The size of quadratic $p$-adic linearization disks

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Abstract

We find the exact radius of linearization disks at indifferent fixed points of quadratic maps in $\mathbb{C}_p$. We also show that the radius is invariant under power series perturbations. Localizing all periodic orbits of these quadratic-like maps we then show that periodic points are not the only obstruction for linearization. In so doing, we provide the first known examples in the dynamics of polynomials over $\mathbb{C}_p$ where the boundary of the linearization disk does not contain any periodic point.

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1 Introduction

Let $p$ be a prime and let $\mathbb{C}_p$ be the completion of an algebraic closure of the field of $p$-adic numbers $\mathbb{Q}_p$. We study a $p$-adic analogue of the Siegel center problem in complex dynamics. A power series

$$f(x) = \lambda x + \langle x^2 \rangle \in \mathbb{C}_p[[x]], \quad |\lambda| = 1, \text{ not a root of unity},$$

has an irrationally indifferent fixed point at $x = 0$, and is said to be analytically linearizable at $x = 0$ if there exists a convergent power series solution $H_f$ to the functional equation

$$H_f \circ f \circ H_f^{-1}(x) = \lambda x.$$  

For a large class of quadratic-like maps we find the exact radius of the corresponding linearization disk about $x = 0$. Localizing the periodic orbits of these maps, we then show that the convergence of $H_f$ stops before the appearance of any periodic point different from $x = 0$. Our starting point is Yoccoz study of quadratic Siegel disks [31]. Let $\alpha$ be irrational and $\{p_n/q_n\}_{n \geq 0}$ be the approximants given by its continued fraction expansion. The Brjuno series $B(\alpha)$ is defined by

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\( B(\alpha) = \sum_{n \geq 0} \log(q_{n+1})/q_n \). Yoccoz proved that \( P_\alpha(z) = e^{2i\pi \alpha}z + z^2 \in \mathbb{C}[z] \) is linearizable at \( z = 0 \) if and only if \( B(\alpha) < \infty \). If \( B(\alpha) < \infty \) and \( C(\alpha) \) is the conformal radius of the Siegel disk, then there exist universal constants \( C_1 \) and \( C_2 \) such that \( C_1 < B(\alpha) + \log C(\alpha) < C_2 \). The lower bound is due to Yoccoz \([31]\), and the upper bound is due to Buff and Chéritat \([5]\). A possible obstruction for linearization is the presence of periodic points different from zero. Indeed, Yoccoz \([23, \text{Theorem V.4.2}]\) proved that if \( \alpha \) is irrational and \( z = 0 \) contains a periodic orbit of \( P_\alpha \) different from \( z = 0 \). However, as shown by Pérez-Marco \([23, \text{Theorem V.4.2}]\), in complex dynamics there exist maps having no periodic point on the boundary of a Siegel disk. Below, we present an analogue of the Yoccoz Brjuno function, \( \tilde{r}_p \), that estimates the radius of \( p \)-adic linearization disks (Theorem \([\text{A}]\)). Using the optimal bound obtained for \( p \)-adic quadratic maps (Theorem \([\text{B}]\)), we obtain an analogue of Pérez-Marco’s result on non-existence of periodic points on the boundary (Corollary \([\text{C}]\)), providing the first \( p \)-adic examples of its kind.

Let \( p \) be a prime and denote by \( \mathcal{O}_p \) the closed unit disk in \( \mathbb{C}_p \). For any \( \lambda \in \mathbb{C}_p \) with \( |\lambda| = 1 \) but \( \lambda \) not a root of unity, we define

\[
\mathcal{G}_\lambda := \lambda x + x^2 \mathcal{O}_p[[x]].
\]

By the Non-Archimedean Siegel Theorem of Herman and Yoccoz \([11]\), \( f \in \mathcal{G}_\lambda \) is always linearizable at \( x = 0 \). Given \( f \in \mathcal{G}_\lambda \), we denote by \( \Delta_f \) the corresponding linearization disk, i.e. the maximal disk about the origin on which \( f \) is analytically conjugate to \( T_\lambda : x \to \lambda x \). We denote by \( \tilde{r}(f) \) the exact radius of \( \Delta_f \) and introduce a function \( \tilde{r} = \tilde{r}(\lambda) \) that estimates \( r(f) \) from below. There is an explicit formula for \( \tilde{r}(\lambda) \), to be given in formula \([\text{11}]\) in Section \([\text{II}]\) with the properties stated in Theorem \([\text{A}]\) and \([\text{B}]\) below.

**Theorem A.** Let \( f \in \mathcal{G}_\lambda \), then \( r(f) \geq \tilde{r}(\lambda) \).

The proof is based on estimates of the coefficients of the conjugacy function \( H_f \) and an application of the Weierstrass Preparation Theorem. As our main result, refining the estimates of \( H_f \) for quadratic maps, we obtain the exact size of quadratic linearization disks.

**Theorem B.** Let \( p \geq 3 \) and \( \lambda \in \mathbb{C}_p \) be not a root of unity and put

\[
P_\lambda(x) := \lambda x + x^2 \in \mathbb{C}_p[x], \quad \text{where } 1/p < |1-\lambda| < 1.
\]

Then, the linearization disk \( \Delta_{P_\lambda} \) is the open disk \( D_{r(P_\lambda)}(0) \) where \( r(P_\lambda) = |1-\lambda|^{-1/p} \tilde{r}(\lambda) \).

In view of Theorem \([\text{B}]\) the general estimate \( \tilde{r}(\lambda) \) is nearly optimal. The work lies in obtaining the optimal estimates of the coefficients of \( H_{P_\lambda} \) carried out in Section \([\text{IV}]\) and \([\text{V}]\). To our knowledge, this is the first known case in \( \mathbb{C}_p \) where the exact linearization disk is known without having an explicit formula for the conjugacy \( H_f \). In fact, the radius \( r(P_\lambda) \) is invariant under power series perturbations.

**Corollary B.** Let \( p \geq 3 \) and \( f \in \mathcal{G}_\lambda \), with \( 1/p < |1-\lambda| < 1 \), be of the form

\[
f(x) = \lambda x + \sum_{i=2}^{\infty} a_i x^i \in \mathcal{O}_p[[x]], \quad \text{where } |a_2| = 1 \text{ and } |a_2^2 - a_3| = 1. \tag{1}
\]

Then, the linearization disk \( \Delta_f \) equals the disk \( \Delta_{P_\lambda} \) of radius \( r(P_\lambda) = |1-\lambda|^{-1/p} \tilde{r}(\lambda) \).
By Corollary C, the exact radius $r(f) = |1 - \lambda|^{-1/p\tilde{r}(\lambda)}$ depends only on $\lambda$ for $f$ in this family. Also note the we only impose an extra condition on the cubic term of $f$ for the generalization to hold. In fact, this condition is the same as the condition for $f$ to be minimally ramified [25]. Using Corollary C and localizing all periodic orbits of $f$, we prove the non-existence of periodic points on the boundary $\partial \Delta_f := \{x \in \mathbb{C}_p : |x| = r(f)\}$.

**Corollary C.** Let $p \geq 3$ and let $f$ be of the form (1). Then, the boundary $\partial \Delta_f$ of the linearization disk does not contain any periodic point of $f$.

**Remark 1.1.** The proof of Corollary C induces a stronger version of the Corollary itself in the sense that in Lemma 6.1 we localize all periodic orbits for every $f$ that $0 < |1 - \lambda| < 1$, $|a_2| = 1$ and $|a_2^3 - a_3| = 1$, using that such maps are minimally ramified [24].

Corollary C, similar in its flavor to the complex field case result of Pérez-Marco [23 Theorem V.4.2], reveals a new phenomenon in the dynamics of $p$-adic polynomials. In contrast to our result obtained for quadratic-like maps, the $p$-adic power functions, which are the only previously known examples where the exact size of the linearization disk is known, all have periodic points on the boundary of the linearization disk [2,21], see figure 3 below. For $g(x) = a_0 + a_1 x + \cdots \in \mathcal{O}_p[[x]]$, denote by $\text{wideg}(g)$ the Weierstrass degree of $g$, i.e. the smallest number $d \in \{0, 1, \ldots \} \cup \{\infty\}$ such that $|a_d| = 1$. If $d$ is finite, then $d$ is the number of zeros of $g$ in the open unit disk, counting multiplicity. In fact, in view of Rivera-Letelier’s classification of $p$-adic Siegel disks [25], every polynomial $f \in \mathcal{G}_\lambda$ with $2 \leq \text{wideg}(f - \text{id}) < \infty$, has infinitely many periodic points in the open unit disk. From this point of view, and the results obtained for power maps, it may be tempting to conjecture that a generic $f \in \mathcal{G}_\lambda$, with $2 \leq \text{wideg}(f - \text{id}) < \infty$, has a periodic point on $\partial \Delta_f$. Corollary C shows that such a conjecture is indeed false.

The paper is organized as follows. In Section 2 we present preliminaries concerning power series and the formal solution, and in Section 3 we consider the geometry of the unit sphere and the arithmetic of the multiplier $\lambda$. Sections 4 and 5 are devoted to explicit, and in several cases optimal, bounds of the conjugacy function $H_f$, from which we obtain Theorem A and B respectively. In Section 6 we give a proof of Corollary C. Section 7 ends with an example illustrating the main idea.

We end this introduction by some remarks given below. For some additional references on non-Archimedean dynamics and its relationship, similarities, and differences with respect to the Archimedean theory of complex dynamics, see [1,9,14,15,28] and references therein.

### 1.1 Further remarks

**Remark 1.2.** Our estimate of $r(f)$ in Theorem A extends results obtained for quadratic polynomials over $\mathbb{Q}_p$ by Ben-Menahem [3], and Thiran, Verstegen, and Weyers [29], and for certain polynomials with maximal multipliers over the $p$-adic integers $\mathbb{Z}_p$ by Pettigrew, Roberts and Vivaldi [24], as well as results on small divisors by Khrennikov [13].

**Remark 1.3.** If $f \in \mathcal{G}_\lambda$ and $2 \leq \text{wideg}(f - \text{id}) < \infty$, then we also have the upper bound $r(f) < 1$, since in this case $f$ has at least one periodic point $x \neq 0$ in the open unit disk.

**Remark 1.4.** There is a number of results on the existence of critical points and/or failing of injectivity on the boundary of complex quadratic linearization disks since the work of Herman [10]. See [4,22] for more results and references. Another possible obstruction for linearization would
be the presence of transient points, i.e. points on \( \partial \Delta_f \) that escape \( \partial \Delta_f \) after some iterations. But neither of these obstructions occur for \( f \in G_{\lambda} \) with \( 2 \leq \deg(f - \text{id}) < \infty \); by Remark 1.3 \( r(f) < 1 \), and since \( f \) is isometric on the open unit disk, it is also isometric on \( \partial \Delta_f \). Hence, it remains open whether or not there is a ‘dynamical’ obstruction for linearization on \( \partial \Delta_f \) for \( f \) of the form \([1]\).

**Remark 1.5.** In contrast to the complex field case \([6, 22]\), the boundary of the linearization disk may not be contained in the topological closure of the post-critical set, see Remark 6.1.

**Remark 1.6.** For irrationally indifferent fixed points, the conjugacy function is related to the iterative Lie-logarithm, \( f_* = \lim_{n \to \infty} (f^{o n} \circ \text{id})/p^n \). Formally, \( H_f = \exp(\int \log(\lambda)/f_* \circ \text{id}) \). See \([16, 21, 25]\) and references therein for studies of \( f_* \) and its connection to the dynamics of \( f \).

**Remark 1.7.** There are two possible definitions of a Siegel disk in the \( p \)-adic setting: the linearization disk, or the maximal domain of quasi-periodicity. In contrast to the complex field case, they do not coincide in the \( p \)-adic setting since \( p \)-adic domains of quasi-periodicity contain periodic points \([25]\). It is standard in \( p \)-adic dynamics to use the latter definition.

**Remark 1.8.** In the multi-dimensional \( p \)-adic case, there exist multipliers \( \lambda \) such that the corresponding Siegel condition is violated and the conjugacy diverges \([11, 30]\). In fields of positive characteristics, the convergence of the linearization series is far from trivial even in the one-dimensional case \([17, 19]\). For results in fields of characteristic zero–equal characteristic case, see \([18]\).

### 2 Mapping properties and the formal solution

By definition, \( \mathbb{C}_p \) is a non-Archimedean field, i.e. complete with respect to a non-trivial absolute value \( | \cdot | \), satisfying the following ultrametric triangle inequality:

\[ |x + y| \leq \max(|x|, |y|), \quad \text{for all } x, y \in \mathbb{C}_p. \tag{2} \]

One useful consequence of ultrametricity is that for any \( x, y \in \mathbb{C}_p \) with \( |x| \neq |y| \), the inequality (2) becomes an equality. The absolute value on \( \mathbb{C}_p \) is normalized so that \( |p| = 1/p \). For \( x \in \mathbb{C}_p \) we denote by \( \nu(x) \) the valuation of \( x \), i.e.

\[ \nu(x) := -\log_p |x|, \quad \text{where } \nu(0) := \infty. \]

Given an element \( x \in \mathbb{C}_p \) and real number \( r > 0 \) we denote by \( D_r(x) \) the open disk of radius \( r \) about \( x \), by \( \overline{D}_r(x) \) the closed disk, and by \( S_r(x) \) the sphere of radius \( r \) about \( x \). Put \( \mathbb{C}_p^* := \mathbb{C}_p \setminus \{0\} \). If \( r \in \mathbb{C}_p^* \), i.e. if there exist \( a \in \mathbb{C}_p^* \) such that \( |a| = r \), we say that \( D_r(x) \) and \( \overline{D}_r(x) \) are rational. If \( r \notin \mathbb{C}_p^* \), then we will call \( D_r(x) = \overline{D}_r(x) \) an irrational disk. Note that all disks are both open and closed as topological sets, because of ultrametricity. However, as we will see below, power series distinguish between rational open, rational closed, and irrational disks.

#### 2.1 Univalent mappings

In this paper we consider univalent mappings fixing zero. Let \( a > 0 \) be a real number and put

\[ U_a := \left\{ \sum_{i \geq 1} a_i x^i \in \mathbb{C}_p[[x]] : |a_1| = 1, \text{ and } 1/\sup_{i \geq 2} |a_i|^{1/(i-1)} \geq a \right\}. \]
Then, \( h \in U_a \) converges on the open disk \( D_{\rho(h)}(0) \) of radius

\[
\rho(h) := \frac{1}{\limsup |a_i|^{1/i}} \geq a.
\]  

(3)

A power series \( h \in U_a \) converges on the sphere \( S_{\rho(h)}(0) \) if and only if

\[
\lim_{i \to \infty} |a_i| \rho(h)^i = 0.
\]

(4)

It is well known that all \( h \in U_a \) are univalent and isometric on the open disk \( D_a(0) \). For future reference, we state these facts in the following proposition.

**Proposition 2.1.** Let \( h \in U_a \). Then, \( h : D_a(0) \to D_a(0) \) is a bijective isometry. In particular, if \( f \in G^\lambda \), then \( f : D_1(0) \to D_1(0) \) is bijective and isometric.

In fact, this result can be proven using the following generalization of the Weierstrass Preparation Theorem, see Benedetto [4, Lemma 2.2], that we will utilize later on, proving Theorem A and Theorem B, respectively.

**Proposition 2.2.** Let \( h \in U_a \). Then the following two statements hold.

1. Suppose that \( h \) converges on the rational closed disk \( \overline{D}_R(0) \). Let \( 0 < r \leq R \) and suppose that

\[
|c_i| r^i \leq r \quad \text{for all } i \geq 2.
\]  

Then, \( h : D_r(0) \to D_r(0) \) is a bijection. Furthermore, if

\[
d = \max \{ i \geq 1 : |c_i| r^i = r \},
\]

then \( h \) maps the closed disk \( \overline{D}_r(0) \) onto itself exactly \( d \)-to-1 (counting multiplicity).

2. Suppose that \( h \) converges on the rational open disk \( D_R(0) \). Let \( 0 < r \leq R \) and suppose that

\[
|c_i| r^i \leq r \quad \text{for all } i \geq 2.
\]

Then, \( h : D_r(0) \to D_r(0) \) is bijective.

### 2.2 The formal solution

With \( f \in G^\lambda \), we associate a conjugacy function \( H_f \in \mathbb{C}_p[[x]] \) such that \( H_f \circ f(x) = \lambda H_f(x) \), and normalized so that \( H_f(0) = 0 \) and \( H'_f(0) = 1 \). The lower bound for the size of linearization disks for \( f \in G^\lambda \) is based on the following lemma.

**Lemma 2.1.** Let \( f \in G^\lambda \). Then, there exists a conjugacy function \( H_f(x) = \sum_{k=1}^\infty b_k x^k \) in \( \mathbb{C}_p[[x]] \). The coefficients of \( H_f \) satisfy, \( b_1 = 1 \) and

\[
|b_k| \leq \left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1},
\]  

(5)

for all \( k \geq 2 \).
Proof. The existence of $H_f$ was proven in [11]. Let $f(x) = \sum_{i=1}^{\infty} a_i x^i \in G_\lambda$ and let $H_f(x) = \sum_{k=1}^{\infty} b_k x^k$. Let $k \geq 2$. Solving the equation $H_f \circ f(x) = \lambda H_f(x)$ for the coefficients of $x^k$ we obtain

$$b_k = \frac{1}{\lambda(1 - \lambda^{k-1})} \sum_{i=1}^{k-1} b_i C_l(k),$$

where $C_l(k)$ is the coefficient of the $x^k$-term in

$$(a_1 x + a_2 x^2 + \ldots)^l.$$ 

Let $\mathbb{N}$ be the set of non-negative integers. For $k \geq 2$ and $l \in [1, k-1]$, put

$$A_l(k) := \{(a_1, \ldots, a_k) \in \mathbb{N}^k : \sum_{i=1}^{k} a_i = l, \text{ and } \sum_{i=1}^{k} i a_i = k\}.$$ 

It follows by the Multinomial Theorem that

$$C_l(k) = \sum_{(a_1, a_2, \ldots, a_k) \in A_l(k)} \frac{l!}{a_1! \cdots a_k!} \lambda^{a_1} a_2^{a_2} \cdots a_k^{a_k}.$$ 

Note that the factorial factors, $l!/(a_1! \cdots a_k!)$, are integers and thus of absolute value less than or equal to 1. Also recall that by definition, $|a_i| \leq 1$. It follows by ultrametricity that $|C_l(k)| \leq 1$ and

$$|b_k| \leq \left(\prod_{n=1}^{k-1} \left|1 - \lambda^n\right|\right)^{-1},$$

for all integers $k \geq 2$ as required.

3 Geometry of the unit sphere and the roots of unity

Let $\Gamma$ be the group of all roots of unity in $\mathbb{C}_p$. It has the subgroup $\Gamma_u$ ($u$ for unramified), where

$$\Gamma_u := \{\xi \in \mathbb{C}_p : \xi^m = 1 \text{ for some integer } m \geq 1 \text{ not divisible by } p\}.$$ 

Proposition 3.1. The unit sphere $S_1(0)$ in $\mathbb{C}_p$ decomposes into the disjoint union

$$S_1(0) = \bigcup_{\xi \in \Gamma_u} D_1(\xi).$$

In particular, $\Gamma_u \cap D_1(1) = \{1\}$ and consequently $|1 - \xi| = 1$ for all $\xi \in \Gamma_u$ such that $\xi \neq 1$. To each $\lambda \in S_1(0)$ there are unique $\xi \in \Gamma_u$ and $h \in D_1(1)$ such that $\lambda = \xi h$.

A proof is given in [26] p. 103. The other important subgroup of $\Gamma$ is $\Gamma_r$ ($r$ for ramified), where

$$\Gamma_r := \{\xi \in \mathbb{C}_p : \xi^p = 1 \text{ for some integer } s \geq 0\}.$$ 

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By elementary group theory $\Gamma_u \cap \Gamma_r = \{1\}$ and $\Gamma$ is the direct product $\Gamma = \Gamma_r \cdot \Gamma_u$. Most importantly, for each $s \geq 1$ the primitive $p^s$-roots of unity in $\Gamma_r$ are located in the sphere about $x = 1$ of radius

$$R(s) := p^{-\frac{1}{m-1}}.$$  

We also put $R(0) := 0$. Note that $R(s)$ is strictly increasing with $s$. For integers $s \geq 0$, we will denote by $A_s$ the group of $p^s$-roots of unity in $\Gamma_r$ and by $B_s$ the set $A_s \setminus A_{s-1}$. We have the following proposition.

**Proposition 3.2** (See Theorem 8.9 in [8]). For integers $s \geq 1$, $B_s$ consists of $p^s - p^{s-1}$ roots of order $p^s$, which all lie in $S_{R(s)}(1)$. Moreover, for every $\zeta \in B_s$, we have $B_s \cap D_{R(s)}(\zeta) = \zeta A_{s-1}$.

The latter statement stems from the fact that multiplication by a root of unity $\zeta$ induces an isometry of $\mathbb{C}_p$. Moreover, by ultrametricity, for every $r > 0$ and $\zeta \in D_r(1)$ we have $D_r(1) = D_r(\zeta)$. As a consequence, the arrangement of roots of unity around $\zeta$ looks exactly the same as the arrangement of roots of unity around 1. By the two propositions above and ultrametricity, we have the following lemma (see also figure [1]).

**Lemma 3.1.** Let $\lambda \in S_1(0)$ be not a root of unity. Then, $m = \min\{n \in \mathbb{Z} : n \geq 1, |1 - \lambda^n| < 1\}$ is well defined and not divisible by $p$. Let $\gamma_0 \in \Gamma_r$ be minimizing $|\gamma_0 - \lambda^m|$ and let $t \geq 0$ be the integer such that

$$R(t) \leq |\gamma_0 - \lambda^m| < R(t+1).$$

Then, for every $\gamma \in \gamma_0 A_t$ we have $|\gamma - \lambda^m| = |\gamma_0 - \lambda^m|$ and for every integer $j \geq t + 1$ and $\zeta \in S_{R(j)}(\gamma_0) \cap \Gamma_r$ we have $|\zeta - \lambda^m| = R(j)$. Moreover, the cardinality $\#S_{R(j)}(\gamma_0) \cap \Gamma_r = p^j - p^{j-1}$. In particular, if $\lambda^m \notin S_{R(s)}(1)$ for any $s \geq 1$, then we may assume $\gamma_0 = 1$.

### 3.1 Arithmetic of the multiplier

**Lemma 3.2.** Let $\lambda \in \mathbb{C}_p$ be not a root of unity and $|\lambda| = 1$. Let $m$, $\gamma_0$ and $t$ be as in Lemma 3.1. Let $s \geq 0$ be the integer for which $R(s) \leq |1 - \lambda^m| < R(s+1)$. Then, for integers $n \geq 1$:

1. if $m$ does not divide $n$, we have $|1 - \lambda^n| = 1$,
2. if $m$ is a divisor of $n$ and $\nu(n) < s$, we have $|1 - \lambda^n| = |1 - \lambda^m|^{p^{\nu(n)}}$,
3. if $m$ is a divisor of $n$ and $\nu(n) \geq s$, we have $|1 - \lambda^n| = p^f|n||\gamma_0 - \lambda^m|^{p^f}$. In particular, if $\lambda^m \notin S_{R(s)}(1)$, then $\gamma_0 = 1$ and $t = s$.

**Remark 3.1.** In the third statement, $|n|p^f \leq 1$ since we assume that $t \leq s \leq \nu(n)$ in this case.

**Proof of Lemma 3.2** Part 1 is a direct consequence of Lemma 3.1. To prove 2 and 3 it is enough to consider the case $m = 1$. Hence, we shall assume that $\lambda \in D_1(1)$. The proof is based on the factorization of the polynomial $\lambda^n - 1$ from which we obtain

$$|\lambda^n - 1| = \prod_{\theta^m = 1} |\lambda - \theta|.$$
Figure 1: (Illustration of Lemmas 3.1 and 3.2). Let $\gamma$ be a primitive 9-root of unity. The filled dots illustrate the roots of unity of order 1, 3, and $3^2$ respectively. Let $\lambda_* = \gamma + 3 \in \mathbb{C}_3$. In this case $|1 - \lambda_*| = |1 - \gamma| = R(2)$ so that $s(\lambda_*) = 2$, $m(\lambda_*) = 1$, $\gamma_0(\lambda_*) = \gamma$, with $|\gamma_0 - \lambda| = 1/3$ so that $t(\lambda_*) = 0$. The orbit of $\lambda_*$ will be infinite but on the larger scale it will resemble that of $\gamma$. Also note that $|\gamma - \gamma^4| = |\gamma||1 - \gamma^3| = R(1)$, whereas for example $|\gamma - \gamma^2| = R(2)$. By ultrametricity we have $S_{R(1)}(\gamma_0) \cap \Gamma_r = \{\gamma^4, \gamma^7\}$, $S_{R(2)}(\gamma_0) \cap \Gamma_r = \{1, \gamma^3, \gamma^6, \gamma^2, \gamma^5, \gamma^8\}$, and $S_{R(3)}(\gamma_0) \cap \Gamma_r = S_{R(3)}(1) \cap \Gamma_r = B_3$. The figure also illustrates the first nine points in the orbit of $\lambda = 1 + 3\frac{3}{10} \in \mathbb{C}_3$. For higher iterates, $\lambda^{10}$ will be close to $\lambda$, and $\lambda^{11}$ close to $\lambda^2$ and so forth. In this case $R(2) < |1 - \lambda| < R(3)$ and hence $s(\lambda) = 2$, $m(\lambda) = 1$, $\gamma_0(\lambda) = 1$ and $t(\lambda) = 2$. 

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As noted in the previous section, the set of roots of unity in \( \mathbb{C} \) is given by the product \( \Gamma = \Gamma_r \cdot \Gamma_u \). This representation enables us to write \(|\lambda^n - 1|\) in the form

\[
|\lambda^n - 1| = \prod_{\zeta = 1}^{\lambda - \zeta} \prod_{(\zeta \xi)^n = 1}^{\zeta - \zeta' \xi}, \tag{7}
\]

where \( \zeta \in \Gamma_r \) and \( \xi \in \Gamma_u \setminus \{1\} \). Recall that we assume that \( \lambda \in D(1) \). In view of Proposition 3.1 for \( \xi \neq 1 \) we have \( \zeta \xi \notin D(1) \) and consequently \( |\lambda - \zeta\xi| = 1 \). Moreover, for \( n \equiv p^{\nu(n)} \), we have that \( \zeta^n = 1 \) if and only if \( \zeta^{p^{\nu(n)}} = 1 \). It follows that \( (7) \) can be reduced to

\[
|\lambda^n - 1| = \prod_{\zeta \in A_{\nu(n)}} |\lambda - \zeta|. \tag{8}
\]

We distinguish between two cases given below.

**Case I:** \( s \geq 1 \) and \( \nu(n) \in [0, s - 1] \). In this case, for every \( \zeta \in A_{\nu(n)} \) we have

\[
|1 - \zeta| < R(s) \leq |\lambda - 1|.
\]

Consequently, by ultrametricity \( |\lambda - \zeta| = |(\lambda - 1) + (1 - \zeta)| = |\lambda - 1| \). As the cardinality \( \#A_{\nu(n)} = p^{\nu(n)} \), we obtain the desired result using (8).

**Case II:** \( s \geq 0 \) and \( \nu(n) \geq s \). We first consider the case \( \gamma_0 = 1 \) so that \( t = s \). In this case \( A_s \subseteq A_{\nu(n)} \) and hence

\[
\prod_{\zeta \in A_s} |\lambda - \zeta| = \prod_{\zeta \in A_s} |\lambda - \zeta| \prod_{s' = s+1}^{\nu(n)} \left( \prod_{\zeta \in B_{s'}} |\lambda - \zeta| \right), \tag{9}
\]

where the product over \( s' \) is set to be identically one if \( \nu(n) = s \). Recall that by assumption \( \gamma_0 = 1 \) and \( t = s \), so that \( |\lambda - \zeta| = |\lambda - 1| \) for all \( \zeta \in A_s \) and

\[
\prod_{\zeta \in A_s} |\lambda - \zeta| = |\lambda - 1|^{p^s}. \tag{10}
\]

Moreover, for every integer \( j > s \) and \( \zeta \in B_j \) we have \( |\lambda - 1| < R(s + 1) \leq |1 - \zeta| \) and hence

\[
|\lambda - \zeta| = |(\lambda - 1) - (1 - \zeta)| = |1 - \zeta| = R(j).
\]

Therefore, in view of Proposition 3.2

\[
\prod_{s' = s+1}^{\nu(n)} \left( \prod_{\zeta \in B_{s'}} |\lambda - \zeta| \right) = \prod_{s' = s+1}^{\nu(n)} R(s')^{p^{s'} - p^{s'-1}} = p^{-(\nu(n)-s)}.
\]

Combined with (8), (9), and (10) this implies the desired result. This completes the proof in the case \( \gamma_0 = 1 \).

As to the remaining case \( s \geq 1 \) and \( \gamma_0 \in B_s \). Recall that as noted after Proposition 3.2 the arrangement of roots of unity about \( \gamma_0 \) looks exactly like the arrangement of roots of unity near 1. It follows by Lemma 3.1 that the arguments used above in case II, carry over to the present case replacing 1 by \( \gamma_0 \), \( s \) by \( t \), \( A_s \) by \( \gamma_0 A_t \), and \( B_j \) by \( S_{R(j)}(\gamma_0) \cap \Gamma_r \). This completes the proof of the lemma.
4 General estimate of linearization disks

Throughout the rest of this paper let \( m, s, \gamma_0 \) and \( t \) be as in Lemma 3.2 and put

\[
\hat{r}(\lambda) := R(s + 1) \left( \frac{s}{mp} + 1 \right) \frac{1}{mp} |1 - \lambda|^s \frac{mp^t}{mp} |\gamma_0 - \lambda|^1. \tag{11}
\]

This section is devoted to prove Theorem 4.1 stated below, from which we obtain Theorem A.

**Theorem 4.1.** Let \( f \in G_\lambda \). Then, the linearization disk \( \Delta_f \supseteq D_{\hat{r}(\lambda)}(0) \). Moreover, if the conjugacy \( H_f \) converges on the closed disk \( D_{\hat{r}(\lambda)}(0) \), then \( \Delta_f \supseteq D_{\hat{r}(\lambda)}(0) \).

**Remark 4.1.** Let \( |\gamma_0 - \lambda| \) be fixed. Then, the estimate \( \hat{r}(\lambda) \to 1 \) as \( m \) or \( s \) goes to infinity. The latter case follows from the fact that both \( R(s + 1) \) and \( |1 - \lambda|^s \) approach one, as \( s \) goes to infinity. On the other hand, if \( s \) and \( m \) are fixed, then \( \hat{r}(\lambda) \to 0 \) as \( |\gamma_0 - \lambda| \to 0 \).

For any non-negative real number \( \alpha \), let \( \lfloor \alpha \rfloor \) denote the integer part of \( \alpha \). The main ingredient in the proof of Theorem 4.1 is Lemma 4.1 below, which also plays a fundamental role in the analysis of the quadratic case in the following section.

**Lemma 4.1.** Let \( \lambda \in S_1(0) \) be not a root of unity and let \( k \geq 2 \) be an integer. Then,

\[
\left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} \leq R(s + 1) \frac{s}{mp} |\gamma_0 - \lambda|^s \frac{mp^t}{mp} \frac{1}{mp} \hat{r}(\lambda)^{(k-1)}. \tag{12}
\]

Second, if \( (k-1)/mp^s \) is a non-negative integer power of \( p \), then

\[
\left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} = p^{-\frac{k-1}{mp^s}} \hat{r}(\lambda)^{(k-1)}. \tag{13}
\]

Third, if \( m = 1 \) and \( p^s \nmid k - 1 \) (so that \( s \geq 1 \)), then

\[
\left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} \leq R(s + 1)|1 - \lambda|^s \frac{mp^t}{mp} \frac{1}{mp} \hat{r}(\lambda)^{(k-1)}. \tag{14}
\]

**Remark 4.2.** Concerning the proof of Theorem that will be given in the next section, the equality \( k-1 \) is essential in proving that for a quadratic map \( P_\lambda \), the conjugacy \( H_{P_\lambda} \) diverges on the sphere of radius \( \tau = |1 - \lambda|^{-1/p} \hat{r}(\lambda) \). The extra factor containing \( |1 - \lambda| \) in the estimate \( \hat{r}(\lambda) \) is essential in the proof that in the quadratic case \( H_{P_\lambda} \) is one-to-one on \( D_{\tau}(0) \).

**Remark 4.3.** Let \( k \geq 2 \). Then, in view of \( \hat{r}(\lambda) \), we have

\[
\left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} < \hat{r}(\lambda)^{(k-1)}. \]

The proof of Lemma 4.1 is below, after the following two lemmas.
Lemma 4.2 (See Lemma 25.5 in [20]). Let \( n \geq 1 \) be an integer and let \( S_n \) be the sum of the coefficients in the \( p \)-adic expansion of \( n \). Then,

\[
\nu(n!) = \frac{n - S_n}{p - 1} \leq \frac{n - 1}{p - 1},
\]

with equality if \( n \) is a power of \( p \). Consequently, for all integers \( a \geq 1 \),

\[
\lim_{n \to \infty} \frac{\nu\left(\left\lfloor \frac{a}{n} \right\rfloor!ight)}{n} = \frac{1}{a(p - 1)}.
\]

Throughout the rest of this section we fix an integer \( k \geq 2 \) and for any integer \( z \geq 1 \) we put

\[
\delta(z) := \frac{k - 1}{z} - \left\lfloor \frac{k - 1}{z} \right\rfloor.
\]

So defined \( 0 \leq \delta < 1 - 1/z \) and \( \delta(z) = 0 \) if and only if \( z \) is a divisor of \( k - 1 \).

Lemma 4.3. Let \( s \geq 1 \) be an integer and put

\[
\Sigma := \sum_{j=0}^{s-1} \left( \left\lfloor \frac{k - 1}{mp^j} \right\rfloor - \left\lfloor \frac{k - 1}{mp^{j+1}} \right\rfloor \right) p^j.
\]

Then,

\[
\Sigma = \frac{k - 1}{m} \frac{p - 1}{p} + \delta(mp^s)p^{s-1} - \delta(m) - \sum_{s'=1}^{s-1} \delta(mp^{s'}) (p^{s'} - p^{s'-1}),
\]

where the sum over \( s' \) is set to be identically zero if \( s = 1 \).

Proof. Rearranging the terms of \( \Sigma \), we obtain

\[
\Sigma = \left\lfloor \frac{k - 1}{m} \right\rfloor - \left\lfloor \frac{k - 1}{mp^s} \right\rfloor p^{s-1} + \sum_{s'=1}^{s-1} \left\lfloor \frac{k - 1}{mp^{s'}} \right\rfloor (p^{s'} - p^{s'-1}).
\]

Consequently, using \( \left\lfloor (k - 1)/mp^{s'} \right\rfloor = (k - 1)/mp^{s'} - \delta(mp^{s'}) \), we have

\[
\Sigma = \left\lfloor \frac{k - 1}{m} \right\rfloor - \left\lfloor \frac{k - 1}{mp^s} \right\rfloor p^{s-1} + (s - 1) \frac{k - 1}{m} \frac{p - 1}{p} - \sum_{s'=1}^{s-1} \delta(mp^{s'}) (p^{s'} - p^{s'-1}),
\]

which implies (17) as required.\( \Box \)
Proof of Lemma 4.1. Suppose that $s = 0$. In this case $\gamma_0 = 1$ and $t = 0$ so $\tilde{r}(\lambda)$ reduces to

$$\tilde{r}_1(\lambda) := p^{-\frac{1}{m(p-1)}}|1 - \lambda^m|^{\frac{1}{m}}.$$ 

By Lemma 3.2 we then have

$$|1 - \lambda^n| = \begin{cases} 1, & \text{if } m \nmid n, \\ |n||1 - \lambda^m|, & \text{if } m | n. \end{cases} \tag{18}$$

Let $N := [(k - 1)/m]$. Then, from (18) we obtain

$$\prod_{n=1}^{k-1} |1 - \lambda^n| = |N!| |1 - \lambda^m|^N. \tag{19}$$

By Lemma 4.2 we have

$$\nu(N!) \leq \frac{N - 1}{p - 1} \leq \frac{k - 1}{m(p-1)} - \frac{1}{p - 1},$$

where each inequality becomes an equality if $(k - 1)/m$ is a non-negative integer power of $p$. Together with (19) and the fact that $N = (k - 1)/m - \delta(m)$ it follows that

$$\prod_{n=1}^{k-1} |1 - \lambda^n| \geq p^{-\frac{1}{m}}|1 - \lambda^m|^{-\delta(m)}\tilde{r}_1(\lambda)^{(k-1)}, \tag{20}$$

with equality if $(k - 1)/m$ is a non-negative integer power of $p$. Identifying $\tilde{r}(\lambda)$ with $\tilde{r}_1(\lambda)$ and using that $\delta(m) = 0$ if $m$ divides $k - 1$, this completes the case $s = 0$.

Now, suppose that $s \geq 1$. By Lemma 3.2 we have

$$|1 - \lambda^n| = \begin{cases} 1, & \text{if } m \nmid n, \\ |1 - \lambda^m|^{p^s(n)}, & \text{if } m | n \text{ but } mp^s \nmid n, \\ |n| p^s|\gamma_0 - \lambda^m| p^s, & \text{if } mp^s | n. \end{cases} \tag{21}$$

Throughout the rest of this proof we put

$$M := \left\lfloor \frac{k - 1}{mp^s} \right\rfloor, \quad \text{and } S_{k-1} := \{ i \in [1,k-1] : mp^s | i \}.$$ 

Note that so defined the cardinality $\#S_{k-1} = M$ and $S_{k-1} = \{ jmp^s : j \in [1,M] \}$. It follows from these observations that we have the following two identities

$$\prod_{n \in S_{k-1}} |n| = |M!| p^{-sM}, \quad \text{and} \tag{22}$$

$$\prod_{n \in S_{k-1}} p^s|1 - \lambda^m| p^s = p^{sM}|1 - \lambda^m|^{p^sM}. \tag{23}$$

Moreover, put

$$T_{k-1} := \{ i \in [1,k-1] : m \mid i \text{ but } mp^s \nmid i \}.$$
and let $\Sigma$ be defined as in Lemma 4.3. Then,
\[
\prod_{n \in T_{k-1}} |1 - \lambda^m|^{p^e(n)} = |1 - \lambda^m|^{\Sigma}.
\] (24)

Using (21) and multiplying the products (22), (23), and (24) we obtain
\[
\prod_{n=1}^{k-1} |1 - \lambda^n| = |M!|^{p^{-\nu(s-t)M}} |1 - \lambda^m|^{\Sigma} |\gamma_0 - \lambda^m|^{p^eM}.
\] (25)

This equation is the starting point of the estimates given below. By Lemma 4.2 we have
\[
\nu(M!) \leq \frac{M - 1}{p - 1} \leq \frac{k - 1}{mp^s(p - 1)} - \frac{1}{p - 1},
\] (26)

where each inequality become an equality if $(k - 1)/mp^s$ is a non-negative integer power of $p$. In view of Lemma 4.3 we have
\[
\Sigma \leq s \frac{k - 1}{m} \frac{p - 1}{p} + \delta(mp^s)p^{s-1},
\] (27)

with equality if $mp^{s-1}$ is a divisor of $k - 1$. Also recall that by definition
\[
\tilde{r}(\lambda) = R(s + 1) \frac{1}{p} p^{-\nu(s-t)} |1 - \lambda^m|^{\frac{1}{mp^s}} |\gamma_0 - \lambda^m|^{1/mp^{s-1}}.
\]
Applying (27) and (26) to the identity (25), and using $M = (k - 1)/mp^s - \delta(mp^s)$, we obtain the following estimate
\[
\prod_{n=1}^{k-1} |1 - \lambda^n| \geq p^{\nu(s-t)\delta(mp^s)} |1 - \lambda^m|^{\delta(mp^s)p^{s-1}} |\gamma_0 - \lambda^m|^{p^{s-1}p^e\delta(mp^s)} \tilde{r}(\lambda)^{k-1},
\] (28)

with equality if $(k - 1)/mp^s$ is a non-negative integer power of $p$. As $\delta(mp^s) = 0$ if $mp^s$ is a divisor of $k - 1$, we obtain the equality (13) as required.

As to the case (12), first note that $p^{(s-t)\delta(mp^s)} \geq 1$. Second, note that for $s \geq 1$ we have
\[
|1 - \lambda^m|^{p^{s-1}} \geq p^{-1/(p-1)},
\]
and $\delta(mp^s) \leq 1 - 1/mp^s$. Together with (28) this implies
\[
\prod_{n=1}^{k-1} |1 - \lambda^n| \geq p^{-\nu(s-t)} |\gamma_0 - \lambda^m|^{-p^e\delta(mp^s)} \tilde{r}(\lambda)^{k-1}.
\] (29)

This completes the proof of the inequality (12).

Concerning the inequality (14), recall that $\gamma_0$ is minimizing $|\gamma_0 - \lambda^m|$. Hence, the case $s = 1$ is a direct consequence of (29) putting $m = 1$ and using $|\gamma_0 - \lambda|^{p^e} \leq |1 - \lambda|$. As to the case $s \geq 2$, suppose that $p^e \not\mid k - 1$ and put $\beta = \nu(k - 1)$. Then for all integers $j \in [\beta + 1, s]$, we have $\delta(p^e) \geq 1/p^e$. Hence, together with (17), for $m = 1$, $s \geq 2$ and $p^e \not\mid k - 1$, we obtain the improved estimate
\[
\Sigma \leq s(k-1)\frac{p - 1}{p} + \delta(mp^s)p^{s-1} - (\beta - s)\frac{p - 1}{p}.
\] (30)
Also note that $\delta(p) \leq (p-1)/p$. Accordingly, using the improved estimate \([20]\) of $\Sigma$, we obtain the following improvement of \([20]\),

$$\prod_{n=1}^{k-1} |1 - \lambda^n| \geq |1 - \lambda|^{-\delta(p)} R(s + 1) |\gamma_0 - \lambda|^{-p'\delta(p')} \tilde{r}(\lambda)^{k-1}.$$  

This completes the proof of \([14]\) and hence of the lemma.

\[ \square \]

**Remark 4.4.** If $\gamma_0 = 1$, then by \([20]\) and \([20]\), $\prod_{n=1}^{k-1} |1 - \lambda^n| \geq R(1) \tilde{r}(\lambda)^{-(k-1)}$.

**Proof of Theorem 4.1.** In view of Lemma 2.1 and Lemma 4.1 we have

\[ (\text{lim sup} |b_j|^{1/j})^{-1} \geq \tilde{r}(\lambda). \]

This implies that $H_f$ converges on the open disk of radius $\tilde{r}(\lambda)$. Let $j \geq 2$ be an integer. By Lemma 2.1 and Remark 4.3 we have

\[ |b_j|^{\tilde{r}(\lambda)} < \tilde{r}(\lambda). \]

It follows by Proposition 2.2 that $H_f : D_{\tilde{r}(\lambda)}(0) \to D_{\tilde{r}(\lambda)}(0)$ is bijective. Moreover, according to the first statement of Proposition 2.2 the strict inequality

\[ |b_j|^{\tilde{r}(\lambda)} < \tilde{r}(\lambda), \]

implies that, if $H_f$ converges on the closed disk $\overline{D_{\tilde{r}(\lambda)}}(0)$, then $H_f : \overline{D_{\tilde{r}(\lambda)}}(0) \to \overline{D_{\tilde{r}(\lambda)}}(0)$ is bijective. Recall that as stated in Proposition 2.1, $f : D_1(0) \to D_1(0)$ is a bijective isometry. Moreover, $\tilde{r}(\lambda) < 1$. Consequently, the linearization disk $\Delta_f$ includes the disk $D_{\tilde{r}(\lambda)}(0)$ or $D_{\tilde{r}(\lambda)}(0)$, depending on whether or not, the conjugacy function $H_f$ converges on the closed disk $\overline{D_{\tilde{r}(\lambda)}}(0)$. This completes the proof of the theorem.

\[ \square \]
5 The size of quadratic linearization disks

The purpose of this section is to determine the exact radius of linearization disks for quadratic polynomials and certain power series perturbations of such maps. Let $p$ be an odd prime and throughout this section let

$$P_{\lambda}(x) := \lambda x + x^2 \in \mathbb{C}_p[x], \quad \text{where } \lambda \text{ is not a root of unity and } 1/p < |1-\lambda| < 1.$$  \hfill (31)

We will prove Theorem $\text{[B]}$ that states that the linearization disk $\Delta_{P_{\lambda}}$, is the open disk $D_{r(P_{\lambda})}(0)$ of radius $r(P_{\lambda}) = |1-\lambda|^{-1/p} \tilde{r}(\lambda)$. Apart from Lemma 4.1, the main ingredient in the proof of Theorem $\text{[B]}$ is the following lemma.

Lemma 5.1. The coefficients of the conjugacy $H_{P_{\lambda}}$ satisfy

$$|b_k| = \frac{|1-\lambda|^{\frac{k-1}{2}}}{\prod_{n=1}^{k-1} |1-\lambda^n|}, \quad \text{for } p \geq 3, \text{ and } k \geq 2.$$  \hfill (32)

A key point in obtaining the exact absolute value of the coefficients of the corresponding conjugacy function $H_{P_{\lambda}}$ is that, using the assumptions $1/p < |1-\lambda| < 1$ and $p \geq 3$, we prove that the coefficients of the conjugacy $\{b_k\}_{k \geq 1}$ form a strictly increasing sequence in $k$. This may not be the case if we drop the condition on $\lambda$ or $p$ respectively, since this may yield cancellation of large terms (this will be explained later in Remark and 5.1 and 5.2).

Let $N$ be the set of non-negative integers. For $k \geq 2$ and $l \in \lceil k/2 \rceil$, put $A_l^1(k) := \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 : \alpha_1 + \alpha_2 = l, \text{ and } \alpha_1 + 2\alpha_2 = k\}$.

Given a rational number $x$, we denote by $\lceil x \rceil$ the smallest integer greater than or equal to $x$. Note that $A_l^1(k)$ is non-empty if and only if $l \in \lceil k/2 \rceil$. Given $k \geq 2$ and $l \in \lceil k/2 \rceil$, the set $A_l^1(k)$ of ordered pairs $(\alpha_1, \alpha_2)$ contains precisely one element,

$$A_l^1(k) = \{(2l-k, k-l)\}.$$  

For $k \geq 2$ and $l \in \lceil k/2 \rceil$, we put

$$C_l(k) := \frac{l!}{(2l-k)!((k-1)!)^2} \lambda^{2l-k}.$$  

It follows that for $k \geq 2$, the coefficients of the conjugacy function $H_{P_{\lambda}}(x) = \sum_{k \geq 1} b_k x^k$ satisfy the recurrence relation

$$b_k = \frac{1}{\lambda(1-\lambda^{k-1})} \sum_{l=\lceil k/2 \rceil}^{k-1} b_l C_l(k),$$  \hfill (33)

starting with $b_1 := 1$.

Proof of Lemma 5.1 First note that $b_1 = 1$ and consequently

$$b_2 = \frac{1}{\lambda(1-\lambda)}.$$  

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Hence, the lemma is clearly true for $k = 2$ and $p \geq 3$. We will proceed by induction in $k$. Without loss of generality we may thus assume that the lemma holds for $p \geq 3$ and $k - 1 \geq 2$. By the induction hypothesis

$$|b_{k-1}| = \frac{|1 - \lambda|^{\frac{k-2}{p}}}{\prod_{n=1}^{k-2} |1 - \lambda^n|}, \quad p \geq 3, \ k \geq 3. \quad (34)$$

Note that since $m = 1$, by Lemma 3.2, we have

$$|1 - \lambda^n| = |1 - \lambda| \text{ if } p \nmid n.$$ 

For future reference, also note that by the induction hypothesis and the fact that for all integers $i \geq 1,$

$$|1 - \lambda|/|1 - \lambda^p| > 1,$$

we then have

$$|b_{k-1}| > |b_j|, \text{ for } j \in [1, k - 2]. \quad (35)$$

Recall that since $C_l(k)$ is always an integer times a power of $\lambda$ we have

$$|C_l(k)| \leq 1. \quad (36)$$

Moreover, $A_{k-1}'(k) = \{(k - 2, 1)\}$ and hence

$$C_{k-1}(k) = (k - 1)\lambda^{k-2}.$$ 

We identify two cases.

**Case I: $p \mid k - 1$.** In this case

$$|C_{k-1}(k)| = 1.$$ 

Hence, by the recurrence relation (33) and the facts (35) and (36), using ultrametricity we obtain

$$|b_k| = \frac{|b_{k-1}|}{|1 - \lambda^{k-1}|}. \quad (37)$$

Moreover, as $p \mid k - 1$ we have $|(k - 1)/p| = |(k - 2)/p|$. By (37) and the induction hypothesis (34) we obtain (32). This completes the induction step in this case.

**Case II: $p \nmid k - 1$.** First note that since $p \geq 3$, we may assume that $k \geq 4$. Moreover, with the induction step for case I completed, using (35) and (37), replacing $k$ by $k - 1$, we have

$$|b_{k-1}| = \frac{|b_{k-2}|}{|1 - \lambda^{k-2}|} > |b_{k-2}| > |b_j|, \text{ for } j \in [1, k - 3]. \quad (38)$$

Also note that by the assumption $p \mid k - 1$ we have

$$|C_{k-1}(k)| \leq |p|.$$ 

Recall that by the condition of the lemma $|p| < |1 - \lambda|$, and hence

$$|C_{k-1}(k)| < |1 - \lambda|. \quad (39)$$

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By the assumption $p | k - 1$ we have $|1 - \lambda^{k-2}| = |1 - \lambda|$, and hence in view of (38) we obtain

$$|C_{k-1}(k)b_{k-1}| < |b_{k-2}|.$$

On the other hand, for $k \geq 4$, $A_{k-2}'(k) = \{(k - 4, 2)\}$ and

$$C_{k-2}(k) = \frac{(k - 2)(k - 3)}{2} \lambda^{k-4}, \quad (39)$$

and hence

$$|C_{k-2}(k)| = 1,$$

since $p | k - 1$, $p \geq 3$ and $k \geq 4$.

It follows that

$$|C_{k-1}(k)b_{k-1}| < |b_{k-2}| = |C_{k-2}(k)b_{k-2}|.$$

Then, by the recurrence relation (33) and the facts (36) and (38), using ultrametricity we have

$$|b_k| = \frac{|b_{k-2}|}{|1 - \lambda^{k-1}|}.$$

Hence, using the induction hypothesis for $b_{k-2}$, we obtain

$$|b_k| = \frac{|1 - \lambda^{k-2}| |1 - \lambda|^{\frac{k-2}{p}}}{\prod_{n=1}^{k-1} |1 - \lambda^n|}.$$ 

Again, since $p | k - 1$ we have $|1 - \lambda^{k-2}| = |1 - \lambda|$ and $|(k - 1)/p| = \lfloor (k - 2)/p \rfloor + 1$. Accordingly,

$$|1 - \lambda^{k-2}| |1 - \lambda|^{\frac{k-2}{p}} = |1 - \lambda|^{\frac{k-1}{p}}.$$

This completes the induction step for case II and together with case I, it completes the proof of the lemma.

Remark 5.1. That the proof does not work for $p = 2$ stems from the fact that, in the case $k = 3$, we then have $C_{k-2}(k) = 0$ in (39). As a consequence, we obtain

$$b_3 = \frac{2}{1 - \lambda} \cdot \frac{1}{\lambda(1 - \lambda^2)}.$$

Moreover, for larger $k$, we encounter a similar problem; if $p = 2$ and $2 | k - 1$, we may have that $|C_{k-2}(k)| = \lfloor (k - 2)(k - 3)/2 \rfloor$ is very small for large $k$.

Remark 5.2. If the multiplier $\lambda$ satisfies $0 < |1 - \lambda| \leq 1/p$, then it is more complicated to calculate the absolute value of the coefficients of the conjugacy since in this case we may have cancellation of large terms. For example, consider the case $p = 3$ and

$$\lambda = 1 + 3, \quad f(x) = \lambda x + x^2.$$

Straightforward calculations of the coefficients of the conjugacy, starting with $b_1 = 1$, give

$$b_2 = 1/|\lambda(1 - \lambda)|, \quad b_3 = 2/|\lambda(1 - \lambda^2)(1 - \lambda)|,$$
and
\[ b_4 = \frac{1}{\lambda(1 - \lambda^3)} (b_2 + 3\lambda^2 b_3). \]

Note that
\[ |3| = |1 - \lambda| = |1 - \lambda^2| = \frac{1}{3}, \]
and hence
\[ |b_2| = |3\lambda^2 b_3|. \]

Thus, by ultrametricity, the sum of these two terms in \( b_4 \) may add up to something small. In fact, we have
\[ b_2 + 3\lambda^2 b_3 = \frac{1 + 5\lambda^2}{\lambda(1 - \lambda^2)(1 - \lambda)} = \frac{3^4}{4 \cdot (5 \cdot 3) \cdot 3} = \frac{3^2}{20}. \]

Consequently, since \( |1 - \lambda^3| = 3^{-2} \) we obtain
\[ |b_4| = 1 < |b_2| < |b_3|. \]

Hence, it seems the coefficients of the conjugacy may not grow as fast as in the case \( 1/p < |1 - \lambda| < 1 \).

This example also indicates that, for large \( k \), in this case it is a delicate problem to find the absolute value of the coefficients \( b_k \) for \( k \) large.

**Lemma 5.2.** Let \( p \geq 3 \) and \( \tau = r(P_\lambda) = |1 - \lambda|^{-1/p} \tilde{r}(\lambda) \). Then, for \( k \geq 2 \) the coefficients of the conjugacy function \( H_{P_\lambda} \) satisfy \( |b_k| \tau^k < \tau \). If \( s \geq 1 \) we have
\[ |b_k| \tau^k \leq R(s + 1) \tau, \quad \text{for } k \geq 2. \]

*In particular, if \( k - 1 = p^{s+\alpha} \) for some integer \( \alpha \geq 1 \), then \( |b_k| \tau^k = p^{-\frac{1}{p-1} \tau} \).

**Proof.** Let \( k \geq 2 \) be an integer. By definition
\[ \tau^{-(k-1)} = |1 - \lambda|^{\frac{k-1}{p-1} \tilde{r}(\lambda)^{-(k-1)}}. \]

Hence, to prove the inequality \((40)\) we need to prove that, for \( s \geq 1 \) and \( k \geq 2 \),
\[ |b_k| \leq R(s + 1)|1 - \lambda|^{\frac{k-1}{p-1} \tilde{r}(\lambda)^{-(k-1)}}. \]

Note that by Lemma 5.1, we have the identity \((32)\). We have two cases. First, suppose that \( p^s \mid k - 1 \). Then, by \((12)\) and \((32)\), using the assumption that \( s \geq 1 \), we obtain \((11)\). Second, suppose that \( p^s \nmid k - 1 \). Then, by \((14)\) and \((32)\), we obtain \((11)\). This completes the proof of the inequality \((40)\).

For \( s = 0 \) we need to prove that \( |b_k| \tau^k < \tau \) or equivalently
\[ |b_k| < |1 - \lambda|^{\frac{k-1}{p-1} \tilde{r}(\lambda)^{-(k-1)}}. \]

In view of \((12)\) and \((32)\), we have
\[ |b_k| \leq |1 - \lambda|^{\frac{k-1}{p-1} p^{-1/(p-1)} \tilde{r}(\lambda)^{-(k-1)}}. \]
The inequality \( |b_k| < |1 - \lambda| \frac{k-1}{p} p^{1/p - 1/(p-1)} \tilde{r}(\lambda)^{-1} \).

This completes the proof of (42).

As to the last statement of the lemma, given \( k \) such that \( k - 1 = p^{\alpha + \alpha} \), for some integer \( \alpha \geq 1 \), we have the equality (13) and hence by Lemma 5.1 we have

\[
|b_k| = p^{\frac{1}{\tau}} \tilde{r}(\lambda)^{-1} |1 - \lambda|^{\frac{k-1}{p}},
\]

for \( \alpha \geq 1 \).

This completes the proof of the lemma.

\[ \square \]

**Theorem 5.1.** Let \( p \geq 3 \) and let \( P_\lambda \) be a quadratic polynomial of the form (67). Then, the linearization disk \( \Delta P_\lambda = D_r(P_\lambda)(0) \), where \( r(P_\lambda) = |1 - \lambda|^{-1/p} \tilde{r}(\lambda) \).

**Proof.** First note that by Lemma 5.2 \( H_{P_\lambda} \) converges on the open disk \( D_\tau(0) \), because \( |b_k|^r \) is bounded, and hence \( |b_k|^r \to 0 \) for all \( r < \tau \). In fact, \( H_{P_\lambda} \) converges on the sphere \( S_\tau(0) \); let \( I \geq s + 1 \) be an integer, then, by Lemma 5.2 we have

\[
|b_{p^I}| = p^{\frac{1}{\tau}} \tilde{r}(\lambda)^{-1} |1 - \lambda|^{\frac{k-1}{p}},
\]

which does not approach zero as \( I \) goes to infinity. This proves that \( H_{P_\lambda} \) converges on the sphere \( S_\tau(0) \). Finally, by Lemma 5.2 we have

\[
|b_k|^r < \tau, \quad k \geq 2.
\]

Consequently, by Proposition 5.2 \( H_{P_\lambda} : D_\tau(0) \to D_\tau(0) \) is a bijection. Recall that, as stated in Proposition 5.1 \( P_\lambda : D_1(0) \to D_1(0) \) is a bijective isometry. Since \( \tau < 1 \), it then follows that the linearization disk of the quadratic polynomial \( P_\lambda \) is the disk \( \Delta P_\lambda = D_\tau(0) \). This completes the proof of the theorem.

\[ \square \]

**Corollary 5.1.** Let \( p \geq 3 \) and \( f \in \mathcal{G}_\lambda \), with \( 1/p < |1 - \lambda| < 1 \), be of the form

\[
f(x) = \lambda x + \sum_{i=2}^{\infty} a_i x^i \in \mathcal{O}_p[[x]], \quad \text{where } |a_2| = 1 \text{ and } |a_2^2 - a_3| = 1.
\]

Then, the linearization disk \( \Delta_f = D_{r(f)}(0) \), where \( r(f) = |1 - \lambda|^{-1/p} \tilde{r}(\lambda) \).

**Proof.** By the conditions imposed on \( a_2 \) and \( a_3 \), the same terms as in the proof of Lemma 5.1 will be strictly larger than all the others in (6). This follows since, for \( f \) we obtain

1) \( A_{k-1}(k) = \{(k-2, 1, 0, \ldots, 0)\} \), and hence we have \( C_{k-1}(k) = (k-1)\lambda^{k-2} a_2 \) (with \( |a_2| = 1 \)),
2) \( A_{k-2}(k) = \{(k-4, 2, 0, \ldots, 0), (k-3, 0, 1, 0, \ldots, 0)\} \), but since \( |a_2^2 - a_3| = 1 \) we then have

\[
|C_{k-2}(k)| = |\lambda^{k-4} a_2^2 (k-2)(k-3)/2 + \lambda^{k-3} a_3 (k-3)| = |\lambda a_2^2[(k-1) - 2]/2 + a_3| = 1,
\]

for \( p \geq 3 \) and \( p \mid k - 1 \). In the last equality we also use that \( 0 < |1 - \lambda| < 1 \). Hence, all steps in the proof of Lemma 5.1 hold true replacing \( P_\lambda \) by \( f \), and by the same arguments as in the proof of Theorem 5.1 we obtain the desired result concerning the size of the linearization disk.

\[ \square \]
6 Periodic points of quadratic-like maps

The purpose of this section is to prove Corollary C that states that for the quadratic polynomials and their power series perturbations studied in the previous section, the boundary of the linearization disk is free from periodic points. Let \( p \geq 3 \), and let \( f \) be of the form

\[
f(x) = \lambda x + \sum_{i=2}^{\infty} a_i x^i \in G_\lambda, \text{ with } |1-\lambda| < 1, |a_2| = 1, \text{ and } |a_2^2 - a_3| = 1. \tag{43}
\]

For convenience, put

\[
\Psi(\lambda) := |1-\lambda|^{-\frac{1}{p-1}} |\gamma_0 - \lambda|^{\frac{1}{p-1}}. \tag{44}
\]

We first show that, apart from \( x = 0 \), a map \( f \) of the form (43), has no periodic point in the open disk \( D_\rho(0) \), where the radius \( \rho = \min\{ |1-\lambda|, \Psi(\lambda) \} \). Second, we prove that if in addition \( |1-\lambda| > 1/p \), then the linearization radius \( r(f) < \rho \). Apart from Corollary B, the main ingredient in the proof is the following lemma in which we localize all periodic orbits of \( f \).

**Lemma 6.1.** Let \( p \geq 3 \) and let \( f \) be of the form (43). Then, the periodic points of \( f \) in the open unit disk, are all of minimal period \( p^n \) for some integer \( n \geq 0 \), and

1. for \( n = 0 \), the periodic fixed points are \( x = 0 \) and \( x_0 \neq 0 \) with \( |x_0| = |1-\lambda| \),
2. for \( n \in [1, s - 1] \) where \( s \geq 2 \), the periodic points of minimal period \( p^n \) are located on the sphere of radius \( |1-\lambda|^{\frac{s-t}{p-1}} \), about the origin,
3. for \( n = s \) where \( s \geq 1 \), the periodic points of minimal period \( p^n \) are located on the sphere of radius \( \Psi(\lambda) \), about the origin, and
4. for \( n \geq s + 1 \), the periodic points of minimal period \( p^n \) are located on the sphere of radius \( \rho^{-1/p^n} \), about the origin.

In particular, apart from zero, \( f \) has no periodic point in the open disk \( D_\rho(0) \), where the radius \( \rho = \min\{ |1-\lambda|, \Psi(\lambda) \} \). Furthermore, \( \rho = \Psi(\lambda) \) if and only if \( s - t \geq 2 \) (or \( s - t = 1 \) and \( |\gamma_0 - \lambda| \leq |1-\lambda|^2 \)).

To localize the periodic orbits we analyze the Newton polygons of iterates of \( f \), using the fact that all the periods of periodic points of \( f \) are powers of the prime \( p \), and that \( f \) is minimally ramified \([25, \text{Exemple 3.19}]\) as well as isometric. The following proposition is known, see e.g. \([21, \text{p. 333}]\) or \([25, \text{p. 190}]\).

**Proposition 6.1.** Let \( f \in \mathcal{O}_p[[x]] \) be such that \( f(0) = 0 \) and \( |f'(0) - 1| < 1 \). Then, the minimal period of each periodic point of \( f \) in the open unit disk is a power of \( p \).

For each integer \( n \geq 0 \), we will associate with \( f \in \mathcal{O}_p[[x]] \) such that \( f(0) = 0 \) and \( |f'(0) - 1| < 1 \), the ramification number

\[
i_n := \text{wideg}(f^{p^n} - \text{id}) - 1.
\]

So defined, \( i_n \) counts the number of fixed points of \( f^{p^n} \) in the open unit disk different from zero, counting multiplicity. By Sen’s Theorem \([27]\), if \( i_n < \infty \), then \( i_n \equiv i_{n-1} \mod p^n \). Note that by
the assumptions on \( f, i_0 \geq 1 \) and \( i_{n+1} \geq i_n + 1 \) and together with Sen’s Theorem this implies that \( i_n \geq 1 + p + \cdots + p^n \). The power series \( f \) is said to be \textit{minimally ramified} if

\[
i_n = 1 + p + \cdots + p^n, \quad \text{for all integers } n \geq 1.
\]

By a result of Keating \[12\], \( f \) is minimally ramified if and only if \( i_0 = 1 \) and \( i_1 = 1 + p \). In fact, Rivera-Letelier \[25\] \textit{Example 3.19} classified minimally ramified power series.

\textbf{Proposition 6.2} (See \[25\]). \textit{A power series } \( f(x) = \sum_{i \geq 1} a_i x^i \in \mathcal{O}_p[[x]] \) \textit{with } \( |f'(0) - 1| < 1 \), \textit{is minimally ramified if and only if } \( p \geq 3 \), \( |a_2| = 1 \) and \( |a_2^2 - a_3| = 1 \).

Let \( g(x) = \sum_{j \geq 1} a_j x^j \in \mathcal{O}_p[[x]] \) be a power series of finite Weierstrass degree of the form \( g(x) = \sum_{j \geq 1} a_j x^j \). We denote by \( \mathcal{N}(g) \) the principal part of the Newton polygon of \( g \), \textit{i.e.} the convex hull in \( \mathbb{R}^2 \) of

\[
\mathcal{D}(g) = \{(j, \nu(a_j)) : j \in \{1, \ldots, \text{wideg}(g)\}\}.
\]

Recall that if a line segment of the Newton polygon \( \mathcal{N}(g) \) has slope \( \kappa \) and the projection of the segment to the \( j \)-axis has length \( l \), then \( g \) has exactly \( l \) roots of absolute value \( p^\kappa \), counting multiplicity. For all integers \( n \geq 1 \), we associate with \( \lambda \) the number

\[
\kappa_n := \frac{-\nu(1 - \lambda^p^n) - \nu(1 - \lambda^{p^{n-1}})}{p^n}.
\]

Note that \( \kappa_n \) is well defined since by assumption, \( \lambda \) is not a root of unity.

\textbf{Lemma 6.2.} Let \( p \geq 3 \) and let \( f \) be of the form \((\nu)\). Then, for every integer \( n \geq 1 \), the Newton polygon \( \mathcal{N}(f^p^n - \text{id}) \) is obtained from that of \( \mathcal{N}(f^p - \text{id}) \) by adding only one single line segment of length \( p^n \) and slope \( \kappa_n \). In particular, all periodic points of \( f \) of minimal period \( p^n \) are located on the sphere of radius \( p^\kappa \) about the origin.

\textbf{Proof.} By Proposition \(6.2\) \( f \) is minimally ramified. Hence, for all integers \( n \geq 1 \), the Newton polygon \( \mathcal{N}(f^p^n - \text{id}) \) is obtained from that of \( \mathcal{N}(f^p - \text{id}) \) by adding line segments of total length \( \text{wideg}(f^p - \text{id}) - \text{wideg}(f^p^{n-1} - \text{id}) = p^n \). That the average weighted slope of these additional segments is \( \kappa_n \), then follows from the fact that for all integers \( n \geq 0 \), \( f^p(x) = \lambda^p x + (x^2) \). Also recall that by Proposition \(6.1\) for all periodic points of \( f \in G_\lambda \), with \( 0 < |1 - \lambda| < 1 \), the minimal period is a power of \( p \). Consequently, since \( f \) is minimally ramified, we add only one single cycle going from the \( p^n \)-th iterate of \( f \) to the \( p^n \)-th iterate of \( f \). Moreover, as stated in Proposition \(2.1\) \( f \) is isometric on the open unit disk. Hence, all the periodic points of such a cycle, of length \( p^n \), must be located on the same sphere about the origin. If follows that this sphere has to be of radius \( p^\kappa \). This also proves that in fact we add only one single line segment. \( \square \)

\textbf{Lemma 6.3.} Let \( p \geq 3 \) and let \( f \) be of the form \((\nu)\). Then,

1. for \( n \geq s + 1 \), we have \( \kappa_n = -1/p^n \),
2. for \( n = s \) where \( s \geq 1 \), \( \kappa_n = -(s - t)p^{-s} - p^{-(s-t)}\nu(\gamma_0 - \lambda) + \frac{1}{p}\nu(1 - \lambda) \), and
3. for \( n \in [1, s - 1] \) where \( s \geq 2 \), we have \( \kappa_n = -\frac{s-1}{p}\nu(1 - \lambda) \).

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Proof. By assumption, $0 < |1 - \lambda| < 1$. Hence, by Lemma 3.2 for all integers $k \geq s$,

$$|1 - \lambda^{p^k}| = p^k|p^k||\gamma_0 - \lambda|^{p^k}.$$ 

Recall that by definition, for all non-zero elements $x, y \in \mathbb{C}_p$, we have $\nu(x) = -\log_p|x|$ and $\nu(xy) = \nu(x) + \nu(y)$. Consequently, for every integer $n \geq s + 1$, we obtain

$$\nu(1 - \lambda^{p^n}) - \nu(1 - \lambda^{p^{n-1}}) = \nu(p^n) - \nu(p^{n-1}) = 1.$$ 

This proves the first statement of the lemma. As to the second statement note that for $k = s$,

$$|1 - \lambda^{p^k}| = p^{-(s-t)}|\gamma_0 - \lambda|^{p^k}.$$ 

Also note that if $k \in [0, s - 1]$, then by Lemma 3.2 we have

$$|1 - \lambda^{p^k}| = |1 - \lambda|^{p^k}.$$ 

Hence, for $n = s$ we obtain

$$\nu(1 - \lambda^{p^n}) - \nu(1 - \lambda^{p^{n-1}}) = (s - t) + p^t\nu(\gamma_0 - \lambda) - p^{s-1}\nu(1 - \lambda).$$ 

Finally, concerning the third statement, provided that $s \geq 2$, for $n \in [1, s - 1]$ we obtain

$$\nu(\lambda^{p^n} - 1) - \nu(1 - \lambda^{p^{n-1}}) = (p^n - p^{n-1})\nu(1 - \lambda).$$ 

This completes the proof of the lemma.

Proof of Lemma 6.7 The fact that the minimal periods are all prime powers is a consequence of Proposition 6.4. The absolute value of the fixed point $x_0 \neq 0$ is $|1 - \lambda|$, since by assumption we have $i_0 = 1$. By the assumptions on $f$, Lemma 6.2 and Lemma 6.3 apply, and we obtain the second, third and fourth statement of the lemma. As to the statements concerning $\rho$, first note that by the definition of $s$ we have

$$|1 - \lambda| < R(s + 1) = p^{-1/p^{s-1}},$$

and hence

$$|1 - \lambda| < |1 - \lambda|^{\frac{p-1}{p}} < p^{-1/p^{s-1}}.$$ 

In addition, by definition

$$p^{-(s-t)/r'} = R(s)^{\frac{s-t}{p-1}} \leq |1 - \lambda|^{\frac{s-t}{p-1}},$$

so that

$$\Psi(\lambda) \leq |1 - \lambda|^{\frac{s-t}{p-1}} |\gamma_0 - \lambda|^{\frac{1}{p^{s-t}}},$$

where the inequalities become an equality if and only if $\gamma_0 = 1$ (so that $t = s$). Accordingly, apart from zero, $f$ has no periodic point in $D_{\rho}(0)$ where $\rho = \min\{|1 - \lambda|, \Psi(\lambda)|$.

Concerning the last statement of the lemma, for the equality $\rho = \Psi(\lambda)$ to hold it is necessary that $\gamma_0 \neq 1$. In other words we must have $s - t > 0$, so that $s \geq 1$ and $|1 - \lambda| = R(s)$. In this case we have $p^{-1/p'} = |1 - \lambda|^{(p-1)/p}$ which implies

$$\Psi(\lambda) = |1 - \lambda|^{\frac{s-t}{p-1}} |\gamma_0 - \lambda|^{\frac{1}{p^{s-t}}}.$$
As $|\gamma_0 - \lambda| < |1 - \lambda|$ we certainly have that $\Psi(\lambda) < |1 - \lambda|$ for $s - t \geq 2$. If $s - t = 1$ we have $\Psi(\lambda) \leq |1 - \lambda|$ if and only if $|\gamma_0 - \lambda| \leq |1 - \lambda|^2$. The latter is a solution since $p \geq 3$ and for $s - t = 1$ we have $R(s - 1) \leq |\gamma_0 - \lambda| < R(s) = |1 - \lambda|$. This completes the proof of the lemma.

**Proof of Corollary** Let $p \geq 3$, $f$ be of the form (13), and suppose $1/p < |1 - \lambda| < 1$. By Corollary [3] the radius of the linearization disk $\Delta_f$ is given by $r(f) = |1 - \lambda|^{-1/p}\gamma(\lambda)$. By the assumptions on $\lambda$, $m = 1$, and

$$r(f) = |1 - \lambda|^{-1/p}R(s + 1)p^{s-1}|1 - \lambda|^{s-1}|\gamma_0 - \lambda|^{1/p-s-1}.$$ Suppose that $s \geq 1$. Then, $R(s + 1) = R(s)^{1/p}$ and

$$r(f) = R(s)^{\frac{1}{p}}|1 - \lambda|^{\frac{s-1}{p}}|\gamma(\lambda)|.$$ By definition $R(s) \leq |1 - \lambda|$. Consequently, for $s \geq 1$ we obtain

$$r(f) \leq |1 - \lambda|\Psi(\lambda),$$

with equality if and only if $s = 1$ and $|1 - \lambda| = R(1)$. By Lemma [6.1] apart from zero, $f$ has no periodic point in the open disk $D_{\rho}(0)$, where the radius $\rho = \min\{|1 - \lambda|, \Psi(\lambda)|$. Together with the observation (16), this completes the proof of the corollary for the case $s \geq 1$.

Finally, if $s = 0$ we have $1/p < |1 - \lambda| < R(1)$ and by Lemma [3.1] $\gamma_0 = 1$ and $s = t$. Accordingly, $r(f) = |1 - \lambda|^{-1/p}R(1)|1 - \lambda|$. Note that $p^{1/p}R(1) = R(2)$, and hence by the assumption $|1 - \lambda| > 1/p$ we obtain $r(f) < R(2)|1 - \lambda|$. As $\rho = |1 - \lambda|$ for $s = 0$, this completes the proof of the corollary.

### 6.1 Example of linearization disk and periodic points of a quadratic map

Let $p = 3$ and put

$$\tilde{P}(x) := \lambda x + x^2 \in \mathbb{C}_3, \quad \text{with } \lambda := 1 + 3^{1/4}.$$ Apart from zero, $\tilde{P}$ has a fixed point of absolute value $|1 - \lambda| = 3^{-1/4}$. Note that $R(1) = 3^{-1/2}$, $R(2) = 3^{-1/6}$, $R(1) < |1 - \lambda| < R(2)$, and hence $s = 1$ and, in view of Lemma [3.2] $\gamma_0 = 1$. By (13) we then have $\Psi(\lambda) = 3^{-\frac{1}{4}}$, and hence by (15), the radius of the corresponding linearization disk $r(\tilde{P}) = 3^{-1/2}$. By the example of Keating [12, p. 321] (and more generally Rivera-Letelier [25, p. 191]), $\tilde{P}$ is minimally ramified and Lemma [6.1] applies. The principal part of the corresponding Newton polygon for the 9th iterate, $N(\tilde{P}^9 - \text{id})$, has three segments shown in figure 2. The distribution of the corresponding periodic points, i.e. roots of $N(\tilde{P}^9 - \text{id})$, outside the linearization disk $\Delta_{\tilde{P}}$ is illustrated in figure 3.

**Remark 6.1.** In the complex field case [6, 22], the boundary of the linearization disk, $\partial\Delta_f$, is contained in the closure of the post-critical set, the union of all forward images $f^k(c)$, where $k \geq 1$ is an integer and where $c$ ranges over all critical points of $f$. With $\tilde{P}$ as above, $c = -\lambda/2 \in S_1(0)$, $\tilde{P}(c) = -\lambda^2/2 \in S_1(0)$, and $\tilde{P}^2(c) = \lambda^3 - 4/16 \in S_{|\lambda - 1|}(0)$. As $\tilde{P}$ is isometric on $S_{|\lambda - 1|}(0)$, the forward iterates will stay on this sphere for $k \geq 2$, and therefore, the intersection between the post-critical set and the boundary of the linearization disk is empty in this case.
Figure 2: The Newton polygon $\mathcal{N}(\tilde{P}^3 - \text{id})$. $\tilde{P}$ has a fixed point of absolute value $|1 - \lambda| = 3^{-\frac{1}{4}}$, three periodic points of minimal period 3 of absolute value $\Psi(\lambda) = 3^{-\frac{1}{6}}$, and nine periodic points of minimal period 9 of absolute value $3^{-\frac{1}{9}}$.

Figure 3: To the left, the linearization disk $\Delta_{\tilde{P}}(0)$ of radius $r(\tilde{P}) = 3^{-\frac{1}{2}}$ and periodic points; $\rho_0 = 3^{-\frac{1}{4}}$ fixed; $\rho_1 = \Psi(\lambda) = 3^{-\frac{1}{6}}$ minimal period 3; $\rho_n = 3^{-\frac{1}{3}}$ (n ≥ 2) minimal period 3”. To the right, $Q(x) = (1 + x)^{p+1} - 1 \in \mathbb{C}_p[x]$; periodic points of minimal period $p^n$, n ≥ 0, distributed on spheres of radius $\sigma_n = p^{-1/(p^{n+1})}$. The linearization of $Q$ is broken by fixed points of absolute value $r(Q) = \sigma_0$. In this case the conjugacy $H_Q(x) = \log_p(1 + x)$. See [2] for more details.
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