New solutions of the Ermakov-Pinney equation in curved space-time

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Abstract
An Ermakov-Pinney-like equation associated with the scalar wave equation in curved space-time is here studied. The example of Schwarzschild space-time considered in the present work shows that this equation can be considered more as a “model equation”, with interesting applications in black hole physics. Other applications considered involve cosmological spacetimes (de Sitter) and pulse of plane gravitational waves: in all these cases the evolution of the Ermakov-Pinney field seems to be consistent with a rapid blow-up, unlike the Schwarzschild case where (damped) oscillations are allowed. Eventually, the phase function is also evaluated in many of the above spacetime models.

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1 Introduction
Let \((M, g)\) denote a four-dimensional space-time with metric \(g\) of Lorentzian signature +2, and with associated scalar product defined as
\[
\langle A, B \rangle \equiv g_{\mu\nu}A^\mu B^\nu = g(A, B).
\]
The scalar wave equation describing a massless scalar field propagating on \((\mathcal{M}, g)\) reads
\[
\square \chi = 0, \tag{1}
\]
where the “box operator” is the familiar wave operator in curved space-time
\[
\square = g^{\mu\nu} \nabla_\mu \nabla_\nu. \tag{2}
\]

Equation \(\square\) has been largely studied in the literature, in various contexts including black holes, cosmological spacetimes, gravitational wave spacetimes, etc. In particular, as shown in Ref. \[1\], the technical difficulty of the coupled nature of Maxwell equations in curved spacetime can be overcome by mapping them into a wave equation for a complex scalar field. The real and imaginary part of such a field are therefore ruled by a scalar wave equation as Eq. \(\square\). This property is eventually applied to the investigation of binary systems in relativistic astrophysics \[1\].

The standard approach to Eq. \(\square\) is the separation of variables when the symmetries of space-time allow for it, or mode-sum decomposition when a Fourier analysis can be performed. In general, the theory in Ref. \[2\] suggests looking for a solution \(\phi\) which, up to a remainder term, consists of a function \(\Phi\) obtained by integrating products of functions depending on all cotangent bundle local coordinates. This occurs because Fourier transforms must be replaced from Fourier-Maslov integral operators on passing from Minkowski spacetime to curved pseudo-Riemannian manifolds \[2\]. However, we here try to understand whether exact solutions of Eq. \(\square\) exist that possess the oscillating behaviour
\[
\chi = \alpha e^{i\varphi}. \tag{3}
\]

Under such an assumption, one finds the coupled set \[3,4\] of equations (here written first in dimensionless units for simplicity)
\[
\operatorname{div}(\alpha^2 \text{grad} \varphi) \equiv \nabla^\mu (\alpha^2 \nabla_\mu \varphi) = 0, \tag{4}
\]
\[
\langle \text{grad} \varphi, \text{grad} \varphi \rangle = \square \alpha / \alpha. \tag{5}
\]

This means that, given the vector field \(\psi\) with covariant components (i.e., its 1-form realization)
\[
\psi_\mu = \alpha^2 \nabla_\mu \varphi, \tag{6}
\]
one can first look for solutions of the divergenceless condition
\[
\operatorname{div}(\psi) = \nabla^\mu \psi_\mu = 0. \tag{7}
\]

As a second step, one obtains from Eq. \(\square\) the equation
\[
\alpha^3 \Box \alpha = \langle \psi, \psi \rangle, \tag{8}
\]
to be solved for \(\alpha\). Third, Eq. \(\psi\) yields the phase function \(\varphi\) by solving the first-order equations
\[
\nabla_\mu \varphi = \alpha^{-2} \psi_\mu. \tag{9}
\]
Hereafter, in order to obtain a physics-oriented scheme, we find it convenient to re-express the covariant components of $\psi$ in the form

$$\psi_\mu = f^c_\mu J_\mu, \tag{10}$$

where $f^c_\mu$ is a freely specifiable coupling constant, and $J_\mu$ is a current covector. The non-linear equation $[\psi]$ bears a clear resemblance with the Ermakov-Pinney non-linear ordinary differential equation $[5,6]

$$y''(x) + p(x)y(x) = q(x)y^{-3}(x), \tag{11}$$

upon setting $p = 0$ therein, which is why we refer to it as the Ermakov-Pinney equation in curved space-time. More precisely, by the Ermakov-Pinney-like equation in a generic curved space-time we mean hereafter the coupled set of equations

$$\alpha^3 \Box \alpha = f^2_c ||J||^2, \quad \nabla_\mu J^\mu = 0, \tag{12}$$

where $||J||^2 = \langle J, J \rangle = \epsilon |J^\mu J_\mu|$, with $\epsilon = -1, 0, 1$ for $J$ timelike, null, spacelike, respectively. Since for $\epsilon = 0$ one finds for $\alpha$ just the scalar wave equation $[11]$ we started from, we shall limit ourselves to studying timelike and spacelike currents. Our $\alpha$ can be seen as a real, self-interacting scalar field, gravitationally interacting with the background space-time $^1$ and sourced by a divergence-free vectorial current $J^\alpha$. The latter is consistently obtained by solving the divergence equation in the assigned background. The equation for $\alpha$ can also be cast in the form

$$\alpha^3 \Box \alpha = f^2_c J^\mu J_\mu \equiv \epsilon f^2_c |J^\mu J_\mu|, \tag{13}$$

In the spacelike and timelike cases non-linearities come into play, leading to new interesting features, as will be shown in the following sections.

Some formal simplification or extensions can still be considered. For example, it can be convenient to separate magnitude and direction of $J$, i.e., $J^\mu = \rho u^\mu$, $u \cdot u = \epsilon$, so that

$$\nabla_u \rho + \rho \nabla_u u = \nabla_u \rho + \rho a(u) = 0, \tag{14}$$

where $a(u) = \nabla_u u$. If $J$ is timelike $(u \cdot u = -1)$ the above equation implies in general

$$a(u) = -\nabla_u \ln \rho, \tag{15}$$

i.e., the timelike family $u$ is accelerated with acceleration $a(u)$. Choosing $u$ timelike and also tangent to a geodesic of the background, $a(u) = 0$, the equation for the evolution of the magnitude of the vector $J^\mu$ becomes

$$\nabla_u \ln \rho = 0, \tag{16}$$

identifying $\rho$ so far. Consequently, with the introduction of $\rho$, the equation for $\alpha$, Eq. (13), reads as

$$\alpha^3 \Box \alpha = \epsilon f^2_c \rho^2. \tag{17}$$

We notice that in view of other extensions, it is possible to consider the counterpart of the above Eq. (17) in a gauge-theory context, where the full

$^1$ Generalizations with $\alpha$ a complex scalar field are also possible.
covariant derivatives receive a contribution from a gauge potential $A^\beta_\mu(x)$, i.e. a Lie-algebra-valued 1-form, according to

$$\partial_\mu \rightarrow \nabla_\mu + G_\beta A^\beta_\mu(x)$$  \hspace{1cm} (18)

with $G_\beta$ the generators of the symmetry group. In the case of a scalar field Eq. (18) becomes

$$\partial_\mu \rightarrow D_\mu = \nabla_\mu + qA_\mu(x),$$  \hspace{1cm} (19)

implying

$$D_\mu D^\mu \alpha = \Box \alpha + 2q\alpha \nabla_\mu A^\mu + qA^\mu \nabla_\mu \alpha + q^2 A_\mu A^\mu \alpha.$$  \hspace{1cm} (20)

On reverting now to the central aim of our paper, a naturally occurring question is whether such a scheme is equally successful as the other direct (standard) approach. Properly speaking, the two methods are equivalent, even if in one case one aims at solving a single, linear equation, Eq. (1), whereas in the other case the equations to be solved are two, Eqs. (7) and (8), coupled and with one of the two (Eq. (8)) which is also non-linear. Recently, the work in Refs. [3,4], following the second approach in a Kasner spacetime, has obtained the exact expression for amplitude $\alpha$ and phase function $\varphi$ in the integral representation of the solution for given initial conditions. The success achieved in the simple Kasner context has (mainly) taken advantage from the fact that one of the two equations, the divergence equation, was particularly easy to solve. This solution, in turn, has been “driving” in a sense the corresponding solution of the second equation too. A naturally occurring question is therefore whether this approach can be equally successful also in other cases. In order to answer this question we have analyzed, in Secs. 2-4 below, three typical exact solutions: Schwarzschild, de Sitter, gravitational wave solving in all cases the Ermakov-Pinney equation. Later on we solve for the phase function that obeys Eq. (9), and hence we plot the solutions $\alpha \cos \varphi$ of the wave equation (1). Concluding remarks are made in Sect. 5.

2 Explicit examples

We will discuss hereafter the cases of Schwarzschild, de Sitter and a single plane gravitational wave space-times. Their examination in specific contexts will contribute to clarify the physical content of the Ermakov-Pinney equation. We will solve the coupled set of equations (13) and (7)

$$\alpha^3 \Box \alpha = f^2 J^\mu J_\mu, \hspace{1cm} \nabla_\mu J^\mu = 0,$$  \hspace{1cm} (21)

also providing (analytically or numerically) the phase function $\varphi$ defined by Eq. (9) above

$$\nabla_\mu \varphi = \alpha^{-2} f c J_\mu,$$  \hspace{1cm} (22)

as well as the product $X = \alpha \cos \phi$, which is the real part of the desired oscillating solution of Eq. (1).

\footnote{We follow the convention according to which Greek indices from the beginning of the alphabet are Lie-algebra indices. When necessary, this index specification is explicitly repeated in the text to avoid confusion.}
2.1 Schwarzschild spacetime

Let us consider the case of a Schwarzschild spacetime with metric written in standard Schwarzschild coordinates \( x^\mu = (t, r, \theta, \phi) \)

\[
ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\] (23)

Looking for particular solutions instead of general formulas, one can provide interesting examples, as shown below.

1. \( J \) timelike

Upon assuming \( J \) timelike (\( \epsilon = -1 \)) and aligned with the time lines, a simple solution of the divergence equation \( \text{div}(J) = 0 \) is given by the vector field

\[
J = A^2 \frac{\partial}{(1 - \frac{2M}{r}) \partial t} \Rightarrow J^\flat = -A^2 dt,
\] (24)

where \( A = A(r) \). Thus, the squared pseudo-norm of \( J \) is equal to

\[
\langle J, J \rangle = -\frac{A^4}{(1 - \frac{2M}{r})}.
\] (25)

On setting \( f_c^2 \to f_c^2 M^{-2} \) (for dimensional reasons) and \( \alpha = A(r) \), Eq. (13) can be rewritten in the form

\[
A^3 \Box A = \frac{f_c^2}{M^2} \langle J, J \rangle = -\frac{f_c^2}{M^2} \frac{A^4}{(1 - \frac{2M}{r})}.
\] (26)

Thus, the Ermakov-Pinney equation becomes

\[
A'' = \frac{2(M - r)}{r^2(1 - \frac{2M}{r})} A' - \frac{f_c^2}{M^2(1 - \frac{2M}{r})} A,
\] (27)

where the prime denotes differentiation with respect to \( r \). This form of the equation suggests defining the independent variable \( x \) via

\[
\rho \equiv \frac{r}{M}, \quad x \equiv \frac{\rho}{2} - 1,
\] (28)

such that \( x \in [0, \infty] \). Hence we obtain the linear second-order equation

\[
A_{xx} + \frac{(1 + 2x)}{x(x + 1)} A_x - \xi^2 \frac{(x + 1)^2}{x^2} A = 0.
\] (29)

where \( \xi = 2if_c \). Note that this equation contains \( \xi^2 \) and then is invariant under the map \( \xi \to -\xi \equiv \bar{\xi} \). A simple consequence of this is that, given a

\[
3 \text{ In general, all our formulae for currents are particular cases of the general expressions}
\]

\[
J = J^\mu \frac{\partial}{\partial x^\mu}, \quad J^\nu = J^\mu g_{\mu\nu}, \quad J^\flat = J^\nu dx^\nu.
\]
solution \( A_1(\xi, x) \), a second, independent solution is \( A_2(\xi, x) = A_1(-\xi, x) \).
Indeed, on denoting by HeunC the confluent Heun function \([7]\), the general solution of such an equation is
\[
A(\xi, x) = C_1 A_1(\xi, x) + C_2 A_2(\xi, x) = e^{-\xi x} \left[ C_1 x^2 H_1(\xi, x) + C_2 x^{-\xi} H_2(\xi, x) \right],
\]
where \( C_1 \) and \( C_2 \) are arbitrary constants, and we have defined
\[
H_1(\xi, x) \equiv \text{HeunC}\left[2\xi, 2\xi, 0, 2\xi^2, -2\xi^2, -x\right],
\]
\[
H_2(\xi, x) \equiv \text{HeunC}\left[2\xi, -2\xi, 0, 2\xi^2, -2\xi^2, -x\right].
\]
One can then prove the above mentioned “doubling” property, \( A_2(\xi, x) = A_1(-\xi, x) \), by virtue of the identity
\[
H_1(-\xi, x) = e^{-2\xi x} H_2(\xi, x).
\]
The dependence on the parameter \( f_c \) in the solutions of Eq. (27) can be studied by integrating numerically the equation. The case \( f_c > 0 \) of interest here corresponds to damped oscillations, as stated above. The situation is illustrated in Fig. 2.

In this case the phase function reduces to
\[
\varphi = -f_c t + \varphi_0,
\]
where \( \varphi_0 \) is an integration constant, and then the associated quantity becomes \( X = \alpha \cos \varphi = A(r) \cos(f_c t - \varphi_0) \).

2. \( J \) spacelike

For the case of spacelike current, we consider
\[
J = \frac{M^2 \cos^2 \omega t \partial}{r^2} \partial_r \implies J^p = \frac{M^2 \cos^2 \omega t}{r^2 (1 - \frac{2M}{r})} dr,
\]
with
\[
\langle J, J \rangle = \frac{M^4 \cos^4 \omega t}{r^4 (1 - \frac{2M}{r})}.
\]
This current satisfies the divergenceless condition, although it is not the most general form of current that satisfies such a property. Now we look for the amplitude function \( \alpha \) in the factorized form
\[
\alpha(r, t) = A(r) \cos \omega t.
\]
The Ermakov-Pinney equation reads
\[
\alpha^3 \Box \alpha = \frac{f_c^2}{M^2} \langle J, J \rangle,
\]
leading to the following non-linear equation for \( A(r) \):
\[
A'' = \frac{2(M - r)}{r^2 (1 - \frac{2M}{r})} A' + \frac{1}{(r - 2M)^2} \left[ \frac{f_c^2 M^2}{r^2 A'} - r^2 A'' \right].
\]
On passing to the variable $\rho$, defined in Eq. (28), Eq. (38) becomes

$$A_{,\rho \rho} = -2 \frac{(\rho - 1)}{\rho(\rho - 2)} A_{,\rho} - \frac{[\rho^2 A \Omega - f_c][\rho^2 A \Omega + f_c]}{\rho^2(\rho - 2)^2 A^3(\rho)},$$

(39)

where $\Omega$ is the dimensionless parameter $M \omega$.

Next, we look for the phase function $\varphi$ that solves the first-order equation

$$\frac{\partial \varphi}{\partial r} = \alpha^{-2} f_c J_r,$$

(40)
The full solution $A(\rho)$ is plotted in the case $f_c = 1$, as an example. Initial conditions are chosen so that $\alpha(3) = 0$ and $\alpha'(3) = 1$. The oscillating and damped behaviour is further enhanced as soon as the value of $f_c$ increases.

and is therefore found to be

$$\varphi = f_c M^2 \int_{r}^{r_0} \frac{dr}{A(r)r^2(1 - \frac{2M}{r})} + \varphi_0. \quad (41)$$

The case $f_c > 0$ of interest here corresponds to damped oscillations, as in the timelike case. The difference is that now the graph of the function stays positive while damping. The larger the value of $f_c$, the more frequent are the oscillations. The behaviour of $A$ is illustrated in Fig. 4 for $f_c = 1$, whereas Fig. 6 shows the behaviour of the phase $\varphi$ as a function of $\rho$, again in the case $f_c = 1$ and $\Omega = 1$.

3 de Sitter spacetime

The de Sitter space-time metric in spherical-like coordinates $x^\mu = (t, r, \theta, \phi)$ has a squared line element

$$ds^2 = -(1 - H^2 r^2)dt^2 + \frac{dr^2}{(1 - H^2 r^2)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (42)$$

and it is formally close to the Schwarzschild metric, except for a different expression for the $g_{tt} = -1/g_{rr}$ metric component and the presence of a cosmological horizon at $r_h = 1/H$, so that in this case $r \in [0, 1/H]$. This metric is a solution of Einstein’s equations in the presence of a cosmological constant $A = 3H^2 \ [8]$. 

Fig. 2 Schwarzschild solution and a timelike current vector: The full solution $A(\rho)$ is plotted in the case $f_c = 1$, as an example. Initial conditions are chosen so that $\alpha(3) = 0$ and $\alpha'(3) = 1$. The oscillating and damped behaviour is further enhanced as soon as the value of $f_c$ increases.
Fig. 3  Schwarzschild solution and a spacelike current vector: The full solution $A(\rho)$ is plotted in the case $f_c = 1$ and $\Omega = 1$ (black online), as an example. The initial conditions are chosen so that $A(3) = 1$ and $A'(3) = 1$.

Fig. 4  Schwarzschild solution and a spacelike current vector: The phase $\varphi(\rho)$ is plotted in the case $f_c = 1$ and $\Omega = 1$ (black online), as an example. The initial conditions are chosen so that $A(3) = 1$ and $A'(3) = 1$. 
1. *J* timelike

We consider a divergenceless current having the vector field description

\[ J = \frac{A^2(r)}{(1 - H^2 r^2)} \frac{\partial}{\partial t} \implies J^t = -A^2(r) dt. \]  

(43)

The Ermakov-Pinney equation for \( \alpha = A(r) \) is

\[ A^3 \Box A = f_c^2 H^2 (J, J), \]  

(44)

and leads, upon defining \( \rho \equiv rH \in (0,1) \), to the linear equation

\[ A_{\rho \rho} = \frac{2(2 \rho^2 - 1)}{\rho(1 - \rho^2)} A_{,\rho} - \frac{f_c^2}{(\rho^2 - 1)^2} A. \]  

(45)

This equation can be solved explicitly in the form

\[ A(\rho) = C_1 A(f_c, \rho) + C_2 A(-f_c, \rho), \]  

(46)

where

\[ A(f_c, \rho) \equiv \left( \frac{\rho - 1}{\rho + 1} \right)^{\frac{1}{2}} (\rho + i f_c), \]  

(47)

and \( C_1 \) and \( C_2 \) are integration constants. Moreover, the phase function \( \varphi \) solves the equation

\[ \frac{\partial \varphi}{\partial t} = \alpha^{-2} f_c J^t = -f_c, \]  

(48)

which therefore yields

\[ \varphi = -f_c t + \varphi_0, \]  

(49)

where \( \varphi_0 \) is an additive constant.

The case \( f_c > 0 \) of interest here corresponds to the “deformed bell” behavior shown in Fig. 5. Since both the phase and associated \( X \) variables are simply related to \( A(\rho) \) we will not display their plots.

2. *J* spacelike

We here assume a divergenceless current vector field in de Sitter reading as

\[ J = \cos^2 \omega t \frac{\partial}{\partial r} \implies J^r = \frac{\cos^2 \omega t}{H^2 r^2 (1 - H^2 r^2)} dr. \]  

(50)

By using the variable \( \rho \equiv rH \) and (dimensionless) parameter \( \Omega \equiv \frac{\omega}{H} \), the equation for \( \alpha(t,r) = A(r) \cos \omega t \) becomes the following non-linear equation for \( A \)

\[ A_{,\rho \rho} = 2 \frac{(2 \rho^2 - 1)}{(1 - \rho^2)} A_{,\rho} - \frac{\Omega A^2 \rho^2 - f_c}{(1 - \rho^2)^2 \rho^2} A. \]  

(51)

The phase \( \varphi \) is then such that

\[ \varphi = \int_0^r \frac{dr}{r^2 (1 - H^2 r^2)} + \varphi_0. \]  

(52)

The behavior of \( A(\rho) \) is shown in Fig. 6, whereas that of the phase \( \varphi(\rho) \) in Fig. 7.

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4 Note that, as discussed above in the Schwarzschild case, the second term here is obtained from the first by replacing \( f_c \to -f_c \).
Fig. 5  de Sitter solution and a timelike current vector: The solution \( A(\rho) \) is plotted in the case \( f_c = 1 \) (black online) and initial conditions \( A(0.5) = 1, \ A'(0.5) = 0 \).

Fig. 6  de Sitter solution and a spacelike current vector: The solution \( A(\rho) \) is plotted in the case \( f_c = 1 \) (black online). The parameters are fixed as \( \Omega = 1 = H \) and the initial conditions are chosen so that \( A(0.5) = 0 \) and \( A'(0.5) = 1 \), as an example.
Fig. 7  de Sitter solution and a spacelike current vector: The solution $\varphi(\rho)$ is plotted in the case $f_c = 1$ (black online). The parameters are fixed as $\Omega = 1 = H$ and the initial conditions are chosen so that $\varphi(0.5) = 0$, as an example.

4 Gravitational plane wave pulse

The last explicit example that we consider is a single pulse of gravitational radiation, associated with coordinates $x^\mu = (u, v, x, y)$ and described by the metric

$$ds^2 = -du dv + \cos^2(u) dx^2 + \cosh^2(u) dy^2,$$

which solves the vacuum Einstein equations. This specific choice of the metric implies $u \in [0, \pi/2]$, with $u = \pi/2$ a coordinate horizon.

1. $J$ timelike

We choose

$$J = -\frac{2}{\cos u \cosh u} \frac{\partial}{\partial u} + 2A(u) \frac{\partial}{\partial v} \implies J^\flat = A(u) du + \frac{1}{\cos u \cosh u} dv,$$

so that

$$\langle J, J \rangle = -\frac{4A(u)}{\cos u \cosh u}.$$  \hspace{1cm} (55)

Assuming then for the variable $\alpha$ the relation

$$\alpha(u, v) = A(u)(4v + 1)^{1/4},$$

the equation satisfied by $A(u)$ is

$$\frac{dA}{du} = -\frac{(-\sin(u) \cosh(u) + \sinh(u) \cos(u))}{2 \cos(u) \cosh(u)} A(u) + \frac{f_c^2}{A(u)^2 \cos(u) \cosh(u)},$$

and can be integrated numerically.
2. $J_{spacelike}$

We choose

$$J = \frac{A(u) \cos^2(K_y y) e^{2 K_v v}}{\cos^4 u} \partial_x \Rightarrow J^b = \frac{A(u) \cos^2(K_y y) e^{2 K_v v} dx}{\cos^2 u} \quad (58)$$

so that

$$\langle J, J \rangle = \frac{A^2(u) \cos^4(K_y y) e^{4 K_v v}}{\cos^2 u}. \quad (59)$$

Assuming then for the variable $\alpha$ the following relation:

$$\alpha(u, v, y) = A(u) \cos(K_y y) e^{K_v v}, \quad (60)$$

the equation satisfied by $A(u)$ is

$$\frac{dA}{du} = -\left( \frac{(\cos(u)K_y^2 + 2\cos(u)\sinh(u)\cosh(u)K_v - 2K_v \sin(u)\cosh^2(u))}{4 \cos(u) \cosh^2(u)K_v} \right) A(u)$$

$$- \frac{f_c^2}{4K_v \cos(u)^2 A(u)} \quad (61)$$

and can be integrated numerically.

The results of our study indicate qualitatively a blow-up of the field at the coordinate horizon $u = \pi/2$. 

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**Fig. 8** Gravitational wave solution and a timelike current vector: The solution $A(u)$ is plotted in the case $f_c = 1$ (black online). The initial condition is chosen so that $A(0) = 1$, as an example.
Fig. 9 Gravitational wave solution and a spacelike current vector: The solution $A(u)$ is plotted in the case $f_c = 1$ (black online). The initial condition is chosen so that $A(0) = 1$, as an example.

5 Concluding remarks

We have studied an Ermakov-Pinney-like equation in curved space-time, elucidating both content and role of non-linearities as associated with a divergence-free current source. We have discussed the explicit examples of Schwarzschild space-time, de Sitter space-time, and the space-time corresponding to single pulse of gravitational radiation in both cases of a timelike and spacelike current source, investigating results from the analytic point of view, or from a numerical perspective when the analytic solution is not directly available. The main difficulty of the problem is that to solve the Ermakov-Pinney equation having already requested $J$ to be divergence-free, a fact that poses strong limitations to any general discussion. We are forced then to explore special cases and we do this in the case of a few physically relevant space-times: black holes (Schwarzschild), cosmological spacetimes (de Sitter), gravitational wave spacetime (the metric of a single gravitational wave pulse).

1. In the Schwarzschild case our analysis shows that a positive coupling constant implies damped oscillations of the field, whereas a negative or null one is generally associated with blowing-up of the solutions. Based on this, in view of the interest in studying more deeply the quantization properties of fields, it seems more relevant to limit considerations to the case of a positive coupling constant, a hot topic to be developed in future works. It would also be of interest to understand the relation (if any) with the asymptotic behaviour of solutions of the scalar wave equation found in Ref. [9].
2. In the de Sitter case, the discussion of analogous situations does not show in general oscillations but parabolic-like behaviour, meaning that, from the point of view of studying quantization properties of fields, all cases can be considered.

3. Last but not least, in the gravitational wave case, the numerical analysis shows the occurrence of blow-up behaviours even before reaching the coordinate horizon.

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