Bayesian uncertainty relation for a joint measurement of canonical variables

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We present a joint-measurement uncertainty relation for a pair of mean square deviations of canonical variables averaged over Gaussian distributed quantum optical states. Our Bayesian formulation is free from the unbiasedness assumption, and enables us to quantify experimentally implemented joint-measurement devices by feeding a moderate set of coherent states. Our result also reproduces the most informative bound for quantum estimation of phase-space displacement in the case of pure Gaussian states.

The canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$ determines a primary noise property on quantum states, and it is highly fascinating if one can characterize the penalty on controlling or measuring a pair of canonical variables in a unified manner. The amplification uncertainty relation represents the penalty of the linear amplification by a gain dependent uncertainty relation.

$$\langle (\hat{x} - \sqrt{G_x x_{\text{in}}})^2 \rangle \langle (\hat{p} - \sqrt{G_p p_{\text{in}}})^2 \rangle \geq \frac{\hbar^2}{4} (G + |G - 1|)^2,$$

where $(G_x, G_p)$ represents the gain of the physical process with $G = \sqrt{G_x G_p}$, and $(x_{\text{in}}, p_{\text{in}})$ stands for the mean value of $(\hat{x}, \hat{p})$ for an arbitrary input state. The amplification map is assumed to satisfy the covariance condition, $\langle \hat{x} \rangle = \sqrt{G_x x_{\text{in}}}$ and $\langle \hat{p} \rangle = \sqrt{G_p p_{\text{in}}}$.

$$\langle (\hat{x} - \sqrt{G_x x_{\text{in}}})^2 \rangle \langle (\hat{p} - \sqrt{G_p p_{\text{in}}})^2 \rangle \geq \frac{\hbar^2}{4} (2G + 1)^2.$$

The right-hand value of Eq. (2) is two units of the shot noise $2 \times (\hbar/2)$ larger than the right-hand side of Eq. (1) in the case of a unit gain $G = 1$. For a measurement process, the joint-measurement uncertainty relation can be expressed as

$$\langle (\hat{Q}_1 - x_{\text{in}})^2 \rangle \langle (\hat{P}_2 - p_{\text{in}})^2 \rangle \geq \hbar^2,$$

where the variables of two meter systems $\hat{Q}_1$ and $\hat{P}_2$ are assumed to satisfy the unbiasedness condition, $\langle \hat{Q}_1 \rangle = x_{\text{in}}$ and $\langle \hat{P}_2 \rangle = p_{\text{in}}$. The right-hand term of Eq. (3) becomes larger-than the right-hand term of Eq. (1) with $G = 1$ by a single unit of the shot noise $\hbar/2$. As a standard interpretation, this extra noise originates from an introduction of the auxiliary system, and another extra unit for entanglement breaking channels in Eq. (2) is due to the state-preparation process.

Unfortunately, neither the covariance condition nor the unbiasedness condition for continuous-variable (CV) systems is realized in experiments since such a condition implies devices executing a perfectly linear response to arbitrary field amplitude. Hence, there is no reason to consider that the inequalities in Eqs. (1), (2), and (3) hold for real physical systems. In addition, the amplitude of input states is practically bounded due to physical conditions in experimental systems, and it is impossible to confirm the linearity. This motivates us to search for theoretical limitations on physical maps assuming a moderate set of input states.

In our approach, we consider the MSDs for a set of coherent states instead of the deviations for an arbitrary state in the left-hand side of Eqs. (1), (2), and (3). In what follows we set $\hbar = 1$ and the canonical commutation relation indicates $[\hat{x}, \hat{p}] = i$ as a standard notation. The mean input quadrature $(x_{\text{in}}, p_{\text{in}})$ for a coherent state
\( \rho_\alpha := |\alpha\rangle\langle \alpha | \) is specified by
\[
x_{\alpha} := \text{Tr}(\hat{x}\rho_\alpha) = \frac{\alpha + \alpha^*}{\sqrt{2}}, \quad p_\alpha := \text{Tr}(\hat{p}\rho_\alpha) = \frac{\alpha - \alpha^*}{\sqrt{2i}}.
\]

Let \( \hat{M} \) and \( \hat{N} \) be self-adjoint operators, and \( \mathcal{E} \) be a quantum channel. We define a pair of the MSDs as
\[
\begin{align*}
\bar{V}^{(x)}_M(\eta_x, \lambda) &:= \text{Tr} \int p_\lambda(\alpha)(\hat{M} - \sqrt{\eta_x}x_{\alpha})^2 \mathcal{E}(\rho_\alpha)d^2\alpha

\bar{V}^{(p)}_N(\eta_p, \lambda) &:= \text{Tr} \int p_\lambda(\alpha)(\hat{N} - \sqrt{\eta_p}p_{\alpha})^2 \mathcal{E}(\rho_\alpha)d^2\alpha
\end{align*}
\]
where \( \eta_x, \eta_p \) is a pair of non-negative number, and
\[
p_\lambda(\alpha) := \frac{\lambda}{\pi} \exp(-\lambda|\alpha|^2)
\]
is a Gaussian prior distribution with \( \lambda > 0 \). If we choose the canonical pair \( \langle \hat{M}, \hat{N} \rangle = (\hat{x}, \pm\hat{p}) \), we can reach the Bayesian amplification uncertainty relation for quantum channels \[27\]
\[\begin{align*}
\bar{V}^{(x)}_x(G_x, \lambda)\bar{V}^{(p)}(G_p, \lambda) &\geq \frac{1}{2} \left( \frac{G}{1 + \lambda} + \frac{G}{1 + \lambda} + 1 \right) \quad (7)
\end{align*}\]
where the lower sign corresponds to the case of phase-conjugate amplification and attenuation. Moreover, if \( \mathcal{E} \) is entanglement breaking, the minimum of the product \( \bar{V}_x\bar{V}_p \) has to satisfy a more restricted condition \[27\]
\[
\bar{V}^{(x)}_x(G_x, \lambda)\bar{V}^{(p)}(G_p, \lambda) \geq \frac{1}{4} \left( \frac{2G}{1 + \lambda} + 1 \right)^2 .
\]
(8)

Notably, Eqs. (7) and (8) reproduce the forms of Eqs. (1) and (2) in the limit of the uniform prior \( \lambda \rightarrow 0 \). In what follows, we will find that a general joint measurement, which is described by a quantum channel \( \mathcal{E} \) and commutable observables \( [\hat{M}, \hat{N}] = 0 \), has to fulfill
\[
\bar{V}^{(x)}_M(G_x, \lambda)\bar{V}^{(p)}_N(G_p, \lambda) \geq \left( \frac{G}{1 + \lambda} \right)^2 .
\]
(9)

This relation will establish a joint-measurement uncertainty relation free from the unbiasedness assumption. The inequality in Eq. (9) also reproduces the form of Eq. (3) in the limit \( \lambda \rightarrow 0 \) for \( G = 1 \).

We will find a bound on the pair of the MSDs based on the method in Refs. \[7, 27\]. Let be \( |\psi_\lambda\rangle = \sqrt{\lambda/(1 + \lambda)} \sum_{n=0}^{\infty} (1 + \lambda)^{-n/2} |n\rangle_A |n\rangle_B \) a two-mode squeezed state, and consider the state after an action of the quantum channel \( \mathcal{E} \) on subsystem \( A \),
\[
J = \mathcal{E}_A \otimes I_B(|\psi_\lambda\rangle\langle \psi_\lambda |),
\]
where \( I \) denotes the identity map. We can observe that the MSD represents the correlation between a measurement observable on \( A \) and a canonical variable on \( B \) for the state \( J \). To be concrete, a straightforward calculation \[27\] from Eqs. (5), (6), and (10), leads to
\[
\begin{align*}
\bar{V}^{(x)}_M(\eta_x, \lambda) &:= \text{Tr} \left[ (\hat{M}_A - \sqrt{\eta_x}\hat{x}_A)^2 J \right] + \frac{\tau_x}{2}

\bar{V}^{(p)}_N(\eta_p, \lambda) &:= \text{Tr} \left[ (\hat{N}_A + \sqrt{\eta_p}\hat{p}_B)^2 J \right] + \frac{\tau_p}{2},
\end{align*}
\]
(11)

where the rightmost terms are responsible for the vacuum fluctuation due to the mapping procedure from the q-numbers to c-numbers \( (\hat{x}, \hat{p}) \rightarrow (x_{\alpha}, p_{\alpha}) \), and we defined
\[
(\tau_x, \tau_p) := (\eta_x, \eta_p)/(1 + \lambda).
\]
(12)

A physical limitation in a product form is directly imposed by using the preparation uncertainty relation for the state \( J \):
\[
\begin{align*}
\text{Tr}[ (\hat{M}_A - \sqrt{\eta_x}\hat{x}_A)^2 J ] \text{Tr}[ (\hat{N}_A + \sqrt{\eta_p}\hat{p}_B)^2 J ] &\geq \langle \Delta^2 (\hat{M}_A - \sqrt{\eta_x}\hat{x}_A) \rangle \langle \Delta^2 (\hat{N}_A + \sqrt{\eta_p}\hat{p}_B) \rangle

&\geq \frac{1}{4} \left( \langle [\hat{M}_A, \hat{N}_A] \otimes I_B - \sqrt{\eta_x}\eta_p \hat{B} \rangle \right)^2 .
\end{align*}
\]
(13)

\[\textbf{Lemma.}\]— Let \( \hat{M} \) and \( \hat{N} \) be a pair of self-adjoint operators, and \( \mathcal{E} \) be a quantum channel. For any given positive numbers \( (\lambda, \eta_x, \eta_p) \), the following relation holds:
\[
\left( \bar{V}^{(x)}_M(\eta_x, \lambda) - \frac{\eta_x}{2(1 + \lambda)} \right) \left( \bar{V}^{(p)}_N(\eta_p, \lambda) - \frac{\eta_p}{2(1 + \lambda)} \right) \geq \frac{1}{4} \left( \langle [\hat{M}_A, \hat{N}_A] \otimes I_B \rangle - \sqrt{\eta_x}\eta_p \right)^2 ;
\]
(14)

where the MSDs \( (\bar{V}^{(x)}_M, \bar{V}^{(p)}_N) \) is defined in Eq. (9), and the state \( J \) is given in Eq. (10)

\[\text{Proof.}\]— Substituting Eqs. (11) and (12) into Eq. (13) with the help of the canonical commutation relation \( [\hat{x}_B, \hat{p}_B] = i \), we obtain the inequality of Eq. (14).

When we set \( \langle \hat{M}, \hat{N} \rangle = (\hat{x}, \pm\hat{p}) \), Lemma leads to the Bayesian amplification uncertainty relation in Eq. (7) \[27\]. Similar setting enables us to derive Eq. (8) where a separable inequality for Einstein-Podolsky-Rosen like operators \[28, 29\] is employed instead of Eq. (10). Our interest here is to address the joint-measurement uncertainty relation in the form of Eq. (9). In a general setup of joint measurements, the signal state in system \( A \) is interacted with an ancilla system \( A' \). Then, a projective measurement \( \mathcal{Q} \) concerning the original, or true position of the signal is performed on the system \( A \), and another projection \( \mathcal{P} \) concerning the momentum of the original signal is carried out on system \( A' \). The measurement observables are typically written as
\[
\hat{M} = \hat{Q}_A \otimes \mathbb{I}_{A'}, \quad \hat{N} = \mathbb{I}_A \otimes \hat{P}_{A'},
\]
and thus commutable \( [\hat{M}, \hat{N}] = 0 \).
Here, we describe the interaction with possible ancillary systems by a quantum channel \(\mathcal{E}\), in which an input state in a single mode system could be transformed into an output state in any physically allowable system of an arbitrary size. As for the measurement observables we only assume that they are commutable \([\hat{M}, \hat{N}] = 0\).

**Proposition.**—Let \((\hat{M}, \hat{N})\) be a pair of commutable observables satisfying \([\hat{M}, \hat{N}] = 0\) and \(\mathcal{E}\) be a quantum channel. For any given positive numbers \((\lambda, \eta_x, \eta_p)\), the following trade-off relation holds

\[
\left( \frac{\bar{V}_M^{(x)}(\eta s, \lambda)}{2} - \frac{\eta s}{1 + \lambda} \right) \times \left( \frac{\bar{V}_N^{(p)}(\eta s, \lambda)}{2} - \frac{\eta s}{1 + \lambda} \right) \geq \frac{1}{4} \left( \frac{\eta}{1 + \lambda} \right)^2, \tag{16}
\]

where the MSDs \((\bar{V}_M^{(x)}, \bar{V}_N^{(p)})\) are defined in Eq. (5). Moreover, the equality of Eq. (16) can be achieved by a joint-measurement setup using a beam splitter and quadrature measurements.

**Proof of Proposition.**—Substituting \((\eta_x, \eta_p) = \eta(s, s^{-1})\) and \([\hat{M}, \hat{N}] = 0\) into our Lemma of Eq. (14), we obtain Eq. (16). In order to prove the attainability, let us consider a half-beam splitter with a vacuum field as an ancilla that transforms the coherent state as

\[
\mathcal{E}(\rho_0) = \rho_0/\sqrt{2} \otimes \rho_0/\sqrt{2}.	ag{17}
\]

Let us set the pair of observables as

\[
(\hat{M}, \hat{N}) = \frac{\sqrt{2\eta}}{1 + \lambda} \left( s^{1/2} \hat{x} \otimes 1, s^{-1/2} 1 \otimes \hat{p} \right). \tag{18}
\]

It clearly fulfills \([\hat{M}, \hat{N}] = 0\). Substituting Eqs. (17) and (18) into Eq. (5), we obtain

\[
(\bar{V}_M(\eta s, \lambda), \bar{V}_N(\eta s, \lambda)) = \frac{\eta}{1 + \lambda}(s, s^{-1}). \tag{19}
\]

This saturates the inequality in Eq. (16). Note that the ratio \(s = \sqrt{\eta_x/\eta_p}\) can be interpreted as a consequence that either the state is squeezed or the measured value is rescaled.

Proposition implies that the product \(\bar{V}_M \bar{V}_N\) is lower bounded as in Eq. (17) when the gain \(\eta = \sqrt{\eta_x/\eta_p}\) is fixed and the ratio \(s = \sqrt{\eta_x/\eta_p}\) is tweaked as in Fig. 1 (See also Sec. 2A.5 in \[22\]). Moreover, the inverse proportional curve due to Eq. (19) can be swept by Eq. (19). Therefore, this completes the final step to reformulate the three uncertainty relations of Eqs. (1), (2), and (3) into experimentally testable relations of Eqs. (7), (8), and (9) based on a unified framework without imposing the linearity assumptions on the physical maps. We can observe that the minimum penalty curve for the measurement process is located in the middle of the curves for the quantum channel and the entanglement breaking channel (See Fig. 1). This is a rigorous example that shows a hierarchy on the trade-off relations of controlling canonical variables for general quantum channels, joint measurements, and entanglement breaking channels suggested from the uncertainty relations in Eqs. (11), (12), and (13).

**FIG. 1:** Interrelation between the minimum uncertainty curves for quantum channels, joint-measurements, and entanglement-breaking maps in the case of unit gain and uniform prior \((G, \lambda) = (1, 0)\) in Eqs. (7), (8), and (9). The curve for the joint-measurement process is obtained by tweaking \(s \in (0, \infty)\) in Eq. (16). Here, the cases of \(s = 1\) and \(s = 2\) are displayed as the dashed curves. Tangent lines of these curves correspond to the bound for the weighted mean-square errors in the multi-parameter estimation.

Thus far, we have described the measurement process by using a quantum channel \(\mathcal{E}\) and a pair of observables, \(\{\mathcal{E}, \hat{M}, \hat{N}\}\). In the following part, we rewrite our measurement model by using the POVM. This will establish a link between our measurement uncertainty relation and a major result of the quantum estimation theory \[12\]. Let us start with recalling the framework of parameter estimation. We consider a set of quantum states called parametric family \(\{\rho_\theta\}_{\theta \in \Theta}\), where \(\theta\) is an unknown parameter belongs to a set \(\Theta\). A prior probability distribution \(\{p_\theta\}_{\theta \in \Theta}\) is assigned to specify an ensemble of states \(\{\rho_\theta\}_{\theta \in \Theta}\) in Bayesian approach. The true value of an operator \(\hat{X}\) for an unknown state \(\rho_\theta\) is defined as \(X_\theta = \text{Tr}(\rho_\theta \hat{X})\). An estimation process is described by a POVM \(\{\hat{m}_i\}\) and a set of real quantities \(\{X_i\}\); The estimator \(\hat{\theta}_i, X_i\) an estimation value \(X_i\) for each measurement outcome \(i\) of the POVM. An estimator is said to be optimal if it minimizes the mean square error (MSE)

\[
V_X := \sum_{\theta \in \Theta} \sum_i p_\theta (X_i - \theta)^2 \text{Tr}(\hat{m}_i \rho_\theta), \tag{20}
\]

where both the set \(\{X_i\}\) and the POVM \(\{\hat{m}_i\}\) are optimized. Roughly speaking, the estimator \(\hat{\theta}_i, X_i\) is more
favorable if the loss function $V_X$ is smaller. For the multi-parameter estimation, we may consider a set of operators \( \{X, \hat{Y}, \hat{Z}, \cdots \} \), and the problem is to minimize a weighted sum of the mean square errors for a set of operators such as $g_X V_X + g_Y V_Y + g_Z V_Z + \cdots$, where $g_X, g_Y, g_Z, \cdots \geq 0$ denotes the weighting factors, and a multi-parameter estimator can be specified by the sequence \( \{m_i, X_i, \hat{Y}_i, \hat{Z}_i, \cdots \} \).

Our Proposition leads to a bound for a product of the MSEs for the canonical observables \( \hat{X}, \hat{Y}, \hat{Z}, \cdots \). Let \( \{m_i\} \) be a POVM. For any multi-parameter estimator described by \( \{m_i, X_i, P_i\} \), the following trade-off relation holds

$$V_X V_P \geq \left( \frac{G}{1+\lambda} \right)^2$$  \hspace{1cm} (21)

where the pair of the MSEs are given by

$$V_X := \sum_i \int p_\lambda(\alpha) (X_i - \sqrt{G} x_\alpha e^{-R})^2 \text{Tr}(\hat{m}_i \rho_\alpha) d^2 \alpha,$$

$$V_P := \sum_i \int p_\lambda(\alpha) (P_i - \sqrt{G} \rho_\alpha e^{R})^2 \text{Tr}(\hat{m}_i \rho_\alpha) d^2 \alpha.$$  \hspace{1cm} (22)

Note that one can drop the parameters \((G, R)\) by considering a rescaled estimation with \((\hat{X}, \hat{P}) = G^{-1/2} (X_i e^R, P_i e^{-R})\). In such a scenario we have $V_X V_P \geq (1+\lambda)^{-2}$ with $(\hat{V}_X, \hat{V}_P) = (G^{-1} (V_X e^{2R}, V_P e^{-2R})$, instead of Eq. (21). This is indeed in the form of Eq. (23).

**Proof.** — Let us define an entanglement breaking channel associated with the POVM \( \{\hat{m}_i\} \) as

$$\mathcal{E}(\rho) = \sum_i \text{Tr}(\hat{m}_i \rho) |u_i \rangle \langle u_i| \otimes |v_i \rangle \langle v_i|,$$ \hspace{1cm} (23)

where \( \{|u_i\}_i \) and \( \{|v_i\}_i \) are orthonormal bases. Let us choose a pair of commutative observables as

$$\hat{M} = \sum_i (X_i |u_i \rangle \langle u_i|) \otimes \mathbb{1}, \ \ \hat{N} = \mathbb{1} \otimes \sum_i P_i |v_i \rangle \langle v_i|.$$ \hspace{1cm} (24)

By substituting Eqs. (23) and (24) into Eq. (15) with \((\eta_x, \eta_p) = (Ge^{-2R}, Ge^{2R})\), we obtain $V_X$ and $V_P$ in Eqs. (22), i.e., we can write $(V_X, V_P) = (V_M(Ge^{-2R}, \lambda), V_N(Ge^{2R}, \lambda))$. Hence, the pair $(V_X, V_P)$ satisfies Eq. (23) as well as Eq. (15). This implies Eq. (21).

In the proof of Corollary we have shown that any multi-parameter estimator \( \{m_i, X_i, P_i\} \) can be described by the measurement model \( \{\mathcal{E}, \hat{M}, \hat{N}\} \). We can show that the converse direction is also true, and the two descriptions are equivalent as a type of Naimark’s theorem.

**Theorem.** — For any joint measurement described by \( \{\mathcal{E}, \hat{M}, \hat{N}\} \) with \([\hat{M}, \hat{N}] = 0\), there exists a multi-parameter estimator \( \{\hat{m}_i', a_i, b_i\} \) that gives the MSDs $(\hat{V}_M, \hat{V}_N)$. The converse direction also holds.

**Proof.** — Since $[\hat{M}, \hat{N}] = 0$, there exists an orthonormal basis \( \{\omega_i\} \) that simultaneously diagonalizes $(\hat{M}, \hat{N})$ such that $\hat{M} = \sum_i a_i |\omega_i \rangle \langle \omega_i|$ and $\hat{N} = \sum_i b_i |\omega_i \rangle \langle \omega_i|$ with the sets of eigenvalues \( \{a_i\} \) and \( \{b_i\} \). Since $\mathcal{E}$ is a quantum channel, we have a Kraus representation $\mathcal{E}(\rho) = \sum_j K_j \rho K_j^\dagger$ with \( \sum_j K_j^\dagger K_j = \mathbb{1} \). Hence, we can write

$$\hat{V}_M^{(x)} (\eta_M, \lambda) = \sum_{i} \int p_\lambda(\alpha) (a_i - \sqrt{\eta_M} x_\alpha)^2 \text{Tr}(\hat{m}_i' \rho_\alpha) d^2 \alpha,$$

$$\hat{V}_N^{(p)} (\eta_N, \lambda) = \sum_{i} \int p_\lambda(\alpha) (b_i - \sqrt{\eta_N} \rho_\alpha)^2 \text{Tr}(\hat{m}_i' \rho_\alpha) d^2 \alpha.$$ \hspace{1cm} (25)

where $\hat{m}_i' := \sum_j K_j |\omega_i \rangle \langle \omega_i| K_j^\dagger \geq 0$. From $\sum_j K_j^\dagger K_j = \mathbb{1}$ and $\sum_i |\omega_i \rangle \langle \omega_i| = \mathbb{1}$ we can readily check that \( \{\hat{m}_i'\} \) fulfills the condition for a POVM. $\sum_i \hat{m}_i' = \mathbb{1}$. This confirms that the pair $(\hat{V}_M, \hat{V}_N)$ represents the MSEs in the form of Eq. (25). Therefore, the pair $(\hat{V}_M, \hat{V}_N)$ can be determined by the property of an estimator $(\{\hat{m}_i', a_i, b_i\})$.

Conversely, for any estimator $(\{\hat{m}_i, X_i, P_i\})$ we can find a set \( \{\mathcal{E}, \hat{M}, \hat{N}\} \) that gives the pair of the MSEs $(V_X, V_P)$ as shown in the proof of Corollary.

Finally, we will reproduce one of the central outcomes in the quantum estimation theory. For the mean-value estimation for Gaussian states with the variances $(\sigma_x^2, \sigma_p^2)$, the most informative bound for weighted MSEs [12] is given by (see Eq. (6.6.65) of Ref. [13])

$$g_x V_x + g_p V_p \geq g_x \sigma_x^2 + g_p \sigma_p^2 + \sqrt{g_x g_p}.$$ \hspace{1cm} (26)

We can immediately reach this relation from Eq. (16) for the case of pure Gaussian states, i.e., $\sigma_x^2 \sigma_p^2 = 1/4$, as follows. Let us set $(\eta, \lambda) = (1, 0)$ and take the square root of Eq. (16). Then, by using the relation $(a b + b t^{-1})/2 \geq \sqrt{ab}$ for positive numbers $(a, b, t)$, we obtain

$$t(\hat{V}_M(s, 0) - s/2) + t^{-1}(\hat{V}_N(s^{-1}, 0) - s^{-1}/2) \geq 1.$$ \hspace{1cm} (27)

We can see that Eq. (27) coincides with Eq. (26) by applying the following replacement: $(\hat{V}_M(s, 0), \hat{V}_M(s^{-1}, 0)) \rightarrow (V_x, V_p)$, $(s, s^{-1}) \rightarrow 2(\sigma_x^2, \sigma_p^2)$, and $t \rightarrow \sqrt{g_x / g_p}$ as long as $\sigma_x^2 \sigma_p^2 = 1/4$ holds. To this end, our approach reveals that the origin of the bound rather directly comes from commutation relations. The case for mixed Gaussian states should be addressed elsewhere. From a geometrical point of view, Eq. (26) corresponds to tangent lines of the curve of Eq. (26) (See Fig. 1).

We have presented a joint-measurement uncertainty relation based on a Bayesian input ensemble of optical states. It reproduces the form of the Arthurs-Kelly relation in Eq. (3) in the uniform prior limit, and the most informative bound for quantum estimation of phase-space displacement in the case of pure Gaussian states. Our measurement model is equivalent to the parameter estimation based on the POVM. Our uncertainty relation is applicable to such a general measurement process and,
one can determine to what extend the performance of a given joint-measurement device is close to the theoretical limit in terms of the MSDs by using a moderate set of input states within realistic assumptions.

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