EXEMPLARY TRICHOTOMY AND \((r,p)\)-ADMISSIBILITY
FOR DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT. The aim of this paper is to present a new and very general method for the study of the uniform exponential trichotomy of nonautonomous dynamical systems defined on the whole axis. We consider a discrete dynamical system and we introduce the property of \((r,p)\)-admissibility relative to an associated control system, where \(r, p \in [1, \infty]\). In several constructive steps, we obtain full descriptions of the sufficient conditions and respectively of the necessary criteria for uniform exponential trichotomy based on the \((r,p)\)-admissibility of the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\). In the same time, we provide a complete diagram of the \(\ell^p\)-spaces which can be considered in the admissible pairs for the study of the uniform exponential trichotomy of discrete dynamical systems. We present illustrative examples in order to motivate the hypotheses and the generality of our method. Finally, we apply the main results to obtain new criteria for uniform exponential trichotomy of dynamical systems modeled by evolution families using the admissibility of various pairs of \(\ell^p\)-spaces.

1. Introduction. One of the most spectacular class of methods for the study of the asymptotic properties of dynamical systems is represented by the admissibility techniques, which allow the characterization of an asymptotic behavior by means of the solvability of an associated control system between two well-chosen spaces. Although these methods have a long history that started several decades ago with the pioneering work of Perron [12] and were consolidated in the sixties by the remarkable contributions of Massera and Schäffer [5] and Coffman and Schäffer [1], in recent years this class of techniques has had an increasing impact on the development of the control type methods in the asymptotic theory of dynamical systems (see Elaydi and Hájek [2], Elaydi and Janglajew [4], Minh [6], Minh and Huy [7], Palmer [10], Pliss and Sell [13], Sasu and Sasu [18]–[25] and the references therein).

It should be noted that in this context, the progress concerning the study of the exponential dichotomy was notable and a large variety of open problems were completely clarified (see Minh [6], Minh and Huy [7], Palmer [9], [10], Pliss and Sell [13], Sasu and Sasu [18], [23], Sasu [19], Sasu [21] and the references therein). For all that, the passing from exponential dichotomy to exponential trichotomy was difficult and represented a new challenge. The notions of exponential trichotomy were introduced in the remarkable works of Sacker and Sell [17] and respectively

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of Elaydi and Hájek [2], [3] and these works have marked the beginning of a new and very interesting topic in the asymptotic theory of dynamical systems. Even if the studies devoted to dichotomy have covered many open problems and the theory was extremely advanced (see Minh [6], Palmer [9], [11], Pötzsche [14]–[16], Sasu and Sasu [23] and the references therein), the methods that had to be considered in order to investigate the exponential trichotomy have proved to be more complicated and new approaches have had to be gradually implemented (see Elaydi and Hájek [2], [3], Elaydi and Janglajew [4], Minh and Wu [8], Palmer [11], Pötzsche [15], Sasu and Sasu [20], [22], [24], [25]).

The first studies were devoted to dynamical systems which admit an exponential trichotomy (see Elaydi and Hájek [2], [3], Elaydi and Janglajew [4], Minh and Wu [8], Palmer [11]). Important works were devoted to robustness of the exponential trichotomy subjected to various perturbations (see Elaydi and Hájek [2], Elaydi and Janglajew [4]). A remarkable step was done by Palmer in [11], where the author has shown that the hyperbolicity of a flow can be expressed in terms of trichotomies and in this context the author used the shadowing theorem to deduce Silnikov’s theorem. Notable results were obtained by Minh and Wu in [8], where the authors proved the existence of center-unstable and center manifolds for nonlinear processes that admit an exponential trichotomy.

The control type methods for exponential trichotomy were developed in progressive stages (see Elaydi and Hájek [2], Elaydi and Janglajew [4], Sasu and Sasu [20], [22], [24], [25] and the references therein). The main difficulty in the study of the exponential trichotomy of a dynamical system, by means of control techniques, has relied in identifying adequate solvability criteria for associated control systems in order to ensure on the one hand the existence of the trichotomy projections, that realize the trichotomic splitting of the state space into stable, unstable and bounded subspace at every moment, and, on the other hand, to deduce the behavior on the ranges of projections and respectively the compatibility properties between projections and coefficients (see Definition 2.1 below). The first input-output conditions for uniform exponential trichotomy of evolution families defined on the whole axis were obtained in [20] using an admissibility property relative to an associated integral equation. After that we have obtained an admissibility criterion for uniform exponential trichotomy of difference equations in infinite-dimensional spaces using input sequences with finite support (see [22]). The continuous variational case was treated in [24] where we have proposed a general method for the detection of the uniform exponential trichotomy of skew-product flows, using an admissibility of a pair of function spaces. A major step in the case of nonautonomous dynamical systems was made in [24], where we have presented a new approach based on a distinct admissibility condition using $\ell^p$-spaces. In [24], using admissibility arguments, we have proved that all the trichotomic properties of a nonautonomous system can be completely recovered from the trichotomic behavior of the associated discrete dynamical system. This result has considerably increased the applicability spectrum of the discrete admissibility criteria for general dynamical systems.

The aim of this paper is to extend the framework and to propose a general admissibility method which on the one side offers a new perspective on the control type techniques in the study of the exponential trichotomy and on the other side includes as particular cases all the previous results in this topic. We consider a discrete dynamical system with arbitrary coefficients, we associate to it an input-output system with input sequences in $\ell^1(\mathbb{Z}, X)$ and output sequences in $\ell^\infty(\mathbb{Z}, X)$.
and we define a new and general concept called \((r,p)\)-admissibility, with \(r, p \in [1, \infty]\).

First we establish the connections between the \((r,p)\)-admissibility and the admissibility properties previously used in the study of the uniform exponential trichotomy. After that, we will prove that if \((r,p) \neq (\infty, 1)\), then the \((r,p)\)-admissibility of the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) is a sufficient condition for the existence of the uniform exponential trichotomy.

In what follows, the natural question arises whether the property of \((r,p)\)-admissibility is also necessary for uniform exponential trichotomy. Another interesting question is which are the minimal requirements that should be imposed to the \(\ell^p\)-spaces considered in the admissible pair, such that the admissibility implies the existence of the exponential trichotomy and respectively which are the minimal hypotheses that assure the equivalence between \((r,p)\)-admissibility and uniform exponential trichotomy. The purpose of the third section of the paper is to answer these questions. First, we show that the uniform exponential trichotomy implies the \((r,p)\)-admissibility if \(r \geq p\). After that, we obtain new characterizations for uniform exponential trichotomy of discrete dynamical systems in terms of the \((r,p)\)-admissibility of the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\). Next, we will establish a complete diagram regarding the eligible \(\ell^p\)-spaces which can be chosen in the admissibility pair for the study of the uniform exponential trichotomy. We will present illustrative examples in order to motivate the method, to clarify the optimal hypotheses and to emphasize the generality degree of this new approach and the impact to further applications.

Finally we present a new application for the case of nonautonomous dynamical systems modeled by evolution families, providing new and general criteria for the study of the uniform exponential trichotomy by means of the \((r,p)\)-admissibility relative to an associated discrete control system. Thus, this new method extends all the previous approaches and provides a complete description of the criteria for uniform exponential trichotomy which use \(\ell^p\)-spaces as input and output spaces in the admissible pair.

2. The property of \((r,p)\) - admissibility and sufficient criteria for uniform exponential trichotomy. In this section we will introduce a new and very general concept of admissibility and we will establish the connections with the previous admissibility properties used in the study of the trichotomy. Using this new admissibility criteria we will prove the existence of the uniform exponential trichotomy.

Throughout this paper we maintain the notations and the basic notions introduced in [24]. For the sake of clarity we start by recalling them briefly.

Let \(X\) be a real or complex Banach space. We denote by \(I_d\) the identity operator on \(X\) and by \(B(X)\) the space of all bounded linear operators on \(X\). The norm on \(X\) and on \(B(X)\) will be denoted by \(\| \cdot \|\).

Let \(Z\) be the set of real integers, \(\mathbb{N} := \{ n \in \mathbb{Z} : n \geq 0 \}\) and \(\mathbb{Z}_- := \{ n \in \mathbb{Z} : n \leq 0 \}\). We consider \(\Delta := \{ (m,n) \in \mathbb{Z} \times \mathbb{Z} : m \geq n \}\). If \(E \subset \mathbb{Z}\), then we denote by \(E^* = E \setminus \{0\}\) and respectively by \(\chi_E\) the characteristic function of the set \(E\).

Let \(\mathcal{S}(Z,X)\) denote the linear space of all sequences \(s : Z \rightarrow X\). As in [24] we shall work with the following spaces:

(i) \(\ell^\infty(Z,X)\) - the space of all bounded sequences \(s \in \mathcal{S}(Z,X), c_0(Z,X)\) - the space of all sequences \(s \in \mathcal{S}(Z,X)\) with \(\lim_{k \to \pm \infty} s(k) = 0\), \(c_0,\infty(Z,X) := \{ s \in \ell^\infty(Z,X) : \)
\[ \lim_{k \to -\infty} s(k) = 0 \] and \( \ell_{\infty}(\mathbb{Z}) := \{ s \in \ell_{\infty}(\mathbb{Z}) : \lim_{k \to -\infty} s(k) = 0 \} \), which are Banach spaces with respect to the norm \( ||s||_\infty := \sup_{k \in \mathbb{Z}} ||s(k)|| \):

\[ (ii) \ \ell^p(\mathbb{Z}, X) := \{ s \in \mathcal{S}(\mathbb{Z}, X) : \sum_{k \in \mathbb{Z}} ||s(k)||^p < \infty \} \] which is a Banach space with respect to the norm

\[ ||s||_p = ( \sum_{k=-\infty}^{\infty} ||s(k)||^p)^{1/p} \quad \text{for each } p \in [1, \infty). \]

Let \( \{ A(n) \}_{n \in \mathbb{Z}} \subset \mathcal{B}(X) \). We consider the discrete dynamical system

\[ (A) \quad x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}. \]

Then the discrete evolution family associated to \((A)\) is defined by

\[ \Phi_A : \Delta \to \mathcal{B}(X), \quad \Phi_A(m, n) := \begin{cases} A(m - 1) \ldots A(n) & , \quad m > n \\ I_d & , \quad m = n \end{cases}. \]

In [24] we have used three categories of representative subspaces: stable subspace, bounded subspace and unstable subspace whose properties were crucial in the description of the trichotomy phenomena.

Indeed, let \( n \in \mathbb{Z} \). We denote by

\[ \mathcal{F}_n(\mathbb{Z}, X) := \{ \varphi \in \ell^{\infty}(\mathbb{Z}, X) : \varphi(k) = A(k - 1)\varphi(k - 1), \forall k \leq n \}. \]

Thus, the linear subspace

\[ \mathcal{S}(n) := \{ x \in X : \lim_{k \to -\infty} \Phi_A(k, n)x = 0 \} \]

is called the stable subspace at the moment \( n \), the linear subspace

\[ \mathcal{B}(n) := \{ x \in X : \sup_{k \geq n} ||\Phi_A(k, n)x|| < \infty \text{ and there is } \varphi \in \mathcal{F}_n(\mathbb{Z}, X) \text{ with } \varphi(n) = x \} \]

is called the bounded subspace at the moment \( n \) and respectively

\[ \mathcal{U}(n) := \{ x \in X : \text{ there exists } \varphi \in \mathcal{F}_n(\mathbb{Z}, X) \text{ with } \varphi(n) = x \text{ and } \lim_{k \to -\infty} \varphi(k) = 0 \} \]

is called the unstable subspace at the moment \( n \).

The concept of exponential trichotomy studied in what follows is in the spirit of the theory of Sacker and Sell (see [17] and also our works [22] and [24]).

**Definition 2.1.** ([24]) We say that the dynamical system \((A)\) has a uniform exponential trichotomy if there exist three families of projections \( \{ P_k(n) \}_{n \in \mathbb{Z}} \subset \mathcal{B}(X) \), \( k \in \{1, 2, 3\} \) and two constants \( K \geq 1 \) and \( \nu > 0 \) such that the following properties hold:

\[ (i) \ A(n)P_k(n) = P_k(n+1)A(n), \text{ for all } n \in \mathbb{Z} \text{ and all } k \in \{1, 2, 3\}; \]

\[ (ii) \ P_k(n)P_j(n) = 0, \text{ for all } k \neq j \text{ and all } n \in \mathbb{Z}; \]

\[ (iii) \ P_1(n) + P_2(n) + P_3(n) = I_d, \text{ for all } n \in \mathbb{Z}; \]

\[ (iv) \ \sup_{n \in \mathbb{Z}} ||P_k(n)|| < \infty, \text{ for all } k \in \{1, 2, 3\}; \]

\[ (v) \ ||\Phi_A(m, n)x|| \leq Ke^{-\nu(m-n)}||x||, \text{ for all } x \in \text{Range } P_1(n) \text{ and all } (m, n) \in \Delta; \]

\[ (vi) \ |x| \leq ||\Phi_A(m, n)x|| \leq K ||x||, \text{ for all } x \in \text{Range } P_2(n) \text{ and all } (m, n) \in \Delta; \]

\[ (vii) \ ||\Phi_A(m, n)x|| \geq \frac{1}{K} e^{\nu(n-m)}||x||, \text{ for all } x \in \text{Range } P_3(n) \text{ and all } (m, n) \in \Delta; \]

\[ (viii) \text{ for each } k \in \{2, 3\}, \text{ the restriction } A(n) : \text{Range } P_k(n) \to \text{Range } P_k(n+1) \text{ is an isomorphism, for all } n \in \mathbb{Z}. \]
In order to study the existence of the uniform exponential trichotomy we associate to the system \((A)\) the input-output system

\[(S_A) \quad \gamma(n + 1) = A(n) \gamma(n) + s(n + 1), \quad \forall n \in \mathbb{Z}\]

with \(s \in \ell^1(\mathbb{Z}, X)\) and \(\gamma \in \ell^\infty(\mathbb{Z}, X)\).

The first concept of admissibility introduced in [24] is the following (see Definition 2.3 in [24]):

**Definition 2.2.** ([24]) The pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is said to be **admissible** for the system \((S_A)\) if for every \(s \in \ell^1(\mathbb{Z}, X)\) there exist a unique \(\gamma_s \in c_{0,\infty}(\mathbb{Z}, X)\) and a unique \(q_s \in c_{\infty,0}(\mathbb{Z}, X)\) such that the pairs \((\gamma_s, s)\) and \((q_s, s)\) satisfy the system \((S_A)\).

**Remark 2.1.** If the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is admissible for \((S_A)\), then we consider the **input-output operators**

\[
\Gamma : \ell^1(\mathbb{Z}, X) \to c_{0,\infty}(\mathbb{Z}, X), \quad \Gamma(s) = \gamma_s \\
Q : \ell^1(\mathbb{Z}, X) \to c_{\infty,0}(\mathbb{Z}, X), \quad Q(s) = q_s
\]

and we have that \(\Gamma\) and \(Q\) are bounded linear operators (see Remark 2.5 in [24]).

The next result was obtained in Theorem 2.5 in [24] and shows that the above admissibility property provides the existence of a *uniform* trichotomic behavior relative to the stable, bounded and unstable subspaces, as follows:

**Theorem 2.1.** ([24]) If the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is admissible for the system \((S_A)\), then there are three families of projections \(\{P_k(n)\}_{n \in \mathbb{Z}}, k \in \{1, 2, 3\}\) such that the following properties hold:

(i) \(\text{Range } P_1(n) = S(n), \text{ Range } P_2(n) = B(n)\) and \(\text{Range } P_3(n) = U(n)\), for all \(n \in \mathbb{Z}\);

(ii) \(P_1(n) + P_2(n) + P_3(n) = I_d\), for all \(n \in \mathbb{Z}\);

(iii) \(P_k(n)P_j(n) = 0\), for all \(k \neq j\) and all \(n \in \mathbb{Z}\);

(iv) \(A(n)P_k(n) = P_k(n + 1)A(n)\), for all \(n \in \mathbb{Z}\) and all \(k \in \{1, 2, 3\}\);

(v) \(\forall k \in \{1, 2, 3\}, \sup_{n \in \mathbb{Z}} \|P_k(n)\| < \infty\);

(vi) \(\|\Phi_A(m,n)x\| \leq ||\Gamma||\|x\|\), for all \(x \in \text{Range } P_1(n) \cup \text{Range } P_2(n)\) and all \((m,n) \in \Delta\);

(vii) \(\frac{1}{||Q||}\|x\| \leq \|\Phi_A(m,n)x\|\), for all \(x \in \text{Range } P_2(n) \cup \text{Range } P_3(n)\) and all \((m,n) \in \Delta\);

(viii) for each \(k \in \{2, 3\}\), the restriction \(A(n) : \text{Range } P_k(n) \to \text{Range } P_k(n + 1)\) is an isomorphism, for all \(n \in \mathbb{Z}\).

According to Theorem 2.1 a central aim will be to identify the minimal requirements that should be added to the admissibility of the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) in order to provide the existence of the exponential trichotomy. The first step in this direction was made in [24] by introducing the following admissibility concept (see Definition 2.4 in [24]):

**Definition 2.3.** ([24]) Let \(p \in (1, \infty)\). The pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is said to be \(p\)-**admissible** for the system \((S_A)\) if the pair is admissible for the system \((S_A)\) and there is \(L > 0\) such that

(i) for every \(s \in \ell^1(\mathbb{Z}, X)\) with \(s(n) \in S(n)\), for all \(n \in \mathbb{Z}\), we have that

\[\|Q(s)\|_\infty \leq L\|s\|_p;\]
(ii) for every $s \in \ell^1(\mathbb{Z}, X)$ with $s(n) \in \mathcal{U}(n)$, for all $n \in \mathbb{Z}$, we have that
\[ ||\Gamma(s)||_\infty \leq L ||s||_p. \]

The central result in [24] (see Theorem 3.6 therein) established the connections between the $p$-admissibility of the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ for the system $(S_A)$ and the uniform exponential trichotomy of the system $(A)$ as follows:

**Theorem 2.2.** ([24]) The following assertions are equivalent:

(i) the system $(A)$ has a uniform exponential trichotomy;

(ii) for every $p \in (1, \infty$) the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is $p$-admissible for the system $(S_A)$;

(iii) there exists $p \in (1, \infty)$ such that the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is $p$-admissible for the system $(S_A)$.

In this paper we propose a new and more general approach. With this aim we will introduce a new admissibility concept and we will obtain in several steps a complete diagram of the pairs of admissible $\ell^p$-spaces that can be used in the study of the uniform exponential trichotomy.

**Remark 2.2.** Suppose that the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is admissible for the system $(S_A)$ and let $\{P_k(n)\}_{n \in \mathbb{Z}, k \in \{1, 2, 3\}}$ be the families of projections given by Theorem 2.1. For every $k \in \{1, 2, 3\}$, we consider the operator
\[ P_k : \ell^1(\mathbb{Z}, X) \rightarrow \ell^1(\mathbb{Z}, X), \quad (P_k(n))(s) = P_k(n)s(n). \]

Then $P_k$ is a linear operator on $\ell^1(\mathbb{Z}, X)$ and denoting by
\[ \alpha_k := \sup_{n \in \mathbb{Z}} ||P_k(n)||, \quad \text{for each } k \in \{1, 2, 3\} \]
we have that
\[ ||(P_k(n))(s)|| \leq \alpha_k ||s(n)||, \quad \forall n \in \mathbb{Z}, \forall k \in \{1, 2, 3\}. \tag{2.1} \]

From (2.1) we deduce that $P_k$ is a bounded operator on $\ell^1(\mathbb{Z}, X)$ and
\[ ||P_k|| \leq \alpha_k, \quad \forall k \in \{1, 2, 3\}. \]

**Definition 2.4.** Let $r, p \in [1, \infty]$. The pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is said to be $(r, p)$-admissible for the system $(S_A)$ if the pair is admissible for the system $(S_A)$ and the following conditions hold:

(i) for every $s \in \ell^1(\mathbb{Z}, X)$ we have that $(Q^{P_1})(s) \in \ell^r(\mathbb{Z}, X)$ and $(\Gamma P_3)(s) \in \ell^r(\mathbb{Z}, X)$;

(ii) there is a constant $L > 0$ such that
\[ \max\{||Q^{P_1}(s)||_r, ||\Gamma P_3(s)||_r\} \leq L ||s||_p, \quad \forall s \in \ell^1(\mathbb{Z}, X). \]

**Remark 2.3.** Suppose that the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is admissible for the system $(S_A)$. Since
\[ Q(s) \in c_{\infty, 0}(\mathbb{Z}, X) \quad \text{and} \quad \Gamma(s) \in c_{0, \infty}(\mathbb{Z}, X), \quad \forall s \in \ell^1(\mathbb{Z}, X) \]
we have in particular that
\[ Q(s) \in \ell^\infty(\mathbb{Z}, X) \quad \text{and} \quad \Gamma(s) \in \ell^\infty(\mathbb{Z}, X), \quad \forall s \in \ell^1(\mathbb{Z}, X) \]
This shows that if $r = \infty$, then condition (i) from Definition 2.4 is satisfied.
Remark 2.4. Let \( p \in [1, \infty] \). From Remark 2.3 it follows that the \((\infty, p)\)-admissibility of the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) for the system \((S_A)\) means that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is admissible for \((S_A)\) and there exist a constant \( L > 0 \) with the property that
\[
\max\{\|Q\mathcal{P}_1(s)\|\infty, \|\Gamma\mathcal{P}_3(s)\|\infty\} \leq L \|s\|_p, \quad \forall s \in \ell^1(Z, X).
\]

Remark 2.5. If the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is admissible for the system \((S_A)\), then (see Remark 2.2) we deduce that there is \( \alpha > 0 \) such that
\[
\|\mathcal{P}_k(s)\|_1 \leq \alpha \|s\|_1, \quad \forall s \in \ell^1(Z, X), \forall k \in \{1, 2, 3\}.
\]
Using relation (2.2) we have that
\[
\|Q\mathcal{P}_1(s)\|_\infty \leq \|Q\| \|\mathcal{P}_1(s)\|_1 \leq \alpha \|Q\| \|s\|_1, \quad \forall s \in \ell^1(Z, X)
\]
and respectively
\[
\|\Gamma\mathcal{P}_3(s)\|_\infty \leq \|\Gamma\| \|\mathcal{P}_3(s)\|_1 \leq \alpha \|\Gamma\| \|s\|_1, \quad \forall s \in \ell^1(Z, X).
\]
Setting \( L := \max\{\|Q\|, \|\Gamma\|\} \) from relations (2.3) and (2.4) it follows that
\[
\max\{\|Q\mathcal{P}_1(s)\|\infty, \|\Gamma\mathcal{P}_3(s)\|\infty\} \leq L \|s\|_1, \quad \forall s \in \ell^1(Z, X).
\]
Thus, from relation (2.5) and Remark 2.3 we obtain that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \((\infty, 1)\)-admissible for the system \((S_A)\).

Remark 2.6. If the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \((\infty, 1)\)-admissible for the system \((S_A)\) we have in particular that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is admissible for \((S_A)\). Then, according to Remark 2.5 it follows that the \((\infty, 1)\)-admissibility of the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) for the system \((S_A)\) is equivalent with the admissibility of the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) for \((S_A)\).

The first connection between the admissibility concepts introduced in Definition 2.3 and Definition 2.4 is given by:

Lemma 2.1. Let \( p \in (1, \infty) \). Then, the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \((\infty, p)\)-admissible for the system \((S_A)\) if and only if the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \(p\)-admissible for \((S_A)\).

Proof. Necessity. Suppose that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \((\infty, p)\)-admissible for the system \((S_A)\) and let \( L > 0 \) be given by Definition 2.4. Then
\[
\max\{\|Q\mathcal{P}_1(s)\|\infty, \|\Gamma\mathcal{P}_3(s)\|\infty\} \leq L \|s\|_p, \quad \forall s \in \ell^1(Z, X).
\]

Let \( s \in \ell^1(Z, X) \) with \( s(n) \in S(n) \), for all \( n \in Z \). Then \( \mathcal{P}_1(s) = s \) and from relation (2.6) we obtain that
\[
\|Q(s)\|\infty = \|(Q\mathcal{P}_1(s))\|\infty \leq L \|s\|_p.
\]
In addition, if \( s \in \ell^1(Z, X) \) with \( s(n) \in U(n) \), for all \( n \in Z \), then \( \mathcal{P}_3(s) = s \) and using relation (2.6) we have that
\[
\|\Gamma(s)\|\infty = \|(\Gamma\mathcal{P}_3(s))\|\infty \leq L \|s\|_p.
\]
From relations (2.7) and (2.8) we deduce that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \(p\)-admissible for the system \((S_A)\).

Sufficiency. Suppose that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \(p\)-admissible for the system \((S_A)\). Then, in particular, the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is admissible for the system \((S_A)\).
Let $\alpha > 0$ be such that

$$||P_k(n)|| \leq \alpha, \quad \forall n \in \mathbb{Z}, \forall k \in \{1, 2, 3\} \tag{2.9}$$

and let $L > 0$ be given by Definition \[2.3\].

Let $s \in \ell^1(\mathbb{Z}, X)$. Then $(P_1(s))(n) \in \mathcal{S}(n)$, for all $n \in \mathbb{Z}$. According to our hypothesis it follows that

$$||Q(P_1(s))||_\infty \leq L ||P_1(s)||_p. \tag{2.10}$$

Using relation (2.9) we have that

$$||(P_1(s))(n)|| \leq \alpha ||s(n)||, \quad \forall n \in \mathbb{Z}$$

which implies that

$$||P_1(s)||_p \leq \alpha ||s||_p. \tag{2.11}$$

From relations (2.10) and (2.11) we obtain that

$$||(Q P_1_1(s))||_\infty \leq \alpha L ||s||_p. \tag{2.12}$$

On the other hand, $(P_2(s))(n) \in \mathcal{U}(n)$, for all $n \in \mathbb{Z}$. Then, according to our hypothesis we have that

$$||\Gamma(P_2(s))||_\infty \leq L ||P_2(s)||_p. \tag{2.13}$$

Using similar arguments with those considered above we deduce that

$$||P_2(s)||_p \leq \alpha ||s||_p. \tag{2.14}$$

From relations (2.13) and (2.14) it follows that

$$||(\Gamma P_2(s))||_\infty \leq \alpha L ||s||_p. \tag{2.15}$$

Setting $\hat{L} := \alpha L$, from relations (2.12) and (2.15) we obtain that

$$\max\{||Q P_1(s)||_\infty, ||\Gamma P_2(s)||_\infty\} \leq \hat{L} ||s||_p, \quad \forall s \in \ell^1(\mathbb{Z}, X). \tag{2.16}$$

From Remark \[2.4\] and relation (2.16) it follows that the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is $(\infty, p)$-admissible for the system $(S_A)$ and the proof is complete. \(\square\)

**Remark 2.7.** Lemma \[2.1\] shows that the admissibility concept introduced in this paper generalizes the admissibility concept used in \[24\]. In this manner we extend the framework for the study of the uniform exponential trichotomy such that our previous results will arise as special cases of this new study.

The main result of this section is:

**Theorem 2.3.** Let $r, p \in [1, \infty]$ with $(r, p) \neq (\infty, 1)$. If the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is $(r, p)$-admissible for the system $(S_A)$, then the system $(A)$ has a uniform exponential trichotomy.

**Proof.** Since $(r, p) \neq (\infty, 1)$ there are two possible cases that will be considered in what follows.

**Case 1.** $r \neq \infty$.

According to our hypothesis, we have in particular that the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is admissible for the system $(S_A)$. Then, denoting by $M := \max\{||\Gamma||, ||Q||\}$ from Theorem \[2.1\] (vi) and (viii) it follows that

$$||\Phi_A(m, n)x|| \leq M ||x||, \quad \forall x \in \text{Range } P_1(n), \forall (m, n) \in \Delta \tag{2.17}$$
In addition, from (2.17) we have that
\[ \|\Phi_A(m, n)x\| \geq \frac{1}{M} \|x\|, \quad \forall x \in \text{Range } P_3(n), \forall (m, n) \in \Delta. \] (2.18)

Let \( L > 0 \) be given by Definition 2.4. Then
\[ \max\{\|QP_1(s)||_r, \|\Gamma P_3(s)||_r\} \leq L \|s\|_p, \quad \forall s \in \ell^1(Z, X). \] (2.19)

Let \( \delta \in \mathbb{N}^* \) be such that
\[ \delta^{1/r} \geq eLM^2 \] (2.20)
and let \( \nu = 1/(2\delta) \) and \( K = eM \).

**Step 1.** We prove that
\[ \|\Phi_A(m, n)x\| \leq Ke^{-\nu(m-n)}\|x\|, \quad \forall x \in \text{Range } P_1(n), \forall (m, n) \in \Delta. \]

Let \( n \in \mathbb{Z} \) and let \( x \in \text{Range } P_1(n) \setminus \{0\} \). If \( \Phi_A(n+\delta, n)x \neq 0 \), then \( \Phi_A(j, n)x \neq 0 \), for all \( j \in \{n+1, \ldots, n+\delta\} \). Then it makes sense to consider the sequences
\[
s : \mathbb{Z} \to X, \quad s(k) = \chi_{(n+1, \ldots, n+\delta)}(k) \frac{\Phi_A(k, n)x}{\|\Phi_A(k, n)x\|} \]
\[
q : \mathbb{Z} \to X, \quad q(k) = \left\{ \begin{array}{ll}
\sum_{j=n+1}^{k} \chi_{(n+1, \ldots, n+\delta)}(j) \frac{\Phi_A(k, n)x}{\|\Phi_A(k, n)x\|} & , \quad k \geq n+1 \\
0 & , \quad k \leq n
\end{array} \right. .
\]

A simple computation shows that the pair \((q, s)\) satisfies the system \((S_A)\). Denoting by
\[ a := \sum_{j=n+1}^{n+\delta} \frac{1}{\|\Phi_A(j, n)x\|} \]
we note that
\[ q(k) = a \Phi_A(k, n)x, \quad \forall k \geq n + \delta. \] (2.21)

Since \( x \in \text{Range } P_1(n) = \delta(n) \), we have that \( \Phi_A(k, n) \to 0 \) as \( n \to \infty \). Then, from (2.21) we deduce that \( q \in c_{\infty,0}(\mathbb{Z}, X) \), so \( q = Q(s) \).

Since \( x \in \text{Range } P_1(n) \) using Theorem 2.1 (iv) we have that
\[ s(k) \in \text{Range } P_1(k), \quad \forall k \in \{n+1, \ldots, n+\delta\}. \]

This implies that \( P_1(k)s(k) = s(k) \), for all \( k \in \mathbb{Z} \), so \( P_1(s) = s \). Using relation (2.19) we successively obtain that
\[ \|q||_r = \|Q(s)||_r = \|(QP_1(s)||_r \leq L\|s\|_p. \] (2.22)

From relations (2.17) and (2.21) we have that
\[ \|\Phi_A(n+2\delta, n)x\| \leq M \|\Phi_A(j, n)x\| = \frac{M}{a} \|q(j)||, \quad \forall j \in \{n+\delta+1, \ldots, n+2\delta\}. \]

It follows that
\[ \|\Phi_A(n+2\delta, n)x\| \chi_{(n+\delta+1, \ldots, n+2\delta)}(j) \leq \frac{M}{a} \|q(j)||, \quad \forall j \in \mathbb{Z}. \] (2.23)

From relations (2.22) and (2.23) we deduce that
\[ \|\Phi_A(n+2\delta, n)x\| \delta^{1/r} \leq \frac{M}{a} \|q||_r \leq \frac{LM}{a} \|s\|_p. \] (2.24)

In addition, from (2.17) we have that
\[ \|\Phi_A(j, n)x\| \leq M\|x\|, \quad \forall j \in \{n+1, \ldots, n+\delta\}. \]
This implies that
\[ a \geq \frac{\delta}{M ||x||}. \] (2.25)

From relations (2.20), (2.24) and (2.25) we successively obtain that
\[ eLM^2 \|\Phi_A(n + 2\delta, n)x\| \leq \frac{LM}{a} \|s\|_p \leq \frac{LM^2}{\delta} ||x|| \|s\|_p \]
which implies that
\[ \|\Phi_A(n + 2\delta, n)x\| \leq \frac{1}{e} \|s\|_p \|x\|. \] (2.26)

We observe that
\[ \|s(k)\| = \begin{cases} 1, & k \in \{n + 1, \ldots, n + \delta\} \\ 0, & k \notin \{n + 1, \ldots, n + \delta\} \end{cases} \]
So, if \( p \in [1, \infty) \), we have that
\[ \|s\|_p = \delta^{1/p} \leq \delta \] (2.27)
and if \( p = \infty \) then
\[ \|s\|_\infty = 1 \leq \delta. \] (2.28)

From relations (2.26)–(2.28) we deduce that
\[ \|\Phi_A(n + 2\delta, n)x\| \leq \frac{1}{e} \|x\|. \] (2.29)

If \( \Phi_A(n + \delta, n)x = 0 \), then obviously relation (2.29) holds. Thus, denoting by \( h = 2\delta \) and since \( h \) does not depend on \( n \) or \( x \) it follows that
\[ \|\Phi_A(n + h, n)x\| \leq \frac{1}{e} \|x\|, \quad \forall x \in \text{Range} P_1(n), \forall n \in \mathbb{Z}. \] (2.30)

Let \( (m, n) \in \Delta \) and \( x \in \text{Range} P_1(n) \). Then there are \( j \in \mathbb{N} \) and \( r \in \{0, \ldots, h - 1\} \) such that \( m = n + jh + r \). Using Theorem 2.1 (iv), from relations (2.17) and (2.30), we have that
\[ \|\Phi_A(m, n)x\| \leq M e^{-j} \|x\| \leq Ke^{-\nu(m - n)} \|x\|. \]

**Step 2.** We prove that
\[ \|\Phi_A(m, n)x\| \geq \frac{1}{K} e^{\nu(m - n)} \|x\|, \quad \forall x \in \text{Range} P_3(n), \forall (m, n) \in \Delta. \]

Let \( n \in \mathbb{Z} \) and let \( x \in \text{Range} P_3(n) \setminus \{0\} \). Then, from Theorem 2.1 (viii) it follows that \( \Phi_A(k, n)x \neq 0 \), for all \( k \geq n \).

We consider the sequence
\[ u : \mathbb{Z} \to X, \quad u(k) = -\chi_{\{n + \delta + 1, \ldots, n + 2\delta\}}(k) \frac{\Phi_A(k, n)x}{\|\Phi_A(k, n)x\|}. \]

Since \( x \in \text{Range} P_3(n) \), from Theorem 2.1 (iv) we deduce that \( u(k) \in \text{Range} P_3(k) \), for all \( k \in \{n + \delta + 1, \ldots, n + 2\delta\} \), so
\[ \mathcal{P}_3(u) = u. \] (2.31)

We denote by
\[ c := \sum_{j=n+\delta+1}^{n+2\delta} \frac{1}{\|\Phi_A(j, n)x\|}. \]
From relation (2.18) we have that
\[ \|\Phi_A(n + 2\delta, n)x\| \geq \frac{1}{M} \|\Phi_A(j, n)x\|, \quad \forall j \in \{n, \ldots, n + 2\delta\}. \]
This implies that
\[ c \geq \frac{\delta}{M \|\Phi_A(n + 2\delta, n)x\|}. \tag{2.32} \]
Since \( x \in Range P_3(n) = 1/(n), \) there is \( \varphi \in \mathcal{F}_n(Z, X) \cap c_{0,\infty}(Z, X) \) with \( \varphi(n) = x. \) We consider the sequence
\[ \gamma : Z \to X, \quad \gamma(k) = \begin{cases} 0, & k \geq n + 2\delta \\ \left( \sum_{j=k+1}^{n+2\delta} \frac{1}{\|\Phi_A(j, n)x\|} \right) \Phi_A(k, n)x, & k \in \{n + \delta, \ldots, n + 2\delta - 1\} \\ c\Phi_A(k, n)x, & k \in \{n, \ldots, n + \delta - 1\} \\ c\varphi(k), & k \leq n - 1 \end{cases} \]
From \( \varphi \in c_{0,\infty}(Z, X) \) we have in particular that \( \gamma \in c_{0,\infty}(Z, X). \) An easy computation shows that the pair \( (\gamma, u) \) satisfies the system \((S_A), \) so \( \gamma = \Gamma(u). \) Then, using relations (2.31) and (2.19) it follows that
\[ \|\gamma\|_r = \|\Gamma(u)\|_r = \|(\Gamma P_3)(u)\|_r \leq L \|u\|_p. \tag{2.33} \]
In addition, using relation (2.18) we have that
\[ \|\gamma(k)\| = c \|\Phi_A(k, n)x\| \geq \frac{c}{M} \|x\|, \quad \forall k \in \{n, \ldots, n + \delta - 1\} \]
which implies that
\[ \|\gamma\|_r \geq \frac{c}{M} \delta^{1/r} \|x\|. \tag{2.34} \]
From relations (2.32)-(2.34) and (2.20) we successively deduce that
\[ L \|u\|_p \geq \|\gamma\|_r \geq \frac{c}{M} \delta^{1/r} \|x\| \geq \frac{\delta \delta^{1/r}}{M^2} \frac{\|x\|}{\|\Phi_A(n + 2\delta, n)x\|} \geq e\delta L \frac{\|x\|}{\|\Phi_A(n + 2\delta, n)x\|} \]
which implies that
\[ \|\Phi_A(n + 2\delta, n)x\| \geq c \frac{\delta}{\|u\|_p} \|x\|. \tag{2.35} \]
Since
\[ \|u(k)\| = \begin{cases} 1, & k \in \{n + \delta + 1, \ldots, n + 2\delta\} \\ 0, & k \in \mathbb{Z} \setminus \{n + \delta + 1, \ldots, n + 2\delta\} \end{cases} \]
we immediately obtain that
\[ \|u\|_p \leq \delta. \tag{2.36} \]
Then, from (2.35) and (2.36) we have that
\[ \|\Phi_A(n + 2\delta, n)x\| \geq c \|x\|. \]
Denoting by \( h = 2\delta \) and taking into account that \( h \) does not depend on \( n \) or on \( x \) it follows that
\[ \|\Phi_A(n + h, n)x\| \geq c \|x\|, \quad \forall x \in Range P_3(n), \forall n \in \mathbb{Z}. \tag{2.37} \]
Let \((m, n) \in \Delta \) and \( x \in Range P_3(n). \) Let \( j \in \mathbb{N} \) and \( r \in \{0, \ldots, h - 1\} \) be such that \( m = n + jh + r. \) Using Theorem 2.1 (iv), from relations (2.18) and (2.37) we have that
\[ \|\Phi_A(m, n)x\| \geq \frac{1}{M} \|\Phi_A(n + jh, n)x\| \leq \frac{1}{M} e^j \|x\| \geq \frac{1}{K} e^{\nu(m-n)} \|x\|. \]
From Step 1, Step 2 and Theorem 2.1 it follows that the system \((A)\) has a uniform exponential trichotomy.

**Case 2.** \(r = \infty\). In this case \(p \in [1, \infty] \).

If \(p \in (1, \infty)\), then according to Lemma 2.1 our hypothesis implies that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \(p\)-admissible for the system \((S_A)\). Then, from Theorem 2.2 it follows that the system \((A)\) has a uniform exponential trichotomy.

If \(p = \infty\), then we have that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is \((\infty, \infty)\)-admissible for the system \((S_A)\). Thus we immediately deduce that the pair \((\ell^\infty(Z, X), \ell^j(Z, X))\) is \((\infty, q)\)-admissible for \((S_A)\), for every \(q \in (1, \infty)\). Then, using Lemma 2.1 and the same arguments with those presented above, we deduce that the system \((A)\) has a uniform exponential trichotomy.

In conclusion, from Case 1 and Case 2 we have that the proof is complete. \(\Box\)

### 3. The diagram of connections between \((r, p)\) - admissibility and uniform exponential trichotomy.

The next natural question is whether the uniform exponential trichotomy of a discrete dynamical system implies the new admissibility property introduced in the previous section. Moreover, it would be interesting to investigate if the properties of admissibility considered in our method remain the minimal requirements in this context and to analyze whether all the working hypotheses on the \(\ell^p\)-spaces involved are indeed necessary. Starting from these questions we will obtain in what follows a complete diagram of the connections between admissibility with pairs of \(\ell^p\)-spaces and uniform exponential trichotomy.

Let \(X\) be a real or a complex Banach space. Let \(\{A(n)\}_{n \in \mathbb{Z}} \subset \mathcal{B}(X)\). We consider the discrete dynamical system

\[
(A) \quad x(n + 1) = A(n)x(n), \quad n \in \mathbb{Z}.
\]

In what follows we work in the hypothesis that the system \((A)\) has a uniform exponential trichotomy with respect to the family of projections \(\{P_k(n)\}_{n \in \mathbb{Z}}, k \in \{1, 2, 3\}\). For every \(j \in \{2, 3\}\) and \((k, n) \in \Delta\), from Definition 2.1 (viii) we have that the operator \(\Phi_A(k, n) : \text{Range } P_j(n) \to \text{Range } P_j(k)\) is an isomorphism and we denote by

\[
\Phi_A^j(k, n)^{-1} : \text{Range } P_j(k) \to \text{Range } P_j(n)
\]

the inverse of this operator.

We associate to the system \((A)\) the input-output system

\[
\gamma(n + 1) = A(n)\gamma(n) + s(n + 1), \quad \forall n \in \mathbb{Z}
\]

with \(s \in \ell^1(Z, X)\) and \(\gamma \in \ell^\infty(Z, X)\).

First, we will establish whether the property of \((r, p)\)-admissibility introduced in the previous section is also a necessary condition for uniform exponential trichotomy. The starting step in this direction was made in Theorem 3.3 in [24] as follows:

**Theorem 3.1.** (24) If the system \((A)\) has a uniform exponential trichotomy with respect to the families of projections \(\{P_k(n)\}_{n \in \mathbb{Z}}, k \in \{1, 2, 3\}\), then the pair \((\ell^\infty(Z, X), \ell^j(Z, X))\) is admissible for the system \((S_A)\). Moreover, if \(\Gamma\) and \(Q\) are the operators introduced in Remark 2.1 then

\[
(\Gamma s)(n) = \sum_{k=-\infty}^{n} \Phi_A(n, k)(P_1(k) + P_2(k))s(k) - \sum_{k=n+1}^{\infty} \Phi_A^3(k, n)^{-1}P_3(k)s(k)
\]

and respectively
\[(Qs)(n) = \sum_{k=\infty}^{n} \Phi_A(n,k)P_1(k)s(k) - \sum_{k=n+1}^{\infty} \Phi_A^2(k,n)\frac{1}{P_2(k)s(k)} - \sum_{k=n+1}^{\infty} \Phi_A^3(k,n)\frac{1}{P_3(k)s(k)}\]

for all \(n \in \mathbb{N}\) and all \(s \in \ell^1(\mathbb{Z}, X)\).

An important role in all what follows is played by the following two lemmas.

**Lemma 3.1.** Let \(p \in [1, \infty]\) and \(\nu > 0\). Then, for every \(s \in \ell^p(\mathbb{Z}, \mathbb{R})\), the sequence

\[q_s : \mathbb{Z} \rightarrow \mathbb{R}_+, \quad q_s(n) = \sum_{k=\infty}^{n} e^{-\nu(n-k)}|s(k)|\]

belongs to \(\ell^p(\mathbb{Z}, \mathbb{R})\). Moreover, we have that

\[||q_s||_p \leq \frac{1}{1 - e^{-\nu}} ||s||_p.\]

**Proof.**

**Case 1.** \(p = \infty\).

Let \(s \in \ell^\infty(\mathbb{Z}, X)\). Then

\[q_s(n) \leq \sum_{k=\infty}^{n} e^{-\nu(n-k)}||s||_\infty = \frac{1}{1 - e^{-\nu}} ||s||_\infty, \quad \forall n \in \mathbb{Z}.\]

This shows that \(q_s \in \ell^\infty(\mathbb{Z}, X)\) and

\[||q_s||_\infty \leq \frac{1}{1 - e^{-\nu}} ||s||_\infty.\]

**Case 2.** \(p \in [1, \infty)\).

Let \(s \in \ell^p(\mathbb{Z}, \mathbb{R})\). For every \(j \in \mathbb{Z}\) we consider the sequence

\[s_j : \mathbb{Z} \rightarrow \mathbb{R}, \quad s_j(n) = s(n - j).\]

We have that \(s_j \in \ell^p(\mathbb{Z}, \mathbb{R})\) and \(||s_j||_p = ||s||_p\), for all \(j \in \mathbb{Z}\). We observe that

\[q_s(n) = \sum_{k=\infty}^{n} e^{-\nu(n-k)}|s(k)| = \sum_{j=0}^{\infty} e^{-\nu j}s(n - j) = \sum_{j=0}^{\infty} e^{-\nu j}|s_j(n)|, \quad \forall n \in \mathbb{Z}.\]

For every \(m \in \mathbb{N}\) we define

\[q_m : \mathbb{Z} \rightarrow \mathbb{R}_+, \quad q_m(n) = \sum_{j=0}^{m} e^{-\nu j}|s_j(n)|.\]

Then, we have that

(i) \(q_m(n) \leq q_{m+1}(n)\), for all \(m \in \mathbb{N}\) and \(n \in \mathbb{Z}\);

(ii) \(q_m(n) \rightarrow q_s(n)\) as \(m \rightarrow \infty\), for all \(n \in \mathbb{Z}\).

Using Beppo Levi’s Theorem it follows that

\[\left( \sum_{n=\infty}^{\infty} (q_s(n))^p \right)^{1/p} = \lim_{m \rightarrow \infty} ||q_m||_p.\] (3.1)
Theorem 3.2. Let

\[\|q_m\|_p \leq \sum_{j=0}^{m} e^{-\nu j} \|s_j\|_p = \left(\sum_{j=0}^{m} e^{-\nu j}\right) \|s\|_p \leq \frac{1}{1-e^{-\nu}} \|s\|_p, \quad \forall m \in \mathbb{N}. \quad (3.2)\]

From relations (3.1) and (3.2) we obtain that \(q_s \in \ell^p(\mathbb{Z}, \mathbb{R})\) and

\[\|q_s\|_p \leq \frac{1}{1-e^{-\nu}} \|s\|_p\]

which completes the proof.

\[\square\]

Lemma 3.2. Let \(p \in [1, \infty]\) and \(\nu > 0\). Then, for every \(s \in \ell^p(\mathbb{Z}, \mathbb{R})\), the sequence

\[\gamma_s : \mathbb{Z} \to \mathbb{R}_+, \quad \gamma_s(n) = \sum_{k=n+1}^{\infty} e^{-\nu(k-n)}|s(k)|\]

belongs to \(\ell^p(\mathbb{Z}, \mathbb{R})\). In addition, we have that

\[\|\gamma_s\|_p \leq \frac{1}{e^\nu - 1} \|s\|_p.\]

Proof. This follows using similar arguments with those in the proof of Lemma 3.1.

\[\square\]

The first main result of this section is given by:

Theorem 3.2. Let \(r, p \in [1, \infty]\) with \(r \geq p\). If the system \((A)\) has a uniform exponential trichotomy, then the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is \((r, p)\)-admissible for the system \((S_A)\).

Proof. We suppose that the system \((A)\) has a uniform exponential trichotomy with respect to the family of projections \(\{P_k(n)\}_{n \in \mathbb{Z}}\), \(k \in \{1, 2, 3\}\) and let \(K, \nu > 0\) be two constants given by Definition 2.1. From Definition 2.1 (iv) we have that there is \(\alpha > 0\) such that

\[\|P_k(n)\| \leq \alpha, \quad \forall n \in \mathbb{Z}, \forall k \in \{1, 2, 3\}. \quad (3.3)\]

For every \(k \in \{1, 2, 3\}\), we consider the operator

\(P_k : \ell^1(\mathbb{Z}, X) \to \ell^1(\mathbb{Z}, X), \quad (P_k(s))(n) = P_k(n)s(n).\)

Using relation (3.3) we deduce that \(P_k\) is a bounded linear operator, for every \(k \in \{1, 2, 3\}\).

Since the system \((A)\) has a uniform exponential trichotomy, from Theorem 3.1 we have that the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is admissible for the system \((S_A)\). It remains to show that the conditions (i) and (ii) from Definition 2.4 are satisfied.

Let \(s \in \ell^1(\mathbb{Z}, X)\). According to Theorem 3.1 we have that

\[((QP_1)(s))(n) = \sum_{k=-\infty}^{n} \Phi_A(n, k)P_1(k)s(k), \quad \forall n \in \mathbb{Z} \quad (3.4)\]

and

\[((\Gamma P_3)(s))(n) = -\sum_{k=n+1}^{\infty} \Phi_A^3(k, n)|^{-1}P_3(k)s(k), \quad \forall n \in \mathbb{Z}. \quad (3.5)\]

From Definition 2.1 (iv) and relations (3.4) and (3.3) we deduce that

\[\|((QP_1)(s))(n)\| \leq \alpha K \sum_{k=-\infty}^{n} e^{-\nu(n-k)}||s(k)||, \quad \forall n \in \mathbb{Z}. \quad (3.6)\]
Moreover, from Definition 2.1 (vii), (viii) and relations (3.5) and (3.3) we have that
\[
\|(Q\mathcal{P}_3)(s))(n)\| \leq \alpha K \sum_{k=n+1}^{\infty} e^{-\nu(k-n)}n, \quad \forall n \in \mathbb{Z},
\]
(3.7)
Since \(s \in \ell^1(\mathbb{Z}, X)\) in particular we have that \(s \in \ell^p(\mathbb{Z}, X)\). Then, from relation (3.6) and Lemma 3.1 it follows that \((Q\mathcal{P}_1)(s)) \in \ell^p(\mathbb{Z}, X)\) and
\[
\|(Q\mathcal{P}_1)(s))\|_p \leq \frac{\alpha K}{1-e^{-\nu}} \|s\|_p.
\]
(3.8)
In addition, from relation (3.7) and Lemma 3.2 we deduce that \((\Gamma \mathcal{P}_3)(s)) \in \ell^p(\mathbb{Z}, X)\) and
\[
\|(\Gamma \mathcal{P}_3)(s))\|_p \leq \frac{\alpha K}{e^{\nu}-1} \|s\|_p.
\]
(3.9)
Moreover, from Definition 2.1 (vii), (viii) and relations (3.5) and (3.3) we have that \(\|u\|_r \leq \delta \|u\|_p, \quad \forall u \in \ell^p(\mathbb{Z}, X)\).
(3.10)
Observing that
\[
\frac{1}{e^{\nu}-1} = \frac{e^{-\nu}}{1-e^{-\nu}} \leq \frac{1}{1-e^{-\nu}}
\]
from relations (3.8)-(3.10) it follows that
\[
\max\{|(Q\mathcal{P}_1)(s)|_r, |(\Gamma \mathcal{P}_3)(s)|_r\} \leq \frac{\alpha K\delta}{1-e^{-\nu}} \|s\|_p, \quad \forall s \in \ell^1(\mathbb{Z}, X).
\]
In conclusion, we have the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is \((r,p)\)-admissible for the system \((S_A)\) and the proof is complete. 

\textbf{Remark 3.1.} The natural questions arises if in Theorem 3.2 the hypothesis \(r \geq p\) can be removed. The answer is negative, as the following example will show.

\textbf{Example 3.1.} Let \(X = \mathbb{R}^3\) with respect to the norm
\[
\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}, \quad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3
\]
and let
\[
A : X \rightarrow X, \quad A(x_1, x_2, x_3) = (e^{-1}x_1, x_2, e x_3).
\]
We consider the discrete dynamical system
\[
x(n+1) = Ax(n), \quad \forall n \in \mathbb{Z}.
\]
The discrete evolution family associated to \((A)\) is
\[
\Phi_A(m, n)(x_1, x_2, x_3) = (e^{-(m-n)}x_1, x_2, e^{m-n}x_3), \quad \forall x = (x_1, x_2, x_3) \in X, \forall (m, n) \in \Delta.
\]
Let
\[
P_1 : X \rightarrow X, \quad P_1(x_1, x_2, x_3) = (x_1, 0, 0)
P_2 : X \rightarrow X, \quad P_2(x_1, x_2, x_3) = (0, x_2, 0)
P_3 : X \rightarrow X, \quad P_3(x_1, x_2, x_3) = (0, 0, x_3).
\]
It is easy to see that the system \((A)\) is uniformly exponentially trichotomic with respect to the families of projections \(\{P_k(n)\}_{n \in \mathbb{Z}}, k \in \{1, 2, 3\}\), where
\[
P_k(n) = P_k, \quad \forall n \in \mathbb{Z}, \forall k \in \{1, 2, 3\}.
\]
We associate to the system \((A)\) the input-output system
\[
\gamma(n + 1) = A\gamma(n) + s(n + 1), \quad \forall n \in \mathbb{Z}
\]
with \(s \in \ell^1(\mathbb{Z}, X)\) and \(\gamma \in \ell^\infty(\mathbb{Z}, X)\).

Since the system \((A)\) has a uniform exponential dichotomy, from Theorem 3.1 we have that the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is admissible for the system \((S_A)\). Let \(\Gamma, Q\) be the input-output operators associated to the system \((S_A)\) (see Remark 2.1) and let \(P_k : \ell^1(\mathbb{Z}, X) \rightarrow \ell^1(\mathbb{Z}, X), \ k \in \{1, 2, 3\}\), be the operators introduced in Remark 2.2.

Let \(r, p \in [1, \infty)\) with \(r < p\). We prove that the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is not \((r, p)\)-admissible for the system \((S_A)\).

Indeed, supposing by contrary that the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is \((r, p)\)-admissible for the system \((S_A)\), it follows that there is \(L > 0\) such that
\[
\max \{(||QP_1(s)||_r, ||(\Gamma P_3(s))||_r\} \leq L \ ||s||_p, \ \forall s \in \ell^1(\mathbb{Z}, X). \tag{3.11}
\]

For every \(j \in \mathbb{N}^*\) let
\[
s_j : \mathbb{Z} \rightarrow X, \quad s_j(n) = (\chi_{\{1, \ldots, j\}}(n), 0, 0).
\]
Then we have that
\[
||s_j(n)|| = \chi_{\{1, \ldots, j\}}(n), \quad \forall n \in \mathbb{Z}, \forall j \in \mathbb{N}^*.
\]
It follows that \(s_j \in \ell^p(\mathbb{Z}, X)\) and
\[
||s_j||_p = \begin{cases} j^{1/p}, & \text{if } p < \infty, \\ 1, & \text{if } p = \infty, \end{cases} \quad \forall j \in \mathbb{N}^*. \tag{3.12}
\]
Moreover, from Theorem 3.1 we obtain that
\[
((QP_1)(s_j))(n) = \sum_{k=-\infty}^{n} \Phi_A(n, k) P_k s_j(k) = (\sum_{k=-\infty}^{n} e^{-(n-k)} \chi_{\{1, \ldots, j\}}(k), 0, 0), \quad \forall n \in \mathbb{Z}, \forall j \in \mathbb{N}^*. \tag{3.13}
\]
Let \(j \in \mathbb{N}^*.\) From (3.13) we have that
\[
||((QP_1)(s_j))(n)|| = \sum_{k=-\infty}^{n} e^{-(n-k)} \chi_{\{1, \ldots, j\}}(k), \quad \forall n \in \mathbb{Z}. \tag{3.14}
\]
From (3.14), in particular, it follows that
\[
||((QP_1)(s_j))(n)|| = \sum_{k=1}^{n} e^{-(n-k)} \geq 1, \quad \forall n \in \{1, \ldots, j\}
\]
which implies that
\[
||((QP_1)(s_j))(n)|| \geq \chi_{\{1, \ldots, j\}}(n), \quad \forall n \in \mathbb{Z}. \tag{3.15}
\]
From relation (3.15) we have that
\[
||((QP_1)(s_j))||_r \geq j^{1/r}, \quad \forall j \in \mathbb{N}^*. \tag{3.16}
\]
Then, from relations (3.16) and (3.11) we deduce that
\[
j^{1/r} \leq L \ ||s_j||_p, \quad \forall j \in \mathbb{N}^*. \tag{3.17}
\]
Since \( r < p \) and taking into account relation (3.12), for \( j \to \infty \) in relation (3.17) we obtain a contradiction.

In conclusion, although the system \((A)\) has a uniform exponential trichotomy, the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) is not \((r,p)\)-admissible for the associated system \((S_A)\).

The central result of this paper is given by:

**Theorem 3.3.** Let \( r, p \in [1, \infty) \) with \((r,p) \neq (\infty,1)\). The following assertions hold:
(i) if the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) is \((r,p)\)-admissible for the system \((S_A)\), then the system \((A)\) has a uniform exponential trichotomy;
(ii) if \( r \geq p \), then the system \((A)\) has a uniform exponential trichotomy if and only if the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) is \((r,p)\)-admissible for the system \((S_A)\).

**Proof.** (i) This follows from Theorem 2.3.
(ii) This follows from (i) and from Theorem 3.2. \(\square\)

**Remark 3.2.** We note that all the results obtained so far are applicable to any nonautonomous discrete dynamical system without any requirement concerning the coefficients.

In order to establish the complete diagram of connections between \((r,p)\)-admissibility and uniform exponential trichotomy, it remains to answer whether the hypothesis \((r,p) \neq (\infty,1)\) from Theorem 3.3 can be removed. This means to study if the \((\infty,1)\)-admissibility of the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) for the system \((S_A)\) can be a sufficient condition for the uniform exponential trichotomy of the system \((A)\). But, according to Remark 2.6 the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) is \((\infty,1)\)-admissible for the system \((S_A)\) if and only if the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) is admissible for \((S_A)\). So, the main question is whether the admissibility of the pair \((\ell^\infty(Z,X), \ell^1(Z,X))\) for the system \((S_A)\) implies the uniform exponential trichotomy of \((A)\). The answer is negative, as the following example shows:

**Example 3.2.** Let \((V, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space. The norm on \(V\) and on \(B(V)\) will be denoted by \(||\cdot||\).

Let \(\{\varphi_n\}_{n \in \mathbb{Z}}\) be an orthonormal basis in \(V\). Then we have that
\[
v = \sum_{n=-\infty}^{\infty} <v, \varphi_n> \varphi_n, \quad \forall v \in V.
\]

For every \(n \in \mathbb{Z}\) we define the operator
\[
T_n : V \to V, \quad T_n(v) = \sum_{k=n}^{\infty} <v, \varphi_k> \varphi_k.
\]

Then, we have that \(T_nT_m = T_m\), for all \(m, n \in \mathbb{Z}\) with \(m \geq n\) and
\[
||T_n|| = 1, \quad \forall n \in \mathbb{Z}.
\] (3.18)

We consider the sequence \((\alpha_n)_{n \in \mathbb{Z}}\), where
\[
\alpha_n := \begin{cases} 
1, & n \in \mathbb{N} \\
2^{-n}, & n \in \mathbb{Z} \setminus \mathbb{N}
\end{cases}
\]

Let \(X = V \times V \times V\) with respect to the norm
\[
||x||_X := \max\{||x_1||, ||x_2||, ||x_3||\}, \quad \forall x = (x_1, x_2, x_3) \in X.
\]
For every \( n \in \mathbb{Z} \) we consider the operator
\[
A(n) : X \to X, \quad A(n)(x_1, x_2, x_3) = \left( \frac{\alpha_{n+1}}{\alpha_n} T_n(x_1), x_2, e x_3 \right).
\]
Since \((\alpha_n)_{n \in \mathbb{Z}}\) is decreasing and using relation (3.18) it is easy to see that \( A(n) \in \mathcal{B}(X) \), for all \( n \in \mathbb{Z} \).

We consider the discrete dynamical system
\[
(A) \quad x(n+1) = A(n)x(n), \quad \forall n \in \mathbb{Z}.
\]
The discrete evolution family associated to \((A)\) is
\[
\Phi_A : \Delta \to \mathcal{B}(X), \quad \Phi_A(m, n)(x_1, x_2, x_3) = \left\{ \left( \frac{\alpha_m}{\alpha_n} T_{m-1}(x_1), x_2, e^{m-n} x_3 \right) \right\} \quad m > n \quad m = n.
\]

We associate to the system \((A)\) the input-output system
\[
(S_A) \quad \gamma(n+1) = A(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z}
\]
with \( s \in \ell^1(\mathbb{Z}, X) \) and \( \gamma \in \ell^\infty(\mathbb{Z}, X) \).

**Step 1.** We prove that the pair \((\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))\) is admissible for the system \((S_A)\).

Let \( s = (s_1, s_2, s_3) \in \ell^1(\mathbb{Z}, X) \). We consider the sequences
\[
\begin{align*}
\gamma_1 : \mathbb{Z} \to V, & \quad \gamma_1(n) = s_1(n) + \sum_{k=-\infty}^{n-1} \frac{\alpha_n}{\alpha_k} T_{n-1} s_1(k) \\
\gamma_2 : \mathbb{Z} \to V, & \quad \gamma_2(n) = \sum_{k=-\infty}^{n} s_2(k) \\
\gamma_3 : \mathbb{Z} \to V, & \quad \gamma_3(n) = - \sum_{k=n+1}^{\infty} e^{-(k-n)} s_3(k).
\end{align*}
\]

Since \( s \in \ell^1(\mathbb{Z}, X) \) it follows that \( s_k \in \ell^1(\mathbb{Z}, V) \), for all \( k \in \{1, 2, 3\} \). Then we immediately deduce that
\[
\gamma_2 \in c_{0, \infty}(\mathbb{Z}, V) \quad \text{and} \quad \gamma_3 \in c_0(\mathbb{Z}, V).
\]

Using (3.18) and the property that \((\alpha_n)_{n \in \mathbb{Z}}\) is decreasing we deduce that
\[
\|\gamma_1(n)\| \leq \|s_1(n)\| + \sum_{k=-\infty}^{n-1} \|s_1(k)\|, \quad \forall n \in \mathbb{Z}.
\]

From relation (3.20) it follows that \( \gamma_1(n) \to 0 \) as \( n \to -\infty \). Moreover, we observe that
\[
\gamma_1(n) = s_1(n) + T_{n-1} \left( \sum_{k=-\infty}^{-1} \frac{s_1(k)}{\alpha_k} \right) + T_{n-1} \left( \sum_{k=0}^{n-1} s_1(k) \right), \quad \forall n \in \mathbb{N}^*.
\]

Denoting by
\[
\begin{align*}
z & = \sum_{k=-\infty}^{-1} \frac{s_1(k)}{\alpha_k} \quad \text{and} \quad w = \sum_{k=0}^{\infty} s_1(k)
\end{align*}
\]
and using relation (3.21) we successively obtain that
\[
\|\gamma_1(n)\| \leq \|s_1(n)\| + \|T_{n-1}(z)\| + \|T_{n-1} \left( w - \sum_{k=n}^{\infty} s_1(k) \right) \| \leq
\]
\[ \leq ||s_1(n)|| + ||T_{n-1}(z)|| + ||T_{n-1}(w)|| + \sum_{k=n}^{\infty} ||s_1(k)||, \quad \forall n \in \mathbb{N}^*. \quad (3.22) \]

From relation (3.22) it follows that \( \gamma_1(n) \longrightarrow 0 \) as \( n \to \infty \). So we have that
\[ \gamma_1 \in c_0(Z, V). \quad (3.23) \]

We consider the sequence
\[ \gamma_s : Z \to X, \quad \gamma_s(n) = (\gamma_1(n), \gamma_2(n), \gamma_3(n)). \]

From relations (3.19) and (3.23) it follows that \( \gamma_s \in c_{0,\infty}(Z, X) \). An easy computation shows that the pair \( (\gamma_s, s) \) satisfies the system \( (S_A) \).

To prove the uniqueness of \( \gamma_s \), let \( \tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3) \in c_{0,\infty}(Z, X) \) be such that the pair \( (\tilde{\gamma}, s) \) satisfies the system \( (S_A) \). Denoting by \( \delta = \tilde{\gamma} - \gamma_s \) we have that 
\[ \delta = (\delta_1, \delta_2, \delta_3) \] with \( \delta_j \in c_{0,\infty}(Z, V) \), for all \( j \in \{1, 2, 3\} \) and 
\[ \delta_1(n) = \frac{\alpha_n}{\alpha_k} T_{n-1} \delta_1(k), \quad \forall n > k \quad (3.24) \]
and
\[ \delta_2(n) = \delta_2(k), \quad \forall n > k \quad (3.25) \]
and
\[ \delta_3(n) = e^{n-k} \delta_3(k), \quad \forall n > k. \quad (3.26) \]

From (3.25) it follows that there is \( u \in V \) such that \( \delta_2(n) = u \), for all \( n \in Z \). But, since \( \delta_2 \in c_{0,\infty}(Z, V) \) it follows that \( u = 0 \), so \( \delta_2 = 0 \).

Let \( n \in Z \). From relation (3.24) we have that 
\[ ||\delta_1(n)|| \leq \frac{\alpha_n}{\alpha_k} ||\delta_1(k)|| \leq \frac{\alpha_n}{\alpha_k} ||\delta_1||_\infty, \quad \forall k < n. \quad (3.27) \]

For \( k \to -\infty \) in (3.27) it follows that \( \delta_1(n) = 0 \). In addition, from (3.26) we deduce that 
\[ ||\delta_3(n)|| = e^{-(m-n)} ||\delta_3(m)|| \leq e^{-(m-n)} ||\delta_3||_\infty, \quad \forall m > n. \quad (3.28) \]

For \( m \to \infty \) in (3.28) we obtain that \( \delta_3(n) = 0 \). Since \( n \in Z \) was arbitrary it follows that \( \delta_1 = 0 \) and \( \delta_3 = 0 \). This implies that \( \tilde{\gamma} = \gamma_s \), so \( \gamma_s \) is uniquely determined.

Let 
\[ q_2 : Z \to V, \quad q_2(n) = -\sum_{k=n+1}^{\infty} s_2(k). \]
Since \( s_2 \in \ell^1(Z, V) \) it is easy to see that \( q_2 \in c_{\infty,0}(Z, V) \). We consider the sequence 
\[ q_s : Z \to X, \quad q_s(n) = (\gamma_1(n), q_2(n), \gamma_3(n)). \]

Then, we have that \( q_s \in c_{\infty,0}(Z, X) \) and an easy computation shows that the pair \( (q_s, s) \) satisfies the system \( (S_A) \). Using analogous arguments with those presented to prove the uniqueness of \( \gamma_s \) we deduce that \( q_s \) is also uniquely determined.

In this way, we obtain that the pair \((\ell^\infty(Z, X), \ell^1(Z, X))\) is admissible for the system \( (S_A) \).

**Step 2.** We prove that the system \((A)\) is not uniformly exponentially trichotomic.

Suppose by contrary that the system \((A)\) has a uniform exponential trichotomy with respect to the families of projections \( \{P_j(n)\}_{n \in Z}, j \in \{1, 2, 3\} \) and let \( K, \nu > 0 \) be two constants given by Definition 2.1. From Definition 2.1(v) we have that
\[ ||\Phi_A(m, n)x|| \leq Ke^{-\nu(m-n)}||x||, \quad \forall x \in Range P_1(n), \nu(m, n) \in \Delta. \quad (3.29) \]
According to the structure theorem - Theorem 3.4 in [24] we have that
\[ \text{Range } P_1(n) = S(n) = \{ x \in X : \lim_{k \to \infty} \Phi_A(k, n)x = 0 \}, \quad \forall n \in \mathbb{Z}. \] (3.30)

From (3.30) it necessarily follows that
\[ \text{Range } P_1(n) = V \times \{ 0 \} \times \{ 0 \}, \quad \forall n \in \mathbb{Z}. \] (3.31)

Then, from relations (3.29) and (3.31) we obtain that
\[ \frac{\alpha_m}{\alpha_n} ||T_{m-1}(v)|| \leq Ke^{-\nu(m-n)||v||}, \quad \forall v \in V, \forall m, n \in \mathbb{Z}, m > n. \] (3.32)

In particular, for \( n = 0 \) in (3.32) we deduce that
\[ ||T_{m-1}(v)|| \leq Ke^{-\nu m ||v||}, \quad \forall v \in V, \forall m \in \mathbb{N}^*. \] (3.33)

For \( v = \varphi_m \) in (3.33) it follows that \( 1 \leq Ke^{-\nu m} \), for all \( m \in \mathbb{N}^* \), which is absurd.

In conclusion, we have that the system (A) is not uniformly exponentially trichotomic.

**Remark 3.3.** The above example shows that the results obtained in Theorem [3.3] are the most general in this topic.

4. **Applications to the study of the uniform exponential trichotomy of evolution families.** In this section, based on the central results of this paper and on a theorem obtained in [24], we will present new applications concerning the study of the exponential trichotomy of evolution families providing very general criteria which are applicable to any nonautonomous dynamical system modeled by evolution families on the whole axis. Indeed, we start with the basic definitions and notations.

Let \( X \) be a Banach space. We denote by \( I_d \) the identity operator on \( X \).

**Definition 4.1.** We say that \( U = \{ U(t, s) \}_{t \geq s} \subset \mathcal{B}(X) \) is an evolution family if the following properties are fulfilled:

(i) \( U(t, t) = I_d \), for all \( t \in \mathbb{R} \), and \( U(t, s)U(s, \tau) = U(t, \tau) \), for all \( t \geq s \geq \tau \);

(ii) there exist \( M \geq 1 \) and \( \omega > 0 \) such that \( ||U(t, s)|| \leq Me^{\omega(t-s)} \), for all \( t \geq s \).

**Definition 4.2.** ([24]) We say that an evolution family \( U = \{ U(t, s) \}_{t \geq s} \) has a uniform exponential trichotomy if there exist three families of projections \( \{ P_k(t) \}_{t \in \mathbb{R}} \subset \mathcal{B}(X), k \in \{ 1, 2, 3 \} \) and two constants \( L \geq 1 \) and \( \nu > 0 \) such that the following properties are satisfied:

(i) \( U(t, t_0)P_k(t_0) = P_k(t_0)U(t, t_0) \), for all \( t \geq t_0 \) and \( k \in \{ 1, 2, 3 \} \);

(ii) \( P_k(t)P_j(t) = 0 \), for all \( k \neq j \) and \( t \in \mathbb{R} \);

(iii) \( P_1(t) + P_2(t) + P_3(t) = I_d \), for all \( t \in \mathbb{R} \);

(iv) \( \sup_{t \in \mathbb{R}} ||P_k(t)|| < \infty \), for all \( k \in \{ 1, 2, 3 \} \);

(v) \( ||U(t, t_0)x|| \leq Le^{-\nu(t-t_0)||x||} \), for all \( x \in \text{Range } P_1(t_0) \) and all \( t \geq t_0 \);

(vi) \( \frac{1}{\nu} ||x|| \leq ||U(t, t_0)x|| \leq L||x|| \), for all \( x \in \text{Range } P_2(t_0) \) and all \( t \geq t_0 \);

(vii) \( ||U(t, t_0)x|| \geq \frac{1}{\nu} e^{\nu(t-t_0)}||x|| \), for all \( x \in \text{Range } P_3(t_0) \) and all \( t \geq t_0 \);

(viii) for each \( k \in \{ 2, 3 \} \), the restriction \( U_k(t, t_0) : \text{Range } P_k(t_0) \to \text{Range } P_k(t) \) is an isomorphism, for all \( t \geq t_0 \).
Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on $X$. We associate to $\mathcal{U}$ the discrete dynamical system
\[ (A_n) \quad x(n+1) = U(n+1,n)x(n), \quad n \in \mathbb{Z}. \]

For the first time in the literature, we have obtained in [24] the following result (see Theorem 4.3 in [24]):

**Theorem 4.1.** ([24]) Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on $X$. Then $\mathcal{U}$ has a uniform exponential trichotomy if and only if the discrete dynamical system $(A_n)$ associated to $\mathcal{U}$ has a uniform exponential trichotomy.

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on $X$. We consider the input-output control system
\[ (S_n) \quad \gamma(n+1) = U(n+1,n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z} \]
with $s \in \ell^1(\mathbb{Z}, X)$ and $\gamma \in \ell^\infty(\mathbb{Z}, X)$.

By applying the main results obtained in the previous section, we obtain new and very general criteria for the existence of the uniform exponential trichotomy:

**Theorem 4.2.** Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on $X$. Let $r, p \in [1, \infty]$ with $(r, p) \neq (\infty, 1)$. The following assertions hold:

(i) if the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is $(r, p)$-admissible for the system $(S_n)$, then the evolution family $\mathcal{U}$ has a uniform exponential trichotomy;

(ii) if $r \geq p$, then the evolution family $\mathcal{U}$ has a uniform exponential trichotomy if and only if the pair $(\ell^\infty(\mathbb{Z}, X), \ell^1(\mathbb{Z}, X))$ is $(r, p)$-admissible for the system $(S_n)$.

**Proof.** This follows from Theorem 3.3 and from Theorem 1.1.

**REFERENCES**

[1] C. V. Coffman and J. J. Schäffer, Dichotomies for linear difference equations, Math. Ann., 172 (1967), 139–166.

[2] S. Elaydi and O. Hájek, Exponential dichotomy of differential systems, J. Math. Anal. Appl., 129 (1988), 362–374.

[3] S. Elaydi and O. Hájek, Exponential dichotomy and trichotomy of nonlinear differential equations, J. Differ. Integral Equ., 3 (1990), 1201–1224.

[4] S. Elaydi and K. Janglajew, Dichotomy and trichotomy of difference equations, J. Math. Anal. Appl., 3 (1998), 417–448.

[5] J. L. Massera and J. J. Schäffer, Linear Differential Equations and Function Spaces, Academic Press, 1966.

[6] N. Van Minh, On the proof of characterisations of the exponential dichotomy, Proc. Amer. Math. Soc., 127 (1999), 779–782.

[7] N. Van Minh and N. T. Huy, Characterizations of dichotomies of evolution equations on the half-line, J. Math. Anal. Appl., 261 (2001), 28–44.

[8] N. Van Minh and J. Wu, Invariant manifolds of partial functional differential equations, J. Differential Equations, 198 (2004), 381–421.

[9] K. J. Palmer, Exponential dichotomies and transversal homoclinic points, J. Differential Equations, 55 (1984), 225–256.

[10] K. J. Palmer, Exponential dichotomies and Fredholm operators, Proc. Amer. Math. Soc., 104 (1988), 149–156.

[11] K. J. Palmer, Shadowing and Silnikov chaos, Nonlinear Anal., 27 (1996), 1075–1093.

[12] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Math. Z., 32 (1930), 703–728.

[13] V. A. Pliss and G. R. Sell, Robustness of the exponential dichotomy in infinite-dimensional dynamical systems, J. Dynam. Differential Equations, 3 (1999), 471–513.

[14] C. Pötzsch, Geometric Theory of Discrete Nonautonomous Dynamical Systems, Lecture Notes in Mathematics, vol. 2002, Springer, 2010.
[15] C. Pötzsche, Fine structure of the dichotomy spectrum, *Integral Equations Operator Theory*, **73** (2012), 107–151.

[16] C. Pötzsche, Dichotomy spectra of triangular equations, *Discrete Contin. Dyn. Syst.*, **36** (2016), 423–450.

[17] R. J. Sacker and G. R. Sell, Existence of dichotomies and invariant splittings for linear differential systems, III, *J. Differential Equations*, **22** (1976), 497–522.

[18] B. Sasu and A. L. Sasu, Exponential dichotomy and \( (\ell^p, \ell^q) \)-admissibility on the half-line, *J. Math. Anal. Appl.*, **316** (2006), 397–408.

[19] B. Sasu, Uniform dichotomy and exponential dichotomy of evolution families on the half-line, *J. Math. Anal. Appl.*, **323** (2006), 1465–1478.

[20] B. Sasu and A. L. Sasu, Exponential trichotomy and p-admissibility for evolution families on the real line, *Math. Z.*, **253** (2006), 515–536.

[21] A. L. Sasu, Exponential dichotomy and dichotomy radius for difference equations, *J. Math. Anal. Appl.*, **344** (2008), 906–920.

[22] A. L. Sasu and B. Sasu, Input-output admissibility and exponential trichotomy of difference equations, *J. Math. Anal. Appl.*, **380** (2011), 17–32.

[23] B. Sasu and A. L. Sasu, On the dichotomic behavior of discrete dynamical systems on the half-line, *Discrete Contin. Dyn. Syst.*, **33** (2013), 3057–3084.

[24] A. L. Sasu and B. Sasu, Discrete admissibility and exponential trichotomy of dynamical systems, *Discrete Contin. Dyn. Syst.*, **34** (2014), 2929–2962.

[25] A. L. Sasu and B. Sasu, Admissibility and exponential trichotomy of dynamical systems described by skew-product flows, *J. Differential Equations*, **260** (2016), 1656–1689.

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