Spin-dependent transport through hybrid structures combining ferromagnets (F) and normal metals has attracted a lot of interest in the recent years. This interest is motivated by the prospect of potential technological applications in the field of spintronics [1]. Particular attention is given to two related effects involving mutual influence between the electric current through a structure and its magnetic configuration. The first is giant magnetoresistance [2] in which the conductance is much larger when different magnetic regions have their magnetic moments aligned than when they are anti-aligned. The opposite effect is the appearance of torques acting on magnetic moments when an electric current flows through the system [3]. These non-equilibrium current-induced torques appear due to non-conservation of spin currents accompanying a flow of charge through ferromagnetic regions. They allow manipulation of the magnetic configuration, including switching between the opposite directions or steady-state precession, without application of magnetic fields [4]. The two effects combined promise important practical applications in nonvolatile memory, programmable logic, and microwave oscillators.

When the multilayer is coupled to a superconductor (S), an additional constraint is added, viz. that the spin current through the superconducting part vanishes [3]. This modifies the non-equilibrium torques, opening the possibility of perpendicular alignment of magnetic moments. A very different situation arises when a magnetic structure is contacted by two superconductors. In this case, the proximity effect may be present, leading to a finite Josephson current through the structure at equilibrium. The torques generated by this current correspond to an equilibrium effective exchange interaction between the magnetic moments which can be controlled by the phase difference between the superconductors [6]. The same mechanism enables the reciprocal effect in which the supercurrent depends on the magnetic configuration.

Naive considerations might suggest that the proximity effect should be suppressed at short distances in the presence of ferromagnets. However, recently it was shown that a long-range effect can exist due to triplet superconducting correlations [7]. This triplet proximity effect (TPE), and in particular, the associated Josephson current, depend essentially on the magnetic configuration of the system [8]. Hence S/F multilayers exhibiting TPE are especially suitable for studying the Josephson-induced magnetic exchange interaction.

By varying the relative magnetization directions of different magnetic regions, one can control the supercurrent flowing through the structure. Then, if the magnetic configuration is allowed to respond to the Josephson-current induced torques, it creates feedback for the supercurrent and considerably modifies it. We show that its main signature is frequency doubling in the current-phase relation.

Below, we consider this feedback for a Josephson junction biased by a dc voltage. In the AC Josephson effect, the time dependence is normally determined by the current-phase relation. TPE in diffusive systems usually leads to the conventional $J = J_c \sin \phi$ relation, except for a special magnetic configuration with mutually perpendicular directions where a transition between “0” and “$\pi$” states occurs [8]. Then the first harmonic vanishes and the current is given by the second harmonic $\sim \sin 2\phi$; however, its amplitude is relatively small. In general, Josephson junctions exhibiting double-periodic behavior, besides being interesting objects in proximity-effect studies, may be useful in flux-qubit design schemes [1]. Josephson frequency doubling was predicted in other types of junctions involving unconventional superconductors, such as s-p [10], s-d-s [9], p-p and d-d junctions with specific misorientation angles of the order parameter [11]. It was observed in experiments involving d-wave grain-boundary junctions [12]. It should be stressed, however, that in all these cases the frequency doubling occurs at isolated points in the parameter space where the first harmonic vanishes. Moreover, the magnitude of the current is suppressed in comparison with the usual value $\sim \Delta/eR_n$, where $R_n$ is the normal-state resistance.

In this work, we consider the magnetic exchange interaction induced by Josephson currents in a dirty S/F heterostructure exhibiting TPE. We show that this interaction may prefer non-collinear magnetic configurations and the preferred direction depends continuously on the superconducting phase difference. Thus, the static magnetic configuration can be controlled by the applied phase...
difference. We then consider the influence of feedback from the magnetic moments on the AC Josephson effect. The magnetic system exhibits a range of different behaviors, from simple harmonic oscillations to fractional-frequency periodic behavior and chaotic motion. A finite zero-frequency deviation from the equilibrium configuration may appear, allowing control of the direction of the average magnetization also by an applied voltage. The magnetic feedback complicates the behavior of the current in the time domain, making it generally impossible to express it in terms of a current-phase relation. On the other hand, we find that both in the low- and high frequency limit such a relation becomes meaningful, with the current exhibiting a double-phase dependence, $J \sim \sin 2\phi(t)$ or $J \sim \cos 2\phi(t)$. The critical current in the low-frequency regime is of the order of the value $E_{Th}/eR_n$, characteristic for diffusive systems. The unusual cosine dependence of the Josephson current appears when Gilbert damping is important in the magnetic dynamics, breaking the time-reversal symmetry. At high frequencies, the magnetization cannot effectively follow the phase variation, leading to a $\sim 1/\omega^2$ suppression of the effective Josephson coupling. At even higher frequencies, the damping is dominant, and the frequency dependence becomes $\sim 1/\omega$. The presence of damping is expressed in the appearance of a dc component of the current leading to a finite resistance.

The system. We consider an S/F heterostructure described in Fig. 1 which is a minimal discrete setup exhibiting the triplet proximity effect. Two magnetic regions 1 and 3 are adjacent to the superconducting reservoirs that induce proximity mini-gaps $\Delta_{1,3}$ in them. Between these regions there is an additional magnetic region 2 whose length is much larger than $\xi_h$ and where triplet superconducting correlations are induced. This region is assumed to be weakly polarized (metallic), so that both spin directions are present at the Fermi surface. The magnetic regions are characterized by the exchange energies $h_i$, while the magnetization directions $n_i$ are specified by the angles $\theta_1$, $\theta_3$ and $\chi$ as shown in Fig. 1. Assuming that the conductances of these regions are much higher than the conductances $g_{i,3}$ of the connectors between them, our system can be described by a circuit-theory model for the triplet proximity effect used in Ref. [8]. Magnetization directions of regions 1 and 2 are assumed fixed, e.g. by pinning to an antiferromagnetic substrate, or by geometrical shaping, with the angle between them being $\theta_1$. On the other hand, magnetization $n_3$ is free to rotate, with region 3 separated by a normal spacer from region 2 in order to avoid exchange coupling between them.

In accordance with the model assumptions, regions 1 and 3 act as effective S-F reservoirs, hence their energies are independent of the magnetic configuration. On the other hand, triplet superconducting correlations extending through region 2, are very sensitive to the magnetization directions. Hence the configuration-dependent part of the energy can be found by integrating over the density of states (DOS) in region 2. The DOS for each spin direction is given by

$$\nu_{1,1}(\varepsilon) = \frac{\nu_0}{2} \text{Re} \left( 1 - \frac{a_1^2 + a_2^2 + 2a_1a_3 \cos(\phi \pm \chi)}{(b_1 + b_3 - i\epsilon/E_{Th})^2} \right)^{\frac{1}{2}},$$

where $\nu_0$ is the normal-state DOS, $a_k = g_k^1|\Delta_k|\sin\theta_k/(g_k^1 + g_k^3)\sqrt{h_k - |\Delta_k|^2}$, $b_k = g_k^2h_k/(g_k^1 + g_k^3)\sqrt{h_k - |\Delta_k|^2}$, $\phi$ is the superconducting phase difference, and $E_{Th}$ is the Thouless energy of the structure. Using this expression, one can see that the energy is given by a logarithmic integral and the main contribution comes from $\epsilon \gg E_{Th}$. In the leading order one obtains

$$E = \frac{\nu_0\nu_2}{2} \log \frac{\Delta_{cut}}{E_{Th}} \left( a_1^2 + a_3^2 + 2a_1a_3 \cos\phi \cos\chi \right),$$

where $\nu_2$ is the volume of the magnetic region 2 and $\Delta_{cut} \approx \min(\Delta_i, h_i - \Delta_i)$ is a cutoff energy. This expression can be written in a form presenting explicitly the dependence on the orientation angles $\theta_3$ and $\chi$,

$$E = p_1^3 \sin^2\theta_3 + 2p_1p_3 \sin\theta_3 \cos\phi \cos\chi,$$

with $p_{1,3}$ being effective exchange couplings for the magnetic vector $n_3$. The stable configuration is achieved when all magnetization directions are in the same plane, denoted in the following as the $x - z$ plane, and $n_3$ is tilted with respect to $n_2$ by a finite angle satisfying

$$\sin\theta_3 = \frac{|p_1|}{p_3} \cos\phi.$$

This angle depends continuously on the applied superconducting phase difference $\phi$, while the angle $\chi$ assumes the values 0 or $\pi$ so that the product $\cos\phi \cos\chi$ is negative. In fact, there are two stable directions, given by the angles $\theta_3$ and $\pi - \theta_3$. In what follows we will treat them
as equivalent, since they correspond to the same current. 

The energy of the stable configuration is given by

\[ E_{\text{min}} = -\gamma_3^2 \cos^2 \phi. \] (5)

Hence allowing the magnetization direction \( \mathbf{n}_3 \) to orient itself along the stable direction leads to the current-phase relation \( J = J_c \sin 2\phi \).

**Low frequencies.** When a small voltage \( V \) is applied to the structure, such that the corresponding frequency \( \omega_J = 2eV/\hbar \) is much smaller than the characteristic frequency of the magnetic system \( \omega_m \) (see below), the vector \( \mathbf{n}_3 \) follows the stable direction given by Eq. (4), performing slow oscillations in the \( x-z \) plane. The alternating Josephson current oscillates with the double frequency

\[ J = \frac{2e}{\hbar} p_1^3 \sin \frac{4eV}{\hbar} t, \] (6)

while the critical current remains of the same order of magnitude as in the case with a fixed magnetic configuration.

For higher Josephson frequencies, the variation of \( \mathbf{n}_3 \) is no more limited to the \( x-z \) plane. Instead, the magnetization performs a variety of non-harmonic motions whose frequency may be a multiple or a fraction of the driving frequency \( \omega_J \) [Fig. 2 (a), (c), (d)]. For certain trajectories the time average of \( \theta_3 \) is finite [Fig. 2 (c)], corresponding to a tilt of \( \mathbf{n}_3 \) away from the equilibrium in response to an applied voltage. Within some frequency intervals, the motion is chaotic, as shown in Fig. 2 (b). In these intermediate regimes, the Josephson current shows a complicated time dependence which is generally not periodic in \( 2\pi/\omega_J \). Hence this dependence cannot be parameterized in terms of the phase. Instead, one can speak of a Josephson current with a time-dependent coupling.

**High frequencies.** When applied voltage is high, the Josephson frequency becomes much higher than the magnetic frequencies. In this case the magnetic vector \( \mathbf{n}_3 \) cannot effectively follow the fast oscillations of the potential, and the time-averaged potential seen by \( \mathbf{n}_3 \) has a minimum for \( \mathbf{n}_3 \parallel z \). The motion of \( \mathbf{n}_3 \) can be determined by expanding \( \mathbf{n}_3 = z + \delta \mathbf{n} \) and using a linearized Landau-Lifshits-Gilbert (LLG) equation,

\[ \dot{\delta \mathbf{n}} = z (-\gamma \times H_{\text{eff}} + \alpha \delta \mathbf{n}), \] (7)

where \( \gamma \) is the gyromagnetic ratio, \( \alpha \) is the effective damping coefficient, \( H_{\text{eff}} = -\partial E/\partial \mathbf{m} \) is the effective field, and \( \mathbf{m} \) is the magnetization density of magnetic region 3.

When Gilbert damping is negligible, the trajectory of \( \mathbf{n}_3 \) has a very low aspect ratio, so that the motion is almost completely confined to the \( y \) axis. It is given by

\[ \delta n_x = \frac{\gamma_3^2 p_1^2 \hbar^2}{e^2 V^2 m_3} \cos \frac{2eVt}{\hbar} ; \]
\[ \delta n_y = \frac{\gamma_3^2 p_1 \hbar}{e V m_3} \sin \frac{2eVt}{\hbar}. \] (8)

Thus at high frequencies, \( \mathbf{n}_3 \) precesses in phase with the voltage pumping. This leads to an increase in the Josephson energy, and, correspondingly, a negative Josephson current,

\[ J = -\frac{2h}{e} \left( \frac{\gamma_3^2 p_1^3}{V m_3} \right)^2 \sin \frac{4eVt}{\hbar}. \] (9)

Hence in the high-frequency regime the system shows not only frequency doubling, but also an effective \( \pi \)-junction behavior. The magnitude of the current is suppressed as \( \sim V^{-2} \) as shown in Fig. 3.

The neglect of damping is justified as long as \( \alpha \omega_J \ll \omega_m = \gamma_3^2 p_1^3/m_3 \). When the voltage is high enough, this condition is not satisfied anymore, and the dissipation starts to be important. As the Josephson frequency becomes so large that the opposite inequality holds, the motion of \( \mathbf{n}_3 \) is determined by the driving against the damping force,

\[ \delta \mathbf{n} = \frac{\gamma_3 p_1 \hbar}{e V m_3} (-\alpha \hat{x} + \hat{y}) \sin \frac{2eVt}{\hbar}. \] (10)

Then the Josephson current is given by

\[ J = \frac{2\alpha \gamma_3^2 p_1^3}{m_3 V} \left( 1 - \cos \frac{4eVt}{\hbar} \right). \] (11)

Note the unusual cosine dependence on the phase. It occurs since the time-reversal symmetry is broken by the...
dissipation in this regime. Due to the same reason, a zero-frequency component of the current appears, signifying the onset of a finite nonlinear resistance across the structure. Since this regime is governed by the damping, the current amplitude is proportional to $\alpha$, while the suppression $\sim 1/V$ is weaker in this regime (Fig. 3).

To estimate the magnetic dynamics frequency $\omega_m$, we use typical values $E_{Th} \sim 1$ meV, $\nu_0 \sim 1/(eV/\text{atom})$, and $m_3 \sim \mu_B/\text{atom}$, where $\mu_B$ is the Bohr magneton. Then $\omega_m \sim \nu_2/\nu_3$ GHz, where $\nu_{2,3}$ are the volumes of the corresponding magnetic regions. As this frequency is quite low, observation of the high-frequency regimes should present no difficulty. On the other hand, the low-frequency AC regime would require extremely low voltages, below 1 $\mu$V. A reasonable alternative would be incorporating the structure in a superconducting loop and measuring the Josephson current as a function of the applied flux.

Applicability of our model requires that any magnetic anisotropy of part 3 should be smaller than the proximity-induced energy, Eq. (2). With the above values of the parameters it is of the order of $10^4 \times \nu_2 J/\text{m}^3$, so one should choose materials with low value of the crystalline anisotropy, such as permalloy. Finally, we emphasize that the properties discussed above are specific for metallic systems. In half-metals, the behavior will be very different. Thus, in the low-frequency regime $n_3$ precesses around $n_2$ at a constant angle $\theta_3$, while the Josephson current vanishes.

**Conclusions.** We have considered the AC Josephson effect in a S/F/S structure with magnetic dynamics coupled to the dynamics of superconducting correlations. The magnetic configuration in the structure was assumed to be non-uniform so that the structure exhibits a triplet proximity effect. Variation of the magnetic configuration is shown to essentially modify the current behavior that can be observed in the appearance of fractional Shapiro steps. Thus measurement of the Josephson current would provide information about the coupling and self-consistent feedback dynamics between the superconducting and magnetic degrees of freedom. The coupling also allows to control the magnetization direction by means of applied voltage or superconducting phase. In the low-frequency limit, the magnetization follows the immediate potential minimum, leading to a $\sim \sin 2\phi$ current-phase relation. The critical current has the same order of magnitude $E_{Th}/eR_m$ as that due to the usual singlet proximity effect in dirty structures. In the high-frequency regime, as long as the damping is not important, the Josephson current is negative, corresponding to a $\pi$-junction behavior. It is suppressed by a factor $\sim (\omega_m/\omega_f)^2$ relative to the low-frequency regime. At even higher frequencies, Gilbert damping starts playing the major role in the dynamics. Then the time-reversal symmetry is broken and the current-phase relation takes an unusual cosine form. In addition, a DC component of the current appears, manifesting itself in a finite resistance. The current suppression becomes weaker in this regime.

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