REDUCTION AND REALIZATION IN TODA AND VOLTERRA

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ABSTRACT. We construct a new symplectic, bi-hamiltonian realization of the KM-system by reducing the corresponding one for the Toda lattice. The bi-hamiltonian pair is constructed using a reduction theorem of Fernandes and Vanhaecke. In this paper we also review the important work of Moser on the Toda and KM-systems.

1. Introduction

Some of the most important work of Moser concerns the Toda and Volterra lattices. These two systems are closely related and Moser gave an explicit construction demonstrating the relationship between the two systems. The Toda lattice was discovered by Morikasu Toda in 1967. Computer experiments by Ford et al. [22] suggested that the Toda lattice is integrable. In two papers, both in the same issue of Physical Review B ([19], [23]) Hénon and Flaschka in 1974 demonstrated the integrability of the system. Hénon provided the required independent constants of motion using combinatorial arguments. Flaschka proved integrability of the lattice via a change of variables and the construction of a Lax pair. A few months later, Manakov [27] established independently similar results. The Toda lattice is a discrete approximation of the KdV equation. In [20] Flaschka used this analogy to solve the system using a discretization of the inverse scattering method of Gardner, Greene, Kruskal and Miura. The method which was developed for partial differential equations was applied for the first time for the infinite Toda lattice with success. Moser’s paper [28] deals with the finite non-periodic Toda lattice. Moser solves the system by constructing action-angle coordinates. His action variables are the eigenvalues of the Lax matrix of Flaschka and the angles are the last components of the normalized eigenvectors. We will describe the solution of the Toda lattice by Moser in Section 2. The Volterra system (also known as the KM system) is closely related to the Toda lattice, and the connection will become even more intimate after the results of the present paper. It was first solved by Kac and van-Moerbeke in [24], using the discrete version of inverse scattering of Flaschka. Moser in [29] solved this system explicitly in a fashion similar to [28], i.e. by construction of action-angle coordinates, using the Weyl function and continued fractions. We will not examine this result in the paper since it is similar (although more complicated) to the corresponding one for the Toda lattice which we describe in Section 2.

The purpose of this paper is to understand completely the following diagram:

\[
\begin{array}{ll}
\text{Toda}(q, p) & \xrightarrow{F} \text{Toda}(a, b) \\
\text{Invol.} & \downarrow \text{Invol.} \\
\text{Volterra}(q) & \xrightarrow{G} \text{Volterra}(a)
\end{array}
\]

Dedicated to H. Flaschka and J. Moser.
The top arrow is the well-known symplectic realization of the Toda lattice where $F$ is the Flaschka transformation. We can think of it as the following symplectic bi-hamiltonian realization:

$$F : (J_1, J_2, h_1, h_2) \rightarrow (\pi_1, \pi_2, H_1, H_2),$$

where $\pi_1$ and $\pi_2$ are the well-known linear and quadratic Toda brackets and $H_1, H_2$ are the trace of the Lax matrix and the Hamiltonian respectively. They form a bi-Hamiltonian pair: $\pi_1 dH_2 = \pi_2 dH_1$. Similarly, $J_1$ is the standard symplectic bracket while $J_2$ is the Das-Okubo bracket in $(q, p)$ coordinates. $h_1$ and $h_2$ correspond to the sum of the momenta and the Hamiltonian respectively. Here, we also have a bi-hamiltonian formulation $J_1 dh_2 = J_2 dh_1$. The multi-hamiltonian structure of the KM-system is obtained by reducing the Toda hierarchy using a Theorem of Fernandes and Vanhaecke [18]. The left hand arrow is similarly a reduction of the multi-hamiltonian structure of the Toda lattice in $(a, b)$ coordinates was obtained in [6], [8] while in $(q, p)$ coordinates in [17] using a Theorem of Oevel. There is a recursion operator defined by $R = J_2 J_1^{-1}$.

The bottom arrow is a new symplectic bi-hamiltonian realization of the KM-system. Let us denote it by

$$G : (w_2, w_3, i_1, i_0) \rightarrow (v_2, v_3, I_1, I_0).$$

The brackets $v_2$ and $v_3$ are the well-known quadratic and cubic brackets for the KM-system defined in [7]. The functions $I_1$ are defined by $I_1 = \frac{1}{2} \text{tr} L^2$, and $I_0 = \log |\det(L)|$, where $L$ is the Lax matrix. The phase space for the KM-system $V_a$ is of odd dimension $(a_1, a_2, \ldots, a_{2n+1})$. Letting $N = 2(n + 1)$ the dimension of $V_a$ is $N - 1$, while the dimension of $T_{(a, b)}$ is $2N - 1$ and the dimension of $T_{(q, p)}$ is $2N$.

We consider the space $V_q$ in the variables $(q_1, q_2, \ldots, q_N)$. On this space we define a symplectic Poisson bracket $w_2$ by the formula

$$\{q_i, q_j\} = 1 \quad \forall \ i < j.$$  

This is a constant symplectic bracket and the matrix $\{q_i, q_j\}$ has determinant one.

This bracket $w_2$ corresponds to the quadratic bracket $v_2$ via the mapping

$$G(q_1, \ldots, q_N) = (e^{q_1-q_2}, \ldots, e^{q_{N-1}-q_N}).$$

We then define a bracket $w_3$ in $R^N$ which is mapped to $v_3$ under the transformation $G$. The bracket $w_3$ is defined by the formula:

$$\{q_i, q_j\} = e^{q_{i-1}-q_i} + (1 - \delta_{i+1,j})e^{q_{i-1}-q_{i+1}} + e^{q_{i-1}-q_i} + e^{q_{i-1}-q_{i+1}}.$$  

Whenever a term is not defined we omit that whole term. Define

$$i_0 = q_1 - q_2 + q_3 - q_4 + \cdots + q_{N-1} - q_N$$

and the Hamiltonian $i_1$ given by

$$i_1 = \sum_{i=1}^{N-1} e^{q_{i}-q_{i+1}}.$$  

Then we have the following bi-hamiltonian pair:

$$w_2 di_1 = w_3 di_0.$$  

Finally, let us comment the two vertical arrows of the diagram. The right hand arrow is described in [10] where the multi-hamiltonian structure of the KM-system is obtained by reducing the Toda hierarchy using a Theorem of Fernandes and Vanhaecke [13]. The left hand arrow is similarly a reduction of the multi-hamiltonian structure of the Toda lattice in $(q, p)$ coordinates to a new multi-hamiltonian structure for the KM-system in $V_q$ space. We use the involution $\psi(q, p) = (q, -p)$ which is a Poisson automorphism of both $J_2$ and $J_4 = R^2 J_2$. The reduction of $J_2$ and $J_4$
produces the brackets $w_2$ and $w_3$ respectively. We should mention that Moser had an algorithm for going in the opposite direction of the right hand side arrow, i.e. a procedure for obtaining Toda equations from the Volterra equations by squaring the Lax matrix and some chopping. This is described briefly in Section 10.

2. Moser’s solution of the Toda lattice

The Toda lattice with Hamiltonian function

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}},$$

can be transformed via a change of variables due to Flaschka [19] to a Lax pair of the form $\dot{L} = [B, L]$, where $L$ is the Jacobi matrix

$$L = \begin{pmatrix}
    b_1 & a_1 & 0 & \cdots & \cdots & 0 \\
    a_1 & b_2 & a_2 & \cdots & \vdots & \\
    0 & a_2 & b_3 & \cdots & \vdots & \\
    \vdots & \vdots & \ddots & \ddots & \vdots & \\
    \vdots & \vdots & \cdots & a_{N-1} & b_N \\
    0 & \cdots & \cdots & a_{N-1} & b_N
  \end{pmatrix},$$

and $B$ is the projection of $L$ into the skew-symmetric part of $L$ in the Lie algebra decomposition, lower triangular plus skew-symmetric. This is an example of an isospectral deformation; the entries of $L$ vary over time but the eigenvalues remain constant. It follows that the symmetric polynomials of the eigenvalues, $H_i = \frac{1}{\text{tr}} \text{tr}^i L^i$ are constants of motion. The differential equations in the case of symmetric tridiagonal Lax matrix are:

$$\dot{a}_i = a_i (b_{i+1} - b_i),$$
$$\dot{b}_i = 2 (a_i^2 - a_{i-1}^2).$$

Moser’s elegant solution of the open Toda lattice uses the Weyl function $f(\lambda)$ and an old (19th century) method of Stieltjes which connects the continued fraction of $f(\lambda)$ with its partial fraction expansion. The key ingredient is the map which takes the $(a, b)$ phase space of tridiagonal Jacobi matrices to a new space of variables $(\lambda_i, r_i)$ where $\lambda_i$ is an eigenvalue of the Jacobi matrix and $r_i$ is the residue of $f(\lambda)$. We present a brief outline of Moser’s construction.

Moser in [28] introduced the resolvent

$$R(\lambda) = (\lambda I - L)^{-1},$$

and defined the Weyl function

$$f(\lambda) = R_{NN}(\lambda) = (R(\lambda)e_N, e_N),$$

where $e_N = (0, 0, \ldots, 0, 1)$.

The function $f(\lambda)$ has a simple pole at $\lambda = \lambda_i$ and so it admits a partial fraction expansion

$$f(\lambda) = \sum_{i=1}^{N} \frac{r_i^2}{\lambda_i - \lambda_i}.$$

with positive residue $r_i^2$. It is clear from the formula for calculating the inverse of a matrix that $f(\lambda) = \frac{\Delta_{N-1}}{\Delta_N}$. We denote by $\Delta_k$ the $k$ by $k$ sub-determinant.
obtained by deleting the last $N - k$ rows and columns of $\lambda I - L$. It follows that
\[ \lim_{\lambda \to \infty} \lambda f(\lambda) = 1 \] and therefore the residue of $f(\lambda)$ at infinity is $-1$. As a result,
\[ \sum_{i=1}^{N} r_i^2 = 1. \]

More generally, one has a recursion formula:
\[ \Delta_k = (\lambda - b_k)\Delta_{k-1} - a_{k-1}^2\Delta_{k-2}. \]

Moser notes that the mapping $\phi$ between
\[ T_{(a,b)} = \{ (a_1, \ldots, a_{N-1}, b_1, \ldots, b_N, \ a_i > 0) \} \]
and
\[ T_{(\lambda,r)} = \{ (\lambda_1, \ldots, \lambda_N, r_1, \ldots, r_N, \ \lambda_1 < \lambda_2 < \cdots < \lambda_N, \ \sum_{i=1}^{N} r_i^2 = 1, \ r_i > 0) \} \]
is one to one and onto. The inverse mapping $\phi^{-1} : T_{(\lambda,r)} \to T_{(a,b)}$ corresponds to the inverse scattering transform.

Moser derives the differential equations in the variables $(\lambda, r)$ using the Lax pair of the system and the recursion formula (5). The equations take the form
\[
\dot{\lambda}_i = 0 \quad \dot{r}_i = -(\lambda_i - \sum_{j=1}^{N} \lambda_j r_j^2) r_i. \tag{6}
\]

If one considers $r_i$ as homogeneous variables then the differential equations (6) become linear
\[ \dot{\lambda}_i = 0 \quad \dot{r}_i = -\lambda_i r_i. \]

The variables $a_i^2$, $b_i$ may be expressed as rational functions of $\lambda_i$ and $r_i$ using a continued fraction expansion of $f(\lambda)$ which dates back to Stieltjes. Since the computation of the continued fraction from the partial fraction expansion is a rational process the solution is expressed as a rational function of the variables $(\lambda_i, r_i)$.

The procedure is as follows:

The $R_{NN}$ element of the resolvent, as defined previously, takes the following continued fraction representation:
\[
f(\lambda) = \frac{1}{\lambda - b_N - \frac{a_{N-1}^2}{\lambda - b_{N-1} - \frac{a_{N-2}^2}{\cdots - \frac{a_1^2}{\lambda - b_1 - \frac{1}{\lambda - b_2 - \frac{1}{\cdots}}}}}}. \tag{7}
\]

Stieltjes described a procedure that allows one to express $a_i$ and $b_i$ in terms of $\lambda_1, \ldots, \lambda_N$ and $r_1, \ldots, r_N$. We briefly describe the method. We expand the partial fraction expansion of $f(\lambda)$ given in (4) in a series of powers of $\frac{1}{\lambda}$.

The coefficient of $\lambda^{j+1}$ is denoted by $c_j$ and equals,
\[ c_j = \sum_{i=1}^{N} r_i^2 \lambda_i^j, \quad j = 0, 1, \ldots \]

The formulas of Stieltjes involve certain $i \times i$ determinants which we now define:
The formulas that give the relation between the variables \((a, b)\) and \((r, \lambda)\) are,

\[
A_i = \begin{bmatrix}
  c_0 & c_1 & \ldots & c_{i-1} \\
  c_1 & c_2 & \ldots & c_i \\
  \vdots \\
  c_{i-1} & c_i & \ldots & c_{2i-2}
\end{bmatrix}, \quad
B_i = \begin{bmatrix}
  c_1 & c_2 & \ldots & c_i \\
  c_2 & c_3 & \ldots & c_{i+1} \\
  \vdots \\
  c_{i+1} & c_{i+1} & \ldots & c_{2i-1}
\end{bmatrix}.
\]

The formulas that give the relation between the variables \((a, b)\) and \((r, \lambda)\) are,

\[
a_{N+1-i}^2 = \frac{A_{i-1}A_{i+1}}{A_i^2}, \quad i = 1, \ldots, N - 1
\]

\[
b_{N+1-i} = \frac{A_iB_{i+1}}{A_iB_{i-1}} + \frac{A_{i-1}B_{i}}{A_iB_{i-1}}, \quad i = 1, \ldots, N
\]

where \(A_0 = 1, B_0 = 1, B_1 = 0\).

For example, in the case \(N = 2\)

\[
A_1 = c_0, \quad A_2 = c_0c_2 - c_1^2, \quad B_1 = c_1, \quad B_2 = c_1c_3 - c_2^2
\]

and therefore

\[
a_1^2 = A_2, \quad b_1 = \frac{A_2}{B_1} + \frac{B_2}{A_2B_1}, \quad b_2 = B_1.
\]

Thus,

\[
a_1^2 = \frac{r_1^2r_2^2(\lambda_2 - \lambda_1)^2}{(r_1^2 + r_2^2)^2}
\]

\[
b_1 = \frac{r_1^2\lambda_2 + r_2^2\lambda_1}{r_1^2 + r_2^2}
\]

\[
b_2 = \frac{r_1^2\lambda_1 + r_2^2\lambda_2}{r_1^2 + r_2^2}.
\]

We have written the solution in this form to show that \(r_i\) are homogeneous coordinates, i.e. the solution does not change if we replace \(r_i\) by a non-zero multiple. As we mentioned earlier in these homogeneous coordinates the differential equations become linear. One can check that the differential equations (6) correspond via transformation (8) to the A\(_2\) Toda equations

\[
\dot{a}_1 = a_1(b_2 - b_1)
\]

\[
\dot{b}_1 = 2a_1^2
\]

\[
\dot{b}_2 = -2a_1^2.
\]

As Moser notes, it is not too hard to obtain explicit expressions for \(N = 3\) but the general case is quite complicated.

3. Background

Consider a differential equation on a manifold \(M\) defined by a vector field \(\chi\). A vector field \(Z\) is a symmetry of the equation if

\[
[Z, \chi] = 0.
\]

A vector field \(Z\) is called a master symmetry if

\[
[[Z, \chi], \chi] = 0,
\]

but

\[
[Z, \chi] \neq 0.
\]

Master symmetries were first introduced by Fokas and Fuchssteiner in [21] in connection with the Benjamin-Ono Equation.
A bi-Hamiltonian system is defined by specifying two Hamiltonian functions $H_1$, $H_2$ and two Poisson tensors $\pi_1$ and $\pi_2$, that give rise to the same Hamiltonian equations. Namely, $\pi_1 \nabla H_2 = \pi_2 \nabla H_1$. The notion of bi-Hamiltonian structures is due to Magri [26]. Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors $\pi_1$, $\pi_2$ and the Hamiltonians $H_1$, $H_2$. Assume that $\pi_1$ is symplectic. We define the recursion operator $R = \pi_2 \pi_1^{-1}$, the higher flows

$$\chi_i = R^{i-1} \chi_1,$$

and the higher order Poisson tensors

$$\pi_i = R^{i-1} \pi_1.$$

For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to Oevel [30].

**Theorem 1.** Suppose that $X_0$ is a conformal symmetry for both $\pi_1$, $\pi_2$ and $H_1$, i.e. for some scalars $\lambda$, $\mu$, and $\nu$ we have

$$\mathcal{L}_{X_0} \pi_1 = \lambda \pi_1, \quad \mathcal{L}_{X_0} \pi_2 = \mu \pi_2, \quad \mathcal{L}_{X_0} H_1 = \nu H_1.$$  

Then the vector fields $X_i = R^i X_0$ are master symmetries and we have,

$$(a) \quad \mathcal{L}_{X_i} H_j = (\nu + (j - 1 + i)(\mu - \lambda)) H_{i+j}
$$

$$(b) \quad \mathcal{L}_{X_i} \pi_j = (\mu + (j - i - 2)(\mu - \lambda)) \pi_{i+j}
$$

$$(c) \quad [X_i, X_j] = (\mu - \lambda)(j - i) X_{i+j}.$$ 

A symplectic realization (see [37]) of a Poisson manifold $(M, \pi)$ is a symplectic manifold $(S, \omega)$ together with a surjective Poisson submersion $f : S \rightarrow M$. In this paper our realizations will have additional structure; they will be symplectic bi-Hamiltonian realizations as in [33], [34]. Suppose on $M$ we have a bi-Hamiltonian pair $\pi_1$, $\pi_2$ and two Hamiltonians $H_1$, $H_2$ such that $\pi_1 dH_1 = \pi_2 dH_2$. A symplectic bi-Hamiltonian realization will be a manifold $S$, of even dimension, together with two Poisson tensors $J_1$, $J_2$ with $J_1$ symplectic and a surjective submersion $F : S \rightarrow M$ which is a Poisson mapping between $J_i$ and $\pi_i$, i.e. $\{F^* f, F^* g\}_{J_1} = F^* \{f, g\}_\pi$, for all $f, g \in C^\infty(M)$. In addition, $J_i dh_1 = J_2 dh_2$ where $h_i = H_i \circ F$.

We shall see that the relation between the Toda and Volterra systems relies on special symmetries of the phase spaces. We will need a theorem of Fernandes and Vanhecke which gives conditions under which the fixed point set of a Poisson action inherits a Poisson bracket.

Although we will be interested mainly in finite symmetries, we have the following general result [18]:

**Theorem 2.** Suppose that $(M, \{\cdot, \cdot\})$ is a Poisson manifold, and $G$ is a compact group acting on $M$ by Poisson automorphisms. Let $N = M^G$ be the submanifold of $M$ consisting of the fixed points of the action and let $i : N \hookrightarrow M$ be the inclusion. Then $N$ carries a (unique) Poisson structure $\{\cdot, \cdot\}_N$ such that

$$i^* \{F_1, F_2\} = \{i^* F_1, i^* F_2\}_N$$

for all $G$-invariant functions $F_1, F_2 \in C^\infty(M)$.

**Remark 3.** The previous result can be seen as a particular case of Dirac reduction (for the general theorem on Dirac reduction, see Weinstein [37], Prop. 1.4) and Courant [15], Thm. 3.2.1).

Since this result applies in particular when $G$ is a finite group, we have:

**Corollary 4.** Suppose that $(M, \{\cdot, \cdot\})$ is a Poisson manifold, and $G$ is a finite group acting on $M$ by Poisson automorphisms. Then the fixed point set $N = M^G$ carries a (unique) Poisson structure $\{\cdot, \cdot\}_N$ satisfying [27].
Let us consider the special case $G = \mathbb{Z}_2$. Then $G = \{I, \phi\}$, where $\phi : M \to M$ is a Poisson involution. We conclude that $N = M^G = \{x : \phi(x) = x\}$ has a unique Poisson bracket satisfying equation (9). So we see that Theorem 2 contains as a special case the following result, which is known as the Poisson involution theorem (see [13, 36]).

**Corollary 5.** Suppose that $(M, \{\cdot , \cdot \})$ is a Poisson manifold, and $\phi : M \to M$ is a Poisson involution. Then the fixed point set $N = \{x \in M : \phi(x) = x\}$ carries a (unique) Poisson structure $\{\cdot , \cdot \}_N$ such that

$$\iota^* \{F_1, F_2\} = \{\iota^* F_1, \iota^* F_2\}_N$$

for all functions $F_1, F_2 \in C^\infty(M)$ invariant under $\phi$.

4. KM-system

The Volterra system, also known as KM system is defined by

$$\dot{a}_i = a_i (a_{i+1} - a_{i-1}) \quad i = 1, 2, \ldots, n,$$

where $a_0 = a_{n+1} = 0$. It was studied originally by Volterra in [35] to describe population evolution in a hierarchical system of competing species. In [29] Moser gave a solution of the system using the method of continued fractions and in the process he constructed action-angle coordinates. Equations (10) can be considered as a finite-dimensional approximation of the Korteweg-de Vries (KdV) equation. They also appear in the discretization of conformal field theory; the Poisson bracket for this system can be thought as a lattice generalization of the Virasoro algebra [16]. The variables $a_i$ are an intermediate step in the construction of the action-angle variables for the Liouville model on the lattice.

The Volterra system is usually associated with a simple Lie algebra of type $A_n$. Bogoyavlensky generalized this system for each simple Lie algebra and showed that the corresponding systems are also integrable. See [2, 3] for more details. The relation between Volterra and Toda systems is also examined in [10, 11].

The Hamiltonian description of system (10) can be found in [15] and [7]. The Lax pair is given by

$$\dot{L} = [B, L],$$

where

$$L = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & 1 & \ddots & \ddots & \vdots \\ 0 & a_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 & a_n \\ 0 & \cdots & \cdots & 0 & a_n & 0 \end{pmatrix}$$

and
I det to the case where $n^{(10)}$ is integrable for any value of $n$. Taking the Lie derivative of the KdV bracket in the continuum limit, it is defined by the formulae,

$$B = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
-a_1 a_2 & 0 & 0 & \ddots & \vdots \\
\vdots & -a_2 a_3 & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & -a_{n-1} a_n & 0 \\
0 & \cdots & \cdots & \cdots & -a_n & 0 \\
\end{pmatrix}.$$  

It follows that the functions $I_i = \frac{1}{2} \text{Tr} L^{2i}$ are constants of motion. The system (10) is integrable for any value of $n$ but in this paper we restrict our attention only to the case where $n$ is odd.

Following [7] we define the following quadratic Poisson bracket,

$$\{a_i, a_{i+1}\} = a_i a_{i+1},$$

and all other brackets equal to zero. We denote this bracket by $v_2$. For this bracket det$L$ is a Casimir and the eigenvalues of $L$ are in involution. Of course, the functions $I_i$ are also in involution. Taking the function $I_1 = \sum_i a_i$ as the Hamiltonian we obtain equations (10), i.e. $v_2 dI_1$ is the KM Hamiltonian vector field.

In [7] one also finds a cubic Poisson bracket which corresponds to the second KdV bracket in the continuum limit. It is defined by the formulae,

$$\{a_i, a_{i+1}\} = a_i a_{i+1} (a_i + a_{i+1}),$$
$$\{a_i, a_{i+2}\} = a_i a_{i+1} a_{i+2},$$

all other brackets are zero. We denote this bracket by $v_3$. In this bracket we still have involution of invariants. We also have Lenard type relations of the form

$$v_3 dI_{2i} = v_2 dI_{2i+2}.$$  

In [7] appears a bracket that is homogeneous of degree one, a rational bracket constructed using a master symmetry. This bracket, denoted by $v_1$, has $I_1 = \frac{1}{2} \text{Tr} L^2$ as Casimir and the Hamiltonian is $H_2 = \frac{1}{2} \text{Tr} L^4$. The definition of the bracket is the following: We define the master symmetry $Y_{-1}$ to be

$$Y_{-1} = \sum_{i=1}^n f_i \frac{\partial}{\partial a_i},$$

where the $f_i$ are determined recursively as follows,

$$f_1 = -1, \quad f_2i = \frac{a_{2i}}{a_{2i-1}} f_{2i-1}, \quad f_{2i-1} = -f_{2i-2} - 1.$$  

Taking the Lie derivative of $v_2$ in the direction of $Y_{-1}$ we obtain $v_1$, a Poisson bracket that is homogeneous of degree 1. For $n = 5$, $v_1$ takes the form:

\begin{align*}
\{a_1, a_2\} &= a_2, & \{a_1, a_3\} &= -a_2, & \{a_1, a_4\} &= \frac{a_2 a_4}{a_3}, & \{a_1, a_5\} &= -\frac{a_2 a_5}{a_3} \\
\{a_2, a_3\} &= a_2, & \{a_2, a_4\} &= -\frac{a_2 a_4}{a_3}, & \{a_2, a_5\} &= \frac{a_2 a_5}{a_3} \\
\{a_3, a_4\} &= a_4, & \{a_3, a_5\} &= -a_4, & \{a_4, a_5\} &= a_4.
\end{align*}

Note that the KM Hamiltonian can be expressed as $v_2 dI_1 = v_1 dI_2$ and this is a bi-hamiltonian formulation of the KM flow. In this paper we rediscover these brackets using a recursion operator.

The higher Poisson brackets are constructed in [7] using a sequence of master symmetries $Y_i$. For example, the bracket $v_2$ is obtained from $v_1$ by taking the
Lie derivative in the direction of the first master symmetry \( Y_1 \). Similarly, the Lie derivative of \( v_2 \) in the direction of \( Y_1 \) gives \( v_3 \).

The brackets \( v_1, v_2 \) and \( v_3 \) are just the beginning of an infinite hierarchy which we will derive again in this paper using a different method.

5. Toda Lattice

In this paper we will deal with the finite, non-periodic version of the Toda lattice. However, there are also other interesting variations worth mentioning. There is a generalization due to Deift, Li, Nanda and Tomei [14] who showed that the system remains integrable when \( L \) is replaced by a full (generic) symmetric \( n \times n \) matrix. The functions \( H_i = \frac{1}{i} \text{Tr} \ L^i \) are still in involution but they are not enough to ensure integrability. In other words, the existence of a Lax pair does not guarantee integrability. There are, however, additional integrals which are rational functions of the entries of \( L \). The method used to obtain these additional integrals is called chopping and was used originally in [14].

The classical Toda lattice was also generalized in another direction. One can define a Toda type system for each simple Lie algebra. The finite, non–periodic Toda lattice corresponds to a root system of type \( A_n \). This generalization is due to Bogoyavlensky [4]. These systems were studied extensively in [25] where the solution of the system was connected intimately with the representation theory of simple Lie groups. There are also studies by Olshanetsky and Perelomov [31] and Adler, van Moerbeke [1]. The description of the systems, following [1] and [25] is as follows:

Let \( \mathfrak{g} \) be any semi-simple Lie algebra, equipped with its Killing form \( \langle \cdot | \cdot \rangle \). One chooses a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \), a root system \( \Delta = \Delta(\mathfrak{h}, \mathfrak{g}) \) of \( \mathfrak{h} \) in \( \mathfrak{g} \), a basis of simple roots \( \Pi \), and a set of positive roots \( \Delta^+ \).

The Lax pair \( (L(t), B(t)) \) in \( \mathfrak{g} \) can be described in terms of the root system as follows:

\[
L(t) = \sum_{i=1}^{l} b_i(t) H_{\alpha_i} + \sum_{i=1}^{l} a_i(t) (X_{\alpha_i} + X_{-\alpha_i}),
\]

\[
B(t) = \sum_{i=1}^{l} a_i(t) (X_{\alpha_i} - X_{-\alpha_i})
\]

where \( H_{\alpha_i} \) is an element of \( \mathfrak{h} \) and \( X_{\alpha_i} \) is a root vector corresponding to the simple root \( \alpha_i \).

We consider now the classical Toda lattice but we will use a Lax pair slightly different than (2).

Let \( D \) be the diagonal matrix with entries \( d_i \) where \( d_1 = 1 \) and \( d_i = a_1 a_2 \ldots a_{i-1} \) \( i = 2, 3, \ldots N \). In [25] Kostant conjugates the matrix \( L \) in (2) by the matrix \( D \) and the resulting matrix \( DLD^{-1} \) has the form

\[
\begin{pmatrix}
    b_1 & 1 & 0 & \cdots & \cdots & 0 \\
    a_1 & b_2 & 1 & \ddots & \vdots & \vdots \\
    0 & a_2 & b_3 & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & 1 & \vdots \\
    0 & \cdots & \cdots & 0 & a_{N-1} & b_N
\end{pmatrix}
\]

(13)

Denoting this matrix again by \( L \) the equations take the Lax form.
\[ \hat{L}(t) = [L(t), P L(t)] \]

where \( P \) is the projection onto the strictly lower triangular part of \( L(t) \). The decomposition here is strictly lower plus upper triangular. This form is convenient in applying Lie theoretic techniques to describe the system. Note that the diagonal elements correspond to the Cartan subalgebra while the subdiagonal elements correspond to the set \( \Pi \) of simple roots. The equations of motion for the Toda lattice (in Kostant form) are:

\[
\begin{align*}
\dot{a}_i &= a_i (b_{i+1} - b_i) \\
\dot{b}_i &= a_i - a_{i-1}.
\end{align*}
\]

The functions \( H_i = \frac{1}{4} \text{Tr} L^i \) are independent invariants in involution.

There exists a Lie-Poisson bracket given by the formula

\[
\{a_i, b_i\} = -a_i, \\
\{a_i, b_{i+1}\} = a_i; \\
\{a_i, b_{i-1}\} = a_i.
\]

all other brackets are zero. \( H_1 = b_1 + b_2 + \cdots + b_N \) is the only Casimir. The Hamiltonian in this bracket is \( H_2 = \frac{1}{2} \text{tr} L^2 \). We also have involution of invariants, \( \{H_i, H_j\} = 0 \). The Lie algebraic interpretation of this bracket can be found in [25]. We denote this bracket by \( \pi_1 \).

The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. It is a Poisson bracket in which the Hamiltonian vector field generated by \( H_1 \) is the same as the Hamiltonian vector field generated by \( H_2 \) with respect to the \( \pi_1 \) bracket. The defining relations are

\[
\begin{align*}
\{a_i, a_{i+1}\} &= a_i a_{i+1} \\
\{a_i, b_i\} &= -a_i b_i \\
\{a_i, b_{i+1}\} &= a_i b_{i+1} \\
\{b_i, b_{i+1}\} &= a_i; \\
\{a_i, b_{i-1}\} &= -a_i a_{i-1} \\
\{b_i, b_{i-1}\} &= a_i (b_i + b_{i-1});
\end{align*}
\]

all other brackets are zero. The bracket \( \pi_2 \) is easily defined by taking the Lie derivative of \( \pi_1 \) in the direction of suitable master symmetry \( X_1 \), see [3] for details. This bracket has \( \det L \) as Casimir and \( H_1 = \text{tr} L \) is the Hamiltonian. The eigenvalues of \( L \) are still in involution. Furthermore, \( \pi_2 \) is compatible with \( \pi_1 \). We also have

\[ \pi_2 dH_1 = \pi_1 dH_{i+1}. \]

Finally, we remark that taking the derivative of \( \pi_2 \) in the direction of \( X_1 \) yields another Poisson bracket, \( \pi_3 \), which is cubic in the coordinates. The defining relations for \( \pi_3 \) are

\[
\begin{align*}
\{a_i, a_{i+1}\} &= 2a_i a_{i+1} b_{i+1} \\
\{a_i, b_i\} &= -a_i b_{i+1}^2 + a_i^2 \\
\{a_i, b_{i+1}\} &= a_i b_i^2 + a_i^2 \\
\{a_i, b_{i+2}\} &= a_i a_{i+1} \\
\{a_{i+1}, b_i\} &= -a_i a_{i+1} \\
\{b_i, b_{i+1}\} &= a_i (b_i + b_{i+1});
\end{align*}
\]

all other brackets are zero. The bracket \( \pi_3 \) is compatible with both \( \pi_1 \) and \( \pi_2 \) and the eigenvalues of \( L \) are still in involution. The Casimir for this bracket is \( \text{tr} L^{-1} \).

In fact there is an infinite hierarchy of Poisson tensors \( \pi_1, \) master symmetries \( X_i \) and invariants \( H_i \) and they obey some deformation relations. We quote the results from refs. [3], [8].
Theorem 6.

i) $\pi_j, \ j \geq 1$ are all Poisson.

ii) The functions $H_i, i \geq 1$ are in involution with respect to all of the $\pi_j$.

iii) $X_i(H_j) = (i + j)H_{i+j}, \ i \geq -1, \ j \geq 1$.

iv) $L_X\pi_j = (j - i - 2)\pi_{i+j}, \ i \geq -1, \ j \geq 1$.

v) $[X_i, \ X_j] = (j - i)X_{i+j}, \ i \geq 0, \ j \geq 0$.

vi) $\pi_j dH_i = \pi_j - 1 dH_{i+1}$.

6. Toda lattice in $(q, p)$ coordinates

Let $J_1$ be the symplectic bracket with Poisson matrix

$$ J_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, $$

where $I$ is the $N \times N$ identity matrix. The bracket $J_1$ is mapped precisely onto the bracket $\pi_1$ under the Flaschka transformation $F$:

$$ a_i = e^{q_i - q_{i+1}}, \quad b_i = -p_i. $$

We define the following tensor, $J_2$, due to Das and Okubo [13]:

$$ J_2 = \begin{pmatrix} A & B \\ -B & C \end{pmatrix}, $$

where $A$ is the skew-symmetric matrix defined by $a_{ij} = 1 = -a_{ji}$ for $i < j$, $B$ is the diagonal matrix $(-p_1, -p_2, \ldots, -p_N)$ and $C$ is the skew-symmetric matrix whose non-zero terms are $c_{i,i+1} = c_{i+1,i} = e^{q_i - q_{i+1}}$ for $i = 1, 2, \ldots, N - 1$. The bracket $J_2$ is mapped precisely onto the bracket $\pi_2$ under the Flaschka transformation (18).

It is easy to see that we have a bi-Hamiltonian pair. We define

$$ h_1 = -(p_1 + p_2 + \cdots + p_N), $$

and $h_2$ to be the Hamiltonian:

$$ h_2 = \sum_{i=1}^{N} \frac{1}{2} b_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}. $$

Then we obtain the bi-Hamiltonian pair

$$ J_1 dh_2 = J_2 dh_1. $$

We define the recursion operator as follows:

$$ R = J_2J_1^{-1}. $$

The matrix form of $R$ is quite simple:

(20)$$ R = \frac{1}{2} \begin{pmatrix} B & -A \\ C & B \end{pmatrix}. $$

Using the recursion operator we obtain the higher order Poisson tensors

$$ J_i = R^{i-1}J_1, \quad i = 2, 3, \ldots. $$

Following [17] we define the conformal symmetry

$$ Z_0 = \sum_{i=1}^{N} (N - 2i + 1) \frac{\partial}{\partial q_i} + \sum_{i=1}^{N} p_i \frac{\partial}{\partial p_i}. $$

It is straightforward to verify that

$$ \mathcal{L}_{Z_0} J_1 = -J_1, $$
\[ L_{Z_0} J_2 = 0 . \]

In addition,
\[ Z_0(h_1) = h_1 \]
\[ Z_0(h_2) = 2h_2 . \]

Consequently, \( Z_0 \) is a conformal symmetry for \( J_1, J_2 \) and \( h_1 \). The constants appearing in Theorem 1 are \( \lambda = -1, \mu = 0 \) and \( \nu = 1 \). According to Oevel’s Theorem we end up with the following deformation relations:
\[ [Z_i, h_j] = (i + j)h_{i+j} \]
\[ L_{Z_i} J_j = (j - i - 2)J_{i+j} \]
\[ [Z_i, Z_j] = (j - i)Z_{i+j} . \]

Switching to Flaschka coordinates, we obtain relations iii)- v) of Theorem 6.

7. FROM TODA TO VOLterra

We consider \( T(a,b) \) the phase space of the Toda lattice in Flaschka coordinates and the space \( V_a \) of KM-system in \( a \) coordinates. Note that \( V_a \) is not a Poisson subspace of \( T(a,b) \). However, as was demonstrated in [10] \( V_a \) is the fixed manifold of the involution \( \phi : T(a,b) \rightarrow T(a,b) \) defined by
\[ \phi(a_1, a_2, \ldots, a_{N-1}, b_1, b_2, \ldots, b_N) \rightarrow (a_1, a_2, \ldots, a_{N-1}, -b_1, -b_2, \ldots, -b_N), \]
and we have the following result:

**Theorem 7.** \( \phi : T(a,b) \rightarrow T(a,b) \) is a Poisson automorphism of \( (T(a,b), \pi_k) \), if \( k \) is even.

Therefore, by Corollary 5 \( V_a \) inherits a family of Poisson brackets \( v_2, v_3, \ldots \). For example, the quadratic bracket can be computed from formulas (19) and is given by
\[ \{a_i, a_j\} = a_ia_j(\delta_{i,j+1} - \delta_{i+1,j}), \]
while the Poisson bracket \( v_3 \) is found to be given by the formulas
\[ \{a_i, a_{i+1}\} = a_ia_{i+1}(a_i + a_{i+1}), \quad (i = 1, \ldots, n - 1) \]
\[ \{a_i, a_{i+2}\} = a_ia_{i+1}a_{i+2}, \quad (i = 1, \ldots, n - 2) \]
i.e. we obtain \( v_2 \) and \( v_3 \) of Section 4. It follows that the restriction of the integrals \( H_{2k} \) to \( V_a \) gives a set of commuting integrals, with respect to these Poisson brackets. Also, the Lax equations (13) lead to Lax equations for the corresponding flows, merely by putting all \( b_i \) equal to zero. We recover the vector field
\[ \dot{a}_i = a_i(a_{i-1} - a_{i+1}), \quad i = 1, \ldots, n, \]
and a family of integrable systems admitting a multiple hamiltonian formulation:
\[ v_k dI_l = v_{k-1} dI_{l+1}, \quad (k = 1, 2, \ldots), \]
where \( v_k \) is the restriction of \( \pi_{2k} \) to \( V_a \) and \( I_l \) is the restriction of \( H_{2l} \) to \( V_a \).
8. Symplectic realization

Let $N = 2(n + 2)$ and consider coordinates $(q_1, q_2, \ldots, q_N)$ in $\mathbb{R}^N$. We define the following transformation $G$ from $\mathbb{R}^{2n+2}$ to $\mathbb{R}^{2n+1}$,

\begin{equation}
    a_i = e^{q_i - q_{i+1}} \quad i = 1, 2, \ldots, N - 1.
\end{equation}

The Hamiltonian in $q$ coordinates is given by

\begin{equation}
    i_1 = \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.
\end{equation}

We define the Poisson bracket

\[ \{q_i, q_j\} = 1 \quad \forall \quad i < j. \]

Let us denote this constant Poisson tensor by $w_2$ (since it is the analogue of $v_2$ in $q$ coordinates).

Hamilton’s equation $w_2 di_1$ become

\begin{equation}
    \dot{q}_i = -e^{q_i - q_{i-1}} - e^{q_i - q_{i+1}}.
\end{equation}

It is straightforward to check that Hamilton’s equations correspond in the $a-$space to the KM-system \([10]\) via the mapping $G : \mathbb{R}^N \to \mathbb{R}^{N-1}$

\begin{equation}
    G(q_1, \ldots, q_N) = (e^{q_1 - q_2}, \ldots, e^{q_{N-1} - q_N}).
\end{equation}

In fact we calculate:

\begin{align*}
    \dot{a}_i &= e^{q_i - q_{i+1}} (\dot{q}_i - \dot{q}_{i+1}) \\
    &= a_i (e^{q_i - q_{i-1}} - e^{q_i - q_{i+1}} + e^{q_i - q_{i+1}} + e^{q_i - q_{i-1}}) \\
    &= a_i (a_{i+1} - a_{i-1}).
\end{align*}

The symplectic bracket $w_2$ in $V_q$ space corresponds to the quadratic bracket $v_2$ in $V_a$ space.

We then define a bracket $w_3$ in $\mathbb{R}^N$ which is mapped to $v_3$ under the transformation $G$. The bracket $w_3$ is defined by the formula:

\[ \{q_i, q_j\} = e^{q_i - q_{i-1}} + (1 - \delta_{i+1,j})e^{q_i - q_{i+1}} + e^{q_j - q_{i-1}} + e^{q_j - q_{i+1}}. \]

Whenever a term is not defined we omit that whole term. This happens only when $i = 1$ or $j = N$. We note that $w_2$ is compatible with $w_3$.

Define

\[ i_0 = \sum_{k=1}^{N} (-1)^{k+1} q_k = q_1 - q_2 + q_3 - q_4 + \cdots + q_{N-1} - q_N. \]

**Remark 8.** It is not difficult to discover $i_0$. According to [12], in the presence of an invertible Nijenhuis tensor a natural choice of functions to form a bi-hamiltonian pair is

\[ \frac{1}{2} \log(\det R) \quad \text{and} \quad \frac{1}{2} \text{tr} R. \]

It turns out that the determinant of the Toda recursion operator restricted to the Volterra phase space is $e^{2i_0}$ and the trace equals $2i_1$.

Then

\[ \dot{q}_i = \{q_i, i_0\}_{w_3} = -e^{q_i - q_{i-1}} - e^{q_i - q_{i+1}}. \]

In other words we have a bi-hamiltonian pair

\[ w_2 di_1 = w_3 di_0. \]
We define a recursion operator as follows:
\[ R = w_3 w_2^{-1}. \]
In \( q \) coordinates, the symbol \( \chi_j \) is a shorthand for \( \chi_{ij} \). It is generated, as usual, by
\[ \chi_i = R^{i-1} \chi_1. \]
Note that \( i_1 \) corresponds under mapping (27) to a constant multiple of \( I_1 = \frac{1}{4} \text{Tr} (L)^2 \). In a similar fashion we obtain the higher order Poisson tensors
\[ w_i = R^{i-2} w_2 \quad i = 3, 4, \ldots. \]
We finally define the conformal symmetry
\[ X_0 = \sum_{i=1}^{N} (N - i + 1) \frac{\partial}{\partial q_i}. \]
The Poisson tensors \( w_2, w_3 \) and the functions \( i_0, i_1 \) define a bi-Hamiltonian pair. It is straightforward to verify that
\[ \mathcal{L}_{X_0} w_2 = 0, \quad \mathcal{L}_{X_0} w_3 = w_3, \quad \mathcal{L}_{X_0} i_1 = i_1. \]
Consequently, \( X_0 \) is a conformal symmetry for \( w_2, w_3 \) and \( i_1 \). The constants appearing in Oevel’s Theorem are \( \lambda = 0, \mu = 1 \) and \( \nu = 1 \). Therefore, we end up with the following deformation relations:
\[ [X_k, i_j] = (k + j)i_{k+j} \]
\[ L_{X_k} v_j = (j - k - 2)v_{k+j} \]
\[ [X_k, X_j] = (j - k)X_{k+j}. \]
Projecting to the \( a-\)space under mapping (27) we obtain the multiple hamiltonian structures of [7].

Remark 9. Note that we may define a Poisson tensor \( w_1 \) by the formula \( w_1 = w_2 w_3^{-1} w_2 \) as in [9]. The projection of \( w_1 \) under transformation (27) gives \( v_1 \) of Section 4. Perhaps a more appropriate bottom arrow in the diagram of the introduction is
\[ G : (w_1, w_2, i_1, i_2) \to (v_1, v_2, I_1, I_2). \]

9. From Toda \((q,p)\) space to Volterra \( q \) space

In this Section we will explain the origin of the symplectic bi-hamiltonian realization of Section 8. The idea is to use a Poisson involution in the Toda \((q,p)\) space and to reduce the equations to the Volterra \( V_q \) space. We consider \( T_{(q,p)} \), the phase space of the Toda lattice in natural coordinates. In Section 6 we have constructed a bi-hamiltonian system given by the Poisson tensors \( J_1, J_2 \) and the Hamiltonians \( h_1 \) and \( h_2 \). We also have a recursion operator \( \mathcal{R} \) which gives rise to a sequence of Poisson tensors \( J_k \quad i = 1, 2, \ldots. \)
Define the involution \( \psi : T_{(q,p)} \to T_{(q,p)} \) by the formula
\[ \psi(q_1, \ldots, q_N, p_1, \ldots, p_N) = (q_1, \ldots, q_N, -p_1, \ldots, -p_N). \]
We have the following result:

Theorem 10. \( \psi : T_{(q,p)} \to T_{(q,p)} \) is a Poisson automorphism of \( (T_{(q,p)}, J_{2k}) \), \( k = 1, 2, \ldots. \)
We will not give the proof of this result since it is entirely analogous to the proof of Theorem 7 which is given in [10].

Therefore using Corollary 5 $V_q$ inherits a family of Poisson tensors $w_2, w_3, \ldots$. For example, the bracket $w_2$ is clearly the $A$ block of the Poisson matrix $J_2$ in (19). On the other hand it is straightforward to compute that in the bracket $J_4$ we have

$$\{q_i, q_j\} = p_i^2 + p_ip_j + p_j^2 + e^{q_{i-1} - q_{i}} + (1 - \delta_{i+1,j})e^{q_{i-1} - q_{i}} + e^{q_{i-2} - q_{i}} + e^{q_{i-1} - q_{i+1}}.$$ 

Therefore the reduction of $J_4$ to $V_q$ is precisely $w_3$ of Section 8.

10. Moser’s recipe

Moser in [29] describes a relation between the KM system [10] and the non–periodic Toda lattice. The procedure is the following: Form $L^2$ which is not anymore a tridiagonal matrix but is similar to one. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and $E_o = \{\text{span } e_{2i-1}, i = 1, 2, \ldots\}$, $E_e = \{\text{span } e_{2i}, i = 1, 2, \ldots\}$. Then $L^2$ leaves $E_o, E_e$ invariant and reduces to each of these spaces to a tridiagonal symmetric Jacobi matrix. For example, if we omit all even columns and all even rows we obtain a tridiagonal Jacobi matrix and the entries of this new matrix define the transformation from the KM–system to the Toda lattice. We illustrate with a simple example where $n = 5$.

We use the symmetric version of the KM system Lax pair (due to Moser) given by

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix}. \tag{28}$$

It is simple to calculate that $L^2$ is the matrix

$$L^2 = \begin{pmatrix} a_1^2 & 0 & a_1a_2 & 0 & 0 \\ 0 & a_1^2 + a_2^2 & 0 & a_2a_3 & 0 \\ a_1a_2 & 0 & a_2^2 + a_3^2 & 0 & a_3a_4 \\ 0 & a_2a_3 & 0 & a_3^2 + a_4^2 & 0 \\ 0 & 0 & a_3a_4 & 0 & a_4^2 \end{pmatrix}. \tag{29}$$

Omitting even columns and even rows of $L^2$ we obtain the matrix

$$L^2_e = \begin{pmatrix} a_1^2 & 0 & a_1a_2 & 0 \\ a_1a_2 & 0 & a_2^2 + a_3^2 & 0 \\ 0 & a_2^2 + a_3^2 & a_3^2 + a_4^2 & 0 \end{pmatrix}. \tag{30}$$

This is a tridiagonal Jacobi matrix. It is natural to define new variables $A_1 = a_1a_2$, $A_2 = a_3a_4$, $B_1 = a_1^2$, $B_2 = a_2^2 + a_3^2$, $B_3 = a_4^2$. The new variables $A_1, A_2, B_1, B_2, B_3$ satisfy the Toda lattice equations [19].

This procedure shows that the KM-system and the Toda lattice are closely related: The explicit transformation which is due to Hénon maps one system to the other. The mapping in the general case is given by

$$A_1 = -\frac{1}{2} \sqrt{a_1a_2}a_{2i-1}, \quad B_1 = \frac{1}{2} (a_{2i-1} + a_{2i-2}). \tag{31}$$

The equations satisfied by the new variables $A_i, B_i$ are given by:

$$\dot{A}_i = A_i (B_{i+1} - B_i), \quad \dot{B}_i = 2 (A_i^2 - A_{i-1}^2).$$

These are precisely the Toda equations in Flaschka’s form [19].
This idea of Moser was applied with success to establish transformations from the generalized Volterra lattices of Bogoyavlensky [2,3] to generalized Toda systems. The relation between the Volterra systems of type $B_n$ and $C_n$ and the corresponding Toda systems is in [10]. The similar construction of the Volterra lattice of type $D_n$ and the generalized Toda lattice of type $D_n$ is in [11]. It turns out that the Volterra $D_n$ system corresponds not to the Toda $D_n$ system but to a special case of the Sklyanin lattice [32].

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