Two-loop scalar diagrams from string theory

Paolo Di Vecchia and Lorenzo Magnea
NORDITA
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Alberto Lerda
Dipartimento di Scienze e Tecnologie Avanzate and Dipartimento di Fisica Teorica, Università di Torino Via P.Giuria 1, I-10125 Torino, Italy and I.N.F.N., Sezione di Torino

Raffaele Marotta
Dipartimento di Scienze Fisiche, Università di Napoli Mostra D’Oltremare, Pad. 19, I-80125 Napoli, Italy

Rodolfo Russo
Dipartimento di Fisica, Politecnico di Torino Corso Duca degli Abruzzi 24, I-10129 Torino, Italy and I.N.F.N., Sezione di Torino

Abstract

We show how to obtain correctly normalized expressions for the Feynman diagrams of $\Phi^3$ theory with an internal $U(N)$ symmetry group, starting from tachyon amplitudes of the open bosonic string, and suitably performing the zero–slope limit by giving an arbitrary mass $m$ to the tachyon. In particular we present explicit results for the two–loop amplitudes of $\Phi^3$ theory, in preparation for the more interesting case of the multiloop amplitudes of non–abelian gauge theories.
1 Introduction

It has been stressed by several people that string theory can not only be used as a laboratory for a realistic unifying theory of all interactions, but can also provide an efficient way of computing perturbative field theory amplitudes in the limit of vanishing Regge slope $\alpha' \rightarrow 0$. This approach, pioneered for gluon amplitudes at tree level by the authors of Ref. [1], was extensively applied to one–loop amplitudes [2], leading to a set of simple string–inspired “Feynman” rules. Subsequently, the first steps have been taken towards a generalization to multiloops [3], mainly using the technology of the multiloop operator formalism for string amplitudes [4]. These string–derived methods have also inspired the development of new techniques in field theory, which make use of world–line path integrals. This latter approach was studied at one loop in Ref. [5], and partially extended to higher loops in Ref. [6]. Recently, a connection between the world–line Green functions used in field theory and the multiloop Green function of string theory has been established in Ref. [7].

In two previous papers, we performed the field theory limit in the amplitudes with two [8], three and four gluons [9] in the open bosonic string, and extracted the corresponding renormalization constants after a suitable off–shell extension. These results were obtained by discarding by hand the contribution of the tachyons, and isolating the contribution of the gluons (the higher string states are always negligible when $\alpha' \rightarrow 0$). In Refs. [8, 9] we also stressed that, in order to compute gluon amplitudes in field theory, one is not forced to start from a consistent string model; indeed, it was practical to use the simplest model containing Yang-Mills theory when $\alpha' \rightarrow 0$, i.e. the open bosonic string.

In this paper we show how these results can be extended to higher loops. As a first step, to avoid the computational difficulties associated with multiloop Yang–Mills amplitudes, which are, however, inessential for the understanding of the field theory limit, we focus on scalar particles instead of gluons. This can be achieved by considering a slightly different zero–slope limit of the bosonic string, in which only the lowest tachyonic state, with a mass $m$ satisfying $\alpha'm^2 = -1$, is kept. To avoid the inconsistencies associated with tachyon propagation, one must carefully modify the measure of integration by replacing the tachyon poles with exponentials of the mass of the lowest state. In the case of tree and one–loop diagrams, this procedure is equivalent to taking the zero–slope limit of an old pre–string dual model characterized by an arbitrary value of the intercept of the Regge trajectory $a$ [11]. This model was recognized to be inconsistent, because of the presence of ghost states unless $a = 1$, in which case, however, the lowest scalar particle becomes a tachyon. The tree and one–loop diagrams of this pre–string dual model were shown to lead to the Feynman diagrams of $\Phi^3$ theory by Scherk [12].

In this paper we explicitly show, up to two loops, that, by performing a zero–slope limit in which we keep only a suitably reinterpreted tachyon state, one corre-

\[^1\]Note that $a$ is related to the mass of the lowest scalar state of the theory by $a + \alpha'm^2 = 0$. 

rectly reproduces the Feynman diagrams of the $\Phi^3$ theory, with the inclusion of an internal symmetry group $U(N)$, and with the right overall normalization. In the process we also develop a precise understanding of which corners of the multiloop moduli space contribute to the field theory limit, and propose an algorithm that allows to obtain all diagrams of the $\Phi^3$ theory in the Schwinger parametrization (after integration of all loop momenta) from the general string amplitudes. We believe that this algorithm will be easily generalizable to Yang–Mills theory, where we can expect extra difficulties due to the complicated numerator structure of the integrands, but the nature of the field theory limit should be unchanged.

2 Scalar amplitudes in string theory

In Ref. [9] we have given the correctly normalized $h$–loop $M$–gluon amplitude in the open bosonic string. That formula can be immediately adapted to the case of scalars. The planar $h$–loop scattering amplitude of $M$ scalars with momentum $p_1, \ldots, p_M$, is

$$A_M^{(h)}(p_1, \ldots, p_M) = N^h \operatorname{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) C_h \left[ 2 g_s (2\alpha')^{(d-2)/4} \right]^M \times \int [dm]^M_h \prod_{i<j} \left[ \frac{\exp \left( G^{(h)}(z_i, z_j) \right)}{\sqrt{V'_i(0) V'_j(0)}} \right]^{2\alpha' p_i \cdot p_j} ,$$  \hspace{1cm} (2.1)

where $N^h \operatorname{Tr}(\lambda^{a_1} \cdots \lambda^{a_M})$ is the appropriate $U(N)$ Chan-Paton factor, $g_s$ is the dimensionless string coupling constant, $C_h$ is a normalization factor given by

$$C_h = \frac{1}{(2\pi)^d} g_s^{2-2h} \frac{1}{(2\alpha')^{d/2}} ,$$  \hspace{1cm} (2.2)

and $G^{(h)}$ is the $h$–loop bosonic Green function

$$G^{(h)}(z_i, z_j) = \log E^{(h)}(z_i, z_j) - \frac{1}{2} \int_{z_i}^{z_j} \omega^\mu (2\pi \text{Im}\tau_{\mu\nu})^{-1} \int_{z_i}^{z_j} \omega^{\nu} ,$$  \hspace{1cm} (2.3)

with $E^{(h)}(z_i, z_j)$ being the prime form, $\omega^\mu (\mu = 1, \ldots, h)$ the abelian differentials and $\tau_{\mu\nu}$ the period matrix. All these objects, as well as the measure on moduli space $[dm]^M_h$, can be explicitly written in the Schottky parametrization of the Riemann surface, and their expressions for arbitrary $h$ can be found for example in Ref. [11]. Here we give only the expression for the measure, which for scalar amplitudes differs slightly from the one of Ref. [9]. It is given by

$$[dm]^M_h = \frac{1}{dV_{abc}} \prod_{i=1}^{M} dV'_i(0) \prod_{\mu=1}^{h} \left[ \frac{dk_\mu d\xi_\mu d\eta_\mu}{k_\mu^2 (\xi_\mu - \eta_\mu)^2 (1 - k_\mu)^2} \right] \times \left[ \det ( -i\tau_{\mu\nu} ) \right]^{-d/2} \prod_{\alpha} \left[ \prod_{n=1}^{\infty} (1 - k_\alpha^n)^{-d} \prod_{n=2}^{\infty} (1 - k_\alpha^n)^2 \right] .$$  \hspace{1cm} (2.4)

\footnote{We adopt the same normalizations and conventions of ref. [10], in particular $\operatorname{Tr} (\lambda^a \lambda^b) = \frac{1}{2} \delta^{ab}$.}
where $k_\mu$ are the multipliers, $\xi_\mu$ and $\eta_\mu$ are the fixed points of the generators of the Schottky group, and $dV_{abc}$ is the projective invariant volume element

$$dV_{abc} = \frac{d\rho_a \ d\rho_b \ d\rho_c}{(\rho_a - \rho_b) (\rho_a - \rho_c) (\rho_b - \rho_c)} ,$$

with $\rho_a, \rho_b, \rho_c$ being any three of the $M$ Koba-Nielsen variables $z_i$, or of the $2h$ fixed points of the generators of the Schottky group, which can be fixed at will. In the last line of Eq. (2.4), the primed product over $\alpha$ denotes the product over the primary classes of elements of the Schottky group, as defined in Ref. [11]. Notice that the measure (2.4) contains an extra factor of $\prod_{i=1}^{M} [V'_i(0)]^{-1}$, which originates from the $N$-reggeon vertex, as explained in Ref. [4]. Finally, like in any planar open string amplitude, here too the Koba-Nielsen variables $z_i$ must be cyclically ordered along one of the boundaries of the world–sheet.

Before proceeding, a few comments are in order. First of all, Eq. (2.1) represents the scattering amplitude of $M$ tachyons if we require that the bosonic string be consistent. In fact, in order not to have ghosts at the excited levels, the intercept of the Regge trajectory, $a$, must be fixed to 1. In this way, however, the mass squared of the scalar states turns out to be negative: $\alpha' p_i^2 = -\alpha' m^2 = 1$. Secondly, the amplitude in Eq. (2.1) depends, in general, on the projective transformations $V_i(z)$, that parametrize a choice of local coordinates around the punctures $z_i$. However, by imposing the mass–shell condition, $\alpha' p_i^2 = 1$, this dependence disappears, just as it did for gluons.

As stressed in the introduction, our main purpose is to study the zero–slope limit of Eq. (2.1). Since in this limit all excited string states (including possible ghost states) become irrelevant, we can try to relax the condition $a = 1$, and verify whether this leads to any inconsistencies in the field theory limit. The analysis of string amplitudes with an arbitrary value of the intercept of the Regge trajectory was actually performed long ago in the context of dual models, at least at tree level and at one loop. Details on these amplitudes can be found, for example, in the old review paper in Ref. [12]. We will thus begin by borrowing some of the results of these models, whose lowest state is a scalar particle with a mass $m$ such that

$$a + \alpha' m^2 = 0$$

The on–shell tree–level $M$–point scalar amplitude is

$$A_M^{(0)} (p_1, \ldots, p_M) = \text{Tr}(\lambda^{a_1} \ldots \lambda^{a_M}) \frac{1}{g^2_s (2\alpha')} \left[ 2g_s (2\alpha')^{(d-2)/4} \right]^M \int \frac{d\rho_1 \ldots d\rho_M}{dV_{abc}} \prod_{i=1}^{M} |z_i - 1 - z_i|^{a-1} \prod_{i<j} (z_i - z_j)^{2\alpha' p_i \cdot p_j} ,$$

(2.7)

For gluons, this extra factor is cancelled since the amplitude is multilinear in every gluon polarization vector $\varepsilon_i$ and each one of these is accompanied by an extra $V'_i(0)$.
which obviously reduces to the $M$-tachyon amplitude in the open bosonic string if $a = 1$. In particular for $M = 3$, we get

$$A_3^{(0)} = \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3}) 8 g_s (2\alpha')^{(d-6)/4}.$$  \hspace{1cm} (2.8)

We observe that this same expression can be obtained by computing the color–ordered 3-point function in the theory defined by the Lagrangian

$$L = \text{Tr} \left[ \partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2 - \frac{g}{3!} \Phi^3 \right],$$  \hspace{1cm} (2.9)

where $\Phi = \Phi^a \lambda^a$, if the coupling constant $g$ and the string coupling $g_s$ are related by

$$g = 16 g_s (2\alpha')^{(d-6)/4}.$$  \hspace{1cm} (2.10)

Thus in the limit $\alpha' \to 0$, the amplitudes in Eq. (2.7) yield the tree–level Feynman diagrams of the $\Phi^3$ theory (2.9).

We now consider the one–loop corrections. The one–loop $M$–point scalar amplitude can also be found in Ref. [12]. We rewrite it here including the Chan-Paton factors, the correct overall normalization, and choosing the fixed points of the generator of the Schottky group $\xi = \infty$ and $\eta = 0$, together with $z_1 = 1$. We find

$$A_M^{(1)}(p_1, \ldots, p_M) = N \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) \frac{1}{(4\pi)^{d/2}} \left( \frac{g}{8} \right)^M (2\alpha')^{M-d/2}$$

$$\times \int_0^1 \frac{dk}{k^{a+1}} \left[ \frac{1}{2} \log k \right]^{-d/2} \prod_{n=1}^{\infty} (1 - k^n)^{-d}$$

$$\times \int_k^1 dz_M \int_{z_M}^1 dz_{M-1} \cdots \int_{z_3}^1 dz_2 \prod_{i=1}^{M} \frac{1}{[V_i'(0)]^a}$$

$$\times \prod_{i<j} \left[ \exp \left( \mathcal{G}^{(h)}(z_i, z_j) \right) \right]^{2\alpha' p_i \cdot p_j} \prod_{i=2}^{M+1} \left| z_{i-1} - z_i \right|^{a-1},$$  \hspace{1cm} (2.11)

where we have defined $z_{M+1} = k$. Notice that Eq. (2.11) is derived by including only orbital degrees of freedom, and without cancelling any unphysical state circulating in the loop (such states will anyway be irrelevant in the field theory limit). If one compares Eq. (2.11) with the corresponding one–loop expression that can be obtained from Eqs. (2.1) and (2.4), one sees two discrepancies. The first one is the absence, in the open string amplitude, of the last term in Eq. (2.11). The second one is the power of $k^{-1-a}$ in the measure in Eq. (2.11), instead of $k^{-2}$. Obviously, the two expressions agree if $a = 1$, as expected. Actually there is a third difference between the two one–loop formulas, because the ghost contribution has been included in one case and not in the other; however, for the scalar amplitudes, this will irrelevant in the field theory limit.

It is now useful to introduce the variables

$$\nu_i = -\frac{1}{2} \log z_i \quad \text{and} \quad \tau = -\frac{1}{2} \log k,$$  \hspace{1cm} (2.12)
so that Eq. (2.11) becomes

\[ A^{(1)}_M(p_1, \ldots, p_M) = N \text{Tr}(\lambda^a_1 \cdots \lambda^a_M) \frac{1}{(4\pi)^{d/2}} \left( \frac{g}{4} \right)^M (2\alpha')^{M-d/2} \]

\times \int_0^\infty dt e^{-2\alpha' m^2 t} t^{-d/2} \prod_{n=1}^\infty \left( 1 - e^{-2n\tau} \right)^{-d}

\times \int_0^\tau d\nu M \int_0^{\nu M} d\nu_{M-1} \cdots \int_0^{\nu_3} d\nu_2 \prod_{i=1}^M \left[ \frac{z_i}{V'_i(0)} \right]^{a-\alpha'\nu_i^2}

\times \prod_{i<j} \left[ \exp(G(\nu_{ji})) \right]^{2\alpha' p_i \cdot p_j} \prod_{i=2}^{M+1} \left[ 1 - \exp(2(\nu_{i-1} - \nu_i)) \right]^{a-1}, \tag{2.13} \]

where \( G(\nu_{ji}) \) is the one–loop bosonic Green function given in Ref. [9] with \( \nu_{ji} = \nu_j - \nu_i \), and in the last product we have used the notation \( \nu_{M+1} = \tau \). As in the case of the one–loop \( M \)--gluon amplitude, here too we find a dependence on the local coordinates \( V'_i(0) \). This can be eliminated either by going on shell, or by choosing \( V'_i(0) = z_i \) and formally staying off shell. We will now show that Eq. (2.13) can be used to obtain in a simple and direct way the one–loop 1PI diagrams of the \( \Phi^3 \) field theory defined by the Lagrangian (2.9).

3 The field theory limit of scalar amplitudes

If we neglect the pinching configurations that yield the one-particle reducible diagrams, as explained in great detail in Ref. [9], the field theory limit of the one-loop amplitude (2.13) is simply obtained by sending both \( \nu_i \) and \( \tau \) to \( \infty \). This follows from the fact that, if we introduce the dimensionful variables

\[ t = 2\alpha' \tau \quad \text{and} \quad x_i = 2\alpha' \nu_i, \tag{3.1} \]

the field theory is recovered in the limit \( \alpha' \to 0 \) keeping the “proper times” \( t \) and \( x_i \) finite. In this case we can approximate the one-loop bosonic Green function with [5, 6]

\[ G(\nu) \to \nu - \frac{\nu^2}{\tau}, \tag{3.2} \]

so that Eq. (2.13) drastically simplifies. In fact, observing that the last terms in Eq. (2.13) are negligible in this limit, we obtain

\[ A^{(1)}_M(p_1, \ldots, p_M) = N \text{Tr}(\lambda^a_1 \cdots \lambda^a_M) \frac{1}{(4\pi)^{d/2}} \left( \frac{g}{4} \right)^M \]

\times \int_0^\infty dt e^{-m^2 t} t^{M-1-d/2}

\times \int_0^1 d\hat{\nu} M \int_0^{\hat{\nu} M} d\hat{\nu}_{M-1} \cdots \int_0^{\hat{\nu}_3} d\hat{\nu}_2 \prod_{i=1}^{M+1} \left[ \exp \left( p_i \cdot p_j t \hat{\nu}_{ji}(1 - \hat{\nu}_{ji}) \right) \right], \tag{3.3} \]
where, as in Ref. [9], we have chosen \( V_i(0) = z_i \) in order to reproduce the field theory limit also off mass–shell, and defined \( \nu_{ji} = \nu_{ji}/\tau \).

Both the integrand and the integration region in Eq. (3.3) agree with the irreducible \( \Phi^3 \) field theory diagram obtained from the Lagrangian (2.9), after the exponentiation of each propagator of the diagram by means of the corresponding Schwinger proper time

\[
\frac{1}{p^2 + m^2} = \int_0^\infty dx \ e^{-x(p^2 + m^2)} ,
\]

and after the integration over the loop momentum. The variable \( t \) corresponds to the total Schwinger proper time of the diagram, \( i.e. \) the sum of all Schwinger proper times, while the variables \( \nu_i \) correspond to the fraction of a partial proper time relative to the total one, namely

\[
t \equiv \sum_{i=1}^M t_i , \quad \nu_i \equiv \frac{t_2 + t_3 + \ldots + t_i}{t} \quad \text{for } i = 2, \ldots, M .
\]

This shows that the field theory limit of the one–loop \( M \)-point scalar amplitudes of the dual model correctly reproduces the one–loop 1PI diagram with \( M \) legs of the \( \Phi^3 \) field theory.

In principle, also multiloop amplitudes could be computed in this pre-string dual model, using for instance the sewing procedure. However, since our main interest here is to understand which regions in moduli space lead to the different field theory diagrams, with the aim to later apply this analysis to Yang-Mills theory, in this paper we will use a simpler ansatz, which is a natural generalization of Eq. (2.13), and leads to the correct results. In practice, we will perform the field theory limit directly in the amplitudes (2.1) (corresponding to the case \( a = 1 \)), keeping only the lowest tachyonic state; we then transform it into a normal scalar particle with arbitrary mass \( m \) by rewriting each quadratic pole of measure (2.4) according to

\[
x^{-2} \rightarrow x^{-1-a} = x^{-1} \exp [-a \log x] = x^{-1} \exp \left[ m^2 \alpha' \log x \right] .
\]

Let us then turn back to Eq. (2.1), and focus on the case \( h = 2 \). Since we want only the scalars to propagate in the loops, it is enough to compute the various quantities that appear in Eq. (2.1) for \( k_1, k_2 \rightarrow 0 \). In this limit, the determinant of the period matrix can be approximated by

\[
det(-i\tau_{\mu\nu}) = \frac{1}{4\pi^2} \left[ \log k_1 \log k_2 - \log^2 S \right] ,
\]

while the two–loop bosonic Green function becomes

\[
\mathcal{G}^{(2)}(z_1, z_2) = \log(z_1 - z_2) + \frac{1}{2} \left[ \log k_1 \log k_2 - \log^2 S \right]^{-1} \times \left[ \log^2 T \log k_2 + \log^2 U \log k_1 \log U \log S \right] ,
\]

6
where we defined the anharmonic ratios

\[
S = \frac{(\eta_1 - \eta_2)(\xi_1 - \xi_2)}{(\xi_1 - \eta_2)(\eta_1 - \xi_2)}, \\
T = \frac{(z_2 - \eta_1)(z_1 - \xi_1)}{(z_2 - \xi_1)(z_1 - \eta_1)}, \\
U = \frac{(z_2 - \eta_2)(z_1 - \xi_2)}{(z_1 - \eta_2)(z_2 - \xi_2)}.
\]

(3.9)

Notice finally that in the field theory limit of the scalar amplitudes, the infinite product over primary classes of the Schottky group is negligible and can be simply replaced by 1. We can thus express the two–loop \(M\)–point scalar amplitude as

\[
A^{(2)}_M(p_1, \ldots, p_M) = N^2 \operatorname{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) \frac{1}{(4\pi)^d 2^{2+M}} g^{2+M} (2\alpha')^{d-M} \left[ \prod_{i=1}^M \frac{dz_i}{V_i'(0)} \right] \frac{1}{d \rho_c} \int \frac{dk_1}{k_1^2} \int \frac{dk_2}{k_2^2} \int \frac{d\xi_1}{(\xi_1 - \eta_1)^2} \int \frac{d\eta_1}{(1 - \eta_1)^2} \int \frac{d\rho_c}{V_i'(0) V_j'(0)} \exp \left[ \frac{1}{4} \left( \log k_1 \log k_2 - \log \left( \frac{2S}{\eta_1} \right) \right) \right]^{-d/2} \prod_{i<j} \frac{\exp \left( G^{(2)}(z_i, z_j) \right)}{\sqrt{V_i'(0) V_j'(0)}}^{2\alpha' \rho_c},
\]

(3.10)

where \(G^{(2)}(z_i, z_j)\) is given by Eq. (3.8), with \(\xi_2 = \infty\) and \(\eta_2 = 0\). For future convenience, we have left undetermined the variable \(\rho_c\), that eventually will be fixed to 1. This formula is the master formula that can be used to calculate all two–loop diagrams in the \(\Phi^3\) field theory, as we will now show.

Let us start from the two–loop vacuum bubbles \((M = 0)\). In this case, choosing \(\rho_c = \xi_1 = 1\), Eq. (3.10) simply becomes

\[
A^{(2)}_0 = \frac{N^3}{(4\pi)^d 2^8} g^{2+2} (2\alpha')^{3-d} \int \frac{dk_1}{k_1^2} \int \frac{dk_2}{k_2^2} \int \frac{d\eta_1}{(1 - \eta_1)^2} \int \frac{d\rho_c}{V_i'(0) V_j'(0)} \exp \left[ \frac{1}{4} \left( \log k_1 \log k_2 - \log \left( \frac{2S}{\eta_1} \right) \right) \right]^{-d/2} \prod_{i<j} \frac{\exp \left( G^{(2)}(z_i, z_j) \right)}{\sqrt{V_i'(0) V_j'(0)}}^{2\alpha' \rho_c}.
\]

(3.11)

As expected, the two multipliers \(k_1\) and \(k_2\) play the same role, since Eq. (3.11) is symmetrical in the exchange of \(k_1\) and \(k_2\). Therefore, to avoid double counting, we order them by choosing, for example, \(k_2 \leq k_1\). The integration region of the third modular parameter, \(\eta_1\), can be deduced by studying in more detail the Schottky representation of the two–annulus, which is shown in Fig. 1. The points \(A, B, C\) and \(D\) have to be identified with \(A', B', C'\) and \(D'\) respectively, under the action of the two generators of the Schottky group.

\^The overall \(N^3\) factor comes from \(N^2 \operatorname{Tr}(1)\).
Fig. 1: In the Schottky parametrization, the two–annulus corresponds to the part of the upper–half plane which is inside the big circle passing through $A'$ and $B'$, and which is outside the circles $K_1$, $K'_1$ and $K_2$.

The position of these points is completely determined once $k_1$, $k_2$ and $\eta_1$ are given; for example, following [11], one can verify that

$$B = \sqrt{k_2}, \quad C = \frac{\eta_1 - \sqrt{k_1}}{1 - \sqrt{k_1}}, \quad D' - D = \frac{1 - \sqrt{k_1}}{1 + \sqrt{k_1}} (1 - \eta_1). \quad (3.12)$$

It is easy to realize that the two segments $(AA')$ and $(DD')$ represent respectively the two inner boundaries of the two–annulus, while the union of $(BC)$ and $(C'B')$ represents the external boundary. With these choices, the interpretation of the three moduli, $k_1$, $k_2$ and $\eta_1$, is particularly simple. In fact, $\sqrt{k_2}$ is the radius of the circle $K_2$, while the radii of $K_1$ and $K'_1$ are proportional to $\sqrt{k_1}$. Furthermore, $\eta_1$ turns out to be inside $K_1$, while the point $\xi_1 = 1$ is inside $K'_1$. Therefore, in this configuration the circle $K'_1$ is fixed while $K_1$ can move, depending on the value of $\eta_1$. In particular, if the point $D'$ is very close to $D$, $\eta_1$ is almost equal to 1, while if $C$ is near to $B$, then $\eta_1$ is slightly bigger than $\sqrt{k_1} \to 0$. Note that in this way we have just found for $k_1$, $k_2$ and $\eta_1$ the same integration region of integration derived by Roland [15] for the closed string, i.e. $0 \leq \sqrt{k_2} \leq \sqrt{k_1} \leq \eta_1 \leq 1$. These facts allow us to interpret $\eta_1$ as the “distance” between the two loops. Thus, when $\eta_1 \to 1$ we expect to eventually find from Eq. (3.11) the reducible vacuum bubble of the $\Phi^3$ field theory with the two loops widely separated, while when $\eta_1 \to \sqrt{k_1} \to 0$ we expect to obtain from Eq. (3.11) the irreducible vacuum bubble with the two loops attached to each other. In what follows we will show that this is indeed what happens.

In fact, when $\eta_1 \to 1$ (together with $k_1, k_2 \to 0$), we have

$$\left. A_0^{(2)} \right|_{\text{red}} = \frac{N^3}{(4\pi)^d} \frac{g^2}{2^8} (2\alpha')^{3-d} \int_{1-\epsilon}^1 \frac{d\eta_1}{(1 - \eta_1)^2} \int_0^\epsilon \frac{dk_2}{k_2^2} \int_{1-\epsilon}^\epsilon \frac{dk_1}{k_1^2} \left[ \frac{1}{4} (\log k_1 \log k_2) \right]^{-d/2}$$

$$= \frac{N^3}{(4\pi)^d} \frac{g^2}{2^5} \int_0^\infty dt_3 \int_0^\infty dt_2 \int_0^{t_2} dt_1 e^{-m^2(t_1 + t_2 + t_3)} (t_1 t_2)^{-d/2}, \quad (3.13)$$

where in the second line we have introduced the mass $m$ as explained in Eq. (3.6),
and the Schwinger proper times $t_i$ according to

$$t_1 = -\alpha' \log k_1, \quad t_2 = -\alpha' \log k_2, \quad t_3 = -\alpha' \log(1 - \eta_1).$$  \hspace{1cm} (3.14)$$

Since Eq. (3.13) is symmetrical in $t_1$ and $t_2$, it is possible to perform the integration over $t_1$ and $t_2$ independently from 0 to $\infty$ by introducing a factor of $1/2$. In this way one obtains exactly the same result of the reducible vacuum bubble of the $\Phi^3$ field theory defined by Eq. (2.3).

In the second case, $\eta_1 \rightarrow 0$, it is more convenient to introduce

$$q_1 = \frac{k_2}{\eta_1}, \quad q_2 = \frac{k_1}{\eta_1}, \quad q_3 = \eta_1,$$

so that Eq. (3.11) becomes

$$A^{(2)}_0 \Big|_{\text{irr}} = \frac{N^3}{(4\pi)^d} \frac{g^2}{2^8} (2\alpha')^{3-d} \int_0^\epsilon \frac{dq_3}{q_3^3} \int_0^{q_3} \frac{dq_2}{q_2^3} \int_0^{q_2} \frac{dq_1}{q_1^3}$$

$$\times \left[ \frac{1}{4} (\log q_1 \log q_2 + \log q_1 \log q_3 + \log q_2 \log q_3) \right]^{-d/2}$$

$$= \frac{N^3}{(4\pi)^d} \frac{g^2}{2^5} \int_0^\infty dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 e^{-m^2(t_1+t_2+t_3)}$$

$$\times (t_1 t_2 + t_1 t_3 + t_2 t_3)^{-d/2}. \hspace{1cm} (3.16)$$

Here, again, we have introduced the Schwinger proper times $t_i$ related to $q_i$ variables as in Eq. (3.14). Since Eq. (3.16) is completely symmetrical, we can introduce a factor of $1/3!$ and perform the three integrals independently from 0 to $\infty$. In this way we correctly reproduce the irreducible vacuum bubble of the $\Phi^3$ theory. Note that by using a single starting formula, namely Eq. (3.11), we have been able to obtain two diagrams which have a different weight; this has been possible because Eq. (3.11) has different symmetry properties in the two regions of the moduli space that yield the two vacuum bubbles of the $\Phi^3$ field theory.

We now turn to the one–point amplitude ($M = 1$). In this case, momentum conservation prevents us from imposing the mass shell condition, so that the final result will depend on the choice of the local coordinates $V'_i(0)$ around the puncture $z_1$. This issue requires a thorough discussion which we leave to a future publication. Here we simply propose a generalization of the one–loop choice ($V'_i(0) = z_i$), that is strongly inspired by Ref. [7] and that leads to consistent results. We propose to define

$$(V'_i(0))^{-1} = \left| \frac{1}{z_i - \rho_a} - \frac{1}{z_i - \rho_b} \right|,$$

where $\rho_a$ and $\rho_b$ depend on the position of $z_i$ and are the two fixed points that stay on the left and on the right of $z_i$. For example, referring to Fig. 1, if $z_i$ is between $D$ and $D'$ we have $\rho_a = \eta_1$ and $\rho_b = \xi_1 = 1$, while if $z_i$ is between $C'$ and $B'$ we have $\rho_a = \xi_1 = 1$ and $\rho_b = \xi_2 = \infty$. Note that at one loop in the standard configuration
we have \( \rho_a = \eta = 0 \) and \( \rho_b = \xi = \infty \), and thus, in this case, Eq. (3.17) reduces to \( V_\ell'(0) = z_i \). We remark that the local coordinates in Eq. (3.17) are chosen at the string level before taking the field theory limit, and as such, they have a general validity. However, since different regions of the moduli space (i.e. different values of \( \rho_a \) and \( \rho_b \)) usually correspond to different Feynman diagrams in the field theory limit, the explicit expression of \( V_\ell'(0) \) may look different for different diagrams, precisely like in the analysis of Ref. [7].

Using the local coordinates (3.17), and the integration region of the moduli space determined from the study of the vacuum bubbles, we can compute the two–loop one–point amplitude explicitly starting from Eq. (3.10) with \( M = 1 \) and \( \rho_c = \xi_1 = 1 \). The result of this calculation, which we will describe in detail elsewhere, is

\[
A_1^{(2)} = -N^2 \text{Tr}(\lambda^{a_1}) \frac{1}{(4\pi)^d} \frac{g^3}{2^{10}} (2\alpha')^{4-d} \int_0^1 \frac{dq_3}{q_3^2} \int_0^{q_3} \frac{dq_2}{q_2^2} \int_0^{q_2} \frac{dq_1}{q_1^2} \left[ \frac{1}{4} \left( \log q_1 \log q_2 + \log q_1 \log q_3 + \log q_2 \log q_3 \right) \right]^{-d/2} \times \left[ \log q_1 + \log q_2 + \log q_3 + \log(1 - q_3) \right],
\]

(3.18)

where \( q_1, q_2 \) and \( q_3 \) are defined in Eq. (3.15). From Eq. (3.18) one can derive the irreducible two–loop one–point function of the \( \Phi^3 \) field theory when \( q_3 \to q_2 \to q_1 \to 0 \), while when \( q_3 \to 1 \) and \( q_2 \to q_1 \to 0 \) one obtains the two reducible diagrams.

We have explicitly verified that this method gives the correct results also when there are two external states. However, since in this case there are many corners of the string moduli space that contribute to the same field theory diagram, the analysis is a bit lengthy, and so we leave it to a forthcoming publication. Here instead, we propose a different procedure inspired by Ref. [14] that also leads to the correct identification of the field theory diagrams starting from Eq. (3.10).

Let us start from the two-loop diagram depicted in Fig. 2.

![Fig. 2: A reducible two–loop diagram in the \( \Phi^3 \) field theory contributing to the two–point function.](image)

First we cut open the two loops of the diagram, and subdivide the resulting tree diagram so that it can be obtained by sewing three-point vertices. Next we fix the legs of each three–point vertex at 0, 1 and \( \infty \), as depicted in Fig. 3.
Then, we reconnect the diagram by inserting between vertices a suitable propagator acting on the external legs as a well–specified projective transformation. This transformation is chosen in such a way that its fixed points are the Koba-Nielsen variables of the two legs that are sewn. For example, if we sew two legs corresponding to the points 0 and ∞, we use the transformation

\[ S(z) = Az, \]  

(3.19)

whereas, if we sew two legs corresponding to 1 and ∞, we use

\[ S(z) = Az + 1 - A. \]  

(3.20)

Finally, we sew together the legs corresponding to \( \xi_1 \) and \( \eta_1 \), and to \( \xi_2 \) and \( \eta_2 \) by means of two projective transformations with multipliers \( k_1 \) and \( k_2 \) respectively. In this way we recover the full two–loop diagram.

After the sewing has been performed, both the Koba-Nielsen variables and the Schottky fixed points become functions of the parameters \( A_i \) that appear in the various projective transformations. For example, for the particular sewing configuration depicted in Fig. 3, we obtain

\[ \xi_1 = A_1, \quad z_2 = A_1(1 - A_2 A_3), \quad \eta_1 = A_1(1 - A_2), \quad z_1 = 1 \]  

(3.21)

together with \( \xi_2 = \infty \) and \( \eta_2 = 0 \).

The parameters \( A_i \) have a simple geometric interpretation, and drive the field theory limit. In fact, as described in Ref. [14], they are related to the length of the
strip connecting two three–point vertices. In the limit \( \alpha' \to 0 \), this length is given by

\[
t_i = -\alpha' \log A_i , \quad i = 1, 2, 3 ,
\]

while the lengths of the two entire loops are similarly related to the multipliers \( k_1 \) and \( k_2 \). The field theory limit is thus the limit \( A_i \to 0 \) and \( k_\mu \to 0 \), with a definite ordering prescribed by the sewing procedure. Notice that, although the result of this procedure does not depend on how a given diagram is cut, for the explicit calculations it is always convenient to choose a particular sewing configuration, like for example the one of Fig. 3 corresponding to the diagram of Fig. 2. Then, the limit \( A_i \ll 1 \) implies that the original variables of the diagram in Fig. 2 must be ordered as follows

\[
\xi_2 = \infty \gg z_1 = 1 \gg \xi_1 \gg z_2 \gg \eta_1 \gg \eta_2 = 0 .
\]

The proper times associated to individual propagators are given by Eq. (3.22), and by

\[
t_4 = -\alpha' \log \frac{k_2}{A_1} , \quad t_5 = -\alpha' \log \frac{k_1}{A_3} .
\]

We now use Eq. (3.6) to eliminate the double poles in the measure of integration which, in terms of the proper times \( t_i \), becomes

\[
[dm]^2 \to -2^5 (2\alpha')^{d-5} \prod_{i=1}^{5} dt_i \left[ (t_3 + t_5)(t_1 + t_4) \right]^{-d/2} e^{-m^2(t_5+t_4)} \left[ V'_1(0)V'_2(0) \right]^{\alpha'm^2}.
\]

On the other hand, in the region Eq. (3.23), the Green function \( G^{(2)} \) simplifies according to

\[
G^{(2)} \to -\frac{1}{2\alpha'} \left[ \frac{t_1^2}{t_1 + t_4} + \frac{t_3^2}{t_3 + t_5} \right] .
\]

If we insert Eqs. (3.25) and (3.26) into Eq. (3.10), with \( M = 2 \) and \( \rho_c = z_1 = 1 \), we see that all factors of \( \alpha' \) cancel, so that we are left with

\[
A_2^{(2)}(p_1, p_2) \to N^2 \Tr(\lambda^{a_1} \lambda^{a_2}) \frac{1}{(4\pi)^d} \frac{g^4}{2^9} \int_0^\infty dt_2 \, e^{-t_2(\rho^2+m^2)}
\]

\[
\times \prod_{i \neq 2} \int_0^\infty dt_i \, e^{-m^2(t_1+t_3+t_4+t_5)} \left[ (t_3 + t_5)(t_1 + t_4) \right]^{-d/2}
\]

\[
\times \exp \left\{ -\rho^2 \left[ \frac{t_1 t_4}{t_1 + t_4} + \frac{t_3 t_5}{t_3 + t_5} \right] \right\}
\]

\[
\times \exp \left\{ (\rho^2 + m^2) \left[ t_1 + t_2 + t_3 + \alpha' \log (V'_1(0)V'_2(0)) \right] \right\} ,
\]

where, by momentum conservation, \( p_1 = -p_2 \equiv p \).

One can easily check that Eq. (3.27) correctly reproduces the field theory diagram in Fig. 2, including the overall factor. In particular, if we put the external
legs on the mass shell, the dependence on $V'_i(0)$ disappears like in the full string amplitude. On the other hand, if we use the coordinates $V'_i(0)$ of Eq. (3.17), we can reproduce the correct field theory result also off the mass–shell.

Let us consider now the 1PI diagram in Fig. 4.

Fig. 4: An irreducible two-loop diagram contributing to the two–point function of the $\Phi^4$ theory.

Applying our procedure, we first open the two loops, and then we sew the corresponding tree diagrams in the configuration shown in Fig. 5.

Fig. 5: The sewing configuration of three-string vertices corresponding to the diagram of Fig. 4.

Following the same steps as in the previous case, we obtain

$$z_1 = 1 - A_1 \ , \quad z_2 = A_2 \ , \quad \eta_1 = A_2 A_3 \ , \quad \xi_1 = 1 \ , \quad \xi_2 = \infty$$

(3.28)

together with $\eta_2 = 0$ and $\xi_2 = \infty$. Since the variables $A_i$ must be constrained to be very close to 0, the variables of the diagram of Fig. 4 are ordered as follows

$$\xi_2 = \infty \gg \xi_1 = 1 \gg z_1 \gg z_2 \gg \eta_1 \gg \eta_2 = 0$$

(3.29)

The proper times are given again by Eq. (3.22), together with

$$t_4 = -\alpha' \log \frac{k_2}{A_2 A_3} \ , \quad t_5 = -\alpha' \log \frac{k_1}{A_1 A_2 A_3}$$

(3.30)
In terms of these variables, the measure becomes

\[ [dm]^2 \rightarrow 2^5(2\alpha')^{d-5} \prod_{i=1}^{5} dt_i \ e^{-m^2(t_3+t_4+t_5)} \prod_{i=1}^{2} \ e^{\alpha'm^2\log V'_i(0)} \times \left[ (t_2 + t_3 + t_4)(t_1 + t_2 + t_3 + t_5) - (t_2 + t_3)^2 \right]^{-d/2}, \]

while the two–loop Green function in the region (3.29) simplifies according to

\[ G^{(2)} \rightarrow \frac{1}{2\alpha'} (t_2 + t_3 + t_4)(t_1 + t_2 + t_3 + t_5) - (t_2 + t_3)^2 \times \left[ 2(t_1 + t_2)(t_2 + t_3) t_2 - (t_2 + t_3 + t_4)(t_1 + t_2)^2 - t_2^2(t_1 + t_2 + t_3 + t_5) \right]. \]

Therefore, inserting Eqs. (3.31) and (3.32) into Eq. (3.10), with \( M = 2 \) and \( \rho_c = \xi_1 = 1 \), we get

\[ A_2^{(2)}(p_1, p_2) \rightarrow N^2 \text{Tr} (\lambda^a \lambda^{a_2}) \frac{1}{(4\pi)^d} \frac{g^4}{2^9} \int_0^\infty \prod_{i=1}^{5} dt_i \ e^{-m^2 \sum_{i=1}^{5} t_i} \times \left[ (t_2 + t_3 + t_4)(t_1 + t_2 + t_3 + t_5) - (t_2 + t_3)^2 \right]^{-d/2} \times \exp \left\{ -p^2 \left[ 2(t_1 + t_2)(t_2 + t_3) t_2 - (t_2 + t_3 + t_4)(t_1 + t_2)^2 - t_2^2(t_1 + t_2 + t_3 + t_5) \right] \right. \]

\[ \left. \left( t_2 + t_3 + t_4)(t_1 + t_2 + t_3 + t_5) - (t_2 + t_3)^2 \right) + (t_1 + t_2) \right\} + (p^2 + m^2) \left[ t_1 + t_2 + \alpha' \log (V'_1(0)V'_2(0)) \right]. \]

Once again, the field theory diagram in Fig. 4 is correctly reproduced, as one can easily check. If the external legs are kept on the mass–shell, the dependence on \( V'_i(0) \) drops out; on the other hand, if they are kept off shell, the correct result is reproduced by choosing again the \( V'_i(0) \)'s of Eq. (3.17). We have also checked that this procedure allows us to recover correctly all other two–loop diagrams with two external legs.

The analysis presented in this paper shows that the open bosonic string can be used to compute explicitly the amplitudes of the \( \Phi^3 \) field theory including their overall normalization. These results are suited for several generalizations, for example by increasing the number of external legs, or by increasing the number of loops. Furthermore, if, in the zero–slope limit, we select the contributions of the vectors instead of those of the scalars, our methods should allow to recover explicitly the multiloop amplitudes of Yang–Mills theory from the open bosonic string.

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