d^R\), there are \(x, y \in Z_d\) and \(|\psi_{\tilde{m}}\rangle \in H^{\otimes L}\) such that \(|\tilde{m} : M\rangle |\phi \rangle \simeq |x : Q_u \rangle \otimes |\psi_{\tilde{m}}\rangle\) and \(|\tilde{m} : M\rangle |\phi' \rangle \simeq |y : Q_u \rangle \otimes |\psi_{\tilde{m}}\rangle\).

In the latter case, make some arbitrary choice of measurements \(M : R \to U(H)\), and expand \(|\phi\rangle\) in their common eigenbases:

\[
|\phi\rangle = \sum_{n \in Z_d} \sum_{\tilde{a} \in Z_d^R} c(n, \tilde{a}) |x : Q_u \rangle \otimes |\tilde{a} : M \rangle \otimes |\phi(a)\rangle. \quad (139)
\]

Then in particular, we have that for any choice \(\tilde{m} \in Z_d^R\),

\[
|\tilde{m} : M |\phi \rangle = \sum_{n \in Z_d} c(n, \tilde{m}) |n : Q_u \rangle \otimes |\phi(n, \tilde{m})\rangle \simeq |x : Q_u \rangle \otimes |\psi_{\tilde{m}}\rangle, \quad (140)
\]

which implies that \(c(n, \tilde{m}) = 0\) whenever \(n \neq x\), and we have \(|\phi\rangle = |x : Q_u \rangle \otimes |\psi_{\tilde{m}}\rangle\), where

\[
|\psi_{\tilde{m}}\rangle = \sum_{\tilde{a} \in Z_d^R} c(x, \tilde{m}) |\tilde{a} : M \rangle \otimes |\phi(x, \tilde{m})\rangle. \quad (141)
\]

Similarly \(|\tilde{m} : M |\phi' \rangle \simeq |y : Q_u \rangle \otimes |\psi_{\tilde{m}}\rangle\). It follows that for any \(\tilde{m} \in Z_d^R\), we must have \(|\tilde{m} : M |\psi_{\tilde{m}}\rangle \simeq |\psi_{\tilde{m}}\rangle\) and \(|\tilde{m} : M |\psi_{\tilde{m}}\rangle \simeq |\psi_{\tilde{m}}\rangle\), so that \(|\tilde{m} : M |\psi_{\tilde{m}}\rangle \simeq |\tilde{m} : M |\psi_{\tilde{m}}\rangle\). Then, by the induction hypothesis, there is \(L \subseteq R\) and \(x, y \in Z_d^L\) such that \(|\phi\rangle = |\psi\rangle \bigotimes_{\tilde{a} \in L \setminus \{u\}} |x : Q_u\rangle\) and \(|\phi'\rangle = |\psi\rangle \bigotimes_{\tilde{a} \in L \setminus \{u\}} |y : Q_u\rangle\), and we are done. \qed
Appendix B. Proof of lemmas 20–22

Lemma 20. If $\Gamma$ is a maximally delayed $\mathbb{Z}_d$-flow for an open graph $(G, I, O, \lambda)$, then

$$\Lambda_0 = O \cup \{ u \in V | \forall v \in V : G_{uv} = 0 \},$$

i.e. the union of the outputs and isolated vertices of $(G, I, O, \lambda)$.

Proof. Let $A := O \cup \{ u \in V | \forall v \in V : G_{uv} = 0 \}$, and define a layer decomposition $\Lambda'$ on $(G, I, O, \lambda)$ by

$$\Lambda' := \Lambda_0 \setminus A \quad \text{for} \quad k > 0 \quad \text{and} \quad \Lambda' = \Lambda_0 \cup A.$$  \hfill (142)

Then it is clear that $\Lambda'$ is more delayed than $\Lambda$. Let $C'$ be the matrix obtained by replacing, for every isolated vertex $u \in V$, the $u$th column of $C$ by $C_{u \ell}(u)$.

We show that $(C', \Lambda')$ is a $\mathbb{Z}_d$-flow for $(G, I, O, \lambda)$.

(a) We have not touched the diagonal elements of $C$ and have only changed the columns corresponding to isolated vertices. Then

$$(GC')_{uv} = \begin{cases} \sum_{\ell} G_{uv} C_{u \ell} = 0 & \text{if } u \text{ is isolated;} \\ (GC)_{uv} & \text{otherwise} \end{cases}$$  \hfill (143)

and condition (a) of the definition is still verified.

(b) Since $C'_{uv} = C_{uv}$ if $u \in I$ or $v \in O$, we have condition (b) of the definition.

(c) For every $m > n \in \mathbb{N}^*$, $C[\Lambda'_{n}, \Lambda'_{m}] = (GC)[\Lambda'_{n}, \Lambda'_{m}] = 0$, since they are submatrices of $C[\Lambda_n, \Lambda_n] = (GC)[\Lambda_n, \Lambda_n] = 0$, and $C[\Lambda'_m, \Lambda'_m], (GC)[\Lambda'_m, \Lambda'_m]$ are diagonal for the same reason. Also,

$$(GC')[\Lambda'_{m}, \Lambda'_{n}]_{uv} = \begin{cases} 0 & \text{if } v \in \Lambda_0 \quad \text{since otherwise } (C, \Lambda) \text{ is not a } \mathbb{Z}_d\text{-flow}; \\ \sum_{\ell \in V} G_{uv} C'_{\ell v} = G_{uv} C_{uv} = 0 & \text{if } v \text{ is isolated;} \\ \sum_{\ell \in V} G_{uv} C'_{\ell v} = \sum_{\ell \in V} G_{uv} C_{\ell v} = 0 & \text{if } v \in O. \end{cases}$$  \hfill (144)

Finally, it is clear that $C'[\Lambda'_{0}, \Lambda'_{0}]$ is diagonal if $C[\Lambda_0, \Lambda_0]$ was, since we have only added zero for outputs or ‘diagonal’ columns for isolated vertices. Therefore we have condition (c).

As a result, $(C', \Lambda')$ is a $\mathbb{Z}_d$-flow for $(G, I, O, \lambda)$ that is more delayed than $(C, \Lambda)$. This implies that we must have $\Lambda \subseteq \Phi_0$ if $\Phi$ is maximally delayed.

Now assume there is some $v \in \Lambda_0 \setminus O$. We know that $C[\Lambda_0, \Lambda_0] = 0$, and that $GC[\Lambda_n, \Lambda_0] = \sum_{\ell \in \Lambda_n} G[\Lambda_n, \Lambda_{\ell}] C[\Lambda_{\ell}, \Lambda_0] = G[\Lambda_n, \Lambda_0] C[\Lambda_0, \Lambda_0]$. If $C_{uv} \neq 0$, then for $GC[\Lambda_0, \Lambda_0]$ to be diagonal and $GC[\Lambda_n, \Lambda_0] = 0$, we must have either $C_{uv} = 0$ or for all $u \in V$, $G_{uv} = 0$ since then $(GC)_{uv} = G_{uv} C_{uv}$ must be $0$ if $u \neq v$. In the latter case, $v$ is isolated in the graph $G$.

In the former case, $G_{uv} C_{uv} = 0$, and we have $(C_{uv}, (GC)_{uv}) = (0, 0)$. But since $u$ is not an output, we must have $(C_{uv}, (GC)_{uv}) = \lambda(u)$, so that $(C, \Lambda)$ is not a $\mathbb{Z}_d$-flow for $(G, I, O, \lambda)$. As a result, there can be no such $u$ if $(C, \Lambda)$ is a valid $\mathbb{Z}_d$-flow. We conclude that $\Lambda_0 = O \cup \{ u \in V | \forall v \in V : G_{uv} = 0 \}$. \hfill \Box

Lemma 21. If $(C, \Lambda)$ is maximally delayed for $(G, I, O, \lambda)$, then

$$\Lambda_1 = \left\{ u \in O^2 | \exists e \in \mathbb{Z}_d^{|V|} \text{ s.t. } \begin{array}{l} (c_u, (GC)_{u}) = \lambda(u) \\ \forall v \not \in O \cup \{ u \}, c_v = (GC)_{v} = 0 \end{array} \right\}. \hfill (70)$$
Proof. Let \((D, \Phi)\) be a maximally delayed \(\mathbb{Z}_d\)-flow for \((G, I, O, \lambda)\) and define \(c^\theta\) as the \(u\)th column of \(D\). The only elements below the diagonal in column \(v \in \Phi_1\) of \(D\) correspond to \(\Phi_1\) or \(\Phi_0\). Since \(D[\Phi_1, \Phi_1] \) and \((GD)[\Phi_1, \Phi_1]\) are diagonal, and \(\Phi_0 = O\) by lemma 20, for any \(v \notin O \cup \{u\}\) we must have \(D_{uv} = c^\theta_v = 0\) and \((GD)_{uv} = (Ge^\theta)_v = 0\). The condition \(\lambda(u) = (e^\theta, (Ge^\theta)_u)\) itself corresponds to part (iii) of the definition of \(\mathbb{Z}_d\)-flow. As a result, every maximally delayed \(\mathbb{Z}_d\)-flow of \((G, I, O, \lambda)\) must verify equation (70), and there can be no layer decomposition \(\Phi\) where \(\Phi_1\) is not contained in \(\Lambda_1\).

Now, assume \((G, I, O, \lambda)\) is an open graph with \(\mathbb{Z}_d\)-flow, that \((D, \Phi)\) is a maximally delayed \(\mathbb{Z}_d\)-flow and let \(u \in \Lambda_1 \setminus \Phi_1\). Let \(E\) be the matrix obtained by replacing the \(u\)th column of \(D\) by \(c^\theta\) and permuting the \(u\)th column to the start of \(\Phi_1\). Then \((E, \Psi)\) where

\[
\Psi_k := \begin{cases} 
\Lambda_1 \cup \{u\} & \text{if } k = 1; \\
\Lambda_k \setminus \{u\} & \text{otherwise};
\end{cases}
\]  

is a more delayed \(\mathbb{Z}_d\)-flow than \((D, \Phi)\). As a result, there can be no such \(u\), so that if \((D, \Phi)\) is maximally delayed, \(\Phi_1 = \Lambda_1\).

Lemma 22. If \((C, \Lambda)\) is a maximally delayed \(\mathbb{Z}_d\)-flow of \((G, I, O, \lambda)\), \((D, \Phi)\) is a maximally delayed \(\mathbb{Z}_d\)-flow of \((G, I, O \cup \Lambda_1, \Lambda |_{(O \cup \Lambda_1)^c})\), where

- \(D\) is the matrix obtained by replacing the columns of \(C\) corresponding to \(\Lambda_1\) with zeros;
- \(\Phi\) is given by

\[
\Phi_k := \begin{cases} 
\Lambda_1 \cup O & \text{if } k = 0; \\
\Lambda_{k+1} & \text{otherwise.}
\end{cases}
\]  

Proof. It is clear that \((D, \Phi)\) is a layer decomposition, since if it were not, this would imply that \((C, \Phi)\) is not either.

There cannot be a more delayed \(\mathbb{Z}_d\)-flow of \((G, I, O \cup \Lambda_1, \lambda)\) since that would immediately imply that there is a layer decomposition of \((G, I, O, \lambda)\) that is more delayed than \((C, \Lambda)\).

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