Semiclassical analysis and the Agmon-Finsler metric for discrete Schrödinger operators

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Abstract
The Agmon estimate for multi-dimensional discrete Schrödinger operators is studied with emphasis on the microlocal analysis on the torus. We first consider the semiclassical setting where semiclassical continuous Schrödinger operators are discretized with the mesh width proportional to the semiclassical parameter. Under this setting, the Agmon estimate for eigenfunctions is described by an Agmon metric, which is a Finsler metric rather than a Riemannian metric. Klein-Rosenberger (2008) proved this by a different argument in the case of a potential minimum. We also prove the Agmon estimate and the optimal anisotropic exponential decay of eigenfunctions for discrete Schrödinger operators in the non-semiclassical standard setting.

1 Introduction
We first discuss the semiclassical Agmon estimate for discrete Schrödinger operators. We start with a continuous semiclassical Schrödinger operator

\[ H^{\text{cont}}(h) = -\hbar^2 \Delta + V(x) \quad \text{on} \quad L^2(\mathbb{R}^d), \]

where \( V \in C^\infty(\mathbb{R}^d; \mathbb{R}) \). The dimension \( d \in \mathbb{Z}_{>0} \) is fixed throughout this paper. If we discretize this operator with mesh width \( \tau > 0 \), we obtain a discrete Schrödinger operator \( H^{\tau}(h) \) on \( \ell^2(\tau \mathbb{Z}^d) \) defined by

\[ H^{\tau}(h)u(x) = - \left( \frac{\hbar}{\tau} \right)^2 \sum_{|x-y|=\tau} (u(y) - u(x)) + V(x)u(x), \]

where \( x, y \in \tau \mathbb{Z}^d \subset \mathbb{R}^d \) and \( u \in \ell^2(\tau \mathbb{Z}^d) \). A rich quantum-classical correspondence is obtained if we discretize \( H^{\text{cont}}(h) \) with the mesh width proportional to the semiclassical parameter \( (\tau \sim h) \). For simplicity, we put \( \tau = h \) and obtain a semiclassical discrete Schrödinger operator \( H(h) \) on \( \ell^2(h \mathbb{Z}^d) \) defined by

\[ H(h)u(x) = - \sum_{|x-y|=h} (u(y) - u(x)) + V(x)u(x), \]

where \( x, y \in h \mathbb{Z}^d \subset \mathbb{R}^d \) and \( u \in \ell^2(h \mathbb{Z}^d) \). This setting was studied in [7] for \( d = 1 \) in the context of the Harper operator and in [10], [16] for general \( d \).
The limit $\tau \to 0$ for fixed $h > 0$ is the problem of the continuum limit and various quantities related to $H^\tau(h)$ converge to those of $H^{cont}(h)$, that is “$\lim_{\tau \to 0} H^\tau(h) = H^{cont}(h)$” (see for instance, [8] [15]). In the limit $h \to 0$ for fixed $\tau > 0$, $H^\tau(h)$ converges to $V(x)$ on $\ell^2(\tau \mathbb{Z}^d)$ since difference operators are bounded. The related rescaled problem of $h^{-2}H^\tau(h)$ when $h \to 0$ for fixed $\tau > 0$ is studied in [3]. It may be interesting to study $\tau = h^\alpha$ for $1 < \alpha < \infty$. The continuum limit formally corresponds to $\alpha = \infty$.

In this paper, the semiclassical discrete Fourier transform $\mathcal{F}_h : \ell^2(h\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$, where $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$, is defined by

$$\mathcal{F}_h u(\xi) = (2\pi)^{-d/2} \sum_{x \in h\mathbb{Z}^d} u(x) e^{i(x,\xi)/h}.$$

Then we have

$$\tilde{H}(h) \overset{\text{def}}{=} \mathcal{F}_h H(h) \mathcal{F}_h^{-1} = \sum_{j=1}^d (2 - 2\cos \xi_j) + V(hD\xi).$$

Here $V(hD\xi)$ denotes the semiclassical pseudodifferential operator on $\mathbb{T}^d$ with the symbol $V(x)$ (see Section 2 for the definition), where $x \in \mathbb{R}^d$ is interpreted as the dual variable of $\xi \in \mathbb{T}^d$ on $T^*\mathbb{T}^d$. Thus $\tilde{H}(h)$ is the semiclassical quantization of the classical Hamiltonian $p(\xi, x) = \sum_{j=1}^d (2 - 2\cos \xi_j) + V(x) \in C^\infty(T^*\mathbb{T}^d)$ on the torus, and it is expected that various quantities related to $H(h)$ are asymptotically described in terms of $p(\xi, x)$, that is “$\lim_{h \to 0} H(h) = p(\xi, x)$”.

For instance, the Weyl law for eigenvalues naturally follows (Proposition 2.1).

We set $\mathcal{G}_E = \{ x \in \mathbb{R}^d | V(x) \leq E \}$ and $\mathcal{G}_{E, \delta} = \{ x \in \mathbb{R}^d | \text{dist}(x, \mathcal{G}_E) < \delta \}$ (dist$(\cdot, \cdot)$ is the usual Euclidean distance). We set $\mathcal{G}_{E, \delta}^c = \mathbb{R}^d \setminus \mathcal{G}_{E, \delta}$. Denote the space of smooth functions which are bounded with their all derivatives by $C^\infty_c(\mathbb{R}^d)$.

**Assumption 1.** The potential $V$ belongs to $C^\infty_c(\mathbb{R}^d; \mathbb{R})$ and there exists $E \in \mathbb{R}$ such that $\inf_{x \in \mathcal{G}_{E, \delta}^c} V(x) > E$ for any $\delta > 0$.

We note that the compactness of $\mathcal{G}_E$ is not assumed. We set (10)

$$L(x, v) = \sup_{\xi \in K_x} \langle \xi, v \rangle,$$

where

$$K_x = \{ \xi \in \mathbb{R}^d | \sum_{j=1}^d \sinh^2 \xi_j/2 \leq \frac{(V(x) - E)x}{4} \}.$$

Here $(\cdot)_+ = \max\{\cdot, 0\}$. We call $L(x, v)$ or $L : T\mathbb{R}^d \to [0, \infty)$ the Agmon-Finsler metric for discrete Schrödinger operators. This gives the length of $v \in T_x\mathbb{R}^d = \mathbb{R}^d$ in this metric. Let $d_E(x, y)$ be the (pseudo-)distance between $x, y \in \mathbb{R}^d$ induced from $L(x, v)$ (see Section 3.2 for details). Set

$$d_E(x) = \inf_{y \in \mathcal{G}_E} d_E(x, y).$$
We state the semiclassical Agmon estimate for discrete Schrödinger operators.

**Theorem 1.** Under Assumption 1 and the above notation, for any $C_0 > 0$, $\delta_0 > 0$ and $\varepsilon > 0$, there exist $C > 0$, $h_0 > 0$, $0 < \delta < \delta_0$, $\chi, \tilde{\chi} \in C^\infty_c(\mathbb{R}^d; [0, 1])$ with

\[
\text{supp}(1 - \chi) \subset \mathcal{G}_{E, \delta}, \quad \text{supp} \tilde{\chi} \subset \mathcal{G}_{E, \delta} \setminus \mathcal{G}_{E, \delta/2}
\]

and $\rho \in C^\infty(\mathbb{R}^d; [0, \infty))$ with

\[
|\rho(x) - (1 - \varepsilon)d_E(x)| \leq \varepsilon \quad \text{for} \quad x \in \mathbb{R}^d
\]

such that for $0 < h < h_0$,

\[
\|\chi e^{i\rho(x)/h}u\|_{\ell^2} \leq C\|\tilde{\chi}u\|_{\ell^2} + C\|\chi e^{\rho(x)/h}(H(h) - z)u\|_{\ell^2}
\]

for any $u \in \ell^2(h\mathbb{Z}^d)$ and any $z \in [E - C_0, E + C_0h] + i[-C_0, C_0]$.

We note that Theorem 1 also provides the exponential decay at infinity of eigenfunctions when $\mathcal{G}_E$ is bounded though it is valid only for small $h > 0$. The case of $h = 1$ is discussed below.

For $d = 1$, the Agmon estimate for $H(h)$ was proved in [7] using the Agmon-type Riemannian metric

\[
ds_E = 2\text{arsinh}\frac{\sqrt{(V(x) - E)_+}}{2}ds,
\]

where $ds$ is the length of the standard metric on $\mathbb{R}$. In higher dimensions, Klein-Rosenberger [10] [12] introduced the same Finsler metric and proved the Agmon estimate in the case of potential minimums, where $\mathcal{G}_E$ consists of finite points. We allow general $\mathcal{G}_E$, which is possibly unbounded. The strategy of the proof in [10] is similar to that in Dimassi-Sjöstrand [4] Section 6 while our proof is similar to that in Nakamura [13] and is more microlocal. Rabinovich [16] also studied the same semiclassical setting for general $d$ and proved the Agmon estimate though the relation with Finsler metric is not discussed in [16].

We note that Klein-Rosenberger [11] constructed WKB solutions for the eigenfunction problem of $H(h)$ near a nondegenerate potential minimum in terms of the Agmon-Finsler metric, which shows that this metric is the natural notion for estimating the tunneling effect for $H(h)$.

We next prove the Agmon estimate and the optimal anisotropic exponential decay of eigenfunctions for the non-semiclassical discrete Schrödinger operator

\[
Hu(x) = -\sum_{|x - y| = 1} (u(y) - u(x)) + V(x)u(x),
\]

where $x, y \in \mathbb{Z}^d$. Namely, we set $H = H(1)$.

**Assumption 2.** The potential $V : \mathbb{Z}^d \to \mathbb{R}$ has a smooth extension $\tilde{V} : \mathbb{R}^d \to \mathbb{R}$ with the following properties. There exists $0 < \theta \leq 1$ such that

\[
|\partial^\alpha \tilde{V}(x)| \leq C_\alpha (1 + |x|)^{-\theta|\alpha|}
\]

for any $\alpha \in \mathbb{Z}^d_{\geq 0}$ and $\lim_{|x| \to \infty} \tilde{V}(x) \geq 0$.  

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Note that any $V \in \ell^\infty_{\text{comp}}(\mathbb{Z}^d)$ satisfies the Assumption 2. We fix $E < 0$. We write $\tilde{V} = V$ without confusion. We note that a necessary and sufficient condition for the existence of an extension $V : \mathbb{R}^d \to \mathbb{R}$ of $V : \mathbb{Z}^d \to \mathbb{R}$ satisfying (1) is given by Nakamura [14, Lemma 2.1]. Although the case of $\theta = 1$ is discussed in [14], the case of $0 < \theta < 1$ is similar.

We set $q(\xi) = 4 \sum_{j=1}^d \sin^2 \frac{\xi_j}{2}$. We also define the Gauss map $G_E : \partial K^E \to \mathbb{S}^{d-1}$ of $K^E = \{ \xi \in \mathbb{R}^d \mid q(\xi) \leq |E| \}$ by $G_E(x) = \partial q(\xi)/|\partial q(\xi)|$ for $\xi \in \partial K^E$. This is bijective since $K^E$ is convex and the Gaussian curvature of $\partial K^E$ does not vanish. We set

$$\rho_E(x) = \sup_{\xi \in K^E} \langle x, \xi \rangle = x \cdot G_E^{-1} \left( \frac{x}{|x|} \right).$$

**Theorem 2.** Under Assumption 2 and the above notation, for any $C_0 > 0$ and $\varepsilon > 0$ there exist $C > 0$ and $1 - \chi$, $\tilde{x} \in \ell^\infty_{\text{comp}}(\mathbb{Z}^d)$ such that

$$\|\chi e^{(1-\varepsilon)\rho_E(x)} u\|_{L^2} \leq C\|\tilde{x} u\|_{L^2} + C\|\chi e^{(1-\varepsilon)\rho_E(x)}(H - z)u\|_{L^2}$$

for any $u \in \ell^2(\mathbb{Z}^d)$ and any $z \in [E - C_0, E] + i[-C_0, C_0]$.

**Corollary 1.** Under Assumption 2 and the above notation, if $(H - E)u = 0$ and $u \in \ell^2(\mathbb{Z}^d)$, then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|u(x)| \leq C_\varepsilon e^{-(1-\varepsilon)\rho_E(x)}$$

for any $x \in \mathbb{Z}^d$.

**Remark 1.1.** We note that $\rho_E(x)$ is the length of the line segment joining $0$ and $x$ with respect to the Agmon-Finsler metric $L(x, v)$ at energy $E$ for $V \equiv 0$. The geodesics with respect to this metric in this case are the straight lines since $L(x, v)$ is independent of $x$, and thus $\rho_E(x)$ coincides with $d_E(x, 0)$ for $V \equiv 0$ (see [2, Section 5.3, 6.6]).

Rabinovich-Roch [17] proved the exponential decay of eigenfunctions for the discrete Schrödinger operator with a slowly oscillating potential. In our notation, their exponential decay corresponds to $|u(x)| \leq C_\varepsilon e^{-(1-\varepsilon)\rho(x)}$ with a condition on $\sup_j |\partial_{x_j} \rho(x)|$. Our condition $\partial \rho(x) \in K^E$ is more precise and is optimal as seen in Subsection 4.2.

The Agmon estimate was introduced by Agmon (see [1]). Our approach to Theorem 1 and Theorem 2 is similar to the arguments in [13]. Since we work in the Fourier space, we need to study the operator conjugated with the exponential of a Fourier multiplier and the calculations are more complicated than those in [13]. See [13] for the history of the semiclassical Agmon estimate for continuous Schrödinger operators.

We note that the viewpoint of microlocal analysis on the torus for discrete Schrödinger operators is also discussed in the context of long-range scattering theory (see [14] [18]).

In Section 2, we recall basic facts about the microlocal analysis on the torus and present the Weyl law. In Section 3, we discuss the Agmon-Finsler metric
for discrete Schrödinger operators and prove Theorem 1. In Section 3, we prove the Agmon estimate for non-semiclassical discrete Schrödinger operators and the optimality of this estimate.

## 2 Preliminaries

In this section, we recall basic facts on microlocal analysis on the torus. We identify functions on $T^*\mathbb{T}^d$ or $\mathbb{T}^d$ with those on $T^*\mathbb{R}^d$ or $\mathbb{R}^d$ which are $2\pi\mathbb{Z}^d$-periodic. We recall the notation $\langle x \rangle = (1 + x^2)^{1/2}$ and

$$S^m_{\theta,0}(T^*\mathbb{T}^d) = \{a(\cdot;h) \in C^\infty(T^*\mathbb{T}^d) | |\partial_{\xi}^\alpha \partial_{\eta}^\beta a(\xi, x; h)| \leq C_{\alpha,\beta} \langle x \rangle^{m-\theta|\beta|}\}.$$  

Here $\alpha$ and $\beta$ range over $\mathbb{Z}^d_{\geq 0}$. We write $S^m_{\theta,0} = S^m_{\theta,0}(T^*\mathbb{T}^d)$, $S^m = S^m_{1,0}$, $S = S^0_{0,0}$ and $S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m$. For $a \in S^m_{\theta,0}$ and for $u \in C^\infty(\mathbb{T}^d)$, we define

$$a(\xi, hD_\xi)u(\xi) = (2\pi h)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(\xi, x)e^{i(\xi - \eta \cdot x)/h}u(\eta)d\eta dx$$

in the sense of oscillatory integral. The corresponding class of pseudodifferential operators is denoted by $\text{Op}S^m_{\theta,0}$.

**Lemma 2.1.** $V(hD_\xi) = \mathcal{F}_h V(x)\mathcal{F}_h^{-1}$ for $V \in C^\infty_0(\mathbb{R}^d)$.

**Proof.** We have

$$V(hD_\xi)u(\xi) = (2\pi h)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V(x)e^{i(\xi - \eta \cdot x)/h}u(\eta)d\eta dx$$

$$= (2\pi h)^{-d} \int_{\mathbb{R}^d} V(x) \left( \sum_{x \in \mathbb{Z}^d} (2\pi)^{d/2} h^d (\mathcal{F}_h^{-1} u)(x)\delta_x \right) e^{i(\xi - x)/h} dx$$

$$= (2\pi)^{-d/2} \sum_{x \in \mathbb{Z}^d} V(x)(\mathcal{F}_h^{-1} u)(x)e^{i(\xi - x)/h}$$

for $u \in C^\infty(\mathbb{T}^d)$, which completes the proof. \qed

Although we use the spacial structure of the torus to define $a(\xi, hD_\xi)$, we can employ the general theory of pseudodifferential operators on manifolds including the functional calculus and the trace formula for pseudodifferential operators (see [19] Chapter 5, 14]). To illustrate these, we give the Weyl law.

**Proposition 2.1.** Assume that $V \in C^\infty(\mathbb{R}^d; \mathbb{R})$, $\lim_{|x| \to \infty} V(x) \geq 0$ and there exists $0 < \theta \leq 1$ such that

$$|\partial^\alpha V(x)| \leq C_\alpha (1 + |x|)^{-\theta|\alpha|}$$

for any $\alpha \in \mathbb{Z}^d_{\geq 0}$. Then for any fixed $a < b < 0$, the number $N_{[a,b]}(h)$ of eigenvalues of $\hat{H}(h)$ in $[a,b]$ satisfies

$$N_{[a,b]}(h) = (2\pi h)^{-d} \text{Vol}(\{(\xi, x) \in T^*\mathbb{T}^d | a \leq p(\xi, x) \leq b\}) + o(h^{-d})$$

when $h \to 0$. 

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Proof of Proposition 2.1. Take small $\varepsilon > 0$ and $\chi_{1,\varepsilon}, \chi_{2,\varepsilon} \in C_c^\infty(\mathbb{R}; [0, 1])$ such that $\chi_{1,\varepsilon} = 1$ on $[a-\varepsilon, b+\varepsilon]$, supp $\chi_{1,\varepsilon} \subset [a-2\varepsilon, b+2\varepsilon]$, $\chi_{2,\varepsilon} = 1$ on $[a+2\varepsilon, b-2\varepsilon]$ and supp $\chi_{2,\varepsilon} \subset [a + \varepsilon, b - \varepsilon]$. Then we have

$$\text{tr}(\chi_{2,\varepsilon}(\tilde{H}(h))) \leq N_{[a,b]}(\tilde{H}(h)) \leq \text{tr}(\chi_{1,\varepsilon}(\tilde{H}(h)))$$

since $N_{[a,b]}(\tilde{H}(h)) = \text{tr}(\chi_{[a,b]}(\tilde{H}(h)))$ and $\chi_{2,\varepsilon} \leq \chi_{[a,b]} \leq \chi_{1,\varepsilon}$.

The functional calculus and the trace formula for pseudodifferential operators imply that

$$\text{tr}(\chi_{j,\varepsilon}(\tilde{H}(h))) = (2\pi h)^{-d} \int_{T^*\mathbb{T}^d} \chi_{j,\varepsilon}(p(\xi, x)) d\xi dx + O_{\varepsilon}(h^{-d+1})$$

for $j = 1, 2$. We note that $\text{Vol}_{2d}(\{(\xi, x) | p(\xi, x) = a, b\}) = 0$, which follows from Fubini’s theorem and the definition of $p(\xi, x)$. Then we have

$$\lim_{\varepsilon \to 0} \int_{T^*\mathbb{T}^d} \chi_{j,\varepsilon}(p(\xi, x)) d\xi dx = \text{Vol}(\{(\xi, x) \in T^*\mathbb{T}^d | a \leq p(\xi, x) \leq b\})$$

for $j = 1, 2$.

Take any $\delta > 0$. Then for sufficiently small $\varepsilon > 0$, the above arguments imply that

$$-\delta - O_{\varepsilon}(h) \leq (2\pi h)^d N_{[a,b]}(\tilde{H}(h)) - \text{Vol}(\{(\xi, x) | a \leq p(\xi, x) \leq b\}) \leq \delta + O_{\varepsilon}(h).$$

Taking $h \to 0$ and then taking $\delta \to 0$, the proof is finished.

The proof followed the standard strategy (see [4]).

3 Semiclassical Agmon estimate

In this section, we prove Theorem 3.1.

3.1 Calculation of exponentially conjugated operator

Take any $\rho \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$. We compute $\tilde{H}_\rho(h) = e^{\rho(hD\xi)/h}\tilde{H}(h)e^{-\rho(hD\xi)/h}$. Since we have $V(\rho(hD\xi)/h) = V(hD\xi)$, we only have to consider $e^{\rho(hD\xi)/h} p_0(\xi)e^{-\rho(hD\xi)/h}$, where $p_0(\xi) = \sum_{j=1}^d (2 - 2\cos \xi_j)$.

Lemma 3.1. For $\rho \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$,

$$e^{\rho(hD\xi)/h} p_0(\xi)e^{-\rho(hD\xi)/h} = a_\rho(\xi, hD\xi; h) \in \text{OpS},$$

where $a_\rho \sim \sum_{k=0}^\infty h^k a_{\rho,k}(\xi, x)$ with $a_{\rho,k} \in S$ and

$$a_{\rho,0}(\xi, x) = p_0(\xi - i\partial_\rho(x), x).$$

If moreover

$$|\partial_\rho^\alpha \rho(x)| \leq C_{\alpha} (x)^{1-|\alpha|} \text{ for any } \alpha \in \mathbb{Z}^d_{\geq 0},$$

then $a_\rho \in S^0$ and $a_{\rho,k} \in S^{-k}$.

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Proof. We set \( g(x) = e^{-|x|^2} \). Then we have
\[
e^{-\rho(hD_n)/h}u(\hat{\eta}) = \lim_{\epsilon \to 0}(2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{i(\hat{\eta} - \eta,x)/h} e^{-\rho(x)/h} u(\eta)g(\varepsilon \eta)g(\varepsilon \eta)d\eta dx
\]
\[
= (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{i(\hat{\eta} - \eta,x)/h} t L_1^N \left( e^{-\rho(x)/h} u(\eta) \right) d\eta dx,
\]
where \( N \geq 2d + 1 \) and
\[
L_1 = \frac{1 - x \cdot hD_n + (\hat{\eta} - \eta) \cdot hD_x}{1 + |x|^2 + |\eta - \hat{\eta}|^2}.
\]
Thus
\[
e^{\rho(hD_i)/h} p_0(\xi) e^{-\rho(hD_i)/h} u(\xi)
\]
\[
= (2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i(\xi - \hat{\eta},y)/h} t L_2^N e^{\rho(y)/h} p_0(\hat{\eta}) \int_{\mathbb{R}^{2d}} e^{i(\hat{\eta} - \eta,x)/h} t L_1^N \left( e^{-\rho(x)/h} u(\eta) \right) d\eta dy dx d\eta
\]
\[
= \lim_{\epsilon \to 0}(2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i(\xi - \hat{\eta},y)/h} e^{i(\hat{\eta} - \eta,x)/h} p_0(\hat{\eta} - i \Phi(x,y)) u(\eta)
\]
\[
g(\varepsilon \eta)g(\varepsilon \eta)g(\varepsilon \eta)g(\varepsilon \eta)d\eta dy dx d\eta,
\]
where
\[
L_2 = \frac{1 - y \cdot hD_n + (\xi - \hat{\eta}) \cdot hD_y}{1 + |y|^2 + |\eta - \xi|^2}.
\]
We set \( \rho(y) - \rho(x) = (y - x) \cdot \Phi(x,y) \) with \( \Phi(x,y) = \int_0^1 \partial_\xi \rho(y + t(x - y)) dt \). We deform the integral and obtain
\[
e^{\rho(hD_i)/h} p_0(\xi) e^{-\rho(hD_i)/h} u(\xi)
\]
\[
= \lim_{\epsilon \to 0}(2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i(\xi - \hat{\eta},y)/h} e^{i(\hat{\eta} - \eta,x)/h} p_0(\hat{\eta} - i \Phi(x,y)) u(\eta)
\]
\[
g(\varepsilon \eta)g(\varepsilon \eta)g(\varepsilon \eta)g(\varepsilon \eta)d\eta dy d\eta dy.
\]
Using \( t L_2^N \) and \( t L_1^N \), we see that
\[
e^{\rho(hD_i)/h} p_0(\xi) e^{-\rho(hD_i)/h} u(\xi)
\]
\[
= \lim_{\epsilon \to 0}(2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i(\xi - \hat{\eta},y)/h} e^{i(\hat{\eta} - \eta,x)/h} p_0(\hat{\eta} - i \Phi(x,y)) u(\eta)
\]
\[
\psi(\varepsilon \eta)\psi(\varepsilon \eta)\psi(\varepsilon \eta)\psi(\varepsilon \eta)d\eta dy d\eta dy
\]
\[
= \lim_{\epsilon \to 0}(2\pi h)^{-2d} \int_{\mathbb{R}^{4d}} e^{i(\xi - \eta,x)/h} e^{-i(\eta,y)/h} p_0(\hat{\eta} + \xi - i \Phi(x,y + x)) u(\eta)
\]
\[
\psi(\varepsilon \eta)\psi(\varepsilon \eta)\psi(\varepsilon \eta)\psi(\varepsilon \eta)dy d\eta dy dx,
\]
where \( \psi \in C_0^\infty(\mathbb{R}^d) \) is a cutoff near 0 with \( \text{supp } \psi \subset \{ x \in \mathbb{R}^d \mid |x| < 1/4 \} \). We also changed the variables from \( y \) and \( \hat{\eta} \) to \( y + x \) and \( \eta + \xi \), respectively.
We next insert
\[
1 = (1 - \psi(y)\psi(\hat{\eta})) + \psi(y)\psi(\hat{\eta})
\]
into the integrand and estimate the \(\lim_{\varepsilon \to 0}(2\pi h)^{-d} \int_{\mathbb{R}^{2d}} \cdots d\hat{\eta}dy\) part. We set
\[
L_3 = \frac{-\hat{\eta}hD_y - yhD_\eta}{|\hat{\eta}|^2 + |y|^2}.
\]
We see that the \(1 - \psi(y)\psi(\hat{\eta})\) term contributes as \(h^{\infty}S^{-\infty}\) if we use \(L_3^N\) with \(N \gg 1\). To estimate the \(\psi(y)\psi(\hat{\eta})\) term, we apply the stationary phase method (see Theorem 7.7.6) with respect to \(\eta, y\). The stationary point \((\hat{\eta}, y) = (0, 0)\) and we have \(\det \frac{\partial^2}{\partial \eta \partial y} \phi = 0\) and \(\det \frac{\partial^2}{\partial \eta \partial y} \phi = 1\) there. Then we obtain an asymptotic expansion with respect to \(h\) in \(S\) with the leading term \(p_0(\xi - i\Phi(x, x))\). Note that \(\Phi(x, x) = \partial p(x)\).

Finally we assume (2) and prove the asymptotic expansion in \(S^0\). For this, we change the variables from \(y\) to \(\langle x \rangle y\) and introduce \(\hat{h} = h(x)^{-1}\). Then we have
\[
e^{\mu(hD_\xi)/h} p_0(\xi) e^{-\nu(hD_\xi)/h} u(\xi)
= \lim_{\varepsilon \to 0} \lim_{\varepsilon' \to 0} (2\pi h)^{-d}(2\pi \hat{\eta})^{-d} \int_{\mathbb{R}^{2d}} e^{i\langle \xi - \eta, x \rangle/h} e^{-i\langle y, \hat{\eta} \rangle/h} p_0(\hat{\eta} + \xi - i\Phi(x, \langle x \rangle y + x))
\]
\[
\cdot u(\eta)\psi(\varepsilon x)\psi(\varepsilon\eta)\psi(\varepsilon' (x) y + \varepsilon' x)\psi(\varepsilon' \hat{\eta} + \varepsilon' \xi) d\hat{\eta}dyd\eta dx.
\]
We insert
\[
1 = (1 - \psi(y)) + \psi(y)(1 - \psi(\hat{\eta})) + \psi(y)\psi(\hat{\eta})
\]
into the integrand and estimate the \(\lim_{\varepsilon \to 0}(2\pi h)^{-d} \int_{\mathbb{R}^{2d}} \cdots d\hat{\eta}dy\) part. We set
\[
\hat{L}_3 = \frac{-\hat{\eta}hD_y - yhD_\eta}{|\hat{\eta}|^2 + |y|^2} \quad \text{and} \quad \hat{L}_4 = \frac{-\hat{\eta}hD_\eta}{|y|^2}.
\]
We see that the \(1 - \psi(y)\) term contributes as \(h^{\infty}S^{-\infty}\) if we use \(\hat{L}_3^{d+1}\) and \(\hat{L}_4^N\) with \(N \gg 1\). We also see that the \(\psi(y)(1 - \psi(\hat{\eta}))\) term contributes as \(h^{\infty}S^{-\infty}\) if we use \(\hat{L}_3^N\) with \(N \gg 1\). To see this, we note that
\[
|\partial^\alpha y \Phi(x, \langle x \rangle y + x)| \leq C_\alpha \quad \text{for any} \quad \alpha \in \mathbb{Z}^d_{\geq 0}
\]
since \(|\langle x \rangle y + x| \geq |x|/2\) for \(|x| \geq 1\) and \(|y| \leq 1/4\). We apply the stationary phase method to the \(\psi(y)\psi(\hat{\eta})\) term and obtain asymptotic expansion with respect to \(h(x)^{-1}\) in \(S^0\) by the above estimate on \(\partial^\alpha y \Phi(x, \langle x \rangle y + x)\). These complete the proof. \(\square\)

Remark 3.1. The use of the Gaussian weight to justify contour deformation in the oscillatory integral is found in the context of the resonance theory (see Galkowski-Zworski [5] Appendix B.1])

Remark 3.2. The second part of Lemma 3.1 is used in Section 4.
This lemma implies that the semiclassical principal symbol is given by
\[ \sigma_h(\tilde{H}_\rho(h)) = p(\xi - i\partial\rho(x), x). \]

In the proof of the Agmon estimate, we treat unbounded \( \rho \in C^\infty(\mathbb{R}^d; \mathbb{R}) \) such that \( \rho \) is lower semibounded and \( \partial \rho \in C^\infty_b(\mathbb{R}^d, \mathbb{R}^d) \). Take \( \nu(t) \in C^\infty(\mathbb{R}; \mathbb{R}) \) with \( 0 \leq \nu'(t) \leq 1 \) and \( \nu''(t) \leq 0 \) such that \( \nu(t) = t \) for \( t < 0.9 \) and \( \nu(t) = 1 \) for \( t > 1.1 \). We set \( \rho_M(x) = M \nu(\rho(x)/M) \). We note that \( \rho_M(x) \nearrow \rho(x) \) when \( M \to \infty \) since \( \nu''(t) \leq 0 \).

The proof of Lemma 3.1 implies that the first statement in Lemma 3.1 with \( \rho \) replaced by \( \rho_M \) is valid uniformly for \( M > 1 \) since \( \partial \rho_M \in C^\infty_b(\mathbb{R}^d, \mathbb{R}^d) \) uniformly for \( M > 1 \). The second statement with \( \rho \) replaced by \( \rho_M \) is also valid uniformly for \( M > 1 \) if we add the assumption that \( \rho(x) \gtrsim |x| \) for large \( |x| \) to ensure that (2) with \( \rho \) replaced by \( \rho_M \) is valid uniformly for \( M > 1 \).

We set
\[ \tilde{H}_M(h) = e^{\rho_M(hD \xi)} \tilde{H}(h)e^{-\rho_M(hD \xi)}. \]

It may be possible to prove that \( \tilde{H}_\rho(h) = \lim_{M \to \infty} \tilde{H}_M(h) \in \text{OpS} \) and that this is given by the integral expression in the proof of Lemma 3.1. In fact, in the proof of the Agmon estimate, we do not use this and we take the limit \( M \to \infty \) in a later step of the proof.

### 3.2 The Agmon-Finsler metric

We recall that
\[ p_0(\xi) = \sum_{j=1}^{d} (2 - 2 \cos \xi_j) = 4 \sum_{j=1}^{d} \sin^2 \frac{\xi_j}{2}. \]

We will find a condition which ensures that the real part of
\[ p_0(\xi - i\partial\rho(x)) + V(x) - E \]
is positive away from \( G_E = \{ x \in \mathbb{R}^d | V(x) \leq E \} \). Since
\[ 4 \sin^2 \frac{\xi + i\lambda}{2} = 4(\sin \frac{\xi}{2} \cosh \frac{\lambda}{2} + i \cos \frac{\xi}{2} \sinh \frac{\lambda}{2})^2, \]
we have
\[ \text{Re} \left( 4 \sin^2 \frac{\xi + i\lambda}{2} \right) \geq -4 \sin^2 \frac{\lambda}{2}. \]

This implies that
\[ \text{Re} \left( p_0(\xi - i\partial\rho(x)) + V(x) - E \right) \geq V(x) - E - 4 \sum_{j=1}^{d} \frac{\sin^2 \partial_j \rho(x)}{2}. \quad (3) \]

We set
\[ K_x = \{ \xi \in \mathbb{R}^d | \sum_{j=1}^{d} \sin^2 \frac{\xi_j}{2} \leq \frac{(V(x) - E)_+}{4} \}. \]
which is interpreted as a subset of \( T^*\mathbb{R}^d \).

We present a construction (which is valid for more general \( K_x \)) of a function \( d(x) \) such that
\[
\partial d(x) \in K_x
\]
for (almost all) \( x \in \mathbb{R}^d \). For this, we define a Finsler metric as the supporting function of \( K_x \) (10):
\[
L(x, v) = \sup_{\xi \in K_x} \langle \xi, v \rangle,
\]
which gives the length of \( v \in T_x\mathbb{R}^d = \mathbb{R}^d \) in this metric.

Remark 3.3. We note that \( K_x \) for \( x \) with \( V(x) > E \) is a strictly convex compact set with 0 \( \in K_x \) such that \( \partial K_x \) is smooth and has non-vanishing Gaussian curvature. This implies that \( \frac{1}{2} \partial_{v_i} \partial_{v_j} L(x, v)^2 \) is positive definite for \( v \neq 0 \) and \( x \) with \( V(x) > E \). Thus \( L(x, v) \) satisfies the conditions of the definition of the Finsler metric (2, Section 1.1) on \( \mathcal{G}_E = \{ x \in \mathbb{R}^d \mid V(x) > E \} \).

We set
\[
d_E(x, y) = \inf_{x(\cdot)} \int_0^1 L(x(t), x'(t))dt,
\]
where \( x(\cdot) : [0, 1] \to \mathbb{R}^d \) ranges over \( C^1 \) curves such that \( x(0) = x \) and \( x(1) = y \).

Note that \( d_E(x, y) = d_E(y, x) \) since \( L(x, v) = L(x, -v) \). Take any closed set \( \mathcal{G} \) in \( \mathbb{R}^d \). We set
\[
d_G(x) = d_{E, \mathcal{G}}(x) = \inf_{y \in \mathcal{G}} d_E(x, y).
\]
Note that \( d_G \) is a Lipschitz continuous function. We then have the following.

**Lemma 3.2.** For almost all \( x \in \mathbb{R}^d \),
\[
\partial d_G(x) \in K_x.
\]

**Proof.** Take \( x \) such that \( d_G(x) \) is differentiable at \( x \). Take any \( v \in T_x\mathbb{R}^d \). By the triangle inequality, we have
\[
|d_G(x) - d_G(x + tv)|/t \leq d_E(x, x + tv)/t.
\]
Taking limit \( t \to 0 \), we obtain
\[
\langle \partial d_G(x), v \rangle \leq L(x, v).
\]
Recall that the compact convex set \( K_x \) is recovered from its supporting function as
\[
K_x = \{ \xi \in \mathbb{R}^d \mid \langle \xi, v \rangle \leq L(x, v) \text{ for any } v \in \mathbb{R}^d \}
\]
(see [3, Section 4.3]). This implies \( \partial d_G(x) \in K_x \).

Then the exponential decay of the eigenfunctions is stated in terms of
\[
d_E(x) = d_{E, \mathcal{G}_E}(x).
\]
By the inequality [3] and Lemma 3.2, we have
\[
\text{Re} \left( p_0(\xi - i\partial d_E(x)) + V(x) - E \right) \geq 0.
\]
outside \( \mathcal{G}_E \).
3.3 Proof of Theorem 1

In the proof of Theorem 1, we should modify $d_E(x)$ as follows. For a given $\varepsilon > 0$, we take a sufficiently small $\delta > 0$. In the following, we fix $\psi_\delta \in C_c^\infty(\mathbb{R}; \mathbb{R}_{\geq 0})$ such that $\text{supp} \psi_\delta \subset \{x \in \mathbb{R}^d | |x| < \delta/30\}$ and $\int_{\mathbb{R}^d} \psi_\delta(x) dx = 1$. Set $\chi = 1_{G_{E,\delta}^\varnothing} \ast \psi_\delta$, $\tilde{\chi}_1 = 1_{G_{E,\delta}^\varnothing} \ast \psi_\delta$ and $\tilde{\chi} = 1_{G_{E,\delta}^\varnothing \setminus G_{E,\delta}^\varnothing} \ast \psi_\delta$. Here $\varnothing$ denotes the indicator function of a set. Then $\chi, \tilde{\chi}, \tilde{\chi}_1 \in C_c^\infty(\mathbb{R}^d; [0, 1])$, $\chi \chi_1 = \chi$, $\chi \partial \chi = \partial \chi$ and $\partial \chi$.

Suppose $u$ belongs to $\mathcal{O}_{p, \delta}$.

By mollifying $(1 - \varepsilon)d_E G_{E,\delta}$, we obtain $\rho \in C^\infty(\mathbb{R}^d; \mathbb{R}_{\geq 0})$ satisfying $|\rho(x) - (1 - \varepsilon)d_E(x)| \leq \varepsilon$ for $x \in \mathbb{R}^d$ and $\partial \rho(x) \in (1 - \varepsilon/2)K_x$ on $\text{supp} \chi_1$. Moreover, $\text{dist}(\text{supp} \rho, \text{supp} \partial \chi) > \delta/10$.

Define $\rho_M$ and $H_M(h)$ from this $\rho$ as in subsection 3.1. Then $\partial \rho_M(x) \in (1 - \varepsilon/2)K_x$ on $\text{supp} \chi_1$.

**Proof of Theorem 1.** Take any $z \in [E - C_0, E + C_0 h] + i[-C_0, C_0]$ for a fixed $C_0$. Lemma 3.1 implies that

$$\chi_1(hD_\xi)(H_M(h) - z)^*(H_M(h) - z)\chi_1(hD_\xi) - \gamma^2 \chi_1(hD_\xi)^2$$

belongs to $\mathcal{O}_{p, \delta}$ uniformly for $M > 1$ and its principal symbol is

$$\chi_1(x)^2|p(\xi - i\partial \rho_M(x), x) - z|^2 - \gamma^2 \chi_1(x)^2.$$ 

Then the inequality (3) and the estimate for $\partial \rho_M(x)$ above imply that this is nonnegative for small $\gamma > 0$ (we replace $z$ with $z - C_0 h$ if $E \leq \text{Re} z \leq E + C_0 h$).

Thus the Gårding inequality implies that there exists $h_0 > 0$ such that

$$\|H_M(h) - z\chi_1(hD_\xi)\hat{\chi}_\delta\|_{L^2(T^d)} \geq \gamma \|\chi_1(hD_\xi)\hat{\chi}_\delta\|_{L^2(T^d)} - \frac{\gamma}{2} \|\hat{\chi}_\delta\|_{L^2(T^d)}$$

for any $\hat{\chi}_\delta \in L^2(T^d)$ and $0 < h < h_0$. Here $h_0$ is independent of $M > 1$ by the uniformity mentioned above. Replacing $\hat{\chi}_\delta$ with $\chi(hD_\xi)\hat{\chi}_\delta$, this implies

$$\|e^{\rho_M(x)/h}(H(h) - z)e^{-\rho_M(x)/h}\chi u\|_{L^2} \geq \frac{\gamma}{2}\|\chi u\|_{L^2}$$

for $u \in \ell^2(h\mathbb{Z}^d)$ and $0 < h < h_0$. Replacing $u$ with $e^{\rho_M(x)/h}u$, we obtain

$$\|e^{\rho_M(x)/h}(H(h) - z)\chi u\|_{L^2} \geq \frac{\gamma}{2}\|e^{\rho_M(x)/h}\chi u\|_{L^2}$$

for $u \in \ell^2(h\mathbb{Z}^d)$ and $0 < h < h_0$. Taking the limit $M \to \infty$, this is valid with $\rho_M(x)$ replaced by $\rho(x)$. Thus we have

$$\|\chi e^{\rho(x)/h}u\|_{L^2} \leq C\|e^{\rho(x)/h}(H(h) - z)\chi u\|_{L^2}$$

$$\leq C\|\chi e^{\rho(x)/h}(H(h) - z)u\|_{L^2} + C\|e^{\rho(x)/h}[H(h), \chi]u\|_{L^2}$$

$$\leq C\|\chi e^{\rho(x)/h}(H(h) - z)u\|_{L^2} + C\|\hat{\chi} u\|_{L^2}.$$ 

The last inequality follows from the fact that $\rho = 0$ near $\text{supp} \partial \chi$ and $\hat{\chi} = 1$ near $\text{supp} \partial \chi$. 

\[\square\]
4 The exponential decay of eigenfunctions for discrete Schrödinger operators

4.1 Proof of Theorem 2

Recall that \( q(\xi) = 4\sum_{j=1}^{d} \sinh^{2} \frac{\xi_j}{2}, \) \( K^E = \{ \xi \in \mathbb{R}^d | q(\xi) \leq |E| \}, \) \( G_E(x) = \partial q(\xi)/|\partial q(\xi)| \) for \( \xi \in \partial K^E \) and

\[
\rho_E(x) = \sup_{\xi \in K^E} \langle x, \xi \rangle = x \cdot G_E^{-1} \left( \frac{x}{|x|} \right).
\]

**Lemma 4.1.** The function \( \rho_E(x) \) satisfies the eikonal equation

\[
q(\partial \rho_E(x)) = |E| \text{ for any } x \in \mathbb{R}^d \setminus \{0\}.
\]

**Proof.** By the definition of \( G_E, \) we have

\[
q(G_E^{-1}(x/|x|)) = |E| \text{ and } (\partial q)(G_E^{-1}(x/|x|)) = x/|x|.
\]

Differentiating the first equality and using the second, we obtain

\[
x \cdot \partial_x(G_E^{-1}(x/|x|)) = 0.
\]

This and the definition of \( \rho_E \) imply that

\[
\partial \rho_E(x) = G_E^{-1}(x/|x|)
\]

and thus

\[
q(\partial \rho_E(x)) = q(G_E^{-1}(x/|x|)) = |E|.
\]

\[
\square
\]

**Remark 4.1.** Set \( \Lambda_0 = T^*_0 \mathbb{R}^d \cap \{ q(\xi) = |E| \}, \) which is a \((d-1)\)-dimensional isotropic submanifold of \( T^* \mathbb{R}^d. \) Then the solution in Lemma 3.1 corresponds to the Lagrangian submanifold \( \Lambda = \bigcup_{t \geq 0} \Lambda_t, \) where \( \Lambda_t \) is the image of \( \Lambda_0 \) under the time \( t \) map of the Hamilton flow generated by \( q(\xi). \)

**Proof of Theorem 2** Take a smooth modification \( \tilde{\rho}_E(x) \) of \( \rho_E(x) \) such that \( \tilde{\rho}_E(x) = \rho_E(x) \) for \( |x| > 1. \) We see that \( |\partial^\alpha \tilde{\rho}_E(x)| \leq C_{\alpha}(x)^{-|\alpha|} \) for any \( \alpha \in \mathbb{Z}_{\geq 0}^d. \) We also note that \( \tilde{\rho}_E(x) \geq \chi(x) \) for large \( |x| \). For a given small \( \varepsilon > 0, \) we define \( \rho_M \) and \( \tilde{H}_M = \tilde{H}_M(1) \) from \( (1 - \varepsilon)\tilde{\rho}_E \) as in subsection 3.1.

Take any \( z \in [E - C_0, E] + i[-C_0, C_0] \) for a fixed \( C_0. \) We also take \( \chi_1 \in C^\infty(\mathbb{R}^d; [0, 1]) \) such that \( \text{supp} \chi_1 \subset \{ x \in \mathbb{R}^d | |x| > R - 2 \} \) and \( \chi_1(x) = 1 \) for \( |x| > R - 1. \) Then Lemma 3.1 implies that

\[
\chi_1(D_\xi)(\tilde{H}_M - z)^*(\tilde{H}_M - z)\chi(D_\xi) - \gamma^2 \chi_1(D_\xi)^2 \]

belongs to \( \text{OpS}^{0}_{\theta, 0} \) uniformly for \( M > 1 \) and its principal symbol is

\[
\chi_1(x)^2 |p(\xi - i\partial \rho_M(x), x) - z|^2 - \gamma^2 \chi_1(x)^2,
\]

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where $0 < \theta \leq 1$ is that in Assumption 2. If $R > 2$ is sufficiently large and $\gamma > 0$ is sufficiently small, this is everywhere nonnegative for any $M > 1$ by the construction of $\tilde{\rho}_E$, Lemma 4.1 and Assumption 2.

Then the sharp Garding inequality implies that

\[
\| (\tilde{H}_M - z) \chi_1(D\xi) \tilde{u} \|_{L^2}^2 - \gamma^2 \| \chi_1(D\xi) \tilde{u} \|_{L^2}^2 \geq -C \| \tilde{u} \|_{H^{-\theta/2}}^2
\]

for any $\tilde{u} \in L^2(\mathbb{T}^d)$. Here $H^{-\theta/2}$ denotes the Sobolev space on $\mathbb{T}^d$. We replace $\tilde{u}$ with $\chi(D\xi) \tilde{u}$, where $\chi \in C^\infty(\mathbb{R}^d; [0, 1])$ satisfies $\text{supp} \chi \subset \{ x \in \mathbb{R}^d \mid |x| > R \}$ and $\chi(x) = 1$ for $|x| > R + 1$. Then we have

\[
\| (\tilde{H}_M - z) \chi_1(D\xi) \tilde{u} \|_{L^2}^2 - \gamma^2 \| \chi_1(D\xi) \tilde{u} \|_{L^2}^2 \geq -C \| \chi_1(D\xi) \tilde{u} \|_{H^{-\theta/2}}^2.
\]

Taking $R > 1$ large enough, we see that

\[
C \| \chi_1(D\xi) \tilde{u} \|_{H^{-\theta/2}}^2 \leq \frac{\gamma^2}{2} \| \chi_1(D\xi) \tilde{u} \|_{L^2}^2.
\]

Here $C$ and thus $R$ are independent of $M > 1$ by the uniformity mentioned above. This implies that

\[
\| e^{\rho_M(x)} (H - z) e^{-\rho_M(x)} \chi(x) u \|_{\ell^2} \geq \frac{\gamma}{2} \| \chi(x) u \|_{\ell^2}
\]

for any $u \in \ell^2(\mathbb{Z}^d)$. This implies that

\[
\| e^{\rho_M(x)} (H - z) \chi(x) u \|_{\ell^2} \geq \frac{\gamma}{2} \| e^{\rho_M(x)} \chi(x) u \|_{\ell^2}
\]

for any $u \in \ell^2(\mathbb{Z}^d)$. Taking the limit $M \to \infty$, we have

\[
\| e^{(1 - \varepsilon) \rho_E(x)} (H - z) \chi(x) u \|_{\ell^2} \geq \frac{\gamma}{2} \| e^{(1 - \varepsilon) \rho_E(x)} \chi(x) u \|_{\ell^2}
\]

for any $u \in \ell^2(\mathbb{Z})$. Then Theorem 2 follows if we calculate the commutator as in the proof of Theorem 1 and take $\tilde{\chi} \in C^\infty_\text{comp}$ which is 1 on $\{ x \in \mathbb{Z}^d \mid R - 1 < |x| < R + 2 \}$. \hfill \Box

### 4.2 The optimality of Theorem 2

We prove that the exponential decay of eigenfunctions in Corollary 1 is optimal for a simple discrete Schrödinger operator. Fix any $E < 0$ and define $u_E \in \ell^2(\mathbb{Z}^d)$ by

\[
u_E(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} (4 \sum_{j=1}^d \sin^2 \frac{\xi_j}{2} + |E|) ^{-1} e^{-i(x,\xi)} d\xi.
\]

Then $(H_0 + |E|) u_E(x) = \delta_0(x)$, where $H_0$ is the free discrete Schrödinger operator and $\delta_0$ is the delta function supported on $0 \in \mathbb{Z}^d$. We note that $u_E(0) > 0$. Thus if we set $V(x) = -u_E(0)^{-1} \delta_0(x)$, we have $(H_0 + V) u_E(x) = E u_E(x)$. We
study the exponential decay of this eigenfunction $u_E$. We note that Corollary 1 for $u_E$ follows from the deformation of the integral in the definition of $u_E$.

Take a bounded domain $0 \in \Omega \subset \mathbb{R}^d$ and set

$$\rho_\Omega(x) = \sup_{\xi \in \Omega} \langle x, \xi \rangle.$$ 

The following proposition gives the optimality of Theorem 2 and Corollary 1. Recall that $K^E = \{ \xi \in \mathbb{R}^d | 4 \sum_{j=1}^d \sinh^2 \frac{\xi_j}{2} \leq |E| \}$.

**Proposition 4.1.** Under the above notation, assume that

$$|u_E(x)| \leq C e^{-\rho_\Omega(x)}$$

for some $C > 0$ and any $x \in \mathbb{Z}^d$. Then $\Omega \subset K^E$.

**Proof.** The Fourier inversion formula implies

$$(4 \sum_{j=1}^d \sin^2 \frac{\xi_j}{2} + |E|)^{-1} = \sum_{x \in \mathbb{Z}^d} u_E(x) e^{i\langle x, \xi \rangle}.$$ 

The assumption on $u_E$ implies that

$$|u_E(x) e^{i\langle x, \xi \rangle}| \leq C e^{-\rho_\Omega(x)} e^{-\langle \text{Im} \xi, x \rangle}.$$ 

This implies that $(4 \sum_{j=1}^d \sin^2 \frac{\xi_j}{2} + |E|)^{-1}$ has an analytic continuation to $\{ \xi \in \mathbb{C}^d/2\pi\mathbb{Z}^d | -\text{Im} \xi \in \Omega \}$. Since $4 \sin^2 \xi_j/2 = -4 \sinh^2 \text{Im} \xi_j/2$ for $\text{Re} \xi_j = 0$, this implies $\Omega \subset K^E$. 

**Remark 4.2.** For $d = 1$, it is known that $u_E(x) = (|E|(4 + |E|))^{-1/2} e^{-\rho_E(x)}$ (Lemma 2.2). Ito-Jensen [9, Theorem 2.1, 2.4] proved that $u_E$ is expressed by a hypergeometric function of several variables for $d \geq 2$ and by a generalized hypergeometric function of one variable for $d = 2$. The precise asymptotics of $u_E(x)$ when $|x| \to \infty$ does not seem to be immediate from these expressions.

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